Patterns of chiral symmetry breaking and a candidate for a $C$-theorem in four dimensions

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We test a candidate for a four-dimensional $C$-function. This is done by considering all asymptotically free, vectorlike gauge theories with $N_f$ flavors and fermions in arbitrary representations of any simple Lie group. Assuming spontaneous breaking of chiral symmetry in the infrared limit and that the value of the $C$-function in this limit is determined by the number of Goldstone bosons, we find that only in the case of a theory with two colors and fermions in one single pseudo-real representation of $SU(2)$ the $C$-theorem seems to be violated. Conversely, this might also be a sign of new constraints, restricting the number of flavors consistent with spontaneous chiral symmetry breaking. For all other groups and representations we find that this candidate $C$-function decreases along the renormalization group flow.

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I. INTRODUCTION

For two-dimensional field theories, Zamolodchikov’s $C$-theorem states that it is possible to construct a $C$-function which decreases monotonically along the renormalization group (RG) flow. This is viewed as a sign of irreversibility of the RG flow. This $C$-function depends on the couplings of the theory and is stationary at conformal fixed points. At these it reduces to the central charge of the theory, and since this may be interpreted as a measure of the degrees of freedom, the $C$-theorem demonstrates the loss of information from short-distance behaviour to the infrared limit.

Since a $C$-theorem measuring irreversibility of RG flows holds in two dimensions, it is natural to ask whether this is also true in four dimensions. A proof that this is indeed the case has been proposed but it has not been entirely accepted. In four dimensions we face the problem that Zamolodchikov’s two-dimensional $C$-function does not generalise in a unique way. In fact there are three possible generalisations as well as combinations of those. In ref. Cardy’s proposal for a $C$-function in four dimensions has been discussed (see also [7-12]). It was seen, that the inequality $c_{UV} \geq c_{IR}$ was satisfied for all combinations compatible with asymptotic freedom under the same assumptions as we will use below. Cardy’s proposed $C$-function is constructed from the Euler term in the trace of the energy-momentum tensor. However, in this paper we will test an alternative candidate for a $C$-function, given by the coefficient proportional to the Weyl tensor in the trace anomaly. The properties of this $C$-function have been investigated previously. In a study of $\mathcal{N} = 1$ supersymmetric gauge theories it was concluded that no linear combination of this $C$-function together with Cardy’s proposal is decreasing in all models (except the trivial combination consisting of only Cardy’s function). Also, perturbative studies around perturbative fixed points have shown that in some cases this $C$-function increases along the flow. However, the only non-supersymmetric gauge theory which have been studied non-perturbatively is QCD where it was found that this $C$-function actually does decrease along the flow. Although there is evidence pointing towards Cardy’s proposal, it is still of interest to study the properties of all possible candidates. We will follow the procedure of.

The theories we are considering are asymptotically free, vector-like gauge theories with $N_f$ massless Dirac-fermions. Our assumption is that the $C$-function is given by the same expression also when we approach the infrared limit, where chiral symmetry is assumed spontaneously broken. We will not consider exotic scenarios where, in this limit, there are other massless states than precisely those required by Goldstone’s theorem. We also assume that there are no fundamental scalars in the theory. Then, for a theory with $N_f$ flavors of fermions the one-loop $\beta$-function takes the form

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[ \frac{11}{3} \ell(G) - \frac{4}{3} \ell(r) N_f \right] + \ldots,$$

and since we require that the theory be asymptotically free

$$N_f < \frac{11 \ell(G)}{4 \ell(r)}.$$

Here $\ell(G)$ and $\ell(r)$ are the indices of the representations of the gauge bosons ($G$) and fermions ($r$), respectively. This bound may be too weak since asymptotic freedom and chiral symmetry breaking may be lost for smaller values of $N_f$.

The number of flavors $N_f$ and the representation $r$ is seen to be constrained by the demand that the theory
be asymptotically free. We now want to compare the values of the $C$-function in the ultraviolet and infrared fixed points, for all irreducible representations of compact, simple Lie groups and for all $N_f$ compatible with asymptotic freedom.

The $C$-function we want to test takes the value

$$c = N_0 + 6N_{1/2} + 12N_1$$

(3)

at the fixed points. Here, $N_0$ is the number of massless real scalars, $N_{1/2}$ is the number of massless Dirac fermions and $N_1$ is the number of massless gauge bosons. In the ultraviolet limit we have

$$c_{UV} = 6N_f d(r) + 12d(G)$$

(4)

where $d(r)$ and $d(G)$ are the dimensions of the representation $r$ and of the gauge group $G$, respectively.

