Dynamical Mode Recognition of Triple Flickering Buoyant Diffusion Flames in Wasserstein Space

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Abstract
Triple flickering buoyant diffusion flames in an isosceles triangle arrangement, as a nonlinear dynamical system of coupled oscillators, were experimentally studied. The focus of the study is two-fold: we established a well-controlled gas-fuel diffusion flame experiment, which well remedies the deficiencies of prevalent candle-flame experiments, and we developed a Wasserstein-space-based methodology for dynamical mode recognition, which is validated in the present triple flame systems but can be readily generalized to the dynamical systems consisting of arbitrary finite number of flames. By use of the present experiment and methodology, seven distinct stable dynamical modes were recognized, such as the in-phase mode, the flickering death mode, the partial flickering death mode, the partial in-phase mode, the rotation mode, the partial decoupled mode, and the decoupled mode. These modes unify the literature results for triple flickering flame system in the straight-line and equal-lateral triangle arrangements. Compared with the mode recognitions in physical space and phase space, the Wasserstein-space-based methodology avoids personal subjectivity and is more applicable in high-dimensional systems, as it is based on the concept of distance between distribution functions of phase points. Consequently, the identification or discrimination of two dynamical modes can be quantified as the small or large Wasserstein distance, respectively.

Keywords: Flickering diffusion flame; Nonlinear dynamical system; Dynamical mode; Phase space; Wasserstein space

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1. Introduction

Diffusion flames are ubiquitous in nature (e.g., wildland and urban fires), domestic applications (e.g., fireplaces and furnaces), and industrial applications (e.g., gas-turbine and rocket engines). Vibratory motion of Bunsen-type diffusion flames was observed and referred to as “the flicker of luminous flames” by Chamberlin and Rose [1], who described “the upper portion of the luminous zone rises to a maximum height ten times per second”. Subsequently, Barr [2] discovered the similar phenomena in a Burke-Schumann diffusion flame, in which the flame vibrates between the maximum and minimum positions and “the vibration is seen to consists of a progressive necking of the flame which can lead to the formation of a flame bubble which burns itself out separated from the anchored flame”. Similar phenomena were also observed in pool fires and often referred to as “puffing flames” [3, 4].

Many studies have been devoted to understanding the physics of flickering diffusion flames. The fact that the flicker of diffusion flames is a self-exciting flow oscillation was substantiated by Chen et al.’s flow visualization of a methane jet diffusion flame [5], in which the small vortices inside of the luminous flame are due to the Kelvin-Helmholtz instability of the fuel jet, and the large toroidal vortices outside the luminous flame are due to the buoyance-induced Kelvin-Helmholtz instability. The frequency of the toroidal vortices was found to well correlate with the flicker frequency [6-12]. Buckmaster and Peters [13] conducted a linear stability analysis to a self-similar solution of an annular burner diffusion flame and confirmed that the Kelvin-Helmholtz instability of the buoyancy-induced flow is responsible for the flicker. The quantitatively poor predictions of the stability analysis based on the quasi-parallel approximation were remedied by Moreno-Boza et al. [14] using the global linear stability analysis.

In recent years, coupled flickering buoyant diffusion flames as a nonlinear dynamical system have gained increasing attentions. In the pioneer work on dual flickering candle flames by Kitahata et al. [15], two distinct dynamical modes at different flame distances were observed, such as the in-phase mode, in which both flames flicker identically with no phase difference, and the anti-phase mode, in which both flames flicker identically but with a phase-shift of $\pi$. Similar phenomena were also reported by Forrester [16] and Manoj et al. [17] for candle flames. In addition, Manoj et al. [17] observed that an amplitude death mode occurs for small candles and that in-phase and anti-phase modes could coexist and transition to each other. The interaction between buoyance-induced vortices generated by each flame plays a significant role in producing the different dynamical modes at different flame distances. This has been confirmed by the experiments of Dange et al. for candle flames [18], of Fujisawa et al. for pipe-
burner diffusion flames [19], of Bunkwang et al. for methane/air jet diffusion flames [20, 21], and by the numerical simulations of Yang et al. for pool flames [22] and of Tokami et al. for buoyancy-induced turbulent diffusion flames [23]. Their results substantiate the vortex-dynamical conjecture that the interaction between two toroidal vortices is responsible for the different flickering modes.

Coupled multiple flickering diffusion flames appear more complex dynamical modes. Okamoto et al. [24] investigated three coupled flickering candle flames in an equilateral triangle arrangement. With increasing the size of the triangle, they observed four distinct dynamical modes, such as the in-phase mode, the partial in-phase mode (two of the three flames flicker in phase but the remaining one flickers with a phase-shift of $\pi$), the rotation mode (all three flames flicker with a constant phase-shift of $2\pi/3$), and the death mode (all three flames fall into stable combustion without any oscillation).

Manoj et al. [25] investigated the coupled four flickering candle flames in a rectangle arrangement and observed the clustering mode, in which the flames separate into two clusters of synchronized flames, the chimera mode, in which the flames separate into synchronized and desynchronized groups of flames, and the weak chimera mode, in which three frequency-synchronized flames coexist with one desynchronized flame. Forrester [16] observed that a ring of flames collectively enhance or suppress the height of a central flame. Manoj et al. [26] experimentally observed very rich dynamical behaviors in a network of flames with various arrangements such as straight line, triangle, square, star, and annular networks. Non-identical asymmetric flames were also studied by Chen et al. [27].

Many studies have attempted to understand the physics of flickering buoyant diffusion flames. It is well understood that the flame flicker is a self-exciting flow oscillation [5]. Specifically, the periodical formation, growth, and shedding of the toroidal vortices (aka vortex rings) due to the buoyance-induced Kelvin-Helmholtz instability is found to well correlate with the flame flicker [12]. The in-phase and anti-phase flicker of coupled dual buoyant diffusion flames was numerically reproduced by Yang et al. [22] for pool flames. Their results substantiate that the interaction between two toroidal vortices is responsible for the different flickering modes. The recent computational work of Yang et al. [28] on triple flame systems reproduced a few typical dynamical modes and identified two important mechanisms of vortex interaction: vortex reconnection and vortex-induced flow.

