LONG TIME BEHAVIORS FOR THE INHOMOGENEOUS NLS WITH A POTENTIAL IN $\mathbb{R}^3$

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ABSTRACT. In this article, we aim to study the scattering of the solution to the focusing inhomogeneous nonlinear Schrödinger equation with a potential of form

$$i\partial_t u + \Delta u - Vu = -|x|^{-b}|u|^{p-1}u$$

in the energy space $H^1(\mathbb{R}^3)$. We prove a scattering criterion, and then we use it together with Morawetz estimate to show the scattering theory, which generalizes the results of Dinh [6] to the non-radial symmetric case.

Key Words: Nonlinear Schrödinger equation; Kato class potential; Virial-Morawetz estimate; Scattering theory.

1. Introduction

We consider the Cauchy problem of the inhomogeneous nonlinear Schrödinger equation with a potential:

$$\begin{cases}
  i\partial_t u + \Delta u - Vu = -|x|^{-b}|u|^{p-1}u, \\
  u(0, x) = u_0(x) \in H^1(\mathbb{R}^3),
\end{cases}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $p > 1$, $0 < b < 1$, and $V$ is a real-valued potential satisfying

$$(1.1) \quad V \in \mathcal{K}_0 \cap L^2,$$

and

$$(1.2) \quad \|V_-\|_{\mathcal{K}_0} < 4\pi.$$  

Here, $\mathcal{K}_0$ denotes the closed space of bounded and compactly supported functions endowed with the Kato norm

$$\|V\|_{\mathcal{K}_0} := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy$$

with $V_- := \min\{V, 0\}$ being the negative part of $V$. We recall from [6,17] that the operator $\mathcal{H} := -\Delta + V$ has no eigenvalues, and the dispersive and Strichartz estimates hold for the corresponding Schrödinger flow $\{e^{-it\mathcal{H}}\}$ under the conditions $(1.1)$ and $(1.2)$.

It is known that solutions to $(\text{INLS}_V)$ with suitable regularity conserve the mass and energy, defined respectively by

$$M(u) := \int_{\mathbb{R}^3} |u(t, x)|^2 dx = M(u_0),$$

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u(t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} \frac{|u(t, x)|^{p+1}}{|x|^b} dx = E(u_0).$$
Firstly, let us recall the existence results for the inhomogenous Schrödinger equation without potentials:

\begin{equation}
\text{(INLS)} \quad \begin{cases}
    i\partial_t u + \Delta u = \lambda |x|^{-b}|u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
    u(0, x) = u_0(x),
\end{cases}
\end{equation}

where \( \lambda = 1 \) (resp. \( \lambda = -1 \)) corresponds to the defocusing (resp. focusing) case. Equation \text{(INLS)} is invariant under the scaling

\[ u_\lambda(x, t) := \lambda^{\frac{2+b}{p-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0 \]

which preserves the \( H^s(\mathbb{R}^d) \)-norm with \( s_c := \frac{d}{2} - \frac{2+b}{p-1} \). Thus, the equation \text{(INLS)} is called \( H^s(\mathbb{R}^d) \)-critical. The case where \( 0 < s_c < 1 \) is called energy-critical, and the case where \( 0 < s_c < 1 \) is called intercritical.

Genoud-Stuart \[13\] firstly obtained the \( H^1 \) well-posedness by using an abstract theory of Cazenave \[4\] for \( d \geq 1 \) and \( 0 < b < \min\{2, d\} \). Later, Guzmán \[14\] used Strichartz estimates and the standard Picard iterative technique to revisit the local well-posedness in energy space \( H^1 \) with \( d \geq 2, 0 < b < b^*(b^* = \frac{d}{2}d = 2, 3 \) or \( b^* = 2 \) for \( d \geq 4 \)). Dinh \[7\] extended the results of Guzmán in dimension three for \( 0 < b < \frac{3}{2} \) and \( \frac{7-2b}{3} < p < \frac{5-2b}{2b-1} \).

In the focusing case where \( \lambda = -1 \), equation \text{(INLS)} admits a global but non-scattering solution

\[ u(t, x) = e^{it}Q(x), \]

where \( Q \) is the ground state, that is, the unique positive radial solution to the elliptic equation (see \[3, 6\])

\begin{equation}
\Delta Q - Q + |x|^{-b}|Q|^{p-1}Q = 0.
\end{equation}

Genoud \[12\] considered \text{(NLS)} of mass critical and proved the global well-posedness for the \( H^1 \) initial data below the ground state, i.e., \( \|u_0\|_{L^2} \leq \|Q\|_{L^2} \). In \[10\], Farah proved the global well-posed in \( H^1 \) below the ground state in the intercritical case. Regarding the scattering, Farah-Guzmán \[11\] proved the energy scattering of radial solution when \( 0 < b < \frac{1}{2}, p = 3 \) and \( d = 3 \). It was extended by Miao-Murphy-Zheng \[22\] to the non-radial setting, based on the concentration-compactness method developed by Kenig-Merle \[20\]. Moreover, Xu-Zhao \[27\] proved the energy scattering with \( 0 < b < 1 \) and \( p > 3 - b \), by adapting a new argument of Arora-Dodson-Murphy \[1\] in dimension two. We also refer to the works by Dinh \[8\] and Farah \[10\] for the finite-time blow-up solutions.

Secondly, we turn to the development of nonlinear Schrödinger equations with potential:

\begin{equation}
\text{(NLS}_V) \quad \begin{cases}
    i\partial_t u + \Delta u - Vu = -|u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
    u(0, x) = u_0(x),
\end{cases}
\end{equation}

whose scattering behavior of solutions has been also studied. Hong \[17\] proved the energy scattering in the case where \( p = 3 \), by applying the concentration-compactness method. And Hamano-Ikeda \[18\] extended it to the whole subcritical regime for \( V \) satisfying \( \|V\|_{L^1} \) and \( \|V\|_{L^2} \). Recently, Miao-Murphy-Zheng \[23\] studied the threshold scattering for this model with a repulsive potential (\( V \) in \text{(NLS}_V) \) satisfies \( \|V\|_{L^1} \geq 0 \) and \( x \cdot \nabla V \leq 0 \)). We also mention that, Miao-Zhang-Zheng \[22\] studied the Cauchy problem with Coulomb potential (\( V = \frac{C}{|x|} \) in \text{(NLS}_V) \) where
1 < p \leq 5 and d = 3. Moreover, the focusing \([\text{NLS}_V]\) with inverse square potential \((V = \frac{a}{x^2})\) and \(a \geq 0\) was studied by Zheng [28].

