Testing non-unitarity of neutrino mixing matrices at neutrino factories

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In this paper we explore the effect of non-unitary neutrino mixing on neutrino oscillation probabilities both in vacuum and matter. In particular, we consider the $\nu_{\mu} \rightarrow \nu_{e}$ channel and using a Neutrino Factory as the source for $\nu_{\mu}$'s discuss the constraints that can be obtained on the moduli and phases of the parameters characterizing the violation of unitarity. We point out how the new CP violation phases present in the case where the non-unitary mixings give rise to spurious “degenerate” solutions in the parameter space and discuss how the true solutions can be extricated by combining measurements at several baselines.

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I. INTRODUCTION

There is a phenomenal increase in our knowledge of neutrino properties in the past few years coming from neutrino oscillation data from solar, atmospheric, accelerator and reactor neutrino experiments. For three neutrino flavours, there are nine parameters characterizing the light neutrino mass matrix, the three masses, three mixing angles and three CP phases. Neutrino oscillation data determines the best-fit values and the 3σ ranges of the mass squared differences and mixing angles as [1]

- Combined analysis of solar and KamLAND reactor neutrino data gives the best-fit values and 3σ ranges of mass and mixing parameters as $\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = 7.9_{-1.0}^{+1.0} \times 10^{-5}$ eV$^2$ and $\sin^2 \theta_{12} = 0.31_{-0.08}^{+0.09}$. The solar data implies $\Delta m_{21}^2 > 0$.
- Global analysis of atmospheric neutrino data from SuperKamiokande and data from accelerator experiments K2K and MINOS gives $|\Delta m_{31}^2| \equiv |m_3^2 - m_1^2| = 2.5_{-0.7}^{+0.7} \times 10^{-3}$ eV$^2$ and $\sin^2 \theta_{23} = 0.5_{-0.16}^{+0.18}$.
- The value of the third leptonic mixing angle $\theta_{13}$ is not yet known and at present it is bounded to be $\sin^2 \theta_{13} < 0.05$ leaving open the possibility of very small or zero value for this.

This tremendous progress has initiated the precision era of neutrino physics, and experiments are planned and proposed to further increase the precision of the known neutrino parameters and to pin-down the value of the mixing angle $\theta_{13}$ and determine the sign of $\Delta m_{31}^2$ (sign($\Delta m_{31}^2$)) $^1$.

A non-zero value of $\theta_{13}$ is intimately related to the possibility of observation of CP phase in the lepton sector. A large value of $\theta_{13}$ would also enable one to determine the sign[$\Delta m_{31}^2$] through observation of large matter effects for neutrinos propagating through earth $^{2}$ $^{3}$ $^{4}$ $^{5}$ $^{6}$. If $\theta_{13}$ is relatively large, $\sin^2 2\theta_{13} \gtrsim 0.01$, then the answers to these questions may be obtained from superbeam $^{6}$ $^{7}$ $^{8}$ and future atmospheric neutrino experiments $^{2}$ $^{3}$ $^{4}$ $^{5}$ $^{6}$ $^{10}$ $^{11}$ $^{12}$. However, if Nature selects $\theta_{13}$ to be smaller than this, then one has to go to either $\beta$-beam or neutrino factory experiments. The R&D for both are actively pursued $^{13}$ $^{14}$. Future facilities also have the potential to discover new physics $^{13}$ $^{14}$ $^{15}$ $^{16}$ $^{17}$ $^{18}$ $^{19}$ $^{20}$.

The best-fit values of masses and mixing angles quoted above are obtained assuming the neutrino mixing matrix (Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix) to be unitary. However, for models with heavy fermionic fields,

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$^1$ Usually $\Delta m_{31}^2 > 0$ and $m_3^2 \geq \Delta m_{31}^2$ $\Rightarrow$ $m_3^2 \geq \Delta m_{31}^2$ $\Rightarrow$ $m_3^2$ is referred to as normal hierarchy (NH), and $\Delta m_{31}^2 < 0$ and $m_3^2 \simeq |\Delta m_{31}^2| + |\Delta m_{21}^2| > m_1^2 \simeq |\Delta m_{21}^2|$ as inverted hierarchy (IH). The three neutrinos can also be quasi-degenerate with $m_3^2 \simeq m_2^2 \simeq m_1^2 \equiv m_\nu^2 \gg |\Delta m_{31}^2|$ in which there is no hierarchy. However, one can still ask what the sign of $\Delta m_{31}^2$ is.
the deviation of the lepton mixing matrix from unitarity is a generic feature \[21, 22, 23\]. A typical example is the type-I seesaw mechanism \[24, 25, 26, 27, 28\] which provides a natural framework of generating small neutrino masses. This requires introduction of one or more heavy right handed singlet neutrino field(s). Although the full mixing matrix at the high scale is expected to be unitary in these cases, the mixing matrix relevant for low energy phenomenology is not unitary as the production of the heavy particles are kinematically forbidden. However the violation from unitarity in the canonical Type-I seesaw mechanism is found to be very small if the mass scale of the heavy neutrinos are of the order of the GUT scale \( \sim 10^{16} \) GeV and the heavy neutrinos decouple and do not influence the physics at low scale. However non-minimal seesaw models have been constructed with heavy neutrinos of mass \( O(1) \) TeV, invoking symmetry arguments to suppress the seesaw term \[29, 30, 31, 32\]. Such models can give rise to significant light-heavy mixing and deviation from unitarity. The TeV scale seesaw models are interesting as these can have signatures in the Large Hadron Colliders (LHC) in the near future \[33, 34, 35\]. Also successful leptogenesis can be generated if the mixing and deviation from unitarity . The TeV scale seesaw models are interesting as these can have signatures in the Large Hadron Colliders (LHC) in the near future \[33, 34, 35\]. Also successful leptogenesis can be generated if the mixing and deviation from unitarity .

This paper we address this question. The non-unitary nature of the neutrino mixing matrix due to mixing with fields heavier than \( M_Z/2 \) can manifest itself in tree level process like \( \pi \rightarrow \mu \nu, Z \rightarrow \bar{\nu} \nu, W \rightarrow \nu \nu \) or in flavour violating rare charged lepton decays like \( \mu \rightarrow e\gamma \), \( \tau \rightarrow \mu \nu \) etc., which proceed via one-loop processes and hence can be constrained from low energy electroweak data \[21, 22, 40, 41, 42, 43, 44, 45, 46, 47, 48\]. Non-unitarity of neutrino mixing matrices can also affect the neutrino oscillation probabilities \[49, 50, 51, 52, 53\]. In this paper, we concentrate on the effect of non-unitarity on neutrino oscillation probabilities and the possibility of probing this in neutrino factories. We show that the effect of non-unitarity can be more pronounced in the appearance channel than in the survival channel. In particular, we look into the effect of deviation from non-unitarity in the \( \nu_\tau - \nu_\tau \) channel since the present constraint on the non-unitarity parameter in this channel is much weaker than the constraint on the \( \nu_e - \nu_\mu \) channel. We consider \( \nu_\tau \) detectors like OPERA \[55\] or ICARUS \[56\] detectors for CERN Neutrinos to Gran Sasso (CNGS) \( \nu_\mu \rightarrow \nu_\tau \) oscillation search programme and discuss the possibility of constraining the moduli and phases parametrising the unitarity violation. These phases characterizing the non-unitarity constitute a new source for CP violation which can be present even in the limit of \( \theta_{13} \rightarrow 0 \). We also discuss the matter effects in the presence of non-unitarity and show that for a non-unitary mixing matrix, matter effect can manifest itself even in the limit of the third leptonic mixing angle \( \theta_{13} \rightarrow 0 \) and in the One Mass Scale Dominance (OMSD) limit of \( \Delta m_{31}^2/\Delta m_{31}^2 \rightarrow 0 \). There is some overlap of our work with Ref. \[54\] who have also constrained non-unitarity violation using the \( \nu_\mu \rightarrow \nu_\tau \) channel. However, we consider the possibility of combining several baselines, reducing the degeneracy of parameter space. To distinguish the non-unitarity signature with that of non-standard interactions, the combination of the baselines is useful. When two or more observations suggest the same parameter region for scenarios with non-unitary lepton mixing matrix, there can be stronger implications to determine the origin of the signal beyond the standard oscillation scenario.

