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EXPLORING THE TREE OF NUMERICAL SEMIGRAPHS

JEAN FROMENTIN

Abstract. In this paper we describe an algorithm visiting all the numerical semigroups up to a given genus using a new representation of numerical semigroups.

1. Introduction

A numerical semigroup \( S \) is a subset of \( \mathbb{N} \) containing 0, close under addition and of finite complement in \( \mathbb{N} \). For example the set

\[
SE = \{0, 3, 6, 7, 9, 10\} \cup [12, +\infty[ \tag{1}
\]

is a numerical semigroup. The genus of a numerical semigroup \( S \), denoted by \( g(S) \), is the cardinality of \( \mathbb{N} \setminus S \), i.e. \( g(S) = \text{card}(\mathbb{N} \setminus S) \). For example the genus of \( SE \) defined in (1) is 6, the cardinality of \( \{1, 2, 4, 5, 8, 11\} \)

For a given positive integer \( g \), the number of numerical semigroups is finite and is denoted by \( n_g \). In J.A. Sloane’s on-line encyclopedia of integer [1] we find the values of \( n_g \) for \( g \leq 52 \). These values have been obtained by M. Bras-Amorós (view [2] for more details for \( g \leq 50 \)). On his home page [3], M. Delgado gives the value of \( n_{55} \) without specifying the values of \( n_{53} \) and \( n_{54} \). M. Bras Amorós used a depth first search exploration of the tree of numerical semigroups up to a given genus. This tree was introduced by J.C. Rosales and al. in [4] and it is the subject of the Section 2.

Starting with all the numerical semigroups of genus 49 she obtained the number of numerical semigroups of genus 50 in 18 days on a pentium D running at 3GHz. In the package NumericalSgs [5] of GAP [6], M. Delgado together with P.A. Garcia-Sanchez and J. Morais used the same method of exploration.

The paper is divided as follows. In section 2 we describe the tree of numerical semigroups and give bounds for some parameters attached to a numerical semigroup. In Section 3 we describe a new representation of numerical semigroups that is well suited to the construction of the tree. In Section 4 we describe an algorithm based on the representation given in Section 3 and give its complexity. Section 5 is more technical, and is devoted to the optimisation of the algorithm introduced in Section 4.

2. The tree of numerical semigroups

We first start by some notations.
Definition 2.1. Let $S$ be a numerical semigroup. We define

i) $m(S) = \min(S \setminus \{0\})$, the multiplicity of $S$;

ii) $g(S) = \text{card}(\mathbb{N} \setminus S)$, the genus of $S$;

iii) $c(S) = 1 + \max(\mathbb{N} \setminus S)$, the conductor of $S$ for $S$ different from $\mathbb{N}$. By convention the conductor of $\mathbb{N}$ is 0.

By definition a numerical semigroup is an infinite object. We need a finite description of such a semigroup. That is the role of generating sets.

Definition 2.2. A subset $X$ of a semigroup is a generating set of $S$ if every element of $S$ can be expressed as a sum of elements in $X$.

Notation 2.3. A numerical semigroup admitting $X = \{x_1 < x_2 < \ldots < x_n\}$ as generating set is denoted by $S = \langle x_1,...,x_\ell \rangle$.

We now introduce a specific generating set.

Definition 2.4. A non-zero element $x$ of a numerical semigroup $S$ is said to be irreducible if it cannot be expressed as a sum of two non-zeros elements of $S$. We denote by $\text{Irr}(S)$ the set of all irreducible elements of $S$.

Proposition 2.5. Let $S$ be a numerical semigroup. Then $\text{Irr}(S)$ is the minimal generating set of $S$.

Proof. Assume for a contradiction, that there exists an integer $x$ in $S$ than cannot be decomposed as a sum of irreducible elements. We may assume that $x$ is minimal with this property. As $x$ cannot be irreducible, there exist $y$ and $z$ in $S \setminus \{0\}$ satisfying $x = y + z$. Since we have $y < x$ and $z < x$, the integers $y$ and $z$ can be expressed as a sum of irreducible elements of $S$ and so $x = y + z$ is a sum of irreducible elements, in contradiction to hypothesis.

