Quantum computation from fermionic anyons on a 1D lattice

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 (Dated: December 18, 2018)

Fermionic linear optics corresponds to the dynamics of free fermions, and is known to be efficiently simulable classically. We define fermionic anyon models by deforming the fermionic algebra of creation and annihilation operators, and consider the dynamics of number-preserving, quadratic Hamiltonians on these operators. We show that any such deformation results in an anyonic linear optical model which allows for universal quantum computation.

I. INTRODUCTION

There are good reasons to study computational models using particles with some kind of fractional statistics. Experimentally, there has been a growing interest in finding such exotic particles in condensed matter systems such as the one exhibiting the fractional quantum hall effect (FQHE) [1], as they can simplify the description of a multitude of complex condensed matter phenomena. They can also be used as a platform for models of quantum computation that are intrinsically robust to decoherence under specific conditions [2–5]. Theoretically, understanding the computational power of models based on such particles can give us more knowledge about the transition from classical to quantum computation and what resources make this transition possible.

In this paper we generalize the model of quantum computation based on the dynamics of non-interacting fermions (known as fermionic linear optics (FLO) [6–8]) to one of anyons on a 1-D lattice. Our anyon system is defined via a deformation of the fermionic anti-commutation relations that introduces non-trivial exchange phases (also called fractional exchange phases).

A system of identical particles with abelian fractional exchange statistics is one where the multiparticle wave function gains an arbitrary complex phase factor under an operation that exchanges particle positions [9, 10]. A non-abelian system is very similar but, in this case, each particle has an internal Hilbert space, and because of it the exchange phases are replaced by arbitrary unitary matrices. Particles with these properties are called abelian anyons in the first case and non-abelian anyons in the second [11]. Such statistics have a topological origin related to the dimensionality of the physical space, and can be observed in two-dimensional many-body systems subject to a process known as transmutation of statistics [11]. This involves many-body interactions described by non-dynamical effective vector potentials, whose only roles are to associate fictitious magnetic fluxes to charged particles, generating the exchange phase factor via the Aharonov-Bohm effect. These fictitious fields are described by topological quantum field theories (TQFT), such as the Chern-Simons theory [12], and used to describe many classes of states of FQHE systems [13, 14].

In contrast to this topological description, known to be valid only in two dimensions, a system is said to have fractional exclusion statistics if the maximum number of particles allowed per mode is a finite integer not equal to one [15]. This is a dimension independent generalization of the Pauli Exclusion Principle which also applies to the description of some subsets of FQHE states giving a consistent description with exclusion statistics. This generalization is unrelated to statistical transmutation and is, in fact, independent of it. For such a definition, spinless fermions and spinless hard-core bosons are both seen as fermions, since both obey the exclusion principle, even though their commutation relations differ if particles are in different states.

Another dimension independent way to describe anyon systems is via a fermion-anyon mapping [16]. This mapping relates creation and annihilation operators for fermions in a lattice with the corresponding operators for anyons in the same lattice. The anyon operators are related to the fermionic ones by a generalized Jordan-Wigner transform [17]. Besides being dimension-independent, this approach to anyons inspired experimental proposals for the simulation of such systems in optical-lattice implementations [18].

We define a computational model similar to the Boson Sampling model [19] where a finite number of photons are input into a circuit made of optical devices such as beam-splitters and phase-shifters, followed by photon-counting detectors. Our anyonic model interpolates between one-dimensional fermions and one-dimensional hard-core bosons. We remark that our proposal is different from the one made in topological quantum computing (TQC), which deals with the computational power of braiding non-abelian anyons and is intrinsically fault-tolerant. [3, 20–24]

This paper is organized as follows. In section II we define and review the properties of all such computational models based on free quadratic number-preserving particle dynamics, including FLO and a model with hard-core bosons [25, 26]. In section III we define and discuss the version of this computational model for our anyon system, showing how to solve the relevant dynamics and defining equivalent "linear-optical" devices. In section IV we construct a universal set of gates inspired by a protocol similar to the one defined in [26], showing that our model is universal for quantum computation. Finally, in section V we offer some concluding remarks.
II. REVIEW

In this section we review how to describe non-interacting bosons and fermions, whose dynamics are respectively known as bosonic (section II A) and fermionic linear optics (section II B). In section II C we review how qubits can be understood as a system of bosons under a hard-core interaction. This will be the basis for defining our model of fermionic anyons on a 1D lattice, and their associated free-particle dynamics, which we study in the following sections.

A. Linear Optics

Photons are described by their degrees of freedom (which we call modes, for short). We consider a system with \( m \) modes described by creation and annihilation operators \( b_i^\dagger \) and \( b_i \) with \( i = 1, \ldots, m \) which respectively, create and destroy a single photon in mode \( i \). These operators obey the canonical bosonic commutation relations

\[
\begin{align}
    & b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij}, \quad (1a) \\
    & b_i b_j - b_j b_i = 0, \quad (1b) \\
    & b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0, \quad (1c)
\end{align}
\]

for all modes \( i, j \). The basis vectors for this system’s Hilbert space can be chosen to be

\[
|n_1^B, \ldots, n_m^B\rangle = \frac{1}{\sqrt{n_1^B! \cdots n_m^B!}} |0_B\rangle , \quad (2)
\]

where \( n_i^B \) is the eigenvalue of the number operator \( N_i^B = b_i^\dagger b_i \). This basis is called the Fock basis or the occupation number basis, and we use this basis throughout this work.

