BETWEENNESS, MONOTONICITY AND ROAD SYSTEMS: A CATEGORICAL INTERPRETATION

J. BRUNO AND A. MCCLUSKEY

Abstract. We develop and investigate a category theoretic framework in which to study betweenness relations. Within this setting, we study several naturally occurring subcategories of ternary relations generated from betweenness relations (R-relations) and provide categorical links in the form of adjunctions between road systems and R-relations. Lastly, we employ the notion of Grothendieck fibrations to illustrate a succinct description of R-relations as lattices with join and meet preserving functions.

1. Introduction and background

In the most abstract interpretation, for a point $b$ to lie between points $a$ and $c$ it must be that any way to get from $a$ to $c$ must inevitably go through $b$. A natural setting for studying betweenness of points can be found in ternary relations; for an arbitrary set $X$, a ternary relation $R \subseteq X^3$ can be interpreted as a betweenness relation by reading $(a, b, c)$ as $b$ lies between $a$ and $c$. There are certain undesirable triplets that would contradict this intuitive notion. For instance, in the spirit of betweenness, one would hope to avoid relations for which $(a, b, a), b \neq a$, are elements of it. One would also demand that if a point $b$ lies between $a$ and $c$ then that it would also lie between $c$ and $a$ (i.e. if $(a, b, c) \in R$ then $(c, b, a) \in R$). It turns out that this primitive notion of betweenness can be captured in a very basic first-order language with only one ternary predicate symbol and equality. The study of betweenness axioms dates back as far the late 1800’s ([8]) with sporadic revivals throughout the last century (see for example, [6] and [5]). An excellent modern approach can be found in [2] where the author explores a large variety of settings in which betweenness relations arise, with particular emphasis on continuum topology. The author makes a strong case for considering the simple notion of a road system on an arbitrary set as a natural approach to generating the intuitive notion of betweenness relations as ternary relations.

The aforementioned work considers sets endowed with certain structures as the focal point of study. This article extends the study of betweenness by adopting monotone functions as morphisms and subsequently revealing a rich categorical structure. In addition, we provide functorial links between road systems and betweenness relations as introduced in [2]. Specifically, we construct a complete and cocomplete category in which to study road systems and whose morphisms, although stronger than monotone maps, closely resemble betweenness preservation. We also provide several categorical results when considering universal closures of ternary relations as betweenness relations. This is achieved by a systematic study of naturally occurring betweenness axioms and the categories they define. In addition, we show that the category of betweenness relations enjoys a succinct description as lattices
with join and meet preserving functions. We establish this result by exploiting the notion of Grothendieck’s fibrations. For this article we restrict our attention to four defining axioms for betweenness relations. We refer the reader to forthcoming work on further properties of betweenness relations as introduced by Bankston in [4], [3], and [2] (see Conclusions for more details).

For the most part this article is self contained and notation is standard. However, we make use of the following: ternary relations will be denoted by $[\cdot, \cdot, \cdot]$ with subscripts to differentiate between them (i.e. $[\cdot, \cdot, \cdot]_{X}$, $[\cdot, \cdot, \cdot]_{R}$, etc). For a set $X$ and a relation $[\cdot, \cdot, \cdot]$ on it, we denote $(a, b, c) \in [\cdot, \cdot, \cdot]$ by $[a, b, c]$; subscripts are also included in this notation (e.g. $(a, b, c) \in [\cdot, \cdot, \cdot]_{R}$, is denoted as $[a, b, c]_{R}$). We will frequently employ the categorical notion of universal arrows when dealing with adjunctions. That is, a functor $F : C \to D$ is right adjoint precisely when for any $D$-object $X$, we can find a $C$-object $X_{0}$ and morphism $X \to F(X_{0})$ so that given any other $X \to F(Y)$ there exists a unique $X_{0} \to Y$ so that $F(X_{0}) \circ X = X \to F(Y)$. For any such $X$, the pair $(F(X_{0}), X \to F(X_{0}))$ is commonly referred to as a universal arrow from $X$ to $F$. The existence of $F$’s left adjoint is then equivalent to the existence of a universal arrow from $X$ to $F$ for each $X$. The dual of a universal arrow from $X$ to $F$ is one from $F$ to $X$ and this is achieved by reversing all arrows on the former. The existence of a universal arrow from $F$ to $X$ for each $X$ yields the existence of $F$’s right adjoint. All other categorical notions are standard and can be found in [7].

2. R-relations

Let $T$ denote the category whose objects are sets endowed with ternary relations and whose morphisms are monotone functions; for objects $(X, [\cdot, \cdot, \cdot]_{X})$ and $(Y, [\cdot, \cdot, \cdot]_{Y})$ a function $f : X \to Y$ is a morphism provided $[a, b, c]_{X} \Rightarrow [f(a), f(b), f(c)]_{Y}$. The forgetful functor $T \to \text{Set}$ is left and right adjoint (the right adjoint maps any $X$ to $(X, X^{\times 3})$ while the left takes $X$ to $(X, \emptyset)$) and, therefore, the underlying sets for limits and colimits in $T$ are those of limits and colimits in $\text{Set}$. In fact, one can readily see that $T$ is complete and cocomplete. For a collection $\{(X_{i}, [\cdot, \cdot, \cdot])_{i} \mid i \in I\}$ their product $(X, [\cdot, \cdot, \cdot])$ is the one for which $X = \coprod X_{i}$, with projections $\pi_{i} : X \to X_{i}$ so that $[a, b, c] \Leftrightarrow [\pi_{i}(a), \pi_{i}(b), \pi_{i}(c)]_{i}$ for all $i \in I$. Similarly, the underlying set of their coproduct is $\coprod X_{i}$. The morphisms into $\coprod X_{i}$ are the usual injections $p_{i} : X_{i} \to \coprod X_{i}$ and the ternary relation $[\cdot, \cdot, \cdot]$ on it is the one for which $[a, b, c]$ if, and only if, $a, b, c \in p_{i}(X_{i})$ and $[p_{i}^{-1}(a), p_{i}^{-1}(b), p_{i}^{-1}(c)]_{i}$ for some $i \in I$. For $f, g : (X, [\cdot, \cdot, \cdot]_{X}) \Rightarrow (Y, [\cdot, \cdot, \cdot]_{Y})$ the equaliser is the usual $E = \{x \in X \mid f(x) = g(x)\}$ with relation $[\cdot, \cdot, \cdot]$ so that $[a, b, c] \Leftrightarrow [a, b, c]_{X}$ and the obvious inclusion mapping $E \hookrightarrow X$. Coequalisers are also simple to construct: given $(X, [\cdot, \cdot, \cdot])$ and an equivalence relation $\sim$ on $X$ we let $Y = X/\sim$ with quotient function $p : X \to Y$ and ternary relation $[\cdot, \cdot, \cdot]_{\sim}$ so that $[a, b, c]_{\sim} \Leftrightarrow [x, y, z]$ for some $x \in a, b \in y$ and $z \in c$.

For a set $X$, a road system $R$ on it is a collection of subsets of $X$ so that: (a) any singleton is in $R$ and (b) for any two points in $X$ there exists a set in $R$ containing both. Any such road system generates a ternary relation $[\cdot, \cdot, \cdot]_{R}$ on $X$ called an $R$-relation (or a
betweenness relation] where $[a, b, c]_R$ if, and only if, $b \in R$ for any $R \in \mathcal{R}$ with $a, c \in R$. Using Bankston’s notation, we let

$$[a, b] := \{c \in X \mid [a, c, b]_R\} = \bigcap_{a, b \in R} R.$$ 

In [2] the author proves that ternary relations arising from road systems are first-order axiomatizable.