Despite the above noteworthy experimental progress in discovering dynamical behaviors of coupled flickering diffusion flames, there are many interesting problems to be solved and the study is still in its infant stage. Consequently, the present study was motivated by
recognizing the following two major deficiencies in the existing experimental studies and data analysis:

First, most previous experimental studies adopted candle flames, which are experimentally accessible but have a major drawback of the imprecise and inadequate controllability of flame parameters. In these experiments, an observed dynamical mode was often unstable and just sustained for a certain percentage of entire experimental duration. We hypothesized that the lack of adequate flame controllability is responsible for the unstable modes and that a well-controlled gas-fuel diffusion flame experiment could enhance their stability to external disturbances. Very recently, Aravind et al. [29] made an interesting attempt to use an ethanol lamp to enhance the tunability of their flame system.

Second, almost all previous experimental studies reported dynamical modes by presenting their most distinct and representative cases. Based on our experimental results to be expatiated in the paper, many cases significantly deviate from the representative cases and appear as “transition modes” that are very difficult to recognize by observing the time-resolved images and analyzing their flicker frequencies [17, 29]. Dynamical mode recognition in physical space or frequency space becomes increasingly difficult for a system consisting of an increasing number of flames. Inspired by Bifurcation theory, we hypothesized that the different modes are actually caused by the bifurcation of the flame dynamical system due to the change of its parameter values (the bifurcation parameters, e.g., the ratio of flame distance to flame diameter). Consequently, the conceptually correct method for mode recognition would focus on examining the topological structures of phase portraits in phase space.

The present study adopted Bunsen burners to produce flickering buoyant diffusion flames of methane. Owing to the precise controllability of flame parameters and the novel data analysis methodology based on Wasserstein Distance, a concept from optimal transport [30] and deep learning [31] theories, we could recognize more atypical dynamical modes in an objective way. It should be emphasized that, although the present study focused on triple flame systems, the proposed methodologies can be readily extended to larger flame systems. Furthermore, the theoretical modelling of the dynamical system, while it is worthy and challenging, will not be considered in the present work.

2. Experimental Methodology

A schematic of the key experimental apparatus established for the present study is shown in Figure 1. Some of the apparatuses have been used in our previous experimental works [32-34]. Each Bunsen burner is 1 cm in diameter and 12 cm in height, and the nozzle outlet is
pinched slightly to reduce the effect of tube thickness. The fuel flow rate $Q$ of each individual burner is accurately regulated by a mass flow controller (Alicat Scientific, MC-Series:5SLPM-D/5M). In a trial, all burners are placed at same level and supplied at same $Q$ to make flames identical. A high-speed camera (Chronos 2.1-HD) with a 105 mm UV lens was used to obtain time-resolved (500 fps) images of front views. As the camera was 2-meter away from the burners, which significantly larger than the flame characteristic scale ($<0.1m$), the influence of depth of field on the physical quantity of each flame is negligible in our image processing. To ensure the consistency in both phenomena observed and experimental data at the same testing conditions, fire-proof curtains and mesh screens were used to minimize any external disturbance, and the ambient environment was kept at a constant condition (25℃ and 1 atm) prior to each trial. Furthermore, each case was carried out for multiple times, and each trial was continuously recorded for 22s after triple flames were fully developed.

Figure 1. Schematic and photograph of the established experimental apparatus consisting of burners, fuel flow controls, and visualization systems.

To unify the previous studies on the straight-line flame system [26] and the equal-lateral triangle flame system [24, 26], we arranged the triple flames in an isosceles triangle with variable leg ($L$) and base ($B$). As a limiting case of $B = 2L$, the isosceles triangle is degenerated...
to a straight line. The case of \( B = L \) belongs to the equal-lateral triangle flame system. It should be noted that the \( D_2 \) symmetry of the isosceles triangle arrangement imposes additional constraints to the flame dynamical system. The present symmetric triple flames retain the nature of the problem (interaction among flames) and avoid unnecessary complexity (multifarious arrangements). The merit will be seen shortly in recognizing flame modes.

![Snapshots of flickering buoyant diffusion flames](image)

Figure 2. Snapshots of flickering buoyant diffusion flames for (a) single flame system and dual flame system in (b) the decoupled mode, (c) the weakly coupled mode, (d) the anti-phase mode, (e) the flickering death mode, (f) the in-phase mode, and (g) the merged case, at the distance of 8.0, 6.0, 4.0, 3.0, 2.0, and 1.5 cm, respectively. All the flames have the fuel flow rate \( Q = 0.55 \) slpm.
3. Dynamical Mode Recognition in Physical Space

3.1 Dynamical Modes of Dual Flickering Buoyant Diffusion Flames

To validate the present experimental setup, we reproduced the previous experimental observations [11, 14-21, 35-38] in single and dual flame systems. As shown in Figure 2(a), a diffusion flame significantly oscillates up and down in accompany with the flame pinch-off (an upper part of flame gets separated from the main flame). The periodic motion, named as the flame flicker here, is clearly observed in the flow rate range of $Q = 0.45 \sim 0.65$ slpm (as seen in Figure S1 in the Supplementary Material). Such flickering flames were used to constitute multiple flame systems in the present study. In Figure 2(b)-(f), by changing the distance between two identical flickering flames, there are five distinct modes such as the decoupled mode (at sufficiently large distance, e.g., $B = 8$ cm), the weakly coupled mode (at relatively large distance, e.g., $B = 6$ cm), the anti-phase (at intermediate distances, e.g., $B = 4$ cm), the flickering death (at intermediate distances of a narrow range, e.g., $B = 3$ cm), and the in-phase modes (at relatively small distances, e.g., $B = 2$ cm). Further reducing the distance (e.g., $B = 1.5$ cm) between the two flames will cause them to merge as shown in Figure 2(g), which is often not considered as a dynamic mode. It is interesting to note that the flickering death mode is characterized by that the flame pinch-off vanishes and the flame tip oscillates with a slight amplitude. The similar amplitude death phenomenon was observed in the candle-flame experiment of Manoj et al. [17] as those flames ceases to oscillate.

