BUILDING STRING FIELD THEORY AROUND
NON-CONFORMAL BACKGROUNDS

Barton Zwiebach*

Center for Theoretical Physics,
Laboratory of Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

ABSTRACT

The main limitations of string field theory arise because its present formulation requires a background representing a classical solution, a background defined by a strictly conformally invariant theory. Here we sketch a construction for a gauge-invariant string field action around non-conformal backgrounds. The construction makes no reference to any conformal theory. Its two-dimensional field-theoretic aspect is based on a generalized BRST operator satisfying a set of Weyl descent equations. Its geometric aspect uses a complex of moduli spaces of two-dimensional Riemannian manifolds having ordinary punctures, and organized by the number of special punctures which goes from zero to infinity. In this complex there is a Batalin-Vilkovisky algebra that includes naturally the operator which adds one special puncture. We obtain a classical field equation that appears to relax the condition of conformal invariance usually taken to define classical string backgrounds.

* E-mail address: zwiebach@irene.mit.edu
Supported in part by D.O.E. cooperative agreement DE-FC02-94ER40818.
1. Introduction and Summary

Light-cone string field theory, developed in the early seventies, was explicitly constructed around Minkowski spacetime background \([1]\). Its framework, however, can be used to build string theory around spacetime backgrounds which include at least two flat dimensions, one of which is time. Covariant string field theory, developed from the middle eighties to the early nineties, removed this restriction. String field theory can now be formulated around general conformal backgrounds, backgrounds encoded by strictly conformal invariant field theories (CFT’s) and representing rather arbitrary classical solutions of string theory. Even though this formulation does not give an \textit{a priori} characterization of classical solutions, CFT’s are clearly classical solutions because the corresponding string field action is quadratic in the fluctuating field. In passing from light-cone to covariant string field theory much was learned about the properties and significance of the BRST operator, the construction of covers of moduli spaces of Riemann surfaces, and the role of Batalin-Vilkovisky algebras. The next logical step would be to achieve a construction of string field theory around completely general backgrounds, backgrounds that need not represent classical solutions. The resulting string field theory would have terms linear in the fluctuating field. It is the purpose of the present work to begin such construction.

Indeed the main limitations of present day string field theory are not due to the use of a ‘background’ to write the theory. The limitations largely arise because the background must correspond to a classical solution. Ideally string field theory should help us find classical solutions, but finding classical solutions using a reference classical solution is not very easy. Moreover, it is not manifest that string field theories formulated using different conformal backgrounds are really the same theory. This is the question of background independence, a property that was proven explicitly in Refs.\([2,3]\). One may wonder if a fully satisfactory string action should be written without using a background at all, but comparison with Einstein’s theory, a familiar and natural analog, suggests otherwise. In gravity, backgrounds play the role of off-shell fields and represent arbitrary metrics on a manifold. The action is a function in the space of backgrounds, a function that is stationary at the backgrounds representing classical solutions. Background independence for Einstein’s theory is simply the fact that the space of backgrounds (the space of metrics) can be described without using a special background (a special metric). It is natural to wonder what is the space of string backgrounds. Present day string field theory provides a partial answer: the space of string backgrounds in the neighborhood of a conformal background is the vector space of all local operators of the
conformal theory.\footnote{Restricted to operators annihilated by $L_0^-$ and $b_0^-$, in the case of closed strings.} It has not proven easy to give a description of the space of backgrounds without using a reference conformal background, but the expected answer has been that this space is something closely related to the space of two-dimensional field theories.

Indeed, we say something closely related to the space of two-dimensional field theories, since subtle complications abound. The field theories we must consider must have all possible couplings, including non-renormalizable ones. This may not be a serious difficulty, if all couplings are present non-renormalizability is really not an issue, the space of theories is just the space of renormalized couplings, a space of infinite dimension. Another issue, possibly more relevant \[4\], is the recent finding that the space of string backgrounds have parameters that are not present in the space of quantum field theories. This is the case of the string coupling. The string coupling arises from the expectation value of the string dilaton and is a parameter of string backgrounds but it is not a parameter of the conformal theories (due to the ghost anomaly). This suggests that in some sense the space of backgrounds is a little more akin to the space of possible two-dimensional Lagrangians.

In this paper the ingredient we will use for the construction of the string action is a two-dimensional quantum field theory, a non-conformal one. How close it is to a general two-dimensional quantum field theory is something that one may learn by further study of the construction to be given here. The string action will be written as a function on the state space of this two-dimensional quantum field theory. This state space represents the tangent space to the space of backgrounds at the particular background encoded by the field theory in question. As such, the state space can only be expected to give a local description of the space of backgrounds. Ideally we would like to have the string action defined directly on the space of backgrounds. Strictly speaking, the construction to be presented here still has a residual background dependence since the string action is not written directly on the space of backgrounds but rather on its tangent space at some background: the identification of the tangent space with the space itself can be background dependent. Thus given two backgrounds, the string actions written in the two different tangent spaces may not be manifestly the same. It seems very plausible that a background independence analysis would enable one to rewrite the action to be built in this paper as a function on the space of backgrounds.

There are two pieces of indirect evidence that suggest that the above construction of a string field theory around non-conformal backgrounds should be possible. First, by giving an arbitrary
expectation value to the string field of a SFT formulated around a conformal background we obtain a well-defined string action which is not stationary in the new fluctuating field [5]. This action is gauge invariant and represents string theory around a background which is not a classical solution. The reason this action is not the desired answer is that the new background is described explicitly in terms of the original conformal background. The second piece of evidence comes from the sigma model approach to string theory. In this approach given fairly arbitrary two-dimensional field theories one can define beta functions whose vanishing appear to represent string field equations. In this approach one does not assume conformal invariance. The sigma model method suffers from several technical complications: it is very difficult to deal with backgrounds that correspond to massive string fields, the definition of the list of background fields is not systematic and, there is no prescription to find the action, nor its gauge invariances.

In the standard formulation of string field theory gauge transformations are defined by the BRST operator, and gauge invariance is a consequence of the nilpotency of this operator. It should be emphasized that string field theory formulated around non-conformal backgrounds must still be gauge invariant, despite the fact that one will not be able to have a BRST operator that squares to zero. Earlier work trying to write a string action as a function in the space of two dimensional field theories has been mostly inconclusive. The functions that were written do not show any clear evidence of string field gauge invariances. While the condition to have a conformal background is the vanishing of the trace $T \equiv T_{\mu}^{\mu}$ of the stress tensor, it is not clear that the equation of motion that selects a classical string background is simply $T = 0$. Such equation of motion need not have string field gauge invariance. In fact we will find that the equation of motion of string theory appears to be non polynomial in $T$, with leading term linear in $T$. Vanishing $T$ would always be a solution, but there might exist solutions with non zero $T$, such solutions would represent classical string backgrounds that are not conformal.

The construction of string field theory around a non conformal background will have two ingredients, one from two-dimensional geometry and the other from two-dimensional field theory. Both ingredients were also present in the case of conformal backgrounds but as we will see there are major departures. Let us first consider the two dimensional geometry ingredient.

---

* With the notable exception of Ref.[6] which addressed the case of open strings using the Batalin-Vilkovisky framework to ensure gauge invariance from the start. Difficulties with this approach have been studied in Ref.[7].
Moduli spaces of two-dimensional surfaces. In the conformal case we found string vertices, collectively denoted by $\mathcal{V}$ which represented pieces of the moduli spaces of Riemann surfaces with punctures and local coordinates at the punctures. The regions of moduli spaces corresponding to the string vertices, and the local coordinates at the punctures had to be chosen carefully in order that the string vertices satisfy a set of recursion relations involving sewing. Such choices were made using an auxiliary problem on Riemann surfaces, that of finding a metric (a Weyl factor) of least area under the condition that all nontrivial closed curves be longer than or equal to $2\pi$. The local coordinates at the punctures could be defined using the behavior of the metric near the punctures. While the metric played a purely auxiliary role in the conformal case, in the non conformal case it is necessary since the two-dimensional field theory is not conformal and correlation functions will depend on the chosen metric. Since the recursion relations hold when the string vertices are equipped with metrics and local coordinates, they can be used in the non conformal case. It will not be possible, however, to attain gauge invariance using only the string vertices. Apart from two-dimensional quantum field theory complications, new moduli spaces of Riemann surfaces equipped with Weyl metrics are needed.

The need arises from the algebraic structure of the theory to be built. Such structure was explored indirectly Ref.[5] sect 4.5. The algebraic structure of standard closed string field theory is based on a homotopy Lie algebra defined by string field products $m_n$ where $n \geq 1$ is the number of string fields to be multiplied. The lowest product $m_1$ is simply action by the BRST operator of the conformal theory. For the present case the homotopy Lie algebra must have a product $m_0$. Such product simply represents a particular (fixed) string field, a grassmann odd, ghost-number +3 string field to be denoted as $F$. The surfaces in the new moduli spaces that we need will have special punctures where the string field $F$ will be inserted. In addition the surfaces will also have ordinary punctures where the dynamical string fields are inserted. The $\mathcal{B}^1$ spaces of background independence [3], now equipped with Weyl metrics, have one special puncture. The special string field $F$ will be inserted at this special puncture, and the dynamical string field is inserted at the ordinary punctures. The resulting function of the string field will define the part of the string action linear in $F$. The identities satisfied by the $\mathcal{B}^1$ spaces that guaranteed background independence in [3] are precisely the identities that guarantee gauge invariance in the new construction. With this linear term included gauge invariance will hold to $O(F)$, but not to quadratic order. Moduli spaces $\mathcal{B}^2$ of surfaces with two special punctures are required. Such kind of moduli spaces were studied in detail in [8] where they were seen to arise from second order background independence
conditions. Again, the identities required for background independence are essentially the same identities that are required to attain gauge invariance in the new construction. The $B^2$ spaces with $F$'s inserted in each of the two special punctures defines the part of the string action quadratic in $F$, and gauge invariance now holds to $O(F^2)$. To achieve full gauge invariance we introduce moduli spaces $B^3, B^4, \cdots$ with all numbers of special punctures. While we work with such spaces directly, we will sketch how such spaces also arise from higher order background independence conditions. We thus see that the geometrical ingredient necessary to build a string field theory around a non-conformal background was encoded in the potentially infinite set of background independence conditions that arise from a string field theory built around a background encoded by a CFT that sits on a CFT space.

We explain that all $B$ spaces $B^1, B^2, \cdots$ should be thought as string vertices, with the standard string vertices $V$, which have no special punctures, identified as the space $B^0$. The complete collection of all string vertices can be formally put together into a single space $B \equiv B^0 + B^1 + B^2 + \cdots$. This element $B$ is an element of the complex which includes the formal sum of moduli spaces of Riemann spheres with Weyl metrics with all possible numbers of ordinary and special punctures. We write $B = \sum_{k,p} B^k_p$ where $B^k_p$ denotes a moduli space of spheres with $k$ special punctures and $p$ ordinary punctures. In this complex we have an antibracket operation, which corresponds to the sewing of ordinary punctures of the corresponding surfaces, and an operator $K$ that acting on a moduli space of surfaces it adds one special puncture to the each of the surfaces [8]. The operator $K$ satisfies the identity $K^2 = 0$, and this is related to the fact that any space in the complex is defined to be antisymmetric under the exchange of the labels on any two of the special punctures. The generalized string vertex $B$ is shown to satisfy an extremely simple relation: $\partial B - KB + \frac{1}{2}\{B, B\} - V_3' = 0$. If expanded in terms of the number of special punctures this equation gives all the recursion relations satisfied by the familiar string vertices $V$ and all the $B^k$ spaces. This equation explains in a simple way why the action we build satisfies the Batalin-Vilkovisky master equation.

