Action functional for $\kappa$-Minkowski Noncommutative Spacetime

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We examine some alternative possibilities for an action functional for $\kappa$-Minkowski noncommutative spacetime, with an approach which should be applicable to other spacetimes with coordinate-dependent commutators of the spacetime coordinates ($[x_\mu, x_\nu] = f_{\mu,\nu}(x)$). Early works on $\kappa$-Minkowski focused on $\kappa$-Poincaré covariance and the dependence of the action functional on the choice of Weyl map, renouncing to invariance under cyclic permutations of the factors composing the argument of the action functional. A recent paper (hep-th/0307149), by Dimitrijevic, Jonke, Moller, Tsouchnika, Wess and Wohlgenannt, focused on a specific choice of Weyl map and, setting aside the issue of $\kappa$-Poincaré covariance of the action functional, introduced in implicit form a cyclicity-inducing measure. We provide an explicit formula for (and derivation of) a choice of measure which indeed ensures cyclicity of the action functional, and we show that the same choice of measure is applicable to all the most used choices of Weyl map. We find that this “cyclicity-inducing measure” is not covariant under $\kappa$-Poincaré transformations. We also notice that the cyclicity-inducing measure can be straightforwardly derived using a map which connects the $\kappa$-Minkowski spacetime coordinates and the spacetime coordinates of a “canonical” noncommutative spacetime, with coordinate-independent commutators.
I. INTRODUCTION

A sizeable literature has been devoted to noncommutative versions of Minkowski spacetime. Certain types of arguments [1] for an “uncertainty principle for localization” lead one to consider the simplest such spacetimes, the “canonical noncommutative spacetimes”

\[ [x_\mu, x_\nu] = i\delta_{\mu,\nu} \, . \]  

(I.1)

Other possible forms of the uncertainty principle for localization may lead [2] to a “Lie-algebra” form of noncommutativity:

\[ [x_\mu, x_\nu] = i\zeta^\sigma_{\mu,\nu} x_\sigma \, . \]  

(I.2)

(Both \( \theta_{\mu,\nu} \) and \( \zeta^\sigma_{\mu,\nu} \) are coordinate-independent.)

It is emerging that the canonical spacetimes may play a role [3] in an effective-theory description of the physics of strings in presence of a spacetime-coordinate-independent \( B \)-tensor external background. For appropriate choice of spacetime dependence of the \( B \)-tensor external background one can instead obtain [4,5] a description in terms of a Lie-algebra spacetime. Recent results [6,7] suggest that Lie-algebra spacetimes may also play a role in the description of some formulations of Loop Quantum Gravity.

These results have motivated a strong interest in the construction of field theories in noncommutative spacetimes. For the case of the simple canonical spacetimes there are already some field-theory proposals for which a rather advanced level of development has been achieved (see, e.g., Refs. [3]). But for the case of Lie-algebra spacetimes (and in general of spacetimes with coordinate-dependent commutators of the spacetime coordinates \( [x_\mu, x_\nu] = f_{\mu,\nu}(x) \)) several additional difficulties are encountered [8–10] in the attempts to construct field theories. The difficulties start already at the level of constructing an action functional, which should be a map from functions of the noncommutative spacetime coordinates to the complex numbers. It appears that one should renounce to some familiar properties that the action functional enjoys in commutative spacetimes, and it is difficult to choose which properties should be maintained and which one should be given up.

These issues concerning the action functional for Lie-algebra noncommutative spacetimes have been most extensively considered for the case of the “\( \kappa \)-Minkowski” noncommutative spacetime” [11,12], whose coordinates satisfy the commutation relations

\[ [x_j, x_0] = i\lambda x_j \, , \quad [x_j, x_k] = 0 \]  

(I.3)

where \( j, k = 1, 2, 3 \) and we denote the dimensionful noncommutativity parameter by \( \lambda \) (other \( \kappa \)-Minkowski studies introduce the parameter \( \kappa \), which is related to the \( \lambda \) of (I.3) by \( \lambda = \kappa^{-1} \)). In most early works on \( \kappa \)-Minkowski the action functional was structured in such a way to reflect fully the underlying \( \kappa \)-Poincaré invariance \([8–10,13]\), even allowing for a possible dependence on the choice of Weyl map (a choice which affects the description of the \( \kappa \)-Poincaré transformations). However, this invariance criterion was found to lead to action functionals which do not enjoy invariance under cyclic permutations of the fields

\[ \int f_1(x)f_2(x)\ldots f_{n-1}(x)f_n(x) \neq \int f_n(x)f_1(x)f_2(x)\ldots f_{n-1}(x) \, . \]  

(I.4)

Without cyclicity of the action functional many familiar features of field theory are immediately lost and some apparently unsurmountable difficulties are encountered. In the recent Ref. [14] Dimitrijevic, Jonke, Moller, Tsouchnika, Wess and Wohlgenannt, while focusing on a specific choice of Weyl map, considered a possible cyclic action functional. The corresponding integration measure was not described explicitly in Ref. [14], and its possible dependence on the choice of Weyl map was not considered. Moreover, the issue of \( \kappa \)-Poincaré invariance of the action functional, which had been central in previous works, was not explored in Ref. [14].

Here we provide explicitly a choice of measure which indeed ensures cyclicity of the action functional, and we find that the same choice of measure is applicable to all the most common choices of Weyl map. We show that this choice of measure can be derived constructively using only \( \kappa \)-Minkowski properties, but it can also be derived more simply by exploiting a map which exists between \( \kappa \)-Minkowski spacetime coordinates and the spacetime coordinates of a canonical spacetime. We observe that this “cyclicity-inducing measure” is not invariant under \( \kappa \)-Poincaré transformations, but there appears to be room for attempting to construct with such a measure a \( \kappa \)-Poincaré invariant theory.

We start, in the next section, by reviewing some of the main properties of \( \kappa \)-Minkowski noncommutative spacetime and of some possible choices of Weyl maps, which can be used to introduce a correspondence between functions in \( \kappa \)-Minkowski and commutative functions of the \( \lambda \rightarrow 0 \) limit. In Section III we introduce the problem of finding a cyclic
action functional and we construct explicitly a measure which ensures cyclicity. We also show that the same choice of measure is compatible with all the Weyl maps considered in Section II. In Section IV we examine the invariance properties of the action functional. In Section V we introduce the concept of a measure induced in $\kappa$-Minkowski via a map which exists between $\kappa$-Minkowski spacetime coordinates and the spacetime coordinates of a canonical spacetime, and we show that our cyclicity-inducing measure can be obtained in that way. Finally in Section VI we comment on our results and on the outlook of this research programme.

II. $\kappa$-MINKOWSKI STAR PRODUCTS

A. Preliminaries on star products and Weyl maps

The natural framework for classical mechanics is a smooth manifold $M$ equipped with a Poisson bracket $\{,\}$. In quantum mechanics the commutative algebra $C^\infty(M)$ of smooth real-valued functions on the manifold is replaced by a noncommutative $C^*$-algebra $A$, and the Lie bracket $[,]$, given by the commutator of two elements of the algebra, replaces the classical Poisson bracket.

A quantization is a one-to-one correspondence $\Omega : C^\infty(M) \rightarrow A$ such that

$$[\Omega(f), \Omega(g)] = i\hbar \Omega(\{f, g\}) + o(\hbar) \quad \forall f, g \in C^\infty(M)$$

This formulation [15,16] of the quantization problem, is called deformation quantization [17], and the role of deformation parameter is played by the Planck constant $\hbar$.

