REALIZING SPACES AS PATH-COMPONENT SPACES

TARAS BANAKH, JEREMY BRAZAS

Abstract. The path component space of a topological space X is the quotient space \( \pi_0(X) \) whose points are the path components of X. We show that every Tychonoff space X is the path-component space of a Tychonoff space Y of weight \( w(Y) = w(X) \) such that the natural quotient map \( Y \to \pi_0(Y) = X \) is a perfect map. Hence, many topological properties of X transfer to Y. We apply this result to construct a compact space \( X \subset \mathbb{R}^3 \) for which the fundamental group \( \pi_1(X, x_0) \) is an uncountable, cosmic, \( k_\omega \)-topological group but for which the canonical homomorphism \( \psi : \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0) \) to the first shape homotopy group is trivial.

1. Introduction

The path-component space of a topological space X is the set \( \pi_0(X) \) of path components of X equipped with the natural quotient topology inherited from X, i.e. the natural map \( X \to \pi_0(X) \) identifying each path component to a point is a quotient map. The path-component space of the space \( \Omega(X, x_0) \) of based loops \( S^1 \to X \) is the usual fundamental group \( \pi_1(X, x_0) \) equipped with a functorial topology that gives it the structure of a quasitopological group. Recent interest and applications of homotopy groups enriched with such topologies has brought more relevance to path-component spaces.

It is a beautiful and surprising result of Douglas Harris that every topological space is a path-component space [19]. In particular, for every space X, Harris constructed a paracompact Hausdorff space \( H(X) \) and a natural homeomorphism \( \pi_0(H(X)) \cong X \). Harris’ result plays a key role in the application of the topological fundamental group to prove that every topological group is isomorphic to the topological fundamental group of some space [5]. In particular, the free Markov topological group \( F_M(X) \) on a space X is isomorphic to the topological fundamental group of the reduced suspension \( \Sigma H(X)_+ \) of \( H(X)_+ = H(X) \cup \{ * \} \) with an isolated basepoint. Harris’ result is used in a similar fashion to prove a topological group analogue of the Nielsen-Schreier Theorem [6].

A remaining problem of relevance to topological fundamental groups is: given a particular class of spaces \( C \), identify a subclass \( D \) such that for every \( Y \in C \), we have \( Y \cong \pi_0(X) \) for some \( X \in D \).

For instance, Banakh, Vovk, and Wójcik show in [2] that every first countable space X is the path-component space of a complete metric space \( \odot X \) called the cobweb of X. The construction of \( \odot X \) is actually quite similar to that of \( H(X) \), however, it is not necessarily compact or separable when X is. These results suggest that analogous constructions might be possible for other classes of spaces.

2010 Mathematics Subject Classification. Primary 55Q52, 58B05, 54B15; Secondary 22A05, 54C10, 54G15.

Key words and phrases. path-component, quotient space, fundamental group, free topological group.
In the current paper, we prove the following Theorem using techniques quite different from those in [19] and [2].

**Theorem 1.1.** Every Tychonoff space $X$ is the path-component space of a Tychonoff space $Y$ of weight $w(Y) = w(X)$ such that the quotient map $q_Y : Y \to \pi_0(Y) = X$ is perfect, i.e. closed with compact preimages of points.

A topological property $\mathcal{P}$ is inversely preserved by perfect maps if whenever $f : Y \to X$ is a perfect map and $X$ has property $\mathcal{P}$, then so does the preimage $f^{-1}(X)$. Such properties are numerous and include compactness, paracompactness, metrizability, Lindelöf number, Čech completeness, Borel type, etc. We refer to [11, Section 3.7] for more on perfect maps as well as the column of inverse invariants of perfect maps in the table on pg. 510 of [11]. Theorem 1.1 implies that any topological property of $X$ that is inversely preserved by perfect maps is also a property enjoyed by the space $Y$ whose path-component space is $X$.

**Corollary 1.2.** Every Tychonoff space $X$ is the path-component space of a Tychonoff space $Y$ such that $Y$ shares any topological property of $X$ that is inversely preserved by perfect maps.

As an application, we apply Theorem 1.1 to identify an interesting phenomenon in topologized fundamental groups. The path-component space of a loop space $\Omega(X,x_0)$ is the fundamental group $\pi_1(X,x_0)$ equipped with its natural quotient topology. In general, $\pi_1(X,x_0)$ need not be a topological group [7], however, it is a quasitopological group in the sense that inversion is continuous and multiplication is continuous in each variable [4]. We refer to [1] for basic theory of quasitopological groups.

Since multiplication in $\pi_1(X,x_0)$ need not be continuous, one cannot take separation axioms for granted. While $T_0 \iff T_1$ holds for all quasitopological groups it need not be the case that $T_1$ implies $T_2$ or that $T_2$ implies higher separation axioms. The relevance of separation axioms in fundamental groups was noted in [2] where it is shown that if $\pi_1(X,x_0)$ is $T_1$, then $X$ has the homotopically path Hausdorff property introduced in [15]. The converse holds if $X$ is locally path connected. Furthermore, if $\pi_1(X,x_0)$ is $T_1$, then $X$ admits a generalized universal covering in the sense of [17].

There are several open questions related to the continuity of multiplication and separation axioms in quasitopological fundamental groups. We note here the open problems from [7, Problem 36].

**Problem 1.3.** Must $\pi_1(X,x_0)$ be normal for every compact metric space $X$?

**Problem 1.4.** Is there a Peano continuum (connected, locally path-connected, compact metric space) $X$ for which $\pi_1(X,x_0)$ is $T_1$ but not $T_4$?

