VOJTA’S CONJECTURE FOR SINGULAR VARIETIES

TAKEHIKO YASUDA

Abstract. We formulate a generalization of Vojta’s conjecture in terms of log pairs and variants of multiplier ideals. In this generalization, a variety is allowed to have singularities. It turns out that the generalized conjecture for a log pair is equivalent to the original conjecture applied to a log resolution of the pair. A special case of the generalized conjecture can be interpreted as representing a general phenomenon that there tend to exist more rational points near singular points than near smooth points. The same phenomenon is also observed in relation between greatest common divisors of integer pairs satisfying an algebraic equation and plane curve singularities, which is discussed in Appendix. As an application of the generalization of Vojta’s conjecture, we also derive a generalization of a geometric conjecture of Lang concerning varieties of general type to singular varieties and log pairs.

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1. Introduction

Vojta’s famous conjecture in the Diophantine geometry was originally stated for a smooth variety $X$ and a simple normal crossing divisor $D$ of it. In a recent paper [Voj12], he generalized it to an arbitrary proper closed subscheme $D \subset X$. The aim of the present paper is to generalize it further by allowing $X$ to have mild singularities. Our formulation is made in terms of log pairs and their singularities, which are basic notions in the birational geometry, in particular, in the minimal model program. We then show that the generalized conjecture is in fact equivalent to the original one in a
similar way as Vojta did in the cited paper. We hope that our formulation will encourage interaction between the Diophantine geometry and the minimal model program.

Let $X$ be a smooth variety over a number field $k$ and $D$ a simple normal crossing divisor of it. The original conjecture of Vojta concerns the inequality

\[(1.1) \quad h_{K_X}(x) - m_D(x) \leq \epsilon h_A(x) + d_k(x) + O(1).\]

See Sections 4 and 5 for details. Recently he generalized the conjecture to an arbitrary proper closed subscheme $D \subset X$. Let $I \subset \mathcal{O}_X$ the multiplier ideal sheaf of $(X, (1-\epsilon)D)$ for sufficiently small $\epsilon > 0$, which was denoted by $\mathcal{J}^-$ in [Voj12]. According to Silverman [Sil87], we can define proximity functions $m_W$ and counting functions $N_W$ of closed subschemes $W \subset X$ and similarly ones of ideal sheaves. In [Voj12], Vojta generalized the conjecture by changing the left hand side of (1.1) to

\[h_{K_X}(x) - m_D(x) - m_I(x),\]

introducing the correction term $-m_I(x)$.

In this paper, we further generalize it, allowing the variety $X$ to have (not necessarily normal) $\mathbb{Q}$-Gorenstein singularities and $D$ to be $\mathbb{Q}_{\geq 0}$-linear combination of closed subschemes. We can similarly define the ideal sheaf $I$ also in this case and define another ideal sheaf $\mathcal{H}$, which is again a variant of the multiplier ideal. We change the left hand side of the inequality to

\[(1.2) \quad h_{K_X}(x) + h_D(x) - N_{\mathcal{H}}(x) - m_I(x).\]

If $X$ is smooth, then we have $m_D \leq h_D - N_{\mathcal{H}}$ and our inequality is slightly stronger than the one of Vojta. However it turns out that Vojta’s and our generalizations are eventually equivalent to the original conjecture. The first two terms $h_{K_X} + h_D$ of (1.2) is then interpreted as the height function of the “canonical divisor” $K_{(X,D)} = K_X + D$ of the log pair $(X,D)$, and the last two terms $-N_{\mathcal{H}}(x) - m_I(x)$ as contribution of singularities of $(X,D)$. In the special case where $D = 0$ and $X$ has only log terminal singularities, then (1.2) is written as

\[h_{K_X}(x) - N_{\text{NonC}(X)}(x),\]

where $\text{NonC}(X)$ is the non-canonical locus of $X$. As an example showing the necessity of the correction term $-N_{\text{NonC}(X)}(x)$, we construct a “rational surface of general type” having only quotient singularities (Section 7).

From our generalization of Vojta’s conjecture, we derive the following geometric conjecture:

**Conjecture 1.1** (See Corollary 6.5 for a more general version). Let $X$ be a $\mathbb{Q}$-Gorenstein variety with a canonical divisor $K_X$ big. Then there exists a proper closed subset $Z \subset X$ such that for every potentially dense closed subvariety $Y \subset X$, either $Y$ is contained in $Z$ or $Y$ intersects $\text{NonC}(X)$.

If $X$ is smooth, then $\text{NonC}(X)$ is empty. The last conjecture in this case was raised by Lang [Lan91].

As a height function $h_D(x)$ as well as counting and proximity functions represents a closeness to $D$, we may regard our generalization of Vojta’s conjecture as a representation of a phenomenon that there tend to be more rational points near singular points...
than near smooth points. As another representation of the same phenomenon, in Appendix, we discuss relation of greatest common divisors and plane curve singularities. Let $C \subset \mathbb{P}^2_k$ be an irreducible plane curve of degree $d$ and let $m$ be the multiplicity of $C$ at $O = (0 : 0 : 1)$. Let $h_O$ be a height function of $O$ regarded as a closed reduced subscheme of $\mathbb{P}^2_k$ and $h$ be the standard height of $\mathbb{P}^2_k$. From the functoriality of height functions and a small computation of resolution of singularities, we observe that when restricted to $C(\mathbb{k})$, $h_O$ approximates $\frac{m}{d} h$. Combining it with Silverman’s interpretation of greatest common divisors in terms of heights, we obtain a not deep but amusing fact that if $k = \mathbb{Q}$, then the greatest common divisor $\gcd(x, y)$ for $(x, y) \in C \cap A^2_\mathbb{Q}$, $x, y \in \mathbb{Z}$, approximates $\max\{|x|, |y|\}^{m/d}$ (for a more general and precise statement, see Corollary A.7).

Throughout the paper, we fix a number field $k$. A variety means a separated reduced scheme of pure dimension and finite type over $k$. We suppose that every morphism of varieties is a morphism of $k$-schemes.

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2. Singularities of log pairs

In this section, we recall the notion of log pairs in a slightly generalized context and basics of their singularities.

**Definition 2.1.** A variety $X$ is is said to be $\mathbb{Q}$-Gorenstein if

1. $X$ satisfies Serre’s condition $S_2$,
2. $X$ is Gorenstein in codimension one, and
3. a canonical divisor $K_X$ is $\mathbb{Q}$-Cartier.

Note that this definition of $\mathbb{Q}$-Gorenstein is more general than the usual one in the sense that we do not assume that $X$ is normal. (However, if he prefers, the reader may safely assume that $X$ is normal.) These conditions appear in the definition of “pair” in [Kol13, Def. 1.5] and the one of semi-log canonical singularities in [HK10, p. 35]. From the first two conditions, which are automatic if $X$ is normal, a canonical divisor $K_X$ exists, is unique up to linear equivalence and is Cartier in codimension one. Therefore the last condition makes sense and is equivalent to that for some $m \in \mathbb{Z}_{>0}$, the reflexive power $\omega_X^{[m]} := (\omega_X^m)^{**}$ of the canonical sheaf $\omega_X$ is invertible.

For a $\mathbb{Q}$-Gorenstein variety $X$ and a proper birational morphism $f : Y \to X$ of varieties with $Y$ normal, the pull-back $f^*K_X$ is defined as a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor of $Y$.

**Definition 2.2.** A $\mathbb{Q}$-subscheme of a variety $X$ is a formal linear combination $D = \sum_{i=1}^n c_i D_i$ of proper closed subschemes $D_i \subseteq X$ with $c_i \in \mathbb{Q}$. Following the terminology for divisors, we say that a $\mathbb{Q}$-subscheme $D = \sum_{i=1}^n c_i D_i$ is effective if every $c_i$ is non-negative; then we write $D \geq 0$. The support of $D$, denoted $\text{Supp}(D)$, is defined to be the closed subset $(\bigcup_{c_i \neq 0} D_i)_{\text{red}}$ and denoted by $\text{Supp}(D)$.

A log pair is the pair $(X, D)$ of a $\mathbb{Q}$-Gorenstein variety $X$ and a $\mathbb{Q}$-subscheme $D$ of $X$. We say that a log pair $(X, D)$ is effective if $D$ is effective. We say that a log pair $(X, D)$ is projective if $X$ is projective.
When $D = 0$, we usually omit $D$ from the notation and identify the log pair $(X, 0)$ with the variety $X$. For instance, the discrepancy $\text{discrep}(X, D)$ defined below will be written also as $\text{discrep}(X)$ when $D = 0$.

**Remark 2.3.** If $X$ is a normal $\mathbb{Q}$-Gorenstein variety and $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor, then $D$ is written as a $\mathbb{Q}$-linear combination $\sum_{i=1}^{n} b_i E_i$ of effective Cartier divisors $E_i$; this allows us to regard the pair $(X, D)$ as a log pair in the sense defined above.

**Definition.** A **resolution** of a variety $X$ is a proper birational morphism $f: Y \to X$ such that $Y$ is smooth over $k$. Let $(X, D = \sum_{i=1}^{n} c_i D_i)$ be a log pair. A **log resolution** of $(X, D)$ is a resolution $f: Y \to X$ of $X$ such that

1. for every $i$, the scheme-theoretic preimage $f^{-1}(D_i)$ is a Cartier divisor (that is, if $\mathcal{I}_{D_i}$ is the defining ideal sheaf of $D_i$, then the pull-back $f^{-1}\mathcal{I}_{D_i}$ as an ideal sheaf is locally principal),
2. if we denote by $\text{Exc}(f)$ the exceptional set of $f$, then $\text{Exc}(f) \cup \bigcup_{i=1}^{n} f^{-1}(D_i)_{\text{red}}$ is a simple normal crossing divisor.

