Conformally Invariant Equations for Graviton

Mohsen Fathi
Department of Physics,
Tehran Central Branch
Islamic Azad University

A thesis submitted for the Master degree
\textit{Master of Science in Physics}
Tehran, Winter 2010
I am grateful to my supervisor Dr. Mohammad Reza Tanhayi for the helps, supports and scientific training, during this work and thereafter.
Abstract

Recent astrophysical data indicate that our universe might currently be in a de Sitter (dS) phase. The importance of dS space has been primarily ignited by the study of the inflationary model of the universe and the quantum gravity. As we know Einstein’s theory of gravitation (with a nonzero cosmological constant) can be interpreted as a theory of a metric field; that is, a symmetric tensor field of rank-2 on a fixed de Sitter background. It has been shown the massless spin-2 Fierz-Pauli wave equation (or the linearized Einstein equation) is not conformally invariant. This result is in contrary with what we used to expect for massless theories. In this thesis we obtain conformally invariant wave equation for the massless spin-2 in the dS space. This study is motivated by the belief that conformal invariance may be the key to a future theory of quantum gravity.
Contents

Introduction

1 The Lorentz and the conformal groups, and the concept of invariance
   1.1 Group theory .................................................. 3
   1.1.1 Orthogonal groups ........................................ 4
   1.1.2 Rotation groups .......................................... 5
   1.2 Invariance under a group action .............................. 7
   1.2.1 Invariance of the laws of physics ....................... 7
   1.3 The Lorentz group ........................................... 8
   1.4 The conformal group ........................................ 8
   1.4.1 Conformal symmetry ..................................... 9
   1.5 Invariance under Lorentz group (Lorentz invariance) ...... 11
   1.6 Conformal invariance ....................................... 12

2 Conformal transformations
   2.1 Conformal mapping ........................................... 14
   2.2 Conformal covariance and conformal invariance, and the scalar wave equations ................................. 23
   2.3 Conformal transformations, and conformal invariance in gravitation ................................. 24
   2.4 Conformal transformations in Einstein gravity ................ 25
   2.5 Explaining some axioms using conformal transformations ......... 27

3 Einstein field equations
   3.1 Introduction ................................................... 29
   3.2 The mathematical form ...................................... 29
   3.3 The equivalent form ......................................... 31
   3.4 The cosmological constant .................................. 31
   3.5 Energy-momentum conservation ............................. 32
| Section | Title | Page |
|---------|-------|------|
| 3.6     | Non-linearity | 33   |
| 3.7     | The equivalence principle | 33   |
| 3.8     | The vacuum field equations | 33   |
| 3.9     | Einstein-Maxwell equations | 34   |
| 3.10    | The solutions | 35   |
| 3.11    | The linearized field equations | 35   |
| 3.11.1  | Derivation of Minkowski metric | 36   |
| 3.12    | The de Sitter space | 37   |
| 3.12.1  | Mathematical definition | 37   |
| 3.12.2  | Properties | 38   |
| 3.13    | De Sitter group and invariance | 38   |
| 3.14    | The coordinate systems in de Sitter spacetime | 38   |
| 3.14.1  | Static coordinate system | 38   |
| 3.14.2  | Flat coordinate system | 39   |
| 3.14.3  | Open slicing | 39   |
| 3.14.4  | Closed slicing | 40   |
| 3.14.5  | De Sitter slicing | 40   |
| 3.15    | Solutions to Einstein equations from Birkhoff’s view point | 41   |
| 4       | The mathematical operators and Dirac’s six cone formalism | 43   |
| 4.1     | Introducing the transverse projector | 43   |
| 4.2     | Applying the transverse projector on operators in de Sitter spacetime | 45   |
| 4.3     | The differentiation operator in de Sitter spacetime | 46   |
| 4.4     | Transversifying the vectors on de Sitter spacetime | 47   |
| 4.5     | Dirac’s six cone formalism | 47   |
| 5       | The conformally invariant equations for graviton | 50   |
| 5.1     | The conformally invariant system of conformal degree 1 | 50   |
| 5.2     | The effect of Casimir operator on a tensor of second rank | 51   |
| 5.3     | Obtaining the conformally invariant field equation using a conformal system of degree 1 | 52   |
| 5.4     | The conformally invariant system of conformal degree 2 | 53   |
| 5.5     | Obtaining the conformally invariant field equation using a conformal system of degree 2 | 53   |

Conclusion | 54
List of Figures

2.1 a complex number in $x - y$ plane .......................... 14
2.2 illustrating two correspondent curves in complex planes ........ 15
2.3 the equipotential curves for $n = -2$ .......................... 19
2.4 the equipotential curves for $n = -1$ .......................... 20
2.5 the equipotential curves for $n = \frac{1}{2}$ .......................... 20
2.6 the equipotential lines for $n = 1$ .............................. 21
2.7 the equipotential curves for $n = \frac{3}{2}$ .......................... 22
2.8 the equipotential curves for $n = 2$ .............................. 22
Introduction

Our intention in this thesis, pursuing Dirac’s work, is to obtain a conformally invariant equation, to describe graviton. Evidently, the Einstein equation is a peculiar one, which in addition to its simplicity in mathematical form, could well describe important astrophysical observations (at least in their classical limits). Nevertheless, this equation has been confronted some ambiguities; for example when an acceptable explanation of Dark Matter or Dark Energy is demanded. Some physicists believe that by substituting the Einstein equation by an alternative one, or finding a generalized theory of gravitation, this problem can be resolved. Recently some interesting theories have been proposed, like $f(R)$ gravity, brane-world gravity, Lovelock gravity and etc., which all of them can be categorized in modified gravitational models.

Indeed our attempts are supposed to result in Maxwell-like equations for gravitation. As we know, the electromagnetic theory is describing a vector boson, namely the photon. This spin-1 boson which travels with the speed of light, contains no mass. One important trait of electromagnetic equations is the conformal invariance, which is in harmony with the massless-ness of the described boson, namely the photon. The Einstein equation are also describing a massless spin-2 boson, namely the graviton. Hence, the gravitational waves should travel with the speed of light. However, Einstein equations which are supposed to describe the graviton, lack in conformal invariance.

It is worth to note that in this thesis, we consider a tensorial field to describe graviton. What is done in this work is to review the paper in [44]. Our method is the one which has been demonstrated by Dirac, called the Dirac’s six cone.

Historically, Dirac tried to find a conformally invariant equation in Minkowski space. He considered a flat six-dimensional space due to the six dimensions of the conformal group $SO(2, 4)$. Afterwards, he derived the field equations in the so-called space, and projected them on a five-dimensional hyper-surface. Finally he projected them again on the flat four-dimensional Minkowski space.

In this thesis however, after projecting the six-dimensional field equations on the five-dimensional hyper-surface, applying some transverse projector operators, called the induced metric, we will project the five-dimensional equations on a four-dimensional curved de Sitter space. The importance of de Sitter space is because of its ability to being in consistent with the recent astrophysical data (like accelerated expansion of the universe), since it possesses a constant (the cosmological constant),
correlated to the vacuum energy theory. Therefore the final curved background in this thesis, would be the de Sitter spacetime.

Moreover, the transverse projector operators will be introduced as follows:

\[ \theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta. \]

Using this operator, we project all the operators like differentiation operators or Casimir operators, from a flat five-dimensional space, on a four-dimensional curved space. Therefore we shall project the equations, derived by the Dirac’s cone, on the de Sitter space. Although the final equations have the conformal invariance, but they are still incapable to be transformed by the irreducible representations of the conformal group and therefore do not have physical interpretations.
Chapter 1

The Lorentz and the conformal groups, and the concept of invariance

1.1 Group theory

In order to explain symmetries in physics, it is common to use groups, therefore the important properties of the groups, must be defined.

Definition 1 A set of elements \{a, b, c, ...\} form a group, if there exist a linear combination \(a \circ b\) of these elements (called the product), such that the following properties are confirmed for the set:

- the group \(G = \{a, b, c, ...\}\) comprises the product \(ab\).
- the set \(G = \{a, b, c, ...\}\) comprises a unit element \(e\) such that \(ea = ae\).
- for every element of the group, there exist an inverse element, such that \(a^{-1} = aa^{-1} = e\).
- the contribution property:
  \[(ba)c = a(bc)\].

Some peculiar virtues of groups have been listed below:

- A group having the property \(ab = ba\) for all of its elements, is an Abelian group.
- In a continuous group, the elements are functions of one or several continuous variables, e.g. \(G = \{a(t), b(t), ...\}\), in which \(t\) is a continuous variable.
• In any sequence, inside a compact group, there exists an infinite number of partial sequences, converging to an element of the group:

\[ \lim_{n \to -\infty} a_n = a, \quad a \in G. \]

• Two groups \( \{a, b, c, \ldots\} \) and \( \{a', b', c', \ldots\} \), are isometric, if there exists a bijective transformation between their elements (say \( a \leftrightarrow a' \) and \( b \leftrightarrow b' \)) such that \([\mathbb{I}]\):

\[ ab \leftrightarrow a'b'. \]

Now we introduce two important groups.

### 1.1.1 Orthogonal groups

In mathematics, an orthogonal group of degree \( n \) on field \( F \) (notated by \( O(n, F) \)), is a group of \( n \times n \) matrices. The elements of these matrices come from \( F \), and the group action is done via matrix multiplication. This group is a subgroup of the general linear group \( GL(n, F) \), which is defined by:

\[ O(n, F) = \{ Q \in GL(n, F) | Q'Q = QQ' \}. \] (1.1)

The classic orthogonal group on real numbers, is notated by \( O(n) \). The determinant of any orthogonal matrix is \( \pm1 \). The matrices with determinant +1, form a normal subgroup of the orthogonal group, named the special orthogonal group \( SO(n, F) \). Both of these groups are algebraic groups. When the field \( F \), is a set of real numbers, these groups are shown simply by \( O(n) \) and \( SO(n) \). Also these groups, form the compact Lie groups of dimension \( \frac{n(n-1)}{2} \).

These geometric properties are comprised by orthogonal and the special orthogonal real groups:

• The orthogonal group is a subgroup of the Euclidian group \( E(n) \), formed by isometries from \( \mathbb{R}^n \), keeping the origin unchanged. This group is the symmetric group of the hyper-sphere \( s^n \), and all the spherically symmetric geometrical objects, if the origin is kept unchanged.

• \( SO(2, \mathbb{R}) \) is a subgroup of \( E(n) \), consist of direct isometries, i.e. the isometries, keeping the origin and directions unchanged. This group is indeed the rotation group for all spherically symmetric geometrical objects, if the origin is considered at their center.
• \{+I, -I\} is a normal subgroup of \(O(n, \mathbb{R})\), and if \(n\) is an even number, this group also is a subgroup of \(SO(n, \mathbb{R})\). If \(n\) is taken to an odd number, \(O(n, \mathbb{R})\) is the direct product of \(SO(n, \mathbb{R})\) and \{+I, -I\}. The rotation group \(C_k\), formed by \(k\) number of rotations (\(k\) is a correct number), is a normal subgroup of \(O(2, \mathbb{R})\) and \(SO(2, \mathbb{R})\).

Proportional to proper orthogonal basis, the isometries have the following matrix form:

\[
\begin{bmatrix}
R_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
R_k & \cdots & \pm 1 \\
0 & \cdots & \pm 1
\end{bmatrix}
\]

The circular symmetric group is \(O(2, \mathbb{R})\), from which \(SO(2, \mathbb{R})\) as a Lie group, is isomorphic to the sphere \(s^1\). This isomorphism, takes the complex number \(\exp(i\phi) = \cos(i\phi) + i\sin(i\phi)\) to the orthogonal matrix [2]:

\[
\begin{bmatrix}
\cos(\phi) & -\sin(\phi) \\
\sin(\phi) & \cos(\phi)
\end{bmatrix}
\]

1.1.2 Rotation groups

In classical mechanics, the isotropy of space, means the invariance of nature’s rules. Let us consider two vectors \(\vec{r}\) and its rotated version \(\vec{r}'\) in three dimensional space. We can write:

\[\vec{r}' = R\vec{r} .\]

One can state that the three basis of space, namely \(e_1, e_2, e_3\), rotate, and are mapped to \(e_1', e_2', e_3'\), i.e.

\[e_i' = R_{ij}e_j'.\]  \hspace{1cm} (1.2)

The 3 \times 3 rotation matrix, must be real. The inverse transformation can be written as:

\[e_i = U_{ij}e_j'.\]  \hspace{1cm} (1.3)

Therefore we will have:

\[\vec{r}' = x_ie_i,\]

\[\vec{r}' = x_ie_i' = x_i'e_i,\]  \hspace{1cm} (1.4)
or

\[ x_i e'_i = x'_i (R^{-1})_{ij} e'_j, \]  
\[ \text{in which} \]

\[ x_i = R_{ij} x_j. \]  
\[ \text{Equation (1.6)} \]

The rotation matrices exhibit orthogonal properties:

\[ R_{ij} R'_{ij} = \delta_{ii'}, \]
\[ R_{ij} R'_{ij} = \delta_{jj'}. \]  
\[ \text{Equation (1.7)} \]

\[ \det(R_{ij} R'_{ij}) = \det(R_{ij}) \det(R'_{ij}) = \det(\delta_{jj'}) \]
\[ \Rightarrow \quad \det^2(R_{ij}) = 1 \quad \Rightarrow \quad \det(R_{ij}) = \pm 1. \]  
\[ \text{Equation (1.8)} \]

Therefore, the rotation will be separated into two distinct sets. One set is a group which is formed by matrices of determinant +1, and another is the matrices of determinant −1, which do not form a group. Since the rotations are described by three independent parameters, the so-called set, constitutes a continuous three parameter group. This rotation group is notated by \( SO(3) \), showing that there exists an orthogonal group in three dimensions, comprising all 3 \( \times \) 3 matrices of determinant +1 [3].

The three dimensional isometries, keeping the origin unchanged, are listed below:

- The identity \( I \).
- Rotation around an axis, which is crossing the origin.
- Rotation around an axis with an angle other that 180°, which is combined with the reflections from the origin surface.
- An inverse in the origin.
- A reflection from an origin crossing surface.

The fourth and the fifth items, in a special case, and also the sixth item, are known as special rotations [4].

As it was mentioned from mathematical view points, a symmetry is represented by a group. If the symmetric properties of a system is represented by a group \( G \), a function (or some functions) describes the system, possessing a part to represent the symmetry, and another part to represent a real system. For example, if a system
A of differential equations, accepts a symmetry group, it would be so rare to reduce this system to a smaller system $B$ with a definite symmetry. Indeed, the system $B$, is derived from $A$, by eliminating the symmetry $\mathfrak{g}$.

1.2 Invariance under a group action

**Definition 2** Let $G$ be a group of transformations on manifold $\mathcal{M}$. The set $S \subset \mathcal{M}$ is called $G$-invariant, and $G$ is called a symmetry group of $S$, if for a defined $g.p$ ($p \in S$ and $g \in \mathcal{M}$) we have $g.p \in S$.

**Definition 3** Let $G$ be a group of transformations on manifold $\mathcal{M}$. The map $F : \mathcal{M} \rightarrow N$ in which $N$ is another manifold, is called a $G$-invariant map, if for all $p \in \mathcal{M}$ and $g \in N$ which $g.p$ has been defined, $F(g.p) = F(p)$. A $G$-invariant real valued function, is simply called invariant.

**Theorem 1** Let $G$ be a group of transformations on manifold $\mathcal{M}$. A real valued function $f : \mathcal{M} \rightarrow \mathbb{R}$ is $G$-invariant, if and only if for all $p \in \mathcal{M}$ and generators $\xi \in \mathbb{R}$ we have $\xi$:

$$\xi_{\mathcal{M}} |_p (f) = 0.$$

1.2.1 Invariance of the laws of physics

In 1872, Felix Klein proposed *Erlangen Program für Geometry*, based on the group of symmetric transformations. In 1952, Fantappiè, having the same idea and using expressions with a relativistic tendency, proposed *Erlangen Program für Physik*, representing a distinct worlds by a symmetric group, keeping the laws of physics, invariant. It must be note that this world, is indeed a physical system which is defined by a symmetric group.

Isotropy of spacetime and its homogeneity with respect to laws of physics, assert that these laws are based on symmetry. Therefore it is believed that the laws of physics can be separated using some symmetric groups, which leave them invariant. To do this, one can use two groups, describing two different physical worlds. The Galilei group and the Lorentz-Poincarè group.

Note that the Galilei group is an special case of Lorentz group in $c \rightarrow \infty$. Mathematically the Lorentz group can be expressed as a rotation-translation group, in a way that a geometrical object, say Minkowski spacetime, is left invariant.
1.3 The Lorentz group

The three dimensional rotations in classical and quantum mechanics, can be investigated using the group of transformations, which keep the measurements constant. In special theory of relativity, the Lorentz transformations of the 4-dimensional coordinates \((x_0, \vec{x})\), obeys the following invariance:

\[
s^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2.
\]

Hence, the cinematics of special relativity, can be reexpressed by the group of transformations, which keep \(s^2\) invariant. This group is the homogenous Lorentz group, consisting of normal rotations, as well as the Lorentz group in Minkowski spacetime. The group of transformations which create the invariance

\[
s^2(x, y) = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2,
\]

are called the inhomogenous Lorentz group or the Poincarè group. This group consists of translations and reflections in spacetime.

The Lorentz group is a subgroup of the Poincarè group, composed of all the isometries, which keep the origin unchanged. Mathematically, the Lorentz group is the generalized orthogonal group \(O(1, 3)\). This group is a 6-dimensional un-compact Lie group, and one of its subgroups is \(O(3)\) \[10\] \[11\].

