Research Article
Some New Results on Trans-Sasakian Manifolds

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In this paper, we classify trans-Sasakian manifolds which are realized as real hypersurfaces in a complex space form. We also investigate trans-Sasakian manifolds whose Reeb vector fields are harmonic-Killing. The above results bring some new characterizations for the property of trans-Sasakian 3-manifolds.

1. Introduction

In differential geometry of almost-contact Riemannian manifolds, the so-called trans-Sasakian manifolds play important roles when studying topology as well as geometry of almost-contact structures. Here, an almost-contact metric manifold $M^{2n+1}$ of dimension $2n+1$, $n \geq 1$, together with its almost-contact metric structure $(\phi, \xi, \eta, g)$, is said to be a trans-Sasakian manifold (it is, often referred to, of type $(\alpha, \beta)$) (see [1–3]) if it satisfies

$$
\nabla_X \phi Y = \alpha (g(X,Y)\xi - \eta(Y)X) + \beta (g(\phi X,Y)\xi - \eta(Y)\phi X),
$$

(1)

for all vector fields $X$ and $Y$, where both $\alpha$ and $\beta$ are smooth functions. The classical Sasakian, Kenmotsu, and cosymplectic manifolds (see [1]) are all its trivial cases.

In general, a trans-Sasakian manifold of type $(\alpha, \beta)$ is said to be proper (see [4–6]) when either $\alpha$ or $\beta$ vanishes identically. Marrero in [7] proved that a trans-Sasakian manifold of dimension greater than 3 must be proper. However, such a property holds not necessarily true for general trans-Sasakian manifolds of dimension three. In the past decade, to determine on what geometric conditions a connected, compact, or complete trans-Sasakian three-manifold is proper has been proposed by Deshmukh in [8] and later considered by many authors (see recent results by De et al. [9–12], Deshmukh et al. [8, 13–19], Wang and Wang and Liu [20, 21], Wang [4, 22, 23], Zhao [5, 6] and Ma and Pei [24].

It is interesting to point out that trans-Sasakian three-manifolds isometrically immersed in the Euclidean four-space $\mathbb{R}^4$ have been studied in [14]. In the present paper, extending Deshmukh’s above results, we consider a trans-Sasakian manifold of an arbitrary dimension immersed in a complex space form realized as a real hypersurface. As an immediate corollary, we also present a new characterization for the property of trans-Sasakian three-manifolds without compactness restriction. On the other hand, Zhao [6] provided a characterization for the property by considering the Reeb vector field of a trans-Sasakian three-manifold being affine Killing. In the present paper, we generalize such a result by weakening the above restriction; namely, we need only to suppose that the Reeb vector field is harmonic-Killing (see its definition in Section 4).

2. Trans-Sasakian Manifolds

Let $(M^{2n+1}, g)$ be a smooth Riemannian manifold of dimension $2n+1$ on which there exist a $(1,1)$-type, $(1,0)$-type, and $(0,1)$-type tensor fields $\phi$, $\xi$, and $\eta$, respectively. According to Blair [1], $M^{2n+1}$ is called an almost-contact metric manifold if
\[ \phi^2 = -1 + \eta \otimes \xi, \]
\[ \eta(\xi) = 1, \]
\[ \eta \circ \phi = 0, \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2} \]
for any vector fields \( X \) and \( Y \). \( \xi \) is said to be the Reeb or structure vector field. An almost-contact metric manifold is said to be normal if \( [\phi, \phi] = -2 d \eta \otimes \xi \), where \( [\phi, \phi] \) denotes the Nijenhuis tensor of \( \phi \). An almost-contact metric manifold is said to be trans-Sasakian if it satisfies equality \( (1) \). A three-dimensional almost-contact metric manifold is trans-Sasakian if and only if it is normal. This is not necessarily true for higher dimension.

