A NEURAL NETWORK WITH LOCAL LEARNING RULES FOR MINOR SUBSPACE ANALYSIS

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ABSTRACT

The development of neuromorphic hardware and modeling of biological neural networks requires algorithms with local learning rules. Artificial neural networks using local learning rules to perform principal subspace analysis (PSA) and clustering have recently been derived from principled objective functions. However, no biologically plausible networks exist for minor subspace analysis (MSA), a fundamental signal processing task. MSA extracts the lowest-variance subspace of the input signal covariance matrix. Here, we introduce a novel similarity matching objective for extracting the minor subspace, Minor Subspace Similarity Matching (MSSM). Moreover, we derive an adaptive MSSM algorithm that naturally maps onto a novel neural network with local learning rules and gives numerical results showing that our method converges at a competitive rate.

Index Terms—artificial neural networks, minor subspace analysis, dimensionality reduction.

1. INTRODUCTION

One of the most straightforward tasks that a neural network (NN) can perform is to learn a low-dimensional space of stimulus features that captures the directions of the lowest variance, forming the minor subspace (Fig. 1B). This task is known as minor subspace analysis, and some datasets are well characterized by the directions of least variation [1, 2, 3]. Minor subspace analysis (MSA) has been used for tasks such as total least square regression [4], direction of arrival estimation [5], and others [6, 7]. MSA is also integral to problems more closely related to neuroscience, such as invariance learning [8], and slow features analysis [9].

While online MSA algorithms with associated NNs exist [8, 10, 11], they are often the result of straightforward adaptations of Oja’s learning rule for Principal Subspace Analysis (PSA). However, these NNs inherit the non-locality of the learning rules characteristic of the Oja’s NNs. Besides understanding and modeling brain functions, increased biological realism in artificial NN can be useful for handling large datasets or streaming tasks [12], and the development of neuromorphic hardware [13]. In particular, biologically plausible NNs operate online, i.e., sample-by-sample, avoiding storage of large datasets in memory. A necessary yet not sufficient learning for learning rules in biological networks is that the learning rules be local because synapses only have access to information about the neurons they connect.

Our two main contributions are the following. Firstly, to overcome the non-local nature of the rules used in existing NNs, we propose a similarity matching objective function for MSA. Secondly, we show that such an objective is optimized by an online algorithm that maps onto a neural network with local learning rules (Fig. 1C). Numerical experiments show that, despite using local learning rules, our neural network performs competitively with existing methods that do not respect such constraints.

2. BACKGROUND AND PROBLEM SETTING

Given $T$ centered input data samples $(x_t)_{t=1}^T \in \mathbb{R}^n$, which defines the input matrix by $X = [x_1, \ldots, x_T] \in \mathbb{R}^{n \times T}$, the MSA problem aims at finding a set of $m$ orthonormal vectors, denoted $W = [w_1, \ldots, w_m] \in \mathbb{R}^{m \times n}$, such that the projections of $x_t$ onto these vectors, denoted by $(y_t := W x_t) \in \mathbb{R}^m$ has minimum variance. We also define the output matrix by $Y = [y_1, \ldots, y_T] \in \mathbb{R}^{m \times T}$. The empirical covariance matrix of $X$, denoted by $C_x = \frac{1}{T}XX^\top$, is assumed to be full rank. One formulation of the MSA problem is thus

$$
\min_{W \in \mathbb{R}^{m \times n}, WW^\top = I_m} \frac{1}{2} \text{Tr}[WC_x W^\top].
$$

Suppose the eigen-decomposition of $C_x = V_x A_x V_x^\top$, where $A_x = \text{diag}(\lambda_1^x, \ldots, \lambda_n^x)$, with $\lambda_1^x \geq \ldots \geq \lambda_n^x > 0$ are the eigenvalues of $C_x$. It is well known that the optimal solution of the problem (1) is the projections of the input dataset $X$ onto its minor subspace. The minor subspace is spanned by the columns of $V_x$ corresponding to the $m$ smallest eigenvalues of $C_x$, denoted by $V_m^{MS} = [v_m^x, v_{m+1}^x, \ldots, v_n^x]$. Standard singular value decomposition and other offline methods exist [14] to extract the $m$ minor subspace.
2.1. Existing Learning Rules are Non-local

