Integrable structures of specialized hypergeometric tau functions

By
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Abstract

Okounkov’s generating function of the double Hurwitz numbers of the Riemann sphere is a hypergeometric tau function of the 2D Toda hierarchy in the sense of Orlov and Scherbin. This tau function turns into a tau function of the lattice KP hierarchy by specializing one of the two sets of time variables to constants. When these constants are particular values, the specialized tau functions become solutions of various reductions of the lattice KP hierarchy, such as the lattice Gelfand-Dickey hierarchy, the Bogoyavlnsky-Itoh-Narita lattice and the Ablowitz-Ladik hierarchy. These reductions contain previously unknown integrable hierarchies as well.

§ 1. Introduction

The notion of hypergeometric tau functions was coined by Orlov and Scherbin in an attempt to capture multi-variate hypergeometric series in the language of the KP and 2D Toda hierarchies [1, 2]. Other sources of this type of special tau functions were the $c = 1$ string theory [3, 4, 5] and the Hurwitz numbers of the Riemann sphere [6]. Since the work of Orlov and Scherbin, many applications of hypergeometric tau functions have been found in random matrix theory, topological string theory, combinatorics and enumerative geometry (see the references cited in our recent review [7]).

In this paper, we search for integrable structures that underlie a family of hypergeometric tau functions of the lattice KP hierarchy (aka the modified KP hierarchy) [8, 9]. These tau functions are obtained from a special hypergeometric tau function $\mathcal{T}(s, t, \bar{t})$.
of the 2D Toda hierarchy by freezing the second set of time variables \( \tilde{t} = (\tilde{t}_k)_{k=1}^\infty \) (the so-called “negative” time variables) to a set of constants \( c = (c_k)_{k=1}^\infty \). \( \mathcal{T}(s, \tilde{t}, \tilde{c}) \) is the tau function constructed by Okounkov as a generating function of the double Hurwitz numbers of the Riemann sphere [6]. As far as \( c_k \)'s are generic values, \( \mathcal{T}(s, \tilde{t}, \tilde{c}) \) is merely a special tau function of the lattice KP hierarchy. If, however, \( c_k \)'s are particular values, \( \mathcal{T}(s, \tilde{t}, \tilde{c}) \) can be a solution of a reduction (i.e., a subsystem) of the lattice KP hierarchy.

We examine four cases of such special choices of \( c \). They are known to have many implications for the Schur polynomials (or the Schur functions) [10]. The special values of the Schur polynomials for these \( c \)'s play important roles in the work of Orlov and Scherbin as well [1, 2, 11]. The specialized hypergeometric tau functions are related to our previous work on topological string theory [12, 13] and our recent work on the Hurwitz numbers and the Hodge integrals [14, 15, 16]. We found therein that integrable hierarchies of the Ablowitz-Ladik, Volterra and Gelfand-Dickey types emerge as underlying integrable structures. Remarkably, the specialized hypergeometric tau functions indicate the existence of further variations of these well known integrable hierarchies. We are thus led to a wide spectrum of integrable structures hidden behind Okounkov’s hypergeometric tau function.

All these integrable structures are realized as reductions of the lattice KP hierarchy. Such a reduction is characterized by a particular shape of the Lax operator. To find the shape of the Lax operator of a specialized hypergeometric tau function, we use the method developed in our study on integrable structures of melting crystal models and topological string theory [17, 18, 19]. This method is based on a factorization problem [20]. The factorization problem enables us to find the initial value of the dressing operator at \( \tilde{t} = 0 \). The initial value of the Lax operator can be computed from the initial value of the dressing operator. We can thereby identify the relevant reduction.

This paper is organized as follows. Section 2 and 3 are a brief review of the lattice KP hierarchy and the notion of hypergeometric tau functions. Section 4 is devoted to the factorization problem and its implications. The initial values of the dressing and Lax operators of the specialized hypergeometric tau functions are computed therein. The four types of specialization of \( c_k \)'s are also examined in detail. The integrable structures hidden behind these initial values are formulated in a general form in Sections 5, 6, 7 and 8. Sections 5 and 6 deal with integrable hierarchies of the Gelfand-Dickey and Volterra types. Sections 7 and 8 are focussed on generalizations and variations of the Ablowitz-Ladik hierarchy.

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1To me more precise, we consider \( \mathcal{T}(s, \tilde{t}, -\tilde{c}) \) rather than \( \mathcal{T}(s, \tilde{t}, \tilde{c}) \).
§ 2. Lattice KP hierarchy

Let $t = (t_k)_{k=1}^{\infty}$ be a set of time variables, $s$ a spacial variable, and $\Lambda$ the shift operator

$$\Lambda = e^{\partial_s}, \quad \partial_s = \partial/\partial s.$$  

The Lattice KP hierarchy may be thought of as a subsystem of the 2D Toda hierarchy with the flows with respect to $\tilde{t} = (\tilde{t}_k)_{k=1}^{\infty}$ being suppressed. In the conventional setup of these integrable hierarchies \[8, 9\], $s$ is a lattice coordinate, hence a discrete variable. In the following, however, we treat $s$ as a continuous variable.

The Lax formalism of the lattice KP hierarchy is given by the Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \ldots,$$

for the difference (or, so to speak, pseudo-difference) Lax operator

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}, \quad u_n = u_n(s, t).$$

The generators $B_k$ of the flows are defined by the Lax operator as

$$B_k = (L^k)_{\geq 0}$$

where $(\quad)_{\geq 0}$ denotes the projection

$$\left( \sum_{n \in \mathbb{Z}} a_n \Lambda^n \right)_{\geq 0} = \sum_{n \geq 0} a_n \Lambda^n$$

to the non-negative power part of a difference operator. One can rewrite the Lax equations as

$$\frac{\partial L}{\partial t_k} = [L, B_k^{-}], \quad B_k^{-} = (L^k)_{< 0}.$$  

$(\quad)_{< 0}$ denotes the projection to the negative power part

$$\left( \sum_{n \in \mathbb{Z}} a_n \Lambda^n \right)_{< 0} = \sum_{n < 0} a_n \Lambda^n.$$

