THE SMALLEST HYPERBOLIC 6-MANIFOLDS

BRENT EVERITT, JOHN RATCLIFFE, AND STEVEN TSCHANTZ

ABSTRACT. By gluing together copies of an all-right angled Coxeter polytope a number of open hyperbolic 6-manifolds with Euler characteristic $-1$ are constructed. They are the first known examples of hyperbolic 6-manifolds having the smallest possible volume.

1. Introduction

The last few decades has seen a surge of activity in the study of finite volume hyperbolic manifolds—that is, complete Riemannian $n$-manifolds of constant sectional curvature $-1$. Not surprisingly for geometrical objects, volume has been, and continues to be, the most important invariant for understanding their sociology. The possible volumes in a fixed dimension forms a well-ordered subset of $\mathbb{R}$, indeed a discrete subset except in 3-dimensions (where the orientable manifolds at least have ordinal type $\omega^\omega$). Thus it is a natural problem with a long history to construct examples of manifolds with minimal volume in a given dimension.

In 2-dimensions the solution is classical, with the minimum volume in the compact orientable case achieved by a genus 2 surface, and in the non-compact orientable case by a once-punctured torus or thrice-punctured sphere (the identities of the manifolds are of course also known in the non-orientable case). In 3-dimensions the compact orientable case remains an open problem with the Matveev-Fomenko-Weeks manifold [16, 30] obtained via (5, $-2$)-Dehn surgery on the the sister of the figure-eight knot complement conjecturally the smallest. Amongst the non-compact orientable 3-manifolds the figure-eight knot complement realizes the minimum volume [17] and the Gieseking manifold (obtained by identifying the sides of a regular hyperbolic tetrahedron as in [14, 20]) does so for the non-orientable ones [1]. One could also add “arithmetic” to our list of adjectives and so have eight optimization problems to play with (so that the Matveev-Fomenko-Weeks manifold is known to be the minimum volume orientable, arithmetic compact 3-manifold, see [2]).

When $n \geq 4$ the picture is murkier, although in even dimensions we have recourse to the Gauss-Bonnet Theorem, so that in particular the minimal volume a $2m$-dimensional hyperbolic manifold could possibly have is when the Euler characteristic $\chi$ satisfies $|\chi| = 1$. The first examples of non-compact 4-manifolds with $\chi = 1$ were constructed in [23] (see also [13]). The compact case remains a difficult unsolved problem, although if we restrict to arithmetic manifolds, then it is known [2, 8] that a minimal volume arithmetic compact orientable 4-manifold $M$.

1991 Mathematics Subject Classification. Primary 57M50.

The first author is grateful to the Mathematics Department, Vanderbilt University for its hospitality during a stay when the results of this paper were obtained.

©1995 American Mathematical Society
has \( \chi \leq 16 \) and \( M \) is isometric to the orbit space of a torsion-free subgroup of the hyperbolic Coxeter group \([5,3,3,3]\). The smallest compact hyperbolic 4-manifold currently known to exist has \( \chi = 8 \) and is constructed in \([8]\). Manifolds of very small volume have been constructed in 5-dimensions \([13, 24]\), but the smallest volume 6-dimensional example hitherto known has \( \chi = -16 \) \([13]\).

In this paper we announce the discovery of a number of non-compact non-orientable hyperbolic 6-manifolds with Euler characteristic \( \chi = -1 \). The method of construction is classical in that the manifolds are obtained by identifying the sides of a 6-dimensional hyperbolic Coxeter polytope.

2. Coxeter polytopes

Let \( C \) be a convex (not necessarily bounded) polytope of finite volume in a simply connected space \( X^n \) of constant curvature. Call \( C \) a Coxeter polytope if the dihedral angle subtended by two intersecting \((n-1)\)-dimensional sides is \( \pi/m \) for some integer \( m \geq 2 \). When \( X^n = S^n \) or the Euclidean space \( E^n \), such polyhedra have been completely classified \([11]\), but in the hyperbolic space \( H^n \), a complete classification remains a difficult problem (see for example \([27]\) and the references there).

