ON THE TOPOLOGICAL ENTROPY OF THE IRREGULAR PART OF V-STATISTICS MULTIFRACTAL SPECTRA

ALEJANDRO M. MESON AND FERNANDO VERICAT
INSTITUTO DE FISICA DE LIQUIDOS Y SISTEMAS BIOLOGICOS (IFLYSIB)
CONICET LA PLATA - UNLP
AND
GRUPO DE APLICACIONES MATEMATICAS Y ESTADISTICAS DE LA FACULTAD DE INGENIERIA (GAMEFI), UNLP, LA PLATA, ARGENTINA

E-MAILS: MESON@IFLYSIB.UNLP.EDU.AR; VERICAT@IFLYSIB.UNLP.EDU.AR

(Received: 05 December 2012, Accepted: 05 January 2013)

Abstract. Let \((X,d)\) be a compact metric space and \(f : X \to X\), if \(X^r\) is the product of \(r\) copies of \(X\), \(r \geq 1\), and \(\Phi : X^r \to \mathbb{R}\), then the multifractal decomposition for \(V\)-statistics is given by

\[
E_\Phi(\alpha) = \left\{ x : \lim_{n \to \infty} \frac{1}{n^r} \sum_{0 \leq i_1 \leq \ldots \leq i_r \leq n - 1} \Phi(f^{i_1}(x), \ldots, f^{i_r}(x)) = \alpha \right\}.
\]

The irregular part, or historic set, of the spectrum is the set points \(x \in X\), for which the limit does not exist.

In this article we prove that for dynamical systems with specification, the irregular part of the \(V\)-statistics spectrum has topological entropy equal to that of the whole space \(X\).

AMS Classification: 37C45, 37B40

Keywords: Topological entropy; \(V\)-statistics; Multifractal spectra

1. INTRODUCTION

Motivated by the problems on convergence of multiple ergodic averages Fan, Schmeling and Wu[5], treated the problem of multifractal analysis of \(V\)-statistics.
In the present paper, we would like to study the irregular part of the multifractal decomposition.

Let us consider a topological dynamical system \((X, f)\), with \(X\) a compact metric space and \(f\) a continuous map. Let \(X^r = X \times \ldots \times X\) be the product of \(r\)-copies of \(X\) with \(r \geq 1\), if \(\Phi : X^r \to \mathbb{R}\) is a continuous map, then let

\[
V_\Phi(n, x) = \frac{1}{n^r} \sum_{1 \leq i_1, \ldots, i_r \leq n} \Phi(f^{i_1}(x), \ldots, f^{i_r}(x)).
\]

These averages are called the \(V\)-statistics of order \(r\) with kernel \(\Phi\). For the idea of \(V\)-statistics from a Statistical point of view and its relationship with the \(U\)-statistics see section 2 of \([5]\)

Ergodic limits of the form

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi(f^{i_1}(x), \ldots, f^{i_r}(x)),
\]

were studied among others by Furstenberg\([8]\], Bergelson\([2]\) and Bourgain\([3]\).

The multifractal spectra of \(V\)-statistics are specified by the decomposition sets

\[
E_\Phi(\alpha) = \left\{ x : \lim_{n \to \infty} V_\Phi(n, x) = \alpha \right\}.
\]

Fan, Schemeling and Wu\([5]\) treated the problem of measuring the sizes of the multifractal sets \(E_\Phi(\alpha)\). They established the following variational principle:

\[
h_{top}(E_\Phi(\alpha)) = \sup \left\{ h_\mu(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\},
\]

where \(h_\mu\) is the measure-theoretic entropy of \(\mu\). This formula is valid for dynamical systems with the specification property. This generalizes the variational formula obtained by Takens and Verbitski for \(r = 1\)\([9]\).

