QUASISYMMETRIC EMBEDDINGS OF SLIT SIERPIŃSKI CARPETS

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Abstract. In this paper we study the problem of quasisymmetrically embedding metric carpets, i.e., spaces homeomorphic to the classical Sierpiński carpet, into the plane. We provide a complete characterization in the case of so called dyadic slit carpets. The main tools used are Oded Schramm’s transboundary modulus and the recent quasisymmetric uniformization results of Bonk [Bon11] and Bonk-Kleiner [BK02].

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1. Introduction

The study of quasiconformal geometry of fractal metric spaces has received much attention recently, cf. [Bon11, BK02, BKM09, BM13, Kle06], etc. In particular, the spaces homeomorphic to the classical Sierpiński carpet $S_{1/3}$, see Fig. 3.1, also known as metric carpets, were studies in the papers of Bonk, Merenkov [Bon11, BM13, Mer10] and others, partly because of problems arising in geometric group theory. One such problem, the Kapovich-Kleiner conjecture, suggests that if the boundary at infinity $\partial_\infty G$ of a Gromov hyperbolic group $G$ is a metric carpet then it is quasisymmetrically equivalent to a round carpet in the plane $\mathbb{R}^2$, i.e., a subset $X$ of the plane homeomorphic to $S_{1/3}$ such that every complimentary component of $X$ is a round disk. A recent breakthrough work of Bonk [Bon11] implies that a metric carpet $X \subset \mathbb{R}^2$ is quasisymmetric to a round carpet provided some mild natural conditions are satisfied. In light of Bonk’s theorem, Kapovich-Kleiner conjecture reduces to the question of quasisymmetrically embedding $\partial_\infty G$.
into the plane provided it is homeomorphic to the Sierpiński carpet. This motivates the general question we study in this paper.

**Problem.** Suppose $X$ is a metric space homeomorphic to the classical Sierpiński carpet. Provide necessary and sufficient conditions so that there is a quasisymmetric embedding of $X$ into $\mathbb{R}^2$.

In this paper we define a class of metric carpets, dyadic slit Sierpiński carpets, and study the problem of quasisymmetric embeddability into the plane in this class. Our main result is a complete characterization of dyadic slit carpets which admit a quasisymmetric embedding into the plane.

To formulate our main theorem we need some notation. Suppose $r = \{r_i\}_{i=0}^\infty$ is a sequence of real numbers such that $r_i \in (0, 1)$, $i \geq 0$. We construct a nested sequence of domains $D_i$ in the plane corresponding to $r$ as follows. To obtain $D_0$ we remove from $(0, 1)^2$ the closed vertical slit (interval) of length $r_0$ centered at $(1/2, 1/2)$, i.e., the center of $(0, 1)^2$. Similarly $D_1$ is obtained by removing from $D_0$ the 4 vertical slits of length $r_1/2$, which are located in the dyadic squares of generation 1 and whose centers at the centers of the corresponding squares. Continuing by induction we obtain a sequence of domains $D_{i+1} \subset D_i$ in the unit square $(0, 1)^2$. Note that the area of $D_i$ is 1 since its complement in the unit square is a union of finitely many straight intervals. Next, consider the sequence of metric spaces $S_i$, where $S_i$ is the completion of the domain $D_i$ in its inner path metric $d_{D_i}$. It turns out that the spaces $S_i$ converge (in an appropriate sense) to a metric carpet, which we denote by $\mathcal{S} = \mathcal{S}_r$ and call the slit carpet corresponding to $r$. The following is the main result of this paper.

**Theorem 1.1.** Suppose $\mathcal{S}_r$ is a dyadic slit carpet corresponding to a sequence $r = \{r_i\}_{i=0}^\infty$. There is a quasisymmetric embedding of $\mathcal{S}_r$ into the plane if and only if $r = \{r_n\}_{n=0}^\infty \in \ell^2$.

It was shown by Bonk [Bon11] that if $X \subset \mathbb{C}$ is a Sierpiński carpet such that the peripheral circles are uniformly relatively separated uniform quasicircles then there is a quasisymmetric mapping $f : \mathbb{C} \to \mathbb{C}$ such that $f(X)$ is a round carpet, i.e., every complementary component of $f(X)$ is a round disk. Thus, combining Theorem 1.1 with Bonk’s theorem we obtain the following.

**Theorem 1.2.** Suppose $\mathcal{S}_r$ is a dyadic slit carpet whose peripheral circles are uniformly relatively separated. Then $\mathcal{S}_r$ is quasisymmetric to a round carpet if and only if $r \in \ell^2$.

Slit carpets were first studied by Merenkov in [Mer10] where the author obtained deep rigidity results about the quasiconformal geometry of $\mathcal{S}_{1/2}$, i.e., the slit carpet corresponding to the constant sequence $r_i = 1/2$, $\forall i \geq 0$. In [MW11] Merenkov and Wildrick showed that Merenkov’s carpet $\mathcal{S}_{1/2}$ does not embed quasisymmetrically into the plane. This was mainly due to the fact that every quasisymmetric image $f(\mathcal{S}_{1/2}) \subset \mathbb{R}^2$ would have to be a porous subset of the plane and thus would have to have zero area. This would contradict the fact that a quasisymmetric image of Merenkov’s carpet has to be Ahlfors 2-regular, due to a result from [Mer10]. Now, if $r \notin \ell^2$ then the image of $\mathcal{S}_r$ in the plane would not have to be porous in general and thus the porosity argument which applied to Merenkov’s carpet would not work.
In this paper we use Oded Schramm’s transboundary modulus and Bonk’s uniformization results from [Bon11] to show that if \( r \notin \ell^2 \) then \( S_r \) cannot be quasisymmetrically embedded in \( \mathbb{R}^2 \). A key point in our proof is the estimate of transboundary modulus which is inspired by the work of the first author [Hak18], where it was shown that if \( r \notin \ell^2 \) then the classical modulus of “non-vertical” curves in \( S_r \) vanishes.

2. Preliminaries

2.1. Notations and Definitions. Given a metric space \((X, d)\), a point \( x \in X \) and \( r > 0 \), we denote by \( B(x, r) \) the open ball of radius \( r \) centered at \( x \), i.e., \( B(x, r) = \{ y \in X : d(x, y) < r \} \).

The closed unit disk and its boundary circle in the Euclidean plane \( \mathbb{R}^2 \) will be denoted by \( D \) and \( \partial D \), respectively. The unit sphere in \( \mathbb{R}^n \) will be denoted by \( S^{n-1} \).

If \( E \subset X \), then the closure, interior and topological boundary of \( E \) will be denoted by \( \overline{E} \), \( \text{int}(E) \), and \( \partial E \), respectively. The diameter of \( E \) in \( X \) and the distance between subsets \( E \) and \( F \) of \( X \) are defined as follows,

\[
\text{diam}(E) = \sup\{d(x, y) : x, y \in E\},
\]
\[
\text{dist}(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.
\]

Sometimes we will write \( \text{diam}_X(E) \) and \( \text{dist}_X(E, F) \) to emphasize the metric with respect to which these quantities are being calculated.

If \( \text{diam}(E) > 0 \) and \( \text{diam}(F) > 0 \), the relative distance between \( E \) and \( F \) is

\[
\Delta(A, B) = \frac{\text{dist}(A, B)}{\min\{\text{diam}(A), \text{diam}(B)\}}
\]

(2.1)

Let \( I \) be a finite or countable indexing set. A family \( K = \{K_i\}_{i \in I} \) of subsets of \( X \) is called \( s \)-relatively separated for \( s > 0 \) if \( \Delta(K_i, K_j) \geq s \) for every \( i, j \in I \), \( i \neq j \). The sets in \( K \) are said to be uniformly relatively separated if they are \( s \)-relatively separated for some \( s > 0 \).

Everywhere in this paper we will denote by \( \mathcal{H}^Q \) the Hausdorff \( Q \)-measure on \( X \), \( Q \geq 0 \). A metric measure space \((X, d)\) is said to be \( \text{Ahlfors } Q \)-regular, \( Q \geq 0 \), if there exists a constant \( C \geq 1 \) such that

\[
\frac{r^Q}{C} \leq \mathcal{H}^Q(B(p, r)) \leq C \cdot r^Q.
\]

(2.2)

for all \( p \in X \) and \( 0 < r \leq \text{diam}(X) \). The constant \( C \) in (2.2) will be called the \( \text{Ahlfors regularity constant} \) of \( X \). The upper and lower estimates of \( \mathcal{H}^Q(B(x, r)) \) in (2.2) often will be written as

\[
\mathcal{H}^Q(B(x, r)) \lesssim r^Q \quad \text{and} \quad \mathcal{H}^Q(B(x, r)) \gtrsim r^Q,
\]

respectively, while if both inequalities hold we will simply write \( \mathcal{H}^Q(B(x, r)) \asymp r^Q \), instead of (2.2).

2.2. Quasiconformal and quasisymmetric mappings. Here we define the various classes of mappings we are going to work with and refer to [Ahl06], [Hei01] and [Vai71] for further details and the properties of these maps.
Let $f : X \to Y$ be a homeomorphism between two metric spaces $(X, d_X)$ and $(Y, d_Y)$. For a point $x \in X$ and $r > 0$, we define the \textit{linear dilatation of $f$ at $x$} as

\begin{equation}
H_f(x) = \limsup_{r \to 0} \frac{L_f(x, r)}{l_f(x, r)},
\end{equation}

where

\begin{align*}
L_f(x, r) &= \sup_y \{d_Y(f(x), f(y)) \mid d_X(x, y) \leq r\}, \\
l_f(x, r) &= \inf_y \{d_Y(f(x), f(y)) \mid d_X(x, y) \geq r\}.
\end{align*}

We say that a homeomorphism $f : X \to Y$ is \textit{(metrically) $K$-quasiconformal} (or $K$-qc) if

\[ \sup_{x \in X} H_f(x) \leq K \]

for some $1 \leq K < \infty$. A map is quasiconformal if it is $K$-qc for some $K$.

A homeomorphism $f : X \to Y$ is called \textit{$\eta$-quasisymmetric}, where $\eta : [0, \infty) \to [0, \infty)$ is a given homeomorphism, if

\[ \frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left( \frac{d_X(x, y)}{d_X(x, z)} \right) \]

for all $x, y, z \in X$ with $x \neq z$. The map $f$ is called \textit{quasisymmetric} if it is $\eta$-quasisymmetric for some \textit{distortion function} $\eta$.

Here are some useful properties of quasisymmetric maps, which will be used repeatedly in the paper.

\begin{lemma}
Suppose $f : X \to Y$ and $g : Y \to Z$ are $\eta$ and $\eta'$-quasisymmetric mappings, respectively.

(1) The composition $f \circ g : X \to Z$ is an $\eta' \circ \eta$-quasisymmetric map.

(2) The inverse $f^{-1} : Y \to X$ is a $\theta$-quasisymmetric map, where $\theta(t) = 1/\eta(1/t)$.

(3) If $A$ and $B$ are subsets of $X$ and $A \subset B$, then

\begin{equation}
\frac{1}{2\eta \left( \frac{\text{diam}(B)}{\text{diam}(A)} \right)} \leq \frac{\text{diam}(f(A))}{\text{diam}(f(B))} \leq \eta \left( \frac{2\text{diam}(A)}{\text{diam}(B)} \right)
\end{equation}

\end{lemma}

An a priori stronger notion than quasisymmetry is that of weak quasisymmetry. We say that $f : X \to Y$ is $H$-\textit{weakly quasisymmetric}, $H \geq 1$, if for all $x, y, z \in X$ the following implication holds:

\begin{equation}
dx(x, y) \leq d_X(x, z) \implies d_Y(f(x), f(y)) \leq H d_Y(f(x), f(z)).
\end{equation}

Note that an $\eta$-quasisymmetric map is weakly-quasisymmetric with $H = \eta(1)$, and while the converse is not always true the two classes often do coincide.

\begin{lemma}
(See [Hei01], p.80). Let $f : X \to Y$ be an $H$-weakly quasisymmetric mapping. If $X$ is connected and both $X$ and $Y$ are $N$-doubling, then $f$ is $\eta$-quasisymmetric with $\eta$ only depending on $N$ and $H$.

Recall that a metric space $(X, d)$ is $N$-\textit{doubling}, where $N \in \mathbb{N}$, if every ball of radius $r > 0$ in $X$ can be covered by at most $N$ balls of radius $r/2$ in $X$.


2.3. Finitely connected domains bounded by quasicircles. A quasicircle is a quasisymmetric image of the unit circle \( \partial \mathbb{D} \). The following well-known result of Tukia and Väisälä \([TV80]\) provides a complete characterization of quasicircles.

**Proposition 2.3.** A simple closed curve \( \gamma \subset X \) is a quasicircle if and only if it is doubling and

\[
\min\{\text{diam}(\gamma_1), \text{diam}(\gamma_2)\} \leq k \cdot d_X(x, y),
\]

where \( \gamma_1 \) and \( \gamma_2 \) are the two subarcs of \( \gamma \) with endpoints \( x \) and \( y \).

A quasicircle is a \( k \)-quasicircle for some \( k \geq 1 \) if it satisfies (2.6). If \( \gamma \) is a \( k \)-quasicircle and is also \( N \)-doubling, then there exists a \( H \)-weak quasisymmetry \( f : \partial \mathbb{D} \to \gamma \), where \( H \) depends only on \( k \) and \( N \). On the other hand, if \( f : \partial \mathbb{D} \to \gamma \) is \( H \)-weakly quasisymmetric then \( \gamma \) satisfies (2.6) with \( k = 2H \).

A family \( \{\gamma_i : i \in I\} \) of quasicircles in \( X \) is said to consist of uniform quasicircles if there exists \( k \geq 1 \) such that \( \gamma_i \) is a \( k \)-quasicircle for each \( i \in I \). Below we will need the following result, cf. \([Bon11, Corollary 4.7]\). In \([Bon11]\) the proof is given in the planar case, but it generalizes to the quite general case below verbatim.

**Proposition 2.4.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces and \( \{\gamma_i\}_{i \in I} \) a family of \( s \)-relatively separated \( k \)-quasicircles in \( X \). If \( f : X \to Y \) is an \( \eta \)-quasisymmetric map, then the family \( \{f(\gamma_i)\}_{i \in I} \) consists of \( s' \)-relatively separated \( k' \)-quasicircles, where \( s' = s'(\eta, s) \) and \( k' = k'(\eta, k) \).

3. Dyadic Slit Carpets

3.1. Metric carpets. The classical Sierpiński carpet \( S_{1/3} \) is the subset of the plane obtained as follows. Divide the unit square \([0,1]^2\) into nine congruent squares of side-length \( 1/3 \) with disjoint interiors, and let \( E_1 \) be the closed set obtained by removing the interior of the middle square from \([0,1]^2\). Continuing by induction, assume that for \( i \geq 1 \) the set \( E_i \) has been constructed and is a union of finitely many closed and essentially disjoint squares with side-length \( 1/3^i \). Dividing each such square in \( E_i \) into 9 subsquares and removing the interiors of middle squares produces the set \( E_{i+1} \subset E_i \). Finally the classical Sierpiński carpet \( S_{1/3} \) is defined as the infinite intersection \( \cap_{i \in \mathbb{N}} E_i \).

![Figure 3.1](image)

**Figure 3.1.** First three steps in the construction of the standard Sierpiński carpet \( S_{1/3} \).
We say a metric space $X$ is a **metric carpet** if it is homeomorphic to $S^1/3$. The following classical theorem of Whyburn [Why58] characterizes the subsets of the plane which are homeomorphic to the classical Sierpiński carpet.

**Theorem 3.1 (Whyburn).** Suppose $D_i \subset S^2$, $i \geq 0$, is a sequence of topological disks satisfying the conditions:

1. $D_i \cap D_j = \emptyset$, for $i \neq j$.
2. $\text{diam}(D_i) \to 0$, as $i \to \infty$.
3. $(\bigcup_i D_i) = S^2$.

Then the compact set $S^2 \setminus \bigcup_i D_i$ is homeomorphic to the standard Sierpiński carpet $S^1/3$.

If $X$ is a metric carpet then a topological circle $\gamma \subset X$ is called a **peripheral circle** if $X \setminus \gamma$ is connected, i.e., $\gamma$ is a non-separating curve in $X$. From Whyburn’s theorem it follows that $\gamma \subset X$ is a non-separating curve if and only if there is a homeomorphism mapping $X$ to $S^1/3$ and $\gamma$ to the boundary of one of the complementary domains of $S^1/3$ in the plane.

### 3.2. Slit carpets.

In this section we construct a class of metric carpets called **dyadic slit carpets** which are the main object of study of this paper. Dyadic slit carpets include the slit carpet considered by Merenkov in [Mer10] and were also considered by the first author in [Hak18].