The Weyl map [18] $\Omega$ establishes an isomorphism between (a suitable subalgebra of) $A$ and $C^\infty(M)$, if we equip the latter with a deformed product, the star product ($\ast$-product or generalized Moyal-Weyl product [19]), implicitly defined by the equality

$$\Omega(f \ast g) = \Omega(f) \cdot \Omega(g) \quad \text{(II.1)}$$

In the study of noncommutative spacetimes an approach based on the star product is widely used. It has proven very fruitful for the construction of theories in canonical noncommutative spacetime, and it is expected that it should be also useful in the analysis of Lie-algebra spacetimes, such as $\kappa$-Minkowski. In Ref. [20] a construction of star products for a generic noncommutative spacetime was presented, generalizing the Moyal-Weyl procedure with which the star product of canonical spacetimes is obtained. An even more general procedure, within an analysis that focused on $\kappa$-Minkowski, was discussed in Ref. [21].

We find useful to limit our analysis to three alternative choices of star product for $\kappa$-Minkowski, so that we can explore the possible Weyl-map dependence of the results, while keeping a bearable level of complexity of the discussion.

B. Time-to-the-right star product

The first star product we consider is the “time-to-the-right” star product, $\ast_R$, which has been widely used in the $\kappa$-Minkowski literature (see for example [11,9,22–24]). The corresponding Weyl map $\Omega_R$ can be introduced on the basis of

$$\Omega_R(f) = \frac{1}{(2\pi)^2} \int d^4 k \tilde{f}(k) e^{ik_j x_j} e^{-ik_0 x_0} , \quad \text{(II.2)}$$

where $\tilde{f}$ is the Fourier transform of the commutative function $f(x)$. This map is “time-to-the-right” in the sense that

$$\Omega_R(e^{ik_j x_j - ik_0 x_0}) = e^{ik_j x_j} e^{-ik_0 x_0} .$$

The star product $\ast_R$ associated with this Weyl map must of course satisfy

$$(f \ast_R g)(x) = \Omega_R^{-1}(\Omega_R(f)\Omega_R(g)) \quad \text{(II.3)}$$

Using the identity

$$e^{ik_j x_j} e^{-ik_0 x_0} e^{ip_j x_j} e^{-ip_0 x_0} = \Omega_R(e^{i\gamma_R(k,p)x_\mu}) \quad \text{(II.4)}$$

where
\[
\gamma_R^\mu(k, p) = (k^0 + p^0, k^i + e^{-\lambda k_0} p^i)
\]  

(II.5)

it is easy to see that the \(*_R^\mu\) product for exponential functions is \(^1\):

\[
e^{ikx} \ast_R e^{ipx} = \Omega_R^{-1}\left(\Omega_R(e^{ikx})\Omega_R(e^{ipx})\right) = \Omega_R^{-1}\left(e^{ikx}e^{-ik0x^0}e^{ip0x}e^{-ip0x^0}\right)
\]

\[
= \Omega_R^{-1}\left(\Omega_R(e^{i\gamma_R^\mu(k, p)x_\mu})\right) = e^{i\gamma_R^\mu(k, p)x_\mu}.
\]

(II.6)

Having specified the \(*_R\) product for exponential functions one of course obtains the \(*_R\) product for generic functions through (II.2).

\section{C. Time-Symmetrized Star Product}

As announced, we intend to explore the possible dependence of the results on the choice of star product by considering a total of three star products. The second in our list, the “time-symmetrized” star product \(*_T\), was first introduced\(^2\) in Ref. [21]. It is based on the Weyl map \(\Omega_T\), introduced through

\[
\Omega_T(f) = \frac{1}{(2\pi)^2} \int d^4k \tilde{f}(k)e^{-i\frac{k0x0}{\pi}}e^{ikx}e^{-i\frac{p0x0}{\pi}}
\]

(II.7)
in which again \(\tilde{f}\) is the Fourier transform of the commutative function \(f(x)\).

The star product \(*_T\) is of course such that

\[
(f \ast_T g)(x) = \Omega_T^{-1}\left(\Omega_T(f)\Omega_T(g)\right),
\]

(II.8)

and in particular for exponentials one finds

\[
\left(e^{ikx} \ast_T e^{ipx}\right) = \Omega_T^{-1}\left(\Omega_T(e^{ikx})\Omega_T(e^{ipx})\right) = \Omega_T^{-1}\left(\Omega_R\left(e^{-ik0x^0}e^{-\lambda k0/\pi}e^{ip0x}e^{-\lambda p0/\pi}\right)\right)
\]

\[
= \Omega_T^{-1}\left(\Omega_R\left(e^{i\gamma_R^\mu(k0, k0)x0}e^{i\gamma_R^\mu(p0, p0)x0}\right)\right) = e^{i\gamma_T^\mu(k0, k0)x0}.
\]

(II.9)

Thus one obtains

\[
\gamma_T^\mu(k, p) = \gamma_R^\mu(k0, k0)e^{\lambda p0/\pi}e^{\lambda k0/\pi} = (k0 + p0, k^i + e^{\lambda k0/\pi} e^{\lambda p0/\pi}),
\]

(II.10)

which also exposes the simple four-momenta transformation that relates the \(\Omega_T\) Weyl map and the \(\Omega_R\) Weyl map:

\[
\Omega_T(e^{ipx}) = \Omega_R\left(e^{-ip0x^0}e^{-\lambda p0/\pi}\right).
\]

(II.11)

\(^1\)Here we are using four-dimensional Greek indices \((\mu, \nu = 0, \ldots, 3)\) and three-dimensional Latin indices \((i, j = 1, \ldots, 3)\). The short notations e. g. \(kx\) stand for the contracted forms \(k0x0\) with the \((-++, ++, +-)\) signature.

\(^2\)Note that in earlier studies comparing the time-symmetrized star product and the time-to-the-right star product, such as the one of Ref. [10], the index “\(S\)” rather than “\(T\)” was associated with the time-symmetrized star product. Since we are here also considering a fully symmetric star product (see later) we reserve the index “\(S\)” for that choice.
D. Symmetric Star Product

The third example of star product which we intend to consider, the symmetric star product $\ast_S$, is the one adopted in Refs. [14,8,20], and can be introduced in association with the Weyl map $\Omega_S$: 

$$\Omega_S(f) = \frac{1}{(2\pi)^2} \int d^4k \, \hat{f}(k) e^{ikx}.$$  

The symmetric star product $\ast_S$ is such that 

$$(f \ast_S g)(x) = \Omega_S^{-1}(\Omega_S(f) \Omega_S(g)).$$  

(II.12)

Using the Baker–Campbell–Hausdorff (BCH) formula for the product of exponentials of noncommuting quantities, one can easily verify the identity 

$$e^{i k^\mu x_{\mu}} e^{i p^\nu x_{\nu}} = e^{i \gamma^\mu_S(k,p)x_{\mu}}$$  

(II.13)

where 

$$\gamma^\mu_S(k,p) = \left( k^0 + p^0, \frac{\phi(k^0) e^{i k^0 j} + \phi(p^0) p^j}{\phi(k^0 + p^0)} \right)$$  

(II.14)

with the function $\phi(a)$ defined by $\phi(a) = \frac{1}{a^\lambda} (e^{a^\lambda} - 1)$.