Various NNs exist for solving MSA in the online setting. It is natural to identify the inputs $x_t \in \mathbb{R}^n$ with the activity of $n$ upstream neurons at time $t$. In response, the NN outputs an activity vector, $y_t \in \mathbb{R}^m$, with $m$ the number of output neurons. For each time $t$, $y_t$ is obtained by multiplying $x_t$ by the corresponding synaptic weights, $W$ [15]. Existing learning rules used to train NNs for MSA result from adaptations of Oja’s original work for principal subspace analysis (PSA). Indeed, PSA is a variance maximization problem formulated as

$$
\max_{W \in \mathbb{R}^{m \times n}, W^T W = I} \frac{1}{2} \text{Tr}[WC_x W^T].
$$

(2)

Oja first proposed [15] a stochastic gradient ascent algorithm for solving PSA (2) leading to the popular Oja’s rule.

Thus, it appears natural to implement a stochastic gradient descent, instead of ascent, of the same objective function to obtain an algorithm for MSA. Oja algorithm for MSA is then

$$
\Delta W \approx -\eta (y_t x_t^T - y_t y_t^T W),
$$

(3)

with $\eta > 0$ the learning rate. However, besides the fact that such an update rule leads to diverging weights [15], implementing it in a single-layer NN architecture requires non-local learning rules [16]. Indeed, the last term in (3) implies that updating the weight of a synapse requires knowledge of output activities of all other neurons, which are not available to the synapse.

2.2. Similarity Matching for Principal Subspace Analysis

To better understand our approach for building an MSA NN with local learning, we recall the similarity matching (SM) approach for deriving single-layer PSA NNs with local learning rules [17]. If the similarity of a pair of vectors is quantified by their scalar product SM leads to the following objective:

$$
\min_{Y \in \mathbb{R}^{m \times T}} \frac{1}{T^2} \|X^T X - Y^T Y\|^2_F.
$$

(4)

Despite a different form, both PSA (2) and SM (4) lead to the same embeddings [18, 19]. Since $C_x$ and $X^T X$, have the same $n$ non-zero eigenvalues and related eigenvectors, SM also projects $X$ onto the subspace of $m$ largest eigenvectors of $C_x$.

A key insight of [17, 20] was that the optimization problem (4) can be converted algebraically to an online-tractable form by introducing dynamical variables $W$ and $M$:

$$
\min_{Y \in \mathbb{R}^{m \times T}} \min_{W \in \mathbb{R}^{m \times n}} \max_{M \in \mathbb{R}^{m \times m}} \frac{1}{T} \text{Tr} \left( -4X^TW^TY + 2Y^TMY \right) + 2 \text{Tr} \left( W^T W \right) - \text{Tr} \left( M^T M \right).
$$

(5)

They also proposed an online algorithm based on alternating optimization [17] with respect to $y_t$ and $(W, M)$ that can be implemented by a single-layer NN (Fig. 1A) as:

$$
\frac{dy_t(\gamma)}{d\gamma} = W x_t - M y_t(\gamma),
$$

(6)

$$
\Delta W := \eta \left( y_t x_t^T - W \right), \Delta M := \eta \left( y_t y_t^T - M \right).
$$

(7)

As before, the activity of the upstream neurons encodes input variables, $x_t$. Output variables, $y_t$, are computed by the dynamics of activity (6) in a single layer of neurons. They also suggested that the elements of matrices $W$ and $M$ are represented by the weights of synapses in feedforward and lateral connections, respectively. Crucially, unlike in (3), the resulting learning rules (7) are local.

3. A SIMILARITY MATCHING APPROACH TO MINOR SUBSPACE ANALYSIS

To overcome the non-locality of existing learning rules for MSA, we propose exploring a similarity matching approach. We present the first similarity matching objective function for MSA, referred to as Minor Subspace Similarity Matching (MSSM) in the following. We also derive an online algorithm for optimizing it.

3.1. A Novel Objective Function for MSA

To develop our MSSM algorithm, we are looking for a similarity matching with the following property: its eigenvectors
associated with its largest non-zero eigenvalues must span the same subspace as the smallest non-zero eigenvalues of the original matrix of similarity $X^\top X$ (Fig. 2A).

