The open Gromov-Witten-Welschinger theory of blowups of the projective plane

Asaf Horev, Jake P. Solomon

October 2012

Abstract

We compute the Welschinger invariants of blowups of the projective plane at an arbitrary conjugation invariant configuration of points. Specifically, open analogues of the WDVV equation and Kontsevich-Manin axioms lead to a recursive algorithm that reconstructs all the invariants from a small set of known invariants. Example computations are given, including the non-del Pezzo case.

Contents

1 Introduction
  1.1 Acknowledgements .................................................2

2 Gromov-Witten invariants and Kontsevich-Manin axioms
  2.1 Open Gromov-Witten invariants ................................4
  2.2 Axioms for open Gromov-Witten invariants .................4
  2.3 WDVV equations ..................................................8

3 Real blowups of the projective plane
  3.1 Homology classes of the blowup ................................9
  3.2 The involution invariant homology groups of the blowup ....10
  3.3 Chern numbers and Maslov index ...............................12

4 Gromov-Witten potentials
  4.1 The open Gromov-Witten potential .............................13
  4.2 The closed Gromov-Witten potential ..........................15
The object of this paper is the calculation of the Welschinger invariants of $\mathbb{CP}^2_{r,s}$, the blowup of a projective plane at $r$ real points and $s$ pairs of conjugate points, for arbitrary $r, s$.

In [13], Kontsevich and Manin gave a simple recursive formula calculating the Gromov-Witten invariants of $\mathbb{CP}^2$. The geometric insight behind the formula is a splitting principle, captured by the WDVV equations [21], and several other properties known as axioms. The WDVV equations and Kontsevich-Manin axioms were then used by Göttsche and Pandharipande [5] to give a set of formulae that recursively compute the Gromov-Witten invariants of $\mathbb{CP}^2_r$, the blowup of $\mathbb{CP}^2$ at $r$ points.

A real symplectic manifold $(X, \omega, \phi)$ is a symplectic manifold $(X, \omega)$ together with an anti-symplectic involution

$$\phi : X \rightarrow X, \quad \phi^*\omega = -\omega.$$ 

The involution $\phi$ generalizes complex conjugation. For a real strongly semi-positive symplectic manifold of dimension 4 or 6, Welschinger [19, 20] defined invariants based on a signed count of real genus 0 $J$-holomorphic curves passing through a $\phi$-invariant set of point constraints. In dimension 4 the sign is given by the parity of the number of isolated real double points and in dimension 6 by spinor states. For a real strongly semi-positive symplectic manifold $X$ of dimension 4 or 6 such that the fixed point set $\text{Fix}(\phi) \subset X$ is orientable, Cho [2] used the moduli space of $J$-holomorphic discs to define open Gromov-Witten invariants. Independently, the second author [17] defined open Gromov-Witten invariants of general real symplectic manifolds of dimension 4 and 6 in arbitrary genus, assuming
orientability of $\text{Fix}(\phi)$ only in dimension 6. He also showed that the open Gromov-Witten invariants recover Welschinger’s invariants in the strongly semi-positive genus 0 case.

The present paper uses analogues of the WDVV equation and Kontsevich-Manin axioms for open Gromov-Witten invariants due to the second author [18]. He showed that the open WDVV equation and axioms give recursive formulae for the Welschinger invariants of $\mathbb{CP}^2$. We prove the following theorem for $\mathbb{CP}_{r,s}^2$.

**Theorem 1.1.** The Welschinger invariants of $\mathbb{CP}_{r,s}^2$ with arbitrary real and complex conjugate point constraints are completely determined by

- the open Kontsevich-Manin axioms (see Subsection 2.2),
- the open WDVV equations (see Subsection 2.7),
- the closed Gromov-Witten invariants of $\mathbb{CP}_{r+2s}^2$ (computed by [3]), and
- a finite set of known initial values (see Section 7).

Our proof will consist of a set of recursive relations and an explicit algorithm for computing the open Gromov-Witten invariants of $X$ starting from a finite set of initial values. Theorem 1.1 is a direct consequence of Section 8 and the initial values are computed in Section 7. A selection of values obtained from the algorithm is given in Section 9.

The Welschinger invariants of $\mathbb{CP}_{r,s}^2$ for $r + 2s \leq 6$ with purely real point constraints were studied previously by Itenberg, Kharlamov and Shustin [6, 7, 8, 9, 10, 11, 12]. They used techniques of tropical geometry, and a real analogue of the Caporaso-Harris formula. In [1], Brugallé and Mikhalkin computed the Welschinger invariants of $CP_{r,s}^2$ for $r + 2s \leq 3$ with arbitrary real and complex conjugate pairs of point constraints. They used the tropical technique of floor diagrams. Our results agree with the previously obtained results in most cases.

### 1.1 Acknowledgements

The authors would like to thank Emmanuel Farjoun for helpful conversations. The authors were partially supported by United States - Israel Binational Science Foundation grant No. 2008314 and Marie Curie International Reintegration Grant No. 239381.
2 Gromov-Witten invariants and Kontsevich-Manin axioms

Genus zero Gromov-Witten invariants of a symplectic manifold \((X, \omega)\) of dimension \(2n\) are a set of maps \(GW_d : H^*(X; \mathbb{Q})^k \to \mathbb{Q}\), one for each \(d \in H_2(X; \mathbb{Z})\) and \(k = 0, 1, 2, \ldots\). Intuitively, the Gromov-Witten invariant \(GW_d(\gamma_1, \ldots, \gamma_k)\) counts the number of \(J\)-holomorphic spheres intersecting submanifolds \(R_1, \ldots, R_k \subset X\) that represent the Poincaré duals of \(\gamma_1, \ldots, \gamma_k\).

The Gromov-Witten invariants admit a set of basic properties, the Kontsevich-Manin axioms [13].

**Proposition 2.1.** (The Kontsevich-Manin axioms.)

1. **Symmetry:** \(GW_d(\gamma_1 \otimes \ldots \otimes \gamma_k)\) is \(\mathbb{Z}_2\)-graded symmetric with respect to permutations of the constraints \(\gamma_1, \ldots, \gamma_k\).
2. **Grading:** \(GW_d(\gamma_1 \otimes \cdots \otimes \gamma_k) = 0\) unless \(\Sigma |\gamma_i| = 2 \dim_{\mathbb{R}} X + 2c_1(d) + 2n - 6\).
3. **Fundamental class:** Let \(t_0 \in H^0(X)\) be the unit. Then

\[
GW_d(\gamma_1 \otimes \ldots \otimes \gamma_k \otimes t_0) = \begin{cases} 
\int_X \gamma_1 \wedge \gamma_2 & \text{if } d=0, k=2, \\
0 & \text{otherwise}.
\end{cases}
\]

4. **Divisor:** For \(\gamma \in H^2(X)\), we have

\[
GW_d(\gamma_1 \otimes \cdots \otimes \gamma_k \otimes \gamma) = \left(\int_d \gamma\right) \cdot GW_d(\gamma_1 \otimes \cdots \otimes \gamma_k).
\]

An expository account of genus 0 Gromov-Witten invariants and the Kontsevich-Manin axioms can be found in [13] Chapter 7.

2.1 Open Gromov-Witten invariants

For a real symplectic manifold of dimension \(2n\) with \(n = 2, 3\), the second author [17] defined Gromov-Witten type invariants using intersection theory on the moduli space of \(J\)-holomorphic discs. In this paper the term *open Gromov-Witten invariant* will refer to this definition. In the following paragraphs we recall the relevant material from [17].

Let \((X, \omega, \phi)\) be a real symplectic manifold and let \(L = \text{Fix}(\phi) \subset X\) be the Lagrangian submanifold given by the fixed points of \(\phi\). The \(\phi\)-*invariant* homology
groups with rational coefficients $H^\phi_{2i}(X; \mathbb{Q})$, $H^\phi_{2i}(X, L; \mathbb{Q})$, are defined as the $(-1)^i$ eigenspaces of
\[ \phi_* : H_{2i}(X; \mathbb{Q}) \to H_{2i}(X; \mathbb{Q}), \quad \phi_* : H_{2i}(X, L; \mathbb{Q}) \to H_{2i}(X, L; \mathbb{Q}) \]
respectively. There is a unique $\phi_*$-invariant projection $H_{2i}(X; \mathbb{Q}) \to H^\phi_{2i}(X; \mathbb{Q})$. The $\phi$-invariant homology group $H^\phi_{2i}(X; \mathbb{Q})$ is the image of the composition
\[ H_{2i}(X; \mathbb{Q}) \to H_{2i}(X, L; \mathbb{Q}) \to H^\phi_{2i}(X; \mathbb{Q}) \]
The relative $\phi$-invariant homology groups $H^\phi_{2i}(X, L; \mathbb{Q})$ are defined in the same way. Similarly, we define the $\phi$-invariant cohomology group $H^\phi_{2i}(X; \mathbb{Q})$ to be the $(-1)^i$ eigenspace of $\phi^* : H^{2i}(X; \mathbb{Q}) \to H^{2i}(X; \mathbb{Q})$. The cup product makes $H^\phi_{2i}(X; \mathbb{Q})$ into a ring.

For simplicity, in the following overview, we restrict to the case $n = 2$, so $X$ is real 4 dimensional. The open Gromov-Witten invariants of $(X, \omega, \phi)$ are maps
\[ OGW_{d,k} : H^\phi_{2i}(X; \mathbb{Q}) \otimes^k \to \mathbb{Q}, \]
one for every choice of $\phi$-invariant homology class $d \in H^\phi_{2i}(X, L; \mathbb{Q})$ and non-negative integers $k, l \geq 0$.

In the following, we work with an $\omega$-tame almost complex structure $J$. We denote by $M^D_{k,l}(X/L, d)$ the moduli space of $J$-holomorphic discs
\[ u : (D^2, \partial D^2) \to (X, L) \]
such that $u_*([D^2, \partial D^2]) \in H_2(X, L; \mathbb{Z})$ projects to $d \in H^\phi_{2i}(X, L; \mathbb{Z})$, with $k$ marked boundary points and $l$ marked interior points, modulo reparametrization. We denote the evaluation maps of the boundary marked points and interior marked points by
\[ ev_{b_i} : M^D_{k,l}(X/L, d) \to L, \quad i = 1, \ldots, k \]
\[ ev_{i_j} : M^D_{k,l}(X/L, d) \to X, \quad j = 1, \ldots, l \]
See [3] and [17] for details.

Throughout this paper, we denote by $\mu : H^\phi_{2i}(X; \mathbb{Z}) \to \mathbb{Z}$ the Maslov index. The following theorem proved in [17] gives a relative orientation of the moduli space of discs.

**Theorem 2.2.** Assume $k = \mu(d) + 1 \mod 2$. A $Pin^-$ structure $p_L$ on $L$ and an orientation on $L$ if $L$ is orientable determine up to homotopy an isomorphism
\[ \det(TM^D_{k,l}(X/L, d)) \sim \otimes_{i=1}^k ev_{b_i}^* \det(TL). \]
We fix a $Pin^-$ structure $\mathfrak{p}_L$ on $L$ and an orientation on $L$ if $L$ is orientable.

Let $\gamma_1, \ldots, \gamma_l \in H^*_c(X; \mathbb{Q})$ be homogeneous $\phi$-invariant cohomology classes.

Choose $R_i : M_i \rightarrow X$ smooth maps representing the Poincaré duals of $\gamma_i$, where $M_i$ are smooth oriented manifolds of dimension $2n - |\gamma_i|$. Choose also $k$ points on $L$ represented by $S_i : pt \rightarrow L$, for $i = 1, \ldots, k$.