From Hironaka’s theorem, any variety has a resolution and any log pair has a log resolution.

**Definition 2.4.** For a $\mathbb{Q}$-Gorenstein variety and a resolution $f: Y \to X$ of $X$, the **relative canonical divisor** $K_{Y/X}$ of $Y$ over $X$ is defined as a $\mathbb{Q}$-divisor of $Y$ supported in $\text{Exc}(f)$ as follows. If $m$ is a positive integer such that $\omega_X^{[m]}$ is invertible, then the natural morphism $f^*\omega_X^{[n]} \to \omega_Y^{\otimes m}$ is injective and its image is written as $\omega_Y^{\otimes n}(\Delta)$ for some ($\mathbb{Z}$-)divisor $\Delta$. We then define

$$K_{Y/X} := -\frac{1}{n} \Delta.$$ 

For a log pair $(X, D)$ and a log resolution $f: Y \to X$ of it, we define the **relative canonical divisor** $K_{Y/(X, D)}$ of $Y$ over $(X, D)$ as the $\mathbb{Q}$-divisor $K_{Y/X} - f^*D$. Here, if we write $D = \sum_{i=1}^{n} c_i D_i$, then we define the pull-back $f^*D$ as $\sum_{i=1}^{n} c_i f^{-1}D_i$, which is a $\mathbb{Q}$-divisor. Let us write

$$K_{Y/(X, D)} = \sum_F a_F \cdot F,$$

$F$ running over the prime divisors of $Y$. We call $a_F$ the **discrepancy** of $F$ with respect to $(X, D)$ and write it as $a(F; X, D)$.

**Definition 2.5.** Let $X$ be a variety. A **divisor over** $X$ is a prime divisor $F$ on $Y$ for a resolution $f: Y \to X$. We say that such an $F$ is called **exceptional** if $f$ is not an isomorphism at the generic point of $F$.

Let $(X, D)$ be a log pair. We define the **discrepancy** of $(X, D)$ by

$$\text{discrep}(X, D) := \inf\{a(F; X, D) \mid F \text{ is an exceptional divisor over } X\},$$

and the **total discrepancy** of $(X, D)$ by

$$\text{totaldiscrep}(X, D) := \inf\{a(F; X, D) \mid F \text{ is a divisor over } X\}.$$
We are mainly interested in the total discrepancy rather than the discrepancy. It is easy to see that for a log resolution \( f : Y \rightarrow X \) of a log pair \((X, D)\), we have
\[
\text{totaldiscrep}(Y, -K_Y/(X, D)) = \text{totaldiscrep}(X, D).
\]

**Lemma 2.6.** We have either
\[
\text{totaldiscrep}(X, D) = -\infty
\]
or
\[
-1 \leq \text{totaldiscrep}(X, D) \leq 0.
\]

**Proof.** Firstly, since a general prime divisor has zero discrepancy, we have
\[
\text{totaldiscrep}(X, D) \leq 0.
\]
The lemma is well known in the case where \( X \) is normal (see [KM98, Cor. 2.31]). The general case follows from (2.1). \( \square \)

We can now define three classes of singularities of log pairs as follows:

**Definition 2.7.** We say that \((X, D)\) is strongly canonical (resp. Kawamata log terminal, log canonical) if
\[
\text{totaldiscrep}(X, D) \geq 0 \quad \text{(resp. } > -1, \geq -1).
\]
We say that \( X \) is canonical (resp. log terminal, log canonical) if \((X, 0)\) is strongly canonical (resp. Kawamata log terminal, log canonical).

**Remark 2.8.** The author does not know whether the notion of strongly canonical has been ever considered, while the other notions are quite standard. This notion is necessary for our reformulation of Vojta’s conjecture. A log pair \((X, D)\) is said to be canonical if \( \text{discrep}(X, D) \geq 0 \); this is standard, but we do not use it in this paper. When \( D = 0 \), strongly canonical and canonical are equivalent notions.

**Remark 2.9.** When \( X \) is normal and \( \mathbb{Q} \)-Gorenstein and \( D \) is a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-Weil divisor, then for a log pair \((X, D)\) being strongly canonical (resp. Kawamata log terminal, log canonical), the multiplicity of every prime divisor in \( D \) needs to be \( \leq 0 \) (resp. \( < 1, \leq 1 \)).

The definition of total discrepancy uses all divisors over a given variety \( X \). However the following lemma allows us to compute it in terms of a single log resolution.

**Lemma 2.10.** Let \( X \) be a smooth variety and let \( D = \sum_{i=1}^l c_i F_i \) be a \( \mathbb{Q} \)-divisor of \( X \) such that \( c_i \in \mathbb{Q} \), \( F_i \) are prime divisors and \( \bigcup_{i=1}^l F_i \) is a simple normal crossing divisor. Then \((X, D)\) is strongly canonical (resp. Kawamata log terminal, log canonical) if and only if \( c_i \leq 0 \) (resp. \( < 1, \leq 1 \)) for every \( i \).

**Proof.** If \( c_i = -a(F_i; X, D) > 1 \) for some \( i \), then \( \text{totaldiscrep}(X, D) < -1 \) and \((X, D)\) is not log canonical. Otherwise, from [KoI13, Cor. 2.11],
\[
\text{discrep}(X, D) = \min\{1, \min_i \{1 - c_i\}, \min_{F_i \cap F_j \neq \emptyset} \{1 - c_i - c_j\}\}.
\]
The lemma follows from
\[
\text{totaldiscrep}(X, D) = \min\{0, \min_i \{-c_i\}, \text{discrep}(X, D)\}.
\]
\[\square\]

The above notions of singularities are local; \((X, D)\) is strongly canonical (resp. Kawamata log terminal, log canonical) if and only if every point \(x \in X\) has an open neighborhood \(U \subset X\) such that \((U, D|_U)\) is so.

**Definition 2.11.** Let \((X, D)\) be a log pair. The non-sc locus (resp. non-lc locus) of \((X, D)\) is the smallest closed subset \(W \subset X\) such that \((X \setminus W, D|_{X \setminus W})\) is strongly canonical (resp. Kawamata log terminal, log canonical). We write it as NonSC\((X, D)\) (resp. NonKLT\((X, D)\), NonLC\((X, D)\)). We call the non-sc locus NonSC\((X, 0)\) of the log pair \((X, 0)\) also as the non-canonical locus of \(X\) and denote it by NonC\((X)\).

Clearly
\[\text{NonLC}(X, D) \subset \text{NonKLT}(X, D) \subset \text{NonSC}(X, D)\]

**Lemma 2.12.** Let \(f: Y \to X\) be a log resolution of \((X, D)\). Then
\[
\text{NonSC}(X, D) = \bigcup_{F \subset Y: a(F; X, D) < 0} f(F),
\]
\[
\text{NonKLT}(X, D) = \bigcup_{F \subset Y: a(F; X, D) \leq -1} f(F)
\]
\[
\text{NonLC}(X, D) = \bigcup_{F \subset Y: a(F; X, D) < -1} f(F),
\]
in each of which \(F\) runs over the prime divisors of \(Y\) satisfying the indicated inequality.

**Proof.** This is a direct consequence of Lemma [2.10] \(\square\)

### 3. Multiplier-like ideal sheaves

In this section, we define two variants of multiplier ideals and study their basic properties.

For an effective log pair \((X, D)\) with \(X\) normal, the **multiplier ideal sheaf** \(J(X, D)\) is usually defined to be \(f_*\mathcal{O}_Y(\lceil K_Y/(X, D) \rceil)\) for a log resolution \(f: Y \to X\) of \((X, D)\) (see [Laz04, Def. 9.3.56]). Here \(\lceil \cdot \rceil\) denotes the round up, while \(\lfloor \cdot \rfloor\) used below denotes the round down. When \(X\) is not normal, \(f_*\mathcal{O}_Y(\lceil K_Y/(X, D) \rceil)\) is no longer an ideal sheaf. To handle this trouble, we replace \(f_*\) with \(f_\bullet\) defined as follows:

**Definition 3.1.** For a proper birational morphism \(f: Y \to X\) of varieties and a divisor \(E\) of \(Y\), we define \(f_\bullet\mathcal{O}_Y(E)\) as the largest ideal sheaf \(\mathcal{I} \subset \mathcal{O}_X\) such that the ideal pullback \(f^{-1}\mathcal{I}\) is contained in \(\mathcal{O}_Y(E)\) as an \(\mathcal{O}_Y\)-submodule of the sheaf of total quotient rings.

We then generalize the multiplier ideal sheaf to the non-normal case as follows:

**Definition 3.2.** Let \((X, D)\) be an effective log pair and \(f: Y \to X\) a log resolution of it. The **multiplier ideal sheaf** \(J(X, D)\) is defined to be \(f_\bullet\mathcal{O}_Y(\lceil K_Y/(X, D) \rceil)\).
We will define two variants $\mathcal{H}(X, D)$ and $\mathcal{I}(X, D)$ of the multiplier ideal to formulate a generalization of Vojta’s conjecture for log pairs. We first define $\mathcal{H}(X, D)$.

**Lemma 3.3.** Let $X$ be a smooth variety, $E$ a (not necessarily effective) $\mathbb{Q}$-divisor and $f : Y \to X$ a proper birational morphism. Then
\[
f_* \mathcal{O}_Y([K_{Y/X} + f^*E]) = \mathcal{O}_X([E]).
\]

**Proof.** First suppose that $[E] = 0$. To show the lemma in this case, it suffices to show that $[K_{Y/X} + f^*E]$ is an effective divisor supported in $\text{Exc}(f)$. Since $K_{Y/X}$ and $E$ are effective, so is $[K_{Y/X} + f^*E]$. On the locus where $f$ is an isomorphism, the two divisors $[K_{Y/X} + f^*E]$ and $[E]$ coincide, the latter being zero by the assumption. This proves the lemma in this case.

