ON A LOWER BOUND FOR THE TIME CONSTANT OF
FIRST-PASSAGE PERCOLATION

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Abstract

We consider the Bernoulli first-passage percolation on \(\mathbb{Z}^d\) \((d \geq 2)\). That is, the edge passage time is taken independently to be 1 with probability \(1 - p\) and 0 otherwise. Let \(\mu(p)\) be the time constant. We prove in this paper that

\[
\mu(p_1) - \mu(p_2) \geq \frac{\mu(p_2)}{1 - p_2} (p_2 - p_1)
\]

for all \(0 \leq p_1 < p_2 < 1\) by using Russo’s formula.

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1 Introduction and statement of the results.

We begin with the general first-passage percolation on \(\mathbb{Z}^d\). Let \(\{t(e) : e \in \mathbb{Z}^d\}\) be a sequence of i.i.d. positive random variables with common distribution \(F\), \(t(e)\) is the random passage time of edge \(e\) and \(F\) is the edge-passage distribution of the model. For any path \(\gamma = \{e_1, e_2, \ldots, e_n\}\), the passage time of \(\gamma\) is

\[
T(\gamma) := \sum_{k=1}^{n} t(e_k).
\]

For any vertices \(u, v \in \mathbb{Z}^d\) and vertex sets \(A, B \subset \mathbb{Z}^d\), let

\[
T(u, v) := \inf_{\gamma \ni u,v} T(\gamma); \quad T(A, B) := \inf_{u \in A, v \in B} T(u, v)
\]

be the passage time from \(u\) to \(v\) and the passage time from \(A\) to \(B\).

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Let 0 be the origin of \( \mathbb{Z}^d \), \( \hat{e}_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^d \) and \( H_n = \{ u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^d : u_1 = n \} \).

Define

\[
a_{0,n} := T(0, n\hat{e}_1), \quad b_{0,n} := T(0, H_n).
\]

To restrict \( a_{0,n}, b_{0,n} \) on cylinders, let

\[
\Gamma_{\text{cyl}}(0, n\hat{e}_1) = \{ \gamma : 0, n\hat{e}_1 \in \gamma \text{ and } \forall u \in \gamma, 0 \leq u_1 \leq n \}\]

\[
\Gamma_{\text{cyl}}(0, H_n) = \{ \gamma : 0 \in \gamma, \gamma \cap H_n \neq \emptyset, \text{ and } \forall u \in \gamma, 0 \leq u_1 \leq n \}\]

and define

\[
t_{0,n} := \inf_{\gamma \in \Gamma_{\text{cyl}}(0, n\hat{e}_1)} T(\gamma); \quad s_{0,n} := \inf_{\gamma \in \Gamma_{\text{cyl}}(0, H_n)} T(\gamma).
\]

The time constant \( \mu \) of the model is the common limit of \( \theta_{0,n}/n \) when \( n \to \infty \) for \( \theta = a, b, t \) or \( s \).

Here we will not introduce all the detailed situations for the above convergence under various moment conditions of \( F \), and only point out that, in most cases, for \( \theta = a, b, t \) or \( s \),

\[
\frac{\theta_{0,n}}{n} \to \mu = \mu(F) \text{ a.s. as } n \to \infty. \tag{1.1}
\]

For the details on the convergence to \( \mu \), one may refer to [4, 6, 7, 8].

It is straightforward that \( \theta_{0,n}, \theta = a, b, t \) or \( s \), depends on the states of infinitely many edges. The following is another limit representation of \( \mu \) given by Grimmett and Kesten [3], from which, \( \mu \) is represented as the limit of random variables which only depend on the states of finitely many edges.

