Lecture 5: Resonance

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Context

- In the last lecture, we discussed simple zeros and poles of a transfer function, and creating approximate Bode plots for hand plotting of phase and magnitude characteristics.
- In this lecture, we will discuss second order transfer functions, circuits which have resonances.
Second Order Circuits

- The series resonant circuit is one of the most important elementary circuits:

![Series Resonant Circuit](image)

- This model is not only useful for physical LCR circuits, but also approximates mechanical resonances, molecular resonance, microwave cavities, transmission lines, buildings, bridges, ...

Time Domain analysis

- The differential equations for this circuit are:

\[
\begin{align*}
    v_r(t) &= i_r(t) \cdot r \\
    i_c(t) &= C \frac{dv_C(t)}{dt} \\
    v_L(t) &= L \frac{di_L(t)}{dt} \\
    v_s(t) &= v_L(t) + v_C(t) + v_R(t) \\
    v_s &= L \frac{d}{dt} \left( C \frac{dv_C(t)}{dt} \right) + v_C(t) + RC \frac{dv_C(t)}{dt}
\end{align*}
\]
So the differential equation for the circuit is:

\[ v_s(t) = LC \frac{d^2}{dt^2} v_c(t) + v_c(t) + RC \frac{dv_c(t)}{dt} \]

Let’s see how this circuit responds to a step input, zero before time \( t=0 \), and \( V_{dd} \) for \( t>0 \)

First of all, note that the steady state solution is

\[ v_s(t \to \infty) = V_{dd} \]

**Transient solution**

To find the transient solution, for \( t>0 \), let’s try a solution of the form:

\[ v_c(t) = Ae^{st} + V_{dd} \]

- Where \( s \) is a complex number

Now we substitute this into our D. E.:

\[ v_s(t) = LC \frac{d^2}{dt^2} v_c(t) + v_c(t) + RC \frac{dv_c(t)}{dt} \]

\[ V_{dd} = LCs^2 Ae^{st} + V_{dd} + Ae^{st} + RCsAe^{st} \]

Giving us a second order equation for \( s \):

\[ 0 = LCs^2 + 1 + RCs \]
We can use the quadratic formula to find solutions for $s$:

$$0 = LCs^2 + 1 + RCs$$

$$s_{1,2} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

- If $s$ is real, then the circuit is **overdamped**, and the voltage will change exponentially to its steady state value.
- If $s$ is complex, the circuit is **underdamped**, and the solution will oscillate around the steady state value before settling down to it.

Underdamped case

For the underdamped case:

$$s_{1,2} = \frac{R}{2L} \pm j \sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}$$

- We will need to take sums of the complex exponentials to get real solutions, solutions are of the form:

$$v_C(t) = V_{dd} + Ae^{-\alpha t} \sin(\omega t + \phi)$$

- Where:

$$\omega = \sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}$$

$$\alpha = \frac{R}{2L}$$

and $A$ and $\phi$ are determined by the boundary conditions.
Response of underdamped circuit to step

- If the circuit is moderately damped:

Underdamped response to step

- And if very underdamped:
Underdamped Oscillations:

- In the very underdamped case (R small), the ringing dies exponentially in a time:
  \[ \alpha^{-1} = \frac{2L}{R} \]

- And each oscillation takes a time:
  \[ (2\pi\omega)^{-1} = \left(2\pi \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right)^{-1} \approx \frac{\sqrt{LC}}{2\pi} \]

- So the number of oscillations of ringing that will occur is approximately:
  \[ N \approx \frac{2L}{R} \frac{\sqrt{LC}}{2\pi} \approx \frac{1}{\pi R} \sqrt{\frac{L}{C}} \]

Frequency domain analysis

- With phasor analysis, this circuit is readily analyzed, for example, the input impedance:
  \[ Z = j\omega L + \frac{1}{j\omega C} + R \]
  \[ Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega L \left(1 - \frac{1}{\omega^2 LC}\right) \]
  You can also write the expression for the voltage across any component
**Series LCR Impedance**

- For example, the voltage across the capacitor:

\[
V_c = I_c Z_C = \frac{1}{j\omega C} \left( j\omega L + \frac{1}{j\omega C} + R \right)^{-1} V_s
\]

**Low frequency behavior**

- At low frequencies, the characteristic of this circuit is dominated by the capacitor.