1.4 The conformal group

The isometric orthogonal transformations (preserving the distances), also keep the angles, and consequently are conformal maps. Nevertheless all the linear conformal transformations, are not orthogonal. In the next chapter, these maps will be discussed explicitly.

The group of conformal mappings in \(\mathbb{R}^n\), is denoted by \(CO(n)\), consisting of the product of the orthogonal group, by the dilation group. If \(n\) is an odd number, these two groups do not intersect, and indeed are the following product:

\[
CO(2n + 1) = O(2n + 1) \times \mathbb{R}.
\]

Whereas if \(n\) is an even number, these two groups will intersect and therefore \([1.10]\) is not a direct product. Instead it will be a product by the subgroup of the dilation group with a positive scalar, i.e.

\[
CO(2n) = O(2n) \times \mathbb{R}^+.
\]
Also one can define \[1\]:

\[
CSO(2n) := CO(n) \cap LG_+(n) = SO(n) \times \mathbb{R}^+.
\] (1.12)

### 1.4.1 Conformal symmetry

It is well-known that the conformal group is more powerful in two dimensions than it in higher ones. In two dimensions, this group possesses infinite dimensions. When the space has an Euclidian metric, any function obeying the Cauchy-Riemann equations, will be locally conformally invariant. If we consider the definition of the conformal symmetries, same as the way that general relativity keeps the light cone invariant, we find out that any two-dimensional metric can be applied in this form. We can find a transformation, to transform the so-called metric, to the same metric, multiplied by a function. This property is not valid for higher dimensions.

One of the definitions for the conformal symmetry, comes through the group theory. This definition coincides the previous one, when the space possesses more than two dimensions. To realize the conformal symmetry, we ought to understand the mathematical field operators, which are satisfying the Lagrange equations, which are used to define the conformal group. These fields can be defined on a Minkowskian manifold \(x_\mu\), or on a six-dimensional imaginary manifold \(\eta_A\), correlated to \(x_\mu\).

The conformal group of spacetime, is organized of coordinate transformations, as listed below:

- dilation

\[
x_\mu = x_\rho \mu \quad \text{where} \quad \rho > 0,
\]

- special conformal transformations

\[
\sigma(x) = 1 - 2cx + c^2x^2,
\]

\[
x'_\mu = \sigma^{-1}(x)(x_\mu - C_\mu x^2),
\]

- inhomogenous Lorentz transformations.

Here, we notate the conformal group by \(SO(d, 2)\), in which \(d\) is dimensions of spacetime. This transformation can be expressed as the complete Poincaré group, which is a re-scaling transformation. Also \(\mu\) is a constant, and the special conformal transformation would be:

\[
x^\mu \rightarrow \frac{x^\mu - b^\mu x^\mu}{1 - 2bx + xb^2x^2}.
\] (1.13)
This group has \(\frac{(d+2)(d+1)}{2}\) parameters and

\[ x^2 = x_\mu x^\mu = b_\mu x^\mu. \]  

These special conformal transformations, are generated by \(R_\mu = P_\mu + IP_\mu I\), where \(P_\mu\) is the translation generator. The relation (1.13), takes \(x\) to \(\frac{-x}{x^2}\) [5, 6].

The conformal group \(SO(d, 2)\) is generated by the Killing equations; like the group of motions, for a five-dimensional pseudo-sphere:

\[ |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 = |\psi_4|^2 = 1 \]

where \(\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{C}^4\),

(1.15)

which has been solved accurately in spherical coordinates. Therefore, the fifteen natural generators of the conformal group, can be figured out, in the natural coordinates of this space. It has been shown that the square operator \(P_\mu P^\mu\), would not be diagonal, when it is used to express the conformal group in the Hilbert space of the harmonic functions on a five-dimensional pseudo-sphere. Hence, to define the conformal group in the Hilbert space of the harmonic functions on a projective five-dimensional space, we write:

\[ \psi^\dagger \beta \psi = 0, \]

and \(\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}\),

(1.16)

contained in the complex space \(\mathbb{C}^4\), and \(\psi\) is a vector. With

\[ \psi = (\psi_1, \psi_2, \psi_3, \psi_4) \]

and \n\[
|\psi_4|^2 = 1,
\]

and the substitutions:

\[ \psi_1 = x^1 + xi^2, \quad \psi_2 = x^3 + xi^6, \quad \psi_3 = x^4 + xi^5, \]

we find out that \(SO\) can be written as follows:

\[ g_\alpha\beta x^\alpha x^\beta - (x^5)^2 + (x^6)^2 = 1, \]  

(1.17)
in which \( g_{\alpha \beta} \) is the Minkowski metric, conventionally having the signs \(+ + + -\).

Therefore \( SO \) exhibits the following fundamental form, in a six-dimensional space:

\[
ds^2 = g_{\alpha \beta} x^\alpha x^\beta - (x^5)^2 + (x^6)^2.
\]

(1.18)

It is known that the conformal group can be described as the group of motions, in a five-dimensional pseudo-spherical space. The most natural generators of this group, can be derived by solving the Killing equations, for one dimensional motions in spherical coordinates. Also the generators of Poincarè group in this coordinates, are created by inserting the Minkowsi space in \( SO \) \(^7\).

1.5 Invariance under Lorentz group (Lorentz invariance)

The Lorentz invariance, is an expression in physics, due to the properties of spacetime, which implies that:

\[
\text{In two different frames, which are having relative motions and observing a same event, the non-gravitational laws of physics, must have the same predictions about that event.}
\]

A physical quantity is called Lorentz covariant, if it is transformed under a given Lorentz representation. Such quantities, if left invariant under the so-called transformations, would be Lorentz invariant.

Due to the representation theory of Lorentz group, the Lorentz covariant quantities, consist of scalars, 4-vectors, 4-tensors and spinors. The spacetime interval, is a Lorentz invariant quantity, just like the Minkowski norm for any 4-vector.

The correct equations in any inertial frame, are also Lorentz covariant, which can be written due to the Lorenz covariant quantities. According to the principal of relativity, all fundamental laws of physics, must be Lorentz covariant.

Just think for a moment that the theory of relativity, would have been derived form the following rational statement:

\[
The \text{newtonian physics, is invariant under rotation } L_i = \epsilon_{ijk} x_j p_k \text{ and the Galilean transformations } \tilde{p}_i = t \tilde{p}_i.
\]
The mentioned generators, obey the following relations:

\[
[ L_i, L_j ] = \epsilon_{ijk} L_k, \\
[ L_i, g_j ] = \epsilon_{ijk} g_k, \\
[ g_i, g_j ] = 0.
\] (1.19)

To investigate these relations, it is easier to add a term proportional to \( \frac{1}{c^2} \epsilon_{ijk} L_k \), to the left hand side of the last equation. The coefficient \( c \) has the velocity dimensions. In fact, \( \vec{L} \) is dimensionless and \( g \) has the inverse velocity dimension. Afterwards one can construct a theory, to be invariant with respect to the new group of motions.

The different of this group from Newtonian mechanics, arises only in speeds closed to \( c \) [12, 22].

### 1.6 Conformal invariance

Conformal invariance, has been proposed correlated to the scale invariance, since the last century. For example, the vacuum Maxwell equations are both scale and conformal invariant. This concept in quantum field theory, is illustrated by a local energy-momentum tensor.

However, the applications of conformal invariance, have been recognized in 1970. Till then it seemed that the consequences of conformal invariance in spaces with arbitrary dimensions, were not interesting. This will be quite different, when a two-dimensional space is considered, since for two dimensions, the conformal group possesses infinite dimensions.

In a word, conformal invariance, is a logical expansion of scale invariance. The scale invariance, is dependent to the invariance of our system, under the changes in homogenous length scales. However, conformal invariance allows inhomogenous and local changes in scales and the only need here is the invariance of angles. This expansion, is indeed coherent, since it can be shown that an invariant system under translations and rotations, at least at continuity limits, would be scale invariant and exposes short range effects. Therefore, the conformal symmetry is spontaneously generated.

For two-dimensional systems, conformal symmetry notably has improved our knowledge about nature. Whereas the scale invariance can only put our system in large parts, dependent to properties like symmetry, space dimensions and number of order parameters, the conformal symmetry proposes and assortment of partition functions.
Currently, researches on theoretical physics, show a considerable focus on conformal invariance. However to use the technical tools to understand it, one should have professional skills; lots of initial concepts in string theory, require adequate knowledge of quantum theory of fields. On the other hand we believe that it is time to apply some methods in physics, based on conformal invariance. While conformal invariance constitutes the basis of higher dimensional string theory, it provides the possibility to study the fields on oscillating metrics. This studies are of great importance in two-dimensional quantum gravity [21].
Chapter 2

Conformal transformations

2.1 Conformal mapping

Conformal mapping, or conformal transformation or the angle preserving transformation, is indeed the transformation \( w = f(z) \), where \( z \) would be the complex number \( z = x + iy \). Here \( r = |z| \) is the amplitude and \( \theta = \arg \) is the phase. \( x \) and \( y \) are the cartesian coordinates (see figure (2.1)).

![Figure 2.1: a complex number in x − y plane](image)

Axiom 1 A function \( f(z) \) is analytic at \( z_0 \), if it is differentiable at \( z = z_0 \) and in its neighborhood.

As long as \( f(z) \) is analytic, we have:

\[
\frac{df}{dz} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{2.1}
\]
Considering the polar version of this equation, it is possible to equate the amplitudes and the phases with each other. For the phases we can write:

\[
\arg \lim_{{\Delta z \to 0}} \frac{\Delta w}{\Delta z} = \lim_{{\Delta z \to 0}} \arg \frac{\Delta w}{\Delta z} = \arg \frac{\Delta w}{\Delta z} = \arg \frac{dw}{dz} = \alpha, \tag{2.2}
\]

in which \( \alpha \) is the argument of the derivative, for a constant definite \( z \), and is independent of the direction from which we have approached \( z \). To declare the importance of relation (2.1), consider two curves, one \( C_z \) in the \( z \) plane, and another curve \( C_w \), corresponding to \( C_z \) in the \( w \) plane. As it is illustrated in figure (2.2), the evolvement \( \Delta z \), makes an angle \( \theta \) by the real axis \( x \) and corresponding to it, the evolvement \( \Delta w \), makes an angle \( \phi \) by the real axis \( u \). From (2.1) we conclude

\[
\phi = \theta + \alpha. \tag{2.3}
\]

In other words, as long as \( w \) is analytic and its differentiations are nonzero, any line in the \( z \) plane, will be rotated by an angle \( \alpha \) in the \( w \) plane. This conclusion is valid for any \( z_0 \)-crossing line. Therefore we can apply it for two lines as well. The angle between these two lines would be:

\[
\phi_2 - \phi_1 = (\theta_2 + \alpha) - (\theta_1 + \alpha) = \theta_2 - \theta_1. \tag{2.4}
\]

This relation asserts that the angle between two lines, will be preserved by analytic transformations. Such transformations, which keep the angles, are called conformal transformations. The rotation angle \( \alpha \), and also \( |f(z')| \), are usually functions of \( z \).
Axiom 2  An analytic function, is conformal everywhere it possesses nonzero differentiations. In reverse, any conformal mapping of a complex variable, having continuous partial differentiations, would be analytic.

As well as lots of other regions in physics, conformal mappings are of great importance in complex analysis.

Definition 4 A map which conserves the angles, but changes the directions, is an isogonal map.

Axiom 3 A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is conformal, if and only if there exist two complex numbers $a \neq 0$ and $b$, such that

$$f(z) = az + b.$$  

Conformal transformations are rather profitable in solving physics problems. Taking $w = f(z)$, the real and complex parts of $w(z)$, must satisfy the Cauchy-Riemann and Laplace equations. Assume $u(x, y)$ and $v(x, y)$ are respectively the real and the imaginary parts of $f(z)$. Then:

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v \quad \text{the Cauchy-Riemann condition},$$

$$\partial_x^2 u + \partial_y^2 u = 0, \quad \partial_x^2 v + \partial_y^2 v = 0 \quad \text{the two-dimensional Laplace equation}.$$  

Hence, spontaneously a scalar potential is provided. For a three-dimensional electrostatic potential we have:

$$\partial_x^2 \varphi + \partial_y^2 \varphi + \partial_z^2 \varphi = 0,$$

which is reliable in a space without any charge density. Generally, any function which satisfies the Laplace equation, is called a harmonic function. Therefore the above electrostatic potential, is a three-dimensional harmonic function.

As stated above, the real and imaginary parts of a complex function are harmonic, therefore complex analysis methods become sometimes advantageous in solving electrostatic problems. Since $\varphi$ satisfies the Laplace equation, one can consider it as a part of the analytic function $w(z)$, which here we call it a complex potential. To obtain the potential for a sequence of charges, firstly we must derive $w(z)$ for a line of charges, which embarks on moving when it is at the origin. If this line is located at $z_0 = x_0 + iy_0$, then it can be shown that:

$$w(z) = 2\lambda \ln(z - z_0),$$
where $\lambda$ is the linear charge density. And in general for a sequence of charges we have:

$$w(z) = 2 \sum_{k=1}^{n} \lambda_k \ln(z - z_k). \quad (2.5)$$

The function $w(z)$ can be used to solve electrostatic problems, with simple charge distributions.

Instead of discussing $w(z)$, let us consider a map from the $z$ plane (or $xy$ plane) to the $w$ plane (or $uv$ plane). In special case, the curves with same potential, are mapped to lines parallel to $z$ axis in the $w$ plane, since these curves are defined to be $u$-constant. Also the $v$-constant curves are mapped to horizontal lines in $w$ plane. This would be a big simplification in geometry:

*The straight lines, specially when they are parallel to the axis, are far simpler than circles and are easier to be analyzed than other geometrical objects. Specially when their centers do not coincide to the origin.*

Therefore let us introduce two different complex worlds; one the $x - y$ plane and the other, which is primed, would be the $x' - y'$ plane. Assume that we are in the $z$ coordinate system$^1$ and we wish to obtain a quantity like the electrostatic potential $\varphi$. If solving this problem was hard in $z$, we could just translate it to $z'$ to simplify the resolving process. To do this, we solve the problem with respect to $(x', y')$ and then, we translate it again to $z$. If any physical problem exists for which this solution is valid, then the problem is solved. Therefore pursuing an inverse procedure, we find a solution, for which if we wanted to obtain straightforwardly, we had could have confronted a sophisticated work. Hence, mappings that relate $z$ and $z'$, should be selected accurately.

We ought to consider two necessary conditions:

- **First Condition:** The differential equation which is describing the physics of our problem, must be simpler than a translation in $z'$. Since the Laplace equation is among the simplest ones, the $z'$ world should obey the Laplace equation.$^1$

$^1$Here the $x - y$ plane is denoted by $z$, and $z'$ is used instead of $w$ and $(x', y')$ instead of $(u, v)$.
• Second Condition: The applied mapping must conserve the angles. This condition is of great importance, since the equipotential curves and the filed lines, are supposed to be perpendicular to both $z$ and $z'$ worlds.

Briefly speaking, our map must be conformal.

**Theorem 2** Consider $\gamma_1$ and $\gamma_2$ be two curves in the complex plane $z$, making an angle $\alpha$ at point $z_0$. Assume that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a map with relation $f(z) = z' = x' + iy'$, which is analytical at $z_0$. Also consider $\gamma'_1$ and $\gamma'_2$ be transformed curves by $f$, making the angle $\alpha'$ with each other. Hence, we will have:

- **a)** $\alpha' = \alpha$, or the map is conformal if 
  \[ \frac{df}{dz}|_{z=z_0} \neq 0, \]
- **b)** if $f$ is harmonic at $(x, y)$, therefore it will be harmonic at $(x', y')$.

Some examples of conformal mapping are listed below:

- $z' = z + a$, where $a$ is an arbitrary constant. This simply can be regarded as a translation from the $z$ axis.
- $z' = bz$
- $z' = \frac{1}{z}$

• combining the formerly mentioned transformations, we obtain the following map:
  \[ z' = az + b \]
  which will be conformal if 
  \[ \frac{dz'}{dz} \neq 0 \neq cz + d. \]
  Also when $az + b \neq 0$.

  Such mappings are often called *homographic* transformations. One of the advantages of these transformations is that they can map an infinite domain from the $z$ plane, to a finite domain in $z'$ plane. In fact using them we can transform high valued points to a neighborhood of point $z' = \frac{a}{z}$.

**Example:** A physical application [17]
Consider the following potential:

\[ w(z) = Az^n = Ar^n e^{in\theta}. \]

The real and the imaginary parts are respectively

\[ \phi = Ar^n \cos(n\theta), \]
\[ \psi = Ar^n \sin(n\theta). \]

For \( n = -2 \): The equipotential curves are shown in figure (2.3) and the two parts of

\[ f(z) = 1/z^2 \]

the potential are derived as:

\[ \phi = \frac{A}{r^2} \cos(2\theta), \]
\[ \psi = -\frac{A}{r^2} \sin(2\theta). \]

For \( n = -1 \): The equipotential curves are shown in figure (2.4) and the two parts of

the potential are derived as:

\[ \phi = \frac{A}{r} \cos(\theta), \]
\[ \psi = -\frac{A}{r} \sin(\theta). \]

This solution is composed of two circular systems, and \( \phi \) is the potential function on two parallel lines of charge with opposite signs.
\[ f(z) = 1/z \]

Figure 2.4: the equipotential curves for \( n = -1 \)

\[ f(z) = z^{1/2} \]

Figure 2.5: the equipotential curves for \( n = \frac{1}{2} \)
For $n = \frac{1}{2}$: The equipotential curves are shown in figure (2.5) and the two parts of the potential are derived as:

$$\phi = Ar^{\frac{1}{2}}\cos\left(\frac{\theta}{2}\right) = A\sqrt{\frac{(x^2 + y^2)^{\frac{1}{2}} + x}{2}},$$

$$\psi = Ar^{\frac{1}{2}}\sin\left(\frac{\theta}{2}\right) = A\sqrt{\frac{(x^2 + y^2)^{\frac{1}{2}} - x}{2}},$$

from which, $\phi$ gives the field near the edge of a thin layer.