**Lemma 1** (Bankston). A relation $[\cdot, \cdot, \cdot]$ on a set $X$ can be generated from a road system if, and only if, the following universal axioms hold:

1. **Symmetry:** $[a, b, c] \Rightarrow [c, b, a]$;
2. **Reflexivity:** $[a, b, b]$;
3. **Minimality:** $[a, b, a] \Rightarrow a = b$; and
4. **Transitivity:** $[a, b, c] \land [a, d, c] \land [b, x, d] \Rightarrow [a, x, c]$.

For instance, (R4) reads as

$$\forall a, b, c, d, x \in X ([a, b, c], [a, d, c], [b, x, d] \Rightarrow [a, x, c]).$$

An $R_1$-relation (resp. $R_2$-relation, $R_3$-relation, $R_4$-relation) is a ternary relation satisfying R1 (resp. R2, R3, R4). Define $\mathcal{R}$ (resp. $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$) to be the full subcategory of $\mathcal{T}$ of all $R$-relations (of ternary relations satisfying R1, R2, R3 and R4, respectively). A very simple $R$-relation (of which we make extensive use of in the sequel) is $\mathcal{T} = (\{1\}, \{(1, 1, 1)\})$. For any set $X$, the smallest ternary relation on it that qualifies as an $R$-relation is $X_\perp := \{(a, b, b), (b, b, a) \mid a, b \in X\}$, while the largest is $X_\top := X^3 \setminus \{(a, b, a) \mid a \neq b\}$. For every set $X$ the collection $R(X)$ of all $R$-relations on $X$ forms a partial order under set inclusion and can be easily shown to be closed under meets. Consequently, it forms a complete lattice when $id_X : (X, [\cdot, \cdot, \cdot]) \to (X, [\cdot, \cdot, \cdot])$ if, and only if, $[\cdot, \cdot, \cdot] \subseteq [\cdot, \cdot, \cdot]$. In the sequel, we use this lattice completeness of $R(X)$ to show that $\mathcal{R}$ is a category fibered in lattices. Unlike the functor $\mathcal{T} \to \text{Set}$ the forgetful functor $\mathcal{R} \to \text{Set}$ is not left adjoint since the obvious, and necessary, assignment $X \mapsto X_\top$ does not define a right adjoint. This simple fact can be noted when considering any function $f : X \to X$ that is not one-to-one for a pair $a \neq b \in X$. Any $R$-relation $[\cdot, \cdot, \cdot]$ on $X$ for which $[a, c, b]$ for some $c \neq a$ cannot admit a monotone function between $(X, [\cdot, \cdot, \cdot])$ and $(X, X_\top)$ since this demands $[f(a), f(c), f(b)]$ of $X_\top$, thus violating R3. In the sequel we show this discrepancy to be caused by coequalisers in $\mathcal{R}_3$ being different from those in $\mathcal{T}$. Furthermore we will see that, for $1 \leq i \leq 4$, these full subcategories $\mathcal{R}_i$ of $\mathcal{T}$ can be recovered via left adjoints from $\mathcal{T}$ while only $\mathcal{R}_1 \hookrightarrow \mathcal{T}$ will be shown to be left adjoint. For $(X, [\cdot, \cdot, \cdot])$ a ternary relation we will interchangeably refer to both $(X, [\cdot, \cdot, \cdot])$ and $[\cdot, \cdot, \cdot]$ as relations. For the latter, the underlying set will always be known from context.
2.1. The inclusions $R_1 \hookrightarrow T$. We begin by noting that $R_1 \hookrightarrow T$ is right and left adjoint; a right adjoint to it is the one for which $(X, [\cdot, \cdot, \cdot]) \mapsto (X, [\cdot, \cdot, \cdot])$ so that
\[
[\cdot, \cdot, \cdot'] = [\cdot, \cdot, \cdot] \setminus \{(a, b, c) \in [\cdot, \cdot, \cdot] \mid (c, b, a) \not\in [\cdot, \cdot, \cdot] \};
\]
while a left adjoint is given by $(X, [\cdot, \cdot, \cdot]) \mapsto (X, [\cdot, \cdot, \cdot'])$ where
\[
[\cdot, \cdot, \cdot'] = [\cdot, \cdot, \cdot] \cup \{(a, b, c) \mid (c, b, a) \in [\cdot, \cdot, \cdot] \}.
\]
It is then simple to check that $R_1$ is complete and cocomplete with limits and colimits coinciding with those from $T$.

The inclusion functor $R_2 \hookrightarrow T$ is, as the following suggests, only right adjoint. Its left adjoint maps any ternary relation $(X, [\cdot, \cdot, \cdot])$ to $(X, [\cdot, \cdot, \cdot'])$ where
\[
[\cdot, \cdot, \cdot'] = [\cdot, \cdot, \cdot] \cup \{(a, b, b) \mid a, b \in X \}.
\]
Thus, $R_2$ is cocomplete but in contrast with $R_1$, coproducts in $R_2$ are finer than those in $T$ since $R_2$ demands $[a, b, b]$ for any pair $a, b$. For instance, given $(X, [\cdot, \cdot, \cdot]_X)$ and $(Y, [\cdot, \cdot, \cdot]_Y)$ their coproduct has for underlying set $Z = X \sqcup Y$, maps $p_x : X \to Z$ and $p_y : Y \to Z$ so that $[a, b, c]$ is true in $Z$ if, and only if, $[p_x^{-1}(a), p_x^{-1}(b), p_x^{-1}(c)]_X$ or $[p_y^{-1}(a), p_y^{-1}(b), p_y^{-1}(c)]_X$ or $b = c$. The inclusion functor $R_2 \hookrightarrow T$ is, thus, only right adjoint. That said, limits and coequalizers in $R_2$ coincide with those in $T$.

The category $R_3$ is also complete and contains all coproducts. Indeed, the reader can verify that all such constructions in $R_3$ coincide with those in $T$. Coequalisers also exist but, as the following example suggests, differ greatly from those in $T$ - where the coequaliser of any diagram $Y \Rightarrow X$ can be constructed as an equivalence relation on $X$, as the underlying set.

Remark 2. When constructing coequalisers in $R_3$ the following example and lemma consider equivalence relations on $X$ instead of the usual diagrams. These two concepts are interchangeable in $R_3$: fix a $R_3$ object $(X, [\cdot, \cdot, \cdot])$ and consider any $\sim \in Eq(X)$. The set $X \times X$ can be naturally turned into an $R_3$-relation as the product relation of $(X, [\cdot, \cdot, \cdot])$ and the set $R \subseteq X \times X$ representing $\sim$ can be endowed with the subset relation from $X \times X$, call it $[\cdot, \cdot, \cdot]_R$. Since the projections $\pi_1, \pi_2 : (R, [\cdot, \cdot, \cdot]_R) \Rightarrow X$ are monotone, the equivalence $\sim$ is then recovered. Thus, constructing coequalisers in $R_3$ is equivalent to forging some $\sim_\omega$ for every any $R_3$ object with some relation $\sim$.

