The nature of the continuous non-equilibrium phase transition of Axelrod’s model

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Abstract – Axelrod’s model in the square lattice with nearest-neighbors interactions exhibits culturally homogeneous as well as culturally fragmented absorbing configurations. In the case in which the agents are characterized by $F = 2$ cultural features and each feature assumes $k$ states drawn from a Poisson distribution of parameter $q$, these regimes are separated by a continuous transition at $q_c = 3.10 \pm 0.02$. Using Monte Carlo simulations and finite-size scaling we show that the mean density of cultural domains $\mu$ is an order parameter of the model that vanishes as $\mu \sim (q - q_c)^\beta$ with $\beta = 0.67 \pm 0.01$ at the critical point. In addition, for the correlation length critical exponent we find $\nu = 1.63 \pm 0.04$ and for Fisher’s exponent, $\tau = 1.76 \pm 0.01$. This set of critical exponents places the continuous phase transition of Axelrod’s model apart from the known universality classes of non-equilibrium lattice models.

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Social influence and homophily (i.e., the tendency of individuals to interact preferentially with similar others) have long been acknowledged as major factors that influence the persistence of cultural diversity in a community [1,2]. The manner in which these factors affect diversity, however, has begun to be understood quantitatively after the proposal of an agent-based model by the political scientist Robert Axelrod in the late 1990s only [3]. In Axelrod’s model, the agents are represented by strings of cultural features of length $F$, where each feature can adopt a certain number $k$ of distinct states (i.e., $k$ is the common number of states each feature can assume). The term “culture” is used to indicate the set of individual attributes that are susceptible to social influence. The homophily factor is taken into account by assuming that the interaction between two agents takes place with probability proportional to their cultural similarity (i.e., proportional to the number of states they have in common), whereas social influence enters the model by allowing the agents to become more similar when they interact. Hence there is a positive feedback loop between homophily and social influence: similarity leads to interaction, and interaction leads to still more similarity. Overall, the conclusion was that the homophilic interactions together with the limited range of the agents’ interactions favor multicultural steady states [3], whereas relaxation of these conditions favors cultural homogenization [4].

In Axelrod’s model, there are two types of absorbing configurations in the thermodynamic limit [5–8]: the ordered configurations, which are characterized by the presence of few cultural domains of macroscopic size, and the disordered absorbing configurations, where all domains are microscopic. In time, a cultural domain is defined as a bounded region of uniform culture. According to the rules of the model, two neighboring agents that do not have any cultural feature in common are not allowed to interact and the interaction between agents who share all their cultural features produces no changes. Hence at the stationary state we can guarantee that any pair of neighbors are either identical or completely different regarding their cultural features. In fact, a feature that sets Axelrod’s model apart from most lattice models that exhibit non-equilibrium phase transitions [9] is that all stationary states of the dynamics are absorbing states, i.e., the dynamics always freezes in one of these states. This contrasts with lattice models that exhibit an active state in addition to infinitely many absorbing states [10] and the phase transition occurs between the active state and the (equivalent) absorbing states. In Axelrod’s model, the competition between the disorder of the initial configuration that favors
cultural fragmentation and the ordering bias of social influence that favors homogenization results in the phase transition between those two classes of absorbing states in the square lattice [5]. Since the transition occurs in the properties of the absorbing states, it is static in nature [11].

Here we address a variant of Axelrod’s model proposed by Castellano et al. that is more suitable for the study of the phase transition [5]. In the original Axelrod’s model, the initial states of the $F$ cultural features of the agents are drawn randomly from a uniform distribution on the integers $1, 2, \ldots, q$. Since both parameters of the model — $q$ and $F$ — are integers, it is not possible to determine whether the transition is continuous or not, let alone to say something meaningful about its class of universality. A way to circumvent this problem is to draw the initial integer values (states) of the cultural features using a Poisson distribution of parameter $q \in [0, \infty),$