For any fixed \( n \geq 1 \), let \( B_n = \{ u \in \mathbb{Z}^d : 0 \leq u_i \leq n, 1 \leq i \leq d \} \) be the box with side length \( n \). Let

\[
\phi_{0,n} = \inf \{ T(\gamma) : \gamma \text{ is a path in } B_n \text{ from } \{0\} \times [0, n]^{d-1} \text{ to } \{n\} \times [0, n]^{d-1} \}
\]

Grimmett and Kesten [3] proved that, if the time-passage distribution \( F \) satisfying:

\[
\int (1 - F(x))^d dx < \infty \text{ for } d = 2; \text{ or } \int x^2 dF(x) < \infty \text{ for } d \geq 3
\]

then

\[
\frac{\phi_{0,n}}{n} \to \mu \text{ a.s. and in } L^1, \text{ as } n \to \infty. \tag{1.2}
\]

The first problem for time constant \( \mu \) is: when will \( \mu > 0 \)? Kesten [5] solved this problem for all \( d \geq 2 \) as:

\[
\mu > 0 \iff F(0) < p_c(d), \tag{1.3}
\]

where \( F(0) = \mathbb{P}(t(e) = 0) \) and \( p_c(d) \) be the critical probability for the general bond percolation on \( \mathbb{Z}^d \).
Further study on $\mu$ is carried out to solve such a problem: How does $\mu = \mu(F)$ depend on the edge-passage distribution $F$? Berg and Kesten solved this problem in part. As our result is a further research in this direction, in the next paragraph, we introduce the results of Berg and Kesten in detail.

Let’s begin with some notations. For any given edge-passage distributions $F$, let $\text{supp}(F) = \{x \geq 0 : F(x) > 0\}$ be the support of $F$, let $\lambda(F) = \inf \text{supp}(F)$. We say $F$ is useful, if

$$\lambda(F) = 0 \text{ and } F(0) < p_c(d), \text{ or } \lambda(F) > 0 \text{ and } F(\lambda) < \tilde{p}_c(d),$$

where $\tilde{p}_c(d)$ is the critical probability for directed bond percolation on $\mathbb{Z}^d$. For two edge-passage distributions $F$ and $\tilde{F}$, we say $\tilde{F}$ is more variable than $F$, if

$$\int \varphi(x)d\tilde{F}(x) \leq \int \varphi(x)dF(x) \quad (1.4)$$

for all increasing convex function $\varphi$. Clearly, by the above definition, “$\tilde{F}$ is more variable than $F$” is a weaker condition than “$\tilde{F}$ is stochastically dominated by $F$”, note that the latter requires equation (1.4) hold for all increasing $\varphi$.

**Theorem 1.1** [Berg and Kesten]

(a) Let $F$ and $\tilde{F}$ be two edge-passage distribution functions, if $\tilde{F}$ is more variable than $F$, then

$$\mu(\tilde{F}) \leq \mu(F);$$

(b) if, in addition, $F$ is useful and $F \neq \tilde{F}$, then

$$\mu(\tilde{F}) < \mu(F).$$

Theorem 1.1 gives sufficient conditions for (strict) inequality between $\mu(\tilde{F})$ and $\mu(F)$, but for the difference $\mu(F) - \mu(\tilde{F})$, no information is provided. One may ask: what can we say for such a difference? In this paper, for the simplest case, i.e., under the following Bernoulli setting, we give a nontrivial lower bound for this difference.

From now on, we take $\{t(e) : e \in \mathbb{Z}^d\}$ to be the i.i.d. random variable sequence such that $t(e) = 1$ with probability $1 - p$ and $t(e) = 0$ with probability $p$, $p \in [0, 1]$. Write $\mathbb{P}_p$ as the percolation measure and $\mathbb{E}_p$ as its expectation. Write $\hat{\mu}(p)$ as the corresponding time constant. By (1.3) and Theorem 1.1, $\mu(p)$ decreases strictly in $p$ when $p \in [0, p_c(d)]$, i.e.,

$$\frac{\mu(p_1)}{\mu(p_2)} > 1 \quad (1.5)$$

for all $0 \leq p_1 < p_2 < p_c(d)$.

Now, we state our main result as follows.
**Theorem 1.2** For the above Bernoulli first-passage percolation model, let $\mu(p)$ be its time constant. We have that $\mu(p)/(1 - p)$ decreases in $p$ and then

$$\mu(p_1) - \mu(p_2) \geq \frac{\mu(p_2)}{1 - p_2}(p_2 - p_1)$$

for all $0 \leq p_1 < p_2 \leq 1$.

**Remark 1.1** By the monotonicity of $\mu(p)/(1 - p)$ and (1.3), when $0 \leq p_1 < p_2 < p_c(d)$, one has

$$\frac{\mu(p_1)}{\mu(p_2)} \geq 1 + \frac{p_2 - p_1}{1 - p_2}$$

(1.7)

This is a concretion of (1.5).