Example 3. Let $X = \{a, b, c\}$ so that $[a, b] = \{a, b\}$, $[b, c] = \{b, c\}$ and $[a, c] = X$. Further, take $\sim \in Eq(X)$ so that $\sim = \{(a, c), \{b\}\}$. Is there any $R_3$-relation $[\cdot, \cdot, \cdot]_\sim$ on $X/\sim$ making $p : X \to X/\sim$ monotone? The answer is no; for $p$ to be a morphism we need $[p(a), p(b), p(c)]_\sim$ since $[a, b, c]$. However, $p(a) = p(c)$ but $[p(a), p(b), p(a)]_\sim$ does not imply $p(b) = p(a)$ and whichever ternary relation we apply to $X/\sim$ in order to make $p$ a morphism will fail to satisfy minimality.

The issue raised in the previous example is that neither the functor $R_3 \hookrightarrow T$ nor $R_3 \to Set$ is left adjoint. For instance, the coequaliser for the above scenario in $R_3$ is simply $T$ while the underlying set for the one in $T$ has two elements. For any diagram $f, g : (Y, [\cdot, \cdot, \cdot]_Y) \Rightarrow (X, [\cdot, \cdot, \cdot])$ in $R_3$ constructing its limiting cone will amount to marrying $[\cdot, \cdot, \cdot]$.
to the equivalence relation \( \sim \) in \textbf{Set} generated by \( f, g : Y \to X \) by constructing a coarser relation \( \sim_\omega \) on \( X \) as follows:

- \( b \sim_1 c \) if, and only if, \( b \sim c \)
- \( b \sim_{n+1} c \) if, and only if, \( b \sim_n c \) or there exists \( \{r_i, s_i, t_i\} \subseteq [\cdot, \cdot, \cdot] \) with
  - \( b \in \{r_m, s_m, t_m\} \) and \( c \in \{r_1, s_1, t_1\} \), and
  - for all \( i \leq m \), \( r_i \sim_n t_i \) and \( x \sim_n y \) with \( x \in \{r_i, s_i, t_i\} \) and \( y \in \{r_{i+1}, s_{i+1}, t_{i+1}\} \).

We demand \( a \sim_\omega b \) precisely when \( a \sim_n b \) for some \( n \in \mathbb{N} \).

**Lemma 4.** The relation \( \sim_\omega \) defined above is an equivalence relation on \( X \).

**Proof.** Reflexivity and symmetry are obvious so let \( a \sim_\omega b \) and \( b \sim_\omega c \). It follows that for some \( k \in \mathbb{N} \) we have \( a \sim_{k+1} b \) and \( b \sim_{k+1} c \). By definition, there exist finite sequences \( \{r_i, s_i, t_i\} \mid i \leq m \) and \( \{|u_i, v_i, w_i| \mid i \leq n\} \) so that

\[
[r_1, s_1, t_1] \land \ldots \land [r_m, s_m, t_m] \quad \text{and} \quad [u_1, v_1, w_1] \land \ldots \land [u_n, v_n, w_n]
\]

with the defining relationships between the variables \( r_i, s_i, t_i, u_i, v_i \) and \( w_i \), and the points \( a, b \) and \( c \). Since \( \{b\} \subseteq \{r_m, s_m, t_m\} \cap \{u_1, v_1, w_1\} \), the two chains above can be concatenated so as to form a chain from \( a \) to \( c \), in which case \( a \sim_{k+1} c \) and transitivity holds. \( \square \)

Next, let \( Z = X/\sim_\omega \) with quotient map \( p : X \to Z \) and ternary relation \([\cdot, \cdot, \cdot]_Z \) on \( Z \) so that \([a, b, c]_Z \Leftrightarrow [x, y, z] \) for some \( x \in a, y \in b \) and \( z \in c \); by design, the quotient map \( p : X \to Z \) is monotone.

**Lemma 5.** For any ternary relation \((X, [\cdot, \cdot, \cdot])\), the above defined \((X/\sim_\omega = Z, [\cdot, \cdot, \cdot]_Z)\) is an \( R_3 \)-relation so that

(a) if \((X, [\cdot, \cdot, \cdot])\) satisfies \( R_1 \) (resp. \( R_2 \)), then so does \((Z, [\cdot, \cdot, \cdot]_\omega)\) and for any other \( R_1 \) (resp. \( R_2 \)) relation \((W, [\cdot, \cdot, \cdot]_W)\) if \( f : (X, [\cdot, \cdot, \cdot]) \to (W, [\cdot, \cdot, \cdot]_W) \) is monotone function, then so is \( f : (Z, [\cdot, \cdot, \cdot]_\omega) \to (W, [\cdot, \cdot, \cdot]_W) \);

(b) the category \( R_3 \) is a complete and cocomplete reflective (but not a coreflective) subcategory of \( T \).

**Proof.** We begin by showing that \((Z, [\cdot, \cdot, \cdot]_Z)\) satisfies \( R_3 \). Let \([a, b, a]_Z \) for some \( a, b \in Z \) and recall that this is true precisely when for some \( x, z \in a \) and \( y \in b \), \([x, y, z] \). Let \( n \in \mathbb{N} \) be the smallest number for which \( x \sim_n z \) (to avoid trivialities we assume \( n > 1 \)). It follows that there exists \( \{r_i, s_i, t_i\}\) for some \( m \in \mathbb{N} \) with \( x \in \{r_1, s_1, t_1\} \), \( z \in \{r_m, s_m, t_m\} \) and so that for all \( i \leq m \), \( s_i \sim_n t_i \) and, \( u \sim_n v \) with \( u \in \{r_i, s_i, t_i\} \) and \( v \in \{r_{i+1}, s_{i+1}, t_{i+1}\} \). By adjoining \([x, y, z] \) to \([r_i, s_i, t_i]\) and since \( x \sim_n z \) we have that \( x \sim_{n+1} y \sim_{n+1} z \) and that \( a = b \).

Next we prove (b) since (a) follows immediately from the definition of \((Z, [\cdot, \cdot, \cdot]_Z)\) and notice that (b) implies (c). Assume that for some \( R_3 \)-relation \((W, [\cdot, \cdot, \cdot]_W)\) with monotone \( f : X \to W \) we have \( a \sim b \Rightarrow f(a) = f(b) \). We first show that for all \( a \in X \) the assignment
for which \( p(a) \mapsto f(a) \) is a well-defined function. Clearly, if \( a \sim_1 b \) (i.e., \( a \sim b \)) for \( a, b \in X \), then \( f(a) = f(b) \). Assume this holds up to some \( k \in \mathbb{N} \) (i.e., \( a \sim_k b \implies f(a) = f(b) \)). If \( a \sim_{k+1} b \) then we can find \( \{r_i, s_i, t_i \in [\cdot, \cdot] \mid i \leq n \} \) with \( a \in \{r_1, s_1, t_1 \} \), \( b \in \{r_m, s_m, t_m \} \) and so that for all \( i \leq m, s_i \sim t_i \) and \( u \sim_n v \) with \( u \in \{r_i, s_i, t_i \} \) and \( v \in \{r_{i+1}, s_{i+1}, t_{i+1} \} \).

