On the Design of Integral Multiplex Control Protocols for Nonlinear Network Systems with Delays

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Abstract

We consider the problem of designing control protocols for possibly nonlinear networks with delays that not only allow the fulfillment of some desired behaviour, but also simultaneously guarantee the rejection of polynomial disturbances and the non-amplification of other classes of disturbances across the network. To address this problem, we propose the systematic use of multiplex architectures to deliver integral control protocols ensuring the desired disturbance rejection and non-amplification properties. We then present a set of sufficient conditions to assess these properties and hence to design the multiplex architecture for both leaderless and leader-follower networks with time-varying references consisting of possibly heterogeneous nonlinearly coupled agents affected by communication delays. The effectiveness of our conditions, which are also turned into an optimisation problem allowing protocol design, is illustrated via both in-silico and experimental validations with a real hardware set-up.

1 Introduction

Driven by the introduction of low-cost, high performance and connected devices, network systems have considerably increased their size and complexity. In this context, we tackle the challenge of designing control protocols for the network that do not only guarantee the fulfillment of some desired behaviour, but also satisfy the following requirements: (i) ensure that certain classes of disturbances are rejected, i.e., steady state errors caused by these disturbances are compensated; (ii) guarantee that the disturbances that are not rejected are not amplified across the network. Vehicle platoons (Stüdli et al., 2017; Feng et al., 2019; Monteil et al., 2019), robot formations (Liu et al., 2018; Li et al., 2019), power grids (Dörfler and Bullo, 2014; Silva et al., 2022), neural (Jafarpour et al., 2021; Revay and Manchester, 2020; Xie et al., 2021) and biochemical (Qian and Del Vecchio, 2018) networks are just few examples of network systems for which these two control requirements cannot be neglected. Motivated by this, in the present paper we study the problem of designing control protocols for networks that not only allow the achievement of some desired behaviours, but also, by means of integral actions, allow rejection of polynomial disturbances and ensure the non-amplification of other classes of disturbances across the network. We capture the fulfillment of these requirements via a scalability property of the network (see Section 3 for definitions).

Related works. The study of multiplex networks has emerged within the physics and control literature. We refer to (Mucha et al., 2010; De Domenico et al., 2013; Tran et al., 2020; Gomez et al., 2013; Burbano Lombana and di Bernardo, 2016) and references therein for a set of results, with literature reviews, on these networks. We now briefly survey some related works on disturbance propagation in networks. The study of how disturbances propagate within a network system is a central topic for the platooning of autonomous vehicles. In particular, the key idea behind several definitions of string stability (Swaroop and Hedrick, 1996) in the literature is that of giving upper bounds on the deviations induced by disturbances that are uniform with respect to the platoon size, see e.g. (Ploeg et al., 2014; Besselink and Johansson, 2017; Monteil et al., 2019). These works assume delay-free inter-vehicle communications and an extension to delayed platoons with linear vehicle dynamics can be found in e.g. (di Bernardo et al., 2015). In this context, we also recall (Xie and Russo, 2023) where networks with heterogeneous delays are considered and network stability (a weaker notion than scalability) is investigated in the presence of disturbances. Consensus stability for (disturbance-free) robotic networks subject to delays is instead considered in (Fan et al., 2021). The...


problem of designing a distributed integral action (Freeman et al., 2006; Andreasson et al., 2014; Seyboth and Allgöwer, 2015) has been recently considered in the context of string stability to design protocols able to compensate constant disturbances for delay-free platoons (Knorn et al., 2014; Silva et al., 2021). For networks with delay-free interconnections and no integral actions, we also recall results on mesh stability (Seiler et al., 1999) where linear dynamics are considered and its extension to nonlinear networks (Pant et al., 2002). Leader-to-formation stability is instead considered in (Tanner et al., 2004) and it characterizes network behaviour with respect to inputs from a leader. For delay-free leaderless networks with regular topology, the scalability property has been recently investigated in (Besselink and Knorn, 2018), where Lyapunov-based conditions are given; for networks with arbitrary topologies and delays, sufficient conditions for scalability can be found in (Xie et al., 2021), which leverages contraction theory arguments for time-delayed systems. Contracting systems exhibit transient and asymptotic behaviors (Lohmiller and Slotine, 1998) that are desirable when designing network systems (Centorrino et al., 2022); we refer to (Aminzare and Sontag, 2014; di Bernardo et al., 2016; Tsukamoto et al., 2021) and references therein for further details. We also recall (Wang and Slotine, 2006) which shows, using the Euclidean metric, how contraction is preserved through certain time-delayed communications and (Shiromoto et al., 2019) where conditions for the synthesis of distributed controls are given by using separable metric structures. For delay-free systems, based on the use of contraction, a sufficient condition for the stability of a feedback loop consisting of an exponentially stable multi-input multi-output nonlinear plant and an integral controller (to compensate constant disturbances) has been obtained in (Simpson-Porco, 2021). The problem of constant output regulation for a class of input-affine multi-input multi-output nonlinear systems with constant disturbances, has also been recently tackled via contraction in (Giaccagni et al., 2021). Other works have also shown that contraction using non-Euclidean metrics can be useful to study a wide range of biological (Russo et al., 2010a), neural (Davydov et al., 2021; Ofir et al., 2022; Centorrino et al., 2022) and engineered (Monteil and Russo, 2017) networks. We also recall (Russo and Wirth, 2022) the recent extension of contraction to dynamical systems on time scales, i.e. systems evolving on arbitrary (potentially non-uniform) time domains.

(i) we propose a novel, systematic, use of multiplex architectures to deliver protocols ensuring a scalability property (also introduced in this paper) for the network. This, besides guaranteeing the fulfillment of a desired network behaviour, implies rejection of the polynomial components of the disturbances and the non-amplification of their, e.g. residual, non-polynomial components;

(ii) our main methodological contribution is a set of sufficient conditions to assess these properties and hence to design the multiplex architecture. Our methodological results are based on showing that the desired network properties can be achieved by designing the control protocols so that a contractivity property is fulfilled in a specific non-Euclidean norm. Then, we give sufficient conditions for contractivity, leveraging non-Euclidean contraction arguments and certain structured norms. This allows to consider networks consisting of nonlinear (possibly, heterogeneous) agents affected by communication delays. Moreover, the results allow to consider leaderless and leader-follower networks, arbitrary topologies, nonlinear protocols and time-varying references;

(iii) we then leverage our conditions in the context of formation control to design protocols allowing the formation to track a reference provided by a leader, reject polynomial disturbances and ensure the non-amplification of other classes of disturbances. Specifically, after discussing certain design implications of the results, we show that the conditions can be conveniently recast into a convex optimisation problem that allows to design the control protocol; finally, we validate our protocols both in-silico and via a real hardware test-bed. All the experiments confirm the effectiveness of our approach.

In summary, with our methodological contributions we tackle the challenge of designing control protocols for (possibly, nonlinear) network systems affected by delays guaranteeing tracking of some desired behavior, rejection of polynomial disturbances and the non-amplification of other disturbances. As such, our results directly extend (Silva et al., 2021; Monteil et al., 2019), which are focused on string stability of platoon systems arranged on a string-like topology, and our prior work (Xie et al., 2021). Specifically: (i) in (Monteil et al., 2019) string stability was considered for networks without delays and their protocols cannot reject polynomial disturbances; (ii) in (Silva et al., 2021) constant disturbances (i.e., zero order polynomials) are compensated under the assumption that the network is delay-free; (iii) in (Xie et al., 2021) delays are instead considered but the protocols cannot guarantee any disturbance rejection. An early version of our results was presented

1. The code, data and all the required information to replicate our results are available at https://tinyurl.com/4vyac7z together with video recordings of the experiments.
Then: (i) $\beta$ Let in the literature on contracting systems. The next lemma
The norm $\| \cdot \|$ is denoted as $[A] := \frac{A + A^T}{2}$. Given a piece-wise continuous signal $w_i(t)$, we let $\|w_i(\cdot)\|_{\mathcal{L}_\infty} := \sup_{t} |w_i(t)|$. We denote by $I_n$ the $n \times n$ identity matrix and by $0_{m \times n}$ the $m \times n$ zero matrix (if $m = n$ we simply write $0_n$). We let $\text{diag}\{a_1, \ldots, a_N\}$ be a diagonal matrix with diagonal elements $a_i$. For a generic set $\mathcal{A}$, its cardinality is denoted by $|\mathcal{A}|$. Let $f$ be a smooth function, then we denote by $f^{(n)}$ the $n$-th derivative of $f$. We recall that a continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $K$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $K_{\infty}$ if $\alpha = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $K_{K}$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $K$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. The following lemma is stated in (Xie et al., 2021) and follows directly from (Russo et al., 2010). We let $| \cdot |_G$ and $\mu_s(\cdot)$ be, respectively, any $p$-vector norm and its induced matrix measure on $\mathbb{R}^n$. In particular, the norm $| \cdot |_G$ is monotone, i.e. for any non-negative N-dimensional vector $x, y \in \mathbb{R}^n, x \leq y$ implies that $|x| \leq |y|$ where the inequality $x \leq y$ is component-wise.

