Distributed Leader-Follower Formation Tracking Control of Multiple Quad-rotors

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The leader-follower formation control analysis for multiple quad-rotor systems is investigated in this paper. To achieve predefined formation in the three-dimensional air space \((x, y, z)\), a novel local tracking control law and a distributed observer are obtained. The local tracking control law starts with finding a bounded continuous yet greater-than-zero control in \(z\), based on which following a feedback linearization (FL) controls derived for errors associated with \(x\) and \(y\). By this design method, we obtain less states to be regulated than the traditional extension FL methodology. Then, the proposed distributed observer solves the problems that only a subset of followers can know the leaders states and only neighboring communication is available. Simulation results validate the proposed formation scheme.

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I. INTRODUCTION

A quad-rotor is a multi-rotor helicopter that is lifted and propelled by four symmetrically mounted rotors [3, 6–8]. It has experienced a great boom in recent years due to potential applications such as aerial photography, geological exploration and disaster relief. To enhance the reliability and safety of these applications, researchers have developed many cooperative strategies of quad-rotor systems, one of which is the leader-follower formation scheme that allows for steering multiple quad-rotors to form a geometric pattern while tracking a leader/reference. For this sake, three classical approaches can be applied, that is, linearization [11, 15, 23, 24, 26, 29], inner-outer loop method [18, 19, 22, 25, 27, 28, 31] and feedback linearization approach [4, 5, 8, 20].

A direct approach for quad-rotor formation consists in linearizing quad-rotor model around maneuvering point. Due to easy implementation of linearized quad-rotor model, the control scheme reported in [11] elaborates the potential functions with its formation control law, achieving the formation pattern with collision avoidance behavior. The control design reported in [15] realizes cooperative formation of multiple heterogeneous agents, including many quad-rotors and linearized differentiable mobile robots moving on the ground. Given the possible interaction fault between adjacent quad-rotors during their formation tracking, two \(H_\infty\)-formation schemes are proposed in [23] and [26] respectively, presenting fault-tolerant capacity during flights of multiple quad-rotors. In [24], the result developed for high-order linear integrators is adapted to solve the formation problem of multiple quad-rotors with external disturbances. A finite-time formation tracking controller can be found in [29]. Some other literature, such as [16, 17, 21, 30], extend the results developed for linear double integrators to achieve formation tracking of multiple quad-rotors directly. These formation tracking control laws, developed by either linearized quad-rotor model or linear double integrators, however, can only solve the formation rendezvous problem or formation tracking problem with a slowly-moving leader. They are incapable of steering quad-rotors to perform agile motions with large roll/pitch angles due to the loss of model nonlinearities in their designs.

As for the inner-outer loop approach, the longitudinal and latitudinal position errors therein are viewed as an outer loop, and the attitude errors in roll and pitch are called inner loop [13]. This approach becomes popular out of two facts. First, the attitude and yaw can be steered independently. Second, the roll and pitch angles can be viewed as virtual control inputs for the dynamics of longitudinal and latitudinal position. Some associated results can be found in [18, 19, 22, 25, 27, 28, 31]. In [18], a discontinuous observer for the formation trajectory is proposed based on neighboring connections and graph theory, which, together with the inner attitude algorithm, makes the formation errors converge to zero asymptotically. A discontinuous formation tracking controller reported in [22] obtains finite-time convergence of the quad-rotor formation errors. The centroid formation, steering the average position of all quad-rotors to track the leader’s position, can be achieved by the control laws proposed in both [18] and [22]. In consideration of inefficiency of GPS during indoor flying, the works [19, 27] propose two vision-based formation control laws. The invertibility of the Laplacian matrix associated with an undirected connected interaction graph is made full use by [25], in which the quad-rotor formation error is proven to be convergent, and this convergent rate is proportional to the smallest eigenvalue of the interaction graph. The non-smooth consensus formation tracking scheme shown in [31] achieves the formation pattern with a constant speed. For agile coordination, a virtual structure approach utilized in [28] views each quad-rotor in the group.

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as a rigid body, achieving swarm with over 200 quad-rotors. However, the inner-outer loop formation schemes of multiple quad-rotors generally lead to difficulties in obtaining the desired attitude derivatives for the inner-loop control. This is because the desired attitude (roll and pitch angles) includes the coordination position and velocities, and the direct calculation of desired attitude derivatives will contain unavailable information of unconnected quad-rotor. To obtain the desired attitude derivatives, the velocity observer approach [18], first derivative method [22] or direct differentiation [19] can be applied. As a result, the overall stability analysis becomes incomplete.

The feedback linearization approach involves coming up with system state transformations into an equivalent linear system through variable changes, state extensions and suitable control input. Although it is proved that the normal twelve-dimensional system of a quad-rotor is not feedback linearizable in [4], an extended system with fourteen system states is feedback linearizable with triple fourth-order position states and a second-order yaw state. This kind of dynamic extension is built upon the flat property of the quad-rotor and viewing the second-order state of thrust force as the control input to be designed. Classical results associated with this approach can be found in [5, 8]. The main inefficiency of feedback linearization based on dynamic extension lies in that the greater-than-zero total thrust control cannot be always ensured, which might obstruct the flight because the quad-rotor needs an upward thrust to hover in the air. To solve the problem for the dynamic-extension-based feedback linearization method associated with quad-rotor formation, a recent literature [20] proposes a control design based on Euler-Poincaré equations, by whose result the obtained thrust force control is kept being greater than zero all the time.