As we will shortly see, any free-particle bosonic dynamics can be expressed in terms only of one- and two-mode passive linear optical elements (as described in e.g. [27]). A phase-shifter is a single-mode passive linear device described as the time-evolution operator \( PS_i(\theta) = \exp\{i\theta H_i^{PS}\} \) of the Hamiltonian \( H_i^{PS} = N_i^B \).

Its effect on Fock states results in an occupation-number dependent phase:

\[
PS_i(\theta) |n_1^B, \ldots, n_m^B\rangle = e^{i\theta n_i^B} |n_1^B, \ldots, n_m^B\rangle, \quad (3)
\]

a phase-shifter acts on creation operators as:

\[
PS_i(\theta)b_j^\dagger PS_i(-\theta) = e^{i\theta \delta_{ij}} b_j^\dagger. \quad (4)
\]

A beam-splitter is a two-mode passive linear device described by the time-evolution operator \( BS_{ij}(\theta) = \exp\{i\theta H_{ij}^{BS}\} \), corresponding to the Hamiltonian \( H_{ij}^{BS} = b_i^\dagger b_j + b_j^\dagger b_i \). Its effect in creation operators is given by the matrix equation

\[
BS_{ij}(\theta) \begin{bmatrix} b_i^\dagger \\ b_j^\dagger \end{bmatrix} BS_{ij}(-\theta) = \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} b_i^\dagger \\ b_j^\dagger \end{bmatrix}. \quad (5)
\]

Another way to understand how these devices act is to consider a system with a single photon which can be in any of the \( m \) modes. The basis states of this system in the occupation number representation are the \( m \) states \( |1, 0, \ldots, 0\rangle, |0, 1, 0, \ldots, 0\rangle, \ldots, |0, 0, \ldots, 1\rangle \). In terms of these states, the actions of beam-splitters and phase-shifters are \( SU(2) \) matrices in the subspaces on which they act. In fact it was proven that any \( SU(m) \) matrix can be constructed in this single photon system using only successive applications of phase-shifters and beam-splitters [28]. The computational model where the computational basis states are the Fock states, the unitaries are arbitrary circuits of phase-shifters and beam-splitters between two arbitrary modes, and measurements are made in the Fock basis is called the Boson Sampling model [19]. This model is not known to be universal for quantum computational, but there is evidence that it is hard to simulate on a classical computer given some complexity-theoretic assumptions.

B. Fermionic Linear Optics

We now turn to the fermionic linear optics (FLO) model as described e.g in [6, 7, 29], and specialize it to our needs. Consider a system of abstract fermionic modes described by creation and annihilation operators \( f_i^\dagger \) and \( f_i \) on \( m \) modes, satisfying the canonical fermionic anti-commutation relations

\[
\begin{align}
    & f_i f_j^\dagger + f_j^\dagger f_i = \delta_{ij}, \quad (6a) \\
    & f_i f_j + f_j f_i = 0, \quad (6b) \\
    & f_i^\dagger f_j^\dagger + f_j^\dagger f_i^\dagger = 0, \quad (6c)
\end{align}
\]

for all modes \( i, j \). The occupation number operator is \( N_i^F = f_i^\dagger f_i \) and the vacuum state is \( |0_F\rangle \). The Fock basis for fermions comprises basis states

\[
|n_1^F, \ldots, n_m^F\rangle = (f_1^\dagger)^{n_1^F} \cdots (f_m^\dagger)^{n_m^F} |0_F\rangle, \quad (7)
\]

where the \( n_j^F \) are eigenvalues of the corresponding number operators which, due to the commutation relations, can only be 0 or 1.

We call passive fermionic linear optical elements, unitaries of the form

\[
\begin{align}
    & PS_i(\theta) = \exp\{i\theta H_i^{PS}\}, \quad (8a) \\
    & BS_{ij}(\theta) = \exp\{i\theta H_{ij}^{BS}\}, \quad (8b)
\end{align}
\]

with hamiltonians given by

\[
\begin{align}
    & H_i^{PS} = N_i^F, \quad (8c) \\
    & H_{ij}^{BS} = f_i^\dagger f_j + f_j^\dagger f_i, \quad (8d)
\end{align}
\]

The action of these elements over creation operators is very similar to the bosonic case, and is given by the equa-
tions

\[ PS_i(\theta) f^\dagger_j PS_i(-\theta) = e^{i\theta\delta_{ij}} f^\dagger_j, \]  
(9a)

\[
BS_{ij}(\theta) \left[ f^\dagger_i f^\dagger_j \right] BS_{ij}(-\theta) = \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} \left[ f^\dagger_i f^\dagger_j \right]. \]  
(9b)

As in the bosonic case, one can construct any \( SU(m) \) operator over the Hilbert space of a single fermion in \( m \) modes using only compositions of beam-splitters and phase-shifters.

By taking as computational basis states the occupation number basis, and as logic gates fermionic linear optical elements between arbitrary modes, and measurements in the occupation number basis, we define the computational model called Fermionic Linear Optics. This model of computation is proven to be exactly efficiently simulable by a classical computer in [7], which means that there is an efficient classical algorithm to evaluate \(|\langle y| U |x\rangle|^2\) for arbitrary computational states \(|x\rangle\) and \(|y\rangle\) and arbitrary unitaries generated by the application of passive optical elements (and any desired marginal probabilities).