The value of the $C$-function at the infrared fixed point is given by the number of massless degrees of freedom, i.e. the dimension of the Goldstone manifold. Thus it depends on the way in which chiral symmetry is spontaneously broken. There are believed to be three ways in which this can happen\[15\], a conjecture which has very recently been investigated in the context of lattice gauge theories\[16\]. The three classes of breaking to consider are:

- The representation of the fermions is pseudo-real. Spontaneous breaking of chiral symmetry in this case is expected to break $SU(2N_f) \rightarrow Sp(2N_f)$.
- The representation is complex. Here we expect the symmetry breaking pattern to be $SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$.
- The representation is real. This case is similar to the pseudo-real case, but here the expected symmetry breaking pattern is $SU(2N_f) \rightarrow SO(2N_f)$

These cases can be labelled by their Dyson-indices, $\beta = 1, 2, 4$ respectively, due to a connection to Random Matrix Theory\[17\]. That these classes of spontaneous symmetry breaking actually do occur in the $N_c \rightarrow \infty$ limit has been proven in ref.\[18\] for the classes $\beta = 2, 4$ with arguments easily extended to $\beta = 1\[5\]$. Each case assumes maximal symmetry breaking consistent with the Vafa-Witten theorem\[18\] and thus gives an upper bound on the value of the $C$-function in the infrared. Determining the number of Goldstone bosons, we have

$$c_{IR} = \begin{cases} N_f(2N_f - 1) - 1 & \text{for } \beta = 1, \\ N_f^2 - 1 & \text{for } \beta = 2, \\ N_f(2N_f + 1) - 1 & \text{for } \beta = 4. \end{cases}$$

(5)

It is useful to note that $c_{IR}(\beta = 2) \leq c_{IR}(\beta = 1) < c_{IR}(\beta = 4)$. This leads to the following lemma\[5\]:

**Lemma:** If $c_{UV} \geq c_{IR}(\beta[r_0])$ for fermions in a representation $r_0$ of $G$ with dimension $d(r_0)$ and index $l(r_0)$, then this will also hold for all other representations $r$ of $G$ with $c_{IR}(\beta[r]) \leq c_{IR}(\beta[r_0])$, $d(r) \geq d(r_0)$ and $l(r) \geq l(r_0)$.

It is in the sense of the last two inequalities, that we will talk about the smallest representation.

## II. GROUPS AND REPRESENTATIONS

We now perform a systematic investigation of simple, compact Lie-groups in the Cartan classification.

$SO(N_c)$: Since $SO(2)$ is abelian, $SO(3) \sim SU(2)$, $SO(5) \sim Sp(4)$, $SO(6) \sim SU(4)$ and since $SO(4)$ is not simple, here we only consider $SO(N_c)$. The dimension of $SO(N_c)$ is $d(G) = N_c(N_c - 1)/2$ while the index of the adjoint representation is $\ell(G) = N_c - 2$.

The defining representations are real ($\beta = 4$) and for these $\ell(r) = 1$ and $d(r) = N_c$. Thus

$$c_{UV} = 6N_c N_f + 6N_c(N_c - 1)$$

$$c_{IR} = N_f(2N_f + 1) - 1.$$  

(6)

Solving the condition $c_{UV} \geq c_{IR}$ with respect to $N_f$ we find, that

$$N_f \leq \frac{1}{4} \left( 6N_c - 1 + \sqrt{3(28N_c^2 - 20N_c + 3)} \right),$$  

(7)

which is well above the bound from the condition of asymptotic freedom

$$N_f < \frac{11}{4}(N_c - 2).$$  

(8)
Again we require that $c_{\ell} \geq c_{\ell}^{}\text{UV}$ (●) and asymptotic freedom (▲).

This is illustrated in Fig. 2. We now use the lemma on all other representations of $SO(N_c)$, since all have $\ell(r) \geq 1$ and $d(r) \geq N_c$. Thus, this function satisfies the $C$-theorem for the group $SO(N_c)$

$$Sp(2N_c):$$ The dimension of $Sp(2N_c)$ is $d(G) = N_c(2N_c + 1)$ and the index of the adjoint representation is $\ell(G) = 2(N_c + 1)$.

The representations of $Sp(2N_c)$ are all real or pseudo-real. The fundamental representations of $Sp(2N_c)$ are all pseudo-real and have $\ell(r) = 1$ and $d(r) = 2N_c$. Thus

$$c_{\ell}^{\text{UV}} = 12N_c(2N_c + 1) + 12N_cN_f$$
$$c_{\ell}^{IR} = N_f(2N_f - 1) - 1.$$ (9)

Again we require that $c_{\ell}^{\text{UV}} \geq c_{\ell}^{IR}$ which corresponds to

$$N_f \leq \frac{1}{4} \left(12N_c + 1 + \sqrt{3(112N_c^2 + 40N_c + 3)}\right).$$ (10)

This should be compared to the condition of asymptotic freedom

$$N_f < \frac{11}{2}(N_c + 1).$$ (11)

This is illustrated in Fig. 3. It is seen that for $N_c = 1$ corresponding to $Sp(2)$, we have a non-trivial bound, $N_f \leq 8$ instead of the bound of $N_f < 11$ obtained from demanding asymptotic freedom. For $N_c = 2$ it looks like there is another non-trivial condition but in this case both conditions give the same bound, namely $N_f \leq 16$. Since all pseudo-real representations have $\ell(r) \geq 1$ and $d(r) \geq 2N_c$ we conclude from the lemma that the $C$-theorem is fulfilled for all pseudo-real representations as long as $N_c \geq 2$.