The irregular part of the spectrum, or historic set, is the set of points \(x\) for which \(\lim_{n \to \infty} V_\Phi(n, x)\) does not exist. We denote this set by \(E_\Phi^\infty\), so that the space \(X\) can be decomposed as

\[
X = \bigcup_{\alpha \in \mathbb{R}} E_\Phi(\alpha) \cup E_\Phi^\infty
\]
An important problem in Multifractal Analysis is to determine the dimension of the irregular part. For $r = 1$ the irregular part of the spectrum has been extensively studied. Fan, Feng and Wu, in reference [6], did it for topological mixing subshifts. Barreira and Schmeling[1] obtained a similar result than [6] but for Hölder continuous maps. More recently the irregular part was studied by Thompson [11] and by Zhou and Chen [13]. Here we propose the study of the irregular part of the spectrum for multiple ergodic averages. The result to be proved is

**Theorem:** Let $(X, f)$ be a dynamical system with the property of specification, let $\Phi \in C(X^r)$, $r \geq 1$, if the irregular part $E_\Phi^\infty$ of the spectrum of multiple ergodic averages $V_\Phi(n, x)$ is non-empty then it has the same topological entropy as the whole space $X$.

The case $E_\Phi^\infty = \emptyset$ can occur in situations like for instance $\Phi$ cohomologous to 0, or when the ergodic limits $V_\Phi(n, x)$ have the same value for any $x$.

2. Preliminary definitions

Firstly let us recall the Bowen definition of topological entropy of sets: Let $f : X \to X$, with $X$ a compact metric space, for $n \geq 1$ the dynamical metric, or Bowen metric, is $d_n(x, y) = \max \{d(f^i(x), f^i(y)) : i = 0, 1, ..., n - 1\}$. We denote by $B_n(x)$ the ball of centre $x$ and radius $\varepsilon$ in the metric $d_n$. Let $Z \subset X$ and let $C(n, \varepsilon, Z)$ be the collection of finite or countable coverings of the set $Z$ by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z, s, n, \varepsilon) = \inf_{B \in C(n, \varepsilon, Z)} \sum_{B_{m,\varepsilon}(x) \in B} \exp(-sm),$$

and set

$$M(Z, s, \varepsilon) = \lim_{n \to \infty} M(Z, s, n, \varepsilon).$$

There is an unique number $\pi$ such that $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z, \varepsilon) = \pi = \sup \{s : M(Z, s, \varepsilon) = +\infty\} = \inf \{s : M(Z, s, \varepsilon) = 0\},$$

and

(2) $$h_{top}(Z) = \lim_{\varepsilon \to 0} H(Z, \varepsilon).$$
The number $h_{\text{top}}(Z)$ is the topological entropy of $Z$.

A dynamical system $(X, f)$ has the specification property if the following condition holds: for $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any finite disjoint collection of integer intervals $I_1 = [a_1, b_1], ..., I_k = [a_k, b_k]$, of length $\geq M(\varepsilon)$ and for any points $x_1, x_2, ..., x_k \in X$, there is a point $z \in X$ which $\varepsilon$-shadows the sequence $\{x_1, x_2, ..., x_k\}$, i.e. $d(f^{\tau_i+n}(z), f^{n}(x_j)) \leq \varepsilon$, for any $n = 0, \ldots, b_j - a_j$ and $j = 0, 1, \ldots, k$.

By $\mathcal{M}(X)$ we denote the space of measures in $X$, and by $\mathcal{M}_{\text{inv}}(X, f)$ the space of $f$-invariant measures on $X$. The space $\mathcal{M}(X)$ can be endowed with a metric $D$ compatible with the metric in $X$, in the sense that $D(\delta_x, \delta_y) = d(x, y)$, where $\delta$ is the point mass measure. More precisely the metric considered in $\mathcal{M}(X)$ will be

$$D(\mu, \nu) = \sup_{\phi_n} \left\| \int \phi_n d\mu - \int \phi_n d\nu \right\|_{\infty},$$

where $\{\phi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center $\mu$ and radius $R$ in the above metric. The topology induced by this metric is the weak star topology, and if $X$ is compact then $\mathcal{M}(X)$ is compact in the weak topology. The weak star convergence is the convergence in the metric which induces the weak star topology.