The plan of the paper is as follows. In the next section discuss the parametrization that we use for non-unitary mixing matrices and present the current constraints on unitarity violation. In section III we give simplified expressions for the oscillation probabilities in vacuum and matter assuming the mixing matrix to be non-unitary. In section IV we discuss the degeneracies in the oscillation probabilities. In section V we give our numerical results on the allowed regions of the parameter space in the model with the non-unitary PMNS matrix. We conclude in section VI.

## II. NON-UNITARY MIXING MATRICES AND CURRENT CONSTRAINTS

Since non-unitarity of mixing matrices is a generic feature of theories with heavy neutrinos we consider a picture with three light and one heavy neutrino. In this case the full \( 4 \times 4 \) mixing matrix is unitary but the \( 3 \times 3 \) light neutrino submatrix is non-unitary. A \( 4 \times 4 \) unitary matrix can be parametrized by 6 angles \( \theta_{12,13,14,23,24,34} \) and three phases \( \delta_{11,24,34} \). If the neutrinos are Majorana in nature then three additional phases can be present. We parametrize the \( 4 \times 4 \) unitary matrix in the usual way in terms of the rotation matrices \( \hat{R}_{ij} \)

\[
\mathcal{U} = \hat{R}_{34} \hat{R}_{24} R_{14} R_{23} \hat{R}_{13} R_{12} P
\]

(1)
where the $R_{ij}$ represent rotations in $ij$ generation space, for instance:

$$\tilde{R}_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} e^{-i\delta_{34}} \\ 0 & 0 & -s_{34} e^{i\delta_{34}} & c_{34} \end{pmatrix} \quad \text{or} \quad R_{14} = \begin{pmatrix} c_{14} & 0 & 0 & s_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14} & 0 & 0 & c_{14} \end{pmatrix},$$

(2)

with the usual notation $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$. The symbol tilde means the mixing matrix including the CP phase. The diagonal matrix $P$ contains the three Majorana phases, which we denote $\alpha, \beta$ and $\gamma$:

$$P = \text{diag} \left(1, e^{-i\alpha/2}, e^{-i(\beta/2 - \delta_{13})}, e^{-i(\gamma/2 - \delta_{34})}\right).$$

(3)

Since the Majorana phases are not important for oscillation studies henceforth we will omit the matrix $P$.

Assuming the mixing of the fourth heavy state to be small, the above equation can be expanded in terms of small parameters $\epsilon_e$, $\epsilon_\mu$ and $\epsilon_\tau$ characterizing the 14, 24 and 34 rotations respectively\(^2\). With this simplification Eq. (3) can be expressed as

$$U = \begin{pmatrix} W & \epsilon_e e^{-i\delta_{24} \epsilon_\mu} \\ e^{i\delta_{34} \epsilon_\tau} & e^{-i\delta_{34} \epsilon_\tau} \end{pmatrix}$$

(4)

where $W$ is the $3 \times 3$ non-unitary mixing matrix. This can be written as

$$W = \begin{pmatrix} U_{e1}(1 - \epsilon_\tau^2/2) & U_{e2}(1 - \epsilon_\tau^2/2) & U_{e3}(1 - \epsilon_\tau^2/2) \\ -e^{-i\delta_{24} \epsilon_\mu} e_{\tau} U_{e1} & -e^{-i\delta_{24} \epsilon_\mu} e_{\tau} U_{e2} & -e^{-i\delta_{24} \epsilon_\mu} e_{\tau} U_{e3} \\ U_{\tau1}(1 - \epsilon_\tau^2/2) & U_{\tau2}(1 - \epsilon_\tau^2/2) & U_{\tau3}(1 - \epsilon_\tau^2/2) \\ -e^{i\delta_{34} \epsilon_\tau} e_{\tau} U_{\mu1} & -e^{i\delta_{34} \epsilon_\tau} e_{\tau} U_{\mu2} & -e^{i\delta_{34} \epsilon_\tau} e_{\tau} U_{\mu3} \end{pmatrix}$$

(5)

were $\phi = \delta_{24} - \delta_{34}$, $U_{\phi k} = -\epsilon_e U_{\phi k} - e^{i\delta_{34}} \epsilon_\tau U_{\mu k} - e^{i\delta_{34}} \epsilon_\tau U_{\tau k}$, and the $3 \times 3$ matrix $U_{\alpha i}$ with $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$ is defined and parameterized as the usual unitary PMNS matrix for three generations.

Bound on the moduli of the unitarity violation parameters can come from electroweak processes and from neutrino oscillations. The bounds obtained from present neutrino oscillation experiments are weaker than those obtained from electroweak decays\(^2\). Constraint on $\sum_{i=1}^{3} |W_{\alpha i}|^2$ comes from rare decays of charged leptons $l_\alpha \to l_\beta \gamma$\(21, 22, 30, 37, 40, 41\) whereas $\sum_{i=1}^{3} |W_{\alpha i}|^2$ can be constrained from processes like $W \to l\nu$, $Z \to l\bar{l}$\(\mu\). Constraints on the diagonal elements of the non-unitary matrix can also come from tests for lepton universality\(21, 22, 40, 41\). At present there is strict constraint on light-heavy mixing in the $e-\mu$ sector coming from non-observation of the decay $\mu \to e\gamma$. For non-unitarity induced through heavy right handed neutrinos the bound quoted in Ref.\(36, 37\) is

$$\sum_{i=1}^{3} W_{\alpha i}^* W_{\alpha i} \equiv \epsilon_e e_\mu \lesssim 1.2 \cdot 10^{-4}$$

(6)

The bound on the $\mu-\tau$ sector is much weaker

$$\sum_{i=1}^{3} W_{\mu i}^* W_{\tau i} \equiv \epsilon_\mu \epsilon_\tau \lesssim 2 \cdot 10^{-2}$$

(7)

The $\epsilon_\alpha$’s are also constrained by electroweak measurements individually as\(34, 48\)

$$\epsilon_e^2 < 0.012, \quad \epsilon_\mu^2 < 0.00096, \quad \epsilon_\tau^2 < 0.016.$$
### III. Calculation of Oscillation Probabilities

#### A. Oscillation Probability in Vacuum

The most general expression of survival/oscillation probability for $\nu_\alpha \rightarrow \nu_\beta$ in vacuum without assuming unitarity of mixing matrices is \[ \left( \sin \phi \right) \]

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \frac{1}{N_\alpha N_\beta} \left\{ \left( \sum_{i=1}^{\text{light}} W_{\beta i} \bar{W}_{\alpha i}^* \right)^2 - 4 \sum_{i<j}^{\text{light}} R_{\alpha \beta}^{ij} \sin^2 \left( \frac{m_2^2 - m_3^2}{2E} L \right) - 2 \sum_{i<j}^{\text{light}} R_{\beta \alpha}^{ij} \sin^2 \left( \frac{m_3^2 - m_2^2}{2E} L \right) \right\}, \quad (9)
\]

where $N_\alpha = \sum_{i=1}^{\text{light}} |W_{\alpha i}|^2$; $R_{\alpha \beta}^{ij} = \Re [W_{\beta i} \bar{W}_{\alpha j}^* W_{\alpha i} W_{\beta j}^*]$; $R_{\beta \alpha}^{ij} = \Im [W_{\beta i} \bar{W}_{\alpha j}^* W_{\alpha i} W_{\beta j}^*]$, and the sum of the mass eigenstate index is taken with the states which concern with the neutrino propagation (which is mentioned as “light” here). Although we consider a $4 \times 4$ mixing matrix (for the three light mass eigenstates and one heavy one) in the previous section and in the rest of the paper, the above expression for probability can be applied to the more general case where $W$ is the part of the larger unitary matrix than $4 \times 4$.