A normal almost-contact metric manifold is said to be an \( \alpha \)-Sasakian manifold if \( d\eta = \alpha \Phi \) and \( d\Phi = 0 \), where \( \alpha \) is a nonzero constant. An \( \alpha \)-Sasakian manifold reduces to a Sasakian manifold when \( \alpha = 1 \). A normal almost-contact metric manifold is called a \( \beta \)-Kenmotsu manifold if it satisfies \( d\eta = 0 \) and \( d\Phi = 2\beta \eta \wedge \Phi \), where \( \beta \) is a nonzero constant. A \( \beta \)-Kenmotsu manifold becomes a Kenmotsu manifold when \( \beta = 1 \). A normal almost-contact metric manifold is said to be a cosymplectic manifold if it satisfies \( d\eta = 0 \) and \( d\Phi = 0 \). Obviously, the set of all \( \alpha \)-Sasakian manifolds (resp., \( \beta \)-Kenmotsu) is a proper subset of that of all trans-Sasakian manifolds of type \( (\alpha, 0) \) (resp., \( (0, \beta) \)).

Putting \( Y = \xi \) into \( (1) \) and using \( (2) \), we have
\[ \nabla_X \xi = -\alpha \phi X + \beta (X - \eta(X)\xi), \tag{3} \]
for any vector field \( X \). In this paper, all manifolds are assumed to be connected.

3. Trans-Sasakian Manifolds as Real Hypersurfaces in Complex Space Forms

Let \( M^n(c) \) be a complete and simply connected complex space form which is complex analytically isometric to the following:

(i) A complex projective space \( CP^n(c) \) if \( c > 0 \)
(ii) A complex Euclidean space \( C^n \) if \( c = 0 \)
(iii) A complex hyperbolic space \( CH^n(c) \) if \( c < 0 \)

Here, \( c \) is the constant holomorphic sectional curvature.

Let \( M \) be a real hypersurface immersed in a complex space form \( M^n(c) \) and \( N \) be a unit normal vector field of \( M \). We denote by \( \nabla \) the Levi-Civita connection of the metric \( g \) of \( M^n(c) \) and \( J \) the complex structure. Let \( g \) and \( \nabla \) be the induced metric from the ambient space and the Levi-Civita connection of the metric \( g \), respectively. Then, the Gauss and Weingarten formulas are given, respectively, as follows:
\[ \nabla_X Y = \nabla_X Y + g(AX, Y)N, \nabla_X N = -AX, \tag{4} \]
for any vector fields \( X \) and \( Y \), where \( A \) denotes the shape operator of \( M \) in \( M^n(c) \). For any vector field \( X \), we put
\[ JX = \phi X + \eta(X)N, \]
\[ JN = -\xi. \tag{5} \]

One can check that \( (2) \) holds and hence, on real hypersurfaces, there exist natural almost-contact metric structures. If the structure vector field \( \xi \) is principal, that is, \( A\xi = \delta \xi \) at each point, where \( \delta \ni = \eta(A\xi) \), then \( M \) is called a Hopf hypersurface and \( \delta \) is called Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., \( \nabla J = 0 \)) of \( M^n(c) \) and using \( (4) \) and \( (5) \), we have
\[ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{6} \]
for any vector fields \( X \) and \( AX, Y \). Let \( R \) be the Riemannian curvature tensor of \( M \). As \( M^n(c) \) is of constant holomorphic sectional curvature \( c \), the Gauss equation of \( M \) in \( M^n(c) \) is given by
\[ R(X, Y)Z = -\zeta \{ g(Y, Z)X - g(X, Z)Y + g(\phi Y, \phi X)X \]
\[ -g(\phi X, Z)Y - 2g(\phi X, \phi Y)Z \]
\[ + g(AY, Z)AX - g(AX, Z)AY, \tag{7} \]
for any vector fields \( X \) and \( XY, Y \).

Because an almost-contact metric structure exists on a real hypersurface, then it is very interesting to ask what almost-contact metric structure can be if it is realized as a real hypersurface in complex space forms? Some authors have studied contact, Sasakian, and generalized Sasakian space form structures on real hypersurfaces (see [25–27]).

