ON A CERTAIN CLASS OF $\mathcal{K}_{\sigma\delta}$ BANACH SPACES

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ABSTRACT. Using a strengthening of the concept of $\mathcal{K}_{\sigma\delta}$ set, introduced in this paper, we study a certain subclass of the class of $\mathcal{K}_{\sigma\delta}$ Banach spaces; the so called strongly $\mathcal{K}_{\sigma\delta}$ Banach spaces. This class of spaces includes subspaces of strongly weakly compactly generated (SWCG) as well as Polish Banach spaces and it is related to strongly weakly $\mathcal{K}$–analytic (SWKA) Banach spaces as the known classes of $\mathcal{K}_{\sigma\delta}$ and weakly $\mathcal{K}$–analytic (WKA) Banach spaces are related.

INTRODUCTION

In this paper we study a certain subclass of $\mathcal{K}_{\sigma\delta}$ Banach spaces ([T1], [A-A-M 1] and [A-A-M 2]) and simultaneously a subclass of strongly weakly $\mathcal{K}$–analytic (SWKA) Banach spaces ([M-S]). Strongly $\mathcal{K}$–analytic topological spaces (a class of spaces contained in $\mathcal{K}$–analytic spaces) were introduced in [M-S] in order to study subspaces of strongly weakly compactly generated (SWCG) Banach spaces; the latter class was introduced by Schlüchtermann and Wheller in [S-W] (see also [P-M-Z-2] and [L-R]). SWKA Banach spaces are those Banach spaces which are strongly $\mathcal{K}$–analytic in their weak topology and include subspaces of SWCG Banach spaces ([M-S]). We recall that weakly $\mathcal{K}$–analytic (WKA) spaces, defined as those Banach spaces which are $\mathcal{K}$–analytic in their weak topology ([T1], [V]), include $\mathcal{K}_{\sigma\delta}$ Banach spaces (i.e. those Banach spaces $X$ which are $\mathcal{K}_{\sigma\delta}$ sets in $(X^{**}, w^*)$). The interrelation and analogies between the above mentioned “strong” classes of spaces and also the structure of SWKA Banach spaces are discussed and investigated in [M-S] (see also [K-M]).

In Section 1 of the present paper we strengthen the $\mathcal{K}_{\sigma\delta}$ property of a subset of a topological space by introducing the concept of a strongly $\mathcal{K}_{\sigma\delta}$ set (Def. 1.1). Our motivation was to find a characterization of strongly $\mathcal{K}$–analytic topological spaces in the sense of Choquet; recall that a topological space is $\mathcal{K}$–analytic according to Choquet, if it is a continuous image of a $\mathcal{K}_{\sigma\delta}$ subset of some compact space (see [J-R]). So we prove that a

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topological space is strongly $\mathcal{K}$–analytic if and only if, it is an image under a (continuous) compact covering map of a strongly $\mathcal{K}_{\sigma\delta}$ subset of some compact space (Th. 1.13). We also prove that any Polish subspace of any compact space $K$ is a strongly $\mathcal{K}_{\sigma\delta}$ subset of $K$ (Th. 1.11). Then we characterize strongly $\mathcal{K}_{\sigma\delta}$ sets in a topological space $X$ as those sets $A \subseteq X$ such that $\mathcal{K}(A)$ (the set of compact non empty subsets of $A$) is a $\mathcal{K}_{\sigma\delta}$ subset of $\mathcal{K}(X)$, where the latter space is endowed with Vietoris topology (Th. 1.14).

In Section 2 we deal with a natural generalization of SWCG Banach spaces. A SWCG Banach space $X$ is one which is strongly generated by a weakly compact set $K \subseteq X$. We consider a superspace $Z$ of $X$ and we let the strongly generating weakly compact set $K$ be contained in $Z$. In that case we say that $X$ is SWCG with respect to $Z$. It is clear that the new class is closed under (closed) subspaces; on the other hand by [M-S, Th. 3.9] the class of SWCG is not. So the new class strictly contains the class of SWCG. Besides this fact, we show that the new class has similar properties with the old one (cf. [S-W]). So we prove a characterization of the new class using the Mackey topology on the dual space (Th. 2.10) and we show that their members are weakly sequentially complete spaces (Th. 2.20). We also prove by an example that the new class is not closed for countable $\ell_p$–sums, $1 < p < +\infty$, even in the case when each space is SWCG (Th. 2.25). Our example, a separable weakly sequentially complete space not isomorphic to a (subspace of) SWCG, has stronger properties than [S-W, Example 2.6].

In Section 3 we introduce the common subclass of $\mathcal{K}_{\sigma\delta}$ and SWKA Banach spaces, which we mentioned in the first lines of this introduction. A Banach space $X$ will be called a strongly $\mathcal{K}_{\sigma\delta}$ space, if it is a strongly $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$. It should be clear that such a Banach space is $\mathcal{K}_{\sigma\delta}$ and SWKA. Note that every separable Banach space is $\mathcal{K}_{\sigma\delta}$, but not necessarily a strong $\mathcal{K}_{\sigma\delta}$ space; indeed, the space $c_0(\mathbb{N})$ is not even SWKA (M-S). The class of strongly $\mathcal{K}_{\sigma\delta}$ Banach spaces is related to the class of SWKA Banach spaces as the familiar classes of $\mathcal{K}_{\sigma\delta}$ and WKA Banach spaces are related. Every Banach space which is SWCG with respect to a superspace and every Polish Banach space is strongly $\mathcal{K}_{\sigma\delta}$ (Ths. 3.2 and 3.4). Moreover the class of strongly $\mathcal{K}_{\sigma\delta}$ Banach spaces is closed under countable $\ell_p$–sums, $1 \leq p < +\infty$ (Th. 3.7). Finally, we investigate locally (in the weak topology of a Banach space) the property of strong $\mathcal{K}$–analyticity and the property of a set to be strongly $\mathcal{K}_{\sigma\delta}$. We conclude with some open questions.
STRONGLY $K_{\sigma \delta}$ SPACES

Preliminaries and notation

We denote by $\Sigma$ the set $\mathbb{N}^\mathbb{N}$ of infinite sequences of positive integers, endowed with the cartesian topology, which makes $\Sigma$ (usually called the "Baire space") a Polish space (i.e., homeomorphic to a complete separable metric space). $S$ stands for the set $\bigcup_{n=0}^{\infty} \mathbb{N}^n$, $\emptyset = \emptyset$ of finite sequences of positive integers. We give $S$ the partial order of "initial segments" which makes $S$ into a tree: for $s = (s_1, \ldots, s_n)$, $t = (t_1, \ldots, t_m)$ members of $S$ we define $s \leq t$ if $n \leq m$ and $s_i = t_i$ for all $i = 1, 2, \ldots, n$. If $s = (s_1, \ldots, s_k) \in S$, $\sigma = (n_1, n_2, \ldots, n_k, \ldots) \in \Sigma$ and $m \in \mathbb{N}$, then we write: (i) $s < \sigma$ if $s_i = n_i$ for all $i = 1, 2, \ldots, k$ (ii) $\sigma \mid m$ for the finite sequence $(n_1, n_2, \ldots, n_m)$. For $\tau$, $\sigma \in \Sigma$ we set $\sigma \leq \tau$ if $\sigma(n) \leq \tau(n)$ for all $n \in \mathbb{N}$.

In the present paper by the term topological space we mean a Hausdorff and completely regular space. For a topological space $X$, $K(X)$ is the set of compact non empty subsets of $X$. If $Y$ is a topological space, a map $F : Y \mapsto K(X)$ is said to be upper semicontinuous (usco) if for every $y \in Y$ and $V$ open subset of $X$ with $F(y) \subseteq V$ there exists a neighbourhood $W$ of $y$ such that $F(W) = \bigcup\{F(t) : t \in W\} \subseteq V$.

A topological space $X$ is called $K$–analytic if there exists an usco map $F : \Sigma \mapsto K(X)$ such that $F(\Sigma) = X$. Moreover, if for each compact subset $L$ of $X$ there exists $\sigma \in \Sigma$ such that $L \subseteq F(\sigma)$ the space $X$ is called strongly $K$–analytic ([M-S]).

A topological space $X$ is called Čech complete if it is a $G_{\delta}$ subset in some (every) compactification $cX$ of $X$.

Let $X$ be a Banach space, $\varepsilon \geq 0$. A bounded subset $M$ of $X$ is called $\varepsilon$–weakly relatively compact if $M^{\varepsilon} \subseteq X + \varepsilon B_X^{**}$.

We use [L-T] and [F-H-H-M-Z] as basic references for the theory, notation and terminology of Banach spaces and [J-R] for the theory of $K$–analytic and countably determined topological spaces. If $X$ is a (real) Banach space, then $B_X$ and $B_{X^*}$ are the closed unit balls of $X$ and its dual $X^*$ respectively.

1. STRONGLY $K_{\sigma \delta}$ TOPOLOGICAL SPACES

We recall that a subset $A$ of a topological space $X$ is called $K_{\sigma \delta}$ if there exists a countable family $\{K_{n,m} : n, m \in \mathbb{N}\}$ of compact subsets of $X$ such that $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$. In other words $A$ is a $K_{\sigma \delta}$ subset of $X$ if $A$ can be written as a countable intersection of $\sigma$–compact subsets of $X$.

In the next definition we strengthen the concept of the $K_{\sigma \delta}$ subset of a topological space.
Definition 1.1. A subset $A$ of a topological space $X$ is called strongly $\mathcal{K}_{\sigma\delta}$ if there exists a countable family $\{K_{n,m} : n, m \in \mathbb{N}\}$ of compact subsets of $X$ such that:

(i) $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$.

(ii) For each compact subset $K$ of $A$ and for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $K \subseteq K_{n,m}$.

Remark 1.2. (i) It is clear that a strongly $\mathcal{K}_{\sigma\delta}$ subset of a topological space $X$ is a $\mathcal{K}_{\sigma\delta}$ subset of $X$.

(ii) It is easy to check that if $A$ is a $\mathcal{K}_{\sigma\delta}$ (resp. strongly $\mathcal{K}_{\sigma\delta}$) subset of a topological space $X$ and $B$ is a relatively closed subset of $A$, then $B$ is a $\mathcal{K}_{\sigma\delta}$ (resp. strongly $\mathcal{K}_{\sigma\delta}$) subset of $X$.

We now give some simple examples of strongly $\mathcal{K}_{\sigma\delta}$ subsets. We first recall that a topological space $X$ is said to be hemicompact if it can be written as $X = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ are compact subsets of $X$ such that each compact set in $X$ is contained in some $K_n$. It is obvious that every hemicompact subspace $A$ of a topological space $X$ is a strongly $\mathcal{K}_{\sigma\delta}$ subset of $X$.

Example 1.3. (a) Every $\sigma$–compact and locally compact space is hemicompact ([E, p. 250]). A considerable class of such spaces is the class of locally compact, metrizable and separable topological spaces. In particular, an open and $F_\sigma$ subset of a compact space is in the relative topology $\sigma$–compact and locally compact space. (b) Every countable Hausdorff space whose compact sets are finite is hemicompact. (c) Every dual Banach space $X^*$, endowed with the $w^*$ topology is hemicompact.

