AN EXPLICIT EXAMPLE OF FROBENIUS PERIODICITY

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Abstract. In this note we show that the restriction of the cotangent bundle $\Omega_{\mathbb{P}^2}$ of the projective plane to a Fermat curve $C$ of degree $d$ in characteristic $p \equiv -1 \mod 2d$ is, up to tensoration with a certain line bundle, isomorphic to its Frobenius pull-back. This leads to a Frobenius periodicity $F^* (\mathcal{E}) \cong \mathcal{E}$ on the Fermat curve of degree $2d$, where $\mathcal{E} = \text{Syz}(U^2, V^2, W^2)(3)$.

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1. Introduction

Let $C$ be a smooth projective curve defined over a field $K$ of characteristic $p > 0$. If $F$ denotes the absolute Frobenius morphism $F : C \to C$, then we say that a vector bundle $\mathcal{E}$ on $C$ admits an $(s, t)$-Frobenius periodicity if there are natural numbers $s$ and $t$, $t > s$, such that $F^{t*}(\mathcal{E}) \cong F^{s*}(\mathcal{E})$. Of particular interest are vector bundles which admit a $(0, t)$-Frobenius periodicity, i.e., $F^{t*}(\mathcal{E}) \cong \mathcal{E}$. By the classical theorem of H. Lange and U. Stuhler [19, Satz 1.4] such a bundle is étale trivializable, i.e., there exists an étale covering $f : D \to C$ such that $f^*(\mathcal{E}) \cong \mathcal{O}_D^r$ where $r = \text{rk}(\mathcal{E})$. Hence a vector bundle $\mathcal{E}$ with a $(0, t)$-Frobenius periodicity comes from a (continuous) representation $\rho : \pi_1(C) \to GL_r(K)$ of the étale fundamental group $\pi_1(C)$ of the curve (see [ibid., Proposition 1.2]). We recall that a vector bundle which can be trivialized under an étale covering does not necessarily admit a Frobenius periodicity (see [9, Example 2.10] or [2, Example below Theorem 1.1]).

Quasicoherent modules over a scheme of positive characteristic allowing a Frobenius periodicity appear under several names ($F$-finite modules, unit $\mathcal{O}_X[F]$-modules) and from several perspectives ($D$-modules, local cohomology, Cartier modules, constructible sheaves on the étale site, Riemann-Hilbert correspondence) in the literature. Beside [19] we mention work of Katz [17, Proposition 4.1.1], Lyubeznik [21], Emerton and Kisin [11], Blickle [3] and Blickle and Böckle [4].

Despite the importance of vector bundles having a Frobenius periodicity, it is not easy to write down non-trivial explicit examples. For a line bundle
the condition becomes \( F^t \mathcal{L} = \mathcal{L}^t = \mathcal{L} \) (with \( q = p^t \)), so \( \mathcal{L} \) must be a torsion element in \( \text{Pic} \ C \) of order \( q - 1 \). For higher rank, a necessary condition is that the bundle \( \mathcal{S} \) has degree 0 and is semistable. By the periodicity it follows that the bundle is in fact strongly semistable, meaning that \( F^t(\mathcal{E}) \) is semistable for all \( t \geq 0 \). On the other hand, if the curve \( C \) and the bundle \( \mathcal{E} \) are defined over a finite field and \( \mathcal{E} \) is strongly semistable of degree 0, then there is necessarily a \((s,t)\)-Frobenius periodicity due to the fact that the number of isomorphism classes of semistable vector bundles of fixed rank and degree is finite ([19, Satz 1.9]). Nevertheless, it is still hard to detect the periodicity \( s \) and \( t \). If we have an extension \( 0 \to \mathcal{O}_C \to \mathcal{S} \to \mathcal{O} \to 0 \) given by \( c \in H^1(C, \mathcal{O}_C) \), then its Frobenius pull-back is given by the class \( F^*(c) \), and one can get (semistable, but not stable) examples by looking at the Frobenius action on \( H^1(C, \mathcal{O}_C) \).

In this note we provide a down to earth example of a stable rank-2 vector bundle \( \mathcal{E} \) on a suitable Fermat curve admitting a \((0,1)\)-Frobenius periodicity. Moreover, this periodicity only depends on a congruence condition of the characteristic of the base field, not on its algebraic structure. Our main tools will be results of P. Monsky on the Hilbert-Kunz multiplicity of Fermat hypersurface rings and the geometric approach to Hilbert-Kunz theory developed independently by the first author in [7] and V. Trivedi in [26].

The results of this article are contained in Chapter 4 of the PhD-thesis [14] of the second author. Related results on the free resolution of Frobenius powers on a Fermat ring can be found in the preprint [18]. We thank Manuel Blickle, Aldo Conca, Neil Epstein and Andrew Kustin for useful discussions.

2. A lemma on global sections

To begin with we recall the notions of a syzygy bundle. Let \( K \) be a field and let \( R \) be a normal standard-graded \( K \)-domain of dimension \( d \geq 2 \). Then homogeneous \( R_+ \)-primary elements \( f_1, \ldots, f_n \) (i.e., \( \sqrt{(f_1, \ldots, f_n)} = R_+ \)) of degrees \( d_1, \ldots, d_n \) define a short exact (presenting) sequence

\[
0 \longrightarrow \text{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_X(-d_i) \overset{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_X \longrightarrow 0
\]

on the projective scheme \( X = \text{Proj} R \). The kernel \( \text{Syz}(f_1, \ldots, f_n) \) is locally free and is called the syzygy bundle for the elements \( f_1, \ldots, f_n \).

In this article we only deal with restrictions of syzygy bundles of the form \( \text{Syz}(X^a, Y^a, Z^a) \), \( a \in \mathbb{N} \setminus \{0\} \), on \( \mathbb{P}^2 = \text{Proj} K[X, Y, Z] \) to a plane curve \( C \). Our main interest will be the case \( a = 1 \) which corresponds via the Euler sequence to the cotangent bundle \( \Omega_{\mathbb{P}^2}|_C \) on the projective plane. Since there will be no confusion in the sequel we also denote the restricted bundle on the curve by \( \text{Syz}(X^a, Y^a, Z^a) \).