Theorem 1. Let \( M^{2n-1} \) be a trans-Sasakian manifold. Then, \( M^{2n-1} \) is realized as a real hypersurface in a complex space form \( M^n(c) \), \( n \geq 2 \), if and only if the following statements are valid:

(1) If \( M^n(c) = CP^n(c) \), \( M^{2n-1} \) is locally congruent to a geodesic hypersphere.
(2) If \( M^n(c) = CH^n(c) \), \( M^{2n-1} \) is locally congruent to
   (i) a horosphere
   (ii) a geodesic hypersphere
   (iii) a tube around a totally geodesic \( CH^{n-1} \)
(3) If \( M^n(c) = C^n \), \( M^{2n-1} \) is locally congruent to
   (i) a hyperplane \( \mathbb{R}^{2n-1} \)
   (ii) a sphere \( S^{2n-1} \)
   (iii) a cylinder over a plane curve \( \gamma \times \mathbb{R}^{2n-2} \)

Proof. If a real hypersurface \( M^{2n-1} \) in a complex space form \( M^n(c) \) is trans-Sasakian, by definition, from (1) and (6), we get
\[ \begin{align*}
\alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)
&= \eta(Y)AX - g(AX, Y)\xi,
\end{align*} \tag{8} \]
for any vector fields \(X, Y, YX, Y\). In the above equality, setting \(Y = \xi\) gives
\[
AX = \eta(A X)\xi - \beta F X + \alpha(\eta(X)\xi - X),
\]
(9)
for any vector field \(X\). Obviously, it follows that \(A\xi = \eta(A\xi)\xi\) and hence, \(M^{2n+1}\) is Hopf. Using this in the previous equality, we get
\[
AX = (\alpha + \eta(A\xi))\eta(X)\xi - \beta F X - \alpha X,
\]
(10)
for any vector field \(X\). Recall that the shape operator is self-adjoint; it follows directly that \(\beta = 0\), and hence,
\[
AX = (\alpha + \eta(A\xi))\eta(X)\xi - \alpha X,
\]
(11)
for any vector field \(X\). Now, the hypersurface is totally \(\eta\)-umbilical. Conversely, the application of the above equality in (6) implies that the hypersurface is always a trans-Sasakian manifold of type \((\alpha, 0)\). Next, we divide our discussions into two cases.

When the ambient space is \(C^p\) or \(CH^n\), following [28, 29], we observe that a totally \(\eta\)-umbilical real hypersurface satisfying (11) is locally congruent to the following:

(i) A geodesic sphere of radius \(r\) in \(C^p\) with \(a = -((\sqrt{c} r/2)\cot(\sqrt{c} r/2))\), where \(0 < r < (\pi/\sqrt{c})\).

(ii) A horosphere in \(CH^n\) with \(a = -((\sqrt{c} r/2))\coth(\sqrt{c} r/2))\), where \(0 < r < \infty\).

(iii) A geodesic sphere of radius \(r\) in \(CH^n\) with \(a = -((\sqrt{c} r/2))(\tanh(\sqrt{c} r/2))\), where \(0 < r < \infty\).

(iv) A tube of radius \(r\) around a totally geodesic complex hyperplane \(CH^n-1\) in \(CH^n\) with \(a = -((\sqrt{c} r/2))(\tanh(\sqrt{c} r/2))\), where \(0 < r < \infty\).

Law the ambient space is \(C^n\), from Gauss equations (7) and (11), we see that the hypersurface is pseudo-Einstein, i.e.,
\[
QX = a((2n - 3)a - \eta(A\xi))X - (2n - 3)a(\alpha + \eta(A\xi))\eta(X)\xi,
\]
(12)
for any vector field \(X\), where \(Q\) denotes the Ricci operator. The remaining proof follows immediately from Proof of Theorem 1 in [27] (see also [30]).

The converse is easy to check.

In view of Theorem 1, a new characterization for the property of trans-Sasakian 3-manifolds is given. We remark that \(\alpha\) in two cases in the proof of Theorem 1 is both constant.

\[\square\]

**Corollary 1.** A trans-Sasakian 3-manifold is an \(\alpha\)-Sasakian manifold if it is realized as a real hypersurface in the complex space form.

As pointed out in the Introduction section, a trans-Sasakian 3-manifolds of type \((\alpha, \beta)\) realized as a hypersurface in \(\mathbb{R}^4\) is isometric to the Sasakian manifold \(S^3\) provided that the hypersurface is compact. Such a situation occurs in our Theorem 1 in view of (12) and (11) for \(n = 2\) (for more details, see [14, Theorem 2]).

