Rates of convergence of ρ-estimators for sets of densities satisfying shape constraints

Y. Baraud and L. Birgé

Abstract. The purpose of this paper is to pursue our study of ρ-estimators built from i.i.d. observations that we defined in Baraud et al. (2014). For a ρ-estimator based on some model $\mathcal{S}$ (which means that the estimator belongs to $\mathcal{S}$) and a true distribution of the observations that also belongs to $\mathcal{S}$, the risk (with squared Hellinger loss) is bounded by a quantity which can be viewed as a dimension function of the model and is often related to the “metric dimension” of this model, as explained in Birgé (2006). This is a minimax point of view and it is well-known that it is pessimistic. Typically, the bound is accurate for most points in the model but may be very pessimistic when the true distribution belongs to some specific part of it. This is the situation that we want to investigate here.

For some models, like the set of decreasing densities on $[0, 1]$, there exist specific points in the model that we shall call extremal and for which the risk is substantially smaller than the typical risk. Moreover, the risk at a non-extremal point of the model can be bounded by the sum of the risk bound at a well-chosen extremal point plus the square of its distance to this point. This implies that if the true density is close enough to an extremal point, the risk at this point may be smaller than the minimax risk on the model and this actually remains true even if the true density does not belong to the model.

1. Introduction

Let us consider the problem of estimating an unknown non-increasing density $s$ (with respect to the Lebesgue measure) on $[a, +\infty)$ from a sample of $n$ observations. If $a$ is known, the Grenander estimator $g_a$ of $s$ is the derivative of the least concave majorant of the empirical cumulative distribution function built on $[a, +\infty)$, see Grenander (1981) and Groeneboom (1985). It is also the maximum likelihood estimator (MLE for short) and it converges (say in $L_1$-norm) at rate at least $n^{-1/3}$ for all non-increasing and bounded $s$ with bounded support $[a, b], b > a$. On the one hand some known lower bounds arguments show that this rate cannot be improved uniformly. On the other hand, it is also known that, if $s$ is uniform on $[a, b]$ or, more generally, if $s$ is non-increasing on $[a, b]$ and piecewise constant with a finite number of pieces, the rate of convergence of the MLE becomes faster, namely $n^{-1/2}$, that is parametric. Moreover the MLE has some robustness property with respect to small deviations from the assumption of monotonicity as shown in Birgé (1997).

Unfortunately, the situation changes drastically if $a$ is unknown since the MLE on the set $\mathcal{S}$ of all decreasing densities on intervals $[a, +\infty)$ with $a$ arbitrary in $\mathbb{R}$ does not exist and one cannot use the MLE with $a$ replaced by the smallest observation $X_{(1)}$. To solve the problem we have to estimate $a$ by some $\hat{a} < X_{(1)}$ and if this estimator strongly underestimates $a$ the resulting MLE based on the assumption that $s$ is non-increasing on $[\hat{a}, +\infty)$ will perform quite poorly. Besides, if one wants to keep the robustness properties of the MLE when $a$ is
known, the estimator \( \hat{a} \) should be robust, in particular to some contamination of the true density by a small amount (of order \( n^{-1} \)) of a probability located far away to the left of \( a \).

One purpose of this paper is to offer a more satisfactory solution to the problem of estimating an unknown element \( s \in \mathcal{S} \). For this, we replace the MLE by a robust alternative, namely a \( \rho \)-estimator as defined in Baraud et al. (2014) and based on the model \( \mathcal{S} \). If \( s \) is the true density and \( \hat{s} \) the \( \rho \)-estimator, we measure its risk by \( \mathbb{E}_{s} \left[ h^2(s, \hat{s}) \right] \), where \( h \) denotes the Hellinger distance. This estimator actually retains all the good above mentioned properties of the MLE, although the parameter \( a \) is now unknown. More precisely, if \( V \) is the set of all piecewise constant (with a finite number of pieces) non-increasing densities, the risk of \( \hat{s} \) at \( \bar{s} \in V \) is bounded by \( c(\bar{s})(\log n)^{3/2}/n \). The robustness properties of the \( \rho \)-estimator implies that

\[
\mathbb{E}_{s} \left[ h^2(s, \hat{s}) \right] \leq C \inf_{\pi \in V} \left[ h^2(s, \pi) + c(\pi) (\log n)^{3} \right]
\]

for some universal constant \( C > 0 \) and all densities \( s \), \( \mathbb{E}_{s} \) denoting the expectation when \( s \) obtains. We shall see that this implies in particular that if \( s \in \mathcal{S} \) is bounded with a bounded support \([a, b]\) \((a < b)\), the risk at \( s \) is bounded by \( c(s)n^{-2/3}(\log n)^{2} \) and therefore coincides (up to the logarithmic factor) with the minimax rate established when \( a \) is known.

This phenomenon of a “super-efficient rate” at points in \( V \) is actually more general. There exist models \( S \) for which the typical rate of estimation is \( r_{n} \) but on a subset \( V \) of \( S \) it becomes \( r_{n}' \) with \( r_{n}' = o(r_{n}) \). In such a situation, the risk of a \( \rho \)-estimator can be bounded, up to a universal numerical factor, by \( \inf_{\pi \in V} \left[ h^2(s, \pi) + c(\pi)r_{n}' \right] \) for all densities \( s \). This implies that the risk at \( s \) will depend on its distance to the set \( V \) of what we call the “extremal points” of \( S \). We shall provide below several examples of such situations: the set of piecewise monotone densities, the set of piecewise concave or convex densities and the set of log-concave densities.

2. The statistical setting

Let \((\mathcal{X}, \mathcal{A})\) be a measurable set, \( \mu \) a \( \sigma \)-finite measure on \((\mathcal{X}, \mathcal{A})\), \( \mathcal{P} \) the set of all probabilities on \((\mathcal{X}, \mathcal{A})\) and \( \mathcal{P}_{\mu} \) the set of all elements of \( \mathcal{P} \) which are absolutely continuous with respect to \( \mu \). We observe \( n \) i.i.d. random variables \( X_{1}, \ldots, X_{n} \) with values in \((\mathcal{X}, \mathcal{A})\) and distribution \( P_{s} \in \mathcal{P} \). To avoid trivialities, we shall always assume, in the sequel, that \( n \geq 3 \), so that \( \log n > 1 \). We shall also consider the set \( \mathcal{L} \) of real-valued functions from \( \mathcal{X} \) to \( \mathbb{R} \) and denote by \( L_{\mu} \) the set of those \( t \in L_{1}(\mu) \) which are non-negative \( \mu \)-a.e. with \( \int t \, d\mu = 1 \), that is the set of probability densities with respect to \( \mu \). An element of \( \mathcal{P}_{\mu} \) with density \( t \) with respect to \( \mu \) will be denoted by \( P_{t} \). If \( P_{s} \in \mathcal{P}_{\mu} \), then \( s = dP_{s}/d\mu \), otherwise the index \( s \) simply means that \( P_{s} \) is the true distribution of the \( X_{i} \).

In the sequel, we shall consider subsets of \( L_{\mu} \) of densities \( t \) sharing some specific property, like monotonicity, defined by their inclusion in a given subset \( \mathcal{L}' \) of \( \mathcal{L} \). This will always mean that there exists \( t' \in \mathcal{L}' \) such that \( t = t' \mu \)-a.e. For instance, if \((\mathcal{X}, \mathcal{A}) = (\mathbb{R}, B(\mathbb{R}))\) and \( \mu \) is the Lebesgue measure, when we say that the density \( t \) is non-increasing on \((0, +\infty)\), this means that there exists a non-increasing function \( t' \) on \((0, +\infty)\) such that \( t = t' \mu \)-a.e. This being said once and for all, we shall systematically, in the sequel, identify an element of \( L_{\mu} \) with its version in \( \mathcal{L} \) with the required property.
We turn $\mathcal{P}$ into a metric space via the Hellinger distance $h$. We recall from Le Cam (1973 or 1986) that the Hellinger distance between two elements $P$ and $Q$ of $\mathcal{P}$ is given by

$$h(P, Q) = \left[ \frac{1}{2} \int_{\mathcal{X}} \left( \sqrt{dP/d\nu} - \sqrt{dQ/d\nu} \right)^2 d\nu \right]^{1/2},$$

where $\nu$ is an arbitrary positive measure which dominates both $P$ and $Q$, the result being independent of the choice of $\nu$ among all such measures. The one-to-one mapping $t \mapsto P_t$ between $\mathbb{L}_\mu$ and $\mathcal{P}_\mu$ allows us to consider $h$ as a distance on $\mathbb{L}_\mu$ as well and to write $h(t, u)$ for $h(P_t, P_u)$ if $t$ and $u$ belong to $\mathbb{L}_\mu \cup \{s\}$.

The estimators $\widehat{s}$ that we shall consider here will be based on models $\overline{S}$ for $s$, that is separable subsets of the metric space $(\mathbb{L}_\mu, h)$, that are chosen in order to approximate the true distribution $P_s$ and may or may not contain $s$. Of course a model $\overline{S}$ is good only if its distance $h(s, \overline{S})$ to $P_s$ is not too large, where we set, for $A \subset \mathbb{L}_\mu, h(t, A) = \inf_{u \in A} h(t, u)$.

Our aim in this paper is to study the performance of a $\rho$-estimator $\widehat{s}$ of $s$ built on $\overline{S}$. The definition and properties of $\rho$-estimators have been described in great details in Baraud et al. (2014) and we shall recall in the next section the facts that are needed here.

2.1. Notations, conventions and definitions. We set $\log_+(x) = \max\{\log x, 1\}, \mathbb{N}^* = \mathbb{N} \setminus \{0\}, a \vee b = \max(a, b), a \wedge b = \min(a, b)$ and, for $x \in \mathbb{R}_+, \lfloor x \rfloor = \inf\{n \in \mathbb{N}, n \geq x\}; |A|$ denotes the cardinality of the finite set $A$ and $C, C', \ldots$ numerical constants that may vary from line to line. For a function $f$ on $\mathbb{R}$, $f(x^+)$ and $f(x^-)$ denote respectively the right-hand and left-hand limits of $f$ at $x$ whenever these limits exist. We shall also use the following conventions: $\sum_{\emptyset} = 0, x/0 = +\infty$ if $x > 0$ and $0/0 = 1$.

