Risk upper bounds for RKHS ridge group sparse estimator in the regression model with non-Gaussian and non-bounded error

HALALEH KAMARI, SYLVIE HUET, MARIE-LUCE TAUPIN

June 20, 2021

Abstract

We consider the problem of estimating a meta-model of an unknown regression model with non-Gaussian and non-bounded error. The meta-model belongs to a reproducing kernel Hilbert space constructed as a direct sum of Hilbert spaces leading to an additive decomposition including the variables and interactions between them. The estimator of this meta-model is calculated by minimizing an empirical least-squares criterion penalized by the sum of the Hilbert norm and the empirical $L^2$-norm. In this context, the upper bounds of the empirical $L^2$ risk and the $L^2$ risk of the estimator are established.

Keywords: meta-model, reproducing kernel Hilbert space, ridge group sparse penalty, risk upper bound.

1 Introduction

Let us consider the following regression model:

$$Y = m(X) + \sigma \varepsilon, \sigma > 0,$$

where the variables $X = (X_1, ..., X_d)$ are independent with a known law $P_X = \otimes_{a=1}^d P_a$ on $\mathcal{X} = \prod_{a=1}^d X_a$, a compact subset of $\mathbb{R}^d$. The number $d$ of components of $X$ may be large. The model $m$ from $\mathbb{R}^d$ to $\mathbb{R}$ may be complex, presenting strong non-linearities, and it is assumed to be square-integrable, i.e. $m \in L^2(\mathcal{X}, P_X)$.

Let $\mathcal{D}$ be the set of densities,

$$\mathcal{D} = \{\pi_\alpha : \pi_\alpha(x) = a_\alpha \exp(-|x|^\alpha), \text{ with } (a_\alpha)^{-1} = \int_\mathbb{R} \exp(-|x|^\alpha) dx, \alpha > 2\}.$$

In this paper, we assume that the error term $\varepsilon$ is equal to $Z/\sigma_\alpha$, where $Z$ is a random variable with density $\pi_\alpha \in \mathcal{D}$ and $\sigma_\alpha^2$ is its variance, i.e. $\text{var}(Z) = \sigma_\alpha^2$.

Based on $n$ data points $\{(X_i, Y_i)\}_{i=1}^n$, a meta-model that approximates the Hoeffding decomposition of $m$ is estimated. This meta-model belongs to a reproducing kernel Hilbert space (RKHS), which is constructed as a direct sum of Hilbert spaces (Durrande, Ginsbourger, Roustant, and Carraro 2013). The estimation of the meta-model is carried out via a penalized least-squares minimization allowing to select the subsets of variables $X$ that contribute to predict the output $Y$ (Huet and Taupin 2017).

Let us be more precise on the Hoeffding decomposition. Let $\mathcal{P}$ be the set of all the subsets of $\{1, ..., d\}$ with dimension 1 to $d$, and for all $v \in \mathcal{P}$ and $X \in \mathcal{X}$, let $X_v$ be the vector with components $X_a$ for all $a \in v$. Let also $|A|$ be the cardinality of a set $A$ and for all $v \in \mathcal{P}$, let $m_v : \mathbb{R}^{|v|} \to \mathbb{R}$ be a function of $X_v$. Then, the Hoeffding decomposition of $m$ is written as (Hoeffding 1948, Sobol 1993, van der Vaart 1998),

$$m(X) = m_0 + \sum_{v \in \mathcal{P}} m_v(X_v),$$

where $m_0$ is a constant.

*Halaleh Kamari, Université Paris-Saclay, France, @ Sylvie Huet, INRAE, France, @ Marie-Luce Taupin, Université Evry Val d’Essonne, France, @.
This decomposition (3) is unique (Sobol 1993), all the functions \( m_v \) are centered, and they are orthogonal with respect to \( L^2(\mathcal{X}, P_X) \).

The Hoeffding decomposition of \( m \) is approximated by the orthogonal projection of \( m \) on a RKHS \( \mathcal{H} \) which is constructed as a direct sum of Hilbert spaces (Durrande, Ginsbourger, Roustant, and Carraro 2013).

Let \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) be the inner product in \( \mathcal{H} \), and let \( k \) and \( k_v \) be the reproducing kernels associated with the RKHS \( \mathcal{H} \) and the RKHS \( \mathcal{H}_v \), respectively. The properties of the RKHS \( \mathcal{H} \) insures that any function \( f \in \mathcal{H}, f : \mathcal{X} \subset \mathbb{R}^d \rightarrow \mathbb{R} \) can be written as the following decomposition:

\[
f(X) = \langle f, k(X, \cdot) \rangle_{\mathcal{H}} = f_0 + \sum_{v \in \mathcal{P}} f_v(X_v),
\]

where \( f_0 \) is a constant, and \( f_v : \mathbb{R}^{|v|} \rightarrow \mathbb{R} \) is defined by,

\[
f_v(X) = \langle f, k_v(X, \cdot) \rangle_{\mathcal{H}}.
\]

For all \( v \in \mathcal{P} \), the functions \( f_v(X_v) \) are centered and for all \( v, v' \in \mathcal{P}, v \neq v' \), the functions \( f_v(X_v) \) and \( f_{v'}(X_{v'}) \) are orthogonal with respect to \( L^2(\mathcal{X}, P_X) \). Therefore, the decomposition of any function \( f \) presented in Equation (4) is unique and is its Hoeffding decomposition.

The meta-model \( f^* \) that approximates the Hoeffding decomposition of \( m \) is defined as follows:

\[
f^* = \arg \min_{f \in \mathcal{H}} \| m - f \|_2^2 = \arg \min_{f \in \mathcal{H}} E_X (m(X) - f(X))^2.
\]

Since the function \( f^* \) belongs to the RKHS \( \mathcal{H} \), its decomposition on \( \mathcal{H} \) is its Hoeffding decomposition:

\[
f^* = f_0^* + \sum_{v \in \mathcal{P}} f_v^*.
\]

And for all \( v \in \mathcal{P} \), the function \( f_v^* \) in Equation (5) approximates the function \( m_v \) in Equation (3).

Decomposition (5) contains |
\( \mathcal{P} \) terms \( f_v^* \) to be estimated. The cardinality of \( \mathcal{P} \) is equal to \( 2^d - 1 \) which may be huge since it raises very quickly by increasing \( d \). In order to deal with this problem, one may estimate \( f^* \) by a sparse estimator \( \hat{f} \in \mathcal{H} \). To this purpose, the estimation of \( f^* \) is done on the basis of \( n \) observations by minimizing an empirical least-squares criterion penalized by the sum of the Hilbert norm and the empirical norm.

This procedure, called ridge group sparse, estimates the groups \( v \) that are suitable for predicting \( f^* \), and the relationship between \( f_v^* \) and \( X_v \) for each group \( v \) ([Huet and Taupin 2017]). The estimator so obtained is called the RKHS ridge group sparse estimator.

Several authors studied the theoretical properties of estimators similar to the RKHS ridge group sparse estimator. Let us briefly review their framework and their results.

([Meier, van de Geer, and Buhlmann 2009]) considered an estimator similar to the RKHS ridge group sparse estimator. Instead of adding two separate sparsity and smoothness penalties, they combine these two terms into a single sparsity and smoothness penalty. In the fixed design regression model with error \( \varepsilon \) that is distributed as a sub-Gaussian random variable, they established upper bounds of the empirical risk for estimating the projection of \( m \) onto the set of univariate additive functions. Afterwards, ([Raskutti, Wainwright, and Yu 2012]) showed (in Section 3.4. of their paper) that the convergence rate of this estimator is sub-optimal.

([Koltchinskii and Yuan 2010]) considered a more general RKHS including the functions that have an additive representation over kernel spaces and obtained an estimator based on a ridge group sparse type procedure. Under a global boundedness condition, they established upper bounds on the excess risk assuming that the function \( m \) has a sparse representation. A global boundedness condition means that the quantity \( \sup_{f \in \mathcal{H}} \sup_{X \in \mathcal{X}} |f(X)| \) is assumed to be bounded independently of dimension \( d \). Their results are valid for a large class of loss functions, and for distributions of the observations \( Y \) such that some defined boundedness conditions on the loss functions are satisfied (see Section 2.1. of their paper). In their framework, the input variables \( X \) are not assumed to be independent and there is no orthogonality assumption between the kernel spaces. Instead, the authors introduced some characteristics related to the degree of dependence of their kernel spaces which insures almost orthogonality between these spaces. Their method to derive their upper bounds relies on the elementary empirical and Rademacher process methods such as symmetrization and concentration inequalities for Rademacher processes and Bernstein type exponential bounds.

([Raskutti, Wainwright, and Yu 2012]) assumed that the function \( m \) has a sparse univariate additive representation, i.e. \( m = \sum_{a \in S} m_a(X_a) \) for \( m_a(X_a) \) being univariate functions and \( |S| < d \), such that each
univariate function $m_a$ lies in a RKHS $H_a$. They used the ridge group sparse procedure to calculate the estimator of $m$, and studied the theoretical properties of their estimator in the Gaussian regression model, i.e. $\varepsilon$ in Equation (1) is distributed as a centered Gaussian random variable. They provided upper bounds for the integrated and the empirical risks and a lower bound for the integrated risk of their estimator over spaces of sparse additive models, including polynomials, splines and Sobolev classes.

(Huet and Taupin 2017) studied the theoretical properties of the RKHS ridge group sparse estimator in the Gaussian regression model. They derived upper bounds with respect to the $L^2$-norm and the empirical $L^2$-norm for the distance between the true function $m$ and its estimation $\hat{f}$ into the RKHS $H$.

(Raskutti, Wainwright, and Yu 2012) and (Huet and Taupin 2017) did not assume the global boundedness condition. Instead, they assumed that each function within the unit ball of the Hilbert space $H_v$ is uniformly bounded by a constant. The proof of their results is based on the probabilistic methods of empirical Gaussian process such as concentration inequalities and Sudakov minoration (e.g. (Pisier 1989), (Massart 2000), (van de Geer, Gill, Ripley, Ross, Silverman, and Stein 2000), (Ledoux 2001)), as well as results on the Rademacher complexity of kernel classes ((Mendelson 2002), (Bartlett, Bousquet, and Mendelson 2005)).

In this paper, the upper bounds of the empirical $L^2$ risk and the $L^2$ risk of the RKHS ridge group sparse estimator are provided, in the regression model (see Equation (1)) with non-Gaussian and non-bounded error $\varepsilon$, and by considering a quadratic loss function. In this case the conditions assumed in (Koltchinskii and Yuan 2010) are not satisfied, and the empirical Gaussian process methods such as concentration inequalities and Sudakov minoration can not be used.

The proof of our results requires different mathematical tools than those used in the works mentioned above:

- a Sudakov type minoration that is satisfied for the non-Gaussian and non-bounded random variables,
- a concentration bound for the lower and upper tails of a convex function of the random variables $\{\varepsilon_i\}_{i=1}^n$ that are non-Gaussian and non-bounded.

To the best of our knowledge, in our context of regression model with non-Gaussian and non-bounded error $\varepsilon$, and with quadratic loss function, the only Sudakov type minoration which allows to obtain the same rate of convergence for the RKHS ridge group sparse estimator as in the Gaussian regression model (see (Huet and Taupin 2017)), is the one obtained by (Talagrand 1994). The minoration obtained by (Talagrand 1994) is specific to the densities $\pi_\alpha \in D$ as defined in Equation (2). This is the reason why this class of densities is considered in this work.

Concerning the concentration bound, it can be shown that the distribution functions associated with the densities $\pi_\alpha \in D$ belong to a class of distribution functions defined by (Adamczak 2005), for which the log-Sobolev inequality ((Gross 1975)) is satisfied. (Shu and Strzelecki 2017) provided bounds for the lower and upper tails of convex functions of independent random variables which satisfy the log-Sobolev inequality. Since the distribution functions associated with the densities $\pi_\alpha \in D$ satisfy the log-Sobolev inequality, the concentration inequality derived by (Shu and Strzelecki 2017) holds for them.

This paper is organised as follows: The RKHS construction and the procedure for estimating a meta-model are presented in Section 2. The theoretical properties of the RKHS ridge group sparse estimator are stated in Theorem 1 and Corollary 2. The proof of Theorem 1 is postponed in Section 5. In Section 4 the main arguments of the proof of Theorem 1 and motivation for the choice $\pi_\alpha$ are detailed.

## 2 Meta-modelling and the RKHS ridge group sparse estimator

The independency between the input variables $X$ allows to write the function $m$ according to its Hoeffding decomposition presented in Equation (3),

$$m(X) = m_0 + \sum_{v \in P} m_v(X_v).$$

The unknown function $m$ is approximated by its orthogonal projection, denoted $f^*$, on a RKHS, denoted $H$, that is constructed as a direct sum of Hilbert spaces. The RKHS $H$ is associated with a so-called ANOVA kernel which is defined in order to obtain the analytical expression of the terms of the Hoeffding decomposition of the functions of $H$. As $f^*$ is the orthogonal projection of $m$ on $H$, each term in its decomposition is an approximation of the associated term in the Hoeffding decomposition of $m$. The construction of the RKHS $H$ has been proposed by (Durrande, Ginsbourger, Roustant, and Carraro 2013) that we recall briefly in the following.
2.1 RKHS construction

Let \( X = X_1 \times \ldots \times X_d \) be a subset of \( \mathbb{R}^d \). For each \( a \in \{1, \ldots, d\} \), we choose a RKHS \( H_a \), and its associated kernel \( k_a \) defined on the set \( X_a \subset \mathbb{R} \) such that the two following properties are satisfied:

(i) \( k_a : X_a \times X_a \to \mathbb{R} \) is measurable,

(ii) \( \mathbb{E}_{X_a} \sqrt{k_a(X_a, X_a)} < \infty \).

The property (ii) depends on the kernel \( k_a \), \( a = 1, \ldots, d \) and the distribution of \( X_a \), \( a = 1, \ldots, d \). It is not very restrictive since it is satisfied, for example, for any bounded kernel.

The RKHS \( H_a \) can be decomposed as a sum of two orthogonal sub-RKHS,

\[
H_a = H_{0a} \perp H_{1a},
\]

where \( H_{0a} \) is the RKHS of zero mean functions,

\[
H_{0a} = \left\{ f_a \in H_a, \mathbb{E}_{X_a}(f_a(X_a)) = 0 \right\},
\]

and \( H_{1a} \) is the RKHS of constant functions,

\[
H_{1a} = \left\{ f_a \in H_a, f_a(X_a) = C \right\}.
\]

The kernel \( k_{0a} \) associated with the RKHS \( H_{0a} \) is defined as follows:

\[
k_{0a}(X_a, X'_a) = k_a(X_a, X'_a) - \frac{E_{U \sim P_a}(k_a(X_a, U))E_{U \sim P_a}(k_a(X'_a, U))}{E_{(U,V) \sim P_a \times P_a} k_a(U, V)}.
\]

Let \( k_v(X_v, X'_v) = \prod_{a \in v} k_{0a}(X_a, X'_a) \), then the ANOVA kernel \( k \) is defined by:

\[
k(X, X') = \prod_{a=1}^d (1 + k_{0a}(X_a, X'_a)) = 1 + \sum_{v \in \mathcal{P}} k_v(X_v, X'_v).
\]

For \( H_v \) being the RKHS associated with the kernel \( k_v \), the RKHS associated with the ANOVA kernel \( k \) is then defined by:

\[
H = \prod_{a=1}^d \left( \mathbb{I} \perp H_{0a} \right) = \mathbb{I} + \sum_{v \in \mathcal{P}} H_v,
\]

where \( \perp \) denotes the \( L^2 \) inner product.

According to this construction, any function \( f \in H \) satisfies the following decomposition,

\[
f(X) = \langle f, k(\cdot, \cdot) \rangle_H = f_0 + \sum_{v \in \mathcal{P}} f_v(X_v),
\]

which is the Hoeffding decomposition of \( f \).

For more background on the RKHS spaces see (Aronszajn 1950), (Saitoh 1988), (Berlinet and Thomas-Agnan 2003).

2.2 Approximating the Hoeffding decomposition of \( m \)

Let \( f^* \in H \) be defined as follows:

\[
f^* = \arg \min_{f \in H} \|m - f\|_2^2 = \arg \min_{f \in H} \mathbb{E}_X (m(X) - f(X))^2.
\]

The function \( f^* = f^*_0 + \sum_{v \in \mathcal{P}} f^*_v \) is the approximation of \( m \) on the RKHS \( H \), and its Hoeffding decomposition is an approximation of the Hoeffding decomposition of \( m \). Therefore, according to Equation (3), for all \( v \in \mathcal{P} \), each function \( f^*_v \) approximates the function \( m_v \).

The number of functions \( f^*_v \) is related to the cardinality of \( \mathcal{P} \), i.e. \( 2^d - 1 \), that may be huge. The idea is to calculate a sparse estimator of \( f^* \) as an estimator of \( m \). To do so, the ridge group sparse procedure as proposed by (Huet and Taupin 2017) is used that we recall in the following.
2.3 Ridge group sparse procedure and associated estimator

Let \( n \) be the number of observations. For all \( v \in \mathcal{P} \), let \( X_v \) be the matrix of variables corresponding to the \( v \)-th group, i.e.

\[
X_v = (X_{vi}, i = 1, ..., n, v \in \mathcal{P}) \in \mathbb{R}^{n \times |\mathcal{P}|}.
\]

For any \( f \in \mathcal{H} \) such that \( f = f_0 + \sum_{v \in \mathcal{P}} f_v \), and for some tuning parameters \( \gamma_v, \mu_v, v \in \mathcal{P} \), the ridge group sparse criterion is defined as follows:

\[
\mathcal{L}(f) = \frac{1}{n} \sum_{i=1}^{n} \left( y_i - f_0 - \sum_{v \in \mathcal{P}} f_v(X_{vi}) \right)^2 + \sum_{v \in \mathcal{P}} \gamma_v \| f_v \|_n + \sum_{v \in \mathcal{P}} \mu_v \| f_v \|_{\mathcal{H}_v},
\]

where \( \| f_v \|_n \) is the empirical \( L^2 \)-norm of \( f_v \) defined by the sample \( \{X_{vi}\}_{i=1}^{n} \) as

\[
\| f_v \|_n^2 = \frac{1}{n} \sum_{i=1}^{n} f_v^2(X_{vi}).
\]

The penalty function in the criterion \( \mathcal{L}(f) \) is the sum of the Hilbert norm and the empirical norm, which allows to select few terms in the additive decomposition of \( f \) over sets \( v \in \mathcal{P} \). Moreover, the Hilbert norm favours the smoothness of the estimated \( f_v, v \in \mathcal{P} \).

Let us define the set of functions,

\[
\mathcal{F} = \left\{ f : f = f_0 + \sum_{v \in \mathcal{P}} f_v, \text{ with } f_v \in \mathcal{H}_v, \text{ and } \| f_v \|_{\mathcal{H}_v} \leq r_v, r_v > 0 \right\}.
\]

Then the RKHS ridge group sparse estimator is defined by,

\[
\hat{f} = \arg \min_{f \in \mathcal{F}} \mathcal{L}(f). \quad (7)
\]

3 Risk upper bounds

In this Section, the upper bounds of the empirical \( L^2 \) risk and the \( L^2 \) risk of the RKHS ridge group sparse estimator are presented in Theorem[1] and Corollary[1] respectively. Before stating these results, let us introduce some notation and assumptions that are needed in the rest of this paper.

For a function \( f \in \mathcal{H} \), let \( S_f \) be its support,

\[
S_f = \{ v \in \mathcal{P} : f_v \neq 0 \}. \quad (8)
\]

The RKHS construction as described in Section[2] insures that the following properties are satisfied:

- for all \( v \in \mathcal{P} \), the functions \( f_v \in \mathcal{H}_v \) are centered and are square-integrable, i.e.

\[
E_X(f_v(X_v)) = 0 \text{ and } E_X(f_v^2(X_v)) < \infty,
\]

- for all \( v, v' \in \mathcal{P}, v \neq v' \), the functions \( f_v \in \mathcal{H}_v \) and \( f_{v'} \in \mathcal{H}_{v'} \) are orthogonal with respect to \( L^2(\mathcal{X}, P_X) \), i.e.

\[
E_X(f_v(X_v)f_{v'}(X_{v'})) = 0.
\]

We assume moreover that,

- for all \( v \in \mathcal{P} \), the functions \( f_v \in \mathcal{H}_v \) are uniformly bounded, i.e.

\[
\exists R > 0 \text{ such that } \| f_v \|_{\infty} = \sup_{X_v} |f_v(X_v)| \leq R.
\]

Each kernel \( k_v, v \in \mathcal{P} \) is associated with an integral operator \( T_{k_v} \) from \( L^2(\mathcal{X}_v, P_v) \) to \( L^2(\mathcal{X}_v, P_v) \) defined by:

\[
\forall f \in L^2(\mathcal{X}_v, P_v), T_{k_v}(f) = \int_{\mathcal{X}_v} k_v(., t)f(t)dP_v(t).
\]
For each \( v \in \mathcal{P} \), let \( \omega_{v,1} \geq \omega_{v,2} \geq \ldots \geq 0 \) be the eigenvalues of the integral operator \( T_{k_v} \) (see Equation (19)). Let us define the function \( Q_{n,v}(t) \) for some positive \( t \) as follows:

\[
Q_{n,v}(t) = \sqrt{\frac{5}{n} \sum_{\ell \geq 1} \min(t^2, \omega_{v,\ell})},
\]

(9)

and for some \( \Delta > 0 \) let \( \nu_{n,v} \) be defined by:

\[
\nu_{n,v} = \inf_{t} \left\{ Q_{n,v}(t) \leq \Delta t^2 \right\}.
\]

(10)

For each \( v \in \mathcal{P} \), \( \nu_{n,v} \) refers to the minimax optimal rate for \( L^2(\mathcal{X}, \mathcal{P}_X) \)-estimation in the RKHS \( \mathcal{H}_v \) (Mendelson 2002).