It is important to note that the label $k$ in $B^k_p$ not only gives the number of special punctures on the surfaces in the moduli space $B^k_p$ but also specifies the dimension of $B^k_p$. We have: $\text{dim } (B^k_p) = k + \text{dim } (V_{k+p})$, namely, for each special puncture the moduli space gains one real dimension above that of the standard string vertex with the same total number of punctures. This is in accord with ghost number conservation, given that $F$ has ghost number three, and

* In this paper we only consider the classical closed string theory and thus only spheres are relevant. The extension to quantum closed strings is not expected to be problematic.
that ordinary string fields giving a nonzero contribution to an ordinary string vertex must have an average ghost number of two. One can therefore think of $B^k_p$ as a space fibered over $M_{k+p}$ where $\pi : B^k_p \rightarrow M_{k+p}$ is a projection that forgets about the Weyl metric on the surface and the choice of local coordinates at the punctures. For every complex structure $\Sigma \in \pi^{-1}(\Delta)$ one has a compact $k$ dimensional space $\pi^{-1}(\Sigma) \in B^k_p$ with the same complex structure. In the conformal context the fibers simply represented different local coordinates at the punctures. Integration over $\mathcal{B}$ spaces includes integration over fibers, and in the present case this is properly thought as integration over Weyl metrics! This represents a surprising realization of an intuitive expectation. If we start with non conformal theories one would expect that integration over metrics would include integration over conformal structures and Weyl metrics. This could not have been strictly true, however. A complete integration over Weyl metrics would very likely give rise to infinities, since the space of inequivalent Weyl metrics is infinite dimensional and non-compact. The standard string vertices have no Weyl integration, but the $B^k_p$ spaces do. Since they are compact finite dimensional spaces they give rise to limited integration over the space of Weyl metrics. As the number of special punctures increases one is integrating over compact subspaces of the space of Weyl metrics of larger and larger dimensionality.

Two dimensional field theory ingredient. The main question here is what is the string field $F$ and what is the proper replacement of the BRST operator, which for the case of conformal backgrounds defines the linearized gauge transformations of the theory. In the conformal context, gauge invariance, to first order, follows from the fact that the BRST operator squares to zero. For two-dimensional non-conformal field theories we do not expect to have a BRST charge that is conserved nor we expect this charge to square to zero.

There is a way to motivate the type of identities that must be satisfied by the generalized BRST charge $Q$ and the string field $F$. The type of identities required for gauge invariance away from conformal backgrounds were given in Ref.[5] sect.4.5, and the first few read

$$Q|\mathcal{F}\rangle = 0,$$

(1.1) $$Q^2|A\rangle = -[\mathcal{F}, A],$$

(1.2) $$Q[A_1, A_2] + [QA_1, A_2] + (-)^{A_1}[A_1, QA_2] = -[\mathcal{F}, A_1, A_2],$$

(1.3) We have used calligraphic symbols to denote $Q$ and $\mathcal{F}$ because these are not the same objects as $Q$ and $F$. We need $Q$ and $\mathcal{F}$ but it seems unlikely that one can give a simple construction
of them starting from a two-dimensional quantum field theory. Indeed, $\mathcal{F} = 0$ is the condition that selects a classical string background, and such equation is probably fairly intricate. Our strategy will be to construct $Q$ and $F$ from simpler objects $\tilde{Q}$ and $\tilde{F}$ whose existence we will postulate. The object $\tilde{Q}$ will correspond to a non-conserved charge and therefore will be contour dependent and denoted as $Q(\gamma)$. We demand that

$$\lim_{r \to 0} Q(\gamma_r) F(0) = 0,$$

$$[Q(\gamma)]^2 = \frac{1}{2\pi i} \int_{\gamma} F^{[1]},$$

$$Q(\gamma_2) - Q(\gamma_1) = -\frac{1}{2\pi i} \int_{M} F^{[2]}.$$  

(1.4)

We claim that these equations define a natural extension of the conformal field theory BRST operator to the non conformal case. The special ghost number three local operator $F(z, \bar{z})$, vanishing in the conformal case, controls the violation of all the standard BRST properties. The first equation says that as we shrink the contour around $F$ the BRST operator will give a result that goes to zero. This is the closest analog of (1.1) given that the BRST operator is not conserved. The next identity equates the failure of $Q(\gamma)$ to square to zero to the line integral of the one-form $F^{[1]}(z, \bar{z})$ associated to $F(z, \bar{z})$. Finally the failure of $Q$ to be conserved is measured by the variation of $Q$ as we change the defining contours. For two homologous contours $\gamma_1$ and $\gamma_2$, namely two contours bounding a cylindrical region $M$: $\partial M = \gamma_2 - \gamma_1$ we find that the change in $Q$ is given by an integral of the two-form $F^{[2]}(z, \bar{z})$ associated to $F$.

We will not attempt in the present paper to give an explicit construction of the operators $Q$ and $F$ in terms of two-dimensional field theory data. This will be the subject of a forthcoming publication [9]. The BRST operator will be built in terms of line integrals involving ghost fields and the stress tensor of the theory (which is not traceless). The two form $F^{[2]}$ appears to be equal to the divergence of the ghost field times the trace $T$ of the stress tensor. Apart from states that have nontrivial ghost dependence, the operator $F$ is also proportional to the trace $T$ of the stress tensor. The ghost-sector of the two-dimensional field theory must be defined so that the BRST operator acts properly on forms on the moduli space of Riemann surfaces equipped with Weyl metrics.

The relation between $Q$ and $F$ on one hand and $\tilde{Q}$ and $\tilde{F}$ on the other is through the family of $B$ spaces discussed earlier. In fact $\tilde{Q}$ can be identified as the operator appearing in the part
of the string action quadratic on the string field and $\mathcal{F}$ as the string field appearing in the part of the action linear on the string field. In the case of $\mathcal{F}$ it turns out that it equals $F$ plus contributions from $\mathcal{B}$ spaces with one ordinary puncture and two or more special punctures, namely $\mathcal{B}^k_1$ spaces with $k \geq 2$. The complete field equation reads:

$$|\mathcal{F}\rangle_1' \equiv |F\rangle_1' + \sum_{k \geq 2} \frac{1}{k!} \int_{\mathcal{B}^k_1} \langle \Omega_{11...k}^{|F\rangle_1'} |F\rangle_1 \cdots |F\rangle_k |S_{11'}\rangle = 0.$$  \hspace{1cm} (1.5)

This is a non-polynomial equation that starts with a term linear in $|F\rangle$ and by construction $|F\rangle = 0$ is a solution. As we mentioned earlier there may be nontrivial solutions of this equation. Such solutions, having $|F\rangle \neq 0$, but $|F\rangle = 0$ would represent classical string backgrounds that do not correspond to conformal field theories.

It is worthwhile to point out briefly the parallel with Einstein’s theory. The utility of Einstein’s equations derives mostly from the fact that they can be written without committing one-self to a reference classical background. The equations of motion for the fluctuating field around flat space would be of little help in finding nontrivial classical solutions. The Einstein action, expanded around an arbitrary background metric $\bar{g}_{\alpha\beta}$ that does not satisfy the field equations reads

$$S(g) = -\frac{1}{\kappa^2} \int \sqrt{\bar{g}} R d^4x + \frac{1}{\kappa^2} \int d^4x \sqrt{\bar{g}} \left[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right] h_{\mu\nu} + O(h^2), \hspace{1cm} (1.6)$$

where $h_{\mu\nu}$ is the fluctuating field. The advantage of expanding around an arbitrary metric is clear. Multiplying the fluctuating field is the fully geometrical Einstein field equation: $\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R} = 0$. Our string analog for this equation is $|\mathcal{F}(F)\rangle = 0$, as shown in (1.5).
Gauge invariance of \((1.6)\) under \(\delta_\epsilon h_{\mu\nu} = \overline{\nabla}_\mu \epsilon_\nu + \overline{\nabla}_\nu \epsilon_\mu\) holds on account of the Bianchi identity \(\overline{\nabla}_\nu \left[ R^\mu_{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R} \right] = 0\). The string field theory analog of the gravitational Bianchi identity is the equation \(Q|\mathcal{F}\rangle = 0\). The field independent term in the expansion \((1.6)\) is simply the Einstein action evaluated at the background metric. It is for this reason that it should be of interest to understand the \(B_k^0\) spaces. These spaces give the string field-independent contributions to the string field action (see conclusions, for further comments).

Organization of this paper In section 2 we first review a few of the results of Ref.[8] and then discuss the systematics of higher order background independence conditions, showing how they arise from the requirement of vanishing antibracket cohomology. In section 3 we explain the identities that should be satisfied by the generalized BRST operator \(Q\) and the special string field \(F\). We explore the consistency of these equations, and discuss \(Q\) action on moduli spaces whose surfaces have both ordinary and special punctures. In section 4 we begin the construction of the new gauge invariant string action as a power expansion in the number of \(F\) insertions. We discuss explicitly the terms with zero, one and two insertions. Such terms use the string vertices \(V\), the \(B^1\) spaces and \(B^2\) spaces. In section 6 we complete the construction by working to all orders in the number of \(F\) insertions. We offer comments and spell out some open questions in section 7.

2. Review and Developments

In this section we review some of the technology developed in [8] dealing with moduli spaces of surfaces with ordinary and special punctures. While the cases of one or two special punctures were the subject of main attention in previous works, in the present paper the number of special punctures will be arbitrary. We define in this general context the homomorphism from surfaces to functions. Finally, we explain how relevant \(B\) spaces with more than two punctures arise from higher order background independence consistency conditions.

2.1 Properties of String Vertices and \(B\) spaces

The moduli spaces of surfaces we must deal with are moduli spaces of Riemann surfaces with labelled punctures. The punctures can be of two types, ordinary punctures, where the dynamical string field is inserted, and special punctures where special states are inserted. The punctures have analytic local coordinates defined around them, and the moduli spaces are symmetric under the exchange of labels of any two ordinary punctures, and antisymmetric
under the exchange of labels of any two special punctures. For the purposes of the present paper one must also define a Weyl metric (or conformal factor $\rho$ with $ds = \rho |dz|$) on every surface of each moduli space. The above moduli spaces, called $\mathcal{B}$ spaces, are indexed by the number of special punctures, sometimes indicated as a superscript. The Batalin Vilkovisky algebra of surfaces with ordinary punctures [3], was extended to this more general case in Ref.[8]. The main results are summarized below.