If we concentrate on baselines and energies such that the OMS approximation can be employed, then the terms containing $\Delta m_{21}^2 L/(4E)$ can be neglected and the expression simplifies to

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \frac{1}{N_\alpha N_\beta} \left\{ \left( \sum_{i=1}^{\text{light}} W_{\beta i} \bar{W}_{\alpha i}^* \right)^2 - 4[R_{\alpha \beta}^{13} + R_{\beta \alpha}^{23}] \sin^2 \left( \frac{\Delta m_{31}^2}{2E} L \right) - 2[R_{\beta \alpha}^{13} + R_{\alpha \beta}^{23}] \sin \left( \frac{\Delta m_{31}^2}{2E} L \right) \right\}, \quad (10)
\]

As mentioned above, there is already strong constraint on the combination of the parameters $\epsilon_\alpha \epsilon_\mu$. Therefore we assume $\epsilon_\alpha = 0$ throughout this article. With this assumption, the deviation of unitarity can occur in the $\nu_\mu \rightarrow \nu_\mu$, $\nu_\mu \rightarrow \nu_\tau$ and $\nu_\tau \rightarrow \nu_\tau$ channel\(^3\). In the limit of $\theta_{13} \rightarrow 0$ and $\Delta m_{21}^2 / \Delta m_{31}^2 \rightarrow 0$, the survival probability $P_{\nu_\mu \rightarrow \nu_\mu}$ can be expressed as

\[
P_{\nu_\mu \rightarrow \nu_\mu} = 1 - \sin^2 2\theta_{23} \sin^2 \left( \frac{\Delta m_{31}^2}{4E} L \right) + O(\epsilon^3). \quad (11)
\]

From this equation, we see that the second order of the non-unitary effects in each term cancel out with the normalization factor $1/N_\mu^2$. In the $\nu_\tau \rightarrow \nu_\tau$ channel, the non-unitary effect comes as a small correction to the standard oscillation term and the standard oscillation term dominates. On the other hand, the oscillation probability for $\nu_\mu \rightarrow \nu_\tau$ is approximated as

\[
P_{\nu_\mu \rightarrow \nu_\tau} = \left\{ \epsilon_\mu^2 \epsilon_\tau^2 + O(\epsilon^3) \right\}
+ \sin 2\theta_{23} \left\{ \sin 2\theta_{23} + 2\epsilon_\mu \epsilon_\tau \cos 2\theta_{23} \cos \phi + O(\epsilon^3) \right\} \sin^2 \left( \frac{\Delta m_{31}^2}{4E} L \right)
+ \left\{ \epsilon_\mu \epsilon_\tau \sin \phi \sin 2\theta_{23} + O(\epsilon^3) \right\} \sin \left( \frac{\Delta m_{31}^2}{2E} L \right)
+ O(s_{13}) + O(\Delta m_{21}^2 / \Delta m_{31}^2). \quad (12)
\]

The term with $\sin \phi$ takes a different energy dependence from the standard oscillation term. Therefore, we can expect that this can be distinguished from the standard oscillation signals. The term of $O(\epsilon^2)$ in the standard oscillation term ($\sin^2 2\theta_{23} \Delta m_{31}^2 L/(4E)$ term) cannot be important because it is always smaller enough than the standard contribution $\sin^2 2\theta_{23}$. Assuming $L = 130$ km, $E = 50$ GeV, and $\epsilon_\mu \epsilon_\tau = 10^{-2}$, the order of each term is calculated to be

- standard oscillation term: $\sin^2 \left( \frac{\Delta m_{31}^2}{4E} L \right) \sim 6.8 \cdot 10^{-5}$,
- $\sin \phi$ term: $\epsilon_\mu \epsilon_\tau \sin \left( \frac{\Delta m_{31}^2}{2E} L \right) \sim 1.7 \cdot 10^{-4}$,
- zero-distance term: $\epsilon_\mu^2 \epsilon_\tau^2 \sim 10^{-4}$.

\(^3\) Alternatively one can study the violation of unitarity in both $\nu_e \rightarrow \nu_\tau$ channel and $\nu_\mu \rightarrow \nu_\tau$ channel \[54\].
and the three terms in Eq. (12) are thus of the same order of magnitude and this channel provides a better option for probing violation of unitarity.

The noteworthy feature of the above equation is the zero-distance term \( \epsilon_\mu^2 \tau \). Consequently for a near detector one gets,

\[
P_{\nu_\mu \rightarrow \nu_\tau}^{\text{near}} = \epsilon_\mu^2 \tau. \tag{16}
\]

It is actually very small. However, there are two positive aspects: (i) a huge number of neutrinos comes into the near detector (ii) the background for this process, i.e., the standard oscillation events, is highly suppressed.

### B. Oscillation Probability in Matter

When we introduce the non-unitary PMNS matrix, neutrinos obtain the additional matter effect mediated by neutral current \[51, 52, 53\]. For non-unitary mixing, the \( \nu_\mu \rightarrow \nu_\tau \) oscillation probability in matter of constant density in the simplifying approximation of \( \theta_{13} \rightarrow 0 \) and \( \Delta m_{21}^2/\Delta m_{31}^2 \rightarrow 0 \) can be expressed as,

\[
P_{\nu_\mu \rightarrow \nu_\tau} = \sin 2\theta_{23} (\sin 2\theta_{23} + 2\epsilon_\mu \epsilon_\tau \cos 2\theta_{23} \cos \phi) \sin^2 \frac{\Delta m_{31}^2 L}{4E} + \epsilon_\mu \epsilon_\tau \sin \phi \sin 2\theta_{23} \sin \frac{\Delta m_{31}^2 L}{2E} \\
- \epsilon_\mu \epsilon_\tau \left( \frac{a_{\text{NC}}}{2E} \right) \sin^3 2\theta_{23} \cos \phi \sin \frac{\Delta m_{31}^2 L}{2E} - 4\epsilon_\mu \epsilon_\tau \left( \frac{a_{\text{NC}}}{\Delta m_{31}^2} \right) \sin 2\theta_{23} \cos^2 2\theta_{23} \cos \phi \sin^2 \frac{\Delta m_{31}^2 L}{4E} \\
- 2 \left( \frac{a_{\text{NC}}}{\Delta m_{31}^2} \right) \sin^2 2\theta_{23} \cos 2\theta_{23} (\epsilon_\mu^2 - \epsilon_\tau^2) \sin^2 \frac{\Delta m_{31}^2 L}{4E} + \left( \frac{a_{\text{NC}}}{\Delta m_{31}^2} \right) \sin^2 2\theta_{23} \cos 2\theta_{23} (\epsilon_\mu^2 - \epsilon_\tau^2) \sin \frac{\Delta m_{31}^2 L}{2E} + O(\epsilon^3) + O(s_{13}) + O(\Delta m_{21}^2/\Delta m_{31}^2), \tag{17}
\]

where \( a_{\text{NC}} \) is the matter effect mediated by neutral current interaction. This is consistent with the result shown in Ref. [52] though the procedures used are somewhat different. Since \( \theta_{23} \simeq \pi/4 \), we can omit the terms which proportional to \( \cos 2\theta_{23} \), and finally, it is reduced to

\[
P_{\nu_\mu \rightarrow \nu_\tau} = \sin^2 2\theta_{23} \sin^2 \frac{\Delta m_{31}^2 L}{4E} + \epsilon_\mu \epsilon_\tau \sin 2\theta_{23} \sin \phi \sin \frac{\Delta m_{31}^2 L}{2E} - \epsilon_\mu \epsilon_\tau \left( \frac{a_{\text{NC}}}{2E} \right) \sin^3 2\theta_{23} \cos \phi \sin \frac{\Delta m_{31}^2 L}{2E}. \tag{18}
\]

This formula can nicely explain the numerical result which will be shown in the following sections. We have an additional term in comparison with Eq. (12), which depends on \( \cos \phi \) differing from the vacuum term. This is the key feature to resolve the degeneracies which will be explained in the next section. The details of the derivation are described in Appendix.