Let
\[
\text{ev} : \mathcal{M}^D_{k,l}(X/L, d) \rightarrow X^l \times L^k,
\]
be the product
\[
\text{ev} = ev_1 \times \cdots \times ev_l \times evb_1 \times \cdots \times evb_k,
\]
and let
\[
R : M_1 \times \cdots \times M_k \times pt \times \cdots \times pt \rightarrow X^l \times L^k,
\]
be the product
\[
R = R_1 \times \cdots \times R_l \times S_1 \times \cdots \times S_k.
\]
Generically, $R$ and $\text{ev}$ are transverse, so their pull-back
\[
\begin{array}{ccc}
\mathcal{M}^D_{k,l}(X/L, d) & \xrightarrow{\text{ev}} & X^l \times L^k \\
\downarrow & & \downarrow \text{ev} \\
M_1 \times \cdots \times M_k \times pt \times \cdots \times pt & \xrightarrow{R} & X^l \times L^k
\end{array}
\]
is a smooth manifold of dimension
\[
\dim P = \dim \mathcal{M}^D_{k,l}(X/L, d) + \dim(M_1 \times \cdots \times M_k) - \dim(X^l \times L^k)
\]
\[
= (n + \mu(d) + k + 2l - 3) - \sum_i |\gamma_i| - nk.
\]

To define open Gromov-Witten invariants we need the following property of relative orientations.

**Lemma 2.3.** Let $f : X \rightarrow A$, $g : Y \rightarrow A$ be smooth transverse maps. A relative orientation $\det(T X) \xrightarrow{\sim} f^* \det(T A)$ of $f$ induces a relative orientation $\det(T(X \times_A Y)) \xrightarrow{\sim} p^* \det(T Y)$ of the projection $p : X \times_A Y \rightarrow Y$.

Assume for the next paragraph that the conditions of Theorem 2.2 hold. Then we have a relative orientation
\[
\det(T \mathcal{M}^D_{k,l}(X/L, d)) \xrightarrow{\sim} \bigotimes_{i=1}^k evb_i^* \det(T L),
\]
so by Lemma 2.3 we have a relative orientation
\[
\det(T P) \xrightarrow{\sim} p_1^* \det(T(M_1 \times \cdots \times M_k \times pt \times \cdots \times pt)).
\]
Therefore, since the manifold $M_1 \times \cdots \times M_k$ is oriented, so is $P$. We denote the orientation by $o$.

We define $OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_k)$ as follows.
• If \( \dim(P) \neq 0 \), set \( OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_k) = 0 \).

• If \( \dim(P) = 0 \) then \( k = \mu(d) + 1 \mod 2 \) as required by Theorem 2.2 and \( P \) is oriented. So, we define \( OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_k) = \sum_{p \in P} o(p) \).

We say an almost complex structure \( J \) is \( \phi \)-invariant if \( \phi^* J = -J \). For generic \( \phi \)-invariant \( J \), the last sum is finite and well defined by the following theorem from [17].

Theorem 2.4. The numbers \( OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_k) \) are finite and independent of the choice of generic \( \phi \)-invariant \( J \), the points \( S_i \) on \( L \) and the maps \( R_i : M_i \to X \) representing the duals of \( \gamma_i \).

2.2 Axioms for open Gromov-Witten invariants

The open Gromov-Witten invariants satisfy the following properties analogous to the Kontsevich-Manin axioms. We use the fact that the natural map

\[
H_2^\phi(X; \mathbb{Q}) \rightarrow H_2^\phi(X, L, \mathbb{Q})
\]

is an isomorphism. This follows from the exact sequence of the pair \((X, L)\) because \( \phi \) acts trivially on \( H_2(L) \).

**Proposition 2.5. (The open Gromov-Witten axioms.)**

1. **Symmetry:** \( OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_l) \) is \( \mathbb{Z}_2 \)-graded symmetric with respect to permutations of the constraints \( \gamma_1, \ldots, \gamma_l \).

2. **Grading:** \( OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_l) = 0 \) unless

\[
\sum |\gamma_i| + k \dim_{\mathbb{R}} L = \dim_{\mathbb{C}} X + \mu(d) + 2l + k - 3.
\]

3. **Fundamental class:** Let \( \tau_0 \in H_0^\phi(X) \) be the unit. Then

\[
OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_l \otimes \tau_0) = \begin{cases} 
1 & \text{if } d=0, k=1 \text{ and } l=0, \\
0 & \text{otherwise}.
\end{cases}
\]

4. **Divisor:** for \( \gamma \in H_2^\phi(X) \), we have

\[
OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_l \otimes \gamma) = \left( \int_d \gamma \right) \cdot OGW_{d,k}(\gamma_1 \otimes \cdots \otimes \gamma_l),
\]

where \( \int_d \gamma \) is defined by identifying \( d \) with an element of \( H_2^\phi(X) \) using isomorphism [11].
2.3 WDVV equations

The WDVV equations are a concise way of writing down further relations among Gromov-Witten invariants that come from non-linear gluing theory. In this section we give the relevant definitions and formulate the WDVV equations for closed and open Gromov-Witten invariants.

Let \((X, \omega)\) be a symplectic manifold of dimension 4. The closed Gromov-Witten potential of \(X\) is a formal power series encoding the closed Gromov-Witten invariants of \(X\).

Let \(t_0, \ldots, t_m \in H^*(X; \mathbb{Q})\) be a basis for the rational cohomology of \(X\). Let \(w = (w_0, \ldots, w_m)\) be a collection of formal variables, and let

\[\delta_w = w_0 t_0 + \cdots + w_m t_m\]

be the corresponding formal cohomology class. Let \(T\) be an additional formal variable. The closed Gromov-Witten potential is the formal power series

\[\Phi(w) = \sum_{d,n} \frac{T^{d}w}{n!} \cdot GW_d(\delta_{w}^n).\] (2)

The variable \(T\) is necessary to take care of convergence issues. We denote by \((g_{ij})\) the matrix with entries

\[g_{ij} = \begin{cases} \int_X t_i \wedge t_j & \text{if } t_i \wedge t_j \in H^4(X), \\ 0 & \text{otherwise}, \end{cases}\]

and we denote by \((g^{ij})\) the inverse matrix.

In the following theorems, we use the Einstein summation convention. Mathematical proofs of the following theorem first appeared in [16] and [14].

**Theorem 2.6.** Let \(i, j, k, l \in \{0, \ldots, m\}\). Then the WDVV equation holds,

\[\partial_i \partial_j \partial_\nu \Phi \cdot g^{\mu \nu} \cdot \partial_\mu \partial_k \partial_l \Phi = \partial_j \partial_k \partial_\nu \Phi \cdot g^{\nu \mu} \cdot \partial_\mu \partial_i \partial_l \Phi,\]

where \(\partial_i\) differentiation with respect to \(w_i\).

For a real symplectic manifold \((X, \omega, \phi)\) of dimension \(\dim X = 4\), the open Gromov-Witten invariants and closed Gromov-Witten invariants together satisfy the open WDVV equations. Let \(\tau_0, \ldots, \tau_m \in H^*_\phi(X; \mathbb{Q})\) be a basis. Let \(w = (w_0, \ldots, w_m)\) be a collection of formal variables, and let

\[\delta_w = w_0 \tau_0 + \cdots + w_m \tau_m\]

be the corresponding formal cohomology class. We still define the closed Gromov-Witten potential \(\Phi(w)\) by equation (2), but now \(\delta_w\) takes values only in the subspace \(H^*_\phi(X; \mathbb{Q}) \subset H^*(X; \mathbb{Q})\).
Let $T$ and $u$ be additional formal variables. The open Gromov-Witten potential is the formal power series

$$\Omega(w, u) = \sum_{d, k, l} \varepsilon(d) \frac{T^d u^k}{k!l!} \text{OGW}_{(d, \alpha), k}(\delta_w^{\otimes l}),$$

where $\varepsilon$ is given by

$$\varepsilon(d) = \begin{cases} +1 & \text{if } \mu(d) = 0 \mod 2, \\ \sqrt{-1} & \text{if } \mu(d) = 1 \mod 2. \end{cases}$$

The following theorem is due to [18].

**Theorem 2.7.** For every $a, b, c \in \{1, \ldots, m\}$ the following open WDVV equations hold.

1. $\partial_a \partial_b \partial_i \Phi g^{ij} \partial_j \partial_c \Omega + \partial_a \partial_b \partial \partial_i \partial_c \Omega = \partial_c \partial_b \partial_i \Phi g^{ij} \partial_j \partial_a \Omega + \partial_c \partial_b \partial \partial_a \partial_c \Omega.$
2. $\partial_a \partial_b \partial_i \Phi g^{ij} \partial_j \partial_i \Omega + \partial_a \partial_b \partial \partial_i \partial_i \Omega = \partial_c \partial_b \partial \partial_a \partial_i \Omega.$

Here $\partial_i$ denotes differentiation with respect to $u$ and $\partial_i$ denotes differentiation with respect to $w_i$.

3 **Real blowups of the projective plane**

Choose a configuration $C$ of $r$ real points $x_1, \ldots, x_r$, and $s$ pairs of conjugate points $y_1, y_2, \ldots, y_s, \bar{y}_s$, on $\mathbb{CP}^2$. Let $X$ be the blowup of $\mathbb{CP}^2$ at the $r + 2s$ points of $C$. The complex manifold $X$ comes with a standard family of Kähler forms. See Chapter 1 of [4]. For the following we take our symplectic form to be any of the standard Kähler forms on $X$ such that complex conjugation on $\mathbb{CP}^2$ lifts to an anti-symplectic involution

$$\phi : X \to X, \quad \phi^* \omega = -\omega.$$ 

For the purposes of this paper, the choice does not matter because Gromov-Witten type invariants depend only on the deformation class of $\omega$, and the family of Kähler forms of a fixed complex structure is convex. Deformation invariance also implies that the choice of blowup points does not matter. By extension, we often call $\phi$ conjugation. We write $L = \text{Fix}(\phi) \subset X$.
3.1 Homology classes of the blowup

It will be useful to choose explicit submanifolds of $X$ that will represent the generators of $H_2(X, L; \mathbb{Z})$. Let $\mathbb{L} \subset X$ be the strict transform of a conjugation invariant line in $\mathbb{C}P^2$ not passing through any of the points of $C$. Let $E_1, \ldots, E_r$, be the exceptional divisors of the $r$ real blowup points $x_1, \ldots, x_r$. Let $E_{r+1}, \ldots, E_{r+s}$, be the exceptional divisors of the $s$ blowup points $y_1, \ldots, y_s$, and let $E_{r+s+1}, \ldots, E_{r+2s}$, be the exceptional divisors of the conjugate blowup points $\overline{y_1}, \ldots, \overline{y_s}$.

The Lagrangian $L$ splits $\mathbb{L}$ into two hemispheres. We denote one of these hemispheres by $H$, and give $H$ the orientation induced by the complex structure of $\mathbb{L}$. We denote by $\overline{H} = \phi(H)$ the image of $H$ under complex conjugation. Similarly, the real submanifold $L$ splits the exceptional divisors $E_1, \ldots, E_r$ into two hemispheres. We denote one of these hemispheres by $F_i$, and give it the orientation induced by the complex structure of the exceptional divisor $E_i$. As before, we denote the image of $F_i$ under complex conjugation by $\overline{F_i}$.

The submanifolds $H, F_i, \overline{F_i}, E_{r+j}, E_{r+j+1}$, represent relative homology classes

$$[H], [\overline{H}], [F_i], [\overline{F_i}], \ldots, [F_r], [\overline{F_r}], [E_{r+1}], \ldots, [E_{r+2s}] \in H_2(X, L; \mathbb{Z}).$$

The relative homology groups $H_*(X, L)$ are given by the following lemma.

