Self-dual cyclic codes over finite chain rings

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Abstract

Let \( R \) be a finite commutative chain ring with unique maximal ideal \( \langle \gamma \rangle \), and let \( n \) be a positive integer coprime with the characteristic of \( R/\langle \gamma \rangle \). In this paper, the algebraic structure of cyclic codes of length \( n \) over \( R \) is investigated. Some new necessary and sufficient conditions for the existence of nontrivial self-dual cyclic codes are provided. An enumeration formula for the self-dual cyclic codes is also studied.

Keywords: Cyclic code, dual code, self-dual cyclic code, chain ring.

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1 Introduction

The study of codes over finite rings has grown tremendously since the seminal work of Hammons et al. It is shown in [8] that some of the best nonlinear codes over \( \mathbb{F}_2 \) can be viewed as linear codes over \( \mathbb{Z}_4 \). It was pointed out in [24, 25] that only finite Frobenius rings are suitable for coding alphabets, in the sense that several fundamental properties of codes over finite fields still hold for codes over such rings. This has motivated numerous authors to research on codes over finite chain rings, as chain rings are Frobenius rings with good algebraic structures.

On the other hand, the class of cyclic codes plays a very significant role in the theory of error-correcting codes. One is that they can be efficiently encoded using shift registers. There is a lot of literature about cyclic codes over finite chain rings (e.g., see [1], [5]-[7], [12], [19]-[21]).

Generally, cyclic codes over finite chain rings can be divided into two classes: simple-root cyclic codes, if the code lengths are coprime with the characteristic of the ring; otherwise, we have the so-called repeated-root cyclic codes. In this paper, we study simple-root cyclic codes over finite chain rings.

Pless and Qian [19] showed that cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) have generators of an interesting form: \( \langle fh, 2gh \rangle \), where \( f, g, h \in \mathbb{Z}_4[X] \) satisfy \( fgh = X^n - 1 \). Pless, Solé and Qian in [20] considered existence conditions for nontrivial self-dual cyclic codes of odd length over \( \mathbb{Z}_4 \). Results of [19, 20] were then extended to simple-root cyclic codes over \( \mathbb{Z}_{p^m} \) [14]. Following that line of research, Wan continued to consider simple-root cyclic codes over Galois rings [22]. Extending the main results of [14] and [22], Dinh and López-Permouth in [5] completely described simple-root cyclic codes over a finite commutative chain ring \( R \). Several necessary and sufficient conditions for the existence of nontrivial self-dual cyclic codes were provided.

Let \( R \) be a finite commutative chain ring with unique maximal ideal \( \langle \gamma \rangle \). Then \( \langle \gamma \rangle \) is nilpotent and we denote its nilpotency index by \( t \). Let \( n \) be a positive integer coprime with the characteristic of \( \mathbb{F}_q = R/\langle \gamma \rangle \). First, we generalize the methods of [13] to obtain the algebraic structure of cyclic codes of length \( n \) over \( R \), which is different from that given in [5]. Using this structure, we show that self-dual cyclic codes of length \( n \) over \( R \) exist if and only if \( t \) is even. Some new necessary and sufficient conditions

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for the existence of nontrivial self-dual cyclic codes are also derived. We show that, when the nilpotency index $t$ is even, the number of self-dual cyclic codes is fully determined by $|\Delta_n|$, the number of reciprocal polynomial pairs in the monic irreducible factorization of $X^n - 1$ over $\mathbb{F}_q$. The counting problem for $|\Delta_n|$ naturally reduces to an equivalent question about counting $|\Omega_n|$, the number of self-reciprocal monic irreducible factors of $X^n - 1$ over $\mathbb{F}_q$. Write $n = 2^m n'$, where $n'$ is odd. It is shown that, the problem of determining the value of $|\Omega_{2^m n'}|$ can be entirely reduced to those of computing $|\Omega_n|$ and $|\Omega_{n'}|$, where $|\Omega_{n'}|$ denotes the number of self-reciprocal monic irreducible factors of $X^n - 1$ over $\mathbb{F}_{q^2}$. In particular, very explicit formulas for the value of $|\Omega_n|$ are obtained when $n$ has exactly two prime divisors.

This paper is organized as follows. After presenting preliminary concepts and results in Section 2, we obtain structure theorems for cyclic codes of length $n$ over $R$ in Section 3. In Section 4, we provide some results concerning the structure, existence conditions and enumeration formula for self-dual cyclic codes. In Section 5, we study an enumeration formula for the number of self-dual cyclic codes.

## 2 Preliminaries

A finite commutative ring with identity is called a **finite chain ring** if it is local and its unique maximal ideal is principal. Throughout this paper, $R$ denotes a finite chain ring. Let $\gamma$ be a fixed generator of the unique maximal ideal of $R$, and assume that $(\gamma) = \{r\gamma \mid r \in R\}$ is the principal ideal of $R$ generated by $\gamma$. Then $(\gamma)$ is nilpotent and we denote its nilpotency index by $t$. We set $\mathbb{F}_q = R/(\gamma)$. Here $\mathbb{F}_q$ is the finite field with $q = p^n$ elements, where $p$ is the characteristic of $\mathbb{F}_q$.

The natural surjective ring homomorphism from $R$ onto $\mathbb{F}_q$ is given as follows:

$$- : R \longrightarrow \mathbb{F}_q, \quad r \mapsto \bar{r}, \text{ for any } r \in R.$$

The map $\overline{\cdot}$ can be extended to a ring homomorphism from $R[X]$ onto $\mathbb{F}_q[X]$ in an obvious way:

$$R[X] \longrightarrow \mathbb{F}_q[X], \quad \sum_{i=0}^{n} a_i X^i \mapsto \sum_{i=0}^{n} \bar{a_i} X^i, \text{ for any } a_0, a_1, \ldots, a_n \in R,$$

which is also denoted by $\overline{\cdot}$ for simplicity.

Two polynomials $f_1(X), f_2(X)$ in $R[X]$ are called **coprime** if there exist polynomials $u_1(X), u_2(X)$ in $R[X]$ such that $u_1(X)f_1(X) + u_2(X)f_2(X) = 1$. The following result is very useful (e.g., see [18, Lemma 2.8] or [23, Lemma 14.19]).

**Lemma 2.1.** Let $f_1(X), f_2(X)$ be two polynomials in $R[X]$. Then $f_1(X), f_2(X)$ are coprime in $R[X]$ if and only if $\bar{f}_1(X), \bar{f}_2(X)$ are coprime in $\mathbb{F}_q[X]$.

A polynomial $f(X) \in R[X]$ is said to be **basic irreducible** if $\overline{f(X)}$ is irreducible in $\mathbb{F}_q[X]$. A polynomial $f(X) \in R[X]$ is called **regular** if it is not a zero divisor. Clearly, monic polynomials are regular polynomials. A polynomial over a field is called **square free** if it has no multiple irreducible divisors in its decomposition.

Hensel’s Lemma [17, Theorem XIII.4] plays a very significant role in the study of finite chain rings as well as codes over finite chain rings. Using Hensel’s Lemma, it is easy to get the next result given in [13, Lemma 2.3].

**Lemma 2.2.** Let $f$ be a monic polynomial over $R$ such that $\overline{f}$ is square free. If $\overline{f} = g_1 g_2 \cdots g_s$ is the unique factorization into a product of pairwise coprime monic irreducible polynomials in $\mathbb{F}_q[X]$, then there exists a unique family of pairwise coprime monic basic irreducible polynomials $f_1, f_2, \cdots, f_s$ over $R$ such that $f = f_1 f_2 \cdots f_s$ and $\overline{f_i} = g_i$ for $1 \leq i \leq s$.

In the rest of this section, we recall some notations and basic facts about codes over rings. Let $n$ be a positive integer. A **code** $C$ of length $n$ over $R$ is a nonempty subset of $R^n$. If, in addition, $C$ is an $R$-submodule of $R^n$, then $C$ is called a **linear code**. A linear code $C$ of length $n$ over $R$ is called **cyclic** if $(c_{n-1}, c_0, \cdots, c_{n-2}) \in C$ for every $(c_0, c_1, \cdots, c_{n-1}) \in C$. 

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Each codeword $c = (c_0, c_1, \cdots, c_{n-1})$ is customarily identified with its polynomial representation $c(X) = c_0 + c_1 X + \cdots + c_{n-1} X^{n-1}$. In this way, any cyclic code of length $n$ over $R$ is identified with exactly one ideal of the quotient algebra $R[X]/(X^n - 1)$.

For any cyclic code $C$ of length $n$ over $R$, the dual code of $C$ is defined as $C^\perp = \{ u \in R^n \mid u \cdot v = 0 \text{ for any } v \in C \}$. Since $u \cdot v$ denotes the standard Euclidean inner product of $u$ and $v$ in $R^n$. The code $C$ is said to be self-orthogonal if $C \subseteq C^\perp$, and self-dual if $C = C^\perp$. It turns out that the dual of a cyclic code is again a cyclic code. The following result is well known (e.g., see [6, Proposition 2.3]).

Lemma 3.3. Let $R$ be a finite chain ring. Then the number of codewords in any linear code $C$ of length $n$ over $R$ satisfies $|C| \cdot |C^\perp| = |R|^n$.

