Perturbation-Free Prediction of Resonance-Assisted Tunneling in Mixed Regular–Chaotic Systems

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For generic Hamiltonian systems we derive predictions for dynamical tunneling from regular to chaotic phase-space regions. In contrast to previous approaches, we account for the resonance-assisted enhancement of regular-to-chaotic tunneling in a non-perturbative way. This provides the foundation for future semiclassical complex-path evaluations of resonance-assisted regular-to-chaotic tunneling. Our approach is based on a new class of integrable approximations which mimic the regular phase-space region and its dominant nonlinear resonance chain in a mixed regular–chaotic system. We illustrate the method for the standard map.

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I. INTRODUCTION

Tunneling is a fundamental effect in wave mechanics, which allows for entering classically inaccessible regions. While textbooks focus on tunneling through potential barriers, tunneling processes in nature often take place in the absence of such energetic barriers. Instead one observes dynamical tunneling [1, 2] between classically disjoint regions in phase space.

In generic Hamiltonian systems dynamical tunneling usually occurs between regions of regular and chaotic motion. For a typical phase space of a mixed regular–chaotic system see Fig. 1(b). In particular, while a classical particle cannot traverse from the regular to the chaotic region, a wave can tunnel from the regular to the chaotic region. This regular-to-chaotic tunneling process manifests itself impressively in chaos-assisted tunneling [3, 4].

Until today the importance of regular-to-chaotic tunneling has been demonstrated in numerous experiments, including optical microcavities [3, 10], microwave billiards [11, 14], and cold atom systems [12, 16]. A recent success being the experimental verification [8, 14] that tiny nonlinear resonance chains within the regular region, as shown in Fig. 1(b), indeed drastically enhance tunneling as predicted in Refs. [17–20]. Furthermore, regular-to-chaotic tunneling is expected to play an important role for atoms and molecules in strong fields, as discussed in Refs. [21–23].

Motivated by these applications regular-to-chaotic tunneling is also a field of intense theoretical research [12, 23, 39], which is mainly focused on periodically driven model systems with one degree of freedom. Here, a major achievement is the combination of (i) direct [39, 53] and (ii) resonance-assisted [17, 21] regular-to-chaotic tunneling in a single prediction [36, 40]. This prediction shows that as function of decreasing effective Planck’s constant \( \hbar \) one has two corresponding regimes:

(i) Regular states localize on a single quantizing torus. In this regime, tunneling is determined by direct tran-
sitions from this regular torus into the chaotic region which can be evaluated semiclassically using complex paths. (ii) For even smaller $h$ a regular state, while still mostly concentrated on the main quantizing torus, acquires resonance-assisted contributions on further quantizing tori located more closely to the border of the regular region, see Fig. 2(c) for an illustration. This resonance-assisted contribution dominates tunneling to the chaotic region. Thus, one observes a resonance-assisted enhancement of regular-to-chaotic tunneling. For an example of this enhancement see Fig. 1(a). Note, that for much smaller $h$ there is even a third regime for which regular states may localize within the resonance chain. This regime is not considered here.

Despite the above achievements, a semiclassical evaluation of resonance-assisted tunneling in mixed regular–chaotic systems remains an open problem. In particular, the state-of-the-art predictions of resonance-assisted tunneling in integrable systems defy a semiclassical evaluation using the techniques developed for integrable systems. More specifically, so far (i) an integrable approximation of the regular region which ignores resonance chains is used to predict the magnitude of direct tunneling transitions from quantizing tori towards the chaotic region. Subsequently, (ii) resonance-assisted contributions are taken into account by perturbatively solving an additional pendulum Hamiltonian which models the relevant resonance chain. However, only a perturbation-free prediction, based on a single integrable approximation which includes the relevant resonance chain will allow for a semiclassical evaluation of resonance-assisted regular-to-chaotic tunneling in the spirit of Refs. [41, 42].

In this paper we derive such perturbation-free predictions of resonance-assisted regular-to-chaotic tunneling. They are based on a new class of integrable approximations $H_{r,s}$ which include the dominant $r:s$ resonance, see Fig. 4(c). In particular, the eigenvalue equation

$$\hat{H}_{r,s} |m_{\text{int}}\rangle = E_m |m_{\text{int}}\rangle,$$

of such integrable approximations $H_{r,s}$ provides eigenstates $|m_{\text{int}}\rangle$ which model the localization of regular states on the regular phase-space region, explicitly including the resonance-assisted contributions on multiple quantizing tori, in a non-perturbative way. Using such states allows for extending the results of Refs. [34, 35] to the case of resonance-assisted tunneling.

In particular, the decay rates $\gamma_m$ of metastable states which localize on the regular phase-space region and decay via regular-to-chaotic tunneling can be predicted according to

$$\gamma_m \approx \Gamma_m(t = 1) := \| \hat{P}_L \hat{U} |m_{\text{int}}\rangle \|^2.$$  (2)

We further show that regular-to-chaotic decay rates can be predicted with similar accuracy, when using a simplified formula which no longer contains the time-evolution operator. Instead it evaluates only the probability of the state $|m_{\text{int}}\rangle$ on the leaky region $L$

$$\gamma_m \approx \Gamma_m(t = 0) := \| \hat{P}_L |m_{\text{int}}\rangle \|^2.$$  (3)

Both perturbation-free predictions, Eqs. (2) and (3), give good results for the standard map, see Fig. 1. In that both predictions provide the foundation for future semiclassical predictions of resonance-assisted regular-to-chaotic tunneling. We remark that the prediction of Eq. (2) has previously been evaluated semiclassically for integrable approximations without resonances [34], using the time-domain techniques of Refs. [24, 25, 31–33] giving predictions for direct regular-to-chaotic tunneling. However, we believe that a future semiclassical prediction of resonance-assisted regular-to-chaotic tunneling would be easier obtained from Eq. (3), since it does not involve any time evolution and thus allows for a semiclassical evaluation using the simpler WKB-like techniques of Refs. [11, 42].

The paper is organized as follows: In Sec. II we introduce the standard map as a paradigmatic Hamiltonian example system with a mixed phase space. We further present regular-to-chaotic decay rates as a measure of regular-to-chaotic tunneling and discuss their numerical evaluation. In Sec. III we derive the predictions, Eqs. (2) and (3). In Sec. IV we illustrate how these predictions are evaluated using the example of the standard map. In Sec. V we present our results and compare them to the perturbative predictions of Refs. [34, 40]. In Sec. VI we discuss the main approximations and limitations of our approach. A summary and outlook is given in Sec. VII.

II. EXAMPLE SYSTEM

In this paper we focus on periodically driven Hamiltonian systems with one degree of freedom, which exhibit all generic features of a mixed phase space. Classically, the stroboscopic map

$$U : (q_n, p_n) \rightarrow (q_{n+1}, p_{n+1}),$$  (4)

describes the evolution of positions and momenta, $(q, p)$, in phase space from time $t = n$ to $t = n + 1$ over one period of the external driving. Quantum-mechanically, the time-evolution is given by the corresponding unitary time-evolution operator $\hat{U}$.

In Sec. II A we introduce the standard map as a paradigmatic example of a periodically driven one-degree-of-freedom system with a mixed phase space. In Sec. II B we introduce regular-to-chaotic decay rates $\gamma$, as the central object of our investigation. Furthermore, we discuss their numerical computation. Particular attention is paid to nonlinear resonance chains and their quantum manifestations.
A. Standard Map

Classically, the standard map originates from a periodically kicked Hamiltonian with one degree of freedom
\( H(q, p, t) = T(p) + V(q) \sum_{n \in \mathbb{Z}} \delta(n - 1) \). Here, \( \delta(\cdot) \) is the Dirac delta function. For the standard map \( T(p) = p^2 / 2 \) and \( V(q) = \kappa / (2\pi)^2 \cos(2\pi q) \), where \( \kappa \) is the kicking strength. Its stroboscopic map \( U \) \([43]\), Eq. (1), in its symmetrized version is given by

\[
q_{n+1} = q_n + p_n + \frac{\kappa}{4\pi} \sin(2\pi q_n), \quad (5a)
\]

\[
p_{n+1} = p_n + \frac{\kappa}{4\pi} \sin(2\pi q_n) + \frac{\kappa}{4\pi} \sin(2\pi q_{n+1}), \quad (5b)
\]

where \( (q_n, p_n) \) represents a phase-space point in the middle of the \( n \)-th kick. For convenience the standard map is considered on a torus \( (q, p) \in [0, 1] \times [-0.5, 0.5] \) with periodic boundary conditions.

In this paper we mainly focus on kicking strength \( \kappa = 3.4 \). Here, the phase space exhibits a large regular region which is centered around an elliptic fixed point, see Fig. (1b). As expected from the KAM theorem \([40–43]\), the regular region consists of one-dimensional invariant tori. Along these tori orbits of regular motion rotate around the fixed point. These tori are interspersed by nonlinear resonance chains, wherever \( s \) rotations of a regular orbit match \( r \) periods of the external driving \([45, 50, 51]\). For example, the standard map at \( \kappa = 3.4 \) has a dominant \( r:s = 6:2 \) resonance, leading to the six regular sub-regions in Fig. (1b). Note that we choose the numbers \( r \) and \( s \) in the ratio \( r:s \) such that \( r \) is the number of sub-regions of the resonance. The region of regular motion is embedded in a region of chaotic motion.