**Lemma 1.** Consider the vector $\eta := [n_1, \ldots, n_N]^T, n_i \in \mathbb{N}$. We let $| \cdot |_{G_1} := \left( |n_1| \cdot |G_1|, \ldots, |n_N| \cdot |G_N| \right)$, such that $| \cdot |_G$ is the norm induced by the diagonal matrix with diagonal elements $a_1, \ldots, a_N$. Finally, let:

1. $A := (A_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times n \times n}$, $A_{ij} \in \mathbb{R}^{n \times n}$;
2. $\hat{A} := (\hat{A}_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times n \times n}$, with $A_{ij} := \mu_G(\hat{A}_{ij})$ and $\hat{A}_{ij} := \|A_{ij}\|_{G_{ij}}$;
3. $A := (A_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times n \times n}$, with $A_{ij} := \|A_{ij}\|_{G_{ij}}$.

Then: (i) $\mu(A) \leq \mu_s(\hat{A})$; (ii) $|A|_G \leq |A|_G$.

The norm $| \cdot |_G$ is also known as structured (vector) norm in the literature on contracting systems. The next lemma is adapted from (Wen et al., 2008 Theorem 2.4).

**Lemma 2.** Let $u : [t_0 - \tau_{\max}, +\infty) \rightarrow \mathbb{R} \geq 0$, $\tau_{\max} < +\infty$ and assume that $D^+ u(t) \leq au(t) + b \sup_{t \in [t_0 - \tau(t) \leq s \leq t]} u(s) + c, \quad t \geq t_0$

with:

1. $\tau(t)$ being bounded and non-negative, i.e. $0 \leq \tau(t) \leq \tau_{\max}, \forall t$;
2. $u(t) = |\varphi(t)|, \forall t \in [t_0 - \tau_{\max}, t_0]$ where $\varphi(t)$ is bounded in $[t_0 - \tau_{\max}, t_0]$;
3. $a < 0, b > 0$ and $c \geq 0$ and that there exists some $\sigma > 0$ such that $a + b \leq -\sigma < 0, \forall t_0$.

Then:

$$u(t) \leq \sup_{t_0 - \tau_{\max} \leq s \leq t_0} u(s) e^{-\lambda(t-t_0) + \frac{c}{\sigma}}$$

where $\lambda := \inf_{t \geq t_0} \{ \lambda(t) + a + be^{\lambda(t)\tau(t)} = 0 \} > 0$ is the convergence rate of the system.

### 3 Statement of the Control Problem

We now describe the proposed control architecture (Section 3.1) and formalize the control goal (Section 3.2).

#### 3.1 Architecture Description

We consider a network system of $N > 1$ agents with the dynamics of the $i$-th agent given by

$$\dot{x}_i(t) = f_i(x_i, t) + u_i(t) + d_i(t), \quad t \geq t_0 \geq 0,$$

$$y_i(t) = g_i(x_i),$$

with initial conditions $x_i(0), i = 1, \ldots, N$, and where:

1. $x_i(t) \in \mathbb{R}^n$ is the state of the $i$-th agent; (ii) $u_i(t) \in \mathbb{R}^m$ is the control input; (iii) $d_i(t) \in \mathbb{R}^m$ is an external disturbance signal on the agent; (iv) $f_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is the intrinsic dynamics of the agent, which is assumed to be smooth; (v) $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the output function for the $i$-th agent. We consider disturbances of the form:

$$d_i(t) = w_i(t) + \tilde{d}_i(t) := \tilde{w}_i(t) + \sum_{k=0}^{m-1} \tilde{d}_{i,k} \cdot t^k,$$

where $w_i(t)$ is a piece-wise continuous signal and $\tilde{d}_{i,k}$’s are constant vectors. Disturbances of the form (2) can be thought of as the superposition of a polynomial disturbance of order $m - 1$ (denoted as $d_i(t)$) and the signal $w_i(t)$. This latter signal might capture residual terms in the disturbance that are not modeled with the polynomial. For example, in the special case where $m = 1$ in (2) we have $d_i(t) = w_i(t) + \tilde{d}_{i,0}$. As noted in e.g. (Silva et al., 2021), in the context of platooning, these types of disturbances model situations when a platoon of vehicles encounters a slope: this gives rise to a constant disturbance on the vehicle acceleration and the term $w_i(t)$ can then model small bumps along the slope. See also Remark 3 below. As we shall see, with our main results, we give
sufficient conditions on the control protocol to guarantee: (i) rejection of the \(d_l(t)\)’s in (2); (ii) non-amplification of the \(w_i(t)\’s\) which do not need to be polynomials for our results to hold. See Section 3.2 for the formulation of the control goal. Our goal is to design the control protocol \(u_i(t)\) in (1) so that the polynomial disturbance in (2) is rejected, while ensuring non-amplification of the residual disturbance \(w_i(t)\) – this is captured via a scalability property, see Section 3.2 for the rigorous definition and statement of the control goal. To do so, we propose the multiplex network architecture schematically shown in Figure 1. In such a figure, the network system is in layer 0 and the multiplex layers (i.e. layer 1, \(\ldots\), \(m\)) concur to build up the control protocol. This is of the form:

\[
\begin{align*}
 u_i(t) &= h_{i,0}(x(t), x_i(t), t) \\
 &+ h_{i,1}(x(t - \tau(t)), x_i(t - \tau(t)), t) + r_{i,1}(t), \\
 \dot{r}_{i,1}(t) &= h_{i,1}(x(t), x_i(t), t) \\
 &+ h_{i,1}^{(\tau)}(x(t - \tau(t)), x_i(t - \tau(t)), t) + r_{i,2}(t), \\
 & \vdots \\
 \dot{r}_{i,m}(t) &= h_{i,m}(x(t), x_i(t), t) \\
 &+ h_{i,m}^{(\tau)}(x(t - \tau(t)), x_i(t - \tau(t)), t),
\end{align*}
\]

where \(r_{i,k}(t)\) is the output generated by the multiplex layer \(k\in\{1, \ldots, m\}\) and where \(\tau(t)\leq \tau_{\text{max}}, \forall t\). As illustrated in Figure 1, the multiplex layer \(k\in\{1, \ldots, m\}\) receives information from the agents (on layer 0) and outputs a signal to the layer immediately below, i.e. layer \(k-1\). In (3): (i) \(x(t) := [x_1^T(t), \ldots, x_m^T(t)]^T\) is the state of the network; (ii) \(x_i(t) := [x_{i,1}^T(t), \ldots, x_{i,m}^T(t)]^T\) is the reference signal, possibly provided by a group of \(M\) leaders; (iii) the functions \(h_{i,k} : \mathbb{R}^{nN} \times \mathbb{R}^{nM} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n\) and \(h_{i,k}^{(\tau)} : \mathbb{R}^{nN} \times \mathbb{R}^{nM} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n, k \in \{0, 1, \ldots, m\}\), include both (leader and leaderless) delayed and delay-free couplings (see Remark 4 for an example). These functions model, on layer \(k\), the possible connections between the agents and leaders. Note that not all the agents necessarily receive information from leaders (if any). In what follows, we simply term the smooth coupling functions \(h_{i,k}(\cdot, \cdot)\) as delay-free coupling functions, while the functions \(h_{i,k}^{(\tau)}(\cdot, \cdot)\) are termed as delayed coupling functions. As noted in e.g. \cite{Xie2021} situations where there is an overlap between delayed and non-delayed communications naturally arise in the context of e.g. platooning, robotics formation control and neural networks. Without loss of generality, in (3) we set, \(\forall s \in [t_0 - \tau_{\text{max}}, t_0], \forall i = 1, \ldots, N, \forall k = 1, \ldots, m, x_i(s) = \varphi_i(s)\) and \(r_{i,k}(s) = \phi_{i,k}(s)\), with \(\varphi_i(s)\) and \(\phi_{i,k}(s)\) being continuous and bounded functions in \([t_0 - \tau_{\text{max}}, t_0]\).

**Remark 1.** As shown in (3) a key assumption is that the time varying delay in the multiplexed control protocol is the same for all the agents. That is, the delays across the network are homogeneous. Network systems affected by these homogeneous delays naturally arise in the context of multi-agent systems. For example, in \cite{Abolfazli2021} string stability of platoon is studied when the agents are affected by homogeneous constant delays. Homogeneous delays are also considered in the context of consensus \cite{Wang2013}, synchronization of networks with linear dynamics \cite{Gao2008} and nonlinear dynamics \cite{Xie2019}.

**Remark 2.** We do not require that the multiplex layers have the same topology. This degree of freedom in the design of the layers can be leveraged, as noted in \cite{Burbano2016}, Remark 12), to e.g. reduce the control interventions across the network.

**Remark 3.** We consider disturbances consisting of a polynomial component and a piece-wise continuous component. Polynomial disturbances are commonly considered in the literature. See e.g. \cite{Kim2010}, where observers for these disturbances are devised and \cite{Park2014}, where the problem of rejecting these disturbances is considered. The signal \(w_i(t)\) can be physically interpreted as a (typically, small) discrepancy between the polynomial disturbance model and the actual disturbance signal. For platooning, rejection of constant disturbances (i.e. the disturbance in (2) when \(m = 1\)) has been considered in \cite{Knorn2014a,Silva2021}.