### IV. DEGENERACIES

From the expression Eq. (12) for the oscillation probability \( P_{\nu_\mu \rightarrow \nu_\tau} \) in vacuum, we see that this is invariant under the following transformations:

1. \( \theta_{23} \) (octant) degeneracy: \( P_{\nu_\mu \rightarrow \nu_\tau}(\theta_{23}) = P_{\nu_\mu \rightarrow \nu_\tau}(\pi/2 - \theta_{23}) \),
2. \( \text{sign}(\Delta m_{31}^2) \cdot \phi \) degeneracy: \( P_{\nu_\mu \rightarrow \nu_\tau}(\Delta m_{31}^2 > 0, \phi) = P_{\nu_\mu \rightarrow \nu_\tau}(\Delta m_{31}^2 < 0, -\phi) \),
3. \( \phi - (\pi - \phi) \) degeneracy: \( P_{\nu_\mu \rightarrow \nu_\tau}(\phi) = P_{\nu_\mu \rightarrow \nu_\tau}(\pi - \phi) \),
4. \( (\epsilon_\mu \epsilon_\tau) \cdot \phi \) correlation (quasi-degeneracy): \( P_{\nu_\mu \rightarrow \nu_\tau}( (\epsilon_\mu \epsilon_\tau), \phi) = P_{\nu_\mu \rightarrow \nu_\tau}( (\epsilon_\mu \epsilon_\tau)' , \phi') \).

Here, the values of oscillation parameters which are not explicitly shown are taken to be the same on both the sides of the equations. These can give rise to degeneracies in the \( (\epsilon_\mu \epsilon_\tau) \cdot \phi \) plane even in the limit \( \theta_{13} \rightarrow 0 \). Below we discuss these degeneracies. If \( \theta_{13} \) is non-zero then the additional degeneracies due to \( \delta_{\text{CP}} \) can also be there. But this will not give rise to any additional degenerate solutions in the \( (\epsilon_\mu \epsilon_\tau), \phi \) plane. Note that in addition to the degeneracies

\[4 \] The non-standard matter effect mediated by neutral current interactions was also discussed in Ref. [55].
\(\epsilon_\mu \epsilon_\tau\) and \(\phi\) occur in a correlated fashion in the oscillation probability shown in Eq. (12). Hence the uncertainty in determination of one of these parameters can affect that of the other even for the same hierarchy. When we assume the maximal mixing for \(\theta_{23}\), the \(\theta_{23}\) octant degeneracy is not present.

The expression Eq. (13) breaks some of the degeneracies. Because of the presence of the \(\cos \phi\) term, induced by matter effect, sign[\(\Delta m^2_{31}\)]-\(\phi\) and \(\phi-(\pi-\phi)\) degeneracies can be resolved if we can see this term. To do so, we have to go to the long baseline because the term is simply proportional to the baseline length. However, in the long baseline region, the standard oscillation term can be order one, and the tiny non-unitarity effect could be easily absorbed by the standard oscillation term. The significance of the non-unitary matter effect should be checked numerically. The \((\epsilon_\mu \epsilon_\tau)-\phi\) correlation is present in the oscillation probability in matter as well.

We can illustrate the occurrence of degeneracies due to the invariance listed above by using the equi-probability plots [58]. Here, the standard oscillation parameters are fixed as,

\[
\begin{align*}
\sin^2 \theta_{12} &= 0.31, \\
\sin^2 2\theta_{13} &= 10^{-2}, \\
\delta_{\text{CP}} &= 0, \\
|\Delta m^2_{31}| &= 2.5 \times 10^{-3} \text{ [eV}^2], \\
\Delta m^2_{21} &= 7.9 \times 10^{-5} \text{ [eV}^2],
\end{align*}
\]

and \(\theta_{23}\) and the sign of the atmospheric mass square difference will be given later. For the non-unitary parameters, we adopt

\[
(\epsilon_\mu \epsilon_\tau)^{\text{true}} = 10^{-2}, \quad \phi^{\text{true}} = \pi/4
\]

as the reference values throughout this paper\(^5\). The equi-probability curves shown in the following mean that the condition

\[
P_{\nu_\mu-\nu_\tau} \left( (\epsilon_\mu \epsilon_\tau)^{\text{fit}}, \phi^{\text{fit}} \right) = P_{\nu_\mu-\nu_\tau} \left( (\epsilon_\mu \epsilon_\tau)^{\text{true}}, \phi^{\text{true}} \right),
\]

is fulfilled on each curve.

FIG. 1: Equi-probability plots for \(\theta_{23}\) degeneracy (left), for sign of \(\Delta m^2_{31}\) degeneracy (centre), and for \(\phi-(\pi-\phi)\) degeneracy and \(\epsilon_\mu \epsilon_\tau-\phi\) correlation (right). The neutrino energy is taken to be 50 GeV and the source-detector distance is 130 km.

The left panel in Fig. 1 is for the \(\theta_{23}\) degeneracy. The plot is done for \(E = 50\) GeV, and the baseline is taken to be 130 km with 2.7 g/cm\(^3\) as the matter density although matter effect is not relevant in this setup. In this plot we draw equi-probability contours in the \((\epsilon_\mu \epsilon_\tau)-\phi\) plane for two values of \(\theta_{23}\),

\[
\sin^2 \theta_{23} = \{0.64, 0.36\}.
\]

Here we assume the NH mass spectrum. The plot shows that the curve with \(\sin^2 \theta_{23} = 0.64\) (thin solid) completely coincides with that of \(\sin^2 \theta_{23} = 0.36\) (thick dashed gray), and these two cannot be distinguished. However, this

\(^5\) More precisely, we take \(\epsilon_\mu^{\text{true}} = \epsilon_\tau^{\text{true}} = 0.1\) in our numerical calculations. This allocation does not affect the results since the leading contribution of the non-unitarity always appears as the combination \(\epsilon_\mu \epsilon_\tau\).
degeneracy does not give rise to any additional regions in the \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) parameter plane because this degeneracy is not due to the non-unitary parameters, i.e., the degenerate solutions take the same values of \((\epsilon_\mu \epsilon_\tau)\) and \(\phi\) in the both side of Eq. (21),

\[
P_{\nu_\mu \to \nu_\tau}(\theta_{23}, (\epsilon_\mu \epsilon_\tau), \phi) = P_{\nu_\mu \to \nu_\tau}(\pi/2 - \theta_{23}, (\epsilon_\mu \epsilon_\tau), \phi).
\]  

(23)

The middle panel in Fig. 1 is for the degeneracy on \(\text{sign}[\Delta m^2_{31}] \cdot \phi\). The solid curve is similar to the curves in the left panel. Here, the standard oscillation parameters are again taken to be the values in Eq. (19) but \(\sin^2 \theta_{23}\) is assumed to be 0.5, and the NH is adopted in the both side of Eq. (21). In the calculation of the dashed (blue) curve, the true probability with NH is fitted by the probability with the IH mass spectrum. Therefore, the condition which is satisfied on the dashed curve is written as

\[
P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau)^{\text{fit}}, \phi^{\text{fit}}, \text{IH}) = P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau)^{\text{true}}, \phi^{\text{true}}, \text{NH}).
\]  

(24)

Although the dashed curve does not pass through the true value point which is shown as the black dot in the plot, the true oscillation probability can also be reproduced on it. The shape of the dashed curve is the reflection of the solid curve at the \(\phi = 0\) point.