**Lemma 3.1.** $H_4(X, L; \mathbb{Z}) \simeq \mathbb{Z}$ with generator the image of the fundamental class under $H_4(X) \to H_4(X, L)$,

$$H_2(X, L; \mathbb{Z}) \simeq \mathbb{Z} \left\{ [H], [\overline{H}], [F_i], [\overline{F_i}], \ldots, [F_r], [\overline{F_r}], [E], \ldots, [E_{r+2s}] \right\}$$

$$[H] - [\overline{H}] = \sum_i [F_i] - [\overline{F_i}]$$

and $\forall n \neq 2, 4 : H_n(X, L; \mathbb{Z}) = 0$.

3.2 The involution invariant homology groups of the blowup

The conjugation $\phi : X \to X$ induces a conjugation map on the relative homology $\phi_* : H_2(X, L; \mathbb{Z}) \to H_2(X, L; \mathbb{Z})$.

**Lemma 3.2.** $\phi_*$ acts on the generators of $H_2(X, L; \mathbb{Z})$ by

$$\phi_* : [F_i] \mapsto [\overline{F_i}]$$

$$\phi_* : [E_{r+j}] \mapsto [\overline{E_{r+j}}] = -[E_{r+j}], \quad j = 1, \ldots, s,$$

$$\phi_* : [E_{r+j+1}] \mapsto [\overline{E_{r+j+1}}] = -[E_{r+j}], \quad j = 1, \ldots, s,$$

$$\phi_* : [H] \mapsto [\overline{H}] = [H] - \sum_i ([F_i] - [\overline{F_i}]).$$
We define generators of $H^{\phi}_{2}(X, L; \mathbb{Z})$ by projecting the generators of $H_{2}(X, L; \mathbb{Z})$ to $H^{\phi}_{2}(X; \mathbb{Q})$:

\[
\tilde{H} = \frac{1}{2}([H] - [\mathcal{H}]),
\]

\[
\tilde{F}_i = \frac{1}{2}([F_i] - [\mathcal{F}_i]), \quad i = 1, \ldots, r
\]

\[
\tilde{E}_j = \frac{1}{2}([E_{r+j}] - [\mathcal{E}_{r+j}]) = \frac{1}{2}([E_{r+s+j}] + [\mathcal{E}_{r+s+j}]), \quad j = 1, \ldots, s,
\]

\[
G_j = \frac{1}{2}([E_{r+j}] + [\mathcal{E}_{r+j}]) = \frac{1}{2}([E_{r+s+j}] - [\mathcal{E}_{r+s+j}]), \quad j = 1, \ldots, s.
\]

The relative homology classes $\tilde{H}, \tilde{F}_i, \tilde{E}_j$, freely generate the relative $\phi$-invariant homology group

\[
H^{\phi}_{2}(X, L; \mathbb{Z}) \simeq \mathbb{Z}\{\tilde{H}, \tilde{F}_1, \ldots, \tilde{F}_r, \tilde{E}_1, \ldots, \tilde{E}_1\}.
\]

They extend to a basis of $H_{2}(X, L; \mathbb{Q})$ by adding $G_j$.

We use the following shorthand notation for elements of the homology groups of $X$:

- For multi-indices
  \[a = (a_1, \ldots, a_r), \quad b = (b_1, \ldots, b_s), \quad c = (c_1, \ldots, c_s), \quad a_i, b_j, c_j \in \mathbb{Z}\]
  and $d \in \mathbb{Z}$ we denote by $(d, a, b, c)$ the homology class
  \[(d, a, b, c) = d[L] - \sum_i a_i[E_i] - \sum_j b_j[E_{r+j}] - \sum_j c_j[E_{r+s+j}] \in H_{2}(X; \mathbb{Z}).\]

- For multi-indices
  \[\alpha = (\alpha_1, \ldots, \alpha_r), \quad \beta = (\beta_1, \ldots, \beta_s), \quad \alpha_i, \beta_j \in \mathbb{Z},\]
  and $d \in \mathbb{Z}$, we denote by $[d, \alpha, \beta]$ the $\phi$-invariant relative homology class
  \[[d, \alpha, \beta] = d\tilde{H} - \sum_i \alpha_i\tilde{F}_i - \sum_j \beta_j\tilde{E}_j \in H^{\phi}_{2}(X, L; \mathbb{Z}).\]

We use Poincaré duality to define a basis for the cohomology groups $X$, and to the $\phi$-invariant cohomology groups. Let $m = \dim H^*(X; \mathbb{Q}) = 2 + r + 2s$. Define a basis for $H^*(X; \mathbb{Q})$ by taking
• $t_0$ Poincaré dual to $[X]$,
• $t_1$ Poincaré dual to $[L]$,
• $t_{1+i}$ Poincaré dual to the exceptional divisor $[E_i]$ for $i = 1 \ldots, r + 2s$,
• and $t_m$ Poincaré dual to a point.

Define a basis for $H^*_\phi(X)$ by

• $\tau_0 = t_0$ Poincaré dual to the fundamental class,
• $\tau_1 = \frac{1}{2}(t_1 - \overline{t_1}) = t_1$,
• $\tau_{1+i} = \frac{1}{2}(t_{1+i} - \overline{t_{1+i}}) = t_{1+i}$ for $i = 1, \ldots, r$,
• $\tau_{1+r+j} = \frac{1}{2}(t_{1+r+j} - \overline{t_{1+r+j}}) = \frac{1}{2}(t_{1+r+j} + t_{1+r+s+j})$ for $j = 1, \ldots, s$,
• $\tau_{1+r+s+j} = \frac{1}{2}(t_{1+r+j} + \overline{t_{1+r+j}}) = \frac{1}{2}(t_{1+r+j} + t_{1+r+s+j})$ for $j = 1, \ldots, s$,
• and $\tau_m = t_m$ Poincaré dual to a point.

The $\phi$-invariant cohomology $H^*_\phi(X)$ is spanned by $\tau_0, \ldots, \tau_{1+r+s}, \tau_m \in H^*_\phi(X)$.

### 3.3 Chern numbers and Maslov index

**Lemma 3.3.** The Chern numbers of the generators of $H_2(X; \mathbb{Z})$ are

$$c_1([L]) = 3, \quad c_1([E_i]) = 1, \quad i = 1, \ldots, r + 2s.$$ 

*Proof.* Use the adjunction formula $c_1([A]) = [A] \cdot [A] + \chi(\Sigma)$ for $A : \Sigma \to X$ an embedding, where $\chi$ is the Euler characteristic. For $A : S^2 \to X$ with image $L$ we get $c_1(L) = 3$ and for $A : S^2 \to X$ with image $E_i$ we get $c_1(L_i) = 1$. 

**Lemma 3.4.** The Maslov indices of the generators of $H_2(X, L; \mathbb{Z})$ are

$$\mu(\tilde{H}) = 3, \quad \mu(\tilde{F}_i) = 1, \quad i = 1, \ldots, r, \quad \mu(\tilde{E}_j) = 2, \quad j = 1, \ldots, s.$$ 

*Proof.* Since

$$\mu(2\tilde{H}) = \mu([H] - [\overline{H}]) = \mu([L]) = 2c_1([L]) = 6,$$

we get $\mu(\tilde{H}) = 3$. The other Maslov indices are calculated in the same way.

For a multi-index $q = (q_1, \ldots, q_k)$ we denote the sum $q_1 + \cdots + q_k$ by $|q|$. The first Chern number and Maslov index are given by

$$c_1(d, a, b, c) = 3d - |a| - |b| - |c|,$$

$$\mu([d, \alpha, \beta]) = 3d - |\alpha| - 2|\beta|.$$
4 Gromov-Witten potentials

In this section we use the axioms to reduce the computation of the open Gromov-Witten invariants from the computation of the multi-linear maps

\[ OGW_{[d,\alpha,\beta],k} : H^*_p(X; \mathbb{Q})^\otimes l \to \mathbb{Q} \]

to the computation of the values

\[ \Gamma_{[d,\alpha,\beta],k} := OGW_{[d,\alpha,\beta],k}(\partial \otimes l w) \]

Specifically, we derive a formula for the open Gromov-Witten potential in terms of \( \Gamma_{[d,\alpha,\beta],k} \). This will be useful in conjunction with Theorem 2.7. We also recall the analogous formula for the closed Gromov-Witten potential, derived in [5], which again will be useful in conjunction with Theorem 2.7.

4.1 The open Gromov-Witten potential

The open Gromov-Witten potential is the formal power series in variables \( T, w = (w_0, \ldots, w_{1+r+s}, w_m) \) and \( u \) given by

\[ \Omega(w, u) = \sum_{[d,\alpha,\beta],k,l} \varepsilon([d,\alpha,\beta]) T^{[d,\alpha,\beta]} \omega u^k \frac{\omega^l}{k!!} \cdot OGW_{[d,\alpha,\beta],k}(\delta_w^\otimes l). \] (3)

Here \( \delta_w \) is a formal cohomology class

\[ \delta_w = w_0 \tau_0 + \cdots + w_{1+r+s} \tau_{1+r+s} + w_m \tau_m, \]

and \( \varepsilon \) is given by

\[ \varepsilon([d,\alpha,\beta]) = \begin{cases} +1 & \text{if } \mu([d,\alpha,\beta]) = 0 \mod 2, \\ \sqrt{-1} & \text{if } \mu([d,\alpha,\beta]) = 1 \mod 2. \end{cases} \]

Using the open Gromov-Witten axioms, we show that

\[ \Omega(w, u) = w_0 \cdot u + \sum_{[d,\alpha,\beta],k} \varepsilon([d,\alpha,\beta]) T^{[d,\alpha,\beta]} \omega u^k \frac{\omega^l}{k!!} \cdot \frac{w_{1+i}^{l_i}}{l_i!} \cdot e^{\frac{1}{2} w_1 - \sum \frac{a_i}{2} w_{1+i} - \sum \frac{b_j}{2} w_{1+r+i}} \cdot \Gamma_{[d,\alpha,\beta],k}. \] (4)
where \( t_U = \frac{3d-|\alpha|-2|\beta|-k-1}{2} \). In our derivation, we separate the summands of \( \mathcal{S}_w \) that depend on \( w_0 \), which we call the classical part, from the rest of the summands, which we call the quantum part.