For all \( i \leq m \), by the inductive hypothesis, \( f(r_i) = f(s_i) = f(t_i) \) and \( f(u) = f(v) \) for some \( u \in \{r_i, s_i, t_i \} \) and some \( v \in \{r_{i+1}, s_{i+1}, t_{i+1} \} \). It follows that \( \forall n \in \mathbb{N}, a \sim_n b \implies f(b) = f(a) \) and that \( a \sim_\omega b \implies f(a) = f(b) \). That is, \( f \) maps elements from the same equivalence class in \( \sim_\omega \) to the same element in \( W \). Hence, the assignment for which \( p(a) \mapsto f(a) \) is a well-defined function; denote it by \( f' \) and notice that \( f = f' \circ p \).

Lastly, we show monotonicity of \( f' \). For \( a, b, c \in Z \), \([a, b, c]_Z \) occurs when for some \( x \in a, y \in b \) and \( z \in c \) we have \([x, y, z]_T \). That \( f \) is monotone yields \( [f(x), f(y), f(z)]_W \), in which case \([f'(a), f'(b), f'(c)]_W \), since \( f = f' \circ p \).

The last claim follows since \( R_3 \leftrightarrows T \) does not preserve coequalisers. 

The reader can easily verify that constructions for limits and coproducts in \( T \) are also those for \( R_4 \). Coequalisers \( R_4 \) do not agree with those in \( T \). Take for instance the set \( X = \{a, a', b, d, x, c\} \) with \([\cdot, \cdot, \cdot] = X_\perp \cup \{[a, b, c], [a', d, c], [b, x, d]\} \). Gluing \( a, b \) and applying \( T \)'s coequalizer to this quotient does not yield an \( R_4 \)-relation. As Lemma 5 shows, in contrast with the discrepancy of coequalisers between \( R_3 \) and \( T \), the underlying set for such constructions in \( R_4 \) and \( T \) remains fixed; for a pair \( f, g : (Y, [\cdot, \cdot, \cdot])_Y \Rightarrow (X, [\cdot, \cdot, \cdot])_X \) the limiting cone will be given by applying a recursive construction to the set theoretic coequaliser of \( f, g : Y \Rightarrow X \). Also, Remark 2 applies to this case. Given any ternary relation \((X, [\cdot, \cdot, \cdot])\) and for all \( a, b \in Z \) we define recursively,

- \([a, b]_1 = [a, b] \)
- \([a, b]_n+1 = [a, b]_n \cup \{x \in X \mid \exists c, d \in [a, b]_n \text{ so that } x \in [c, d]_n \} \).

Next, let \([a, b]_\omega = \bigcup_{n \in \omega} [a, b]_n \) and \((X, [\cdot, \cdot, \cdot])_\omega \) be the one generated by the intervals \([a, b]_\omega \).

**Lemma 6.** For any \( T \) object \((X, [\cdot, \cdot, \cdot])\) the ternary relation \([\cdot, \cdot, \cdot]_\omega \) on \( X \) as described above satisfies \( R_4 \) and

(a) if \((X, [\cdot, \cdot, \cdot])\) satisfies \( R_1 \) (resp. \( R_2, R_3 \)) then so does \((Z, [\cdot, \cdot, \cdot])_\omega \) and for any other \( R_1 \) (resp. \( R_2, R_3 \)) relation \((W, [\cdot, \cdot, \cdot])_W \) if \( f : (X, [\cdot, \cdot, \cdot])_X \rightarrow (W, [\cdot, \cdot, \cdot])_W \) is monotone then so is \( f : (X, [\cdot, \cdot, \cdot])_\omega \rightarrow (W, [\cdot, \cdot, \cdot])_W \);

(b) the category \( R_4 \) is a complete and cocomplete reflective (but not a cocoreflective) subcategory of \( T \).

**Proof.** First we show \((X, [\cdot, \cdot, \cdot])_\omega \) is an \( R_4 \)-relation: let \( b, d \in [a, c]_\omega \) and \( x \in [b, d]_\omega \) and, by design, let \( n \in \mathbb{N} \) be any number so that \( b, d \in [a, c]_n \) and \( x \in [b, d]_n \). It follows that \( x \in [a, c]_{n+1} \) and that \( x \in [a, c]_\omega \). For (a), that \([\cdot, \cdot, \cdot]_\omega \) preserves \( R_1 \) and \( R_2 \) is clear and if \( b \in [a, a]_1 \) then \( b = a \). Assume this is true up to \( k \in \mathbb{N} \). If \( b \in [a, a]_{k+1} \) then \( b \in [a, a]_k \) (in which case we are done) or we can find \( c, d \in [a, a]_k \) so that \( b \in [c, d]_k \). Immediately we get that \( a = c = b = d \). Thus, \([a, b, a]_\omega \) implies \( b = a \) and that \([a, a]_\omega = \{a\} \) for all \( a \in X \). Consequently, we have that \([\cdot, \cdot, \cdot]_\omega \) satisfies \( R_3 \).
Next, we show (b). Let \( \sim \in Eq(X) \) and assume that for some \( R_4 \)-relation \( (W, [\cdot, \cdot, \cdot]_W) \) with monotone \( f : X \to W \) we have \( a \sim b \Rightarrow f(a) = f(b) \). We need only show that the canonical function \( f' : X/\sim \to W \) for which \( f' \circ p = f \) is monotone. That \([p(a), p(b), p(c)]_1 \) implies \([a, b, c] \) and thus \([f(a), f(b), f(c)]_W \) for any \( a, b, c \in X \). Assume this to hold up to some \( k \in \mathbb{N} \) and let \([p(a), p(b), p(c)]_{k+1} \). Obviously, for \([p(a), p(b), p(c)]_k \) we have our result. Otherwise, there exist \( p(c), p(d) \in [p(a), p(c)]_k \) with \( p(b) \in [p(c), p(d)]_k \). By the inductive hypothesis, \([f(c), f(d)] \in [f(a), f(c)]_W \) and \([f(b)] \in [f(c), f(d)]_W \) and since \([\cdot, \cdot, \cdot]_W \) satisfies \( R_4 \) then \([f(b)] \in [f(a), f(c)]_W \). Lastly, that \( R_4 \hookrightarrow T \) is right adjoint follows directly from (b) by letting \( \sim \) be the discrete relation on \( X \).

One can readily verify that the previous result provides us with an explicit construction for joins in \( R(X) \), for any \( X \). The adjoints to the inclusions \( R_i \hookrightarrow T \) are, in a sense, equivalent to closure operations for turning ternary relations that do not satisfy a particular axiom into ones that do. These closures are natural in the sense that the obvious diagrams commute. By design, these operators are idempotent. What is certainly not obvious is that the order in which these operators are applied does not commute. Denote the closure operators related to the left adjoints with \( L_i \), for \( i \leq 4 \), and notice that the compositions \( L_1 \circ L_2 \) and \( L_2 \circ L_1 \) are not the same; consider \( X = \{a, b\} \) with \([\cdot, \cdot, \cdot] = \{[a, a, a], [b, b, b], [a, b, b]\} \) and notice that applying \( L_1 \circ L_2 \) and \( L_2 \circ L_1 \) yields hugely different relations. The discrepancy can be explained by the operator \( L_2 \) not preserving \( R_1 \). In fact, more is true: the operator \( L_1 \circ L_2 \) defines the left adjoint to the inclusion \( R_{1.2} \hookrightarrow T \), where \( R_{1.2} \) denotes the - complete and cocomplete - full subcategory of \( T \) of relations satisfying \( R_1 \) and \( R_2 \). A less trivial scenario is given by \( L_3 \) and \( L_4 \): \( L_4 \circ L_3 \) defines the left adjoint to the inclusion \( R_{3.4} \hookrightarrow T \), where \( R_{3.4} \) defines the - also complete and cocomplete - full category of \( T \) of relations satisfying \( R_3 \) and \( R_4 \); this is a simple consequence of Lemmas \( \Box \) and \( \Box \). In contrast, \( L_3 \) guarantees an \( R_3 \)-relation as an output and it does not, however, guarantee that the new \( R_3 \)-relation will satisfy \( R_4 \) (even if the original relation did). For instance, consider \( X = \{a_1, a_2, b, c, d\} \) with \([\cdot, \cdot, \cdot] = \{[a_1, a_2, a_1], [c, b, a_1], [b, d, a_2]\} \) and apply \( L_3 \) to it. That would glue \( a_1 \) and \( a_2 \) together, but leave a non-\( R_4 \) relation on the underlying 4 point set. In view of the above and Lemmas \( \Box \) and \( \Box \) it might not surprising to learn that \( L_4 \circ L_3 \circ L_1 \circ L_2 \) defines the left adjoint to \( R \hookrightarrow T \). The following theorem justifies the commutative diagram below, where the arrows going up are inclusions and the ones pointing down represent adjoints.

