ON THE HYPER-LYAPUNOV MATRIX INCLUSION

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ABSTRACT. Disks in the complex plane, which are closed under inversion, are introduced. They turn to be centered on the real axis and bounded away from the imaginary axis. These disks are shown to be intimately linked to the Matrix Sign Function iteration scheme, used in matrix computations.

The set of all matrices whose spectrum lies within such a disk are said to satisfy a Hyper-Lyapunov inclusion. It is shown that these families can be characterized through Quadratic Matrix Inequalities.

The analogous Hyper-Stein sets, are of matrices whose spectrum lies in a sub-unit disk.

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1. INTRODUCTION

In the complex plane, the Cayley transform maps the unit disk to \( \mathbb{C}_R \), the right-half plane. Here we address respective subsets: Disks centered at the origin with radius less than one; and their image under the Cayley transform, which turn to be disks, closed under inversion.

In passing, it is pointed out that these invertible disks are natural objects in demonstrating the convergence of the Matrix Sign Function iteration scheme, used in matrix computations, see e.g. [10, Chapter 5], [16, Chapter 22], [13] and also [5, 6, 17, 21].
Extending the framework from scalars to matrices, leads to the classical Stein and Lyapunov inclusions, again related through the Cayley transform, see e.g. [2], [16, Section 5.3], [25].

As mentioned, we here focus on the respective subsets: The Hyper-Stein (e.g. matrices whose spectral radius is strictly less than one) and the Hyper-Lyapunov inclusions. A family of matrices satisfying a given Hyper-Lyapunov inclusion, can be equivalently characterized by Quadratic Matrix Inequalities.

By introducing a parameter \( \eta \in (1, \infty] \) we show that whenever \( \eta_b > \eta_a \), the set associated with \( \eta_a \) is contained in the set associated with \( \eta_b \). Moreover, when \( \eta \) approaches infinity, the classical Lyapunov case (associated with Linear Matrix Inequalities, see e.g. [4]) is recovered.

This work is outlined as follows: Relevant background is reviewed in Section 2. In Section 3 we address scalar disks. Stein inclusions, i.e. families of matrices, all with spectrum in a sub-unit disk, are studied in 4. The corresponding image under the Cayley transform, the Lyapunov inclusions, are explored in Section 5.

2. Background

In this section we summarize the background relevant to the sequel.

Let \( \mathbb{H}_n \) (\( \mathbb{H}_n \)), denote the set of \( n \times n \) Hermitian (non-singular) matrices and let \( \mathbb{P}_n \), (\( \mathbb{P}_n \)) be the subsets of positive (semi)-definite matrices. We shall follow the convention that \( \mathbb{P}_n \) is the closure of the open set \( \mathbb{P}_n \).

For a prescribed \( H \in \mathbb{H}_n \) let us define the set of all matrices satisfying the following Lyapunov inclusion,

\[
L_H = \{ A \in \mathbb{C}^{n \times n} : HA + A^*H \in \mathbb{P}_n \}
\]

(2.1)

We consider \( L_H \) as the closure of the open set \( L_H \), see e.g. [5].

We now have an analogous family of the Stein inclusion,

\[
S_H = \{ A \in \mathbb{C}^{n \times n} : H - A^*HA \in \mathbb{P}_n \}
\]

(2.2)

We consider \( S_H \) as the closure of the open set \( S_H \). A characterization through structure of the set \( S_H \), was given by T. Ando in [2] Theorem 3.5].

We next resort to the Cayley transform

**Definition 2.1.** We denote by \( \mathcal{C}(A) \) the Cayley transform of a matrix \( A \in \mathbb{C}^{n \times n} \),

\[
\mathcal{C}(A) := (I_n - A)(I_n + A)^{-1} = -I_n + 2(I_n + A)^{-1} -1 \not\in \text{spect}(A).
\]

Recall that the Cayley transform is involutive in the sense that, whenever well defined,

\[
\mathcal{C}(\mathcal{C}(A)) = A.
\]

The following is classical, see e.g. [2], [25].
Proposition 2.2. For a given $H \in \mathbb{H}_n$ let $L_H$ and $S_H$ (and the respective closures) be as in Eqs. (2.1) and (2.2) respectively. Then one has that

\begin{equation}
C(S_H) = L_H.
\end{equation}

We next examine the structure of the two sets in Eqs. (2.1), (2.2) and (2.3). To this end, we need the following.

Definition 2.3. A family $A$, of square matrices $1$ (of various dimensions) is said to be matrix-convex of level $n$, if for all $\nu = 1, \ldots, n$:

For all natural $k$,

\begin{equation}
\sum_{j=1}^{k} v_j^* A_j v_j, \quad \forall v_j \in \mathbb{C}^{\gamma_j \times \nu}, \quad \gamma_j \in [1, \nu],
\end{equation}

one has that having $A_1, \ldots, A_k$ (of various dimensions $1 \times 1$ through $\nu \times \nu$) within $A$, implies that

\begin{equation}
\sum_{j=1}^{k} v_j^* A_j v_j,
\end{equation}

belongs to $\in A$ as well.

If the above holds for all $n$, we say that the set $A$ is matrix-convex. \hfill \Box

Background to matrix-convex sets $[8]$ and more recently, $[9]$, $[15]$ and $[23]$. See also $[18$, Sections 2,3].