C. Qubits as hard-core bosons

To make a point of comparison we now give an alternate description of the usual circuit model of computation on qubits. We will call this model qubit linear optics. Consider a system of \( m \) qubits, with the Pauli matrices denoted by \( X_i, Y_i, Z_i \). Following [25] we can write the usual computational basis states as a Fock space basis given by the equations

\[
|n_1^Q, ..., n_m^Q\rangle = (\pi^\dagger_1)^{n_1^Q} \cdots (\pi^\dagger_m)^{n_m^Q} |0\rangle,
\]  
(10)

with the creation and annihilation operators given by \( \pi^\dagger_i = \frac{1}{\sqrt{2}}(X_i + iY_i) \) and \( \pi_i = \frac{1}{\sqrt{2}}(X_i - iY_i) \) acting on state \( |0\rangle = |0, ..., 0\rangle \), and number operator \( N_i^Q = \pi^\dagger_i \pi_i = \frac{1}{2}(1 + Z_i) \) whose eigenvalues \( n_i^Q \) can be only 0 or 1. These operators must obey the algebra

\[
\begin{align*}
\pi_i \pi_j - \pi_j \pi_i &= 0, \\
\pi_i \pi_j - \pi_j \pi_i &= 0, \\
\pi^\dagger_i \pi^\dagger_j - \pi^\dagger_j \pi^\dagger_i &= 0,
\end{align*}
\]  
(11a) (11b) (11c)

for all modes \( i, j \) with \( i \neq j \), [compare with Eqs. (1)] and

\[
\begin{align*}
\pi_i \pi^\dagger_i + \pi^\dagger_i \pi_i &= 1, \\
(\pi^\dagger_i)^2 = (\pi_i)^2 &= 0,
\end{align*}
\]  
(11d) (11e)

for each mode \( i \) [compare with Eqs.(6)]. This algebra forbids more than one boson in the same mode, which is why this system can be understood as bosons with a hard-core interaction.

Similarly to bosonic and fermionic linear optics, qubit passive linear optical elements are defined as unitary operators of the form

\[
\begin{align*}
PS_i(\theta) &= \exp\{i\theta H^P_i\}, \\
BS_{ij}(\theta) &= \exp\{i\theta H^BS_{ij}\},
\end{align*}
\]  
(12a) (12b)

with hamiltonians given by

\[
\begin{align*}
H^P_i &= N_i^Q, \\
H^BS_{ij} &= \pi^\dagger_i \pi_j + \pi^\dagger_j \pi_i.
\end{align*}
\]  
(12c) (12d)

Using a bit of algebra we can write the beam-splitter Hamiltonian in terms of Pauli operators as \( H^BS_{ij} = X_iX_j + Y_iY_j \). In [26], Kempe and Whaley showed that this interaction acting between nearest and next-nearest neighbours on a 1D chain, along with Pauli \( Z \) rotations, can perform universal quantum computation in an encoded subspace of the Fock space (next-to-nearest neighbour interactions are essential for the protocol to work). This result suggest that qubit linear optics is computationally more powerful than fermionic linear optics, even though the only differences are the signs of commutators of operators in different sites.

III. LINEAR OPTICS OF 1D FERMIONIC ANYONS

In this section we define what we call fermionic anyons on a 1D lattice, in terms of a deformation of the fermionic algebra of creation and annihilation operators. In analogy with free fermions, we define the dynamics corresponding to anyonic linear optics. We solve the equations of motion for the dynamics which is analogous to phase shifts and beam-splitters, and show how these Hamiltonians result in a novel effect when beam-splitters act on non-neighboring modes. We conclude this section by showing the existence of a one-dimensional analogue of the Aharonov-Bohm effect, and argue that it is responsible for the differences in the beam-splitter action, with respect to free fermions.

A. Definition of the model and dynamical equations

The anyons we consider are particles defined via creation and annihilation operators \( a^\dagger_i \) and \( a_i \) satisfying the deformed anti-commutation relations

\[
\begin{align*}
a_i a^\dagger_j + e^{i\epsilon_{ij}} a^\dagger_i a_j &= \delta_{ij}, \\
ak_i a_j + e^{i\epsilon_{ij}} a^\dagger_j a_i &= 0,
\end{align*}
\]  
(13a) (13b)

where the symbol \( \epsilon_{ij} \) is given by

\[
\epsilon_{ij} = \begin{cases} 
1 & \text{if } i < j, \\
0 & \text{if } i = j, \\
-1 & \text{if } i > j.
\end{cases}
\]  
(13c)
This dependence on the $\epsilon_{ij}$ function defines an order over the lattice, coming from the way the deformation is defined [see Eqs. (14)]. When $\varphi = 0$ or $\pi$ this order is irrelevant and we re-obtain a fermionic system [see Eqs. (1)], and hard-core bosons [or qubits, see Eqs. (11d) and (11e)]. For all $0 < \varphi < \pi$ we have a non-trivial anyonic model. The deformed anti-commutation relations come from the generalized Jordan-Wigner transformation below

$$a_i^\dagger = JW_i^{(\varphi)^\dagger} f_i^\dagger,$$

$$a_i = JW_i^{(\varphi)} f_i,$$

$$JW_i^{(\varphi)} = \exp\left\{i\varphi \sum_{k=1}^{i-1} f_k^\dagger f_k \right\}.$$  