$$P_k = \exp(-q) \frac{q^k}{k!}$$

with $k = 0, 1, 2, \ldots$. As in the case in which the states are chosen from a uniform distribution, Castellano et al. showed that the Poisson variant exhibits a phase transition in the square lattice with the bonus that they were also able to show that the transition is continuous for $F = 2$ and discontinuous for $F > 2$ [5]. Here we focus on the continuous transition for $F = 2$ in the square lattice of size $L \times L$ with periodic boundary conditions using extensive Monte Carlo simulations of lattices of linear size up to $L = 1200$. We show that this transition takes place at $q = q_c = 3.10 \pm 0.02$ and determine the critical exponents that characterize the model near the critical point.

The Poisson variant differs from the original Axelrod model only by the procedure used to generate the cultural states of the agents at the beginning of the simulation. Once the initial configuration is set, the dynamics proceeds as in the original model [3]. In particular, at each time we pick an agent at random (this is the target agent) as well as one of its neighbors. These two agents interact with probability equal to their cultural similarity, defined as the fraction of identical features in their cultural strings. An interaction consists of selecting at random one of the distinct features, and making the selected feature of the target agent equal to the corresponding feature of its neighbor. This procedure is repeated until the system is frozen into an absorbing configuration.

Once an absorbing state is reached we count the number of cultural domains ($N$) and record the size of the largest one ($S_{\text{max}}$). The average of these quantities over a large number of independent runs, which differ by the choice of the initial cultural states of the agents as well as by their update sequence, yields the measures we use to characterize the statistical properties of the absorbing configurations.

Let us consider first the mean density of domains $\mu = \langle N \rangle / L^2$. This quantity is important because it determines whether the number of domains is extensive or not in the thermodynamic limit. In the standard percolation, which exhibits a similar static phase transition, $\mu$ is continuous and non-zero at the threshold [12]. The situation is quite different in Axelrod’s model as illustrated in the upper panel of fig. 1, which shows the mean density of domains as a function of the Poisson parameter $q$. The data suggest that for $q$ less than some critical value $q_c$, the density of domains vanishes in the thermodynamic limit and so that there must exist a few macroscopic domains in that region. For $q > q_c$ the number of domains scales linearly with the number of sites in the lattice and so the average domain size $\langle S \rangle = L^2 / N$ is finite in this region. Since fig. 1 indicates that the first derivative of $\mu$ is discontinuous at $q_c$ and that $\mu$ behaves as an order parameter of the model, we will assume that $\mu \sim (q - q_c)^\beta$ near the critical point, where $\beta > 0$ is a critical exponent. In addition, for finite but large $L$ the finite-size scaling theory yields [13]

$$\mu \sim L^{-\beta/\nu} f[L^{1/\nu}(q - q_c)],$$

where the scaling function is $f(x) \propto x^x$ for $x \gg 1$ and $\nu > 0$ is a critical exponent that determines the size of the critical region for finite $L$ and governs the divergence of the correlation length as $q \rightarrow q_c^\pm$.

Use of the finite-size scaling equation (2) allows us to produce quantitative estimates for the critical point $q_c$ and for the critical exponents $\beta$ and $\nu$, as well. For instance, according to that equation, $\mu$ should decrease to zero as a power law of $L$ at $q = q_c$, and in fig. 2 we explore this fact to determine $q_c$ and the ratio $\beta/\nu$. In particular, we fit the data for different values of $q$ with the function $\mu = B L^{\beta/\nu}$ in the range $L \in [400, 1200]$ and gauge how the fitting curves deviate from the data for $L \in [15, 300]$ in order to pick the critical value $q_c$. This is necessary because for large $L$ all fittings are bona fide straight lines in the log-log scale of fig. 2 in the range $q \in [0.308, 0.312]$. The data for $q = 3.12$ exhibits a definite convexity and the fitting with the exponent $(\beta/\nu)_{q=0.312} = 0.35 \pm 0.01$ deviates from the