The set $L_H$ where $H \in \mathbb{H}_n$, was first explored in $[5]$. Here we summarize its main properties.

Theorem 2.4. $[18$, Theorems 2.4 and 4.1$] The following statements are true.

(i) The set $L_H$ where $H \in \mathbb{H}_n$, is a cone closed under inversion. It is maximal open convex set of non-singular matrices, containing the matrix $H$.

(ii) The set $L_I$ is in addition matrix-convex.

(iii) Conversely, a cone closed under inversion and a maximal open matrix-convex set of non-singular matrices, containing the matrix $I$, is the set $L_I$.

Similarly, here are the main properties of $S_H$.

Theorem 2.5. $[19$, Theorem 2.1$] For $H \in \mathbb{H}_n$ the set $S_H$ is open, convex, closed under multiplication by $c \in \mathbb{C}$, $1 \geq |c|$ and under product among its elements, i.e. whenever $A \in \frac{1}{\alpha} S_H$ and $B \in \frac{1}{\beta} S_H$, for some $\alpha, \beta > 0$, then their product satisfies $AB \in \frac{1}{\alpha \beta} S_H$.

When $H = I$, the set $S_I$ is matrix-convex,

To gain intuition, we temporarily confine the discussion to scalar set-up.

\footnote{We do not assume that $A \subset \mathbb{H}$.}
3. Disks

3.1. Sub-Unit Disks. We shall use the following notation for disks in the complex plane

\[ D(\text{Center}, \text{Radius}) = \{ x \in \mathbb{C} : \text{Radius} \geq |x - \text{Center}| \} \]

A key-parameter in this work is \( \eta \), which assumes values in \((1, \infty]\). We start describing sub-unit disks of the form,

\[ D_{\text{origin}}(\eta) := D\left(0, \sqrt{\frac{\eta - 1}{\eta + 1}}\right) \quad \eta \in (0, \infty]. \]

This is illustrated in Figure 1. We next examine properties of these disks.

![Figure 1](image-url)

**Figure 1.** \( D_{\text{origin}}\left(\sqrt{\frac{\eta - 1}{\eta + 1}}\right) \) — red, \( D_{\text{origin}}\left(\frac{\eta - 1}{\eta + 1}\right) \) — blue, \( \eta \in (1, \infty] \).

**Lemma 3.1.** For arbitrary scalars \( \eta_a, \eta_b \in (1, \infty] \) let

\[ \eta_c = \frac{1 + \eta_a \eta_b}{\eta_a + \eta_b}. \]

Then the following is true.

(i) Let \( \eta_c \) be as in Eq. (3.2), then

\[ \eta_c = \eta_a + (1 - \theta) \frac{\eta_a}{\eta_b}, \quad \theta = \frac{\eta_a}{\eta_a + \eta_b}. \]

Alternatively,

\[ \frac{1}{\eta_c} = \frac{1}{\eta_a} + \frac{1}{\eta_b} + \frac{1}{\eta_a}. \]
(ii) For $\eta_b \geq \eta_a$, let $\eta_c$ be as in Eq. (3.2). Then,

$$\frac{1}{2} \left( \eta_b + \frac{1}{\eta_b} \right) \geq \eta_c \geq \frac{1}{2} \left( \eta_a + \frac{1}{\eta_a} \right).$$

(iii) The product of a pair of disks is given by,

$$D_{\text{origin}} \left( \frac{\sqrt{\eta_a} - 1}{\sqrt{\eta_a} + 1} \right) \cdot D_{\text{origin}} \left( \frac{\sqrt{\eta_b} - 1}{\sqrt{\eta_b} + 1} \right) = D_{\text{origin}} \left( \frac{\sqrt{\eta_c} - 1}{\sqrt{\eta_c} + 1} \right),$$

with $\eta_c$ as in Eq. (3.2).

**Proof:** (i) This is immediate.

(ii) Bounding $\eta_c$ from above and from below,

$$\eta_b(\eta_b^2 - 1) \geq \eta_a(\eta_a^2 - 1)$$

by assumption

$$\eta_a + \eta_b^3 \geq \eta_a \eta_b^2 + \eta_b$$

re - arranging terms

$$(\eta_b^2 + 1)(\eta_a + \eta_b) \geq 2\eta_b(\eta_a \eta_b + 1)$$

adding the same terms to both sides

$$\frac{1}{2}(\eta_b + \frac{1}{\eta_b}) \geq \eta_c$$

dividing each side by $2\eta_b(\eta_a + \eta_b)$

so the right-hand side is obtained. Similarly, for the left-hand side,

$$\eta_b(\eta_a^2 - 1) \geq \eta_a(\eta_a^2 - 1)$$

by assumption

$$\eta_a^2 \eta_b + \eta_a \geq \eta_a^3 + \eta_b$$

re - arranging terms

$$2\eta_a(\eta_a \eta_b + 1) \geq (\eta_a^2 + 1)(\eta_a + \eta_b)$$

adding the same terms to both sides

$$\eta_c \geq \frac{1}{2}(\eta_a + \frac{1}{\eta_a})$$

dividing each side by $2\eta_a(\eta_a + \eta_b)$,

establishes, this part of the claim.

