FREE TWO-STEP NILPOTENT GROUPS WHOSE AUTOMORPHISM GROUP IS COMPLETE

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Abstract. Dyer and Formanek (1976) proved that if $N$ is a free nilpotent group of class two and of rank $\neq 1, 3$, then the automorphism group $\text{Aut}(N)$ of $N$ is complete. The main result of this paper states that the automorphism group of an infinitely generated free nilpotent group of class two is also complete.

Introduction

According to the result by J. Dyer and E. Formanek [4], the automorphism group of a finitely generated free two-step nilpotent group is complete except in the case when this group is a one- or three-generator (the three-generator groups have automorphism tower of height two). The purpose of this paper is to prove that the automorphism group of an infinitely generated free two-step nilpotent group is also complete. (Recall that a group $G$ is said to be complete if $G$ is centreless and every automorphism is inner.)

The paper may be considered as a contribution to the study of automorphism towers of relatively free groups. The study arose from conjectures of G. Baumslag that automorphism towers of finitely generated absolutely free and free nilpotent groups must be very short (strictly speaking, his conjecture in the nilpotent case concerns finitely generated torsion-free nilpotent groups [7 problem 4.9]). The case of absolutely free groups was considered in the paper [2] by Dyer and Formanek: they proved the automorphism group of a finitely generated non-abelian free group $F$ is complete, that is $\text{Aut}(\text{Aut}(F)) \cong \text{Aut}(F)$ (in fact, the automorphism group of any non-abelian free group is complete [12]). In [3] Dyer and Formanek obtained the following generalization of the results from [2]: the automorphism group of any group $F/R'$, where $F$ is absolutely free of finite rank and $R$ is a characteristic subgroup of $F$ lying in the commutator subgroup $F'$ of $F$, is complete provided that $F/R$ is approximated by torsion-free nilpotent groups. In particular, any finitely generated non-abelian free solvable group has complete automorphism group. The above cited paper [4] and the present paper give full description of automorphism towers of free two-step nilpotent groups.

Let $N$ denote an infinitely generated free two-step nilpotent group. The ideas we use in the present paper are closely related to those of Dyer and Formanek [4]. Thus, like the cited authors, we prove in the last section that after multiplication by a suitable inner automorphism of $\text{Aut}(N)$ any automorphism of $\text{Aut}(N)$ preserves the elements of the subgroup $\text{Inn}(N)$ and a fixed automorphism of $N$ which inverts all members of some basis of $N$ (we call such automorphisms of $N$ symmetries; note also that in [4] a similar result, in a part concerning the symmetry, is formulated a bit weaker). However, instead of further analysis of the action of the transformed automorphism, say, $\Delta$ of $\text{Aut}(N)$ on all generators of $\text{Aut}(N)$, we prefer to prove that $\Delta$ preserves all $\text{IA}$-automorphisms, and hence the elements of the conjugacy class of all symmetries (Theorem 5.3). This enables us to prove that $\Delta$ preserves all elements of $\text{Aut}(N)$. (Generally speaking, if $K$ is a conjugacy class of a group $G$}
such that the centralizer of $K$ in $G$ is trivial, then any automorphism of $G$ which fixes all elements of $K$ necessarily fixes all elements of $G$.

Statements which are formulated similarly or exactly the same as some statements from [4] can be also found in Sections 2 and 3; all such statements, mostly with different proofs, will be specially indicated in the main body of the paper. The proof of the completeness of $\text{Aut}(\mathcal{N})$ given in this paper follows, nevertheless, an alternative general plan.

The specific feature of our proofs is the method we usually use to show invariance of a subset of $\text{Aut}(\mathcal{N})$ under automorphisms of the group. The method is based on the following general observation that came from model theory: if a subset of an algebraic structure is the set of realizations of a formula of a certain logic with parameters in the structure, or, in model-theoretic terms, if this subset can be defined by the mentioned formula, then every automorphism of the structure, which fixes each of the parameters, setwise fixes the subset. In particular, if a subset is definable by a formula without parameters then the subset is invariant under all automorphisms of the structure. In such situations algebraists used to say that the subset of the structure can be characterized in terms of basic operations. Thus, the reader who is not familiar with model-theoretic terminology can substitute his or her own arguments where necessary. We should stress, however, that this paper does not assume familiarity with model theory.

Usually, to define subsets we shall use formulae of the first-order logic or the monadic second-order logic (which allows quantification by arbitrary subsets of a structure). A subset of a structure definable by means of first-order logic is simply called a definable subset. The use of monadic second-order is not actually particularly deep: we are just trying to express the fact that characterization of some subsets in $\text{Aut}(\mathcal{N})$ requires higher-order relations.

After preliminary Section 1 outlining terminology and background material, we begin by showing that a family of all involutions of $\mathcal{N}$ which are symmetries modulo the subgroup $\text{IA}(\mathcal{N})$ is definable in $\text{Aut}(\mathcal{N})$ (Lemma 2.1). Then we prove that the subgroup $\text{IA}(\mathcal{N})$ itself is a definable, and hence a characteristic subgroup of $\text{Aut}(\mathcal{N})$ (Proposition 2.2). In the same section we prove definability in $\text{Aut}(\mathcal{N})$ modulo $\text{IA}(\mathcal{N})$ for one more family of involutions of $\mathcal{N}$, for extremal involutions (we call an automorphism $\varphi$ of $\mathcal{N}$ an extremal involution, if there is a basis of $\mathcal{N}$ such that $\varphi$ inverts some element of this basis element and fixes others; the term is chosen in analogy with classical group theory).

In Section 3 we prove definability of conjugations and symmetries in $\text{Aut}(\mathcal{N})$. First, using involutions extremal modulo $\text{IA}(\mathcal{N})$ we prove that the set of all conjugations by powers of primitive elements is definable in $\text{Aut}(\mathcal{N})$ (Lemma 3.1; note that a similar result holds for the automorphism groups of non-abelian free groups [12]). Lemma 3.1 implies that the subgroup $\text{Inn}(\mathcal{N})$ is characteristic in $\text{Aut}(\mathcal{N})$. Next, by means of monadic second-order logic we define in $\text{Aut}(\mathcal{N})$ symmetries; this involves symmetries modulo $\text{IA}(\mathcal{N})$ and normalizers of free generating sets of the (free abelian) group $\text{Inn}(\mathcal{N})$ (Lemma 3.3).

The main result of Section 4, the next to the last in this paper, states that the subgroup $\text{IA}_\tau(\mathcal{N})$ of $\text{Aut}(\mathcal{N})$ consisting of all $\text{IA}$-automorphisms which stabilize a given primitive element $x$ of $\mathcal{N}$ (in fact, any element in $x\mathcal{N}'$, where $\mathcal{N}'$ is the commutator subgroup of $\mathcal{N}$) is definable in $\text{Aut}(\mathcal{N})$ by means of monadic second-order logic with the parameter $\tau$, where $\tau$ is conjugation by $x$ (Theorem 4.1). The stabilizers $\text{IA}_\tau(\mathcal{N})$ are involved in a second-order modelling in $\text{Aut}(\mathcal{N})$ the primitive elements of $\mathcal{N}$ and play a crucial role in the proofs in the last section, which was briefly described above.

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1. Basic concepts and notation

Everywhere in this paper \( N \) denotes an infinitely generated free two-step nilpotent group, \( N' \) stands for the commutator subgroup of \( N \) and \( A \) for the free abelian group \( N/N' \). We denote by \( \sim \) the natural homomorphism \( N \to A \), and use the same symbol to denote the corresponding induced homomorphism \( \text{Aut}(N) \to \text{Aut}(A) \).