Let $U$ denote the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$. We say that $\Delta \subset U$ is a **dyadic square of generation** $n$ if there exist $i, j \in \{0,1,2,\ldots,2^n-1\}$ such that

$$\Delta = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right].$$

We will denote by $\mathcal{D}_n$ be the collection of all dyadic squares of generation $n$ and by $\mathcal{D} = \cup_{n=0}^{\infty} \mathcal{D}_n$ the collection of all dyadic squares in $[0,1]^2$. The sidelength of a dyadic square $\Delta$ will be denote by $l(\Delta)$. Thus, if $\Delta \in \mathcal{D}_n$ then $l(\Delta) = 1/2^n$.

Given a sequence $r = \{r_n\}_{n=0}^{\infty}$, such that $r_n \in (0,1)$ for $n = 0, 1, 2, \ldots$, we construct the corresponding sequence of “slit” domains $S_n = S_n(r)$ in $U$ as follows. For every dyadic square $\Delta$ of generation $n$ we denote by $s(\Delta)$ the closed vertical slit in $\Delta$ of length $r_n l(\Delta)$, whose center coincides with the center of $\Delta$. More precisely if $(x,y)$ is the center of $\Delta \in \mathcal{D}_n$ then

$$s(\Delta) = \{x\} \times \left[y - \frac{r_n}{2^{n+1}}, y + \frac{r_n}{2^{n+1}}\right].$$

We say that a slit $s = s(\Delta) \subset \Delta$ is a **slit of generation** $n$ if $\Delta \in \mathcal{D}_n$, for some $n \geq 0$. For $n \geq 0$ let

$$K_n = K_n(r) = \{s(\Delta) : \Delta \in \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_n\} \quad \text{and} \quad K_n = \bigcup_{s \in K_n} s = \bigcup_{i=0}^{n} \bigcup_{\Delta \in \mathcal{D}_i} s(\Delta)$$

be the collection of all slits of generation $n$ or less and their union, respectively. We will also use the following convention: $K_{-1} = \emptyset$. 
Similarly, we will use the following notation for the the collection of all slits and their union,

\[ K = K(r) = \{ s(\Delta) : \Delta \in D \} \text{ and } \\
K = \bigcup_{s \in K} s = \bigcup_{\Delta \in D} s(\Delta) \]

Finally, for \( n \geq 0 \) define the slit domains \( S_n \subset \mathbb{U} \) by letting \( S_0 = [0, 1]^2 \) and for \( n \geq 1 \), letting

\[ S_n = (0, 1)^2 \setminus K_{n-1} = (0, 1)^2 \setminus \bigcup_{i=0}^{n-1} \bigcup_{\Delta \in D_i} s(\Delta). \]

In particular \( S_1 = \mathbb{U} \setminus s_0 \), where \( s_0 \) is the central slit of length \( r_0 \) with the midpoint at \((1/2, 1/2)\).

To define the metric carpet \( \mathcal{S} \) we first let \( \mathcal{S}_n \) be the completion of the domain \( S_n \) in its path metric \( d_{S_n} \). Recall that the path metric \( d_\Omega \) on a domain \( \Omega \subset \mathbb{R}^n \) is defined as follows: if \( x, y \in \Omega \), let

\[ d_\Omega(x, y) = \inf \{ l(\gamma) : \gamma \subset \Omega \text{ s.t. } \gamma(0) = x, \gamma(1) = y \}, \]

where \( l(\gamma) \) denotes the length of a rectifiable curve \( \gamma \) in \( \Omega \), and the infimum is over all rectifiable curves in \( \Omega \) connecting \( x \) and \( y \). The metric on \( \mathcal{S}_n \) will be denoted by \( d_{\mathcal{S}_n} \). Note that \( \mathcal{S}_0 \) is isometric to \([0, 1]^2\) equipped with the Euclidean metric.

A boundary component of \( \mathcal{S}_n \) corresponding to a slit of a dyadic square \( \Delta \in D_m \) of generation \( m \leq n-1 \) will be called a slit of \( \mathcal{S}_n \) of generation \( m \). The slit of generation \( 0 \) in \( \mathcal{S}_n \) will be called the central slit of \( \mathcal{S}_n \). The boundary component of \( \mathcal{S}_n \) corresponding to \( \partial([0, 1]^2) \) will be called the outer square of \( \mathcal{S}_n \).

For every \( m, n \in \mathbb{N} \cup \{0\} \) with \( m \leq n \) there exists a natural 1-Lipschitz projection

\[ \pi_{m,n} : \mathcal{S}_n \rightarrow \mathcal{S}_m \]

obtained by identifying the points on the slits of \( \mathcal{S}_n \) that correspond to the same point of \( \mathcal{S}_m \). More precisely, if \( p, q \in \mathcal{S}_n \) then \( \pi_{m,n}(p) = \pi_{m,n}(q) \), whenever \( d_{\mathcal{S}_n}(p, q) = 0 \). Note that all the boundary components of \( \mathcal{S}_n \) are topological circles. Moreover, every slit of diameter \( d > 0 \) in \( \mathcal{S}_n \) is isometric to the square \( \partial([0, d/2] \times [0, d/2]) \subset \mathbb{R}^2 \) equipped with the metric induced from the \( \ell^1 \) norm on \( \mathbb{R}^2 \).

As a topological space the dyadic slit Sierpiński carpet corresponding to \( r \) is defined as the inverse limit of the system \((\mathcal{S}_n, \pi_{m,n})\), and is denoted by \( \mathcal{S} \). More explicitly,

\[ \mathcal{S} = \{ (p_0, p_1, \ldots) : p_i \in \mathcal{S}_i \text{ and } p_i = \pi_{i,i+1}(p_{i+1}) \} . \]

If the sequence \( r \) is understood from the context, we will denote \( \mathcal{S} \) simply by \( \mathcal{S} \).

The inverse limits of the slits and outer squares of \( \mathcal{S}_n \) are topological circles and will be called the slits and outer square of \( \mathcal{S} \), respectively. Clearly, the slits are dense in \( \mathcal{S} \), i.e., for every point \( p \) in \( \mathcal{S} \) and every neighborhood \( U \) of \( p \), there exists a slit of \( \mathcal{S} \) that intersects \( U \).

The diameter of each \( \mathcal{S}_n \) is clearly bounded by 2. If \( x = (x_0, x_1, \ldots) \) and \( y = (y_0, y_1, \ldots) \) are points in \( \mathcal{S} \), we define a distance between them by

\[ d_\mathcal{S}(x, y) = \lim_{n \rightarrow \infty} d_{\mathcal{S}_n}(x_n, y_n) . \]

Since every \( \pi_{m,n} \) is 1-Lipschitz, \( (d_{\mathcal{S}_n}(p_n, q_n)) \) is a monotone increasing bounded sequence, and thus \( d_\mathcal{S}(p, q) \) exists and defines a metric on \( \mathcal{S} \).
For each $n \geq 0$, there is a natural projection map

$$\pi_n : \mathcal{S} \to \mathcal{S}_n.$$ 

To simplify notations, we denote $\pi_0 : \mathcal{S} \to [0, 1]^2$ simply by $\pi$. We will often make use of the mappings $\pi_{0,n}$ which map $\mathcal{S}_n$ onto $\mathcal{S}_0 = [0, 1]^2$.

It was shown in [Hak18] (see also [Mer10]) that a slit carpet $\mathcal{S}$ corresponding to a general collections of slits $\{s_i\}_{i=1}^\infty \subset (0, 1)^2$ is homeomorphic to the Sierpiński carpet $\mathcal{S}_{1/3}$, provided the slits are uniformly relatively separated, dense in $[0, 1]^2$ and $\text{diam}(s_i) \to 0$ as $i \to \infty$. The same proof shows that $\mathcal{S}_r$ is homeomorphic to $\mathcal{S}_{1/3}$ for arbitrary sequence $r$ between 0 and 1.

For studying quasisymmetric geometry of $\mathcal{S}_r$ it will be important to know when the peripheral circles of $\mathcal{S}_r$ are uniformly relatively separated. It will be shown that this condition holds provided the sequence $r_i$ is bounded away from 1. We start with the following easy observation.

**Lemma 3.2.** If there is a constant $r \in (0, 1)$ such that $r_i \leq r < 1$ for $i \geq 0$, then the collection of slits $\{s(\Delta)\}_{\Delta \in D} \cup \{\partial U\}$ is uniformly relatively separated in $[0, 1]^2$ with relative separation constant $\delta = \min\{1/2, 1-r\}$.

**Proof.** Suppose $n \geq m$, and pick two slits $s', s'' \subset \mathcal{S}$ of generations $n$ and $m$, respectively. If $s'$ and $s''$ do not belong to the same vertical line, then $\min\{\text{diam}(s'), \text{diam}(s'')\} = r_n 2^{-n} < 2^{-n}$ while $\text{dist}(s', s'') \geq 2^{-n}/2$. Therefore $\Delta(s', s'') > 1/2$. On the other hand, if $s'$ and $s''$ are on the same vertical line then $m = n$ and $\text{dist}(s', s'') = (1-r_n) 2^{-n} \geq (1-r) 2^{-n}$. Thus, $\Delta(s', s'') \geq \text{dist}(s', s'')/2^{-n} \geq 1 - r$. Similarly, since $\text{dist}(s', \partial U) \geq 2^{-(n+1)}$ it follows that $\Delta(s', \partial U) \geq 1/2$. Therefore $\{s(\Delta)\}_{\Delta \in D}$ is uniformly relatively separated with the required separation constant.

Combining this result with the fact that the projections $\pi_{0,n}$ are Lipschitz we obtain the following.

**Lemma 3.3.** If there is a constant $r \in (0, 1)$ such that $r_i \leq r < 1$ for $i \geq 0$, then the boundary components of $\mathcal{S}_n$ and peripheral circles of $\mathcal{S} = \mathcal{S}_r$ are uniformly relatively separated with relative separation constant which does not depend of $n$. 

**Figure 3.2.** Slit domains corresponding to sequences $(1/2, 1/2, 1/2, 1/2)$ (left) and $(1/10, 2/5, 1/8, 1/2)$ (right).
Proof. Note that for every \( m \geq 0 \) and a slit \( s \subset (0,1)^2 \) of generation \( m \) we have
\[
\text{diam}_{\mathcal{S}}(s) = r_m 2^{-m} = \text{diam}_{\mathcal{S}}(\pi_{0,n}^{-1}(s)) = \text{diam}_{\mathcal{S}}(\pi^{-1}(s)),
\]
for \( m < n \). Moreover, for every \( n \geq 0 \) we have
\[
\sqrt{2} = \text{diam}(\partial U) \leq \text{diam}_{\mathcal{S}}(\pi_{0,n}^{-1}(\partial U)) \leq 2.
\]
Therefore, if \( \gamma \) is a boundary component of \( \mathcal{S}_n \) then
\[
\text{diam}(\gamma) \leq \text{diam}_{\mathcal{S}_n}(\gamma) \leq 2\text{diam}(\gamma).
\]

Let \( \gamma' \) be another boundary component of \( \mathcal{S}_n \). Since \( \pi_{0,n} \) is a 1-Lipschitz function, we have \( \text{dist}(\gamma, \gamma') \geq \text{dist}(\pi_{0,n}(\gamma), \pi_{0,n}(\gamma')) \). Combining this with (3.3) and Lemma 3.2 we obtain
\[
\Delta(\gamma, \gamma') \geq 2^{-1} \Delta(\pi_{0,n}(\gamma), \pi_{0,n}(\gamma)) \geq \min\{1/4, (1-r)/2\} > 0,
\]
which completes the proof. \( \square \)

Let \( L, R, T, B \) denote the left, right, top and bottom sides of the outer square of \( \mathcal{S} \) respectively, i.e.,
\[
L = \pi^{-1}(\{(0, y) : 0 \leq y \leq 1\}) \quad R = \pi^{-1}(\{(1, y) : 0 \leq y \leq 1\})
\]
\[
T = \pi^{-1}(\{(0, x) : 0 \leq x \leq 1\}) \quad B = \pi^{-1}(\{(1, x) : 0 \leq x \leq 1\}).
\]
Sometimes we will use this notation to denote the corresponding sides of \( \mathcal{S}_n, U \) or other slit domains if there is no chance of confusion.

When talking about a dyadic square of generation \( n \) in \( \mathcal{S} \), we mean the subset of \( \pi^{-1}(\Delta), \Delta \subset D_n \), which can be thought of as a slit carpet with respect to \( \{r_i\}_{i=1}^\infty \) constructed in \( \Delta \) instead of \( U \). More precisely, we say that \( T \subset \mathcal{S} \) is a dyadic square of generation \( n \) in \( \mathcal{S} \), if there is a dyadic square \( \Delta \subset D_n \) such that
\[
T_\Delta = \pi^{-1}(\text{int}(\Delta)).
\]
We will also use the following notation
\[
\partial T_\Delta := T_\Delta \setminus \pi^{-1}(\text{int}(\Delta)).
\]
Thus \( \partial T_\Delta \) is the “outer square” of \( T_\Delta \). A dyadic square of generation \( m \) in \( \mathcal{S}_n \) is the image of a dyadic square of generation \( m \) in \( \mathcal{S} \) under \( \pi_n \). Note that for \( m > n \) dyadic squares of generation \( m \) in \( \mathcal{S}_n \) do not contain slits in their interiors and therefore are isometric to Euclidean squares.

Define a projection map \( \text{proj}(x, y) = x \) for \( \forall (x, y) \in U \). A curve \( \gamma \subset U \) is called vertical if \( \text{proj}(\gamma) \) is a constant, i.e., the first coordinate of \( \gamma \) is constant. A curve which is not vertical is called nonvertical.

The following properties are from [Mer10] and [Hak18]. We state them without proof.

**Lemma 3.4.** There exists a constant \( 0 < c < 1 \) independent of \( n \) such that for every \( p \in \mathcal{S} \) and \( 0 < r < \text{diam}(\mathcal{S}) \) there exists a point \( q \in \mathcal{S}_n, n \geq 0 \) such that
\[
B(q, cr) \subset \pi_n(B(p, r)) \subset B(\pi_n(p), r).
\]

**Lemma 3.5.** There exists a constant \( C \geq 1 \), independent of \( n \geq 1 \) such that for any Borel set \( E \subset \mathcal{S} \) we have
\[
\frac{1}{C} \mathcal{H}^2(\pi_n(E)) \leq \mathcal{H}^2(E) \leq C \mathcal{H}^2(\pi_n(E)).
\]
In addition, \( S \) and \( S_n \) are Ahlfors 2-regular with the same Ahlfors regularity constant and \( N \)-doubling with the same doubling constant for every \( n \).

**Lemma 3.6.** The metric space \( S \) equipped with \( H^2 \) is a metric Sierpiński carpet which is doubling and Ahlfors 2-regular.

### 4. Modulus and Transboundary Modulus

In this section we recall the notions of modulus and transboundary modulus and formulate some of their properties which will be used in the proof of Theorem 1.1. In particular we prove quasiconformal quasi-invariance of transboundary modulus in \( \mathbb{R}^n \) and recall a formula of Bonk calculating the transboundary modulus relative to a family of \( \mathcal{C}^* \)-squares.

#### 4.1. Lengths of curves

A curve in a metric space \( X \) is a continuous function \( \gamma : J \to X \) where \( J \) is an interval in \( \mathbb{R} \), i.e., there are reals \( a < b \) such that \( J \) has one of the following forms \( [a, b] \), \( (a, b) \), \( [a, b) \) or \( (a, b] \). We will often denote the image \( \gamma(J) \) simply by \( \gamma \). We say the curve \( \gamma \) is rectifiable if it has finite length: \( l(\gamma) < \infty \).

If every compact subcurve of \( \gamma \) is rectifiable, we say that \( \gamma \) is locally rectifiable.

If \( \Gamma \) is a family of curves in \( X \) and \( f : X \to Y \) is a homeomorphism, we denote by \( f(\Gamma) = \{ f \circ \gamma : \gamma \in \Gamma \} \).

Let \( E, F \) be subsets of \( X \). We will denote by \( \Gamma(E, F; X) \) the family of curves \( \gamma \) in \( X \) "connecting" \( E \) and \( F \). More precisely,

\[
\Gamma(E, F; X) = \{ \gamma \subset X : \gamma(0) \in E \text{ and } \gamma(1) \in F \}\]

where \( \overline{\gamma} : [0, 1] \to \overline{X} \), the closure of \( \gamma \), is a closed curve.

For a rectifiable curve \( \gamma : J \to X \), the associated length function, \( s : \gamma : J \to [0, l(\gamma)] \) is defined by \( s(\gamma(t)) = l(\gamma([0, t])) \). The arclength parametrization of \( \gamma \) is the unique 1-Lipschitz function \( s_\gamma : [0, l(\gamma)] \to X \) that satisfies the equation \( \gamma = \gamma_s \circ s_\gamma \).

