We study medial centrality measures that assess the role of a node in connecting others in the network. We focus on a setting with one target node $t$ and several source nodes. We consider four classic measures adapted to this setting: Betweenness Centrality, Stress Centrality, Random Walk Betweenness Centrality and PageRank. While Betweenness and Stress Centralities assume that the information in the network follows the shortest path, Random Walk Betweenness Centrality and PageRank assume it moves randomly along the edges. We develop the axiomatic characterizations of all four measures. Our analysis shows that, while conceptually different, all four measures share several common properties.

Introduction

Identifying key elements in complex interconnected systems is one of the fundamental challenges of network science. For this goal, more than a hundred different methods have been proposed [Brandes and Erlebach, 2005, Jackson, 2008]. Such methods, called centrality measures, are the subject of this paper.

Medial centralities [Borgatti and Everett, 2006], along with the (radial) distance-based centralities and feedback centralities, constitute one of the major classes of centrality measures. These centralities assess a node by the role it plays as an intermediary in a network; hence, they are sometimes called betweenness-like centralities after their canonical example: Betweenness Centrality. Medial centralities have been used in many applications, including the analysis of protein interaction networks [Joy et al., 2005], identifying gatekeepers in the social or covert networks (i.e., nodes with the ability to control information flow) [Coffman et al., 2004], or assessing monitoring capabilities in computer networks [Dolev et al., 2010].

Classic medial centralities first assess the role in connecting two nodes, a source and a target, and then sum these values over all pairs of nodes. In the case of Betweenness and Stress Centralities the number of shortest paths is counted. In turn, Random Walk Betweenness Centrality focuses on the visits of the random walk and Flow Betweenness Centrality on the amount of flow the node is responsible for.

The choice of a suitable centrality measure for a specific goal out of multiple similar concepts is often hard. On one hand, real-world situations are often hard to translate to graph notions. In particular, in the citation network it is not clear should we focus on the shortest paths or the random walk. On the other hand, intuitive interpretations of centralities are often misleading. To give an example, Betweenness Centrality is often highly correlated simply with the degree of a node, as high degree imposes high number of shortest paths a node is on.

That is why, in recent years, the axiomatic approach has gained popularity in centrality analysis [Boldi and Vigna, 2014, Skibski et al., 2019, Bloch et al., 2016]. In this approach, simple properties called axioms are defined that highlight specific behaviors of a measure. A carefully designed set of axioms allows to uniquely characterize a measure.

There are several papers that consider distance-based centralities [Garg, 2009, Skibski and Sosnowska, 2018] and even more that concern feedback centralities [Dequiedt and Zenou, 2014, Was and Skibski, 2021]. Medial centralities, however, are understudied. Only PageRank, which does not fit in the classic sum-over-pairs scheme, but can be considered a medial centrality, has been axiomatically characterized [Was and Skibski, 2020, Altman and Tennenholtz, 2005]. In particular, no axiomatization of Betweenness Centrality and its variants has been proposed so far. The main reason is the fact that for betweenness-like measures it is especially hard to find any graph operations that do not change the centralities and such operations are usually the basis of axiomatic characterizations.
To cope with this challenge, in this paper we focus on a setting with one target node $t$ and arbitrary many source nodes. In the World Wide Web, centrality analysis for such a setting can indicate the role of websites in directing users to a specific page. In the financial network, centralities can identify top intermediaries responsible for transferring money to a specific bank account. In the communication networks, they can assess the role in controlling the flow of information going to one specific entity.

We consider four classic measures adapted to this setting: Betweenness, Stress, Random Walk Betweenness Centralities and PageRank and create an axiom system that enables us to characterize them. Our analysis shows that, while conceptually different, all four measures share several common properties. We identify four such axioms satisfied by all considered centralities: Locality, Additivity, Node Redirect and Target Proxy. To give an example, Target Proxy states that if all paths to the target node go through one specific “proxy” node, then the role in connecting source nodes to the target node is the same as the role in connecting source nodes to the proxy node.

We also identify specific differences between studied measures. We introduce two axioms for measures based on the shortest paths, Symmetry and Direct Link Domination and adopt two axioms from the axiomatization of PageRank for those based on the random walks: Edge Swap and Edge Multiplication [Was and Skibski, 2020]. Also, in Baseline axioms, we concentrate on a simple graph with only two nodes on which these centralities differ.

**Preliminaries**

In our work we consider directed multigraphs with possible self-loops. Such a graph is a pair $G = (V, E)$, where $V$ is the set of nodes and $E$ is the multiset of edges, that is ordered pairs of nodes $(v, u) \in V \times V$. From now on, we will refer to directed multigraphs simply as graphs. We will denote the number of occurrences of an element $e$ in the multiset $E$ by $m_e(E)$. An edge $(v, u)$ is outgoing from the node $v$, which is the start of the edge and incoming to the node $u$, which is the end of the edge. If $v = u$, the edge is a self-loop.

Let $\Gamma^+(V, E)) = \{(v, u) \in E\}$ denote the multiset of edges outgoing from the node $v$ and let its cardinality be the out-degree of the node $v$. Let $N^+_v(V, E) = \{u \in V : (v, u) \in E\}$ denote the set of direct successors of $v$. Similarly, let $\Gamma^{-}(V, E)) = \{(u, v) \in E\}$ denote the multiset of edges incoming to the node $v$ and let its cardinality be the in-degree of the node $v$. Let $N^{-}_v(V, E) = \{u \in V : (u, v) \in E\}$ denote the set of direct predecessors of $v$. Also, let $\Gamma_v(G) = \Gamma^+_v(G) \cup \Gamma^{-}_v(G)$.

Two nodes $v$ and $u$ are out-twins if they have the same outgoing edges, that is for every $w \in V$ there holds $m(v, w)(E) = m(u, w)(E)$.

A path $p = (e_1, e_2, ..., e_k)$ of length $k$ is a sequence of edges of the graph (i.e., $e_1, ..., e_k \in E$) in which every edge starts with a node with which the previous edge ended, that is for every $i \in \{1, ..., k - 1\}$ there are some nodes $v, u, w \in V$ such that $e_i = (v, u)$, $e_{i+1} = (u, w)$. The start of the first edge is the start of the path and the end of the last edge is the end of the path. A cycle is a path that starts and ends in the same node.

A node $v$ is reachable from a node $v$ if there is a path that starts with $v$ and ends with $u$. The distance from $v$ to $u$, denoted by $dist_{v,u}(G)$, is the length of a shortest path that starts with $v$ and ends with $u$. The number of shortest paths from $s$ to $t$ in $G$ is denoted by $\sigma_{s,t}(G)$ and the number of shortest paths from $s$ to $t$ in $G$ that contain $v$ is denoted by $\sigma_{s,t}(G, v)$.

**Our Setting**

In this paper, we treat graphs as the information networks. We will assume there is one target node, denoted by $t$, which is the destination of all data traveling through the network. We will restrict our attention to graphs in which node $t$ is reachable from every node; the set of all such graphs will be denoted by $G_t$.

To specify which nodes are the sources of information and how much information they send, we will consider node weight functions $b : V \rightarrow \mathbb{R}_{\geq 0}$. The simplest case is when there is only one source: $s \in V$. To describe such situations, for a constant $x \in \mathbb{R}_{\geq 0}$ we denote by $x \cdot 1^s$ a node weight function such that $(x \cdot 1^s)(s) = x$ and $(x \cdot 1^s)(v) = 0$ for $v \in V - \{s\}$.

We define some operations on weighted graphs. The sum of two graphs is obtained by summing the corresponding node sets, edge multisets and weight functions. Formally, for two weighted graphs $(G, b)$, $(G', b')$ with $G = (V, E), G' = (V', E')$ we have:

$$(G, b) + (G', b') = (V \cup V', E + E', b + b'),$$

where $E + E'$ denotes the sum of multisets $E, E'$ and $(b + b')(v) = b(v) + b'(v)$ for every $v \in V \cap V'$, $(b + b')(v) = b(v)$ for every $v \in V - V'$ and $(b + b')(v) = b'(v)$, otherwise.

We will also use a shorthand notation for adding and deleting edges $E'$ from the graph $G = (V, E)$: we define $G + E' = (V, E + E')$ and $G - E' = (V, E - E')$. 

2
For two different nodes $v, u \in V$ we define *merging* and *redirecting*. *Merging* $v$ into $u$ deletes $v$ from the graph and moves its weight and all incident edges to $u$. Formally: $M_{v \rightarrow u}(G, b) = ((V - \{v\}, E'), b')$, where $E' = E - \Gamma_v(G) + \{(f_{v \rightarrow u}(w), f_{v \rightarrow u}(w')) : (w, w') \in \Gamma_v(G)\}$ where $f_{v \rightarrow u}(v) = u$ and $f_{v \rightarrow u}(w) = w$ for $w \in V - \{v\}$ and $b'(u) = b(v) + b(u)$ and $b'(w) = b(w)$ for $w \in V - \{v, u\}$. Now, *redirecting* $v$ into $u$ deletes outgoing edges of $v$ (except for self-loops) and merges $v$ into $u$: $R_{v \rightarrow u}(G, b) = M_{v \rightarrow u}((V, E - (\Gamma_v^+_t(G) - \Gamma_v^-_t(G))), b)$.

### Centralities

A *centrality measure* $F$ is a function that for every node $v$ in a weighted graph $(G, b)$ assigns a non-negative real value, denoted by $F_v(G, b)$.

In our setting, centrality measures capture how often a node conveys the data to the target node. Hence, we will consider centrality measures parameterized with the target node $t$ defined on the class of graphs $G_t$. We will call them *t-centricity measures*. We will consider four t-centrality measures which are direct counterparts of the classic measures for the general setting.

The simplest and chronologically the first medial centrality was proposed by [Shimbel 1953](#) under the name *Stress Centrality*. This measure simply counts the number of shortest paths that go through a specific node. We adjust Stress Centrality to our setting by fixing the target node of all paths and considering paths from different sources with different weights.

**Definition 1.** For a graph $G = (V, E) \in G_t$ with weights $b$, t-Stress Centrality of node $v \in V$ is defined as follows:

$$S^t_v(G, b) = \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, v).$$

### Betweenness Centrality

[Freeman, 1977](#), arguably the most widely used medial centrality, can be considered a relative version of Stress Centrality: In Betweenness Centrality the number of shortest paths from $s$ to $t$ that goes through $v$ is divided by the total number of shortest paths from $s$ to $t$. We define t-Betweenness Centrality accordingly.

**Definition 2.** For a graph $G = (V, E) \in G_t$ with weights $b$, t-Betweenness Centrality of node $v \in V$ is defined as follows:

$$B^t_v(G, b) = \sum_{s \in V} b(s) \cdot \frac{\sigma_{s,t}(G, v)}{\sigma_{s,t}(G)}.$$  

The standard Betweenness and Stress Centralities are the sum of the t-Betweenness and t-Stress Centralities, respectively, over all target nodes $t \in V$ with unit node weights.

Both measures have the following interpretation. Imagine $b(s)$ packets are sent from each node $s \in V$ through the shortest path to the target. Now, if one of the shortest paths is chosen uniformly at random, then the expected number of packets that visits a specific node is equal to t-Betweenness Centrality. In turn, if the packet is sent through all the shortest paths at once, then the number of packets that visits a specific node equals t-Stress Centrality.

So far, we considered measures which are based on the underlying assumption that the information travels the network through shortest paths. Such an assumption makes sense in settings in which the whole structure is known beforehand and the process is optimized. However, this is not the case in many real-world applications.

The opposite approach is proposed in measures based on the random walk. Assume the information packet starts from a source node $s$ and moves randomly through the network. In each step, it chooses one of the outgoing edges of a node $t$ it is in uniformly at random and moves along this edge. To measure the role in transferring the information to one specific target node $t$, node $t$ is treated as an absorbing node in which all packets end their travel (technically, this is achieved by deleting outgoing edges of $t$). Now, the role in connecting node $s$ to $t$ in the random-walk version of Betweenness Centrality is defined as the expected number of times a packet visits a specific node.

Formally, let us denote by $P(\omega_{G,s}(k) = v)$ the probability that the random walk on graph $G$ that starts in node $s$ after $k$ steps will be at node $v$. We define t-Random Walk Betweenness Centrality as follows.

**Definition 3.** For a graph $G = (V, E) \in G_t$ with weights $b$, t-Random Walk Betweenness Centrality of node $v \in V$ is defined as follows:

$$RW_{t}(G, b) = \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G-s-t}(G, s)(k) = v).$$

We note that t-Random Walk Betweenness Centrality is a counterpart of the measure defined by [Blöchl et al., 2011](#) for directed graphs and not of the measure defined by [Newman, 2005](#) for undirected graphs under the same name.
Now, to define t-PageRank, consider the random walk with decay \cite{Was and Skibski 2020}. In this model, an additional decay factor \( a \in [0, 1) \) is introduced. Now, in each step, with probability \( a \) the random walk moves further and with probability \( 1 - a \) it ends its travel. Let us denote by \( P(\omega_{G,s}^a(k) = v) \) the probability that the random walk with decay factor \( a \) on graph \( G \) that starts in node \( s \) after \( k \) steps will be at node \( v \). We define t-PageRank as follows.

**Definition 4.** For a graph \( G = (V, E) \in \mathcal{G}_t \) with weights \( b \), t-PageRank with decay factor \( a \in [0, 1) \) of node \( v \in V \) is defined as follows:

\[
PR_t^{a,t}(G, b) = \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G,s}^a(k) = v).
\]

Hence, t-Random Walk Betweenness Centrality can be considered a borderline case of t-PageRank for \( a = 1 \).

**Axioms**

We will now present the set of simple properties that we will use to uniquely characterize all four medial centralities. First, we introduce common axioms, namely Locality, Additivity, Node Redirect, Target Proxy, which are satisfied by all four measures. Then, we present Symmetry and Direct Link Domination which are specific for measures based on shortest paths: t-Stress and t-Betweenness Centrality. Furthermore, we discuss Edge Swap and Edge Multiplication which are specific for measures based on the random walk: t-Random Walk Betweenness Centrality and t-PageRank. Finally, we propose three versions of the Baseline axiom that specify centralities in a simple borderline case: Baseline 1-1, 1-\( \alpha \) and \( k \). See Table 1 for a summary. Each centrality measure satisfies exactly one Baseline axiom, hence for clarity of presentation in the last row we simply state which variant is satisfied.

**Common Axioms**

We start with axioms satisfied by all four t-centralities. For an illustration of these axioms see Figure 1.

**Axiom 1.** (Locality) For every two graphs \( G = (V, E), G' = (V', E') \in \mathcal{G}_t \) with weights \( b, b' \) such that \( V \cap V' = \{t\} \) and node \( w \in V - \{t\} \):

\[
F^t_w((G, b) + (G', b')) = F^t_w(G, b)
\]

and \( F^t_w((G, b) + (G', b')) = F^t_w(G, b) + F^t_w(G', b') \).

**Axiom 2.** (Additivity) For every graph \( G = (V, E) \in \mathcal{G}_t \) with weights \( b, b' \) and node \( w \in V \):

\[
F^t_w(G, b + b') = F^t_w(G, b) + F^t_w(G, b').
\]
Axiom 3. (Node Redirect) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights $b$ and out-twins $v, u \in V - \{t\}$:

$$F^t_w(R_{u \rightarrow v}(G, b)) = F^t_w(G, b) + F^t_w(G, b)$$

and

$$F^t_w(R_{w \rightarrow v}(G, b)) = F^t_w(G, b)$$

for every $w \in V - \{v, u\}$.