For $n = 1$: The equipotential curves are shown in figure (2.6) and the two parts

$$f(z) = z$$

of the potential are derived as:

$$\phi = Ar \cos(\theta),$$

$$\psi = Ar \sin(\theta),$$

which are two straight lines.

For $n = \frac{3}{2}$: $\phi$ gives the field near and outside of a corner, made of perpendicular plates. The equipotential curves are shown in figure (2.7). Also the potential is given by:

$$w = Ar^{\frac{3}{2}} e^{\frac{3}{2}i\theta}.$$

For $n = 2$: The potential and its real and imaginary parts are:
\[ f(z) = z^{3/2} \]

Figure 2.7: the equipotential curves for \( n = \frac{3}{2} \)

\[ f(z) = z^2 \]

Figure 2.8: the equipotential curves for \( n = 2 \)
\[ w = A(x + iy)^2 = A[(x^2 - y^2) + 2ixy], \]
\[ \phi = A(x^2 - y^2) = Ar^2 \cos(2\theta), \]
\[ \psi = 2Axy = Ar^2 \sin(2\theta), \]

The equipotential curves are shown in figure (2.8), which are two perpendicular hyperboloids, and \( \phi \) would be the potential at the vicinity of the middle point, between two separated charges. Also \( \phi \) can be regarded as the potential on the open part of a rectangular conductor [9, 16, 23].

### 2.2 Conformal covariance and conformal invariance, and the scalar wave equations

A new formulation of scalar field equations is given by Klein-Gordon equation, when there exist a metric tensor \( g^{ij}(x) \) and a scalar field \( C(x) \). This equation is covariant with respect to the gauge transformations

\[ A(x) \rightarrow A_i(x) + \nabla_i V(x), \]  
(2.6)

and the conformal transformations of the tensor field, i.e.

\[ g^{ij} \rightarrow \exp[\theta(x)]g^{ij}, \]  
(2.7)

in which \( V(x) \) and \( \theta(x) \) are arbitrary functions of \( x = x(x^1, x^2, ..., x^n) \). In this case, the square of the remnant mass \( m^2(x) \) is defined is a function of given fields, in the form:

\[ m^2(x) = c - \frac{n-2}{4(n-1)} R - \frac{1}{4} A^i A_i - \frac{1}{2} \nabla_i A^i, \]  
(2.8)

which is transformed as follows:

\[ m^2(x) \rightarrow \exp[-\theta(x)]m^2. \]  
(2.9)

In equation (2.8), \( R \) is the scalar curvature, and \( \nabla_i \) is the covariant derivative in the Reimannian space \( V_n \). Also for the metric tensor \( g_{ij}(x) \) (the inverse of \( g^{ij}(x) \)) we have \( \bar{A}(x) \rightarrow g_{ij}A^j \).

Considering a class of wave equations with constant mass, it has been shown that we shall confront the transformation matrices \( \bar{g}^{ij}(x) \rightarrow m^2(x)g_{ij}(x) \), depending directly on the given tensor and scalar and vector fields. For the vector field potential
\[ \nabla_i A_j = \nabla_j A_i, \] we have equation (2.10) which is the equation of motion of a free particle in Riemannian space \( V_n \).

\[
g^{ij} \nabla_i \nabla_j \phi + \frac{n-2}{4(n-1)} R \phi + m_0^2 \phi = 0. \tag{2.10}
\]

This equation exposes the complete characteristics of a group, and is comparable to the equation

\[
g^{ij} \nabla_i \nabla_j \phi + m_0^2 \phi = 0, \tag{2.11}
\]

which is considerable in quantum mechanical expansions and the quantum theory of fields, where the space curvature is assumed to be nonzero \[18\].

### 2.3 Conformal transformations, and conformal invariance in gravitation

The conformal transformations are used frequently in studying the relations between diversified theories of gravitation, and Einstein relativity. Therefore inevitably, it is important to consider the instructions of conformal transformations for geometrical quantities in general relativity.

In special case, we concern about the conformal transformations of the energy-momentum tensor, and as a must, we should proceed with the delicate and exact principal of conservation (or the Bianchi identity) in one of the conformal systems, as an origin for the others. The importance of this principal, goes back to this fact that the conformal transformations create a matter term, composed of the conformal factor, and insert it into the conservation principal.

In the outstanding case of flat spacetime, this matter is created from the work, done by the conformal transformations to curve the spacetime.

We also have to note the structure of conformal gravity. In this case which is the simplest one, we should concern about the Brans-Dicke theory, taking its parameter to be \( \frac{3}{2} \).

The conformal transformations are applied to investigate gravitation in higher dimensional theories. In this way, we obtain the laws of conformal transformations for scalar invariants, namely \( R^2 \), \( R_{ab} R^{ab} \) and \( R_{abcd} R^{abcd} \), and consequently for the Gauss-Bonnet invariant in arbitrary dimensions.

Conformal transformations of a metric tensor, exhibits some interesting characteristics of gravitational theories, which are based on scalars. Here the point is that
we can demonstrate these theories, in two different systems with conformal relations:

1) The Jordan system, in which the scalar field is coupled by the metric tensor, non-minimally.

2) The Einstein system, where this field has a minimum coupling with the metric tensor.

Note that the scalar based gravitational theories, are the lower limits of the superstring theory. It has been shown that some physical processes like expansion of the universe or the perturbations in density, are seemed to be different in related conformal systems. Therefore, it seems to be crucial to investigate these transformations, through diversified gravitational theories.

Lots of studies have been devoted to the problem of changes in geometrical quantities under conformal transformations. However the transformation laws, have not always been clarified and some simple traits of them have not been investigated in details. Therefore, we should discuss these laws explicitly, and also concern about conformal transformations in higher-dimensional curvatures.

2.4 Conformal transformations in Einstein gravity

Let us consider the spacetime \((\mu, g_{ab})\), where \(\mu\) is a smooth manifold and \(g_{ab}\) is the Lorentz metric on \(\mu\). The conformal transformation

\[
\bar{g}_{ab}(x) = \Omega^2 g_{ab}(x),
\]

in which \(\Omega\) is an uniform and nonzero function of spacetime. The transformation in (2.12) is a congruence of the metric, independent of points, and is called the conformal factor. This factor relies in the interval \(0 < \Omega < \infty\).

The conformal transformations, increase or decrease the distances between two definite points, in the same coordinate systems \(x^a\) on manifold \(\mu\), but leave the angles between the vectors, unchanged. This leads to preservation of manifold’s structure. Assuming a constant \(\Omega\), we will have a congruent transformation. In fact, conformal transformations are local congruences. We have:

\[
\Omega = \Omega(x).
\]
On the other hand, the coordinate transformations like $x^a \rightarrow \bar{x}^a$, only change the coordinate systems, not the geometry. Consequently, such transformations differ from conformal transformations. This is an accurate fact, since conformal transformations, lead to new physical conditions. Whereas this concept, relates to different couplings between physical fields and gravitation, we investigate different systems, in which the physics of our problem is studied.

In a $D$-dimensional spacetime, the determinant of metric, i.e. $g = \det(g_{ab})$ is transformed in follows:

$$\sqrt{-\bar{g}_{ab}} = \sqrt{-g_{ab}} \Omega^D. \quad (2.13)$$

From (2.13) we have:

$$\bar{g}^{ab} = \Omega^{-2} g^{ab}, \quad (2.14)$$

and

$$\bar{ds}^2 = \Omega^2 ds^2. \quad (2.15)$$

Finally, the concept of conformal flatness will be:

$$\bar{g}_{ab} \Omega^{-2}(x) = \eta_{ab}, \quad (2.16)$$

where $\eta_{ab}$ is the Minkowski metric. Using conformal transformations of the metric, we are able to calculate such transformations of the Einstein tensor, after obtaining the Christoffel symbols and Riemann and Ricci tensors, in $D$-dimensional spacetime. We have:

$$\bar{G}_{ab} = G_{ab} + \frac{D - 2}{2\Omega^2} [4\Omega_{,a} \Omega_{,b} + (D + 5)\Omega_{,c} \Omega^{c} g_{ab}] - \frac{D - 2}{\Omega} [\Omega_{,ab} - g_{ab} \Box \Omega]. \quad (2.17)$$

In this equation, the D’alembertian $\Box$ acts with respect to $g_{ab}$. An important feature of conformal transformation is that they preserve the Weyl curvature tensor

$$C_{abcd} = R_{abcd} + \frac{2}{D - 2} [g_{a[d} R_{c]b} + g_{b[c} R_{d]a}] + \frac{2}{(D - 2)(D - 1)} R g_{a[c} g_{d]} b. \quad (2.18)$$

This means that from (2.12) we conclude

$$\bar{C}_{abcd} = C_{abcd}. \quad (2.19)$$

Using (2.19) together with relations (2.12) and (2.14), one can simply derive the Weyl Lagrangian. This Lagrangian is also invariant under conformal transformations, i.e.

$$\bar{L}_W = -\alpha(\bar{g})^{\frac{1}{2}} \bar{C}^{abcd} \bar{g}_{abcd} = -\alpha(\bar{g})^{\frac{1}{2}} C^{abcd} C_{abcd} = L_W. \quad (2.20)$$
2.5 Explaining some axioms using conformal transformations

**Axiom 4** Assume $\mathcal{M}$ be a complete Einstein manifold on which there exists a vector field, generating a 1-parameter group of conformal transformations. Thereupon $\mathcal{M}$ is isometric to a continuous space with positive curvature. In spacial case, $\mathcal{M}$ is homeomorphic to the sphere $S^n$.

On the other hand, a pseudo-circular transformation from Riemannian manifold $\mathcal{M}$ with metric $g_{\mu\lambda}$, to the Riamannian manifold $\mathcal{M}'$ with metric $g'_{\mu\lambda}$, is a conformal transformation, i.e.

$$g'_{\mu\lambda} = \rho^2 g_{\mu\lambda}. \quad (2.21)$$

This relation, takes the circular geodesics in $\mathcal{M}$ to circular geodesics in $\mathcal{M}'$. This process can be denoted by the following equation:

$$\nabla_{\mu} \rho_{\lambda} - \rho_{\mu} \rho_{\lambda} = \psi g_{\mu\lambda}, \quad (2.22)$$

where $\rho$ and $\psi$ are real-valued functions on $\mathcal{M}$. Also

$$\rho_{\lambda} = \nabla_{\lambda} \log \rho. \quad (2.23)$$

**Axiom 5** Assume $\mathcal{M}$ and $\mathcal{M}'$ be two Riemannian manifolds with constant curvatures $k$ and $k'$. Consider $\mathcal{M}'$ to be complete and also there exists a pseudo-circular transformation from $\mathcal{M}$ to $\mathcal{M}'$. Therefore $\mathcal{M}$

- is the Euclidean space if $k = 0$,
- is a spherical space if $k > 0$,
- is a hyperbolic space if $k < 0$.

**Axiom 6** In addition to the assumptions in axiom 5, assume that, is also complete and the pseudo-circular transformation, is a homeomorphic transformation from $\mathcal{M}$ to $\mathcal{M}'$. Therefore the constant curvatures $k$ and $k'$ must be positive and $\mathcal{M}$ and $\mathcal{M}'$ are spherical spaces.

We should note that:

**Theorem 3** A conformal transformation, which transforms an Einstein manifold to another one, is a pseudo-circular transformation.
**Theorem 4** If a complete Einstein manifold $\mathcal{M}$ is transformed conformally to another Einstein manifold $\mathcal{M}'$, therefore $\mathcal{M}$

- is the Euclidean space if $k = 0$,
- is a spherical space if $k > 0$,
- is a hyperbolic space if $k < 0$.

**Theorem 5** If a complete Einstein manifold $\mathcal{M}$ accepts its conformal transformation to itself, thereupon $\mathcal{M}$ would be the spherical space $[20]$. 

Chapter 3

Einstein field equations

3.1 Introduction

The Einstein field equations are a set of 10 equations in general theory of relativity, which explain the gravitational reactions, as a consequence of the spacetime curvature under the influence of mass or energy [25].

Einstein equation was first proposed by Einstein in 1915, as a tensorial equation. This equation equates the spacetime curvature (which is described by Einstein tensor) to the energy (which is described by energy-momentum tensor) [26].

Just like the procedure in which the electromagnetic fields are determined by Maxwell’s equations, using the charges and currents, Einstein field equations are applied to determine the geometry of spacetime, as a consequence of the presence of energy, mass and linear momentum. This means that these equations, give the metric for a definite form of energy in spacetime.

The correlations between the metric tensor and Einstein tensor, demonstrates the field equations, as a set of partial differential equations. The solutions to these equations, would be the metric tensor components. Also the trajectories due to particle motions, can be derived from the consequent geometry. Since Einstein field equations are locally obeying energy-momentum conservation, these equations reduce to Newtonian field equations, in weak field limits.

3.2 The mathematical form

Einstein field equation can be written as follows [25]:

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (3.1) \]
where \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) the scalar curvature, \( g_{\mu\nu} \) the metric tensor, \( \Lambda \) the cosmological constant, \( G \) the gravitational constant, \( T_{\mu\nu} \) the energy-momentum tensor and \( c \) is the speed of light.

Einstein field equation, is a tensorial equation which relates a set of \( 4 \times 4 \) tensors. Each tensor possesses 10 independent components. Considering our freedom in choosing the spacetime coordinates, the number of independent equations decreases to 6.

Albeit Einstein equations had been initially based on four-dimensional spacetime, some theorists have expanded the results in \( n \) dimensions. These equations, which are technically regarded to be out of the region of general relativity, are still referred to Einstein equations. The vacuum field equations, define the concept of Einstein manifold.

Despite the simple form, Einstein equations are indeed rather complicated. For an ordinary distribution of matter and energy, Einstein field equations become a set of equations for the metric tensor \( g_{\mu\nu} \), on which the Ricci tensor and the Ricci scalar, will non-linearly depend.

In fact, Einstein field equations, when fully written, are combined of ten coupled non-linear hyperbolic-elliptical differential equations.

Introducing the Einstein tensor
\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,
\]
where
\[
\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu},
\]
which is a symmetric tensor of rank two, one can write the equations in the following compact form:
\[
G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]
where the cosmological term has been moved into the energy-momentum tensor, pertaining the Dark energy concept. In geometrical units \( G = c = 1 \), this equation reduces to
\[
G_{\mu\nu} = 8\pi T_{\mu\nu},
\]
in which the left hand side (lhs), indicates the spacetime curvature, caused by the metric tensor. The right hand side (rhs) stands for the matter/energy contents of spacetime. Einstein field equations can be expressed as a set of equations, demonstrating how the spacetime will curve, related to the matter/energy constituting the universe.

These equations, beside the geodesic equation, are the foundations of general theory of relativity.
3.3 The equivalent form

Einstein field equations can also be written in the following equivalent form:

\[ R_{\mu\nu} - g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}), \]

which becomes useful when we are interested in weak-field limits, where \( g_{\mu\nu} \) can be substituted by the Minkowski metric, with an acceptable accuracy.

3.4 The cosmological constant

Historically, Einstein had included a cosmological constant in his equation, relating to the metric.

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi T_{\mu\nu}. \quad (3.5) \]

Since \( \Lambda \) is a constant, the conservation of energy is still valid. The cosmological term, initially introduced by Einstein to describe a static universe (a universe with no expansion or contraction). This attempt was unsuccessful because of two reasons; first, the so-called static universe was unstable, and second, the Hubble’s observations of distant galaxies, confirmed that our universe is not static, and is indeed expanding. Therefore since Einstein had called \( \Lambda \) as his biggest blunder, it was taken to be zero, during the next decades [27].

Despite this misleading for introducing the cosmological constant, recent observational instruments, have assigned a definite value for it which is confirmed by some observational data [28, 29].