3 Structure of cyclic codes over finite chain rings

Starting from this section till the end of this paper, we always assume that $n$ is a positive integer coprime with the characteristic of $\mathbb{F}_q$. We set $\mathcal{R}_n := R[X]/(X^n - 1)$. Recall that $R$ is a finite chain ring with maximal ideal $\langle \gamma \rangle$, and that $\mathbb{F}_q = R/\langle \gamma \rangle$ is the residue field of order $q = p^s$. We adopt the following notations.

Notation 3.1. Let $r_0$ be an element in $R$ such that $r_0 = 1 + \mu \gamma$, where $\mu$ is a unit in $R$. It follows that $X^n - r_0 = X^n - 1$ in $\mathbb{F}_q[X]$. Since $q$ is coprime with $n$, the irreducible factors of $X^n - 1$ in $\mathbb{F}_q[X]$ can be described by the $q$-cyclotomic cosets. Let $I$ be a fixed complete set of representatives of all $q$-cyclotomic cosets modulo $n$. Then the polynomial $X^n - 1$ factors uniquely into pairwise coprime monic irreducible polynomials in $\mathbb{F}_q[X]$ as $X^n - 1 = \prod_{i \in I} h_i$ (e.g., see [3, Theorem 4.1.1]). Using Lemma 2.2, $X^n - r_0$ has a unique decomposition as a product $\prod_{i \in I} f_i$ of pairwise coprime monic basic irreducible polynomials in $R[X]$ with $\bar{f}_i = h_i$ for each $i \in I$. Using Lemma 2.2 again, $X^n - 1 \in R[X]$ also has a unique decomposition as a product $\prod_{i \in I} g_i$ of pairwise coprime monic basic irreducible polynomials in $R[X]$ with $\bar{g}_i = h_i$ for each $i \in I$.

For any commutative ring $S$, elements $s_1, s_2 \in S$ are said to be associates if there is a unit $\lambda \in S$ with $\lambda s_1 = s_2$. For a monic polynomial $h(X)$ of degree $k$ in $S[X]$ with $h(0)$ being a unit in $S$, its reciprocal polynomial $h(0)^{-1}X^k h(X^{-1})$ is denoted by $h(X)^*$. Note that $h(X)^*$ is a monic polynomial over $S$. Following [11], if $h(X) = h(X)^*$, then $h(X)$ is said to be self-reciprocal over $S$; otherwise, we say that $h(X)$ and $h(X)^*$ form a reciprocal polynomial pair. For a polynomial $f(X) \in S[X]$ with leading coefficient $a_0$, being a unit of $S$, let $\bar{f}(X) = a_0^{-1}f(X)$, which is a monic polynomial over $S$.

Lemma 3.2. ([6, Lemma 3.1]) Let $R$ be a finite chain ring with maximal ideal $\langle \gamma \rangle$, and let $t$ be the nilpotency index of $\gamma$. If $g$ is a basic irreducible polynomial of the ring $R[X]$, then $R[X]/\langle g \rangle$ is also a finite chain ring, whose maximal ideal is generated by $\gamma + \langle g \rangle$. The nilpotency index of $\gamma + \langle g \rangle$ is equal to $t$.

The next lemma shows that the element $f_i + \langle g_i \rangle$ generates the maximal ideal of $R[X]/\langle g_i \rangle$.

Lemma 3.3. Let the notation be as above. Then, for each $i \in I$, we have that $\langle f_i^{k_i} + \langle g_i \rangle \rangle = \langle g_i \rangle$ as ideals in $R[X]/\langle g_i \rangle$, where $k_i$ is any nonnegative integer.

Proof. It suffices to show that $f_i + \langle g_i \rangle$ and $\gamma + \langle g_i \rangle$ are associates in $R[X]/\langle g_i \rangle$. By Lemma 2.1, $\bar{f}_i = X^n - r_{f_i}$ is coprime with $g_i$ in $R[X]$, since $f_i$ is coprime with $\bar{r}_f$ in $\mathbb{F}_q[X]$. Thus there exist elements $u_i, v_i$ in $R[X]$ such that

$$u_i \bar{f}_i + v_i g_i = 1.$$ 

Then in $R[X]/\langle g_i \rangle$,

$$(u_i + \langle g_i \rangle) (\bar{f}_i + \langle g_i \rangle) = 1 + \langle g_i \rangle,$$ 

(3.1)
which means \( u_i + \langle g_i \rangle \) is a unit in \( R[X]/\langle g_i \rangle \). Multiplying by \( f_i + \langle g_i \rangle \) on both sides of (3.1) gives \( u_i(X^n - r_0) + \langle g_i \rangle = f_i + \langle g_i \rangle \). Then

\[
(u_i + \langle g_i \rangle)^{-1}(f_i + \langle g_i \rangle) = X^n - r_0 + \langle g_i \rangle = X^n - 1 - r_0 + \langle g_i \rangle = 1 - r_0 + \langle g_i \rangle = -\gamma \mu + \langle g_i \rangle.
\]

We are done.

By the Chinese Remainder Theorem, we have the following \( R \)-algebra isomorphism:

\[
\varphi : R_n = R[X]/\langle X^n - 1 \rangle \cong \bigoplus_{i \in I} R[X]/\langle g_i \rangle.
\]

Making use of (3.2), Dinh and López-Permouth obtained the structure of cyclic codes of length \( n \) over \( R \). It was shown that any ideal in \( R_n \) is a sum of ideals of the form \( \langle \gamma^j \hat{g}_i + \langle X^n - 1 \rangle \rangle \) (see [5, Theorem 3.2]). In the light of Lemma 3.3, we have another characterization of cyclic codes of length \( n \) over \( R \) with polynomial generators in terms of \( f_i, i \in I \).

**Theorem 3.4.** Let the notation be the same as before. Let \( C \) be a cyclic code of length \( n \) over \( R \). Then \( C \cong \bigoplus_{i \in I} \langle \gamma^{k_i} + \langle g_i \rangle \rangle \) under the map given by (3.2) if and only if \( C = \langle \prod_{i \in I} f_i^{k_i} \rangle \), where \( \langle \gamma^{k_i} + \langle g_i \rangle \rangle \) is an ideal of \( R[X]/\langle g_i \rangle \) with \( 0 \leq k_i \leq t \). In this case, \( |C| = q^{\sum_{i \in I} \deg f_i} \). Moreover, for any ideal \( C \) in \( R_n \), there exists a unique sequence \( (k_i)_{i \in I} \), with \( 0 \leq k_i \leq t \), such that \( C = \langle \prod_{i \in I} f_i^{k_i} \rangle \).

**Proof.** From Lemma 3.3

\[
\bigoplus_{i \in I} \langle \gamma^{k_i} + \langle g_i \rangle \rangle = \bigoplus_{i \in I} \langle f_i^{k_i} + \langle g_i \rangle \rangle.
\]

It is readily seen that, under the map \( \varphi \) given by (3.2),

\[
\varphi \left( \langle \prod_{j \in I} f_j^{k_j} \rangle \right) = \bigoplus_{i \in I} \langle \prod_{j \in I} f_j^{k_j} + \langle g_i \rangle \rangle = \bigoplus_{i \in I} \langle f_i^{k_i} + \langle g_i \rangle \rangle.
\]

The last equality holds because \( \prod_{j \in I \setminus \{i\}} f_j^{k_j} + \langle g_i \rangle \) is a unit in \( R[X]/\langle g_i \rangle \). We have also shown that, for any cyclic code \( C \) of length \( n \) over \( R \), there exists a sequence \( (k_i)_{i \in I} \) with \( 0 \leq k_i \leq t \) such that \( C = \langle \prod_{i \in I} f_i^{k_i} \rangle \).

Uniqueness can be proved as follows: if \( \langle \prod_{i \in I} f_i^{k_i} \rangle = \langle \prod_{i \in I} f_i^{k_i} \rangle \) with \( 0 \leq k_i, k'_i \leq t \) for all \( i \in I \), then

\[
\bigoplus_{i \in I} \langle \gamma^{k_i} + \langle g_i \rangle \rangle = \varphi \left( \langle \prod_{i \in I} f_i^{k_i} \rangle \right) = \varphi \left( \langle \prod_{i \in I} f_i^{k_i} \rangle \right) = \bigoplus_{i \in I} \langle \gamma^{k_i} + \langle g_i \rangle \rangle.
\]

This forces \( k_i = k'_i \) for all \( i \in I \).

