In this paper, we use the Gauss Bonnet theorem to obtain the deflection angle by the photons coupled to Weyl tensor in a Schwarzschild black hole and Schwarzschild-like black hole in bumblebee gravity in the weak limit approximation. To do so, we first calculate the corresponding optical metrics, and then we find the Gaussian curvature to use it in Gauss-Bonnet theorem, which is first done by Gibbons and Werner. Hence, in the leading order terms we compute the deflection angle, which is affected by the coupling between the photon and Weyl tensor. It is shown that there is a difference in the deviations of light passing near to the Schwarzschild black hole and to the Schwarzschild-like black hole in the bumblebee gravity. Moreover, we compare the deflection angles formed by Einstein-Rosen type wormhole in Weyl gravity and in bumblebee gravity. Remarkably, the deflection angle by Einstein-Rosen type wormhole in bumblebee gravity is found as being to be larger than the deflection angle by Einstein-Rosen type wormhole in Weyl gravity.

PACS numbers: 95.30.Sf, 04.20.Dw, 04.70.Bw, 98.62.Sb
Keywords: Relativity and Gravitation; Gravitational lensing; Classical black holes; Deflection angle; Gauss-Bonnet theorem; Wormholes; Einstein-Rosen

I. INTRODUCTION

Over the past decade, generalized Einstein-Maxwell theories have been receiving great attention. Those theories consider higher derivative interactions and thus reveal more information about the features and effects of the electromagnetic (em) fields. In general, we can split the generalized Einstein-Maxwell theories into two classes: (i) minimally coupled gravity-em in which there exists no coupling between the Maxwell tensor and the curvature in the action. For example, Born-Infeld theory [2] is of this class, which eliminates the divergent self energy of the electron by modifying Maxwell’s theory and give good physical results such as the absence of shock waves and birefringence phenomena [3–5]. (ii) non-minimal coupling between the gravitational and Maxwell fields in the action [6, 7]. Such non-minimal couplings in the Lagrangian changes the coefficients of the second-order derivatives appeared in the Maxwell and Einstein equations. Therefore, the propagation of gravitational and em waves in the manifold has time delays [6]. In this way, the physics of the evolution of the early Universe is expected to be explained: quantum fluctuations of the em fields and inflation [8–19].

One of the generalized Einstein-Maxwell theories is the Weyl corrected electrodynamics that involves a coupling between the Weyl tensor and the Maxwell field [20, 21]. Namely, the Lagrangian density of the electrodynamics is modified with the Weyl tensor. In other words, this theory [20], is a special kind of em theory, which contains a coupling between the gravitational and em fields. In fact, QED (quantum electrodynamics) of the light effective action for one-loop vacuum polarization on a curved background [21] admits such couplings of the Weyl and Maxwell tensors. Besides, Weyl corrected electrodynamics could play a role on the supermassive black holes located at the center of galaxies [22, 23]. The effects of the Weyl corrections on black hole physics are explored in [20], which considers the holographic conductivity and diffusion in the presence of the Weyl corrections for the AdS spacetime. It was shown that Weyl corrections terminate the central charge seen at the leading order, tunes the critical temperature at which holographic superconductors occur, and modifies the order of the phase transition of the holographic superconductor [24–30]. Moreover, Weyl corrections have a significant influence on the stability of the Schwarzschild black hole

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Since our current technology is not enough to directly observe the black holes, one of the promising physical phenomena that can prove the validity of Einstein’s general relativity theory is the gravitational lensing. In the recent years, the most commonly used method to investigate the weak gravitational lensing is the Gauss-Bonnet theorem [41–43] or the so-called the Gibbons-Werner method (GWM) [44, 45, 69]. This method dramatically changed the usual calculation method for getting the deflection angle. Gibbons and Werner proved that by applying the Gauss-Bonnet theorem to the corresponding optical metric of the spacetime considered, one can straightforwardly compute the deflection angle. The latter remark highlights that the bending of light ray has a global effect which is contrary to the popular opinion. Because, the bending of light is usually computed for a compact region having a radius at the order of the impact parameter. In fact GWM focuses on a non-singular domain, which is outside of the light ray.