Motivated by the facts and challenges stated above, this paper makes further endeavors to consider the leader-follower formation issue for a team of quad-rotors. To deal with this formation problem, we first establish a local tracking control law via non-regular feedback linearization method, given any reference signal with bounded derivatives and a reference altitude acceleration no more than gravitational acceleration. Then, a distributed observer is investigated by the reduced-order and linear time-varying techniques, solving the problem that only neighboring connection is available and only partial followers can know the leaders states. The combination of the local tracking control law and the distributed reduced-order observer leads to the formation tracking scheme. Compared with previous research, the main innovation points of proposed formation scheme are as follows:

- the safe maneuvering can be ensured as the total thrust of each quad-rotor is kept being greater than zero all the time, and simultaneously, the roll and pitch are strictly limited in $(-\pi/2, \pi/2)$;
- the local tracking controller derived from non-regular feedback linearization method allows quad-rotors for admissible agile motions with large roll and pitch angles;
- the fully distributed coordination without global interaction will not cause large communication burden when adding cooperative quad-rotors to achieve complex formation pattern.

The rest is organized as follows. Section 2 contains quad-rotor modeling, basic graph theory and problem formulation. Section 3 presents the main results. Section 4 considers the numerical simulation. Section 5 concludes the work briefly.

**Notations:** The norm $\| \cdot \|$ refers to Euclidean norm, the letter ’e’ without subscript/superscript denotes exponent, ’diag(·)’ means diagonalization and $I_n$ denotes a $n$-dimensional identity matrix.

### II. PRELIMINARIES AND PROBLEM FORMULATION

#### A. Model Description

![FIG. 1: The illustration of a quad-rotor.](image)

The quad-rotor model used for control formulation and validation displays a symmetrical configuration, see an example in Figure 1, where $F^I(O_1x_1y_1z_1)$ denotes the inertial frame and $F^B(x_by_bz_b)$ denotes the body-fixed frame. Suppose that there are $n$ quad-rotors with index belonging to $N = \{1, 2, ..., n\}$. For $i \in N$, let $p_i = [x_i, y_i, z_i]^T$ be the position of the $i$-th quad-rotor in inertial frame, and $\eta_i = [\phi_i, \theta_i, \psi_i]^T$ denote the roll angle $\phi_i$, pitch angle $\theta_i$ and yaw angle $\psi_i$, respectively. The common assumption below can be used to simplify the quad-rotor modeling.

**Assumption 1** The quad-rotor body is rigid and with an invariant structure and aerodynamic parameters, and the thrust and drag are proportional to the square of the propeller speed.
Based on Assumption 1, the quad-rotor dynamics can be described by \[ [3, 29]
\begin{align}
\dot{\eta}_i &= R_i^T T_{IB}/m_i + G
\dot{M}_i(\eta_i) \dot{\eta}_i + C_i(\eta_i, \dot{\eta}_i) \eta_i = \tau_i c,
\end{align}
where \( T_{IB} = [0, 0, F_i]^T \) is the thrust with respect to the body-fixed frame with \( F_i \) the total lift, \( m_i \) the mass, \( G = [0, 0, -g]^T \) denotes the gravity vector with respect to the inertial frame and \( \tau_i c = [\tau_i, \tau_i, \tau_i, \tau_i]^T \) is the control torque. Moreover, the specific definitions of \( M_i(\eta_i) \) and \( C_i(\eta_i, \dot{\eta_i}) \) can be found in \([3, 29]\).

To present the main idea concisely, define virtual control inputs by \( u_{i,1} = F_i/m_i \) and \( \tau_i = [u_{i,2}, u_{i,3}, u_{i,4}]^T \), where \( \tau_i = M_i^{-1}(\eta_i)(\tau_i c - C_i(\eta_i, \dot{\eta}_i)\dot{\eta}_i). \) Then, rewrite the quad-rotor dynamics as follows:
\[ \begin{align}
\dot{x}_i &= u_{i,1} \left( \cos \psi_i \sin \theta_i \phi + \sin \psi_i \sin \phi \right)
\dot{y}_i &= u_{i,1} \left( \sin \psi_i \sin \theta_i \phi - \cos \psi_i \sin \phi \right)
\dot{z}_i &= u_{i,1} \cos \theta_i \cos \phi - g
\phi_i &= u_{i,2}
\dot{\theta}_i &= u_{i,3}
\dot{\psi}_i &= u_{i,4}
\end{align} \] (2)
Suppose that the leader agent with index 0 is time-parameterized and defined by \( p_0(t) = [x_0(t), y_0(t), z_0(t)]^T \), where \( (x_0, y_0) \) denotes the coordinate of latitude and longitude and \( z_0 \) is the altitude. Steering multiple quad-rotors to form a pattern while tracking a leader is related to many potential applications such as cooperative patrolling and geometrical prospecting, it is therefore very important to maintain quad-rotors moving with fixed altitude and geometrical prospecting, it is therefore very important to maintain quad-rotors moving with fixed altitude and geometrical prospecting while tracking a leader.

**Assumption 2** The latitudinal and longitudinal positions of the leader, \( x_0 \) and \( y_0 \), are fourth-order differentiable with bounded derivatives; and the altitude \( z_0 \) is a constant.