The antibracket of $\mathcal{B}$ spaces satisfies the following identities [8]:

\[
\{ \mathcal{B}_1, \mathcal{B}_2 \} = - (-)^{(\mathcal{B}_1 + \bar{n}_1 + 1)(\mathcal{B}_2 + \bar{n}_2 + 1)} \{ \mathcal{B}_2, \mathcal{B}_1 \},
\]

\[
\partial \{ \mathcal{B}_1, \mathcal{B}_2 \} = \{ \partial \mathcal{B}_1, \mathcal{B}_2 \} + (-)^{\mathcal{B}_1 + \bar{n}_1 + 1} \{ \mathcal{B}_1, \partial \mathcal{B}_2 \},
\]

\[
0 = (-)^{(\mathcal{B}_1 + \bar{n}_1 + 1)(\mathcal{B}_3 + \bar{n}_3 + 1)} \{ \{ \mathcal{B}_1, \mathcal{B}_2 \}, \mathcal{B}_3 \} + \text{Cycl}.
\]

These identities work as if $\mathcal{B}$ spaces with $\bar{n}$ special punctures had an effective dimensionality equal to $\text{dim}(\mathcal{B}) + \bar{n}$. The operators $\mathcal{K}$ and $\mathcal{I}$ satisfy the relations

\[
\mathcal{K} (\{ \mathcal{B}_1, \mathcal{B}_2 \}) = (-)^{\mathcal{B}_2 + \bar{n}_2 + 1} \{ \mathcal{K} \mathcal{B}_1, \mathcal{B}_2 \} + \{ \mathcal{B}_1, \mathcal{K} \mathcal{B}_2 \},
\]

\[
[\partial, \mathcal{K}] \mathcal{B} = (-)^{\mathcal{B} + \bar{n}} \{ \mathcal{V}_3', \mathcal{B} \},
\]

\[
\mathcal{K} \mathcal{K} = 0,
\]

and,

\[
\mathcal{I} (\{ \mathcal{B}_1, \mathcal{B}_2 \}) = (-)^{\mathcal{B}_2 + \bar{n}_2 + 1} \{ \mathcal{I} \mathcal{B}_1, \mathcal{B}_2 \} + \{ \mathcal{B}_1, \mathcal{I} \mathcal{B}_2 \},
\]

\[
[\partial, \mathcal{I}] = 0,
\]

\[
\mathcal{I} \mathcal{I} = 0.
\]

The anticommutator of $\mathcal{K}$ and $\mathcal{I}$ is given by

\[
\mathcal{K} \mathcal{I} + \mathcal{I} \mathcal{K} = \{ \mathcal{T}_1^2 \},
\]

where $\mathcal{T}_1^2 = \tau_1^2(0) - P \tau_1^2(0)$. Here $P$ is the operator that exchanges the labels of the special punctures, and $\tau_1^2(0)$ is a collection of three punctured spheres $\tau_1^2(0) = \left\{ S(t) \mid t \in [0,1] \right\}$ (see Figure 1, and Ref.[8] for further details). This space satisfies

\[
\partial \mathcal{T}_1^2 = \mathcal{I} \mathcal{V}_3'.
\]
Figure 2. The one dimensional moduli space of three puncture spheres that defines the space \( \mathcal{T}_1^2 \) having two special punctures and one ordinary puncture. The first space is \( \tau_1^2(0) \) and the second one is \( P\tau_1^2(0) \), the same space with the labels for the special punctures exchanged. One of the special punctures travels from the point \( w = 0 \), where it coincides with the other special puncture, up to the point \( w = 1 \).

Defining \( \mathcal{M} = \mathcal{K} - \mathcal{I} \) one has the following identities

\[
\mathcal{M} \left( \{ B_1, B_2 \} \right) = (-)^{B_2 + \hat{r}_2 + 1} \{ MB_1, B_2 \} + \{ B_1, MB_2 \},
\]

\[
[\partial, \mathcal{M}] B = (-)^{B + \hat{r}} \{ V_3', B \},
\]

\[
\mathcal{M} \mathcal{M} = - \{ , \mathcal{T}_1^2 \}. \tag{2.6}
\]

\( \mathcal{V}, B^1, \) and \( B^2 \) Spaces. At genus zero, the string vertices \( \mathcal{V} \) include moduli spaces of punctured spheres with three or more punctures. They satisfy the identities

\[
\partial \mathcal{V} + \frac{1}{2} \{ \mathcal{V}, \mathcal{V} \} = 0. \tag{2.7}
\]

\[
\delta_\mathcal{V}^2 = 0, \quad \text{with} \quad \delta_\mathcal{V} \equiv \partial + \{ \mathcal{V}, \}
\]

\[
\delta_\mathcal{V} \mathcal{V} = \frac{1}{2} \{ \mathcal{V}, \mathcal{V} \}. \tag{2.8}
\]

As defined in \([3]\), genus zero \( B^1 \) spaces begin with a moduli space of three punctured spheres, one of which is a special puncture. The moduli spaces \( B^2 \) at genus zero also begin with a moduli space of three punctured spheres, but this time two of the punctures are special. Their recursion relations take the form \([3,8]\):

\[
\delta_\mathcal{V} B^1 = \mathcal{V}_3' + \mathcal{M} \mathcal{V}, \quad \delta_\mathcal{V} B^2 = \mathcal{T}_1^2 + \mathcal{M} B^1 - \frac{1}{2} \{ B^1, B^1 \}. \tag{2.10}
\]

In proving the consistency of the above equations one uses the following equalities, valid for
arbitrary $\mathcal{B}$ spaces

\[
[\delta_\mathcal{V}, \mathcal{K}]\mathcal{B} = (-)^{\mathcal{B}+\bar{n}} \{ \mathcal{V}_{\mathcal{B}} + \mathcal{K} \mathcal{V}, \mathcal{B} \}, \\
[\delta_\mathcal{V}, \mathcal{I}]\mathcal{B} = (-)^{\mathcal{B}+\bar{n}} \{ \mathcal{I} \mathcal{V}, \mathcal{B} \}, \\
[\delta_\mathcal{V}, \mathcal{M}]\mathcal{B} = (-)^{\mathcal{B}+\bar{n}} \{ \mathcal{V}_{\mathcal{M}} + \mathcal{M} \mathcal{V}, \mathcal{B} \}.
\]  

(2.11)

The spaces $\mathcal{V}, \mathcal{B}^1, \mathcal{B}^2$, and the higher spaces $\mathcal{B}^k$ we will introduce later have something in common. Their dimensionality exceeds that of the corresponding moduli space of Riemann surfaces by the number of special punctures. As a consequence they are effectively Grassmann even. They will all be thought eventually as string vertices. It follows from the results listed earlier that they satisfy the following identities

\[
\{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \} = \{ \mathcal{B}^{k_2}, \mathcal{B}^{k_1} \},
\]

(2.12)

\[
\mathcal{M}\{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \} = -\{ \mathcal{M}\mathcal{B}^{k_1}, \mathcal{B}^{k_2} \} + \{ \mathcal{B}^{k_1}, \mathcal{M}\mathcal{B}^{k_2} \},
\]

(2.13)

\[
\left\{ \{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \}, \mathcal{B}^{k_3} \right\} + \text{Cyclic} = 0.
\]

(2.14)

2.2 Mapping to string functionals

Throughout this paper we will insert the ghost number three, Grassmann odd $\mathcal{F}$ states at every special puncture. It is therefore convenient to define

\[
f(\mathcal{B}^k_n) = \frac{1}{n!} \frac{1}{\mathcal{B}^k_n!} \int_{\mathcal{B}^k_n} \langle \Omega | \Psi \rangle_1 \cdots | \Psi \rangle_n | \mathcal{F} \rangle_1 \cdots | \mathcal{F} \rangle_{\bar{k}}.
\]

(2.15)

Note that we have included a symmetry factor both for the ordinary punctures and for the special punctures. Moreover, for string vertices $\mathcal{B}^k_n$, the function $f(\mathcal{B}^k_n)$ is Grassmann even, as it should be in order to be a candidate for a term in the string action. The subscript on each $\mathcal{F}$ denotes the label of the puncture where the state is inserted. Note that the ordering of the $\mathcal{F}$ states, ascending from left to right, is important since the $\mathcal{F}$ states are Grassmann odd.

The main homomorphism identity arises as we consider the antibracket of two functions
arising from $\mathcal{B}$ spaces. Using the primed antibracket of Ref.[8], we find
\[
\{ f(\mathcal{B}^{k_1}), f(\mathcal{B}^{k_2}) \} = -(-)^{k_1(k_2+1)} f(\{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \}'),
\]
where the sign factor requires careful consideration.* The normalization works out correctly as a consequence of the inclusion of the factor $(1/k!)$ in the definition (2.15). As a consistency check notice that both the rhs and the lhs, are symmetric under the exchange $k_1 \leftrightarrow k_2$. Recalling the relation between the primed and unprimed antibracket
\[
\{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \} \equiv (-)^{k_1+k_2} \{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \},'
\]
we find
\[
\{ f(\mathcal{B}^{k_1}), f(\mathcal{B}^{k_2}) \} = -f(\{ \mathcal{B}^{k_1}, \mathcal{B}^{k_2} \}').
\]
This equation is valid for arbitrary $\mathcal{B}$ spaces. Defining now
\[
B^{(2)}_O \equiv \langle \omega_{12} | \mathcal{O}_1 | \Psi \rangle_2 ,
\]
we obtain another useful homomorphism identity is
\[
f(I\mathcal{B}^k) = \{ f(\mathcal{B}^k), B^{(2)}_F \} .
\]

2.3 Vanishing Antibracket cohomology classes

Starting from the master equation $\{ S, S \} = 0$, covariant differentiation with a symplectic connection immediately yields $\{ S, D_\mu S \} = 0$. This means $D_\mu S$ is $S$-closed. We expect, however, that $D_\mu S$ is actually $S$-exact. If it were not so, $D_\mu S$ could be added to the string action to define an inequivalent string action, something we do not expect physically to be true. Therefore, there should be a $B_\mu$ such that
\[
D_\mu S = \{ S , B_\mu \}.
\]
Finding the explicit form of $B_\mu$ was the subject of Ref.[2]. This last equation can be written briefly by introducing a covariant derivative acting on functions on the vector bundle [8]
\[
\mathcal{D}_\mu S = 0 , \quad \mathcal{D}_\mu \equiv D_\mu + \{ B_\mu , \} .
\]
Note that this covariant derivative acts nicely on the antibracket: $\mathcal{D}_\mu \{ A, B \} = \{ \mathcal{D}_\mu A, B \} + \{ A, \mathcal{D}_\mu B \}$. Recalling that $[D_\mu, D_\nu] A = -\{ A, R_{\mu\nu} \}$, we find that the commutator of two

---

* It arises as follows: there is one minus sign from the sign factor in the right hand side of Eqn.(2.10) of Ref.[8]. There are $(k_1k_2 + k_1 + k_2)$ minus signs that arise from moving all the $F$ states to the right. Finally there are $k_2$ minus signs from moving the sewing ket $|S\rangle$ in between the operator valued forms representing the moduli spaces, as in Ref.[3] Eqn.(3.12).
covariant derivatives gives

\[ [\mathcal{D}_\mu, \mathcal{D}_\nu] = - \{ H_{\mu\nu} , \} , \quad H_{\mu\nu} \equiv D_\mu B_\nu - D_\nu B_\mu + \{ B_\mu, B_\nu \} + R_{\mu\nu} . \]  \hspace{1cm} (2.22)

It follows from (2.21) that \([\mathcal{D}_\mu, \mathcal{D}_\nu] S = 0\), and this together with (2.22) implies that \(H_{\mu\nu}\) is \(S\)-closed:

\[ \{ S, H_{\mu\nu} \} = 0 . \]  \hspace{1cm} (2.23)

Since we do not expect nontrivial antibracket cohomology classes we are led to write

\[ H_{\mu\nu} = \{ S, B_{\mu\nu} \} . \]  \hspace{1cm} (2.24)

Much of the work in [8] went into constructing the hamiltonian \(B_{\mu\nu}\) and showing that it was defined by moduli spaces of surfaces having two special punctures. We now show that higher conditions arise and could be used to define moduli spaces of surfaces having more than two special punctures.