Axiom 4. (Target Proxy) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights $b$ such that $\Gamma^-_t(G) = \{(v, t)\}$ and $\Gamma^+_t(G) = \{(v, t)\}$, $b(t) = 0$ and node $w \in V - \{t\}$:

$$F^t_w(M_{t \rightarrow v}(G, b)) = F^t_w(G, b).$$

Locality describes the operation of joining two networks with the common target. The axiom states that the centrality of all nodes other than the target does not change and the centrality of the target is the sum of its centralities from both separate graphs. This means that the behavior of data packets is not affected by the existence of another independent part of the network with the same target. Locality is a natural translation to class $\mathcal{G}_t$ of Locality proposed by [Skibski et al., 2019] that concerns adding separate connected graphs. It is also similar to Sink Merging from [Was et al., 2019], as we assume that data packets end their travel in the target node.

Additivity states that the centrality as the function of the node weights is additive. This means that the packets travel independently from each other.

Node Redirect formalizes the intuition that if the packet from two nodes has the same possible further routes, then redirecting one of these nodes into the other would not change the centralities of other nodes. Moreover, the centrality of the combined node will be the sum of centralities of both nodes in the original graph. This axiom was proposed in [Was and Skibski, 2020], but in our version we do not allow redirecting of (and to) the target node.

Target Proxy can be understood in the following way: if every path to the target node goes through a proxy, $v$, then the role in transferring data to the target is the same as the role in transferring data to node $v$. More precisely, consider a graph in which the target has zero weight and one incoming edge from $v$ which is also the only outgoing edge of node $v$. The axiom states that in such a graph nodes $v$ and $t$ can be merged and this would not change the centralities of the remaining nodes. The name was inspired by the Proxy axiom by [Altman and Tennenholz, 2005]. The meaning of our axiom is different: Proxy considered adding a proxy between arbitrary nodes inside the graph and such an operation does not preserve centralities based on shortest paths.

**Axioms for Centralities Based on Shortest-Paths**

The next two axioms are satisfied by t-Betweenness and t-Stress Centralities, but not by t-PageRank and t-Random Walk Betweenness Centrality.

Axiom 5. (Symmetry) For every graph $G = (V, E) \in \mathcal{G}_s$, source node $s \in V$ and node $w \in V$ such that $G' = (V, \{(u, v) : (v, u) \in E\}) \in \mathcal{G}_s$:

$$F^s_w(G', 1^s) = F^s_w(G, 1^s)$$

Axiom 6. (Direct Link Domination) For every graph $G = (V, E) \in \mathcal{G}_s$ with weights $b$ such that $(v, t), (v, u) \in E$, $u \neq t$ and node $w \in V$:

$$F^s_w(G - \{(v, u)\}, b) = F^s_w(G, b)$$

Symmetry states that if there is only one source node, then reversing the graph and swapping the source and target nodes does not change centralities in the graph. Note that this operation applies only if the reversed graph belongs to $\mathcal{G}_s$, i.e., node $s$ which is the target in this graph is reachable from every node. It is suitable when data packets would be transferred through the same paths from the target to the source in the reversed network. This is true if data packets goes through the shortest paths. However, it is not the case for the random walk, for which some paths would be used more often in the reversed network than in the original one.

Figure 2: An illustration of axioms for centrality measures based on shortest-paths. Node $s$ is the only source: $b = 1^s$ and the target is marked with a double line. The values according to t-Betweenness Centrality are placed next to the nodes in the first graph and they do not change in further graphs.
Direct Link Domination states that if from \( v \) there is a direct connection to \( t \), then we can delete other outgoing edges of \( v \). This captures the assumption that a node which can send a data packet directly to the target will do so. At the same time, the axiom does not impose any restrictions on the behavior of packets in other nodes. This property is clearly satisfied if the whole route of the data packet is planned optimally, as in t-Betweenness and t-Stress Centralities, but is violated if the data packets travel randomly and could miss a direct connection, as in t-Random Walk Betweenness Centrality and t-PageRank.

Figure 2 illustrates both axioms. While t-Betweenness Centrality does not change in all graphs, t-PageRank of node \( v \) does change. Specifically, we have \( PR_v^t(G, b) = PR_v^t(G_2, b) = 1/2 \), but \( PR_v^t(G_1, b) = 2/3 \).

### Axioms for Centralities Based on the Random Walk

Edge Swap and Edge Multiplication, that we present in this section, are satisfied by t-PageRank and t-Random Walk Betweenness Centrality, but not by t-Betweenness and t-Stress Centralities. Both axioms were introduced in the axiomatization of PageRank in [Was and Skibski, 2020]. For an illustration see Figure 3.

**Axiom 7. (Edge Swap)** For every graph \( G = (V, E) \in G_t \) with weights \( b \) and edges \( (v, v'), (u, u') \in E \) such that \( v, u \neq t \), \( F_v(G, b) = F_u(G, b) \), \( |\Gamma_v^+(G)| = |\Gamma_u^+(G)| \), \( G' = G - \{(v, v'), (u, u')\} + \{(v, u'), (u, v')\} \in G_t \) and node \( w \in V \):

\[
F_w^t(G', b) = F_w^t(G, b).
\]

**Axiom 8. (Edge Multiplication)** For every graph \( G = (V, E) \in G_t \) with weights \( b \), number \( k \in \mathbb{Z}_{\geq 0} \) and nodes \( v, w \in V \):

\[
F_w^t(G + k \cdot \Gamma_v^+(G), b) = F_w^t(G, b)
\]

Edge Swap states that swapping ends of two outgoing edges of nodes with equal centralities and out-degrees does not affect the centrality of any node. This means that if information packets are equally often in two nodes, then it does not matter from which of them a node has an incoming edge. Such an assumption makes sense for centralities based on the random walk, but if the data packets go through the shortest paths, then the edge from a node which is closer to the source of information is more profitable, even if it is responsible for the same proportion of paths. The only difference between our version of the axiom and the original one from [Was and Skibski, 2020] is the fact that swapped edges cannot start in the target node.

Edge Multiplication states that creating additional copies of the outgoing edges of a node does not affect the centrality of any node. Such an operation does not affect the random walk as it preserves the transition probabilities between nodes, but it changes the number of shortest paths.

### Baseline Axioms

Let us present our final three axioms.

**Axiom 9. (Baseline 1-1)** For every node \( s \), natural number \( k \in \mathbb{N}_+ \) and graph \( G = (\{s, t\}, k \cdot \{(s, t)\}) \):

\[
F_s^t(G, \mathbb{1}^s) = 1 = F_t^t(G, \mathbb{1}^t).
\]

**Axiom 10. (Baseline k-k)** For every node \( s \), natural number \( k \in \mathbb{N}_+ \), and graph \( G = (\{s, t\}, k \cdot \{(s, t)\}) \):

\[
F_s^t(G, \mathbb{1}^s) = k = F_t^t(G, \mathbb{1}^t).
\]

**Axiom 11. (Baseline 1-\( \alpha \))** For every node \( s \), natural number \( k \in \mathbb{N}_+ \) and graph \( G = (\{s, t\}, k \cdot \{(s, t)\}) \):

\[
F_s^t(G, \mathbb{1}^s) = 1 > F_t^t(G, \mathbb{1}^t).
\]

[Figure 3: An illustration of axioms for centrality measures based on the random walk. Nodes \( s_1 \) and \( s_2 \) are the only sources: \( b = \mathbb{1}^s_1 + \mathbb{1}^s_2 \) and the target is marked with a double line. The values according to t-PageRank are placed next to the nodes in the first graph and they do not change in further graphs.]
Each Baseline axiom specifies the centrality in a simple graph with two nodes, the source \( s \) and the target \( t \), and \( k \) edges from \( s \) to \( t \). Hence, these axioms are disjoint and each centrality measure satisfies exactly one of them.

Baseline 1-1 states that both nodes should have centrality equal to 1, as this is the amount of information packets both nodes are responsible for. Baseline 1-1 is satisfied by both t-Betweenness and t-Random Walk Betweenness Centralities.

Baseline \( k-k \) states that both nodes have centrality equal to the number of edges between them. Such an approach makes sense if we assume the information packets can be duplicated and sent through each edge. Then, if we measure not the relative importance of nodes, but the absolute number of packets that go through the node we get these values. Baseline \( k-k \) is satisfied only by t-Stress Centrality.

Finally, Baseline 1-\( a \) states that the centrality of the source is one, but the centrality of the target is smaller than one. This is the case when there is a risk that the information will be lost along the way. Baseline 1-\( a \) is satisfied by t-PageRank.

**Axiomatization of t-Betweenness and t-Stress Centrality**

In this section we present the axiomatic characterization of t-Betweenness and t-Stress Centralities.

**Theorem 1.** A t-centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1 if and only if it is t-Betweenness Centrality.

**Theorem 2.** A t-centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline \( k-k \) if and only if it is t-Stress Centrality.

The full proofs can be found in the appendix. Here we present only a sketch.

It is easy to check that t-Betweenness Centrality and t-Stress Centrality satisfy their axioms (Lemmas 1, 2). Hence, in what follows, we will show that both sets of axioms uniquely characterize a centrality measure.

We begin by showing three simple properties.

- **(Anonymity):** Any node other than the target can be renamed without changing any centralities (Lemma 3).
- **(Target Self-Loop):** Deleting a self-loop of the target does not change any centralities (Lemma 4).
- **(No Target Outlet):** Deleting any outgoing edge of the target does not change any centralities (Lemma 5).

No Target Outlet is especially useful property as it allows us to add edges from \( t \) to all other nodes in the graph before using Symmetry. In this way, we make sure that after reversing the graph from each node there will be a path to the new target, possibly through \( t \).

For now, let us concentrate on a graph with one source \( s \) with weight 1. We proceed by induction on the distance from \( s \) to \( t \). In the base case we consider \( \text{dist}_{s,t}(G) \leq 2 \):

- For \( \text{dist}_{s,t}(G) = 0 \) (Lemma 5), we present the graph as the sum of two graphs: the original graph with weight of \( t \) changed to zero and the second graph with only one node \( t \) with unitary weight. From Locality and Additivity we get that \( F^1_t(G, I^t) = F^1_t(\{\{t\}\}, I^t) \) and \( F^v_t(G, I^t) = 0 \) for \( v \in V - \{t\} \). Now, the centrality of \( t \) in graph \( \{\{t\}\}, \{\} \) with weights \( I^t \) can be determined based on Baseline axioms by using Target Proxy and Target Self-Loop.
Theorem 3. A t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline while satisfied by both measures, is implied by other axioms. Hence, it does not appear in the theorem statements.

This section presents our axiomatic characterizations of t-PageRank and t-Random Walk Betweenness. Note that Additivity, while satisfied by both measures, is implied by other axioms. Hence, it does not appear in the theorem statements.

Figure 5: The key inductive step from the proofs of Theorems 3 and 4. We transform the original acyclic graph G in a way that the number of edges is decreased, but the centrality of v remains unchanged. First, we create copies of nodes without incoming edges, each with one outgoing edge (G1). Then, using Node Redirect, we merge copies of direct predecessors of v into one node s (G2). Now, using the new node technique, we add a node v’ with the same number of outgoing edges as v, all to t, and the weight equal to the centrality of v (G3). Next, we use Edge Swap to exchange the outgoing edges of v with the outgoing edges of v’ (G4). Finally, using Locality and Edge Multiplication, we split the graph in two graphs, one which has one edge less than G (G5) and another simple graph for which centralities follow from previous lemmas (G′5).

- For dists,t(G) = 1 (Lemma 7), using Node Redirect and Locality we decompose graph G into the original graph with weight of s changed to zero and a graph with two nodes, source s and target t, and k edges from s to t, with unitary weight of s. Centralities in the first graph all equal zero from Additivity and centralities in the second graph are known from Baseline axioms.
- For dists,t(G) = 2 (Lemmas 8, 9), we first use Node Redirect to split each successor of s into several nodes so that each has only one incoming edge. Then, using Symmetry and Node Redirect for the reversed graph we split predecessors of t so that each copy has only one outgoing edge. In the resulting graph all nodes which are both successors of s and predecessors of t are isomorphic, which allows us to argue they have equal centralities. To argue what are the centralities of other nodes, we merge isomorphic nodes using Node Redirect and use Target Proxy to obtain the case where the distance from s to t equals one.

Let us discuss the inductive step (Lemma 10). Fix graph G with dists,t(G) ≥ 3 and some node v. Since dists,v(G) + distv,t(G) ≥ dists,t(G) ≥ 3, we either have dists,v(G) ≥ 2 or distv,t(G) ≥ 2. Let us assume the former; in the other case, we reverse the graph and based on Symmetry proceed in the same way. Now, we show that we can transform the graph in a way that the distance from the source to the target decreases, but the centrality of v remains unchanged. We present this key construction in Figure 4.

So far, we have considered only one source. If there are multiple sources, by using Additivity we split the graph into several copies, each with a single unitary source (Lemma 11):

\[ F_s^t(G, b) = \sum_{s \in V} b(s) \cdot F_s^t(G, 1^s). \]

Based on that, we show that if a centrality measure satisfies Baseline 1-1, then it is Betweenness Centrality (Lemma 12). In turn, if it satisfies Baseline k-k, then it is Stress Centrality (Lemma 13).

Axiomatization of t-PageRank and t-Random Walk Betweenness

This section presents our axiomatic characterizations of t-PageRank and t-Random Walk Betweenness. Note that Additivity, while satisfied by both measures, is implied by other axioms. Hence, it does not appear in the theorem statements.

**Theorem 3.** A t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-a if and only if it is t-PageRank for some decay factor \( a \in [0, 1) \).

**Theorem 4.** A t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 if and only if it is t-Random Walk Betweenness.

The full proofs can be found in the appendix. Here we present only a sketch. Our proof was inspired by [Was and Skibski, 2020]. However, since all graphs considered by us must belong to the class \( G' \), techniques used in our proof are different.

Roughly speaking, t-Random Walk Betweenness can be considered t-PageRank with \( a = 1 \). Hence, the proof of both theorems is joint. Specifically, in the main part of the proof, stretching the definition of t-PageRank, we show that axioms imply that the measure is t-PageRank for some \( a \in [0, 1] \).
We begin by considering simple graphs with two nodes: the source $s$ and the target $t$.

- First, we show that if there is only one edge, from $s$ to $t$, then there exists a constant $a \in [0, 1]$ such that the centrality of the source $s$ equals $b(s)$ and the centrality of the target $t$ equals $a \cdot b(s)$ (Lemma 16).
- Second, we show that outgoing edges from $t$ does not affect the centralities of $s$ and $t$ (Lemma 17).

Building upon this, we show that if we add to a graph a new node $v$ with several edges $\{v, t\}$, then its centrality will be equal to its weight and centralities of nodes other than $t$ will not change (Lemma 18). This operation, that we call new node technique, is used frequently in the remainder of the proof. In particular, we use it to show that every node $v$ without incoming edges has centrality equal to its weight (Lemma 19). Furthermore, assuming such $v$ has $k$ outgoing edges, if we replace it with $k$ copies, each with one outgoing edge and weight $b(v)/k$, then the centralities of other nodes will not change (Lemma 20).

In the next part of the proof, we show that on acyclic graphs the centrality is equal to $t$-PageRank. The proof proceeds by induction on the number of edges (Lemma 21, 22).

- If $|E| \leq 2$, then the thesis follows from previous lemmas.
- If $|E| \geq 3$, but all edges ends in $t$, then this result follows from Locality.

Assume otherwise, i.e., $|E| \geq 3$ and there exists a node with incoming edges other than the target. Let $v$ be a node, other than $t$, such that all its incoming edges come from nodes without incoming edges. Now, we show that we can transform a graph in a way that the number of edges decreases, but the centrality of $v$ remains unchanged. This operation is described in Figure 5.

Acyclic graphs are the basis for the induction on the number of cycles which constitutes the top level of the proof. Assume that a graph contains some cycles. We can also assume that the target node does not have any outgoing edges, as otherwise they can be deleted without changing any centralities (Lemma 23).