The Lax equations can be converted to the evolution equations

$$\frac{\partial W}{\partial t_k} = (W \Lambda^k W^{-1})_{\geq 0} W - W \Lambda^k = -(W \Lambda^k W^{-1})_{< 0} W$$

for the dressing operator

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}.$$
$L$ is thereby expressed in the dressed form

$$L = W\Lambda W^{-1}.$$ 

The wave function

$$\Psi = \left(1 + \sum_{n=1}^{\infty} w_n z^{-n}\right) \exp\left(\sum_{k=1}^{\infty} t_k z^k\right)$$

satisfies the auxiliary linear equations

$$\frac{\partial \Psi}{\partial t_k} = B_k \Psi, \quad L \Psi = z \Psi.$$ 

The amplitude part of $\Psi$ is related to the tau function $T(s, t)$ as

$$1 + \sum_{n=1}^{\infty} w_n z^{-n} = \frac{T(s - 1, t - [z^{-1}])}{T(s - 1, t)}, \quad [z] = \left(\frac{z^k}{k}\right)_{k=1}^{\infty}.$$ 

We can define the logarithm $\log L$ and the fractional power $L^\alpha$ of $L$ as

$$\log L = W \log \Lambda W^{-1}, \quad L^\alpha = W \Lambda^\alpha W^{-1}.$$ 

Since

$$\log \Lambda = \partial_s, \quad \Lambda^\alpha = e^{\alpha \partial_s},$$

log $L$ is a differential-difference operator

$$\log L = \partial_s - \frac{\partial W}{\partial s} W^{-1},$$

and $L^\alpha$ is a difference operator with fractional shift

$$L^\alpha = W \cdot W(s + \alpha)^{-1} e^{\alpha \partial_s} = (1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \cdots) e^{\alpha \partial_s},$$

where $W(s + \alpha)$ means substituting $s \to s + \alpha$ in the coefficients of $W$.

§ 3. Hypergeometric tau functions

§ 3.1. General form

For the moment, let us restrict the value of $s$ to $\mathbb{Z}$. We call the tau function

$$(3.1) \quad T(s, t) = \sum_{\lambda \in \mathcal{P}} S_\lambda(t) h_\lambda(s) S_\lambda(c)$$

a hypergeometric tau function of the lattice KP hierarchy. Let us specify the notations used here.

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3For consistency with our previous work, the variable $s$ of the tau function is shifted to $s - 1$. 

• $\mathcal{P}$ denotes the set of all partitions $\lambda = (\lambda_i)_{i=1}^\infty$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l(\lambda) > 0$, $\lambda_i = 0$ for $i > l(\lambda)$.

• $S_\lambda(t)$'s are the Schur polynomials
\[
S_\lambda(t) = \det(S_{\lambda_1-i+j}(t))_{i,j=1}^\infty,
\]
\[
\sum_{n=0}^\infty S_n(t)z^n = \exp\left(\sum_{k=1}^\infty t_k z^k\right).
\]
The determinant is understood to be the stable limit
\[
det(S_{\lambda_1-i+j}(t))_{i,j=1}^n = \lim_{n \to \infty} \det(S_{\lambda_1-i+j}(t))_{i,j=1}^n = \det(S_{\lambda_1-i+j}(t))_{i,j=1}^n \quad \text{for } n \geq l(\lambda)
\]
of the $n$-th principal minor. By substituting
\[
t_k = \frac{1}{k} \sum_{i=1}^\infty x_i^k,
\]
$S_\lambda(t)$'s turn into the Schur functions $s_\lambda(x)$, $x = (x_k)_{k=1}^\infty$, in the theory of symmetric functions [10].

• $c$ is a set of constants $c = (c_k)_{k=1}^\infty$.

• $h_\lambda(s)$'s are defined by a diagonal matrix $\text{diag}(h_n)_{n \in \mathbb{Z}}$ of non-zero constants as
\[
(3.2) \quad h_\lambda(s) = \frac{\prod_{i=1}^\infty h_{\lambda_1-i+s+1}}{\prod_{i=1}^\infty h_{-i+1}}.
\]
The right hand side is a somewhat formal expression. Actually, since all but a finite number of terms in the numerator and the denominator pairwise cancel out, this formal expression can be reduced to a finite expression, e.g.,
\[
(3.3) \quad h_0(s) = \begin{cases} 
\prod_{n=1}^s h_n & \text{for } s > 0, \\
1 & \text{for } s = 0, \\
\prod_{n=s+1}^0 h_n^{-1} & \text{for } s < 0.
\end{cases}
\]

Moreover, the ratio $h_\lambda(s)/h_0(s)$ can be cast into the so called contents product
\[
(3.4) \quad \frac{h_\lambda(s)}{h_0(s)} = \prod_{(i,j) \in \lambda} r_{j-i+s+1}, \quad r_n = \frac{h_n}{h_{n-1}},
\]
where $(i,j) \in \lambda$ means that $(i,j)$ is a box in the $i$-th row and the $j$-th column of the Young diagram of shape $\lambda$. 
(3.1) is a specialization of the tau function

(3.5) \[ \mathcal{T}(s,t,\bar{t}) = \sum_{\lambda \in \mathcal{P}} S_\lambda(t) h_\lambda(s) S_\lambda(-\bar{t}) \]
of the 2D Toda hierarchy to $\bar{t} = -c$. It is the tau functions of this form that are studied by Orlov and Scherbin [1, 2]. The definition (3.2) of $h_\lambda(s)$'s stems from a fermionic description of the tau functions. The diagonal matrix $\text{diag}(h_n)_{n \in \mathbb{Z}}$ corresponds to the element

\[ h = \exp \left( \sum_{n \in \mathbb{Z}} \log h_n : \psi_{-n} \psi_n^* : \right) \]
of the GL($\infty$) group that acts on the Fock space of 2D complex free fermion fields

\[ \psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}. \]

$h_\lambda(s)$'s are the diagonal matrix elements $\langle \lambda, s|h|\lambda, s \rangle$ of $h$ with respect to a basis $|\lambda, s\rangle$, $\lambda \in \mathcal{P}$, $s \in \mathbb{Z}$, of the Fock space. $|s\rangle = |\emptyset, s\rangle$ is the ground state in the charge-$s$ sector, and the excited states are labelled by the partition $\lambda$. Thus being a specialization of the tau function (3.5) of the 2D Toda hierarchy, (3.1) becomes a tau function of the lattice KP hierarchy.

