ON THE CONVERGENCE OF DENJOY-WOLFF POINTS

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Abstract. If $\varphi$ is an analytic function from the unit disk $D$ to itself, and $\varphi$ is not a conformal automorphism, we denote by $\lambda_\varphi$ its Denjoy-Wolff point, that is, the limit of the iterates $\varphi(\cdots \varphi(0)\cdots))$. A result of Heins shows that, given a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of such analytic functions that convergence pointwise to $\varphi$, it follows that $\lim_{n\to\infty} \lambda_{\varphi_n} = \lambda_\varphi$. This allows us to improve results about the continuous extensions of the subordination functions that arise in the study of free convolutions. We also offer an alternate proof of the result of Heins.

1. Introduction

Denote by $\mathbb{D}$ the unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{T}$ be its boundary. Suppose that $\varphi: \mathbb{D} \to \overline{\mathbb{D}}$ is an analytic map. Denjoy and Wolff discovered independently [8, 14] that, if $\varphi$ does not have a fixed point in $\mathbb{D}$, then it must have an attracting fixed point in $\mathbb{T}$. More precisely, in this case, there exists a unique point $\lambda_\varphi \in \mathbb{T}$ with the property that

$$\lim_{r\uparrow 1} \varphi(r \lambda_\varphi) = \lambda_\varphi,$$

and a certain nonnegative quantity, denoted $\varphi'(\lambda_\varphi)$ and called the Julia-Carathéodory derivative of $\varphi$ at $\lambda_\varphi$, exists and satisfies

$$\varphi'(\lambda_\varphi) \leq 1.$$

It follows that for an analytic map $\varphi: \mathbb{D} \to \overline{\mathbb{D}}$ one, and only one, of the following alternatives occurs.

1. $\varphi$ is the identity map $\varphi(z) = z$.
2. $\varphi(z) = \gamma$ for every $z \in \mathbb{D}$, where $\gamma \in \mathbb{T}$. In this case, we set $\lambda_\varphi = \gamma$ and this point can be thought of as a generalized fixed point for $\varphi$.
3. $\varphi$ is not the identity map but it has a fixed point $\lambda_\varphi \in \mathbb{D}$ such that $|\varphi'(\lambda_\varphi)| = 1$. In this case $\varphi$ is a (hyperbolic) rotation about the point $\lambda_\varphi$ and it has no other fixed points in $\mathbb{D}$.
4. $\varphi$ has a fixed point $\lambda_\varphi \in \mathbb{D}$ and $|\varphi'(\lambda_\varphi)| < 1$. In this case, $\lambda_\varphi$ is the limit of the iterates $\varphi(\cdots (\varphi(z))\cdots))$ for every $z \in \mathbb{D}$.
5. $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi$ has no fixed point in $\mathbb{D}$. In this case, the Denjoy-Wolff result provides the point $\lambda_\varphi \in \mathbb{T}$ which is, again, the limit of the iterates $\varphi(\cdots (\varphi(z))\cdots))$ for every $z \in \mathbb{D}$. The function $\varphi$ may have other fixed points on $\mathbb{T}$, but the Julia-Carathéodory derivative at such points will be greater than one.

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Thus, the point $\lambda_{\varphi}$ is defined for every analytic selfmap of $D$ other than the identity map. For our purposes, Denjoy-Wolff points on $T$ are best understood in the context of the conformally equivalent domain

$$H = \{x + iy \in \mathbb{C} : y > 0\}.$$

This is mapped to $D$ via the function $z \mapsto (z - i)/(z + i), z \in H$. This identification extends to a homeomorphism of the closure

$$\overline{H} = \overline{H} \cup \overline{R} \cup \{\infty\}$$

in the Riemann sphere to

$$\overline{D} = D \cup T$$

that sends $\infty$ to 1. Thus, any point in $R \cup \{\infty\}$ can be the Denjoy-Wolff point of some analytic map $\psi : H \to H$. According to Nevanlinna [10], an arbitrary such map $\psi$ can be written under the form

$$\psi(z) = \beta + \int_{R \cup \{\infty\}} \frac{1 + tz}{t - z} \, d\sigma(t), \quad z \in H,$$