**Remark 1.** The rate \( \nu_{n,v}, v \in \mathcal{P} \), depends on the regularity of the RKHS via the decreasing rate of the eigenvalues \( \{\omega_{v,\ell}\}_{\ell=1}^{\infty} \). When RKHS is of high regularity, i.e. when the eigenvalues \( \{\omega_{v,\ell}\}_{\ell=1}^{\infty} \) decrease quickly, then the rate \( \nu_{n,v}, v \in \mathcal{P} \) will be close to the parametric rate of convergence (see Section 3.1).

The choice of tuning parameters in the criterion \( \mathcal{L}(f) \) is specified in terms of the following quantity:

\[
\lambda_{n,v} = \max \left( \nu_{n,v}, \sqrt{\frac{d}{n}} \right).
\]

(11)

**Theorem 1.** Consider the regression model defined at Equation (4) with \( \sigma = 1 \). Let \( \{(Y_i, X_i)\}_{i=1}^{n} \) be a \( n \)-sample with the same law as \((Y, X)\), and let \( \{\epsilon_i\}_{i=1}^{n} \) be the random errors that are independent and identically distributed (i.i.d.) like \( \epsilon \). Let also \( \hat{f} \) be defined by (7) with \( r_v = 1 \) in (6), and let the tuning parameters \( \mu_{v}'s \) and \( \gamma_v's \) be chosen as follows:

For some constant \( C_1 > 10 + 4\Delta \),

\[
\forall v \in \mathcal{P}, \quad \mu_{v} = C_1 \lambda_{n,v}^2, \quad \gamma_{v} = C_1 \lambda_{n,v}.
\]

(12)

If there exists positive constants \( C_2, C_3, \) and \( 0 < \beta < 1/\alpha \) such that the following assumptions are satisfied:

\[
\forall v \in \mathcal{P}, \quad n \lambda_{n,v}^2 \geq -C_2 \log \lambda_{n,v},
\]

(13)

and

\[
\forall f \in \mathcal{F}, \quad \sum_{v \in \mathcal{S}_f} \lambda_{n,v}^2 \leq C_3 n^{2\beta - 1},
\]

(14)

then, there exists \( 0 < \eta < 1 \) depending on constants \( \{C_1\}_{i=1}^{\infty}, \beta, \) and \( n \) (\( \eta \) tends to 0 as \( n \) increases), such that with probability greater than \( 1 - \eta \), we have for some constant \( C \),

\[
\|m - \hat{f}\|^2_n \leq C \inf_{f \in \mathcal{F}} \left\{ \|m - f\|^2_n + \sum_{v \in \mathcal{S}_f} (\mu_{v} + \gamma_{v}^2) \right\}.
\]

(15)

Let us now comment on the theorem.

**Remark 2.** Let \( f' \) be the function in \( \mathcal{F} \) such that the infimum of the right hand side of the inequality (15) is realized. The term \( \|m - f'\|^2_n \) is the usual bias term. It quantifies both the approximation properties of the RKHS \( \mathcal{H} \), and the bias-variance trade-off.

**Remark 3.** This result is similar to the one obtained in the Gaussian regression model at the cost of the additional Assumption (14). This assumption allows to obtain the same rate of convergence for the RKHS ridge group sparse estimator as in the Gaussian regression model (see Huet et Taupin 2017). However, it implies some restrictions on the regularity of the RKHS \( \mathcal{H} \). Indeed, as for all \( v \in \mathcal{P}, \lambda_{n,v} \geq \nu_{n,v} \) (see Equation (19)), it follows that \( \sum_{v \in \mathcal{S}_f} \nu_{n,v}^2 \leq C_3 n^{2\beta - 1} \), which implies some restrictions on the regularity of the RKHS: if \( \beta \) is small, which will be the case if \( \alpha \) is large, then the RKHS should be of high regularity.

**Remark 4.** By Equation (14), we also have that for all \( v \in \mathcal{P}, \lambda_{n,v} \geq \sqrt{d/n} \). This assumption allows to control the probability of the \(|\mathcal{P}| \) events (see Equation (48)), where \( \log(|\mathcal{P}|) \) is of order \( d \).
Remark 5. The result in Theorem 1 can be generalized to the case where \( \sigma \neq 1 \) in Equation (1), and where \( r_v \neq 1 \) in (9).

Let \( \hat{g} \) be defined as follows:

\[
\hat{g} = \arg\min_{g \in \mathcal{F}} \left\{ \frac{\|Y\|}{\sigma} - g \|n\| \frac{1}{\sigma} \sum_v \gamma_v \|g_v\| + \frac{1}{\sigma} \sum_v \mu_v \|g_v\|_{\mathcal{H}_v} \right\},
\]

with

\[
\mathcal{F}' = \left\{ g : g = g_0 + \sum_v g_v, \text{ with } g_v \in \mathcal{H}_v, \text{ and } \|g_v\|_{\mathcal{H}_v} \leq \frac{r_v}{\sigma} \right\}.
\]

We have \( \hat{f} = \sigma \hat{g} \) for \( \hat{f} \) being defined by (7).

For all \( u > 0 \), let \( \mathcal{H}_v^u \) be the RKHS associated with the kernel \( u_k \). If \( u = r_v^2/\sigma^2 \), then

\[
\hat{g} = \arg\min_{g \in \mathcal{F}'} \left\{ \frac{\|Y\|}{\sigma} - g \|n\| \frac{1}{\sigma} \sum_v \gamma_v \|g_v\| + \frac{1}{\sigma} \sum_v \mu_v r_v \|g_v\|_{\mathcal{H}_v^u} \right\},
\]

where

\[
\mathcal{F}'' = \left\{ g : g = g_0 + \sum_v g_v, \text{ with } g_v \in \mathcal{H}_v^u, \text{ and } \|g_v\|_{\mathcal{H}_v^u} \leq \frac{1}{\sigma} \right\}.
\]

We apply Theorem 1 with \( Y/\sigma \) and \( m/\sigma \) in place of \( Y \) and \( m \), to \( \hat{g} \) defined as above.

Let

\[
Q_{n,v}(t) = \sqrt{\frac{\sigma}{n}} \sum_{\ell \geq 1} \min(t^2, u_{\ell,v}, t),
\]

and for \( \Delta' > 0 \), let

\[
\nu_{n,v}(\Delta') = \inf_t \left\{ Q_{n,v}(t) \leq \Delta' t^2 \right\}.
\]

Let also

\[
\lambda_{n,v} = \max \left( \nu_{n,v}, \sqrt{\frac{d}{n}} \right).
\]

For some constant \( C_1 > 10 + \Delta' \), take

\[
\frac{\mu_v r_v}{\sigma^2} = C_1 \left( \lambda_{n,v} \right)^2, \quad \frac{\gamma_v}{\sigma} = C_1 \lambda_{n,v}.
\]

Then, for \( S_g \) being defined as follows

\[
S_g = \{ v \in \mathcal{P} : g_v \neq 0 \},
\]

we have

\[
\frac{m}{\sigma} - \hat{g} \|n\| \leq C \inf_{g \in \mathcal{F}''} \left\{ \frac{m}{\sigma} - g \|n\| \frac{1}{\sigma^2} \sum_v (\mu_v r_v + \gamma_v^2) \right\},
\]

or, multiplying both sides by \( \sigma^2 \), and taking \( u = r_v^2/\sigma^2 \),

\[
\|m - \sigma \hat{g}\|_2^2 \leq C \inf_{g \in \mathcal{F}''} \left\{ \|m - \sigma g\|_n^2 + \sum_{v \in S_g} (\mu_v r_v + \gamma_v^2) \right\}.
\]

Corollary 1. Under the same assumptions as Theorem 1, we have with high probability for some constant \( C' \) that,

\[
\|m - \hat{f}\|_2^2 \leq C' \inf_{f \in \mathcal{F}} \left\{ \|m - f\|_n^2 + \|m - f\|_2^2 + \sum_{v \in S_f} (\mu_v + \gamma_v^2) \right\}.
\]

Remark 6. The result in Corollary 1 can be generalized to the case where \( \sigma \neq 1 \) in Equation (1), and where \( r_v \neq 1 \) in (9). It suffices to apply Corollary 1 with \( Y/\sigma \) and \( m/\sigma \) in place of \( Y \) and \( m \), to \( \hat{g} \) as defined in Equation (10). Then, with similar demonstration as in Remark 5 we obtain,

\[
\|m - \sigma \hat{g}\|_2^2 \leq C' \inf_{g \in \mathcal{F}''} \left\{ \|m - \sigma g\|_n^2 + \|m - \sigma g\|_2^2 + \sum_{v \in S_g} (\mu_v r_v + \gamma_v^2) \right\},
\]

where \( \mathcal{F}' \) and \( S_g \) are defined in Equations (17) and (18), respectively.
3.1 Rate of convergence

**Corollary 2.** Under the same assumptions as Theorem 7 we have

\[
\|m - \hat{f}\|_n^2 \leq C \inf_{f \in \mathcal{F}} \left\{ \|m - f\|_n^2 + \left( \sum_{v \in S_f} \nu_{n,v}^2 + \frac{d|S_f|}{n} \right) \right\}.
\]

This Corollary highlights that the upper bound is relevant when the infimum is reached for functions \( f \) that have a sparse decomposition in \( \mathcal{H} \), i.e. \(|S_f|\) is small, and when \( d \) is small face to \( n \). When \( d \) is large, the decomposition of functions in \( \mathcal{H} \) should be limited to interactions of a limited order, so that the number of elements in the estimated meta-model is of order smaller than \( d^r \) for some small \( r \), say \( r = 2 \) for example. In such a case, the cardinality of \( \mathcal{P} \) will be smaller than \( d^r \). As we mentioned in Remark 3 the assumption \( \lambda_{n,v} \geq \sqrt{d/n} \) is needed to control the value \( \log(|\mathcal{P}|) \), which will be now smaller than \( 2 \log(d) \). Therefore, the value \( d \) in the definition of \( \lambda_{n,v} \) (see Equation (11)) as well as the term \( d|S_f|/n \) in the infimum above will be replaced by \( 2 \log(d) \) and \( 2 \log(d)|S_f|/n \), respectively.

Let us discuss the rate of convergence given by \( \sum_{v \in S_f} \nu_{n,v}^2 \). For the sake of simplicity we consider the case where the variables \( X_1, \ldots, X_d \) have the same distribution \( P_1 \) on \( \mathcal{X}_1 \subset \mathbb{R} \), and where the unidimensional kernels \( k_{0a} \) are all identical, such that \( k_0(X_v, X'_v) = \prod_{a \in v} k_{0}(X_a, X'_a) \). The kernel \( k_0 \) admits an eigen expansion given by

\[
k_0(X_a, X'_a) = \sum_{\ell_a \geq 1} \omega_{0,\ell_a} \phi_{\ell_a}(X_a) \phi_{\ell_a}(X'_a),
\]

where the eigenvalues \( \{\omega_{0,\ell_a}\}_{\ell_a=1}^\infty \) are non-negative and ranged in the decreasing order, and where the \( \{\phi_{\ell_a}\}_{\ell_a=1}^\infty \) are the associated eigenfunctions, orthonormal with respect to \( L^2(\mathcal{X}_1, P_1) \). Therefore, the kernel \( k_v \) admits the following expansion,

\[
k_v(X_v, X'_v) = \sum_{\ell=(\ell_1, \ldots, \ell_v)} \prod_{a=1}^{|v|} \omega_{0,\ell_a} \phi_{\ell_a}(X_a) \phi_{\ell_a}(X'_a).
\]

Consider the case where the eigenvalues \( \{\omega_{0,\ell_a}\}_{\ell_a=1}^\infty \) are decreasing at a rate \( \ell_a^{-2\alpha'} \) for some \( \alpha' > 1/2 \), i.e. the \( \omega_{0,\ell} \) are of order \( \ell^{-2\alpha'} = (\prod_{a=1}^{|v|} \ell_a)^{-2\alpha'} \). It is shown in Section 8.3. of (Huet and Taupin 2017), that

\[
\nu_{n,v} \propto n^{-\frac{\alpha'}{2\alpha' + 11} (\log n)^{\gamma'}},
\]

where the \( \nu_{n,v} \) is defined at Equation (10) and

\[
\gamma' \geq (|v| - 1) \frac{\alpha'}{2\alpha' - 1}.
\]

For all \( f \in \mathcal{F} \) we have then,

\[
\sum_{v \in S_f} \nu_{n,v}^2 \propto |S_f| n^{-\frac{\alpha'}{2\alpha' + 11} (\log n)^{2\gamma'}}.
\]

Note that in this particular case, the rate of convergence depends on \( |v| \) through the logarithmic term \((\log n)^{2\gamma'}\), and that up to this logarithmic term the rate of convergence has the same order than the usual non-parametric rate for unidimensional functions. It follows that the RKHS space \( \mathcal{H} \) should be chosen such that the unknown function \( m \) is well approximated by sparse functions in \( \mathcal{H} \) with low order of interactions.

Besides, the rate \( \nu_{n,v} \) should satisfy assumption (14),

\[
\sum_{v \in S_f} \nu_{n,v}^2 \leq C_3 n^{2\beta - 1},
\]

which holds if

\[
\alpha' > \frac{1 - 2\beta}{4\beta} > \alpha - \frac{2}{4}.
\]

This shows that for the large values of \( \alpha \) the assumption (14) implies some restrictions on the regularity of the RKHS chosen: If \( \alpha < 4 \), then all \( \alpha' \) greater than \( 1/2 \) satisfy Equation (20), since \((\alpha - 2)/4 < 1/2\). If \( \alpha \geq 4 \), then we have \( \alpha' > (\alpha - 2)/4 > 1/2 \). As \( \alpha \) increases, i.e. \( \beta \) decreases (recall that \( 0 < \beta < 1/\alpha \)), and assumption (14) implies that the RKHS chosen should be of high regularity.
4 Main arguments of the proof of Theorem 1 and motivation for the choice $\pi_\alpha$

The proof of Theorem 1 starts in the same way as the proof of Theorem 2.1. in (Huet and Taupin 2017) where they considered the Gaussian regression model. However, it differs in two essential points:

1. Sudakov type minoration,
2. Concentration inequality.

In the following Section, we give a sketch of the proof of Theorem 1, we highlight the two points above that differs the proof from the proof in the Gaussian regression model, and we provide a detailed comparison to the related works. In Section 4.2 we give a brief introduction to the Sudakov type minoration context, we explain the motivation for choosing densities $\pi_\alpha \in \mathcal{D}$ defined in Equation (2), and we state in Corollary 3 the appropriate Sudakov minoration used in the proof of Theorem 1. In Section 4.3 we present the concentration inequality context, and we state in Corollary 4 the appropriate concentration inequality used in the proof of Theorem 1.

4.1 Sketch of the proof

We give here a sketch of the proof of Theorem 1 and we postpone to Section 5 for complete statements. We begin by introducing some notation.

We denote by $C$ constants that vary from an equation to the other. For each $v \in \mathcal{P}$, and for a function $\phi : \mathbb{R}^{[n]} \to \mathbb{R}$, we denote by $V_{n,v}$ the empirical process defined as,

$$V_{n,v}(\phi) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \phi(X_{v,i}).$$

For all $v \in \mathcal{P}$, let $\mathcal{H}_v$ be the RKHS associated with the reproducing kernel $k_v$. For any function $g_v \in \mathcal{H}_v$, $v \in \mathcal{P}$, and $V_{n,v}$ being defined in Equation (21), we consider two following processes,

$$W_{n,2,v}(t) = \sup \left\{ |V_{n,v}(g_v)|, \|g_v\|_{\mathcal{H}_v} \leq 2, \|g_v\|_2 \leq t \right\},$$

$$W_{n,n,v}(t) = \sup \left\{ |V_{n,v}(g_v)|, \|g_v\|_{\mathcal{H}_v} \leq 2, \|g_v\|_n \leq t \right\}.$$  

(22)

(23)

Starting from the definition of $\hat{f}$, some simple calculations give that for all $f \in \mathcal{F}$,

$$C \|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + |V_{n,v}(\hat{f} - f)| + \sum_{v \in \mathcal{S}_f} [\gamma_v \|\hat{f}_v - f_v\|_n + \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v}]$$

$$- \sum_{v \in \mathcal{S}_f} [\mu_v \|\hat{f}_v\|_{\mathcal{H}_v} + \gamma_v \|\hat{f}_v\|_n],$$

$$\leq \|m - f\|_n^2 + |V_{n,v}(\hat{f} - f)| + \sum_{v \in \mathcal{S}_f} [\gamma_v \|\hat{f}_v - f_v\|_n + \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v}].$$

(24)

If we set $g = \hat{f} - f$, then $g \in \mathcal{H}$, $g = g_0 + \sum_v g_v$, with $g_v = \hat{f}_v - f_v$, and for each $v$, $\|g_v\|_{\mathcal{H}_v} \leq 2$. The main problem is now to control the empirical process $V_{n,v}$. For each $v$, letting $\lambda_{n,v}$ as in (14), we state (see Lemma 3, page 17) that, with high probability,

$$|V_{n,v}(g_v)| \leq C \lambda_{n,v}^2 \|g_v\|_{\mathcal{H}_v} + C \lambda_{n,v} \|g_v\|_n.$$  

(25)

One of the key points in the proof of Lemma 3 is to find an upper bound for the two following quantities:

$$|W_{n,n,v}(t) - E_{\varepsilon}(W_{n,n,v}(t))|, \text{ and } |W_{n,2,v}(t) - E_{\varepsilon}(W_{n,2,v}(t))|.$$  

In the Gaussian regression model, one uses the isoperimetric inequality for Gaussian processes in (Massart and Picard 2007). When dealing with errors that are not distributed as a Gaussian distribution, different tools are needed to obtain the upper bounds for the quantities in Equation (25) (see Section 4.3 for a complete discussion of this point of the proof). Let us continue the sketch of the proof before coming back to this point.
If for all $v, \mu_v$ and $\gamma_v$ satisfying Equation (12), by using Equation (24) we deduce that with high probability,

$$C\|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + \sum_{v \notin S_f} [\gamma_v \|g_v\|_n + \mu_v \|g_v\|_{\mathcal{H}_v}] + \sum_{v \notin S_f} [\gamma_v \|\hat{f}_v\|_n + \mu_v \|\hat{f}_v\|_{\mathcal{H}_v}].$$

Besides, we can express the decomposability property of the penalty as follows (see lemma [4] page 18):

Putting the things together, and using that $\|g_v\|_{\mathcal{H}_v} \leq 2$, we obtain the following upper bound:

$$C\|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + \sum_{v \notin S_f} [\mu_v + \gamma_v \|g_v\|_n].$$

The last important step consists in comparing $\sum_{v \in S_f} \|g_v\|_n$ to $\sum_{v \in S_f} \|g_v\|_n$. To do so, we show first (see lemma [5] page 18) that for all $v \in \mathcal{P}$, with high probability,

$$\|g_v\|_n \leq 2\|g_v\|_2 + \gamma_v.$$

Using inequality above and that for all positive $K$, $2ab \leq (1/K)a^2 + Kb^2$ we obtain,

$$C\|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + \sum_{v \notin S_f} (\mu_v + \gamma_v^2) + \sum_{v \in S_f} \|g_v\|_2^2.$$  
$$\leq \|m - f\|_n^2 + \sum_{v \notin S_f} (\mu_v + \gamma_v^2) + \sum_{v \in \mathcal{P}} \|g_v\|_2^2.$$  

Then we use the orthogonality assumption between the spaces $\mathcal{H}_v$,

$$\sum_{v \in \mathcal{P}} \|g_v\|_2^2 = \|\sum_{v \in \mathcal{P}} g_v\|_2^2 = \|g\|_2^2,$$

which allows us to obtain the following result:

$$C\|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + \sum_{v \notin S_f} (\mu_v + \gamma_v^2) + \|\hat{f} - f\|_2^2.$$  

It remains now to consider different cases according to the rankings of $\|\hat{f} - f\|_2^2$ and $\|\hat{f} - f\|_n^2$ to get the result of Theorem [1].

If $\|\hat{f} - f\|_2^2 \leq \|\hat{f} - f\|_n$ the result is obtained by a simple rearrangement of the terms.