If \( \Gamma \) is the group generated by reflections in the \((n-1)\)-dimensional sides of \( C \), then \( \Gamma \) is a discrete cofinite subgroup of the Lie group \( \text{Isom} X^n \), and every discrete cofinite reflection group in \( \text{Isom} X^n \) arises in this way from some Coxeter polytope, which is uniquely defined up to isometry. The Coxeter symbol for \( C \) (or \( \Gamma \)) has nodes indexed by the \((n-1)\)-dimensional sides, and an edge labeled \( m \) joining the nodes corresponding to sides that intersect with angle \( \pi/m \) (label the edge joining the nodes of non-intersecting sides by \( \infty \)). In practice the labels 2 and 3 occur often, so that edges so labeled are respectively removed or left unlabeled.

Let \( \Lambda \) be a \((n+1)\)-dimensional Lorentzian lattice, that is, an \((n+1)\)-dimensional free \( \mathbb{Z} \)-module equipped with a \( \mathbb{Z} \)-valued bilinear form of signature \((n,1)\). For each \( n \), there is a unique such \( \Lambda \), denoted \( I_{n,1} \), that is odd and self-dual (see \([25, \text{Theorem V.6}]\), or \([18, 19]\)). By \([4]\), the group \( O_{n,1} \mathbb{Z} \) of automorphisms of \( I_{n,1} \) acts discretely, cofinitely by isometries on the hyperbolic space \( H^n \) obtained by projectivising the negative norm vectors in the Minkowski space-time \( I_{n,1} \otimes \mathbb{R} \) (to get a faithful action one normally passes to the centerless version \( \text{PO}_{n,1} \mathbb{Z} \)).

Vinberg and Kaplinskaja showed \([28, 29]\) that the subgroup \( \text{Reflec} \) of \( \text{PO}_{n,1} \mathbb{Z} \) generated by reflections in positive norm vectors has finite index if and only if \( n \leq 19 \), thus yielding a family of cofinite reflection groups and corresponding finite volume Coxeter polytopes in the hyperbolic spaces \( H^n \) for \( 2 \leq n \leq 19 \). Indeed, Conway and Sloane have shown \([9, \text{Chapter 28}]\) or \([10]\), that for \( n \leq 19 \) the quotient of \( \text{PO}_{n,1} \mathbb{Z} \) by \( \text{Reflec} \) is a subgroup of the automorphism group of the Leech lattice. Borcherds \([3]\) showed that the (non self-dual) even sublattice of \( I_{21,1} \) also acts cofinitely, yielding the highest dimensional example known of a Coxeter group acting cofinitely on hyperbolic space.

When \( 4 \leq n \leq 9 \) the group \( \Gamma = \text{Reflec} \) has Coxeter symbol,
with \( n + 1 \) nodes and \( C \) a non-compact, finite volume \( n \)-simplex \( \Delta^n \) (when \( n > 9 \), the polytope \( C \) has a more complicated structure).

Let \( v \) be the vertex of \( \Delta^n \) opposite the side \( F_1 \) marked on the symbol, and let \( \Gamma_v \) be the stabilizer in \( \Gamma \) of this vertex. This stabilizer is also a reflection group with symbol as shown, and is finite for \( 4 \leq n \leq 8 \) (being the Weyl group of type \( A_4, D_5, E_6, E_7 \) and \( E_8 \) respectively) and infinite for \( n = 9 \) (when it is the affine Weyl group of type \( \tilde{E}_6 \)). Let

\[
P_n = \bigcup_{\gamma \in \Gamma_v} \gamma(\Delta^n),
\]

a convex polytope obtained by gluing \(|\Gamma_v|\) copies of the simplex \( \Delta^n \) together. Thus, \( P_n \) has finite volume precisely when \( 4 \leq n \leq 8 \), although it is non-compact, with a mixture of finite vertices in \( H^n \) and cusped ones on \( \partial H^n \). In any case, \( P_n \) is an all right-angled Coxeter polytope: its sides meet with dihedral angle \( \pi/2 \) or are disjoint. This follows immediately from the observation that the sides of \( P_n \) arise from the \( \Gamma_v \)-images of the side of \( \Delta^n \) opposite \( v \), and this side intersects the other sides of \( \Delta^n \) in dihedral angles \( \pi/2 \) or \( \pi/4 \). Vinberg has conjectured that \( n = 8 \) is the highest dimension in which finite volume all-right angled polytopes exist in hyperbolic space.

