Hidden Symmetry of the Racah and Clebsch-Gordan Problems for the Quantum Algebra $sl_q(2)$

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**Abstract**

The Askey-Wilson algebra $AW(3)$ with three generators is shown to serve as a hidden symmetry algebra underlying the Racah and (new) generalized Clebsch-Gordan problems for the quantum algebra $sl_q(2)$. On the base of this hidden symmetry a simple method to calculate corresponding coefficients in terms of the Askey-Wilson polynomials is proposed.

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1 Introduction

As is well known, the quantum algebra $sl_q(2)$ possesses many remarkable properties closely related to those of the ordinary $sl(2)$ Lie algebra. In particular, the Clebsch-Gordan and Racah problems for $su_q(2)$ and $su_q(1,1)$ algebras can be formulated and resolved in such a manner that the corresponding coefficients are expressed in terms of the Askey-Wilson polynomials [1-7]. What is the origin of this "experimental" result? In other words, why can these coefficients be calculated explicitly in terms of $q$-orthogonal polynomials? For the ordinary $su(2)$ and $su(1,1)$ algebras the answer was found in Ref. [8] where the quadratic Racah algebra $QR(3)$ with three generators was shown to serve as a hidden symmetry algebra underlying the corresponding Racah and Clebsch-Gordan problems. By the way, this result explains why the same Racah polynomials arise as $6j$ symbols for such strictly different algebras as (compact) $su(2)$ and (non-compact) $su(1,1)$.

It is worth mentioning that the representation theory for the quadratic Racah algebra $QR(3)$ is very simple and can be constructed independently of a concrete realization in terms of $su(2)$ or $su(1,1)$ generators. So, really we are dealing with an extra-symmetry of some non-linear combinations of Lie algebra generators (namely, intermediate Casimir operators - for details see [8]) of the added Lie algebras. It is interesting to note that it was Racah who first introduced such a manner some non-linear (cubic) algebra in order to study the representations of the $SU(3)$ group [9]. In fact the structure of the algebra introduced by Racah is closely related (up to some additional term) to that of $QR(3)$. This justifies the name of this algebra.

The purpose of this paper is to present an analogous algebraic treatment of the Racah and Clebsch-Gordan problems associated with the quantum algebra $sl_q(2)$. The paper is organized as follows. In Sec.II, we recall the addition rule for different types of $sl_q(2)$ algebras in accordance with Ref.[10, 11]. In Sec.III, the AW(3) algebra is shown to be the hidden symmetry algebra for the Racah problem of $sl_q(2)$; this allows one to express all types of Racah coefficients in terms of Askey-Wilson polynomials. In Sec.IV we show how the generalized Clebsch-Gordan problem for $sl_q(2)$ can be obtained from the Racah one by a simple contraction procedure having no any classical analogue. This procedure allows one to obtain the corresponding hidden symmetry algebra and the explicit expression for the Clebsch-Gordan coefficients directly from the Racah case.

2 Different types of $sl_q(2)$ and their addition rule

We adopt the standard notation for the $sl_q(2)$ algebra [10, 11, 12]

$$
\begin{align*}
[A_0, A_\pm] &= \pm A_\pm, \\
[A_-, A_+] &= uq^{-A_0} + vq^{A_0},
\end{align*}
$$

(1)

where $q = \exp(-\omega), \omega > 0$.

We are restricted ourselves to the real forms [1] of $sl_q(2)$, i.e. the parameters $u$ and $v$ are assumed to be real. Moreover, we shall assume that some unitary representation of $sl_q(2)$ is chosen where the operators $A_-$ and $A_+$ are Hermitian conjugated whereas the operator $A_0$ is Hermitian. In what follows we shall denote the algebra $sl_q(2)$ with commutation relations (1) by the symbol $(u,v)$. 

The special cases of \((u,v)\) algebra are:

(i) \(su_q(2)\) if \(u = -v < 0\);
(ii) \(su_q(1,1)\) if \(u = -v > 0\);
(iii) \(cu_q(2)\) if \(u = v > 0\);
(iv) \(eu_q^+\) if \(u > 0, v = 0\);
(v) \(eu_q^-\) if \(u = 0, v > 0\).