§ 3.2. Specialization of $h_n$'s

Our consideration in this paper is focussed on the special case

(3.6) \[ h_n = \exp \left( \frac{\beta}{2} \left( n - \frac{1}{2} \right)^2 \right) Q^{n-1/2}, \]
where $\beta$ and $Q$ are non-zero constants. One can use (3.3) and (3.4) to compute $h_\lambda(s)$ explicitly as

(3.7) \[ h_\lambda(s) = \exp \left( \frac{\beta}{2} \left( \kappa(\lambda) + 2s|\lambda| + \frac{4s^3 - s}{12} \right) \right) Q^{(|\lambda|+s^2)/2}. \]

$|\lambda|$ and $\kappa(\lambda)$ denote the parts sum and twice the contents sum:

\[ |\lambda| = \sum_{i=1}^{\infty} \lambda_i, \quad \kappa(\lambda) = 2 \sum_{(i,j) \in \lambda} (j - i). \]

$\kappa(\lambda)$ has the following alternative expression:

\[ \kappa(s) = \sum_{i=1}^{\infty} \lambda_i (\lambda_i - 2i + 1) = \sum_{i=1}^{\infty} \left( \left( \lambda_i - i + \frac{1}{2} \right)^2 - \left( -i + \frac{1}{2} \right)^2 \right). \]
In the fermionic interpretation, the diagonal matrix $\text{diag}(h_n)_{n \in \mathbb{Z}}$ amounts to the operator

$$(3.8) \quad h = e^{\beta K/2}Q^{L_0},$$

where

$$K = \sum_{n \in \mathbb{Z}} \left( n - \frac{1}{2} \right)^2 : \psi_{-n} \psi_n^* :,$$

$$L_0 = \sum_{n \in \mathbb{Z}} \left( n - \frac{1}{2} \right) : \psi_{-n} \psi_n^* :.$$

One can derive (3.7) from the formulae

$$\langle \lambda, s | K | \lambda, s \rangle = \kappa(\lambda) + 2s|\lambda| + \frac{4s^3 - s}{12},$$

$$\langle \lambda, s | L_0 | \lambda, s \rangle = |\lambda| + \frac{s^2}{2}$$

of the matrix elements of $K$ and $L_0$. The tau function $\mathcal{T}(s, t, \bar{t})$ of the 2D Toda hierarchy defined by the same $h$ is nothing but Okounkov’s generating function of the double Hurwitz numbers $[6]$.

Although we have assumed that the variable $s$ takes values in $\mathbb{Z}$, the expression of $h_\lambda(s)$ in (3.7) is meaningful for $s \in \mathbb{R}$ (or even for $s \in \mathbb{C}$) as well. Actually, one can prove that the shifted function $\mathcal{T}(s + a, t)$, too, is a tau function of the 2D Toda hierarchy for any constant $a$. $[7]$ This implies that $\mathcal{T}(s, t)$ is a tau function in the sense explained in Section 2.

§ 4. Factorization problem and its implications

§ 4.1. Factorization problem and initial value of $W$

The dressing operator $W$ can be captured by the factorization problem $[20]

$$(4.1) \quad \exp \left( \sum_{k=1}^{\infty} t_k \Lambda^k \right) U \exp \left( \sum_{k=1}^{\infty} c_k \Lambda^{-k} \right) = W^{-1} \bar{W},$$

where $\bar{W}$ is a difference operator of the form

$$\bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n,$$

$^5$We have modified the definition of $L_0$ in our previous work.

$^7$The proof is parallel to the case of the tau function of the Hodge integrals $[16]$. 
and $U$ is an invertible operator that does not depend on $t$. The problem is to find the pair $W, \bar{W}$ from the given data $U$. This factorization problem is a specialization of the factorization problem for the 2D Toda hierarchy.

In the case where $h_n$’s are specialized as shown in (3.6), $U$ is the difference operator

$$U = e^{\beta(s-1/2)^2/2}Q^{s-1/2}$$

obtained from (3.8) by the correspondence

$$K \leftrightarrow \text{diag}((n-1/2)^2)_{n \in \mathbb{Z}} \leftrightarrow (s-1/2)^2,$$

$$L_0 \leftrightarrow \text{diag}(n-1/2)_{n \in \mathbb{Z}} \leftrightarrow (s-1/2)$$

among the fermion bilinear operators, the infinite matrices and the difference operators.

When $t$ is further specialized to $t = 0$, the factorization problem (4.1) can be solved immediately, because the left hand side is then factorized in an almost final form. This yields the expression

$$W_0 = U \exp \left( -\sum_{k=1}^{\infty} c_k \Lambda^{-k} \right) U^{-1}$$

of the initial value

$$W_0 = W|_{t=0}$$

of the dressing operator.

