On Orthogonal Approximate Message Passing

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Abstract—Approximate Message Passing (AMP) is an efficient iterative parameter-estimation technique for certain high-dimensional linear systems with non-Gaussian distributions, such as sparse systems. In AMP, a so-called Onsager term is added to keep estimation errors approximately Gaussian. Orthogonal AMP (OAMP) does not require this Onsager term, relying instead on an orthogonalization procedure to keep the current errors uncorrelated with (i.e., orthogonal to) past errors. In this paper, we show the generality and significance of the orthogonality in ensuring that errors are “asymptotically independently and identically distributed Gaussian” (AIDG). This AIDG property, which is essential for the attractive performance of OAMP, holds for separable functions. We present a simple and versatile procedure to establish the orthogonality through Gram-Schmidt (GS) orthogonalization, which is applicable to any prototype. We show that different AMP-type algorithms, such as expectation propagation (EP), turbo, AMP and OAMP, can be unified under the orthogonal principle. The simplicity and generality of OAMP provide efficient solutions for estimation problems beyond the classical linear models. As an example, we study the optimization of OAMP via the GS model and GS orthogonalization. More related applications will be discussed in a companion paper where new algorithms are developed for problems with multiple constraints and multiple measurement variables.

Index Terms—Expectation propagation (EP), turbo, belief propagation (BP), approximate message passing (AMP), vector AMP (VAMP), unified framework, state evolution, Haar matrices.

I. INTRODUCTION

A. Motivation and Insight

Fig. 1(a) illustrates a message passing scheme for estimating the random vector $x$ in a Bayesian setting. Here, $\Gamma$ and $\Phi$ represent statistical information about $x$ such as observations or known distributions. They are called constraints for brevity. Message passing repeatedly applies each constraint in turn, aiming to converge to an estimate that optimally combines both pieces of statistical information. This is attractive when applying both pieces of information at once is computationally infeasible. This paper provides new insight into when and how message passing can be made to work.

The following example explains some of the insights. Let the elements of $x$ be source symbols chosen uniformly at random from the finite alphabet $\Omega_x \subset \mathbb{R}$. Additional statistical information comes from observing the received signal

$$y = Ax + n,$$

where $A \in \mathbb{R}^{M \times N}$ is a fat or square sensing matrix of size $M \times N$ with $M \leq N$, and $n \sim \mathcal{N}(0, \sigma_n^2 I)$ represents additive white Gaussian noise (AWGN). Finding the optimal maximum a posteriori (MAP) estimate of $x$ is computationally prohibitive when $N$ is large.

Directly mapping this example to Fig. 1(a) leads to $\Phi$ being the constraint $x_i \sim \text{uniform}(\Omega_x)$ ($1 \leq i \leq N$), the matrix $V$ being the sensing matrix $A$, and $\Gamma$ being the constraint $y = \mathcal{X} + n$. Naively, given an estimate $\hat{x}$ of $x$, the constraint $\Phi$ can improve on it by bringing it closer to an element of $\Omega_x$ (e.g., by soft thresholding), while an estimate $\mathcal{X}$ of $\mathcal{X}$ can be improved on by bringing it closer to the observation $y$. This naive approach will not work due to data incest; the incoming estimate at a local processor in Fig. 1(b) is treated as statistically independent of the statistical constraint imposed by the processor, but in general it is not.

The original way of message passing was made to work for this example, under the strong assumption that $A$ “looked like” it was generated at random with independently and identically distributed (IID) Gaussian entries, was by including an Onsager correction term to mitigate the data incest problem by filtering out the dependent information [1]. The idea of using an Onsager correction term comes from simulating spin glasses in physics and does not provide an elementary understanding of how to make message-passing schemes work in general.

B. Iterative Detection

For the estimation of $x$ in Fig. 1(a), jointly processing $\Gamma$ and $\Phi$ generally requires excessively high complexity. Fig. 1(b) illustrates a low-cost iterative detector. Two local processors, denoted by $\gamma$ and $\phi$, process the two constraints...
and $\Phi$ separately. In each iteration, the estimates for $X$ and $x$, denoted by $\hat{X}^{\text{out}}$ and $\hat{x}^{\text{out}}$, are generated by $\gamma$ and $\phi$ respectively. They are referred to as output messages and, after transforming them by $V^T$ or $V$ respectively, they form the input (i.e., a priori) messages $\hat{x}^{\text{in}}$ and $\hat{X}^{\text{in}}$ for the next iteration. In this way, $\hat{X}^{\text{out}}$ and $\hat{x}^{\text{out}}$ are refined iteratively. Although omitted to reduce notational clutter, the distributions of the estimates are tracked. Each local processor treats the distribution of its input as a prior distribution. The computed (approximate) posterior distribution of the outgoing $\hat{X}^{\text{out}}$ or $\hat{x}^{\text{out}}$ serves as the incoming prior for the next processor.

The local estimators $\gamma$ and $\phi$ can be constructed using minimum mean square error (MMSE) or MAP principles, which are broadly referred to as direct methods. Implementations of such methods include expectation maximization (EM)-based techniques and MAP filtering [2]–[4]. Directly coupling two local estimators as in Fig. 1 (b) may suffer from an error correlation problem. Specifically, errors at the input of a local estimator may become correlated with its previous output errors during iterative processing. Such correlated errors are difficult to track and may have detrimental effects.

Various solutions have been proposed before. In the message passing decoding techniques for turbo and LDPC codes [5], [6], an output of an estimator is “extrinsic”, meaning it is independent of the related input message. This extrinsic principle can be asymptotically met by forward error control (FEC) codes designed on large sparse graphs.

The turbo principle has been extended to the linear system in [1] involving FEC codes [7]. The latter is not sparse when $A$ is full [8]–[12]. Expectation propagation (EP) is a closely related technique based on the Gaussian approximation of messages [13], [14]. The work on EP was initially heuristic but analysis techniques have been derived recently [16]. Approximate message passing (AMP) treats the correlation problem using a so-called Onsager term [1], AMP assumes IID Gaussian (IIDG) entries for the sensing matrix $A$ in (1).

State-evolution (SE), derived heuristically in [1] and proved rigorously in [18], is an analysis tool for AMP as well as other related algorithms [16], [17], [19]–[22]. The discussions on SE in [16], [22] focus on mathematical rigor but are less intuitive. It is not straightforward to see a common core among different approaches. Recently, [23] shows an interesting direction toward a unified framework. However, [23] heavily relies on [16]. The latter involves detailed algorithmic operations of orthogonal AMP (OAMP) [17] (categorized as a special case of EP), which cannot be easily generalized.

AMP has been successfully applied to various communication systems, including grant-free machine-type communications (MTC) [32], massive random access [33], compressed coding [34], synchronous/asynchronous massive connectivity [35], [36], and reconfigurable intelligent surface (RIS) aided multi-user multi-input multi-output (MIMO) systems [37].

C. Orthogonal Approximate Message Passing (OAMP)

The assumption of IID sensing matrices in AMP is relaxed in OAMP [17], which offers a solution to systems with non-IID sensing matrices. SE for OAMP is conjectured in [17] and proved in [16]. The key to OAMP is to maintain error orthogonality during iterative processing [17]. Based on SE, the optimality of OAMP is derived in [17] under certain conditions. Vector AMP (VAMP) [19] is algorithmically similar to OAMP. An elegant proof of SE is given in [19] for VAMP. It is shown that AMP and OAMP-based decoding can outperform the turbo algorithm in coded linear systems [21], [22].

OAMP has been extensively investigated for communication and signal processing applications with non-IID (such as ill-conditioned and correlated) channel/pilot matrices, including massive MIMO [38], clipped sparse regression codes [39], coded linear systems [40], orthogonal-frequency-division-multiplexing (OFDM) [41]–[43], grant-free MTC [44], massive access [45] and orthogonal time frequency space (OTFS) modulation [46]. The connection between unfolded AMP/OAMP and artificial neural networks is studied in [24]–[26]. It has been demonstrated that the orthogonalization parameters can be acquired via deep learning, resulting in deep unfolded AMP/OAMP algorithms with improved performance [24]–[26].

D. The Haar Distribution of $V$

Throughout this paper, we assume that $V$ in Fig. 1 is randomly selected and its size approaches infinity. Consequently, we may expect that there exists a large law of large numbers that the errors $\hat{X}^{\text{in}} = V\hat{x}^{\text{in}}$ and $\hat{x}^{\text{in}} = V^T\hat{X}^{\text{out}}$ are close to being IID Gaussian. We however have to scrutinize this issue carefully as the errors may become correlated with $V$ during iterative processing. To make the problem tractable, we will focus on the average behavior of OAMP over all possible $V$ in a Haar ensemble. This means, although a fixed $V$ is being used to generate the sequence of estimates, the fact that it was originally chosen at random and will be exploited at each step to simplify the analysis. The hope is that the behavior with a fixed $V$, the one of interest, will be similar to that predicted by SE using the “ensemble”. Clearly, this is not always the case, e.g. $V = I$ is a bad choice. It is left to numerical studies to confirm this estimate.

“Average performance” is also implied in other works on AMP family of algorithms [1], [16], [18], [19], [23]. This paper uses this argument explicitly.

E. Contributions of This Paper

This paper aims at a comprehensive understanding on the impact of orthogonality in OAMP and other AMP-type algorithms. Following the basic concept in [17] and inspired by the works in [19], [23], we take a high-level approach. We separate the overall problem into two inter-coupled sub-ones:

(i) the impacts of orthogonality, and
(ii) how to establish orthogonality.

For (i), the orthogonal principle leads to a unified framework for different AMP-type algorithms [1], [16]–[19], [23]. This is implied in [23]. We make it explicit in this paper. For (ii), we outline a Gram-Schmidt orthogonalization (GSO) procedure to establish orthogonality, which leads to a general realization technique for OAMP.
The separated approach above reveals new insights and new treatments/applications, as the contributions listed below.

- OAMP discussed in this paper is based on a class of orthogonal local estimators that are more general than those used in the standard EP/AMP/OAMP/VAMP [1], [16]–[19], [23]. Hence, new realization methods are revealed. For example, we can optimize performance by selecting the best among the class of orthogonal local estimators. (See Section III.) Another example is the integral approach discussed in the next bullet.

- GSO does not require the differentiability required by the standard EP/AMP/OAMP/VAMP. In sub-section IV-C we outline an integral approach to OAMP via GSO, which works well when, e.g., a local estimator is realized by a software package in a black-box manner without guaranteed differentiability. Empirical advantages of integral-based OAMP can be found in [29].

- The derivation of SE for OAMP in this paper is inspired by the techniques used [16], [18], [19], [23]. The new approach is more concise for the following reasons.
  - Under the orthogonal principle, the behavior of SE can be directly characterized as the Bolthausen’s conditioning problem, which avoids the lengthy step-by-step tracking as in [16], [18], [19].
  - We analyze the Bolthausen’s conditioning problem using a few conjectures that can be bridged by more rigorous treatments in [16], [19], [23]. Our aim is conciseness, which can provide useful insights.
  - The algorithm discussed in this paper is symmetric, i.e., it consists of two local estimators under the same orthogonal principle. Hence we only need to analyze one local estimator during induction, since the result is applicable to the other one.

The following are more implications of the findings in this paper. Interested readers are referred to the references listed.

- Most discussions on AMP-family algorithms are for simple single measurement vector (SMV) systems [1], [13], [14], [16]–[22]. GSO-based OAMP can be generalized to more complex multiple measurement vector (MMV) systems, as reported in [45] for correlated massive-access channels. GSO also allows the extension of OAMP to systems with multiple Haar matrices. We are currently working on this issue [27].

- Based on GSO, we can prove that OAMP together with a properly designed decoder is capacity approaching. Such information-theoretic optimality is reported in [30], [31].

Overall, we expect that the findings in this paper provide justifications for the applications of OAMP in [24–31], [38–46], as well as guidelines for new treatments in communications and signal processing.

