Single-mode nonclassicality criteria via Holstein–Primakoff transformation

Mehmet Emre Tasgin

Institute of Nuclear Sciences, Hacettepe University, 06800, Ankara, Turkey
E-mail: metasgin@hacettepe.edu.tr and metasgin@gmail.com

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Abstract
We study different representations of a many-particle system. We reveal the connections among the nonclassicalities of many-particle, single-mode and two-mode systems in the infinite particle number limit. First, we demonstrate that (separable) atomic coherent states, of a many-particle system, mimic the coherent states of single-mode light as the number of particles goes to infinity. This enables the utilization of a many-particle entanglement (MPE) criterion as a single-mode nonclassicality (SMNc) condition via Holstein–Primakoff transformation in this limit. As an example, we show that spin-squeezing criterion, for MPE, can be transformed into quadrature-squeezing, a nonclassicality condition for a single-mode system. Second, we demonstrate a similar connection between the two-mode separable states and separable atomic coherent states of a many-particle system for the infinite particle number limit. The second connection makes us be able to obtain an SMNc condition from a two-mode entanglement (TME) criterion, using the many-particle system as a bridge. As an example, we obtain Mandel’s $Q$-parameter from the TME criterion of Hillery and Zubairy.

Keywords: quantum entanglement, quantum nonclassicality, many-particle entanglement, two-mode entanglement, single-mode nonclassicality, entanglement criteria

1. Introduction
Nonclassical quantum states are the basis for quantum technologies which are believed to reshape the new century [1]. Technologies like quantum teleportation [2], measurements beyond the standard quantum limit [3, 4], quantum radars [5], quantum computation [6] etc, all, require nonclassical states. Hence, detection and quantification of the nonclassicality (Nc) of such states possess fundamental importance in the field.

Although Nc may appear in different forms, they actually can be converted into each other. Single-mode nonclassicality (SMNc) of an incident light mode, e.g. squeezing in quadrature or number fluctuations, can be converted into two-mode entanglement (TME) at the output of a beam splitter (BS). Moreover, a conservation-like relation holds between SMNc and TME in a BS [7–11], a passive device which is unable to generate extra Nc [12]. That is, not all of the SMNc of the incident (input) mode can be converted into TME in the BS, but some SMNc remains in both output modes. A single-mode light can also transfer its Nc (in the form of SMNc), e.g. squeezing, into an ensemble of atoms [13–15]. Quadrature-squeezing of the incident light transforms into spin-squeezing [16, 17] in the ensemble. Spin-squeezing creates a collective (many-particle, MP) entanglement (MPE) between the constituent atoms [18–22]. Similarly, interaction of an entangled two-mode light with an ensemble creates MPE in the ensemble [23, 24]. Hence, Nc is like ‘energy’: it can be transformed into different forms.

Not only can different forms of Ncs be converted into each other, but also criteria witnessing different forms of Ncs can be transformed into each other. For instance, SMNc criteria can be obtained from TME criteria using a BS transformation, see references [25–28] and equations (35) and (36) in the present paper. A single-mode nonclassical state, put into a BS, generates entanglement between the output modes [12, 29]. This makes a TME criterion work as a SMNc criterion [26, 28] via entanglement potential [29].

In reference [19], we further demonstrate (i) how one can obtain an SMNc criterion from an MPE criterion and (ii) how one can anticipate a new MPE criterion from the form of an SMNc criterion, e.g. a new MPE criterion from Mandel’s $Q$ parameter. Such a connection appears because an atomic
coherent state (ACS)\(^1\) of an ensemble, a separable state [30], becomes the coherent state of a single-mode light, a classical state, for \(N \to \infty\) [31, 32]. Reference [33], in a similar manner, shows that some forms for ensemble–ensemble entanglement criteria can be anticipated from the structure of TME criteria.

The quantum state of an ensemble of identical (indistinguishable) particles has to be in a symmetric form with respect to the exchange of particles. In this case, the dynamics of the system can be equivalently described via operators \(\hat{c}_g^\dagger\) and \(\hat{c}_e\) which annihilates/creates a particle in the ground and excited states, respectively [34]. In such a system, a connection between the entanglement of these two modes, \(\hat{c}_g^\dagger\hat{c}_e\), and the entanglement among the atoms of the ensemble (an MPE) has been intrigued in references [35, 36]. More specifically, reference [35] shows that the spin-squeezing (MPE) criterion [18] cannot be satisfied unless the two-modes describing this \(N\)-particle two-level system are entangled.

As a counter-example, for the general case, however, it is straightforward to show that a separable two-mode state, e.g. \(|n_g\rangle |n_e\rangle\), demonstrates a strong MPE. Here, \(n_g\) and \(n_e\) denote the occupation of the modes associated with the ground and excited states. \(|n_g\rangle |n_e\rangle\) is actually the mode representation of a Dicke state [30], where \(n_g\) number of excitation is equally and symmetrically distributed among \(N = n_g + n_e\) number of atoms. \(|n_g\rangle |n_e\rangle\) are the Fock states. Besides the two-mode representation, a Dicke state can as well be represented by spin states, e.g. as \(|S, m\rangle\) with \(S = N/2 = (n_g + n_e)/2\) and \(m = (−S + n_e)\) [30]. The counter-example demonstrates that MPE, in an ensemble of \(N\) identical two-level particles, is possible to emerge via the presence of an SMNc in one of the two modes, besides the TME.