The so called empirical measures on $X$ associated to the dynamical system $(X, f)$ are

$$E_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

We denote the weak limits of the sequence $\{E_n(x)\}$ by $V(x)$. Since $X$ is compact, $V(x) \neq \emptyset$. If $\mu$ is a measure on $X$ then a point $x \in X$ is $\mu$-generic if $V(x) = \{\mu\}$, by $G(\mu)$ is denoted the set of $\mu$-generic points. A result by Bowen[4] is that if $\mu$ is ergodic then

$$h_{\text{top}}(G(\mu)) = h_{\mu}(f).$$

For general measures, not necessarily ergodic, the equality holds for dynamical systems with the specification property[7]. This result is the key point in the proof of variational theorem of Fan, Schemeling and Wu[5].
3. Proof of the theorem

Let
\[ M_{\Phi}(\alpha) = \left\{ \mu \in M_{inv}(X) : \int \Phi \mu^\otimes r = \alpha \right\}, \]
and let
\[ G_{\Phi}(\alpha) = \left\{ x : \text{there is } \{n_k\} \text{ such that } w^* - \lim_{k \to \infty} E_{n_k}(x) = \mu \in M_{\Phi}(\alpha) \right\}, \]
here \( w^* - \) means weak star convergence.

For \( \alpha_1 \neq \alpha_2 \in \mathbb{R} \), we shall find a set \( G \subset G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2) \).

Before proving the theorem we give some lemmas.

**Lemma 1:** If \( \alpha_1 \neq \alpha_2 \) then \( G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2) \subset E_{\Phi}^\infty \).

**Proof:** In [5] was established, as a consequence of the Stone-Weierstrass theorem, that for any \( \Phi \in C(X^r) \) and for any \( \varepsilon > 0 \) there is a map \( \tilde{\Phi} : X^r \to \mathbb{R} \) of the form
\[ \tilde{\Phi} = \sum_j \varphi_j^{(1)} \otimes \ldots \otimes \varphi_j^{(r)}, \]
with \( \varphi_j^{(i)} \in C(X) \) such that \( \|\Phi - \tilde{\Phi}\|_\infty < \varepsilon \). Let \( x \in G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2) \), so there are sequences \( \{n_k\}, \{m_k\} \) such that

\[ \mu = w^* - \lim_{k \to \infty} E_{n_k}(x) ; \mu \in M_{\Phi}(\alpha_1) \]
\[ \nu = w^* - \lim_{k \to \infty} E_{m_k}(x) ; \nu \in M_{\Phi}(\alpha_2) , \]

We have
\[ V_{\Phi}(n, x) = \sum_j \prod_{i=1}^r \frac{1}{n} S_n \left( \varphi_j^{(i)}(x) \right), \]
where \( S_n \left( \varphi_j^{(i)}(x) \right) = \sum_{k=0}^{n-1} \varphi_j^{(i)}(f^k(x)) \). Therefore, by Eqs.(3-4)
\[
\lim_{k \to \infty} V_{\Phi}(n_k, x) = \int \Phi d\mu^\otimes r
\]
\[
\lim_{k \to \infty} V_{\Phi}(m_k, x) = \int \Phi d\nu^\otimes r.
\]

By the above argument of approximation we get in the same way of [5] that
\[
\lim_{k \to \infty} V_{\Phi}(n_k, x) = \int \Phi d\mu^\otimes r = \alpha_1 \text{ and } \lim_{k \to \infty} V_{\Phi}(m_k, x) = \int \Phi d\nu^\otimes r = \alpha_2, \text{ with } \alpha_1 \neq \alpha_2.
\]

Then \( x \in E_{\Phi}^\infty. \)

We have that
\[
G_{\Phi}(\alpha) \subset \left\{ x : \exists \mu \in V(x), \text{ such that } h_{\mu}(f) \leq \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^\otimes r = \alpha \right\} \right\},
\]
and so, by the Bowen lemma
\[
h_{\text{top}}(G_{\Phi}(\alpha)) \leq \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^\otimes r = \alpha \right\}
\]