![Fig. 1: (Colour on-line) Upper panel: mean density of domains $\mu$ as a function of the Poisson parameter $q$ for lattices of linear size $L = 200$ (△), $L = 400$ (▼), $L = 800$ (○) and $L = 1000$ (×). Lower panel: log-log plot of $q$ against $q - q_c$ for $q_c = 3.1$. The solid line in both panels is the two-parameters fitting function $\mu = A(q - q_c)^\beta$ with $A = 0.331 \pm 0.003$, and $\beta = 0.67 \pm 0.01$. The error bars are smaller than the symbol sizes.](image-url)
data already for \( L < 300 \), whereas the data for \( q = 3.08 \) exhibits a very light concavity and the fitting with the exponent \((\beta/\nu)_{q=3.08} = 0.43 \pm 0.01\) deviates from the data only for \( L < 50 \). Finally, for \( q = 0.310 \) the fitting function with the exponent \( \beta/\nu = 0.41 \pm 0.01 \) fits the data very well in the entire range of \( L \) shown in the figure. Hence we conclude that \( q_c = 3.10 \pm 0.02 \).

Once we have a good estimate for \( q_c \), the best strategy is to return to fig. 1 and fit the data for \( L = 1000 \) in the region near \( q_c = 3.1 \) using the fitting function \( \mu = A(q - q_c)^\beta \), where \( A \) and \( \beta \) are the two adjustable parameters of the fitting. This procedure yields \( \beta = 0.67 \pm 0.01 \) for the order parameter critical exponent. The goodness of the resulting fitting is shown in the lower panel of fig. 1, which plots \( \mu \) as function of the distance to the critical point \( q - q_c \) in a log-log scale.

Finally, since \( \beta = 0.67 \pm 0.01 \) and \( \beta/\nu = 0.41 \pm 0.01 \) imply \( \nu = 1.63 \pm 0.04 \) we can validate our estimates of the critical quantities by checking whether the scaled mean density of domains \( L^{\beta/\nu} \mu \) is independent of the lattice size \( L \) when plotted against the scaled distance to the critical parameter \( L^{1/\nu}(q - q_c) \) as predicted by eq. (2). This is shown in fig. 3 and the quality of the resulting data collapse confirms the soundness of our estimate of the critical exponents.

Let us consider now the standard order parameter of Axelrod’s model, namely, the mean fraction of lattice sites that belong to the largest domain \( \rho = (S_{\text{max}})/L^2 \) (see, e.g., [4,5,11]). Figure 4 shows the dependence of \( \rho \) on the Poisson parameter \( q \). The finite-size effects on \( \rho \) are truly perplexing: for \( q \to q_c^- \) the results for different \( L \) seem to converge quickly to some limiting value as \( L \) increases but, surprisingly, as \( q \) departs from \( q_c \) in the region \( q < q_c \), those results begin to diverge, and only for small values of \( q \) (typically \( q < 1 \)) for the lattice sizes shown in the figure we regain the independence on the lattice size, as expected. This means that for finite \( L \) the measure \( \rho \) exhibits a plateau separating the small-\( q \) region from the critical region. It is interesting that this plateau is a finite-size effect of the periodic boundary conditions since our simulations for free boundary conditions (i.e., agents in the corners and in the sides of the lattice have two and three neighbors, respectively), also shown in fig. 4, exhibit a commoner approach to the critical regime. In particular, for free boundary conditions \( \rho \) becomes independent of the lattice size already for \( q < 2.5 \). We observed the same effect of the boundary conditions in the one-dimensional lattice as well. However, for both boundary conditions a plot of \( \rho \) against \( 1/L \) for fixed \( q \) near \( q_c \), as shown in fig. 2 for \( \mu \), shows a tendency of the data to level off at intermediate values of \( L \) and then resume their decrease towards their limiting values as \( L \) becomes very large.

