On Logic of Formal Provability and Explicit Proofs

Elena Nogina*

BMCC CUNY, Department of Mathematics
199 Chambers Street, New York, NY 10007
E.Nogina@gmail.com

Abstract

In 1933, Gödel considered two modal approaches to describing provability. One captured formal provability and resulted in the logic GL and Solovay’s Completeness Theorem. The other was based on the modal logic S4 and led to Artemov’s Logic of Proofs LP. In this paper, we study introduced by the author logic GLA, which is a fusion of GL and LP in the union of their languages. GLA is supplied with a Kripke-style semantics and the corresponding completeness theorem. Soundness and completeness of GLA with respect to the arithmetical provability semantics is established.

1 Introduction

Gödel in [11] suggested a provability reading of modal logic S4, which is axiomatized over the classical logic by the following list of postulates:

\[ \square (F \rightarrow G) \rightarrow (\square F \rightarrow \square G) \]
\[ \square F \rightarrow \square \square F \]
\[ \square F \rightarrow F \]

*Deductible Closure/Normality
*Positive Introspection/Transitivity
*Reflection

*Supported by PSC CUNY Research Awards program.
and the *Necessitation Rule*: \( \vdash F \Rightarrow \vdash \Box F \).

Gödel considered the interpretation of \( \Box F \) as the formal provability predicate

\[
F \text{ is provable in Peano Arithmetic } PA
\]

and noticed that this semantics is inconsistent with \( S4 \).

Indeed, \( \Box(\Box F \rightarrow F) \) can be derived in \( S4 \). On the other hand, interpreting \( \Box \) as the predicate “Provable” of formal provability in Peano Arithmetic \( PA \) and \( F \) as falsum \( \bot \), converts this formula into the false statement that the consistency of \( PA \) is internally provable in \( PA \):

\[
\text{Provable}(\text{Consis } PA).
\]

### 1.1 Formal provability spills over to non-standard proofs

Let \( \text{Proof}(x, F) \) be a standard proof predicate (cf. [4, 8, 9]) \( x \) is a proof for \( F \); \( \text{Provable } F \) be \( \exists x \text{Proof}(x, F) \).

Peano Arithmetic \( PA \) cannot distinguish between standard and nonstandard numbers; given \( \exists x \text{Proof}(x, F) \), \( x \) may be a nonstandard number, hence not a code of any derivation in \( PA \). It means that \( \text{Provable } F \rightarrow F \) can fail in a model, and hence is not derivable in \( PA \).

Indeed, consider a theory \( T = PA + \text{Provable } \bot \). \( T \) is consistent, since \( PA \) does not prove \( \neg \text{Provable } \bot \). Hence \( T \) has a model \( M \) in which \( \text{Provable } \bot \) holds, but \( \bot \) does not.

So, the formal provability interpretation of \( S4 \) does not work; a provability calculus was left without a semantics and a provability semantics was left without a calculus thus opening two problems:

1. Find a precise provability semantics for \( S4 \);
2. Find a modal logic of formal provability \( \text{Provable} \).

Problem 2 was solved in 1976 by Solovay [24], who proved the completeness of Gödel-Löb logic \( GL \) with respect to the formal provability in arithmetic \( PA \).

In 1995, Problem 1 found its solution in Artemov’s Logic of Proofs \( LP \) which provided a semantics of explicit proofs for \( S4 \) ([2], [3]).
1.2 Gödel-Löb logic of formal provability

Logic of Formal Provability GL (standing for Gödel-Löb) is given by the following list of postulates:

1. Axioms and rules of classical propositional logic

2. $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$ \hspace{1cm} Deductive Closure/Normality

3. $\Box F \rightarrow \Box \Box F$ \hspace{1cm} Verification/Transitivity

4. $\Box(\Box F \rightarrow F) \rightarrow \Box F$ \hspace{1cm} Löb Axiom

5. Necessitation Rule: \[ \vdash F \quad \vdash \Box F \]

Formal provability interpretation of a modal language is a mapping $\ast$ from the set of modal formulas to the set of arithmetical sentences such that $\ast$ agrees with Boolean connectives and constants and

$$(\Box G)^\ast = \text{Provable } G^\ast.$$ 

Solovay's completeness theorem ([8, 24]):

$$\text{GL} \vdash F \text{ iff for all formal provability interpretations } \ast, \text{ PA } \vdash F^\ast.$$ 

In 1938, Gödel outlined a way to provide a provability semantics for S4 ([12]): modality there should be read explicitly as proof assertions $t:F$ interpreted as

$t$ is a proof of $F$ in Peano Arithmetic PA.