where $\beta \in \mathbb{R}$ and $\sigma$ is a finite Borel measure on the one-point compactification of $R$. The fraction above must, of course, be understood to be $z$ when $t = \infty$, so the Nevanlinna formula can also be written as

$$\psi(z) = \alpha z + \beta + \int_{R} \frac{1 + tz}{t - z} \, d\sigma(t), \quad z \in H,$$

where $\alpha = \sigma(\{\infty\})$. Clearly, $\psi$ is the identity map precisely when $\alpha = 1, \beta = 0$, and $\sigma(\mathbb{R}) = 0$. We have $\lambda_{\psi} = \infty$ precisely when $\alpha \geq 1$ and $\psi$ is not the identity map. (The number $1/\alpha$ corresponds to the Julia-Carathéodory derivative of $\psi$ at $\infty$.)

The following result is proved in [9].

**Theorem 1.1.** Suppose that $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of analytic self-maps of $D$ that converges pointwise to $\varphi : D \to D$, and suppose that $\varphi$ is not the identity map. Then

$$\lim_{n \to \infty} \lambda_{\varphi_n} = \lambda_{\varphi}.$$

It is, of course, obvious that $\varphi_n$ is not the identity map for large $n$, and then $\lambda_{\varphi_n}$ is indeed defined eventually. Note also that the functions $\varphi_n$ converge to $\varphi$ locally uniformly by the Vitali-Montel theorem (see, for instance [11]). An alternate proof of Theorem 1.1 is provided in Section 2. In Section 2, we take advantage of the fact that one can use another space of parameters in place of $\mathbb{N}$ to derive applications to the subordination functions that occur in free probability. Many of these applications were known, though the original proofs are more involved and rely on the functional equations that subordination functions satisfy (see [12, 2, 3, 4]).

We wish to thank Marco Abate for bringing the work of Heins [9], as well as the existence of subsequent developments, to our attention. Theorem 1.1 was not previously known to us and we included it as an original result. Our proof is perhaps more elementary than the original and we decided to retain it in this note.

2. **Proof of the convergence theorem**

Throughout this section, we assume that $\varphi_n$ and $\varphi$ are given maps $D \to D$ that satisfy the hypotheses of Theorem 1.1. Since $\overline{D}$ is a compact metric space, the sequence $(\lambda_{\varphi_n})_{n \in \mathbb{N}}$ converges if and only if it has a unique subsequential limit. Therefore, for the proof of Theorem 1.1 we may assume that the limit $\mu = \lim_{n \to \infty} \lambda_{\varphi_n}$.
exists, and we must prove that \( \mu = \lambda_{\varphi} \). We distinguish two cases, according to whether \( \mu \in \mathbb{D} \) or \( \mu \in \mathbb{T} \).

**Lemma 2.1.** If \( \mu \in \mathbb{D} \) then \( \mu = \lambda_{\varphi} \).

*Proof.* The functions \( \varphi_n \) converge to \( \varphi \) uniformly in some neighborhood of \( \mu \). Since \( \lambda_{\varphi_n} \) belongs eventually to that neighborhood, this implies that

\[
\mu = \lim_{n \to \infty} \lambda_{\varphi_n} = \lim_{n \to \infty} \varphi_n(\lambda_{\varphi_n}) = \varphi(\mu).
\]

Thus \( \mu \) is the (necessarily) unique fixed point of \( \varphi \), that is, \( \lambda_{\varphi} = \mu \). \( \square \)

**Lemma 2.2.** If \( \mu \in \mathbb{T} \) then \( \mu = \lambda_{\varphi} \).

*Proof.* We write

\[
\lambda_{\varphi_n} = r_n \mu_n, \quad n \in \mathbb{N},
\]

where \( \mu_n \in \mathbb{T} \) converge to \( \mu \). Consider the new maps \( \tilde{\varphi}_n, \tilde{\varphi} : \mathbb{D} \to \mathbb{D} \) defined by

\[
\tilde{\varphi}_n(\lambda) = \varphi_n(\mu_n \lambda)/\mu_n, \quad \tilde{\varphi}(\lambda) = \varphi(\mu \lambda)/\mu, \quad n \in \mathbb{N}, \lambda \in \mathbb{D}.
\]