III. CYCLIC ACTION FUNCTIONAL FOR $\kappa$-Minkowski

In order to construct a field theory one needs a linear functional, which will be used to introduce an action functional. In the commutative case the action functional is a map $I$ 

$$I : C^\infty(\mathbb{R}^4) \to \mathbb{C},$$  

(III.1)

and the familiar field theories in commutative spacetime are based on the natural choice of action functional that coincides with the ordinary integral: 

$$I(f) = \int d^4 x f(x)$$  

(III.2)

In the case of $\kappa$-Minkowski the generalization of the action functional will be given by a linear map $I$: 

$$I : \kappa\text{-Minkowski} \to \mathbb{C}.$$  

Since any element $\hat{f}(x)$ of $\kappa$-Minkowski can be written in terms of an invertible Weyl map $\Omega$, 

$$\hat{f}(x) = \Omega(f(x)),$$  

a natural generalization of (III.2) would be 

$$I(\Omega(f)) = \int d^4 x \mu(x) \Omega^{-1}(\hat{f})(x) = \int d^4 x \mu(x) f(x)$$  

(III.3)

where $\mu(x)$ is an appropriate integration measure.

In most of the works using the Weyl maps discussed in the previous section the simple choice $\mu(x) = 1$ is adopted: 

$$I_1(\hat{f}) = \int d^4 x \hat{f}(x) \equiv \int d^4 x f(x),$$  

(III.4)

where $\hat{f}(x) = \Omega_R(f(x))$ in the cases in which one adopts the time-to-the-right star product, $\hat{f}(x) = \Omega_T(f(x))$ for the time-symmetrized star product, and $\hat{f}(x) = \Omega_S(f(x))$ in the cases with the symmetric star product.
The key objection to this choice of the trivial integration measure, \( \mu(x) = 1 \), is that the resulting integral does not satisfy the cyclic property:

\[
I_1(\hat{f}\hat{g}) \neq I_1(\hat{g}\hat{f}) \tag{III.5}
\]

By renouncing to the cyclicity of the action functional one loses a large number of familiar properties of a field theory, and it is perhaps for this reason that the development of field theories in \( \kappa \)-Minkowski spacetime has not been very successful so far.

It is easy to see what should be required of the measure \( \mu(x) \) in order to achieve cyclicity. The requirement can be naturally expressed in terms of the “star commutator” \([f, g]_* \equiv f \star g - g \star f\). In fact, using the prescription (III.3) and the star-product definition (II.1), the action functional for the product of two functions \( \hat{f}, \hat{g} \) of \( \kappa \)-Minkowski is

\[
I(\hat{f}\hat{g}) = \int d^4x \mu(x) \Omega^{-1}(\hat{f}\hat{g})(x) = \int d^4x \mu(x) \Omega^{-1}(\Omega(\hat{f} \star \Omega g))(x)
\]

and therefore the cyclicity condition is

\[
I([\hat{f}, \hat{g}]) = \int d^4x \mu(x) [f, g]_* = 0 \tag{III.7}
\]

A “cyclicity-inducing measure” should satisfy the requirement (III.7). Since the star commutator depends on the particular choice of star product, we should contemplate the possibility that the cyclicity-inducing measure, if it exists, may also depend on the choice of star product.

In Ref. [14] it was claimed that a cyclicity-inducing measure \( \mu_S \) for the symmetric star product \( \star_S \) should exist, and that at first order in \( \lambda \) this measure should satisfy the requirement \( \tilde{\nabla}(\vec{x} \mu_S(\vec{x})) = 0 \). We will give an explicit exact (valid at all orders in \( \lambda \)) expression of a cyclicity-inducing measure, and show that the same choice of measure is applicable to all three examples of star product we are considering. In the later sections we will also show that, while the measure can take a form that is clearly invariant under space rotations, it is not possible to achieve the desired \( \kappa \)-Poincaré invariance. And we will observe that the existence of a cyclicity-inducing measure could be shown straightforwardly exploiting the fact that there is a map between \( \kappa \)-Minkowski spacetime coordinates and the spacetime coordinates of a canonical spacetime.

A. Cyclic action functional for the time-to-the-right star product

We start by deriving the cyclicity-inducing measure \( \mu_R(x) \) for the case of the time-to-the-right star product. The derivation is actually applicable to the general case of a \( D+1 \)-dimensional \( \kappa \)-Minkowski spacetime, and therefore in the following we adopt conventions such that vectors \( \vec{x} \) are \( D \)-dimensional and latin index \( j, k \) take values in \( \{1, \ldots, D\} \).

The cyclicity condition (III.7) must be satisfied for any \( f \) and \( g \), and, since the functions \( x^N = x_0, \ldots, x_D \) form a basis, it is sufficient for a cyclic action functional to satisfy the following conditions

\[
\int d^{D+1}x \mu_R(x)[x_j^n, g(x)]_{\star_R} = 0 \quad \int d^{D+1}x \mu_R(x)[x_0^n, g(x)]_{\star_R} = 0 \tag{III.8}
\]

for any natural number \( n \).

The \( \star \)-commutators can be analyzed considering the function \( g(x) = e^{ipx} \) and then extending the result to any function by linearity, using the Fourier transform. Noticing that

\[
x_j \star_R e^{ipx} = \lim_{q \to 0} (-i\partial_q e^{ipx} \star_R e^{ipx}) = \lim_{q \to 0} (-i\partial_q e^{i\gamma_R(q,p)x})
\]

\[
x_0 \star_R e^{ipx} = \lim_{q \to 0} (i\partial_q e^{ipx} \star_R e^{ipx}) = \lim_{q \to 0} (i\partial_q e^{i\gamma_R(q,p)x})
\]

\(^3\)We are assuming that the product \( x^N \star g(x) \) is integrable.
one finds that the \([x_j^n, e^{ipx}]_{*R}\) and \([x_0^n, e^{ipx}]_{*R}\) star commutators can be written in differential form as follows:

\[
[x_j^n, e^{ipx}]_{*R} = (-i)^n \lim_{q \to 0} \partial^n_q (e^{i\gamma_R(q,p)x} - e^{i\gamma_R(p,q)x}) \\
[x_0^n, e^{ipx}]_{*R} = (i)^n \lim_{q \to 0} \partial^n_q (e^{i\gamma_R(q,p)x} - e^{i\gamma_R(p,q)x})
\]

Using the explicit expression (II.5) of \(\gamma_R(p, q)\) one then obtains

\[
[x_j^n, e^{ipx}]_{*R} = [1 - e^{-in\lambda p_0}(-i\partial_0)^n] e^{ipx} \\
[x_0^n, e^{ipx}]_{*R} = \{( -i\partial_{p_0} + i\lambda p_1 \partial_{p_1})^n - ( -i\partial_{p_0})^n\} e^{ipx}
\]

and therefore the commutators in (III.8) can be written as

\[
[x_j^n, g(x)]_{*R} = x_j^n (1 - e^{i\lambda \partial_0}) g(x) \tag{III.10} \\
x_0^n, g(x)]_{*R} = \{( t + i\lambda x_j \partial_{x_j} )^n - t^n\} g(x) \tag{III.11}
\]

Substituting these expressions in (III.8) and integrating by parts we obtain the following differential equations for the cyclicity-inducing measure \(\mu_R(x)\)

\[
(1 - e^{-in\lambda \partial_0})\mu_R(x) = 0 \tag{III.12} \\
\left\{ ( t + i\lambda \nabla : \vec{x} )^n - t^n \right\} \mu_R(\vec{x}) = 0 \quad \forall \ n \geq 1 \tag{III.13}
\]