The right panel in Fig. 1 illustrates the \(\phi-(\pi - \phi)\) degeneracy and the \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) quasi-degeneracy. On each curve, the condition

\[
P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau)^{\text{fit}}, \phi^{\text{fit}}, E) = P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau)^{\text{true}}, \phi^{\text{true}}, E),
\]  

(25)

is satisfied, where the values of the standard oscillation parameters are again taken as shown in Eq. (19), and the maximal mixing for \(\theta_{23}\) and the NH are assumed in both side of Eq. (25). We plot the curves of three cases with the following neutrino energies:

\[
E = \{10, 30, 50\} \quad \text{[GeV]}.
\]  

(26)

For a fixed energy all the points on the curve give the same probability reflecting the \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) degeneracy. However, if one considers other illustrative values of energies and draws the corresponding equi-probability curves passing through the true value point, then in a large region of parameter space, the equi-probability curves trace different paths. Consequently in these regions the \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) degeneracy can be removed by adding the spectral information. However, the figure also shows that the three curves cross at two points; one is the true value point (shown as the black dot), and the other is the fake solution which is referred as the \(\phi-(\pi - \phi)\) degeneracy above for each curve. We can also find that at the region between the true solution and the fake solution, all three curves take a quite similar path indicating in this region the different probabilities for different energies have very little dependence on parameters. This means that it is hard to resolve the solutions at this region even with spectral information and hence we mention this as the quasi-degeneracy. We can draw a similar plots as Fig. 1 with IH as the true hierarchy.

In Fig. 2 we plot the equi-probability plot for neutrino of energy 50 GeV and \(L = 3000\) km. The true hierarchy is assumed to be NH and the true value point is again shown as a black dot in the figure. The dashed line shows the plot for the IH fit, on which the true oscillation probability can be reproduced. There are two points at which the NH and IH probability crosses each other. The conditions for obtaining these points can be worked out from the expression Eq. (18). In general the condition for degeneracy on these curves can be written as

\[
P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau), \phi, \text{NH}) = P_{\nu_\mu \to \nu_\tau}((\epsilon_\mu \epsilon_\tau)', \phi', \text{IH}),
\]  

(27)

At the point where the NH and IH curves cross the \(\epsilon_\mu \epsilon_\tau\) and \(\phi\) are same for both NH and IH. This gives,

\[
\tan \phi = \frac{a_{\text{NC}} L}{2E}.
\]  

(28)

For \(L = 3000\) km and \(E = 50\) GeV, the above gives \(\phi \approx 140^\circ\) and \(\pi + 140^\circ\) as obtained in the figure.

V. NUMERICAL RESULTS: ALLOWED REGION ON THE \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) PLANE

In this section we present the results of our numerical analysis. We first present the allowed regions in the \((\epsilon_\mu \epsilon_\tau) \cdot \phi\) plane for an OPERA-like detector at a distance of 130 km from a neutrino factory source and describe how the degeneracies are realized. This experimental setup have already been examined in Ref. [54]. However, we will pay attention to the degeneracy of the solutions. Later, we will see that how this degenerate solutions are resolved including information of matter effect.
In Fig. 3 we plot the $\chi^2$ function which is defined as:

$$
\chi^2((\epsilon_\mu \epsilon_\tau)^{\text{fit}}, \phi^{\text{fit}}) = \min_{\lambda^{\text{fit}}} \sum_i |N_i(\lambda^{\text{true}}, (\epsilon_\mu \epsilon_\tau)^{\text{true}}, \phi^{\text{true}}) - N_i(\lambda^{\text{fit}}, (\epsilon_\mu \epsilon_\tau)^{\text{fit}}, \phi^{\text{fit}})|^2 / V_i,
$$

where $N_i$ is the neutrino event number in the $i$-th energy bin, $\lambda$ represents the standard oscillation parameters and $V_i$ is the variance which are appropriately defined to include the statistical and systematic errors. Here we adopt the values shown in Eq. (19) for the standard oscillation parameters. Since it is not possible to resolve the $\theta_{23}$ degeneracy in this experiment (in the $\nu_\mu \to \nu_\tau$ channel), we take the reference true values for $\theta_{23}$ as the maximal. The true mass hierarchy is assumed to be NH. The parameters for the non-unitary nature are taken as shown in Eq. (20). The left panel in Fig. 3 shows the allowed region in the $(\epsilon_\mu \epsilon_\tau)-\phi$ plane. As discussed earlier, since the probability in this case is a function of $\sin^2 2\theta_{23}$, the $\theta_{23}$ octant degeneracy does not give rise to any additional regions in the $(\epsilon_\mu \epsilon_\tau)-\phi$ plane and the solutions for true $\theta_{23}$ and wrong $\theta_{23}$ occur in the same place. Here the two solutions (two crescent regions) correspond to two choices for the sign of $\Delta m_{31}^2$ in the fit event. The figure also shows that for each hierarchy there is the $\phi-(\pi-\phi)$ degeneracy. The spurious solution corresponding to $(\epsilon_\mu \epsilon_\tau)-\phi$ degeneracy is removed by using the spectrum information. There is a weak negative correlation between $\epsilon_\mu \epsilon_\tau$ and $\phi$ for each allowed zone. We next discuss how one can eliminate the degenerate solutions in the $(\epsilon_\mu \epsilon_\tau)-\phi$ plane by combining the experiments at various baselines. The remaining degeneracies are the sign($\Delta m_{31}^2$)-$\phi$ degeneracy, the $\phi-(\pi-\phi)$ degeneracy, and the $(\epsilon_\mu \epsilon_\tau)-\phi$ quasi-degeneracy. The right plot of Fig. 3 shows the combining results of an OPERA-like detector at 130 km baseline and a 0.1 kton liquid Argon (LAr) type near detector which is located at $L = 2$ km. The probability at the near detector depends on $\epsilon_\mu \epsilon_\tau$ only. Thus combining with this experiment helps to narrow down the allowed region but the degeneracies still exist. The correlation between $\phi$ and $\epsilon_\mu \epsilon_\tau$ is now almost vanishing.

---

6 In the actual implementation, we adopt the Poisson distribution, add the appropriately defined priors, and marginalize also over the systematic parameters, following GLoBES software [54, 55].
7 A 0.1 kt LAr detector as a near detector has been discussed in Ref. [62].
FIG. 3: Allowed regions in the \((\epsilon_\mu \epsilon_\tau) - \phi\) plane for an OPERA-like detector at a distance of 130 km from a Neutrino Factory source (left), and for the same setup but with a 0.1 kt Liquid Argon near detector (right).

