UNIFORM SUBCONVEX BOUNDS FOR RANKIN-SELBERG
L-FUNCTIONS

QINGFENG SUN

Abstract. Let \( f \) be a Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \) with Laplace eigenvalue \( 1/4 + \mu_f^2 \), \( \mu_f > 0 \). Let \( g \) be an arbitrary but fixed holomorphic or Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \). In this paper, we establish the following uniform subconvexity bound for the Rankin-Selberg \( L \)-function \( L(s, f \otimes g) \)

\[
L\left(\frac{1}{2} + it, f \otimes g\right) \ll (\mu_f + |t|)^{9/10 + \varepsilon},
\]

where the implied constant depends only on \( \varepsilon \) and \( g \).

1. Introduction

The subconvexity problems of Rankin-Selberg \( L \)-functions in various aspects are of great interest and importance and have been intensively studied by many authors (see [KMV02], [Mic04], [HM05], [LLY], [MV10], [BR10], [HT14], [ASS20], [BJN21], [Nel21] and the references therein).

For the Rankin-Selberg \( L \)-function \( L(s, \pi_1 \otimes \pi_2) \) associated to two irreducible cuspidal automorphic representations \( \pi_1 \) and \( \pi_2 \) of \( \text{GL}_2 \) with analytic conductor \( Q(s, \pi_1 \otimes \pi_2) \), the subconvexity problem of \( L(s, \pi_1 \otimes \pi_2) \) is aimed at obtaining estimates of the form

\[
L(s, \pi_1 \otimes \pi_2) \ll Q(s, \pi_1 \otimes \pi_2)^{1/4 - \delta}
\]

for some \( \delta > 0 \) when \( \text{Re}(s) = 1/2 \), while the estimate

\[
L(s, \pi_1 \otimes \pi_2) \ll Q(s, \pi_1 \otimes \pi_2)^{1/4 + \varepsilon}
\]

with \( \varepsilon > 0 \) arbitrarily small, which follows from the functional equation and the Phragmén-Lindelöf principle, is referred to as the convexity bound.

Let \( f \) be a Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \) with normalized Fourier coefficients \( \lambda_f(n) \) and Laplace eigenvalue \( 1/4 + \mu_f^2 \), \( \mu_f > 0 \). Let \( g \) be an arbitrary but fixed holomorphic or Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \) with normalized Fourier coefficients \( \lambda_g(n) \). We consider the Rankin-Selberg \( L \)-function

\[
L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s},
\]

where \( \text{Re}(s) > 1 \) and \( \zeta(s) \) is the Riemann zeta-function. In this paper, we are concerned with uniform subconvex estimates for the Rankin-Selberg \( L \)-function \( L(s, f \otimes g) \) in the \( t \) and \( \mu_f \) aspects. In this case, the analytic conductor is \( (\mu_f + |t| + 1)^2(\mu_f - |t| + 1)^2 \) and the convexity bound is \( O((\mu_f + |t| + 1)^{1/2 + \varepsilon}(\mu_f - |t| + 1)^{1/2}) \). The Lindelöf hypothesis asserts that

\[
L\left(\frac{1}{2} + it, f \otimes g\right) \ll_{g, \varepsilon} (\mu_f + |t| + 1)^{\varepsilon}
\]
which is still out of reach at the present. In 2006, Jutila and Motohashi [JM06] proved that
\[
L(1/2 + it, f \otimes g) \ll_{g, \varepsilon} \begin{cases} 
\mu_f^{2/3 + \varepsilon}, & \text{for } 0 \leq t \ll \mu_f^{2/3}, \\
\mu_f^{1/2 + 1/4 + \varepsilon}, & \text{for } \mu_f^{2/3} \leq t \ll t_f, \\
t^{3/4 + \varepsilon}, & \text{for } \mu_f \ll t \ll \mu_f^{3/2 - \varepsilon},
\end{cases}
\] (1.1)
which, however, does not cover all cases of \( t \) and \( \mu_f \).

Our main result states as follows.

**Theorem 1.1.** We have
\[
L(1/2 + it, f \otimes g) \ll (t + \mu_f)^{9/10 + \varepsilon},
\]
where the implied constant depends only on \( \varepsilon \) and \( g \).

Uniform subconvexity estimates in the \( t \) and spectral aspect for \( L \)-values have proven to be very difficult to establish through current methods although there have been a few results. For the \( \mathrm{GL}_2 \) \( L \)-function \( L(s, f) \), Jutila and Motohashi [JM05] established by the moment method the uniform subconvexity bound
\[
L(1/2 + it, f) \ll \varepsilon (\mu_f + |t|)^{1/3 + \varepsilon},
\]
and then extended their result to \( L(1/2 + it, f \otimes g) \) (see (1.1)). Recently, Huang [Hua21b] used the method of Munshi [Mun15] to study the \( \mathrm{GL}_3 \times \mathrm{GL}_2 \) case and proved that
\[
L(1/2 + it, \pi \otimes f) \ll_{\pi, \varepsilon} (\mu_f + |t|)^{27/20 + \varepsilon},
\]
where \( f \) is as before and \( \pi \) is a Hecke-Maass cusp form for \( \mathrm{SL}_3(\mathbb{Z}) \). It is worth noting that in terms of \( t \)-aspect alone the best record bound for \( L(1/2 + it, f \otimes g) \) is the Weyl type bound
\[
L(1/2 + it, f \otimes g) \ll (1 + |t|)^{2/3 + \varepsilon}
\]
due to Blomer, Jana and Nelson [BJN21] by combining in a substantial way representation theory, local harmonic analysis, and analytic number theory.

The paper is organized as follows. In Section 2, we provide a quick sketch and key steps of the proof. In Section 3, we review some basic materials of automorphic forms on \( \mathrm{GL}_2 \) and estimates on exponential integrals. Sections 4 and 5 give details of the proof for Theorem 1.1.

**Notation.** Throughout the paper, the letters \( \varepsilon \) and \( A \) denote arbitrarily small and large positive constants, respectively, not necessarily the same at each occurrence. Implied constants may depend on \( \varepsilon \) as well as on \( g \). The letters \( q, m \) and \( n \), with or without subscript, denote integers. We use \( A \asymp B \) to mean that \( c_1B \leq |A| \leq c_2B \) for some positive constants \( c_1 \) and \( c_2 \) and the symbol \( q \sim C \) means \( C < q \leq 2C \).

2. OUTLINE OF THE PROOF

In this section, we provide a quick sketch of the proof for Theorem 1.1. For simplicity, we assume \( t > 0 \) and \( t + \mu_f \asymp t - \mu_f \asymp T \). By the approximate functional equation, we have
\[
L\left(\frac{1}{2} + it, f \otimes g\right) \ll T^{-1+\varepsilon} S, 
\] (2.1)
where
\[
S = \sum_{n \sim T^2} \lambda_f(n)\lambda_g(n)n^{-it}.
\]
The first step is writing
\[ S = \sum_{n \sim T^2} \lambda_g(n) \sum_{m \sim T^2} \lambda_f(m) m^{-it} \delta(m - n), \]
and using the \( \delta \)-method to detect the Kronecker delta symbol \( \delta(m - n) \). As in [LS], we use the Duke-Friedlander-Iwaniec's \( \delta \)-method (4.5) to write
\[ S = \frac{1}{Q} \sum_{q \sim Q} \frac{1}{q} \int_{T \epsilon}^{-T \epsilon} e\left(-\frac{na}{q}\right) e\left(-\frac{n\zeta}{qQ}\right) \sum_{m \sim T^2} \lambda_f(m) e\left(\frac{ma}{q}\right) m^{-it} e\left(\frac{m\zeta}{qQ}\right) \text{d}\zeta, \] (2.2)
where the \( * \) in the sum over \( a \) means that the sum is restricted to \((a, q) = 1\).

Next we use the Voronoi summation formulas to dualize the \( m \) and \( n \) sums. The \( n \)-sum can be transformed into the following
\[ \leftrightarrow T \sum_{\pm \sim n \sim T^2/Q^2} \lambda_g(n) e\left(\frac{n\bar{a}}{q}\right) \Phi^\pm(n, q, \zeta), \] (2.3)
where
\[ \Phi^\pm(n, q, \zeta) = \int_{x \approx 1} x^{-1/4} e\left(-\frac{\zeta T^2 x}{q Q} \pm 2\sqrt{mn} q \frac{T^2}{q}ight) \text{d}x. \]

Repeated integration by parts shows that \( \Phi^-(n, q, \zeta) \ll T^{-A} \) and a stationary phase analysis shows that (note that \( n \sim T^2/Q^2 \))
\[ \Phi^+(n, q, \zeta) \asymp \frac{q^{1/2}}{(nT^2)^{1/4}} e\left(\frac{nQ}{q \zeta}\right) U^2\left(\frac{nQ^2}{X\zeta^2}\right) \asymp \frac{Q}{T} e\left(\frac{nQ}{q \zeta}\right) U^2\left(\frac{nQ^2}{T^2\zeta^2}\right). \] (2.4)

Plugging (2.3) and (2.4) into (2.2), we are led to the sum
\[ S^* = \sum_{q \sim Q} \frac{1}{q} \sum_{\pm \sim n \sim T^2/Q^2} \lambda_g(n) e\left(\frac{n\bar{a}}{q}\right) \sum_{m \sim T^2} \lambda_f(m) e\left(\frac{ma}{q}\right) m^{-it} \mathcal{K}(m, n, q), \]
where
\[ \mathcal{K}(m, n, q) = \int_{-T \epsilon}^{T \epsilon} U^2\left(\frac{nQ^2}{N\zeta^2}\right) e\left(\frac{nQ}{q \zeta} + \frac{m\zeta}{qQ}\right) \text{d}\zeta. \]

We evaluate the integral \( \mathcal{K}(m, n, q) \) using the stationary phase method and get
\[ \mathcal{K}(y; n, q) \asymp \frac{n^{1/4} q^{1/2} Q}{T^{3/2}} e\left(\frac{2\sqrt{mn}}{q}\right) F\left(\frac{m}{T^2}\right) \asymp \frac{Q}{T} e\left(\frac{2\sqrt{mn}}{q}\right) F\left(\frac{m}{T^2}\right). \]
for some smooth compactly supported function $F(y)$. Thus
\[
S^* = \frac{Q}{T^2} \sum_{q \sim Q} \frac{1}{q} \sum_{a \mod q} \sum_{n \sim T^2/q^2} \lambda_g(n) e \left( \frac{n\bar{a}}{q} \right) \\
\times \sum_{m \sim T^2} \lambda_f(m) e \left( \frac{ma}{q} \right) m^{-it} F \left( \frac{m}{T^2} \right) e \left( \frac{2\sqrt{mn}}{q} \right). \tag{2.5}
\]
Applying the Voronoi formula to the sum over $m$, we have
\[
\sum_{m \sim T^2} \lambda_f(m) e \left( \frac{ma}{q} \right) m^{-it} F \left( \frac{m}{T^2} \right) e \left( \frac{2\sqrt{mn}}{q} \right) \\
\leftrightarrow q \sum_{\pm m \sim Q^2} \lambda_f(m) e \left( \pm \frac{\pi m}{q} \right) \Psi^\pm \left( \frac{m}{q^2}, n, q \right), \tag{2.6}
\]
where
\[
\Psi^\pm (x, n, q) \asymp x^{-it} (T^2 x)^{1/2} e \left( -\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e} + \frac{2\tau_0 n^{1/2} T}{q} \right).
\]
with
\[
\tau_0 := \tau_0(m) = (T_1|T_2|/(4T^2 m))^{1/2} q < 1.
\]
Then by plugging the dual sum (2.6) back into (2.5) and writing the Ramanujan sum $S(m-n,0;q)$ as $\sum_{d|\gcd(m-n,q)} d\mu(q/d)$, we roughly get
\[
S^* \approx Q e \left( -\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e} \right) \sum_{q \sim Q} \frac{1}{q^{1+2it}} \sum_{d|q} d\mu \left( \frac{q}{d} \right) \\
\times \sum_{m \sim Q^2} \frac{\lambda_f(m)}{m^{1/2-it}} \sum_{n \sim T^2/q^2} \lambda_g(n) e \left( 2\tau_0 n^{1/2} T/q \right). \tag{2.7}
\]
To prepare for an application of the Poisson summation in the $m$-variable, we now apply the Cauchy-Schwarz inequality to smooth the $m$-sum and put the $n$-sum inside the absolute value squared to get
\[
S^* \ll \frac{1}{Q} \sum_{q \sim Q} \sum_{d|q} d \left( \sum_{m \sim Q^2} |\lambda_f(m)|^2 \right)^{1/2} \left( \sum_{m \sim Q^2} \sum_{n \sim T^2/q^2} \lambda_g(n) e \left( 2\tau_0 n^{1/2} T/q \right) \right)^{1/2} \\
\ll \sum_{q \sim Q} \sum_{d|q} d \left( \sum_{m \sim Q^2} \sum_{n \sim T^2/q^2} \lambda_g(n) \mathcal{f}^* (m,n,q) \right)^{1/2}.
\]
where
\[
\mathcal{f}^* (m,n,q) \asymp e \left( 2\tau_0 n^{1/2} T/q \right).
\]
Remark 1. If we open the absolute value squared, by the Rankin-Selberg estimate for \( \lambda_f(n) \) and the trivial estimate \( \mathcal{f}^* (m,n,q) \ll 1 \), the contribution from the diagonal term $n = n'$ is
given by
\[
S_{\text{diag}} \ll \sum_{q \sim Q} \sum_{d | q} \left( \sum_{n \sim Q^2} \sum_{n \equiv \pm m \mod d} |\lambda_g(n)|^2 |\mathcal{F}^* (m, n, q)|^2 \right)^{1/2}
\]
\[
\ll Q^{3/2} T,
\]
which will be fine for our purpose (i.e., \(S_{\text{diag}} = o(T^2)\)) as long as \(Q \ll T^{2/3}\).