The inductor looks like a short, at low frequencies.

The \(\omega\) in the denominator of the term for the capacitor makes it the major contribution.
High frequency behavior

- At high frequencies, the characteristics of this circuit are dominated by the inductor.

The capacitor looks like a short at low frequencies.

The $\omega$ proportionality of the term for the inductor makes it the major contribution.

\[ Z = j\omega L + \frac{1}{j\omega C} + R \]

Near resonance

- Near resonance, energy will oscillate between the capacitor and the inductor.

Notice that the terms for the capacitor and the inductor have opposite sign, so they can add up to zero impedance at one frequency.

At that frequency ($\omega_0 = \frac{1}{\sqrt{LC}}$), energy is perfectly oscillating between the inductor and the capacitor, →The only impedance left at that frequency is the resistor.
At resonance

\[ \text{Im}[Z] = \left( j\omega L + \frac{1}{j\omega C} \right) = 0 \]

\[ \omega^2 = \frac{1}{LC} \]

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**Power sidebar**

The power into a circuit is just the voltage times the current, but remember this is not linear, so we’ll go back to complete notation:

\[ v(t) = \frac{1}{2}(\hat{V}e^{j\omega t} + \hat{V}^* e^{-j\omega t}) \]

\[ i(t) = \frac{1}{2}(\hat{I}e^{j\omega t} + \hat{I}^* e^{-j\omega t}) \]

\[ \frac{\hat{V}}{\hat{I}} = Z \]
Convert from phasors, then multiply

- Power:
  \[ P = v(t) \cdot i(t) = \frac{1}{4} (\hat{V}_e^{j\omega} + \hat{V}^* e^{-j\omega})(\hat{I}_e^{j\omega} + \hat{I}^* e^{-j\omega}) \]

  \[ P = \frac{1}{4} (\hat{V}_e^{j\omega} + \hat{I}^* Z * e^{-j\omega})(\hat{V}_e^{j\omega} + \hat{I}^* e^{-j\omega}) \]

  \[ P = \frac{1}{4} \left( (\hat{V}_e^{j\omega} \hat{I}_e^{j\omega}) + (\hat{V}_e^{j\omega} \hat{I}^* e^{-j\omega}) + (\hat{V}^* \hat{I}_e^{j\omega} e^{j\omega}) + (\hat{V}^* \hat{I}^* e^{-j\omega} e^{j\omega}) \right) \]

  \[ P = \frac{1}{4} \left( (\hat{V}_e^{j\omega} \hat{I}_e^{j\omega}) + (\hat{V}^* \hat{I}^* e^{-j\omega}) + (\hat{V}^* \hat{I}^* e^{-j\omega}) + (\hat{V}^* \hat{I}^* e^{-j\omega}) \right) \]

Expression for Power from Z

- Power:
  \[ P = \frac{1}{4} \left( (\hat{V}_e^{j\omega} + \hat{V}^* \hat{I}) + (\hat{V}_e^{j\omega} + \hat{V}^* \hat{I} e^{-j\omega}) \right) \]

  \[ P = \frac{1}{4} \left( (\hat{I} Z \hat{I}^*) + (\hat{I} Z \hat{I}^*) + (\hat{I} Z \hat{I}^*) + (\hat{I} Z \hat{I}^*) \right) \]

- Notice that the second and fourth terms are just the complex conjugate of the first and third, so we can write:
  \[ P = \frac{1}{2} RE \{ \hat{I} Z \hat{I}^* \} + \frac{1}{2} RE \{ \hat{I} Z \hat{I}^* \} \]

- The second term averages to zero, so the first term gives us the average power
Reactive Power