To verify that $\pi_1(X,x_0)$ is Hausdorff, it is sufficient to know that $X$ is $\pi_1$-shape injective, meaning the canonical homomorphism $\psi : \pi_1(X,x_0) \to \bar{\pi}_1(X,x_0)$ to the first shape homotopy group is injective. Motivated by a construction in [4], we construct what is apparently the first example of a compact metric space for which $\pi_1(X,x_0)$ has been verified to be Hausdorff but for which $\psi$ is not injective. In doing so, we show in a strong way that $\pi_1$-shape injectivity is not a necessary condition for the $T_4$ separation axiom in fundamental groups of compact metric spaces.
Theorem 1.5. There exists a compact metric space \( X \subset \mathbb{R}^3 \) such that \( \pi_1(X,x_0) \) is an uncountable, cosmic, \( k_\pi \)-topological group such that \( \psi : \pi_1(X,x_0) \to \pi_1(X,x_0) \) is the trivial homomorphism. In particular, \( \pi_1(X,x_0) \) is \( T_4 \).

2. Path-component spaces

A path in a topological space \( X \) is a continuous function \( \alpha : [0,1] \to X \) from the closed unit interval \([0,1]\). The path component of a point \( x \in X \) is set

\[
[x] = \{ y \in X \mid \exists \text{ a path } \alpha : [0,1] \to X \text{ with } \alpha(0) = x \text{ and } \alpha(1) = y \}.
\]

The relation \([x] = [y] \), \( x, y \in X \) is an equivalence relation on \( X \); an equivalence class is called a path component of \( X \).

Definition 2.1. The path-component space \( \pi_0(X) \) of a topological space \( X \), is the set \( \pi_0(X) = \{ [x] \mid x \in X \} \) of path components of \( X \) equipped with the quotient topology with respect to the map \( q_X : X \to \pi_0(X) \), \( q(x) = [x] \) that collapses each path component to a point.

If \( f : X \to Y \) is a map, then \( f([x]) \subseteq [f(x)] \). Thus \( f \) determines a well-defined map \( f_0 : \pi_0(X) \to \pi_0(Y) \) given by \( f_0([x]) = [f(x)] \). Moreover, \( f_0 \) is continuous since \( f_0 \circ q_X = q_Y \circ f \) and \( q_X \) is quotient. Altogether, we obtain an endofunctor of the usual category \( \text{Top} \) of topological spaces.

Proposition 2.2. \( \pi_0 : \text{Top} \to \text{Top} \) is a functor and the quotient maps \( q_X : X \to \pi_0(X) \) are the components of a natural transformation \( q : id \to \pi_0 \).

Certainly, if \( X \) is locally path connected, then \( \pi_0(X) \) is a discrete space. More generally, \( \pi_0(X) \) is discrete \( \Leftrightarrow \) every path component of \( X \) is open \( \Leftrightarrow X \) is semi-locally 0-connected in the sense that for every point \( x \in X \), there is a neighborhood \( U \) of \( x \) such that the inclusion \( i : U \to X \) induces the constant map \( i_0 : \pi_0(U) \to \pi_0(X) \).

From the following basic examples, we begin to see that separation axioms are not passed to path-component spaces even if \( X \) is a compact metric space.

Example 2.3. If \( P_1 = \{0\} \times [-1,1] \) and \( P_2 = \{(x, -\sin(\pi/|x|)) \mid x \in (0,1] \} \) so that \( T = P_1 \cup P_2 \) is the closed topologist's sine curve, then \( \pi_0(T) = \{P_1, P_2\} \) is the two-point Sierpinski space with topology \( \{\emptyset, \{P_2\}, T\} \). Thus \( \pi_0(X) \) is \( T_0 \) but not \( T_1 \).

Example 2.4. Given any discrete space \( V \), let \( \Gamma \) be a maximal tree in the complete graph with vertex set \( V \). We construct the space \( X \) from \( V \) as follows: for each edge \([x, y]\) of \( \Gamma \), attach two copies of \( T \) (as in the previous example) to \( V \), the first by \((0, 0) \sim x \) and the second by \((0, 0) \sim y \) and \((1, 0) \sim x \). It is easy to see that the resulting space \( X \) has indiscrete path-component space \( \pi_0(X) = \{[v] \mid v \in V \} \) with the same cardinality as \( V \).

Example 2.5. A compact metric space has cardinality at most that of the continuum and so its path-component space must be restricted in the same way. Consider the following path-connected subspaces of the \( xy \)-plane where \([a, b]\) denotes the line segment from \( a \) to \( b \).

- \( P_1 = \{(x, 2 - \sin(\pi/x)) \mid 0 < x \leq 1\} \cup \{(1, 2), (0, -2)\} \cup \{\{0\} \times [-3, -1]\} \)
- \( P_2 = \{(-x, -2 + \sin(\pi/x)) \mid 0 < x \leq 1\} \cup \{(-1, -2), (0, 2)\} \cup \{\{0\} \times [1, 3]\} \)

Let \( D = P_1 \cup P_2 \) (See Figure 1) and note that \( \pi_0(D) = \{P_1, P_2\} \) is indiscrete since each path component is neither open nor closed. Now let \( C \subset [0,1] \) be a Cantor set in the \( z \)-axis and take \( X = (D \times C) \cup \{(0, 0) \times [1, 3] \times [0, 1]\} \). Including the rectangle
The space $D = P_1 \cup P_2$.

At the other extreme, the natural quotient map $q_X : X \to \pi_0(X)$ is a homeomorphism if and only if $X$ is totally path disconnected, i.e. admits no non-constant paths. In general, $\pi_0(X)$ need not be totally path-disconnected. The existence of non-constant paths in $\pi_0(X)$ seems strange at first since it may permit cases where $\pi_0(\pi_0(X))$ is non-trivial. We make clear the relevance of this situation in our discussion on fundamental groups.