In this paper, we investigate the connection between an SMNc, TME and MPE criteria in the limit \(N \to \infty\). (a) We show that in the \(N \to \infty\) limit, entanglement of the modes \(\hat{c}_g^\dagger\hat{c}_e\) is sufficient for the presence of the (collective) MPE of the particles in the ensemble. In demonstrating this connection, it is required to show the absence of SMNc in both of the modes, \(\hat{c}_g^\dagger\hat{c}_g\) and \(\hat{c}_e^\dagger\hat{c}_e\), while taking the limit \(N \to \infty\). (b) Next, we use this, TME \(\rightarrow\) MPE, connection to be able to obtain an SMNc criterion from a TME criterion. This provides a new method, alternative to the BS approach [26, 28], for obtaining SMNc conditions from TME criteria. Obtaining an analytical form does not always become possible via the BS approach. (c) What is more important is; our study brings a clarification to the connections between Ncs associated with many-particle, single-mode and two-mode systems. (d) In reference [19], we have already demonstrated that an MPE criterion becomes an SMNc condition for \(N \to \infty\). Here, for completeness, in section 1, we additionally demonstrate that spin-squeezing criterion for MPE [18] becomes the quadrature-squeezing condition, an SMNc criterion.

Our approach, despite presenting an intriguing TME–SMNc connection and being able to lead to a better anticipation of MPE criteria, has a drawback. It can transform the TME criteria into an SMNc condition only when, at least in the present form, the TME criterion involves particle number conserving terms, e.g. like \(\hat{c}_g^\dagger\hat{c}_g\) but not like \(\hat{c}_g^\dagger\hat{c}_g\).

TME criteria are greater in number compared to the MPE and SMNc criteria. Hence, an approach similar to reference [19] can be used to anticipate a form for an MPE criterion from the form of a TME criterion. For a better visualization: reference [19] anticipates one of the possible forms for an MPE criterion from the SMNc condition Mandel’s \(Q\)-parameter, by examining [18] the uncertainty of the operator \(\hat{R} = S_+S_-\) which becomes \(\hat{R} = Nb^\dagger b\) in the \(N \to \infty\) limit via Holstein–Primakoff transformation. A given TME criterion can also be used for (a detailed) anticipation of a form for an MPE criterion.

The connections between different type Nc criteria—though they work in the \(N \to \infty\) limit—can also stimulate the inquiry of interesting Nc phenomena [11, 37, 38] in corresponding systems, i.e. similar to references [19, 33].

The paper is organized as follows: derivations of the connections MPE \(\rightarrow\) SMNc and TME \(\rightarrow\) MPE are intimately related to each other. So, for the coherence, we demonstrate the connections among the three systems in section 2 and present our examples in section 3. In section 2.1 we overview the ACSs, separable many-particle states, the Dicke states, and highly entangled many-particle states. In section 2.2, we present the spin representation for Dicke states and ACSs, and show that they mimic the Fock and coherent states of light [31, 32], respectively, as the number of the particles becomes very large \(N \to \infty\). In section 3.1, we show that spin-squeezing (an MPE) criterion becomes the quadrature-squeezing (an SMNc) condition. In section 2.4, we introduce the two-mode representation of a many-particle system. In section 2.5 we show that the presence of entanglement between the two modes, \(\hat{c}_g^\dagger\hat{c}_e\) and \(\hat{c}_e^\dagger\hat{c}_g\), of the ensemble implies the presence of MPE in the \(N \to \infty\) limit. Here, different to reference [19], we try to show that an ACS does not possess Nc in both modes. The presence of an MPE in the ensemble of these identical particles, however, does not imply the presence of entanglement even in the \(N \to \infty\) limit. In section 3.2, we demonstrate how one can obtain an SMNc condition from a TME criterion using the path MPE \(\rightarrow\) MPE. As an example, we obtain the Mandel’s \(Q\)-parameter (\(Q < 1\)) condition from Hillery–Zubairy criterion [39], where both work well for number-squeezed-like states [40]. In section 3.3, for completeness, we also provide the alternative definition of Mandel’s \(Q\)-parameter (an SMNc condition) from the Hillery and Zubairy (HZ) TME criterion, this time, using the BS approach. Section 4 contains our summary and discussions.

2. Connections between many-particle, single-mode and two-mode systems

2.1. Atomic and single-mode coherent states

The quantum state of an ensemble of two-level particles can be represented in several ways. One can simply write the ensemble state in terms of single-particle states, \(|g\rangle\), or \(|e\rangle\), where the

\(^1\) Also known as spin coherent states.
ith particle is in the ground and excited states. Alternatively, the ensemble states can be represented by angular momentum addition theorem for $N$ spin-1/2 particles [30], known as Dicke states, see figure 1 in reference [41]. Below, we refer to both representations for gaining understanding on the nonclassical properties of a many-particle system.

The ground state of such an ensemble, where all particles are in the ground state, can be written as

$$|\psi_N^{(\text{ground})}\rangle = |S, -S\rangle = |g\rangle_1 \otimes |g\rangle_2 \otimes \ldots \otimes |g\rangle_N.$$  (1)

Here, the total-spin is $S = N/2$, $|S, m\rangle$, with $m = -S \ldots S$, are the Dicke states and are highly (collectively) many-particle entangled for $m \neq \pm S$. According to the addition of angular momentum, $S$ can assign values from $0 \ldots N/2$. Here, however, we deal only with a symmetric set of Dicke states where $S = N/2$. Dicke states are usually employed in describing collective phenomena in ensembles such as super-radiance [42–46]. When the particles are identical, either bosons [72] or fermions [47], only the symmetric set (maximum cooperation $r = N/2$) of Dicke (internal) states can be occupied [48, 49].

In order to represent the occupation of the excited state, one can equivalently represent (a symmetric) Dicke state as $|S, m\rangle \equiv |N, n_e\rangle$ where $n_e$ is the occupation of the excited state. Here $n_e = S + m$ and for $m = -S$ all particles are in the ground state. The Dicke states become the well-known Fock states of single-mode light $|N, n_e\rangle \rightarrow |n_e\rangle$ in the limit $N \rightarrow \infty$ [19, 31, 32].