In this method, one use the optical geometry, and then calculate the Gaussian optical curvature $K$ to find the asymptotic bending angle which can be calculated as follows [45, 46]:

$$\hat{\alpha} = -\int \int_{D_{\infty}} K dS,$$

(1.1)

which gives exact results for bending angle.

In addition to this original GWM computes the deflection angle for the asymptotically flat spacetimes. For non-asymptotically flat (NAF) spacetimes, GWM is valid only when a finite distance corrections are considered [45, 47, 48]. Soon after, Werner [46] extended the GWM to cover rotating (Kerr) black holes in which Finsler-Randers metric is used. Then, GWM was studied for the gravitational lensing problems in the rotating/non-rotating geometries of wormholes [49, 50]. Today, GWM has been using by numerous studies (see for example [40, 47–69]).

In this paper, our main motivation is to explore the effects of the extended gravitational theories on the gravitational lensing. Nowadays, gravitational lensing for photons coupled to the Weyl tensor has gained much attention because of the observation on the supermassive black hole at the Galactic center, Sgr A*, by the Event Horizon Telescope [74]. For this purpose, we consider the Schwarzschild black hole with Weyl [20] and bumblebee [70] corrections. We shall apply the GWM to the both black holes and analyze the effects of the Weyl and bumblebee corrections on the gravitational lensing.

The paper is organized as follows: in the following section we introduce the Weyl corrected Schwarzschild black hole and derive the deflection angle in the context of the GWM. In Sec.III, the modified Schwarzschild black hole in the bumblebee gravity is summarized and study the change of the deflection angle due to the bumblebee corrections. Moreover, Einstein-Rosen type wormhole solutions and their gravitational lensings are thoroughly studied in Secs. IV and V, respectively. Finally, Sec. VI concludes the paper.

II. WEYL CORRECTION OF A SCHWARZSCHILD BLACK HOLE AND WEAK GRAVITATIONAL LENSING

In this section, to study the effect of the WT correction on the deflection angle we use the action of the Einstein-EMF coupled to WT in the 4-dimensional static and spherical symmetric spacetime as follows [20]:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{4} \left( F_{\mu\nu}F^{\mu\nu} - 4\alpha C_{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \right) \right],$$

(2.1)

where $C_{\mu\nu\rho\sigma}$ stands for the WT as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} \left( g_{\mu\rho} R_{\nu\sigma} - g_{\nu\sigma} R_{\mu\rho} \right) + \frac{R g_{\mu\rho} g_{\nu\sigma}}{6},$$

(2.2)

where the brackets around indices show the antisymmetric part. Moreover, the electromagnetic tensor $F_{\mu\nu}$ is equal to $F_{\mu\nu} = A^\eta_{\mu\nu} - A_{\mu\nu}$. Note that $\alpha$ is the coupling constant with a dimension of length-squared.

The equation of motion for photon coupled to WT is found as

$$k_\mu k_\nu \phi^{\nu} + 8\alpha C_{\mu\nu\rho\sigma} k_\sigma k_\rho a_\phi = 0.$$

(2.3)

Clearly, propagation of the coupled photon is affected by coupling term $\alpha$ with WT. Hence, the coupled photons move non-geodesically in the curved spacetime. Normally, it is known that photons should follow null geodesics
\( \gamma_{\mu \nu}, \text{i.e., } \gamma^{\mu \nu} k_\mu k_\nu = 0 \) [78]. Using the Einstein field equations with above equations, one can define the effective metric for the coupled photon as follows [34]:

\[
ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + C(r)W(r)^{-1}d\Omega^2. \tag{2.4}
\]

Note that \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( A(r) = 1 - \frac{2M}{r} \) and \( C(r) = r^2 \). Furthermore, for two different polarizations of the photon respectively along \( \hat{l}_\mu \) (PPL) and \( \eta_\mu \) (PPM) the quantity \( W(r) \) are

\[
W(r)_{\text{PPL}} = \frac{r^3 - 8aM}{r^3 + 16aM'}, \tag{2.5}
\]

and

\[
W(r)_{\text{PPM}} = \frac{r^3 + 16aM}{r^3 - 8aM'}. \tag{2.6}
\]

After we assume that both the observer and the source are located in the equatorial plane as well as the trajectory of the null photon is restricted on the same plane with \( \theta = \frac{\pi}{2} \), the metric is reduced to this form:

\[
ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + C(r)W(r)^{-1}d\phi^2. \tag{2.7}
\]