The quantum-mechanical analogue of the stroboscopic map \( U \) is the unitary time-evolution operator \([52–57]\)

\[
\hat{U} = \exp \left( -i \frac{V(q)}{2\hbar} \right) \exp \left( -i \frac{T(p)}{\hbar} \right) \exp \left( -i \frac{V(q)}{2\hbar} \right).
\]

Here, \( \hbar = 2m\hbar \) is the effective Planck constant and \( \hat{q} \) and \( \hat{p} \) are the operators of position and momentum, respectively. Similarly to the classical case we consider \( \hat{U} \) on a toric phase space, which leads to grids in position and momentum space \([52–57]\)

\[
\begin{align*}
\overline{q}_n &= \hbar(n + \theta_p), & \text{with} & \quad \overline{q}_n \in [0, 1] \quad (7a) \\
\overline{p}_n &= \hbar(n + \theta_q), & \text{with} & \quad \overline{p}_n \in [-0.5, 0.5], \quad (7b)
\end{align*}
\]

with \( n \in \mathbb{N} \). This implies that the inverse of the effective Planck constant is a natural number \( 1/\hbar = N \in \mathbb{N} \), giving the dimension of the Hilbert space. For the standard map, we choose the Bloch phase \( \theta_p = 0 \), while \( \theta_q = 0 \) if \( N \) is even and \( \theta_q = 0.5 \) if \( N \) is odd. This gives the finite-dimensional time-evolution operator in position representation

\[
\langle \overline{q}_m | \hat{U} | \overline{q}_k \rangle = e^{-i\pi/4} \sqrt{N} \exp \left( i 2\pi N \left[ -\frac{V(\overline{q}_m)}{2} + \frac{(\overline{q}_m - \overline{q}_k)^2}{2} - \frac{V(\overline{q}_k)}{2} \right] \right),
\]

with \( n, k = 0, \ldots, N - 1 \).

In the following it is fundamental that eigenstates of a mixed regular–chaotic system can be classified according to their semiclassical localization on the regular or chaotic region, respectively. More specifically, chaotic states spread across the chaotic region \([55, 60]\), while regular states localize on a torus \( \tau_m \) of the regular region which has quantizing action \([61–63]\)

\[
J_m := \frac{1}{2\pi} \int_{\tau_m} p(q) \, dq = (m + 1/2)\hbar,
\]

labeled by an index \( m \in \mathbb{N} \). In order to account for resonance-assisted tunneling it is further indispensable to consider the finer structure of regular states. In particular, it will be crucial that a regular state \( m \) localizes not only on a dominant quantizing torus \( J_m \). Instead, an \( r:s \) resonance induces additional contributions on the tori \( J_{m+kr} \) with \( k \in \mathbb{Z} \), see Refs. \([17, 20, 64–66]\) and references therein.

B. Regular-to-Chaotic Decay Rates in the Standard Map

In this section we introduce regular-to-chaotic decay rates \( \gamma \) of an open system for quantifying regular-to-chaotic tunneling. Note that in closed systems chaos-assisted tunnel splittings \([4]\) are an often-used alternative \([32]\).

Our general approach for defining regular-to-chaotic decay rates proceeds in three steps: (a) We introduce a leaky region \( \mathcal{L} \) within the chaotic part of phase space, (b) we determine the decay rates of its regular states, and (c) we classify the corresponding decay rates as regular-to-chaotic decay rates. Step (c) is justified because each regular state of the open system decays by regular-to-chaotic tunneling towards the chaotic region and subsequently entering the leaky region within the chaotic part of phase space.

More specifically, we proceed by (a) introducing a projector \( \hat{P}_\mathcal{L} \) which absorbs probability on a phase-space region \( \mathcal{L} \) within the chaotic part of phase space. Based on this projector and the unitary time-evolution operator \( \hat{U} \) of the closed system we define the time-evolution operator of the open system as

\[
\hat{U}_o = (\mathbb{1} - \hat{P}_\mathcal{L}) \hat{U} (\mathbb{1} - \hat{P}_\mathcal{L}).
\]

(b) We solve its eigenvalue equation

\[
\hat{U}_o | m \rangle = \exp \left( i \phi_m - \frac{\gamma_m}{2} \right) | m \rangle.
\]

Here, \( | m \rangle \) represents a metastable, right eigenvector of the sub-unitary operator \( \hat{U}_o \). The corresponding eigenvalue is determined by an eigenphase \( \phi_m \) and a decay rate \( \gamma_m \). The latter describes the exponential decay of \( | m \rangle \) in time. (c) We assign to each regular state \( | m \rangle \)
a label \( m \) according to its dominant localization on the quantizing torus \( J_m \) and refer to its decay rate \( \gamma_m \) as the regular-to-chaotic decay rate.

Specifically, for the standard map \((a)\) we use
\[
\mathcal{L} := \{(q,p) \mid q < q_l \text{ or } q > q_r := 1 - q_l\},
\]
and define the projector
\[
\hat{P}_\mathcal{L}|q\rangle = \chi(q)|q\rangle \quad \text{with } \chi(q) = \begin{cases} 1 & \text{for } (q,\cdot) \in \mathcal{L} \\ 0 & \text{for } (q,\cdot) \notin \mathcal{L}. \end{cases}
\]
(13)

Here, we choose \( q_l \) close to the regular–chaotic border. This ensures that \( \gamma_m \), which depends on the choice of the leaky region \( \mathcal{L} \), is dominated by tunneling from the regular towards the chaotic region. For a more detailed discussion see Sec. \( \nabla A \) \((b)\) We compute the finite-dimensional matrix representation of \( \hat{U}_\mathcal{L} \) for each value of \( 1/\hbar \in \mathbb{N} \). To this end we set all those entries in Eq. \( \nabla \) equal to zero, for which either \( \vartheta_m \) or \( \vartheta_q \) are in the leaky region \( \mathcal{L} \). We diagonalize the resulting \( \hat{U}_\mathcal{L} \) numerically. (c) The regular-to-chaotic decay rates \( \gamma_m \) are labeled according to the dominant localization of \( |m\rangle \) on the quantizing tori \( J_m = \hbar(m + 1/2) \).

We present the numerically obtained regular-to-chaotic decay rates \( \gamma_0 \) of the standard map at \( \kappa = 3.4 \) as a function of the inverse effective Planck constant \( (\text{gray} \) dots) in Fig. \( \nabla 1(a) \). The numerical results are consistent with the expectations due to Refs. \( \nabla 3 \nabla 4 \nabla 10 \) \((i)\) For \( 1/\hbar \lesssim 35 \) the state \( |0\rangle \) localizes on the torus \( J_0 \) such that the direct tunneling from \( J_0 \) to \( \mathcal{L} \) dominates. In this regime, \( \gamma_0 \) decreases exponentially for decreasing \( \hbar \) which is a characteristic feature of direct transitions, see Ref. \( \nabla 3 \nabla 4 \nabla 33 \nabla 67 \). \((ii)\) In the regime \( 1/\hbar \gtrsim 35 \) tunneling is enhanced by the 6:2 resonance. For \( 35 \lesssim 1/\hbar \lesssim 80 \) the resonance contribution of the state \( |0\rangle \) on \( J_0 \) is significant such that direct tunneling transition from \( J_0 \) to \( \mathcal{L} \) dominates \( \gamma_m \). This leads to a peak at \( 1/\hbar = 53 \), where the state \( |0\rangle \) has half its weight on \( J_0 \). Finally, for \( 1/\hbar \gtrsim 80 \) the resonance contribution of \( |0\rangle \) on \( J_{12} \) is significant such that direct tunneling from \( J_{12} \) to \( \mathcal{L} \) dominates the decay rate \( \gamma_m \), with a peak at \( 1/\hbar = 98 \). In Fig. \( \nabla 3(a,c) \) we show similar numerical rates \( (\text{gray} \) dots) for the standard map at \( \kappa = 2.9 \) and \( \kappa = 3.5 \) with a dominating 10:3 and 6:2 resonance, respectively.

III. PERTURBATION-FREE PREDICTIONS OF RESONANCE-ASSISTED REGULAR-TO-CHAOTIC TUNNELING

In this section we derive the perturbation-free predictions for resonance-assisted regular-to-chaotic decay rates. In Sec. \( \nabla A \) we derive Eq. \( \nabla 2 \) which uses the time-evolution operator. In Sec. \( \nabla B \) we derive Eq. \( \nabla 3 \) which does not use the time-evolution operator.

### A. Derivation of Eq. \( \nabla 2 \) with Time Evolution

The starting point for deriving Eq. \( \nabla 2 \) is the definition of the regular-to-chaotic decay rate \( \gamma_m \) from the appropriate eigenvalue problem. We use the same definitions as for the numerical determination of regular-to-chaotic decay rates, see Eqs. \( \nabla 10 \) and \( \nabla 11 \) of Sec. \( \nabla B \). They are repeated for convenience, namely a general sub-unitary operator
\[
\hat{U}_\omega := (1 - \hat{P}_\mathcal{L})\hat{U}(1 - \hat{P}_\mathcal{L}),
\]
and its eigenvalue equation
\[
\hat{U}_\omega |m \rangle = \exp \left(i\phi_m - \frac{\gamma_m}{2} \right) |m \rangle.
\]
(15)

Here, the unitary operator \( \hat{U}_\omega \) describes the time evolution of a mixed regular–chaotic system over one unit of time. Furthermore, \( \hat{P}_\mathcal{L} \) is a projection operator which absorbs probability on the leaky region \( \mathcal{L} \) within the chaotic part of phase space.

For decay rates of such systems, it can be shown, that the following formula applies, see App. \( \nabla A \) for details,
\[
\gamma_m = - \log \left(1 - \|\hat{P}_\mathcal{L}\hat{U}|m \rangle\|^2\right) \approx \|\hat{P}_\mathcal{L}\hat{U}|m \rangle\|^2.
\]
(16)

i.e., a regular-to-chaotic decay rate \( \gamma_m \) (for which \( \gamma_m \ll 1 \)) is given by the probability transfer from the regular state \( |m \rangle \) into the leaky region \( \mathcal{L} \) via the unitary time-evolution operator \( \hat{U} \). Equation \( \nabla 16 \) is as such not useful, since it still contains the unknown eigenvector \( |m \rangle \). In particular, it would require to solve Eq. \( \nabla 15 \) which defines \( \gamma_m \) in the first place. Hence, we proceed in the spirit of Refs. \( \nabla 31 \nabla 37 \), i.e., we approximate \( |m \rangle \) using the eigenstates \( |m_{\text{int}} \rangle \) of an integrable approximation \( H_{r,s} \), leading to our prediction Eq. \( \nabla 2 \).