Let \( K \) be a field and consider a smooth plane curve of the form

\[
V_+(Z^d - P(X, Y)) \subset \mathbb{P}^2 = \text{Proj} K[X, Y, Z],
\]
where $P(X,Y) \in K[X,Y]$ denotes a homogeneous polynomial of degree $d$. In this situation we can compute global sections of a rank-2 syzygy bundle of the form $\text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})$ by the following lemma which is a slight improvement over [6, Lemma 1]. It relates the sheaves $\text{Syz}(X^{a_1}, Y^{a_2}, P(X,Y)^k)$ with the sheaves $\text{Syz}(X^{a_1}, Y^{a_2}, P(X,Y)^k)$ which come from $\mathbb{P}^1$ via the Noetherian normalization $C = V_+(Z^d - P(X,Y)) \to \mathbb{P}^1 = \text{Proj} K[X,Y]$. We will use this result several times in the proof of our main theorem in the next section.

**Lemma 2.1.** Let $K$ be a field and let $P(X,Y) \in K[X,Y]$ be a homogeneous polynomial of degree $d$. Suppose the plane curve

$$C := \text{Proj}(K[X,Y,Z]/(Z^d - P(X,Y)))$$

is smooth. Further, fix $a_1, a_2, a_3 \in \mathbb{N}_+$ and write $a_3 = dk + t$ with $0 \leq t < d$. Then we have for every $m \in \mathbb{Z}$ a surjective sheaf morphism

$$\varphi_m : S_k(m-t) \oplus S_{k+1}(m) \to \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m)$$

$$(f_1, f_2, f_3)(g_1, g_2, g_3) \mapsto (Z^d f_1 + g_1, Z^d f_2 + g_2, f_3 + Z^{d-t} g_3)$$

for every $m \in \mathbb{Z}$, where $S_i := \text{Syz}(X^{a_1}, Y^{a_2}, P(X,Y)^i)$ for $i \geq 0$. Moreover, the corresponding map on global sections

$$\Gamma(C, S_k(m-t)) \oplus \Gamma(C, S_{k+1}(m)) \to \Gamma(C, \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m))$$

is surjective for every $m \in \mathbb{Z}$.

**Proof.** We consider the sheaf morphism

$$\mathcal{O}_C(m-t-a_1) \oplus \mathcal{O}_C(m-t-a_2) \oplus \mathcal{O}_C(m-t-dk)$$

$$\oplus$$

$$\mathcal{O}_C(m-a_1) \oplus \mathcal{O}_C(m-a_2) \oplus \mathcal{O}_C(m-dk-d)$$

$$\mathcal{O}_C(m-a_1) \oplus \mathcal{O}_C(m-a_2) \oplus \mathcal{O}_C(m-a_3)$$

which maps $(s_1, s_2, s_3, s_4, s_5, s_6) \mapsto (Z^d s_1 + s_4, Z^d s_2 + s_5, s_3 + Z^{d-t} s_6)$. Clearly, $\varphi_m$ maps $S_k(m-t) \oplus S_{k+1}(m)$ into $\text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m)$. Hence, the map $\varphi_m$ is obtained from $\tilde{\varphi}_m$ via restriction to $S_k(m-t) \oplus S_{k+1}(m)$ and is therefore a morphism of sheaves. It is enough to prove that $\varphi_m$ is surjective on global sections for all $m$. Let $s := (F,G,H) \in \Gamma(C, \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m))$ be a non-trivial global section, i.e., $FX^{a_1} + GY^{a_2} + HZ^{a_3} = 0$ and $\deg(F) + a_1 = \deg(G) + a_2 = \deg(H) + a_3 = m$. We write

$$F = F_0 + F_1 Z + F_2 Z^2 + \ldots + F_{d-1} Z^{d-1}$$

$$G = G_0 + G_1 Z + G_2 Z^2 + \ldots + G_{d-1} Z^{d-1}$$

$$H = H_0 + H_1 Z + H_2 Z^2 + \ldots + H_{d-1} Z^{d-1}$$
with \( F_i, G_i, H_i \in K[X, Y] \) for \( i = 0, \ldots, d - 1 \). We have \( Z^{a_3} = Z^{dk + t} = (Z^d)^k Z^t = P(X, Y)^k Z^t \). Since \( s \) is a syzygy we obtain (by considering the \( K[X, Y] \))-component corresponding to \( Z^t \) a system of equations

\[
F_i Z^i X^{a_1} + G_i Z^i Y^{a_2} + H_{j(i)} Z^{i(i)} Z^{a_3} = 0,
\]

where \( j(i) \equiv i - t \mod d \). Thus \( s = (F, G, H) \) is the sum of the syzygies

\[
s_i := (F_i Z^i, G_i Z^i, H_{j(i)} Z^{i(i)}) \in \Gamma(C, \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m))
\]

We show that each of these summands does either come from \( \Gamma(C, S_{k+1}(m)) \) or from \( \Gamma(C, S_k(m-t)) \). We fix one equation

\[
F_{i_0} Z^{i_0 X^{a_1} + G_{i_0} Z^{i_0 Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} Z^{a_3} = 0}
\]

with \( j(i_0) \equiv i_0 - t \mod d \). First, we treat the case where \( i_0 < t \), hence \( j(i_0) = i_0 - t + d \). Factoring out \( Z^{i_0} \) and replacing \( Z^{a_3} \) by \( P(X, Y)^k Z^t \) yields

\[
0 = Z^{i_0} (F_{i_0} X^{a_1} + G_{i_0} Y^{a_2} + H_{j(i_0)} Z^{d-t} P(X, Y)^k Z^t) = Z^{i_0} (F_{i_0} X^{a_1} + G_{i_0} Y^{a_2} + H_{j(i_0)} P(X, Y)^{k+1})
\]