(iii) In principle, a product of two disks centered at the origin satisfies,

$$\frac{\sqrt{\eta_a} - 1}{\sqrt{\eta_a} + 1} = \frac{\sqrt{\eta_b} - 1}{\sqrt{\eta_b} + 1} \cdot \frac{\sqrt{\eta_c} - 1}{\sqrt{\eta_c} + 1},$$

which (after taking squares) implies that

$$\frac{(1 - \frac{2}{\eta_c + 1})}{\text{Radius}_c^2} = \frac{(1 - \frac{2}{\eta_a + 1})}{\text{Radius}_a^2} \cdot \frac{(1 - \frac{2}{\eta_b + 1})}{\text{Radius}_b^2},$$

and in turn,

$$\frac{1}{\eta_c + 1} = \frac{\eta_a + \eta_b}{(\eta_a + 1)(\eta_b + 1)},$$

so Eq. (3.2) is obtained. □

Item (i) says that $\eta_c$ may be formulated in terms of $\eta_a$ and $\eta_b$ in two equivalent ways: (a) a convex combination or (b) a special kind of a double harmonic mean.

Recall that in item (iii) of Lemma 3.1 we described the product of a pair of sub-unit disks. Of a special interest is the case where their radii are equal ($\eta_a = \eta_b = \eta$), i.e. when Eq.
\( \eta_{\eta_1 = \eta_2 = \eta} = h_1(\eta) \) see Eq. (3.6)

For example, in Figure 1 the blue disk is a result of the product of a pair of identical red disks.

In Section 4 we extend the above discussion to matricial framework:

\[ D_{\text{origin}}(\eta) \text{ Eq. (3.1)} \rightarrow \left\{ A \in \mathbb{C}^{n \times n} : \sqrt{\eta^2 - 1} > \|A\|_2 \right\} \]

3.2. Disks closed under inversion. In this subsection we exploit the previous disks to describe special disks in the open half planes, \( \mathbb{C}_L \) and \( \mathbb{C}_R \). To this end we find it convenient to introduce the following disks (where the subscript abbreviates “invertible”) see Figures 2 and 3.

\[ D_{\text{inv}}(\eta) := D(i \theta + \sqrt{\eta^2 - 1}, \eta), \quad \eta \in (1, \infty], \]

As always, \(-D_{\text{inv}}(\eta) = D(i \theta - \sqrt{\eta^2 - 1}, \eta) \subset \mathbb{C}_L\), without loss of generality, we can focus in the sequel on \( D_{\text{inv}} \) in \( \mathbb{C}_R \). Here is our motivation to introducing these disks.

**Lemma 3.2.** For a parameter \( \eta \in (1, \infty] \), let \( D_{\text{inv}}(\eta) \) be as in Eq. (3.4).

Then the following statements are true.

(i) \( D_{\text{inv}}(\eta) = \{ c \in \mathbb{C}_R : \eta^2 - 1 \geq |c - \eta|^2 \} \).

(ii) Through inversion, the disk \( D_{\text{inv}}(\eta) \) is mapped onto itself, i.e. \( (D_{\text{inv}}(\eta))^{-1} = D_{\text{inv}}(\eta) \).

(iii) For a “half iteration” function \( \frac{1}{2} \left( D_{\text{inv}}(\eta) + \frac{1}{D_{\text{inv}}(\eta)} \right) = D_{\text{inv}} \left( \frac{1}{2}(\eta + \frac{1}{\eta}) \right) \).

(iv) \( C(D_{\text{inv}}(\eta)) = D_{\text{origin}}(\eta) \).

Indeed, item (i) follows from Eq. (3.4) where for some \( \eta \in (1, \infty] \),

\[ D_{\text{inv}}(\eta) = \{ c \in \mathbb{C}_R : \eta^2 - 1 \geq |c - \eta|^2 \} \]

\[ = \{ c \in \mathbb{C}_R : \eta^2 - 1 \geq \eta^2 - 2\eta \text{Re}(c) + |c|^2 \} \]

\[ = \{ c \in \mathbb{C}_R : 2\eta \text{Re}(c) \geq \eta^2 + 1 \} \]

Item (ii) is obtained upon substituting \( c \rightarrow \frac{1}{c} \), in item (i).

As to item (iii) note that

\[ C(\partial D(i \theta + \sqrt{\eta^2 - 1}, \eta)) = \{ C \left( \sqrt{\frac{2 \eta - 1}{\eta^2 - 1}} \cdot e^{i\theta} \right) : \theta \in [0, 2\pi) \} \]

\[ = \{ c = \frac{1-i\sqrt{\eta^2 - 1} \sin(\theta)}{\eta + (\sqrt{\eta^2 - 1}) \cos(\theta)} : \theta \in [0, 2\pi) \} \]

and thus \( \text{Re}(c) = \frac{1}{\eta + (\sqrt{\eta^2 - 1}) \cos(\theta)} \) and \( |c|^2 + 1 = 2\eta \text{Re}(c) \), so indeed \( \frac{|c|^2 + 1}{2\text{Re}(c)} = \eta. \)
Direct substitution in the Cayley transform, see Definition 2.1 of \( D_{\text{inv}}(\eta) \) from Eq. (3.4), yields item (iv).

To illustrate item (iv) of Lemma 3.2 we have the following.

**Example 3.3.** The red (blue) disk in Figure 2 is the image, through the Cayley transform, of the red (blue) disk from Figure 1.