F. Notation

For convenience, a vector is said IID (resp. IIDG) if its entries are IID (resp. IIDG). A vector is said joint-Gaussian if its entries are jointly Gaussian. Boldface lowercase letters represent column vectors and boldface uppercase symbols denote matrices. \( I \) denotes the identity matrix of appropriate size, \( \mathcal{U}^N \) the set of all \( N \times N \) orthogonal matrices, \( \mathbb{R}^N \) the set of all length-\( N \) vectors, \( \mathbf{0} \) the zero matrix, \( \mathbf{a}^T \) the transpose of \( \mathbf{a} \), \( \| \mathbf{a} \| \) the \( \ell_2 \)-norm of \( \mathbf{a} \), \( N(\mu, \Sigma) \) the Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \), and \( \text{Diag}[\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M] \) the block-diagonal matrix with diagonal blocks \( \{ \mathbf{A}_n \} \). \( E \{ \cdot \} \) is the expectation operation over all random variables involved in the brackets, except when otherwise specified. \( \mathcal{E}\{ \mathbf{a} | b \} \) is the expectation of \( \mathbf{a} \) conditional on \( b \), \( \text{Var} \{ \mathbf{a} | b \} \), \( \text{mse} \{ \mathbf{a} | b \} = E \{ (\mathbf{a} - E \{ \mathbf{a} | b \})^2 | b \} \), and \( \eta'(r) = \frac{\partial}{\partial r} \eta(r) \). \( \text{Var} \{ \mathbf{x} \} \) is the common variance of the entries in \( \mathbf{x} \), where \( \mathbf{x} \) is a vector of IID entries. \( \mathbf{a} \xrightarrow{d} \mathbf{b} \) denotes that the distribution of \( \mathbf{a} \) converges to that of \( \mathbf{b} \) as the length goes to infinity.

To aid the reader in finding a referenced Definition, Property, Theorem and so forth, a single counter is used for labelling (e.g., Definition 1, Property 2, ...).

II. PRELIMINARIES

A. Polar Coordinates

Let the Cartesian coordinate of \( \mathbf{a} \in \mathbb{R}^N \) be \( \mathbf{a} = \{ a_n, n = 1, \ldots, N \} \). Define
\[
\rho(\mathbf{a}) \equiv \| \mathbf{a} \| = \sqrt{\sum_{n=1}^N a_n^2} \quad \text{and} \quad \zeta(\mathbf{a}) = \mathbf{a}/\| \mathbf{a} \|. \tag{2}
\]
We call \( \rho(\mathbf{a}) \) and \( \zeta(\mathbf{a}) \) the amplitude and angle of \( \mathbf{a} \) respectively, and call \( (\rho(\mathbf{a}), \zeta(\mathbf{a})) \) jointly as the polar coordinate of \( \mathbf{a} \). (Note that \( \zeta(\mathbf{a}) \) has \( N - 1 \) degrees of freedom since \( \| \zeta(\mathbf{a}) \| = 1 \).) Denote by \( S^N(\rho) \) a sphere of amplitude \( \rho \) over \( \mathbb{R}^N \) in an \( N \)-dimensional Euclidean space. Any \( \mathbf{a} \in S^N(\rho) \) has a fixed norm \( \| \mathbf{a} \| = \rho \). Clearly, \( \zeta(\mathbf{a}) \in S^N(1) \), i.e., it is a point on the unit sphere.

There are other ways to define the polar coordinates [48]. Our definition above serves the purpose in this paper.

B. Haar Distribution and Haar Transform

\textbf{Definition 1 (Haar Distribution):} An \( N \times N \) matrix \( \mathbf{V} \) is said Haar distributed, denoted by \( \mathbf{V} \sim \mathcal{H}^N \), if \( \mathbf{Vc} \) is uniformly distributed over \( S^N(\rho) \) for any fixed \( \mathbf{c} \in S^N(\rho) \).

For brevity, we say \( \mathbf{V} \) is Haar to mean \( \mathbf{V} \) is Haar distributed. (This should not be confused with the concept of deterministic Haar matrices used in Fourier analysis.)

It can be verified that Definition 1 is consistent with the conventional definition of the Haar distribution [49]. Some common linear transforms, such as discrete cosine transform (DCT) [50] and Hadamard transform [51], can be regarded as samples of a Haar transform. The following follows from Definition 1 directly.

\textbf{Lemma 2:} \( \Omega \in \mathcal{U}^N, \mathbf{V} \sim \mathcal{H}^N \) and \( \mathbf{c} \in \mathbb{R}^N \). Then, \( \Omega \mathbf{V} \sim \mathcal{H}^N \) and \( \mathbf{V} \Omega \sim \mathcal{H}^N \). Furthermore, \( \zeta(\mathbf{Vc}) \) and \( \zeta(\mathbf{c}) \) are mutually independent.

\footnote{Although we only consider a square \( \mathbf{V} \), the results of this paper are still valid for non-square \( \mathbf{V} \). For example, \( \mathbf{V} \) is \( M \times N \) with \( M < N \), we can expand \( \mathbf{V} \) to a square one without destroying the Haar property and add constraints \( \{ x_i = 0, i = M + 1, \ldots, N \} \) at \( \mathbf{V} \). The same method applies for \( M > N \). Then the problem is the same as the square case.}
C. Pseudo-IID (PIID) Variables

We will prove in III-E that the distortions in OAMP are pseudo IID Gaussian (PIIDG) defined below.

**Definition 3:** A sequence \( a = \{a_n\} \) of size \( N \) is said to be pseudo IID (PIID) if any subset of size \( M \) in \( a \) is asymptotically IID when \( N \to \infty \) and \( M \) remains fixed.

**Lemma 4:** Let \( a = V a' \) where \( V \) is Haar of size \( N \times N \) and \( a' \) is an arbitrary fixed vector. From Theorem 2.8 in [52], the entries of \( a \) are PIIDG. The marginal symbol distribution of \( a_n, \forall n \) converges to \( \mathcal{N}(0, v) \) with \( v = \frac{1}{N} E(\|a'\|^2) \) as \( N \to \infty \).

Some properties/conjectures of PIID/PIIDG (corresponding to those of IID/IIDG) are made in Appendix A.

D. Separable Function

Let \( \pi = \pi(a) \) be a vector function between two length-\( N \) vectors \( \pi = \{\pi_n\} \) and \( a = \{a_n\} \). We say that \( \pi \) is separable if we can write \( \pi_n = \pi_n(a_n), \forall n \). We say that \( \pi \) is separable-IID if each \( \pi_n \) is IID drawn from an ensemble of random scalar functions. (See III-E for more explanations.)

E. Assumptions

Throughout this paper, unless stated otherwise, we will assume that (i) the length of a vector is \( N \) and (ii) \( \mathcal{X} \) and \( x \) are normalized, i.e., \( \frac{1}{N} E(\|x\|^2) = \frac{1}{N} E(\|x\|^2) = 1. \) Furthermore, we assume that (i) the entries of \( x \) are IID, (ii) the estimation functions \( \gamma_t \) and \( \phi_t \) are separable-IID, and (iii) the size of \( V \) is infinity, i.e., \( N \to \infty \). Hence \( \mathcal{X} \) is PIIDG.

F. Orthogonality under Law of Large Numbers (LLN)

**Definition 5 (LLN Orthogonal):** Let \( a = \{a_n\} \) and \( b = \{b_n\} \) be two sequences of length \( N, E(\|a\|^2) \neq 0, E(\|b\|^2) \neq 0, E(a_n b_n) = 0 \) and \( \text{Var}(a_n b_n) \) finite. For any fixed \( \delta > 0 \) and \( \varepsilon > 0 \), if there is a fixed \( N' \) such that

\[
\Pr\left(\frac{(a^T b)^2}{E(\|a\|^2)E(\|b\|^2)} < \varepsilon\right) \geq 1 - \delta, \quad \text{for} \quad N > N',
\]

we then say that \( a \) and \( b \) are LLN-orthogonal, denoted as

\[
\frac{1}{N} a^T b \overset{\text{LLN}}{\longrightarrow} 0.
\]

In the sequel, we will use (5) as a measure for the convergence of \( a^T b \). Eq. (3) implies the following,

\[
E(\|a\|^2)E(\|b\|^2) \gg (a^T b)^2 \quad \text{in probability as} \quad N \to \infty.
\]

This is useful for our later discussions. Note that \( E(\|a\|^2) = 0 \) alone cannot guarantee (4) when \( a \) and/or \( b \) are not IID. For example, consider: (i) \( \Pr(a_n = +1) = \Pr(a_n = -1) = 0.5 \) and all \( \{a_n\} \) are equal in each trial (i.e., fully correlated), (ii) \( \{b_n\} \) follow the same distribution as \( \{a_n\} \), and (iii) \( \{a_n\} \) and \( \{b_n\} \) are independent of each other. Then \( E(a^T b) = 0 \) but \( E(\|a\|^2\|b\|^2) = (a^T b)^2 = N^2 \), so (5) does not hold.

III. OAMP Principles

In this section, we define OAMP using an orthogonal constraint, based on which we derive the state evolution procedure for the analysis of OAMP. We will discuss the implementation of the orthogonal constraint in the next section.

AMP algorithms are analysed in a random framework [18]: their “average behaviour” is found by taking expectations with respect to the Gaussian-distributed matrix \( A \) in (1). In this paper, we analyze OAMP in a similar way: the average behaviour of OAMP is studied with respect to the \( N \times N \) Haar-distributed matrix \( V \). Precisely, we study the average behaviour of a sequence of trials, where each trial involves choosing \( x \) and \( V \) at random, generating the “observations” that determine the constraints \( \Gamma \) and \( \Phi \), running the OAMP algorithm, and recording the squared-error. Since the analysis studies average behaviour, simulation studies are needed to ascertain whether a specific system exhibits the predicted average behaviour. This is true of all AMP-type algorithms. The primary use of the analysis is for designing AMP-type algorithms: appropriate if not optimal messages to pass are derived under the assumption that the average behaviour holds.

A. Gram-Schmidt (GS) Model

Let \( \hat{x} \) be an arbitrary observation of \( x \). We generate a new vector \( \xi = \hat{x} - \alpha x \) using GS orthogonalization [53], where

\[
\alpha = \frac{1}{\xi} E(\|x\|^2)
\]

is a scalar. We will call (7) below the GS model of \( \hat{x} \) with respect to \( x \):

\[
\hat{x} = \alpha x + \xi.
\]

We treat \( \xi \) as an error term, referred to as GS error, which is different from the common definition of an error \( \hat{x} - x \). Its average entry-wise power is

\[
v = \frac{1}{N} E(\|\xi\|^2) = \frac{1}{N} E(\xi^T \xi).
\]

It can be verified that \( \xi \) is orthogonal to \( x \), i.e.

\[
E(x^T \xi) = 0.
\]

**Normalized Model:** When \( \alpha = 1 \), i.e., \( \hat{x} = x + \xi \), we call (7) the normalized model.

**MMSE Model:** When \( v = \alpha(1 - \alpha) \), we call (7) the MMSE model. In this case, \( \hat{x} = E(x|\hat{x}) \) is the MMSE solution.

In the existing AMP-type algorithms, the MMSE-NLE is MMSE modeling and their LES are all normalized, which are unified under the GS model. Apart from that, the GS model was used to circumvent the difficulty in the achievable rate analysis and capacity-optimality proof of OAMP [30], [31].

B. Generic Iterative Process

Return to the system: \( \mathcal{X} = Vx, x \sim \Phi \) and \( \mathcal{X} \sim \Gamma \). Our aim is to use the AMP-type iterative approach in Fig. 1(b) to find the \textit{a posteriori} mean of \( x \):

\[
\hat{x} \equiv E(x|\mathcal{X} = Vx, \Gamma, \Phi),
\]

which minimises the mean-square error \( \frac{1}{N} E(\|\hat{x} - x\|^2) \).

Similarly, we define \( \mathcal{X} \).

We add an iteration index \( t \) to the estimates and formally define the iterative process in Fig. 1(b) as follows. Note that, given \( V \), and \( \{\gamma_t\} \) and \( \{\phi_t\} \), the messages...
\{\hat{X}_t^{\text{in}}, \hat{X}_t^{\text{out}}, \hat{x}_t^{\text{out}}, \gamma_t\} in (11) are completely determined by initial values.

**Generic Iterative Process (GIP):** Initializing from arbitrary \(\{\hat{X}_0^{\text{in}}, \hat{X}_0^{\text{out}}\}\) and \(t = 1\),

\[
\hat{X}_t^{\text{out}} = \gamma_t(\hat{X}_{t-1}^{\text{in}}), \quad \hat{x}_t^{\text{out}} = \phi_t(\hat{x}_{t-1}^{\text{in}}),
\]

(11a)

\[
\hat{X}_t^{\text{in}} = V^T \hat{X}_t^{\text{out}}, \quad \hat{X}_t^{\text{in}} = V \hat{x}_t^{\text{out}}.
\]

(11b)

The trivial initialization \(\hat{X}_0^{\text{in}} = \hat{x}_0^{\text{in}} = 0\) are generally adopted. In this case, the iterative process is actually kick-started using the observations within \(\Gamma\) and \(\Phi\).