Recall \(\tau_0 = (T_1T_2/(4Nm))^{1/2} \approx 1\). Note that the oscillation in the \(n\)-variable is of size \(2\tau_0 n T/q \approx T^2/Q^2\). So opening the absolute value squared and applying the Poisson summation formula in the \(m\)-variable, we have
\[
\sum_{m \sim Q^2} \mathcal{F}^* (m, n, q) \mathcal{F}^* (m, n', q) \leftrightarrow \frac{Q^2}{d} \sum_{\tilde{m} \ll \frac{2T^2}{Q^2}} \mathcal{H} \left( \frac{\tilde{m} Q^2}{d} \right),
\]
where
\[
\mathcal{H}(x) = \int_{\mathbb{R}} \mathcal{F}^* (Q^2 \xi, n, q) \mathcal{F}^* (Q^2 \xi, n', q) e (-x \xi) \, d\xi.
\]
(2.9)

The contribution to \(S^*\) from the zero-frequency \(\tilde{m} = 0\) will roughly correspond to the diagonal contribution \(S_{\text{diag}}\) in \(2.8\). For the non-zero frequencies from the terms with \(\tilde{m} \neq 0\), we note that by performing stationary phase analysis, when \(|x|\) is “large”, the expected estimate for the triple integral \(\mathcal{H}(x)\) in \(2.9\) is
\[
\mathcal{H}(x) \ll |x|^{-1/2},
\]
(2.10)
which comes from the square-root cancellation of the two inner integrals and the square-root cancellation in the \(\xi\)-variable. Note that this estimate does not hold for “small” \(|x|\). In fact, for these exceptional cases the “trivial” bound \(\mathcal{H}(x) \ll 1\) will suffice for our purpose. (These are the content of Lemma 4.2). We ignore these exceptions and plug the expected estimate \(2.10\) for \(\mathcal{H}(x)\) into \(S\). It turns out that the non-zero frequencies contribution \(S_{\text{off-diag}}\) from \(\tilde{m} \neq 0\) to \(S\) is given by
\[
S_{\text{off-diag}} \ll \sum_{q \sim Q} \sum_{d | q} \left( \sum_{n \sim T^2/Q^2} |\lambda_g(n)|^2 \sum_{n' \sim T^2/Q^2} \sum_{n' \equiv \pm n \mod d} \frac{Q^2}{d} \sum_{0 \neq \tilde{m} \ll dT^2/Q^4} \frac{d^{1/2}}{|	ilde{m}|^{1/2} Q} \right)^{1/2}
\]
\[
\ll T^{5/2}/Q + T^{3/2} Q^{1/2} \ll T^{5/2}/Q
\]
provided that \(Q < T^{2/3}\). Hence combining this with the diagonal contribution \(S_{\text{diag}}\) in \(2.8\), we get
\[
S \ll Q^{3/2} T + T^{5/2}/Q.
\]
Plugging this estimate into \(2.1\), one has
\[
L \left( \frac{1}{2} + it, f \otimes g \right) \ll T^{\varepsilon} \left( Q^{3/2} + T^{3/2}/Q \right)
\]
By choosing \(Q = T^{3/5}\) we conclude that
\[
L \left( \frac{1}{2} + it, f \otimes g \right) \ll T^{9/10 + \varepsilon}.
\]
3. Preliminaries

First we recall some basic results on automorphic forms for \( GL_2 \).

3.1. Holomorphic cusp forms for \( GL_2 \). Let \( f \) be a holomorphic cusp form of weight \( \kappa \) for \( SL_2(\mathbb{Z}) \) with Fourier expansion

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(\kappa-1)/2} e(nz)
\]

for \( \text{Im} \, z > 0 \), normalized such that \( \lambda_f(1) = 1 \). By the Ramanujan-Petersson conjecture proved by Deligne [Del74], we have \( \lambda_f(n) \ll \tau(n) \ll n^\varepsilon \) with \( \tau(n) \) being the divisor function.

For \( h(x) \in C_c(0, \infty) \), we set

\[
\Phi_h(x) = 2\pi i^\kappa \int_{0}^{\infty} h(y) J_{\kappa-1}(4\pi \sqrt{xy}) dy,
\]

where \( J_{\kappa-1} \) is the usual \( J \)-Bessel function of order \( \kappa - 1 \). We have the following Voronoi summation formula (see [KMV02, Theorem A.4]).

**Lemma 3.1.** Let \( q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) be such that \( (a, q) = 1 \). For \( X > 0 \), we have

\[
\sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{an}{q} \right) h \left( \frac{n}{N} \right) = N \sum_{n=1}^{\infty} \lambda_f(n) e \left( -\frac{n}{q} \right) \Phi_h \left( \frac{n N}{q^2} \right),
\]

where \( \overline{a} \) denotes the multiplicative inverse of \( a \) modulo \( q \).

The function \( \Phi_h(x) \) has the following asymptotic expansion when \( x \gg 1 \) (see [LS], Lemma 3.2).

**Lemma 3.2.** For any fixed integer \( J \geq 1 \) and \( x \gg 1 \), we have

\[
\Phi_h(x) = x^{-1/4} \int_{0}^{\infty} h(y) y^{-1/4} \sum_{j=0}^{J} c_j e \left( 2 \sqrt{xy} \right) + d_j e \left( -2 \sqrt{xy} \right) \left( \frac{x}{y} \right)^{j/2} dy + O_{\kappa,J} \left( x^{-J/2-3/4} \right),
\]

where \( c_j \) and \( d_j \) are constants depending on \( \kappa \).

3.2. Maass cusp forms for \( GL_2 \). Let \( f \) be a Hecke-Maass cusp form for \( SL_2(\mathbb{Z}) \) with Laplace eigenvalue \( 1/4 + \mu_f^2 \). Then \( f \) has a Fourier expansion

\[
f(z) = \sqrt{\pi} \sum_{n \neq 0} \lambda_f(n) K_{i\mu_f}(2\pi |n|y) e(nz),
\]

where \( K_{i\mu} \) is the modified Bessel function of the third kind. The Fourier coefficients satisfy

\[
\lambda_f(n) \ll n^\vartheta \tag{3.2}
\]

where, here and throughout the paper, \( \vartheta \) denotes the exponent towards the Ramanujan conjecture for \( GL_2 \) Maass forms. The Ramanujan conjecture states that \( \vartheta = 0 \) and the current record due to Kim and Sarnak [KS03] is \( \vartheta = 7/64 \). On average we have the following Rankin-Selberg estimate (see Proposition 19.6 in [DF102])

\[
\sum_{n \leq x} |\lambda_f(n)|^2 \ll \varepsilon x(x|\mu_f|)^\varepsilon \tag{3.3}
\]
For $h(x) \in C_c^\infty(0, \infty)$, we define the integral transforms

$$
\Phi_h^+(x) = \frac{-\pi}{\sin(\pi i \mu_f)} \int_0^\infty h(y) \left( J_{2i \mu_f}(4\pi \sqrt{xy}) - J_{-2i \mu_f}(4\pi \sqrt{xy}) \right) dy,
$$

$$
\Phi_h^-(x) = 4\varepsilon_f \cosh(\pi \mu_f) \int_0^\infty h(y) K_{2i \mu_f}(4\pi \sqrt{xy}) dy,
$$

where $\varepsilon_f$ is an eigenvalue under the reflection operator. We have the following Voronoi summation formula (see [KMV02, Theorem A.4]).

**Lemma 3.3.** Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ be such that $(a, q) = 1$. For $X > 0$, we have

$$
\sum_{n=1}^\infty \lambda_f(n) e \left( \frac{an}{q} \right) h \left( \frac{n}{N} \right) = \frac{N}{q} \sum_{n=1}^\infty \sum_{\pm} \lambda_f(n) e \left( \mp \frac{\overline{an}}{q} \right) \Phi_h^\pm \left( \frac{nN}{q^2} \right),
$$

where $\overline{a}$ denotes the multiplicative inverse of $a$ modulo $q$.

For $x \gg 1$, we have (see (3.8) in [LS])

$$
\Phi_h^-(x) \ll_{\mu,A} x^{-A}.
$$

For $\Phi_h^+(x)$ and $x \gg 1$, we have a similar asymptotic formula as for $\Phi_h(x)$ in the holomorphic case (see [LS], Lemma 3.4).

**Lemma 3.4.** For any fixed integer $J \geq 1$ and $x \gg 1$, we have

$$
\Phi_h^+(x) = x^{-1/4} \int_0^\infty h(y) y^{-1/4} \sum_{j=0}^J c_j e(2\sqrt{xy}) + d_j e(-2\sqrt{xy}) \frac{d}{(xy)^{j/2}} dy + O_{\mu,J} \left( x^{-J/2-3/4} \right),
$$

where $c_j$ and $d_j$ are some constants depending on $\mu$.

**Remark 2.** For $x \gg X^\varepsilon$, we can choose $J$ sufficiently large so that the contribution from the $O$-terms in Lemmas 3.2 and 3.3 is negligible. For the main terms we only need to analyze the leading term $j = 1$, as the analysis of the remaining lower order terms is the same and their contribution is smaller compared to that of the leading term.

To deal with both $t$ and $\mu_f$ aspects, it is more convenient to use the Voronoi formula of following form (see [MS06], Eqs. (1.12),(1.15)).

**Lemma 3.5.** Let $\varphi(x) \in C_c^\infty(0, \infty)$. Let $a, \overline{a}, q \in \mathbb{Z}$ with $q \neq 0$, $(a, q) = 1$ and $a\overline{a} \equiv 1$ (mod $q$). Then

$$
\sum_{n \geq 1} \lambda_f(n) e \left( \frac{an}{q} \right) \varphi(n) = q \sum_{n \geq 1} \frac{\lambda_f(n)}{n} e \left( \pm \frac{\overline{an}}{q} \right) \Psi_{\varphi}^\pm \left( \frac{n}{q^2} \right),
$$

where for $\sigma > -1$,

$$
\Psi_{\varphi}^\pm(x) = \frac{1}{4\pi^2 i} \int_0^\infty \left( \pi^2 x \right)^{-\sigma} \gamma_f^\pm(s) \overline{\varphi}(-s) ds,
$$

with

$$
\gamma_f^\pm(s) = \prod_{\pm} \frac{\Gamma \left( \frac{1+s+i\mu_f}{2} \right)}{\Gamma \left( \frac{-s+i\mu_f}{2} \right)} \pm \prod_{\pm} \frac{\Gamma \left( \frac{2+s+i\mu_f}{2} \right)}{\Gamma \left( \frac{1-s+i\mu_f}{2} \right)}.
$$

Here $\overline{\varphi}(s) = \int_0^\infty \varphi(u) u^s \overline{du}$ is the Mellin transform of $\varphi$. 
By Stirling asymptotic formula (see [Olv97], Section 8.4, in particular (4.03)), for \( |\arg s| \leq \pi - \varepsilon \) for any \( \varepsilon > 0 \) and \( |s| \gg 1 \),
\[
\ln \Gamma(s) = \left(s - \frac{1}{2}\right) \ln s - s + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{K_1} \frac{B_{2j}}{2j(2j-1)s^{2j-1}} + O_{K_1, \varepsilon} \left(\frac{1}{|s|^{2K_1+1}}\right),
\]
where \( B_j \) are Bernoulli numbers. Thus for \( s = \sigma + it \), \( \sigma \) fixed and \( |\tau| \geq 2 \),
\[
\Gamma(\sigma + i\tau) = \sqrt{2\pi}(i\tau)^{\sigma-1/2}e^{-\pi|\tau|/2} \left(\frac{|\tau|}{e}\right)^i\tau \left(1 + \sum_{j=1}^{K_2} \frac{c_j}{\tau^j} + O_{\sigma,K_2,\varepsilon} \left(\frac{1}{|\tau|^{K_2+1}}\right)\right),
\]
where the constants \( c_j \) depend on \( j, \sigma \) and \( \varepsilon \). Thus for \( \sigma \geq -1/2 \),
\[
\gamma_j^\pm(\sigma + i\tau) = \prod_{\pm} \frac{\Gamma\left(\frac{1+\sigma+i(\tau \pm \mu_j)}{2}\right)}{\Gamma\left(\frac{-\sigma-i(\tau \pm \mu_j)}{2}\right)} \pm \prod_{\pm} \frac{\Gamma\left(\frac{2+\sigma+i(\tau \pm \mu_j)}{2}\right)}{\Gamma\left(\frac{-\sigma-i(\tau \pm \mu_j)}{2}\right)} \ll (|\tau + \mu_f||\tau - \mu_f|)^{\sigma+1/2}.
\]