§ 4.2. Initial values of $L$, $L^\alpha$ and $\log L$

We can use (4.3) to compute the initial value

$$L_0 = L|_{t=0}$$

of the Lax operator $L$ and its logarithm and fractional power as follows:

**Proposition 4.1.**

$$L_0 = \Lambda \exp \left( \sum_{k=1}^{\infty} c_k Q^k (1 - e^{-\beta k}) e^{-\beta k(k+1)/2} e^{\beta ks} \Lambda^{-k} \right),$$

$$L_0^\alpha = \Lambda^\alpha \exp \left( \sum_{k=1}^{\infty} c_k Q^k (1 - e^{-\alpha \beta k}) e^{-\beta k(k+1)/2} e^{\beta ks} \Lambda^{-k} \right),$$

$$\log L_0 = \log \Lambda + \beta \sum_{k=1}^{\infty} k c_k Q^k e^{-\beta k(k+1)/2} e^{\beta ks} \Lambda^{-k}. $$
Proof. Let us rewrite (4.3) using (4.2) as
\[ W_0 = e^{\beta(s-1/2)^2/2} \exp \left( -\sum_{k=1}^{\infty} c_k Q^k \Lambda^{-k} \right) e^{-\beta(s-1/2)^2/2} \]
and compute
\[ L_0 = W_0 \Lambda W_0^{-1} \]
step by step. Since
\[ e^{-\beta(s-1/2)^2/2} \Lambda e^{\beta(s-1/2)^2/2} = e^{-\beta(s-1/2)^2/2} e^{\beta(s+1/2)^2/2} \Lambda = e^{\beta s} \Lambda \]
and
\[ \exp \left( -\sum_{k=1}^{\infty} c_k Q^k \Lambda^{-k} \right) e^{\beta s} \Lambda \exp \left( \sum_{k=1}^{\infty} c_k Q^k \Lambda^{-k} \right) = e^{\beta s} \Lambda \exp \left( \sum_{k=1}^{\infty} c_k Q^k (1 - e^{-\beta k}) \Lambda^{-k} \right), \]
\( L_0 \) can be expressed as
\[ L_0 = e^{\beta(s-1/2)^2/2} e^{\beta s} \Lambda \exp \left( \sum_{k=1}^{\infty} c_k Q^k (1 - e^{-\beta k}) \Lambda^{-k} \right) e^{-\beta(s-1/2)^2/2}. \]
Moreover, since
\[ e^{\beta(s-1/2)^2/2} e^{\beta s} \Lambda e^{-\beta(s-1/2)^2/2} = \Lambda \]
and
\[ e^{\beta(s-1/2)^2/2} \Lambda^{-k} e^{-\beta(s-1/2)^2/2} = e^{-\beta(k+1)/2} e^{\beta k s} \Lambda^{-k}, \]
the last expression of \( L_0 \) boils down to (4.4). The fractional power
\[ L_0^{\alpha} = W_0 \Lambda^{\alpha} W_0^{-1} \]
can be treated in much the same way and cast into the expression shown in (4.5). Lastly, we can compute \( \log L_0 \) from (4.5) as
\[ \log L_0 = \frac{\partial}{\partial \alpha} L_0^{\alpha} \bigg|_{\alpha=0}. \]
This yields (4.6).

Remark. One can use (4.7) to rewrite (4.6) as
\[ L_0^{\alpha} = \Lambda^{\alpha} e^{\beta(s-1/2)^2/2} \exp \left( \sum_{k=1}^{\infty} c_k Q^k (1 - e^{-\alpha \beta k}) \Lambda^{-k} \right) e^{-\beta(s-1/2)^2/2}, \]
(4.8)\]
\[ \log L_0 = \log \Lambda + \beta e^{\beta(s-1/2)^2/2} \sum_{k=1}^{\infty} k c_k Q^k \Lambda^{-k} e^{-\beta(s-1/2)^2/2}. \]
(4.9)
§ 4.3. Special values of $c$

Inspired by the work of Orlov [11], we choose the following four values of $c = (c_k)_{k=1}^\infty$ as the base points of our quest for new integrable structures:

(a) $c = t_\infty = (1, 0, 0, \ldots)$.

(b) $c = t(a) = \left(\frac{a}{k}\right)_{k=1}^\infty$.

(c) $c = t(\infty, q) = \left(\frac{1}{k(1-q^k)}\right)_{k=1}^\infty$.

(d) $c = t(a, q) = \left(\frac{1-q^{ak}}{k(1-q^k)}\right)_{k=1}^\infty$.

$a$ and $q$ are non-zero constants, and $q$ is assumed to be in the range $|q| < 1$. The Schur polynomials are known to take particular values at these special points [10]. For example,

$$S_\lambda(t_\infty) = \frac{\dim \lambda}{|\lambda|!} = \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)}.$$  

where $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group $S_d$, $d = |\lambda|$, of type $\lambda$, in other words, the number of standard Young tableaux of shape $\lambda$. $h(i,j)$ is the hook length of the box $(i,j)$ of the Young diagram. $S_\lambda(t(\infty, q))$ is a $q$-deformation of $S_\lambda(t_\infty)$:

$$S_\lambda(t(\infty, q)) = \prod_{(i,j) \in \lambda} \left(q^{-\kappa(\lambda)/4-|\lambda|/2_\lambda} \right).$$

Moreover, $S_\lambda(t(a))$ and $S_\lambda(t(a, q))$ are obtained from these values by multiplying simple factors (which are also contents product).

In each case, the initial value $L_0$ of the Lax operator turns out to take a significant form, which indicates what kind of integrable structure is hidden behind.

(a) This case is related to the single Hurwitz numbers of $\mathbb{CP}^1$ [6, 21]. $\log L_0$ can be computed with the aid of (4.6) as

$$\log L_0 = \log \Lambda + \beta Qe^\beta(s-1)\Lambda^{-1}.$$  

As shown in our previous work [14], this differential-difference operator is related to the continuum limit [22, 23] of the Bogoyavlensky-Itoh-Narita lattice [24, 25, 26].

(b) This case is a kind of one-parameter deformation of case (a). $\log L_0$ can be computed with the aid of (4.9) as

$$\log L_0 = \log \Lambda + \beta aQe^\beta(s-1)\Lambda^{-1}(1 - Qe^\beta(s-1)\Lambda^{-1})^{-1}.$$
(4.10) can be recovered from this operator in the scaling limit as $a \to \infty$, $Q \to 0$ and $aQ$ is fixed to a non-zero constant (which becomes the new constant $Q$).