Let us discuss the inductive step of the induction on the number of cycles (Lemma 24). Fix a node $v$ that belongs to at least one cycle. Using again the new node technique we add a node $v'$ with the same number of outgoing edges as $v$, all to $t$, and the weight equal to the centrality of $v$. Now, using Edge Swap, we swap all outgoing edges of $v$ with all outgoing edges of $v'$. In the new graph, nodes $v$ and $v'$ do not belong to any cycles, hence the number of cycles has decreased.

This concludes the proof that axioms imply that the measure is $t$-PageRank for some $a \in [0, 1]$. Now, if the measure satisfies Baseline 1-a, then $a < 1$ so it is $t$-PageRank (Lemma 25). In turn, if it satisfies Baseline 1-1, then $a = 1$ and we get $t$-Random Walk Betweenness (Lemma 26).

**Conclusions**

We proposed a joint axiomatization of four medial centralities: Betweenness, Stress, Random Walk Betweenness Centralities and PageRank. We focused on a setting with one target node and arbitrary many source nodes. This allowed us to specify several properties which are satisfied by all four centrality measures. Also, we defined axioms specific for measures that concentrate on shortest paths and the random walk which highlights the differences between these two approaches. Our characterization could help in choosing a centrality measure for a specific application at hand.

Our work can be extended in many ways. The ultimate goal would be to create an axiomatization of considered measures in a setting with arbitrary many targets. This is, however, challenging as it precludes the use of axioms that do not apply to target nodes. Another interesting question is how to include other, less popular medial centralities based on different concepts, such as Flow Betweenness Centrality. Finally, our work can be extended to a setting with undirected or edge-weighted graphs.

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References

Alon Altman and Moshe Tennenholtz. Ranking systems: the PageRank axioms. In Proceedings of the 6th ACM Conference on Electronic Commerce (ACM-EC), pages 1–8, 2005.

Francis Bloch, Matthew O. Jackson, and Pietro Tebaldi. Centrality measures in networks. arXiv preprint arXiv:1608.05845, 2016.

Florian Blöchl, Fabian J. Theis, Fernando Vega-Redondo, and Eric O’N. Fisher. Vertex centralities in input-output networks reveal the structure of modern economies. Physical Review E, 83(4):046127, 2011.

Paolo Boldi and Sebastiano Vigna. Axioms for centrality. Internet Mathematics, 10(3-4):222–262, 2014.

Stephen P. Borgatti and Martin G. Everett. A graph-theoretic perspective on centrality. Social Networks, 28(4):466–484, 2006.

Ulrik Brandes and Thomas Erlebach. Network analysis: Methodological foundations. Springer-Verlag, 2005. ISBN 3540249796.

Thayne Coffman, Seth Greenblatt, and Sherry Marcus. Graph-based technologies for intelligence analysis. Communications of the ACM, 47(3):45–47, 2004.

Vianney Dequiedt and Yves Zenou. Local and consistent centrality measures in networks. CEPR Discussion Paper No. DP10031, 2014.

Shlomi Dolev, Yuval Elovici, and Rami Puzis. Routing betweenness centrality. Journal of the ACM (JACM), 57(4):1–27, 2010.

Linton C. Freeman. A set of measures of centrality based on betweenness. Sociometry, 40(1):35–41, 1977.

Manuj Garg. Axiomatic foundations of centrality in networks. unpublished, 2009.

Matthew O. Jackson. Social and Economic Networks, volume 3. Princeton university press, 2008.

Maliackal Poulo Joy, Amy Brock, Donald E Ingber, and Sui Huang. High-betweenness proteins in the yeast protein interaction network. Journal of Biomedicine and Biotechnology, 2005(2):96, 2005.

Mark E. J. Newman. A measure of betweenness centrality based on random walks. Social Networks, 27(1):39–54, 2005.

Alfonso Shimbel. Structural parameters of communication networks. Bulletin of Mathematical Biophysics, 15(4):501–507, 1953.

Oskar Skibski and Jadwiga Sosnowska. Axioms for distance-based centralities. In Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI), pages 1218–1225, 2018.

Oskar Skibski, Talal Rahwan, Tomasz P. Michalak, and Makoto Yokoo. Attachment centrality: Measure for connectivity in networks. Artificial Intelligence, 274:151–179, 2019.

Tomasz Wąs and Oskar Skibski. Axiomatic characterization of pagerank. arXiv preprint arXiv:2010.08487, 2020.

Tomasz Wąs and Oskar Skibski. An axiom system for feedback centralities. In Zhi-Hua Zhou, editor, Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI), pages 443–449. International Joint Conferences on Artificial Intelligence Organization, 2021.

Tomasz Wąs, Talal Rahwan, and Oskar Skibski. Random walk decay centrality. In Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI), pages 2197–2204, 2019.
Proof of Theorem 1 and Theorem 2

In this section, we present the joint full proof of Theorem 1 and Theorem 2. We start with showing that the shortest-path based centralities satisfy their respective axiom sets. The main part is dedicated to showing that the axioms uniquely characterize a centrality measure.

t-Betweenness, t-Stress Centrality ⇒ Axioms

We will now consider each from the axioms: Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and show that both t-Stress and t-Betweenness centralities satisfy it. Finally we calculate values of t-Stress and t-Betweenness centralities on the Baseline graph.

Lemma 1. t-Betweenness Centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1.

Lemma 2. t-Stress Centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline k-k.

Let $\Pi_{v,u}(G)$ denote the set of shortest paths from $v$ to $u$ in $G$ (so $\sigma_{v,u}(G) = |\Pi_{v,u}(G)|$).

Locality:

Let us notice that for every $s \in V$ no shortest path from $s$ to $t$ in $G + G'$ could include any edge from $E'$. Let us assume otherwise and let $(v', u')$ be first such edge. So $v' \in V'$, but because up to this point the path was in $G$, also $v' \in V$, so $v' = t$ and the path ending at this point would be shorter. This means for any $s \in V$ that $\Pi_{s,t}(G + G') = \Pi_{s,t}(G)$. In particular, for every $w \in V$, $w' \in V' - \{t\}$ we have:

$$\sigma_{s,t}(G + G', w) = \sigma_{s,t}(G, w),$$

$$\sigma_{s,t}(G + G', w') = 0,$$

$$\sigma_{s,t}(G + G') = \sigma_{s,t}(G).$$

This gives:

$$S'_{w}(((G, b) + (G', b'))) = \sum_{s \in V \cup V'} (b + b')(s) \cdot \sigma_{s,t}(G + G', w)$$

$$= \sum_{s \in V - \{t\}} b(s) \cdot \sigma_{s,t}(G, w) + (b(t) + b'(t)) \cdot 0$$

$$+ \sum_{s' \in V' - \{t\}} b'(s') \cdot 0 = S'_{w}(G, b)$$

and

$$B'_{w}(((G, b) + (G', b'))) = \sum_{s \in V \cup V'} (b + b')(s) \cdot \frac{\sigma_{s,t}(G + G', w)}{\sigma_{s,t}(G + G')}$$

$$= \sum_{s \in V - \{t\}} b(s) \cdot \frac{\sigma_{s,t}(G, w)}{\sigma_{s,t}(G)} + (b(t) + b'(t)) \cdot 0$$

$$+ \sum_{s' \in V' - \{t\}} b'(s') \cdot \frac{0}{\sigma_{s',t}(G + G')} = B'_{w}(G, b).$$

As for the centrality of $t$, we also note that there is always exactly one shortest path from $t$ to $t$, the empty path:
\[
S^t_w((G, b) + (G', b')) = \sum_{s \in V \cup V'} (b + b')(s) \cdot \sigma_{s,t}(G + G', t)
\]
\[
= \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, t) + b(t) + b'(t)
\]
\[
+ \sum_{s' \in V'} b'(s') \cdot \sigma_{s',t}(G', t)
\]
\[
= S^t_w(G, b) + S^t_w(G', b')
\]

and
\[
B^t_w((G, b) + (G', b')) = \sum_{s \in V \cup V'} (b + b')(s) \cdot \sigma_{s,t}(G + G', t)
\]
\[
= \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, t) + b(t) + b'(t)
\]
\[
+ \sum_{s' \in V'} b'(s') \cdot \sigma_{s',t}(G', t)
\]
\[
= B^t_w(G, b) + B^t_w(G', b').
\]

Additivity:

From definitions we have:

\[
S^t_w(G, b + b') = \sum_{s \in V} (b(s) + b'(s)) \cdot \sigma_{s,t}(G, w)
\]
\[
= \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, w) + \sum_{s \in V} b'(s) \cdot \sigma_{s,t}(G, w)
\]
\[
= S^t_w(G, b) + S^t_w(G, b')
\]

and

\[
B^t_w(G, b + b') = \sum_{s \in V} (b(s) + b'(s)) \cdot \sigma_{s,t}(G, w)
\]
\[
= \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, w) + \sum_{s \in V} b'(s) \cdot \sigma_{s,t}(G, w)
\]
\[
= B^t_w(G, b) + B^t_w(G, b').
\]

Node Redirect:

Let \((G', b') = R_{u \to v}(G, b)\). We know shortest paths from \(v\) to \(t\) are the same in \(G\) and \(G'\), as going through the out-twin \(u\) of \(v\) could only produce longer paths and edges not incident to \(u\) are preserved. So \(\Pi_{v,t}(G') = \Pi_{v,t}(G)\). Moreover, since \(u\) is the out-twin of \(v\), they have the same shortest paths to \(t\) in \(G\), up to appropriately changing first edge, that is \(\Pi_{v,t}(G') = \{((v, v_1), ..., (v_k, t)) : ((u, v_1), ..., (u_k, t)) \in \Pi_{u,t}(G)\}\). In particular, for every \(w \in V\) we have:

\[\sigma_{v,t}(G', w) = \sigma_{v,t}(G, w) = \sigma_{u,t}(G, w),\]
\[\sigma_{v,t}(G') = \sigma_{v,t}(G) = \sigma_{u,t}(G).\]

As for the other nodes \(s \in V - \{v, u\}\), paths from \(s\) to \(t\) not going through \(v\) or \(u\) are the same in \(G\) and \(G'\). Any path going through \(v\) or \(u\) first passes one of their incoming edges. For any path from \(s\) to \(t\) in \(G\) that goes into \(v\) through some edge there
is a path from $s$ to $t$ in $G'$ that also goes through this edge. For any path from $s$ to $t$ in $G$ that goes through an edge $(w, u)$ there is a path from $s$ to $t$ in $G'$ that goes through the corresponding edge $(w, v)$ and through node $v$. And similarly, for any edge from $s$ to $t$ in $G'$ that goes into $v$ with edge $(w, v)$ there is a corresponding path from $s$ to $t$ that is either the same, or the same apart from going through $(w, u)$ instead. Let $R_{u \rightarrow v}(((w_1, w_2), \ldots, (w_i, u), (u, w_{i+2}), \ldots,(w_{k-1}, w_k)) = ((w_1, w_2), \ldots, (w_i, v), (v, w_{i+2}), \ldots,(w_{k-1}, w_k)). We get $\Pi_{v, i}(G') = \Pi_{v, t}(G) + \{R_{u \rightarrow v}(\pi) : \pi \in \Pi_{u, t}(G)\}. In particular, for every $w \in V - \{v, v'\}$ we have:

$$
\sigma_{s, t}(G', w) = \sigma_{s, t}(G, w),
\sigma_{s, t}(G', v) = \sigma_{s, t}(G, v) + \sigma_{s, t}(G, u),
\sigma_{s, t}(G') = \sigma_{s, t}(G).
$$

This gives:

$$
S'_v(G', b') = \sum_{s \in V - \{v, u\}} b'(s) \sigma_{s, t}(G', v) + b'(v) \sigma_{v, t}(G', v)
= \sum_{s \in V - \{v, u\}} b(s)(\sigma_{s, t}(G, v) + \sigma_{s, t}(G, u)) + (b(v) + b(u)) \sigma_{v, t}(G, v)
= S'_v(G, b) + S'_u(G, b)
$$

and

$$
B'_v(G', b') = \sum_{s \in V - \{v, u\}} b'(s) \frac{\sigma_{s, t}(G', v)}{\sigma_{s, t}(G')} + b'(v) \frac{\sigma_{v, t}(G', v)}{\sigma_{v, t}(G')}
= \sum_{s \in V - \{v, u\}} b(s) \frac{\sigma_{s, t}(G, v) + \sigma_{s, t}(G, u)}{\sigma_{s, t}(G)} + (b(v) + b(u)) \frac{\sigma_{v, t}(G, v)}{\sigma_{v, t}(G)}
= B'_v(G, b) + B'_u(G, b)
$$

and for every $w \in V - \{v, u\}$:

$$
S'_w(G', b') = \sum_{s \in V - \{v, u\}} b'(s) \sigma_{s, t}(G', w) + b'(v) \sigma_{v, t}(G', w)
= \sum_{s \in V - \{v, u\}} b(s) \sigma_{s, t}(G, w) + (b(v) + b(u)) \sigma_{v, t}(G, w)
= S'_w(G, b)
$$

and

$$
B'_w(G', b') = \sum_{s \in V - \{v, u\}} b'(s) \frac{\sigma_{s, t}(G', w)}{\sigma_{s, t}(G')} + b'(v) \frac{\sigma_{v, t}(G', w)}{\sigma_{v, t}(G')}
= \sum_{s \in V - \{v, u\}} b(s) \frac{\sigma_{s, t}(G, w)}{\sigma_{s, t}(G)} + (b(v) + b(u)) \frac{\sigma_{v, t}(G, w)}{\sigma_{v, t}(G)}
= B'_w(G, b).
$$
**Target Proxy:**

Let \( G' = M_{t \rightarrow v}(G, b) \). Let us notice for any \( s \in V - \{t\} \) that every path from \( s \) to \( t \) has \( (v, t) \) as the last edge, because it is the only edge ending in \( t \). So for every shortest path from \( s \) to \( t \) in \( G' \) there is a path from \( s \) to \( v \) in \( G' \) that lacks only the last edge and for every shortest path from \( s \) to \( v \) in \( G' \) there is a path from \( s \) to \( t \) in \( G \) that is the same apart from the additional edge \( (v, t) \) at the end. This means for any \( s \in V \) that \( \Pi_{s,v}(G') = \{(e_1, \ldots, e_{k-1}) : (e_1, \ldots, e_{k-1}, e_k) \in \Pi_{s,t}(G)\} \). In particular, for every \( w \in V - \{t\} \) we have:

\[
\begin{align*}
\sigma_{s,v}(G', w) &= \sigma_{s,t}(G, w), \\
\sigma_{s,v}(G') &= \sigma_{s,t}(G).
\end{align*}
\]

This gives for every \( w \in V - \{t\} \):

\[
S^w_w(G', b) = \sum_{s \in V - \{t\}} b(s) \cdot \sigma_{s,v}(G', w)
\]

\[
= \sum_{s \in V - \{t\}} b(s) \cdot \sigma_{s,t}(G, w) + 0 \cdot \sigma_{t,t}(G', w)
\]

\[
= S^t_w(G, b)
\]

and

\[
B^w_w(G', b) = \sum_{s \in V - \{t\}} b(s) \cdot \frac{\sigma_{s,v}(G', w)}{\sigma_{s,v}(G')}
\]

\[
= \sum_{s \in V - \{t\}} b(s) \cdot \frac{\sigma_{s,t}(G, w)}{\sigma_{s,t}(G)} + 0 \cdot \frac{\sigma_{t,t}(G', w)}{\sigma_{t,t}(G')}
\]

\[
= B^t_w(G, b).
\]

**Symmetry:**

For any path \( \pi = (\langle v_1, v_2 \rangle, \ldots, \langle v_{k-1}, v_k \rangle) \) let \( \pi^R = (\langle v_k, v_{k-1} \rangle, \ldots, \langle v_2, v_1 \rangle) \). Note that for any path \( \pi \) from \( s \) to \( t \) in \( G \) there is a path \( \pi^R \) from \( t \) to \( s \) in \( G' \). This means for any \( s \in V \) that \( \Pi_{t,s}(G') = \{\pi^R : \pi \in \Pi_{s,t}(G)\} \). In particular, for every \( w \in V \) we have:

\[
\begin{align*}
\sigma_{t,s}(G', w) &= \sigma_{s,t}(G, w), \\
\sigma_{t,s}(G') &= \sigma_{s,t}(G).
\end{align*}
\]

This gives for every \( w \in V \):

\[
S^w_w(G', \mathbb{1}^t) = \sigma_{t,s}(G', w) = \sigma_{s,t}(G, w) = S^t_w(G, \mathbb{1}^s).
\]

and

\[
B^w_w(G', \mathbb{1}^t) = \frac{\sigma_{t,s}(G', w)}{\sigma_{t,s}(G')} = \frac{\sigma_{s,t}(G, w)}{\sigma_{s,t}(G)} = B^t_w(G, \mathbb{1}^s).
\]

**Direct Link Domination:**

Let \( G' = G - \{(v, u)\} \). We know deleted edge \( (v, u) \) is not a part of any shortest path from any \( s \) to \( t \), because if it was, then the path having the same edges from \( s \) to \( v \) and then edge \( (v, t) \) would be shorter. This means for any \( s \in V \) that \( \Pi_{s,t}(G') = \Pi_{s,t}(G) \). In particular, for every \( w \in V \) we have:

\[
\begin{align*}
\sigma_{s,t}(G', w) &= \sigma_{s,t}(G, w), \\
\sigma_{s,t}(G') &= \sigma_{s,t}(G).
\end{align*}
\]

This gives for every \( w \in V \):

\[
S^t_w(G', b) = \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G', w)
\]

\[
= \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, w) = S^t_w(G, b)
\]
Now we split $F$ into $F'$. Let us split $v$. We will now prove that common axioms: Locality, Additivity, Node Redirect and Target Proxy together with Symmetry and Direct Link Domination from the axioms for shortest path based centralities and either Baseline 1-1 or Baseline $k$-$k$ uniquely define centrality. Combined with the fact that $t$-Betweenness and $t$-Stress Centralities satisfy the axiom set including respectively Baseline 1-1 and Baseline $k$-$k$, we get that the axiom set determines the centrality to be respectively $t$-Betweenness and $t$-Stress Centralities.