While in Figure 1 the blue disk is the (point-wise) square of the red disk; in Figure 2 the (point-wise) sum of red disk together with its inverse divided by two, yields the blue disk, see Example 3.3.

From Figure 2 it is easy to see that under inversion, the leftmost and rightmost points of the disk exchange places. Item (ii) of Lemma 3.2 says that \( D_{\text{inv}}(\eta) \) is closed under inversion, as a set.

\[ \eta_1 = \frac{1}{2}(\eta + \frac{1}{\eta}) \]

**Figure 2.** \( D_{\text{inv}}(\eta) \) red \( D_{\text{inv}}(\eta_1) \) blue

As a side remark, recall that, Andres Rantzer already addressed (in the context of weak Kharitonov Theorem) convex sets closed under inversion, see [24] and for perspective [3].

In [1] the sets \( D_{\text{inv}}(\eta) \) are used in yet in another area: Absolute stability of feedback-loops, for background see e.g. [14, Section 7.1]

In Section 5 we extend the above discussion to matrical framework:

\[ D_{\text{inv}}(\eta) \text{ Lemma 3.2 item (i)} \rightarrow \{ A \in L_n : \eta > \rho((A^* A + I)(A + A^*)^{-1}) \} \]
3.3. Matrix Sign Function Iterations. In this subsection we review the Matrix Sign Function iteration scheme, used in matrix computations, see e.g. [10, Chapter 5], [16, Chapter 22], [13] and also [5], [6], [17], [21].

Recall that for a matrix \( A \in \mathbb{C}^{n \times n} \) whose spectrum avoids the imaginary axis, one can define \( E_A := \text{Sign}(A) \) as a matrix satisfying,

\[
(\text{Sign}(A))^2 = I_2 \quad A \cdot \text{Sign}(A) = \text{Sign}(A) \cdot A \quad \text{spect}(\text{Sign}(A) \cdot A) \in \mathbb{C}_R.
\]

The readers interested in learning on the advantages of obtaining the Sign of a matrix, can look at the above references.

It is known that for an arbitrary matrix \( A \in \mathbb{C}^{n \times n} \) whose spectrum avoids the imaginary axis, the following iterative scalar function,

\[
h_o(x) = x \quad \text{and} \quad h_j(x) = \frac{1}{2} \left( h_{j-1}(x) + \frac{1}{h_{j-1}(x)} \right) \quad j = 1, 2, \ldots,
\]

is so that

\[
\lim_{j \to \infty} h_j(A) = \text{Sign}(A).
\]

As illustrated below, this convergence is actually “fast”.

The following is straightforward, yet useful.

**Lemma 3.4.** Let \( \mathcal{C}(A) \) be the Cayley transform of \( A \in \mathbb{C}^{n \times n} \) so that \(-1, 0, \pm i \not\in \text{spect}(A)\), then

(i) \( -\mathcal{C}(A) = \mathcal{C}(A^{-1}) \).

(ii) \( -(\mathcal{C}(A))^2 = \mathcal{C} \left( \frac{1}{2} (A + A^{-1}) \right) \).

Namely, under the Cayley transform \( \mathcal{C} \) (see Definition [2.1]), the operation of inversion takes the form of minus.

From the above discussion we arrive at the main observation of this subsection: An illustrative description of the convergence of the “half iterations” from Eq. \((3.6)\), applied to invertible disks.

**Observation 3.5.** For \( \eta \in (1, \infty) \) let \( D_{\text{origin}}(\eta) \) and \( D_{\text{inv}}(\eta) \) be as in Eqs. \((3.1)\) and \((3.4)\) respectively. Then

\[
\mathcal{C} \left( D_{\text{inv}} \left( \frac{1}{2} (\eta + \frac{1}{\eta}) \right) \right) = -\left( D_{\text{origin}}(\eta) \right)^2,
\]

and more generally, for \( h_j \) from Eq. \((3.6)\),

\[
\mathcal{C} \left( h_j \left( D_{\text{inv}}(\eta) \right) \right) = (-1)^j D_{\text{origin}} \left( \left( \frac{\eta - 1}{\eta + 1} \right)^{\frac{1}{j}} \right) \quad j = 1, 2, \ldots
\]

In Figure 3 below, this iterative map of the disk \( D_{\text{inv}}(\eta) \) is illustrated

- \( \eta = 2 + \sqrt{3} \)
- \( h_1(\eta) = 2 \)
- \( h_2(\eta) = \frac{3}{2} \)
- \( h_3(\eta) = \frac{41}{40} \)
- \( \text{Radius}_0 = \sqrt{6 + 4\sqrt{3}} \)
- \( \text{Radius}_1 = \sqrt{3} \)
- \( \text{Radius}_2 = \frac{5}{2} \)
- \( \text{Radius}_3 = \frac{9}{2} \)

Roughly spectrum of a given matrix is contained in the two union of the two blue disks in Figure 3 after three “half iterations” the spectrum is contained in the silver disks.
Figure 3. $D_{\text{inv}}(\eta)$: $\eta = 2 + \sqrt{3}$ – blue, $h_1(\eta) = 2$ – red, $h_2(\eta) = \frac{5}{4}$ – green, $h_3(\eta) = \frac{41}{40}$ – gray

4. Matrices whose spectrum is within a sub-unit disk

We now take the scalar parameter $\eta$ (where $\eta \in (1, \infty]$) from the previous section, and introduce it to the matricial sets $S_H$ in Eq. (2.2).