As irreducible elements of $S$ cannot be decomposed as the sum of two non-zeros integers in $S$, they must occur in each generating set of $S$. \qed

We recall that Apéry elements of a numerical semigroup $S$ associated to $m(S)$ are the integers $x$ in $S$ such that $x - m(S)$ is no longer in $S$. We denote by $\text{App}(S)$ the set of these elements. It is well known that the cardinality of $\text{App}(S)$ is exactly $m(S)$ (see [7] for example). Note that the set $\text{Irr}(S)$ is in included in $\text{App}(S)$. In particular, the set $\text{Irr}(S)$ is finite and its cardinality is at most $m(S)$.

If we reconsider the numerical semigroup of (1), we obtain

$$S_E = \{0,3,6,7,9,10\} \cup [12, +\infty[ = \{3,7\}$$

(2)

Let $S$ be a numerical semigroup. The set $T = S \cup \{c(S) - 1\}$ is also a numerical semigroup and its genus is $g(S) - 1$. As $[c(S) - 1, +\infty[ $ is included in $T$ we have $c(T) \leq c(S) - 1$. Therefore every semigroup $S$ of genus $g$ can be obtained from a semigroup $T$ of genus $g - 1$ by removing an element of $T$ greater than or equal to $c(T)$.

Proposition 2.6. Let $S$ be a numerical semigroup and $x$ an element of $S$. The set $S^x = S \setminus \{x\}$ is a numerical semigroup if and only if $x$ is irreducible in $S$. 

**Proof.** If \( x \) is not irreducible in \( S \), then there exist \( a \) and \( b \) in \( S \setminus \{0\} \) such that \( x = a + b \). Since \( a \neq 0 \) and \( b \neq 0 \) hold, the integers \( a \) and \( b \) belong to \( S \setminus \{x\} \). Since \( x = a + b \) and \( x \notin S^\circ \), it follows that \( S^\circ \) is not stable under addition.

Conversely, assume that \( x \) is irreducible in \( S \). As \( 0 \) is never irreducible, the set \( S^\circ \) contains \( 0 \). Let \( a \) and \( b \) be two integers belonging to \( S^\circ \). The set \( S \) is stable under addition, hence \( a + b \) lies in \( S \). As \( S \) is equal to \( S^\circ \cup \{x\} \), the integer \( a + b \) also lies in \( S^\circ \) except if it is equal to \( x \). The latter is impossible since \( a \) and \( b \) are different from \( x \) and \( x \) is irreducible. \( \square \)

Proposition 2.6 implies that every semigroup \( T \) of genus \( g \) can be obtained from a semigroup \( S \) by removing a generator \( x \) of \( S \) that is greater than \( c(S) \). Hence relations \( T = S^\circ = S \setminus \{x\} \) hold.

We construct the tree of numerical semigroups, denoted by \( T \) as follows. The root of the tree is the unique semigroup of genus 0, i.e., (1), that is equal to \( \mathbb{N} \). If \( S \) is a semigroup in the tree, the sons of \( S \) are exactly the semigroup \( S^\circ \) where \( x \) belongs to \( \text{Irr}(S) \cap [c(s), +\infty] \). By convention, when depicting the tree, the numerical semigroup \( S^\circ \) is in the left of \( S^\circ \) if \( x \) is smaller than \( y \).

The above remarks imply that a semigroup \( S \) has depth \( g \) in \( \mathcal{T} \) if and only if its genus is \( g \), see Figure 1. We denote by \( T^{\leq g} \) the subtree of \( T \) restricted to all semigroup of genus \( \leq g \).

As the reader can check, the main difficulty to characterize the son of a semigroup is to determine its irreducible elements. In [5], the semigroup are given by their Apéry set \( \text{App}(S) \) and then the main difficulty is to describe \( \text{App}(S_c) \) from \( \text{App}(S) \). This approach is elegant but not sufficiently basic for our optimisations.

We conclude this section with some basic results on numerical semigroups of a given genus. Let \( S \) be a numerical semigroup. We first prove:

\[
x \in \text{Irr}(S) \quad \text{implies} \quad x \leq c(S) + m(S).
\]  

(3)

If \( y \) is a positive integer with \( y \geq c(S) + m(S) \) then \( y \) lies in \( S \) (as \( y \geq c(S) \) holds). Moreover, we always have \( y - m(S) \geq c(s) \) and so \( y \) is not irreducible.