**Lemma 3. (Anonymity) If $F$ satisfies Node Redirect, then for every graph $G = (V, E) \in \mathcal{G}_t$, node $v \neq t$, new identifier $v' \notin V$ with renaming function $f$ such that $f(v) = v'$ and $f(w) = w$ for $w \in V \setminus \{v\}$, $G' = (\{f(w) : w \in V\}, \{(f(w), f(w')) : (w, w') \in E\})$ and for every node $u$:

$$F_{t(u)}^t(G', b \cdot f^{-1}) = F_{u}^t(G, b).$$

**Proof.** Let us split $v$ into two nodes: $v$ with original weight and incoming edges and its out-twin with desired identifier $v'$, zero weight and no incoming edges: $G'' = (V \cup \{v'\}, E + \{(v', w) : (v, w) \in E\})$, $b'' = b'[v' \rightarrow 0]$. From Node Redirect we know that for every $u \in V \setminus \{v\}$:

$$F_u^t(G, b) = F_u^t(G'', b'),$$

(1)

$$F_v^t(G, b) = F_v^t(G'', b'') + F_v^t(G''', b''').$$

(2)

Now we redirect $v$ back into $v'$, obtaining $R_{v \rightarrow v'}(G'', b'') = (G', b \cdot f^{-1})$. From Node Redirect we know that for every node $u \in V \setminus \{v\}$:

$$F_u^t(G', b'') = F_u^t(G', b \cdot f^{-1}),$$

(3)

$$F_v^t(G', b'') + F_v^t(G'', b''') = F_v^t(G', b \cdot f^{-1}).$$

(4)

From equations (1) and (3) we get for every $u \in V \setminus \{v\}$ that $F_u^t(G', b \cdot f^{-1}) = F_u^t(G, b)$ and from equations (2) and (4) we get $F_v^t(G', b \cdot f^{-1}) = F_v^t(G, b)$. 

**Lemma 4. (Target Self-Loop) If $F$ satisfies Locality, Node Redirect and Target Proxy then for every graph $G = (V, E) \in \mathcal{G}_t$ such that $(t, t) \in E$ and node $v$:

$$F_v^t(G - \{(t, t)\}, b) = F_v^t(G, b).$$

**Proof.** We start with separating the graph $(\{t\}, \{(t, t)\})$ with zero weight from the rest of the graph. From Locality we know that for every node $v \neq t$ there holds $F_v^t(G, b) = F_v^t(G - \{(t, t)\}, b)$, which is the thesis for $v$ and $F_t^t(G, b) = F_t^t(G - \{(t, t)\}, b) + F_t^t(\{(t, t)\}, 0)$.

We add a new target $r$. From Target Proxy we get $F_t^r(\{(t, t)\}, 0) = F_t^r(\{(t, r), (t, r)\}), 0)$. For now we know that:

$$F_t^t(G, b) = F_t^t(G - \{(t, t)\}, b) + F_t^t(\{(t, r), (t, r)\}, 0).$$

(5)

Now we split $t$ into out-twins $t$ and $t'$, obtaining $G' = \{(t, t'), \{(t, r), (t', r)\}\}$. From Node Redirect we get:

$$F_t^t(\{(t, r), (t, r)\}, 0) = F_t^t(G', 0) + F_t^r(G', 0).$$

(6)
Equations 9 and 10 give the thesis for \( t \) from Additivity for each node \( v \). We separate a graph consisting only of the node \( v \) and from Lemma 3 (Anonymity) also \( F^t_v((t, r), \{(t', r')\}, 0) = F^t_v((t, r), \{(t, r)\}, 0) \), so:

\[
F^t_v(G', 0) = F^t_v((t, r), \{(t, r)\}, 0).
\]

Combining equations 8 with 7 and 8 we get \( F^t_v((t, r), \{(t, r)\}, 0) = F^t_v((\{(t, r)\}, \{(t, r)\}), 0) + F^t_v((\{(t, r)\}, \{(t, r)\}), 0) \), that is \( F^t_v((\{(t, r)\}, \{(t, r)\}), 0) = 0 \). This combined with equation 5 gives the thesis for \( t \).

**Lemma 5.** (No Target Outlet) If \( F \) satisfies Locality, Node Redirect, Target Proxy and Direct Link Domination then for every graph \( G = (V, E) \in G_t \) such that \( t, v \in E \) and node \( u \):

\[
F^t_u(G - \{(t, v)\}, b) = F^t_u(G, b).
\]

**Proof.** If \( v = t \), then thesis follows from Lemma 4 (Target Self-Loop). Assume otherwise.

We start with adding a \( t \) self-loop. From Lemma 3 (Target Self-Loop) we know that \( F^t_u(G, b) = F^t_v((V, E + \{(t, t)\}), b) \). Now we delete edge \( (t, v) \). From Direct Link Domination (because \( t \) has now an edge to \( t \)) we know that \( F^t_v((V, E + \{(t, t)\}), b) = F^t_v((V, E + \{(t, t)\}) - \{(t, v)\}, b) \). Finally we delete \( t \) self-loop and again from Lemma 4 (Target Self-Loop) we get the thesis.

**Lemma 6.** (Distance 0) If \( F \) satisfies Locality, Additivity, Node Redirect, Target Proxy and Baseline 1-1 or Baseline k-k, then for every graph \( G \in G_t \) and node \( v \neq t \):

\[
F^t_v(G, 1^t) = 0
\]

and \( F^t_v(G, 1) = 1 \).

**Proof.** We separate a graph consisting only of the node \( t \) with unitary weight from the rest of the graph. From Locality we know that \( F^t_v(G, 1^t) = F^t_v(G, 0) \) and \( F^t_v(G, 1) = F^t_v(G, 0) + F^t_v((\{(t, t)\}), 1^t) \). Centralities in \( (G, 0) \) are equal zero, because from Additivity for each node \( v \) it holds \( F^t_v(G, 0) = F^t_v(G, 0) + F^t_v(G, 0) \). This gives the thesis for \( v \neq t \). This also means:

\[
F^t_v(G, 1^t) = F^t_v((\{(t, t)\}), 1^t).
\]

From Baseline 1-1 or Baseline k-k we know that \( F^t_v((\{(t, r)\}, \{(t, r)\}), 1^t) = 1 \). We merge \( r \) into \( t \) and from Target Proxy we know that \( F^t_v((\{(t, r)\}, \{(t, t)\}), 1^t) = F^t_v((\{(t, t)\}), 1^t) \). But from Lemma 3 (Target Self-Loop) we already know that \( F^t_v((\{(t, t)\}), 1^t) = F^t_v((\{(t, t)\}), 1^t) \). Which results in:

\[
F^t_v((\{(t, t)\}), 1^t) = 1.
\]

Equations 9 and 10 give the thesis for \( t \).

**Lemma 7.** (Distance 1) If \( F \) satisfies Locality, Additivity, Node Redirect, Direct Link Domination and Baseline 1-1 or Baseline k-k, then for every graph \( G = (V, E) \in G_t \), source node \( s \neq t \), such that \( (s, t) \in E \) and node \( v \in V - \{s, t\} \):

\[
F^t_v(G, 1^s) = 0,
\]

\( F^t_s(G, 1^s) \) and \( F^t_v(G, 1^s) \) are determined.

**Proof.** We delete all edges outgoing from \( s \) to nodes other than \( t \), obtaining graph \( G_1 = (V_1, E_1) = (V, E - \{(s, u) : (s, u) \in E \land u \neq t\}) \). From Direct Link Domination for every node \( v \in V \) we know that:

\[
F^t_v(G_1, 1^s) = F^t_v(G_1, 1^s).
\]

Let \( k = |\Gamma^+_1(G_1)| \). We split \( s \) into out-twins \( s \) with all incoming edges and zero weight and \( s' \) with no incoming edges and unit weight, obtaining \( G_2 = (V_2, E_2) = (V \cup s', E_1 \cup k \cdot \{(s', t)\}) \). From Node Redirect for every node \( v \in V - \{s\} \) we know that:

\[
F^t_v(G_2, 1^s) = F^t_v(G_2, 1^s),
\]

\[
F^t_s(G_1, 1^s) = F^t_s(G_2, 1^s) + F^t_s(G_2, 1^s).
\]

We further separate \( s' \) into another graph, leaving again \( G_1 \), but this time with zero weights. From Locality for every node \( v \in V - \{t\} \) we know that \( F^t_v(G_2, 1^s') = F^t_v(G_2, 1^s) \), \( F^t_s(G_2, 1^s') = F^t_s((\{(s', t)\}, k \cdot \{(s', t)\}), 1^s') \) and \( F^t_v(G_2, 1^s') =
Now we proceed as before. We merge all nodes from \( F \) and from equations 11, 12 and 16 we get that \( v \) knows not only that from \( s \) we know that: 

\[
F^v(F_1(G_0, 0) + F^v_1((s', t), k \cdot \{(s', t)\}), 1^{s'}) = 0, \tag{14}
\]

\[
F^v(G_2, 1^{s'}) = F^v_1((s', t), k \cdot \{(s', t)\}), 1^{s'}), \tag{15}
\]

\[
F^v(G_2, 1^{s'}) = F^v_1((s', t), k \cdot \{(s', t)\}), 1^{s'}). \tag{16}
\]

Combining equations 11, 12 and 14 we get for \( v \in V - \{s, t\} \) that \( F^v(v, 1^s) = 0 \). Centralities in \( (s', t), k \cdot \{(s', t)\}, 1^{s'} \) are determined from Baseline 1-1 or Baseline \( k-k \). From equations 11, 12 and 13 we get that \( F^v(F_1(G_1, 1^s) \) is determined and from equations 11, 12 and 16 we get that \( F^v(F_1(G_1, 1^s) \) is determined.

**Lemma 8. (Distance 2 Simple)** If \( F \) satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1 or Baseline \( k-k \), then for every graph \( G = (V, E) \in G \) with source node \( s \neq t \), such that \( (s, t) \notin E \), which satisfies following assumptions:

- For every node \( u \in S_1^1(G) \) it holds \( \Gamma^u_u(G) = \{(s, u)\} \).
- For every node \( u \in P_1^1(G) \) it holds \( \Gamma^u_u(G) = \{(u, t)\} \).
- \( W = S_1^1(G) \cap P_1^1(G) \) is not empty,

for every node \( v \in V - W - \{s, t\} \):

\[
F^v(G, 1^s) = 0,
\]

and centralities of other nodes are determined.

**Proof.** We will use for graph \( G = (V, E) \) with weights \( b \) and nodes \( S \subseteq V \) a shorthand \( F^v_S(G, b) = \sum_{v \in S} F^v_v(G, b) \).

Let us assume \( t \) has no self-loops, as we could delete them and from Lemma 4 (Target Self-Loop) we know that centralities would not change.

Let \( T = P_1^1(G) \) and \( S = S_1^1(G) - \{s\} \). We merge all nodes from \( T \) into one node, \( t' \), obtaining \( G_1 = (V_1, E_1) = (V - T \cup \{t'\}, E - \bigcup_{u \in T} \Gamma_u(G) + \{(t', t)\} + \{(w, t') : (w, u) \in E \wedge u \in T\}) \). From Node Redirect for every \( v \in V - T \) we know that:

\[
F^v(G_1, 1^s) = F^v_v(G_1, 1^s), \tag{17}
\]

\[
F^v(G_1, 1^s) = F^v_v(G_1, 1^s). \tag{18}
\]

Now target has only one predecessor, \( t' \). We merge \( t \) into \( t' \) obtaining \( G_2 = (V_2, E_2) = M_{t \rightarrow t'}(G_1) \). From Target Redirect for every \( v \in V_2 \) we know that:

\[
F^v(G_2, 1^s) = F^v_v(G_2, 1^s). \tag{19}
\]

We obtained a graph in which distance from the source \( s \) to the target \( t' \) is equal one. So from Lemma 7 (Distance 1) for every \( v \in V_2 - \{s, t'\} \) we get that \( F^v_v(G_2, 1^s) \) is determined and \( F^v_v(G_2, 1^s) \) are determined. Combining this with equations 17 and 19 for every \( v \in V - T - \{s, t\} \) we know that:

\[
F^v_v(G, 1^s) = 0 \tag{20}
\]

and \( F^v_v(G, 1^s) \) is determined. Combining with equations 18 and 19 we get that \( F^v_v(G, 1^s) \) is determined.

Let us now go back to graph \( G \) and for each node \( u \in V - S \) add an edge \( (t, u) \), obtaining graph \( G_3 = (V, E_3) = (V, E + \{(t, u) : u \in V - S\}) \). From Lemma 5 (No Target Outlet) for every \( v \in V \) we know that \( F^v_v(G_3, 1^s) = F^v_v(G_3, 1^s) \). In \( G_3 \) we know not only that from \( s \) there is an edge to each node from \( S \) and a path to \( t \), but also from \( t \) there is an edge to every node \( u \in V - S \), so from \( s \) there is a path to every node. This means that the reversed graph \( G_4 = (V, E_4) = (V, \{(u, w) : (w, u) \in E_4\}) \) is in the class \( G \). From Symmetry for every \( v \in V \) we know that \( F^v_v(G_3, 1^s) = F^v_v(G_4, 1^t) \). We get that for every \( v \in V \):

\[
F^v_v(G, 1^s) = F^v_v(G_4, 1^t). \tag{21}
\]

Now we proceed as before. We merge all nodes from \( S \) into one node, \( s' \), obtaining \( G_5 = (V_5, E_5) = (V - S \cup \{s'\}, E_4 - \bigcup_{u \in S} \Gamma_u(G) + \{(s', s)\} + \{(w, s') : (w, u) \in E_4 \wedge u \in S\}) \). From Node Redirect for every \( v \in V - S \) we know that:

\[
F^v_v(G_5, 1^t) = F^v_v(G_5, 1^t), \tag{22}
\]

\[
F^v_v(G_4, 1^t) = F^v_v(G_5, 1^t). \tag{23}
\]
Now target $s$ has only one predecessor, $s'$. We merge $s$ into $s'$ obtaining $G_6 = (V_6, E_6) = M_{s \to s'}(G_5)$. From Target Redirect for every $v \in V_6$ we know that:

$$F^*_v(G_5, 1^t) = F^*_v(G_6, 1^t).$$  \hfill (24)

We obtained a graph in which distance from the source $t$ to the target $s'$ is equal one. So from Lemma 7 (Distance 1) for every $v \in V_6 - \{s', t\}$ we know that $F^*_v(G_6, 1^s_v) = 0$ and $F^*_v(G_6, 1^t)$ are determined. Combining this with equations 21, 22 and 24 for every $v \in V - S - \{s, t\}$ we know that:

$$F^*_v(G, 1^s) = 0$$  \hfill (25)

and $F^*_v(G, 1^s)$ is determined. Combining with equations 21, 22 and 24 we get that $F^*_v(G, 1^s)$ is determined.