**Remark 4.** Protocols of the form of (3) naturally arise in a wide range of applications, where agents might have access to a mixture of delay-free and delayed information. For example, see e.g. \cite{Abolfazli2021,diBernardo2015,Peters2014}, in the context of platooning certain information (e.g., separation from nearby vehicles obtained by radar) might be available to an agent with a negligible/no delay, while other information such as separation from more distant agents (obtained via vehicle-to-infrastructure commu-

\footnote{We leave the study of scalability in the case of heterogeneous delays for our future research. See also Section 7.}
nication) may be subject to delays. Moreover, protocols of the form of (3) also arise within popular neural network models [Zhang and Yu 2016]. In this case, agents are neurons and a delay-free communication coupling models an activation function from the closest neurons while the delayed communication can instead model interactions from neurons farther away. Finally, in formation control, typical choices for the coupling functions [Laouton et al. 2003; Olfati-Saber and Murray 2004; Tanner et al. 2002], are $h_{i0}(x(t), x(t), t) = \sum_{j \in \mathcal{N}_i} h_{ij}(x(t), x(t), t) + \sum_{l \in \mathcal{L}_i} h_{il}(x(t), x(t), t)$ and $h_{i\tau}(x(t - \tau(t)), x(t - \tau(t)), t) = \sum_{j \in \mathcal{N}_i} h_{ij}(x(t - \tau(t)), x(t - \tau(t)), t) + \sum_{l \in \mathcal{L}_i} h_{il}(x(t - \tau(t)), x(t - \tau(t)), t)$, where $\mathcal{N}_i$ and $\mathcal{L}_i$ denote, respectively, the set of neighbours of the $i$-th robot and the set of leaders to which the $i$-th robot is connected. Also, the coupling functions are typically of the diffusive type and model delayed and delay-free communications.

#### 3.2 Control Goal

We now formulate the control goal. To this aim, we first formally define the notions of desired solution (Definition 1) and of scalability (Definition 2). We let $u(t) = [u_1^T(t), \ldots, u_N^T(t)]^T$ be the stack of the control inputs, $d(t) = [d_1^T(t), \ldots, d_N^T(t)]^T$ be the stack of the disturbances, $w(t) = [w_1^T(t), \ldots, w_N^T(t)]^T$ be the stack of the residual disturbances, $\bar{d}(t) = [\bar{d}_1^T(t), \ldots, \bar{d}_N^T(t)]^T$ be the stack of the polynomial disturbances and $r_i(t) = [r_{i1}^T(t), \ldots, r_{im}^T(t)]^T$. Our control goal is expressed in terms of the so-called desired solution of the disturbance-free (or unperturbed in what follows) network system [Monteil et al. 2019]. The desired solution is the solution of the network system characterized by: (i) the state of the agents keeping some desired configuration; (ii) the multiplex layers giving no contribution to the $u_i$'s. This is formalized next:

**Definition 1.** Consider the network system (1) controlled by (3) with $d_i(t) = 0$, $\forall i$ and $\forall t$. We say that $[x_1^T(t), \ldots, x_{\tau_{\max}}^T(t)]$ is the desired solution for the network system if:

- $x_i^T(t) = f_i(x_i^T(t), t)$ with $x_i^T(s) = x_i^T(t_0), s \in [t_0 - \tau_{\max}, t_0]$;
- $r_{i\tau}(t) = 0$, $\forall i$, $\forall k$ and $\forall t$.

We term $x_{\tau_{\max}}^T(t)$ as the desired state of the $i$-th agent and we let $x^T(t) := [x_1^T(t), \ldots, x_{\tau_{\max}}^T(t)]^T$. In the special case where the closed-loop system has no multiplex layers, then Definition 1 yields as special case the definition used in [Xie et al. 2021; Monteil et al. 2019; Besselink and Knorn 2018] to formalize their control goal. When the closed-loop system has one multiplex layer, then Definition 1 yields the notion of desired solution used in [Silva et al. 2021] to characterize string stability. It is intrinsic in this definition that when the desired solution is achieved it must hold that $u_i(t) = 0$; note that this property is satisfied by e.g. any diffusive-type control protocol and implicitly used in e.g. [Besselink and Knorn 2018; Mirabilio et al. 2021]. In what follows, for the sake of brevity, we simply say that $x^T(t)$ is the desired solution. Analogously, the desired output of (1) is $y^T(t) := [y_1^T(t), \ldots, y_{\tau_{\max}}^T(t)]^T$, with $y_i^T(t) = g_i(x_i^T(t)), \forall i$ (leaving it implicit that $r_{i\tau}^T(t) = 0$, $\forall i$, $\forall k$ and $\forall t$). We aim at designing the control protocol (3) so that the closed-loop system rejects the polynomial disturbances $\bar{d}(t)$ while guaranteeing that the residual disturbance $w(t)$ is not amplified within the network system. This requirement is captured with the definition of scalability with respect to $w(t)$ formalized next:

**Definition 2.** Consider the closed-loop system (1) - (3) with disturbance $d(t) = w(t) + \bar{d}(t)$. The system is

- $\mathcal{L}_\infty$-Input-to-State Scalable with respect to $w(t)$: if there exists class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$, such that for any initial condition and $\forall t \geq t_0$,

$$\max_i |x_i(t) - x_i^T(t)|_p \leq \gamma \left( \max_i \|w_i(\cdot)\|_{\mathcal{L}_\infty} \right) + \beta \left( \sup_{t_0 - \tau_{\max} \leq s \leq t_0} \sum_{k=1}^m |r_{i,k}(s) + \bar{d}_{i,k}^{(k-1)}(s)|_p, t - t_0 \right)$$

$$+ \alpha \left( \max_i \sup_{t_0 - \tau_{\max} \leq s \leq t_0} |x_i(s) - x_i^T(s)|_p, t - t_0 \right)$$

holds $\forall N$;

- $\mathcal{L}_\infty$-Input-Output Scalable with respect to $w(t)$: if there exists class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$, such that for any initial condition and $\forall t \geq t_0$,

$$\max_i |y_i(t) - y_i^T(t)|_p \leq \gamma \left( \max_i \|w_i(\cdot)\|_{\mathcal{L}_\infty} \right) + \beta \left( \sup_{t_0 - \tau_{\max} \leq s \leq t_0} \sum_{k=1}^m |r_{i,k}(s) + \bar{d}_{i,k}^{(k-1)}(s)|_p, t - t_0 \right)$$

$$+ \alpha \left( \max_i \sup_{t_0 - \tau_{\max} \leq s \leq t_0} |x_i(s) - x_i^T(s)|_p, t - t_0 \right)$$

holds $\forall N$.

Note that, for a $\mathcal{L}_\infty$-Input-to-State Scalable ($\mathcal{L}_\infty$-Input-Output Scalable) system, if $w(t) = 0$, then $\lim_{t \to +\infty} |x_i(t) - x_i^T(t)|_p = 0$, $\forall i$ ($\lim_{t \to +\infty} |y_i(t) - y_i^T(t)|_p = 0$, $\forall i$). In the special case when $d(t) = 0$ and there are no multiplex layers, Definition 2 becomes the definition for scalability given in [Xie et al. 2021]. In this context we note that the bounds in Definition 2 are uniform in $N$ and this in turn guarantees that the residual disturbances are not amplified within the network system. Intuitively, scalability is a stronger property than stability. In what follows, whenever it is clear from the context, we simply say that the network system is
\( \mathcal{L}_\infty \)-Input-to-State Scalable (\( \mathcal{L}_\infty \)-Input-Output Scalable) if Definition 2 is fulfilled. In the special case where \( p = 2 \) we simply say that the the network system is \( \mathcal{L}_\infty \)-Input-to-State Scalable (\( \mathcal{L}_\infty \)-Input-Output Scalable).

We can finally formulate our control goal. Given the network system (1), the control goal is to design the protocol (3) for the agents so that the closed-loop network system fulfills Definition 2. Specifically, as we shall see in Section 4, with Proposition 1 we give a sufficient condition on the control protocol to guarantee that the closed-loop system is \( \mathcal{L}_\infty \)-Input-to-State Scalable. Then, in Corollary 1 we give a sufficient condition guaranteeing that the closed-loop system is \( \mathcal{L}_\infty \)-Input-Output Scalable.

4 Main Methodological Results

For the network system (1) we give a sufficient condition on the control protocol (3) guaranteeing that the closed-loop system affected by disturbances of the form (2) is \( \mathcal{L}_\infty \)-Input-to-State Scalable (see Definition 2). We also give a corollary for \( \mathcal{L}_\infty \)-Input-Output Scalability of the closed-loop system. The results are stated in terms of the block diagonal coordinate transformation matrix \( T := \text{diag}\{T_1, \ldots, T_N\} \in \mathbb{R}^{N \cdot (m+1) \times N \cdot (m+1)} \) with

\[
T_i := \begin{bmatrix}
I_n & \alpha_{i,1} \cdot I_n \\
I_n & \ddots \\
\vdots & \ddots & \ddots \\
\alpha_{i,m} \cdot I_n & \ddots & \ddots & I_n
\end{bmatrix} \in \mathbb{R}^{n \cdot (m+1) \times n \cdot (m+1)},
\]

where \( \alpha_{i,k} \in \mathbb{R}, \forall k \).