For the general case, we write $\{E\} := E - [E]$. Obviously, $\{E\} = 0$. From the projection formula and the case considered above, we have
\[
f_* \mathcal{O}_Y([K_{Y/X} + f^*E]) = f_* \mathcal{O}_Y([K_{Y/X} + f^*\{E\}] + f^*[E])
= f_* (\mathcal{O}_Y([K_{Y/X} + f^*\{E\}]) \otimes_{\mathcal{O}_Y} f^* \mathcal{O}_X([E]))
= \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X([E])
= \mathcal{O}_X([E]).
\]

We have completed the proof. $\square$

**Proposition 3.4.** Let $(X, D)$ be an effective log pair and $f : Y \to X$ a log resolution of $(X, D)$. Then $f_\bullet \mathcal{O}_Y([K_{Y/(X,D)}])$ is an ideal sheaf contained in $\mathcal{J}(X, D)$ and independent of $f$.

**Proof.** Since $[K_{Y/(X,D)}] \leq [K_{Y/(X,D)}]$, we have
\[
\mathcal{O}_Y([K_{Y/(X,D)}]) \subset \mathcal{O}_Y([K_{Y/(X,D)}]).
\]
Applying $f_\bullet$, we obtain the first assertion.

To show the second assertion, we consider another log resolution $f' : Y' \to X$ of $(X, D)$. Without loss of generality, we may suppose that $f'$ factors as $f \circ g$ with a morphism $g : Y' \to Y$. Then
\[
K_{Y'/(X,D)} = K_{Y'/Y} + g^* K_{Y/(X,D)}.
\]
From the above lemma,
\[
f_\bullet \mathcal{O}_{Y'}([K_{Y'/(X,D)}]) = f_\bullet (g_* \mathcal{O}_{Y'}([K_{Y'/Y} + g^* K_{Y/(X,D)}]))
= f_\bullet \mathcal{O}_Y([K_{Y/(X,D)}]).
\]
We have proved the second assertion. $\square$

**Definition 3.5.** For an effective log pair $(X, D)$, we define an ideal sheaf $\mathcal{H}(X, D)$ on $X$ by
\[
\mathcal{H}(X, D) := f_\bullet \mathcal{O}_Y([K_{Y/(X,D)}])
\]
for a log resolution $f : Y \to X$ of $(X, D)$. 
Proposition 3.6. For an effective log pair \((X, D)\) and a point \(x \in X\), we have \(\mathcal{H}(X, D)_x = \mathcal{O}_{X,x}\) if and only if \((X, D)\) is strongly canonical around \(x\). Equivalently, the support of

\[
\mathrm{Supp}(\mathcal{O}_X / \mathcal{H}(X, D)) = \text{NonSC}(X, D).
\]

Proof. We write \(\mathcal{H}(X, D)\) as \(\mathcal{H}\). We first show the “if” part. We suppose that \((X, D)\) is strongly canonical. Let \(f : Y \rightarrow X\) be a log resolution of \((X, D)\). By the definition of strongly canonical, \(K_{Y/(X, D)} \geq 0\) and \(\lfloor K_{Y/(X, D)} \rfloor \geq 0\). By the definition of \(f_{\bullet}\), we have

\[
\mathcal{H} = f_{\bullet} \mathcal{O}_Y(\lfloor K_{Y/(X, D)} \rfloor) = \mathcal{O}_X.
\]

Next we show the “only if” part. Suppose that \((X, D)\) is not strongly canonical around \(x\). Then there exists a prime divisor \(F\) on \(Y\) such that \(x \in f(F)\) and the multiplicity of \(F\) in \(K_{Y/(X, D)}\) is negative. Therefore,

\[
f^{-1} \mathcal{O}_X = \mathcal{O}_Y \not\subset \mathcal{O}_Y(\lfloor K_{Y/(X, D)} \rfloor).
\]

This remains true even if we replace \(X\) with any open neighborhood of \(x\). Again by the definition of \(f_{\bullet}\), \(\mathcal{H}_x \neq \mathcal{O}_{X,x}\).

\[\square\]

Corollary 3.7. Let \((X, D)\) be an effective log pair. If \((X, D)\) is log canonical, then \(\mathcal{H}(X, D)\) is the defining ideal sheaf of the closed subset \(\text{NonSC}(X, D)\).

Proof. Let \(\mathcal{N}\) be the defining ideal of \(\text{NonSC}(X, D)\). From Proposition 3.6 we have \(\mathcal{H} \subset \mathcal{N}\). To see the opposite inclusion, let \(U \subset X\) be an open subset and \(g \in \mathcal{N}(U)\). For a log resolution \(f : Y \rightarrow X\) of \((X, D)\), \(f^* g\) vanishes along the closed set \(f^{-1}(\text{NonSC}(X, D))\). The last set contains every prime divisor \(F\) on \(Y\) having a negative coefficient in \(\lfloor K_{Y/(X, D)} \rfloor\), which is equal to \(-1\) since \((X, D)\) is log canonical. Therefore, \(f^* g \in \mathcal{O}_Y(\lfloor K_{Y/(X, D)} \rfloor)(f^{-1} U)\) and hence \(g \in \mathcal{H}(U)\). Thus \(\mathcal{N} \subset \mathcal{H}\), proving the corollary.

\[\square\]

Next we will define the other variant, denoted by \(\mathcal{I}(X, D)\), of the multiplier ideal.

Lemma 3.8. Let \((X, D)\) be an effective log pair. Suppose that \(D\) is effective. There exists a positive rational number \(\epsilon_0\) such that for every \(\epsilon \in (0, \epsilon_0] \cap \mathbb{Q}\),

\[
\mathcal{J}(X, (1 - \epsilon)D) = \mathcal{J}(X, (1 - \epsilon_0)D).
\]

Proof. Let \(f : Y \rightarrow X\) be a log resolution of \((X, D)\). For a rational number \(\epsilon > 0\),

\[
K_{Y/(X,(1-\epsilon)D)} = K_{Y/(X,D)} + \epsilon f^* D.
\]

We choose so small \(\epsilon_0 > 0\) that every coefficient of \(\epsilon_0 f^* D\) is smaller than the fractional part \(\{x\} := x - \lfloor x \rfloor\) of any non-integral coefficient \(x\) of \(K_{Y/(X,D)}\). Then, for every \(\epsilon \in (0, \epsilon_0]\),

\[
\lfloor K_{Y/(X,(1-\epsilon)D)} \rfloor = \lfloor K_{Y/(X,(1-\epsilon_0)D)} \rfloor,
\]

which shows the lemma.

\[\square\]

Definition 3.9. For an effective log pair \((X, D)\), we define an ideal sheaf \(\mathcal{I}(X, D)\) on \(X\) as the multiplier ideal sheaf \(\mathcal{J}(X, (1 - \epsilon)D)\) for a sufficiently small rational number \(\epsilon > 0\).
Remark 3.10. Let $a \subset \mathcal{O}_X$ be the defining ideal sheaf of $D$. When $X$ is smooth, then the ideal $\mathcal{I}(X, D)$ was denoted by $J^-(a)$ in [Voj12] and Vojta used it in a generalization of his own conjecture.

Multiplier-like ideal sheaves which we saw above satisfy the following inclusion relations,

$$\mathcal{H}(X, D) \subset J(X, D) \subset \mathcal{I}(X, D).$$

Proposition 3.11 (cf. [Laz04, Def. 9.3.9]). Let $(X, D)$ be an effective log pair. Then

(3.1) $\text{NonLC}(X, D) \subset \text{Supp}(\mathcal{O}_X/\mathcal{I}(X, D)) \subset \text{NonKLT}(X, D)$.

Moreover, if $X$ is log terminal outside $\text{Supp}(D)$, then

(3.2) $\text{NonLC}(X, D) = \text{Supp}(\mathcal{O}_X/\mathcal{I}(X, D))$.

Proof. Let $f : Y \to X$ be a log resolution of $(X, D)$ and $\epsilon > 0$ a sufficiently small rational number. The closed subset defined by $\mathcal{I}(X, D)$ is expressed as

$$\bigcup_{F \subset Y: \text{mult}_F([K_Y/(X, D) + \epsilon f^*D]) < 0} f(F),$$

where $\text{mult}_F(E)$ denotes the multiplicity of $F$ in $E$, while $\text{NonKLT}(X, D)$ and $\text{NonLC}(X, D)$ have similar expressions as in Lemma 2.12. We have the following implications among conditions on a prime divisor $F$ of $Y$,

$$a(F; X, D) < -1 \Rightarrow \text{mult}_F([K_Y/(X, D) + \epsilon f^*D]) < 0 \Rightarrow a(F; X, D) \leq -1.$$  

This shows (3.1).

To show (3.2), it suffices to show

$$a(F; X, D) \geq -1 \Rightarrow \text{mult}_F([K_Y/(X, D) + \epsilon f^*D]) \geq 0.$$

We consider the case $a(F; X, D) > -1$ and the case $a(F; X, D) = -1$ separately. In the former, we obviously have $\text{mult}_F([K_Y/(X, D) + \epsilon f^*D]) \geq 0$. In the latter, since $X$ is log terminal outside $\text{Supp}(D)$, $F$ is contained in $\text{Supp}(f^*D)$. Hence $\text{mult}_F(K_Y/(X, D) + \epsilon f^*D) > -1$ and $\text{mult}_F([K_Y/(X, D) + \epsilon f^*D]) \geq 0$. This completes the proof. 