Proposition 1.4. Every strongly $\mathcal{K}_{\sigma\delta}$ subset $A$ of a topological space $X$ is in the relative topology strongly $\mathcal{K}$–analytic.

Proof. Let $(K_{n,m})$ be a double sequence of compact subsets of $X$, which satisfies conditions (i), (ii) of Def. Let $(K_{n,m})_m$ is increasing. For each $s = (m_1, m_2, \ldots, m_n) \in S$, put $B_s = \bigcap_{i=1}^{n} K_{i,m_i}$. Let $K$ be a compact subset of $A$. Then according to condition (ii) of Def for each $i \in \mathbb{N}$ there exists $m_i \in \mathbb{N}$ such that $K \subseteq K_{i,m_i}$. If we set $\sigma = (m_1, m_2, \ldots, m_i, \ldots)$, then we have

$$K \subseteq \bigcap_{i=1}^{\infty} K_{i,m_i}, \text{ hence } K \subseteq \bigcap_{n=1}^{\infty} B_{\sigma^n} \subseteq A.$$
It is not difficult to see that the map
\[ F: \Sigma \to K(A) \quad \text{with} \quad F(\sigma) = \bigcap_{n=1}^{\infty} B_{\sigma|n} \]
is usco with \( F(\Sigma) = A \), hence the space \( A \) is strongly \( K \)-analytic. \( \square \)

**Corollary 1.5.** Let \( A \) be a metrizable and strongly \( K_{\sigma} \) subset of a topological space \( X \). Then \( A \) is a Polish space.

*Proof.* It is immediate from Prop. 1.4 that \( A \) is a strongly \( K \)-analytic and metrizable. Then the result follows from [M-S, Prop. 1.11.1]. \( \square \)

We proceed to prove a characterization of strongly \( K \)-analytic spaces analogous to the Choquet characterization of \( K \)-analytic spaces. We need some results that seems to have independent interest.

**Proposition 1.6.** Let \((A_i)\) be a sequence of (strongly) \( K_{\sigma} \) subsets of a topological space \( X \). Then \( A = \bigcap_{i=1}^{\infty} A_i \) is a (strongly) \( K_{\sigma} \) subset of \( X \).

*Proof.* Given \( i \in \mathbb{N} \) there exists a double sequence \((K_{n,m}^{i})\) of compact subsets of \( X \) with \( A_i = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}^{i} \) (such that for each pair \( i, n \in \mathbb{N} \) and for each compact subset \( L \) of \( A_i \) there exists \( m \in \mathbb{N} \) such that \( L \subseteq K_{n,m}^{i} \)). Then \( \bigcap_{i=1}^{\infty} A_i = \bigcap_{i,n} \bigcup_{m=1}^{\infty} K_{n,m}^{i} \) and the conclusion follows. \( \square \)

**Proposition 1.7.** Let \( X \) be a topological space, \( A, Y \) subsets of \( X \) such that \( A \subseteq Y \) and \( A \) is a (strongly) \( K_{\sigma} \) subset of \( X \). If \( Y \) is a (hemicompact subset of \( X \)) \( \sigma \)-compact subset of \( X \), then \( A \) is a (strongly) \( K_{\sigma} \) subset of \( Y \).

*Proof.* It is enough to notice the following. Let \( K_{n,m} \) and \( \Omega_l \) be compact subsets of \( X \) such that \( A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m} \) and \( Y = \bigcup_{l=1}^{\infty} \Omega_l \). Then \( A \cap Y = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m} \cap \Omega_l \) and the conclusion follows. \( \square \)

In the sequel we prove that the product of two (strongly) \( K_{\sigma} \) subsets is also (strongly) \( K_{\sigma} \), hence inductively we have that the class of (strongly) \( K_{\sigma} \) subsets is closed under finite products. Moreover, if we consider a sequence \((A_i)\) of (strongly) \( K_{\sigma} \) subsets of compact spaces \((X_i)\) respectively, then the product \( \prod_{i=1}^{\infty} A_i \) is a (strongly) \( K_{\sigma} \) subset of the space \( \prod_{i=1}^{\infty} X_i \).

**Proposition 1.8.** Let \( A_1, A_2 \) be (strongly) \( K_{\sigma} \) subsets of the topological spaces \( X_1 \) and \( X_2 \) respectively. Then the set \( A_1 \times A_2 \) is a (strongly) \( K_{\sigma} \) subset of the topological space \( X_1 \times X_2 \).

*Proof.* It is enough to prove only the case when \( A_1, A_2 \) are \( K_{\sigma} \) sets. For each \( i = 1, 2 \) there exists a double sequence \((K_{n,m}^{i})_{n,m} \) of compact subsets of
\(X_i\) which satisfies the requirements for a \(K_{\sigma \delta}\) set. For each \(n \in \mathbb{N}\) we consider the countable family \(\{K_{i,m_1}^n \times K_{n,m_2}^n : m_1, m_2 \in \mathbb{N}\}\) of compact subsets of \(X_1 \times X_2\). Then \(A_1 \times A_2 = \bigcap_{n=1}^{\infty} \bigcup \{K_{i,m_1}^n \times K_{n,m_2}^n : m_1, m_2 \in \mathbb{N}\}\) and the conclusion follows immediately. \(\square\)

**Proposition 1.9.** Let \(A_i\) be a (strongly) \(K_{\sigma \delta}\) subset of a compact space \(X_i\) for each \(i \in \mathbb{N}\). Then the set \(\prod_{i=1}^{\infty} A_i\) is a (strongly) \(K_{\sigma \delta}\) subset of the space \(\prod_{i=1}^{\infty} X_i\).

**Proof.** Let us prove this time the strongly \(K_{\sigma \delta}\) case. For each \(n \in \mathbb{N}\) by Prop. [1.8] the set \(\prod_{i=1}^{n} A_i\) is a strongly \(K_{\sigma \delta}\) subset of the space \(\prod_{i=1}^{n} X_i\). The space \(\prod_{i=n+1}^{\infty} X_i\) is compact, so the set \(\prod_{i=1}^{n} A_i \times \prod_{i=n+1}^{\infty} X_i\) is a strongly \(K_{\sigma \delta}\) subset of the topological space \(\prod_{i=1}^{n} X_i \times \prod_{i=n+1}^{\infty} X_i\), that is, of the space \(\prod_{i=1}^{\infty} X_i\). By Proposition [1.6] a countable intersection of strongly \(K_{\sigma \delta}\) subsets of a topological space is a strongly \(K_{\sigma \delta}\) subset, so the set

\[
\bigcap_{n=1}^{\infty} \left( \prod_{i=1}^{n} A_i \times \prod_{i=n+1}^{\infty} X_i \right)
\]

is a strongly \(K_{\sigma \delta}\) subset of \(\prod_{i=1}^{\infty} X_i\). Furthermore

\[
\bigcap_{n=1}^{\infty} \left( \prod_{i=1}^{n} A_i \times \prod_{i=n+1}^{\infty} X_i \right) = \prod_{i=1}^{\infty} A_i,
\]

and the conclusion follows. \(\square\)

**Remark 1.10.** The above result fails to be true if compactness of \(X_i\) is omitted from the hypotheses. Indeed, let \(A_i = \mathbb{N}\) and \(X_i = \mathbb{R}\) for each \(i \in \mathbb{N}\). Then \(\prod_{i=1}^{\infty} A_i = \mathbb{N}^{\mathbb{N}} = \Sigma \subseteq \prod_{i=1}^{\infty} X_i = \mathbb{R}^{\mathbb{N}}\). But the Baire space \(\Sigma\) is not a \(K_{\sigma \delta}\) subset of \(\mathbb{R}^{\mathbb{N}}\). If it was, then it would be \(\sigma\)-compact, which contradicts Baire’s theorem. We note that a \(K_{\sigma \delta}\) subset of a topological space \(X\) is contained in some \(\sigma\)-compact subset of \(X\) and the space \(\mathbb{N}^{\mathbb{N}}\) is a closed subspace of \(\mathbb{R}^{\mathbb{N}}\).

**Theorem 1.11.** Let \(M\) be a Polish subspace of a compact space \(K\). Then \(M\) is a strongly \(K_{\sigma \delta}\) subset of \(K\).

**Proof.** It is clear that we can assume that \(M\) is dense in \(K\). Consider a complete metric \(d\), that induces the topology of \(M\). For each \(n \in \mathbb{N}\) and for each \(x \in M\) choose an open subset \(V_n(x)\) of \(K\) such that \(x \in V_n(x)\) and \(\text{diam}(V_n(x) \cap M) \leq \frac{1}{n}\). Define \(V_n = \bigcup_{x \in M} V_n(x)\). It is rather easy to see from completeness of \((M, d)\) that \(M = \bigcap_{n=1}^{\infty} V_n\) (see [ Chr, Lemma 2, p. 40]). For each \(x \in M\) consider an open subset \(W_n(x)\) of \(K\) such that

\[
x \in W_n(x) \subseteq \overline{W_n(x)} \subseteq V_n(x).
\]
The space $M$ is a separable metric space, hence Lindelöf. Consequently the open covering $\{ V_n(x) : x \in M \}$ of $M$ has a countable subcovering. Then for each $n \in \mathbb{N}$ choose a sequence $(x_{n,m})_m$ of $M$ such that $M \subseteq \bigcup_{m=1}^{\infty} W_n(x_{n,m})$. Then $M \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} W_n(x_{n,m}) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} V_n(x_{n,m}) \subseteq M$, hence $M = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} W_n(x_{n,m})$. For each $n, m \in \mathbb{N}$ put $G_{n,m} = \bigcup_{k=1}^{m} W_n(x_{n,k})$ and then $M = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} G_{n,m}$. Let $L$ be a compact subset of $M$ and $n \in \mathbb{N}$. Then the open covering $(W_n(x_{n,m}))_{m=1}^{\infty}$ of $L$ has a finite subcovering, hence there exists $m \in \mathbb{N}$ such that $L \subseteq \bigcup_{k=1}^{m} W_n(x_{n,k})$, so $L \subseteq G_{n,m}$ and the proof is complete.

**Remark 1.12.** A similar argument proves that every Čech complete and Lindelöf space $X$ is strongly $K_{\sigma\delta}$ in every compact superspace of $X$ (cf. the proof of \cite[Th. 3.9.2]{E}).

We recall that a continuous map $f: X \to Y$ is called compact covering if for every compact subset $L$ of $Y$ there exists a compact subset $K$ of $X$ such that $f(K) = L$. It is clear that a compact covering map is surjective.

**Theorem 1.13.** Let $X$ be a topological space. The following are equivalent:

(i) $X$ is strongly $K$–analytic.

(ii) There exists a compact topological space $\Omega$, a strongly $K_{\sigma\delta}$ subset $C$ of $\Omega$ and a compact covering map $f: C \to X$.

