Stability of a vacuum non-singular black hole

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Abstract

This is the first of series of papers in which we investigate stability of the spherically symmetric spacetime with de Sitter centre. Geometry, asymptotically Schwarzschild for large $r$ and asymptotically de Sitter as $r \to 0$, describes a vacuum non-singular black hole for $m \geq m_{cr}$ and particle-like self-gravitating structure for $m < m_{cr}$ where a critical value $m_{cr}$ depends on the scale of the symmetry restoration to the de Sitter group in the origin. In this paper, we address the question of stability of a vacuum non-singular black hole with de Sitter centre to external perturbations. We specify first two types of geometries with and without changes of topology. Then we derive the general equations governing polar perturbations, specify criteria of stability for a regular black hole with de Sitter centre, and study in detail the case of the density profile $\rho(r) = \rho_0 e^{-r^3/\kappa r_0^2}$ where $\rho_0$ is the density of de Sitter vacuum at the centre, $r_0 = \sqrt{3/\kappa \rho_0}$ is the de Sitter radius and $r_g$ is the Schwarzschild radius.

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1. Introduction

The idea of replacing a Schwarzschild singularity with de Sitter vacuum goes back to 1965 papers by Sakharov who considered $p = -\rho$ as the equation of state for superhigh density [1] and by Gliner who interpreted $p = -\rho$ as corresponding to a vacuum and suggested that it could be a final state in a gravitational collapse [2].

In 1968, Bardeen presented the spherically symmetric metric of the same form as the Schwarzschild and Reissner–Nordström metric, describing a non-singular black hole (BH) without specifying the behaviour at the centre [3]. The very important point was noted in [3] for the first time: that the considered spacetime exhibits smooth changes in topology of space-like hypersurfaces.
Direct matching of the Schwarzschild metric to the de Sitter metric within a short transitional space-like layer of the Planckian depth [4–8] has resulted in metrics typically with a jump at the junction surface.

The situation with transition to de Sitter as \( r \to 0 \) was analysed in 1988 by Poisson and Israel who found it necessary to introduce a transitional layer of ‘non-inflationary material’ of uncertain depth at the characteristic scale \( (r_0^2 r_g)^{1/3} \) (\( r_0 \) is the de Sitter radius, and \( r_g \) is the Schwarzschild radius), where geometry can be self-regulatory and describable semiclassically down to a few Planckian radii by the Einstein equations with a source term representing vacuum polarization effects [9].

Generic properties of ‘non-inflationary material’ were considered in 1990 in [10]. For a smooth de Sitter–Schwarzschild transition a source term satisfies [10]

\[
T^t_t = T^\phi_\phi; \quad T^\theta_\theta = T^\phi_\phi (1.1)
\]

and the equation of state, following from \( T^{\mu\nu} = 0 \), is

\[
p_r = -\rho; \quad p_\perp = -\rho - \frac{r}{2} \rho'.
\]

Here \( \kappa = 8\pi G \) (we adopted \( c = 1 \) for simplicity), \( \rho(r) = T^t_t \) is the energy density, \( p_r(r) = -T^r_r \) the radial pressure and \( p_\perp(r) = -T^\theta_\theta = -T^\phi_\phi \) is the tangential pressure for anisotropic perfect fluid [11].

The stress–energy tensor with the algebraic structure (1.1) has an infinite set of comoving reference frames and is identified therefore as describing a spherically symmetric anisotropic vacuum [10], invariant under boosts in the radial direction and defined by the symmetry of its stress–energy tensor (for a review see [12–17]).

The exact analytical solution was found in 1990 for the case of the density profile [10]

\[
\rho(r) = \rho_0 e^{-r^2/2r_0^2}; \quad r_0^2 = 3/\kappa \rho_0; \quad r_g = 2Gm
\]

(1.3)

which models gravitational vacuum polarization in a simple semiclassical model for vacuum polarization [18] applied to the case of the spherically symmetric gravitational field [19]. Existence of a de Sitter centre follows from the algebraic structure of the source term (1.1) in the case when it is regular and satisfies the weak energy condition [20].

In 1991, Morgan considered a black hole in a simple model for quantum gravity with quantum effects represented by an upper cut-off on the curvature, and obtained de Sitter-like past and future cores replacing singularities [21]. In 1992 Strominger demonstrated the possibility of natural, not \textit{ad hoc}, arising of a de Sitter core inside a black hole in the model of two-dimensional dilaton gravity conformally coupled to \( N \) scalar fields [22].

In 1996, it was shown that in the course of Hawking evaporation a vacuum non-singular black hole evolves towards a self-gravitating particle-like vacuum structure without horizons [19], a kind of gravitational vacuum soliton called a G-lump [20]. The form of temperature–mass diagram is generic for a de Sitter–Schwarzschild black hole [19] and dictated by the Schwarzschild asymptotic and by the existence of two horizons—when decreasing during evaporation mass reaches a certain critical value \( m_{cr} \) and evaporation stops [19, 23].

In 1997 in [24] it was shown that in a large class of spacetimes that satisfy the weak energy condition (WEC), the existence of a regular black hole requires a topology change. The Bardeen metric and the metric generated by the density profile (1.3) belong to this class.

In 2000, studying the quantum gravitational effects by the effective average action with the running Newton constant, and improving the Schwarzschild black hole with renormalization group, Bonnano and Reuter [25] constructed a non-singular black-hole metric and confirmed the results of [19] concerning the form of the temperature–mass diagram and the fundamental
fact that evaporation stops when the mass approaches the critical value \( m_{cr} \). In 2002, Hawking radiation from a non-singular black hole in two dimensions was studied by Easson [26].

The regular BH solution with a charged de Sitter core has been considered by Kao [27] by using the density profile (1.3) for distribution of a charged material.

The regular magnetic black hole and monopole solutions were found by Bronnikov [28] in nonlinear electrodynamics (NED) coupled to gravity with the stress–energy tensor of the algebraic structure (1.1).

Existence of regular electrically charged structures in nonlinear electrodynamics coupled to general relativity was proved recently in [29], where it was shown that in NED coupled to GR and satisfying WEC, regular charged structures must have a de Sitter centre.

The idea of a universe inside a black hole was proposed by Farhi and Guth in 1987 [6] in the context of direct de Sitter–Schwarzschild matching, it was developed by Farhi, Guth and Guven [30], by Frolov, Markov and Mukhanov [31] and by Easson and Brandenberger [32]. Quantum birth of causally disconnected baby universes inside a non-singular black hole due to quantum instability of the de Sitter vacuum in its interior, was considered in [33].

In 2001, the non-singular quasi-black-hole model representing a compact object without horizons was constructed by Mazur and Mottola [34] by extending the concept of Bose–Einstein condensation to gravitational systems. An interior de Sitter condensate phase is matched to an exterior Schwarzschild geometry of arbitrary mass through a phase boundary of small but finite thickness with equation of state \( p = \rho \).

In 2002, a non-singular BH solution was found by Nashed [35] as a general solution of Möller tetrad theory of gravitation by assuming the same specific form of the vacuum stress–energy tensor as in [10] with the density profile (1.3). Later it was extended to the case of teleparallel theory of gravitation [36]. The stability condition of geodesic motion in the field of a vacuum non-singular black hole described by the regular analytic solution [10] with the density profile (1.3) was considered in [37].

Let us note that a vacuum black hole with de Sitter core is only one of the configurations described by a spherically symmetric spacetime with de Sitter centre. Existence of such spacetimes follows from imposing in GR equations certain general requirements on a source term (specified below for two cases which differ in the absence or presence of smooth changes in topology of space-like hypersurfaces). All these geometries describe anisotropic vacuum defined by symmetry of its stress–energy tensor which approaches the Einstein cosmological term as its asymptotic limit(s) [38, 20, 39]. This point relates them to the problem of dark energy which dominates our universe [40] and is specified by the equation of state \( p = -w \rho \) with the best fit \( w = 1 \) [41] corresponding to cosmological constant \( \lambda = \kappa \rho_{vac} \) (constant by the contracted Bianchi identities). This suggests a need in time-dependent and spatially inhomogeneous version of cosmological vacuum energy, and this role can be played by anisotropic vacuum with de Sitter centre [38, 39, 17]. The Einstein cosmological term \( \Lambda g_{\mu\nu} = \kappa \rho_{vac} g_{\mu\nu} \) is associated with a vacuum stress–energy tensor of maximal symmetry. The variable cosmological term \( \Lambda_{\mu\nu} \) [38] is introduced by reducing its symmetry in the frame of possibilities given by the Petrov classification for stress–energy tensors [42]. The full symmetry remains asymptotically; in between it is reduced to the Lorentz boosts in a certain space direction.

The vacuum behaviour of \( \Lambda_{\mu\nu} \) follows from its symmetry and variability follows from the contracted Bianchi identities [38]. Dependent on parameters and choice of coordinate frame, it describes cosmological models with smoothly evolving vacuum density [38, 33, 39, 16], and localized objects with de Sitter vacuum trapped inside: non-singular black and white holes, and self-gravitating particle-like structures (for a recent review [17]). In particular, in the nonlinear electrodynamics coupled to general relativity, the weak energy condition leads
to the existence of regular electrically charged structures, both black hole and electrovacuum soliton [29]. In the asymptotically flat case the mass of objects with de Sitter centre is related to smooth breaking of spacetime symmetry from the de Sitter group at the origin to the Poincaré group at infinity, and the standard formula for the ADM mass relates it to de Sitter vacuum trapped inside an object [20]. This has been tested by evaluating the gravito-electroweak unification scale from the measured mass-squared differences for neutrinos which predicted TeV unification scale [43].

Model-independent analysis of the Einstein spherically symmetric minimally coupled equations has shown [20] which geometry they can describe if certain general requirements are satisfied: (a) regularity of density; (b) finiteness of the ADM mass; (c) dominant energy condition (DEC) for $T_{\mu\nu}$. These conditions lead to the existence of regular structures with de Sitter centre including regular black holes without topological changes4.