For \( \rho_1, \rho_2, \ldots, \rho_k \in \mathcal{M}(X) \) and positive numbers \( R_1, R_2, \ldots, R_k \), let \( x_1, x_2, \ldots, x_k \in X, n_1, n_2, \ldots, n_k \in \mathbb{N} \) such that \( \mathcal{E}_{n_j}(x_j) \in B_{R_j}(\rho_j), j = 1, 2, \ldots, k. \), for a given \( \rho_1, \rho_2, \ldots, \rho_k \in \mathcal{M}(X) \) and \( R_1, R_2, \ldots, R_k \). Let \( \varepsilon_1 > 0, \varepsilon_2 > 0, \ldots, \varepsilon_k > 0 \), if \( n_i > M(\varepsilon_i) \) (the number of specification), \( i = 1, 2, \ldots, k, \) then by specification property
\[
\bigcap_{j=1}^{k} f^{-M_{j-1}}(B_{n_j, \varepsilon_j}(x_j)) \neq \emptyset, \text{with } M_j = n_1 + n_2 + \ldots + n_j.
\]

**Lemma 2:** Let \( z \in \bigcap_{j=1}^{k} f^{-M_{j-1}}(B_{n_j, \varepsilon_j}(x_j)) \), then for any \( \rho \in \mathcal{M}(X) \) holds
\[
D(\mathcal{E}_{M_k}(z), \rho) \leq \frac{1}{M_k} \sum_{j=1}^{k} n_j (\overline{R_j} + D(\rho_j, \rho)),
\]
where \( \overline{R_j} = R_j + \varepsilon_j, j = 1, 2, \ldots, k. \)

**Remark:** It can replace an uniform \( \varepsilon \) for all balls \( B_{n_i, \varepsilon}(x_j) \), by the \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \)

**Proof:** We have
ON THE TOPOLOGICAL ENTROPY OF THE ...

\[ E_{M_k}(z) = \frac{1}{M_k} \sum_{j=1}^{k} n_j E_{n_j}(f^{M_j-1}(z)), \]

and

\[ D(E_{n_j}(x_j), E_{n_j}(f^{M_j-1}(z))) \leq \frac{1}{n_j} \sum_{l=0}^{n_j-1} d(f^l(x_j), f^{-M_j-1-l}(z)). \]

Therefore

\[
D(E_{M_k}(z), \rho) \\
\leq \frac{1}{M_k} \sum_{j=1}^{k} \left[ D(E_{n_j}(x_j), E_{n_j}(f^{M_j-1}(z))) + D(E_{n_j}(x_j), \rho) + D(\rho_j, \rho) \right] \\
\leq \frac{1}{M_k} \sum_{j=1}^{k} [R_j + \varepsilon_j + D(\rho_j, \rho)]
\]

\[ \blacksquare. \]

**Lemma 3:** Let \( \alpha_1 \neq \alpha_2 \) with \( M_\Phi(\alpha_1) \neq \emptyset, M_\Phi(\alpha_2) \neq \emptyset \) then

\[ h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) = \min \{ h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2)) \}. \]

**Proof:** Since \( G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_1) \) and \( G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_2) \), by the monotonicity of the entropy we have

\[ h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) \leq \min \{ h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2)) \}. \]

To prove the other inequality we shall find a set \( G \subset G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \) with \( h_{top}(G) \geq \min \{ h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2)) \}. \)

To construct \( G \), let us choose sequences \( \{n_k\}, \{R_k\}, \{\varepsilon_k\} \) with \( R_k \downarrow 0 \) and \( \varepsilon_k \downarrow 0 \) and, for a given sequence \( \{\rho_1, \rho_2, ..., \rho_k\} \subset \mathcal{M}(X) \), for \( \tau > \varepsilon_1 \) let us consider \( (n_k, \tau) \) -sets \( \Gamma_k \subset \{ x : E_{n_k}(x) \in B_{R_k}(\rho_k) \} \), so that (by the Lemma 2)

\[ x \in \Gamma_k, z \in B_{n_k, \varepsilon_k}(x) \implies E_{n_k}(z) \in B_{R_k + \varepsilon_k}(\rho_k). \]

Let us choose now a strictly increasing sequence \( \{N_k\} \) such that
\[ n_{k+1} \leq R_k \sum_{j=1}^{k} n_j N_j \]

and
\[ \sum_{j=1}^{k-1} n_j N_j \leq R_k \sum_{j=1}^{k} n_j N_j. \]

We consider stretched sequences \( \{ n'_j \}, \{ \epsilon'_j \}, \{ \Gamma'_j \} \) such that if \( j = N_1 + \ldots + N_{k-1} + q \) with \( 1 \leq q \leq N_k \) then \( n'_j = n_k, \ \epsilon'_j = \epsilon_k \) and \( \Gamma'_j = \Gamma_k \).