4. Weil functions

In this section, we summerize basic properties of Weil functions (local height functions) of arbitrary closed subschemes, and define associated height functions, counting functions and proximity functions.

We denote by $M_k$ the set of places of $k$. In what follows, we fix a finite set $S \subset M_k$ containing all infinite places. We also fix an algebraic closure $\bar{k}$ of $k$.

Let $X$ be a projective variety. To an ideal sheaf $a \subset \mathcal{O}_X$, we associate a Weil function

$$\lambda_a : X(\bar{k}) \times M_k \to [0, +\infty],$$

following [Sil87], which is unique up to addition of $M_k$-bounded functions. If $Z$ is the closed subscheme defined by an ideal sheaf $a$, then we write $\lambda_a$ also as $\lambda_Z$. If $Z$ is a Cartier divisor (that is, $a$ is locally principal), then it is the usual Weil function for an effective Cartier divisor.
There are several ways to normalize Weil functions. We follow the one adopted in \[Voj11, \text{Def. 8.6}\]. Namely, for \(v \in M_k\), if \(p\) is the place of \(\mathbb{Q}\) under \(v\), then we denote by \(\| \cdot \|_v\) the norm on \(\bar{k}\) extending the one on \(k\) defined by
\[
\|a\|_v := |\cdot|_{\bar{k}}^{[\bar{k}:\mathbb{Q}_p]},
\]
where \(|\cdot|_p\) denotes the usual \(p\)-adic absolute value and \(k_v\) denotes the \(v\)-adic completion of \(k\). When a Cartier divisor \(D\) is locally defined by a rational function \(f\), then a Weil function \(\lambda_D\) of \(D\) should be locally of the form
\[
\lambda_D(x, v) = -\log \|f(x)\|_v + \alpha(x)
\]
for a continuous locally \(M_k\)-bounded function \(\alpha\).

Basic properties of Weil functions are as follows; in this proposition, comparison (equality or inequality) of Weil functions are made up to addition of \(M_k\)-bounded functions:

**Proposition 4.1** ([Sil87, Th. 2.1]).

1. For a morphism \(f: Y \to X\) of varieties and a closed subvariety, we have
\[
\lambda_Z \circ f = \lambda_{f^{-1}Z}.
\]
2. For \(Z \subset Z' \subset X\),
\[
\lambda_Z \leq \lambda_{Z'}.
\]
3. For closed subvarieties \(Z, Z' \subset X\),
\[
\lambda_{Z + Z'} = \lambda_Z + \lambda_{Z'}.
\]
   Here, if \(Z\) and \(Z'\) are defined by ideal sheaves \(a\) and \(a'\) respectively, then \(Z + Z'\) is the closed subscheme defined by the product \(aa'\).
4. For closed subvarieties \(Z, Z' \subset X\),
\[
\lambda_{Z \cap Z'} = \min\{\lambda_Z + \lambda_{Z'}\}.
\]
   Here, if \(Z\) and \(Z'\) are defined by ideal sheaves \(a\) and \(a'\) respectively, then \(Z \cap Z'\) is the closed subscheme defined by the sum \(a + a'\).

For later use, we need the following explicit description of a Weil function in a special case:

**Proposition 4.2** (cf. [Voj11, Th. 8.8, (c)], [HS00, Ex. B.8.4]). Let \(X = \mathbb{P}^n_k\) be a projective space of dimension \(n\) with homogeneous coordinates \(x_0, \ldots, x_n\) and \(D\) the Cartier divisor defined by a homogeneous polynomial \(f \in k[x_0, \ldots, x_n]\) of degree \(d\). Then the function
\[
\lambda_D ((x_0: \cdots: x_n), v) := -\log \frac{\|f(x_0, \ldots, x_n)\|_v}{\max\{\|x_0\|_v, \ldots, \|x_n\|_v\}^d}
\]
is a Weil function with respect to \(D\).

For \(v \in M_k\) and \(x \in X(\bar{k})\), we write
\[
\lambda_{a,v}(x) := \lambda_a(x, v).
\]
For a finite extension $L/k$ and a place $w$ of $L$, if $v \in M_k$ is the place under $w$, then we define
\[
\lambda_{a,w}(x) := [L_w : k_v] \cdot \lambda_{a,v}(x).
\]

**Definition 4.3.** We define the height function $h_a$, the counting function $N_a$ and the proximity function $m_a$ on $X(\bar{k})$ relative to $\lambda$ and $k$ as follows. For $x \in X(\bar{k})$, we denote by $k(x)$ its residue field. Let $L/k$ be a finite extension containing $k(x)$ and let $T \subset M_L$ be the set of places over places in $S$. Then,
\[
\begin{align*}
    h_a(x) &:= \frac{1}{[L : k]} \sum_{w \in M_L} \lambda_{a,w}(x), \\
    N_a(x) &:= \frac{1}{[L : k]} \sum_{w \in M_L \setminus T} \lambda_{a,w}(x), \\
    m_a(x) &:= \frac{1}{[L : k]} \sum_{w \in T} \lambda_{a,w}(x).
\end{align*}
\]

When $Z$ is the closed subscheme defined by $a$, we write these also as $h_Z, N_Z, m_Z$ respectively.

These definitions are independent of the choice of such a field $L$. Obviously $h_a = N_a + m_a$. Note that counting and proximity functions depend on the choice of $S$, and their symbols are often accompanied with the subscript $S$, which we however omit.

That $\lambda_{Z,w}(x)$ is large means that the point $x$ is $w$-adically close to $Z$. Thus the function $h_Z$ (resp. $N_Z, m_Z$) expresses total closeness all over the places $w$ in $M_L$ (resp. $M_L \setminus T, T$).

**Definition 4.4.** If $Z = \sum_i c_i Z_i$ is a $\mathbb{Q}$-scheme, then we define
\[
\begin{align*}
    h_Z := \sum_i c_i h_{Z_i}, & \quad N_Z := \sum_i c_i N_{Z_i}, & \quad m_Z := \sum_i c_i m_{Z_i}
\end{align*}
\]
as functions on $(X \setminus \text{Supp}(Z))(\bar{k})$.

For a (not necessarily effective) Cartier divisor $D$, the height function $h_D$ on $(X \setminus \text{Supp}(D))(\bar{k})$ defined as above extends to the whole set $X(\bar{k})$ and defines a unique function $h_D$ up to addition of bounded functions. The function class $h_D$ modulo bounded functions depends only on the linear equivalence class of $D$. Furthermore, we can easily generalize this to $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisors; if $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor and if $n$ is a positive integer such that $nD$ is Cartier, then a height function $h_D$ is defined as $\frac{1}{n} h_{nD}$. In particular, for a $\mathbb{Q}$-Gorenstein variety $X$, we can define a height function $h_{K_X}$ of a canonical divisor $K_X$.

**Definition 4.5.** For an effective projective log pair $(X, D)$, we define $h_{K(X,D)}$ as
\[
h_{K(X,D)} := h_{K_X} + h_D : X(\bar{k}) \to (-\infty, +\infty).
\]

When $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor, then $h_{K(X,D)}$ is a height function of $K_X + D$, which is considered as a “canonical divisor” of $(X, D)$, hence the notation $h_{K(X,D)}$.

We need one more definition to formulate Vojta’s conjecture.
Definition 4.6. For a number field $F$, let $D_F \in \mathbb{Z}$ denote the absolute discriminant of $F$. For a finite extension $L/k$, we define its logarithmic discriminant $d_k(L)$ by
\[ d_k(L) := \frac{1}{[L:k]} \log |D_L| - \log |D_k|. \]

For a point $x \in X(\bar{k})$ of a $k$-variety $X$, we define its logarithmic discriminant by
\[ d_k(x) := d_k(k(x)). \]

5. A generalization of Vojta’s conjecture to log pairs

The original form of Vojta’s conjecture for algebraic points is as follows:

Conjecture 5.1. Let $X$ be a smooth complete variety, $A$ a big divisor of $X$ and $D$ a reduced simple normal crossing divisor of $X$. Let $r$ be a positive integer and $\epsilon$ a positive real number. Then there exists a proper closed subset $Z \subset X$ depending only on $X, D, A, \epsilon$ such that for all $x \in (X \setminus Z)(\bar{k})$ with $[k(x) : k] \leq r$, we have
\[
(5.1) \quad h_{K_X}(x) + m_D(x) \leq d_k(x) + \epsilon h_A(x) + O(1).
\]

If we set $r = 1$ in this conjecture, then $d_k(x)$ is always zero and can be removed from the inequality. The conjecture in this case is called Vojta’s conjecture for rational points.

Using log pairs and multiplier-like ideals introduced in Section 3, we formulate a generalization of this conjecture as follows:

Conjecture 5.2. Let $(X, D)$ be an effective projective log pair, $A$ a big divisor of $X$. Let $r$ be a positive integer and $\epsilon$ a positive real number. Then there exists a proper closed subset $Z \subset X$ depending only on $X, D, A, \epsilon$ such that for all $x \in (X \setminus Z)(\bar{k})$ with $[k(x) : k] \leq r$, we have
\[
(5.2) \quad h_{K_X(D)}(x) - N_{H(X,D)}(x) - m_{I(X,D)}(x) \leq d_k(x) + \epsilon h_A(x) + O(1).
\]

We can view the left hand side of the above inequality as follows. The main term is $h_{K_X(D)}(x)$ and the other two terms are correction terms arising from singularities of $(X, D)$. Indeed, from Proposition 3.6, the term $-N_{H(X,D)}(x)$ can be thought of as contribution of the non-sc locus $\text{NonSC}(X, D)$. From Proposition 3.11, $-m_{I(X,D)}(x)$ can be thought of as contribution of $\text{NonKLT}(X, D)$ (and also one of $\text{NonLC}(X, D)$ if $X$ is log terminal outside $\text{Supp}(D)$).