Proposition 1. Consider the closed-loop network system (1) - (3) with \( y_i(t) = x_i(t) \) affected by disturbances (2). Let the matrices \( \bar{A}_{ii}(t), \bar{A}_{ij}(t), \bar{B}_{ij}(t) \) be defined as in (4) and assume that, \( \forall t \geq t_0 \), the following set of conditions are satisfied for some \( 0 \leq \sigma < \sigma < +\infty \):

\[
C1 \quad h_{i,k}(x^*, x_i, t) = h_{i,k}^{(r)}(x^*, x_i, t) = 0, \forall i, k;
\]

\[
C2 \quad \mu_p(T_i \bar{A}_{ii}(t) T_i^{-1}) + \sum_{j \neq i} \| T_i \bar{A}_{ij}(t) T_j^{-1} \|_p \leq -\bar{\sigma}, \forall i \text{ and } \forall x \in \mathbb{R}^{nN}, \forall x_i \in \mathbb{R}^{nM};
\]

\[
C3 \quad \sum_{j=1}^{N} \| T_i \bar{B}_{ij}(t) T_j^{-1} \|_p \leq \sigma, \forall i \text{ and } \forall x \in \mathbb{R}^{nN}, \forall x_i \in \mathbb{R}^{nM}.
\]

Then, the system is \( \mathcal{L}_\infty \)-Input-to-State Scalable. In particular,

\[
\begin{align*}
\max_{i} |x_i(t) - x_i^*(t)| & \leq \frac{\kappa_G(T)}{\sigma - \bar{\sigma}} \max_{i} \| u_i(\cdot) \|_{\mathcal{L}_p}^p \\
& + \kappa_G(T) e^{-\lambda(t-t_0)} \max_{i} \sup_{t_0-\tau_{\max} \leq s \leq t_0} |x_i(s) - x_i^*(s)|_p \\
& + \kappa_G(T) e^{-\lambda(t-t_0)} \max_{i} \sup_{t_0-\tau_{\max} \leq s \leq t_0} \sum_{k=1}^{m} |r_{i,k}(s)| \\
& + \sum_{b=0}^{m-k} \frac{\prod_{b=0}^{m-k} \lambda(t-t_0) - \sigma + \sigma e^{-\lambda(t-t_0)}}{\lambda(t-t_0)} |\bar{d}_{i,m-1-b} \cdot \bar{d}_{i,m-1-b}^T|_p, \forall N,
\end{align*}
\]

where \( \kappa_G(T) := \|T_i\|_{\mathcal{L}_G} \|T_i^{-1}\|_{\mathcal{L}_G} \) and

\[
\dot{\lambda} = \inf_{t \geq t_0} \{ \lambda|\lambda(t) - \bar{\sigma} + \sigma e^{\lambda(t)\tau(t)} = 0 \}.
\]

Proposition 1 allows to consider situations where some of the coupling functions in (3) are equal to 0. This is useful in the special case when, for example, it is assumed that all the information to which the agents have access is delay-free. In this special case, the absence of delays implies that condition C3 is satisfied with \( \bar{\sigma} = 0 \) (when there are no delays the \( \bar{B}_{ij}(t) \)’s are matrixes of all zeros). The proof of the result is given in Appendix A and we now make the following considerations on the conditions.

Remark 5. From the design viewpoint, if one wants to design a control protocol to guarantee rejection of polynomial disturbances of order up to \( m - 1 \), then Proposition 1 says that \( m \) multiplex layers need to be foreseen – see
the last term of the upper bound in (5). That is, the control protocol in (3) for the \( i \)-th agent needs to foresee the dynamics for the \( r_{ij} \)'s with \( j = 1, \ldots, m \). In accordance with Proposition 1, the protocol also guarantees that the \( w_i \)'s in (2) are not be amplified across the network.

**Remark 6.** Condition C1 implies that \( u_i(t) = 0 \) at the desired solution. This rather common condition (see e.g. [di Bernardo et al., 2015; Xie et al., 2021]) guarantees that \( x^\dagger(t) \) is a solution of the unperturbed dynamics. This assumption is satisfied in e.g. all consensus/synchronization dynamics with diffusive-type couplings. Condition C2, giving an upper bound on the matrix measure of the Jacobian of the delay-free part of the closed-loop network dynamics, is a diagonal dominance condition. Instead, condition C3 gives an upper bound on the norm of the Jacobian of the dynamics containing delays. Intuitively, Proposition 1 implies that the matrix measure should be negative enough to compensate the presence of delays.

**Remark 7.** Conditions C2 and C3 can be leveraged to shape the coupling functions between the agents and to determine the maximum number of neighbours for each agent in the network. Moreover, if C2 and C3 are satisfied, then the network is also connective stable in the sense of [Siljak, 2014, Chapter 2.1]. Intuitively, a network is connective stable if the removal of couplings preserves stability. This property and the related \( \gamma \)-scalability from [Knorn and Besselink, 2020] have been investigated for delay-free networks. In Section 5 we show that C2 and C3 can be effectively checked by recasting the fulfillment of these conditions into an optimization problem that allows to design the control protocol for each agent independently on the other agents.

**Remark 8.** Following Proposition 1, the convergence rate of the closed-loop network is at least \( \lambda \). We note that \( \lambda \) depends on \( \tau(t) \) and (6) highlights that the larger \( \tau_{\text{max}} \), the lower \( \lambda \). In particular, \( \lambda = \bar{\sigma} \) when \( \tau_{\text{max}} = 0 \) (i.e. when there are no delays) and decreases as the delay increases. From the design viewpoint, the expression in (6) can be used to design the protocols so that the network exhibits a given, desired, convergence rate.

**Remark 9.** Proposition 1 generalises a number of results in the literature. Specifically, in the special case when the network topology is a string, there are no delays and \( d_i(t) \) is constant \( \forall i \), then our conditions yield these from [Silva et al., 2021]. That is, we extend the results in [Silva et al., 2021; Monteil et al., 2019] by considering dynamic compensation of polynomial disturbances, general network topologies and delays. We also extend our prior work [Xie et al., 2021] in which scalability is considered without rejection of polynomial disturbances.

Before illustrating (Section 5) how our approach can be effectively used for protocol design, we give here the next result that immediately follows from Proposition 1.

**Corollary 1.** Consider the closed-loop network system (1) - (3) affected by disturbances (2). Assume that all the conditions in Proposition 1 are satisfied and that, in addition, the output functions \( g_i(\cdot) \) are Lipschitz. Then the system is \( \mathcal{L}_\infty \), Input-Output Scalable.

**Proof.** The proof, directly following from the Lipschitz hypothesis on \( g_i(\cdot) \) and from the upper bound in (5), is omitted here for brevity.

## 5 Multiplex Protocols for Formation Scalability

We consider the problem of designing a control protocol guaranteeing that a network of \( N \) unicycle robots is \( \mathcal{L}_\infty \), Input-to-State Scalable. Our goal is to design a protocol allowing the formation to: (i) track a reference provided by a leader; (ii) reject certain polynomial disturbances; (iii) ensure the non-amplification of residual disturbances. Before presenting the validation results, we describe here the robot dynamics and how Proposition 1 was used to synthesize the protocol. The results from the \textit{in-silico} and experimental validations are presented in Section 6. We made use of the Robotarium ([Wilson et al., 2020]) for our experimental validations.

### Agent dynamics.

We consider the following dynamics for unicycle robots (see e.g. [Wilson et al., 2020] and references therein):

\[
\begin{aligned}
\dot{p}_i^x(t) &= v_i(t) \cos \theta_i(t) + d_i^x(t), \\
\dot{p}_i^y(t) &= v_i(t) \sin \theta_i(t) + d_i^y(t), \\
\dot{\theta}_i(t) &= \Omega_i(t),
\end{aligned}
\]

\( \forall i \), where the state variables \( p_i(t) = [p_i^x(t), p_i^y(t)]^T \) represent the inertial position of the \( i \)-th robot and \( \theta_i(t) \) is its heading angle. The control input is denoted by \( u_i(t) = [v_i(t), \Omega_i(t)]^T \) with \( v_i(t) \) being the linear velocity and \( \Omega_i(t) \) being the angular velocity. We consider the case where the disturbances affecting the robots, i.e. \( d_i(t) = [d_i^x(t), d_i^y(t)]^T \), are of the form \( d_i^x(t) := d_{i,0}^x + d_{i,1}^x \cdot t + w_i^x(t) \) and \( d_i^y(t) := d_{i,0}^y + d_{i,1}^y \cdot t + w_i^y(t) \). The terms \( d_{i,0}^x \) and \( d_{i,0}^y \) model constant disturbances on the velocity of the robots. These disturbances also naturally arise in the context of unicycle-like marine robots, the dynamics of which is also captured by (7). For these robots, the constant terms in \( d_{i}(t) \) model the disturbances due to the ocean current ([Panagou and Kyriakopoulos, 2011] and \( w_i^x(t), w_i^y(t) \) model e.g. transient variations of the current. The ramp terms in the disturbance can instead model ramp attack signals ([Sridhar and Govindarasu, 2014]). In what follows, we make use of the compact notation \( \omega_i(t) := [w_i^x(t), w_i^y(t)][t] \), \( d_i(t) := [d_{i,0}^x + d_{i,1}^x \cdot t, d_{i,0}^y + d_{i,1}^y \cdot t][t] \). Following [Lawton

\footnote{The Robotarium by Georgia Tech is an open research infrastructure to perform experiments on real robots. Experiments can be launched remotely via a web interface.}
the dynamics for the robot hand position is given by
\[ \dot{h}_i(t) = \begin{bmatrix} \cos \theta_i(t) & -l_i \sin \theta_i(t) \\ \sin \theta_i(t) & l_i \cos \theta_i(t) \end{bmatrix} u_i(t) + d_i(t), \]
where \( l_i \in \mathbb{R}_{>0} \) is the distance of the hand position to the wheel axis. The dynamics can be feedback linearised by
\[ u_i(t) = \begin{bmatrix} \cos \theta_i(t) & -l_i \sin \theta_i(t) \\ \sin \theta_i(t) & l_i \cos \theta_i(t) \end{bmatrix}^{-1} \nu_i(t), \]
which yields
\[ \dot{h}_i(t) = \nu_i(t) + d_i(t), \quad \forall i \]  
Next we leverage Proposition 1 to design \( \nu_i(t) \) so that network (9) is \( L_\infty \)-Input-to-State Scalable.