Eq. (III.12) implies that \(\mu_R(\vec{x})\) does not depend on the variable \(t\), while (III.13) gives rise to the following series of equations:

\[
\nabla (\vec{x} \mu_R(\vec{x})) = 0 \\
\lambda^2 \nabla \vec{x} \nabla \vec{x} \mu_R(\vec{x}) + 2t \lambda \nabla (\vec{x} \mu_R(\vec{x})) = 0 \\
i\lambda^3 \nabla \vec{x} \nabla \vec{x} \nabla \vec{x} \mu_R(\vec{x}) - 3\lambda^2 t \nabla \vec{x} \nabla \vec{x} \mu_R(\vec{x}) - 3i\lambda^2 \nabla (\vec{x} \mu_R(\vec{x})) = 0 \\
\ldots = 0
\]

A cyclicity-inducing measure will be obtained only if all the equations of the series are satisfied, and this happens when \(\mu_R(\vec{x})\) is such that

\[
\nabla (\vec{x} \mu_R(\vec{x})) = 0. \tag{III.14}
\]

This equation (III.14) can be written in the form

\[
\vec{x} \nabla \mu_R(\vec{x}) = -D \mu_R(\vec{x}), \tag{III.15}
\]

which, if we search for a measure that preserves space-rotational invariance \((\mu = \mu(|\vec{x}|))\), is equivalent to

\[
|\vec{x}| \frac{\partial}{\partial |\vec{x}|} \mu_R(|\vec{x}|) = -D \mu_R(|\vec{x}|). \tag{III.16}
\]

The only solution is, up to a multiplicative constant,

\[
\mu_R(\vec{x}) = |\vec{x}|^{-D}. \tag{III.16}
\]

We therefore can formulate an action functional which enjoys the sought cyclicity\(^4\):

\[
\mathcal{I}(\Omega_R(f)) = \int \frac{1}{|\vec{x}|^{D+1}} f(x) d^{D+1}x \tag{III.17}
\]

\(^4\)Clearly this choice of measure is only acceptable when the function \(f(t, \vec{x})\) is such that \(f(t, 0) = 0\). The reader can however easily verify that the measure can be extended to the case \(f(t, 0) \neq 0\). We shall not dwell on this point since, for independent reasons, one naturally considers [14] for theories in \(\kappa\)-Minkowski the choice of “Lagrangian densities” that vanish for \(\vec{x} = 0\). We shall also comment on this point in our closing remarks.
B. Cyclic action functional for the time-symmetrized star product

We now show that the same choice of measure that induces cyclicity of the action functional for the time-to-the-right star product also induces cyclicity of the action functional for the time-symmetrized star product. We proceed searching for a cyclicity-inducing $\mu_T(x)$ and then verify that $\mu_T(x) = \mu_R(x)$.

Cyclicity requires a $\mu_T(x)$ such that

$$\int d^4x \mu_T(x)[f(x), g(x)]_{*T} = 0 \quad \text{(III.18)}$$

i.e.

$$\int d^4x \mu_T(x)[x^0_j, g(x)]_{*T} = 0 \quad \int d^4x \mu_T(x)[x^n_j, g(x)]_{*T} = 0$$

for all the integer $n$.

As already stressed in (II.9), the time-to-the-right Weyl map $\Omega_R$ and the time-symmetrized Weyl map $\Omega_T$ are connected in the following way:

$$\Omega_R(e^{ipx}) = \Omega_T(e^{ip'x}) \quad \text{(III.19)}$$

where $p' = (p_0, e^{\lambda p_0/2}p_j)$.

This can be used to show that from

$$[x^n_j, e^{ipx}]_{*R} = (1 - e^{-n\lambda p_0})(-i\partial_{p_j})^n e^{ipx} \quad \text{(III.20)}$$

it follows that

$$[x^n_j, e^{ipx}]_{*T} = (1 - e^{-n\lambda p_0})e^{n\lambda p_0/2}(-i\partial_{p_j})^n e^{ipx}$$

and therefore

$$[x^n_j, g(x)]_{*T} = x^n_j(1 - e^{in\lambda \partial_t})e^{-in\lambda \partial_t/2}g(x) \quad \text{(III.21)}$$

The form of this equation differs slightly from the corresponding equation obtained for the time-to-the-right star product, but the associated requirement for $\mu_T$

$$e^{-in\lambda \partial_t/2}(1 - e^{-in\lambda \partial_t})\mu_T(x) = 0 \quad \text{(III.22)}$$

still leads to the conclusion that the cyclicity-inducing measure should not depend on $t$.

Once this $t$-independence is taken into account, in order to ensure cyclicity of the integral we are left with the task of enforcing

$$\int d^4x \mu_T(x)[x^n_0, g(x)]_{*T} = 0 \quad \text{(III.23)}$$

or equivalently (using Fourier series for $x^n_0, g(x)$)

$$(-i)^n \int d^4x \mu_T(x) d^4k d^4p \left( \partial_{k_0} \tilde{g}^{(4)}(k) \right) \tilde{g}(p)[e^{ikx}, e^{ipx}]_{*T} = 0 \quad \text{(III.24)}$$

Using

$$\gamma^\mu_T(k, p) = (k^0 + p^0, e^{\lambda p_0/2}k^j + e^{-\lambda k_0/2}p_j)$$

the condition (III.24) can be rewritten as

\textbf{We are going back to focusing on the case }D = 3, \textbf{but, as in the previous star-product case, the discussion can be easily generalized to an arbitrary number of dimensions.}
\[ (-i)^n \int d^4x \mu(\vec{x}) d^4k d^4p \left( \partial_{k_0}^n \delta^4(k) \right) \tilde{g}(p) \left[ e^{i\gamma_T(k,p)x} - e^{i\gamma_T(p,k)x} \right] = 0 \quad \text{(III.26)} \]

Then, upon integration in \( dt \) and in \( d^4k \), one obtains

\[
\int d\vec{x} \mu(\vec{x}) d^4p \left( \partial_{k_0}^n \delta(k_0) \right) \bigg|_{k_0 = -p_0} \tilde{g}(p) e^{ip \cdot x} = 0
\]

Performing the substitution \( x^i \to e^{-\lambda p_0/2} x^i \) on the left-hand side, and the substitution \( x^i \to e^{\lambda p_0/2} x^i \) on the right-hand side, one then obtains

\[
\int d\vec{x} d^4p \left[ e^{-\lambda p_0/2} \mu( e^{-\lambda p_0/2} \vec{x} ) - e^{\lambda p_0/2} \mu( e^{\lambda p_0/2} \vec{x} ) \right] \left( \partial_{k_0}^n \delta(k_0) \right) \bigg|_{k_0 = -p_0} \tilde{g}(p) e^{ip \cdot x} = 0
\]

which can be satisfied if and only if

\[
\mu_T(a\vec{x}) = a^{-3} \mu_T(\vec{x}) \quad \text{(III.27)}
\]

for all \( a \in \mathbb{R}^+ \).

If we search for a measure that preserves space-rotational invariance, the only solution is, up to a multiplicative constant,

\[
\mu_R(\vec{x}) = |\vec{x}|^{-3}.
\]

The form of \( \mu_R(\vec{x}) \) discussed earlier is therefore also an acceptable cyclicity-inducing choice of \( \mu_T(\vec{x}) \).

**C. Cyclic action functional for the symmetric star product**

Next we turn to the case of the symmetric star product \( \ast_S \). As mentioned, this is the case considered in Ref. [14], where the first remarks on the possibility of a cyclicity-inducing measure were made.

The cyclicity condition (III.7) for the symmetric star product

\[
\int d^4x \mu_S(x)[f(x), g(x)]_{\ast_S} = 0 \quad \text{(III.28)}
\]

can be satisfied by imposing

\[
\int d^4x \mu_S(x)[x^n, g(x)]_{\ast_S} = 0 \quad \int d^4x \mu_S(x)[x^n, g(x)]_{\ast_S} = 0
\]

for all the integers \( n \).

