REPRESENTATIONS OF MATROIDS

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ABSTRACT. In this paper we give a necessary and sufficient criterion for representability of a matroid over an algebraic closed field. This leads to an algorithm, based on an extension of Gröbner Bases, in order to decide if a given matroid is representable over such a field.

1. Introduction

A matroid $M$ on a finite set $S$ is said to be representable over a field $k$ if there exists a vector space $V$ over $k$ and an injection $\phi: S \rightarrow V$ such that a subset $X$ of $S$ is independent in $M$ if and only if the vectors of $\phi(X)$ are linearly independent over $k$.

The problem of the existence of such representations has been largely studied or fields of characteristic 2, 3 and 4. In these cases, the main result is that there exists a finite list of non-representable matroids, such that, for each matroid $M$ having a minor in the list, $M$ is not representable.

There are also results, connecting representability in finite characteristic, with representability in characteristic 0.

In this paper the point of view is quite different; an algorithm, based on Gröbner bases, is given, in order to establish if there exists a linear representation for a given matroid over some field. This transform the problem of representability over such fields in a purely algebraic question.

There are two problems in this kind of approach: the first one is computational, due to the large amount of calculus involved in the determination of the Gröbner bases. This step may be exceeded by finding an appropriate algebraic presentations for the given matroid. This question leads to a sufficient algorithm for the representability of a matroid. The second obstacle is theoretic, it depends from the fact that a-priori it is not known over which field a representation may be given. For example, there are matroids that are representable only over a field of characteristic 2, for such a matroid, the response of a representability test will be negative if the field has a characteristic different from 2. This lead us to develop a Gröbner bases algorithm over $\mathbb{Z}$.

2. Representability and Algorithms

In order to find a representation of the matroid $M$, we are lead to construct an $r \times n$ matrix, whose entries belong to some unknown field $k$:

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Consider the set of bases of $M$. If $v_{\sigma(1)}, \ldots, v_{\sigma(r)}$ is one such a basis ($\sigma$ is a choose of $r$ numbers in $\{1, \ldots, n\}$), then take the determinant of the corresponding $r \times r$ minor.

Let $\bar{x} = x_{1,1} \cdots x_{r,n}$ and let $P_1(\bar{x}) \cdots P_s(\bar{x})$ be the polynomials obtained from all the bases of $M$. Let $Q_1(\bar{x}) \cdots Q_l(\bar{x})$ be polynomials obtained by the determinant of the minors arising from the circuits and let $I$ be the ideal of $k[x_{1,1} \cdots x_{r,n}]$ generated by the $Q_i$. With this notation, we have the following:

**Theorem 2.1.** A matroid $M$ of cardinality $n$ and rank $r$ is representable over an algebraic closed field $k$ if and only if

$$\prod_{i=1}^{s} P_i(\bar{x}) \notin \text{Rad}(I)$$

where $< Q_1(\bar{x}), \ldots, Q_l(\bar{x}) >$ is the ideal of $k[\bar{x}]$ generated by $Q_1, \ldots, Q_l$.

**Proof.** To find a representation of $M$ over $k$ is equivalent to find $\bar{x}_0$ such that $Q_1(\bar{x}_0) = \cdots = Q_l(\bar{x}_0) = 0$ and $P_i(\bar{x}_0) \neq 0$ for each $i \in \{1, \ldots, s\}$. This means that the hypersurface given by the zero locus of $\prod_{i=1}^{s} P_i(\bar{x}) = 0$ does not contain the variety given by the intersection of the $Q_i$. By Hilbert’s Nullstellensatz, this is equivalent to ask that $\prod_{i=1}^{s} P_i(\bar{x})$ does not belong to the radical of the ideal $I$ generated by the $Q_i$’s. \qed

In order to apply the preceding proposition we need to solve an ideal membership problem. This may be done by using Gröbner basis. First of all, we may suppose that the first $r$ elements of $M$ are a basis of the matroid, and hence in matrix 1 we may take the first $r \times r$ minor to be the identity matrix.

**Proposition 2.2.** A matroid $M$ is completely determined by the set of circuits of order less than or equal to its rank.

**Proof.** Each basis is determined by this set, since if a set of $r$ elements ($r = \text{rank}(M)$) is dependent, then it must contain a circuit with $l \leq r$ elements. \qed

This allows us to restrict our attention to the $Q_i(\bar{x})$ coming from such circuits. Equations coming from circuits with more than rank($M$) elements are automatically satisfied when one search a representation in a vector space of dimension $\leq \text{rank}(M)$.

The first algorithm, allows one to determine if a matroid $M$ is representable over a given algebraically closed field $k$.