Thirdly, there are also several works for the present \([\text{NLS}_V]\). Deng-Lu-Meng [5] studied the blow-up versus global well-posedness of \([\text{NLS}_V]\) with inverse-square potential \(V = \frac{a}{|x|^2}\) for \(a > -\frac{(d-2)^2}{4}\). Dinh [6] and Guo-Wang-Yao [16] proved the global dynamics for a class \(L^2\)-supercritical nonlinear inhomogeneous Schrödinger equation with a potential in dimension three with radially symmetric initial data. Inspired by Campos and Cardoso [3] and Murphy [25], we study the scattering theory of \([\text{NLS}_V]\) in the non-radial setting here.

To show the scattering theory of Theorem 1.2 our strategy in this paper is to divide the whole proof into two steps. And the first step is to establish a Marrowetz type estimate stated as in Lemma 5.2 and the second step is the scattering criterion as follows.

**Theorem 1.1** (Scattering criterion). For \(0 < b < 1, 1 + \frac{4-2b}{3} < p < 1 + 4 - 2b,\) and \(V : \mathbb{R}^3 \to \mathbb{R}\) satisfies (1.1), (1.2) in \([\text{NLS}_V]\). If \(u \in H^1\) is a solution to (1.1) defined on \([0, +\infty)\) with the following priori bound

\[
\sup_t \|u(t)\|_{H^1} := E < +\infty,
\]

then there exist two constants \(R > 0\) and \(\epsilon > 0\), depending only on \(E, p\) and \(b\) (but not on \(u\) or \(t\)), such that if

\[
\liminf_{t \to +\infty} \int_{B(0,R)} |u(t,x)|^2 dx < \epsilon^2,
\]

then \(u\) scatters forward in time in \(H^1(\mathbb{R}^3)\), that is, there exists a function \(u_+ \in H^1\) such that

\[
\liminf_{t \to +\infty} \|u(t) - e^{-it\mathcal{H}}u_+\|_{H^1} = 0.
\]

Next, we formulate the scattering result for the solutions to \([\text{NLS}_V]\). We shall use the Generalized Sobolev norms

\[
\|\Gamma f\|_{L^2}^2 := \int_{\mathbb{R}^3} |\nabla f|^2 dx + \int_{\mathbb{R}^3} V|f|^2 dx.
\]

**Theorem 1.2** (Scattering). Consider equation \([\text{NLS}_V]\) with \(0 < b < 1\) and \(1 + \frac{4-2b}{3} < p < 1 + 4 - 2b\). Assume that \(V : \mathbb{R}^3 \to \mathbb{R}\) satisfies (1.1), and in addition \(V' \geq 0, x \cdot \nabla V \leq 0, x \cdot \nabla V \in L^r,\) where \(r \in \left(\frac{3}{2}, \infty\right)\). Let \(u_0 \in H^1\) satisfy

\[
E(u_0)[M(u_0)]^{\frac{1}{2-p}} < \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{\frac{4}{2-p}}.
\]

Then, if

\[
\|\Gamma u_0\|_{L^2}^2 \|u_0\|_{L^2}^{\frac{4}{2-p}} < \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{\frac{4}{2-p}},
\]

we have that the corresponding solution \(u\) to \([\text{NLS}_V]\) exists globally in time and satisfies

\[
\|\Gamma u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{\frac{4}{2-p}} < \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{\frac{4}{2-p}}
\]

for all \(t \in \mathbb{R}\). Moreover, the global solution scatters in \(H^1\) in both time directions.
Remark 1.3. In fact, the Theorem 1.2 not only extends the results of Dinh [6] to the non-radial case but also extends the integrability condition of \( x \cdot \nabla V \in L^r \) to any \( r \in \left[ \frac{3}{2}, \infty \right) \). Because of the absence of translation symmetry in the potential, the non-radial case cannot be achieved directly by the method of Miao–Murphy–Zheng [22] and Dodson–Murphy [9]. Here, we construct a special class of cut off functions whose derivatives are bounded and do not depend on compact set \( B_R(0) \), and use the decay of the factor \( |x|^{-b} \) in the nonlinear term which can replace the radial Sobolev embedding in some sense. Based on this, we establish the non-radial scattering criterion and more complicated Morawetz estimate with potential for non-radial. In the Morawetz estimate, we find that the error term

\[
\left| \int_{|x| \geq \frac{1}{2}} \nabla a \cdot \nabla V |u|^2 dx \right| \leq C \| x \cdot \nabla V \|_{L^r(|x| \geq \frac{1}{2})} \| u \|_{L^{2^*,r}}^2 \nabla a \cdot \nabla V \nabla u
\]
tends to 0 as \( R \to \infty \). From this, we can extend the integrability of \( x \cdot \nabla V \in L^r(\mathbb{R}^3) \) to \( r \in \left[ \frac{3}{2}, \infty \right) \) and generalize the results of Dinh [6] to the non-radial symmetric case.

Outline. This paper is organized as follows. First, in Sect. 2 we recall some preliminaries for the Schrödinger operator. Then, in Sect. 3, we recall the variational analysis of the ground state. In Sect. 4, we prove the scattering criterion in Theorem 1.1. Finally, in Sect. 5, we apply the scattering criterion, together with the Virial-Morawetz estimate, to prove Theorem 1.2.

2. Preliminaries

Notations. We use the notation \( X \lesssim Y \) to denote \( X \leq CY \) for some constant \( C > 0 \) and use the notation \( X \sim Y \) to denote \( X \lesssim Y \) and \( Y \lesssim X \) at the same time. Similarly, \( X \lesssim_u Y \) means that there exists a constant \( C := C(u) \) depending on \( u \) such that \( X \leq C(u)Y \). We also use the notation \( A = O(B) \), which means that there exists a constant \( l \neq 0 \) such that \( \lim_{B \to 0} = l \). The derivative operator \( \nabla \) refers to the spatial variable only. Let \( L^r(\mathbb{R}^3) \) denote the usual space of integrable functions \( f : \mathbb{R}^3 \to \mathbb{C} \) endowed with the norm

\[
\| f \|_r := \| f \|_{L^r} = \left( \int_{\mathbb{R}^3} |f(x)|^r dx \right)^{\frac{1}{r}},
\]

with the usual modification when \( r = \infty \). For any non-negative integer \( k \), \( H^{k,r}(\mathbb{R}^3) \) denotes the Sobolev space, defined as the closure of smooth compactly supported functions under the norm

\[
\| f \|_{H^k,r} := \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_r.
\]

In particular, when \( r = 2 \), we denote it by \( H^k \). Besides, we use the norm

\[
\| f \|_{H^s} := \left( \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\]

to denote the homogeneous Sobolev space \( H^s \), where \( s \) is a real number. And for any time slab \( I \), \( L^r_t(I, L^s_x(\mathbb{R}^3)) \) means the space of \( L^s_x \)-valued functions with the
norm
\[ \|f\|_{L^q_t L^r_x(I \times \mathbb{R}^3)} = \left( \int_I \|f(t, x)\|_{L^q_x}^r \, dt \right)^{\frac{1}{r}}, \]
with the usual modifications when \(q\) or \(r\) is infinite. For simplicity, we also use the shorthand \(\|f\|_{L^q_t L^r_x(I \times \mathbb{R}^3)}\) or \(\|f\|_{L^q_t L^r_x(\mathbb{R}^3)}\).