- Let’s take the phase angle of the current to be zero, to make it simpler:
  \[ P = \left| I \right|^2 \left( \frac{1}{2} Z \cos \omega t - \frac{1}{2} \text{RE} \left\{ Ze^{j2\omega t} \right\} \right) \]

- Now look at the second term:
  \[ P_2 = \frac{1}{2} \left| I \right|^2 \text{RE} \left\{ Z \left( \cos(2\omega t) + j \sin(2\omega t) \right) \right\} \]

\[ P_2 = \frac{1}{2} \left| I \right|^2 \left( Z_r \cos(2\omega t) - Z_i \sin(2\omega t) \right) \]

- The first term is unavoidable ripple in the power, since this is AC, but the imaginary part of \( Z \) is power which goes in at one part of the cycle, and then is returned at a later point.
  This is called reactive power.

Resonance

- At resonance the circuit impedance is purely real

- The reactive power for the Cap is supplied from the inductor, and visa versa.

- Imaginary components of impedance cancel out

- For a series resonant circuit, the current is maximum at resonance
**Series Resonance Voltage Gain**

- At *resonance*, the voltage across the inductor, and across the capacitor, can be larger than the voltage of the voltage source:

  \[ V_L = I \frac{1}{j\omega_0 C} = \frac{V_s}{Z(\omega_0)} \frac{1}{j\omega_0} = \frac{V_s}{R \omega_0} j\omega_0 L = jQV_s \]

  \[ V_C = I \frac{1}{j\omega_0 L} = \frac{V_s}{Z(\omega_0)} \frac{\omega_0 L}{j} = -\frac{V_s}{R} j\omega_0 L = -jQV_s \]

- Remember:

  \[ Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C} = \frac{\sqrt{LC}}{R} = \sqrt{\frac{L}{C}} \frac{1}{R} = \frac{Z_0}{R} \]

**Second Order Transfer Function**

- So we have:

  \[ H(j\omega) = \frac{V_0}{V_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} \]

- To find the poles/zeros, let’s put the \( H \) in canonical form:

  \[ H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC} \]

- One zero at DC frequency \( \rightarrow \) can’t conduct DC due to capacitor
Poles of 2nd Order Transfer Function

- Denominator is a quadratic polynomial:

\[ H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC} = \frac{j\omega R}{L} \]

\[ H(j\omega) = \frac{j\omega R}{\omega_0^2 + (j\omega)^2 + j\omega \frac{R}{L}} \]

\[ \alpha_0^2 \equiv \frac{1}{LC} \]

\[ H(j\omega) = \frac{j\omega \omega_0}{Q} \]

\[ \frac{\omega_0^2}{Q} + (j\omega)^2 + j\omega \frac{\omega_0}{Q} \]

\[ Q \equiv \frac{\omega_0 L}{R} \]

Finding the poles...

- Let’s factor the denominator:

\[ (j\omega)^2 + j\frac{\omega_0 \omega_0}{Q} + \omega_0^2 = 0 \]

\[ \omega = -\frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \omega_0^2} = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1 - \frac{1}{4Q^2}} \]

- Poles are complex conjugate frequencies
Resonance without Loss

- The transfer function can be parameterized in terms of loss. First, take the lossless case, $R=0$:

$$\omega = \left( -\frac{\alpha_0}{2Q} \pm \sqrt{\frac{\alpha_0^2}{4Q^2} - \omega_0^2} \right)_{Q \to \infty} = \pm j\omega_0$$

- When the circuit is lossless, the poles are at real frequencies, so the transfer function blows up!

- At this resonance frequency, the circuit has zero imaginary impedance and thus zero total impedance

- Even if we set the source equal to zero, the circuit can have a steady-state response

Magnitude Response

- How peaked the response is depends on $Q$

$$H(j\omega) = \frac{j\omega_0 \frac{R}{L}}{\omega_0^2 - \omega^2 + j\omega_0 \frac{R}{L}} = \frac{j\omega_0 \frac{R}{Q}}{\omega_0^2 - \omega^2 + j\omega_0 \frac{R}{Q}}$$

$$H(j\omega_0) = \frac{j\omega_0^2}{Q} \frac{\omega_0^2}{\omega_0^2 - \omega_0^2 + j\omega_0^2} = 1$$
How Peaky is it?

