Merging percolation on $\mathbb{Z}^d$ and classical random graphs: Phase transition

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Abstract

We study a random graph model which is a superposition of the bond percolation model on $\mathbb{Z}^d$ with probability $p$ of an edge, and a classical random graph $G(n, c/n)$. We show that this model, being a homogeneous random graph, has a natural relation to the so-called ”rank 1 case” of inhomogeneous random graphs. This allows us to use the newly developed theory of inhomogeneous random graphs to describe the phase diagram on the set of parameters $c \geq 0$ and $0 \leq p < p_c$, where $p_c = p_c(d)$ is the critical probability for the bond percolation on $\mathbb{Z}^d$. The phase transition is similar to the classical random graph, it is of the second order. We also find the scaled size of the largest connected component above the phase transition.

1 Introduction.

We consider a graph on the set of vertices $B(N) := \{-N, \ldots, N\}^d$ in $\mathbb{Z}^d$, $d \geq 1$, with two types of edges: the short-range edges connect independently with probability $p$ each pair $u$ and $v$ if $|u - v| = 1$, and the long-range edges connect independently any pair of two vertices with probability $c/|B(N)|$. (Here for any set $A$ we denote $|A|$ the number of the elements in $A$.) This graph, call it $G_N(p, c)$ is a superposition of the bond percolation model (see, e.g., [4]), where each pair of neighbours in $\mathbb{Z}^d$ is connected with probability $p$, and a random graph model $G_{n,c/n}$ (see, e.g., [6]) on $n$ vertices, where each vertex is connected to any other vertex with probability $c/n$; all the edges in both models are independent. By this definition there can be one or two edges between two vertices in graph $G_N(p, c)$, and in the last case the edges are of different types.

The introduced model is a simplification of the most common graphs designed to study natural phenomena, in particular, biological neural networks [12]. Notice the difference between $G_N(p, c)$ and the so-called ”small-world” models intensively studied after [13]. In the ”small-world” models where edges from the grid may be kept or removed, only finite number (often at most $2d$) of the long-range edges may come out of each vertex, and the probability of those is a fixed number.

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We are interested in the connectivity of the introduced graph $G_N(p, c)$ as $N \to \infty$. We say that two vertices are connected, if there is a path of edges, no matter of which types, between them. Clearly, if $c = 0$, we have a purely bond percolation model on $\mathbb{Z}^d$, where any edge from the grid is kept (i.e., ”is open” in the terminology of percolation theory) with a probability $p$, or, alternatively, removed with a probability $1 - p$. Let us recall some basic facts from the percolation theory which we need here. Denote $C$ an open cluster containing the origin of $\mathbb{Z}^d$ in the bond percolation model. It is known (see, e.g., [4]) that for any $d \geq 1$ there is $p_c(p_c(d))$ such that

$$
P\{|C| = \infty\} =
\begin{cases}
0, & \text{if } p < p_c, \\
1, & \text{if } p > p_c,
\end{cases}
$$

where $0 < p_c < 1$, unless $d = 1$, in which case, obviously, $p_c = 1$. We shall assume here that $0 < p < p_c$, in which case the connected components formed by the short-range edges only, are finite with probability one. Recall also that for all $0 < p < p_c$ the limit

$$
\zeta(p) = \lim_{n \to \infty} \left(-\frac{1}{n} \log P\{|C| = n\}\right)
$$

exists and satisfies $\zeta(p) > 0$ (Theorem (6.78) from [4]).

Let further $C_1(G)$ denote the size (the number of vertices) of the largest connected component in a graph $G$.

**Theorem 1.1.** Assume, that $d \geq 1$ and $0 \leq p < p_c(d)$. Define

$$
c^{cr}(p) = \frac{1}{E|C|}.
$$

i) If $c < c^{cr}(p)$ define $y$ to be the root of $E c|C| e^{c|C|y} = 1$, and set

$$
\alpha(p, c) := (c + cy - E ce^{c|C|y})^{-1}.
$$

Then for any $\alpha > \alpha(p, c)$

$$
P \left\{ C_1(G_N(p, c)) > \alpha \log |B(N)| \right\} \to 0.
$$

as $N \to \infty$.

ii) If $c \geq c^{cr}(p)$ then

$$
\frac{C_1(G_N(p, c))}{|B(N)|} \to_P \beta
$$

as $N \to \infty$. 

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as \( N \to \infty \), with \( \beta = \beta(p, c) \) defined as the maximal solution to

\[
\beta = 1 - \mathbb{E} \{ e^{-c \beta |C|} \}.
\]

(1.6)

In view of (1.1) it is obvious that \( \chi(p) := \mathbb{E} |C| < \infty \) for all \( 0 \leq p < p_c \). It is also known (see Theorem (6.108) and (6.52) in [4]) that \( \chi(p) \) is analytic function of \( p \) on \( [0, p_c) \) and \( \chi(p) \to \infty \) as \( p \to p_c \). This implies that \( c^{cr}(p) \) is continuous, strictly decreasing function on \( [0, p_c) \) with \( c^{cr}(0) = 1 \) and \( c^{cr}(p_c) = 0 \). Hence, \( c^{cr} \) has inverse, i.e., for any \( 0 < c < 1 \) there is a unique \( 0 < p^{cr}(c) < p_c = p_c(d) \) such that \( c^{cr}(p^{cr}(c)) = c \). This leads to the following duality of the result in Theorem 1.1.

**Corollary 1.1.** For any \( 0 < c < 1 \) there is a unique \( 0 < p^{cr}(c) < p_c \) such that for any \( p^{cr}(c) < p < p_c \) graph \( G_N(p, c) \) has a giant component with a size \( O(|B(N)|) \) whp (i.e., with probability tending to one as \( N \to \infty \)), but for any \( p < p^{cr}(c) \) the size of the largest connected component in \( G_N(p, c) \) whp is at most \( O(\log |B(N)|) \). \( \square \)

Hence, Theorem 1.1 may also tell us something about the “distances” between the components of a random graph when it is considered on the vertices of \( \mathbb{Z}^d \).

It is worth mentioning that the symmetry between \( c^{cr} \) and \( p^{cr} \) is most spectacular in the dimension one case, when \( p_c(1) = 1 \). Notice, that \( d = 1 \) case is exactly solvable, and this is the only case when we know the distribution of \( |C| \):

\[
\mathbb{P} \{|C| = k\} = (1 - p)^2 kp^{k-1}, \quad k \geq 1.
\]

(1.7)

Hence, if \( d = 1 \) we compute for all \( 0 \leq p < 1 = p_c(1) \)

\[
c^{cr}(p) = \frac{1 - p}{1 + p},
\]

(1.8)

which also yields

\[
p^{cr}(c) = \frac{1 - c}{1 + c},
\]

for all \( 0 \leq c < 1 \). (For more details on \( d = 1 \) case we refer to [11].)

**Remark 1.1.** For any fixed \( c \) function \( \beta(p, c) \) is continuous at \( p = 0 \): if \( p = 0 \), i.e., when our graph is merely a classical \( G_{n,c/n} \) random graph, then \( |C| \equiv 1 \) and (1.6) becomes a well-known relation.
Furthermore, for any fixed \( c \), if \( p = 0 \), it is not difficult to derive from (1.3) that
\[
\alpha(0, c) = \frac{1}{c - 1 + |\log c|}.
\] (1.9)

But \( \log n/(c - 1 + |\log c|) \) is known (see Theorem 7a in [3]) to be the principal term in the asymptotics (in probability) of the largest connected component of \( G_{n,c/n} \). This inevitably leads to the following (open) question. Will
\[
P \left\{ C_G \left( G_N(p, c) \right) < \alpha_1 \log |B(N)| \right\} \to 0
\] (1.10)
hold also for all \( \alpha_1 < \alpha(p, c) \) when \( 0 < p < p_c \) and \( c < c^{cr}(p) \)?

It is easy to check that if \( c \leq c^{cr} \) then the equation (1.6) does not have a strictly positive solution, while \( \beta = 0 \) is always a solution to (1.6). This allows us to derive
\[
\beta_c' \big|_{c = c^{cr}} = 2 \frac{E|C|}{E(|C|^2)} > 0,
\] (1.11)
which confirms that the phase transition remains to be of the second order for any \( p < p_c \), as it is for \( p = 0 \), i.e., in the case of classical random graph.

