RELATIVE LOG-SYMPELCTIC STRUCTURE ON A SEMI-STABLE DEGENERATION OF MODULI OF HIGGS BUNDLES

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Abstract

In a recent paper [3], a semi-stable degeneration of moduli space of Higgs bundles on a curve has been constructed. In this paper, we show that there is a relative log-symplectic form on this degeneration, whose restriction to the generic fibre is the classical symplectic form discovered by Hitchin. We compute the Poisson ranks at every point and describe the symplectic foliation on the closed fibre. We also show that the closed fibre, which is a variety with normal crossing singularities, acquires a structure of an algebraically completely integrable system.

1. Introduction

Degenerations of moduli spaces of bundles on curves have had several interesting applications usually combined with the induction on the genus of the curve. For instance, proof of Newstead-Ramanan conjecture and proof of the factorisation theorem. It is, therefore, a natural question to ask whether there is a semi-stable degeneration of the moduli space of stable Higgs bundles of rank \( n \geq 2 \) on a smooth projective curve of genus \( g \geq 2 \). Such a degeneration has been recently constructed by Balaji et al. in [3], extending the techniques of Gieseker [14] and Nagaraj-Seshadri [31]. We recall that the construction begins with a choice of a degeneration of the smooth curve, i.e., a flat family of curves \( X \) over a complete discrete valuation ring \( S \), whose generic fibre is a smooth projective curve of genus \( g \geq 2 \) and the closed fibre is an irreducible nodal curve with a single node. Then one can construct a flat family of varieties over the discrete valuation ring \( S \) such that

1. the generic fibre is isomorphic to the moduli of stable Higgs bundles (vector bundles) on the generic curve,
2. the total space of the family is regular over \( C \), and the closed fibre is a normal crossing divisor,
3. the closed fibre has a modular description; namely, the objects are certain admissible (2.10) Higgs bundles (admissible vector bundles) on certain semi-stable models of the nodal curve.

Let us denote the degeneration of moduli of vector bundles by \( \mathcal{M}_{GVB, S} \) and the closed fiber by \( \mathcal{M}_{GVB} \). Let us denote the degeneration of moduli of Higgs bundles by \( \mathcal{M}_{GHB, S} \) and the closed fibre by \( \mathcal{M}_{GHB} \).

In [18], Hitchin proved that the moduli space of stable Higgs bundles on a compact Riemann surface has a natural holomorphic symplectic structure. The existence of the symplectic form can be seen in the
following way. The cotangent bundle of the moduli space of vector bundles is a dense open subset of the moduli of Higgs bundles. Moreover, its complement has co-dimension two. Therefore the naturally occurring symplectic form on the cotangent bundle extends to the moduli space of Higgs bundles. Later in [9], Biswas and Ramanan and in [10], Bottacin studied the algebraic version of the symplectic form on the moduli of vector bundles on a smooth projective curve.

Let $\mathcal{M}_S \to S$ be a semi-stable degeneration of a holomorphic-symplectic variety. We call it a good degeneration if there exists a relative log-symplectic form (definition 2.29) on $\mathcal{M}_S$, whose restriction to the generic fibre is the given symplectic form.

We began with the following question. Is the semi-stable degeneration of moduli of Higgs bundles good in the above sense? To prove this, we compute the relative log-cotangent and log-tangent space of $\mathcal{M}_{GVB,S}$ at a given point in terms of the first-order infinitesimal logarithmic deformations of the objects of the moduli. Then we observe that the relative log-cotangent bundle $\Omega_{\mathcal{M}_{GVB,S}/S}(\mathcal{M}_{GVB})$ of $\mathcal{M}_{GVB,S}$ is a dense open subset of $\mathcal{M}_{GHB,S}$. By generality, $\Omega_{\mathcal{M}_{GVB,S}/S}(\mathcal{M}_{GVB})$ has a relative log symplectic form. Using the explicit description of the log-tangent space, we show that there is a skew-symmetric, non-degenerate bilinear form on the relative log-tangent space at any given point. We show that this form also coincides with the classical symplectic form on the generic fibre and also with the natural relative log symplectic form on $\Omega_{\mathcal{M}_{GVB,S}/S}(\mathcal{M}_{GVB})$. We can summarise the above discussion in the following theorem from §5.