We have then

\[
\lambda_{\tilde{\varphi}} = \lambda_{\varphi}/\mu, \quad \lambda_{\tilde{\varphi}_n} = r_n, \quad n \in \mathbb{N},
\]

and the sequence \( (\tilde{\varphi}_n)_{n \in \mathbb{N}} \) converges pointwise to \( \tilde{\varphi} \). To conclude the proof, it suffices to show that \( \lambda_{\tilde{\varphi}} = 1 \). At this point, we use the conformal map

\[
u(z) = \frac{z - i}{z + i}, \quad z \in \mathbb{H},
\]

to reformulate the problem. We define analytic maps \( \psi_n, \psi : \mathbb{H} \to \mathbb{H} \) by

\[
\psi_n(z) = u^{-1}(\tilde{\varphi}_n(u(z))), \quad \psi(z) = u^{-1}(\tilde{\varphi}(u(z))) \quad z \in \mathbb{H}.
\]

The sequence \( (\psi_n)_{n \in \mathbb{N}} \) converges pointwise to \( \psi \). Since \( \{u(iy) : y > 0\} = (-1, 1) \), it follows that \( \lambda_{\psi_n} \) is either \( \infty \) or of the form \( iy_n \), for some \( y_n > 0 \), and moreover \( \lim_{n \to \infty} y_n = \infty \). We conclude the proof by showing that \( \lambda_{\psi} = \infty \). To do this, we write the Nevanlinna representations

\[
\psi_n(z) = \beta_n + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + tz}{t - z} d\sigma_n(t) = \alpha_n z + \beta_n + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\sigma_n(t),
\]

\[
\psi(z) = \beta + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + tz}{t - z} d\sigma(t) = \alpha z + \beta + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\sigma(t), \quad z \in \mathbb{H}.
\]

The convergence of \( \psi_n \) to \( \psi \) amounts to \( \lim_{n \to \infty} \beta_n = \beta \) and to the convergence of \( \sigma_n \) to \( \sigma \) in the weak*-topology (obtained by viewing these measures as linear functionals on the Banach space \( C(\mathbb{R} \cup \{\infty\}) \)). Clearly, if \( \lambda_{\psi_n} = \infty \) for infinitely many values of \( n \), then we have \( \alpha_n = \sigma_n(\{\infty\}) \geq 1 \) for infinitely many values of \( n \), and weak convergence implies \( \alpha \geq 1 \) as well. We can therefore restrict ourselves to the case in which \( \lambda_{\psi_n} = iy_n \) with \( y_n \in (0, +\infty) \). Thus, \( iy_n \) is an actual fixed point of \( \psi_n \), that is

\[
iy_n = \beta_n + \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + tiy_n}{t - iy_n} d\sigma_n(t).
\]

Taking imaginary parts in this equation (and dividing by \( y_n \)) yields

\[
1 = \int_{\mathbb{R} \cup \{\infty\}} \frac{1 + t^2}{t^2 + y_n^2} d\sigma_n(t).
\]
Suppose that \( y_n \geq 1 \). For fixed \( T > 0 \), we have
\[
1 = \int_{[-T,T]} \frac{1 + t^2}{t^2 + y_n^2} \, d\sigma_n(t) + \int_{[-T,T] \cup \{\infty\}} \frac{1 + t^2}{t^2 + y_n^2} \, d\sigma_n(t) \\
\leq \int_{[-T,T]} \frac{1 + t^2}{t^2 + y_n^2} \, d\sigma_n(t) + \sigma_n([-T,T]) + \sigma_n(\{\infty\}) \\
\leq \int_{[-T,T]} \frac{1 + t^2}{t^2 + y_n^2} \, d\sigma_n(t) + \infty \\
= \int_{[-T,T]} \frac{1 + t^2}{t^2 + y_n^2} \, d\sigma_n(t) + \infty.
\]
Since \( y_n \to \infty \), the last integral above tends to zero as \( n \to \infty \), and thus
\[
\liminf_{n \to \infty} \sigma_n([-T,T]) \geq 1
\]
for every \( T > 0 \). We conclude (using, for instance, the portmanteau theorem) that \( \alpha = \sigma(\{\infty\}) \geq 1 \), thus reaching the desired conclusion that \( \lambda_0 = \infty \).

