Adopting the Uncertainty Principle for the Entropy Estimation of
Black Holes, de Sitter Space and Rindler Space

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By a simple physical consideration and uncertain principle, we derive that temperature is proportional to the surface gravity and entropy is proportional to the surface area of the black hole. We apply the same consideration to de Sitter space and estimate the temperature and entropy of the space, then we deduce that the entropy is proportional to the boundary surface area. By the same consideration, we estimate the temperature and entropy in the uniformly accelerated system (Rindler coordinate). The cases in higher dimensions are considered.

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§1. Introduction

Although it has passed almost 30 years since the discussion of the thermodynamics of black hole began, there are still many problems about the fundamental concepts.1)–3) But several striking concepts are accepted. One of them is that the entropy of the black hole is proportional to the surface which characterizes the thermodynamics of black hole.1), 4) Such statements have been discussed under the quantum field theory5), 6) and superstring theory7)–9).

Some results could be understood if heuristic assumptions are adopted. One of the key concept is the uncertainty principle.10) As the black hole has the gravitational radius, there is the corresponding momentum. If the black hole is formed by such photons or particles corresponding to this momentum, it could be speculated that the temperature of the black hole is proportional to the surface gravity and it could be explained that the entropy of the black hole is proportional to the surface of the black hole.

Recent Type Ia supernovae and WMAP observations confirm the existence of cosmological constant.11)–13) In the space with the cosmological constant (de Sitter space), each observer is surrounded by the event horizon of the cosmological scale and it is discussed that the entropy of the universe is proportional to the horizon area.14) In this case there is the characteristic length derived from the cosmological constant, so it could be speculated the temperature of the horizon from the uncertainty principle. If we assume that the universe is composed of particles with such de Broglie length, it could be derived that the entropy of the universe is related to

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the surface of the horizon.

The similar consideration is applied to the uniformly accelerated coordinate (Rindler coordinate), where it has the characteristic length due to its acceleration. If we assume that the acceleration is due to the gravity of the gravitational mass of particles, it is derived that the entropy per surface is constant.

The detailed derivations about the black hole, de Sitter space and Rindler space are given in § 2, 3 and 4, respectively. The cases in higher dimensions are considered in § 5. The conclusions and problems are discussed in § 6. In the following, it is an order of magnitude argument and units with $c = h = G = k_B = 1$ are used, however in many cases the physical constants are expressed.

§ 2. Entropy of Black Hole

Black hole could be formed by photons or relativistic particles.$^{15)-17)}$ Here we try to derive the entropy of the black hole simply by adopting a few assumptions. Denoting the mass and radius of black Hole by $M$ and $r_g = 2GM/c^2$, the energy of the particle $\Delta E$ which is constrained within this scale is estimated as

$$\Delta E \simeq \Delta pc \simeq \frac{hc}{2\Delta x} \simeq k_BT,$$

where $k_B$ is the Boltzmann constant and we use the relation $\Delta E \simeq \Delta pc$, $\Delta x\Delta p \simeq h/2$, $\Delta E \simeq k_BT$. Now putting $\Delta x \simeq c_1r_g$, the temperature becomes

$$T = \frac{hc}{2c_1k_B(2GM/c^2)} \simeq \frac{hc^3}{4c_1GMk_B},$$

which shows that temperature is proportional to the surface gravity $a = GM/r_g^2 = c^4/GM$ as $T = ha/c_1k_B$. If we take as $c_1 = 2\pi$, it becomes

$$T = \frac{hac}{2\pi k_B} = \frac{hc^3}{8\pi GMk_B} (= T_{BH}),$$

which is the one that Hawking has derived.$^{3)}$

If the black hole of mass $M$ is formed by the black body radiation of temperature $T$, the volume $V$, total photon number $N$ and total entropy $S$ are given by

$$V = \frac{Mc^2}{\epsilon} = \frac{15(8\pi G)^4M^5}{\pi^2 h^3},$$

$$N = nV = \frac{30\zeta(3)}{\pi^4} \frac{8\pi G}{hc}M^2 \simeq \frac{240}{\pi^3} \frac{G}{hc}M^2,$$

$$\frac{S}{k_B} = \frac{sV}{k_B} = \frac{32\pi}{3} \left( \frac{GM^2}{hc} \right),$$

where the following radiation density $\epsilon$, number density $n$, and entropy density $s$ of the radiation temperature $T$ are used

$$\epsilon = \tilde{a}T^4$$
Adopting the Uncertainty Principle for the Entropy Estimation of Black Holes,