**Theorem 1.1.** There is a relative logarithmic-symplectic form on $\mathcal{M}_{GHB,S}$, whose restriction to the generic fibre is the classical symplectic form.

For any variety $Z$, let us denote its singular locus by $\partial Z$. We see that the closed fibre of our degeneration has the following natural stratification.

$$\mathcal{M}_{GHB} \supset \partial \mathcal{M}_{GHB} \supset \partial^2 \mathcal{M}_{GHB} \supset \ldots$$

(1.1)

The log-symplectic form induces a Poisson structure on the closed fibre and every successive singular locus. In §6 and §7, we compute the Poisson rank at every point of $\mathcal{M}_{GHB}$ and show that the stratification by Poisson ranks coincides with the stratification given by the successive singular loci. To compute the Poisson rank, we first show that every smooth stratum is isomorphic, as a Poisson scheme, to a torus-quotient of a smooth variety equipped with an equivariant symplectic form (Corollary 6.16). Then we compute the drop in the Poisson rank because of the torus-quotient (Lemma 7.2).

**Theorem 1.2.** The stratification of the Poisson variety $\mathcal{M}_{GHB}$ given by the successive degeneracy loci of the Poisson structure is the same as the stratification given by the successive singular loci. Moreover, $\partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB}$ is a smooth Poisson sub-variety of dimension $2(n^2(g-1)+1)-r$ with constant Poisson rank $2(n^2(g-1)+1)-2r$. In particular, the most singular locus is a smooth Poisson variety of dimension $2(n^2(g-1)+1)-n$ with constant Poisson rank $2(n^2(g-1)+1)-2n$. 2
Let $\mathcal{M}_{\text{GH}\text{B}}$ denote the normalisation of the closed fibre $\mathcal{M}_{\text{GH}\text{B}}$ and $\partial \mathcal{M}_{\text{GH}\text{B}}$ denote the inverse image of $\partial \mathcal{M}_{\text{GH}\text{B}}$. The pullback form equips $\mathcal{M}_{\text{GH}\text{B}}$ with a log-symplectic structure. In [30], Matviichuk et al. showed that the log-cotangent bundle of a variety with a normal-crossing divisors has many natural log-symplectic forms other than the tautological one and any such form differs by a bi-residue, called the magnetic term. Moreover, any log-symplectic manifold is stably equivalent to the log-cotangent bundle of a normal-crossing divisor. We show that all the magnetic terms of the log-symplectic form on $\mathcal{M}_{\text{GH}\text{B}}$ are zero. As a consequence, we obtain the local normal form of the Poisson structure on $\mathcal{M}_{\text{GH}\text{B}}$.

In section §8, we describe the Casimir functions of the symplectic leaves of every strata $\partial^r, o\mathcal{M}_{\text{GH}\text{B}} := \partial^r \mathcal{M}_{\text{GH}\text{B}} \setminus \partial^{r+1} \mathcal{M}_{\text{GH}\text{B}}$ of $\mathcal{M}_{\text{GH}\text{B}}$. For the notation, we refer to §8. We show that every such strata $\partial^r, o\mathcal{M}_{\text{GH}\text{B}}$ is a free torus quotient of a smooth variety equipped with an equivariant symplectic form. Moreover, the latter variety has an equivariant momentum map. In this case, the momentum map descends to the stratum $\partial^r, o\mathcal{M}_{\text{GH}\text{B}}$ because the co-adjoint action of any torus is trivial. The Casimir functions of the strata $\partial^r, o\mathcal{M}_{\text{GH}\text{B}}$ are precisely the coordinate functions of the descended map. The following is the precise statement.