More similarities and differences between our model and the "mean-field" case one can see in the following example. Let \( d = 1 \), in which case (1.7) holds. Introducing a random variable \( X \) with the first-success distribution
\[
P \{ X = k \} = (1 - p)p^{k-1}, \quad k = 1, 2, \ldots,
\] one can rewrite (1.6) as follows (see the details in [11])
\[
\beta = 1 - \frac{1}{EX} E \{ X e^{-cX} \beta \}.
\] This equation looks somewhat similar to the equation obtained in [2] for the "volume" (the sum of degrees of the involved vertices) of the giant component in the graph with a given sequence of the expected degrees. Note, however, that in our model the critical mean degree when \( c = c^{cr} \) and \( N \to \infty \) is given according to (1.8) by
\[
2p + c^{cr} = 2p + \frac{1 - p}{1 + p} = 1 + \frac{2p^2}{1 + p},
\] (1.12)
which is strictly greater than 1 for all positive \( p < 1 \). This is in a contrast with the model studied in [2], where the critical expected average degree is still 1 as in the classical random graph.
Although our model (it can be considered on a torus, in the limit the result is the same) is a perfectly *homogeneous* random graph, in the sense that the degree distribution is the same for any vertex, we study it via *inhomogeneous* random graphs, making use of the recently developed theory from [1]. The idea is the following. First, we consider the subgraph induced by the short-range edges, i.e., the edges which connect two neighbouring nodes with probability $p$. It is composed of the connected clusters (which may consist just of one single vertex) in $B(N)$. Call a *macro-vertex* each of the connected components of this subgraph. We say that a macro-vertex is of type $k$, if $k$ is the number of vertices in it. Conditionally on the set of macro-vertices, we consider a graph on these macro-vertices induced by the long-range connections. Two macro-vertices are said to be connected if there is at least one (long-range type) edge between two vertices belonging to different macro-vertices. Thus the probability of an edge between two macro-vertices $v_i$ and $v_j$ of types $x$ and $y$ correspondingly, is

$$\tilde{p}_{xy}(N) := 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy}. \quad (1.13)$$

Below we argue that this model fits the conditions of a general inhomogeneous graph model defined in [1], find the critical parameters and characteristics for the graph on macro-vertices, and then we turn back to the original model. We use essentially the results from [1] to derive (1.6). The result in the subcritical phase (part i) of Theorem 1.1 does not follow by the theory in [1]; we discuss this in the end of Section 2.4.

Notice also that in order to analyze the introduced model, we derive here some result on the joint distribution of the sizes of clusters in the percolation model (see Lemma 2.1 below), which may be of interest on its own.

The principle of treating some local structures in a graph as new vertices ("macro-vertices"), and then considering a graph induced by the original model on these vertices appears to be rather general. For example, in [5] a different graph model was also put into a framework of inhomogeneous graphs theory by certain restructuring. This method should be useful for analysis of a broad class of complex structures, whenever one can identify local and global connections. Some examples of such models one can find in [7].

Finally we comment that our result should help to study a model for the propagation of the neuronal activity introduced in [12]. Here we show that a giant component in the graph can emerge from two sources, none of which can be neglected, but each of which may be in the subcritical phase, i.e., even when both $p < p_c$ and $\ c < 1$. In particular, for any $0 < c < 1$ we can find $p < p_c$ which allows with a positive probability the propagation of impulses through the large part of the network due to the local activity.
2 Proof

2.1 Random graph on macro-vertices.

Consider now the subgraph on $B(N)$ induced by the short-range edges only, which is a purely bond percolation model. By the construction this subgraph, call it $G_N^{(s)}(p)$, is composed of a random number of clusters (of connected vertices) of random sizes. We call here the size of a cluster the number of its vertices (it may be just one). We recall here more results from percolation theory which we shall use later on.

Let $K_N$ denote the number of the connected components (clusters) in $G_N^{(s)}(p)$, and let

$$X = \{X_1, X_2, \ldots, X_{K_N}\} \quad (2.1)$$

denote the collection of all connected clusters $X_i$ in $G_N^{(s)}(p)$. We shall also use $X_i$ to denote the set of vertices in the $i$–th cluster. By this definition $\sum_{i=1}^{K_N} |X_i| = |B(N)|$.

**Theorem** [4, (4.2) Theorem, p. 77]

$$\frac{K_N}{|B(N)|} \to \kappa(p) := \mathbf{E} \frac{1}{|C|} \quad (2.2)$$

a.s. and in $L^1$ as $N \to \infty$.

Note (see, e.g., [4]) that $\kappa(p)$ is strictly positive and finite for all $0 < p < p_c$. Furthermore, in [14] the large deviations property of $K_N$ is formulated as follows

**Theorem** [14, Theorem 2] Given $\kappa(p) > \varepsilon > 0$, there exist $\sigma_j(\varepsilon, p) > 0$ for $j = 1, 2$ such that

$$\lim_{n \to \infty} \frac{-1}{|B(n)|} \log \mathbf{P} \left( \frac{K_N}{|B(n)|} \geq \kappa(p) + \varepsilon \right) = \sigma_1(\varepsilon, p)$$

and

$$\lim_{n \to \infty} \frac{-1}{|B(n)|} \log \mathbf{P} \left( \frac{K_N}{|B(n)|} \leq \kappa(p) - \varepsilon \right) = \sigma_2(\varepsilon, p).$$

This theorem implies that for all $0 < \delta < \kappa(p)$ and all large $N$ there is a positive constant $\sigma = \sigma(\delta, p)$ such that

$$\mathbf{P} \left\{ \left| \frac{K_N}{|B(N)|} - \kappa(p) \right| > \delta \right\} \leq e^{-\sigma |B(N)|}. \quad (2.3)$$

Define for any $k \geq 1$ and $x \geq 0$ an indicator function:

$$I_k(x) = \begin{cases} 1, & \text{if } x = k, \\ 0, & \text{otherwise.} \end{cases}$$
Proposition 2.1. For any fixed $k \geq 1$

$$
\frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) \to \frac{1}{\kappa(p)} \frac{1}{k} \frac{P\{|C| = k\}}{k} =: \mu(k)
$$

(2.4)

a.s. and in $L^1$ as $N \to \infty$.

Proof. Let $C(z), z \in B(N)$, denote a connected (in $G^{(s)}_N(p)$) cluster which contains vertex $z$. Then we write

$$
\frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) = \frac{|B(N)|}{K_N} \frac{1}{k} \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|). \tag{2.5}
$$

By the ergodic theorem

$$
\frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) \to P\{|C| = k\} \tag{2.6}
$$

a.s. as $N \to \infty$. This in turn implies that convergence (2.6) holds in $L^1$ as well, since

$$
0 \leq \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) \leq 1.
$$

Hence, statement (2.4) follows by (2.5) and (2.2). \qed

Given a collection of clusters $X$ defined in (2.1), we introduce another graph $\tilde{G}_N(X, p, c)$ as follows. The set of vertices of $\tilde{G}_N(X, p, c)$ we denote $\{v_1, \ldots, v_{K_N}\}$. Each vertex $v_i$ is said to be of type $|X_i|$, which means that $v_i$ corresponds to the set of $|X_i|$ connected vertices in $B(N)$. We shall also call any vertex $v_i$ of $\tilde{G}_N(X, p, c)$ a macro-vertex, and write sometimes

$$
v_i = X_i. \tag{2.7}
$$

With this notation the type of a macro-vertex $v_i$ is simply the cardinality of set $v_i = X_i$. The space of the types of macro-vertices is $S = \{1, 2, \ldots\}$. According to (2.4) the distribution of type of a (macro-)vertex in graph $\tilde{G}_N(X, p, c)$ converges to measure $\mu$ on $S$. The edges between the vertices of $\tilde{G}_N(X, p, c)$ are presented independently with probabilities induced by the original graph $G_N(p, c)$. More precisely, the probability of an edge between any two vertices $v_i$ and $v_j$ of types $x$ and $y$ correspondingly, is $\tilde{p}_{xy}(N)$ introduced in (1.13). Clearly, this construction provides a one-to-one correspondence between the connected components
in the graphs $\tilde{G}_N(X, p, c)$ and $G_N(p, c)$: the number of the connected components is the same for both graphs, as well as the number of the involved vertices from $B(N)$ in two corresponding components. In fact, considering conditionally on $X$ graph $\tilde{G}_N(X, p, c)$ we neglect only those long-range edges from $G_N(p, c)$, which connect vertices within each $v_i$, i.e., the vertices which are already connected through the short-range edges.

Consider now

$$\tilde{p}_{xy}(N) = 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} =: \frac{\kappa_N'(x, y)}{|B(N)|}.$$  \hspace{1cm} (2.8)

Observe that if $x(N) \to x$ and $y(N) \to y$ then

$$\kappa_N'(x(N), y(N)) \to c_{xy}$$  \hspace{1cm} (2.9)

for all $x, y \in S$. In order to place our model into the framework of the inhomogeneous random graphs from [1] let us introduce another (random) kernel

$$\kappa_{K_N}(x, y) = \frac{K_N}{|B(N)|}\kappa_N'(x, y),$$

so that we can rewrite the probability $\tilde{p}_{xy}(N)$ in a graph $\tilde{G}_N(X, p, c)$ taking into account the size of the graph:

$$\tilde{p}_{xy}(N) = \frac{\kappa_{K_N}(x, y)}{K_N},$$  \hspace{1cm} (2.10)

(We use notations from [1] whenever it is appropriate.) According to (2.2) and (2.9), if $x(N) \to x$ and $y(N) \to y$ then

$$\kappa_{K_N}(x(N), y(N)) \xrightarrow{a.s.} \kappa(x, y) := c\kappa(p)_{xy}$$  \hspace{1cm} (2.11)

as $N \to \infty$ for all $x, y \in S$.