The analysis proceeds in close analogy to the case of the time-symmetrized star product considered in the previous subsection. In particular, for the first commutator one finds that

\[
[x^n, g(x)]_{\ast_S} = (1 - e^{-n\lambda p_0}) \phi^{-n}(p_0)(-i\partial_{p_j})^n g(x) = x^n \left(1 - e^{-in\lambda \partial} \right) \phi^{-n}(-i\partial) g(x),
\]

and the corresponding condition for \( \mu_S(x) \),

\[
\phi^{-n}(-i\partial)(1 - e^{-in\lambda \partial}) \mu_S(x) = 0 \quad \text{(III.30)}
\]

implies that \( \mu_S(x) \) does not depend on \( t \).

The residual requirement for \( \mu_S(x) \) can be written in the form

\[
\int d^4x \mu_S(\vec{x})[x^n, g(x)]_{\ast_S} = 0, \quad \text{(III.31)}
\]

which is equivalent to
\[
\int d^4p d^3x \mu_S(\vec{x}) \tilde{g}(p)[\partial_{k_0} \delta(k_0)]_{k_0=-p_0 e^{ip' \phi(p')} x} = \\
= \int d^3k d^3x \mu_S(\vec{x})[\partial_{k_0} \delta(k_0)]_{k_0=-p_0 e^{ip' e^{\lambda p_0} \phi(p')} x},
\]
where we have taken into account that \( \phi(-p_0) = e^{\lambda p_0} \phi(p_0) \) and \( \phi(0) = 1 \).

Therefore \( \mu_S(x) \) must be such that

\[
\int d^4p d^3x \phi^{-3}(p_0) \left[ \mu_S(\phi^{-1}(p_0) \vec{x}) - e^{-3\lambda p_0} \mu_S(\phi^{-1}(p_0) e^{-\lambda p_0} \vec{x}) \right] \tilde{g}(p) \partial_{k_0} \delta(k_0)_{k_0=-p_0 e^{ip' x}} = 0
\]
and this condition can be satisfied if and only if

\[
\mu_S(a \vec{x}) = a^{-3} \mu_S(\vec{x}) \tag{III.32}
\]
for all \( a \in \mathbb{R}^+ \). This is the same requirement encountered already for the cases of the time-symmetrized star products. If we require rotational invariance, for all three examples \( \ast_R, \ast_T \) and \( \ast_S \) of star products the integration measure is (III.16).

**IV. SYMMETRY ANALYSIS**

**A. Symmetry operators in the commutative case**

One of the most studied aspects of \( \kappa \)-Minkowski is the possibility that it might be invariant under an Hopf-algebra of symmetries known as \( \kappa \)-Poincaré. This is a rather technical subject which has been discussed extensively in the literature (see, e.g., Refs. [8–13]). Here it is sufficient for us to introduce the \( \kappa \)-Poincaré Hopf-algebra transformations in an elementary way. For definiteness we focus on the case of the time-to-the-right star product (but analogous results hold for the other choices of star product which we are considering), and we also find sufficient to consider the 1+1-dimensional \( \kappa \)-Minkowski spacetime.

Just as a way to fix notations and terminology, in this subsection we review briefly the situation in the commutative-spacetime case. If \( x \to x' \) is an element of a fixed transformation group \( G \), a function \( f \) is called *scalar* for \( G \) if it transforms as

\[
f \to f', \quad f'(x') = f(x) \tag{IV.1}
\]

Given a scalar Lagrangian (that for us is simply a scalar function), in order to construct a \( G \)-invariant action, we need an invariant integral, *i.e.* an integral satisfying

\[
\int f'(x) = \int f(x)
\]
for all transformations of \( G \) and for all the scalar functions \( f \).

If \( f \) is a function of the classical Minkowski spacetime coordinates, and one adopts the standard integral \( \int d^{D+1}x f(x) \), then the relevant symmetry group is the Poincaré group.

For an integral with a non-trivial measure one can observe that

\[
\int f' = \int d^{D+1}x \mu(x) f'(x) = \int d^{D+1}x' \mu(x') f'(x')
\]
and that, using (IV.1),

\[
\int f' = \int d^{D+1}x' \mu(x') f(x)
\]
Therefore the integral is invariant if and only if \( \mu(x)d^{D+1}x \) is invariant.

For a generic \( G \), if

\[
x \to x' = x + A(x),
\]
with $A : \mathbb{R}^4 \to \mathbb{R}^4$ smooth, is a transformation of coordinates, the operators $A^\nu(x)\partial_\nu$ form a Lie algebra and their exponentiation $e^{iA^\nu(x)\partial_\nu}$ give us a Lie group which describes the transformation rule of scalar functions

$$f'(x) = e^{-iA^\nu(x)\partial_\nu}f(x) \iff f'(x + A(x)) = f(x)$$

For $T = A^\nu(x)\partial_\nu$ a generator of the group, if we denote

$$\Delta T = T \otimes 1 + 1 \otimes T$$

we can rewrite the Leibniz rule as

$$T(f \cdot g) = (T(1)f)(T(2)g)$$

where $\Delta T = T(1) \otimes T(2)$ is the Sweedler notation. The operation $\Delta$ (the coproduct) is extended to all the universal enveloping algebra of $G$ by the request that it should be an algebra-morphism and gives a structure of (trivial) Hopf-algebra.

A coproduct such as (IV.2) is “called trivial”. When we consider a noncommutative algebra of functions (as in the case of the study of theories in noncommutative spacetimes), in general the equation (IV.3) cannot be satisfied by a trivial coproduct, but still a Hopf-algebra structure can emerge.

It appears [10] that a description of symmetries in terms of Hopf algebras is appropriate both for commutative and for noncommutative spacetimes (but when the spacetime is commutative the co-algebra sector is trivial, and a description in terms of a Lie algebra would suffice).

**B. $\kappa$-Poincaré and the symmetries of the non-cyclic action functional**

The analysis of the symmetries of the action functional in the $\kappa$-Minkowski noncommutative spacetime can be set up in complete analogy with the commutative case discussed in the previous subsection. In $\kappa$-Minkowski, analyzed in terms of the $\Omega_R$ time-to-the-right Weyl map, the action functional

$$\mathcal{I}(\Omega_R(f)) = \int dxdt \mu(x,t) f(x,t) ,$$

where $f(x)$ is a scalar function, is invariant under a transformation $T$ if and only if

$$\int dx dt \mu(x) f'(x) = \int dx dt \mu(x) f(x) .$$

For each generator $T$ of the symmetry transformations, we define

$$[\Omega_R(f)\Omega_R(g)] = [T(1)\Omega_R(f)][T(2)\Omega_R(g)] ,$$

which is the analogue of (IV.3).

In order to have a genuine symmetry algebra, an algebra that can be used to describe all aspects of the symmetries of the action functional, it is necessary [10] that $T(1)$ and $T(2)$ involve only operators of the algebra. For the commutative case considered in the previous subsection this request is automatically satisfied, but (as it will become apparent later on in our analysis) the noncommutativity of the spacetime coordinates can change the situation significantly. When $T(1)$ and $T(2)$ (for all $T$ in the symmetry algebra) involve only operators of the algebra one inevitably obtains [10] the structure of a Hopf algebra of symmetries. We will therefore adopt terminology such that the requirement that the would-be symmetry generators close a Hopf algebra is identified with the description of a symmetry algebra.