Hence \( g_{i_0} := (Z^{i_0} F_{i_0}, Z^{i_0} G_{i_0}, Z^{i_0} H_{j(i_0)}) \in \Gamma(C, S_{k+1}(m)) \) and \( \varphi_m(g_{i_0}) = s_{i_0} \). Next, we consider the case \( i_0 \geq t \), hence \( j(i_0) = i_0 - t \). We factor out \( Z^t \) and replace \( Z^{a_3} \). This gives

\[
0 = F_{i_0} Z^{j(i_0) + t X^{a_1} + G_{i_0} Z^{j(i_0) + t Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} P(X, Y)^k Z^t} = Z^t (F_{i_0} Z^{j(i_0) X^{a_1} + G_{i_0} Z^{j(i_0) Y^{a_2} + H_{j(i_0)} Z^{j(i_0)} P(X, Y)^k})
\]

Hence we have \( h_{i_0} := (F_{i_0} Z^{j(i_0)}, G_{i_0} Z^{j(i_0)}, H_{j(i_0)} Z^{j(i_0)}) \in \Gamma(C, S_k(m-t)) \) and \( \varphi_m(h_{i_0}) = s_{i_0} \). \( \square \)

**Remark 2.2.** It is easy to see that the morphisms \( \varphi_m, m \in \mathbb{Z} \), are injective on both summands, i.e., the induced mappings

\[
S_k(m-t) \rightarrow \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m), \ (f_1, f_2, f_3) \mapsto (Z^t f_1, Z^t f_2, f_3)
\]

and

\[
S_{k+1}(m) \rightarrow \text{Syz}(X^{a_1}, Y^{a_2}, Z^{a_3})(m), \ (g_1, g_2, g_3) \mapsto (g_1, g_2, Z^{d-t} g_3)
\]

are both injective.

**Remark 2.3.** The sheaves \( S_k \) and \( S_{k+1} \) are the pull-backs

\[
\pi^*(\text{Syz}_{p_1}(X^{a_1}, Y^{a_2}, P(X, Y)^k)) \text{ and } \pi^*(\text{Syz}_{p_1}(X^{a_1}, Y^{a_2}, P(X, Y)^{k+1}))
\]

respectively under the Noetherian normalization \( \pi : C \rightarrow \mathbb{P}^1 = \text{Proj } K[X, Y] \). In particular, \( S_k \) and \( S_{k+1} \) split as a direct sum of line bundles. If \( t = 0 \) we have \( \text{Syz}_C(X^{a_1}, Y^{a_2}, Z^{a_3}) \cong \text{Syz}_C(X^{a_1}, Y^{a_2}, P(X, Y)^k) \) and the bundle is therefore already defined on \( \mathbb{P}^1 \).
3. Frobenius periodicity up to a twist

Let \( C \) be a smooth projective curve defined over a field of positive characteristic. It is a well-known fact that the pull-back of a semistable vector bundle under the (absolute) Frobenius morphism is in general not semistable anymore; see for instance the example of Serre in [13, Example 3.2]. Using syzygy bundles on Fermat curves one can produce fairly easy examples of this phenomenon.

**Example 3.1.** Let \( C := \text{Proj}(\mathbb{F}_3[X, Y, Z]/(X^4 + Y^4 - Z^4)) \) be the Fermat quartic in characteristic 3. The cotangent bundle \( \Omega_{\mathbb{P}^2} \) is stable on the projective plane (see for instance [8, Corollary 6.4]) and so is the restriction \( \Omega_{\mathbb{P}^2}|_C = \text{Syz}(X, Y, Z) \) by Langer’s restriction theorem [20, Theorem 2.19]. Its Frobenius pull-back is the syzygy bundle \( \text{Syz}(X^3, Y^3, Z^3) \). The curve equation yields the relation \( X \cdot X^3 + Y \cdot Y^3 - Z \cdot Z^3 = 0 \) and thus we obtain a non-trivial global section of \((F^*(\Omega_{\mathbb{P}^2}|_C))(4)\). But the degree of this bundle equals \(-4\) and therefore \( F^*(\Omega_{\mathbb{P}^2}|_C) \) is not semistable.

A vector bundle \( \mathcal{E} \) such that \( F^e*(\mathcal{E}) \) is semistable for all \( e \geq 0 \) is called **strongly semistable**. This notion goes back to Miyaoka (cf. [22, Section 5]). Before we state our main theorem, we prove the following Lemma separately.

**Lemma 3.2.** Let \( d \geq 2 \) be an integer and let \( K \) be a field of characteristic \( p \equiv -1 \mod 2d \). Then \( \Omega_{\mathbb{P}^2}|_C \) is strongly semistable on the Fermat curve \( C := \text{Proj}(K[X, Y, Z]/(X^d + Y^d - Z^d)) \).

**Proof.** We use Hilbert-Kunz theory and its geometric interpretation developed in [7] and [26]. The Hilbert-Kunz multiplicity \( e_{HK}(R) \) of the homogeneous coordinate ring \( R := K[X, Y, Z]/(X^d + Y^d - Z^d) \) of the Fermat curve equals \( \frac{d^2}{4} \) in characteristic \( p \equiv -1 \mod 2d \) by Monsky’s result [24, Theorem 2.3]. By [7, Corollary 4.6] this is equivalent to the strong semistability of \( \Omega_{\mathbb{P}^2}|_C \) in these characteristics. \( \square \)

**Remark 3.3.** Note that for \( d = 1 \) we have \( C \cong \mathbb{P}^1 \) and \( \Omega_{\mathbb{P}^2}|_C \cong \mathcal{O}_C(-2) \oplus \mathcal{O}_C(-1) \), i.e., \( \Omega_{\mathbb{P}^2}|_C \) is not even semistable. For a general characterization of strong semistability of \( \Omega_{\mathbb{P}^2}|_C \) on the Fermat curve of degree \( d \) depending on the characteristic of the base field see [14, Chapter 4]. The restriction of \( \mathcal{S} \) to every smooth projective curve of degree \( \geq 7 \) is stable by Langer’s restriction theorem [20, Theorem 2.19].