To complete the proof, we have

\[
|C| = \prod_{i \in I} |\langle \gamma^{k_i} + \langle g_i \rangle \rangle| = \prod_{i \in I} q^{(t-k_i)\deg g_i} = q^{\sum_{i \in I} (t-k_i)\deg f_i}.
\]

\[\Box\]

**Remark 3.5.** The pairwise coprime monic basic irreducible factors of \( X^n - r_0 \) in \( R[X] \) can be easily derived from the pairwise coprime monic basic irreducible factors of \( X^n - 1 \) in \( R[X] \). To see this, observe that \( r_0 = 1 + \gamma \mu \) is an element in the Sylow p-subgroup of \( R^* \), where \( R^* \) stands for the unit group of \( R \). Let \( P \) be the Sylow p-subgroup of \( R^* \). Since \( \gcd(n, p) = 1 \), then \( \theta : P \to P^* \), defined by \( \theta(y) = y^n \), is actually an automorphism. Thus, we can find a unique element \( \delta \) of \( P \) such that \( \delta^{n_0} = 1 \). Therefore, if we already have the pairwise coprime monic basic irreducible factorization of \( X^n - 1 \) in \( R[X] \):

\[
X^n - 1 = \prod_{i \in I} g_i(X),
\]
then substitute $X$ for $\delta X$ to obtain the pairwise coprime monic basic irreducible factorization of $X^n - r_0$ in $R[X]$: 

$$X^n - r_0 = \prod_{i \in I} \left( \delta^{-\deg g_i} g_i(\delta X) \right).$$

### 4 Dual cyclic codes

We can also characterize the dual code of $C$ in terms of the polynomials $f_i, i \in I$. Recall that $X^n - 1 = \prod_{i \in I} h_i$ gives the monic irreducible factorization of $X^n - 1$ in $\mathbb{F}_q[X]$, and that $X^n - r_0 = \prod_{i \in I} f_i$ is the pairwise coprime monic basic irreducible factorization in $R[X]$ with $f_i = h_i$ for subscripts in this range. Observe that $h_i^*$ is also a monic divisor of $X^n - 1$ in $\mathbb{F}_q[X]$. Thus, for each $i \in I$, there exists a unique $i' \in I$ such that $h_i = h_i^*$, This implies that ' is a bijection from $I$ onto $I$, which satisfies $(i')' = i$ for all $i \in I$.

**Lemma 4.1.** With respect to the above notation, let $C = \left\{ \prod_{i \in I} (f_i^*)^{k_i} \right\}$ be a cyclic code of length $n$ over $R$ with $0 \leq k_i \leq t$. We then have $C = \left\{ \prod_{i \in I} f_{i'}^{k_i} \right\}$.

**Proof.** We know that $\prod_{i \in I} f_i^* = \prod_{i \in I} h_i = h_{i'} = \prod_{i \in I} f_i$, which implies that there exists $q_i \in R[X]$ such that $f_i = f_i^* + \gamma q_i$. Now in $\mathcal{R}_n$, 

$$r_0 \prod_{i \in I} f_i^* = r_0(X^n - r_0^{-1}) = 1 + \mu \gamma - 1 = \mu \gamma$$

and

$$\prod_{i \in I} f_i^* = X^n - r_0 = 1 - 1 - \mu \gamma = -\mu \gamma.$$

It follows that

$$\prod_{i \in I} f_i^{k_i} = \prod_{i \in I} (f_i^* + \gamma q_i)^{k_i} = \prod_{i \in I} (f_i^* + \mu^{-1} r_0 q_i) \prod_{j \in I} f_j^{k_j},$$

$$\prod_{i \in I} (f_i^*)^{k_i} = \prod_{i \in I} (f_i^* - \gamma q_i)^{k_i} = \prod_{i \in I} (f_i^* - \mu^{-1} q_i) \prod_{j \in I} f_j^{k_j}.$$ 

Clearly $\prod_{i \in I} (f_i^*)^{k_i}$ is a divisor of $\prod_{i \in I} f_i^{k_i}$ and vice versa. We have obtained the desired result. 

**Lemma 4.2.** Let $C = \left\{ \prod_{i \in I} (f_i^*)^{k_i} \right\}$ be a cyclic code of length $n$ over $R$, where the polynomials $f_i$ are the pairwise coprime monic basic irreducible factors of $X^n - r_0$ in $R[X]$ and $0 \leq k_i \leq t$ for each $i \in I$. Then $C^\perp = \left\{ \prod_{i \in I} f_{i'}^{-k_i} \right\}$ and $|C^\perp| = q^{\sum_{i \in I} \deg f_i}.$

**Proof.** By Theorem 8.31

$$|C^\perp| = \frac{|R|^n}{|C|} = \frac{q^{nt}}{q^{\sum_{i \in I} (t - k_i) \deg f_i}} = q^{\sum_{i \in I} k_i \deg f_i - \sum_{i \in I} k_i \deg f_{i'}} = q^{\sum_{i \in I} k_i \deg f_i} = \left| \prod_{i \in I} f_{i'}^{-k_i} \right|.$$ 

The fourth equality holds because $\deg f_{i'} = \deg f_i$. From Lemma 4.1, $\left\langle \prod_{i \in I} (f_i^*)^{t - k_i} \right\rangle = \left\langle \prod_{i \in I} f_{i'}^{t - k_i} \right\rangle$, it remains to prove that $\left\langle \prod_{i \in I} (f_i^*)^{t - k_i} \right\rangle \subset C^\perp$. Following Proposition 2.12, it suffices to show that $\prod_{i \in I} f_i^{k_i} \cdot \left\langle \prod_{i \in I} (f_i^*)^{t - k_i} \right\rangle^* = 0$ in $\mathcal{R}_n$. Indeed,

$$\prod_{i \in I} f_i^{k_i} \cdot \left\langle \prod_{i \in I} (f_i^*)^{t - k_i} \right\rangle^* = \delta \prod_{i \in I} f_i^t = \delta (X^n - 1 - \mu \gamma)^t = 0,$$

where $\delta$ is a suitable unit of $\mathcal{R}_n$. 

□
We now produce a criterion to determine whether or not a given cyclic code of length $n$ over $R$ is self-dual.

**Theorem 4.3.** Let $C = \langle \prod_{i \in I} f_i^{k_i} \rangle$ be a cyclic code of length $n$ over $R$, where $f_i$ are the pairwise coprime monic basic irreducible factors of $X^n - r_0$ in $R[X]$ and $0 \leq k_i \leq t$. Then $C$ is self-dual if and only if $k_i + k_{i'} = t$ for all $i \in I$.

**Proof.** Recall that $'$ is a bijection from $I$ onto $I$, which satisfies $(i')' = i$ for all $i \in I$. Then

$$C^\perp = \langle \prod_{i \in I} f_i^{-k_i} \rangle = \langle \prod_{i \in I} f_{i'}^{-k_{i'}} \rangle = \langle \prod_{i \in I} f_i^{-k_{i'}} \rangle.$$ 

Comparing with $C = \langle \prod_{i \in I} f_i^{k_i} \rangle$, it follows that $C = C^\perp$ if and only if $k_i + k_{i'} = t$ for all $i \in I$. \hfill $\square$

From the criterion above, we are led to a simple condition for the existence of self-dual cyclic codes over finite chain rings.

**Theorem 4.4.** Let the notation be the same as before. Then there exists a self-dual cyclic code of length $n$ over $R$ if and only if $t$, the nilpotency index of $R$, is even.

**Proof.** If $t$ is even, then $\langle \prod_{i \in I} f_i^\mp \rangle$ is a self-dual cyclic code of length $n$ over $R$.

Conversely, assume that there exists a self-dual cyclic code $C = \langle \prod_{i \in I} f_i^{k_i} \rangle$ of length $n$ over $R$. From Theorem 4.3, $k_i + k_{i'} = t$ for all $i \in I$. In particular, 0 is always an element in $I$ with $0' = 0$. It follows that $2k_0 = t$, which gives the desired result. \hfill $\square$

Recall that $I$ is a fixed complete set of representatives of all $q$-cyclotomic cosets modulo $n$. Let $\Omega_n$ and $\Delta_n$ be the sets $\Omega_n = \{ i \in I | i' = i \}$ and $\Delta_n = \{ i \in I | i' \neq i \} = \{ i_1, i_1', \ldots, i_s, i_s' \}$ respectively. Clearly $I$ is the disjoint union of $\Omega_n$ and $\Delta_n$, $I = \Omega_n \cup \Delta_n$. It follows that $X^n - r_0 = \prod_{i \in \Omega_n} f_i \cdot \prod_{j = 1}^s f_{i_j} f_{i_j}'$. Similar to [11, Theorem 2] and [11, Corollary 1], we can characterize all self-dual cyclic codes according to the sets $\Omega_n$ and $\Delta_n$.

**Corollary 4.5.** With respect to the above notation, assume that $t$ is even. We then have that $C$ is a self-dual cyclic code of length $n$ over $R$ if and only if $C$ can be expressed as the form $\langle \prod_{i \in I} f_i^{k_i} \rangle \cdot \prod_{j = 1}^s f_{i_j} f_{i_j}'$, where $k_j$ are integers with $0 \leq k_j \leq t$. In particular, there are exactly $(t+1)^s = (t+1)^{\Delta_n}$ self-dual cyclic codes of length $n$ over $R$.

When the nilpotency index $t$ is even, the self-dual cyclic code $\langle \prod_{i \in I} f_i^\mp \rangle$ is called trivial self-dual code.

In order to investigate the existence conditions for nontrivial self-dual cyclic codes, we need the following observation.

**Lemma 4.6.** Let $g \in R[X]$ be a monic basic irreducible factor of $X^n - 1$. Let $h \in F_q[X]$ be the image of $g$ under the surjective ring homomorphism $\xrightarrow{\cdot \mp}$ from $R[X]$ onto $F_q[X]$, namely $\bar{g} = h$. Then $g$ and $g^*$ are associates in $R[X]$ if and only if $h$ and $h^*$ are associates in $F_q[X]$.