If $\|\hat{f} - f\|_2^2 \geq \|\hat{f} - f\|_n$, under some suitable assumptions it is shown (see Lemma [6] page 19) that with high probability we have

$$\|\hat{f} - f\|_2 \leq \sqrt{2}\|\hat{f} - f\|_n.$$  

One of the steps to prove the inequality above is to lower bound the expectation of the supremum of the empirical process, i.e. $\mathbb{E}_\varepsilon \sup_X |V_{n,\varepsilon}(g)|$ by a function of the covering number of the functional class under study, say $\mathcal{G}$. In order to solve this step in the Gaussian regression model one may use the Sudakov minoration in [Pisier 1989], for which the minoration is obtained thanks to the Slepian’s Lemma. The Slepian’s Lemma is specific to the Gaussian setting, and it does not hold when dealing with errors that are not distributed as a centered Gaussian distribution.

In the regression model (see Equation (11)) with error $\varepsilon$ that is distributed with density proportional to $\pi_\alpha \in \mathcal{D}$, the proof of the upper bound stated in Theorem [1] needs two following mathematical tools:

Point 1. a Sudakov type minoration to link the covering number on a class $\mathcal{G}$ to the expectation of the supremum of the empirical process over this class $\mathcal{G}$, $\mathbb{E}_\varepsilon \sup_{g \in \mathcal{G}} |V_{n,\varepsilon}(g)|$, and conclude Lemma [6].

Point 2. a concentration inequality to bound the quantities defined in Equation (25) which leads to bound the empirical process $V_{n,\varepsilon}$ and conclude Lemma [3].
The Point 1. is solved using a Sudakov type minoration which is a consequence of the result obtained by (Talagrand 1994). More precisely, it can be shown (see Corollary 3 page 14 that for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ being i.i.d. random variables distributed with density $\pi_\alpha \in D$ (see Equation (2)) and for all $\delta > 0$, we have,

$$
\frac{1}{K} \log N(\delta, \mathcal{G}, ||.||) \leq \left( \frac{2nE_{\varepsilon} \sup_{g \in \mathcal{G}} |V_{n,\varepsilon}(g)|}{\delta} \right)^2 1_{\{2nE_{\varepsilon} \sup_{g \in \mathcal{G}} |V_{n,\varepsilon}(g)| < \infty\}(\delta)} + \left( \frac{2nE_{\varepsilon} \sup_{g \in \mathcal{G}} |V_{n,\varepsilon}(g)|}{\delta} \right)^\alpha 1_{\{0,2nE_{\varepsilon} \sup_{g \in \mathcal{G}} |V_{n,\varepsilon}(g)|\}(\delta)},
$$

(26)

where $K$ is a constant that depends on $\alpha$ only, $||.||$ is the Euclidean norm, $N(\delta, \mathcal{G}, ||.||)$ is the $\delta$-covering number of the metric space $(\mathcal{G}, ||.||)$, and $1_A : A \rightarrow \{0, 1\}$ is the indicator function of $A \subset \mathcal{A}$, i.e.

$$
1_A(a) = \begin{cases} 
1 & \text{if } a \in A, \\
0 & \text{if } a \notin A.
\end{cases}
$$

The proof of Lemma 6 proceeds using Equation (24) and is concluded under the Hypothesis (12) and (13).

The Point 2. is solved using a concentration inequality (see Corollary 3 page 16) which is a consequence of the result obtained by (Shu and Strzelecki 2017).

The appropriate results to solve Point 1. and Point 2. are stated in Corollary 3 in Section 4.2.2 and Corollary 4 in Section 4.3.2 respectively.

4.1.1 Comparison with related works

- (Meier, van de Geer, and Buhlmann 2009) considered a least-squares criterion penalized by a penalty function similar to the one we consider in our work. Their estimator of the unknown function $m$ has an univariate additive decomposition, i.e. decomposition (7) limited to the main effects.

They used a compatibility condition to compare the sum of the empirical $L^2$-norm of the univariate functions to the empirical $L^2$-norm of the sum of the univariate functions. More precisely,

Let $S^* = \{a \in \{1, \ldots, d\}, ||f_a||_n \neq 0\}$, then for $C(f_a)$ being a term depending on the functions $f_a$, $a \in S^*$,

$$
\sum_{a \in S^*} ||f_a||_n^2 \leq \sum_{a=1}^d ||f_a||_n^2 + C(f_a).
$$

The control of the Empirical process is done in their Lemma 1. This lemma is proved using Lemma 8.4 in (van de Geer, Gill, Ripley, Ross, Silverman, and Stein 2000), for which the errors should have sub-Gaussian tails, i.e.

$$
\max_i E\left(\exp\left(\frac{\varepsilon_i^2}{C_1}\right)\right) \leq C_2,
$$

where $C_1$ and $C_2$ are constants.

Afterwards, it was shown by (Raskutti, Wainwright, and Yu 2012) (see Section 3.4. of their paper) that the convergence rate of this estimator is sub-optimal.

- (Koltchinskii and Yuan 2010) considered a large class of loss functions, called loss functions of quadratic type, which satisfies the boundedness conditions. More precisely, for $l$ being a loss function, they assume that $l(Y, \cdot)$ is uniformly bounded from above by a numerical constant. So for a given distribution of the observations $Y$, there may exists a loss function that belongs to the class of the loss functions of quadratic type (see Section 2.1. of their paper for some examples).

They consider the input variables $X$ that may be not independent, and they do not assume that there is orthogonality between their RKHS, therefore $\|\sum_v f_v\|_2 \neq \sum_v ||f_v||_2$. Instead, in their Section 2.2., they introduce some geometric characteristics related to the degree of dependence of their RKHS, which insures almost orthogonality between these spaces.
The control of the empirical process is done in their Lemma 9. This lemma is proved under the global boundedness condition and the assumptions of the loss functions of quadratic type.

We consider the quadratic loss function to obtain an estimator of the function $m$ in the regression model defined in Equation (1), with error $\varepsilon$ that is non-bounded. This case is not included in the class of the loss functions of quadratic type. We do not impose the global boundedness condition. Instead, we assume that for all $v \in \mathcal{P}$ the functions $f_v$ are uniformly bounded. More precisely, the quantity $\sup_{X \in \mathcal{X}} |f_v(X)|$ is bounded from above by a constant. This assumption is easily satisfied as soon as the kernel $k_{\nu}$ is bounded on the compact set $\mathcal{X}$,

$$\sup_{X \in \mathcal{X}} |f_v(X)| \leq \sup_{X \in \mathcal{X}} \sqrt{k_{\nu}(X_v, X_v)} \|f_v\|_{H_v}.$$  

For a detailed discussion on this subject, we refer to the paper by (Raskutti, Wainwright, and Yu 2012).

- In the Gaussian regression model,
  - (Raskutti, Wainwright, and Yu 2012) assumed that the unknown function $m$ has a sparse univariate decomposition, where each component in its decomposition lies in a RKHS. They obtained an estimator for $m$, based on a ridge group sparse type procedure. They established upper and lower bounds on the risk in the $L^2$-norm and upper bound on the risk in the empirical $L^2$-norm.
  - (Huet and Taupin 2017) assumed that the unknown function $m$ admits a Hoeffding decomposition involving the main effects and interactions. They obtained a RKHS ridge group sparse estimator of a meta-model that approximates the Hoeffding decomposition of $m$. They established upper bounds on the risk in the $L^2$-norm and the empirical $L^2$-norm.

(Raskutti, Wainwright, and Yu 2012) and (Huet and Taupin 2017) do not assume global boundedness condition. Instead, they assume that for all $v \in \mathcal{P}$ the functions $f_v$ are uniformly bounded. The proof of their results relies on the empirical Gaussian process methods such as Sudakov minoration (Pisier 1989) and concentration inequalities for Gaussian processes.

As we are not in the Gaussian regression model, these methods could not be used in our work. We require new tools that we describe in details in the two next Sections.

### 4.2 Sudakov minoration

In the following Section, we recall the definition of the covering numbers, the statement of the classical Sudakov minoration, which is specific to the Gaussian process, and the generalized Sudakov minoration known also as the Sudakov minoration principal, which could be applied to some other processes. In Section 4.2.2 we state the appropriate Sudakov type minoration to the process associated with the random variables that are distributed with density $\pi_\alpha \in \mathcal{D}$ (see Equation (2) in Corollary 3).

#### 4.2.1 Introduction

Let $T$ be a set of square-integrable functions, i.e. $T \subset L^2$, and $\|\cdot\|$ be the Euclidean norm. For any $\delta > 0$, we denote by $C(\delta, T, \|\cdot\|)$ the $\delta$-covering set of the metric space $(T, \|\cdot\|)$:

$$C(\delta, T, \|\cdot\|) = \left\{ f^1, \ldots, f^N : \forall f \in T, \exists k \in \{1, \ldots, N\} \text{ such that } \|f - f^k\| \leq \delta \right\}.$$  

The $\delta$-covering number of $(T, \|\cdot\|)$, denoted $N(\delta, T, \|\cdot\|)$, is the cardinal of the smallest covering set. A proper covering restricts the covering to use only elements in the set $T$. It can be shown that the covering numbers and the proper covering numbers are related by the following inequality:

$$N(\delta, T, \|\cdot\|) \leq N_{\text{proper}}(\delta, T, \|\cdot\|) \leq N\left(\frac{\delta}{2}, T, \|\cdot\|\right).$$  

(27)

Consider a random variable $Z$ such that $E(Z^2) < \infty$, and consider an i.i.d. sequence $\{Z_i\}_{i=1}^n$ distributed like $Z$. To each $t = (t_1, \ldots, t_n)$ of $T \subset L^2$ one can associate the process $V_t = \sum_{i=1}^n Z_i t_i$, $t \in T$. 

12
In order to link the covering number on a class $T$, i.e. $N(\delta, T, \|\cdot\|)$, to the expectation of the supremum of the process $V_t = \sum_{i=1}^{n} Z_i t_i$ in the Gaussian setting, the classical Sudakov minoration could be used ([Pisier 1989]):

$$\frac{1}{K} \log N(\delta, T, \|\cdot\|) \leq \left( \frac{n E_Z \sup_{t \in T} \sum_{i=1}^{n} Z_i t_i}{\delta} \right)^2. \quad (28)$$

When dealing with the processes $V_t = \sum_{i=1}^{n} Z_i t_i$, $t \in T$ associated with the random variables $\{Z_i\}_{i=1}^{n}$ that are not Gaussian, a generalized Sudakov minoration, known also as the Sudakov minoration principal, could be used to lower bound the value $E_Z \sup_{t \in T} \sum_{i=1}^{n} Z_i t_i$. Let us recall this inequality.

**Definition 1.** (Definition 1.1. in [Latała 2014]) Let $Z = (Z_1, ..., Z_n)$ be a random vector in $\mathbb{R}^n$. We say that $Z$ satisfies the $L_p$-Sudakov minoration principle with a constant $K' > 0$, $\text{SMP}_p(K')$, if for any set $T \subset \mathbb{R}^n$ with $|T| > \exp(p)$ such that

$$\left( E_Z \sum_{t,s \in T} | \sum_{i=1}^{n} (t_i - s_i)Z_i |^p \right)^{1/p} := \| \sum_{i=1}^{n} (t_i - s_i)Z_i \|_p \geq \delta, \forall s, t \in T, s \neq t, \quad (29)$$

we have

$$K' \delta \leq E_Z \sup_{t,s \in T} \sum_{i=1}^{n} (s_i - t_i)Z_i.$$

A random vector $Z$ satisfies the Sudakov minoration principle with a constant $K'$, $\text{SMP}(K')$, if it satisfies $\text{SMP}_p(K')$ for any $p \geq 1$.

If $\{Z_i\}_{i=1}^{n}$ are independent symmetric ±1 random variables or equivalently if the vector $Z = (Z_1, ..., Z_n)$ is uniformly distributed on the cube $[-1, 1]$ the Sudakov minoration principal with universal $K'$ was proven by [Talagrand 1993].

[Latała 2014] proved the Sudakov minoration principal for the independent log-concave random variables. A measure on $\mathbb{R}^n$ with the full dimensional support is log-concave if and only if it has a density of the form $\exp(-\phi(x))$, where $\phi : \mathbb{R}^n \to (-\infty, \infty]$ is convex ([Borell 1974]). In the dependent setting the Sudakov minoration principal for the log-concave random variables was proven by [Bednorz 2014].

As we are in the independent setting and the densities $\pi_\alpha \in D$ (see Equation (2)) are log-concave, the Sudakov minoration obtained by [Latała 2014] holds in our context. However, we could not deduce from the result obtained by [Latała 2014] the adapted Sudakov type minoration that leads to obtain the optimal rate of convergence for our estimator. By optimal we mean the same rate of convergence as in the Gaussian regression setting (see [Huet and Taupin 2017]). This is the reason why we restricted ourselves to the densities $\pi_\alpha \in D$ for which there exists a result given by [Talagrand 1994].

In the next Section we provide in Corollary 3 the appropriate Sudakov type minoration for the random variables that are distributed with density $\pi_\alpha \in D$. This Corollary is a consequence of the result obtained by [Talagrand 1994].

### 4.2.2 Sudakov minoration for density $\pi_\alpha$

In this Section we state in Corollary 3 the Sudakov minoration appropriate for the random variables that are distributed with density $\pi_\alpha \in D$ (see Equation (2)). This Corollary is a consequence of the Sudakov minoration stated in Theorem 3.1. in [Talagrand 1994]. We start by introducing some notation that we need in the rest of this Section.

Let us denote by $\tilde{\alpha}$ the conjugate exponent of $\alpha$, i.e. $1/\alpha + 1/\tilde{\alpha} = 1$. So, for all $\alpha > 2$ we have $1 < \tilde{\alpha} < 2$.

We consider the sets $B_{\tilde{\alpha}}$ and $U_{\tilde{\alpha}}(u)$, $u \geq 0$ defined as follows:

$$B_{\tilde{\alpha}} = \left\{ x \in \mathbb{R}^n : \sum_{k=1}^{n} |x_k|^{\tilde{\alpha}} \leq 1 \right\}, \quad (30)$$

and

$$U_{\tilde{\alpha}}(u) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \eta_{\tilde{\alpha}}(x_i) \leq u, \; u \geq 0 \right\}, \quad (31)$$

13
Corollary 3. Under the same assumptions as for Theorem 2 we have for all $K$ where $\eta_\alpha(x_i) = x_i^2 1_{[-1,1]}(x_i) + |x_i|^{\alpha} 1_{(-\infty,-1]\cap[1,\infty)}(x_i)$.

For $T \subset L^2$ and $u \geq 0$, let $D(T, U_\tilde{\alpha}(u))$ be a covering set of translates of $T$ by $U_\tilde{\alpha}(u)$:

$$D(T, U_\tilde{\alpha}(u)) = \left\{ f^1, ..., f^N : \forall f \in T, \exists k \in \{1, ..., N\} \text{ such that } f - f^k \in U_\tilde{\alpha}(u) \right\},$$

$$= \left\{ f^1, ..., f^N : \forall f \in T, \exists k \in \{1, ..., N\} \text{ such that } \sum_{i=1}^{N} \eta_\tilde{\alpha}(f_i - f^k) \leq u \right\}.$$  

We denote by $N(T, U_\tilde{\alpha}(u))$ the minimum number of translates of $U_\tilde{\alpha}(u)$ by elements of $T$ needed to cover $T$.

**Lemma 1.** For all $\tilde{\alpha} \leq 2$ and $u \geq 0$, it is shown that ([Talagrand 1994]):

$$U_\tilde{\alpha}(u) \subset (u^{1/2}B_2 + u^{1/\tilde{\alpha}}B_\tilde{\alpha}).$$

(32)

**Remark 7.** If $\tilde{\alpha} \leq 2$ and $u \geq 0$, then

$$U_\tilde{\alpha}(u) \subset 2 \times \text{max}(u^{1/2}, u^{1/\tilde{\alpha}})B_2.$$

The proof of Remark 7 is given in Section 7.1 page 36.

**Theorem 2.** (Theorem 3.1. in [Talagrand 1994]) Let $Z = (Z_1, ..., Z_n)$ be i.i.d. random variables distributed with density $\pi_\alpha \in \mathcal{D}$ defined in Equation 32, $U_\tilde{\alpha}(u)$, $u \geq 0$ be defined by (31) and $T \subset L^2$. Set

$$M = E_Z \sup_{t \in T} \sum_{i=1}^{n} t_i Z_i,$$

then it is shown that:

$$N(T, U_\tilde{\alpha}(M)) \leq \exp(KM),$$

(34)

where $K$ is a constant that depends on $\alpha$ only.

**Remark 8.** According to Theorem 2 and Remark 7 for all $u \geq 0$ we have,

$$N(2 \times \text{max}(u^{1/2}, u^{1/\tilde{\alpha}}), T, \|\|) \leq N(T, U_\tilde{\alpha}(u)) \leq \exp(Ku).$$

(35)

To be more precise, since $1 < \tilde{\alpha} < 2$ we have

(i) For $u \leq 1$, $u^{1/\tilde{\alpha}} \leq u^{1/2}$ and $N(2u^{1/2}, T, \|\|) \leq \exp(Ku)$.

(ii) For $u \geq 1$, $u^{1/\tilde{\alpha}} \geq u^{1/2}$ and $N(2u^{1/\tilde{\alpha}}, T, \|\|) \leq \exp(Ku)$.

**Corollary 3.** Under the same assumptions as for Theorem 2 we have for all $\delta > 0$,

$$\frac{1}{K} \log N(\delta, T, \|\|) \leq \left( \frac{2M}{\delta} \right) \eta_1(0,2M)(\delta) + \left( \frac{2M}{\delta} \right)^2 \eta_1(2M,\infty)(\delta),$$

which is exactly Equation 20 with $M$ defined in Equation 33.

The proof of Corollary 3 is given in Section 7.2 page 36.

**4.3 Concentration inequality**

We start this Section with a small introduction on the concentration inequalities context in Section 4.3.1 and we detail the concentration inequality used in our work in Section 4.3.2.
4.3.1 Introduction

Let $Z = (Z_1, ..., Z_n)$ be a random vector in $\mathbb{R}^n$, and the function $\phi$ from $\mathbb{R}^n$ to $\mathbb{R}$ be convex and 1–Lipschitz with respect to the Euclidean norm on $\mathbb{R}^n$, i.e.

$$\|\phi(Z) - \phi(Z')\| \leq \|Z - Z'\|, \ Z, Z' \in \mathbb{R}^n.$$ 

We are interested in the concentration inequalities of order two that provide bounds on how $\phi(Z)$ deviates from its expected value. More precisely, for $P$ being the probability measure on $\mathbb{R}^n$, and for all $u \geq 0$,

$$P\left(\|\phi(Z) - E(\phi(Z))\| \geq u\right) \leq C_1 \exp\left(-\frac{u^2}{C_2}\right),$$

(36)

where $C_1$, and $C_2$ are constants.

It was shown by (Bobkov and Ledoux 1997) that, if $Z = (Z_1, ..., Z_n)$ is a centered Gaussian random vector in $\mathbb{R}^n$, then:

$$P\left(\|\phi(Z) - E(\phi(Z))\| \geq u\right) \leq 4 \exp\left(-\frac{u^2}{2}\right).$$

This result could be proved using an inequality established by geometric arguments and an induction on the number of coordinates.

After that, an alternative approach to some of Talagrand’s inequalities was proposed by (Ledoux 1997) based on the log-Sobolev inequalities. He showed that if the probability measure $P$ on $[0, 1]^n$ satisfies the log-Sobolev inequality then it satisfies the concentration inequalities of the form (36), i.e. the log-Sobolev inequality implies the deviation inequality.

We say that the probability measure $P$ satisfies the log-Sobolev inequality for a class of functions $\Psi$ with loss function $R : \mathbb{R}^n \rightarrow [0, +\infty)$, if for every $\psi \in \Psi$ we have,

$$\text{Ent}(\exp(\psi)) \leq CE(R(\nabla \psi) \exp(\psi)),$$

where $\nabla \psi$ is the usual gradient of $\psi$, and $\text{Ent}(\exp(\psi))$ is the usual entropy of $\exp(\psi)$, i.e.

$$\text{Ent}(\exp(\psi)) = E(\psi \exp(\psi)) - E(\exp(\psi)) \log(E(\exp(\psi))).$$

This inequality was first introduced by (Gross 1975) with $R(x) = \|x\|^2, \ x \in \mathbb{R}^n$ and $\Psi$ being the class of $C^1$ functions. A lot of work has been done with different loss and class of functions, see for example (Bobkov and Ledoux 1997), (Gentil, Guillin, and Miclo 2005, Gentil, Guillin, and Miclo 2007).