Example 5.3. If $(X, D)$ is Kawamata log terminal or if $(X, D)$ is log canonical and $X$ is log terminal outside $\text{Supp}(D)$, then from Proposition 3.11, we can remove the term $-m_{I(X,D)}(x)$. From Corollary 3.7, if we give the reduced scheme structure to the non-sc locus $\text{NonSC}(X, D)$, then
\[
N_{H(X,D)}(x) = N_{\text{NonSC}(X,D)}(x).
\]

Thus inequality (5.2) is written as
\[
(5.3) \quad h_{K(X,D)}(x) - N_{\text{NonSC}(X,D)}(x) \leq d_k(x) + \epsilon h_A(x) + O(1).
\]
**Example 5.4.** Let $X$ be a smooth projective variety and $D$ a reduced simple normal crossing divisor on $X$. Then $\text{NonSC}(X, D) = \text{Supp}(D)$ and the left hand side of (5.2) is equal to

$$h_{K_{(X,D)}}(x) - N_D(x) = h_{K_X} + m_D.$$ 

Thus Conjecture 5.2 is the same as Conjecture 5.1 in this situation.

Let $Y \subset X$ be a smooth closed subscheme of codimension $r$ with transversally intersects $D$. Let us next consider the log pair $(X, D + (r - 1)Y)$. Looking at the blowup of $X$ along $Y$, we can see that this log pair is log canonical and

$$\text{NonSC}(X, D + (r - 1)Y) = \text{Supp}(D).$$

Therefore (5.3) in this case becomes

$$h_{K_X}(x) + h_D(x) + (r - 1)h_Y(x) - N_D(x) \leq d_k(x) + ch_A(x) + O(1).$$

When $D$ is linearly equivalent to $-K_X$, then this is written also as

$$(5.4) \quad h_Y(x) \leq \frac{1}{r - 1} (d_k(x) + ch_A(x) + N_D(x)) + O(1).$$

This slightly refines Silverman’s result [Sil05, Th. 6], which was stated in relation to a problem of bounding greatest common divisors, by removing $\delta \epsilon$ appearing there (cf. Appendix).

Although Conjecture 5.2 is more general than Conjecture 5.1, they are in fact equivalent:

**Proposition 5.5.** Let $(X, D)$ be an effective projective log pair and $f: Y \rightarrow X$ a log resolution of $(X, D)$. Suppose that Conjecture 5.2 holds for $Y$ and the reduced simple normal crossing divisor

$$[K_Y / (X, D)] + \epsilon f^* D - [K_Y / (X, D)]$$

for $0 < \epsilon \ll 1$. Then Conjecture 5.2 holds for $(X, D)$. In particular, Conjectures 5.1 and 5.2 are equivalent.

**Proof.** The proof here is similar to the one of Vojta’s similar result [Voj12 Prop. 4.3]. By definition,

$$f^{-1}\mathcal{H}(X, D) \subset \mathcal{O}_Y([K_Y / (X, D)])$$

and for $0 < \epsilon \ll 1$,

$$f^{-1}\mathcal{I}(X, D) \subset \mathcal{O}_Y([K_Y / (X, D)] + \epsilon f^* D]).$$

These imply

$$N_{\mathcal{H}(X, D)} \circ f \geq N_{-[K_Y / (X, D)]},$$

$$m_{\mathcal{I}(X, D)} \circ f \geq m_{-[K_Y / (X, D) + \epsilon f^* D]}.$$


We have
\[
(h_{K_{(X,D)}} - N_{H_{(X,D)}} - m_{I_{(X,D)}}) \circ f \\
\leq h_K - h_{K_{(Y,D)}} - N_{-}[-K_{(X,D)}] - m_{-}[K_{(X,D)} + f^*D] \\
\leq h_K + h_{-}[K_{(X,D)}] - N_{-}[K_{(X,D)}] - m_{-}[K_{(X,D)} + f^*D] \\
= h_K + m_{K_{(X,D)} + f^*D} \circ [K_{(X,D)}].
\]
To show the first assertion of the proposition, it suffices to recall that for a big divisor $A$ on $X$, $f^*A$ is also big. The second assertion is now obvious. □

Remark 5.6. If $X$ is smooth and $D \subset X$ is a genuine closed subscheme, then $-m_{I_{(X,D)}}$ is the same correction term as the one used in [Voj12] (see Remark 3.10). In this case, for a log resolution $f: Y \rightarrow X$ of $(X,D)$, $K_{Y/X} = K_{X} - f^{-1}D$ is a $\mathbb{Z}$-divisor. Since $K_{Y/X} \geq 0$,
\[
H_{(X,D)} = f_*O_Y(K_{Y/X} - f^{-1}D) \supset f_*O_Y(f^{-1}D) \supset I_{D},
\]
where $I_D$ is the defining ideal sheaf of $D$. Therefore,
\[
h_{K_{(X,D)}} - N_{H_{(X,D)}} - m_{I_{(X,D)}} \geq h_K + h_D - N_D - m_{I_{(X,D)}} = h_K + m_D - m_{I_{(X,D)}}.
\]
It follows that for such a pair $(X,D)$, Conjecture 5.2 is slightly stronger than Conjecture 4.2 of [Voj12] except that Vojta considers more general base fields as well as non-projective complete varieties.

Since Conjecture 5.1 is known to hold if dim $X = 1$ and $r = 1$ (for instances, see [BG06 Rem. 14.3.5] or [Voj92]), Proposition 5.5 implies:

Corollary 5.7. Suppose that $X$ has dimension one. If we set $r = 1$, then Conjecture 5.2 holds.

6. Log pairs of general type

In this section, we specialize Vojta’s conjecture to log pairs $(X,D)$ of general type and derive geometric consequences about potentially dense subvarieties.

Definition 6.1. A variety $X$ over $k$ is said to be potentially dense if for some finite extension $L/k$, $X(L)$ is Zariski dense in $X$.

For instance, an irreducible curve which is birational over $\bar{k}$ to either $\mathbb{P}^1$ or an elliptic curve is potentially dense. More generally, the image of a rational map $G \dashrightarrow X$ of a group variety $G$ is potentially dense; this follows from the facts that every connected group variety is an extension of an abelian variety by a connected linear algebraic group (see [Con02]), that every abelian variety is potentially dense [Has03, Prop. 4.2] and that every linear algebraic group is unirational, in particular, it has the Zariski dense set of rational points over any number field [Bor91, Th. 18.2 and Cor. 18.3]. Lang [Lan91, p. 17] conjectured that for a smooth variety $X$ of general type (that is, $K_X$ is big), there exists a proper closed subset $Z \subsetneq X$ such that every potentially dense subvariety $Y \subset X$ is contained in $Z$. This conjecture follows from Vojta’s conjecture 5.1. We can generalize it a little to varieties with canonical singularities as follows:

It is said that Shinichi Mochizuki [Moc] announced a proof of Conjecture 5.1 for an arbitrary $r$ in August, 2012 and it is now under a process of verification.
Proposition 6.2. Suppose that Conjecture 5.1 holds. Let $X$ be a $\mathbb{Q}$-Gorenstein canonical projective variety such that $K_X$ is big. Then there exists a proper closed subset $Z \subset X$ such that every potentially dense subvariety $Y \subset X$ is contained in $Z$.

Proof. Let $Z \subset X$ be a proper closed subset as in Conjecture 5.1 applied to $D = 0$ and $A = K_X$. For any fixed finite extension $L/k$ and for all $x \in (X \setminus Z)(L)$, we have

$$(1 - \epsilon)h_A(x) \leq O(1).$$

It follows that $h_A$ is bounded from above on $(X \setminus Z)(L)$. Since $A$ is big, for any ample divisor $A'$, there exists a constant $C > 0$ such that

$$h_{A'}(x) \leq Ch_A(x) + O(1)$$

for all $x \in X(\bar{k})$. Therefore $h_{A'}$ is also bounded from above on $(X \setminus Z)(L)$. From Northcott’s theorem, $(X \setminus Z)(L)$ is a finite set, which shows the proposition. \qed

It would be natural to ask what is expected for log pairs $(X, D)$ with $K_{(X,D)} = K_X + D$ big. First we make clear the meaning of “$K_X + D$ is big”.

Definition 6.3. A $\mathbb{Q}$-subscheme $D$ of a projective variety $X$ is big if for a log resolution $f : Y \to X$ of $(X, D)$, the $\mathbb{Q}$-divisor $f^* D$ is big. We say that a projective log pair $(X, D)$ is of general type if for some (hence every) expression of $K_X$ as a $\mathbb{Q}$-subscheme, $K_X + D$ is big.

Proposition 6.4. Let $(X, D)$ be an effective projective log pair of general type, let $A$ be a big divisor on $X$. Suppose that Conjecture 5.1 holds. Then there exists a proper closed subset $Z \subset X$ and a constant $C > 0$ such that for every $x \in (X \setminus Z)(k)$,

$$h_A(x) \leq Ch_{\text{NonSC}(X,D)}(x) + O(1).$$

Furthermore, if $(X, D)$ is log canonical, then $h_{\text{NonSC}(X,D)}(x)$ in the above inequality can be replaced with $N_{\text{NonSC}(X,D)}(x)$.