Using the multi-linearity of the open Gromov-Witten invariants, the symmetry axiom and the divisor axiom, we see that the quantum part of \( OGW_{[d,\alpha,\beta],k}(\tau^\otimes m) \) is given by

\[
\sum_{l_1+\cdots+l_{i+r+s}+l_m=l} \left( \frac{l}{l_1! \cdots l_m!} \right) \cdot OGW_{[d,\alpha,\beta],k}(\tau^\otimes l_1 \otimes \cdots \otimes \tau^\otimes l_m) =
\]

\[
= \sum_{l_1+\cdots+l_{i+r+s}+l_m=l} \frac{l!}{l_1! \cdots l_m!} \cdot \prod_i \left( \frac{\alpha_i}{2} \right)^{l_{i+1}} \times \prod_j \left( \frac{-\beta_j}{2} \right)^{l_{1+r+j}} \cdot OGW_{[d,\alpha,\beta],k}(\tau^\otimes l_m)
\]

\[
= l! \sum_{l_1+\cdots+l_{i+r+s}+l_m=l} \frac{(d\omega_1)^{l_1}}{l_1!} \cdot \prod_i \left( \frac{-\alpha_i w_{1+i}}{2} \right)^{l_{i+1}} \times \prod_j \left( \frac{-\beta_j w_{1+r+j}}{2} \right)^{l_{1+r+j}} \cdot \frac{w_{l_m}^{l_m}}{l_m!} \cdot OGW_{(d,\alpha),k}(\tau^\otimes l_m).
\]

By the grading axiom, \( OGW_{[d,\alpha,\beta],k}(\tau^\otimes l_m) \) vanishes unless

\[
l \cdot |\tau_m| + k \cdot \dim_R(L) = \mu([d, \alpha, \beta]) + \dim_C(X) + 2l + k - 3
\]

\[
\Leftrightarrow l = \frac{3d-|\alpha|-2|\beta|-k-1}{2}.
\]

Hence setting \( t_U = \frac{3d-|\alpha|-2|\beta|-k-1}{2} \), the quantum part of the open potential \( \Omega(w, u) \) is given by

\[
\sum_{[d,\alpha,\beta],k} \sum_{l_1+\cdots+l_{i+r+s}+l_m} \varepsilon([d, \alpha, \beta]) T_{[d,\alpha,\beta]}^{l_1} \omega^k \cdot \frac{(d\omega_1)^{l_1}}{k!} \cdot \prod_i \left( \frac{-\alpha_i w_{1+i}}{2} \right)^{l_{i+1}} \times \prod_j \left( \frac{-\beta_j w_{1+r+j}}{2} \right)^{l_{1+r+j}} \cdot \frac{w_{l_m}^{l_m}}{l_m!} \cdot OGW_{(d,\alpha),k}(\tau^\otimes l_m) =
\]

\[
= \sum_{[d,\alpha,\beta],k} \sum_{l_m} \varepsilon([d, \alpha, \beta]) T_{[d,\alpha,\beta]}^{l_1} \omega^k \cdot \frac{w_{l_m}^{l_m}}{l_m!} \times \prod_j \left( \frac{-\beta_j w_{1+r+j}}{2} \right)^{l_{1+r+j}} \cdot \frac{w_{l_m}^{l_m}}{l_m!} \cdot OGW_{[d,\alpha,\beta],k}(\tau^\otimes l_m)
\]

\[
= \sum_{[d,\alpha,\beta],k} \varepsilon([d, \alpha, \beta]) T_{[d,\alpha,\beta]}^{l_1} \omega^k \cdot \frac{w_{l_m}^{l_m}}{l_m!} \cdot e^{\frac{d\omega_1}{2} - \sum_i \frac{\alpha_i}{2} w_{1+i} - \sum_j \frac{\beta_j}{2} w_{1+r+j}} \cdot OGW_{[d,\alpha,\beta],k}(\tau^\otimes l_m)
\]

\[
= \sum_{[d,\alpha,\beta],k} \varepsilon([d, \alpha, \beta]) T_{[d,\alpha,\beta]}^{l_1} \omega^k \cdot \frac{w_{l_m}^{l_m}}{l_m!} \cdot e^{\frac{d\omega_1}{2} - \sum_i \frac{\alpha_i}{2} w_{1+i} - \sum_j \frac{\beta_j}{2} w_{1+r+j}} \cdot \Gamma_{[d,\alpha,\beta],k}.
\]
We turn to the classical part. By the fundamental class axiom there is only one non-vanishing open Gromov-Witten invariant involving $\tau_0$, namely

$$OGW_{[0,0,0],1}(\tau_0) = 1.$$  

So the classical part is

$$\varepsilon([0,0,0]) \frac{T_0u}{1!} OGW_{[0,0,0],1}(w_0\tau_0) = (+1) \cdot u \cdot w_0 \cdot OGW_{[0,0,0],1}(\tau_0) = w_0u.$$

Combining the classical and quantum part we obtain expression (4).

### 4.2 The closed Gromov-Witten potential

The closed Gromov-Witten potential is the formal power series

$$\Phi(w) = \sum_{(d,a,b,c),n} \frac{T^{l_f}_{(d,a,b,c)} w}{n!} GW_{(d,a,b,c)}(\delta_w^\otimes n)$$

in the formal variables $T$ and $w = (w_0, \ldots, w_m)$. We combine these formal variables to a formal cohomology class $\delta_w = w_0t_0 + \cdots + w_mt_m$.

Using the closed Gromov-Witten axioms, a formula for the closed Gromov-Witten potential can be derived like the formula for the open Gromov-Witten potential. Namely,

$$\Phi(w) = \frac{1}{2} \left\{ w_0^2w_m + w_0w_m^2 - \sum_{i=1}^{r+2s} w_0w_i^2 \right\} + \sum_{(d,a,b,c)} T^{l_f}_{(d,a,b,c)} w_m^{l_F} \times$$

$$\times e^{dw_1 - \sum_{i=1}^{r} a_i w_i + \sum_{j=1}^{s} b_j w_{i+j} - \sum_{j=1}^{s} c_j w_{i+j+s+j}} N_{(d,a,b,c)},$$

where $l_F = 3d - |a| - |b| - |c| - 1$, and

$$N_{(d,a,b,c)} = GW_{(d,a,b,c)}(t_m^{\otimes l_F}).$$

See [3] for more details.

### 5 Empty moduli spaces and vanishing open Gromov-Witten invariants

In this section we show that many of the open Gromov-Witten invariants of $X = \mathbb{C}P^2_{r,s}$ vanish. More specifically, we show that for many homology classes $[d, \alpha, \beta]$, the moduli space $\mathcal{M}_{k,l}(X/L, [d, \alpha, \beta])$ is empty for generic $J$ by the positivity of
intersections of $J$-holomorphic curves. See [15] Theorem E.1.5 for reference. Also, we use the open grading axiom to show that for $d = 0, 1$, many of the open invariants $\Gamma_{[d, \alpha, \beta], k}$ vanish. We denote by $[q]$ the multi-index with 1 for the $q^{th}$ entry and 0 everywhere else.

**Lemma 5.1.** For generic $J$, the moduli space $\mathcal{M}_{k,l}^D(X/L, [d, \alpha, \beta])$ is empty if one or more of the following conditions hold:

1. $d < 0$,
2. $\alpha_q > d$ for some $q = 1, \ldots, r$,
3. $2\beta_q > d$ for some $q = 1, \ldots, s$ and $[d, \alpha, \beta] \neq [1, 0, [q]]$,
4. $\alpha_q < 0$ for some $q = 1, \ldots, r$ and $[d, \alpha, \beta] \neq [0, -[q], 0]$,
5. $\beta_q < 0$ for some $q = 1, \ldots, s$.

**Proof.** Assume $\mathcal{M}_{k,l}^D(X/L, [d, \alpha, \beta]) \neq \emptyset$. Then there exists a $J$-holomorphic disc $f : (D^2, \partial D^2) \to (X, L)$ with the projection of $f_*([D^2, \partial D^2]) \in H_2(X, L)$ to $H_2^D(X, L)$ equal to $[d, \alpha, \beta]$. Doubling $f$ we get a $J$-holomorphic curve $u : \mathbb{CP}^1 \to X$ of degree

$$[u] = d \cdot [L] - \sum \alpha_i [E_i] - \sum \beta_j ([E_{r+j} + E_{r+s+j}]) \in H_2(X).$$

For each case of the lemma, we show there exists a $J$-holomorphic curve $c$ with $[c] \cdot [u] < 0$

and $[c] \neq [u]$. By [15] Lemma 2.4.3, the curves $c, u$, do not agree on any open set. This contradicts the positivity of intersections, so $\mathcal{M}_{k,l}^D(X/L, [d, \alpha, \beta]) = \emptyset$.

Denote by $W_{\zeta, l}$ the Welschinger invariant counting real rational $J$-holomorphic curves of degree $\zeta$ on $X$ passing through $l$ pairs of conjugate points and $k = c_1(\zeta) - 2l - 1$ real points. See [19]. To treat a given case of the lemma, it suffices to construct a homology class $\zeta$ such that

$$W_{\zeta, 0} \neq 0, \quad \zeta \cdot [u] < 0, \quad [\zeta] \neq [u].$$

We apply this strategy for the first three cases of the lemma.

1. Take $\zeta = [L]$. Then $W_{\zeta, 0} = 1$ and

$$\zeta \cdot [u] = [L] \cdot \left(d[L] - \sum \alpha_i [E_i] - \sum \beta_j ([E_{r+j} + E_{r+s+j}]) \right)$$

$$= d[L] \cdot [L] = d < 0.$$
2. Take $\zeta = [L] - [E_q]$. Then $W_{\zeta,0} = 1$ and

$$\zeta \cdot [u] = ([L] - [E_q]) \cdot \left( d[L] - \sum \alpha_i[E_i] - \sum \beta_j(E_{r+j} + E_{r+s+j}) \right)$$

$$= d - \alpha_q < 0.$$ 

3. Take $\zeta = [L] - [E_{r+q}] - [E_{r+s+q}]$. Then $W_{\zeta,0} = 1$ and

$$\zeta \cdot [u] = ([L] - [E_{r+q}] - [E_{r+s+q}]) \cdot$$

$$\left( d[L] - \sum \alpha_i[E_i] - \sum \beta_j(E_{r+j} + E_{r+s+j}) \right)$$

$$= d - 2\beta_q < 0.$$ 

For the remaining cases, we use the fact that if $\zeta$ is the homology class of an exceptional divisor, it admits a possibly reducible stable $J$-holomorphic representative for any $\omega$-tame $J$. See Example 7.1.15 in [15]. By taking $\omega$ such that $\int_\zeta \omega$ is the minimal positive value of $\omega$ on $H_2(X, \mathbb{Z})$, we can make sure the $J$-holomorphic representative is irreducible. So, it suffices to find $\zeta$ the homology class of an exceptional divisor with $\zeta \cdot [u] < 0$ and $\zeta \neq [u]$.

4. Take $\zeta = [E_q]$. Then

$$\zeta \cdot [u] = [E_q] \cdot \left( d[L] - \sum \alpha_i[E_i] - \sum \beta_j(E_{r+j} + E_{r+s+j}) \right)$$

$$= -\alpha_q[E_q] \cdot [E_q] = \alpha_q < 0.$$ 

5. Take $\zeta = [E_{r+q}]$. Then

$$\zeta \cdot [u] = [E_{r+q}] \cdot \left( d[L] - \sum \alpha_i[E_i] - \sum \beta_j(E_{r+j} + E_{r+s+j}) \right)$$

$$= -\beta_q[E_{r+q}] \cdot [E_{r+q}] = \beta_q < 0.$$ 

By the definition of $\Gamma_{[d,\alpha,\beta],k} = OGW_{[d,\alpha,\beta],k}(r_\omega^{\otimes l})$, we have proved the following corollary.

**Corollary 5.2.** If $\Gamma_{[d,\alpha,\beta],k} \neq 0$ then $d \geq 0$ and $0 \leq \alpha_i, \beta_j \leq d$ for all $i = 1, \ldots, r$, and $j = 1, \ldots, s$, except, perhaps, for $[d, \alpha, \beta] = [0, -[i], 0]$. 

17
We now show that for $d = 0, 1$ there are only few possible values of $\alpha, \beta, k$ for which the open Gromov-Witten invariants do not vanish.

**Lemma 5.3.** If $\Gamma_{[1,\alpha,\beta],k} \neq 0$ then one of the following holds:

1. $\alpha = 0, \beta = 0, k = 2$.
2. $\alpha = 0, \beta = 0, k = 0$.
3. $\alpha = [i], \beta = 0, k = 1$, for some $i = 1, \ldots, r$.
4. $\alpha = [i] + [j], \beta = 0, k = 0$, for some $i \neq j$.
5. $\alpha = 0, \beta = [j]$, for some $j = 1, \ldots, s$.

**Proof.** Since $\Gamma_{[1,\alpha,\beta],k}$ does not vanish, by Corollary 5.2 we have $\alpha_i = 0, 1, \beta_j = 0, 1$, for all $i, j$. By the grading axiom we have

$$l = \frac{3d - |\alpha| - 2|\beta| - k - 1}{2} \iff 2l + k = 2 - |\alpha| - 2|\beta|.$$ 

Since $k, l, |\alpha|, |\beta|$, are all positive integers, one of the specified conditions must hold.