An ACS $|S, z\rangle$ can be generated from the ground state $|S, -S\rangle$ by applying the collective atomic displacement operator

$$\hat{D}_z(\xi) = e^{\xi \hat{S}_z}$$  as  \hspace{1cm} (2)

$$|S, z\rangle = \hat{D}_z|S, -S\rangle$$

$$= \frac{1}{(1 + |z|^2)^{N/2}} \sum_{m=0}^{N/2} \binom{N}{S + m} \frac{1}{\sqrt{m!}} \xi^{S + m} |S, m\rangle,$$  \hspace{1cm} (3)

where $\hat{S}_z$ are ladder operators with

$$\hat{S}_z|S, m\rangle = (S + m)|S, m\rangle,$$  \hspace{1cm} (4)

and $z = \tan[\xi]e^{\text{arg}(\xi)}$ [30].

An ACS has a special property. It is a separable state, that is, it can be written as a product of single-atom states

$$|S, z\rangle = C \left(|g\rangle_1 + z|e\rangle_1\right) \otimes \ldots \otimes \left(|g\rangle_N + z|e\rangle_N\right),$$  \hspace{1cm} (5)

similar to equation (1), where $C$ stands for the normalization constant $C = 1/(1 + |z|^2)^{N/2}$. Hence, an ACS is a subset of mixed separable states

$$\hat{\rho} = \sum_k P_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes \ldots \otimes \hat{\rho}_k^{(N)},$$  \hspace{1cm} (6)

studied, e.g. by Sørensen et al [18, 64]. Any mixed many-particle state which cannot be written in the form (6) are entangled and thus nonclassical. We do not refer to Glauber–Sudarshan P-function representations of many-particle states [50].

In the large number of particles limit $N \rightarrow \infty$, Dicke states transform to Fock states of a single-mode light, i.e. $|S, m\rangle \equiv |N, n_e\rangle \rightarrow |n_e\rangle$. When the summation in equation (3) is carried over an infinite number of particles, the coefficients of the ACS, $|S, z\rangle$, converge to $\alpha^2/\sqrt{m!}$ [19, 31, 32], to the coefficients of the well-known coherent states of light. Here, for $N \rightarrow \infty$ one finds $\alpha = \sqrt{N}\xi$.

Therefore, an ACS transforms to a coherent state of single-mode light at the infinite-$N$ limit [31, 32], i.e.

$$|S = N/2, z\rangle \rightarrow |\alpha\rangle,$$  \hspace{1cm} (7)

where $N \rightarrow \infty$ and $\xi \rightarrow 0$, but keeping $\alpha$ as a finite value. Below, using this phenomenon we obtain a connection between the strengths of MPE in an ensemble and the Nc of a single-mode light.

2. Many-particle entanglement and single-mode nonclassicality

An $N$-particle state of indistinguishable particles can be written as a superposition of ACSs, in the angular momentum representation,

$$|\psi_N\rangle = \sum_{i=1}^r \kappa_i |N/2, z_i\rangle,$$  \hspace{1cm} (8)

or equivalently in terms of separable (product) single particle states [18]

$$|\psi_N\rangle = \sum_{i=1}^r \kappa_i \left(|g\rangle_1 + z_i|e\rangle_1\right) \otimes \ldots \otimes \left(|g\rangle_N + z_i|e\rangle_N\right).$$  \hspace{1cm} (9)

Here, both rank $r$ and the coefficients $\kappa_i$ determine the non-classical features of both the many-particle system and the single-mode system [19] which becomes

$$|\psi\rangle = \sum_{i=1}^r \kappa_i |\alpha_i\rangle.$$  \hspace{1cm} (10)

Therefore, the strength of the MPE in $|\psi_N\rangle$ determines also the strength of the Nc of the single-mode field it maps, as $N \rightarrow \infty$. This observation can be used for deriving an SMNc condition from a given MPE criterion [19].

Nc (all MPE, TME and SMNc) criteria, however, adapt the expectation values of observables: the operators. So, for our connection

$$\sum_{i=1}^r \kappa_i |N/2, z_i\rangle \stackrel{N \rightarrow \infty}{\longrightarrow} \sum_{i=1}^r \kappa_i |\alpha_i\rangle$$  \hspace{1cm} (11)

to be useful, we need to establish a relation also between the operators belonging to the many-particle system, e.g. $\hat{S}_z$, and the ones belonging to the single-mode system $\hat{a}$ and $\hat{a}^\dagger$ (or $\hat{b}$ and $\hat{b}^\dagger$). The operators for the two systems can also be mapped via Holstein–Primakoff transformation.
2.3. Holstein–Primakoff transformation

The operators of a symmetric many-particle system and a single-mode system can be related via Holstein–Primakoff transformation [42, 51, 52]

\[
\hat{S}_+ = \hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}}, \quad \hat{S}_- = \sqrt{N - \hat{b}^\dagger \hat{b}} \hat{b}
\]

and

\[
\hat{S}_z = \hat{b}^\dagger \hat{b} - N/2
\]

without referring to states explicitly. Here, for finite \( N \), we distinguish the operator \( \hat{b} \) from the annihilation operator \( \hat{a} \) of a single-mode light. When \( N \) is sufficiently large, the ladder operators transform to single-mode annihilation/creation operators

\[
\hat{S}_+ \rightarrow \sqrt{N} \hat{b}^{\dagger}, \quad \hat{S}_- \rightarrow \sqrt{N} \hat{b} \quad \text{and} \quad \hat{S}_z \rightarrow -N/2 + \sqrt{N} \hat{b},
\]

where, now, \( \hat{b} \) stands for \( \hat{a} \) of a single-mode state, i.e. \( \hat{a} \equiv \hat{b} \). (In the following subsection, we also show that in the two-mode representation it is \( \hat{b} \equiv \hat{c}_g \) in the \( N \rightarrow \infty \) limit.) In section 2.1, using the maps introduced in sections 2.2 and 2.3, we demonstrate that the quadrature-squeezing (a SMNc) condition can be obtained from spin-squeezing (an MPE) criterion [18].