Einstein have considered his cosmological constant, to be an independent variable. However, the corresponding term in the field equation (3.1) can be algebraically moved to the other side, to become a part of the energy-momentum tensor.

\[ T^{(\text{vac})}_{\mu\nu} = -\frac{\Lambda c^4}{8\pi G} g_{\mu\nu}. \quad (3.6) \]

The vacuum energy is constant and is given by

\[ \rho^{(\text{vac})} = \frac{\Lambda c^2}{8\pi G}. \quad (3.7) \]

Therefore the existence of the cosmological constant, is equivalent to the existence of a nonzero vacuum energy. These terms are currently being used in general relativity.
3.5 Energy-momentum conservation

In general relativity, the energy-momentum conservation law is expressed in the following form:

$$\nabla_b T^{ab} = T^b_{;b} = 0.$$  \hspace{1cm} (3.8)

This relation can be derived from Bianchi identity, which can be summarized as follows:

$$R_{ab[cd;e]} = 0,$$  \hspace{1cm} (3.9)

for which, a multiplication by $g_{ab}$, knowing that the metric tensor is a covariant constant, gives:

$$R^c_{bed;e} + R^c_{bec;d} + R^c_{bde;c} = 0.$$  \hspace{1cm} (3.10)

The antisymmetric property of the Riemann tensor, provides this opportunity to rewrite the second term of (3.10), in the following way:

$$R^c_{bed;e} - R^c_{bec;d} + R^c_{bde;c} = 0,$$  \hspace{1cm} (3.11)

which is equivalent to

$$R_{bd;e} - R_{be;d} + R^c_{bde;c} = 0.$$  \hspace{1cm} (3.12)

To obtain (3.12), we used the definition of the Ricci tensor. Now multiplying both sides of (3.12) to the metric tensor, once again we make another contraction.

$$g^{bd}(R_{bd;e} - R_{be;d} + R^c_{bde;c}) = 0,$$  \hspace{1cm} (3.13)

getting

$$R^d_{;e} - R^d_{;e} + R^{cd}_{dec} = 0.$$  \hspace{1cm} (3.14)

Applying Riemann tensor and Ricci scalar definitions, one obtains:

$$R_{;e} - 2R^c_{e;c} = 0,$$  \hspace{1cm} (3.15)

which can be rewritten as

$$(R^c_{e} - \frac{1}{2} g^c_{e} R)_{;c} = 0.$$  \hspace{1cm} (3.16)

One more, subtraction by $g^{ab}$ yields:

$$(R^{cd} - \frac{1}{2} g^{cd} R)_{;c} = 0,$$  \hspace{1cm} (3.17)

from which, the symmetry in the parenthesis and the definition of Einstein tensor implies that:

$$G^{ab}_{;b} = 0.$$  \hspace{1cm} (3.18)
Using Einstein equations, we immediately conclude
\[ \nabla_b T^{ab} = T^{ab} = 0, \]
which is the same as equation (3.8). This relation locally expresses the energy-momentum conservation. This conservation law is of physical interest. In his field equations, Einstein confirms that general relativity possesses such conservation law.

### 3.6 Non-linearity

The non-linearity of Einstein equations, makes general relativity a distinguishable theory, among the others in physics. For example, Maxwell’s equations for electromagnetism, are linear in electric and magnetic fields and corresponding distributions of charge and current (since a linear combination of two solutions, is itself a solution). Another example is the Schrödinger equation in quantum mechanics, which is linear with respect to the wave function.

### 3.7 The equivalence principle

Einstein field equations and its approximations for weak fields and low speeds, lead to Newton’s law of gravitation. In fact the constant which appears in Einstein equations, is determined from these two approximations.

### 3.8 The vacuum field equations

If the energy-momentum tensor \( T_{\mu\nu} \) becomes zero in a definite region, as a consequence, the field equations transform to the vacuum case. Taking \( T_{\mu\nu} = 0 \) in all filed equations, one can write the vacuum field equations as follows:
\[ R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}. \]  
(3.19)

Deriving the trace of this equation (contraction by \( g_{\mu\nu} \)) and knowing that \( g^{\mu\nu} g_{\mu\nu} = 4 \), we will have:
\[ R = \frac{1}{2} R \times (4) = 2R, \]  
(3.20)

or
\[ R = 0. \]  
(3.21)
Substituting (3.21) in (3.19) provides a corresponding form for the vacuum equations.

\[ R_{\mu\nu} = 0. \]  
\[ (3.22) \]

When a nonzero cosmological constant is included, we will have:

\[ R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}, \]  
\[ (3.23) \]

from which one obtains

\[ R = 4\Lambda. \]  
\[ (3.24) \]

This relation also has a corresponding vacuum form like

\[ R_{\mu\nu} = \Lambda g_{\mu\nu}. \]  
\[ (3.25) \]

The solutions to these equations are called *vacuum solutions*. Minkowski spacetime is the simplest vacuum solution to Einstein equations. Some other nontrivial solutions are the Schwarzchild and Kerr solutions.

Manifolds with \( R_{\mu\nu} = 0 \), are called *Ricci-flat manifolds*, and the manifolds in which Ricci tensor is proportional to the metric, are called *Einstein manifolds*.

### 3.9 Einstein-Maxwell equations

If the energy-momentum tensor is considered to be

\[ T^{\alpha\beta} = -\frac{1}{\mu_0}(F^{\alpha\beta}F_{\psi}^{\beta} + \frac{1}{4}g^{\alpha\beta}F_{\psi\tau}F^{\psi\tau}), \]  
\[ (3.26) \]

corresponding to an electromagnetic field in free space, then the resultant equations from (3.1), get the Einstein-Maxwell field equations.

\[ R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + g^{\alpha\beta} \Lambda = -\frac{8\pi G}{c^4 \mu_0} (F^{\alpha\beta}F_{\psi}^{\beta} + \frac{1}{4}g^{\alpha\beta}F_{\psi\tau}F^{\psi\tau}). \]  
\[ (3.27) \]

Furthermore, the covariant form of Maxwell’s equations are also applicable in free space.

\[ F^{\alpha\beta}_{;\beta} = 0, \]  
\[ (3.28) \]

and

\[ F_{[\alpha\beta\gamma]} = \frac{1}{3} (F_{\alpha\beta\gamma} + F_{\beta\gamma\alpha} + F_{\gamma\alpha\beta}) = 0, \]  
\[ (3.29) \]

in which ; denotes covariant differentiation, and the brackets represents the antisymmetry. The first equation asserts that a four-dimensional divergence of \( F \), vanishes.
The second equation, equates exterior differentiation to zero. Therefore from Poincaré lemma, we can define a vector potential $A_\alpha$ so that

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha} = A_{\alpha,\beta} - A_{\beta,\alpha},$$

where $\alpha$ stands for partial differentiations. Equations (3.30) sometimes is regarded as covariant Maxwell equation [30]. However, there are some generalized solutions to this equation, without possessing a generalized definite potential [31].

### 3.10 The solutions

The solutions of Einstein equations are the spacetime structures. Hence, this solutions are named *metrics*. These metrics describe the spacetime structure, including the inertial characteristics of the contained objects. Since the field equations are nonlinear, it is not possible to solve them completely (inevitably we must make approximations). For example, there is no known complete solution for the spacetime, containing two separated masses (which is the theoretical model for dipole stars). However in such problems, it is common to consider some sort of approximations, called *post-Newtonian approximations*. Although in several cases, the field equations have been solved completely, resulting in *exact solutions* [32].

Probing the exact solutions of Einstein equations, is one of the most important activities in cosmology. These investigations, have led to anticipations like black-hole existence, and also to proposing some different models of universe’s evolution.

### 3.11 The linearized field equations

The linearized Einstein equation is an approximation, which is valid for weak-filed limits and is applied to simplifying the problems in general relativity or discussing the concept of gravitational waves. This approximation can also be used to deduce Newtonian gravitation from general relativity, as the weak-field limit.

These approximations can be derived, by considering that the spacetime metric, differs only slightly from the Minkowskian one. Then the difference between the metrics can be regarded as a field on the original metric (background metric), and its behavior is investigated by a set of linear equations.
3.11.1 Derivation of Minkowski metric

Let us consider the following metric for our spacetime:

\[ g_{ab} = \eta_{ab} + h_{ab}, \quad (3.31) \]

where \( \eta_{ab} \) is the Minkowski metric and \( h_{ab} \) correlated to a field located on the background metric. \( h \) has to be ignorable versus \( \eta \). That is \( |h_{\mu\nu}| \ll 1 \) (and also for all derivatives of \( h \)). Therefore one can ignore all the multiplications of \( h \) by itself (and all its derivatives with respect to \( h \)). Then we shall assume that all the indices of \( h \) can be raised or lowered by a \( \eta \).

\( h \) is always symmetric, and the condition \( g_{ab}g^{bc} = \delta_a^c \) implies that

\[ g^{ab} = \eta^{ab} + h^{ab}, \quad (3.32) \]

from which the Christoffel symbols can be calculated as follows:

\[ 2\Gamma^a_{bc} = (h^a_{b,c} + h^a_{c,b} - h^a_{bc}), \quad (3.33) \]

where

\[ h^a_{bc} := \eta^{ar}h_{bc,r} \]

is used to calculate the Riemann tensor

\[ 2R^a_{bcd} = 2(\Gamma^a_{bd,c} + \Gamma^a_{bc,d}) \]

\[ = \eta^{ac}(h_{eb,dc} + h_{ed,bc} - h_{bd,ec} - h_{eb,cd} - h_{bc,ed}) \]

\[ = \eta^{ac}(h_{ed,be} - h_{bd,ec} - h_{ec,bd} + h_{bd,ce}) \]

\[ = (h^a_{d,be} + h^a_{e,bd} + h^a_{d,be} - h^a_{e,be}). \quad (3.34) \]

Using \( R_{ab} = \delta^c_a R^a_{bcd} \) we have:

\[ 2R_{bd} = h^r_{d,br} + h^r_{b,dr} - h_{bd} - h_{bd,rs} \eta^{rs}. \quad (3.35) \]

Therefore the linearized Einstein equations take the form \([1]\)

\[ 8\pi T_{ab} = R_{bd} - \frac{1}{2} R \eta^{ac} \eta_{bd}. \quad (3.36) \]

\^[1]Stephani, Hans (1990), *General Relativity: An Introduction to the Theory of Gravitation Filed*, Cambridge University Press. ISBN 0-521-37941-5.
3.12 The de Sitter space

In mathematical physics, the $n$-dimensional de Sitter space, denoted by $dS_n$, is the Lorentz analogy of a $n$-dimensional sphere (with a canonic metric). This space is a maximally symmetric Lorentz manifold, with a constant positive curvature, which would be continuous for $n \geq 3$. From general relativity viewpoint, the de Sitter space is a maximally symmetric vacuum solution to Einstein equations, possessing a positive cosmological constant (repelling) corresponding to a positive density of vacuum energy with negative pressure. When $n = 4$, de Sitter space is also a cosmological model. This space has been proposed by Willem de Sitter, and independently by Tullio Levi-Civita in 1917.

Formerly, this space was considered as a basis of general relativity, instead of Minkowski space, forming a formalism called de Sitter relativity [33].

3.12.1 Mathematical definition

De Sitter space is a sub-manifold of Minkowski space with one additional dimension. Let us consider a Minkowski space $\mathbb{R}^{1,n}$ with the standard metric

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2.$$  \hfill (3.37)

The so-called sub-manifold is a hyperbolic surface, defined by

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = \alpha^2,$$  \hfill (3.38)

in which $\alpha$ is a positive constant having the dimension of length. The induced metric which is defined on de Sitter space, is deduced from Minkowski ambient space. One can inspect that this induced metric, is non-degenerate exhibits a Lorentz form. Note that, if $\alpha^2$ is substituted by $-\alpha^2$ in the above definition, a two-surfac ed hyperboloid is achieved. The so-called induced metric in this case, is definite and positive, and every surface, is a copy of a $n$-dimensional hyperboloid.

De Sitter space can also be regarded as the quotient of the fraction $O(1,n)/O(1,n-1)$ of two indefinite nonorthogonal groups, which shows that this space is symmetric and non-Riemannian. From topological viewpoint, de Sitter space is $\mathbb{R} \times s^{n-1}$, therefore for $n \geq 3$, is continuous [34, 35, 36].
3.12.2 Properties

The isometry group of de Sitter space, is the Lorentz group $O(1, n)$. Therefore its metric is maximally symmetric and has $\frac{n(n+1)}{2}$ independent Killing vectors. Every maximally symmetric spaces, have a constant curvature. The corresponding Riemann tensor is given by

$$R_{\rho\mu\nu\lambda} = \frac{1}{\alpha^2} (g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}).$$  \hspace{1cm} (3.39)

Since the Ricci tensor is proportional to the metric, de Sitter space is an Einstein manifold.

$$R_{\mu\nu} = \frac{n-1}{\alpha^2} g_{\mu\nu}. \hspace{1cm} (3.40)$$

This means that de Sitter space is a vacuum solution to Einstein equations, with the cosmological constant

$$\Lambda = \frac{(n-1)(n-2)}{2\alpha^2}. \hspace{1cm} (3.41)$$

The constant curvature of de Sitter space is given by the following relation:

$$R = \frac{n(n-1)}{\alpha^2} = \frac{\Lambda}{n-2}. \hspace{1cm} (3.42)$$

For $\Lambda = 4$ we have $\Lambda = \frac{3}{\alpha^2}$ and $R = 4\alpha$ \cite{37, 38}.

3.13 De Sitter group and invariance

The de Sitter group $SO_0(1, 4)$ is composed of all $g_{5\times5}$ matrices with determinant 1, preserving the following quadratic form:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The compact maximally symmetric subgroup $k$ of $SO_0(1, 4)$ is isometric to $SO(4)$ and is composed of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},$$

and $k \in SO(4)$ \cite{13, 15, 24}.

3.14 The coordinate systems in de Sitter space-time

3.14.1 Static coordinate system

It is possible to define the static coordinates $(t, r, ...)$ in de Sitter space as follows:

$$x_0 = \sqrt{\alpha^2 - r^2} \sinh\left(\frac{t}{\alpha}\right),$$

$$x_i = \sqrt{\alpha^2 - r^2} \cosh\left(\frac{t}{\alpha}\right), \quad i = 1, 2, 3.$$
\[ x_1 = \sqrt{\alpha^2 - r^2 \cosh(t/\alpha)}, \]
\[ x_i = rz_i \quad 2 \leq i \leq n, \]

where \( z_i \) is the contained space in a \((n - 2)\)-dimensional sphere in \( \mathbb{R}^{n-1} \). In this system, the de Sitter metric has the form

\[ ds^2 = -(1 - \frac{r^2}{\alpha^2}) dt^2 + (1 - \frac{r^2}{\alpha^2})^{-1} dr^2 + r^2 d\Omega^2_{n-2}. \tag{3.43} \]

This metric is also known as the Einstein-de Sitter metric. Note that in this coordinate system, there would be an event horizon (cosmological horizon) at \( r = \alpha \), as a result of the singularity in this point.

### 3.14.2 Flat coordinate system

If we have
\[ x_0 = \alpha \sinh(t/\alpha) + \frac{r^2}{2\alpha} e^{\frac{t}{\alpha}}, \]
\[ x_1 = \alpha \cosh(t/\alpha) - \frac{r^2}{2\alpha} e^{\frac{t}{\alpha}}, \]
\[ x_i = e^{\frac{t}{\alpha}} y_i \quad 2 \leq i \leq n, \]

where \( r^2 = \sum_i y_i^2 \), therefore for this system, the de Sitter metric in \((t, y_i)\) coordinates will be

\[ ds^2 = -dt^2 + e^{\frac{2t}{\alpha}} dy^2, \tag{3.44} \]

in which \( dy^2 = \sum_i dy_i^2 \) is the flat metric on \( y_i \).

### 3.14.3 Open slicing

In this system we have
\[ x_0 = \alpha \sinh(t/\alpha) \cosh(\xi), \]
\[ x_1 = \alpha \cosh(t/\alpha), \]
\[ x_i = \alpha z_i \sinh(t/\alpha) \sinh(\xi) \quad 2 \leq i \leq n, \]

where \( \sum_i z_i^2 = 1 \), forms a \( s^{n-1} \) with the standard metric \( \sum_i dz_i^2 = d\Omega^2_{n-2} \). In this case the de Sitter metric takes the form

\[ ds^2 = -dt^2 + \alpha^2 \sinh(\frac{t}{\alpha}) dH^2_{n-1}, \tag{3.45} \]

where
\[ dH^2_{n-1} = d\xi^2 + \sinh^2(\xi) d\Omega^2_{n-1} \]

is a hyperbolic Euclidean space.
3.14.4 Closed slicing

Assume that
\[ x_0 = \alpha \sinh \left( \frac{t}{\alpha} \right), \]
\[ x_i = \alpha \cosh \left( \frac{t}{\alpha} \right) z_i \quad 1 \leq i \leq n, \]
where \( z_i \) describes a \( s^{n-1} \). Therefore the metric will be
\[ ds^2 = -dt^2 + \alpha^2 \cosh^2 \left( \frac{t}{\alpha} \right) d\Omega^2_{n-1}. \] \hspace{1cm} (3.46)

Changing the time variable to the conformal time using \( \tan \left( \eta \frac{t}{2} \right) = \tanh \left( \frac{t}{2\alpha} \right) \) (or equivalently \( \cos(\eta) = \frac{\cosh(t/\alpha)}{\cosh(\eta)} \)), we obtain a metric, which is conformally equivalent to a static Einstein universe.
\[ ds^2 = \frac{\alpha^2}{\cos^2(\eta)} (-d\eta^2 + d\Omega^2_{n-1}). \] \hspace{1cm} (3.47)

This metric is of benefit when we concern about the Penrose diagrams of de Sitter spacetime.

3.14.5 De Sitter slicing

If we have
\[ x_0 = \alpha \sin \left( \frac{\chi}{\alpha} \right) \sinh \left( \frac{t}{\alpha} \right) \cosh(\xi), \]
\[ x_1 = \alpha \cos \left( \frac{\chi}{\alpha} \right), \]
\[ x_2 = \alpha \sin \left( \frac{\chi}{\alpha} \right) \cosh \left( \frac{t}{\alpha} \right), \]
\[ x_i = \alpha z_i \sin \left( \frac{\chi}{\alpha} \right) \sinh \left( \frac{t}{\alpha} \right) \sinh(\xi) \quad 3 \leq i \leq n, \]
where \( z_i \) describes a \( s^{n-1} \), the metric takes the form
\[ ds^2 = d\chi^2 + \sin^2 \left( \frac{\chi}{\alpha} \right) ds^2_{s,\alpha,n-1}, \] \hspace{1cm} (3.48)
in which
\[ ds^2_{s,\alpha,n-1} = -dt^2 + \alpha^2 \sinh^2 \left( \frac{t}{\alpha} \right) dH^2_{n-2} \]
is the metric of a \((n-1)\)-dimensional de Sitter space with \( \alpha \) as the radius of curvature in open coordinate system. The hyperbolic metric is given by the following relation:
\[ dH^2_{n-2} = d\xi^2 + \sinh^2(\xi) d\Omega^2_{n-3}. \] \hspace{1cm} (3.49)
Continuing the open coordinate system analysis, this metric is derived from the transformation
\[(t, \xi, \theta, \phi_1, \phi_2, \ldots, \phi_{n-3}) \rightarrow (i\chi, \xi, iT, \phi_2, \ldots, \phi_{n-4}),\]
and also the substitution \(x_0\) by \(x_2\), since these two, replace their timelike/spacelike natures \[39\].