**Proof.** (ii) $\Rightarrow$ (i) It is an immediate consequence of Prop. \cite[Prop. 3.4]{K-M}

(i) $\Rightarrow$ (ii) From \cite[Prop. 3.2]{K-M} $X$ is the image through a compact covering map $f$ of a closed subset $C$ of a space $M \times K$, where $M$ is a Polish space and $K$ is a compact space. Let $K_0$ be a compactification of $M$. From Th. \cite{1.11} $M$ is a strongly $K_{\sigma\delta}$ subset of $K_0$, consequently by Prop. \cite{1.8} the space $M \times K$ is a strongly $K_{\sigma\delta}$ subset of the compact space $\Omega = K_0 \times K$. It follows that $C$ is a strongly $K_{\sigma\delta}$ subset of $\Omega$, as a closed subset of $M \times K$.

(cf. Remarks \cite{1.2} and \cite{1.12})

The following result is analogous to \cite[Th. 1.12]{M-S} and \cite[Th. 3.1]{K-M}.

We recall that if $X$ is any topological space then the Vietoris topology on $\mathcal{K}(X)$ has a basis consisting of sets of the form

$$
\beta(V_1, \ldots, V_n) = \left\{ K \in \mathcal{K}(X) : K \subseteq \bigcup_{i=1}^{n} V_i \text{ and } K \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, n \right\},
$$

where $n \in \mathbb{N}$ and $V_1, \ldots, V_n$ are open non empty subsets of $X$ (cf. \cite[p. 162]{E}).
Theorem 1.14. Let $X$ be a topological space and $A \subseteq X$. The following are equivalent:

(i) $A$ is a strongly $K_{\sigma \delta}$ subset of $X$.
(ii) $K(A)$ is a strongly $K_{\sigma \delta}$ subset of $\mathcal{K}(X)$.
(iii) $K(A)$ is a $K_{\sigma \delta}$ subset of $\mathcal{K}(X)$.

Proof. (i)$\Rightarrow$(ii) As $A$ is a strongly $K_{\sigma \delta}$ subset of $X$, there exists a double sequence $K_{n,m}$, $n$, $m \in \mathbb{N}$ of compact subsets of $X$ such that $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$ and for each compact subset $L$ of $A$ and for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, such that $L \subseteq K_{n,m}$. We shall prove that the double sequence $K(K_{n,m})$, $n$, $m \in \mathbb{N}$ of compact subsets of $\mathcal{K}(X)$, makes $K(A)$ into a strongly $K_{\sigma \delta}$ subset of $\mathcal{K}(X)$.

Step 1. Claim: $K(A) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K(K_{n,m})$.
Let $K \in K(A)$. As $A$ is a strongly $K_{\sigma \delta}$ subset of $X$, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $K \subseteq K_{n,m}$. In other words $K \in K(K_{n,m})$, hence $K \in \bigcup_{n=1}^{\infty} K(K_{n,m})$. Then $K(A) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K(K_{n,m})$. Assume $K \in \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K(K_{n,m})$. For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, such that $K \in K(K_{n,m})$, that is $K \subseteq K_{n,m}$, hence $K \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$, that is, $K \subseteq A$. Consequently $\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K(K_{n,m}) \subseteq K(A)$ and the proof of the claim is complete.

Step 2. Claim: For each $n \in \mathbb{N}$ and for each compact subset $L$ of $K(A)$ there exists $m \in \mathbb{N}$ such that $L \subseteq K(K_{n,m})$.
Let $n \in \mathbb{N}$ and $L$ a compact subset of $K(A)$. Consider the set $X_L = \bigcup L = \bigcup\{ K \mid L \subseteq K \}$. Let $\{K_{n,m} : n, m \in \mathbb{N}\}$ be a double sequence of compact sets in $K(A)$ such that $K(A) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Omega_{n,m}$, then setting $Y_s = \bigcap_{i=1}^{n} \Omega_{i,m}$ for $s = (m_1, m_2, \ldots, m_n)$ we get that $K(A) = \bigcup_{\sigma} \bigcap_{n} Y_{\sigma|n}$ and the map $\sigma \rightarrow \bigcap_{n=1}^{\infty} Y_{\sigma|n}$ is usco. For each $n$, $m \in \mathbb{N}$ the set $K_{n,m} = \bigcup\{ K : K \in \Omega_{n,m} \}$ is compact in $X$. Now let $B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$; it is enough to prove that $A = B$. It is obvious that $A \subseteq B$; let $x \in B$. Then there exist sequences $\sigma = (m_n) \in \Sigma$ and $(K_n) \subseteq K(X)$ such that $x \in K_n$ and $K_n \in \Omega_{n,m}$ for all $n \in \mathbb{N}$. Then the sequence $(K_n)$ has a limit point, say $K \in \bigcap_{n=1}^{\infty} \Omega_{n,m_n} = \bigcap Y_{\sigma|n}$ (see [A-A-M 1, Lemma 3.1]). Therefore $K$ is a compact subset of $A$ and $x \in K \subseteq A$. \qed
Remark 1.15. A $\mathcal{K}_{\sigma\delta}$ subset of a compact space is not necessarily a strongly $\mathcal{K}_{\sigma\delta}$ set. Indeed, let $X = [0, 1]$ and $A = \mathbb{Q} \cap X$; then $A$ is a (countable and hence) $\sigma$–compact set, but it is not a strongly $\mathcal{K}$–analytic space (see Prop. 1.7 and [M-S, Prop. 1.15]). Note that there exist countable topological spaces, which are not even strongly countably determined (see [K-M, Remark 4.4] and [M-S, Example 2.9]).

Question 1.16. Let $K$ be a compact space and $X \subseteq K$.

(i) Assume that $X$ is a strongly $\mathcal{K}$–analytic subspace of $K$. Is then $X$ a strongly $\mathcal{K}_{\sigma\delta}$ or at least a $\mathcal{K}_{\sigma\delta}$ subset of $K$?

(ii) Assume that $X$ is a strongly $\mathcal{K}_{\sigma\delta}$ subset of $K$. Is then $X$ strongly $\mathcal{K}_{\sigma\delta}$ in every compact space, which contains $X$ homeomorphically?

Remark 1.17. The analogous questions in the $\mathcal{K}_{\sigma\delta}$ case have both negative answers. Indeed, concerning the first question, consider a Borel (or an analytic non Borel) non $\mathcal{K}_{\sigma\delta}$ subset of $[0, 1]$. For the second, note that Talagrand ([T2]) was the first who constructed a counterexample; still another counterexample can be found in [A-A-M 1].

2. Banach spaces SWCG with respect to a superspace

We recall that a Banach space $X$ is called strongly weakly compactly generated (SWCG) if there exists a weakly compact subset $K$ of $X$ that strongly generates $X$, that is, for every weakly compact subset $L$ of $X$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $L \subseteq nK + \varepsilon B_X$. The class of SWCG Banach spaces is introduced and studied by Schlüchtermann and Wheeler in [S-W]. In the next definition we generalize the concept of SWCG Banach space, considering the weakly compact subset $K$, that strongly generates $X$ in a superspace $Z$ of $X$. Thus, we are led to the following definition.

Definition 2.1. Let $X, Z$ be Banach spaces with $X \subseteq Z$. We shall say that $X$ is strongly weakly compactly generated (SWCG) with respect to (or relatively to) $Z$ if there exists a weakly compact (convex and symmetric) subset $K$ of $Z$, such that for each $\varepsilon > 0$ and for each weakly compact subset $L$ of $X$ there exists $n \in \mathbb{N}$ with $L \subseteq nK + \varepsilon B_Z$.

Remark 2.2. (i) The previous definition is equivalent to the following: A Banach space $X$ is SWCG with respect to a superspace $Z$ if there exists a sequence $(K_n)$ of weakly compact, convex symmetric subsets of $Z$ such that for every weakly compact subset $L$ of $X$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $L \subseteq K_n + \varepsilon B_Z$ (cf. [S-W, Th. 2.1]).
(ii) Let \( Z \) be a SWCG Banach space, according to the definition given in [S-W], and let \( K \) be a weakly compact set, that strongly generates \( Z \). It is clear that if \( X \) is a closed linear subspace of \( Z \), then \( K \) strongly generates \( X \) according to Defin. \[2.1\]. Moreover as there exists a closed linear subspace \( X = L_1[0,1] \), that is not SWCG ([M-S, Cor. 3.10], [R1]), Defin. \[2.1\] makes sense. If \( X = Z \), then Defin. \[2.1\] gives us the concept of the SWCG Banach space according to Schlüchterman and Wheeler ([S-W]).

(iii) Let \( X, Z \) be Banach spaces such that \( X \subseteq Z \) and \( X \) is SWCG with respect to \( Z \). If \( Y \) is a closed linear subspace of \( X \), then it is SWCG with respect to \( Z \). Thus the class of spaces introduced by Defin. \[2.1\] is closed under subspaces.

**Proposition 2.3.** If a Banach space \( X \) is SWCG with respect to a Banach space \( Z \), then there exists a WCG Banach space \( Y \), with \( X \subseteq Y \subseteq Z \), such that \( X \) is SWCG with respect to \( Y \).

**Proof.** Consider a weakly compact subset \( K \) of \( Z \), that strongly generates \( X \). Put \( Y = \langle K \rangle \) (the closed linear hull of \( K \) in \( Z \)). It is evident that \( Y \) is a proper space. \(\square\)

**Remark 2.4.** (i) Let \( X \subseteq Z_1 \subseteq Z_2 \) be Banach spaces. If \( X \) is SWCG with respect to \( Z_1 \), then it is SWCG with respect to \( Z_2 \).

(ii) Let \( X \subseteq Z \) be Banach spaces, such that \( X \) is SWCG with respect to its superspace \( Z \). By Prop. \[2.3\] \( Z \) may be assumed WCG, consequently \( X \) as a subspace of \( Z \) is WKA. Later we shall prove that \( X \) satisfies stronger conditions. More precisely, we shall prove that \( X \) is SWKA (Cor. \[2.19\]) and moreover is a strongly \( K_{\sigma \delta} \) subset of \((X^{**},w^*)\).

In the sequel we prove that the above property remains invariant under isomorphisms. In fact, we prove something more general. This property is preserved by weakly–compact covering linear operators.

**Proposition 2.5.** Let \( X, Y, Z \) be Banach spaces such that \( X \) is SWCG with respect to \( Z \). If there exists a weakly–compact covering linear operator \( T: X \to Y \), then there exists a Banach space \( Z_1 \), such that \( Y \) is SWCG with respect to \( Z_1 \).