All these case were intensively studied in the literature. Note, that the cases (iv) and (v) describing two types of q-oscillator algebra can be obtained from the cases (i)-(iii) by a simple contraction procedure [12]. It is worth mentioning that the cases (i)-(iii) describe three different types of \(sl_q(2)\) algebra, i.e. the unitary representations of these can not be obtained from one another by simple analytic continuation and renormalization of the initial generators \(A_0, A_\pm\) (for example, the \(su_q(2)\) algebra has only finite-dimensional representations, whereas the algebras \(su_q(1,1)\) and \(cu_q(2)\) have only infinite-dimensional representations - see below).

The Casimir operator of the \((u,v)\) algebra has the expression

\[ \hat{\kappa} = A_+ A_- + (vq^{A_0} - uq^{1-A_0})/(1 - q) \]  

Fixing the value of the Casimir operator

\[ \kappa(\alpha) = (vq^\alpha - uq^{1-\alpha})/(1 - q), \]  

where \(\alpha\) is some parameter, we get a unitary representation of the \((u,v)\) algebra. In this paper we restricted ourselves to the representations of the positive discrete series \(D^+_\alpha\) in some canonical basis \(|n; \alpha\rangle\):

\[ A_0 |n; \alpha\rangle = (\alpha + n) |n; \alpha\rangle, \quad n = 0, 1, 2, ... \]  

\[ A_- |n; \alpha\rangle = r_n |n - 1; \alpha\rangle, \]  

\[ A_+ |n; \alpha\rangle = r_{n+1} |n + 1; \alpha\rangle, \]  

where

\[ r_n^2 = (1 - q^n)(vq^\alpha + uq^{1-n-\alpha})/(1 - q) \]  

Note that because \(r_0 = 0\) the state \(|0; \alpha\rangle\) is the vacuum of the representation \(D^+_\alpha\).

It is seen from (7) that the representation \(D^+_\alpha\) exists provided that

\[ vq^\alpha + uq^{1-n-\alpha} > 0 \]  

for all values of \(n\). The condition (8) is fulfilled for \(su_q(1,1)(\alpha > 0), cu_q(2), eu_q^+\) (\(\alpha\) is arbitrary real parameter) algebras. Otherwise, if (8) is fulfilled for \(n \leq N\) but

\[ vq^\alpha + uq^{N-\alpha} = 0, \]  

then one obtains a \(N + 1\) dimensional representation (this takes place, e.g. for \(su_q(2)\)).

The \((u,v)\) algebra possesses an addition property that allows to add different types of \(sl_q(2)\) algebras [10, 11, 12]:

\[ A_0^{(3)} = A_0^{(1)} + A_0^{(2)}, \]  

\[ A_\pm^{(3)} = A_\pm^{(1)} \exp(\omega A_0^{(2)}) + A_\pm^{(2)} \exp(-\omega A_0^{(1)}) \]  

\[ \hat{A}_\alpha \]
The addition rule (10) is the same as for ordinary $sl_q(2)$ algebra [13], however the algebras $(u_1, v_1)$ and $(u_2, v_2)$ in (10) may have different types. It is easily seen that in order for the operators $A_0^{(3)}, A_{\pm}^{(3)}$ to form new $(u_3, v_3)$ algebra, the following relations must be fulfilled:

$$u_3 = u_1, \quad v_3 = v_2, \quad u_2 = -v_1$$

(11)

In symbolic form the addition rule (10) can be written as

$$(u, v)_1 \oplus (-v, w)_2 = (u, w)_3$$

(12)

It is worth mentioning that this addition rule does not destroy the unitarity of the representation, i.e. the operator $A_0^{(3)}$ is Hermitian and the operators $A_{\alpha_1}^{(3)}$ and $A_{\alpha_2}^{(3)}$ are Hermitian conjugated. So, if the representations $D_{\alpha_1}^+$ and $D_{\alpha_2}^+$ are given, then one can construct the Clebsch-Gordan decomposition

$$|n_3; \alpha_3\rangle = \sum_{n_1, n_2} (n_1 \alpha_1 n_2 \alpha_2; n_3 \alpha_3) |n_1; \alpha_1\rangle \otimes |n_2; \alpha_2\rangle,$$

(13)

where the symbol $(n_1 \alpha_1 n_2 \alpha_2; n_3 \alpha_3)$ stands for the Clebsch-Gordan coefficients (CGC). Obviously, in (13) the relation

$$n_1 + n_2 = n_3 + 3 - \alpha_1 - \alpha_2$$

(14)

is satisfied.