Finally, we can define
\[ G_k := \bigcap_{j=1}^{k} \left( \bigcup_{x_j \in \Gamma'_j} f^{-M_{j-1}} \left( B_{n'_j, \epsilon'_j} (x_j) \right) \right), \]

with \( M_j = n'_1 + n'_2 + \ldots + n'_j \) and
\[ G := \bigcap_{k \geq 1} G_k. \]

Any element of \( G \) can be labelled by a sequence \( x_1 x_2 \ldots, \) with \( x_j \in \Gamma_j \). According to Pfister and Sullivan [10] the following holds: Let \( x_j, y_j \in \Gamma'_j, x_j \neq y_j, \) if \( x \in B_{n_j, \epsilon_j} (x_j), y \in B_{n_j, \epsilon_j} (y_j) \) then \( \max \{ d \left( f^k(x), f^k(y) \right) : k = 0, \ldots, n_j - 1 \} > 2\epsilon, \)
with \( \epsilon > \epsilon_1/4. \)

We see that \( G \subset G_\Phi (\alpha_1) \cap G_\Phi (\alpha_2). \) Let \( z \in G, \) and let \( \mu_0 \in M_\Phi (\alpha_1), \nu_0 \in M_\Phi (\alpha_2), \) it can be considered sequences[13] \( \{ \mu_k \}, \{ \nu_k \} \) such that \( D(\mu_0, \mu_k) < R_k \) and \( D(\nu_0, \nu_k) < R_k, \) then form the sequence
\[ \{ \rho_k \} = \{ \mu_1, \mu_1, \nu_1, \nu_1, \mu_2, \mu_2, \nu_2, \nu_2, \ldots \}. \]

Let \( \rho \in \{ \mu_0, \nu_0 \}, \) and \( \sum_{i=1}^{j} n_i N_i \leq M_k \leq \sum_{i=1}^{j+1} n_i N_i, \) thus
\[ D \left( E_{M_k} (z), \rho \right) \leq \frac{1}{M_k} \sum_{i=1}^{j-1} n_i N_i D \left( E_{i+1} \sum_{i=1}^{n_i N_i} (z), \rho \right) + \frac{n_j N_j}{M_k} D \left( E_{n_j N_j} (z), \rho \right) + \]
\[ \frac{M_k - \sum_{i=1}^{j+1} n_i N_i}{M_k} D \left( E_{n_j N_j N_{j+1}} (z), \rho \right). \]

Therefore
\[ D(E_{M_k}(z), \rho) \]
\[ \leq R_j + D(E_{n_jN_j}(z), \rho_j) + D(\rho_j, \rho) + D(E_{n_j+1N_j+1}(z), \rho) + D(\rho_{j+1}, \rho) \]
\[ \leq 2R_j + \varepsilon_j + D(\rho_j, \rho) + D(\rho_{j+1}, \rho). \]

Thus, choosing subsequences \( t_k = 4k + 1 \) and \( s_k = 4k + 3 \), we get

\[ \mu_0 = w^* - \lim_{k \to \infty} E_{M_k}(z) \]
\[ \nu_0 = w^* - \lim_{k \to \infty} E_{M_k}(z), \]
so that \( z \in G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2). \)

To complete the proof it must be proved that

\[ h_{\text{top}}(G) \geq \min \{ h_{\text{top}}(G_\Phi(\alpha_1)), h_{\text{top}}(G_\Phi(\alpha_2)) \}. \]

For this, we follow [10]. Let \( s < \overline{h} := \min \{ h_{\text{top}}(G_\Phi(\alpha_1)), h_{\text{top}}(G_\Phi(\alpha_2)) \} \), the set \( G \) is closed, and so it is compact, let us consider a finite covering \( \mathcal{U} \) by balls \( B_{m, \varepsilon}(x) \) having non-empty intersection with \( G \). Now

\[ M(G, s, N, \varepsilon) = \inf_{\mathcal{U} \in \mathcal{C}(n, \varepsilon, G)} \sum_{B_{m, \varepsilon}(x) \in \mathcal{U}} \exp(-sm). \]