**Proof.** The Banach space \( Y \) is contained isometrically in the Banach space \( \ell_\infty(\Gamma) \) for some set \( \Gamma \). The space \( \ell_\infty(\Gamma) \) is injective, so there exists a bounded linear operator \( \overline{T}: Z \to \ell_\infty(\Gamma) \), that extends \( T \). Put \( Z_1 = \ell_\infty(\Gamma) \) and \( \Omega = \overline{T}(K) \), where \( K \) is any weakly compact subset of \( Z \) that strongly generates \( X \). We are going to prove that \( \Omega \) strongly generates \( Y \). Consider
a weakly compact subset $L$ of $Y$ and $\varepsilon > 0$. As the operator $T$ is weakly compact covering, there exists a weakly compact subset $L_1$ of $X$, such that $T(L_1) = L$. The set $K$ strongly generates $X$, hence there exists $n \in \mathbb{N}$ such that $L_1 \subseteq nK + \frac{\varepsilon}{\|T\|}B_Z$, hence $\tilde{T}(L_1) \subseteq n\tilde{T}(K) + \frac{\varepsilon}{\|T\|}\tilde{T}(B_Z)$. Then $L \subseteq n\Omega + \frac{\varepsilon}{\|T\|}\tilde{T}B_{Z_1}$, that is, $L \subseteq n\Omega + \varepsilon B_{Z_1}$. Thus the weakly compact set $\Omega$ strongly generates $Y$, so $Y$ is SWCG with respect to $Z_1$. □

**Proposition 2.6.** Let $X$, $Z$ be Banach spaces and $X$ is SWCG with respect to $Z$. If $Y$ is isomorphic to $X$, then there exists a Banach space $Z_1$, such that $Y$ is SWCG with respect to $Z_1$.

**Proof.** The proof is immediate by Prop. 2.5 □

The next result is the analogue of [S-W, Th. 2.7] (and also of [K-M, Prop.4.2]).

**Proposition 2.7.** Let $X$, $Z$ be Banach spaces and $X$ is SWCG with respect to $Z$. If $Y$ is a reflexive subspace of $X$, then the space $X/Y$ is SWCG with respect to a Banach space $Z_1$.

**Proof.** The natural map $\pi: X \to X/Y$ with $\pi(x) = x + Y$ is weakly–compact covering (cf. [K-M, Prop. 4.2]). So our claim is immediate from Prop. 2.5 □

**Proposition 2.8.** Let $X$, $Z$ be Banach spaces and $X$ is SWCG with respect to $Z$. If $X$ separable, then there exists a separable Banach space $Y$, such that $X$ is SWCG with respect to $Y$.

**Proof.** As we proved above $X$ is SWCG with respect to a WCG Banach space, so we can assume that $Z$ is WCG. As $X$ is a separable subspace of a (non separable) WCG Banach space there exists a projection

$$P: Z \to Z \quad \text{with} \quad \|P\| = 1, \quad P(Z) \quad \text{separable and} \quad X \subseteq P(Z).$$

(cf. [L, Th. 3.1]). We are going to prove that $X$ is SWCG with respect to the separable Banach space $Y = P(Z)$. Consider a weakly compact subset $K$ of $Z$, that strongly generates $X$ and put $\Omega = P(K)$.

Let $L$ be a weakly compact subset of $X$ and $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $L \subseteq nK + \varepsilon B_Z$, hence $P(L) \subseteq nP(K) + \varepsilon P(B_Z)$. As $P$ is a projection with $\|P\| = 1$ and $L \subseteq X \subseteq P(Z)$, by the last relation follows $L \subseteq n\Omega + \varepsilon B_Y$. Therefore $X$ is SWCG with respect to the separable Banach space $Y$. □
Remark 2.9. The separable space $Y$ of Prop. 2.8 can be chosen to be the space $C(B_{X^*})$. Indeed, $Y$ is embedded into $C(B_{Y^*})$, which in its turn is isomorphic (by Milyutin’s theorem) to the space $C(B_{X^*})$. We do not know whether this result holds true in the case where $X$ is not separable.

In the sequel we state a characterization of the class introduced by Defin. 2.1, which is analogous to the characterization of the class of SWCG Banach spaces ([S-W, Th. 2.1]. Since its proof is analogous to the corresponding proof for SWCG spaces, we give a brief outline of it.

Theorem 2.10. Let $X$ be a Banach space. The following are equivalent:

(i) $X$ is SWCG with respect to a Banach space.

(ii) $X$ is contained in a Banach space $Z$ and exists a metrizable topology $\tau_d$ in $B_{Z^*}$ such that:

(a) The Mackey topology $\tau$ on $B_{Z^*}$ is finer than $\tau_d$.

(b) The topology $\tau_d$ is finer than topology $\tau_X$ on $B_{Z^*}$ of uniform convergence on weakly compact subsets of $X$.

Proof. (i)⇒ (ii) Let $X \subseteq Z$ and $K \subseteq Z$ be a weakly compact convex and symmetric set that strongly generates $X$. A metric $d$ whose topology $\tau_d$ satisfies our requirements is the following

$$d(x^*, y^*) = \max\{|x^*(x) - y^*(x)| : x \in K\} \quad \text{for all} \quad x^*, y^* \in B_{Z^*}.$$  

(ii) ⇒ (i) Let $Z$ be a Banach space with $X \subseteq Z$, that satisfies conditions (a) and (b). Consider a neighbourhood base $(U_n)$ of 0 for $(B_{Z^*}, \tau_d)$. As the Mackey topology $\tau$ is finer than the metric topology $\tau_d$, there exists a sequence $(K_n)$ of weakly compact convex symmetric subsets of $Z$ such that $K_n^0 \cap B_{Z^*} \subseteq U_n$ for all $n \in \mathbb{N}$. Let $L$ be a weakly compact subset of $X$ and $0 < \varepsilon < 1$. Put $c = \frac{1}{\varepsilon}$. As the metric topology $\tau_d$ is finer than $\tau_X$ there exists $n \in \mathbb{N}$ such that $U_n \subseteq (cL)^0 \cap B_{Z^*}$. We continue as in the proof of implication (a)⇒(b) of [S-W, Th. 2.1].

Remark 2.11. The metric $d$, that appears in the proof of direction (i)⇒(ii) of Th. 2.10 is nothing else but the metric of supremum norm of the space $C(K)$. Indeed, the operator $T: Z^* \to C(K)$ defined by $T(z^*) = z^*|K$ is bounded, linear and 1–1 (as $K$ can be assumed total in $Z$). Moreover $T$ is $(w^*, \tau_p)$ continuous (where $\tau_p$ is the topology of pointwise convergence), so $\Omega = T(B_{Z^*})$ is $\tau_p$–compact set homeomorphic with the unit ball $(B_{Z^*}, w^*)$. It is also bounded, so Grothendieck’s theorem implies that it is weakly compact. Therefore the space $\Omega$ endowed with the metric of supremum
norm is a complete metric space, as a closed subset of the Banach space $C(K)$.

**Corollary 2.12.** Let $X \subseteq Z$ be Banach spaces and $X$ is SWCG with respect to $Z$. If $X$ is separable, then the space $(B_{X^*}, \tau)$ is analytic, where $\tau$ is the Mackey topology on $X^*$.

**Proof.** We can assume and we do that $Z$ is separable, so by the previous remark $C(K)$ is a separable Banach space (as $K$ is weakly compact metrizable set). It follows that the closed subspace $\Omega = T(B_{Z^*})$ of $C(K)$ endowed with the metric $d$ of supremum norm is a complete separable metric space. By assertion (b) of (ii) of Th. 2.10 the map

$$\Phi: (B_{Z^*}, \tau_d) \to (B_{X^*}, \tau) \quad \text{with} \quad \Phi(z^*) = z^*|X$$

is continuous and onto, hence the conclusion follows. \qed

**Remark 2.13.** A separable Banach space $X$ such that the space $(B_{X^*}, \tau)$ is analytic is not necessarily SWCG with respect to some Banach space. Indeed, consider a non reflexive Banach space $X$ with separable dual. Then by Cor. 2.22 (or 2.23) $X$ does not satisfy Def. 2.1; on the other hand since the norm topology is finer than the Mackey topology on $B_{X^*}$ we get that $(B_{X^*}, \tau)$ is an analytic space.

As we have noted (Rem. 2.2) a closed linear subspace $X$, of a SWCG Banach space $Z$, is SWCG with respect to $Z$. But we do not know the answer to the next question.

**Question 2.14.** Let $X \subseteq Z$ be Banach spaces and $X$ is SWCG with respect to $Z$. Is then $X$ isomorphic to a subspace of a SWCG Banach space? In other words can we always assume in Def. 2.1 that $Z$ is SWCG? (According to Prop. 2.3 we can always assume that $Z$ is WCG.)

In the sequel, we are going to prove that the class defined by Def. 2.1 is contained in the class of SWKA Banach space. We generalize Def. 2.1 and we shall prove that the (at least formally) larger class has the same property. Firstly, we recall an internal characterization of subspaces of WCG Banach spaces, due to Fabian, Montesinos and Zizler ([F-M-Z-1, Th. 1]). Namely, it is proved that the following significant result holds:

**Theorem** (Fabian, Montesinos, Zizler). A Banach space $X$ is a subspace of a WCG Banach space if and only if for each $p \in \mathbb{N}$ there exists a sequence $(M_{n,p})_n$ consisting of $\frac{1}{p}$–weakly relatively compact subsets of $X$ such that $X = \bigcup_{n=1}^{\infty} M_{n,p}$. 


A natural question which arises from the previous theorem is whether we can give an analogous characterization for subspaces of SWCG Banach spaces. Presently, we do not have an answer to this question, but a necessary condition for a space to be SWCG with respect to a Banach space, according to Def. 2.1 is given by the following proposition. We note that in this proposition we follow the method of proof of [F-M-Z-1, Th. 1].

**Proposition 2.15.** If the Banach space $X$ is SWCG with respect to a Banach space $Z$, then there exists a family $\{M_{n,p} : n, p \in \mathbb{N}\}$ of subsets of $X$ with the following properties:

(i) For every $n \in \mathbb{N}$ and for every $p \in \mathbb{N}$ the set $M_{n,p}$ is $\frac{1}{p}$–weakly relatively compact.

(ii) For every $p \in \mathbb{N}$ and for every weakly compact subset $L$ of $X$ there exists $n \in \mathbb{N}$ such that $L \subseteq M_{n,p}$.

**Proof.** Let $K$ be a weakly compact subset of $Z$, that strongly generates $X$. Then for each $n, p \in \mathbb{N}$ the set $nK + \frac{1}{p}B_Z$ is $\frac{1}{p}$–weakly relatively compact subset of $Z$. Indeed,

\[ nK + \frac{1}{p}B_Z \subseteq nK^* + \frac{1}{p}B_Z = nK + \frac{1}{p}B_Z^{**} \subseteq Z + \frac{1}{p}B_Z^{**}. \]

For every $n, p \in \mathbb{N}$ put $M_{n,p} = X \cap (nK + \frac{1}{4p}B_Z)$. Then, by [H-M-V-Z, Prop. 3.62], the set $M_{n,p}$ is an $\frac{1}{p}$–weakly relatively compact subset of $X$ for every $n, p \in \mathbb{N}$. Additionally, for every weakly compact subset $L$ of $X$ and for each $p \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $L \subseteq nK + \frac{1}{4p}B_Z$, so $L \subseteq M_{n,p}$. Therefore the family $\{M_{n,p} : n, p \in \mathbb{N}\}$ has the desired properties. \(\square\)

So we arrive at the following definition.