(c) This case is related to the cubic Hodge integrals and the topological vertex \[27, 28, 29\] if $Q$ and $\beta$ are chosen as

\begin{equation}
Q = q^{1/2}, \quad \beta = (f + 1) \log q,
\end{equation}

where $f$ is a non-negative integer called the framing number. After some algebra based on (4.8), the fractional power of $L_0$ of order $1/(f + 1)$ turns out to take the simple form

\begin{equation}
L_0^{1/(f+1)} = (1 - q^{(f+1)(s-1)+1/2} \Lambda^{-1})\Lambda^{1/(f+1)}.
\end{equation}

As pointed out in our recent work \[16\], this difference operator of fractional order (for $f > 0$) is related to the Bogoyavlensky-Itoh-Narita lattice. The case of $f = 1$ amounts to the Volterra lattice. This interpretation can be extended to the case where $f$ is a positive rational number. Actually, in the context of the Hodge integrals \[30, 31\], $f$ can be replaced by an arbitrary real value $\tau \neq 0, -1$. The counterpart

\begin{equation}
L_0^{1/(\tau+1)} = (1 - q^{(\tau+1)(s-1)+1/2} \Lambda^{-1})\Lambda^{1/(\tau+1)}.
\end{equation}

of the foregoing expression (4.13) persists to hold as far as $\tau > -1$, in particular, in the new regime $-1 < \tau < 0$. If $\tau$ is a rational number in this negative interval, one can choose two positive coprime integers $M, N$ with $M < N$ such that

\begin{equation}
\frac{1}{\tau + 1} = \frac{N}{M}.
\end{equation}

(4.14) then implies that the $N$-th power of $L_0$ takes the special form

\begin{equation}
L_0^N = \Lambda^N + v_{10}\Lambda^{N-1} + \cdots + v_{M0}\Lambda^{N-M}.
\end{equation}

The regime $-1 < \tau < 0$ is thus related to the lattice version of the Gelfand-Dickey hierarchy \[32\].

(d) This case, as well as case (c), is related to topological string theory on toric Calabi-Yau threefolds \[12, 13, 18\]. The target space therein is the resolved conifold, whereas that of case (c) is $\mathbb{C}^3$. The extra factor $q^a$ in the definition of $t(a, q)$ is a geometric parameter of the resolved conifold. Under the same specialization (4.12) as case (c), the fractional power of $L_0$ of order $1/(f + 1)$ develops yet another operator factor:

\begin{equation}
L_0^{1/(f+1)} = (1 - q^{(f+1)(s-1)+1/2} \Lambda^{-1})(1 - q^{(f+1)(s-1)+a+1/2} \Lambda^{-1})^{-1} \Lambda^{1/(f+1)}.
\end{equation}

\[10\] The case where $f = 0$ is trivial in the context of integrable structures.
(4.13) can be recovered from this operator in the limit as \( a \to \infty \). In the case where \( f = 0 \), (4.16) emerges in the Ablowitz-Ladik (or relativistic Toda) hierarchy \cite{33,17}.

Thus the expressions of \( L_0 \) or \( \log L_0 \) indicate the existence of various underlying integrable structures. These integrable structures should be realized as reductions of the lattice KP hierarchy. The reduction conditions on the Lax operator have to be consistent with the flows of the lattice KP hierarchy.

In the subsequent sections, we present these reductions in a general form and check the consistency.

§ 5. Reductions of Gelfand-Dickey type

§ 5.1. Lattice version of Gelfand-Dickey hierarchy

The reductions of this type are defined by the condition that a positive integral power of the Lax operator consists of a finite number of terms:

\[ L^N = \Lambda^N + v_1 \Lambda^{N-1} + \cdots + v_M \Lambda^{N-M}, \quad v_n = v_n(s,t). \]

Following Frenkel’s work \cite{32} \cite{11}, we call this reduction a lattice version of the Gelfand-Dickey hierarchy.

If \( M \) is greater than \( N \), this reduction is a subsystem of the bigraded Toda hierarchy \cite{34} obtained by suppressing the flows with respect to \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \). On the other hand, as mentioned in Sect. 4.3, a Lax operator of this type with \( M < N \) emerges in the context of the cubic Hodge integrals.

One can directly verify that (5.1) is a consistent reduction condition, namely, preserved by the flows of the lattice KP hierarchy, for any \( M > 0 \). The reasoning is based on the Lax equations

\[ \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}] = [\mathcal{L}, B_k^+] \]

satisfied by \( \mathcal{L} = L^N \). The two expressions of the Lax equations imply that the right hand side is an operator of the form

\[ [B_k, \mathcal{L}] = [\mathcal{L}, B_k^-] = f_{1k} \Lambda^{N-1} + \cdots + f_{Mk} \Lambda^{N-M}. \]

The Lax equations thus turn out to be equivalent to the evolution equations

\[ \frac{\partial v_n}{\partial t_k} = f_{nk}, \quad n = 1, \ldots, M, \]

\footnote{Frenkel uses q-shift operators rather than shift operators. This is not an essential difference.}
for the coefficients \(v_1, \ldots, v_M\) of \(L^N\). Therefore, if the initial value of \(L^N\) at an initial time (e.g., \(t = 0\)) is an operator of the form (5.1), \(L^N\) itself takes the same form at all times.

It may happen, however, that \(v_1, \ldots, v_M\) are not the full degrees of freedom of the reduced system. An extreme situation arises in the case where \(M = 0\) — no degree of freedom is left in \(L^N = \Lambda^N\). Of course, all degrees of freedom are carried by \(L\) itself. In particular, as we show below, the coefficient \(u_1\) of \(L\) is related to the usual Gelfand-Dickey hierarchy in a continuous space.