\[ n = \frac{2\zeta(3)}{\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3 = 0.244 \left( \frac{k_B T}{\hbar c} \right)^3, \]

\[ s = \frac{4}{3} \tilde{a} T^3, \]

\[ \frac{n}{k_B} = \frac{2 \zeta(3)}{\pi} \frac{\pi^4}{15} = \frac{45 \zeta(3)}{2\pi^4} \approx 0.2776, \]

beeing \( \tilde{a} = \pi^2 k_B^4 / (15 \hbar^3 c^3) \) and \( \zeta(3) = 1.202 \) the radiation constant and zeta function.\(^{18} \)

Using the surface of the black hole \( A = 4\pi r_g^2 = 16\pi G^2 M^2 / c^4 \), and considering the case not satisfying \( c_1 \neq 2\pi \), the following relation is derived

\[ \frac{S}{k_B} = \frac{2\epsilon^2 A}{3\epsilon h} = \frac{2}{3} \left( \frac{c_1}{2\pi} \right) \frac{A}{\ell_p^2} = \frac{1}{4} \left( \frac{c_1}{3\pi/4} \right) \frac{A}{\ell_p^2} = \frac{1}{4} \frac{A}{\ell_p^2}, \]

where \( \ell_p = \sqrt{\hbar / c^3} \) is the Planck length. The factor \( c_1 \) should be taken as \( c_1 = \frac{3\pi}{4} \) when we consider entropy/energy.

In the above derivation, it is assumed implicitly that gravitational mass \( M \) is equal to the proper mass \( (M_p = \epsilon V / c^2) \). If we take the gravitational mass as \( M = c_2 M_p \), the relation of the entropy to the area becomes

\[ \frac{S}{k_B} = \frac{c_1}{3\pi c_2} \frac{A}{\ell_p^2} = \frac{1}{4} \frac{A}{\ell_p^2}, \]

then we take

\[ \frac{c_1}{c_2} = \frac{3\pi}{4}, \]

so we get \( c_2 = \frac{8}{3} \), if we take \( c_1 = 2\pi \). For the calculation of the neutron star, the gravitational mass is smaller than the proper mass \( c_2 < 1 \), and it is opposite for this case. Even though the factor could not be well interpreted, it is derived the relation that entropy \( S \) is proportional to the area of the black hole.

The problem is the volume \( V \), which is much greater than the expected black hole volume \( V_{BH} \sim \frac{4}{3}\pi r_g^3 \). The ratio is

\[ \frac{V}{V_{BH}} = \frac{V}{\frac{4}{3}\pi r_g^3} = \frac{15 (8\pi G)^4 M^5}{\hbar c^7} \]

\[ = \frac{45}{4} \frac{G}{\pi^3 \hbar c} M^2 (8\pi)^4 \frac{M^2}{2^3} = \frac{45}{4} 8^3 \pi \left( \frac{M}{m_p} \right)^2 = 90 \times 8^2 \pi \left( \frac{M}{m_p} \right)^2, \]

then \( V \gg V_{BH} \) for \( M \gg m_p \), where \( m_p = \sqrt{\hbar c / G} \) is the Planck mass.

In the above derivation, we estimate the temperature from the uncertainty principle and assume that the black hole is formed from the gravitational force of the radiation of temperature \( T \). Then it is derived the relation that the entropy is proportional to the area of the black hole. In the normal star, the entropy is the order
of $\frac{S}{k_B} \sim N \sim 10^{57} (M/M_\odot)$, being $N$ the total number of particles, whereas the entropy of the black hole is order of $\frac{S}{k_B} \sim \frac{A}{(4\ell_p^2)} \sim 10^{77} (M/M_\odot)^2$. This huge difference is the annoying problem of the entropy problem of the black hole. However it is interesting to note the following point.