Hence, in view of Proposition 2.1 we conclude that conditionally on $K_N = t(N)$, where $t(N)/|B(N)| \to \mathbb{E}(|C|^{-1})$, our model falls into the so-called "rank 1 case" of the general inhomogeneous random graph model $G^V(t(N), \kappa_{t(N)})$ with a vertex space

$$V = (S, \mu, (v_1, \ldots, v_{t(N)})_{N \geq 1})$$

(see [1], Chapter 16.4). Note, that according to (1.1) function $\mu(k)$ (defined in (2.4)) decays exponentially, which implies

$$\kappa \in L^1(S \times S, \mu \times \mu).$$  \hspace{1cm} (2.12)
Furthermore, it is not difficult to verify with a help of (2.2) and Proposition 2.1 that for any $t(N)$ such that $t(N)/|B(N)| \to \mathbb{E}(|C|^{-1})$

$$
\frac{1}{t(N)} \mathbb{E}\{e(\tilde{G}_N(X, p, c))|K_N = t(N)\} \to \frac{1}{2} \sum_{y=1}^{\infty} \sum_{x=1}^{\infty} \kappa(x, y)\mu(x)\mu(y), \quad (2.13)
$$

where $e(G)$ denotes the number of edges in a graph $G$. According to Definition 2.7 from [1], under the conditions (2.13), (2.12) and (2.11) the sequence of kernels $\kappa_{t(N)}$ (on the countable space $S \times S$) is called **graphical** on $V$ with limit $\kappa$.

### 2.2 A branching process related to $\tilde{G}_N(X, p, c)$.

Here we closely follow the approach from [1]. We shall use a well-known technique of branching processes to reveal the connected component in graph $\tilde{G}_N(X, p, c)$. Recall first the usual algorithm of finding a connected component. Conditionally on the set of macro-vertices, take any vertex $v_i$ to be the root. Find all the vertices $\{v_{i1}, v_{i2}, \ldots, v_{in}\}$ connected to this vertex $v_i$ in the graph $\tilde{G}_N(X, p, c)$, call them the first generation of $v_i$, and then mark $v_i$ as "saturated". Then for each non-saturated but already revealed vertex, we find all the vertices connected to them but which have not been used previously. We continue this process until we end up with a tree of saturated vertices.

Denote $\tau_N(x)$ the set of the macro-vertices in the tree constructed according to the above algorithm with the root at a vertex of type $x$.

It is plausible to think (and in our case it is correct, as will be seen below) that this algorithm with a high probability as $N \to \infty$ reveals a tree of the offspring of the following multi-type Galton-Watson process with type space $S = \{1, 2, \ldots\}$: at any step, a particle of type $x \in S$ is replaced in the next generation by a set of particles where the number of particles of type $y$ has a Poisson distribution $Po(\kappa(x, y)\mu(y))$. Let $\rho(x)$ denote the probability that a particle of type $x$ produces an infinite population.

**Proposition 2.2.** The function $\rho(x)$, $x \in S$, is the maximum solution to

$$
\rho(x) = 1 - e^{-\sum_{y=1}^{\infty} \kappa(x, y)\mu(y)\rho(y)}. \quad (2.14)
$$

**Proof.** We have

$$
\sum_{y=1}^{\infty} \kappa(x, y)\mu(y) = c\mathbb{E}(|C|^{-1})x \frac{1}{\mathbb{E}(|C|^{-1})} = cx < \infty \text{ for any } x,
$$

which together with (2.12) verifies that the conditions of Theorem 6.1 from [1] are satisfied, and the result (2.14) follows by this theorem. \qed
Notice that it also follows by the same Theorem 6.1 from [1] that \( \rho(x) > 0 \) for all \( x \in S \) if and only if
\[
\sum_{y=1}^{\infty} y^2 \mu(y) = c \mathbb{E}(|C|^{-1}) \sum_{y=1}^{\infty} \frac{1}{y} \mathbb{P}\{|C| = y\} = c \mathbb{E}|C| > 1;
\] (2.15)
otherwise, \( \rho(x) = 0 \) for all \( x \in S \). Hence, formula (1.2) for the critical value \( c_{cr}(p) \) follows from (2.15).

As we showed above, conditionally on \( K_N \) so that \( K_N/|B(N)| \to \mathbb{E}(|C|^{-1}) \), the sequence \( \kappa_{K_N} \) is graphical on \( \mathcal{V} \). Hence, the conditions of Theorem 3.1 from [1] are satisfied and we derive (first, conditionally on \( K_N \), and therefore unconditionally) that
\[
\frac{C_1(\tilde{G}_N(X, p, c))}{K_N} \xrightarrow{p} \rho,
\]
where \( \rho = \sum_{x=1}^{\infty} \rho(x)\mu(x) \). This together with (2.2) implies
\[
\frac{C_1(\tilde{G}_N(X, p, c))}{|B(N)|} \xrightarrow{p} \mathbb{E}(|C|^{-1}) \rho.
\] (2.16)

Notice that here \( C_1(\tilde{G}_N(X, p, c)) \) is the number of macro-vertices in the largest connected component of \( \tilde{G}_N(X, p, c) \).

### 2.3 On the distribution of types of vertices in \( \tilde{G}_N(X, p, c) \).

Given a collection of clusters \( X \) (see (2.1)) we define for all \( 1 \leq k \leq |B(N)| \)
\[
\mathcal{N}_k = \mathcal{N}_k(X) = \sum_{i=1}^{K_N} I_k(|X_i|).
\]
In words, \( \mathcal{N}_k \) is the number of (macro-)vertices of type \( k \) in the set of vertices of graph \( \tilde{G}_N(X, p, c) \). We shall prove here a useful result on the distribution of \( \mathcal{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_{K_N}) \).

**Lemma 2.1.** Set
\[
\tilde{\mu}(k) = \sum_{n=k}^{\infty} \mathbb{P}\{|C| = n\}
\]
and fix \( \nu > 2 \) arbitrarily. Then for any fixed \( \varepsilon > 0 \)
\[
\mathbb{P}\left\{|\mathcal{N}_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq |B(N)|\right\} = o(1) \quad (2.17)
\]
as \( N \to \infty \).
Proof. Let us fix \( \varepsilon > 0 \) arbitrarily. Define a constant \( L_0 \) so that \( \varepsilon L_0^\nu = \mathbb{E}(|C|^{-1}) \). Then for all \( k > L_0 \)
\[
\varepsilon k^\nu \tilde{\mu}(k) > \mu(k),
\]
and for any \( L > L_0 \)
\[
P\{|N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq |B(N)|\} \leq P\{ \max_{1 \leq i \leq K_N} |X_i| > L \}
+ P\{|N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq L\}.
\]
We shall choose later on an appropriate \( L = L(N) \) so that we will be able to bound from above by \( o(1) \) (as \( N \to \infty \)) each of the summands on the right in (2.19).

First we derive
\[
P\{ \max_{1 \leq i \leq K_N} |X_i| > L \} = P\{ \max_{z \in B(N)} |C(z)| > L \} \leq |B(N)| P\{|C| > L\}.
\]
where \( C(z) \) is an open cluster containing \( z \). For a further reference we note here, that according to (1.1) for any \( 0 < \alpha < \zeta(p) \) there is constant \( b > 0 \) such that
\[
P\{|C| \geq L\} \leq be^{-\alpha L}
\]
for all \( L \geq 1 \), which together with (2.20) implies, in particular, that
\[
P\{ \max_{1 \leq i \leq K_N} |X_i| > \frac{2}{\zeta(p)} \log |B(N)|\} \to 0
\]
as \( N \to \infty \).

Now we consider the last term in (2.19). Let us define for any \( 0 < \delta < \mathbb{E}(|C|^{-1}) \) an event
\[
A_{\delta,N} = \left\{ \left| \frac{K_N}{|B(N)|} - \mathbb{E}(|C|^{-1}) \right| \leq \delta \right\}.
\]
Recall that according to (2.3)
\[
P(A_{\delta,N}) \geq 1 - e^{-\sigma|B(N)|} = 1 - o(1)
\]
as \( N \to \infty \). Then we can bound the last term in (2.19) as follows
\[
P\{|N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq L\} \leq P\{|N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq L_0\}
+ P\{ \left( N_k/K_N > \varepsilon k^\nu \tilde{\mu}(k) + \mu(k) \text{ for some } L_0 < k \leq L\} \cap A_{\delta,N} \} + P\{ \overline{A_{\delta,N}} \} \leq P\{ \left( N_k/K_N > \varepsilon k^\nu \tilde{\mu}(k) + \mu(k) \text{ for some } L_0 < k \leq L\} \cap A_{\delta,N} \} + o(1),
\]
as \( N \to \infty \) where the last inequality follows by Proposition \ref{prop:2.1} and bound (2.24). Write

\[
P(k) := P\left\{ \left( \frac{N_k}{K_N} - \mu(k) > \varepsilon k^\nu \tilde{\mu}(k) \right) \cap A_{\delta,N} \right\}.
\]

(2.26)

Clearly, we have by (2.25):

\[
P\left\{ |N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq L \right\} \leq \sum_{k=L_0+1}^L P(k) + o(1),
\]

(2.27)

as \( N \to \infty \). Substituting now (2.27) and (2.20) into (2.19) we derive

\[
P\left\{ |N_k/K_N - \mu(k)| > \varepsilon k^\nu \tilde{\mu}(k) \text{ for some } 1 \leq k \leq |B(N)| \right\}
\]

(2.28)

\[
\leq |B(N)| P\{|C| > L\} + \sum_{k=L_0+1}^L P(k) + o(1) \leq |B(N)| \tilde{\mu}(L) + \sum_{k=L_0+1}^L P(k) + o(1)
\]

as \( N \to \infty \).