For any two-step nilpotent group the commutator subgroup is contained in the centre of this group; for a free two-step nilpotent group the centre, a free abelian group, is exactly the commutator subgroup (§9, ch. 3, §1]). We denote by \( \tau_a \) the inner automorphism of \( N \), or conjugation, determined by an element \( a \in N \). Since \( N' \) is the centre of the group \( N \), then \( \tau_a = \tau_b \) if and only if \( a \equiv b(\mod N') \).

Hence \( \text{Inn}(N) \), the group of all inner automorphisms of \( N \), is isomorphic to \( N/N' \), and, in particular, is a free abelian group.

**Theorem 1.1.** A set \( \{x_i : i \in I\} \) is a basis (free generating set) of \( N \) if and only if the set \( \{\tau_i : i \in I\} \) is a basis of the free abelian group \( N/N' \).

Recall that the sufficiency part of the Theorem can be proved by using the following two results: (1) if a set \( X \cup G' \) generates a nilpotent group \( G \), then \( G \) is generated only by \( X \) itself [5, Cor. 10-3-3] and (2) if \( X \) generates a free nilpotent group \( G \) and \( \overline{X} \) is a basis of \( G/G' \), then \( \overline{X} \) is a basis of \( G \) [10, §4] (see also [8, ch 3, §1, ch. 4, §2]).

**Corollary 1.2.** Every automorphism of \( A \) is induced by an automorphism of \( N \).

The kernel of the induced homomorphism \( \text{Aut}(N) \to \text{Aut}(A) \), the subgroup of \( \text{IA-automorphisms} \), is denoted as usual by \( \text{IA}(N) \). Considering the action of the elements of \( \text{IA}(N) \) on a fixed basis of \( N \), one sees that \( \text{IA}(N) \) is isomorphic to an infinite Cartesian power of \( N' \), and therefore is a torsion-free abelian, but not free abelian group as in the case of finitely generated free two-step nilpotent groups [4].

We shall work with involutions in the automorphism group of the abelianization \( A \). Suppose that \( f \) is an involution of the group \( A \). Write \( A_f^+ \) (in the manner of R. Baer) for the fixed-point subgroup of \( f \) and \( A_f^- \) for the subgroup of elements \( \{a\} \) such that \( fa = -a \). An involution \( f \in \text{Aut}(A) \) is diagonalizable in some basis of \( A \) if and only if

\[
A = A_f^+ \oplus A_f^-
\]

(note that the latter property does not hold for all involutions in \( \text{Aut}(A) \); see Theorem 1.3 below). We call a diagonalizable involution \( f \) a \( \kappa \)-involution, where \( \kappa \) is a cardinal, if

\[
\kappa = \min(\text{rank } A_f^+, \text{rank } A_f^-).
\]

A standard argument proves that

**Lemma 1.3.** Diagonalizable involutions \( f \) and \( g \) from \( \text{Aut}(A) \) commute if and only if

\[
A = (A_f^+ \cap A_g^+) \oplus (A_f^+ \cap A_g^-) \oplus (A_f^- \cap A_g^+) \oplus (A_f^- \cap A_g^-).
\]

In general, the structure of involutions in \( \text{Aut}(A) \) is described by the following

**Theorem 1.4.** Any involution \( f \) in the group \( \text{Aut}(A) \) has a basis \( B \) of \( A \) such that \( fb = \pm b \) or \( fb \in B \) for each \( b \in B \).

The result is essentially known for free abelian groups of finite rank (it immediately follows from Lemma 1 in [6] by L. K. Hua and I. Reiner); we give a sketch of the proof in the case of infinite rank.

**Proof of Theorem 1.4.** The fixed-point subgroup \( A_f^+ \) is a free summand of \( A \). Let \( R \) be a direct complement \( \text{Fix}(\varphi) \) to \( A \) and let \( \{r_i : i \in I\} \) be a basis of \( R \). For every
For every transfinite induction a family \( \{ k : k \in K \} \) of elements of \( A \) such that

(i) the set \( \{ s_k : k \in K \} \cup \{ s_j : j \in J \} \) generates a direct complement of \( \text{Fix}(A) \) to \( A \);

(ii) \( fs_k = -s_k + u_k \), where \( u_k = 0 \) or \( u_k \) is a unimodular element of \( A \) (that is, can be found in some basis of \( A \)) and

(iii) the family consisting of all non-zero elements \( u_k \) forms a basis of a direct summand of \( \text{Fix}(\varphi) \).

Let \( I_1 = \{ i \in I : fs_i = -s_i \} \) and \( I_2 = I \setminus I_1 \). By the construction the subgroup \( \langle fs_i + s_i : i \in I_2 \rangle \) is a direct summand of \( \text{Fix}(\varphi) \):

\[
\text{Fix}(\varphi) = \langle fs_i + s_i : i \in I_2 \rangle \oplus V.
\]

Suppose that \( V \) is a basis of \( V \). Then the following basis of \( A \)

\[
\{ s_i : i \in I_1 \} \cup \{ s_i, fs_i : i \in I_2 \} \cup V
\]

satisfies the conditions of the Theorem.

2. Coming down to the abelianization

We shall use throughout the paper two conjugacy classes of involutions in the group \( \text{Aut}(N) \). The first conjugacy class consists of involutions similar to involutions one can find in standard generating sets of automorphism groups of finitely generated two-step free nilpotent group. We shall call \( \varphi \in \text{Aut}(N) \) an extremal involution if there is a basis of \( N \) such that \( \varphi \) inverts exactly one element of this basis and fixes other elements; we shall also call any basis of \( N \) on which \( \varphi \) acts in such a way a canonical basis for \( \varphi \). Any symmetry \( \theta \), a member of the second conjugacy class, has a basis of \( N \) such that \( \theta \) takes each element of this basis to the inverse (a canonical basis for \( \theta \)).

Note that both described classes are used in the cited paper [4] by Dyer and Formanek. In this section we obtain a first-order characterization of both symmetries and extremal involutions in \( \text{Aut}(N) \) modulo \( \text{IA}(N) \).

**Lemma 2.1.** (a) Let \( \theta \) be a symmetry. Then for every \( \text{IA-automorphism} \ \alpha \)

\[ \theta \alpha \theta = \alpha^{-1}; \]

in particular, the automorphism \( \theta \alpha \) is an involution in \( \text{Aut}(N) \).

(b) Let \( \theta \) be a symmetry, \( \sigma \in \text{Aut}(N) \) and \( \alpha \in \text{IA}(N) \). If both \( \sigma \) and \( \sigma \alpha \) commute with \( \theta \), then \( \alpha = \text{id} \).

(c) An involution \( \theta \in \text{Aut}(N) \) is a symmetry modulo \( \text{IA}(N) \) (that is, has the form \( \theta \sigma \beta \) for some symmetry \( \theta \sigma \) and \( \beta \in \text{IA}(N) \)) if and only if

\[ \text{a product of any three conjugates of} \ \theta \ \text{is an involution}. \]

(d) The family of all involutions which are symmetries modulo \( \text{IA}(N) \) is definable in \( \text{Aut}(N) \).

**Proof.** (a) Since \( \theta \) inverts all elements of some basis \( B \) of \( N \), then \( \theta \) inverts modulo \( N' \) all elements of \( N \). This implies that \( \theta \) preserves any commutator \( [a, b] \in N' \):

\[ \theta[a, b] = [\theta a, \theta b] = [a^{-1}, b^{-1}] = [a, b]^{-1} = ([a, b]^{-1})^{-1} = [a, b], \]

and hence all elements of \( N' \).

Let \( x \) be an element of \( B \) and \( \alpha x = xc \), where \( c \in N' \). Then we have that

\[ \theta \alpha \theta x = \theta \alpha (x^{-1}) = \theta (x^{-1}c^{-1}) = xc^{-1} = \alpha^{-1}x. \]
For the group $GL(2, \mathbb{Z})$, for the automorphism group of a two-generator free abelian group, or, equivalently, the theorem that the latter statement follows from the fact that the similar result holds the conjugacy relation. Then it suffices to prove that a central element of $Aut(A)$ is an involution. Further, an involution from $Aut(N)$ is a symmetry modulo $IA(N)$ if and only if its image under the induced homomorphism $Aut(N) \to Aut(A)$ (recall that $A = N/N'$) is equal to $-\text{id}_A$. Then a product of any three (conjugate) symmetries modulo $IA(N)$ is again a symmetry modulo $IA(N)$, and therefore an involution.