As mentioned, a large literature has been devoted to the possibility that field theories in $\kappa$-Minkowski might provide a realization of $\kappa$-Poincaré Hopf-algebra symmetries. In order to see how $\kappa$-Poincaré can naturally emerge let us start by introducing in $\kappa$-Minkowski translation and rotation transformations naturally obtained by acting with the Weyl map on the corresponding transformations of the commutative limit:

$$P_\mu \Omega_R(f) = \Omega_R(-i\partial_\mu f)$$

$$M_j \Omega_R(f) = \Omega_R(i\epsilon_{jkl}x_k \partial_l f)$$

One can easily verify [10], imposing (IV.5), that $P_0$ and $M_j$ have trivial coproduct, while
\[ \Delta P_j = P_j \otimes 1 + e^{-\lambda P_0} \otimes P_j \]

A key point in the analysis of \( \kappa \)-Minkowski is the observation that these descriptions of translations and space rotations, in terms of Weyl-map quantizations of the corresponding classical transformations, are incompatible with the description of boost transformations given by Weyl-map quantization

\[ \hat{N}_j \Omega_R(f) = \Omega_R(i[x_0 \partial_j - x_j \partial_0]f) \]  

(IV.8)

In fact acting with (IV.8) on the product of two element of \( \kappa \)-Minkowski implicitly requires \[10\] a description in terms of an operator which is external to the algebra. Consistency with the Hopf-algebra requirements leads to the introduction \[10–12\] of the “deformed boost action”

\[ N_j = it \partial_j - x_j \left( \frac{1 - e^{2i\lambda \partial_0}}{2\lambda} - \frac{\lambda}{2} \nabla^2 \right) - i\lambda x_i \partial_i \partial_j \]  

(IV.9)

The elements \( P_\mu \), \( M_j \) and \( N_j \) generate the \( \kappa \)-Poincaré Hopf-algebra (in the Majid-Ruegg basis \[11\]).

As a first step in the exploration of the possibility of a \( \kappa \)-Poincaré invariant field theory in \( \kappa \)-Minkowski, we can ask if \( \kappa \)-Poincaré can be realized as symmetry group of the action functional. This is basically the reason that motivated the choice of measure \( \mu = 1 \) in most of the early works on \( \kappa \)-Minkowski. In fact, the choice \( \mu = 1 \) leads to a \( \kappa \)-Poincaré invariant action functional. For our purposes here it is sufficient to revisit this result considering only the case in which one adopts the time-to-the-right star product, and focusing on the case of a 1+1-dimensional spacetime.

In the noncommutative case the symmetry analysis involves several new elements of complexity, especially in light of the fact that, since the coproduct is deformed, it is no longer true that \( f'(x') = f(x) \), with \( x' = e^{-i\alpha T}x \), if \( T \) is a generator of a one-parameter transformation and \( f' = e^{i\alpha T}f \). This equality holds if and only if \( T \) has a trivial coproduct. In fact, only when \( \Delta T \) is trivial, the invariance condition

\[ \int f'(x, t) \mu(x, t) dx dt = \int f(x, t) \mu(x, t) dx dt \]  

(IV.10)

is equivalent\(^6\) to the invariance of the integration measure: \( \mu(x', t') dx' dt' = \mu(x, t) dx dt \).

It is actually easy to see, in the analysis of the symmetries of the action functional

\[ \int f(x, t) dx dt \]

in the sense of Eq. (IV.10), that time translations and space translations, generated by \( P_\mu \), are symmetries; indeed

\[ \int \{e^{i\alpha P}f\}(x, t) dx dt = \int f(x, t) dx dt \]

In other words, the infinitesimal variation is zero:

\[ \int (P_\mu f)(x, t) dx dt = 0 \]

It is also easy to verify that the deformed boost is a symmetry, indeed integration by parts gives

\[ \int (N_j f)(x, t) dx dt = 0 \]

The requirement that the generators of a genuine symmetry algebra should close on to a Hopf-algebra is also satisfied since \( P_\mu \) and \( N_j \) are the generators of the well-known \( \kappa \)-Poincaré Hopf algebra.

\(^6\)In particular, in our \( \kappa \)-Poincaré context the space-rotation generators \( M_j \) have a trivial coproduct, so it makes sense to state that a rotationally-invariant measure \( \mu(|\vec{x}|) \) gives a rotationally-invariant integral. (Of course, this remark is relevant when working with more than one space dimension, since in 1+1 dimensions there are no space rotations).
C. Symmetries of the cyclic action functional

The fact that the simple choice of measure \( \mu = 1 \) leads to a \( \kappa \)-Poincaré invariant action functional has garnered a strong interest in the literature. Some of the reasons for this interest originate from a possible “\( \kappa \)-Poincaré phenomenology” \[25,26\] and some other reasons of interest reside deep in the rather rich mathematical structures involved. Here we just want to stress one obvious attractive aspect of the \( \kappa \)-Poincaré symmetries: in the commutative \( \lambda \to 0 \) limit they reduce to standard (classical, Lie-algebra) Poincaré symmetries. Therefore the \( \kappa \)-Poincaré invariance is fully compatible with the fact that our low-energy experiments (involving distance scales much larger than \( \lambda \)) are all consistent, within their available accuracy, with classical Poincaré invariance.

However, as stressed above, the fact that with the measure \( \mu = 1 \) the action functional is “non-cyclic” leads to several problems. With our cyclic action functional these problems would be avoided, but we intend to observe in this subsection that the covariance properties of the cyclic action functional appear to be somewhat peculiar. We can establish this point already by working again with \( 1 + 1 \)-dimensional \( \kappa \)-Minkowski, where the cyclic action functional takes the form

\[
\mathcal{I}(\Omega_R(f)) = \int f(x,t) \frac{dx}{|x|} dt
\]

The Weyl-map quantization of the time translation is clearly a symmetry, and is generated by \( P_0 = -i \partial_t \). Indeed, it keeps invariant the element

\[
\mu(x)dxdt = |x|^{-1}dxdt
\]

and, since it has a trivial coproduct, it leaves invariant also the integral, in the sense of Eq. (IV.10).

On the other hand, it is equally easy to verify that the space translation and the boost (both the classical boost and the \( \kappa \)-Poincaré deformed boost) are not symmetries of the cyclic action functional. These two symmetries are replaced by two other, possibly unwanted, symmetries. One is the Weyl-map quantization of an \( x \)-dilatation, with generator \( D = -ix\partial_x \). The finite transformation obtained with \( D \) is \( f \to f' = e^{-i(x \partial_0 - \lambda \partial_0)} f \), with \( a \in \mathbb{R}^+ \). In order to establish the form of \( \Delta D \), the coproduct of \( D \), it is sufficient to take \( f(x,t) = e^{ipx} \) and \( g(x,t) = e^{iqx} \) with \( p \) and \( q \) arbitrary, so that \( (f * g)(x,t) = e^{\gamma_R(p,q)x} \), where \( \gamma_R(p,q) \) is the one given in Eq. (II.5). One can then easily check that

\[
\partial_x(f * g) = (\partial_x f) * g + (e^{-\lambda \partial_0} f) * (\partial_x g)
\]

Moreover, \( x * f = xf \) and therefore (by associativity of the \(*\)-product) \( (xf) * g = x * f * g = x(f * g) \), while \( e^{ip\partial_0} * x = x * e^{ip\partial_0} e^{-\lambda \partial_0} \). Thus

\[
f * (xg) = f * x * g = e^{-\lambda \partial_0} x * f * g = x[(e^{-\lambda \partial_0} f) * g]
\]

and from this one concludes that

\[
x\partial_x(f * g) = x[(\partial_x f) * g] + x[(e^{-\lambda \partial_0} f) * (\partial_x g)] = (x\partial_x f) * g + f * (x\partial_x g),
\]

i.e. \( \Delta D = \Delta(x\partial_x) \) is trivial.