**Definition 2.16.** We shall say that a Banach space $X$ has property (*), if there exists a countable family $\{M_{n,p} : n, p \in \mathbb{N}\}$ of (convex and symmetric) subsets of $X$, that satisfies assertions (i) and (ii) of Prop. 2.15.

It is clear, from Prop. 2.15 that every Banach space $X$ that is SWCG with respect to a Banach space $Z$ has property (*). If $A, B$ are $\varepsilon$–weakly relatively compact subsets of a Banach space $X$, then clearly the set $A \cup B$ is $\varepsilon$–weakly relatively compact. Thus in Def. 2.16 we can assume that $M_{n,p} \subseteq M_{n+1,p}$ for each $n, p \in \mathbb{N}$. Moreover we can assume that the sets $M_{n,p}$ are convex and symmetric (see [H-M-V-Z, Th. 3.64]).

**Remark 2.17.** (a) If a Banach space $Z$ has property (*), then every closed linear subspace of $X$ has property (*). Indeed, if a countable family
\{M_{n,p} : n, p \in \mathbb{N}\} of subsets of \(Z\) witnesses that \(Z\) has property (*), then by \[H-M-V-Z\] Prop. 3.62 the sets \(K_{n,p} = X \bigcap M_{n,4p}\), \(n, p \in \mathbb{N}\) ensure that \(X\) has property (*).

(b) It is easy to see that a Banach space \(X\) has property (*) if there exists a countable family \(\{M_{n,p} : n, p \in \mathbb{N}\}\) of (convex and symmetric) subsets of \(B_X\) such that assertions (i) and (ii) of Prop. 2.15 are satisfied for \(B_X\).

**Theorem 2.18.** Let \(X\) be a Banach space with property (*). Then \(X\) is SWKA and subspace of a WCG Banach space.

**Proof.** Let \(\{M_{n,p} : n, p \in \mathbb{N}\}\) be a family of subsets of \(X\) that witnesses property (*) of \(X\).

The fact that the space \(X\) is a subspace of a WCG Banach space follows immediately by [F-M-Z-1, Th. 1]. To show that \(X\) is SWKA consider the map \(F : \Sigma \rightarrow K(X)\) with \(F(\sigma) = \bigcap_{p=1}^{\infty} M_{\sigma(p),p}\). It is not difficult to see that \(F\) is well defined and usco. The proof is similar to the proof of direction (ii) \(\Rightarrow\) (iii) of [K-M] Prop. 3.1, so we omit it. \(\square\)

**Corollary 2.19.** Every Banach space \(X\), that is SWCG with respect to a Banach space \(Z\) is SWKA.

**Proof.** It is immediate from Th. 2.18 and Def. 2.16. \(\square\)

The next result extends a result of Schlüchterman and Wheeler ([S-W, Th. 2.5]) in the wider class of Banach spaces with property (*).

**Theorem 2.20.** Let \(X\) be a Banach space with property (*). Then \(X\) is weakly sequentially complete.

**Proof.** Let \(\{K_{m,p} : m, p \in \mathbb{N}\}\) be a family of closed convex symmetric subsets of \(X\) that witnesses property (*) and \(K_{m,p} \subseteq K_{m+1,p}\) for all \(m, p \in \mathbb{N}\).

Let \((x_n)\) be a weakly Cauchy sequence of \(X\). Then there exists \(x^{**} \in X^{**}\) such that \(x_n \rightharpoonup x^{**}\). Assume that the set \(\{x_n : n \in \mathbb{N}\}\) is not weakly relatively compact. The family \(\{2K_{m,p} : m, p \in \mathbb{N}\}\) consists of bounded, convex symmetric subsets and satisfies the following conditions:

(a) For every \(m, p\) the set \(2K_{m,p}\) is weakly relatively compact.

(b) For every \(m, p \in \mathbb{N}\) holds \(2K_{m,p} \subseteq 2K_{m+1,p}\).

(c) For every weakly compact subset \(L\) of \(X\) and for every \(p \in \mathbb{N}\) there exists \(m \in \mathbb{N}\) such that \(L \subseteq 2K_{m,p}\).

Then there exists \(p \in \mathbb{N}\) such that \(\{x_n : n \in \mathbb{N}\} \not\subseteq 2K_{m,p}\) for all \(m \in \mathbb{N}\). Thus for every \(m \in \mathbb{N}\) there exists \(n_m \in \mathbb{N}\) such that \(x_{n_m} \not\in 2K_{m,p}\). Moreover for each \(m \in \mathbb{N}\) infinitely many terms of the sequence do not belong to
2K_{m,p}$. For every $n \in \mathbb{N}$ define $m_1(n) = \min\{m \in \mathbb{N} : x_n \in 2K_{m,p}\}$ and $m_2(n) = \min\{m \in \mathbb{N} : x_n \in K_{m,p}\}$. Clearly $m_1(n) \leq m_2(n)$ for every $n \in \mathbb{N}$, $x_n \in 2K_{m_1(n),p}$ and $x_n \notin 2K_{\lambda,p}$ if and only if $\lambda < m_1(n)$.

We can choose inductively a subsequence $(x_{k_n})$ of $(x_n)$ such that (1) $m_1(k_{n+1}) > m_2(k_n)$ for all $n \in \mathbb{N}$. Choose $k_1 = 1$. Assume that $k_1 < k_2 < \cdots < k_n$ have been chosen. For $m = m_2(k_n)$ there exists $n_m$ such that $x_{n_m} \notin 2K_{m,p}$ and $n_m > k_n$. Put $k_{n+1} = n_m$ and then we have $m_1(k_{n+1}) > m_{m_2(k_n)}$. The sequence $(x_{k_n})$ is weakly Cauchy, hence there exists $m_0 \in \mathbb{N}$ such that $x_{k_n+1} - x_{k_n} \in K_{m,p}$ for all $n \in \mathbb{N}$. Then it follows $x_{k_n+1} = (x_{k_n+1} - x_{k_n}) + x_{k_n} \in K_{m,p} + K_{m_2(k_n),p}$ for all $n \in \mathbb{N}$. Choose $m_2(k_n) \geq m_0$. Then $K_{m_2(k_n)} + K_{m_2(k_n),p} \subseteq 2K_{m_2(k_n),p}$, so $x_{k_n+1} \in 2K_{m_2(k_n),p}$; consequently $m_1(k_{n+1}) \leq m_2(k_n)$, which contradicts (1). We conclude then that the set $\{x_n : n \in \mathbb{N}\}$ is weakly relatively compact, hence $x^{**} \in X$ and the space $X$ is weakly sequentially complete.

\begin{corollary}
Every Banach space $X$ that is SWCG with respect to a Banach space $Z$ is weakly sequentially complete.
\end{corollary}

\begin{proof}
It is immediate from Th. 2.20 as $X$ has property (*).
\end{proof}

\begin{corollary}
Let $X$ be a Banach space not containing $\ell_1(\mathbb{N})$. Then $X$ has property (*) if and only if it is reflexive.
\end{corollary}

\begin{proof}
If $X$ is reflexive, then its closed unit ball $B_X$ is a weakly compact set, hence $X$ is SWCG and consequently has property (*).

Assume that $X$ has property (*). Then according to Th. 2.20 $X$ is weakly sequentially complete and as $\ell_1(\mathbb{N})$ is not contained in $X$, by Rosenthal’s $\ell_1$-theorem follows that $X$ is reflexive.
\end{proof}

\begin{corollary}
Let $X$ be an Asplund Banach space (in particular $X$ has separable dual). If $X$ has property (*), then it is reflexive.
\end{corollary}

\begin{proof}
Every separable subspace of $X$ has separable dual, so $X$ does not contain $\ell_1(\mathbb{N})$. According to Cor. 2.22 $X$ is reflexive.
\end{proof}

We recall that a Banach space is called Polish if its closed unit ball $(B_X, w)$ is a Polish topological space. We also say that a Banach space $X$ has Čech complete ball if $B_X$ is a $G_\delta$ subset of $(B_X^{**}, w^*)$. These classes of Banach spaces are introduced and studied by Edgar and Wheeler in [E-W] (see also [R2]).

\begin{corollary}
Let $X$ be a non reflexive Banach space with Čech complete ball $(B_X, w)$ (in particular, $X$ is a non reflexive Polish Banach space). Then $X$ does not have property (*).
\end{corollary}
Proof. Every Banach space with Čech complete ball is Asplund (\cite{E-W}). So the result follows from Cor. 2.23.

It is known that an $\ell_p$–direct sum ($1 \leq p < +\infty$) of SWKA Banach spaces is SWKA (cf. \cite{K-M}). We shall show now that property (*) is not preserved by $\ell_p$–direct sums ($p > 1$), even if each space is SWCG.

**Theorem 2.25.** The Banach space

$$X = \left( \sum_{m=1}^{\infty} \oplus \ell_1(N \times \{m\}) \right)_2$$

does not have property (*), although it is SWKA and weakly sequentially complete. In particular it is not SWCG with respect to any Banach space.

Proof. The space $X$ is by \cite{K-M} Prop. 5.1 SWKA, as $\ell_1(N)$ is SWCG. Furthermore

$$X = Y^* \text{ with } Y = \left( \sum_{m=1}^{\infty} \oplus c_0(N \times \{m\}) \right)_2$$

and $X$ has as an unconditional basis, the double sequence

$$(e_{(n,m)} = (0, \ldots, 0, \bar{e}_{(n,m)}, 0, \ldots, 0) \quad n, m \in \mathbb{N},$$

where $\bar{e}_{(n,m)}$, $n \in \mathbb{N}$ is the usual basis of $\ell_1(N) = \ell_1(N \times \{m\})$. (In the sequel we identify $\bar{e}_{(n,m)} \in \ell_1(N)$ with $e_{(n,m)} \in X$.) The space $X$ does not contain $c_0(N)$, as it is SWKA (cf. \cite{M-S} Prop. 1.9, Cor 1.10) (ii), thus it is weakly sequentially complete (\cite{L-T} Th. 1.c.10).

Assume, towards a contradiction that $X$ has property (*). Then there exists a family $\{M_{n,p} : n, p \in \mathbb{N}\}$ consisting of closed, convex and symmetric subsets of $B_X$ such that

(i) For every $n, p \in \mathbb{N}$ the set $M_{n,p}$ is $\frac{1}{p}$–weakly relatively compact.
(ii) For every $p \in \mathbb{N}$ and for every weakly compact subset $L$ of $B_X$ there exists $n \in \mathbb{N}$ with $L \subseteq M_{n,p}$.

For every $\sigma \in \Sigma$ consider the space

$$X_{\sigma} = \left( \sum_{m=1}^{\infty} \oplus \ell_1(\{1, 2, \ldots, \sigma(m)\}) \right)_2,$$

which is reflexive and satisfies $X_{\sigma} \subseteq X$. By $B_{\sigma}$ we denote the closed unit ball of $X_{\sigma}$. Then $\bigcup_{\sigma \in \Sigma} B_{\sigma} \subseteq B_X$ and for every $\sigma \in \Sigma$ the set $B_{\sigma}$ is weakly compact.