A similar problem was raised when considering the covariance matrix, $C_x$, for which at least two methods exist for transforming the smallest eigenvalues into the largest eigenvalues. One is by considering the eigenvalue of $C_x^{-1}$, the other is by considering $\sigma I_n - C_x$, with $\sigma > \lambda_1^*$. However, neither of these tricks work when the matrix of similarity $X^\top X$ is considered. Indeed, $X^\top X$ is not invertible if $T > n$, and has $T - n$ zero eigenvalues. Also, shifting the spectrum of $X^\top X$ by considering $\sigma I_n - X^\top X$ makes the null eigenvalues of $X^\top X$ the largest eigenvalues of the resulting similarity matrix (Fig. 2B), which is not the $m$ minor subspace of $C_x$.

Let us now consider the matrix $\sigma X^\top C_x^{-1} X$. The aforementioned matrix is the scaled matrix of similarity of whitened input. It has $n$ non-zero eigenvalues, all equal to $\sigma$, and the same eigenvectors as $X^\top X$. It is simply resulting from the fact that $C_x^{-1/2}X$ has all singular values equal to 1. Assuming that $\sigma > \lambda_1$, $\sigma X^\top C_x^{-1} X - X^\top X$, has the following spectrum, $\sigma - \lambda_1 \geq \ldots \geq \sigma - \lambda_1 > 0$, with $T - n$ null eigenvalues (Fig. 2C). This matrix is thus the perfect candidate for the MSSM objective.

Our MSSM objective for discovering a low-dimensional subspace spanning the $m$-MS of $C_x$, with $\sigma \geq \lambda_1$, is thus

$$\min_{Y \in \mathbb{R}^{m \times T}} \frac{1}{T} \| \sigma X^\top C_x^{-1} X - X^\top X - Y^\top Y \|_F^2,$$  \hspace{1cm} (8)

from the following proposition with proof in Appendix A.

**Proposition 1.** Optimal solutions $Y^* \in \mathbb{R}^{m \times T}$ of MSSM (8) are projections of the dataset $X$ onto the $m$-dimensional minor subspace of $C_x$, spanned by $V_x^M$, defined in Section 2

### 3.2. MSSM as a min-max optimization problem

We propose a tractable min-max formulation of MSSM instrumental for deriving an algorithm that maps onto NN with local learning rules. We start by expanding the squared Frobenius norm and discard the terms independent of $Y$ to obtain

$$\min_{Y \in \mathbb{R}^{m \times T}} \frac{2}{T^2} \text{Tr} \left( X^\top \left( \sigma I_n - C_x \right) C_x^{-1} X Y^\top Y \right) + \frac{1}{T^2} \text{Tr} \left( Y^\top Y Y^\top Y \right).$$ \hspace{1cm} (9)

We then introduce dynamical matrix variables $W$ and $M$ in place of $\sigma Y^\top X C_x^{-1}$, and $Y^\top Y$, respectively. Similar substitution tricks are detailed in [20]. We can now rewrite (9) as the following min-max optimization problem

$$\min_{W \in \mathbb{R}^{m \times n}} \min_{M \in \mathbb{R}^{m \times n}} \max_{Y \in \mathbb{R}^{m \times T}} L(W, M, Y)$$ \hspace{1cm} (10)

with

$$L(W, M, Y) := \frac{1}{T} \text{Tr} \left( -4X^\top \left( \sigma I_n - C_x \right) W^\top Y \right) + \frac{1}{T} \text{Tr} \left( 2Y^\top M Y \right) + \frac{1}{T} \text{Tr} \left( 2WC_x W^\top - M^\top M \right).$$

In the offline setting, we can solve (10) by alternating optimization [21]. We first minimize with respect to $Y$ while holding $(W, M)$ fixed, which admits a closed-form solution

$$Y = M^{-1} (\sigma WX - WC_x X).$$ \hspace{1cm} (11)

Holding $Y$ fixed, we then perform a gradient descent-ascent step with respect to $(W, M)$:

$$W \leftarrow W + 2\eta_y \left( \frac{1}{T} Y \left( \sigma I_n - C_x \right) X^\top - WC_x \right);$$ \hspace{1cm} (12)

$$M \leftarrow M + \frac{\eta}{\tau} \left( \frac{1}{T} Y Y^\top - M \right).$$ \hspace{1cm} (13)

Here, $\eta > 0$ is the learning rate for both $W$, and $\tau > 0$ is the ratio of the learning rates of $W$ and $M$. The stability of similar learning rules is investigated in [22, 23].