The condition (c) can be lost to (c2): weak energy condition for $T_{\mu\nu}$ and regularity of pressures [44]. WEC, which is contained in DEC, in both cases is needed for the de Sitter asymptotic at the centre.

The requirements (a)–(c), or (a)–(c2), define the family of asymptotically flat solutions with regular centre which includes the class of metrics asymptotically de Sitter as $r \rightarrow 0$. The class of metrics with de Sitter centre includes two subclasses with and without topological changes. Cases (c)–(c2) differ by behaviour of the 4-curvature scalar $R(r)$ and 3-curvature scalar $P(r)$. In the case (c) both are non-negative which evidences the existence of regular black holes without topological changes. In the case (c2) both scalars can change sign somewhere, and spacetime can experience smooth changes in topology of space-like hypersurfaces.

In this paper, we specify the conditions of existence of two types of geometries with the de Sitter centre and derive equations describing stability of localized vacuum structures with de Sitter centre to external polar perturbations. We apply the approach of directly studying perturbations in the metric coefficients using Einstein equations linearized about the unperturbed spacetime [45].

The perturbation analysis is given in the general case of an arbitrary density profile and equations governing polar perturbations are obtained for any regular spherically symmetric geometry specified by (1.1). We find criteria of stability of a regular black hole with de Sitter centre, and study in detail the case of the density profile $\rho(r) = \rho_0 e^{-r^3/r_s^3}$. Results are valid for geometries of both types. The particular model (1.3) is considered as a particular case.

A black hole with de Sitter centre differs from a Schwarzschild black hole by the presence of the internal Cauchy horizons. The global structure of spacetime is similar to that for the Reissner–Nordström black hole but with the essential difference that the surface $r = 0$ is the time-like regular surface. Cauchy horizons in the Reissner–Nordström and Kerr spacetimes are unstable [46–48]. Small perturbations grow infinitely on approaching the Cauchy horizon; radiation falling into a black hole from the external region is infinitely blue-shifted there, so that an observer crossing the Cauchy horizon would have to experience the impact of an infinite flux of radiation even if perturbations crossing the event horizon are of compact support [45]. On the other hand, in the case of the regular background metric, the back reaction of matter on the metric can prevent the formation of a singularity in place of the Cauchy horizon [8], e.g. by self-regulatory effect of vacuum polarization [9]. In any case the method of small perturbations is not applicable in the neighbourhood of the Cauchy horizon. The question of its stability will be addressed in one of our next papers on stability of a vacuum black hole with de Sitter centre.

4 An example of such a case is the exact analytic solution [29] describing in a certain mass range a regular charged black hole with de Sitter centre.
2. Spherically symmetric spacetime with de Sitter centre

A static spherically symmetric line element can be written in the form [11]
\[ ds^2 = e^{\nu(r)} dr^2 - e^{\nu(r)} dr^2 - r^2 d\Omega^2, \]
where \( d\Omega^2 \) is the metric of a unit 2-sphere. Integration of the Einstein equations gives
\[ e^{-\nu(r)} = 1 - \frac{2GM(r)}{r}; \quad M(r) = 4\pi \int_0^r \rho(x)x^2 dx \]
whose asymptotic for large \( r \) is \( e^{-\nu} = 1 - 2Gm/r \), with the mass parameter
\[ m = 4\pi \int_0^\infty \rho(r)r^2 dr. \]

Requirement of regularity of density, \( \rho_0 = \rho(r \to 0) < \infty \), leads to the behaviour of mass function \( M(r) \sim r^3 \) as \( r \to 0 \) and thus \( \nu(0) = 0 \).

To outline the conditions of existence of spherically symmetric spacetime with de Sitter centre, we need the Oppenheimer equation [49]
\[ T'_r - T'_r = p_r + \rho = \frac{1}{\kappa} e^{-\nu}(v' + \mu'). \]

The dominant energy condition \( T'^{ab} \geq |T^{ab}| \) for each \( a, b = 1, 2, 3 \), holds if and only if [50] \( \rho \geq 0; \rho + p_k \geq 0; \rho - p_k \geq 0; k = 1, 2, 3 \). It includes the weak energy condition which implies \( \rho \geq 0; \rho + p_k \geq 0 \). Together with the condition of regularity of density, DEC (via \( p_k \leq \rho \)) leads to \( \mu' + v' = 0 \) as \( r \to 0 \) [20].

The same result can be achieved by the requirement of regularity of pressure (the subclass satisfying (a)–(c2)).

In the limit \( r \to \infty \), the condition of finiteness of the mass (2.3) requires density profile \( \rho(r) \) to vanish at infinity quicker than \( r^{-3} \). In the case (c) the dominant energy condition requires pressures to vanish as \( r \to \infty \). Then \( \mu' = 0 \) and \( \mu = \text{const} \) at infinity. Rescaling the time coordinate allows one to put the standard boundary condition \( \mu \to 0 \) as \( r \to \infty \) which ensures the asymptotic flatness needed to identify (2.3) as the ADM mass [51].

The same result can be achieved in the case (c2) by postulating regularity of pressures including vanishing of \( p_r \) at infinity sufficient to get \( \mu' = 0 \) needed for asymptotic flatness.

The weak energy condition requires \( \mu' + v' \geq 0 \). The function \( \mu + v \) is growing from \( \mu = \mu(0) \) at \( r = 0 \) to \( \mu = 0 \) at \( r \to \infty \), which gives \( \mu(0) \leq 0 \) (in the region between horizons, the radial coordinate \( r \) is timelike and \( T'_r \) represents a tension, \( p_r = -T'_r \), along the axes of the spacelike 3-cylinders of constant time \( r \), then \( T'_r - T'_{\mu} = -(p_r + \rho) \), and WEC demands \( \mu' + v' \geq 0 \) also in the \( T \)-region between horizons, for more detail see [20]).

The range of family parameter \( \mu(0) \) includes \( \mu(0) = 0 \). In this case the function \( v(r) + \mu(0) \) is zero at \( r = 0 \) and at \( r \to \infty \), its derivative is non-negative (by WEC via \( \mu' + v' \geq 0 \)); it follows that \( v(r) = -\mu(r) \) everywhere.
A source term for this class of metrics corresponds to anisotropic perfect fluid which satisfies the r-dependent equation of state (1.2), and the weak energy condition \( p_\perp + \rho \geq 0 \) demands monotonic decreasing of a density profile, \( \rho' \leq 0 \) [20].

Behaviour at \( r \to 0 \) is dictated by the WEC [20, 29]. The equation of state near the centre becomes \( p = -\rho \), which gives de Sitter asymptotic as \( r \to 0 \)

\[
\text{ds}^2 = \left( 1 - \frac{r^2}{r_0^2} \right) \, dt^2 - \frac{dr^2}{\left( 1 - \frac{r^2}{r_0^2} \right)} - r^2 \, d\Omega^2 \tag{2.5}
\]

\[
T_{\mu\nu} = \rho_0 g_{\mu\nu}; \quad r_0^2 = \frac{3}{\Lambda}; \quad \Lambda = \kappa \rho_0. \tag{2.6}
\]

where \( \rho_0 = \rho \left( r \to 0 \right) \) and \( \Lambda \) is the cosmological constant which appeared at the origin although it was not present in the basic equations.

Requirements (a)–(c) (or (a)–(c2)) thus lead to the existence of the class of metrics

\[
\text{ds}^2 = g(r) \, dt^2 - \frac{dr^2}{g(r)} - r^2 \, d\Omega^2 \tag{2.7}
\]

\[
g(r) = 1 - \frac{R_g(r)}{r}; \quad R_g(r) = 2G M(r); \tag{2.8}
\]

\[
M(r) = 4\pi \int_0^r \rho(x) x^2 \, dx \tag{2.9}
\]

which are asymptotically de Sitter as \( r \to 0 \), and asymptotically Schwarzschild at large \( r \)

\[
\text{ds}^2 = \left( 1 - \frac{r_s}{r} \right) - \frac{dr^2}{\left( 1 - \frac{r}{r_s} \right)} - r^2 \, d\Omega^2; \quad r_s = 2Gm. \tag{2.10}
\]

The weak energy condition defines the form of the metric function \( g(r) \). In the region \( r > 0 \) it has only a minimum and the geometry can have no more than two horizons: a black-hole horizon \( r_+ \) and an internal horizon \( r_- \) [20].

The scalar curvature \( R(r) \), proportional to the trace of stress–energy tensor \( T \), is proportional to \( \rho - p_\perp \) for geometries satisfying (1.2), and the 3-curvature scalar is given by \( \mathcal{P}(r) = \kappa \left( (\rho - p_\perp) + \rho \right) \). Therefore conditions (a)–(c) and (a)–(c2) distinguish two types of geometries. In the case (a)–(c) satisfying the DEC requirement, both scalar curvatures remain non-negative, since DEC requires \( \rho - p_\perp \geq 0 \). The subclass satisfying (a)–(c) does not exhibit changes of topology by virtue of DEC and can be specified as a DEC subclass. The dominant energy condition requires that each principal pressure does not exceed the density which guarantees that speed of sound cannot exceed speed of light. In nonlinear electrodynamics coupled to gravity, photons do not follow null geodesics of background geometry but propagate along null geodesics of an effective geometry, and propagation of photons resembles propagation inside a dielectric medium [52]. In the case of the regular NED structure satisfying DEC [29], it allows one to avoid problems with the speed of sound exceeding the speed of light.

In the case (a)–(c2) 3-scalar curvature \( \mathcal{P}(r) \) changes sign somewhere and geometry experiences smooth changes in topology of space-like hypersurfaces. This subclass satisfying only weak energy condition (needed in both cases for de Sitter behaviour at the centre) can be specified as a WEC subclass.