For arbitrary $H \in \mathbf{H}$ and $\eta \in (1, \infty]$ the set

$$S_H(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : \left( \eta \frac{n+1}{n} H - A^*HA \right) \in \mathbf{P}_n \right\},$$

(4.1)

$$S_H(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : \left( \eta \frac{n+1}{n} H - A^*HA \right) \in \mathbf{P}_n \right\},$$

is well defined, e.g. for all $\theta \in [0, 2\pi)$ the matrix $A = e^{i\theta} \sqrt{n+1} H$ belongs to $S_H(\eta)$. As before, $S_H(\eta)$ is the closure of the open set $S_H(\eta)$.

For simplicity, in the rest of this section, we confine the discussion to $H \in \mathbf{P}_n$ and to emphasize that for $P \in \mathbf{P}_n$ and $\eta \in (1, \infty]$ we shall consider

$$S_P(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : \left( \eta \frac{n+1}{n} P - A^*PA \right) \in \mathbf{P}_n \right\},$$

(4.2)

and $S_P(\eta)$.

Eq. (4.1) offers a more detailed examination of the set $S_P$, in the following sense,

$$\infty > \eta > \eta_1 > 1 \implies S_P(\eta_1) \subset S_P(\eta) \subset S_P,$$

where each inclusion is strict, and

$$\lim_{\eta \to \infty} S_P(\eta) = S_P.$$

Here are two basic properties of this set.

**Proposition 4.1.** For parameters $P \in \mathbf{P}_n$ and $\eta \in (1, \infty]$, the matricial set $S_P(\eta)$ in Eq. (4.2) satisfies the following,
For all \( c \in \mathbb{D}_{\text{origin}}(\eta) \), see Eq. (3.1), the set \( S_P(\eta) \) contains all elements of the form \( cI \), and

\[
A \in S_P(\eta) \implies cA \in S_P(\eta)
\]

(ii) This set is matrix-product-contractive, i.e. for some \( \eta_a, \eta_b \in (1, \infty] \)

\[
A_a \in S_P(\eta_a) \land A_b \in S_P(\eta_b) \implies A_aA_b \in S_P(\eta_c) \quad \text{with} \quad \eta_c = \frac{1+\eta_a\eta_b}{\eta_a+\eta_b}, \quad \text{see Eq. (3.2)}.
\]

The above properties are easy to verify.

By multiplying Eq. (4.2) by \( P^{-\frac{1}{2}} \) from both sides, one can equivalently write

\[
S_P(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : \sqrt{\eta - \frac{1}{\eta + 1}} > \| P^{\frac{1}{2}}AP^{-\frac{1}{2}} \|_2 \right\},
\]

where \( \| \cdot \|_2 \) denotes the spectral (a.k.a. Euclidean) norm, see e.g. [11, item 5.6.6]. As a motivation, one has the following model of a stability robustness problem.

**Difference inclusion stability problem**

Let \( x(\cdot) \) be a real vector-valued sequence satisfying,

\[
x(k+1) = A(k, x(k))x(k) \quad k = 0, 1, 2, \ldots
\]

where the actual sequence \( \{ A(0, x(0)), A(1, x(1)), A(2, x(2)), \ldots \} \) can be arbitrary, for perspective see e.g. [19], [22].

**Proposition 4.2.** If in the above difference inclusion,

\[
A(\cdot, \cdot) \in S_I(\eta),
\]

for some \( \eta \in (1, \infty) \), then,

\[
\| x(0) \|_2 \cdot \left( \frac{\eta - 1}{\eta + 1} \right)^k \geq \| x(k) \|_2 \quad \forall x(0) k = 0, 1, 2, \ldots
\]

We end this section by mentioning the following property of the set \( S_P(\eta) \).

**Proposition 4.3.** For \( P = I \) and arbitrary \( \eta \in (1, \infty] \) the matricial set \( S_I(\eta) \) in Eqs. (4.1) and (4.3), is matrix-convex.

This can be deduced from: (a) part (I) of [18, Observation 2.2]; or (b) Lemma 5.8 below.

5. Hyper-Lyapunov Inclusions

5.1. Hyper-Lyapunov Inclusion - Properties. We start with a basic fact, whose verification is straightforward.

**Proposition 5.1.** For \( H \in \mathbb{H}_n \) and \( \eta \in (1, \infty] \), both arbitrary, let \( S_H(\eta) \) be as in Eq. (4.1). Then, one has that

\[
C(S_H(\eta)) = L_H(\eta),
\]

where,

\[
L_H(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : -\frac{1}{\eta}(A^*HA + H) + HA + A^*H) \in \mathbb{P}_n \right\}.
\]

**Remark 5.2.** Note that in the Hyper-Stein inclusion (4.1), (5.1) and in the Hyper-Lyapunov inclusion (5.2), the parameters \( H \) and \( \eta \), are indeed the same. \( \square \)
Here is the first result concerning this set.

**Proposition 5.3.** For arbitrary \( H \in \mathbb{H}_n \) let \( \text{Sign}(H) \) be as defined in Eq. (3.5) and let \( \eta \in (1, \infty) \).

(i) The set \( \mathbf{L}_{H(\eta)} \) is closed under inversion.