This Gödel's suggestion was realized in Artemov's Logic of Proofs ([2, 3]).

1.3 Artemov's Logic of Proofs

Proof terms in LP are built from constants and variables by two binary operations application "·" and sum "+", and one unary operation proof checker "!". Formulas of LP are built as the usual propositional formulas with an additional formation rule: whenever $F$ is a formula and $t$ a proof terms, $t:F$ is a formula.
Axioms and rules of the Logic of Proofs LP are those of classical propositional logic plus axioms
\[
\begin{align*}
\forall s(F \rightarrow G) & \rightarrow (t:F \rightarrow [s \cdot t]:G) & & \text{Application} \\
\forall t:F & \rightarrow !t(t:F) & & \text{Proof Checker} \\
\forall s:F \rightarrow [s + t]:F, t:F \rightarrow [s + t]:F & & \text{Sum} \\
\forall t:F & \rightarrow F & & \text{Explicit Reflection}
\end{align*}
\]

Each axiom \( A \) is assumed internally provable, which is represented by formula \( c:A \) where \( c \) is a proof constant. The fundamental property of LP is given by Artemov’s Realization Theorem ([2, 3]): for each theorem \( F \) of S4 one could recover a witness (proof term) to each occurrence of \( \square \) in \( F \) in such a way that the resulting formula \( F^r \) is derivable in LP. This theorem embeds S4 into LP. Further interpretation of LP proof terms as formal proofs in PA ([2, 3]) provided a Gödelian provability semantics for LP and S4 and completed Gödel’s project of 1933. Nowadays, the Logic of Proofs has evolved into a general logical theory of justification [5, 6, 7].

1.4 Comparing two Gödel approaches to provability

Logic of formal provability GL formalizes Gödel’s second incompleteness theorem
\[
\neg \square (\neg \square \bot),
\]
Löb’s theorem
\[
\square (\square F \rightarrow F) \rightarrow \square F,
\]
and a number of other meaningful provability principles.

Logic of Proofs LP represents proofs explicitly, naturally extends typed \( \lambda \)-calculus, modal logic, and modal \( \lambda \)-calculus.

GL and S4/LP complement each other by addressing different areas of application. GL finds applications in traditional proof theory. LP targets areas of mathematical theories of knowledge and justification, foundations of verification, typed theories and lambda-calculi, etc.

1.5 Mixture of provability and explicit proofs

Certain principles require a mixture of both provability and explicit proofs. Consider the negative introspection principle. Its purely modal formulation
\[ \neg \Box F \rightarrow \Box \neg \Box F \] is not valid as a provability principle. Indeed, let \( F \) be \( \bot \). Then \( \neg \Box \bot \) reads as \( \text{Consis PA} \) and the whole formula as

\[ \text{Consis PA} \rightarrow \text{Provable}(\text{Consis PA}), \]

which is false, by Gödel’s Second Incompleteness Theorem.

There is no explicit negative introspection either. The principle \( \neg p;S \rightarrow t;(\neg p;S) \), where \( p \) and \( t \) are proof terms and \( S \) is a propositional variable, is not valid. Indeed, fix an interpretation \( * \) of \( p \) and \( t \) and the standard Gödel proof predicate. There are infinitely many arithmetical instances of \( S \) for which the antecedent holds. Hence \( t^* \) should be a proof of infinitely many theorems, which is impossible. However, the mixed language of proofs and provability fits this version of negative introspection:

\[ \neg p;F \rightarrow \Box (\neg p;F) \]

is arithmetically provable, by \( \Sigma \)-completeness of \( \text{PA} \), according to which for each \( \Sigma \)-formula \( \sigma \),

\[ \text{PA} \vdash \sigma \rightarrow \text{Provable} \sigma. \]

We develop introduced in \[18\] a joint logic of formal provability and explicit proofs \( \text{GLA} \) (Gödel-Löb-Artëmov logic) in the language with provability assertions \( \Box F \) and proof assertions \( t;F \), find Kripke semantics for \( \text{GLA} \) and establish the arithmetical completeness of this logic.