For the photon moving in the equatorial plane \( (\theta = \frac{\pi}{2}) \), we have \( k_2 = 0 \) and in this way solving for null geodesics with \( ds^2 = 0 \), the optical line element becomes

\[
\frac{\text{d}t^2}{A(r)^2} + \frac{C(r)\text{d}\phi^2}{W(r)A(r)}. \tag{2.8}
\]

the optical metric \( \tilde{g}_{ab} \) in terms of the new coordinate \( r^* \) as

\[
\text{d}l^2 = \tilde{g}_{ab} \text{d}x^a \text{d}x^b = \text{d}r^2 + f^2(r^*) \text{d}\phi^2, \tag{2.9}
\]

where the function \( f(r^*) \) is given by

\[
f(r^*) = \frac{\sqrt{C(r)}}{\sqrt{W(r)A(r)}}. \tag{2.10}
\]

Note that determinant is equal to \( \det \tilde{g}_{ab} = f^2(r^*) \). Using the optical metric (3.3), only non-vanishing Christoffel symbols are calculated as \( \Gamma^r_{\phi \phi} = -f(r^*)f'^{*a} \) and \( \Gamma^r_{\phi \phi} = f^{*a} / f(r^*) \). Using the (3.3), we calculate the Gaussian optical curvature \( K \) [45]:

\[
K = -\frac{R_{r\theta \phi \phi}}{\det \tilde{g}_{ab}} = -\frac{1}{f(r^*)} \frac{\text{d}^2 f(r^*)}{\text{d}r^*}. \tag{2.11}
\]

Optical curvature \( K \) is also written in terms of \( r \) [77];

\[
K = -\frac{1}{f(r^*)} \left[ \frac{\text{d}r}{\text{d}r^*} \frac{\text{d}f}{\text{d}r} + \left( \frac{\text{d}r}{\text{d}r^*} \right)^2 \frac{\text{d}^2 f}{\text{d}r^2} \right]. \tag{2.12}
\]

First we consider the PPL case, after we use the equation (3.4) into (2.12), the optical curvature is obtained as follows:

\[
K = -\frac{(C''(A)^2)}{2C} + \frac{(A'')}{2} A + \frac{(W'')(A)^2}{2W} + \frac{(C')^2(A)^2}{4(C)^2} \nonumber \]  

\[
+ \frac{(C')(W')(A)^2}{2CW} - \frac{(4A')^2}{4(W)^2} \frac{2(W')^2(A)^2}{4(W)^2}. \tag{2.13}
\]
\[ K_{\text{PPL}} = \frac{M}{(r^3 + 8\alpha M)^2 r^4 (r^3 + 16\alpha M)^2} \]  

\[ \times (3Mr^2 - 2r^3 + 336M^2r^9 - 320M\alpha r^{10} + 72\alpha r^{11} - 6912M^3\alpha^2\rho^6 + 6720M^2\alpha^2r^7 \]

\[ - 1584M^2r^8 + 67584M^4\alpha^3r^3 - 69632M^3\alpha^3r^4 + 18432M^2\alpha^3r^5 + 49152M^5\alpha^4 - 32768M^4\alpha^4r), \]

\[ K_{\text{PPL}} \simeq -2M \frac{3M^2}{r^4} - \frac{72M\alpha}{r^3}. \tag{2.15} \]

**A. Calculation of Deflection angle**

Now, we calculate the deflection angle using the Gauss-Bonnet theorem. GBT provides a relation between the intrinsic geometry of the spacetime and its topology of the region \( D_R \) in \( M \), with boundary \( \partial D_R = g_{\hat{g}} \cup C_R \) [45]:

\[
\int_{D_R} K \, dS + \oint_{\partial D_R} \kappa \, dt + \sum_i \epsilon_i = 2\pi\chi(D_R), \tag{2.16}
\]

where \( K \) is the Gaussian curvature, \( \kappa \) the geodesic curvature, given by \( \kappa = \hat{g}(\nabla_i \gamma, \gamma) \), such that \( \hat{g}(\gamma, \gamma) = 1 \), with the unit acceleration vector \( \gamma \) and \( \epsilon_i \) the corresponding exterior angle at the \( i \)th vertex. As \( R \to \infty \), both jump angles are \( \pi/2 \), so that \( \hat{\theta}_O + \hat{\theta}_S \to \pi \). Since \( D_R \) is non-singular, than the Euler characteristic is \( \chi(D_R) = 1 \), hence we have

\[
\int_{D_R} K \, dS + \oint_{\partial D_R} \kappa \, dt + \theta_i = 2\pi\chi(D_R). \tag{2.17}
\]