- Let’s find the points when the transfer function squared has dropped in half:

\[
|H(j\omega)|^2 = \frac{\left(\frac{\omega}{Q}\right)^2}{\left(\frac{\omega}{Q}\right)^2 + \left(1 - \frac{\omega}{Q}\right)^2} = \frac{1}{2}
\]

\[
|H(j\omega)|^2 = \frac{1}{\left(\frac{\omega}{Q}\right)^2 + 1} = \frac{1}{2}
\]

\[
\left(\frac{\omega}{Q}\right)^2 + 1 = 1
\]

\[
\left(\frac{\omega}{Q}\right)^2 = 0
\]

\[
\omega = \pm \frac{\omega_0 \pm \sqrt{\frac{\omega_0^2}{4Q^2} + \omega_0^2}}{2Q} = \pm a \pm b
\]

Four solutions!

\[
+ a + b > 0
\]

\[
- a + b > 0
\]

\[
+ a - b < 0
\]

\[
- a - b < 0
\]

Take positive frequencies:

\[
\Delta \omega = \omega_+ - \omega_- = \frac{\omega_0}{Q}
\]

\[
\frac{\Delta \omega}{\omega_0} = \frac{1}{Q}
\]
More “Notation”

- Often a second-order transfer function is characterized by the “damping” factor as opposed to the “Quality” factor

\[ \omega_0^2 + (j \omega)^2 + j \frac{\omega_0 \omega}{Q} = 0 \]
\[ 1 + (j \omega \tau)^2 + j \frac{\omega \tau}{Q} = 0 \]
\[ 1 + (j \omega \tau)^2 + (j \omega \tau) 2 \zeta = 0 \]
\[ Q = \frac{1}{2 \zeta} \]

Second Order Circuit Bode Plot

- Quadratic poles or zeros have the following form:

\[ (j \omega \tau)^2 + (j \omega \tau) 2 \zeta + 1 = 0 \]

- The roots can be parameterized in terms of the damping ratio:

\[ \zeta = 1 \Rightarrow (j \omega \tau)^2 + (j \omega \tau) 2 + 1 = (1 + j \omega \tau)^2 \]
\[ \zeta > 1 \Rightarrow (j \omega \tau)^2 + (j \omega \tau) 2 \zeta + 1 = (1 + j \omega \tau_1)(1 + j \omega \tau_2) \]
\[ j \omega \tau = -\frac{\zeta \pm \sqrt{\zeta^2 - 1}}{2} \]

- Two equal poles
- Two real poles
Bode Plot: Damped Case

- The case of $\zeta > 1$ and $\zeta = 1$ is a simple generalization of simple poles (zeros). In the case that $\zeta > 1$, the poles (zeros) are at distinct frequencies. For $\zeta = 1$, the poles are at the same real frequency:

\[
\zeta = 1 \quad \Rightarrow \quad (j\omega \tau)^2 + (j\omega \tau)2 + 1 = (1 + j\omega \tau)^2
\]

\[
\| (1 + j\omega \tau) \|^2 = | 1 + j\omega \tau |^2
\]

\[
20 \log | 1 + j\omega \tau | = 40 \log | j\omega |.
\]

\[
\angle (1 + j\omega \tau)^2 = \angle (1 + j\omega \tau) + \angle (1 + j\omega \tau) = 2 \angle (1 + j\omega \tau)
\]

Asymptotic Slope is 40 dB/dec

Asymptotic Phase Shift is 180°

Underdamped Case

- For $\zeta < 1$, the poles are complex conjugates:

\[
(j\omega \tau)^2 + (j\omega \tau)2\zeta + 1 = 0
\]

\[
j\omega \tau = -\zeta \pm \sqrt{\zeta^2 - 1} = \zeta \pm j\sqrt{1 - \zeta^2}
\]