As \( \mathbb{N} \setminus S \) contains \( \{1, \ldots, m(S) - 1\} \) we have

\[
m(S) \leq g(S) + 1.
\]  

(4)

Let \( x \) be an element of \( S \cap [0, c(S) - 1] \). The integer \( y = c(S) - 1 - x \) lies in \( [0, c(S) - 1] \). Moreover \( y \) is not in \( S \), for otherwise we write \( c(S) - 1 = y + x \) where \( x, y \) are elements of \( S \) implying \( c(S) - 1 \in S \). In contradiction with the definition of conductor. Thus we define an involution

\[
\psi : [0, c(S) - 1] \to [0, c(S) - 1]
\]

\[
x \mapsto c(S) - 1 - x
\]

that send \( S \cap [0, c(S) - 1] \) into \( [0, c(S) - 1] \setminus S \). The cardinality of \( [0, c(S)] \setminus S \) is exactly \( g(S) \). Let us denote by \( k \) the cardinality of \( S \cap [0, c(S) - 1] \). Since \( \psi \) is injective we must have \( k \leq g(S) \). From the relation

\[
[0, c(S)] = S \cap [0, c(S)] \cup [0, c(S)] \setminus S
\]
Figure 1. The first five layers of the tree $T$ of numerical semigroups, corresponding to $T_{\leq 5}$. A generator of a semigroup is in gray if it is not in $[c(S), +\infty]$. An edge between a semigroup $S$ and its son $S'$ is labelled by $x$ is $S'$ is obtained from $S$ by removing $x$, that is $S' = S \setminus x$.

we obtain $c(S) = k + g(S)$ and so

$$c(S) \leq 2g(S).$$

(5)

3. Decomposition number

The aim of this section is to describe a new representation of numerical semigroups, which is well suited to an efficient exploration of the tree $T$ of numerical semigroups.

Definition 3.1. Let $S$ be a numerical semigroup. For every $x$ of $\mathbb{N}$ we set

$$D_S(x) = \{ y \in S \mid x - y \in S \text{ and } 2y \leq x \}$$

and $d_S(x) = \text{card} \, D_S(x)$. We called $d_S(x)$ the $S$-decomposition number of $x$. The application $d_S : \mathbb{N} \to \mathbb{N}$ is the $S$-decomposition numbers function.

Assume that $y$ is an element of $D_S(x)$. By very definition of $D_S(x)$, the integer $z = x - y$ also belongs to $S$. Then $x$ can be decomposed as $x = y + z$ with $y$ and $z$ in $S$. Moreover the condition $2y \leq x$ implies $y \leq z$. In other words if we define $D'_S(x)$ to be the set of all $(y, z) \in S \times S$ with $x = y + z$ and $y \leq z$ then $D_S(x)$ is the image of $D'_S(x)$ under the projection on the
first coordinate. Hence \( D_S(x) \) describes how \( x \) can be decomposed as sums of two elements of \( S \). This is why \( d_S(x) \) is called the \( S \)-decomposition number of \( x \).

**Example 3.2.** Reconsider the semigroup \( S_E \) given at (1). The integer 14 admits three decompositions as sums of two elements of \( S \), namely 14 = 0+14, 14 = 3+11 and 14 = 7+7. Thus the set \( D_{S_E}(14) \) is equal to \( \{0, 3, 7\} \) and \( d_{S_E} \) equals 3.

**Lemma 3.3.** For every numerical semigroup \( S \) and every integer \( x \in \mathbb{N} \), we have \( d_S(x) \leq 1 + \left\lfloor \frac{x}{2} \right\rfloor \) and the equality holds for \( S = \mathbb{N} \).

**Proof.** As the set \( D_S(x) \) is included in \( \{0, \ldots, \left\lfloor \frac{x}{2} \right\rfloor \} \), the relation \( d_S(x) \leq 1 + \left\lfloor \frac{x}{2} \right\rfloor \) holds. For \( S = \mathbb{N} \) we have the equality for \( D_S(x) \) and so for \( d_S(x) \). \( \square \)

**Proposition 3.4.** Let \( S \) be a numerical semigroup and \( x \in \mathbb{N} \setminus \{0\} \). We have:

i) \( x \) lies in \( S \) if and only if \( d_S(x) > 0 \).

ii) \( x \) is in \( \text{Irr}(S) \) if and only if \( d_S(x) = 1 \).