**Theorem 3.4.** Let \( d \geq 2 \) be an integer and let \( K \) be a field of characteristic \( p \equiv -1 \mod 2d \). Then \( \mathcal{E} := \Omega_{\mathbb{P}^2}|_C \) is strongly semistable on the Fermat curve \( C := \text{Proj}(K[X, Y, Z]/(X^d + Y^d - Z^d)) \) and

\[
F^*(\mathcal{E}) \cong \mathcal{E}(-\frac{3(p-1)}{2}).
\]
Proof. The strong semistability of $E$ in characteristics $p \equiv -1 \mod 2d$ has already been proved in Lemma 3.2. So we have to show that $F^*(E) \cong \text{Syz}(X_p, Y_p, Z_p) \cong E(-\frac{3(p-1)}{2})$. Since the proof is quite long, we divide it into several steps. Note that, since semistability is preserved under base change, we may assume without loss of generality that $K$ is algebraically closed.

Step 1. We write $p = dk + (d - 1)$ with $k$ odd. Accordingly, we set $t = d - 1$. Further, we follow the notation of Lemma 2.1 and define the bundles

$$S_k := \text{Syz}(X_p, Y_p, (X^d + Y^d)^k), \quad S_{k+1} := \text{Syz}(X_p, Y_p, (X^d + Y^d)^{k+1}).$$

We show that the surjective morphism

$$\varphi_{2p+1} : S_k(\frac{3p+1}{2} - t) \oplus S_{k+1}(\frac{3p+1}{2}) \to \text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2})$$

defined in Lemma 2.1 can be identified with

$$(\mathcal{O}_C(-d + 2) \oplus \mathcal{O}_C) \oplus \mathcal{O}_C^2 \to \text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}).$$

We consider the vector bundle $\text{Syz}(U^{k+1}, V^{k+1}, (U + V)^k)(\frac{3k+1}{2})$ on the projective line $\mathbb{P}^1 = \text{Proj} K[U, V]$. Since the degree of this bundle is $-1$, it has to have a non-trivial global section. Substituting $U = X^d$ and $V = Y^d$ yields a non-trivial syzygy

$$FX^{dk+d} + GY^{dk+d} + H(X^d + Y^d)^k = (FX)X^p + (GY)Y^p + H(X^d + Y^d)^k = 0$$

of total degree $\frac{3dk+d}{2}$. That is, we have a non-trivial global section of $S_k(\frac{3dk+d}{2})$ on the curve $C$. We have $\Gamma(C, S_k(\frac{3dk+d}{2} - 1)) = 0$ because otherwise the twisted semistable Frobenius pull-back $\text{Syz}(X^p, Y^p, Z^p)(\frac{3dk+d}{2} + d - 2)$ of degree $-d$ would have a non-trivial global section too (see Remark 2.2) which is impossible by semistability. Since deg($S_k(\frac{3dk+d}{2})$) = $(-d + 2)d$, we obtain the splitting (rewrite $\frac{3dk+d}{2} = \frac{3p+1}{2} - (d - 1) = \frac{3p+1}{2} - t$)

$$S_k(\frac{3p+1}{2} - t) \cong \mathcal{O}_C(-d + 2) \oplus \mathcal{O}_C.$$

The other summand $S_{k+1}(\frac{3dk+d}{2} + d - 1)$ has degree 0. It follows once again from Lemma 2.1 and the semistability of $\text{Syz}(X^p, Y^p, Z^p)$ that

$$\Gamma(C, S_{k+1}(\frac{3dk+d}{2} + d - 2)) = 0,$$

i.e., $S_{k+1}(\frac{3dk+d}{2} + d - 1)$ splits as (rewrite $\frac{3dk+d}{2} + d - 1 = \frac{3p+1}{2}$)

$$S_{k+1}(\frac{3p+1}{2}) \cong \mathcal{O}_C^2.$$

Step 2. Let $(FX, GY, H)$ be the non-trivial global section of $S_k(\frac{3p+1}{2} - t)$ constructed above (corresponding to the component $\mathcal{O}_C$). We show that $H(P) \neq 0$ for every point $P = (x, y, z) \in C$ satisfying $z^d = x^d + y^d = 0$. 


The last component $H$ of the section $(FX, GY, H)$ is a homogeneous polynomial in $X^d$ and $Y^d$ (it stems by construction from a syzygy on $\mathbb{P}^1$ in $U$ and $V$). Let $P = (x, y, z) \in C$ be a point on the curve such that $z^d = x^d + y^d = 0$. Then $x^d = -y^d$ which implies $x = \zeta y$ where $\zeta$ is a $d$th root of $-1$. In particular, $P = (\zeta y, y, 0)$. Since $K$ is algebraically closed, $\text{char}(K) \neq 2$ and $p \equiv -1 \mod 2d$, the group $\mu_{2d}(K)$ of $(2d)$th roots of unity in $K$ has order $2d$. Hence, we have

$$X^d + Y^d = \prod_\zeta (X - \zeta Y),$$

where $\zeta \in \mu_{2d}(K)$ runs through the elements with the property $\zeta^d = -1$ (there are exactly $d$ such roots). Now assume $H(P) = 0$. Then $H(P') = 0$ for all points $P'$ of the form $P' = (\zeta y, y, 0)$. So $X^d + Y^d$ has to divide $H$, i.e., $H = \tilde{H}(X^d + Y^d)$ with a homogeneous polynomial $\tilde{H} \in K[X, Y]$. So we have a relation

$$(FX)X^p + (GY)Y^p + \tilde{H}(X^d + Y^d)^{k+1} = 0$$

of total degree $\frac{3p+1}{2} - t$. That is, we have a non-trivial section of the bundle $S_{k+1}(\frac{3p+1}{2} - t)$. This section maps by Lemma 2.1 and Remark 2.2 to a non-trivial global section of $\text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2} - t)$. But

$$\deg(\text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2} - t)) = (3p + 1 - 2t - 3p)d = (1 - 2t)d < 0.$$ 

Hence, the section contradicts the semistability of $\text{Syz}(X^p, Y^p, Z^p)$.

Step 3. We show that in the surjective sheaf homomorphism

$$\varphi_{\frac{3p+1}{2}} : (\mathcal{O}_C(-d + 2) \oplus \mathcal{O}_C) \oplus \mathcal{O}_C^2 \longrightarrow \text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2})$$

the summand $\mathcal{O}_C(-d + 2)$ is not necessary, i.e.,

$$\varphi_{\frac{3p+1}{2}} : \mathcal{O}_C^3 = \mathcal{O}_C \oplus \mathcal{O}_C^2 \longrightarrow \text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2})$$

is also surjective.