**Lemma 5.4.** If $\Gamma_{[0,\alpha,\beta],k} \neq 0$, then $\alpha = -[i], \beta = 0, k = 0$, for some $i = 1, \ldots, r$.

**Proof.** Since $\Gamma_{[0,\alpha,\beta],k} \neq 0$, the corresponding moduli space is non-empty, hence $\alpha_i, \beta_j \leq 0$ for all $i, j$ by Lemma 5.1. In addition,

$$l = \frac{3d - |\alpha| - 2|\beta| - k - 1}{2} \iff 2l + k = -|\alpha| - 2|\beta| - 1$$

by the grading axiom. Therefore, $|\alpha|$ must be negative. But Lemma 5.1 implies this is only possible if $[d, \alpha, \beta] = [0, -[i], 0]$. Turning our attention back to the grading axiom, we have $2l + k = 0$ for $k, l$, non negative integers. So $k, l = 0$.

### 6 Application of open WDVV equations

We now apply the open WDVV equations to formulae (4) and (5) for the Gromov-Witten potentials to derive explicit relations between the open and closed Gromov-Witten invariants.

As the WDVV equations use the matrix $(g^{ij})$, we note that for $X = \mathbb{CP}^2_{r,s}$ the inverse intersection matrix of the invariant homology basis

$$[pt], \hat{H}, \hat{F}_1, \ldots, \hat{F}_r, \hat{E}_1, \ldots, \hat{E}_s, [X],$$
is given by

\[
(g^{ij}) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & -1 & 0 & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -1 \\
1 & & & & &
\end{pmatrix}.
\]

### 6.1 Results of open WDVV

In this section we apply Theorem 2.7 with specific choices of \(a, b, c\), and extract relations between \(N(d, \alpha, \beta, \gamma)\), \(\Gamma[d, \alpha, \beta, k]\), by examining the coefficients of the relevant power series.

In the following equations we use several shorthand notations. Given \([d, \alpha, \beta, k]\), we write \(l = \frac{1}{2}(\mu([d, \alpha, \beta]) - k - 1)\). We denote by \([d, \alpha, \beta, k]\) the set of all integers \(d', k', l', d'', k'', l''\), and integral multi-indices \(\alpha' = (\alpha'_1, \ldots, \alpha'_{r'})\), \(\alpha'' = (\alpha''_1, \ldots, \alpha''_{r''})\), \(\beta' = (\beta'_1, \ldots, \beta'_{s'})\), \(\beta'' = (\beta''_1, \ldots, \beta''_{s''})\), satisfying

\[
d' + d'' = d, \quad \alpha' + \alpha'' = \alpha, \quad \beta' + \beta'' = \beta, \quad 0 \leq d', d'', k', k'', l', l'',
\]

\[-1 \leq \alpha'_i \leq d', \quad -1 \leq \alpha''_i \leq d'', \quad 0 \leq 2\beta'_j \leq d' + 1, \quad 0 \leq 2\beta''_j \leq d'' + 1,
\]

\[
l' = \frac{1}{2}(\mu([d', \alpha', \beta']) - k' - 1), \quad l'' = \frac{1}{2}(\mu([d'', \alpha'', \beta'']) - k'' - 1).
\]

We take multinomial coefficients to be zero if there are non-integer or negative indices.

For \(d \in \mathbb{Z}\), and multi-indices

\[
\alpha = (\alpha_1, \ldots, \alpha_r), \quad \alpha_i \in \mathbb{Z}, \quad \beta = (\beta_1, \ldots, \beta_s), \quad \beta_j \in \frac{1}{2}\mathbb{Z},
\]

we define

\[
\tilde{N}_{[d, \alpha, \beta]} = \sum_c N_{(d, \alpha, \beta + c, \beta - c)}.
\]

19
where we sum over all half integral multi-indices \( c = (c_1, \ldots, c_s), c_j \in \frac{1}{2}\mathbb{Z} \). Note that this sum is finite because \( N_{(d, \alpha, \beta+c, \beta-c)} \) vanishes unless

\[-1 \leq \beta_j + c_j, \beta_j - c_j \leq d, \quad \forall j = 1, \ldots, s,\]

by \([5]\). We extend the definition of \( \tilde{N}_{(d, \alpha, \beta)} \) to the case that \( d \) and \( \alpha \) are not integral by setting it equal to zero.

Similarly, the notation \( |d, \alpha, \beta| \) denotes the set of integers \( d_F, l_F, d_U, l_U, \) integral multi-indices

\[\alpha_F = ((\alpha_F)_1 \ldots (\alpha_F)_r), \quad \alpha_U = ((\alpha_U)_1 \ldots (\alpha_U)_r), \quad \beta_U = ((\beta_U)_1 \ldots (\beta_U)_s)\]

and half integral multi-indices \( \beta_F = ((\beta_F)_1 \ldots (\beta_F)_s) \) satisfying

\[d_F + d_U = \frac{d}{2}, \quad \alpha_F + \frac{\alpha_U}{2} = \frac{\alpha}{2}, \quad \beta_F + \frac{\beta_U}{2} = \frac{\beta}{2}\]

\[d_F, l_F, d_U, l_U \geq 0,\]

\[l_F = c_1(d_F, \alpha_F, \beta_F, 0) - 1, \quad l_U = \frac{1}{2}(\mu([d_U, \alpha_U, \beta_U]) - k - 1).\]

Finally, we write

\[\alpha_F \cdot \alpha_U = \sum_{i=1}^r ((\alpha_F)_i) \cdot ((\alpha_U)_i), \quad \beta_F \cdot \beta_U = \sum_{j=1}^s ((\beta_F)_j) \cdot ((\beta_U)_j).\]

**Theorem 2.7(1) with \( a=b=1, c=m \)**

Applying Theorem 2.7(1) with \( a = b = 1, c = m \), for \( l \geq 2 \) we get

\[
\Gamma_{[d, \alpha, \beta], k} = \sum_{l \geq 2} \frac{\varepsilon([d', \alpha', \beta']) \varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d, \alpha, \beta])} \binom{k}{k', k'' - 1} \times (\text{OGW1}) \]

\[\times \left[ \frac{d'}{2} \left( \frac{l - 2}{l' - 1, l''} \right) \frac{d''}{2} - \left( \frac{l - 2}{l', l'' - 1} \right) \frac{d''}{2} \right] \Gamma_{[d', \alpha', \beta'], k'} \Gamma_{[d'', \alpha'', \beta''], k''} + \]

\[+ \sum_{l \geq 2} \frac{\varepsilon([d_U, \alpha_U, \beta_U])}{\varepsilon([d, \alpha, \beta])} d_F \left( \frac{d_U}{2} - \frac{1}{2} \alpha_F \cdot \alpha_U - \beta_F \cdot \beta_U \right) \times \]

\[\times \left[ \frac{d_U}{2} \left( \frac{l - 2}{l_F - 1, l_U} \right) \frac{d_U}{2} - \left( \frac{l - 2}{l_F, l_U - 1} \right) \frac{d_U}{2} \right] \tilde{N}_{[d_F, \alpha_F, \beta_F], k} \Gamma_{[d_U, \alpha_U, \beta_U], k}.\]
Theorem 2.7(2) with \(a=b=1\)

Applying Theorem 2.7(2) with \(a=b=1\), for \(l \geq 1, k \geq 1\), we get

\[
\Gamma_{[d,\alpha,\beta],k} = \sum_{r=[d,\alpha,\beta],k} \frac{\varepsilon([d',\alpha',\beta'])\varepsilon([d'',\alpha'',\beta''])}{\varepsilon([d,\alpha,\beta])} \left( l - 1 \right) \left( l' l'' \right) \times \]  

\[
\times \frac{d'}{2} \left[ \left( \frac{k-1}{k' - 1, k'' - 1} \frac{d''}{2} \right) - \left( \frac{k-1}{k', k'' - 2} \frac{d'}{2} \right) \right] \Gamma_{[d',\alpha',\beta',k'],[d'',\alpha'',\beta'',k'']} - 
\]

\[- \sum_{|\neq[d,\alpha,\beta]} \frac{\varepsilon([d_U,\alpha_U,\beta_U])}{\varepsilon([d,\alpha,\beta])} \left( \left( l - 1 \right) \left( l_F, l_U \right) \right) \times 
\times \frac{d_F^2}{2} \left( \frac{d_U}{2} \frac{1}{2} \alpha_F \alpha_U - \beta_F \beta_U \right) \hat{N}_{[d_F,\alpha_F,\beta_F],k'} \Gamma_{[d_U,\alpha_U,\beta_U],k''} \]

\[
- \delta_{k,1} \frac{d^2}{4} \tilde{N}_{\left[\frac{1}{2}, \frac{3}{2} \right]}.
\]

Theorem 2.7(1) with \(a=b=1, c=1+i\) for \(i=1,...,r\)

Fixing \(i=1,...,r\), we apply Theorem 2.7(1) with \(a=b=1, c=1+i\). For \(l \geq 1\) we get

\[
\left( -\frac{\alpha_i}{2} \right) \cdot \Gamma_{[d,\alpha,\beta],k} = \sum_{r=[d,\alpha,\beta],k} \frac{\varepsilon([d',\alpha',\beta'])?\varepsilon([d'',\alpha'',\beta''])}{\varepsilon([d,\alpha,\beta])} \left( \left( \frac{k}{k', k'' - 1} \left( l - 1 \right) \left( l' l'' \right) \right) \times 
\times \frac{d'}{2} \left[ \frac{d''}{4} - \frac{d'}{4} \right] \Gamma_{[d',\alpha',\beta',k'],[d'',\alpha'',\beta'',k'']} + 
\]

\[- \sum_{|\neq[d,\alpha,\beta]} \frac{\varepsilon([d_U,\alpha_U,\beta_U])}{\varepsilon([d,\alpha,\beta])} \left( \left( l - 1 \right) \left( l_F, l_U \right) \right) \frac{d_F}{2} \left( \frac{d_U}{2} - \frac{d_U}{2} \right) \right] \times 
\times \left( -\frac{d_F d_U}{2} + \frac{1}{2} \alpha_F \alpha_U + \beta_F \beta_U \right) \hat{N}_{[d_F,\alpha_F,\beta_F],k'} \Gamma_{[d_U,\alpha_U,\beta_U],k''} \]

In our recursive algorithm, we apply this relation for \(i\) with \(\alpha_i \neq 0\) in order to compute \(\Gamma_{[d,\alpha,\beta],k}\).
Theorem 2.7(1) with \( a=b=1, c=1+r+j \) for \( j=1,\ldots,s \)

Fixing \( j = 1,\ldots,s \) we apply Theorem 2.7(1) with \( a = b = 1, c = 1 + r + j \). For \( l \geq 1 \) we get

\[
\left( -\frac{\beta_j}{2} \right) \cdot \Gamma_{[d,\alpha,\beta],k} = \sum_{l=\epsilon([d,\alpha,\beta])} \varepsilon([d,\alpha,\beta]) \varepsilon([d',\alpha',\beta']) \varepsilon([d'',\alpha'',\beta'']) \Gamma_{[d',\alpha',\beta'],k'} \Gamma_{[d'',\alpha'',\beta''],k''} \times \\
\frac{k}{k', k'' - 1} \binom{l - 1}{l', l''} \times \\
\left( \frac{d'\beta'_j}{2} - \frac{d''\beta''_j}{4} \right) \Gamma_{[d,\alpha,\beta],k} + \sum_{l=\epsilon([d,\alpha,\beta])} \varepsilon([d,\alpha,\beta]) \binom{l - 1}{l_F, l_U} d_F \left( \frac{\beta_F j}{2} - \frac{\beta_U j}{2} \right) \times \\
\left( -d_F \frac{d_U}{2} + \frac{1}{2} \alpha_F \bullet \alpha_U + \beta_F \bullet \beta_U \right) \tilde{N}_{[d,\alpha,\beta],k}. 
\]