The novel point of this paper is the use of an integrable approximation \( H_{r,s} \), which includes the dominant \( r,s \) resonance. This ensures that \( |m_{\text{int}} \rangle \) models not only the localization of \( |m \rangle \) on the main quantizing torus \( J_m \) but also accounts for the resonance-assisted contributions on the tori \( J_{m+k}\omega \). Precisely this extends Eq. \( \nabla 2 \), as previously used in \( \nabla 30 \nabla 33 \) for direct tunneling, to the regime of resonance-assisted regular-to-chaotic tunneling in a non-perturbative way.

### B. Derivation of Eq. \( \nabla 3 \) without Time Evolution

In this section we derive Eq. \( \nabla 3 \). It predicts regular-to-chaotic decay rates from the localization of the mode \( |m_{\text{int}} \rangle \) on the leaky region \( \mathcal{L} \). In contrast to Eq. \( \nabla 2 \) it does not use the time-evolution operator. In that, Eq. \( \nabla 3 \) is an ideal starting point for future semiclassical predictions of regular-to-chaotic decay rates \( \nabla 4 \) in the spirit of Refs. \( \nabla 41 \nabla 42 \). In particular, it avoids the complications which arise in a semiclassical evaluation of Eq. \( \nabla 2 \) due...
to the time-evolution operator. We further remark that predictions like Eq. (3) are common for open systems. For regular-to-chaotic decay rates they have heuristically been used, e.g. in Refs. [20, 26, 27, 40]. Here, the main purpose of deriving Eq. (3) is to explicitly point out the involved approximations.

The derivation starts from an alternative definition of the sub-unitary time-evolution operator

$$\tilde{U}_o' := \tilde{U}(1 - \tilde{P}_L),$$

which satisfies the eigenvalue equation

$$\tilde{U}_o' |m'\rangle = \exp \left( i\phi_m - \frac{\gamma_m}{2} \right) |m'\rangle.$$

Compare with Eqs. (14) and (15). As shown in App. B, the operators $\tilde{U}_o$ and $\tilde{U}_o'$ are isospectral. Therefore, they exhibit the same eigenvalues, which give rise to the same regular-to-chaotic decay rates $\gamma_m$. Furthermore, the corresponding normalized right eigenvectors can be transformed into each other, see App. B. We find,

$$|m\rangle = \frac{1}{\exp \left( i\phi_m - \frac{2\pi m}{2} \right)} (1 - \tilde{P}_L) |m'\rangle,$$

which implies that $|m\rangle$ and $|m'\rangle$ localize on the quantizing tori $J_{m+kr}$ of the regular region with equal probability (for $\gamma_m \ll 1$). On the other hand, $|m'\rangle$ is the time-evolved mode $|m\rangle$ according to

$$|m'\rangle = \tilde{U} |m\rangle.$$

Inserting Eq. (20) into Eq. (16) gives

$$\gamma_m = -\log \left( 1 - \|\tilde{P}_L |m'\rangle\| \right) \approx \|\tilde{P}_L |m'\rangle\|^2,$$

which shows that a regular-to-chaotic decay rate $\gamma_m$ (for which $\gamma_m \ll 1$) is equivalent to the probability to find $|m'\rangle$ on the leaky region $L$. Similar to Eq. (16), Eq. (21) is as such not helpful, because it still contains the eigenvector $|m'\rangle$. In particular, it would require to solve Eq. (15) which defines $\gamma_m$ in the first place. Hence, we approximate the mode $|m'\rangle$ using the more accessible eigenstates $|m_{int}\rangle$ of an integrable approximation $H_{r,s}$, leading to our prediction Eq. (3).

Here, the key point is again the use of integrable approximations $H_{r,s}$ which includes the relevant $r:s$ resonance. Therefore, $|m_{int}\rangle$ models not only the localization of $|m'\rangle$ on the main quantizing torus $J_m$ but also its resonance-assisted contributions on the tori $J_{m+kr}$. Precisely this allows for predicting resonance enhanced regular-to-chaotic decay rates from Eq. (3) in a non-perturbative way.

An application of the predictions, Eqs. (2) and (3), for the standard map is demonstrated in Sec. IV. The key approximation, i.e., modeling metastable regular states $|m\rangle$ (or $|m'\rangle$) in terms of eigenstates $|m_{int}\rangle$ of an integrable approximation $H_{r,s}$, is discussed in Sec. VI B.

Moreover, a comparison of the non-perturbative predictions, Eqs. (2) and (3), to the perturbative predictions of Refs. [26, 40] is given in Sec. VI B.

IV. PERTURBATION-FREE PREDICTION OF TUNNELING IN THE STANDARD MAP

In this section we illustrate our approach by applying it to the standard map. In Sec. IV A we determine the $r:s$ resonance which dominates tunneling. In Sec. IV B we set up an integrable approximation including the nonlinear resonance chain using the iterative canonical transformation method [43, 68] as presented in Ref. [43]. In Sec. IV C we quantize the integrable approximation and determine its eigenstates $|m_{int}\rangle$ from Eq. (11). Finally, the results will be discussed in the next section, Sec. V.

A. Choosing the Relevant Resonance

In order to apply our prediction it is crucial to first identify the $r:s$ resonance which dominates the tunneling process. A detailed discussion as to which resonance dominates tunneling in which regime, can be found in Ref. [39]. Here, we focus on the $r:s$ resonance of lowest order $r$, which dominates the numerically and experimentally relevant regime where $\gamma > 10^{-15}$.

The area covered by the sub-regions of such a resonance can be very small, see the inset of Fig. 3(a). Therefore, it is necessary to search for resonances systematically. To this end we determine the frequencies of orbits within the regular region, as described in Ref. [43]. We then identify the $r:s$ resonance of lowest order $r$, by searching for the rational frequencies $2\pi s/r$ with smallest possible denominator.

Specifically, for the standard map parity implies that $r$ has to be an even number in order to reflect the correct number of subregions forming the resonance chain. For the examples we consider in this paper we find a dominant 10:3 resonance for $r = 2.9$ and a dominant 6:2 resonance for both $\kappa = 3.4$ and $\kappa = 3.5$.

B. Integrable Approximation of a Regular Region including a Resonance Chain

In order to determine an integrable approximation of the regular region which includes the dominant $r:s$ resonance, we use the method introduced in Ref. [42]. Here, we briefly summarize the key points.

The integrable approximations $H_{r,s}(q,p)$ of Ref. [42] is generated in two steps. First the normal-form Hamiltonian is defined as

$$H_{r,s}(\theta, I) = H_0(I) + 2V_{r,s} \left( \frac{I}{I_{r,s}} \right)^{r/2} \cos(r\theta + \phi_0),$$

$$H_0(I) = \frac{(I - I_{r,s})^2}{2M_{r,s}} + \sum_{n=3}^{N_{r,s}} h_n(I - I_{r,s})^n.$$

It contains the essential information on the regular region in the co-rotating frame of the resonance. This Hamiltonian is precisely the effective pendulum Hamiltonian...
used in Ref. [34, 40]. Here, $\mathcal{H}_0(I)$ is a low order polynomial, chosen such that its derivative fits the actions and frequencies of the regular region in the co-rotating frame of the resonance. The action of the resonant torus is $I_{r,s}$. The parameters $M_{r,s}$ and $V_{r,s}$ are determined from the size of the resonance regions in the mixed system as well as the stability of its central orbit [23]. Finally, $\phi_0$ is used to control the fixed-point locations of the resonance chain.

In a second step, a canonical transformation

$$\mathcal{T} : (\theta, I) \mapsto (q, p)$$

is used to adapt the tori of the effective pendulum Hamiltonian to the shape of the regular region in $(q, p)$-space, giving the Hamilton function

$$H_{r,s}(q, p) = \mathcal{H}_{r,s}(\mathcal{T}^{-1}(q, p)).$$

The transformation $\mathcal{T}$ is composed of: (i) a harmonic oscillator transformation to the fixed point of the regular region $\mathcal{T}^0$, Eq. [C1], which provides a rough integrable approximation and (ii) a series of canonical near-identity transformations $\mathcal{T}^1, \ldots, \mathcal{T}^{N_r}$, Eq. [C2], which improve the agreement between the shape of tori of the mixed system and the integrable approximation.

Note that a successful prediction of decay rates requires an integrable approximation which provides a smooth extrapolation of tori into the chaotic region [34, 35], see insets of Fig. 3. This is ensured by using simple near-identity transformations $\mathcal{T}^1, \ldots, \mathcal{T}^{N_r}$, i.e., low orders $N_q, N_p$ in Eq. [C2]. For further details the reader is referred to Ref. [43] and Appendix C where it is described how the integrable Hamiltonians for the standard map at $\kappa = 2.9$, $\kappa = 3.4$ and $\kappa = 3.5$, see insets of Fig. 4, are generated.

C. Quantization of the Integrable Approximation

In the following, we summarize the quantization procedure for the integrable approximation. The details are discussed in App. C2. In its final form, this quantization procedure is almost identical to the approach presented in Ref. [34]. It consists of two steps: (Q1) The integrable approximation without resonance is used to construct states which localize along a single quantizing torus of the regular region. (Q2) The mixing of states, localizing along a single quantizing torus, is described by solving the quantization of the effective pendulum Hamiltonian, Eq. [22], introduced in Ref. [10]. Combining (Q1) and (Q2) gives the sought-after eigenstate $|m_{\text{int}}\rangle$ of the integrable approximation which includes the resonance.