Set $m := \frac{3p+1}{2}$. By the Nakayama lemma, we can check surjectivity pointwise over the residue field $K$ at every point $P = (x, y, z) \in C$. For this we have to find two linearly independent vectors in the image. First we treat the case $z \neq 0$. We show that we even have a surjective map

$$S_{k+1}(m) = \mathcal{O}_C^2 \longrightarrow \text{Syz}(X^p, Y^p, Z^p)(m).$$

We take basic sections

$$f = (f_1, f_2, f_3), g = (g_1, g_2, g_3) \in \Gamma(C, S_{k+1}(m)) \cong \Gamma(C, \mathcal{O}_C^2).$$

Their images are $\tilde{f} = (f_1, f_2, Z f_3)$ and $\tilde{g} = (g_1, g_2, Z g_3)$. Assume there is a relation $\tilde{f}(P) + \lambda \tilde{g}(P) = 0$ with $\lambda \in K^\times$. Looking at each component this
such that \( v, w \)

therefore we would obtain a relation \( f(P) + \lambda g(P) = 0 \) which contradicts the assumption.

Now we deal with the case \( z = 0 \), i.e., \( P = (x, y, 0) \). Let

\[
f = (FX, GY, H) \in \Gamma(C, S_k(m - t)) \cong \Gamma(C, \mathcal{O}_C(-d + 2) \oplus \mathcal{O}_C)
\]

be the section (corresponding to \( \mathcal{O}_C \) which we have found in step 1. The image of \( f \) in the bundle \( \text{Syz}(X^p, Y^p, Z^p)(m) \) is the section \( (Z^iFX, Z^iGY, H) \). Evaluated at \( P \) we obtain the vector \( v = (0, 0, H(P)) \). Since \( 0 = z^d = x^d + y^d \)
we have \( H(P) \neq 0 \) by step 2. Now we take a section \( 0 \neq g = (g_1, g_2, g_3) \in \Gamma(C, S_k+1(m)) \cong \Gamma(C, \mathcal{O}_C^3) \). The image of \( g \) equals \( (g_1, g_2, Zg_3) \). Evaluation at \( P \) gives the vector \( w = (g_1(P), g_2(P), 0) \), where either \( g_1(P) \) or \( g_2(P) \) is not 0 (otherwise \( g_3(P) \) would be 0 as well). Hence we have found a vector \( w \) such that \( v, w \) are linearly independent over \( K \).

Step 4. So far we have shown that we have a surjective morphism

\[
\mathcal{O}_C^3 \twoheadrightarrow \text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}) \to 0.
\]

Since \( \det(\text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2})) \cong \mathcal{O}_C(-1) \) we have a short exact sequence

\[
0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C^3 \to \text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}) \to 0.
\]

Dualizing and tensoring with \( \mathcal{O}_C(-1) \) gives

\[
0 \to (\text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}))^\vee(-1) \to \mathcal{O}_C^3(-1) \to \mathcal{O}_C \to 0,
\]

where the map \( \mathcal{O}_C^3(-1) \to \mathcal{O}_C \) is given by some linear forms \( L_1, L_2, L_3 \)
in the homogeneous coordinate ring \( R = K[X, Y, Z]/(X^d + Y^d - Z^d) \). In particular, we have \((\text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}))^\vee(-1) \cong \text{Syz}(L_1, L_2, L_3) \). We show that \( \{L_1, L_2, L_3\} \) and \( \{X, Y, Z\} \) generate the same ideal in \( R \). Assume to the contrary that \( L_1, L_2, L_3 \) are linearly dependent. Such an equation yields a non-trivial section of \( \text{Syz}(L_1, L_2, L_3)(1) \). This bundle has degree \( \deg(\text{Syz}(L_1, L_2, L_3)(1)) = (2 - 3)d = -d < 0 \). But since \( \text{Syz}(X^p, Y^p, Z^p) \) is semistable, so is \((\text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}))^\vee \) and thus also \( \text{Syz}(L_1, L_2, L_3) \). So the section contradicts the semistability.

Step 5. We have already proved that

\[
\mathcal{E} \cong \text{Syz}(X, Y, Z) \cong (\text{Syz}(X^p, Y^p, Z^p)(\frac{3p + 1}{2}))^\vee(-1).
\]
Since $\text{Syz}(X^p, Y^p, Z^p)$ is a bundle of rank 2, we have
\[ \text{Syz}(X^p, Y^p, Z^p) \cong \text{Syz}(X^p, Y^p, Z^p) \otimes O_C(-3p). \]
So finally we obtain
\[
\mathcal{E} \cong \text{Syz}(X, Y, Z) \\
\cong (\text{Syz}(X^p, Y^p, Z^p)(\frac{3p+1}{2}))^\vee(-1) \\
\cong \text{Syz}(X^p, Y^p, Z^p)^\vee \otimes O_C(-\frac{3p+1}{2}) \otimes O_C(-1) \\
\cong \text{Syz}(X^p, Y^p, Z^p) \otimes O_C(3p) \otimes O_C(-\frac{3p+1}{2}) \otimes O_C(-1) \\
\cong \text{Syz}(X^p, Y^p, Z^p)(\frac{3p-1}{2}),
\]
and consequently $F^*(\mathcal{E}) \cong \text{Syz}(X^p, Y^p, Z^p) \cong \mathcal{E}(\frac{-3p-1}{2})$ which finishes the proof.