### 3.15 Solutions to Einstein equations from Birkhoff’s viewpoint

The Schwarzschild metric
\[ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,\]  
(3.50)
is known to be directly related to the spherically symmetric metric, as the original result of the Birkhoff’s theorem in 1923 \[40\]. This metric is the vacuum solution to Einstein field equations, explaining the exterior geometry of a spherical object of mass \(M\). One can consider the Birkhoff’s theorem as follows:

*A spacetime is spherically symmetric, if there exists a SO(3) group of isometries on it, which are isomorphic to a \(S^2\)-sphere.*

Nevertheless, the Schwarzschild solution is supposed to be static, the results of Birkhoff’s theorem are not confined to static spherically symmetric metrics. According to the previous section, the Einstein-de Sitter metric
\[ds^2 = -(1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2)dt^2 + (1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2)^{-1}dr^2 + r^2d\Omega^2\]  
(3.51)
where \(\Lambda\) is the cosmological constant, is also capable to be considered as a result of Birkhoff’s theorem. This metric has to be the trivial solution to vacuum Bach’s conformal field equations, derived from conformal Weyl gravity \[41\], i.e.
\[W_{\alpha\beta} = \nabla^\rho \nabla_\alpha R_{\beta\rho} + \nabla^\rho \nabla_\beta R_{\alpha\rho} - \Box R_{\alpha\beta} - g_{\alpha\beta} \nabla_\rho \nabla^\lambda R^{\rho\lambda} \]
\[-2R_{\rho\beta} R^{\rho\alpha} + \frac{1}{2} g_{\alpha\beta} R_{\rho\lambda} R^{\rho\lambda} - \frac{1}{3} \left(2\nabla_\alpha \nabla_\beta R - 2g_{\alpha\beta} \Box R - 2RR_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R^2\right) = 0.\]  
(3.52)
This holds, since de Sitter spacetime is conformally flat and obeys the \(SO(d, 2)\) conformal algebra, and therefore it would be conformally symmetric. What we are about to do in this work, is to derive conformally invariant equations for massless particles.
with spin-2, on a de Sitter background. Previously, the conformal invariance of such particles, beside investigating this property for scalars and partially massless particles in de Sitter and Anti-de Sitter spacetimes, have been studied [42]. In this work however, we shall use a different approach, namely the Dirac’s six cone formalism to reconsider this, which will be introduced in the next chapter.
Chapter 4

The mathematical operators and Dirac’s six cone formalism

4.1 Introducing the transverse projector

Let’s consider a spherical surface in 3-dimensional space. Cartesian expression of this surface would be

\[ x^2 + y^2 + z^2 = R^2 = 1, \]

indicating a spherical shell with a unit radius. Now let us introduce the transverse projector operator on this shell, in three-dimensional space as

\[ \theta^{ij} A_j = \bar{A}^i \quad (4.1) \]

where \( \bar{A}^i \) are the components of \( \vec{A} \) on the sphere’s surface. Since \( \bar{A}^i \) is a tangent vector, it is perpendicular to the radius vector of the sphere, so

\[ r_i \theta^{ij} A_j = r_i \bar{A}^i = 0. \quad (4.2) \]

generally this operator can be defined as below:

\[ \theta_{ij} = \delta_{ij} - r^{-2} r_i r_j. \quad (4.3) \]

Corresponding to the operator defined in (4.1), we introduce a transverse projector, to project a vector from de Sitter ambient (flat) space to de Sitter inherent (curved-hyperbolic) space.

\[ \theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta, \quad (4.4) \]

in which \( \alpha, \beta = 1, 2, 3, 4, 5 \) and

\[ \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1). \]
The metric will be
\[ ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = g_{\mu\nu}dX^\mu dX^\nu. \]

It can be shown that
\[ x_\alpha \tilde{A}^\alpha = x^\alpha \tilde{A}_\alpha = 0. \quad (4.5) \]

Note that
\[ x_\alpha x^\alpha = -H^{-2} \quad (4.6) \]

which is deduced from de Sitter space definition, and is equivalent to the following relation:
\[ x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -H^{-2}. \quad (4.7) \]

The differentiation operators are also capable to be projected from ambient space onto curved space. In general relativity, this kind of differentiation is called covariant differentiation. We have
\[ \bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha (x.\partial). \quad (4.8) \]

One can easily show that $\bar{\partial}_\alpha$ is the transverse component of differentiation in the curved space, i.e.
\[ x_\alpha \bar{\partial}^\alpha = x^\alpha \bar{\partial}_\alpha = 0. \quad (4.9) \]

Note that
\[ \frac{\partial x^\alpha}{\partial x^\beta} = \delta_\beta^\alpha = \eta_\beta^\alpha \quad (4.10) \]

and
\[ \bar{\partial}^\alpha x_\beta = \theta_\beta^\alpha \Rightarrow \bar{\partial}^\alpha x_\alpha = 4. \quad (4.11) \]

It can be shown that for the Hubble parameter we have
\[ \bar{\partial}_\alpha H^{-2} = 0, \quad (4.12) \]

and
\[ \partial_\alpha H^{-2} = -2x_\alpha. \quad (4.13) \]

The vector components can be differentiated in the following way:
\[ \bar{\partial}_\alpha x^\beta = \theta_\alpha^\beta. \quad (4.14) \]

Note that
\[ \partial_\gamma x^\beta = \eta_\gamma^\beta. \quad (4.15) \]

Therefore it can proved that
\[ \bar{\partial}_\beta x^\beta = 4. \quad (4.16) \]
4.2 Applying the transverse projector on operators in de Sitter spacetime

Defining
\[ M_{\alpha\beta} = i(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \]
It can be shown that
\[ (x_\alpha \partial_\beta - x_\beta \partial_\alpha)(x^\alpha \partial^\beta - x^\beta \partial^\alpha) = (x_\alpha \bar{\partial}_\beta - x_\beta \bar{\partial}_\alpha)(x^\alpha \bar{\partial}^\beta - x^\beta \bar{\partial}^\alpha). \]  
(4.17)
The scalar Casimir operator of de Sitter group is defined as
\[ Q_0 = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}, \]
(4.18)
from which
\[ Q_0 = H^{-2} \bar{\partial}_\alpha \bar{\partial}_\alpha. \]  
(4.19)
Applying the transverse projector on itself, yields
\[ \theta_{\alpha\beta} \theta^{\alpha\beta} = \theta^\gamma_\gamma. \]  
(4.20)
It is also possible to calculate covariant differentiations of the transverse projector.
\[ \bar{\partial}_\beta \theta_{\alpha\sigma} = H^2 \theta_{\beta\sigma} x_\sigma + H^2 x_\alpha \theta_{\beta\sigma}. \]  
(4.21)
For an arbitrary function \( f \), we have
\[ \bar{\partial}_\beta (x_\gamma f) = \theta_{\beta\gamma} f + x_\gamma \bar{\partial}_\beta f. \]  
(4.22)
One can prove that
\[ Q_0 x^\beta = -4 x^\beta. \]  
(4.23)
Differentiation in curved space, is not commutative and indeed has the following commutation relation:
\[ [\bar{\partial}_\alpha, \bar{\partial}_\beta] = H^2 (x_\beta \bar{\partial}_\alpha - x_\alpha \bar{\partial}_\beta). \]  
(4.24)
Also some other commutation relations are valid and provable.
\[ [Q_0, x_\alpha] = -4 x_\alpha - 2H^{-2} \bar{\partial}_\alpha, \]  
(4.25)
\[ [Q_0, \bar{\partial}_\alpha] = 6 \bar{\partial}_\alpha + 2H^2 (Q_0 + 4) x_\alpha. \]  
(4.26)
We should mention that
\[ x_\alpha \theta_{\alpha\gamma} = 0. \]  
(4.27)
4.3 The differentiation operator in de Sitter space-time

The differentiation operator is defined like

\[ D_{1\alpha} \equiv H^{-2} \tilde{\partial}_\alpha, \] (4.28)

or

\[ D_1 \equiv H^{-2} \tilde{\partial}. \] (4.29)

It is proved that

\[ Q_1 D_1 k_\alpha = D_1 Q_0 \phi, \] (4.30)

in which \( \phi \) is an arbitrary scalar field like the formerly introduced function \( f \). Note that

\[ Q_1 k_\alpha = (Q_0 - 2)k_\alpha + 2x_\alpha \tilde{\partial}.k, \] (4.31)

where \( k_\alpha \) is a vector field. Therefore in general, for a vector field we have

\[ Q_1 k_\alpha = (Q_0 - 2)k_\alpha + 2x_\alpha \tilde{\partial}.k - 2\tilde{\partial}x.k. \] (4.32)

And since \( k \) is supposed to be transverse, therefore

\[ x.k = 0. \] (4.33)

The Casimir operators are classified as

- \( Q_0 \) operates on scalars,
- \( Q_1 \) operates on vectors,
- \( Q_2 \) operates on tensors (two components).

If \( \tilde{z}_\alpha \) is an arbitrary constant vector, its derivatives are zero, however note that

\[ \tilde{z}_\alpha = \theta^\beta_\alpha \tilde{z}_\beta. \] (4.34)

That is because of the existence of \( \theta^\beta_\alpha \) its derivatives do not vanish.

\[ \tilde{\partial}.\tilde{z}\phi = \partial^\alpha \tilde{z}_\alpha \phi = \partial^\alpha \theta^\beta_\alpha \tilde{z}_\beta \phi \]
\[ = 4H^2(x.z)\phi + z^\beta \tilde{\partial}_\beta \phi = 4H^2(x.z)\phi + z.\tilde{\partial}\phi. \] (4.35)

Also we have

\[ Q_0(x.z)\phi = (x.z)(Q_0 - 4)\phi + 2H^{-2}z.\tilde{\partial}\phi. \] (4.36)

One can prove

\[ Q_0(z.\tilde{\partial})\phi = z.\tilde{\partial}(Q_0 + 1)\phi + 2H^2(x.z)Q_0 \phi. \] (4.37)
4.4 Transversifying the vectors on de Sitter spacetime

The Laplace-Belzami operator is defined as
\[ \Box \equiv -H^{-2}Q_0. \] (4.38)

Since we are now familiar with the basics concepts of \( \theta_{\alpha\beta}, Q_0, Q_1 \) and \( D_1 \) and their commutation relations, it is worth to know that the transposition of a differentiation between de Sitter ambient space and coherent space, is done in the following way:
\[ \nabla_{\mu}A_{\nu} \rightarrow \theta_{\alpha'}^{\alpha} \theta_{\beta'}^{\beta} \partial_{\alpha'} k_{\beta'}. \] (4.39)

Since we possessed two indices, we had to differentiate twice. This is called transverse projection, which is noted by TrPr. We have:
\[ \nabla_{\rho} \nabla_{\lambda} h_{\mu\nu} \rightarrow (\text{TrPr}) \partial_{\alpha}(\text{TrPr}) \partial_{\beta} k_{\mu\nu}. \] (4.40)

The term \( \text{TrPr}(\partial_{\beta} k_{\mu\nu}) \) must results in an expressions, which vanishes when multiplied by an \( x \); in other words the result must be transverse.
\[ \text{TrPr}(\partial_{\beta} k_{\mu\nu}) = \bar{\partial}_{\beta} k_{\mu\nu} - x_{\mu} k_{\beta\nu} - x_{\nu} k_{\mu\beta}. \] (4.41)

Since \( k \) is transverse itself \( (x.k = 0) \), the rhs of (4.41) results in zero when it is multiplied by \( x_{\beta}, x_{\mu} \) or \( x_{\nu} \).
\[ x_{\beta} \text{TrPr}(\partial_{\beta} k_{\mu\nu}) = x_{\mu} \text{TrPr}(\partial_{\beta} k_{\mu\nu}) = x_{\nu} \text{TrPr}(\partial_{\mu} k_{\mu\nu}) = 0. \] (4.42)

The other calculations for TrPr, can be done in the same way. For example it can be proved that
\[ \nabla_{\mu}A_{\nu} \rightarrow \text{TrPr}(\partial_{\alpha} k_{\beta}) = \bar{\partial}_{\alpha} k_{\beta} - H^2 x_{\beta} k_{\alpha}. \] (4.43)

Note that it is common to consider \( H^2 = 1 \).

4.5 Dirac’s six cone formalism

The Dirac’s six cone is a procedure through which the conformally invariant equations are derived in six-dimensional space. Here the variables are denoted by \( u^a \), where:
\[ u^a, \quad a = 0, 1, 2, 3, 4, 5. \]
From differential calculus, we know that $x.\partial$ indicates the degree of a differentiable function. For example for $f(x) = x^5$, this operator results in $5$. i.e.

$$x.\partial f(x) = 5f(x). \quad (4.44)$$

Similarly in Dirac’s six cone formalism, the operator $u^a\partial_a$ is regarded as the conformal degree operator. The filed is denoted by $\psi$ and the following relation is applied to find the equations in six-dimensional space:

$$\begin{cases} N_5\psi = (p - 2)\psi \\ (\partial_a\partial^a)\psi = 0 \end{cases} \quad (4.45)$$

where $N_5 = u_a\partial^a$ and the whole set are conformally invariant. The relation between $\partial_a\partial^a$ and the Casimir operators of de Sitter group is

$$(\partial_a\partial^a)^p = -x_5^{-2p} \prod_{j=1}^{p} \left( Q_0 + (j + 1)(j - 2) \right). \quad (4.46)$$

As an example, for the simplest case of $p = 1$ we have

$$(\partial_a\partial^a)^1 = -\frac{1}{x_5}(Q_0 - 2), \quad (4.47)$$

where $Q_0$ is the Casimir operator of de Sitter group. Also for $p = 2$:

$$(\partial_a\partial^a)^2 = -\frac{1}{x_5^2}(Q_0 - 2)Q_0. \quad (4.48)$$

The following relation can be regarded as the connection between six-dimensional space and de Sitter ambient space:

$$\begin{cases} x^\alpha = (u^5)^{-1}u^\alpha, \quad \alpha = 0, 1, 2, 3, 4 \\ x^5 = u^5 \end{cases} \quad (4.49)$$

Note that $x^5$ is an extra component; it is not one of the 5 components of de Sitter ambient space and will vanish in our final equations.

For the scalar field $\psi$ (the simplest case), the conformally invariant system is

$$\begin{cases} (\partial_a\partial^a)\psi = 0 \\ N_5\psi = -\psi \end{cases} \quad (4.50)$$

Now since $\psi$ does not possess indices, and is invariant in all systems, defining $\phi = x_5\phi$, we can construct a de Sitter scalar in the form

$$(Q_0 - 2)\phi = 0. \quad (4.51)$$
This solution is valid for a massless scalar field in de Sitter space.