Proof. From the assumption and Proposition 5.5 Conjecture 5.2 holds. Applying it to $(X, D)$, the given divisor $A'$ and $r = 1$, we obtain the inequality

$$(6.1) \quad h_{K_{(X,D)}}(x) - \epsilon h_{A'}(x) \leq N_{H(X,D)}(x) + m_{\mathcal{I}(X,D)}(x) + O(1)$$

holding for all $x \in (X \setminus W)(k)$ for a proper closed subset $W$. For $0 < \epsilon \ll 1$, we have

$$(6.2) \quad 2\epsilon h_{A'}(x) \leq h_{K_{(X,D)}}(x) + O(1)$$

for $x \in (X \setminus \text{Supp}(D))(k)$.

If $(X, D)$ is log canonical, then $m_{\mathcal{I}(X,D)}(x) = 0$. In the general case, since $\mathcal{H}(X, D) \subset \mathcal{I}(X, D)$, we have

$$N_{H(X,D)}(x) + m_{\mathcal{I}(X,D)}(x) \leq N_{H(X,D)}(x) + m_{H(X,D)}(x) = h_{H(X,D)}(x).$$

For an integer $C \gg 0$, the $C$-th power $H(X, D)^C$ of $H(X, D)$ is contained in the defining ideal sheaf of the closed subset $\text{NonSC}(X, D)$, and

$$(6.3) \quad N_{H(X,D)} \leq CN_{\text{NonSC}(X,D)} \quad \text{and} \quad h_{H(X,D)} \leq Ch_{\text{NonSC}(X,D)}.$$

Combining (6.1), (6.2) and (6.3), we obtain the proposition. \qed
We may interpret this result as that most rational points lie near \( \text{NonSC}(X, D) \), which sounds a little surprising, in particular, if \( \text{NonSC}(X, D) \) has codimension \( \geq 2 \).

**Corollary 6.5.** Let \((X, D)\) be an effective log pair of general type. Suppose that Conjecture 5.1 holds. Then there exists a proper closed subset \( Z \subset X \) such that for every potentially dense closed subvariety \( Y \subset X \), either \( Y \) is contained in \( Z \), or \( Y \) intersects \( \text{NonSC}(X, D) \).

**Proof.** Let \( Z \subset X \) be a closed subvariety as in Proposition 6.4. To obtain a contradiction, suppose that there exists a potentially dense closed subvariety \( Y \subset X \) such that \( Y \cap \text{NonSC}(X, D) = \emptyset \) and \( Y \not\subset Z \). If necessary enlarging \( k \), we may suppose that \( Y(k) \) is Zariski dense in \( Y \). From the lemma below, the height function \( h_{\text{NonSC}(X, D)} \) is bounded on \( Y(k) \). On the other hand, from Proposition 6.4 and Northcott’s theorem, the same function cannot be bounded on \( Y(k) \), a contradiction. The corollary follows. \( \square \)

**Lemma 6.6.** Let \( X \) be a projective variety and \( C, D \subset X \) proper closed subschemes with \( C \cap D = \emptyset \). Let \( h_D : X(k) \to \mathbb{R} \cup \{\infty\} \) be a height function of \( D \). Then its restriction \( h_D|_{C(k)} \) is a bounded function.

**Proof.** From the functoriality of the Weil function, \( h_D|_{C(k)} \) is a height function of \( D \cap C \) as a closed subscheme of \( C \). In our situation, it is empty and any height function of it is bounded. \( \square \)

In Section 7, we construct rational projective surfaces of general type. If we restrict ourselves to smooth projective varieties, being rational and being of general type are completely opposite properties and cannot hold at the same time. However, they can be compatible with each other for singular varieties.

**Definition 6.7.** An irreducible variety \( X \) over \( \bar{k} \) is said to be **rationally connected** if general two points \( x, y \in X \) are connected by a rational curve.

Rational varieties are rationally connected. The next proposition shows that the assertions of Proposition 6.4 and Corollary 6.5 never hold if \( X \) is rationally connected and \( \text{NonSC}(X, D) \) is replaced with a smooth closed subset of codimension \( \geq 2 \) which is contained in the smooth locus of \( X \).

**Proposition 6.8.** Let \( X \) be a rationally connected projective variety over \( \bar{k} \), let \( Z \varsubsetneq X \) be a proper closed subset and let \( W \subset X \) be a smooth closed subvariety of codimension \( \geq 2 \) with \( W \subset X_{\text{sm}} \). Then there exists a rational curve \( C \subset X \) such that \( C \not\subset Z \) and \( C \cap W = \emptyset \).

**Proof.** Taking a resolution, we may suppose that \( X \) is smooth. Let \( x \in X(\bar{k}) \) be a point outside \( Z \cup W \). By [KMM92, 2.1], there exists a morphism

\[
f : \mathbb{P}^1 \to X
\]

such that

1. \( f \) is an immersion in the sense that for every closed point \( p \in \mathbb{P}^1, T_p \mathbb{P}^1 \to T_{f(p)}X \) is injective,
(2) \( x \in f(\mathbb{P}^1) \),

(3) \( f^*T_X \) is ample.

Since

\[
\text{Ext}^1(f^*\Omega_X, \mathcal{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1, f^*T_X) = 0,
\]

from [Kol96] Ch. I, Th. 2.16, the moduli space of morphisms \( \mathbb{P}^1 \to X \), Hom(\( \mathbb{P}^1, X \)),

is smooth at \([f]\) and the tangent space \( T_{[f]}\text{Hom}(\mathbb{P}^1, Y) \) is identified with \( H^0(\mathbb{P}^1, f^*T_Y) \).

Let \( B := f^{-1}W \). If \( B \) is empty, then \( f(\mathbb{P}^1) \) is a desired rational curve. Otherwise, there exists a subvariety \( V \subset H^0(\mathbb{P}^1, f^*T_X) \) of dimension \( \dim X - 1 \) such that for every \( b \in B \), the image of \( V \) in \( T_{f(b)}X \) is transversal to the image of \( T_b\mathbb{P}^1 \). We take a smooth irreducible subvariety \( W \subset \text{Hom}(\mathbb{P}^1, Y) \) of dimension \( \dim X - 1 \) passing through \([f]\) such that \( T_{[f]}W = V \). The induced morphism

\[
G: \mathbb{P}^1 \times W \to Y
\]

is then an immersion around \( B \times \{[g]\} \). Therefore \( G^{-1}W \) has codimension \( \geq 2 \) in \( \mathbb{P}^1 \times W \) around \( \mathbb{P}^1 \times \{[g]\} \). This shows that \( G^{-1}W \) does not surject onto \( W \). We conclude from this that for a general point \([h] \in V \), the image of \( h: \mathbb{P}^1 \to X \) does not intersect \( W \). Similarly, \( G^{-1}Z \) has codimension \( \geq 1 \). Hence, for a general point \([h] \in V \), the fiber \( \mathbb{P}^1 \times \{[h]\} \) is not contained in \( G^{-1}Z \). This means that the image of \( h: \mathbb{P}^1 \to Y \) is not contained in \( Z \). Thus, for a general \([h] \in W \), \( h(\mathbb{P}^1) \) is a rational curve satisfying the desired condition. \( \square \)

**Example 6.9.** The arguably easiest way to construct a rationally connected variety of general type is to take the closure of the image of a morphism \( \mathbb{A}^n_k \to \mathbb{P}^{n+1}_k \). For an irreducible polynomial \( f(x_1, \ldots, x_n) \) of degree \( d \), we consider a closed embedding

\[
g: \mathbb{A}^n_k \to \mathbb{A}^{n+1}_k, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n)).
\]

The closure \( X \) of \( g(\mathbb{A}^n_k) \) in \( \mathbb{P}^{n+1}_k \) is a hypersurface of degree \( d \). Therefore \( X \) is Gorenstein and rational. By the adjunction formula, if \( d \geq n + 2 \), then \( X \) is also of general type. However the author does not know what kind of singularities such an \( X \) would have. In the next section, we will construct a rational surface of general type having only log terminal singularities.

**7. Rational surfaces of general type**

In this section, we show the following proposition.

**Proposition 7.1.** Suppose that \( k \) contains a primitive \( n \)-th root for some \( n \geq 5 \). Then there exists a projective rational surface \( X \) having only log terminal singularities such that \( K_X \) is ample.

**Proof.** We construct such an \( X \) as the quotient of a Fermat hypersurface by a finite group action. Let \( n \geq 5 \) be an integer such that \( k \) contains a primitive \( n \)-th root \( \zeta \) of 1. Consider the Fermat hypersurface of degree \( n \),

\[
F := V(x_0^n + x_1^n + x_2^n + x_3^n) \subset \mathbb{P}^3_k.
\]
This is a smooth irreducible projective surface. From the condition \( n \geq 5 \) and the adjunction formula, the canonical divisor \( K_F \) is ample. Let \( G = \langle g \rangle = \mathbb{Z}/n\mathbb{Z} \) be a cyclic group of order \( n \) generated by an element \( g \). We define a \( G \)-action on \( \mathbb{P}^3 \) by
\[
g((x_0 : x_1 : x_2 : x_3)) = (\zeta x_0 : \zeta x_1 : x_2 : x_3).
\]
Clearly \( F \) is preserved by the action. We see that the fixed point locus \( F^G \) in \( F \) is
\[
\{x_0 = x_1 = x_2^n + x_3^n = 0\} \cup \{x_2 = x_3 = x_0^n + x_1^n = 0\},
\]
which has dimension zero and \( 2n \) points. Let \( X := F/G \) be the associated quotient variety and \( \pi : F \to X \) the natural morphism, which is étale in codimension one. Since \( K_F = \pi^*K_X \) and \( K_F \) is ample, \( K_X \) is also ample. The variety \( X \) has only quotient singularities. As is well known, quotient singularities are log terminal (for instance, see [Kol13, Cor. 2.43 or p. 103]).