**Proof.** We start with i). If \( x \) is an element of \( S \) then \( x \) equals 0 + \( x \). The relation \( 2 \times 0 \leq x \) and \( 0 \in S \) imply that \( D_S(x) \) contains 0, and so \( d_S(x) > 0 \) holds. Conversely, the relation \( d_S(x) > 0 \) implies that \( D_S(x) \) is non-empty. Let \( y \) be an element of \( D_S(x) \). As \( y \) and \( x - y \) belong to \( S \), by definition, the integer \( x = (x - y) + y \) is in \( S \) (since \( S \) is stable by addition).

Let us show ii). Assume \( x \) is irreducible in \( S \). There cannot exist \( y \) and \( z \) in \( (S \setminus \{0\})^2 \) such that \( x = y + z \). The only possible decomposition of \( x \) as a sum of two elements of \( S \) is \( x = 0 + x \). Hence, the set \( D_S(x) \) is equal to \( \{0\} \) and we have \( d_S(x) = 1 \). Conversely, let \( x \) such that \( d_S(x) = 1 \). By i) the integer \( x \) must be in \( S \). As \( x = 0 + x \) is always a decomposition of \( x \) as a sum of two elements in \( S \), we obtain \( D_S(x) = \{0\} \). If there exist \( y \) and \( z \) in \( S \) such that \( y \leq z \) and \( x = y + z \) hold then \( y \) lies in \( D_S(x) \). This implies \( y = 0 \) and \( z = x \). Hence \( x \) is irreducible in \( S \). \( \square \)

We note that 0 is never irreducible despite the fact \( d_S(0) \) is 1 for all numerical semigroup \( S \).

We now explain how to compute the \( S \)-decomposition numbers function of a numerical semigroup from these of its father.

**Proposition 3.5.** Let \( S \) be a numerical semigroup and \( x \) be an irreducible element of \( S \). Then for all \( y \in \mathbb{N} \setminus \{0\} \) we have

\[
\begin{align*}
   d_{S^x}(y) &= \begin{cases}
   d_S(y) - 1 & \text{if } y \geq x \text{ and } d_S(y - x) > 0, \\
   d_S(y) & \text{otherwise}.
   \end{cases}
\end{align*}
\]

**Proof.** Let \( y \) be in \( \mathbb{N} \setminus \{0\} \). We have

\[
D_{S^x}(y) = \{ z \in S^x \mid y - z \in S^x \text{ and } 2z \leq y \}
\]
and $D_{S^x}(y)$ is a subset of $D_S(y)$. We have $D_S(y) \setminus D_{S^x}(y) = E \cup F$ where

$$E = \begin{cases} \{x\} & \text{if } y - x \in S^x \text{ and } 2x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$F = \{ z \in S \mid y - z = x \text{ and } 2z \leq y \}$$

Since the relation $y - x \in S^x$ can be rewritten as the conjunction of $y - x \in S$ and $y \neq 2x$, we obtain

$$E = \begin{cases} \{x\} & \text{if } y - x \in S \text{ and } 2x < y, \\ \emptyset & \text{otherwise.} \end{cases}$$

By Proposition 3.4, the relation $y - x \in S$ is equivalent to $y \geq x$ and $d_S(y - x) > 0$, we have

$$E = \begin{cases} \{x\} & \text{if } y \geq x \text{ and } d_S(y - x) > 0 \text{ and } 2x < y, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$F = \{ z \in S \mid y - z = x \text{ and } 2z \leq y \}
= \{ z \in S \mid z = y - x \text{ and } y \leq 2x \}
= \begin{cases} \{y - x\} & \text{if } y - x \in S \text{ and } y \leq 2x, \\ \emptyset & \text{otherwise.} \end{cases}$$

As in the case of $E$, we get

$$F = \begin{cases} \{y - x\} & \text{if } y \geq x \text{ and } d_S(y - x) > 0 \text{ and } y \leq 2x, \\ \emptyset & \text{otherwise.} \end{cases}$$

In $E$ we have the constraint $2x < y$ while in $F$ we have $y \leq 2x$, hence only one of the sets $E$ or $F$ can be non-empty. This implies

$$D_S(y) \setminus D_{S^x}(y) = \begin{cases} \{x\} & \text{if } y \geq x \text{ and } d_S(y - x) > 0 \text{ and } y \leq 2x, \\ \{y - x\} & \text{if } y \geq x \text{ and } d_S(y - x) > 0 \text{ and } y > 2x, \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$d_{S^x}(y) = \begin{cases} d_S(y) - 1 & \text{if } y \geq x \text{ and } d_S(y - x) > 0, \\ d_S(y) & \text{otherwise.} \end{cases}$$
advantage in our approach is the small memory needs. The cost to pay is
that, if we want to explore the tree deeper, we must restart from the root.