**Theorem 7.** For \( 1 \leq i \leq 4 \), the categories \( R_i \) are reflective subcategories of \( T \) with only \( R_1 \) being also coreflective. The complete and cocomplete category \( R \) is only a reflective subcategory of each \( R_i \) and \( T \).

**Proof.** The first part of the theorem is proved by the preceding lemmas and examples. Products and equalizers in \( R \) agree with those from \( T \). Coproducts in \( R \) are constructed by applying \( L_1 \) to coproducts in \( R_2 \) and coequalizers arise from applying \( L_4 \circ L_3 \) to coequalizers in \( R_{1.2} \). To this end, we have only to notice that the operators \( L_i \) are idempotent and that: \( L_1 \) preserves \( R_2 \), \( L_3 \) preserves both \( R_1 \) and \( R_2 \), and \( L_4 \) preserves all three axioms \( R_1 \)-\( R_3 \).
2.2. Weak and strong R-relations. The aim of this section is to facilitate the reading of Section 2.3 on Grothendieck fibrations of \( \mathbb{R} \). Since \( \mathbb{R} \) is complete, the notion of weak and strong \( \mathbb{R} \)-relations is well defined. Here we present explicit constructions of such. Given a function \( f : X \to Y \) and \([\cdot, \cdot, \cdot] \in R(X)\), one can easily verify that

\[
\bigwedge \{[\cdot, \cdot, \cdot]' \in R(Y) \mid f : [\cdot, \cdot, \cdot] \to [\cdot, \cdot, \cdot]' \text{ is monotone}\}
\]

is the smallest relation on \( Y \) making \( f \) monotone. Not so obvious is the fact that given a function \( f : X \to Y \) and \([\cdot, \cdot, \cdot] \in R(Y)\) the relation

\[
\bigvee \{[\cdot, \cdot, \cdot]' \in R(X) \mid f : [\cdot, \cdot, \cdot]' \to [\cdot, \cdot, \cdot] \text{ is monotone}\},
\]

is the largest relation on \( X \) making \( f \) monotone. The proof of this and of further results from Section 2.3 relies on the following lemmas, where the operator \( L := L_4 \circ L_3 \).

**Lemma 8.** For any collection \( \{[\cdot, \cdot, \cdot]_i \mid i \in I\} \in \text{ob}(T) \) of relations on a set \( X \),

\[
L \left( \bigcup_{i \in I} L_i \right) = L \left( \bigcup_{i \in I} \right).
\]

**Proof.** Obviously, \((\supseteq)\) holds and we prove the other direction. For any \( i \), by universality of \( L \)

\[
\begin{array}{ccc}
[\cdot, \cdot, \cdot]_i & \xrightarrow{id_X} & L[\cdot, \cdot, \cdot]_i \\
\downarrow{id_X} & & \downarrow{id_X} \\
L \left( \bigcup_{i \in I} \right) & &  \\
\end{array}
\]

commutes and all arrows are monotone. In turn, we have that

\[
\begin{array}{ccc}
\bigcup_{i \in I} L[\cdot, \cdot, \cdot]_i & \xrightarrow{id_X} & L \left( \bigcup_{i \in I} L[\cdot, \cdot, \cdot]_i \right) \\
\downarrow{id_X} & & \downarrow{id_X} \\
L \left( \bigcup_{i \in I} [\cdot, \cdot, \cdot]_i \right) & &  \\
\end{array}
\]

also commutes and all arrows are monotone. Thus, \( L \left( \bigcup_{i \in I} \right) \supseteq L \left( \bigcup_{i \in I} \right) \) and the proof is complete. \( \Box \)

**Lemma 9.** For any function \( f : X \to Y \) and any relation \([\cdot, \cdot, \cdot] \in \text{ob}(T) \) on \( X \), we have

\[
L(f[\cdot, \cdot, \cdot]) = L(f[L[\cdot, \cdot, \cdot]]).
\]

**Proof.** Obviously, \((\subseteq)\) holds and we prove the other direction. By universality of \( L \)

\[
\begin{array}{ccc}
[\cdot, \cdot, \cdot] & \xrightarrow{id_X} & L[\cdot, \cdot, \cdot] \\
\downarrow{f} & & \downarrow{f} \\
L(f[\cdot, \cdot, \cdot]) & &  \\
\end{array}
\]
commutes and all arrows are monotone. Consequently, \( f(L[\cdot, \cdot, \cdot]) \subseteq L(f[\cdot, \cdot, \cdot]) \),

\[
\begin{array}{ccc}
  f(L[\cdot, \cdot, \cdot]) & \xrightarrow{\text{id}_X} & L(f(L[\cdot, \cdot, \cdot])) \\
\downarrow \text{id}_X & & \downarrow \text{id}_X \\
L(f \bigcup_{i \in I} [\cdot, \cdot, \cdot]) & \subseteq & L(f[\cdot, \cdot, \cdot])
\end{array}
\]

and \( L(f(L[\cdot, \cdot, \cdot])) \subseteq L(f \bigcup_{i \in I} [\cdot, \cdot, \cdot]). \)

\[ \square \]

The following corollary follows directly from the above lemmas.

**Corollary 10.** For any collection of \( R \)-relations \( (X_i, [\cdot, \cdot, \cdot], i) \) and functions

\[
f^i : (X_i, [\cdot, \cdot, \cdot], i) \rightarrow Y
\]

\[
f_i : Y \rightarrow (X_i, [\cdot, \cdot, \cdot], i),
\]

then

\[ \bigwedge \{[\cdot, \cdot, \cdot] \in R(Y) \mid f^i : [\cdot, \cdot, \cdot], i \rightarrow [\cdot, \cdot, \cdot]' \text{ is monotone for some } i \} \]

is the coarsest \( R \)-relation on \( Y \) making all \( f_i \) monotone, while

\[ \bigvee \{[\cdot, \cdot, \cdot] \in R(X) \mid f : [\cdot, \cdot, \cdot] \rightarrow [\cdot, \cdot, \cdot] \text{ is monotone for some } i \} \]

is the finest \( R \)-relation on \( Y \) making all \( f^i \) monotone.