**Theorem 1.3.**  
(1) The map

$$\mu_r : \mathcal{M}^{n,e,ad}_{\text{GH}\text{B}, X_r} \to (T_{A_r}, e)^\vee$$

(1.2)

defined by

$$\mu_r(\mathcal{E}, \phi)(X_\psi) = \lambda(t(X_\psi)) = \text{Trace}(\phi \circ t(X_\psi)),$$

for $X_\psi \in H^0(X_r, T_{X_r})$

(1.3)

is a momentum map, where

(a) $\iota : H^0(X_r, T_{X_r}) \to \mathcal{H}^1(\mathcal{E}_*)$ denotes the differential of the orbit map $A_r \to \mathcal{M}^{n,e,ad}_{\text{GH}\text{B}, X_r}$ at the point $(\mathcal{E}, \phi)$.

(b) $\lambda$ denotes the symplectic potential on $\mathcal{M}^{n,e,ad}_{\text{GH}\text{B}, X_r}$ (2.35.1 and remark 2.36).

(2) $\mu_r(\mathcal{E}, \phi) = (\text{Trace} \phi|_{\mathcal{O}(_{R[r]}_{i})}^{(1)\oplus a_i}, \ldots, \text{Trace} \phi|_{\mathcal{O}(_{R[r]}_{i})}^{(1)\oplus a_i})$, where $\mathcal{E}|_{R[r]}_{i} \equiv \mathcal{O}|_{R[r]}_{i}^{(1)\oplus a_i} \oplus \mathcal{O}|_{R[r]}_{i}^{(1)\oplus b_i}$ for every $i = 1, \ldots, r$.

(3) The coordinate functions of $\mu_r$ are the Casimir functions of $\mathcal{M}^{n,e,ad}_{\text{GH}\text{B}}$ (6.15). In particular, the variety $\mu_r^{-1}(0) \cap \mathcal{M}^{n,e,ad}_{\text{GH}\text{B}}$ is a symplectic leaf of $\mathcal{M}^{n,e,ad}_{\text{GH}\text{B}}$ containing $\Omega, \mathcal{M}^{n,e,ad}_{\text{GH}\text{B}}$. Moreover, it consists of triples $(X_r, \mathcal{E}, \phi)$ such that the trace of $\phi|_{\mathcal{O}(_{R[r]}_{i})}^{(1)\oplus a_i} : \mathcal{O}|_{R[r]}_{i}^{(1)\oplus a_i} \to \mathcal{O}|_{R[r]}_{i}^{(1)\oplus a_i}$ is zero for all $i = 1, \ldots, r$.

In [3], Balaji et al. showed that there exists a proper Hitchin map $h : \mathcal{M}_{\text{GH}\text{B}} \to B := \oplus_{i=1}^n H^0(X_0, \omega_{X_0}^{\oplus i})$ on the moduli space of stable Gieseker-Higgs bundles. In the final section of this article, we recall the definition of an algebraically completely integrable system structure (ACIS) on a variety with normal-crossing singularities. Following the strategy of [29] and [26], we prove the following.

**Theorem 1.4.** The general fibre $h^{-1}(\xi)$ corresponding to a spectral vine curve ramified outside the nodes is Lagrangian in a symplectic leaf for the log-symplectic structure on $\mathcal{M}_{\text{GH}\text{B}}$. Therefore the Hitchin map $h : \mathcal{M}_{\text{GH}\text{B}} \to B$ is an algebraically completely integrable system (9.4).
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1.6. **Notation and convention.**