Next we shall find an upper bound for \( P(k) \). Due to the definition (2.23) of \( A_{\delta,N} \), we have

\[
P(k) \leq P\left\{ N_k > (\kappa(p) - \delta)B(N)\left( \varepsilon k^\nu \tilde{\mu}(k) + \mu(k) \right) \right\}.
\]

(2.29)

We shall use the following special case of the Talagrand’s inequality.

**Proposition 2.3.** For every \( r \in \mathbb{R} \) and \( t \geq 0 \)

\[
P\left\{ N_k \leq r - t \right\} P\left\{ N_k \geq r \right\} \leq \exp\left\{ -\frac{t^2}{8dkr} \right\}.
\]

(2.30)

**Proof.** We shall derive this result as a corollary to the Talagrand’s inequality \cite{Talagrand} which we cite here from the book \cite{Boucheron}, p. 40.

**Theorem.** [Talagrand’s Inequality] Suppose that \( Z_1, \ldots, Z_n \) are independent random variables taking their values in some sets \( \Lambda_1, \ldots, \Lambda_n \), respectively. Suppose further that \( W = f(Z_1, \ldots, Z_n) \), where \( f : \Lambda_1 \times \ldots \times \Lambda_n \to \mathbb{R} \) is a function such that, for some constants \( c_k \), \( k = 1, \ldots, n \), and some function \( \Psi \), the following two conditions hold:

1) If \( z, z' \in \Lambda = \prod_1^n \Lambda_i \) differ only in the \( i \)-th coordinate, then \( |f(z) - f(z')| \leq c_i \).

2) If \( z \in \Lambda \) and \( r \in \mathbb{R} \) with \( f(z) \geq r \), then there exists a set \( J \subseteq \{1, \ldots, n\} \) with \( \sum_{i \in J} c_i^2 \leq \Psi(r) \), such that for all \( y \in \Lambda \) with \( y_i = z_i \) when \( i \in J \), we have \( f(y) \geq r \).
Then, for every \( r \in \mathbb{R} \) and \( t \geq 0 \),

\[
P(W \leq r - t)P(W \geq r) \leq e^{-t^2/4\Psi(r)}.
\]

(2.31)

We shall show now that function \( N_k \) satisfies the conditions of this theorem. Let \( \{e_1, \ldots, e_n\} \) be the set of all edges from the lattice \( \mathbb{Z}^d \) which have both end points in \( B(N) \). Define

\[
Z_i = \begin{cases} 
1, & \text{if } e_i \text{ is open in } G_N^s(p), \\
0, & \text{if } e_i \text{ is closed in } G_N^s(p).
\end{cases}
\]

According to the definition of our model, \( Z_i \in Bc(p), i = 1, \ldots, n \), are independent random variables, and

\[
N_k = N_k(Z_1, \ldots, Z_n)
\]

since \( N_k \) is the number of the components of size \( k \) (open \( k \)-clusters) in \( G_N^s(p) \), which is defined completely by \( Z_1, \ldots, Z_n \). Furthermore, it is clear that removing or adding just one edge in \( G_N^s(p) \), may increase or decrease by at most one the number of \( k \)-clusters. Hence, the first condition of the Talagrand’s inequality is satisfied with \( c_i = 1 \) for all \( 1 \leq i \leq n \): if configurations \( z, z' \in \{0, 1\}^n \) differ only in the \( i^{th} \) coordinate, then

\[
|N_k(z) - N_k(z')| \leq 1.
\]

Next we check that the second condition is fulfilled as well, and we shall determine the function \( \Psi \). Assume, \( z \in \{0, 1\}^n \) corresponds such configuration of the edges in \( B(N) \) that \( N_k(z) \geq r \), for some \( r \geq 1 \), i.e., there are at least \( r \) clusters of size \( k \). Let \( \{e_j, j \in J\} \subset \{e_1, \ldots, e_n\} \) be a set of edges which have at least one common vertex with a set of exactly \( r \) (arbitrarily chosen out of \( N_k(z) \)) clusters of size \( k \). Clearly, \( |J| \leq 2dkr \), and for any \( z' \in \{0, 1\}^n \) with \( z'_j = z_j \) if \( j \in J \), we have

\[
N_k(z') \geq r,
\]

proving that the second condition of the Talagrand’s inequality is satisfied as well with \( \Psi(r) = 2dkr \), since

\[
\sum_{i \in J} c_i^2 = |J| \leq 2dkr.
\]

(2.32)

Hence, the inequality (2.30) follows by (2.31). \( \square \)

Set now

\[
k_N = (\kappa(p) - \delta)|B(N)|,
\]

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and consider the inequality \(2.30\) with
\[
\begin{align*}
    r &= k_N \left( \varepsilon k^\nu \tilde{\mu}(k) + \mu(k) \right), \\
    r - t &= k_N \left( \frac{\varepsilon}{2} k^\nu \tilde{\mu}(k) + \mu(k) \right) .
\end{align*}
\] \(2.33\)
First we derive for any \(k > L_0\) (in which case \(\varepsilon k^\nu \tilde{\mu}(k) \geq \mu(k)\))
\[
\begin{align*}
    \mathbb{P}\{ N_k \leq r - t \} &\geq \mathbb{P}\{ N_k \leq \frac{3}{2} k_N \mu(k) \} \geq 1 - \frac{\mathbb{E}N_k}{\frac{3}{2} k_N \mu(k)}, \tag{2.34}
\end{align*}
\]
where we used the Chebyshev’s inequality. Recall that by Proposition \(2.1\) (and \(2.2\))
\[
\frac{\mathbb{E}N_k}{|B(N)|} \to \kappa(p) \mu(k)
\]
as \(N \to \infty\). Hence, choosing \(0 < \delta \leq \kappa(p)/10\) we have
\[
\frac{\mathbb{E}N_k}{\frac{3}{2} k_N \mu(k)} = \frac{\mathbb{E}N_k}{\frac{3}{2} (\kappa(p) - \delta)|B(N)| \mu(k)} \leq 3/4
\]
for all large \(N\), which together with \(2.31\) implies
\[
\mathbb{P}\{ N_k \leq r - t \} \geq \frac{1}{4} \tag{2.35}
\]
for all large \(N\). Using the last bound in the Talagrand’s inequality \(2.30\) with \(r\) and \(t\) defined in \(2.33\), we derive for all large \(N\) when \(k > L_0\) (and therefore \(\varepsilon k^\nu \tilde{\mu}(k) \geq \mu(k)\))
\[
\begin{align*}
    \mathbb{P}\{ N_k \geq r \} &\leq \left( \mathbb{P}\{ N_k \leq r - t \} \right)^{-1} \exp \left\{ - \frac{t^2}{8dkr} \right\} \tag{2.36}
\end{align*}
\]
\[
\leq 4 \exp \left\{ - \frac{\left( \frac{\varepsilon}{2} k_N k^\nu \tilde{\mu}(k) \right)^2}{8dk \left( k_N \left( \varepsilon k^\nu \tilde{\mu}(k) + \mu(k) \right) \right)} \right\}
\]
\[
\leq 4 \exp \left\{ - \frac{\varepsilon k_N k^\nu \tilde{\mu}(k)}{64dk} \right\} = 4 \exp \left\{ - \frac{\varepsilon (\kappa(p) - \delta)}{64d} |B(N)| k^{\nu-1} \tilde{\mu}(k) \right\} .
\]
Substituting \(2.36\) into \(2.29\) we get
\[
P(k) \leq 4 \exp \left\{ -a |B(N)| k^{\nu-1} \tilde{\mu}(k) \right\} ,
\]
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where
\[ a := \frac{\varepsilon (\kappa(p) - \delta)}{64d}. \]
The last bound combined with (2.28) yields
\[
P\left\{ |N_k/K_N - \mu(k)| > \varepsilon k^{\nu} \bar{\mu}(k) \text{ for some } 1 \leq k \leq |B(N)| \right\}
\leq |B(N)| \bar{\mu}(L) + 4 \sum_{k=L_0}^{L} \exp \left\{ -a |B(N)| k^{\nu-1} \bar{\mu}(k) \right\} + o(1), \tag{2.37}
\]
as \( N \to \infty \) for any \( L \geq L_0 \).

Next we shall show that for any \( \Delta > 0 \) one can choose a finite constant \( L_0 \) and numbers \( L = L(N) \) such that
\[
\sum_{k=L_0}^{L(N)} \exp \left\{ -a |B(N)| k^{\nu-1} \bar{\mu}(k) \right\} < \Delta, \tag{2.38}
\]
for all large \( N \), and
\[
|B(N)| \bar{\mu}(L(N)) \to 0, \tag{2.39}
\]
as \( N \to \infty \). This together with (2.37) will clearly imply the statement of Lemma.