Algorithm 1 Depth search first exploration of the tree of numerical semi-
groups

1: procedure Explore(G)
2: Stack stack → the empty stack
3: S ← {1}
4: while stack is not empty do
5: S ← stack.top()
6: stack.pop()
7: if g(S) < G then
8: for x from c(S) to c(S) + m(S) do
9: if x ∈ Irr(S) then
10: S.push(Sx)
11: end if
12: end for
13: end if
14: end while
15: end procedure

In Algorithm 1 we do not specify how to compute c(S), g(S) and m(S)
from S neither how to test if an integer is irreducible. It also miss the char-
acterisation of $S^x$ from S. These items depend heavily of the representation
of S. Our choice is to use the S-decomposition numbers function. The
first task is to use a finite set of such numbers to characterise the whole
semigroup.

Proposition 4.1. Let G be an integer and S be a numerical semigroup of
genus $g \leq G$. Then S is entirely described by $\delta_S = (d_S(0),...,d_S(3G))$.
More precisely we can obtain $c(S)$, $g(S)$, $m(S)$ and $\text{Irr}(S)$ from $\delta_S$.

Proof. By (5) we have $c(S) \leq g(S)$ and so the S-decomposition number of
$c(S)$ occurs in $\delta_S$. Since $c(S)$ is equal to $\text{max}(\mathbb{N} \setminus S)$, Proposition 4.1 implies
$q(S) = 1 + \text{max}\{i \in [0,...,3G], \ d_S(i) = 0\}$.

As all elements of $\mathbb{N} \setminus S$ are smaller than $c(S)$, their S-decomposition numbers
are in $\delta_S$ and we obtain

$g(S) = \text{card}\{i \in [0,...,3G], \ d_S(i) = 0\}$.

By (4) the relation $m(S) \leq g(S) + 1$ holds. This implies that the S-decomposition number of $m(S)$ appears in $\delta_S$:

$m(S) = \text{min}\{i \in [0,...,3G], \ d_S(i) > 0\}$.

Since, by (3), all irreducible elements are smaller than $c(S) + m(S) - 1$, which is itself smaller than $3G$, equations (4) and (5) give

$\text{Irr}(S) = \{i \in [0,...,3G], \ d_S(i) = 1\}$.
Even though it is quite simple, the computation of $c(S), m(S)$ and $g(S)$ from $\delta_S$ has a non negligible cost. We represent a numerical semigroup $S$ of genus $g \leq G$ by $(c(S), g(S), c(S), \delta_S)$. In an algorithmic context, if the variable $S$ stands for a numerical semigroup we use:

- $S.c, S.g$ and $S.m$ for the integers $c(S), g(S)$ and $m(S)$;
- $S.d[i]$ for the integer $d_S(i)$.

For example the following Algorithm initializes a representation of the semigroup $N$ ready for an exploration up to genus $G$.

**Algorithm 2** Return the root of $T$ for an exploration up to genus $G$

```
function Root(G)
R.c ← 1 ▷ R stands for N
R.g ← 0
R.m ← 1
for x from 1 to 3G do
    R.d[x] ← 1 + ⌊x/2⌋
end for
return R
end function
```

We can now describe an algorithm that returns the representation of the semigroup $S^x$ from that of the semigroup $S$ where $x$ is an irreducible element of $S$ greater than $c(S)$.