The reciprocal decomposition has the form

$$|n_1; \alpha_1\rangle \otimes |n_2; \alpha_2\rangle = \sum_{\alpha_3 = \alpha_1 + \alpha_2} (n_1 \alpha_1 n_2 \alpha_2; n_3 \alpha_3) |n_3; \alpha_3\rangle.$$

(15)

The explicit expression of these CGC (in the case of the representations of discrete positive series) in terms of Hahn polynomials was found in [3, 5, 6] for $su_q(2)$ and $su_q(1, 1)$ algebras and in [11] for the case when the algebras of different types are added (note that in [11] addition rule (10) was presented in another - but equivalent - form).

It is worth mentioning that among the addition rules (10) there exists some that can not be obtained from well-known addition rules for $su_q(2)$ or $su_q(1, 1)$ by any contraction (with real parameters).

As an example we present some non-trivial addition

$$(u, 0)_1 \oplus (0, u)_2 = (u, u)_3, \quad u > 0,$$

(16)

mapping two different q-oscillator algebras $eu_q^+(2)$ and $eu_q^-(2)$ onto the third q-oscillator algebra $cu_q(2)$. Obviously, such an addition rule can not be obtained form $su_q(1, 1)$ or $su_q(2)$ cases by means of ordinary contraction (obviously, contraction with complex parameters destroys the unitarity of the representations). Note, that the addition rule (16) allows one to solve the problem of finding the addition rules for the q-oscillator algebras: there is no such addition involving the same q-oscillator algebras, instead, all the added and resulting algebras should be different (see also [14]).

Of course, there are many other special types of addition rule having no any classical analogs: for example one can obtain from (12) a new (non-Schwinger) realization of $su_q(2)$ and $su_q(1, 1)$ algebras in terms of two q-oscillator; this realization takes place only in a q-domain and disappears in the classical limit $q \to 1$ (for details see [13]).
3 \textit{AW}(3) algebra and the Racah problem

The addition rule \[(\text{II})\] possesses an associativity property when three algebras are added. Indeed, let us have three (mutually commuting) sets of the \(sl_q(2)\) generators \(A_0^{(i)}, A_\pm^{(i)}, i = 1, 2, 3\) with corresponding indicators \((u_i, v_i)_i\). If the relations

\[v_1 = -u_2 = v, v_2 = -u_3 = w, u_1 = u, v_3 = z,\]  \hspace{1cm} (17)

hold, then one can obtain the same resulting algebra \((u, z)_4\) in two different ways:

i) one can first add the algebras \(i = 1, 2\) obtaining the algebra

\[(u, w)_{12} = (u, v)_1 \oplus (-v, w)_2,\]  \hspace{1cm} (18)

and then getting the complete sum by adding \(i = 3\) algebra:

\[(u, z)_4 = (u, w)_{12} \oplus (-w, z)_3;\]  \hspace{1cm} (19)

ii) or, one can first add the \(i = 2, 3\) algebras

\[(-v, z)_{23} = (-v, w)_2 \oplus (-w, z)_3,\]  \hspace{1cm} (20)

to get the complete sum by adding the \(i = 1\) algebra:

\[(u, z)_4 = (u, v)_1 \oplus (-v, z)_{23}.\]  \hspace{1cm} (21)

According to the schemes i) and ii) we can introduce two intermediate Casimir operators \(\hat{\kappa}_{12}\) and \(\hat{\kappa}_{23}\) corresponding to the \((u, w)_{12}\) and \((-v, z)_{23}\) algebras:

\[K_1 = \hat{\kappa}_{12} = A_+^{(1)} A_-^{(2)} \exp(\omega(A_0^{(2)} - A_0^{(1)}) - 1)) + h.c. + \kappa_1 \exp(2\omega A_0^{(2)})
+ \kappa_2 \exp(-2\omega A_0^{(1)}) - v \coth \omega \exp(2\omega(A_0^{(2)} - A_0^{(3)})),\]  \hspace{1cm} (22)

\[K_2 = \hat{\kappa}_{23} = A_+^{(2)} A_-^{(3)} \exp(\omega(A_0^{(3)} - A_0^{(2)}) - 1)) + h.c. + \kappa_2 \exp(2\omega A_0^{(3)})
+ \kappa_3 \exp(-2\omega A_0^{(2)}) - w \coth \omega \exp(2\omega(A_0^{(3)} - A_0^{(2)})),\]  \hspace{1cm} (23)

where \(\kappa_i, (i = 1, 2, 3)\) stands for the values of corresponding Casimir operators \(\hat{\kappa}_i\) obviously commuting with \(K_1\) and \(K_2\).