**B. Graph Theory**

A graph \( G = (\mathcal{N}, \mathcal{E}, \mathcal{A}) \) is used to describe the interaction among multiple quad-rotors, where \( \mathcal{N} = \{1, 2, \ldots, n\} \) denotes the node set, \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) is the edge set and \( \mathcal{A} \) is adjacent matrix [35]. Each node \( i \in \mathcal{N} \) represents one quad-rotor, and an edge \( (i, j) : i \neq j \in \mathcal{E} \) denotes that the quad-rotor \( j \) can send information to quad-rotor \( i \) via wireless module. The adjacent matrix is defined by \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n} \), where \( a_{ij} = 1 \) if \( (i, j) \in \mathcal{E} \), otherwise \( a_{ij} = 0 \). Self connection is forbidden by setting \( a_{ii} = 0, \forall i \in \mathcal{N} \).

For an undirected graph, \( a_{ij} = 1 \Leftrightarrow a_{ji} = 1 \) holds, denoting that the quad-rotor \( i \) and quad-rotor \( j \) can transmit information to each other. A path of graph \( G \) is an edge sequence \( \{(i, j_1), (j_2, j_3), \ldots, (j_s, j)\} \). The in-degree matrix of graph \( G \) is given by \( \mathcal{D} = \text{diag} \{l_{11}, l_{22}, \ldots, l_{nn}\}, \) where \( l_{ii} = \sum a_{ij}, \forall i, j \in \mathcal{N} \), and the Laplacian matrix can then be obtained as \( \mathcal{L} = \mathcal{D} - \mathcal{A} \).

As reported in [14], the matrix \( \mathcal{L} \) is semi-positive definite and has only one zero eigenvalue and \( n - 1 \) positive eigenvalues provided that \( G \) is undirected and connected. Define \( a_{i0} = 1 \) if there is a valid information flow from the leader to the \( i \)-th quad-rotor, otherwise \( a_{i0} = 0 \), which then leads to the matrix given by \( \mathcal{H} = \mathcal{L} + \mathcal{B} \).

**Remark 1** \( \mathcal{B} \neq 0 \) means that there at least one quad-rotor can know the leader’s position and derivatives up to appropriate orders.

**C. Problem Formulation**

In this note, the focus is set on achieving a fixed formation pattern. Define a constant vector by \( \Delta_i = [d_{i,x}, d_{i,y}, d_{i,z}]^T \), \( i \in \mathcal{N} \), and formation error by \( \eta_i = \begin{bmatrix} x_i - x_0 - d_{i,x} \\ y_i - y_0 - d_{i,y} \\ z_i - z_0 - d_{i,z} \end{bmatrix} \) .

The control objective can then be stated as: Based on the quad-rotor model (2) and Assumptions 1-3, find control laws \( u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4} \) so that \( \lim_{t \to \infty} \eta_i = 0, \forall i \in \mathcal{N} \).

**III. THE MAIN RESULTS**

The formation scheme includes a local controller and a distributed observer. Given any smooth reference trajectory with bounded derivatives and a less-than-\( g \) altitude
acceleration, the local controller is proposed firstly with the help of non-regular feedback linearization technique, steering the tracking errors converge to zero asymptotically. The distributed tracking observer is then investigated via interaction between connected agents and is viewed as virtual reference trajectory. The leader-follower formation can be realized via applying the local control law on each follower quad-rotor to track its reference signal, see Figure 2 for an illustration.

FIG. 2: The illustration of formation control scheme.

A. Local Tracking Control Design

A lemma is needed to formulate the control design.

**Lemma 1** [9] The system

$$\ddot{\xi} = -a_1 \tanh(\xi + a_2 \xi) - a_3 \tanh(\xi)$$  \hspace{1cm} (8)

is globally asymptotically stable with respect to $\xi \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^m$, provided that $a_1, a_2, a_3 > 0$.

Let $p_{id} = [x_{id}, y_{id}, z_{id}]^T$ be the reference trajectory of the $i$-th quad-rotor and suppose that $p_{id}$ is fourth-order differentiable with bounded derivatives and $|z_{id}(t)| < g, \forall t \geq 0$. The assumption $\ddot{z}_{id}(t) < g$ is made here due to the altitude coordinates between quad-rotors will be addressed and $z_{id}$ can be time-varying. Define errors

$$\begin{align*}
\epsilon_{i,x} &= x_i - x_{id} - d_{i,x} \\
\epsilon_{i,y} &= y_i - y_{id} - d_{i,y} \\
\epsilon_{i,z} &= z_i - z_{id} - d_{i,z}
\end{align*}$$

(9)

and calculate the second-order derivative of (9) as follows,

$$\begin{align*}
\ddot{\epsilon}_{i,x} &= u_{i,1}(\cos \psi_i \sin \theta_i \cos \phi_i + \sin \psi_i \sin \phi_i) - \ddot{x}_{id} \\
\ddot{\epsilon}_{i,y} &= u_{i,1}(\sin \psi_i \sin \theta_i \cos \phi_i - \cos \psi_i \sin \phi_i) - \ddot{y}_{id} \\
\ddot{\epsilon}_{i,z} &= u_{i,1}\cos \theta_i \cos \phi_i - g - \ddot{z}_{id}
\end{align*}$$

(10)

In view of (10), the non-regular feedback linearization method can be applied. The specific steps are:

