Anomalous quantum mechanics *

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The fractional operators together with exponential quantum in coordinate and momentum space corresponding to the power of observables are introduced. Based on an exponential relation between energy and momentum, the fractional Schrödinger equations for the free particle and the one in potential fields in heterogeneous complex media are found. The fractional equation of motion and the fractional virial theorem for anomalous quantum mechanics are then developed. Applying the fractional virial theorem, we derive an anomalous hydrogen atom whose transition energy values are much higher than that of Bohr hydrogen atom. The anomalous Heisenberg picture being equivalent to the fractional Schrödinger picture is also discussed.

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1. Introduction

In classical mechanics, the general requirements imposed by the homogeneity and isotropy of space and by Galileo’s relativity principle lead to a quadratic dependence of the energy of the particle on its momentum: $E = p^2/2m$, where the constant $m$ is the mass of the particle.[1–4] However, most of the actual systems are nonhomogeneous and in which the relations $E = p^2/2m$ does not still hold for the energy and momentum. In recent years anomalous properties in disordered systems and the nonquadratic dependences of the energy on momentum attracted growing attention.[5–15]

In 2000, Laskin first discussed the anomalous properties in quantum physics and derived the fractional Schrödinger equation with anomalous exponent $1 < \alpha \leq 2$ by applying the path integrals over the Lévy paths.[10]
The approach to quantum dynamics describing the state of a particle by state function $\varphi(x,t)$ is the Schrödinger equation, known as the Schrödinger picture. Another formulation of quantum dynamics is the Heisenberg picture where observables or the corresponding operators depend on time and obey the Heisenberg equation of motion. The Heisenberg picture is a particularly useful device in relativistic quantum mechanics.

In this article we shall introduce anomalous quantum mechanics by introducing fractional operators and exponential quanta. Besides, based on exponential hypothesis that in heterogeneous complex media the energy $E$ and momentum $p$ of a particle of mass $m$ satisfy $E = \frac{p^{2\alpha}}{(2m)^{\alpha}}$, we derived the fractional Schrödinger equation for the free particle and the one in the potential $V(x^\beta)$. We also derive the fractional equation of motion and the fractional virial theorem for anomalous quantum mechanics. Moreover, we give an anomalous hydrogen atom whose transition energy values are much higher and can match up the anomalous X–ray spectrums observed by Wang, etc.. Finally, we will discuss anomalous Heisenberg picture being equivalent to the fractional Schrödinger equation.

2. Fractional operators and exponential quanta

2.1. Anomalous operators in coordinate space

We start by introducing the fractional operators and exponential quanta in coordinate space for anomalous quantum mechanics. For simplicity, in what follows we shall consider a particle moving in $x$ direction.

For a state normalized according to condition $\int \psi^*(x,t)\psi(x,t)dx = 1$, the average value of the power of coordinate $x^\alpha$ with the exponent $\alpha > 0$ is

$$\langle x^\alpha \rangle = \int \psi^*(x,t)x^\alpha\psi(x,t)dx \quad (1)$$

where $\psi^*(x,t)\psi(x,t) = |\psi(x,t)|^2$ denotes the probability of the various values of coordinate, and $x^\alpha$ is regarded as the operator corresponding to $x^\alpha$ that multiplies the wave function by $x^\alpha$, that is,

$$\hat{x}^\alpha = x^\alpha. \quad (2)$$

We now consider the operator $\hat{p}^\alpha$ corresponding to the power of momentum $p^\alpha$ in coordinate space. Note that this operator must be defined so that the average value of $p^\alpha$ can be represented
in the form
\[ \langle p^\alpha \rangle = \int \psi^*(x,t) \hat{p}^\alpha \psi(x,t) dx. \] (3)

On the other hand, this average value is determined from the momentum wave function by
\[ \langle p^\alpha \rangle = \int \phi^*(p,t) p^\alpha \phi(p,t) dp. \] (4)

where \( \phi(p,t) \) is related to \( \psi(x,t) \) by the reciprocal Fourier transforms:
\[ \psi(x,t) = \frac{1}{(2\pi \hbar)^{1/2}} \int \phi(p,t) \exp(\frac{i}{\hbar} p \cdot x) dp, \] (5)

and
\[ \phi(p,t) = \frac{1}{(2\pi \hbar)^{1/2}} \int \psi(x,t) \exp(-\frac{i}{\hbar} p \cdot x) dx. \] (6)