**Protocol design.** We denote by \( \eta_i(t) \) the hand position provided by a virtual leader. Robots are required to keep a desired offset from the leader (\( \delta_{ji}^0 \)) and from neighbours (\( \delta_{ji}^2 \)) while tracking a reference velocity from the leader, say \( v_l(t) \). Following these requirements, we pick the desired solution (see Section 3.2) for the \( i \)-th robot dynamics, say \( \eta^*_i(t) \), so that: (i) the robot keeps the desired offsets from the leader and from the neighbours, i.e. \( \eta^*_i(t) - \eta^*_i(t) = \delta_{ji}^0 \) and \( \eta^*_i(t) - \eta^*_j(t) = \delta_{ji}^2 \); (ii) the reference velocity is tracked, i.e. \( \dot{\eta}^*_i(t) = v_l(t) \). Inspired by Liu et al. [2019] and Li et al. [2013], we consider protocols of the form:
\[ \nu_i(t) = \tilde{\nu}_i(t) + v_l(t) \]  
with
\[ \tilde{\nu}_i(t) = h_{i,0}(\eta(t), \eta_i(t), t) + h_{i,0}^r(\eta(t - \tau(t)), \eta(t - \tau(t)), t) + r_{i,1}(t), \]
\[ \dot{r}_{i,1}(t) = h_{i,1}(\eta(t), \eta_i(t), t) + h_{i,1}^r(\eta(t - \tau(t)), \eta(t - \tau(t)), t) + r_{i,2}(t), \]
\[ \dot{r}_{i,2}(t) = h_{i,2}(\eta_i(t), t) + h_{i,2}^r(\eta(t - \tau(t)), \eta_i(t), t), \]
In the above protocol, \( r_{i,1}(t) \) and \( r_{i,2}(t) \) are generated by two multiplex layers (say, layer 1 and layer 2). With this protocol, the closed-loop dynamics is given by
\[ \dot{\eta}_i(t) = \nu_i(t) + \tilde{\nu}_i(t) + d_i(t), \quad \forall i \]
Hence, we recast the problem of designing the protocol \( \nu_i(t) \) for (9) into the problem of designing \( \tilde{\nu}_i(t) \) in (11) for the dynamics (12). To this aim, we can apply Proposition 1. The coupling functions in (11) are (time dependence inside these functions omitted in what follows):
\[ h_{i,0}(\eta, \eta_i, t) = k_0(\eta - \eta_i - \delta_{ji}^0), \]
\[ h_{i,0}^r(\eta(t), \eta_i(t), t) = \sum_{j \in N_i} \psi(\eta_j - \eta_i - \delta_{ji}^0), \]
\[ h_{i,1}(\eta, \eta_i, t) = k_1(\eta_i - \eta), \]
\[ h_{i,1}^r(\eta(t), \eta_i(t), t) = \sum_{j \in N_i} \psi(\eta_j - \eta_i - \delta_{ji}^0), \]
\[ h_{i,2}(\eta, \eta_i, t) = k_2(\eta - \eta_i), \]
\[ h_{i,2}^r(\eta(t), \eta_i(t), t) = \sum_{j \in N_i} \psi(\eta_j - \eta_i - \delta_{ji}^0), \]  
with \( \psi(x) := \tanh(k^\psi x) \) being inspired from Monteil et al. [2019]. In the above expression, \( N_i \) is the set of neighbours of robot \( i \). The cardinality of \( N_i \) is bounded, that is, \( |N_i| \leq N, \forall i \). Also, the control gains \( k_0, k_1, k_2, k_0^r, k_1^r, k_2^r, k^\psi \) are non-negative scalars designed next. Specifically, we make use of Proposition 1 to select the control gains so that the robotic network is \( L_\infty \)-Input-to-State Scalable. In particular, we note that the choice of the control protocol (10) with coupling functions (13) guarantees the fulfillment of C1. Hence, we now only need to find a set of control gains that fulfills C2 and C3. To this aim, in Proposition 1, we pick all transformation matrices to be the same, say \( T_i = T_j = T \), \( \forall i, j \). Then, as shown in Appendix B, the problem of finding control gains that fulfill these two conditions can be recast as the following optimisation problem:
\[ \min_\xi J \]
\[ s.t. \quad k_0 \geq 0, k_1 \geq 0, k_2 \geq 0, g_0 \geq 0, \tilde{g}_1 \geq 0, \tilde{g}_2 \geq 0, \]
\[ k_0 + \tilde{g}_0 > 0, k_1 + \tilde{g}_1 > 0, k_2 + \tilde{g}_2 > 0, \]
\[ \sigma > 0, \sigma \geq 0, \sigma - \bar{\sigma} > 0, \| \bar{T}A_{i}\bar{T}^{-1} \|_s \leq -\bar{\sigma} I_6, \]
\[ \begin{bmatrix} \bar{\sigma} I_6 & (\bar{T}B_{i}\bar{T}^{-1})^T \\ \bar{T}B_{i}\bar{T}^{-1} & \bar{\sigma} I_6 \end{bmatrix} \geq 0, \]
\[ \begin{bmatrix} \bar{\sigma} I_6 & (\bar{T}B_{i}\bar{T}^{-1})^T \\ \bar{T}B_{i}\bar{T}^{-1} & \bar{\sigma} I_6 \end{bmatrix} \geq 0. \]  
(15)
The decision variables are \( \xi := [k_0, k_1, k_2, \tilde{g}_0, \tilde{g}_1, \tilde{g}_2, \sigma, \bar{\sigma}] \) with \( \tilde{g}_0 = k^\psi k_0^r, \tilde{g}_1 = k^\psi k_1^r \) and \( \tilde{g}_2 = k^\psi k_2^r \). The matrices in the above expression are defined in (14). In (15) we decided to use the cost \( J := -\tilde{g}_0 - \tilde{g}_1 - \tilde{g}_2 \), which was chosen in accordance to Monteil et al. [2019] with the aim of maximizing the upper bound of the inter-robot coupling functions (other cost functions could be chosen as the steps described in Appendix B formalizing the optimisation problem are not dependent on \( J \)).

**Remark 10.** For concreteness, we used Proposition 1 to
6 Validation

We now validate the effectiveness of the protocol obtained in Section 5. We do this by showing that: (i) robots are able to keep a given desired formation; (ii) the protocol prohibits the amplification of residual disturbances; (iii) the polynomial disturbances are rejected by the control protocol. Robots need to keep a formation consisting of concentric circles (the \( k \)-th circle consists of \( 4k \) robots) and their hand positions need to move following a reference trajectory. Robots receive the velocity and position signals from the virtual leader and each robot is connected to a maximum of \( N = 3 \) neighbours (i.e. the closest robots). Specifically, a given robot on the \( k \)-th circle is connected to the robots immediately ahead and behind on the same circle and with the closest robot on circle \( k - 1 \) (if any). The set-up we consider in our validation experiments is schematically illustrated in Figure 2, together with the reference trajectory from the virtual leader. In the figure, for clarity, 3 concentric circles are shown. We used a delay of \( \tau(t) = 0.33s \) in both our simulations and hardware experiments. We first illustrate the results from the simulations and then the results obtained from the experiments on the Robotarium. The code and data to replicate all the experiments of Section 6 can be found at [https://tinyurl.com/4wyacf7z](https://tinyurl.com/4wyacf7z).

In-silico validation. We consider a formation of 30 circles, with two robots on circle 1 (say, robot 1 and robot 3) affected by disturbances:

\[
\begin{align*}
\tau_1(t) &= \begin{bmatrix} 0.04 + 0.4 \sin(0.5t) e^{-0.1t} \\ 0.04 + 0.4 \sin(0.5t) e^{-0.1t} \end{bmatrix}, \\
\tau_2(t) &= \begin{bmatrix} -0.05t + 0.4 \sin(0.5t) e^{-0.1t} \\ -0.05t + 0.4 \sin(0.5t) e^{-0.1t} \end{bmatrix}.
\end{align*}
\]  

We computed the gains of the control protocol in (10) - (13) by solving the optimisation problem in (15) for a grid of parameters \( \alpha_1 \) and \( \alpha_2 \). We then selected the control parameters as the ones returning the lowest cost for each fixed pair of \( \alpha \)'s. By doing so, we obtained the gains \( k_0 = 1.4155, k_1 = 1.5103, k_2 = 0.4803, k^{(r)} = 0.642, k^{(r)} = 0.872, k^{(r)} = 0.425, k^{(r)} = 0.1 \) (corresponding to \( \alpha_1 = -0.6, \alpha_2 = -1.6 \)). In Figure 3 the maximum hand position deviation is shown when the number of robots in the formation is increased, starting with a formation of 1 circle only to a formation with 30 circles (i.e. 1860 robots). The figure was obtained by starting with a formation of 1 circle and increasing at each simulation the number of circles. We recorded at each simulation the maximum hand position deviation for each robot on a given circle and finally plotted the largest deviation on each circle across all the simulations. The figure clearly shows that the polynomial components of the disturbances in (16) are compensated by the integral control protocol and the residual disturbances are not amplified through the formation. We also report the behaviour of the full formation with 30 circles in Figure 4. Both panels of the figure confirm that, in accordance with our theoretical results, the protocol allows the robots to keep the desired formation, while compensating the polynomial disturbances. 

\[
\begin{align*}
\mathcal{A}_{ii} := & \begin{bmatrix} -k_0 I_2 & I_2 & 0_2 \\ -k_1 I_2 & 0_2 & I_2 \\ -k_2 I_2 & 0_2 & 0_2 \end{bmatrix}, \quad \mathcal{B}_{ii} := -\bar{N} \begin{bmatrix} g_0 I_2 & 0_2 & 0_2 \\ g_1 I_2 & 0_2 & 0_2 \\ g_2 I_2 & 0_2 & 0_2 \end{bmatrix}, \quad \bar{B}_{ij} := \begin{bmatrix} g_0 I_2 & 0_2 & 0_2 \\ g_1 I_2 & 0_2 & 0_2 \\ g_2 I_2 & 0_2 & 0_2 \end{bmatrix}, \quad \bar{T} := \begin{bmatrix} I_2 & \alpha_1 I_2 & 0_2 \\ 0_2 & I_2 & \alpha_2 I_2 \\ 0_2 & 0_2 & I_2 \end{bmatrix}.
\end{align*}
\]
components of the disturbances and prohibiting the amplification of the residual disturbances.