In (11a), \(\gamma_t(\hat{X}_{t-1}^{\text{in}})\) and \(\phi_t(\hat{x}_{t-1}^{\text{in}})\) respectively generate refined estimates of \(X\) and \(\hat{X}\). Proper statistical models of \(\hat{X}_{t-1}^{\text{in}}\) and \(\hat{x}_{t-1}^{\text{in}}\) are required for the design of \(\gamma_t\) and \(\phi_t\), respectively. Tracking such models is in general a prohibitively difficult task due to the correlation problem. We will resolve this difficulty in III-D by introducing an orthogonal principle, and hence define OAMP.

We call (11) a parallel algorithm. Tracking (11), we obtain two processes as follows:

\[
\hat{X}_0^{\text{in}} \rightarrow \hat{X}_1^{\text{out}} \rightarrow \hat{x}_1^{\text{out}} \rightarrow \hat{X}_2^{\text{out}} \rightarrow \hat{X}_3^{\text{out}} \rightarrow \cdots, \quad (12a)
\]

\[
\hat{x}_0^{\text{in}} \rightarrow \hat{X}_1^{\text{out}} \rightarrow \hat{x}_1^{\text{out}} \rightarrow \hat{X}_2^{\text{out}} \rightarrow \hat{x}_2^{\text{out}} \rightarrow \hat{x}_3^{\text{out}} \rightarrow \cdots. \quad (12b)
\]

There are no common variables (considering both superscripts and subscripts) in these two processes, so they are uncoupled. We call (12a) and (12b) two serial algorithms. They are initialized by \(\hat{X}_0^{\text{in}} = 0\) and \(\hat{x}_0^{\text{in}} = 0\), respectively. It can be verified that the EP/OAMP/VAMP algorithm discussed in [16] is the one in (12a). Hence, the parallel and serial algorithms are equivalent for the purpose of analysis, and so the results from one can be applied to another. On the other hand, only one of the two serial algorithms is necessary for implementation. Later we will see that (12) leads to more concise analysis due to its symmetry.

### C. GS Errors in GIP

Note that OAMP defined above is symmetric since \(\gamma_t\) and \(\phi_t\) in (11) are under the same orthogonal constraint. Hence, in Theorem 7 below, we only need to analyze one local estimator and the result is applicable to the other one.

Let the messages in (11) be expressed in their GS models:

\[
\hat{X}_t^{\text{out}} = \alpha_t^{\gamma_x} X + g_t^{\text{out}}, \quad \hat{x}_t^{\text{out}} = \alpha_t^{\phi_{\gamma_x}} x + f_t^{\text{out}},
\]

(13a)

\[
\hat{X}_t^{\text{in}} = \alpha_{t+1}^{\gamma_{\gamma_t}} X + g_t^{\text{in}}, \quad \hat{x}_t^{\text{in}} = \alpha_{t+1}^{\phi_{\gamma_t}} x + f_t^{\text{in}}.
\]

(13b)

Let the average powers of the GS errors \(g_t^{\text{out}}, g_t^{\text{in}}\), \(f_t^{\text{out}}\) and \(f_t^{\text{in}}\) be \(\gamma_t^{\text{out}}, \gamma_t^{\text{in}}, \phi_{\gamma_t}^{\text{out}}\) and \(\phi_{\gamma_t}^{\text{in}}\), respectively. Since \(V\) is orthogonal, combining (11b) and (13), then the following relationships hold:

\[
\alpha_t^{\gamma_t} = \alpha_{t+1}^{\gamma_t}, \quad \alpha_t^{\phi_t} = \alpha_{t+1}^{\phi_t}, \quad \gamma_t^{\text{out}} = \gamma_t^{\text{in}}, \quad \phi_{\gamma_t}^{\text{out}} = \phi_{\gamma_t}^{\text{in}}.
\]

(14a)

\[
\gamma_t^{\text{out}} = \gamma_t^{\text{in}}, \quad \phi_{\gamma_t}^{\text{out}} = \phi_{\gamma_t}^{\text{in}}.
\]

(14b)

We will use (14) to simplify the derivation of SE in III-G.

Let \(g_0^{\text{out}} = V f_0^{\text{in}}\) and \(f_0^{\text{out}} = V^T g_0^{\text{in}}\). The following matrices contain the GS errors in (13) up to iteration \(t\):

\[
G_t^{\text{out}} = \begin{bmatrix} g_0^{\text{out}}, \cdots, g_{t-1}^{\text{out}} \end{bmatrix}, \quad F_t^{\text{out}} = \begin{bmatrix} f_0^{\text{out}}, \cdots, f_{t-1}^{\text{out}} \end{bmatrix},
\]

(15a)

\[
G_t^{\text{in}} = \begin{bmatrix} g_0^{\text{in}}, \cdots, g_{t-1}^{\text{in}} \end{bmatrix}, \quad F_t^{\text{in}} = \begin{bmatrix} f_0^{\text{in}}, \cdots, f_{t-1}^{\text{in}} \end{bmatrix}.
\]

(15b)

Denote

\[
A_t = \begin{bmatrix} X, G_t^{\text{in}}, G_t^{\text{out}} \end{bmatrix}, \quad B_t = \begin{bmatrix} x, F_t^{\text{out}}, F_t^{\text{in}} \end{bmatrix}.
\]

(15c)

Combining (11), (13) and (15), we have

\[
A_t = V B_t.
\]

(16)

We will discuss the behavior of GS errors in III-F based on the constraint in (16).

### D. Orthogonal AMP (OAMP)

**Definition 6:** OAMP is a special case of the GIP in III-B when the following orthogonal constraint holds for \(N \rightarrow \infty\), \(t \geq 1\) and \(0 \leq t' < t\),

\[
\frac{1}{N} (f_{t'}^{\text{in}})^T f_t^{\text{out}} \xrightarrow{\text{LLN}} 0, \quad \frac{1}{N} (f_t^{\text{out}})^T f_t^{\text{out}} \xrightarrow{\text{LLN}} 0,
\]

(17a)

\[
\frac{1}{N} X^T g_t^{\text{out}} \xrightarrow{\text{LLN}} 0, \quad \frac{1}{N} x^T f_t^{\text{out}} \xrightarrow{\text{LLN}} 0.
\]

(17b)

In words, in OAMP, the output errors of \(\gamma_t\) and \(\phi_t\) in (11) are LLN-orthogonal to their respective current and previous input errors, as well as to \(X\) and \(x\). The orthogonality in (17) is the key to solve the correlation problem, as will be discussed below.

Here are some intuitions. Return to the GIP in Fig. 1(b) and focus on \(\phi_t\). Notice the following:

- The error \(f_{t-1}^{\text{in}}\) into \(\phi_t\) comes from the output error of \(\gamma_{t-1}\) in the previous iteration.
- The output error \(f_t^{\text{out}}\) of \(\phi_t\) becomes the error into \(\gamma_{t+1}\) in the next iteration.

The correlation between \(f_{t-1}^{\text{in}}\) and \(f_t^{\text{out}}\) implies error circulation from the output of \(\gamma\) back to its input, which may lead to positive feedback and instability. In general, we should avoid this effect. This gives a heuristic reason for the orthogonality requirement between \(f_t^{\text{out}}\) and \(f_{t-1}^{\text{in}}\).

The LLN-orthogonality in (17a) is between \(f_t^{\text{out}}\) and \(f_{t'}^{\text{in}}\). \(t' = 1, 2, \ldots, t-1\), which is referred to as global orthogonality.

Definition 3 is a more general definition of OAMP. Existing AMP-type algorithms are restricted to particular orthogonality realizations, which are sufficient but not necessary conditions of (17). We will discuss the specific realizations of (17) in IV.

### E. Average MSE Performance

Return to the original problem of estimating \(x\) after \(t\) iterations in (11). We adopt the following method:

(i) We track the GS models in (7) recursively for \(t' = 1, 2, \ldots, t\) and obtain (see (13a) and (14a)):

\[
\hat{x}_t^{\text{out}} = \alpha_t^{\gamma_x} x + f_t^{\text{out}}.
\]

(18)
(ii) We treat $f_{t}^{\text{out}}$ above as a pure error. We use $\omega \hat{x}_{t}^{\text{out}}$ as an estimate of $x$, where $\omega$ is a scaling factor. Based on the above discussions, we have

$$
\text{MSE} \equiv \frac{1}{N} \mathbb{E}_{r,x} \left\{ \| \omega \hat{x}_{t}^{\text{out}} - x \|^2 \right\} = \nu_{t}^{2} / \left( (\alpha_{t}^{2})^2 + \nu_{t}^{2} \right),
$$

(19)

where $\omega = \alpha_{t}^{2} / \left( (\alpha_{t}^{2})^2 + \nu_{t}^{2} \right)$ minimizes the mean square error (MSE) and $\nu_{t}^{2}$ is defined in (14). The expectation in (19) is over the distribution of the observation $r$. The latter is in turn determined by the distributions of $x$ and possible distortions in $r$. In particular, if the distortion is an additive noise vector $\eta$, then the distribution of $r$ is jointly determined by $x$ and $\eta$.

The GS model in (18) is not an optimal estimation of $x$. The GS error $f_{t}^{\text{out}}$ may actually contain useful information about $x$, so it is possible to compute a lower MSE as that in (19), but the related cost can be high in iterative processing. The approach based on GS models has a distinguished advantage of low cost. It can be seen from (14) that the parameters of the GS models are not affected by transforms involving $V$ and $V^{T}$. This makes it computationally efficient. Incidentally, the above method may still lead to global optimality in OAMP. See [16] for more details.

In general, evaluating (19) is still a difficult task. To overcome this difficulty, we borrow a technique from [1]; we redefine the MSE as follows:

$$
\text{MSE} \equiv \frac{1}{N} \mathbb{E}_{r,x,V} \left\{ \| \omega \hat{x}_{t}^{\text{out}} - x \|^2 \right\}.
$$

(20)

In (20), we compute the GS model of $\hat{x}_{t}^{\text{out}}$ using the joint distribution of $r$ and $V$. We study the average behavior over a sequence of trials, where each trial involves choosing $r$ and $V$ at random. (See the second paragraph of this section.)

F. Error Behavior

We now consider the details in tracking the GS models in (18). As an example, consider the GS model $\hat{x}_{t}^{\text{out}} = \alpha_{t} x + f_{t}^{\text{out}}$. From (6), (11), (13) and (14), we generate $\alpha_{t}^{\phi}$ as follows

$$
\alpha_{t}^{\phi} = \frac{1}{N} \mathbb{E}_{r,x,V} \left\{ \phi_{1}(\alpha_{t-1}^{\gamma} x + f_{t-1}^{\text{in}})^{T} x \right\}.
$$

(21)

where $\alpha_{t-1}^{\gamma} x + f_{t-1}^{\text{in}}$ is the GS model for $\hat{x}_{t-1}^{\text{in}}$. Assume that $\alpha_{t-1}^{\gamma}$ is known. In the following, we give the distribution of $f_{t}^{\text{in}}$ in order to evaluate (21).

Theorem 7 is the main result of this paper. For convenience, we adopt the following notations. A matrix is said columnwise PIIDG and row-wise joint-Gaussian (CPIIDG-RJG) if each column is PIIDG and each row is joint Gaussian. Note two subtle points here: (i) as mentioned for (20), $V$ is randomly selected for each trial, and (ii) $V$ remains unchanged in all iterations during one iterative process. In other words, $V$ is randomly selected in Fig. 1(a) but, once selected, it remains unchanged throughout an iterative process in Fig. 1(b). Point (ii) means that, for $t > 0$, $V$ is constrained by $A = V B$ in (16). This is referred to as the Bolthausen’s conditioning problem in (15), (18), (19), (23). Taking these points into consideration, we prove Theorem 7 in Appendix B.

**Theorem 7:** Assume that $V$ is Haar distributed and OAMP is initialized at $t = 1$ with IIDG $g_{0}^{\text{in}}$ and $f_{0}^{\text{in}}$ independent of $x$ and $x$, respectively. Then, when $t$ is finite, $N \to \infty$ and over the Haar ensemble of $V$, the following hold for the errors in (15) for OAMP asymptotically:

(a) $G_{t}^{\text{in}}$ is CPIIDG-RJG and independent of $x$ and any $z$ provided that $z$ is independent of $V$;

(b) $F_{t}^{\text{in}}$ is CPIIDG-RJG and independent of $x$ and any $z$ provided that $z$ is independent of $V$.