\( \text{GLA} \) proved to be useful for applications in formal epistemology where it became a template for a family of epistemic logics with justifications (cf. \[6, 7\]). An elaborate proof theory of \( \text{GLA} \) and another version of Kripke models for \( \text{GLA} \) were offered by Kurokawa in \[16, 17\].

2 Description and basic properties of \( \text{GLA} \)

The following two systems are predecessors of \( \text{GLA} \):

- system \( B \) from \[1\], which does not have operations on proofs;

- system \( \text{LPP} \) from \[23, 25\] in an extension of languages of the logic of formal provability \( \text{GL} \) and the Logic of Proofs \( \text{LP} \).

Immediate successors of \( \text{GLA} \) are the logic \( \text{GrzA} \) of strong provability and explicit proofs \[20\], and symmetric logic of proofs and provability \[21\].
Language of GLA.

Proof terms are built from proof variables \( x, y, z, \ldots \) and proof constants \( a, b, c, \ldots \) by means of two binary operations: application ‘‘\( \cdot \)’’ and union ‘‘\( + \)’’, and one unary proof checker ‘‘\(!\)''.

Formulas of GLA are defined by the grammar

\[
A = S \mid A \rightarrow A \mid A \land A \mid A \lor A \mid \neg A \mid \Box A \mid t.A ,
\]

where \( t \) stands for any proof term and \( S \) for any sentence letter.

Axioms and rules of both Gödel-Löb logic \( \mathsf{GL} \) and \( \mathsf{LP} \), together with three specific principles connecting explicit proofs with formal provability, constitute \( \mathsf{GLA} \).

I. Axioms of classical propositional logic

Standard axioms of the classical logic (e.g., A1-A10 from [13])

II. Axioms of Provability Logic \( \mathsf{GL} \)

- \( \mathsf{GL1} \quad \Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G) \)  
  Deductive Closure/Normality
- \( \mathsf{GL2} \quad \Box F \rightarrow \Box \Box F \)  
  Positive Introspection/Transitivity
- \( \mathsf{GL3} \quad \Box(\Box F \rightarrow F) \rightarrow \Box F \)  
  Löb Principle

III. Axioms of the Logic of Proofs \( \mathsf{LP} \)

- \( \mathsf{LP1} \quad s:(F \rightarrow G) \rightarrow (t:F \rightarrow [s \cdot t]:G) \)  
  Application
- \( \mathsf{LP2} \quad t:F \rightarrow !t:(t:F) \)  
  Proof Checker
- \( \mathsf{LP3} \quad s:F \rightarrow [s+t]:F, \ t:F \rightarrow [s+t]:F \)  
  Sum
- \( \mathsf{LP4} \quad t:F \rightarrow F \)  
  Explicit Reflection

IV. Axioms connecting explicit and formal provability

- \( \mathsf{C1} \quad t:F \rightarrow \Box F \)  
  Explicit-Implicit connection
- \( \mathsf{C2} \quad \neg t:F \rightarrow \Box \neg t:F \)  
  Explicit-Implicit Negative Introspection
- \( \mathsf{C3} \quad t: \Box F \rightarrow F \)  
  Explicit-Implicit Reflection

V. Rules of inference

- \( \mathsf{R1} \quad F \rightarrow G, \ F \vdash G \)  
  Modus Ponens
- \( \mathsf{R2} \quad \vdash F \rightarrow \vdash \Box F \)  
  Necessitation
- \( \mathsf{R3} \quad \vdash \Box F \rightarrow \vdash F \)  
  Reflection Rule

A Constant Specification \( \mathsf{CS} \) for GLA is the set of formulas

\[
\{c_1:A_1, c_2:A_2, c_3:A_3, \ldots \},
\]
where each $A_i$ is an axiom of $\text{GLA}_\emptyset$ and each $c_i$ is a proof constant.

$$\text{GLA}_{cs} = \text{GLA}_\emptyset + CS,$$

$$\text{GLA} = \text{GLA}_{cs} \text{ with the "total" } CS.$$

**Theorem 1 (Internalization Theorem).**

If $\text{GLA} \vdash F$ then for some proof term $p$, $\text{GLA} \vdash p:F$.

**Proof.** Induction on a derivation of $F$.

Base: $F$ is an axiom. Then use Constant Specification. In this case, $p$ is a proof constant.

Induction steps: by internalized rules of GLA.

Internalization of *Modus Ponens* immediately follows from the Application axiom LP1.