As a first step we prove a “Bianchi identity” for \(H_{\mu\nu}\). Note the trivially satisfied identity

\[ [\mathcal{D}_\mu, \mathcal{D}_\nu] \mathcal{D}_\rho + \text{Cycl} = \mathcal{D}_\mu [\mathcal{D}_\nu, \mathcal{D}_\rho] + \text{Cycl} , \]  \hspace{1cm} (2.25)

where ‘Cycl’ denotes adding the two cyclic permutations of \((\mu \nu \rho)\) Letting both sides of the equation act on an arbitrary function \(A\), one deduces that

\[ \mathcal{D}_\mu H_{\nu\rho} + \mathcal{D}_\nu H_{\rho\mu} + \mathcal{D}_\rho H_{\mu\nu} = 0 . \]  \hspace{1cm} (2.26)

It follows now from (2.23) and (2.26) that

\[ \{ S, H_{\mu\nu\rho} \} = 0 , \quad H_{\mu\nu\rho} \equiv \mathcal{D}_\mu B_{\nu\rho} + \mathcal{D}_\nu B_{\rho\mu} + \mathcal{D}_\rho B_{\mu\nu} , \]  \hspace{1cm} (2.27)

where we have introduced a three-index field strength \(H_{\mu\nu\rho}\). Again, expecting no nontrivial antibracket cohomology, we are led to write

\[ H_{\mu\nu\rho} = \{ S, B_{\mu\nu\rho} \} . \]  \hspace{1cm} (2.28)

The new hamiltonian \(B_{\mu\nu\rho}\) would be expected to arise from moduli spaces of surfaces with three special punctures.
The idea to go to higher orders is simple, we take a covariant derivative $\mathcal{D}_\mu$ of the equation $H_{\mu_1\ldots\mu_k} = \{S, B_{\mu_1\ldots\mu_k}\}$, and antisymmetrize on the $(k+1)$ indices. The left hand side is written in the form $\{S, \cdot \}$ with the help of lower order identities, and a new identity of the form $\{S, H_{\mu_0\ldots\mu_k}\} = 0$ follows. Rather than introducing general notation to do this efficiently we limit ourselves to consider the next case. Defining

$$\mathcal{D}_\mu H_{\nu\rho\sigma} \pm \text{Cycl.} \equiv \mathcal{D}_\mu H_{\nu\rho\sigma} - \mathcal{D}_\nu H_{\rho\sigma\mu} + \mathcal{D}_\rho H_{\sigma\mu\nu} - \mathcal{D}_\sigma H_{\mu\nu\rho},$$

it then follows from (2.27) and (2.22) that

$$\mathcal{D}_\mu H_{\nu\rho\sigma} \pm \text{Cycl.} = - \{H_{\mu\nu}, B_{\rho\sigma}\} - \{H_{\mu\rho}, B_{\sigma\nu}\} - \{H_{\mu\sigma}, B_{\nu\rho}\} - \{H_{\rho\nu}, B_{\sigma\mu}\} - \{H_{\sigma\nu}, B_{\mu\rho}\} - \{H_{\rho\sigma}, B_{\mu\nu}\}. \tag{2.30}$$

We now use (2.23) and the Jacobi identity to find

$$\mathcal{D}_\mu H_{\nu\rho\sigma} \pm \text{Cycl.} = - \left\{ S, \left\{ B_{\mu\nu}, B_{\rho\sigma} \right\} + \left\{ B_{\mu\rho}, B_{\sigma\nu} \right\} + \left\{ B_{\mu\sigma}, B_{\nu\rho} \right\} \right\}. \tag{2.31}$$

It now follows from (2.28) that

$$\left\{ S, H_{\mu\nu\rho\sigma} \right\} = 0. \tag{2.32}$$

where the four-index antisymmetric field strength $H_{\mu\nu\rho\sigma}$ is given by

$$H_{\mu\nu\rho\sigma} \equiv \mathcal{D}_\mu B_{\nu\rho\sigma} \pm \text{Cycl.} + \left\{ B_{\mu\nu}, B_{\rho\sigma} \right\} + \left\{ B_{\mu\rho}, B_{\sigma\nu} \right\} + \left\{ B_{\mu\sigma}, B_{\nu\rho} \right\}. \tag{2.33}$$

Equation (2.32) suggests the existence of a four index hamiltonian function $B_{\mu\nu\rho\sigma}$ such that

$$H_{\mu\nu\rho\sigma} = \left\{ S, B_{\mu\nu\rho\sigma} \right\}. \tag{2.34}$$

The hamiltonian function $B_{\mu\nu\rho\sigma}$ is expected to be defined by moduli spaces of surfaces with four special punctures.

The general structure of the set of equations is now apparent. We have a family of hamiltonians $B_\mu, B_{\mu\nu}, \cdots$ that can be thought of as gauge fields, and a family of field strengths $H_{\mu\nu}, H_{\mu\nu\rho}, \cdots$ built out of the gauge fields. The typical field strength $H_{\mu_1\ldots\mu_n}$ involves a covariant derivative $D$ acting on the gauge field with one less index, plus all possible antibrackets of gauge fields. All field strengths are $S$-closed: $\{S, H_{\mu_1\ldots\mu_n}\} = 0$, and turn out to be $S$-exact; $H_{\mu_1\ldots\mu_n} = \{S, B_{\mu_1\ldots\mu_n}\}$.  

16
3. Generalized BRST operator

A string field theory formulated around an arbitrary background must have an action linear in the fluctuating string field and would be expected to look like

\[ S(\Psi) = S_0 + \langle \Psi, F \rangle + \frac{1}{2} \langle \Psi, Q \Psi \rangle + \frac{1}{3!} \langle \Psi, [\Psi, \Psi] \rangle \cdots. \]  

(3.1)

This action is expected to be invariant under gauge transformations of the form

\[ \delta_\Lambda \Psi = Q\Lambda + [\Psi, \Lambda] + \cdots, \]  

(3.2)

In the above equations \( F \) is a ghost number three, grassmann odd, string field, and \( Q \) is a grassmann odd, ghost number one operator. The first few conditions for gauge invariance of the above string action read (Ref. [5], sect. 4.5)

\[ Q|F\rangle = 0, \]  

(3.3)

\[ Q^2|A\rangle = -[F, A], \]  

(3.4)

\[ Q[A_1, A_2] + [QA_1, A_2] + (-)^{A_1} [A_1, QA_2] = -[F, A_1, A_2]. \]  

(3.5)

When \( F = 0 \) we recover the structure arising from conformal backgrounds and \( Q \) becomes the BRST operator of the conformal field theory. To first approximation (in the departure from conformality) one expects the string product \([, ,]\) to be defined by the symmetric three punctured sphere that gives rise to the three string vertex. Similarly the product \([, , ,]\) should be defined, to first approximation, by the collection of four punctured spheres that comprises the four string vertex.

The first of the above identities says that \( Q \) annihilates the special state \(|F\rangle\). This is the analog of the Bianchi identity in Einstein’s gravity, as explained in the introduction. The second identity says that as an operator \( Q^2 \) is an operator roughly represented by the two punctured sphere obtained by filling one of the punctures of the three string vertex with the state \(|F\rangle\). It is hard to see how the square of an operator defined by a conventional contour integral could be represented by the action of a symmetric three punctured sphere. The last equation is also peculiar in that the failure of \( Q \) to act like a derivation of the string product defined (roughly) by the three-string vertex is given by the insertion of \( F \) on roughly the four-string vertex. A direct construction of these objects using 2-dimensional field theory seems difficult.
In this section we will use the above discussion to motivate a set of identities that could be expected to arise in a two-dimensional field theory. These identities will involve an operator $Q$, to be referred to as the generalized BRST operator, and a special string field denoted as $F$. The consistency of the postulated identities will be examined. Finally we will examine BRST action on moduli spaces of surfaces with ordinary and special punctures. The explicit construction of $Q$ and $F$ using two-dimensional field theory will be discussed in [9].

3.1 $(Q, F)$ - Descent Equations

If we are to use two-dimensional field theory ingredients to build a generalized BRST charge, we expect to build it by integration over some closed curve $\gamma$ of an operator valued one-form on a two-dimensional surface. This surface is equipped with a conformal structure and a Weyl factor. Similarly we can expect $|F\rangle$ to correspond to some ghost number three local operator $F(z, \bar{z})$ of the two-dimensional field theory.

This generalized BRST charge will lose many of the properties standard in the conformal case, but it will do so in a well-defined fashion. A fundamental change is that the BRST charge will be contour dependent, and we thus write $Q(\gamma)$ to indicate explicitly this dependence. We demand, however, that it should not depend on the parametrization of the contour, nor on the local coordinates used to do the line integral: the charge arises as the integral of a well-defined (current) one-form. The failure of contour independence for homologous contours is equivalent to lack of conservation for the current one-form whose integral defines the BRST charge.

According to Eqn.(3.3) we should expect the BRST charge to annihilate the local operator $F(z, \bar{z})$ in some suitable sense. Assume $F$ is inserted at point $P$ using a local coordinate $z$ vanishing at $P$: $z(p) = 0$. Consider then the family of contours $\gamma_r = \{|z| = r\}$ that surround the point $P$ (see Figure 3). We demand that

$$\lim_{r \to 0} Q(\gamma_r) F(0) = 0.$$  \hspace{1cm} (3.6)

This is the best we can do given that the BRST operator is not conserved.

The next identity we postulate has to do with the failure of $Q$ to square to zero and is thus related to (3.4). This property must again refer to some chosen contour. Let $F^{[1]}(z, \bar{z})$ denote the operator-valued one-form arising from the local operator $F(z, \bar{z})$. We now demand

$$(Q(\gamma))^2 = \frac{1}{2\pi i} \int_{\gamma} F^{[1]}.$$  \hspace{1cm} (3.7)
Figure 3. (a) The operators $Q(\gamma_r)$, in the limit $r \to 0$ annihilate the operator $F$. (b) The BRST charge is not conserved, the difference between the charge evaluated for two homologous contours $\gamma_1$ and $\gamma_2$ is given by a surface integral of $F$ over the region $M$.

The integral in the right hand side is over the contour that defines the BRST charge in the left hand side. Note that ghost number works out, $F^{[1]}(z, \bar{z})$ is of ghost number two, the same ghost number as that of $Q^2$.

Finally, we must consider Eqn.(3.5). This equation addresses the curve dependence of the BRST charge. As the right hand side indicates, the local operator $F(z, \bar{z})$ is required to control the contour dependence. Therefore, given two homologous contours $\gamma_1$ and $\gamma_2$, namely two contours bounding a cylindrical region $M$: $\partial M = \gamma_2 - \gamma_1$ we postulate that

$$Q(\gamma_2) - Q(\gamma_1) = -\frac{1}{2\pi i} \int_M F^{[2]}, \quad (3.8)$$

where $F^{[2]}(z, \bar{z})$ is the two-form associated to $F$. The ghost number of $F^{[2]}$ is +1, and thus both sides of the above equation have the same ghost number.

The three relevant equations (3.6)(3.7), and (1.4) resemble descent equations, since they involve the various forms that arise from the local operator $F(z, \bar{z})$. We will now rewrite these equations in a language which is more suitable for string field theory. This language will also facilitate the discussion of consistency checks between the various equations.
3.2 \((Q, F)\) \textbf{Equations in Symplectic Notation}

We will now rewrite equations (3.6), (3.7), and (1.4) in the language of antibrackets and Hamiltonian functions. For this purpose we must introduce a symplectic structure in the space of local operators of the two-dimensional field theory. This is defined, in analogy to the conformal case, by a correlator on a special two-punctured sphere.

**Canonical Cylinder.** This is the canonical two-punctured Riemann sphere equipped with a special Weyl metric. We recall that the canonical two-punctured sphere is the sphere with local coordinates \(z_1\) and \(z_2\), satisfying \(z_1 z_2 = 1\), and with the punctures at \(z_1 = 0\) and \(z_2 = 0\). The Weyl metric \(\rho\), with \(ds = \rho |dz|\), is given by \(\rho(z_1) = 1/|z_1|\) (and \(\rho(z_2) = 1/|z_2|\)). This metric makes the surface into a flat infinite cylinder of circumference \(2\pi\).