Since \( g_{\hat{g}} \) is geodesic, \( \theta_i = \pi \) is total jump angle, \( \kappa(\gamma_{\hat{g}}) = 0 \), and Euler characteristic number \( \chi \) is 1, so that remain part is calculated as \( \kappa(C_R) = |\nabla C_R \hat{C}_R| \) as \( R \to \infty \). At very large \( R \), \( C_R := r(\phi) = R = \text{const.} \), the radial component of the geodesic curvature

\[
\left( \nabla C_R \hat{C}_R \right)^r = \hat{C}_R^r \partial_\phi \hat{C}_R^r + \Gamma^r_{\theta\phi} \left( \hat{C}_R^\theta \right)^2, \tag{2.18}
\]

Note that first term is zero and \( (\hat{C}_R^\theta)^2 = 1/f^2(r^*) \), recall that \( \Gamma^r_{\theta\phi} = -f(r^*)f''^* \), so it becomes:

\[
\left( \nabla C_R \hat{C}_R \right)^r \to -\frac{1}{R}. \tag{2.19}
\]

At very large \( r(\phi) = R = \text{const.} \), it follows that the geodesic curvature is independent of topological defects, \( \kappa(C_R) \to R^{-1} \), however from the optical metric (3.3), it’s not difficult to see that \( dt = R \, d\phi \), and hence

\[
\kappa(C_R) dt = \frac{1}{R} R \, d\phi. \tag{2.20}
\]

After we substitute above results,

\[
\int_{D_R} K \, dS + \int_{C_R} \kappa \, dt \bigg|_{R \to \infty} = \int_{S_\infty}^{\pi + \hat{\kappa}} \int_0 \frac{\pi + \hat{\kappa}}{R} d\phi. \tag{2.21}
\]

In the weak deflection limit, we may assume that the light ray is given by \( r(t) = b/ \sin \varphi \) at zeroth order, using (2.14) and (2.21) it follows that the deflection angle is given by

\[
\hat{\kappa} = \frac{\pi}{2} \sqrt{\det \hat{g}} \, d\varphi \, d\varphi. \tag{2.22}
\]
where
\[ \sqrt{\det \tilde{g}} d\rho = r dr \left( 1 + \frac{3M}{r} + \ldots \right) \] (2.23)
and the following equation for the deflection angle [48]
\[ \frac{1}{r_\gamma} = \frac{\sin \varphi}{b} + \frac{M(3 + \cos(2\varphi))}{2b^2} \]
\[ + \frac{M^2(37 \sin \varphi + 30(\pi - 2\varphi) \cos \varphi - 3 \sin(3\varphi))}{16b} \] (2.24)
Substituting the leading terms of the Gaussian curvature (2.14) into the last equation we find the deflection angle up to second order terms as follows:
\[ \hat{\alpha}_{PPL} \approx \pi \int_0^\infty \int_{r_\gamma} \frac{1}{\sqrt{\det \tilde{g}}} dr^* d\varphi \]
\[ \approx \frac{4M}{b} + \frac{15\pi M^2}{4b^2} + \frac{32M_a}{b^3} + \frac{261\pi M_a}{4b^4} . \] (2.25)
As a second case, we consider the PPM case and find
\[ K_{PPM} \simeq -\frac{2M}{r^3} + \frac{3M^2}{r^4} + \frac{72\alpha}{r^5} , \] (2.26)
which admits the following deflection angle:
\[ \hat{\alpha}_{PPM} \approx \pi \int_0^\infty \int_{r_\gamma} \frac{1}{\sqrt{\det \tilde{g}}} dr^* d\varphi \]
\[ \approx \frac{4M}{b} + \frac{15\pi M^2}{4b^2} - \frac{32M_a}{b^3} - \frac{261\pi M_a}{4b^4} . \] (2.27)

