Non-tangential limits for analytic Lipschitz functions

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Abstract

Let $U$ be a bounded open subset of the complex plane. Let $0 < \alpha < 1$ and let $A_\alpha(U)$ denote the space of functions that satisfy a Lipschitz condition with exponent $\alpha$ on the complex plane, are analytic on $U$ and are such that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $z, w \in U$, $|f(z) - f(w)| \leq \epsilon |z - w|^\alpha$ whenever $|z - w| < \delta$. We show that if a boundary point $x_0$ for $U$ admits a bounded point derivation for $A_\alpha(U)$ and $U$ has an interior cone at $x_0$ then one can evaluate the bounded point derivation by taking a limit of a difference quotient over a non-tangential ray to $x_0$. Notably our proofs are constructive in the sense that they make explicit use of the Cauchy integral formula.

1 Background and statement of results

In this paper, we consider the behavior of Lipschitz functions which are analytic on a bounded open subset of the complex plane and how much analyticity extends to the boundary of the domain. Let $U$ be an open subset in the complex plane and let $0 < \alpha < 1$. A function $f : U \to \mathbb{C}$ satisfies a Lipschitz condition with exponent $\alpha$ on $U$ if there exists $k > 0$ such that for all $z, w \in U$

$$|f(z) - f(w)| \leq k|z - w|^\alpha$$ (1)

Let $\text{Lip}_\alpha(U)$ denote the space of functions that satisfy a Lipschitz condition with exponent $\alpha$ on $U$. $\text{Lip}_\alpha(U)$ is a Banach space with norm given by $||f||_{\text{Lip}_\alpha(U)} = \sup_U |f| + k(f)$, where $k(f)$ is the smallest constant that satisfies (1). If we let $||f||'_{\text{Lip}_\alpha(U)} = k(f)$ then $||f||'_{\text{Lip}_\alpha(U)}$ is a seminorm on $\text{Lip}_\alpha(U)$.

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An important subspace of Lip$_\alpha(U)$ is the little Lipschitz class, lip$_\alpha(U)$, which consists of those functions in Lip$_\alpha(U)$ that also satisfy the additional property that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $z, w$ in $U$, $|f(z) - f(w)| \leq \epsilon |z - w|^\alpha$ whenever $|z - w| < \delta$.

The importance of lip$_\alpha(U)$ is illustrated by the following result of De Leeuw [1]. Let $\Delta$ be a closed disk. Then the restriction spaces Lip$_\alpha(\Delta) = \{ f|\Delta : f \in$ Lip$_\alpha(\mathbb{C}) \}$ and lip$_\alpha(\Delta) = \{ f|\Delta : f \in$ lip$_\alpha(\mathbb{C}) \}$ are Banach spaces and lip$_\alpha^{**}(\Delta)$ is isometrically isomorphic to Lip$_\alpha(\Delta)$. Thus the weak-star topology can be applied to Lip$_\alpha(\Delta)$ as the dual of lip$_\alpha^*(\Delta)$.

Let $U$ be a bounded open subset of the complex plane. We will restrict our study to those functions in lip$_\alpha(\mathbb{C})$ which are analytic on $U$. Let $A_\alpha(U) = \{ f \in$ lip$_\alpha : \overline{\partial}f = 0$ on $U \}$, where $\overline{\partial}f = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$. For an arbitrary set $E \subset \mathbb{C}$, let $A_\alpha(E) = \bigcup \{ A_\alpha(U) : U$ open , $E \subset U \}$.

While the functions in $A_\alpha(U)$ are differentiable on the interior of $U$, they need not be differentiable on the boundary of $U$. In this paper, we consider the question of how close the functions in $A_\alpha(U)$ come to being differentiable at boundary points of $U$. To answer this question we will make use of the concept of a bounded point derivation. For $x_0 \in \mathbb{C}$, it is known that $A_\alpha(U \cup \{ x_0 \})$ is dense in $A_\alpha(U)$. [4, Lemma 1.1] Thus we say that $A_\alpha(U)$ admits a bounded point derivation at $x_0$ if the map $f \to f'(x_0)$ extends from $A_\alpha(U \cup \{ x_0 \})$ to a bounded linear functional on $A_\alpha(U)$. Equivalently, $A_\alpha(U)$ admits a bounded point derivation at $x_0$ if and only if there exists a constant $C > 0$ such that

$$|f'(x_0)| \leq C \| f \|_{\text{Lip}_\alpha(\mathbb{C})}, \tag{2}$$

for all $f$ in $A_\alpha(U \cup \{ x_0 \})$.