- For $\omega \tau << 1$, this quadratic is negligible (0dB)
- For $\omega \tau >> 1$, we can simplify:

\[
20 \log | (j\omega \tau)^2 + (j\omega \tau)2\zeta + 1 | \approx 20 \log | (j\omega \tau)^2 | = 40 \log | \omega \tau |
\]

- In the transition region $\omega \tau \sim 1$, things are tricky!
Underdamped Mag Plot

The phase for the quadratic factor is given by:

\[
\angle((j\omega\tau)^2 + (j\omega\tau)2\zeta + 1) = \tan^{-1}\left(\frac{2\omega\tau\zeta}{1-(\omega\tau)^2}\right)
\]

- For \(\omega\tau < 1\), the phase shift is less than 90°
- For \(\omega\tau = 1\), the phase shift is exactly 90°
- For \(\omega\tau > 1\), the argument is negative so the phase shift is above 90° and approaches 180°
- Key point: argument shifts sign around resonance
Phase Bode Plot

\[ \zeta = \begin{cases} 0.01 \\ 0.1 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \end{cases} \]

Bode Plot Guidelines

- In the transition region, note that at the breakpoint:
  \[ (j\omega)^2 + (j\omega)2\zeta + 1 = (j)^2 + (j)2\zeta + 1 = 2\zeta = \frac{1}{Q}. \]

- From this you can estimate the peakiness in the magnitude response.

- Example: for \( \zeta = 0.1 \), the Bode magnitude plot peaks by 20 \( \log(5) \sim 14 \) dB.

- The phase is much more difficult. Note for \( \zeta = 0 \), the phase response is a step function.

- For \( \zeta = 1 \), the phase is two real poles at a fixed frequency.

- For \( 0 < \zeta < 1 \), the plot should go somewhere in between!
**Energy Storage in “Tank”**

- At resonance, the energy stored in the inductor and capacitor are
  
  \[ w_L = \frac{1}{2} L (i(t))^2 = \frac{1}{2} LI_M^2 \cos^2 \omega_0 t \]
  
  \[ w_C = \frac{1}{2} C (v(t))^2 = \frac{1}{2} C \left( \frac{1}{C} \int i(\tau) d\tau \right)^2 \]
  
  \[ = \frac{1}{2} C \frac{I_M^2}{\omega_0^2 C^2} \sin^2 \omega_0 t = \frac{1}{2} \frac{I_M^2}{\omega_0^2 C} \sin^2 \omega_0 t \]
  
  \[ w_s = w_L + w_C = \frac{1}{2} I_M^2 \left( L \cos^2 \omega_0 t + \frac{1}{\omega_0^2 C} \sin^2 \omega_0 t \right) = \frac{1}{2} I_M^2 L \]
  
  \[ W_{L,\text{max}} = W_S = \frac{1}{2} I_M^2 L \]

**Energy Dissipation in Tank**

- Energy dissipated per cycle:
  
  \[ w_D = P \cdot T = \frac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0} \]
  
  - The ratio of the energy stored to the energy dissipated is thus:
  
    \[ \frac{w_s}{w_D} = \frac{\frac{1}{2} LI_M^2}{\frac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{2\pi R} \frac{1}{\omega_0} = Q \]
    
    \[ \frac{w_s}{w_D} = \frac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0} = \frac{\omega_0 L}{2\pi R} \]
    
    \[ Q = \frac{\omega_0 L}{2\pi R} \]

Physical Interpretation of Q-Factor

- For the series resonant circuit we have related the Q factor to very fundamental properties of the tank:
  \[ Q = 2\pi \frac{W_s}{W_D} \]

- The tank quality factor relates how much energy is stored in a tank to how much energy loss is occurring.
- If \( Q \gg 1 \), then the tank pretty much runs itself … even if you turn off the source, the tank will continue to oscillate for several cycles (on the order of Q cycles)
- Mechanical resonators can be fabricated with extremely high Q

thin-Film Bulk Acoustic Resonator (FBAR)

- Agilent Technologies (IEEE ISSCC 2001)
- \( Q > 1000 \)
- Resonates at 1.9 GHz
- Can use it to build low power oscillator