- $k := \mathbb{C} = \text{the field of complex numbers}$.
- $k[\epsilon] := \text{the ring of dual numbers over } k$.
- $\mathbb{N} := \text{monoid of all positive integers with respect to } ^{+}$.
- $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}, \text{monoid with respect to } ^{+}$.
- **Standard Log Point:** A monoid structure on $\text{Spec } k$ given by a morphism of monoids $\overline{\mathbb{N}} \to k$ which maps $0 \to 1$ and $n \to 0$ for all $n \neq 0$.
- $S := \{\eta, \eta_0\}$ denotes the spectrum of a complete discrete valuation ring, where $\eta$ denotes the generic point and $\eta_0$ denotes the closed point.
- $r := \text{rank, } d := \text{the degree and } \chi := \text{the Euler characteristic of the vector bundles. We will assume throughout that } (r, d) = 1 \text{ or equivalently } (r, \chi) = 1$.
- $\mathcal{X}$ denotes a flat family of curves whose generic fibre is smooth projective and the closed fibre is a nodal curve with a single node. We denote the nodal curve by $X_0$ and the node by $x$. We denote its normalisation by $q : \tilde{X}_0 \to X_0$ and the two preimages of the node $x$ by $\{x^+, x^\}$.
- **for any local ring $\mathcal{O}$ we will denote by $\mathcal{O}^h$ the Henselization at the maximal ideal.**

2. **Preliminaries**

2.1. **On Moduli of Gieseker-Higgs bundles.** Let $X_0$ be a projective irreducible nodal curve of genus $g \geq 2$ with a single node $x$. Let $q : \tilde{X}_0 \to X_0$ be the normalisation and $q^{-1}(x) = \{x^+, x^\}$.

**Definition 2.2.** The dualising sheaf of the nodal curve $X_0$ is the kernel of the following morphism of $\mathcal{O}_{X_0}$-modules.

$$ q_* \Omega^{\frac{1}{2}}_{\tilde{X}_0}(x^+ + x^-) \to \mathbb{C}_x, $$

(2.1)

where

(1) $\mathbb{C}_x$ denotes the sky-scraper sheaf at the point $x$. 
the map \( q_* \Omega_{\tilde{X}_0}(x^+ + x^-) \rightarrow \mathbb{C}_x \) is given by
\[
s \mapsto \text{Res}(s; x^+) + \text{Res}(s; x^-)
\] (2.2)
We denote it by \( \omega_{\tilde{X}_0} \). Here, \( \text{Res}(s; x) \) denotes the residue of a form \( s \) at a point \( x \).

**Remark 2.3.** Notice that the fibres \( \omega_{\tilde{X}_0}(x^+ + x^-)_{x^+} \) and \( \omega_{\tilde{X}_0}(x^+ + x^-)_{x^-} \) can be identified with \( \mathbb{C} \), using Poincare adjunction formula. More precisely, for any coordinate function \( z^+ \) around \( x^+ \) with \( z^+(x^+) = 0 \), the image of \( \frac{dz^+}{z^+} \) in \( \omega_{\tilde{X}_0}(x^+ + x^-)_{x^+} \) is independent of the choice of the coordinate function; the above identification between \( \omega_{\tilde{X}_0}(x^+ + x^-)_{x^+} \) and \( \mathbb{C} \) sends this independent image to \( 1 \in \mathbb{C} \). Similarly, at \( x^- \). Therefore the map (2.2) makes sense.

**Remark 2.4.** The dualising sheaf can be defined for any nodal curve similarly. To be more precise, let \( C \) be a nodal curve and \( D \) denote the set of nodes. Let \( q : \tilde{C} \rightarrow C \) denote the normalisation and \( \tilde{D} \) denote the preimage \( q^{-1}(D) \). Then the dualising sheaf \( \omega_C \) is the kernel of the map
\[
q_* \Omega_{\tilde{C}}(\tilde{D}) \rightarrow \oplus_{x \in D} \mathbb{C}_x,
\] (2.3)
where the map is constituted out of the maps (2.2) at every point \( x \in D \).

**Definition 2.5.** Let \( C \) be a nodal curve. A Higgs bundle on \( C \) is a pair \((\mathcal{E}, \phi)\), where

1. \( \mathcal{E} \) is a vector bundle on \( C \), and
2. \( \phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C \) any \( \Theta_C \)-module homomorphism.