The fact that \( D \) has a trivial coproduct implies that \( f \) is scalar under \( D \) when \( f'(x,t) = f(ax,t) \). Since \( a > 0 \), the fact that \( D \) is a symmetry follows from the corresponding invariance of the integration measure: \( dx/|x| = d(ax)/|ax| \).

While the form of the cyclicity-inducing measure immediately suggests that dilatations and time translations should be symmetries, the identification of the third symmetry, which we denote by \( K \), requires more work. However, writing \( K \) as a formal series in the coordinates and derivatives one can straightforwardly (but tediously) reconstruct the form of this third symmetry, obtaining the result

\[
K = -itx\partial_x + \lambda \log |x| \left( \frac{1 - e^{2it\partial_0}}{2} + \frac{1}{2} (x\partial_x)^2 \right).
\]

\[7\] One could formally consider also \( a < 0 \) but that would not be a genuine dilatation. It combines a dilatation with an inversion, \( x \to -x \). The cases with \( a < 0 \) will be obtained combining an \( a > 0 \) dilatation and a space-rotation such that \( x \to -x \).
The fact that this \( K \) generates a symmetry of the cyclic-integral, i.e. \( \mathcal{I}(\Omega(e^{i\xi K} f)) = \mathcal{I}(\Omega(f)) \) for all \( \xi \) (or equivalently \( \mathcal{I}(\Omega(K f)) = 0 \)), is quickly verified observing that integration by part gives

\[
\int (K f)(x,t) \frac{dx}{|x|} \ dt = \int f(x,t) \left[ \partial_x \frac{i t x}{|x|} + \frac{\lambda}{2} \left( (1 - e^{-2i\lambda \partial_0}) \log |x| \right) + \partial_x^2 \frac{x^2 \log |x|}{|x|} - \partial_x \frac{x \log |x|}{|x|} \right] \ dx \ dt = 0.
\]

We do have three symmetry candidates, \( P_0, D \) and \( K \); however, these are not clearly the symmetries we were looking for, since in the commutative \( \lambda \to 0 \) limit they do not recover the classical Poincaré algebra. On the other hand this should be expected since in the \( \lambda \to 0 \) limit the cyclicity-inducing measure does not reduce to the Poincaré invariant measure \( \mu = 1 \).

Also alarming is the fact that the three operators \( P_0, D \) and \( K \) do not generate a genuine symmetry algebra, not in the sense needed when dealing with a noncommutative spacetime. In fact, the expression of their co-products requires operators external to the triad \( P_0, D \) and \( K \), so \( P_0, D \) and \( K \) do not generate a Hopf algebra. This is an automatic consequence of the fact that both \( P_0 \) and \( D \) have trivial coproduct, while the commutator \( [K, D] \) is nonlinear:

\[
[K, D] = i \lambda \left( \frac{1 - e^{-2\lambda P_0}}{2} \right) - \frac{i}{2} D^2
\]

In fact, the Hopf-algebra axioms imply that \( \Delta K \) is of the form

\[
\Delta K = K \otimes a(P_0, D) + b(P_0, D) \otimes K
\]

with \( a \) and \( b \) to be determined. But the condition \( [\Delta K, \Delta P_0] = \Delta[K, P_0] = i \lambda \Delta D \) (where we used the observation that \( [K, P_0] = i \lambda D \)) implies \( a = b = 1 \), and for \( a = b = 1 \) the condition \( [\Delta K, \Delta D] = \Delta[K, D] \) is not satisfied

\[
[\Delta K, \Delta D] - \Delta[K, D] = \frac{i}{2} \lambda \left( 1 - e^{-2\lambda P_0} \right) \otimes \left( 1 - e^{-2\lambda P_0} \right) - i D \otimes D \neq 0
\]

V. CYCLICITY-INDUCING MEASURE FOR \( \kappa \)-MINKOWSKI DERIVED FROM CANONICAL-SPACETIME ACTION FUNCTIONALS

In the previous sections we have shown that a cyclicity-inducing measure for \( \kappa \)-Minkowski can be obtained explicitly using an analysis which only relies on properties of \( \kappa \)-Minkowski itself. In this section we intend to show that the same results can be obtained even more easily by exploiting the properties of maps which allow to formulate the \( \kappa \)-Minkowski spacetime coordinates in terms of the spacetime coordinates of a canonical spacetime.

This may prove also valuable as a starting point for future use of our results on the action functional for the formulation of field theories in \( \kappa \)-Minkowski. In fact, canonical spacetimes are rather well understood and perhaps could be exploited as a starting point for further analysis of \( \kappa \)-Minkowski.

There are actually two ways to obtain a cyclicity-inducing measure for \( \kappa \)-Minkowski starting from canonical spacetimes: one which is based on a direct one-to-one description of \( \kappa \)-Minkowski coordinates in terms of canonical coordinates and one which is based on a description of the \( \kappa \)-Minkowski coordinates in terms of a higher-dimensional canonical spacetime.

A. A procedure involving simply a Jacobian

Let us start by observing that the relations

\[
x_j = e^{q_j}
\]

give us an isomorphism of the first quadrant of \( \kappa \)-Minkowski with \( \mathbb{R}^{D+1} \) equipped with commutation relations:

\[
[q_j, q_k]_* = 0 \quad [q_j, t]_* = i \lambda
\]

For any given star product on \( \kappa \)-Minkowski there is a corresponding star product on the spacetime with \( (q, t) \) coordinates. Since the \( (q, t) \) algebra is canonical, the action functional \( \int F(q, t) d^D q \ dt \) is cyclic for any choice of star product.

We seek a space-rotations invariant action functional, so, for a \( D + 1 \)-dimensional \( \kappa \)-Minkowski, we can divide the \( D \)-dimensional space in \( 2^D \) quadrants. In each quadrant all the coordinates \( x_j \) have a fixed sign. If we determine \( \mathcal{I}(f) \)
for \( f \) zero everywhere except when all \( x_j > 0 \), then by rotational invariance we also determine \( \mathcal{I}(\hat{f}) \) for a function different from zero in any other quadrant. We can therefore focus on the first quadrant:

\[
\mathcal{I}(\hat{f}) = \int_{x_j \geq 0 \forall j} f(x) \mu(|\vec{x}|) d^{D+1}x , \quad f(0, t) = 0
\]

Using the points made above about the map \( x_j \to e^{Q_j} \) we can describe a function \( f(x_1, \ldots, x_D, t) \) of the \( \kappa \)-Minkowski coordinates as a function \( F(\vec{q}, t) = f(e^{Q_1}, \ldots, e^{Q_D}, t) \) of the \((q, t)\) coordinates. Then the classical integral in the \((q, t)\) coordinates corresponds to a cyclic action functional in \( \kappa \)-Minkowski:

\[
\int F(\vec{q}, t)d^Dq dt = \int_{x_j \geq 0 \forall j} f(x) J(\vec{x}) d^{D+1}x
\]

where

\[
J(\vec{x}) = \frac{1}{|x_1| \cdots |x_D|}
\]

is the Jacobian of the transformation. Note that \( J(\vec{x}) \) is a particular solution of (III.15), as one should expect.