Let us prove the converse. It follows from Corollary 1.2 that if $\sigma \in Aut(N)$, then any conjugate of $\sigma$ in $Aut(A)$ can be lifted to a conjugate of $\sigma$ in $Aut(N)$: if $s' \sim \sigma$ in $Aut(A)$, then there is $\sigma' \in Aut(N)$ such that $\sigma' \sim \sigma$ and $\sigma' = s'$ (sim denotes the conjugacy relation). Then it suffices to prove that $-\text{id}$, or a unique non-trivial central element of $Aut(A)$, is the only involution with the property (1) in the group $Aut(A)$.

We shall base our argument on Theorem 1.4. It is quite clear in view of this theorem that the latter statement follows from the fact that the similar result holds for the automorphism group of a two-generator free abelian group, or, equivalently, for the group $GL(2, \mathbb{Z})$. Indeed, if $f$ is an involution in $Aut(A)$ and $B$ is a basis of $A$ with the property described in Theorem 1.4 then there are two elements $u, v$ of $B$ such that both subgroups $\langle u, v \rangle$ and $\langle B \setminus \{u, v\} \rangle$ are $f$-invariant. Thus, one can make conjugates of $f$ changing its action on $\langle u, v \rangle$, but preserving the action on $\langle B \setminus \{u, v\} \rangle$.

Theorem 1.4 implies also that any matrix of order two from $GL(2, \mathbb{Z})$ which is not in the centre of this group is conjugate to the matrix

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or to the matrix

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

One easily checks that neither the product of the following three conjugates of $X$

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix},$$

nor the product of the following conjugates of $Y$

$$Y_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has order two.

(d) By (c).  

\[\square\]

As an immediate consequence of the previous Lemma we have:

**Proposition 2.2.**

(a) An automorphism $\sigma \in Aut(N)$ is in the subgroup $IA(N)$ if and only if $\sigma$ can be written as a product of two involutions with (1).

(b) $IA(N)$ is a definable subgroup of $Aut(N)$.

Let us obtain one more consequence.

**Proposition 2.3.** The group $Aut(N)$ is centreless.

**Proof.** Since the induced homomorphism $Aut(N) \to Aut(A)$ is surjective, then an element $\sigma$ from the centre of $Aut(N)$ should induce a central element of the group $Aut(A)$. The only elements in the centre of $Aut(A)$ are $\pm \text{id}_A$, and therefore $\sigma = \pm \text{id}_A$. In the case when $\sigma = \text{id}_A$ we have $\sigma \in IA(A)$, but there is no non-trivial $IA$-automorphism which commutes with symmetries (Lemma 2.1 (a)). Assuming $\sigma = -\text{id}_A$ we obtain that $\sigma$ is a symmetry modulo $IA(N)$, that is, has the form.
\(\theta \beta\) for some symmetry \(\theta\) and an \(IA\)-automorphism \(\beta\). But again by Lemma 2.1 we have for every \(IA\)-automorphism \(\alpha\) that
\[
\theta \beta \alpha \beta^{-1} \theta = (IA(N) \text{ is an abelian group}) = \theta \alpha \theta = \alpha^{-1},
\]
and the result follows.

Next is a first-order characterization of extremal involutions modulo \(IA(N)\) in the group \(\text{Aut}(N)\). The fact that \(IA(N)\) is a definable subgroup of \(\text{Aut}(N)\) (Proposition 2.2) enables us to involve the structures related to \(IA(N)\) and the subgroup \(IA(N)\) itself into first-order characterisations of subsets of \(\text{Aut}(N)\).

**Proposition 2.4.** Let \(K(f)\) denote the conjugacy class of an automorphism \(f\) of \(A\) in the group \(\text{Aut}(A)\). An involution \(\varphi \in \text{Aut}(N)\) is an extremal modulo \(IA(N)\) if and only if
\[
\begin{align*}
(i) & \quad \rho \text{ is not a square in } \text{Aut}(A); \\
(ii) & \quad \text{the set } K^2(\rho) = K(\rho)K(\rho) \text{ contains no elements of order three and} \\
(iii) & \quad \text{all involutions in } K^2(\rho) \text{ are conjugate.}
\end{align*}
\]
The family of involutions extremal modulo \(IA(N)\) is definable in \(\text{Aut}(N)\).

**Proof.** If \(\varphi\) is an extremal involution modulo \(IA(N)\), then there is a basis \(\{x\} \cup Y\) of \(N\) such that
\[
\begin{align*}
\varphi x & \equiv x^{-1} \pmod{N'}, \\
\varphi y & \equiv y \pmod{N'}, \quad \forall y \in Y.
\end{align*}
\]
Let \(f = \varphi\). Then \(\pm \varphi\) are the only unimodular elements in \(A^{-} = \{a : fa = -a\}\). Suppose that \(f = g^2\), where \(g \in \text{Aut}(A)\). We then have \(f(g\varphi) = -g\varphi\), since \(g\) commutes with \(f\). Hence \(g\varphi = \pm \varphi\) and the equation \(f = g^2\) is impossible.

Let us check (ii). Consider a natural homomorphism \(\hat{\cdot}\) from \(A\) onto \(A/2A\), the quotient group of \(A\) by the subgroup of even elements; let \(\hat{\cdot}\) denote also the corresponding induced homomorphism \(\text{Aut}(A) \to \text{Aut}(A/2A)\). Take an automorphism \(s \in \text{Aut}(A)\) of order three. We claim that the image of \(s\) under the homomorphism \(\hat{\cdot}\) is non-trivial. This will imply that a product of any two conjugates of \(\varphi\), an element of \(K^2(\rho)\), cannot have order three, since the image of \(\varphi\) in \(\text{Aut}(A/2A)\) under \(\hat{\cdot}\) is trivial.

The kernel of the endomorphism \(s^2 + s + \text{id}\) of the group \(A\) has non-trivial elements modulo \(2A\) (it contains, in particular, the subgroup \((s - \text{id})A\)). Let \(a\) be such an element. Suppose that \(\hat{s} = \text{id}\). Hence we have
\[
0 = \hat{s}^2\hat{a} + \hat{s}\hat{a} + \hat{a} = 3\hat{a} = \hat{a},
\]
a contradiction.

To check the last condition (iii) we use Lemma 1.3. The image of \(\varphi\) in \(\text{Aut}(A)\) is a 1-involution (see Section 1 for definitions), and according to Lemma 1.3 a product of any two conjugate and commuting 1-involutions is a 2-involution whose fixed point-subgroup has rank equal to rank \(A\) (recall that rank \(A\) is an infinite cardinal). Thus, any two involutions from \(K^2(\rho)\) are conjugate.

Conversely, suppose that \(\varphi\), where \(\varphi\) is an involution from \(\text{Aut}(N)\), satisfies the conditions (i-iii). The condition (ii) imply that \(f = \varphi\) is diagonalizable. Indeed, suppose, towards a contradiction, that there is a basis \(B\) of \(A\) such that \(fB \subseteq \pm B\) and there exist two elements \(u, v\) of \(B\) taken by \(f\) to one another (Theorem 1.4). It is easy then to find conjugates \(f', f''\) of \(f\) such that \(f'f''\) has order three: we require that \(f'b = f''b = fb\) for each \(b \in B \setminus \{u, v\}\) and
\[
\begin{align*}
\begin{cases}
f'u = -v, \\
f'v = -u,
\end{cases} & \quad \begin{cases}
f''u = u + v, \\
f''v = -v.
\end{cases}
\end{align*}
\]
Thus, we have that $f$ is a diagonalizable, and hence a $\infty$-involution for some cardinal $\kappa$. If $\kappa > 1$, then one finds in $K^2(f)$ not only $2$-involutions, but also, for instance, $4$-involutions and (iii) fails.