3. Applications

All the results in this section arise from free probability considerations. These connections are well-known and we refer to \cite{12, 13, 7, 4} for their closer examination. The approach using Denjoy-Wolff points was described earlier \cite{5}. The new features here are continuity at the boundary and the two-variable results.

**Corollary 3.1.** Suppose that \( \varphi : \mathbb{D} \to \mathbb{D} \) is an analytic function. There exists a unique continuous function \( \omega : \mathbb{D} \to \mathbb{D} \) such that \( \omega \) is analytic in \( \mathbb{D} \), \( \omega(0) = 0 \), and
\[
\omega(z) = z \varphi(\omega(z)), \quad z \in \mathbb{D}.
\]

**Proof.** For every \( z \in \mathbb{D} \) we define a map \( \varphi_z : \mathbb{D} \to \mathbb{D} \) by
\[
\varphi_z(\lambda) = z \varphi(\lambda).
\]
Clearly, the map \( z \mapsto \varphi_z(\lambda) \) is continuous for fixed \( \lambda \), and therefore Theorem \cite{11} shows that the function \( \omega(z) = \lambda \varphi_z \) is continuous, unless \( \varphi_z \) is the identity function for some \( z \). This last situation can only occur when \( z \in T \) and \( \varphi(\lambda) = \lambda/z \), \( z \in \mathbb{D} \). The corollary is proved by direct computation in this special case. In all other cases, analyticity in \( \mathbb{D} \) follows because \( \omega(z) \) is the limit of the iterates \( \varphi_z(\varphi_z(\cdots(\varphi_z(0)) \cdots)) \). Uniqueness follows because \( \varphi_z \) is not the identity map for any \( z \in \mathbb{D} \).

The preceding result, proved differently, is instrumental in the arguments of \cite{4}, showing that the free (multiplicative) convolution powers of a probability measure on the unit circle have certain regularity properties. More precisely, such measures are absolutely continuous relative to arclength measure, with the exception of a finite number of atoms, and their densities are locally analytic where positive.

**Corollary 3.2.** Suppose that \( \varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{D} \) are analytic functions. Then there exist unique continuous functions \( \omega_1, \omega_2 : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \) that are analytic on \( \mathbb{D} \times \mathbb{D} \), \( \omega_1(0,0) = \omega_2(0,0) = 0 \), and
\begin{align*}
\omega_1(z_1, z_2) &= z_2 \varphi_2(\omega_2(z_1, z_2)), \\
\omega_2(z_1, z_2) &= z_1 \varphi_1(\omega_1(z_1, z_2)), \quad z_1, z_2 \in \mathbb{D}.
\end{align*}

**Proof.** Define \( \omega_2(z_1, z_2) = \lambda \varphi_{z_1, z_2} \) and use \cite{6, 11} to define \( \omega_1 \), where \( \varphi_{z_1, z_2}(\lambda) = z_1 \varphi_1(z_2 \varphi_2(\lambda)) \) for \( \lambda \in \mathbb{D} \) and \( z_1, z_2 \in \mathbb{D} \). One must worry again about the possibility that \( \varphi_{z_1, z_2} \) is the identity map for some \( z_1, z_2 \in \mathbb{D} \). This can happen for only one pair \( (z_1, z_2) \in \mathbb{T}^2 \) and for functions \( \varphi_1, \varphi_2 \) that are automorphisms of \( \mathbb{D} \). This case is again treated by direct computation; see for instance \cite{1}.
For applications to free probability, a special case of the preceding result is useful. For the proof, we just observe that the functions \( z \mapsto \omega_j(z, z) \) satisfy the requirements. Uniqueness of these functions is an easy consequence of the uniqueness of Denjoy-Wolff points.