Step 3 may be hard to finish, due to the great amount of calculation (the polynomial $P_i$ may be very bigger!). In order to avoid this obstacle it is possible to test if $P_i \equiv 0 \pmod{I}$ for each bases equation $P_i$. But this idea does not work well, since in almost all cases it happens that some
Algorithm 1  Representation of matroids over an algebraically closed field

Require: \( Q_1 \cdots Q_l, P_1 \cdots P_r, \) the field \( k \) and \( \bar{x} = [\cdots x_{i,j} \cdots] \)
\[
1: \quad I := (Q_1 \cdots Q_l), P := \prod P_i, p = \text{char}(k) \\
2: \quad \text{F} \text{ind} \text{ a Gröbner basis of } I \text{ over } \mathbb{Z}_p[\bar{x}] \\
3: \quad \text{if } P \equiv 0 \pmod{\text{Rad}(I)} \text{ then} \\
4: \quad \text{The matroid in not representable over } k. \\
5: \quad \text{end if}
\]

It is possible to consider the problem of the representation of the given matroid over every field. The idea is to consider Gröbner bases of polynomials over the ring of integers. The following algorithm gives a sufficient condition for non-representability of matroids over every field.

Algorithm 2  Representation of matroids over an algebraically closed field

Require: \( Q_1 \cdots Q_l, P_1 \cdots P_r, \) char(\( k \)) and \( \bar{x} = [\cdots x_{i,j} \cdots] \)
\[
1: \quad I := (Q_1 \cdots Q_l, 1 - t \prod x_{i,j}), p = \text{char}(k) \\
2: \quad \text{F} \text{ind} \text{ a Gröbner basis of } I \text{ over } \mathbb{Z}_p[\bar{x}, t] \\
3: \quad \text{f} \text{or all } P \in \{P_1 \cdots P_r\} \text{ do} \\
4: \quad \text{if } P \equiv 0 \pmod{I} \text{ then} \\
5: \quad \text{The matroid in not representable over } k. \\
6: \quad \text{end if} \\
7: \quad \text{end for}
\]

Proposition 2.3. Given a basis for \( M \), there exists only one circuit containing a given element and elements from the basis.

Proof. Clearly one such circuits must exists. Suppose that there is another circuit with this property. Then the element \( v \) may be expressed in two ways as a linear combination of elements of the given basis. By substitution, this would give a linear dependence between the basis elements.

This means that it is possible to establish which of the \( x_{i,j} \) vanish and which not: it is sufficient to consider the circuit given by the first \( r \) columns and by the column containing \( x_{i,j} \). For each one of the non vanishing \( x_{i,j} \), there is a corresponding bases \( B \), such that the corresponding polynomial is \( x_{i,j} \). This means that if we ask that the \( x_{i,j} \) do not vanish, then we already take in account many bases equation. The easiest way to do this is to change the ideal generated by the \( Q_i \), by adding another term: \( 1 - t \prod x_{i,j} \). This lead to the following:
Algorithm 3 Representation of matroids over any field

Require: $Q_1 \cdots Q_l, P_1 \cdots P_r, L$ and $\bar{x} = [\cdots x_{i,j} \cdots]$
1: $I := < Q_1 \cdots Q_l, 1 - t \prod x_{i,j} >, L := \emptyset$
2: Find a Gröbner basis $G$ of $I$ over $\mathbb{Z}[\bar{x}, t]$
3: for all division by $n$ performed to obtain $G$ do
4: for all $p$ prime, $p | n$ do
5: use Algorithm 2 to test representability in char$(p)$
6: end for
7: end for
8: if $M$ is not representable over the preceding fields and $1 \in I$ then
9: The matroid in not representable over any field
10: end if
11: for all $P \in \{P_1 \cdots P_r\}$ do
12: if $M$ is not representable over the preceding fields and $P \equiv 0 \pmod{I}$ then
13: The matroid in not representable over any field
14: end if
15: end for

3. Examples

As an example of non-representable matroid, we consider the non-Pappus matroid, in this case there is a contradiction, between the conditions imposed by the circuits and one base.