The Jordan-Wigner operator $JW_i^{(\varphi)}$ transmute statistics in a way similar to the Chern-Simons field $[30]$. Number operators for this system are as in the previous cases, and we represent them by $N_i^A = a_i^\dagger a_i$ which have eigenvalues $n_i^A$ being either 0 or 1. The Fock basis states for anyons are, therefore

$$|n_1^A, ..., n_m^A\rangle = (a_1^\dagger)^{n_1^A} ... (a_m^\dagger)^{n_m^A} |0, A\rangle,$$

Anyonic passive linear optical elements are the unitaries

$$PS_i(\theta) = \exp\{i\theta H_i^{PS}\}$$

$$BS_{ij}(\theta) = \exp\{i\theta H_{ij}^{BS}\}$$

with hamiltonians given by

$$H_i^{PS} = N_i^A,$$

$$H_{ij}^{BS} = a_i^\dagger a_j + a_j^\dagger a_i.$$  

The computational model is as before: the computational basis states are the Fock states, the unitaries are all possible combinations of phase-shifters and beam-splitters for arbitrary pairs of modes (this arbitrariness is essential for of our result) and measurements are allowed in the Fock basis only.

From now on we consider only the dynamics of these Hamiltonians. In the anyonic case, the evolution of creation operators is more involved, and this is what we discuss now. For phase-shifters we have that

$$PS_i(\theta)a_j^\dagger PS_i(-\theta) = e^{i\theta H_i^{PS}} a_j^\dagger,$$

just like the bosonic and fermionic cases. The action of beam-splitters, on the other hand, requires more attention. We will solve the dynamical problem defined by the $H_{ij}^{BS}$ hamiltonian. The Heisenberg equations of motion are

$$i \frac{da_i^\dagger}{d\theta} = [H_{ij}^{BS}, a_i^\dagger],$$

$$i \frac{da_j}{d\theta} = [H_{ij}^{BS}, a_j],$$

where the commutators are calculated using the creation and annihilation operator algebra for anyons [Eqs. (13)]:

$$[H_{ij}^{BS}, a_i^\dagger] = a_j^\dagger \{ 1 - (1 - e^{i\varphi}) N_i^A \},$$

$$[H_{ij}^{BS}, a_j] = a_i^\dagger \{ 1 - (1 - e^{-i\varphi}) N_j^A \}.$$  

### B. Solving the dynamical equations

Let us start by rewriting the equation for mode $i$ in a more suggestive form:

$$i \frac{da_i^\dagger}{d\theta} = a_j^\dagger W_i^{(\varphi)},$$

where we have introduced a short-hand notation for the non-linear term $W_i^{(\varphi)} = 1 - (1 - e^{i\varphi}) N_i^A$. By computing the commutator $[H_{ij}^{BS}, a_i^\dagger W_i^{(\varphi)}]$ we find that the equation of motion for this operator is given by

$$i \frac{d(a_j^\dagger W_i^{(\varphi)})}{d\theta} = a_i^\dagger,$$

therefore this is a coupled linear system of equations for the operators $a_i^\dagger$ and $a_j^\dagger W_i^{(\varphi)}$, which is exactly solvable. The equation of motion for mode $j$ has a similar property giving the system

$$i \frac{da_j^\dagger}{d\theta} = a_i^\dagger W_j^{(\varphi)},$$

$$i \frac{d(a_i^\dagger W_j^{(\varphi)})}{d\theta} = a_j^\dagger.$$  

By linearity, the solutions of the equations for $a_i^\dagger$ and $a_j^\dagger$ must be:

$$BS_{ij}(\theta)a_k^\dagger BS_{ij}(-\theta) = \cos \theta a_k^\dagger + i \sin \theta a_j^\dagger W_i^{(\varphi)},$$

$$BS_{ij}(\theta)a_j^\dagger BS_{ij}(-\theta) = \cos \theta a_j^\dagger + i \sin \theta a_i^\dagger W_j^{(\varphi)}.$$  

Notice that these solutions are very similar to the corresponding dynamics of fermions and bosons [Eqs.(5),(9b)].

To complete our description we need to discuss what happens with the modes that do not appear in the Hamiltonian. In the fermionic and bosonic cases, these modes commute with $H_{ij}^{BS}$, but in the anyonic case creation operators $a_k^\dagger$ with $i < k < j$ do not commute with $H_{ij}^{BS}$. In fact, they satisfy the relation

$$\exp\{i\theta(a_k^\dagger a_j + a_j^\dagger a_i)\} a_k^\dagger = a_k^\dagger \exp\{i\theta(e^{i2\varphi} a_k^\dagger a_j + e^{-i2\varphi} a_j^\dagger a_i)\},$$

which we can treat as an effective beam-splitter that acts on states by introducing a phase correction dependent on the number of modes occupied between $i$ and $j$. Or putting it in a more compact notation

$$BS_{ij}(\theta)a_k^\dagger = a_k^\dagger BS_{ij}^{(2\varphi)}(\theta),$$
with the new effective beam-splitter unitary defined by

$$BS_{ij}^{(\alpha)}(\theta) = \exp \left\{ i \theta (e^{i\alpha} a_i^\dagger a_j + e^{-i\alpha} a_j^\dagger a_i) \right\} ,$$