In our recursive algorithm, we apply this relation for \( j \) with \( \beta_j \neq 0 \) in order to compute \( \Gamma_{[d,\alpha,\beta],k} \).
6.2 Results of combining relations (OGW1)-(OGW3b)

Combining (OGW1) and (OGW2)

We compute \( \Gamma_{[d+1,\alpha,\beta],k-1} \) using (OGW1) and (OGW2). Subtracting (OGW1) from (OGW2), for \( k \geq 2 \) we get

\[
\frac{\varepsilon([d, \alpha, \beta]) \varepsilon([1,0,0])}{\varepsilon([d+1, \alpha, \beta])} \cdot \frac{d+1}{4} \cdot \Gamma_{[1,0,0],0} \Gamma_{[d,\alpha,\beta],k} = (OGW4)
\]

\[
= \sum_{|([d', \alpha', \beta'])| \leq k-1, ([d', \alpha', \beta'], k') \neq ([d, \alpha, \beta], k)} \frac{\varepsilon([d', \alpha', \beta']) \varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d+1, \alpha, \beta])} \left( \frac{l+1}{2} \right) \times
\]

\[
\frac{d'}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d''}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d'}{2} \right] \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\sum_{|[d, \alpha, \beta]|} \frac{\varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d+1, \alpha, \beta])} \left( \frac{k-1}{2} \right) \times
\]

\[
\frac{d''}{2} \left[ \left( \frac{l'}{l' - 1, l'' - 1} \right) \frac{d'}{2} - \left( \frac{l'}{l', l'' - 1} \right) \frac{d''}{2} \right] \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\delta_{k,2} \left( \frac{d+1}{4} + \frac{1}{2} \right)
\]

\[
\frac{d}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d''}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d}{2} \right] \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\sum_{|[d, \alpha, \beta]|} \frac{\varepsilon([d, \alpha, \beta])}{\varepsilon([d+1, \alpha, \beta])} \left( \frac{l}{l' - 1, l'' - 1} \right) \times
\]

\[
\frac{d}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d'}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d}{2} \right] \Gamma_{[d, \alpha, \beta]} \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\frac{d}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d}{2} \right] \Gamma_{[d, \alpha, \beta]} \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\delta_{k,2} \left( \frac{d+1}{4} + \frac{1}{2} \right)
\]

\[
\sum_{|[d, \alpha, \beta]|} \frac{\varepsilon([d, \alpha, \beta])}{\varepsilon([d+1, \alpha, \beta])} \left( \frac{l}{l' - 1, l'' - 1} \right) \times
\]

\[
\frac{d}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d'}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d}{2} \right] \Gamma_{[d, \alpha, \beta]} \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\delta_{k,2} \left( \frac{d+1}{4} + \frac{1}{2} \right)
\]

\[
\sum_{|[d, \alpha, \beta]|} \frac{\varepsilon([d, \alpha, \beta])}{\varepsilon([d+1, \alpha, \beta])} \left( \frac{l}{l' - 1, l'' - 1} \right) \times
\]

\[
\frac{d}{2} \left[ \left( \frac{l}{l' - 1, l'' - 1} \right) \frac{d}{2} - \left( \frac{l}{l', l'' - 1} \right) \frac{d}{2} \right] \Gamma_{[d, \alpha, \beta]} \Gamma_{[d', \alpha', \beta'], k} \Gamma_{[d'', \alpha'', \beta''], k''} -
\]

\[
\delta_{k,2} \left( \frac{d+1}{4} + \frac{1}{2} \right)
\]
Combining (OGW2) and (OGW3.a[i])

Compute $\Gamma_{[d+1, \alpha, \beta], k+1}$ using (OGW2) and (OGW3.a[i]). Subtracting (OGW3.a[i]) from (OGW2), for $\alpha_i \neq 0$ we get

$$\frac{\varepsilon([d, \alpha, \beta])\varepsilon([1, 0, 0])}{\varepsilon([d + 1, \alpha, \beta])} \cdot \frac{d^2 + (1 - k)d - k}{4} \cdot \Gamma_{[1,0,0],2}\Gamma_{[d,\alpha,\beta],k} = (OGW5.a[i])$$

$$= \sum_{\delta[d+1,\alpha,\beta],k+1} \frac{\varepsilon([d', \alpha', \beta'])\varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d + 1, \alpha, \beta])} \left( l \mid l', l'' \right) \times$$

$$\times \frac{d'}{2} \left[ \frac{k}{k' - 1, k'' - 1} \frac{d''}{2} - \frac{k}{k', k'' - 2} \frac{d'}{2} \right] \Gamma_{[d',\alpha',\beta'],k'}\Gamma_{[d'',\alpha'',\beta''],k''} -$$

$$- \sum_{|\delta[d+1,\alpha,\beta]|} \frac{\varepsilon([d_U, \alpha_U, \beta_U])}{\varepsilon([d + 1, \alpha, \beta])} \left( l \mid l_F, l_U \right) \times$$

$$\times d_F^2 \left( \frac{d_U}{2} - \frac{1}{2} \alpha_F \cdot \alpha_U - \beta_F \cdot \beta_U \right) \tilde{N}_{[d_U,\alpha_F,\beta_F]} \Gamma_{[d_U,\alpha_U,\beta_U],k+1} -$$

$$- \delta[k,0] \frac{(d + 1)^2}{4} \tilde{N}_{[\alpha, \beta], \frac{d+1}{2}, \frac{d}{2}} +$$

$$+ \left( \frac{2}{\alpha_i} \right) \cdot \sum_{\delta[d+1,\alpha,\beta],k+1} \frac{\varepsilon([d', \alpha', \beta'])\varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d + 1, \alpha, \beta])} \left( k + 1 \right) \left( l \mid l', l'' \right) \times$$

$$\times \frac{d'}{2} \left[ -\frac{d''}{4} + \frac{d''}{4} \frac{d'}{4} \right] \Gamma_{[d',\alpha',\beta'],k'}\Gamma_{[d'',\alpha'',\beta''],k''} -$$

$$+ \left( \frac{2}{\alpha_i} \right) \cdot \sum_{|\delta[d+1,\alpha,\beta]|} \frac{\varepsilon([d_U, \alpha_U, \beta_U])}{\varepsilon([d + 1, \alpha, \beta])} \left( l \mid l_F, l_U \right) d_F \left[ (\alpha_F), \frac{d_U}{2} - \frac{d_F (\alpha_F)}{2} \right] \times$$

$$\times \left( -d_F \frac{d_U}{2} + \frac{1}{2} \alpha_F \cdot \alpha_U + \beta_F \cdot \beta_U \right) \tilde{N}_{[d_F,\alpha_F,\beta_F]} \Gamma_{[d_U,\alpha_U,\beta_U],k+1}.$$
Combining (OGW2) and (OGW3.b[j])

Compute $\Gamma_{[d+1, \alpha, \beta], k+1}$ using (OGW2) and (OGW3.b[j]). Subtracting (OGW3.b[j]) from (OGW2), for $\beta_j \neq 0$ we get

$$\frac{\varepsilon([d, \alpha, \beta]) \varepsilon([1, 0, 0])}{\varepsilon([d + 1, \alpha, \beta])} \cdot \frac{d^2 + (1-k)d - k}{4} \cdot \Gamma_{[1, 0, 0], 2} \Gamma_{[d, \alpha, \beta], k} = \quad \text{(OGW5.b[j])}$$

$$= \sum_{l \geq 0} \frac{\varepsilon([d', \alpha', \beta']) \varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d + 1, \alpha, \beta])} \left( \begin{array}{c} l \\ l_F, l_U \end{array} \right) \times$$

$$\times \frac{d'}{2} \left[ \begin{array}{c} k \\ k' - 1, k'' - 1 \end{array} \right] \frac{d''}{2} \left( \begin{array}{c} k \\ k', k'' - 2 \end{array} \right) \frac{d''}{2} \Gamma_{[d', \alpha', \beta'], k'} \Gamma_{[d'', \alpha'', \beta''], k''} -$$

$$- \sum_{l \geq 0} \frac{\varepsilon([d_U, \alpha_U, \beta_U])}{\varepsilon([d + 1, \alpha, \beta])} \left( \begin{array}{c} l \\ l_F, l_U \end{array} \right) \times$$

$$\times d_F^2 \left( d_{U, \alpha_U - \beta_U} - d_{F, \alpha_U - \beta_U} \right) - N_{[d, \alpha_U, \beta_U], k+1}$$

$$- \delta_{k, 0} \frac{(d + 1)^2}{4} \tilde{N}_{[d, \frac{d - 1}{2}, \frac{d}{2}]} + \left( \begin{array}{c} 2 \\ \beta_j \end{array} \right) \cdot \sum_{l \geq 0} \frac{\varepsilon([d', \alpha', \beta']) \varepsilon([d'', \alpha'', \beta''])}{\varepsilon([d + 1, \alpha, \beta])} \left( \begin{array}{c} k + 1 \\ k', k'' - 1 \end{array} \right) \left( \begin{array}{c} l \\ l_F, l_U \end{array} \right) \times$$

$$\times \frac{d'}{2} \left[ \begin{array}{c} d' \beta_j' \\ \frac{d' \beta_j'^2}{4} \end{array} \right] \Gamma_{[d', \alpha', \beta'], k'} \Gamma_{[d'', \alpha'', \beta''], k''}$$

$$+ \left( \begin{array}{c} 2 \\ \beta_j \end{array} \right) \cdot \sum_{l \geq 0} \frac{\varepsilon([d_U, \alpha_U, \beta_U])}{\varepsilon([d + 1, \alpha, \beta])} \left( \begin{array}{c} l \\ l_F, l_U \end{array} \right) d_F \left[ \begin{array}{c} (\beta_U)_j \frac{d_U}{2} - d_F \frac{(\beta_U)_j}{2} \end{array} \right] \times$$

$$\times \left( -d_F \frac{d_U}{2} + 1 \frac{\alpha_F \cdot \alpha + \beta_F \cdot \beta}{2} \right) \tilde{N}_{[d, \alpha_U, \beta_U], k+1}.$$

7 Welschinger invariants and initial calculations of open Gromov-Witten invariants

The open Gromov-Witten invariants are related to the Welschinger invariants by a simple formula. In this section we recall this formula, due to the second author in [17, 18], and use it to compute some of the invariants $\Gamma_{[d, \alpha, \beta], k}$ for $d = 0, 1$.

We first relate every $\phi$-invariant relative homology class $\theta \in H^2_2(X, L; \mathbb{Q})$ to a $\phi$-invariant homology class $\tilde{\theta} \in H^2_2(X; \mathbb{Q})$ by imitating the doubling of $J$-holomorphic
discs. Denote by 
\[ j : H_2^p(X; \mathbb{Q}) \to H_2^p(X, L; \mathbb{Q}) \]
the map induced by the long exact sequence of \((X, L)\). Since the conjugation \(\phi\) acts trivially on \(L\), the map \(j\) is an isomorphism. Therefore, for every \(\theta \in H_2^p(X, L; \mathbb{Q})\) we can define \(\tilde{\theta} \in H_2^p(X; \mathbb{Q})\) by \(j: \tilde{\theta} \mapsto 2\theta\).