**Example 2.6.** Let $L$ be a linearly ordered space and $X = L \times [0, 1]$ have the order topology induced by the lexicographical ordering. Certainly, the sets $\{\ell\} \times [0, 1]$ are path-connected. If $\alpha : [0, 1] \to X$ is a path from $(\ell_1, t_1)$ to $(\ell_2, t_2)$ where $\ell_1 < \ell_2$, then by the intermediate value theorem the image of $\alpha$ contains all $(\ell, t)$ with $(\ell_1, t_1) \leq (\ell, t) \leq (\ell_2, t_2)$. By composing with the projection $L \times [0, 1] \to L$, we see that $[\ell_1, \ell_2]$ is path connected and Hausdorff and therefore must be uncountable. Thus the image of $\alpha$ contains the uncountable family of disjoint open sets: $\{(\ell, t) | 1/3 < t < 2/3, \ell \in (\ell_1, \ell_2)\}$, which is not possible. It follows that the sets $\{\ell\} \times [0, 1]$ are the path components of $X$ and that $\pi_0(X) \cong L$ where $q_X : X \to \pi_0(X)$ may be identified with the projection onto the first coordinate. Taking $L = [0, 1]$ or $L = \mathbb{R}$ provides examples of non-metrizable $X$ for which $\pi_0(X)$ admits non-constant paths.

It is less clear how often $\pi_0(X)$ is totally path disconnected when $X$ is a metric space. A natural question one may then ask is: what spaces can be realized as path-component spaces? A beautiful and surprising result due to Douglas Harris shows that, in fact, every topological space is a path-component space in a functorial way.

**Theorem 2.7** (Harris [19]). For every space $X$, there is a paracompact Hausdorff space $H(X)$ and a natural homeomorphism $\pi_0(H(X)) \cong X$.

An analogue for metric spaces was identified by Banakh, Vovk, and Wójcik. Their most general result relating to path-component spaces is the following. Recall that *premetric* on set $X$ is a function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$ for all $x \in X$. Any premetric generates a topology on $X$ by basic open sets $B_r(x) = \{y \in X : d(x, y) < r\}$. For example, $[0] \times [1, 3] \times [0, 1]$ has the effect of combining the path components $P_1 \times \{c\} \in \pi_0(D \times C)$, $c \in C$ into a single path component. Thus $X \subset \mathbb{R}^3$ is a compact metric space such that $\pi_0(X)$ is uncountable and indiscrete.
A \textit{premetric space} is a topological space whose topology is generated by a premetric on \(X\).

**Theorem 2.8** (Banakh, Vovk, Wójcik [2]). \textit{Every premetric space} \(X\) \textit{is the path-component space of a complete metric space} \(\Theta(X)\) \textit{called the cobweb space of} \(X\).

**Corollary 2.9.** \textit{Every metric space} is the path-component space of a complete metric space.

3. **Realizing the unit interval as a path-component space**

The closed unit interval may be realized as the quotient of any infinite connected compact Hausdorff space \(X\) using any Uryshon function \(f : X \to [0, 1]\) where \(f(a) = 0\) and \(f(b) = 1\) for \(a \neq b\). We take a more specific approach here since our goal in this section is to realize the unit interval naturally as the path-component space of a space \(K\) where we wish for \(K\) to be as “nice” as possible.

**Remark 3.1.** We first recall an elementary fact from decomposition theory [9]. An onto map \(f : X \to Y\) is said to be \textit{monotone} if the preimage \(f^{-1}(y)\) is compact and connected for every \(y \in Y\). It is well-known that if an onto map \(q : [0, 1] \to Z\) is monotone, then \(Z \approx [0, 1]\).

Let \(C = \left\{\sum_{i=1}^{\infty} \frac{x_i}{3^i} \mid x_i \in \{0, 2\}\right\}\) be the standard middle third Cantor set in \([0, 1]\). Let \(C\) be the set of open intervals which are the connected components of \([0, 1]\setminus C\). For each \(k \geq 1\), let \(C_k \subseteq C\) be the set of connected components of \([0, 1]\setminus C\) which are open intervals of length \(\frac{1}{3^k}\).

Let \(K_{odd} = \bigcup_{k \ odd} C_k\) and \(K_{even} = \bigcup_{k \ even} C_k\). Define

\[
K = (C \times [0, 1]) \cup (K_{even} \times \{0\}) \cup (K_{odd} \times \{1\})
\]

(See Figure 2).

**Figure 2.** The space \(K\).

Alternatively, let \(K_0 = [0, 1]^2\). For each \(n \geq 1\) the set

\[
U_n = \bigcup_{1 \leq j \leq n} \bigcup_{\substack{\text{odd} \ j \ \text{odd} \ j \ \text{odd}}} J \times [0, 1] \cup \bigcup_{1 \leq j \leq n} \bigcup_{\substack{\text{even} \ j \ \text{odd} \ j \ \text{even}}} J \times (0, 1)
\]

is open in \([0, 1]^2\) and \(K_n = [0, 1]^2 \setminus U_n\) is a 2-dimensional, contractible, compact CW-complex. Therefore, \(K = \bigcap_{n \in \mathbb{N}} K_n\) is a compact metric space shape equivalent to a point.
Definition 3.2. We say a map \( f : X \to Y \) is perfect if \( X \) is Hausdorff, \( f \) is a closed map, and the point-preimage \( f^{-1}(y) \) is compact for each \( y \in Y \).

Theorem 3.3. \( \pi_0(K) \cong [0, 1] \) where the quotient map \( q_K : K \to \pi_0(K) \) is a perfect map.

Proof. First, we identify the path components of \( K \). Let \( \pi : K \to [0, 1] \) be the projection onto the first coordinate. Let \( E \subseteq C \) be the set containing the endpoints of the intervals \( I \in C \) and set \( D = C \setminus E \). For \( c \in D \), \( Y_c = \pi^{-1}(c) = [c] \times [0, 1] \) is path-connected and for each \( I \in C \), \( Y_I = \pi^{-1}(\bar{I}) \equiv [0, 1] \) is path-connected. For convenience, we say \( I \in C \) is of odd-type if \( I \subsetneq K_{odd} \) and of even-type if \( I \subsetneq K_{even} \).

For any two distinct elements of \( \{Y_c|c \in D\} \cup \{Y_I|I \in C\} \), there are infinitely many intervals \( I \in C \) of odd-type and of even-type lying between them (as subsets of \([0, 1])\).