In the rest of this paper, we assume that $\Psi$ is the class of convex functions, and we consider only the quadratic loss $R(x) = \|x\|^2, \ x \in \mathbb{R}^n$. Therefore, the probability measure $P$ satisfies the convex log-Sobolev inequality if,

$$E(\psi \exp(\psi)) - E(\exp(\psi)) \log(E(\exp(\psi))) \leq CE(\|\nabla \psi\|^2 \exp(\psi)).$$

(37)

(Adamczak 2004) found a sufficient condition for a class of probability distributions, denoted $\mathcal{M}(m, \rho^2)$ with $m > 0$ and $\rho \geq 0$, on the real line, to satisfy the convex log-Sobolev inequality. He deduced then the following concentration inequality which is satisfied for all probability distributions belonging to $\mathcal{M}(m, \rho^2)$:

$$P\left(\phi(Z) - E(\phi(Z)) \geq u\right) \leq \exp\left(-\frac{u^2}{4C(m, \rho^2)}\right).$$

(38)

We show in Lemma 2 that the probability distributions associated with the densities $\pi_\alpha \in \mathcal{D}$ defined in Equation 2 belong to $\mathcal{M}(m, \rho^2)$, and so they satisfy the convex log-Sobolev inequality. As a consequence the concentration inequality (38) holds for them.

Recall that (see Section 4.1 page 9) we need concentration bounds for the lower and upper tails of $\phi(Z)$, while the concentration inequality (38) does not contain these two sides.

(Shu and Strzelecki 2017) gave a sufficient and necessary condition for a probability measure on the real line to satisfy the convex log-Sobolev inequality. They obtained concentration bounds for the lower and upper tails of convex functions of independent random variables which satisfy the convex log-Sobolev inequality.

The result obtained by (Shu and Strzelecki 2017) allows us to state in Corollary 4 the appropriate concentration inequality for the probability distributions associated with the densities $\pi_\alpha \in \mathcal{D}$. 

15
4.3.2 Concentration inequality for density $\pi_\alpha$

In this Section we give the definition of the class of probability distributions $\mathcal{M}(m, \rho^2)$ and some of its properties. We show in Lemma \[3\] that the probability distributions associated with the densities $\pi_\alpha \in \mathcal{D}$ (see Equation \[2\]) belong to $\mathcal{M}(m, \rho^2)$, and so they satisfy the convex log-Sobolev inequality \[37\]. Finally, we state in Corollary \[4\] the appropriate concentration inequality for our work which is a consequence of the concentration inequality stated in Corollary 1.7. of the paper by (Shu and Strzalecki 2017).

**Definition 2.** (Definition 4 in \[Adameczak 2005\]) For $m > 0$ and $\rho \geq 0$ let $\mathcal{M}(m, \rho^2)$ denote the class of probability distributions $\Pi$ on $\mathbb{R}$ for which

$$v^+(A) \leq \rho^2 \Pi(A),$$

for all sets $A$ of the form $A = [x, \infty)$, $x \geq m$ and

$$v^-(A) \leq \rho^2 \Pi(A),$$

for all sets $A$ of the form $A = (-\infty, x]$, $x \geq m$, where $v^+$ is the measure on $[m, \infty)$ with density $x \Pi([x, \infty))$ and $v^-$ is the measure on $(-\infty, -m]$ with density $-x \Pi((-\infty, x])$.

**Example 1.** (Example page 5 in \[Adameczak 2005\]) The absolutely continuous distributions $\Pi$ that satisfy for $t \geq m$,

$$\frac{d}{dt} \log \Pi([t, \infty)) \leq -\frac{t}{\rho^2} \text{ and } \frac{d}{dt} \log \Pi((-\infty, -t]) \leq -\frac{t}{\rho^2}.$$  \hspace{1cm} (39)

belong to $\mathcal{M}(m, \rho^2)$. In particular, if $\Pi$ has density of the form $\exp(-V(x))$ with $dV(x)/dx \geq x/\rho^2$ and $dV(-x)/dx \leq -x/\rho^2$ then $\Pi \in \mathcal{M}(1, \rho^2)$.

It is shown by (Adameczak 2005) that the probability distributions belonging to $\mathcal{M}(m, \rho^2)$ satisfy the convex log-Sobolev inequality \[37\]. Let us denote by $\Pi_\alpha$ the probability distribution associated with the density $\pi_\alpha \in \mathcal{D}$ defined in Equation \[2\]. In the following Lemma we will show that $\otimes \Pi_\alpha$ satisfies the convex log-Sobolev inequality \[37\].

**Lemma 2.** There exists some $m$ such that $\Pi_\alpha \in \mathcal{M}(m, \rho^2)$, and therefore $\otimes \Pi_\alpha$ satisfies the convex log-Sobolev inequality \[37\].

The proof of Lemma \[2\] is given in Section \[5.1\] page 36.

As $\Pi_\alpha \in \mathcal{M}(m, \rho^2)$ and they satisfy the convex log-Sobolev inequality \[37\], so the concentration bound \[38\] holds for them. Recall that (see Section \[4.1\] page 4), we need a concentration bound for the both upper and lower tails of a convex function of the random variables that are distributed as $\Pi_\alpha$. Therefore, the concentration bound \[38\] is not sufficient for our work. We state in Corollary \[4\] the appropriate concentration inequality for our work which is a consequence of the concentration inequality obtained by (Shu and Strzalecki 2017). This result holds under a supplementary condition that we will state in the following Remark.

**Remark 9.** Let $Z$ be a random variable distributed as $\Pi_\alpha$, then for every $s > 0$ the quantity $E(\exp(s|Z|))$ exists and is finite.

The proof of Remark \[9\] is given in Section \[5.2\] page 37.

Note that, if $\alpha < 2$ then $E(\exp(s|Z|)) \not< \infty$.

**Corollary 4.** Let $Z = (Z_1, ..., Z_n)$ be i.i.d. random variables distributed as $\Pi_\alpha$. Then there exists $A, B < \infty$ (depending only on $C$ in the log-Sobolev inequality \[37\]), such that for any convex (or concave) function $\phi : \mathbb{R}^n \to \mathbb{R}$ which is 1-Lipschitz (with respect to the Euclidean norm on $\mathbb{R}^n$) we have:

$$P\left(|\phi(Z) - E(\phi(Z))| \geq u\right) \leq 2B \exp \left(-\frac{u^2}{8A}\right), \hspace{1cm} u \geq 0.$$  \hspace{1cm} (40)

Corollary \[4\] is a consequence of the concentration inequality shown by (Shu and Strzalecki 2017):

$$P\left(|\phi(Z) - M(\phi(Z))| \geq u\right) \leq B \exp \left(-\frac{u^2}{A}\right), \hspace{1cm} u \geq 0,$$  \hspace{1cm} (41)

where $M$ is the median of $\phi(Z)$.

The proof of Corollary \[4\] is given in Section \[5.3\] page 38 and is based on the fact that the concentration inequalities around the mean and the median are equivalent up to a numerical constant (Milman and Schechtman 1986).
5 Proof of Theorem 1

The proof is based on four main lemmas proved in Section 5.2. In Section 5.1, other lemmas used all along the proof are stated.

Let us first establish inequalities that will be used in the following. Let $f \in \mathcal{H}$ and $v \in S_f$ (see (8)).

Using that for any $v \in S_f$, and any norm $\| \cdot \|$ in $\mathcal{H}_v$, $\|f_v\| - \|f_{\hat{v}}\| \leq \|f_v - f_{\hat{v}}\|$ and that for any $v \notin S_f$, $\|f_v\| = 0$, we get,

\[
\sum_{v \in P} \mu_v \|f_v\|_{\mathcal{H}_v} - \sum_{v \notin S_f} \mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} \leq \sum_{v \in S_f} \mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} - \sum_{v \notin S_f} \mu_v \|f_v\|_{\mathcal{H}_v},
\]

(42)

and,

\[
\sum_{v \in P} \gamma_v \|f_v\|_n - \sum_{v \notin S_f} \gamma_v \|f_v - f_{\hat{v}}\|_n \leq \sum_{v \in S_f} \gamma_v \|f_v - f_{\hat{v}}\|_n - \sum_{v \notin S_f} \gamma_v \|f_{\hat{v}}\|_n.
\]

(43)

Combining (42), and (43), to the fact that for any function $f \in \mathcal{H}$, $\mathcal{L}(\hat{f}) \leq \mathcal{L}(f)$, we obtain,

\[
\|m - \hat{f}\|_n^2 \leq \|m - f\|_n^2 + B,
\]

with

\[
B = 2V_{n,v}(\hat{f} - f) + \sum_{v \in S_f} [\mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} + \gamma_v \|f_v - f_{\hat{v}}\|_n] - \sum_{v \notin S_f} [\mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} + \gamma_v \|f_v - f_{\hat{v}}\|_n].
\]

(44)

If $\|m - f\|_n^2 \geq B$, we immediately get the result since in that case

\[
\|m - \hat{f}\|_n^2 \leq 2\|m - f\|_n^2 \leq 2\|m - f\|_n^2 + \sum_{v \in S_f} \mu_v + \sum_{v \notin S_f} \gamma_v^2.
\]

If $\|m - f\|_n^2 < B$, we get that

\[
\|\hat{f} - m\|_n^2 \leq 2B
\]

(45)

\[
\leq 4|V_{n,v}(\hat{f} - f)| + 2 \sum_{v \in S_f} [\mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} + \gamma_v \|f_v - f_{\hat{v}}\|_n].
\]

(46)

The control of the empirical process $|V_{n,v}(\hat{f} - f)|$ is given by the following lemma (proved in Section 5.2.4).

**Lemma 3.** Let $V_{n,v}$ be defined in (37). For any $f$ in $\mathcal{F}$, we consider the event $\mathcal{T}$ defined as

\[
\mathcal{T} = \left\{ \forall f \in \mathcal{F}, \forall v \in \mathcal{P}, |V_{n,v}(\hat{f} - f_a)| \leq \kappa \lambda_{n,v}^2 \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \kappa \lambda_{n,v} \|\hat{f}_v - f_v\|_n \right\},
\]

(47)

where $\lambda_{n,v}$ is defined in Equation (11) and where $\kappa = 10 + 4\Delta$. Then, for some positive constants $c_1$, $c_2$,

\[
P_{X,v}(\mathcal{T}) \geq 1 - c_1 \sum_{v \in \mathcal{P}} \exp(-nc_2\lambda_{n,v}^2).
\]

(48)

Conditioning on $\mathcal{T}$, Inequality (44) becomes

\[
\|\hat{f} - m\|_n^2 \leq 4\kappa \sum_{v \in \mathcal{P}} [\lambda_{n,v}^2 \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \lambda_{n,v} \|\hat{f}_v - f_v\|_n] + 2 \sum_{v \in S_f} [\mu_v \|f_v - f_{\hat{v}}\|_{\mathcal{H}_v} + \gamma_v \|f_v - f_{\hat{v}}\|_n],
\]

(49)

which may be decomposed as follows

\[
\|\hat{f} - m\|_n^2 \leq \sum_{v \in S_f} [4\kappa \lambda_{n,v}^2 + 2\mu_v] \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \sum_{v \in S_f} [4\kappa \lambda_{n,v} + 2\gamma_v] \|\hat{f}_v - f_v\|_n + 4 \sum_{v \notin S_f} \kappa \lambda_{n,v} \|\hat{f}_v - f_v\|_n.
\]

(50)
If we choose $C_1 \geq \kappa$ in Theorem 1 then $\kappa \lambda_{a,v}^2 \leq \mu_v$ and $\kappa \lambda_{n,v} \leq \gamma_v$ and the previous inequality becomes
\[
\|\hat{f} - m\|^2_n \leq 6 \sum_{v \in S_f} [\mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \gamma_v \|\hat{f}_v - f_v\|_n] + 4 \sum_{v \notin S_f} [\mu_v \|\hat{f}_v\|_{\mathcal{H}_v} + \gamma_v \|\hat{f}_v\|_n]. \tag{49}
\]

Next we use the decomposability property of the penalty expressed in the following lemma (proved in Section 5.2.2 page 23).

**Lemma 4.** For any $f \in \mathcal{F}$, under the assumptions of Theorem 1 conditionally on $\mathcal{T}$ (see (47)), we have:
\[
\sum_{v \in S_f} \mu_v \|\hat{f}_v\|_{\mathcal{H}_v} + \sum_{v \notin S_f} \gamma_v \|\hat{f}_v\|_n \leq 3 \sum_{v \in S_f} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + 3 \sum_{v \in S_f} \gamma_v \|\hat{f}_v - f_v\|_n. \tag{50}
\]

Hence, by combining (49) and Lemma 4 we obtain
\[
\|\hat{f} - m\|^2_n \leq 18 \sum_{v \in S_f} [\mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \gamma_v \|\hat{f}_v - f_v\|_n].
\]

For each $v$, $\|\hat{f}_v - f_v\|_{\mathcal{H}_v} \leq 2$ (because the functions $\hat{f}_v$ et $f_v$ belong to the class $\mathcal{F}$, see (6)), and consequently, for some constant $C$,
\[
\|\hat{f} - m\|^2_n \leq C \left\{ \sum_{v \in S_f} \mu_v + \sum_{v \notin S_f} \gamma_v \right\}. \tag{51}
\]

To finish the proof it remains to compare the two quantities $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_n$ and $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_n$. For that purpose we show that $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_n$ is less than $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_2$ plus an additive term coming from concentration results (see the Lemma given below). Next, thanks to the orthogonality of the spaces $\mathcal{H}_v$ with respect to $L^2(P_X, \mathcal{X})$, $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_2 = \sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_2$. To conclude, it remains to consider several cases, according to the rankings of $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_2$ and $\sum_{v \in S_f} \|\hat{f}_v - f_v\|^2_n$. This is the subject of the following lemma whose proof is given in Section 5.2.3 page 23.

**Lemma 5.** For $f \in \mathcal{H}$, let $\mathcal{A}$ be the event
\[
\mathcal{A} = \left\{ \forall f \in \mathcal{F}, \forall v \in \mathcal{P}, \|\hat{f}_v - f_v\|_n \leq 2\|\hat{f}_v - f_v\|_2 + \gamma_v \right\}. \tag{52}
\]

Then, for some positive constant $c_2$,
\[
P_{X,v}(\mathcal{A}) \geq 1 - \sum_{v \in \mathcal{P}} \exp(-nc_2 \gamma^2_v).
\]

On the set $\mathcal{A}$, Inequality (51) provides that, for all $K > 0$
\[
\frac{1}{C} \|\hat{f} - m\|^2_n \leq \sum_{v \in S_f} [\mu_v + 2\gamma_v \|\hat{f}_v - f_v\|_2 + \gamma^2_v],
\]
\[
\leq \sum_{v \in S_f} [\mu_v + (1 + K) \gamma^2_v + \frac{1}{K} \|\hat{f}_v - f_v\|^2_2], \tag{53}
\]
\[
\leq \sum_{v \in S_f} [\mu_v + (1 + K) \gamma^2_v] + \frac{1}{K} \sum_{v \in \mathcal{P}} \|\hat{f}_v - f_v\|^2_2,
\]
\[
\leq \sum_{v \in S_f} [\mu_v + (1 + K) \gamma^2_v] + \frac{1}{K} \sum_{v \in \mathcal{P}} \|\hat{f}_v - f_v\|^2_2. \tag{54}
\]

Inequality (53) uses the inequality $2ab \leq \frac{1}{K} a^2 + Kb^2$ for all positive $K$, and Inequality (54) uses the orthogonality with respect to $L^2(P_X)$.

In the following we have to consider several cases, according to the rankings of $\|\sum_{v \in \mathcal{P}} \hat{f}_v - f_v\|_2$ and $\|\sum_{v \in \mathcal{P}} \hat{f}_v - f_v\|_n$. More precisely, we consider two following cases:
Case 1: If $\|\sum_{v \in P} \hat{f}_v - f_v\|_2 \leq \|\sum_{v \in P} \hat{f}_v - f_v\|_n$.

Case 2: If $\|\sum_{v \in P} \hat{f}_v - f_v\|_2 \geq \|\sum_{v \in P} \hat{f}_v - f_v\|_n$.

Case 1: From (54), for any $f \in \mathcal{H}$, we get

$$\frac{1}{C} \|\hat{f} - m\|_n^2 \leq \sum_{v \in S_f} [\mu_v + (1 + K) \gamma_v^2] + \frac{1}{K} \|\hat{f} - f\|_n^2.$$ 

Hence, using that for all $K' > 0$,

$$\|\hat{f} - f\|_n^2 \leq (1 + K')\|\hat{f} - m\|_n^2 + (1 + \frac{1}{K'})\|f - m\|_n^2,$$ 

we obtain for a suitable choice of $K'$, say $1 + K' < K/C$, that, for some positive constant $C'$,

$$\|\hat{f} - m\|_n^2 \leq C' \left\{ \|f - m\|_n^2 + \sum_{v \in S_f} [\mu_v + \sum_{v \in S_f} \gamma_v^2] \right\}.$$ 

This shows the result in Case 1.

Case 2: This case is solved by applying the following Lemma (proved in Section 5.2.4 page [24]), which states that with high probability, $\|\hat{f} - f\|_2 \leq \sqrt{2} \|\hat{f} - f\|_n$.

**Lemma 6.** Let $f = \sum_{v} f_v \in \mathcal{F}$ with support $S_f$, $\lambda_{n,v}$ be defined by (17), and let $\mathcal{G}(f)$ be the class of functions written as $g = \sum_{v \in P} g_v$, such that $\|g_v\|_{\mathcal{H}_v} \leq 2$ satisfying for all $f \in \mathcal{F}$

\[
\begin{align*}
C_1 &\quad \sum_{v \in P} \mu_v \|g_v\|_{\mathcal{H}_v} + \sum_{v \in S_f} \gamma_v \|g_v\|_n \leq 4 \sum_{v \in S_f} \mu_v \|g_v\|_{\mathcal{H}_v} + 4 \sum_{v \in S_f} \gamma_v \|g_v\|_n \\
C_2 &\quad \sum_{v \in S_f} \gamma_v \|g_v\|_n \leq 2 \sum_{v \in S_f} \gamma_v \|g_v\|_2 + \sum_{v \in S_f} \gamma_v^2 \\
C_3 \quad \|g\|_n \leq \|g\|_2
\end{align*}
\]

Then the event

$$\left\{ \|g\|_n^2 \geq \frac{\|g\|_2^2}{2} \right\},$$

have probability greater than $1 - c_1 \exp(-nc_3 \sum_{v \in S_f} \lambda_{n,v}^2)$ for some constants $c_1$ and $c_3$.

If $f$ is such that $|S_f| = 0$, then Condition $C_1$ is not satisfied except if $g_v = 0$ for all $v \in P$. Because we will apply Lemma 6 to $g_v = \hat{f}_v - f_v$, this event has probability 0. If $f$ is such that $|S_f| \geq 1$, then Condition $C_1$ is satisfied:

from Equation (50) in Lemma 1 we have,

\[
\begin{align*}
\sum_{v \in S_f} \mu_v \|\hat{f}_v - \mathcal{H}_v\|_n &+ \sum_{v \in S_f} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \sum_{v \in S_f} \gamma_v \|\hat{f}_v\|_n + \sum_{v \in S_f} \gamma_v \|\hat{f}_v - f_v\|_n \\
& \leq 3 \sum_{v \in S_f} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \sum_{v \in S_f} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + 3 \sum_{v \in S_f} \gamma_v \|\hat{f}_v\|_n + \sum_{v \in S_f} \gamma_v \|\hat{f}_v - f_v\|_n, \\
& \iff \sum_{v \in P} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + \sum_{v \in S_f} \gamma_v \|\hat{f}_v - f_v\|_n \leq 4 \sum_{v \in S_f} \mu_v \|\hat{f}_v - f_v\|_{\mathcal{H}_v} + 4 \sum_{v \in S_f} \gamma_v \|\hat{f}_v\|_n.
\end{align*}
\]

Moreover, Assumption $n\lambda_{n,v}^2 \geq -C_2 \log(\lambda_{n,v})$ implies that

$$\lambda_{n,v} = \frac{K_{n,v}}{\sqrt{n}} \text{ with } K_{n,v} \to \infty.$$ 

Then,

$$\exp(-nc_3 \sum_{v \in S_f} \lambda_{n,v}^2) \leq \exp(-c_3 |S_f| \min K_{n,v}^2).$$
Lemma 8. Let \( \Omega \) be the event defined as
\[
\left\{ g \in H_v, \|g\|_{H_v} \leq 2, \|g_v\|_2 \leq t, \|g_v\|_\infty \leq b \right\}. 
\]
Let \( \Omega_{v,t} \) be the event defined as
\[
\Omega_{v,t} = \left\{ \sup_{g_v \in \mathcal{G}(t)} \{|\|g_v\|_2 - \|g_v\|_\infty|\} \leq \frac{bt}{2} \right\}. 
\]
Then for any \( t \geq \nu_{n,v} \), the event \( \Omega_{v,t} \) has probability greater than \( 1 - \exp(-c_2 nt^2) \), for some positive constant \( c_2 \).

Its proof is given in Section 5.3.2, page 31.

Lemma 9. For any function \( g_v \in H_v \) satisfying \( \|g_v\|_{H_v} \leq 2, \|g_v\|_\infty \leq b \) and \( \|g_v\|_2 \geq t \), for all \( t \geq \nu_{n,v} \) and \( b \geq 1 \), the event
\[
(1 - \frac{b}{2})\|g_v\|_2 \leq \|g_v\|_\infty \leq (1 + \frac{b}{2})\|g_v\|_2 
\]
has probability greater than \( 1 - \exp(-c_2 nt^2) \) for some positive constant \( c_2 \).