1. Determine $u_{i,1}$ and $u_{i,4}$;

2. Basing on the designed $u_{i,1}$ and the closed-loop $e_{i,z}$-dynamics, compute and obtain the feedback linearizable form of $[e^{(3)}_{i,x}, e^{(3)}_{i,y}]$, in which the control inputs $(u_{i,2}, u_{i,3})$ will be obviously contained;

3. Find a suitable feedback linearized control law for $(e_{i,x}, e_{i,y})$-dynamics;

It is admissible to steer the yaw angle $\psi_i$ to zero for formation purpose [22, 25] and determine $u_{i,4}$ in a PD-form by

$$u_{i,4} = -k_{1,\psi} \psi_i - k_{2,\psi} \dot{\psi}_i,$$  \hspace{1cm} (11)

where $k_{1,\psi}, k_{2,\psi} > 0$. Suppose that $\theta_i, \phi_i \in (-\pi/2, \pi/2)$ and design $u_{i,1}$ by

$$u_{i,1} = \frac{1}{\cos \theta_i \cos \phi_i} \ddot{u}_{i,1},$$

$$\ddot{u}_{i,1} := g + \ddot{z}_{id} - k_{1,z} \tanh(\dot{e}_{i,z} + k_{2,\epsilon_{i,z}}) - k_{3,z} \tanh(\dot{e}_{i,z}),$$  \hspace{1cm} (12)

where $k_{1,z}, k_{2,z}, k_{3,z} > 0$ and $k_{1,z} + k_{3,z} < g - |\ddot{z}_{id}|$. The term $\ddot{u}_{i,1}$ is an intermediate variable for calculation convenience. Via substituting (12) into (10), $\dot{e}_{i,z}$ becomes

$$\dot{e}_{i,z} = -k_{1,z} \tanh(\dot{e}_{i,z} + k_{2,\epsilon_{i,z}}) - k_{3,z} \tanh(\dot{e}_{i,z}),$$  \hspace{1cm} (13)

which shows that, according to Lemma 1, $e_{i,z}$ is asymptotically stable. Additionally, it can be seen that $\ddot{u}_{i,1} > 0, \forall t \geq 0$, which helps to formulate the control inputs $[u_{i,2}, u_{i,3}]^T$. By (12), organize the column $[\dot{e}_{i,x}, \dot{e}_{i,y}]^T$ by

$$\begin{bmatrix}
\dot{e}_{i,x} \\
\dot{e}_{i,y}
\end{bmatrix} = \ddot{u}_{i,1} R_2(\psi_i) S \begin{bmatrix}
tan \theta_i \\
\tan \phi_i / \cos \theta_i
\end{bmatrix} - \begin{bmatrix}
\ddot{x}_{id} \\
\ddot{y}_{id}
\end{bmatrix},$$

(14)

where

$$R_2(\psi_i) = \begin{bmatrix}
\cos \psi_i & -\sin \psi_i \\
\sin \psi_i & \cos \psi_i
\end{bmatrix}, S = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.$$  \hspace{1cm} (15)

Differentiating (14) results

$$\begin{bmatrix}
e^{(3)}_{i,x} \\
e^{(3)}_{i,y}
\end{bmatrix} = \begin{bmatrix}
\ddot{u}_{i,1} R_2(\psi_i) + \ddot{u}_{i,1} \frac{dR_2(\psi_i)}{dt} \\
\frac{dR_2(\psi_i)}{dt}
\end{bmatrix} S \begin{bmatrix}
tan \theta_i \\
\tan \phi_i / \cos \theta_i
\end{bmatrix} + \ddot{u}_{i,1} R_2(\psi_i) SM_i \begin{bmatrix}
\dot{\theta}_i \\
\dot{\phi}_i
\end{bmatrix} - \begin{bmatrix}
x^{(3)}_{id} \\
y^{(3)}_{id}
\end{bmatrix},$$

(16)

where

$$\ddot{u}_{i,1} = \ddot{z}_{id} - k_{1,z} \left(1 - \tanh^2(\dot{e}_{i,z})\right) \dot{e}_{i,z} - k_{3,z} \left(1 - \tanh^2(\dot{e}_{i,z} + k_{2,\epsilon_{i,z}})\right) (\dot{e}_{i,z} + k_{2,\epsilon_{i,z}}),$$

$$\frac{dR_2(\psi_i)}{dt} = \psi_i R_2(\dot{\theta}_i),$$

$$M_i = \begin{bmatrix}
0 & \sec^2 \phi_i \sec \theta_i \tan \phi_i \sec \theta_i \\
\sec^2 \phi_i \sec \theta_i & 0
\end{bmatrix}.$$  \hspace{1cm} (17)
Given any virtual reference trajectory \( \phi_v \), the control inputs \( \phi_i \) and \( \theta_i \) are obtained as follows:

\[
\begin{align*}
\phi_i &= \arctan\left(\frac{\dot{y}_i - \dot{z}_i}{\sqrt{\dot{x}_i^2 + (\dot{y}_i + \dot{z}_i)^2}}\right), \\
\theta_i &= \arctan\left(\frac{\dot{x}_i + \dot{z}_i}{\dot{y}_i + \dot{z}_i}\right).
\end{align*}
\]

Based on (25) and Lemma 1 as well as classical linear stability theorem, it is direct to obtain (24). Moreover, the latitudinal and longitudinal states \( e_{i,x}, e_{i,y}, \dot{e}_{i,x}, \dot{e}_{i,y} \) and \( e_{i,y}, \dot{e}_{i,y} \) globally uniformly exponentially converge to zero. The altitude errors \( \dot{e}_{i,z}, \ddot{e}_{i,z}, \) and \( \dddot{e}_{i,z} \) are globally uniformly asymptotically convergent.