3.3. Rankin-Selberg L-function. Let \( f \) be a Maass cusp form for \( SL_2(\mathbb{Z}) \) with Laplace eigenvalue \( 1/4 + \mu_f^2 \), \( \mu_f > 0 \), and parity \( \delta_f = 0 \) or 1. Let \( g \) be either a holomorphic cusp form of weight \( 2\kappa \) or a Maass cusp form for \( SL_2(\mathbb{Z}) \) with Laplace eigenvalue \( 1/4 + \mu_g^2 \), \( \mu_g > 0 \), and parity \( \delta_g = 0 \) or 1. For \( \text{Re}(s) > 1 \), the Rankin-Selberg L-function is defined as
\[
L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s},
\]
which can be meromorphically continued to the whole complex plane except for a simple pole at \( s = 1 \) if \( g = f \) and satisfies the functional equation (see [IK04], Section 5.11 and [JM06], Lemma 1)
\[
\Lambda(s, f \otimes g) = \Lambda(1 - s, f \otimes g),
\]
where
\[
\Lambda(s, f \otimes g) = \gamma(s, f \otimes g)L(s, f \otimes g)
\]
with
\[
\gamma(s, f \otimes g) = (2\pi)^{-2s}\Gamma\left(s + \frac{\kappa - 1}{2} + i\mu_f\right)\Gamma\left(s + \frac{\kappa - 1}{2} - i\mu_f\right) \text{ for } g \text{ holomorphic,}
\]
\[
\gamma(s, f \otimes g) = \pi^{-2s}\Gamma\left(s + \frac{\delta + i(\mu_f + \mu_g)}{2}\right)\Gamma\left(s + \frac{\delta + i(\mu_f - \mu_g)}{2}\right)
\times \Gamma\left(s + \frac{\delta - i(\mu_f + \mu_g)}{2}\right)\Gamma\left(s + \frac{\delta - i(\mu_f - \mu_g)}{2}\right) \text{ for } g \text{ a Maass form.}
\]
Here \( \delta = 0 \) or 1 according to whether \( \delta_f = \delta_g \) or not.
3.4. Estimates for exponential integrals. Let

\[ I = \int_{\mathbb{R}} w(y)e^{iy}dy. \]

Firstly, we have the following estimates for exponential integrals (see [BKY13 Lemma 8.1] and [AHLQ20 Lemma A.1]).

Lemma 3.6. Let \( w(x) \) be a smooth function supported on \([a, b]\) and \( g(x) \) be a real smooth function on \([a, b]\). Suppose that there are parameters \( Q, U, Y, Z, R > 0 \) such that

\[ g^{(i)}(x) \ll_i Y/Q^i, \quad w^{(j)}(x) \ll_j Z/U^j, \]

for \( i \geq 2 \) and \( j \geq 0 \), and

\[ |g'(x)| \geq R. \]

Then for any \( A \geq 0 \) we have

\[ I \ll_A (b - a)Z\left(\frac{Y}{R^2Q^2} + \frac{1}{RQ} + \frac{1}{RU}\right)^A. \]

Next, we need the following evaluation for exponential integrals which are Lemma 8.1 and Proposition 8.2 of [BKY13] in the language of inert functions (see [KPY19 Lemma 3.1]).

Let \( \mathcal{F} \) be an index set, \( Y: \mathcal{F} \to \mathbb{R}_{\geq 1} \) and under this map \( T \mapsto Y_T \) be a function of \( T \in \mathcal{F} \). A family \( \{w_T\}_{T \in \mathcal{F}} \) of smooth functions supported on a product of dyadic intervals in \( \mathbb{R}^d_{> 0} \) is called \( Y \)-inert if for each \( j = (j_1, \ldots, j_d) \in \mathbb{Z}^d_{\geq 0} \) we have

\[ C(j_1, \ldots, j_d) = \sup_{T \in \mathcal{F}} \sup_{(y_1, \ldots, y_d) \in \mathbb{R}^d_{> 0}} Y_T^{-j_1-\cdots-j_d}\left|y_1^{j_1} \cdots y_d^{j_d}w_T^{(j_1, \ldots, j_d)}(y_1, \ldots, y_d)\right| < \infty. \]

Lemma 3.7. Suppose that \( w = w_T(y) \) is a family of \( Y \)-inert functions, with compact support on \([Z, 2Z]\), so that \( w^{(j)}(y) \ll (Z/X)^{-j} \). Also suppose that \( g \) is smooth and satisfies \( g^{(j)}(y) \ll Y/Z^j \) for some \( H/X^2 \geq R \geq 1 \) and all \( y \) in the support of \( w \).

1. If \( |g'(y)| \gg Y/Z \) for all \( y \) in the support of \( w \), then \( I \ll_A ZR^{-A} \) for \( A \) arbitrarily large.
2. If \( g''(y) \gg Y/Z^2 \) for all \( y \) in the support of \( w \), and there exists \( y_0 \in \mathbb{R} \) such that \( g'(y_0) = 0 \) (note \( y_0 \) is necessarily unique), then

\[ I = \frac{e^{iy_0}}{\sqrt{g''(y_0)}}F(y_0) + O(AZR^{-A}), \tag{3.11} \]

where \( F(y_0) \) is an \( Y \)-inert function (depending on \( A \)) supported on \( y_0 \ll Z \).

We also need the second derivative test (see [Hux96 Lemma 5.1.3]).

Lemma 3.8. Let \( g(x) \) be real and twice differentiable on the open interval \([a, b]\) with \( g''(x) \gg \lambda_0 > 0 \) on \([a, b]\). Let \( w(x) \) be real on \([a, b]\) and let \( V_0 \) be its total variation on \([a, b]\) plus the maximum modulus of \( w(x) \) on \([a, b]\). Then

\[ I \ll \frac{V_0}{\sqrt{\lambda_0}}. \]
4. Proof of the main theorem

4.1. Reduction. Without loss of generality, we assume $t > 0$. In view of (3.10), by the approximate functional equation (see [IK04], Theorem 5.3, Proposition 5.4), we have
\[
L \left( \frac{1}{2} + it, f \otimes g \right) \ll T_1^\varepsilon \sup_{N \ll T_1^{1+\varepsilon}T_2} \frac{1}{\sqrt{N}} \left| \sum_{n \geq 1} \lambda_f(n) \lambda_g(n)n^{-it}V \left( \frac{n}{N} \right) \right| + 1,
\]
where $V \in C_\infty^\infty(1,2)$ satisfying $V^{(j)}(x) \ll_j 1$ for any integer $j \geq 0$, and
\[
T_1 = t + \mu_f, \quad T_2 = t - \mu_f.
\]
By Cauchy-Schwarz inequality and (3.3), we have
\[
\sup_{N \ll T_1^{9/5}} \frac{1}{\sqrt{N}} \left| \sum_{n \geq 1} \lambda_f(n) \lambda_g(n)n^{-it}V \left( \frac{n}{N} \right) \right| \ll \sup_{N \ll T_1^{9/5}} \frac{1}{\sqrt{N}} \left( \sum_{n \leq N} |\lambda_f(n)|^2 \right)^{1/2} \left( \sum_{n \leq N} |\lambda_g(n)|^2 \right)^{1/2} \ll \sup_{1 \ll N \ll T_1^{9/5}} N^{1/2} \ll T_1^{9/10}.
\]
It follows that
\[
L \left( \frac{1}{2} + it, f \otimes g \right) \ll T_1^\varepsilon \sup_{T_1^{9/5} \ll T_2 \ll T_1^{1+\varepsilon}} \frac{1}{\sqrt{N}} |S(N)| + T_1^{9/10},
\]
where
\[
S(N) = \sum_{n \geq 1} \lambda_f(n) \lambda_g(n)n^{-it}V \left( \frac{n}{N} \right).
\]
Note that the first term above is vanishing unless
\[
|T_2| \gg T_1^{4/5-\varepsilon},
\]
which we shall henceforth assume. Note that the trivial upper bound of $S(N)$ is $O(N)$ by Cauchy-Schwarz inequality and (3.3). In the rest of the paper, we are devoted to proving a nontrivial estimate for $S(N)$.

4.2. Duke-Friedlander-Iwaniec $\delta$-method. Let
\[
\delta(n) = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{otherwise.}
\end{cases}
\]
The $\delta$-method of Duke, Friedlander and Iwaniec (see [IK04], Chapter 20) states that for any $n \in \mathbb{Z}$ and $Q \in \mathbb{R}^+$,
\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \text{ mod } q} e \left( \frac{na}{q} \right) \int_{\mathbb{R}} g(q, \zeta) e \left( \frac{n\zeta}{qQ} \right) d\zeta,
\]
where the $\star$ on the sum indicates that the sum over $a$ is restricted to $(a, q) = 1$. The function $g$ has the following properties (see (20.158) and (20.159) of [IK04] and Lemma 15 of [Hua21])
\[
g(q, \zeta) \ll |\zeta|^{-A}, \quad g(q, \zeta) = 1 + O \left( \frac{Q}{q} \left( \frac{q}{Q} + |\zeta| \right)^A \right)
\]
for any $A > 1$ and
\[
\frac{\partial^j}{\partial \zeta^j} g(q, \zeta) \ll |\zeta|^{-j} \min \left( |\zeta|^{-1}, \frac{Q}{q} \right) \log Q, \quad j \geq 1. \tag{4.7}
\]

We write (4.3) as
\[
S(N) = \sum_{n \geq 1} \lambda_g(n) U \left( \frac{n}{N} \right) \sum_{m \geq 1} \lambda_f(m) m^{-it} V \left( \frac{m}{N} \right) \delta(m - n),
\]
where $U(x) \in C^\infty_c(1/2, 5/2)$ satisfying $U(x) = 1$ for $x \in [1, 2]$ and $U^{(j)}(x) \ll_j 1$ for any integer $j \geq 0$. Plugging the identity (4.5) for $q \sim C$ with $1 \ll C \ll Q$ and write $S(N)$ as
\[
S(N) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \int_{\mathbb{R}} g(q, \zeta) W \left( \frac{\zeta}{\Xi} \right) \sum_{a \mod q} \sum_{n \geq 1}^* \lambda_g(n) e \left( -\frac{na}{q} \right) U \left( \frac{n}{N} \right) e \left( -\frac{n\zeta}{qQ} \right)
\]
\[
\sum_{m \geq 1} \lambda_f(m) e \left( \frac{ma}{q} \right) m^{-it} V \left( \frac{m}{N} \right) e \left( \frac{m\zeta}{qQ} \right) d\zeta.
\]

Note that the contribution from $|\zeta| \leq N^{-G}$ is negligible for $G > 0$ sufficiently large. Moreover, by the first property in (4.6), we can restrict $\zeta$ in the range $|\zeta| \leq N^\varepsilon$ up to an negligible error. So we can insert a smooth partition of unity for the $\zeta$-integral and write $S(N)$ as
\[
\sum_{N^{-G} \ll \Xi \ll N^\varepsilon \text{ dyadic}} \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \int_{\mathbb{R}} g(q, \zeta) W \left( \frac{\zeta}{\Xi} \right) \sum_{a \mod q} \sum_{n \geq 1}^* \lambda_g(n) e \left( -\frac{na}{q} \right) U \left( \frac{n}{N} \right) e \left( -\frac{n\zeta}{qQ} \right)
\]
\[
\times \sum_{m \geq 1} \lambda_f(m) e \left( \frac{ma}{q} \right) m^{-it} V \left( \frac{m}{N} \right) e \left( \frac{m\zeta}{qQ} \right) d\zeta + O_A(N^{-A}),
\]
where $W(x) \in C^\infty_c(1, 2)$ satisfying $W^{(j)}(x) \ll_j 1$ for any integer $j \geq 0$. Without loss of generality, we only consider the contribution from $\zeta > 0$ (the proof for $\zeta < 0$ is entirely similar). By abuse of notation, we still write the contribution from $\zeta > 0$ as $S(N)$.