Internalization of Necessitation rule $\vdash F \Rightarrow \vdash \Box F$:

For each $F$ there is $t(x)$ such that $\text{GLA} \vdash x:F \rightarrow t(x):\Box F$

1. $x:F \rightarrow \Box F$ - axiom Explicit-Implicit Connection C1;
2. $a(x:F \rightarrow \Box F)$ - , from 1, by Constant Specification;
3. $x:F \rightarrow !xx:F$ - axiom Proof Checker LP2;
4. $!xx:F \rightarrow (a!x)\Box F$ - from 2, by Application LP1;
5. $x:F \rightarrow (a!x)\Box F$ - from 3,4, by propositional logic.

Now put $t(x) = a!x$.

Internalization of Reflection rule $\vdash \Box F \Rightarrow \vdash F$:

For each $F$ there is $s(x)$ such that $\text{GLA} \vdash x\Box F \rightarrow s(x):F$

1. $x\Box F \rightarrow F$ - axiom Explicit-Implicit Reflection C3;
2. $b(x\Box F \rightarrow F)$ from 1, by Constant Specification;
3. $x\Box F \rightarrow !xx\Box F$ - Proof Checker LP2;
4. $!xx\Box F \rightarrow (b!x)\Box F$ - from 2, by Application LP1;
5. $x\Box F \rightarrow (b!x)\Box F$ - from 3,4, by propositional logic.

Now put $s(x) = b!x$. Note that in 2, we need an *internalized* Explicit-Implicit Reflection!

\[\square\]

The list of postulated axioms and rules of $\text{GLA}$ contains some principles which are derivable from the rest of the system. Such redundancies are
generally acceptable to make exposition more readable. For example, in GLA (as well as in the Provability Logic GL) the positive introspection axiom \( GL2 \) is derivable from the rest of the system (cf. \[8\]). In GLA the same holds for Reflection Axiom \( LP4 \), Necessitation Rule \( R2 \) and Reflection Rule \( R3 \). In all these cases we decide to postulate the corresponding principles for the sake of more concise definitions of important subsystems of GLA.

Note that for any finite constant specification \( CS \) the rule of necessitation is not redundant in \( GLA_{CS} \) since to emulate \( R2 \) one needs an infinite constant specifications.

Here is an example of a yet more delicate dependency in GLA: even though Explicit-Implicit Reflection Axiom \( C3 \) is derivable from the rest of \( GLA_\emptyset \) (Proposition 1 below), proof constants corresponding to \( C3 \) are needed to guarantee the Internalization Property of GLA (cf. Theorem \[4\]). Hence, we keep \( C3 \) as a basic postulate of GLA.

**Proposition 1** \( C3 \) is derivable from the rest of \( GLA_\emptyset \).

**Proof.** The following is a derivation of \( t: \Box F \rightarrow F \) in \( GLA_\emptyset \) without \( C3 \).

1. \( \neg \Box F \rightarrow \neg t: \Box F \), contrapositive of \( LP4 \);
2. \( \neg t: \Box F \rightarrow \Box (\neg t: \Box F) \), axiom \( C2 \);
3. \( \Box (\neg t: \Box F) \rightarrow \Box (t: \Box F \rightarrow F) \), by reasoning in GL;
4. \( \neg \Box F \rightarrow \Box (t: \Box F \rightarrow F) \), from 1, 2, and 3;
5. \( \Box F \rightarrow \Box (t: \Box F \rightarrow F) \), by reasoning in GL;
6. \( \Box (t: \Box F \rightarrow F) \), from 4 and 5;
7. \( t: \Box F \rightarrow F \), by \( R3 \).

GLA is closed under substitutions of proof terms for proof variables and formulas for propositional variables, enjoys the deduction theorem, and contains both GL and LP.

**2.1 Some principles of GLA**

**Positive Introspection:** \( GLA \vdash t:F \rightarrow \Box t:F \)

1. \( t:F \rightarrow \! t:t:F \) - Proof Checker axiom \( LP2 \);
2. \( \! t:t:F \rightarrow \Box t:F \) - Explicit-Implicit Connection axiom \( C1 \);
3. $t:F \rightarrow \Box t:F$ - from 1,2, by propositional logic.

Stability of proof assertions: $\text{GLA} \vdash \Box t:F \lor \Box \neg t:F$

4. $\neg t:F \rightarrow \Box \neg t:F$ - Explicit-Implicit Negative Introspection $\text{C2}$;

5. $\Box t:F \lor \Box \neg t:F$ - from 3,4, by propositional logic.