3.2 Dynamical Modes of Triple Flickering Buoyant Diffusion Flames

By using the present flame system, we investigated 125 cases of triple flames in an isosceles triangle by varying the base ($B/D = 4 \sim 8$), the leg ($L/D = 2 \sim 10.8$) and the flow rate ($Q = 0.45 \sim 0.65$ slpm). As three flames are too close to each other, they can merge into a bigger one with a smaller flickering frequency, as shown in Figure S2 in the Supplementary Material, which renders a case of no interest to the present study. After examining all experimental cases, seven representative dynamical modes, as shown in Figure 3, were readily identified by observing the different phase and amplitude of the flames and described in detail below. Their flame setups are given in Table S1 and videos are provided in the Supplementary Material.

Mode I: the in-phase mode appears as the three flames flicker synchronously with negligible phase difference, when the flame burners are sufficiently separated (e.g., $B = 4.0$ cm, $L = 2.8$ cm, $Q = 0.55$ slpm). It is noted that Manoj et al. [26] named the mode as “clustering”, where three flames exhibit the same frequency and maintain a constant phase difference. However, they found the case of vanishing phase difference (i.e., the in-phase mode)
was unstable in their candle experiments. In the recent computational work of Yang et al. [28], a “trefoil” vortex is formed around the flames due to the vortex reconnection. The periodic shed-off of the “trefoil” vortex results in the necking and pinch-off of the three flames in a synchronous manner.

Figure 3. Distinct stable dynamical modes of triple flickering buoyant diffusion flames. Mode I: in-phase, Mode II: flickering death, Mode III: partial flickering death, Mode IV: partial in-phase, Mode V: rotation, Mode VI: partial decoupled, and Mode VII: decoupled.

Mode II: the flickering death mode appears as the three flames oscillate with a small amplitude without the flame flicker, if flow rate is slightly smaller (e.g., $B = 4.0$ cm, $L = 2.8$ cm, $Q = 0.45$ slpm) than that in Mode I. The death mode reported by Okamoto et al. [24] is a
special case of Model II when the oscillation amplitude is sufficiently suppressed. According to Yang et al. [28], this mode occurs when the formed “trefoil” vortex sheds off in the downstream of the flames, resulting in no necking and pinch-off of the flames.

Mode III: partial flickering death mode occurs when the distances of three flame burners are further increased (e.g., \( B = 5.0 \) cm, \( L = 3.2 \) cm, \( Q = 0.45 \) slpm). The two base flames flicker in an anti-phase way while the vertex flame oscillates without the flame flicker.

Mode IV: the partial in-phase mode appears as the two base flames are in-phase to each other but anti-phase to the vertex flame, if the vertex flame is farther compared with that in Mode I (e.g., \( B = 4.0 \) cm, \( L = 4.5 \) cm, \( Q = 0.50 \) slpm). This mode was interpreted by Yang et al. [28] as that the vorticity reconnection occurs between the toroidal vortices around two flames. This mode was reported by Okamoto et al. [24] as an unstable one. Manoj et al. [26] categorized it into a “rotating cluster” mode predominantly observed for the straight-line flame configuration, in which the in-phase flame pairs transition in time.

Mode V: the rotation mode appears as the flames alternatively flicker with a fixed phase difference, if the three flames are arranged to an equal-lateral triangle (e.g., \( B = 5.0 \) cm, \( L = 5.0 \) cm, \( Q = 0.45 \) slpm). Computational results of Yang et al. [28] showed that each toroidal vortex alternatively sheds off but without apparent vorticity reconnection and that the vortex-induced flows are responsible for the vortex interaction. The phase difference is \( 2\pi/3 \) for an equal-lateral triangle due to its \( D_3 \) symmetry, but it can slightly deviate from \( 2\pi/3 \) for the isosceles triangle with \( D_2 \) symmetry (for example, the case of \( B = 5.0 \) cm, \( L = 4.7 \) cm, \( Q = 0.50 \) slpm).

Mode VI: the partial decoupled mode appears as the two base flames are anti-phase while the vertex flame flickers independently, if the vertex flame is sufficiently away from the two base flames (e.g., \( B = 4.0 \) cm, \( L = 10.2 \) cm, \( Q = 0.50 \) slpm).

Mode VII: the decoupled mode occurs if the two base flames are also sufficiently away from each other (e.g., \( B = 8.0 \) cm, \( L = 10.8 \) cm, \( Q = 0.50 \) slpm), so that the three flames flicker independently. It is noted that Manoj et al. [26] named Mode VI as “weak chimera” and Mode VII as “complete desynchrony”.

It should be emphasized that the above seven representative modes can exist within almost the entire time duration of recorded flame videos and were highly repeatable for any longer duration. However, in the previous candle flame experiment [24, 26], dynamical modes were often unstable as different modes were found during different time durations in the same test run. We hypothesized that the unstable behavior was due to the use of undisciplinable candle
flames and can be completely eliminated by using independent fuel control system for each flame in the present experiment. In fact, our experimental results verify that dynamical modes of coupled flickering flame system should be stable in a well-controlled gas-fuel diffusion flame experiment.

4. Dynamical Mode Recognition in Phase Space

To verify our second hypothesis that different modes of the flame system correspond to the different topological structures of the phase portrait in phase space, we proposed to reduce the present infinite-dimensional dynamical system, which is governed by the partial differential equations describing the time-space evolution of the chemically reacting flow, to a finite-dimensional dynamical system, whose temporal evolution can be described by ordinary differential equations. The crucial procedure of the dimension reduction is to choose a certain number of time-dependent variables that can characterize each flickering flames and their interaction.