The second stage, is a vector field for which \( p = 1 \). We have

\[
\begin{cases}
  (\partial_a \partial^a) \psi_a = 0 \\
  N_5 \psi_a = -\psi_a
\end{cases}.
\]  

(4.52)

\( \psi_a \) possesses six components; it has six degrees of freedom. Now \( k_\alpha \), the vector filed in de Sitter ambient space, must be deduced from \( \psi_\alpha \) such that it is transverse.

\[
k_\alpha = x_5 (\psi_\alpha + x_\alpha x. \psi).
\]  

(4.53)

The vector in (4.53) would be transverse. By applying the operator defined in (4.51) on the vector \( k_\alpha \) we get [43, 44]:

\[
(Q_0 - 2)(\psi_\alpha + x_\alpha x. \psi) = 0.
\]  

(4.54)

The next step, is to derive an equation for \( k_{\alpha\beta} \), which is the last part of this thesis and we shall concern about in the next chapter.
Chapter 5

The conformally invariant equations for graviton

5.1 The conformally invariant system of conformal degree 1

Working with $p = 1$, the formalism of Dirac’s six becomes

\[
\begin{align*}
(\partial_A \partial^A)\psi_{AB} &= 0 \\
N_5 \psi_{AB} &= -\psi_{AB}.
\end{align*}
\]  

(5.1)

We introduce the second rank tensor $k_{\alpha\beta}$ from $\psi_{\alpha\beta}$ in de Sitter ambient space.

\[
k_{\alpha\beta} = \psi_{\alpha\beta} + Sx_\beta \psi_\alpha.x + x_\alpha x_\beta \psi.x.x,
\]

(5.2)

where $S$ denotes that the next expression is added to itself with a commutation on its coefficients. Also the differentiations are done like

\[
\bar{\partial}.k_\alpha = 3(x.\psi_\alpha + x_\alpha x.\psi.x).
\]

(5.3)

Now let $\partial_A \partial^A$ operate on the tensor defined in (5.2). From (4.51) we know that $\partial_A \partial^A$ is equivalent to $(Q_0 - 2)$. We have

\[
(Q_0 - 2)k_{\alpha\beta} = (Q_0 - 2)\psi_{\alpha\beta} + (Q_0 - 2)Sx_\beta \psi_\alpha.x + (Q_0 - 2)x_\alpha x_\beta \psi.x.x.
\]

(5.4)

We rewrite the conformally invariant system as

\[
\begin{align*}
(Q_0 - 2)\psi_{\alpha\beta} &= 0 \\
(Q_0 - 2)\psi_{55}, \psi_{aa} &= 0.
\end{align*}
\]

(5.5)
The transversality condition implies that

\[ u^A \psi_{AB} = 0 \Rightarrow (Q_0 - 2) x^A \psi_B = 0. \]  \hspace{1cm} (5.6)

Since \( k \) is traceless, therefore

\[ k' = 0 \Rightarrow (Q_0 - 2) x^A \psi_A = 0 \] \hspace{1cm} (5.7)

### 5.2 The effect of Casimir operator on a tensor of second rank

Multiplying a \( x^A \) to both sides of (5.6) yields

\[ Q_0 x^A \psi_A + 2x^A \psi_A + 2 \partial_A \psi_A = 0. \]

Using (5.7) we obtain

\[ \partial_A \psi_A = -2x^A \psi_A. \] \hspace{1cm} (5.8)

Substituting (5.7) in (5.8) we have

\[ (Q_0 - 2) \partial_A \psi_A = 0. \] \hspace{1cm} (5.9)

In chapter 4 we mentioned that

\[ (Q_0 - 2) x^A = x^A Q_0 - 6x^A - 2 \partial_A, \]

hence, from (5.6),

\[ (Q_0 - 2) x^A \psi_B = -2(\partial_A + 2x_A) \psi_B. \] \hspace{1cm} (5.10)

Therefore the effect of \( (Q_0 - 2) \) on a tensor of second rank, can be summarized as follows:

\[ (Q_0 - 2) k_{\alpha \beta} = -2(\partial_\alpha + 2x_\alpha) \psi_\beta. x - 2(\partial_\beta + 2x_\beta) \psi_\alpha. x - 2(\partial_\alpha + 2x_\alpha) x_\beta \psi_\alpha. x. \] \hspace{1cm} (5.11)

Using (5.3), this can be rewritten as

\[ (Q_0 - 2) k_{\alpha \beta} = -2(\partial_\beta + 2x_\beta) \psi_\alpha. x - \frac{2}{3}(\partial_\alpha + 2x_\alpha) \partial_\beta k_\beta. \] \hspace{1cm} (5.12)
5.3 Obtaining the conformally invariant field equation using a conformal system of degree 1

From (5.5) and (5.6) we have

\[(Q_0 - 2)\psi_{\alpha\beta} = 0 \Rightarrow 2\psi_{\beta} = -\bar{\partial}\psi_{\beta}, \quad (5.13)\]

\[(Q_0 - 2)\psi_{\beta}.x = 0 \Rightarrow 2\psi_{\beta}.x = -\bar{\partial}\psi_{\beta}.x. \quad (5.14)\]

Let us multiply (5.4) by \(x_{\alpha}\).

\[
2\bar{\partial}k_{\beta} = Q_0\psi_{\beta} - 2\psi_{\beta} + S\psi_{\beta}.x + Q_0x_{\beta}\psi_{\beta}.x + 2x_{\beta}\psi_{\beta}.x
+ 2\bar{\partial}(x_{\beta}\psi_{\alpha}) - Q_0x_{\beta}\psi_{\beta}.x - 2x_{\beta}\psi_{\beta}.x + 8x_{\beta}\psi_{\beta}.x. \quad (5.15)
\]

Now multiply (5.4) by \(\bar{\partial}_{\alpha}\). We have

\[-\frac{2}{3}Q_0\bar{\partial}k_{\beta} + 2\bar{\partial}k_{\beta} = 4\psi_{\beta}.x. \quad (5.16)\]

If (5.12) is multiplied by \(x_{\beta}\), we obtain

\[
\bar{\partial}k_{\alpha} = 3\psi_{\alpha}.x, \quad (5.17)
\]

or

\[
\psi_{\alpha}.x = \frac{1}{3}\bar{\partial}k_{\alpha}. \quad (5.18)
\]

Substitute this equation in (5.12).

\[
(Q_0 - 2)k_{\alpha\beta} = -\frac{2}{3}(\bar{\partial}_{\beta} + 2x_{\beta})\bar{\partial}k_{\alpha} - \frac{2}{3}(\bar{\partial}_{\alpha} + 2x_{\alpha})\bar{\partial}k_{\beta}. \quad (5.19)
\]

Also from (5.16) and (5.18) we have

\[
(Q_0 - 1)\bar{\partial}k_{\beta} = 0,
\]

from which we obtain the conformally invariant system of degree 1, for tensorial field.

\[
(Q_0 - 2)k_{\alpha\beta} = -\frac{2}{3}S(\bar{\partial}_{\beta} + 2x_{\beta})\bar{\partial}k_{\alpha}, \quad (5.20)
\]

and

\[
(Q_0 - 1)\bar{\partial}k_{\beta} = 0. \quad (5.21)
\]

Nevertheless Equation (5.20) is conformal, it does not possess physical descriptions, since it is not capable to be transformed by the irreducible representations of conformal group. This equation was firstly proposed by Barut and Xu in 1982 by varying the coefficient in the standard equation. This equation in de Sitter inherent space has the following form:

\[
(\square + 4)h_{\mu\nu} = -\frac{2}{3}S\nabla_{\mu}\nabla_{\nu}h_{\nu} = 0. \quad (5.22)
\]

\[\text{1A.O. Barut, B.W. Xu, J. Phys. A: 15 (1982) 207.}\]
5.4 The conformally invariant system of conformal degree 2

Generally we can write
\[ \bar{\partial} k_{\alpha} = 4(\psi_{\alpha}.x + x_{\alpha}.\psi.x), \] (5.23)
and
\[ (Q_2 + 6)k_{\alpha\beta} = Q_0 k_{\alpha\beta} + 2S x_{\alpha} \bar{\partial} k_{\beta} + 2\eta_{\alpha\beta} k' - 2S \partial_{\alpha} x_{\beta}. \] (5.24)

If \( p = 2 \), the formalism of Dirac’s six cone becomes
\[
\begin{cases}
(Q_0 - 2)Q_0 \psi_{AB} = 0 \\
N_5 \psi_{AB} = 0
\end{cases}
\] (5.25)

Let us rewrite our information.
\[
\begin{align*}
(Q_0 - 2)Q_0 x.\psi.x &= 0, \\
(Q_0 - 2)Q_0 \psi_{\alpha\beta} &= 0, \\
(Q_0 - 2)Q_0 \psi_{55} &= 0, \\
(Q_0 - 2)Q_0 x.\psi_B &= 0, \\
(Q_0 - 2)Q_0 \bar{\partial}.\psi.x &= 0.
\end{align*}
\] (5.26-5.30)

The effect of \((Q_0 - 2)Q_0\) on the tensor defined in (5.2) is
\[ (Q_0 - 2)Q_0 k_{\alpha\beta} = (Q_0 - 2)Q_0 S x_{\alpha} \psi_{\beta}.x + (Q_0 - 2)Q_0 x_{\alpha} x_{\beta}.x.\psi.x. \] (5.31)

5.5 Obtaining the confomally invariant field equation using a conformal system of degree 2

It can be shown that
\[ x_{\alpha}(Q_0 - 2)Q_0 = (Q_0 - 2)(Q_0 x_{\alpha} + 4x_{\alpha} + 4\bar{\partial}_{\beta}). \] (5.32)

Also we have
\[ (Q_0 - 2)Q_0 x_{\alpha} \psi_{\beta}.x = x_{\alpha}(Q_0 - 2)Q_0 \psi_{\beta}.x - 4(3x_{\alpha} + \bar{\partial}_{\alpha})(Q_0 - 2)x_{\beta}.x.\psi.x, \] (5.33)
and
\[ (Q_0 - 2)Q_0 x_{\alpha} x_{\beta}.x.\psi.x = x_{\alpha} x_{\beta}(Q_0 - 2)Q_0 x.\psi.x \]
\[ -4 x_\alpha (3 x_\beta + \bar{\partial}_\beta)(Q_0 - 2)x.\psi.x - 4(3 x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)x_\beta x.\psi.x. \quad (5.34) \]

Therefore, an initial equation is achieved to demonstrate the effect of \((Q_0 - 2)Q_0\) on \(k_{\alpha\beta}\).

\[ (Q_0 - 2)Q_0k_{\alpha\beta} = -4 x_\alpha (3 x_\beta + \bar{\partial}_\beta)(Q_0 - 2)x.\psi.x - 4(3 x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)x_\beta x.\psi.x \]
\[ - (3 x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)\bar{\partial}.k_\beta. \quad (5.35) \]

If \((5.5)\) is multiplied by \(x_\beta\), we will have

\[ 4(Q_0 - 2)\bar{\partial}.k_\alpha = 12(Q_0 - 2)\psi_\alpha.x - 3 x_\alpha(2\bar{\partial}.\bar{\partial}.k) - 2\bar{\partial}_\alpha\bar{\partial}.\bar{\partial}.k \]
\[ + (Q_0 - 2)\bar{\partial}.k_\alpha + x_\alpha(2\bar{\partial}.\bar{\partial}.k) + x_\alpha(Q_0 - 2)\bar{\partial}.\bar{\partial}.k. \quad (5.36) \]

Therefore

\[ (Q_0 - 2)\psi_\alpha.x = \frac{1}{4}(Q_0 - 2)\bar{\partial}.k_\alpha + \frac{1}{3} x_\alpha\bar{\partial}.\bar{\partial}.k - \frac{1}{12} x_\alpha(Q_0 - 2)\bar{\partial}.\bar{\partial}.k + \frac{1}{6}\bar{\partial}_\alpha\bar{\partial}.\bar{\partial}.k. \quad (5.37) \]

Substitution in \((5.5)\) yields

\[ (Q_0 - 2)Q_0k_{\alpha\beta} = -Q_0Sx_\beta\bar{\partial}.k_\alpha - Q_0S\bar{\partial}_\alpha\bar{\partial}.k_\beta + 2Sx_\alpha\bar{\partial}.k_\beta \]
\[ + 2S\bar{\partial}_\alpha\bar{\partial}.k_\beta - 4 x_\alpha x_\beta\bar{\partial}.\bar{\partial}.k - \frac{1}{3}S\bar{\partial}_\beta\bar{\partial}_\alpha\bar{\partial}.\bar{\partial}.k \]
\[ - \frac{5}{3}Sx_\alpha\bar{\partial}_\beta\bar{\partial}.\bar{\partial}.k - 2\theta_{\alpha\beta}\bar{\partial}.\bar{\partial}.k + \frac{1}{3}\theta_{\alpha\beta}Q_0\bar{\partial}.\bar{\partial}.k. \quad (5.38) \]

And since

\[ - \frac{1}{3}S\bar{\partial}_\beta\bar{\partial}_\alpha = -\frac{2}{3}\bar{\partial}_\beta\bar{\partial}_\alpha + \frac{1}{3}x_\alpha\bar{\partial}_\beta - \frac{1}{3}x_\beta\bar{\partial}_\alpha, \quad (5.39) \]

we obtain the following relation

\[ (Q_2 + 4)[(Q_2 + 6)k_{\alpha\beta} + D_2\bar{\partial}.k_\alpha] + \frac{1}{3}D_2D_1\bar{\partial}.\bar{\partial}.k - \frac{1}{3}\theta_{\alpha\beta}(Q_0 + 6)\bar{\partial}.\bar{\partial}.k = 0. \quad (5.40) \]

Equation \((5.40)\) is the conformally invariant equation of conformal degree 2, for the second ranked tensor \(k_{\alpha\beta}\) [3].
Conclusion

The obtained equation in (5.40), is a conformally invariant equation for graviton. As we know, graviton is an spin-2 elementary particle, described by Einstein field equations. Since this particle is supposed to be massless, it must be described by a conformally invariant equation, which is the property that Einstein equation lacks. The attempts in this thesis, were to obtain a conformally invariant equation to describe graviton, or a tensorial field of rank 2. These attempts led to equation (5.40).

Despite its conformal characteristics, this equation however does not have physical descriptions, since it is not transformed by irreducible representations of conformal group $SO(2, 4)$. Such problem, occurs for Weyl gravity, which is regarded as a gravitational theory of higher order with respect to Ricci scalar.

Therefore it is considerable to relate a tensor of higher rank to graviton, and try to find the conformal equation using this new tensorial field. Pursuing this assumption, led to obtaining an equation, which beside its conformal invariance, is capable to be transformed by the conformal group $[15]$. Theoretical physicists are still trying to find the corresponding conformally invariant Lagrangian.
Appendix A

Proof of important relations in chapter 4

Proof of relation (4.12):
\[ \bar{\partial}_\alpha H^{-2} = (\partial_\alpha + H^2 x_\alpha (x, \bar{\partial})) H^{-2} \]
\[ \partial_\alpha H^{-2} + H^2 x_\alpha (x, \bar{\partial}) H^{-2} - 2x_\alpha + H^2 x_\alpha (x_\beta \partial^\beta) H^{-2} \]
\[ = -2x_\alpha - 2H^2 x_\alpha x_\beta x^\beta = -2x_\alpha - 2x_\alpha H^2 (-H^{-2}) \]
\[ = 2x_\alpha = 0. \]

Proof of relation (4.13):
\[ \partial_\alpha H^{-2} = \partial_\alpha (-x^\beta x_\beta) = -2x_\alpha. \]

Proof of relation (4.14):
\[ \bar{\partial}_\alpha x^\beta = (\partial_\alpha + H^2 x_\alpha (x, \bar{\partial})) x^\beta \]
\[ = \eta_\alpha^\beta + H^2 x_\alpha x^\gamma \partial_\gamma x^\beta \]
\[ = \eta_\alpha^\beta + H^2 x_\alpha x^\beta \equiv \theta_\alpha^\beta, \quad \partial_\gamma x^\beta = \eta_\alpha^\beta. \]

Proof of relation (4.17):
\[ x_\beta \tilde{\partial}_\beta - x_\beta \bar{\partial}_\alpha = x_\alpha [\partial_\beta + H^2 x_\beta (x, \bar{\partial})] - x_\beta [\partial_\alpha + H^2 (x, \bar{\partial})] \]
\[ = x_\alpha \partial_\beta - x_\beta \partial_\alpha + H^2 x_\alpha x_\beta (x, \bar{\partial}) - H^2 x_\beta x_\alpha (x, \bar{\partial}) \]
\[ = x_\alpha \partial_\beta - x_\beta \partial_\alpha \quad \Rightarrow \quad x_\alpha \tilde{\partial}^\beta - x_\beta \bar{\partial}^\alpha = x_\alpha \tilde{\partial}^\beta - x_\beta \bar{\partial}^\alpha. \]

Proof of relation (4.19):
\[ -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -\frac{1}{2} [x_\alpha \tilde{\partial}^\beta - x_\beta \bar{\partial}^\alpha][x^\alpha \tilde{\partial}^\beta - x^\beta \bar{\partial}^\alpha] \]
\[ = -\frac{1}{2} [x_\alpha \tilde{\partial}^\beta x^\alpha \tilde{\partial}^\beta - x_\alpha \tilde{\partial}^\beta x^\beta \bar{\partial}^\alpha - x_\beta \bar{\partial}^\alpha x^\alpha \tilde{\partial}^\beta + x_\beta \bar{\partial}^\alpha x^\beta \bar{\partial}^\alpha] \]
\[ = -\frac{1}{2} [x_\alpha \tilde{\partial}^\beta x^\alpha \tilde{\partial}^\beta - x_\alpha (\tilde{\partial} x) \bar{\partial}^\alpha - x_\alpha x^\alpha \tilde{\partial}^\beta \bar{\partial}^\alpha \]
\[ - x_\beta (\bar{\partial} x) \tilde{\partial}^\beta + x_\beta x^\alpha \tilde{\partial}^\beta \bar{\partial}^\alpha + x_\beta \theta_\beta^\alpha \tilde{\partial}^\alpha - x_\beta x^\beta \tilde{\partial}^\alpha \bar{\partial}^\alpha \]
\[ = H^{-2} \tilde{\partial}_\alpha \bar{\partial}^\alpha = H^{-2} \tilde{\partial}^2. \]
Proof of relation (4.20):
\[
\theta_{\alpha\beta}\theta^{\alpha\gamma} = (\eta_{\alpha \beta} + H^2 x_{\alpha} x_{\beta})(H^2 x^\alpha x^\gamma)
\]
\[
= \eta_{\alpha \beta} \eta^{\alpha\gamma} + H^2(x_{\alpha} x^\gamma + x^\gamma x_{\beta}) + H^2 x_{\alpha} x_{\beta} x^\gamma
\]
\[
= \eta_{\alpha \beta} \eta^{\alpha\gamma} + 2H^2 x_{\beta} x^\gamma - H^2 x_{\beta} x^\gamma = \eta_{\beta}^\alpha + H^2 x_{\beta} x^\gamma = \theta_{\beta}^\alpha.
\]

Proof of relation (4.21):
\[
\bar{\partial}_{\beta}\theta_{\alpha\sigma} = \bar{\partial}_{\beta}(\eta_{\alpha\sigma} + H^2 x_{\alpha} x_{\sigma}) = \bar{\partial}_{\beta}(H^2 x_{\alpha} x_{\sigma}) = H^2 \theta_{\beta\alpha} x_{\sigma} + H^2 x_{\sigma} \theta_{\beta\sigma}.
\]