Explicit version of L"ob Principle. In $\square(\square F \rightarrow F) \rightarrow \square F$ both modalities of the depth 1 can be read explicitly as

$$x:(\square F \rightarrow F) \rightarrow l(x):F$$

for some proof term $l(x)$. Indeed,

1. $x:(\square F \rightarrow F) \rightarrow t(x):(\square F \rightarrow F)$ - by Internalized Necessitation Rule;
2. $c:(\square F \rightarrow F) \rightarrow \square F$ - from L"ob Principle $\text{GL3}$ by Constant Specification;
3. $t(x):(\square F \rightarrow F) \rightarrow (c \cdot t(x)):(\square F)$ - from 1,2 by Application $\text{LP1}$;
4. $(c \cdot t(x)):(\square F \rightarrow s(c \cdot t(x)):F$ - by Internalized Reflection Rule;
5. $x:(\square F \rightarrow F) \rightarrow s(c \cdot t(x)):F$ - from 1,3,4.

L"ob Principle cannot be realized in full. Suppose for some proof polynomsials $u$ and $v$, $\text{GLA} \vdash x:(u:\perp \rightarrow \perp) \rightarrow v:\perp$,

hence $\text{GLA} \vdash x:(u:\perp \rightarrow \perp) \rightarrow \perp$ and so $F = \neg x:(u:\perp \rightarrow \perp)$ is derivable in $\text{GLA}$.

Consider a $\text{GLA}$-derivable formula $G = c(u:\perp \rightarrow \perp)$.

Let us perform a substitution $\tau = [c/x]$ to both $F$ and $G$. Then $F$ becomes $\neg c(\tau u;\perp \rightarrow \perp)$ and $G$ yields $c(\tau u;\perp \rightarrow \perp)$, which is impossible.

2.2 Realizable provability principles.

A Franco Montagna’s question which theorems of $\text{GL}$ are realizable in $\text{GLA}$, has been answered by Evan Goris in [13, 14].

It follows from the realization theorem for $\text{LP}$ that all formulas of $\text{GL} \cap \text{S4}$ are realizable in $\text{LP}$, and the question was actually whether proof terms of $\text{GLA}$ were capable of realizing some other modal theorems of $\text{GL}$. Goris’ Theorem yields that it is not the case.
Theorem \[13, 14\]. Only those theorems of GL are realizable in GLA which are from S4.

3 Models for GLA

In this section, we build Kripke-style models for GLA, which were described in \[19\].

A frame is a standard GL-frame \((W, \prec, \text{root})\) with the root node \(\text{root}\), where \(W\) is a non-empty set of possible worlds, \(\prec\) is a binary transitive and conversely well-founded accessibility relation on \(W\) (a relation \(\prec\) is conversely well-founded if any increasing chain \(a_1 \prec a_2 \prec a_3 \prec \ldots\) is finite).

Possible evidence relation (first considered by Mkrtychev and then by Fitting) is a relation \(\mathcal{E}\) between proof terms and formulas such that the following closure conditions are met:

1. **Application**: \(\mathcal{E}(s, F \rightarrow G)\) and \(\mathcal{E}(t, F)\) implies \(\mathcal{E}(s \cdot t, G)\).
2. **Proof Checker**: \(\mathcal{E}(t, F)\) implies \(\mathcal{E}(!t, (t:F))\).
3. **Sum**: \(\mathcal{E}(s, F)\) or \(\mathcal{E}(t, F)\) implies \(\mathcal{E}(s + t, F)\).

Model is a structure \(M = (W, \prec, \text{root}, \mathcal{E}, \models)\); here \(\models\) is a relation between worlds and formulas such that

1. \(\models\) respects Boolean connectives at each world
   \((u \models F \land G\) iff \(u \models F\) and \(u \models G\); \(u \models \neg F\) iff \(u \not\models F\), etc.);
2. \(u \models \Box F\) iff \(v \models F\) for every \(v \in W\) with \(u \prec v\);
3. \(u \models t:F\) iff \(\mathcal{E}(t, F)\) and \(v \models F\) for every \(v \in W\).