Let $p > 2$. For every $n \in \mathbb{N}$ put $A_n = \{\sigma \in \Sigma : B_{\sigma} \subseteq M_{n,p}\}$; then we have $\Sigma = \bigcup_{n=1}^{\infty} A_n$. Using a Baire category argument, due to Talagrand
(cf. [11] Th. 4.3] we find \( n_0 \in \mathbb{N} \) and an infinite subset \( D = \{ \sigma_k : k \in \mathbb{N} \} \) of \( A_{n_0} \) such that for some \( s_0 \in S \) holds \( s_0 < \sigma_k \) and \( \sigma_k(m_0 + 1) = k \) for all \( k \in \mathbb{N} \), where \( m_0 = |s_0| = \) the length of the finite sequence \( s_0 \). As \( D \subseteq A_{n_0} \), it follows that \( B_{\sigma_k} \subseteq M_{n_0,p} \) for every \( k \in \mathbb{N} \). Consequently

\[
\{ e_{(n,m_0+1)} : 1 \leq n \leq k = \sigma_k(m_0 + 1) \} \subseteq B_{\sigma_k} \subseteq M_{n_0,p} \quad \text{for every} \quad k \in \mathbb{N}.
\]

Thus, we conclude that \( M = \{ e_{(n,m_0+1)} : n \in \mathbb{N} \} \subseteq M_{n_0,p} \). But this means that the usual basis of \( \ell_1(\mathbb{N}) \) is \( \frac{1}{p} \)-weakly relatively compact subset of \( X \), so \( \frac{2}{p} \)-weakly relatively compact subset of \( \ell_1(\mathbb{N}) \), which is false as \( \frac{2}{p} < 1 \) and the usual basis of \( \ell_1(\mathbb{N}) \) is not \( \varepsilon \)-weakly relatively compact subset of \( \ell_1(\mathbb{N}) \) for \( 0 < \varepsilon < 1 \).

\[ \square \]

**Remark 2.26.** (a) The previous result provides us with still another example with the properties of \([S-W, \text{Example 2.6}]\); that is a weakly sequentially complete and separable space that is not SWCG. Note that as it has been proved in \([M-S, \text{Example 2.9}]\) the example in \([S-W]\), in contrast with the present example, is not a SWKA space (see also \([K-M, \text{Remark 4}]\)).

(b) The result described in Th. 2.25 is generalized, with the same essentially proof for a direct sum of the form

\[
X = \left( \sum_{m=1}^{\infty} \oplus \ell_1(\mathbb{N} \times \{m\}) \right)_p , \quad \text{where} \quad 1 < p < +\infty.
\]

**Corollary 2.27.** Let \((X_n)\) be a sequence of Banach spaces such that \( \ell_1(\mathbb{N}) \) embeds in \( X_n \) for infinitely many \( n \in \mathbb{N} \). Then the space

\[
X = \left( \sum_{n=1}^{\infty} \oplus X_n \right)_p , \quad \text{where} \quad 1 < p < +\infty,
\]

does not have property (*).

**Proof.** It follows immediately from Th. 2.25 and Remark 2.26 \( \square \)

**Corollary 2.28.** Let \((X_n)\) be a sequence of Banach spaces, such that \( X_n \) has property (*) for every \( n \in \mathbb{N} \). If \( X_n \) is not reflexive for infinitely many \( n \in \mathbb{N} \), then the space

\[
X = \left( \sum_{n=1}^{\infty} \oplus X_n \right)_p , \quad \text{where} \quad 1 < p < +\infty
\]

does not have property (*).

**Proof.** If \( X_n \) is not reflexive, then from Cor 2.22, it contains \( \ell_1(\mathbb{N}) \). By Cor. 2.27 the conclusion follows. \( \square \)
Corollary 2.29. Let \((X_n)\) be a sequence of SWCG Banach spaces, such that \(X_n\) is not reflexive for infinitely many \(n \in \mathbb{N}\). Then the space

\[
X = \left( \sum_{n=1}^{\infty} \oplus X_n \right)_p,
\]

where \(1 < p < +\infty\)

is not isomorphic to any subspace of a SWCG Banach space.

Proof. It is clear by Cor. 2.22 and Cor. 2.28. \(\square\)

In contrast to Th. 2.25 an \(\ell_1\)–direct sum of a sequence \((X_n)\) of SWCG Banach spaces is SWCG ([S-W, Prop. 2.9]). An analogous result can be proved if every \(X_n\) is SWCG with respect to some superspace of it.

Proposition 2.30. Let \((X_n), (Z_n)\) be sequences of Banach spaces, such that \(X_n \subseteq Z_n\), for all \(n \in \mathbb{N}\) and every \(X_n\) is SWCG with respect to \(Z_n\). Then the space \(X = (\sum_{n=1}^{\infty} \oplus X_n)_1\) is SWCG with respect to the space \(Z = (\sum_{n=1}^{\infty} \oplus Z_n)_1\).

Proof. According to Th. 2.10 for every \(n \in \mathbb{N}\) there exists a metrizable topology \(\tau_{d_n}\) in \(B_{Z_n^*}\) such that:

(i) \(\tau_{d_n}\) is coarser than the Mackey topology \(\tau_n\) on \(B_{Z_n^*}\).

(ii) \(\tau_{d_n}\) is finer than the topology \(\tau_{X_n}\) on \(B_{Z_n^*}\) (of uniform convergence on weakly compact subsets of \(X_n\)).

As \(Z\) is an \(\ell_1\)–direct sum of Banach spaces it follows that \(B_{Z^*} = \prod_{n=1}^{\infty} B_{Z_n^*}\). Furthermore it is known by classical results about Mackey topology (cf. [S-W, Prop. 1.2]) that Mackey topology \(\tau\) on \(B_{Z^*}\) is identified with the product topology of the spaces \((B_{Z_n^*}, \tau_n)\). Consider \(B_{Z^*}\) endowed with the product topology of the topologies \(\tau_{d_n}\), which is denoted by \(\tau_d\). It is clear that \(\tau_d\) is metrizable and coarser than \(\tau\) (by (i)). If \(\tau_X\) is the topology on \(B_{Z^*}\) of uniform convergence on weakly compact subsets of \(X\), then we can easily prove that \(\tau_X\) is identified with the product topology of \(\tau_{X_n}, n \in \mathbb{N}\), thus it is coarser than \(\tau_d\) (the proof is analogous to the proof of the fact that the space \((B_{Z^*}, \tau)\) is identified with the topological product of the spaces \((B_{Z_n^*}, \tau_n)\)). It follows from Th. 2.10 that \(X\) is SWCG with respect to \(Z\). \(\square\)

Remark 2.31. It is worth noting that results similar to those described in the above proposition or the analogue of Schlüchtermann and Wheeler, are due to the fact that \(\ell_1\) norm essentially does not introduce new compact sets in an \(\ell_1\)–direct sum of Banach spaces. So (with analogous techniques) we can prove that an \(\ell_1\)–direct sum of Banach spaces with property (*), has property (*) as well.
**Question 2.32.** Let $X$ be a Banach space with property (*). Is then $X$ SWCG with respect to some superspace?

3. **Strongly $K_{σδ}$ Banach spaces.**

In this section we introduce a subclass of SWKA Banach spaces, that is, those Banach spaces which are strongly $K_{σδ}$ subsets of $(X^{**}, w^*)$. As we shall see this subclass contains any other subclass of SWKA Banach spaces considered in this article. We recall that the class of $K_{σδ}$ Banach spaces $X$ (that is, Banach spaces $X$, which are $K_{σδ}$ subset of $(X^{**}, w^*)$) contains the closed subspaces of WCG Banach spaces ([I]) and is contained properly in WKA Banach spaces ([A-A-M II]). Let $X$ be a SWCG Banach space and $K$ a weakly compact (convex and symmetric) subset of $X$, that strongly generates $X$, that is, for every $ε > 0$ and for every weakly compact subset $L$ of $X$ there exists $n ∈ \mathbb{N}$, such that $L ⊆ nK + εB_X$. Then

$$X = \bigcap_{p=1}^{∞} \bigcup_{m=1}^{∞} \left( mK + \frac{1}{p}B_{X^{**}} \right),$$

that is, $X$ is a $K_{σδ}$ subset of $(X^{**}, w^*)$. Moreover for each $p ∈ \mathbb{N}$ and for each weakly compact subset $L$ of $X$ there exists $m ∈ \mathbb{N}$ such that

$$L ⊆ mK + \frac{1}{p}B_{X^{**}},$$

which means that every SWCG Banach space is a strongly $K_{σδ}$ subset of its second dual $(X^{**}, w^*)$.

As shown below the same is true and for those Banach spaces which have property (*) (consequently for those Banach spaces which are SWCG with respect to some superspace). This leads to the following definition.

**Definition 3.1.** A Banach space $X$ is called **strongly $K_{σδ}$** if it is a strongly $K_{σδ}$ subset of its second dual $(X^{**}, w^*)$.

It is clear that every strongly $K_{σδ}$ Banach space is SWKA (see Prop. [I.4]).

**Theorem 3.2.** Every Banach space $X$ with property (*) is strongly $K_{σδ}$.

**Proof.** Let $\{K_{n,p} : n, p ∈ \mathbb{N}\}$ be a family of convex symmetric subsets of $X$ with the following properties:

(i) For every $n, p ∈ \mathbb{N}$ the set $K_{n,p}$ is $\frac{1}{p}$–weakly relatively compact.
(ii) For every $p ∈ \mathbb{N}$ and for every weakly compact subset $L$ of $X$ there exists $n ∈ \mathbb{N}$ such that $L ⊆ K_{n,p}$.
For every $n, p \in \mathbb{N}$ we have
\[
K^*_{n,p} \subseteq X + \frac{1}{p} B_{X^{**}}, \quad \text{so} \quad X \subseteq \bigcup_{n=1}^{\infty} K_{n,p} \subseteq \bigcup_{n=1}^{\infty} K^*_{n,p} \subseteq X + \frac{1}{p} B_{X^{**}},
\]
hence
\[
X \subseteq \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} K^*_{n,p} \subseteq \bigcap_{p=1}^{\infty} \left(X + \frac{1}{p} B_{X^{**}}\right) \subseteq X.
\]
Consequently
\[
X = \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} K^*_{n,p}.
\]
Moreover for every $p \in \mathbb{N}$ and for every weakly compact subset $L$ there exists $n \in \mathbb{N}$ such that $L \subseteq K^*_{n,p}$, hence $X$ is a strongly $K_{\sigma \delta}$ Banach space. \qed

Another class of strongly $K_{\sigma \delta}$ Banach spaces is the class of Polish Banach spaces.

**Lemma 3.3.** Let $X$ be a separable Banach space. Then:

(i) Every open (or closed) subset of $(X^*, w^*)$ is in $w^*$ topology a hemicompact space, so a strongly $K_{\sigma \delta}$ subset of $(X^*, w^*)$.