To complete the proof we have to show that a $1$-involution $g$ such that rank $A^*_g =$ rank $A$ is a square in Aut($A$). Assume $\{b\} \cup C$ is a basis of $A$ such that $gb = b$ and $gc = −c$ for each $c \in C$. Then we construct $h \in$ Aut($A$) whose square is $g$ putting $h b = b$ and having in mind a well-known relation

$$
\begin{pmatrix}
0 & −1 \\
1 & 0
\end{pmatrix}^2 = \begin{pmatrix}
−1 & 0 \\
0 & −1
\end{pmatrix},
$$

in order to define the action of $h$ on $C$. This completes the proof of the Proposition.

We close this section with a simple result on the conjugation action of extremal involutions on IA($N$).

**Lemma 2.5.** Let $\varphi$ be an extremal involution and $B = \{x\} \cup Y$ a canonical basis for $\varphi$ (that is $\varphi x = x^{-1}$ and $\varphi y = y$ for each $y \in Y$). Suppose that $\alpha$ is an IA-automorphism. Then

(a) $\varphi \alpha \varphi = \alpha$ if and only if $\alpha$ moves the elements of $B$ as follows:

$$
\begin{aligned}
\alpha x &= x[x, t], \\
\alpha y &= yd_y, & y &\in Y,
\end{aligned}
$$

where $t$ is an element of the subgroup $\langle Y \rangle$ generated by $Y$ and $d_y \in \langle Y \rangle'$ for each $y \in Y$;

(b) $\varphi \alpha \varphi = \alpha^{-1}$ if and only if $\alpha$ moves the elements of $B$ as follows:

$$
\begin{aligned}
\alpha x &= xc, \\
\alpha y &= y[x, a_y], & y &\in Y,
\end{aligned}
$$

where $c \in \langle Y \rangle'$ and $a_y \in \langle Y \rangle$ for each $y \in Y$.

**Proof.** The important point is that the restriction of $\varphi$ on $N'$ is a diagonalizable involution, and $[x, N]$ and $\langle Y \rangle'$ are its $(-)$- and $(+)$-subgroup, respectively. Really, $\varphi$ fixes or inverts natural generators of $N'$, basis commutators $[u, v]$, where $u, v$ are distinct elements of $B$. Clearly $\varphi[u, v] = [u, v]$ if neither $u = x$, nor $v = x$. On the other hand, if $v \neq x$, then $\varphi[x, v] = [x, v]^{-1}$.

Let us prove now, for example, (a). Assume that $\alpha x = xd$, where $d \in N'$. We have

$$
\varphi \alpha \varphi x = \alpha x \iff x \varphi(d^{-1}) = xd \iff \varphi d = d^{-1} \iff d \in [x, N].
$$

Similarly, if $\alpha y = yd_y$, then

$$
\varphi \alpha \varphi y = \alpha y \iff \varphi d_y = d_y \iff d_y \in \langle Y \rangle',
$$

as required.

**Remark.** Our starting point, the proposition stating that IA($N$) is a characteristic subgroup of Aut($N$), is the same as in the paper [4] by Dyer and Formanek. They prove (the proof of Lemma 3 on page 273) that IA($N_k$) is the Hirsch-Plotkin radical of Aut($N_k$), that is the maximal locally nilpotent subgroup of Aut($N_k$), where $N_k$ is a $k$-generator ($k < \infty$) free two-step nilpotent group; therefore IA($N_k$) is characteristic in Aut($N_k$). On the same page of [4] one finds a statement on the conjugation action of a symmetry on IA($N_k$) and the proof that Aut($N_k$) is centreless. These facts correspond to Lemma 2.1 (a) and Proposition respectively.
3. Characterizing conjugations and symmetries

We begin with the proof of definability of conjugations by powers of primitive elements of \( N \) in \( \text{Aut}(N) \) (recall that an element of the group \( N \) is said to be primitive if it is a member of some basis of this group). The mentioned conjugations generate the subgroup \( \text{Inn}(N) \), and hence the latter is characteristic in \( \text{Aut}(N) \). The rest of the section is devoted to a characterization of symmetries and consideration of the ways in which the basis sets of \( N \) and the primitive elements of \( N \) can be modelled in \( \text{Aut}(N) \).

**Lemma 3.1.** An automorphism \( \alpha \in \text{IA}(N) \) is conjugation by a power of a primitive element of \( N \) if and only if there is an involution \( \varphi \) extremal modulo \( \text{IA}(N) \) such that

(i) \( \varphi \alpha \varphi = \alpha^{-1} \);

(ii) if \( \varphi \) and \( \psi \), where \( \psi \) is an extremal involution modulo \( \text{IA}(N) \), are distinct and commuting modulo \( \text{IA}(N) \), then \( \psi \) commutes with \( \alpha \).

The family of conjugations by powers of primitive elements is definable in \( \text{Aut}(N) \).

**Proof.** If \( \tau \) is conjugation by, say, an \( m \)th power a given primitive element \( x \in N \), then one easily finds an extremal involution \( \varphi \) which inverts \( x \), whence \( \varphi \tau \varphi = \tau^{-1} \).

If further \( \psi \) is an extremal involution modulo \( \text{IA}(N) \) satisfying the conditions of the Lemma, then by Lemma 1.3 there exist a basis of the free abelian group \( B \) in which both \( \varphi \) and \( \psi \) are diagonalizable. Since \( \varphi \neq \psi \), then \( \psi \tau \psi = \tau \).

Let us prove the converse. The conditions (i) and (ii) deal with the conjugation action on \( \text{IA}(N) \) and the commutativity modulo \( \text{IA}(N) \); this then allow us to assume that \( \varphi \) is an extremal involution. Let \( B = \{ x \} \cup Y \) be a canonical basis of \( N \) for \( \varphi \). According to Lemma 2.3 an IA-automorphism \( \alpha \) with (i) moves the elements of \( B \) as follows:

\[
\alpha x = xc_x, \\
\alpha y = y[x, a_y], \quad y \in Y,
\]

where \( c_x \in \langle Y \rangle' \) and \( a_y \in \langle Y \rangle \) for each \( y \in Y \).

Any extremal involution \( \varphi_y \) which inverts \( y \in Y \) and whose action on \( B \) is canonical commutes with \( \varphi \). Therefore if (ii) holds, then, again by Lemma 2.3

\[ \varphi_y c_x = c_x \quad \text{and} \quad \varphi_y[x, a_y] = [x, a_y]^{-1} \]

for each \( y \in Y \). As for \( c_x \), we have that

\[ c_x \in \bigcap_{y \in Y} \langle B \setminus \{ y \} \rangle', \]

and this, along with \( c_x \in \langle B \setminus \{ x \} \rangle' \), implies that \( c_x = 1 \). For every \( y \in Y \) the commutator \( [x, a_y] \) must be equal to \( [y, b_y] \) for some \( b_y \) in \( \langle B \setminus \{ y \} \rangle \): \([x, a_y] = [y, b_y]\).

It follows that \([x, a_y] = [x, y]^{k_y} \) for a suitable integer \( k_y \in Z \). So the action of \( \alpha \) on \( B \) looks like

\[
\alpha x = x, \\
\alpha y = x^{k_y}y^{k_y}x^{-k_y}.
\]

To prove that \( \alpha \) is conjugation we have to prove that the integers \( k_y \) are the same.