### 4. Harmonic-Killing Reeb Vector Field

From [31], a vector field \(V\) on a Riemannian manifold \((M, g)\) is called affine Killing if
\[
\mathcal{L}_V V = 0,
\]
(13)
where \(V\) denotes the Levi-Civita connection of the metric \(g\) (see also [32]). According to [33, 34], a vector field \(V\) on a Riemannian manifold is called harmonic-Killing if each local parameter group of infinitesimal transformations associated to \(V\) is a group of harmonic maps. For any harmonic-Killing vector field \(V\), from Theorem 2.1 in [33], we have
\[
\text{tr}(\mathcal{L}_V V) = 0.
\]
(14)

By considering the Reeb vector field of trans-Sasakian three-manifolds being affine Killing, Zhao [6] studied the property of trans-Sasakian three-manifolds. In this section, we consider a weaker condition on trans-Sasakian manifolds of arbitrary dimensions.

**Lemma 1.** If the Reeb vector field of trans-Sasakian manifolds \(M^{2n+1}\) of type \((\alpha, \beta)\) is harmonic-Killing, then we have
\[
\xi(\beta) = -2\beta^2.
\]
(15)

Moreover, if \(n > 1\), we have \(d\beta = -2\beta^2\eta\).

**Proof.** Recall that on any differentiable manifold, there holds (see Yano ([35], pp. 23))
\[
(\nabla X L_{\xi} g - \nabla Y L_{\xi} g - \nabla [X,Y]g)(Y, Z) = -g((L_{\xi} V)(X, Y), Z) - g((L_{\xi} V)(X, Z), Y),
\]
(16)
for any vector fields \(X, Y,\) and \(Z\). Notice that in our case, the Riemannian metric \(g\) is parallel and it follows that
\[
(\nabla X L_{\xi} g)(Y, Z) = g((L_{\xi} V)(X, Y), Z) + g((L_{\xi} V)(X, Z), Y),
\]
(17)
for any vector fields \(X, Y,\) and \(Z\). Cyclically interchanging the roles of \(X, Y,\) and \(Z\) in the above equality, we obtain
\[
(\nabla Y L_{\xi} g)(Z, X) = g((L_{\xi} V)(Y, Z), X) + g((L_{\xi} V)(Y, X), Z),
\]
(18)
\[
(\nabla Z L_{\xi} g)(X, Y) = g((L_{\xi} V)(Z, X), Y) + g((L_{\xi} V)(Z, Y), X),
\]
(19)
for any vector fields \(X, Y,\) and \(Z\). The addition of (17) with (18) gives an equality; subtracting this equality from (19), with the aid of the symmetry of \(L_{\xi} V\), we have
for any vector fields \( X, Y \), and \( ZX, Y, Z \).

From (3), we have
\[
(\mathcal{L}_\xi g)(Y, Z) = 2\beta (g(Y, Z) - \eta(Y)\eta(Z)),
\]
for any vector fields \( X, Y \), and \( ZX, Y, Z \). By a direct calculation, taking the covariant derivative of the above equality, with the aid of (3), we have
\[
(\nabla_X L_\xi g)(Y, Z) = 2X(\beta) (g(Y, Z) - \eta(Y)\eta(Z)) + 2\alpha\beta g(\phi X, Y)\eta(Z)
\]
\[
+ 2\alpha\beta g(\phi X, Z)\eta(Y) - 2\beta^2 g(X, Y)\eta(Z)
\]
\[
- 2\beta^2 g(X, Z)\eta(Y) + 4\beta^2 \eta(X)\eta(Y)\eta(Z),
\]
for any vector fields \( X, Y \), and \( ZX, Y, Z \).

We consider a local orthonormal frame \( \{e_1, \ldots, e_{2n+1}\} \) of the tangent space at each point. By a direct calculation, from (22), we have
\[
\sum_{i=1}^{2n+1} \left( \nabla_{e_i} L_\xi g \right)(e_i, Z) = 2Z(\beta) - 2\beta(\beta)\eta(Z) - 4n\beta^2 \eta(Z),
\]
(23)

\[
\sum_{i=1}^{2n+1} \left( \nabla_{e_i} L_\xi g \right)(e_i, e_i) = 4nZ(\beta),
\]
(24)
for any vector field \( Z \).