**Corollary 3.3.** Suppose that \( \varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{D} \) are analytic functions. Then there exist unique continuous functions \( \omega_1, \omega_2 : \mathbb{D} \to \mathbb{W} \) that are analytic on \( \mathbb{D} \), \( \omega_1(0) = \omega_2(0) = 0 \), and

\[
\omega_1(z)\varphi_1(\omega_1(z)) = \omega_2(z)\varphi_2(\omega_2(z)) = \frac{\omega_1(z)\omega_2(z)}{z}, \quad z \in \mathbb{D}\setminus\{0\}.
\]

The existence of functions \( \omega_1, \omega_2 \) defined in \( \mathbb{D} \) and satisfying the conditions of the preceding corollary was first proved in [7]. The new information here is that these functions extend continuously to \( \mathbb{T} \) (but see [1] for the case in which the functions \( \varphi_j \) continue analytically through some arc and the continuations map that arc to \( \mathbb{T} \)). As before, this result can be used to study the regularity of free multiplicative convolutions of Borel probability measures on \( \mathbb{T} \); see [1] for measures on \( \mathbb{T} \) whose support is not the entire circle.

For the counterparts of these results in the case of additive free convolution, we recall the notation \( \mathbb{H} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\} \) for the closure of the complex upper half-plane in the Riemann sphere.

**Corollary 3.4.** Suppose that \( \psi_1, \psi_2 : \mathbb{H} \to \mathbb{H} \) are two analytic functions such that

\[
\lim_{y \to \infty} \frac{\psi_j(iy)}{iy} = 0, \quad j = 1, 2.
\]

Then there exist continuous functions

\[ \omega_1, \omega_2 : (\mathbb{H} \cup \mathbb{R}) \times (\mathbb{H} \cup \mathbb{R}) \to \mathbb{H} \]

that are finite and analytic on \( \mathbb{H} \times \mathbb{H} \), and

\[
\omega_1(z_1, z_2) = z_2 + \psi_2(\omega_2(z_1, z_2)), \\
\omega_2(z_1, z_2) = z_1 + \psi_1(\omega_1(z_1, z_2)), \quad z_1, z_2 \in \mathbb{H}.
\]

The proof is almost identical with that of Corollary 3.2 using the family of maps

\[ \varphi_{z_1, z_2}(\lambda) = z_1 + \psi_1(z_2 + \psi_2(\lambda)), \quad \lambda \in \mathbb{H}, z_1, z_2 \in \mathbb{H} \cup \mathbb{R}. \]

If none of these maps is the identity on \( \mathbb{H} \), Theorem [1] applies. If one of these maps is the identity, which can only occur for one pair \((z_1, z_2)\) \( \in \mathbb{R}^2 \) and only for fractional linear maps \( \psi_j \), the corollary is verified by a direct calculation that can be found essentially in [1]. The asymptotic condition on the functions \( \psi_j \) ensures that \( \omega_j \) is not equal to \( \infty \) at some point in \( \mathbb{H} \times \mathbb{H} \) (for instance \((i, i)\)). The existence of the analytic functions \( \omega_j(z, z), z \in \mathbb{H} \), was known earlier [12, 7, 13].

The maps \( \omega_j \) can often be extended continuously to pairs \((z_1, z_2)\) \( \in \mathbb{H} \times \mathbb{H} \) for which one or both coordinates are infinite. For instance, for every \( z_2 \in \mathbb{H} \) we have

\[
\lim_{z_1 \to \infty} \varphi_{z_1, z_2}(\lambda) = \infty, \quad \lambda \in \mathbb{H},
\]

and thus

\[
\lim_{z_1 \to \infty} \omega_1(z_1, z_2) = \lim_{z_1 \to \infty} \lambda \varphi_{z_1, z_2} = \infty.
\]

If the limit

\[
\psi_1(\infty) = \lim_{\lambda \to \infty, \lambda \in \mathbb{H}} \psi_1(\lambda)
\]

is

exists and is not zero, we have
\[ \lim_{z_2 \to \infty} \varphi_{z_1, z_2}(\lambda) = z_1 + \psi_1(\infty), \quad \lambda \in \mathbb{H}, \]
so
\[ \lim_{z_2 \to \infty} \omega_1(z_1, z_2) = z_1 + \psi_1(\infty), \quad z_1 \in \mathbb{H}. \]
Similar considerations apply to \( \omega_2 \).