The existence of a bounded point derivation at $x_0$ shows that the functions in $A_\alpha(U)$ possess some semblance of analytic structure at $x_0$. If, in addition, $U$ has an interior cone at $x_0$, a more explicit description of this analytic structure can be obtained. We say that $U$ has an interior cone at $x_0$ if there is a segment $J$ ending at $x_0$ and a constant $k > 0$ such that $\text{dist}(x, \partial U) \geq k|x - x_0|$ for all $x$ in $J$. The segment $J$ is called a non-tangential ray to $x_0$. It is a result of O'Farrell [5] that if $U$ has an interior cone at a boundary point $x_0$, then a bounded point derivation on $A_\alpha(U)$ at $x_0$ can be evaluated by taking the limit of the difference quotient over a non-tangential ray to $x_0$. To be precise, O'Farrell has proven the following theorem.
**Theorem 1.** Let $0 < \alpha < 1$, and let $U$ be an open set with $x_0$ in $\partial U$. Suppose that $U$ has an interior cone at $x_0$ and that $J$ is a non-tangential ray to $x_0$. If $A_\alpha(U)$ admits a bounded point derivation $D$ at $x_0$, then for every $f \in A_\alpha(U)$,

$$Df = \lim_{x \to x_0, x \in J} \frac{f(x) - f(x_0)}{x - x_0}. $$

Thus the difference quotient for boundary points that admit bounded point derivations for $A_\alpha(U)$ converges when taken over a non-tangential ray to the point. This illustrates the additional analytic structure of functions in $A_\alpha(U)$ at these points.

O’Farrell comments that the methods used in his proof of Theorem 1 are non-constructive, involving abstract measures and duality arguments from functional analysis as opposed to using the Cauchy integral formula directly, and suggests that it should be possible to give a proof using constructive techniques. In this paper we present a constructive proof of Theorem 1 which confirms O’Farrell’s conjecture. In Section 2 we review some key properties of $A_\alpha(U)$ and in Section 3 we prove Theorem 1 using constructive techniques.

## 2 Preliminary Results

We begin by reviewing the Hausdorff content of a set, which is defined using measure functions. A measure function is a monotone nondecreasing function $h : [0, \infty) \to [0, \infty)$. For example, $r^\beta$ is a measure function for $0 \leq \beta < \infty$. If $h$ is a measure function then the Hausdorff content $M_h$ associated to $h$ is defined by

$$M_h(E) = \inf \sum h(\text{diam } B),$$

where the infimum is taken over all countable coverings of $E$ by balls and the sum is taken over all the balls in the covering. If $h(r) = r^\beta$ then we denote $M_h$ by $M^\beta$. The lower $1 + \alpha$ dimensional Hausdorff content $M_{*}^{1+\alpha}(E)$ is defined by

$$M_{*}^{1+\alpha}(E) = \sup M_h(E),$$

where the supremum is taken over all countable coverings of $E$ by balls and the sum is taken over all the balls in the covering.
where the supremum is taken over all measurable functions $h$ such that $h(r) \leq r^{1+\alpha}$ and $r^{-1-\alpha}h(r)$ converges to 0 as $r$ tends to 0. The lower $1+\alpha$ dimensional Hausdorff content is a monotone set function; i.e. if $E \subseteq F$ then $M_\ast^{1+\alpha}(E) \leq M_\ast^{1+\alpha}(F)$.