**2.3. Grothendieck fibrations of \( R \).** Let \( \text{CJLat} \) (resp. \( \text{CMLat} \)) be the category of complete lattices with arbitrary join preserving and nonempty meet preserving functions (resp. the category of complete lattices with arbitrary meet preserving and nonempty join preserving functions). Our ultimate goal here is to show that \( \mathbb{R} \) is fibered in lattices, and that these are Grothendieck fibrations. We begin by letting

\[
F^* : \text{Set} \rightarrow \text{CJLat}, \text{ for which } X \mapsto R(X) \text{ and } f : X \rightarrow Y \mapsto f^* : R(X) \rightarrow R(Y) \text{ so that for any } [\cdot, \cdot, \cdot] \in R(X) \text{ we let}
\]

\[
f^*[\cdot, \cdot, \cdot] = \bigwedge \{[\cdot, \cdot, \cdot]' \in R(Y) \mid f : [\cdot, \cdot, \cdot] \rightarrow [\cdot, \cdot, \cdot]' \text{ is monotone},
\]

\[
F_* : \text{Set}^{op} \rightarrow \text{CMLat}, \text{ for which } X \mapsto R(X) \text{ and } f : X \rightarrow Y \mapsto f_* : R(X) \rightarrow R(Y) \text{ so that for any } [\cdot, \cdot, \cdot] \in R(X) \text{ we let}
\]

\[
f_*[\cdot, \cdot, \cdot] = \bigvee \{[\cdot, \cdot, \cdot]' \in R(Y) \mid f : [\cdot, \cdot, \cdot] \rightarrow [\cdot, \cdot, \cdot]' \text{ is monotone},
\]

and proceed to show that \( F_* \) and \( F^* \) are functors. The following are easy consequences of Lemma 8 and Lemma 9.

**Lemma 11.** Let \( f : X \rightarrow Y \) be any functions between sets.

1. The functions \( f_* \) and \( f^* \) are monotone.

2. For any \( [\cdot, \cdot, \cdot] \in R(X) \) we have \( f : [\cdot, \cdot, \cdot] \rightarrow f^*[\cdot, \cdot, \cdot] \) is monotone.
(3) For any \([\cdot, \cdot, \cdot] \in R(Y)\) we have \(f : f_*[\cdot, \cdot, \cdot] \to [\cdot, \cdot, \cdot]\) is monotone.

(4) The functions \(f_*\) and \(f^*\) preserve all non-empty joins and all non-empty meets.

(5) The functions \(f_*\) and \(f^*\) preserve all meets and all joins, respectively, and \(f^* \dashv f_*\)

Remark 12. Neither \(f^*\) preserves empty meets nor does \(f_*\) preserve empty joins. To see the
former, take any constant function \(X \to Y\) so that \(x \mapsto a \in Y\) for all \(x\), then \([\cdot, \cdot, \cdot] \mapsto Y_{\perp}\) for all \([\cdot, \cdot, \cdot] \in R(X)\). The other case follows easily by also considering constant functions.

Recall that there exists a natural adjunction \(\text{CJLat} \rightleftarrows \text{CMLat}\) bijective on objects, and
so that any arrow is sent to its adjoint. For instance, an arrow \(V \to W\) in \(\text{CMLat}\) is mapped
to its left adjoint (since \(V \to W\) preserves all of the meets that \(V\) contains, this left adjoint
exists).

Theorem 13. The assignments \(F_*\) and \(F^*\) are functors so that

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{F^*} & \text{CJLat} \\
\downarrow & & \downarrow \\
\text{Set}^{\text{op}} & \xleftarrow{F_*} & \text{CMLat}
\end{array}
\]

commutes. Moreover, \(F_*\) (resp. \(F^*\)) defines a Grothendieck fibration (resp. opfibration)
on \(R \to \text{Set}\).

Proof. To show that \(F_*\) and \(F^*\) are functors we need only show that they preserve composition. We prove functoriality of \(F^*\) since both proofs are almost identical: let \(h : X \to Z = g : Y \to Z \circ f : X \to Y\) and notice that by Lemma [9]

\[
\begin{align*}
g^* \circ f^*[\cdot, \cdot, \cdot] &= L(g(L(f[\cdot, \cdot, \cdot)))) \\
&= L(g \circ f[\cdot, \cdot, \cdot]) \\
&= L(h[\cdot, \cdot, \cdot]) \\
&= h^*[\cdot, \cdot, \cdot].
\end{align*}
\]

Next, we show that for each \(f : X \to Y\) with R-relation \([\cdot, \cdot, \cdot]\) on \(X\), \(f : (X, [\cdot, \cdot, \cdot]) \to (Y, f^*[\cdot, \cdot, \cdot])\) is cartesian. We know that \(f : (X, [\cdot, \cdot, \cdot]) \to (Y, f^*[\cdot, \cdot, \cdot])\) is monotone. In fact, \(f^*[\cdot, \cdot, \cdot] = L(f[\cdot, \cdot, \cdot])\). Hence, given any monotone function \(g : (Z, [\cdot, \cdot, \cdot]z) \to (Y, f^*[\cdot, \cdot, \cdot])\) we have \(L(g[\cdot, \cdot, \cdot]z) \subseteq f^*[\cdot, \cdot, \cdot] = L(f[\cdot, \cdot, \cdot])\). If any function filler \(h\) exists so that \(f \circ h = g\), then it must be that \(L(f(L(h[\cdot, \cdot, \cdot]z))) = L(g[\cdot, \cdot, \cdot]z) \subseteq L(f[\cdot, \cdot, \cdot])\) or that \(h\) is monotone.

\(\square\)
Capturing the abstract notion of road systems as a category requires a suitable definition for morphisms between road systems. Moreover, these potential definitions must agree with monotone functions when considering R-relations generated by such road systems. The most natural options are functions between underlying sets with some extra properties; possibly involving forwards and backwards images of roads. In what follows, we explore several candidates for morphisms between road systems and select the one that better approximates monotone functions. One option is to consider forward images of roads to match some requirement of roads on the codomain. Since singletons are always roads, demanding images of roads to be contained in (resp. match or contain) unions of roads yields that any function would be a morphism, and thus not a great candidate for morphisms. Continuing with images of roads, one can demand for images of roads to be contained, match or contain intersection of roads on the codomain. The latter option fails since, again, singletons are roads. For the remaining two options, consider $X = \{a, b, c\}$ with $\mathcal{R}_1 = \{(a, b), (b, c), X\}$ and $\mathcal{R}_2 = \{(a), \{b\}, \{c\}, \{a, b\}\}$. The identity $id_X : (X, \mathcal{R}_1) \rightarrow (X, \mathcal{R}_2)$ satisfies the remaining two options for a road morphism. However, the identity is not monotone between the R-relations generated by the road systems $\mathcal{R}_1$ and $\mathcal{R}_2$. We are left with pre-images of roads, where (by now obvious reasons) we discard inverse of roads involving (i.e. $\subseteq, \supseteq, =$) unions of roads in the domain. The remaining feasible options for a function $f : (X, \mathcal{R}) \rightarrow (Y, \mathcal{S})$ are: for all $S \in \mathcal{S}$, (a) $f^{-1}(S) \subseteq \bigcap R_i$ for $R_i \in \mathcal{R}$, (b) $f^{-1}(S) \subseteq R \in \mathcal{R}$, (c) $f^{-1}(S) \in \mathcal{R}$ and (d) $f^{-1}(S) = \bigcap R_i$ with $R_i \in \mathcal{R}$. The first two can be disregarded by considering the example $X = \{a, b, c\}$ with $\mathcal{R}_1$ and $\mathcal{R}_2$ and the identity $id_X : (X, \mathcal{R}_1) \rightarrow (X, \mathcal{R}_2)$ as above. The weaker (i.e. (c) $\Rightarrow$ (d)) of the remaining options will turn out to be our choice of morphism. Next we show that (d) yields monotone functions (and thus, (c) does as well). First we must, ever so slightly, tweak our definition of road systems for these definitions to fully apply.