Given a Borel function \( \rho : X \to [0, \infty) \) we define the \( \rho \)-length of a rectifiable curve \( \gamma \) as follows

\[
l_\rho(\gamma) := \int_\gamma \rho ds = \int_0^{l(\gamma)} \rho(s_\gamma(t))dt.
\]

For \( f : X \to Y \) and \( x \in X \) let

\[
L_f(x) := \limsup_{r \to 0} \left( \frac{L_f(x, r)}{r} \right),
\]

where \( L_f(x, r) \) is the distortion of \( f \) at \( x \) at scale \( r \) defined in Section 2.2.

The following is Theorem 5.3 in [Vai71] and will be crucial in the proof of quasi-invariance of transboundary modulus below.

**Theorem 4.1.** Suppose \( D \subset \mathbb{R}^n \) and \( f : D \to \mathbb{R}^n \) is a continuous map. If \( \gamma \in D \) is a locally rectifiable curve and \( f \) is absolutely continuous on every closed subcurve of \( \gamma \), then \( f(\gamma) \) is locally rectifiable, and for every Borel function \( \rho : Y \to [0, \infty) \) we have

\[
\int_{f(\gamma)} \rho ds \leq \int_\gamma (\rho \circ f) \cdot L_f ds.
\]
4.2. Modulus. Everywhere below we will assume that \((X,d)\) is a metric space equipped with a Borel measure \(\mu\).

Let \(\Gamma\) be a family of curves in \(X\). A Borel function \(\rho : X \to [0,\infty)\) is called \textit{admissible for} \(\Gamma\), denoted by \(\rho \wedge \Gamma\), if

\[
\int_\gamma \rho \, ds \geq 1, \quad \forall \gamma \in \Gamma,
\]

where, as in (4.1), \(ds\) is the arclength measure of \(\gamma\). For \(p \geq 1\), the \(p\)-\textit{modulus} of \(\Gamma\) is defined as

\[
\text{mod}_p \Gamma = \inf_{\rho \wedge \Gamma} \int_X \rho^p d\mu.
\]

The following lemma summarizes some of the most important properties of modulus which will be used in this paper. We say \(\Gamma_1\) \textit{minorizes} \(\Gamma_2\) and write \(\Gamma_1 < \Gamma_2\), if every curve \(\gamma \in \Gamma_2\) contains a subcurve \(\delta \subset \gamma\) which belongs to \(\Gamma_1\).

**Lemma 4.2.** Suppose \((X,d,\mu)\) is a metric measure spaces, \(p \geq 1\) and \(\Gamma_i, i = 1,2,\ldots\) are curve families in \(X\). Then

(1) \(\text{(Monotonicity)}\) \(\text{mod}_p \Gamma \leq \text{mod}_p \Gamma'\), if \(\Gamma \subset \Gamma'\),

(2) \(\text{(Subadditivity)}\) \(\text{mod}_p \Gamma \leq \sum_{i=1}^n \text{mod}_p \Gamma_i\), if \(\Gamma = \bigcup_{i=1}^n \Gamma_i\),

(3) \(\text{(Overflowing)}\) \(\text{mod}_p (\Gamma_1) \geq \text{mod}_p (\Gamma_2)\) if \(\Gamma_1 < \Gamma_2\).

4.3. Transboundary modulus. Suppose \(\mathcal{K} = \{K_1, \ldots, K_n\}\) is a finite collection of pairwise disjoint compact connected subsets of a metric measure space \((X,d,\mu)\) and let \(K = \bigcup_{i=1}^n K_i\).

A \textit{transboundary mass distribution} (or just \textit{mass distribution}) on \((X,\mathcal{K})\) is an \((n+1)\)-tuple \((\rho; \rho_1, \ldots, \rho_n)\), where \(\rho : X \setminus K \to [0,\infty)\) is a Borel function, while \(\rho_i\)'s, \(i = 1, \ldots, n\), are non-negative weights corresponding to \(K_i\)'s. The \(p\)-\textit{mass} of \((\rho; \{\rho_i\})\) is

\[
A_p((\rho; \{\rho_i\})) = \int_{X \setminus K} \rho^p d\mu + \sum_{i=1}^n \rho_i^p
\]

Given a family of curves \(\Gamma\) in \(X\), we say that the mass distribution \((\rho; \{\rho_i\})\) is \textit{admissible for} \(\Gamma\), and write \((\rho; \{\rho_i\}) \wedge (\Gamma, \mathcal{K})\), if

\[
l_{(\rho; \{\rho_i\})}(\gamma) := \int_{\gamma \cap X \setminus K} \rho \, ds + \sum_{\gamma \cap K, \neq \emptyset} \rho_i \geq 1, \quad \text{for all } \gamma \in \Gamma.
\]

Note that \(l_\rho(\gamma)\) where \(\varrho = (\rho; \{\rho_i\})\) is defined if \(\gamma\) is locally rectifiable on each component of \(\gamma \setminus K\). For \(p \geq 1\), the \textit{transboundary \(p\)-modulus} of a curve family \(\Gamma\) in \(X\) relative the family of subsets \(\mathcal{K}\) is defined as

\[
\text{Mod}_{p,X,\mathcal{K}}(\Gamma) = \inf_{(\rho; \{\rho_i\}) \wedge (\Gamma, \mathcal{K})} \left\{ \int_{X \setminus K} \rho^p d\mu + \sum_{i=1}^n \rho_i^p \right\}.
\]

Alternatively, transboundary modulus can be defined via the quotient space \(X_\mathcal{K}\) obtained by identifying each of the subsets \(K_i\) of \(X\) to a point, i.e.,

\[X_\mathcal{K} = X/\sim,
\]

where

\[x \sim y \text{ if and only if } x,y \in K_i \text{ for some } 1 \leq i \leq n.
\]
Proposition 4.3. Let \( q : X \to X_\mathcal{K} \) be the quotient map and let \( k_i := q(K_i) \in X_\mathcal{K} \). Since \( q \) is the identity on \( X \setminus K \) we will slightly abuse the notation and denote by \( \mu\big|_{X\setminus K} \) the pushforward by \( q \) of the restriction of \( \mu \) to \( X \setminus K \). We equip \( X_\mathcal{K} \) with a Borel measure \( \mu_\mathcal{K} \) defined by

\[
(4.6) \quad \mu_\mathcal{K} = \mu\big|_{X\setminus K} + \sum_{i=1}^{n} \delta_{k_i}.
\]

A transboundary mass distribution

\[
(q) = (\rho; \rho_1, \ldots, \rho_n),
\]

then is just a Borel function \( q : X_\mathcal{K} \to [0, \infty] \), such that \( \rho = q\big|_{X_\mathcal{K} \cup \bigcup_{i=1}^{n} k_i} \) and \( \rho_i = q(k_i) \), while the \( p \)-mass defined above is equal to

\[
(4.7) \quad A_p(q) = \int q^p d\mu_\mathcal{K}
\]

Given a mass distribution \( q \) and a curve \( \gamma \in X_\mathcal{K} \) such that \( \tilde{\gamma} := q^{-1}(\gamma) \setminus K \) is locally rectifiable in \( X \), the \( \varrho \)-length of \( \gamma \) relative \( \mathcal{K} \) is

\[
(4.8) \quad l_\varrho(\gamma) := \int_{\tilde{\gamma} \cap X \setminus K} (q \circ q) \, ds + \sum_{k_i \in \gamma} \rho(k_i).
\]

Finally, if \( \Gamma \) is a family of curves in \( X_\mathcal{K} \) we say that a Borel function \( q : X \to [0, \infty] \) is admissible for \( \Gamma \) (and write \( q \wedge \Gamma \)), if \( l_\varrho(\gamma) \geq 1 \) and we may define the transboundary \( p \)-modulus of \( \Gamma \) as

\[
(4.9) \quad \text{Mod}_{p, X_\mathcal{K}}(\Gamma) = \inf_{q \wedge \Gamma} \int q^p d\mu_\mathcal{K}.
\]

Note that if \( \Gamma \) is a family of curves in \( X \) and \( \tilde{\Gamma} := q(\Gamma) \) is the image of \( \Gamma \) in \( X_\mathcal{K} \), then \( \text{Mod}_{p, X_\mathcal{K}}(\Gamma) = \text{Mod}_{p, X_\mathcal{K}}(\tilde{\Gamma}) \).

Some of the properties of transboundary modulus can be proved exactly the same way as for the regular modulus of curve families. However the property of overflowing can be somewhat strengthened. Indeed, we say that \( \Gamma_1 \) minorizes \( \Gamma_2 \) relative \( \mathcal{K} \), and write \( \Gamma_1 \prec_\mathcal{K} \Gamma_2 \), if for every \( \gamma \in \Gamma_2 \) there is a curve \( \delta \in \Gamma_1 \) such that for the images of the curves \( \delta \) and \( \gamma \) under the quotient map \( q : X \to X_\mathcal{K} \) we have \( q(\delta) \subset q(\gamma) \subset X_\mathcal{K} \).

**Proposition 4.3.** Let \( (X, d, \mu) \) be a metric measure space, and \( \mathcal{K} = \{ K_i \}_{i=1}^{n} \) be a finite collection of pairwise disjoint compact connected subsets of \( X \). Then for every \( p \geq 1 \) the following properties are satisfied:

1. **(Monotonicity in \( \Gamma \))** \( \text{Mod}_{p, X_\mathcal{K}}(\Gamma) \leq \text{Mod}_{p, X_\mathcal{K}}(\Gamma') \), if \( \Gamma \subset \Gamma' \).
2. **(Subadditivity)** \( \text{Mod}_{p, X_\mathcal{K}}(\Gamma) \leq \sum_{j=1}^{\infty} \text{Mod}_{p, X_\mathcal{K}}(\Gamma_j) \), if \( \Gamma = \bigcup_{j=1}^{\infty} \Gamma_j \).
3. **(Overflowing)** \( \text{Mod}_{p, X_\mathcal{K}}(\Gamma_1) \geq \text{Mod}_{p, X_\mathcal{K}}(\Gamma_2) \), if \( \Gamma_1 \prec_\mathcal{K} \Gamma_2 \).

**Proof.** To prove the properties of overflowing (and therefore of monotonicity) note that if \( \Gamma_1 \prec_\mathcal{K} \Gamma_2 \), then any mass distribution \( (\rho, \{ \rho_i \}) \) admissible for \( \Gamma_1 \) is also admissible for \( \Gamma_2 \). So \( \text{Mod}_{p, X_\mathcal{K}}(\Gamma_1) \geq \text{Mod}_{p, X_\mathcal{K}}(\Gamma_2) \).
To prove subadditivity assume without loss of generality that $\sum_j \text{Mod}_{p,X,\mathcal{K}}(\Gamma_j) < \infty$. Fix $\varepsilon > 0$. Then for every $j \geq 1$ there is a mass distribution $(\rho_j, \{\rho_{i,j}\}_{i=1}^n) \land (\Gamma_j, \mathcal{K})$ so that

$$A_p(\rho_j, \{\rho_{i,j}\}) < \text{Mod}_{p,X,\mathcal{K}}(\Gamma_j) + \frac{\varepsilon}{2j}.$$ 

Let $\tilde{\rho} = (\sum_j \rho_j)\sharp$ and $\tilde{\rho}_i = (\sum_j \rho_{i,j})\sharp$ for $1 \leq i \leq n$. Then $(\tilde{\rho}, \{\tilde{\rho}_i\})$ is admissible for $\Gamma$ since $\rho \geq \rho_j$, and $\tilde{\rho}_i \geq \rho_{i,j}$ for every $i \in \{1, \ldots, n\}$ and every $j \geq 1$. Therefore,

$$(4.10) \quad \text{Mod}_{p,X,\mathcal{K}}(\Gamma) \leq A_p(\tilde{\rho}, \{\tilde{\rho}_i\}) < \sum_{j=1}^{\infty} \text{Mod}_{p,X,\mathcal{K}}(\Gamma_j) + \varepsilon.$$ 

Letting $\varepsilon \to 0$ finishes the proof. \hfill \Box

We will also need the following property.

**Proposition 4.4.** Let $(X,d,\mu)$ be a metric measure space. Suppose $\mathcal{K} = \{K_i\}_{i=1}^n$ and $\mathcal{K}' = \{K'_i\}_{i=1}^m$ are finite collections of pairwise disjoint compact connected subsets of $X$. Then for every $p \geq 1$ the following property holds:

(4) (Monotonicity in $\mathcal{K}$) $\text{Mod}_{p,X,\mathcal{K}}(\Gamma) \leq \text{Mod}_{p,X,\mathcal{K}'}(\Gamma)$, if $\mathcal{K}' \subset \mathcal{K}$.

**Proof.** This follows immediately from the fact that if $\varrho$ is admissible for $\Gamma$ relative $\mathcal{K}'$ then it is also admissible for $\Gamma$ relative $\mathcal{K}$. \hfill \Box

If $X$ is an Ahlfors $Q$-regular metric measure space, $Q \geq 1$, we will denote the transboundary $Q$-modulus of a family $\Gamma$ in $X$ (or in $X_{\mathcal{K}}$) by $\text{Mod}_{X,\mathcal{K}}(\Gamma)$.

One of the most important properties of transboundary modulus is that it is a conformal invariant, cf. [Bon11] [Sch95]. Next we show that transboundary modulus is a quasiconformal quasi-invariant. This fact is crucial in the proof of Theorem 1.1. The following result can be proved in a greater generality, but we only need it in the case of quasiconformal mapping between finitely connected domains in $\mathbb{R}^n$.

**Theorem 4.5** (Quasiconformal Quasi-invariance of transboundary modulus). Let $\Omega$ and $\Omega'$ be domains in $\mathbb{R}^n$ and $\mathcal{K} = \{K_i\}_{i=1}^n$ and $\mathcal{K}' = \{K'_i\}_{i=1}^m$ be finite collections of pairwise disjoint compact connected subsets of $\Omega$ and $\Omega'$, respectively. If $f : \Omega_{\mathcal{K}} \to \Omega'_{\mathcal{K}'}$, is a homeomorphism, such that $f : \Omega \setminus \cup_{i=1}^m K_i \to \Omega' \setminus \cup_{i=1}^m K'_i$ is an $H$-quasiconformal mapping then there exists a constant $C \geq 1$ depending only on $H$ and $n$ such that for every curve family $\Gamma$ in the quotient space $X_{\mathcal{K}}$ we have

$$(4.11) \quad \frac{1}{C} \text{Mod}_{\Omega,\mathcal{K}}(\Gamma) \leq \text{Mod}_{\Omega',\mathcal{K}'}(f(\Gamma)) \leq C \text{Mod}_{\Omega,\mathcal{K}}(\Gamma).$$

**Proof.** Since the inverse of an $H$-quasiconformal map between domains in $\mathbb{R}^n$ is $H$-quasiconformal it is enough to show only the first inequality in (4.11). We first show that it is enough to prove the inequality

$$(4.12) \quad \text{Mod}_{\Omega,\mathcal{K}}(\Gamma) \leq C \text{Mod}_{\Omega',\mathcal{K}'}(f(\Gamma)),$$

assuming that for every $\gamma \in \Gamma$ the mapping $f$ is absolutely continuous on every closed sub curve of $\gamma \cap \Omega \setminus K$. For this, let $\Gamma$ be an arbitrary curve family in $\Omega_{\mathcal{K}}$ and let

$\Gamma_1 = \{\gamma \in \Gamma : f$ is absolutely continuous on every closed subcurve of $\gamma \cap \Omega \setminus K\},$
where $K = \bigcup_{i=1}^{m} K_i$. For every $\gamma \in \Gamma \setminus \Gamma_1$ there exists a closed subcurve $\gamma_f \subset \Omega \setminus K$ so that $f$ is not absolutely continuous on it. Let $\Gamma_0 = \{ \gamma_f \subset \Omega \setminus K : \gamma \in \Gamma \}$, then $\Gamma_0 <_{K} \Gamma \setminus \Gamma_1$. Since $f$ is a quasiconformal map between domains in $\mathbb{R}^n$, the partial derivatives of $f$ are locally $L^p$-integrable, or $f \in ACL^n$, cf. [Vai84, page 111]. Therefore, Fuglede’s theorem, cf. [Vai84, page 95], implies that $\text{mod}_n(\Gamma_0) = 0$.