The volume of the polytope \( P_n \) is given by

\[
\text{vol}(P_n) = |\Gamma_v|\text{vol}(\Delta^n) = |\Gamma_v| |\text{PO}_{n,1}Z : \Gamma| \text{covol}(\text{PO}_{n,1}Z),
\]

where \( \text{covol}(\text{PO}_{n,1}Z) \) is the volume of a fundamental region for the action of \( \text{PO}_{n,1}Z \) on \( H^n \) (and for \( 4 \leq n \leq 9 \) the index \(|\text{PO}_{n,1}Z : \Gamma| = 1\)). When \( n \) is even, we have by \( [20] \) and \( [22] \),

\[
\text{covol}(\text{PO}_{n,1}Z) = \frac{(2^n \pm 1)\pi^{n/2}}{n!} \prod_{k=1}^{n/2} |B_{2k}|,
\]

with \( B_{2k} \) the \( 2k \)-th Bernoulli number and with the plus sign if \( n \equiv 0, 2 \mod 8 \) and the minus sign otherwise.

Alternatively (when \( n \) is even), we have recourse to the Gauss-Bonnet Theorem, so that \( \text{vol}(P_n) = \kappa_n|\Gamma_v|\chi(\Gamma) \), where \( \chi(\Gamma) \) is the Euler characteristic of the Coxeter group \( \Gamma \) and \( \kappa_n = 2^n(n!)^{-1}(-\pi)^{n/2}(n/2)! \). The Euler characteristic of Coxeter groups can be easily computed from their symbol (see \( [3, 7] \) or \( [13] \) Theorem 9). Indeed, when \( n = 6 \), \( \chi(\Gamma) = -1/Z \) where \( Z = 2^{10}3^45 \) and so \( \text{vol}(P_6) = 8\pi^3|E_6|/15Z = \pi^3/15 \).

The Coxeter symbol for \( P_n \) has a nice description in terms of finite reflection groups. If \( v' \) is the vertex of \( \Delta^n \) opposite the side \( F_2 \), let \( \Gamma_e \) be the pointwise stabilizer of \( \{v, v'\} \): the elements thus stabilize the edge \( e \) of \( \Delta^n \) joining \( v \) and \( v' \).

Now consider the Cayley graph \( \mathcal{C}_v \) for \( \Gamma_v \) with respect to the generating reflections in the sides of the symbol for \( \Gamma_v \). Thus, \( \mathcal{C}_v \) has vertices in one to one correspondence with the elements of \( \Gamma_v \) and for each generating reflection \( s_\alpha \), an undirected edge labeled \( s_\alpha \) connecting vertices \( \gamma_1 \) and \( \gamma_2 \) if and only if \( \gamma_2 = \gamma_1 s_\alpha \) in \( \Gamma_v \). In particular, \( \mathcal{C}_v \) has \( s_2 \) labeled edges corresponding to the reflection in \( F_2 \). Removing these \( s_2 \)-edges decomposes \( \mathcal{C}_v \) into components, each of which is a copy of the Cayley graph \( \mathcal{C}_v \) for \( \Gamma_e \), with respect to the generating reflections.

Take as the nodes of the symbol for \( P_n \) these connected components. If two components have an \( s_2 \)-labeled edge running between any two of their vertices in \( \mathcal{C}_v \), then leave the corresponding nodes unconnected, otherwise, connect them by
an edge labeled $\infty$. The resulting symbol (respectively the polytope $P_n$) thus has $|\Gamma_n|/|\Gamma_e|$ nodes (resp. sides). The number of sides of $P_n$ for $n = 4, 5, 6, 7, 8$ is 10, 16, 27, 56 and 240 respectively.

3. Constructing the manifolds

We now restrict our attention to the case $n = 6$. We work in the hyperboloid model of hyperbolic 6-space

$$H^6 = \{ x \in \mathbb{R}^7 : x_1^2 + x_2^2 + \cdots + x_6^2 - x_7^2 = -1 \text{ and } x_7 > 0 \}$$