Next we break the $q$-sum $\sum_{1 \leq q \leq Q}$ into dyadic segments $q \sim C$ with $1 \ll C \ll Q$ and write
\[
S(N) = \sum_{N^{-G} \ll \Xi \ll N^\varepsilon \text{ dyadic}} \sum_{1 \ll C \ll Q \text{ dyadic}} \mathscr{S}(C, \Xi) + O_A(N^{-A}), \tag{4.8}
\]
where $\mathscr{S}(C, \Xi) = \mathscr{S}(N, C, \Xi)$ is
\[
\mathscr{S}(C, \Xi) = \frac{1}{Q} \sum_{q \sim C} \frac{1}{q} \int_{\mathbb{R}} g(q, \zeta) \chi \left( \frac{\zeta}{\Xi} \right) \sum_{a \mod q} \sum_{n \geq 1}^* \lambda_g(n) e \left( -\frac{na}{q} \right) U \left( \frac{n}{N} \right) e \left( -\frac{n\zeta}{qQ} \right)
\]
\[
\sum_{m \geq 1} \lambda_f(m) e \left( \frac{ma}{q} \right) m^{-it} V \left( \frac{m}{N} \right) e \left( \frac{m\zeta}{qQ} \right) d\zeta. \tag{4.9}
\]

4.3. Applying Voronoi summation formulas. In this subsection, we shall apply Voronoi summation formulas to the $n$- and $m$-sums in (4.9). We first consider the sum over $n$. Depending
on whether \( g \) is holomorphic or Maass, we apply Lemma 3.1 or Lemma 3.3 respectively with \( h(x) = U(x)e(-\zeta N x/(qQ)) \), to transform the \( n \)-sum in (4.9) into

\[
\frac{N}{q} \sum_{\pm} \sum_{n \geq 1} \lambda_g(n) e \left( \pm \frac{n\pi}{q} \right) \Phi_{h}^\pm \left( \frac{nN}{q^2} \right),
\]

where if \( g \) is holomorphic, \( \Phi_{h}^+(x) = \Phi_{h}(x) \) with \( \Phi_{h}(x) \) given by (3.1) and \( \Phi_{h}^-(x) = 0 \), while for \( g \) a Hecke–Maass cusp form, \( \Phi_{h}^\pm(x) \) are given by (3.4).

Assume

\[
Q < N^{1/2-\varepsilon}.
\]

Then we have \( nN/q^2 \gg N^\varepsilon \). In particular, by (3.3), the contribution from \( \Phi_{h}^-(nN/q^2) \) is \( O_A(N^{-A}) \). For \( \Phi_{h}^+(nN/q^2) \) we apply Lemma 3.2, Lemma 3.4 and Remark 2 and find that evaluating of the sum (4.10) is reduced to dealing with the sum

\[
\frac{N^{3/4}}{q^{1/2}} \sum_{\pm} \sum_{n \geq 1} \lambda_g(n) e \left( \frac{n\pi}{q} \right) \Phi^\pm (n, q, \zeta),
\]

where

\[
\Phi^\pm (n, q, \zeta) = \int_0^\infty U(x)x^{-1/4} e \left( -\frac{\zeta N x}{qQ} \pm \frac{2\sqrt{nN x}}{q} \right) dx.
\]

Note that by (4.11), the first derivative of the phase function in \( \Phi^- \) is

\[
-\frac{\zeta N}{qQ} - \frac{\sqrt{nN x}}{q} \gg N^\varepsilon.
\]

By applying integration by parts repeatedly, one finds that the contribution from \( \Phi^- \) is negligible. Moreover, for \( \zeta \asymp \Xi, N \Xi/(CQ) \ll N^\varepsilon \), the first derivative of the phase function in \( \Phi^+ \) is

\[
-\frac{\zeta N}{qQ} + \frac{\sqrt{nN x}}{q} \gg N^\varepsilon
\]

which implies the contributions from these \( \zeta \) for \( \Phi^+ \) are also negligible. So in the following, we only need to consider \( \Phi^+(n, q, \zeta) \) with \( \zeta \) in the range

\[
\zeta \asymp \Xi, \quad N \Xi/(CQ) \gg N^\varepsilon.
\]

In this case, we apply a stationary phase analysis to \( \Phi^+ \). The stationary point \( x_0 \) is given by \( x_0 = nQ^2/(N\zeta^2) \). Applying Lemma 3.7 (2) with \( X = Z = 1 \) and \( Y = R = \sqrt{nN}/q \gg N^\varepsilon \), we obtain

\[
\Phi^+ (n, q, \zeta) = \frac{q^{1/2}}{(nN)^{1/4}} e \left( \frac{nQ}{q\zeta} \right) U^2 \left( \frac{nQ^2}{N\zeta^2} \right) + O_A \left( N^{-A} \right),
\]

where \( U^2 \) is an 1-inert function (depending on \( A \)) supported on \( x_0 \asymp 1 \). In particular, this implies, up to a negligible error, we only need to consider those \( n \) in the range \( n \asymp N\Xi^2/Q^2 \).
Plugging the above asymptotic formula for $\Phi^+(n,q,\zeta)$ and $(4.12)$ into $(4.9)$ and switching the order of integrations and summations, we are led to the sum

$$S^∗(C,\Xi) := \frac{N^{1/2}}{Q} \sum_{q \sim C} \sum_{a \mod q} \sum_{n \sim \sqrt{N} \atop n \neq q^2} \lambda_q(n) e\left(\frac{na}{q}\right)$$

$$\times \sum_{m \geq 1} \lambda_f(m) e\left(\frac{ma}{q}\right) m^{-it} V\left(\frac{m}{N}\right) K(m,n,q,\Xi),$$

(4.15)

where $K(m,n,q,\Xi)$ is given by

$$K(m,n,q,\Xi) = \int_\mathbb{R} g(q,\zeta) W\left(\frac{\zeta}{\Xi}\right) U^\sharp\left(\frac{nQ^2}{N\zeta^2}\right) e\left(\frac{nQ}{q\zeta} + \frac{m\zeta}{qQ}\right) d\zeta.$$

Next, we derive an asymptotic expansion for $G(m,n,q,\Xi)$. By making a change of variable $nQ^2/(N\zeta^2) \to \zeta$,

$$K(m,n,q,\Xi) = \frac{n^{1/2}Q}{N^{1/2}} \int_0^\infty \phi(\zeta) \exp(i\varpi(\zeta)) d\zeta,$$

where

$$\phi(\zeta) := -\frac{1}{2} \zeta^{-3/2} U^\sharp(\zeta) g\left(q,\frac{n^{1/2}Q}{\zeta^{1/2}N^{1/2}}\right) W\left(\frac{n^{1/2}Q}{\zeta^{1/2}N^{1/2}\Xi}\right)$$

and the phase function $\varpi(\zeta)$ is given by

$$\varpi(\zeta) = \frac{2\pi n^{1/2}N^{1/2}}{q}\left(\frac{m}{N}\zeta^{-1/2} + \zeta^{1/2}\right).$$

Note that

$$\varpi'(\zeta) = \frac{\pi n^{1/2}N^{1/2}}{q}\left(-\frac{m}{N}\zeta^{-3/2} + \zeta^{-1/2}\right),$$

and for $j \geq 2$,

$$\varpi^{(j)}(\zeta) = \left(-\frac{3}{2}\right) \cdots \left(\frac{1}{2} - j\right) \frac{\pi n^{1/2}N^{1/2}}{q}\left(-\frac{m}{N}\zeta^{-2-j} + \frac{1}{2j-1}\zeta^{1/2-j}\right).$$

Thus the stationary point is $\zeta_0 = m^{-1}N$ and $\varpi^{(j)}(\zeta) \ll_j n^{1/2}N^{1/2}/q$ for $j \geq 2$. By (4.7), we have $\phi^{(j)}(\zeta) \ll_j N^\epsilon$ (Here we note that for $C \leq Q^{1-\epsilon}$ and $\Xi \ll N^{-\epsilon}$, we can replace $g\left(q,\frac{n^{1/2}Q}{\zeta^{1/2}N^{1/2}}\right)$ by 1 at the cost of a negligible error). Applying Lemma 3.7 (2) with $X = Z = 1$ and $Y = R = n^{1/2}N^{1/2}/q \gg N^\epsilon$, we obtain

$$K(m,n,q,\Xi) = \frac{n^{1/4}q^{1/2}Q}{N^{3/4}} e\left(\frac{2\sqrt{mn}}{q}\right) F\left(\frac{m}{N}\right) + O_A\left(N^{-A}\right),$$

(4.16)

where $F(x) = F(x;\Xi)$ is an inert function (depending on $A$ and $\Xi$) supported on $x \times 1$. 
Substituting (4.16) into (4.15), we obtain
\[ S^*(C, \Xi) = N^{-1/4-it} \sum_{q \sim \sigma} \frac{1}{q^{1/2}} \sum^* \sum_{n \sim N^2/Q^2} \frac{\lambda_g(n)}{n^{1/4}} e \left( \frac{n\alpha}{n} \right) \]
\[ \times \sum_{m \geq 1} \lambda_f(m) e \left( \frac{ma}{q} \right) \left( \frac{m}{N} \right)^{-it} \bar{V} \left( \frac{m}{N} \right) e \left( \frac{2\sqrt{mn} \epsilon}{q} \right) + O_A \left( N^{-A} \right), \]  
(4.17)

where \( \bar{V}(x) = V(x)F(x) \in C_c^\infty(1, 2) \) satisfying \( \bar{V}^{(j)}(x) \ll j \) for any integer \( j \geq 0 \).

Now we apply Lemma 3.5 with \( \varphi(x) = (x/N)^{-it} \bar{V}(x/N) e (2\sqrt{nx}/q) \) to the \( m \)-sum in (4.17) and get
\[ m\text{-sum} = q^2 \sum_{\pm} \sum_{m \geq 1} \frac{\lambda_f(m)}{m} e \left( \pm \frac{\alpha m}{q} \right) \Psi_\varphi^\pm \left( \frac{m}{q^2}, n, q \right), \]  
(4.18)

where by (3.6), for \( \sigma > -1 \),
\[ \Psi_\varphi^\pm(x, n, q) = \frac{N^{it}}{4\pi^2} \int_0^\infty (\pi^2 x)^{-s} I(s) \left( \int_0^\infty \bar{V} \left( \frac{y}{N} \right) e \left( \frac{2n^{1/2}y^{1/2}}{q} \right) y^{-s-it-1}dy \right) ds. \]  
(4.19)

Further substituting (4.18) into (4.17) and writing the Ramanujan sum
\[ S(m \pm n, 0; q) = \sum_{d|m \pm n, q} d\mu(q/d), \]
then we have
\[ S^*(C, \Xi) = N^{-1/4-it} \sum_{q \sim \sigma} \frac{1}{q^{1/2}} \sum_{d|q} d\mu \left( \frac{q}{d} \right) \sum_{m \geq 1} \frac{\lambda_f(m)}{m} \]
\[ \times \sum_{n \sim N^2/Q^2} \sum_{n \equiv \pm m \mod d} \frac{\lambda_g(n)}{n^{1/4}} \Psi_\varphi^\pm \left( \frac{m}{q^2}, n, q \right) + O_A \left( N^{-A} \right). \]  
(4.20)

We will show in Section 5 that the integral \( \Psi_\varphi^\pm(x, n, q) \) has the following properties.

**Lemma 4.1.** Let \( B = 2n^{1/2}N^{1/2}/q \).

(1) If \( T_1^{1-\varepsilon} \ll B \ll T_1^{1+\varepsilon} \), then \( \Psi_\varphi^\pm(x, n, q) = \Psi_1 + \Psi_2 \), where \( \Psi_1 \) is negligibly small unless \( Nx \ll T_1^{1+\varepsilon} \), in which case
\[ \Psi_1 \ll (BNx)^{1/2}, \]
and \( \Psi_2 \) is negligibly small unless \( Nx \ll BT_1^{1+\varepsilon} \), in which case
\[ \Psi_2 \ll (Nx)^{1/2}. \]

(2) If \( B \ll T_2^{1-\varepsilon} \), then \( \Psi_\varphi^\pm(x, n, q) \) is negligibly small unless \( x \times T_1|T_2|/N \), in which case
\[ \Psi_\varphi^\pm(x, n, q) = (Nx)^{1/2+it} V_\varepsilon^\pm(\tau_s) e \left( -\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e} + B\tau_0 \frac{T_1}{\pi} \right) \]
\[ + \frac{B}{2\pi} \sum_{j=1}^K g_j \left( \frac{B}{T_1}; \frac{B}{T_2}; \frac{\tau_j+1}{\tau_0} \right) + O_A(N^{-A}), \]
where $K \geq 1$ is an integer, $A > 0$ is a large constant depends on $K$, $\tau_0 = (T_1|T_2|/(4Nx))^{1/2}$, $V_\pm^\tau(\tau)$ is some inert function supported on $\tau \times 1$, $\tau_*$ is defined in [5.12], and $g_j(y_1, y_2)$ are some homogeneous polynomials of degree $j$ and satisfy $g_j(y_1, y_2) \ll_j y_2^j$ for any integer $j \geq 1$.

(3) If $B \gg T_1^{1+\varepsilon}$, then $\Psi_\varphi^\pm(x, n, q)$ is negligibly small unless $N x \asymp T_1|T_2|$, in which case $\Psi_\varphi^\pm(x, n, q) \ll (Nx)^{1/2}$.