Finally, before presenting the validation results on the Robotarium, we benchmark the performance of our protocol to control (12) with these obtained following (Xie et al., 2021) and (Silva et al., 2021). Since the design conditions from (Silva et al., 2021) are tailored towards networks with a (bi-directional) string topology, we compared performance of the protocols using such a topology. To this aim we again considered the formation control problem for the formation of Figure 2 with 30 circles this time with each robot bidirectionally coupled to the robots before and after (that is, robot i in the formation was coupled to robot \(i-1\) and \(i+1\), if any). The disturbance considered in the simulation was again the one in (16). The time evolution of the hand position deviations for the robots controlled by our control protocol (10)–(11) is shown in the bottom panel of Figure 5. The simulation results, consistently with our theoretical findings, show that the control goal is achieved. The top panel of the figure instead shows the time evolution of the hand position deviations when a control protocol designed according to (Xie et al., 2021) is used. In this case, as shown in the panel, the protocol is not able to compensate the polynomial disturbances. Finally, we also benchmarked the performance of our protocol with a protocol designed following (Silva et al., 2021). To do so, we considered a situation where the network had no delays (the results from [Silva et al., 2021] apply to delay-free networks). In the middle panel of Figure 5 the time evolution of the deviation of the hand position is shown when a protocol designed according to (Silva et al., 2021) is used. The panel clearly shows that the first order polynomial disturbance is not compensated by this protocol.

Experimental validation. We further validate our results by carrying out experiments on the Robotarium, which provides both hardware infrastructure and a high-fidelity simulator of the hardware. In the experiments, a formation of 2 concentric circles (hence with 12 robots) is considered and, for consistency with our previous set of simulations, 2 robots on circle 1 are perturbed by the disturbances \(d_1(t)\) and \(d_3(t)\) given in (16). The Robotarium documentation (Robotarium, 2022) reports a nominal step size (for both the simulator and the hardware infrastructure) of 0.033s. Since the step size is used to implement the multiplex layers of the integral control protocol in (10) – (13) as a first step we measured the actual step size in the hardware infrastructure. The result is given in Figure 6. Such a figure reports the average step size we measured using built-in timing functions across 10 experiments. In the same figure, the shaded area represents the confidence interval corresponding to the standard deviation. As illustrated in the figure, while the average step size is indeed around 0.033s and consistent with the nominal value, it also introduces some variability in the experiments. Such variability leads to the observation of a gap between the results obtained from the Robotarium simulator and the actual experimental results. This phenomenon is essentially due to the fact that in the Robotarium experiments we could

\footnote{as also remarked in our statement of contributions these are the two works that are most related to our results.}

\footnote{See our code at \url{https://tinyurl.com/4wyacf7z} for the details on how these measurements were performed.}
only use the nominal step-size of 0.033s (and not the actual step size) for the implementation of the dynamics of the multiplex layers in the protocol. We decided to mitigate this simulation-to-reality gap by reducing the gains for the couplings of the multiplex layers. Hence, in the experimental results presented next, we impose that the control gains of the multiplex layers (i.e., layer 1 and layer 2) are smaller than the gains of layer 0. This was done by solving the optimisation problem in (15) this time with the following additional constraints: \( k_0 \geq 2k_1, k_0 \geq 2k_2, g_0 \geq 2g_1, g_0 \geq 2g_2 \). As an outcome of this process, we obtained the following gains: \( k_0 = 1.2674, k_1 = 0.6312, k_2 = 0.133, k_{1}^{(\tau)} = 0.325, k_{1}^{(\tau)} = 0.162, k_{2}^{(\tau)} = 0.06, k_\psi = 0.1 \) (which correspond to \( \alpha_1 = -1.1, \alpha_2 = -2.6 \)). We then validated the control protocol with this choice of parameters by first leveraging the Robotarium simulator and the results, consistent with our theoretical findings, are shown in the top panel of Figure 7. Next, we validated the control protocol on the Robotarium hardware infrastructure and the outcome from these experiments are shown in the bottom panel of Figure 7. In the figure, which was obtained from a set of 10 experiments, the solid lines are the robots’ average hand position deviations and the shaded area represents the confidence interval corresponding to the standard deviation. The behaviour of the hardware experiments is in agreement with the one obtained from the simulator. Both panels show that, in accordance with Proposition 1, our control protocol allowed the robots to keep the desired formation, while compensating the polynomial components of the disturbances and ensuring non-amplification of the residual disturbances.

7 Conclusions and Future Work

We considered the problem of designing distributed control protocols for possibly nonlinear networks affected by delays that not only allow the network to achieve some desired behaviour, but also, by means of integral actions delivered via multiplex architecture, guarantee a scalability property of the network. This property implies the rejection of polynomial disturbances and the non-amplification of the residual disturbances. To tackle this problem, we presented a set of sufficient conditions for scalability enabling the design of integral multiplex protocols for both leader-follower and leaderless networks consisting of possibly heterogeneous agents affected by communication delays. After turning the results into an optimisation problem, we experimentally evaluated the effectiveness of our approach via in-silico and hardware validations. Besides considering networks with heterogeneous delays, we plan to build on the results of this paper to consider network systems evolving over arbitrary time domains. These time scales dynamics (Russo and Wirth [2022]) can be used to model discrete-time systems with non-uniform sampling times. Hence, the time scales formalism might be useful to tackle situations, also observed in Section 6, where the step size is non-uniform and not known a-priori. We will also explore the effectiveness of our results on the design of integral actions for biochemical networks (Qian and Del Vecchio [2018]).

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A Proof of Proposition 1

We start with augmenting the state of the original dynamics by defining

\[ z_i(t) := [x_i^T(t), \zeta_{i,1}(t), \zeta_{i,2}(t), \ldots, \zeta_{i,m}(t)]^T, \]

and where

\[ \zeta_{i,k}(t) = r_{i,k}(t) + \sum_{b=0}^{m-k} \frac{(m-1-b)!}{(m-k-b)!} \cdot d_{i,m-1-b} \cdot t^{m-k-b}, \]

\[ k = 1, \ldots, m. \] In these new coordinates the dynamics of the network system becomes

\[ \dot{z}_i(t) = \tilde{f}_i(z_i, t) + \tilde{v}_i(z, t) + \tilde{w}_i(t), \tag{A.1} \]

where \( \tilde{f}_i(z_i, t) = \left[ f_i^T(x_i(t), 0_{1 \times n}, \ldots, 0_{1 \times n})^T \right], \) \( \tilde{w}_i(t) = \left[ w_i^T(t), 0_{1 \times n}, \ldots, 0_{1 \times n}\right]^T, \) \( \tilde{v}_i(z, t) = v_i(z, t) + v_i^{(\tau)}(z, t) \) with

\[ v_i(z, t) = \begin{bmatrix} h_{i,0}(x(t), x_i(t), t) + \zeta_{i,1}(t) \\ h_{i,1}(x(t), x_i(t), t) + \zeta_{i,2}(t) \\ \vdots \\ h_{i,m-1}(x(t), x_i(t), t) + \zeta_{i,m}(t) \\ h_{i,m}(x(t), x_i(t), t) \end{bmatrix} \]

and

\[ v_i^{(\tau)}(z, t) = \begin{bmatrix} h_{i,0}^{(\tau)}(x(t-\tau(t)), x_i(t-\tau(t)), t) \\ h_{i,1}^{(\tau)}(x(t-\tau(t)), x_i(t-\tau(t)), t) \\ \vdots \\ h_{i,m}^{(\tau)}(x(t-\tau(t)), x_i(t-\tau(t)), t) \end{bmatrix} \]

Condition C1 implies that \( x_i^*(t) \) is a solution of the unperturbed network dynamics, i.e. \( x_i^*(t) \) is a solution of (1) when there are no disturbances. Moreover, when there are no disturbances, in the augmented dynamics, the solution \( z_i^*(t) := [x_i^*(t), 0_{1 \times n}, \ldots, 0_{1 \times n}]^T \) satisfies \( \dot{z}_i(t) = \tilde{f}_i(z_i^*, t), \) with \( \tilde{f}_i(z_i^*, t) := \left[ f_i^T(x_i^*(t), 0_{1 \times n}, \ldots, 0_{1 \times n})^T \right]. \) Hence, the dynamics of state deviation (i.e. the error) \( e_i(t) = z_i(t) - z_i^*(t) \) is given by

\[ \dot{e}_i(t) = \tilde{f}_i(z_i, t) - \tilde{f}_i(z_i^*, t) + \tilde{v}_i(z, t) + \tilde{w}_i(t). \tag{A.2} \]