It is discussed the formation of black holes in the early time of the universe due to the large amplitude of fluctuations.\textsuperscript{15)–17)} From this point of view, the radiation energy should be enough to form the black hole within the volume $V_{BH}$. Then the temperature $T_\gamma$ must satisfy

$$Mc^2 = \tilde{a}T_\gamma^4V_{BH}.$$ 

As $V_{BH} \propto M^3$, it is derived $T_\gamma \propto M^{-\frac{1}{2}}$. The total entropy within the volume $V_{BH}$ for the temperature $T_\gamma$ is given by $S_\gamma = \frac{4}{3} \tilde{a}T_\gamma^3V_{BH}$, so it is derived $S_\gamma \propto M^\frac{3}{2}$. If we put $S_{BH}/k_B = \frac{A}{(4\ell_p^2)}$, the ratio is

$$\frac{S_{BH}}{S_\gamma} = \frac{3}{2} \left( \frac{45\pi}{2} \right)^{\frac{1}{4}} \left( \frac{M}{m_{pl}} \right)^{\frac{1}{2}} \left( \frac{T_\gamma}{T_{BH}} \right).$$

Then the relation $S_{BH}T_{BH} \sim S_\gamma T_\gamma \sim U(\sim Mc^2)$ is deduced. However it is difficult to understand in the microscopic process why the entropy has increased by such amount.\textsuperscript{19)}

### §3. Universe with $\Lambda$ term

The SNe Ia and WMAP observations confirm that our universe is now accelerating. For simplicity, we consider the universe with $\Lambda$ term as de Sitter space with metric\textsuperscript{20)}

$$ds^2 = -\left(1 - \frac{A}{3} r^2\right)c^2 dt^2 + \left(1 - \frac{A}{3} r^2\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right).$$

The characteristic point of this space is that there is the horizon with the radius of $\ell_A = \sqrt{3/\Lambda}$ (de Sitter horizon). Applying the uncertainty principle for this length, the energy $\Delta E$ for the photon or particle is estimated as

$$\Delta E \simeq \Delta pc \simeq \frac{hc}{2\Delta x} \simeq k_B T.$$ 

Taking $\Delta x = c_3 \ell_A$, the temperature $T$ is given by

$$k_B T = \frac{hc}{2c_3 \ell_A} \simeq \frac{hc}{2c_3 \sqrt{3} \sqrt{\Lambda}}.$$

When we put $c_3 = \pi$ and the acceleration of the space as $a = c^2 \sqrt{\Lambda}/3$, it becomes the one $k_B T = \frac{hc\sqrt{\Lambda}}{(2\pi\sqrt{3})} = ah/(2\pi c)$ what Gibbons and Hawking have derived.\textsuperscript{20)}

Because the cosmological constant is related to the vacuum energy density $\rho_\Lambda$ as

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G},$$
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the number density of the particle is given by

\[ n_A \frac{\Delta E}{c^2} = \rho_A. \]

Assuming that particle energy is given by \( \Delta E = \hbar c / 2c_3 \ell_A \), the number density \( n_A \) becomes as

\[ n_A = \frac{\rho_A c^2}{\Delta E} = \frac{\Lambda c^4}{8\pi G} \frac{2c_3 \ell_A}{\hbar c} = \frac{3c_3}{4\pi G} \frac{c^3}{\hbar \ell_A}. \]

Taking the volume of the universe \( V \) as

\[ V = \frac{4}{3} \pi \ell_A^3, \]

the total number is given by

\[ N_A = n_A V = \frac{3c_3}{4\pi G} \frac{c^3}{\hbar \ell_A^3} = c_3 \left( \frac{c^3}{G \hbar} \right) \ell_A^3 \ell_A^2 = c_3 \frac{\ell_A^3}{\ell_p^3}. \]

If we assume the particle as Bose particle such as photon, the total entropy is proportional to the total number as (see eq.(2.1) ; \( N/(S/k_B) = 0.2776 \))

\[ \frac{S}{k_B} \sim \frac{N}{0.2776} \sim 4c_3 \frac{\ell_A^3}{\ell_p^3}. \]

Using the area of the de Sitter horizon \( A = 4\pi \ell_A^2 \), it is expressed as

\[ \frac{S}{k_B} \sim \frac{c_3}{\pi} \frac{A}{\ell_p^2}, \]

where the entropy is proportional to the horizon area \( A \). If we take \( c_3 = \pi / 4 \), it becomes \( S/k_B = A/(4\ell_p^2) \) which is derived by Gibbons and Hawking.\(^{20}\)

It must be noted that the equation of state for the matter corresponding to cosmological constant is \( p = -\rho c^2 \), which is different from the photons. It might be some scalar fields. The problem of the above derivation is that the energy density due to the cosmological constant is very different from the inferred radiation energy density \( \epsilon_\gamma = \tilde{a} T^4 \) of temperature \( T \). The ratio of \( \rho_A \) to \( \rho_\gamma \) is

\[ \frac{\rho_A}{\rho_\gamma} = \frac{\rho_A}{\rho_\gamma} = \frac{A c^2}{8\pi G} \frac{15}{\tilde{a} T^4} \left( \frac{c}{\hbar \ell_A} \right)^2 \left( \frac{\ell_A}{\ell_p} \right)^2. \]

(3.1)

As \( \ell_A \gg \ell_p \), the relation \( \rho_A \gg \rho_\gamma \) is derived.