### 3.3. Derivation of the Local Learning Rules

We now propose an online implementation of (8) by observing that (10) can be decomposed so that optimal outputs at different time steps can be computed independently as

$$\min_{W \in \mathbb{R}^{m \times n}} \max_{M \in \mathbb{R}^{m \times n}} \frac{1}{T} \sum_{t=1}^T \left[ \text{Tr} \left( 2WC_x W^\top - M^\top M \right) \right]$$

$$\text{subject to} \quad \min_{Y \in \mathbb{R}^{m \times T}} \min_{W \in \mathbb{R}^{m \times n}} \max_{M \in \mathbb{R}^{m \times n}} l_t(W, M, y_t)$$ \hspace{1cm} (14)

with $l_t(W, M, y_t) := -4z_t x_t^\top W^\top y_t + 2y_t^\top M y_t$. \hspace{1cm} (15)

with $z_t = (\sigma - \|x_t\|^2)$. The approximation of $C_x$ by $x_t x_t^\top$ is essential in the online setting as the true covariance matrix is not available and should be approximated at each $t$.

We can thus solve (14) sample-by-sample, i.e., online, by first minimizing (15) with respect to the output variables, $y_t$. To do so, we run the following neural dynamics obtained by gradient-descent until convergence, while keeping $(W, M)$ fixed:

$$\frac{dy_t(\gamma)}{d\gamma} = z_t W x_t - M y_t(\gamma).$$ \hspace{1cm} (16)

After the convergence of $y_t$, we update $(W, M)$ by gradient descent-ascent as

$$W \leftarrow W + 2\eta_y (z_t y_t - W x_t) x_t^\top,$$ \hspace{1cm} (17)

$$M \leftarrow M + \frac{\eta}{\tau} (y_t y_t^\top - M).$$ \hspace{1cm} (18)

Similarly to PSA SM, our algorithm can be implemented by a NN with feedforward, $W$, and lateral connections $M$ (Fig. 1C). Here, however, the output and $W$ update rules are gated by the global factor $z_t$. Global gating factors like $z_t$ have been used outside of the similarity matching framework for PCA and ICA [24, 25].
4. NUMERICAL EXPERIMENTS

As an illustration of the capability of the NN derived from SMMS we provide experimental results of our algorithm against two popular algorithms proposed in [26] and [27], denoted by CAL and DKA. The competing algorithms use the following update rules

\[
\text{CAL} : \Delta W = -\eta (WW^T yy^T - yy^T W) \; ; \tag{19}
\]
\[
\text{DKA} : \Delta W = -\eta ((WW^T)yy^T - yy^T W) \; . \tag{20}
\]

We evaluate our algorithm on two artificially generated datasets, \(X \in \mathbb{R}^{50 \times 10,000}\), with a linear spectrum, \((\lambda_k = k/10, \forall k \in \{1, 50\})\) (Fig. 3A,B and C), and with a randomly generated spectrum (Fig. 3D and E), respectively shown Inset of Fig. 3A and Fig. 3C.

The performance of the online algorithms are measured based on the subspace alignment error. Given matrices \((W, M)\), we define the projection \(F = M^{-1}W(\sigma I_n - C_x)\). The subspace alignment error is then generally defined by the relative difference in Frobenius norm square between the true normalized projector \(V_M^{MS} = (V_{m}^MV_{ms}^MV_{ms}^M)^{-1}V_{m}^M\) and the learned normalized projector \(F(F^TF)^{-1}F^T\).

In Fig. 3, we show that after convergence, the subspace spanned by the synaptic connections learned by our online algorithms, is the same as the true basis vectors. In the experiments, our algorithm appears to be converging faster than CAL and DKA. However, no MSA algorithm, including those used here, have known provable convergence rates.

5. DISCUSSION

In this work, we proposed a novel similarity matching objective function, and showed that the online optimization of such an objective leads to the extraction of the minor subspace of the input covariance matrix. The online algorithm we derived maps naturally onto a NN using only local learning rules.

Generalizing our work to the learning of other minor subspace analysis based tasks, such as slow feature analysis [9], will open a path towards principled biologically plausible for invariance learning [8], complementing the work on transformation learning from [28].

A. PROOF OF PROPOSITION

Our result is an extension of the work of Mardia that connected PSA and similarity matching used in [22] with proofs in [29]. The result states that similarity matching is optimized by the projections of inputs onto the principal subspace of their covariance, i.e., performing PSA [18, 29].