Its proof is given in Section 5.3.3, page 32.

Lemma 10. If \( E_x \) denotes the expectation with respect to the distribution of \( x \), we have
\[
P_{x,z}(|W_{n,v}(t) - E_x(W_{n,v}(t))| \geq \delta t) \leq 2 \exp(-\frac{\delta t^2}{8A}). 
\]

Its proof is given in Section 5.3.4, page 32.

Lemma 11. Conditionally on the space \( \Omega_{v,t} \) defined by (56), we have the following inequalities:
\[
P_{x,z}(|W_{n,2,v}(t) - E_x(W_{n,2,v}(t))| \geq \delta t) \leq 2 \exp(-\frac{\delta t^2}{32A}), 
\]
\[
P_x\left(E_x W_{n,2,v}(t) - E_{x,z}(W_{n,2,v}(t)) \geq x \right) \leq \exp(-\frac{n x^2}{Q_{n,v}(t)}). 
\]

Its proof is given in Section 5.3.5, page 32.

Lemma 12. Let \( \lambda_{n,v} \) be defined at Equation (17), \( \Delta \) at Equation (10) and \( \kappa = 10 + 4\Delta \). Conditionally on the space \( \Omega_{v,\lambda_{n,v}} \) defined at Equation (55), for some positive constants \( c_1, c_2 \), with probability greater than \( 1 - c_1 \exp(-c_2 n\lambda_{n,v}^2) \), we have
\[
W_{n,v}(\lambda_{n,v}) \leq \kappa \lambda_{n,v}^2 \quad \text{and} \quad E_x W_{n,v}(\lambda_{n,v}) \leq \kappa \lambda_{n,v}^2. 
\]

Its proof is given in Section 5.3.6, page 34.
5.2 Proof of lemma 3 to 6

5.2.1 Proof of lemma 3

For \( f \in F \) and \( v \in P \), let \( g_v = \hat{f}_v - f_v \). Note that \( \|g_v\|_{\mathcal{H}_v} \leq 2 \). Let us show that

\[
|V_{n,\varepsilon}(g_v)| \leq \kappa \left( \lambda^2_{n,v} \|g_v\|_{\mathcal{H}_v} + \lambda_{n,v} \|g_v\|_n \right).
\]

(63)

We start by writing that

\[
|V_{n,\varepsilon}(g_v)| = \|g_v\|_{\mathcal{H}_v} \left| V_{n,\varepsilon} \left( \frac{g_v}{\|g_v\|_{\mathcal{H}_v}} \right) \right| \leq \|g_v\|_{\mathcal{H}_v} W_{n,n,v} \left( \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}} \right).
\]

(64)

Consider the two following cases:

**Case A:** \( \|g_v\|_n \leq \lambda_{n,v} \|g_v\|_{\mathcal{H}_v} \),

**Case B:** \( \|g_v\|_n > \lambda_{n,v} \|g_v\|_{\mathcal{H}_v} \).

**Case A:** Since \( \|g_v\|_n \leq \lambda_{n,v} \|g_v\|_{\mathcal{H}_v} \), we have

\[
W_{n,n,v} \left( \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}} \right) \leq W_{n,n,v} (\lambda_{n,v}).
\]

We then apply Lemma 12 page 20 and conclude that (63) holds in Case A for each \( v \in P \) since, with high probability

\[
|V_{n,\varepsilon}(g_v)| \leq \kappa \lambda^2_{n,v} \|g_v\|_{\mathcal{H}_v} \leq \kappa \lambda^2_{n,v} \|g_v\|_{\mathcal{H}_v} + \kappa \lambda_{n,v} \|g_v\|_n.
\]

(65)

**Case B:** Consider now the case \( \|g_v\|_n > \lambda_{n,v} \|g_v\|_{\mathcal{H}_v} \) and let us show that for any \( v \in P \),

\[
W_{n,n,v} \left( \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}} \right) \leq \kappa \lambda_{n,v} \|g_v\|_n.
\]

Let \( r_v \) be a deterministic number such that \( r_v > \lambda_{n,v} \). Our first step relies on the study of the process \( W_{n,n,v}(r_v) \), for \( r_v > \lambda_{n,v} \). In that case we state two results:

**R1** For any deterministic \( r_v \geq \lambda_{n,v} \), with probability greater than \( 1 - c_1 \exp(-c_2 n \lambda^2_{n,v}) \),

\[
W_{n,n,v}(r_v) \leq \kappa r_v \lambda_{n,v}.
\]

(66)

**R2** Inequality (66) continues to hold for random \( r_v \) of the form

\[
r_v = \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}}.
\]

Combining these two points implies that, with probability greater than \( 1 - c_1 \exp(-c_2 n \lambda^2_{n,v}) \),

\[
\|g_v\|_{\mathcal{H}_v} W_{n,n,v} \left( \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}} \right) \leq \kappa \|g_v\|_n \lambda_{n,v}.
\]

Consequently, in Case B, according to (64), for each \( v \), Inequality (63) holds because

\[
|V_{n,\varepsilon}(g_v)| \leq \kappa \|g_v\|_n \lambda_{n,v} \leq \kappa \lambda^2_{n,v} \|g_v\|_{\mathcal{H}_v} + \kappa \lambda_{n,v} \|g_v\|_n.
\]

This ends up the proof of Lemma 3.
Proof of R1. From Lemma 10 page 20 with $t = r_v$ and $\delta = \lambda_{n,v}$, we get that with probability greater than $1 - 2B \exp(-n\lambda^2_{n,v}/8A)$,

$$W_{n,n,v}(r_v) \leq E_\varepsilon(W_{n,n,v}(r_v)) + r_v \lambda_{n,v}$$

(67)

Next we prove that for some positive $r_v$, with probability greater than $1 - \exp(-nc\lambda^2_{n,v})$, we have

$$E_\varepsilon(W_{n,n,v}(r_v)) \leq \kappa r_v \lambda_{n,v}.$$ 

(68)

Let $\hat{\nu}_{n,v}$ be defined as the smallest solution of $E_\varepsilon(W_{n,n,v}(t)) \leq \kappa t^2$. For $W_{n,n,v}$, defined by (24), we write

$$E_\varepsilon(W_{n,n,v}(r_v)) = \frac{r_v}{\hat{\nu}_{n,v}} E_\varepsilon \sup \left\{|V_{n,v}(g_v)|, \|g_v\|_{\mathcal{H}_v} \leq 2\left(\frac{\hat{\nu}_{n,v}}{r_v}\right), \|g_v\|_n \leq \hat{\nu}_{n,v}\right\}.$$ 

Besides, Lemma 12 stated that on the event $\Omega_v, \lambda_{n,v}$, $E_\varepsilon(W_{n,n,v}(\lambda_{n,v})) \leq \kappa \lambda^2_{n,v}$. It follows from the definition of $\hat{\nu}_{n,v}$, and Lemma 8 that $\hat{\nu}_{n,v} \leq \lambda_{n,v}$ for all $v \in \mathcal{P}$ with probability greater than $1 - \exp(-nc^2 \sum_{v \in \mathcal{P}} \lambda^2_{n,v})$. Consequently, for any deterministic $r_v$ such that $r_v \geq \lambda_{n,v}$, we have

$$\hat{\nu}_{n,v} \leq \lambda_{n,v} \leq r_v \iff \frac{\hat{\nu}_{n,v}}{r_v} \leq 1,$$

and so,

$$E_\varepsilon(W_{n,n,v}(r_v)) \leq \frac{r_v}{\hat{\nu}_{n,v}} E_\varepsilon \sup \left\{|V_{n,v}(g_v)|, \|g_v\|_{\mathcal{H}_v} \leq 2, \|g_v\|_n \leq \hat{\nu}_{n,v}\right\},$$

$$\leq \frac{r_v}{\hat{\nu}_{n,v}} E_\varepsilon(W_{n,n,v}(\hat{\nu}_{n,v})) \leq \frac{r_v}{\hat{\nu}_{n,v}} \kappa \hat{\nu}^2_{n,v} = \kappa r_v \hat{\nu}_{n,v} \leq \kappa r_v \lambda_{n,v}.$$ 

Proof of R2. Let us prove R2 by using a peeling-type argument. Our aim is to prove that (69) holds for any $r_v$ of the form

$$r_v = \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}}.$$ 

Since $\|g_v\|_\infty/\|g_v\|_{\mathcal{H}_v} \leq 1$, we have $\|g_v\|_n/\|g_v\|_{\mathcal{H}_v} \leq 1$. We thus restrict ourselves to $r_v$ satisfying $r_v = \|g_v\|_n/\|g_v\|_{\mathcal{H}_v}$ with $\|g_v\|_n/\|g_v\|_{\mathcal{H}_v} \in (\lambda_{n,v}, 1]$.

We start by splitting the interval $(\lambda_{n,v}, 1]$ into $M$ disjoint intervals such that

$$(\lambda_{n,v}, 1] = \cup_{k=1}^{M} [2^{k-1}\lambda_{n,v}, 2^k \lambda_{n,v}],$$

for some $M$ that will be chosen later. Consider the event $\mathcal{D}^c$ defined as follows:

$$\mathcal{D}^c = \left\{\exists v \in \mathcal{P} \text{ and } \exists g_v, \text{ such that } |V_{n,v}(g_v)| \geq \kappa \lambda_{n,v} \|g_v\|_n, \text{ with } \|g_v\|_{\mathcal{H}_v} \in (\lambda_{n,v}, 1]\right\}.$$ 

We prove that, for some positive constants $c_1, c_2$,

$$P(\mathcal{D}^c) \leq c_1 \exp(-c_2 n\lambda^2_{n,v}).$$

For $g_v \in \mathcal{D}^c$, let $\bar{k}$ be the integer in $\{1, \cdots, M\}$, such that

$$2^{\bar{k}-1}\lambda_{n,v} \leq \|g_v\|_n/\|g_v\|_{\mathcal{H}_v} \leq 2^{\bar{k}} \lambda_{n,v}.$$ 

This $\bar{k}$ satisfies

$$\|g_v\|_{\mathcal{H}_v} W_{n,n,v}(2^{\bar{k}} \lambda_{n,v}) \geq \|g_v\|_{\mathcal{H}_v} W_{n,n,v}(\|g_v\|_n/\|g_v\|_{\mathcal{H}_v}) \geq |V_{n,v}(g_v)| \geq \kappa \lambda_{n,v} \|g_v\|_n.$$ 

Therefore, we get

$$W_{n,n,v}(2^{\bar{k}} \lambda_{n,v}) \geq \kappa \lambda_{n,v} \frac{\|g_v\|_n}{\|g_v\|_{\mathcal{H}_v}} \geq \kappa \lambda^2_{n,v} 2^{\bar{k}-1} \geq \kappa \frac{\lambda_{n,v}}{2} 2^{\bar{k}} \lambda_{n,v}.$$
By taking \( r_v = 2^k \lambda_{n,v} \) in \( (66) \), we have
\[
P(\{ W_{n,v}(2^k \lambda_{n,v}) \geq \kappa \lambda_{n,v} 2^k \lambda_{n,v} \}) \leq c_1 \exp(-c_2 n \lambda_{n,v}^2).
\]

Now let us write \( \mathcal{D}^c \) as follows:
\[
\mathcal{D}^c = \bigcup_{k=1}^{M} \{ \exists v \text{ and } \exists \mathcal{F}_v \text{ such that } |V_{n,v}(\mathcal{F}_v)| \geq \kappa \lambda_{n,v} ||\mathcal{F}_v||_n, \text{ with } \frac{||\mathcal{F}_v||_n}{||\mathcal{F}_v||_H} \in (2^{k-1} \lambda_{n,v}, 2^k \lambda_{n,v}] \}.
\]
The set \( \mathcal{D}^c \) has probability smaller than \( c_1 M \exp(-c_2 n \lambda_{n,v}^2) \). If we choose \( M \) such that \( \log M \leq (c_2/2) n \lambda_{n,v}^2 \), then the probability of the set \( \mathcal{T} \) is greater than
\[
1 - \sum_{v \in \mathcal{P}} c_1 \exp(-\frac{c_2}{2} n \lambda_{n,v}^2).
\]

It follows that \( \mathbf{R2} \) is proved which ends up the proof of Lemma 4.

\[\square\]

### 5.2.2 Proof of lemma 4

Starting from \((15)\) with \( B \) defined by Equation \((44)\), we write
\[
\frac{1}{2} \| \hat{f} - m \|^2_n \leq 2 |V_{n,v}(\hat{f} - f)| + \sum_{v \in S_f} [\mu_v \| \hat{f}_v - f_v \|_{\mathcal{H}_v} + \gamma_v \| \hat{f}_v - f_v \|_n] - \sum_{v \notin \mathcal{S}_f} [\mu_v \| \hat{f}_v \|_{\mathcal{H}_v} + \gamma_v \| \hat{f}_v \|_n].
\]

On the event \( \mathcal{T} \) defined in \((14)\), we have
\[
\frac{1}{2} \| \hat{f} - m \|^2_n \leq 2 \kappa \sum_{v \in \mathcal{P}} \lambda_{n,v}^2 \| \hat{f}_v - f_v \|_{\mathcal{H}_v} + 2 \kappa \sum_{v \in \mathcal{P}} \lambda_{n,v} \| \hat{f}_v - f_v \|_n + \sum_{v \in \mathcal{S}_f} [\mu_v \| \hat{f}_v - f_v \|_{\mathcal{H}_v} + \gamma_v \| \hat{f}_v - f_v \|_n] - \sum_{v \notin \mathcal{S}_f} [\mu_v \| \hat{f}_v \|_{\mathcal{H}_v} + \gamma_v \| \hat{f}_v \|_n].
\]

Rearranging the terms we obtain that
\[
\frac{1}{2} \| \hat{f} - m \|^2_n \leq \sum_{v \in \mathcal{S}_f} (2 \kappa \lambda_{n,v}^2 + \mu_v) \| \hat{f}_v - f_v \|_{\mathcal{H}_v} + \sum_{v \in \mathcal{S}_f} (2 \kappa \lambda_{n,v} + \gamma_v) \| \hat{f}_v - f_v \|_n + \sum_{v \notin \mathcal{S}_f} (2 \kappa \lambda_{n,v}^2 - \mu_v) \| \hat{f}_v \|_{\mathcal{H}_v} + \sum_{v \notin \mathcal{S}_f} (2 \kappa \lambda_{n,v} - \gamma_v) \| \hat{f}_v \|_n.
\]

Now, thanks to Assumption \((12)\) with \( C_1 \geq \kappa \) we have \( \kappa \lambda_{n,v}^2 \leq \mu_v \) and \( 2 \kappa \lambda_{n,v} \leq \gamma_v \) and Lemma 4 is shown since
\[
0 \leq \frac{1}{2} \| \hat{f} - m \|^2_n \leq 3 \sum_{v \in \mathcal{S}_f} \mu_v \| \hat{f}_v - f_v \|_{\mathcal{H}_v} + 3 \sum_{v \in \mathcal{S}_f} \| \hat{f}_v - f_v \|_n - \sum_{v \notin \mathcal{S}_f} \mu_v \| \hat{f}_v \|_{\mathcal{H}_v} - \sum_{v \notin \mathcal{S}_f} \gamma_v \| \hat{f}_v \|_n.
\]

\[\square\]

### 5.2.3 Proof of lemma 5

Let us consider the following two cases:
Step 1

Let and showed that the event \( \| \hat{f}_v - f_v \|_n \leq \| \hat{f}_v - f_v \|_2 + \gamma_v \).

It follows that, for some positive \( c_2 \), with probability greater than \( 1 - \exp(-nc_2 \gamma_v^2) \),

\[
\| \hat{f}_v - f_v \|_2 \leq 2 \| \hat{f}_v - f_v \|_2.
\]

\[ \bullet \quad \| \hat{f}_v - f_v \|_2 \geq \gamma_v. \]

We apply Lemma 8 (page 20) to the function \( g_v = \hat{f}_v - f_v \) with \( b = 2 \). It follows that, for some positive \( c_2 \), with probability greater than \( 1 - \exp(-nc_2 \gamma_v^2) \),

\[
\| \hat{f}_v - f_v \|_2 \leq 2 \| \hat{f}_v - f_v \|_2.
\]

\[ \square \]

5.2.4 Proof of Lemma 6

Throughout the proof, we make use of the quantity \( d_n \) defined as follows:

For \( \beta < 1/\alpha \) and some constant \( \eta' \),

\[
d_n^2 \geq \eta' n^{\alpha \beta - 1}.
\]

Let \( \mathcal{G}(f) \) and \( \mathcal{G}'(f) \) be the following sets:

\[
\mathcal{G}(f) = \left\{ g = \sum_{v \in P} g_v, \text{ satisfying } \| g_v \|_{H_v} \leq 2, \text{ and Conditions C1, C2, C3} \right\},
\]

\[
\mathcal{G}'(f) = \left\{ g \in \mathcal{G}(f), \text{ such that } \| g \|_2 = d_n \right\}.
\]

In order to prove this lemma we consider two cases: if \( \sum_{v \in P} \hat{f}_v - f_v \|_2 \geq d_n \), and if \( \sum_{v \in P} \hat{f}_v - f_v \|_2 \leq d_n \).

First, we suppose that \( \sum_{v \in P} \hat{f}_v - f_v \|_2 \geq d_n \), and we consider the two events \( \mathcal{B} \) and \( \mathcal{B}' \) defined as follows:

\[
\mathcal{B} = \left\{ \forall h \in \mathcal{G}, \| h \|_2^2 \geq \frac{\| h \|_2^2}{2}, \text{ and } \| h \|_2 \geq d_n \right\},
\]

and

\[
\mathcal{B}' = \left\{ \forall h \in \mathcal{G}', \| h \|_2^2 \geq d_n^2 \right\}.
\]

If \( h \in \mathcal{B}' \), then \( h \in \mathcal{G}, \| h \|_2 = d_n \) and \( \| h \|_n \geq d_n^2/2 \). It follows that \( \| h \|_n^2 \geq \| h \|_2^2/2 \) and \( \| h \|_2 \geq d_n \). We just showed that the event \( \mathcal{B}' \) is included into the event \( \mathcal{B} \). So, this case is proved if the event \( \mathcal{B}' \) holds with high probability. Consider

\[
Z_n(\mathcal{G}') = \sup_{g \in \mathcal{G}'} \left\{ d_n^2 - \| g \|_n^2 \right\}.
\]

We show that the event \( Z_n(\mathcal{G}') \leq d_n^2/2 \) has probability greater than \( 1 - c_1 \exp(-nc_3d_n^2) \).

Consider a \( d_n/8 \)-covering of \( \mathcal{G} \). So that, for all \( g \) in \( \mathcal{G} \) there exists \( g^k \) such that

\[
\| g - g^k \|_n \leq \frac{d_n}{8}.
\]

The associated proper covering number is:

\[
N_{pr} = N_{pr}(\frac{d_n}{8}, \mathcal{G}', \| \cdot \|_n).
\]

Now, for all \( g \in \mathcal{G} \), we write:

\[
d_n^2 - \| g \|_n^2 = T_1 + T_2,
\]

with \( T_1 = \| g^k \|_n^2 - \| g \|_n^2 \) and \( T_2 = d_n^2 - \| g^k \|_n^2 \). The proof is splitted into four steps:

Step 1 The first step consists in showing that

\[
T_1 = \| g^k \|_n^2 - \| g \|_n^2 \leq \frac{d_n^2}{4}.
\]

24
Step 2 The second step consists in proving that, for $N_{pr}$ given at Equation (71) and for some constant $C$,

$$P_X \left( \max_{k \in \{1, \ldots, N_{pr} \}} \left[ d_n^2 - \| g^k \|^2_n \right] \geq \frac{d_n^2}{4} \right) \leq \exp \left( \log N_{pr} - Cn d_n^2 \right).$$

Step 3 The third step concerns the control of $N_{pr}$. Let $\sigma_n^2$ be the variance of a random variable distributed with density $\pi_n \in \mathcal{D}$ (see Equation (2)), then for some $K > 0$,

$$\frac{1}{K} \log N_{pr} \leq \left( 32 \sigma_n \sqrt{\pi(E_0 \sup_{g \in G} |V_{n,e}(g)|)/d_n} \right)^\alpha_1 \{ (0, 32 \sigma_n \sqrt{\pi E_0 \sup_{g \in G} |V_{n,e}(g)|}) (d_n) + \}

1_{\{32 \sigma_n \sqrt{\pi E_0 \sup_{g \in G} |V_{n,e}(g)|}, \infty \}} (d_n).$$

Step 4 The last step consists in bounding from above the Gaussian complexity. For some $\kappa > 0$

$$E_0 \sup_{g \in G} \sum_{v \in P} |V_{n,e}(g_v)| \leq \frac{4K}{C_1} \left\{ \sum_{v \in S_f} (2 \mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} d_n \right\}.$$

Let us conclude the proof of the lemma before proving these four steps.