We claim that both (2.38) and (2.39) hold with
\[
L(N) = \min \left\{ k : k^\alpha \bar{\mu}(k) < \frac{1}{|B(N)|} \right\}, \tag{2.40}
\]
where
\[
\alpha = \frac{\nu - 2}{2}
\]
is positive by the assumption of Lemma. Observe, that \( k^\alpha \bar{\mu}(k) \to 0 \), as \( k \to \infty \) for any fixed \( \alpha \) due to the exponential decay (1.1). This yields that \( L(N) \to \infty \) as \( N \to \infty \), which in turn implies that there exists
\[
\lim_{N \to \infty} \bar{\mu}(L(N)) |B(N)| < \lim_{N \to \infty} L(N)^{-\alpha} = 0,
\]
and (2.39) follows.

To prove (2.38) first we note that by the definition (2.40) of \( L(N) \)
\[
(L(N) - 1)^\alpha \bar{\mu}(L(N) - 1) \geq \frac{1}{|B(N)|}. \tag{2.41}
\]
Recall that according to Lemma 6.102 from [4] (p.139), for all \( n, m \geq 0 \)
\[
\frac{1}{m+n} \mathbb{P}(|C| = n + m) \geq p(1 - p)^{-2} \frac{1}{m} \mathbb{P}(|C| = m) \frac{1}{n} \mathbb{P}(|C| = n).
\]
(2.42)

When \( m = 1 \) the inequality (2.42) implies
\[
\mathbb{P}(|C| = n + 1) \geq p(1 - p)^{2(d-1)} \mathbb{P}(|C| = n),
\]
(2.43)

for all \( n \geq 0 \). This clearly yields
\[
\tilde{\mu}(L(N)) \geq p(1 - p)^{2(d-1)} \tilde{\mu}(L(N) - 1).
\]
(2.44)

Notice that \( \gamma := p(1 - p)^{2(d-1)} \leq p < 1 \) for all \( d \geq 1 \). Combining (2.44) with (2.44) we immediately get
\[
L(N)^{\alpha} \tilde{\mu}(L(N)) \geq \frac{\gamma}{|B(N)|},
\]
(2.45)

and also by the definition (2.40) for all \( k < L(N) \)
\[
k^{\alpha} \tilde{\mu}(k) \geq \frac{1}{|B(N)|} \geq \frac{\gamma}{|B(N)|}.
\]
(2.46)

Making use of (2.45) and (2.46) we derive
\[
\sum_{k=L_0}^{L(N)} \exp \{-a|B(N)|k^{\nu-1} \tilde{\mu}(k)\} \leq \sum_{k=L_0}^{L(N)} \exp \{-a \gamma k^{\nu-1-\alpha}\} \leq a_1 \exp \{-a \gamma L_0^{\nu-2-\alpha}\},
\]
(2.47)

where \( a_1 \) is some positive constant independent of \( L_0 \). It is clear now, that for any \( \Delta > 0 \) we can fix \( L_0 \) so large that (2.47) would imply (2.38), and in the same time \( L_0 \) will satisfy (2.18) and \( L_0 < L(N) \). This completes the proof of the lemma.

\[\square\]

2.4 Proof of Theorem 1.1 in the subcritical case \( c < c^{cr}(p) \).

Let us fix \( 0 < p < p_c \) and then \( c < c^{cr}(p) \) arbitrarily. Given \( X \) let again \( v_i \) denote the macro-vertices with types \( |X_i|, i = 1, 2, \ldots \), respectively, and let \( \tilde{L} \) denote a connected component in \( \tilde{G}_N(X, p, c) \). Consider now for any positive constant \( a \) and a function \( w = w(N) \)
\[
\mathbb{P} \left\{ C_1 \left(G_N(p_c) \right) > aw \right\} = \mathbb{P} \left\{ \max_{v_i \in \tilde{L}} \sum_{i} X_i > aw \right\}.
\]
(2.48)
We know already from (2.16) that in the subcritical case the size (the number of macro-vertices) of any $\tilde{L}$ is $o(N)$ with probability tending to one as $N \to \infty$. Note that when the kernel $\kappa(x, y)$ is not bounded uniformly in both arguments, which is our case, it is not granted that the largest component in the subcritical case is at most of order $\log |B(N)|$ (see, e.g., discussion of Theorem 3.1 in [1]). Therefore first we shall prove the following intermediate result.

**Lemma 2.2.** If $c < c^\ast(p$ then

$$\mathbf{P} \left\{ C_1 \left( \tilde{G}_N(X, p, c) \right) > |B(N)|^{1/2} \right\} = o(1),$$

(2.49)

as $N \to \infty$.

**Proof.** Let us fix $\varepsilon > 0$ and $\delta > 0$ arbitrarily and introduce the following event

$$B_N = \mathcal{A}_{\delta, N} \cap \left( \max_{1 \leq i \leq K_N} |X_i| \leq \frac{2}{\zeta(p)} \log |B(N)| \right)$$

(2.50)

$$\cap \left( \bigcap_{k=1}^{\lfloor B(N) \rfloor} \left\{ \left| \frac{N_k}{K_N} - \mu(k) \right| \leq \varepsilon k^\nu \tilde{\mu}(k) \right\} \right).$$

According to (2.21), (2.22) and (2.17) we have

$$\mathbf{P} \{ B_N \} = 1 - o(1)$$

(2.51)

as $N \to \infty$.

Recall that $\tau_N(x)$ denote the set of the macro-vertices in the tree constructed according to the algorithm of revealing of connected component described above. Let $|\tau_N(x)|$ denote the number of macro-vertices in $\tau_N(x)$. Then we easily derive

$$\mathbf{P} \left\{ C_1 \left( \tilde{G}_N(X, p, c) \right) > |B(N)|^{1/2} \right\} \leq \mathbf{P} \left\{ \max_{1 \leq i \leq K_N} |\tau_N(|X_i|)| > |B(N)|^{1/2} \mid B_N \right\} + o(1)$$

(2.52)

$$\leq |B(N)| \left( \delta + \mathbf{E}(|C|^{-1}) \right) \sum_{k=1}^{\lfloor B(N) \rfloor} (\mu(k) + \varepsilon k^\nu \tilde{\mu}(k)) \mathbf{P} \left\{ |\tau_N(k)| > |B(N)|^{1/2} \mid B_N \right\} + o(1)$$

as $N \to \infty$. We shall use the multi-type branching process introduced above (Section 2.2) to approximate the distribution of $|\tau_N(k)|$. Let further $X^{c-p}(k)$ denote the total number of the particles (including the initial one) produced by the branching process starting with a single particle of type $k$. Observe that at each step of the exploration algorithm, the number
of new neighbours of $x$ of type $y$ has a binomial distribution $\text{Bin}(N'_y, \tilde{p}_{xy}(N))$ where $N'_y$ is the number of remaining vertices of type $y$, so that $N'_y \leq N'_y$.

We shall explore the following obvious relation between the Poisson and the binomial distributions. Let $Y_{n,p} \in \text{Bin}(n, p)$ and $Z_{a} \in \text{Po}(a)$, where $0 < p < 1/4$ and $a > 0$. Then for all $k \geq 0$

$$\mathbb{P}\{Y_{n,p} = k\} \leq (1 + C p^2)^{n} \mathbb{P}\{Z_{n\frac{p}{1-p}} = k\}; \quad (2.53)$$

where $C$ is some positive constant (independent of $n$, $k$ and $p$). Notice that for all $1 \leq x, y \leq 2 \zeta(p)$

$$\tilde{p}_{xy}(N) = 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} = \frac{c}{|B(N)|} xy (1 + o(1)), \quad (2.54)$$

and clearly, $\tilde{p}_{xy}(N) \leq 1/4$ for all large $N$. Therefore for any fixed positive $\varepsilon_1$ we can choose small $\varepsilon$ and $\delta$ in (2.50) so that conditionally on $B_N$ we have

$$N'_y \frac{\tilde{p}_{xy}(N)}{1 - \tilde{p}_{xy}(N)} \leq (\mu(y) + y^\nu \varepsilon_1 \tilde{\mu}(y)) \kappa(x, y) \quad (2.55)$$

for all large $N$.