III. DEFLECTION ANGLE OF SCHWARZSCHILD-LIKE SOLUTION IN A BUMBLEBEE GRAVITY

The metric of Schwarzschild-like solution in a bumblebee gravity spherically symmetric is given by solution [70]
\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 + \ell \right) \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 \]
\[ + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 , \] (3.1)
where we have conveniently identified \( \rho_0 \equiv 2M \) (\( M = G_N m \) is the usual geometrical mass) such that in the limit \( \ell \to 0 \) (\( b^2 \to 0 \)) we recover the usual Schwarzschild metric. The metric (3.1) represents a purely radial LSB solution outside a spherical body characterizing a modified black hole solution.

For the photon moving in the equatorial plane (\( \theta = \frac{\pi}{2} \)), we have \( k_2 = 0 \) and in this way solving for null geodesics with \( ds^2 = 0 \), the optical line element becomes
\[ dt^2 = \left( 1 + \ell \right) \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)^2} + \frac{r^2 d\varphi^2}{\left( 1 - \frac{2M}{r} \right)^2} . \] (3.2)
The optical metric \( \tilde{g}_{ab} \) in terms of the new coordinate \( r^* \) as
\[ dr^2 = \tilde{g}_{ab} dx^a dx^b = dr^{*2} + f^2(r^*) d\varphi^2 , \] (3.3)
where the function $f(r^*)$ is given by

$$f(r^*) = \frac{r}{\sqrt{1 - \frac{2M}{r}}}.$$  \hspace{1cm} (3.4)

Using the (3.3), we calculate the Gaussian optical curvature $K$ [45]:

$$K = \frac{M(3M - 2r)}{(1 + l)r^4}.$$  \hspace{1cm} (3.5)

The geodesic curvature is found to be

$$\kappa \to \frac{1}{R\sqrt{1 + l}},$$  \hspace{1cm} (3.7)

yielding

$$\kappa dt \to \frac{1}{\sqrt{1 + l}} d\varphi.$$  \hspace{1cm} (3.8)

It follows that the deflection angle is given by

$$\hat{\alpha} = \pi \left( \sqrt{1 + l} - 1 \right) - \sqrt{1 + l} \int_0^\infty \frac{K \sqrt{\det \bar{g}}}{r} dr^* d\varphi.$$  \hspace{1cm} (3.9)

expanding

$$\sqrt{1 + l} = 1 + \frac{l}{2} + ...$$  \hspace{1cm} (3.10)

We find the total deflection angle

$$\hat{\alpha}_{BSCH} \approx \frac{\pi l}{2} + \frac{4M}{b} - \frac{2Ml}{b}.$$  \hspace{1cm} (3.11)

**IV. LENSING BY EINSTEIN-ROSEN TYPE WORMHOLE IN WEYL GRAVITY**

We introducing a new coordinate transformation $u^2 = r - 2M$ to the spacetime metric 2.4 to remove the singularity and make it Einstein-Rosen wormhole (WERW) in Weyl gravity as follows:

$$ds^2 = -\frac{u^2}{u^2 + 2M} dt^2 + 4(u^2 + 2M) du^2 + \frac{(u^2 + 2M)^2}{B(u)} d\Omega^2,$$

$$B(u) = \frac{(u^2 + 2M)^3 - 8\alpha M}{(u^2 + 2M)^3 + 16\alpha M}.$$  \hspace{1cm} (4.1)

Note that the radius of the throat is located at $u_{throat} = 0$ and non-singular in the interval $u \in (-\infty, \infty)$. Then we write optical metric of WERW as follows:

$$dt^2 = \frac{4(u^2 + 2M)^2}{u^2} du^2 + \frac{(u^2 + 2M)^3}{u^2 B(u)} d\varphi^2.$$  \hspace{1cm} (4.1)

The Gaussian optical curvature is calculated as:

$$K \approx \frac{1}{4u^4} - 4 \frac{\left( u^4 + 27 \alpha \right) M}{u^{16}} + \frac{(25u^8 + 1512 \alpha u^4 + 432 \alpha^2) M^2}{u^{16}}.$$  \hspace{1cm} (4.2)