The full Casimir operator

\[
\hat{\kappa}_4 = A_+^{(4)} A_-^{(4)} + (z \exp(-2\omega A_0^{(4)}) - u \exp(2\omega(A_0^{(4)} - 1))/(1 - q) + K_1 \exp(2\omega A_0^{(3)}) + K_2 \exp(-2\omega A_0^{(1)}) - \kappa_2 \exp(2\omega(A_0^{(3)} - A_0^{(1)}))
+ [A_+^{(1)} A_-^{(3)} \exp(\omega(A_0^{(3)} - A_0^{(1)}) - 1)) + h.c.]
\]  \hspace{1cm} (24)

also commute with both \(K_1\) and \(K_2\) and can be replaced by the constant \(\kappa_4\).

The Racah problem consists in finding the overlaps between eigenstates of the intermediate Casimir operators \(K_1\) and \(K_2\) in the space with fixed values of \(\kappa_i, i = 1, 2, 3, 4\). This problem is non-trivial because the operators \(K_1\) and \(K_2\) do not commute with one another.

The crucial observation in our considerations is that the operators \(K_1\) and \(K_2\) are closed in frames of simple algebra with three generators.
In order to see this let us introduce the procedure of ”q-mutation" for arbitrary operators $L, M$

$$[L, M]_\omega \equiv e^\omega LM - e^{-\omega} ML$$  \hfill (25)

A direct calculation shows that operators $K_1, K_2$ together with their q-mutator $K_3$ obey the following algebra

\[
\begin{align*}
[K_1, K_2]_\omega &= K_3, \\
[K_2, K_3]_\omega &= BK_2 + C_1 K_1 + D_1, \\
[K_3, K_1]_\omega &= BK_1 + C_2 K_2 + D_2,
\end{align*}
\hfill (26)
\]

where $B, C_{1,2}, D_{1,2}$ are the structure constants of the algebra (26):

\[
\begin{align*}
B &= 4 \sinh^2 \omega (\kappa_1 \kappa_3 + \kappa_2 \kappa_4), \\
C_1 &= 4vz \cosh^2 \omega, \\
C_2 &= -4uv \cosh^2 \omega, \\
D_1 &= -2 \sinh 2\omega (z\kappa_1 \kappa_2 + v\kappa_3 \kappa_4), \\
D_2 &= 2 \sinh 2\omega (u\kappa_2 \kappa_3 - w\kappa_1 \kappa_4).
\end{align*}
\hfill (27)
\]

The algebra (26) is known as the Askey-Wilson algebra with three generators $AW(3)$. It was introduced and studied in [15, 16]. The Casimir operator $\hat{Q}$ commuting with all the generators $K_1, K_2, K_3$ of the $AW(3)$ algebra has the expression

\[
\begin{align*}
\hat{Q} &= \frac{1}{2} \{K_3, \tilde{K}_3\} + \cosh 2\omega (C_1 K_1^2 + C_2 K_2^2) \\
&\quad + B \{K_1, K_2\} + 2 \cosh^2 \omega (D_1 K_1 + D_2 K_2),
\end{align*}
\hfill (28)
\]

where the symbol $\{.,.\}$ stands for the anticommutator and $\tilde{K}_3$ is the ”dual” generator:

$$\tilde{K}_3 = [K_1, K_2]_{-\omega} \equiv e^{-\omega} K_1 K_2 - e^{\omega} K_2 K_1$$  \hfill (29)