Definition 1. A partition of the interval $(a, b)$ ($-\infty \leq a < b \leq +\infty$) of size $k + 1$ with $k \in \mathbb{N}$ is either $\emptyset$ when $k = 0$ or a finite set $\mathcal{I} = \{x_1, \ldots, x_k\}$ of real numbers with $a < x_1 < x_2 < \cdots < x_k < b$ if $k \geq 1$. We shall call endpoints of the partition $\mathcal{I}$ the numbers $x_j$ and intervals of the partition the open intervals $I_j = (x_j, x_{j+1}), 0 \leq j \leq k$ with $x_0 = a$ and $x_{k+1} = b$. A partition $\mathcal{I}$ will also be identified to the set of its intervals and we shall equally write $\mathcal{I} = \{I_0, \ldots, I_k\}$ or $\mathcal{I} = \{x_1, \ldots, x_k\}$.

The set of all partitions of $\mathbb{R}$ with $k$ endpoints or $k + 1$ intervals is denoted by $\mathcal{J}(k + 1)$ and the length of $I_j$ by $\ell(I_j)$. If $\mathcal{I} = \{x_1, \ldots, x_k\}$ and $\mathcal{I}' = \{x_1', \ldots, x_k'\}$, $\mathcal{I} \vee \mathcal{I}' = \{x_1, \ldots, x_k\} \cup \{x_1', \ldots, x_k'\}$ and $\mathcal{I} \supset \mathcal{I}'$ means that $\{x_1, \ldots, x_k\} \supset \{x_1', \ldots, x_k'\}$.

3. Monotone densities

In view of illustrating the main result of this paper to be presented in Section 4, let us consider the example of the model $\overline{S}$ consisting of all the densities with respect to the Lebesgue measure $\mu$ that are non-increasing on some arbitrary interval of $\mathbb{R}$ and vanish elsewhere. In this case $(\mathcal{X}, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\overline{S}$ is the set of all densities of the form $t = f 1_{(a, +\infty)}$ where $a \in \mathbb{R}$ and $f$ is a non-increasing and non-negative function on $(a, +\infty)$ (which may be unbounded in the neighbourhood of $a$) such that $\int_a^{+\infty} f(x) dx = 1$. The results would be similar for the set of all densities which are non-decreasing on some interval of $\mathbb{R}$ and vanish elsewhere. For $D \in \mathbb{N}^*$ we define $\overline{\mathcal{V}}(D)$ to be the set of all densities of the form $\sum_{j=1}^D a_j 1_{(x_j, x_{j+1})}$ with $\mathcal{I} = \{x_1, \ldots, x_{D+1}\} \in \mathcal{J}(D + 2)$ and $a_j \geq 0$ for $1 \leq j \leq D$. Note that the densities in $\overline{\mathcal{V}}(D)$ necessarily take the value 0 on the two unbounded extremal
intervals $I_0$ and $I_{D+1}$ of the partition $\mathcal{I}$. For instance, $\overline{V}(1)$ corresponds to the family of uniform densities on intervals $(\theta_0, \theta_0 + \theta_1]$, that is

$$\overline{V}(1) = \{ t(\cdot) = \theta_1^{-1} \mathbb{1}_{[0, \theta_1]}(\cdot - \theta_0), \ \theta_1 > 0, \ \theta_0 \in \mathbb{R} \}.$$ 

In such a situation, we can prove the following result.

**Theorem 1.** We can build a $p$-estimator $\hat{s}$ on $\overline{S}$ such that for all $P_s \in \mathcal{P}$,

$$C \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{D \geq 1} \left[ h^2(s, \overline{V}(D) \cap \overline{S}) + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right]$$

for some universal constant $C \in (0, 1]$.

**Remark.** Since $C \leq 1$, the left-hand side is always bounded by one so that it is useless to consider values of $D$ that lead to a bound which is not smaller than one, in particular $D \geq n$, and (1) is actually equivalent to

$$C \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{1 \leq D < n} \left[ h^2(s, \overline{V}(D) \cap \overline{S}) + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right].$$

Although we shall not repete it systematically, the same remark will hold for all our subsequent results.

Bound (1) means that the risk function $P_s \mapsto \mathbb{E}_s \left[ h^2(s, \hat{s}) \right]$ of $\hat{s}$ over $\mathcal{P}$ can be quite small in the neighbourhood of some specific densities $t \in \overline{S}$: if $s$ is piecewise constant on a partition of $\mathbb{R}$ into $D + 2 < n + 2$ intervals or $P_s$ is close enough to a probability which has a density of this form, the risk of $\hat{s}$ is of order $D/n$, up to logarithmic factors. More precisely,

$$\sup_{s \in \overline{V}(D) \cap \overline{S}} \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq C' \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \text{ for } 1 \leq D < n.$$ 

When $n$ becomes large and $D$ remains fixed, the rate of convergence of $\hat{s}$ towards an element of $\overline{V}(D) \cap \overline{S}$ is therefore almost parametric.

Of particular interest are the densities $t$ which are bounded, supported on a compact interval $[a, b]$ of $\mathbb{R}$ (for numbers $a < b$ depending on $t$) and non-increasing on $(a, b)$. Given $\overline{M} \geq 0$, we introduce the set $\overline{S}(\overline{M})$ of densities $t$ of this form and for which

$$(2) \quad (b - a) V^2_{[a, b]} \left( \sqrt{t} \right) = M(t) \leq \overline{M},$$

where the variation $V_{[a, b]} \left( \sqrt{t} \right)$ of $\sqrt{t}$ on $[a, b]$ is defined in the following way:

**Definition 2.** Let $f$ be defined on some interval $I$ (with positive length) of $\mathbb{R}$ and monotone on $I$. Its variation on $I$ is given by

$$V_I(f) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x) \in [0, +\infty].$$

Note that $\overline{S}(0)$ is the set of uniform densities on some finite interval, so that $\overline{S}(0) = \overline{V}(1)$, and that $\overline{S}(\overline{M})$ is not compact and contains densities that can be arbitrarily large in sup-norm. The functional $M$ remains invariant by translation and scaling: if $u(\cdot) = \lambda t(\lambda(\cdot - \tau))$ with $\lambda > 0$ and $\tau \in \mathbb{R}$, then $M(u) = M(t)$ which implies that $\overline{S}(\overline{M})$ is invariant by translation and scaling. It turns out that the densities lying in $\overline{S}(\overline{M})$ can be well approximated by elements of $\overline{V}(D)$. More precisely, the following approximation result holds.
Proposition 1. For all $D \in \mathbb{N}^*$ and $t \in \bigcup_{M \geq 0} \mathcal{S}(M)$,

$$h^2(t, \overline{\mathcal{V}}(D) \cap \overline{\mathcal{S}}) \leq [M(t)/(2D)^2] \wedge 1.$$ 

Using the triangular inequality, the right-hand side of (1) can be bounded from above in the following way: for all $M \geq 0$ and $t \in \overline{\mathcal{S}(M)}$,

$$\inf_{D \geq 1} \left[ h^2(s, \overline{\mathcal{V}}(D) \cap \overline{\mathcal{S}}) + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right]$$

$$\leq 2h^2(s, t) + \inf_{D \geq 1} \left[ 2h^2(t, \overline{\mathcal{V}}(D) \cap \overline{\mathcal{S}}) + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right]$$

$$\leq 2h^2(s, t) + \inf_{D \geq 1} \left[ \frac{M}{2D^2} + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right].$$

Finally, since $t$ is arbitrary in $\overline{\mathcal{S}(M)}$,

$$\inf_{D \geq 1} \left[ h^2(s, \overline{\mathcal{V}}(D) \cap \overline{\mathcal{S}}) + \frac{D}{n} \log^3 \left( \frac{cn}{D} \right) \right] \leq 2h^2(s, \overline{\mathcal{S}}(M)) + \inf_{D \geq 1} \left[ \frac{M}{2D^2} + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right].$$

Optimizing the right-hand side with respect to $D$ and using the facts that $\overline{M}$ is arbitrary and $\log n > 1$, we derive the following corollary of Theorem 1.

Corollary 1. For all probabilities $P_s \in \mathcal{P}$, the $\hat{s}$-estimator $\hat{s}$ of $s$ on $\overline{\mathcal{S}}$ satisfies for some constant $C \in (0, 1]$,

$$CE_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{M \geq 0} \left[ h^2(s, \overline{S}(M)) + \left( \overline{M}^{1/3} n^{-2/3} (\log n)^2 \right) \vee (n^{-1} (\log n)^3) \right].$$

In particular, if $s \in \overline{\mathcal{S}(M)}$ for some $\overline{M} \geq n^{-1} (\log n)^3$, the risk bound of the estimator is not larger (up to a universal constant) than $\overline{M}^{1/3} n^{-2/3} (\log n)^2$ while for smaller values of $\overline{M}$ it is bounded by $n^{-1} (\log n)^3$. Up to logarithmic factors, this rate (with respect to $n$) is optimal since it corresponds to the lower bound of order $n^{-2/3}$ for the minimax risk on the subset of $\overline{\mathcal{S}(M)}$ consisting of the non-increasing densities supported in $[0, 1]$ and bounded by $\overline{M}$. This lower bound follows from the proof of Proposition 1 of Birgé (1987). The result was actually stated in this paper for the $\mathbb{L}_1$-distance but its proof shows that it applies to the Hellinger distance as well. This property means that, although the set $\overline{\mathcal{S}(M)}$ is not compact because the support of the densities is unknown, the minimax risk is finite. We do not know any other estimator with the same performance which is also robust with respect to the Hellinger distance.

Note that Corollary 1 can also be used to determine the rate of estimation for decreasing densities $s$ with possibly unbounded support and maximum value provided that we have some assumption about the behaviour of the function $\overline{M} \mapsto h(s, \overline{S}(M))$ when $\overline{M}$ goes to infinity.

4. The main result

Let us first recall the following classical definition.

Definition 3. A class $\mathcal{C}$ of subsets of $\mathcal{X}$ is said to shatter a finite subset $A = \{x_1, \ldots, x_m\}$ of $\mathcal{X}$ if the class of subsets

$$\mathcal{C} \cap A = \{C \cap A, \ C \in \mathcal{C}\}$$
is equal to the class of all subsets of $A$ or, equivalently, if $|\mathcal{C} \cap A| = 2^m$. A non-empty class $\mathcal{C}$ of subsets of $\mathcal{X}$ is a VC-class with dimension $d \in \mathbb{N}$ if there exists some integer $m$ such that no finite subset $A \subset \mathcal{X}$ with cardinality $m$ can be shattered by $\mathcal{C}$ and $d+1$ is the smallest $m$ with this property.

We may now introduce the main property to be used in this paper.