More specifically: (Q1) We use the canonical transformation, Eq. [23], in order to define the function $I(q, p)$. Its contours approximate the tori of the regular phase-space region, ignoring the resonance chain. It thus resembles the role of the integrable approximation, previously used in Refs. [34, 35, 36]. The Weyl-quantization of this function on a phase-space torus gives a Hermitian matrix

$$\langle q_n | \hat{\mathcal{H}}_I | q_m \rangle = \frac{1}{2^N} \sum_{l=0}^{2N-1} \exp \left( i \frac{l}{\hbar} (\mathcal{T}_n - \mathcal{T}_m) \hat{p}_I \right) \times \langle q_n | \mathcal{H}_0 | q_m \rangle \left( I \left( \frac{\mathcal{T}_n + \mathcal{T}_m + M_k}{2} \right) + (-1)^l I \left( \frac{\mathcal{T}_n + \mathcal{T}_m + M_k}{2} \right) \right).$$

(25)

Solving its eigenvalue equation gives states $|\mathcal{T}_l | I_n \rangle$ which localize along a single contour of $I(q, p)$ with quantizing action $I_n = \hbar(n + 1/2)$. These states model the localization of states along the tori of quantizing action $J_n$ in the mixed system. For an illustration see Fig. 2(a, b).

(Q2) In the second step, we model the mixing of states $|\mathcal{T}_l | I_n \rangle$ due to the nonlinear resonance chain. To this end, we follow Refs. [34, 40] and consider the quantization of the effective pendulum Hamiltonian, Eq. [22], given by

$$\langle I_m | \hat{\mathcal{H}}_{r,s} | I_n \rangle = \mathcal{H}_0(I_n) \delta_{m,n} + V_{r,s} \left( \frac{\hbar}{I_{r,s}} \right)^{r/2} \left( e^{-i\phi_0} \sqrt{\frac{n!}{(n-r)!}} \delta_{m,n-r} + e^{i\phi_0} \sqrt{\frac{(n+r)!}{n!}} \delta_{m,n+r} \right).$$

(26)

Solving this eigenvalue problem gives the sought-after state in the basis of quantizing actions $|I_n | m_{\text{int}} \rangle$. Note that the matrix in Eq. [26] couples basis states $|I_n \rangle$ and $|I_{n+r} \rangle$ only if $|n' - n| = kr$. Thus, the coefficients $\langle I_n | m_{\text{int}} \rangle$ are non-zero, only if $n = m + kr$. This is called the selection rule of resonance-assisted tunneling. Combining (Q1) and (Q2) results in the mode expansion

$$|\mathcal{T}_l | m_{\text{int}} \rangle = \sum_k \langle \mathcal{T}_l | I_{m+kr} \rangle |I_{m+kr} | m_{\text{int}} \rangle.$$  

(27)

For an illustration of a state $|m_{\text{int}} \rangle$ see Fig. 2(c). Note that its Husimi-function exhibits exactly the morphology discussed in Ref. [63].

We now make a couple of remarks: (a) We use the above quantization procedure, rather than directly applying the Weyl-rule to $H_{r,s}(q, p)$, Eq. [23], in order to explicitly enforce the selection rule of resonance-assisted tunneling. (b) The ad-hoc two step quantization scheme avoids the problem of defining the quantum counterpart for the canonical transformations $\mathcal{T}^1, \ldots, \mathcal{T}^{N_r}$, Eq. [C2], used in the classical construction of the integrable approximation, see App. C2 for details. (c) The above quantization is almost identical to the procedure used in Refs. [34, 40]. This allows for a direct comparison to the results of Refs. [34, 40], see Sec. V B. (d) The quantization procedure cannot determine the relative phase between the terms in the mode expansion of Eq. [27].

In order to understand the relative phase recall: (i) The coefficient vector $|I_{m+kr} | m_{\text{int}} \rangle$ is determined by solving the eigenvalue problem of Eq. [26]. Hence, it is determined up to a global phase $\varphi_{m+kr}$. Therefore: (i)
has no consequences for predicting decay rates. However, (ii) changing the phases $\varphi_{m+kr}$ of each coefficient vector $\langle q | I_{m+kr} \rangle$, changes the relative phase of contributions in Eq. (27). This changes the interference between the contributions to the sum in Eq. (27) and affects the predicted decay rates.

So far the phase issue was avoided by neglecting interference terms in the tunneling predictions [36, 40]. For the symmetrized standard map, we propose to define the phases as follows: (i) Eq. (26) gives a real symmetric matrix. This allows for choosing real coefficients $\langle I_n | m_{\text{int}} \rangle$ such that $\langle I_m | m_{\text{int}} \rangle > 0$. (ii) Eq. (25) also gives a real symmetric matrix. This allows for choosing real coefficients $\langle q_n | I_n \rangle$. Choosing the sign of these coefficients is discussed in App. C 2. The main idea is to exploit the eigenstates of the harmonic oscillator which approximates the central fixed point of the regular region. For these harmonic oscillator states the relative phase is well-defined. Then we choose the sign of $\langle q_n | I_n \rangle$ such that its overlap with the corresponding eigenstate of the harmonic oscillator is positive, Eq. (C29).

V. RESULTS

We now apply the above procedure to the standard map at $\kappa = 2.9, 3.4, \text{ and } 3.5$. This gives eigenstates $| m_{\text{int}} \rangle$ which we insert into our predictions, Eqs. (2) and (3). The necessary time-evolution operator, used in Eq. (2), is given by Eq. (8). The projector is defined by Eq. (13) using $q_l = 0.27, 0.26, 0.25$, respectively. The results are shown in Fig. 3. The numerically determined rates and the predicted rates are overall in good qualitative agreement. In both cases they deviate from the exact numerical rates by at most two orders of magnitude. In that the accuracy of the perturbation-free predictions, Eq. (2) and Eq. (3), is equivalent to perturbative predictions from Refs. [36, 40]. This establishes Eqs. (2) and (3) as state of the art perturbation-free predictions of resonance-assisted regular-to-chaotic tunneling. See Sec. V B for a detailed comparison.

A. Incoherent Predictions and Quantum Phase

As discussed in Sec. IV C our quantization scheme cannot determine the relative phases between the contributions of Eq. (27) for a system without time-reversal symmetry. In the following, we discuss the consequences of such an undetermined phase for the prediction of decay rates. To this end we summarize our predictions, Eqs. (2) and (3), in the following compact form

$$\Gamma_m(t) := \| \hat{P}_E \hat{U}^t | m_{\text{int}} \rangle \|^2,$$

where $t = 1$ denotes the prediction based on time-evolution and $t = 0$ denotes the prediction without time evolution. Now we insert the mode expansion, Eq. (27),
FIG. 3. (color online) Decay rates for the standard map at (a) $\kappa = 2.9$, (b) $\kappa = 3.4$, and (c) $\kappa = 3.5$ versus the inverse effective Planck constant $1/\hbar$. Numerically determined rates (dots) are compared to predicted rates, using Eq. (2) ([red] lines) and chaotic orbits (dots) with tori of the integrable approximation ([gray] lines) and Eq. (3) ([magenta] squares). The insets show the corresponding phase space with regular tori ([gray] lines) and chaotic orbits (dots) with tori of the integrable approximation ([red] lines).

and average over the undetermined phases $\varphi_{m+kr}$ of the coefficient vectors $\langle \tilde{Q}_l | I_{m+kr} \rangle$. This gives the incoherent prediction

$$\Gamma_{m,n}^{\text{inc}}(t) := \sum_k \Gamma_{m,m+kr}^{\text{diag}}(t)$$

(29)

where the diagonal term $\Gamma_{m,n}^{\text{diag}}(t)$ is the contribution of the state $|I_n\rangle$ to the incoherent prediction as

$$\Gamma_{m,n}^{\text{diag}}(t) := |\langle I_n | m_{\text{int}} \rangle|^2 \Gamma_n^{\text{d}}(t)$$

(30)

and

$$\Gamma_n^{\text{d}}(t) := \| \tilde{P}_L \tilde{U}^t | I_n \rangle \|^2.$$ 

(31)

is the rate of direct regular-to-chaotic tunneling as previously introduced in Refs. [30, 35].

The results based on Eq. (29) are shown in Fig. 4. As expected the incoherent predictions, Eq. (29), and the full predictions, Eq. (28), agree very well in the regime where a single diagonal contribution dominates, i.e., in the regime of direct tunneling as well as the peak region. However, in between these regions there are always two diagonal contributions of similar magnitude, which can interfere. It is in these regions that we observe clear deviations between the predictions of Eq. (28) and the incoherent predictions of Eq. (29). In particular, for $\kappa = 3.4$ and $\kappa = 3.5$ the prediction of Eq. (28) predicts destructive interference, while the incoherent results describe the numerical rates much better.

These results highlight the relevance of the phase factor $\varphi_{m+kr}$ for obtaining an accurate description of decay rates even between the resonance-assisted tunneling peaks. In previous studies of resonance-assisted tunneling in systems with a mixed phase space [35] this phase factor has been ignored by directly employing the incoherent predictions. Hence, a satisfactory theoretical treatment of the phase factor $\varphi_{m+kr}$ does so far not exist. Clearly, our current approach is also insufficient. The precise reason is not clear to us. We expect that exploiting the symmetry of the integrable approximation in order to find a real representation of the approximate mode $|\tilde{Q}_l | m_{\text{int}} \rangle$ is too naive. In particular, because it is used for approximating the metastable state $|\tilde{Q}_l | m \rangle$ of the open standard map, which can never admit an entirely real representation. For a detailed discussion of this point see Sec. VI C. Another possibility is that the phase factor in a non-integrable system is beyond an integrable approximation.

B. Perturbative Predictions

In this section, we compare our results to the perturbative predictions of Refs. [34, 40]. This perturbative prediction is obtained by approximating the coefficient $\langle I_{m+kr} | m_{\text{int}} \rangle$ in the incoherent prediction Eq. (29) by solving Eq. (26) perturbatively, [40],

$$\langle I_{m+kr} | m_{\text{int}} \rangle \approx \mathcal{A}_{m,m+kr}^{(rs)} := \prod_{l=1}^k \frac{\langle I_{m+kr} | \hat{H}_{rs} | I_{m+(l-1)kr} \rangle}{\mathcal{H}_0(I_m) - \mathcal{H}_0(I_{m+kr})},$$

(32)
Note that $\mathcal{H}_0(I)$ is considered in the co-rotating frame. This leads to

$$
\Gamma_{m}^{\text{per}}(t) := \sum_k |A_{m,m+kr}^{(r,s)}|^2 \Gamma_{m}(t).
$$

A slight difference of the above expression as compared to Ref. [36, 40] is the use of the projector $\hat{P}_L$ rather than a projector on the whole chaotic region. Thus our prediction eliminates a free parameter from the perturbative predictions of Refs. [36, 40]. The results of the perturbative predictions are presented in Fig. 5. They agree with the prediction obtained from Eq. [24], with the slight difference that the perturbative results deviate around the peak region.