If \( A, B \) are connected sets in \([0, 1]\) and \( a < b \) for all \( a \in A \) and \( b \in B \), we write \( A < B \). Suppose \( \alpha : [0, 1] \to K \) where \( \alpha(0) \) and \( \alpha(1) \) lie respectively within distinct elements \( A \) and \( B \) of \( \{Y_c|c \in D\} \cup \{Y_I|I \in C\} \). Without loss of generality, suppose \( A < B \).

Inductively applying the intermediate value theorem, we may find an increasing sequence \( 0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < 1 \) and sets \( A < I_1 < I_2 < \ldots < I_n \subseteq B \) where \( I_n \in C \) is of odd-type, \( J_n \in C \) of even-type, \( \pi \circ \alpha(s_n) = a_n \) is the midpoint of \( I_n \), and \( \pi \circ \alpha(t_n) = b_n \) is the midpoint of \( J_n \). Note \( \alpha(s_n) = (a_n, 1) \) and \( \alpha(t_n) = (b_n, 0) \). If \( x = \sup \{s_n|n \geq 1\} = \sup \{t_n|n \geq 1\} \), then by continuity of \( \alpha \), we have \( \alpha(x) = \lim_{n \to \infty} (a_n, 1) \) in \([0, 1] \times \{1\} \) and \( \alpha(x) = \lim_{n \to \infty} (b_n, 0) \) in \([0, 1] \times \{0\} \); a contradiction. We conclude that \( \pi_0(K) = \{Y_c|c \in D\} \cup \{Y_I|I \in C\} \) as a set.

Consider the map \( f : [0, 1] \to Z \) which, for each \( I \in C \), collapses \( \bar{I} \) to a point. Certainly, \( f \) is monotone so \( Z \equiv [0, 1] \) by Remark 3.1. The fibers of \( f \circ \pi \) are exactly the fibers of the quotient map \( q_K : K \to \pi_0(K) \) so there is a unique continuous bijection \( g : \pi_0(K) \to Z \) such that \( g \circ q_K = f \circ \pi \). Since \( K \) is compact, so is its quotient \( \pi_0(K) \). Since \( Z \) is Hausdorff, \( g \) is a closed map. We conclude that \( g \) is a homeomorphism and thus \( \pi_0(K) \cong Z \equiv [0, 1] \).

Example 3.4. The space \( K \) can also be used to construct reasonably nice spaces whose path-component spaces satisfy only weak separation axioms. The space \( K' = (K \times [0, 1]) \setminus ([1] \times [0, 1] \times [0, 1])) \subset \mathbb{R}^3 \) is not compact but is a separable metric space. The path components of \( K' \) are precisely the sets:

1. \( C \times [0, 1] \) where \( C \subset [0, 1] \times [0, 1] \) is a path component of \( K \),
2. \( C_0 = [1] \times [0, 1] \times [0] \),
3. \( C_1 = [1] \times [0, 1] \times [1] \).

Based on Theorem 3.3 it follows that \( \pi_0(K') \cong [0, 1] \times [0, 1]/\sim \) where \((t, 0) \sim (t, 1)\) if \( t < 1 \). Hence, \( \pi_0(K') \) is the \( T_1 \) but non-Hausdorff “closed unit interval with two copies of 1.”
4. Realizing Tychonoff spaces as path-component spaces

This section is dedicated to proving Theorem 4.1 as well as a functorial version of the construction.

**Proposition 4.1.** Let \( \{X_\lambda\} \) be a family of spaces and \( X = \prod_\lambda X_\lambda \). Let \( q_\lambda : X_\lambda \to \pi_0(X_\lambda) \) and \( q_X : X \to \pi_0(X) \) be the canonical quotient maps and \( \prod_\lambda q_\lambda : X \to \prod_\lambda \pi_0(X_\lambda) \) be the product map. There is a natural continuous bijection \( \phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda) \) such that \( \phi \circ q_X = \prod_\lambda q_\lambda \), which is a homeomorphism if and only if the product map \( \prod_\lambda q_\lambda : \prod_\lambda X_\lambda \to \prod_\lambda \pi_0(X_\lambda) \) is a quotient map.

**Proof.** The projections \( r_\lambda : X \to X_\lambda \) induces maps \( (r_\lambda)_0 : \pi_0(X) \to \pi_0(X_\lambda) \) which in turn induce the natural map \( \phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda), \phi([x_\lambda]) = ([x_\lambda]) \). Clearly \( \phi \) is surjective. If \([x_\lambda] = [y_\lambda]\) for each \( \lambda \), then there is a path \( \alpha_\lambda : [0,1] \to X_\lambda \) from \( x_\lambda \) to \( y_\lambda \). These maps induce a path \( \alpha : [0,1] \to X \) such that \( r_\lambda \circ \alpha = \alpha_\lambda \). Since \( \alpha \) is a path from \( (x_\lambda) \) to \( (y_\lambda) \), we have \([x_\lambda] = [y_\lambda]\). Thus \( \phi \) is injective.

\[
\begin{array}{ccc}
\prod_\lambda X_\lambda & \xrightarrow{\phi} & \prod_\lambda \pi_0(X_\lambda) \\
\downarrow q_X & & \downarrow \prod_\lambda q_\lambda \\
\pi_0(\prod_\lambda X_\lambda) & \xrightarrow{\phi} & \prod_\lambda \pi_0(X_\lambda)
\end{array}
\]

The last fact follows directly from the fact that \( q_X \) is a quotient map, \( \phi \) is a bijection, and the commutativity of the triangle above. \( \square \)

**Corollary 4.2.** If, for all \( \lambda \), \( X_\lambda \) is compact and \( \pi_0(X_\lambda) \) is Hausdorff, then \( \phi : \pi_0(\prod_\lambda X_\lambda) \to \prod_\lambda \pi_0(X_\lambda) \) is a homeomorphism.