If we take that the energy density \( \rho_{pl} \) of the vacuum due to the zero-point energy is\(^{21,22}\)

\[ \rho_{pl} \simeq \frac{m_{pl}^2}{\ell_p^3} \simeq \frac{\hbar}{\ell_p^4} \]

\[ \simeq \frac{\hbar}{\ell_p^4 c^2}, \]
and the ratio with $\rho_\Lambda$ is

$$\frac{\rho_\Lambda}{\rho_{pl}} = \frac{\frac{h}{8\pi\ell_p^2}}{\hbar} \approx \frac{3}{8\pi} \left( \frac{\ell_p}{\ell_\Lambda} \right)^2.$$  

If the above relations are considered, the following relation is derived

$$\frac{\rho_\Lambda}{\rho_\gamma} \times \frac{\rho_\Lambda}{\rho_{pl}} \approx \frac{90}{\pi^2} c_3^4 \left( \frac{\ell_\Lambda}{\ell_p} \right)^2 \times \frac{3}{8\pi} \left( \frac{\ell_p}{\ell_\Lambda} \right)^2 \approx O(1),$$  

and it is induced

$$\rho_\Lambda \approx \sqrt{\rho_\gamma \rho_{pl}}.$$  

It means that the energy density $\rho_\Lambda$ of the cosmological constant is the geometrical mean of the radiation energy density $\rho_\gamma$ of the temperature $T$ originated from the uncertainty principle and the vacuum energy density $\rho_{pl}$ from the zero-point energy.

### §4. Uniformly accelerating coordinate

It has been said as Unruh Effect that to the observer in the uniformly accelerating coordinate it seems that he or she is in a bath of blackbody radiation at the temperature $T$ which is related to the acceleration $\kappa (= a)$ as $k_B T = \hbar \kappa/(2\pi c).$  

There is a characteristic length $\ell_\kappa = c^2/\kappa$ due to the acceleration $\kappa$. Applying the uncertainty principle to this length, the energy $\Delta E$ of the particle is given by

$$\Delta E \approx \Delta pc \approx \frac{\hbar c}{2\Delta x} \approx c \frac{\hbar c}{2 c_4 \ell_\kappa} \approx \frac{\hbar \kappa}{2 c_4 c} \approx k_B T,$$

where we put $\Delta x = c_4 \ell_\kappa$. The relation between the temperature and the acceleration $\kappa$ is given by

$$k_B T \approx \frac{\hbar \kappa}{2 c_4 c}.$$  

If we take $c_4 = \pi$, the result is the same derived by Unruh.$^{24}$

In the following we consider the relation of this temperature to the entropy as

$$k_B S = \frac{A}{4\ell_p^2}.$$  

One way of the derivation is to assume that the acceleration $\kappa$ is the gravitational acceleration by the mass $M$, which is composed of the blackbody radiation of temperature $T$. If we take the volume of the considering region as $V$, the mass is given as

$$M = \frac{\tilde{a} T^4}{c^2} V.$$  

The acceleration $\kappa$ due to this mass is given by
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\[ \kappa = \frac{GM}{V^\frac{4}{3}} = \frac{G\tilde{a}T^4 V}{c^2 V^\frac{4}{3}} = \frac{G\pi^2 k_B^4 T^4 V^\frac{4}{3}}{15 \hbar^3 c^3 \ell_p^4}. \]

Here we use the relations \( k_B T = \hbar c/(2c_4 \ell_p) \), the above equation becomes

\[ \kappa = \frac{c^2}{\ell_p} = \frac{\pi^2}{15} \frac{1}{16 c_4^4} \frac{\hbar c V^\frac{1}{3}}{\ell_p^4}. \]

Then the region size is given by

\[ V^\frac{1}{3} = \frac{15 \times 16 c_4^4}{\pi^2} \frac{\ell_p^2}{\hbar^3 c^3}. \]