Notes: (i) Theorem 7 is defined over a Haar ensemble of $V$. It may not hold for a specific sample of $V$. (ii) An example of $z$ is additive thermal noise in the system.

G. State Evolution (SE) for OAMP

Return to the error behavior of OAMP for given $\{f_{t}^{\text{in}}\}$ and $\{g_{t}^{\text{in}}\}$ in the asymptotic case of $N \to \infty$. For this purpose, we track the distributions of the messages in OAMP and evaluate MSEs accordingly, as detailed below.

Theorem 7 states that any separable-IID function of GS errors $F_{t}^{\text{in}}$ or $G_{t}^{\text{in}}$ in OAMP are asymptotically CPIIDG-RJG. We still need the GS parameters to determine the GS model of messages.

Recall from (14) that the GS parameters are not affected by the transform by an orthogonal $V$. This greatly simplifies the problem; we only need to focus on $\phi_{t}$ and $\gamma_{t}$. Assume that $\phi_{t}$ and $\gamma_{t}$ are both separable-IID. Let $z_{t}^{\phi} \sim \mathcal{N}(0,1)$. From Theorem 7, $f_{t}^{\text{in}}$ is PIIDG. Hence following Conjecture 20 (see Appendix A-B), we can rewrite (21) as

$$
\alpha_{t}^{\phi} = \mathbb{E}_{x,n,z} \left\{ \phi_{t,n}(\alpha_{t-1}^{\gamma} x + \sqrt{\nu_{t-1}^{2}} z_{t}^{\phi}) \cdot x_{n} \right\}.
$$

(22)

We can apply similar reasoning to generate $\alpha_{t}^{\gamma}, \nu_{t}^{\gamma}, \alpha_{t}^{\phi}$ and $\nu_{t}^{\phi}$ defined in (13) and (14). We then obtain the following recursion for the GS parameters.

$$
\alpha_{t}^{\gamma} = \mathbb{E}_{r,n,z} \left\{ \gamma_{t,n}(\alpha_{t-1}^{\phi} x + \sqrt{\nu_{t-1}^{2}} z_{t}^{\phi}) \cdot x_{n} \right\},
$$

(23a)

$$
\nu_{t}^{\gamma} = \mathbb{E}_{r,n,z} \left\{ \{ \gamma_{t,n}(\alpha_{t-1}^{\phi} x + \sqrt{\nu_{t-1}^{2}} z_{t}^{\phi}) - \alpha_{t-1}^{\phi} x_{n} \}^2 \right\},
$$

(23b)

$$
\alpha_{t}^{\phi} = \mathbb{E}_{x,n} \left\{ \phi_{t,n}(\alpha_{t-1}^{\gamma} x + \sqrt{\nu_{t-1}^{2}} z_{t}^{\phi}) \cdot x_{n} \right\},
$$

(23c)

$$
\nu_{t}^{\phi} = \mathbb{E}_{x,n} \left\{ \{ \phi_{t,n}(\alpha_{t-1}^{\gamma} x + \sqrt{\nu_{t-1}^{2}} z_{t}^{\phi}) - \alpha_{t-1}^{\phi} x_{n} \}^2 \right\},
$$

(23d)

where $\{z_{t}^{\phi}, z_{t}^{\gamma}\}$ are IID drawn from $\mathcal{N}(0,1)$. The expectations are based on the assumption that $\gamma_{t,n}$ contains observation $r_{n}$ and there is no observation in $\phi_{t,n}$. Similar results can be obtained if $\phi_{t,n}$ also contains observation. The key here is replacing the GS errors using IIDG variables, which makes the problem tractable.

We re-write the functions in (23) into a more concise form in (24) below, where $\gamma_{SE}$ and $\phi_{SE}$ may or may not have explicit expressions, but they can always be numerically tabulated.

**State Evolution for OAMP:** Starting with $t = 1$ and $\{\alpha_{0}, \alpha_{0}^{\gamma}, \nu_{0}^{\gamma}, \nu_{0}^{\phi}\}$,

$$
\begin{align*}
(\alpha_{t}^{\gamma}, \nu_{t}^{\gamma}) &= \gamma_{SE}(\alpha_{t-1}^{\phi}, \nu_{t-1}^{\phi}), \\
(\alpha_{t}^{\phi}, \nu_{t}^{\phi}) &= \phi_{SE}(\alpha_{t-1}^{\gamma}, \nu_{t-1}^{\gamma}).
\end{align*}
$$

(24a)

(24b)
Consider the messages in (15) under their GS models. Assume that the distribution of $x$ is given. Lemma 8 and Theorem 7 give the distributions of $\mathbf{X}$ and the GS errors. Hence we can find the distributions of the messages using the GS parameters generated in (24). This answers the question on how to generate the distributions in OAMP.

The performance of an estimation is not determined by a single GS parameter $\alpha$ or $v$. Instead, it is determined by their ratio $\alpha^2/v$, i.e., an effective SNR. As a result, the SE in (24) consists of dual-input and dual-output (DIDO) functions of GS model parameters. This is slightly different from the conventional single-input and single-output (SISO) SE functions for AMP-type algorithms [4], [17], but the spirits are the same. We can convert (24) into a SISO recursion via proper scaling so that $\alpha_i^2 = \alpha_v^2 = 1$ (scaling does not change the performance of an estimation). However, the singular situations of $\alpha_i^2 = 0$ or $\alpha_v^2 = 0$ in the trivial initialization may cause problems in normalization.

Using (22)-(24), we can assess the MSE of OAMP via (20). This provides a convenient tool for analysis and optimization.

H. Distributions of $\{\phi_n\}$ and $\{\gamma_n\}$

In (15) and (24), we assume that $\{\phi_n\}$ and $\{\gamma_n\}$ are IID samples from their respective ensembles. We now clarify this assumption using two examples.

Example: Consider $\gamma_n = \gamma, \forall n$. In this special case, the ensemble for $\{\gamma_n\}$ has only one element.

Next, recall that (23) and (24) are derived over the Haar ensemble for $V$. Let $Q$ be a permutation matrix. Clearly, $QV$ is in the Haar ensemble if $V$ is. Then we have Lemma 8.

Lemma 8: Let $Q$ be a permutation matrix. The SE for GIP in (11) remains unchanged if $V$ is replaced by $QV$, or equivalently, $\gamma(X)$ is replaced by $\gamma(QX) = Q^T \gamma(X)Q$.

Example: Consider $r = D\mathbf{X} + \eta$, where $D$ is diagonal. We construct a separable $\gamma$ using the so-called LMMSE estimator (see (12) in (17)):

$$\gamma_n(X_n) = \frac{1}{N} \sum_{m=1}^N D_{mn}^2 \gamma_n + \left(1 + \frac{N}{M} \sum_{m=1}^N D_{mn}^2 + \sigma^2 \right) \lambda_n, \quad \forall n. \quad (25)$$

It can be verified that such $\gamma$ is orthogonal. We further consider a special case of $D$ with its diagonal entries given by

$$D_{nn} = \begin{cases} 1, & \text{ones} \\ 0, & \text{zeros} \end{cases}, \quad \forall n. \quad (26)$$

We cannot regard $\{D_{nn}\}$ as a random sequence. However, from Lemma 8, we can randomly permute $\{D_{nn}\}$ without affecting the related SE. In this sense, when $M \to \infty, N \to \infty$ and ratio $M/N$ fixed, the entries in a randomly permuted $\{D_{nn}\}$ asymptotically follow a binary distribution $D_{nn} = 1$ of probabilities $M/N$ and $D_{nn} = 0$ of probabilities $(N-M)/N$.

In conclusion, the separable-IID assumption is applicable to $\gamma$ in (25). Incidentally, (26) corresponds to $A$ formed by the first $M$ rows in an $N \times N$ Haar distributed matrix.

Such initialization ($\alpha = 0$) carries no useful information of the true signal. However, once the GIP started, the estimations in general carries useful information of the true signal, i.e., $\alpha \neq 0$. In this case, we can use a normalized GS model (e.g., $\alpha = 1$) determined by a single parameter $v$.

I. A Brief Summary

The following is a summary on the discussions so far.

- We presented a message passing process named GIP (see Fig. 1(b) and (11)) to solve the problem in Fig. 1(a).
- We introduced the GS model to characterize the messages in GIP. The related GS parameters are not affected by transform by an orthogonal matrix $V$, which facilitates a SE technique to track GS errors.
- We defined OAMP as an orthogonalized GIP.
- We showed that the separable-IID function of GS errors in OAMP is asymptotically the same as that of PIIDG random vectors when $N \to \infty$. We developed a SE technique to track GS parameters in OAMP.

Hence, using SE, we can approximately determine the MSE of OAMP for the system in Fig. 1(a) when $N$ is large.

IV. GRAM-SCHMIDT ORTHOGONALIZATION

In this section, we discuss techniques to realize the orthogonality required in (17). For simplicity, we may sometimes omit the subscript $t$.

A. Gram-Schmidt Orthogonalization

Definition 9: Consider the GS models: $\pi = \pi_{\text{out}} x + \xi_{\text{out}}$ and $\hat{x} = \pi_{\text{in}} x + \xi_{\text{in}}$. We say that $\pi = \pi(\hat{x})$ is an orthogonal estimator if $E\{\langle \xi_{\text{in}} \rangle^T \xi_{\text{out}} \} = 0$.

Let $\pi = \pi(\hat{x})$ be an arbitrary prototype. We construct an orthogonal $\pi(\hat{x})$ as follows

$$\pi = \pi(\hat{x}) = \hat{\pi}(\hat{x}) - B\hat{x}. \quad (27)$$

Then we can rewrite the orthogonal requirement as

$$E\{\langle \xi_{\text{in}} \rangle^T \xi_{\text{out}} \} = 0 \quad (28a)$$

$$E\{\langle \xi_{\text{in}} \rangle^T \pi \} = 0 \quad (28b)$$

$$E\{\langle \xi_{\text{in}} \rangle^T \langle \hat{\pi}(\hat{x}) - B\hat{x} \rangle \} = 0 \quad (28c)$$

where (a) follows the definition of $\xi_{\text{out}}$, (b) follows (9) for the GS model of $\hat{x}$, and (c) due to (27). Noting that $E\{\langle \xi_{\text{in}} \rangle^T \hat{x} \} = 0$, we can rewrite (28) as

$$E\{\langle \xi_{\text{in}} \rangle^T \langle \pi_{\text{out}} x + \xi_{\text{in}} \rangle \} = 0 \quad (29)$$

It’s interesting to note that GSO is a fundamental component of the conjugate gradient approach, which is used to solve the high-complexity challenges of the LMMSE estimator in OAMP [54]. We think there are some similarities between the GSO for OAMP in this paper and the GSO for the conjugate gradient approach.

The orthogonality discussed above is looser than LLN-orthogonality required in [4]. We will consider the latter next.

B. LLN-Orthogonality

The GSO in [V-A] establishes the orthogonality between the current input and output errors. The following theorem gives a sufficient condition for the LLN-orthogonality in (17), where...
the current output errors are orthogonal to the current and previous input errors.

**Theorem 10:** The orthogonality in (17) is satisfied if \( \gamma_t \) and \( \phi_t \), \( \forall t \), are orthogonal and separable-IID.

**Proof:** See Appendix C.

GSO and Theorem 10 together show a way to construct an OAMP algorithm based on two arbitrary separable \( \gamma_t \) and \( \phi_t \).

C. Computational Aspects for GSO

The key to GSO is to find \( B \) in (29). We first consider a special case when \( \phi \) is separable and, and \( x \) IID. From Theorem 7 and Conjecture 20 (see Appendix A-B), (29) reduces to

\[
B = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^{N} \left( \frac{\partial \phi_i(\hat{x}^i)}{\partial \hat{x}^i} \right) d\hat{x}^i dx}{v_{fn}},
\]

where \( z_f \sim \mathcal{N}(0, v_{fn}) \) and \( v_{fn} = \frac{1}{\phi} \| f^i \|^2 \). Assume that \( p_x(x) \) is known. Then (30) can be evaluated numerically. We may pre-calculate a table for \( B \) as a function of \( v_{fn} \) off-line. Then the cost is low for on-line processing.

In some cases, \( \phi \) (or \( \gamma \) or both) may not be explicitly given. For example, \( \phi \) can be a black-box type estimator in a software package. In this case, we can still generate \( B, \phi_t \) and \( \phi_t^0 \) numerically by the Monte Carlo method.