**Definition 2.6.** Let \( r \) be a positive integer.

1. A chain of projective lines is a scheme \( R[r] \) of the form \( \bigcup_{i=1}^j R[r]_i \) such that
   a. \( R[r]_i \equiv \mathbb{P}^1 \),
   b. for any \( i < j, R[r]_i \cap R[r]_j \) consists of a single point \( p_j \) if \( j = i + 1 \) and empty otherwise.

   We call \( r \) the length of the chain \( R[r] \). Let us choose and fix two smooth points \( p_1 \) and \( p_{r+1} \) on \( R[r]_1 \) and \( R[r]_r \), respectively.

   \[ \begin{array}{c}
   p_1 \\
   \ldots \ldots \\
   \ldots \ldots \\
   \ldots \ldots \\
   p_{r+1}
   \end{array} \]

2. A Gieseker curve \( X_r \) is the categorical quotient of the disjoint union of the curves \( \tilde{X}_0 \) and \( R[r] \) obtained by identifying \( x^+ \) with \( p_1 \) and \( x^- \) with \( p_{r+1} \).

**Remark 2.7.** There is a natural morphism \( \pi_r : X_r \rightarrow X_0 \) that contracts the chain \( R[r] \) to the node \( x \) and that is isomorphism outside. It is easy to see that the pullback of the dualising sheaf \( \omega_{X_0} \) to a Gieseker curve
\( X_r \) is isomorphic to the dualising sheaf \( \omega_{X_r} \) of \( X_r \). The sheaf \( \omega_{X_r} \) can be constructed by glueing \( \omega_{X_0}(x^+ + x^-) \) and \( \omega_{X_r}|_{R[r]} \cong \mathcal{O}_{R[r]} \) by the following identifications

\[
P_x^+ : \omega_{X_0}(x^+ + x^-) \xrightarrow{d_+ - 1} \mathcal{O}_{R[r]}, \quad P_x^- : \omega_{X_0}(x^+ + x^-) \xrightarrow{d_- - 1} \mathcal{O}_{R[r]}
\]

(2.4)

Such a curve is called a semi-stable model of the stable curve \( X_0 \). In the literature, a semi-stable curve is also referred to as a pre-stable curve.

2.7.1. Choice of a degeneration of curves. Let us choose a flat family of projective curves \( X \to S \), such that

1. the generic fibre \( X_\eta \) is a smooth curve of genus \( g \geq 2 \),
2. the closed fibre is the nodal curve \( X_0 \), and
3. the total space \( X \) is regular over \( \text{Spec} \mathbb{C} \).

The existence of such a family follows from \([26, \text{Theorem B.2 and Corollary B.3, Appendix B}]\). Let us denote the relative dualising sheaf by \( \omega_{X/S} \).

Moreover, it follows from \([16, 17.16.3 (ii)]\) that there exists an étale neighbourhood \( S' \to S \) of \( \eta_0 \) (the closed point of \( S \)) such that the morphism \( X' : X \times_S S' \to S' \) has a section \( \sigma : S' \to X' \) which passes through smooth points of the morphism \( X' \to S' \).

**Definition 2.8.** For every \( S \)-scheme \( T \), a modification is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_T^{\text{mod}} & \xrightarrow{\pi_T} & \mathcal{X}_T := \mathcal{X} \times_S T \\
p_T & & \text{proj} \\
T & & \text{proj}
\end{array}
\]

(2.5)

such that

1. \( p_T : \mathcal{X}_T^{\text{mod}} \to T \) is flat,
2. the horizontal morphism is finitely presented which is an isomorphism when \( (\mathcal{X}_T)_t \) is smooth,
3. over each closed point \( t \in T \) over \( \eta_0 \in S \), we have \( (\mathcal{X}_T^{\text{mod}})_t = X_r \) for some integer \( r \) and the horizontal morphism restricts to the morphism which contracts the \( \mathbb{P}^1 \)'s on \( X_r \).