Following [Desoer and Haneda (1972)], we let \( \eta_i(\rho) = \rho z_i + (1-\rho)z_i^*, \) \( \eta(\rho) = [\eta_1(\rho), \ldots, \eta_N(\rho)]^T \) and then rewrite the error dynamics as

\[ \dot{e}(t) = A(t)e(t) + B(t)e(t - \tau(t)) + \tilde{w}(t), \] \( \tag{A.3} \)

where \( \tilde{w} = [\tilde{w}_1^T(t), \ldots, \tilde{w}_N^T(t)]^T \) and \( A(t) \) has entries: (i) \( A_{ii}(t) = \int_0^1 \left( J_{\tilde{f}_i}(\eta_i(\rho), t) + J_{\tilde{v}_i}(\eta_i(\rho), t) \right) \, d\rho; \)
(ii) \( A_{ij}(t) = \int_0^1 J_{\tilde{v}_j}(\eta_i(\rho), t) \, d\rho. \) Similarly, \( B(t) \) has entries: \( B_{ij}(t) = \int_0^1 J_{\tilde{v}_j}(\eta_i(\rho), t) \, d\rho. \) In the above expressions, the Jacobian matrices are defined as \( J_{\tilde{f}_i}(\eta_i, t) := \frac{\partial \tilde{f}_i(\eta_i, t)}{\partial \eta_i}, J_{\tilde{v}_i}(\eta_i, t) := \frac{\partial \tilde{v}_i(\eta_i, t)}{\partial \eta_i}, J_{\tilde{v}_j}(\eta_i, t) := \frac{\partial \tilde{v}_j(\eta_i, t)}{\partial \eta_i}. \) Now, consider the coordinate transformation \( \tilde{z}(t) := Tz(t) \) and \( \tilde{e}(t) := Te(t), \) we have

\[ \dot{\tilde{e}}(t) = TA(t)T^{-1}\tilde{e}(t) + TB(t)T^{-1}\tilde{e}(t - \tau(t)) + T\tilde{w}(t). \tag{A.4} \]

Let \( |x|_G := ||x_1||_p, \ldots, ||x_N||_p \). Then, by taking the Dini derivative of \( |\tilde{e}(t)|_G \) we may continue as follows

\[ D^+|\tilde{e}(t)|_G = \limsup_{h \to 0^+} \frac{1}{h} (|\tilde{e}(t) + h\tilde{T}A(t)T^{-1}\tilde{e}(t) + h\tilde{T}B(t)T^{-1}\tilde{e}(t - \tau(t)) + h\tilde{T}\tilde{w}(t)|_G - |\tilde{e}(t)|_G) \]

\[ = \limsup_{h \to 0^+} \frac{1}{h} (|\tilde{e}(t) + h\tilde{T}A(t)T^{-1}\tilde{e}(t) + h\tilde{T}B(t)T^{-1}\tilde{e}(t - \tau(t)) + h\tilde{T}\tilde{w}(t)|_G - |\tilde{e}(t)|_G) \]

\[ \leq \limsup_{h \to 0^+} \frac{1}{h} (||I + h\tilde{T}A(t)T^{-1}||_G - 1) |\tilde{e}(t)|_G + |T\tilde{w}(t)|_G \]

\[ + ||\tilde{T}B(t)T^{-1}||_G |\tilde{e}(t - \tau(t))|_G \]

\[ \leq \mu_G(TA(t)T^{-1}) |\tilde{e}(t)|_G + ||\tilde{T}B(t)T^{-1}||_G \sup_{t - \tau_{\max} \leq s \leq t} |\tilde{e}(s)|_G \]

\[ + ||\tilde{T}||_G \max_{t} \|\tilde{w}_i(\cdot)\|_{C^0}. \]

Next, we find upper bounds for \( \mu_G(TA(t)T^{-1}) \) and \( ||\tilde{T}B(t)T^{-1}||_G \) which allow us to apply Lemma 2. First, we give the expression of the matrix \( \tilde{A}(t) \) which have entries: (i) \( \tilde{A}_{ii}(t) = J_{\tilde{f}_i}(z_i(t), t) + J_{\tilde{v}_i}(z_i(t), t) \); (ii) \( \tilde{A}_{ij}(t) = J_{\tilde{v}_j}(z_i(t), t), \) and \( B(t) \) has entries: \( B_{ij}(t) = J_{\tilde{v}_j}(z_i(t), t). \) Then, by sub-additivity of matrix measures and matrix norms, we get \( \mu_G(TA(t)T^{-1}) \leq \int_0^1 \mu_G(T\tilde{A}(t)T^{-1}) \, dp \) and \( ||\tilde{T}B(t)T^{-1}||_G \leq \int_0^1 ||\tilde{T}B(t)T^{-1}||_G \, dp \) (see also Lemma 3.4 in [Russo and Wirth (2022)]. Moreover, from Lemma 1 it then follows that

\[ \mu_G(T\tilde{A}(t)T^{-1}) \leq \max_{i} \left\{ \mu_{\rho}(T, T_{\tilde{A}_i}(t))^{-1} + \sum_{j \neq i} ||T, T_{\tilde{A}_j}(t)||_{\rho} \right\}, \]
and

\[ \|TB(t)T^{-1}\|_G \leq \max_i \left\{ \sum_j \|T_i B_{ij}(t)T_j^{-1}\|_p \right\}. \]

Condition C2 and C3 yield

\[
\max_i \left\{ \mu_p(T_i \tilde{A}_{ii}(t)T_i^{-1}) + \sum_j \|T_i \tilde{A}_{ij}(t)T_j^{-1}\|_p \right\} \leq -\sigma,
\]

and

\[
\max_i \left\{ \sum_j \|T_i B_{ij}(t)T_j^{-1}\|_p \right\} \leq \sigma,
\]

for some \(0 \leq \sigma < \sigma < +\infty\). This implies that

\[
\mu_G(TA(t)T^{-1}) + \|TB(t)T^{-1}\|_G \leq -\sigma + \sigma := -\sigma,
\]

and Lemma 2 then yields

\[
|\tilde{e}(t)|_G \leq \sup_{t_0 - \tau_{\text{max}} \leq s \leq t_0} |\tilde{e}(s)|_G e^{-\tilde{\lambda}(t-t_0)} + \frac{\|T\|_G}{\sigma - \sigma} \max_i \|\tilde{w}_i(\cdot)\|_{L^\infty},
\]

with \(\tilde{\lambda}\) defined as in the statement of the proposition. Since \(\tilde{e}(t) = T e(t)\) we get \(|\tilde{e}(t)|_G \leq \|T^{-1}\|_G|e(t)|_G\) and \(|\tilde{e}(t)|_G \leq \|T\|_G|e(t)|_G\). We also notice that the definition of \(\tilde{w}_i(\cdot)\) implies that \(\|\tilde{w}_i(\cdot)\|_{L^\infty} = ||w_i(\cdot)\|_{L^\infty}\). Hence

\[
|e(t)|_G \leq \|T^{-1}\|_G|e(t)|_G \left( \sup_{t_0 - \tau_{\text{max}} \leq s \leq t_0} |e(s)| e^{-\tilde{\lambda}(t-t_0)} + \frac{1}{\sigma - \sigma} \max_i \|\tilde{w}_i(\cdot)\|_{L^\infty} \right).
\]

We note that \(e_i(t)_p = |x_{1i}(t) - x_{2i}(t)|, \ldots, x_{ni}(t)\)_p \geq |x_{1i}(t) - x_{2i}(t)|, 0, \ldots, 0 = \tilde{x}_i(t) - \tilde{x}_i(t)|_p,\)

and \(e_i(t)|_p = |\tilde{x}_i(t)|_p = |\tilde{x}_i(0) - \tilde{x}_i(0)|_p + \sum_{k=1}^m |y_{i,k}(t)|_p \leq \|\tilde{x}_i(t) - \tilde{x}_i(0)|_p + \sum_{k=1}^m |y_{i,k}(t)|_p \). We then finally obtain the upper bound of the state deviation

\[
\max_i \|x_i(t) - x_i^*(t)\|_p \leq \kappa_G(T) e^{-\tilde{\lambda}(t-t_0)} \max_i \|w_i(\cdot)\|_{L^\infty} + \kappa_G(T) e^{-\tilde{\lambda}(t-t_0)} \max_i \sup_{t_0 - \tau_{\text{max}} \leq s \leq t_0} |x_i(s) - x_i^*(s)|_p
\]

\[ + \kappa_G(T) e^{-\tilde{\lambda}(t-t_0)} \max_i \sup_{t_0 - \tau_{\text{max}} \leq s \leq t_0} \sum_{k=1}^m |r_{i,k}(s)|_p \]

\[ + \sum_{b=0}^{m-k} \frac{(m - k - b)!}{(m - k)!} \cdot d_{i,m-1-b} \cdot s^{m-k-b} |p, \forall N. \]

\[ \text{B Recasting the fulfillment of C2 and C3 as an optimisation problem} \]