Observe equations (12)–(20), one may find out that it is essential to bound \( \phi_i \) and \( \theta_i \) in \((-\pi/2, \pi/2)\) regarding the designs of (12) and (20). To demonstrate the condition under which \( \phi_i, \theta_i \in (-\pi/2, \pi/2) \) establishes, the proposition below is introduced.

**Proposition 1** Given any virtual reference trajectory \( \bar{p}_{id} = [x_{id}, y_{id}, z_{id}]^T \) with bounded derivatives and satisfying \( |\dot{z}_{id}(t)| < g \), applying any control law \([u_{i,x}, u_{i,y}]^T\) capable of achieving \((\dot{e}_{i,x}, \dot{e}_{i,y}) \in L_\infty, \) and \( u_{i,1} \) defined in (11) and any \( u_{i,1} \in L_\infty \) on the quad-rotor (2) ensures that \( \phi_i(t), \theta_i(t) \in (-\pi/2, \pi/2), \forall t \geq 0. \)

**Proof.** By (14), one has

\[
\begin{pmatrix}
\dot{\phi}_i \\
\dot{\theta}_i
\end{pmatrix} = R^{-1}_2(\psi_i) S^{-1}(\xi_i, \eta_i, \zeta_i) \begin{pmatrix}
\dddot{e}_{i,x} \\
\dddot{e}_{i,y}
\end{pmatrix} + \begin{pmatrix}
\dddot{x}_{id} \\
\dddot{y}_{id}
\end{pmatrix},
\]

which, together with \( \dot{e}_{i,x}, \dot{e}_{i,y} \in L_\infty \) drawn from Theorem 1 and \( u_{i,1} > 0 \) due to \( k_{1,x} + k_{3,x} - k_{3,y} < g - |\dot{z}_{id}|, \) shows that tan \( \theta_i \in L_\infty \) and tan \( \phi_i \in L_\infty. \) Hence, \( \phi_i(t), \theta_i(t) \in (-\pi/2, \pi/2), \forall t \geq 0. \)

The Proposition 1 illustrates the sufficient condition to avoid singularity when calculating the control laws (12)(20)(23), that is, basically, ensuring \( \dot{e}_{i,x}, \dot{e}_{i,y} \in L_\infty. \)

**Remark 2** By (26), further computation can obtain the roll and pitch angles of steady state, with trivial tracking errors, as follows:

\[
\begin{pmatrix}
\phi_i \\
\theta_i
\end{pmatrix} = \arctan\left(-\frac{\dot{y}_i - \dot{z}_i}{\sqrt{\dot{x}_i^2 + (\dot{y}_i + \dot{z}_i)^2}}\right).
\]

**Remark 3** Note that the control inputs \([u_{i,2}, u_{i,3}]\) are directly designed with position error states rather than steering the roll and pitch to track their reference signals as traditional inner-outer loop methods do. One of such reference signals refers to the virtual roll and pitch angles \((\phi_{id}, \theta_{id})\) solved according to

\[
\begin{pmatrix}
\tan \theta_{id} \\
\tan \phi_{id}
\end{pmatrix} = R^{-1}_2(\psi_i) S^{-1}(\xi_i, \eta_i, \zeta_i) \begin{pmatrix}
\dddot{e}_{i,x} \\
\dddot{e}_{i,y}
\end{pmatrix} + \begin{pmatrix}
\dddot{x}_{id} \\
\dddot{y}_{id}
\end{pmatrix} + \begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} \begin{pmatrix}
\dddot{e}_{i,x} \\
\dddot{e}_{i,y}
\end{pmatrix},
\]

with suitable \( b_1, b_2 > 0. \) Steering \((\phi_i, \theta_i)\) to \((\phi_{id}, \theta_{id})\) can then lead to \( [\dot{e}_{i,x}, \dot{e}_{i,y}]^T = -b_1 [\dddot{e}_{i,x}, \dddot{e}_{i,y}]^T - b_2 [\dddot{e}_{i,x}, \dddot{e}_{i,y}]^T \) and the position error \([\dot{e}_{i,x}, \dot{e}_{i,y}]^T\) would converge to zero.
\[
\begin{align*}
\begin{bmatrix}
\dot{e}^{(4)}_i \\
e^{(4)}_i
\end{bmatrix}
= & \ddot{\xi}_{i,1} + \left[ \ddot{u}_{i,1} R_2(\psi_i) SM_i + \ddot{u}_{i,1} \frac{dR_2(\psi_i)}{dt} \right] S \left[ \tan \theta_i \tan \phi_i \right] + \ddot{u}_{i,1} R_2(\psi_i) SM_i \left[ \frac{u_{i,2}}{u_{i,3}} \right] - x^{(4)}_i
\end{align*}
\]

Remark 4 Different form traditional feedback linearization results having fourteen states to be regulated [18, 19, 22, 25, 27, 28, 31], there are only twelve control states to be regulated in our design.

Remark 5 By (22), the latitudinal and longitudinal position errors are converted into two fourth-order integrators via viewing \((u_{i,x}, u_{i,y})\) as control inputs. It is therefore able to design \((u_{i,x}, u_{i,y})\) with the help of classical linear techniques, such as saturated method [33, 34].