Substituting the expression (6) for \( \phi^*(p,t) \), we have
\[ \langle p^\alpha \rangle = \int \int \psi^*(x,t) \frac{1}{(2\pi \hbar)^{1/2}} \exp(\frac{i}{\hbar} p \cdot x) p^\alpha \phi(p,t) dx dp. \] (7)

Let us write
\[ \exp(\frac{i}{\hbar} p \cdot x) p^\alpha = [(-i\hbar)^\alpha \hat{D}_x^\alpha] \exp(\frac{i}{\hbar} p \cdot x), \]
where \( \hat{D}_x^\alpha \) is a type of fractional derivative\cite{18}, satisfying
\[ \hat{D}_x^\alpha \exp(ax) = a^\alpha \exp(ax), \] (8)

and integration by parts formula
\[ (\varphi, \hat{D}_x^\alpha \psi) = (\hat{D}_x^\alpha \varphi, \psi). \] (9)

The integral (7) then becomes
\[ \langle p^\alpha \rangle = \int \int \psi^*(x,t) \frac{1}{(2\pi \hbar)^{1/2}} \exp(\frac{i}{\hbar} p \cdot x) p^\alpha \phi(p,t) dx dp. \] (10)

Using Eq.(5), we find
\[ \langle p^\alpha \rangle = \int \psi^*(x,t) [(-i\hbar)^\alpha \hat{D}_x^\alpha] \psi(x,t) dx. \] (11)

Comparing this with (3), we see that the operator of the power of momentum \( p^\alpha \) in the coordinate representation is
\[ \hat{p}^\alpha = (-i\hbar)^\alpha \hat{D}_x^\alpha, \] (12)

which is a fractional operator times exponential quantum. When the power \( \alpha \) is a positive integer, the operator \((-i\hbar)^\alpha \hat{D}_x^\alpha\) reduces to the ordinary operator \((-i\hbar \frac{\partial}{\partial x})^n\). More generally,
the same technique gives for analytic function of \( p^\alpha \) that can be expressed as a power series
\[
f(p^\alpha) = \sum C_n (p^\alpha)^n
\]
where \( n \) denotes positive integer, then yields
\[
\langle f(p^\alpha) \rangle = \int \psi^*(x, t) \{ \sum C_n [(-ih)\alpha D^\alpha_x]^n \} \psi(x, t) dx,
\]
from which we have the function \( f(p^\alpha) \) has a corresponding operator \( f(\hat{p}^\alpha) \), which we call anomalous operator in coordinate space.

### 2.2. Anomalous operators in momentum space

We now propose the fractional operators and exponential quanta in momentum space for anomalous quantum mechanics. One can find that the operator corresponding to \( p^\alpha \) in momentum space is represented as \( p^\alpha \) for \( \alpha > 0 \), since the expectation value of the power of momentum \( p^\alpha \) is
\[
\langle p^\alpha \rangle = \int \phi^*(p, t)p^\alpha \phi(p, t)dp.
\]
In analogy with the evaluation of \( p^\alpha \) in \( x \) space, we get
\[
\langle x^\alpha \rangle = \int \psi^*(p, t)[(ih)\alpha D^\alpha_p] \psi(p, t)dp,
\]
which implies that the operator corresponding to \( x^\alpha \) in momentum space is a fractional operator times exponential quantum
\[
x^\alpha = (ih)\alpha D^\alpha_p
\]
Here \( D^\alpha_p \) is a type of fractional derivative, satisfying
\[
D^\alpha_p \exp(ap) = a^\alpha \exp(ap),
\]
and integration by parts formula
\[
(\varphi, D^\alpha_p \psi) = (D^\alpha_p \varphi, \psi).
\]
If \( f(x^\alpha) = \sum C_n (x^\alpha)^n \), it can then be shown that
\[
\langle f(x^\alpha) \rangle = \int \phi^*(p, t) \{ \sum C_n [(ih)\alpha D^\alpha_p]^n \} \phi(p, t)dp,
\]
from which we get
\[
f(\hat{x}^\alpha) = f[(ih)\alpha D^\alpha_p].
\]
We call this generalized operator, anomalous operator in momentum space.
2.3. Linearity and Hermiticity of anomalous operators

Because of

\[ (-i\hbar)^\alpha D_\alpha^x (\psi_1(x) + \psi_2(x)) = (-i\hbar)^\alpha D_\alpha^x (\psi_1(x)) + (-i\hbar)^\alpha D_\alpha^x (\psi_2(x)), \]

and

\[ (-i\hbar)^\alpha D_\alpha^x (C \psi(x)) = C (-i\hbar)^\alpha D_\alpha^x (\psi(x)), \]

we obtain that the operator \( \hat{p}^\alpha = (-i\hbar)^\alpha D_\alpha^x \) is a linear operator. Here \( C \) is an arbitrary constant.