It is worth also noting that collective atomic displacement operator \( \hat{D}_s(\xi) \), in equation (2), transforms to a single-mode displacement operator [53]

\[
\hat{D}(\beta) = e^{i\beta \hat{a}^\dagger - \beta \hat{a}}
\]

for \( N \rightarrow \infty \). There is a striking common feature between the ensemble operator \( \hat{D}_r(\xi) \) (subscript ‘a’ stands for atoms) and the displacement operator for single-mode light \( \hat{D}(\beta) \), with \( \beta = \sqrt{N}\xi \). In generating the separable ACSs, in equations (3) and (5), from the separable ground state, in equation (1), \( \hat{D}_r(\xi) \) does not change the \( Nc \) features of the state it operates on. Both initial and final states are separable many-particle states. Similarly, as is well-known from quantum optics courses [34], the displacement operator \( \hat{D}(\beta) \) does not change the noise features of the single-mode system [54]. In the following subsection this relation is enriched with an extension to the two-mode system.

2.4. Two-mode representation of a many-particle system

Another (a third) representation for Dicke states, alternative to single-particle states \( |e\rangle \), of total-spin states \( |s,m\rangle \), is the two-mode representation \( |n_g, n_e\rangle \) with the particle number super-selection [35] constraint \( n_g + n_e = N \). In the infinite \( N \) limit of the Holstein–Primakoff transformation, one replaces \( \hat{c}_g \rightarrow \sqrt{N} \), as in parametric approximation, where the reduction in the number of, e.g. laser photons is negligibly small. Here, this corresponds to: number of particles is infinite so a finite \( n_e \) alters the \( n_g \) from \( N \) only negligibly. When one sets \( \hat{c}_g \rightarrow \sqrt{N} \) in the \( N \rightarrow \infty \) limit, one can realize \( \hat{c}_g \) behaves as the \( \hat{b} \) operator in the Holstein–Primakoff transformation.

We remark that separable atomic coherent states are exchange-symmetric states. Hence, they cannot represent fermionic many-particle states which are already entangled under the anti-symmetry constraint.

2.5. Two-mode entanglement and many-particle inseparability

In this subsection, we show that a TME criterion can be transformed into an SMNc condition by using the many-particle system as a bridge. First, we transform a TME criterion into an MPE criterion. Then, using the method introduced in sections 2.2 and 2.3 and in reference [19], we obtain an SMNc condition via Holstein–Primakoff transformation.

We follow the same strategy we do in section 2.2. We show that a superposition of ACSs can be written as a superposition of separable two-mode states. Doing this, however, one needs to be careful. Because an intra-mode \( Nc \) existing in one of the two modes, may also be responsible for the MPE in the mapping.

We aim to show

\[
|\psi\rangle_N = \sum_r |\psi^{(r)}_{\text{ACS}}\rangle^{N-\infty} |\psi\rangle = \sum_r |\ldots \rangle_g \otimes |\alpha^{(r)}\rangle_e
\]

in the infinite particle number limit. All ACSs are generated from the symmetric ground state \( |\psi^{(\text{ground})}\rangle = |S, -S\rangle = |n_g = N, n_e = 0\rangle \), as \( |\psi^{(r)}_{\text{ACS}}\rangle = e^{\alpha \hat{a}^\dagger} |\psi\rangle \), see equation (3). Hence, first we need to consider the \( Nc \) of \( |\psi^{(\text{ground})}\rangle \). (i) The state \( |\psi\rangle = |n_g = N, n_e = 0\rangle \) is obviously separable. That is, an Nc associated with TME does not exist in this state. (ii) The second mode, \( |n_e = 0\rangle_e \), also does not possess any SMNc. (iii) However, we need to be sure (show) whether \( |\psi\rangle = |n_g = N, n_e = 0\rangle \), a Fock state for finite \( N \), does not possess SMNc as \( N \rightarrow \infty \). That is, we also need to show \( |\psi^{(r)}_{\text{ACS}}\rangle^{N-\infty} |\psi\rangle = |\ldots \rangle_g \otimes |\alpha^{(r)}\rangle_e \) does not possess any \( Nc \). Actually, this (i.e. (iii)) is the main step in demonstrating \( |\psi^{(r)}_{\text{ACS}}\rangle^{N-\infty} \rightarrow |\ldots \rangle_g \otimes |\alpha^{(r)}\rangle_e \).