Proof of relation (4.24):
\[
\bar{\partial}_{\alpha}\bar{\partial}_{\beta} = \bar{\partial}_{\alpha}(\theta_{\beta\gamma} \partial_{\gamma}) = (\bar{\partial}_{\alpha}\theta_{\beta\gamma}) \partial_{\gamma} + \theta_{\beta\gamma} \bar{\partial}_{\alpha} \partial_{\gamma}
\]
\[
= \theta_{\beta\gamma} \theta_{\alpha\lambda} \partial_{\lambda} \partial_{\gamma} + H^2 \theta_{\alpha\gamma} x_{\beta} \delta^\alpha_\delta \partial_{\gamma} + H^2 \theta_{\beta\gamma} x_{\alpha} \delta^\alpha_\beta \partial_{\gamma}
\]
\[
= \theta_{\beta\gamma} \theta_{\alpha\lambda} \partial_{\lambda} \partial_{\gamma} + (H^2 \theta_{\alpha\gamma} x_{\beta} + H^2 \theta_{\beta\alpha} x_{\gamma}) \partial_{\gamma}
\]
\[
= \theta_{\beta\gamma} \theta_{\alpha\lambda} \partial_{\lambda} \partial_{\gamma} + H^2 x_{\beta} \bar{\partial}_{\alpha} + H^2 \theta_{\alpha\beta}(x.\partial)
\]
\[
= \partial_{\beta}\partial_{\alpha} - H^2 \theta_{\beta\gamma} - H^2 \theta_{\beta\gamma}[\delta_{\gamma\alpha} x_{\lambda} + \delta_{\lambda\alpha} x_{\gamma}] \partial_{\gamma} + H^2[x_{\beta} \bar{\partial}_{\alpha} + \theta_{\alpha\beta}(x.\partial)]
\]
\[
\Rightarrow \quad \bar{\partial}_{\alpha}\bar{\partial}_{\beta} - \bar{\partial}_{\beta}\bar{\partial}_{\alpha} = -H^2 x_{\alpha} \bar{\partial}_{\beta} + H^2 x_{\beta} \bar{\partial}_{\alpha}
\]
\[
\Rightarrow \quad [\bar{\partial}_{\alpha}, \bar{\partial}_{\beta}] = H^2(x_{\beta}\bar{\partial}_{\alpha} - x_{\alpha}\bar{\partial}_{\beta}).
\]

Proof of relation (4.25):
\[
Q_0 x_{\alpha} f = -H^{-2} \partial^\gamma \partial_{\gamma} x_{\alpha} f = -H^{-2} \partial^\gamma[f \theta_{\gamma\alpha} f + x_{\alpha} \bar{\partial}_{\gamma} f]
\]
\[
= -H^{-2}[(\bar{\partial}^\gamma \theta_{\gamma\alpha}) f + \theta_{\gamma\alpha} \bar{\partial}^\gamma f + (\bar{\partial}^\gamma x_{\alpha}) \bar{\partial}_{\gamma} f x_{\alpha} \bar{\partial}^\alpha \partial_{\gamma} f]
\]
\[
\Rightarrow \quad Q_0 x_{\alpha} = -4x_{\alpha} - 2H^{-2} \bar{\partial}_{\alpha} + x_{\alpha} Q_0
\]
\[
\Rightarrow \quad [Q_0, x_{\alpha}] = -4x_{\alpha} - 2H^{-2} \bar{\partial}_{\alpha}.
\]

Proof of relation (4.26):
\[
Q_0 \bar{\partial}_{\alpha} f = -H^{-2} \bar{\partial}^\beta \bar{\partial}_{\beta}(\bar{\partial}_{\alpha} f) = -H^{-2} \bar{\partial}^\beta \bar{\partial}_{\beta} \bar{\partial}_{\alpha} f
\]
\[
= -H^{-2} \bar{\partial}^\beta[\bar{\partial}_{\alpha} \bar{\partial}_{\beta} - H^2 x_{\beta} \bar{\partial}_{\alpha} + H^2 x_{\alpha} \bar{\partial}_{\beta}]
\]
\[
= -H^{-2} \bar{\partial}^\beta \bar{\partial}_{\alpha} \bar{\partial}_{\beta} - H^2 x_{\beta} \bar{\partial}_{\alpha} + H^2 x_{\alpha} \bar{\partial}_{\beta} + (\bar{\partial}^\beta x_{\beta}) \bar{\partial}_{\alpha}
\]
\[
+ x_{\beta} \bar{\partial}^\beta \bar{\partial}_{\alpha} - [(\bar{\partial}^\beta x_{\alpha}) \bar{\partial}_{\beta} + x_{\alpha} \bar{\partial}^\beta \bar{\partial}_{\beta}]
\]
\[
= \bar{\partial}_{\alpha} Q_0 + (\bar{\partial}_{\alpha} x_{\beta} - \theta_{\alpha}^\beta \partial_{\beta} + 2H^2(Q_0 x_{\alpha} + 4x_{\alpha} + 2H^{-2} \bar{\partial}_{\alpha}) + 3\bar{\partial}_{\alpha}
\]
\[
= 57
\]
\[ Q_0 \bar{\partial}_a = \partial_a Q_0 + 6 \bar{\partial}_a + 2H^2(Q_0 + 4)x_a \]
\[ [Q_0, \bar{\partial}_a] = 6 \bar{\partial}_a + 2H^2(Q_0 + 4)x_a. \]

Proof of relation (4.30):

\[ D_1 Q_0 \phi = H^{-2} \bar{\partial}_a Q_0 \phi = H^{-2} Q_0 \bar{\partial}_a \phi - 6H^{-2} \bar{\partial}_a \phi - 2(Q_0 + 4)x_a \phi \]
\[ = H^{-2} Q_0 \bar{\partial}_a \phi - 6H^{-2} \bar{\partial}_a \phi - 8x_a \phi - 2x_a Q_0 \phi + 8x_a \phi + 4H^{-2} \bar{\partial}_a \phi \]
\[ \Rightarrow \quad D_1 Q_0 \phi = H^{-2} Q_0 \bar{\partial}_a \phi - 2H^{-2} \bar{\partial}_a \phi - 8x_a \phi - 2x_a Q_0 \phi. \]

On the other hand

\[ Q_1 D_1 \alpha \phi = (Q_0 - 2) D_1 \alpha \phi - 2x_a \bar{\partial}_\beta D_1 \beta \phi \]
\[ = Q_0 D_1 \phi - 2D_1 \phi - 2x_a Q_0 \phi. \]

Therefore

\[ Q_1 D_1 \phi = D_1 Q_0 \phi. \]

Proof of relation (4.36):

\[ Q_0 x.z \phi = Q_0 x^\alpha z_\alpha \phi = z_\alpha Q_0 x^\alpha \phi = z_\alpha [x_\alpha Q_0 - 4x_\alpha - 2H^{-2} \bar{\partial}_a] \phi \]
\[ = (x.z)(Q_0 - 4) \phi - 2H^{-2} z. \bar{\partial} \phi. \]

Proof of relation (4.53)

\[ x.k = x^\alpha k_\alpha = x_5 (x^\alpha \psi_\alpha + x^\alpha x_\alpha x. \psi) \]
\[ x_5 (x. \psi - x. \psi) = 0. \]
Appendix B

Detailed mathematical calculations of chapter 5

For \( p = 2 \) The Conformal system due to Dirac’s six cone is

\[
(Q_0 - 2)Q_0 \psi_{AB} = 0,
\]

\[
\hat{N}_5 \psi_{AB} = 0.
\]

Or

\[
(Q_0 - 2)Q_0 \psi' = 0,
\]

\[
(Q_0 - 2)Q_0 \psi_{\alpha\beta} = 0,
\]

\[
(Q_0 - 2)Q_0 \psi_{55} = 0.
\]

In general

\[
(Q_2 + 6)k_{\alpha\beta} = Q_0 k_{\alpha\beta} + 2S x_\alpha \, \partial \beta \, k_{\beta} + 2\eta_{\alpha\beta} k' - 2S \partial_\alpha x. k_{\beta},
\]

where

\[
k_{\alpha\beta} = \psi_{\alpha\beta} + S x_\beta \psi_\alpha x + x_\alpha x_\beta x. \psi x.
\]

The transversality condition yields

\[
u^5 \psi_{AB} = 0 \quad \Rightarrow \quad x^5 (\psi_{5B} + x. \psi_B) = 0
\]

\[
\Rightarrow \quad (Q_0 - 2)Q_0 \psi_{5B} + (Q_0 - 2)Q_0 x. \psi_B = 0
\]

\[
\Rightarrow \quad (Q_0 - 2)Q_0 x. \psi_B = 0.
\]

And the tracelessness implies yields

\[
k' = 0 \Rightarrow \psi_{\alpha\alpha} + x. \psi x = 0.
\]

Therefore

\[
(Q_0 - 2)Q_0 x. \psi_\beta = 0 \quad (I)
\]

\[
(Q_0 - 2)Q_0 x. \psi x = 0 \quad (II)
\]

We can show that

\[
x_\alpha (Q_0 - 2)Q_0 = (Q_0 - 2)(Q_0 x_\alpha + 4x_\alpha + 4\tilde{\alpha}) = (Q_0 - 2)Q_0 x_\alpha + 4(3x_\alpha + \tilde{\alpha})(Q_0 - 2).
\]
Multiply $x_\beta$ by (I):

$$(Q_0 - 2)(Q_0 x_\beta + 4 x_\beta + 4 \bar{\partial} x_\beta) x. \psi_\beta = 0$$

$$\Rightarrow (Q_0 - 2)Q_0 x. \psi. x + 4(Q_0 - 2)x. \psi. x + 4(Q_0 - 2)\bar{\partial} x. \psi. x = 0.$$  

From (II) we have

$$(Q_0 - 2)x. \psi. x = -(Q_0 - 2)\bar{\partial} x. \psi. x$$

$$\Rightarrow (Q_0 - 2)(x. \psi. x + \bar{\partial} x. \psi. x) = 0.$$  

(III)

The divergence of $\psi_{\alpha\beta}$ is

$$\nabla_a \psi^{ab} = 0,$$

therefore

$$\partial x. \psi_B = -x. \partial x. \psi_B,$$

and

$$\partial x. \psi_5 = -x. \partial x. \psi_5.$$  

Also the transversality condition $u_{\alpha} \psi^{ab} = 0$ results in

$$x^5(\psi_{5b} + x. \psi_b) = 0,$$

therefore

$$\partial x. \psi. x + x. \partial x. \psi. x = 0.$$  

The divergence of $k_{\alpha\beta}$ is

$$\bar{\partial} k_\alpha = \bar{\partial} \psi_\alpha + 4 \psi_\alpha x + \theta_{\alpha\beta} \psi_\beta x + x_\alpha \bar{\partial} \psi. x + \theta_{\alpha\beta} x_\beta x. \psi. x + 4 x_\alpha x. \psi. x$$

$$\Rightarrow \bar{\partial} k_\alpha = 4(\psi_\alpha x + x_\alpha x. \psi. x) + (\bar{\partial} \psi_\alpha + x. \psi_\alpha + x_\alpha \bar{\partial} \psi. x + x_\alpha x. \psi. x).$$

Since our conformal degree degree is 1, we have:

$$\bar{\partial} \psi_\beta = -x. \bar{\psi}_\beta,$$

and

$$\bar{\partial} x. \psi. x = -x. \psi. x.$$  

Therefore

$$\bar{\partial} k_\alpha = 4(\psi_\alpha x + x_\alpha x. \psi. x).$$  

(IV)

Let us combine (III) and (IV). This yields have

$$(Q_0 - 2)(\frac{1}{12} \bar{\partial} \bar{\partial} k - x. \psi. x) = 0.$$  

(V)
Now multiply both sides of equation (I) by $\bar{\partial}_\beta$. We obtain

$$(Q_0 - 2)Q_0\bar{\partial}_\beta \cdot x - 4x_\beta(Q_0 - 2)Q_0 x_\beta = 0.$$  

Using (I) we have

$$(Q_0 - 2)Q_0\bar{\partial}_\beta \cdot x = 0.$$  

The effect of $(Q_0 - 2)Q_0$ on our tensor field, will be

$$(Q_0 - 2)Q_0k_{\alpha\beta} = (Q_0 - 2)Q_0S_{\alpha\beta} \cdot x + (Q_0 - 2)Q_0x_\alpha x_\beta \cdot x \cdot x. \quad (\ast)$$

We have

$$(Q_0 - 2)Q_0x_\alpha \psi_\beta \cdot x = x_\alpha(Q_0 - 2)Q_0\psi_\beta \cdot x - 4(3x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)x_\beta x_\beta.$$

and

$$(Q_0 - 2)Q_0x_\alpha x_\beta \cdot x \cdot x = x_\alpha x_\beta(Q_0 - 2)Q_0 x_\beta \cdot x - 4x_\alpha(3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)x_\beta \cdot x \cdot x
- 4(3x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)x_\beta \cdot x \cdot x.$$

Therefore using equation (IV), equation $(\ast)$ by substitution yields

$$(Q_0 - 2)Q_0k_{\alpha\beta} = -4(3x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)\bar{\partial}_\alpha \cdot x - 4(3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)\bar{\partial}_\beta.$$

Finally using equation (V) we get

$$(Q_0 - 2)k_{\alpha\beta} = -4(3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)\bar{\partial}_\alpha \cdot x - 3x_\alpha(2\bar{\partial}_\alpha \bar{\partial}_\beta) + \bar{\partial}_\alpha \bar{\partial}_\beta.$$

Now let us multiply both sides of above equation by $x_\beta$.

$$4(Q_0 - 2)\bar{\partial}_\alpha \cdot x = 12(Q_0 - 2)\psi_\alpha \cdot x - 3x_\alpha x_\beta(Q_0 - 2)\bar{\partial}_\alpha \bar{\partial}_\beta - \bar{\partial}_\alpha x_\beta(Q_0 - 2)\bar{\partial}_\beta$$

$$\Rightarrow 4(Q_0 - 2)\bar{\partial}_\alpha \cdot x = 12(Q_0 - 2)\psi_\alpha \cdot x - 3x_\alpha(2\bar{\partial}_\alpha \bar{\partial}_\beta) - 2\bar{\partial}_\alpha \bar{\partial}_\beta$$

$$+ (Q_0 - 2)\bar{\partial}_\alpha \cdot x + x_\alpha(2\bar{\partial}_\alpha \bar{\partial}_\beta) + x_\alpha(Q_0 - 2)\bar{\partial}_\beta$$

$$\Rightarrow (Q_0 - 2)\psi_\alpha \cdot x = \frac{1}{4}(Q_0 - 2)\bar{\partial}_\alpha \cdot x + \frac{1}{3}x_\alpha \bar{\partial}_\beta - \frac{1}{12}x_\alpha(Q_0 - 2)\bar{\partial}_\beta \bar{\partial}_\alpha \bar{\partial}_\beta$$

$$+ \frac{1}{6}\bar{\partial}_\alpha \bar{\partial}_\beta \bar{\partial}_\alpha \cdot x.$$
Substitution in (**) yields

\[(Q_0-2)Q_0 k_{\alpha \beta} = -4(3x_\beta + \bar{\partial}_\beta)\left\{\frac{1}{4}(Q_0-2)\bar{\partial}.k_\alpha + \frac{1}{3} x_\alpha \bar{\partial}.\bar{\partial}.k - \frac{1}{12} x_\alpha (Q_0-2)\bar{\partial}.\bar{\partial}.k + \frac{1}{6} \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k \right\}
\]

\[-(3x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)\bar{\partial}.k_\beta - \frac{1}{3} x_\alpha (3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)\bar{\partial}.\bar{\partial}.k
\]

\[\Rightarrow \quad (Q_0 - 2)Q_0 k_{\alpha \beta} = -(3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)\bar{\partial}.k_\alpha - \frac{4}{3} (3x_\beta + \bar{\partial}_\beta)x_\alpha \bar{\partial}.\bar{\partial}.k
\]

\[+\frac{1}{3} (3x_\beta + \bar{\partial}_\beta)x_\alpha (Q_0 - 2)\bar{\partial}.\bar{\partial}.k - \frac{2}{3} (3x_\beta + \bar{\partial}_\beta)\bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k
\]

\[-(3x_\alpha + \bar{\partial}_\alpha)(Q_0 - 2)\bar{\partial}.k_\beta - \frac{1}{3} x_\alpha (3x_\beta + \bar{\partial}_\beta)(Q_0 - 2)\bar{\partial}.\bar{\partial}.k
\]

\[\Rightarrow \quad (Q_0 - 2)Q_0 k_{\alpha \beta} = -3x_\beta (Q_0 - 2)\bar{\partial}.k_\alpha - \bar{\partial}_\beta (Q_0 - 2)\bar{\partial}.k_\alpha - 4x_\beta x_\alpha \bar{\partial}.\bar{\partial}.k
\]

\[-\frac{4}{3} \bar{\partial}_\beta x_\alpha \bar{\partial}.\bar{\partial}.k + x_\beta x_\alpha (Q_0 - 2)\bar{\partial}.\bar{\partial}.k + \frac{1}{3} \bar{\partial}_\beta x_\alpha (Q_0 - 2)\bar{\partial}.\bar{\partial}.k - 2x_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k
\]

\[-\frac{2}{3} \bar{\partial}_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k - 3x_\alpha (Q_0 - 2)\bar{\partial}.k_\beta - \bar{\partial}_\alpha (Q_0 - 2)\bar{\partial}.k_\beta
\]

\[-x_\alpha x_\beta (Q_0 - 2)\bar{\partial}.\bar{\partial}.k - \frac{1}{3} x_\alpha \bar{\partial}_\beta (Q_0 - 2)\bar{\partial}.k_\beta
\]