## 2 Proof of Theorem 1.2

To use the Russo’s formula, we first give the definition of pivotal edges according to Grimmett [2]. For any edge $e$ and configuration $\omega$, let $\omega_e$ be the configuration such that $\omega_e(f) = \omega(f)$ for all $f \neq e$ and $\omega_e(e) = 1 - \omega(e)$.

Recall that $B_n = [0,n]^d \cap \mathbb{Z}^d$. Suppose that $A$ be an event which only depends on edges of $B_n$. We say edge $e \in B_n$ is pivotal for pair $(A, \omega)$, if

$$I_A(\omega) \neq I_A(\omega_e),$$

where $I_A$ be the indicator function of $A$. Write $S_e(A)$ as the event that $e$ is a pivotal edge for $A$, i.e.

$$S_e(A) = \{\omega : e \text{ is pivotal for pair } (A, \omega)\}.$$  

(2.1)

By the above definition, $S_e(A)$ is independent of $t(e)$. Denote by $N(A)$ the number of pivotal edges of $A$, i.e.

$$N(A)(\omega) = |\{e \in B_n : \omega \in S_e(A)\}|.$$  

(2.2)

Event $A$ is called increasing if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega' \in A$, where $\omega \leq \omega'$ means $\omega(e) \leq \omega'(e)$ for all $e$. The Russo’s formula says that (in our setting), if $A$ is increasing, then

$$\frac{d\mathbb{P}_p(A)}{dp} = -\mathbb{E}_p(N(A)).$$

(2.3)

**Proof of Theorem 1.2** Firstly, by equation (1.2), we have

$$\mu(p) = \lim_{n \to \infty} \frac{E_p \phi_{0,n}}{n}$$

(2.4)
for all \( p \in [0, 1] \).

For any integer \( k \geq 1 \), let \( A_{n,k} = \{ \phi_{0,n} \geq k \} \). Clearly, \( A_{n,k} \) is increasing and only depends on edges in \( B_n \). Rewrite \( \mathbb{E}_p(\phi_{0,n}) \) as

\[
\mathbb{E}_p(\phi_{0,n}) = \sum_{k=1}^{\infty} \mathbb{P}_p(A_{n,k}).
\]

(2.5)

For any \( 0 \leq p_1 < p_2 \leq 1 \), by (2.4) and (2.5), we have

\[
\mu(p_1) - \mu(p_2) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \left( \mathbb{P}_p(A_{n,k}) - \mathbb{P}_{p_2}(A_{n,k}) \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} \frac{d\mathbb{P}_p(A_{n,k})}{dp}. \tag{2.6}
\]

Using the Russo’s formula and the fact that \( A_{n,k} \) is increasing, we have

\[
\frac{d\mathbb{P}_p(A_{n,k})}{dp} = -\mathbb{E}_p(N(A_{n,k})) = -\sum_{e \in B_n} \mathbb{P}_p(S_e(A_{n,k}))
\]

\[
= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(\{t(e) = 1\} \cap S_e(A_{n,k}))
\]

\[
= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(A_{n,k} \cap S_e(A_{n,k})) \tag{2.7}
\]

\[
= -\frac{1}{1-p} \sum_{e \in B_n} \mathbb{P}_p(S_e(A_{n,k} | A_{n,k}) \mathbb{P}_p(A_{n,k})
\]

\[
= -\frac{1}{1-p} \mathbb{E}_p(N(A_{n,k}) | A_{n,k}) \mathbb{P}_p(A_{n,k}).
\]

Note that the third equality comes from the independence of \( t(e) \) and \( S_e(A_{n,k}) \).

To finish the proof, we have to give appropriate lower bound for \( \mathbb{E}_p(N(A_{n,k}) | A_{n,k}) \). To this end, for any configuration \( \omega \in A_{n,k} \), we give lower bounds to \( N(A_{n,k})(\omega) \) in the following two cases respectively: 1) \( \phi_{0,n}(\omega) \geq k + 1 \); 2) \( \phi_{0,n}(\omega) = k \).