Therefore, due to the somewhat pathological dependence of the standard order parameter \( \rho \) on the lattice size \( L \) and on the Poisson parameter \( q \), a study of the nature
of the phase transition of Axelrod’s model based on this parameter only would be practically impossible: it is no wonder that the authors of ref. [5] refrained even from offering an estimate for $q_c$. In addition, since the dynamics takes a very long time to relax to absorbing configurations characterized by macroscopic cultural domains (see, e.g., [14] for the quantification of this finding for the one-dimensional lattice), the simulations are typically much slower in the region $q < q_c$ where $\rho$ is non-zero than in the region $q > q_c$ where $\mu$ is non-zero. Interestingly, for $q < q_c$, the simulations with free boundary conditions are way faster than with periodic conditions, perhaps because the translational invariance of the lattice is broken in the former case.

Castellano et al. offered an insight on the nature of the phase transition of Axelrod’s model by focusing on the probability distribution of domain sizes [5]. Consider the average domain size

$$\langle S \rangle = \sum_{s=1}^{\infty} s P_L(s, q),$$  \hspace{1cm} (3)

where $P_L(s, q)$ is the probability distribution of the size $s$ of domains in a lattice of linear size $L$. Of course, $P_L(s, q) = 0$ for $s > L^2$. In the limit $L \to \infty$ and for $q > q_c$, this probability can be written in the scaling form $P_L(s, q) = s^{-\tau}g(s/s_c)$, where $\tau > 0$ is the Fisher exponent and the scaling function $g(x)$ tends to a constant for $x \ll 1$ and decays very rapidly for $x \gg 1$. As in the standard percolation [12], the transition occurs through the divergence of the divergence of the cutoff scale $s_c \sim (q - q_c)^{-1/\sigma}$ and hence of a correlation length $\xi$ since $s_c \sim \xi^D \sim (q - q_c)^{-\nu D}$. Here $\sigma > 0$ is a critical exponent and $D < 2$ is the fractal dimension of the incipient macroscopic domain. Clearly, $\sigma = 1/\nu D$. We note that the divergence of $\langle S \rangle = L^2/N$ as $q \to q_c^+$ implies that $\tau < 2$ [5] and fig. 5, which shows the critical distribution $P_L(s, q_c)$, leads to the estimate $\tau \approx 1.76 \pm 0.01$. (The estimate of Castellano et al. is $\tau \approx 1.64$ for $L = 100$.)

The finding that the density of domains vanishes at $q = q_c$, as $\mu \sim (q - q_c)^{\beta}$ makes the continuous transition of Axelrod’s model depart markedly from the percolation transition, since the exponent $\beta$ has no counterpart in that case. In fact, we need to derive the relations between $\beta$ and the other critical exponents anew. In particular, noting that $\langle S \rangle = 1/\mu$ we can use eq. (3) to obtain the relation $\beta = (2 - \tau)/\sigma$, which together with $\sigma = 1/\nu D$ results in the estimates $\sigma = 0.36 \pm 0.02$ and $D = 1.71 \pm 0.02$.

We note that not only our order parameter ($\mu$) is different from the order parameter ($\rho$) considered in the original analysis of the continuous phase transition of Axelrod’s model [5], but the investigated regimes differ as well. In particular, here we focus on the regime $q > q_c$, where $\mu > 0$ and $\rho = 0$ in the thermodynamic limit, and thus we describe the onset of order by focusing on the process of agglutination of the domains. Although this is different from observing the growing of a macroscopic domain in the regime $q < q_c$ as done by Castellano et al. [5], both perspectives describe the same critical phenomenon.

Our aim here was to offer a quantitative characterization of the continuous non-equilibrium phase transition of the Poisson variant of Axelrod’s model that was first reported in 2000 [5]. The transition is static in nature and separates two types of absorbing configurations that differ on their distributions of domain sizes. Because of the two distinctive features — both phases correspond to absorbing configurations and the density of domains vanishes at the critical point — the continuous phase transition of Axelrod’s model is characterized by a set of critical exponents that sets it apart from the known universality classes of non-equilibrium lattice models [9].

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