Given the realization (22), (23) of the $AW(3)$ algebra the Casimir takes the value

\[
\begin{align*}
Q &= 4[-uvwz \cosh^4 \omega \sinh^{-2} \omega + \sinh^2 \omega (\kappa_1^2 \kappa_2^2 + \kappa_2^2 \kappa_4^2) - 2 \sinh^2 \omega \cosh 2\omega \kappa_1 \kappa_2 \kappa_3 \kappa_4 \\
&\quad \quad \quad \quad + \cosh^2 \omega (-wz\kappa_1^2 + uz\kappa_2^2 + uv\kappa_3^2 - v\kappa_4^2)]
\end{align*}
\hfill (30)
\]

So, the operators $\hat{\kappa}_{12}, \hat{\kappa}_{23}$ together with their q-mutator form a realization of the $AW(3)$ algebra with fixed values of the structure constants (27) and Casimir operator (30).

So far, we have not made any assumptions on the concrete type of the $(u_i, v_i)_i$ representations. Now let us assume that all the added algebras $(i = 1, 2, 3)$ belong to the positive discrete series $D^+_{\alpha_i}$ (the finite-dimensional case $D^0$ is also admitted). Then the resulting algebra $i = 4$ has also the representation of $D^+_{\alpha_4}$. It is easily seen that on the space with fixed values $\alpha_i, i = 1, 2, 3, 4$ the operators $K_1$ and $K_2$ become finite-dimensional matrices:

\[
\begin{align*}
K_1 \psi_p &= \kappa_{12}(p) \psi_p, \\
K_2 \phi_s &= \kappa_{23}(s) \phi_s
\end{align*}
\hfill (31, 32)
\]
where $\psi_p$ and $\phi_s$ are some eigenstates of the operators $K_1$ and $K_2$; the corresponding eigenvalues are

\begin{align}
\kappa_{12}(p) &= (wq^p - uq^{1-p})/(1 - q), \tag{33} \\
\kappa_{23}(s) &= (zq^s + vq^{1-s})/(1 - q) \tag{34}
\end{align}

The discrete parameters $p$ and $s$ take the values:

\begin{align}
p &= \alpha_{12} = \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 1, ..., \alpha_4 - \alpha_3, \\
s &= \alpha_{23} = \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + 1, ..., \alpha_4 - \alpha_1. \tag{35}
\end{align}

It is clear that

\begin{align}
p_{\text{max}} - p_{\text{min}} = s_{\text{max}} - s_{\text{min}} = N, \tag{36}
\end{align}

where

\begin{align}
N &= \alpha_4 - \alpha_1 - \alpha_2 - \alpha_3 = 0, 1, 2, ...
\end{align} \tag{37}

so $N + 1$ is the dimension of the space where the operators $K_1$ and $K_2$ act, in other words, $N + 1$ is the dimension of $\text{AW}(3)$ representation (strictly speaking, the relation (37) is valid when all the representations $D_\alpha$ are infinite-dimensional; if some of these are finite-dimensional (i.e. for $\text{su}_q(2)$) then apart from (37) there are another possibilities; we shall assume, however, that the relation (37) is fulfilled also in this case).

As was shown in [15], all finite-dimensional representations of $\text{AW}(3)$ algebra are easily obtained and have the following important properties:

(i) If $\psi_p$ are the eigenstates of the operator $K_1$

\begin{align}
K_1 \psi_p = \lambda_p \psi_p, \tag{38}
\end{align}

then the operator $K_2$ is three-diagonal in this basis:

\begin{align}
K_2 \psi_p &= a_p \psi_{p+1} + a_p \psi_{p-1} + b_p \psi_p, \tag{39}
\end{align}

where the spectrum $\lambda_p$ and the matrix coefficients of the representation $a_p, b_p$ are expressed by the formulae:

\begin{align}
\lambda_p &= C_2q^{-p} + q^p/(q - q^{-1})^2, \tag{40} \\
b_p &= (B\lambda_p + D_2)/g_p, \tag{41} \\
\Omega_p^2 &= (B\lambda_p + D_2)(B\lambda_{p-1} + D_2)/g_p^2 \\
&\quad + C_1\lambda_p\lambda_{p-1} + D_1(\lambda_p + \lambda_{p-1}) - Q, \tag{42}
\end{align}

where

\begin{align}
g_p &= \lambda_p - \lambda_{p-1}, \quad \Omega_p = \lambda_{p+1} - \lambda_{p-1} \tag{43}
\end{align}