The optimisation problem in (15) was obtained by noticing that C2 and C3 can be fulfilled by solving the following optimisation problem:

\[
\min_{\xi} J
\]

\[ \text{s.t. } k_0 \geq 0, k_1 \geq 0, k_2 \geq 0, k_0^{(r)} \geq 0, k_1^{(r)} \geq 0, k_2^{(r)} \geq 0, \]

\[ k_0 > 0, k_0 + k_0^{(r)} > 0, k_1 + k_1^{(r)} > 0, k_2 + k_2^{(r)} > 0, \]

\[ \sigma > 0, \sigma \geq 0, \sigma - \sigma > 0, \mu_2(\tilde{T} \tilde{A}_i \tilde{T}^{-1}) \leq -\sigma, \]

\[ \sum_{j \in N_i} ||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 + ||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq \sigma, \]

\[ (B.2) \]

where the decision variables are \(\xi := [k_0, k_1, k_2, k_0^{(r)}, k_1^{(r)}, k_2^{(r)}, k_0^{(s)}, k_0^{(s)}, k_1^{(s)}, k_2^{(s)}] \) and the cost is defined as in Section 5. The matrices \(B_i, B_j\) are given in (B.1) and \(A_{ii}\) is given in (14), in accordance with Proposition 1 while the transformation matrix \(T\) is also given in (14). In order to find the control gains, we propose to solve the optimisation problem for fixed \(\alpha_1, \alpha_2\). Further, in order to obtain a suitable formulation for the optimisation, we recast the constraints in (B.2) as LMIs as follows. First, by definition, \(\mu_2(\tilde{T} \tilde{A}_i \tilde{T}^{-1}) \leq -\sigma\) is equivalent to \([\tilde{T} \tilde{A}_i \tilde{T}^{-1}] \leq -\sigma I_k\). Moreover, the constraint \(\sum_{j \in N_i} ||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 + ||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq \sigma\) is satisfied if we impose that \(||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq \frac{\sigma}{\sqrt{\gamma}}\) and, simultaneously, \(||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq \frac{\sigma}{\sqrt{\gamma}}, \forall j \in N_i\). In turn, since \(-1 \leq \frac{\sigma}{\sqrt{\gamma}} \leq 0\) and \([N_i] = \tilde{N}\)

we have (by means of the absolutely homogeneous property for matrix norms) that \(||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq 1\). Analogously, we have that \(||T \tilde{B}_{ij}(t)\tilde{T}^{-1}||_2 \leq 1\), with \(B_i\) defined in (14). Hence, the constraints on the norm in (B.2) are satisfied if \(\|T \tilde{B}_i \tilde{T}^{-1}\|_2 \leq \frac{\sigma}{\sqrt{\gamma}}\) and \(\|T \tilde{B}_i \tilde{T}^{-1}\|_2 \leq \frac{\sigma}{\sqrt{\gamma}}, \forall j \in N_i\), which, following (Boyd and Vandenberghe [2004] Example 4.6.3), can be written as \((\frac{\sigma}{\sqrt{\gamma}})^2 I_N - (T \tilde{B}_i \tilde{T}^{-1})^T (T \tilde{B}_i \tilde{T}^{-1}) \geq 0\) and \((\frac{\sigma}{\sqrt{\gamma}})^2 I_N - (T \tilde{B}_i \tilde{T}^{-1})^T (T \tilde{B}_j \tilde{T}^{-1}) \geq 0\). Now, by means of Schur complement (Horn and Johnson [1985] Theorem 7.7.7), this pair of inequalities is equivalent to

\[
\begin{bmatrix}
\frac{\sigma}{\sqrt{\gamma}} I_N & (T \tilde{B}_i \tilde{T}^{-1})^T \\

T \tilde{B}_i \tilde{T}^{-1} & \frac{1}{\sqrt{\gamma}} I_N
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
\frac{\sigma}{\sqrt{\gamma}} I_N & (T \tilde{B}_j \tilde{T}^{-1})^T \\

T \tilde{B}_j \tilde{T}^{-1} & \frac{1}{\sqrt{\gamma}} I_N
\end{bmatrix} \geq 0.
\]

Since the cost is quadratic and the constraints are LMIs, the optimisation problem in (15) is convex for fixed \(\alpha\)’s.
Rejecting Higher Order Polynomials

The optimisation setting of Section 5 is suitable to design control protocols able to reject polynomials of arbitrary (say, \( m - 1 \)) order. We recall that, to reject such a disturbance the protocol needs to have \( m \) multiplex layers (Remark 5). Hence, we design a protocol of the form of (13) but with \( m \) layers. The coupling functions for the \( n \)-th layer, \( n \in \{0, \ldots, m\} \), are, in analogy to (13):

\[
\begin{align*}
    h_{i,n}(\eta, \eta, t) &= k_n(\eta_i - \eta_n - \delta_{i,n}^c), \\
    h_{i,n}^{(r)}(\eta, \eta, t) &= k_n^{(r)} \sum_{j \in \mathcal{N}_i} \psi(\eta_j - \eta_n - \delta_{j,n}^c),
\end{align*}
\]

(B.4)

where \( \psi(x) := \tanh(k^\psi x) \) as in (13) and \( k_n, k_n^{(r)}, k^\psi \) are the control parameters to be designed. Now, C2 and C3 can be fulfilled by solving the following problem:

\[
\begin{align*}
    \min_{\xi} & \quad J \\
    \text{s.t.} & \quad k_n \geq 0, \bar{g}_n \geq 0, k_n + \bar{g}_n > 0, n \in \{0, \ldots, m\}, \\
    & \quad k^\psi > 0, \bar{\sigma} > 0, \bar{\sigma} \geq 0, \bar{\sigma} - \bar{\sigma} > 0, \\
    & \quad \mu_2(\bar{T}A_{ii}\bar{T}^{-1}) \preceq -\bar{\sigma}, \\
    & \quad \sum_{j \in \mathcal{N}_i} ||\bar{T}B_{ij}(t)\bar{T}^{-1}||_2 + ||\bar{T}B_{ii}(t)\bar{T}^{-1}||_2 \leq \bar{\sigma},
\end{align*}
\]

(B.5)

where \( A_{ii}, \bar{B}_{ii}, \bar{B}_{ij} \) are given by (4) – the explicit expressions for these matrices together with the expression of the transformation matrix \( \bar{T} \) are given, for completeness, in (B.3) where \( \bar{g}_n := k^\psi k_n^{(r)} \), \( n \in \{0, \ldots, m\} \). The decision variables are \( \xi := [k_0, \ldots, k_m, k_0^{(r)}, \ldots, k_m^{(r)}, k^\psi, \bar{\sigma}, \bar{\sigma}, \alpha_1, \ldots, \alpha_m] \). As in Section 5, the cost function can be chosen to maximize the upper bound of the interrobot coupling functions (see the discussion in Section 5). Then, following the same steps used to obtain (B.2) the constraints in (B.5) can be recast using LMIs and this yields the analogous of (15):

\[
\begin{align*}
    \min_{\xi} & \quad J \\
    \text{s.t.} & \quad k_n \geq 0, \bar{g}_n \geq 0, k_n + \bar{g}_n > 0, n \in \{0, \ldots, m\}, \\
    & \quad \bar{\sigma} > 0, \bar{\sigma} \geq 0, \bar{\sigma} - \bar{\sigma} > 0, \\
    & \quad [\bar{T}A_{ii}\bar{T}^{-1}] \preceq -\bar{\sigma} I_{(m+1)n}, \\
    & \quad \left[ \frac{\bar{\sigma}}{2} I_{(m+1)n} (\bar{T}B_{ij}\bar{T}^{-1})^T \right] \preceq 0, \\
    & \quad \left[ \frac{\bar{\sigma}}{2} I_{(m+1)n} (\bar{T}B_{ij}\bar{T}^{-1})^T \right] \preceq 0, \\
\end{align*}
\]

(B.6)

where \( \xi = [k_0, \ldots, k_m, \bar{g}_0, \ldots, \bar{g}_m, \bar{\sigma}, \bar{\sigma}] \) and

\[
\begin{align*}
    \bar{B}_{ii} &= -\bar{N}, \\
    \bar{B}_{ij} &= \begin{bmatrix} \bar{g}_0 I_n & 0 & \cdots & 0_n \\ \vdots & \ddots & \vdots & \vdots \\ \bar{g}_m I_n & 0 & \cdots & 0_n \end{bmatrix}, \\
    \bar{B}_{ij} &= \begin{bmatrix} \bar{g}_0 I_n & 0 & \cdots & 0_n \\ \vdots & \ddots & \vdots & \vdots \\ \bar{g}_m I_n & 0 & \cdots & 0_n \end{bmatrix}
\end{align*}
\]

with \( \bar{N} \) defined as in (14).

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\[ T = \begin{bmatrix} I_n & \alpha_1 I_n & 0 & \cdots & 0_n \\ 0_n & I_n & \alpha_2 I_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_n & 0_n & 0_n & \cdots & \alpha_m I_n \\ 0_n & 0_n & 0_n & \cdots & I_n \end{bmatrix}, \quad \bar{A}_{ij} = \begin{bmatrix} -k_0 I_n & I_n & 0_n & \cdots & 0_n \\ -k_1 I_n & 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -k_{m-1} I_n & 0_n & 0_n & \cdots & I_n \\ -k_m I_n & 0_n & 0_n & \cdots & 0_n \end{bmatrix}, \]

\[ B_{ii}(t) = |\mathcal{N}_i| \begin{bmatrix} \bar{g}_0 I_n & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \vdots \\ \bar{g}_m I_n & 0_n & \cdots & 0_n \end{bmatrix} \frac{\partial \tanh(\eta_j - \eta_i - \delta_{ji}^*)}{\partial \eta_i}, \quad \bar{B}_{ij}(t) = \begin{bmatrix} \bar{g}_0 I_n & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \vdots \\ \bar{g}_m I_n & 0_n & \cdots & 0_n \end{bmatrix} \frac{\partial \tanh(\eta_j - \eta_i - \delta_{ji}^*)}{\partial \eta_j}. \]
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