Extending the operators on the vectors and partial differentials, leads to

\[(Q_0 - 2)Q_0 k_{\alpha \beta} = -3Q_0 x_\beta \bar{\partial}.k_\alpha - 6x_\beta \bar{\partial}.k_\alpha - 6\bar{\partial}_\beta \bar{\partial}.k_\alpha - Q_0 \bar{\partial}_\beta \bar{\partial}.k_\alpha + 8\bar{\partial}_\beta \bar{\partial}.k_\alpha
\]

\[+ 2Q_0 x_\beta \bar{\partial}.k_\alpha + 8x_\beta \bar{\partial}.k_\alpha - 4x_\alpha x_\beta \bar{\partial}.\bar{\partial}.k - \frac{4}{3} \theta_{\alpha \beta} \bar{\partial}.\bar{\partial}.k
\]

\[-\frac{4}{3} x_\alpha \bar{\partial}_\beta \bar{\partial}.\bar{\partial}.k + \frac{1}{3} \theta_{\alpha \beta} (Q_0 - 2)\bar{\partial}.\bar{\partial}.k + \frac{1}{3} x_\alpha \bar{\partial}_\beta (Q_0 - 2)\bar{\partial}.\bar{\partial}.k
\]

\[-2x_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k - \frac{2}{3} \bar{\partial}_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k - 3Q_0 x_\alpha \bar{\partial}.k_\beta - 6x_\alpha \bar{\partial}.k_\beta
\]

\[-6\bar{\partial}_\alpha \bar{\partial}.k_\beta - Q_0 \bar{\partial}_\alpha \bar{\partial}.k_\beta + 8\bar{\partial}_\alpha \bar{\partial}.k_\beta + 2Q_0 x_\alpha \bar{\partial}.k_\beta
\]

\[+ 8x_\alpha \bar{\partial}.k_\beta - \frac{1}{3} x_\alpha \bar{\partial}_\beta Q_0 \bar{\partial}.\bar{\partial}.k + \frac{2}{3} x_\alpha \bar{\partial}_\beta \bar{\partial}.\bar{\partial}.k.
\]

After some simplifications, we will have

\[(Q_0 - 2)Q_0 k_{\alpha \beta} = -Q_0 x_\beta \bar{\partial}.k_\alpha - Q_0 x_\alpha \bar{\partial}.k_\beta - Q_0 \bar{\partial}_\beta \bar{\partial}.k_\alpha - Q_0 \bar{\partial}_\alpha \bar{\partial}.k_\beta
\]

\[+ 2x_\beta \bar{\partial}.k_\alpha + 2x_\alpha \bar{\partial}.k_\alpha + 2\bar{\partial}_\beta \bar{\partial}.k_\alpha + 2\bar{\partial}_\alpha \bar{\partial}.k_\alpha
\]

\[-4x_\alpha x_\beta \bar{\partial}.\bar{\partial}.k - \frac{4}{3} x_\alpha \bar{\partial}_\beta \bar{\partial}.\bar{\partial}.k - 2x_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k
\]

\[-2\theta_{\alpha \beta} \bar{\partial}.\bar{\partial}.k + \frac{1}{3} \theta_{\alpha \beta} Q_0 \bar{\partial}.\bar{\partial}.k - \frac{2}{3} \bar{\partial}_\beta \bar{\partial}_\alpha \bar{\partial}.\bar{\partial}.k.
\]
Finally, the conformal equation for tensor fields, with $p = 2$, is derived, i.e.

\[
(Q_0 - 2)Q_0 k_{\alpha\beta} = -Q_0 S x_\beta \partial_\alpha k_\beta - Q_0 S \partial_a \partial_\alpha k_\beta + 2 S x_\alpha \partial_\alpha k_\beta \\
+ 2 S \partial_\alpha \partial_\beta k_\beta - 4 x_\alpha x_\beta \partial_\alpha \partial_\beta k - \frac{1}{3} S \partial_\beta \partial_\alpha \partial_\alpha \partial_\beta k \\
- \frac{5}{3} S x_\alpha \partial_\beta \partial_\alpha \partial_\beta k - 2 \theta_{\alpha\beta} \partial_\alpha \partial_\beta k + \frac{1}{3} \theta_{\alpha\beta} Q_0 \partial_\alpha \partial_\beta k. \tag{***}
\]

Note that

\[
- \frac{1}{3} S \partial_\beta \partial_\alpha = \frac{1}{3} (\partial_\beta \partial_\alpha + \partial_\alpha \partial_\beta) = - \frac{1}{3} (2 \partial_\beta \partial_\alpha - x_\alpha \partial_\beta + x_\beta \partial_\alpha) \\
\Rightarrow \quad - \frac{1}{3} S \partial_\beta \partial_\alpha = - \frac{2}{3} \partial_\beta \partial_\alpha + \frac{1}{3} x_\alpha \partial_\beta - \frac{1}{3} x_\beta \partial_\alpha.
\]

Another representation of (*** is

\[
(Q_2 + 4)[(Q_2 + 6) k_{\alpha\beta} + D_2 \partial_\alpha] + \frac{1}{3} D_2 D_1 \partial_\alpha \partial_\beta k - \frac{1}{3} \theta_{\alpha\beta} (Q_0 + 6) \partial_\alpha \partial_\beta k = 0.
\]

Now let us prove the recursion relation between (5.20) and (5.22). The transverse projections of all the parts of Barut and Xu conformal equation, are as below

\[
\Box h_{\mu\nu} = \text{TrPr} \partial_\alpha \partial_\beta k_{\mu\nu} = \partial_\alpha (\partial_\alpha k_{\mu\nu} - x_\mu k_{\alpha\nu} - x_\nu k_{\alpha\mu}) - x_\alpha (\partial_\alpha k_{\mu\nu} - x_\mu k_{\alpha\nu} - x_\nu k_{\alpha\mu}) \\
- x_\mu (\partial_\alpha k_{\alpha\nu} - x_\nu k_{\alpha\alpha}) - x_\nu (\partial_\alpha k_{\alpha\mu} - x_\mu k_{\alpha\alpha}) \\
= \partial_\alpha \partial_\alpha k_{\mu\nu} - \partial_\alpha \partial_{\alpha\nu} - \partial_\alpha \partial_{\alpha\mu} - x_\mu \partial_\alpha k_{\alpha\nu} - x_\nu \partial_\alpha k_{\alpha\mu} \\
= (Q_0 - 2) k_{\mu\nu} - 2 S x_\mu \partial_\nu k_{\nu}. \tag{VI}
\]

\[
\nabla_\mu \nabla_\nu h = \text{TrPr} \partial_\mu \partial_\lambda k_{\lambda\nu} = \partial_\mu (\partial_\lambda k_{\lambda\nu} - x_\lambda k_{\lambda\nu} - x_\nu k_{\lambda\lambda}) - x_\lambda (\partial_\mu k_{\lambda\nu} - x_\lambda k_{\mu\nu} - x_\nu k_{\lambda\mu}) \\
- x_\lambda (\partial_\lambda k_{\mu\nu} - x_\mu k_{\mu\nu} - x_\nu k_{\mu\lambda}) - x_\nu (\partial_\lambda k_{\mu\nu} - x_\lambda k_{\mu\nu}) \\
= \partial_\mu \partial_\lambda k_{\lambda\nu} + x_\lambda \partial_\lambda k_{\mu\nu} - x_\nu \partial_\lambda k_{\mu\lambda} - x_\nu \partial_\lambda k_{\mu\lambda} \\
- \partial_\mu \partial_\nu k_{\nu}. \tag{VII}
\]

\[
\nabla_\mu \nabla_\nu h = \text{TrPr} \partial_\mu \text{TrPr} \partial_\nu = \partial_\mu (\partial_\nu k_{\mu\nu} - x_\mu k_{\mu\nu} - x_\nu k_{\mu\mu}) - x_\nu (\partial_\mu k_{\mu\nu} - x_\mu k_{\mu\nu} - x_\nu k_{\mu\mu})
\]

63
\[-x_\mu (\bar{\partial}_\nu k_{\mu\nu} - x_\mu k_{\nu\nu} - x_\nu k_{\mu\mu}) - x_\nu (\bar{\partial}_\nu k_{\mu\mu} - x_\mu k_{\mu\nu} - x_\mu k_{\mu\nu})\]
\[= \bar{\partial}_\mu \bar{\partial}_\nu k_{\mu\nu} - \theta_{\mu\nu} k_{\mu\nu} = \bar{\partial} \bar{\partial} k - k_{\mu\nu}. \quad (VIII)\]

Note the following transformation:
\[g_{\mu\nu} \rightarrow \theta_{\mu\nu} \quad (IX)\]

Therefore
\[(\Box + 4) h_{\mu\nu} - \frac{2}{3} S \nabla_\mu \nabla h_{\nu} + \frac{1}{3} g_{\mu\nu}^{\text{ds}} \nabla \nabla h = 0.\]

\[ (Q_0 - 2) k_{\mu\nu} - 2 S x_\mu \bar{\partial} k_{\nu} - \frac{2}{3} [S \bar{\partial}_\mu \bar{\partial} k_{\nu} - S x_\nu \bar{\partial} k_{\mu}] + \frac{1}{3} \theta_{\mu\nu} [\bar{\partial} \bar{\partial} k - k_{\mu\nu}] = 0 \]
\[\Rightarrow \quad (Q_0 - 2) k_{\mu\nu} - \frac{2}{3} S (\bar{\partial}_\mu + 2 x_\mu) \bar{\partial} k_{\nu} + \frac{1}{3} \theta_{\mu\nu} \bar{\partial} \bar{\partial} k = 0. \]
Bibliography

[1] Walter Greiner, Berndt Müller, *Quantum Mechanics - Symmetries*. Springer (1994). ISBN 3-540-58080-8.

[2] Larry C. Grove, Classical groups and geometric algebra, Graduate Studies in Mathematics, 39, Providence, R.I.: American Mathematical Society, MR1859189, (2002). ISBN 978-0-8218-2019-3.

[3] Walter Greiner, Andreas Schfer, *Quantum Chromodynamics*, Springer (1994). ISBN 0-387-57103-5.

[4] James E Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Berlin, New York: Springer-Verlag, MR0396773, (1972). ISBN 978-0-387-90108-4.

[5] G. Mack, A. Salam, *Finite-component field representation of the conformal group*, Ann. Phys. 53 (1969) 174-202.

[6] M. Hortacsu, *Explicit Examples on Conformal Invariance*, Int. J. Theor. Phys. 42 (2003) 49.

[7] A. Maduemezia, *On the Conformal Group Algebra and Its Unitary Representations in the Space of Harmonic Functions*, International Center Of Theoretical Physics (1967).

[8] Ivan Kachuryk and Anatoliy Klimyk, *Eigenfunction Expansions of Functions Describing Systems with Symmetries*, SIGMA 3 (2007), 055, 84 pages.

[9] Sadri Hassani, *Mathematical Physics: A Modern Introduction to Its Foundations*, Berlin, Springer-Verlag, (1999). ISBN 0387985794.

[10] John David Jackson, *Classical Electrodynamics (3rd ed.).* New York: Wiley. (1999) ISBN 0-471-30932-X.
[11] Gregory Naber, *The Geometry of Minkowski Spacetime*, New York: Springer-Verlag (1992). ISBN 0-486-43235-1 (Dover reprint edition). An excellent reference on Minkowski spacetime and the Lorentz group.

[12] A. N. Leznov, *What our world might be like*, arXiv:0803.4289.

[13] I. Licata, *Universe Without Singularities. A Group Approach to De Sitter Cosmology*, Electron. J. Theor. Phys. 3:211-224,(2006).

[14] Yu. A. Neretin, *Plancherel formula for Berezin deformation of (L2) on Riemannian symmetric space*, arXiv:math/9911020.

[15] M. Bertola, F. Corbetta, U. Moschella, *Massless scalar field in two-dimensional de Sitter universe*, (2006), math-ph/0609080.

[16] George B. Arfken, Hans J. Weber, *Mathematical Methods for Physicists*. Elsevier Academic Press. p. 743. (2005). ISBN 0120598760.

[17] Weisstein, Eric W. *Conformal Mapping*, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ConformalMapping.html

[18] I. I. Tugov, *Conformal covariance and invariant formulation of scalar wave equations*. Annales de l’I. H. P., section A, tome 11, no 2 (1969), p. 207-220.

[19] M. P. da Browski, J. Garecki, D. B. Blaschke, *conformal transformations and conformal invariance in gravitation*, Annalen der Physik, vol. 18, issue 1 (2008), pp. 13-32.

[20] Yoshihiro Tashiro, *Remarks on a theorem concerning conformal transformations*, Proc. Japan Acad. Volume 35, Number 8 (1959), 421-422.

[21] Philippe Christie, Malte Henkel, *Introduction to Conformal Invariance and its Applications to Critical Phenomena*, Springer Lecture Notes in Physics, Vol. m16 (1993).

[22] Steven Weinberg,(1972), *Gravitation and Cosmology*, New York, Wiley (1972), ISBN 0-471-925.

[23] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. 2. Redwood City, CA: Addison-Wesley, (1989).
[24] Joseph Wolf, *Spaces of constant curvature*, (1967) p. 334.

[25] Albert Einstein, *The Foundation of the General Theory of Relativity*. Annalen der Physik, (1916). [http://www.alberteinstein.info/gallery/gtext3.html](http://www.alberteinstein.info/gallery/gtext3.html)

[26] Albert Einstein, Albert, *Die Feldgleichungen der Gravitation*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin, (November 25, 1915) 844847.

[27] George Gamow, *My World Line: An Informal Autobiography*, Viking Adult (April 28, 1970). ISBN 0670503762. [http://www.jb.man.ac.uk/~jpl/cosmo/blunder.html](http://www.jb.man.ac.uk/~jpl/cosmo/blunder.html) (Retrieved 2007-03-14).

[28] Nicolle Wahl, *Was Einstein’s ‘biggest blunder’ a stellar success?*, (2005-11-22). [http://www.news.utoronto.ca/bin6/051122-1839.asp](http://www.news.utoronto.ca/bin6/051122-1839.asp) (Retrieved 2007-03-14).

[29] Michael S. Turner, *A Spacetime Odyssey*, Int. J. Mod. Phys. A. (2001) 17S1: 180196. [http://arxiv.org/abs/astro-ph/0202008](http://arxiv.org/abs/astro-ph/0202008) (Retrieved 2007-03-14).

[30] Harvey Brown, *Physical Relativity*, Oxford University Press (2005) p. 164.

[31] Andrzej Trautman, *Solutions of the Maxwell and Yang-Mills equations associated with hopf fibrings*, International Journal of Theoretical Physics 16 (9): 561565 (1977). doi:10.1007/BF01811088.

[32] Hans Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt. *Exact Solutions of Einstein’s Field Equations*, Cambridge University Press, (2003). ISBN 0-521-46136-7.

[33] Qingming Cheng (2001), ”De Sitter space”, in Hazewinkel, Michiel, Encyclopaedia of Mathematics, Kluwer Academic Publishers, (2001). ISBN 978-1556080104. [http://ecom.springer.de/d/d110040.htm](http://ecom.springer.de/d/d110040.htm)

[34] W. de Sitter, *On the relativity of inertia: Remarks concerning Einstein’s latest hypothesis*, Proc. Kon. Ned. Acad. Wet. (1917) 19: 12171225.

[35] W. de Sitter, *On the curvature of space*, Proc. Kon. Ned. Acad. Wet. (1917) 20: 229243.

[36] K. Nomizu, *The Lorentz-Poincar metric on the upper half-space and its extension*, Hokkaido Mathematical Journal 11 (3): 253261 (1982).
[37] H. S. M. Coxeter, *A geometrical background for de Sitter’s world*, American Mathematical Monthly 50: 217-228 (1943), doi:10.2307/2303924.

[38] L. Susskind, J. Lindesay, *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*, p. 119 (2005).

[39] Tullio Levi-Civita, *Realt fisica di alconi spaz normali del Bianchi*, Rendiconti, Reale Accademia Dei Lincei 26: 51931 (1917).

[40] H. Goenner, *Einstein tensor and generalizations of Birkhoffs theorem*, Commun. Math. Phys. 16, 34 (1970).

[41] P.D. Mannheim and D. Kazanas, *Exact vacuum solution to conformal Weyl gravity and galactic rotation curves*, Astrophysical Journal 342: 635 (1989).

[42] see S. Deser, A. Waldron, *Conformal invariance of partially massless higher spins*, Physics Letters B 603 (2004) 3034, arXiv:hep-th/0408155.

S. Deser, A. Waldron, *Partial masslessness of higher spins in (A)dS*, Nuclear Physics B 607 [FS] (2001) 577604; and other related works by same authors.

[43] S. Behroozi, S. Rouhani, M. V. Takook, and M. R. Tanhayi, *Conformally invariant wave equations and massless fields in de Sitter spacetime*, Phys. Rev. D 74, 124014 (2006).

[44] S. Behroozi, S. Rouhani, M. V. Takook, and M. R. Tanhayi, *Conformally invariant massless spin-2 field in the de Sitter universe*, Phys. Rev. D 74, 124014 (2008).

[45] M.V. Takook, M.R. Tanhayi, S. Fatemi, *Conformal linear gravity in de Sitter space*, J. Math. Phys. 51: 032503 (2010).