**Definition 4.** Let $\mathcal{F}$ be a class of real-valued functions on $\mathcal{X}$. We shall say that an element $f(\mathcal{F}) \in \mathcal{F}$ is extremal in $\mathcal{F}$ (or is an extremal point of $\mathcal{F}$) with degree $d(\mathcal{F}) \in \mathbb{N}$ if the class $\mathcal{C}(\mathcal{F}, f)$ of subsets of $\mathcal{X}$ given by

$$\mathcal{C}(\mathcal{F}, f) = \{ \emptyset, \mathcal{X}, \{ x \in \mathcal{X} \mid f(x) > \lambda f(x) \}, \lambda \geq 0, f \in \mathcal{F} \}$$

is a VC-class with dimension $d(\mathcal{F})$.

Since the class $\mathcal{C}(\mathcal{F}, f)$ contains $\{ \emptyset, \mathcal{X} \}$, it can at least shatter one point and the degree $d(\mathcal{F})$ is therefore at least 1. Our main result is as follows.

**Theorem 2.** Let $\mathcal{S}$ be a model with a non-void set $\mathcal{X}$ of extremal points. There exists a $\rho$-estimator on $\mathcal{S}$ for some universal constant $C \in (0, 1)$ and all $\xi > 0$,

$$\mathbb{P}_s \left[ Ch^2(s, \mathcal{S}) \leq \inf_{\pi \in \mathcal{X}} \left[ h^2(s, \mathcal{S}) + \frac{d(\mathcal{S})}{n} \log_+ \left( \frac{n}{d(\mathcal{S})} \right) \right] + \frac{\xi}{n} \right] \geq 1 - e^{-\xi}$$

whenever the true distribution $P_s$. Consequently,

$$C \mathbb{E}_s \left[ h^2(s, \mathcal{S}) \right] \leq \inf_{\pi \in \mathcal{X}} \left[ h^2(s, \mathcal{S}) + \frac{d(\mathcal{S})}{n} \log_+ \left( \frac{n}{d(\mathcal{S})} \right) \right].$$

Note that the boundedness of $h$ implies that values of $d(\mathcal{S}) \geq n$ lead to a trivial bound so that the infimum could be reduced to those $\mathcal{S}$ such that $d(\mathcal{S}) < n$.

5. Applications

In this section we shall use the following lemma:

**Lemma 1.** Let $\mathcal{C}$ be a class of subsets of $J \subset \mathbb{R}$ such that each element of $\mathcal{C}$ is the union of at most $k$ sub-intervals of $J$, then it is a VC-class with dimension at most $2k$.

**Proof.** Let $x_1 < x_2 < \ldots < x_{2k+1}$ be $2k + 1$ points of $J$. It is easy to check that elements of the form $J_1 \cup \cdots \cup J_l$, where the $J_j$ are disjoint sub-intervals of $J$ and $l \leq k$, cannot pick up the subset of points $\bigcup_{i=0}^{k} \{ x_{2i+1} \}$.

5.1. Piecewise constant densities. Let us now consider the model $\overline{\mathcal{V}}(D)$ of Section 3 to build a $\rho$-estimator. If $f$ and $\overline{\mathcal{F}}$ belong to $\overline{\mathcal{V}}(D)$, for all $\lambda \geq 0$, $f - \lambda \overline{\mathcal{F}}$ is of the form $\sum_{j=1}^{k} a_j I_{(x_j, x_{j+1}]}$ with $k < 2(D+1)$ so that $\{ x \in \mathcal{X} \mid f(x) > \lambda \overline{\mathcal{F}}(x) \}$ is the union of at most $D + 1$ intervals. This implies by Lemma 1 that all elements of $\overline{\mathcal{V}}(D)$ are extremal with dimension bounded by $2(D+1)$. It then follows from Theorem 2 that

$$\sup_{s \in \overline{\mathcal{V}}(D)} \mathbb{E}_s \left[ h^2(s, \mathcal{S}) \right] \leq C \frac{D}{n} \log_+ \left( \frac{n}{D} \right),$$

which, up to the logarithmic factor, corresponds to a parametric rate (with respect to $n$) although the partition that defines $s$ can be arbitrary in $\mathcal{J}(D+2)$ and the support of $s$
is unknown. It follows from Birgé and Massart (1998), Proposition 2 that a lower bound for the minimax risk on $\mathcal{V}(D)$ is of the form $C'(D/n)\log_+(n/D)$, which shows that some power of $\log_+(n/D)$ is necessary in (6). We suspect that the power three for the logarithm is not optimal.

5.2. Piecewise monotone densities. Let us now see how Theorem 2 can be applied in the simple situation of piecewise monotone densities.

**Definition 5.** Given $k \in \mathbb{N}^*$ and a partition $\mathcal{I} = \{I_0, \ldots, I_{k-1}\} \in \mathcal{J}(k)$, a real-valued function $f$ on $\mathbb{R}$ will be called piecewise monotone (with $k$ pieces) based on $\mathcal{I}$ if $f$ is monotone on each $I_j$, $0 \leq j \leq k-1$. The set of all such functions will be denoted by $\mathcal{G}_k$. For $k \geq 2$ (since no density is monotone on $\mathbb{R}$), $\mathcal{F}_k$ is the set of densities (with respect to the Lebesgue measure) that belong to $\mathcal{G}_k$.

Clearly, $\mathcal{G}_k \subset \mathcal{G}_l$ and $\mathcal{F}_k \subset \mathcal{F}_l$ for all $l > k$.

**Proposition 2.** For $D \geq 1$ and $k \geq 2$, any element $\mathcal{F}$ of $\mathcal{F}_k \cap \mathcal{V}(D)$ is extremal in $\mathcal{F}_k$ with degree not larger than $3(k + D + 1)$.

**Proof.** Let $f$ be a piecewise monotone density on $\mathbb{R}$ based on a partition $\mathcal{I}_0 \in \mathcal{J}(k)$, therefore with $k - 1$ endpoints, let $\mathcal{F} \in \mathcal{V}(D)$ be based on a partition $\mathcal{I}_1 \in \mathcal{J}(D + 2)$ (with $D + 1$ endpoints) and let $\mathcal{I}_2 = \mathcal{I}_1 \cup \mathcal{I}_0$. It is a partition of $\mathbb{R}$ with at most $k + D$ endpoints, therefore at most $k + D + 1$ intervals and on each such interval $f$ is monotone and $\mathcal{F}$ is constant which implies that $f - \lambda \mathcal{F}$ belongs to $\mathcal{G}_{k+D+1}$ for all $\lambda \geq 0$. It then follows from Lemma 2 below that $\{x \in \mathcal{H} | f(x) > \lambda \mathcal{F}(x)\}$ is a union of at most $(3/2)(k + D + 1)$ intervals. The conclusion follows from Lemma 1.

**Lemma 2.** If $f \in \mathcal{G}_k$, whatever $a \in \mathbb{R}$ the set $\{x \mid f(x) > a\}$ can be written as a union of at most $k + \lceil (k - 1)/2 \rceil \leq 3k/2$ intervals and the set $\{x \mid f(x) \leq a\}$ as well.

**Proof.** If $I_j$ is an interval of the partition $\mathcal{I}$ that defines $f$, $\{x \mid f(x) > a\} \cap I_j$ is either $\emptyset$ or a non-void interval. Let us now focus on the points $x_j$ such that $f(x_j) > a$ since only these matter. The only case for which $\{f > a\} \cap (x_{j-1}, x_{j+1})$ is the union of three disjoint intervals occurs when $f(x_{j-1}) > a$, $f(x_j) \leq a$, $f(x_{j+1}) > a$, in which case the same configuration cannot occur for the points $x_{j-1}$ and $x_{j+1}$. It implies that the maximal number of points for which this configuration can occur is $\lceil (k - 1)/2 \rceil$. This means that at most $\lceil (k - 1)/2 \rceil$ new intervals can be added to the ones of the form $\{x \mid f(x) > a\} \cap I_j$. The result follows by combining all these intervals and the proof for $\{x \mid f(x) \leq a\}$ is the same.

An application of Theorem 2 then leads to

**Corollary 2.** For all $k \geq 2$ there exists a $\rho$-estimator on $\mathcal{F}_k$ satisfying, whatever the true distribution $P$,

$$C \mathbb{E}_s \left[ h^2(s, \hat{s}) \right] \leq \inf_{D \geq 1} \left[ \inf_{\mathcal{F} \in \mathcal{F}_k \cap \mathcal{V}(D)} h^2(s, \mathcal{F}) + \frac{k + D}{n} \log_+^3 \left( \frac{n}{k + D} \right) \right]$$

where $C \in (0, 1]$ is a universal constant.

Note that using $D$ with $k + D \geq n$ leads to a trivial upper bound so that we should restrict ourselves to $D < n - k$. 7
Since, for \( t \in \mathcal{F}_k \),
\[
\inf_{\overline{s} \in \mathcal{F}_k \cap \mathcal{V}(D)} h(s, \overline{s}) \leq \inf_{t \in \mathcal{F}_k} \left[ h(s, t) + \inf_{\overline{s} \in \mathcal{F}_k \cap \mathcal{V}(D)} h(t, \overline{s}) \right],
\]
to go further with our analysis it will be necessary to evaluate \( \inf_{\overline{s} \in \mathcal{F}_k \cap \mathcal{V}(D)} h(t, \overline{s}) \) for \( t \in \mathcal{F}_k \). In order to do this we shall use an approximation result based on the following functional \( M_k \).

**Definition 6.** Let \( t \in \mathcal{F}_k \) be based on the partition \( I = \{I_0, \ldots, I_{k-1}\} \in \mathcal{J}(k) \). Using the convention \((+\infty) \times 0 = 0\), we define the functional \( M_k(t) \) on \( \mathcal{F}_k \) by
\[
(7) \quad M_k(t) = \left[ \sum_{j=0}^{k-1} \left[ \ell(I_j) V_{I_j}^2 \left( \sqrt{t} \right) \right]^{1/3} \right]^3 \leq +\infty,
\]
where \( V_{I_j} (\sqrt{t}) \) is the variation of \( \sqrt{t} \) on \( I_j \) given by (3).

For \( 0 \leq M < +\infty \) and \( k \in \mathbb{N}^* \), we denote by \( \mathcal{F}_{k+2}(M) \) the subset of \( \mathcal{F}_{k+2} \) of those densities \( t \) based on a partition \( I = \{I_0, \ldots, I_{k+1}\} \) such that \( M_{k+2}(t) \leq M \).

Note that with our convention, if \( M_k(t) < +\infty \), \( t \) is equal to zero on \( I_0 \cup I_{k-1} \), in which case the summation in (9) can be restricted to \( 1 \leq j \leq k - 2 \), and that \( \mathcal{F}_{k+2}(0) \) is equal to \( \mathcal{V}(k) \) (in the \( \mathbb{L}_1 \) sense).