Considering now the real representations, the smallest of these has $d(r) = N_c(2N_c - 1)$ and $\ell(r) = 2(N_c - 1)$ (note that this representation is trivial for $Sp(2)$). In this case

$$c_{\ell}^{\text{UV}} = 12N_c(2N_c + 1)$$
$$+ 6N_f(N_c(2N_c - 1) - 1)$$
$$c_{\ell}^{IR} = N_f(2N_f + 1) - 1.$$ (12)

The condition $c_{\ell}^{\text{UV}} \geq c_{\ell}^{IR}$ is now equivalent to

$$N_f \leq \frac{1}{4} \left(12N_c^2 - 6N_c - 7 + \sqrt{3}ight.$$ 
$$\times \sqrt{48N_c^4 - 48N_c^3 + 20N_c^2 + 60N_c + 19}\).$$ (13)

The condition (3) ensuring asymptotic freedom becomes

$$N_f < \frac{11}{4} \frac{N_c + 1}{N_c - 1}.$$ (14)
and the situation is illustrated in Fig. 3. It is seen, that for $N_c \geq 2$ we have $c_{\ell}^{\text{UV}} \geq c_{\ell}^{IR}$ and since this was the smallest real representation, by the lemma we conclude that the $C$-theorem is valid for all real representations of $Sp(2N_c)$.

$SU(N_c)$: The group $SU(2)$ plays a special role, so we begin by considering $N_c \geq 3$. For $SU(N_c)$ we have $d(G) = N_c^2 - 1$ and $\ell(G) = 2N_c$. The fundamental representations are complex, they all have $\ell(r) = 1$ while the dimension is $d(r) = N_c$. The $C$-function in the two limits thus becomes

$$c_{\ell}^{\text{UV}} = 12(N_c^2 - 1) + 6N_cN_f$$
$$c_{\ell}^{IR} = N_f^2 - 1.$$ (15)
Thus \( \ell (\text{the representations have larger dimensions and indices than} \) \result was already noticed in ref. [14]. All other complex
is automatically satisfied as illustrated in Fig. 4. This
and including \( SU(4) \) and \( SU(6) \) have representations that are both smaller
than those of the adjoint representations and satisfy the
constraint from asymptotic freedom. For \( SU(4) \) there
are three relevant real representations, out of which the
smallest has \( \ell (r) = 2 \) and \( d(r) = 6 \). For this representation we get
\[
\begin{align*}
c_{UV} &= 36N_f + 180 \\
c_{IR} &= N_f(2N_f + 1) - 1,
\end{align*}
\tag{19}
\]
and thus
\[
N_f \leq \frac{1}{4}(35 + 9\sqrt{3}) \tag{20}
\]
which is above the bound of \( N_f < 11 \) from the condition (3). We now use the lemma on the other two real representations.
\( SU(6) \) only has one relevant representation which is
pseudo-real and has \( \ell (r) = 6 \) and \( d(r) = 20 \). The condition \( c_{UV} \geq c_{IR} \) is automatically satisfied, since it translates into \( N_f \leq (121 + 3\sqrt{2001})/4 \) which is much above the bound of \( N_f < 11/2 \) from (3).

Turning now to the group \( SU(2) \) the smallest pseudo-
real representation is the fundamental, which has \( \ell (r) = 1 \) and \( d(r) = 2 \). The bound of \( N_f < 11 \) from the
requirement of asymptotic freedom is seen to be above the
bound of \( N_f \leq (13 + \sqrt{465})/4 \approx 8.6 \). However, this is as
expected since \( SU(2) \sim Sp(2) \) so this is in fact the same
representation which had this feature for \( Sp(2) \). For the
real representations the smallest has \( \ell (r) = 4 \), \( d(r) = 3 \)
so that the bound of \( N_f \leq (17 + 3\sqrt{65})/4 \) from the
requirement of \( c_{UV} \geq c_{IR} \) is much above the bound of
\( N_f < 11/4 \) from (3).

The exceptional groups: For the exceptional groups we
only need to calculate the bound on \( N_f \ell (r) \) from the
requirement of asymptotic freedom and then check the
possible representations. This is done based on the tables of [24].