The case of the density profile (1.3) belongs to the WEC subclass satisfying (c2). The metric function and the mass function are given by [10]

\[
g(r) = 1 - \frac{r_s}{r} \left( 1 - e^{-r/r_s^0} \right); \quad M(r) = m \left( 1 - e^{-r/r_s^0} \right). \tag{2.11}
\]
The dominant energy condition is not satisfied so that a surface of zero scalar curvature exists at which \( R(r) = 0 \). Zero curvature surface \( r = r_s \) is shown in figure 1 together with two horizons (a black-hole event horizon \( r_+ \) and an internal horizon \( r_- \)), and the characteristic surface of any geometry with de Sitter centre: a zero-gravity surface \( r = r_c \) beyond which the strong energy condition of singularity theorems [50] is violated (the zero-gravity surface is defined by \( 2\rho + r\rho' = 0 \) [19]).

Two horizons come together at the value of a mass parameter \( m_{cr} \), which puts a lower limit on a black-hole mass (see figure 2). For the case of a density profile (1.3) the critical mass is given by [19]

\[
m_{cr} \simeq 0.3 m_{Pl} \sqrt{\rho_{Pl}/\rho_0}.
\]  

(2.12)

The temperature-mass diagram is shown in figure 3. Its form does not depend on a particular choice of density profile. The temperature drops to zero at \( m = m_{cr} \), while the Schwarzschild asymptotic requires \( T_s \to 0 \) as \( m \to \infty \). As a result the temperature–mass diagram should have a maximum between \( m_{cr} \) and \( m \to \infty \) [19]. At a maximum, at \( m = m_{cr2} \), a specific heat is broken and changes sign testifying to a second-order phase transition in the course of Hawking evaporation [23].
Figure 3. Temperature–mass diagram for a vacuum non-singular black hole with de Sitter centre.

For \( m \geq m_{cr} \), de Sitter–Schwarzschild geometry describes the vacuum non-singular black hole, and the global structure of spacetime shown in figure 4 [19] contains an infinite sequence of black and white holes whose future and past singularities are replaced with regular cores asymptotically de Sitter as \( r \to 0 \) [19].

3. Basic equations for perturbations

Our task is to investigate stability of the localized spherically symmetric vacuum structures with de Sitter centre to external perturbations. The source term for this case is identified as anisotropic vacuum specified by symmetry of its stress–energy tensor (1.1) invariant under radial boosts [10, 38, 20].

The class of metrics with de Sitter centre and a source term of the algebraic structure (1.1) is extended to the case of non-zero cosmological constant \( \Lambda \) at infinity [53]. The variable cosmological term [38, 20]

\[
\Lambda_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.1}
\]

corresponding to anisotropic vacuum (1.1) connects smoothly two de Sitter vacua with different values of cosmological constant. In this approach a constant scalar \( \Lambda \) related to a vacuum density, \( \Lambda = \kappa \rho_{vac} \), becomes a tensor component \( \Lambda'_{\mu} \) associated explicitly with a density component of a perfect fluid tensor whose vacuum properties follow from its symmetry (1.1) and whose variability follows from the contracted Bianchi identities [38, 20].

Since an anisotropic fluid with the stress–energy tensor of type (1.1) admits identification as a vacuum-like medium associated with a time-evolving and spatially inhomogeneous cosmological term we can write the Einstein equations in the form [38, 20, 39],

\[
G_{\mu\nu} + \Lambda_{\mu\nu} = 0 \tag{3.2}
\]

(for a discussion of where to put the cosmological term, see [20]). Then the quantities \( \rho, p_k \) are treated as corresponding (in a one-to-one way) components of the variable cosmological term \( \Lambda'_{\mu} = \kappa \rho, \Lambda'_{k} = -\kappa p_k, k = 1, 2, 3 \) [38].

Here we investigate stability for the case when \( \lambda = 0 \) and spherically symmetric spacetime with de Sitter centre describes localized structures in the asymptotically flat space.

Since we apply the approach of studying direct perturbations in the metric coefficients via Einstein equations, we consider the behaviour of small perturbations for both the Einstein tensor \( G_{\mu\nu} \) and the components of a stress–energy tensor associated with \( \Lambda_{\mu\nu} \). In the Petrov
classification, its algebraic structure is \[(II) (II)\] for the background geometry. In the applied approach stability to external perturbations is studied in the manner [45] in which gravitational waves incident from infinity\(^5\) are scattered and absorbed by localized objects static for an outside observer (an object is stable if no growing modes develop in the process of scattering). Therefore we admit for perturbations of \(\Lambda_{\mu\nu}\) only those which preserve its vacuum identity, i.e. \([(II) II]\) which corresponds to

\[
\begin{align*}
T_{tt} &= e^{2\nu} p_t; \\
T_{rr} &= e^{2\nu} p_r; \\
T_{\theta\theta} &= e^{2\omega} p_\theta; \\
T_{\phi\phi} &= e^{2\psi} p_\phi,
\end{align*}
\]

\[(3.3)\]

where \(p_r, p_\theta, p_\phi\) are the principal pressures.

\(^5\) For this reason we consider only modes with \(l \geq 2\) [45].
The perturbations of a spherically symmetric system are in essence time-dependent axially symmetric modes (the reason is the absence of preferred axes in a spherically symmetric background) which are described by the line element [45]

$$ds^2 = e^{2g} dr^2 - e^{2\psi}(d\phi - \omega dt - q_2 dr - q_3 d\theta)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2. \tag{3.4}$$

The metric functions $v$, $\psi$, $\mu_2$, $\mu_3$, $\omega$, $q_2$, $q_3$ are functions of $t$, $r$, $\theta$.

We obtain the perturbation equations by linearizing the Einstein equations around the spherically symmetric solution with de Sitter centre. This solution considered as a special case of the line element (3.4) with

$$\mu_2 = -v(r); \quad \psi = \ln(r \sin(\theta)); \quad \mu_3 = \ln(r), \tag{3.5}$$

has the form (2.7) with

$$g(r) = e^{2v(r)} = 1 + \frac{C_1}{r} - \frac{\kappa}{r} \int \rho(r) r^2 dr. \tag{3.6}$$

The particular solution (3.6) is specified by such a choice of the constant $C_1$ which gives unperturbed metric (2.8).

A general perturbation of a background geometry will result in $\omega$, $q_2$, $q_3$ becoming small quantities of the first order and the functions $v$, $\mu_2$, $\mu_3$, $\psi$ and $\rho$, $p_k$ experiencing small increments $\delta v$, $\delta \mu_2$, $\delta \mu_3$, $\delta \psi$ and $\delta \rho$, $\delta p_k$.

The perturbations leading to non-vanishing values of $\omega$, $q_2$ and $q_3$ involve rotation, are called axial perturbations [45] and we consider them in a separate paper. Perturbations which do not involve rotation are called polar perturbations [45]. In the considered case they lead to increments in $v$, $\mu_2$, $\mu_3$, $\psi$ and $\rho$, $p_k$. The equations for the axial and polar perturbations decouple.

The Einstein equations for the polar perturbations are

$$-R_{rr} = (\psi + \mu_3)_{,rr} + \psi_{,r}(\psi - \mu_2)_{,r} + \mu_3\mu_2(\mu - \mu_2)_{,r} - v_{,r}(\psi + \mu_3)_{,r} = 0, \tag{3.7}$$

$$-R_{t\theta} = (\psi + \mu_2)_{,\theta t} + \psi_{,\theta}(\psi - \mu_3)_{,\theta} + \mu_2\mu_3(\mu - \mu_3)_{,\theta} - v_{,\theta}(\psi + \mu_2)_{,\theta} = 0, \tag{3.8}$$

$$-R_{\theta \theta} = (\psi + \mu_3)_{,\theta\theta} + \psi_{,\theta}(\psi - \mu_2)_{,\theta} + v_{,\theta}(\psi - \mu_2)_{,\theta} - \mu_3\psi_{,\theta} = 0, \tag{3.9}$$

$$e^{-2\mu_2}[(\psi + \mu_3)_{,rr} + \psi_{,r}(\psi - \mu_2)_{,r} + \mu_3\mu_2(\mu - \mu_2)_{,r}] + e^{-2\mu_1}[\psi_{,\theta}(\psi + \mu_2)_{,\theta} + \mu_2\mu_3(\mu - \mu_3)_{,\theta}] - e^{-2\psi}[\psi_{,r}(\psi + \mu_3)_{,r} + \psi_{,\theta}(\psi + \mu_2)_{,\theta}] = -\kappa\rho, \tag{3.10}$$

$$e^{-2\mu_2}[v_{,r}(v - \mu_2)_{,r} + \psi_{,r}(v - \mu_2)_{,r} + \mu_3\mu_2(\mu - \mu_2)_{,r}] + e^{-2\psi}[v_{,\theta}(v - \mu_3)_{,\theta} + \mu_2\mu_3(\mu - \mu_3)_{,\theta}] - e^{-2\psi}[v_{,r}(v + \mu_3)_{,r} + \psi_{,\theta}(v + \mu_3)_{,\theta}] = \kappa p_0. \tag{3.11}$$

$$e^{-2\mu_1}[\psi_{,r}(v + \mu_3)_{,r} + \psi_{,\theta}(v + \mu_3)_{,\theta}] + e^{-2\mu_2}[\psi_{,r}(v + \mu_2)_{,r} + \psi_{,\theta}(v + \mu_2)_{,\theta}] - e^{-2\psi}[\psi_{,r}(v + \mu_3)_{,r} + \psi_{,\theta}(v + \mu_3)_{,\theta}] = \kappa p_r. \tag{3.12}$$

$$e^{-2\mu_1}[\psi_{,r}(v + \mu_2)_{,r} + \psi_{,\theta}(v + \mu_2)_{,\theta}] + e^{-2\mu_2}[\psi_{,r}(v + \mu_2)_{,r} + \psi_{,\theta}(v + \mu_2)_{,\theta}] - e^{-2\psi}[\psi_{,r}(v + \mu_2)_{,r} + \psi_{,\theta}(v + \mu_2)_{,\theta}] = \kappa p_0. \tag{3.13}$$