As the total entropy is \( S = \frac{4}{3} \tilde{a} T^3 V \), the entropy per surface \( S/V^\frac{4}{3} \) is given by

\[ \frac{S}{k_B} V^\frac{4}{3} = \frac{4 \pi^2 k_B^4 T^4 V^\frac{1}{3}}{3 \times 15 \hbar^3 c^3} = \frac{4 \pi^2}{3 \times 15} \frac{15}{8 \pi^2} \frac{16 c_4^4 \ell_p^2}{8 \pi^2 \hbar^3 c^3 \ell_p^4} \frac{1}{8 \pi^3} = \frac{3 c_4^4}{4 \ell_p^2}. \]

If we take \( c_4 = \frac{3}{32} \) and \( A = V^\frac{2}{3} \), the following relation is derived

\[ \frac{S}{k_B} A = \frac{1}{4 \ell_p^2}, \]

which means that the entropy per surface is \( 1/(4 \ell_p^2) \).

Even though we consider by the order estimation, it is derived the relation \( S/(k_B A) = 1/(4 \ell_p^2) \), which is the common relation of the entropy to the surface of the event horizon of the black hole and de Sitter universe.

§5. Consideration on higher dimensions

In this section, we consider the above 3 cases in the 4+n dimensions \((n \geq 0)\).

5.1. Spherical symmetric case

The Schwarzschild metric for 4+n dimension is given by

\[ ds^2 = -h(r)c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\Omega_{2+n}^2, \]

where \( d\Omega_{2+n}^2 \) is the surface metric of unit sphere in 3+n dimensions, so \( h(r) \) is written as

\[ h(r) = 1 - \left( \frac{r_H}{r} \right)^{n+1} = 1 - \frac{16 \pi}{n+2} \frac{G(n) M_{BH}}{A_{n+2}} \frac{1}{c^2 r^{n+1}}. \]

\( G(n) \) is the gravitational constant in 4+n dimension and \( A_{n+2} \) is the surface area of unit sphere as

\[ A_{n+2} = \frac{2\pi^{n+3}}{\Gamma\left(\frac{n+3}{2}\right)}, \]
where it becomes $A_2 = 4\pi$ for $n = 0$, because $\Gamma(3/2) = \sqrt{\pi}/2$.

The Planck mass, length, and time are given by

$$m_{\text{pl}}(n) = \left(\frac{h^{n+1}c^{1-n}}{G(n)}\right)^{\frac{1}{2+n}}, \quad \ell_{\text{pl}}(n) = \left(\frac{G(n)h}{c^3}\right)^{\frac{1}{2+n}}, \quad t_{\text{pl}}(n) = \left(\frac{G(n)h}{c^{3+n}}\right)^{\frac{1}{2+n}},$$

respectively. So $r_H$ is expressed as

$$r_H = \frac{1}{\sqrt{\pi}} \frac{1}{m_{\text{pl}}(n)} \frac{1}{c} \left(\frac{M_{\text{BH}}}{m_{\text{pl}}(n)}\right)^{\frac{1}{2+n}} \left(\frac{8\Gamma(n+3)}{n+2}\right)^{\frac{1}{n+1}}.$$

Due to the uncertainty principle, the energy of the wave which is confined to the black hole is given as

$$\Delta E \sim \Delta pc \sim \frac{hc}{\Delta r} \sim kT \implies kT \sim \frac{hc}{r_H},$$

The radiation energy density $\epsilon(T)$ and the entropy density $s(T)$ are given by

$$\epsilon(T) = \frac{A_{n+2}}{(2\pi)^{n+3}k_B} T(n+2) \left(\frac{k_B T}{hc}\right)^{n+3} \Gamma(n+4)\zeta(n+4) \sim (k_B T)^{n+4}$$

and

$$s(T) = \frac{(n+4)(n+2)}{n+3} \frac{A_{n+2}}{(2\pi)^{3+n}k_B} \left(\frac{k_B T}{hc}\right)^{n+3} \Gamma(n+4)\zeta(n+4) \sim (k_B T)^{n+3},$$

respectively. Then the volume could be estimated as

$$V = \frac{M c^2}{\epsilon} \propto \frac{M c^2}{k_B T\left(\frac{k_B T}{hc}\right)^{n+3}}$$

and the total entropy becomes

$$S = sV = k_B \left(\frac{k_B T}{hc}\right)^{n+3} \frac{M c^2}{k_B T\left(\frac{k_B T}{hc}\right)^{n+3}} \sim \frac{M c^2}{T} \sim \frac{k_B r_H M c^2}{hc} \sim \frac{kr_H}{hc} \left(\frac{r_H}{\ell_{\text{pl}}(n)}\right)^{n+1} m_{\text{pl}} c \sim k \left(\frac{r_H}{\ell_{\text{pl}}}(n)\right)^{n+2},$$

which means that the entropy in the 4+n dimension is proportional to the 2+n dimensional hyper surface in n+3 space dimension. Therefore the entropy of black hole is proportional to the surface.
5.2. de Sitter space