We shall use the following property of measure

$$\mu(k) = \frac{1}{\mathbb{E}(|C|^{-1})} \frac{\mathbb{P}\{|C| = k\}}{k}$$

defined in Proposition 2.1. Recall, that along with the result (1.1) it is also proved in [4] that for all $0 < p < p_c$

$$\zeta(p) = lim_{n \to \infty} \left(-\frac{1}{n} \log \mathbb{P}\{|C| \geq n\}\right). \quad (2.56)$$

Hence, (1.1) and (2.56) immediately imply the existence and equality of the following limits for all $0 < p < p_c$

$$\zeta(p) = lim_{n \to \infty} \left(-\frac{1}{n} \log \mu(n)\right) = lim_{n \to \infty} \left(-\frac{1}{n} \log \tilde{\mu}(n)\right), \quad (2.57)$$

i.e., that both $\mu(n)$ and $\tilde{\mu}(n)$ decay exponentially fast, and moreover with the same exponent in the limit. Let us write further

$$\mu(y) = \mu_p(y), \quad \tilde{\mu}(y) = \tilde{\mu}_p(y), \quad \kappa(x, y) = \kappa_{c,p}(x, y) = c \kappa(p) xy,$$
emphasizing dependence on \( p \) and \( c \). The result (2.57) allows us to choose for any positive \( \varepsilon_2 \) and \( p < p' < p_c \) a positive \( \varepsilon_1 = \varepsilon_1(\varepsilon_2, p') \) such that

\[
\mu_p(y) + y'\varepsilon_1 \mu_{p'}(y) \leq (1 + \varepsilon_2) \mu_{p'}(y).
\]  

(2.58)

Setting now \( c' := (1 + \varepsilon_2)^{\kappa(p)/\kappa(p')} c \) we derive from (2.55) with a help of (2.58), that conditionally on \( \mathcal{B}_N \) with an appropriate choice of constants

\[
\frac{N'}{N} \frac{\tilde{p}_{xy}(N)}{1 - \tilde{p}_{xy}(N)} \leq (1 + \varepsilon_2) \mu_{p'}(y) \kappa_{c,p}(x, y) = \mu_{p'}(y) \kappa_{c', p'}(x, y).
\]  

(2.59)

Recall that above we fixed \( p \) and \( c < c^*(p) \), where \( c^*(p) \) is strictly decreasing and continuous in \( p \). Furthermore, function \( \kappa(p) \) is analytic on \([0, p_c)\). Hence, we can choose \( p' > p \) and \( c' := (1 + \varepsilon_2)^{\kappa(p)/\kappa(p')} c \) so that

\[
c < c' < c^*(p') < c^*(p),
\]  

(2.60)

and moreover \( c' \) and \( p' \) can be chosen arbitrarily close to \( c \) and \( p \), respectively. Now according to (2.53) and (2.59)

\[
P\{Y_{N'-\tilde{p}_{xy}(N)} \geq k\} \leq (1 + C\tilde{p}_{xy}(N)^2)^{N'} P\{Z_{N'-\tilde{p}_{xy}(N)} \geq k\} \]  

(2.61)

\[
\leq (1 + C\tilde{p}_{xy}(N)^2)^{|B(N)|} P\{Z_{\mu_{p'}(y)\kappa_{c',p'}(x, y)} \geq k\}.
\]

Hence, if conditionally on \( \mathcal{B}_N \) at each (of at most \(|B(N)|\)) step of the exploration algorithm which reveals \( \tau_N(k) \), we replace the \( \text{Bin}(N', \tilde{p}_{xy}(N)) \) variable with the \( \text{Po}(\mu_{p'}(y)\kappa_{c',p'}(x, y)) \) one, we arrive at the following bound using branching process:

\[
P\left\{ |\tau_N(k)| > |B(N)|^{1/2} \mid \mathcal{B}_N \right\} \]  

(2.62)

\[
\leq \left(1 + C \left( \max_{x, y \leq \log |B(N)|/\kappa(p)} \tilde{p}_{xy}(N) \right)^2 \right)^{|B(N)|^2} P\left\{ \chi^{c', p'}(k) > |B(N)|^{1/2} \right\}.
\]

This together with (2.54) implies

\[
P\left\{ |\tau_N(k)| > |B(N)|^{1/2} \mid \mathcal{B}_N \right\} \leq e^{b_1|B(N)|^4} P\left\{ \chi^{c', p'}(k) > |B(N)|^{1/2} \right\},
\]  

(2.63)

where \( b_1 \) is some positive constant. Substituting the last bound into (2.52) we derive with a help of (2.58)

\[
P\left\{ C_1(\tilde{G}_N(X, p, c)) > |B(N)|^{1/2} \right\}
\]  

(2.64)
as \( N \to \infty \), where \( b_2 \) is some positive constant. By the Markov’s inequality

\[
P \left\{ X^{c', p'}(k) > |B(N)|^{1/2} \right\} \leq z^{-|B(N)|^{1/2}} E_z X^{c', p'}(k)
\]

for all \( z \geq 1 \). Denote \( h_z(k) = E_z X^{c', p'}(k) \); then with a help of (2.65) we get from (2.64)

\[
P \left\{ C_1 \left( \tilde{G}_N(X, p, c) \right) > |B(N)|^{1/2} \right\}
\]

\[
\leq b_2 |B(N)| \varepsilon^b (\log |B(N)|)^4 z^{-|B(N)|^{1/2}} \sum_{k=1}^{\infty} k \mu_{p'}(k) h_z(k) + o(1).
\]

Now we will show that there exists \( z > 1 \) such that the series

\[
D_z(c', p') = \sum_{k=1}^{\infty} k \mu_{p'}(k) h_z(k) = \kappa(p')^{-1} \sum_{k=1}^{\infty} P_{p'}\{|C| = k\} h_z(k)
\]

converge. This together with (2.66) will clearly imply the statement of the lemma.

Note that function \( h_z(k) \) (as a generating function for a branching process) satisfies the following equation

\[
h_z(k) = z \exp \left\{ \sum_{x=1}^{\infty} \kappa_{c', p'}(k, x) \mu_{p'}(x) (h_z(x) - 1) \right\}
\]

\[
= z \exp \left\{ c' \kappa(p') k \left( \sum_{x=1}^{\infty} x \mu_{p'}(x) h_z(x) - \kappa(p')^{-1} \right) \right\}
\]

\[
= z \exp \left\{ c' \kappa(p') k (B_z(c', p') - \kappa(p')^{-1}) \right\}.
\]

Multiplying both sides by \( k \mu_{p'}(k) \) and summing up over \( k \) we find

\[
D_z(c', p') = \sum_{k=1}^{\infty} k \mu_{p'}(k) z \exp \left\{ c' \kappa(p') k (D_z(c', p') - \kappa(p')^{-1}) \right\}
\]

\[
= \sum_{k=1}^{\infty} \kappa(p')^{-1} P_{p'}\{|C| = k\} z \exp \left\{ c' \kappa(p') k (D_z(c', p') - \kappa(p')^{-1}) \right\},
\]

where we also used the definition of \( \mu(k) \). Let us write for simplicity \( D_z = D_z(c, p) \). Hence, as long as \( D_z \) is finite, it should satisfy equation

\[
D_z = \kappa(p)^{-1} z E_c e^{C(k(p)D_z-1)},
\]

(2.68)
which implies in turn that $D_z$ is finite for some $z > 1$ if and only if (2.68) has at least one solution (for the same value of $z$). Notice that by the definition (2.67)

$$D_z = D_1 = \kappa(p)^{-1} = (E(|C|^{-1}))^{-1}$$

(2.69)

for $z \geq 1$. Let us fix $z > 1$ and consider equation

$$y/z = E e^{c|C|(y-1)} =: F(y)$$

(2.70)

for $y \geq 1$. Using the property (1.1) of the distribution of $|C|$ it is easy to derive that function $F(y)$ is defined on $[0, \zeta(p)/c]$ where it is finite, increasing and has positive second derivative. Compute now

$$\frac{\partial}{\partial y} F(y)|_{y=1} = cE|C| = \frac{c}{c^\tau}.$$  

(2.71)

Hence, if $c < c^{\tau}$ then there exists $z > 1$ such that there is a finite solution $y$ to (2.70), and therefore (2.68) also has at least one solution for some $z > 1$. Taking into account condition (2.60), we find that $D_z(c', p')$ is also finite for some $z > 1$, which finishes the proof of the lemma.

Now we are ready to complete the proof of (1.4), following almost the same arguments as in the proof of the previous lemma. Let $S_N(x) = \sum_{X_i \in \tau_N(x)} |X_i|$ denote the number of vertices from $B(N)$ which compose the macro-vertices of $\tau_N(x)$. Denote

$$B'_N := B_N \cap \left( C_1 \left( \tilde{G}_N(X, p, c) \right) \leq |B(N)|^{1/2} \right).$$

According to (2.51) and Lemma 2.2 we have

$$P \{ B'_N \} = 1 - o(1).$$

This allows us to derive from (2.48)

$$P \left\{ C_1 \left( G_N(p, c) \right) > aw \right\} \leq P \left\{ \max_{1 \leq i \leq K_N} S_N(X_i) > aw \mid B'_N \right\} + o(1)$$

(2.72)

$$\leq |B(N)| \left( \delta + E(|C|^{-1}) \right) \sum_{k=1}^{|B(N)|} (\mu(k) + \varepsilon k'' \mu'(k)) P \{ S_N(k) > aw \mid B'_N \} + o(1).$$

Let now $S^{c,p}(y)$ denote the sum of types (including the one of the initial particle) in the total progeny of the introduced above branching process starting with initial particle of type
y. Repeating the same argument which led to (2.62), we get the following bound using the introduced branching process:

\[ P\{S_N(k) > aw \mid B'_N\} \]

\[
\leq \left(1 + C \left(\max_{x,y \leq 2 \log |B(N)|} \frac{\tilde{p}_{xy}(N)}{\zeta(p)} \right)^2 \right)^b_{\|B(N)\|\sqrt{|B(N)|}} P\{S^{c',p'}(k) > aw\}
\]

as \( N \to \infty \), where we take into account that we can perform at most \( \sqrt{|B(N)|} \) steps of exploration (the maximal possible number of macro-vertices in any \( L \) conditioned on \( B'_N \)). This together with (2.54) implies

\[
P\{\tau_N(k) > aw \mid B'_N\} \leq (1 + o(1))P\{S^{c',p'}(k) > aw\}
\]