**Proof.** Given the assumptions, \( \prod_\lambda X_\lambda \) is compact by the Tychonoff Theorem and \( \prod_\lambda \pi_0(X_\lambda) \) is Hausdorff. Thus \( \prod_\lambda q_\lambda \) is a closed surjection and therefore quotient and we may apply the final statement of Proposition 4.1. \( \square \)

**Remark 4.3.** Certainly every onto perfect map is quotient. Moreover, a perfect map \( f : X \to Y \) has the property that if \( S \subseteq Y \), then the restriction \( f|_{f^{-1}(S)} : f^{-1}(S) \to S \) is a perfect map [11] Proposition 3.7.6).

**Lemma 4.4.** Suppose \( A \) is a compact Hausdorff space such that \( \pi_0(A) \) is Hausdorff and \( X \subseteq \pi_0(A) \). If \( B = q_A^{-1}(X) \subseteq A \), then there is a canonical homeomorphism \( \pi_0(B) \cong X \) and \( q_B : B \to \pi_0(B) = X \) is a perfect map.

**Proof.** Since \( A \) and \( \pi_0(A) \) are compact Hausdorff, the quotient map \( q_A : A \to \pi_0(A) \) is a perfect map. According to Remark 4.3 the restriction of \( q_A \) to \( B \) is a perfect map \( q_{A|B} : B \to X \). Since onto perfect maps are quotient, \( q_{A|B} \) is a quotient map which makes the same identifications as \( q : B \to \pi_0(B) \). Thus there is a canonical homeomorphism \( \pi_0(B) \cong X \). \( \square \)

Recall that the weight \( w(X) \) of a topological space \( X \) is the minimal cardinality of a basis generating the topology of \( X \).

**Proof of Theorem 4.1.** Suppose \( X \) is a Tychonoff space of weight \( m = w(X) \). Recall that for any cardinal \( m \geq \aleph_0 \), the direct product \( [0,1]^m \) (of weight \( m \)) is universal for all Tychonoff spaces \( X \) of weight \( m \) [11] 2.3.23], i.e. \( X \) homeomorphically embeds as a subspace of \( [0,1]^m \). Hence, we may identify \( X \) as a subspace of \( [0,1]^m \).
Recalling the compact space $K \subset [0,1]^2$ constructed in the previous section, we may identify $\pi_0(K) = [0,1]$ by Theorem 3.3. By Corollary 4.2, the product map $Q = (q_K)^m : K^m \to [0,1]^m$ is a quotient map whose fibers are the path-components of $K^m$ and the canonical bijection $\phi : \pi_0(K^m) \to [0,1]^m$ such that $\phi \circ q_{K^m} = Q$ is a homeomorphism. Moreover, the Hausdorff spaces $K^m$ and $[0,1]^m$ are compact by the Tychonoff Theorem. Thus $Q$ is a closed map and hence a perfect map.

\[
\begin{array}{c}
pi_0(K^m) \\
\downarrow \phi \\
[0,1]^m \\
\downarrow q^m \\
K^m \\
\end{array}
\]

Set $Y = Q^{-1}(X) \subseteq K^m$. Note that since $K$ is second countable, $w(Y) \leq w(K^m) = m = w(X)$. According to Remark 4.3, the restriction $Q|_Y : Y \to X$ of $Q$ is an onto perfect map. In general, if $f : Y \to X$ is an onto perfect map, then $w(X) \leq w(Y)$ [11, 3.7.19]. Hence $w(Y) = w(X)$. Since $\phi \circ q_{K^m} = Q$, the quotient map $q_Y : Y \to \pi_0(Y)$ makes the same identifications as $Q|_Y$ and hence, $\phi$ restricts to a homeomorphism $\phi|_{\pi_0(Y)} : \pi_0(Y) \cong X$. Since $\phi|_{\pi_0(Y)} \circ q_Y = Q|_Y$ where $\phi|_{\pi_0(Y)}$ is a homeomorphism and $Q|_Y$ is perfect, $q_Y$ is also a perfect map. □

The construction of $Y$ in the proof of Theorem 4.1 is not functorial since a choice of basis for the topology of $X$ is required for the embedding $X \subseteq [0,1]^m$. We now show that if one is willing to give up the equality $w(Y) = w(X)$, the construction can be made functorial.

**Theorem 4.5.** Let $\text{Tych} \subset \text{Top}$ be the full subcategory of Tychonoff spaces. Then there is a functor $\Phi : \text{Tych} \to \text{Tych}$ and a natural isomorphism $\pi_0 \circ \Phi \cong \text{Id}_{\text{Tych}}$ such that the natural maps $q_{\Phi(X)} : \Phi(X) \to \pi_0(\Phi(X))$ are onto perfect maps.

**Proof.** Let $X$ be a Tychonoff space and $C(X, I)$ denote the set of all continuous functions $X \to [0,1]$. Since this set of functions separates points and closed sets, the natural map $i_X : X \to [0,1]^{C(X, I)}$, $i_X(f) = f(x)$ to the direct product is an embedding. For convenience, we identify $X = i(X)$. We follow the same line of argument used in the proof of Theorem 4.1 except that we replace $[0,1]^m$ and $K^m$ with the compact spaces $[0,1]^{C(X, I)}$ and $K^{C(X, I)}$ respectively. The product map $Q = (q_K)^{C(X, I)} : K^{C(X, I)} \to [0,1]^{C(X, I)}$ is perfect and we define $\Phi(X) = Q^{-1}(X)$. It follows that $\pi_0(\Phi(X)) \cong X$ where $q_{\Phi(X)} : \Phi(X) \to \pi_0(\Phi(X))$ is a perfect map.