Note that $B \asymp N\Xi/(CQ) \gg N^\varepsilon$ (see (4.14) and the range of $n$ in (4.20)). So by Lemma 4.1, the properties of $\Psi_\varphi^\pm(x, n, q)$ depend on the size of $C$. By Lemma [4.1] we distinguish two cases according to $C \geq N^{1+\varepsilon}\Xi/(Q|T_2|)$ or not.

4.4. The case of large modulus. In this section, we consider the case $C \geq N^{1+\varepsilon}\Xi/(Q|T_2|)$ which is equivalent to the condition $B \ll T_1^{1-\varepsilon}$. In this case, we use the second statement of Lemma 4.1. By (4.20) and Lemma 4.1 (2),

\[
\mathcal{J}_*(C, \Xi) = N^{1/4} e\left(-\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e}\right) \sum_{m \asymp C^2T_1|T_2|/N} \frac{\lambda_f(m)}{m^{1/2-\varepsilon}} \sum_{n \asymp N\Xi/Q^2 \mid n \equiv \pm m \mod d} \sum_{d \mid q} d \mu\left(\frac{q}{d}\right)
\times \sum_{\tau_0 \asymp \Xi} \sum_{q \sim C} \frac{1}{q^{1/2+2\varepsilon}} \sum_{d \mid q} d \frac{\lambda_g(n)}{n^{1/4}} \mathfrak{J}_\mp(m, n, q) + O_A\left(N^{-A}\right),
\]

where

\[
\mathfrak{J}_\mp(m, n, q) = V_\pm^\tau(\tau_*) e\left(\frac{2\tau_0 n^{1/2} N^{1/2}}{\pi q} + \frac{B}{2\pi} \sum_{j=1}^K g_j\left(\frac{B}{T_1}, \frac{B}{T_2}\right) \tau_0^{j+1}\right)
\]

with $\tau_0 = (T_1|T_2|/(4Nm))^{1/2}$.

4.4.1. Cauchy-Schwarz and Poisson summation. Applying the Cauchy-Schwarz inequality to (4.21) and using the Rankin-Selberg estimate (??), one sees that

\[
\mathcal{J}_*(C, \Xi) \ll \frac{N^{3/4}}{CT_1^{1/2}|T_2|^{1/2}} \sum_{\tau_0 \sim \Xi} \sum_{q \sim C} q^{-1/2} \sum_{d \mid q} d \left(\sum_{m \asymp C^2T_1|T_2|/N} |\lambda_f(m)|^2\right)^{1/2}
\times \left(\sum_{m \asymp C^2T_1|T_2|/N} \sum_{n \asymp N\Xi/Q^2 \mid n \equiv \pm m \mod d} \lambda_g(n) n^{-1/4} \mathfrak{J}_\mp(m, n, q)^2\right)^{1/2}
\ll N^{1/4} \sum_{\tau_0 \asymp \Xi} \sum_{q \sim C} q^{-1/2} \sum_{d \mid q} \sqrt{\Omega(q, d)}
\]

where

\[
\Omega(q, d) = \sum_{m \in \mathbb{Z}} \omega\left(\frac{m}{C^2T_1|T_2|/N}\right) \sum_{n \asymp N\Xi/Q^2 \mid n \equiv \pm m \mod d} \lambda_g(n) n^{-1/4} \mathfrak{J}_\mp(m, n, q)^2.
\]

Here $\omega$ is a nonnegative smooth function on $(0, +\infty)$, supported on $[2/3, 3]$, and such that $\omega(x) = 1$ for $x \in [1, 2]$. 

Opening the absolute square, we break the $m$-sum into congruence classes modulo $d$ and apply the Poisson summation formula to the sum over $m$ to get

$$
\Omega(q, d) = \sum_{n_1 \approx N\Xi^2/Q^2} \lambda_g(n_1)n_1^{-1/4} \sum_{n_2 \approx N\Xi^2/Q^2} \lambda_g(n_2)n_2^{-1/4} \times \sum_{m \equiv \pm n_1 \mod d} \omega\left(\frac{m}{C^2T_1|T_2|/N}\right) \mathfrak{I}(m, n_1, q) \mathfrak{I}(m, n_2, q)
$$

where the integral $\mathcal{H}(x) = \mathcal{H}(x; n_1, n_2, q)$ is given by

$$
\mathcal{H}(x) = \int_{\mathbb{R}} \omega(\xi) \mathfrak{I}(C^2T_1|T_2|\xi/N, n_1, q) \mathfrak{I}(C^2T_1|T_2|\xi/N, n_2, q) e(-x\xi) \, d\xi.
$$

We have the following estimates for $\mathcal{H}(x)$, whose proofs we postpone to Section 5.

**Lemma 4.2.** Assume $C$ satisfies $C \geq N^{1+\varepsilon}(Q|T_2|)$ and $n_i \approx N\Xi^2/Q^2$, $i = 1, 2$.

1. We have $\mathcal{H}(x) \ll 1$ for any $x \in \mathbb{R}$.
2. For $x \gg N^{1+\varepsilon}\Xi/(CQ)$, we have $\mathcal{H}(x) \ll A N^{-A}$.
3. For $x \neq 0$, we have $\mathcal{H}(x) \ll |x|^{-1/2}$.
4. $\mathcal{H}(0)$ is negligibly small unless $|n_1 - n_2| \ll N^\varepsilon$.

With estimates for $\mathcal{H}(x)$ ready, we now continue with the treatment of $\Omega(q, d)$ in (4.25). By Lemma 4.2 (2), the contribution from the terms with

$$
|m| \gg \frac{dN^{2+\varepsilon}\Xi}{C^3QT_1|T_2|} := N_1
$$

is negligible. So we only need to consider the range $0 \leq |\tilde{m}| \ll N_1$.

We treat the cases where $\tilde{m} = 0$ and $\tilde{m} \neq 0$ separately and denote their contributions to $\Omega(q, d)$ by $\Omega_0$ and $\Omega_{\neq 0}$, respectively.

### 4.4.2. The zero frequency.

Let $\Sigma_0$ denote the contribution of $\Omega_0$ to (4.25). Correspondingly, we denote its contribution to (4.23) by $\Sigma_0$.

**Lemma 4.3.** We have

$$
\Sigma_0 \ll N^\varepsilon Q^{3/2}T_1^{1/2}|T_2|^{1/2}.
$$

**Proof.** Splitting the sum over $n_1$ and $n_2$ according as $n_1 = n_2$ or not, and applying Lemma 4.2 (4), the Rankin-Selberg estimate (4.22) and using the inequality $|\lambda_g(n_1)\lambda_g(n_2)| \leq |\lambda_g(n_1)|^2 +$
\[ |\lambda_g(n_2)|^2, \quad \text{we have} \]

\[
\Omega_0 \ll \frac{M}{d t N_1^{1/2}} \sum_{n_1, n_2 \in \mathbb{Z}^2 / Q^2} |\lambda_g(n_1)||\lambda_g(n_2)|
\ll \frac{C^2 T_1 |T_2| Q}{d N N_1^{1/2}} \sum_{n_1 \in \mathbb{Z}^2 / Q^2} |\lambda_g(n_1)|^2 \sum_{n_2 \in \mathbb{Z}^2 / Q^2} \frac{1}{|n_1 - n_2| \in \mathbb{N}^4}
\ll N^\varepsilon \frac{C^2 T_1 |T_2| \Xi}{d N^{1/2} Q}.
\]

This bound when substituted in place of \( \Omega(q, d) \) into (4.23) yields that

\[
\Sigma_0 \ll N^{1/4 + \varepsilon} \sum_{\pm} \sum_{q \sim C} q^{-1/2} \sum_d \frac{C T_1^{1/2} |T_2|^{1/2} \Xi^{1/2}}{d^{1/2} N^{1/4} Q^{1/2}} \ll N^\varepsilon Q^{3/2} T_1^{1/2} |T_2|^{1/2}.
\]

This proves the lemma. \( \square \)

4.4.3. The non-zero frequencies. Recall \( \Omega_{\neq 0} \) denotes the contribution from the terms with \( \tilde{m} \neq 0 \) to \( \Omega(q, d) \) in (4.25). Correspondingly, we denote its contribution to (4.23) by \( \Sigma_{\neq 0} \). Using the inequality \( |\lambda_g(n_1)\lambda_g(n_2)| \leq |\lambda_g(n_1)|^2 + |\lambda_g(n_2)|^2 \), we have

\[
\Omega_{\neq 0} \ll \frac{C^2 Q T_1 |T_2|}{d N^{3/2} \Xi} \sum_{n_1 \in \mathbb{Z}^2 / Q^2} |\lambda_g(n_1)|^2 \sum_{n_2 \in \mathbb{Z}^2 / Q^2} \sum_{\tilde{m} \equiv N_1 \pmod{d}} \left| \mathcal{H} \left( \frac{C^2 T_1 |T_2| \tilde{m}}{d N} \right) \right|, \tag{4.28}
\]

where \( N_1 \) is defined in (4.21).

**Lemma 4.4.** Assume

\[
Q < N^{1/3}. \tag{4.29}
\]

We have

\[
\Sigma_{\neq 0} \ll N^{5/4 + \varepsilon} / Q.
\]

**Proof.** For \( x = C^2 T_1 |T_2| \tilde{m} / (dN) \), we split the sum over \( \tilde{m} \) according to \( x \ll N^\varepsilon \) or not. Set

\[
N_2 := \frac{d N^{1+\varepsilon}}{C^2 T_1 |T_2|} \tag{4.30}
\]
For $0 \neq \tilde{m} \ll N_2$, we use the bound $\mathcal{H}(x) \ll 1$ in Lemma 4.2 (1), and for the remaining part we apply the bound $\mathcal{H}(x) \ll |x|^{-1/2}$ in Lemma 4.2 (3). By (4.28), we have

$$\Omega \neq 0 \ll \frac{C^2 QT_1 T_2}{dN^{3/2}} \left( \sum_{n_1 \gg N^{-2}/Q^2} |\lambda_g(n)|^2 \sum_{n_2 \gg N^{-2}/Q^2} \sum_{n \equiv n_1 \mod d} 1 \right)$$

$$+ \frac{C^2 QT_1 T_2}{dN^{3/2}} \left( \sum_{n_1 \gg N^{-2}/Q^2} |\lambda_g(n)|^2 \sum_{n_2 \gg n_1 \mod d} \sum_{N_2 \ll \tilde{m}} \frac{(C^2 T_1 T_2 |\tilde{m}|)}{dN} \right)^{-1/2}$$

$$\ll \frac{C^2 T_1 T_2 |\Xi N_2|}{dQ N^{1/2}} \left( 1 + \frac{N^{-2}}{dQ^2} \right) + \frac{C T_{1/2}^2 |T_2|^{1/2} |\Xi N_2|}{d^{1/2} Q} \left( \frac{N^{1/2}}{C T_{1/2}^3 T_2^{1/2}} + \frac{N^{-1/2}}{C^{3/2} Q^{1/2} T_{1/2}^{1/2} |T_2|^{1/2}} \right)$$

Here we have applied (3.3). By (4.27) and (4.30),

$$\Omega \neq 0 \ll \frac{C T_{1/2}^2 |T_2|^{1/2} |\Xi N_2|}{Q} \left( 1 + \frac{N^{-2}}{dQ^2} \right) \left( \frac{N^{1/2}}{C T_{1/2}^3 T_2^{1/2}} + \frac{N^{-1/2}}{C^{3/2} Q^{1/2} T_{1/2}^{1/2} |T_2|^{1/2}} \right)$$

since $\Xi \ll N^{-2}$ and $Q < N^{1/2-\varepsilon}$ by (4.11). This bound when substituted in place of $\Omega(q, d)$ in (4.23) gives that

$$\Sigma \neq 0 \ll N^{1/4} \sum_{q \sim C} q^{-\varepsilon} q \sum_{d \mid q} \frac{d N^{1/2+\varepsilon}}{C^{1/4} Q^{3/4}} \left( 1 + \frac{N^{1/2}}{d^{1/2} Q} \right)$$

$$\ll N^{3/4} Q^{-3/4} C^{3/4} \left( C^{1/2} + N^{1/2} / Q \right)$$

$$\ll N^{3/4+\varepsilon} \left( Q^{1/2} + N^{1/2} / Q \right)$$

$$\ll N^{5/4+\varepsilon} / Q$$

provided that $Q < N^{1/3}$.

\[ \square \]

