Winding numbers and $SU(2)$-representations of knot groups

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Abstract

Given an abelian group $A$ and a Lie group $G$, we construct a bilinear pairing from $A \times \pi_1(R)$ to $\pi_1(G)$, where $R$ is a subvariety of the variety of representations $A \to G$.

In the case where $A$ is the peripheral subgroup of a torus or two-bridge knot group, $G = S^1$ and $R$ is a certain variety of representations arising from suitable $SU(2)$-representations of the knot group, we show that this pairing is not identically zero. We discuss the consequences of this result for the $SU(2)$-representations of fundamental groups of manifolds obtained by Dehn surgery on such knots.

1 Introduction

The real algebraic variety of representations from a 3-manifold group $\pi_1(M)$ to $SU(2)$ or $SO(3)$ has long been a subject of interest, giving rise as it does to useful invariants such as the Casson invariant and the $A$-polynomial \cite{3}.

In the case where $\partial M$ is a torus – in particular, where $M$ is the exterior of a knot in $S^3$ – there is a particular interest in finding representations which vanish on a given slope $\alpha \in \mathbb{Q} \cup \{\infty\}$ on $\partial M$, and hence give rise to
a representation of $\pi_1(M(\alpha))$, where $M(\alpha)$ is the manifold obtained from $M$ by Dehn filling along $\alpha$.

A description of the character variety in the case of a 2-bridge knot is given by Burde in [1]. For twist knots, a more detailed description is given by Uygur and Azcan in [8].

Burde [1] used this description to show that nontrivial representations $\pi_1(M(+1)) \to SU(2)$ exist for any nontrivial 2-bridge knot exterior $M$, and deduced the Property P Conjecture for 2-bridge knots. More recently, Kronheimer and Mrowka [5] proved the Property P Conjecture in full by showing that nontrivial representations $\pi_1(M(+1)) \to SO(3)$ exist for an arbitrary nontrivial knot exterior $M$.

In another article [6], the same authors proved that there is an irreducible representation $\pi_1(M(r)) \to SU(2)$ (that is, a representation with nonabelian image), for any nontrivial knot exterior $M$ and any slope $r \in \mathbb{Q}$ such that $|r| \leq 2$. One consequence of this (see [2, 4]) is that every nontrivial knot has a nontrivial $A$-polynomial.

In the present note, we construct a bilinear pairing $\pi_1(C) \times \pi_1(\partial M) \to \mathbb{Z}$ for suitable subsets $C$ of the variety $R$ of representations $\pi_1 M \to SU(2)$, and apply it to Burde’s description [1] of $R$ in the case of 2-bridge knots, to show that the restriction $|r| \leq 2$ in [6] can be weakened in this case:

**Theorem 1.1** Let $M$ be the exterior of a nontrivial 2-bridge knot in $S^3$ which is not a torus knot, and let $\alpha$ be any non-meridian slope in $\partial M$. Then there exists an irreducible representation $\pi_1(M(\alpha)) \to SU(2)$.

Since there are many examples of lens spaces obtainable by Dehn surgery on nontrivial knots, it is clear that the above theorem cannot possibly extend from 2-bridge knots to arbitrary knots. However, by varying the subset $C$ of the representation in our construction, we can adapt the technique to consider also reducible representations.

As an example, we prove the following result for torus knots.

**Theorem 1.2** Let $X$ be the exterior of the $(p, q)$ torus knot, where $1 < p < q$, and $X(\alpha)$ the manifold obtained from $X$ by Dehn filling along a non-meridian slope $\alpha \in \mathbb{Q}$. Then

1. if $\alpha = pq$ and $p > 2$, then $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$;
(2) if $\alpha = pq$ and $p = 2$, then $\pi_1(X(\alpha))$ admits no irreducible representation to $SU(2)$, but admits a representation to $SO(3)$ with nonabelian image;

(3) if $\alpha = pq \pm \frac{1}{n}$ for some positive integer $n$, then every representation from $\pi_1(X(\alpha))$ to $SO(3)$ has abelian image;

(4) for any other value of $\alpha$, $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$.

Results of [7] indicate that this result is in a sense best possible: for example, in Case (3) the Dehn surgery manifold $X(\alpha)$ is a lens space.