Following Solovay, we define

\[
\mathcal{H}(F) = \{ \Box F \rightarrow G \mid \Box F \text{ is a subformula of } F \};
\]

for a set of formulas \(X\),

\[
\mathcal{H}(X) = \bigcup_{F \in X} \mathcal{H}(F).
\]

A model \(M\) is called \(F\)-sound if \(\text{root} \models \mathcal{H}(F)\). For a set of formulas \(X\), \(M\) is \(X\)-sound if \(M\) is \(F\)-sound for each \(F \in X\).
For a given constant specification $CS$, a model $M$ is a $CS$-model if $M$ is $CS$-sound and $CS$ holds in $M$.

**Theorem 2 (Soundness)** For any formula $F$ and any constant specification $CS$, if $F$ is derivable in $GLA_{CS}$ then $F$ holds in each $F$-sound $CS$-model.

**Theorem 3 (Completeness)** For any finite constant specification $CS$ if $F$ is not derivable in $GLA_{CS}$, then there is an $F$-sound $CS$-model with a finite frame where $F$ does not hold.

Proof goes by a canonical model construction with the use of technique developed by Solovay [24], Artemov [1], and Fitting [10]. $GLA_{\emptyset}$ exhibits some sort of a finite model property, which also yields the decidability of $GLA_{CS}$ for any given finite constant specification:

**Theorem 4** For any finite constant specification $CS$, the logic $GLA_{CS}$ is decidable.

### 4 Provable semantics for $GLA$, completeness

In what follows, all proof predicates are assumed normal (3), i.e., satisfying two properties.

1. Finiteness of proofs.
   For every $k$ set $T(k) = \{ \varphi \mid \text{Proof}(k, \varphi) \}$ is finite, the function from $k$ to $T(k)$ is computable.

2. Conjoinability of proofs.
   For any $k$ and $l$ there is $n$ such that
   $$T(k) \cup T(l) \subseteq T(n).$$

Prime example: Gödel’s proof predicate.

**Arithmetical interpretation** of $GLA$ is the sum of the intended arithmetical interpretations for $GL$ and $LP$. In particular,

$$\square G^* = \text{Provable } G^*;$$

$$(pF)^* = \text{Proof}(p^*, F^*).$$
Theorem 5 (Soundness of GLA with respect to arithmetical provability)
For any Constant Specification CS and any arithmetical interpretation ∗ respecting CS, if GLA_{CS} ⊢ F then PA ⊢ F∗.

Proof. It is immediate that Reflection Rule is valid: if Provable F is derivable in PA, then Provable F is true hence F is provable.
Validity of C1 and C2 immediately follows from Σ-completeness of PA.
Soundness of Explicit-Implicit Reflection takes place since \( t: \Box F \rightarrow F \) is derivable from other principles of GLA, which is already proved sound.

Arithmetical completeness of GLA_{∅} could be established following arithmetical completeness proofs from [1, 2, 3] (cf. also [25]).

Theorem 6 (Arithmetic completeness) For any finite constant specification CS, if GLA_{CS} \not\vdash F, then there exists a CS-interpretation * such that PA \not\vdash F∗.

Proof. The claim of the theorem follows from the arithmetical completeness of GLA_{∅}.

4.1 Explicit-Implicit Reflection vs. Implicit-Explicit Reflection
Explicit-Implicit Reflection \( x: \Box F \rightarrow F \), as we have seen in Theorem 5, is arithmetically valid. However, the Implicit-Explicit Reflection

\[ IER = \Box x: P \rightarrow P \]

is not a provable principle.

1. A proof via GLA.
   It suffices to establish that IER is not derivable in GLA_{∅}. For this we will use an appropriate Kripke model. Take

   \[ W = \{1, 2\}, 1 \prec 2, P \text{ is false at 1 and 2, } \mathcal{E}(t, F) \text{ is always false.} \]

   \[ 2 \quad \neg P, \neg x: P, \Box x: P, \neg (\Box x: P \rightarrow P) \quad (\text{i.e., } \neg IER) \]

   \[ \uparrow \]

   \[ 1 \quad \neg P, \neg x: P, \neg \Box x: P, \Box x: P \rightarrow x: P \quad (IER\text{-soundness}) \]
Therefore, IER is false at node 2 of the model.

2. An arithmetical proof.

If $P = \bot$, then $x:P$ is provably equivalent to $\bot$. Therefore, this instance of IER is equivalent to $\square \bot \rightarrow \bot$, which is the consistency statement, not provable in PA.

For other reflection principles of PA see our paper [22].