Choose an element \( z \in Y \) and consider a basis

\[ B' = \{ x \} \cup \{ z \} \cup \{ yz : y \in Y, y \neq z \} \]

of the group \( N \). The action of \( \varphi \) on \( B' \) is also canonical and by applying the above arguments one can deduce that

\[ \alpha(yz) = x^{m_y}(yz)x^{-m_y} \]
for each \( y \in Y \setminus \{ z \} \). Thus, for every \( y \in Y \setminus \{ z \} \) we have \( m_y = k_y = k_z, \) and \( \alpha \) is conjugation by a power of \( x, \) as desired. \( \square \)

**Corollary 3.2.** The subgroup of all conjugations \( \text{Inn}(N) \) is definable in \( \text{Aut}(N) \).

**Proof.** Every element of an infinitely generated free abelian group (in particular, every element of \( \text{Inn}(N) \)) can be written as a product of two unimodular (primitive) elements.

**Remark.** It can been seen quite easily that every element of a free abelian group is actually a product of at most three unimodular elements \([14]\).

The fact that a set \( B \) is a basis of a free abelian group \( \langle G, + \rangle \) can be easily expressed by a formula of monadic second-order logic. This formula may be chosen as a ‘translation’ of the following statement: a subset \( B \) of \( G \) is a basis of \( G \) if and only if

\[
G = \langle b \rangle \oplus (B \setminus \{ b \})
\]

for each \( b \in B \).

We shall call a basis \( B \) of the free abelian group \( \text{Inn}(N) \) a basis set of conjugations.

By Theorem [13] there is a basis \( B \) of \( N \) such that conjugations in \( B \) are determined by the elements of \( B: B = \{ \tau_b : b \in B \} \). Consider a symmetry \( \theta^* \) which inverts all elements of \( B \) and let \( N(B) \) denote the normalizer of \( B \) in \( \text{Aut}(N) \).

**Lemma 3.3.** Let \( \theta \) be a symmetry modulo \( \text{IA}(N) \). Then the following statements are equivalent:

1. \( \theta \) has the form \( \theta^* \alpha^2 \), where \( \alpha \in \text{IA}(N) \);
2. \( \theta \) commutes modulo the subgroup \( \text{IA}^2(N) \) with each element of \( N(B) \), where \( \text{IA}^2(N) \) is the subgroup of \( \text{IA}(N) \) generated by squares of the elements of \( \text{IA}(N) \).

One immediately deduces from the Lemma that

**Proposition 3.4.** There is a monadic second-order formula which is satisfied in \( \text{Aut}(N) \) exactly by symmetries.

**Proof.** Any involution of the form \( \theta^* \alpha^2 \) is a symmetry, namely a conjugate of the symmetry \( \theta^* \): \( \theta^* \alpha^2 = \alpha^{-1} \theta^* \alpha \) (Lemma 2.1 (a)). \( \square \)

**Remark.** We use an idea Dyer and Formanek use in [4] in order to identify the image of a chosen symmetry under an automorphism of \( \text{Aut}(N_k) \), where \( N_k \) is a free two-step nilpotent of finite rank \( k \). In the case of infinite rank this idea can be applied, however, with an optimal effect, since one finds among the realizations in \( \text{Aut}(N_k) \) of the condition (ii) from Lemma 3.3 involutions which are not necessarily symmetries (for example, if \( \{ x_1, \ldots, x_k \} \) is a basis of \( N_k \) corresponding to our basis \( B \), then an involution \( \theta^* \tau_{x_1} \ldots \tau_{x_n} \) is such a realization).

**Proof of Lemma 3.3** Every element in the normalizer of \( B \) in \( \text{Aut}(N) \) can be written in the form \( \pi \beta \), where \( \pi \) acts on the basis \( B \) as a permutation (and hence commutes with \( \theta^* \)) and \( \beta \in \text{IA}(N) \). Therefore

\[
\pi \beta(\theta^* \alpha^2)\beta^{-1} \pi^{-1} = \pi \theta^* (\alpha \beta^{-1})^2 \pi^{-1} = \theta^* \pi (\alpha \beta^{-1})^2 \pi^{-1} \equiv \theta^* \alpha^2 (\mod \text{IA}(N)).
\]

Conversely, preserving notation we have just introduced, suppose that for any \( \pi \) and \( \beta \)

\[
\pi \beta(\theta^* \gamma)\beta^{-1} \pi^{-1} \equiv \theta^* \gamma (\mod \text{IA}(N)),
\]

where \( \gamma \in \text{IA}(N) \). It then follows that for any \( \pi \)

\[
(2) \quad \pi \gamma \pi^{-1} \equiv \gamma (\mod \text{IA}(N)).
\]
We claim that $\gamma$ is a square in $IA(N)$.

The assumption of the existence of $b \in B$ such that $\gamma b = bd_2^2$, where $d_2 \in N'$ trivially guarantees the conclusion. Really, let $t \in B \setminus \{b\}$ and $\gamma t = tc_t$, where $\gamma \in N'$. Taking $\pi$ such that $\pi b = t$ we have

$$\gamma t \equiv \pi \gamma \pi^{-1} t \pmod{N'^2} \Rightarrow tc_t \equiv t \pi (d_2^2) \pmod{N'^2} \Rightarrow c_t \equiv 1 \pmod{N'^2}.$$ 

Therefore, we assume that for each $b \in B$, $\gamma b = b c_b$, where $c_b$ is an element of $N'$ which is not a square in $N'$. Take an arbitrary $b \in B$ and suppose that the word $c_b$ (in the letters $B$) has non-trivial occurrences of an element $a \in B$:

$$c_b = [b, a]^k [b, u_a][a, v_b]d_b,$$

where the words $u_b, v_b,$ and $d_b \in N'$ contain no occurrences of both $a$ and $b$, and $v_b \notin N'$ or $k \neq 0$. Write in analogous way the action of $\gamma$ on the element $a$:

$$\gamma a = a[a, b]^m [a, u_a][b, v_b]d_a.$$ 

Assume that the elements $u_a, v_a, d_a, u_b, v_b, d_b$ are the words in the letters $b_1, \ldots, b_n \in B$. Since $B$ is infinite, there is an automorphism $\pi \in Aut(N)$ which preserves $B$, takes $a$ and $b$ to each other, and such that

$$\pi \{b_1, \ldots, b_n\} \cap \{b_1, \ldots, b_n\} = \emptyset.$$ 

By (2), $\gamma b \equiv \pi \gamma \pi^{-1} b \pmod{N'^2}$, and then

$$[b, a]^k [b, u_a][a, v_b]d_b \equiv [b, a]^m [b, \pi u_a][a, \pi v_b]d_a \pmod{N'^2}.$$ 

This implies that the elements $u_a, v_a, u_b, v_b$ are all squares modulo $N'$ and $d_a, d_b \in N'^2$. Hence $c_b \equiv [b, a]^k \pmod{N'^2}$.

Consider $t \in B \setminus \{b, a\}$. We have $\gamma t \equiv \ell[t, s]^n \pmod{N'^2}$ for some $s \in B$. To prove that the element $[b, a]^k b$ is a square, one may choose $\pi$ such that $\pi$ takes $t$ to $b$ and fixes all elements in $B \setminus \{b, t\}$ (if $s \neq a$), or $\pi$ which acts on $\{a, b, t\}$ as a cycle $(b, a, t)$ if $s = a$.

The proof of Lemma 3.3 is now completed. □

In view of Lemma 3.3 a symmetry $\theta$ satisfying the condition (ii) from this Lemma for a given basis set of conjugations $B$ will be called attached to $B$. One can then associate with a pair $(B, \theta)$ a basis $C = B(B, \theta)$ of $N$ uniquely determined by the following conditions:

(i) $B = \{\tau z : z \in C\}$;

(ii) $\theta$ inverts all elements of $C$ (that is, acts on $C$ canonically in terms introduced in Section 2).