§ 5.2. Relation to usual Gelfand-Dickey hierarchy

Let us consider the spectral problem

\[
(\Lambda^N + v_1 \Lambda^{N-1} + \cdots + v_M \Lambda^{N-M}) \Psi = L^N \Psi = z^N \Psi
\]

implied by the reduction condition (5.1). One can use the auxiliary linear equation

\[
\frac{\partial \Psi}{\partial t_1} = B_1 \Psi = (\Lambda + u_1) \Psi
\]

of the first flow to rewrite the terms \(\Lambda^m \Psi\)'s on the left hand side of (5.3) as

\[
\Lambda^m \Psi = (\partial_1 - u_1 (s + m - 1)) \cdots (\partial_1 - u_1 (s)) \Psi, \quad \partial_1 = \partial / \partial t_1.
\]

Thus (5.3) turn into the spectral problem

\[
(\partial_1^N + a_2 \partial_1^{N-2} + \cdots + a_N) \Psi = z^N \Psi
\]

of the Gelfand-Dickey type. The auxiliary linear equations of the other flows can be similarly transformed to differential equations of the form

\[
\frac{\partial \Psi}{\partial t_k} = \tilde{B}_k \Psi, \quad \tilde{B}_k = \partial_1^k + \tilde{b}_{k2} \partial_1^{k-2} + \cdots + \tilde{b}_{kk}.
\]

These equations can be identified with the auxiliary linear equations of the usual Gelfand-Dickey hierarchy, in which \(t_1\) plays the role of the spatial variable.

§ 6. Reductions of Volterra type

§ 6.1. Bogoyavlensky-Itoh-Narita lattice and its continuum limit

Let us formulate the reduction to the Bogoyavlensky-Itoh-Narita lattice in a form generalizing (4.13). The reduction can be characterized by the condition that

\[\text{To be more precise, we encounter the problem of non-locality in the definition of } B_k \text{'s. A remedy is to introduce a parameter (Planck constant) } \epsilon \text{ in the construction of the hierarchy } \text{[34].}\]

\[\text{This is also related to the non-locality of the flows.}\]
the fractional power of $L$ of order $1/(f+1)$ consists of two terms:

\begin{equation}
L^{1/(f+1)} = (1 - v\Lambda^{-1})\Lambda^{1/(f+1)} = \Lambda^{1/f+1} - v\Lambda^{-f/(f+1)}, \quad v = v(s, t).
\end{equation}

The Lax equations for $\mathcal{L} = L^{1/(f+1)}$ take the same form as (5.2), and implies that $v$ satisfies the evolution equations

$$\frac{\partial v}{\partial t_k} = f_k,$$

where $f_k$’s are the coefficients of

$$[B_k, \mathcal{L}] = [\mathcal{L}, B_k^-] = f_k\Lambda^{-f/(f+1)}.$$

The Bogoyavlensky-Itoh-Narita lattice is a “hungry”\footnote{This term was coined by S. Tsujimoto.} generalization of the well known Volterra lattice. Bogoyavlensky and Itoh also considered its continuum limit as $f \to \infty$ \cite{22, 23}. The continuum version can be characterized in the lattice KP hierarchy by the reduction condition

\begin{equation}
\log L = \log \Lambda - \tilde{v}\Lambda^{-1}, \quad \tilde{v} = \tilde{v}(s, t).
\end{equation}

The continuum limit is actually a scaling limit letting $f \to \infty$ and keeping $\tilde{v} = (f+1)v$ finite. The terms $\log L$ and $\log \Lambda$ in (6.2) emerge in the scaling limit as

$$\log L = \lim_{f \to \infty} (f+1)(L^{1/(f+1)} - 1), \quad \log \Lambda = \lim_{f \to \infty} (f+1)(\Lambda^{1/(f+1)} - 1).$$

We have seen in Section 4.3 that the hypergeometric tau function specialized to $c = t_\infty$ yields the operator (4.10) of this form.

Note that $L$, hence $B_k$’s, in these two reductions can be expressed in terms of $v$ and its $s$-shifts only. The right hand side of the evolution equations for $v$ are local in this sense. Thus $v$ carries full degrees of freedom of the reduced system. This is in contrast with the reduction of the Gelfand-Dickey type.

\section*{6.2. Generalizations}

We find further generalizations of these reductions of the Volterra type. Let us consider the generalization

$$c = (c_1, \ldots, c_N, 0, 0, \ldots)$$

of the special value $c = t_\infty$ of Sect. 4.3. The associated hypergeometric tau functions (in particular, with $c_1 = \cdots = c_{N-1} = 1$ and $c_N = 0$) are known to be related to generalized Hurwitz numbers \cite{35, 36}. (4.6) then implies that $\log L_0$ is an operator of the form

$$\log L_0 = \log \Lambda + \sum_{k=1}^N v_k \Lambda^{-k}.$$
This indicates the existence of a reduction defined by the condition
\[
(6.3) \quad \log L = \log \Lambda + \sum_{n=1}^{N} v_n \Lambda^{-n}, \quad v_n = v_n(s, t).
\]

The consistency of this reduction condition can be verified in the same manner as the case of (6.2).

In much the same way, the special value \( c = t(\infty, q) \) of Section 4.3 can be generalized as
\[
c_k = \frac{1}{k(1 - q^k)} \sum_{n=1}^{N} q^{b_n k},
\]
where \( b_n \)'s are new constants. If \( Q \) and \( \beta \) are chosen as shown in (4.12), the associated fractional power of \( L_0 \) of order \( 1/(f + 1) \) turns out to take the product form
\[
L_0^{1/(f+1)} = \prod_{n=1}^{N} (1 - q^{(f+1)(s-1)+b_n+1/2} \Lambda^{-1}) \cdot \Lambda^{1/(f+1)}.
\]

We are thus led to the possibility of a new type of reduction defined by the condition
\[
(6.4) \quad L^{1/(f+1)} = B \Lambda^{1/(f+1)},
\]
where \( B \) is a difference operator of the form
\[
B = 1 + \sum_{n=1}^{N} v_n \Lambda^{-n}, \quad v_n = v_n(s, t).
\]

The foregoing reduction condition (6.3) can be derived from (6.4) by a suitable scaling limit as \( f \to \infty \). The consistency of the reduction condition (6.4) will be explained in a broader context below.