**Remark 3.5.** We also comment on the case $p \equiv 1 \mod 2d$. Then we write $p = dk + 1$ with $k$ even and set $t = 1$. The syzygy bundle $\text{Syz}(U^k, V^k, (U + V)^{k+1})(\frac{3k}{2})$ on $\mathbb{P}^1 = \text{Proj} K[U, V]$ has degree $-1$ and therefore has to have a non-trivial global section. Substituting $U = X^d$ and $Y^d$ and multiplying with $XY$ gives then a syzygy
\[
(FY)X^p + (GX)Y^p + (HXY)(X^d + Y^{d+1}) = 0,
\]
i.e., a global section of $\mathcal{S}_{k+1}(\frac{3dk}{2} + 2)$ on the Fermat curve. As in the proof of Theorem 3.4 we obtain the splittings (rewrite $\frac{3dk}{2} + 2 = \frac{3p+1}{2}$)
\[
\mathcal{S}_{k+1}(\frac{3p+1}{2}) \cong O_C(-d + 2) \oplus O_C \text{ and } \mathcal{S}_k(\frac{3p+1}{2} - t) \cong O_C^2.
\]
Unfortunately, we do not know how to prove an analog of step 2 in these characteristics, i.e., to show that $H(P) \neq 0$ for every point $P = (x, y, z) \in C$ with $x^d = x^d + y^d = 0$. Here the reasoning of the proof (step 2) of Theorem 3.4 would lead to a section of $\text{Syz}(X^p, Y^p, (X^d + Y^{d+1})^{k+2})$ which is not helpful to get a contradiction.

**Remark 3.6.** We cannot expect that Theorem 3.4 holds in every characteristic $p$ where $\Omega_{xz}|_C$ is strongly semistable. For example, consider in characteristic 2 the Fermat cubic $C = \text{Proj}(K[X, Y, Z]/(X^3 + Y^3 - Z^3))$, which is an elliptic curve. It is a well-known fact that semistable vector bundles on elliptic curves are strongly semistable (see for instance [27, appendix]). Hence $\Omega_{xz}|_C \cong \text{Syz}(X, Y, Z)$ is strongly semistable by [5, Proposition 6.2]. The pullback $F^*(\Omega_{xz}|_C) \cong \text{Syz}(X^2, Y^2, Z^2)$ has for the first time non-trivial global sections in total degree 3, namely the (only) syzygy $(X, Y, -Z)$ which comes
from the equation of the curve. This section gives rise to the short exact sequence

\[ 0 \rightarrow \mathcal{O}_C \rightarrow \text{Syz}(X^2, Y^2, Z^2)(3) \rightarrow \mathcal{O}_C \rightarrow 0, \]

i.e., Syz\((X^2, Y^2, Z^2)(3)\) is the bundle \(F_2\) in Atiyah’s classification [1]. Since the Hasse invariant of \(C\) is 0, we have \(F^*(F_2) \cong \mathcal{O}_C^2\) and therefore \(F^*(\Omega_{p2}|_C) \not\cong \Omega_{p2}|_C(-\frac{3(p-1)}{2})\). We have \(F^{2*}(\Omega_{p2}|_C) \cong \mathcal{O}_C(-6) \oplus \mathcal{O}_C(-6)\) and we obtain (up to a twist) the periodicity \(F^{3*}(\Omega_{p2}|_C) \cong (F^{2*}(\Omega_{p2}|_C))(-6)\).

4. A computation of the Hilbert-Kunz function

We recall that the Hilbert-Kunz function of a standard graded ring \(R\) of characteristic \(p > 0\) with graded maximal ideal \(m\) is the function

\[ e \mapsto \varphi_R(e) := \text{length}(R/m^{\lceil pe \rceil}), \]

where \(m^{\lceil pe \rceil}\) denotes the extended ideal under the \(e\)-th iteration of the Frobenius endomorphism on \(R\); see for instance [23] for this rather complicated function and its properties. As a consequence of Theorem 3.4 we obtain the complete Hilbert-Kunz function of the Fermat ring \(R = K[X, Y, Z]/(X^d+Y^d-Z^d)\) in characteristics \(p \equiv -1 \mod 2d\). The following result is implicitly contained in [12, Lemma 5.6] of P. Monsky and C. Han.

**Corollary 4.1.** Let \(d \geq 2\) be a positive integer and let \(K\) be a field of characteristic \(p \equiv -1 \mod 2d\). Then the Hilbert-Kunz function of the Fermat ring \(R = K[X, Y, Z]/(X^d+Y^d-Z^d)\) is

\[ \varphi_R(e) = \frac{3d}{4}p^{2e} + 1 - \frac{3d}{4} \]

**Proof.** Since the length of \(R/m^{\lceil pe \rceil}\), \(m = (X, Y, Z)\), does not change if one enlarges the base field, we may assume that \(K\) is algebraically closed. Hence, \(\varphi_R(e) = \sum_{m=0}^{\infty} \text{dim}_K(R/m^{\lceil pe \rceil})_m\) (this sum is in fact finite since the algebras \(R/m^{\lceil pe \rceil}\) have finite length). It follows from the presenting sequence of \(\text{Syz}(X, Y, C)\) on the Fermat curve \(C = \text{Proj} R\) that (setting \(q := p^e\))

\[ (\ast) \quad \text{dim}_K(R/m^{[q]})_m = h^0(C, \mathcal{O}_C(m) - 3h^0(C, \mathcal{O}_C(m-q)) + h^0(C, \text{Syz}(X^q, Y^q, Z^q)(m)). \]

By Theorem 3.4 we have \(\text{Syz}(X^p, Y^p, Z^p) \cong \text{Syz}(X, Y, Z)(-\frac{3(p-1)}{2})\) and consequently \(\text{Syz}(X^q, Y^q, Z^q) \cong \text{Syz}(X, Y, Z)(-\frac{3(q-1)}{2})\) for all \(q = p^e, e \geq 1\). The global evaluation of the presenting sequence of \(\mathcal{E}(k) := \text{Syz}(X, Y, Z)(k)\) gives the exact sequence

\[ 0 \rightarrow \Gamma(C, \mathcal{E}(k)) \rightarrow \Gamma(C, \mathcal{O}_C(k-1)^3) \rightarrow \Gamma(C, \mathcal{O}_C(k)) \rightarrow K \rightarrow 0 \]

for \(k = 0\) and the short exact sequence

\[ 0 \rightarrow \Gamma(C, \mathcal{E}(k)) \rightarrow \Gamma(C, \mathcal{O}_C(k-1)^3) \rightarrow \Gamma(C, \mathcal{O}_C(k)) \rightarrow 0 \]
for $k \geq 1$. Hence we obtain

$$h^0(C, \mathcal{E}(k)) = \begin{cases} 3h^0(C, \mathcal{O}_C(k - 1)) - h^0(C, \mathcal{O}_C(k)) + 1 & \text{if } k = 0, \\ 3h^0(C, \mathcal{O}_C(k - 1)) - h^0(C, \mathcal{O}_C(k)) & \text{if } k \neq 0. \end{cases}$$