We now consider the transposed operator

\[ (-i\hbar)^\alpha D_\alpha^x = (i\hbar)^\alpha D_\alpha^x. \]

According to the definition of transposed operator, we get

\[
\int_{-\infty}^{+\infty} \varphi^*(x,t)(-i\hbar)^\alpha D_\alpha^x \psi(x,t)dx = \int_{-\infty}^{+\infty} \psi(x,t)(i\hbar)^\alpha D_\alpha^x \varphi^*(x,t)dx
\]

\[
= \int_{-\infty}^{+\infty} ((-i\hbar)^\alpha D_\alpha^x \varphi(x,t))^* \psi(x,t)dx,
\]

where the superscript * denotes the complex conjugate. Thus,

\[ ((-i\hbar)^\alpha D_\alpha^x)^* = (-i\hbar)^\alpha D_\alpha^x. \]

This means that the operator \( \hat{p}^\alpha = (-i\hbar)^\alpha D_\alpha^x \) is a Hermitian or self-adjoint operator. We also have any anomalous operator \( \hat{O}^\alpha \) that can be expressed as a power series \( \sum_{i=1}^n C_n (\hat{p}^\alpha)^n \) is Hermitian.

Similarly, one can obtain that the operator \( x^\alpha \) and the function \( f(\hat{x}^\alpha) \) in momentum space is linear and Hermitian.

3. Fractional Schrödinger equation

We assume that in heterogeneous complex media the energy \( E \) and momentum \( p \) of a particle of mass \( m \) are related by an exponential relation

\[ E = \frac{p^{2\alpha}}{(2m)^\alpha}. \]
where the constant $m$ is the mass of the particle. Combining Eq.(18) with the de Broglie hypothesis

$$P = \hbar v/c,$$

and the Einstein relation

$$E = \hbar v,$$

where $v$ denotes the frequency of the incident, $c$ is the velocity of light, $h$ is Planck’s constant, we obtain the complete (time-dependent) wave function as

$$\psi(x, t) = Ce^{i\hbar(px - p^2 \alpha / (2m) \alpha t) / \hbar},$$ \hspace{1cm} (19)

where $C$ is a constant. Each such function, a plane wave, describes a state in which the free particle has a definite energy $p^\alpha / (2m)\alpha$ and momentum $p$. We now look for a differential equation having (19) as its solution. We have from (19)

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \frac{p^{2\alpha}}{(2m)^\alpha} \psi(x, t),$$ \hspace{1cm} (20)

and

$$(-i\hbar)^\alpha D_x^\alpha \psi(x, t) = p^\alpha \psi(x, t),$$ \hspace{1cm} (21)

so what we can write the differential equation

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \left( (-i\hbar)^\alpha D_x^\alpha \right)^2 \psi(x, t),$$ \hspace{1cm} (22)

Similarly, we can find Eq.(22) for a free particle with a wave packet,

$$\psi(x, t) = \frac{1}{(2\pi\hbar)^{1/2}} \int \phi(p, t) e^{i\hbar(px - p^2 \alpha / (2m) \alpha t) / \hbar} dp.$$

A generalization of the free particle wave equation to the case of motion of a particle acted by a force given by a potential energy function $V(x^\beta)$, is the fractional Schrödinger equation:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \left( (-i\hbar)^\alpha D_x^\alpha \right)^2 \psi(x, t) + V(x^\beta) \psi(x, t).$$ \hspace{1cm} (23)

Noting that $((-i\hbar)^\alpha D_x^\alpha)^2 = (p^\alpha)^2 = p^{2\alpha}$, we can write Eq.(23) in the form

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \hat{H}_{\alpha, \beta} \psi(x, t),$$ \hspace{1cm} (24)

where $\hat{H}_{\alpha, \beta}$ represents the corresponding operator of total energy or Hamiltonian

$$H_{\alpha, \beta} = T + V = \frac{p^{2\alpha}}{(2m)^\alpha} + V(x^\beta).$$
By the Hermiticity property, in combination with the fact that \( \hat{p}^{\alpha} \) is Hermitian, we have the Hamiltonian \( \hat{H}_{\alpha,\beta} \) is also a Hermitian operator.