In [4], Lord and O’Farrell gave necessary and sufficient conditions for the existence of bounded point derivations on $A_\alpha(U)$ in terms of Hausdorff contents. There are similar conditions for bounded point derivations defined on other function spaces. ([2], [3])

**Theorem 2.** Let $U$ be an open subset of the complex plane with $x_0$ on the boundary of $U$. Let $0 < \alpha < 1$. Then $A_\alpha(U)$ has a bounded point derivation at $x_0$ if and only if

$$\sum_{n=1}^{\infty} 4^n M_\ast^{1+\alpha}(A_n(x_0) \setminus U) < \infty.$$ 

Another key lemma is the following Cauchy theorem for Lipschitz functions which also appears in the paper of Lord and O’Farrell [4, pg.110].

**Lemma 3.** Let $\Gamma$ be a piecewise analytic curve bounding a region $\Omega \in \mathbb{C}$, and suppose that $\Gamma$ is free of outward pointing cusps. Let $0 < \alpha < 1$ and suppose that $f \in \text{lip}_\alpha(\mathbb{C})$. Then there exists a constant $\kappa > 0$ such that

$$\left| \int f(z)dz \right| \leq \kappa \cdot M_\ast^{1+\alpha}(\Omega \cap S) \cdot ||f||_{\text{Lip}_\alpha(\Omega)}.$$ 

The constant $\kappa$ only depends on $\alpha$ and the equivalence class of $\Gamma$ under the action of the conformal group of $\mathbb{C}$. In particular this means that $\kappa$ is the same for any curve obtained from $\Gamma$ by rotation or scaling.

**3 The proof of the main theorem**

To prove Theorem [1], we first note that by translation invariance we may suppose that $x_0 = 0$. Moreover by replacing $f$ by $f - f(0)$ if needed, we may suppose that $f(0) = 0$. In addition, we may suppose that $U$ is contained in the unit disk. Let $J$ be a non-tangential ray to $x_0$ and for each $x$ in $J$, define a linear functional $L_x$ by $L_x(f) = \frac{f(x)}{x} - Df$. Then to prove Theorem [1] it suffices to show that $L_x$ tends to the 0 functional as $x \to 0$ through $J$. We make the following claim.
Lemma 4. The collection \( \{L_x : x \in J\} \) is a family of bounded linear functionals on \( A_\alpha(U) \); that is there exists a constant \( C > 0 \) that does not depend on \( x \) or \( f \) such that \( |L_x(f)| \leq C \|f\|_{\text{Lip}(\Omega)} \) for all \( f \) in \( A_\alpha(U) \) and all \( x \in J \).

Proof. We will first prove Lemma 4 for the case when \( f \) belongs to \( A_\alpha(U \cup \{0\}) \) and then extend to the general case. It follows from (2) that it is enough to show that \( \frac{f(x)}{x} \leq C \|f\|_{\text{Lip}(\Omega)} \) where the constant \( C \) does not depend on \( f \) or \( x \). If \( f \) belongs to \( A(U \cup \{0\}) \), then there is a neighborhood \( \Omega \) of 0 such that \( f \) is analytic on \( \Omega \). We can further suppose that \( U \subseteq \Omega \). Let \( B_n \) denote the ball centered at 0 with radius \( 2^{-n} \). Then there exists an integer \( N > 0 \) such that \( \Omega \) contains \( B_N \) and hence \( f \) is analytic inside the ball \( B_N \). In addition, there exists an integer \( M \) such that \( \Omega \subseteq B_M \). Since \( J \) is a non-tangential ray to \( x_0 \), it follows that there is a sector in \( \dot{U} \) with vertex at \( x_0 \) that contains \( J \). Let \( C \) denote this sector. It follows from the Cauchy integral formula that

\[
\frac{f(x)}{x} = \frac{1}{2\pi i} \int_{\partial(C \cup B_N)} \frac{f(z)}{z(z-x)} dz
\]

where the boundary is oriented so that the interior of \( C \cup B_N \) lies always to the left of the path of integration. (See Figure 1.) Let \( D_n = A_n \setminus C \). Then

\[
\frac{f(x)}{x} = \frac{1}{2\pi i} \sum_{n=M}^{N} \int_{\partial D_n} \frac{f(z)}{z(z-x)} dz + \frac{1}{2\pi i} \int_{|z|=2^{-M}} \frac{f(z)}{z(z-x)} dz.
\]