For $k \geq d - 2$ we have by Riemann-Roch $h^0(C, \mathcal{O}_C(k)) = dk - g + 1$, where $g$ is the genus of the curve. Since $p \equiv -1 \mod 2d$, this holds in particular for $k \geq \frac{p+1}{2}$. So the geometric formula for the Hilbert-Kunz function (*) gives (in order to obtain an easier calculation we sum up to $2q$):

$$\varphi_R(e) = \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m - q))$$

$$+ 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m - \frac{3(q-1)}{2} - 1))$$

$$- 2 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m - \frac{3(q-1)}{2})) + 1$$

$$= \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m))$$

$$+ 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) + 1$$

$$= \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) - 3 \sum_{m=0}^{2q} h^0(C, \mathcal{O}_C(m)) + 1$$

$$= \sum_{m=\frac{q+3}{2}}^{2q} (dm - g + 1) - 3 \sum_{m=\frac{q+3}{2}}^{2q} (dm - g + 1) + 1$$

$$= d \left( q(2q+1) - \frac{(q+3)(q+5)}{8} \right) - \frac{3g(q-1)}{2} + \frac{3(q-1)}{2}$$

$$- 3d \left( \frac{q(q+1)}{2} - \frac{(q+1)(q+3)}{8} \right) + \frac{3g(q-1)}{2} - \frac{3(q-1)}{2} + 1$$

$$= \frac{3d}{4} q^2 + 1 - \frac{3d}{4}.$$

Thus we have obtained the desired formula for the Hilbert-Kunz function of the ring $R$. \hfill \square

**Remark 4.2.** Corollary 4.1 matches for $d = 3$ with the result [10, Theorem 4] of Buchweitz and Chen, which says that the Hilbert-Kunz function of the
homogeneous coordinate ring of a plane elliptic curve defined over a field $K$ of odd characteristic $p$ equals $\frac{4}{3}p^2e - \frac{3}{4}$.

5. Examples of a $(0,1)$-Frobenius periodicity on Fermat curves

In this section, we show how to get via Theorem 3.4 non-trivial examples of $(0,1)$-Frobenius periodicities, i.e., we give explicit examples of vector bundles $E$ on certain Fermat curves such that $E \cong F^*(E)$.

Example 5.1. Let $d \geq 2$ and let $K$ be a field of characteristic $p \equiv -1 \mod 2d$. The ring homomorphism

$$K[X, Y, Z]/(X^d + Y^d - Z^d) \longrightarrow K[U, V, W]/(U^{2d} + V^{2d} - W^{2d})$$

which sends $X \mapsto U^2$, $Y \mapsto V^2$, and $Z \mapsto W^2$ induces a finite cover $f : C^{2d} \rightarrow C^d$, where $C^d$ denotes the Fermat curve of degree $i$. Since $f^*(\mathcal{O}_{C^{2d}}(1)) \cong \mathcal{O}_{C^{2d}}(2)$, we see that $\deg(f) = 4$. The group $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ acts on $C^{2d}$ by sending $(u, v, w)$ either to $(u, v, w)$, $(-u, v, w)$, $(u, -v, w)$, or $(u, v, -w)$, and $C^d$ is the quotient of this action. Moreover, $f$ is a finite separable morphism and therefore $f$ preserves semistability. Theorem 3.4 gives, via pull-back under $f$, the isomorphic vector bundles

$$\text{Syz}_{C^{2d}}(U^{2p}, V^{2p}, W^{2p}) \cong f^*(\text{Syz}_{C^d}(X^p, Y^p, Z^p))$$
$$\cong f^*(\text{Syz}_{C^d}(X, Y, Z)(-\frac{3(p - 1)}{2}))$$
$$\cong f^*(\text{Syz}_{C^d}(X, Y, Z)) \otimes f^*(\mathcal{O}_{C^d}(-\frac{3(p - 1)}{2}))$$
$$\cong \text{Syz}_{C^{2d}}(U^2, V^2, W^2)(-3(p - 1))$$

on the Fermat curve $C^{2d}$. In particular, we have the periodicity

$$\text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \cong F^*(\text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3))$$

(note that this bundle has degree 0 and is not trivial since there are no non-trivial global sections below the degree of the curve).

Remark 5.2. By the classical result [19, Satz 1.4] of H. Lange and U. Stuhler the periodicity $\text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \cong F^*(\text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3))$ in Example 5.1 implies the existence of an étale cover

$$g : D \longrightarrow C^{2d}$$

such that

$$g^*(\text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3)) \cong \mathcal{O}_D^2.$$
of the algebraic fundamental group \( \pi_1(C^{2d}) \). It would be interesting to see how the étale trivialization \( g \) and the representation \( \rho \) look explicitly in this example.