Putting together Steps 3 and 4 we have:

If $d_n \in \{32 \sigma_n \sqrt{\pi E_0 \sup_{g \in G} |V_{n,e}(g)|}, \infty \}$, then

$$\frac{1}{K} \log N_{pr} \left( d_n \frac{1}{8}, G', \| \cdot \|_n \right) \leq 1.$$

Thanks to Step 2,

$$P_X \left( T_2 \geq \frac{d_n^2}{4} \right) \leq P_X \left( \max_{k \in \{1, \ldots, N_{pr} \}} \left[ d_n^2 - \| g^k \|^2_n \right] \geq \frac{d_n^2}{4} \right) \leq K \exp \left( - Cn d_n^2 \right),$$

and, therefore

$$P_X \left( Z_n(g') \leq \frac{d_n^2}{2} \right) \leq K \exp \left( - Cn d_n^2 \right). \tag{74}$$

If $d_n \in (0, 32 \sigma_n \sqrt{\pi E_0 \sup_{g \in G} |V_{n,e}(g)|})$, then

$$\frac{1}{K} \log N_{pr} \left( d_n \frac{1}{8}, G', \| \cdot \|_n \right) \leq (32 \sigma_n)^\alpha n^{\frac{\alpha}{2}} \left( \frac{E_0 \sup_{g \in G} |V_{n,e}(g)|}{d_n} \right)^{\frac{\alpha}{2}} \leq \left( \frac{128 \kappa \sigma_n}{C_1} \right)^\alpha n^{\frac{\alpha}{2}} \left( \sum_{v \in S_f} (2 \mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} d_n \right)^{\frac{\alpha}{2}}.$$

We have to show that $\log N_{pr} - Cn d_n^2 \leq -c_3 n d_n^2$ or equivalently that $\log N_{pr} \leq \tilde{C} n d_n^2$, where $\tilde{C} = C - c_3$.

Let $A = K(128 \kappa \sigma_n/C_1)^\alpha$. We have,

$$\log N_{pr} \leq \tilde{C} n d_n^2 \iff A \left( \sum_{v \in S_f} (2 \mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} \right)^{\frac{\alpha}{2}} \leq \tilde{C} n d_n^2,$$

$$\iff \left( \sum_{v \in S_f} (2 \mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} \right)^{\frac{\alpha}{2}} \leq \left( \frac{\tilde{C}}{A} \right)^{\frac{\alpha}{2}} n^{\frac{\alpha}{2}} - \frac{3}{2} d_n^{2 \frac{\alpha}{2}},$$

$$\iff \sum_{v \in S_f} (2 \mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} \leq \left( \frac{\tilde{C}}{A} \right)^{\frac{\alpha}{2}} n^{\frac{\alpha}{2}} - \frac{3}{2} d_n^{2 \frac{\alpha}{2}}.$$

Because $\gamma_v = C_1 \lambda_{n,v}$ and $\mu_v = C_1 \lambda_{n,v}^2$,

$$\log N_{pr} \leq \tilde{C} n d_n^2 \iff C_1 (2 + C_1) \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{\alpha}{2}} + 2C_1 \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{\alpha}{2}} \leq \left( \frac{\tilde{C}}{A} \right)^{\frac{\alpha}{2}} n^{\frac{\alpha}{2}} - \frac{3}{2} d_n^{2 \frac{\alpha}{2}}.$$
Therefore, the inequality

\[ \sum_{v \in S_f} \lambda_{n,v}^2 \leq B n^{\frac{1}{n} - \frac{\alpha}{2}} d_n^2 \Leftrightarrow d_n^2 \geq B' \frac{m}{2} \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{2}{na}} n^{\frac{n}{na + 2}}. \]

As \( \sum_{v \in S_f} \lambda_{n,v}^2 \leq C_3 n^{2\beta - 1} \) (see Equation (14)), we get

\[ B' \frac{2m}{2} \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{2}{na}} n^{\frac{n}{na + 2}} \leq \left( \frac{B}{C_3} \right)^{\frac{2}{na}} n^{\frac{4\alpha \beta - 1}{na + 2}}. \]

Therefore, the inequality

\[ C_1 (2 + C_1) \frac{\sum_{v \in S_f} \lambda_{n,v}^2}{d_n^2} \leq \frac{1}{2} \frac{\tilde{C}}{A} n^{\frac{1}{n} - \frac{\alpha}{2}} d_n^2, \]

will be satisfied if

\[ d_n^2 \geq \left( \frac{C_1}{B} \right)^{\frac{2}{na}} n^{\frac{4\alpha \beta - 1}{na + 2}}. \]

For the second term, let

\[ B' = \frac{1}{2} \times \frac{1}{2C_1} \left( \frac{\tilde{C}}{A} \right)^{\frac{1}{n}}, \]

then

\[ \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{1}{n}} \leq B' n^{\frac{1}{n} - \frac{\alpha}{2}} d_n^{\frac{1}{n}} \Leftrightarrow d_n^2 \geq B' \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{1}{n}} n^{\frac{n}{na + 2}}. \]

As \( \sum_{v \in S_f} \lambda_{n,v}^2 \leq C_3 n^{2\beta - 1} \) (see Equation (14)), then

\[ B' \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{1}{n}} n^{\frac{n}{na + 2}} \leq \left( \frac{C_3}{B'} \right)^{\frac{n}{na + 2}} n^{\alpha \beta - 1}. \]

Therefore the inequality

\[ 2C_1 \left( \sum_{v \in S_f} \lambda_{n,v}^2 \right)^{\frac{1}{n}} \leq \frac{1}{2} \left( \frac{\tilde{C}}{A} \right)^{\frac{1}{n}} n^{\frac{1}{n} - \frac{\alpha}{2}} d_n^2, \]

will be satisfied if

\[ d_n^2 \geq \left( \frac{C_3}{B'} \right)^{\frac{n}{na + 2}} n^{\alpha \beta - 1}. \]

As \( \alpha > 2 \), \( 4\alpha \beta/(\alpha + 2) < \alpha \beta \). Therefore, there exists a constant \( \eta' \), take for example

\[ \eta' = \max \left( \left( \frac{C_3}{B'} \right)^{\frac{n}{na + 2}} \right) \]

such that if \( d_n^2 \geq \eta' n^{\alpha \beta - 1} \), then \( \log N_{pr} \leq \tilde{C} n d_n^2 \), and Step 2 states that

\[ P_X \left( T_2 \geq \frac{d_n^2}{4} \right) \leq P_X \left( \max_{k \in \{1, \ldots, N_{pr}\}} [d_n^2 - \|g_k\|^2_n] \geq \frac{d_n^2}{4} \right) \leq \exp \left( - c_3 n d_n^2 \right). \]

Now, we have

\[ P_X \left( Z_n(g') \leq \frac{d_n^2}{2} \right) = P_X \left( \max_{g_k \in \{g_1, \ldots, g_N\}} [d_n^2 - \|g_k\|^2_n] \geq \frac{d_n^2}{4} \right) \leq \exp \left( - c_3 n d_n^2 \right). \] (75)

Finally, we obtain for \( c_1 = \max(K, 1) \) and \( c_3 \leq C \) (see Equations (74) and (75)):

\[ P_X \left( Z_n(g') \leq \frac{d_n^2}{2} \right) \leq c_1 \exp \left( - c_3 n d_n^2 \right). \]
Moreover, for \( n \) large enough, we have \( \sum_{v \in S_f} \lambda^2_{n,v} \leq d_n^2 \leq 1 \) (see Equations \((14)\) and \((69)\)), and

\[
1 - c_1 \exp\left(-c_3 nd_n^2\right) \geq 1 - c_1 \exp\left(-c_3 n \sum_{v \in S_f} \lambda^2_{n,v}\right).
\]

Therefore,

\[
P_X\left(Z_n(G') \leq \frac{d_n^2}{2}\right) \leq c_1 \exp\left(-c_3 n \sum_{v \in S_f} \lambda^2_{n,v}\right).
\]

Before proving the Steps 1 to 4 let us solve the second case: if \( \|\sum_{v \in P} \tilde{f}_v - f_v\|_2 \leq d_n \) then we consider the event \( B'' \) defined as follows:

\[
B'' = \{ \forall h \in G, \|h\|_n^2 \geq \frac{\|h\|_2^2}{2}, \text{ and } \|h\|_2 \leq d_n \}.
\]

We have that the event \( G'' \) defined in Equation \((70)\) is included in \( B'' \) and the same proof as in the first case applies.

**Proofs of Steps 1 to 4**

The proofs of Step 1 and Step 2 are strictly the same as in the Gaussian case. More precisely

**Proof of Step 1:**

It is easy to see that,

\[
T_1 = \|g^k\|_n^2 - \|g\|_n^2 = \frac{1}{n} \sum_{i=1}^{n} [((g^k(X_i))^2 - (g(X_i))^2)]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} [g^k(X_i) - g(X_i)] [g^k(X_i) + g(X_i)]
\]

\[
\leq \|g^k - g\|_n \left(\frac{1}{n} \sum_{i=1}^{n} [g^k(X_i) + g(X_i)]^2\right)^{\frac{1}{2}}
\]

where in the inequality above we used Cauchy Schwarz inequality. Using the inequality \((a + b)^2 \leq 2a^2 + 2b^2\), \( g \in G' \), and the property that \( g \) satisfies Condition C3, we get

\[
\frac{1}{n} \sum_{i=1}^{n} [g^k(X_i) + g(X_i)]^2 \leq 2\|g^k\|_n^2 + 2\|g\|_n^2 \leq 4d_n^2.
\]

Besides, the covering set is constructed such that \( \|g^k - g\|_n \leq d_n/8 \). It follows that Step 1 is proved.

**Proof of Step 2:**

We prove that for some constant \( C \),

\[
P_X\left(T_2 \geq \frac{d_n^2}{4}\right) \leq P_X\left(\max_{1 \leq k \leq n_{pr}} [d_n^2 - \|g^k\|_n^2] \geq \frac{d_n^2}{4}\right) \leq \exp \left(\log n_{pr} - Cn_{pr}d_n^2\right).
\]

As \( g^k \in G' \), \( d_n = \|g^k\|_2 \). Then

\[
\max_{1 \leq k \leq n_{pr}} [d_n^2 - \|g^k\|_n^2] = \max_{1 \leq k \leq n_{pr}} [\|g^k\|_2^2 - \|g^k\|_n^2].
\]

Applying Theorem 3.5. in \(\text{Chung and Lu 2006}\) with \( X = \sum_i (g^k(X_i))^2 \), for all positive \( \lambda \) we have:

\[
P_X\left(\sum_{i=1}^{n} (g^k(X_i))^2 \leq n\mathbb{E}(g^k(X_i))^2 - \lambda\right) \leq \exp \left(-\frac{\lambda^2}{2n\mathbb{E}(g^k(X))^4}\right),
\]

or equivalently,

\[
P_X\left(\|g^k\|_2^2 - \|g^k\|_n^2 \geq \frac{\lambda}{n}\right) \leq \exp \left(-\frac{\lambda^2}{2n\mathbb{E}(g^k(X))^4}\right).
\]
Taking \( \lambda = nd_n^2/4 \) and using that \( \|g^k\|_n^2 = d_n^2 \) we get

\[
P_X \left( d_n^2 - \|g^k\|_n^2 \geq \frac{d_n^2}{4} \right) \leq \exp \left( - \frac{nd_n^2}{32 \mathbb{E}(g^k(X))^4} \right).
\]

It follows that

\[
P_X \left( \max_{1 \leq k \leq N_{pr}} \left[ d_n^2 - \|g^k\|_n^2 \right] \geq \frac{d_n^2}{4} \right) \leq \sum_{k=1}^{N_{pr}} \exp \left( - \frac{nd_n^4}{32 \mathbb{E}(g^k(X))^4} \right)
\]

\[
\leq \exp \left( \log N_{pr} - \frac{nd_n^4}{32 \max_k \mathbb{E}(g^k(X))^4} \right). \tag{76}
\]

Moreover, \( g \in \mathcal{H} \), so \( g = \sum_{v \in P} g_v \), where the functions \( g_v \) are centered and orthogonal in \( L^2(P_X) \). Therefore \( \mathbb{E}(g(X))^4 \) is the sum of the following terms:

\[
A_1 = \sum_{v \in P} E_X g_v^4(X_v),
\]

\[
A_2 = \left( \frac{4}{2} \right) \sum_{v \not= v'} E_X g_v^2(X_v)g_{v'}^2(X_{v'}),
\]

\[
A_3 = \left( \frac{4}{3} \right) \sum_{v_1 \not= v_2 \not= v_3} E_X g_{v_1}^2(X_{v_1})g_{v_2}(X_{v_2})g_{v_3}(X_{v_3}),
\]

\[
A_4 = \left( \frac{4}{3} \right) \sum_{v_1 \not= v_2} E_X g_{v_1}^3(X_{v_1})g_{v_2}(X_{v_2}),
\]

\[
A_5 = \left( \frac{4}{1} \right) \sum_{v_1 \not= v_2 \not= v_3 \not= v_4} E_X g_{v_1}(X_{v_1})g_{v_2}(X_{v_2})g_{v_3}(X_{v_3})g_{v_4}(X_{v_4}).
\]

Using the Cauchy Schwartz inequality and the fact that \( \|g_v\|_\infty \leq \|g_v\|_{\mathcal{H}_v} \leq 2 \) and \( \|g\|_2 = d_n \) (because \( g \in \mathcal{G}' \)), we get that \( A_1 \) is proportional to \( d_n^2 \), \( A_2, A_3, A_5 \) to \( d_n^4 \), and \( A_4 \) to \( d_n^6 \). For example,

\[
A_1 = \sum_{v \in P} E_X g_v^4(X_v) \leq \|g\|_\infty^2 \sum_{v \in P} \|g_v\|_2^2 \leq \|g\|_\infty^2 \sum_{v \in P} \|g_v\|_2^2 \leq 4d_n^2.
\]

After calculation of the terms \( A_i \), since \( d_n^2 \) is assumed to be smaller than one, we get that:

\[
\max_k E_X(g^k(X))^4 \leq C d_n^2 (1 + O(d_n^2)). \tag{77}
\]

Step 2 is proved by combining (76) and (77).

We now focus on Step 3 and Step 4:

**Proof of Step 3:**

Let \( N_{pr} \) be defined at Equation (71). We prove that

\[
\frac{1}{K} \log N_{pr} \left( \frac{d_n}{8}, \mathcal{G}', \| \cdot \|_n \right) \leq \left( 32 \sigma_\alpha \sqrt{n} (E_{x \sup_{g \in \mathcal{G}'}} |V_{n,x}(g)|) / d_n \right) \alpha \mathbf{1}_{(0,32 \sigma_\alpha \sqrt{n} E_{x \sup_{g \in \mathcal{G}'}} |V_{n,x}(g)|) / d_n} +
\]

\[
\mathbf{1}_{[32 \sigma_\alpha \sqrt{n} E_{x \sup_{g \in \mathcal{G}'}} |V_{n,x}(g)|, \infty)} (d_n).
\]

We start from Equation (27) and write that:

\[
\log N_{pr} \left( \frac{d_n}{8}, \mathcal{G}', \| \cdot \|_n \right) \leq \log N \left( \frac{d_n}{16}, \mathcal{G}', \| \cdot \|_n \right).
\]

Next, we use Corollary [3]

Let \( Z = (Z_1, ..., Z_n) \) be i.i.d. random variables distributed with density \( \pi_\alpha \in \mathcal{D} \) defined in Equation (2) with \( \sqrt{\text{var}(Z)} = \sigma_\alpha \). Set \( T = \mathcal{G}', \delta = \sqrt{nd_n}/16 \) and \( M = n \times E_{Z \sup_{g \in \mathcal{G}'}} |V_{n,Z}(g)| \), then for all \( \alpha \geq 2 \) we
have,
\[
\log N \left( \frac{d_n}{16}, G', \|\cdot\|_n \right) = \log N \left( \frac{\sqrt{n}d_n}{16}, G', \|\cdot\|_n \right),
\]
\[
\leq K \left( \frac{32nE_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^\alpha 1_{\left(0, \frac{2n \times E_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)} + \frac{nE}{16},
\]
\[
K \left( \frac{32nE_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^2 1_{\left(0, \frac{2n \times E_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)},
\]
or equivalently,
\[
\log N \left( \frac{d_n}{16}, G', \|\cdot\|_n \right) \leq K \left( \frac{32nE_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^\alpha 1_{\left(0, \frac{32 \times E_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)} + \frac{nE}{16},
\]
\[
K \left( \frac{32nE_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^2 1_{\left(0, \frac{32 \times E_{g} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)}.
\]

Take \( \varepsilon_i = Z_i / \sigma_\alpha = h(Z_i) \) for \( i = 1, \ldots, n \), then \( \text{var} \varepsilon = 1 \) and,
\[
E_{\varepsilon}(\varepsilon_i) = E_{\varepsilon}(h(Z_i)) = \int h(Z_i) \pi_\alpha(Z_i) dZ_i = \frac{1}{\sigma_\alpha} \int Z_i \pi_\alpha(Z_i) dZ_i = \frac{1}{\sigma_\alpha} E_{\varepsilon}(Z_i).
\]

Therefore, \( E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)| = \sigma_\alpha E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)| \) and,
\[
\log N \left( \frac{d_n}{16}, G', \|\cdot\|_n \right) \leq K \left( \frac{32n\sigma_\alpha E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^\alpha 1_{\left(0, \frac{32 \times \sqrt{\pi} E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)} + \frac{nE}{16},
\]
\[
K \left( \frac{32n\sigma_\alpha E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)^2 1_{\left(0, \frac{32 \times \sqrt{\pi} E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)|}{\sqrt{\|n\|_n}} \right)}.
\]

Proof of Step 4:

This Step consists in bounding from above the quantity \( E_{\varepsilon} \sup_{g \in G'} |V_{n,z}(g)| \). According to Inequality [63] we have,
\[
\sum_{v \in P} |V_{n,z}(g_v)| \leq K \left\{ \sum_{v \in P} \lambda_v^2 \|g_v\|_{\mathcal{H}} + \sum_{v \in P} \lambda_v \|g_v\|_n \right\},
\]
with \( \lambda_v \) defined by Equation [11] satisfying Equation [12] for all \( v \in \mathcal{P} \). It follows
\[
\sup_{g \in G'} \sum_{v \in P} |V_{n,z}(g_v)| \leq K \sup_{g \in G'} \left\{ \sum_{v \in P} \lambda_v^2 \|g_v\|_{\mathcal{H}} + \sum_{v \in P} \lambda_v \|g_v\|_n \right\},
\]
\[
\leq \frac{K}{C_1} \sup_{g \in G'} \left\{ \sum_{v \in P} \mu_v \|g_v\|_{\mathcal{H}} + \sum_{v \in P} \gamma_v \|g_v\|_n \right\}.
\]

Thanks to Condition C1 and using \( \|g_v\|_{\mathcal{H}} \leq 2 \) we obtain then:
\[
\sup_{g \in G'} \sum_{v \in P} |V_{n,z}(g_v)| \leq \frac{4K}{C_1} \sup_{g \in G'} \left\{ \sum_{v \in P} \mu_v \|g_v\|_{\mathcal{H}} + \sup_{g \in G'} \sum_{v \in S_f} \gamma_v \|g_v\|_n \right\},
\]
\[
\leq \frac{4K}{C_1} \left\{ 2 \sum_{v \in S_f} \mu_v + \sup_{g \in G'} \sum_{v \in S_f} \gamma_v \|g_v\|_n \right\}.
\]
Now, according to Condition C2, we get
\[
\sup_{g \in \mathcal{G}'} \sum_{v \in \mathcal{P}} |V_{n,\varepsilon}(g_v)| \leq \frac{4k}{C_1} \left\{ \sum_{v \in S_f} \left[ 2 \sum_{v \in S_f} \mu_v + 2 \sup_{g \in \mathcal{G}'} \sum_{v \in S_f} \gamma_v \|g_v\|_2 + \sum_{v \in S_f} \gamma_v^2 \right] \right\},
\]
\[
\leq \frac{4k}{C_1} \left\{ \sum_{v \in S_f} (2\mu_v + \gamma_v^2) + 2 \sup_{g \in \mathcal{G}'} \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} \left( \sum_{v \in S_f} \|g_v\|_2^2 \right)^{1/2} \right\},
\]
\[
\leq \frac{4k}{C_1} \left\{ \sum_{v \in S_f} (2\mu_v + \gamma_v^2) + 2 \left( \sum_{v \in S_f} \gamma_v^2 \right)^{1/2} d_n \right\},
\]
where in the second inequality we used Cauchy-Schwarz inequality and the third inequality coming from the fact that for all \(g \in \mathcal{G}', \|g\|_2^2 = d_n^2 \geq \sum_{v \in S_f} \|g_v\|_2^2 \).

\[ \square \]

5.3 Proofs of intermediate Lemmas

5.3.1 Proof of Lemma \[\text{[7]}\]

The kernel \(k_v\) is written as:
\[
k_v(X_v, X_v') = \sum_{\ell \geq 1} \omega_{v,\ell} \phi_{v,\ell}(X_v) \phi_{v,\ell}(X_v')
\]
where \(\{\phi_{v,\ell}\}_{\ell \geq 1}\) is an orthonormal basis of \(L^2(P_v)\) with \(P_v = \prod_{a \in v} P_a\).