(2.73) as \( N \to \infty \). Substituting the last bound into (2.72) we derive

\[
P\{C_1(G_N(p,c)) > aw\} \leq b|B(N)| \sum_{k=1}^{\|B(N)\|} k \mu_{c',p'}(k)P\{S^{c',p'}(k) > aw\} + o(1)
\]

(2.74) as \( N \to \infty \), where \( b \) is some positive constant. Denote \( g_z(k) = \mathbb{E} z^{S^{c',p'}(k)} \); then similar to (2.66) we derive from (2.74)

\[
P\{C_1(G_N(p,c)) > aw(N)\} \leq b|B(N)| \sum_{k=1}^{\|B(N)\|} k \mu_{c',p'}(k)g_z(k)z^{-aw(N)} + o(1).
\]

(2.75)

We shall search for all \( z \geq 1 \) for which the series

\[
A_z(c',p') = \sum_{k=1}^{\infty} k \mu_{c',p'}(k)g_z(k) = \frac{1}{\kappa(p')} \sum_{k=1}^{\infty} P_{c',p'}[|C| = k]g_z(k)
\]

converge. Function \( g_z(k) \) (as a generating function for a certain branching process) satisfies the following equation

\[
g_z(k) = z^k \exp \left\{ \sum_{x=1}^{\infty} \kappa_{c',p'}(k,x) \mu_{c',p'}(x)(g_z(x) - 1) \right\}
\]

\[
= z^k \exp \left\{ c'\kappa(p') k \sum_{x=1}^{\infty} x \mu_{c',p'}(x)(g_z(x) - 1) \right\}
\]

\[
= z^k \exp \left\{ c'k(\kappa(p') A_z(c',p') - 1) \right\}.
\]
Multiplying both sides by \( k \mu_p(k) \) and summing up over \( k \) we find
\[
A_z(c', p') = \sum_{k=1}^{\infty} k \mu_p(k) z^k \exp \{ c' k (\kappa(p) A_z(c', p') - 1) \}.
\]

Denoting for simplicity \( A_z = A_z(c, p) \), we can rewrite the last equation as follows:
\[
A_z = \sum_{k=1}^{\infty} k \mu_p(k) z^k \exp \{ c k (\kappa(p) A_z - 1) \}.
\]

(2.76)

It follows from here (and the fact that \( A_z \geq 1/\kappa(p) = A_1 \) for all \( z \geq 1 \) that if there exists \( z > 1 \) for which the series \( A_z \) converge, it should satisfy by (1.1)
\[
z < e^{\zeta(p)}.
\]

(2.77)

According to (2.76), as long as \( A_z \) is finite it satisfies the equation
\[
A_z = (\kappa(p))^{-1} E \left( z^{\lvert C \rvert} e^{\lvert C \rvert (\kappa(p) A_z - 1)} \right),
\]
which implies that \( A_z \) is finite for some \( z > 1 \) if and only if the last equation has at least one solution
\[
A_z \geq A_1 = 1/\kappa(p).
\]

(2.78)

Let us fix \( z > 1 \) and consider equation
\[
y = E \left( z^{\lvert C \rvert} e^{\lvert C \rvert (y-1)} \right)
\]

(2.79)

for \( y > 1 \). It is easy to check that at least for some \( y > 1 \) and \( z > 1 \) function
\[
f(y, z) := E \left( z^{\lvert C \rvert} e^{\lvert C \rvert (y-1)} \right)
\]
is increasing, it has all the derivatives of the second order, and \( \frac{\partial^2}{\partial y^2} f(y, z) > 0 \). Compute now
\[
\frac{\partial}{\partial y} f(y, z)_{|y=1, z=1} = c E |C| = \frac{c}{c^{cr}}.
\]

(2.80)

Hence, if \( c > c^{cr} \) there is no solution \( y \geq 1 \) to (2.79) for any \( z > 1 \). On the other hand, if \( c < c^{cr} \) then there exists \( 1 < z_0 < e^{\zeta(p)} \) such that for all \( 1 \leq z < z_0 \) there is a finite solution \( y \geq 1 \) to (2.79). We shall find \( z_0 \) as the (unique!) value for which function \( y \) is tangent to \( f(y, z_0) \) if \( y \geq 1 \).
First we rewrite (2.79) as follows. Set
\[ a = \frac{1}{c} \log z, \]
then (2.79) is equivalent to
\[ y = E e^{c |C| (y - 1 + a)}, \tag{2.81} \]
which after the change \( x = y - 1 + a \) becomes
\[ x + 1 - a = E e^{c |C||x|}. \tag{2.82} \]

Here on the right we have a convex function with a positive second derivative (for all \( x < \zeta(p)/c \)). Notice also that by the assumption
\[ \frac{\partial}{\partial x} E e^{c |C||x|} \bigg|_{x=0} = E e^{c |C|} < 1. \]

Hence, there exists unique \( y_0 > 0 \) such that
\[ \frac{\partial}{\partial x} E e^{c |C||x|} \bigg|_{x=y_0} = E c |C| = 1. \tag{2.83} \]

Define now
\[ a_0 := 1 + y_0 - E e^{c |C||y_0|}, \tag{2.84} \]
which is strictly positive due to the preceding argument. Clearly, function \( x + 1 - a_0 \) is tangent to \( E e^{c |C||x|} \). Hence, for all \( a \leq a_0 \) equation (2.82) has at least one solution, which implies due to (2.81) that for all
\[ z \leq z_0 := e^{c a_0} = \exp\{c(1 + y_0 - E e^{c |C||y_0|})\} \tag{2.85} \]
equation (2.79) has also at least one finite solution \( y > 1 \). This yields in turn that \( A_z \) is finite for all \( z \leq z_0 \).

Now taking into account that \( c' > c \) and \( p' > p \) can be chosen arbitrarily close to \( c \) and \( p \), respectively, we derive from (2.75) that for all \( 1 < z < z_0 \)
\[ P \left\{ C_1 \left( G_N(p,c) \right) > a w(N) \right\} \leq b(z) |B(N)| z^{-a w(N)} + o(1) \tag{2.86} \]
as \( N \to \infty \), where \( b(z) < \infty \). This implies that for any
\[ a > 1/ \log z_0 = (c + c y_0 - E e^{c |C||y_0|})^{-1} \]
and \( w(N) = \log |B(N)| \)

\[
P \left\{ C_1 \left( G_N(p, c) \right) > a \log |B(N)| \right\} = o(1) \tag{2.87}
\]
as \( N \to \infty \), which proves (1.4). \( \square \)

To conclude this section we comment on the methods used here. It is shown in [9] that in the subcritical case of the classical random graph model \( G_{n,c/n} \) (i.e., \( p = 0 \) in terms of our model) the same method of generating functions leads to a constant which is exactly \( \alpha(0, c) \) (see (1.9)). The last constant is known to be the principal term for the asymptotics of the size of the largest component (scaled to \( \log N \)) in the subcritical case. This gives us hope that the constant \( \alpha(p, c) \) is close to the optimal one also for \( p > 0 \).

Similar methods were used in [10] for some class of inhomogeneous random graphs, and in [1] for a general class of models. Note, however, some difference with the approach in [1]. It is assumed in [1], Section 12, that the generating function for the corresponding branching process with the initial state \( k \) (e.g., our function \( g_z(k), k \geq 1 \)) is bounded uniformly in \( k \). As we prove here this condition is not always necessary: we need only convergence of the series \( A_z \), while \( g_z(k) \) is unbounded in \( k \) in our case. Furthermore, our approach allows one to construct constant \( \alpha(p, c) \) as a function of the parameters of the model.

### 2.5 Proof of Theorem 1.1 in the supercritical case.

Let \( C_k \) denote the set of vertices in the \( k \)-th largest component in graph \( G_N(p, c) \), and conditionally on \( X \) let \( \tilde{C}_k \) denote the set of macro-vertices in the \( k \)-th largest component in graph \( \tilde{G}_N(X, p, c) \) (ordered in any way if there are ties). Let also \( C_k \) and \( \tilde{C}_k \) denote correspondingly, their sizes. According to our construction for any connected component \( \tilde{L} \) in \( \tilde{G}_N(X, p, c) \) there is a unique component \( L \) in \( G_N(p, c) \) such that they are composed of the same vertices from \( B(N) \), i.e., in the notations (2.7)

\[
L = \bigcup_{x_i \in \tilde{L}} \bigcup_{z \in X_i} \{ z \} =: V(\tilde{L}).
\]

Next we prove that with a high probability the largest components in both graphs consist of the same vertices.

**Lemma 2.3.** For any \( 0 \leq p < p_c \) if \( c > c^*(p) \) then

\[
P \{ C_1 = V(\tilde{C}_1) \} = 1 - o(1) \tag{2.88}
\]
as \( N \to \infty \).