Note that

$$\text{Mod}_{\Omega, \mathcal{K}}(\Gamma \setminus \Gamma_1) \leq \text{Mod}_{\Omega, \mathcal{K}}(\Gamma_0) = \text{mod}_n(\Gamma_0)$$

by Proposition 4.3. Therefore $\text{Mod}_{\Omega, \mathcal{K}}(\Gamma \setminus \Gamma_1) = 0$ and in particular by subadditivity of transboundary modulus we have

$$\text{Mod}_{\Omega, \mathcal{K}}(\Gamma) = \text{Mod}_{\Omega, \mathcal{K}}(\Gamma_1).$$

In particular, since $\text{Mod}_{\Omega', \mathcal{K}'}(f(\Gamma_1)) \leq \text{Mod}_{\Omega', \mathcal{K}'}(f(\Gamma))$, in order to obtain (4.12) it is enough to show the following inequality

$$\text{Mod}_{\Omega, \mathcal{K}}(\Gamma_1) \leq C\text{Mod}_{\Omega', \mathcal{K}'}(f(\Gamma_1)).$$

Thus, from now on we assume that $f$ is absolutely continuous on every closed subcurve of $\gamma \cap \Omega \setminus K$, whenever $\gamma \in \Gamma$.

Suppose $(\rho'; \{ \rho_i' \})$ is a mass distribution on $(\Omega', \mathcal{K}')$ admissible for $f(\Gamma)$. Define a mass distribution $(\rho; \{ \rho_i \})$ on $(\Omega, \mathcal{K})$ as follows,

\[
\rho(x) = \rho'(f(x)) \cdot L_f(x), \quad \text{for } x \in \Omega \setminus K
\]
\[
\rho_i = \rho_i', 1 \leq i \leq m.
\]

Since $f$ is absolutely continuous on every subcurve of $\gamma \in \Gamma$ we have

\[
L_{(\rho; \{ \rho_i \})}(\gamma) \geq L_{(\rho'; \{ \rho_i' \})}(f \circ \gamma) \geq 1,
\]

by Theorem 5.3 in [Vai84]. Thus, $(\rho; \{ \rho_i \})$ is an admissible mass distribution for $\Gamma$ and we have

\[
\text{Mod}_{\Omega, \mathcal{K}}(\Gamma) \leq \int_{\Omega \setminus K} \rho^n d\mu + \sum_{i=1}^{n} \rho_i^n
\]
\[
= \int_{\Omega \setminus K} \rho'(f(x))^n L_f(x) dx + \sum_{i=1}^{m} (\rho_i')^n
\]
\[
\leq H^{n-1} \left( \int_{\Omega_0} \rho'(f(x))^n |J_f(x)| dx + \sum_{i=1}^{m} (\rho_i')^n \right)
\]
\[
\leq H^{n-1} \left( \int_{\Omega' \setminus K'} \rho'(y)^n dy + \sum_{i=1}^{m} (\rho_i')^n \right).
\]

The second to last inequality above holds because a quasiconformal map between domains in $\mathbb{R}^n$ is differentiable almost everywhere and at a point $x \in \Omega \setminus K$ of differentiability of $f$ we have

\[
|Df(x)|^n = L_f(x)^n \leq H^{n-1}|J_f(x)|.
\]

Taking infimum over all mass distributions $(\rho'; \{ \rho_i' \})$ admissible for $f(\Gamma)$ we obtain (4.12), which completes the proof. \qed
4.4. $C^*$-cylinder and $C^*$-squares. In general it is hard to estimate transboundary modulus in terms of regular modulus or to calculate these quantities explicitly. In this section, following Bonk [Bon11], we define a class of subsets of round annuli in $\mathbb{C}$, such that the transboundary modulus relative to these subsets is equal to the classical modulus and can be easily calculated explicitly.

If we equip $\mathbb{C} = \mathbb{C} \setminus \{0\}$ with the flat metric $d_{C^*}$ which is induced by the length element

$$ds_{C^*} = \frac{|dz|}{|z|},$$

then $\mathbb{C}^*$ may be identified with the infinite cylinder $Z = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the Riemannian metric induced from $\mathbb{R}^3$. Let $A_{C^*}$ be the corresponding measure on $\mathbb{C}^*$ induced by the volume element $dA_{C^*}(z) = \frac{dm}{|z|^2}$, where $m$ is the Lebesgue measure in the plane.

A finite $C^*$-cylinder is a round annulus

$$A = A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$$

in $(\mathbb{C}^*, d_{C^*})$ where $0 < r < R < \infty$. We will denote by $\partial_i A$ and $\partial_o A$ the inner and outer boundary components of $A$, respectively. The height $h_A$ of $A(r, R)$ is $\log(R/r)$.

A $C^*$-square is a set $Q$ in $(\mathbb{C}^*, d_{C^*})$ of the form

$$(4.13) \quad Q = \{r e^{it} : \alpha \leq t \leq \beta \text{ and } r \leq \rho \leq R\}$$

where $\alpha \leq \beta, \beta - \alpha < 2\pi, 0 < r < R$, and $\beta - \alpha = \log(R/r)$. Note that $Q$ is the image of the square $[\alpha, \beta] \times [\alpha, \beta]$ under the exponential map. The side length $l(Q)$ of $Q$ is defined as

$$l(Q) = \beta - \alpha = \log(R/r)$$

Note that $0 < l(Q) < 2\pi$.

The following result of Bonk [Bon11] states that transboundary modulus of the family of curves connecting the inner and outer boundary components of a finite $C^*$-cylinder relative a family of $C^*$-squares is the same as the classical modulus of the same family.

**Proposition 4.6** (Bonk). Let $A = A(r, R)$ be a finite $C^*$-cylinder, and $Q = \{Q_i\}_{i=1}^n$ be a finite family of pairwise disjoint $C^*$-squares in $A$. If $\Gamma$ is the family of curves connecting the two boundary components of $A$, then

$$(4.14) \quad \text{Mod}_{A, Q}(\Gamma) = \frac{2\pi}{\log(R/r)}.$$

In this paper we will need a related estimate for the modulus of a family of radial curves connecting subsets of the boundary components of a $C^*$-cylinder. We will use the following notation. If $E \subset \mathbb{C}$ and $t > 0$ we let $tE = \{t \cdot z : z \in E\}$.

**Proposition 4.7.** Let $A = A(r, R)$ be a finite $C^*$-cylinder, and $Q = \{Q_i\}_{i=1}^n$ be a finite family of pairwise disjoint $C^*$-squares in $A$. Given a measurable subset $E \subset \partial D = \{|z| = 1\}$, let $\Gamma_E = \Gamma(rE, RE; A)$ be the curve family in $A$ connecting $rE \subset \{|z| = r\}$ to $RE \subset \{|z| = R\}$. Then

$$(4.15) \quad \text{Mod}_{A, Q}(\Gamma_E) \geq \frac{\mathcal{H}^1(E)}{\log(R/r)}.$$
Proof. Given a $z = e^{i\varphi} \in \partial \mathbb{D}$, let $\gamma_{\varphi}$ be the corresponding “radial curve in $A$”, i.e., $\gamma_{\varphi}(t) = te^{i\varphi}$ for $r < t < R$. Also, let $\Gamma'_{E} = \{ \gamma_{\varphi} : e^{i\varphi} \in E \}$ be the collection of the radial curves in $A$ corresponding to the set $E$. Since $\Gamma'_{E} \subset \Gamma_{E}$ we have $\operatorname{Mod}_{A,Q}\Gamma_{E} \geq \operatorname{Mod}_{A,Q}\Gamma'_{E}$ and we only need to show that

$$\operatorname{Mod}_{A,Q}\Gamma'_{E} \geq \frac{\mathcal{H}^{1}(E)}{\log(R/r)}.$$ 

For this let $A_{E} = A \cap \bigcup_{e^{i\varphi} \in E, \gamma_{\varphi}} \gamma_{\varphi}$ and let $I_{E} = \{ 1 \leq i \leq n : Q_{i} \cap A_{E} \neq \emptyset \}$. Note that without loss of generality we may assume that a mass distribution which is admissible for $\Gamma'_{E}$ consists of a non-negative Borel function $\rho : A \to [0, \infty)$ and weights $\rho_{i} \geq 0$ such that

$$\rho(x) = 0 \text{ for } x \in A \setminus (A_{E} \cup Q),$$

$$\rho_{i} = 0 \text{ for } i \notin I_{E}$$

and for every $\gamma_{\varphi} \in \Gamma'_{E}$ we have

$$\int_{\gamma_{\varphi} \setminus Q} \rho ds_{C} + \sum_{\gamma_{\varphi} \cap Q_{i} \neq \emptyset} \rho_{i} \geq 1,$$

where $Q = \bigcup_{i=1}^{n} Q_{i}$. By Fubini’s theorem, integrating over $e^{i\varphi} \in E$ we have

$$\mathcal{H}^{1}(E) \leq \int_{A_{E} \setminus Q} \rho dA_{C} + \sum_{i \in I_{E}} \mathcal{H}^{1}(E_{i})\rho_{i},$$

where $E_{i} = \{ \varphi \in E : Q_{i} \cap \gamma_{\varphi} \neq \emptyset \}$. By Cauchy-Schwartz inequality we then have

$$\mathcal{H}^{1}(E) \leq \left( A_{C^{*}}(A_{E} \setminus Q) + \sum_{i \in I_{E}} \mathcal{H}^{1}(E_{i}) \right)^{1/2} \left( \int_{A_{E} \setminus Q} \rho^{2} dA_{C^{*}} + \sum_{i \in I_{E}} \rho_{i}^{2} \right)^{1/2} \mathcal{H}^{1}(E_{i}) \leq \mathcal{H}^{1}(E)\mathcal{h}_{A}.$$ 

Since $\mathcal{H}^{1}(E_{i}) \leq l(Q_{i})$ for every $i \in I_{E}$, we have that

$$A_{C^{*}}(A_{E} \setminus Q) + \sum_{i \in I_{E}} \mathcal{H}^{1}(E_{i})^2 \leq A_{C^{*}}(A_{E} \setminus Q) + \sum_{i \in I_{E}} l(Q_{i})\mathcal{H}^{1}(E_{i}) \leq A_{C^{*}}(A_{E} \setminus Q) + \sum_{i \in I_{E}} A_{C^{*}}(A_{E} \cap Q_{i}) \leq A_{C^{*}}(A_{E} \setminus Q) + A_{C^{*}}(A_{E} \cap Q) = A_{C^{*}}(A_{E}) \leq \mathcal{H}^{1}(E)\mathcal{h}_{A},$$

Therefore

$$\frac{\mathcal{H}^{1}(E)}{\mathcal{h}_{A}} \leq \int_{A_{E} \setminus Q} \rho^{2} dA_{C^{*}} + \sum_{i \in I_{E}} \rho_{i}^{2},$$

thus completing the proof. \qed

Based on the above definitions, we have the following uniformization theorem of Bonk, cf. [Bon11, Proposition 11.6].
Theorem 4.8. Let $n \geq 1$, and suppose that $D_0, \ldots, D_{n-1}$ are pairwise disjoint Jordan domains, all contained in another Jordan domain $D_n \subset \mathbb{C}$. Then there exist a $C^*$-cylinder $A$, pairwise disjoint $C^*$-squares $Q_1, \ldots, Q_{n-1} \subset A$ and a homeomorphism $f : \overline{\Omega} \to \overline{U}$ where

$$\Omega = D_n \setminus (D_0 \cup \ldots \cup D_{n-1}) \text{ and } U = A \setminus (Q_1 \cup \ldots \cup Q_{n-1}),$$

that is conformal on $\Omega$ and maps $\partial D_0$ to $\partial_i A$ and $\partial D_n$ to $\partial_o A$.

Furthermore, if the curves $\partial D_0, \ldots, \partial D_n$ are $s$-relatively separated $k$-quasicircles, $\text{diam}(\Omega) \leq c$, and

$$\min\{\text{diam}(\partial D_0), \text{diam}(\partial D_n)\} \geq d > 0,$$

then $f$ is an $\eta$-quasisymmetric map from $\Omega$ equipped with Euclidean metric to $U$ equipped with flat metric on $C^*$, where $\eta$ only depends on $s, k, c$, and $d$.

Bonk’s original theorem deals with Jordan domains in $\hat{\mathbb{C}}$ equipped with the chordal metric, however the following proposition combined with [Bon11, Proposition 11.6] immediately implies the result stated above.

Proposition 4.9. Let $| \cdot |$ be the Euclidean metric and $\sigma$ be the chordal metric on the plane. Then the mapping $id : (\mathbb{C}, | \cdot |) \to (\mathbb{C}, \sigma)$ is conformal. Furthermore, $id$ is $\eta$-quasisymmetric when restricted on a bounded set $X \subset \mathbb{C}$ and $\eta$ only depends on $\text{diam}(X)$.

We omit the proof since it is quite easy and well-known.

5. A necessary condition for a QS embedding

In this section we provide a necessary condition for existence of a quasisymmetric embedding of the slit carpet $\mathcal{S}_r$ into the plane. This condition is an estimate on the transboundary modulus relative to the collection of slits $\mathcal{K}_n$. Below we use the notations introduced in Section 3. In particular, $U = [0,1]^2$ and $s_0 = s(Q_0)$ is the “central slit” of generation 0. Moreover, we denote

$$\mathcal{U}_0 = \text{int}(U) \setminus s_0,$$

$$\mathcal{K}_n = \mathcal{K}_n \setminus \{s_0\}$$

$$\delta_0 = \{(1,y) : 0 \leq y \leq 1\} \subset \partial \mathcal{U}_0,$$

$$\Gamma = \Gamma(s_0, \delta_0; \mathcal{U}_0).$$

(5.1)

Lemma 5.1. Suppose there is an $\eta$-quasisymmetric embedding $f : \mathcal{S}_r \hookrightarrow \mathbb{R}^2$ of the slit carpet $\mathcal{S} = \mathcal{S}_r$ into the plane. Let $\Gamma = \Gamma(s_0, \delta_0; \mathcal{U}_0)$ be the family of curves in the unit square connecting the central slit $s_0$ to the right vertical edge of $\mathcal{U}$. Then there is a constant $c > 0$ which depends on $\eta$ but not on $n \geq 0$, such that, using the notation in (5.1), for every $n > 0$ we have

$$\text{Mod}_{\mathcal{U}_0, \mathcal{K}_0} \Gamma \geq c.$$

(5.2)

We will first show that a quasisymmetric embedding $f : \mathcal{S} \hookrightarrow \mathbb{R}^2$ descends to mappings $f_n : \mathcal{S}_n \hookrightarrow \mathbb{R}^2$ which have certain nice properties listed in Lemma 5.3 below.

For $n \geq 1$, we will denote by $\Pi_n$ and $\tilde{\Pi}_n$ the preimages of the dyadic grid of generation $n$ in $U = [0,1]^2$ under the projections $\pi_0$ and $\pi$ in $\mathcal{S}_n$ and $\mathcal{S}$,
respectively. In other words we have

$$\Pi_n = \pi_{0,n}^{-1}\left( \bigcup_{\Delta \in D_n} \partial \Delta \right) \subset \mathcal{S}_n, \quad \tilde{\Pi}_n = \pi^{-1}\left( \bigcup_{\Delta \in D_n} \partial \Delta \right) \subset \mathcal{S}.$$ 

From the definitions it follows that $\pi_n|\tilde{\Pi}_n$ is a homeomorphism. In fact more is true.

**Lemma 5.2.** For every $n \geq 0$, the mapping $\pi_n|\tilde{\Pi}_n$, i.e., the restriction of the projection maps $\pi_n : \mathcal{S} \to \mathcal{S}_n$ to $\tilde{\Pi}_n$ is bi-Lipschitz. More precisely, if $p, q \in \tilde{\Pi}_n$ then

$$d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)) \leq d_{\mathcal{S}}(p, q) \leq 3d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)). \tag{5.3}$$

**Proof.** The left inequality in (5.3) follows from the fact that the sequence $d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q))$ is non decreasing in $n$.

To obtain the second inequality in (5.3) we will use the following notation. Suppose $n \geq 0$ and pick a dyadic square $\Delta \in D_n$. Let $T = T_\Delta$ be the corresponding “dyadic square” in $\mathcal{S}$, i.e.,

$$T_\Delta = \pi^{-1}(\text{int}(\Delta)),$$

where the closure is in $d_{\mathcal{S}}$ metric.

First, assume that $p, q \in \partial T_\Delta$ for some $\Delta \in D_n$. If $\pi(p)$ and $\pi(q)$ belong to the same edge of the square $\partial \Delta$ then

$$d_{\mathcal{S}}(p, q) = d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)) = |\pi(p) - \pi(q)|.$$

On the other hand, if $\pi(p)$ and $\pi(q)$ belong to different edges of the Euclidean square $\partial \Delta \subset \mathbb{R}^2$ then there are at most two corner points $z_1, z_2$ of $\partial \Delta$ between $\pi(p)$ and $\pi(q)$ on $\partial \Delta$ such that

$$|\pi(p) - \pi(z_1)| + |\pi(z_1) - \pi(z_2)| + |\pi(z_2) - \pi(q)| \leq 3|\pi(p) - \pi(q)|.