Indeed, if $B$ is a basis of $N$ which satisfies (i) and $\theta_B$ is the symmetry whose action on $B$ is canonical, then by Lemma 3.3 $\theta = \alpha^{-1} \theta_B \alpha$ for some $IA$-automorphism $\alpha$. Hence $\theta$ inverts all elements of the basis $\alpha^{-1} B$ of $N$, and, moreover, this basis satisfies (i). Finally, if $x$ is a primitive element of $N$ (in particular, an element of $B$) and $\theta(xc) = x^{-1} c^{-1}$ for some $c \in N'$, then $xc$ is the only element of $x N'$ taken by $\theta$ to the inverse: assuming $\theta(xd) = x^{-1} d^{-1}$, where $d \in N'$, we have that

$$\theta(xd) = \theta(x) \theta(d) = x^{-1} c^{-1} d;$$

therefore $c^{-2} = d^{-2}$, or $c = d$.

In a more general setting any triplet $(\tau, B, \theta)$, where $B$ is a basis set of conjugations, $\theta$ a symmetry attached to $B$ and $\tau \in B$, codes a primitive element of $N$: we assign to each such a triplet a unique element $x$ of the basis $B(B, \theta)$ such that $\tau = \tau_x$, or, equivalently a unique element in $yN'$, where $y$ is any primitive such that $\tau = \tau_y$, taken by $\theta$ to the inverse.

Two triplets $(\tau, B, \theta)$ and $(\tau', B', \theta')$ will code the same primitive element of $N$ if and only if the following conditions hold

(PE1) $\tau = \tau'$ (and hence there is a primitive $x \in N$ such that $\tau = \tau = \tau_x$);

(PE2) $\theta$ and $\theta'$ invert the same element in $x N'$. 

It is easy to see that if an IA-automorphism $\alpha$ preserves a primitive $y \in N$, then it preserves all elements in $yN'$. Then the condition (PE2) is equivalent to the condition which states that the IA-automorphism $\theta \alpha$ fixes all elements in $xN'$ (this will imply that $\theta x = \theta' x$, and hence (PE2) will hold).

Let $\tau$ be conjugation by a primitive element, and let $\text{IA}_\tau(N)$ denote the subgroup of all IA-automorphisms which fix any $z \in N$ such that $\tau = \tau_z$. Thus, in order to prove that the condition (PE2) can be expressed in $\text{Aut}(N)$ by means of group theory it suffices to obtain a characterization of the subgroups $\text{IA}_\tau(N)$. This is the main subject of the next section.

4. IA-stabilizers

**Theorem 4.1.** Let $B$ be a basis set of conjugations and $\theta$ a symmetry attached to $B$. Then for each $\tau \in B$ the subgroup $\text{IA}_\tau(N)$ is definable with the parameters $\tau, \theta, B$ in $\text{Aut}(N)$ by means of monadic second-order logic.

**Remark.** In fact $\text{IA}_\tau$, where $\tau$ is conjugation by a primitive element, is definable in $\text{Aut}(N)$ by means of monadic second-order logic only with the parameter $\tau$. The use of other parameters in the Theorem is a little more convenient for the proofs in the next section.

**Proof.** As we saw in the previous section there is a unique basis $\mathcal{B} = \mathcal{B}(B, \theta)$ of $N$ such that the set of conjugations by elements of $\mathcal{B}$ is $B$ and $\theta$ inverts all elements of $\mathcal{B}$. Assume $x$ is an element of $\mathcal{B}$ such that $\tau = \tau_x$ and write $\mathcal{B}$ in the form $\{x\} \cup Y$.

In order to prove that $\text{IA}_\tau(N)$ is definable by means of monadic second-order logic, we shall define by means of monadic second-order logic the sets of IA-automorphisms $\text{IA}_\tau^+(N)$ and $\text{IA}_\tau^-(N)$. Here $\text{IA}_\tau^+(N)$ denotes the set of IA-automorphisms of the form

$$\alpha x = x,$$
$$\alpha y = y d_y, \quad y \in Y,$$

where $d_y \in \langle Y \rangle^\prime$, and $\text{IA}_\tau^-(N)$ the set of IA-automorphisms of the form

$$\alpha x = x,$$
$$\alpha y = y[x, a_y], \quad y \in Y,$$

where $a_y \in \langle Y \rangle$.

Clearly, $\text{IA}_\tau(N)$ is a direct product of $\text{IA}_\tau^+(N)$ and $\text{IA}_\tau^-(N)$.

We shall call extremal involutions whose action on $\mathcal{B}$ is canonical basis extremal involutions; a basis extremal involution taken an element $b$ from $\mathcal{B}$ to the inverse will be denoted by $\varphi_b$. The automorphisms of $N$ which act on $\mathcal{B}$ as permutations will be called basis permutations. By Lemma 2.7 (b) $\varphi$ is a basis extremal involution if and only if $\varphi$ is extremal modulo $\text{IA}(N)$, commutes with $\theta$ and $\varphi B \varphi \subseteq B^{\pm 1}$. Similarly, the basis permutations are those elements in the normalizer of $B$ which commute with $\theta$.

**1. Characterization of $\text{IA}_\tau^+$.** The natural superset of $\text{IA}_\tau^+$ is the set $C$ of IA-automorphisms in the centralizer of the basis extremal involution $\varphi_x$. By Lemma 2.5 if an IA-automorphism $\gamma$ commutes with $\varphi_x$, then $\gamma$ acts on $\mathcal{B} = \{x\} \cup Y$ as follows

$$(3) \quad \gamma x = x[t, x],$$
$$\gamma y = y d_y, \quad y \in Y,$$

where $t$ is an element of $\langle Y \rangle$ and $d_y \in \langle Y \rangle^\prime$ for each $y \in Y$; it is easily seen that every basis extremal involution normalizes $C$.

Thus, we have to choose those automorphisms $\gamma$ with (3) that have $d(\gamma) = [t, x]$ equal to 1.
Let $b$ be an element of $Y = \mathcal{B} \setminus \{x\}$ and $\varphi_b$ the corresponding basis extremal involution. For an IA-automorphism $\gamma_b = \sqrt{\varphi_b\gamma^{-1}\varphi_b\gamma}$ obtained from an element $\gamma \in C$, the word $d(\gamma_b)$ has as a word in the letters $\mathcal{B}$ only occurrences of $x$ and $b$. Really, write the action of $\gamma$ on $\mathcal{B}$ in the following form
\[
\gamma x = x[b^ku, x],
\gamma y = y[b, e_y]f_y, \quad y \in Y \setminus \{b\},
\gamma b = b[b, e_b]f_b,
\]
where $u \in \langle Y \setminus \{b\}\rangle$, $e_y, e_b \in \langle Y \setminus \{b\}\rangle$ and $f_y, f_b$ are in $\langle Y \setminus \{b\}\rangle'$. One easily verifies that
\[
\begin{align*}
\varphi_b\gamma^{-1}\varphi_b\gamma x &= x[b^k, x]^2, \\
\varphi_b\gamma^{-1}\varphi_b\gamma y &= y[b, e_y]^2, \quad y \in Y \setminus \{b\}, \\
\varphi_b\gamma^{-1}\varphi_b\gamma b &= bf_b^2. 
\end{align*}
\]
We can therefore conclude that $\gamma \in C$ preserves $x$ if and only if for each $b \in Y$ so does $\gamma_b$.

Let us now fix $b$. The automorphism $\gamma_b$ is an element of the set
\[
D = D(b) = \{\delta \in C : \varphi_b\delta\varphi_b = \delta^{-1}\}
\]
(according to Lemma 2.5 $D$ is equal to the set of square roots of the elements of the form (4)). Then in order to explain when $\gamma_b$ preserves $x$, it suffices to obtain a general characterization of the subgroup of all elements of $D$ which preserve $x$.