4.1 Characteristic Quantities of Flickering Flames

As we have discussed in the Introduction, the flickering of a buoyant diffusion flame results from the formation, growth and shed-off of toroidal vortex around the flame. Apparently, there is quite freedom in selecting and acquiring characteristic quantities from the flames, for example, the pressure or temperature of flickering flame at a certain position [11]; the flame luminosity at a certain height [39]; the flame morphology information obtained from high-speed images. The first two contain local flame information of the flickering flames, while the last one contains both local (e.g., flame height and amplitude [16, 19]) and global (e.g., flame size and brightness [15, 29, 40]) information of the flickering flames. In the present study, we adopted the simplest approach by acquiring one time-dependent variable for each flame directly from the experimental high-speed images. Similar approaches have been widely used in previous studies [15-19, 24-27, 29, 40].

On each grayscale high-speed image at a certain time instant, the brightness of each pixel can be represented by an integer $b(t)$ from 0 (pure black) to 255 (pure white). A truncation value of $b = 50$ was used to obtain the bright contour for each flame in current experiments, which is equivalently the Otsu method for binarizing grayscale image [40, 41]. As shown in Figure 4, four time-varying quantities (height, width, size, and brightness) of each flickering flame can be defined: the flame height is the vertical distance between the flame tip and the nozzle; the flame width is the horizontal distance of flame contour at the height of three nozzle
diameters; the flame size is the contour area of each flame; the flame brightness is the integration of all brightness values within a flame contour. Our results show that all of the quantities exhibit the same frequency with the flickering flame, while there are phase differences among them (see more details in the Supplementary Material).

Figure 4. The schematic of establishing phase portraits in a three-dimensional phase space. Each grayscale image is processed and extracted for four time-varying quantities (height, width, size, and brightness) of each flickering flame. The flame brightness is taken as an example here. The 22s duration of triple flickering flames yields 11000 phase points in phase space.

By use of the time-varying characteristic quantities for each flame, we can obtain a representation of the dynamical behavior of the flame system in phase space. For instance, the flame brightness is expressed as $B_i(t) = \int b(t) \, dA$, where $i$ indicates the flame 1-3 and $A$ is the contour area of each flame. Consequently, the dynamical state of the triple flame system at time $t$ is represented by a phase point in a three-dimensional phase space with coordinates $(B_1(t), B_2(t), B_3(t))$. The time-evolution of the triple flame system generates a continuous phase trajectory, and the phase trajectory within a sufficiently long-time duration generates an approximately continuous phase portrait. In addition, to facilitate the recognition of topological
structure of phase portraits, the two-dimensional phase portraits are generated by projecting the
three-dimensional phase portraits to each coordinate planes, as shown in Figure 4.

By comparing the phase portraits based on the flame height, width, size, and brightness
respectively, we found that they have similar topological structures except that based on the
flame width (see the Supplementary Material for details). The reason can be explained from
the perspective of vortex dynamics. The flickering flame is a periodic phenomenon due to the
streamwise evolution of toroidal vortices, while the flame width just manifests the local
transverse information. Consequently, the global quantity of flame brightness may be better to
classify the flames than the other local flame quantities, particularly under the presence of
flame interaction. It is noted that the flame brightness is calculated from the front-view
snapshot and neglects the three-dimensional effect of flame collective behavior [16], because
we found that the flame brightness of each one in triple flickering flames compared with that
in single flickering flame has no significant difference in magnitude. Based on the above
considerations, the flame brightness was adopted in the following data analysis, and this choice
is consistent with previous studies [24-27].

4.2 Phase Portraits for Dynamical Modes of Dual Flickering Flames

To facilitate the dynamical mode recognition of triple flickering flames and to validate the
phase-space-based mode recognition, we analyzed the experimental results for dual flicking
flames, as shown in Figure 5, which correspond to the cases in Figure 2(b)-(f), namely the
decoupled, weakly decoupled, anti-phase, flickering death, and in-phase modes, respectively,
with decreasing the distance between two flames.

![Figure 5](image)

**Figure 5.** Two-dimensional phase portraits for dual flickering buoyant diffusion flames,
corresponding to the cases in Figure 2(b)-(f). There are five synchronized flickering modes
with the distance decreasing. All phase spaces, plotted by the flame brightness, have the
same ranges of values in all dimensions.

The phase portrait for dual decoupled flames, i.e., the case in Figure 2(b), shows a square-
shape, probably due to the “ergodic” nature of the decoupled dynamical system where the two
flames have no interaction. As a result, the flames can exhibit almost all dynamical states, and
the phase points can fill up almost the entire accessible phase space. The phase portrait for dual anti-phase flames, i.e., the case in Figure 2(d), has a butterfly-like shape along the (1, -1) direction. The anti-phase synchronization can be seen as the coordinates of phase points vary temporarily in an opposite way. The phase portrait for dual weakly decoupled flames, i.e., the case in Figure 2(c), has a mixed shape of a butterfly and a square, implying an intermediate mode between the decoupled and anti-phase modes. The phase portrait for dual flickering-death flames, i.e., the case in Figure 2(e), occupies a significantly smaller phase volume in the phase space than other modes. If the small-amplitude oscillation of the flames ceases, this phase volume may be vanishingly small. The phase portrait for dual in-phase flames, i.e., the case in Figure 2(f), is a slender ellipse along the (1, 1) direction, indicating that the coordinates of the phase points vary temporarily in the same way.

It is noted that previous studies [17, 29] adopted the correlation coefficient to distinguish the dynamical modes because of the directional features of phase-point distribution. But the approach cannot be suitable for complex synchronized modes in more than two coupled flickering flames. The significance of the present approach of identifying distinct shapes (topological structures) for the five typical flickering modes of dual flames is based on the following hypotheses to be verified in the following section. The phase portraits of larger dynamical systems are substantially more complex, but their two-dimensional projections can be readily analyzed by comparing them with the five typical phase portraits of the dual flame systems.