**Algorithm 3** Returns the son $S^x$ of $S$ with $x \in \text{Irr}(S) \cap [c(S), c(S)+m(S)]$

```
1: function Son(S,x,G)
2:    T.c ← x + 1 ▷ T stands for $S^x$
3:    T.g ← S.g + 1
4:    if x > S.m then
5:        T.M ← S.m
6:    else
7:        T.M ← S.m + 1
8:    end if
9:    T.d ← S.d
10:   for y from x to 3G do
11:       if S.d[y-x] > 0 then
12:           T.d[y] ← T.d[y] - 1
13:       end if
14:   end for
15:   return T
16: end function
```
Proposition 4.2. Running on $(S, x, G)$ with $g(S) \leq G$, $x \in \text{Irr}(S)$ and $x \leq c(S)$, Algorithm 3 returns the numerical semigroup $S_x$ in time $O(\log(G) \times G)$.

Proof. By construction $S_x$ is the semigroup $S \setminus \{x\}$. Thus the genus $S_x$ is $g(S) + 1$, see Line 1. Every integer of $I = [x + 1, +\infty[$ lies in $S$ since $x$ is greater than $c(S)$, so the interval $I$ is included in $S_x^c$. As $x$ does not belong to $S$, the conductor of $S_x^c$ is $x + 1$, see Line 2. For the multiplicity of $S_x^c$ we have two cases. First, if $x > m(S)$ holds then $m(S)$ is also in $S_x^c$ and so $m(S_x^c)$ is equal to $m(S)$. Assume now $x = m(S)$. The relation $x(S) \geq c(S)$ and the characterisation of $m(S)$ implies $x = m(S) = c(S)$. Thus $S_x^c$ contains $m(S) + 1$ which is $m(S_x^c)$. The initialisation of $m(S_x^c)$ is done by Lines 4 to 8. The correctness of the computation of $\delta_{S^c}$ (see Proposition 4.1) done from Line 9 to Line 15 is a direct consequence of Proposition 3.4.

Let us now prove the complexity statement. Since by (5) and (4) we have $x \leq 3G$ together with $m(S) \leq G + 1$, each line from 2 to 8 is done in time $O(\log(G))$. The for loop needs $O(G)$ steps and each step is done in time $O(\log(G))$. Summarizing, these results give that the algorithm runs in time $O(\log(G) \times G)$.

Algorithm 4 Returns an array containing the value of $n_g$ for $g \leq G$

1: function Count($G$)
2:   \textbf{n} $\leftarrow$ [0, ..., 0] $\triangleright$ $n[g]$ stands for $n_g$ and is initialised to 0
3:   \textbf{Stack stack} $\triangleright$ the empty stack
4:   \textbf{S} $\leftarrow$ Root($G$)
5:   \textbf{while} stack is not empty \textbf{do}
6:     \textbf{S} $\leftarrow$ stack.top()
7:     \textbf{stack.pop()}
8:     \textbf{n}[S.g] $\leftarrow$ \textbf{n}[S.g] + 1
9:     \textbf{if} S.g < G \textbf{then}
10:        \textbf{for} x from S.c to S.c + S.m \textbf{do}
11:           \textbf{if} S.d[x] = 1 \textbf{then}
12:              S.push(SON(S, x, G))
13:        \textbf{end if}
14:     \textbf{end if}
15:   \textbf{end while}
16:   \textbf{return n}
17: end function

Proposition 4.3. Running on $G \in \mathbb{N}$, Algorithm Count returns the array $[n_0, ..., n_G]$ in time

$$O\left(\log(G) \times G \times \sum_{g=0}^{G} n_g\right)$$
and its space complexity is $O(\log(G) \times G^3)$.

Proof. The correctness of the algorithm is a consequence of Proposition 4.2 and of the description of the tree $T$ of numerical semigroups.

For the time complexity, let us remark that Algorithm Son is called for every semigroup of the tree $T^{\leq G}$ (the restriction of $T$ to semigroup of genus $\leq G$). Since there are exactly $N = \sum_{g=0}^{G} n_g$ such semigroups, the time complexity of Son established in Proposition 4.2 guarantees that the running time of Count is in $O(\log(G) \times G \times N)$, as stated.