We will extend the definition of a road system on an arbitrary set to include the empty set and the set itself so as to create a complete and cocomplete category. Notice that no information is gained or lost by demanding such requirement on road systems; in particular, no betweenness information is either lost or gained. The category $\mathcal{B}$ will have as objects pairs $(X, \mathcal{R})$, where $X$ is a set and $\mathcal{R}$ is a road system (in the newly defined sense) on it. A morphism between two objects $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ is a function $f : X \rightarrow Y$ so that for any road $S \in \mathcal{S}$ we have $f^{-1}(S) = \bigcap_{i \in I} R_i$, for some collection $\{R_i\} \subseteq \mathcal{R}(\prod)$. The choice of such functions as arrows is not arbitrary. As demonstrated above, purely in terms of roads, this choice of arrows is as close as one can get to monotone functions between ternary relations. Next we show that $\mathcal{B}$-morphisms are also monotone between the R-relations generated by the respective road systems and that the converse is not true. The following lemma is simple to prove.

**Lemma 14.** Any $f : (X, [\cdot, \cdot, \cdot])_X \rightarrow (Y, [\cdot, \cdot, \cdot])_Y$ is monotone if, and only if, $\forall a, b \in X$, 
\[ f^{-1}([f(a), f(b)]_Y) \supseteq [a, b]_X. \]

**Lemma 15.** Any $\mathcal{B}$-morphism $(X, \mathcal{R}) \rightarrow (Y, \mathcal{S})$ is also a monotone function $(X, [\cdot, \cdot, \cdot]_{\mathcal{R}}) \rightarrow (Y, [\cdot, \cdot, \cdot]_{\mathcal{S}})$. 

Proof. Take any \( a, c \in X \) and notice

\[
f^{-1}([f(a), f(c)]_S) = f^{-1} \left( \bigcap_{f(a), f(b) \in S} S \right) = \bigcap_{f(a), f(b) \in S} f^{-1}(S) = \bigcap_{f(a), f(b) \in S} R \supseteq [a, c]_\mathcal{R},
\]

where for each \( S, \bigcap_S R = f^{-1}(S) \) for some collection of roads \( R \) in \( \mathcal{R} \). \( \square \)

This gives rise to the obvious functor \( \mathcal{B} \to \mathcal{R} \) where a road system is mapped to the \( R \)-relation it generates. In the sequel we highlight many similarities between \( \mathcal{B} \) and \( \mathcal{R} \) by showing that \( \mathcal{B} \) is complete and cocomplete, and that this surjective functor is right adjoint (whose left adjoint creates limits and colimits). The discrepancy between morphisms within \( \mathcal{B} \) and \( \mathcal{R} \) will become apparent when comparing coequalisers from both categories.

**Corollary 16.** The functor \( \mathcal{B} \to \mathcal{R} \) is right adjoint.

Proof. Given an arbitrary \((X, [\cdot, \cdot, \cdot])\) let \( \mathcal{R}_{[\cdot, \cdot, \cdot]} := \{ A \subseteq X \mid \forall a, b \in A, [a, b] \subseteq A \} \); obviously, \( \mathcal{R}_{[\cdot, \cdot, \cdot]} \) is mapped to \([\cdot, \cdot, \cdot]\) via \( \mathcal{B} \to \mathcal{R} \). We claim that for any such \((X, [\cdot, \cdot, \cdot]), (X, \mathcal{R}_{[\cdot, \cdot, \cdot]}), id_X)\) is a universal arrow from \( \mathcal{R}_{[\cdot, \cdot, \cdot]} \) to \( \mathcal{B} \to \mathcal{R} \). In other words, we claim that for any \((Y, \mathcal{S})\) with monotone \( f : (X, [\cdot, \cdot, \cdot]) \to (Y, [\cdot, \cdot, \cdot]) \), the function \( f : (X, \mathcal{R}_{[\cdot, \cdot, \cdot]}) \to (Y, \mathcal{S}) \) is a road morphism. Indeed, let \( S \in \mathcal{S} \) and pick any \( a, b \in f^{-1}(S) \). By monotonicity of \( f \) and Lemma 14 \([a, b] \subseteq f^{-1}(S) \). \( \square \)

By construction, the functor \( \mathcal{B} \to \mathcal{R} \) and its left adjoint, denote them temporarily by \( G \) and \( F \) respectively, generate an equivalence of categories \( G : F(\mathcal{R}) \cong \mathcal{R} : F \) and thus \( F(\mathcal{R}) \) becomes a reflective subcategory of \( \mathcal{B} \). The forgetful functor \( \mathcal{B} \to \mathcal{Set} \) is only right adjoint: the assignment \( X \mapsto (X, \mathcal{P}(X)) \) defines the left adjoint and it cannot be left adjoint since \( \mathcal{R} \to \mathcal{B} \) preserves colimits (it’s left adjoint to \( \mathcal{B} \to \mathcal{R} \)) but the forgetful functor \( \mathcal{R} \to \mathcal{Set} \) does not. Since \( \mathcal{B} \to \mathcal{Set} \) is right adjoint then for any \((X, \mathcal{R})\) consider some \( Y \subseteq X \). As one might suspect, the obvious road system on \( Y \) making the inclusion a morphism is \( \mathcal{R}_Y = \{ Y \cap R \mid R \in \mathcal{R} \} \) (i.e., the restriction of \( \mathcal{R} \) to \( Y \)). It is simple to verify that \((Y \mapsto X, (Y, \mathcal{R}_Y))\) represents the equaliser. For a collection \((X_i, \mathcal{R}_i)\) of road systems the smallest road system on \( X = \bigsqcup X_i \) making all projections \( \pi_i : X \to X_i \) morphisms is \( \mathcal{R} = \{ \bigcap_i \pi_i^{-1}(R) \mid R \in \mathcal{R}_i \text{ for all } i \in I \} \). Similarly, for a collection \((X_i, \mathcal{R}_i)\) of road systems the largest road system on \( X = \bigsqcup X_i \) making all injections \( i_i : X_i \to X \) morphisms is \( \mathcal{R} = \{ \bigcup_i i_i(R) \mid R \in \mathcal{R}_i \text{ for all } i \in I \} \).

**Remark 17.** None of the above constructions in \( \mathcal{B} \) is unique. For instance, when constructing products one can choose \((X, \mathcal{S})\) so that \( \mathcal{S} \) is the join-closure of \( \mathcal{R} \). In other words, the reflection of \((X, \mathcal{R})\) under the adjunction \( \mathcal{B} \cong \mathcal{R} \). Something similar occurs with equalisers. As for coproducts, the choice of \((X, \mathcal{R})\) is far from unique. Consider \( \mathcal{S} \) to be the meet-closure.
of $\mathcal{R}$. Since inverse images of roads must be equal to intersections of roads then all injections are $\mathcal{B}$-morphisms. It is not difficult to prove that had the morphisms in $\mathcal{B}$ been option (c) (found in the introductory paragraph), these constructions would have been unique.