2.1. Strichartz estimates. Let us start this section by introducing the notation used throughout the paper. We recall some Strichartz estimates associating with the linear Schrödinger propagator.

We say the pair \((q, r)\) is \(L^2\)-admissible or simply admissible pair if \((q, r)\) satisfies
\[ \frac{2}{q} = \frac{3}{2} - \frac{3}{r}, \quad 2 \leq r \leq 6. \]
The pair is called \(\dot{H}^s\)-admissible, if
\[ \frac{2}{q} = \frac{3}{2} - \frac{3}{r} - s. \]
And \((q, r)\) is called \(\dot{H}^{-s}\)-admissible, if
\[ \frac{2}{q} = \frac{3}{2} - \frac{3}{r} + s. \]

Given \(s \in (0, 1)\), let
\[ (2.1) \quad \Lambda_s = \left\{ (q, r) \text{ is } \dot{H}^s \text{ - admissible} \mid \left( \frac{6}{3 - 2s} \right)^+ \leq r \leq 6^- \right\} \]
and
\[ (2.2) \quad \Lambda_s = \left\{ (q, r) \text{ is } \dot{H}^{-s} \text{ - admissible} \mid \left( \frac{6}{3 - 2s} \right)^+ \leq r \leq 6^- \right\}, \]
where the notations \(a^+\) and \(a^-\) denote, respectively, \(a + \epsilon\) and \(a - \epsilon\), for fixed \(0 < \epsilon \ll 1\).

We define the following Strichartz norm
\[ \|u\|_{S(\dot{H}^s, I)} = \sup_{(q, r) \in \Lambda_s} \|u\|_{L^q_t L^r_x(I)}, \]
and dual Strichartz norm
\[ \|u\|_{S'(\dot{H}^{-s}, I)} = \inf_{(q, r) \in \Lambda_s} \|u\|_{L^q_t L^r_x'(I)}. \]
If \(I = \mathbb{R}\), \(I\) is omitted usually. For more details see [4] and [19].

Lemma 2.1 (Dispersive estimate and Strichartz estimate, [17]). The following statements hold: (i) Let \(V : \mathbb{R}^3 \to \mathbb{R}\) satisfy \([-1, 1]\) and \([-1, 2]\). Then it holds that
\[ (2.3) \quad \|e^{-it\mathcal{H}}\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{s}{2}}, \]
(ii) Let \(V : \mathbb{R}^3 \to \mathbb{R}\) satisfy \([-1, 1]\) and \([-1, 2]\). Then it holds that
\[ (2.4) \quad \|e^{-it\mathcal{H}}f\|_{S(\dot{H}^s)} \lesssim \|f\|_{\dot{H}^s}, \quad s \geq 0, \]
and
\[ (2.5) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{H}}g(\cdot, s)ds \right\|_{S(\dot{H}^s)} \lesssim \|g\|_{S'(\dot{H}^{-s})}, \quad s \in (0, 1). \]
2.2. **Equivalence of Sobolev norms.** In this subsection, we define the homogeneous and inhomogeneous Sobolev spaces associated to $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^3)$ under the norms

$$
\|f\|_{\dot{W}^\gamma,r} := \|\Gamma \gamma f\|_{L^r}, \quad \|f\|_{W_0^\gamma,r} := \|(\Gamma) \gamma f\|_{L^r}, \quad \Gamma := \sqrt{\mathcal{H}}.
$$

To simplify the notation, we denote $\dot{H}^\gamma_r := \dot{W}^\gamma_2$ and $H^\gamma_r := W_0^\gamma.$

**Lemma 2.2** (Sobolev inequalities, [17]). Let $V : \mathbb{R} \to \mathbb{R}$ satisfy (1.1) and (1.2). Then it holds that

$$
\|f\|_{L^q} \lesssim \|f\|_{\dot{W}^\gamma,r}, \quad \|f\|_{L^q} \lesssim \|f\|_{W_0^\gamma,r},
$$

where $1 < p < q < \infty$, $1 < p < \frac{3}{\gamma}$, $0 \leq \gamma \leq 2$ and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{4}.$

**Lemma 2.3** (Equivalence of Sobolev spaces, [17]). Let $V : \mathbb{R} \to \mathbb{R}$ satisfy (1.1) and (1.2). Then it holds that

$$
\|f\|_{W_0^\gamma,r} \sim \|f\|_{\dot{W}^\gamma,r}, \quad \|f\|_{\dot{W}^\gamma,r} \sim \|f\|_{W_0^\gamma,r},
$$

where $1 < r < \frac{3}{\gamma}$ and $0 \leq \gamma \leq 2.$

2.3. **Some nonlinear estimates.** We recall some interpolation estimates for nonlinearities, which plays an important role in proving scattering theory.

**Lemma 2.4** (Nonlinear estimates, [2, 14]). Let $u, v \in C_0^\infty(\mathbb{R}^{3+1})$, $1 + \frac{4 - 2b}{3} < p < 1 + \frac{4 - 2b}{3}$ and $0 < b < 1$. Then there exists $0 \leq \theta = \theta(p, b) < p - 1$ such that the following estimates hold

\begin{align*}
(2.6) \quad & \| |x|^{-b} |u|^{p-1} v\|_{S(H^{-\epsilon}, I)} \lesssim \| u\|_{L^p_t H^\gamma_x(I)} \| u\|_{S(H^{-\epsilon}, I)}^p, \\
(2.7) \quad & \| |x|^{-b} |u|^{p-1} u\|_{S(L^2, I)} \lesssim \| u\|_{L^p_t H^\gamma_x(I)} \| u\|_{S(L^2, I)}^p, \\
(2.8) \quad & \| \nabla (|x|^{-b} |u|^{p-1} u)\|_{S(L^2, I)} \lesssim \| u\|_{L^p_t H^\gamma_x(I)} \| u\|_{S(L^2, I)}^p, \\
(2.9) \quad & \| |x|^{-b} |u|^{p-1} u\|_{L^p_t L^\gamma_x(I)} \lesssim \| u\|_{L^p_t L^\gamma_x(I)}^p,
\end{align*}