For example, for \(X = \mathbb{C}P^2_{r,s}\) and the relative homology class \(\theta = [d, \alpha, \beta] = d - \sum_i \alpha_i E_i - \sum_j \beta_j E_j \in H_2^p(X, L; \mathbb{Z})\), we have
\[ \tilde{\theta} = (d, \alpha, \beta, \beta) = d - \sum_i \alpha_i E_i - \sum_j \beta_j E_{r+j} - \sum_j \beta_j E_{r+s+j} \in H_2^p(X; \mathbb{Z}). \]

Next, following [18], there is a one-to-one correspondence between \(\text{Pin} \) structures \(p\) on \(L\), and functions \(t_p : H_1(L; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}\) that satisfy
\[ t_p(x + y) = t(x) + t(y) + x \cdot y + w_1(x)w_1(y) \mod 2. \]
Here \(w_1 \in H^1(L; \mathbb{Z}/2\mathbb{Z})\) is the first Stiefel-Whitney class of the tangent bundle \(TL\). The function \(t_p\) is determined by its values on the generators of \(H_1(L; \mathbb{Z}/2\mathbb{Z})\).

We use \(t_p\) to define another function
\[ s_p : H_2^p(X, L) \to \mathbb{Z}/2\mathbb{Z}, \quad s_p(\theta) = \frac{\mu(\theta) - \tilde{\theta} \cdot \tilde{\theta} - 2 + t_p(\partial \theta) + 1}{2}. \]

We denote by \(W_{\zeta, l}\) the Welschinger invariant counting real rational \(J\)-holomorphic curves of degree \(\zeta\) on \(X\) passing through \(l\) pairs of conjugate points and \(k = c_1(\zeta) - 2l - 1\) real points. See [19]. Then the open Gromov-Witten invariants and the Welschinger invariants are related by the formula
\[ OGW_{\theta, k}(\tau_m^{\infty}) = (-1)^{s_p - 1} W_{\theta, l}, \quad (6) \]
where \(\tau_m \in H^4(X)\) is the Poincaré dual of a point.

We now use relation (6) to calculate a small set of \(\Gamma_{[d,\alpha,\beta], k}\) values, which serve as the initial conditions for the recursive calculation of Section 8.

**Lemma 7.1.** There is a choice of \(\text{Pin} \) structure \(p\) such that
\[
\begin{align*}
\Gamma_{[0,\ldots,0],0} &= 2, & \Gamma_{[1,0,0],2} &= 2, & \Gamma_{[1,0,0],0} &= 1, \\
\Gamma_{1,\ldots,0},1 &= 2, & i &= 1, \ldots, r, & \Gamma_{[1,0,\ldots,0],0} &= 2, & j &= 1, \ldots, s.
\end{align*}
\]
Proof. We choose $p$ such that $t_p : H^1(L; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ evaluates to 1 on the generators $[\partial H], [\partial F_1], \ldots, [\partial F_r] \in H^2(L; \mathbb{Z}/2\mathbb{Z})$. A simple calculation shows that $t_p(\partial H + i\partial F_i) = 1$. Therefore, we have

$$s_p([0, -[i], 0]) = 0, s_p([1, 0, 0]) = 0, s_p([1, [i], 0]) = 0, s_p([1, 0, [j]]) = 0.$$ 

Applying relation (6), we have

$$\Gamma_{[d, \alpha, \beta], k} = (-1)^s p^2 1^{-l} W_{(d, \alpha, \beta, \beta), l} = \frac{1}{2} (3d - |\alpha| - 2 |\beta| - k - 1).$$

The results now follow from the known fact that $W_{(d, \alpha, \beta, \beta), l} = 1$ for each choice of $d, \alpha, \beta, l$, considered in the lemma.

For the remainder of the paper, we use the $Pin$ structure $p$ given by Lemma 7.1.

8 Proof of recursion

We will show that the $\Gamma_{[d, \alpha, \beta], k}$ calculated in Section 7 and the relations [(6)](OGW5.b)](OGW5.b), together with the closed Gromov-Witten invariants, determine all the open Gromov-Witten invariants $\Gamma_{[d, \alpha, \beta], k}$ of $\mathbb{P}^2_{r,s}$ by induction on the integers $d, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$, and $k$. We can perform induction on these a priori signed integers since by Corollary 5.2 we can assume that $d \geq 0$ and $\alpha_i \geq 0$ unless $[d, \alpha, \beta] = (0, -[i], 0)$ for some $i$, which we take as one of the initial conditions.

Formally, we define a partial order on the indices $([d, \alpha, \beta], k)$ where $d, k$, are integers and $\alpha = (\alpha_1, \ldots, \alpha_r), \beta = (\beta_1, \ldots, \beta_s)$, are integral multi-indices. We say that $([d', \alpha', \beta'], k') < ([d, \alpha, \beta], k)$ if

- $d' < d$, or
- $d' = d$, and $\alpha'_i \leq \alpha_i, \beta'_j \leq \beta_j$ for all $i = 1, \ldots, r, j = 1, \ldots, s$, with at least one the inequalities strict, or
- $d' = d, \alpha' = \alpha, \beta' = \beta$, and $k' < k$.

We also write $0 \leq ([d, \alpha, \beta], k)$ for $0 \leq d, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, k$. The following lemma is immediate from the open grading axiom.

Lemma 8.1. Let $([d, \alpha, \beta], k)$ be an index as above. The set

$$\{([d', \alpha', \beta'], k') : 0 \leq ([d', \alpha', \beta'], k') < ([d, \alpha, \beta], k), \Gamma_{[d', \alpha', \beta'], k'} \neq 0\}$$

is finite.
Lemma 8.2. \[(OGW1)\] relates \(\Gamma_{[d, \alpha, \beta], k} \) (with \(l \geq 2\)) only to open Gromov-Witten invariants \(\Gamma_{[d, \alpha', \beta'], k'} \) with \(d' < d\).

2. \((OGW2)\) relates \(\Gamma_{[d, \alpha, \beta], k} \) (with \(l \geq 1, k \geq 1\)) only to open Gromov-Witten invariants \(\Gamma_{[d, \alpha', \beta'], k'} \) with \(d' < d\).

3. If \(\alpha_i \neq 0\), then \((OGW3.a[i])\) relates \(\Gamma_{[d, \alpha, \beta], k} \) (with \(l \geq 1\)) only to open Gromov-Witten invariants \(\Gamma_{[d, \alpha', \beta'], k'} \) with \(d' < d\).

4. If \(\beta_j \neq 0\), then \((OGW3.b[j])\) relates \(\Gamma_{[d, \alpha, \beta], k} \) (with \(l \geq 1\)) only to open Gromov-Witten invariants \(\Gamma_{[d, \alpha', \beta'], k'} \) with \(d' < d\).

Proof. Examine the first sum in \((OGW1)\). It vanishes for \(d' = 0\), so \(d' + d'' = d\) implies \(d'' < d\). Assume that \(d' = d, \frac{d''}{2} = 0\). Then by Lemma 5.4 we must have \(\Gamma_{[0, \alpha', \beta''], k''} = 0\). In both cases the summand vanishes, in the first case because the open Gromov-Witten invariant vanishes, and in the second case because the binomial coefficient \(\binom{k}{k'}\) vanishes.

Examine the second sum. Assume \(d_U = d\). Then \(d_F + \frac{d_U}{2} = \frac{d}{2}\) implies \(d_F = 0\), so the sum vanishes. Applying the same arguments gives the analogous results for \((OGW2), (OGW3.a[i]), (OGW3.b[j])\). \(\square\)

Next we show that applying \((OGW4)\) reduces either \(d\), one of the \(\alpha_i\)'s, one of the \(\beta_j\)'s, or \(k\).

Lemma 8.3. \((OGW4)\) relates \(\Gamma_{[d, \alpha, \beta], k} \) (with \(k \geq 2\) boundary points) only to open Gromov-Witten invariants \(\Gamma_{[d', \alpha', \beta'], k'} \) with

\[([d', \alpha', \beta'], k') < ([d, \alpha, \beta], k).\]

Proof. We prove that \(d', d'', d_U \leq d\) in all the sums involved by following the argument in Lemma 8.2. After all, \((OGW1)\) is just the difference of \((OGW4)\) and \((OGW2)\) applied to \(\Gamma_{[d+1, \alpha, \beta], k'-1}\). The sums involving closed invariants do not include summands with \(d_U = d\), since \(d_F + \frac{d_U}{2} = \frac{d+1}{2}\) would imply \(d_F = \frac{1}{2}\) which is not integral. We ignore these sums for the rest of the proof, focusing on the sums not containing closed invariants.

Next we show that \(\alpha_i', \alpha_i'' \leq \alpha_i\) for all \(i\) and \(\beta_j', \beta_j'' \leq \beta_j\) for all \(j\). We already know that \(d'', d'' \leq d\) and \(d' + d'' = d + 1\), so \(d', d'' \geq 1\), and by Corollary 5.2 we have \(\alpha_i', \alpha_i'' \leq \alpha_i\) for all \(i, j\). Our claim now follows from \(\alpha' + \alpha'' = \alpha, \beta' + \beta'' = \beta\).

So far we showed that \(d', d'' \leq d, \alpha_i', \alpha_i'' \leq \alpha_i\) for all \(i\), and that \(\beta_j', \beta_j'' \leq \beta_j\) for all \(j\). To finish the proof we have to show that no summand contains \(\Gamma_{[d, \alpha, \beta], k}\).
with $K \geq k$. Assume $[d', \alpha', \beta'] = [d, \alpha, \beta]$ (the argument for $[d'', \alpha'', \beta''] = [d, \alpha, \beta]$ is the same). Then $[d'', \alpha'', \beta''] = [1, 0, 0]$ and by Lemma 5.3 either $\Gamma_{d', \alpha', \beta', k''}$ vanishes or $k'' = 0, 2$. The binomial coefficients then imply that $k' + k'' = k$, therefore $k' = k, k - 2$. The case $([d', \alpha', \beta'], k') = ([d, \alpha, \beta], k)$ was explicitly removed from the sum, we are left with $k' = k - 2$ and we’re done.

**Lemma 8.4.** The relation (OGW5.a[i]) relates $\Gamma_{[d, \alpha, \beta], k}$ having $l = 0$ and $\alpha_i \neq 0$ for some $i$ only to open Gromov-Witten invariants $\Gamma_{[d', \alpha', \beta'], k'}$ with $([d', \alpha', \beta'], k') < ([d, \alpha, \beta], k)$. The relation (OGW5.b[j]) relates $\Gamma_{[d, \alpha, \beta], k}$ having $l = 0$ and $\beta_j \neq 0$ for some $j$ only to open Gromov-Witten invariants $\Gamma_{[d', \alpha', \beta'], k'}$ with $([d', \alpha', \beta'], k') < ([d, \alpha, \beta], k)$.

**Proof.** Showing that

$$d', d'' \leq d, \quad d_U < d, \quad \alpha'_i, \alpha''_i \leq \alpha_i, \quad \beta'_j, \beta''_j \leq \beta_j,$$

is the same as the last proof. It remains to prove that all summands containing $\Gamma_{[d, \alpha, \beta], K}$ with $K \geq k$ vanish.

For $[d', \alpha', \beta'] = [d, \alpha, \beta]$ we have $[d'', \alpha'', \beta''] = [1, 0, 0]$. Therefore $\Gamma_{d', \alpha', \beta', k''}$ vanishes or $k'' = 0, 2$. The binomial coefficients now imply $k' + k'' = k + 2$, so $k' = k + 2, k$. As the case $k' = k$ is explicitly removed from the sum, we are left with proving that $\Gamma_{[d, \alpha, \beta], k+2}$ vanishes. Applying the grading axiom, we note that $0 = \frac{1}{2}(\mu([d, \alpha, \beta]) - k - 1)$ since $\Gamma_{[d, \alpha, \beta], k}$ has 0 interior points. Now we apply the grading axiom to $\Gamma_{[d, \alpha, \beta], k+2}$ and we get

$$l = \frac{1}{2}(\mu([d, \alpha, \beta]) - (k + 2) - 1) = \frac{1}{2}(\mu([d, \alpha, \beta]) - k - 1) - 1 = -1,$$

so $\Gamma_{[d, \alpha, \beta], k+2}$ vanishes.