If \( t \in \mathcal{F}_k \) is based on \( I \) and \( l > k \), \( t \in \mathcal{F}_l \) based on any partition \( I' \in \mathcal{J}(l) \) with \( I' \supseteq I \). The following proposition compares the values of \( M_k(t) \) and \( M_l(t) \) for two such partitions.

**Lemma 3.** If \( t \in \mathcal{F}_k \) is based on \( I \) and \( t \in \mathcal{F}_l \) based on \( I' \supseteq I \), then \( M_l(t) \leq M_k(t) \).

**Proof.** To show that \( M_l^3(t) \leq M_k^3(t) \) it suffices to show that this is true when we simply divide an interval \( J \) of length \( L \) of \( I \) into \( m \) intervals \( J_1, \ldots, J_m \) of respective lengths \( L_1, \ldots, L_m \). The initial contribution of \( J \) to \( M_k^3(t) \) which is \( [L V_J^2 (\sqrt{t})]^{1/3} \) becomes
\[
\sum_{j=1}^{m} \left[ L_j V_{J_j}^2 \left( \sqrt{t} \right) \right]^{1/3} \text{ with } \sum_{j=1}^{m} L_j = L \text{ and } \sum_{j=1}^{m} V_{J_j} (\sqrt{t}) \leq V_J (\sqrt{t}).
\]
Setting \( L_j = \alpha_j L \) and \( V_{J_j} (\sqrt{t}) = \beta_j V_J (\sqrt{t}) \), this amounts to show that \( \sum_{j=1}^{m} \alpha_j^{1/3} \beta_j ^{2/3} \leq 1 \) which follows from Hölder’s Inequality.

The approximation of elements of \( \mathcal{F}_{k+2}(M) \) by elements of \( \mathcal{V}(D) \) is controlled in the following way.

**Proposition 3.** Let \( k \geq 1 \) and \( t \in \mathcal{F}_{k+2} \) with \( M_{k+2}(t) < +\infty \). Then, for all \( D \geq 1 \),
\[
h^2 \left( t, \mathcal{V}(D + k) \cap \mathcal{F}_{k+2} \right) \leq (2D)^{-2} M_{k+2}(t).
\]

Applying Corollary 2 leads to the following bound which is valid for all \( t \in \mathcal{F}_{k+2} \) with \( M_{k+2}(t) < +\infty \) and whatever the distribution \( P_\pi \) of the observations:
\[
(8) \quad C \mathbb{E}_\pi \left[ h^2(s, \overline{s}) \right] \leq h^2(s, t) + \inf_{D \geq 1} \left[ \frac{M_{k+2}(t)}{D^2} + \frac{k + D}{n} \log \left( \frac{n}{k + D} \right) \right].
\]
A final optimization with respect to \( D \) leads to
\[
C \mathbb{E}_\pi \left[ h^2(s, \overline{s}) \right] \leq h^2(s, t) + [M_{k+2}(t)]^{1/3} n^{-2/3} (\log n)^2 + kn^{-1} (\log n)^3.
\]
Since this result is valid for all densities $t \in \mathcal{F}_{k+2}$, we can again optimize it with respect to $t$ which finally leads to:

**Theorem 3.** There exists a $\rho$-estimator $\hat{s}$ based on the model $\mathcal{F}_{k+2}$ and such that, whatever $P_s$,

$$C\mathbb{E}[h^2(s, \hat{s})] \leq \inf_{M > 0} \left[ h^2(s, \mathcal{F}_{k+2}(M)) + (\log n)^2 \left( (Mn^{-2})^{1/3} \sqrt{(kn^{-1}\log n)} \right) \right].$$

In particular

$$\sup_{s \in \mathcal{F}_{k+2}(M)} \mathbb{E}[h^2(s, \hat{s})] \leq C(\log n)^2 \left( (Mn^{-2})^{1/3} \sqrt{(kn^{-1}\log n)} \right).$$

If we want to estimate a bounded unimodal density $s$ with support of finite length $L$, we may build a $\rho$-estimator on $\mathcal{F}_4$. In such a case, $M_4(s)$ can be bounded by $4L\|s\|_{\infty} \geq 1$ (since $s$ is a density, $L\|s\|_{\infty} \geq 1$) and the performance of the $\rho$-estimator for such a unimodal density $s$ will be given by

$$\mathbb{E}[h^2(s, \hat{s})] \leq C \left( L\|s\|_{\infty}n^{-2} \right)^{1/3}(\log n)^2.$$

### 5.3. Piecewise concave-convex densities.

In the previous sections we considered densities $t$ which were piecewise monotone or constant which implied the same properties for $\sqrt{t}$ but it follows from Proposition 3 that it is actually the approximation properties of $\sqrt{t}$ that matter. When going to more sophisticated properties than monotonicity, it is no more the same to state them for $t$ or for $\sqrt{t}$ which accounts for the slightly more complicated structure of this section.

**Definition 7.** Let $I \in \mathcal{J}(k)$ be a partition with $k$ intervals. A function $f$ is piecewise convex-concave based on $I$ if it is either convex or concave on each interval $I_j$ of the partition. The set of all such functions when $I$ varies in $\mathcal{J}(k)$ will be denoted by $\mathcal{G}_k^1$. For $D \in \mathbb{N}^*$ we denote by $\overline{W}_1(D)$ the set of all functions $\gamma$ of the form $\gamma = \sum_{j=1}^D \gamma_j \mathbb{1}_{(x_j, x_{j+1}]}$ with $x_1 < x_2 < \cdots < x_{D+1}$ where $\gamma_j$ is an affine function for all $j$. The sets $\mathcal{F}_k^1$ and $\overline{W}_1(D)$ are the sets of those densities $t$ such that $\sqrt{t}$ belongs to $\mathcal{G}_k^1$ and $\overline{W}_1(D)$ respectively.

We recall that $f$ is either concave or convex on some open interval $I$ if and only if it is absolutely continuous on $I$ with a monotone derivative $f'$ on $I$.

In order to find extremal points of $\mathcal{F}_k^1$ we need the following result.

**Lemma 4.** If the function $f$ is absolutely continuous with derivative $f' \in \mathcal{G}_k$, then $f \in \mathcal{G}_{2k}$.

*Proof.* There exists $I \in \mathcal{J}(k)$ such that $f'$ is monotone on each interval $I_j$ of $I$ and therefore can only change its sign once on each $I_j$. It follows that there exists a partition $I' \in \mathcal{J}(K)$ with $K \leq 2k$ such that $f'$ has a constant sign on each interval of $I'$ and therefore $f$ is monotone on each interval of $I'$.

**Proposition 4.** For all $D, k \in \mathbb{N}^*$, the elements $\overline{g} \in \mathcal{G}_k^1 \cap \overline{W}_1(D)$ with are extremal in $\mathcal{G}_k^1$ with $d(\overline{g}) \leq 6(D + k + 1)$. Consequently the elements $\overline{f} \in \mathcal{F}_k^1 \cap \overline{W}_1(D)$ are extremal in $\mathcal{F}_k^1$ with $d(\overline{f}) \leq 6(D + k + 1)$.

*Proof.* Let us consider $g \in \mathcal{G}_k^1$. The derivatives $g'$ and $\overline{g}'$ are piecewise monotone based on partitions $I_0$ with $k - 1$ endpoints and $I_1$ with $D + 1$ endpoints respectively, so that $I_2 = I_0 \cup I_1$ is a partition with at most $k + D + 1$ intervals and, on each interval of this
partition, \( g' \) is monotone and \( \overline{g}' \) is constant. It follows that, whatever \( \lambda \geq 0, g' - \lambda \overline{g}' \) is monotone on each such interval and therefore belongs to \( \mathcal{G}_{k+D+1} \). Applying Lemma 4, we conclude that \( g - \lambda \overline{g} \in \mathcal{G}_k \) and by Lemma 2 that the set \( \{ x \mid g(x) > \lambda \overline{g}(x) \} \) is the union of at most \( 3(k + D + 1) \) intervals. Finally Lemma 1 allows to conclude that \( \overline{g} \) is extremal with dimension bounded by \( 6(k + D + 1) \). The last result follows from Lemma 5 of Section 7.1 below with \( \alpha = 1/2 \). \hfill \Box 

This leads to the following corollary of Theorem 2.

**Corollary 3.** For all \( k \geq 2 \) there exists a \( \rho \)-estimator on \( \mathcal{F}^1_k \) satisfying, whatever the true distribution \( P_s \),

\[
C \mathbb{E}_s \left[ h^2(s, \overline{s}) \right] \leq \inf_{D \geq 1} \left[ \inf_{\pi \in \mathcal{V}_1(D) \cap \mathcal{F}^1_k} h^2(s, \pi) + \frac{k + D}{n} \log^3 \left( \frac{n}{k + D} \right) \right]
\]

where \( C \in (0, 1) \) is a universal constant.

In order to control the approximation term \( \inf_{\pi \in \mathcal{V}_1(D) \cap \mathcal{F}^1_k} h(s, \pi) \), we shall use the following results.

**Definition 8.** Let \( t \in \mathcal{F}^1_k \) be based on the partition \( \mathcal{I} = \{ I_0, \ldots, I_{k-1} \} \in \mathcal{J}(k) \) so that \( \sqrt{t} \) is absolutely continuous on each \( I_j \) with monotone derivative \( (\sqrt{t})' \). Using the convention \( (\infty) \times 0 = 0 \), we define the functional \( M_{k,1} \) on \( \mathcal{F}^1_k \) by

\[
M_{k,1}(t) = \left[ \sum_{j=0}^{k-1} \left( \ell(I_j)^3 V_j^2 \left( (\sqrt{t})' \right) \right]^{1/5} \right)^5 \leq +\infty,
\]

where \( V_j \) is the variation of \( (\sqrt{t})' \) on \( I_j \).

For \( 0 \leq \overline{M} < +\infty \) and \( k \in \mathbb{N}^* \), we denote by \( \mathcal{F}^1_{k+2}(\overline{M}) \) the subset of \( \mathcal{F}^1_{k+2} \) of those densities \( t \) based on a partition \( \mathcal{I} = \{ I_0, \ldots, I_{k+1} \} \) such that \( M_{k,2,1}(t) \leq \overline{M} \).

An analogue of Lemma 3 holds for the functional \( M_{k,1}(t) \) with a similar proof. We omit the details.