(ii) The analogous statement is valid for the dual basis:

\begin{align}
K_2 \phi_s &= \mu_s \phi_s, \tag{44} \\
K_1 \phi_s &= c_{s+1} \phi_{s+1} + c_s \phi_{s-1} + d_s \phi_s, \tag{45}
\end{align}
where the spectrum \( \mu_s \) and the matrix elements \( c_s, d_s \) are obtained from the expressions (40), (41), (42) by the substitutions:

\[
p \to s, \lambda_p \to \mu_s, a_p \to c_s, b_p \to d_s, C_{1,2} \to C_{2,1}, D_{1,2} \to D_{2,1}.
\]

The procedure (46) expresses the symmetry property of the \( AW(3) \) algebra.

Note that the discrete variables \( p \) and \( s \) are defined up to arbitrary additive constant. However, if a \( N + 1 \)-dimensional representation of \( AW(3) \) is considered then one has

\[
p = p_1 + n, \quad s = s_1 + k, \quad n, k = 0, 1, ..., N,
\]

with the obvious conditions

\[
a_{p_1} = d_{s_1} = a_{p_1+N+1} = d_{s_1+N+1} = 0
\]

(iii) The overlaps between two eigenbases \( \psi_p \) and \( \phi_s \) are expressed in terms of the Askey-Wilson polynomials \([17]\):

\[
\langle \phi_s | \psi_p \rangle = W_k h_n 4\Phi_3 \left( q^{-n}, q^{-k}, \beta q^{n-N}, \gamma \delta q^{k+1} \left| q^{-N}, \beta \delta q, \gamma q \right. \right),
\]

where \( W_k = \langle \phi_s | \psi_p \rangle \) is the “vacuum amplitude”, \( h_n \) is some normalization factor and \( 4\Phi_3 \) are the Askey-Wilson polynomials (AWP) expressed in terms of basic hypergeometric function \([17]\); the parameters \( \beta, \gamma, \delta \) of the AWP are expressed via the representations parameters \( B, C_{1,2}, D_{1,2}, Q \) of \( AW(3) \) (for details see \([13]\)).

The formula (49) (found in \([13]\)) provides the solution of the Racah problem because for the realization (22), (23) the overlaps \( \langle \phi_s | \psi_p \rangle \) coincide with the Racah coefficients. Omitting the details of calculation, we present the final result concerning the connection between the AWP and the \( sl_q(2) \) parameters:

\[
N = \alpha_4 - \alpha_3 - \alpha_2 - \alpha_1, \quad \beta = -\frac{w}{u} q^{\alpha_1 + \alpha_2 + \alpha_4 - \alpha_3 - 1},
\]
\[
\gamma = \frac{w}{v} q^{2\alpha_2 - 1}, \quad \delta = \frac{z}{w} q^{2\alpha_3 - 1},
\]
\[
n = \alpha_1 - \alpha_2, \quad k = \alpha_2 - \alpha_3 - \alpha_1
\]

(50)

It may be verified that for the \( su_q(2) \) and \( su_q(1,1) \) algebras the expressions (49) for the Racah coefficients coincide with those obtained in \([1, 4, 5, 6]\). However, the formula (49) contains much more information allowing one to obtain all classes of Askey-Wilson polynomials as the Racah coefficients corresponding to addition different real forms of \( sl_q(2) \).

Indeed, the Racah coefficients for \( su_q(2) \) and \( su_q(1,1) \) algebras correspond to the Askey-Wilson polynomials with cosh-like spectra \( \lambda_p \) and \( \mu_s \), because in these cases \( uw < 0 \) and \( vz > 0 \) (see (40) and (27)). The polynomials with other types of spectra can be obtained only if the algebras with different structures are added. Consider, for example, the case \( u = -v = w = z > 0 \). This leads to somewhat unusual (but fully justified in our approach!) addition

\[
su_q(1,1) \oplus cu_q(2) \oplus su_q(2) = cu_q(2),
\]

(51)

involving three non-degenerate distinct types of \( sl_q(2) \). Such an addition rule has never been considered in the literature (note that the addition (51) can not be obtained from the ordinary \( su_q(2) \)
or $su_q(1, 1)$ additions by renormalization of the generators with real parameters. Corresponding intermediate Casimir operators have both sinh-like spectrum. In the "classical" limit $q \to 1$ the Racah coefficients for the addition (51) are expressed in terms of ordinary Krawtchouk polynomials, whereas the Racah coefficients for $su_q(2)$ or $su_q(1, 1)$ algebras become Racah polynomials.