for $\frac{2(3-b)}{3+4-2b} < r < \frac{2(3-b)}{3+2-2b}.$

2.4. **Local well-posedness in $H^1$.** In this subsection, we recall the local well-posedness in $H^1$, the global well-posedness of small initial data and its corresponding scattering theory under the assumptions (1.1) and (1.2). Particularly, we denote the $\Lambda_0$ the Strichartz norm for any time interval $I \subset \mathbb{R}

$$
(2.10) \quad \|u\|_{S(L^2, I)} := \sup_{(q, r) \in \Lambda_0} \|u\|_{L^q(I), L^r}, \quad \|u\|_{S'(L^2, I)} := \inf_{(q', r') \in \Lambda_0} \|u\|_{L^{q'}(I), L^{r'}},
$$

where $(q, q')$ and $(r, r')$ are Hölder’s conjugate pairs. The condition $2 \leq r < 3$ can ensure $\dot{W}^1_r \sim \dot{W}^1$, and $\dot{W}^1_r \sim \dot{W}^1.$

**Theorem 2.5** (Local well-posedness [36]). Let $0 < b < 1$ and $1 + \frac{4 - 2b}{3} < p < 1 + 4 - 2b$. Let $V : \mathbb{R} \to \mathbb{R}$ satisfy (1.1) and (1.2). Then the equation is locally well-posed in $H^1$. 
Theorem 2.6 (Small initial data, [36]). Let $0 < b < 1$ and $1 + \frac{4 - 2b}{3} < p < 1 + 4 - 2b$. Let $V : \mathbb{R}^3 \to \mathbb{R}$. Suppose $\|u_0\|_{H^1} \leq E$. Then there exists $\delta_{sc} = \delta_{sc}(E) > 0$ such that if
\[
\|e^{-it\mathcal{H}}u_0\|_{S(\mathcal{H}^{\infty, [0, +\infty)})} \leq \delta_{sc},
\]
then there exists a unique global solution $u$ to (INLS) with initial date $u_0$ satisfying
\[
\|u\|_{S(\mathcal{H}^{\infty, [0, +\infty)})} \leq \|e^{-it\mathcal{H}}u_0\|_{S(\mathcal{H}^{\infty, [0, +\infty])}}.
\]

Furthermore, $u$ scatters forward in time in $H^1$, i.e., there exists $u_+ \in H^1$ such that
\[
\lim_{t \to -\infty} \|u(t) - e^{-it\mathcal{H}}\|_{H^1} = 0.
\]

3. Variational analysis

Let us recall some properties related to the ground state $Q$ which is the unique positive radial decreasing solution to the elliptic equation (for more detail, see [36])
\[
\Delta Q - Q + |x|^{-b}|Q|^{p-1}Q = 0.
\]

Lemma 3.1 (Coercivity I, [236]). Let $0 < b < 1$ and $1 + \frac{4 - 2b}{3} < p < 1 + 4 - 2b$. Let $V : \mathbb{R}^3 \to \mathbb{R}$ satisfy (1.1), $V > 0$. Let $u_0 \in H^1$ satisfy (1.7) and (1.8), then the corresponding solution to the focusing problem (INLS) exists globally in time. Moreover, there exists $\delta = \delta(u_0, Q) > 0$ such that
\[
\|
\begin{bmatrix}
\Gamma u(t) \\
u(t)
\end{bmatrix}
\|_{L^2} \leq \|\nabla Q\|_{L^2} \|Q\|_{L^2}^\frac{b}{2}, \quad \forall \ t \in \mathbb{R}.
\]

In particular, the corresponding solution to the focusing problem (INLS) satisfies
\[
\|
\begin{bmatrix}
\Gamma u(t) \\
u(t)
\end{bmatrix}
\|_{L^2} \leq \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\frac{1}{2}}, \quad \forall \ t \in \mathbb{R}.
\]

Remark 3.2. By the assumption $V \geq 0$ and the same argument as above, we see that if $u_0 \in H^1$ satisfies (1.7) and (1.8), then the corresponding solution to the focusing problem (INLS) satisfies
\[
\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\frac{1}{2}} \leq \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\frac{b}{2}}, \quad \forall \ t \in \mathbb{R}.
\]

In particular, the corresponding solution to the focusing problem (INLS) exists globally in time. Moreover, there exists $\delta = \delta(u_0, Q) > 0$ such that
\[
\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\frac{1}{2}} \leq (1 - \delta)\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\frac{1}{2}}, \quad \forall \ t \in \mathbb{R}.
\]

Lemma 3.3 (Coercivity II, [236]). Suppose $f \in H^1(\mathbb{R}^3)$, that
\[
\|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2} \leq (1 - \delta)\|Q\|_{L^2}^{\frac{1}{2}} \|\nabla Q\|_{L^2}.
\]

then there exists $\delta' = \delta'(\delta) > 0$ so that
\[
\begin{aligned}
\int_{\mathbb{R}^3} \|
\begin{bmatrix}
\nabla f \\
f
\end{bmatrix}
\|^2 dx &+ \left(\frac{3 - b}{p + 1} - \frac{3}{2}\right) \int_{\mathbb{R}^3} \frac{|f|^{p+1}}{|x|^b} dx \\
&\geq \delta' \int_{\mathbb{R}^3} \frac{|f|^{p+1}}{|x|^b} dx.
\end{aligned}
\]

Lemma 3.4 (Kinetic energy on balls, [23]). Let $\chi$ be a smooth cutoff to the set $\{|x| \leq 1/2\}$ and set $\chi_R(x) := \chi(x)$. If $f \in H^1$, then
\[
\int_{\mathbb{R}^3} |\nabla (\chi_R f)|^2 dx = \int_{\mathbb{R}^3} \chi_R^2 |\nabla f|^2 dx - \int_{\mathbb{R}^3} \chi_R \Delta(\chi_R) |f|^2 dx.
\]
In particular, by Lemma 3.3, there exists \( \delta \) satisfying (3.4).

Let \( \delta, M[u_0], Q, s_c > 0 \) such that, for any \( R \geq R_0 \),

\[
1.2 \text{ to the verification of condition (1.5) on local mass in long time. The history of }
\]

where we obtain Theorem 1.1 comes back to Dodson-Murphy [9] for classical nonlinear Schrödinger equation in 3D and then F. Meng [21] for the nonlinear Hartree equation in 5D.