Therefore,

$$d_{\mathcal{S}}(p, q) \leq d_{\mathcal{S}}(p, z_1) + d_{\mathcal{S}}(z_1, z_2) + d_{\mathcal{S}}(z_2, q)
= |\pi(p) - \pi(z_1)| + |\pi(z_1) - \pi(z_2)| + |\pi(z_2) - \pi(q)| \leq 3d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)) \leq 3d_{\mathcal{S}}(\pi_n(p), \pi_n(q)).$$

More generally, suppose $p, q \in \tilde{\Pi}_n$. Consider a curve $\gamma$ connecting $\gamma(0) = \pi_n(p)$ and $\gamma(1) = \pi_n(q)$ in $\mathcal{S}_n$ of minimal length. It is easy to see that such a curve exists, and it is, in fact, a preimage of a piecewise linear curve in $U$ under $\pi_{0,n}$. Observe that there are points $\zeta_j, j = 0, \ldots, k + 1$, on $\gamma$ such that: (i) $\zeta_0 = \pi_n(p)$, $\zeta_{k+1} = \pi_n(q)$, (ii) for every $j$ the two consecutive points $\zeta_j$ and $\zeta_{j+1}$ belong to the boundary of the same dyadic square $\pi_n(T_\Delta) \subset \mathcal{S}_n$ for some $\Delta \in D_{n+1}$, and (iii) the following equality holds $d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)) = \sum_{j=0}^{k} d_{\mathcal{S}_n}(\zeta_j, \zeta_{j+1})$. Indeed, this can be achieved by letting $\zeta_1$ be the “last point of exit” of $\gamma$ from the (closed) square $T_\Delta$ containing $\zeta_0 = \gamma(0)$, and continuing by induction.

Finally, letting $p_j = \pi_{n}^{-1}(\zeta_j), j = 0, \ldots, k + 1$, and using the estimate above, we obtain

$$d_{\mathcal{S}}(p, q) \leq \sum_{j=0}^{k} d_{\mathcal{S}}(p_j, p_{j+1}) \leq \sum_{j=0}^{k} 3d_{\mathcal{S}_n}(\zeta_j, \zeta_{j+1}) = 3d_{\mathcal{S}_n}(\pi_n(p), \pi_n(q)).$$
which completes the proof. □

The following result shows that any quasisymmetric map $f: \mathcal{I} \to \mathbb{R}^2$ gives rise to a sequence of uniformly quasiconformal embeddings of the “precarpets” $\mathcal{I}_n$ into the plane which are quasisymmetric on the sets $\Pi_n \subset \mathcal{I}_n$.

**Lemma 5.3.** Suppose there is a $\eta$-quasisymmetric mapping $f: \mathcal{I} \to \mathbb{R}^2$. Then there are constants $H = H(\eta)$, $C = C(\eta)$ and embeddings $f_n: \mathcal{I}_n \to \mathbb{R}^2$ such that the following conditions hold.

(a). For every $n \geq 1$ the mapping $f_n$ is $H$-quasiconformal.
(b). $f_n|_{\Pi_n}$ is an $\eta'$-quasisymmetric mapping for every $n$, where $\eta'$ depends only on $\eta$.

**Remark 5.4.** It is possible to show that the mappings $f_n$ constructed below are in fact uniformly quasisymmetric on $\mathcal{I}_n$, however the details are not illuminating and we do not use this fact in the proof of Lemma 5.1.

To prove Lemma 5.3 we will need an extension result of Bonk, cf. Proposition 5.3 in [Bon11], which is a generalization of the classical Beurling-Ahlfors extension.

**Theorem 5.5.** Let $D, D' \subset \mathbb{C}$ be Jordan domains and $f: \partial D \to \partial D'$ be an $\eta$-quasisymmetric mapping. Suppose that $\partial D$ is a $k$-quasicircle. If

$$\min\{\text{diam}(D), \text{diam}(D')\} \leq \delta$$

for some $\delta > 0$, then $f$ can be extended to an $\eta'$-quasisymmetric mapping $F: D \to D'$ where $\eta'$ only depends on $\delta, k$ and $\eta$.

The original theorem in [Bon11] deals with Jordan regions in $\mathbb{C}$, however Theorem 5.5 is easily obtained by using Proposition 4.9.

**Proof of Lemma 5.3.** To define the embeddings $f_n: \mathcal{I}_n \to \mathbb{R}^2$ we will first define them locally on lifts of (closed) dyadic squares $\Delta \subset [0,1]^2$ using Bonk’s extension result above. The definition will be such that it will be consistent along the common parts of boundaries of such lifts in $\mathcal{I}_n$.

For $n \geq 0$ and a dyadic square $\Delta \in \mathcal{D}_n$ let $T = T_\Delta$ be the “dyadic square” in $\mathcal{I}$ as in (5.4).

Observe that if $\Delta \in \mathcal{D}_n$ then $\Delta$ does not contain a slit of $S_n$ in its interior and hence the path metric on $\mathcal{I}_n$ restricted to $\pi_n(T) \subset \mathcal{I}_n$ coincides with the Euclidean metric on $\Delta = \pi_{0,n}(\pi_n(T))$. Therefore $\pi_n(T)$ is isometric to a closed Jordan domain in $\mathbb{C}$ with the boundary which is a $\sqrt{2}$-quasicircle (since it is a square). On this boundary curve we define the following mapping

$$f_n^{\partial T} := f|_{\partial T} \circ (\pi_n|_{\partial T})^{-1}: \partial \pi_n(T) \to \mathbb{R}^2.$$

Since $f$ is $\eta$-quasisymmetric and by Lemma 5.2 $\pi_n|_{\partial T}$ is bi-Lipschitz, it follows that $f_n^{\partial T}$ is an $\eta_1$-quasisymmetric map, where $\eta_1(t)$ only depends on $\eta$, but not on the particular choice of the dyadic square (in fact $\eta_1(t) = \eta(3t)$, but this is not important for us). It follows that all the conditions of Theorem 5.5 are satisfied and applying it to $f_n^{\partial T}$ and $\pi_n(T)$ we obtain that for every $\Delta \in \mathcal{D}_n$ there is a quasisymmetric map $f_n^{\Delta} = f_n^{\Delta_2}: \pi_n(T) \to \mathbb{R}^2$ which extends $f_n^{\partial T}$. Moreover, $f_n^{\Delta}$ is $\eta_2$-quasisymmetric, where $\eta_2$ depends only on $\eta_1$, the quasiconformal constant of the boundary curves (i.e., $\sqrt{2}$ in this case), and diameters of these circles, which
are bounded by \( \text{diam}(\mathcal{S}) \). Thus, \( \eta_2 \) is independent of \( n \) as well as of the particular dyadic square \( \Delta \subset D_n \) (or \( T = T_\Delta \)).

Combining the functions \( f_n^T \) produces a homeomorphism \( f_n : \mathcal{S} \to \mathbb{R}^2 \). More precisely, if \( \xi \in \mathcal{S}_{n+1} \) is such that \( \pi_{0,n}(\xi) \in \Delta \in D_{n+1} \) we let

\[
(5.5) \quad f_n(\xi) = f_n^{T_\Delta}(\xi).
\]

Note that \( f_n \) is well defined since the squares \( \{\pi_n(T_\Delta)\}_{\Delta \in D_{n+1}} \) cover \( \mathcal{S} \) and the maps \( f_n^{T_\Delta} \) coincide at points which are common to different dyadic squares of generation \( n+1 \) in \( \mathcal{S} \).

For part (a) note that \( f_n \) is a homeomorphism, which is \( \eta_2(1) \)-quasiconformal at every point \( \xi \in \mathcal{S} \) such that \( \pi_{0,n}(\xi) \in \text{int}(\Delta) \) for some \( D \in D_{n+1} \). Next, suppose \( \xi \in \Pi_{n+1} \) and \( 0 < r < 2^{-(n+1)} \). Denote by \( \zeta_M \) and \( \zeta_m \) the points at which the quantity \( |f(\zeta) - f(\xi)| \) on the circle \( \partial B(\xi,r) \subset \mathcal{S} \) is maximized and minimized, respectively. Then, by continuity we have

\[
\frac{L_{f_n}(\xi,r)}{L_{f_n}(\xi,r)} = \frac{\sup\{|f(\zeta) - f(\xi)| : d_{\mathcal{S}}(\zeta,\xi) \leq r\}}{\inf\{|f(\zeta) - f(\xi)| : d_{\mathcal{S}}(\zeta,\xi) \geq r\}} = \frac{|f(\zeta_M) - f(\xi)|}{|f(\zeta_m) - f(\xi)|},
\]

where \( \zeta_M, \zeta_m \in \Pi_{n+1} \cap \partial B(\xi,r) \) belong to the boundaries of same dyadic squares in \( \mathcal{S} \) as \( \zeta_M \) and \( \zeta_m \). Therefore we have \( H_{f_n}(\xi) \leq \eta_2(1)^2 \eta_1(1) \), and \( f_n \) is \( H \)-quasiconformal with \( H = \eta_2(1)^2 \eta_1(1) \) independent of \( n \).

To prove (b) note that \( f_n|_{\Pi_{n+1}} = f \circ (\pi_{0,n}^{-1}|_{\Pi_{n+1}}) \). Since \( f \) is quasisymmetric and by Lemma 5.2 \( \pi_{n}^{-1} \) is \( 3 \)-bi-Lipschitz it follows that \( f_n|_{\Pi_{n+1}} \) is \( \eta' \)-quasisymmetric with \( \eta' \)-depending only on \( \eta \).

**Proof of Lemma 5.1.** Assume there exists a quasisymmetric embedding \( f : \mathcal{S} \to \mathbb{R}^2 \). By Lemma 5.3 there exists an \( H \)-quasiconformal map \( f_n : \mathcal{S} \to \mathbb{R}^2 \) such that \( f_n(\mathcal{S}_n) = \Omega_n = D_{n,m_n} \setminus (D_{n,0} \cup \ldots \cup D_{n,m_n-1}) \subset \mathbb{R}^2 \), where \( m_n = 1 + 4 + \ldots + 4^{n-1} \), and \( D_{n,i} \), \( i = 0, \ldots, m_n - 1 \), are pairwise disjoint Jordan domains compactly contained in the Jordan domain \( D_{n,m_n} \). We would like to show that without loss of generality \( \Omega_n \) satisfies the conditions of Bonk’s cylinder uniformization theorem, possibly after post-composing by a suitable quasisymmetric mapping of the plane.

First, we observe that possibly by post-composing \( f_n \) with an appropriate Möbius transformation of the plane and denoting the resulting mapping by \( f_n \) again, we may assume that the image of the “outer square” of \( \mathcal{S}_n \) under \( f_n \) is the “outermost” boundary component \( \gamma_n^o \) of \( \Omega_n \), i.e., \( \gamma_n^o = f_n(\pi_{0,n}^{-1}(\partial \mathcal{U})) \) is the boundary of the unbounded component of \( \mathbb{R}^2 \setminus \Omega_n \). Indeed, we may first post-compose \( f_n \) with a scaling so that \( \text{diam}(f_n(\pi_{0,n}^{-1}(\partial \mathcal{U}))) = 1 \). Then, one can compose the result with a reflection in the boundary of a largest disk, of radius say \( \alpha_n \), inscribed in the domain bounded by \( f_n(\pi_{0,n}^{-1}(\partial \mathcal{U})) \). Since \( f_n(\pi_{0,n}^{-1}(\partial \mathcal{U})) \) is a \( k' \)-quasicircle, the radius \( \alpha_n \) of the disk is bounded from below by a constant depending only on \( k' \). Therefore, the resulting mapping will be uniformly quasisymmetric on \( f(\mathcal{S}_n) \), since

\[
\text{diam} f_n(\mathcal{S}_n) \leq \eta'(1) \cdot \text{diam}(f_n(\pi_{0,n}^{-1}(\partial \mathcal{U}))) = \eta'(1).
\]

Since \( f_n|_{\Pi_{n+1}} \) is \( \eta' \)-quasisymmetric, the boundary components of the domain \( f_n(\mathcal{S}_n) \) are uniformly \( s' \)-separated, uniform \( k' \)-quasicircles where \( s' \) and \( k' \) do not depend on \( n \) by Propositions 2.4 and 3.2. Moreover, the ratio of diameters of the “outer square” \( \pi_{0,n}^{-1}(\partial \mathcal{U}) \) and “central slit” \( \pi_{0,n}(s_0) \) is also independent of

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n (it is comparable to \(1/\text{diam}(s_0)\)). Therefore, letting \(\gamma'_n = f_n(\pi_{0,n}^{-1}(\partial U))\) and \(\gamma''_n = f_n(\pi_{0,n}^{-1}(s_0))\), it follows from (2.4) that the quantity \(\text{diam}(\gamma'_n)\) is also independent of \(n\). In particular, assuming that \(\text{diam}(f_n(\pi_{0,n}^{-1}(\partial U))) = 1\), as explained above, we obtain that \(\text{diam}(f_n(\pi_{0,n}^{-1}(s_0))) \geq d\) where \(d > 0\) is independent of \(n\).

Thus, all the conditions of Theorem 4.8 are satisfied and applying it to \(f_n(S_n)\) we obtain an \(\eta''\)-quasisymmetric map \(\psi_n\) of the plane and a \(\mathbb{C}^*\)-cylinder

\[A_n = \{r_n < |z| < R_n\},\]

such that \(\psi_n \circ f_n\) maps the central slit and the outer square of \(S_n\) to \(\partial A_n\) and \(\partial_0 A_n\), respectively and moreover

\[(\psi_n \circ f_n)(S_n) = A_n \setminus (Q_n^1 \cup \ldots \cup Q_n^m),\]

where \(Q_n^1, \ldots, Q_n^m\) are disjoint \(\mathbb{C}^*\)-squares.

Since the distortion functions \(\eta'\) and \(\eta''\) are independent of \(n\) it follows that the maps \(\psi_n \circ f_n\) are uniformly quasisymmetric with the same distortion function \(\vartheta = \eta'' \circ \eta'\), cf. Proposition 10.6 in [Hei01]. Letting \(Q_n = \{Q_n^1, \ldots, Q_n^m\}\), \(E_n = \psi_n(f_n(\delta_n))\), where \(\delta_n\) is the “right side” of the outer boundary component of \(S_n\), and \(\Gamma_n = \Gamma(\partial A_n, E_n, A_n)\) and applying Proposition 4.7 and inequality (2.4) we obtain

\[
\text{Mod}_{A_n, Q_n}(\Gamma_n) \geq \frac{\mathcal{H}^1(\left\{ \overline{B(z_n, t)} \mid z \in E_n \right\})}{h_{A_n}} \geq \frac{1}{\eta_{A_n}} \frac{\text{diam}(E_n)}{\text{diam}(\partial_0 A_n)} \geq \frac{1}{\pi h_{A_n}} \frac{1}{2\vartheta} \left( \frac{\text{diam}(\partial_0 A_n)}{\text{diam}(\delta_n)} \right) \geq \frac{1}{2\pi \log \frac{R_n}{r_n}} \frac{1}{\vartheta(2)},
\]

since \(\text{diam}(\delta_n) = 1\) and \(\text{diam}(\pi_{0,n}^{-1}(\partial U))) \leq 2\). Moreover, by inequality (2.4) again, we have

\[
\frac{R_n}{r_n} \leq \vartheta \left( \frac{2\text{diam}(\pi_{0,n}^{-1}(\partial U)))}{\text{diam}(s_0)} \right) \leq \vartheta \left( \frac{4}{r_0} \right).
\]

Combining the estimates above we obtain that for every \(n \geq 0\) we have

\[
(5.6) \quad \text{Mod}_{A_n, Q_n}(\Gamma_n) \geq \frac{1}{2\pi \vartheta(2) \log(\vartheta(4/r_0))} > 0.
\]

On the other hand note that the “identity map” \(\text{id}_n\) from the domain \(S_n\) equipped with the Euclidean metric to \((S_n, d_{S_n})\) is conformal, i.e., 1-quasiconformal. Therefore, by letting \(\phi_n := \psi_n \circ f_n \circ \text{id}_n\), we have that \(\phi_n : S_n \to A_n \setminus (Q_n^1 \cup \ldots \cup Q_n^m)\) is a \(K\)-quasiconformal map between domains in \(\mathbb{R}^2\), with \(K = \vartheta(1)\). Observe that the mapping \(\phi_n\) descends to a homeomorphism between the quotient spaces

\[
\tilde{\phi}_n : (U_0)_{K_n} \to (A_n)_{Q_n},
\]

and if \(\tilde{\Gamma}\) and \(\tilde{\Gamma}_n\) are the images of \(\Gamma\) and \(\Gamma_n\) under the quotient maps, then \(\tilde{\phi}_n(\tilde{\Gamma}) = \tilde{\Gamma}_n\). By quasiconformal quasi-invariance of transboundary modulus, cf. Theorem
4.5, there exists a constant $C$ independent of $n$ so that
\begin{equation}
\text{Mod}_{U_n,K_n}(\Gamma) = \text{Mod}_{\check{U}_n,\check{K}_n}(\check{\Gamma}) \\
\geq \frac{1}{C} \text{Mod}_{A_n,Q_n}(\check{\Gamma}_n) = \frac{1}{C} \text{Mod}_{A_n,Q_n}(\Gamma_n).
\end{equation}
Combining (5.6) and (5.7) we obtain (5.2) with $c = 1/(C^2 \pi \theta(2) \log(\theta(4r_0)))$. \qed

6. Transboundary Modulus and Quasisymmetric Non-Embeddings

In this section we prove the “only if” direction of Theorem 1.1. For this we estimate the transboundary modulus of curve families connecting the vertical sides of the unit square in dyadic slit domains. In particular we show that if the sequence of relative sizes $r_i$ of slits is square summable then the transboundary modulus approaches 0. Combining with the results of Section 5 we show that if $\mathbf{r} \notin \ell^2$ then there is no QS embedding of $\mathcal{K}$ into the plane, cf. Theorem 6.3.