Remark 6 The application of control law (11)(12)(20)(23) on a real quad-rotor should consider more practical scenarios such as available thrust, mass and reference acceleration as well as other requirements. This note does not take these factors into account to make the main idea be presented in a concise manner.

B. Distributed Observer (Virtual Reference Trajectory)

To achieve formation via a local tracking control law, as stated above, the virtual reference trajectory \(p_{id} = [x_{id}, y_{id}, z_{id}]^T\) should have bounded derivatives and satisfy \(\lim_{t \to \infty} [x_{id}, y_{id}, z_{id}]^T = [x_0, y_0, z_0]^T\) and \(|\dot{z}_{id}(t)| < g\).

Two problems obstruct the design of such reference trajectory, that is, only partial quad-rotors can know the leader’s states and only neighboring interaction is available. Concerning these problems and the fact that \(z_{id}\) needs an acceleration less than \(g\) while there is no such restriction about \([x_{id}, y_{id}]^T\), the designs of \((x_{id}, y_{id})\) and \(z_{id}\) are separately presented in two lemmas below.

Let \(\zeta_{id} = [x_{id}, y_{id}]^T\) and \(\zeta_0 = [x_0, y_0]^T\) and extend the second-order observer reported in our previous work [32], propose the following lemma.

Lemma 2 Given Assumptions 2-3, the fourth-order dynamics described by
\[
\zeta^{(4)}_{id} = -g_3 \zeta^{(3)}_{id} - g_2 \zeta^{(2)}_{id} - g_1 \zeta_{id} - g_4 c_{i} - Q_i c_{i}
\]
with
\[
c_i = \begin{bmatrix} c_{i,x} & c_{i,y} \end{bmatrix}^T
\]

\[
\begin{align*}
&= \sum_{j=1}^{n} a_{ij} \left( \zeta^{(3)}_{id} - \zeta^{(3)}_{jd} \right) + a_{i0} \left( \zeta^{(3)}_{id} - \zeta_{0i} \right) \\
&+ g_3 \sum_{j=1}^{n} a_{ij} \left( \zeta_{id} - \zeta_{jd} \right) + g_3 a_{i0} \left( \zeta_{id} - \zeta_0 \right) \\
&+ g_2 \sum_{j=1}^{n} a_{ij} \left( \zeta_{id} - \zeta_{jd} \right) + g_2 a_{i0} \left( \zeta_{id} - \zeta_{0} \right) \\
&+ g_1 \sum_{j=1}^{n} a_{ij} \left( \zeta_{id} - \zeta_{jd} \right) + g_1 a_{i0} \left( \zeta_{id} - \zeta_0 \right)
\end{align*}
\]

\[
Q_i = \text{diag}\left\{ \frac{g_{5,x}}{|c_{i,x}| + \gamma e^{-\lambda t}}, \frac{g_{5,y}}{|c_{i,y}| + \gamma e^{-\lambda t}} \right\}
\]

and gain selections
\[
\begin{align*}
g_2 & > 0, g_3 > 0, g_2 g_3 > g_1 > 0, g_4 > 0, \gamma > 0, \lambda > 0, \\
g_{5,x} & \geq g_{0,x} := \sup_{t \geq 0} \left| x^{(4)}_0 + g_3 a_{i0} \right| + g_2 \dot{x}_0 + g_1 x_0, \\
g_{5,y} & \geq g_{0,y} := \sup_{t \geq 0} \left| y^{(4)}_0 + g_3 y_0 + g_2 \dot{y}_0 + g_1 y_0 \right|
\end{align*}
\]

ensures that \(\zeta_{id}, \zeta^{(2)}_{id}\) and \(\zeta^{(3)}_{id}\) are bounded, and
\[
\begin{align*}
\lim_{t \to \infty} \zeta_{id} = \zeta_0, \lim_{t \to \infty} \dot{\zeta}_{id} = \dot{\zeta}_0, \lim_{t \to \infty} \ddot{\zeta}_{id} = \ddot{\zeta}_0, \lim_{t \to \infty} \zeta^{(3)}_{id} = \zeta^{(3)}_0.
\end{align*}
\]
Given Assumptions 1-3, applying the local limits, one can solve that $h$ with $t \to \infty$. 

The theorem below concludes the formation scheme briefly.

**Theorem 2** Given Assumptions 1-3, applying the local tracking control laws (11)(12)(20)(23) and distributed observers (29)(33) on a team of quad-rotors described by (2) achieves

$$\lim_{t \to \infty} \begin{bmatrix} x_i - x_0 \\ y_i - y_0 \\ z_i - z_0 \end{bmatrix} = \begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix}, \forall i \in \mathcal{N}. \quad (36)$$

**Proof.** The actual formation error $p_i - p_0 - \Delta_i$ satisfies

$$\|p_i - p_0 - \Delta_i\| \leq \|p_i - p_0 - \Delta_i + p_0 - p_0\| \leq \|p_i - p_0 - \Delta_i\| + \|p_0 - p_0\|, \quad (37)$$

which, together with Theorem 1, Lemma 2 and Lemma 3, shows that $\lim_{t \to \infty} p_i - p_0 - \Delta_i = [0,0,0]^T$. The claim (36) follows.

**Remark 7** Inspired by the analysis above, one can solve other cooperative problems of quad-rotor systems (such as time-varying formation and communication delay) via modifying associated control protocols developed for linear integrators into virtual reference trajectories meeting the requirements of our local controller.