We perturb equations (3.7)–(3.13) up to the first order, and in equations (3.10)–(3.13) we disturb both left- and right-hand sides. As a result we obtain a linear system of seven partial differential equations for the polar perturbations, and a linear system of two equations for the axial perturbations.
4. Polar perturbations

4.1. General equations

Linearizing (3.7)–(3.13) about the background geometry, we get the equations for the polar perturbations

\[
\begin{align*}
(\delta \psi + \delta \mu_3),_r - \left( v_r - \frac{1}{r} \right) (\delta \psi + \delta \mu_3) - \frac{2}{r} \delta \mu_2 = 0, \\
[(\delta \psi + \delta \mu_2),_\theta + \cotan(\theta)(\delta \psi - \delta \mu_3)],_r = 0, \\
[\delta \psi,\theta + \cotan(\theta)(\delta \psi - \delta \mu_3)],_r + \delta v,_{\theta \theta} + \left( v_r - \frac{1}{r} \right) \delta v,_{\theta} - \left( v_r + \frac{1}{r} \right) \delta \mu_{2,\theta} = 0,
\end{align*}
\]

(4.1)

\[
\begin{align*}
\left[ 2 \delta \psi,_{\theta \theta} + \cotan(\theta)(2\delta \psi - \delta \mu_3),_\theta + 2\delta \mu_3 + \delta \mu_{2,\theta} + \cotan(\theta)\delta \mu_{2,\theta} \right] = \kappa \delta p_r, \\
e^{2\nu} \left[ (\delta \psi + \delta \mu_3),_{rr} + \left( v_r + \frac{1}{r} \right) (\delta \psi + \delta \mu_3),_r - 2 \left( v_{rr} + 2v_r^2 + \frac{2}{r} v_r \right) \delta \mu_2 \right] \\
- \frac{1}{r^2} [\delta \psi,_{\theta \theta} + \cotan(\theta)(2\delta \psi - \delta \mu_3),_\theta + 2\delta \mu_3 + \delta \mu_{2,\theta} + \cotan(\theta)\delta \mu_{2,\theta}] = \kappa \delta p_r,
\end{align*}
\]

(4.2)

\[
\begin{align*}
e^{2\nu} \left[ 3v_r + \frac{1}{r} \right] \delta v,_{rr} + 2 \left( v_r + \frac{1}{r} \right) \delta \mu_{3,rr} \\
- \frac{1}{r^2} (\delta \psi + \delta \mu_2),_{\theta \theta} - e^{-2\nu}(\delta \mu_2 + \delta \mu_3),_{rr} = \kappa \delta p_r,
\end{align*}
\]

(4.3)

\[
\begin{align*}
e^{2\nu} \left[ \frac{2}{r} \delta v,_{r} + \left( v_r + \frac{1}{r} \right) (\delta \psi + \delta \mu_3),_r - 2 \left( \frac{2}{r} v_r + \frac{1}{r^2} \right) \delta \mu_2 \right] \\
+ \frac{1}{r^2} [\delta \psi,_{\theta \theta} + \cotan(\theta)(2\delta \psi - \delta \mu_3),_\theta + 2\delta \mu_3 + \delta \mu_{2,\theta} + \cotan(\theta)\delta \mu_{2,\theta}] \\
e^{2\nu}(\delta \psi + \delta \mu_3),_{rr} = \kappa \delta p_r,
\end{align*}
\]

(4.4)

\[
\begin{align*}
e^{2\nu} \left[ (\delta \psi + \delta \mu_3),_{\theta \theta} + \left( v_r + \frac{1}{r} \right) (\delta \psi + \delta \mu_3),_r - 2 \left( v_{rr} + \frac{2}{r} v_r \right) \delta \mu_2 \right] \\
- \frac{1}{r^2} \cotan(\theta)(\delta \psi + \delta \mu_2),_{\theta \theta} - e^{-2\nu}(\delta \mu_2 + \delta \psi),_{rr} = \kappa \delta p_r.
\end{align*}
\]

(4.5)

Equations (4.1)–(4.7) form a system of seven linear partial differential equations of first order for eight quantities: four small perturbations of the metric tensor and four small perturbations of the stress–energy tensor (associated in the considered case with a variable cosmological term) whose unperturbed components are related by the equation of state, in our case (1.2). To investigate this system we should make an assumption concerning perturbation of \( p_r \) valid for the case of small perturbations. Since for the background geometry we have \( p_r = -\rho \), i.e. \( p_r = p_r(\rho) \), we can assume for small perturbations

\[
\delta p_r = \frac{dp_r}{d\rho} \delta \rho.
\]

(4.6)
which results in

$$\delta p_r = -\delta \rho.$$  

(4.9)

Relation (4.9) is valid only for small perturbations. If we prove that the system is stable, i.e., growing perturbation modes are absent, this will justify (4.9).

Taking into account (4.4) and (4.6), we can write equation (4.9) in the form

$$e^{2\nu} \left[-(\delta \psi + \delta \mu_3),rr + \frac{2}{r} (\delta \psi + \delta \mu_3),r + \frac{2}{r} \delta \nu,rr + \frac{2}{r} \delta \mu_2,rr \right]$$

$$+ \frac{1}{r^2} \left[ \delta v,\theta \theta + \cotan(\theta) \delta v,\theta - \delta \mu_2,\theta \theta - \cotan(\theta) \delta \mu_2,\theta \right]$$

$$- e^{-2\nu} (\delta \psi + \delta \mu_3),tt = 0.$$  

(4.10)

In this way we obtain a system of seven equations for seven unknown functions which splits into a uniform system of four linear partial differential equations (4.1), (4.2), (4.3), (4.10) for four small perturbations of the metric tensor, $\delta v(r, \theta, t)$, $\delta \mu_2(r, \theta, t)$, $\delta \mu_3(r, \theta, t)$, $\delta \psi(r, \theta, t)$; and three linear algebraic equations (4.5)–(4.7), determining $\delta p_r$, $\delta p_\phi$, $\delta p_\theta$ through expressions for metric perturbations.

The problem ultimately reduces to investigation of the uniform linear system (4.1), (4.2), (4.3), (4.10).

Following Chandrasekhar [45], we assume the time dependence $e^{i\sigma t}$ which corresponds to the Fourier analysis of perturbations. The variables $r$ and $\theta$ are separated by the Friedman substitutions [45].

As a result we present perturbations as the series

$$\delta v(r, \theta, t) = \sum_{l=2}^{+\infty} N_l(r) P_l(\cos \theta) e^{i\sigma l t},$$  

(4.11)

$$\delta \mu_2(r, \theta, t) = \sum_{l=2}^{+\infty} L_l(r) P_l(\cos \theta) e^{i\sigma l t},$$  

(4.12)

$$\delta \mu_3(r, \theta, t) = \sum_{l=2}^{+\infty} \left[ T_l(r) P_l(\cos \theta) + V_l(r) P_{l,\theta \theta}(\cos \theta) \right] e^{i\sigma l t},$$  

(4.13)

$$\delta \psi(r, \theta, t) = \sum_{l=2}^{+\infty} \left[ T_l(r) P_l(\cos \theta) + V_l(r) P_{l,\theta}(\cos \theta) \cotan \theta \right] e^{i\sigma l t},$$  

(4.14)

$$\delta \rho(r, \theta, t) = \sum_{l=2}^{+\infty} C_l(r) P_l(\cos \theta) e^{i\sigma l t},$$  

(4.15)

$$\delta p_r(r, \theta, t) = \sum_{l=2}^{+\infty} D_l(r) P_l(\cos \theta) e^{i\sigma l t},$$  

(4.16)

$$\delta p_\phi(r, \theta, t) = \sum_{l=2}^{+\infty} \left[ E_l(r) P_l(\cos \theta) + H_l(r) P_{l,\theta \theta}(\cos \theta) \cotan \theta \right] e^{i\sigma l t},$$  

(4.17)

$$\delta p_\theta(r, \theta, t) = \sum_{l=2}^{+\infty} \left[ E_l(r) P_l(\cos \theta) + H_l(r) P_{l,\theta}(\cos \theta) \cotan \theta \right] e^{i\sigma l t}.$$  

(4.18)

Now we introduce the function $X_l$ useful in further reductions

$$X_l(r) = n V_l(r),$$  

(4.19)

6 The possibility of connecting perturbations $\delta p_r$ and $\delta \rho$ is implied by our system which contains seven equations for eight functions.
where

\[ n = l(l + 1)/2 - 1; \quad l = 2, 3, \ldots; \quad n = 2, 5, 9, \ldots. \]

Using the properties of the Legendre polynomials

\[ (\sin \theta P_l, \theta) + l(l + 1) \sin \theta P_l(\cos \theta) = 0, \quad P_{l, l} + P_l \cot \theta = -l(l + 1) P_l(\cos \theta), \]

we get from equations (4.1)–(4.3), (4.10), after some algebra, the following relations between amplitudes:

\[ T_l(r) = V_l(r) - L_l(r); \]

\[ (X_l(r) + L_l(r))_r + \frac{1}{r} N_l = (\nu, r) \left( N_l(r) + 2X_l(r) + 3L_l(r) \right)_r = 0; \]

\[ e^{2\nu} \left[ (X_l(r) + L_l(r))_r, r \right] + \frac{1}{r} N_l = \frac{1}{r} X_l + \frac{2}{r} L_l \]