6.1. Estimates for Transboundary Modulus. The following lemma is the main result of this section. Below we use the same notation as in Section 5.

Lemma 6.1. Let $\Gamma$ be the collection of all the curves in the unit square $[0,1]^2$ connecting the vertical edges of the square. Suppose $\mathbf{r} = \{r_i\}_{i=0}^\infty$ is a sequence of numbers in $(0,1)$ such that $\mathbf{r} \notin \ell^2$. Then for every $0 < \epsilon < 1$ we have
\begin{equation}
\text{Mod}_{U,K_n}(\Gamma) \leq \prod_{i=0}^n (1 - \frac{1}{8} \epsilon r_i^2) + 3\epsilon,
\end{equation}
for every $n \geq 0$. In particular, if $\{r_i\}_{i=0}^\infty \notin \ell^2$ then
\begin{equation}
\lim_{n \to \infty} \text{Mod}_{U,K_n}(\Gamma) = 0.
\end{equation}
Proof. The proof below is similar to proofs in [Hak18], where estimates for the classical modulus in slit domains were obtained. However, transboundary modulus is in general larger than the classical modulus and therefore the results in this section do not follow directly from [Hak18].

6.1.1. Constructing mass distribution $\check{\rho}_n$. We will first prove the estimate (6.1) assuming that the sequence $r_i$ is such that for every $i \geq 0$ we have $r_i = 2^{-j_i}$ for some $j_i \geq 1$, and $\epsilon = 2^{-m}$ for some $m \geq 1$. The estimate is obtained by defining a particular mass distribution for the pair $(U,K_n)$. In order to do that, new notations are introduced below.

Given a slit $s = s(\Delta) = \{x\} \times [a,b] \subset \mathbb{U}$ of length $l(s) = b-a$ and first coordinate $x$, and $0 < \epsilon \leq 1$ the $\epsilon$-collar of $s$ is the rectangle $s^\epsilon = (x,x+\epsilon l(s)) \times s$. Equivalently,
\begin{equation}
s^\epsilon = s + (0,\epsilon l(s)) = \{t+x : t \in s, x \in (0,\epsilon l(s))\}.
\end{equation}

Let $t(s^\epsilon), b(s^\epsilon), \ell(s^\epsilon), r(s^\epsilon)$ be the top, bottom, left, and right sides of $s^\epsilon$, respectively. Note that $\ell(s^\epsilon) = s$. As mentioned above we will assume that $\epsilon = 2^{-m}$ for some fixed $m \geq 1$.

Observe that under the assumptions on $r_i$ and $\epsilon$ we have that the $\epsilon$-collars of any two slits are either disjoint, or one is completely contained in the other. Indeed, if $s = s(\Delta) = \{x\} \times [a,b]$ with $\Delta \in \mathcal{D}_n$, $r_n = 2^{-j_n}$ and $\epsilon = 2^{-m}$ then $s^\epsilon$ is a rectangle that can be written as a union of $\epsilon^{-1} = 2^m$ dyadic squares of generation $N = n + j_n + m$. Therefore, if $\Delta'$ is a dyadic subsquare of $\Delta$ of generation $k \geq N$ then it is either disjoint from $s^\epsilon$ or is completely contained in it and the same is true
for \( s' = s(\Delta') \). On the other hand, if \( \Delta' \) is a dyadic square of generation \( k \leq N + 1 \) in \( \Delta \), and \( s' = s(\Delta') = \{x'\} \times [a', b'] \), then the distance between \( x \) and \( x' \) is at least the half of the sidelength of \( \Delta' \) and therefore

\[
|x - x'| \geq \frac{1}{2} 2^{-k} \geq 2^{-1-(N-1)} = 2^{-N}.
\]

Since the width of \( s \) is exactly \( 2^{-N} \) and \((s')'\) is located to the right of the slit \( s' \), it follows that the \( \epsilon \) collars of \( s \) and \( s' \) are disjoint if \( x' > x \). In the case \( x' < x \) there is nothing to prove since any dyadic square \( \Delta' \) contained in the left half of \( \subset \Delta \) does not intersect \( s' \).

From the above it follows that it is possible to select an infinite subsequence \( \mathcal{K}_{\epsilon} = \{ s_{i_n} \} \) in \( \mathcal{K} \) for which the \( \epsilon \)-collars are disjoint (i.e., the “smaller” collars which are contained in “larger” ones are not enumerated). Indeed, we may first enumerate \( \mathcal{K} = \{ s_i \}_{i=0}^{\infty} \) so that the lengths of the slits are non-increasing, i.e., \( l(s_i) \geq l(s_{i+1}) \) for every \( i \geq 0 \). Then, choose the sequence \( v_n := s_{i_n} \) by induction as follows. Let \( v_0 = s_0 \). Suppose for \( n \geq 1 \) the sequence \( v_0, \ldots, v_{n-1} \) has been defined, and let \( v_n = s_{i_n} \), where

\[
i_n = \min \left\{ j : s_j' \cap \left( \bigcup_{i}^{n-1} v_i' \right) = \emptyset \right\}.
\]

Since the set \([0, 1]^2 \setminus \left( \bigcup_{i}^{n-1} v_i' \right)\) always contains a dyadic square (it has a nonempty interior) the process never ends and the collars \( \{ v_i' \}_{i=0}^{\infty} \) are disjoint by our construction. Let

\[
\mathcal{K}_{\epsilon} = \{ v_i \}_{i=0}^{\infty}
\]
denote this subsequence. Moreover, for \( n \geq 0 \) let
\[
\mathcal{K}_{\epsilon,n} = \mathcal{K}_\epsilon \cap \mathcal{K}_n = \{ v_i \}_{i=0}^{N_\epsilon},
\]
where \( N_\epsilon = |\mathcal{K}_\epsilon \cap \mathcal{K}_n| \) is the cardinality of \( \mathcal{K}_{\epsilon,n} \).

**Figure 6.2.** A slit \( v_i \) in the unit square \( U \). The white, dark grey and light grey regions on \( U \) are the \( \epsilon \)-omitted, buffer, and residual subsets corresponding to \( v_i \).

For \( \epsilon \) as above, we denote by \( B_\epsilon^i \), the \( \epsilon \)-buffer of the slit \( v_i \), the union of the top and bottom squares in \( v_\epsilon^i \). More precisely,
\[
B_\epsilon^i = \{ x \in v_\epsilon^i : \text{dist}(x, t(v_\epsilon^i)) \leq \epsilon l(v_i) \text{ or dist}(x, b(v_\epsilon^i)) \leq \epsilon l(v_i) \}.
\]
The sets \( O_\epsilon^i = v_\epsilon^i \setminus B_\epsilon^i \) and \( R_\epsilon^i = U \setminus v_\epsilon^i = U \setminus (B_\epsilon^i \cup O_\epsilon^i) \) will be called the \( \epsilon \)-omitted and residual regions of \( v_i \), respectively.

We also define the \( \epsilon \)-buffer, omitted and residual sets in \( U \), denoting them by \( B_\epsilon^n, O_\epsilon^n, R_\epsilon^n \), respectively, as follows:

\[
(6.3) \quad B_\epsilon^n = \bigcup_{v_j \in \mathcal{K}_{\epsilon,n}} B_\epsilon^j, \quad O_\epsilon^n = \bigcup_{v_j \in \mathcal{K}_{\epsilon,n}} O_\epsilon^j, \quad R_\epsilon^n = U \setminus (B_\epsilon^n \cup O_\epsilon^n).
\]

Finally, we define a Borel function \( \rho_\epsilon^n : U \setminus \mathcal{K}_n \to [0, \infty] \) and weights \( \{ \rho_{n,j} \} = \{ \rho_n(s_j) \} \) on \( \mathcal{K}_n \) as follows:

\[
\rho_\epsilon^n = \chi_{B_\epsilon^n \cup R_\epsilon^n} = \chi_{U \setminus O_\epsilon^n}
\]
\[
(6.4) \quad \rho_{n,j} : = \rho_n(s_j) = \begin{cases} 
\epsilon l(s_j) + \epsilon r_i^2, & s_j \in \mathcal{K}_n \cap \mathcal{K}_\epsilon, \\
\epsilon l(s_j), & s_j \in \mathcal{K}_n \setminus \mathcal{K}_\epsilon.
\end{cases}
\]

where \( \chi_E \) denotes the characteristic function of the set \( E \), and let
\[
\rho^n = (\rho_\epsilon^n; \rho_{n,1}, \ldots, \rho_{n,N}),
\]
where \( N = 1 + \ldots + 4^n \) is the number of slits of generation at most \( n \). In other words, \( \rho^n_\epsilon \) vanishes on the omitted set and is equal to 1 otherwise, and \( \rho_{n,j} \) is equal to the width of the \( \epsilon \)-collar for each slit \( s_j \).
6.1.2. Admissibility of $\varrho_n^\epsilon$ relative $K_n$. Next, we show that $\varrho_n^\epsilon$ is admissible for $\Gamma$ relative $K_n$, i.e., the estimate

$$l_{\varrho_n^\epsilon}(\gamma) = \int_{\gamma} \rho_n^\epsilon ds + \sum_{\gamma \cap s_i \neq \emptyset} \rho_{n,i} \geq 1,$$

holds for every $\gamma \in \Gamma$.

In [Hak18] it was shown that if $\gamma \in \Gamma$ does not intersect any of the slits of $K_n$ then $\rho_n^\epsilon$-length of $\gamma$ (i.e., $\int_{\gamma} \rho_n^\epsilon ds$) is at least 1. The idea and the reason for defining the discrete weights $\rho_{n,j}$ as in (6.4), is to ensure that when a curve $\gamma \in \Gamma$ intersects a slit $s_j \in K_n$ its “horizontal-length” does not decrease too much. Indeed, if $\gamma$ intersects a slit $s_i$ the integral $\int_{\gamma} \rho_n^\epsilon ds$ may decrease by the amount equal to the width of the corresponding collar (or more), but the second term in $l_{\varrho_n^\epsilon}(\gamma)$ would increase by $\rho_{n,j} = \ell(s_j)$, which is the “width” of the collar of $s_i$. This balance implies that the $\varrho_n^\epsilon$-lengths of the curves stays bounded below by 1. Next we provide the details of this argument.

![Figure 6.3](image)

Figure 6.3. Examples showing the subsets $O_n^\epsilon, B_n^\epsilon$ and $R_n^\epsilon$ (in white, dark grey and light grey, respectively), for the "standard" collection of slits corresponding to the sequence $r_i = 1/2, i \geq 0$. Here $\epsilon = 1/4$, and $n = 1, 2$.

To prove (6.5) we will show that for every $\gamma \in \Gamma$ there is a subset $\gamma' \subset U$, which is not necessarily a curve, such that

$$l_{\varrho_n^\epsilon}(\gamma) \geq l_{\varrho_n^\epsilon}(\gamma') \text{ and } l_{\varrho_n^\epsilon}(\gamma') \geq 1.$$

Pick a curve $\gamma \in \Gamma$. Without loss of generality, we may assume that $\gamma$ is oriented so that it starts at the left and ends at the right vertical edge of the unit square $U$. Given two disjoint subsets $E$ and $F$ in $U$, we say that $\gamma$ meets $E$ before $F$ if there exists $t \in (0, 1)$ so that $\gamma(t) \in E$ and $\gamma(s) \notin F$ for any $s < t$ and $\gamma$ meets $E$ after $F$ if $\gamma$ meet $F$ before $E$. Before constructing $\gamma'$, we modify $\gamma$ inductively around every slit $v_i \in K_n \cap K_n$ as follows.

Denote $\gamma_{-1} := \gamma$. For $0 \leq i \leq N_\epsilon$, suppose the subsets $\gamma_0, \ldots, \gamma_{i-1}$ of $U$ have been defined and define $\gamma_i$ as follows:
Figure 6.4. A curve $\gamma = \gamma_{-1}$ and its first modification $\gamma_0$. Since $\gamma$ meets the omitted region of the slit $v_0$ before exiting the collar through the right side $r(v_0^i)$, we have $\gamma_{-1} = \gamma \setminus v_0^i \cup (t(v_0^i) \setminus v_0)$.

(a) If $\gamma \cap v_i = \emptyset$, then

$$
\gamma_i = \begin{cases} 
\gamma_{i-1} \setminus v_i^i \cup (t(v_i^i) \setminus v_i) & \text{if } \gamma \cap O_i^i = \emptyset, \\
\gamma_{i-1} \setminus O_i^i & \text{if } \gamma \text{ meets } O_i^i \text{ before } r(v_i^i), \\
\gamma_{i-1} \setminus O_i^i & \text{if } \gamma \text{ meets } O_i^i \text{ after } r(v_i^i).
\end{cases}
$$

(b) If $\gamma \cap v_i \neq \emptyset$ then

$$
\gamma_i = (\gamma_{i-1} \setminus (v_i^i \cup v_i)) \cup (t(v_i^i) \setminus v_i),
$$

where $t(v_i^i)$ and $r(v_i^i)$ as before denote the top and the right sides of the collar $v_i^i$, respectively.

This is a finite induction. Thus, we only construct $\gamma_i$ for $i = 0, \ldots, N_\varepsilon$ and let $\gamma' = \gamma_{N_\varepsilon}$.

Note that $\gamma' \subset B_{N_\varepsilon}^\varepsilon \cup R_{N_\varepsilon}^\varepsilon$. Moreover, since at every step of the construction above the curves are modified so that $\text{proj}_x(\gamma_i) = [0, 1]$, we also have $\text{proj}_x(\gamma') = [0, 1]$, where $\text{proj}_x$ denotes the projection onto the $x$ axis in the plane. Therefore

$$
l_{\phi_n}(\gamma') = \int_{\gamma'} \rho_n' ds = \mathcal{H}^1(\gamma') \geq \mathcal{H}^1(\text{proj}_x(\gamma')) = \mathcal{H}^1([0, 1]) = 1,
$$

and it would be sufficient to prove $l_{\phi_n}(\gamma') \geq l_{\phi_n}(\gamma)$. Since $\gamma = \gamma_{-1}$ and $\gamma' = \gamma_{N_\varepsilon}$ it is enough to show that for every $0 \leq i \leq N_\varepsilon$ we have

$$
l_{\phi_n}(\gamma_{i-1}) \geq l_{\phi_n}(\gamma_i).
$$

By the definition of mass distribution $\phi_n$ in (6.4), we have

$$
l_{\phi_n}(\gamma_{i-1}) = \mathcal{H}^1(\gamma_{i-1} \cap R_n^\varepsilon) + \mathcal{H}^1(\gamma_{i-1} \cap B_n^\varepsilon) + \sum_{\{j : \gamma_{i-1} \cap v_j \neq \emptyset\}} \rho_{n,j}
$$

$$
= \mathcal{H}^1(\gamma_{i-1} \cap R_n^\varepsilon) + \sum_{j=0}^{N_\varepsilon} \mathcal{H}^1(\gamma_{i-1} \cap B_j^\varepsilon) + \sum_{\{j : \gamma_{i-1} \cap v_j \neq \emptyset\}} \rho_{n,j}.
$$
Figure 6.5. Further modifications of the curve $\gamma$ from Fig. 6.4. Since $\gamma$ does not intersect the collars of $v_1$ and $v_2$ (the two slits of generation 1 not pictured), we have $\gamma_2 = \gamma_1 = g_0$. The curves $\gamma_3$ and $\gamma_4$ are constructed according to method in the proof of admissibility of $\varphi_n$ relative $K_n$.