The paper is organised as follows. In Section 2 below we recall some basic properties of the $SU(2)$ representation and character varieties of a knot group. In Section 3 we describe our bilinear pairing, in a fairly general context. We then apply this in Sections 4 and 5 to prove Theorems 1.1 and 1.2 respectively.

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2 The $SU(2)$ representation and character varieties

If $\Gamma$ is any finitely presented group, and $G$ is a (real) algebraic matrix group, then the set of representations $\Gamma \to G$ forms a real affine algebraic variety $\mathcal{R}$ on which $G$ acts by conjugation, giving rise to a quotient character variety $\mathcal{X}$.

For the purposes of the present paper, $\Gamma$ will always be a knot group, and $G = SU(2)$. In this case $\mathcal{R}$ is naturally expressed as a union of two closed $SU(2)$-invariant subvarieties $\mathcal{R}_{\text{red}} \cup \mathcal{R}_{\text{irr}}$, and hence also $\mathcal{X}$ is a union of subvarieties $\mathcal{X}_{\text{red}} \cup \mathcal{X}_{\text{irr}}$. Here $\mathcal{R}_{\text{red}}$ denotes the variety of reducible representations $\rho : \Gamma \to SU(2)$, in other words those for which the resulting $\Gamma$-module $\mathbb{C}^2$ splits as a direct sum of two 1-dimensional modules. This happens precisely when the image of $\rho$ is abelian, in other words when $\rho$ is induced from a representation of $\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}$. Hence $\mathcal{R}_{\text{red}}$ is canonically homeomorphic to $SU(2) \cong S^3$. The corresponding character subvariety $\mathcal{X}_{\text{red}}$ is canonically homeomorphic to the closed interval $[-2, 2] \subset \mathbb{R}$, parametrised by the trace $Tr(\rho(\mu))$, where $\rho$ is a representative of a conjugacy class of
reducible representations, and $\mu \in \Gamma$ is a fixed meridian element. The complement of $\mathcal{R}_{red}$ in $\mathcal{R}$ is not closed, but its closure is a subvariety $\mathcal{R}_{irr}$ which is $SU(2)$-invariant and hence gives rise to a closed subvariety $\mathcal{X}_{irr}$ of $\mathcal{X}$.

Now fix once and for all a meridian $\mu \in \Gamma$, and consider the following subset $\mathcal{C}$ of $\mathcal{R}$. A representation $\rho : \Gamma \to SU(2)$ belongs to $\mathcal{C}$ if and only if

$$\rho(\mu) = \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}$$

with $x, y \in \mathbb{R}$, $x^2 + y^2 = 1$ and $y \geq 0$. Note that every representation in $\mathcal{R}$ is conjugate to one in $\mathcal{C}$, so the quotient map $\mathcal{R} \to \mathcal{X}$ restricts to a surjection on $\mathcal{C}$ (and to a homeomorphism $\mathcal{C} \cap \mathcal{R}_{red} \to \mathcal{X}_{red}$).

3 Winding numbers

Let $A$ be an abelian group, $G$ a (connected) Lie group, and $\mathcal{D}$ a subset of the variety of representations $A \to G$. Given any path $P = \{\rho_t, 0 \leq t \leq 1\}$ in $\mathcal{D}$, and any $a \in A$, we obtain a path $P(a) = \{\rho_t(a), 0 \leq t \leq 1\}$ in $G$.

Clearly, if $P'$ is homotopic (rel end points) to $P$, then $P'(a)$ is homotopic (rel end points) to $P(a)$, for any $a \in A$. Hence we obtain a pairing

$$\nu : A \times \pi_1(\mathcal{D}) \to \pi_1(G), \quad \nu(a, [P]) := [P(a)].$$

**Remark** Recall that, if $f, g : [0, 1] \to G$ are closed paths in the topological group $G$, based at the identity element $1 \in G$, then $[f][g] = [f.g] = [g][f]$ in $\pi_1(G, 1)$, where $f.g$ denotes the pointwise product $t \mapsto f(t)g(t) \in G$. This can easily be seen, for example, from the diagram below, representing the map $[0, 1]^2 \to G$, $(s, t) \mapsto f(s)g(t)$.
In particular, \( \pi_1(G,1) \) is abelian, so the above definition of \( \nu \) is unaffected by base-point choices.