**Proof.** Obviously, if $g$ and $g^*$ are associates in $R[X]$ then $h$ and $h^*$ are associates in $F_q[X]$.

Conversely, assume that $h$ and $h^*$ are associates in $F_q[X]$. Suppose otherwise that $g$ and $g^*$ are not associates in $R[X]$. Consider the homomorphism $\rho$ as given in the proof of Lemma 3.2

$$\rho : R[X]/\langle g \rangle \to F_q[X]/\langle h \rangle, \quad \sum_{j=0}^n a_j X^j + \langle g \rangle \mapsto \sum_{j=0}^n \bar{a}_j X^j + \langle h \rangle,$$

for any $a_0, a_1, \ldots, a_n$ in $R$.

On the one hand, $g^*$ is coprime with $g$ in $R[X]$. This implies that $g^* + \langle g \rangle$ is a unit in $R[X]/\langle g \rangle$, and so is $\rho(g^* + \langle g \rangle)$ in $F_q[X]/\langle h \rangle$. On the other hand, $\rho(g^* + \langle g \rangle) = h^* + \langle h \rangle = 0$ in $F_q[X]/\langle h \rangle$. This is a contradiction. \hfill $\square$
Remark 4.7. It follows from Theorem 4.4 and Corollary 4.5 that nontrivial self-dual cyclic codes of length \( n \) over \( R \) exist if and only if \( t \) is even and \( |\Delta_n| > 0 \). Clearly, the condition \( |\Delta_n| > 0 \) holds if and only if there exists a monic irreducible factor \( \eta \in \mathbb{F}_q[X] \) of \( X^n - 1 \) such that \( \eta \) and \( \eta^* \) are not associates. Thanks to Lemma 4.6, nontrivial self-dual cyclic codes of length \( n \) over \( R \) exist if and only if there exists a monic basic irreducible factor \( g \in R[X] \) of \( X^n - 1 \) such that \( g \) and \( g^* \) are not associates. In conclusion, we have the following result.

Theorem 4.8. Assume that the nilpotency index \( t \) is even. The following five statements are equivalent to one another:

(i) Nontrivial self-dual cyclic codes of length \( n \) over \( R \) exist.

(ii) The cardinality of the set \( \Delta_n \) is nonzero, i.e., \( |\Delta_n| > 0 \).

(iii) \( q^i \neq -1 \mod n \) for all positive integer \( i \), where \( q \) is the order of the residue field \( \mathbb{F}_q = R/\langle \gamma \rangle \).

(iv) There exists a monic irreducible factor \( \eta \in \mathbb{F}_q[X] \) of \( X^n - 1 \) such that \( \eta \) and \( \eta^* \) are not associates.

(v) There exists a monic basic irreducible factor \( g \in R[X] \) of \( X^n - 1 \) such that \( g \) and \( g^* \) are not associates.

Note that the equivalence of (i), (iii) and (v) appeared previously in [5, Theorem 4.3] and [5, Theorem 4.4].

5 Enumeration of self-dual cyclic codes

In this section, we study an enumeration formula for self-dual cyclic codes of length \( n \) over \( R \). It follows from Corollary 4.6 that, if the nilpotency index \( t \) is even, this number is fully determined by \( |\Delta_n| \), the number of reciprocal polynomial pairs in the monic irreducible factorization of \( X^n - 1 \) over \( \mathbb{F}_q \). Recall that the value \( |\Delta_n| + |\Omega_n| \) is exactly equal to the number of all monic irreducible factors of \( X^n - 1 \) over \( \mathbb{F}_q \), where \( |\Omega_n| \) is the cardinality of all self-reciprocal monic irreducible factors of \( X^n - 1 \) over \( \mathbb{F}_q \). Meanwhile, one knows that the number of monic irreducible factors of \( X^n - 1 \) over \( \mathbb{F}_q \) can be explicitly given by \( \sum_{d|n} \phi(d) \delta_d(n) \), where \( \phi \) is Euler’s function. Thus, the counting problem for \( |\Delta_n| \) naturally reduces to the equivalent question of determining the size of \( \Omega_n \).

5.1 An enumeration formula for \( |\Omega_{2^m}| \)

We first consider the case when the code length \( n \) is a power of 2, \( n = 2^m \). The value \( |\Omega_{2^m}| \) can be easily determined. In fact, the irreducible factorization of \( X^{2^m} - 1 \) over \( \mathbb{F}_q \) has been given explicitly (e.g., see [15] or [4] Theorem 3.1 for the case \( q \equiv 1 \mod 4 \), and see [2] Corollary 4 or [3] Lemma 2.2 for the case \( q \equiv -1 \mod 4 \)). For convenience, we reproduce these results below.

Lemma 5.1. Assume that \( q \equiv 1 \mod 4 \). Write \( q - 1 = 2^vc \) with \( \gcd(2, c) = 1 \) and \( v \geq 2 \). Let \( \eta \) be a primitive \( 2^v \)th root of unity in \( \mathbb{F}_q \). Then

\[
X^{2^m} - 1 = \begin{cases} 
\prod_{k=0}^{2^m-1} (X - \eta^k) \cdot \prod_{j=1}^{m-v} \prod_{i=1}^{2^j-1} (X^{2^j} - \eta^i), & \text{if } m > v; \\
\prod_{k=0}^{2^m-1} (X - \delta^k), & \text{if } m \leq v,
\end{cases}
\]

where \( \delta \) is a primitive \( 2^m \)th root of unity in \( \mathbb{F}_q \) for \( m \leq v \). All the factors on the right hand side of the equation above are irreducible over \( \mathbb{F}_q \).

Next is the case \( q \equiv -1 \mod 4 \). Note that \( 4 \mid (q+1) \) in this case, hence there is a unique integer \( a \geq 2 \) such that \( 2^a \mid (q+1) \), where the notation \( 2^a \mid (q+1) \) means \( 2^a \mid (q+1) \) but \( 2^{a+1} \nmid (q+1) \).

Lemma 5.2. Assume that \( q \equiv -1 \mod 4 \). Set \( H_1 = \{0\} \); recursively define

\[
H_i = \left\{ \pm \left( \frac{h+1}{2} \right)^{\frac{a_i}{2}} \mid h \in H_{i-1} \right\},
\]

where \( a_i \) is a divisor of \( a \).
for $i = 2, 3, \cdots, a - 1$; and set

$$H_a = \left\{ \pm \left( \frac{d-1}{2} \right)^{d+1} \mid h \in H_{a-1} \right\}.$$  

Then for $1 \leq i \leq a$, $H_i$ has cardinality $2^{i-1}$. The irreducible factorization of $X^{2^m} - 1$ over $\mathbb{F}_q$ is given as follows:

If $1 \leq m \leq a$, then

$$X^{2^m} - 1 = (X - 1)(X + 1) \prod_{h \in H_i} \prod_{1 \leq i \leq 2^{i-1}} (X^2 - 2hX + 1); \quad (5.1)$$

if $m \geq a + 1$, then

$$X^{2^m} - 1 = (X - 1)(X + 1) \prod_{h \in H_i} \prod_{1 \leq i \leq 2^{i-1} \left. \frac{x}{x} \right| (a-1)} (X^2 - 2hX + 1) \prod_{h \in H_i} \prod_{0 \leq k \leq (m-a-1)} (X^{2^k+1} - 2aX^{2^k} - 1). \quad (5.2)$$

The above two lemmas combine to give the following result.

**Proposition 5.3.** The number of self-reciprocal monic irreducible factors of $X^{2^m} - 1$ over $\mathbb{F}_q$ is explicitly given by

$$|\Omega_{2^m}| = \begin{cases} 1, & \text{if } m = 0; \\ 2^{\min(m,a)-1} + 1, & \text{if } m \geq 2 \text{ and } 4 \mid (q - 1); \\ 2, & \text{if } m = 1 \text{ or } m \geq 2 \text{ and } 4 \nmid (q - 1). \\ \end{cases} \quad (5.3)$$

### 5.2 A reduction formula for $|\Omega_{2^m}|$

We turn our attention to the more general case. Let $n = 2^m n'$ with $n' = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $p_j$ are distinct odd primes and $r_j$ are positive integers for $1 \leq j \leq k$. Our major goal is to show that, the problem for determining the value of $|\Omega_{2^m n'}|$ can be entirely reduced to computing $|\Omega_{n'}|$ and $|\Omega_{n'}|$, where $|\Omega_{n'}|$ denotes the number of self-reciprocal monic irreducible factors of $X^{n'} - 1$ over $\mathbb{F}_{q^n}$.

**Lemma 5.4.** Let $\ell$ be an odd prime integer coprime with $q$, and let $s$ be a positive integer. Then $|\Omega_{\ell^s}| = 1$ if and only if $q$ has odd order in the multiplicative group of integers modulo $\ell$, i.e., $2 \nmid \text{ord}_{\ell}(q)$.