We conclude this section with a short list of advantages and disadvantages of the perturbation-free and perturbative predictions:

(i) The perturbation-free framework, Eqs. (2) and (3), as well as their incoherent version, Eq. (29), predict nu-
merical rates with similar accuracy as the perturbative framework of Refs. [36, 40].

(ii) One advantage of the perturbative prediction is the possibility to evaluate the terms \( \langle |m| m_{\text{int}} \rangle \) analytically, Eq. (33). Yet, for practical use even the perturbative approach requires an integrable approximation for predicting the direct rates \( \Gamma_m^d \). Hence, both predictions are equally challenging in their implementation.

(iii) Another advantage of the perturbative prediction is the possibility to include multiple resonances into Eq. (33), which is not yet possible for the perturbation-free predictions presented in this paper. Note that this restriction is not too severe, because decay rates in the experimentally and numerically accessible regimes \( \gamma > 10^{-15} \) are typically affected by a single resonance only. Nevertheless, an extension of the perturbation-free results to the multi-resonance regime is of theoretical interest and requires normal-form Hamiltonians \( \mathcal{H}_{r,s} \) which include multiple resonances.

(iv) The main advantage of the perturbation-free framework is that it provides the foundation for deriving a future semiclassical prediction of resonance-assisted regular-to-chaotic tunneling [43].

VI. DISCUSSION

In this section, we discuss several aspects of our results in detail. In Sec. VI A we discuss the dependence of decay rates on the choice of the leaky region. In Sec. VI B we compare the metastable states \( |m\rangle \) and \( |m'\rangle \) to the eigenstate \( |m_{\text{int}}\rangle \) of an integrable approximation. In Sec. VI C we analyze the approximation of \( |m\rangle \) and \( |m'\rangle \) via \( |m_{\text{int}}\rangle \) more systematically. In Sec. VI D we comment on the predictability of peaks.

A. Dependence of Decay Rates on the Leaky Region

This paper focuses entirely on situations where the leaky region \( \mathcal{L} \) is chosen close to the regular–chaotic border region. However, in generic Hamiltonian systems like the standard map, the chaotic region is interspersed with partial barriers [64, 72]. This leads to sticky motion in a hierarchical region surrounding the regular region. Furthermore, the chaotic component might be inhomogeneous and exhibit slow classical transport.

In view of these classical phenomena, it is not surprising that the numerical decay rates of the standard map, defined via Eqs. (15), depend on the choice of the leaky region via the parameter \( \kappa \). In order to illustrate this phenomenon, we show the numerically determined decay rate \( \gamma_{\text{st}} \) of the standard map for two choices of the leaky region and two different \( \kappa \) parameters in Figs. 6 and 7, respectively.

In Fig. 6 we show results for the standard map at \( \kappa = 3.4 \). Here, we compare (i) the regular-to-chaotic decay rates obtained for \( \kappa = 0.26 \) (parameter used in this paper, [gray] dots) to (ii) decay rates obtained for \( \kappa = 0.1 \) ([magenta] squares). While the decay rates for \( \kappa = 0.26 \) exhibit a rather smooth behavior the decay rates for \( \kappa = 0.1 \) clearly exhibit additional oscillations and some overall suppression.

An even stronger deviation between regular-to-chaotic decay rates with varying leaky regions is observed in Fig. 7 for the standard map at \( \kappa = 2.9 \). Here, (i) the decay rates as obtained for \( \kappa = 0.27 \) (parameter used in this paper, [gray] dots) are compared to (ii) the decay rates obtained for \( \kappa = 0.1 \) ([magenta] squares). In addition to oscillations, the decay rates for \( \kappa = 0.1 \) exhibit a clear suppression of their average value.

The origin of these deviations is unclear. The suppression of decay rates for leaky regions far from the regular–chaotic border could be due to slow transport through an inhomogeneous chaotic region from the regular–chaotic border towards the leaky region.

So far a quantitative prediction of decay rates with leaky regions far from the regular–chaotic border remains an open problem. While varying the leaky region \( \mathcal{L} \) close to the regular–chaotic border can be accounted for by our approach, predicting decay rates with leaky region far from the regular–chaotic border is beyond our framework. In particular, while we observe that the numerical decay rates stabilize when pushing the leaky region away from the regular–chaotic border, the predicted rates con-
Note that, accurately predicting decay rates based on Eqs. (2) and (3), even for leaky regions far from the regular region, requires modes \(|m_{\text{int}}\rangle\) which model the localization of \(|m\rangle\) and \(|m'\rangle\) even in the chaotic region. We expect that this is beyond the framework of an integrable approximation.

### B. Metastable States and Integrable Eigenstates

We now discuss the key approximation of our predictions. To this end we compare the metastable states \(|m\rangle\) and \(|m'\rangle\) to the corresponding approximate state \(|m_{\text{int}}\rangle\), which originates from an integrable approximation \(H_{\text{int}}\) including the relevant resonance. We focus on a typical example using the states \(m = m' = m_{\text{int}} = 0\) of the standard map at \(\kappa = 3.4\) with \(1/h = 55\) close to the first resonance peak in Fig. 11(c). The absolute squared values of the states in position representation are shown in Fig. 8. Here we compare (a) \(|m\rangle\) to \(|m_{\text{int}}\rangle\), (b) \(|m'\rangle\) to \(|m_{\text{int}}\rangle\), and (c) \(|m'\rangle = \hat{U} |m\rangle\) to \(\hat{U} |m_{\text{int}}\rangle\), depicting them by (gray) dots and (magenta) squares, respectively.

As a first conclusion we see that the metastable states are well approximated by their integrable partners within the non-leaky region, i.e., the region between the dashed lines in Figs. 8(a-c). In particular, both the metastable states and their integrable approximations exhibit the generic structure which is determined by the regular region and the dominant 6:2 resonance \([19, 20]\): (i) A main Gaussian-like hump at \(q = 0.5\) marks the main localization of the modes on the torus \(J_0\). (ii) The decrease of the hump is interrupted at two side humps, which correspond to the resonance-assisted contribution of each mode on the torus \(J_0\). From there, the Gaussian-like exponential decrease continues towards the leaky region, which is outside the dashed lines in Figs. 8(a-c).

As a second conclusion from Fig. 8 we infer that beyond the regular–chaotic border, i.e., within the leaky region the metastable states deviate from their integrable counterparts. Here, the integrable states continue to decrease exponentially. In contrast, the state \(|m\rangle\) vanishes, see Fig. 8(a), while the state \(|m'\rangle = \hat{U} |m\rangle\), Eq. (20), does not decrease much slower, see Fig. 8(b,c).

Finally, we emphasize that \(|m'\rangle\) and \(|m_{\text{int}}\rangle\) agree for positions close to the regular–chaotic border. Furthermore, these contributions dominate the probability of \(|m'\rangle\) and \(|m_{\text{int}}\rangle\) on the leaky region. Precisely this ensures that replacing \(|m'\rangle\) in the exact prediction, Eq. (21), by \(|m_{\text{int}}\rangle\) results in a meaningful prediction according to Eq. (3)

An analogous argument explains why replacing \(\hat{U} |m\rangle\) in the exact result, Eq. (10), by \(\hat{U} |m_{\text{int}}\rangle\) gives meaningful predictions according to Eq. (2).
FIG. 8. (color online) For the standard map at $\kappa = 3.4$ with $1/h = 55$ we compare the position representation of (a) $|q(m)|^2$, (b) $|q(m')|^2$, and (c) $|q(U)(m')|^2$ with $m = m' = m_{\text{int}} = 0$. The dashed lines mark the positions $q_i$ and $1 - q_i$ of the leaky region, as given in the text.

C. Error Analysis

In this section, we investigate the approximation of the metastable states $|m\rangle$ in the exact result Eq. (16) via the mode $|m_{\text{int}}\rangle$ in Eq. (8) from the perspective of Eq. (27), i.e., (i) we investigate the basis states $|I_n\rangle$ and (ii) the expansion coefficients $\langle I_n|m_{\text{int}}\rangle$. We focus on the standard map at $\kappa = 3.4$.

(i) In order to investigate our basis set $|I_n\rangle$, we expand the metastable state $|m\rangle$ in this basis and insert this expansion into the exact result (16). This gives

$$\gamma_m = \sum_n \langle I_n|m\rangle^2 \| \hat{P}_z I_n \| ^2 + \sum_{n,n'} \langle I_n|m\rangle \langle I_n|\hat{U}^\dagger \hat{P}_z^2 \hat{U} I_n \rangle \langle I_n|m\rangle .$$

(34)

Since the diagonal terms

$$\gamma_{m,n}^{\text{diag}} := \langle I_n|m\rangle^2 \| \hat{P}_z I_n \| ^2 = \langle I_n|m\rangle^2 \Gamma_n(1)$$

(35)

provide a bound to the off-diagonal terms according to Cauchy’s inequality

$$\left| \langle I_n|m\rangle \langle I_n|\hat{U}^\dagger \hat{P}_z^2 \hat{U} I_n \rangle \langle I_n|m\rangle \right| \leq \sqrt{\gamma_{m,n}^{\text{diag}} \gamma_{m,n'}^{\text{diag}}}$$

(36)

we can interpret them as a way to quantify the contribution of the $n$th basis state $|I_n\rangle$ to the decay rate $\gamma_m$. In that $\gamma_{m,n}^{\text{diag}}$ takes a similar role as the contribution spectrum, discussed in Ref. [38].