By using these relations we transform our starting system to the system of three differential equations in the normal form for the functions \( N_l(r), L_l(r), X_l(r) \)

\[ N_l, r = \left( n + 1 \right) a^1 N_l + \left( \nu, r + 1 \right) X_l - \left( \nu, r + 1 \right) X_l \]

\[ L_l, r = \left( \nu, r + 1 \right) X_l - \left( \nu, r + 1 \right) X_l \]

\[ X_l, r = \left( \nu, r + 1 \right) X_l - \left( \nu, r + 1 \right) X_l \]

where

\[ a^1(r) = e^{2\nu(r)}; \quad b^1(r) = -r \nu, r - r \nu, r; \quad c^1(r) = r e^{-2\nu(r)}, \]

and four equations which define amplitudes \( D_l(r), E_l(r), H_l(r) \) and \( C_l(r) \) through solutions of (4.20)
\[ H_l(r) = \frac{1}{nK} \left[ e^{2\nu} \left[ \frac{X_{l,rr}}{X_{l,r}} + 2 \left( \nu + \frac{1}{2} \right) X_{l,r} \right] + \frac{n}{r^2} (N_l + L_l) + \sigma_l^2 e^{-2\nu} X_l \right]. \] (4.24)

\[ C_l(r) = -D_l(r). \] (4.25)

Let us now introduce the dimensionless variables

\[ x = \frac{r}{r_1}; \quad \rho \rightarrow \frac{\rho}{\rho_0}; \quad \text{where} \quad r_1^3 = r_0^3 r_g \] (4.26)

and the characteristic parameter

\[ \alpha = \frac{r_g}{r_1}. \] (4.27)

In this notation the unperturbed solution (2.7) reads

\[ dx^2 = g(x) \, dr^2 - \frac{dx^2}{g(x)} - x^2 \, d\Omega^2 \] (4.28)

In terms of \( g(x) \) our basic system (4.20) takes the form

\[ N_{l,x} = \left( \frac{n + 1}{xg} \right) N_l + \left[ \frac{1}{2} g' \left( g' \right)^2 - \frac{x g''}{2g} - \frac{(n + 1)}{xg} + \frac{\sigma_l^2}{g} \right] L_l \]
\[ + \left[ \frac{x}{4} \left( g' \right)^2 - \frac{x g''}{2g} + \frac{\sigma_l^2}{g} \right] X_l, \] (4.29a)

\[ L_{l,x} = \left[ \frac{1}{2} g' - \frac{1}{x} + \frac{(n + 1)}{xg} \right] N_l + \left[ \frac{1}{2} g' - \frac{1}{x} - \frac{x}{4} \left( g' \right)^2 + \frac{x g''}{2g} + \frac{(n + 1)}{xg} + \frac{\sigma_l^2}{g} \right] L_l \]
\[ + \left[ \frac{x}{4} \left( g' \right)^2 - \frac{x g''}{2g} + \frac{\sigma_l^2}{g} \right] X_l, \] (4.29b)

\[ X_{l,x} = \left[ \frac{1}{2} g' - \frac{1}{x} - \frac{x}{4} \left( g' \right)^2 + \frac{x g''}{2g} + \frac{(n + 1)}{xg} - \frac{\sigma_l^2}{g} \right] L_l \]
\[ + \left[ \frac{1}{2} g' - \frac{1}{x} - \frac{x}{4} \left( g' \right)^2 + \frac{x g''}{2g} - \frac{\sigma_l^2}{g} \right] X_l. \] (4.29c)

This system is transformed to the equivalent form

\[ x g N_{l,x} = (n + 1) g N_l + \left( \frac{x}{2} g' - (n + 1) \right) L_l + x^2 \left( \frac{1}{4} \left( g' \right)^2 - \frac{x g''}{2g} + \frac{\sigma_l^2}{g} \right) X_l \] (4.30a)

\[ x g L_{l,x} + \left( \frac{x}{2} g' + g \right) L_l = x g N_{l,x} + \left( \frac{x}{2} g' - g \right) N_l \] (4.30b)

\[ x g X_{l,x} = -g L_l + \left( \frac{x}{2} g' - g \right) X_l, \] (4.30c)

where

\[ X_l = X_l + L_l, \] (4.31)

which can be compared with the analogous Chandrasekhar system ([45], chapter 4, equations (46), (47), (50)). Our equations (4.30b), (4.30c) coincide with Chandrasekhar equations (46), (47), while our equation (4.30a) coincides with the Chandrasekhar equation (50) if and only if \( \rho' = 0 \) which is equivalent to \( x^2 g''/2 - g + 1 = 0 \).
The basic system (4.30) can be directly applied to study the extreme black-hole case. In the next subsection we investigate first the case of a simple horizon to make clear the peculiarity of the case of the double horizon.

4.2. Extreme black-hole case

In the neighbourhood of a simple horizon \( x_+ \) we have \( g(x) = g'(x_+)(x - x_+) + \frac{1}{2}g''(x_+)(x - x_+)^2 + \cdots \). To study behaviour in the limit \( x \to x_+ + 0 \) we introduce the variable \( z = x - x_+ \).

In a small neighbourhood of \( z = 0 \) the limiting system for (4.30) reads

\[ x_+ (g'(x_+))^2 z^2 N_{l,z} = (n + 1) g'(x_+) z N_l \]
\[ + g'(x_+) z \left( \frac{x_+}{2} g'(x_+) - (n + 1) \right) L_l + x_+^2 \left( \frac{1}{4} (g'(x_+))^2 + \sigma_l^2 \right) \tilde{X}_l \]  
(4.32)

\[ z(N_l - L_l) + \frac{1}{2} (N_l - L_l) = 0 \]  
(4.33)

\[ x_+ z \tilde{X}_l,z = -z L_l + \frac{1}{2} x_+ \tilde{X}_l. \]  
(4.34)

One immediately sees from (4.33) that the solutions restricted near \( z = 0 \) should satisfy \( N_l(z) = L_l(z) \). Then (4.32) and (4.34) form a system of two first-order equations for the functions \( N_l, L_l \)

\[ (g'(x_+))^2 z^2 N_{l,z} = \frac{1}{2} (g'(x_+))^2 z N_l + x_+ (\frac{1}{2} (g'(x_+))^2 + \sigma_l^2) \tilde{X}_l, \]
(4.35)

\[ x_+ z \tilde{X}_l,z = -z N_l + \frac{1}{2} x_+ \tilde{X}_l. \]  
(4.36)

This system reduces to one second-order equation for \( \tilde{X}_l \)

\[ z^2 \tilde{X}_l,z - z \tilde{X}_l,z + \left( 1 + \frac{\sigma_l^2}{(g'(x_+))^2} \right) \tilde{X}_l = 0. \]  
(4.37)

This is the Euler equation, and solutions to (4.32)–(4.34) restricted near zero have the form

\[ \tilde{X}_l(z) = \left[ B_{lH} \cos \left( \frac{\sigma_l}{g'(x_+)} \ln z \right) + B_{lL} \sin \left( \frac{\sigma_l}{g'(x_+)} \ln z \right) \right] z, \]  
(4.38)

\[ N_l = L_l = -x_+ \left[ \left( \frac{1}{2} B_{lH} + \frac{\sigma_l}{g'(x_+)} B_{lL} \right) \cos \left( \frac{\sigma_l}{g'(x_+)} \ln z \right) \right. \]
\[ + \left. \left( \frac{1}{2} B_{lH} - \frac{\sigma_l}{g'(x_+)} B_{lL} \right) \sin \left( \frac{\sigma_l}{g'(x_+)} \ln z \right) \right], \]  
(4.39)

where \( B_{lH}, B_{lL} \) are arbitrary constants. As a result in the small neighbourhood of a simple horizon, the restricted solutions exist for all real values of \( \sigma_l \).

4.2.1. Behaviour near the double horizon. The double horizon \( x_{\pm} \) corresponds to the case \( \alpha = \alpha_c \) in (4.28). For the case of the density profile (1.3)

\[ \alpha_c \approx 1.456. \]  
(4.40)

In a small neighbourhood of the point \( x = x_{\pm} \), the metric function is written as

\[ g(x) = \gamma (x - x_{\pm})^2 + \cdots, \gamma = \frac{1}{2} g''(x_{\pm}). \]
In the variable $z = x - x_{\pm}$, in a small neighbourhood of $z = 0$ the limiting system for (4.30) reads

$$x_{\pm} \gamma^2 z^4 N_{l,z} = (n + 1) \gamma^2 (N_l - L_l) + x_{\pm}^2 \sigma_l^2 \tilde{X}_l,$$  \hspace{1cm} (4.41)

$$z (N_l - L_l)_{z} + (N_l - L_l) = 0,$$  \hspace{1cm} (4.42)

$$x_{\pm} z \tilde{X}_l_{z} = -z L_l + x_{\pm} \tilde{X}_l.$$  \hspace{1cm} (4.43)

As follows from (4.42), for a restricted solution it should be $N_l = L_l$. Then the equations (4.41) and (4.43) form a system of two first-order equations for $N_l, L_l$:

$$z^4 N_{l,z} = x_{\pm} \sigma_l^2 \tilde{X}_l,$$  \hspace{1cm} (4.44)

$$z \tilde{X}_l_{z} = -\frac{z}{x_{\pm}} N_l + \tilde{X}_l.$$  \hspace{1cm} (4.45)

This system reduces to one second-order equation for $N_l$

$$z^4 N_{l,zz} + 3 z^3 N_{l,z} + \frac{\sigma_l^2}{\gamma^2} N_l = 0$$  \hspace{1cm} (4.46)

which differs essentially from the analogous equation (4.37) for a simple horizon case. The general solution to (4.46) is given by [55]

$$N_l(z) = \frac{1}{z} \left[ C_{1l} J_1 \left( \frac{\sigma_l}{\gamma} z \right) + C_{2l} Y_1 \left( \frac{\sigma_l}{\gamma} z \right) \right],$$  \hspace{1cm} (4.47)

where $C_{1l}, C_{2l}$ are arbitrary constants, and $J_1, Y_1$ are the Bessel functions.