**Proposition 3.1** The pairing \( \nu : (a, [P]) \mapsto [P(a)] \) defined above is bilinear.

*Proof.* For a fixed element \( a \in A \), if \( P,Q \) is the concatenation of paths \( P,Q \) in \( D \), then \( (P,Q)(a) \) is the concatenation of \( P(a), Q(a) \) in \( G \), so \( [P] \mapsto [P(a)] \) is a homomorphism \( \pi_1(D) \to \pi_1(G) \).

For \( a,b \in A \) and a fixed path \( P \) in \( D \), we have \( P_t(ab) = P_t(a)P_t(b) \) for each \( t \in [0,1] \), since \( P_t \) is a representation \( A \to G \). By the above remark, \( [P(ab)] = [P(a)][P(b)] \) in \( \pi_1(G) \). In other words, \( a \mapsto [P(a)] \) is a homomorphism \( A \to \pi_1(G) \).

We apply Proposition 3.1 in the following restricted context. Let \( X \) be the exterior of a nontrivial knot in \( S^3 \), and let \( A = \pi_1(\partial X) \cong \mathbb{Z}^2 \). Let \( \mu, \lambda \in A \) denote a fixed meridian and longitude respectively.

Let \( G \) be the Lie group \( S^1 \). The subset \( D \) of the variety of representations \( A \to S^1 \) arises as follows. We regard \( S^1 \) as the subgroup of \( SU(2) \) consisting of diagonal matrices. Recall that \( R \) is the variety of representations \( \pi_1(X) \to SU(2) \), and that \( C \) is the subvariety of \( R \) consisting of representations \( \rho : \pi_1(X) \to SU(2) \) such that \( \rho(\mu) \) is diagonal, and the imaginary part of the \( (1,1) \) entry of \( \rho(\mu) \) is non-negative. Since \( A \) is abelian and \( \pi_1(X) \) is generated by conjugates of \( \mu \), it follows that \( \rho(A) \) contains only diagonal matrices whenever \( \rho \in C \). We define \( D \) to be the set of representations \( A \to S^1 \) that arise as restrictions of representations in \( C \).

Note that \( \pi_1(S) \cong \mathbb{Z} \), so the bilinear pairing \( \nu : A \times \pi_1(D) \to \pi_1(S^1) \) is integer-valued.

**Proposition 3.2** For each \( \gamma \in \pi_1(D) \) let \( K_\gamma \subset A \) denote the kernel of the homomorphism \( A \to Z, a \mapsto \nu(a,\gamma) \). Then either \( K_\gamma = A \) or \( K_\gamma = \mathbb{Z}\mu \), the subgroup of \( A \) generated by \( \mu \).

*Proof.* Certainly \( \mu \) belongs to \( K_\gamma \) for all \( \gamma \in \pi_1(D) \), since for \( \rho \in C \) we have \( \rho(\mu) \) contained in an open interval in \( S^1 \) (so the winding number of \( \rho(\mu) \) as \( \rho \) travels around \( C \) is zero).

On the other hand, let \( c = \nu(\lambda,\gamma) \). Then by bilinearity, for any \( m,n \in \mathbb{Z} \) we have \( \nu(m\mu+n\lambda,\gamma) = cn \). If \( cn = 0 \) for some \( n \) then either \( c = 0 \) or \( n = 0 \). In the first case \( \nu(m\mu+n\lambda,\gamma) = 0 \) for all \( m,n \). In the second case, \( m\mu+n\lambda = m\mu \in \mathbb{Z}\mu \).  

\( \square \)
Corollary 3.3 If the pairing $\nu : A \times \pi_1(D) \to \mathbb{Z}$ is not uniformly vanishing, and $\alpha$ is any non-meridian slope on $\partial X$, then $\pi_1(X(\alpha))$ admits a nontrivial representation to $SU(2)$, where $X(\alpha)$ is the 3-manifold obtained from $X$ by Dehn-filling along $\alpha$.

Proof. By hypothesis, $K_\gamma \neq A$ for some $\gamma \in \pi_1(D)$, so $K_\gamma = \mathbb{Z}\mu$ by the Proposition. Since $\alpha \notin \mathbb{Z}\mu$, it follows that $\nu(\alpha, \gamma) \neq 0$. Hence the map $S^1 \to S^1$ defined by $t \mapsto \gamma_t(\alpha)$, has nonzero winding number, and hence in particular is surjective. Thus we may choose $t \in S^1$ such that $\gamma_t(\alpha) = 1 \in SU(2)$ and hence $\sigma$ induces a nontrivial representation

$$
\tau : \pi_1(X(\alpha)) = \pi_1(X)/\langle\langle \alpha \rangle\rangle \to SU(2).
$$

□

In practice, to find suitable closed paths in $D$ we may find a closed path in $C$ and project it to $D$ using the restriction map $\rho \mapsto \rho|_A$. The next result shows that it is equally valid to work in the character variety $X$ rather than $C$.