We call (30) the integral approach to \( B \). The advantage of this approach over the derivative one can be found in (29) for an application in image processing. The integral approach requires \( p_X(x) \). This condition is usually met in communication applications, but not in some signal-processing applications. The derivative approach discussed next provides an alternative approach.

D. Derivative Approach to B

We next consider a special case when \( \phi \) is separable and derivable, and \( x \) IID. The following is an alternative derivative approach to OAMP, first introduced in (17) and inspired by (1).

From Theorem 7, Conjecture 20 (see Appendix A-B) and Stein’s Lemma (55),

\[
\frac{1}{N} \mathbb{E}\{ (f^i)^T \phi(\hat{x}^i) \} = v_{fn} \mathbb{E}\{ \partial \phi(\hat{x}^i) / \partial \hat{x}^i \},
\]

where \( v_{fn} = \frac{1}{\phi} \| f^i \|^2 \). Hence, (29) can be rewritten as

\[
B = \mathbb{E}\{ \partial \phi(\hat{x}^i) / \partial \hat{x}^i \} = \mathbb{E}\{ \phi_t^0 \}.
\]

For an orthogonal \( \phi \), substituting (32) into \( \phi(\hat{x}^i) = \phi(\hat{x}^i) - B\hat{x}^i \), the derivative of \( \phi \) is equal to zero. On the other hand, if the derivative of \( \phi \) is zero, we have \( B = 0 \) from (32) and also \( \mathbb{E}\{ (f^i)^T \phi(\hat{x}^i) \} \) from (29), i.e., \( \phi \) is orthogonal. Thus, we have a necessary and sufficient condition below for the orthogonal requirement in OAMP.

**Proposition 11:** Under the IID \( x \) and the GS error \( f^i \) in OAMP, an estimator \( \phi \) is orthogonal if and only if

\[
\mathbb{E}\{ \partial \phi(\hat{x}^i) / \partial \hat{x}^i \} = 0.
\]

In particular, if \( \phi \) can be expressed as \( \phi = \Delta \hat{x}^i \), where \( \Delta \) is a diagonal matrix, then (33a) is equivalent to

\[
\text{tr}\{ \Delta \} = 0.
\]

Finally, when \( \phi \) is separable, (32) can be rewritten to

\[
B = \frac{1}{N} \mathbb{E}\{ \phi^0(\hat{x}^i) \} / \partial [\hat{x}^i],
\]

Eqn. (34) does not require the distribution of \( x \), which makes it attractive in many signal processing problems. This advantage was first pointed out in (1).

E. Alternative Form of OAMP

The original OAMP algorithm was derived in (17) for solving (1) via the recursion of a linear estimator (LE) and a non-linear estimator (NLE): (3a)

\[
\text{LE: } r_t = s_t + \frac{1}{\mathbb{I}_t[\hat{W}]} W_t(y - A s_t),
\]

(3b) \( \text{NLE: } s_{t+1} = \phi_t(r_t) = C_t(\hat{\phi}_t(r_t) - \mathbb{E}\{ \hat{\phi}_t \} \cdot r_t), \)

where \( W_t \) is an arbitrary prototype matrix and \( C_t \) a proper scalar. Eqn. (35) does not involve SVD and so is usually convenient for implementation. It is equivalent to the OAMP in III-D, which can be seen from the discussions below.

From (1), \( A = \mathbb{U}^T D \mathbb{V} \) and \( r = U y \). Denote \( \hat{x}^i_t = q_t, \hat{x}^i_{t+1} = s_{t+1}, \hat{x}^i_t = V s_t, \hat{x}^i_t = V q_t, \Delta_t = V W_t U^T \) and \( \lambda_t = \frac{N}{\lambda_t} \). We can rewrite (35) as

\[
\text{LE: } \hat{x}^i_t = V \hat{x}^i_{t+1},
\]

(36a) \( \hat{x}^i_t = \gamma_t(\hat{x}^i_t) = \lambda_t \Delta_t r + (I - \lambda_t \Delta_t D) \hat{x}^i_t, \)

(36b) \( \hat{x}^i_t = \mathbb{V}^T \hat{x}^i_t, \)

(36c) \( \hat{x}^i_t = \phi_t(\hat{x}^i_t) = C_t(\hat{\phi}_t(\hat{x}^i_t) - \mathbb{E}\{ \hat{\phi}_t \} \cdot \hat{x}^i_t). \)

It can be verified that \( \gamma_t \) and \( \phi_t \) in (36) are both orthogonal, \( \alpha_t = 1, \forall t \) and \( q_t^{out} = \hat{x}^i_t - x \). Hence (36) is consistent with OAMP in III-D.

Eqns. (35) and (36) give two equivalent forms of OAMP. Eqn. (35) does not involve SVD so it is convenient for implementation. Eqn. (36) is convenient for performance analysis, as seen in Section III. Its implementation involves SVD, which is very costly when the size of \( A \) is large.

F. Optimality of OAMP

The MMSE optimality of OAMP for the system in (1) is analyzed in (17) based on the assumption that \( \phi_t \) and \( \gamma_t \) are respectively linear and nonlinear, and they are separable, Lipschitz continuous and locally optimal. The SE fixed point of OAMP satisfies the following replica equation (56): (37)

\[
v_{\infty}^{-1} = \sigma^{-2} \cdot R_{A^T A} \left( -\sigma^{-2} \cdot \text{mmse}(x|x + \sqrt{\sigma^{-2}} \cdot z) \right),
\]

where \( \sigma^2 \) is the noise variance in (1) and \( z \sim \mathcal{N}(0, 1) \). \( R_{A^T A} \) denotes the R-transform w.r.t. the eigenvalue distribution of \( A^T A \) (49). Following (56), OAMP achieves MMSE when (37) has only one solution.

In a similar way, the replica optimality of VAMP for the system (1) is shown in (19). The result is extended to a system with both \( \phi_t \) and \( \gamma_t \) being nonlinear, separable and locally MMSE-optimal (14), (28). This conclusion also applies to OAMP under the unified orthogonal framework discussed in (4-E). These results are summarized in Proposition 12.
Proposition 12: For the system in Fig. [1]a, OAMP converges to the MMSE provided that (i) both \( \hat{\phi}_t \) and \( \hat{\gamma}_t \) are separable and locally MMSE-optimal, and (ii) the fixed-point equation (37) for \( \hat{\phi}_t \) and \( \hat{\gamma}_t \) has only one solution.

V. COMPARISON WITH RELATED ALGORITHMS

In this section, we compare OAMP-GSO with other related algorithms in [1], [13], [14], [17], [19]. We will show that orthogonality is a common feature underpinning these algorithms. This observation provides useful insights into the turbo/EP/AMP family of iterative signal processing techniques.

A. Connection to EP

When \( \hat{\phi} \) achieves MMSE, \( B \) in (29) can be calculated using Proposition 13, which is proved in [23, Appendix B].

Proposition 13: Assume that \( \hat{x}^{in} = x + f^{in} \) (GS model) and \( \hat{\phi} \) achieves local MMSE. Then

\[
B = v^\phi/v_{2in},
\]

(38)

where \( v^\phi \equiv 1/N_E \mathbb{E}\{\|\hat{\phi}(\hat{x}^{in})-x\|^2\} \) and \( v_{2in} \equiv 1/N_E \mathbb{E}\{\|f^{in}\|^2\} \).

Recall the EP updating rule (13), (14):

\[
\hat{x}^{out} = \hat{\phi}(\hat{x}^{in}) = \frac{1}{1-v^\phi/v_{2in}} \left( \hat{\phi}(\hat{x}^{in}) - v^\phi/v_{2in} \hat{x}^{in} \right).
\]

(39)

Clearly, (39) is equivalent to (27) with \( B \) given in (38). Therefore EP and OAMP are equivalent when local estimators achieve MMSE. Otherwise, they do not. For example, \( [\hat{\phi}(\hat{x}^{in})]_i = (\hat{x}^{in})^2_1 \), where \( \hat{x}^{in} = x + n \) and \( x \) is zero mean and independent of the Gaussian noise \( n \). In this case, following (29) or (32), we have \( B = 0 \), i.e., (38) does not hold. Therefore, OAMP provides a new treatment for sub-optimal local estimators.

After the introduction of OAMP in [17], a derivative form of EP is discussed in [28]. These two algorithms are equivalent.

B. Connection to Conventional Turbo or Belief Propagation

The celebrated turbo principle [5], [7]–[9], which has been regarded as the de facto solution to coded linear systems [10]–[12], is based on the concept of belief propagation (BP), where extrinsic messages are used to avoid the correlation problem of an iterative process. Intuitively speaking, the conventional turbo requires the input and output errors to be independent, while OAMP requires the input and output errors to be orthogonal. Therefore, turbo is a special case of OAMP because independence is a subset of orthogonality. OAMP may potentially outperform turbo for non-Gaussian constraints. For more details, refer to [22, Section III-D].

Since EP is also a special case of OAMP (see Subsection V-A), the results in this paper are applicable to the combined BP-EP message-passing algorithms in [57].

C. Connection to AMP

Early discussions on AMP did not involve orthogonality, but new insight can be gained from the recent results in [23]. In this part, we outline the connection of OAMP and AMP in [23], explicitly emphasizing the orthogonality in AMP.

AMP can be rewritten as the following iteration [1] for solving (1). Starting with \( t = 1 \), \( s_t = s_{t-1}^{\text{Onsager}} = 0 \),

\[
\begin{align}
\text{LE : } \hat{x}^{in}_t &= s_t + A^T(y - As_t) + s_{t-1}^{\text{Onsager}}, \\
\text{NLE : } s_{t+1} &= \hat{\phi}_t(\hat{x}^{in}_t),
\end{align}
\]

(40a)

where \( s_{t-1}^{\text{Onsager}} \) is an “Onsager term” [1] defined as

\[
s_{t-1}^{\text{Onsager}} = B_{t-1} \cdot (\hat{x}^{in}_t - s_{t-1}),
\]

(40c)

\[
\begin{align}
B_{t-1} &= E\left\{\partial [\hat{\phi}_{t-1}(\hat{x}^{in}_{t-1})]/\partial [\hat{x}^{in}_{t-1}]\right\} \\
&\approx 1/N \sum_{i=1}^N \partial [\hat{\phi}_{t-1}(\hat{x}^{in}_{t-1})]/\partial [\hat{x}^{in}_{t-1}].
\end{align}
\]

(40d)

Let \( A = U^*D^*V \) and \( y \equiv U y \), we rewrite (40) as

\[
\begin{align}
S_t &= V s_t, \\
\hat{x}^{out}_t &= \hat{\gamma}_t(S_1, \cdots, S_t), \\
\hat{x}^{out}_t &= V^* \hat{x}^{out}_t, \\
S_{t+1}^{\text{Onsager}} &= V s_{t+1}^{\text{Onsager}}.
\end{align}
\]

(41a)

(41b)

(41c)

(41d)

We now define an intermediate variable as follows [23].

\[
\hat{x}^{out}_{t+1} = \hat{\phi}_t(\hat{x}^{in}_{t+1}) = A_t(\hat{\phi}_t(\hat{x}^{in}_{t-1}) - B_t \hat{x}^{in}_{t}),
\]

(42)

where \( A_t = 1/(1-B_t) \) and \( B_t \) is defined in (40e). Define \( \hat{X}^{out}_t = V \hat{x}^{out}_t \). Furthermore, we can express \( \hat{X}^{out}_t \) by a function of \( \gamma_t(\hat{X}^{in}_1, \cdots, \hat{X}^{in}_t) \). Then, we rewrite (40) (i.e. AMP) into a GIP form with memory (GIP-M) as follows.

GIP-M: Starting with \( t = 1 \) and \( \hat{x}^{out}_1 = 0 \),

\[
\begin{align}
\hat{X}^{out}_t &= V \hat{x}^{out}_t, \\
\hat{X}^{out}_t &= \gamma_t(\hat{X}^{in}_1, \cdots, \hat{X}^{in}_t), \\
\hat{x}^{out}_t &= V^* \hat{X}^{out}_t, \\
\hat{x}^{out}_{t+1} &= \hat{\phi}_t(\hat{x}^{in}_{t+1}).
\end{align}
\]

(43a)

(43b)

The above can be seen as an extension of GIP in [11] by allowing memories in \( \gamma_t \) and \( \phi_t \). Corresponding to Theorems 7 and 10, we have the following properties for GIP-M.