§ 7. Variation of rational reductions

§ 7.1. Reduction conditions

The initial value (4.16) for the special value \( c = t(a, q) \) in Section 4.3 suggests a variation of (6.1) of the form
\[
(7.1) \quad L^{1/(f+1)} = (1 - v \Lambda^{-1})(1 - u \Lambda^{-1})^{-1} \Lambda^{1/(f+1)}
\]
with one more field \( u = u(s, t) \). This also is a variation of the reduction condition for the Ablowitz-Ladik hierarchy [33 [17].
The Ablowitz-Ladik reduction is a special case of the so-called “rational reductions” of the 2D Toda hierarchy [37]. The consistency of the reduction condition (7.1) can be verified by the same method as used for those rational reductions. Moreover, from the point of view of rational reductions, (7.1) can be further generalized as follows.

Let us recall that a generalization of the Ablowitz-Ladik hierarchy emerges in topological string theory on a generalization of the resolved conifold [18, 37]. This case amounts to choosing the constants $c_k$ as

$$c_k = \frac{1}{k(1 - q^k)} \left( \sum_{n=1}^{N} q^{b_n k} - \sum_{n=1}^{N} q^{a_n k} \right).$$

$q^{b_n}$ and $q^{a_n}$ are geometric parameters of the target space geometry. The associated fractional power $L_0^{1/(f+1)}$ of $L_0$ then takes the “rational” form

$$L_0^{1/(f+1)} = \prod_{i=1}^{N} (1 - q^{(f+1)(s-1)+b_n+1/2} \Lambda^{-1})$$

$$\cdot \prod_{n=1}^{N} (1 - q^{(f+1)(s-1)+a_n+1/2} \Lambda^{-1}) \cdot \Lambda^{1/(f+1)}.$$

This example suggests to consider a generalization of (7.1) of the form

$$(7.2) \quad L^{1/(f+1)} = BC^{-1} \Lambda^{1/(f+1)},$$

where $B$ is the same difference operator as used in (6.4), and $C$ is another operator [17]

$$C = 1 + \sum_{n=1}^{N} u_n \Lambda^{-n}, \quad u_n = u_n(s, t).$$

§ 7.2. Consistency of reduction condition

To verify the consistency of the reduction condition (7.2), we prove a few propositions.

Let us introduce the difference operators

$$P_k = \left( (BC^{-1} \Lambda^{1/(f+1)})^{k(f+1)} \right)_{\geq 0} = (L^k)_{\geq 0},$$

$$Q_k = \left( (C^{-1} \Lambda^{1/(f+1)} B)^{k(f+1)} \right)_{\geq 0} = (B^{-1} L^k B)_{\geq 0},$$

$$R_k = \left( (\Lambda^{1/(f+1)} BC^{-1})^{k(f+1)} \right)_{\geq 0} = (\Lambda^{1/(f+1)} L^k \Lambda^{-1/(f+1)})_{\geq 0}$$

for $k = 1, 2, \ldots$.

---

17Note that the coefficients $u_n$ are different from those of $L$. 
Proposition 7.1. $P_k$ and $R_k$ satisfy the identity

\begin{equation}
R_k \Lambda^{1/(f+1)} = \Lambda^{1/(f+1)} P_k.
\end{equation}

\textbf{Proof.} We have the general operator identity

$$
\left( \Lambda^{1/(f+1)} \cdot \sum_{n \in \mathbb{Z}} a_n(s) \Lambda^n \cdot \Lambda^{-1/(f+1)} \right)_{\geq 0} = \sum_{n \geq 0} a_n(s + 1/(f + 1)) \Lambda^n \\
= \Lambda^{1/(f+1)} \cdot \sum_{n \geq 0} a_n(s) \Lambda^n \cdot \Lambda^{-1/(f+1)} \\
= \Lambda^{1/(f+1)} \left( \sum_{n \in \mathbb{Z}} a_n(s) \Lambda^n \right)_{\geq 0} \cdot \Lambda^{-1/(f+1)}.
$$

Consequently,

$$
R_k = \Lambda^{1/(f+1)} (L^k)_{\geq 0} \Lambda^{-1/(f+1)} = \Lambda^{1/(f+1)} P_k \Lambda^{-1/(f+1)}.
$$

\hfill \square

Proposition 7.2. If $B$ and $C$ satisfy the equations

\begin{equation}
\frac{\partial B}{\partial t_k} = P_k B - B Q_k, \quad \frac{\partial C}{\partial t_k} = R_k C - C Q_k,
\end{equation}

$L$ satisfies the Lax equations of the lattice KP hierarchy.

\textbf{Proof.} Differentiating $\mathcal{L} = L^{1/(f+1)}$ with respect to $t_k$ yields

$$
\frac{\partial \mathcal{L}}{\partial t_k} = \frac{\partial B}{\partial t_k} C^{-1} \Lambda^{1/(f+1)} - B C^{-1} \frac{\partial C}{\partial t_k} C^{-1} \Lambda^{1/(f+1)}.
$$

By (7.4) and (7.3), one can rewrite the right hand side as

$$
\frac{\partial \mathcal{L}}{\partial t_k} = P_k \mathcal{L} - \mathcal{L} \Lambda^{-1/(f+1)} R_k \Lambda^{1/(f+1)} = [P_k, \mathcal{L}].
$$

Since $P_k = (L^k)_{\geq 0}$, these equations imply that $L = \mathcal{L}^{f+1}$ satisfies the Lax equations of the lattice KP hierarchy. \hfill \square

The Lax equations of the lattice KP hierarchy can be thus reduced to the equations (7.4) for $B$ and $C$. Moreover, we can rewrite these equations as

$$
\frac{\partial B}{\partial t_k} = B Q_k - P_k B, \quad \frac{\partial C}{\partial t_k} = C Q_k - R_k C,
$$
where

\[ P_k^- = \left( (BC^{-1}\Lambda^{1/(f+1)})^{k(f+1)} \right)_{<0}, \]
\[ Q_k^- = \left( (C^{-1}\Lambda^{1/(f+1)}B)^{k(f+1)} \right)_{<0}, \]
\[ R_k^- = \left( (\Lambda^{1/(f+1)}BC^{-1})^{k(f+1)} \right)_{<0}. \]

These equations and (7.4) imply that the right hand side of (7.4) are linear combinations of \( \Lambda^{-1}, \ldots, \Lambda^{-N} \). Thus (7.4) can be reduced to evolution equations of the form

\[ \frac{\partial v_n}{\partial t_k} = f_{nk}, \quad \frac{\partial u_n}{\partial t_k} = g_{nk} \]

for the coefficients of \( B \) and \( C \). The right hand side of these equations are local because the coefficients of \( L \), hence those of \( P_k, Q_k \) and \( R_k \), can be expressed in terms of \( v_n, u_n \) and their \( s \)-shifts. This completes the verification of the consistency of the reduction condition (7.2).