The same argument gives the analogous result for (OGW5.b[j]).

We now collect the results of the last lemmas and the calculations of $\Gamma_{[d, \alpha, \beta], k}$ for $d = 0, 1$, to prove the main theorem.

**Theorem 8.5.** The open Gromov-Witten invariants $\Gamma_{[d, \alpha, \beta], k}$ are recursively determined by the relations (OGW1) - (OGW5.b[i]), the vanishing results of Corollary 5.2, the closed Gromov-Witten invariants $N_{[d, \alpha, \beta]} = \sum c N_{[d, \alpha, \beta + c, \beta - c]}$, and the following initial values:

1. $\Gamma_{[0, i], 0, 0} = 2$ for $i = 1, \ldots, r$,
2. $\Gamma_{[1, 0, 0], 2} = 2$,
3. $\Gamma_{[1, 0, 0], 0} = 1$,
4. $\Gamma_{[1,i],0,1} = 2$ for $i = 1, \ldots, r,$

5. $\Gamma_{[1,0,j],0} = 2$ for $j = 1, \ldots, s.$

**Proof.** We prove by induction on $([d, \alpha, \beta], k).$

**Proof of induction base.** All the initial conditions were verified in Lemma 7.1.

By Corollary 5.2 and the grading axiom, $\Gamma_{[d, \alpha, \beta], k} = 0$ for $0 \not\leq (\alpha, \beta, 1), k = ([0, -[i], 0, 0]).$

For $d = 0, 1,$ we know by Lemmas 5.4 and 5.3 that the invariants vanish except for the cases $\Gamma_{[0, -[i], 0, 0], 0} = 2$ and $\Gamma_{[1,0,0],0} = 2$ and $\Gamma_{[1,0,0],0} = 2$ and

These invariants are all initial values except $\Gamma_{[1,0,0],0} = 2$ and $\Gamma_{[1,0,0],0} = 2$ and $\Gamma_{[1,0,0],0} = 2$ and $\Gamma_{[1,0,0],0} = 2$.

**Proof of induction step.** We show that for $d \geq 2$ we can always apply one of the relations (OGW1)- (OGW5.b[j]). By Lemmas 8.2, 8.3, and 8.4, the relations (OGW1)- (OGW5.b[j]) determine $\Gamma_{[d, \alpha, \beta], k}$ as a function of $\Gamma_{[d', \alpha', \beta'], k'}$ for $0 \leq (\alpha', \beta', k') < (\alpha, \beta, k)$ and of $\Gamma_{[0, -[i], 0, 0]}.$ Since the set of such invariants which do not vanish is finite by Lemma 8.1, this completes the proof.

Let $([d, \alpha, \beta], k)$ be an index with $d \geq 2.$ We cover all possible cases and show that we can apply a recursive relation.

1. If $l = \frac{1}{2}(\mu([d, \alpha, \beta]) - k - 1 \geq 2$ we can apply (OGW1).

2. If $l = 1$ then either $k \geq 1$ and we can apply (OGW2) or $k = 0.$ In the latter case we use the grading axiom to see that $|\alpha| + |\beta| = 3d - 3 > 0,$ so there exists some $\alpha_i \neq 0$ or some $\beta_j \neq 0,$ and therefore we can apply (OGW3.a[i]) or (OGW3.b[j]).

3. For the case $l = 0$ we will use (OGW1), (OGW5.a[i]), and (OGW5.b[j]). To calculate $\Gamma_{[d, \alpha, \beta], k}$ using these relations we need its coefficient to be non zero. By Lemma 7.1 we have $\Gamma_{[1,0,0],0} = 1,$ so for $k \geq 2$ we can apply (OGW1). Therefore, we only have to deal with the cases $k = 0, 1.$ We do this using relations (OGW5.a[i]) and (OGW5.b[j]). The coefficient of $\Gamma_{[d, \alpha, \beta], k}$ does not vanish since $\Gamma_{[1,0,0],2} = 2$ by Lemma 7.1 and $d^2 + (1 - k)d - k = 0$ only when $k = 0, d = 0, -1$ or $k = 1, d = \pm 1.$ By the grading axiom, we have $|\alpha| + |\beta| = 3d - k - 1 > 0.$ So, there exists $\alpha_i \neq 0$ or $\beta_j \neq 0,$ and we can apply either (OGW5.a[i]) or (OGW5.b[j]).

Theorem 1.1 is an immediate corollary.
9 Example computations

In this section we list some calculated values of open Gromov-Witten invariants. The invariants were calculated by a Maple program implementing the recursive algorithm implied by Theorem 8.5.

We first reiterate our notations. For an integer \(d \in \mathbb{Z}\) and integral multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_r), \beta = (\beta_1, \ldots, \beta_s)\), we write \([d, \alpha, \beta] \in H_2^\phi(X, L; \mathbb{Z})\) for the relative homology class

\[
[d, \alpha, \beta] = d\tilde{H} - \sum_i \alpha_i \tilde{F}_i - \sum_j \tilde{E}_j.
\]

The homology classes \(\tilde{H}, \tilde{F}_i, \tilde{E}_j \in H_2^\phi(X, L; \mathbb{Z})\) are the \(\phi\)-invariant projections of relative homology classes represented by \(H\) a hemisphere of a real line with real boundary, \(F_i\) a hemisphere of the \(i^{th}\) exceptional divisor with real boundary, and \(E_{r+j}\) an imaginary exceptional divisor.

For notational convenience we write 0 for the empty multi-index and use exponential notation for multi-indices of repetitive values, for example

\[
[10, (2^3), (3^2)] = [10, (2, 2, 2), (3, 3)] = 10\tilde{H} - 2\tilde{F}_1 - 2\tilde{F}_2 - 3\tilde{E}_1 - 3\tilde{E}_2.
\]

We denote by \(\Gamma_{[d, \alpha, \beta], k} = \text{OGW}_{[d, \alpha, \beta], k}(\tau_{\alpha}^\phi)\) the open Gromov-Witten invariant of degree \([d, \alpha, \beta] \in H_2^\phi(X, L; \mathbb{Z})\) and \(k = 0, 1, 2, \ldots\) boundary points and \(l = \frac{1}{2}(3d - |\alpha| - 2|\beta| - k - 1)\) internal points.

Table 1: Open Gromov-Witten invariants of a \(\mathbb{CP}^2\) blowup at \(r\) real points.

| \(d = 6, \beta = 0\) | \(d = 7, \beta = 0\) | \(d = 8, \beta = 0\) |
|----------------------|----------------------|----------------------|
| \(\Gamma_{[6,(2^6)],0},7\) = 8320 | \(\Gamma_{[7,(2^7)],0},10\) = 4224960 | \(\Gamma_{[8,(2^9)],0},13\) = -2824394880 |
| \(\Gamma_{[6,(2^6)],0},5\) = -2000 | \(\Gamma_{[7,(2^7)],0},8\) = -1226256 | \(\Gamma_{[8,(2^9)],0},11\) = 906723840 |
| \(\Gamma_{[6,(2^7)],0},3\) = 448 | \(\Gamma_{[7,(2^8)],0},6\) = 348054 | \(\Gamma_{[8,(2^9)],0},9\) = -287936880 |
| \(\Gamma_{[6,(2^8)],0},1\) = -96 | \(\Gamma_{[7,(2^9)],0},4\) = -96256 | \(\Gamma_{[8,(2^9)],0},7\) = 90364160 |
| \(\Gamma_{[6,(2^7)],0},2\) = 25820 | \(\Gamma_{[7,(2^{10})],0},0\) = -6672 | \(\Gamma_{[8,(2^{10})],0},5\) = -27996424 |
| \(\Gamma_{[6,(2^9)],0},1\) = -2571612 | | \(\Gamma_{[8,(2^{10})],0},3\) = 8551776 |
| | | \(\Gamma_{[8,(2^{11})],0},1\) = -2571612 |
Recall that the open Gromov-Witten invariant $\Gamma_{[d,\alpha,\beta],k}$ is equivalent to the Welschinger invariant counting curves of degree $d[L] - \sum_i [E_i] - \sum_j ([E_{r+j}] + [E_{r+j+s}])$ passing through $k$ real points and $l = \frac{1}{2}(3d - |\alpha| - 2 |\beta| - k - 1)$ complex conjugate pairs of points by the following relation

$$\Gamma_{[d,\alpha,\beta],k} = \pm 2^{1-l} W_{d[L] - \sum_i [E_i] - \sum_j ([E_{r+j}] + [E_{r+j+s}]),l}.$$ 

A precise formula for the sign is given in Section 7.

### References

[1] E. Brugallé, G. Mikhalkin, *Floor decompositions of tropical curves: the planar case*, Proceedings of Gökova Geometry-Topology Conference (2008), 64-90.

[2] Cho, C. H., *Counting real pseudo-holomorphic discs and spheres in dimension four and six*, J. Korean Math. Soc. 45 (2008), no. 5, 1427-1442.

[3] U. Frauenfelder, *Gromov convergence of pseudoholomorphic disks*, J. Fixed Point Theory and Application 3 (2008), no. 2, 215-271.
[4] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, (1978), 813 pp.

[5] L. Göttsche, R. Pandharipande, *The quantum cohomology of blowups of $\mathbb{P}^2$ and enumerative geometry*, J. Diff. Geom. 48 (1998), no. 1, 61-90.

[6] I. Itenberg, V. Kharlamov, E. Shustin, *Welschinger invariant and enumeration of real rational curves*, Internat. Math. Res. Notices 49 (2003), 2639-2653.

[7] I. Itenberg, V. Kharlamov, E. Shustin, *Logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Uspekhi Mat. Nauk 59 (2004), no. 6, 85-110 (Russian); English translation: Russian Math. Surveys 59 (2004), no. 6, 1093-1116.

[8] I. Itenberg, V. Kharlamov, E. Shustin, *Logarithmic asymptotics of the genus zero Gromov-Witten invariants of the blown up plane*, Geometry and Topology 9 (2005), paper no. 14, 483-491.

[9] I. Itenberg, V. Kharlamov, E. Shustin, *New cases of logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Proc. Steklov Math. Inst. 258 (2007), 65-73.

[10] I. Itenberg, V. Kharlamov, E. Shustin, *A Caporaso-Harris type formula for Welschinger invariants of real toric Del Pezzo surfaces*, Comment. Math. Helv. 84 (2009), no. 1, 87-126.

[11] I. Itenberg, V. Kharlamov, E. Shustin, *Welschinger invariance of small non-toric Del Pezzo surfaces*, [arXiv:1002.1399](http://arxiv.org/abs/1002.1399), to appear in Journal of EMS, 2012.

[12] I. Itenberg, V. Kharlamov, E. Shustin, *Welschinger invariants of real Del Pezzo surfaces of degree $\geq 3$*, [arXiv:1108.3369](http://arxiv.org/abs/1108.3369).

[13] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. 164 (1994), no. 3, 525-562.

[14] D. McDuff, D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, 6, American Mathematical Society, Providence, RI, (1994), 207 pp.

[15] D. McDuff, D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52, American Mathematical Society, Providence, RI, (2004), 669 pp.
[16] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, J. Diff. Geom. 42 (1995), no. 2, 259-367.

[17] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, [arXiv:math.SG/0606429](http://arxiv.org/abs/math.SG/0606429).

[18] J. Solomon, *A differential equation for the open Gromov-Witten potential*, preprint.

[19] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. 162 (2005), 195-234.

[20] J.-Y. Welschinger, *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants*, Duke Math. J. 127 (2005), no. 1, 89–121.

[21] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. 1, (1991), 243-310.

Institute of Mathematics
Hebrew University, Givat Ram
Jerusalem, 91904, Israel