Proposition 14: The following holds for the equivalent form of AMP in (43) if \( A \) is IIDG and \( \phi_t \) is separable-IID.

(i) The error orthogonality in (17) holds.

(ii) All the conclusions of Theorem 7 apply.

Proposition [14] can be verified as follows. It is shown in [23] that \( E\{\partial [\gamma_t(\hat{X}^{in}_t)],/\partial [\hat{X}^{in}_t]\} = 0 \) (see (4) and Theorem 2 in [23]). Furthermore, \( E\{\partial [\hat{\phi}_t(\hat{x}^{in}_t)]/\partial [\hat{x}^{in}_t]\} = 0 \) can be derived strictly from (40e) and (42). Then (i) follows from Proposition 14. The proof of Theorem 7 requires the orthogonality in (17) on the input and output errors of \( \gamma \) and \( \phi \) only, which does not involve their internal structures with or without memory. Hence (ii) holds.

The above shows some interesting connections between AMP and OAMP as follows.

- There are different ways to establish orthogonality such as using GSO or Onsager term.
- Orthogonality provides useful insights into the mechanism of the AMP-family of algorithms, based on which many findings can be derived more concisely. This can
be seen by comparing the proof of SE for AMP in [18] and that of OAMP in this paper. (Note: Proposition 14 together with Theorems 7 and 10 in this paper form an alternative proof for the SE for AMP.) Despite that they are both underpinned by orthogonality, AMP is not equivalent to OAMP without memory. In particular, Proposition 14 holds for AMP only when the sensing matrix \(A\) in [1] has IIDG entries [1]. Otherwise, the performance of AMP may degrade noticeably. OAMP is not subjected to these restrictions.

D. Connection to OAMP in the Differentiation Form

OAMP in the differentiation form (OAMP-DF) is introduced in [17]. Its equivalence to OAMP-GSO is clear from the discussions in V-E. OAMP-DF can be implemented using (35) without knowing the distribution of \(x\). This has an advantage in many signal processing problems where the distributions of \(x\) are not known a priori. (In communication problems, the distributions of \(x\) are usually determined by modulation methods and can be assumed to be known.)

E. Connection to VAMP

The derivative version of OAMP was originally derived in [17] in the form of (35) and (36). Its equivalence to VAMP can be seen by comparing (36) with Algorithm 2 in [19]. The latter is a special case of (35) using \(C_t = (1 - E\{\phi_t^2\})^{-1}\) and an MMSE form of (35) with \(W_t = v_x A^T (v_x A A^T + \sigma^2 I)^{-1}\), where \(v_x = \frac{1}{N} E[||s_t - x||^2]\).

The SE for OAMP is conjectured based on the Haar distribution of \(V\) in [17]. (See Definition 1 and Proposition 1 in [17].) The key contribution in [19] is an elegant proof of SE for VAMP based on the same assumption on \(V\). The proof of SE for OAMP in this paper is inspired by the work in [19]. There is a difference though. The proof in this paper relies solely on the orthogonality of local estimators. It is easier to extend the proof in this paper to a broader class of applications, which will be discussed elsewhere [27].

F. Summary

We summarize the characteristics of the existing EP, turbo, AMP, OAMP, and VAMP below.

- EP and OAMP are equivalent when local estimators are optimal. They are different otherwise.
- Turbo is a special case of OAMP since orthogonality includes independence.
- VAMP is equivalent to the derivative form of OAMP originally proposed in [17].
- AMP is not equivalent to the standard form of OAMP without memory. However, AMP falls in the general class of OAMP with memory.

Finally, the behavior of all these algorithms can be analyzed using the orthogonal principle. In general, the performance of an algorithm deteriorates if such orthogonality no longer holds. This is the case, e.g., for AMP when the sensing matrix is not IID, or for EP when the estimators are not locally optimal.

VI. Optimization via GS Model and GSO

A. Motivation

This section provides an example to demonstrate an advantage of the GS model and GSO. It results from the freedom in tuning \(\gamma\) and \(\phi\) in [17] under the orthogonal principle in [17]. Specifically, we consider the standard linear model in [1]. Assume that \(A\) in [1] is not IID, such as in the applications in [24, 25, 38, 39]. Then (1) can be solved by EP, OAMP, and VAMP, but not AMP. The standard EP/OAMP/VAMP solution is given in (35). Below, we modify (35a) for performance optimization facilitated by the GS model and GSO. This is useful when sub-optimal \(\phi\) or \(\gamma\) or both are used due to complexity concerns [1]. Significant performance improvement over standard EP/OAMP/VAMP is demonstrated.

B. Tuning OAMP via GS Model and GSO

Replace (35a) by the following linear operation:

\[
\gamma_t(s_t) = C_t \hat{\gamma}_t(s_t) - B_t s_t,
\]

where \(\hat{\gamma}_t(s_t) = W_t y + (I - W_t A) s_t\), the GSO coefficient \(B_t\) is defined in (27) and is given by \(B_t = \frac{1}{N} \text{Tr}(\xi(I - W_t A))\) (following (29)), \(C_t\) is a normalization coefficient given by \(C_t = \frac{1}{\text{Tr}(W_t A^T)}\), and \(\xi\) is a variable for optimization. Then, (44) can be rewritten as

\[
r_t = \gamma_t(s_t) = \xi s_t + \frac{N}{\text{Tr}(W_t A^T)} W_t (y - \xi A s_t).
\]

We can see that (35a) is a special case of (45) when \(\xi = 1\). More generally, the MSE of \(r_t\) (i.e., \(v_r\)) is a concave function of \(\xi\). Hence, the optimal \(\xi\) that minimizes \(v_r\) can be obtained by \(d v_r(\xi)/d\xi = 0\).

Assume that \(s_t\) is characterized by the GS model: \(s_t = \phi_t x + f_t\) with \(f_t\) the variance of \(f_t\). The optimal \(\xi\) is

\[
\xi = \alpha_t/(\alpha_t^2 + v_{f_t}),
\]

and the corresponding MSE is given by

\[
v_r = \frac{1}{N} \text{Tr}(B_t B_t^T) \frac{v_{f_t}}{\sigma^2} + \frac{1}{N} \text{Tr}(W_t W_t^T) \sigma^2,
\]

where \(B_t = I - W_t A\) and \(W_t = \frac{N}{\text{Tr}(W_t A^T)} W_t\). Some interesting observations are as follows.

(i) The \(v_r\) in (46b) is not larger than the MSE of (35a) (see (32a) in [17]) in the standard EP/OAMP/VAMP.
(ii) The complexity of the optimized OAMP in (44) is the same as the standard EP/OAMP/VAMP in (35).
(iii) For MMSE \(\phi\), we can show that (35a) is equivalent to (44).

We omit the proof as it is quite straightforward.

Since MSE \(\phi\) may be hard to obtain or the computational complexity is practically prohibitive, sub-optimal solutions are commonly employed. The optimized LE in (44) for OAMP is useful in this case.

C. Numerical Example

Let us consider a special case of (1) for the following compressed sensing problem [53]. Let the SVD of \(A\) be \(A = U^H D V\). The eigenvalues \(\{d_i\}\), i.e. diagonal elements...
of $D_i$ are generated as: $d_{i+1}/d_i = \kappa^{1/M}$ for $i = 1, \ldots, M - 1$ and $\sum_{i=1}^{M} d_i^2 = N$ \cite{59}. Here, $\kappa$ is the condition number of $A$. The entries of $x$ are IID and follow the Bernoulli-Gaussian (i.e., sparse) distribution, i.e. for all $i$,

$$x_i \sim \begin{cases} 0, & \text{probability } 1 - \lambda, \\ CA(0, \lambda^{-1}), & \text{probability } \lambda. \end{cases}$$  

(47)

The variance of $x_i$ is normalized to 1. The applications of this example can be seen \cite{60}.

Consider employing the prototypes in \cite{36} for the above problem. To reduce complexity, assume that $\hat{\phi}_i$ is constructed using a low-cost thresholding estimator \cite{61}:

$$\hat{\phi}_i(r_i) = \max(\|r_i\| - \vartheta_i, 0) \cdot \text{sign}(r_i),$$  

(48)

where $\vartheta_i$ is a threshold.

Assuming that $\phi$ is obtained by plugging \cite{48} into \cite{36} with $C_i = 1/(1 - E(\hat{\phi}_i))$ (commonly used in EP, OAMP and VAMP), we consider two LMMSE-LEs \cite{35a} and \cite{44} (with $W_t = v_{a_n} A^T (v_{a_n} A A^T + \sigma^2 I)^{-1}$) to get $\gamma_t$ for the standard EP/OAMP/VAMP and optimized OAMP, respectively. Note that $v_{a_n} = N^{-1/2} \|s_t - x\|^2$ is used in the standard EP/OAMP/VAMP, while in the optimized OAMP, $V_{a_n}$ is set as $N^{-1/2} \|s_t - x\|^2$. Fig. 2 shows that the optimized LE via the GS model and GSO can bring significant performance improvement over the standard EP/OAMP/VAMP in this example. Additionally, there is a good agreement between simulation results for standard/optimized OAMPs and their SEs.

VII. CONCLUSION

This paper studied the impact of orthogonality in OAMP and other AMP-type algorithms. Specifically, orthogonality suppresses the correlated error component in the iterative process and hence ensures the correctness of SE. This provides useful insights into the mechanism of OAMP. We also developed a GSO procedure to establish orthogonality. GSO offers a simple and versatile realization technique for OAMP. Various turbo/EP/AMP-type algorithms can be transformed into equivalent forms with orthogonal local estimators. Hence they can be unified under a common orthogonal framework.

A householder dice method has recently been proposed for effectively simulating the dynamics on Haar random matrices \cite{62}. It will be fascinating to see how the householder dice technique may be used to demonstrate the SE of OAMP.

APPENDIX A

PROPERTIES/CONJECTURES OF IID/IIDG AND PIID/PIIDG VARIABLES

A. LLN Orthogonality

Let $a = \{a_n\}$ and $b = \{b_n\}$ be two sequences of length $N$. Assume that both $a$ and $b$ are IID. Then $\{a_n b_n\}$ are also IID and so we have

$$E(\|a\|^2) = N E(a_n^2),$$  

(49a)

$$E(\|b\|^2) = N E(b_n^2),$$  

(49b)

$$\text{Var}(\frac{1}{N} a^T b) = \text{Var}(a_n b_n)/N.$$  

(49c)

The following is related to the convergence of $a^T b$ when $N \to \infty$, which follows the law of large numbers \cite{63}.

**Lemma 15**: Let $E(\|a\|^2) \neq 0, E(\|b\|^2) \neq 0$ and both $a$ and $b$ be IID with $E\{a_n b_n\} = 0$ and $\text{Var}(a_n b_n) \to \infty$. Then $\frac{1}{N} a^T b \xrightarrow{LLN} 0$.

**Proof**: From Chebyshev’s inequality \cite{63}, we have

$$\text{Pr}\left(\left|\frac{1}{N} a^T b\right| < \frac{\text{Var}(a_n b_n)}{\delta}\right) \geq 1 - \delta.$$  

(50)

Using \cite{49}, we rewrite \cite{50} as

$$\text{Pr}\left(\left|\frac{1}{N} a^T b\right| < \frac{\text{Var}(a_n b_n)}{\delta}\right) \geq 1 - \delta.$$  

(51)

Then \cite{3} holds provided that

$$N \geq N' \equiv \frac{\text{Var}(a_n b_n)}{\text{E}(\|a_n^2\| \cdot \text{E}(\|b_n^2\|) \cdot \delta \cdot \varepsilon).}$$  

(52)

This completes the proof.

The following lemma follows Lemma \cite{15}.

**Lemma 16 (LLN Orthogonality of Separable-IID Function)**: Assume that $\pi = \{\pi_n\}$ is separable-IID, and $a = \{a_n\}$ IID, both of length $N \to \infty$. Then (i) $\{a_n \pi_n(a_n)\}$ are IID and (ii) $\frac{1}{\sqrt{N}} a^T \pi(a) \xrightarrow{LLN} 0$ if $E[a_n \pi(a_n)] = 0$.

Similar to Lemma \cite{15} we can verify Property \cite{17} for PIID variables using the concept of group IID variables. The details are omitted due to space limitations.

**Property 17**: Let both $a$ and $b$ be PIID with $E\{a_n b_n\} = 0$ and $\text{Var}(a_n b_n) \to \infty$. Then $\frac{1}{N} a^T b \xrightarrow{LLN} 0$.