If the system is conservative and the Hamiltonian operator \( \hat{H}_{\alpha,\beta} \) does not depend on time \( t \), the solution of (24) may be written as

\[
\psi(x, t) = e^{-iE t / \hbar} \psi_E(x).
\]

(25)

A particle in this state of this type has a well defined energy \( E \), since \( E / \hbar \) is the time rate of change of the phase of the wave function. Substituting (25) into (24) we find the time-independent equation fractional Schrödinger equation

\[
\hat{H}_{\alpha,\beta} \psi_E(x) = E \psi_E(x).
\]

or, more explicitly,

\[
\frac{((-i\hbar)^\alpha D^\alpha)}{(2m)^\alpha} \psi(x, t) + V(x^\beta) \psi(x, t) \psi_E(x) = E \psi_E(x).
\]

For only certain values of \( E \) will this equation have solutions. These values are the possible energy values of the particle, and the corresponding \( \psi_E(x) \) is the wave function of the particle when it has energy \( E \).

### 4. Fractional equation of motion

We then derive the fractional equation of motion for anomalous quantum mechanics.

From the expectation value of one anomalous operator

\[
\langle \hat{O}_\alpha \rangle = \langle \psi(x, t), \hat{O}_\alpha \psi(x, t) \rangle,
\]

we have

\[
\hbar \frac{d}{dt} \langle \hat{O}_\alpha \rangle = i\hbar \langle \frac{\partial \psi(x, t)}{\partial t}, \hat{O}_\alpha \psi(x, t) \rangle + i\hbar \langle \psi(x, t), \hat{O}_\alpha \frac{\partial \psi(x, t)}{\partial t} \rangle + i\hbar \langle \psi(x, t), \frac{\partial \hat{O}_\alpha}{\partial t} \psi(x, t) \rangle.
\]

(27)

From Eq.(24), we find

\[
\hbar \frac{d}{dt} \langle \hat{O}_\alpha \rangle = -\langle \hat{H}_{\alpha,\beta} \psi(x, t), \hat{H}_{\alpha,\beta} \psi(x, t) \rangle + \langle \psi(x, t), \hat{O}_\alpha \hat{H}_{\alpha,\beta} \psi(x, t) \rangle + i\hbar \langle \frac{\partial \hat{O}_\alpha}{\partial t} \psi(x, t) \rangle.
\]

(28)

Since the operator \( \hat{H}_{\alpha,\beta} \) is Hermitian, one obtains

\[
\langle \hat{H}_{\alpha,\beta} \psi(x, t), \hat{O}_\alpha \psi(x, t) \rangle = \langle \psi(x, t), \hat{H}_{\alpha,\beta} \hat{O}_\alpha \psi(x, t) \rangle.
\]
Therefore,
\[
\frac{\hbar}{i} \frac{d}{dt} \langle \hat{O}_{\alpha} \rangle = \langle \hat{O}_{\alpha} \hat{H}_{\alpha,\beta} - \hat{H}_{\alpha,\beta} \hat{O}_{\alpha} \rangle + \hbar \langle \frac{\partial \hat{O}_{\alpha}}{\partial t} \rangle
\]
\[
= \langle [\hat{O}_{\alpha}, \hat{H}_{\alpha,\beta}] \rangle + \hbar \langle \frac{\partial \hat{O}_{\alpha}}{\partial t} \rangle. \tag{29}
\]
In the second series we used the fractional poisson bracket with exponential quanta
\[
[\hat{O}_{\alpha}, \hat{H}_{\alpha,\beta}] = \hat{O}_{\alpha} \hat{H}_{\alpha,\beta} - \hat{H}_{\alpha,\beta} \hat{O}_{\alpha}.
\]
Noting that \( \langle \hat{O}_{\alpha} \rangle = \int \psi^*(x,t) \hat{O}_{\alpha} \psi(x,t) dx \) is independent of the position \( x \), we consider
\[
\frac{d}{dt} \langle \hat{O}_{\alpha} \rangle = \langle \frac{d}{dt} \hat{O}_{\alpha} \rangle = \langle \int \psi^*(x,t) \frac{d\hat{O}_{\alpha}}{dt} \psi(x,t) dx \rangle. \tag{30}
\]
Eq.(29) then becomes
\[
\frac{\hbar}{i} \frac{d}{dt} \hat{O}_{\alpha} = [\hat{O}_{\alpha}, \hat{H}_{\alpha,\beta}] + \hbar \frac{\partial \hat{O}_{\alpha}}{\partial t}, \tag{31}
\]
which is the fractional equation of motion for anomalous quantum mechanics. If the operator \( \hat{O}_{\alpha} \) is independent of time and \( \alpha \) is integer, then Eq.(31) reduces to the usual equation of motion
\[
\frac{\hbar}{i} \frac{d}{dt} \hat{O}_{\alpha} = [\hat{O}_{\alpha}, \hat{H}_{\alpha,\beta}]. \tag{32}
\]