For any finite covering \( \mathcal{U} \) of \( G \), we can construct a covering \( \mathcal{U}_0 \) in the following way: each ball \( B_{m, \varepsilon}(x) \) is replaced by a ball \( B_{M_r, \varepsilon}(x) \) with \( M_r \leq m \leq M_{r+1} \). Thus

\[ M(G, s, N, \varepsilon) = \inf_{\mathcal{U} \in \mathcal{C}(n, \varepsilon, G)} \sum_{B_{m, \varepsilon}(x) \in \mathcal{U}} \exp(-sm) \geq \inf_{\mathcal{U} \in \mathcal{C}(N, \varepsilon, G)} \sum_{B_{M_r, \varepsilon} \in \mathcal{U}_0} \exp(-sm_{r+1}). \]

Now we can consider a covering \( \mathcal{U}_0 \) in which

\[ m = \max \{ r : \text{there is a ball } B_{M_r, \varepsilon}(x) \in \mathcal{U}_0 \}. \]

We set

\[ W_k := \prod_{i=1}^{k} \Gamma_i, \quad W_m = \bigcup_{k=1}^{m} W_k. \]
Let \( x_j, y_j \in \Gamma_j, x_j \neq y_j, \) as we pointed out earlier, if \( x \in B_{N_j, \varepsilon_j} (x_j), y \in B_{N_j, \varepsilon_j} (y_j) \) then \( d (f^l(x), f^l(y)) > 2 \varepsilon \) for any \( l = 0, \ldots, N_j - 1, \) and with \( \varepsilon > \varepsilon_1/4. \) Now for any \( x \in B_{M_r, \varepsilon} (z) \cap G \) there is a, uniquely determined \( z = z(x) \in W_r. \) A word \( \varpi \in W_j, \) with \( j = 1, 2, \ldots, k, \) is called a prefix of a word \( w \in W_k \) if the first \( j-\)letters of \( \varpi \) agree with the first \( j-\)letters of \( w. \) The number of times that each \( w \in W_k \) is a prefix of a word in \( W_m \) is

\[
\text{card} W_m / \text{card} W_k, \ \text{thus if} \ W \ \text{is a subset of} \ \overline{W}_m \ \text{then}
\]

\[
\sum_{k=1}^{m} \frac{\text{card} (W \cap W_k)}{\text{card} (W_k)} \geq \text{card} (W_m).
\]

If each word in \( W_m \) has a prefix contained in a \( W \subset \overline{W}_m \) then

\[
\sum_{k=1}^{m} \frac{\text{card} (W \cap W_k)}{\text{card} (W_k)} \geq 1,
\]

and since \( U_0 \) is a covering each point of \( W_m \) has a prefix associated to a ball in \( U_0. \) By this and because \( \text{card} W_k \geq \exp (\kappa M_r), \) we obtain

\[
\sum_{B_{M_r, \varepsilon} \in U_0} \exp (-sM_r) \geq 1.
\]

Thus if \( r \) is taken such that \( k \geq r \) then \( sM_{k+1} \leq \kappa M_k, \) for \( N \geq M_r, \ U \in G (N, \varepsilon, G). \)

Therefore

\[
\sum_{B_{m, \varepsilon} \in U} \exp (-sm) \geq 1,
\]

and so

\[
M (G, s, N, \varepsilon) \geq 1.
\]

By this \( h_{top} (G) \geq \kappa. \)

We are now in condition of giving the proof of the theorem. Let
\[ \Psi = \psi_{r, \Phi} : \mathcal{M}(X) \to \mathbb{R} \]
\[ \Psi (\mu) = \int \Phi d\mu \otimes r \]
and let
\[ h = h_{\text{top}}(X) \]
be the topological entropy of the whole space \( X \). By the classical variational principle and by the variational principle of [5]

\[ h = \sup \{ h_\mu (f) : \mu \in \mathcal{M}_{\text{inv}}(X, f) \} = \sup \{ h_\mu (f) : \mu \in \mathcal{M}_\Phi (\alpha) \} \]
\[ = \sup_{\alpha \in \text{Im}(\Psi)} \{ \text{h}_{\text{top}}(E_\Phi (\alpha)) \} . \]