Let us consider the class of functions \(\mathcal{K}(t)\) defined as
\[
\mathcal{K}(t) = \{g_v \in \mathcal{H}_v, \|g_v\|_{\mathcal{H}_v} \leq 2, \|g_v\|_2 \leq t\}.
\]

It comes that
\[
g_v = \sum_{\ell} a_{\ell} \phi_{v,\ell}, \text{ with } \|g_v\|_{\mathcal{H}_v}^2 = \sum_{\ell} \frac{a_{\ell}^2}{\omega_{v,\ell}} \leq 4, \text{ and } \|g_v\|_2^2 = \sum_{\ell} a_{\ell}^2 \leq t^2
\]

In the following, we set \(\mu_{v,\ell}(t) = \min\{t^2, \omega_{v,\ell}\}\). Hence
\[
\sum_{\ell} \frac{a_{\ell}^2}{\mu_{v,\ell}(t)} \leq \frac{1}{t^2} \sum_{\ell} a_{\ell}^2 + \sum_{\ell} \frac{a_{\ell}^2}{\omega_{v,\ell}} = \frac{1}{t^2} \|g_v\|_2^2 + \|g_v\|_{\mathcal{H}_v}^2 \leq 5, \tag{78}
\]
as soon as \(g_v \in \mathcal{K}(t)\).

Now, let us prove the lemma:
\[
E_{X,\varepsilon} W_{n,2,\varepsilon}(t) = E_{X,\varepsilon} \sup_{g \in \mathcal{K}(t)} \left[ \frac{1}{n} \sum_{i=1}^n \varepsilon_i \sum_{\ell} a_{\ell} \phi_{v,\ell}(X_{vi}) \right] |
\]
\[
= E_{X,\varepsilon} \sup_{g \in \mathcal{K}(t)} \left[ \frac{1}{n} \sum_{\ell} \frac{a_{\ell}}{\sqrt{\mu_{v,\ell}(t)}} \sum_{i=1}^n \varepsilon_i \sqrt{\mu_{v,\ell}(t)} \phi_{v,\ell}(X_{vi}) \right],
\]
\[
\leq \sqrt{5} \left( E_{X,\varepsilon} \sum_{\ell} \left( \frac{1}{n} \sum_{i=1}^n \varepsilon_i \sqrt{\mu_{v,\ell}(t)} \phi_{v,\ell}(X_{vi}) \right)^2 \right).
\]
The last inequality follows from the Cauchy-Schwartz inequality and Inequality (78). Now, simple calculation leads to
\[
E_{X,\varepsilon} W_{n,2,\varepsilon}(t) \leq \sqrt{5} \left( \frac{1}{n} \sum_{\ell} \mu_{v,\ell}(t) \right).
\]

\[ \square \]
5.3.2 Proof of Lemma 8

Using that \( |\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|} \), we get

\[
\|g_v\|_2^2 - \|g_v\|_n^2 \leq \sqrt{\|g_v\|_2^2 - \|g_v\|_n^2}.
\]

Hence

\[
\left\{ \|g_v\|_\infty \leq b, \|g_v\|_2 - \|g_v\|_n \geq \frac{bt}{2} \right\} \subset \left\{ \|g_v\|_2^2 - \|g_v\|_n^2 \geq \frac{b^2t^2}{4} \right\}.
\]

The centered process

\[
\|g_v\|_2^2 - \|g_v\|_n^2 = \frac{1}{n} \sum_{i=1}^{n} g_v^2(X_{v,i}) - \mathbb{E}(g_v^2(X_v))
\]

satisfies a concentration inequality given, for example, by Theorem 2.1 in [Bartlett, Bousquet, and Mendelson 2005]: if \( C \) is a class of functions \( f \) such that \( \|f\|_\infty \leq B \) and \( \mathbb{E}f(X) = 0 \), and if there exists \( \gamma > 0 \) such that for every \( f \in C \), \( \text{Var}f(X) \leq \gamma^2 \). Then for every \( x > 0 \), with probability at least \( 1 - e^{-x} \),

\[
\sup_{f \in C} \frac{1}{n} \sum_{j=1}^{n} f(X_j) \leq \inf_{\alpha > 0} \left\{ 2(1 + \alpha)\mathbb{E}(\sup_{f \in C} \frac{1}{n} \sum_{j=1}^{n} f(X_j)) + \sqrt{\frac{2x}{n}\gamma} + B\left( \frac{1}{3} + \frac{1}{\alpha} \right) x \right\}.
\]

(79)

For any \( t > 0 \), for \( G(t) \) defined by (57), let us consider the class of functions \( C(t) \) defined as follows

\[
C(t) = \left\{ f \text{ such that } f = g_v^2 - \mathbb{E}(g_v^2), \text{ with } g_v \in G(t) \right\}.
\]

Note that if \( f \in C(t) \), \( \mathbb{E}f(X_v) = 0 \) and \( \|f\|_\infty \leq b^2 \). We have to study

\[
\gamma^2(t) = \sup_{g_v \in G(t)} \mathbb{E}_X \left( g_v^2(X) - \|g_v\|_2^2 \right)^2 \quad \text{and} \quad \Gamma(t) = \mathbb{E}_X \left( \sup_{g_v \in G(t)} \|g_v\|_2^2 - \|g_v\|_n^2 \right).
\]

It is easy to see that

\[
\gamma^2(t) \leq b^2 \sup_{g_v \in G(t)} \mathbb{E}_X (g_v(X) + \|g_v\|_2)^2 \leq 4b^4t^2.
\]

Let \( \zeta_i \) be i.i.d. Rademacher random variables and let \( \mathbb{E}_{X,\zeta} \) denotes the expectation with respect to the law of \( (X, \zeta) \). By a symmetrization argument,

\[
\Gamma(t) \leq 2\mathbb{E}_{X,\zeta} \sup_{g_v \in G(t)} \frac{1}{n} \sum_{i=1}^{n} \zeta_ig_v^2(X_i).
\]

Since \( \|g_v\|_\infty \leq b \), applying the contraction principal (see [Ledoux and Talagrand 1991]) we get that, for \( Q_{n,v}(t) \) defined by (59),

\[
\mathbb{E}_{X,\zeta} \sup_{g_v \in G(t)} \frac{1}{n} \sum_{i=1}^{n} \zeta_ig_v^2(X_i) \leq 4b\mathbb{E}_{X,\zeta} \sup_{g_v \in G(t)} \frac{1}{n} \sum_{i=1}^{n} \zeta_ig_v^2(X_i) \leq 4bQ_{n,v}(t).
\]

The last inequality was proved by [Mendelson 2002], Theorem 41 (see the proof of Lemma 7). Now, thanks to (79) we get that for all \( x > 0 \), with probability greater than \( 1 - e^{-x} \)

\[
\sup_{g_v \in G(t)} \|g_v\|_n^2 - \|g_v\|_2^2 \leq \inf_{\alpha > 0} \left\{ 16(1 + \alpha) bQ_{n,v}(t) + \sqrt{\frac{2x}{n}} 2bt + b^2 \left( \frac{1}{3} + \frac{1}{\alpha} \right) x \right\}.
\]

Taking \( x = c_2nt^2, t \geq \nu_n \), we have that with probability greater than \( 1 - e^{-c_2nt^2} \)

\[
\sup_{g_v \in G(t)} \|g_v\|_n^2 - \|g_v\|_2^2 \leq \inf_{\alpha > 0} \left\{ 16(1 + \alpha) b\Delta + \sqrt{2c_2} 4b + b^2 \left( \frac{1}{3} + \frac{1}{\alpha} \right) c_2 \right\}.
\]
The infimum of the right hand side is reached in \( \alpha = \sqrt{c_2b/16\Delta} \), and equals
\[
\frac{b^2c_2}{3} + 8\sqrt{\Delta c_2b^{3/2}} + 4(4\Delta + \sqrt{2c_2})b.
\]
The constants \( \Delta \) and \( c_2 \) should satisfy that this infimum is strictly smaller than \( b^2/4 \). For example, if \( 16\Delta < b/8 \), it remains to choose \( c_2 \) small enough such that
\[
b\left( \frac{c_2}{3} + \frac{\sqrt{2c_2}}{2} \right) + 4\sqrt{2c_2} < \frac{b}{8},
\]
\[
\square
\]

5.3.3 Proof of Lemma 9

Let \( t > \nu_{n,v} \) and \( h \) be defined as
\[
h = \frac{tg_v}{\|g_v\|_2}.
\]
If \( g_v \) satisfies the assumptions of the lemma, then \( h \) satisfies \( \|h\|_2 = t \), \( \|h\|_H \leq 2 \) and \( \|h\|_\infty \leq b \). Applying Lemma 8 (page 20) to the function \( h \), we obtain that for all \( t \geq \nu_{n,v} \), with probability greater than \( 1 - \exp(-c_2nt^2) \), we have
\[
|t - \|h\|_n| \leq \frac{bt}{2} \text{ for all } h \in G(t).
\]
This concludes the proof of the lemma.

\[
\square
\]

5.3.4 Proof of Lemma 10

We apply Corollary 4 to \( \phi(\varepsilon_1, \ldots, \varepsilon_n) = \frac{\sqrt{n}}{t} W_{n,n,v}(t) \).

Using Cauchy-Schwarz Inequality and the fact that \( \|g_v\|_n \leq t \),
\[
|\phi(\varepsilon) - \phi(\varepsilon')| \leq \frac{\sqrt{n}}{t} \sup_{\|g_v\|_n \leq t} \|g_v\|_n \|\varepsilon - \varepsilon'\|_n \leq \frac{\sqrt{n}}{t} t \|\varepsilon - \varepsilon'\|_n,
\]
leading to \( \|\phi\|_L = 1 \). So,
\[
P_{X,\varepsilon}\left( \left| \frac{\sqrt{n}}{t} W_{n,n,v}(t) - \frac{\sqrt{n}}{t} E_v W_{n,n,v}(t) \right| \geq u \right) \leq 2B \exp\left( -\frac{u^2}{8A} \right),
\]
and Lemma 10 is proved by taking \( \delta = u/\sqrt{n} \).

\[
\square
\]

5.3.5 Proof of Lemma 11

We start with the proof of (60) in Lemma 11 by applying once again Corollary 4 to the function
\[
\phi(\varepsilon) = \phi(\varepsilon_1, \ldots, \varepsilon_n) = \frac{\sqrt{n}}{2t} W_{n,2,v}(t).
\]
On the event \( \Omega_{v,t} \) defined by (58), we have
\[
\|g_v\|_n \leq \frac{bt}{2} + \|g_v\|_2.
\]
Besides if \( \|g_v\|_H \leq 2 \), then \( \|g_v\|_\infty \leq 2 \). Therefore applying Lemma 8 with \( b = 2 \), we get that if \( \|g_v\|_2 \leq t \),
\[
|\phi(\varepsilon) - \phi(\varepsilon')| \leq \frac{\sqrt{n}}{2t} \sup_{\|g_v\|_n \leq 2t} \|g_v\|_n \|\varepsilon - \varepsilon'\|_n \leq \frac{\sqrt{n}}{2t} 2t \|\varepsilon - \varepsilon'\|_n,
\]
32
leading to $\|\phi\|_L = 1$. So,

$$P_{X,\varepsilon} \left\{ \left\{ \frac{\sqrt{n}}{2t} W_{n,2,v}(t) - \frac{\sqrt{n}}{2t} E_{\varepsilon}(W_{n,2,v}(t)) \right\} \cap \Omega_{v,t} \right\} \leq 2B \exp \left( -\frac{u^2}{8A} \right),$$

and inequality (60) in Lemma 11 is proved by taking $\delta = 2u/\sqrt{n}$.

We now come to the proof of the inequality (61) in Lemma 11 using a Poissonian inequality for self-bounded processes (see [Boucheron, Lugosi, and Massart 2000]) and Theorem 5.6, p 158 in [Massart and Picard 2007]). Let us recall it in the particular case we are interested in:

**Theorem 3.** Let $X_1, \ldots, X_n$ be $n$ i.i.d. random variables. For $i \in \{1, \ldots, n\}$ let

$$X_{(i)} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n).$$

Let $h$ be a non-negative and bounded measurable function of $X = (X_1, \ldots, X_n)$. Assume that for all $i \in \{1, \ldots, n\}$, there exists a measurable function $h_i$ of $X_{(i)}$ such that $0 < h - h_i \leq 1$, and $\sum_{i=1}^{n} (h - h_i) \leq h$. Then, for all $x > 0$, we have

$$P(h \geq E(h) + x) \leq \exp \left( -\frac{x^2}{2E(h)} \right).$$

We apply this result to $h$ defined as

$$h = h(X_1, \ldots, X_n) = nE_{\varepsilon} W_{n,2,v}(t) = nE_{\varepsilon} \sup \left\{ |V_{n,\varepsilon}(g_v)|, \|g_v\|_2 \leq t, \|g_v\|_{\mathcal{H}_v} \leq 2 \right\}.$$

The variable $h$ is positive, and because the distribution of $(\varepsilon_1, \ldots, \varepsilon_n)$ is symmetric, we have that

$$h = E_{\varepsilon} \sup \left\{ nV_{n,\varepsilon}(g_v), \|g_v\|_2 \leq t, \|g_v\|_{\mathcal{H}_v} \leq 2 \right\}.$$

Let $\tau$ be the function in $\mathcal{H}_v$ such that $h = E_{\varepsilon} nV_{n,\varepsilon}(\tau)$ (note that $\tau$ depends on $(X_1, \ldots, X_n)$ and on $(\varepsilon_1, \ldots, \varepsilon_n)$), and let

$$h_i = E_{\varepsilon} \sup_{g_v} \sum_{j \neq i} \varepsilon_j g_v(X_j).$$

We show that $h$ and $h_i$ satisfy the assumptions of Theorem 3

$$h - h_i = E_{\varepsilon} \left( \varepsilon_i \tau(X_i) + \sum_{j \neq i} \varepsilon_j \tau(X_j) - \sup_{g_v} \sum_{j \neq i} \varepsilon_j g_v(X_j) \right),$$

$$\leq E_{\varepsilon} \left( \varepsilon_i \tau(X_i) \right),$$

$$\leq E_{\varepsilon} \left( |\varepsilon_i| \sup_{x \in \mathcal{X}} |\tau(X)| \right),$$

$$\leq 2E_{\varepsilon} \left( |\varepsilon_i| \right),$$

where the last inequality comes from the fact that $\sup_{x \in \mathcal{X}} |\tau(X)| \leq \|\tau\|_{\mathcal{H}_v} \leq 2$.

Let $Z = (Z_1, \ldots, Z_n)$ be i.i.d. random variables distributed with density $\pi_{\alpha} \in \mathcal{D}$ defined in Equation (2) with $\sqrt{\text{var}(Z_i)} = \sigma_i$. Take $\varepsilon_i = Z_i/\sigma_i$ for $i = 1, \ldots, n$, then $\|\varepsilon_i\| = 1$ and $E_{\varepsilon}(|\varepsilon_i|) = E_{Z}(|Z_i|)/\sigma_i$. We have:

$$E_Z(|Z_i|) = \int |Z_i| a_{\alpha} \exp \left( -|Z_i|^\alpha \right) dZ_i.$$

Take $|Z_i| = u^{1/\alpha}$ and

$$dZ_i = \left\{ \begin{array}{ll} \frac{1}{\alpha} u^{\frac{1}{\alpha} - 1} du & \text{if } Z_i \geq 0, \\ -\frac{1}{\alpha} u^{\frac{1}{\alpha} - 1} du & \text{if } Z_i \leq 0. \end{array} \right.$$
Therefore,
\[ E_Z(|Z_i|) = \int_0^{+\infty} a_u u^{\frac{1}{\alpha}} \exp(-u) \frac{1}{\alpha} u^{\frac{1}{\alpha} - 1} du - \int_{+\infty}^{0} a_u u^{\frac{1}{\alpha}} \exp(-u) \frac{1}{\alpha} u^{\frac{1}{\alpha} - 1} du, \]
\[ = 2 \int_0^{+\infty} a_u u^{\frac{1}{\alpha}} \exp(-u) \frac{1}{\alpha} u^{\frac{1}{\alpha} - 1} du, \]
\[ = a_\alpha \int_0^{+\infty} \frac{2}{\alpha} u^{\frac{1}{\alpha} - 1} \exp(-u) du, \]
\[ = a_\alpha \frac{2}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) = a_\alpha \Gamma\left(1 + \frac{2}{\alpha}\right), \]

where \( \Gamma(.) \) is the gamma function.

It follows that,
\[ h - h_i \leq \frac{2a_\alpha}{\sigma_\alpha} \Gamma\left(1 + \frac{2}{\alpha}\right). \]

Moreover, \( h - h_i \geq 0 \) since
\[ h = E_\varepsilon \left(\sup_{g\nu} \sum_{j=1}^{n\nu} \varepsilon_j g_\nu(\text{X}_j)\right) = E_\varepsilon \left(\sum_{j=1}^{n\nu} \varepsilon_j g_\nu(\text{X}_j)\right) \geq E_\varepsilon \left(\sup_{g\nu} \sum_{j=1}^{n\nu} \varepsilon_j g_\nu(\text{X}_j)\right) = h_i. \]

Finally we have:
\[ \sum_i (h - h_i) = \sum_{i=1}^{n} E_\varepsilon \left(\varepsilon_i \tau(\text{X}_i) + \sum_{j \neq i} \varepsilon_j \tau(\text{X}_j) - \sup_{g\nu} \sum_{j \neq i} \varepsilon_j g_\nu(\text{X}_j)\right) \leq \sum_{i=1}^{n} E_\varepsilon \varepsilon_i \tau(\text{X}_i) = h. \]

Therefore, following Theorem 3 we get that for all positive \( u \)
\[ P_{X,\varepsilon}\left(E_\varepsilon W_{n,2,v}(t) - E_{X,\varepsilon} W_{n,2,v}(t) \leq \frac{u}{n}\right) \leq \exp\left(-\frac{u^2}{E_{X,\varepsilon} W_{n,2,v}(t)}\right). \]

As \( E_{X,\varepsilon} W_{n,2,v}(t) \leq Q_{n,v}(t) \), see Lemma 7 page 20 we get the expected result since for all positive \( x \)
\[ P_X \left( E_\varepsilon W_{n,2,v}(t) \geq E_{X,\varepsilon} W_{n,2,v}(t) + x \right) \leq \exp\left(-\frac{nx^2}{Q_{n,v}(t)}\right). \]

\[ \square \]

### 5.3.6 Proof of Lemma 12

From Lemma 10 page 20 with \( t = \lambda_{n,v} = \delta \), with probability greater than \( 1 - 2B \exp(-n\lambda^2_{n,v}/8a) \), we get that:
\[ E_\varepsilon(W_{n,v,n}(\lambda_{n,v})) \leq \lambda^2_{n,v} + W_{n,v,n}(\lambda_{n,v}) \] (80)

The next step consists in comparing \( W_{n,v,n}(\lambda_{n,v}) \) and \( W_{n,2,v}(2\lambda_{n,v}) \). Recall that \( \lambda_{n,v} \geq \nu_{n,v}, \) see 11.

Let \( g_\nu \) such that \( \|g_\nu\| \leq \lambda_{n,v} \).

- When \( \|g_\nu\| \leq \lambda_{n,v} \), according to Lemma 8 (page 20), taking \( b = 2 \), since \( \|g_\nu\| \leq \lambda_{n,v} \), we get that with probability greater than \( 1 - \exp(-c_2n\lambda^2_{n,v}) \),
\[ \|g_\nu\| \leq \lambda_{n,v} \leq \|g_\nu\| \leq \|g_\nu\| + \lambda_{n,v} \leq 2\lambda_{n,v}. \]

- When \( \|g_\nu\| \geq t \), we apply Lemma 9 (page 20) with \( b = 2 \). For any function \( g_\nu \) such that \( \|g_\nu\| \leq 2 \), and \( \|g_\nu\| \geq \lambda_{n,v} \), we have \( \|g_\nu\| \leq 2\|g_\nu\| \leq 2\lambda_{n,v}. \)

This implies that, with probability greater than \( 1 - \exp(-c_2n\lambda^2_{n,v}) \) we have
\[ W_{n,v,n}(\lambda_{n,v}) \leq W_{n,2,v}(2\lambda_{n,v}). \]

34
We now study the process \( W_{n,2,v}(\lambda_{n,v}) \). By applying (63) in Lemma 11 on page 20 with \( \delta = t = \lambda_{n,v} \) we get that with probability greater than \( 1 - 2B \exp(-n\lambda_{n,v}^2/32A) \)
\[
W_{n,2,v}(\lambda_{n,v}) \leq \lambda_{n,v}^2 + E_\varepsilon(W_{n,2,v}(\lambda_{n,v})).
\]

It follows that
\[
E_\varepsilon W_{n,2,v}(\lambda_{n,v}) \leq \lambda_{n,v}^2 + W_{n,2,v}(\lambda_{n,v}),
\]
\[
\leq \lambda_{n,v}^2 + W_{n,2,v}(2\lambda_{n,v}),
\]
\[
\leq 5\lambda_{n,v}^2 + E_\varepsilon(W_{n,2,v}(2\lambda_{n,v})).
\]

Next, we apply (61) in Lemma 11 with \( t = 2\lambda_{n,v} \) and \( x = 4\lambda_{n,v} \). We get that
\[
E_\varepsilon W_{n,2,v}(2\lambda_{n,v}) \leq 4\lambda_{n,v}^2 + E_{X,\varepsilon}(W_{n,2,v}(2\lambda_{n,v})),
\]
with probability greater than
\[
1 - 2\exp(-16n\lambda_{n,v}^4/Q_{n,v}(2\lambda_{n,v})) \geq 1 - 2\exp(-4n\lambda_{n,v}^2/\Delta).
\]
The last inequality comes from the definition of \( \nu_{n,v} \), see (10), and from the fact that \( \lambda_{n,v} \geq \nu_{n,v} \), see (11).