Proposition A. For \(X \in \mathbb{R}^{n \times t}\), and fixed \(m (1 \leq m \leq n)\), amongst all projections of \(X\) onto \(m\)-dimensional subspaces of \(\mathbb{R}^n\), the objective (4) is minimized when \(X\) is projected onto its principal coordinates in \(m\) dimensions. (Mardia et al. [29] Theorem 14.4.1).

Now, to show our results we need the following two results. Firstly, that \(X^TX\) and \(X^TC_x^{-1}X\) are simultaneously diagonalizable in the basis formed by the eigenvector of \(X^TX\). Secondly, that the subspace associated with \(m\) largest eigenvalues of \(\sigma X^TC_x^{-1}X - X^TX\) is the same as the \(m\)-subspace by the \(m\) smallest non-zero eigenvalues of \(X^TX\). Finally, we apply Prop.A to the MSSM similarity objective.

Step 1: By the spectral theorem \(\exists U \in \mathcal{O}_T(\mathbb{R})\), composed of the eigenvectors of \(X^TX\), such that \(X^TX = U\Lambda U^T\), with \(\Lambda\) a diagonal matrix of eigenvalues of \(X^TX\) sorted by decreasing order. We can now show that \(X^TC_x^{-1}X\) is diagonalizable in the basis formed by the columns of \(U\). Indeed, for all \(i \in \{1, n\}\), \(u_i\) be the \(i\)-th column of \(U\) by definition we have that \(X^TXu_i = \lambda_i u_i\). As a result we have that

\[
X^TC_x^{-1}Xu_i = \frac{T}{\lambda_i}X^T \left[ C_x^{-1} \frac{1}{T}XX^T \right] X u_i = \frac{T}{\lambda_i}X^TX u_i = Tu_i \; , \tag{S.1}
\]

which proves that all eigenvectors of \(X^TX\) are eigenvectors of \(X^TC_x^{-1}X\). We can now rewrite the difference between the two matrices of similarities in the basis of \(U\) as

\[
\sigma X^TC_x^{-1}X - X^TX = \tilde{U} \begin{pmatrix} \hat{\sigma} - \lambda_n & \cdots & \hat{\sigma} - \lambda_1 & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & \hat{\sigma}
\end{pmatrix} \tilde{U}^T
\]

with \(\tilde{U} = [u_n, \ldots, u_1, u_{n+1}, \ldots, u_T]\) and \(\tilde{\sigma} = \sigma T\). We can then use Prop.A on the new similarity matrix to prove Prop.1.
References