Let us now prove the space complexity statement. For this we need to describe the stack through the run of the algorithm. Since the stack is filled with a depth first search algorithm, it has two properties. The first one is that reading the stack from the bottom to the top, the genus of semigroup increases. The second one is that, for all $g \in [0, G]$, every semigroup of genus $g$ in the stack has the same father. As the number of sons of a semigroup $S$ is the number of $S$-irreducible elements in $[c(S), c(S) + m(S) - 1]$, a semigroup $S$ has at most $m(S)$ sons. By $(4)$, this implies that a semigroup of genus $g$ as at most $g + 1$ sons. Therefore the stack contains at most $g + 1$ semigroup of genus $g + 1$ for $g \leq G$. So the size of the stack is bounded by

$$S = \sum_{g=0}^{G} g = \frac{G(G+1)}{2}$$

A semigroup is represented by a quadruple $(c(S), g(S), m(S), \delta_S)$. By equations $(5)$ and $(4)$, we have $c \leq 2g(S)$ and $m \leq g(S) + 1$. As $g(S) \leq G$ holds, the integers $c$, $g$ and $m$ of the representation of $S$ require a memory space in $O(\log(G))$. The size of $\delta_S = (d_S(0), \ldots, d_S(3G))$ is exactly $3G + 1$. Each entry of $\delta_S$ is the $S$-decomposition number of an integer smaller than $3G$ and hence requires $\log(G)$ bytes of memory space. Therefore the space complexity of $\delta_S$ is in $O(\log(G) \times G)$, which implies that the space complexity of the Count algorithm is

$$O(\log(G) \times G \times S) = O(\log(G) \times G^3).$$

□

5. Technical optimizations and results

Assume for example that we want to construct the tree $T^{\leq 100}$ of all numerical semigroup of genus smaller than 100. In this case, the representation of numerical semigroup given in Section 3 uses decomposition numbers of integers smaller than 300. By Lemma 3.3, such a decomposition number is smaller than 151 and requires 1 byte of memory. Thus at each for step of Algorithm Son, the CPU actually works on 1 byte. However current CPUs work on 8 bytes. The first optimization uses this point.

To go further we must specify that the array of decomposition numbers in the representation of a semigroup corresponds to consecutive bytes in memory. In the for loop of Algorithm Son we may imagine two cursors:
the first one, denoted \textit{src} pointing to the memory byte of \texttt{S.d[0]} and the second one, denoted \textit{dst} pointing to the memory byte \texttt{T.d[y]}. Using these two cursors, Lines 10 to 14 of Algorithm SON can be rewritten as follows
\begin{verbatim}
src ← address(S.d[0])
dst ← address(T.d[x])
i ← 0
while i ≤ 3G − x do
    if content(src) > 0 then
        decrease content(dst) by 1
    end if
    increase src,dst,i by 1
end while
\end{verbatim}
In this version we can see that the cursors \textit{src} and \textit{dst} move at the same time and that the modification of the value pointed by \textit{dst} only needs to access the values pointed by \textit{src} and \textit{dst}. We can therefore work in multiple entries at the same time without collision. Current CPUs allow this thanks to the SIMD technologies as MMX, SSE, etc. The acronym SIMD stands for Single Operation Multiple Data.

The MMX technology permits to work on 8 bytes in parallel while the SSE works in 16 bytes. As the rest of the CPU works on 8 bytes, the SSE technology needs some constraint on memory access than cannot be fulfilled in our algorithm. This motivate our choice to use the MMX technology. More precisely, we use three commands: \texttt{pcmpeqb}, \texttt{pandn} and \texttt{psubb}. These commands work on two arrays of 8 bytes, called \texttt{s} and \texttt{d} here. In each case the array \texttt{d} is modified. We denote bytes by \{0, 1\}-words of length 8. After a call to \texttt{pcmpeqb(s,d)} the \texttt{i}th entry of \texttt{d} contains \texttt{11111111} if \texttt{s[i]} = \texttt{d[i]} holds and \texttt{00000000} otherwise. The command \texttt{pand(s,d)} store in \texttt{d[i]} the value of the byte-logic operation \texttt{s[i] and not d[i]}. When \texttt{pand(s,d)} is called the new value of \texttt{d[i]} is \texttt{s[i]} − \texttt{d[i]}. Glueing all the pieces together we obtain the following version of the \texttt{for} loop of Algorithm SON :
\begin{verbatim}
1: src ← address(S.d[0])
2: dst ← address(T.d[x])
3: i ← 0
4: while i ≤ 3G − x do
5:    t ← [00000000, ..., 00000000] ▷ 8 bytes equal to 00000000
6:    pcmpeqb(src,t)
7:    pandn([00000001,...,00000001],t)
8:    psub(dst,t)
9:    dst ← t ▷ copy the array [t[0],...,t[7]] to [d[0],...,d[7]]
10:   increase src,dst,i by 8
11: end while
\end{verbatim}
Let us explain how this works in more details. Let \( i \) be an integer in \( \{0, ..., 7\} \). After Line 5, the value of \( t[i] \) is 00000000. After Line 6, we have

\[
t[i] = \begin{cases} 
11111111 & \text{if src}[i] = 00000000 \text{ (corresponding to 0)} \\
00000000 & \text{otherwise}
\end{cases}
\]