(ii) Every $G_\delta$ subset of $(X^*, w^*)$ is a strongly $K_{\sigma \delta}$ subset of $(X^*, w^*)$.

(iii) Every strongly $\mathcal{K}$–analytic subset of $(X^*, w^*)$ is a $G_\delta$ (hence a strongly $\mathcal{K}_{\sigma \delta}$) subset of $(X^*, w^*)$.

**Proof.**

(i) As $(X^*, w^*)$ is a hemicompact space it follows immediately that every closed subset of it is also a hemicompact space.

Let $U$ be an open subset of $(X^*, w^*)$. For every $n \in \mathbb{N}$ put
\[
U_n = U \cap nB_{X^*} = U \cap B_{X^*}[0, n].
\]
It is clear that $U_n$ is an open subset of the compact metric space $B_{X^*}[0, n]$ (since $X$ is separable). Thus $U_n$ is a locally compact separable metric space, consequently it is a hemicompact topological space (cf. Example 1.3). $U_n$ can be written in the form $U_n = \bigcup_{m=1}^{\infty} K_{n,m}$, where every $K_{n,m}$ is a compact subset of $(X^*, w^*)$ and in addition for every $w^*$ compact subset $K$ of $U_n$ there exists $m \in \mathbb{N}$ such that $K \subseteq K_{n,m}$. We remark that
\[
U = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m},
\]
that is, $U$ is written as a countable union of compact subsets of it. Let $K$ be a $w^*$ compact subset of $U$. Since $K$ is bounded it is contained in a closed ball $B_{X^*}[0, n]$, hence $U \subseteq U_n$. Then there exists $m \in \mathbb{N}$, such that $K \subseteq K_{n,m}$, which means that the countable family of compact sets
\{K_{n,m} : n, m \in \mathbb{N}\} dominates the compact subsets of \(U\), thus \(U\) is in \(w^*\) topology a hemicompact space.

(ii) By (i) and by Prop. 1.6, the conclusion follows.

(iii) Let \(A\) be a strongly \(K\)-analytic subset of \((X^*, w^*)\). Then each \(A_n = A \cap nB_{X^*}\) is a strongly \(K\)-analytic subset of \(nB_{X^*}\) and so a \(G_\delta\) subset of this space. Since \(X^* \setminus A = \bigcup_{n=1}^{\infty} (nB_{X^*} \setminus A_n)\) and each of the sets \(nB_{X^*} \setminus A_n\) is a \(\sigma\)-compact subset of \(nB_{X^*}\) we get that \(A\) is a \(G_\delta\) set in \(X^*\). \(\square\)

Theorem 3.4. Every Polish Banach space is strongly \(K_{\sigma\delta}\).

Proof. Let \(X\) be a Polish Banach space. The closed unit ball \(B_X\) of \(X\) is a \(G_\delta\) subset of the closed unit ball \((B_{X^{**}}, w^*)\) of \(X^{**}\). Moreover each ball \(nB_X\) is a \(G_\delta\) subset of \(nB_{X^{**}}\) for \(n \geq 1\). So, as in the proof of assertion (iii) of Lemma 3.3 (and since \(X^*\) is separable) we get that \(X\) is a \(G_\delta\) subset of \((X^{**}, w^*)\). The conclusion follows from assertion (ii) of the same Lemma. \(\square\)

Remark 3.5. (a) From Cor. 2.22 if a Polish Banach space is non reflexive (for example, the predual of JT space of James), then it does not have property (*), especially it is not SWCG with respect to any superspace of it.

(b) A weak*-Polish Banach space (according to Rosenthal’s definition \([R2]\)) is a separable Banach space \(X\), that embeds to the dual of a (separable) Banach space \(Y\) such that \(B_X\) is a Polish space in the weak* topology of \(Y^*\). It is clear that a Polish Banach space \(X\) is weak*-Polish in \(X^{**}\). This wider class of Banach spaces is studied by Rosenthal in \([R2]\). We note that the argument of the previous theorem with Lemma 3.3 imply that every weak*-Polish Banach space is a strongly \(K_{\sigma\delta}\) subset of \((Y^*, w^*)\).

(c) The result stated in Th. 3.4 is also true and for Banach spaces with \((B_X, w) \check{C}ech complete space, because every such space is isomorphic to a direct sum of a Polish and a reflexive Banach space (cf. \([E-W]\) and Prop. \(3.6\) below).

Proposition 3.6. The class of strongly \(K_{\sigma\delta}\) Banach spaces is closed under closed subspaces and finite products.

Proof. It follows immediately from Rem. 1.2 (ii) and from Prop. 1.8 \(\square\)

It is also proved the analogue of \([K-M\] Prop. 5.1]. (cf. also Th. 2.25).

Theorem 3.7. Let \((X_n)\) be a sequence of (strongly) \(K_{\sigma\delta}\) Banach spaces and \(1 \leq p < \infty\). Then the space \(X = (\sum_{n=1}^{\infty} \oplus X_n)_p\) is (strongly) \(K_{\sigma\delta}\).
Proof. For every \( n \in \mathbb{N} \) put \( Y_n = \left( \sum_{k=1}^{n} X_k \right)_p \) and \( Z_n = \left( \sum_{k=n+1}^{\infty} X_k \right)_p \). So \( X = Y_n \oplus Z_n \) and \( X^{**} = Y^{**} \oplus Z^{**} \), where \( Y^{**} = \left( \sum_{k=1}^{n} X_k^{**} \right)_p \). More precisely \( X = (Y_n \oplus Z_n)_p \) and \( X^{**} = (Y_n^{**} \oplus Z_n^{**})_p \).

Let \( p > 1 \). It follows from Prop. 1.8 that \( Y_n \oplus Z_n^{**} \) is a (strongly) \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\), thus \( Y_n + \frac{1}{n} B_{Z_n^{**}} \) is also a (strongly) \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\) for every \( n \in \mathbb{N} \). Let \( x^{**} \in \bigcap_{n=1}^{\infty} (Y_n + \frac{1}{n} B_{Z_n^{**}}) \). Then there exist \( y_n \in Y_n \) and \( z_n^{**} \in B_{Z_n^{**}} \) such that \( x^{**} = y_n + \frac{1}{n} z_n^{**} \) for all \( n \in \mathbb{N} \), hence \( \|x^{**} - y_n\| \leq \frac{1}{n} \) for all \( n \in \mathbb{N} \). It follows that \( y_n \rightarrow x^{**} \). Moreover \( \{y_n : n \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{\infty} Y_n \) and the subspace \( \bigcup_{n=1}^{\infty} Y_n \) is dense in \( X = \left( \sum_{n=1}^{\infty} X_n \right)_p \), so \( x^{**} \in X \). Thus \( \bigcap_{n=1}^{\infty} (Y_n + \frac{1}{n} B_{Z_n^{**}}) = X \) and from Prop. 1.6 \( X \) is a (strongly) \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\).

Let \( p = 1 \). Put \( E = (\sum_{k=1}^{\infty} X_k)_1 \); clearly \( E \) is a subspace of \( X^{**} \). We claim that \( E \) is a strongly \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\). We first prove that

\[
E = \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} \left( Y_n^{**} + \frac{1}{p} B_{Z_n^{**}} \right).
\]

Since each \( Y_n^{**} + \frac{1}{p} B_{Z_n^{**}} \) is a \( \sigma \)-compact subset of \((X^{**}, w^*)\), if equality (1) holds true, then \( E \) is a \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\). Let \( x^{**} = (x_n^{**})_n \in E \). Then \( \sum_{n=1}^{\infty} \|x_n^{**}\| < +\infty \). So for every \( p \in \mathbb{N} \) there exists \( n(p) \in \mathbb{N} \) such that \( \sum_{k=n(p)+1}^{\infty} \|x_k^{**}\| < \frac{1}{p} \). It then follows that given \( p \in \mathbb{N} \) we have that,

\[
x^{**} = (x_1^{**}, x_2^{**}, \ldots, x_{n(p)}^{**}, 0, 0, \ldots) + (0, 0, \ldots, 0, x_{n(p)+1}^{**}, \ldots) \in Y_n^{**} + \frac{1}{p} B_{Z_n^{**}},
\]

hence \( x^{**} \in \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} \left( Y_n^{**} + \frac{1}{p} B_{Z_n^{**}} \right) \).

Assume that \( x^{**} \in \bigcap_{p=1}^{\infty} \bigcup_{n=1}^{\infty} \left( Y_n^{**} + \frac{1}{p} B_{Z_n^{**}} \right) \). Then for each \( p \in \mathbb{N} \) there exists \( n(p) \in \mathbb{N} \) such that \( x^{**} \in Y_n^{**} + \frac{1}{p} B_{Z_n^{**}} \), so for every \( p \in \mathbb{N} \) there exist \( y_p^{**} \in Y_n^{**} \) and \( z_p^{**} \in B_{Z_n^{**}} \), such that \( x^{**} = y_p^{**} + \frac{1}{p} z_p^{**} \). It follows that \( \|x^{**} - y_p^{**}\| \leq \frac{1}{p} \) for all \( p \in \mathbb{N} \), which implies that \( y_p^{**} \rightarrow x^{**} \in X^{**} \). As for every \( p \in \mathbb{N} \) we have \( y_p^{**} \in Y_n^{**} \subseteq E = (\sum_{k=1}^{\infty} X_k)_1 \subseteq X^{**} \), it follows that \( x^{**} \in E \) and equality (1) has been proved. Now let \( \Omega \subseteq E \) be a \( w^* \)-compact subset in \( X^{**} \). Then using a ”gliding hump” argument we can prove that for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( \sum_{n=N}^{\infty} \|x_n^{**}\| < \varepsilon \) for all \( x^{**} = (x_n^{**}) \in \Omega \) (cf. [S-W] Prop. 1.2). This fact together with equality (1), easily imply that \( E \) is a strongly \( \mathcal{K}_{\sigma\delta} \) subset of \((X^{**}, w^*)\). It is now clear that for every \( n \in \mathbb{N} \) the space \( (\sum_{k=1}^{n} X_k)_1 \) is a strongly \( \mathcal{K}_{\sigma\delta} \) subset of \( Z_n^{**} \).

As the space \( Y_n = (\sum_{k=1}^{n} X_k)_1 \) is a (strongly) \( \mathcal{K}_{\sigma\delta} \) subset of \((Y_n^{**}, w^*)\), we conclude that for every \( n \in \mathbb{N} \) the space \([ (\sum_{k=1}^{n} X_k)_1 \oplus (\sum_{k=n+1}^{\infty} X_k)_1 ]_1 \)
is a (strongly) $K_{\sigma\delta}$ subset of $(Y_n^{**} \oplus Z_n^{**})_1 = X^{**}$. It then follows that the space
\[
\bigcap_{n=1}^{\infty} \left[ Y_n \oplus \left( \sum_{k=n+1}^{\infty} X_k^{**} \right) \right]_1 = X
\]
is a (strongly) $K_{\sigma\delta}$ subset of $X^{**}$.