[1] C. K. Williams and F. V. Agakov, “Products of Gaussians
and probabilistic minor component analysis,” Neural
Computation, vol. 14, no. 5, pp. 1169–1182, 2002.
[2] M. Welling, C. Williams, and F. V. Agakov, “Extreme
components analysis,” in Advances in Neural Information
Processing Systems, 2004, pp. 137–144.
[3] Y. Weiss and W. T. Freeman, “What makes a good
model of natural images?,” in 2007 IEEE Conference on
Computer Vision and Pattern Recognition. IEEE, 2007,
pp. 1–8.
[4] Y. Gao, X. Kong, C. Hu, H. Zhang, and L. Hou, “Conver-
genesis analysis of M³ller algorithm for estimating minor
component,” Neural Processing Letters, vol. 42, no. 2,
pp. 355–368, 2015.
[5] X. Kong, C. Hu, and C. Han, “A self-stabilizing MSA
algorithm in high-dimension data stream,” Neural net-
works, vol. 23, no. 7, pp. 865–871, 2010.
[6] X. Kong, C. Hu, and C. Han, “A dual purpose principal
and minor subspace gradient flow,” IEEE Transactions on
Signal Processing, vol. 60, no. 1, pp. 197–210, 2011.
[7] T. D. Nguyen and I. Yamada, “A unified convergence
analysis of normalized PAST algorithms for estimating
principal and minor components,” Signal processing, vol.
93, no. 1, pp. 176–184, 2013.
[8] N. N. Schraudolph and T. J. Sejnowski, “Competitive
anti-Hebbian learning of invariants,” in Advances in
Neural Information Processing Systems, 1992, pp. 1017–
1024.
[9] L. Wiskott and T. J. Sejnowski, “Slow feature analysis:
Unsupervised learning of invariances,” Neural computa-
tion, vol. 14, no. 4, pp. 715–770, 2002.
[10] F.-L. Luo, R. Unbehauen, and A. Cichocki, “A minor
component analysis algorithm,” Neural Networks, vol.
10, no. 2, pp. 291–297, 1997.
[11] G. Cirrincione, M. Cirrincione, J. Hé rault, and S.
Van Huffel, “The MCA EXIN neuron for the minor
component analysis,” IEEE Transactions on Neural Net-
works, vol. 13, no. 1, pp. 160–187, 2002.
[12] A. Giovannucci, V. Minden, C. Pehelevan, and D. B.
Chklovskii, “Efficient principal subspace projection of
streaming data through fast similarity matching,” in 2018
IEEE International Conference on Big Data (Big Data).
IEEE, 2018, pp. 1015–1022.
[13] C. Pehelevan, “A spiking neural network with local learn-
ing rules derived from nonnegative similarity matching,”
in ICASSP 2019-2019 IEEE International Conference
on Acoustics, Speech and Signal Processing (ICASSP).
IEEE, 2019, pp. 7958–7962.
[14] G. Allaire and S. M. Kaber, Numerical linear algebra,
vol. 55, Springer, 2008.
[15] E. Oja, “Principal components, minor components, and
linear neural networks,” Neural Networks, vol. 5, no. 6,
pp. 927–935, 1992.
[16] C. Pehelevan and D. B. Chklovskii, “Neuroscience-
inspired online unsupervised learning algorithms: Artifi-
cial neural networks,” IEEE Signal Processing Magazine,
vol. 36, no. 6, pp. 88–96, 2019.
[17] C. Pehelevan and D. Chklovskii, “A normative theory of
adaptive dimensionality reduction in neural networks,”
in Advances in Neural Information Processing Systems,
2015, pp. 2269–2277.
[18] T. F. Cox and M. A. Cox, Multidimensional scaling,
Chapman and hall/CRC, 2000.
[19] C. K. Williams, “On a connection between kernel PCA
and metric multidimensional scaling,” in Advances in
Neural Information Processing Systems, 2001, pp. 675–
681.
[20] C. Pehelevan, A. M. Sengupta, and D. B. Chklovskii,
“Why do similarity matching objectives lead to Hebbian/anti-Hebbian networks?,” Neural Computation,
vol. 30, no. 1, pp. 84–124, 2018.
[21] B. A. Olshausen and D. J. Field, “Emergence of simple-
cell receptive field properties by learning a sparse code
for natural images,” Nature, vol. 381, pp. 607–609, 1996.
[22] C. Pehelevan, T. Hu, and D. B. Chklovskii, “A
Hebbian/anti-Hebbian neural network for linear subspace
learning: A derivation from multidimensional scaling of
streaming data,” Neural computation, vol. 27, no. 7, pp.
1461–1495, 2015.
[23] D. Lipshutz, Y. Bahroun, S. Golkar, A. M. Sengupta,
and D. B. Chklovskii, “A biologically plausible neural
network for multi-channel canonical correlation analysis,”
arXiv preprint arXiv:2010.00525, 2020.
[24] T. Isomura and T. Toyozumi, “Error-gated hebbian rule:
A local learning rule for principal and independent com-
ponent analysis,” Scientific reports, vol. 8, no. 1, pp.
1–11, 2018.
[25] T. Isomura and T. Toyozumi, “Multi-context blind
source separation by error-gated hebbian rule,” Scientific
reports, vol. 9, no. 1, pp. 1–13, 2019.
[26] T. Chen, S. I. Amari, and Q. Lin, “A unified algorithm
for principal and minor components extraction,” Neural
networks, vol. 11, no. 3, pp. 385–390, 1998.
[27] S. C. Douglas, S.-Y. Kung, and S.-i. Amari, “A self-
stabilized minor subspace rule,” IEEE Signal Processing
Letters, vol. 5, no. 12, pp. 328–330, 1998.
[28] Y. Bahroun, A. Sengupta, and D. B. Chklovskii, “A
similarity-preserving network trained on transformed
images recapitulates salient features of the fly motion
detection circuit,” in Advances in Neural Information
Processing Systems, 2019, pp. 14178–14189.
[29] K. V. Mardia, J. T. Kent, and J. M. Bibby, Multivari-
ate analysis (probability and mathematical statistics),
Academic Press London, 1980.