5. Fractional virial theorem

We will further derive the fractional virial theorem for the particle with Hamiltonian
\[
H_{\alpha,\beta} = T + V = \frac{p^{2\alpha}}{(2m)^\alpha} + V(x^\beta). \tag{33}
\]
By Eq.(32), one gets
\[
\frac{\hbar}{i} \frac{d}{dt} (\hat{x} \cdot \hat{p}) = \langle [\hat{x} \cdot \hat{p}, \hat{H}_{\alpha,\beta}] \rangle.
\]
Applying the rules of commutator bracket notation, we obtain
\[
\frac{d}{dt} (\hat{x} \cdot \hat{p}) = \frac{1}{i\hbar} \{ \langle [\hat{x} \cdot \hat{p}, \frac{p^{2\alpha}}{(2m)^\alpha}] \rangle + \langle [\hat{x} \cdot \hat{p}, V(x^\beta)] \rangle \}.
\]
\[
= \frac{1}{i\hbar} \{ 2i\hbar \alpha \langle \frac{p^{2\alpha}}{(2m)^\alpha} \rangle - \hbar \langle \hat{x} \cdot \frac{\partial}{\partial x} V(x^\beta) \rangle \}.
\]
\[
= 2\alpha \langle \frac{p^{2\alpha}}{(2m)^\alpha} \rangle - \langle x \cdot \frac{\partial}{\partial x} V(x^\beta) \rangle. \tag{34}
\]
For a stationary state all expectation values in (34) are constant in time, and it follows that
\[ 2\alpha\langle \hat{T} \rangle = \langle x \cdot \frac{\partial}{\partial x} V(x^\beta) \rangle, \tag{35} \]
where \( \hat{T} \) denotes the anomalous operator for kinetic energy. We call Eq.(34) and (35) the fractional virial theorems. When \( \alpha = 1, \beta = 1 \), Eq.(35) yields the classical virial theorem
\[ 2\langle \frac{\hat{p}^2}{2m} \rangle = \langle x \cdot \frac{\partial}{\partial x} V(x) \rangle. \tag{36} \]

6. Anomalous hydrogen atom

We will consider one kind of anomalous hydrogen atom whose total energy is
\[ H_{\alpha,\beta} = \frac{p^{2\alpha}}{(2m)^\alpha} - \frac{e^2}{4\pi \varepsilon_0 r}, \]
where \( \frac{p^{2\alpha}}{(2m)^\alpha} \) is the kinetic energy, and \( -\frac{e^2}{4\pi \varepsilon_0 r} \) is the potential energy.

According to the fractional virial theorem (35), we obtain
\[ 2\alpha \frac{p^{2\alpha}}{(2m)^\alpha} = \frac{e^2}{4\pi \varepsilon_0 r}. \]
Using the first Niels Bohr’s postulate \( p a_n = n \hbar \), one gets
\[ \frac{2\alpha}{(2m)^\alpha} \left( \frac{n\hbar}{a_n} \right)^{2\alpha} = \frac{e^2}{4\pi \varepsilon_0 a_n}, \tag{37} \]
where \( a_n \) denotes circular orbits of radius. From (37), we find
\[ a_n = \left[ \frac{8\pi \varepsilon_0 h^{2\alpha}}{e^2 (2m)^\alpha} \right]^{1/(2\alpha-1)} n^{2\alpha/(2\alpha-1)} \]. \tag{38} \]
Thus, the total energy levels of the anomalous hydrogen atom is
\[ H_{\alpha,\beta} = (1 - 2\alpha) \left( \frac{me^4}{8h^2 \varepsilon_0^2 \alpha^2} \right)^{\alpha/(2\alpha-1)} n^{-2\alpha/(2\alpha-1)}. \tag{39} \]
When \( \alpha = 1 \), we find the well-known results of the Bohr theory:
\[ a_n = \frac{4\pi \varepsilon_0 h^2}{e^2 m} n^2 = 0.529 \times 10^{-10} m, \tag{40} \]
and
\[ E_n = -\frac{me^4}{8h^2 \varepsilon_0^2} n^2 = 13.6 ev \tag{41} \]
Let us now consider a transition from $k$ to $n (k > n)$. According to Einstein’s relation $\Delta E = hv$, we get the transition energy