Subsequently we examine locally (in the weak topology of a Banach space) the property of strong $K$–analyticity and the property of a set to be strongly $K_{\sigma\delta}$.

**Proposition 3.8.** Let $X$ be a strongly $K_{\sigma\delta}$ Banach space. Then every ball (open or closed) of $X$ is a strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$.

**Proof.** Clearly, every closed ball of $X$ is a strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$, as a closed subset of $(X, w)$.

We are going to prove that the open unit ball $B_X^o$ of $X$ is a strongly $K_{\sigma\delta}$ subset of $X^{**}$. It holds that $B_X^o = X \cap B_{X^{**}}$ and $B_{X^{**}} = \bigcup_{m=1}^{\infty} (1 - \frac{1}{m})B_{X^{**}}$. The space $(B_X^o, w^*)$ is hemicompact, because for each $w^*$–compact subset $L$ of $B_X^o$, there exists $m \in \mathbb{N}$ such that $L \subseteq (1 - \frac{1}{m})B_{X^{**}}$. Consequently $(B_X^o, w^*)$ is a strongly $K_{\sigma\delta}$ subset of $X^{**}$, so the open unit ball of $X$ is a strongly $K_{\sigma\delta}$ subset of $X^{**}$, as intersection of two strongly $K_{\sigma\delta}$ subsets.

In a similar way we can prove that any open ball of $X$ is a strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$.

**Remark 3.9.** In the same manner we can prove that if $X$ is $K_{\sigma\delta}$, then every ball of $X$ is a $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$. If in addition $X$ is separable, then it is easy to see that every norm $G_\delta$ subset of $X$ is a $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$.

**Proposition 3.10.** Let $X$ be a Banach space and $A \subseteq X$. Then the following are equivalent:

(i) $(A, w)$ is a strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$.

(ii) There exists a family $\{K_{n,m} : n, m \in \mathbb{N}\}$ of $w^*$–compact subsets of $X^{**}$ such that:

(a) $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}$

(b) For each sequence $(x_k)$ of $A$, which converges weakly to some $x$ in $A$ and for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\{x_k : k \in \mathbb{N}\} \subseteq K_{n,m}$.

**Proof.** The proof uses the sequential compactness of a weakly compact set and as it is simple we omit it.
Lemma 3.11. Let $X$ be a separable Banach space and $V$ an open subset of $X$. Then there exists a countable family $\{V_n : n \in \mathbb{N}\}$ of closed balls of $X$ such that $V = \bigcup_{n=1}^{\infty} V_n$ with the following property: For every sequence $(x_n)$ of $X$, $x \in X$ with $x_n \xrightarrow{\|\cdot\|} x$ and $L = \{x_n : n \in \mathbb{N}\} \cup \{x\} \subseteq V$ there exists $N \in \mathbb{N}$ such that $L \subseteq \bigcup_{n=1}^{N} V_n$.

Proof. The proof is simple, so we omit it. $\square$

Lemma 3.12. Let $X$ be a Polish space and $\{F_\sigma : \sigma \in \Sigma\}$ be a family of closed subsets of $X$ such that:

(i) If $\sigma \leq \tau$, then $F_\sigma \subseteq F_\tau$.

(ii) For every compact subset $K$ of the space $Y = \bigcup_{\sigma \in \Sigma} F_\sigma$ there exists $\sigma \in \Sigma$ such that $K \subseteq F_\sigma$.

Then $Y$ is Polish.

Proof. As $X$ is a Polish space there exists a family $\{X_\sigma : \sigma \in \Sigma\}$ of compact subsets of $X$ such that:

(a) If $\sigma \leq \tau$, then $X_\sigma \subseteq X_\tau$

(b) For each compact subset $K$ of $X$ there exists $\sigma \in \Sigma$ such that $K \subseteq X_\sigma$ (see [Chr, Th. 3.3] or [M-S, Th. 1.8]). Put $K^\sigma_{\tau} = F_\sigma \cap X_\tau$ for all $\sigma, \tau \in \Sigma$. Clearly every $K^\sigma_{\tau}$ is compact and also for every $\sigma \in \Sigma$ we have $F_\sigma = \bigcup_{\tau \in \Sigma} (F_\sigma \cap X_\tau) = \bigcup_{\tau \in \Sigma} K^\sigma_{\tau}$, so $Y = \bigcup_{\sigma \in \Sigma} F_\sigma = \bigcup_{\sigma, \tau \in \Sigma} K^\sigma_{\tau}$. We remark that:

(a) If $\tau_1 \leq \tau_2$ and $\sigma_1 \leq \sigma_2$, then $K^\sigma_{\tau_1} = F_{\sigma_1} \cap X_{\tau_1} \subseteq F_{\sigma_2} \cap X_{\tau_2} = K^\sigma_{\tau_2}$

(b) If $K$ is a compact subset of $Y$, then there exist $\sigma, \tau \in \Sigma$ such that $K \subseteq K^\sigma_{\tau}$. (From (ii) and (b) there exists $\sigma \in \Sigma$ such that $K \subseteq F_\sigma$ and $\tau \in \Sigma$ such that $K \subseteq F_\sigma \cap X_\tau = K^\sigma_{\tau}$.)

Then by [Chr] Th. 3.3] the conclusion follows. $\square$

Proposition 3.13. Let $X$ be a separable Banach space and $A \subseteq X$. If $(A, w)$ is strongly $\mathcal{K}$–analytic, then $(A, \|\cdot\|)$ is strongly $\mathcal{K}$–analytic, equivalently $A$ is a norm $G_\delta$ subset of $X$.

Proof. As $(A, w)$ is strongly $\mathcal{K}$–analytic, there exist compact sets $\{A_\sigma : \sigma \in \Sigma\}$ such that:

(i) If $\sigma \leq \tau$, then $A_\sigma \subseteq A_\tau$.

(ii) For every weakly compact subset $K$ of $A$ there exists $\sigma \in \Sigma$, such that $K \subseteq A_\sigma$. 


Then the sets $A_{\sigma}$ are weakly compact, so norm–closed in $X$. As every norm compact set is weakly compact, by Lemma 3.12 the space $(A, \| \cdot \|)$ is Polish, so $A$ is a norm $G_\delta$ subset of $X$. □

The following results assure that in certain classes of separable Banach spaces $X$, a set $A \subseteq X$ is strongly $K$–analytic in $(X, w)$ if and only if $A$ is a strongly $K_{\sigma\delta}$ subset in $(X^{**}, w^*)$.

**Theorem 3.14.** Let $X$ be a separable Banach space with Schur property and $A \subseteq X$. Then the following are equivalent:

(i) $(A, w)$ is strongly $K$–analytic

(ii) $(A, \| \cdot \|)$ is a $G_\delta$ subset of $(X, \| \cdot \|)$ (equivalently a Polish space)

(iii) $A$ is a strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$.

**Proof.** (i) ⇒ (ii) It is immediate from Prop. 3.13.

(ii)⇒ (iii) It is enough to prove that the result holds for norm open subsets of $X$. Let $V$ be a norm open subset of $X$. As $X$ is separable $V$ is written as a countable union of closed balls, as in Lemma 3.11. Let $V = \bigcup_{n=1}^{\infty} B[y_n, r_n] = \bigcup_{n=1}^{\infty} V_n$. Then $V = X \cap (\bigcup_{n=1}^{\infty} B_{X^{**}}[y_n, r_n])$. The space $X$ is separable Banach space with Schur property, so it is SWCG. Let $K$ be a weakly compact subset of $X$, that strongly generates $X$. Then

$$X = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (mK + \frac{1}{n} B_{X^{**}})$$

and

$$V = \bigcap_{n=1}^{\infty} \left[ \bigcup_{m=1}^{\infty} (mK + \frac{1}{n} B_{X^{**}}) \right] \cap \left[ \bigcup_{n=1}^{\infty} \left( \bigcup_{m=1}^{n} B_{X^{**}}[y_m, r_m] \right) \right].$$

Assume $x_n \xrightarrow{w} x$ and $L = \{x_n : n \in \mathbb{N}\} \cup \{x\} \subseteq V$. The set $L$ is weakly compact, so for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $L \subseteq mK + \frac{1}{n} B_{X^{**}}$. Furthermore $X$ has Schur property, so $x_n \xrightarrow{\| \cdot \|} x$. Then according to Lemma 3.11 there exists $m \in \mathbb{N}$ such that $L \subseteq \bigcup_{n=1}^{m} V_n$, so $L \subseteq \bigcup_{n=1}^{m} B_{X^{**}}[y_n, r_n]$, which completes the proof.

(iii) ⇒ (i) Every strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$ is strongly $K$–analytic space. □

The following result is an immediate consequence of Lemma 3.3.

**Proposition 3.15.** Let $X$ be a Banach with separable dual and $A \subseteq X$. Then the following are equivalent:

(i) $(A, w)$ is strongly $K$–analytic.

(ii) $(A, w)$ is strongly $K_{\sigma\delta}$ subset of $(X^{**}, w^*)$. 
Corollary 3.16. Let $X$ be a Banach space with separable dual. The following are equivalent:

(i) $X$ is SWKA.
(ii) $B_X$ is strongly $\mathcal{K}_{\sigma\delta}$ subset of $(B_{X^{**}}, w^*)$.
(iii) $X$ is Polish.
(iv) $X$ is strongly $\mathcal{K}_{\sigma\delta}$.

Proof. It is immediate from the previous proposition and Th. 3.4.

We now mention some open questions.

(A) Let $X$ be a Banach space.

(i) If $X$ is SWKA, is then strongly $\mathcal{K}_{\sigma\delta}$ or at least $\mathcal{K}_{\sigma\delta}$?
(ii) If $X$ is SWKA with unconditional basis, is then $X$ strongly $\mathcal{K}_{\sigma\delta}$?

We note that there exists a WKA Banach space, that is not $\mathcal{K}_{\sigma\delta}$ and in addition every WKA Banach space with unconditional basis is $\mathcal{K}_{\sigma\delta}$ ([A-A-M 1], [A-A-M 2]).

(B) Let $X$ be a Banach space.

(i) Assume that the closed unit ball $B_X$ of $X$ is a $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$. Is then $X$ itself a $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$? (If $X$ is separable, then it is of course a $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$.)
(ii) Assume that the closed unit ball $B_X$ of $X$ is a strongly $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$. Is then $X$ itself a strongly $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$?

(C) Let $X$ be a separable Banach space.

(i) Assume that $X$ is strongly $\mathcal{K}_{\sigma\delta}$. Does there exists a norm $G_{\delta}$ subset of $X$, that is not a strongly $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$? (cf. Remark 3.9)
(ii) Let $X = C[0,1]$. Does there exists a (necessarily norm $G_{\delta}$) strongly $\mathcal{K}$–analytic subset of $(X, w)$, that is not a strongly $\mathcal{K}_{\sigma\delta}$ subset of $(X^{**}, w^*)$? (cf. Question 1.16)

It is worth noting that questions (A) and (B)(ii) make sense even in the class of separable Banach spaces.

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