In [3], it was proved for the (INLS) without potential. For the (INLS) with a potential, it was proved in [6] under the radial assumption. Here, we prove the result for non-radial initial data and the full subcritical range in dimension three.

**Lemma 4.1.** Let \( 0 < b < 1 \) and \( 1 + \frac{4 - 2b}{3} < p < 1 + 4 - 2b \). Let \( V : \mathbb{R}^3 \to \mathbb{R} \) satisfy (1.1), (1.2) and \( u \) a (possibly non-radial) \( H^1 \)-solution to (1.4). If \( u \) satisfies (1.5) for some \( 0 < \epsilon < 1 \), then there exist \( \gamma, T > 0 \) such that

\[
\| e^{-i(t-T)H}u(T) \|_{S(H^{s_c};[0,\infty))} \lesssim \epsilon^\gamma.
\]

**Proof.** First, we fix the parameters \( \alpha, \gamma > 0 \) (to be chosen later). By Strichartz estimate, there exists \( T_0 > \epsilon^{-\alpha} \) such that

\[
\| e^{-itH}u_0 \|_{S(H^{-\alpha};[T_0,\infty))} \lesssim \epsilon^\gamma.
\]

For \( T \geq T_0 \) to be chosen later, define \( I_1 := [T - \epsilon^{-\alpha}, T] , I_2 := [0, T - \epsilon^{-\alpha}] \) and let \( \eta \) denote a smooth, spherically symmetric function which equals 1 on \( B(0, 1/2) \) and 0 outside \( B(0, 1) \). Set \( \eta_R(x) := \eta(x/R) \) for any \( R > 0 \).

From Duhamel’s formula

\[
u(T) = e^{iTH}u_0 - i \int_0^T e^{i(t-s)H} \frac{|u|^{p-1}u}{|x|^b} ds,
\]

we obtain

\[
e^{i(t-T)H}u(T) = e^{-itH}u_0 + iG_1 + iG_2,
\]

where

\[
G_j(t) := \int_{I_j} e^{i(t-s)H} \frac{|u|^{p-1}u}{|x|^b} ds, \quad j = 1, 2.
\]

**Estimate on \( G_1 \).** By hypothesis (1.5), we can fix \( T \geq T_0 \) such that

\[
\int_{\mathbb{R}^3} \eta_R(x)|u(T,x)|^2 dx \lesssim \epsilon^2.
\]
Multiplying (INLS), by \( \eta R \hat{u} \), taking the imaginary part and integrating by parts, we obtain
\[
\partial_t \int_{\mathbb{R}^3} \eta_R(x) |u(t,x)|^2 \, dx \lesssim \frac{1}{R},
\]
so that, by (1.2), for \( t \in I_1 \),
\[
\int_{\mathbb{R}^3} \eta_R(x) |u(t,x)|^2 \, dx \lesssim \epsilon^2 + \frac{\epsilon^{-\alpha}}{R}.
\]
If \( R \geq \epsilon^{-(\alpha+2)} \), then we have \( \| \eta_R u \|_{L^\infty_t L^2_x} \lesssim \epsilon \).

We choose \((s, l) \in \Lambda_{s_c} \) as
\[
s = \frac{4(p-1)(p+1-\theta)}{(p-1)(3p-1 + 2b) - \theta(3p-1) - 4 + 2b}, \quad l = \frac{3(p-1)(p+1-\theta)}{(p-1)(3-\theta) - \theta(2-\theta)}.
\]
By Hölder’s inequality and Sobolev embedding, for \( t \in I_1 \),
\[
(4.3) \quad \left\| \eta_R \left| \frac{u^{p-1} u}{|x|^b} \right|_{L^l_t L^s_x} \right\| \lesssim \| u(t) \|_{H^s_t}^q \| u(t) \|_{L^{l+1}_x}^{p-1-\theta} \| \eta_R u(t) \|_{L^p_t L^q_x} \lesssim \| \eta_R u(t) \|_{L^p_t L^q_x}.
\]

Now, let \( 0 < \mu < 1 \) satisfy \( \frac{1}{l} = \frac{\mu}{2} + \frac{1-\mu}{6} \), we have
\[
(4.4) \quad \| \eta_R u(t) \|_{L^p_t L^q_x} \lesssim \| u(t) \|_{L^p_t L^q_x}^{1-\epsilon} \| \eta_R u(t) \|_{L^p_t L^q_x}^\mu \lesssim \epsilon^\mu,
\]
uniformly for \( t \in I_1 \). We now exploit the decay of the nonlinearity. By Hölder’s inequality and Sobolev embedding again, for \( R \geq 0 \) sufficiently large (depending on \( \epsilon \)) and \( t \in I_1 \),
\[
\left\| (1 - \eta_R) \left| \frac{u^{p-1} u}{|x|^b} \right|_{L^{l+1}_t L^{s-\theta}_x} \right\| \lesssim \left\| \frac{1}{|x|^b} \right\|_{L^{l+1}_t L^{s-\theta}_x} \left\| u \right\|_{H^s_t}^\theta \| u \|_{L^{l+1}_x}^{p-\theta}
\]
(4.5)
\]
where \( l_1 \) and \( l_2 \) are such that \( bl_1 > 3, 2 < bl_2 < \frac{3(p-1)}{2-\theta} \) and
\[
\frac{1}{\tilde{l}} = \frac{1}{l_1} + \frac{1}{l_2} + \frac{p-\theta}{l}.
\]

Using the Strichartz estimates, together with estimates (4.3), (4.4), and (4.5), we obtain
\[
\left\| \int_{I_1} e^{i(t-s)R} \left| \frac{u^{p-1} u(s)}{|x|^b} \right| ds \right\|_{S(H^-; \mathcal{T}, \infty)} \lesssim \| u \|_{L^\infty_t L^2_x(\mathcal{I}_1)} \lesssim \| u \|_{L^\infty_t L^2_x(\mathcal{I}_1)} \lesssim \| \eta_R \left| \frac{u^{p-1} u}{|x|^b} \right|_{L^{l+1}_t L^{s-\theta}_x} + \left\| (1 - \eta_R) \left| \frac{u^{p-1} u}{|x|^b} \right|_{L^{l+1}_t L^{s-\theta}_x} \right\| \lesssim |I_1|^{1/\hat{s}} \epsilon^\mu = \epsilon^{\mu - \alpha/\hat{s}} = \epsilon^{\mu/2},
\]
where we choose \( \alpha := s' \mu/2 \).