5 Acknowledgements

The author is grateful to Sergei Artemov, Melvin Fitting, Evan Goris, Gerhard Jäger, Makoto Kikuchi, Taishi Kurahashi, Hidenori Kurokawa, Franco Montagna, Anil Nerode, Thomas Strahm, Thomas Studer, Tatiana Yavorskaya, Junhua Yu, Ren-June Wang, logic groups in Bern University, Nihon University in Tokyo, Kobe University, Academia Sinica and National Chung Cheng University of Taiwan for useful discussions.

References

[1] S. Artemov. Logic of proofs. *Annals of Pure and Applied Logic*, 67(1):29–59, 1994.

[2] S. Artemov. *Operational modal logic*. Technical Report MSI 95-29, Cornell University, 1995.

[3] S. Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, 2001.

[4] S. Artemov and L. Beklemishev. Provability logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2nd ed.*, volume 13, pages 189–360. Springer, Dordrecht, 2005.

[5] S. Artemov and M. Fitting. Justification Logic. *Stanford Encyclopedia of Philosophy*, 2011.

[6] S. Artemov and E. Nogina. On epistemic logic with justification. In R. van der Meyden, editor, *Theoretical Aspects of Rationality and Knowledge. Proceedings of the Tenth Conference (TARK 2005), June 10-12, 2005, Singapore.*, pages 279–294. National University of Singapore, 2005.
[7] S. Artemov and E. Nogina. Introducing justification into epistemic logic. *Journal of Logic and Computation*, 15(6):1059–1073, 2005.

[8] G. Boolos. *The Logic of Provability*. Cambridge University Press, Cambridge, 1993.

[9] S. Feferman. Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae*, 49:35–92, 1960.

[10] M. Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, 2005.

[11] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse Math. Kolloq.*, 4:39–40, 1933. English translation in: S. Feferman et al., editors, *Kurt Gödel Collected Works*, Vol. 1, pages 301–303. Oxford University Press, Oxford, Clarendon Press, New York, 1986.

[12] K. Gödel. Vortrag bei Zilsel/Lecture at Zilsel’s (*1938a). In Solomon Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons, and Robert M. Solovay, editors, *Unpublished essays and lectures*, volume III of *Kurt Gödel Collected Works*, pages 86–113. Oxford University Press, 1995.

[13] E. Goris. Explicit proofs in formal provability logic. In *Logical Foundations of Computer Science ’07*, Lecture Notes in Computer Science, v. 4514, pp. 241–253, Springer, 2007.

[14] E. Goris. A modal provability logic of explicit and implicit proofs. *Annals of Pure and Applied Logic*, 161(3):388–403, 2009.

[15] S. Kleene. *Introduction to Metamathematics*. Van Norstrand, 1952.

[16] H. Kurokawa. *Tableaux and Hypersequents for Modal and Justification Logics*. City University of New York, 2012.

[17] H. Kurokawa. Prefixed Tableau Systems for Logic of Proofs and Provability. *Automated Reasoning with Analytic Tableaux and Related Methods*. Springer Berlin Heidelberg, pp. 203–218, 2013.

[18] E. Nogina. On logic of proofs and provability. *Bulletin of Symbolic Logic*, 12(2):356, 2006.
[19] E. Nogina. Epistemic completeness of GLA. *Bulletin of Symbolic Logic*, 13(3):407, 2007.

[20] E. Nogina. Logic of Strong Provability and Explicit Proofs. *Bulletin of Symbolic Logic*, 15(1):124–125, 2009.

[21] E. Nogina. Symmetric Logic of Proofs and Provability. 2010 Spring AMS Eastern Sectional Meeting May 22-23, 2010 New Jersey Institute of Technology, Newark, NJ, 2010. [http://www.ams.org/meetings/sectional/1060-03-29.pdf](http://www.ams.org/meetings/sectional/1060-03-29.pdf)

[22] E. Nogina. On a Hierarchy of Reflection Principles in Peano Arithmetic. arXiv preprint, 2014.

[23] T. Sidon. Provability logic with operations on proofs. In S. Adian and A. Nerode, editors, *Logical Foundations of Computer Science’ 97, Yaroslavl*, volume 1234 of *Lecture Notes in Computer Science*, pages 342–353. Springer, 1997.

[24] R.M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 28:33–71, 1976.

[25] T. Yavorskaya (Sidon). Logic of proofs and provability. *Annals of Pure and Applied Logic*, 113(1-3):345–372, 2002.