4.5. **The case of small modulus.** In this section, we deal with the case $1 \ll C \leq N^{1+\varepsilon} / (Q |T_2|)$ which is equivalent to the condition $B \gg T_2^{1-\varepsilon}$ (see Lemma 4.1). In this case, we will use the first and third statement of Lemma 4.20. By (4.20), (5.2) and (1) and (3) of Lemma 4.20,

$$\mathcal{F}^* (C, \Xi) \ll \frac{C^{1/2}}{N^{1/4}} \left( \frac{N^{-2}}{Q^2} \right)^{\theta - 1/4} \sum_{q \sim C} q^{-\theta} \sum_{d \mid q} \frac{d}{m \geq 1} \frac{\lambda_g(m)}{m} \sum_{n \gg N^{-2}/Q^2} \frac{1}{n \equiv \pm m \mod d} \left| \frac{\psi_q (m)}{m} \frac{1}{q^2} \right|^{1/2} (n) (4.31)$$

$$\ll R_1 + R_2 + 1_{C \leq N^{1+\varepsilon} / (Q T_1^{1+\varepsilon})} R_3,$$  (4.32)
where \( 1_S = 1 \) is true and equals 0 otherwise,

\[
R_1 = \frac{C^{1/2}}{N^{1/4}} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta - 1/4} \sum_{q \sim C} \sum_{d \mid q} d \sum_{m \ll C^2 T_1^{1+\varepsilon}/N} \frac{|\lambda_f(m)|}{m} \sum_{n \ll N \Xi^2/Q^2} \left( \frac{B m}{q^2} \right)^{1/2},
\]

\[
R_2 = \frac{C^{1/2}}{N^{1/4}} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta - 1/4} \sum_{q \sim C} \sum_{d \mid q} d \sum_{m \ll B C^2 T_1^{1+\varepsilon}/N} \frac{|\lambda_f(m)|}{m} \sum_{n \ll N \Xi^2/Q^2} \left( \frac{N m}{q^2} \right)^{1/2},
\]

and

\[
R_3 = \frac{C^{1/2}}{N^{1/4}} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta - 1/4} \sum_{q \sim C} \sum_{d \mid q} d \sum_{m \ll C^2 T_1 | T_2|/N} \frac{|\lambda_f(m)|}{m} \sum_{n \ll N \Xi^2/Q^2} \left( \frac{N m}{q^2} \right)^{1/2}.
\]

Recall that \( B \asymp N \Xi/(CQ) \), \( \Xi \ll N^\varepsilon \) and \( C \leq N^{1+\varepsilon} \Xi/(Q|T_2|) \). By (3.3), we have

\[
R_1 \ll \frac{B^{1/2} N^{1/4}}{C^{1/2}} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta - 1/4} \sum_{q \sim C} \sum_{d \mid q} d \left( \frac{C^2 T_1^{1+\varepsilon}}{N} \right)^{1/2} \left( 1 + \frac{N \Xi^2}{dQ^2} \right)
\]

\[
\ll T_1^{1/2+\varepsilon} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta} C \left( C + \frac{N \Xi^2}{Q^2} \right)
\]

\[
\ll T_1^{1/2+\varepsilon} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta} \frac{N \Xi}{Q|T_2|} \left( \frac{N \Xi}{Q|T_2|} + \frac{N \Xi^2}{Q^2} \right)
\]

\[
\ll T_1^{1/2+\varepsilon} \left( \frac{N^{1+\varepsilon}}{Q^2} \right)^{\vartheta + 1} \frac{N^{1+\varepsilon}}{Q|T_2|}
\]

assuming

\[
Q < |T_2|.
\]

Thus

\[
R_1 \ll \frac{N^{2+\vartheta+\varepsilon} T_1^{1/2}}{Q^{3+2\vartheta}|T_2|}.
\]

Similarly,

\[
R_2 \ll \frac{N^{1/4}}{C^{1/2}} \left( \frac{N \Xi^2}{Q^2} \right)^{\vartheta - 1/4} \sum_{q \sim C} \sum_{d \mid q} d \left( \frac{BC^2 T_1^{1+\varepsilon}}{N} \right)^{1/2} \left( 1 + \frac{N \Xi^2}{dQ^2} \right)
\]

\[
\ll \frac{N^{2+\vartheta+\varepsilon} T_1^{1/2}}{Q^{3+2\vartheta}|T_2|}
\]
and for $C \leq N\Xi/(QT_1^{1+\varepsilon})$, we have

\[ R_3 \ll \frac{N^{1/4}}{C^{1/2}} \left( \frac{N\Xi^2}{Q^2} \right)^{\vartheta-1/4} \sum_{q \sim C} \sum_{d | q} \left( \frac{C^2 T_1 |T_2|}{N} \right)^{1/2} \left( 1 + \frac{N\Xi^2}{dQ^2} \right) \]

\[ \ll \frac{T_1^{1/2+\varepsilon} |T_2|^{1/2} Q^{1/2}}{N^{1/2} \Xi^{1/2}} \left( \frac{N\Xi^2}{Q^2} \right)^{\vartheta} \left( C + \frac{N\Xi^2}{Q^2} \right) \]

\[ \ll \frac{T_1^{1/2+\varepsilon} |T_2|^{1/2} Q^{1/2}}{N^{1/2} \Xi^{1/2}} \left( \frac{N\Xi^2}{Q T_1^{1+\varepsilon}} \right)^{3/2} \left( \frac{N\Xi}{QT_1^{1+\varepsilon}} + \frac{N\Xi^2}{Q^2} \right) \]

\[ \ll \frac{N^{2+\vartheta+\varepsilon} T_1^{1/2}}{Q^{3+2\vartheta}|T_2|} \]

(4.36)

under the assumption in (4.33).

By (4.31) and (4.34)-(4.36), we conclude that for $1 \ll C \leq N^{1+\varepsilon}\Xi/(Q|T_2|)$,

\[ \mathcal{S}^*(C, \Xi) \ll \frac{N^{2+\vartheta+\varepsilon} T_1^{1/2}}{Q^{3+2\vartheta}|T_2|} \]

(4.37)

4.6. Conclusion. By inserting the upper bounds in Lemmas 4.3 and 4.4 into (4.23), we have

\[ \mathcal{S}^*(C, \Xi) \ll N^\varepsilon \left( Q^{3/2} T_1^{1/2} |T_2|^{1/2} + N^{5/4}/Q \right) \]

under the assumption $N^{1+\varepsilon}\Xi/(Qt) \leq C \ll Q$ and

\[ Q < N^{1/3} \]

(4.38)

which is a combination of (4.11) and (4.29). We set $Q = N^{1/2}/(|T_1| |T_2|)^{1/5}$ to balance the contribution. Then this $Q$ also satisfies (4.33) and for $N^{1+\varepsilon}\Xi/(Qt) \leq C \ll Q$, \n
\[ \mathcal{S}^*(C, \Xi) \ll N^{3/4+\varepsilon} (|T_1| |T_2|)^{1/5} \ll N^{3/4+\varepsilon} T_1^{2/5} \]

(4.39)

provided $N < (|T_1| |T_2|)^{6/5}$, which is satisfactory since we only need this estimate in the range $N < T_1^{1+\varepsilon}/T_2$. Moreover, for this choice of $Q$, when $C \leq X^{1+\varepsilon}\Xi/(Qt)$, by (4.37), $\mathcal{S}^*(C, \Xi)$ is bounded by

\[ \frac{N^{2+\vartheta+\varepsilon} T_1^{1/2}}{Q^{3+2\vartheta}|T_2|} = N^{1/2+\varepsilon} T_1^{11/10+2\vartheta/5} |T_2|^{-2/5+2\vartheta/5} \ll N^{1/2+\varepsilon} T_1^{39/50+2\vartheta/25} \]

(4.40)

by the condition $|T_2| \gg T_1^{4/5-\varepsilon}$ in (4.4). Substituting the estimates in (4.39) and (4.40) for $\mathcal{S}^*(C, \Xi)$ into (1.8), we obtain

\[ S(N) \ll N^{3/4+\varepsilon} T_1^{2/5} + N^{1/2+\varepsilon} T_1^{39/50+2\vartheta/25} \]

Then by (1.2),

\[ L \left( \frac{1}{2} + it, f \otimes g \right) \ll T_1^\varepsilon \sup_{T_1^{3/5} \ll N \ll T_1^{1+\varepsilon} |T_2|} N^{1/4+\varepsilon} T_1^{2/5} + T_1^{39/50+2\vartheta/25} + T_1^{9/10} \]

\[ \ll T_1^{9/10+\varepsilon} + T_1^{39/50+2\vartheta/25+\varepsilon} \]

Note that the second term is dominated by the first term since we can take $\vartheta = 7/64$ by [KS03]. This completes the proof of Theorem 1.3.
5. Estimation of integrals

We first prove Lemma 4.1.

Proof of Lemma 4.1. The proof is similar as Huang [Hua21b]. Let \( s = \sigma + i\tau \). Making changes of variables \( \tau \rightarrow \tau - t \) and \( y = Ny^2 \) in (4.19), one has

\[
\Psi_{\varphi}^{\pm}(x, n, q) = \frac{1}{2\pi^2} \int_{\mathbb{R}} \left( \pi^2 N x \right)^{-\sigma - i\tau + it} \gamma_f^{\pm}(\sigma + i\tau - it) \phi(n, q, \tau) d\tau,
\]

where

\[
\phi(n, q, \tau) = \int_0^\infty \tilde{V}(y^2) y^{-2\sigma - 1} \exp(i\varphi(y)) dy
\]

with \( \varphi(y) = 2\pi B y - 2\tau \log y \) and \( B = 2n^{1/2}N^{1/2}/q \). Note that

\[
\varphi'(y) = 2\pi B - 2\tau/y,
\]

\[
\varphi^{(j)}(y) = 2\tau(-1)^j(j - 1)!y^{-j} \asymp |\tau|, \quad j = 2, 3, \ldots.
\]

By repeated integration by parts one shows that \( \phi(n, q, \tau) \) is negligibly small unless \( \tau > 0 \) and \( \tau \asymp \epsilon \). The stationary point is \( y_0 = \tau/(\pi B) \). Recall that \( B \gg N\Xi/(CQ) \gg N^{\epsilon} \). Applying Lemma 3.7 (2) with \( X = Z = 1 \) and \( Y = R = \tau \gg N^{\epsilon} \), we obtain

\[
\phi(n, q, \tau) = \tau^{-1/2}V^2_{\sigma}(\frac{\tau}{\pi B}) e\left( -\frac{\tau}{\pi} \log \frac{\tau}{\pi eB} \right) + O_A \left( N^{-A} \right),
\]

where \( V^2_{\sigma}(x) \) is an inert function (depending on \( A \) and \( \sigma \)) supported on \( x \asymp 1 \). Assembling these results, we obtain

\[
\Psi_{\varphi}^{\pm}(x, n, q) = \frac{1}{2\pi^2} \int_0^\infty \left( \pi^2 N x \right)^{-\sigma - i\tau + it} \gamma_f^{\pm}(\sigma + i\tau - it)
\]

\[
\times \tau^{-1/2}V^2_{\sigma}(\frac{\tau}{\pi B}) e\left( -\frac{\tau}{\pi} \log \frac{\tau}{\pi eB} \right) d\tau + O_A \left( N^{-A} \right).
\]

Making a change of variable \( \tau \rightarrow B\tau \),

\[
\Psi_{\varphi}^{\pm}(x, n, q) = \frac{B^{1/2}}{2\pi^2} \int_0^\infty \left( \pi^2 N x \right)^{-\sigma - iB\tau + it} \gamma_f^{\pm}(\sigma + iB\tau - it)
\]

\[
\times \tau^{-1/2}V^2_{\sigma}(\frac{\tau}{\pi}) e\left( -\frac{B\tau}{\pi} \log \frac{B\tau}{\pi e} \right) d\tau + O_A \left( N^{-A} \right),
\]

where by (5.2),

\[
\gamma_f^{\pm}(\sigma + iB\tau - it) = \prod_{j=1,2} \frac{\Gamma\left( \frac{1+i\sigma+(B\tau-T_j)}{2} \right)}{\Gamma\left( \frac{-\sigma-i(B\tau-T_j)}{2} \right)} \pm \prod_{j=1,2} \frac{\Gamma\left( \frac{2+i\sigma+(B\tau-T_j)}{2} \right)}{\Gamma\left( \frac{-2\sigma-i(B\tau-T_j)}{2} \right)}.
\]

(1) For \( T^{1-\epsilon}_2 < B < T^{1+\epsilon}_1 \), we divide the range of \( \tau \) into two pieces:

\( (0, \infty) = \{ \tau \mid |T_1 - B\tau| \leq T^\epsilon_2 \} \cup \{ \tau \mid |T_1 - B\tau| > T^\epsilon_2 \} := I_1 + I_2 \) and correspondingly denote by the integral over \( I_j \) by \( \Psi_j \), \( j = 1, 2 \). Then by (3.9),

\[
\Psi_1 \ll B^{1/2}(Nx)^{-\sigma} \int_{I_1} \left( \|T_1 - B\tau\|_2 - B\tau \right)^{\sigma+1/2} \left| V^2_{\sigma}(\frac{\tau}{\pi}) \right| d\tau
\]

\[
\ll B^{1/2}T^{1/2+\epsilon}_1(Nx/T^{1+\epsilon}_1)^{-\sigma}.
\]
By taking \(\sigma\) sufficiently large, one sees that \(\Psi_1\) is negligibly small unless \(Nx \ll T_1^{1+\varepsilon}\), in which case by taking \(\sigma = -1/2\) we have the estimate
\[
\Psi_1 \ll (BNx)^{1/2}. \tag{5.3}
\]