Lemma 3.4 The restriction map $C \to D$, $\rho \mapsto \rho|_A$, factors through $X$.

Proof. Given $\rho, \rho' \in C$ with the same image in $X$, we know that $\rho, \rho'$ are conjugate by some matrix $M \in SU(2)$. If $\rho(\mu) \in Z(SU(2)) = \{\pm I\}$, then the image of $\rho$ is central and so $\rho' = \rho$. Otherwise, $\rho(\mu) = \rho'(\mu)$ is a diagonal matrix with non-real diagonal entries, so the conjugating matrix $M$ must also be diagonal. But in this case $\rho(A)$ consists only of diagonal matrices, which therefore commute with $M$, so the restrictions of $\rho$ and $\rho'$ to $A$ coincide. □

An immediate consequence of Lemma 3.4 is that any path in $R$ between two conjugate representations gives rise to a closed path in $D$ by first projecting to $X$ and then applying the restriction map $X \to D$.

4 Two-bridge knots

In this section we prove the following result.

Theorem 4.1 Let $R_{\text{irr}}$ be the variety of irreducible $SU(2)$-representations of a two-bridge knot group $G$, and let $A$ be a peripheral subgroup of $G$. Then there is a closed curve $\gamma$ in $R_{\text{irr}}$ such that the pairing $\nu : \pi_1(\gamma) \times A \to \mathbb{Z}$ is not identically zero.
Proof. A two-bridge knot group $G$ has a presentation of the form

$$G = \langle x, y \mid Wx = yW \rangle,$$

where $W = W(x, y)$ is a word of the form $x^{\varepsilon(1)}y^{\varepsilon(2)} \cdots y^{\varepsilon(2n)}$ with $\varepsilon(i) = \pm 1$ for each $i$. Here $x$ and $y$ are meridians. The symmetry of the presentation ensures that $xW^* = W^*y$ in $G$, where $W^*(x, y) := W(y, x)$. Hence $\beta = W^*W$ commutes with the meridian $x$, so is a peripheral element and represents a slope on the boundary torus of the knot exterior.

The exponents $\varepsilon(i)$ can be more explicitly described. There is an odd integer $k$ coprime to $2n + 1$ such that

$$\varepsilon(i) = (-1)^{\left\lfloor \frac{ik}{2n+1} \right\rfloor} \left\lfloor \frac{(2n+1 - i)k}{2n+1} \right\rfloor$$

for each $i$. In particular, since

$$\frac{ik}{2n+1} - 1 < \left\lfloor \frac{ik}{2n+1} \right\rfloor < \frac{ik}{2n+1}$$

for each $i$, we have

$$\left\lfloor \frac{ik}{2n+1} \right\rfloor + \left\lfloor \frac{(2n+1 - i)k}{2n+1} \right\rfloor = k - 1 \equiv 0 \bmod 2$$

for each $i$, so that $\varepsilon(2n+1 - i) = \varepsilon(i)$. From this, it follows that

$$W(x^{-1}, y^{-1}) = W(y, x)^{-1} = W^{*-1} \quad \text{and} \quad W^{*}(x^{-1}, y^{-1}) = W^{-1}.$$

The following construction is essentially due to Burde (see [2, p.116]). Under the action of $SU(2)$ by rotations on $S^2$, we may choose fixed points of $\rho(x), \rho(W), \rho(y), \rho(W^*)$ as the vertices $A, B, C, D$ respectively of a spherical rhombus, such that $\rho(W)(A) = C$ and $\rho(W^*)(C) = A$. (There are degenerate cases: possibly $A = C$ if $\rho(G)$ is abelian; possibly $B = D$ if $\rho(W) = \rho(W^*)$ with $\rho(W)^2 = -I.$) It follows that the angle of rotation of $W^*W$ is $2\theta$ modulo $2\pi$, where $\theta$ is the angle $\hat{DAB}$ of the rhombus.
Conjugacy in $SU(2)$ allows us freedom to place this rhombus where we wish. Let us choose to place it with $A = (1, 0, 0)$, and $C = (\cos \psi, \sin \psi, 0)$ with $0 < \psi < \pi$.