Putting everything together, we get that with probability greater than \( 1 - c_1 \exp(-c_2 n\lambda_{n,v}^2) \) for some positive constants \( c_1, c_2 \),
\[
E_\varepsilon W_{n,2,v}(\lambda_{n,v}) \leq 9\lambda_{n,v}^2 + E_{X,\varepsilon}(W_{n,2,v}(2\lambda_{n,v})),
\]
\[
\leq 9\lambda_{n,v}^2 + Q_{n,v}(2\lambda_{n,v}), \text{ thanks to Lemma 7 on page 20}
\]
\[
\leq 9\lambda_{n,v}^2 + 4\Delta \lambda_{n,v}^2.
\]

Applying once again Lemma 10 on page 20 we get that
\[
W_{n,2,v}(\lambda_{n,v}) \leq E_\varepsilon W_{n,2,v}(\lambda_{n,v}) + \lambda_{n,v}^2 \leq \left(10 + 4\Delta\right)\lambda_{n,v}^2.
\]

This ends the proof of the lemma by taking \( \kappa = 10 + 4\Delta \).

\[\square\]

6 Proof of Corollary 1

According to Theorem 1 we have with high probability,
\[
\|\hat{f} - m\|^2_n \leq C \inf_{f \in \mathcal{F}} \left\{ \|m - f\|^2_n + \sum_{v \in S_f} (\mu_v + \gamma_v^2) \right\}.
\]

Besides, for all \( K > 0 \),
\[
\|\hat{f} - m\|^2_n \leq (1 + K)\|\hat{f} - f\|^2_n + (1 + \frac{1}{K})\|m - f\|^2_n.
\]

We consider once again two cases defined in page 19.

Case 1: \( \|\hat{f} - f\|_2 \leq \|\hat{f} - f\|_n \).

In this case Equation (82) gives,
\[
\|\hat{f} - m\|^2_n \leq (1 + K)\|\hat{f} - f\|^2_n + (1 + \frac{1}{K})\|m - f\|^2_n.
\]

Then, using Equations (52) and (81) we obtain the result.

Case 2: \( \|\hat{f} - f\|_2 \geq \|\hat{f} - f\|_n \).

Apply Lemma 6 (page 19) and conclude that conditioning on the events \( \mathcal{T} \) and \( \mathcal{A} \), defined by (47) and (52), \( \hat{f} - f \) belongs to \( \mathcal{G}(f) \) defined in Lemma 6. Now, conditioning on the event \( \mathcal{C} \) we get the result as in Case 1 since,
\[
\|\hat{f} - f\|_2 \leq \sqrt{2}\|\hat{f} - f\|_n.
\]

\[\square\]
7 Proofs of Section 4.2

7.1 Proof of Remark 7

From Lemma 1 we have $U_{\alpha}(u) \subset (u^{1/2}B_2 + u^{1/\alpha}B_{\delta})$. It suffices to show that $(u^{1/2}B_2 + u^{1/\alpha}B_{\delta}) \subseteq 2 \times \max(u^{1/2}, u^{1/\alpha})B_2$.

Consider $x \in u^{1/2}B_2 + u^{1/\alpha}B_{\delta}$, $x = y + z$ with $y \in u^{1/2}B_2$, means $\sum_{i=1}^n y_i^2 \leq u$, and $z \in u^{1/\alpha}B_{\delta}$, means $\sum_{i=1}^n z_i^2 \leq u$. Moreover, we know that $\|x\| \leq \|y\| + \|z\|$ which leads to $\|x\| \leq u^{1/2} + u^{1/\alpha}$ and $\|x\|^2 \leq 2(u + u^{2/\alpha}) \leq 4 \times \max(u, u^{2/\alpha})$.

\[\square\]

7.2 Proof of Corollary 3

From Equation (33) we have $N(2 \times \max(M^{1/2}, M^{1/\alpha}), T, \|\cdot\|) \leq \exp(KM)$. Using this on $\varepsilon T$ for $\varepsilon > 0$ we have $sM = E_Z \sup_{T \in T} \sum_{i=1}^n t_i Z_i$ and,

$$N(2 \times \max((sM)^{1/2}, (sM)^{1/\alpha}), T, \|\cdot\|) \leq \exp(KsM).$$

Moreover,

$$N(2 \times \max((sM)^{1/2}, (sM)^{1/\alpha}), T, \|\cdot\|) = N\left(\frac{2}{s} \times \max((sM)^{1/2}, (sM)^{1/\alpha}), T, \|\cdot\|\right),$$

since for all $t_1, t_2 \in T$ and some constant $C$, $\|st_1 - st_2\| \leq C$ is equivalent to $\|t_1 - t_2\| \leq C/s$.

We obtain then,

$$N\left(\frac{2}{s} \times \max((sM)^{1/2}, (sM)^{1/\alpha}), T, \|\cdot\|\right) \leq \exp(KsM).$$

As in Remark 4 for $u = sM$ we consider two following cases (recall that $1 < \alpha < 2$):

(i) If $sM \leq 1$ we have $(sM)^{1/\alpha} \leq (sM)^{1/2}$ and so,

$$N\left(\frac{M}{s}^{1/2}, T, \|\cdot\|\right) \leq \exp(KsM).$$

Take $\delta = 2(M/s)^{1/2}$ and thus $s = 4M/\delta^2$. Moreover, $sM \leq 1$ (i.e. $(4M/\delta^2) \times M \leq 1$) and so $\delta \geq 2M$. Finally, we obtain in this case:

$$\forall \delta \geq 2M, \log N(\delta, T, \|\cdot\|) \leq K\left(\frac{2M}{\delta}\right)^2.$$

(ii) If $sM \geq 1$ we have $(sM)^{1/\alpha} \leq (sM)^{1/2}$ and so,

$$N\left(\frac{2}{s} (sM)^{1/\alpha}, T, \|\cdot\|\right) \leq \exp(KsM).$$

Take $\delta = (2/s)(sM)^{1/\alpha}$ and thus $s = (2/\delta)^{\alpha/(\alpha-1)} M^{1/(\alpha-1)}$. Moreover, $sM \geq 1$ (i.e. $(2M/\delta)^{\alpha/(\alpha-1)} \geq 1$) and so $0 < \delta \leq 2M$. Finally, we obtain in this case:

$$\forall 0 < \delta \leq 2M, \log N(\delta, T, \|\cdot\|) \leq K\left(\frac{2M}{\delta}\right)^{\alpha/(\alpha-1)} = K\left(\frac{2M}{\delta}\right)^\alpha.$$

\[\square\]

8 Proofs of Section 4.3

8.1 Proof of Lemma 2

In order to prove this Lemma it suffices to show that $\Pi_{\alpha} \in \mathcal{M}(m, \rho^2)$ for some $m$. To do so, we use Example 1.
First show \( d \log \Pi_\alpha([t, \infty))/dt \leq -t/\rho^2 \):

We know that

\[
\frac{d}{dt} \log \Pi_\alpha([t, \infty)) = \frac{d}{dt} \log(1 - \Pi_\alpha((-\infty, t])) = -\frac{\pi_\alpha(t)}{1 - \Pi_\alpha((-\infty, t))} = -\frac{\pi_\alpha(t)}{\Pi_\alpha([t, \infty))}.
\]

For all \( t > 0 \) we have,

\[
\Pi_\alpha([t, \infty)) = \int_t^{\infty} a_\alpha \exp(-|x|^\alpha)dx = \int_t^{\infty} a_\alpha \exp(-x^\alpha)dx.
\]

Take \( x = u^{1/\alpha} \), so \( dx = (1/\alpha)u^{(1/\alpha)-1}du \), and

\[
\Pi_\alpha([t, \infty)) = \int_t^{\infty} \frac{a_\alpha}{\alpha} u^{(1/\alpha)-1} \exp(-u)du,
\]

where \( \Gamma(\frac{1}{\alpha}, t^\alpha) \) is incomplete gamma function. Moreover, for \( s \in \mathbb{R} \) as \( x \to \infty \),

\[
\frac{\Gamma(s, x)}{x^{s-1} \exp(-x)} \to 1.
\]

Therefore,

\[
\Pi_\alpha([t, \infty)) = \frac{a_\alpha}{\alpha} t^{1-\alpha} \exp(-t^\alpha).
\]

Since \( t > 0 \) so \( \pi_\alpha(t) = a_\alpha \exp(-t^\alpha) \), and

\[
\frac{d}{dt} \log \Pi_\alpha([t, \infty)) = -\frac{a_\alpha a_\alpha \exp(-t^\alpha)}{a_\alpha t^{1-\alpha} \exp(-t^\alpha)} = -\alpha t^{\alpha-1}.
\]

The inequality \(-\alpha t^{\alpha-1} \leq -t/\rho^2 \) (i.e. \( t^{\alpha-2} \geq 1/\alpha \rho^2 \)) holds for all \( \alpha > 2 \) and \( t \geq (1/\alpha \rho^2)^{1/(\alpha-2)} \).

Second show \( d \log \Pi_\alpha((-\infty, -t))/dt \leq -t/\rho^2 \):

The probability distribution \( \Pi_\alpha \) is symmetric, therefore \( \Pi_\alpha((-\infty, -t]) = \Pi_\alpha([t, \infty)) \), and

\[
\frac{d}{dt} \log \Pi_\alpha((-\infty, -t]) = \frac{d}{dt} \log \Pi_\alpha([t, \infty)) = -\alpha t^{\alpha-1},
\]

which is smaller than \(-t/\rho^2 \) if \( \alpha > 2 \) and \( t \geq (1/\alpha \rho^2)^{1/(\alpha-2)} \).

Take \( m = (1/\alpha \rho^2)^{1/(\alpha-2)} \), then for \( x \geq m \), \( \Pi_\alpha \) verifies the Equations \( (39) \). That is \( \Pi_\alpha \in \mathcal{M}((1/\alpha \rho^2)^{1/(\alpha-2)}, \rho^2) \).

\[
\square
\]

8.2 Proof of Remark \[ \]

If \( \alpha = 2 \), according to the Laplace transform of the Gaussian function we have

\[
E(\exp(s|Z|)) = 2a_\alpha \sqrt{\pi} \exp\left(\frac{s^2}{4}\right).
\]

(83)

If \( \alpha > 2 \) we have,

\[
E(\exp(s|Z|)) = \int_{-\infty}^{+\infty} \exp(s|z|)a_\alpha \exp(-|z|^\alpha)dz = 2a_\alpha S,
\]

where

\[
S = \int_0^{+\infty} \exp(sz - z^\alpha)dz = \int_0^1 \exp(sz - z^\alpha)dz + \int_{s_1}^{+\infty} \exp(sz - z^\alpha)dz + \int_{s_2}^{s_1} \exp(sz - z^\alpha)dz.
\]

37
For $z \in [0,1]$ we have $\exp(-z^\alpha) \leq 1$ and so
\[ S_1 \leq \int_0^1 \exp(sz)dz = \frac{\exp(s)-1}{s}. \]
For $z \geq 1$ we have $\exp(z^2 - z^\alpha) < 1$ and so
\[ S_2 = \int_1^{+\infty} \exp(sz - z^2 + z^2 - z^\alpha)dz < \int_1^{+\infty} \exp(sz)dz < \sqrt{\pi} \exp\left(\frac{s^2}{4}\right), \]
where the last inequality is obtained using Equation (83). Finally, we obtain
\[ S < \frac{\exp(s) - 1}{s} + \sqrt{\pi} \exp\left(\frac{s^2}{4}\right), \]
and therefore
\[ E(\exp(s|Z|)) < 2a_n\left(\frac{\exp(s) - 1}{s} + \sqrt{\pi} \exp\left(\frac{s^2}{4}\right)\right). \]

\[ \Box \]

8.3 Proof of Corollary 4

We suppose that the inequality (41) holds and we want to find an upper bound for $P\left(|\phi(Z) - E(\phi(Z))| \geq u\right)$. Using the Markov’s inequality we have,
\[ P\left(|\phi(Z) - E(\phi(Z))| > u\right) = P\left(\exp(\lambda|\phi(Z) - E(\phi(Z))|^2) > \exp(\lambda u^2)\right), \]
\[ \leq \exp(-\lambda u^2)E\left(\exp(\lambda|\phi(Z) - E(\phi(Z))|^2)\right). \]  \hfill (84)

To demonstrate the result of the Theorem, it suffices to find an upper bound for the following quantity
\[ E\left(\exp(\lambda|\phi(Z) - E(\phi(Z))|^2)\right). \]
Let $Z_1$ and $Z_2$ be two independent random variables distributed with the same law, then for all $u' > 0$ we have:
\[ P\left(|Z_1 - Z_2| > u'\right) \leq P\left(|Z_1 - M(\phi(Z_1))| > \frac{u'}{2}\right) + P\left(|Z_2 - M(\phi(Z_2))| > \frac{u'}{2}\right). \]  \hfill (85)
Furthermore, for all convex function $\psi$ we have:
\[ E\left(\psi(Z_1 - E(Z_1))\right) = \int \psi\left(\int (z_1 - z_2)dP(z_2)\right)dP(z_1). \]
Applying the Jensen’s inequality we obtain then,
\[ E\left(\psi(Z_1 - E(Z_1))\right) \leq \int \left(\int \psi(z_1 - z_2)dP(z_2)\right)dP(z_1), \]
\[ \leq E\left(\psi(Z_1 - Z_2)\right). \]  \hfill (86)
Set $\psi(t) = \exp(\lambda t^2)$ for $\lambda > 0$, $Z_1 = \phi(Z)$ and $Z_2 = \phi(Z')$. Since $\psi(t)$ is convex, then Equation (86) gives:
\[ E\left(\exp(\lambda|\phi(Z) - E(\phi(Z))|^2)\right) \leq E\left(\exp(\lambda(\phi(Z) - \phi(Z'))^2)\right). \]  \hfill (87)
For all non-negative random variables $Z$ we have $E(Z) = \int_{[0,\infty)} P(Z \geq z)dz$. So, we obtain from Equation (87):
\[ E\left(\exp(\lambda|\phi(Z) - E(\phi(Z))|^2)\right) \leq \int_0^{\infty} P\left(\exp(\lambda(\phi(Z) - \phi(Z'))^2) > t\right)dt. \]
Using Equation (85) and simple calculations leads to:

\[ E\left( \exp(\lambda |\phi(Z) - E(\phi(Z))|^2) \right) \leq \int_{1}^{\infty} P\left( |\phi(Z) - \phi(Z')| > \sqrt{\frac{\log(t)}{\lambda}} \right) dt, \]

\[ \leq 2\int_{1}^{\infty} P\left( |\phi(Z) - M(\phi(Z))| > \frac{1}{2} \sqrt{\frac{\log(t)}{\lambda}} \right) dt. \]  

(88)

In this step we can use the result in Equation (41), from which we obtain:

\[ E\left( \exp(\lambda |\phi(Z) - E(\phi(Z))|^2) \right) \leq 2B \int_{1}^{\infty} \exp\left( -\frac{\log(t)}{4\lambda A} \right) dt, \]  

(89)

and, therefore,

\[ E\left( \exp(\lambda |\phi(Z) - E(\phi(Z))|^2) \right) \leq \frac{8\lambda AB}{1 - 4\lambda A}, \quad \forall \lambda < \frac{1}{4A}. \]  

(90)

The proof is complete by taking \( \lambda = 1/8A. \)
References

Adamczak, R. (2005, 06). Logarithmic sobolev inequalities and concentration of measure for convex functions and polynomial chaos. Bulletin of the Polish Academy of Sciences Mathematics 53.

Aronszajn, N. (1950). Theory of reproducing kernels. Transactions of the American Mathematical Society 68(3), 337–404.

Bartlett, P. L., O. Bousquet, and S. Mendelson (2005, 08). Local rademacher complexities. Ann. Statist. 33(4), 1497–1537.

Bednorz, W. (2014). Some remarks on the sudakov minoration. ArXiv e-prints.

Berlinget, A. and C. Thomas-Agnan (2003). Reproducing Kernel Hilbert Spaces in Probability and Statistics. Springer US.

Bobkov, S. and M. Ledoux (1997, Mar). Poincaré’s inequalities and talagrand’s concentration phenomenon for the exponential distribution. Probability Theory and Related Fields 107(3), 383–400.

Borell, C. (1974, 12). Convex measures on locally convex spaces. Ark. Mat. 12(1-2), 239–252.

Boucheron, S., G. Lugosi, and P. Massart (2000). A sharp concentration inequality with applications. Random Struct. Algorithms 16, 277–292.

Chung, F. and L. Lu (2006). Concentration inequalities and martingale inequalities: a survey. Internet Math. 3(1), 79–127.

Durrande, N., D. Ginsbourger, O. Roustant, and L. Carraro (2013). Anova kernels and rkhs of zero mean functions for model-based sensitivity analysis. Journal of Multivariate Analysis 115, 57 – 67.

Gentil, I., A. Guillin, and L. Miclo (2005). Modified logarithmic sobolev inequalities and transportation inequalities. Probability Theory and Related Fields 133 (3), 409–436.

Gentil, I., A. Guillin, and L. Miclo (2007, 04). Modified logarithmic sobolev inequalities in null curvature. Rev. Mat. Iberoamericana 23(1), 235–258.

Gross, L. (1975). Logarithmic sobolev inequalities. American Journal of Mathematics 97(4), 1061–1083.

Hoeffding, W. (1948, 09). A class of statistics with asymptotically normal distribution. Ann. Math. Statist. 19(3), 293–325.

Huet, S. and M.-L. Taupin (2017). Metamodel construction for sensitivity analysis. ESAIM: Procs 60, 27–69.

Koltchinskii, V. and M. Yuan (2010, 12). Sparsity in multiple kernel learning. Ann. Statist. 38(6), 3660–3695.

Latała, R. (2014). Sudakov-type minoration for log-concave vectors. Studia Mathematica 223(3), 251–274.

Ledoux, M. (1997). On talagrand’s deviation inequalities for product measures. ESAIM: Probability and Statistics 1, 63–87.

Ledoux, M. (2001). The Concentration of Measure Phenomenon. Mathematical surveys and monographs. American Mathematical Society.

Ledoux, M. and M. Talagrand (1991, May). Probability in Banach Spaces: isoperimetry and processes. Berlin: Springer.

Massart, P. (2000, 04). About the constants in talagrand’s concentration inequalities for empirical processes. Ann. Probab. 28(2), 863–884.

Massart, P. and J. Picard (2007). Concentration Inequalities and Model Selection: Ecole d’Eté de Probabilités de Saint-Flour XXXIII - 2003. Lecture Notes in Mathematics. Springer Berlin Heidelberg.

Meier, L., S. van de Geer, and P. Buhlmann (2009, 12). High-dimensional additive modeling. Ann. Statist. 37(6B), 3779–3821.

Mendelson, S. (2002). Geometric parameters of kernel machines. In Computational learning theory (Sydney, 2002), Volume 2375 of Lecture Notes in Comput. Sci., pp. 29–43. Springer, Berlin.

Milman, V. D. and G. Schechtman (1986). Asymptotic Theory of Finite Dimensional Normed Spaces. New York, NY, USA: Springer-Verlag New York, Inc.
Pisier, G. (1989). *The volume of convex bodies and Banach space geometry*, Volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

Raskutti, G., M. J. Wainwright, and B. Yu (2012, February). Minimax-optimal rates for sparse additive models over kernel classes via convex programming. *J. Mach. Learn. Res.* 13(1), 389–427.

Saitoh, S. (1988). *Theory of reproducing kernels and its applications*. Pitman research notes in mathematics series. Longman Scientific & Technical.

Shu, Y. and M. Strzelecki (2017, 02). A characterization of a class of convex log-sobolev inequalities on the real line. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 54.

Sobol, I. M. (1993). Sensitivity estimates for nonlinear mathematical models. In *Sensitivity Estimates for Nonlinear Mathematical Models*.

Talagrand, M. (1993, 01). Regularity of infinitely divisible processes. *Ann. Probab.* 21(1), 362–432.

Talagrand, M. (1994). The supremum of some canonical processes. *American Journal of Mathematics* 116(2), 283–325.

van de Geer, S., R. Gill, B. Ripley, S. Ross, B. Silverman, and M. Stein (2000). *Empirical Processes in M-Estimation*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.