**Symplectic Structure.** Acting on two local operators of the theory it furnishes a number. This number is the correlator on the canonical cylinder of the local operators, inserted at the punctures, and a line integral of the ghost field producing the \(c_0^-\) insertion familiar in the conformal case.

**BRST Operator.** We must now define a BRST operator and the associated BRST Hamiltonian function. We will denote both the BRST operator and the BRST Hamiltonian by the same symbol \(Q\); which one is being used should be clear by the context. Given that we have contour dependent BRST operators \(Q(\gamma)\), defining a fixed BRST operator entails a choice of contour in some chosen surface. We define the BRST operator \(Q\) to be the operator \(Q(\gamma)\) that arises when we choose \(\gamma\) to be the closed geodesic \(|z_1| = |z_2| = 1\) in the canonical cylinder. This will be the BRST operator to be used in string field theory.

We will use the notation of bras and kets. The symplectic form will be denoted as \(\langle \omega_{12} \rangle\), with \(\langle \omega_{12} \rangle = -\langle \omega_{12} \rangle\), and its inverse as \(|S_{12}\rangle\) with \(|S_{12}\rangle = |S_{21}\rangle\). The Hamiltonian function \(Q\) is given as \(Q = \frac{1}{2} \langle \omega_{12} | Q^{(2)} | \Psi \rangle_1 | \Psi \rangle_2\), where the \(Q\) appearing in the right is the BRST operator. All these results are familiar in the conformal case. Most results involving antibrackets will carry from the conformal case to the present case, one must only make sure the Weyl metrics of all surfaces are well defined.

We now claim that the following three equations encode in symplectic language the content of Eqns. (3.6), (3.7), and (3.8)

\[
\{Q, B_F^{(2)}\} = -f(T^2_F), \quad (3.9)
\]

\[
\frac{1}{2} \{Q, Q\} = f(V_3'), \quad (3.10)
\]
\{Q, f(\Sigma)\} = f(\mathcal{K}\Sigma) \cdot \tag{3.11}

Let us begin from the last equation. In the conformal case the right hand side is equal to zero, as a result of the identity \langle\Sigma| \sum Q = 0 which is proven by contour deformation. This time \Sigma is assumed to be equipped with a Weyl metric such that the coordinate disks are semiinfinite canonical cylinders. This is necessary for sewing compatibility since the BRST operator is defined on the canonical cylinder. The right hand side of (3.11) is intuitive, it simply indicates that we must integrate the insertion of \(F\) over the two-dimensional region outside of the coordinate disks. This equation follows from (3.8) rather simply. Consider the definition of the function \(f(\Sigma)\)

\[ f(\Sigma) = \frac{1}{n!} (-2\pi i)^{3-n} \langle\Sigma|\Psi_1 \cdots |\Psi\rangle_n, \tag{3.12} \]

A straightforward calculation gives

\[ \{Q, f(\Sigma)\} = -\frac{1}{n!} (-2\pi i)^{3-n} \langle\Sigma|\left(\sum Q\right)|\Psi_1 \cdots |\Psi\rangle_n. \tag{3.13} \]

On the other hand, (3.8) implies that \(\sum Q = \frac{1}{2\pi i} \int_{\Sigma-\cup D_i} F^{[2]}\) (recall the contours for \(Q\) are oriented oppositely to the boundary of the surface minus the unit disks). We thus find

\[ \{Q, f(\Sigma)\} = \frac{1}{n!} (-2\pi i)^{3-(n+1)} \int_{\Sigma-\cup D_i} \langle\Sigma; p|F^{[2]}|\Psi_1 \cdots |\Psi\rangle_n \]

\[ = \frac{1}{n!} \int_{\Sigma-\cup D_i} \langle\Omega^{[2]n+1} F|\Psi\rangle_n \int_{\Sigma-\cup D_i} \langle F^{[1]} |\Psi\rangle_n = f(\mathcal{K}\Sigma). \tag{3.14} \]

Let us now consider Eqn.(3.10). It expresses the failure of the BRST hamiltonian to have zero bracket with itself in terms of the insertion of a single \(F\) in the special puncture of the three punctured sphere \(\mathcal{V}_3\). This three punctured sphere is taken to be the canonical cylinder with the third puncture considered special and located at \(z_1 = 1\). The coordinate at that puncture must be chosen such that the cylinder is symmetric under the exchange of the two ordinary punctures. To verify Eqn.(3.10) we begin with the computation of the left hand side. One finds

\[
\frac{1}{2}\{Q, Q\} = \frac{1}{2} \langle\omega_{12}|(QQ|\Psi\rangle_1|\Psi\rangle_2 = \frac{1}{2} \langle\omega_{12}| \left(\frac{1}{2\pi i} \int F^{[1]} |\Psi\rangle\right)_1 |\Psi\rangle_2, \tag{3.15} \]
where use was made of (3.7). We now claim that

\[ 1 \cdot \left( \oint F^{[1]} \right)_{1} = 2\pi i \langle \mathbf{V}_{123}' | F \rangle_{3} | \mathbf{S}_{12} \rangle, \tag{3.16} \]

where the left hand side refers to the line integral over the central geodesic of the canonical cylinder using the local coordinates induced by the special puncture of \( \mathbf{V}'_{3} \). The right hand side creates this integration by inserting the state \( F \) in the special puncture and then by rotating it via the twist-sewing ket \( | \mathbf{S} \rangle \). In (3.16) the state \( F \) is arbitrary. The only part of (3.16) that needs verification is the constant of proportionality. The right hand side can be compared to the left hand side by separating out the \( b_{0}^{-(2)} \) factor in the twist sewing ket and expressing \( \langle \mathbf{V}_{123}' | b_{0}^{-(2)} \rangle \) in terms of antighost operators in the state space of \( | F \rangle \). This small calculation confirms the value of the proportionality constant in (3.16). We can now use (3.16) in (3.15) and find

\[ \frac{1}{2} \{ Q, Q \} = \frac{1}{2} \langle \omega_{12} | (\mathbf{V}_{123}' | F \rangle_{3} | \Psi \rangle_{1} | \Psi \rangle_{2}, \]

\[ = \frac{1}{2} \langle \mathbf{V}_{123} | F \rangle_{3} | \Psi \rangle_{1} | \Psi \rangle_{2} \]

\[ = f(\mathbf{V}'_{3}), \tag{3.17} \]

as we wanted to show. We now consider equation (3.9), which as we will see is a consequence of (3.6) and (3.8). A short calculation shows that

\[ \{ Q, B_{F}^{(2)} \} = \langle \omega_{12} | (Q | F \rangle \rangle_{1} = -\frac{1}{2\pi i} \langle \omega_{12} | \Psi \rangle_{1} \left( \int_{D} F^{[2]} | F \rangle \right)_{2}, \tag{3.18} \]

where \( D \) denotes the punctured disk \( 0 < | z_{2} | \leq 1 \) corresponding to the semiinfinite portion of the canonical cylinder that contains puncture number two, where \( | F \rangle \) is inserted. In obtaining this we have used the contour deformation property (3.8) to relate the BRST operator, defined on the central geodesic, to the operator \( Q(\gamma_{r}) \) where \( \gamma_{r} \) is the geodesic \( | z_{2} | = r \) in the limit as \( r \rightarrow 0 \) (see Figure 4). In this limit \( Q(\gamma_{r}) \) annihilates \( F \) by virtue of (3.6) and we simply get the surface integral of the \( F \) insertion.

In order to proceed further we now claim that for arbitrary states \( | F \rangle \) and \( | \mathcal{O} \rangle \) the following equation holds

\[ \left( \int_{D} F^{[2]} | \mathcal{O} \rangle \right)_{1} = 2\pi i \int_{\tau_{1}^{(0)}} \langle \Omega_{112}^{[1]} | F \rangle_{1} | \mathbf{S}_{12} \rangle | \mathcal{O} \rangle_{2}, \tag{3.19} \]

where \( \tau_{1}^{(u)}(u) \) is the one (real) dimension space of three punctured spheres introduced in [8] and reviewed in section two. Roughly speaking the integration over \( \tau_{1}^{(0)}(0) \) generates the radial part
Figure 4. In calculating \( \{ Q, B_F^{(2)} \} \) we find the BRST insertion on the central geodesic. This operator can be moved towards \( z_2 = 0 \), where \( F \) is inserted, and in the process we obtain the surface integral of \( F \) over the shaded region.

of the integration over the disk, while the twist-sewing ket \( |S \rangle \) generates the angular part of the integration. Equation (3.19) can be verified by explicit construction of both the left hand side and the right hand side (for the right hand side, part of the construction can be found following Eqn.(4.24) of Ref.[8]). Back in (3.18) we now obtain

\[
\{ Q, B_F^{(2)} \} = - \langle \omega_{12} | \Psi \rangle_1 \int_{\tau_1^2(0)} (\Omega_{012}^{[1]} |F\rangle_1 |S_{02} \rangle |F\rangle_2 = - \int_{\tau_1^2(0)} (\Omega_{112}^{[1]} |\Psi \rangle_1 |F\rangle_1 |F\rangle_2
\]

\[
= - \frac{1}{2} \int_{\tau_1^2} (\Omega_{112}^{[1]} |\Psi \rangle_1 |F\rangle_1 |F\rangle_2 = -f(T_1^2).
\]

This concludes our discussion of the basic BRST/Weyl identities in the antibracket formalism.

3.3 \( Q \)-Action on moduli spaces of surfaces

We must now consider the generalization of the property \( \{ Q, f(\Sigma) \} = f(K\Sigma) \), describing BRST action on the correlators on a fixed surface, to the case when we deal with correlators integrated over moduli spaces of surfaces. We will consider first the case of moduli spaces of surfaces with ordinary punctures only. We then turn to the case of moduli spaces of surfaces with both ordinary and special punctures.

Moduli spaces with ordinary punctures. Let \( \mathcal{A} \) be a a moduli space of surfaces with just ordinary punctures. Recall that surfaces now means Riemann surfaces with punctures, local coordinates at the punctures, and a Weyl factor. We will always assume that the coordinate disks around ordinary punctures are semiinfinite flat cylinders of circumference \( 2\pi \).
demand that

\[ \{ Q, f(A) \} = -f(\partial A) + (-)^A f(KA) \]

\[ = -f(\partial A - (-)^A K A). \] (3.21)

This equation cannot be fully established starting from the BRST/Weyl equations we have postulated so far. It must be derived from an explicit construction of \( Q \) and the explicit construction of forms in the moduli space of punctured Riemann surfaces equipped with Weyl metrics. These topics will be discussed in [9]. In the conformal limit \( F = 0 \) the second term in the right hand side vanishes, and we recover the familiar correspondence \( Q \leftrightarrow -\partial \) that establishes that BRST acts as an exterior derivative on CFT-valued forms on moduli spaces of Riemann surfaces. The second term in the right hand side is the natural extension of the term appearing in (3.11). Its sign factor arises from the sign factor in Eqn.(2.10) of Ref.[8].

The \( Q/\partial \) correspondence does not hold in the non-conformal case. Still, \( Q \) has a moduli space interpretation as the operator \(-\partial \pm \mathcal{K}\), where \( \mathcal{K} \) inserts the special state \( F \). Note that the \( \mathcal{K} \) insertion of \( F \) roughly amounts to a covariant derivative \( D_F(\Gamma) \) along the direction \( F \) with the special connection \( \Gamma \). Thus we have the correspondence \( Q - D_F \leftrightarrow -\partial \).