The metric is given by

$$ds^2 = -\left(1 - \frac{A}{3}r^2\right)c^2 dt^2 + \left(1 - \frac{A}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega_n^2.$$  

Taking $\ell_A = (3/A)^{1/2}$, the temperature of the universe $T$ is

$$\Delta E \sim \Delta pc \sim \frac{hc}{\ell_A} \sim k_BT.$$  

The cosmological constant $\Lambda$ is related to the density as

$$\rho(n) = \frac{Ac^2}{8\pi G(n)},$$

then the particle density $n_A$ is given by

$$n_A \frac{\Delta E}{c^2} \sim \rho_A \sim \frac{Ac^2}{8\pi G(n)}.$$  

Taking $V \sim \ell_A^{3+n}$, the total particle number becomes as

$$N_A = n_A V \sim \frac{Ac^4}{8\pi G(n)} \frac{\ell_A \ell_A^{3+n}}{hc} \sim \frac{\ell_A^{2+n}}{\ell_{pl}^{2+n}} = \left(\frac{\ell_A}{\ell_{pl}}\right)^{2+n}.$$  

As the total entropy $S$ is proportional to $N_A$, the following relation is derived,

$$S \propto \left(\frac{\ell_A}{\ell_{pl}}\right)^{2+n},$$

which means that the total entropy is proportional to the horizon surface.

5.3. Rindler space

Denoting the constant acceleration as $\kappa$, the characteristic length becomes $\ell_\kappa = c^2/\kappa$ and the temperature $T$ is estimated,

$$\Delta E \sim \Delta pc \sim \frac{hc}{2\ell_\kappa} \sim k_BT.$$  

We tentatively assume that the acceleration is due to the gravity of mass $M$. Taking the volume $V = L^{3+n}$ and the mass $M$ as

$$M = \frac{\epsilon}{c^2} V \sim \frac{k_BT}{c^2} \left(\frac{k_BT}{hc}\right)^{3+n} L^{3+n},$$

the acceleration $\kappa$ is given by

$$\frac{c^2}{\ell_\kappa} \sim \kappa = \frac{G(n)M}{L^{2+n}} \sim \frac{G(n)k_BT}{L^{2+n}c^2} \left(\frac{k_BT}{hc}\right)^{3+n} L^{3+n} \sim G(n)\left(\frac{1}{\ell_\kappa}\right)^{4+n} \frac{h}{cL}.$$
Then the length $L$ becomes

$$L = \frac{c^3}{G(n)\hbar} \ell_\kappa^{3+n} = \left( \frac{\ell_\kappa}{\ell_{pl}} \right)^{2+n} \ell_\kappa.$$

As the total entropy is $S = sV = sL^{3+n}$, the entropy per unit surface is

$$\frac{S}{L^{2+n}} \sim sL \sim k_B \left( \frac{1}{\ell_\kappa} \right)^{2+n} \ell_\kappa \left( \frac{\ell_\kappa}{\ell_{pl}} \right)^{2+n} \sim k_B \left( \frac{1}{\ell_{pl}} \right)^{2+n},$$

or it is written as

$$\frac{S}{\left( \ell_{pl} \right)^{2+n}} \sim k_B.$$

It is also the natural extension of the constant of entropy per surface to 4+n dimension case.

§6. Conclusions and discussion

By using the uncertainty principle, it is estimated the temperature of the black hole, de Sitter space, and Rindler space are commonly expressed by the acceleration $a$ as $T = a/2\pi$. Considering the black body radiation of this temperature, it is derived the common relation that the entropy is proportional to the surface of the event horizon for each case as $S = A/4$. It is also derived the curious relation that the energy density related to cosmological constant $\rho_A = 3\Lambda/(8\pi G)$ is the geometrical mean of the radiation density $\rho_\gamma$ and the vacuum energy density $\rho_{pl}$ as $\rho_A = \sqrt{\rho_\gamma \rho_{pl}}$.

There are many points to be investigated further in the above derivation. It is not treated here that the problems about the general second law of thermodynamics, black hole evaporation and the surface increase of the cosmological event horizon with $\Lambda$ term. It is interesting to investigate these problems under these simplified assumptions.

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