Finally we know that both $F^*_v(G, 1^s)$ and $F^*_v(G, 1^s)$ are determined. We also know from equations 20 and 25 for every $v \in V - W - \{s, t\}$ that $F^*_v(G, 1^s) = F^*_v(G, 1^s) - F^*_v(G, 1^s) = F^*_v(G, 1^s)$ is determined. But each of the nodes in $W$ has exactly one incoming edge, from $s$, and one outgoing edge, to $t$. So from Lemma 3 (Anonymity) they all have the same centrality. This means they divide determined $F^*_v(G, 1^s)$ equally among them.

\[\square\]

**Lemma 9.** (Distance 2) If $F$ satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1 or Baseline $k$-$k$, then for every graph $G \in G_k$ and source node $s$, such that $\text{dist}_{s,t}(G) = 2$, centrality $F^*(G, 1^s)$ is determined.

\[\text{Proof.}\] Let us assume $t$ has no self-loops, otherwise we could delete them and from Lemma 4 (Target Self-Loop) we know that centralities would not change.

We start by splitting every successor $u_i$ of $s$ (apart from possibly $s$) into out-twins $u_i$ and $u_i^k$ for $k_i = m_{i,s,u_i}(E)$, such that each of the nodes $u_i^k$ has only one incoming edge, from $s$, and $u_i$ has the rest of the edges originally incoming to $u_i$, that is ones not outgoing from $s$. More precisely, for every $u_i \in S^1_s(G) - \{s\}$ let nodes $V'_i = u_i^1, ..., u_i^{k_i}$ and edges $E'_i = \{(s, u_i^1), ..., (s, u_i^{k_i})\} \cup \{(u_i^k, w) : (u_i, w) \in E\} \cup \{(u_i^k, w) : (u_i, w) \in E\}$. Let $k = |S^1_s(G) - \{s\}|$. Let $G_1 = (V_1, E_1) = (V \cup V'_1 \cup ... \cup V'_k, E - k \cdot \{(s, u_i^1)\} - ... - k \cdot \{(s, u_k^k)\} + E'_1 + ... + E'_k)$. From Node Redirect for every $v \in V - (S^1_s(G) - \{s\})$ we know that:

$$F^*_v(G, 1^s) = F^*_v(G_1, 1^s)$$  \hfill (26)

and for every $u_i \in S^1_s(G) - \{s\}$ we know that:

$$F^*_u(G, 1^s) = F^*_u(G_1, 1^s)$$  \hfill (27)

Now every successor of $s$ (apart from possibly $s$), has exactly one incoming edge, the one from $s$. Note also that because $\text{dist}_{s,t}(G) = 2$ there were some $u_i \in S^1_s(G) - \{s\}$ such that $(u_i, t) \in E$. But this means that without loss of generality there is some $u_i^k \in S^1_s(G_1)$ such that $(u_i^k, t) \in E$, so $\text{dist}_{s,t}(G_1) = 2$.

Let us now for each node $u \in V - S^1_s(G)$ add an edge $(t, u)$, obtaining graph $G_2 = (V_2, E_2) = (V_1, E_1 + \{(t, u) : u \in V - S^1_s(G_1)\})$. From Lemma 5 (No Target Outlet) for every $v \in V_1$ we know that $F^*_v(G_1, 1^s) = F^*_v(G_2, 1^s)$. In $G_2$ we know not only that from $s$ there is an edge to each node from $S^1_s(G_1)$ and a path to $t$, but also from $t$ there is an edge to every node $u \in V - S^1_s(G_1)$, so from $s$ there is a path to every node. This means that the reversed graph $G_3 = (V_1, E_3) = (V_1, \{(u, w) : (w, u) \in E_2\})$ is in the class $G_s$. From Symmetry for every $v \in V_1$ we know that $F^*_v(G_2, 1^s) = F^*_v(G_3, 1^s)$. We get for every $v \in V_1$ that:

$$F^*_v(G_1, 1^s) = F^*_v(G_3, 1^t)$$  \hfill (28)

Now every predecessor of $s$ (apart from possibly $s$), has exactly one outgoing edge, the one to $s$.

Again, let us assume $s$ has no self-loops, as we could delete them and from Lemma 4 (Target Self-Loop) we now that centralities would not change, because $s$ is now the target. Now every predecessor of $s$ has only one outgoing edge, to $s$.

Let us now proceed similarly as before and split every successor $u_i$ of $t$ into out-twins $u_i$ and $u_i^k$ for $k_i = m_{t,u_i}(E_3)$, such that each of the nodes $u_i^k$ has only one incoming edge, from $t$ and $u_i$ has remaining incoming edges, obtaining $G_4 = (V_4, E_4)$. From Node Redirect for every $v \in V_1 - S^1_s(G_3)$ we know that:

$$F^*_v(G_3, 1^t) = F^*_v(G_4, 1^t)$$  \hfill (29)
and for every $u_i \in S_t^1(G_3)$ we know that:
\[
F^s_{u_i}(G_3, \mathbb{1}^t) = F^s_{u_i}(G_4, \mathbb{1}^t) + F^s_{u_i}(G_4, \mathbb{1}^t) + \ldots + F^s_{u_i}(G_4, \mathbb{1}^t).
\]  
(30)

Now every successor of $t$ has exactly one incoming edge, the one from $t$.

We obtained a graph in which every successor of the source $t$ has exactly one outgoing edge, from $t$ and every predecessor of the target $s$ has exactly one outgoing edge, to $s$. Moreover, $s \neq t, (t, s) \notin E_4$ and $W = S_t^2(G_4) \cap P^s_t(G_4)$ is not empty. So centrality of each node in $G_4$ is determined from Lemma 8 (Distance 2 Simple). Combining this with equations 29 and 30 we see that centrality of each node in $(G_3, \mathbb{1}^t)$ is determined. Further combining with equations 26, 27 and 28 we see that centrality of each node in $(G, \mathbb{1}^s)$ is also determined.

\[\square\]

Lemma 10. (Single Source) If $F$ satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1 or Baseline k-k, then for every graph $G \in G_i$ and source node $s$ centrality $F^s_i(G, \mathbb{1}^s)$ is determined.

Proof. The proof will be inductive from the $dist_{s,t}(G)$. If $dist_{s,t}(G) < 3$, then the thesis follows from Lemmas 5 (Distance 0), 7 (Distance 1) and 9 (Distance 2).

Let us discuss the inductive step. Let $(V, E) = G$ and fix a node $v \in V$. We know $dist_{s,v}(G) + dist_{v,t}(G) \geq dist_{s,t}(G) \geq 3$, so either $dist_{s,v}(G) \geq 2$ or $dist_{v,t}(G) \geq 2$.

If $dist_{s,v}(G) \geq 2$, for each node $u \in V$ we address an edge $(t, u)$, obtaining graph $G' = (V, E') = (V, E + \{(t, u) : u \in V\})$. From Lemma 5 (No Target Outlet) we know that $F^s_i(G, \mathbb{1}^s) = F^s_i(G', \mathbb{1}^s)$. In $G'$ we know not only that from $s$ there is a path to $t$, but also from $t$ there is an edge to every node $u$, so from $s$ there is a path to $u$. This means the reversed graph $G'' = (V, E'') = (V, \{(u, w) : (w, u) \in E'\})$ is in the class $G_s$. From Symmetry we know that $F^s_i(G', \mathbb{1}^s) = F^s_i(G'', \mathbb{1}^s)$. So we reduced this case to the case where the distance from the source to the fixed node $v$ is greater or equal two (and the distance from the source to the target is the same).

Assume now $dist_{s,v}(G) \geq 2$. We start with splitting every successor $u_i$ of $s$ (apart from possibly $s$) into out-twins $u_i$ and $u_i^k$, for $k_i = m_{i,s,v}(E)$, such that each of the nodes $v_i, v_i^k$ has only one incoming edge, from $s$ and $u_i$ has remaining incoming edges. More precisely, for every $u_i \in S_t^1(G) - \{s\}$ let nodes $V_i' = u_i^1, \ldots, u_i^k$ and edges $E_i' = \{(s, u_i^1), \ldots, (s, u_i^k)\} \cup \{(u_i^k, w) : (u_i, w) \in E\} \cup \{(u_i^k, w) : (u_i, w) \in E\}$. Let $k = |S_t^1(G) - \{s\}|$. Let $G_1 = (V_1, E_1) = (V \cup V_i' \cup \ldots \cup V_k', E - \Gamma^+_s(G) + E_i' + \ldots + E_k')$. From Node Redirect we know that:
\[
F^s_i(G, \mathbb{1}^s) = F^s_i(G_1, \mathbb{1}^s),
\]  
(31)

because $v \notin S_t^1(G)$. Now every successor of $s$ (apart from possibly $s$), has exactly one incoming edge, the one from $s$.

Let us now for each node $u \in V - S_t^1(G_1)$ add an edge $(t, u)$, obtaining graph $G_2 = (V_1, E_2) = (V_1, E_1 + \{(t, u) : u \in V - S_t^1(G_1)\})$. From Lemma 3 (No Target Outlet) we know that $F^s_i(G_1, \mathbb{1}^s) = F^s_i(G_2, \mathbb{1}^s)$. In $G_2$ we know not only that from $s$ there is an edge to each node from $S_t^1(G_1)$ and a path to $t$, but also from $t$ there is an edge to every node $u \in V - S_t^1(G_1)$, so from $s$ there is a path to every node. This means the reversed graph $G_3 = (V_1, E_3) = (V_1, \{(u, w) : (w, u) \in E_2\})$ is in the class $G_s$. From Symmetry we know that $F^s_i(G_2, \mathbb{1}^s) = F^s_i(G_3, \mathbb{1}^s)$. We get that:
\[
F^s_i(G_1, \mathbb{1}^s) = F^s_i(G_3, \mathbb{1}^t)
\]  
(32)

We delete self-loops, obtaining $G_4 = (V_4, E_4) = (V_1, E_3 - \{m_{s,v}(E_3) \cdot (s, s)\}$. Node $s$ is now the target, so from Lemma 4 (Target Self-Loop) we obtain:
\[
F^s_i(G_4, \mathbb{1}^t) = F^s_i(G_4, \mathbb{1}^t).
\]  
(33)

Now every predecessor of $s$ has only one outgoing edge, to $s$. We merge them into one node, $s'$, obtaining $G_5 = (V_5, E_5) = (V_4 - P_d^s(G_4) \cup \{s'\}, E_4 - \Gamma^+_s(G_4) + \{(s', s)\} + \{(w, s') : (w, u), (u, s) \in E\})$. From Node Redirect we know that:
\[
F^s_i(G_4, \mathbb{1}^t) = F^s_i(G_5, \mathbb{1}^t).
\]  
(34)

Finally $s$ has only one incoming edge, from $s'$ (and $s'$ has only one outgoing edge, to $s$). Let us merge $s$ into $s'$, obtaining $G_6 = M_{s \to s'}(G_5, \mathbb{1}^t)$. From Target Proxy we know that:
\[
F^s_i(G_5, \mathbb{1}^t) = F^s_i(G_6, \mathbb{1}^t).
\]  
(35)

But $dist_{s,t}(G) = dist_{s,t}(G_6) + 1$, so from the inductive assumption we know that $F^s_i(G_6, \mathbb{1}^t)$ is determined. This combined with equations 31, 35 shows that also $F^s_i(G, \mathbb{1}^s)$ is uniquely determined.

\[\square\]
Lemma 11. If \( F \) satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1 or Baseline \( k-k \), then it is determined.

Proof. Fix graph \( G = (V, E) \) with weights \( b \) and a node \( v \in V \). If \( b = 0 \), then centrality of \( v \) is equal zero, because from Additivity it holds \( F^t_v(G, 0) = F^t_v(G, 0) + F^t_v(G, 0) \).

Otherwise we decompose weight function into non-empty sum of functions positive for only one node: \( b = \sum_{s \in V} b(s) \cdot 1^s \).

From Additivity we know that:
\[
F^t_v(G, b) = \sum_{s \in V} F^t_v(G, b(s) \cdot 1^s).
\] (36)

Moreover, from Additivity function \( F^t_v(G, b(s) \cdot 1^s) \) is additive from \( b(s) \) and from the definition it is non-negative, so we know it is linear, that is:
\[
F^t_v(G, b(s) \cdot 1^s) = b(s) \cdot F^t_v(G, 1^s).
\] (37)

We know that \( F^t_v(G, 1^s) \) is determined from Lemma 10 (Single Source). This combined with equations 36 and 37 shows that also \( F^t_v(G, b) \) is uniquely determined.

Lemma 12. If \( F \) satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline 1-1, then it is t-Betweenness Centrality.

Proof. From Lemma 11 we know that \( F \) is uniquely determined. But from Lemma 1 we know that t-Betweenness satisfies these axioms, so this is our determined centrality.

Lemma 13. If \( F \) satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Baseline \( k-k \), then it is t-Stress Centrality.

Proof. From Lemma 11 we know that \( F \) is uniquely determined. But from Lemma 2 we know that t-Stress satisfies these axioms, so this is our determined centrality.
Proof of Theorem 3 and Theorem 4

In this section, we present the joint full proof of Theorem 3 and Theorem 4. We start with showing that the random-walk based centralities satisfy their respective axiom sets. The main part is dedicated to showing that the axioms uniquely characterize a centrality measure (up to the decay factor). We use our stretched definition of t-PageRank that allows $a = 1$, to present the joint proofs for t-PageRank and t-Random Walk Betweenness.

$t$-PageRank, $t$-Random Walk Betweenness $\Rightarrow$ Axioms

We will now consider each from the axioms: Locality, Additivity, Node Redirect, Target Proxy, Edge Swap and Edge Multiplication and show that t-PageRank for stretched decay factor $a \in [0, 1)$ satisfies it. Finally we calculate values of t-Random Walk Betweenness and t-PageRank centralities on the Baseline graph.

Lemma 14. t-PageRank for every decay factor $a \in [0, 1)$ satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline $1 - a$.

Lemma 15. t-Random Walk Betweenness satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline $1 - 1$.

Let us define a shorthand $G_t = G - \Gamma_t^+ (G)$.