(ii) The matrix \( A = c\text{Sign}(H) \) belongs to \( \mathbf{L}_{H(\eta)} \) for all \( c \in \mathbb{D}_{\text{inv}(\eta)} \), see Eq. (3.4).

**Proof:** (i) We shall show inveribility in two ways:

(a) Multiply Eq. (5.2) by \((A^{-1})^*\) and \(A^{-1}\) from the left and from the right, respectively. Then,

\[
(A^{-1})^* \left( -\frac{1}{\eta} (A^*PA + P) + PA + A^*P \right) A^{-1} = \left( -\frac{1}{\eta} ((A^{-1})^*PA^{-1} + P) + PA^{-1} + (A^{-1})^*P \right) = (A^{-1})^* QA^{-1} \in \mathbb{P}_n.
\]

(b) Invertibility can be indirectly deduced by using the relation \( C(A^{-1}) = -C(A) \) which is immediate from Definition 2.1 along with Eqs. (4.1) and (4.3).

(ii) By item (i) of Lemma 3.2

\[
c \in \mathbb{D}_{\text{inv}(\eta)} \implies \left( 2\text{Re}(c) - \frac{1}{\eta}(1 + |c|^2) \right) H\text{Sign}(H) \in \mathbb{P}_n.
\]

Now, substituting \( A = c\text{Sign}(H) \) in the right-hand side of Eq. (5.2), yields

\[
2\text{Re}(c) H\text{Sign}(H) - \frac{1}{\eta}(1 + |c|^2)H,
\]

which by the above, must be positive definite. \( \square \)

For simplicity, in the rest of this section, we confine the discussion to \( H \in \mathbb{P}_n \) and to emphasize that for \( P \in \mathbb{P}_n \) and \( \eta \in (1, \infty) \) we shall consider

\[
(5.3) \quad \mathbf{L}_{P(\eta)} = \{ A \in \mathbb{C}^{n \times n} : \left( -\frac{1}{\eta} (A^*PA + P) + PA + A^*P \right) \in \mathbb{P}_n \}.
\]

Expectedly, in Eq. (5.3) we have

\[
\lim_{\eta \to \infty} \mathbf{L}_{P(\eta)} = \lim_{\eta \to \infty} \{ A \in \mathbb{C}^{n \times n} : \left( -\frac{1}{\eta} (A^*PA + P) + PA + A^*P \right) \in \mathbb{P}_n \} = \mathbf{L}_{P},
\]

where the lower part is as in Eq. (2.1). Furthermore, for a prescribed \( P \in \mathbb{P}_n \) one has that,

\[
(5.4) \quad \infty > \eta > \eta_n > 1 \implies \mathbf{L}_{P(\eta_n)} \subset \mathbf{L}_{P(\eta)} \subset \mathbf{L}_{P},
\]

and each inclusion is strict.

This family can also be written as,

\[
\mathbf{L}_{P(\eta)} = \{ A \in \mathbb{C}^{n \times n} : \left( -1 + \frac{1}{\eta} \right) (A^*PA + P) + (A + I_n)^*P(A + I_n) \} \in \mathbb{P}_n.
\]

which illustrates the fact that \( \mathbf{L}_{P(\eta)} \) is (linear in \( P \) and) quadratic in \( A \), so this set may contain \( A \) when \( \|A\| \) is neither “too small” nor “too large”.

We next present yet another description of the set \( \mathbf{L}_{P(\eta)} \). To this end, denote by \( \rho(M) \) the spectral radius of a square matrix \( M \).
Corollary 5.4. For $P \in \mathbb{P}_n$ and $\eta \in (1, \infty]$ one has that,

\[ L_P(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : (A^*PA + P) = Q \in \mathbb{P}_n \right\} . \]

Proof: Writing Eq. (5.2) explicitly, one has the following chain of relations

\[ L_P(\eta) = \left\{ A \in \mathbb{C}^{n \times n} : (A^*PA + P) = Q \in \mathbb{P}_n \right\} . \]

Let \( \eta \in (1, \infty] \), be prescribed.

(i) Then, the open set \( L_P(\eta) \) in Eq. (5.2) is convex and closed under inversion.

(ii) For \( \eta = \infty \) the set \( L_P \) is a maximal convex set of matrices whose spectrum is in \( \mathbb{C} \).

(iii) For \( P = I_n \), the set \( L_{I_n}(\eta) \), is in addition matrix-convex.

(iv) For arbitrary \( \hat{P} \in \mathbb{P}_n \) there exists a non-singular matrix \( T \) so that

\[ T^{-1}L_P(\eta)T = L_{\hat{P}}(\eta) . \]

Proof: (i) Invertibility was shown in item (i) of Proposition 5.3.