**Estimate on \( G_2 \).** Let \((q, r) \in \Lambda_{s_c} \). Define, for small \( \delta \geq 0 \),
\[
\frac{1}{\hat{h}} = \left( \frac{1}{1 - s_c} \right) \left[ \frac{1}{q} - \delta s_c \right],
\]
We see that \((h, k) \in \Lambda_0\). By interpolation,
\[
\|G_2\|_{L_t^1(\mathbb{T}^1, L_x^q)} \lesssim \|G_2\|_{L_t^1(\mathbb{T}^1, L_x^q)}^{1-s_c} \|G_2\|_{L_t^\infty(\mathbb{T}^1, L_x^{\frac{6}{3s_c}})}^{s_c}.
\]
By the dispersive estimate (2.3) and Lemma 2.4, one can get
\[
\|G_2\|_{L_t^{\frac{6}{3s_c}}(\mathbb{T}^1, L_x^{\frac{6}{3s_c}})} \lesssim \|u\|_{L_t^p H_x^s} \left\| (t - T + \epsilon^\alpha) \right\|_{L_t^{\frac{6}{3s_c}}(\mathbb{T}^1, L_x^{\frac{6}{3s_c}})} \lesssim \epsilon^{\alpha\delta}.
\]
Using Duhamel’s principle, rewrite
\[
iG_2 = e^{it\mathcal{H}} \left[ e^{-i(\mathcal{H}(T - \epsilon^\alpha))} u(T - \epsilon^\alpha) - u_0 \right],
\]
thus, by the Strichartz estimate and (4.7),
\[
\|G_2\|_{L_t^{1}(\mathbb{T}^1, L_x^{q})} \lesssim \left\| e^{it\mathcal{H}} \left[ e^{-i(\mathcal{H}(T - \epsilon^\alpha))} u(T - \epsilon^\alpha) - u_0 \right] \right\|_{L_t^{1}(\mathbb{T}^1, L_x^{q})}^{1-s_c} \|G_2\|_{L_t^\infty(\mathbb{T}^1, L_x^{\frac{6}{3s_c}})}^{s_c} \lesssim \epsilon^{\alpha\delta}.
\]
Therefore, defining \(\gamma := \min\{\mu/2, \alpha \delta s_c\}\) and recalling that
\[
e^{-i(t-T)\mathcal{H}} u(T) = e^{-i(t-T)\mathcal{H}} u_0 + iG_1 + iG_2,
\]
we have
\[
\|e^{-i(t-T)\mathcal{H}} u(T)\|_{S(H^{s_c}, [T, \infty)}) \lesssim \epsilon^\gamma.
\]
Then, we complete the proof of Lemma 4.1. \(\square\)

**Proof of Theorem 1.1.** Choose \(\epsilon > 0\) sufficiently small, by Theorem 2.5 and 2.6
\[
\|e^{-iH} u(T)\|_{S(H^{s_c}, [0, \infty))} = \|e^{-i(t-T)\mathcal{H}} u_0 \|_{S(H^{s_c}, [T, \infty))} \lesssim \epsilon^\gamma \lesssim \delta s_c,
\]
where \(\delta s_c\) is given in Theorem 2.6. Thus by small data scattering theory, \(u\) scatters forward in time in \(H^1\), as desired. \(\square\)

## 5. Proof of Theorem 1.2

In this section, we will prove the Morawetz estimate, and use it together with scattering criterion to obtain the main result Theorem 1.2.

**Lemma 5.1** (Virial/Morawetz identity). Let \(a : \mathbb{R}^3 \to \mathbb{R}\) be a real-valued weight. If \(\|\nabla a\|_{L^\infty} < \infty\), define
\[
Z(t) = 2 \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla u \cdot \nabla a dx.
\]
Then, if $u$ is a solution to (INLS), we have the following identity
\[
\frac{d}{dt} Z(t) = \left( \frac{4}{p+1} - 2 \right) \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|^b} \Delta u dx - \frac{4b}{p+1} \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|^{b+2}} x \cdot \nabla u dx
\]
\[
- \int_{\mathbb{R}^3} |u|^2 \Delta u dx + 4Re \int_{\mathbb{R}^3} a_i \bar{u}_j u_j dx - 2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla |u|^2 dx.
\]
(5.1)

Proof. We omit the proof of Lemma 5.1 here for it is classical and standard. \qed

Let $\phi : [0, \infty) \to [0, 1]$ be a smooth function satisfying
\[
\phi(r) = \begin{cases} 
1, & r \leq \frac{1}{2}, \\
0, & r \geq 1.
\end{cases}
\]
(5.2)

Define $\omega : [0, \infty) \to [0, \infty)$ by
\[
\omega(r) := \int_0^r \int_0^s \phi(t) dt ds.
\]

For given $R > 0$, we define a radial function
\[
a(r) := R^2 \omega (\frac{r}{R}), \quad r = |x|.
\]

It is easy to see $a(x) \sim |x|^2$ for $|x| \leq \frac{R}{2}$. Moreover, we have the following properties:
\[
0 \leq a''(r) \leq 2, \quad 0 \leq \Delta a \leq 10, \quad \forall r \geq 0, \quad \forall x \in \mathbb{R}^3.
\]

Lemma 5.2 (Morawetz estimate). For $0 < b < 1$ and $1 + \frac{4-2b}{3} < p < 1 + 4 - 2b$ in (INLS), let $V : \mathbb{R}^3 \to \mathbb{R}$ satisfy (1.1), $V \geq 0$, $x \cdot \nabla V \leq 0$. If $u_0 \in H^1(\mathbb{R}^3)$ satisfy (1.7), (1.8), and (1.9), then for any $T \geq 0$ and any $R \geq \bar{R}$ as in Lemma 3.5, the corresponding global solution to the focusing problem (INLS) satisfies
\[
\frac{1}{T} \int_0^T \int_{|x| \leq \frac{R}{2}} \frac{|u(t, x)|^{p+1}}{|x|^b} dx dt \lesssim_{u_0, Q} R^4 + \frac{1}{R^6} + o_R(1).
\]
(5.3)

Proof. Using the Cauchy-Schwarz inequality and the definition of $Z(t)$, we have
\[
\sup_{t \in \mathbb{R}} |Z(t)| \lesssim R.
\]
(5.4)

Denote the angular derivative as
\[
\nabla u = \nabla u - \frac{x \cdot \nabla u}{|x|^2} x.
\]