$$\Delta E_{kn} = hv = (2\alpha - 1)\left(\frac{me^4}{8\hbar^2\varepsilon_0^2}\right)^{\alpha/(2\alpha - 1)}[n^{-2\alpha/(2\alpha - 1)} - k^{-2\alpha/(2\alpha - 1)}].$$

(42)

Let $\beta = 2\alpha$. Then Eq.(42) can be written as

$$\Delta E_{kn} = (\beta - 1)\left(\frac{2\sqrt{13.6 \times 1.6 \times 10^{-19}}}{\beta}\right)^{\beta/(\beta - 1)}[n^{-\beta/(\beta - 1)} - k^{-\beta/(\beta - 1)}]$$

Assuming $\beta = 2.3566$, we find

- $\Delta E_{2,1} = 2.0357$ kev
- $\Delta E_{3,1} = 2.4767$ kev
- $\Delta E_{4,1} = 2.6464$ kev
- $\Delta E_{20,1} = 2.8920$ kev
- $\Delta E_{120,1} = 2.9073$ kev

which are much higher than what are usual for the Bohr atom. And this theoretical results are in good agreement with the anomalous experimental data detected by Wang, etc. for one new hydrogen atom.\[19–21\] Note also that the first three radiuses of the anomalous hydrogen atom are as following:

- $a_1 = 1.8613 \times 10^{-16} m,$
- $a_2 = 6.205 \times 10^{-16} m,$
- $a_3 = 1.2550 \times 10^{-15} m,$

which are much smaller than the Bohr radius.

7. Anomalous Heisenberg picture.

7.1. Matrix representation of an anomalous operator

When a basis is given, an anomalous operator $\hat{O}_\alpha$ in a given representation can be characterized by its effect on the basis vectors $\{\psi_n\}$. Indeed, being a vector in the form of series $\psi = \sum_n a_n \psi_n$ in the space, $\hat{O}_\alpha \psi$ can obviously be expanded as

$$\hat{O}_\alpha \psi_j = \sum_i \psi_i O_{ij},$$

(43)
where
\[ O_{ij} = (\psi_i, \hat{O}_\alpha \psi_j) \]  \hspace{1cm} (44)
which, owing to the linearity of \( \hat{O}_\alpha \), completely specify the effect of \( \hat{O}_\alpha \) on any vector \( \psi \). To see this explicitly, we note,
\[ \phi = \hat{O}_\alpha \psi = \hat{O}_\alpha \sum_j a_j \psi_j = \sum_j \sum_i a_j \psi_i O_{ij} = \sum_i \psi_i (\sum_j O_{ij} a_j). \]  \hspace{1cm} (45)
Since \( \phi = \sum_i b_i \psi_i \), we find
\[ b_i = \sum_j O_{ij} a_j, \]  \hspace{1cm} (46)
proving the contention that the effect of \( \hat{A}_\alpha \) on any vector is known if all \( O_{ij} \) are known.

### 7.2. Anomalous Heisenberg picture

Let us define a time translation operator
\[ U(t, t_0) = e^{-\frac{i}{\hbar} \hat{H}_{\alpha,\beta}(t-t_0)}, \]  \hspace{1cm} (47)
where \( \hat{H}_{\alpha,\beta} \) is an anomalous Hamiltonian of the system. Using the Dirac notation, we then express the state vector \( |\psi(x, t)\rangle \) in terms of \( |\psi(x, t_0)\rangle \) by the relation: \( |\psi(x, t)\rangle = U(t, t_0)|\psi(x, t_0)\rangle \).