It is clear that by choosing the other possible values for $u, v, w, z$, we exhaust all the possible types of the Askey-Wilson polynomials in discrete variables (in accordance with classification scheme of Ref. [18]). Indeed, there are 9 types of the polynomials corresponding to possible types of the spectra $\lambda_p$ and $\mu_s$: cosh-, sinh- or exp-. These types correspond to 9 ways of combining the $(u, w)_{12}$ algebra with the $(v, z)_{23}$ one (with additional requirement that only the representations $D_{\alpha_i}^+$ are admitted).

4 Generalized Clebsch-Gordan problem and its hidden symmetry

In the previous Section we established the hidden symmetry algebra $AW(3)$ underlying the Racah problem for $sl_q(2)$ algebra. In this Section we show how one can obtain a new (generalized) Clebsch-Gordan problem and the corresponding Clebsch-Gordan coefficients (GCGC) from the Racah scheme by means of some simple contraction procedure.

Consider again the Racah scheme (22), (23) and suppose that $u \geq 0$ (if $u = 0$ we suppose in addition that $v \geq 0$). This condition means that the algebra $(u, v)_1$ has the representation $D_{\alpha_1}^+$, so the spectrum $n + \alpha_1$ is unbounded ($n = 0, 1, ..., \infty$). Consider the limit $n \to \infty$. It is easily verified that the operators $\Xi_+ = \exp(-\omega A_0^{(1)})A_+^{(1)}$ and $\Xi_- = A_-^{(1)}\exp(-\omega A_0^{(1)})$ commute with one another in this limit and hence can be replaced by constants:

$$\Xi_+ \to \xi, \quad \Xi_- \to \xi^*, \quad (52)$$

where

$$|\xi|^2 = uq/(1 - q) \quad (53)$$

(note that other physical implications of the degeneration (52) of the $sl_q(2)$ algebra were studied in [19]).

The value $\kappa_1$ of the Casimir operator $\hat{\kappa}_1$ remains a real constant (3) in this limit.

Then the algebra $(u, v)_1$ is completely degenerate and the operator $K_1$ (22) becomes in this limit

$$K_1 = \xi A_+^{(2)}\exp(\omega A_0^{(3)}) + \xi^* \exp(\omega A_0^{(3)})A_+^{(2)} + \kappa_1 \exp(2\omega A_0^{(2)}). \quad (54)$$

The operator $K_2$ (23) remains obviously unchanged, and the full Casimir operator (25) becomes

$$\hat{\kappa}_4 = K_1 \exp(2\omega A_0^{(3)}) + \xi A_-^{(3)}\exp(\omega A_0^{(3)}) + \xi^* \exp(\omega A_0^{(3)})A_-^{(3)}$$

$$= \xi A_-^{(3)}\exp(\omega A_0^{(23)}) + \xi^* \exp(\omega A_0^{(23)})A_-^{(3)} + \kappa_1 \exp(2\omega A_0^{(23)}) \quad (55)$$

Thus we get a generalized Clebsch-Gordan problem for two $sl_q(2)$ algebras $(i = 2, 3)$, because diagonalization of the operator $K_1$ (54) corresponds to the choosing of some “twisted” eigenbasis $\psi_p$ for the representation space of the algebra $(-v, w)_{12}$, whereas diagonalization of the operator $K_2$ corresponds to choosing the connected basis $\phi_s$ on the representation space of the algebra $(u, w)_{12}$. Then the operator $\hat{\kappa}_4$ (23) is exact analogue of the operator $K_1$ for the algebra $(u, w)_{12}$ and plays...
the role of the projection of the total momentum (in terminology of the ordinary Clebsch-Gordan problem), obviously commuting with $K_1$ and $K_2$.

It is clear, by the construction, that $AW(3)$ remains to be a hidden symmetry algebra underlying the generalized Clebsch-Gordan problem for $sl_q(2)$.