It is simple to extend (3.21) to the case when one of the ordinary punctures has a special state \( \mathcal{O} \) inserted in it (rather than the string field). The operator \( B^{(2)}_\mathcal{O} \) defined in (2.18) does the insertion. The Jacobi identity applied to \( \{ Q, \{ f(A), B^{(2)}_\mathcal{O} \} \} \) yields the equation

\[ \{ Q, f_\mathcal{O}(I A) \} = \{ \{ Q, f(A) \}, B^{(2)}_\mathcal{O} \} + (-)^A f_{Q \mathcal{O}}(IA), \] (3.22)

and as a result we obtain

\[ \{ Q, f_\mathcal{O}(I A) \} = -f_\mathcal{O}(I \partial A) + (-)^A f_{Q \mathcal{O}}(IA) + (-)^A f_{F \mathcal{O}}(IKA), \] (3.23)

where the \( \mathcal{K} \) inserts the \( F \) state. Here \( f_{Q \mathcal{O}}(IA) \) represents the correlator where \( \mathcal{O} \) is inserted in the special puncture created by \( I \), and there is a BRST insertion on the boundary of the coordinate disk.

**Moduli spaces of surfaces with special punctures** Consider a general \( B \)-type space having a number of special punctures. The case of interest in the present paper is that where the state \( F \) is inserted in all of the special punctures. The Weyl metric on the coordinate disk of a special puncture taken to be arbitrary. In analogy to (3.21) we now find

\[ \{ Q, f(B) \} = -f(\partial B) + (-)^B f_{F,...}(KB). \] (3.24)

This equation requires explanation. Recall that \( Q \) makes its way into \( f(B) \) through the ordinary
punctures, the antibracket introducing the sum of correlators each of which has a $Q(\partial D_i)$, with $D_i$ the disk associated to an ordinary puncture. Imagine now introducing a small contour $\gamma_\epsilon$ around each of the special punctures, with $\epsilon$ chosen small enough that all such contours are disjoint. Now add and subtract to the left hand side of (3.24) correlators where we have a $Q(\gamma_\epsilon)$ acting on each of the special punctures. As we now do contour deformation through the surface to cancel out all the $Q$’s we obtain the insertion of $F$ over the surface minus the coordinate disks $D_i$ and minus the $\epsilon$-disks around the special punctures. In addition to this, since we added and subtracted terms, we get a sum of correlators each with a $Q(\gamma_\epsilon)$ around a special puncture. We now take the limit as $\epsilon \to 0$ and the latter terms vanish because the BRST integrals are closing into $F$ states (recall (3.6)). The former term gives the $F$ insertion over the surface minus the unit disks around the ordinary punctures. This is precisely the definition of the operator $K$ on moduli spaces with special punctures; insertion ignoring the presence of the special punctures. This explains Eqn.(3.24) except for the sign factor and the dots. By the dots we mean that the $F$ state to be inserted by the $K$ insertion appears to the left of all other $F$ states. This is the reason the sign factor has not changed relative to (3.21). Since in our convention $K$ uses the last label and the $F$ states are arranged in increasing label value from left to right, we must move the last $F$ state across the other $F$’s. Letting $k$ denote the number of special punctures, the sign factor in (3.24) becomes

$$\{Q, f(B)\} = -f(\partial B - (-)^{k+B}KB),$$

and this holds for arbitrary $B$ spaces. If the $B$ space is a string vertex, then $k = \dim B \pmod{2}$ and we find

$$\{Q, f(B)\} = -f(\partial B - KB).$$

### 3.4 Consistency checks

It is useful to perform a few consistency checks with the equations we have introduced in the last two subsections. These checks basically use the Jacobi identity (2.1) for the antibracket, and the properties of the operators $K$ and $I$.

The state $|F\rangle$ being of ghost number three cannot couple to itself through the antibracket. Thus $\{B_F^{(2)}, B_F^{(2)}\} = 0$. This result is compatible with (3.9) and the Jacobi identity:

$$0 = \left\{ \{B_F^{(2)}, B_F^{(2)}\}, Q \right\} \sim \left\{ \{Q, B_F^{(2)}\}, B_F^{(2)} \right\} \sim \{f(T_1^2), B_F^{(2)}\} \sim f(I^2) = 0,$$

where the final equality follows because of ghost number: the sphere needs seven units of ghost.
number, the three $F$’s provide nine.

Equation (3.10) is tested by evaluating $\{\frac{1}{2}\{Q, Q\}, B_F^{(2)}\}$ in two different ways. Direct evaluation gives

$$\left\{ \frac{1}{2}\{Q, Q\}, B_F^{(2)} \right\} = \{f(V_3'), B_F^{(2)}\} = f(\mathcal{I}V_3'),$$  \hspace{1cm} (3.28)$$
on the other hand, using the Jacobi identity

$$\left\{ \frac{1}{2}\{Q, Q\}, B_F^{(2)} \right\} = \left\{ \{Q, Q\}, B_F^{(2)} \right\},$$
$$= -\{Q, f(T^2_1)\},$$
$$= f(\partial T^2_1 + \mathcal{K}T^2_1),$$ \hspace{1cm} (3.29)$$
where use was made of (3.25) in the last step. Using (2.5) we see that the first term in the right hand side gives the desired answer. On the other hand we expect $\mathcal{K}T^2_1 = 0$. As we will explain precisely in sect.5.2, one can think of $T^2_1$ as $\mathcal{K}\mathcal{B}^1_1$, where $\mathcal{B}^1_1$ is a canonical cylinder with one special and one ordinary puncture. The vanishing of $\mathcal{K}T^2_1$ then follows from $\mathcal{K}^2 = 0$. This implies consistency.

As a final, and more general consistency check we now evaluate $\{\frac{1}{2}\{Q, Q\}, f(B)\}$ in two different ways. By direct evaluation we find

$$\left\{ \frac{1}{2}\{Q, Q\}, f(B) \right\} = \{f(V_3'), f(B)\} = -f(\{V_3', B\}).$$  \hspace{1cm} (3.30)$$
on the other hand, using the Jacobi identity and recalling that $f(B)$ is Grassmann even we get:

$$\left\{ \frac{1}{2}\{Q, Q\}, f(B) \right\} = \left\{ \{Q, Q\}, f(B) \right\},$$
$$= -\left\{ Q, f((\partial - \mathcal{K})B) \right\},$$
$$= f \left( (\partial + \mathcal{K})(\partial - \mathcal{K})B \right),$$
$$= -f \left( [\partial, \mathcal{K}]B \right),$$
$$= -f(\{V_3', B\}),$$ \hspace{1cm} (3.31)$$
in agreement with (3.30). In deriving (3.31) we made use of (3.25), of the properties $\partial^2 = \mathcal{K}^2 = 0$ and of the second equation in (2.2).
4. Constructing the New String Action: First Few Terms

We will now take the first few steps in the construction of the string field action. The construction will be perturbative in $F$. Moreover, it will make no reference to any conformal theory. We will denote by $S_0$ the $F$ independent terms in the complete action, by $S_1$ the action including $F$ independent terms and terms linear in $F$, by $S_2$ the action including all terms of order less than or equal to quadratic in $F$, and so on.

4.1 The Construction to Zeroth-order

The discussion in the previous section showed that the new BRST operator fails to satisfy the identities of the standard CFT BRST operator, but that the failure is through terms linear in $F$. We can therefore begin the construction by taking $S_0$ to have the same form it had in the conformal case:

$$S_0 = Q + f(V).$$  \hspace{1cm} (4.1)

It should be emphasized that this $Q$ is the one defined for the chosen two-dimensional non conformal theory, and similarly the correlators in $f(V)$ are correlators in the non conformal theory. $S_0$ has nothing to do with a conformal theory. In fact the chosen non conformal theory need not be close to any CFT.

We compute the antibracket of this action with itself

$$\{S_0, S_0\} = \{Q, Q\} + 2\{Q, f(V)\} + \{f(V), f(V)\},$$

$$= 2f(\mathcal{V}_3) - 2f(\partial V - K V) - f(\{V, V\}), \hspace{1cm} (4.2)$$

where use was made of Eqns. (3.10) and (3.21). The result is neatly organized as

$$\{S_0, S_0\} = -2f(\partial V + \frac{1}{2}\{V, V\}) + 2f(\mathcal{V}_3 + K V), \hspace{1cm} (4.3)$$

where the first line in the right hand side involves moduli spaces with no special punctures, and the second line involves moduli spaces with one special puncture. The geometrical recursion relations (2.7) for the string vertices $\mathcal{V}$ imply that the first line in the above right hand side vanishes. We find, as expected, that $S_0$ is correct to zeroth order in the string field $F$. We
therefore obtain
\[ \{S_0, S_0\} = 2 f \left( V_3^I + \mathcal{K} V \right). \] (4.4)

This nonvanishing right hand side signals that \( S_0 \) is not fully consistent. We must change the action by terms having \( F \) insertions. The interpolating \( \mathcal{B}^1 \) spaces of background independence are the natural candidates.

### 4.2 The construction to first order

We have emphasized earlier that the \( \mathcal{B}^1 \) spaces are natural for the insertion of the ghost number three state \( F \). This state is inserted on the special puncture, and string fields are inserted on the ordinary punctures. Correlators will exist when the total ghost number of the string fields is two times the number of ordinary punctures, just as it is the case for string vertices. We therefore write
\[
S_1 = S_0 + \left( -B_F^{(2)} + f_F(\mathcal{B}^1) \right),
\]
\[
= Q + f(V) + \left( -B_F^{(2)} + f_F(\mathcal{B}^1) \right).
\] (4.5)

and claim that \( S_1 \) is the correct action to order \( \mathcal{O}(F) \). To verify this we now calculate \( \{S_1, S_1\} \)
\[
\{S_1, S_1\} = \{S_0, S_0\} + 2 \{ S_0, -B_F^{(2)} + f(\mathcal{B}^1) \} \]
\[ + \left\{ -B_F^{(2)} + f(\mathcal{B}^1), -B_F^{(2)} + f(\mathcal{B}^1) \right\}.
\] (4.6)

We must calculate the various pieces in this equation. The first line has already been computed. The second line requires the evaluation of two terms, the first of which is
\[
\{S_0, B_F^{(2)}\} = \{Q, B_F^{(2)}\} + \{f(V), B_F^{(2)}\} = f(-\mathcal{T}_I^2 + TV),
\] (4.7)
where use was made of (3.9) and (2.19). Note that in the right hand side the first contribution is from a space with two special punctures, and the second contribution is from a space with a single special puncture. The second term we need in this line is
\[
\{S_0, f(\mathcal{B}^1)\} = \{Q, f(\mathcal{B}^1)\} + \{f(V), f(\mathcal{B}^1)\} \]
\[ = -f(\partial\mathcal{B}^1 - \mathcal{K}\mathcal{B}^1 + \{V, \mathcal{B}^1\}),
\] (4.8)
\[ = -f(\delta_V\mathcal{B}^1 - \mathcal{K}\mathcal{B}^1). \]
where use was made of (3.26). The third line gives
\[
\{-B_F^{(2)} + f(B^1), -B_F^{(2)} + f(B^1)\} = -2 \{B_F^{(2)}, f(B^1)\} + \{f(B^1), f(B^1)\},
\]
\[
= -2f(\mathcal{I}B^1) - f(\{B^1, B^1\}),
\]
given that the antibracket of $B_F^{(2)}$ with itself is zero, and making use of (2.19) and (2.17).