In left panel of Fig. 4 we plot the allowed regions for the combination of a neutrino factory and a 100 kton LAr type far detector which is located at \(L = 3000\) km. A comparison of this figure with the equi-probability plot Fig. 2 reveals that the middle region in Fig. 4 corresponds to the IH fit. As discussed in the previous section the main contribution of matter effect depends on \(\cos \phi\) which is different from the case in vacuum. Therefore, the \(\phi - (\pi - \phi)\) degeneracy as well as \(\text{sign}[\Delta m^2_{31}]-\phi\) degeneracy can be removed by this matter term. This is reflected in the figure. However the probabilities for NH and IH can still be equal when the condition Eq. (27) is satisfied. This gives rise to the middle region in Fig. 4. There is a positive correlation in this case between \(\epsilon_\mu \epsilon_\tau\) and \(\phi\) for each allowed region. The result combining the near detector is shown as the right panel in Fig. 4. This helps to reduce the uncertainty in the \(\epsilon_\mu \epsilon_\tau\) and since \(\phi\) is a variable correlated with \(\epsilon_\mu \epsilon_\tau\), the uncertainty on \(\phi\) is also reduced. Therefore, the allowed regions for
VI. CONCLUSIONS

Non-unitary mixing matrix is a generic feature for theories with mixing between neutrinos and heavy states and provides a window to probe physics at high scale. In this paper we have studied the possibility of probing non-unitarity of neutrino mixing matrix at neutrino factories. We considered the $\nu_\mu \to \nu_\tau$ channel and detectors at a distance of 2 km, 130 km and 3000 km from the source. We show that for the $\nu_\mu \to \nu_\tau$ channel at 130 km, there can be degenerate solutions even for $\theta_{13}=0$ in the $(\epsilon_\mu \epsilon_\tau)\phi$ plane where $\phi$ and $\epsilon_\mu \epsilon_\tau$ are the phase and moduli of the unitarity violation parameter. The degenerate solutions in the $(\epsilon_\mu \epsilon_\tau)\phi$ plane are due to

- $(\Delta m^2_{31} > 0, \phi) \to (\Delta m^2_{31} < 0, -\phi)$
- $\phi \to \pi - \phi$
- $(\epsilon_\mu \epsilon_\tau, \phi) \to ((\epsilon_\mu \epsilon_\tau)', \phi')$

For a detector at distance 130 km from a neutrino factory source the last degeneracy can be removed using spectral information and no additional disconnected solution appear. By adding an experiment at 2 km the correlation between $\phi$ and $\epsilon_\mu \epsilon_\tau$ can be reduced and the allowed ranges narrow down. For the 3000 km experiment the matter effects are relevant and this removes the first and second degeneracy listed above. However although the hierarchy degeneracy listed above gets removed, there can still be the degeneracy where probabilities for NH and IH give same values. If we consider only the 3000 km experiment then there is a greater correlation between $\epsilon_\mu \epsilon_\tau, \phi$ and the allowed regions are larger as compared to the 130 km experiment. However, addition of the 2 km experiment to this reduces this correlation and the allowed regions become narrower. Although we have concentrated on the $\nu_\mu \to \nu_\tau$ channel in this study, if we combine the other channel like $\nu_\tau \to \nu_\mu$, we can obtain information on the hierarchy and then the allowed regions further reduce in size.

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APPENDIX A: ANALYTIC FORMULAE

In this section, we derive the expression of $\nu_\mu \to \nu_\tau$ oscillation probability in matter of constant density under some simplifying assumptions. If the neutrino mixing matrix is non-unitary then, although we can get the canonical form of the kinetic energy in the mass basis in terms of the flavour states the kinetic term is not diagonal. Therefore it is more appropriate here to consider the neutrino propagation equation in the mass basis. The neutrino propagation Hamiltonian in matter can be generally represented in the vacuum mass eigenbasis as follows,

$$H_{ij} = \frac{1}{2E} \left\{ \begin{array}{c} 0 \\ \Delta m^2_{21} \\ \Delta m^2_{31} \end{array} \right\} + a_{\text{CC}} \left( \begin{array}{ccc} |W_{e1}|^2 & W_{e1}^* W_{e2} & W_{e1}^* W_{e3} \\ W_{e2}^* W_{e1} & |W_{e2}|^2 & W_{e2}^* W_{e3} \\ W_{e3}^* W_{e1} & W_{e3}^* W_{e2} & |W_{e3}|^2 \end{array} \right) + a_{\text{NC}} \left( \begin{array}{c} \sum_{\gamma=e,\mu,\tau} |W_{\gamma 1}|^2 \\ \sum_{\gamma=e,\mu,\tau} W_{\gamma 1}^* W_{\gamma 2} W_{\gamma 3} \\ \sum_{\gamma=e,\mu,\tau} W_{\gamma 3}^* W_{\gamma 1} W_{\gamma 2} \end{array} \right) \right\}, \quad (A1)$$
where $a_{CC} = 2\sqrt{2} E_F n_e$, $a_{NC} = - \sqrt{2} E_F n_N = - a_{CC}/2$ are the charged and neutral current potentials respectively. We obtain simplified analytic expressions for the probability by solving the above equations in the limit $\theta_{13} \to 0$, $\Delta m_{21}^2 / \Delta m_{31}^2 \to 0$. In this limit the propagation Hamiltonian in the vacuum mass eigenbasis is

$$
H_{ij} = (H_0)_{ij} + (H_{\epsilon_{\mu\tau}})_{ij} + (H_{\epsilon_{\mu}})_{ij} + (H_{\epsilon_{\tau}})_{ij}
$$

where

$$
(H_0)_{ij} = \frac{1}{2E} \left\{ \begin{array}{ccc} 0 & c_{12} & c_{12} s_{12} 0 \\ \Delta m_{31}^2 & 0 & 0 \\
\end{array} \right\} + a_{CC} \left\{ \begin{array}{ccc} c_{12} & c_{12} s_{12} 0 \\ 0 & s_{12}^2 & 0 \\
0 & 0 & 0 \\
\end{array} \right\},
$$

$$
(H_{\epsilon_{\mu\tau}})_{ij} = \frac{a_{NC}}{2E} \epsilon_{\mu\tau} \left( \begin{array}{ccc} 2e_{23} s_{23} s_{12} c_{\phi} & -2e_{23} s_{23} c_{12} s_{12} c_{\phi} & s_{12} (c_{23}^2 e^{-i\phi} - s_{23}^2 e^{i\phi}) \\
-2e_{23} s_{23} c_{12} s_{12} c_{\phi} & 2e_{23} s_{23} c_{12}^2 c_{\phi} & -c_{12} (c_{23}^2 e^{i\phi} - s_{23}^2 e^{-i\phi}) \\
s_{12} (c_{23}^2 e^{-i\phi} - s_{23}^2 e^{i\phi}) & -c_{12} (c_{23}^2 e^{i\phi} - s_{23}^2 e^{-i\phi}) & -2e_{23} s_{23} c_{\phi} \\
\end{array} \right),
$$

$$
(H_{\epsilon_{\mu}})_{ij} = \frac{a_{NC}}{2E} \epsilon_{\mu} \left( \begin{array}{ccc} s_{12}^2 c_{23} & -s_{12} c_{12} c_{23} & -s_{12} s_{23} c_{23} \\
-s_{12} c_{12} c_{23} & c_{12}^2 c_{23} & c_{12} s_{23} c_{23} \\
-s_{12} s_{23} c_{23} & c_{12} s_{23} c_{23} & s_{23}^2 \\
\end{array} \right),
$$

$$
(H_{\epsilon_{\tau}})_{ij} = - \frac{a_{NC}}{2E} \epsilon_{\tau} \left( \begin{array}{ccc} s_{12}^2 & -s_{12} c_{12} c_{23} & s_{12} s_{23} c_{23} \\
-s_{12} c_{12} c_{23} & c_{12}^2 & -c_{12} s_{23} c_{23} \\
-s_{12} s_{23} c_{23} & c_{12} s_{23} c_{23} & c_{23}^2 \\
\end{array} \right),
$$

up to the second order of the epsilon parameters. In writing the above a part proportional to unit matrix is omitted as it contributes to overall phase. The Hamiltonian is separated into two parts — the zeroth order part $H_0$ which includes $\Delta m_{31}^2$ and $a_{CC}$, and perturbations $H_{\epsilon_{\mu\tau}}, H_{\epsilon_{\mu}},$ and $H_{\epsilon_{\tau}},$ induced by the non-unitarity. Note that the non-unitarity effects appear always at the second order (or higher than that) of the $\epsilon_{\alpha}$ parameters.