The corresponding GCGC are obtained from the Racah coefficients by using the procedure (52), (53): we again get the expression in terms of Askey-Wilson polynomials (13) where the parameters are given by the expressions (50). By the way, we automatically obtain the explicit expression for the spectrum (13) of the operator $K_1$ (54).

The "standard" Clebsch-Gordan problem for $sl_q(2)$ is obtained if we put $u = 0$. Then $\xi = 0$ and the operator $K_1$ (54) is reduced to one term

$$K_1 = \kappa_1 \exp(2\omega A_0^{(2)}),$$

(56)

It is clear that the diagonalization of the operator (56) corresponds to choosing the canonical basis $|n;\alpha_2\rangle$ (3). Then the overlaps between the eigenstates of the operators $K_1$ and $K_2$ are nothing else than ordinary Clebsch-Gordan coefficients for adding two different types of $sl_q(2)$ algebra. In this case one of the AWP parameters tends to infinity: $\beta \to \infty$ (50) and the basic function $4\Phi_3$ is reduced to $3\Phi_2$ (as is easily seen from (13)) and we get expression of CGC in terms of the Hahn q-polynomials. For the $su_q(2)$ and $su_q(1,1)$ algebras this result is well known [1, 2, 3, 5, 6]. However for adding of different types of $sl_q(2)$ this result is new (in another form it was presented in Ref. [10]).

What is the "classical" ($q = 1$) analogue of the GCGC? It is seen from (54) that for $q \to 1$ the operator $K_1$ becomes (up to a constant term) the simple linear combination of the Lie algebra's generators:

$$K_1 = \xi a^{(2)}_- + \xi^* a^{(2)}_+ + \eta a^{(2)}_0 + const,$$

(57)

where $\eta = \lim_{q \to 1} (-2\omega \kappa_1)$ and $a^{(2)}$ are the generators of the corresponding Lie algebra.

The diagonalization of the operator $K_1$ (57) corresponds to choosing of "rotated" basis $\psi_p$ in the space of $D^+_\alpha$, i.e. $\psi_p = U|p;\alpha_2\rangle$, where $U$ is a unitary automorphism of corresponding Lie group (say, rotation for $su(2)$). Then in the "classical" limit the GCGC coincide with ordinary CGC because unitary automorphisms of Lie algebra can not destroy the CGC. However, for $q \neq 1$ the operator $K_1$ (54) can not be obtained from (56) by means of any unitary transformation because these operators have essentially different spectra. So in a q-domain GCGC do not coincide with standard CGC and the Clebsch-Gordan problem for $sl_q(2)$ becomes non-trivial and essentially depends on the choice of appropriate basis. Perhaps, that is why the GCGC were not found earlier. Note, that "twisted" GCGC for $su_q(2)$ algebra were calculated (by means of $AW(3)$ symmetry) in Ref. [20].

It is worth mentioning that the contraction procedure (52) takes place only in q-domain and has no classical analogue. So for the q-case the transition from Racah problem to generalized Clebsch-Gordan one is a much simpler procedure than in the "classical" case (it is instructive to compare (52) with limiting procedure allowing to get CGC from Racah ones for ordinary $su(2)$ [21] and for $su_q(2)$ [4]; that procedure is quite different from the considered here). We plan to discuss this strange phenomenon elsewhere.
5 Conclusion

We have shown that the same $AW(3)$ algebra serves as a hidden symmetry underlying both the Racah and generalized Clebsch-Gordan problems. The $AW(3)$ algebra provides very simple way to obtain explicit expressions for the corresponding coefficients in terms of the Askey-Wilson polynomials. Most properties of these coefficients (symmetry, generating functions, recurrent relations) can be automatically derived from representations of $AW(3)$ (see, e.g. Ref. [8] where similar analysis was applied to explain the symmetry properties of $6j$-symbols of the ordinary $su(2)$ algebra).

In this paper we restricted ourselves to the representations of discrete series $D^+_α$. However the formulae (26), (27) describing the realization of $AW(3)$ algebra are equally valid for all the possible representation series. So one can obtain the expressions for Racah and Clebsch-Gordan coefficients connecting the representations of different series (negative discrete $D^-_α$, principal $C$ etc.): then we obtain the Askey-Wilson polynomials with continuous or mixed spectrum. This will be considered separately.

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