Line 7 performs a logical and between 00000001 and not \( t[i] \). Hence the result byte is 00000001 if the last byte of \( t[i] \) is 0 and 00000000 otherwise:

\[
t[i] = \begin{cases} 
00000000 & \text{if src}[i] = 00000000 \text{ (corresponding to 0)} \\
00000001 & \text{otherwise}
\end{cases}
\]

Line 8 subtracts \( t[i] \) (which is equal to 0 or 1) from \( \text{dst}[i] \) according to Proposition 2.6. Finally we shift the cursor \( \text{src} \) and \( \text{dst} \) by eight cases.

Our second optimization is to use parallelism on exploration of the tree. Today, CPU of personal computer have several cores (2, 4 or more). The given version of our exploration algorithm uses a single core and so a fraction only of the power of a CPU.

Our method of parallelism is very simple: for \( G \in \mathbb{N} \), we cut the tree \( T \leq G \) in sub-trees \( T_1, ..., T_n \) and we launch our algorithm on these sub-trees. The advantage of this method is that there is no communication between the instances of our algorithm. The disadvantage is that the cutting controls the efficiency of the parallelism. Assume for example that we want to explore the tree \( T \leq 40 \). We first determine all numerical semigroups \( S_1, ..., S_{n_{20}} \) of genus 20. To explore \( T \) it remains to explore the tree \( T_i \) rooted in \( S_i \) for \( i = 1, ..., n_{20} \). This works but the time to explore \( T_i \) is similar to the time needed to explore the tree \( T \leq 40 \). And in this case, using many cores does not reduce the time to explore the tree.

Let us now explain in more detail how we cut the tree \( T \leq G \) in order to use parallelism. Semigroups of the form \( \langle g + 1, ..., 2g + 1 \rangle \), which are of genus \( g \), are called ordinary in [8]. Each ordinary semigroup has a unique son that is also ordinary. We define \( X \) to be the set of all the non-ordinary sons of an ordinary semigroup and \( X \leq G \) the restriction of \( X \) to semigroup of genus \( \leq G \). We then denote by \( T_i \) the tree rooted on \( S_i \) where \( X \leq G = \{ S_1, ..., S_n \} \). The time needed to explore \( T_i \) is very heterogeneous but there are many tree \( T_i \) with maximal time. This cutting is more efficient than the previous one.

Figure 5 summarizes the time complexity of various exploration algorithms.

The version \( \text{depth} - \text{expl} - \delta - \text{mmx} \) of our algorithm compute the value of \( n_g \) for \( g \leq 50 \) in 196 minutes on the i5-3570K CPU while the parallel version running on the 4 cores of the same CPU end the work in 50 minutes.

Using two i7 based computers and our parallel algorithm we computed in two days the values of \( n_g \) for \( g \leq 60 \), confirming the values given by M.Bras-Amorós and M.Delgado:
In [9], A. Zhai establishes that the limit of the quotient $\frac{n_g}{n_{g-1}}$, when $g$ goes to $+\infty$, is the golden ratio $\phi \approx 1.618$. As the reader can see, the convergence is very slow.

**References**

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[3] M. Delgado, “Homepage.”
[4] J. C. Rosales, “Fundamental gaps of numerical semigroups generated by two elements,” *Linear Algebra Appl.*, vol. 405, pp. 200–208, 2005.
Figure 2. Comparison of time executions of different exploration algorithms of the tree $\mathcal{T}^<g$ on an Intel i5-3570K CPU with 8GB of memory. The scale is logarithmic in time. The first version is based on a breadth-first search exploration and the peak at genus 35 is due to the consumption of all the memory and use of swap. The second version uses a depth first search exploration. The third is based on the second one with use of decomposition numbers, it corresponds to the algorithm given in Section 4. The last one is an optimisation of the third one with MMX.