By taking into account asymptotic behaviour of the Bessel functions for large values of their arguments, we find the behaviour of function $N_l(z)$ for $z \to 0$

$$N_l(z) = \frac{1}{z^3} \left[ C_{1l} \cos \left( \frac{\sigma_l}{\gamma} z - \frac{3 \pi}{4} \right) + C_{2l} \sin \left( \frac{\sigma_l}{\gamma} z - \frac{3 \pi}{4} \right) \right].$$  \hspace{1cm} (4.48)

We see that solutions to (4.46) are unbounded as $z \to 0$ for all real values of the parameter $\sigma_l$. From (4.44) we get

$$X_l(z) = -\frac{\gamma z}{(x_{\pm}) \sigma_l} \left[ C_{1l} J_0 \left( \frac{\sigma_l}{\gamma} z \right) + C_{2l} Y_0 \left( \frac{\sigma_l}{\gamma} z \right) \right],$$  \hspace{1cm} (4.49)

which gives in the limit $z \to 0$

$$X_l(z) = -\frac{1}{(x_{\pm})} \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \left( \frac{\gamma z}{\sigma_l} \right)^{\frac{1}{2}} \left[ C_{1l} \cos \left( \frac{\sigma_l}{\gamma} z - \frac{\pi}{4} \right) + C_{2l} \sin \left( \frac{\sigma_l}{\gamma} z - \frac{\pi}{4} \right) \right].$$  \hspace{1cm} (4.50)

Analysis of our basic system (4.30) in a small neighbourhood of double horizon $x = x_{\pm}$ shows that for all real values of the parameter $\sigma_l$, there exist unbounded solutions as $x \to x_{\pm}$. Therefore the method of linear perturbations, as well as the assumption (4.8), is not suitable in this case, but the behaviour of perturbations suggests instability of the extreme configuration. It should be investigated separately using nonlinear analysis, and we are currently working on this [56].
4.3. The reduction of the system to a one-dimensional wave equation

Now our goal is to reduce the system (4.30) to a single second-order equation. We introduce the new functions \( z_1, z_2, z_3 \) using the linear transformations

\[
N_l = \left[ \frac{1}{x} z_1 + \left( \frac{g'}{2} - \frac{x}{4g} \left( \frac{g'}{2} - \sigma_l^2 \frac{x}{g} \right) \right) z_2 + z_3 \right] g^{\frac{1}{2}},
\]

(4.51a)

\[
L_l = \left[ -\frac{x}{2} g'' + \frac{g'}{2} \right] z_2 + z_3 \right] g^{\frac{1}{2}},
\]

(4.51b)

\[
X_l = \left[ \left( \frac{n+1}{x} - \frac{g}{x} + \frac{g''}{2g} \right) z_2 - z_3 \right] g^{\frac{1}{2}}.
\]

(4.51c)

The inverse transformation to (4.51) reads

\[
z_1(x) = \frac{x}{\sqrt{gb}} \left( b(x)N_l(x) + \left( b(x) - n - 1 \right) \frac{x^2 g'' - \sigma_l^2 x^2 g(x)}{2} \right) L_l
\]

(4.52a)

\[
z_2(x) = \left[ L_l + X_l \right] \frac{x}{\sqrt{gb}(x)},
\]

(4.52b)

\[
z_3(x) = \left[ \frac{1}{x} b(x) + b_x \right] L_l + b_x X_l \frac{x}{\sqrt{gb}(x)},
\]

(4.52c)

where

\[
b(x) = n + 1 + \frac{1}{2} g'(x) - g(x) = n + \frac{3\alpha}{2x} (M(x) - x^3 \rho).
\]

(4.53)

The sum of (4.51b) and (4.51c) gives

\[
X_l + L_l = \left[ \frac{n+1}{x} + \frac{1}{2} g' - \frac{1}{x} g \right] g^{1/2} z_2.
\]

(4.54)

As a result we get the following system:

\[
z_{1,x} = \left( \frac{2}{x} - \frac{g'}{g} \right) z_1 - \left( \frac{1}{2} x^2 g'' + x g'' - g' \right) z_2 + \left[ 2 + \frac{x^2}{b(x)} \left( \frac{g''}{2} - \frac{(g')^2}{4g} - \sigma_l^2 \frac{1}{g} \right) \right] z_3,
\]

(4.55a)

\[
z_{2,x} = -\frac{1}{b(x)} z_3,
\]

(4.55b)

\[
z_{3,x} = \frac{b(x) g^{-1}}{x^2} z_1 - \left[ \frac{2}{x} + \frac{(x g' - g')}{2b(x)} \right] z_3 + \frac{1}{x} \left( \frac{x^2}{2} g'' + x g'' - g' \right) z_2.
\]

(4.55c)

It is easy to prove that

\[
\frac{1}{2} x^2 g'' + x g'' - g' = -\frac{3\alpha}{2} (x^3 \rho)' = 3\alpha x^2 p_{\perp}
\]

(4.56)

so that in the case when the density profile satisfies the condition

\[
(x^3 \rho)' = 0,
\]

(4.57a)

the system (4.55) splits into the system of two equations (4.55a), (4.55c) for \( z_1, z_3 \) and equation (4.55b).
In the particular case \((x^3 \rho') = \text{const} = 0\) this is the necessary and sufficient condition for our system \((4.30)\) to coincide with the Chandrasekhar system ([45], equations \((46), (47), (50)\) chapter 4).

Condition \((4.57a)\) is equivalent to
\[p_\perp' = 0.\]

Differentiating \((4.55c)\), we come to the system which includes one first-order equation, \((4.55b)\), and one second-order equation
\[z_{3l,xx} + 2 \left(\frac{g'}{g} + \frac{1}{x}\right) z_{3l,x} + q_l(x) z_{3l} = r_l(x) z_{2l},\]
(4.58)
where
\[q_l(x) = \sigma_l^2 \frac{1}{g^2} - \frac{2(n + 1)}{x^2 g} - \frac{1}{g} \left(\frac{g'}{g}\right)^2 + \frac{3}{x} \frac{g'}{g} - \frac{(xg'' - g')}{b(x)} \left[\left(xg'' - g'\right) - \frac{g'}{2b(x)} - \frac{1}{x} \right] + \frac{3\alpha x}{b(x)} p_\perp',\]
(4.59)
\[r_l(x) = -3\alpha p_\perp' \left[\frac{(n + 1)}{g} - \frac{3x}{2} \frac{g'}{b(x)} + \frac{x}{2b(x)} (xg'' - g')\right] + \frac{3\alpha}{x} (x^2 p_\perp')'.\]
(4.60)

It is easy to see that in the case when the condition \((4.57)\) is satisfied, two equations \((4.55b)\) and \((4.58)\) split. Introducing the new function \(\omega_{3l}(x)\) by
\[z_{3l}(x) = \frac{1}{xg} \omega_{3l}(x)\]
(4.61)
we reduce equation \((4.58)\) to a form which does not contain the first derivative:
\[\omega_{3l,xx} + \left[\sigma_l^2 \frac{1}{g^2} - V_{1l}(x)\right] \omega_{3l} = xg r_l z_{2l},\]
(4.62)
where the potential \(V_{1l}(x)\) is given by
\[V_{1l}(x) = \frac{l(l + 1)}{x^2} \frac{1}{g} + \frac{3}{2} \frac{g''}{g} - \frac{1}{g} \frac{g'}{x} - \frac{(xg'' - g')}{b(x)} \left[\left(xg'' - g'\right) - \frac{g'}{2b(x)} - \frac{1}{x} \right] + \frac{1}{2} \frac{x^2 g'' + xg'' - g'}{b(x)}.\]
(4.63)

By taking into account \((4.55b)\) and \((4.61)\), equation \((4.62)\) can be rewritten as an integro-differential equation of the form
\[\omega_{3l,xx} \left[\sigma_l^2 \frac{1}{g^2} - V_{1l}(x)\right] \omega_{3l}(x) = -xg(x) r_l(x) \int \frac{\omega_{3l}(x)}{xg(x)b(x)} \, dx.\]
(4.64)

In this form \((g^{-2}(x))\) scales the spectral parameter \(\sigma_l^2\) equation \((4.64)\) corresponds to the generalized spectral problem with the non-local potential
\[-\omega_{3l,xx} + V_l(x) \omega_{3l}(x) - T_l \omega_{3l}(x) = \sigma_l^2 \frac{1}{g^2} \omega_{3l}(x),\]
(4.65)
where
\[T_l u(x) = xg(x) r_l(x) \int_d^x \frac{u(z) \, dz}{z b(z) g(z)}\]
(4.66)
is the integral Vol’terra operator. The lower limit is \(d = x_*\) for a black-hole case.
Condition (4.57) leads to \( r_l = 0 \), so we conclude that condition (4.57) is the necessary and sufficient condition to reduce the problem of polar perturbations to the Schrödinger equation with the local potential.

Introducing ‘the tortoise coordinate’ \( x_\ast(x) = \int dx / g(x) \) and the function \( w(x_\ast) \) by

\[
 w_{3l}(x_\ast) = x \sqrt{g(x)} z_{3l}(x_\ast) \quad (4.67)
\]

we reduce the system (4.58), (4.55b) to the form

\[
 w_{3l,x_\ast} + \left( \frac{\sigma_l^2 - W_l(x)}{x^2} \right) w_{3l}(x_\ast) = xg'\frac{z_{3l}}{x^2} r_l(x) z_{3l}(x_\ast) \quad (4.68)
\]

\[
 z_{2l,x_\ast} = -\frac{g_1(x)}{x^2 b(x)} w_{3l}(x_\ast), \quad (4.69)
\]

where

\[
 W_l(x) = g \left[ \frac{l(l+1)}{x^2} + g'' - \frac{1}{x} g' + \frac{g(xg'' - g')}{b(x)} \left( \frac{xg'' - g'}{2b(x)} - \frac{g'}{g} + \frac{1}{x} \right) \right.
\]

\[
 -\frac{g(xg'' - g')}{2b(x)} \left( 1 - \frac{g(xg'' - g')}{2b(x)} - \frac{g'}{g} + \frac{2}{x} \right) \right]. \quad (4.70)
\]

In the limit \( x \to x_\ast \), the integral term in (4.66) tends to zero, in essence due to \( z_{2l} \to 0 \). Indeed, when \( x \to x_\ast \), we get \( z_{2l} \sim \sqrt{x - x_\ast} \) by using (4.52b) and taking into account the asymptotic behaviour of \( (L_l + X_l) \sim (x - x_\ast) \) which follows from (4.38).