**Remark 8** The formation scheme is distributed since only neighboring communication is available. It is therefore direct to add cooperative quad-rotors as required.

### IV. NUMERICAL SIMULATION

To validate the proposed distributed formation control algorithm we employ a scenario for simulation, concerning four quad-rotors tracking a leader while performing a fixed square pattern. Without loss of generality, the undirected interaction network is described by the figure below.

![FIG. 3: The interaction network.](image)

The observers proposed in Lemma 2 and Lemma 3 solve the problem that the leader’s states are not available to all quad-rotors. They act as interactions among quad-rotors, and can be generated by on-board computer and transmitted by wireless modules. Moreover, the derivatives of the proposed observer are bounded up to the fourth order and satisfy $z_{id} < g$. It is therefore admissible to view the observers (29)(33) as virtual reference trajectory for each quad-rotor and apply the local control law stated in previous subsection.
• **Case 1:** $p_0(t) = [100 \sin 0.1t, -100 \cos 0.1t, 100]^T$,

• **Case 2:** $p_0(t) = [0, 0, 50]^T$.

For convenience, we select the identical initial states by $x_1(0) = -10, y_1(0) = 12, z_1(0) = 0, \phi_1(0) = 0, \psi_1(0) = 0, \theta_1(0) = \pi/8, x_2(0) = 40, y_2(0) = -12, z_2(0) = 5, \phi_2(0) = 0, \psi_2(0) = \pi/2, x_3(0) = 20, y_3(0) = 10, z_3(0) = 6, \phi_3(0) = 0, \theta_3(0) = 0, \psi_3(0) = \pi, x_4(0) = -20, y_4(0) = 45, z_4(0) = 7, \phi_4(0) = 0, \theta_4(0) = 0, \psi_4(0) = \pi/5$.

The control gains are also selected to be identically for two cases as $k_{1x} = 1, k_{2x} = 0.5, k_{1y} = 0.5, k_{1z} = 0.2, k_{2x} = 1.6, k_{3x} = 3.6, k_{1y} = 3.2, k_{1z} = 2, k_{2y} = 1.6, k_{3y} = 3.6, k_{4y} = 3.2, k_{1\psi} = 0.5, k_{2\psi} = 0.5, g_1 = 0.125, g_2 = 0.75, g_3 = 0.85, g_4 = 0.1, g_{5x} = 2.1, g_{5y} = 2.1, \gamma = 15, \lambda = 0.1, h_1 = 0.5, h_2 = 0.5, h_3 = 0.5, h_4 = 0.5, h_5 = 0.5, h_6 = 1$. To start the simulation, the initial values of the virtual trajectory is chosen as $\zeta_{id} = [x_{id}(0), y_{id}(0), z_{id}(0)]^T = [x_i(0), y_i(0), z_i(0)]^T$ and $z_{id}(0) = z_i(0)$ with $i = \{1, 2, 3, 4\}$ and their derivatives are supposed to be zero.

Three sub-figures are depicted for each case with sub-figure (a) being the geometric position paths of quad-rotors, sub-figure (b) being the norm of formation error $p_i - p_0 - \Delta_i$, and sub-figure (c) being the attitude angles. The simulation results of these two cases are shown in Figure 4 and Figure 5 respectively. It can be seen from sub-figure (a) of both two cases that the follower quad-rotors form the square pattern while tracking the leader with the predefined position displacements. The sub-figure (b) shows that the formation error norm is asymptotically convergent. The attitudes of all followers are kept in reasonable ranges that can be demonstrated by sub-figure (c). All of the simulation results illustrate the effectiveness of the proposed formation algorithm.

V. CONCLUSION

This note solves the leader-follower formation problem for multiple quad-rotors. The whole formation control scheme involves two parts, namely, a local tracking control law and a distributed observer. Given a smooth reference trajectory with bounded derivatives, a novel local tracking control law is proposed with the help of non-regular feedback linearization approach. In view of the fact that the leader’s states are not available to all followers, we propose a distributed observer and feed it into the local tracking control law via viewing it as virtual reference trajectory. As for future research, we will take into account more practical problem associated quad-rotor formation such as communication failure, wind turbulence and inter-agent collision avoidance.
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APPENDIX A: PROOF OF LEMMA 2

Proof. Define

\[ \xi_i = \zeta_{id} - \zeta_0 \in \mathbb{R}^2, \quad s_i = \xi_i^{(3)} + g_3 \dot{\xi}_i + g_2 \ddot{\xi}_i + g_1 \xi_i \in \mathbb{R}^2, \]

and note that \( s_i = 0 \) leads to \( c_n^{(3)} = -g_3 \ddot{\xi}_i - g_2 \dot{\xi}_i - g_1 \xi_i \). As a result, by gains \( g_2 > 0, g_3 > 0, g_2g_3 > g_1 > 0 \), the objective (32) can be realized. Hence, let us prove \( s_i \to 0, \forall i \in \mathcal{N} \) and define some helpful 2n-dimensional vectors by

\[ c = [c_1^T, c_2^T, ..., c_n^T]^T, \quad \xi = [\xi_1^T, \xi_2^T, ..., \xi_n^T]^T, \quad s = [s_1^T, s_2^T, ..., s_n^T]^T. \]