If we have a path $\rho_t$ ($0 \leq t \leq 1$) of representations, then this gives rise to a path $A_tB_tC_tD_t$ of rhombi, and a path $\theta_t \in \mathbb{R}/(2\pi\mathbb{Z})$ of corresponding angles. Parameters $t$ with $\theta_t \in 2\pi\mathbb{Z}$ correspond to degenerate rhombi with $B_t = D_t$, and hence to representations $\rho_t$ with $\rho_t(W) = \rho_t(W^*)$.

Among all $SU(2)$ representations of $G$, a special rôle is played by those whose image in $SO(3)$ is dihedral, in other words where $\rho(x)^2 = \rho(y)^2 = -I$. In this case, the points $B, D$ of our rhombus coincide with the north and south poles $N, S = (0, 0, \pm 1)$. Burde [11 pp. 116-117] explains that, if $\Gamma$ is the group of a two-bridge knot which is not a torus knot, then there is a path $\rho_t$ of irreducible representations joining two dihedral representations $\rho_0, \rho_1$, such that $B_t = D_t$ switch poles on travelling from $t = 0$ to $t = 1$. In other words, the change of angle $\theta_1 - \theta_0$ on traversing this path is an odd multiple of $2\pi$ (in particular nonzero). Replacing the path $\rho_t$ by a smooth approximation if necessary, we may assume that $\theta_t$ is differentiable as a function of $t$, and express this as

$$\int_0^1 \frac{\partial \theta_t}{\partial t} dt \neq 0.$$ 

Now consider another path of representations $\bar{\rho}_t$, defined by $\bar{\rho}_t(x) = -\rho_t(x^{-1})$, $\bar{\rho}_t(y) = -\rho_t(y^{-1})$. The equation $W(x^{-1}, y^{-1}) = W^{*^{-1}}$ enables us to verify that $\bar{\rho}_t$ is indeed a representation for each $t$. Moreover, since
\( \rho_t(x)^2 = \rho_t(y)^2 = -I \) for \( t = 0,1 \), it follows that \( \tilde{\rho}_t = \rho_t \) for \( t = 0,1 \). Finally, since \( \tilde{\rho}_t(W^*W) = \rho_t(W^*W)^{-1} \), the change in \( \theta \) along the path \( \tilde{\rho} \) is the negative of the change along the path \( \rho_t \):

\[
\frac{\partial \tilde{\theta}_t}{\partial t} = -\frac{\partial \theta_t}{\partial t}.
\]

If \( \gamma \) is the closed curve formed by concatenating the paths \( \rho_t \) and \( \tilde{\rho}_{1-t} \), the change in \( \theta \) around \( \gamma \) is precisely twice that along \( \rho_t \), namely an odd multiple of \( 4\pi \):

\[
\int_{\gamma} \frac{\partial \theta_t}{\partial t} dt = \int_0^1 \frac{\partial \theta_t}{\partial t} dt + \int_1^0 \frac{\partial \theta_t}{\partial t} dt = 2 \int_0^1 \frac{\partial \theta_t}{\partial t} dt \neq 0.
\]

In particular \( \nu([\gamma], W^*W) \neq 0. \)

\[\square\]

**Corollary 4.2** Let \( X \) be the exterior of a two-bridge knot in \( S^3 \), and let \( X(\alpha) \) be the manifold formed from \( X \) by Dehn filling along a non-meridian slope \( \alpha \) in \( \partial X \). Then \( \pi_1(X(\alpha)) \) admits an irreducible representation to \( SU(2) \).

**Proof.** By Theorem 4.1, there is a closed curve \( \gamma \) of irreducible representations \( \pi_1(X) \to SU(2) \) such that the pairing \( \nu \) on \( \pi_1(\gamma) \times \pi_1(\partial X) \) is not identically zero.