If the Reeb vector field of a trans-Sasakian manifold is harmonic-Killing, from (14) and (20), we have
\[
\sum_{i=1}^{2n+1} \left( \nabla_{e_i} L_\xi g \right)(e_i, Z) + \sum_{i=1}^{2n+1} \left( \nabla_{e_i} L_\xi g \right)(Z, e_i)
\]
\[
= \sum_{i=1}^{2n+1} \left( \nabla_{e_i} L_\xi g \right)(e_i, e_i),
\]
for any vector field \( Z \), which is simplified by using (23) and (24) yielding
\[
(n - 1)D\beta = - (\beta(\beta) + 2n\beta^2)\eta,
\]
(26)
where \( D \) denotes the gradient operator. Obviously, taking the inner product of the above equality with \( \xi \) implies \( \beta(\beta) = -2\beta^2 \). Moreover, if \( n > 1 \), substituting \( \beta(\beta) = -2\beta^2 \) into (26) gives \( D\beta = -2\beta^2 \xi \).

\textbf{Lemma 2} (see [36]). \textit{If on a Riemannian manifold \( M \) there exists a Killing vector field \( \xi \) of constant length satisfying
\[
k^2(\nabla_X \xi - \nabla_\xi Y) = g(Y, \xi)X - g(X, Y)\xi,
\]
for a nonzero constant \( k \) and any vector fields \( X \) and \( Y \), then \( M \) is homothetic to a Sasakian manifold.}

Based on the above two lemmas, one of our main results is given.

\textbf{Theorem 2.} \textit{If the Reeb vector field of a compact and simply connected trans-Sasakian three-manifold of type \((\alpha, \beta)\) is harmonic-Killing, then the manifold is homothetic to a Sasakian 3-manifold.}

\textbf{Proof.} Taking into account (15), we have
\[
\text{div}\beta^3 \xi = 3\beta^2 \xi(\beta) + \beta^3 \text{div}\xi = -4\beta^4,
\]
(28)
where we have used (3). As the manifold is assumed to be compact, applying Stokes’ theorem on the above equality yields \( \beta = 0 \). Moreover, now from (21), we observe that \( \xi \) is Killing of constant length one. We also claim that \( \alpha \) is a constant and such an assertion is the same with the proof of Theorem 3.1 in [18]. If the constant \( \alpha = 0 \), the manifold is cosymplectic. However, this is impossible. In fact, if \( \alpha = 0 \), with the help of (3), we see that
\[
\text{div}(\xi, Y) = g(\nabla_\xi \eta, Y) - g(\nabla_Y \eta, X) = 0,
\]
(29)
for any vector fields \( X \) and \( Y \). Then, \( \eta \) is closed. Since the manifold is assumed to be simply connected, then \( \eta \) is exact; i.e., there exists a smooth function \( f \) on the manifold such that \( \eta = df \). Consequently, \( \xi = Df \) and there exists a point on the manifold on which \( Df \) vanishes, where we have used the compactness of the manifold. However, as seen in Section 2, \( \xi \) is always a unit vector field, contradicting the above statement. Thus, we conclude that \( \alpha \) is a nonzero constant. Finally, by (1), it is easy to check that (27) is valid. In fact, now the manifold is isometric to a three-sphere \( S^3(\alpha^2) \) (see [19]). This completes the proof.

\textbf{Theorem 2} is an extension of Corollary 3.7.1 in [6].

In Lemma 1, we have obtained a property, i.e., \( d\beta \wedge \eta = 0 \). In fact, such an equality is just one of the requirements when defining a local conformal cosymplectic manifold in the sense of Olszak (for more details, see [37]).

\textbf{Data Availability}

The data used to support the findings of this study are available from the corresponding author upon request.

\textbf{Conflicts of Interest}

The authors declare that there are no conflicts of interest regarding the publication of this article.

\textbf{Authors’ Contributions}

All authors contributed equally and significantly in writing this article. All authors have read and approved the final manuscript.

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