We must show that \( h_{\text{top}}(E_\Phi^\infty) \geq h \). For any \( \gamma > 0 \), there is an \( \alpha_1 \in \text{Im}\Psi \) such that
\[ h_{\text{top}}(E_\Phi (\alpha_1)) > h - \gamma, \]
let \( \alpha_2 \in \text{Im}\Psi \) and let \( \mu_1, \mu_2 \in \mathcal{M}(X, f) \) with \( \Psi (\mu_1) = \alpha_1 \), \( \Psi (\mu_2) = \alpha_2 \). The map \( \lambda \mapsto \Psi ((1 - \lambda) \mu_1 + \lambda \mu_2) \) is continuous. Recall that

\[ h_{\text{top}}(G_\Phi (\alpha_1) \cap G_\Phi ((1 - \lambda) \alpha_1 + \lambda \alpha_2)) \]
\[ = \min \{ h_{\text{top}}(G_\Phi (\alpha_1), h_{\text{top}}(G_\Phi ((1 - \lambda) \alpha_1 + \lambda \alpha_2)) \} , \]
then, by the continuity of \( \Psi \) as a function of \( \lambda \), we have

\[ h_{\text{top}}(E_\Phi^\infty) \geq \lim_{\lambda \to 0} h_{\text{top}}(G_\Phi (\alpha_1) \cap G_\Phi ((1 - \lambda) \alpha_1 + \lambda \alpha_2)) \geq \]
\[ h_{\text{top}}(G_\Phi (\alpha_1)) \geq h_{\text{top}}(E_\Phi (\alpha_1)) > h - \gamma. \]

Since \( \gamma \) is arbitrary the result follows.

\[ \blacksquare \]

We wish to acknowledge the referees for the valuable remarks and suggestions to improve this paper. Support of this work by Consejo Nacional de Investigaciones Científicas y Técnicas (PIP 112-200801-01192), Universidad Nacional de La Plata (Grant 11/1108) and Agencia Nacional de Promoción Científica y Tecnológica of Argentina (PICT 2007-00908) is greatly appreciated. F.V. is a member of CONICET.
REFERENCES

[1] L- Barreira and J. Schmeling, Invariant sets with zero measure and full Hausdorff dimension, 
Electron. Res. Announc. Amer. Math. Soc. 3 (1997), 114-118

[2] V. Bergelson, Weakly mixing PET, Ergod. Th. and Dynam. Sys. 7, (1987) 337-349.

[3] J. Bourgain, Double recurrence and almost sure convergence, J. Reine Angew Math 404, 
(1990) 140-161.

[4] R. Bowen, Topological entropy for non-compact sets, Trans. Amer. Math. Soc., 184, (1973) 
125-136.

[5] A. H. Fan, J. Schmeling and J. Wu, The multifractal spectra of $V$–statistics, preprint, arXiv:1206.3214v1 (2012)

[6] A. Fan, D. J. Feng and J. Wu , Recurrence, dimension and entropy, J. London. Math. Soc., 64, (2001) 229-244.

[7] A. Fan, I. M. Liao and J. Peyrière, Generic points in systems of specification and Banach valued Birkhoff averages, Disc. Cont. Dynam. Sys. 21, (2008) 1103-1128.

[8] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, J. d’Analyse Math 31, (1977) 204-256.

[9] F. Takens and E. Verbitski, On the variational principle for the topological entropy of certain non-compact sets, Ergod. Th. and Dynam. Sys. 23, (2003) 317-348.

[10] C. E. Pfister and W.G. Sullivan, On the topological entropy of saturated sets, Ergod. Th. and Dynam. Sys. 27, (2007) 1-29.

[11] D. Thompson, The irregular set for maps with the specification property has full topological pressure, Dynam Sys: An International Journal’, 25(1) (2010) 25-51.

[12] P. Walters, An introduction to Ergodic Theory, (Springer-Verlag, Berlin,1982)

[13] X. Zhou and E. Chen, Topological pressure of historic set for $\mathbb{Z}^d$–actions, J. Math. Analysis and its applications. 389, (2012) 394-402.