Note that the time translation operator \( U(t, t_0) \) is unitary, it is sufficient to perform the unitary transformation associated with the operator \( U^\dagger(t, t_0) \) to obtain a constant transformed vector
\[ U^\dagger(t, t_0)|\psi(x, t)\rangle = |\psi(x, t_0)\rangle. \]  \hspace{1cm} (48)
Every operator \( \hat{O}_\alpha \) can be transformed into time-dependent operator
\[ \hat{O}_{\alpha}^H = U^\dagger(t, t_0) \hat{O}_\alpha U(t, t_0). \]  \hspace{1cm} (49)
Differentiating the expression (49) with respect to time, we obtain
\[ \frac{\partial}{\partial t} \hat{O}_{\alpha}^H = \frac{i}{\hbar} [\hat{H}_{\alpha,\beta}, \hat{O}_{\alpha}^H] + \frac{\partial \hat{O}_{\alpha}^H}{\partial t}, \]  \hspace{1cm} (50)
where the last term is vanishing when the operators \( \hat{O}_\alpha \) does not depend on \( t \). Note that Eq.(50) is similar in form to (31) but has a somewhat different significance: the expression (31) defines the operator corresponding to the physical quantity \( \langle \hat{O}_\alpha \rangle \), while the left-hand side of equation (50) is the time derivative of the operator of the quantity \( \hat{O}_{\alpha}^H \) itself.
Let $\psi = \Sigma_i a_i(t) \Psi_i$ be the expansion of an arbitrary wave function in terms of the basis $\Psi_i$ in coordinate representation. If we substitute this expansion in Eq. (23) for free particle, we obtain

$$ih \sum_i \dot{a}_i(t) \Psi_i = \sum_i a_i(t) \frac{(-ih)^\alpha D^\alpha}{(2m)^\alpha} \Psi_i.$$  \hspace{1cm} (51)

Now multiplying this equation by $\psi_j^*$, and using normalization and orthogonality of the $\Psi_j$, we then find

$$ih \dot{a}_j(t) \Psi_i = \sum_i H_{ji} a_i(t),$$ \hspace{1cm} (52)

where

$$H_{ji} = (\Psi_j, \frac{(-ih)^\alpha D^\alpha}{(2m)^\alpha} \Psi_i).$$

This equation completely defines how the $a_i(t)$ change, whenever the $a_i(t)$ are known at anyone time.

### 7.3. Fractional secular equation

We now consider the the eigenvalue equation in a given representation as

$$\hat{O}_\alpha \phi = \lambda \phi,$$ \hspace{1cm} (53)

where $\phi = \sum_i c_i \psi_i$ is an eigenvector in terms of the basis $\psi_i$ with the eigenvalue $\lambda$. Since the operator $\hat{O}_\alpha$ is Hermitian, any eigenvalue $\lambda$ of $\hat{O}_\alpha$ is a real number. Substituting the matrix representation of $\hat{O}_\alpha$ and $\phi$ in the equation (53), we get

$$\sum_j O_{ij} c_j = \lambda c_i$$

or

$$\sum_j (O_{ij} - \lambda \delta_{ij}) c_j = 0.$$  

Here $\delta_{ij}$ satisfies $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Such a system has solutions which are different from zero only if

$$|O_{ij} - \lambda \delta_{ij}| = 0.$$ \hspace{1cm} (54)

Eq. (54) is called fractional characteristic equation (or fractional secular equation) which enables us to determine all the eigenvalues of the operator $\hat{A}_\alpha$, that is, its spectrum.
8. Conclusion

In this paper, we propose anomalous quantum mechanics from operator method. We first introduce anomalous operators with fractional derivative and exponential quantum associated with power observables in coordinate space and in momentum space, respectively. Using the definition of anomalous operator and the exponential hypothesis that in heterogeneous complex media the energy $E$ and momentum $p$ of a particle of mass $m$ satisfy $E = \frac{p^{2\alpha}}{(2m)^{\alpha}}$, we then derived the fractional Schrödinger equation (22) for the free particle and (23) in the potential $V(x^\beta)$. Besides, we derive the fractional equation (32) of motion for anomalous quantum mechanics. Furthermore, we obtain the fractional virial equations (34) and (35) from which we obtain the anomalous hydrogen atom whose transition energy values are much higher than that for the Bohr hydrogen atom and match up the anomalous X−ray spectrums observed by Wang, etc.. Finally, we give the anomalous Heisenberg picture which equivalent to the fractional Schrödinger equation. There are so many studies that one does not done the anomalous quantum mechanics in relativistic case from operator method and others.

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