(25b)

and the solution to the equations of motion for modes $i$ and $j$ is:

$$a^{(\alpha)}_i(\theta) = \cos \theta a_i^\dagger + ie^{i\alpha} \sin \theta a_j^\dagger W_i^{(\varphi)},$$

(25c)

$$a^{(\alpha)}_j(\theta) = \cos \theta a_j^\dagger - ie^{-i\alpha} \sin \theta a_i^\dagger W_j^{(\varphi)}.$$  

(25d)

**Example.** To illustrate this phase correction, consider a system with 3 modes, and choose $i = 1$ and $j = 3$. Let us act with a balanced beam-splitter ($\theta = \pi$) on state $|0, 1, 1\rangle = a_2^\dagger a_3^\dagger |0\rangle_A$. The first step of the calculation is to use Eq. (25a):

$$BS_{13} \left(\frac{\pi}{4}\right) a_2^\dagger a_3^\dagger |0\rangle_A = a_2^\dagger \left[ BS_{13}^{(2\varphi)} \left(\frac{\pi}{4}\right) \right] a_3^\dagger |0\rangle_A .$$

(26)

Now we need to analyze the dynamics of $a_3^\dagger |0\rangle_A$ under the action of $BS_{13}^{(2\varphi)} \left(\frac{\pi}{4}\right)$. Using Eqs. (25c) and (25d), and the fact that $W_i^{(\varphi)} |0\rangle_A = |0\rangle_A$ for every mode $i$ and angle $\varphi$ we obtain:

$$a_2^\dagger \left[ BS_{13}^{(2\varphi)} \left(\frac{\pi}{4}\right) \right] a_3^\dagger |0\rangle_A = \frac{1}{\sqrt{2}} a_2^\dagger (ie^{2i\varphi} a_1^\dagger + a_3^\dagger) |0\rangle_A .$$

(27)

We now use the commutation relation $a_2^\dagger a_1^\dagger = -e^{-i\varphi} a_1^\dagger a_2^\dagger$ to write the final equation in normally ordered form

$$\frac{1}{\sqrt{2}} a_2^\dagger (-ie^{2i\varphi} a_1^\dagger + a_3^\dagger) |0\rangle_A = \frac{1}{\sqrt{2}} (-ie^{i\varphi} a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger) |0\rangle_A .$$

(28)

If we compare this result with the same Hamiltonian applied on state $a_1^\dagger a_2^\dagger |0\rangle_A$, we see that the effect of the phase-corrected Hamiltonian is to guarantee unitarity of the time evolution operator. To see this use the solution of the equations of motion to obtain

$$BS_{13} \left(\frac{\pi}{4}\right) a_1^\dagger a_2^\dagger |0\rangle_A = \frac{1}{\sqrt{2}} (a_1^\dagger + ia_2^\dagger) a_2^\dagger |0\rangle_A .$$

(29)

Finally, using $a_3^\dagger a_2 = -e^{-i\varphi} a_2^\dagger a_3^\dagger$ we get

$$\frac{1}{\sqrt{2}} (a_1^\dagger + ia_3^\dagger) a_3^\dagger |0\rangle_A = \frac{1}{\sqrt{2}} (a_1^\dagger a_2^\dagger - ie^{-i\varphi} a_2^\dagger a_3^\dagger) |0\rangle_A .$$

(30)

Therefore the total effect on the Fock basis is given by

$$BS_{13} \left(\frac{\pi}{4}\right) |1, 1, 0\rangle = \frac{1}{\sqrt{2}} ((1, 1, 0) - ie^{-i\varphi} |0, 1, 1\rangle),$$

(31a)

$$BS_{13} \left(\frac{\pi}{4}\right) |0, 1, 1\rangle = \frac{1}{\sqrt{2}} (-ie^{i\varphi} |1, 1, 0\rangle + |0, 1, 1\rangle).$$

(31b)

which is manifestly unitary.

If the anyon in mode 1 tunnels to mode 3 when another anyon occupies mode 2, a relative phase factor between modes appear, and it is easy to see that if mode 2 was empty this factor would not appear. This dynamics can be understood as an one-dimensional analogue of the Aharonov-Bohm effect, where the magnetic flux carried by the particle is given by $\pi - \varphi$ and the $\pi$ factor account for the $-1$ fermionic phase appearing when $\varphi = 0$. This extra phase factors are crucial to understanding the difference in computational power between each model, as shown in the next section.

**IV. THE COMPUTATIONAL POWER OF ANYONS**

In [26], it was shown how to construct an encoded, entangling two-qubit gate using only nearest and next-nearest neighbour Hamiltonians ($XY$ interaction) between physical qubits, to achieve universal quantum computation. In this section we will generalize this construction to show that quadratic dynamics of fermionic anyons is also capable of universal quantum computation. We begin by defining a qubit encoding that is preserved by dynamics of either free fermions, hard-core bosons, or free fermionic anyons. We will then describe linear-optical circuits on fermionic anyons and its action on encoded states. We prove that the two-qubit logical gate implemented by this circuit is deterministic and entangling for any value of the statistical parameter $\varphi \neq 0$. This, together with single-qubit unitaries, shows that linear optics on fermionic anyons is universal for quantum computation.