For \(\tau \in I_2\), by (5.2) and Stirling’s approximation in (3.8), we have
\[
\gamma_{\ell}^{+} (\sigma + iB\tau - it) = \left( \prod_{j=1,2} \left( \frac{|T_j - B\tau|}{2\pi} \right)^{i(B\tau - T_j)} |T_j - B\tau|^{\sigma + 1/2} \right) \times (h_{\sigma,1}(B\tau - T_1)h_{\sigma,1}(B\tau - T_2) \pm h_{\sigma,2}(B\tau - T_1)h_{\sigma,2}(B\tau - T_2)) + O_{\sigma,K_3}(T_1^{-K_3}). \tag{5.4}
\]

Then by (5.1) and (5.4),
\[
\Psi_2 = \frac{B^{1/2}}{2\pi^2} \int_{0}^{\infty} \left( \frac{\pi^2 Nx}{2\pi} \right)^{-\sigma-iB\tau+it} \left( \prod_{j=1,2} \left( \frac{|T_j - B\tau|}{2\pi} \right)^{i(B\tau - T_j)} |T_j - B\tau|^{\sigma + 1/2} \right) \times (h_{\sigma,1}(B\tau - T_1)h_{\sigma,1}(B\tau - T_2) \pm h_{\sigma,2}(B\tau - T_1)h_{\sigma,2}(B\tau - T_2)) \times \tau^{-1/2} V_{\sigma}^{\gamma} \left( \frac{T_j}{\pi} \right) e \left( -\frac{B\tau}{2\pi} \log \frac{\tau}{\pi e} \right) d\tau + \Psi_3, \tag{5.5}
\]
where \(h_{\sigma,j}(x), j = 1, 2\), satisfy \(h_{\sigma,j}(x) \ll_{\sigma,j,K_4} 1\) and \(x^{\ell} h_{\sigma,j}(x) \ll_{\sigma,j,\ell,K_4} 1\) for any integer \(\ell \geq 1\), and
\[
\Psi_3 \ll B^{1/2}(Nx)^{-\sigma} \int_{I_1} \left( |T_1 - B\tau||T_2 - B\tau| \right)^{\sigma + 1/2} \left| V_{\sigma}^{\gamma} \left( \frac{T_j}{\pi} \right) \right| d\tau \ll B^{1/2}T_1^{1/2+\varepsilon}(Nx/T_1^{1+\varepsilon})^{-\sigma}
\]
which can be negligibly small unless \(Nx \ll T_1^{1+\varepsilon}\), in which case by taking \(\sigma = -1/2\) we have
\[
\Psi_3 \ll (BNx)^{1/2}. \tag{5.6}
\]

Denote the first term in (5.5) by \(\Psi_2^0\). Then
\[
\Psi_2^0 \ll B^{1/2}(Nx)^{-\sigma} \int_{\tau > 1} \left( |T_1 - B\tau||T_2 - B\tau| \right)^{\sigma + 1/2} d\tau \ll BT_1^{1/2+\varepsilon}(\frac{Nx}{BT_1^{1+\varepsilon}})^{-\sigma}
\]
which can be negligibly small unless \(Nx \ll BT_1^{1+\varepsilon}\), in which case by taking \(\sigma = -1/2\),
\[
\Psi_2^0 = B^{1/2}(Nx)^{1/2+it} \int_{0}^{\infty} G(\tau) \exp \left( i\eta(\tau) \right) d\tau,
\]
where, temporarily,
\[
G(\tau) = \frac{\pi^{2i\ell-1}}{2\sqrt{\pi}} V_{\sigma}^{\gamma} \left( \frac{T_j}{\pi} \right) \left( h_{\sigma,1}(B\tau - T_1)h_{\sigma,1}(B\tau - T_2) \pm h_{\sigma,2}(B\tau - T_1)h_{\sigma,2}(B\tau - T_2) \right)
\]
and
\[
\eta(\tau) = -B\tau \log \frac{Nx}{e^2} + (B\tau - T_1) \log \frac{|T_1 - B\tau|}{2e} + (B\tau - T_2) \log \frac{|T_2 - B\tau|}{2e} - 2B\tau \log \tau.
\]
Note that
\[
\eta'(\tau) = B \log \frac{|T_1 - B\tau||T_2 - B\tau|}{4Nx\tau^2},
\]
\[
\eta''(\tau) = B \left( \frac{1}{\tau - T_1/B} + \frac{1}{\tau - T_2/B} - \frac{2}{\tau} \right).
\]
and
\[ \int_{\min\{|B\tau-T_1|,|B\tau-T_2|\} > \sqrt{B}} \left| \frac{dG(\tau)}{d\tau} \right| \, d\tau \ll \max_{\tau > 1} \left\{ \frac{B}{|B\tau-T_1|^2}, \frac{B}{|B\tau-T_2|^2} \right\} \ll 1. \]

Moreover, for \( \min_{\tau < 1} \{|B\tau-T_1|,|B\tau-T_2|\} > \sqrt{B} \),
\[ \eta''(\tau) \times \begin{cases} B \max_{\tau > 1} |\tau - T_1/B|^{-1}, & \text{if } T_1^{1-\epsilon} \ll B \ll T_1^{1+\epsilon} \\ B, & \text{if } T_2^{1+\epsilon} \ll B \ll T_1^{1-\epsilon} \\ B \max_{\tau > 1} |\tau - T_2/B|^{-1}, & \text{if } T_2^{1-\epsilon} \ll B \ll T_2^{1+\epsilon}. \end{cases} \]

Then by Lemma 3.3,
\[ B^{1/2}(N_\chi)^{1/2+it} \int_{|B\tau-T_1| > \sqrt{B}, \tau > 1} G(\tau) \exp (i\eta(\tau)) \, d\tau \ll (N_\chi)^{1/2}. \]

Trivially, we have
\[ B^{1/2}(N_\chi)^{1/2+it} \int_{\min\{|B\tau-T_1|,|B\tau-T_2|\} \leq \sqrt{B}} G(\tau) \exp (i\eta(\tau)) \, d\tau \ll (N_\chi)^{1/2}. \]

Assembling the above results, we conclude that
\[ \Psi_2 \ll (N_\chi)^{1/2}. \] (5.7)

Then the first statement follows from (5.3), (5.5), (5.6) and (5.7).

(2) For the second statement in Lemma 4.1, we take \( \sigma = -1/2 \) in (5.1), we have
\[ \Psi_\nu^\pm (x, n, q) = \left( \frac{B^{1/2}}{2\pi} \right)^{\nu} \int_0^\infty (2 \pi^2 N x)^{1/2-iB\tau+it} \gamma_f^\pm \left( -\frac{1}{2} + iB\tau - it \right) \times \tau^{-1/2} V^2 \left( \frac{\tau}{\pi} \right) e \left( -\frac{B\tau}{\pi} \log \frac{\tau}{\pi} \right) d\tau + O_A \left( N^{-A} \right). \] (5.8)

where \( V^2(x) = V_{-1/2}^2(x) \) and by (5.2),
\[ \gamma_f^\pm \left( -\frac{1}{2} + iB\tau - it \right) = \prod_{j=1,2} \Gamma \left( \frac{1/2+i(B\tau-T_j)}{2} \right) \pm \prod_{j=1,2} \Gamma \left( \frac{3/2+i(B\tau-T_j)}{2} \right). \]

Since \( B \ll T_2^{1-\epsilon} \), using Stirling’s approximation in (3.8), we derive
\[ \gamma_f^\pm \left( -\frac{1}{2} + iB\tau - it \right) = \left( \frac{T_1-B\tau}{2e} \right)^{(i(B\tau-T_1))} \left( \frac{T_2-B\tau}{2e} \right)^{(i(B\tau-T_2))} \times (h_1(B\tau-T_1)h_1(B\tau-T_2) \pm h_2(B\tau-T_1)h_2(B\tau-T_2)) + O_{K_4}(T_1^{-K_4}), \] (5.9)
where \( h_j(x), j = 1, 2, \) satisfying \( h_j(x) \ll j 1 \) and \( x^\ell h_j^{(\ell)}(x) \ll j, \ell, K_j x^{-1} \) for any integer \( \ell \geq 1. \) Plugging (5.9) into (5.8), one has

\[
\Psi_\tau^\pm(x, n, q) = B^{1/2}(N x)^{1/2 + i t} \int_0^\infty V_0^\pm(\tau) \exp(i \varrho_0(\tau)) \, d\tau + O_A(N^{-A}), \tag{5.10}
\]

where

\[
V_0^\pm(\tau) = \frac{\pi^{2it-1}}{2\sqrt{\tau}} V^\prime_z \left( \frac{T}{\pi} \right) (h_1(B \tau - T_1) h_1(B \tau - T_2) \pm h_2(B \tau - T_1) h_2(B \tau - T_2))
\]
satisfying \( d^n V_0^\pm(\tau)/d\tau^n \ll 1 \) for any integer \( n \geq 1, \) and

\[
\varrho_0(\tau) = -B \tau \log \frac{N x}{e^2} + (B \tau - T_1) \log \frac{T_1 - B \tau}{2e} + (B \tau - T_2) \log \frac{T_2 - B \tau}{2e} - 2B \tau \log \tau. \tag{5.11}
\]

We compute

\[
\varrho_0'(\tau) = B \log \frac{(T_1 - B \tau)(T_2 - B \tau)}{4N x \tau^2},
\]

\[
\varrho_0^{(j)}'(\tau) = B(-1)^j(j-2)! \left( \frac{1}{(T - T_1/B)^{j-1}} + \frac{1}{(T - T_2/B)^{j-1}} - \frac{2}{\tau^{j-1}} \right) \times B, \quad j = 2, 3, \ldots.
\]

By repeated integration by parts one shows that \( \Psi_\tau^\pm(x, n, q) \) is negligibly small unless \( N x \asymp T_1 |T_2|. \) Denote \( C_j^\alpha = \alpha(\alpha - 1) \cdots (\alpha - j + 1)/j! \). By an iterative argument, the stationary point \( \tau_* \) which is the solution to the equation \( \varrho_0'(\tau) = 0, \) i.e., \( 4N x \tau^2 = T_1 |T_2|(1 - T_1^{-1}B \tau)(1 - T_2^{-1}B \tau), \) can be written as

\[
\tau_* = \left( \frac{T_1 |T_2|}{4N x} \right)^{1/2} \left( 1 - \frac{B}{T_1} \tau_* \right)^{1/2} \left( 1 - \frac{B}{T_2} \tau_* \right)^{1/2}
\]

\[
= \left( \frac{T_1 |T_2|}{4N x} \right)^{1/2} \left( \sum_{j=0}^{K_5} C_1^{j/2} \left( \frac{-B}{T_1} \right)^j \tau_*^j + O_{K_5}((B/T_1)^{K_5+1}) \right)
\]

\[
\times \left( \sum_{j=0}^{K_6} C_1^{j/2} \left( \frac{-B}{|T_2|} \right)^j \tau_*^j + O_{K_6}((B/|T_2|)^{K_6+1}) \right)
\]

\[
= \sum_{j=0}^{K} \tau_j + O_{K_7}((B/|T_2|)^{K_7+1}), \tag{5.12}
\]

where

\[
\tau_0 = \left( \frac{T_1 |T_2|}{4N x} \right)^{1/2} \asymp 1,
\]

\[
\tau_1 = -\frac{1}{2} \left( \frac{B}{T_1} + \frac{B}{|T_2|} \right) \tau_0^2 \asymp \frac{B}{|T_2|},
\]

\[
\tau_2 = -\frac{1}{8} \left( \frac{B^2}{T_1^2} - \frac{2B^2}{T_1 |T_2|} + \frac{B^2}{|T_2|^2} \right) \tau_0^3 \asymp \frac{B^2}{|T_2|^2},
\]

\[
\tau_j = f_j \left( \frac{B}{T_1}, \frac{B}{|T_2|} \right) \tau_0^{j+1} \asymp \left( \frac{B}{|T_2|} \right)^j, j = 3, 4, \ldots,
\]
for some homogeneous polynomials $f_j(y_1, y_2)$ of degree $j$. Since $B \ll T_2^{1-\epsilon}$, the $O$-term in (5.12) is $O(N^{-\epsilon K})$, which can be arbitrarily small by taking $K_7$ sufficiently large.