**Proposition 5.** Let \( k \geq 1 \) and \( t \in \mathcal{F}^1_{k+2} \) with \( M_{k,2,1}(t) < +\infty \). Then, for \( D \geq 1 \),

\[
h^2 \left( t, \mathcal{V}_1(2(D + k)) \cap \mathcal{F}^1_{k+2} \right) \leq (D/2)^{-4} M_{k,2,1}(t).
\]

Now arguing as we did in the previous section we derive from Corollary 3 and Proposition 5 our concluding result.

**Theorem 4.** There exists a \( \rho \)-estimator \( \widehat{s} \) based on the model \( \mathcal{F}^1_{k+2} \) and such that, whatever \( P_s \),

\[
C \mathbb{E}_s \left[ h^2(s, \overline{s}) \right] \leq \inf_{\overline{M}>0} \left[ h^2 \left( s, \mathcal{F}^1_{k+2}(\overline{M}) \right) + \left( (\overline{M} n^{-4})^{1/5} (\log n)^{12/5} \right) \vee \left( k n^{-1} (\log n)^3 \right) \right].
\]

If \( s \in \mathcal{F}^1_{k+2}(\overline{M}) \), then

\[
\mathbb{E}_s \left[ h^2(s, \overline{s}) \right] \leq C \left[ \left( (\overline{M} n^{-4})^{1/5} (\log n)^{12/5} \right) \vee \left( k n^{-1} (\log n)^3 \right) \right].
\]
5.4. Log-concave densities. We now want to investigate a situation which is close to the previous one, the case of log-concave densities on the line. These are densities of the form $\mathbb{1}_I \exp(g)$ for some interval $I$ of $\mathbb{R}$, possibly of infinite length, and some concave function $g$ on $I$. Let us denote by $\mathcal{F}'$ the set of all such densities and by $\overrightarrow{\mathcal{V}}(D)$ the subset of $\mathcal{F}'$ of those densities for which $g$ is piecewise affine on $I$ with $D$ pieces. For instance, the exponential density belongs to $\overrightarrow{\mathcal{V}}(1)$ while the Laplace density belongs to $\overrightarrow{\mathcal{V}}(2)$. Also note that if $\mathbb{1}_I \exp(g)$ is log-concave, the same holds for its square root $\mathbb{1}_I \exp(g/2)$.

**Proposition 6.** For all $D \in \mathbb{N}^*$, the elements of $\overrightarrow{\mathcal{V}}(D)$ are extremal in $\mathcal{F}'$ with dimension bounded by $6(D + 2) + 2$.

*Proof.* Let us consider $\mathbb{1}_I \exp(g) \in \mathcal{F}'$ and $\mathbb{1}_J \exp(\overline{g}) \in \overrightarrow{\mathcal{V}}(D)$. Then the set on which $\mathbb{1}_I \exp(g) > \lambda \mathbb{1}_J \exp(\overline{g})$, with $\lambda \geq 0$ is the subset of $I$ on which $g > \log \lambda + \log \mathbb{1}_J + \overline{g}$, with the convention that $\log 0 = -\infty$. If $\lambda = 0$ it is the set $I$ itself. Otherwise it is equal to the union of $I \cap J^c$ and $I \cap J \cap \{g - \overline{g} > \log \lambda\}$. Since $g'$ is non-increasing and $\overline{g}'$ is piecewise constant with $D$ pieces, $(g' - \overline{g}')\mathbb{1}_{I \cap J}$ is piecewise monotone on $I \cap J$ with at most $D$ pieces and therefore piecewise monotone on $\mathbb{R}$ with at most $D + 2$ pieces. It follows from Lemma 4 that $(g - \overline{g})\mathbb{1}_{I \cap J} \in \mathcal{G}_{3(D + 2)}$ and by Lemma 2 that the set $I \cap J \cap \{x \mid g(x) - \overline{g}(x) > \log \lambda\}$ is the union of at most $3(D + 2)$ intervals and the set on which $\mathbb{1}_I \exp(g) > \lambda \mathbb{1}_J \exp(\overline{g})$ the union of at most $3(D + 2) + 1$ intervals. Finally Lemma 1 allows to conclude that $\overline{g}$ is extremal with dimension bounded by $6(D + 2) + 2$.  

We can now conclude from Theorem 2.

**Corollary 4.** There exists a $\rho$-estimator on $\mathcal{F}'$ satisfying, whatever the true distribution $P_s$,
\[
C \mathbb{E}_s [h^2(s, \hat{s})] \leq \inf_{D \geq 1} \left[ \inf_{\overrightarrow{\mathcal{V}}(D)} \sup_{s \in \mathcal{V}} h^2(s, \overline{s}) + \frac{D}{n} \log^3 \left( \frac{n}{D} \right) \right],
\]
where $C \in (0, 1]$ is a universal constant.

In particular, if $s \in \overrightarrow{\mathcal{V}}(D)$,
\[
\mathbb{E}_s [h^2(s, \hat{s})] \leq C \frac{D}{n} \log^3 \left( \frac{n}{D} \right),
\]
which means that the elements of $\overrightarrow{\mathcal{V}}(D)$ can be estimated by the $\rho$-estimator at a parametric rate, up to some $(\log n)^3$ factor. This is the case for all uniform densities, for exponential densities and their translates and for the Laplace density among many others.

6. Model selection

All previous results were based on the use of a single model: $\overrightarrow{\mathcal{V}}(D)$ in Section 5.1, $\mathcal{F}_{k+2}$ in Section 5.2 and $\mathcal{F}_k$ in Section 5.3, which implies that our risk bounds depend on $D$ in the first case and on $k$ in the other cases. In order to get the best possible value of either $D$ or $k$ for the unknown $P_s$, we may use a selection procedure. Let us focus here on the case of models $\mathcal{F}_{k+2}$, $k \geq 1$, the other cases being similar.

We may proceed according to two different strategies. The first one consists in splitting the sample $X = (X_1, \ldots, X_n)$ in two parts $X_1$ and $X_2$, using $X_1$ to build a family
\( \{ \mathcal{s}_{k+2}, 1 \leq k < n \} \) of \( \rho \)-estimators based respectively on the models \( \mathcal{F}_{k+2} \) and selecting the “best” estimator with the sample \( X_2 \). If we do this with a suitable estimator selection procedure, we shall actually get in the end the following extension of the bound provided by Corollary 2:

\[
C \mathbb{E}_s \left[ h^2(s, \mathcal{s}) \right] \leq \inf_{k \geq 1} \inf_{D \geq 1} \left[ \inf_{\mathcal{F} \in \mathcal{F}_{k+2} \cap \mathcal{V}(D)} h^2(s, \mathcal{F}) + \frac{k + D}{n} (\log n)^3 \right].
\]

It then leads, in view of our approximation results, to a final risk equivalent (up to a constant factor) to the minimal risk in the family, that is, according to Theorem 3,

\[
C \inf_{k \geq 1, M > 0} \left[ h^2(s, \mathcal{F}_{k+2}(M)) + \left( \frac{Mn^{-2/3}(\log n)^2}{3} \right) \right].
\]

Appropriate estimator selection procedures are described in Birgé (2006, Section 9) or Baraud (2011, Section 6.2).

An alternative method would be to use all models \( \mathcal{F}_{k+2} \) simultaneously and a penalized \( \rho \)-estimator as indicated in Section 7 of Baraud et al. (2014) with

\[
\text{pen}(t) = \gamma \inf \left\{ k(\log n)^3 \text{ such that } \mathcal{F}_{k+2} \ni t \right\} \quad \text{and} \quad \Delta(\mathcal{F}_{k+2}) = k
\]

for some large enough constant \( \gamma \). Using Theorem 12 of Baraud et al. (2014) and (12) below, we can get the same extension (10) of the bound provided by Corollary 2 and therefore (11). We omit the details.

7. Proofs

7.1. Preliminaries. In the sequel, we shall use the following elementary properties.

**Lemma 5.**

1) If \( \mathcal{C} \) is a VC-class of subsets of \( \mathcal{X} \) with dimension not larger than \( d \) and \( A \subset \mathcal{X} \), then the same holds for the class \( \mathcal{C} \cap A \) defined by (5).

2) Let \( \mathcal{G} \) be a class of real-valued functions on a set \( \mathcal{X} \), \( \mathcal{y} \) an extremal point of \( \mathcal{G} \) with degree \( d(\mathcal{y}) \) and \( \phi(x) = x^\alpha \) for some positive \( \alpha \). Let \( \mathcal{F} \) be a class of non-negative functions on \( \mathcal{X} \) such that

i) \( \phi(\mathcal{F}) = \{ \phi(f) \mid f \in \mathcal{F} \} \subset \mathcal{G} \);

ii) there exists \( \mathcal{f} \in \mathcal{F} \) such that \( \phi(\mathcal{f}) = \mathcal{y} \).

Then \( \mathcal{f} \) is extremal in \( \mathcal{F} \) with degree not larger than \( d(\mathcal{y}) \).

3) Let \( \mathcal{C} \) be a class of subsets of \( \mathcal{X} \) and \( A_1, \ldots, A_k \) be a partition of \( \mathcal{X} \). If for all \( j \in \{ 1, \ldots, k \} \), \( \mathcal{C} \cap A_j \) is a VC-class with dimension not larger than \( d_j \) then \( \mathcal{C} \) is a VC-class with dimension not larger than \( d = \sum_{j=1}^k d_j \).

**Proof.** Let \( B \subset \mathcal{X} \) be a set with cardinality \( d + 1 \). Either \( B \subset A \) and \( C \subset A \cap B = C \cap B \) for all \( C \in \mathcal{C} \) so that \( B \) cannot be shattered by \( \mathcal{C} \) or \( B \cap A^c \) is not empty and cannot be of the form \( C \cap A \cap B \), which proves our first statement. The second statement follows from the fact that \( \mathcal{C}(\mathcal{F}, \mathcal{f}) = \mathcal{C}(\phi(\mathcal{F}), \mathcal{y}) \subset \mathcal{C}(\mathcal{G}, \mathcal{y}) \). For the third one, we argue as follows: if \( \mathcal{C} \) could shatter \( d + 1 \) points, there would exist some \( j \in \{ 1, \ldots, k \} \) and \( d_j + 1 \) points of \( A_j \) that could be shattered by \( \mathcal{C} \) and hence by \( \mathcal{C} \cap A_j \). This would be contradictory with the fact that \( \mathcal{C} \cap A_j \) is a VC-class with dimension not larger than \( d_j \). \( \square \)
We recall from Baraud et al. (2014) that the construction of \( \rho \)-estimators actually involves the following strictly increasing function \( \psi \) from \([0, +\infty]\) onto \([-1, 1]\):

\[
\psi(u) = \frac{u - 1}{\sqrt{1 + u^2}} \quad \text{for } u \in [0, +\infty) \quad \text{and} \quad \psi(+\infty) = 1.
\]

7.2. Proof of Theorem 2. For an even integer \( d \geq 2 \), let \( \Lambda(d) = \{ \overline{s} \in \overline{\Lambda}, \; d(\overline{s}) = d/2 \} \).