**Property 18**: Assume that $\pi = \{\pi_n\}$ is separable-IID, and $a = \{a_n\}$ PIID, both of length $N \to \infty$. Then (i) $\frac{1}{\sqrt{N}} a^T \pi(a) \xrightarrow{LLN} 0$ if $E[a_n \pi(a_n)] = 0$.

B. Inner Product of Separable-IID Functions

**Property 19**: Let $\pi_1$ and $\pi_2$ be separable-IID, $\xi = \{\xi_1n\}$ and $\xi = \{\xi_2n\}$ be IID variables. It is easy to verify the following.

$$\frac{1}{\sqrt{N}} a^T \xi_1 \pi_1(\xi_1) \pi_2(\xi_2) = \frac{1}{\sqrt{N}} a^T \xi_1(\xi_1) \pi_2(\xi_2) \pi_1(\xi_1) \pi_2(\xi_2).$$  

(53)

where $r$ denotes the potential IID variables except $\xi_1$ and $\xi_2$ in function $\pi_1$ or $\pi_2$. 

Fig. 2. MSE comparison between the standard LMMSE-EP/OAMP/VAMP and the optimized LMMSE-OAMP, where $N = 1500, M/N = 0.65, \kappa = 10, \lambda = 0.25$, $\vartheta_i = v_{r_1}$ and SNR = $\sigma_n^2 = 45$ dB.

![Fig. 2. MSE comparison between the standard LMMSE-EP/OAMP/VAMP and the optimized LMMSE-OAMP, where $N = 1500, M/N = 0.65, \kappa = 10, \lambda = 0.25$, $\vartheta_i = v_{r_1}$ and SNR = $\sigma_n^2 = 45$ dB.](image-url)
The conjecture below extends Property [19] to the PIIDG case.

**Conjecture 20:** Let \( \pi_1 \) and \( \pi_2 \) be separable-IID, \( \xi = \{\xi_{1n}\} \) and \( \xi = \{\xi_{2n}\} \) be PIID variables. Then,
\[
\frac{1}{N} \mathbb{E}_{r, \xi_1, \xi_2} \left\{ \pi_1(\xi_1)^T \pi_2(\xi_2) \right\} = \frac{1}{N} \mathbb{E}_{r, \xi_{1n}, \xi_{2n}} \left\{ \pi_1(\xi_{1n})\pi_2(\xi_{2n}) \right\},
\]
where \( r \) denotes the potential random variables except for \( \xi_1 \) and \( \xi_2 \) in function \( \pi_1 \) or \( \pi_2 \).

From Conjecture 20 in evaluating \( \frac{1}{N} \mathbb{E}_{r, x, \xi} \left\{ \pi(a x + \xi)^T x \right\} \) and \( \frac{1}{N} \mathbb{E}_{r, x, \xi} \left\{ \pi(a x^T + \xi)^T x \right\} \), we can replace the PIIDG \( \xi \) by an IIDG vector \( \sqrt{\text{Var}(\xi)} \mathbf{z} \), where \( z \sim N(0, I) \). This is useful in deriving the state evolution method for OAMP.

C. Incomplete Orthogonal Transform

**Property 21:** Let \( a = Ub \) where (i) \( U \) is orthogonal, (ii) \( b \) IIDG, and (iii) \( U \) and \( b \) are mutually independent. Then (i) \( a \) is IIDG, and (ii) \( a \) and \( b \) are mutually independent.

**Property 22:** Let \( U_1 \) be an \( N \times (N - m) \) block in \( U \in \mathcal{U}^N \), \( b_1 \in \mathbb{R}^{N - m} \) be IIDG, and \( U \) and \( b_1 \) mutually independent. Then the following claims hold when \( m/N \to 0 \):
(i) \( U_1 b_1 \rightarrow \text{IIDG} \); (ii) \( E\|U_1 b_1\|^2 \rightarrow E\|b_1\|^2 \); and (iii) \( U_1 b_1 \) is asymptotically independent of \( U_1 \).

**Proof:** Consider partitions \( U = [U_1, U_2] \) and \( b = \begin{bmatrix} b_1 \end{bmatrix} \) such that \( Ub = U_1 b_1 + U_2 b_2 \), where \( U_1 \) has \( N - m \) columns and \( b \) is IIDG. Then
\[
\begin{align*}
E_1 \{\|U_1 b_1\|^2\} &= \frac{N-m}{N} \cdot E\|b_1\|^2, \\
E_1 \{\|U_2 b_2\|^2\} &= \frac{m}{N} \cdot E\|b_2\|^2.
\end{align*}
\]
When \( m \ll N \), (55) leads to
\[
\frac{E\{\|U_2 b_2\|^2\}}{E\{\|U_1 b_1\|^2\}} = \frac{m}{N-m} \to 0,
\]
where \( U_1 b_1 \xrightarrow{a.s.} Ub \),
\[
E\{\|U_1 b_1\|^2\} \to E\{\|b_1\|^2\}.
\]
When \( b \) is IIDG, both \( Ub \) and \( U_1 b_1 \) are IIDG and independent of \( U \) and \( U_1 \). Therefore, we obtain Property 22.

Intuitively, Property 22 hinges on the factors that \( \|b_2\| \) and \( \|U_2 b_2\| \) are statistically negligible compared with \( \|b_1\| \) and \( \|U_1 b_1\| \) when \( m \ll N \).

The conjectures below extend the above two properties to the PIIDG case.

**Conjecture 23:** Let \( a = Ub \) where \( U \) is orthogonal, \( b \) PIIDG, and \( U \) and \( b \) are mutually independent. Then \( a \) is PIIDG, and \( a \) and \( U \) are mutually independent.

**Conjecture 24:** Let \( U_1 \) be an \( N \times (N - m) \) block in \( U \in \mathcal{U}^N \), \( b_1 \in \mathbb{R}^{N - m} \) be PIIDG, and \( U \) and \( b_1 \) mutually independent. Then the following claims hold when \( m/N \to 0 \):
(i) \( U_1 b_1 \rightarrow \text{PIIDG} \); (ii) \( E\|U_1 b_1\|^2 \rightarrow E\|b_1\|^2 \); and (iii) \( U_1 b_1 \) is asymptotically independent of \( U_1 \).

D. Extended Stein’s Lemma

The following is a variation of the generalized Stein’s Lemma [64]. It will be useful in the proof of Theorem 10.

We say that \( a \) and \( u \) are entry-wise jointly Gaussian if every \( \{u_i, a_i\} \) pair are jointly Gaussian.

**Lemma 25 (Extended Stein’s Lemma):** Let \( \pi = \pi(a) \) where \( a \) is IIDG and \( \pi(a) \) is separable-IID. Let \( u \) be any IIDG vector that is entry-wise jointly Gaussian with \( a \). Let \( E\{a\} = E\{u\} = 0 \) and \( E\{a^T \pi\} = 0 \). Then (i) \( \frac{1}{N} a^T \pi \xrightarrow{\text{LLN}} 0 \); and (ii) \( \frac{m}{N} u^T \pi \xrightarrow{\text{LLN}} 0 \).

**Proof:** Since \( a \) is IIDG and \( \pi(a) \) is separable-IID, \( \pi \) is IID (but not necessarily Gaussian). Then claim (i) follows Lemma 15 directly. For claim (ii), denote \( \xi = u - a a \),
\[
\xi = u - a a,
\]
where \( a = E\{u^T a\}/E\{a^T a\} \) and so \( E\{a^T \pi\} = 0 \). (This can be compared to (6) for the GS model.) Then \( \xi \) and \( a \) are uncorrelated and so mutually independent due to their Gaussianity. (When \( u \) and \( a \) are both IIDG, so is \( \xi \).) Then \( \xi \) is independent of \( \pi \) due to the Markov chain \( \xi \to a \to \pi \). Therefore, \( E\{\xi^T \pi\} = 0 \). Furthermore, from (57) and \( E\{a^T \pi\} = 0 \), we have \( E\{u^T \pi\} = 0 \). It can be seen that \( \{u_{1n}, \pi_{1n}\} \) are IID. Then from Lemma 15 \( \frac{m}{N} u^T \pi \xrightarrow{\text{LLN}} 0 \). This completes the proof.

The conjecture below extends Lemma 25 to the PIIDG case.

**Conjecture 26:** Let \( \pi = \pi(a) \) where \( a \) is PIIDG and \( \pi(a) \) is separable-IID. Let \( u \) be any PIIDG vector that is entry-wise jointly Gaussian with \( a \). Let \( E\{a\} = E\{u\} = 0 \). Assume \( E\{a^T \pi\} = 0 \). Then (i) \( \frac{1}{N} a^T \pi \xrightarrow{\text{LLN}} 0 \); and (ii) \( \frac{m}{N} u^T \pi \xrightarrow{\text{LLN}} 0 \).

**APPENDIX B**

**PROOF OF THEOREM 7**

The asymptotically IID Gaussian (AIIDG) property of OAMP was conjectured in [17] and rigorously proved in [16], [23]. A similar proof is given in [19] for VAMP, which is also applicable to OAMP due to the algorithmic equivalence between the two. The discussions below are inspired by these background works. In particular, Proposition 28 and Lemma 29 are inspired by [19, Lemmas 3 and 5], [16, Lemmas 1 and 3] and [23, Lemmas 1 and 3]. Also, the use of (64) follows the Bolthausen’s conditioning technique used in [16, 19, 23].

We will focus on half of an iteration in the symmetric model in Fig. 1 which is much simpler than tracking a full iteration as in [16, 19, 23]. This, together with several conjectures (i.e., Conjectures 22, 24 and 26) which can be bridged by the rigorous treatments in [16, 19, 23], results in conciseness in derivation. Our aim is to provide a clear insight into the mechanism of OAMP. Combining the discussions in Appendices A and B we show that orthogonality suppresses the correlated error component in the iterative process, and hence ensures the correctness of SE.

A. Haar Distribution under a Linear Constraint

Fix \( A \in \mathbb{R}^{N \times m} \) and \( B \in \mathbb{R}^{N \times m} \). We first assume that \( A \) and \( B \) are fixed. Let \( V \) be a size \( N \times N \) Haar distributed matrix. Denote by \( U_{A \rightarrow VB} \) the subset of \( V \) in the Haar ensemble meeting the following linear constraint \( A = VB \). The following lemma is straightforward.


Lemma 27: Let $V$ be Haar distributed independent of $A$ and $B$. The elements in $U_{A-VB}$ are uniformly distributed.

Let $U_A = [U_A^\parallel, U_A^\perp]$ and $U_B = [U_B^\parallel, U_B^\perp]$ be two orthogonal matrices with $U_A^\parallel \in \text{Sp}(A)$, $U_A^\perp \in \text{Sp}(A)^\perp$, $U_B^\parallel \in \text{Sp}(B)$ and $U_B^\perp \in \text{Sp}(B)^\perp$. Let

$$U_A = V U_B.$$  \hspace{1cm} (58)

From $A = VB$ and (58), $(U_A^\parallel)^T V U_B^\parallel = I$, $(U_A^\perp)^T V U_B^\perp = 0$, so

$$U_A^\parallel V U_B^\parallel = \left[ \begin{array}{c} (U_A^\parallel)^T \\ (U_A^\perp)^T \end{array} \right] V \left[ \begin{array}{c} U_B^\parallel \\ U_B^\perp \end{array} \right] = \left[ \begin{array}{c} I \\ 0 \\ V \end{array} \right], \hspace{1cm} (59a)

where

$$\tilde{V} = (U_A^\parallel)^T V U_B^\parallel.$$  \hspace{1cm} (59b)

Left- and right multiplying (59a) by $U_A$ and $U_B^\top$ gives

$$V = \left[ \begin{array}{c} U_A^\parallel \\ U_A^\perp \end{array} \right] \left[ \begin{array}{c} I \\ 0 \\ V \end{array} \right] \left[ \begin{array}{c} (U_B^\parallel)^T \\ (U_B^\perp)^T \end{array} \right] = U_A^\parallel (U_B^\parallel)^T + U_A^\perp \tilde{V} (U_B^\perp)^T.$$  \hspace{1cm} (60a)

From (59b), if $V$ is orthogonal, so is $\tilde{V}$ and vice versa. Thus (59b) and (60a) establish a one-to-one affine mapping between $V \in U_{A-VB}$ and $\tilde{V} \in U_{N-m}$. Denote as $V \leftrightarrow \tilde{V}$. Fix a pair $V \leftrightarrow \tilde{V}$. For any $\Omega$ such that $V \Omega \in U_{A-VB}$, we can always find a $\Omega$ such that $\tilde{V} \Omega \in U_{N-m}$, and vice versa. From Lemma 27, $V$ is uniformly distributed over $U_{A-VB}$. Then for $\Omega$ and $\tilde{\Omega}$ mentioned above, $p_V(V) = p_{\tilde{V}}(\tilde{V})\Omega$ and, due to the one-to-one affine mapping,

$$p_V(V) = p_{\tilde{V}}(\tilde{V}).\hspace{1cm} (61)$$

Eqn. (61) holds $\forall \Omega \in U_{N-m}$, so $\tilde{V}$ is Haar. This is summarized as follows.