4.3 Phase Portraits for Dynamical Modes of Triple Flickering Flames

The three-dimensional phase portraits and their two-dimensional projections for seven representative dynamical modes, corresponding to the cases of Mode I-Mode VII in Figure 3 respectively, are shown in Figure 6. The 2D projections (i.e., $B_1 - B_2$, $B_1 - B_3$ and $B_2 - B_3$ phase portraits) are convenient to recognize the topological structures of 3D phase portraits, which imply the interaction between each pair of flames. It should be noted that all phase portraits have the same time duration (i.e., 22s physical time and about 250 periods) and that all phase spaces have the same ranges of coordinates in all dimensions. Subsequently, several important observations for the triple flame system can be made as follows:

First, the phase volume occupied by a phase portrait can be used as a measure of the dynamic amplitude of triple flickering flames. After unifying same ranges of coordinates in all dimensions, we obtain that the flickering death mode (Mode II) has the smallest phase volume while the decoupled mode (Mode VII) has the largest phase volume.
Second, the in-phase mode (Mode I) has a 3D phase portrait in the shape of slender ellipsoid, whose major principal axis is along the direction (1,1,1) in the phase space. Their three 2D projections are all in the shape of slender ellipse, whose major axis is along the direction of (1,1). There is a consistency for the phase spaces in dual and triple flickering flame systems.

Third, the flickering death mode (Mode II) has a 3D phase portrait in the nearly spherical shape and all its three 2D projections in the nearly round disk shape. It means that the triple
flickering flames cease flicker. The partial flickering death mode (Mode III) has two horizontal 2D projections and a butterfly-shaped 2D projection. The horizontal phase portrait for $\mathcal{B}_1 - \mathcal{B}_3$ or $\mathcal{B}_2 - \mathcal{B}_3$ suggests that one of flames stops flickering. The butterfly phase portrait for $\mathcal{B}_1 - \mathcal{B}_2$ is a closed loop spreading out along the direction (1, -1) and reflectionally symmetric with respect to the axis (1,1), which is a typical phase portrait for two anti-phase flames.

Fourth, the partial in-phase mode (Mode IV) has an ellipse-shaped 2D projection and two butterfly-shaped 2D projections. It is easy to understand that the two vertex-base flame pairs must have the same dynamical behaviors due to the $D_2$ symmetry of isosceles triangle. As shown in the two rightest columns of Figure 6, the symmetry constraint results in the same topological structure of 2D phase portraits for $\mathcal{B}_1 - \mathcal{B}_3$ and $\mathcal{B}_2 - \mathcal{B}_3$ in all cases.

Fifth, the rotation mode (Mode V) has three 2D projections in the shape of triangle. It is seen that some phase points deviate from the triangular closed loop due to external disturbance but will be attracted back to the closed loop. In fact, the nearly same interactions between toroidal vortices around each two flames are through the vortex-induced flow, instead of the vortex reconnection [28].

Finally, the decoupled mode (Mode VII) has three square-shaped 2D projections. All the phase points tend to homogenously spread out in the square probably due to the ergodicity of the decoupled system. The triangular patterns can be barely recognized in the phase portraits as the result of the very weakly flame interaction at large flame separation distance. The partial decoupled mode (Mode VI) has a butterfly phase portrait for $\mathcal{B}_1 - \mathcal{B}_2$ and two square phase portraits for $\mathcal{B}_1 - \mathcal{B}_3$ and $\mathcal{B}_2 - \mathcal{B}_3$ because the two base flames are in anti-phase but without dependence with the vertex flame.

5. Mode Recognition Based on Wasserstein Distance

5.1 Motivations for Wasserstein-distance-based Mode Recognition

Here, seven representative dynamical modes have been recognized in the present experiment of triple flickering flames, but the above proposed methodology for dynamical mode recognition in phase space lacks sufficient generality and precision if the phase portraits are different from those of the representative cases (see some untypical cases in the Supplementary Material).

There are a few considerations about the model recognition in general situation. First, many experimental cases do not generate the typical phase portraits due to the occurrence of “intermittent” behaviors. Namely, two or more typical modes transition in time within a certain
time duration, for example the case shown in Figure 7. Second, the mode recognition based on the human visual perception of shapes in phase space, is unavoidably subjective and imprecise. In addition, different understanding and cognition usually cause different choices of terminology, such as the shapes of phase portraits for $\mathcal{B}_2 - \mathcal{B}_3$ during 4.0-4.2s, $\mathcal{B}_1 - \mathcal{B}_2$ and $\mathcal{B}_1 - \mathcal{B}_3$ during 12.0-12.2s, and $\mathcal{B}_1 - \mathcal{B}_3$ during 20.0-20.2s in Figure 7. Third, the shape of a phase portrait may not be a useful concept in a higher-dimensional phase space, which certainly emerges in a system with more than three flames. Therefore, the above insights on the phase-space recognition naturally motivate us to seek a more objective methodology for model recognition.

Figure 7. The time-varying topological structures of phase portraits. The 2D phase portraits for $\mathcal{B}_1 - \mathcal{B}_2$, $\mathcal{B}_1 - \mathcal{B}_3$ and $\mathcal{B}_2 - \mathcal{B}_3$ are unstable during the 22s and not recognized readily. All phase spaces, plotted by the flame brightness, have the same ranges of coordinates in all dimensions.

Inspired by the Manifold Distribution Hypothesis from Generative Adversarial Networks [31], we proposed to identify different dynamical modes according to the probability distributions of phase points instead of the shapes of phase portraits. Specifically, any two dynamical modes of the multiple flame systems are the same if the probability distributions of their phase points are close to each other. In mathematics, the Wasserstein distance (or Kantorovich-Rubinstein metric) is a natural way to quantify the closeness of two probability distributions [42, 43]. Therefore, if any dynamical mode of the flame system is chosen as the reference, the closeness of other modes to it can be quantified by the Wasserstein distance between their probability distributions of phase points. As a result, we obtain a metric space
with probability distributions as points and Wasserstein distance as the metric; this metric space is called Wasserstein space.