Setting \( z_1 = z_2 \) in the above result yields the following result, first proved in [2, 3].

**Corollary 3.5.** Suppose that \( \psi_1, \psi_2 : \mathbb{H} \to \mathbb{H} \) are two analytic functions such that
\[ \lim_{y \uparrow \infty} \psi_j(iy) = 0, \quad j = 1, 2. \]
Then there exist continuous functions \( \omega_1, \omega_2 : \mathbb{H} \cup \mathbb{R} \to \mathbb{H} \cup \mathbb{R} \) that are finite and analytic on \( \mathbb{H} \), and
\[ \omega_1(z) + \psi_1(\omega_1(z)) = \omega_2(z) + \psi_2(\omega_2(z)) = \omega_1(z) + \omega_2(z) - z, \quad z \in \mathbb{H}. \]
Of course, we also have
\[ \lim_{z \to \infty} \omega_j(z) = \infty, \]
where \( \lim \) indicates a nontangential limit. This can be deduced from the fact that
\[ \lim_{z \to \infty} \varphi_{z, z}(\lambda) = \infty, \lambda \in \mathbb{H}, \]
or, more simply, from the inequality \( \Im \omega_j(z) \geq \Im z \). Corollary 3.5 is useful in the proof [2, 3] that the free additive convolution \( \mu_1 \boxplus \mu_2 \) of two Borel probability measures on \( \mathbb{R} \), both different from unit point masses, is absolutely continuous relative to Lebesgue measure, except for finitely many atoms, and that the density is locally analytic where positive. (The corresponding result for convolution powers is in [3].)

There is one more operation on measures to which our main result applies, namely the free multiplicative convolution of Borel probability measures on \( [0, +\infty) \). The regularity of such free convolutions was examined in [3] and Theorem 1.1 provides an easier approach. The application of Theorem 1.1 involves the simply connected domain
\[ \Omega = \mathbb{C} \setminus [0, +\infty) \]
and only part of its prime end compactification. We recall that each point \( r \in (0, +\infty) \) corresponds to two distinct prime ends of \( \Omega \), denoted \( r_+ \) and \( r_- \) which can be identified within this compactification as
\[ r_{\pm} = \lim_{\theta \downarrow 0} r e^{\pm i\theta}. \]
We consider the class \( \mathcal{F} \) consisting of those analytic functions \( \eta : \Omega \to \mathbb{C} \) that satisfy the following conditions:

1. \( \eta((-\infty, 0)) \subset (-\infty, 0) \). In particular, \( \overline{\eta(\lambda)} = \eta(\lambda) \) for every \( \lambda \in \Omega \).
2. If \( \lambda \in \mathbb{H} \), we have \( \eta(\lambda) \in \mathbb{H} \) and \( \arg(\eta(\lambda)) \geq \arg \lambda \), where \( \arg \lambda \) represents the principal value of the argument, that is, \( \arg \lambda \in (0, \pi) \).
3. \( \lim_{x \uparrow 0} \eta(x) = 0 \).
Moreover, there exists $\gamma$. With the notation above, we have these maps as $z$, where $\gamma$ is infinite, we have $\varphi \equiv \infty$. Clearly, the map $\varphi$ is continuous for every $\lambda \in \mathbb{H}$. We discuss the limit of these maps as $z \to 0$ and $z \to \infty$ in order to also define maps $\varphi_0 = 0$ and $\varphi_\infty = \gamma/f_2$, where $\gamma = \lim_{z \to \infty} \eta_2(x)$. When the limit $\gamma$ is infinite, we have $\varphi \equiv \infty$.

**Lemma 3.6.** With the notation above, we have $\lim_{z \to 0} \varphi_z(\lambda) = 0$ for every $\lambda \in \mathbb{H}$. Moreover, there exists $\gamma \in \mathbb{H}$ such that $\lim_{z \to \infty} \varphi_z(\lambda) = \gamma/f_2(\lambda)$ for every $\lambda \in \mathbb{H}$.