Here, the angular derivative is not necessarily zero, since the solution is not radial. So, we have
\[
\frac{d}{dt} Z(t) = 8 \int_{|x| \leq \frac{R}{2}} |\nabla u|^2 + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{|x| \leq \frac{R}{2}} \frac{|u|^{p+1}}{|x|^b} dx
\]
\[
+ \int_{|x| \geq \frac{R}{2}} \left[ \left( \frac{4}{p+1} - 2 \right) \Delta a - \frac{4b}{1+p} \frac{x \cdot \nabla a}{|x|^b} |u|^{p+1} \right] dx
\]
\[
+ \frac{4}{3} \int_{|x| \geq \frac{R}{2}} \Delta^2 u dx + 4 \int_{|x| \geq \frac{R}{2}} \frac{\partial^2 a}{|x|} \nabla u^2 dx
\]
\[
+ 4 \int_{|x| \geq \frac{R}{2}} \partial^2 a |\partial_x u|^2 dx + 4 \int_{|x| \geq \frac{R}{2}} \frac{\partial^2 a}{|x|} \nabla u^2 dx
\]
\[
- 4 \int_{|x| \geq \frac{R}{2}} x \cdot \nabla V |u|^2 dx - 2 \int_{|x| \geq \frac{R}{2}} \nabla a \cdot \nabla |u|^2 dx.
\]
(5.5)
Observing that \( \| \Delta \Delta u \|_{L^\infty} \lesssim \frac{1}{R^2} \), so we have
\[
\frac{d}{dt} Z(t) \geq 8 \left[ \int_{|x| < \frac{3}{4}} |\nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{|x| < \frac{3}{4}} |u|^{p+1} |x|^b dx \right] - \frac{cE^{\frac{p+1}{2}}}{R^b} - \frac{c}{R^2} M[u_0] - 2 \int_{|x| > \frac{3}{2}} \nabla a \cdot \nabla V |u(t,x)|^2 dx.
\]
(5.6)

Define a smooth cut-off function satisfies
\[
\chi^\rho(x) = \begin{cases} 
1, & |x| \leq \frac{3}{4}, \\
0, & |x| \geq \frac{1}{2} + \frac{1}{p}.
\end{cases}
\]

It is not difficult to observe the identity as follow:
\[
\int_{|x| < \frac{3}{4}} |\nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{|x| < \frac{3}{4}} |u|^{p+1} |x|^b dx = \left[ \int_{R^3} (\chi^\rho)^2 |\nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{R^3} (\chi^\rho)^2 |u|^{p+1} |x|^b dx \right]
- \left[ \int_{\frac{3}{4} < |x| < \frac{3}{4} + \frac{R}{2}} (\chi^\rho)^2 |\nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{\frac{3}{4} < |x| < \frac{3}{4} + \frac{R}{2}} (\chi^\rho)^2 |u|^{p+1} |x|^b dx \right]
- \left( \frac{3-b}{p+1} \right) \int_{R^3} (\chi^\rho)^2 |u|^{p+1} |x|^b dx
\]
\[(5.7)\]

\[
= \left[ \int_{R^3} |\chi^\rho \nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{R^3} |\chi^\rho u|^{p+1} |x|^b dx \right] - I_\rho - II_\rho.
\]

According to Lemma \ref{lemma-3.4} we get
\[
\int_{R^3} |\chi^\rho \nabla u|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{R^3} |\chi^\rho u|^{p+1} |x|^b dx \geq \int_{R^3} |\nabla (\chi^\rho u)|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{R^3} |\chi^\rho u|^{p+1} |x|^b dx - \frac{c}{R^2} M[u_0],
\]
(5.8)

The inequalities (5.6), (5.8) and (5.9) can be rewritten as
\[
\frac{d}{dt} Z(t) \geq 8 \left[ \int_{R^3} |\nabla (\chi^\rho u)|^2 dx + \left( \frac{3-b}{p+1} - \frac{3}{2} \right) \int_{R^3} |\chi^\rho u|^{p+1} |x|^b dx \right] - \frac{cE^{\frac{p+1}{2}}}{R^b} - \frac{c}{R^2} M[u_0] - 8I_\rho - 8II_\rho - 2 \int_{|x| > \frac{3}{2}} \nabla a \cdot \nabla V |u|^2 dx.
\]
(5.10)

Noting the fact
\[
\nabla a = R \frac{x}{|x|} \omega'(\frac{R}{|R|}), \quad \text{when } |x| \geq \frac{R}{2},
\]
and the derivative of $\omega$ is bounded which is independent of $R$, by Sobolev embedding
and interpolation, we obtain
\[
\|u\|_{L^m} \lesssim \|u\|_{L^2}^\theta \|u\|_{L^6}^{1-\theta} \lesssim \|u\|_{H^1}, \quad 2 \leq m \leq 6.
\]
It is easy to see that $2 \leq 2r' \leq 6$ when $r \geq \frac{4}{3}$. Therefore,
\[
\left| \int_{|x| \geq \frac{4}{3}} \nabla a \cdot \nabla V |u(t,x)|^2 \, dx \right| \lesssim \int_{|x| \geq \frac{4}{3}} |\nabla a \cdot \nabla V| \|u(t,x)|^2 \, dx
\]
\[
\lesssim \int_{\mathbb{R}^3} |x \cdot \nabla V| \|u(t,x)|^2 \, dx
\]
\[
\lesssim \|x \cdot \nabla V\|_{L^6(\mathbb{R}^3)} \|u(t,x)\|_{L^2}^2
\]
\[
< \infty, \quad \forall \ t \in \mathbb{R}.
\]
Hence,
\[
\lim_{R \to \infty} \sup_{t \in \mathbb{R}} \left| \int_{|x| \geq \frac{4}{3}} \nabla a \cdot \nabla V |u(t,x)|^2 \, dx \right| = 0.
\]
Using Lemma 3.5 and recalling that $0 < b < 1$, we can write (5.10) as
\[
\int_{\mathbb{R}^3} \frac{|X_n u(t,x)|^{p+1}}{|x|^b} \, dx \lesssim u_{0,Q} \frac{d}{dt} Z(t) + \frac{1}{R^6} + 8I_\rho + 8I_{1\rho} + o_R(1).
\]
Now, by Dominated Convergence Theorem, we obtain $I_\rho + II_{1\rho} \to 0$ as $\rho \to +\infty$. Hence,
\[
\int_{|x| \leq \frac{4}{3}} \frac{|u(t,x)|^{p+1}}{|x|^b} \, dx \lesssim u_{0,Q} \frac{d}{dt} Z(t) + \frac{1}{R^6} + o_R(1).
\]
Then, integrating over time and using (5.11), we have
\[
\frac{1}{T} \int_0^T \int_{|x| \leq \frac{4}{3}} \frac{|u(t,x)|^{p+1}}{|x|^b} \, dx \, dt \lesssim u_{0,Q} \frac{1}{T} \sup_{t \in [0,T]} |Z(t)| + \frac{1}{R^6} + o_R(1)
\]
\[
\lesssim u_{0,Q} \frac{R}{T} + \frac{1}{R^6} + o_R(1).
\]
Therefore, the proof is completed. \qed