5.2 Calculation of Wasserstein Distance

In the present study, we adopted the simplest 1-Wasserstein distance (aka the earth mover’s distance [44]), defined by

\[ W(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int d(x, y) \, d\gamma(x, y) \]  

(1)

where \( \mu \) and \( \nu \) are two distribution functions on a metric space \((M, d)\), \( \Gamma(\mu, \nu) \) is the set of all joint distributions whose marginals are \( \mu \) and \( \nu \). In the present problem, \( M \) is the set of indexed phase points, and \( d(x, y) \) is the natural distance between two phase points \( x \) and \( y \) by counting the difference of their indices. In addition, the distance satisfies the three basic axioms for a metric [30]: 1) \( W(\mu, \nu) \geq 0 \) and the equality holds only for \( \mu = \nu \); 2) \( W(\mu, \nu) = W(\nu, \mu) \); 3) \( W(\mu, \nu) < W(\mu, \gamma) + W(\gamma, \nu) \). The details about the theory of Wasserstein distance can be found in [30]. In short, the way to compute the distance between the distributions is analogous to shovel dirt from several piles into several holes [45], which can be well explained by the throwing-dice toy model given in the Supplementary Materials.

The key procedures for calculating the Wasserstein distance between two probability distributions of phase points are as follows,

1) to obtain the histogram of phase portraits. For example, two different cases are in the in-phase mode and one case is in the anti-phase mode, of which the three different 2D phase portraits are shown in Figure 8(a)-(c), respectively. Each phase portrait is embedded into a \( 10 \times 10 \) mesh, as shown in Figure 8(d)-(f).

2) to count and sort the number of phase points in each mesh cell for the discrete probability distributions, as shown in Figure 8(g)-(i).

3) to calculate the earth mover’s distance between the two discrete probability distributions. For example, the Wasserstein distance for the phase portraits in Figure 8(a) and Figure 8(b), denoted by \( W^{ab} \), is 11; \( W^{bc} \) is 48; \( W^{ca} \) is 42.

Prior to the Wasserstein calculation, all cases are unified, namely phase points have same number and their physical variable is normalized. The Matlab code used in the present work was based on [46] and given in Supplementary Materials. As a result, all calculated values are nonnegative, and the Wasserstein distance between similar phase portraits is much smaller than that between unlike phase portraits. For example, \( W^{ab} < W^{ac} \), namely the phase portraits in Figure 8(a) and Figure 8(b) are closer than those in Figure 8(a) and Figure 8(c). Apparently,
the distance between the same phase portraits is zero, such as \( W^{aa} = 0 \), \( W^{bb} = 0 \), and \( W^{cc} = 0 \). The other two axioms of metric are also satisfied, for example the symmetry law \( W^{ab} = W^{ba} = 11 \) and the triangle inequality \( W^{bc} < W^{ab} + W^{ac} \).

![Phase portraits and histograms](image)

Figure 8. Phase portraits of in-phase, in-phase and anti-phase modes of two adjacent flames (a)-(c), the corresponding phase point distributions in \( 10 \times 10 \) cells (d)-(f) and the corresponding discrete probability distributions of the number of phase points in each cell (g)-(i). The Wasserstein distance between the two probability distributions of (a) and (b), denoted by \( W^{ab} \), is 11. \( W^{bc} \) of (b) and (c) is 48, and \( W^{ca} \) of (c) and (a) is 42.

5.3 Dynamical Mode Recognition in Wasserstein Space

To validate the applicability of 1-Wasserstein distance in the present problem, we selected seven experimental cases, C1–C7, as shown in Figure 9, which are different from those cases in Figure 6 but can be respectively categorized into Mode I – Mode VII. To facilitate the comparison with the mode recognition based on the shape of phase portraits, we calculated the 1-Wasserstein distance between two corresponding 2D projections of 3D portraits, namely \( W^{12} \) for the base flames, \( W^{13} \) for the leg flames, and \( W^{23} \) for the other leg flames. As a result, for any two sets of three 2D phase portraits from two experimental cases, we obtained a triplet \( (W^{12}, W^{13}, W^{23}) \) of 1-Wasserstein distances. Generally, for the two cases which are classified into the same mode, their triplet of 1-Wasserstein distances should be sufficiently small.
Therefore, we can calculate the root mean square of $W_1^2$, $W_1^3$, and $W_2^3$ as a mathematically convenient indicator for the smallness of the triplet of 1-Wasserstein distances.

Table 1. The calculated triplets ($W_1^2, W_1^3, W_2^3$) and their root mean square (shown in parentheses) of 1-Wasserstein distance between Mode I – Mode VII and the stable cases C1-C7.

| Mode | C1  | C2  | C3  | C4  | C5  | C6  | C7  |
|------|-----|-----|-----|-----|-----|-----|-----|
| I    | 17  | 18  | 31  | 15  | 19  | (15)|     |
|      | (15)|     |     |     |     |     |     |
| II   | 15  | 19  | 29  | (22)|     |     |     |
|      | (22)|     |     |     |     |     |     |
| III  | 27  | 50  | 36  | (39)|     |     |     |
|      | (39)|     |     |     |     |     |     |
| IV   | 30  | 46  | 40  | (39)|     |     |     |
|      | (39)|     |     |     |     |     |     |
| V    | 62  | 41  | 53  | (53)|     |     |     |
|      | (53)|     |     |     |     |     |     |
| VI   | 43  | 80  | 70  | (66)|     |     |     |
|      | (66)|     |     |     |     |     |     |
| VII  | 105 | 92  | 82  | (93)|     |     |     |
|      | (93)|     |     |     |     |     |     |

All the calculation results for the distance between C1–C7 and Mode I – Mode VII are given in Table 1. The corresponding results are diagrammatized in Figure 9. Although the flame setup parameters ($B$, $L$ or $Q$) of C1-C7 are different from those of Mode I – Mode VII, they have very similar topological structures of 2D projections as shown in Figure 9. Specifically, the in-phase mode (C1) has all the 2D projections in the shape of slender ellipse along the direction (1,1), the flickering death mode (C2) has all the 2D projections in the shape of a round or rectangle disk, the partial flickering death mode (C3) has two nearly horizontal rectangle-shaped and one butterfly-shaped 2D projections, the partial in-phase mode (C4) has two butterfly-shaped and a nearly elliptic 2D projections, the rotation mode (C5) has all the 2D projections in the triangle shape, the partial decoupled mode (C6) has one butterfly-shaped and two square-shaped 2D projections, and the decoupled mode (C7) has all the 2D projections in the shape of square. It is clearly seen in Figure 9 (the rightmost column) that the same categories of modes are consistent with the “closeness” of their phase portraits, measured by the Wasserstein distance.