Locality:

Let us notice that for any $s, v, u \in V$ random walk starting in $s$ will have in each step $k$ the same transition probability from $v$ to $u$ in $(G + G')_t$ as in $G_t$, because edges outgoing of $v$ are the same. Also, transition probability to any $u' \in V' - \{t\}$ will be zero, because there are no edges from $v$ to $u'$. This means:

$$P(\omega_a^{G+G'}, s, (k) = w) = P(\omega_a^{G_t, s} (k) = u),$$

$$P(\omega_a^{(G+G'), s} (k) = u') = 0.$$

We obtain:

$$PR^a_{w,t}(G + G', b + b')$$

$$= \sum_{s \in V - \{t\}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_a^{G+G'}, s, (k) = w)$$

$$+ \sum_{k=0}^{\infty} (b(t) + b'(t)) \cdot 0$$

$$+ \sum_{s' \in V' - \{t\}} \sum_{k=0}^{\infty} b'(s') \cdot P(\omega_a^{(G+G'), s'} (k) = w)$$

$$= \sum_{s \in V - \{t\}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_a^{G_t, s} (k) = w) + \sum_{k=0}^{\infty} b(t) \cdot 0$$

$$= PR^a_{w,t}(G, b)$$

21
and

\[
P R_{t}^{a,t}(G + G', b + b')
= \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{(G+G')}_{t,s}(k) = t) + (b(t) + b'(t))
+ \sum_{s' \in V \setminus \{t\}} \sum_{k=0}^{\infty} b'(s') \cdot P(\omega_{(G+G')}_{t,s'}(k) = t)
= \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{(G')}_{t,s}(k) = t) + b(t)
+ \sum_{s' \in V \setminus \{t\}} \sum_{k=0}^{\infty} b'(s') \cdot P(\omega_{(G')}_{t,s'}(k) = t) + b'(t)
= PR_{t}^{a,t}(G, b) + PR_{t}^{a,t}(G', b').
\]

Additivity:

From the definition:

\[
P R_{w}^{a,t}(G, b + b')
= \sum_{s \in V} \sum_{k=0}^{\infty} (b(s) + b'(s)) \cdot P(\omega_{G_{t,s}}(k) = w)
= \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G_{t,s}}(k) = w)
+ \sum_{s \in V} \sum_{k=0}^{\infty} b'(s) \cdot P(\omega_{G_{t,s}}(k) = w)
= PR_{w}^{a,t}(G, b) + PR_{w}^{a,t}(G, b').
\]

Node Redirect:

Let \((G', b') = R_{v \to v}(G, b)\). Let us denote by \(p_{v,u}((V, E)) = a \cdot \frac{m_{v,u}}{|E(G)|}\) the probability of direct transition from node \(v\) to node \(u\) by the random walk on \(G\).

We will describe a few properties of these transition probabilities in our graphs. Firstly, for every \(w \in V\) we have \(p_{u,w}(G_{t}) = p_{v,w}(G_{t})\), because \(v\) and \(u\) are out-twins. Next, we know that \(p_{v,v}(G_{t}) = p_{v,v}(G_{t})\), because outgoing edges of \(v\) stays the same.

As for the transition probability to \(v\) but from some other node \(v \in V \setminus \{v, u\}\), we know that \(p_{u,v}(G_{t}) = p_{v,v}(G_{t}) + p_{r,u}(G_{t})\), because each outgoing edge of \(r\) ending in \(v\) in \(G_{t}\) has corresponding edge ending in \(v\) or \(u\) in \(G_{t}\). Finally, we know for every \(r \in V \setminus \{u\}\) and \(w \in V \setminus \{u, v\}\) that \(p_{r,w}(G_{t}) = p_{r,w}(G_{t})\) because both number of edges \((r, w)\) is the same and out-degree of \(r\) is the same.

Now we want to consider whole random walks. Firstly, for \(w \in V \setminus \{v, u\}\) and every number \(k\) we know that:

\[
P(\omega_{G_{t,v}}(k) = w) = P(\omega_{G_{t,v}}(k) = w),
\]

because \(v\) and \(u\) are out-twins. Moreover:

\[
P(\omega_{G_{t,u}}(k) = v) + P(\omega_{G_{t,u}}(k) = u) = P(\omega_{G_{t,v}}(k) = v) + P(\omega_{G_{t,v}}(k) = u),
\]

because in step zero both sides are equal one, and in further steps the random walks behave the same.
Knowing how the random walk behaves, we deduce that:

\[ P(\omega_{G_t,s}^{a}(0) = w) = [s = w] = P(\omega_{G_t,s}^{a}(0) = w), \]

\[ P(\omega_{G_t,s}^{a}(0) = v) = [s = v] = P(\omega_{G_t,s}^{a}(0) = v) + P(\omega_{G_t,s}^{a}(0) = u), \]

because \( s \neq u \).

Let us show the inductive step.

We would like to prove for any \( s \in V - \{ u \} \) and any number \( k \) that \( P(\omega_{G_t,s}^{a}(k) = w) = P(\omega_{G_t,s}^{a}(k) = w) \) and \( P(\omega_{G_t,s}^{a}(k) = v) = P(\omega_{G_t,s}^{a}(k) = v) + P(\omega_{G_t,s}^{a}(k) = u) \). The proof will be inductive. For \( k = 0 \) we have:

\[ P(\omega_{G_t,s}^{a}(0) = w) = [s = w] = P(\omega_{G_t,s}^{a}(0) = w), \]

\[ P(\omega_{G_t,s}^{a}(0) = v) = [s = v] = P(\omega_{G_t,s}^{a}(0) = v) + P(\omega_{G_t,s}^{a}(0) = u), \]

and

\[ P(\omega_{G_t,s}^{a}(k + 1) = w) = \sum_{r \in V - \{ v, u \}} p_{r,w}(G_t) \cdot P(\omega_{G_t,s}^{a}(k) = r) \]

\[ + p_{v,w}(G_t) \cdot P(\omega_{G_t,s}^{a}(k) = v) \]

\[ = \sum_{r \in V - \{ v, u \}} p_{r,w}(G_t) \cdot P(\omega_{G_t,s}^{a}(k) = r) \]

\[ + p_{v,w}(G_t) \cdot (P(\omega_{G_t,s}^{a}(k) = v) + P(\omega_{G_t,s}^{a}(k) = u)) \]

\[ = P(\omega_{G_t,s}^{a}(k + 1) = w) + P(\omega_{G_t,s}^{a}(k + 1) = u). \]

Knowing how the random walk behaves, we deduce that:

\[ PR_{w}^{a,t}(G', b') \]

\[ = \sum_{s \in V - \{ v \}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G_t,s}^{a}(k) = w) \]

\[ + \sum_{k=0}^{\infty} (b(v) + b(u)) \cdot P(\omega_{G_t,s}^{a}(k) = w) \]

\[ = \sum_{s \in V - \{ v \}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G_t,s}^{a}(k) = w) \]

\[ + \sum_{k=0}^{\infty} b(v) \cdot P(\omega_{G_t,v}(k) = w) + \sum_{k=0}^{\infty} b(u) \cdot P(\omega_{G_t,u}(k) = w) \]

\[ = PR_{w}^{a,t}(G, b). \]
and

\[ PR^{a,t}_w(G', b') \]
\[ = \sum_{s \in V - \{v\}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G',s}^a(k) = v) \]
\[ + \sum_{k=0}^{\infty} (b(v) + b(u)) \cdot P(\omega_{G',v}^a(k) = v) \]
\[ = \sum_{s \in V - \{v\}} \sum_{k=0}^{\infty} b(s) \cdot (P(\omega_{G,t,s}^a(k) = v) + P(\omega_{G,t,s}^a(k) = u)) \]
\[ + \sum_{k=0}^{\infty} b(v) \cdot (P(\omega_{G,t,v}^a(k) = v) + P(\omega_{G,t,v}^a(k) = u)) \]
\[ + \sum_{k=0}^{\infty} b(u) \cdot (P(\omega_{G,t,u}^a(k) = v) + P(\omega_{G,t,u}^a(k) = u)) \]
\[ = PR^{a,t}_w(G, b) + PR^{a,t}_w(G, b). \]

Target Proxy:

Let \((G', b) = M_{t \rightarrow v}(G, b)\). Let us notice that transition probability from \(v\) to any node \(w \in V - \{t\}\) is equal zero in \(G_t\), because the only outgoing edges from \(v\) leads to \(t\). It is also equal zero in \(G'_v\), because outgoing edges of \(v\) are deleted. Outgoing edges of every other node from \(V - \{t\}\) are the same in \(G_t\) and \(G'_v\), so the transition probability from it is also the same. This means for any \(s, w \in V - \{t\}\) and number \(k\) that:

\[ P(\omega_{G',s}^a(k) = w) = P(\omega_{G,t,s}^a(k) = w). \]

We obtain:

\[ PR^{a,v}_w(G', b) = \sum_{s \in V - \{t\}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G',s}^a(k) = w) \]
\[ = \sum_{s \in V - \{t\}} \sum_{k=0}^{\infty} b(s) \cdot P(\omega_{G,t,s}^a(k) = w) \]
\[ + \sum_{k=0}^{\infty} 0 \cdot P(\omega_{G,t,t}^a(k) = w) \]
\[ = PR^{a,t}_w(G, b). \]

Edge Swap:

The proof is the straightforward adaption to our setting of the proof from the [Was and Skibski 2020](#). We will need an alternative definition of t-PageRank. Namely, t-PageRank is the unique function satisfying for every graph \(G = (V, E) \in \mathcal{G}_t\) with weights \(b\) a system of equations, one for every node \(w \in V\):

\[ PR^{a,t}_w(G, b) = b(w) + \sum_{(r, w) \in E} a \cdot \frac{PR^{a,t}_r(G_t, b)}{\Gamma_w^+(G_t)}. \]

We obtain this equivalent definition from the fact that our t-PageRank for \(a \in [0, 1)\) is the special case of the original PageRank having such a definition and for \(a = 1\) is the special case of [Bösch et al. 2011](#) betweenness-like centrality. Let \(x_w = PR^{a,t}_w(G, b)\). We will show that these values satisfy also system of equations for \((G', b)\). Since it has only one solution, this will imply \(x_w = PR^{a,t}_w(G', b)\).
Let us note that out-degree of weight of any node has not changed. For \( w \in V - \{ v', u' \} \), also its incoming edges has not changed, so we obtain its equation in \((G', b)\):

\[
x_w = PR^{a,t}_w(G, b) = b(w) + \sum_{(r,w) \in \Gamma^+_w(G_t)} a \cdot \frac{PR^{a,t}_w(G_t, b)}{\Gamma^+_r(G_t)}
\]

To obtain the equation for \( v' \) (equation for \( u' \) is analogous) in \((G', b)\) we will also use the assumption that \( v \) and \( u \) have equal \( t \)-PageRank in \((G, b)\):

\[
x_{v'} = PR^{a,t}_{v'}(G, b) = b(v') + a \cdot PR^{a,t}_{v'}(G_t, b)
\]

**Edge Multiplication:**

Let \( G' = (V, E') = G + k \cdot \Gamma^+_v(G) \). Let us notice that for every node \( u \) and number \( k \) the probability of the random walk being in \( u \) in step \( k + 1 \) if it was in \( v \) in step \( k \) is the same in \( G' \) as in \( G \), that is for every \( s \in V \):

\[
P(\omega^{a}_{G_t,s}(k + 1) = u | \omega^{a}_{G_t,s}(k) = v) = a \cdot \frac{m(v,u)(E')}{\Gamma^+_v(G')}
\]

This means the transition probabilities from any node to any node in \( G' \) are the same as in \( G \) and weights \( b \) are also the same, so:

\[
PR^{a,t}_w(G', b) = PR^{a,t}_w(G, b).
\]

**Baseline:**

We know that that the random walk from \( s \) will be in \( s \) in step zero and in \( t \) in step one, where it will end its travel. Random walk from \( t \) does not matter, as \( b(t) = 0 \). So:

\[
PR^{a,t}_w(G, 1^*) = \sum_{k=0}^{\infty} P(\omega^{a}_{G_t,s}(k) = s) = P(\omega^{a}_{G_t,s}(0) = s) = 1
\]
and

\[ PR^a_t(G, 1^s) = \sum_{k=0}^{\infty} P(\omega^a_{G_t,s}(k) = t) \]

\[ = P(\omega^a_{G_t,s}(1) = t) = a. \]

Which for t-Random Walk Betweenness Centrality gives \( RW B_t^1(G, 1^s) = 1 \) and for t-PageRank gives \( PR^a_t(G, 1^s) < 1. \)

**Axioms ⇒ t-PageRank, t-Random Walk Betweenness**

We will now prove that Locality, Node Redirect and Target Proxy from the common axioms, Edge Swap and Edge Multiplication from the axioms for random-walk based centralities and either Baseline 1-1 or Baseline 1-a uniquely define centrality up to the decay factor \( a^* = F_t^1((\{s, t\}, \{(s, t)\}), 1^s) \in [0, 1]. \) Combined with the fact that t-Random Walk Betweenness and t-PageRank satisfy the axiom set including respectively Baseline 1-1 and Baseline 1-a, we get that the axiom set determines the centrality to be respectively t-Random Walk Betweenness and t-PageRank.

**Lemma 16.** (1-Arrow Graph) If \( F \) satisfies Locality, Node Redirect, Target Proxy and either Baseline 1-1 or Baseline 1-a then there exists such a constant \( a^* \in [0, 1], \) that for every graph \( G = \{(s, t), \{(s, t)\}\} \) and weights \( b = x \cdot 1^s \) for \( x \in \mathbb{R}_{\geq 0}:\)

\[ F^b_t(G, x \cdot 1^s) = x \]

and \( F^b_t(G, x \cdot 1^s) = a^* \cdot x. \)

**Proof.** Let us assume \( x = 1. \) If we have Baseline 1-a, for \( t \) we only know that \( F^b_t((\{s, t\}, \{(s, t)\}), 1^s) < 1. \) We would also like to know that this value does not depend on the names of \( s \) or \( t. \) The first part we know from Lemma (Anonymity), that is for any \( s: \)

\[ F^b_t((\{s', t\}, \{(s', t)\}), 1^s) = F^b_t((\{s, t\}, \{(s, t)\}), 1^s). \]

(38)

As for the second part, we add a \( t \) self-loop and from Lemma (Target Self-Loop) we have \( F^b_t((\{s, t\}, \{(s, t)\}), 1^s) = F^b_t((\{s, t\}, \{(s, t), (t, t)\}), 1^s) \). Now we add a new node \( r \) and from Target Proxy we know that \( F^b_t((\{s, t\}, \{(s, t), (t, r)\}), 1^s) = F^b_r(((s, t, r), \{(s, t), (t, r)\}), 1^r) \). Because \( t \) is no longer the target, we rename it to arbitrary \( t' \) and from Lemma (Anonymity) we know that \( F^b_t((\{s, t, r\}, \{(s, t, r)\}), 1^s) = F^b_t((\{s, t', r\}, \{(s, t'), (t', r)\}), 1^s). \) Now we revert our changes, that is delete \( r \) and from Target Proxy we have \( F^b_t((\{s, t', r\}, \{(s, t'), (t', r)\}), 1^s') = F^b_t((\{s, t', (t', t')\}, 1^s')) \). Finally we delete \( t' \) self-loop and from Lemma (Target Self-Loop) we have \( F^b_t((\{s, t', \}, \{(s, t'), (t', t')\}), 1^s) = F^b_t((\{s, t', \}, \{(s, t'), \}) \). We obtained that:

\[ F^b_t((\{s, t', \}, \{(s, t')\}), 1^s) = F^b_t((\{s, t\}, \{(s, t)\}), 1^s). \]

(39)

From equations(38) and (39) we know that there is a constant \( a < 1 \) such that for every \( s, t \) it holds:

\[ a = F^b_t((\{s, t\}, \{(s, t)\})) \]

(40)

Now let us assume the weight of the source is arbitrary and \( F \) satisfies either Baseline 1-1 or Baseline 1-a. Let \( G_1 = ((s, s', t), \{(s, t), (s', t)\}) \). Let us notice that for every \( y, z \in \mathbb{R}_{\geq 0} \) from Node Redirect we have \( F^b_t(G, (y + z) \cdot 1^s) = F^b_t(G_1, y \cdot 1^s + z \cdot 1^s) + F^b_t(G_1, (y + z) \cdot 1^s) = F^b_t(G_1, (y + z) \cdot 1^s + z \cdot 1^s) \). Now from Locality \( F^b_t(G, y \cdot 1^s + z \cdot 1^s) = F^b_t(G, (y + z) \cdot 1^s) \).