Putting it all together
\[
\{S_1, S_1\} = 2f\left( V'_3 + (\mathcal{K} - \mathcal{I})V - \delta y B^1 \right)
+ 2f\left( T_1^2 + (\mathcal{K} - \mathcal{I})B^1 - \frac{1}{2}\{B^1, B^1\} \right).
\]
In the right hand side we show in the first line the terms with one special puncture. We see that as desired they cancel out by virtue of the recursion relations (2.10) for the $B^1$ spaces. Thus we have succeeded in defining an $S_1$ that satisfies the master equation to first nontrivial order in $F$. It is not a complete answer because we have
\[
\{S_1, S_1\} = 2f\left( T_1^2 + (\mathcal{K} - \mathcal{I})B^1 - \frac{1}{2}\{B^1, B^1\} \right) \neq 0.
\]
It is clear that we must add to $S_1$ a term corresponding to surfaces with two special punctures.

4.3 Construction to second order

To continue to second order in $F$ we now simply include the $B^2$ spaces discussed in [8] and reviewed in sect.2. We now claim that
\[
S_2 = S_1 + f(B^2),
\]
is the correct action to order $O(F^2)$. To show this we compute $\{S_2, S_2\}$
\[
\{S_2, S_2\} = \{S_1, S_1\} + 2\{S_1, f(B^2)\} + O(F^4),
\]
\[
= \{S_1, S_1\} + 2\{S_0, f(B^2)\} + O(F^3),
\]
\[
= \{S_1, S_1\} - 2f(\delta y B^2) + O(F^3)
\]
since the antibracket of $f(B^2)$ with itself has four special punctures and the antibracket of $f(B^2)$ with $f(B^1)$, or $B_F^{(2)}$ has three special punctures. We therefore have
\[
\{S_2, S_2\} = 2f\left( T_1^2 + (\mathcal{K} - \mathcal{I})B^1 - \frac{1}{2}\{B^1, B^1\} - \delta y B^2 \right) + O(F^3).
\]
The recursion relation (2.10) for $B^2$ spaces implies that the part of $\{S_2, S_2\}$ quadratic in $F$ vanishes. Thus $S_2$ is accurate to order $F^2$. 

29
5. Constructing the New String Action: All orders

In order to complete the construction of a gauge invariant action we need moduli spaces with more than two special punctures. We have indicated in sect.2.3 how such spaces would arise from higher order background independence consistency conditions. It is simpler, however, to derive the equations that such spaces must satisfy directly from gauge invariance, and, then show that the spaces can be consistently defined. This is how we will proceed. We will see how string vertices and $B$ spaces can be naturally grouped together into a single “space” satisfying simple recursion relations. Finally, we will introduce the moduli space $B^1_1$, and show how it simplifies the writing of recursion relations.

5.1 Construction to all orders

Consider now general $B^k$ moduli spaces of surfaces with $k$ special punctures. As before we will define $B^k = \sum_n B^k_n$ where each of the $B^k_n$ spaces has $k$ special labelled punctures $(\bar{1}, \bar{2}, \cdots \bar{k})$, and $n$ ordinary punctures. When $k = 0$ the moduli spaces in $B^0$ are identified with the string vertices, and in this case $n \geq 3$. When $k = 1$ we have the moduli spaces that appeared in the analysis of background independence and here $n \geq 2$ (this will change in the present analysis). Our studies in Ref.[8] show that for $k = 2$ we have $n \geq 1$. The real dimensionality of $B^k_n$ exceeds by $k$ that of the moduli space $\mathcal{M}_{k+n}$ of punctured spheres. It follows that all moduli spaces in $B^k$ are odd if $k$ is odd, and are all even if $k$ is even.

We now claim that the complete gauge invariant action is given as

$$S = S_0 - B^{(2)}_F + \sum_{k=1}^{\infty} f(B^k). \quad (5.1)$$

To verify this claim we must check the master equation, and we must therefore compute

$$\{S, S\} = \{S_0, S_0\} - 2\{S_0, B^{(2)}_F\}$$

$$+ 2 \sum_{k=1}^{\infty} \{S_0, f(B^k)\} - 2 \sum_{k=1}^{\infty} \{f(B^k), B^{(2)}_F\}$$

$$+ \sum_{k_1, k_2=1}^{\infty} \{f(B^{k_1}, f(B^{k_2})\}. \quad (5.2)$$
We now use (4.4), (4.7), (2.19) and (2.17) to rewrite the above equation as

\[
\{ S , S \} = + 2 f \left( V'_3 + K V + T^2_1 - T V \right) \\
+ 2 \sum_{k=1} \left( - \delta V B^k + (K - I) B^k \right) \\
- \sum_{k_1, k_2} f \left( \{ B^{k_1}, B^{k_2} \} \right).
\]  

(5.3)

It is useful to separate out terms with one and two special punctures, since such terms are not completely generic. Doing so and rearranging somewhat the above equation we find

\[
\{ S , S \} = - 2 f \left( \delta V B^1 - (V'_3 + MV) \right) \\
- 2 f \left( \delta V B^2 - (T^2_1 + MB^1) \right) \\
- 2 \sum_{k=3} \left( \delta V B^k \right) - 2 \sum_{k=2} \left( - MB^k \right) \\
- \sum_{k=1}^{\infty} \sum_{l=1}^{k} \left( \{ B^l, B^{k+1-l} \} \right).
\]  

(5.4)

and with a final rearrangement we write

\[
\{ S , S \} = - 2 f \left( \delta V B^1 - (V'_3 + MV) \right) \\
- 2 f \left( \delta V B^2 - (T^2_1 + MB^1 - \frac{1}{2} \{ B^1, B^1 \}) \right) \\
- 2 f \left( \sum_{k=2}^{\infty} \left[ \delta V B^{k+1} - MB^k + \frac{1}{2} \sum_{l=1}^{k} \{ B^l, B^{k+1-l} \} \right] \right).
\]  

(5.5)

We recognize the familiar recursion relations for \( B^1 \) and \( B^2 \) spaces, and find the required recursion relations for \( B^k \) spaces with \( k \geq 3 \):

\[
\delta V B^1 = V'_3 + MV, \\
\delta V B^2 = T^2_1 + MB^1 - \frac{1}{2} \{ B^1, B^1 \}, \\
\delta V B^{k+1} = MB^k - \frac{1}{2} \sum_{l=1}^{k} \{ B^l, B^{k+1-l} \}.
\]  

(5.6)

The above identities are better appreciated if we append the recursion relations for string
vertices, and we separate out the \{\mathcal{V}, \cdot \} term in each \(\delta \mathcal{V}\). We find
\[
\begin{align*}
\partial \mathcal{V} &= - \frac{1}{2} \{\mathcal{V}, \mathcal{V}\}, \\
\partial \mathcal{B}^1 &= \mathcal{V}^\prime_3 + \mathcal{M} \mathcal{V} - \{\mathcal{V}, \mathcal{B}^1\}, \\
\partial \mathcal{B}^2 &= \mathcal{T}^2_1 + \mathcal{M} \mathcal{B}^1 - \frac{1}{2} \{\mathcal{B}^1, \mathcal{B}^1\} - \{\mathcal{V}, \mathcal{B}^2\}, \\
\partial \mathcal{B}^{k+1} &= \mathcal{M} \mathcal{B}^k - \frac{1}{2} \sum_{l=1}^{k} \{\mathcal{B}^l, \mathcal{B}^{k+1-l}\} - \{\mathcal{V}, \mathcal{B}^{k+1}\}.
\end{align*}
\] (5.7)

We now see that it is natural to think of the string vertices \(\mathcal{V}\) as the lowest member of the family of \(\mathcal{B}\) spaces, a \(\mathcal{B}\) space with zero number of special punctures
\[\mathcal{B}^0 \equiv \mathcal{V}.\] (5.8)

We now introduce the complete formal sum of \(\mathcal{B}^k\) spaces
\[\mathcal{B} \equiv \sum_{k=0}^{\infty} \mathcal{B}^k = \mathcal{V} + \mathcal{B}^1 + \mathcal{B}^2 + \cdots.\] (5.9)

The whole set of recursion relation (5.7) is then summarized as a single equation for the total \(\mathcal{B}\) space
\[\partial \mathcal{B} = \mathcal{V}^\prime_3 + \mathcal{T}^2_1 + \mathcal{M} \mathcal{B} - \frac{1}{2} \{\mathcal{B}, \mathcal{B}\}.\] (5.10)

This is a fairly simple equation, simple enough to allow the main consistency check. This is the check that \(\partial (\partial \mathcal{B}) = 0\), or that the right hand side of the above equation has zero boundary. If this consistency check works out one should be able to define the \(\mathcal{B}\) spaces recursively, as was done in [3,8].

Before starting this verification it is useful to remark that previous equations derived for single \(\mathcal{B}^k\) spaces now hold for the sum \(\mathcal{B}\). For example, the Jacobi identity (2.14) implies that
\[\{\{\mathcal{B}, \mathcal{B}\}, \mathcal{B}\} = 0.\] (5.11)

Moreover, (2.6) implies that
\[[\partial, \mathcal{M}] \mathcal{B} = \{\mathcal{V}^\prime_3, \mathcal{B}\}.\] (5.12)

Let us now verify the consistency condition. Since \(\mathcal{V}^\prime_3\) has no boundary we must verify that
\[\partial \mathcal{T}^2_1 + [\partial, \mathcal{M}] \mathcal{B} + \mathcal{M} (\partial \mathcal{B}) - \{\partial \mathcal{B}, \mathcal{B}\} = 0.\] (5.13)
The left hand side equals:

\[ + \mathcal{I} \nu_3' + \{ \nu_3', \mathcal{B} \} + \mathcal{M} \left( \nu_3' + T_1^2 + MB - \frac{1}{2} \{ \mathcal{B}, \mathcal{B} \} \right) - \left\{ \nu_3' + T_1^2 + MB - \frac{1}{2} \{ \mathcal{B}, \mathcal{B} \}, \mathcal{B} \right\} \]

(5.14)

Since \( K \nu_3' = 0 \) (the unit disks around the ordinary punctures cover the sphere) we have \( \mathcal{M} \nu_3' = -\mathcal{I} \nu_3' \), and this cancels the first term. Moreover \( \mathcal{M} T_1^2 = K T_1^2 - \mathcal{I} T_1^2 = 0 \). The vanishing of \( K T_1^2 \) follows from the fact that \( T_1^2 = KB_1^1 \) (to be explained below), and the vanishing of \( \mathcal{I} T_1^2 \) follows from ghost number, as explained after (3.27). Using the Jacobi identity, we now find

\[ + \{ \nu_3', \mathcal{B} \} - \{ \mathcal{B}, T_1^2 \} + \{ MB, \mathcal{B} \} - \left\{ \nu_3' + T_1^2 + MB - \frac{1}{2} \{ \mathcal{B}, \mathcal{B} \}, \mathcal{B} \right\} \]

(5.15)

and since the space \( T_1^2 \) is effectively odd (dimension one and two special punctures) \( \{ \mathcal{B}, T_1^2 \} = -\{ T_1^2, \mathcal{B} \} \), and we see that all terms cancel out. This completes our verification of the consistency conditions.

5.2 Introducing \( B_1^1 \)

Equation (5.1) for the string action would simply read \( S = Q + f(\mathcal{B}) \) were it not for the \( B_F^{(2)} \) term. This term is linear in \( F \) and would be natural to try to include it as part of the \( B_1^1 \) spaces. Since \( B_F^{(2)} \) is also linear in the string field it must correspond to a \( B \)-space with one special puncture and one ordinary puncture. We will denote such space as \( B_1^1 \). The reason this space was not introduced earlier is that being a two punctured sphere it is somewhat peculiar. The function associated to \( B_1^1 \) is not just the corresponding surface state with an insertion of a string field and an \( F \) state, one must include a ghost insertion. We simply define

\[ f(B_1^1) \equiv -B_F^{(2)} = \langle \omega_{12} | \Psi \rangle_1 | F \rangle_2, \]

(5.16)

where the last equality follows from (2.18). It is consistent to declare that for an arbitrary \( B \) space

\[ \{ \mathcal{B}, B_1^1 \} = \mathcal{I} \mathcal{B}, \]

(5.17)

since (2.19) and (2.17) imply that \( f(\mathcal{I} \mathcal{B}) = \{ f(\mathcal{B}), B_F^{(2)} \} = -\{ f(\mathcal{B}), f(B_1^1) \} = f(\{ \mathcal{B}, B_1^1 \}) \). Since \( \mathcal{I} \) does not change the dimensionality of a space, and the antibracket adds one unit
to the dimensionality, it follows from (5.17) that $\mathcal{B}_1^1$ should be thought as a moduli space of dimension minus one. Since the surfaces in the moduli space have one special puncture, the space is effectively Grassmann even, as it should be if it is to be thought as a string vertex.