\[\text{Note that coefficients} \, \hat{c}_g \, \text{and} \, \hat{c}_e \, \text{produce} \, \sqrt{N} \, \text{and} \, \sqrt{N - n_e}, \text{for instance in the} \, \hat{S}_+ \, \text{operator, are also mimicked by} \, \hat{b} \, \text{and} \, \sqrt{N - \hat{b}^\dagger \hat{b}}, \text{operators in the Holstein–Primakoff transformation in equation (2), respectively. Hence, one may consider to assign} \, \hat{c}_g \equiv \hat{b} \, \text{and} \, \hat{c}_e = \sqrt{N - \hat{b}^\dagger \hat{b}} \, \text{via a comparison of the Holstein–Primakoff transformation for} \, \hat{S}_+ \, \text{this,} \, \text{however, is not correct for a finite} \, N. \text{Because in such an assignment} \, \hat{c}_e \, \text{does not satisfy the commutation relation for bosonic operators, i.e.} \, [\hat{c}_g, \hat{c}_e] = 1 \text{but} \, [\sqrt{N - \hat{b}^\dagger \hat{b}}, \sqrt{N - \hat{b}^\dagger \hat{b}}] = 0. \text{We gratefully thank the anonymous person for warning us about this issue.} \]
Such a relation can be demonstrated safely as follows. \( \tilde{D}_N = e^{\tilde{S}_+ - \tilde{S}_-} \equiv e^{\tilde{c}^\dagger \tilde{c} - \tilde{c}^\dagger \tilde{c}} \), generating the ACSs, is a BS transformation, even for finite \( N \). If \( |n_2(N)\rangle = |n_1(N)\rangle = 0 \), from which all ACSs are generated via \( \tilde{D}_N \), or the ACSs themselves, \( \xi_{ACS}\rangle_{N-\infty} \) had demonstrated any kind of \( N_c \), i.e. TME or SMNC; then at some value of \( \xi \) there would exist SMNC in \( \langle \alpha \rangle_c \). Because, \( \tilde{D}_N \) is a BS operator and a BS operator transforms TME and SMNC in a two-mode system [7]. However, we know that in the \( N \rightarrow \infty \) limit the second mode does not possess any SMNC [19, 31, 32]. Hence, none of the two modes \( \langle \tilde{c}_{\pm} \rangle \), representing \( \xi_{ACS}\rangle_{N-\infty} \), contain any \( N_c \). This further proves that \( \xi_{ACS}\rangle_{N-\infty} \rightarrow |\ldots \rangle_{\pm} \otimes |\alpha \rangle_c \) possesses no \( N_c \) at all.

Therefore, if there is TME between \( \tilde{c}_g - \tilde{c}_e \), for \( N \rightarrow \infty \), then there is MPE in \( |\psi_N\rangle \). In other words, when the two-mode state in equation (16) cannot be written as a single term, i.e. \( r \neq 1 \), then the many-particle state is also a superposition of ACSs. Hence, the many-particle state is entangled for \( N \rightarrow \infty \).

Using this relation, we can also derive a SMNC condition from a TME criterion tracking the following path. (i) We can transform \( \tilde{c}_{\pm}^{\dagger} \rightarrow \tilde{c}_e^{\dagger} \rightarrow \tilde{S}_e \), which checks whether there is MPE in the many-particle system, e.g., \( \tilde{c}_g^{\dagger} \rightarrow \tilde{c}_e^{\dagger} \rightarrow \tilde{S}_e \), which checks whether there is MPE in the many-particle system. (ii) Then, we can make a Holstein–Primakoff transformation as \( N \rightarrow \infty \); \( \tilde{S}_+ \rightarrow \sqrt{N} \hat{b} \). (iii) Finally, we obtain an SMNC from the TME. This is an alternative to BS transformation which can be used to derive SMNC criteria from TME criteria [28].

We note that an MPE criterion cannot be used for detecting the TME. Because, the \( N_c \) MPE demonstrates, could be also due to SMNC of the one of the two modes, it could also originate from the entanglement between the two modes.

In section 3.2, as a demonstration of the method, we obtain Mandel’s \( Q \)-parameter (squeezing in the number fluctuations), an SMNC, condition from the HZ (TME) criterion [39].

3. Transformations among nonclassicality criteria: examples

In this section, we present examples for obtaining an SMNC (quadrature-squeezing) condition from an MPE (spin-squeezing) criterion [18], in section 3.1, and for obtaining an SMNC (Mandel’s \( Q \) parameter) condition from a TME (HZ) criterion [39], in section 3.2. We also present derivation of Mandel’s \( Q \) parameter from HZ criterion, but this time, using BS transformation in section 3.3.

3.1. Single-mode squeezing from spin-squeezing criterion

Sørensen et al [18] introduced an inseparability criterion witnessing the entanglement of \( N \) particles

\[
\nu^2 \equiv \frac{N(\Delta \tilde{S}_{n_1})^2}{\langle \tilde{S}_{n_1} \rangle^2 + \langle \tilde{S}_{n_2} \rangle^2} < 1. \tag{17}
\]

They demonstrate that for a general mixed separable many-particle state, given in equation (6), \( \nu \) is larger than 1. Hence, \( \nu \) witnesses the MPE, i.e. the many-particle state cannot be written as equation (6). In equation (17), \( \tilde{S}_{n_i} \) are the operators for any orthogonal spin components, see equation (12).

One can choose \( \tilde{S}_{n_1} = \tilde{S}_e \) and \( \tilde{S}_{n_2,n_3} = \tilde{S}_{\alpha e} \), where \( \tilde{S}_e = (\tilde{S}_e + \tilde{S}_e) / 2 \). In the \( N \rightarrow \infty \) limit, collective spin operators transform as \( \tilde{S}_e \rightarrow \sqrt{N}(b^\dagger + b) / 2 = \sqrt{N}b^\dagger / \sqrt{2} \) and \( \tilde{S}_e \rightarrow \sqrt{N}(b^\dagger - b) / 2 = \sqrt{N}b / \sqrt{2} \). In this limit, \( \tilde{S}_e = (b^\dagger - b) / \sqrt{2} \rightarrow - \tilde{S}_2 / 2 \). Thus, performing the cancellations, MPE criterion (17) transforms to

\[
(\Delta \tilde{S}_b)^2 < 1 / 2, \tag{18}
\]

that is the well-known quadrature-squeezing condition for the single-mode field \( b \), with \( \tilde{x}_b = (b^\dagger + b) / \sqrt{2} \).

Here, we show that an SMNC condition can be obtained from an MPE criterion. In reference [19], we do the reverse. We anticipate one of the possible forms of an MPE criterion by examining the form of a Mandel’s \( Q \)-parameter. The MPE criterion we obtain in reference [19] and the spin-squeezing criterion in equation (17) work well for states of a different nature. Hence, the method we introduce in this subsection, can be used also for anticipating forms for MPE criteria.