Then \( \nu([\gamma], \cdot) : \pi_1(\partial X) \to \mathbb{Z} \) has kernel \( \mu\mathbb{Z} \). Since \( \alpha \notin \mu\mathbb{Z} \), \( \nu([\gamma], \alpha) \neq 0 \). In other words, the closed curve \( t \mapsto \gamma_t(\alpha) \in S^1 \) has non-zero winding number, and so is surjective. There exists a point \( \rho \in \gamma \) such that \( \rho(\alpha) = 1 \) in \( SU(2) \). Since \( \pi_1(X(\alpha)) \) is the quotient of \( \pi_1(X) \) by the normal closure of \( \alpha \), \( \rho \) induces a representation \( \pi_1(X(\alpha)) \to SU(2) \) with nonabelian image. \[\square\]

## 5 Torus knots

In this section we demonstrate that the pairing \( \nu \) is not identically zero on suitable curves in the \( SU(2) \)-representation variety of a torus knot. We then apply this to the fundamental group of any manifold obtained by nontrivial Dehn surgery on a torus knot, and study its representations to \( SU(2) \).

The \((p,q)\)-torus knot has fundamental group \( \Gamma = \langle x, y | x^p = y^q \rangle \). In particular, it has nontrivial centre, generated by \( \zeta = x^p = y^q \). If \( \{\mu, \lambda\} \) is any meridian-longitude pair, then \( \zeta \) belongs to the peripheral subgroup generated by \( \{\mu, \lambda\} \), since it commutes with \( \mu \).
The character variety $X$ of $\text{Hom}(\Gamma, \text{SU}(2))$ splits into a number of arcs as follows. As for all knots, the subvariety $X_{\text{red}}$ corresponding to reducible representations is isomorphic to the closed interval $[-2, 2]$, parametrised by the trace of $\rho(\mu)$.

If $\rho : \Gamma \to \text{SU}(2)$ is an irreducible representation, then $\rho(x), \rho(y)$ are non-commuting matrices with $\rho(x)^p = \rho(y)^q$. This can arise only if $\rho(x)^p = \rho(y)^q = \pm I$, where $I$ is the identity matrix. Hence $\rho(x)$ has trace $2 \cos(a\pi/pq)$ for some integers $a, b$ of the same parity. There are $(p-1)(q-1)/2$ open arcs $A_{(a,b)}$ in the irreducible character variety, one corresponding to each pair $(a, b)$ of integers with $1 \leq a \leq p-1, 1 \leq b \leq q-1, a \equiv b$ modulo 2. Each open arc $A_{(a,b)}$ is the interior of a closed arc $\overline{A}_{(a,b)}$ in the whole character variety, whose endpoints are reducible characters.

**Lemma 5.1** The endpoints of $\overline{A}_{(a,b)}$ are the points

$$2 \cos(c\pi/pq), 2 \cos(d\pi/pq) \in [-2, 2] \cong X_{\text{red}},$$

where where $c, d \in \{1, \ldots, pq-1\}$ are the unique solutions to the congruences

$c, d \equiv \pm a \mod 2p; \quad c, d \equiv \pm b \mod 2q.$

**Proof.** On $A_{(a,b)}$, the trace of $\rho(x)$ is constant at $2 \cos(a\pi/pq)$, so the same will hold at each endpoint of $A_{(a,b)}$, which corresponds to a reducible representation. But $x \equiv \mu \pm q \mod \text{commutator subgroup}$, so for any reducible representation $\rho$ we have $\rho(x) = \rho(\mu)^{\pm q}$. If $z$ is a complex $q$-th root of $\cos(a\pi/pq) \pm i \sin(a\pi/pq)$, then $z = \cos(c\pi/pq) + i \sin(c\pi/pq)$ where $c \equiv \pm a \mod 2p$. Hence, for a reducible representation $\rho$ at an endpoint of $A_{(a,b)}$, the trace of $\rho(\mu)$ must be $2 \cos(c\pi/pq)$ with $c \equiv \pm a \mod 2p$.

A similar analysis using $\rho(y) = \rho(\mu)^p$ gives the congruence $c \equiv \pm b \mod 2q$.