To show convexity, for a pair of \( n \times n \) matrices \( A_o, A_1 \) consider the following identity,

\[ \theta \left( -\frac{1}{\eta}(A_o^*PA_1 + P)A_1^*PA_1 + PA_1 + A_1^*P \right) \]

\[ + (1 - \theta) \left( -\frac{1}{\eta}(A_o^*PA_o + P)A_o^*PA_o + PA_o + A_o^*P \right) \]

\[ + \frac{\theta(1 - \theta)}{\eta} (A_o - A_1)^*P(A_o - A_1) \]

\[ = \left( -\frac{1}{\eta}(A_o^*PA_o + P) + PA_o + A_o^*P \right) \]

\[ \quad + \frac{\theta(1 - \theta)}{\eta} (A_o - A_1)^*P(A_o - A_1) \]

\[ A_\theta := \theta A_1 + (1 - \theta) A_o . \]

The condition \( A_o, A_1 \in L_P(\eta) \) is equivalent to having \( Q_o, Q_1 \in \mathbb{P}_n \). Since by assumption \( P \in \mathbb{P}_n \), it implies that \( (A_o - A_1)^*P(A_o - A_1) \in \mathbb{P}_n \) and thus one has that for all \( \theta \in [0, 1] \), also \( A_\theta := (\theta A_1 + (1 - \theta) A_o) \) belongs to \( L_P(\eta) \), so this part of the claim is established.
(ii) This is shown in two ways. 
(a) Combine item (i) along with Eq. (5.4).
(b) See e.g. [2] Lemma 3.5.

(iii) Matrix convexity for \( P = I \), substitute in Lemma 5.8 below, \( \alpha = \gamma = -\frac{1}{n} \) and \( \beta = 1 \).

(iv) Similarity. Consider Eq. (5.2)

\[
P^{-\frac{1}{2}} \left( -\frac{i}{n} (A^*PA + P) + PA + A^*P \right) P^{-\frac{1}{2}} = P^{-\frac{1}{2}} Q P^{-\frac{1}{2}} \in \mathcal{P}_n
\]

\[
= \left( -\frac{i}{n} (I_n + (P_{\frac{3}{2}}^*AP^{-\frac{1}{2}})^* P_{\frac{3}{2}}^*AP^{-\frac{1}{2}}) + (P_{\frac{3}{2}}^*AP^{-\frac{1}{2}})^* + P_{\frac{3}{2}}^*AP^{-\frac{1}{2}} \right) = P^{-\frac{1}{2}} Q P^{-\frac{1}{2}} \in \mathcal{P}_n
\]

Since \( \hat{A} \in \mathcal{L}_{I_n}(\eta) \), it follows that to obtain Eq. (5.6) one can take \( T = P^{-\frac{1}{2}} \hat{P}^\frac{1}{2} \). Thus, the proof is complete. \( \square \)

We conclude this subsection by further relating the new concept of Hyper-Lyapunov inclusion in Eq. (5.2) to the classical notion of Matrix Sign Function iterations, from Section 3.3, see e.g. [10, Chapter 5], [16, Chapter 22], [13] and also [5], [6], [17], [21].

**Observation 5.6.** Let \( A \) be a matrix whose spectrum is in \( \mathbb{C}_R \) (a.k.a. “positively stable”). Then, there exist parameters \( \eta \in (1, \infty) \) and (non-unique) \( P \in \mathcal{P}_n \) so that \( A \in \mathcal{L}_P(\eta) \). Moreover, the matrix \( \frac{n}{2}(A + A^{-1}) \) is well defined and (with the same \( P \) and \( \eta \)) belongs to \( \mathcal{L}_P \left( \frac{n}{2} \left( \eta + \frac{1}{\eta} \right) \right) \).

**Proof:** For a positively stable \( A \), it is well known, see e.g. [12] Theorem 2.2.3 that for all \( Q \in \mathcal{P}_n \) there exists \( P \in \mathcal{P}_n \) so that \( PA + A^*P = Q \). Then the following spectral radius is well defined \( \rho ((A^*PA + P)(PA + A^*P)^{-1}) \), and by Eq. (5.5) one can find a corresponding \( \mathcal{L}_P(\eta) \).

It is also well known, see e.g. [10, Chapter 5], [16, Chapter 22], [13] and also [5], [6], [17], [21] that \( A_1 := \frac{n}{2}(A + A^{-1}) \) is well defined and by item (I) of Theorem 5.5 one has that \( A_1 \in \mathcal{L}_{H}(\eta) \).

To show that \( A_1 \) in fact belongs to \( \mathcal{L}_P \left( \frac{n}{2} \left( \eta + \frac{1}{\eta} \right) \right) \), we return to the Cayley transform and recall that from Eq. (5.1) that \( \mathcal{C}(A) \in \mathcal{S}_P(\eta) \) and by items (ii) and (iii) of Proposition 4.1 it follows that \( - (\mathcal{C}(A))^2 \) belongs to \( \mathcal{S}_P \left( \frac{n}{2} \left( \eta + \frac{1}{\eta} \right) \right) \). Applying again the Cayley transform, by item (ii) of Lemma 3.4, the construction is complete. \( \square \)

See again Figure 3

The formulation of Observation 5.6 was chosen to be simple. In fact, it can be extended to the case where \( A_1 = \theta A + (1-\theta)A^{-1} \), for arbitrary \( \theta \in [0, 1] \). For details, see items (i) and (iii) in Lemma 3.1

5.2. Quadratic Inclusions - a Unifying Framework. We now introduce a unifying framework for the four inclusions presented thus far. To this end let \( M \in \mathcal{H}_{n+m} \) be a matricial parameter (i.e. \( (n + m)^2 \) real scalar parameters)

\[
\begin{pmatrix}
A & R \\
M & I_m
\end{pmatrix}
\begin{pmatrix}
W & R \\
Y & I_m
\end{pmatrix}
\begin{pmatrix}
A & W \\
M & I_m
\end{pmatrix} = \begin{pmatrix}
A^*WA + RA + (RA)^* + Y \\
Q \in \mathcal{H}_m
\end{pmatrix}
\]
i.e. \( A \in \mathbb{C}^{n \times m} \) and \( n \) may be larger, equal or smaller than \( m \).