Since \( x \) lies on \( J \), which is a non-tangential ray to \( x_0 \), there exists a constant \( k > 0 \) such that for \( z \notin U \), \( \frac{|x|}{|z-x|} \leq k^{-1} \). Thus for \( z \notin U \), \( \frac{|z|}{|z-x|} \leq 1 + \frac{|x|}{|z-x|} \leq 1 + k^{-1} \). Hence
\[
\frac{1}{|z| \cdot |z - x|} \leq \frac{1 + k^{-1}}{|z|^2}
\] and therefore

\[
\frac{|f(x)|}{|x|} \leq \frac{1}{2\pi} \sum_{n=M}^{N} \left| \int_{\partial D_n} \frac{f(z)}{z(z - x)} \, dz \right| + \frac{4M(1 + k^{-1})}{2\pi} \|f\|_{\infty}.
\] (3)

Since \( \frac{f(z)}{z(z - x)} \) is analytic on \( D_n \setminus U \) for \( M \leq n \leq N \), an application of Lemma 3 shows that

\[
\left| \int_{\partial D_n} \frac{f(z)}{z(z - x)} \, dz \right| \leq \kappa M^{1+\alpha}(D_n \setminus U) \cdot \left\| \frac{f(z)}{z(z - x)} \right\|'_{\text{Lip}(D_n)}.
\] (4)

Recall that the constant \( \kappa \) is the same for curves in the same equivalence class. Since the regions \( D_n \) differ from each other by a scaling it follows that \( \kappa \) doesn’t depend on \( n \) in (4).

We now show that \( \left\| \frac{f(z)}{z(z - x)} \right\|'_{\text{Lip}(D_n)} \) can be bounded by a constant independent of \( f \) and \( x \). It follows from the definition of the Lipschitz seminorm that

\[
\left\| \frac{f(z)}{z(z - x)} \right\|'_{\text{Lip}(D_n)} = \sup_{z \neq w; z, w \in D_n} \left| \frac{f(z)}{z(z - x)} - \frac{f(w)}{w(w - x)} \right| \frac{|z - w|^\alpha}{|z - w|}
\]

\[
= \sup_{z \neq w; z, w \in D_n} \left| \frac{|w(w - x)f(z) - z(z - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \right|.
\]

Thus it follows from the triangle inequality that

\[
\left\| \frac{f(z)}{z(z - x)} \right\|'_{\text{Lip}(D_n)} \leq \sup_{z \neq w; z, w \in D_n} \left| \frac{|w(w - x)f(z) - w(w - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \right|
\]

\[
+ \sup_{z \neq w; z, w \in D_n} \left| \frac{|w(w - x)f(w) - z(z - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \right|.
\] (5)

We first bound the first term on the right of (5)

\[
\sup_{z \neq w; z, w \in D_n} \left| \frac{|w(w - x)f(z) - w(w - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \right|
\]

\[
\leq \sup_{z \in D_n} \frac{1}{|z| \cdot |z - x|} \cdot \left\| f \right\|'_{\text{Lip}(D_n)}.
\]
Since \( z \notin U \), \( \frac{1}{|z| \cdot |z - x|} < \frac{1 + k^{-1}}{|z|^2} \), and therefore,

\[
\sup_{z \neq w; z, w \in D_n} \frac{|w(w - x)f(z) - w(w - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \leq C4^n ||f||_{Lip(D_n)}'.
\] (6)

We now bound the second term on the right side of (5). Since \( f(0) = 0 \) it follows that for \( w \in \mathbb{C} \), \( \frac{|f(w)|}{|w|^\alpha} \leq ||f||'_{Lip(\mathbb{C})} \). Moreover, a computation shows that \( w(w - x) - z(z - x) = (w - z)(z + w - x) \). Hence

\[
\sup_{z \neq w; z, w \in D_n} \frac{|w(w - x)f(w) - z(z - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \leq \left( \sup_{z \neq w; z, w \in D_n} \frac{|w - z|^{1-\alpha}}{|z - x| \cdot |w|^{1-\alpha} \cdot |w - x|} + \frac{|w - z|^{1-\alpha}}{|z| \cdot |z - x| \cdot |w|^{1-\alpha}} \right) \cdot ||f||'_{Lip(\mathbb{C})}.
\] (7)