and represent the isometries of $H^6$ by Lorentzian $7 \times 7$ matrices that preserve $H^6$. The right-angled polytope $P_6$ has 27 sides each congruent to $P_5$. We position $P_6$ in $H^6$ so that 6 of its sides are bounded by the 6 coordinate hyperplanes $x_i = 0$ for $i = 1, \ldots, 6$ and these 6 sides intersect at the center $e_7$ of $H^6$. Let $K_6$ be the group of 64 diagonal Lorentzian $7 \times 7$ matrices diag$(\pm 1, \ldots, \pm 1, 1)$. The set $Q_6 = K_6P_6$, which is the union of 64 copies of $P_6$, is a right-angled convex polytope with 252 sides. We construct hyperbolic 6-manifolds, with $\chi = -8$, by gluing together the sides of $Q_6$ by a proper side-pairing with side-pairing maps of the form $rk$ with $k$ in $K_6$ and $r$ a reflection in a side $S$ of $Q_6$. The side-pairing map $rk$ pairs the side $S' = kS$ to $S$ (see §11.1 and §11.2 of [21] for a discussion of proper side-pairings). We call such a side-pairing of $Q_6$ simple. We searched for simple side-pairings of $Q_6$ that yield a hyperbolic 6-manifold $M$ with a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 64 copies of $P_6$ making up $M$ in such a way that the resulting quotient manifold is obtained by gluing together 8 copies of $P_6$. Such a quotient manifold has $\chi = -8/8 = -1$. This is easier said than done, since the search space of all possible side-pairings of $Q_6$ is very large. We succeeded in finding desired side-pairings of $Q_6$ by employing a strategy that greatly reduces the search space. The strategy is to extend a side-pairing in dimension 5 with the desired properties to a side-pairing in dimension 6 with the desired properties.

Let $Q_5 = \{ x \in Q_6 : x_1 = 0 \}$. Then $Q_5$ is a right-angled convex 5-dimensional polytope with 72 sides. Note that $Q_5$ is the union $K_5P_5$ of 32 copies of $P_5$ where $P_5 = \{ x \in P_6 : x_1 = 0 \}$ and $K_5$ is the group of 32 diagonal Lorentzian $7 \times 7$ matrices diag$(1, \pm 1, \ldots, \pm 1, 1)$. A simple side-pairing of $Q_6$ that yields a hyperbolic 6-manifold $M$ restricts to a simple side-pairing of $Q_5$ that yields a hyperbolic 5-manifold which is a totally geodesic hypersurface of $M$. All the orientable hyperbolic 5-manifolds that are obtained by gluing together the sides of $Q_5$ by a simple side-pairing are classified in [24].

We started with the hyperbolic 5-manifold $N$, numbered 27 in [21], obtained by gluing together the sides of $Q_5$ by the simple side-pairing with side-pairing code $2B7JB47JG81$. The manifold $N$ has a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 32 copies of $P_5$ making up $N$ in such a way that the resulting quotient manifold is obtained by gluing together 4 copies of $P_5$. A generator of the $\mathbb{Z}/8$ symmetry group of $N$ is represented by the Lorentzian $6 \times 6$ matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 \\
-1 & 0 & -1 & -1 & 0 & 2
\end{pmatrix}
\]
The strategy is to search for simple side-pairings of $Q_6$ that yield a hyperbolic 6-manifold with a freely acting $\mathbb{Z}/8$ symmetry group with generator represented by the following Lorentzian $7 \times 7$ matrix that extends the above Lorentzian $6 \times 6$ matrix.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & -1 & 0 & -1 & -1 & 0 & 2
\end{pmatrix}
$$

For such a side-pairing the resulting quotient manifold can be obtained by gluing together 8 copies of $P_6$ by a proper side-pairing. By a computer search we found 14 proper side-pairings of 8 copies of $P_6$ in this way, and hence we found 14 hyperbolic 6-manifolds with $\chi = -1$. Each of these 14 manifolds is noncompact with volume $8\text{vol}(P_6) = 8\pi^3/15$ and five cusps. These 14 hyperbolic 6-manifolds represent at least 7 different isometry types, since they represent 7 different homology types. Table 1 lists side-pairing codes for 7 simple side-pairings of $Q_6$ whose $\mathbb{Z}/8$ quotient manifold has homology groups isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/4)^c \oplus (\mathbb{Z}/8)^d$ for nonnegative integers $a, b, c, d$ encoded by $abcd$ in the table. In particular, all 7 manifolds in Table 1 have a finite first homology group.

All of our examples, with $\chi = -1$, can be realized as the orbit space $H^6/\Gamma$ of a torsion-free subgroup $\Gamma$ of $PO_{6,1}\mathbb{Z}$ of minimal index. These manifolds are the first examples of hyperbolic 6-manifolds having the smallest possible volume. All these manifolds are nonorientable. In the near future, we hope to construct orientable examples of noncompact hyperbolic 6-manifolds having $\chi = -1$.