Two major findings for mode recognition can be concluded: 1) any two cases belonging to the same mode have relatively small distances. It can be clearly seen in Table 1 that the diagonal triplet values are significantly smaller than the off-diagonal triplets. This naturally confirms that the recognitions of C1–C7 belong to Mode I – Mode VII, respectively. 2) any two cases belonging to different modes have relatively large distances. It can be clearly seen in Table 1
that the off-diagonal triplet values are relatively large. It is reasonably inferred that, if a sufficient number of cases for each mode are available, the “smallness” and “largeness” of distance can be determined by a learning process, in which the Wasserstein distance plays a crucial role in establishing a discriminator.

Figure 9. Three-dimensional phase portraits, three two-dimensional projections, flame setup parameters (unit of $B$ and $L$: cm; unit of $Q$: slpm), and Wasserstein distances of seven experimental cases, C1–C7, which respectively correspond to Mode I – Mode VII but are different from those cases in Figure 6. All phase spaces, plotted by the flame brightness, have the same ranges of values in all dimensions. The Wasserstein distance (i.e., the radial distance) is the root mean square of $W_{12}$, $W_{13}$, and $W_{23}$. 
6. Concluding Remarks

The present work was motivated by two hypotheses that made by the authors about the existing experimental studies on multiple flickering buoyant diffusion flames as a nonlinear dynamical system of coupled oscillators. The first hypothesis is that the lack of adequate flame controllability is responsible for the unstable modes reported in the previous candle-flame experiments. The second hypothesis is that the different dynamical modes can be discriminated by recognizing their different topological structures of phase portraits in phase space. We successfully verified the two hypotheses in the system of triple flickering buoyant diffusion flames in isosceles triangle arrangement and therefore proposed a new methodological framework for studying dynamical systems of multiple flickering flames.

Bunsen burners were used to produce three identical flickering buoyant diffusion flames of methane, which were precisely controlled by each individual flow rate controller. By minimizing all possible external disturbances, we identified seven distinct stable modes: the in-phase mode, the flickering death mode, the partial flickering death mode, the partial in-phase mode, the rotation mode, the partial decoupled mode, and the decoupled mode. These modes can exist within almost the entire time duration (22 seconds) of recorded flame videos and were highly repeatable for any longer duration. These modes unify all the previously discovered modes for triple candle flames in a straight-line and equal-lateral triangle arrangement.

The proposed new methodology for dynamical mode recognition follows the following procedures. First, the coupled triple flame oscillators constitute an infinite-dimensional dynamical system, which principally should be described by partial differential equations of conservation laws for chemically reacting flows. Second, the infinite-dimensional system can be reduced to a 3D dynamical system by choosing an appropriate characteristic scalar quantity (e.g., the “flame brightness” adopted by the present study) for each flame, Third, the time evolution of the 3D dynamical system generates phase portraits in a 3D phase space, and each phase portrait generates a distribution function. Fourth, the Wasserstein distance quantifies the “closeness” of two distribution functions as such a small Wasserstein distance indicates the similarity of two phase portraits and the identification of the same dynamical mode. The present calculation results validate the proposed methodology, whose real potential should be shown in larger dynamical systems of flame and will be verified in future work.

Acknowledgement

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Supplementary Material to Dynamical Mode Recognition of Triple Flickering Buoyant Diffusion Flames in Wasserstein Space

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1. **Single Flickering Flames**

![Image](image_url)

Figure S1. Single flickering buoyant diffusion flames. The flow rates (a-e) are fixed at $Q = 0.45$, 0.50, 0.55, 0.60 and 0.65 slpm, respectively.
2. Merged Flickering Flames

Figure S2. Merged triple flickering buoyant diffusion flames at $Q = 0.45$ slpm.
3. Synchronicity of flame height, width, size and brightness

Figure S3. Time-varying graphs of scalar quantities (height, width, size, and brightness) for each flickering flame. The time corresponds to 10-12s in Figure 5. The height and width are normalized by the nozzle diameter, the size by the nozzle area, and the brightness by the product of 255 and the nozzle area.
Figure S4. Comparison of phase portraits based on the flame height, width, size, and brightness in Figure S3.
4. Untypical mode cases

Figure S5. Three-dimensional phase portraits, their two-dimensional projections, and the flame setup parameters (unit of $B$ and $L$: cm; unit of $Q$: slpm) for untypical dynamical modes. All phase spaces, plotted by the flame brightness, have the same ranges of coordinates in all dimensions.
5. A toy model of Wasserstein calculation

In the toy model of Wasserstein calculation[1], a dice has seven sides labeled from 0 to 6. By rolling the dice for two matches (first match called “dirt” and second match called “holes”; throwing many and same times in two matches), two probability distributions are given below:

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 20% | 10% | 0%  | 0%  | 30% | 0%  | 40% |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

Following the principle of optimal transport, the minimal amount of work needed to move the dirt distribution to the holes distribution contains five parts:

The first work is $0.2 \times (1 - 0) = 0.2$;

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 0%  | 30% | 0%  | 0%  | 30% | 0%  | 40% |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

The second work is $0.2 \times (4 - 1) = 0.6$;

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 0%  | 50% | 0%  | 0%  | 10% | 0%  | 40% |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

The third work is $0.1 \times (4 - 2) = 0.2$;

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 0%  | 50% | 10% | 0%  | 0%  | 0%  | 40% |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

The fourth work is $0.2 \times (6 - 2) = 0.8$;