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Proof. In a view of the argument preceeding this lemma we have

\[ \mathbb{P}\{C_1 \neq V(\tilde{C}_1)\} = \mathbb{P}\{C_1 = V(\tilde{C}_k) \text{ for some } k \geq 2\}. \]

According to Theorem 12.6 from \[1\], conditions of which are satisfied here, in the supercritical case conditionally on \( K_N \) such that \( K_N/|B(N)| \to E(|C|^{-1}) \), we have \text{whp} \( \tilde{C}_2 = O(\log(K_N)) \), which by Proposition 2.2 implies \( \tilde{C}_2 = O(\log |B(N)|) \) \text{whp}. Also we know already from (2.10) that in the supercritical case \( \tilde{C}_1 = O(|B(N)|) \) \text{whp}, and therefore \( C_1 = O(|B(N)|) \) \text{whp}. Hence, for some positive constants \( a \) and \( b \)

\[ \mathbb{P}\{C_1 \neq V(\tilde{C}_1)\} = \mathbb{P}\{C_1 = V(\tilde{C}_k) \text{ for some } k \geq 2\} \quad (2.89) \]

\[ \leq \mathbb{P}\left\{ \left( \max_{k \geq 2} |V(\tilde{C}_k)| > b|B(N)| \right) \cap \left( \max_{k \geq 2} \tilde{C}_k < a \log |B(N)| \right) \right\} + o(1). \]

It follows from (2.22) that

\[ \mathbb{P}\left\{ \max_{1 \leq i \leq K_N} |X_i| \geq \sqrt{|B(N)|} \right\} = o(1) \]
as \( N \to \infty \). Now we derive

\[ \mathbb{P}\left\{ \left( \max_{k \geq 2} |V(\tilde{C}_k)| > b|B(N)| \right) \cap \left( \max_{k \geq 2} \tilde{C}_k < a \log |B(N)| \right) \right\} \quad (2.90) \]

\[ \leq \mathbb{P}\left\{ \left( \max_{k \geq 2} |V(\tilde{C}_k)| > b|B(N)| \right) \cap \left( \max_{k \geq 2} \tilde{C}_k < a \log |B(N)| \right) \cap \left( \max_{1 \leq i \leq K_N} |X_i| < \sqrt{|B(N)|} \right) \right\} \quad (2.91) \]

\[ + o(1) \leq \mathbb{P}\left\{ \sqrt{|B(N)|} \log |B(N)| > b|B(N)| \right\} + o(1) = o(1). \]

Substituting this bound into (2.89) we immediately get (2.88). \( \square \)

Conditionally on \( C_1 = V(\tilde{C}_1) \) we have

\[ \frac{C_1}{|B(N)|} = \frac{1}{|B(N)|} \sum_{i=1}^{K_N} |X_i| \mathbb{1}\{X_i \in \tilde{C}_1\} \]

\[ = \frac{1}{|B(N)|} \sum_{i=1}^{K_N} \sum_{k=1}^{|B(N)|} k \mathbb{1}\{|X_i| = k\} \mathbb{1}\{X_i \in \tilde{C}_1\} \]

\[ = \frac{K_N}{|B(N)|} \sum_{k=1}^{|B(N)|} k \frac{1}{K_N} \#\{X_i \in \tilde{C}_1 : |X_i| = k\}. \]

Note that Theorem 9.10 from \[1\] (together with (2.2) in our case) implies that

\[ \nu_N(k) := \frac{1}{K_N} \#\{X_i \in \tilde{C}_1(N) : |X_i| = k\} \overset{P}{\to} \rho(k) \mu(k) \]

(2.92)
for each $k \geq 1$, where $\rho(k)$ is the maximal solution to (2.14).

We shall prove below that also

$$W_N := \sum_{k=1}^{\lfloor |B(N)| \rfloor} k \nu_N(k) \xrightarrow{P} \sum_{k=1}^{\infty} k \rho(k) \mu(k) =: \beta \left( \mathbb{E}(|C|^{-1}) \right)^{-1}.$$  \hfill (2.93)

Observe that according to (2.14) constant $\beta$ (defined above) is the maximal solution to

$$\beta = \mathbb{E}(|C|^{-1}) \sum_{k=1}^{\infty} k \rho(k) \mu(k) = \mathbb{E}(|C|^{-1}) \sum_{k=1}^{\infty} k \left(1 - e^{-\sum_{y=1}^{\infty} \sigma(k, y) \mu(y) \rho(y)}\right) \mu(k)$$

$$= 1 - \mathbb{E}\left(e^{-d|C|} \right).$$

This proves that $\beta$ is the maximal root of (1.6). Then (2.93) together with (2.2) will allow us to derive from (2.91) that for any positive $\varepsilon$

$$P\left\{ \frac{C_1(G_N(p, c))}{|B(N)|} - \beta > \varepsilon \mid C_1 = V(\tilde{C}_1) \right\} \to 0$$

as $N \to \infty$. This combined with Lemma 2.3 would immediately imply

$$\frac{C_1(G_N(p, c))}{|B(N)|} \xrightarrow{P} \beta,$$  \hfill (2.94)

and hence the statement of the theorem follows.

Now we are left with proving (2.93). For any $1 \leq R < |B(N)|$ write $W_N := W_N^R + w_N^R$, where

$$W_N^R := \sum_{k=1}^{R} k \nu_N(k), \quad w_N^R := \sum_{k=R+1}^{\lfloor |B(N)| \rfloor} k \nu_N(k).$$

By (2.92) we have for any fixed $R \geq 1$

$$W_N^R \xrightarrow{P} \sum_{k=1}^{R} k \rho(k) \mu(k)$$  \hfill (2.95)

as $N \to \infty$. Consider $w_N^R$. Note that for any $k \geq 1$

$$\mathbb{E} \nu_N(k) \leq \mathbb{E} \frac{1}{K_N} \sum_{i=1}^{K_N} I_k(|X_i|) = \mathbb{E} \frac{|B(N)|}{K_N} \frac{1}{k} \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|),$$  \hfill (2.96)
where \( C(z) \) denotes again a connected cluster in the bond percolation model on \( B(N) \) with a probability \( p \) of bound. Using events \( A_{\delta,N} \) together with bound (2.24), we obtain from (2.96) for any fixed \( 0 < \delta < \mathbb{E}(|C|^{-1})/2 \) and \( k \geq 1 \)

\[
\mathbb{E} \nu_N(k) \leq \mathbb{E} \left( \frac{|B(N)|}{K_N} \mathbb{1}_{A_{\delta,N}} \frac{1}{|B(N)|} \sum_{z \in B(N)} I_k(|C(z)|) \right) + \mathbb{E} \left( \frac{|B(N)|}{K_N} \mathbb{1}_{A_{\delta,N}} \right)
\]

\[
\leq \frac{1}{\mathbb{E}(|C|^{-1}) - \delta} \mathbb{P}\{ |C| = k \} + |B(N)| \mathbb{P}\{ A_{\delta,N} \}.
\]

Bound (2.24) allows us to derive from here that

\[
\mathbb{E} \nu_N(k) \leq A_1 \mathbb{P}\{ |C| = k \} + e^{-a_1 |B(N)|}\] (2.97)

for some positive constants \( A_1 \) and \( a_1 \) independent of \( k \) and \( N \). This together with (1.1) yields

\[
\mathbb{E} w^R_N = \sum_{k=R+1}^{\frac{|B(N)|}{2}} k \mathbb{E} \nu_N(k) \leq A_2 e^{-a_2 R}\] (2.98)

for some positive constants \( A_2 \) and \( a_2 \).

Clearly, for any \( \varepsilon > 0 \) we can choose \( R_0 \) so that for all \( R \geq R_0 \)

\[
\sum_{k=R+1}^{\infty} k \rho(k) \mu(k) < \varepsilon/3,
\]

and then we have

\[
\mathbb{P}\{ |W_N - \sum_{k=1}^{\infty} k \rho(k) \mu(k)| > \varepsilon \} \quad (2.99)
\]

\[
= \mathbb{P}\{ |(W^R_N - \sum_{k=1}^{R} k \rho(k) \mu(k)) + w^R_N - \sum_{k=R+1}^{\infty} k \rho(k) \mu(k)| > \varepsilon \}
\]

\[
\leq \mathbb{P}\{ |W^R_N - \sum_{k=1}^{R} k \rho(k) \mu(k)| > \varepsilon/3 \} + \mathbb{P}\{ w^R_N > \varepsilon/3 \}.
\]

Markov's inequality together with bound (2.98) gives us

\[
\mathbb{P}\{ w^R_N > \varepsilon/3 \} \leq \frac{3 \mathbb{E} w^R_N}{\varepsilon} \leq \frac{3 A_2 e^{-a_2 R}}{\varepsilon}.\] (2.100)
Making use of (2.100) and (2.95) we immediately derive from (2.99)
\[
P\{|W_N - \sum_{k=1}^{\infty} k \rho(k) \mu(k)| > \varepsilon}\leq o(1) + \frac{3A_2 e^{-a_2 R}}{\varepsilon}
\] (2.101)
as \(N \to \infty\). Hence, for any given positive \(\varepsilon\) and \(\varepsilon_0\) we can choose finite \(R\) so large that
\[
\lim_{N \to \infty} P\{|W_N - \sum_{k=1}^{\infty} k \rho(k) \mu(k)| > \varepsilon\} < \varepsilon_0.
\] (2.102)
This clearly proves statement (2.93), and therefore finishes the proof of the theorem. \(\square\)

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