In Fig. 9(a), we consider all contributions $\gamma_{0,n}^{\text{diag}}$ (lines) in comparison with the decay rate $\gamma_0$ (dots) for the standard map at $\kappa = 3.4$. While most contributions are two to three orders of magnitude smaller than $\gamma_0$, we find that the contributions $\gamma_{0,0}^{\text{diag}}, \gamma_{0,6}^{\text{diag}},$ and $\gamma_{0,12}^{\text{diag}}$ dominate. In order to further test whether the modes $|I_n\rangle$ with $n = 0, 6, 12$ are sufficient for describing $\gamma_0$ we sum the contributions $n, n' \in \{0, 6, 12\}$ of the dominant terms in Eq. (34). This gives the red curve of Fig. 9(b). From this numerical observations we conclude that a reasonable description of $\gamma_0$ can be extracted using an approximate mode exclusively composed of states $|I_n\rangle$ with $n = 0, 6, 12, \ldots$, as used in Eq. (27). However, it should be noted that the difference between $\gamma_0$ and its reduced version, based on contributions $n, n' \in \{0, 6, 12\}$ in Eq. (34), is already of the order of $\gamma_0$ itself. See the region $70 < 1/h < 100$ of Fig. 9(b) in particular. Hence, reducing the metastable state $|m\rangle$ to an approximate mode $|m_{\text{int}}\rangle$ using only basis states $|I_{n+kr}\rangle$ as in Eq. (27) can at best provide a reasonable backbone for describing the structure of $\gamma_0$. On the other hand, for our example a prediction of $\gamma_0$ where the remainder is smaller than the decay rate based on a reduced set of basis states $|I_n\rangle$ is only possible when summing over many additional contributions, even including $n \neq m + kr$. The precise origin of such contributions $\gamma_{m,n}^{\text{diag}}$ with $n \neq m + kr$ is currently under debate [38]: From the framework of resonance-assisted tunneling [19, 20, 36, 10], we expect that the overlap $\langle I_n|m\rangle$ vanishes for $n \neq m + kr$. Hence, one might argue that the contributions $\gamma_{m,n}^{\text{diag}}$ with $n \neq m + kr$ arise in our example only because our basis $|I_n\rangle$ is insufficiently accurate to decompose $|m\rangle$ according to the theoretical expectation of resonance-assisted tunneling. On the other hand, the authors of Ref. [38] observe non-vanishing contributions $\langle I_n|m\rangle$ also for $n \neq m + kr$ even for a near-integrable situation, where an excellent integrable approximations exist. They argue that non-vanishing $\langle I_n|m\rangle$ should always occur and claim their treatment is beyond the current framework of resonance-assisted tunneling. Independent of the origin of the non-zero contributions $\gamma_{m,n}^{\text{diag}}$ for $n \neq m + kr$, their theoretical description is beyond the scope of this paper. In our examples the irrelevance of these contributions is ensured by choosing leaky regions close to the regular–chaotic border. However, for leaky regions far from the regular–chaotic border the contributions $\gamma_{m,n}^{\text{diag}}$ with $n \neq m + kr$ become relevant.

(ii) In the next step we evaluate the errors introduced by replacing the expansion coefficients $\langle I_{m+kr}|m\rangle$...
by $\langle I_{m+kr} | m_{\text{int}} \rangle$ in Eq. (33). We focus on the corresponding diagonal contributions $\gamma_{0,n}^\text{diag}$ and $\gamma_{m,m}^\text{diag}$, which represent the squared norm of the expansion coefficients $\langle I_{m+kr} | m \rangle$ and $\langle I_{m+kr} | m_{\text{int}} \rangle$ up to a multiplica-

tion by the direct rate $\Gamma_{m+kr}^d(1)$. See lines and symbols in Fig. [9] c), respectively. From this data we conclude that the norm of $\langle I_{m+kr} | m_{\text{int}} \rangle$ provides a reasonable approximations for the norm of the expansion coefficients $\langle I_{m+kr} | m \rangle$. The deviations before each peak could be due to neglecting the higher order action dependencies discussed in Ref. [10] in the Hamilton function of Eq. (22).

Furthermore, we expect that the slightly broader peaks in the numerical rates $\gamma_{m,m}^\text{diag}$ as compared to the sharper peaks of $\Gamma_{m,m}^\text{diag}(1)$ observed for the integrable approximation, are related to the openness of the mixed system.

Finally, in Fig. [9] d) we compare the phases $\text{Arg}(\langle I_{m+kr} | m \rangle)$ and $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$, for $m = 0, 1, 2$, respectively. Here, $\text{Arg}(\cdot) \in (-\pi, \pi]$ is the principal value of the complex argument function. Note that the global phase of $|m\rangle$ is fixed by setting $\text{Arg}(\langle I_{m} | m \rangle) = 0$. The phases $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$ are fixed as described in App. [C]. While the phases of $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$ jump from $\pi$ to zero upon traversing the peak for decreasing $1/\hbar$ (change from destructive to constructive interference) their counterparts for $\text{Arg}(\langle I_{m+kr} | m \rangle)$ seem to follow this jump in a smoothed out way. Compare symbols and lines in Fig. [9] d).

We attribute this phase detuning to the openness of the system, i.e.: (a) The symmetries of the integrable approximation allow for choosing a real representation of the coefficient $\langle I_{m+kr} | m_{\text{int}} \rangle$. Its phase can thus only take values $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle) \in \{0, \pi\}$. In contrast (b) the mode $|m\rangle$ originates from an open system and thus the coefficient $\langle I_{m+kr} | m \rangle$ are usually complex such that $\text{Arg}(\langle I_{m+kr} | m \rangle)$ might take any value.

While the deviation between the numerically determined phases $\text{Arg}(\langle I_{m+kr} | m \rangle)$ and the theoretically predicted phases $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$ are seemingly small in Fig. [9] d), their deviation has huge effects on the predicted decay rate, i.e.: (a) $\text{Arg}(\cdot)$ predicts destructive interference of the diagonal terms in the region before each peak. This leads to strong deviations from the numerical decay rate, see Fig. [9] b). On the other hand, (b) already the minimal detuning of $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$ from our prediction $\text{Arg}(\langle I_{m+kr} | m_{\text{int}} \rangle)$ is sufficient to lift the destructive interference. We assume that this explains why the incoherent prediction, Eq. (2A), as illustrated in Fig. [9] b) describe the numerical rates much better than predictions according to, Eq. (2), see Fig. [9] b).

D. Predictability of Peak Positions

Finally, we discuss the predictability of peak positions. To this end we recall that $\mathcal{H}_0(I)$ in Eq. (22A), is determined by fitting its derivative to the numerically determined actions and frequencies $(\omega, J)$ of the regular phase-space region in the co-rotating frame. For an illustration see Fig. [10]. In particular, the data of the mixed system has a maximal action $J_{\text{max}}$, see (gray) dotted line in Fig. [10]. Hence, $\mathcal{H}_0$ can be well controlled in the regular region $I < J_{\text{max}}$. However, for $I > J_{\text{max}}$ the function $\mathcal{H}_0$
The key point is the use of an integrable approximation to show (a) the fit of \( \mathcal{H}_0(I) \) (line) to the actions and frequencies of the regular region \((J, \bar{\omega})\) (crosses). (b) The function \( \mathcal{H}_0(I) \) is shown as a (red) line. The two (black) dots show \((I_m, \mathcal{H}_0(I_m))\) and \((I_{m+kr}, \mathcal{H}_0(I_{m+kr}))\) at \(1/\hbar = 53\). (a, b) The dotted line shows the position of \( J_{\text{max}} \).

is only an extrapolation to the chaotic region. Furthermore, the integrable approximation predicts a peak for \( \gamma_m \) [19, 20, 36, 40], if

\[
\mathcal{H}_0(I_m) = \mathcal{H}_0(I_{m+kr}),
\]

where \( I_m = \hbar(m + 1/2) \) and \( I_{m+kr} = \hbar(m + kr + 1/2) \). This resonance conditions follows from Eq. (32).

However, for all examples presented in this paper the resonant torus \( I_{m+kr} \) is always located outside of the regular region, where \( \mathcal{H}_0(I) \) is only given by an extrapolation. See Fig. 2(c) for an example of this situation. The (black) dots in Fig. 10(b) show the corresponding situation for \( \mathcal{H}_0(I) \). In such a situation our approach cannot guarantee an accurate prediction of the peak position. Usually, this problem is not too severe and the extrapolation is good enough. An example where this problem appears can be seen in the second peak of Fig. 3(a) where the peak of the numerical decay rates and the predicted rates is shifted by \(1/\hbar = 1\).

VII. SUMMARY AND OUTLOOK

In this paper we present two perturbation-free predictions of resonance-assisted regular-to-chaotic decay rates, Eqs. (2) and (3). Both predictions are based on eigenstates \( |m_{\text{int}}\rangle \) of an integrable approximation \( H_{r:s} \), Eq. (1).

The key point is the use of an integrable approximation \( H_{r:s} \) of the mixed regular–chaotic system which includes the relevant nonlinear resonance chain. Therefore \( |m_{\text{int}}\rangle \) models the localization of regular modes on the regular region, including resonance-assisted contributions in a non-perturbative way. This allows for extending the validity of Eq. (4), previously used for direct tunneling in Refs. [91, 93], to the regime of resonance-assisted tunneling. Furthermore, we introduce a second prediction, Eq. (6), which no longer requires the time-evolution operator. Instead it allows for predicting decay rates using the localization of the approximate mode on the leaky region. In that Eq. (6) provides an excellent foundation for a future semiclassical prediction of resonance-assisted regular-to-chaotic decay rates [44] in the spirit of Refs. [11, 42]. The validity of the presented approach is verified for the standard map, where predicted and numerically determined regular-to-chaotic decay rates show good agreement.