A word of caution. Our notation, aimed at being consistent throughout the work, is not common in the Riccati equation circles. Nevertheless, there is no contradiction: When in eq. (5.7) \( m = n \), it does conform with conventional Riccati equation, and

\[
\begin{pmatrix}
I_n & -A^* \\
A & I_n
\end{pmatrix}
\begin{pmatrix}
R \\
W
\end{pmatrix}
= 0_n
\]

see e.g. [16]. As a specific example, the classical Linear Quadratic Regulator control problem is obtained when in Eq. (5.7) \( m = n \), so that the pair \( R, Y \) is detectable and \( W = -BW^{-1}B \), where \( B \in \mathbb{C}^{n \times m} \) is so that the pair \( R, B \) is stabilizable and \( \hat{W} \in \mathbb{P}_m \), see e.g. [7] Section 6.2, [16] Chapter 16.

We next focus on the case where in Eq. (5.7) \( n = m \), and on the right-hand side \( Q \in \mathbb{P}_n \). Namely, Eq. (5.7) takes the form

\[
\begin{pmatrix}
A \\
I_n
\end{pmatrix}^* M \begin{pmatrix}
A \\
I_n
\end{pmatrix} = Q \in \mathbb{P}_n
\]

More specifically, with parameters \( H \in \mathbb{H}_n, P \in \mathbb{P}_n \) and \( \eta \in (1, \infty] \), the four previous cases can be recovered,

| \( M \) | Inclusion name     |
|---|----------------|
| \(
\begin{pmatrix}
0 & H \\
H & 0
\end{pmatrix}
\) | Lyapunov, Eq. (2.1) |
| \(
\begin{pmatrix}
-H & 0 \\
0 & H
\end{pmatrix}
\) | Stein, Eq. (2.2) |
| \(
\begin{pmatrix}
-P & 0 \\
0 & \frac{1}{\eta+1}P
\end{pmatrix}
\) | Hyper – Stein, Eq. (4.1) |
| \(
\begin{pmatrix}
-P & P \\
0 & -\frac{1}{\eta}P
\end{pmatrix}
\) | Hyper – Lyapunov, Eq. (5.2). |

**Remark 5.7.** In each of the four above cases, the matrix \( M \) is a non-singular Hermitian, furthermore it in Eq. (5.8) has \( n \) positive and \( n \) negative eigenvalues (since by assumption, \( H \in \mathbb{H}_n \)). In fact, in Eq. (5.8) the \( 2n \times n \) matrix \( \begin{pmatrix} A \\ I_n \end{pmatrix}^* M \begin{pmatrix} A \\ I_n \end{pmatrix} \) corresponds to a non-negative subspace of \( M \).

In [20] the idea of the above table is extended to four variants state-space realization arrays.

The following technical result is useful in showing matrix-convexity of various sets.

**Lemma 5.8.** Let the real parameters \( \alpha, \beta \) and \( \gamma \), be so that the set of matrices \( A \in \mathbb{C}^{n \times n} \) satisfying,

\[
\begin{pmatrix}
A \\
I_n
\end{pmatrix}^* \begin{pmatrix}
\alpha I_n & \beta I_n \\
\beta I_n & \gamma I_n
\end{pmatrix} \begin{pmatrix}
A \\
I_n
\end{pmatrix} \in \mathbb{P}_n ,
\]

is not empty. Then, the family of all matrices \( A \) satisfying that, is matrix-convex.
Now, by Definition 2.3

\[
\begin{pmatrix} A_j v_j \\ I_n \end{pmatrix}^* \begin{pmatrix} \alpha I_n \\ \beta I_n \end{pmatrix} \begin{pmatrix} A_j v_j \\ I_n \end{pmatrix} = \alpha A_j^* A_j + \beta (A_j A_j^*) + \gamma I_n \end{pmatrix} \in \overline{P}_n.
\]

Proof: Indeed for \( j = 1, \ldots, k \) let,

\[
\begin{pmatrix} \sum_{j=1}^k v_j^* A_j v_j \\ I_n \end{pmatrix}^* \begin{pmatrix} \alpha I_n \\ \beta I_n \end{pmatrix} \begin{pmatrix} \sum_{j=1}^k v_j^* A_j v_j \\ I_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^k v_j^* A_j v_j \\ I_n \end{pmatrix} \begin{pmatrix} \alpha I_n \\ \beta I_n \end{pmatrix} \begin{pmatrix} A_j^* A_j \\ I_n \end{pmatrix} v_j = \sum_{j=1}^k v_j^* Q_j v_j.
\]

In a similar way one can show a slightly stronger statement

Corollary 5.9. For given real parameters \( \alpha, \beta, \gamma \) and \( \delta \), whenever not empty, the family of all matrices \( A \in \mathbb{C}^{n \times n} \) satisfying,

\[
\begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} \alpha I_n \\ \beta I_n \\ \delta I_n \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} = \alpha A^* A + \beta AA^* + \gamma A + \delta I_n \end{pmatrix} \in \overline{L}_I_n.
\]

is matrix-convex.

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