Since \( \overline{S} \) is assumed to be separable, \( \Lambda(d) \) which is a subset of \( \overline{S} \) is also separable and we may therefore choose a countable and dense subset \( \Lambda(d) \) of \( \overline{\Lambda}(d) \) for all \( d \geq 2 \). Let us now choose a countable and dense subset \( S \) for \( \overline{S} \). Possibly changing \( S \) into \( \bigcup_{d \geq 2} \Lambda(d) \cup S \), we may assume with no loss of generality that \( \Lambda(d) \subset S \) for all \( d \geq 2 \). Finally, we define our estimator as (any) \( \rho \)-estimator \( \hat{s} \) of \( s \) based on \( S \) following the construction described in Section 4.2 of Baraud et al. (2014) as well as the notations of this paper.

For \( y \geq 1 \) and \( \overline{s} \in \overline{\Lambda} \), we set

\[
\mathcal{B}^S(s, \overline{s}, y) = \{ t \in S \mid h^2(s, t) + h^2(s, \overline{s}) \leq y^2 / n \}.
\]

Note that \( \mathcal{B}^S(s, \overline{s}, y) \) may be empty. We start with the following lemma.

Lemma 6. For all \( y \geq 1 \) and \( \overline{s} \in \overline{\Lambda} \)

\[
\mathcal{F}(S, \overline{s}, y) = \left\{ \psi \left( \sqrt{\frac{t}{\overline{s}}} \right), \; t \in \mathcal{B}^S(s, \overline{s}, y) \right\}
\]

is a VC-major class with dimension not larger than \( 2d(\overline{s}) \).

Proof. Since \( \mathcal{B}^S(s, \overline{s}, y) \subset S \), it suffices to show that

\[
\mathcal{C}'(S, \overline{s}) = \left\{ C(t, u) = \left\{ \psi \left( \sqrt{\frac{t}{\overline{s}}} \right) > u \right\}, \; u \in \mathbb{R}, \; t \in S \right\}
\]

is a VC-class with dimension not larger than \( 2d(\overline{s}) \). Let \( A = \{ \overline{s} > 0 \} \). Using Lemma 5, it suffices to prove that \( \mathcal{C}'(S, \overline{s}) \cap A \) and \( \mathcal{C}'(S, \overline{s}) \cap A^c \) are two VC-classes with dimension not larger than \( d(\overline{s}) \). To do so, it is actually enough to prove that

\[
\mathcal{C}'(S, \overline{s}) \cap A \subset \mathcal{C}(\overline{S}, \overline{s}) \cap A \quad \text{and} \quad \mathcal{C}'(S, \overline{s}) \cap A^c \subset \mathcal{C}(\overline{S}, \overline{s}) \cap A^c
\]

since \( \mathcal{C}(\overline{S}, \overline{s}) \) is a VC-class with dimension not larger than \( d(\overline{s}) \).

Let \( t \in S \subset \overline{S} \). The function \( \psi \) taking its values in \([-1, 1]\), \( C(t, u) = \mathcal{C} \) for \( u < -1 \) and \( C(t, u) = \emptyset \) for \( u \geq 1 \). In both cases, \( C(t, u) \in \mathcal{C}(\overline{S}, \overline{s}) \) hence \( C(t, u) \cap A \in \mathcal{C}(\overline{S}, \overline{s}) \cap A \) and \( C(t, u) \cap A^c \in \mathcal{C}(\overline{S}, \overline{s}) \cap A^c \). Let us now take \( u \in [-1, 1] \). The function \( \psi \) being one to one from \([0, +\infty)\) onto \([-1, 1]\), \( \psi \left( \sqrt{t(x)/\overline{s}(x)} \right) > u \) is equivalent to \( \sqrt{t(x)/\overline{s}(x)} > \psi^{-1}(u) \) (with the conventions \( 0/0 = 1 \) and \( a/0 = +\infty \) for \( a > 0 \)). In particular, for all \( x \in A \),

\[
t(x)/\overline{s}(x) > \left[ \psi^{-1}(u) \right]^2 \quad \text{is equivalent to} \quad t(x) > \left[ \psi^{-1}(u) \right]^2 \overline{s}(x)
\]

showing thus that \( C(t, u) \cap A \subset \mathcal{C}(\overline{S}, \overline{s}) \cap A \). Let us now turn to the case where \( x \notin A \) (i.e. \( \overline{s}(x) = 0 \) so that \( \psi \left( \sqrt{t(x)/\overline{s}(x)} \right) \) is either 1 or \(+\infty\)). If \( u \in [0, 1) \), \( \psi^{-1}(u) \geq 1 \) and, with our convention,

\[
t(x)/\overline{s}(x) > \left[ \psi^{-1}(u) \right]^2 \quad \text{is equivalent to} \quad t(x) > 0.
\]

Hence, for \( u \in [0, 1) \)

\[
C(t, u) \cap A^c = \{ t > 0 \} \cap A^c \in \mathcal{C}(\overline{S}, \overline{s}) \cap A^c.
\]
Finally, for \( u < 0, \psi^{-1}(u) < 1 \) and, with our convention, the three following conditions are equivalent:

\[
t(x)/\bar{\pi}(x) > \left[\psi^{-1}(u)\right]^2, \quad t(x) \geq 0 \quad \text{and} \quad x \in A^c = \mathcal{F} \cap A^c.
\]

This shows that \( C(t, u) \cap A^c \in \mathcal{C}(S, \bar{\pi}) \cap A^c \) and concludes the proof.

Let us now finish the proof of Theorem 2. We fix \( y \geq 1, d \geq 2 \) and \( \bar{\pi} \in \Lambda(d) \subset S \) and set \( c_0 = (\sqrt{2} - 1)/(2\sqrt{2}) \). It follows from Baraud (2011, Proposition 3 on page 386 with \( \psi/\sqrt{2} \) in place of \( \psi \)) and the definition of \( \mathcal{B}^S(s, \bar{\pi}, y) \) that, for all \( t \in \mathcal{B}^S(s, \bar{\pi}, y) \),

\[
E_s \left[ \psi^2 \left( \frac{t}{S}(X_1) \right) \right] \leq \left[ 6 \left( h^2(s, t) + h^2(s, \bar{\pi}) \right) \right] \wedge 1 \leq \left( \frac{6y^2}{n} \right) \wedge 1.
\]

Since \( S \) is countable and \( \psi \) is bounded by 1, the family \( \mathcal{F}(S, \bar{\pi}, y) \) is also countable and its elements are bounded by 1. Besides, Lemma 6 ensures that \( \mathcal{F}(S, \bar{\pi}, y) \) is a VC-major class with dimension not larger than \( d \geq 2 \). We may therefore apply Theorem 1 of Baraud (2014) to the family \( \mathcal{F}(S, \bar{\pi}, y) \) with \( \sigma^2 = (6y^2/n) \wedge 1 \). We get

\[
w^S(s, \bar{\pi}, y) = E_s \left[ \sup_{f \in \mathcal{F}(S, \bar{\pi}, y)} \left| \sum_{i=1}^n (f(X_i) - E_s[f(X_i)]) \right| \right]
\leq 2 \sqrt{2n}\Gamma(d) \times \sigma \log \left( \frac{e}{\sigma} \right) + 8\Gamma(d)
\leq 4 \sqrt{3}\Gamma(d) \times y \log \left( e \left( \frac{n}{6y^2} \vee 1 \right) \right) + 8\Gamma(d),
\]

with

\[
\Gamma(d) = \log \left( 2 \sum_{j=0}^{d \wedge n} \left( \frac{n}{j} \right) \right) \leq \log 2 + (d \wedge n) \log \left( \frac{en}{d \wedge n} \right).
\]

In particular, if \( y^2 \geq \Gamma(d)/6 \geq (d \wedge n)/6 \) then \( \Gamma(d) \leq y \sqrt{6\Gamma(d)} \) and

\[
w^S(s, \bar{\pi}, y) \leq 4y \sqrt{3\Gamma(d)} \left[ \log \left( \frac{en}{d \wedge n} \right) + 2\sqrt{2} \right].
\]

Consequently, \( c_0 y^2 < w^S(s, \bar{\pi}, y) \) implies that either \( y < \sqrt{\Gamma(d)/6} \) or

\[
y \leq (c_0 y)^{-1} w^S(s, \bar{\pi}, y) \leq 4c_0^{-1} \sqrt{3\Gamma(d)} \left[ \log \left( \frac{en}{d \wedge n} \right) + 2\sqrt{2} \right] = B.
\]

We recall that the quantity \( D^S(s, \bar{\pi}) \) is defined in Section 4.3 of Baraud et al. (2014) by

\[
D^S(s, \bar{\pi}) = y_0^2 \vee 1 \quad \text{with} \quad y_0 = \sup \{ y \geq 0 \mid w^S(s, \bar{\pi}, y) > c_0 y^2 \} \quad \text{and} \quad c_0 = \frac{\sqrt{2} - 1}{2\sqrt{2}}.
\]

Since in all cases, \( y^2 \leq \max \{ \Gamma(d)/6; B^2 \} = B^2 \), we deduce that all \( P_s \in \mathcal{P} \) and \( \bar{\pi} \in S \),

\[
D^S(s, \bar{\pi}) \leq \kappa \left( d \wedge n \right) \log^2 \left( \frac{en}{d \wedge n} \right) \leq \kappa d \log^2 \left( \frac{n}{d} \right)
\]

for some numerical constant \( \kappa > 0 \). We now use Theorem 1 in Baraud et al. (2014) for which we recall that the notation \( h^2(t, t') \) defined for densities \( t, t' \in \mathbb{L}_\mu \) means \( nh^2(t, t') \).
Since $\bigcup_{d \geq 2} \Lambda(d) \subset S$, we obtain that for all $\xi > 0$, with probability $1 - e^{-\xi}$

$$C' h^2(s, \overline{s}) \leq \inf_{s \in S} \left[ h^2(s, \overline{s}) + \frac{D^2(s, \overline{s})}{n} \right] + \frac{\xi}{n} \leq \inf_{d \geq 2} \left[ \inf_{s \in \Lambda(d)} h^2(s, \overline{s}) + \frac{C}{n} \log^3 \left( \frac{n}{d} \right) \right] + \frac{\xi}{n}.$$

Finally, $\Lambda(d)$ being dense in $\overline{\Lambda(d)}$

$$\inf_{s \in \Lambda(d)} h^2(s, \overline{s}) = \inf_{s \in \overline{\Lambda(d)}} h^2(s, \overline{s}) \text{ for all } d \geq 2$$

and the first term of right-hand side of (13) becomes

$$\inf_{d \geq 2} \left[ \inf_{s \in \overline{\Lambda(d)}} h^2(s, \overline{s}) + \frac{C}{n} \log^3 \left( \frac{n}{d} \right) \right] = \inf_{s \in \overline{\Lambda}} \left[ h^2(s, \overline{s}) + \frac{C}{n} \log^3 \left( \frac{n}{d} \right) \right].$$

Our conclusion follows.