To construct $\Phi$ on morphisms, suppose $h : X \to Y$ is a map of Tychonoff spaces. Let $m : K^{C(X, I)} \to K^{C(Y, I)}$ and $M : [0,1]^{C(X, I)} \to [0,1]^{C(Y, I)}$ be the canonical induced maps. Set $A = K^{C(X, I)}$ and $B = K^{C(Y, I)}$ so we can simply write the quotient maps identifying path components as $q_A$ and $q_B$ respectively. The naturality of the embeddings $i_X$ and quotient maps $q_X$ indicates that the following diagram
commutes.

\[
\begin{array}{ccc}
K^{C(Y)} & \xrightarrow{m} & K^{C(Y)} \\
\downarrow q & & \downarrow q & \\
[0,1]^{C(X)} & \xrightarrow{M} & [0,1]^{C(Y)} \\
\downarrow i & & \downarrow i & \\
X & \xrightarrow{f} & Y
\end{array}
\]

Since \(M(i_X(X)) = i_Y(f(X)) \leq i_Y(Y)\), we have \(q_B(m(\Phi(X))) = q_B(m(q_A^{-1}(i(X)))) = M(i_X(X)) = i_Y(f(X)) \leq i_Y(Y)\) and thus \(m(\Phi(X)) \leq q_B^{-1}(i_Y(Y)) = \Phi(Y)\). Thus we may define \(\Phi(h) : \Phi(X) \to \Phi(Y)\) to be the restriction of \(m\) to \(\Phi(X)\). The rest of the details of confirming functoriality of \(\Phi\) and naturality of the homeomorphisms \(\pi_0(\Phi(X)) \to X\) are straightforward. \(\Box\)

5. Application to topologized fundamental groups

For a path-connected space \(X\) with basepoint \(x_0 \in X\), let \(\Omega(X,x_0)\) denote the space of based maps \((S^1, b_0) \to (X, x_0)\) with the compact-open topology. The path-component space \(\pi_1(X,x_0) = \pi_0(\Omega(X,x_0))\) is the fundamental group equipped with the natural quotient topology. It is known that \(\pi_1(X,x_0)\) is a quasitopological group in the sense that inversion is continuous and the group operation is continuous in each variable. However, \(\pi_1(X,x_0)\) can fail to be a topological group \([4, 13, 14]\). A general study of fundamental groups with the quotient topology appears in \([7]\).

The following construction is taken from the paper \([4]\). For \(X\) a compact metric space let \(W(X) = X \times S^1 / X \times \{b_0\}\) where the image of \(X \times \{b_0\}\) is the basepoint \(w_0\). Certainly \(W(X)\) is also a compact metric space. Equivalently, if \(X_+ = X \cup \{\ast\}\) is \(X\) with an added isolated basepoint, then \(W(X) \cong \Sigma(X_+)\) is the reduced suspension of \(X_+\). We think of \(W(X)\) as a wedge of circles where the circles are topologically parameterized by the space \(X\). In particular, if \(X = \bigsqcup_{j \in J} A_j\) is a topological sum of contractible spaces \(A_j\), then \(W(X)\) is homotopy equivalent to \(\bigvee_{j \in J} S^1\).

We recall the notion of free topological group in the senses of Markov \([22]\) and Graev \([18]\). Since the groups \(\pi_1(X,x_0)\) need not be Hausdorff, we do not assume any separation axioms.

**Definition 5.1.** The free Markov topological group \(F_M(Y)\) on the space \(Y\) is the topological group equipped with a map \(\sigma : Y \to F_M(Y)\) such that every map \(f : Y \to G\) to a topological group \(G\) extends uniquely to a continuous homomorphism \(\tilde{f} : F_M(Y) \to G\) such that \(\sigma \circ \tilde{f} = f\).

The free Graev topological group \(F_G(Y, y)\) on the based space \((Y, y)\) is the topological group equipped with a map \(\sigma_* : Y \to F_G(Y, y)\) taking \(y\) to the identity element \(e\) such that every map \(f : (Y, y) \to (G, 1_G)\) to a topological group \(G\) extends uniquely to a continuous homomorphism \(\tilde{f} : F_G(Y, y) \to G\) such that \(\sigma_* \circ \tilde{f} = f\).

**Remark 5.2.** By the Adjoint Functor Theorem, the free Markov (Graev) topological group exists for all (based) spaces \(Y\). The underlying group of \(F_M(Y)\) is the free group on the underlying set of \(Y\) and the underlying group of \(F_G(Y, y)\) is the free group on \(Y \setminus \{y\}\). The groups \(F_M(Y)\) and \(F_G(Y, y_0)\) are Hausdorff if \(Y\) is functionally...
1.1. In this case, we may replace $Y$ groups on compact Hausdorff for the case Remark 5.7. such that $w$ Hausdorff Markov topological group $F$. If $X$ is a $k$-space, this must be the entire group $F$. Since the path-component of the identity in a topological group is a subgroup, this must be the entire group $F$. Proof. The image of $\sigma : Y \to F_G(Y, y)$ is path-connected and algebraically generates the entire group. Since the path-component of the identity in a topological group is a subgroup, this must be the entire group $F_G(Y, y)$. □

A topological space $X$ is said to be a $k_\omega$-space if $X$ is the inductive limit of compact subspaces $X_1 \subset X_2 \subset \ldots$, i.e. $X = \bigcup_{n \geq 1} X_n$ and $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed in $X_n$ for all $n$.

Theorem 5.4 (Mack, Morris, Ordman [20]). If $X$ is a Hausdorff $k_\omega$-space and $x_0 \in X$, then $F_M(X)$ and $F_C(X, x_0)$ are Hausdorff $k_\omega$-spaces.

The following Theorem follows from Corollaries 1.2 and 4.23 of [4].

Theorem 5.5. If $X$ is a $k_\omega$-space, then $\pi_1(W(X), \omega_0)$ is naturally isomorphic to the free Markov topological group $F_M(\pi_0(X))$ on the path-component space $\pi_0(X)$.

By combining Theorem 5.5 with Theorem 1.1, we realize all free topological groups on compact Hausdorff spaces as the fundamental group of a compact Hausdorff space. We remark that results in [4] had no control over the compactness or metrizability of the construction since they depended on Harris’ construction.