Hence, the deflection angle of WERW is found

$$\hat{\alpha}_{WERW} \approx \frac{\pi}{16b^2} + \frac{3\pi M}{8b^4}.$$  \hspace{1cm} (4.3)
V. LENSING BY EINSTEIN-ROSEN TYPE WORMHOLE IN BUMBLEBEE GRAVITY

We introduce a new coordinate transformation \( u^2 = r - 2M \) to spacetime metric 3.1 to remove the singularity and make it Einstein-Rosen wormhole (BERW) in bumblebee gravity as follows:

\[
    ds^2 = -\frac{u^2}{u^2 + 2M} dt^2 + (1 + \ell) \frac{4(u^2 + 2M)du^2}{u^2} + \frac{(u^2 + 2M)^2}{u^2} d\Omega^2.
\]  

(5.1)

Note that the radius of the throat is located at \( u_{\text{throat}} = 0 \) and non-singular in the interval \( u \in (-\infty, \infty) \). Then we write optical metric of BERW as follows:

\[
    dt^2 = (1 + \ell) \frac{4(u^2 + 2M)^2}{u^2} du^2 + \frac{(u^2 + 2M)^3}{u^2} d\varphi^2.
\]

(5.2)

The Gaussian optical curvature is calculated as:

\[
    K = \frac{u^4 - 8Mu^2 - 4M^2}{4(u^2 + 2M)^4(1 + \ell)}
\]

(5.3)

\[
    K \approx \frac{1}{4u^4(1 + \ell)} - 4\frac{M}{u^6(1 + \ell)} + 25\frac{M^2}{u^8(1 + \ell)}.
\]

(5.4)

The geodesic curvature is calculated as

\[
    \kappa \to \frac{1}{u^2\sqrt{1 + \ell}}.
\]

(5.5)

and

\[
    \kappa dt \to u^2 d\varphi
\]

(5.6)

yielding

\[
    \kappa dt \to \frac{1}{\sqrt{1 + \ell}} d\varphi
\]

(5.7)

Finally the deflection angle is given by

\[
    \hat{\alpha}_{BERW} = \pi \left( \sqrt{1 + \ell} - 1 \right) - \sqrt{1 + \ell} \int_0^\infty K \sqrt{\det \bar{g}} \, dr^* \, d\varphi,
\]

(5.8)

in which \( \sqrt{\det \bar{g}} \, dr^* \, d\varphi \approx 2\sqrt{1 + \ell} u \, du \, d\varphi \). Hence, the deflection angle of BERW is found as follows:

\[
    \hat{\alpha}_{BERW} \approx \frac{\pi \ell}{2} + \frac{\pi}{16b^2} + \frac{3\pi M}{8b^4}.
\]

(5.9)

VI. CONCLUSION

In this paper, we have studied the effect of the Weyl tensor by using the photons coupled to Weyl tensor in a Schwarzschild black hole in two different cases, namely PPL and PPM. For this purpose we have used the GBT and GWM to calculate the weak gravitational lensing. We have found the following deflection angles:

\[
    \hat{\alpha}_{PPL} \approx \frac{4M}{b} + \frac{15\pi M^2}{4b^2} + \frac{32M\alpha}{b^3} + \frac{261\pi M\alpha}{4b^4},
\]

(6.1)

and
Then we have applied the GBT to Schwarzschild-like black hole in bumblebee gravity and we have obtained the total deflection angle as follows:

$$\hat{\alpha}_{BSCH} \approx \frac{\pi l^2}{2} + \frac{4M}{b} - \frac{2Ml}{b^3}.$$

(6.3)

Afterwards, we have constructed Einstein-Rosen bridges using the photons coupled to Weyl tensor in the Schwarzschild black hole and the Schwarzschild-like black hole in bumblebee gravity to obtain the deflection angles:

$$\hat{\alpha}_{WERW} \simeq \frac{\pi l^2}{2} + \frac{3\pi M}{16b^4},$$

(6.4)

and

$$\hat{\alpha}_{BERW} \simeq \frac{\pi l^2}{2} + \frac{3\pi M}{16b^4}.$$  

(6.5)

In conclusion, we show how the effect of Weyl and bumblebee parameters can lead to deviation of deflection angle as seen in Fig. 1.
Acknowledgments

This work was supported by the Chilean FONDECYT Grant No. 3170035 (A. Ö.).
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