Finally, we list future challenges: (a) The presented approach is so far limited to periodically driven systems with one degree of freedom. An extension to autonomous or periodically driven systems with two or more degrees of freedom is an interesting open problem. (b) The perturbation-free approach applies to the experimentally and numerically relevant regime, where a single resonance dominates regular-to-chaotic tunneling. Its extension to the semiclassical regime where multiple-resonances affect tunneling is of theoretical interest. (c) The suppression of decay rates due to partial barriers is so far treated by choosing leaky regions close to the regular–chaotic region. Explicitly predicting the additional suppression of decay rates due to slow chaotic transport through inhomogeneous chaotic regions remains an open question.

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Appendix A: Derivation of Eq. (16)

In this appendix we derive Eq. (16) starting from Eqs. (14) and (15). Taking the norm of the eigenvalue equation (15) for a normalized state \( |m\rangle \) one finds

\[
\exp(-\gamma_m) = \|\hat{U}_0|m\rangle\|^2 = \langle m|\hat{U}_0^\dagger\hat{U}_0|m\rangle = \langle m|\hat{1} - \hat{P}_E\rangle\dagger\hat{U}_0^\dagger(\hat{1} - \hat{P}_E)\hat{U}(\hat{1} - \hat{P}_E)|m\rangle,
\]

with
where in the last step the definition of $\hat{U}_o$, Eq. (14), is used. We simplify this expression using
\[ (\hat{1} - \hat{P}_L) |m\rangle = |m\rangle, \tag{A2} \]
which follows from Eqs. (14) and (15), giving
\[ \exp(-\gamma_m) = \langle m | \hat{U}^\dagger (\hat{1} - \hat{P}_L)^\dagger (\hat{1} - \hat{P}_L) \hat{U} | m \rangle. \tag{A3} \]
Finally, exploiting the idempotence and hermiticity of the projector $\hat{P}_L$ gives
\[ \exp(-\gamma_m) = \langle m | \hat{U}^\dagger \hat{U} | m \rangle - \langle m | \hat{U}^\dagger \hat{P}_L \hat{U} | m \rangle = 1 - \| \hat{P}_L \hat{U} | m \| ^2, \tag{A4} \]
where in the last step the unitarity of $\hat{U}$ is used. From this follows the expression for regular-to-chaotic tunneling rates, Eq. (16).

Appendix B: Isospectrality

In this appendix, we demonstrate the isospectrality of the sub-unitary operators $\hat{U}_o$ and $\hat{U}_o'$ as defined by Eqs. (14) and (17), respectively. Furthermore, we discuss the transformation relating their eigenmodes. For convenience, we repeat the corresponding eigenvalue equations (15) and (18)
\[ \hat{U}_o | m \rangle = \lambda_m | m \rangle, \tag{B1} \]
\[ \hat{U}_o' | m' \rangle = \lambda'_m | m' \rangle, \tag{B2} \]
where the eigenvalues have been denoted by $\lambda_m$ and $\lambda'_m$.

We now demonstrate the isospectrality of $\hat{U}_o$ and $\hat{U}_o'$. To this end we show:

(a) For each eigenstate $| m \rangle$ of $\hat{U}_o$ with eigenvalue $\lambda_m$, $\hat{U}_o | m \rangle$ is an eigenstate of $\hat{U}_o'$ with the same eigenvalue $\lambda_m$.

This further shows that the normalized eigenmode $| m \rangle$ of $\hat{U}_o$ with eigenvalue $\lambda_m$ gives a normalized eigenmode $| m' \rangle$ of $\hat{U}_o'$ according to Eq. (19).

Appendix C: Details of the Integrable Approximation

In this appendix we summarize some technical aspects on the integrable approximation. Computational details of the classical integrable approximation as well as slight changes as compared to Ref. [43] are given in Sec. C1. Details of the quantization are discussed in Sec. C2.

1. Details of the Classical Integrable Approximation

We now summarize the modifications of the algorithm described in [43] in order to account for the symmetries of our system. Then we give a list of relevant computational parameters.

a. Symmetrization

In agreement with Ref. [43] the canonical transformation $T$, Eq. (23), is composed of (i) an initial canonical transformation
\[ T^0 : (\theta, I) \mapsto \left( Q, P \right) \tag{C1} \]
which provides a rough integrable approximation of the regular phase-space region and (ii) a series of canonical near-identity transformations
\[ T' \equiv T^{N_r} \circ \ldots \circ T^1 : (Q, P) \mapsto (q, p), \tag{C2} \]
which improve the agreement between the shape of the tori of the mixed system and the integrable approximation.

In contrast to Ref. [43] we use the symmetrized standard map in this paper. In order to account for this symmetry, we specify the canonical transformation, Eq. (21), as
\[ T^0 : \left( \frac{\theta}{I} \right) \mapsto \left( \frac{Q}{P} \right) = \left( q^*, \sqrt{2I/s \cos(\theta)} \right) \left( p^* - \sqrt{2I/s \cos(\theta)} \right) \tag{C3} \]
Here, $(q^*, p^*) = (0.5, 0.0)$ are the coordinates of the central fixed point in the standard map. The parameter $s$ is determined from the stability matrix of the standard map at $(q^*, p^*)$
\[ M = \begin{pmatrix} 1 - \frac{s}{2} & 1 \\ -\kappa(1 - \frac{s}{2}) & 1 - \frac{s}{2} \end{pmatrix} \tag{C4} \]
as [68]
\[ \sigma^2 = \left| 1 + \frac{s}{2} - 1 - \frac{s}{2} \right| \left| 1 + \frac{s}{2} \right|, \tag{C5} \]
Furthermore, the symmetry of the systems is imposed on the transformations $T^1, \ldots, T^{N_T}$, Eq. (C2), by specifying their generating function as

$$F^n(q, p') = \left(\sum_{n=1}^{N_\gamma} \sum_{m=1}^{N_p} a_{m,n} \sin(2\pi n[q - q^*]) \sin(2\pi m[p' - p^*]), \right.$$ \(\text{(C6)}\)

rather than using the more general form of Ref. [43, Eq. (31)].

b. Algorithmic Overview

(Ai) We determine the parameters $I_{r,s}$, $M_{r,s}$, $V_{r,s}$, $\phi_0$, Eq. (22) as described in Ref. [28].

(Aii) We determine $H_0(I)$, Eq. (22b), by fitting it to $N_{\text{disp}}$ tuples of action and frequency $(J, \omega)$ describing the tori of the regular region in the co-rotating frame of the resonance.

(Aiii) We determine the near-identity transformations of Eq. (C2). Initially, this requires sampling of the regular region using $N_{\text{ang}}$ points along $N_{\text{tori}}$ tori. The invertibility of the near-identity transformations in a certain phase-space region is ensured by rescaling the coefficients $a_{m,n} \rightarrow \eta a_{m,n}$ in Eq. (C6) using a damping factor $\eta$. If $N_\gamma, N_p$ in Eq. (C6) are too large, the tori of the integrable approximation form curls and tendrils in the chaotic region. In that case the integrable approximation cannot predict decay rates. We control this problem by choosing the largest possible parameters $N_\gamma, N_p$ for which the tori of the integrable approximation provide a smooth extrapolation into the chaotic phase-space region. After a finite amount of steps $N_T$, the canonical transformations do not improve the agreement between the regular region and the integrable approximation. At this point we terminate the algorithm.

c. Computational Parameters

In the following we list the important parameters of the integrable approximation.

For $\kappa = 2.9$ we use $I_{r,s} = 0.009223$, $M_{r,s} = 0.06243$, $V_{r,s} = -1.65 \cdot 10^{-7}$, and $\phi_0 = \pi$. For $H_0$ in Eq. (22b) we used $N_{\text{disp}} = 4$ and fit its derivative to $N_{\text{disp}} = 120$ tori of noble frequency. We use $N_T = 40$ near-identity transformations, Eq. (C2), generated from Eq. (C6) with $N_\gamma = N_p = 2$ and coefficients rescaled by $\eta = 0.05$. The regular region was sampled using $N_{\text{ang}} = 200$ points along $N_{\text{tori}} = 120$ tori, equidistantly distributed in action.

For $\kappa = 3.4$ we use $I_{r,s} = 0.01026$, $M_{r,s} = -0.047$, $V_{r,s} = -1.612 \cdot 10^{-5}$, and $\phi_0 = 0$. For $H_0$ in Eq. (22b) we use $N_{\text{disp}} = 6$ and fit its derivative to $N_{\text{disp}} = 120$ tori, equidistantly distributed in action. We use $N_T = 15$ near-identity transformations, Eq. (C2), generated from Eq. (C6) with $N_\gamma = N_p = 2$ and coefficients rescaled by $\eta = 0.25$. The regular region is sampled using $N_{\text{ang}} = 300$ points along $N_{\text{tori}} = 120$ tori, equidistantly distributed in action.

For $\kappa = 3.5$ we use $I_{r,s} = 0.01244$, $M_{r,s} = -0.048$, $V_{r,s} = -2.98 \cdot 10^{-5}$, and $\phi_0 = 0$. For $H_0$ in Eq. (22b) we use $N_{\text{disp}} = 4$ and fit its derivate to $N_{\text{disp}} = 120$ tori, equidistantly distributed in action. We use $N_T = 15$ near-identity transformations, Eq. (C2), generated from Eq. (C6) with $N_\gamma = N_p = 2$ and coefficients rescaled by $\eta = 0.25$. The regular region is sampled using $N_{\text{ang}} = 300$ points along $N_{\text{tori}} = 120$ tori, equidistantly distributed in action.

d. Robustness

After fixing all parameters as described above the final integrable approximation might differ, depending on the sampling of the regular region. In order to show that this does not affect the final prediction, we evaluate Eq. (2), for three integrable approximations which are based on slightly different sets of sample points. The result is illustrated in Fig. 11. It shows that the prediction is clearly robust.