Example 3.1. Consider the non-Pappus matroid, defined by it’s circuits:

| $n^o$ | Circuit |
|-------|---------|
| 1)    | 1 0 0 0 0 1 0 1 0 |
| 2)    | 0 1 0 0 1 0 0 1 0 |
| 3)    | 1 0 0 1 0 0 0 0 1 |
| 4)    | 0 1 1 0 0 0 0 0 1 |
| 5)    | 0 0 1 0 0 1 1 0 0 |
| 6)    | 0 0 0 1 1 0 1 0 0 |
| 7)    | 0 1 0 1 0 1 0 0 0 |
| 8)    | 1 0 1 0 1 0 0 0 0 |

A possible representation should have the form:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & x_{2,4} & 0 & x_{2,6} & x_{2,7} & x_{2,8} & 1 \\
0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & x_{3,9}
\end{pmatrix}
$$
The 0 and the 1 on the second line, depends respectively by circuits 8 and 4. The conditions expressed by the other circuits are:

\[
\begin{align*}
    x_{2,6}x_{3,8} - x_{3,6}x_{2,8} &= 0 & (1) \\
    x_{3,5} - x_{3,8} &= 0 & (2) \\
    x_{2,4}x_{3,9} - x_{3,4} &= 0 & (3) \\
    x_{2,6} - x_{2,7} &= 0 & (5) \\
    x_{2,4}x_{3,5} - x_{2,4}x_{3,7} - x_{3,5}x_{2,6} + x_{3,4}x_{2,6} &= 0 & (6) \\
    x_{3,4} - x_{3,6} &= 0 & (7)
\end{align*}
\]

From this we obtain:

\[
x_{2,4}x_{3,5} - x_{2,4}x_{3,7} - x_{3,5}x_{2,7} + x_{3,4}x_{2,7} = 0.
\]

Now, consider the base 00000111, this leads to \(x_{3,9}x_{2,8} - x_{3,8} - x_{2,7}x_{3,9} + x_{3,7} \neq 0\). By making the substitutions: \(x_{2,7} \rightarrow x_{2,6}, \ x_{3,9} \rightarrow x_{3,4}/x_{2,4}, \ x_{3,8} \rightarrow x_{3,5}, \ x_{2,8} \rightarrow x_{2,6}x_{3,5}/x_{3,4}\) the preceding expression becomes: \(- (x_{2,4}x_{3,5} - x_{2,4}x_{3,7} - x_{3,5}x_{2,7} + x_{3,4}x_{2,7})/x_{2,4}\) and this is impossible.

Between matroids of order 9 and rank 3, there are other three non-representable one's. These may be obtained by adjoining other circuits to the non-Pappus matroid. For example 001100010 and 110000100. Clearly these circuits does not change the contradiction just showed.

In the following example we show how the algorithm shows that the given matroid is representable only over a fields of characteristic \(p\).

**Example 3.2.** Consider the Fano matroid:

| \(n^\circ\) | Circuit       |
|------------|--------------|
| 1)         | 0 0 1 0 1 1 0 |
| 2)         | 0 1 0 0 1 0 1 |
| 3)         | 1 0 0 0 0 1 1 |
| 4)         | 0 0 1 1 0 0 1 |
| 5)         | 0 1 0 1 0 1 0 |
| 6)         | 1 0 0 1 1 0 0 |
| 7)         | 1 1 1 0 0 0 0 |

A possible representation should have the form:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & x_{2,3} & 0 & 0 & 1 & x_{2,7} \\
0 & 0 & 0 & 1 & x_{3,5} & x_{3,6} & x_{3,7}
\end{pmatrix}
\]

The remaining conditions are:
\begin{equation}
\begin{aligned}
x_{2,3}x_{3,6} + x_{3,5} &= 0 \quad (1) \\
x_{3,5} - x_{3,7} &= 0 \quad (2) \\
x_{3,6}x_{2,7} - x_{3,7} &= 0 \quad (3) \\
x_{2,3} - x_{2,7} &= 0 \quad (4)
\end{aligned}
\end{equation}

This implies that $x_{2,3}x_{3,6} + x_{2,3}x_{3,6} = 0$ and since the variables must be different from 0, this means that the characteristic of the field must be 2. This leads to the well known representation of the Fano matroid over $\mathbb{Z}_2$.

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

A similar situations happens in the following example:

**Example 3.3.** Consider the following matroid of order 9 and rank 4

| \(n^0\) | Circuit |
|---|---|
| 1) | 0 0 0 0 1 1 1 1 0 |
| 2) | 0 0 0 1 0 1 1 1 0 |
| 3) | 1 0 0 0 0 1 1 0 1 |
| 4) | 0 0 0 1 1 0 1 1 0 |
| 5) | 0 1 0 0 1 0 1 0 1 |
| 6) | 0 0 1 1 0 0 1 0 1 |
| 7) | 0 0 0 1 1 1 0 1 0 |
| 8) | 0 0 1 0 1 1 0 0 1 |
| 9) | 0 1 0 1 0 1 0 0 1 |
| 10) | 1 0 0 1 1 0 0 0 1 |
| 11) | 0 0 0 1 1 1 1 0 0 |
| 12) | 0 1 1 0 0 0 0 1 0 |
| 13) | 1 0 1 0 0 0 0 1 0 |
| 14) | 1 1 0 0 0 0 0 1 0 |
| 15) | 1 1 1 0 0 0 0 0 0 |