Since \( x \) lies on \( J \), there exists a constant \( k > 0 \) such that \( \frac{1}{|z - x|} < \frac{1 + k^{-1}}{|z|} \) and \( \frac{1}{|w - x|} < \frac{1 + k^{-1}}{|w|} \). Hence

\[
\sup_{z \neq w; z, w \in D_n} \frac{|w - z|^{1-\alpha}}{|z - x| \cdot |w|^{1-\alpha} \cdot |w - x|} \leq C \frac{2^n \cdot (2^n)^{2-\alpha}}{(2^n)^{1-\alpha}} = C4^n,
\] (8)

and

\[
\sup_{z \neq w; z, w \in D_n} \frac{|w - z|^{1-\alpha}}{|z| \cdot |z - x| \cdot |w|^{1-\alpha}} \leq C \frac{4^n \cdot (2^n)^{1-\alpha}}{(2^n)^{1-\alpha}} = C4^n.
\] (9)

Then (7), (8), and (9) yield

\[
\sup_{z \neq w; z, w \in D_n} \frac{|w(w - x)f(w) - z(z - x)f(w)|}{|z| \cdot |z - x| \cdot |w| \cdot |w - x| \cdot |z - w|^\alpha} \leq C4^n ||f||'_{Lip(\mathbb{C})},
\] (10)

and it follows from (5), (6), and (10) that

\[
\left\| \frac{f(z)}{z(z - x)} \right\|'_{Lip(D_n)} \leq C4^n ||f||'_{Lip(\mathbb{C})},
\] (11)
Thus (3), (4), and (11) together yield

$$\frac{|f(x)|}{|x|} \leq C \sum_{n=1}^{\infty} 4^n M_{1+\alpha}^x \left( D_n \setminus U \right) \cdot \|f\|_{\text{Lip}_x^\alpha}.$$ 

Since Hausdorff content is monotone, $M_{1+\alpha}^x \left( D_n \setminus U \right) \leq M_{1+\alpha}^x \left( A_n \setminus U \right)$ and hence

$$\frac{|f(x)|}{|x|} \leq C \sum_{n=1}^{\infty} 4^n M_{1+\alpha}^x \left( A_n \setminus U \right) \cdot \|f\|_{\text{Lip}_x^\alpha},$$

and it follows from Theorem 2 that

$$\frac{|f(x)|}{|x|} \leq C \|f\|_{\text{Lip}_x(U)},$$

where $C$ does not depend on $x$ or $f$. Thus $L_x(f) \leq C \|f\|_{\text{Lip}_x(U)}$ for $f \in A_\alpha(U \cup \{0\})$ and since $A_\alpha(U \cup \{0\})$ is dense in $A_\alpha(U)$, it follows that $L_x$ is a family of uniformly bounded linear functionals on $A_\alpha(U)$.

\[ \square \]

To complete the proof of Theorem 1 since $A_\alpha(U \cup 0)$ is dense in $A_\alpha(U)$, there exists a sequence $\{f_j\}$ in $A_\alpha(U \cup 0)$ such that $f_j \to f$ in the Lipschitz norm. Since each $f_j$ is analytic in a neighborhood of 0 and since $Df_j = f'_j(0)$, it follows that for each $j$, $L_x(f_j) \to 0$ as $x \to 0$. It follows from the claim that $|L_x(f) - L_x(f_j)| \leq C \|f - f_j\|_{\text{Lip}_x(U)}$. By first choosing $j$ sufficiently large, the right hand side can be made arbitrarily small. Then by choosing $x$ sufficiently close to 0, $L_x(f_j)$ can be made arbitrarily close to 0. Thus $L_x(f) \to 0$ as $x \to 0$ through $J$, which proves Theorem 1.

\[ \square \]

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