It remains to show that \( X \) is rational. Consider the following locally closed subvariety of \( \mathbb{A}^3_k \) with coordinates \( x, y, z \),
\[
W = \left\{ x^n + 1 \neq 0, \ z^n + \frac{y^n + 1}{x^n + 1} = 0 \right\} \subset \mathbb{A}^3_k,
\]
and the morphism
\[
\phi : W \to \mathbb{P}^3_k \quad (x, y, z) \mapsto (xz : z : y : 1).
\]
We see that its image is contained in \( F \cap \{x_3 \neq 0\} \); we denote the induced morphism \( W \to F \) by \( \psi \). The ring homomorphism associated to
\[
W \to \mathbb{A}^3_k = \{x_3 \neq 0\}
\]
is given by
\[
\alpha : k[u_0, u_1, u_2] \to k \left[ x, y, z, \frac{1}{x^n + 1} \right] \quad \begin{align*}
u_0 &\mapsto xz \\
u_1 &\mapsto z \\
u_2 &\mapsto y.
\end{align*}
\]
Since the image of \( \alpha \) together with \( k \) generates the function field of \( W \) as a field, the morphism \( \psi : W \to F \) is birational.

We define a \( G \)-action on \( W \) by
\[
g : (x, y, z) \mapsto (x, y, \zeta z).
\]
Then \( \psi \) is \( G \)-equivariant. Taking quotients, we obtain a morphism
\[
\tilde{\psi} : W/G \to X,
\]
which is again birational. The variety \( W/G \) is naturally isomorphic to the open subvariety \( \{x^n + 1 \neq 0\} \) of \( \mathbb{A}^3_k \). In particular, it is rational. We conclude that \( X \) is also rational. We have proved Proposition 7.1. \( \square \)
Remark 7.2. For a suitable local coordinates \( r, s \) around each point of \( F \), the group action is given by

\[
g(r, s) = (\zeta r, \zeta s) \text{ or } (\zeta^{-1} r, \zeta^{-1} s).
\]
Namely every singular point of \( X \) is the quotient singularity of type \( \frac{1}{n}(1, 1) \). It follows that

\[
\text{totaldiscrep}(X) = \text{discrep}(X) = \frac{2}{n} - 1 < 0
\]
(for instance, see [Yas06, Cor. 6]). In particular, \( X \) is log terminal, but not canonical. Furthermore, the singular locus of \( X \) coincides with the non-canonical locus \( \text{NonC}(X) \).

Remark 7.3. The above proposition shows the necessity of the correction term \(-N_{\text{H}(X,D)}\) in Conjecture 5.2. Indeed, for such an \( X \), if we set \( D = 0, \ r = 1 \) and \( A = K_X \), then inequality (5.2) is written as

\[
(1 - \epsilon) h_{K_X}(x) - N_{\text{NonC}(X)}(x) \leq O(1).
\]
Since \( X \) is rational, the \( k \)-point set \( X(k) \) is Zariski dense. If there was no correction term \( N_{\text{NonC}(X)} \), this fact would contradicts Northcott’s theorem.

Remark 7.4. From Proposition 6.8, inequality (7.1) does not hold if we replace \( \text{NonC}(X) \) with any collection of finitely many smooth points and if we replace \( k \) with some finite extension of it.

Remark 7.5. From Corollary 6.5, if Vojta’s conjecture is true, then all but finitely many irreducible curves on \( X \otimes_k \bar{k} \) birational to \( \mathbb{P}^1 \) or an elliptic curve would pass through one of the singular points of \( X \).

Appendix A. Greatest common divisors and plane curves

Bugeaud, Corvaja and Zannier [BCZ03, CZ05] obtained an upper bound for \( \gcd(a - 1, b - 1) \) for certain families of integer pairs \((a, b)\). To explain their result in relation to Vojta’s conjecture, Silverman [Sil05] observed that the greatest common divisor is essentially a height function associated to a subscheme of codimension \( \geq 2 \), although he uses the blowup along the subscheme and a height function associated to the exceptional divisor instead (see also [Yas12, Yas11]). He then formulated a conjectural generalization of the result of Bugeaud, Corvaja and Zannier. It was in this work that a slightly weaker version of inequality (5.4) appeared.

In this Appendix, as an application of Silverman’s observation, we relate estimation of \( \gcd(a, b) \) for integer pairs \((a, b)\) satisfying an algebraic equation with the multiplicity of the corresponding plane curve at the origin. The only ingredients necessary to do so is basic properties of heights and a simple analysis of resolution of curves.

Lemma A.1. Let \( Z \subset \mathbb{P}^n_\mathbb{Q} \) be the closed subscheme defined by the ideal \( \langle f_1, \ldots, f_l \rangle \subset \mathbb{Q}[x_0, \ldots, x_n] \) generated by homogenous polynomials \( f_1, \ldots, f_l \in \mathbb{Z}[x_0, \ldots, x_n] \). For a point \( x \in \mathbb{P}^n_\mathbb{Q}(\mathbb{Q}) \), we write \( x = (x_0 : x_1 : \cdots : x_n) \) in terms of integers \( x_i \) with \( \gcd(x_0, x_1, \ldots, x_n) = 1 \) and define \( f_i(x) := f_i(x_0, \ldots, x_n) \in \mathbb{Z} \). Then

\[
N_Z(x) := \log \gcd(f_1(x), \ldots, f_l(x))
\]
is a counting function of \( Z \) with respect to \( S = \{\infty\} \), and
\[ h_Z(x) := \log \gcd(f_1(x), \ldots, f_l(x)) - \max_{1 \leq i \leq l} \log \left( \frac{|f_i(x)|_\infty}{\max\{|x_0|_\infty, \ldots, |x_n|_\infty\}^{\deg f_i}} \right) \]

is a height function of \( Z \).

**Proof.** We first note that for integers \( a_i \),
\[ \log \gcd(a_1, \ldots, a_l) = -\sum_{p \in M\mathbb{Q}; p \neq \infty} \max_i |a_i|_p. \]

From Propositions 4.1 and 4.2,
\[ \lambda_{Z,p}(x) := \min_{1 \leq i \leq l} \left( -\log \frac{|f_i(x)|_p}{\max\{|x_0|_p, \ldots, |x_n|_p\}^{\deg f_i}} \right) \quad (p \in M\mathbb{Q}) \]

is a Weil function of \( Z \). For \( p \neq \infty \), since \( \gcd(x_0, \ldots, x_n) = 1 \), we have
\[ \max\{|x_0|_p, \ldots, |x_n|_p\} = 1 \]

and
\[ \lambda_{Z,p}(x) = -\log \max_i |f_i(x)|_p. \]

We conclude that
\[ N_Z(x) := \sum_{p \in M\mathbb{Q}; p \neq \infty} \lambda_{Z,p}(x) \]
\[ = -\sum_{p \in M\mathbb{Q}; p \neq \infty} \log \max_i |f_i(x)|_p \]
\[ = \log \gcd(f_1(x), \ldots, f_l(x)) \]

is a counting function of \( Z \) and
\[ h_Z(x) := N_Z(x) + \lambda_{Z,\infty}(x) \]
\[ = \log \gcd(f_1(x), \ldots, f_l(x)) - \max_{1 \leq i \leq l} \log \left( \frac{|f_i(x)|_\infty}{\max\{|x_0|_\infty, \ldots, |x_n|_\infty\}^{\deg f_i}} \right) \]

is a height function of \( Z \). \( \square \)

**Example A.2.** For \( l < n \), let \( Z \) be the linear subspace defined by
\[ x_0 = x_1 = \cdots = x_l = 0. \]

Then
\[ N_Z(x) = \log \gcd(x_0, \ldots, x_l) \]

is a counting function of \( Z \). Since
\[ \max_{0 \leq i \leq l} \max\{|x_0|_\infty, \ldots, |x_n|_\infty\} \]
\[ = \max\{|x_0|_\infty, \ldots, |x_l|_\infty\} \]
\[ = \min \left\{ 1, \frac{\max\{|x_0|_\infty, \ldots, |x_l|_\infty\}}{\max\{|x_{l+1}|_\infty, \ldots, |x_n|_\infty\}} \right\}, \]
the function
\[ h_Z(x) = \log \gcd(x_0, \ldots, x_l) - \log \min \left\{ 1, \frac{\max \{|x_0|_\infty, \ldots, |x_l|_\infty\}}{\max \{|x_{l+1}|_\infty, \ldots, |x_n|_\infty\}} \right\} \]
is a height function of \( Z \).