We note that reference [56] questions if polarization-squeezing in two-mode states does imply quadrature-squeezing in ‘one of the two modes’, without referring to many-particle states. They neither refer many-particle states, nor use a Holstein–Primakoff transformation-like treatment, this utilizes a relation between coherent atomic states and single-mode coherent states studied in references [31, 32]. They show that there are polarization squeezed states without quadrature-squeezing.

3.2. Mandel’s \( Q \)-parameter from Hillery and Zubairy criterion

Next, we show that TME criteria can be practically transformed into SMNC criteria using many-particle representation as a bridge, as introduced in section 2.5. In particular, we show that Mandel’s \( Q \)-parameter, an SMNC criterion, can be obtained from the HZ (a TME) criterion [39].

A set of sufficient criteria for TME have been introduced by Hillery and Zubairy [39] as

\[
|\langle \tilde{c}_g^{\dagger} \tilde{c}_e \rangle |^2 > |\langle \tilde{c}_g^{\dagger} \tilde{c}_e \rangle |^2, \tag{19}
\]

which witnesses the inseparability of a two-mode system for any integer values of \( n, m = 1, 2, \ldots, \tilde{c}_g \) and \( \tilde{c}_e \) are annihilation operators for the two modes under consideration. One can obtain the Mandel’s \( Q \)-parameter, i.e. \( Q < 1 \) witnesses the SMNC, by setting \( m = n = 2 \)

\[
|\langle \tilde{c}_g^{\dagger} \tilde{c}_e \rangle |^2 > |\langle \tilde{c}_g^{\dagger} \tilde{c}_e \rangle |^2, \tag{20}
\]

as follows.

As demonstrated in section 2.5, for \( N \rightarrow \infty \) a two-mode entangled state implies an inseparable many-particle system. Thus, a TME criterion implies an MPE criterion for \( N \rightarrow \infty \). First, we express the two-mode operators in terms of collective spin operators of the many-particle system, e.g. \( \tilde{c}_g^{\dagger} \tilde{c}_e = \tilde{S}_{\alpha e} \). We note that such an expression can be obtained easily since the HZ TME criterion contains only particle number conserving terms, e.g. not terms like \( \tilde{c}_g^{\dagger} \tilde{c}_e \). Next, we perform a Holstein–Primakoff transformation on the MPE criterion which
is valid merely for $N \to \infty$, e.g. $\hat{S}_- = \sqrt{N - b^\dagger b} \to \sqrt{N} \hat{b}$. Therefore, in sum, we obtain an SMNc condition from a TME criterion, using the many-particle system or representation as a bridge.

For the sake of obtaining the SMNc condition, we transform the term on the right-hand side (rhs) of equation (20) to

$$
(c^\dagger e_g)^2 c_e^2 (c^\dagger e_g)^2 c_e^2 = c^\dagger e_g c^\dagger e_g c_e^2 c_e^2 - 4 c^\dagger e_g c^\dagger e_g c^\dagger c_e^2 c_e^2 + 2 (c^\dagger c_e)^2 c_e^2,
$$

(21)

using the commutation relations $[c^\dagger e_g, c_e] = 1$. The first term on the rhs of equation (21) can be identified as $\bar{S}_g^2 \bar{S}_e^2$. We work in the regime where $N$ is very large compared to $(c^\dagger c_e)$, i.e. the number of particles in the excited state. We notice that the first term on the rhs of equation (21) is proportional to $N^2$ while the second and third terms are in the orders $N$ and 1, respectively. Then, the last two terms can be neglected for $N \to \infty$. So, equation (20) takes the form

$$\langle (c^\dagger e_g)^2 c_e^2 \rangle > \langle (c^\dagger c_e)^2 c_e^2 \rangle = \langle \bar{S}_g^2 \bar{S}_e^2 \rangle.
$$

(22)

The $\bar{S}_g^2 \bar{S}_e^2$ term transforms to the desired form, $(\hat{b}^\dagger \hat{b})^2$, after the Holstein–Primakoff transformation. The term on the left-hand side (lhs) can be related to the number operator $\hat{b}^\dagger \hat{b}$, which corresponds to the number of quasiparticle excitations [19], using the Cauchy–Schwartz inequality

$$\langle \hat{c}^\dagger \hat{c} \rangle \bar{c}_e^2 \bar{c}_e \langle \hat{c}^\dagger \hat{c} \rangle \langle \bar{c}_e^2 \bar{c}_e \rangle \geq \langle \bar{c}_e^2 \bar{c}_e \rangle^2,
$$

(23)

Where the lhs can be written in terms of collective spin operators as $\langle \hat{S} \hat{S} \rangle = \langle \hat{S}_g \hat{S}_e \rangle$ which becomes $(\bar{S}_g \bar{S}_e)$ neglecting the term $\sim N$ compared to the $N^2$ term. Hence, equation (20) now becomes

$$\langle \hat{b}^\dagger \hat{b} \rangle^2 > \langle \bar{S}_g^2 \bar{S}_e^2 \rangle
$$

(24)

for sufficiently large $N$.

Thus, now, the TME criterion is expressed in terms of the many-particle, collective spin, operators for $N \to \infty$. Then, as the next step, we use the relation MPE $\to$ SMNc, see sections 2.2 and 2.3, where an entangled many-particle state is shown to imply a nonclassical single-mode state as $N \to \infty$. Applying the Holstein–Primakoff transformation, summarized in equation (12), inequality (24) becomes

$$\langle \hat{b}^\dagger \hat{b} \rangle^2 > \langle \hat{b}^\dagger \hat{b} \rangle^2,
$$

(25)

that is the Mandel’s $Q$-parameter [57] for a single-mode field. Mandel’s $Q$-parameter can alternatively be expressed as

$$\langle (\Delta \hat{n}_b)^2 \rangle < \langle \hat{n}_b \rangle,
$$

(26)

where number fluctuations reduce below the shot-noise limit. Here, $\hat{n}_b = b^\dagger b$.