**A. Encoding**

We use $2n$ modes to encode $n$ qubits such that each logical qubit corresponds to a pair of neighboring modes as in the equations

$$|0_L\rangle = |1, 0\rangle$$

(32a)

$$|1_L\rangle = |0, 1\rangle$$

(32b)

So, for example, a two-qubit system needs four modes and the logical states are given by

$$|00\rangle_L = |1, 0, 1, 0\rangle ,$$

(33a)

$$|01\rangle_L = |1, 0, 0, 1\rangle ,$$

(33b)

$$|10\rangle_L = |0, 1, 1, 0\rangle ,$$

(33c)

$$|11\rangle_L = |0, 1, 0, 1\rangle ,$$

(33d)

where the right-hand-side of these equations are Fock states. This encoding is independent of the parameter $\varphi$, since all of these particles obey Pauli Exclusion Principle, which allows the direct comparison of logical gates between models.
B. Encoded one- and two-qubit gates

With this encoding it is possible to do any logical one-qubit gate using only phase-shifters and beam-splitters on the two corresponding modes. To prove this, consider a single qubit encoded in modes 1 and 2, and notice that a phase-shifter in mode 2 acts in the logical basis states as

\[ PS_2(\theta) |1,0\rangle = |1,0\rangle, \]  
\[ PS_2(\theta) |0,1\rangle = e^{i\theta} |0,1\rangle, \]  

which is equivalent to a logical Z rotation in the Bloch sphere by $\theta$ degrees. Note also that a beam-splitter between modes 1 and 2 acts in the logical basis states as

\[ BS_{12}(\theta) |1,0\rangle = \cos \theta |1,0\rangle + i \sin \theta |0,1\rangle, \]  
\[ BS_{12}(\theta) |0,1\rangle = i \sin \theta |1,0\rangle + \cos \theta |0,1\rangle, \]

which is equivalent to a logical X rotation in the Bloch sphere, by an angle $\theta$. With arbitrary rotations around two axes in the Bloch sphere, we can perform an arbitrary single-qubit gate, as in Fig.1.

![Fig. 1. Single-qubit unitary decomposed in optical elements: A single-qubit unitary needs four parameters ($\alpha, \beta, \gamma, \delta$). The first one is a global phase and the others are realized by the optical elements in the figure.](image)

To implement an encoded two-qubit gate we generalize the XY-interaction protocol found in [26] (see Fig. 2), adapting the construction to our anyonic model.

![Fig. 2. Two-qubit gate: sequence of beam-splitters that generate our entangling gate](image)

When the particles considered are hard-core bosons (or spins), we recover the original result of [26] and the circuit executes the logical gate $\sqrt{-ZZ}$ which is a maximally entangling gate.

In Appendix A we calculate the effect of this circuit on fermionic anyons characterized by the deformed fermionic algebra of Eqs. (13), for any value of the deformation parameter $\varphi$. Its action on the encoded qubits is the gate

\[ C(\varphi) = R_3(\frac{\pi}{2}) \otimes |0\rangle \langle 0| + R_6(\frac{\pi}{2}) \otimes |1\rangle \langle 1|, \]  

where $R_3(\varphi)$ is a rotation of $\varphi$ around the Z axis in the Bloch sphere, and $R_6(\varphi)$ is a $\frac{\pi}{2}$ rotation around the azimuthal angle $\hat{a} = (-\sin \varphi, 0, \cos \varphi)$ in the Bloch sphere. So this is a controlled rotation gate whose action depends continuously on the parameter $\varphi$ characterizing our anyonic model. We have, for the special cases of fermions and hard-core bosons,

\[ C(\varphi) = \begin{cases} 
Z \otimes 1, & \text{if } \varphi = 0 \\
\sqrt{-ZZ}, & \text{if } \varphi = \pi, 
\end{cases} \]

which in the fermionic case is a local gate, and for hard-core bosons is as described above. Therefore, this gate interpolates between the corresponding ones for the other two models.

C. The entangling power of $C(\varphi)$

We claim that gate $C(\varphi)$ [Eq. (36)], together with arbitrary single-qubit gates, form a set that is universal for quantum computing whenever $\varphi \neq 0$. To prove this it is sufficient to show that $C(\varphi)$ has a non-zero entangling power (as defined by [31]).

The entangling power $e_p(U)$ of a unitary gate $U$ is defined as the average entanglement of formation generated by the action of $U$ on product states $|\psi_1\rangle \otimes |\psi_2\rangle$

\[ e_p(U) = E(U |\psi_1\rangle \otimes |\psi_2\rangle)(|\psi_1\rangle,|\psi_2\rangle), \]

where the bar denotes average with respect to some probability distribution $p(\psi_1, \psi_2)$. It can be shown that, if the average is taken over the uniform distribution, the entangling power is both local invariant and SWAP invariant (that is, it remains the same if $U$ is conjugated by SWAP or by single-qubit gates). In fact, this invariant can be easily calculated in terms of simpler invariants, which was done in [32]. Two-qubit gates have two local invariant quantities, given by

\[ G_1 = \frac{\text{Tr}^2 U_B^T U_B}{16 \det(U)}, \]  
\[ G_2 = \frac{\text{Tr}^2 U_B^T U_B - \text{Tr}\{(U_B^T U_B)^2\}}{4 \det(U)}, \]

where $U_B$ is the matrix representation of the gate $U$ written in the Bell basis. With these invariants, the (normalized) entangling power $e_p(U)$ of a two-qubit gate $U$ over the uniform distribution is just given by [32]

\[ e_p(U) = 1 - |G_1|. \]
With this in hands, the entangling power of $C(\varphi)$ [Eq. (36)] is:
\[
e_p(C(\varphi)) = 1 - \cos^4 \frac{\varphi}{2}
\] (42)

This shows that any $\varphi \neq 0$ results in a fermionic anyon model that allows for universal quantum computation. In fact, since this gate generates entanglement, it can be used to construct encoded CNOT gates using the argument given in [33], with the number of required $C(\varphi)$ gates depending on the value of $e_p(C(\varphi))$.