Direct computations result with

\[ c = (\mathcal{H} \otimes I_2) s, \]

\[ s = \xi^{(3)} + g_3 \ddot{\xi}_i + g_2 \dot{\xi}_i + g_1 \xi_i, \]

where ‘\( \otimes \)’ denotes Kronecker product. By Assumption 3, the matrix \( \mathcal{H} \) is symmetric, invertible and positive definite [14], and hence, so is \( \mathcal{H} \otimes I_2 \). The convergence \( c \to 0_{2n} \) is equivalent with \( s \to 0_{2n} \). Choose a positive definite function

\[ W = 0.5e^T (\mathcal{H} \otimes I_2)^{-1} c, \]

whose derivative can be obtained as \( W = c^T (\mathcal{H} \otimes I_2) \dot{c} = c^T \dot{s} \). By the fact

\[ \dot{s} = \zeta_{id}^{(4)} - 1_n \otimes \zeta_0^{(4)} + g_3 \left( \zeta_{id}^{(3)} - 1_n \otimes \zeta_0^{(3)} \right) + g_2 \left( \ddot{\zeta}_{id} - 1_n \otimes \ddot{\zeta}_0 \right) + g_1 \left( \dot{\zeta}_{id} - 1_n \otimes \dot{\zeta}_0 \right) \]

\[ = -g_4 c - Q c - 1_n \otimes \left( \zeta_0^{(4)} + g_3 \zeta_0^{(3)} + g_2 \dot{\zeta}_0 + g_1 \ddot{\zeta}_0 \right), \]

with \( Q = \text{diag}\{Q_1, Q_2, ..., Q_n\} \in \mathbb{R}^{2n \times 2n} \), rewrite \( \dot{W} \) by,

\[ \dot{W} = -g_4 c^T \dot{c} - c^T G c - c^T 1_n \otimes (\zeta_0^{(4)} + g_3 \zeta_0^{(3)} + g_2 \dot{\zeta}_0 + g_1 \ddot{\zeta}_0) \]

\[ = -\sum_{i=1}^{n} \frac{g_5 \xi_i^2}{|c_{i,x}| + \gamma e^{-\lambda t}} - \sum_{i=1}^{n} c_{i,x} \left( x_0^{(4)} + g_3 x_0^{(3)} + g_2 \dot{x}_0 + g_1 \ddot{x}_0 \right) \]

\[ - \sum_{i=1}^{n} \frac{g_5 \xi_i^2}{|c_{i,y}| + \gamma e^{-\lambda t}} - \sum_{i=1}^{n} c_{i,y} \left( y_0^{(4)} + g_3 y_0^{(3)} + g_2 \dot{y}_0 + g_1 \ddot{y}_0 \right) \]

\[ \leq -g_4 ||c||^2 - \sum_{i=1}^{n} \left( \frac{g_5 \xi_i^2}{|c_{i,x}| + \gamma e^{-\lambda t}} - |c_{i,x}| \sigma_{0,x} \right) \]

\[ - \sum_{i=1}^{n} \left( \frac{g_5 \xi_i^2}{|c_{i,y}| + \gamma e^{-\lambda t}} - |c_{i,y}| \sigma_{0,y} \right), \]

(43)
which, combined with the fact $W \geq \frac{\|c\|^2}{2\lambda_{\max}(\mathcal{H})}$, implies

$$W \leq -2g_4\lambda_{\min}(\mathcal{H})W + \gamma(n(\sigma_{0,x} + \sigma_{0,y})e^{-\lambda t})$$

$$= -qW + \sigma_0 e^{-\lambda t}, \quad (44)$$

where $q := 2g_4\lambda_{\min}(\mathcal{H})$, $\sigma_0 := \gamma(n(\sigma_{0,x} + \sigma_{0,y})$ and inequality $\frac{a_1x^2}{|x|} + a_2 - a_3|x| \geq -a_2a_3$, with real numbers $a_1 \geq a_3 \geq 0$ and $a_2 > 0$, is applied. By comparison principle [10], integrating both sides of (44) results

$$W(t) \leq \begin{cases} W_1(t) = e^{-qt}W(0) + \sigma_0e^{-\lambda t}, & \text{if } q \neq \lambda; \\ W_2(t) = e^{-qt}W(0) + \sigma_0te^{-qt}, & \text{if } q = \lambda. \end{cases} \quad (45)$$

For $W_1(t)$, it satisfies

$$W_1(t) \leq e^{-qt}W(0) + \frac{\sigma_0}{|\lambda - q|}e^{-\min(q,\lambda)t}. \quad (46)$$

For $W_2(t)$, by the fact $te^{-\alpha t} \leq \frac{1}{\alpha}, \forall t \geq 0, a = 0.5\lambda$, it satisfies

$$W_2(t) \leq e^{-qt}W(0) + \frac{\sigma_0}{\lambda}e^{-\frac{1}{2}t}. \quad (47)$$

Hence, both $W_1$ and $W_2$ converge to zero globally exponentially, which means that $c$ and $s$ are globally exponentially stable (GES). By (40), one has $\xi^{(3)} = -g_3\xi - g_2\xi - g_1\xi + s$. The vectors $\xi, \dot{\xi}, \ddot{\xi}$ and $\xi^{(3)}$ are GES due to fact that both $\xi^{(3)} = -g_3\xi - g_2\xi - g_1\xi$ and $s$-dynamics are GES[36, 37]. Therefore, the claims (32) follow. □

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