A possible representation should have the form:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
x_{2,1} & 0 & 1 & 0 & 1 & 0 & x_{2,1} & x_{2,8} & 0 \\
0 & x_{2,1}x_{3,3} & x_{3,3} & 0 & 0 & 1 & x_{2,1}x_{3,3} & x_{3,8} & 0 \\
x_{4,1} & -x_{4,3}x_{3,8} & x_{4,3} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The remaining conditions are:
These equations may be satisfied only on a field of characteristic 2, but in GF(2) there is no solution. The first solution is in GF(4) and is given by:

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\epsilon & 0 & 1 & 0 & 1 & 0 & \epsilon & 1 + \epsilon & 0 \\
0 & \epsilon & 1 & 0 & 0 & 1 & \epsilon & 1 & 0 \\
1 + \epsilon & \epsilon & 1 + \epsilon & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

where $\epsilon^2 + \epsilon + 1 = 0$.

The equations obtained from the basis, may not admit solutions in any field, as is shown in the following example.

**Example 3.4.** Consider the following matroid of order 9 and rank 4

| Circuit | Circuit |
|---------|---------|
| 1) 1 0 0 0 0 1 1 1 0 |
| 2) 0 1 0 0 1 1 0 1 0 |
| 3) 0 0 1 1 0 1 0 1 0 |
| 4) 1 0 1 0 1 0 0 1 0 |
| 5) 1 1 0 0 1 0 0 0 1 |
| 6) 1 0 1 1 0 0 0 0 1 |
| 7) 1 1 0 1 0 0 0 1 0 |
| 8) 0 0 1 0 1 1 1 0 0 |
| 9) 0 1 0 1 0 1 1 0 0 |
| 10) 1 1 1 0 0 1 0 0 0 |

A possible representation should have the form:

$$\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & x_{2,3} & 0 & 1 & 1 & x_{2,7} & 0 & x_{2,9} \\
0 & 1 & x_{3,3} & x_{3,4} & x_{3,5} & 0 & 0 & 0 & x_{3,9} \\
0 & 0 & 0 & x_{4,4} & x_{4,5} & 0 & x_{4,7} & 1 & x_{4,9}
\end{pmatrix}$$
The remaining conditions are:

\[
\begin{align*}
  x_{3,3} - x_{3,4} &= 0 & (3) \\
  x_{2,3}x_{3,5} - x_{3,3} &= 0 & (4) \\
  x_{4,9} - x_{4,5}x_{2,9} &= 0 & (5) \\
  x_{2,3}x_{3,4}x_{4,9} - x_{2,3}x_{4,4}x_{3,9} + x_{3,3}x_{4,4}x_{2,9} &= 0 & (6) \\
  x_{3,5}x_{4,7} + x_{3,3}x_{4,5} &= 0 & (8) \\
  x_{4,4} - x_{4,7} &= 0 & (9)
\end{align*}
\]

After substitutions, one obtain the equation \(x_{2,3}x_{4,4}x_{3,9} = 0\), which is impossible.

4. Results

We have adopted algorithm 3 for matroids of order 8 and 9. Excluded matroids are those which are non-simple or which have an element contained in only one circuit of cardinality less than or equal to the rank. Matroids representable only over a finite characteristic field are listed in the last column.

The following proposition allow us to restrict our attention to matroids of rank \(\leq \) order/2.

**Proposition 4.1.** A matroid \(M\) is representable over a field \(k\) if and only if its dual is.

*Proof.* See [7].

The next proposition allow us to restrict our attention to matroids which do not contain circuits of size \(\leq 2\).

**Proposition 4.2.** A matroid \(M\) is representable over a field \(k\) if and only if its simplification \(\tilde{M}\) is.

*Proof.* See [7].

In the following table we give a list of the results obtained by algorithm 3. In the first three columns we specify which matroids were considered and how many of them were analyzed (third columns). In the fourth column there is the number of non representable matroids that the algorithm has found. In the fifth column there is the number of matroids that may be representable only over a finite characteristic field. For these matroids we do not know if they are representable (since our algorithm test only a sufficient condition for non representability) and in the affirmative case, over which field.

| order | rank | matroids | non-rep. | finite characteristic |
|-------|------|----------|----------|-----------------------|
| 8     | 3    | 18       | 0        | 1                     |
| 8     | 4    | 416      | 44       | 11                    |
| 9     | 3    | 149      | 4        | 5                     |
| 9     | 4    | 179107   | 23860    | 1254                  |
| 10    | 3    | 2951     | 137      | 48                    |
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