**Lemma 5.3 (Energy evacuation).** For $0 < b < 1$ and $1 + \frac{4-2b}{3} < p < 1 + 4 - 2b$ in (NLSE), let $V : \mathbb{R}^3 \to \mathbb{R}$ satisfy (1.1), $V \geq 0$, $x \cdot \nabla V \leq 0$. If $u_0 \in H^1(\mathbb{R}^3)$ satisfy (1.1), (1.5), and (1.4), then there exist a sequence of times $t_n \to \infty$ and a sequence of $R_n \to \infty$ such that
\[
\lim_{n \to \infty} \int_{|x| \leq R_n} \frac{|u(t_n)|^{p+1}}{|x|^b} \, dx = 0.
\]

**Proof.** Choosing $T_n = R_n^3$ and applying Lemma 5.2 we have
\[
\frac{1}{R_n^3} \int_0^{R_n^3} \int_{|x| \leq \frac{4}{3}} \frac{|u(t)|^{p+1}}{|x|^b} \, dx \, dt \lesssim u_{0,Q} \frac{1}{R_n^2} + \frac{1}{R_n^6} + o_R(1).
\]
According to the Mean Value Theorem, there exist sequences $t_n \to \infty$ and $R_n \to \infty$ such that (5.13) holds. \qed
Proof of Theorem 1.3. Take $t_n \to \infty$, and a sequence of $R_n \to \infty$ as in Lemma 5.3. Fix $\epsilon \geq 0$ and $R \geq 0$ as in Theorem 1.1. Choosing $n$ sufficiently large, such that $R_n \geq R$, by Hölder’s inequality yields
\[
\int_{|x|\leq R_n} |u(x,t_n)|^2 \, dx \lesssim R^{\frac{2b+3(p-1)}{p+1}} \left( \int_{|x|\leq R_n} \frac{|u(t_n)|^{p+1}}{|x|^b} \, dx \right)^{\frac{2}{p+1}} \to 0
\]
as $n \to \infty$. Therefore, by Theorem 1.1, $u$ scatters forward in time. \qed

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References

[1] A. K. Arora, B. Dodson and J. Murphy, Scattering below the ground state for the 2D radial nonlinear Schrödinger equation. Proc. Amer. Math. Soc. 148, 1653-1663, 2020.

[2] L. Campos, Scattering of radial solutions to inhomogeneous nonlinear Schrödinger equation. Nonlinear Anal. 202, 1-17, 2021.

[3] L. Campos and M. Cardoso, A virial-Morawetz approach to scattering for the non-radial inhomogeneous NLS. arXiv:2104.11266v1, 2021.

[4] T. Cazenave, Semilinear Schrödinger equations. American Mathematical Society, 2003.

[5] M. Deng, J. Lu, and F. Meng, Blow-up versus global well-posedness for the focusing INLS with inverse-square potential. Math. Meth. Appl. Sci. 46(3), 3285-3293, 2023.

[6] V. D. Dinh, Global dynamics for a class of inhomogeneous nonlinear Schrödinger equations with potential. Mathematische Nachrichten. 294(4), 672-716, 2021.

[7] V. D. Dinh, Scattering theory in a weighted $L^2$ space for a class of the defocusing inhomogeneous nonlinear Schrödinger equation. Commun. Pure Appl. Anal. 18, 2735-2755, 2019.

[8] V. D. Dinh, Blow up of $H^1$ solutions for a class of the focusing inhomogeneous nonlinear Schrödinger equation. Nonlinear Anal. 174, 169-188, 2018.

[9] B. Dodson and J. Murphy, A new proof of scattering below the ground state for the 3d radial focusing cubic NLS. Proceedings of the American Mathematical Society, 145(11):4859-4867, 2017.

[10] L. G Farah, Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation. J. Evol. Equ. 16(1), 193-208.

[11] L. G Farah and C.M. Guzmán, Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation. J. Differential Equations. 262(8), 4175-4231, 2017.

[12] F. Genoud and C. A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves. Discrete Contin. Dyn. Syst. 21(1), 137-186, 2008.

[13] C. M. Guzmán, On well posedness for the inhomogeneous nonlinear Schrödinger equation. Nonlinear Anal. Real World Appl. 37, 249-286, 2017.

[14] T. S. Gill, Optical guiding of laser beam in nonuniform plasma, Pramana. 55(2000), no. 5-6, 835-842.

[15] Q. Guo, H. Wang, X. Yao, Scattering and blow-up criteria for 3D cubic focusing nonlinear inhomogeneous NLS with a potential. arXiv: 1801.05165.

[16] Y. Hong, Scattering for a nonlinear Schrödinger equation with a potential. Commun. Pure Appl. Anal. 15 (5), 1571-1601, 2016.

[17] M. Hamano and M. Ikeda, Global dynamics below the ground state for the focusing Schrödinger equation with a potential. J. Evol. Equ. 20, 1131-1172, 2020.

[18] M. Keel and T. Tao, Endpoint Strichartz estimates. American Journal of Mathematics, 120(5), 955-980, 1998.
[20] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*. Inventiones mathematicae, 166(3), 645-675, 2006.

[21] F. Meng, *A new proof of scattering for the 5D radial focusing Hartree equation*. Applicable Analysis, 2020. DOI: 10.1080/00036811.2020.1859491.

[22] C. Miao, J. Murphy, and J. Zheng, *Scattering for the non-radial inhomogeneous NLS*. To appear in Math. Res. Let. [arXiv:1912.01318v1], 2019.

[23] C. Miao, J. Murphy, and J. Zheng, *Threshold scattering for the focusing NLS with a repulsive potential*. arXiv: 2102.07163v1, 2021.

[24] C. Miao, J. Zhang, and J. Zheng, *Nonlinear Schrödinger equation with Coulomb potential*. arXiv:1809.06685v1, 2018, 9, 18.

[25] J. Murphy, *A simple proof of scattering for the intercritical inhomogeneous NLS*. arXiv: 2101.04811, (2021).

[26] C. Liu and V. Tripathi, *Laser guiding in an axially nonuniform plasma channel*, Physics of Plasmas. 1(1994), no. 9, 3100-3103.

[27] C.Xu and T. Zhao, *A remark on the Scattering theory for the 2D radial focusing INLS*. arXiv: 1908.00743, 2019. 3.

[28] J. Zheng, *Focusing NLS with inverse square potential*. J. Math. P, 59, 111502(2018).