V. CONCLUSION

We have generalized the model of passive Fermionic Linear Optics by studying anyonic systems defined by deformations of fermionic anti-commutation relations. We have taken quadratic, number-preserving Hamiltonians as the analogue of fermionic linear-optical dynamics, describing their action on Fock states. We have shown that the difference to fermionic dynamics is due to an one-dimensional analogue of the Aharonov-Bohm effect, and that this is required to preserve unitarity.

We have generalized a scheme for quantum computing with nearest- and next-nearest-neighbour spin-$\frac{1}{2}$ interactions, showing that the analogous interactions in our model allow for universal quantum computation. This happens for any value of the deformation parameter $\varphi \neq 0$, that is, as long as our anyons differ from fermions. Given that free fermions can be simulated efficiently, this means the transition in computing power is abrupt, going from classically simulable to universal for quantum computation when $\varphi \neq 0$.

This raises the question of whether or not such interactions arising from statistics alone can give computational advantages in other settings, such as Boson Sampling, which is hard to simulate classically given some complexity-theoretic assumptions, but is not known to be universal for quantum computation. Another open problem involves investigating what restrictions can be imposed on this model to make it hard to simulate classically (but not necessarily universal for quantum computation).

VI. ACKNOWLEDGEMENTS

This work was supported by project Instituto Nacional de Ciência e Tecnologia de Informação Quântica (INCT-IQ/CNPq).

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Appendix A: Explicit expression of $C(\varphi)$

In this Appendix we calculate the two-qubit unitary implemented by the anyonic linear-optical circuit of Fig. 2. For this, we need to calculate the matrix elements of the various beam-splitters in the two-particle basis, and see if our protocol preserves the encoding of qubits. Given the four modes of Fig. 2 the two-particle basis is given by the states

$$
|1, 1, 0, 0\rangle, |1, 0, 1, 0\rangle, |1, 0, 0, 1\rangle, |0, 1, 1, 0\rangle, |0, 1, 0, 1\rangle, |0, 0, 1, 1\rangle,
$$

(A1a)

(A1b)



There are three kinds of beam-splitters in the circuit, $BS_{12}$, $BS_{23}$ and $BS_{13}$, where the first two appear more then once, with two different angles (defining the beamsplitting ratios). First let us calculate the matrix representation of the general beam-splitter $BS_{12}(\theta)$ in our two-particle basis. The idea is to use the representation in terms of creation operators and use the solutions to the equations of motion to find the matrix elements. For brevity, we will do the calculation for the more involved basis states, as the others will be similar in form, and easier to calculate.

The first case will be $a_{1}^{+}a_{2}^{+}|0_{A}\rangle$,

$$
BS_{12}(\theta)a_{1}^{+}a_{2}^{+}|0_{A}\rangle =
$$

$$=[BS_{12}(\theta)a_{1}^{+}BS_{12}(-\theta)]BS_{12}(\theta)a_{2}^{+}|0_{A}\rangle,
$$

(A2)

where $BS_{12}(\theta)|0_{A}\rangle = |0_{A}\rangle$. We can now use the solutions to the equations of motion to obtain

$$
[BS_{12}(\theta)a_{1}^{+}BS_{12}(-\theta)]BS_{12}(\theta)a_{2}^{+}|0_{A}\rangle =
$$

$$=(\cos\theta a_{1}^{+} + i\sin\theta a_{2}^{+} W_{1}^{(\varphi)})\times
$$

$$\times (\cos\theta a_{2}^{+} + i\sin\theta a_{1}^{+} W_{2}^{(\varphi)})|0_{A}\rangle,
$$

(A3)

Using that $|W_{1}^{(\varphi)}a_{2}^{+}\rangle = 0$ we can rewrite this as:

$$
\cos^{2}\theta a_{1}^{+} a_{2}^{+}|0_{A}\rangle - \sin^{2}\theta a_{2}^{+} W_{1}^{(\varphi)} a_{1}^{+}|0_{A}\rangle,
$$

(A4)

in the next step, we use the identity $N_{A}^{a_{1}} a_{1}^{+} = a_{1}^{+}$ to obtain

$$
\cos^{2}\theta a_{1}^{+} a_{2}^{+}|0_{A}\rangle - \sin^{2}\theta a_{2}^{+} [1 - (1 - e^{i\varphi})N_{A}^{a_{1}}]|a_{1}^{+}|0_{A}\rangle =
$$

$$= \cos^{2}\theta a_{1}^{+} a_{2}^{+}|0_{A}\rangle - e^{i\varphi} \sin^{2}\theta a_{1}^{+} a_{1}^{+}|0_{A}\rangle,
$$

(A5)