Finally, note that, since $a \equiv b \mod 2$ and since $p, q$ are coprime, each of the four pairs of simultaneous congruences

$c \equiv \pm a \mod 2p; \quad c \equiv \pm b \mod 2q$

has a unique solution modulo $2pq$. Moreover, if $c$ is the solution of one of these pairs of congruences, then $2pq - c$ is the solution of another, so precisely two of the four solutions lie in the indicated range $\{1, \ldots, pq-1\}$. \hfill $\square$

**Proposition 5.2** Let $\gamma$ be the closed curve in $X$ formed by the arc $\overline{A}_{(a,b)}$ together with the subinterval $[2 \cos(c\pi/pq), 2 \cos(d\pi/pq)]$ of $[-2, 2] \cong X_{\text{red}}$. Then $\nu(\gamma, \zeta) \neq 0$.
Proof. The knot is embedded in an unknotted torus $T \subset S^3$. Each component of $S^3 \setminus T$ is an open solid torus. Moreover, $x, y$ are represented by the cores of these solid tori, and $\zeta = x^p = y^q$ represents a curve on $T$ parallel to the knot. In particular, $\zeta \in A$, i.e., $\zeta$ is a peripheral curve. Now $\rho(\zeta) = \pm I$ for any irreducible representation $\rho$, and so $\rho(\zeta)$ is constant for $\rho \in A_{(a,b)}$.

Let $z = \exp(i\pi/pq)$, a primitive $(2pq)$-th root of unity. Then the endpoints of $A_{(a,b)}$ correspond to the reducible representations $\mu \mapsto z^c$ and $\mu \mapsto z^d$, where $c, d$ are given by Lemma 5.1.

Now, as $\rho$ moves continuously through reducible representations from $\mu \mapsto z^c$ to $\mu \mapsto z^d$, the argument of $\rho(\mu)$ changes by $(d - c)\pi/pq$, so the argument of $\rho(\zeta) = \rho(\mu)^{pq}$ changes by $(d - c)\pi$, whence $\nu([\gamma], \zeta) = (d - c)/2 \neq 0$. □

**Corollary 5.3** Let $X$ be the exterior of a torus knot in $S^3$, and $X(\alpha)$ the manifold obtained from $X$ by Dehn filling along a non-meridian slope $\alpha$. Then $\pi_1(X(\alpha))$ admits a nontrivial representation to $SU(2)$.

Proof. If $\gamma$ is the curve in the Theorem, then $\nu([\gamma], \zeta) \neq 0$, and so the kernel of the homomorphism $A \to \mathbb{Z}$, $\beta \mapsto \nu([\gamma], \beta)$, is precisely $\mu\mathbb{Z}$. But by hypothesis $\alpha \notin \mu\mathbb{Z}$, so $\nu([\gamma], \alpha) \neq 0$. Thus the closed curve $t \mapsto \gamma_t(\alpha)$ has nonzero winding number on $S^1$, so is surjective. There is a representation $\rho \in \gamma$ such that $\rho(\alpha) = 1$ in $SU(2)$. This choice of $\rho$ induces a nontrivial representation $\pi_1(X(\alpha)) \to SU(2)$.

Of course, the above corollary is neither new nor surprising. For example, almost all the groups $\pi_1(X(\alpha))$ have nontrivial abelianisation, so admit representations to $SU(2)$ that are reducible but nontrivial. Of more interest is the question of which $\pi_1(X(\alpha))$ admit irreducible representations to $SU(2)$. This question can also be readily answered using the known classification of 3-manifolds obtained by Dehn surgery on torus knots [1]. Here we present an alternative approach using an adaptation of our winding-number technique.

**Theorem 5.4** Let $X$ be the exterior of the $(p, q)$ torus knot, where $1 < p < q$, and $X(\alpha)$ the manifold obtained from $X$ by Dehn filling along a non-meridian slope $\alpha \in \mathbb{Q} \cup \{\infty\}$. Then

1. if $\alpha = pq$ and $p > 2$, then $\pi_1(X(\alpha))$ admits an irreducible representation to $SU(2)$;
(2) If \( \alpha = pq \) and \( p = 2 \), then \( \pi_1(X(\alpha)) \) admits no irreducible representation to \( SU(2) \), but admits a representation to \( SO(3) \) with nonabelian image;

(3) If \( \alpha = pq \pm \frac{1}{n} \) for some positive integer \( n \), then every representation from \( \pi_1(X(\alpha)) \) to \( SO(3) \) has abelian image;

(4) For any other value of \( \alpha \), \( \pi_1(X(\alpha)) \) admits an irreducible representation to \( SU(2) \).

**Remark** The statement of this theorem fits the classification of [7], where it is proved that \( X(\alpha) \) is a lens space in Case (3); a connected sum of two lens spaces in Cases (1) and (2); and a Seifert fibre space in Case (4).