**Lemma A.3.** Let \( X \) be an irreducible projective variety of dimension one over a number field \( k \) and \( \pi : \tilde{X} \to X \) the normalization. Let \( Z \subset X \) be a proper closed subscheme and \( l \in \mathbb{Z} \) the degree of the scheme-theoretic pull-back \( \pi^{-1} Z \) naturally regarded as a divisor. Let \( D \) be a divisor of \( X \) of degree \( l \) supported in the smooth locus of \( X \). Then, for every \( \epsilon > 0 \), their exist constants \( C_1, C_2 > 0 \) such that for all \( x \in (X \setminus Z)(k) \),
\[(A.1) \quad (1 - \epsilon) h_D(x) - C_1 \leq h_Z(x) \leq (1 + \epsilon) h_D(x) + C_2.\]
Moreover, if \( X \) is rational (that is, birational to \( \mathbb{P}^1_k \)), then
\[ h_Z(x) = h_D(x) + O(1). \]

**Proof.** Let \( \tilde{Z} := \pi^{-1} Z \) and \( \tilde{D} := \pi^* D \). Since they are divisors of equal degree, height functions \( h_{\tilde{Z}} \) and \( h_{\tilde{D}} \) are quasi-equivalent (see [Lan83, Cor. 3.5, Ch. 4]), hence so are \( h_D \) and \( h_Z \); it exactly means \((A.1)\). If \( X \) is rational, then \( \tilde{Z} \) and \( \tilde{D} \) are linearly equivalent. Therefore \( h_{\tilde{Z}} \) and \( h_{\tilde{D}} \) differs only by a bounded function, and the same holds for \( h_Z \) and \( h_D \). \( \square \)

**Theorem A.4.** Let \( X \subset \mathbb{P}^2_k \) be an integral plane curve of degree \( d \) and let \( O := (0 : 0 : 1) \in \mathbb{P}^2_k(k) \). Suppose that \( X \) has multiplicity \( m \) at \( O \), that is, \( m \) is the largest integer \( n \) such that \( \mathcal{I}_{X,O} \subset m^\mathbb{O}_{\hat{k}} \), where \( \mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^2_k} \) is the defining ideal sheaf of \( X \), \( \mathcal{I}_{X,O} \) is its stalk at \( O \) and \( m_O \) is the maximal ideal of the local ring \( \mathcal{O}_{\mathbb{P}^2_k} \). Let \( h \) be the standard logarithmic height on \( \mathbb{P}^2_k \) given by
\[ h((x : y : z)) = \sum_{w \in \mathbb{M}_L} \log \max\{\|x\|_w, \|y\|_w, \|z\|_w\} \]
for so large finite extention \( L/k \) that \( x, y, z \in L \). Then, for every \( \epsilon > 0 \), their exist constants \( C_1, C_2 > 0 \) such that for all \( x \in (X \setminus \{O\})(\bar{k}) \),
\[ \left( \frac{m}{d} - \epsilon \right) h(x) - C_1 \leq h_O(x) \leq \left( \frac{m}{d} + \epsilon \right) h(x) + C_2. \]
Moreover, if \( X \) is rational, then
\[ h_O(x) = \frac{m}{d} h(x) + O(1). \]

**Proof.** The standard height \( h \) is a height function of a line in \( \mathbb{P}^2_k \). Take a general line \( L \) which does not meet any singularity of \( X \). We regard the closed point \( O \) as a reduced scheme and apply Lemma [A.3] to \( Z = O \) and \( D = L \cap X \). To see the assertion, we need to show that \( m \) is equal to \( l \) as in Lemma [A.3]. Since these numbers are stable under extension of the base field, we consider a plane curve germ \( \tilde{X} = \text{Spec} \, \bar{k}[[x,y]]/(f) \) defined over \( \bar{k} \). The multiplicity is then equal to the order of \( f \). If \( \tilde{X}_i, \ i = 1, \ldots, r, \) are
the irreducible components of $\hat{X}$ and if $m_i$ and $l_i$ are the numbers similarly defined for $\hat{X}_i$, then

$$m = \sum_{i=1}^{r} m_i \text{ and } l = \sum_{i=1}^{r} l_i.$$  

Therefore, we may assume that $\hat{X}$ is irreducible. Then $\hat{X} \cong \text{Spec } \bar{k}[[g, h]]$, where $g, h \in \bar{k}[[t]]$ are power series of distinct orders such that $\text{Spec } \bar{k}[[t]] \to \hat{X}$ is birational. Now it is easy to see that

$$m = \min\{\text{ord}(g), \text{ord}(f)\} = l.$$

We have completed the proof. \hfill \Box

Note that the theorem is valid even if $O \notin X$; then $m = 0$ and $h_O$ is bounded (Lemma 6.6). The theorem asserts that a singular point has more rational points around it more than a smooth point does and that its extent is determined by the multiplicity, the most fundamental invariant of plane curve singularities.

**Remark A.5.** Theorem A.4 is non-trivial only when $X$ has infinitely many $k$-points; it means from Faltings’ theorem that $X$ has a geometric irreducible component birational to $\mathbb{P}^1$ or an elliptic curve. If $X$ is smooth, then this is possible only when $d \leq 3$. However, if we allow singularities, then there exist plane curves of arbitrary degree having infinitely many $k$-points.

Specializing the theorem to the case $k = \mathbb{Q}$ and to $\mathbb{Q}$-rational points, we obtain:

**Corollary A.6.** Let $f(x, y) \in \mathbb{Q}[x, y]$ be an irreducible polynomial and let $d$ and $m$ be the degree and the order of $f$ respectively. Then, for every $\epsilon > 0$, their exist positive constants $C_1, C_2$ such that for all triplets $(x, y, z) \neq (0, 0, 0), (0, 0, 1)$ of integers satisfying $\gcd(x, y, z) = 1$ and $f(x, y, z) = 0$, we have

$$\begin{align*}
\left(\frac{m}{d} - \epsilon\right) \log \max\{|x|, |y|, |z|\} - C_1 &\leq \log \gcd(x, y) - \log \min \left\{1, \frac{\max\{|x|, |y|\}}{|z|}\right\} \\
&\leq \left(\frac{m}{d} + \epsilon\right) \log \max\{|x|, |y|, |z|\} + C_2.
\end{align*}$$

Moreover, if $X$ is rational, then

$$\log \gcd(x, y) - \log \min \left\{1, \frac{\max\{|x|, |y|\}}{|z|}\right\} = \frac{m}{d} \log \max\{|x|, |y|, |z|\} + O(1).$$

Furthermore, excluding points close to the origin relative to the Euclidean topology, we obtain the following simpler estimation.

**Corollary A.7.** With the same notation as above, for every $\epsilon, \delta > 0$, their exist positive constants $C'_1, C'_2$ such that for all triplets $(x, y, z) \neq (0, 0, 0), (0, 0, 1)$ of integers satisfying $\gcd(x, y, z) = 1$, $f(x, y, z) = 0$ and $\max\{|x/z|, |y/z|\} \geq \delta$, we have

$$C'_1 \max\{|x|, |y|\}^{m/d - \epsilon} \leq \gcd(x, y) \leq C'_2 \max\{|x|, |y|\}^{m/d + \epsilon}.$$

Moreover, if $X$ is rational, then we can replace $\epsilon$ with zero.
\textbf{Proof.} From the condition $\max\{|x/z|, |y/z|\} \geq \delta$, the term

$$- \log \min \left\{1, \max\{|x|, |y|\}/|z|\right\}$$

in (A.2) is bounded and hence can be eliminated. If $\delta \geq 1$, then the condition $\max\{|x/z|, |y/z|\} \geq \delta$ implies

$$\log \max\{|x|, |y|, |z|\} - \log \max\{|x|, |y|\} = 0.$$ 

If $\delta < 1$, then

$$0 \leq \log \max\{|x|, |y|, |z|\} - \log \max\{|x|, |y|\} \leq - \log \delta.$$ 

Therefore $\log \max\{|x|, |y|, |z|\}$ in (A.2) can be replaced with $\log \max\{|x|, |y|\}$. Writing the resulting inequalities multiplicatively, we obtain the corollary. \hfill \Box

Note that the condition imposed in the last corollary on triplets $(x, y, z)$ are satisfied by $(x, y, 1)$ for integer pairs $(x, y)$ with $f(x, y) = 0$.

\textbf{Example A.8.} Let $X \subset \mathbb{A}^2$ be the affine plane curve defined by $x^d = y^m$ for coprime positive integers $d, m$ with $d > m$. This curve is rational and has degree $d$ and multiplicity $m$ at $O$. An integral point $p$ of $X$ is of the form $(a^m, a^d)$ for an integer $a$. With $O = (0, 0)$, we have

$$\gcd(a^m, a^d) = |a^m| = \max\{|a^m|, |a^d|\}^{m/d}.$$ 

Next consider the affine plane curve $Y$ defined by $(x+1)^d = (y+1)^m$ for the same $d, m$ as above. This is a translation of $X$. Note that $Y$ contains $O$ as a smooth point, namely $Y$ has multiplicity one at $O$. An integral point $p$ of $Y$ is of the form $(a^m - 1, a^d - 1)$ for an integer $a$. We claim that for $|a| > 1$,

$$\gcd(a^m - 1, a^d - 1) = |a - 1|.$$ 

To show this, we need to show that

$$\gcd(a^{m-1} + a^{m-1} + \cdots + 1, a^{d-1} + a^{d-1} + \cdots + 1) = 1,$$ 

which can be proved by induction and using the fact that

$$\gcd(a^{m-1} + a^{m-1} + \cdots + 1, a^{d-1} + a^{d-1} + \cdots + 1) = \gcd(a^{m-1} + a^{m-1} + \cdots + 1, a^{(d-m)-1} + a^{(d-m)-1} + \cdots + 1).$$

From the claim,

$$\gcd(a^m - 1, a^d - 1) \sim \max\{|a^m - 1|, |a^d - 1|\}^{1/d} \quad (|a| \to \infty).$$

Finally consider the curve $Z$ defined by $x^d = (y + 1)^m$. This curve does not pass through the origin, equivalently it has multiplicity $m = 0$ at $O$. An integral point $p$ of $Z$ is of the form $(a^m, a^d - 1)$ for an integer $a$. Clearly

$$\gcd(a^m, a^d - 1) = 1 = \max\{|a^m|, |a^d - 1|\}^{0/d}.$$
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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan, tel:+81-6-6850-5326, fax:+81-6-6850-5327  
E-mail address: takehikoyasuda@math.sci.osaka-u.ac.jp, highernash@gmail.com