Treating $H_{\epsilon_{\mu\tau}}, H_{\epsilon_{\mu}},$ and $H_{\epsilon_{\tau}}$ as perturbations, the amplitude of the neutrino oscillation from a vacuum mass eigenstate $\nu_i$ to the other vacuum mass eigenstate $\nu_j$ can be written as

$$
S_{ji} = (S_0)_{ji} + (S_{\epsilon_{\mu\tau}})_{ji} + (S_{\epsilon_{\mu}})_{ji} + (S_{\epsilon_{\tau}})_{ji},
$$

where $S_0$ is the zeroth order part, and $S_{\epsilon_{\mu\tau}}, S_{\epsilon_{\mu}},$ and $S_{\epsilon_{\tau}}$ correspond to the amplitudes with perturbations of $H_{\epsilon_{\mu\tau}}, H_{\epsilon_{\mu}},$ and $H_{\epsilon_{\tau}}$ respectively, which are calculated to be

$$
(S_0)_{ji} = (e^{-iH_0 L})_{ji},
$$

$$
(S_{\epsilon_{\mu\tau}})_{ji} = (e^{-iH_{\epsilon_{\mu\tau}} L})_{jk} (-i) \int_0^L dx (e^{iH_{\epsilon_{\mu\tau}} x})_{kl} (H_{\epsilon_{\mu\tau}})_{lm} (e^{-iH_{\epsilon_{\mu\tau}} x})_{mi},
$$

$$
(S_{\epsilon_{\mu}})_{ji} = (e^{-iH_{\epsilon_{\mu}} L})_{jk} (-i) \int_0^L dx (e^{iH_{\epsilon_{\mu}} x})_{kl} (H_{\epsilon_{\mu}})_{lm} (e^{-iH_{\epsilon_{\mu}} x})_{mi},
$$

$$
(S_{\epsilon_{\tau}})_{ji} = (e^{-iH_{\epsilon_{\tau}} L})_{jk} (-i) \int_0^L dx (e^{iH_{\epsilon_{\tau}} x})_{kl} (H_{\epsilon_{\tau}})_{lm} (e^{-iH_{\epsilon_{\tau}} x})_{mi},
$$

Note that these amplitudes describe the transition between two vacuum mass eigenstates, $\nu_i$ and $\nu_j$ and a transition between flavour states can be obtained by sandwiching them by the flavour states$^8$ which are described as $^{21, 53, 62}$

$$
|\nu_\alpha\rangle = \frac{1}{\sqrt{\sum_{j=1}^{\text{light}} |W_{\alpha j}|^2}} \sum_{j=1}^{\text{light}} W_{\alpha j}^* |\nu_j\rangle
$$

$^8$ We underscore that, strictly speaking, this method has to be followed as it is not correct to write the the neutrino propagation in the flavour basis because the flavour states do not form a complete set for the propagation Hamiltonian.
The oscillation probability between two flavour states $\nu_\alpha$ and $\nu_\beta$ is derived as

$$P_{\nu_\alpha \to \nu_\beta} = \frac{1}{\sum_{\ell=1}^{\text{light}} |W_{\beta\ell}|^2} \left| W_{\beta j} \begin{pmatrix} (S_0 + S_{\epsilon\epsilon} + S_{\epsilon^2} + S_{e^2})_{ji} & \frac{1}{\sqrt{\sum_{k=1}^{\text{light}} |W_{\alpha k}|^2}} (W^\dagger)_{i\alpha} \end{pmatrix} \right|^2$$

$$= \frac{1}{N_\alpha N_\beta} \left[ |(S_0)_{\beta\alpha}|^2 + 2 \text{Re}[|S_0^*\beta\alpha(S_{\epsilon\epsilon})_{\beta\alpha}] + 2 \text{Re}[|S_0^*\beta\alpha(S_{\epsilon^2})_{\beta\alpha}] + 2 \text{Re}[|S_0^*\beta\alpha(S_{e^2})_{\beta\alpha}] \right] + O(\epsilon^4), \quad (A13)$$

up to the first order perturbations. In the following, we will calculate each oscillation amplitude.

Diagonalizing the zeroth order Hamiltonian $H_0$, we obtain the mass squared eigenvalues and the mixing matrix $(V_0)_{ij}$ which connects the vacuum mass eigenbasis $\nu_i$ with the mass eigenbasis in matter $\nu_j$, and in the limit which we adopt here, they take the following simple forms

$$(H_0)_{ij} = \text{diag}(a_{CC}, 0, \Delta m_{31}^2) = (V_0^\dagger)_{ki}(H_0)_{ji}(V_0)_{ik}, \quad (A14)$$

where

$$(V_0)_{ij} = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}. \quad (A15)$$

Therefore, the zeroth order amplitude in the vacuum mass eigenbasis becomes

$$(S_0)_{ji} = (V_0)_{j\ell} \begin{pmatrix} e^{-i\frac{\Delta m_{31}^2}{2E}} & 0 \\ 0 & e^{-i\frac{\Delta m_{31}^2}{2E}} \end{pmatrix} (V_0^\dagger)_{ki}, \quad (A16)$$

and that for the transition between two flavour states is

$$(S_0)_{\beta\alpha} = W_{\beta j}(S_0)_{ji}(W^\dagger)_{i\alpha}. \quad (A17)$$

The oscillation probability at the zeroth order becomes

$$P_{\nu_\alpha \to \nu_\beta}^{0\text{th}} = \sin 2\theta_{23} (\sin 2\theta_{23} + 2\epsilon_\mu \epsilon_\tau \cos 2\theta_{23} \cos \phi) \sin^2 \frac{\Delta m_{31}^2 L}{4E} + \epsilon_\mu \epsilon_\tau \sin \phi \sin 2\theta_{23} \sin \frac{\Delta m_{31}^2 L}{2E} + O(\epsilon^3), \quad (A18)$$

which is the same as the formula in the vacuum case.

Next, let us turn to the perturbation terms. First one is the amplitude of $S_{\epsilon\epsilon}$. According to Eq. (A9), we can calculate it as

$$(S_{\epsilon\epsilon})_{ji} = \epsilon_\mu \epsilon_\tau \left( \frac{\alpha_{NC}}{2E} \right) (V_0)_{ji} \begin{pmatrix} 0 & 0 \\ 0 & -s_{2\times23}c_\phi L \\ 0 & A \frac{2E}{i\Delta m_{31}^2} \left( 1 - e^{-i\frac{\Delta m_{31}^2 L}{2E}} \right) \\ A^* \frac{2E}{i\Delta m_{31}^2} \left( 1 - e^{-i\frac{\Delta m_{31}^2 L}{2E}} \right) \\ s_{2\times23}c_\phi L e^{-i\frac{\Delta m_{31}^2 L}{2E}} \end{pmatrix} (V_0^\dagger)_{ki}, \quad (A19)$$

where the parameters $A$ is defined as

$$A \equiv (e_{23}^2 e^{-i\phi} - s_{23}^2 e^{i\phi}), \quad (A20)$$

and $s_{2\times23} \equiv \sin 2\theta_{23}$. The amplitude for $\nu_\mu \to \nu_\tau$ transition is reduced to

$$(S_{\epsilon\epsilon})_{\tau\mu} = \epsilon_\mu \epsilon_\tau \left[ \frac{\alpha_{NC} L}{4E} s_{2\times23} c_\phi \left( 1 + e^{-i\frac{\Delta m_{31}^2 L}{2E}} \right) + \frac{\alpha_{NC}}{\Delta m_{31}^2} (e^{i\phi} - s_{2\times23} c_\phi) \left( 1 - e^{-i\frac{\Delta m_{31}^2 L}{2E}} \right) \right] + O(\epsilon^3). \quad (A21)$$