**Proof.** Note that $\text{ord}_q(q)$ is odd if and only if $\text{ord}_q(q)$ is odd. Indeed, if $\text{ord}_q(q) = e$ is odd, then there is an integer $k$ such that $q^e \equiv 1 + \ell k \pmod{\ell^e}$. Now the desired result follows from the fact that the natural surjective homomorphism $\pi : \mathbb{Z}_\ell^* \to \mathbb{Z}_\ell^*$ with $\text{Ker}\pi$ being odd, and $1 + \ell k \ell^s \in \text{Ker}\pi$.

Let $C_i = \{ i \cdot q^j \pmod{\ell^s} \mid j = 0, 1, \ldots \}$ be the $q$-cycloidal coset modulo $\ell^s$ containing $i$. Equivalently, we need to prove that $C_i \neq C_{-i}$ for any integer $i \neq 0 \pmod{\ell^s}$ if and only if $\text{ord}_q(q)$ is odd.

We first assume that $C_i \neq C_{-i}$, for any integer $i \neq 0 \pmod{\ell^s}$. Recall that $\mathbb{Z}_{\ell^s}^* = \{ [k]_{\ell^s} \mid \text{gcd}(k, \ell) = 1 \}$ is a cyclic group, which implies that $[-1]_{\ell^s}$ is the unique element of $\mathbb{Z}_{\ell^s}^*$, with order $2$. If $f = \text{ord}_q(q)$ is odd, then $q^{f/2} \equiv -1 \pmod{\ell^s}$. This is a contradiction, since we would obtain $C_1 = C_{-1}$.

Conversely, assume that $f = \text{ord}_q(q)$ is odd. Suppose otherwise that there exists an integer $i_0$ with $i_0 \neq 0 \pmod{\ell^s}$ satisfying $C_{i_0} = C_{-i_0}$. That is to say, an integer $j$ can be found so that $q^j i_0 \equiv -i_0 \pmod{\ell^s}$. We write $i_0 = \ell^s a_0 \text{gcd}(\ell, a_0) = 1$. Clearly $s > 0$. We then have $q^j a_0 \equiv -a_0 \pmod{\ell^s}$, which leads to $q^{j_0} a_0^j \equiv -a_0^j \pmod{\ell^s}$, implying that $q^{j_0} a_0^j \equiv -a_0^j \pmod{\ell^s}$. It follows that $a_0^j \equiv -a_0^j \pmod{\ell^s}$, and thus $\ell \mid a_0$. This is a contradiction. \qed

**Remark 5.5.** From the proof of Lemma 5.4, one can easily deduce that all the monic irreducible factors of $X^{\ell^s} - 1$ over $\mathbb{F}_q$ are self-reciprocal if and only if $\text{ord}_q(q)$ is even.

At this point, we point out that, for any odd prime $\ell$ coprime with $q$, the value $|\Omega_{\ell^s}|$ can be determined easily. Indeed, if $\text{ord}_q(q)$ is odd, then $|\Omega_{\ell^s}| = 1$; otherwise $|\Omega_{\ell^s}| = \sum_{d=0}^{s} \phi(\ell^d) / \text{ord}_q(q)$, the number of all monic irreducible factors of $X^{\ell^s} - 1$ over $\mathbb{F}_q$.  

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For computing the value of $|\Omega_n|$, the following lemma asserts that the odd prime divisor $\ell$ of $n$ can be ruled out once $\text{ord}_\ell(q)$ is odd.

**Lemma 5.6.** Let $\ell$ be an odd prime divisor of $n$, so that $n = \ell^n n_1$ with $\gcd(\ell, n_1) = 1$. Then $|\Omega_n| = |\Omega_{n_1}|$ if and only if $\text{ord}_\ell(q)$ is odd.

**Proof.** Suppose first that $\text{ord}_\ell(q)$ is odd. By Lemma 5.4, we can assume, therefore, that $C_0 = \{0\}, C_{i_1}, C_{i_2}, \ldots, C_{i_r}, C_{i_\rho}, \ldots$ are all the distinct $q$-cyclotomic cosets modulo $\ell^n$. Taking a primitive $\ell^n$th root of unity $\eta$ in a suitable extension field of $\mathbb{F}_q$, we get

$$X^{\ell^n} - 1 = (X - 1)M_{i_1}(X)M_{i_2}(X)\cdots M_{i_\rho}(X)M_{i_\rho}(X),$$

with

$$M_{i_h}(X) = \prod_{j \in C_{i_h}} (X - \eta^j), \quad M_{i_h}(X) = \prod_{j \in C_{i_h}} (X - \eta^j), \quad h = 1, \ldots, \rho,$$

all being monic irreducible in $\mathbb{F}_q[X]$. It follows that

$$X^n - 1 = (X^{n_1})^{\ell^n} - 1 = (X^{n_1} - 1)M_{i_1}(X^{n_1})M_{i_2}(X^{n_1})\cdots M_{i_\rho}(X^{n_1})M_{i_\rho}(X^{n_1}).$$

Clearly,

$$M_{i_h}(X^{n_1}) = M_{i_h}(X^{n_1}), \quad \text{for all } 1 \leq h \leq \rho.$$ 

This implies that the polynomials $M_{i_1}(X^{n_1}), \ldots, M_{i_\rho}(X^{n_1})$ contribute nothing to the value of $|\Omega_n|$. We are done for this direction.

Conversely, assume that $|\Omega_n| = |\Omega_{n_1}|$. Observe that $X^n - 1$ and $\frac{X^{\ell^n} - 1}{X^{\ell} - 1}$ are both divisors of $X^n - 1$, and that $\gcd(X^n - 1, \frac{X^{\ell^n} - 1}{X^{\ell} - 1}) = 1$. This actually means that $\frac{X^{\ell^n} - 1}{X^{\ell} - 1}$ contributes nothing to the value of $|\Omega_n|$. We get the desired result from Lemma 5.4 directly. 

For the value $|\Omega_n|$, now we can assume that $n = 2^n n'$ with $n' = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where $p_j$ are distinct odd primes and $r_j$ are positive integers such that $\text{ord}_{p_j}(q)$ are even for all $1 \leq j \leq k$. The following lemma characterizes the relationship between the $q$-cyclotomic cosets modulo $n'$ and the $q^2$-cyclotomic cosets modulo $n'$.

**Lemma 5.7.** Let $n' = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where $p_j$ are distinct odd primes and $r_j$ are positive integers such that $\text{ord}_{p_j}(q)$ are even for all $1 \leq j \leq k$. Suppose that there are exactly $s$ distinct $q$-cyclotomic cosets modulo $n'$. Then the number of distinct $q^2$-cyclotomic cosets modulo $n'$ is precisely given by $2s - 1$.

**Proof.** Let

$$C_z = \{z \cdot q^j \pmod{n'} | j = 0, 1, \ldots\}$$

be any nonzero $q$-cyclotomic coset modulo $n'$. It is clear that

$$D_z = \{z \cdot q^{2j} \pmod{n'} | j = 0, 1, \ldots\} \quad \text{and} \quad D_{zq} = \{zq \cdot q^{2j} \pmod{n'} | j = 0, 1, \ldots\}$$

are $q^2$-cyclotomic cosets modulo $n'$. Obviously, $C_z = D_z \cup D_{zq}$. To complete the proof, it suffices to show that $D_z \neq D_{zq}$. Suppose otherwise that an integer $j$ can be found such that $z \equiv zq \cdot q^{2j} \pmod{n'}$. This implies that $\xi^z$ is an element in $\mathbb{F}_{q^{2j} + 1}$, where $\xi$ is a primitive $n'$th root of unity. This is impossible: Without loss of generality, we can assume that $p_1$ is a prime divisor of $\text{ord}(\xi^z)$, i.e., $\mathbb{F}_{q^{2j+1}}$ contains a primitive $p_1$th root of unity. On the other hand, by our assumption, $f_1 = \text{ord}_{p_1}(q)$ is even, which gives $f_1 \mid (2j + 1)$, a contradiction. 

Let

$$C_{i_h} = \{i_h \cdot q^j \pmod{n'} | j = 0, 1, \ldots\}, \quad 1 \leq h \leq \rho,$$

be all the distinct nonzero $q$-cyclotomic cosets modulo $n'$. From Lemma 5.7, we know that each $q$-cyclotomic coset $C_{i_h}$ is a disjoint union of two $q^2$-cyclotomic cosets:

$$D_{i_h} = \{i_h \cdot q^{2j} \pmod{n'} | j = 0, 1, \ldots\} \quad \text{and} \quad D_{ihq} = \{i_hq \cdot q^{2j} \pmod{n'} | j = 0, 1, \ldots\}.$$
At this point, we can get the irreducible factorization of $X^{n'} - 1$ over $\mathbb{F}_{q^2}$, where $\mathbb{F}_{q^2}$ is the extension field over $\mathbb{F}_q$ such that $[\mathbb{F}_{q^2} : \mathbb{F}_q] = 2$. Let $\eta$ be a primitive $n'$th root of unity in some extension field of $\mathbb{F}_{q^2}$. Then

$$X^{n'} - 1 = (X - 1)N_i(X)N_{i,q}(X)N_{i_2}(X)\cdots N_{i_p}(X)N_{i_0,q}(X),$$

with

$$N_{i,h}(X) = \prod_{u \in D_{i,h}} (X - \eta^u), \quad N_{i_h,q}(X) = \prod_{u \in D_{i,h}} (X - \eta^u), \quad 1 \leq h \leq \rho,$$

all being monic irreducible in $\mathbb{F}_{q^2}[X]$. Note that $X^{n'} - 1 = (X - 1)\prod_{h=1}^{\rho} M_{i,h}(X)$ gives the monic irreducible factorization of $X^{n'} - 1$ over $\mathbb{F}_q$, where $M_{i,h}(X) = N_{i,h}(X)N_{i_0, q}(X)$, $1 \leq h \leq \rho$.