### 7.3. Proof of Theorem 1

Let $D \geq 1$ and $s \in \overline{\mathcal{V}}(D)$. Then there exists a partition $J = \{x_1, \ldots, x_{D+1}\} \in \mathcal{J}(D+2)$ such that $s = \sum_{j=1}^{D} a_j 1_{(x_j, x_{j+1}]}$. Since each $t \in S$ is non-increasing on some interval of $\mathbb{R}$ and vanishes elsewhere, $\{t \in \mathcal{V}\} \cap (x_j, x_{j+1}]$ is either $\emptyset$ or a subinterval of $(x_j, x_{j+1}]$ with left end-point $x_j$ whatever $\lambda \geq 0$ and $0 \leq j \leq D + 1$. Hence $\mathcal{V}(s, \overline{s}) \cap (x_j, x_{j+1}]$ only consists of sub-intervals of $(x_j, x_{j+1}]$ with left end-point $x_j$ and is therefore a VC-class on $(x_j, x_{j+1}]$ with dimension not larger than 1. Using Lemma 5, we obtain that $\mathcal{V}(s, \overline{s})$ is a VC-class on $\mathbb{R}$ with dimension not larger than $D + 2$ hence that $\overline{\mathcal{V}}(D)$ is a subset of the extremal points of $S$ with dimension bounded by $D + 2$. Theorem 1 then follows from Theorem 2.

### 7.4. Proof of Proposition 1

It relies on a series of approximation lemmas that shall also prove useful in the sequel.

**Lemma 7.** Let $f$ be a monotone function with finite variation $V_I(f)$ on some interval $I$ of finite length $l$. Then

$$\int_I \left[ f(x) - \overline{f} \right]^2 \, dx \leq \frac{l^2 [V_I(f)]^2}{4} \quad \text{with} \quad \overline{f} = \frac{1}{l} \int_I f(x) \, dx$$

and the factor $1/4$ is optimal.

**Proof.** Assuming, without loss of generality that $f$ is non-increasing, let us observe that one can replace $f$ by $g$ with $g(x) = f(x - c) - \overline{f}$ where $c$ is the left-hand point of $I$. This amounts to assume that $c = \overline{f} = 0$. Let $f(0+) = a$, $f(1-) = -b$, $a + b = V_I(f) = R$ and $\lambda = l^{-1} \sup\{x \mid f(x) > 0\} \in (0, 1)$. Then

$$\int_0^M f(x) \, dx = -\int_M^l f(x) \, dx = Al \leq l \min\{a\lambda, b(1 - \lambda)\} \leq l \min\{(R - b)\lambda, b(1 - \lambda)\}.$$ 

A maximization with respect to $b$ and $\lambda$ shows that $A \leq R/4$ and it follows that

$$\int_0^l f^2(x) \, dx \leq (a + b)Al = RA \leq \frac{lR^2}{4}.$$ 

The optimality follows by considering the case of $f = (R/2) \left( 1_{(0,l/2]} - 1_{(l/2,l]} \right)$. \qed
Our next lemma involves the norm in $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$ hereafter denoted by $\|\cdot\|$.

**Lemma 8.** Let $f$ be a non-increasing function on $(a, b)$ with finite variation $V_{(a,b)}(f) < R$. For all $D \geq 1$, there exists a partition $I$ of $(a, b)$ into at most $D$ intervals and a function $f_I$ which is piecewise constant on each element of the partition $I$ and non-increasing such that $f(b-) \leq f_I \leq f(a+)$,

$$
\int_a^b f_I(x) \, dx = \int_a^b f(x) \, dx, \quad \left\| (f - f_I) 1_{(a,b)} \right\| \leq \frac{R \sqrt{b-a}}{2D} \quad \text{and} \quad \left\| f_I 1_{(a,b)} \right\| \leq \left\| f 1_{(a,b)} \right\|.
$$

Besides, there exists a partition $I'$ of $(a, b)$ into at most $2D$ intervals of length not larger than $(b-a)/D$ such that for all $I \in I'$, $V_I(f) \leq RD^{-1}$. The same result holds for non-decreasing functions on $(a, b)$.

**Proof.** Clearly, the results remain valid if we replace $f$ by $g$ with $g = f$ almost everywhere (with respect to the Lebesgue measure), $g(a+) = f(a+)$ and $g(b-) = f(b-)$. Since $f$ is non-increasing on $(a, b)$, for all $x \in (a, b)$ $f(x+)$ exists and $f$ admits an at most countable number of discontinuities. We may therefore assume that $f$ is actually defined on $[a, b]$, right-continuous on $(a, b)$ and left-continuous at $b$.

Starting from $x_0 = a$, define recursively for all $j \geq 1$,

$$
x_j = \sup \{ x \in [x_{j-1}, b], f(x_{j-1}) - f(x) \leq RD^{-1} \}.
$$

If $k \geq 1$ and $x_k < b$, $f(x_{k-1}) - f(x) > RD^{-1}$ for all $x > x_k$ hence $f(x_k) - f(x_{k-1}) \geq RD^{-1}$ since $f$ is right-continuous. In particular for such a $k$, we necessarily have

$$
R > f(a) - f(x_k) = \sum_{j=1}^k f(x_{j-1}) - f(x_j) \geq kRD^{-1},
$$

which implies that $k < D$. The process therefore results in a finite number of distinct points $x_0 = a < x_1 < \ldots < x_{K+1} = b$ with $K + 1 \leq D$. It also follows from the definition of the $x_j$ that $f(x_{j-1}) - f(x_j) \leq RD^{-1}$ for $1 \leq j \leq K + 1$. Let us now set

$$
\overline{T}_j = (x_j - x_{j-1})^{-1} \int_{x_{j-1}}^{x_j} f(x) \, dx \quad \text{and} \quad f_I = \sum_{j=1}^{K+1} \overline{T}_j 1_{(x_{j-1}, x_j)}.
$$

Note that $f(b-) \leq f_I \leq f(a+)$, $\int_a^b f_I(x) \, dx = \int_a^b f(x) \, dx$ and that $f_I$ is non-increasing and piecewise constant on a partition of $(a, b)$ into $K$ intervals. Since, for all $j$, $0 \leq f(x_{j-1}) - f(x_j) \leq RD^{-1}$, it follows from Lemma 7 that

$$
\left\| (f - f_I) 1_{(a,b)} \right\|^2 = \sum_{j=1}^{K+1} \int_{x_{j-1}}^{x_j} (f - \overline{T}_j)^2 \, dx \leq \left( \frac{R}{2D} \right)^2 \sum_{j=1}^{K+1} (x_j - x_{j-1}) = \left( \frac{R}{2D} \right)^2 (b-a).
$$

Moreover Jensen's Inequality implies that

$$
\int_{x_{j-1}}^{x_j} f_I^2(x) \, dx = (x_j - x_{j-1}) \left( \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} f(x) \, dx \right)^2 \leq \int_{x_{j-1}}^{x_j} f^2(x) \, dx,
$$

which shows that $\|f_I 1_{(a,b)}\| \leq \|f 1_{(a,b)}\|$ and proves the first part of the lemma.

For the second part, define $I'$ as follows: for each element $I \in I$ with length $\ell(I)$ larger than $(b-a)/D$ divide $I$ into $[D\ell(I)/(b-a)]$ intervals of length not larger than $(b-a)/D$. For
The process results in a new partition \( \mathcal{I}' \) thinner than \( \mathcal{I} \) and its cardinality is not larger than

\[
\sum_{l \in \mathcal{I}} \left[ \frac{D\ell(I)}{b-a} \right] \leq \sum_{l \in \mathcal{I}} \left[ \frac{D\ell(I)}{b-a} + 1 \right] \leq \left[ \frac{D}{b-a} \sum_{l \in \mathcal{I}} \ell(I) \right] + |\mathcal{I}| \leq 2D.
\]

Since by construction \( V_I(f) \leq RD^{-1} \) for all \( I \in \mathcal{I} \), this property is also true for the elements \( I \) of the partition \( \mathcal{I}' \) which is thinner than \( \mathcal{I} \). For non-decreasing functions, change \( f \) to \(-f\). \( \square \)

**Lemma 9.** Let \( t, u \) be two probability densities with respect to \( \mu \) with \( h^2(t, u) = 1 - \cos \alpha, \) \( 0 < \alpha \leq \pi/2 \). Then

\[
\inf_{\lambda > 0} \left\| \sqrt{t} - \lambda \sqrt{u} \right\| = \left\| \sqrt{t} - (\cos \alpha) \sqrt{u} \right\| = \sin \alpha.
\]

Moreover, if \( v = (\cos \alpha) \sqrt{u} \),

\begin{equation}
(15) \quad h(t, u) = \left\| \sqrt{t} - v \right\| \frac{\sqrt{1 - \cos \alpha}}{\sin \alpha} \leq \left\| \sqrt{t} - v \right\| \leq \left\| \sqrt{t} - \lambda \sqrt{u} \right\| \quad \text{for all } \lambda > 0.
\end{equation}

As a consequence, if \( f \) is a non-negative element in \( \mathbb{L}_2(\mu) \) such that \( \|f\| > 0 \) and \( u = (f/\|f\|)^2 \), then \( h(t, u) \leq \|\sqrt{t} - f\| \) for any probability density \( t \) with respect to \( \mu \).