Therefore, letting

$$\delta_{i,j} = \begin{cases} 1, & \text{if } \gamma_i \cap v_j = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$l_{\varphi_n}(\gamma_{i-1}) = H^1(\gamma_{i-1} \cap R^*_n) + \sum_{j=1}^{N} \left( H^1(\gamma_{i-1} \cap B^*_j) + \delta_{i-1,j} \cdot \rho_{n,j} \right). \tag{6.7}$$

Since $\gamma_i$ is obtained by modifying $\gamma_{i-1}$ only within $(v_i^*)$, we have that the two curves coincide on the residual set $R^*_n$ (note that $t(v_i^*)$ is in the complement of $R^*_n$), and therefore

$$H^1(\gamma_{i-1} \cap R^*_n) = H^1(\gamma_i \cap R^*_n), \tag{6.8}$$

and for every $j \in \{0, \ldots, N\}$ with $j \neq i$ we have

$$H^1(\gamma_{i-1} \cap B^*_j) + \delta_{i-1,j} \cdot \rho_{n,j} = H^1(\gamma_i \cap B^*_j) + \delta_{i,j} \cdot \rho_{n,j}. \tag{6.9}$$

Therefore, by (6.7) and since $\delta_{i,i} = 0$, to prove (6.6) we only need to show the following estimate

$$H^1(\gamma_{i-1} \cap B^*_i) + \delta_{i-1,i} \cdot \rho_{n,i} \geq H^1(\gamma_i \cap B^*_i). \tag{6.10}$$

Corresponding to the definition of $\varphi_n$ in (6.4), there are several cases to consider:

(a) If $\gamma_{i-1} \cap v_i = \emptyset$, i.e., $\delta_{i-1,i} = 0$, then three possibilities can occur:

- If $\gamma \cap O_i = \emptyset$ then $\gamma_{i-1} \cap B^*_i = \gamma_i \cap B^*_i$. In particular $H^1(\gamma_{i-1} \cap B^*_i) = H^1(\gamma_i \cap B^*_i)$.
- If $\gamma$ meets $O_i^*$ before $R(v_i^*)$ then $\gamma_{i-1}$ connects the top and bottom of an $\epsilon$-buffer and therefore $H^1(\gamma_{i-1} \cap B^*_i) \geq \ell(v_i) = H^1(\gamma_i \cap B^*_i)$.
- If $\gamma$ meets $O_i^*$ after $R(s_i^*)$ then $H^1(\gamma_{i-1} \cap B^*_i) \geq 0 = H^1(\gamma_i \cap B^*_i)$.
(b) If \( \gamma_{i-1} \cap v_i \neq \emptyset \) then
\[
\mathcal{H}^1(\gamma_{i-1} \cap B_i) + \rho_{n,i} \geq \rho_{n,i} = e_1(s_i) = \mathcal{H}^1(t(v_i)) = \mathcal{H}^1(\gamma_i \cap B_i).
\]
Thus (6.10) holds in all the cases. Combining (6.7), (6.8), (6.9) and (6.10) we obtain (6.6). Therefore \( l_{\varepsilon r}(\gamma) \geq 1 \) and \( \phi_n \) is admissible for \( \Gamma \) relative \( K_n \).

6.1.3. Estimating the mass of \( \phi_n \). To estimate \( A(\phi_n^\varepsilon) \) note that
\[
A(\phi_n^\varepsilon) = \int_{R_n^\varepsilon \cup B_n^\varepsilon} (\rho_n^\varepsilon)^2 d\mathcal{H}^2 + \sum_{s_j \in K_n} \rho_{n,j}^2
= \mathcal{H}^2(R_n^\varepsilon) + \mathcal{H}^2(B_n^\varepsilon) + \sum_{v_j \in K_{n,n}} (e(v_j))^2.
\]
Since \( e(v_j) \) is the side length of each of the buffer squares, we have that
\[
\mathcal{H}^2(B_j^\varepsilon) = 2(e(v_j))^2 = 2e\mathcal{H}^2(v_j^\varepsilon)
\]
and therefore
\[
A(\phi_n^\varepsilon) = \mathcal{H}^2(R_n^\varepsilon) + (3/2)\mathcal{H}^2(B_n^\varepsilon) = \mathcal{H}^2(R_n^\varepsilon) + 3e\mathcal{H}^2\left( \bigcup_{v_j \in K_{n,n}} v_j^\varepsilon \right)
\leq \mathcal{H}^2(R_n^\varepsilon) + 3e,
\]
where the last inequality holds since \( v_j^\varepsilon \)'s are pairwise disjoint and \( \bigcup_j v_j^\varepsilon \subset U \).

To estimate \( \mathcal{H}^2(R_n^\varepsilon) \), we first note that \( \mathcal{H}^2(R_0^\varepsilon) = 1 - e\mathcal{H}(s_0) = 1 - e\varepsilon_0 \). Next, assume that for some \( n \geq 1 \) we have \( \mathcal{H}^2(R_{n-1}^\varepsilon) \leq \prod_{i=1}^{n-1} (1 - e\varepsilon_i^2) \). From the definition of \( R_n^\varepsilon \) and the disjointness properties of the collars we have
\[
R_n^\varepsilon = [0, 1]^2 \setminus \bigcup_{v_j \in K_n} v_j^\varepsilon = [0, 1]^2 \setminus \bigcup_{s_i \in K_n} s_i^\varepsilon.
\]
Next, we observe that if \( \Delta \in D_n, n > 1 \), then
\[
R_n^\varepsilon \cap \Delta = (R_n^\varepsilon \cap \Delta) \setminus s^\varepsilon(\Delta),
\]
where \( s(\Delta) \) is the slit corresponding to \( \Delta \). Indeed, as noted above either \( s^\varepsilon(\Delta) \) is contained in a previously removed collar, or it does not intersect any such collar. If \( s^\varepsilon(\Delta) \) is contained in a previously removed collar then, since \( e \) is a power of \( 1/2 \), the dyadic square \( \Delta \) is also in the complement of \( R_{n-1}^\varepsilon \) and both sides of (6.12) are empty. On the other hand if \( s^\varepsilon(\Delta) \cap R_{n-1}^\varepsilon \neq \emptyset \) then \( s^\varepsilon(\Delta) \subset R_{n-1}^\varepsilon \) and (6.12) follows from the definition of \( R_n^\varepsilon \).

From (6.12) we have that if \( \Delta \in D_n \) is such that \( R_{n-1}^\varepsilon \cap \Delta \neq \emptyset \) then
\[
\mathcal{H}^2(R_n^\varepsilon \cap \Delta) = \mathcal{H}^2(R_{n-1}^\varepsilon \cap \Delta) - \mathcal{H}^2(s^\varepsilon(\Delta)).
\]
But
\[
\mathcal{H}^2(s^\varepsilon(\Delta)) = e(s(\Delta))^2 = e\left( \frac{r_n}{2^m} \right)^2 = \varepsilon r_n^2 \mathcal{H}^2(\Delta) \geq e\varepsilon r_n^2 \mathcal{H}^2(R_{n-1}^\varepsilon \cap \Delta)
\]
and therefore if \( s(\Delta) \cap R_{n-1}^\varepsilon \neq \emptyset \) we have
\[
\mathcal{H}^2(R_n^\varepsilon \cap \Delta) \leq (1 - \varepsilon r_n^2) \mathcal{H}^2(R_{n-1}^\varepsilon \cap \Delta).
\]
Moreover, as explained above, (6.13) holds even if \( s(\Delta) \cap R_{n-1}^\varepsilon = \emptyset \), in which case both sides are 0. Summing (6.13) over all dyadic cubes of generation \( n \) we obtain
\[
\mathcal{H}^2(R_n^\varepsilon) \leq (1 - \varepsilon r_n^2) \mathcal{H}^2(R_{n-1}^\varepsilon).
\]
By induction hypothesis we have \( \mathcal{H}^n(R_k^\epsilon) \leq \prod_{i=0}^{k-1}(1-\epsilon r_i^2) \), and therefore by (6.11) we obtain

\[
A(g^\epsilon_n) \leq \prod_{i=0}^{n}(1-\epsilon r_i^2) + 3\epsilon.
\]

Since \( g^\epsilon_n \) is admissible for \( \Gamma \) relative \( K_n \) we obtain (a stronger version of) inequality (6.1) in the case when \( r_i \)'s and \( \epsilon \) are powers of 2.

To prove (6.1) in general, assume \( r_i, \epsilon \geq 0 \), and \( \epsilon \) are arbitrary numbers in \((0,1)\). Then there are integers \( j_i \geq 1 \) and \( m \geq 1 \) such that \( 2^{-j_i} \leq r_i < 2^{-j_i+1} \) and \( 2^{-m} \leq \epsilon < 2^{-m+1} \). Let \( \epsilon' = 2^{-m} \), \( r'_i = 2^{-j_i} \), and let \( F_n, n = 0, 1, \ldots, \) be the families of dyadic slits corresponding to the sequence \( \{r'_i\}_{i=0}^\infty \), cf. Section 3. Since \( r'_i \leq r_i \leq 2r'_i \) for \( 0 \leq i \leq n \) we have that every element of \( F_n \) is a subset of an element of \( K_n \). By monotonicity in \( K \), cf. Proposition 4.4, and by the case considered above, we have that

\[
\text{Mod}_{U,K_n} \Gamma \leq \text{Mod}_{U,F_n} \Gamma \leq \prod_{i=0}^{n}(1-\epsilon'(r'_i)^2) + 3\epsilon'.
\]

Since \( \epsilon r_i^2 \leq 2\epsilon' (2r'_i)^2 \) and \( \epsilon' \leq \epsilon \), the last inequality implies (6.1) in general.

Finally, if \( r \not\in \ell^2 \) then the product in the right hand side of (6.1) approaches 0 as \( n \) approaches \( \infty \). Therefore, for every \( \epsilon > 0 \) we have

\[
\limsup_{n \to \infty} \text{Mod}_{U,K_n} \Gamma \leq 3\epsilon,
\]

which implies (6.2) and completes the proof.

The proof given above yields a more general result which we state next.

\[ \textbf{Lemma 6.2.} \] Suppose \( R = [\frac{j-1}{2^k}, \frac{j}{2^k}] \times [0,1] \), where \( j \in \{0, \ldots, 2^k - 1\} \) and let \( \Gamma \) be the family of curves in the rectangle \( R \) connecting its vertical sides. Then for \( 0 < \epsilon < 1 \) and every \( n > k \) we have

\[
\text{Mod}_{R,K_n} (\Gamma) \leq 2^k \left[ \prod_{i=k}^{n}(1-8^{-1}\epsilon r_i^2) + 3\epsilon \right].
\]

In particular, if \( \sum_{i=k}^{\infty} r_i^2 = \infty \) then \( \lim_{n \to \infty} \text{Mod}_{R,K_n} (\Gamma) = 0. \)

\[ \textbf{Proof.} \]

Note that the tansboundary modulus of \( \Gamma \subset R \) with respect \( K_n \) is the same as the one with respect to the subfamily of \( K_n \) contained in \( R \), i.e., only the slits of generation \( k \) or greater make a contribution.

Let \( \phi(z) \) be the conformal map of the plane such that \( \phi(R) = [0,1] \times [0,2^k] \), taking vertices of \( R \) to vertices of \( \phi(R) \). By conformal invariance of transboundary modulus we have

\[
\text{Mod}_{R,K_n}(\Gamma) = \text{Mod}_{\phi(R),\phi(K_n)}(\phi(\Gamma)).
\]

Now, the subfamily \( \phi(K_n) \) contained in \( \phi(R) \) is the collection of slits \( s'(\Delta) \) corresponding to the dyadic subsquares \( \Delta \) of \([0,1] \times [0,2^k] \) such that

\[
\text{diam}(s'(\Delta)) = \frac{r_{i-k}}{2^k},
\]

if \( \Delta \in \mathcal{D}_k \).

Just like in the proof of Lemma 6.1 we may first consider the case when the length of the slits and the widths of the buffers are powers of 2. With this assumption, let \( g^\epsilon_n \) be the mass distribution on \([0,1] \times [0,2^k] \) obtained by using the translates of
the mass distribution \( \varphi_n^* \), see (6.4) in the proof of Lemma 6.1, for the squares \([0,1] \times [j,j+1], \) for \( j \in \{0, \ldots, 2^k-1\} \). More precisely, for \((x,y) \in [0,1] \times [0,2^k]\) we let \( \tilde{\varphi}_n^*(x,y) = \varphi_n^*(x,y \mod 1) \), while to the slits in \( \phi(K_n) \) with the same lengths we assign weights equal to the “widths” of the corresponding buffers as in (6.4). Then just like before \( \tilde{\varphi}_n \) is admissible for \( \phi(\Gamma) \) relative \( \phi(K_n) \) and \( A(\tilde{\varphi}_n) \leq 2^k(\prod_{i=k}^n (1-\epsilon r_i^2) + 3\epsilon) \). The case of general \( r_i \)’s and \( \epsilon \) is done the same way as in Lemma 6.1 and immediately implies (6.14). \( \square \)

6.2. Necessity in Theorem 1.1. Here we combine the results of Sections 5 and 6.1 to show the following.

**Theorem 6.3.** If \( \sum_{i=0}^\infty r_i^2 = \infty \) then there is no quasisymmetric embedding of \( \mathcal{S} = \mathcal{S}_r \) into the plane \( \mathbb{R}^2 \).

**Proof.** Let \( \Gamma_0 = \Gamma(s_0, \delta_0; U \setminus s_0) \) be the family of curves in \( U \) connecting the central slit \( s_0 \) to the right vertical side of \( U \). Also, let \( \Gamma \) be the family of curves connecting the vertical sides of the rectangle \( R = [1/2,0] \times [0,1] \). If \( r \notin \ell^2 \) then by Lemma 6.2 \( \text{Mod}_{R,K_n} \Gamma \to 0 \) as \( n \to \infty \). But, by overflowing and monotonicity properties of transboundary modulus we have

\[
\text{Mod}_{U_0,K_0} \Gamma_0 \leq \text{Mod}_{R,K_n} \Gamma_0 \leq \text{Mod}_{R,K_n} \Gamma.
\]

Therefore \( \text{Mod}_{U_0,K_0} \Gamma_0 \to 0 \) as \( n \to \infty \) and by Lemma 5.1 there is no quasisymmetric embedding of \( \mathcal{S}_r \) into the plane. \( \square \)

7. Embeddings of Slit Carpets

In this section, we prove the “if” direction in Theorem 1.1.

**Theorem 7.1.** If \( r = \{r_i\}_{i=0}^\infty \in \ell^2 \) then there is a quasisymmetric mapping \( F : \mathcal{S} = \mathcal{S}_r \hookrightarrow \mathbb{R}^2 \).

**Proof.** The idea is to show that there is a metric 2 sphere \( \mathcal{S} \) which contains \( \mathcal{S} \) and is quasisymmetric to the standard sphere \( S^2 \). The surface \( \mathcal{S} \) will be obtained by “gluing in” topological disks along the peripheral circles of the slit carpet \( \mathcal{S} \). We will then use Bonk and Kleiner’s uniformization theorem, cf. [BK02], to show that \( \mathcal{S} \) is quasisymmetric to \( S^2 \).

7.1. Pillowcases. For \( l \in (0,1) \) consider the rectangle \( R = R(l) = [-l,l] \times [0,1] \). Define an equivalence relation on \( \partial R \) by identifying \((x,0)\) with \((-x,0)\), and \((x,1)\) with \((-x,1)\) for \( x \in [0,1] \). The quotient space

\[
\mathcal{P} = \mathcal{P}(l) = R(l) / \sim,
\]

can be thought of as a “square pillowcase” with an open “mouth”, which corresponds to the vertical sides of the rectangle \( R \). For this reason we will call \( \mathcal{P} \) a *square pillowcase of side-length \( l \). The image of a point \( z \in R \) in \( \mathcal{P} \) under the quotient map will be denoted by \([z]\). We will also use the following notation,

\[
T(\mathcal{P}) = \{([0,t]) : 0 \leq t \leq l\},
L(\mathcal{P}) = \{([t,0]) : 0 \leq t \leq l(s)\},
U(\mathcal{P}) = \{([l(s),t]) : 0 \leq t \leq l(s)\},
\]
and will call these sets the *top, lower and upper edges of* \( \mathcal{S} \), respectively. Clearly, \( \mathcal{S} \) is a topological disk and \( \partial \mathcal{S} \) is a topological circle corresponding to the vertical sides of \( R \).

As a metric space, \( \mathcal{S} \) is equipped with the quotient of the Euclidean metric on \( R \), cf. [BBI01].

### 7.2. Gluing

Next we show how one can glue a pillowcase to a slit of the slit carpet \( \mathcal{S} \). Suppose \( s \subset \mathcal{S} \) is a slit such that \( \pi(s) = \{x\} \times [a, a + l] \subset \text{int}(U) \). Given a point \( z = (x, a + t) \in \pi(s) \) we will denote by \( p^+_z \) and \( p^-_z \) the preimages of \( z \) in \( \mathcal{S} \). Suppose \( s \) is a slit such that \( \pi(s) = \{x\} \times [a, a + l] \subset \text{int}(U) \).