Coequalisers are significantly more involved and since $\mathcal{R}$ is complete and $\mathcal{B} \to \mathcal{R}$ preserves colimits, the underlying sets of colimits in $\mathcal{B}$ in many cases will differ from those in $\text{Set}$. Given a road system $\mathcal{R}$ on a set $X$ and a partition $P$ on $X$ a family $F \subset \mathcal{R}_\cap$ for which

- $\cup F = X$,
- $\forall p, q \in F$ we have $p \cap q = \emptyset$, and
- $\forall p \in F$, $p = \bigcup_{i \in I} r_i$ for some collection $\{r_i\} \subseteq P$

will be referred to as an $\mathcal{R}_P$-partition of $X$. The collection of all such partitions, denoted by $\mathcal{R}_P$, can be partially ordered as follows: for $F, G \in \mathcal{R}_P$ we let $F \leq G$ and say that $F$ refines $G$ if, and only if, $\forall p \in F$ we can find a $q \in G$ so that $p \subseteq q$. This way, $\{X\}$ becomes the coarsest partition in $(\mathcal{R}_P, \leq)$. Next we show that $(\mathcal{R}_P, \leq)$ is closed under meets. Since any part of a partition can be uniquely identified by any one of the elements it contains, for any $a \in X$ and $F \in \mathcal{R}_P$ define $F_a \in F$ to be the one that contains $a$. This is then enough to prove the existence of meets in $(\mathcal{R}_P, \leq)$. Indeed, for any collection of $G_i \in \mathcal{R}_P$, with $i \in I$, their meet $G \in \mathcal{R}_P$ is the one for which $G_a := \bigcap (G_i)_a$ for each $a \in X$. To see this, first notice that for each $a \in X$, $G_a \in \mathcal{R}$ and defining $G := \{G_a \mid a \in X\}$ yields that $\cup G = X$. If $G_a \neq G_b$, then for some $i \in I$ we have $(G_i)_a \neq (G_i)_b$ (i.e., $(G_i)_a \cap (G_i)_b = \emptyset$) and, by construction, $G_a \cap G_b = \emptyset$. Lastly, for each $a \in X$ if $b \in G_a$, then $b \in (G_i)_a$ for each $i \in I$ and, by design, $P_b \subseteq (G_i)_a$ for each $i \in I$. Thus, $P_b \subseteq G_a$ and $G \in \mathcal{R}_P$. Obviously, $G \leq G_i$, for each $i$ and $G$ is their meet. Given an equivalence relation $\sim \in \text{Eq}(X)$ (i.e., a particular partition $P$) the above constructed set $G$ will represent the underlying set of the coequaliser for the pair $((X, \mathcal{R}), \sim)$. The road system $\mathcal{R}_G$ on $G$ will be the obvious one: $A \in \mathcal{R}_G$ if, and only if, $p^{-1}(A) \in \mathcal{R}_\cap$ (this is but one, of several, coequalisers for the given pair; only saying this since I’ll use this fact to show that $F$ does not create colimits).

Remark 18. The object for the above equaliser is not unique. One can just as well define $\mathcal{R}_Y = \{R \cap Y \mid R \in \mathcal{R}_\cap\}$ and the identity on $Y$ is the unique isomorphism between the two equalisers. Both objects are mapped to the same coequaliser in $\mathcal{R}$ and thus $F$ does not create limits.

Lemma 19. The pair $((G, \mathcal{R}_G), p : X \to G)$ is the coequaliser of $((X, \mathcal{R}), \sim)$.

Proof. Since the quotient function $p$ is clearly a morphism, let $(Y, \mathcal{S})$ with morphism $f : X \to Y$. We first must show that the obvious assignment $G_a \mapsto f(a)$ is a well-defined functions. This shouldn’t be too hard: $f^{-1}(Y)$ generates a partition $P$ of $X$ (based on its fibers) so that for any $a \in X$, $[a] \subseteq p$ for some $p \in P$. Also, it’s easy to verify that any $p \in P$ is a union of equivalence classes from $\sim$ and that each such $p$ also belongs to $\mathcal{R}_\cap$ (this follows $f$ being a morphism). Since $G$ is the finest of all such partitions then $G_a \mapsto f(a)$ is well-defined;
denote such function as \( h \). Next, let \( S \in \mathcal{S} \). It follows that \( h^{-1}(S) \in \mathcal{R}_G \) since \( f^{-1}(S) \in \mathcal{R}_G \) and \( G \) is finer than \( P \).

\[ \square \]

Even though coequalisers in \( \mathbf{B} \) exist, the functor \( \mathbf{B} \to \mathbf{R} \), in general, neither lifts nor preserves limits.

**Example 20.** Let \( X = \{a, b, c, d\} \) with \( \mathcal{R} = \{\{x, y\} \subset X\} \cup \{X, \emptyset\} \) and \( \sim = \{\{a, b, c\}, \{d\}\} \). Let \( q : X \to X/\sim \) and notice that the coequaliser of the above pair is then a singleton: since \( q^{-1}(\{a, b, c\}) \) is not an intersection of roads from \( \mathcal{R} \), \( d \) must be glued to \( \{a, b, c\} \). As for \( (X, [\cdot, \cdot, \cdot]_\mathcal{R}) \) with \( \sim \), the underlying set of this coequaliser is \( \{\{a, b, c\}, \{d\}\} \).

**Lemma 21.** The continuous functor \( \mathbf{B} \to \mathbf{R} \) preserves all coproducts while its left adjoint is continuous and cocontinuous.

**Proof.** This claim follows directly from the constructions of limits and colimits in \( \mathbf{B} \) and the adjunction \( \mathbf{B} \dashv \mathbf{R} \).

\[ \square \]

4. **Conclusions**

As a young research topic the possibilities for further research are overarching rather than specific. For example, consider the following first-order axioms

(i) **Antisymmetry:** the property that if \( c \in [a, b] \) and \( b \in [a, c] \) then \( c = b \).

(ii) **Disjunctivity:** the property that if \( x \in [a, b] \) then for all \( c, x \in [a, c] \) or \( [c, x, b] \).

(iii) **Slenderness:** the property that if \( c \in [a, b] \), then \( [a, c] \cap [c, b] = \{c\} \).

(iv) **Reciprocity:** the property that if \( c, d \in [a, b] \) and \( c \in [a, d] \), then \( d \in [c, b] \).

(v) **Uniqueness of centroids:** the property that the set \( [abc] := [a, b] \cap [b, c] \cap [a, c] \) - the centroids of \( \{a, b, c\} \) - has at most one element.

(vi) **Uniqueness of bracket point sets:** the property that if \( [a, b] = [c, d] \), then \( \{a, b\} = \{c, d\} \).

Each of the above axioms has a road system equivalent and it is known that axioms (iii)-(vi) are equivalent to antisymmetry under disjunctivity [2] but are completely unexplored categorically. In the sequel, we explore the above as closures of \( \mathcal{R} \)-relations and their relationship to their equivalent for road systems.

Links between road morphisms and preservation of connectedness in Hausdorff continua remains to be explored. There is strong evidence to suggest this topic for further research: monotone functions are the right type of morphisms between \( \mathcal{R} \)-relation generated by the C, Q and K-interpretations of betweenness [4] (i.e. those generated from connected and continuumwise connected topological spaces).
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