A ‘transvection’ $U_b \in \text{Aut}(N)$ such that
\[
\begin{align*}
U_b x &= xb, \\
U_b y &= y, \quad y \in Y
\end{align*}
\]
can be modulo IA$(N)$ characterized as one of the automorphisms $U \in \text{Aut}(N)$ with
\[
U\tau U^{-1} = \tau\tau_b, \\
U\nu U^{-1} = \nu, \quad \nu \in B \setminus \{\tau\}.
\]
That is all we need at the moment, since we are going to act by $U_b$ on a set of IA-automorphisms (namely on $D$) by conjugation:
\[
(U_b\beta)\alpha(U_b\beta)^{-1} = U_b\alpha U_b^{-1}
\]
for any $\alpha, \beta \in \text{IA}(N)$.

We shall use below a family of automorphisms $S = S(b)$ such that any member of $S$ fixes all elements in $\mathcal{B} \setminus \{b\}$ and takes $b$ to an element of the form $bf$, where $f \in \langle Y \setminus \{b\}\rangle'$.

Let us obtain a description of the members of $S$. Suppose $\delta$ is an automorphism from $D$ such that
\[
\begin{align*}
\delta x &= x[b^k, x], \\
\delta b &= bf,
\end{align*}
\]
where $f \in \langle Y \setminus \{b\}\rangle'$. One then readily verifies that the automorphism $\delta^* = U_b\delta^{-1}U_b^{-1}\delta$ takes $x$ to $x\delta$ and fixes all elements of $Y$. Therefore $\pi_{x, b}\delta^*\pi_{x, b}$, where $\pi_{x, b}$ is the basis permutation taking $x$ and $b$ to each other and preserving all other elements of $\mathcal{B}$, is in $S$.

We claim now that the subgroup $D_0$ of the elements of $D$ which stabilize $x$ can be described as a unique subgroup $E$ of $D$ satisfying the following conditions (a-c):

(a) $D$ is a direct product of $E$ and the subgroup generated by conjugation by the basis element $b$: $D = E \times \langle \tau_b \rangle$;
(b) any basis permutation fixing both $x$ and $b$ (or equivalently commuting with both $\tau_x$ and $\tau_b$) normalizes $E$;
(c) \( E \supseteq S \).

It is quite clear that \( D_0 \) satisfies the conditions (a-c). Let us prove the converse. Assume the conditions (a-c) hold for a subgroup \( E \) of \( D \).

It can be deduced from (a) that for any \( \delta \in D_0 \) there is an integer \( k \in \mathbb{Z} \) such that

\[
\tau_b^k \delta \in E.
\]

To apply (b) we will prove

**Lemma.** For any automorphism \( \delta \in D_0 \) there exist \( \mu \in D_0 \), a basis permutation \( \pi \) which fixes \( x \) and \( b \), and \( \sigma \in S \) such that

\[
\delta = \mu^\pi \mu^{-1} \sigma
\]

(\( \mu^\pi \) denotes \( \pi \mu \pi^{-1} \)).

This will imply that \( \delta \in E \) and hence \( D_0 \subseteq E \); therefore \( D_0 = E \), since \( D = D_0 \times \langle \tau_b \rangle \). Indeed, as we noted above there is \( m \in \mathbb{Z} \) such that \( \tau^m_b \mu \in E \).

Using (c), we obtain that \( \delta = \mu^\pi \mu^{-1} \sigma \) is in \( E \).

Let us prove the latter Lemma. Write the (infinite) set \( Y \setminus \{b\} \) in the form

\[
Y \setminus \{b\} = \{y_{i,k} : i \in I, k \in \mathbb{Z}\},
\]

where the cardinality of \( I \) is equal to rank \( N \).

Suppose that

\[
\delta y_{i,k} = y_{i,k}[b, z_{i,k}], \quad i \in I, \quad k \in \mathbb{Z},
\]

where \( z_{i,k} \in \langle Y \setminus \{b\} \rangle \), and consider a basis permutation \( \pi \) which fixes \( x \) and \( b \), and which acts on \( Y \setminus \{b\} \) as a permutation with infinite cycles:

\[
\pi y_{i,k} = y_{i+1,k}
\]

for every \( i \in I \) and \( k \in \mathbb{Z} \).

Therefore if \( \mu \in D_0 \) is defined as follows

\[
\mu y_{i,k} = y_{i,k}[b, t_{i,k}], \\
\mu b = b,
\]

then the equation \( \delta = \mu^\pi \mu^{-1} \sigma \) holds for a suitable \( \sigma \in S \) if, for instance,

\[
\pi t_{i,k-1} = z_{i,k} \quad \forall i \in I \quad \forall k \in \mathbb{Z}.
\]

An easy way to satisfy the latter equations is to set all the elements of the form \( t_{i,0} \) equal, say, to 1, and to define other elements \( t_{i,k} \) (for any given \( i \)) by induction: to define the elements \( d \) with positive second indices one can use the formula

\[
t_{i,k} = z_{i,k}^{-1} \pi t_{i,k-1}, \quad k \geq 1
\]

and for \( d \) with negative second indices the formula

\[
t_{i,k-1} = \pi^{-1}(z_{i,k} t_{i,k}), \quad k \leq 0.
\]

**II. Characterization of \( IA^-_\tau \).** As above we start with a natural superset of \( IA^-_\tau \) related to the involution \( \varphi_x \): the set of IA-automorphisms

\[
L = \{ \lambda \in \text{IA}(N) : \varphi_x \lambda \varphi_x = \lambda^{-1} \}.
\]

By Lemma 2.5 a member \( \lambda \) of \( L \) acts on the basis \( B \) as follows:

\[
\lambda x = xc, \\
\lambda y = y[x, a_y], \quad y \in Y,
\]

where \( c \in \langle Y' \rangle \) and \( a_y \in \langle Y \rangle \). This time the problem of determining whether \( c = c(\lambda) \) in (5) is equal to 1 is solved quite easily. It turns out that \( IA^-_\tau \) is a subset
of the elements of \( L \) such that conjugation by one more involution \( \psi \) take them to the inverse:

\[
\tilde{IA} = \{ \lambda \in L : \psi \lambda \psi = \lambda^{-1} \}.
\]

This \( \psi \) can be modulo \( IA(N) \) characterized as a basis permutation with respect to a basis \( \{ b, xb \} \cup (Y \setminus \{ b \}) \) of \( N \), where \( b \in Y \):

- \( \psi^2 = \text{id} \);
- \( \psi \tau_b \psi = \tau_x \tau_b \) (this easily implies that \( \psi \tau_x \psi = \tau_x^{-1} \));
- \( \psi \tau_y \psi = \tau_y \) for each \( y \in Y \setminus \{ b \} \).

As usual without loss of generality we may suppose that our \( \psi \) acts on the basis \( B \) such that it moves \( x \) to the inverse, \( b \) to \( xb \) and preserves all elements in \( Y \setminus \{ b \} \). Let us now prove (6). The involution \( \psi \) inverts any commutator of the form \([x, a]\), where \( a \in \langle Y \rangle \). This easily implies that \( \psi \lambda \psi = \lambda^{-1} \) for each \( \lambda \in IA_B^- \). Conversely, choose \( \lambda \in L \) and suppose that \( \psi \lambda \psi = \lambda^{-1} \). Compare the action of both automorphisms \( \psi \lambda \psi \) and \( \lambda^{-1} \) on the element \( b \). We have

\[
\psi \lambda \psi b = \psi \lambda(xb) = \psi(xc(\lambda)b[x, a_b]) = x^{-1} \psi(x(\lambda) xb[x, a_b]^{-1} = b \psi c(\lambda) [x, a_b]^{-1};
\]

the latter element must be equal to \( \lambda^{-1} b \), that is, to \( b[x, a_b]^{-1} \). Hence \( c(\lambda) = 1 \), \( \lambda \in IA_B^- \), and the theorem is proved. \( \square \)

5. It fixes all conjugations and some symmetry

Suppose \( \Delta \) is an automorphism of the group \( Aut(N) \). Consider a basis set \( B \) of conjugations. Since the property of being a basis set of conjugations can be expressed by means of monadic second-order logic, \( \Delta \) takes \( B \) to other basis set of conjugations \( B' \). On the other hand, there is an automorphism \( \sigma \in Aut(N) \) such that \( \sigma B = B' \). Thus, if we will follow \( \Delta \) by conjugation by \( \sigma^{-1} \) we obtain an automorphism of the group \( Aut(N) \) which fixes all members of the set \( B \), and therefore all conjugations from the subgroup \( Inn(N) \).