### References

1. C Adams. *The non-compact hyperbolic 3-manifold of minimum volume*. Proc. AMS, 100 (1987), 601-606.
2. M Belolipetsky. *On volumes of arithmetic quotients of $SO(1,n)$*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (to appear).
3. R Borcherds. *Automorphism groups of Lorentzian lattices*. J. Algebra 111(1) (1987), 133-153.
4. A Borel and Harish-Chandra. *Arithmetic subgroups of algebraic groups*. Ann. of Math., 75(3) (1962), 485-535.
5. T Chinburg, E Friedman, K N Jones, and A W Reid. *The arithmetic hyperbolic 3-manifold of smallest volume*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), 1-40.
6. I M Chiswell. *The Euler characteristic of graph products and Coxeter groups.* in Discrete Groups and Geometry, W J Harvey and Colin Maclachlan (Editors), London Math. Soc. Lect. Notes, 173 (1992), 36-46.

7. I M Chiswell. *Euler characteristics of groups.* Math. Z., 147 (1976), 1–11.

8. M D E Conder and C Maclachlan. *Small volume compact hyperbolic 4-manifolds.* to appear, Proc. Amer. Math. Soc.

9. J Conway and N J A Sloane. *Sphere Packings, Lattices and Groups.* Second Edition, Springer 1993.

10. J Conway and N J A Sloane. *Leech roots and Vinberg groups.* Proc. Roy. Soc. London Ser. A 384 (no. 1787) (1982), 233-258.

11. H S M Coxeter. *Discrete groups generated by reflections.* Ann. of Math, 35(2) (1934), 588-621.

12. M W Davis. *A hyperbolic 4-manifold.* Proc. Amer. Math. Soc., 93 (1985), 325-328.

13. B Everitt. *Coxeter groups and hyperbolic manifolds.* Math. Ann. 330(1), (2004), 127-150.

14. B Everitt. *3-Manifolds from Platonic solids.* Top. App. 138 (2004), 253-263.

15. B Everitt, J Ratcliffe and S Tschantz. *Arithmetic hyperbolic 6-manifolds of smallest volume* (in preparation).

16. V Matveev and A Fomenko. *Constant energy surfaces of Hamilton systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds.* Russian Math. Surveys, 43 (1988), 3-24.

17. C Cao and G R Meyerhoff. *The orientable cusped hyperbolic 3-manifolds of minimum volume.* Invent. Math. 146(3) (2001), 451-478.

18. J Milnor and D Husemoller. *Symmetric Bilinear Forms.* Springer, 1973.

19. A Neumaier and J J Seidel. *Discrete Hyperbolic Geometry.* Combinatorica 3 (1983), 219-237.

20. I Prok. *Classification of dodecahedral space forms.* Beiträge Algebra Geom. 39(2) (1998), 497-515.

21. J Ratcliffe. *Foundations of hyperbolic manifolds.* Graduate Texts in Mathematics 149, Springer 1994.

22. J Ratcliffe and S Tschantz. *Volumes of integral congruence hyperbolic manifolds.* J. Reine Angew. Math. 488 (1997), 55-78.

23. J Ratcliffe and S Tschantz. *The volume spectrum of hyperbolic 4-manifolds.* Experiment. Math. 9 (2000), no. 1, 101-125.

24. J Ratcliffe and S Tschantz. *Integral congruence two hyperbolic 5-manifolds.* Geom. Dedicata 107 (2004), 187-209.

25. J-P Serre. *A Course in Arithmetic.* Graduate Texts in Mathematics 7, Springer 1973.

26. C L Siegel. *Über die analytische Theorie der quadratischen Formen II.* Ann. Math. 37 (1936), 230-263.

27. E B Vinberg. *Hyperbolic Reflection Groups.* Uspekhi Mat. Nauk 40:1 (1985), 29-66, = Russian Math. Surveys 40:1 (1985), 31-75.

28. E B Vinberg and I M Kaplinskaja. *On the groups O_{19,1}(Z) and O_{19,1}(Z).* SMD 19(1) (1978), 194-197, = DAN 238(6) (1978), 1273-1275.

29. E B Vinberg. *On the groups of units of certain quadratic forms.* Sb. 16(1) (1972), 17-35, = Mat. Sb. 87(1) (1972), 18-36.

30. J Weeks. *Hyperbolic structures on 3-manifolds.* PhD thesis, Princeton University.