\( E_6 \) has adjoint index 24 and the condition (3) thus
becomes \( N_f \ell (r) < 66 \). The relevant representations are the
fundamental and the adjoint. The fundamental representation
is complex and has \( d(r) = 27 \), \( \ell (r) = 6 \) and the
adjoint has \( d(r) = 78 \). In both cases \( c_{UV} \geq c_{IR} \).

In \( E_7 \) the adjoint index is 36 and thus \( N_f \ell (r) < 99 
from (3) \). There are two relevant representations, the
fundamental is pseudo-real and has \( d(r) = 56 \) and \( \ell (r) = 12 \),
while the adjoint representation has dimension \( d(r) = 133 \). Again we find that \( c_{UV} \geq c_{IR} \).

\( E_8 \) only has one relevant representation, the fundamental
which coincides with the adjoint. Here \( d(r) = 248 \)
while \( \ell (r) = 60 \), and again it is seen that \( c_{UV} \geq c_{IR} \).

For the group \( F_4 \) the adjoint index is \( \ell (G) = 18 \) and the condition for asymptotic freedom is \( N_f \ell (r) < 99/2 
Again there are two relevant representations. The fundamental
representation is real and has \( d(r) = 26 \) and \( \ell (r) = 6 \) while the adjoint has \( d(r) = 52 \). By the lemma,
since \( c_{UV} \geq c_{IR} \) for the fundamental representation this is also the case for the adjoint.

The last group, \( G_2 \), has adjoint index \( \ell (G) = 8 \) and thus \( N_f \ell (r) < 22 \). There are three relevant representations,
all real. The adjoint representation has \( d(r) = 14 \).
while the fundamental is the smallest with $d(r) = 7$, $\ell(r) = 2$. Since $c_{UV} \geq c_{IR}$ for the fundamental we need not check the others.

III. CONCLUSION

To conclude, we have performed a systematic investigation of all representations of simple, compact Lie groups. We have shown that in only one case, the requirement of asymptotic freedom is insufficient to ensure that the candidate $C$-function considered in this paper decreases from the ultraviolet to the infrared. This one case is for one pseudo-real representation of the gauge group $SU(2) \sim Sp(2)$. In all other cases, demanding asymptotic freedom alone guarantees that the inequality $c_{UV} \geq c_{IR}$ is fulfilled.

This result supports the general belief that Cardy’s proposed $C$-function is the most likely candidate if Zamolodchikov’s $C$-theorem is to be extended to four dimensions. Alternatively, this may be viewed as an indication of a non-trivial upper limit on the number of flavors in the $SU(2)$ theory consistent with spontaneous breaking of chiral symmetry.

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[1] A. B. Zamolodchikov, JETP Lett. 43 (1986) 730.
[2] S. Forte and J. I. Latorre, Nucl. Phys. B 535 (1998) 709 [hep-th/9805015], [hep-th/9811121].
[3] H. Osborn and G. M. Shore, Nucl. Phys. B 571 (2000) 287 [hep-th/9909043].
[4] A. Cappelli, R. Guida, and N. Magnoli, [hep-th/0103237].
[5] R. D. Ball and P. H. Damgaard, Phys. Lett. B 510 (2001) 341 [hep-th/0103249].
[6] J. L. Cardy, Phys. Lett. B 215 (1988) 749.
[7] I. Jack and H. Osborn, Nucl. Phys. B 343 (1990) 647. [hep-th/9401059].
[8] A. Cappelli, D. Friedan, and J. I. Latorre, Nucl. Phys. B 352 (1991) 616.
[9] A. Cappelli, G. D’Appollonio, R. Guida, and N. Magnoli, [hep-th/0009113].
[10] F. Bastianelli, Phys. Lett. B 369 (1996) 249 [hep-th/9511065].
[11] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, Nucl. Phys. B 526 (1998) 543 [hep-th/9708042].
[12] D. Anselmi, J. Erlich, D. Z. Freedman, and A. A. Johansen, Phys. Rev. D 57 (1998) 7570 [hep-th/9711035].
[13] M. J. Duff, Nucl. Phys. B 125 (1977) 344.
[14] G. M. Shore, Phys. Lett. B 256 (1991) 407.
[15] M. E. Peskin, Nucl. Phys. B 175 (1980) 197.
[16] P. H. Damgaard, U. M. Heller, R. Niclasen, and B. Svetitsky, [hep-lat/0110028].
[17] J. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531 [hep-th/9410076].
[18] S. Coleman and E. Witten, Phys. Rev. Lett. 45 (1980) 100.
[19] C. Vafa and E. Witten, Nucl. Phys. B 234 (1984) 173.
[20] W.G. McKay and J. Patera, Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras (Marcel Dekker Inc., New York 1981).