We first deal with the case of \( \phi_{0,n}(\omega) \geq k + 1 \). For any \( e \in B_n \), because \( \omega_e \) only differs from \( \omega \) in edge \( e \), the change from \( \omega \) to \( \omega_e \) at most decrease \( \phi_{0,n} \) by 1, this implies that \( \phi_{0,n}(\omega_e) \geq k \), and \( \omega_e \in A_{n,k} \). By the definition of pivotal edges, we know that \( e \) is not pivotal for \( (A_{n,k}, \omega) \). So

\[
N(A_{n,k})(\omega) = 0. \tag{2.8}
\]

Now, we consider the case of \( \phi_{0,n}(\omega) = k \). For any \( e \in B_n \), if \( e \) is pivotal for \( (A_{n,k}, \omega) \), we declare that \( \omega(e) = 1 \). Actually, if \( \omega(e) = 0 \), then the change from \( \omega \) to \( \omega_e \) will increase \( \phi_{0,n} \), so we have \( \phi_{0,n}(\omega_e) \geq \phi_{0,n}(\omega) = k \) and \( \omega_e \in A_{n,k} \), this leads to a contradiction.

Suppose \( \gamma \) be a path in \( B_n \) from \( \{0\} \times [0, n]^{d-1} \) to \( \{n\} \times [0, n]^{d-1} \) with \( T(\gamma)(\omega) = \phi_{0,n}(\omega) = k \). If \( e \in \gamma \) satisfying \( \omega(e) = 1 \), then \( T(\gamma)(\omega_e) = k - 1 \). This implies that \( \phi_{0,n}(\omega_e) \leq k - 1 \) and \( \omega_e \notin A_{n,k} \). Thus, by the definition of pivotal edges, \( e \) is pivotal for pair \( (A_{n,k}, \omega) \).
By the arguments in the last two paragraphs, we have
\[ N(A_{n,k})(\omega) \geq |\{e \in \gamma : \omega(e) = 1\}| = T(\gamma)(\omega) = k \] (2.9)
of all \( \omega \in A_{n,k} \).

Combining (2.8) and (2.9), we have
\[
\mathbb{E}_p(N(A_{n,k}) \mid A_{n,k})\mathbb{P}_p(A_{n,k}) = \sum_{\omega \in A_{n,k}} N(A_{n,k})(\omega) \frac{\mathbb{P}_p(\omega)}{\mathbb{P}_p(A_{n,k})} \mathbb{P}_p(A_{n,k}) \\
\geq \sum_{\{\omega : \phi_{0,n}(\omega) = k\}} k \cdot \mathbb{P}_p(\omega) \\
= k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\}).
\] (2.10)

Finally, by (2.6), (2.7) and (2.10), using the Fubini’s theorem and the Fatou’s lemma, we have
\[
\mu(p_1) - \mu(p_2) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} 1 - p \mathbb{E}_p(N(A_{n,k}) \mid A_{n,k})\mathbb{P}_p(A_{n,k})dp \\
\geq \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \int_{p_1}^{p_2} 1 - p k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\})dp \\
= \lim_{n \to \infty} \frac{1}{n} \int_{p_1}^{p_2} 1 - p \sum_{k=1}^{\infty} k \cdot \mathbb{P}_p(\{\omega : \phi_{0,n}(\omega) = k\})dp \\
= \lim_{n \to \infty} \frac{1}{n} \int_{p_1}^{p_2} 1 - p \mathbb{E}_p(\phi_{0,n})dp \\
\geq \int_{p_1}^{p_2} \frac{1}{1 - p} \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}_p(\phi_{0,n})dp \\
= \int_{p_1}^{p_2} \frac{\mu(p)}{1 - p} dp
\] (2.11)
for all \( 0 \leq p_1 < p_2 < 1 \). Clearly, the inequality (2.11) is equivalent to the following differential inequality
\[
\frac{d[\mu(p)/(1 - p)]}{dp} \leq 0, \quad 0 \leq p < 1.
\] (2.12)

This gives that
\[
\int_{p_1}^{p_2} \frac{\mu(p)}{1 - p} dp \geq \frac{\mu(p_2)}{1 - p_2}(p_2 - p_1)
\]
for all \( 0 \leq p_1 < p_2 < 1 \) and we finish the proof of Theorem 1.2.

\[\square\]

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