The potential (4.70) vanishes as \( x_\ast \to +\infty \) as \( x^{-2} \), while for \( x_\ast \to -\infty \), it vanishes exponentially. Therefore solutions to (4.68) have asymptotic \( e^{\pm i\sigma_l x_\ast} \) as \( x_\ast \to \pm \infty \), so that we have to look for solutions satisfying boundary conditions

\[
 w_l \to e^{i\sigma_l x_\ast} + R_l^{(w)} e^{-i\sigma_l x_\ast} \quad \text{as} \quad x_\ast \to \infty
\]

\[
 w_l \to T_l^{(w)} e^{i\sigma_l x_\ast} \quad \text{as} \quad x_\ast \to -\infty. \quad (4.71)
\]

In the particular case of validity of (4.57), \( r_l(x) = 0 \), \( p'_\perp = 0 \), the system (4.68), (4.69) splits and we get the Schrödinger equation

\[
 -w_{3l,x_\ast} + W_{3l}(x) w_{3l}(x_\ast) = \sigma_l^2 w_{3l}(x_\ast)
\]

with the potential

\[
 W_{3l}(x) = g \left[ \frac{l(l+1)}{x^2} + g'' - \frac{1}{x} g' + \frac{g(xg'' - g')}{b(x)} \left( \frac{xg'' - g'}{2b(x)} - \frac{g'}{g} + \frac{2}{x} \right) \right],
\]

which for the Schwarzschild geometry coincides with the potential in the Zerilli equation ([45], chapter 4, equation (63)).

We have reduced the basic system of three first-order linear equations (4.29) to a single second-order equation for the particular combination of these functions, \( w_{3l}(x_\ast) \). In our case this is the Schrödinger equation (4.72) with non-local potential. Its non-local part vanishes when condition (4.57) is satisfied. As in the Schwarzschild case, this empirically found reducibility (resulting from a sequence of mysterious cancellations) follows in fact from the existence of some particular solution to the system which we have to reduce [45].

In our case condition (4.57) is the necessary and sufficient condition for vanishing non-local part in (4.72) and thus for reducing our system (4.29) to the standard Schrödinger equation. Actually, condition (4.57) is also the necessary and sufficient condition for the
existence of a particular solution which guarantees such a reduction (it is shown in detail in our paper [56]). If condition (4.57) is satisfied, then the particular solution reads
\[
N_p^2 = \sqrt{g} \left[ -\frac{xg''}{2} - \frac{x(g')^2}{4g} - \frac{\sigma_l^2}{g} - b'(x) \right]
\]
\[
L_p^2 = -\sqrt{gb''(x)};
X_p^2 = \sqrt{g} \left( \frac{b(x)}{x} + b'(x) \right).
\]

The linear transformation (4.51) is needed to reduce the system of three linear equations to one equation, if the particular solution to (4.29) exists. Remarkable luck in our case is that this transformation works not only in the case when (4.57) is satisfied (i.e. the second-order equation obtained is the standard Schrödinger equation with the local potential), but also in the general case. This fact allowed us to reduce our system (4.29) to the Schrödinger equation (4.72) with the non-local potential.

We see that the problem of stability of the spherically symmetric solution (2.7) to polar perturbations reduces to investigation of the spectral problem (4.68), (4.69) with the potential of the form (4.70) and with the boundary conditions (4.71).

Equation (4.68), by taking into account (4.69), can be written in the form corresponding to the spectral problem with a non-local potential
\[
-w_{3l} + W_l(x)w_{3l}(x) = \sigma_l^2 w_{3l}(x),
\]
(4.72)

where
\[
\tilde{T}_l u(x) = x(x_e)g^2(x(x_e)) \frac{d}{dz_e} \int_{z_e}^{x_e} \frac{g^2(x(z_e))u(z_e) dz_e}{x(z_e)b(x(z_e))},
\]
(4.73)
is the integral Volterra operator.

Ultimately the case reduces to investigation of the eigenvalue problem with integro-differential operator (4.72), which is the Schrödinger equation with non-local potential.

The spectrum of eigenvalues contains all the values of the parameter \(\sigma_l^2\), at which solutions exist that satisfy the imposed boundary conditions.

If such solutions exist only for real values of the time parameter \(\sigma_l\) and if, in addition, they form the complete basic set of functions, then any smooth initial perturbation on the finite interval of variable \(x_e\) (with compact measure) can be expanded on this basic set, and since dependence of each particular mode on time is given by \(\exp(i\sigma_l t)\), it testifies for stability of geometry.

Indeed, a considered static configuration is stable if there are no integrable modes with negative \(\sigma_l^2\). The appearance of negative eigenvalues \(\sigma_l^2\) would lead to the existence of exponentially growing modes of perturbations.

In the next subsection, we study in detail the integro-differential operator governing polar perturbations.

### 4.4. Schrödinger equation with non-local potential

A system governed by a Schrödinger equation with a non-local potential obeys the following theorems:

(i) If in the standard one-dimensional Schrödinger equation the potential is non-negative, then negative eigenvalues are absent (see, e.g., [54]).
(ii) The Weyl theorem [57] for self-conjugate operators: the essential spectrum conserves under relatively compact perturbations [58].
The essential spectrum is defined as follows: if we remove from the spectrum of self-conjugate operator all isolated points which are eigenvalues of finite multiplicity, then the remaining spectrum is the essential spectrum.

The essential spectrum of the non-perturbed (local) potential is continuous and represented by positive semi-axis \([0, \infty)\); isolated points are absent in the case when negative values are excluded by non-negativity of the potential.

It follows that the essential spectrum of the problem with the non-local potential \((4.72)\) is the same as the essential spectrum of the non-perturbed (local) potential. The essential spectrum of the local potential is this total spectrum, since isolated points are absent, i.e., the essential spectrum of the perturbed problem coincides with the total spectrum of the non-perturbed problem.

The non-local part of a potential represents the perturbation of the local potential. To not spoil an essential spectrum, this perturbation should be relatively compact.

So, our task now is to prove that the non-local part represents a compact perturbation and to deduce the criterion of non-negativity of a local potential.

In our case the non-local part (perturbation of a local potential) is given by the integral Vol’terra operator \((4.73)\). Such an operator is totally continuous, if it has a smooth square integrable kernel.

Square integrability requires

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, z) \, dx \, dz < \infty. \tag{4.74}
\]

The kernel of our Vol’terra operator \((4.73)\) is

\[
K(x, z) = xg''(x) + \frac{g'(x)}{bx} e^{\frac{x}{bx}}. \tag{4.75}
\]

Its smoothness is evident. The sufficient condition for square integrability of the kernel \(K(x, y)\) is the condition on the behaviour of \(K^2\) at infinity:

\[
K^2(x, y) < \frac{1}{x^{1+\delta_1}} \frac{1}{y^{1+\delta_2}},
\]

where positive \(\delta_1, \delta_2\) are arbitrarily small.

For \(K(x, z)\) given by \((4.75)\), for \(z \to -\infty\), \(K^2\) vanishes as \(g(x(z_*))\). When \(x_* \to -\infty\), then \(K^2\) vanishes as \(g'(x)\). The metric function \(g(x(x_*))\) near the horizon behaves as \(g(x(x_*)) \sim g'(x_*) e^{\frac{x}{bx}}\). \(g'\) is positive, so that the metric as a function of \(x_*\) vanishes exponentially on approaching the horizon.

When \(z_\to \infty\), \(K^2\) vanishes as \(x^{-2}(z_*).\) From definition \(z_*\) we see the main contribution at infinity is \(z_* \sim x.\) When \(x_* \to \infty\), \(K^2\) vanishes as \(x^2(p_{1/2}^2)\). The tangential pressure \(p_{1/2}\) for de Sitter–Schwarzschild geometry vanishes at infinity quicker than \(x^{-3}\), because it is related to density by the equation of state \(p_{1/2} = -\rho - x\rho'/2\), and density vanishes quicker than \(x^{-3}\) to guarantee the finiteness of a mass. Hence in this limit \(K^2\) vanishes quicker than \(x^{-6}\).

So, for a BH case the kernel is square integrable.

As a result the totally continuous operator \((4.73)\) gives a relatively compact perturbation to the local potential in the integro-differential equation \((4.72)\).

4.4.1. Criterion of non-negativity of local potential. Introducing the function \(p(x) = xg''(x) - g'(x)\), we write the potential \((4.72)\) in the form

\[
W_l(x) = g \left[ \frac{1}{2} \left( \frac{p}{b} \right)^2 + \frac{1}{2b(x)} \left( g'(x) \right)^2 + \frac{2(n+1)}{x^2} - \frac{1}{bx} I_l(x) \right]. \tag{4.76}
\]
\[ I_l(x) = x^2 \left( \frac{1}{2} x \rho' + g g'' \right) - (n + 1 - g) p(x). \]  

(4.77)

In (4.76) we should investigate the term \( I_l \). The rest is positive, since \( b(x) \geq n \) in a BH case.