This particular cyclicity-inducing measure is not space-rotation invariant. One can obtain a space-rotation invariant measure for \( \kappa \)-Minkowski starting from a cyclic integral in the canonical spacetime that is different from the classical one. Formulating the star product as a pseudo-differential operator, and integrating by parts, one finds that an integration measure \( \eta(\vec{q}, t) \) gives a cyclic action functional if it satisfies:

\[
(\partial_{q_1} + \ldots + \partial_{q_D}) \eta(\vec{q}, t) = 0 \quad \partial_t \eta(\vec{q}, t) = 0.
\]

It should be stressed that this is not simply the equation (III.15) rewritten in the new coordinates. However, if we choose the particular solution

\[
\eta(\vec{q}) = \exp \left( q_1 + q_2 + \ldots + q_D - \frac{D}{2} \log[e^{2q_1} + \ldots + e^{2q_D}] \right)
\]

we do obtain our favoured, cyclicity-inducing and space-rotation invariant, measure for \( \kappa \)-Minkowski:

\[
\mu(\vec{x}) = J(x) \cdot \exp \left( \log |x_1| + \ldots + \log |x_D| - \frac{D}{2} \log |\vec{x}|^2 \right) = J(x) \exp \left( \log \left( \frac{|x_1| \cdots |x_D|}{|\vec{x}|^D} \right) \right) = \frac{1}{|\vec{x}|^D}
\]

### B. Cyclic action functionals by reduction

It is possible to view [27] a \( D + 1 \)-dimensional \((D \geq 1)\) noncommutative spacetime \( \mathcal{M} \) as a subspace of a \( 2D \) dimensional symplectic phase space \( \mathcal{C} \), endowed with the usual Poisson Bracket, and a Moyal star product. A star product on \( \mathcal{M} \) can be defined by first lifting the functions from the \((D + 1)\)-dimensional space \( \mathcal{M} \) to functions on the \( 2D \)-dimensional phase space \( \mathcal{C} \), then multiplying them in such a phase space using the Moyal product and finally projecting back to the original space \( \mathcal{M} \). The lift consists in establishing a (generalized Jordan-Schwinger) map between the coordinates of the space \( \mathcal{M} \) and the coordinates of the phase space \( \mathcal{C} \).

Examples of such maps for \( \kappa \)-Minkowski have been known for some time (see, e.g., Refs. [9,21]) and they have recently been used [21] to introduce star products in \( \kappa \)-Minkowski. In this section we consider the case in which \( \mathcal{C} \) is equipped with \( 2D \) canonical coordinates \((x_j, p_j)\) and the Groenewold [28] star product

\[
(F * G)(x, p) = F(x, p)e^{i(\hat{\partial}_{x_j} \hat{\partial}_{p_k} - \hat{\partial}_{x_j} \hat{\partial}_{p_k})} G(x, p)
\]

(V.1)

The commutation rules are those of the Heisenberg algebra

\[
[x_j, p_k]_* = i\delta_{jk} \quad [x_j, x_k]_* = [p_j, p_k]_* = 0
\]

Introducing the map

\[
x_0 = \lambda \vec{x} \vec{p}
\]
one can easily to verify that the $x_\mu$ satisfy the $\kappa$-Minkowski commutation relations

$$[x_j, x_0] = \lambda [x_j, \tilde{x} \tilde{p}] = i \lambda x_j$$

The star product in $\kappa$-Minkowski induced by this map is different [21] from the three star products that we have so far considered.

In the phase space $C$, the standard integral $\int F(\vec{x}, \vec{p}) d^D \vec{x} d^D \vec{p}$ of a function $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ is known to be cyclic with respect to the star product (V.1).

A function $f(\vec{x}, t)$ can be viewed as a function $F(\vec{x}, \vec{p}) = f(\vec{x}, \vec{x} \vec{p})$ on the $2D$ dimensional space. For such a function, the standard integral is clearly divergent, but we can regularize it. For each $\vec{x}$ we fix an orthonormal basis $\{v_1(\vec{x})\}$ for $\mathbb{R}^D$ with positive orientation and with $\vec{v}_1 = \vec{x} / |\vec{x}|$. Then we consider

$$C(\vec{x}) = \{ \vec{u} \in \mathbb{R}^D \text{ such that } |\vec{u} \vec{v}_j| \leq L |\vec{x}|^{-1} \forall \, j = 1, \ldots, D \}$$

which is an hypercube of edge $2L |\vec{x}|^{-1}$ and one of the edges parallel to $\vec{x}$. The regularized integral in $d^D \vec{p}$ is obtained restricting the integration to the interior of $C(\vec{x})$ and normalizing

$$\mathcal{I}'(\vec{f}) = \lim_{L \rightarrow +\infty} \frac{1}{(2L)^{D-1}} \int d^D \vec{x} \int_{C(\vec{x})} d^D \vec{p} f(\vec{x}, \vec{x} \vec{p})$$

The normalization factor has been chosen in such a way to remove divergencies.

In order to put $\mathcal{I}'(\vec{f})$ in explicit form let us start by examining the integration in $d^D \vec{p}$. We introduce $\Lambda \in SO(D)$ as the change of basis from the canonical one to the $\vec{v}_j$ basis. If we make the substitution $\vec{p}' = \Lambda \vec{p}$, we then find that in the new basis $\vec{x} \vec{p} = |\vec{x}| \vec{p}_1'$ and the $p$-integration takes the form

$$\int_{C(\vec{x})} f(\vec{x}, \vec{x} \vec{p}) d^D \vec{p} = \int_{|p_1'| \leq L |\vec{x}|^{-1}} f(\vec{x}, |\vec{x}| \vec{p}_1') d^D \vec{p}' = \frac{(2L)^{D-1}}{|\vec{x}|^D} \int_{-L}^L f(\vec{x}, t) dt$$

which implies

$$\mathcal{I}'(\vec{f}) = \lim_{L \rightarrow +\infty} \int d^D \vec{x} \frac{1}{|\vec{x}|^D} \int_{-L}^L f(\vec{x}, t) dt = \int \frac{1}{|\vec{x}|^D} f(\vec{x}, t) d^D \vec{x} \, dt .$$

So, once again, we encounter the same choice of measure, (III.15), which we had independently obtained in Section 3, using an analysis that relied only on properties of $\kappa$-Minkowski.

VI. CLOSING REMARKS

We have here considered alternative strategies for the introduction of an action functional for $\kappa$-Minkowski. If one is guided by the intuition that $\kappa$-Poincaré invariance should be a key property of the action functional then the $\mu = 1$ choice of measure is natural and cyclicity is lost. If one is guided by the intuition that the action functional should necessarily be cyclic, then $\kappa$-Poincaré invariance cannot be achieved and one is also confronted with an unexpected small-$x$ singularity of the measure.

Both the lack of $\kappa$-Poincaré invariance and the small-$x$ singularity of the measure are not necessarily alarming. These features may not be transferred to the equations of motion (where the physical properties of a theory should be investigated) if one finds a clever way to introduce fields in the action functional. Think for example of a “Lagrangian density” of the type $\mathcal{L} = x \Phi(x) x \Phi(x) x \Phi(x)$, in which case clearly the small-$x$ singularity of the measure would not affect the equations of motion (but clearly this specific example of Lagrangian density is affected by several other pathologies). An intriguing challenge for future studies could be the search of $\kappa$-Poincaré covariant non-singular equations of motion derived from an action functional with our cyclicity-inducing measure.

On the other hand it is tempting to attach a deeper meaning to the peculiar small-$x$ singularity of the cyclicity-inducing measure. In fact, already in the study of the better understood canonical noncommutative spacetimes we have become familiar with an unexpected source of singularities (infrared singularities which arise through the so-called IR/UV mixing [3,29,30]). It is therefore plausible that also in $\kappa$-Minkowski the small-$x$ singularity we discussed might encode some fundamental aspect of the formalism.
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