Using the inequality set (19), one can also identify higher order $Nc$ conditions

$$\langle |\hat{b}^\dagger \hat{b}|^2 \rangle > \langle |\hat{b}|^2 \rangle |\hat{b}|^2,
$$

(27)

other than the standard form equation (25), for higher-order correlation functions $g^{(f)}$ [34, 58], with $f$ integer. In addition to the set of inequality (27), one also obtains

$$\langle |\hat{b}|^2 \rangle^2 > \langle |\hat{b}| \rangle^2 |\hat{b}|^2,
$$

(28)

for $m = n = 1$ in equation (19), which is forbidden by the Cauchy–Schwartz inequality to satisfy.

Unfortunately, two entanglement criteria, Simon–Peres–Horodecki (SPH) [59] and Duan–Giedke–Cirac–Zoller (DGCZ) [60],—which are both necessary and sufficient criteria for two Gaussian modes—require the evaluation of non-number conserving terms like $(\langle \hat{c}^\dagger e_g \rangle)$. Hence, this kind of criteria cannot be transformed into SMNc conditions using the Holstein–Primakoff approach. For this reason, in reference [28] we use the BS approach to deduce the degree of SMNc from SPH [59, 61, 62]. One may also expect a further connection between the entanglement depth [63] in many-particle inseparability and the degree of SMNc (of quasiparticles) due to Holstein–Primakoff transformation.

We kindly remark that both HZ criteria and Mandel’s $Q$-parameter can be derived [25] from criteria based on the matrices of moments, i.e. the Shchukin–Richter–Vogel Nc criterion [64], the Shchukin–Vogel entanglement criterion [65, 66] and its generalizations [25, 67].

3.3. Mandel’s $Q$-parameter using BS approach

Additionally, we show that Mandel’s $Q$-parameter condition can alternatively be obtained using the BS formalism [26, 28] using a ‘lower-order’ HZ criterion. One could naturally raise the question: if the two methods, i.e. TME $\to$ MPE $\to$ SMNc, somehow have a link in between implicitly, that is not apparent to the naked eye at first glance? Our observation, that the two methods achieve the Mandel’s $Q$-parameter using HZ criteria of not the same order, however, reduces the possibility for the existence of such a link.

One can also determine the Nc of a single-mode field according to the TME it creates at the output of a BS. The input beam is mixed either with vacuum noise or a coherent state in order to guarantee that the entanglement generated at the output of the BS originates from the Nc of the input beam [12, 29, 68]. Input modes are transformed [12, 69, 70] via the BS operator

$$\hat{B}(\beta) = D_{TM} = e^{i\beta |\hat{a}^\dagger - \beta^* \hat{a}|^2}.
$$

(29)

One acts $\hat{B}(\beta)$ on the initially separable state of the two-mode state, $|\psi_{12}\rangle = |\psi_1\rangle_1 \otimes |0\rangle_2$, (30) where $|\psi_1\rangle$ is the state of the single-mode field $(\hat{a})$ incident to the BS, whose Nc is to be determined, and $|0\rangle_2$ is the vacuum field mixing with $\hat{a}$. In the language of operators, $\hat{a}$ is the single-mode field operator which is the initial to the $\hat{a}_1$ operator to be transformed in equation (29). The vacuum input is the initial of the $\hat{a}_2$ operator.

In the language of states, Schrödinger picture, $|\psi_1\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ is the initial state of the first mode which transforms via equation (29). The initial state of the second mode, on which $\hat{a}_2$ operator acts, is a vacuum. This method is equivalent to $|\psi_{12}\rangle = f(\mu_1 \hat{a}_1 + \mu_2 \hat{a}_2)|0\rangle_1 \otimes |0\rangle_2$ transform [70] on the two-mode wave function, where the function $f$ is defined by the single-mode wave function $|\psi_1\rangle = (\sum_{n=0}^{\infty} d_n (\hat{a}_1^\dagger)^n) |0\rangle_1$.
as \( f(z) = \sum_{\alpha=0}^{\infty} a_\alpha e^{\alpha z} \). Here, \( \mu_1 = t e^{i\phi} \) and \( \mu_2 = r \) are the coefficients in the well-known form \([12, 69]\).

\[
\hat{a}_1(\beta) = \hat{B}(\beta)\hat{a}_1 \hat{B}(\beta) = t e^{i\phi} \hat{a}_1(0) + r \hat{a}_2(0), \\
\hat{a}_2(\beta) = \hat{B}(\beta)\hat{a}_2 \hat{B}(\beta) = -r \hat{a}_1(0) + t e^{-i\phi} \hat{a}_2(0),
\]

(31a)

(31b)

of the BS transformation, in the language of operators, i.e. the Heisenberg picture. \( t^2 \) and \( r^2 \) correspond to the transmission and reflection coefficients, respectively, and \( \phi \) is the phase of the BS. We note that \( \hat{a}_1(0) = \hat{a} \) where \( \hat{a} \) refers to the input operator to be examined. \( \hat{a}_2(0) \) refers to the second input (vacuum) mode. As usual, it is easier to work in the Heisenberg picture, where operators transform.

An expectation value including \( \hat{a}_1, \hat{a}_2 \) operators, for instance,

\[
\langle \hat{a}_1 \hat{a}_2 \rangle = \langle 0 \rangle \langle \psi_n | \hat{B}(\xi)\hat{a}_1 \hat{a}_2 \hat{B}(\xi) | \psi_n \rangle \otimes | 0 \rangle_2,
\]

(32)

can be evaluated to

\[
\langle \hat{a}_1 \hat{a}_2 \rangle = \langle 0 \rangle \langle \psi_n | (t e^{i\phi} \hat{a} + r \hat{a}_2) \langle t e^{i\phi} \hat{a} + r \hat{a}_2)(0) \rangle \times (-r \hat{a}^\dagger + t e^{i\phi} \hat{a}_2(0)) | \psi_n \rangle_1 | 0 \rangle_2,
\]

(33)

performing the transformations (31a) and (31b), where \( \hat{a}_2(0) \) is the not-evolved input operator referring to the vacuum noise.