The first part we know from Lemma (Anonymity), that is for every \( s': \)

\[ F^b_t((\{s', t\}, \{(s', t)\}), 1^s) = F^b_t((\{s', t\}, \{(s', t)\}), 1^s'). \]

To summarize, we now know for \( v \in \{s, t\} \) that:

\[ F^b_v(G, (y + z) \cdot 1^s) = F^b_v(G, y \cdot 1^s) + F^b_v(G, z \cdot 1^s), \]

(41)

that is the function \( F^b_v(G, x \cdot 1^s) \) is additive over variable \( x. \) Equation(41) combined with our definition of centralities as non-negative gives that the function is linear, that is:

\[ F^b_v(G, x \cdot 1^s) = x \cdot F^b_v(G, 1^s). \]

(42)

If \( F \) satisfies Baseline 1-1, from this axiom and equation(42) for both nodes we know that \( F^b_v(G, x \cdot 1^s) = x \) and \( F^b_v(G, x \cdot 1^s) = x, \) which gives the thesis for \( a^* = 1. \)

If \( F \) satisfies Baseline 1-a, from this axiom and equation(42) for \( s \) we know that \( F^b_v(G, x \cdot 1^s) = x. \) Moreover, from equations(40) and (41) we know that \( F^b_v(G, x \cdot 1^s) = a \cdot x. \)
Lemma 17. (Almost 1-Arrow Graph) If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Multiplication and Baseline 1-1 or Baseline 1-a then there exists such a constant $a^* \in [0, 1]$ that for every graph $G = (\{s, t\}, E)$, such that $E - \Gamma_1^+(G) = \{(s, t)\}$ and weights $b = x \cdot \mathbb{I}^x$ for $x \in \mathbb{R}_{\geq 0}$:

$$F^i_s(G, x \cdot \mathbb{I}^x) = x$$

and $F^i_s(G, x \cdot \mathbb{I}^x) = a^* \cdot x$.

**Proof.** Firstly, let us assume there are no $t$ self-loops, as they could be deleted and from Lemma 4 (Target Self-Loop) we know the centralities would not change. Let $k = m(t_i)(E)$, that is $G = \{(s, t), \{(s, t), k \cdot (t, s)\}\}$.

Let us consider $x = 0$. Firstly we double edges from $t$ to $s$, from Edge Multiplication we know $F^i_s(\{(s, t), k \cdot (t, s)\}, 0) = F^i_s(\{(s, t), \{(s, t), 2k \cdot (t, s)\}\}, 0)$. We now split $s$ into out-twins $s$ with half of the incoming edges and $s'$ with the other half, separate them and rename $s'$. From Node Redirect, Locality and Lemma 3 (Anonymity) we have that $F^i_s(\{(s, t), \{(s, t), k \cdot (t, s)\}\}, 0) = F^i_s(\{(s, t), \{(s, t), k \cdot (t, s)\}\}, 0) + F^i_s(\{(s, t), \{(s, t), k \cdot (t, s)\}\}, 0)$. This means:

$$F^i_s(G, 0) = 0.$$  \hfill (43)

In the general case we again split $s$ into out-twins $s$ with the original weight and no incoming edges and $s'$ with zero weight and all the incoming edges, separate them and rename $s'$ to $s$. From Node Redirect, Locality and Lemma 3 (Anonymity) we have that:

$$F^i_s(G, x \cdot \mathbb{I}^x) = F^i_s(G, 0) + F^i_s(\{(s, t), \{(s, t)\}\}, x \cdot \mathbb{I}^x).$$  \hfill (44)

To get the thesis, in equation (44) we substitute the first term with equation (43) and the second with Lemma 16. \hfill \Box

Lemma 18. (k-Arrow Adjoin) If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every graph $G = (V, E) \in G_t$, new node $w \notin V$ with weight $x$, numbers $k, l \in \mathbb{Z}_{\geq 0}$ and node $v \in V - \{t\}$, let $G^r = (V \cup \{w\}, E + k \cdot \{(w, t)\} + l \cdot \{(t, w)\})$ and $b' = b + x \cdot \mathbb{I}^w$:

$$F^i_v(G^r, b') = F^i_v(G, b).$$

**Proof.** From Lemma 17 (Almost 1-Arrow Graph) for $G_1 = (\{(w, t), (w, t), l \cdot (t, w)\})$, $b_1 = x \cdot \mathbb{I}^w$ we know that $F^i_w(G_1, b_1) = x$ and $F^i_t(G_1, b_1) = a^* \cdot x$. Now we multiply the edge from $w$ to $t$ by $k$ and from Edge Multiplication for $G_2 = (\{(w, t), k \cdot (w, t), (t, t)\})$ we know that $F^i_w(G_2, b_1) = x$ and $F^i_t(G_2, b_1) = a^* \cdot x$. Finally we join $G$ and $G_2$ and from Locality we obtain the thesis. \hfill \Box

Lemma 19. (Siphon) If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every graph $G \in G_t$, and node $v$, such that $\Gamma_v^-(G) = \emptyset$:

$$F^i_v(G, b) = b(v).$$

**Proof.** Firstly let us assume $v = t$, that is $G = (\{t\}, \{\})$, $b = x \cdot \mathbb{I}^t$. From Lemma 16 (1-Arrow Graph) we know that $F^i_t((\{t, r\}, \{\{t, r\}\}, \mathbb{I}^t) = x$. We merge $r$ into $t$ and from Target Proxy we know that $F^i_t((\{t, r\}, \{\{t, r\}\}, x \cdot \mathbb{I}^t) = F^i_t((\{t\}, \{\})), \mathbb{I}^t)$. But from Lemma 4 (Target Self-Loop) we already know that $F^i_t((\{t\}, \{\})), x \cdot \mathbb{I}^t) = F^i_t(\{(t, \{\})\}, x \cdot \mathbb{I}^t)$. Which results in:

$$F^i_t((\{t\}, \{\})) = x.$$  \hfill (45)

Let us now assume $v \neq t$. Let $G = (V, E)$. We add a new node $w$ with weight $F^i_v(G, b)$ and out-degree equal $|\Gamma_v^+(G)|$, obtaining graph $G_1 = (V \cup \{w\}, E + \{\Gamma_v^+(G) \cdot (w, t)\})$, $b_1 = b + F^i_v(G, b) \cdot \mathbb{I}^w$ and we know from Lemma 18 (k-Arrow Adjoin) that:

$$F^i_v(G, b) = F^i_v(G_1, b_1)$$

and $F^i_v(G_1, b_1) = b_1(w) = F^i_v(G, b)$, that is $v$ and $w$ have equal centralities in $(G_1, b_1)$. So we swap all edges outgoing from $v$ and from $w$ and for $G_2 = (V \cup \{w\}, E - \Gamma_v^+(G) + \{(w, u) : (v, u) \in E\} + \{\Gamma_v^+(G) \cdot (v, t)\})$ from Edge Swap we know that:

$$F^i_v(G_1, b_1) = F^i_v(G_2, b_1).$$

Finally we detach $v$ from the rest of the graph obtaining from Lemma 18 (k-Arrow Adjoin) and equations 45 and 46 that $F^i_v(G_2, b_1) = b_1(v) = b(v)$. Together with equation 45 it gives the thesis. \hfill \Box

Lemma 20. (Siphon Split) If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every graph $G = (V, E) \in G_t$, and node $s \in V - \{t\}$, such that $\Gamma_s^-(G) = \{\}$ and $\Gamma_s^+(G) = \{(s, v_1), \ldots, (s, v_k)\}$, let:

...
\[
\begin{align*}
\cdot V' &= V - \{s\} \cup \{s_1, \ldots, s_k\}, \\
\cdot E' &= E - \{(s, v_1), \ldots, (s, v_k)\} + \{(s_1, v), \ldots, (s_k, v)\}, \\
\cdot b' &= b - \left( b(s) \cdot 1^s + \frac{b(s)}{k} \cdot 1^s \right) + \ldots + \left( b(s) \cdot 1^s \right)
\end{align*}
\]

then for every node \( u \in V - \{s\} \):
\[
F'_u((V', E'), b') = F'_u(G, b).
\]

**Proof.** Firstly we split \( s \) into out-twins \( s_1, s_2, \ldots, s_k \), each with weight \( b(s)/k \), obtaining graph \( G_1 = (V', E - \{(s, v_1), \ldots, (s, v_k)\} + \{(s_1, v), \ldots, (s_k, v)\}) \) and from Node Redirect we know that \( F'_u(G_1, b) = F'_u(G_1, b'). \) Moreover, from Lemma 19 (Siphon) we know for each \( i \in 1, \ldots, k \) that \( F'_u(G_1, b') = \frac{b'_{s_i}}{b'_{s_i}}. \) So for each \( i, j \in 1, \ldots, k, i \neq j \) we swap edges \( (s_i, v_j) \) and \( (s_j, v_i) \), obtaining graph \( G_2 = (V', E - \{(s, v_1), \ldots, (s, v_k)\} + \{(s_1, v), \ldots, (s_k, v)\}) \) and from Edge Swap we know that \( F'_u(G_1, b') = F'_u(G_2, b') \). Finally we divide edges outgoing of every \( s_i \) by \( k \) and from Edge Multiplication we get the thesis. \( \square \)

**Lemma 21.** (Long-Arrow Graph) If \( F \) satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every \( x \in \mathbb{R}_{\geq 0} \), graph \( G = \{(s, v, t), \{(s, v), (v, t)\}\} \) and weights \( b = x \cdot 1^s \) centralities of all nodes are determined.

**Proof.** We merge \( t \) into \( v \) and from Target Proxy we know for \( G_1 = \{(s, v), \{(s, v), (v, t)\}\} \) that:
\[
\begin{align*}
F'_u(G_1, x \cdot 1^s) &= F'_u(G_1, x \cdot 1^s), \\
F'_v(G_1, x \cdot 1^s) &= F'_v(G_1, x \cdot 1^s).
\end{align*}
\]

(48)

From Lemma 17 (Almost 1-Arrow Graph) combined with equation 48 we know that \( F'_u(G, x \cdot 1^s) = x \), which is determined. Combined with equation 19 we know that \( F'_v(G, x \cdot 1^s) = a^* \cdot x \), which also is determined.

We add a \( t \) self-loop and a new target node \( r \) to \( G \) and from Lemma 4 (Target Self-Loop) and Target Proxy for \( G_2 = \{(s, v, t, r), \{(s, v), (v, t), (t, r)\}\} \) we know that \( F'_u(G_2, x \cdot 1^s) = F'_u(G_2, x \cdot 1^s) \) and \( F'_v(G_2, x \cdot 1^s) = F'_v(G_2, x \cdot 1^s) \). Let \( y = F'_u(G_2, x \cdot 1^s) = F'_v(G_2, x \cdot 1^s) = a^* \cdot x \). Now we add a new node \( v' \) with an edge to \( r \) and with weight \( y \) and from Lemma 18 (k-Arrow Adjoin) we know for \( G_3 = \{(s, v, v', t, r), \{(s, v), (v, t), (v', t), (t, r)\}\} \) that \( F'_u(G_3, x \cdot 1^s) = F'_u(G_3, b_3) \) and that nodes \( v \) and \( v' \) have equal centralities \( y \) in \( G_3, b_3 \). We swap edges \( (v, t), (v', t) \) and from Edge Swap for \( G_4 = \{(s, v, v', t, r), \{(s, v), (v, t), (v', t), (t, r)\}\} \) we know that \( F'_u(G_4, b_3) = F'_u(G_4, b_3) \). Now we separate graph \( G_5 = \{(v, t', t), \{(v, t), (t, r)\}\} \) and from Locality we know that \( F'_u(G_5, b_3) = F'_u(G_5, y \cdot 1^{v'}) \). We obtained that:
\[
F'_u(G, x \cdot 1^s) = F'_u(G_5, y \cdot 1^{v'})
\]

(50)

Right side of the equation 50 is equal \( a^* \cdot y = (a^*)^2 \cdot x \), because it is the centrality of the middle node in Long-Arrow Graph, so \( F'_u(G, x \cdot 1^s) \) is determined. \( \square \)

**Lemma 22.** (DAG) If \( F \) satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every acyclic \( G \in G \), weights \( b \) and node \( v \) centrality \( F'_v(G, b) \) is determined.

**Proof.** Let \( G = (V, E) \). We will do an induction on \( |E| \).

\[
\begin{align*}
\cdot & \text{ If } |E| = 0, \text{ then the thesis follows from Lemma 19 (Siphon).} \\
\cdot & \text{ If } |E| = 1, \text{ then } G = (\{(s, t), \{(s, t)\}\}), \text{ for from the Locality we know that } F'_u(G, b) = F'_u(G, x \cdot 1^s) \text{ and } F'_v(G, b) = F'_v(G, x \cdot 1^s) + F'_v(\{(t), \{\}\}), y \cdot 1^t \text{. Centralities in } (G, x \cdot 1^s) \text{ are determined from Lemma 16 (1-Arrow Graph) and centrality of } t \text{ in } ((\{(t), \{\}\}), y \cdot 1^t) \text{ is determined from Lemma 19 (Siphon).} \\
\cdot & \text{ If } |E| = 2 \text{ and one edge does not end in } t, \text{ then } G = (\{(s, v, t), \{(s, v), (v, t)\}\}), b = x \cdot 1^s + y \cdot 1^v + z \cdot 1^t. \text{ We split } v \text{ into } v \text{ with incoming edge and zero weight and } v' \text{ with no incoming edges and weight } y. \text{ We also separate graphs } G_1 = (\{(s, v, t), \{(s, v), (v, t)\}\}), b_1 = x \cdot 1^s \text{ and } G_2 = (\{(v', t), \{(v', t)\}\}), b_2 = y \cdot 1^v + z \cdot 1^t. \text{ From Node Redirect and Locality we know that } F'_u(G, b) = F'_u(G_1, b_1), F'_v(G, b) = F'_v(G_1, b_1) + F'_v(G_2, b_2) \text{ and } F'_u(G, b) = F'_u(G_1, b_1) + F'_u(G_2, b_2). \text{ Centrality values in } (G_1, b_1) \text{ are known from Lemma 21 (Long-Arrow) and in } (G_2, b_2) \text{ are known from the previous case.}
\end{align*}
\]
• If $|E| \geq 2$ and all edges end in $t$, then $G = \{(s_1, \ldots, s_{n-1}, t), (k_1 \cdot (s_1, t), \ldots, k_{n-1} \cdot (s_{n-1}, t))\}$, for $n = |V|$. We make each of the edges unique and from Edge Multiplication we know for $G' = \{(s_1, \ldots, s_{n-1}, t), (s_1, t), (s_{n-1}, t)\}$ and every $i \in 1, \ldots, n - 1$ that $F_i^f(G, b) = F_i^f(G', b)$ and $F_i^f(G, b) = F_i^f(G', b)$. We separate each of the nodes $s_1, \ldots, s_{n-1}$ into an independent graph and from Locality for $G_i = \{(s_i, t), (s_i, t)\}, b_i = b(s \cdot \mathbb{I}^s + \frac{b(t)}{n-1} \cdot \mathbb{I}^t$ we have $F_i^f(G', b) = F_i^f(G, b_i)$ and $F_i^f(G, b) = \sum_{i \in 1, \ldots, n-1} F_i^f(G, b_i)$. Centralities in $(G, b_i)$ are determined from the second case.