The picture of the $\mathcal{B}_1^1$ space as a surface is clear, it is the canonical cylinder with one puncture declared special (in some sense something as $\mathcal{D}V_2$, if we had defined a space $V_2$ with two punctures). What happens if $\mathcal{K}$ acts on $\mathcal{B}_1^1$? By our geometrical definition of $\mathcal{K}$ we would expect it to add a new special puncture throughout the unit disk of the original special puncture. Due to the conformal isometry of the sphere this two-dimensional insertion region can be thought simply as an insertion over a (one-dimensional) line starting from the central geodesic and going all the way towards the special puncture. But this is nothing else that the space $\mathcal{T}_1^2$ discussed earlier. We therefore claim that

$$\mathcal{KB}_1^1 = \mathcal{T}_1^2. \quad (5.18)$$

Note that given that $\mathcal{T}_1^2$ is of dimension one, and $\mathcal{K}$ adds two units of dimension, we confirm that $\mathcal{B}_1^1$ must be thought of dimension minus one. Let us now confirm (5.18) by more explicit means. We recall Eqn.(3.19) which taking $|O\rangle = |F\rangle$ reads

$$\left( \int_D F^{[2]} |F\rangle \right)_1 = 2\pi i \int_{\mathcal{T}_1^2} \langle \Omega_1^{[1]} | F\rangle_1 |S_{11}\rangle |F\rangle_2, \quad (5.19)$$

and readily find that

$$\langle \omega_1 | \left( \int_D F^{[2]} |F\rangle \right)_1 |\Psi\rangle_2 = (-2\pi i) f(\mathcal{T}_1^2). \quad (5.20)$$

On the other hand, the operator $\mathcal{K}$ can be thought as $\mathcal{K} = \frac{1}{2\pi i} \int F^{[2]}$ acting from the right, and we would then write, using (5.16)

$$f(\mathcal{KB}_1^1) = \langle \omega_1 | \Psi\rangle_1 |F\rangle_2 \frac{1}{(-2\pi i)} \int F^{[2]},$$

$$= \frac{1}{(-2\pi i)} \langle \omega_1 | \left( \int D F^{[2]} |F\rangle \right)_1 |\Psi\rangle_2 = f(\mathcal{T}_1^2), \quad (5.21)$$

in agreement with our claim. One can also verify directly that $f(\mathcal{KT}_1^2) = 0$ in agreement to what we would expect given that $\mathcal{KT}_1^2 = \mathcal{K}^2\mathcal{B}_1^1 = 0$. 

34
5.3 String action and field equation for classical backgrounds

Having justified the introduction of the $\mathcal{B}_1^1$, we now include this space into the definition of the collection $\mathcal{B}^1$ by taking $\mathcal{B}^1 = \mathcal{B}_1^1 + \mathcal{B}_2^2 + \cdots$. In this way the string action given in (5.1) takes the simple form

$$S = Q + f(\mathcal{B}) ,$$

and the recursion relations (5.10) become

$$\partial \mathcal{B} = \mathcal{V}_3' + \mathcal{KB} - \frac{1}{2} \{\mathcal{B}, \mathcal{B}\} ,$$

where we see that the $(-\mathcal{I} \mathcal{B})$ term in $\mathcal{MB}$ appears now in the antibracket because of (5.17), and the space $\mathcal{T}_1^2$ is now included in $\mathcal{KB}$ because of (5.18). At this stage the recursion relations have become quite simple.

The verification of consistency is now a trivial computation. It is also manifest that the recursion relations guarantee that the action satisfies the BV master equation. This implies that the action is gauge invariant.

Let us now examine the action in order to read the explicit expressions for $\mathcal{F}$ and $Q$. The terms in the action linear in the string field read

$$S_{lin} = f(\mathcal{B}_1^1) + \sum_{k=2}^{\infty} f(\mathcal{B}_1^k) ,$$

$$= \langle \omega_{11} | \Psi \rangle |F\rangle_1 + \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathcal{B}_1^k} \langle \Omega[k]_{11...k} | \Psi \rangle |F\rangle_1 \cdots |F\rangle_k .$$

By definition (Eqn.(3.1)) $S_{lin} = \langle \omega_{11} | \Psi \rangle |F\rangle_1$, and therefore we find

$$|\mathcal{F}\rangle_{1'} = |F\rangle_{1'} + \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathcal{B}_1^k} \langle \Omega[k]_{11...k} |F\rangle_1 \cdots |F\rangle_k |S_{11'}\rangle .$$

The condition that $|\mathcal{F}\rangle$ vanishes is the equation that selects a classical string background. This is a complicated nonlinear equation for the state $|F\rangle$. While $|F\rangle = 0$ is clearly a solution, there may be solutions with $|F\rangle \neq 0$. The first $\mathcal{B}$-space relevant to this equation is the space $\mathcal{B}_1^0$. This space satisfies $\partial \mathcal{B}_1^0 = \mathcal{T}_1^2 - \mathcal{I} \mathcal{B}_2^2$ and was constructed, except for the specification of the Weyl metric, in Ref.[ 8] sect.7.1.

---

* It is tempting to simplify even more the recursion relations by absorbing the $\mathcal{V}_3'$ term in the right hand side. This may be possible by setting $\mathcal{V}_3' = \mathcal{K} \mathcal{V}_2$, where $\mathcal{V}_2$ would be some formal space of two punctured spheres of dimension minus two. We will not explore this possibility here.
The action also has terms quadratic in the string field, such terms define the operator $Q$ (see Eqn.(3.1)). We can readily read

$$Q = Q + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathcal{B}_k^g} \langle \Omega_{12\cdots k}^{[k]} | F \rangle_{1} \cdots | F \rangle_{k} | S_{12} \rangle,$$

as an operator from the state space $2$ to the state space $2'$. We have therefore confirmed that the pair $(Q, F)$ can be constructed using the pair $(Q, F)$ and $\mathcal{B}$-spaces. By construction we are guaranteed that the “Bianchi identity” $Q|F \rangle = 0$ holds.

6. Comments and Open Questions

In this paper we have sketched the construction of a string field theory formulated around backgrounds that are not represented by two-dimensional conformal field theories. The construction is not complete in all its details. This should not be too surprising, passing from conformal to non-conformal backgrounds is a major departure, and many issues arise.

We have not been completely explicit about Weyl metrics on the surfaces that make up $\mathcal{B}$ spaces. While the minimal area problem [5] determines consistent Weyl metrics on the surfaces defining the string vertices $\mathcal{V}$, we have not discussed a minimal area problem for surfaces defining $\mathcal{B}$ spaces. The required problem may be of the type proposed in [10] where one creates interpolating moduli spaces by requiring that curves homotopic to the special punctures be longer than or equal to a quantity $l$ that varies from zero to $2\pi$.

Our analysis throughout this paper has been at the level of classical string theory. We have made no attempt to include the higher genus contributions to the closed string action. No major difficulties are expected here, but as usual, the vacuum contributions are probably quite subtle. The case of open strings has not been addressed.

Certainly the most important matter that has not been addressed is that of an explicit construction of the pair $(Q, F)$ satisfying the postulated descent equations. Nor we have discussed the precise construction of operator-valued forms on the moduli spaces of Riemann surfaces equipped with Weyl metrics. Moreover, since the $\mathcal{B}$ spaces allow the antisymmetrized collision of special punctures, the generality of our construction hinges on the finiteness of the antisymmetrized collision of $F$ states. We hope to address these issues in Ref.[9].
We have obtained the field equation that characterizes a classical string background. Such equation involved the spaces $B^k_1$, spaces with one ordinary puncture and any number of special punctures. It is certainly of interest to understand the $B^k_0$ spaces, that is, the $B$ spaces that have only special punctures. These spaces contribute constants to the string action, constants that are essential for background independence. Since the constants will be background dependent they ought to give us insight into the function on theory space that represents the string action.

We have indicated in the table below the various string vertices with $k$ denoting the number of special punctures and $n$ the number of ordinary punctures.

| $n$ | 0   | 1   | 2   | 3   | 4   | ... | $k$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
|     | $B^4_0$ | $B^3_0$ | $B^3_1$ | $B^2_0$ | $B^2_1$ | $B^2_2$ | $B^2_3$ | $B^1_0$ | $B^1_1$ | $B^1_2$ | $B^1_3$ | $B^1_4$ | ... | $B^1_k$ |
|     | $V_0$ | $V_1$ | $V_2$ | $V_3$ | $V_4$ | ... | $V_k$ |

A few of the entries in this table are somewhat unclear. Conventionally string vertices $V_n$ exist only for $n \geq 3$. It may be useful to define the space $V_2$, as we mentioned in the last section. Probably $V_0$ and $V_1$ are really zero. Similarly while $B^1_p$ spaces were formerly defined for $p \geq 2$ we have seen that $B^1_1$ is naturally defined and useful. The space $B^1_0$ may be zero. It seems likely, however, that all $B^k_0$ with $l \geq 2$ are non-vanishing.

Much was learned in the process of discovering how to write string field theory around conformal backgrounds. It is not unreasonable to believe that as much will be learned by exploration of string field theory around arbitrary backgrounds.
REFERENCES

1. M. Kaku and K. Kikkawa, “Field theory of relativistic strings. I. Trees”, Phys. Rev. D10 (1974) 1110; “Field theory of relativistic strings. II. Loops and Pomerons”, Phys. Rev. D10 (1974) 1823.

2. A. Sen and B. Zwiebach, ‘Local background independence of classical closed string field theory’, Nucl. Phys.B414 (1994) 649, hep-th/9307088.

3. A. Sen and B. Zwiebach, ‘Quantum background independence of closed string field theory’, Nucl. Phys.B423 (1994) 580, hep-th/9311009.

4. O. Bergman and B. Zwiebach ‘The dilaton theorem and closed string backgrounds’ Nucl. Phys. B441 (1995) 76, hep-th/9411047.

5. B. Zwiebach, ‘Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation’, Nucl. Phys. B390 (1993) 33, hep-th/9205075.

6. E. Witten, “On background independent open-string field theory” Phys. Rev. D46 (1992) 5467; hep-th/9208027; “Some computations in background independent open-string field theory”, Phys. Rev. D47 (1993) 3405; hep-th/9210063; K. Li and E. Witten, “Role of short distance behavior in off-shell open-string field theory”, Phys. Rev. D48 (1993) 853; hep-th/9303067.

7. S. Shatashvili, “On the problems with background independence in string theory”, hep-th/9311177.

8. B. Zwiebach, ‘New moduli spaces from string background independence consistency conditions’, MIT preprint MIT-CTP-2527, hep-th/9605075.

9. B. Zwiebach, ‘Generalized BRST operator and Weyl-descent equations’, MIT-CTP-2533, to appear.

10. B. Zwiebach, “Interpolating string field theories”, Mod. Phys. Lett. A7 (1992) 1079, hep-th/9202013.