**Proof.** We notice that \( \sqrt{t} \) and \( \sqrt{u} \) are two vectors of norm one in \( \mathbb{L}_2(\mu) \) and their scalar product is \( \cos \alpha \). It implies that \( v \) is the orthogonal projection of \( \sqrt{t} \) on the linear space generated by \( \sqrt{u} \) and (15) follows from elementary geometry. The last result is merely a rewriting of (15) with \( \lambda = \|f\| \). \( \square \)

To complete the proof of Proposition 1, we apply Lemma 8 with \( f = \sqrt{t} \) and \( R = \sqrt{t(a+)} - \sqrt{t(b-)} \). The resulting function \( f_x \) is then nonnegative, non-increasing on \( (a, b) \) and satisfies \( 0 < \|f_x\| \). Setting \( \sqrt{\mu} = f_x^2 / \|f_x\|^2 \), which is an element of \( \mathbb{V}(D) \), we may apply the last part of Lemma 9 with \( f = f_x \) which gives \( h(t, \sqrt{\mu}) \leq \|f - f_x\| \leq R\sqrt{b-a}/(2D) \). The conclusion follows by letting \( R \) converge to \( V_{[a,b]}(\sqrt{t}) \).

7.5. **Proof of Proposition 3.** Let \( t \) be based on \( \mathcal{I} = \{I_0, \ldots, I_{k+1}\}, \) \( R_j > V_{I_j}(\sqrt{t}) \) for \( 1 \leq j \leq k \) and let \( D_1, \ldots, D_k \) be positive integers. On the intervals \( I_0 \) and \( I_{k+1}, t \) is equal to 0 and for all other intervals of \( \mathcal{I} \), one can apply Lemma 8 to find an approximation \( f_j \) of \( \sqrt{I_j} \) which is monotone, piecewise constant with \( D_j \) pieces on \( I_j \) and satisfies, according to (14), \( \| (f_j - \sqrt{I_j}) \mathbb{1}_{I_j} \| \leq R_j \sqrt{l(I_j)}/(2D_j) \). Therefore, if \( f = \sum_{j=1}^k f_j \) and \( u = (f/\|f\|)^2 \), we derive from Lemma 9 that

\[
h^2(t, u) \leq \left\| f - f_x \right\|^2 = \sum_{j=1}^k \left\| (f_j - \sqrt{I_j}) \mathbb{1}_{I_j} \right\|^2 \leq \sum_{j=1}^k \ell(I_j)R_j^2 4D_j^2 = M.
\]

Moreover, we can always assume (modifying it on a negligible set if necessary) that \( u \) belongs to \( \mathbb{V}(D') \) with \( D' = \sum_{j=1}^k D_j \). Given \( D \), a formal minimization with respect to the \( x_j > 0 \) of \( \sum_{j=1}^k \ell(I_j)R_j^2 x_j^{-2} \) under the condition that \( \sum_{j=1}^k x_j \leq D \) leads to

\[
x_j = \lambda (\ell(I_j)R_j^2)^{1/3} \quad \text{with} \quad \sum_{j=1}^k x_j = D,
\]

17
so that $\lambda^{-1} = D^{-1} \sum_{j=1}^{k} \left( \ell(I_j) R_j^2 \right)^{1/3}$. Taking into account the fact that the $D_j$ should belong to $\mathbb{N}^{*}$, we finally set

$$D_j = \left[ D \left[ \sum_{j=1}^{k} (\ell(I_j) R_j^2)^{1/3} \right]^{-1} (\ell(I_j) R_j^2)^{1/3} \right],$$

which implies that $\sum_{j=1}^{k} D_j \leq D + k$ and

$$M \leq \frac{1}{4D^2} \left[ \sum_{j=1}^{k} (\ell(I_j) R_j^2)^{1/3} \right]^2 \left[ \sum_{j=1}^{k} (\ell(I_j) R_j^2)^{1/3} \right] = \frac{1}{4D^2} \left[ \sum_{j=1}^{k} (\ell(I_j) R_j^2)^{1/3} \right]^3.$$

The corresponding function $u$ belongs to $\nabla(D + k)$ so that

$$h^2(t, \nabla(D + k)) \leq \frac{1}{4D^2} \left[ \sum_{j=1}^{k} (\ell(I_j) R_j^2)^{1/3} \right]^3.$$ 

The conclusion follows by letting each $R_j$ converge to $V_{\ell_j} (\sqrt{t})$.

7.6. Proof of Proposition 5.

**Lemma 10.** Let $f$ be an absolutely continuous density on $[a,b]$ with a monotone derivative $f'$ (in the sense of absolute continuity). Let $g$ be affine on $[a,b]$ with $g(a) = f(a)$ and $g(b) = f(b)$. Then

$$\sup_{a \leq x \leq b} |f(x) - g(x)| \leq \frac{b-a}{4} V_{(a,b)}(f')$$

and the factor $1/4$ is optimal.

**Proof.** Changing $f$ to $-f$ and $g$ to $-g$ if necessary, we may assume that $f'$ is non-increasing hence $f$ is concave so that $h(x) = f(x) - g(x) \geq 0$ for $x \in [a,b]$. Since $h$ is continuous on $[a,b]$ with $h(a) = h(b) = 0$, there exists some $c \in (a,b)$ such that

$$\sup_{a \leq x \leq b} h(x) = h(c) = \int_{a}^{c} (f'(u) - \ell) \, du = \int_{c}^{b} (\ell - f'(u)) \, du \quad \text{with} \quad \ell = \frac{f(b) - f(a)}{b-a}.$$

The function $f'$ being non-increasing,

$$h(c) \leq [(f'(a) - \ell)(c-a)] \land [(\ell - f'(b))(b-c)]$$

and in particular,

$$h(c) \leq \left[ (f'(a) - \ell) (c-a) \frac{b-c}{b-a} \right] + \left[ (\ell - f'(b)) (b-c) \frac{c-a}{b-a} \right]
= \frac{(c-a)(b-c)}{b-a} \left[ f'(a) - \ell + \ell - f'(b) \right]
= (b-a) \left[ \frac{c-a}{b-a} \left( 1 - \frac{c-a}{b-a} \right) \right] \left[ f'(a) - f'(b) \right] \leq \frac{b-a}{4} V_{(a,b)}(f').$$

The constant $1/4$ cannot be improved since it is reached for $f(x) = 1 - |x|$ on $[-1,1]$. □
Let $f'$ be the function of Lemma 10 and $R > V_{(a,b)}(f')$. By Lemma 8, one can partition $(a, b)$ into $K \leq 2D$ intervals $J_j$, $1 \leq j \leq K$ of length not larger than $D^{-1}(b - a)$ with $V_{J_j}(f') < R D^{-1}$. Using this partition to approximate $f$ by a piecewise affine function $g_K$ with $K$ pieces and applying Lemma 10, we derive that

$$
\sup_{a \leq x \leq b} \left| f(x) - g_K(x) \right| \leq (1/4) R D^{-1} [(b - a)/D] = (R/4)(b - a)D^{-2},
$$

hence

$$
\int_a^b \left| f(x) - g_K(x) \right|^2 dx \leq (R/4)^2 (b - a)^3 D^{-4}.
$$

If we replace $f$ by $\sqrt{t}$ which satisfies the assumptions of Lemma 10 on each of the $k$ non-extremal intervals of the partition $\mathcal{I}$ that defines $t$, we get an approximation $v$ of $\sqrt{t}$ with $D' = 2 \sum_{j=1}^k D_j$ pieces with

$$
\left\| \sqrt{t} - v \right\|^2 \leq \frac{1}{16} \sum_{j=1}^k R_j^2 \ell(I_j)^3 D_j^{-4} \quad \text{if} \quad R_j > V_{I_j} \left( \left( \sqrt{t} \right)' \right) \quad \text{for} \quad 1 \leq j \leq k.
$$

Renormalizing $v$ as in Lemma 9, we conclude that there exists $u$ which belongs to $\mathcal{V}(D') \cap \mathcal{S}_{k+2}$ and

$$
h^2(t, u) \leq M = \frac{1}{16} \sum_{j=1}^k R_j^2 \ell(I_j)^3 D_j^{-4}.
$$

We now mimic the proof of Proposition 3 to optimize the $D_j$ and get

$$
D_j = \left[ D \left( \sum_{j=1}^k \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right)^{-1/4} \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right],
$$

so that finally $\sum_{j=1}^k D_j \leq D + k$ and

$$
M \leq \frac{1}{16 D^4} \left[ \sum_{j=1}^k \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right]^4 \left[ \sum_{j=1}^k \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right] = \frac{1}{16 D^4} \left[ \sum_{j=1}^k \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right]^5.
$$

The corresponding function $u$ belongs to $\mathcal{V}(2(D + k))$ so that

$$
h^2 \left( t, \mathcal{V}(2(D + k)) \right) \leq \frac{1}{16 D^4} \left[ \sum_{j=1}^k \left( \ell(I_j)^3 R_j^2 \right)^{1/5} \right]^5.
$$

The conclusion follows by letting $R_j$ converge to $V_{I_j} \left( \left( \sqrt{t} \right)' \right)$ for each $j$.

References

Baraud, Y. (2011). Estimator selection with respect to Hellinger-type risks. *Probab. Theory Related Fields*, 151(1-2):353–401.

Baraud, Y. (2014). Bounding the expectation of the supremum of an empirical process over a (weak) vc-major class.

Baraud, Y., Birgé, L., and Sart, M. (2014). A new method for estimation and model selection: $\rho$-estimation. [http://arxiv.org/abs/1403.6057](http://arxiv.org/abs/1403.6057).
Birgé, L. (1987). Estimating a density under order restrictions: nonasymptotic minimax risk. *Ann. Statist.*, 15(3):995–1012.

Birgé, L. (1997). Estimation of unimodal densities without smoothness assumptions. *Ann. Statist.*, 25(3):970–981.

Birgé, L. (2006). Model selection via testing: an alternative to (penalized) maximum likelihood estimators. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(3):273–325.

Birgé, L. and Massart, P. (1998). Minimum contrast estimators on sieves: exponential bounds and rates of convergence. *Bernoulli*, 4(3):329–375.

Grenander, U. (1981). *Abstract inference*. John Wiley & Sons, Inc., New York. Wiley Series in Probability and Mathematical Statistics.

Groeneboom, P. (1985). Estimating a monotone density. In *Proceedings of the Berkeley conference in honor of Jerzy Neyman and Jack Kiefer, Vol. II (Berkeley, Calif., 1983)*, Wadsworth Statist./Probab. Ser., pages 539–555. Wadsworth, Belmont, CA.

Le Cam, L. (1973). Convergence of estimates under dimensionality restrictions. *Ann. Statist.*, 1:38–53.

Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory*. Springer Series in Statistics. Springer-Verlag, New York.

Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France.

E-mail address: baraud@unice.fr

Univ. Paris VI, CNRS, LPMA, UMR 7599, 75005 Paris, France.

E-mail address: lucien.birge@upmc.fr