In order to determine if the input state \( | \psi_n \rangle \) possesses SMNc, we check if the BS generates entanglement at the output, by calculating the HZ criterion

\[
|\langle \hat{a}_1 \hat{a}_2 \rangle|^2 > |\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle|.
\]

(34)

Using the transformations (31a) and (31b) one can relate the terms in equation (34) to the ones for the single-mode field \( \hat{a} \) as

\[
\langle \hat{a}_1 \hat{a}_2 \rangle = -rt e^{i\phi} \langle \hat{a}^2 \rangle, \\
\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle = r^2 t^2 \langle \hat{a}^2 \hat{a}^2 \rangle,
\]

(35)

(36)

which simply give the condition for the single-mode Nc of the input mode \( \hat{a} \) as

\[
|\langle \hat{a}^2 \rangle|^2 > |\langle \hat{a}^2 \hat{a}^2 \rangle|.
\]

(37)

Unlike the TME \( \rightarrow \) MPE \( \rightarrow \) SMNc approach, introduced in sections 2.5 and 3.2; using the BS approach, it is possible to obtain SMNc conditions from TME criteria involving particle number non-conserving terms \([26, 28]\). However, we could not manage to obtain an easy analytical form for an SMNc condition from the TME criterion introduced by Duan, Giedke, Cirac and Zoller \([60]\). In the case of HZ criterion, the coefficients \( r, t \) and \( e^{i\phi} \) could be cancelled in section 3.2. However, in the DGCZ case numerical minimization with respect to \( r \) and \( \phi \) is required \([29]\). Still, we anticipate the SMNc condition could be obtained from DGCZ TME criterion, as the quadrature-squeezing condition.

We note that the beam-splitter approach of Asboth et al \([29]\) that we utilize here, can also be directly employed to a (single) spin system \([11, 38]\).

4. Summary and discussions

In summary, we study different representations of a symmetric many-particle system of two-level atoms in the large particle number limit \( N \rightarrow \infty \). We utilize these connections for obtaining Nc criteria from one another, i.e. among many-particle, two-mode and single-mode systems. In particular, we obtain SMNc conditions both from many-particle and TME criteria.

An ensemble of \( N \) identical and indistinguishable atoms can occupy the (exchange) symmetric Dicke states (see figure 1 in reference \([41]\)). Symmetric Dicke states can be represented both by single-mode and two-mode operators/states. In particular, Dicke (number) states and ACSs converge to single-mode Fock number states and coherent states, respectively, at the large particle limit. This map can be used to connect the entanglement of a symmetric many-particle state to the Nc of a single-mode field. Hence, an entangled many-particle state implies a nonclassical state in this mapping \([19]\) for \( N \rightarrow \infty \). Therefore a MPE criterion can be used as an SMNc condition in this limit. Since the states are experimentally not accessible in most cases, one maps/transforms the operators alternatively. An MPE criterion can be transformed into an SMNc condition via Holstein–Promakoff transformation, e.g., \( \hat{S}_- = \sqrt{N} - \hat{b}^\dagger \hat{b} \) which becomes \( \hat{S}_- \rightarrow \sqrt{N} \hat{b} \) in the infinite particle limit. \( \hat{b} \) accounts for the quasiparticle excitations of the many-particle system. We demonstrate that spin-squeezing (MPE) criterion becomes the quadrature-squeezing condition in this limit.

One can also represent a symmetric many-particle system in terms of two-modes, e.g. \( \hat{S}_- = \hat{c}_e^\dagger \hat{c}_h \) where \( \hat{c}_e^\dagger \hat{c}_h \) creates an identical particle in the ground/excited state. Here, \( \hat{c}_h \) actually is equivalent to the \( \hat{b} \) operator of the single-mode representation. We notice that separable (many-particle) ACSs are generated via

\[
e^{\hat{S}_+ - \hat{S}_-} \equiv e^{\hat{c}_e^\dagger \hat{c}_h - \hat{c}_h^\dagger \hat{c}_e = N^{\infty} a^{b \dagger - a}} \equiv e^{a^{b \dagger - a}} \hat{M}_{-\infty} \] (38)

and the BS operator conserves the Nc in the two-mode representation. (Here, \( \alpha = \sqrt{N} \).) This makes us introduce a connection also between TME and MPE in the \( N \rightarrow \infty \) limit.

We use the TME \( \rightarrow \) MPE connection, also in the \( N \rightarrow \infty \) limit, to develop a method for deriving SMNc conditions from TME criteria, i.e. following the path TME \( \rightarrow \) MPE \( \rightarrow \) SMNc which works in the \( N \rightarrow \infty \) limit. This method is an alternative to the BS approach, which is also used for TME \( \rightarrow \) SMNc derivation, and demonstrates the transformations among three different systems, see section 2. The new method, however, is limited with the number conserving terms only. Two well-known entanglement criteria \([59, 60]\), for instance, cannot be used to obtain an SMNc condition with the new method, since they include particle nonconserving terms. This connection can also be used to anticipate a form for a nonexisting MPE criteria.

We believe that the intriguing connections, we demonstrate in this paper, will stimulate new research on the intersection of many-particle, single-mode and two-mode systems.
Disclosures

The authors declare no conflicts of interest.

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ORCID iDs

Mehmet Emre Tasgin https://orcid.org/0000-0001-8483-6881

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