Before giving our results, we make the following observation. Assume that

$$X^{n'} - 1 = (X - 1)M_{i,1}(X)\cdots M_{i,\ell}(X)N_{j,1}(X)\cdots N_{j,\ell}(X)M_{j,1}(X)\cdots M_{j,\ell}(X),$$

where $M_{i,k}(X)$ are self-reciprocal monic irreducible factors for $1 \leq k \leq u$, while $M_{j,s}(X)$ and $M_{j,j-s}(X)$ are reciprocal polynomial pairs for $1 \leq s \leq v$. We can further assume that $X - 1, N_{i,1}, N_{i,1,q}, \ldots, N_{i,\rho,1}, N_{i,\rho, q}$, are self-reciprocal monic irreducible factors of $X^{n'} - 1$ over $\mathbb{F}_{q^2}$. That is to say, $2b + 1$ is the number of all self-reciprocal monic irreducible factors of $X^{n'} - 1$ over $\mathbb{F}_{q^2}$, i.e., $|\Omega^{n'}| = 2b + 1$. Now the irreducible factorization of $X^{n'} - 1$ over $\mathbb{F}_{q^2}$ can be given as follows:

$$X^{n'} - 1 = (X - 1)\prod_{i=1}^{\rho} M_{i,j}(X)\cdot \prod_{i=1}^{\rho} N_{i,j}(X)\cdot \prod_{i=1}^{\rho} N_{i,j,q}(X).$$

We assert that $N_{i,j} = N_{i,j,q}$ for $(b + 1) \leq j \leq u$; this is because for $(b + 1) \leq j \leq u$, $N_{i,j,q} = M_{i,j} = M_{i,j}^{*} = N_{i,j}N_{i,j,q}$, and $N_{i,j} \neq N_{i,j,q}$ by assumption.

Let $f(X)$ be a polynomial in $\mathbb{F}_q[X]$ with leading coefficient $a_n \neq 0$. Recall from Notation 3.1 that $\hat{f}(X) = a_n^{-1} f(X)$ is a monic polynomial over $\mathbb{F}_q$.

**Theorem 5.8.** With respect to the above notation, we then have

$$|\Omega_n| = |\Omega^{2m^2n'}| = \begin{cases} 2|\Omega^{n'}|, & \text{if } m = 1, \text{ or } m \geq 2 \text{ and } 4 \nmid (q - 1); \\ 2|\Omega^{n'}| + (2\min\{m,a\} - 1 - 1)(2|\Omega^{n'}| - |\Omega^{n'}|), & \text{if } m \geq 2 \text{ and } 4 \nmid (q - 1). \end{cases}$$

Here, for the case $4 \nmid (q - 1)$, $a$ is the unique integer such that $2a \equiv (q + 1) \mod 4$.

**Proof.** If $m = 1$, the result follows trivially. Indeed, from $X^n - 1 = (X^{n'} - 1)(X^{n'} + 1)$, we easily get $|\Omega_n| = 2|\Omega^{n'}|$. We prove by induction on $m$ for the case $m \geq 2$ and $q \equiv 1 \mod 4$. If $m = 2$, then $X^{2n'} - 1 = (X^{n'} - 1)(X^{n'} + 1)(X^{n'} - \alpha)(X^{n'} + \alpha)$, where $\alpha$ is a primitive fourth root of unity in $\mathbb{F}_q$. Observe that $(X^{n'} - \alpha)^* = X^{n'} - \alpha^{-1} = X^{n'} + \alpha$, which implies that $X^{n'} - \alpha$ and $X^{n'} + \alpha$ contribute nothing to the value of $|\Omega_n|$. Hence the required result follows directly. For the inductive step, we write

$$X^n - 1 = (X^{2m-1n'} - 1)(X^{2m-2n'} - \alpha)(X^{2m-2n'} + \alpha).$$

Similar reasoning then shows that $X^{2m-2n'} - \alpha$ and $X^{2m-2n'} + \alpha$ contribute nothing to the value of $|\Omega_n|$. Thus $|\Omega_n| = |\Omega^{2m-1n'}| = 2|\Omega^{n'}|$ by induction.

We are left with the case $m \geq 2$ and $4 \nmid (q - 1)$. We use Lemma 5.2 to prove this result. Assume first that $2 \leq m \leq a$. From (5.1),

$$X^{2m} - 1 = (X - 1)(X + 1) \prod_{i=1}^{\ell} \prod_{h \in H_i} (X^2 - 2hX + 1).$$

Then

$$X^{2m+1} - 1 = (X^{n'} - 1)(X^{n'} + 1) \prod_{i=1}^{\ell} \prod_{h \in H_i} (X^{2n'} - 2hx^{n'} + 1).$$
The irreducible factorization of $X^{2n' - 2hX^{n'}} + 1$ over $\mathbb{F}_q^2$, $1 \leq i \leq m - 1$ and $h \in H_i$, can be described via the $q^2$-cyclotomic cosets modulo $n'$, as we will show shortly. Since $X^2 - 2hX + 1$ is an irreducible factor of $X^{2n'} - 1$ over $\mathbb{F}_q$, there exists an element $\beta_h$ in the Sylow 2-subgroup of $\mathbb{F}_q^2$ such that $X^2 - 2hX + 1 = (X - \beta_h)(X - \beta_h^{-1})$. Note that $\beta_h^q = \beta_h^{-1}$. We then have

$$X^{2n' - 2hX^{n'}} + 1 = (X^i - \beta_h)(X^{n'} - \beta_h^{-1}).$$

On the one hand, for each element $\beta_h$, there exists a unique element $\lambda_h$ in the Sylow 2-subgroup of $\mathbb{F}_q^2$ such that $\lambda_h^q \beta_h = 1$. We also note that $\lambda_h^q = \lambda_h^{-1}$ for each $h \in H_i$, because $\lambda_h^q = \beta_h^q = \lambda_h^{-n'}$. On the other hand, (5.4) gives the monic irreducible factorization of $X^{n'} - 1$ over $\mathbb{F}_q^2$. We substitute $X$ for $\lambda_h X$ in (5.4) to obtain the monic irreducible factorization of $X^{n'} - \beta_h$ over $\mathbb{F}_q^2$:

$$X^{n'} - \beta_h = (X - \lambda_h^{-1})\hat{N}_{i_1}(\lambda_h X) \cdots \hat{N}_{i_{q-1}}(\lambda_h X)\hat{N}_{i_{q+1}}(\lambda_h X) \cdots \hat{N}_{i_{q-1}}(\lambda_h X)\hat{N}_{i_{q+1}}(\lambda_h X) \cdots \hat{N}_{i_{q-1}}(\lambda_h X).$$

Similarly,

$$X^{n'} - \beta_h^{-1} = (X - \lambda^{-1})\hat{N}_{i_1}(\lambda^{-1} X) \cdots \hat{N}_{i_{q-1}}(\lambda^{-1} X)\hat{N}_{i_{q+1}}(\lambda^{-1} X) \cdots \hat{N}_{i_{q-1}}(\lambda^{-1} X)\hat{N}_{i_{q+1}}(\lambda^{-1} X) \cdots \hat{N}_{i_{q-1}}(\lambda^{-1} X).$$

Now it is easy to check that the polynomial $(X - \lambda^{-1})(X - \lambda)$ is self-reciprocal monic irreducible over $\mathbb{F}_q$. For $1 \leq k \leq b$, we assert that $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ and $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ are irreducible over $\mathbb{F}_q$ and form a reciprocal polynomial pair:

$$\left(\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)\right)^* = \hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X).$$

Assuming that $\eta$ is a primitive $n'$th root of unity in some extension field of $\mathbb{F}_q^2$, then

$$\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X) = \prod_{j \in D_{i_k}} (X - \lambda_h^{-1} \eta^j) \cdot \prod_{j \in D_{i_k}} (X - \lambda_h \eta^j).$$

For every $j \in D_{i_k}$, $\lambda_h^q \eta^j = \lambda_h \eta^j$ is a root of $\hat{N}_{i_k}(\lambda^{-1} X)$; for every $j \in D_{i_k}$, $\lambda_h^q \eta^j = \lambda_h^{-1} \eta^j$ is a root of $\hat{N}_{i_k}(\lambda X)$. In particular, the roots of $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ are invariant under the action of the Galois group $\text{Gal}(\mathbb{F}_q^2/\mathbb{F}_q)$. It follows that $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ is a polynomial over $\mathbb{F}_q$. Moreover, $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ is irreducible over $\mathbb{F}_q$ since $\hat{N}_{i_k}(\lambda_h X) \notin \mathbb{F}_q[X]$ and $\hat{N}_{i_k}(\lambda^{-1} X) \notin \mathbb{F}_q[X]$. Similar reasoning shows that $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ is irreducible over $\mathbb{F}_q$. We are left with proving Formula (5.5). Note that

$$\hat{N}_{i_k}(\lambda_h X)^* = \left(\prod_{j \in D_{i_k}} (X - \lambda_h^{-1} \eta^j)\right)^* = \prod_{j \in D_{i_k}} (X - \lambda_h \eta^j) = \hat{N}_{i_k}(\lambda_h X).$$

The third equality holds because $D_{-i_k} = D_{i_k}$ for $1 \leq k \leq b$. Similarly, $\hat{N}_{i_k}(\lambda^{-1} X)^* = \hat{N}_{i_k}(\lambda_h X)^*$. Thus, Formula (5.5) has been established.