**Proof:** We consider first the limit at 0. Fix $\lambda \in \mathbb{H}$ and suppose that $(z_n)_{n \in \mathbb{N}} \subset \mathbb{H} \cup (0, +\infty)$ is a sequence converging to 0. Suppose, in addition, that

$$\lim_{n \to \infty} \arg(z_n) = \theta \in [0, \pi]$$

exists. The points $w_n = z_nf_2(\lambda)$ tend to zero and their arguments approximate $\theta + \arg f_2(\lambda) \in (0, 2\pi)$. It follows from (3.3) that $\lim_{n \to \infty} \eta_1(w_n) = 0$, and this implies immediately that

$$\lim_{n \to \infty} \varphi_z(\lambda) = \lim_{n \to \infty} \frac{\eta_1(w_n)}{f_2(\lambda)} = 0.$$

Compactness of the Riemann sphere then helps us to get rid of the assumption on the arguments of $z_n$.

Consider next a sequence $(z_n)_{n \in \mathbb{N}} \subset \mathbb{H} \cup (0, +\infty)$ that tends to infinity and $\lim_{n \to \infty} \arg(z_n) = \theta \in [0, \pi]$ exists. Then the points $w_n = z_nf_2(\lambda)$ tend to infinity and their arguments approximate $\theta + \arg f_2(\lambda)$. Now (3.3) implies that $\lim_{n \to \infty} \eta_1(w_n) = \lim_{x \to \infty} \eta_1(x)$. If $\gamma = \lim_{x \to \infty} \eta_1(x)$ is infinite, we conclude immediately that $\lim_{n \to \infty} \varphi_z(\lambda) = \infty$. On the other hand, if $\gamma$ is finite, we have $\lim_{n \to \infty} \varphi_z(\lambda) = \gamma/f_2(\lambda)$.

**Corollary 3.7.** Suppose that $\eta_1, \eta_2 \in \mathcal{F}$ are as above, and let $\omega : \Omega \to \Omega$ be the analytic function satisfying (3.2). Then $\omega|\mathbb{H}$ has a continuous extension to $\mathbb{H} \cup [0, +\infty]$. This extension satisfies $\omega(0) = 0$ and $\omega(\infty) = \infty$.,
Proof. We use the family of maps
\[ \varphi_z(\lambda) = z f_j(z f_2(\lambda)), \quad \lambda \in \mathbb{H}, z \in \mathbb{H} \cup [0, +\infty). \]
considered in the preceding lemma. As in the preceding results, the case in which one of these maps is the identity map—in which case \( z \in \mathbb{R} \)—is treated by direct calculation. In general, a difficulty arises from the fact that \( \varphi_z \) does not map \( \mathbb{H} \) to itself. However, if we set
\[ G_z = \{ \lambda \in \mathbb{H} : \arg \lambda > \arg z \}, \]
we have \( zf_j(\lambda) \in G_z \) for every \( \lambda \in G_z \), \( j = 1, 2 \), and therefore \( \varphi_z(G_z) \subset G_z \).
Denote by \( u_z : \mathbb{H} \to G_z \) the conformal homeomorphism defined by
\[ u_z(\lambda) = (-\lambda)^{1-\frac{z}{r}}, \lambda \in \mathbb{H}, z = re^{i\theta} \in \mathbb{H} \cup (0, +\infty), \]
where \( \theta \in [0, \pi) \), \( r > 0 \), and the power is calculated using the principal branch of the logarithm (that is, choosing \( \arg(-\lambda) \in (-\pi, 0) \) for \( \lambda \in \mathbb{H} \)). Then the map \( \lambda \mapsto \psi_z(\lambda) = u_z(\varphi_z(u_z^{-1}(\lambda))) \), sends \( \mathbb{H} \) to \( \mathbb{H} \) and depends continuously on \( z \). Theorem 14.1 shows that the map \( z \mapsto \lambda_{\psi_z} \) is continuous on \( \mathbb{H} \cup [0, +\infty] \). The corollary follows easily because \( \omega(z) = u_z(\lambda_{\psi_z}) \) for \( z \in \mathbb{H} \), so \( \omega(z) \) tends to \( u_r(\lambda_{\psi_z}) = \lambda_{\psi_z} \) as \( \lambda \in \mathbb{H} \) tends to \( r > 0 \). \( \square \)

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