Expressing \( g(x) \), its derivatives, \( p(x) \) and \( b(x) \) in terms of mass function \( M(x) \), density \( \rho(x) \) and its derivatives, we transform \( I_l(x) \) to the form

\[ I_l(x) = \alpha \left[ -4 \alpha \frac{M^2}{x^3} + \frac{9}{2} \alpha x^4 \rho \rho' + 3(n - 1)x^2 \rho' \right. 
\left. + \frac{3}{2} \alpha x \rho' M - 3n x \rho - 3g x (x^2 \rho'' + 2 \rho) + \frac{3(n + 2)}{x^2} M \right]. \]  

(4.77a)

For a BH case \( g(x) > 0 \) while the weak energy condition gives \( \rho' < 0 \). Then the sufficient condition for \( W_l \geq 0 \) is the condition on the equation of state

\[ x^2 \rho''(x) + 2 \rho(x) \geq 0, \]  

(4.78a)

which constrains the growth of the derivative of \( p_\perp + \rho \)

\[ x (p_\perp + \rho)' \leq \rho + (p_\perp + \rho). \]  

(4.78b)

This condition is actually relevant also for the case without horizons (then \( g(x) > 0 \) for all \( x \)).

When (4.78) is satisfied, then proof of non-negativity of (4.76) reduces to proof of non-negativity of the function

\[ \phi(x) = \frac{2(n + 1)}{x^2} - \frac{3\alpha(n + 2)M}{x^3 b(x)}. \]  

(4.79)

It is bounded from below as follows:

\[ \phi(x) = \frac{2}{x^2} \left[ n + 1 - \frac{3\alpha M (n + 2)}{2 x b(x)} \right] \geq \frac{2}{x^2} \left[ n + 1 - \frac{3(n + 2)}{2} \frac{\alpha M}{n x} \right] = \frac{2}{x^2} \left[ n + 1 - \frac{3(n + 2)}{2} \frac{1 - g}{n} \right] \]

\[ > \frac{2}{x^2} \left[ n - \frac{1}{2} - \frac{3}{n} \right] \geq \frac{2}{x^2} (n - 2) \geq 0. \]  

(4.80)

As a result, we find the sufficient condition (4.78) for non-negativity of the potential (4.70) in all ranges of argument for which \( g(x) > 0 \).

For the density profile (1.3) this condition is satisfied.

We can conclude that the essential spectrum of the integro-differential operator (4.72) is the same as the essential spectrum of its local potential. Now the key point is to find the condition on a perturbation of a local potential which guarantees the absence of isolated points in the total spectrum (negative values of \( \sigma^2_l \)) of the integro-differential operator (4.72).

4.4.2. Non-local contribution. Multiplying (4.68) by \( w_{3l}^7 \) and integrating by parts by taking into account asymptotic behaviour of (4.60) at infinity, we obtain the following relation:

\[ \sigma^2_l \int_{-\infty}^{+\infty} |w_{3l}(x_\ast)|^2 \, dx_\ast + w_{3l,x_\ast} w_{3l,x_\ast} \bigg|_{-\infty}^{+\infty} \]

\[ = \int_{-\infty}^{+\infty} \left[ |w_{3l,x_\ast}|^2 + W_l(x) |w_{3l}(x_\ast)|^2 + \psi_{l,x} g(x) |z_{3l}(x_\ast)|^2 \right] \, dx_\ast, \]  

(4.81)

\footnote{We denote the complex conjugate by * for convenience of comparison with the classical results presented in [45].}
where

\[ \psi_l(x) = \frac{x^2}{2} g^2(x) b(x) \rho_l(x). \]  

(4.82)

The Wronskian \( W_{l,x} w_{l,x}^{+\infty} \) of two independent solutions \( w_{l,x} \) and \( w_{l,x}^{+\infty} \) is constant (see [45]).

The contribution to the spectrum from the non-local part of the potential is given by

\[ N = \int_{x_c}^{+\infty} \psi_{l,x}(x)|z_{2l}(x)|^2 \, dx. \]  

(4.83)

If the condition of non-negativity of a local potential \( W_l(x) \) is satisfied, then the requirement

\[ N = \int_{x_c}^{+\infty} \psi_{l,x}(x)|z_{2l}(x)|^2 \, dx \geq 0 \]  

(4.84)

gives the sufficient condition for the absence of negative eigenvalues \( \sigma_l^2 \) of the considered spectral problem.

Fortunately, the non-local contribution given by (4.83) does not grow with the mode number \( n \), since \( |z_{2l}|^2 \) is constrained from above by the function proportional to \( n^{-2} \). This constraint is valid for any density profile and follows from (4.69) by taking into account that in a BH case \( b(x) \geq n \).

In the case when the metric function \( g(x) \) satisfies condition (4.57), the sufficient condition (4.84) is trivially satisfied \( (N = 0) \), and negative eigenvalues do not appear in the spectrum. As a result spherically symmetric metrics satisfying (4.57) and (4.78) are stable to polar perturbations.

5. The case of density profile (1.3)

In the case of the density profile \( \rho(x) = e^{-x^2} \) the metric function in (4.28) reads

\[ g(x) = e^{2(\alpha x)} = 1 - \frac{\alpha}{x}(1 - e^{-x^2}). \]  

(5.1)

The local potential governing polar perturbations given by (4.70) is shown in figures 5 and 6 for the density profile (1.3) and two values of parameter \( \alpha \) (denoted in the figures as \( \alpha \)).

For bigger values of \( \alpha \) the potentials become similar to those for the Schwarzschild case [45]. Local potentials are smooth short-range potentials, so that the their integrals are finite over all the region of variable \( x \).
The potential (4.70) for the density profile (1.3) satisfies the criterion of non-negativity (4.78), but condition (4.57) is not satisfied, so that the appearance of negative eigenvalues $\sigma_l^2$ is in principle possible.

The question of existence of isolated points with negative values $\sigma_l^2$ in the total spectrum of the integro-differential operator (4.72) requires a complicated numerical analysis which is in progress. Preliminary results suggest that non-local contribution (4.83) does not lead to negative values $\sigma_l^2$ for the masses $m > m_{cr2}$.

This result looks natural. The second critical mass value $m_{cr2}$ is distinguished for the unperturbed geometry. The value $m_{cr2}$ marks the point in the temperature–mass diagram at which specific heat is broken and changes its sign, so that a second-order phase transition starts when in the course of Hawking evaporation the mass approaches the value $m_{cr2}$. For the density profile (1.3) it is given by

$$m_{cr2} \simeq 0.38 m_{Pl} \sqrt{\rho_{Pl}/\rho_0}.$$  

The extreme state of a non-singular black hole ($m = m_{cr}$) can be unstable since some perturbations modes grow unlimited at the double horizon for any density profile. If the considered configuration would develop instability before achieving the extreme state, the most appropriate range gets beyond $m_{cr2}$ where a phase transition starts.

The critical value $m_{cr2}$ corresponds to the maximum at the temperature–mass curve (see figure 3). It is calculated from the condition $dT/dm = 0$. In units normalized to the de Sitter radius $r_0$ (which is the characteristic scale related to de Sitter vacuum trapped in the origin)

$$y_+ = \frac{r_+}{r_0}; \quad s = \frac{r_g}{r_0}$$

the temperature on a BH event horizon is given by [19]

$$T = \frac{1}{y_+} - \frac{3}{y_+} \left(1 - \frac{y_+}{s}\right).$$

The density profile and metric in these units read

$$\rho(y) = e^{-y/s}; \quad g(y) = 1 - \frac{s}{y} (1 - e^{-y/s}).$$

From $dT/ds = 0$ and $g(y_+) = 0$, we get the critical value $s_2$ and the value $y_+$ corresponding to $m = m_{cr2}$

$$s_2 \simeq 2.226; \quad y_+ \simeq 2.166.$$  

For comparison, the critical values for the extreme case $m_{cr}$ of the double horizon $r_\pm$ are [19]

$$s_{cr} \simeq 1.7576; \quad y_{\pm} \simeq 1.4957.$$
6. Summary and discussion

The problem of stability to polar perturbation reduces to a one-dimensional Schrödinger equation with a non-local potential given by the Volterra integral operator with square integrable smooth kernel representing a compact perturbation to the local potential.

We derived the criterion of non-negativity of the local potential which defines the essential spectrum of integro-differential operator governing polar perturbations in general case.

We derived the criterion of vanishing of the non-local part of the potential which distinguishes the class of geometries for which the problem of stability reduces to a standard one-dimensional Schrödinger equation.

For the case when perturbations are described by the Schrödinger equation with non-local potential, we found a sufficient condition for the absence of negative eigenvalues in the spectrum which guarantees the stability of the investigated geometry.

For an extreme black hole, the method of small perturbations is not applicable due to the existence of unlimited perturbation modes on approaching the double horizon. The standard linear perturbation analysis fails here. Nonlinear analysis is needed, in particular, nonlinear numerical simulations which are in progress now [56].

Asymptotic behaviour of the basic system near the double horizon suggests instability of the extreme configuration. The behaviour in this regime is very special; unrestricted solutions for perturbations exist for positive values of the spectral parameter $\sigma_l^2$. The limiting equation for perturbations near the double horizon is essentially different from that for the one-horizon case in which unrestricted solutions do not appear for positive values $\sigma_l^2$ and which cannot be smoothly continued to the two-horizon case. The question arises: what is the place of the metric with $m = m_{cr}$, at the set of metrics whose stability we investigate as a one-parametric set of solutions to the Einstein unperturbed equations with a given density profile.

The critical value of the mass parameter $m_{cr}$ is calculated from two transcendental equations: $g(r_±) = 0$; $g'(r_±) = 0$. This is the unique point for considered metric function $g(r)$ because it is the minimum of $g(r)$ and $g(r)$ has only one minimum [20]. Therefore the transcendental system for $(r_±, m_{cr})$ has a unique solution for each particular one-parametric set with a given density profile.

The metric with double horizon represents an isolated singular point at the set of metrics $g(r)$ for each given density profile. It resembles fixed point attractor behaviour which is currently the key point of our efforts [56].

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