Finally, using the commutation relation $a_{1}^{+} a_{2}^{+} = -e^{i\varphi} a_{2}^{+} a_{1}^{+}$ we conclude that

$$
BS_{12}(\theta)a_{1}^{+} a_{2}^{+}|0_{A}\rangle =
$$

$$= \cos^{2}\theta a_{1}^{+} a_{2}^{+}|0_{A}\rangle - e^{i\varphi} \sin^{2}\theta a_{2}^{+} a_{1}^{+}|0_{A}\rangle =
$$

$$= (\cos^{2}\theta + \sin^{2}\theta) a_{1}^{+} a_{2}^{+}|0_{A}\rangle = a_{1}^{+} a_{2}^{+}|0_{A}\rangle,
$$

(A6)

So the matrix element $\langle 1, 1, 0, 0|BS_{12}(\theta)|1, 1, 0, 0\rangle$ is 1. Similarly $\langle 0, 0, 1, 1|BS_{12}(\theta)|0, 0, 1, 1\rangle$ is also 1, because the beam-splitter has no action on these modes. Now we illustrate one more case, and then give the expression for $BS_{12}(\theta)$. Consider the state $a_{1}^{+} a_{3}^{+}|0_{A}\rangle$, we can proceed in pretty much the same way we did before, and obtain

$$
BS_{12}(\theta)a_{1}^{+} a_{3}^{+}|0_{A}\rangle =
$$

$$= [BS_{12}(\theta)a_{1}^{+}BS_{12}(-\theta)]BS_{12}(\theta)a_{3}^{+}|0_{A}\rangle =
$$

$$= [BS_{12}(\theta)a_{1}^{+}BS_{12}(-\theta)]a_{3}^{+}|0_{A}\rangle =
$$

$$= (\cos\theta a_{1}^{+} + i\sin\theta a_{2}^{+} W_{1}^{(\varphi)})a_{3}^{+}|0_{A}\rangle =
$$

$$= \cos\theta a_{1}^{+} a_{3}^{+}|0_{A}\rangle + i\sin\theta a_{2}^{+} a_{3}^{+}|0_{A}\rangle,
$$

(A7)

which tells us that $\langle 1, 0, 1, 0|BS_{12}(\theta)|1, 0, 1, 0\rangle = \cos\theta$ and $\langle 0, 1, 1, 0|BS_{12}(\theta)|1, 0, 1, 0\rangle = i\sin\theta$. Doing the cal-
calculation of the other matrix elements we obtain
\[
[BS_{12}(\varphi)] = \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & i \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & 0 & i \sin \theta & 0 \\
i \sin \theta & 0 & \cos \theta & 0 & 0 & 0 \\
0 & 0 & i \sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\] (A8)

The matrix for \(BS_{23}(\theta)\) is very similar, since most of the calculations of matrix are elements repeated with different indices. The matrix is
\[
[BS_{23}(\theta)] = \\
\begin{bmatrix}
\cos \theta & i \sin \theta & 0 & 0 & 0 & 0 \\
i \sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & i \sin \theta & 0 \\
0 & 0 & 0 & 0 & \sin \theta & \cos \theta \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\] (A9)

The remaining matrix is the one which marks a departure between fermions and anyons, as the 1D analogue of the Aharonov-Bohm phase appears explicitly. To calculate it we must use the result of Example III B for the matrix elements \(\langle 1100|BS_{13}(\theta)|1100\rangle\), \(\langle 0110|BS_{13}(\theta)|1100\rangle\), \(\langle 1100|BS_{13}(\theta)|0110\rangle\), and \(\langle 0110|BS_{13}(\theta)|0110\rangle\). The other matrix elements are trivial. The result of this calculation is
\[
[BS_{13}(\theta)] = \\
\begin{bmatrix}
\cos \theta & 0 & 0 & i \sin \theta & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta & 0 & 0 & -ie^{-i\varphi} \sin \theta \\
i \sin \theta & 0 & 0 & \cos \theta & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -ie^{i\varphi} \sin \theta & 0 & 0 & \cos \theta
\end{bmatrix}.
\] (A10)

Now, combining all of these results to evaluate the matrix products indicated by the circuit (Fig. 2) we obtain the matrix
\[
[C(\varphi)] = \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
e^{-i\frac{\varphi}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 - i \cos \varphi & 0 & i \sin \varphi & 0 \\
0 & 0 & 0 & e^{i\frac{\varphi}{2}} & 0 & 0 \\
0 & 0 & i \sin \varphi & 0 & 1 + i \cos \varphi & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\] (A11)

To stay in the encoding defined in section IV, the image of the encoded states under the unitary must stay in the encoded subspace, and to guarantee this we only need to show that \(|1,1,0,0\rangle\) and \(|0,0,1,1\rangle\) are eigenstates of \(C(\varphi)\), which is easily seen in the matrix above. In fact \(C(\varphi)\) in the encoded basis is
\[
\begin{bmatrix}
e^{-i\frac{\varphi}{2}} & 0 & 0 & 0 \\
0 & 1 - i \cos \varphi & 0 & i \sin \varphi \\
0 & 0 & e^{i\frac{\varphi}{2}} & 0 \\
i \sin \varphi & 0 & 0 & 1 + i \cos \varphi \\
0 & 0 & 0 & 0
\end{bmatrix},
\] (A12)

showing that \(C(\varphi) = R_{z}\left(\frac{\varphi}{2}\right) \otimes |0\rangle \langle 0| + R_{\hat{n}}\left(\frac{\pi}{2}\right) \otimes |1\rangle \langle 1|\), with \(\hat{n} = (-\sin \varphi, 0, \cos \varphi)\) as claimed.