**Remark 5.3.** In this remark we show that \( \mathcal{E} := \text{Syz}(U^2, V^2, W^2)(3) \) is not étale trivializable in characteristic 0. We consider this bundle on the smooth projective relative curve

\[
C^{2d} := \text{Proj}(\mathbb{Z}[U, V, W]/(U^{2d} + V^{2d} - Z^{2d})) \to \text{Spec} \mathbb{Z}_{2d}.
\]

For a prime number \( p \equiv 2d \) the special fiber \( C_p^{2d} \) over \( (p) \) is the (smooth) Fermat curve over the finite field \( \mathbb{F}_p \). The generic fiber \( C_0^{2d} \) over \( (0) \) is the Fermat curve over \( \mathbb{Q} \). To prove that \( \mathcal{E}_0 := \mathcal{E}|_{C_0^{2d}} \) is not étale trivializable on \( C_0^{2d} \) we use once again Hilbert-Kunz theory (cf. also the proof of Lemma 3.2). Note that \( \mathcal{E}_0 \) is semistable by [5, Proposition 2]. If \( d \geq 4 \) is even (the case \( d = 2 \) is trivial) we consider prime numbers \( p \equiv d \pm 1 \) mod \( 2d \) and if \( d \geq 3 \) is odd we look at prime numbers \( p \equiv d \) mod \( 2d \). In these characteristics, [24, Theorem 2.3] yields that the Hilbert-Kunz multiplicity \( e_{HK}(R_p) \) of the homogeneous coordinate ring \( R_p \) of the Fermat curve \( C^d \to \text{Spec} \mathbb{F}_p \) of degree \( d \) equals

\[
e_{HK}(R_p) = \frac{3d}{4} \frac{(d(d-3))^2}{4dp^2} \quad \text{if } d \text{ is even and } e_{HK}(R_p) = \frac{3d}{4} + \frac{d^3}{4p^2} \quad \text{if } d \text{ is odd}.
\]

Hence, \( \Omega_{\mathbb{Z}^2|C^d} \) is not strongly semistable by [7, Corollary 4.6]. Since we can realize the fibers \( C_p^{2d} \), as in Example 5.1, as coverings \( f : C_p^{2d} \to C^d \), the bundles \( \mathcal{E}_p := \mathcal{E}|_{C_p^{2d}} \cong f^*(\Omega_{\mathbb{Z}^2|C^d})(3) \) are not strongly semistable either. Note that by the well-known theorem of Dirichlet (cf. [25, Chapitre VI, §4, Théorème et Corollaire]) there are infinitely many such fibers. Therefore, there is no étale cover \( g : D \to C_0^{2d} \) such that \( g^*(\mathcal{E}_0) \cong \mathcal{O}_D \).

This observation is somehow related to the Grothendieck-Katz \( p \)-curvature conjecture [16, (I quat)] which states the following: Let \( R \) be a \( \mathbb{Z} \)-domain of finite type, \( \mathbb{Z} \subseteq R \), and \( \mathcal{X} \to \text{Spec} R \) a smooth projective morphism of relative dimension \( d \geq 1 \). If \( \mathcal{E} \) is a vector bundle on \( \mathcal{X} \to \text{Spec} R \) equipped with an integrable connection \( \nabla \) such that \( \nabla|_{\mathcal{X}_p} \) has \( p \)-curvature 0 on the special fiber \( \mathcal{X}_p \) for almost all closed points \( p \in \text{Spec} R \), then there exists an étale cover \( g : Y \to \mathcal{X}_0 \) of the generic fiber \( \mathcal{X}_0 \) such that \((g^*(\mathcal{E}_0), g^*(\nabla_0))\) is trivial. For a detailed account on integrable connections and \( p \)-curvature see [15] and [16].

In our example of the relative Fermat curve \( C^{2d} \), we have for infinitely many prime numbers \( p \equiv -1 \) mod \( 2d \) the Frobenius descent \( F^*(\mathcal{E}_p) \cong \mathcal{E}_p \) on \( C_p^{2d} \). By the so-called Cartier-correspondence [15, Theorem 5.1] this is equivalent to the existence of an integrable connection \( \nabla_p \) on \( \mathcal{E}_p \) with vanishing \( p \)-curvature. If one could establish a connection on \( \mathcal{E} \) (since \( \mathcal{E} \) is a vector bundle over a curve, this connection would be automatically integrable) which is compatible with the connections on the special fibers \( C_p^{2d}, p \equiv -1 \) mod \( 2d \), then our example would show that the Grothendieck-Katz conjecture does not hold if one only requires vanishing \( p \)-curvature for infinitely many closed points.
Remark 5.4. In this remark we assume that the base field is algebraically closed. We consider the Verschiebung 

\[ V : \mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}}) \rightarrow \mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}}), \ [\mathcal{E}] \mapsto [F^*\mathcal{E}] \]

induced by the Frobenius morphism on \( C^{2d} \). We recall that the Verschiebung is a rational map from the moduli space \( \mathcal{M}_{C^{2d}}(2, \mathcal{O}_{C^{2d}}) \) parametrizing (up to \( S \)-equivalence) semistable vector bundles on \( C^{2d} \) of rank 2 and trivial determinant to itself. The vector bundle \( S := \text{Syz}(U^2, V^2, W^2) \) is stable on the projective plane \( \mathbb{P}^2 = \text{Proj} K[U, V, W] \) by [8, Corollary 6.4]. Since the discriminant of this bundle equals \( \Delta(S) = 4c_2(S) - c_1(S)^2 = 12 \), the restriction of \( S \) to every smooth projective curve of degree \( \geq 7 \) remains stable by Langer’s restriction theorem [20, Theorem 2.19]. In particular, \( S|_{C^{2d}} \cong \text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \) is stable on the Fermat curve \( C^{2d} \) for \( d \geq 4 \). Hence, for \( d \geq 4 \) the bundle \( \text{Syz}_{C^{2d}}(U^2, V^2, W^2)(3) \) defines a closed point of \( \mathcal{M}_{C^{2d}}(2, \mathcal{O}_C) \) which is fixed under the Verschiebung \( V \).

Remark 5.5. We may pull-back the vector bundle \( E = \text{Syz}(U^2, V^2, W^2)(3) \) along the cone mapping

\[ p : T = \text{Spec} K[U, V, W]/(U^{2d} + V^{2d} - W^{2d}) \setminus \{m\} \rightarrow C^{2d} \]

to obtain the bundle \( G = p^*(E) \) on the punctured spectrum with the property \( F^*(G) \cong G \). This can however not be extended to get a Frobenius periodicity on the module level, since \( F^*\Gamma(T, G) \neq \Gamma(T, F^*G) \). A Frobenius periodicity for a coherent \( R \)-module \( M \), where \( R \) is a local noetherian domain, implies that \( M \) is free. This observation follows by looking at Fitting ideals of a free resolution (we thank Manuel Blickle and Neil Epstein for this remark).

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