Corollary 5.6. [4 Corollary 1.2] For every Tychonoff space $X$, there is a Tychonoff space $Y$ such that $\omega(Y) = w(X)$, $q : Y \to \pi_0(Y) = X$ is a perfect map, and $\pi_1(W(Y), \omega_0) \cong F_M(X)$.

Remark 5.7. For the case $X = [0, 1]^n$, we need not use the full power of Theorem 1.1. In this case, we may replace $Y$ with $K^n$ since $\pi_0(K^n) \cong [0, 1]^n$. Since $K^n \subseteq \mathbb{R}^{2n}$, the space $W(K^n)$ embeds as a subspace of $\mathbb{R}^{2n+1}$ when considered as a union of circles (indexed over $K^n$) with a single point in common.

Lemma 5.8. [3 Lemma 4.4] If a $2$-cell $e^2$ is attached to a path-connected space $Y$, then the inclusion $Y \to Y \cup e^2$ induces a homomorphism $\pi_1(Y, y_0) \to \pi_1(Y \cup e^2, y_0)$ which is a topological quotient map.

Theorem 5.9. Suppose $X$ is a compact Hausdorff space which is well-pointed at $x_0 \in X$, i.e. $[x_0] \to X$ is a cofibration. Then $\pi_1(\Sigma X, w_0) \cong F_{C}(\pi_0(X), [x_0])$.

Proof. Since $[x_0] \to X$ is a cofibration, $\Sigma X$ is homotopy equivalent to $W(X) \cup e^2$ where the $2$-cell is attached along the loop $\ell : S^1 \to W(X)$ where $\ell(t) = (x_0, t)$ in $W(X)$. Recall by Theorem 5.5 that $\pi_1(W(X), \omega_0) \cong F_M(\pi_0(X))$ and that $[x_0] \in \pi_0(X)$ denotes the path component of $x_0$ in $X$. According to Remark 5.2 there is an isomorphism of topological groups $F_{C}(\pi_0(X), [x_0]) \cong F_M(\pi_0(X))/\{[[x_0]]\}$ where $N$ is the normal subgroup generated by $\{[x_0]\}$. The inclusion $W(X) \to W(X) \cup e^2$ induces a homomorphism $\pi_1(W(X), \omega_0) \to \pi_1(W(X) \cup e^2, \omega_0)$, which according to Lemma 5.8 is a topological quotient map whose kernel is precisely $N$. Thus $\pi_1(W(X) \cup e^2, \omega_0) \cong F_{C}(\pi_0(X), [x_0])$. □
Corollary 5.10. [4 Corollary 1.2] For every compact Hausdorff (resp. metric) space \( X \) and \( x_0 \in X \), there is a path-connected compact Hausdorff (resp. metric) space \( Z \) and \( w_0 \in Z \) such that \( \pi_1(Z, w_0) \equiv F_C(X, x_0) \).

Proof. Using Theorem 1.1, construct a compact Hausdorff space \( Y \) such that \( \pi_0(Y) \equiv X \). Pick a point \( y \in Y \) and attach a copy of the unit interval to \( Y \) where \( 1 \sim y \). Call the resulting space \( Y' \) and let \( y_0 \) be the image of 0, which we take to be the basepoint. Now \( (Y', y_0) \) is well-pointed, and \( \pi_0(Y') \equiv \pi_0(Y) = X \). Set \( Z = \Sigma Y' \) and apply Theorem 5.9. \( \square \)

Example 5.11. As in Remark 5.7, we may perform a more direct construction in the case \( X = [0, 1]^n \) and \( x_0 = 0 \in X \). Set \( y_0 = 0 \in Y = K^n \) and recall \( \pi_0(Y) = [0, 1]^n \).

Let \( Y' \) be the space obtained by attaching a copy of the unit interval to \( Y \) where \( 1 \sim y_0 \). Now we have the desired isomorphisms \( \pi_1(\Sigma Y', w_0) \equiv F_C(\pi_0(Y'), [x_0]) \equiv \pi_0(Y), [x_0]) \equiv F_C((0, 1]^n, 0) \).

Since \( Y' \) embeds in \( \mathbb{R}^{2n} \), \( \Sigma Y' \) may be embedded as a compact subspace of \( \mathbb{R}^{n+1} \). Moreover, since \( K^n \) is the intersection of contractible polyhedra, so are both of the spaces \( Y' \) and \( \Sigma Y' \). Hence, \( \Sigma Y' \) is shape equivalent to a point. These examples are remarkable, because \( \Sigma Y' \) is a finite dimensional compact metric space and yet \( \pi_1(\Sigma Y', w_0) \) is a path-connected topological group.

The first shape homotopy group of a paracompact Hausdorff space is the inverse limit

\[
\hat{\pi}_1(x, x_0) = \lim_{\longrightarrow} \pi_1(N(\mathcal{U}), v_0)
\]

of fundamental groups of geometric realizations of nerves of open covers \( \mathcal{U} \) of \( X \). Since we only require general properties of this group, we refer the reader to [21] for details of the construction. The nerve \( N(\mathcal{U}) \) is a simplicial complex and hence has discrete fundamental group [3]. Hence \( \hat{\pi}_1(x, x_0) \) is naturally topologized as an inverse limit of discrete groups. It is known that there is a canonical homomorphism \( \psi : \hat{\pi}_1(x, x_0) \to \hat{\pi}_1(x, x_0) \) which is continuous with respect to the quotient topology on \( \hat{\pi}_1(x, x_0) \) [7]. The easiest way to ensure that \( \hat{\pi}_1(x, x_0) \) is Hausdorff is to check that \( \psi \) is injective (since any space which injects into a Hausdorff space is Hausdorff). For example, \( \psi \) is known to be injective for all one-dimensional spaces [10] and planar spaces [16]. In the case when \( \psi \) is injective, one may conclude that \( \hat{\pi}_1(x, x_0) \) is functionally Hausdorff since it continuously injects into the functionally Hausdorff space \( \pi_1(X, x_0) \). However, \( \phi \) is not typically an embedding [12], so even when it is injection one can not immediately conclude that \( \pi_1(x, x_0) \) enjoys stronger separation axioms.