The contribution to the oscillation probability is calculated to be

$$2\text{Re}[(S_0^*)_{\tau\mu}(S_{\epsilon\epsilon})_{\tau\mu}] = -\epsilon_\mu \epsilon_\tau \left( \frac{\alpha_{NC} L}{2E} \right) s_{2\times23}^3 c_\phi \sin \frac{\Delta m_{31}^2 L}{2E} - 4\epsilon_\mu \epsilon_\tau \left( \frac{\alpha_{NC}}{\Delta m_{31}^2} \right) s_{2\times23}^2 c_\phi \sin^2 \frac{\Delta m_{31}^2 L}{4E}, \quad (A22)$$

where $s_{2\times23}^3 = s_{23}^2 \sin 2\theta_{23}$.
up to the second order of the $\epsilon$ parameters. The contributions from $S_{1\tau}$ and $S_{\tau\tau}$ can also be calculated with the same way, which are

$$2\text{Re}[(S_{1\tau}^\ast)\tau\mu(S_{\tau\tau}^\ast)\tau\mu] + 2\text{Re}[(S_{\tau\tau}^\ast)\tau\mu(S_{\tau\tau}^\ast)\tau\mu] = \left(\frac{a_{\text{NC}}L}{4E}\right)s_{2\times23}c_{2\times23}(\epsilon_\mu^2 - \epsilon_\tau^2)\sin\frac{\Delta m_{31}^2L}{2E} - 2\left(\frac{a_{\text{NC}}}{\Delta m_{31}^2}\right)s_{2\times23}c_{2\times23}(\epsilon_\mu^2 - \epsilon_\tau^2)\sin\frac{\Delta m_{31}^2L}{4E}. \quad (A23)$$

From Eqs. (A18), (A22) and (A23), the oscillation probability for $\nu_\mu \rightarrow \nu_\tau$ in matter can be expressed as

$$P_{\nu_\mu \rightarrow \nu_\tau} = \sin^2 2\theta_{23}(\sin 2\theta_{23} + 2\epsilon_\mu \epsilon_\tau \cos 2\theta_{23} \cos \phi \sin 2\theta_{23} \sin \theta_{23} \sin \frac{\Delta m_{31}^2 L}{2E} + \epsilon_\mu \epsilon_\tau \sin \phi \sin 2\theta_{23} \sin \frac{\Delta m_{31}^2 L}{2E}$$

$$- \epsilon_\mu \epsilon_\tau \left(\frac{a_{\text{NC}}L}{2E}\right)\sin^3 2\theta_{23} \cos \phi \sin \frac{\Delta m_{31}^2 L}{2E} - 4\epsilon_\mu \epsilon_\tau \left(\frac{a_{\text{NC}}}{\Delta m_{31}^2}\right)\sin 2\theta_{23} \cos^2 2\theta_{23} \cos \phi \sin^2 \frac{\Delta m_{31}^2 L}{4E}$$

$$- 2\left(\frac{a_{\text{NC}}}{\Delta m_{31}^2}\right)\sin^2 2\theta_{23} \cos 2\theta_{23}(\epsilon_\mu^2 - \epsilon_\tau^2)\sin^2 \frac{\Delta m_{31}^2 L}{4E} + \left(\frac{a_{\text{NC}}L}{4E}\right)\sin^2 2\theta_{23} \cos 2\theta_{23}(\epsilon_\mu^2 - \epsilon_\tau^2)\sin \frac{\Delta m_{31}^2 L}{2E}$$

$$+ O(\epsilon^3) + O(s_{13}) + O(\Delta m_{21}^2/\Delta m_{31}^2). \quad (A24)$$

Since $\theta_{23} \approx \pi/4$, we can omit the terms which are proportional to $\cos 2\theta_{23}$, and finally, it reduces to

$$P_{\nu_\mu \rightarrow \nu_\tau} = \sin^2 2\theta_{23} \sin^2 \frac{\Delta m_{31}^2 L}{4E} + \epsilon_\mu \epsilon_\tau \sin 2\theta_{23} \sin \phi \sin \frac{\Delta m_{31}^2 L}{2E} - \epsilon_\mu \epsilon_\tau \left(\frac{a_{\text{NC}}L}{2E}\right)\sin^3 2\theta_{23} \cos \phi \sin \frac{\Delta m_{31}^2 L}{2E}. \quad (A25)$$

**APPENDIX B: EXPERIMENTAL SETUPS IN NUMERICAL CALCULATIONS**

The numerical work is performed using GLoBES software [53, 60] which is modified for our purpose. We consider a neutrino factory as the source for $\nu$ setups. A neutrino factory as the source for $\nu$ setups. We consider an OPERA-like detector at a distance of 130 km from a Neutrino Factory beam, which was examined in Ref. [34]. The detector mass is assumed to be 5.0 kton. The matter profile is assumed to be constant with the density 2.7 g/cm$^3$ although the matter effect itself is not significant in this setup.

For the signal detection efficiency, the errors, and the backgrounds, we follow the glb-file OPERA.glb. Since this glb-file is designed for the CNGS beam source, the numbers should be modified for the neutrino factory beam source. Here, we use the numbers shown in Tab. [1].

1. **NuFACT beam + OPERA-like detector with $L = 130$ km**

We consider an OPERA-like detector at a distance of $L = 130$ km from a Neutrino Factory beam, which was examined in Ref. [34]. The detector mass is assumed to be 5.0 kton. The matter profile is assumed to be constant with the density 2.7 g/cm$^3$ although the matter effect itself is not significant in this setup.
2. NuFACT beam + LAr near detector

In this setup we consider a 0.1 kt liquid Argon detector at 2 km far away from the beam source, which has been discussed in Ref. [61]. Here we follow the glb-file, ICARUS.glb but modify the background estimation.

| $\nu_\mu \to \nu_\tau$ Appearance | $\sigma_{\text{norm}}$ | $\sigma_{\text{cal}}$ |
|------------------------------|----------------|----------------|
| Signal | $0.1056 \otimes (\nu_\mu \to \nu_\tau)_{\text{CC}}$ | $0.05 \times 10^{-4}$ |
| Background | $3.414 \times 10^{-5} \otimes (\nu_\mu \to \nu_x)_{\text{NC}}$ | $0.05 \times 10^{-4}$ |

TABLE I: Rules for the experimental setup NuFACT+OPERA-like detector.

| $\nu_\mu \to \nu_\tau$ Appearance | $\sigma_{\text{norm}}$ | $\sigma_{\text{cal}}$ |
|------------------------------|----------------|----------------|
| Signal | $0.0758 \otimes (\nu_\mu \to \nu_\tau)_{\text{CC}}$ | $0.05 \times 10^{-4}$ |
| Background | $8.502 \times 10^{-5} \otimes (\nu_\mu \to \nu_x)_{\text{NC}}$ | $0.05 \times 10^{-4}$ |

TABLE II: Rules for the experimental setup NuFACT+LAr detector.

3. NuFACT beam + large LAr far detector

In order to solve the $\phi-(\pi-\phi)$ degeneracy and the $(\epsilon_\mu \epsilon_\tau)\phi$ quasi-degeneracy, it is effective to observe the matter effect coming from the non-unitarity effect. To get the matter effect, we need a long baseline. Here, we set $L = 3,000$ km and adopt $3.3 \text{ g/cm}^3$ as the matter density. However, in such a long baseline setup, we need a huge detector to collect enough event rates. We assume 100 kton LAr detector whose rules are taken from ICARUS.glb, which is modified as the same manner as the LAr near detector setup.

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