Proposition 28: Let $A$ and $B$ be two $N \times m$ matrices of full column rank. Let $U_A^\parallel$ and $U_B^\parallel$ be respectively fixed orthonormal bases of $\text{Sp}(A)$ and $\text{Sp}(B)$. Assume that $V$ is uniformly distributed over $U_{A-VB}$. Then $V \equiv (U_A^\parallel)^T V U_B^\parallel$ is Haar over $U_{N-m}$.

There is no loss of generality in the full-rank assumption in Proposition 28. Let the ranks of $A_{N \times m}$ and $B_{N \times m}$ be $m'$. When $m' < m$ (not full rank), we can obtain $A' = VB'$ with full-rank $A'$ and $B'$ by removing the linear correlated columns in $A$ and $B$. Proposition 28 still applies to $A' = VB'$.

B. Random Part of a Constraint Haar Transform

Let $V \sim \mathcal{H}^N$, $f \in \mathbb{R}^N$ and

$$g = V f.$$  \hspace{1cm} (62)

From (60), we write $g = g_{\text{fixed}} + g_{\text{random}}$ under $A = VB$ with

$$g_{\text{fixed}} = U_A^\parallel (U_B^\parallel)^T f,$$  \hspace{1cm} (63a)

Recall from Proposition 28 that $V \sim \mathcal{H}^{N-m}$. Hence $g_{\text{random}}$ is the random part of $g$. Let $N \to \infty$ and $m$ remain fixed. Then from Lemma 4, $V (U_B^\parallel)^T f$ is PIIDG with zero mean. Then from Conjecture 24 (see Appendix A-C), and (63a) and (63b) (since $U_A^\parallel$ is of size $N \times (N-m)$), $g_{\text{random}}$ is PIIDG with zero mean.

The above is for fixed $A$ and $B$. We now consider random $A$ and $B$. In this case, we define $U_{A-VB}$ using samples of $A$ and $B$. From Proposition 28, $V$ is not a function of $A$ and $B$ except the size of $V$. Let $N \to \infty$ and $m$ remain finite. Then from Property 21, $g_{\text{random}}$ in (63) is independent of $B$. From Conjecture 24 (see Appendix A-C), $g_{\text{random}}$ in (63) is asymptotically independent of the columns of $U_A^\parallel$ and so also independent of the columns of $A$. Hence $g_{\text{random}}$ is asymptotically independent of both $A$ and $B$. Furthermore, due to the one-to-one mapping between $V \in U_{A-VB}$ and $\tilde{V}$ (see the discussions below (60)), any $z$ independent of $V$ is also independent of $\tilde{V}$ and $g_{\text{random}}$. We summarize the above in a lemma below.

Lemma 29: Let $V$ be Haar distributed under the constraint $A = VB$. Let $N \to \infty$ and $m$ remains finite. Then $g_{\text{random}}$ is PIIDG with zero mean and asymptotically independent of $A$ and $B$. Furthermore, $g_{\text{random}}$ is asymptotically independent of any $z$ if $z$ is independent of $V$.

Lemma 29 is a generalization of Lemma 4. The latter is for $A$ and $B$ being empty sets, so $g_{\text{fixed}}$ vanishes and $V f = g_{\text{random}}$. Note the subtle difference in the treatments of $A$, $B$ and $z$ in Lemma 29. We cannot claim both $A$ and $B$ are independent of $V$ since $A = VB$. For any finite $N$, $g_{\text{random}}$ is a function of $A$ and $B$, but the related influence becomes negligible when $N \to \infty$.

C. Proof of Theorem 7

The equations below are based on (11), (13), (15c) and (16):

$$A_t = [\mathcal{X}, G_t^{\text{out}}, G_t^{\text{in}}], \hspace{1cm} B_t = [x, f_t^{\text{in}}, f_t^{\text{out}}],$$  \hspace{1cm} (64a)

$$A_t = VB_t,$$  \hspace{1cm} (64b)

$$g_t^{\text{in}} = V f_t^{\text{out}}, \hspace{1cm} f_t^{\text{in}} = V g_t^{\text{out}}.$$  \hspace{1cm} (64c)

Combining (60), (64a) and (64b), we have

$$V = [u_{\mathcal{X}}, U_{G_t^{\text{out}}}] [u_x, U_{F_t^{\text{in}}}]^T + U_{G_t^{\text{in}}} (U_{F_t^{\text{out}}})^T$$

$$+ U_{A_t} \tilde{V} (U_{B_t}^\parallel)^T.$$  \hspace{1cm} (65)

Substitute (65) into (64c):

$$g_t^{\text{in}} = V f_t^{\text{out}}$$

$$= [u_{\mathcal{X}}, U_{G_t^{\text{out}}}] [u_x, U_{F_t^{\text{in}}}]^T f_t^{\text{out}} + U_{G_t^{\text{in}}} (U_{F_t^{\text{out}}})^T f_t^{\text{out}}$$

$$+ U_{A_t} \tilde{V} (U_{B_t}^\parallel)^T f_t^{\text{out}}.$$  \hspace{1cm} (66b)

For OAMP under (17), $\frac{1}{N} [x, F_t^{\text{in}}] f_t^{\text{out}} \overset{LLN}{\to} 0$. Hence in (66)

$$\delta_{t_1} = o(1)$$

and

$$\delta_{t_2} = \delta_{t_2} + \delta_{t_2}^{\text{random}}.$$  \hspace{1cm} (67)

Since $U_{G_t^{\text{in}}}^{\parallel} \in \text{Sp}(G_t^{\text{in}})$, we can find a $c_t$ such that

$$\delta_{t_2} = G_t^{\text{in}} c_t.$$  \hspace{1cm} (68)

From Lemma 29, $\delta_{t_2}^{\text{random}}$ is PIIDG and independent of $A_t, B_t$ and $z$. (Note: $\mathcal{X}$ is a column in $A_t$). Following (67) and (68), the GS error $g_t^{\text{in}}$ is a linear combination of independent PIIDG vectors $\{\delta_t^{\text{random}}, \ldots, \delta_t^{\text{random}}\}$, which are zero mean and independent of $\mathcal{X}$ and $z$. Then, we have Lemma 30.
Lemma 30: We can find an upper triangular $C_t$ such that
\[ G_t^{in} \rightarrow [\delta_0^{\text{random}} \ldots \delta_{t-1}^{\text{random}}] C_t, \]  
(69)
where $\{\delta_0^{\text{random}}, \ldots, \delta_{t-1}^{\text{random}}\}$ are independent PIIDG vectors with zero mean and are independent of $X$ and $z$.

Let $a$ and $b$ be two PIIDG vectors and $c = a + b$ where $\alpha$ and $\beta$ are two constants. It can be shown that $c$ is PIIDG. Then following Lemma 30 we obtain the claim (a) in Theorem 7. Due to the symmetry of the problem, we can prove claim (b) in a similar way. This completes the proof for Theorem 7.

D. Discussions

Here are some intuitions. Recall from (14) and (11) that
\[ g_t = \gamma_t (\alpha_t \gamma_t + g_t^{in}), \]  
(70)
where $g_t^{out}$ and $g_t^{in}$ are errors. Heuristically, in an iterative process, we should avoid correlation between $g_t^{in}$ and $X$, otherwise the design of $\gamma_t$ becomes very complicated. Also, we should avoid correlation between $g_t^{in}$ and $G_t^{out}$. Such correlation may result in positive feedback in an iterative process and cause stability problem.

OAMP achieves such desirable correlation avoidance. To see this, rewrite (66) as
\[ g_t^{in} = \left[ u_x^U, U_{g_t^{out}}, \|u_x^U\| \right]^T f_t^{out} + \delta_{t2} + \delta_t^{\text{random}}. \]  
(71)
Clearly, $\delta_{t2}$ is potentially correlated with $X$ and $G_t^{out}$ since $\delta_{t2} \in \text{Sp}([X, G_t^{out}])$. In OAMP, $\delta_{t2}$ is statistically suppressed by a properly designed $\phi_t$, such that $1/N \|u_x^U\|^T f_t^{out} \xrightarrow{\text{LLN}} 0$ as shown below (66). Furthermore, we showed that $\delta_{t2}$ and $\delta_t^{\text{random}}$ will not cause the correlation problem above.

In summary, the orthogonality of $\phi_t$ prevents the output of $\gamma_t$ from circulating back (partially or fully) to the input of $\gamma_{t+1}$. The similar can be said for $\phi_t$. Being orthogonal, the two local estimators help each other; each stops the input-output error circulation for the opposite one.

APPENDIX C  
PROOF OF THEOREM 10

We prove Theorem 10 by induction.

1) Eqn. (17) holds for $t = 1$: Since $\gamma_1$ and $\phi_1$ are orthogonal estimators, we have (17a) for $t = 1$. From the definition of the GS model in (7), we have
\[ E[X^T g_t^{out}] = 0, \quad E[x^T f_t^{out}] = 0. \]  
(72)
Furthermore, $f_t^{out} = \phi_1(0) - x$ is IID since $\phi_1$ is separable-IID and $x$ is IID. Then, from (72) and Lemma 15, we have
\[ \frac{1}{N} x^T f_t^{out} \xrightarrow{\text{LLN}} 0. \]  
(73)
In addition, $X$ and $g_t^{in}$ are independent PIIDG vectors. Then $g_t^{out} = \gamma_1(0) - X$ is PIID since $\gamma_1$ is separable-IID. Then, from (72) and Property 17, we have
\[ \frac{1}{N} X^T g_t^{out} \xrightarrow{\text{LLN}} 0. \]  
(74)
Therefore, (17b) holds for $t = 1$.

2) Eqn. (17) holds for $t - 1$ $\Rightarrow$ Eqn. (17) holds for $t$: Since (17) holds for $t - 1$, from Theorem 7 (see also the proof in Appendix B), we have the following:
(a) $[g_t^{in}]_{t=1}^{\infty}$ is CPIIDG-RJG and independent of $X$;
(b) $f_t^{in}]_{t=1}^{\infty}$ is CPIIDG-RJG and independent of $x$.

Since $\gamma_t$ and $\phi_t$ are orthogonal estimators, we have
\[ E[(g_t^{in})^T g_t^{out}] = 0, \quad E[(f_t^{in})^T f_t^{out}] = 0. \]  
(75)
From (a) and (b), $g_t^{in}$ and $f_t^{out}$ are respectively PIIDG. Since $\gamma_t$ and $\phi_t$ are separable-IID, $g_t^{out} = \phi_t(g_t^{in}) - x$ and $f_t^{out} = \gamma_t(f_t^{in}) - X$ are PIID. Therefore, from (75) and Property 17 we have
\[ \frac{1}{N} (g_t^{in})^T g_t^{out} \xrightarrow{\text{LLN}} 0, \quad \frac{1}{N} (f_t^{in})^T f_t^{out} \xrightarrow{\text{LLN}} 0. \]  
(76)
Then, from Conjecture 20 and the joint Gaussianity in (a) and (b), the following hold for all $t' < t - 1$:
\[ \frac{1}{N} (g_{t'}^{in})^T g_{t'}^{out} \xrightarrow{\text{LLN}} 0, \quad \frac{1}{N} (f_{t'}^{in})^T f_{t'}^{out} \xrightarrow{\text{LLN}} 0. \]  
(77)
Therefore, (17a) holds for $t$.

From the definition of the GS model in (7), we have
\[ E[X^T g_t^{out}] = 0, \quad E[x^T f_t^{out}] = 0. \]  
(78)
Since $x$, $X$, $f_t^{out}$ and $g_t^{out}$ are PIID, from Property 17 and (78), we have
\[ \frac{1}{N} x^T f_t^{out} \xrightarrow{\text{LLN}} 0, \quad \frac{1}{N} X^T g_t^{out} \xrightarrow{\text{LLN}} 0. \]  
(79)
Therefore, (17b) holds for $t$.

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