Clearly, \( \mathcal{S} \) is homeomorphic to the Sierpiński carpet. Moreover, the path metric \( D \mathcal{S} \) naturally induces a metric on \( D \mathcal{S} \), which we will denote by \( d_{D \mathcal{S}} \).

Let \( \mathcal{D} \mathcal{K} \) denote the collection of all slits in \( D \mathcal{S} \), and let \( \mathcal{D} \mathcal{K} = \{s_j\}_{j=0}^{\infty} \) be an enumeration of the slits. To each slit \( s_j \) in \( \mathcal{D} \mathcal{K} \) we assign a pillowcase \( \mathcal{P}_j \) of sidelenge equal to \( \text{diam}(s_i) = l(s_j) \) and a gluing function \( g_j = g(s_j) : \partial \mathcal{P}_i \to s_j \) as defined in (7.1) and (7.2).

Thus, for every slit carpet \( \mathcal{S} \) we may define the topological space \( \mathcal{D} \) as follows. Consider the quotient space

\[
\mathcal{D} = (D \mathcal{S} \sqcup (\sqcup_{j=0}^{\infty} \mathcal{P}_j)) / \sim,
\]

obtained by gluing the pillowcase \( \mathcal{P}_i \) to \( D \mathcal{S} \) via \( g_i \), i.e., for \( i \geq 0 \), if \( x \in \partial \mathcal{P}_j \) then we have that \( x \sim g_i(x) \). Thus, we cover every slit with a square pillowcase by gluing its boundary with the corresponding slit isometrically.

Note that \( \mathcal{D} \) is homeomorphic to \( S^2 \) since every \( \mathcal{P}_i \) is a topological disk and \( D \mathcal{S} \) is homeomorphic to \( S_{1/3} \) by Whyburn’s Theorem 3.1.

The space \( \mathcal{D} \) can be equipped with a natural metric as follows, cf. [Hai15]. First, define a quasimetric \( \tau \) on \( \mathcal{D} \) by setting

\[
\tau(p, q) = \begin{cases} 
    d_{D \mathcal{S}}(p, q), & \text{if } p, q \in D \mathcal{S}, \\
    d_i(p, q), & \text{if } p, q \in \mathcal{P}_i, i \geq 1, \\
    \inf_{\xi \in s_i} \{d_{D \mathcal{S}}(p, \zeta) + d_i(\zeta, q)\}, & \text{if } p \in D \mathcal{S}, q \in \mathcal{P}_i, \\
    \inf_{\xi \in s_i, \zeta \in s_j} \{d_i(p, \zeta) + d_{D \mathcal{S}}(\zeta, \xi) + d_j(\xi, q)\}, & \text{if } p \in \mathcal{P}_i, q \in \mathcal{P}_j, i \neq j,
\end{cases}
\]
where \( d_i, i \geq 1 \), denotes the metric on \( \mathcal{P}_i \). Furthermore, for \( p, q \in \mathcal{D} \) let

\[
d_{\mathcal{D}}(p, q) = \inf \sum_{k=1}^{n} \tau(\zeta_k, \xi_k),
\]

where the infimum is taken over all sequences \( \zeta_1, \zeta_1, \ldots, \zeta_n, \xi_n \in D \mathcal{J} \cup_{j=1}^{n} \mathcal{P}_j \) such that \( \zeta_1 \in p, \xi_n \in q \) and \( \xi_k \sim \zeta_{k+1} \) for \( i = 1, \ldots, n-1 \). By Theorem 2.2 in [Hai15], \( d_{\mathcal{D}} \) is a metric provided the mappings \( g_i \) are uniformly quasisymmetric and \( \text{diam}_d, \mathcal{P}_i \leq C \text{diam}_d, \partial \mathcal{P}_i \), for all \( i \geq 1 \). Since in our case the mappings \( g_i \) are all isometries, and the inequality above holds with \( d \) and \( \text{diam} \) such that \( S \subset S \) in \([2n] \), we have for every \( x, z \)

\[
\text{diam} \subset \text{diam} = \frac{3}{4} r.
\]

Theorem 7.2 (Bonk, Kleiner, [BK02]). Let \( X \) be an Ahlfors 2-regular compact connected metric space homeomorphic to \( S^2 \). Then \( X \) is quasisymmetric to \( S^2 \) if and only if \( X \) is linearly locally connected.

Recall that a metric space \((X, d)\) is called linearly locally connected (or LLC) if there is a constant \( \lambda \geq 1 \) so that for every \( z \in X \) and \( r > 0 \) the following conditions hold:

1. \((\text{LLC}_1)\) If \( x, y \in B(z, r) \), then there exists a continuum \( E \subset B(z, \lambda r) \) containing \( x \) and \( y \).
2. \((\text{LLC}_2)\) If \( x, y \notin B(z, r) \), then there exists a continuum \( E \subset X \setminus B(z, r/\lambda) \) containing \( x \) and \( y \).

Thus, by Theorem 7.2, to complete the proof we need to show that \( \mathcal{D} \) is LLC and Ahlfors 2-regular.

To show that \( \mathcal{D} \) is LLC we use Theorem 2.6.2 in [Hai15] which implies that \( \mathcal{D} \) is LLC if \( D \mathcal{J} \) and all \( \mathcal{P}_i, i \geq 1 \) are uniformly LLC. Since \( \mathcal{P}_i \) are all uniformly LLC (with \( \lambda = 1 \)) it is enough to show that \( D \mathcal{J} \) is LLC.

Lemma 7.3. The double \( D \mathcal{J} \) of the slit carpet \( \mathcal{J} = \mathcal{J}_r \) is LLC.

Proof. Note that if \( x \in B(z, r) \) and \( \gamma_{xz} \) denotes a length minimizing curve connecting \( x \) and \( z \), then for every \( p \in \gamma_{xz} \) we have \( d_{D \mathcal{J}}(z, p) \leq d_{D \mathcal{J}}(z, x) \) and therefore \( \gamma_{xz} \subset B(z, r) \). Therefore if \( x, y \in B(z, r) \) then \( \gamma_{xz} \cup \gamma_{zy} \subset B(x, r) \) is a continuum connecting \( x \) and \( y \). Therefore \( D \mathcal{J} \) is LLC with \( \lambda = 1 \).

To show that \( D \mathcal{J} \) is LLC let \( x, y \in D \mathcal{J} \setminus B(z, r) \), where \( 2^{-n-1} \leq r < 2^{-n} \). Let

\[
T' = \bigcup_{\Delta \in \mathcal{D}_{n+3} \setminus B(z, 2^{-n-3})} T_\Delta,
\]

where, as before \( T_\Delta = \pi^{-1}(\text{int}(\Delta)) \) denotes a “dyadic square” in \( \mathcal{J} \) corresponding to some dyadic square \( \Delta \subset \mathcal{U} \). Note that, since for \( \Delta \in \mathcal{D}_{n+3} \) we have \( \text{diam}_\mathcal{J} T_\Delta \leq 2 \cdot 2^{-n-3} \), we have for every \( x \in T' \) the following inequalities,

\[
d_{D \mathcal{J}}(x, z) \leq 2^{-(n+3)} + \text{diam} T_\Delta \leq 3 \cdot 2^{-(n+3)} \leq \frac{3}{4} r.
\]
Therefore
\[ B \left( z, \frac{r}{8} \right) \subset B \left( z, \frac{1}{2^n + 3} \right) \subset T' \subset B \left( z, \frac{3}{4^n} \right). \]

Finally, since \( x, y \in D \setminus \partial T' \) there is a continuum connecting \( x \) and \( y \) without intersecting \( B(z, r/8) \). Indeed, if \( x \) and \( y \) belong to the same “dyadic” square \( T_\Delta \) for some \( \Delta \in D_{n+3} \) then there is a curve \( \gamma_{xy} \subset T_\Delta \) connecting \( x \) and \( y \), since \( T_\Delta \) is path connected. On the other hand, if \( x \in T_\Delta \) and \( y \in T_{\Delta'} \) then connecting \( x \) and \( y \) to “outer squares” of \( T_\Delta \) and \( T_{\Delta'} \), respectively, and then connecting these outer squares through the preimages of the grids \( \tilde{\Pi}_{n+3} \), cf. Section 5, without intersecting \( \text{int}(T') \), gives a continuum \( \gamma_{xy} \subset D \setminus B(z, r/8) \) and \( D \) is LLC. \( \square \)

**Lemma 7.4.** If \( r \in \ell^2 \) then \( D \) is Ahlfors 2-regular.

**Proof.** Note that it is enough to show that the space
\[ D' = \mathcal{D} \cup (\bigcup_{s_j \subset \mathcal{D}} \mathcal{P}(s_j))/\sim \]

is Ahlfors regular. Indeed, \( \mathcal{D} \) can be obtained by gluing two copies of \( \mathcal{D}' \) along the outer square of \( \mathcal{D} \) by the identity, and therefore if \( \mathcal{D}' \) is Ahlfors 2-regular with constant \( C \) then \( \mathcal{D} \) is Ahlfors regular with \( 2C \).

Below we use the same notation \( T = T_\Delta \subset \mathcal{D} \) as above for the dyadic squares in \( \mathcal{D} \). Moreover, for a dyadic square \( \Delta \in D_n \) in \( U \) we let \( \tilde{T}_\Delta \) denote the portion of \( \mathcal{D}' \) “over” \( T \), i.e.,
\[ \tilde{T} := \tilde{T}_\Delta = T_\Delta \cup \bigcup_{s_j \subset T_\Delta} \mathcal{P}(s_j). \]

Next, suppose \( \Delta \) is a dyadic square of generation \( n \geq 1 \). Then, by Lemma 3.5, there is a constant \( C \geq 1 \) which does not depend on \( n \), so that the following inequalities hold:
\[ \mathcal{H}^2(\tilde{T}_\Delta) = \mathcal{H}^2(T_\Delta) + \sum_{s_j \subset T_\Delta} \mathcal{H}^2(\mathcal{P}(s_j)) \leq C(2^{-n})^2 + \sum_{k \geq n} \left( \sum_{s(\Delta') \subset T_\Delta, \Delta' \in D_k} l(s(\Delta'))^2 \right). \]

(7.5)

The number of generation \( k \geq n \) slits (or equivalently dyadic subsquares) contained in \( \Delta \) is equal to \( 4^{k-n} \). Therefore, since \( l(s(\Delta')) = r_k 2^{-k} \) for \( \Delta' \in D_k \), the following equality holds for every \( k \geq n \):
\[ \sum_{s(\Delta') \subset T_\Delta, s_j \subset D_k} l(s(\Delta'))^2 = 4^{k-n}(r_k 2^{-k})^2. \]

(7.6)

Hence, combining (7.5) and (7.6) we obtain
\[ \mathcal{H}^2(\tilde{T}_\Delta) \leq C4^{-n} + \sum_{k \geq n} 4^{k-n}(r_k^2 4^{-k}) = 4^{-n}(C + \sum_{k \geq n} r_k^2). \]
Since $2^{-n} \leq \text{diam} T_\Delta \leq 2^{-n+1}$ we obtain that for every $\Delta \in \mathcal{D}$ the following inequalities hold:
\[
\frac{1}{4C} (\text{diam} T_\Delta)^2 \leq \mathcal{H}^2(T_\Delta) \leq C_1 (\text{diam} T_\Delta)^2,
\]
where $C_1 = C + \sum_{k=1}^{\infty} t_k^2$, with $C$ being the constant from Lemma 3.5.

Now, if $x \in \mathcal{I}$ and $2^{-n-1} < r < 2^{-n}$ then considering a dyadic square $T_\Delta$ for some $\Delta \in \mathcal{D}_{n+3}$ such that $B(x, r/8 \cap T_\Delta \neq \emptyset$, we have (like in Lemma 7.3) $T_\Delta \subset B(x, r)$ and
\[
\mathcal{H}^2(B(x, r)) \geq \mathcal{H}^2(T_\Delta) \geq \frac{1}{4C} (\text{diam} T_\Delta)^2 \geq \frac{1}{4C} \left( \frac{r}{2^3} \right)^2 = \frac{r^2}{28C}.
\]
On the other hand, since $\pi(B(x, r)) \subset B(\pi(x), r)$, there are at most 9 dyadic squares of generation $n$ intersecting $B(\pi(x), r)$ such that $\cup_{i=1}^9 \Delta_i$ is a Euclidean square in $U$. It follows that there are at most 9 dyadic squares $\Delta_1, \ldots, \Delta_9 \in \mathcal{D}_n$ such that $B(x, r) \cap T_\Delta, \neq \emptyset, i = 1, \ldots, 9$. Let
\[
\tilde{T} = \cup_{i=1}^9 \tilde{T}_\Delta.
\]
Then, we have
\[
\mathcal{H}^2(B(x, r) \cap \tilde{T}) \leq \sum_{i=1}^9 \mathcal{H}^2(\tilde{T}_\Delta, i) \leq 9 C_1^2 (\text{diam}(\tilde{T}_\Delta))^2 \leq 9 C_1^2 (2 \cdot 2^{-n})^2
\]
\[
\leq 9 \cdot 2^4 C_1^2 = 2^8 C_1.
\]
Next, if $y \in B(x, r) \setminus \tilde{T}$ then $y$ belongs to a pillowcase $\mathcal{P}(s_j)$ over a slit $s_j$ of generation $n - 1$, thus $l(s_j) \geq 2^{n-1} > r$. Note that if $z \in \partial \mathcal{P}(s_j)$ is the closest point in $\mathcal{P}(s_j)$ to $x \in \mathcal{I}$, we have that $\mathcal{P}(s_j) \cap B(z, r)$ is contained in $\mathcal{P}(s_j) \cap B(z, r)$. Therefore
\[
\mathcal{H}^2(B(x, r) \cap \mathcal{P}(s_j)) \leq \mathcal{H}^2(B(z, r) \cap \mathcal{P}(s_j)) \leq \pi r^2/2,
\]
which $z \in \partial \mathcal{P}(s_j)$.

On the other hand, from the construction of $\tilde{T}$ it follows that there are at most 8 such “large pillowcases” $\mathcal{P}(s_j)$’s intersecting $\tilde{T}$, (two for every “vertical curve” containing a vertical side of some $\tilde{T}_\Delta, \subset \tilde{T}$). Therefore,
\[
\mathcal{H}^2(B(x, r) \setminus \tilde{T}) \leq 4 \pi r^2.
\]
Combining (7.7), (8.7) and (9.7) we obtain that for every $x \in \mathcal{I}$ and $0 < r \leq \text{diam} \mathcal{I}$ the following holds:
\[
\mathcal{H}^2(B(x, r)) \approx r^2.
\]
Finally, for $x \in \mathcal{P}_j$ there are three possibilities:

1. If $r < l(s_j)$ then there is a point $y \in B(x, r)$ such that $B(y, r/2) \subset \mathcal{P}_j$ and therefore $\mathcal{H}^2(B(x, r) \geq r^2$. To get the upper estimate, first note that if $B(x, r) \cap s_j = \emptyset$ then $\mathcal{H}^2(B(x, r)) \leq \pi r^2$. On the other hand, if there exists $y \in B(x, r) \cap s_j$, then $B(x, r) \subset B(y, 2r)$ and therefore by (7.10) we have $\mathcal{H}^2(B(x, r)) \leq r^2$.

2. If $l(s_j) \leq r \leq 2l(s_j)$ then
\[
\mathcal{H}^2(B(x, r)) \geq \mathcal{H}^2(B(x, r/2)) \geq r^2,
\]
by part (1), since $r/2 < l(s_j)$. On the other hand, since $\mathcal{D}$ is easily seen to be a metric doubling space, every ball $B(x, r)$ can be covered by $N$ balls $B_i = B(x_i, r/2)$ of radius $r/2 < l(s_j)$, with $N$ independent of $x$. Therefore, $H^2(B(x, r)) \leq \sum_{i=1}^{N} H^2(B(x_i, r/2)) \lesssim r^2$ by part (1) again.

(3). If $r > 2l(s_j) > \text{diam}(\mathcal{P})$, then there is a point $y \in B(x, r) \cap s_j$ such that $B(y, r/2) \subset B(x, r) \subset B(y, 2r)$.

Therefore $H^2(B(x, r)) \asymp r^2$ by (7.10).

Combining Lemma 7.3 and Lemma 7.4 with Theorem 7.2 we obtain a quasisymmetric mapping $g : \mathcal{D} \to \mathbb{S}^2$. By [Hai15] $d_\mathcal{D}$ is comparable to the semi-metric $\vartheta$ (cf. Section 7.2) when restricted to $\mathcal{S} \subset \mathcal{D}$. Since $\vartheta$ on $\mathcal{S}$ is equal to $d_\mathcal{S}$, it follows that $id : (\mathcal{S}, d_\mathcal{S}) \to (\mathcal{S}, d_\mathcal{D})$ is a bi-Lipschitz mapping. Therefore $f = g \circ id : \mathcal{S} \to \mathbb{S}^2$ is quasisymmetric. □

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