This automorphism, say, \( \Delta_1 \), preserves surely the normalizer of \( B \). Therefore if \( \theta \) is a symmetry attached to \( B \), then the image of \( \theta \) under \( \Delta_1 \) is a conjugate of \( \theta \) by an \( IA \)-automorphism \( \beta \) (Lemma 5.2). Follow then \( \Delta_1 \) by conjugation by \( \beta^{-1} \) we get an automorphism of the group \( Aut(N) \) which fixes all conjugations and the chosen symmetry \( \theta \).

Let us denote the latter automorphism again by \( \Delta \). We are going then to prove that \( \Delta \) preserves all \( IA \)-automorphism (Proposition 5.3 below). This will enable us to prove the main result of the paper.

**Theorem 5.1.** The automorphism group of an infinitely generated free two-step nilpotent group is complete.

**Proof (assuming Proposition 5.3).** We saw above that the group \( Aut(N) \) is centreless (Proposition 2.3); we prove now that \( \Delta \) acts trivially on \( Aut(N) \).

Let \( \theta' \) be an arbitrary symmetry in \( Aut(N) \). Then the automorphism \( \Delta \) must preserve \( \theta' \), because the product \( \theta' \theta \) is an \( IA \)-automorphism, it must preserve by Proposition 5.3 and because it must preserve \( \theta \) by the construction:

\[
\theta' \theta = \Delta(\theta' \theta) = \Delta(\theta') \Delta(\theta) = \Delta(\theta') \theta \Rightarrow \Delta(\theta') = \theta'.
\]

**Lemma 5.2.** If an automorphism of the group \( Aut(N) \) preserves all conjugations, then it preserves all elements of \( Aut(N) \) modulo the subgroup \( IA(N) \).

**Proof.** Let \( \Gamma \) be an automorphism of \( Aut(N) \) preserving all conjugations. Let further \( \sigma \) be an automorphism of \( N \) and \( z \) an arbitrary element of \( N \). We then have

\[
\tau_{\sigma z} = \Gamma(\tau_{\sigma z}) = \Gamma(\sigma \tau_z \sigma^{-1}) = \Gamma(\sigma) \tau_z \Gamma(\sigma^{-1}) = \tau_{\Gamma(\sigma) z}.
\]

\( \square \)
Take an arbitrary element $\sigma$ in $\text{Aut}(N)$. The automorphism $\sigma \theta \sigma^{-1}$ is a symmetry and we have that
\begin{equation}
\Delta(\sigma \theta \sigma^{-1}) = \sigma \theta \sigma^{-1}.
\end{equation}
On the other hand, by Lemma 5.2 \(\Delta(\sigma) = \sigma \eta_{\sigma}\), where $\eta_{\sigma} \in \text{IA}(N)$ and hence
\begin{equation}
\Delta(\sigma \theta \sigma^{-1}) = \sigma \eta_{\sigma} \theta \sigma_{\eta_{\sigma}}^{-1} \sigma^{-1} = \sigma \eta_{\sigma}^{-2} \sigma^{-1}.
\end{equation}
This along with (7) implies that $\eta_{\sigma} = \text{id}$. The latter means that $\Delta$ stabilizes all elements of the group $\text{Aut}(N)$, and therefore any automorphism of the group $\text{Aut}(N)$ is inner. \(\Box\)

**Proposition 5.3.** The automorphism $\Delta$ fixes all $\text{IA}$-automorphisms.

**Proof.** First note that each subgroup $\text{IA}_x$, where $x$ is in $B$, the basis set of conjugations chosen above, is invariant under the automorphism $\Delta$, since $\text{IA}_x(N)$ is definable in $\text{Aut}(N)$ with the parameters $\tau, \theta$ and $B$ by means of monadic second-order logic (Theorem 4.1) and $\Delta$ preserves each of the parameters.

Let $B = B(B, \theta)$ denote the basis of $N$ uniquely determined by a pair $(B, \theta)$ (see Section 3). Take $\tau \in B$ and suppose that $\tau = \tau_b$, where $b \in B$. Let $J_\tau$ denote the subgroup of all $\text{IA}$-automorphisms which fix each element in $B \setminus \{b\}$. Clearly any $\text{IA}$-automorphism $\eta$ is fully determined by its projections on the subgroups $J_\tau$ parallel to $\text{IA}_\tau(N)$: for each $\tau \in B$ there exists a unique $\eta_{\tau} \in J_\tau$ such that
\begin{equation}
\eta_{\tau}^{-1} \in \text{IA}(N).
\end{equation}
Therefore in order to complete the proof of the Proposition it suffices to prove that $\Delta$ stabilizes the elements in all subgroups $J_\tau$. These subgroups are all contained in the finitary automorphism group $\text{Aut}_{f, B}(N)$ of $N$ consisting of all automorphisms of $N$ which fix all but finitely many elements of $B$.

The automorphism group of a finitely generated free two-step nilpotent group $N_k$ with a basis $\mathcal{X} = \{x_1, x_2, \ldots, x_k\}$ can be generated by a set consisting of one basis extremal involution, two basis permutations (associated with $\mathcal{X}$) and an automorphism $U$ of $N_k$ such that $U x_i = x_1 x_2$ and $U x_i = x_i$, where $i \geq 2$ (see, e.g., [4] pp. 272-273). This implies that the group $\text{Aut}_{f,B}(N)$ is generated by basis extremal involutions (with respect to the basis $B$), basis permutations acting on $B$ as finite cycles and $U \in \text{Aut}(N)$ such that
\begin{align*}
U x &= x z, \\
U b &= b, \quad \forall b \in B \setminus \{x\}
\end{align*}
where $x$ and $z$ are some distinct elements of $B$.

We claim that $\Delta$ preserves all elements of the subgroup $\text{Aut}_{f,B}(N)$. There are no problems with basis extremal involutions and basis permutations, since they commute with $\theta$: if $\theta \sigma \theta = \sigma$ and $\Delta(\sigma) = \sigma \eta$, where $\eta \in \text{IA}(N)$ (Lemma 5.2), then by applying $\Delta$ to the both sides of the first equation we have that $\eta = \text{id}$.

It is easy to verify that the triplet $(\tau_\sigma, \tau_\theta, B')$, where $B' = \{\tau_\sigma\} \cup \{\tau_\theta \tau : \tau \in B, \tau \neq \tau_\sigma\}$ codes (as it was described in Section 3) the element $xz$ of $N$. Let us denote by $\equiv$ the equivalence relation defined by the conditions (PE1) and (PE2) (both equivalent to group-theoretic ones by Theorem 4.1) from Section 3. Then $U$ is the unique automorphism $\sigma$ of $N$ satisfying the following conditions with the parameters $\theta$ and $B$:
\begin{align*}
(\sigma \tau_\sigma^{-1}, \sigma \theta \sigma^{-1}, \sigma B \sigma^{-1}) &\equiv (\tau_\sigma \tau_\sigma, \tau_\sigma \theta, B'), \\
(\sigma \tau_\sigma^{-1}, \sigma \theta \sigma^{-1}, \sigma B \sigma^{-1}) &\equiv (\tau, \theta, B), \quad \forall \tau \in B \setminus \{\tau_\sigma\},
\end{align*}
and hence $\Delta(U) = U$. \(\Box\)
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