As a corollary, we find the consistency of the reduction condition (6.4) as well. This condition amounts to the case where \( C = 1 \). The condition \( C = 1 \) is preserved by the time evolutions of \( B \) and \( C \) under (7.4). The reduced equations for \( B \) thereby take the form

\[ \frac{\partial B}{\partial t_k} = P_kB - BQ_k, \]

with apparently the same definition of \( P_k \) and \( Q_k \) as those of (7.4). These equations can be reduced to evolution equations of \( v_n \)'s.

§ 8. Further variation of rational reductions

§ 8.1. Reduction condition

We now turn to the problem of elucidating the status of (4.11). This is the most mysterious case among the four specializations examined in Sect. 4.3. (4.11) looks like the initial value of an operator of the form

\[ \log L = \log \Lambda - \tilde{u}\Lambda^{-1}(1 - u\Lambda^{-1})^{-1}. \]

This may be thought of as a scaling limit of (7.1) as \( f \rightarrow \infty \). Let us rewrite (7.1) as

\[ L^{1/(f+1)} = \Lambda^{1/(f+1)} - (v - u)\Lambda^{-1}(1 - u\Lambda^{-1})^{-1}\Lambda^{1/(f+1)}. \]

Letting \( f \rightarrow \infty \) while keeping \( \tilde{u} = (f + 1)(v - u) \) finite, we obtain (8.1) in the limit.

The same idea can be applied to the more general reduction condition (7.2). We are thus led to the following reduction condition:

\[ \log L = \log \Lambda + \tilde{C}C^{-1}, \]
where \( \tilde{C} \) is an operator of the form

\[
\tilde{C} = \sum_{n=1}^{N} \tilde{u}_n \Lambda^{-n}, \quad \tilde{u}_n = \tilde{u}_n(s, t),
\]

that amounts to the limit of \((f + 1)(B - C)\) as \( f \to \infty \).

§ 8.2. Consistency of reduction condition

The consistency of the reduction condition (8.2) can be verified with the aid of evolution equations for \( C \) and \( \tilde{C} \). Heuristically, those equations can be derived as a scaling limit of (7.4). The \( t_k \)-derivative of \( \tilde{C} \) can be computed as

\[
\frac{\partial \tilde{C}}{\partial t_k} = \lim_{f \to \infty} (f + 1) \left( \frac{\partial B}{\partial t_k} - \frac{\partial C}{\partial t_k} \right) = \lim_{f \to \infty} (f + 1) \left( (P_k - R_k)B + R_k(B - C) - (B - C)Q_k \right).
\]

After some algebra, \( C \) and \( \tilde{C} \) turn out to satisfy the evolution equations

(8.3) \[
\begin{align*}
\frac{\partial \tilde{C}}{\partial t_k} &= -\frac{\partial P_k}{\partial s}C + P_k \tilde{C} - \tilde{C}Q_k, \\
\frac{\partial C}{\partial t_k} &= P_k C - CQ_k,
\end{align*}
\]

where \( P_k = (L^k)_{\geq 0}, \quad Q_k = (C^{-1}L^kC)_{\geq 0}. \)

We can directly confirm the correctness of these equations.

**Proposition 8.1.** If \( C \) and \( \tilde{C} \) satisfy (8.3), \( L \) satisfies the Lax equations of the lattice KP hierarchy.

**Proof.** The \( t_k \)-derivative of \( \mathcal{L} = \log \Lambda + \tilde{C}C^{-1} \) yields

\[
\frac{\partial \mathcal{L}}{\partial t_k} = \frac{\partial \tilde{C}}{\partial t_k} C^{-1} - \tilde{C}C^{-1} \frac{\partial C}{\partial t_k} C^{-1}.
\]

By (8.3), one can rewrite the right hand side as

\[
\frac{\partial \mathcal{L}}{\partial t_k} = -\frac{\partial P_k}{\partial s} + [P_k, \tilde{C}C^{-1}] = [P_k, \mathcal{L}].
\]

Since \( P_k = (L^k)_{\geq 0}, \) these equations imply that \( L = e^\mathcal{L} \) satisfies the Lax equations of the lattice KP hierarchy.

(8.3) can be cast into the dual form

(8.4) \[
\begin{align*}
\frac{\partial \tilde{C}}{\partial t_k} &= \frac{\partial P_k}{\partial s}C + \tilde{C}Q_k - P_k \tilde{C}, \\
\frac{\partial C}{\partial t_k} &= CQ_k - P_k C,
\end{align*}
\]
where
\[ P_k^- = (L^k)_{<0}, \quad Q_k^- = (C^{-1} L^k C)_{<0}. \]
This equation is equivalent to (8.3) because of the commutation relation
\[ \frac{\partial L^k}{\partial s} = [\log \Lambda, L^k] = -[\tilde{C} C^{-1}, L^k], \]
and the obvious identities
\[ P_k^+ + P_k^- = L^k, \quad Q_k^+ + Q_k^- = C^{-1} L^k C. \]
The commutation relation is a consequence of the relation
\[ \log L = \mathcal{L} = \log \Lambda + \tilde{C} C^{-1}. \]
(8.3) and (8.4) show that (8.3) can be reduced to evolution equations for the coefficients \( v_n, u_n \) of \( C \) and \( \tilde{C} \). The consistency of the reduction condition (8.2) can be thus verified.

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