Finally, we complete the proof of Theorem 1.5. Recall that a topological space is cosmic if it is the continuous image of a separable metric space.

Proof of Theorem 1.5. Using the case \( n = 1 \) from Example 5.11 we may construct a compact suspension space \( X \subset \mathbb{R}^3 \) such that \( \pi_1(X, x_0) \equiv F_C([0, 1], 0) \). Since \( [0, 1] \) is functionally Hausdorff, \( F_C([0, 1], 0) \) is Hausdorff. In fact, it is a result of M. Zarichnyi [27] that \( F_C([0, 1], 0) \) is homeomorphic to \( \mathbb{R}^\infty \) topologized as the direct limit of finite Euclidean space \( \mathbb{R}^n \), which is a locally convex linear topological space. Since \( [0, 1] \) is a \( k_\omega \)-space, it follows from Theorem 5.4 that \( F_C([0, 1], 0) \) is a \( k_\omega \)-space. Lastly, since \( X \) is a compact metric space, \( \Omega(X, x_0) \) is a separable metric space.
[11] 4.2.17 and 4.2.18] and it follows that the quotient \( \pi_1(X, x_0) \) is cosmic. Every Hausdorff cosmic topological group is regular and Lindel"of and is therefore \( T_4 \).

Finally, recall that the first shape homotopy group \( \tilde{\pi}_1(X, x_0) \) is topologized as an inverse limit of discrete groups and is therefore totally path disconnected. But \( \pi_1(X, x_0) \) is path-connected by Proposition 5.3. Hence, the canonical continuous homomorphism \( \psi : \pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0) \) must be the trivial homomorphism. □

References

[1] A. Arhangel’skii, M. Tkachenko, Topological Groups and Related Structures, Series in Pure and Applied Mathematics, Atlantis Studies in Mathematics, 2008.
[2] T. Banakh, M. Vovk, M.R. Wójcik, Connected economically metrizable spaces, Fund. Math. 212 (2011), 145–173.
[3] J. Brazas, Homotopy Mapping Spaces, Ph.D. Dissertation, University of New Hampshire, 2011.
[4] J. Brazas, The topological fundamental group and free topological groups, Topology Appl. 158 (2011) 779–802.
[5] J. Brazas, The fundamental group as a topological group, Topology Appl. 160 (2013) 170–188.
[6] J. Brazas, Open subgroups of free topological groups, Fundamenta Mathematicae 226 (2014) 17–40.
[7] J. Brazas, P. Fabel, On fundamental groups with the quotient topology, J. Homotopy and Related Structures 10 (2015) 71–91.
[8] J. Calcut, J. McCarthy, Discreteness and homogeneity of the topological fundamental group, Topology Proc. 34 (2009) 339–349.
[9] R.J. Daverman, Decompositions of manifolds, Pure and Applied Mathematics, vol. 124, Academic Press Inc., Orlando, FL, 1986.
[10] K. Eda, K. Kawamura, The fundamental groups of one-dimensional spaces, Topology Appl. 87 (1998) 163–172.
[11] R. Engelking, General topology, Heldermann Verlag Berlin, 1989.
[12] P. Fabel, The topological hawaiian earring group does not embed in the inverse limit of free groups, Algebraic & Geometric Topology 5 (2005) 1585–1587.
[13] P. Fabel, Multiplication is discontinuous in the hawaiian earring group, Bull. Polish Acad. Sci. Math. 59 (2011) 77–83
[14] P. Fabel, Compactely generated quasitopological homotopy groups with discontinuous multiplication, To appear in Topology Proc.
[15] H. Fischer, D. Repovš, Z. Virk, A. Zastrow, On semilocally simply connected spaces, Topology Appl. 158 (2011), no. 3, 397–408.
[16] H. Fischer, A. Zastrow, The fundamental groups of subsets of closed surfaces inject into their first shape groups, Algebraic and Geometric Topology 5 (2005) 1655–1676.
[17] H. Fischer, A. Zastrow, Generalized universal covering spaces and the shape group, Fund. Math. 197 (2007), 167–196.
[18] M.I. Graev, Free topological groups. Amer. Math. Soc. Transl. 8 (1962) 305–365.
[19] D. Harris, Every space is a path component space, Pacific J. Math. 91 (1980) 95–104.
[20] J. Mack, S.A. Morris, E.T. Ordman, Free topological groups and projective dimension of locally compact abelian subgroups. Proc. Amer. Math. Soc. 40 (1973) 303–308.
[21] S. Mardešić, J. Segal, Shape theory, North-Holland Publishing Company, 1982.
[22] A.A. Markov, On free topological groups. Izv. Akad. Nauk. SSSR Ser. Mat. 9 (1945) 3-64 (in Russian); English Transl.: Amer. Math. Soc. Transl. 30 (1950) 11-88; Reprint: Amer. Math. Soc. Transl. 6 (1962) 195–272.
[23] E. Michael, Bi-quotient maps and Cartesian products of quotient maps, Ann. Inst. Fourier, Grenoble. 18, no. 2 (1968) 287–302.
[24] P. Samuel, On universal mappings and free topological groups. Bull. Am. Math. Soc. 54 (1948) 591–598.
[25] O.V. Sipacheva, The Topology of Free Topological Groups. J. Math. Sci. Vol. 131, No. 4, (2005), 5765-5838.
[26] B.V.S Thomas, Free topological groups. General Topology and its Appl. 4 (1974) 51–72.
[27] M.M. Zarichnyi, Free topological groups of absolute neighbourhood retracts and infinite-dimensional manifolds, Soviet Math. Dokl. 26 (1982) 367-371.