Let us discuss the inductive step. Let $s_1, s_2, \ldots, s_k$ be all the nodes without incoming edges. For every $i \in 1, \ldots, k$ from Lemma[19] (Siphon) we know that:

$$F_i^f(G, b) = b(s_i).$$

(51)

Each $s_i$ with outgoing edges $(s_i, v_{i,1}), \ldots, (s_i, v_{i,t})$ we split it into a series of $l_i$ nodes $V_i = \{(s_{i,1}, \ldots, s_{i,l_i})\}$ with only one outgoing edge each, $E_i = \{(s_{i,1}, v_{i,1}), \ldots, (s_{i,l_i}, v_{i,1})\}$, and equal weights, $b_i = \frac{b(s_i)}{l_i} \cdot \mathbb{I}^{s_{i,1}} + \ldots + \frac{b(s_i)}{l_i} \cdot \mathbb{I}^{s_{i,l_i}}$. From Lemma[20] (Siphon Split) for graph $G_1 = (V_1, E_1) = (V - \{s_1, \ldots, s_k\} \cup V_1 \cup \ldots \cup V_k, E - \Gamma_{s_1} - \ldots - \Gamma_{s_k} + E_1 + \ldots + E_k)$, weights $b_1 = b - b(s_i \cdot \mathbb{I}^{s_i} - \ldots - b(s_k \cdot \mathbb{I}^{s_k} + b(s_i) + \ldots + b(s_k)) \cdot \mathbb{I}^t$ and for every node $u \in V - \{s_1, \ldots, s_k\}$ we have:

$$F_i^u(G_1, b) = F_i^u(G_1, b_1),$$

(52)

$$|E_1| = |E_1|. $$

(53)

In the obtained graph $G_1$ every node without incoming edges has exactly one outgoing edge.

Let $v \neq t$ be the topologically greatest node that has some incoming edges $\{(s_1, v), \ldots, (s_k, v)\}$. Because it is topologically greatest, all its predecessors have no incoming edges. So each of them has only one outgoing edge, to $v$. Now merge them into one node, $s$, and from Node Redirect for $G_2 = (V_2, E_2) = (V - \{s_1, \ldots, s_k\} \cup \{s\}, E_1 - \{(s_1, v), \ldots, (s_k, v)\} + \{(s, v)\})$, $b_2 = b - b(s_1 \cdot \mathbb{I}^{s_1} - \ldots - b(s_k \cdot \mathbb{I}^{s_k} + b(s) + \ldots + b(s_k)) \cdot \mathbb{I}^t$ and for every node $u \in V - \{s_1, \ldots, s_k\}$ we know that:

$$F_i^u(G_2, b_2) = F_i^u(G_2, b_2),$$

(54)

$$|E_1| = |E_2|. $$

(55)

Now let us add new node $v'$, obtaining graph $G_3 = (V_3, E_3) = (V_2 \cup \{v'\}, E_2 + \{\Gamma_{3}^+(G_2) \cdot (v', t)\})$, $b_3 = b_2 + F_i^f(G_2, b_2) \cdot v'$ (for the yet unknown $F_i^f(G_2, b_2)$). From Lemma[18] (K-Arrow Adjoin) for every node $u \in V_2 - \{t\}$ we have $F_i^u(G_2, b_2) = F_i^u(G_3, b_3)$, and $F_i^f(G_2, b_2) = F_i^f(G_3, b_3) \cdot a^* \cdot F_i^f(G_2, b_2)$ is determined if $F_i^f(G_2, b_2)$ is determined. And $F_i^f(v, G_3, b_3) = F_i^f(G_3, b_3)$, that is nodes $v$ and $v'$ have equal centralities in $(G_3, b_3)$. As for the number of edges, $|E_2| = |E_3| = |E_3|$. We exchange all the edges outgoing from $v$ with all the edges outgoing from $v'$, obtaining graph $G_4 = (V_4, E_4) = (V_3, E_3 - \Gamma_3^+(G_3) - \Gamma_3^-(G_3) + \{v', w \cdot (v, w) \in E_3\} + \{\Gamma_3^+(G_3) \cdot (v, t)\})$. From Edge Swap we know that for every node $u \in V_3$ we have $F_i^u(G_3, b_3) = F_i^u(G_4, b_3)$. And $|E_4| = |E_4|$. Multiple edges from $v$ to $t$ are unified, obtaining graph $G_5 = (V_5, E_5) = (V_3, E_2 - \{\{v, t\} \cdot (v, t)\})$. From Edge Multiplication for every node $u \in V_5$ we know that $F_i^u(G_4, b_3) = F_i^u(G_5, b_3)$. The number of edges decreased by $\Gamma_{v}^+(G_3) - 1$, that is $|E_4| = |\Gamma_{v}^+(G_3) - 1| = |E_5|$. In total, we obtained for every node $u \in V_2 - \{t\}$ that:

$$F_i^u(G_2, b_2) = F_i^u(G_5, b_3),$$

(56)

$$F_i^f(G_2, b_2) = F_i^f(G_5, b_3) - a^* \cdot F_i^f(G_5, b_3),$$

(57)

$$|E_2| + 1 = |E_3|. $$

(58)

Finally, we separate graphs $G_6 = \{(s, v), (s, v'), t\}$, $b_6 = b_4(s) \cdot \mathbb{I}^s + b_3(v) \cdot \mathbb{I}^v$ and $G_7 = (V_5 - \{s, v\}, E_5 - \{(s, v), (v, t)\}, b_7 = b_3 - b_3(s) \cdot \mathbb{I}^s - b_3(v) \cdot \mathbb{I}^v$. From Locality for every node $u \in V_5 - \{s, v, t\}$ we know that:

$$F_i^u(G_5, b_3) = F_i^u(G_7, b_7),$$

(59)

$$F_i^f(G_5, b_3) = F_i^f(G_6, b_6),$$

(60)

$$F_i^f(G_5, b_3) = F_i^f(G_6, b_6) + F_i^f(G_7, b_7).$$

(61)

$$|E_5| - 2 = |E_7|. $$

(62)

The centralities in $(G_6, b_6)$ are known from the case $E = 2$. This combined with equations[56] for $v$ and $60$ causes the weight $b_7(v') = b_7(v') = F_i^f(G_2, b_2)$ to stop being unknown. Now centralities in $(G_7, b_7)$ are also determined, from the inductive assumption, because from equations[53] 55 58 and 62 we have $|E_7| = |E_2| - 1 \leq |E| - 1$.

This combined with equations[52] 54 56 and[59] gives for $u \in V - \{v, t, s_1, \ldots, s_k\}$ that $F_i^u(G, b)$ is determined. Combined with equations[52] 54 56 and 60 gives that $F_i^f(G, b)$ is determined. And combined with equations[52] 54 57 60 and 61 gives that $F_i^f(G, b)$ is determined. Together with equation[51] we obtained that centrality of every node in acyclic $(G, b)$ is determined. □
Lemma 23. (No Target Outlet) If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a, then for every graph $G = (V, E) \in \mathcal{G}$ such that $(t, v) \in E$ and node $u$:
\[
F_u^t((V, E - \{(t, v)\}), b) = F_u^v(G, b).
\]

Proof. If $v = t$, then the thesis follows from Lemma 24 (Target Self-Loop). Let us again assume $v \neq t$.

Let us split $v$ into out-twins $v'$ with zero weight and an incoming edge from $t$ and $v$ with the original weight and the rest of incoming edges, obtaining $G_1 = (V_1, E_1) = (V \cup \{v\}, E - \{(t, v)\} + \{(v, v')\} + \{(v', w) : (v, w) \in E\})$. From Node Redirect for every $u \in V - \{v\}$ we know that:
\[
F_u^v(G, b) = F_u^v(G_1, b),
\]
\[
F_u^t(G, b) = F_u^t(G_1, b) + F_u^v(G_1, b).
\]

Let $x = F_u^v(G_1, b)$.

We add node $v''$ that has the same out-degree $k = \Gamma^+_v(G_1)$ as $v'$ and weight $x$, obtaining graph $G_2 = (V_2, E_2) = (V_1 \cup \{v''\}, E + k \cdot \{(v'', t)\})$, $b_2 = b + x \cdot 1_{v''}$. From Lemma 18 (k-Arrow Adjoin) for every $u \in V_1 - \{t\}$ we know that $F_u^v(G_1, b) = F_u^v(G_2, b_2)$, $F_u^t(G_1, b) = F_u^t(G_2, b_2) - a^* \cdot x$. We also know that $F_u^v(G_2, b_2) = x = F_u^v(G_2, b_2)$, that is nodes $v'$ and $v''$ have the same centralities in $G_2$. Now we swap edges outgoing of $v'$ and of $v''$, obtaining graph $G_3 = (V_2, E_3) = (V_2, E_2 - \Gamma^+_v(G_2) + \{(v'', w) : (v', w) \in E_2\} - \Gamma^+_v(G_2) + k \cdot \{(v', t)\})$. From Edge Swap for every $u \in V_2$ we know that $F_u^v(G_2, b_2) = F_u^v(G_3, b_2)$. In total, for every $u \in V_1 - \{t\}$ we know that:
\[
F_u^v(G_1, b) = F_u^v(G_3, b_2),
\]
\[
F_u^t(G_1, b) = F_u^t(G_3, b_2) - a^* \cdot x,
\]
\[
F_u^{v''}(G_3, b_2) = F_u^{v''}(G_3, b_2).
\]

Let us now delete node $v'$, obtaining graph $G_4 = (V_4, E_4) = (V_2 - \{v''\}, E_3 - \{(t, v')\} - k \cdot \{(v', t)\})$. From Lemma 18 (k-Arrow Adjoin) for $u \in V_4 - \{t\}$ we know that:
\[
F_u^v(G_3, b_2) = F_u^v(G_4, b_2),
\]
\[
F_u^t(G_3, b_2) = F_u^t(G_4, b_2) + a^* \cdot b_2(v'') = F_u^t(G_4, b_2),
\]
\[
F_u^{v''}(G_3, b_2) = F_u^{v''}(G_4, b_2).
\]

But now from equations (65) for $v''$ and (68) for $u$ we know that $x = F_u^{v''}(G_3, b_2) = 0$.

Let us use this fact in previous equations. From equations (64), (65) for $v$ and (68) for $u$ we have $F_u^v(G, b) = F_u^v(G_4, b_2)$. As for $t$, from equations (63), (66) and (69) we obtained that $F_t^v(G, b) = F_t^v(G_4, b_2)$. Moreover, from equations (65), (65) and (68) for $u \in V - \{v, t\}$ we know that $F_u^v(G, b) = F_u^v(G_4, b_2)$. So for every node $u \in V$ we obtained that:
\[
F_u^v(G, b) = F_u^v(G_4, b_2).
\]

Finally we merge $v''$ into $v$, obtaining $G_5 = (V, E - \{(t, v)\})$. From equations (67) and (68) for $v''$ we know that $F_u^{v'}(G_4, b_2) = 0$. Combining this with Node Redirect and equation (71) we get the thesis.  

Lemma 24. If $F$ satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 or Baseline 1-a then it is determined.

Proof. Fix graph $G = (V, E)$ and $b$. We will prove the thesis by induction on the number of cycles in $G$. If there are no cycles, then the thesis follows from Lemma 22 (DAG). Assume otherwise. Let us also assume there are no edges from $t$, as they could be deleted and from Lemma 23 (No Target Outlet) centralities would not change.

Fix a node $v$, that belongs to at least one cycle (note that $v \neq t$, because $t$ has no outgoing edges) and let $k = |\Gamma^+_v(G)|$. Let $a^* = F_t^v((\{s, t\}, \{(s, t)\}), 1^*_v) \leq 1$ and $x = PR_{v^*}^a(G, b)$.

Let us now consider another graph $(G', b')$, where $G' = (V', E')$ and:
\[
V' = V \cup \{v'\},
\]
\[
E' = E - \Gamma^+_v(G) + \{(v', u) : (v, u) \in E\} + k \cdot \{(v, t)\},
\]
\[
b' = b + x \cdot 1_{v'}.
\]
Let us note, that $G' \in \mathcal{G}_t$, that is $t$ is reachable from every node. If in $G$ from a node $u \in V$ there was a path to $t$ without node $v$, then all of its edges are still present in $G'$. If every path from $u$ to $t$ passed through $v$, then the path up to this point is present in $G'$, and then $v$ has direct edges to $t$. Finally, $v'$ has edges to some other nodes, from which we already know $t$ is reachable.

All the cycles that are present in $G'$ are present also in $G$. Note that in $G'$ there are no cycles involving $v'$, because it has no incoming edges. There are also no cycles involving $v$, because it has only outgoing edges to $t$, and $t$ has no outgoing edges itself. The cycles on nodes $V' - \{v, v'\}$ have all their edges present also in $G'$.

In turn, cycles through $v$ in $G$ are not present in $G_2$. This means the number of cycles in $G'$ is strictly smaller than in $G$.

From the inductive assumption we know that in $G'$ centrality is determined for every $a^*$, so for every $u \in V'$ we have:

$$F^t_u(G', b') = PR^{a^*, t}_u(G', b').$$

(72)

Now we notice that $G'$ is obtained from $G$ by firstly adding a new node $v'$ with out-degree $k$ and weight $x$, obtaining intermediary $G_1 = (V_1, E_1) = (V \cup \{v'\}, E + k \cdot \{(v', t)\})$ with weights $b'$ and then by swapping edges outgoing from $v$ and $v'$, obtaining $(G', b')$. Because $PR^{a^*, t}$ satisfies our axioms, from Lemma 25 (k-Arrow Adjoin) we know that $PR^{a^*, t}_u(G, b) = PR^{a^*, t}_u(G, b_1)$ and $PR^{a^*, t}_u(G_1, b_1) = x$, that is $v$ and $v'$ have equal $PR^{a^*, t}$ in $(G_1, b_1)$. From Edge Swap for every $u \in V_1$ we know that $PR^{a^*, t}_u(G_1, b_1) = PR^{a^*, t}_u(G', b')$. In total:

$$PR^{a^*, t}_u(G', b') = PR^{a^*, t}_u(G, b) = x,$$

(73)

$$PR^{a^*, t}_u(G', b') = x,$$

(74)

From equations 72 for $v$ and 75 we obtained that $F^t_v(G', b') = x$. But from equations 72 for $v'$ and 74 we obtained that $F^t_{v'}(G', b') = x$, that is $v$ and $v'$ have equal $F^t$ centralities in $(G', b')$.

Now we revert our changes, that is swap edges outgoing from $v$ and $v'$ and delete $v'$. From Edge Swap for every $u \in V_1$ we know that $F^t_u(G', b') = F^t_u(G_1, b_1)$. From Lemma 25 (k-Arrow Adjoin) for every $u \in V - \{t\}$ we know that $F^t_u(G_1, b_1) = F^t_u(G, b)$ and $F^t_{v'}(G_1, b_1) = a^* \cdot x = F^t_{v'}(G, b)$. In total, for every $u \in V - \{t\}$:

$$F^t_u(G, b) = F^t_u(G', b'),$$

(75)

$$F^t_v(G, b) = F^t_v(G', b') - a^* \cdot x = F^t_v(G, b') - a^* \cdot F^t_{v'}(G', b').$$

(76)

Equations 75 and 76 together with inductive assumption determine values of $F^t$ on $(G, b)$ up to $a^* = F^t_t((s, t), \{(s, t)\}, 1^s)$.

Lemma 25. If t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-a then it is equal t-PageRank for some decay factor $a \in [0, 1)$.

Proof. From Lemma 24 we know for every value $a^* = F^t_t((s, t), \{(s, t)\}, 1^s) \in [0, 1)$ that t-centrality $F$ is determined. Baseline 1-a further specifies that $a^* \in [0, 1)$. But from Lemma 14 we know that for every given value $a \in [0, 1)$ t-PageRank with decay factor $a$ satisfies these axioms and $PR^{a^*, t}_u((s, t), \{(s, t)\}, 1^a) = a$, so this is our determined centrality.

Lemma 26. If t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Baseline 1-1 then it is equal t-Random Walk Betweenness.

Proof. From Lemma 24 we know for every value $a^* = F^t_t((s, t), \{(s, t)\}, 1^s) \in [0, 1)$ that t-centrality $F$ is determined. Baseline 1-1 specifies that $a^* = 1$, so the axioms uniquely define t-centrality. But from Lemma 15 we know that t-Random Walk Betweenness satisfies these axioms, so this is our determined centrality.