2. Derivation of Quantization

In the following we sketch the basic ideas leading to the quantization procedure presented in Sec. IV C. To this end we first present the quantization of the Hamilton-function obtained after the transformation $T^\gamma$, Eqs. (C1) and (C2), in Sec. C2 a. In Sec. C2 b we present how we extend these results to the full transformation $T$. 

![Fig. 11](image-url) (color online) Decay rates $\gamma_0$ for the standard map at $\kappa = 3.4$ versus the inverse effective Planck constant. Numerically determined rates ([gray] circles) are compared to predicted rates according to Eq. (2) ([colored] symbols) based on three slightly different integrable approximations.
To quantize the Hamilton-function \( H_{r;s}(Q, P) \) obtained after the canonical transformation \( T^0 \), Eqs. (C1) and (C5), we follow Ref. [40] by starting with the transformed Hamilton-function

\[
H'_{r;s}(Q, P) = H_0 \left( \frac{Q^2 + P^2}{2} \right) + \frac{V_{r;s}}{(2I_{r;s})^{1/2}} \times \left[ \exp(i\phi_0) \left( \sigma^{1/2}[Q - q^*] - i\frac{P}{\sigma^{1/2}} \right)^r + \exp(-i\phi_0) \left( \sigma^{1/2}[Q - q^*] + i\frac{P}{\sigma^{1/2}} \right)^r \right].
\]  

(C7)

In order to quantize this function we replace the coordinates \((Q, P)\) by operators

\[
\hat{Q} \rightarrow \hat{\mathcal{Q}} \quad \text{and} \quad P \rightarrow \hat{\mathcal{P}}
\]

(C8a)

(C8b)

and demand the usual commutation relation

\[
[\hat{\mathcal{Q}}, \hat{\mathcal{P}}] = i\hbar.
\]

(C9)

This allows for introducing the corresponding ladder operators as

\[
\hat{a} := \frac{1}{(2\hbar)^{1/2}} \left( \sigma^{1/2}[\hat{\mathcal{Q}} - \hat{q}^*] + i\frac{\hat{\mathcal{P}}}{\sigma^{1/2}} \right)
\]

(C10a)

\[
\hat{a}^\dagger := \frac{1}{(2\hbar)^{1/2}} \left( \sigma^{1/2}[\hat{\mathcal{Q}} - \hat{q}^*] - i\frac{\hat{\mathcal{P}}}{\sigma^{1/2}} \right)
\]

(C10b)

which admit the commutator

\[
[\hat{a}, \hat{a}^\dagger] = 1,
\]

(C11)

such that we get the number operator

\[
\hat{n} := \hat{a}^\dagger \hat{a}.
\]

(C12)

Based on these operators the quantization of \( H_{r;s}^{(0)} \) takes the form

\[
\hat{H}_{r;s}^{(0)} = \mathcal{H}_0(\hat{I}) + \frac{\hbar}{I_{r;s}} \left[ \hat{a}^\dagger \exp(i\phi_0) + \hat{a} \exp(-i\phi_0) \right],
\]

(C13)

where

\[
\hat{I} := \hbar(\hat{n} + 1/2)
\]

(C14)

is the operator replacing the unperturbed action \( I \). Finally, in order to define the basis states, we identify them with the eigenstates of the number operator leading to

\[
\hat{I} |I_n^{(0)}\rangle = I_n |I_n^{(0)}\rangle,
\]

(C15)

where the eigenvalues become quantizing actions \( I_n = \hbar(n + 1/2) \) and the basis states \( |I_n^{(0)}\rangle \) fulfill

\[
\hat{a} |I_n^{(0)}\rangle = 0 \quad \text{and} \quad |I_n^{(0)}\rangle = \frac{1}{\sqrt{n!}} \hat{a}^\dagger^n |I_n^{(0)}\rangle.
\]

(C16a)

(C16b)

With respect to this position basis \( |I_n^{(0)}\rangle \) become the eigenstates of the harmonic oscillator

\[
\langle Q |I_n^{(0)}\rangle = \left( \frac{\sigma}{\pi \hbar} \right)^{1/2} \frac{1}{\sqrt{2^n n!}} H_n \left( \frac{Q}{\sqrt{\hbar/\sigma}} \right) \exp \left( -\frac{\sigma Q^2}{2\hbar} \right).
\]

(C17)

where \( H_n(\cdot) \) are the Hermite polynomials.

b. Quantization after \( T \)

Our final goal is of course to obtain the quantization of the Hamilton-function \( H_{r;s}(q, p) \) which is related to \( H_{r;s}(Q, P) \) via the canonical transformation \( T' \), Eq. (C2). In order to obtain its quantization we assume that \( T' \) quantum-mechanically corresponds to a unitary operator \( \hat{U}_{T'} \) which has the following properties:

\[
\hat{U}_{T'}^{-1} = \hat{U}_{T'}^{-1}, \quad \hat{Q}' = \hat{U}_{T'} \hat{Q} \hat{U}_{T'}^{-1}, \quad \hat{P}' = \hat{U}_{T'} \hat{P} \hat{U}_{T'}^{-1}
\]

(C18a)

(C18b)

(C18c)

Such an operator exists at least within a semiclassical approximation [71]. Note that \( \hat{Q}', \hat{P}' \) represent the operators \( \hat{Q}, \hat{P} \) within the final coordinate frame \((q, p)\). However, they must not be confused with the operators \( \hat{q}, \hat{p} \) which give rise to the position and momentum basis in the final coordinate frame \((q, p)\). In particular, while \( \hat{q}' |q\rangle = q |q\rangle, \hat{Q}' |q\rangle \neq q |q\rangle \).

Under the above assumption the transformed operators preserve the commutation relation

\[
[\hat{Q}', \hat{P}'] = i\hbar.
\]

(C19)

Furthermore, we get the transformed ladder operators as

\[
\hat{a}' := \hat{U}_{T'} \hat{a} \hat{U}_{T'}^{-1}, \quad \hat{a}'^\dagger := \hat{U}_{T'} \hat{a}^\dagger \hat{U}_{T'}^{-1}
\]

(C20a)

(C20b)

which admit the same commutator

\[
[\hat{a}', \hat{a}'^\dagger] = 1.
\]

(C21)

such that we get the transformed number operator

\[
\hat{n}' = \hat{U}_{T'} \hat{n} \hat{U}_{T'}^{-1}
\]

(C22)
and the transformed action operator

\[ \hat{I} := \hbar (\hat{n}' + 1/2). \]  

(C23)

Based on these operators we can define the transformation of the quantization of \( H_{r,s}^{(0)}(Q, P) \) which we identify with the quantization of \( H_{r,s}(q, p) \). It takes the form [10]

\[ \hat{H}_{r,s} = \hat{H}_0(\hat{I}) + V_{r,s} \left( \frac{\hbar}{I_{r,s}} \right) \left[ \hat{a}^{\dagger} \exp(i\phi_0) + \hat{a} \exp(-i\phi_0) \right]. \]  

(C24)

Finally, in order to define the basis states \( |I_n\rangle \), we identify them with the eigenstates of the number operator \( \hat{n}' \), such that

\[ \hat{I} |I_n\rangle = I_n |I_n\rangle \]  

(C25)

with the basis states \( |I_n\rangle \) which admit the property

\[ \hat{a}' |I_0\rangle = 0 \]  

(C26a)

\[ |I_n\rangle = \frac{1}{\sqrt{n!}} \hat{a}^{n} |I_0\rangle. \]  

(C26b)

(Evaluating \( \hat{H}_{r,s} \), Eq. (C24) in the basis of \( |I_n\rangle \), based on Eqs. (C26) gives the matrix representation of Eq. (26).)

Finally, for connecting \( \hat{H}_{r,s} \) and \( \hat{U} \) we require the basis states with respect to the basis \( |q\rangle \). To this end, one can show from the above equations that

\[ |I_n\rangle = \hat{U}_T |I_n^{(0)}\rangle, \]  

(C27)

such that

\[ \langle q | I_n \rangle = \int dQ \langle q | \hat{U}_T | Q \rangle \langle Q | I_n^{(0)} \rangle. \]  

(C28)

In principle, the operator \( \langle q | \hat{U}_T | q' \rangle \) can be evaluated semiclassically, using the techniques described in Ref. [71]. However, this does not give an analytical closed form result and its evaluation is numerically extremely tedious. Furthermore, \( \hat{U}_T \) is usually so close to an identity transformation such that a semiclassical evaluation of \( \langle q | \hat{U}_T | Q \rangle \) contains too many turning points.

Hence, we take an alternative approach, which is numerically feasible: (i) We recognize that the states \( |I_n\rangle \) are the eigenstates of the operator \( \hat{I} \), originating from the phase-space coordinate \( I \). (ii) We define the function \( I(q, p) \) which is obtained after the full canonical transformation \( T \). (iii) We define the Weyl-quantization of this function on a phase-space torus giving the hermitian matrix of Eq. (26). (iv) We diagonalize this matrix numerically, yielding the states \( |\bar{I}_n\rangle \).

Finally, obtaining the modes \( |\bar{I}_n| I_n \rangle \) from an eigenvalue equation comes at the cost that their relative phase (usually ensured via Eq. (C26) or alternatively via Eqs. (C16) and (C27)) is lost. For the standard map we try to restore this phase by exploiting the symmetry of Eq. (25), which for our system becomes a real symmetric matrix. In that we can ensure that the coefficient vector \( (\bar{I}_n| I_n \rangle \) can be chosen real. Finally, we fix the sign of this coefficient vector, by aligning it with the mode \( (Q| I_n^{(0)} \rangle \) defined via Eq. (C17). This means, we choose the sign of the coefficient vector \( (\bar{I}_n| I_n \rangle \) such that the following relation is fulfilled

\[ \sum_n \langle I_n | \bar{I}_n \rangle \left[ \langle Q | I_n^{(0)} \rangle \right] Q = \bar{I}_n > 0. \]  

(C29)

This assumes that the unitary operator representing the quantum canonical transformation in Eq. (28) is sufficiently close to an identity transformation \( \hat{U}_T \approx 1 \).

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