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 0%  | 30% | 30% | 0%  | 0%  | 0%  | 20% |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

The fifth work is $0.2 \times (6 - 4) = 0.4$;

| Dice Side | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Dirt distribution | 0%  | 30% | 30% | 0%  | 20% | 0%  | 0%  |
| Holes distribution | 0%  | 50% | 30% | 0%  | 20% | 0%  | 0%  |

The earth mover’s distance/1-Wasserstein distance, namely the “closeness” of distributions, is the total work, which can be obtained by summing the product of flow and distance.

$$W_1 = 0.2 + 0.6 + 0.2 + 0.8 + 0.4 = 2.2$$
6. Matlab code of Wasserstein calculation

The MATLAB Code is used to calculate the Wasserstein distance between two discrete probability measures from phase portraits of triple flickering flames. The function WS_DISTANCE comes from https://github.com/nklb/wasserstein-distance

```matlab
clc; clear;

%%---Read Data1 for the phase portrait of triple flickering flames---
data1=importdata('data1filename');
[m1,n1]=size(data1); % the data matrix information

%% Read Data2 for the phase portrait of triple flickering flames
data2=importdata('data2filename');
[m2,n2]=size(data2);

%% Show the phase portraits
figure(1);

%% Three 2D projections of 3D phase portrait from Data1
subplot(2,3,1);
colormap jet;
scatter(data1(:,2),data1(:,4),2,data1(:,1),'filled');
title('L-R');
grid on;

subplot(2,3,2);
scatter(data1(:,2),data1(:,3),2,data1(:,1),'filled');
title('L-C');
grid on;

subplot(2,3,3);
scatter(data1(:,3),data1(:,4),2,data1(:,1),'filled');
title('C-R');
grid on;

%% Three 2D projections of 3D phase portrait from Data2
subplot(2,3,4);
scatter(data2(:,2),data2(:,4),2,data2(:,1),'filled');
```
title('L-R'); grid on; subplot(2,3,5); scatter(data2(:,2),data2(:,3),2,data2(:,1),'filled'); title('L-C'); grid on; subplot(2,3,6); scatter(data2(:,3),data2(:,4),2,data2(:,1),'filled'); title('C-R'); grid on;

%%---Check the input data---
if n1~=n2||m1~=m2
disp('STOP!!! Two groups are not consistent'); return;
end

%%---Calculate the probability distribution of phase points in a 10x10 mesh---
%% W^12 Ws_distance for base flames
Ndistribution1=hist3([data1(:,2),data1(:,4)],'Nbins',[10 10]);
Ndistribution2=hist3([data2(:,2),data2(:,4)],'Nbins',[10 10]);
%% The Wasserstein Calculation
%% WS_DISTANCE(u,v,p) returns the p-Wasserstein distance between the
discrete probability measures u and v, p must be 1 or 2.
myvalue(1)=ws_distance(Ndistribution1(:), Ndistribution2(:), 1);
disp('Ws_distance for base:',num2str(myvalue(1)));

%% W^13 and W^23 Ws_distance for leg flames
for i=2:1:3
Ndistribution1=hist3([data1(:,i),data1(:,i+1)]); 
Ndistribution2=hist3([data2(:,i),data2(:,i+1)]); 
myvalue(i)=ws_distance(Ndistribution1(:,), Ndistribution2(:,), 1); 
disp('Ws_distance for leg:',num2str(myvalue(i)));
end

%%% WD Calculation Done%%%
%%----Show the Histogram and Contour----
%phase portrait of base flames for Data1
x=data1(:,2);
y=data1(:,4);

%Histogram
figure(2);
hist3([x,y], 'Nbins', [10 10]);
hold on;

%Contour
N1 = hist3([x,y], 'Nbins', [10 10]);
N_pcolor = N1';
N_pcolor(size(N_pcolor,1)+1,size(N_pcolor,2)+1) = 0;
xl = linspace(min(x), max(x), size(N_pcolor,2)); %Columns of N_pcolor
yl = linspace(min(y), max(y), size(N_pcolor,1)); %Rows of N_pcolor
h = pcolor(xl, yl, N_pcolor);
colormap hot; %Change color scheme
colorbar %Display colorbar
h.ZData = -max(N_pcolor(:))*ones(size(N_pcolor));
ax = gca;
ax.ZTick(ax.ZTick < 0) = [];
xlabel('Number');
set(gca, 'xtick', [], 'xticklabel', []);
set(gca, 'ytick', [], 'yticklabel', []);
set(gca, 'ztick', [0 500 1000 1500], 'zticklabel', [0 500 1000 1500]);
grid off;
title('Histogram and Contour of Phase Portrait');
view(-30,25);

%% The subroutine of ws_distance(u_samples, v_samples, p) is available at Matlab code[2].
7. The specific parameters of representative cases in Figure 3, 6 and 9

Table S1 The flames setup parameters in an isosceles triangle for Mode I – Mode VII, and stable cases C1 – C7 (unit of $B$ and $L$: cm; unit of $Q$: slpm).

| Mode | $B$ | $L$ | $Q$ | Case | $B$ | $L$ | $Q$ |
|------|-----|-----|-----|------|-----|-----|-----|
| I    | 4.0 | 2.8 | 0.55| C1   | 4.0 | 2.8 | 0.65|
| II   | 4.0 | 2.8 | 0.45| C2   | 4.0 | 2.8 | 0.50|
| III  | 5.0 | 3.2 | 0.45| C3   | 6.0 | 3.6 | 0.45|
| IV   | 4.0 | 4.5 | 0.50| C4   | 7.0 | 4.0 | 0.50|
| V    | 5.0 | 5.0 | 0.45| C5   | 5.0 | 5.0 | 0.50|
| VI   | 4.0 | 10.2| 0.50| C6   | 4.0 | 8.2 | 0.50|
| VII  | 8.0 | 10.8| 0.50| C7   | 8.0 | 8.9 | 0.50|

Videos of the corresponding cases are included in the Supplementary Material.

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