Using similar arguments, for $(b + 1) \leq k \leq u$, $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ and $\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)$ are self-reciprocal monic irreducible polynomials over $\mathbb{F}_q$:

$$\left(\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)\right)^* = \hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X),$$

$$\left(\hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X)\right)^* = \hat{N}_{i_k}(\lambda_h X)\hat{N}_{i_k}(\lambda^{-1} X).$$

Finally for $1 \leq k \leq v$,

$$\left(\hat{N}_{j_k}(\lambda_h X)\hat{N}_{j_k}(\lambda^{-1} X)\right)^* = \hat{N}_{-j_k}(\lambda_h X)\hat{N}_{-j_k}(\lambda^{-1} X),$$

$$\left(\hat{N}_{j_k}(\lambda_h X)\hat{N}_{j_k}(\lambda^{-1} X)\right)^* = \hat{N}_{-j_k}(\lambda_h X)\hat{N}_{-j_k}(\lambda^{-1} X).$$

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It is readily seen that, for every $1 \leq i \leq m - 1$ and $h \in H_i$, there are exactly $1 + 2(u - b)$ self-reciprocal monic irreducible factors of $X^{2n} - 2hX^n + 1$ over $\mathbb{F}_q$. Consequently, $\prod_{h \in H_i}^{m-1} (X^{2n} - 2hX^n + 1)$ contributes $(2^{m-1} - 1)(2|\Omega_n'| - |\overline{\Omega}_n'|)$ self-reciprocal monic irreducible factors to the value of $|\Omega_n|$, since $|\Omega_n'| = 1 + u$ and $|\overline{\Omega}_n'| = 1 + 2b$.

We are left to consider the case $m \geq a + 1$. It follows from (5.2) that

$$X^{2m-n} - 1 = (X^n - 1)(X^{n'} + 1) \prod_{k \in \mathbb{N}, \ 1 \leq k \leq (a-1)} (X^{2n} - 2hX^n + 1) \prod_{k \in \mathbb{N}, \ 0 \leq k \leq (m-a-1)} (X^{2k+1-n'} - 2hX^{2k+n'} - 1).$$

We just note that the last term contributes nothing to the value of $|\Omega_n|$. Similar reasoning yields our desired result.

As an immediate application of Theorem 5.8, a general formula for the value of $|\Omega_{2^m,r^e}|$ can be given explicitly, as we show below.

**Corollary 5.9.** Let $\ell$ be an odd prime integer coprime with $q$, and let $s$ be a positive integer. If $\ord_{\ell}(q)$ is odd, then $|\Omega_{2^m,r^e}| = |\Omega_{2^m}|$, where $|\Omega_{2^m}|$ was explicitly given by (5.3). Otherwise, we have:

- $(i)$ If $m = 1$, or $m \geq 2$ and $4 \nmid (q - 1)$, then $|\Omega_{2^m,r^e}| = 2|\Omega_{r^e}| - 2 \sum_{d=0}^{s} \frac{\phi(d \ell^e)}{\ord_{d \ell^e}(q)} - 2 \sum_{d=0}^{s} \frac{\phi(d \ell^e)}{\ord_{d \ell^e}(q)} - |\overline{\Omega}_{r^e}|)$.

Here, if $2 \ord_{d \ell^e}(q)$, then $|\overline{\Omega}_{r^e}| = 1$; otherwise, $|\overline{\Omega}_{r^e}| = 2 \sum_{d=0}^{s} \frac{\phi(d \ell^e)}{\ord_{d \ell^e}(q)} - 1$.

### 5.3 An enumeration formula for $|\Omega_{t_1^e,t_2^e}|$

In this subsection, we give a general formula for the value of $|\Omega_{t_1^e,t_2^e}|$, where $t_1, t_2$ are distinct odd primes coprime with $q$, and $r_1, r_2$ are positive integers. We set $\ord_{t_i^e}(q) = 2^{a_i} f_i$ with $\gcd(2, f_i) = 1$, $i = 1, 2$. By Lemma 5.3, we can assume that $a_1 \geq 1$ and $a_2 \geq 1$.

If $a_1 = a_2 = 1$, we claim that all monic irreducible factors of $X^{t_1^e t_2^e} - 1$ over $\mathbb{F}_q$ are self-reciprocal, and hence $|\Omega_{t_1^e,t_2^e}| = \sum_{d | t_1^e t_2^e} \frac{\phi(d \ell^e)}{\ord_{d \ell^e}(q)}$. To this end, it suffices to prove that there exists some integer $t_0$ such that $q^{t_0} \equiv 1 \pmod{\ell_1^e \ell_2^e}$. Since $q^{2^{a_1} f_1 - 1} \equiv -1 \pmod{\ell_1^e}$ and $q^{2^{a_1} f_2 - 1} \equiv -1 \pmod{\ell_2^e}$, it follows that $q^{2^{a_1} f_1 f_2} \equiv -1 \pmod{\ell_1^e \ell_2^e}$. Then we have $q^{2^{a_1} f_1 f_2} \equiv -1 \pmod{\ell_1^e \ell_2^e}$, as claimed.

Thus, without loss of generality, we need to consider the case $1 \leq a_1 < a_2$.

To compute $|\Omega_{t_1^e,t_2^e}|$, we need to know the relationship between $q$-cyclotomic cosets modulo $\ell_2^e$ and $q^{2^{a_1} f_1}$-cyclotomic cosets modulo $\ell_1^e$. In fact, for any nonzero $q$-cyclotomic coset modulo $\ell_2^e$,

$$C_{j_k} = \{ j_k \cdot q^j \pmod{\ell_2^e} \mid j = 0, 1, \cdots \},$$

we assert that $C_{j_k}$ is a disjoint union of $q^{2^{a_1} f_1}$-cyclotomic cosets modulo $\ell_1^e$:

$$C_{j_k} = D_{j_k} \cup D_{j_k q^d} \cup \cdots \cup D_{j_k q^{d_{a_1} - 1}},$$

where $D_{j_k}, D_{j_k q^d}, \cdots, D_{j_k q^{d_{a_1} - 1}}$ are $q^{2^{a_1} f_1}$-cyclotomic cosets modulo $\ell_2^e$ and $d_k$ is the smallest positive integer such that $D_{j_k q^{d_k}} = D_{j_k}$. This can be seen as follows. We can always divide $C_{j_k}$ into unions of $q^{2^{a_1} f_1}$-cyclotomic cosets (not necessary disjoint):

$$C_{j_k} = D_{j_k} \cup D_{j_k q^d} \cup \cdots \cup D_{j_k q^{d_{a_1} - 1}} \cup \cdots \cup D_{j_k q^{2^{a_1} f_1 - 1}}.$$
Note that $D_{j_k q^r i_1} = D_{j_k}$ as $q^{2^{s_1} i_1}$-cyclotomic cosets modulo $\ell_2^2$. Now assume that $d_k$ is the smallest positive integer such that $D_{j_k q^r h_k} = D_{j_k}$. It is clear that every term between $D_{j_k q^r h_k}$ and $D_{j_k q^{2^{s_1} i_1} h_k}$ is exactly equal to one term of $D_{j_k}, D_{j_k q}, \ldots, D_{j_k q^{d_k - 1}}$. Thus, we get the desired decomposition.

In the following, we first give the irreducible factorization of $X^{\ell_1 \ell_2^2} - 1$ over $\mathbb{F}_{q^{n_1} \ell_1}$. Then we recombine the irreducible factors such that each of them is actually irreducible over $\mathbb{F}_{q^r}$, the following well-known fact from Galois theory will be used (e.g., see [10, Theorem 4.14]): Let $\mathbb{E}$ be a finite extension field over $\mathbb{F}_q$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be distinct elements of $\mathbb{E}$ such that $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} = \{\alpha_1^q, \alpha_2^q, \ldots, \alpha_k^q\}$. Then $f(X) = (X - \alpha_1)(X - \alpha_2)\cdots(X - \alpha_k)$ is a monic polynomial over $\mathbb{F}_q$; if, in addition, for any two elements $\alpha_i, \alpha_j$, there exists an integer $s$ such that $\alpha_i^q = \alpha_j$, then $f(X)$ is a monic irreducible polynomial over $\mathbb{F}_q$.