Applying Lemma 3.7 (2) with $X = Z = 1$ and $Y = R = \tau \gg N^\epsilon$, we obtain

$$\int_0^\infty V_0^\pm(\tau) \exp(i\varrho_0(\tau)) \, d\tau = B^{-1/2}V_z^\pm(\tau_s)e^{i\varrho_0(\tau_s)} + O_A(N^{-A}),$$

where $V_z^\pm(\tau)$ is some inert function supported on $\tau \asymp 1$. From (5.11) and (5.12) and using Taylor series expansion $\log(1-y) = -\sum_{j=1}^\infty y^j/j, y \in (-1, 1)$, we have

$$\varrho_0(\tau_\ast) = B\tau_\ast \frac{(T_1 - B\tau_\ast)(|T_2 - B\tau_\ast| - T_1 \log \frac{T_1 - B\tau_\ast}{2e} - T_2 \log \frac{|T_2 - B\tau_\ast|}{2e})}{4Nx^2}\left|\tau_\ast\right|$$

$$= -T_1 \log \frac{T_1 - B\tau_\ast}{2e} - T_2 \log \frac{|T_2 - B\tau_\ast|}{2e},$$

for some homogeneous polynomials $g_j(y_1, y_2)$ of degree $j$ and satisfying $g_j(y_1, y_2) \asymp y_j^j$ for any integer $j \geq 0$. In particular, $\varrho_0(y_1, y_2) = 2$. Hence,

$$\int_0^\infty V_0(\tau) \exp(i\varrho_0(\tau)) \, d\tau = B^{-1/2}V_z^\pm(\tau_s)e^{i\varrho_0(\tau_s)} \left( -\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e} \right) + O_A(N^{-A}),$$

where $A > 0$ is a large constant depends on $K$. By (5.10) and (5.13),

$$\Psi_{\nu}^\pm(x, n, q) = (Nx)^{1/2 + it}V_z^\pm(\tau_s)e^{i\varrho_0(\tau_s)} \left( -\frac{T_1}{2\pi} \log \frac{T_1}{2e} - \frac{T_2}{2\pi} \log \frac{|T_2|}{2e} \right) + O_A(N^{-A}).$$

This proves the second statement of the lemma.

(3) For $B \gg T_1^{1+\epsilon}$, the proof is similar as that of (2) and we will be brief. In this case, the formula (5.10) still holds. Thus repeated integration by parts shows that $\Psi_{\nu}^\pm(x, n, q)$ is negligibly small unless $Nx \asymp T_1 |T_2|$. Note that the total variation of $V_0^\pm(\tau)$ is bounded by 1 and the second derivative of the phase function is of size $B$. By the second derivative test in Lemma 3.8 we have

$$\Psi_{\nu}^\pm(x, n, q) \ll (Nx)^{1/2}.$$
Next we prove Lemma 4.2.

**Proof of Lemma 4.2.** The proof is similar to [HSZ21, Lemma 4.2]. Recall (4.26) which we relabel as

\[ \mathcal{H}(x) = \int_{\mathbb{R}} \omega(x) \mathcal{J}^\pm (C^2 T_1 | T_2 | \xi/N, n_1, q) \mathcal{J}^\pm (C^2 T_1 | T_2 | \xi/N, n_2, q) e(-x \xi) d\xi, \quad (5.14) \]

where by (4.22),

\[ \mathcal{J}^\pm (C^2 T_1 | T_2 | \xi/N, n, q) = V_k^\pm (\tau_s) e \left( \frac{n_1^{1/2} N^{1/2}}{\pi C \xi^{1/2}} + \frac{B}{2\pi} \sum_{j=1}^{K} g_j \left( \frac{B}{T_1}, \frac{B}{T_2} \right) \left( \frac{q}{2C} \right)^{j+1} \xi^{-(j+1)/2} \right). \quad (5.15) \]

Trivially, one has

\[ \mathcal{H}(x) \ll 1. \]

This proves the first statement of Lemma 4.2.

Plugging (5.15) into (5.14), we obtain

\[ \mathcal{H}(x) = \int_{\mathbb{R}} \omega(x) V_k^\pm (\tau_s) V_k^\pm (\tau_{s}^*) e \left( -x \xi + \frac{(n_1^{1/2} - n_2^{1/2}) N^{1/2}}{\pi C \xi^{1/2}} \right) \]

\[ \times e \left( \frac{1}{2\pi} \sum_{j=1}^{K} \left( B g_j \left( \frac{B}{T_1}, \frac{B}{T_2} \right) - B' g_j \left( \frac{B'}{T_1}, \frac{B'}{T_2} \right) \right) \left( \frac{q}{2C} \right)^{j+1} \xi^{-(j+1)/2} \right) d\xi, \]

where \( \tau_s \) are as in (5.12) with \( B = 2n_1^{1/2} N^{1/2}/q \) and \( B' = 2n_2^{1/2} N^{1/2}/q \). Note that the first derivative of the phase function in the above integral equals

\[ -x - \frac{(n_1^{1/2} - n_2^{1/2}) N^{1/2}}{2\pi C \xi^{3/2}} \]

\[ - \frac{1}{2\pi} \sum_{j=1}^{K} \left( \frac{j + 1}{2} \right) \left( B g_j \left( \frac{B}{T_1}, \frac{B}{T_2} \right) - B' g_j \left( \frac{B'}{T_1}, \frac{B'}{T_2} \right) \right) \left( \frac{q}{2C} \right)^{j+1} \xi^{-(j+3)/2} \quad (5.16) \]

which is \( \gg |x| \gg N^\varepsilon \) if \( |x| \gg N^{1+\varepsilon} \Xi/(CQ) \) since \( n_i \sim N \Xi^2/Q^2 \), \( i = 1, 2 \). Then repeated integration by parts shows that the contribution from \( x \gg N^{1+\varepsilon} \Xi/(CQ) \) is negligible. Thus the second statement of Lemma 4.2 is clear.

Moreover, the second term in (5.16) is of size

\[ \frac{N^{1/2}}{C} |n_1^{1/2} - n_2^{1/2}| \times \frac{Q}{C \Xi} |n_1 - n_2| \]

since \( n_i \sim N \Xi^2/Q^2 \), \( i = 1, 2 \). Thus repeated integration by parts shows that \( \mathcal{H}(x) \) is negligibly small unless \( |x| \ll \frac{Q}{C \Xi} |n_1 - n_2| \). Now by applying the second derivative test in Lemma 3.8 we infer that for \( x \neq 0 \),

\[ \mathcal{H}(x) \ll |x|^{-1/2}. \]

This proves (3).
Finally, for $x = 0$, using the fact $g_j(y_1, y_2)$ are some homogeneous polynomials of degree $j$ and satisfy $g_j(y_1, y_2) \ll_j y_2^j$ for any integer $j \geq 1$ and the identity $a^{j+1} - b^{j+1} = (a - b)(a^j + a^{j-1}b + \cdots + ab^{j-1} + b^j)$, one sees that, for $j \geq 1$,

$$B g_j \left( \frac{B}{T_1}, \frac{B}{T_2} \right) - B' g_j \left( \frac{B'}{T_1}, \frac{B'}{T_2} \right)$$

$$= B \sum_{j_1 + j_2 = j} \left( \frac{B}{T_1} \right)^{j_1} \left( \frac{B}{T_2} \right)^{j_2} - B' \sum_{j_1 + j_2 = j} \left( \frac{B'}{T_1} \right)^{j_1} \left( \frac{B'}{T_2} \right)^{j_2}$$

$$= (B^{j+1} - B'^{j+1}) \sum_{j_1 + j_2 = j} T_1^{-j_1} T_2^{-j_2}$$

$$\ll \frac{|B - B'|(B_j + B'^j)T_2^{-j}}{T_1^{-j_1} T_2^{-j_2}}$$

$$\ll |B - B'| N^{-\varepsilon}.$$

Thus the first derivative of the phase function in (5.16) is

$$\gg |B - B'| \times \frac{Q}{C\Xi} |n_1 - n_2|.$$

By repeated integration by parts, $\mathcal{H}(0)$ is negligible small unless $|n_1 - n_2| \ll C\Xi N^{\varepsilon}/Q$. Since $\Xi \ll N^{\varepsilon}$ and $C \ll Q$, we have that $\mathcal{H}(0)$ is negligibly small unless $|n_1 - n_2| \ll N^{\varepsilon}$. This completes the proof of Lemma 1.2.

\section*{References}

[ASS20] R. Acharya, P. Sharma, and S. K. Singh, \textit{t-aspect subconvexity for GL(2) \times GL(2) L-function} (2020). [arXiv:2011.01172]

[AHLQ20] K. Aggarwal, R. Holowinsky, Y. Lin, and Z. Qi, \textit{A Bessel delta-method and exponential sums for GL(2)}, Q. J. Math. \textbf{71} (2020), no. 3, 1143–1168, DOI 10.1093/qmathj/haaa026.

[BR10] J. Bernstein and A. Reznikov, \textit{Subconvexity bounds for triple L-functions and representation theory}, Ann. of Math. (2) \textbf{172} (2010), no. 3, 1679–1718.

[BJN21] V. Blomer, S. Jana, and P. Nelson, \textit{The Weyl bound for triple product L-functions} (2021). [arXiv:2101.12106]

[BKY13] V. Blomer, R. Khan, and M. Young, \textit{Distribution of mass of holomorphic cusp forms}, Duke Math. J. \textbf{162} (2013), no. 14, 2609–2644, DOI 10.1215/00127094-2380967.

[Del74] P. Deligne, \textit{La conjecture de Weil. I}, Inst. Hautes Études Sci. Publ. Math. \textbf{43} (1974), 273–307 (French).

[DFI02] W. Duke, J. B. Friedlander, and H. Iwaniec, \textit{The subconvexity problem for Artin L-functions}, Invent. Math. \textbf{149} (2002), no. 3, 489–577.

[HM06] G. Harcos and P. Michel, \textit{The subconvexity problem for Rankin–Selberg L-functions and equidistribution of Heegner points. II.}, Invent. Math. \textbf{163} (2006), no. 3, 581–655.

[HT14] R. Holowinsky and N. Templier, \textit{First moment of Rankin–Selberg central L-values and subconvexity in the level aspect}, Ramanujan J. \textbf{33} (2014), no. 1, 131–155.

[Hua21a] B. Huang, \textit{On the Rankin-Selberg problem}, Math. Ann., posted on 2021, DOI 10.1007/s00208-021-02186-7.

[Hua21b] \textit{Uniform subconvex bounds for GL(3) \times GL(2) L-functions} (2021). [arXiv:2104.13025]

[HHSZ21] B. Huang, Q. Sun, and H. Zhang, \textit{Analytic twists of GL2 \times GL2 automorphic forms} (2021). [arXiv:2108.09410]

[Hux96] M. N. Huxley, \textit{Area, lattice points, and exponential sums}, London Mathematical Society Monographs. New Series, vol. 13, The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[IK04] H. Iwaniec and E. Kowalski, \textit{Analytic number theory}, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.
[JM05] M. Jutila and Y. Motohashi, *Uniform bound for Hecke L-functions* 195 (2005), 61–115.

[JM06] ________, *Uniform bounds for Rankin-Selberg L-functions, Multiple Dirichlet series, automorphic forms, and analytic number theory* 75 (2006), 243–256.

[KS03] H. Kim and P. Sarnak, *Appendix 2 in Functoriality for the exterior square of GL4 and the symmetric fourth of GL2*, J. Amer. Math. Soc. 16 (2003), no. 1, 139–183.

[KPY19] E. M. Kiral, I. Petrow, and M. P. Young, *Oscillatory integrals with uniformity in parameters*, J. Théor. Nombres Bordeaux 31 (2019), no. 1, 145–159 (English, with English and French summaries).

[KMV02] E. Kowalski, Ph. Michel, and J. VanderKam, *Rankin-Selberg L-functions in the level aspect*, Duke Math. J. 114 (2002), no. 1, 123–191, DOI 10.1215/S0012-7094-02-11416-1.

[LLY] Y.-K. Lau, J. Liu, and Y. Ye, *A new bound $k^{2/3-\epsilon}$ for Rankin-Selberg L-functions for Hecke congruence sub-groups*, Int. Math. Res. Pap. 2006. Art. ID 35090, 78 pp.

[LS] Y. Lin and Q. Sun, *Analytic twists of GL3 × GL2 automorphic forms*, Int. Math. Res. Not. 2021, 15143–15208, DOI 10.1093/imrn/rnaa348.

[Mic04] P. Michel, *The subconvexity problem for Rankin–Selberg L-functions and equidistribution of Heegner points*, Ann. of Math. (2) 160 (2004), no. 1, 185–236.

[MV10] P. Michel and A. Venkatesh, *The subconvexity problem for GL2*, Publ. Math. Inst. Hautes Études Sci. 111 (2010), 171–271.

[MS06] S. D. Miller and W. Schmid, *Automorphic distributions, L-functions, and Voronoi summation for GL(3)*, Ann. of Math. (2) 164 (2006), no. 2, 423–488.

[Mun15] R. Munshi, *The circle method and bounds for L-functions—III: $t$-aspect subconvexity for GL(3) L-functions*, J. Amer. Math. Soc. 28 (2015), no. 4, 913–938, DOI 10.1090/jams/843.

[Ne21] P. D. Nelson, *Bounds for standard L-functions* (2021), arXiv:2109.15230.

[Olv97] F. W. J. Olver, *Asymptotics and special functions*, AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997.

School of Mathematics and Statistics, Shandong University, Weihai, Weihai, Shandong 264209, China

Email address: qfsun@sdu.edu.cn