**Proof.**

(1) Since \( 2 < p < q \), one of the components of \( X_{irr} \) is the arc \( A_{(2,2)} \). But any point on \( A_{(2,2)} \) corresponds to a representation \( \rho \) with \( \rho(x^p) = \rho(y^q) = I \).

(2) In this case \( \pi_1(X(\alpha)) \cong \mathbb{Z}_2 \ast \mathbb{Z}_q \). Since the only element of order 2 in \( SU(2) \) is the central element \( -I \), the image of any representation \( \mathbb{Z}_2 \ast \mathbb{Z}_q \to SU(2) \) is abelian. However, corresponding to any point on \( A_{(1,1)} \) is a representation \( \rho \) with \( \rho(x^2) = \rho(y^n) = -I \), so composing this with the quotient map \( SU(2) \to SO(3) \) gives a representation of \( \pi_1(X(\alpha)) \) to \( SO(3) \) with nonabelian image.

(3) Let \( \zeta \) be the curve \( x^p = x^q \) of slope \( pq \). Then \( \zeta = \mu^{pq} \lambda \), so \( \alpha = \mu^{npq \pm 1} \lambda^n = \mu^{\pm 1} \zeta^n \) in \( \pi_1(\partial X) \). Now any representation from \( \pi_1(X(\alpha)) \) to \( SO(3) \) with nonabelian image arises from a representation of \( \pi_1(X) \) with nonabelian image, which therefore lifts to an irreducible representation \( \rho : \pi_1(X) \to SU(2) \), such that \( \rho(\alpha) = \pm I \). But \( \rho \) corresponds to a point on one of the open arcs \( A_{(a,b)} \), so \( \rho(\zeta) = (-I)^a \) and hence \( \rho(\mu) = (\rho(\alpha)\rho(\zeta)^{-n})^{\pm 1} = \pm I \), contradicting the assumption that \( \rho \) is irreducible.

(4) As in the previous case, let \( \zeta = \mu^{pq} \lambda \) denote the curve with slope \( pq \). Then \( \pi_1(\partial X) \) is generated by \( \zeta \) and \( \mu \), so we can write \( \alpha = \mu^g \zeta^h \). If \( |g| \leq 1 \) then we are in one of the previous cases, so we have \( |g| \geq 2 \).

Suppose first that \( pq \) is even. Then the endpoints of \( A_{1,1} \) are reducible representations \( \rho \) in which the trace of \( \rho(\mu) \) is \( \pm 2 \cos(\pi/pq) \). Choose \( \theta \in [\pi/pq, (pq-1)\pi/pq] \) such that \( \theta \) is an odd multiple of \( \pi/|g| \). Then by continuity of trace, we can choose \( \rho \in A_{(1,1)} \) such that the trace of \( \rho(\mu) \) is \( 2 \cos(\theta) \). Provided \( h \) is odd, this gives \( \rho(\mu)^g = -I = \rho(\zeta)^{-h} \), so \( \rho(\alpha) = I \). If \( h \) is even
then $|g|$ is odd, since $\alpha$ is a slope. In particular $|g| > 2$. In this case, we take $\theta$ to be an even multiple of $\pi/|g|$, and the argument goes through as before.

Now consider the case where $pq$ is odd. Precisely one of the two positive integers $(q \pm p)/2$ is odd. Call it $c$, and note that $c \in \{1, \ldots, q-1\}$. Let $a$ be the unique odd integer with $1 \leq a \leq p - 1$ and $a \equiv \pm c \mod p$. Then the endpoints of $A_{a,c}$ are reducible representations $\rho$ where the trace of $\rho(\mu)$ is $2\cos(c\pi/pq)$ and $2\cos((pq - q + c)\pi/pq)$ respectively. Now the interval $[c\pi/pq, (pq - q + c)\pi/pq]$ contains at least one odd multiple of $\pi/|g|$, and (if $|g| > 2$) at least one even multiple of $\pi/|g|$. Arguing as before, we can choose $\rho \in A_{a,c}$ such that $\rho(\mu)^g = \rho(\zeta)^{-h}$, and so $\rho(\alpha) = I$, except possibly if $|g| = 2$ and $h$ is even (which does not arise, since $\alpha$ is a slope).

□

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