Assume that $C_{h_k} = \{ih \cdot q^j \pmod{\ell_1}\} | j = 0, 1, \ldots\}$, $1 \leq h \leq \rho$, are all the distinct nonzero $q$-cyclotomic cosets modulo $\ell_1^2$. Let

$$X^{\ell_1^2} - 1 = (X - 1)M_{i_1}(X) \cdots M_{i_\rho}(X)$$

be the monic irreducible factorization of $X^{\ell_1^2} - 1$ over $\mathbb{F}_q$, where $M_{i_h}(X) = \prod_{j \in C_{i_h}} (X - \eta^j)$ and $\eta$ is a primitive $\ell_1$th root of unity in $\mathbb{F}_{q^{n_1} \ell_1}$. It follows that

$$X^{\ell_1^2 \ell_2^2} - 1 = (X^{\ell_2^2} - 1)M_{i_1}(X^{\ell_2^2}) \cdots M_{i_\rho}(X^{\ell_2^2}).$$

Note that $M_{i_h}(X^{\ell_2^2}) = M_{i_h}(X^{\ell_2^2})$ for all $1 \leq h \leq \rho$, because $M_{i_h}(X)^* = M_{i_h}(X)$ by Remark 5.5. We need to answer this question: how many self-reciprocal monic irreducible factors of each $M_{i_h}(X^{\ell_2^2})$ contribute to $[\Omega_{\ell_1^2 \ell_2^2}]^*$? The answer is precisely equal to 1, as we will show shortly.

Now, assuming that $\deg M_{i_h}(X) = t_{i_1}$, in $\mathbb{F}_{q^{n_1} \ell_1}[X],$

$$M_{i_h}(X^{\ell_2^2}) = \prod_{s \in C_{i_1}} (X^{\ell_2^2} - \eta^s) = (X^{\ell_2^2} - \eta^{i_1})(X^{\ell_2^2} - \eta^{i_1 + 1}) \cdots (X^{\ell_2^2} - \eta^{i_1 q^{t_{i_1}} - 1}).$$

Let $C_{j_k}$, $1 \leq k \leq \nu$, be all the distinct nonzero $q$-cyclotomic cosets modulo $\ell_2^2$, and let $N_{j_k}(X)$ be the monic irreducible factors over $\mathbb{F}_q$ corresponding to $C_{j_k}$. It follows that $N_{j_k}(X)$ splits into $d_k$ irreducible factors over $\mathbb{F}_{q^{n_1} \ell_1}$, $N_{j_k}(X) = D_{j_k}(X)D_{j_k q}(X)\cdots D_{j_k q^{d_k - 1}}(X)$, where $D_{j_k}(X), \ldots, D_{j_k q^{d_k - 1}}(X)$ are monic irreducible factors over $\mathbb{F}_{q^{n_1} \ell_1}$, corresponding to the $q^{n_1} \ell_1$-cyclotomic cosets $D_{j_k}, \ldots, D_{j_k q^{d_k - 1}}$, respectively. We then have the monic irreducible factorization of $X^{\ell_2^2} - 1$ over $\mathbb{F}_{q^{n_1} \ell_1}$ as follows:

$$X^{\ell_2^2} - 1 = (X - 1)\prod_{k=1}^{\nu} D_{j_k}(X)D_{j_k q}(X)\cdots D_{j_k q^{d_k - 1}}(X).$$

We see that for each $s \in C_{i_1}$, there exists a unique element $\zeta_s$ in the Sylow $\ell_1$-subgroup of $\mathbb{F}_{q^{n_1} \ell_1}^*$ such that $\zeta_s^{\ell_2^2} \eta^s = 1$. Indeed, this is because $\eta$ is an element of the Sylow $\ell_1$-subgroup of $\mathbb{F}_{q^{n_1} \ell_1}^*$ and $\gcd(\ell_1, \ell_2) = 1$. Thus, we substitute $X$ for $\zeta_s X$ in (5.3.6) to obtain

$$X^{\ell_2^2} - \eta^s = (X - \zeta_s^{-1})\prod_{k=1}^{\nu} D_{j_k}(\zeta_s X)D_{j_k q}(\zeta_s X)\cdots D_{j_k q^{d_k - 1}}(\zeta_s X),$$

the monic irreducible factorization of $X^{\ell_2^2} - \eta^s$ over $\mathbb{F}_{q^{n_1} \ell_1}$. At this point, the monic irreducible factorization of $M_{i_1}(X^{\ell_2^2})$ over $\mathbb{F}_{q^{n_1} \ell_1}$ is given by

$$M_{i_1}(X^{\ell_2^2}) = \prod_{s \in C_{i_1}} (X - \zeta_s^{-1})\prod_{k=1}^{\nu} D_{j_k}(\zeta_s X)D_{j_k q}(\zeta_s X)\cdots D_{j_k q^{d_k - 1}}(\zeta_s X).$$
For any integer $k$, since $\zeta_{i,k}^{q^k} = 1$ and $\zeta_{i,k}^{q^k} \eta^{i,k} = 1$, so $(\zeta_{i,k}^{q^k})^e = \eta^{-k}i = \zeta_{i,k}^{q^k}$, which implies $\zeta_{i,k}^{q^k} = \zeta_{i,k}^{q^k}$. It follows that

$$
\prod_{s \in C_{i,k}} (X - \zeta_{s}^{-1}) = (X - \zeta_{i,k}^{-1})(X - \zeta_{i,k}^{-1}) \cdots (X - \zeta_{i,k}^{-1})
$$

is a monic irreducible polynomial over $\mathbb{F}_q$. Moreover, $\prod_{s \in C_{i,k}} (X - \zeta_{s}^{-1})$ is self-reciprocal, since it is a divisor of $X^{t_i} - 1$ over $\mathbb{F}_q$. Now for any positive integer $k$, $1 \leq k \leq \nu$, we analyze the polynomial $F_k(X) = \prod_{s \in C_{i,k}} (D_{i,k}(\zeta_s X) \cdots D_{j,k} q^{s-1} (\zeta_s X))$. The polynomial $F_k(X)$ can be rewritten as follows:

$$
F_k(X) = \prod_{r=0}^{t_i-1} (D_{i,k}(\zeta_i q^{r-i} X) \cdots D_{j,k} q^{s-1}(\zeta_i q^{r-i} - 1 - 1 X)).
$$

Here, $\zeta_i q^{r-i} = \zeta_i$, namely the exponents of $q$ are calculated modulo $t_i$. Recall that $D_{j,k} q^{s} = D_{j,k}$. We deduce that, for any $0 \leq r \leq t_i-1$,

$$
G_r(X) = D_{j,k}(\zeta_i q^{r-i} X) \cdots D_{j,k} q^{s-1}(\zeta_i q^{r-i} X).
$$

is irreducible over $\mathbb{F}_q$. Since by assumption, $1 \leq a_1 < a_2$, it follows that $\text{ord}_{\zeta_i} (q^{a_1 f_i})$ is even. By Remark, $D_{j,k} q^{s} = D_{j,k} q^{s}$ for every $0 \leq i \leq d_k - 1$. We then have

$$
G_r(x)^* = D_{j,k}(\zeta_i q^{r-i} X) \cdots D_{j,k} q^{s-1}(\zeta_i q^{r-i} X).
$$

We claim that $G_r(x)^* \neq G_s(x)$ for all $0 \leq r \leq t_i - 1$. Because $\eta^{i,q^{r-i}}$ is neither 1 nor -1, we see that $\zeta_i q^{r-i}$ is neither 1 nor -1. This implies that $D_{j,k}(\zeta_i q^{r-i} X) \neq D_{j,k}(\zeta_i q^{r-i} X)$. Suppose otherwise that $D_{j,k}(\zeta_i q^{r-i} X) = D_{j,k}(\zeta_i q^{r-i} X)$. This leads to $\zeta_i q^{r-i} = \zeta_i q^{r-i}$, and hence $D_{j,k}(\zeta_i q^{r-i} X) = D_{j,k}(\zeta_i q^{r-i} X)$. This is a contradiction, since we would obtain $D_{j,k} = D_{j,k}$.

Each of the remaining factors of $G_r(x)^*$ does likewise in turn, proving the claim.

Summarizing the discussions above, we have the following.

**Theorem 5.10.** Let $\ell_1, \ell_2$ be distinct odd primes coprime with $q$, and let $r_1, r_2$ be positive integers. Put $\text{ord}_{\zeta_i} (q) = 2a_i f_i$, with $\gcd(2, f_i) = 1$, $i = 1, 2$. We then have:

(i) If $a_1 = 0$ (resp. $a_2 = 0$), then $\Omega_{\ell_1, e_1}^{r_1} \Omega_{\ell_2, e_2}^{r_2} = \Omega_{\ell_2, e_2}^{r_1} \Omega_{\ell_1, e_1}^{r_2}$. (resp. $\Omega_{\ell_1, e_1}^{r_1} \Omega_{\ell_2, e_2}^{r_2} = \Omega_{\ell_1, e_1}^{r_2} \Omega_{\ell_2, e_2}^{r_1}$).

(ii) If $a_1 = a_2, a_1 \geq 1$, then $\Omega_{\ell_1, e_1}^{r_1} \Omega_{\ell_2, e_2}^{r_2} = \sum_{d|e_1 e_2} \phi(d) \text{ord}_{\zeta_i}r_1^d + \phi(d) \text{ord}_{\zeta_i}r_2^d - 1$.

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