We will also alternatively call such modifications as Gieseker curves. We call two such modifications \( \mathcal{X}_T^{\text{mod}} \) and \( \mathcal{X}'_T^{\text{mod}} \) isomorphic if there exists an isomorphism \( \sigma_T : \mathcal{X}_T^{\text{mod}} \to \mathcal{X}'_T^{\text{mod}} \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{X}_T^{\text{mod}} & \xrightarrow{\sigma_T} & \mathcal{X}'_T^{\text{mod}} \\
\pi_T & & \pi'_T \\
\mathcal{X}_T & & \mathcal{X}'_T
\end{array}
\]

(2.6)

**Remark 2.9.** By definition, a modification is a pre-stable curve over the base \( T \). From remark 2.7, it follows that the pullback of the relative dualising sheaf of \( \mathcal{X}_T/T \) is isomorphic to the relative dualising sheaf of \( \mathcal{X}_T^{\text{mod}}/T \). We denote it by \( \omega_{\mathcal{X}_T^{\text{mod}}/T} \).
**Definition 2.10.** A vector bundle $E$ of rank $n$ on $X_r$ with $r \geq 1$ is called a Gieseker vector bundle if

1. $E|_{R[r]}$ is a strictly standard vector bundle on $X_r$, i.e., for each $i = 1, \ldots, r$, $\exists$ non-negative integers $a_i$ and $b_i$ such that $E|_{R[r]} \cong \mathcal{O}^{a_i} \oplus \mathcal{O}(1)^{b_i}$, and
2. the direct image $(\pi_r)_*(E)$ is a torsion-free $\mathcal{O}_{X_0}$-module.

Any vector bundle on $X_0$ is called a Gieseker vector bundle. In the literature, a Gieseker vector bundle is also called an admissible vector bundle.

A Gieseker vector bundle on a modification $\mathcal{X}_T^{\text{mod}}$ is a vector bundle such that its restriction to each $(\mathcal{X}_T^{\text{mod}})_t$ is a Gieseker vector bundle.

**Definition 2.11.** A Gieseker–Higgs bundle on $\mathcal{X}_T^{\text{mod}}$ is a pair $(E_T, \phi_T)$, where $E_T$ is a vector bundle on $\mathcal{X}_T^{\text{mod}}$, and $\phi_T : E_T \to E_T \otimes \omega_{\mathcal{X}_T^{\text{mod}}/T}$ is an $\mathcal{O}_{\mathcal{X}_T^{\text{mod}}}$-module homomorphism satisfying the following

1. $E_T$ is a Gieseker vector bundle on $\mathcal{X}_T^{\text{mod}}$,
2. for each closed point $t \in T$ over $\eta_0 \in S$, the direct image $(\pi_t)_*(E_t)$ is a torsion-free sheaf on $X_0$ and $(\pi_t)_*(E_t) \to (\pi_t)_*(E_t) \oplus \omega_{X_0}$ is an $\mathcal{O}_{X_0}$-module homomorphism. We refer to such a pair $((\pi_t)_*(E_t), (\pi_t)_*(\phi_t))$ as a torsion-free Higgs pair on the nodal curve $X_0$.

**Definition 2.12.** A Gieseker–Higgs bundle $(E_T, \phi_T)$ is called stable if the direct image $(\pi_T)_*(E_T, \phi_T)$ is a family of stable torsion-free Higgs pairs on $\mathcal{X}_T$ over $T$.

We define

$$Aut(X_r/X_0) = \left\{ \text{automorphisms of } X_r, \text{ which commute with the projection morphism to } X_0 \right\}$$

(2.7)

Notice that $Aut(X_r/X_0)$ is also the subgroup of $Aut(X_r)$, which consists of all the automorphisms, which are the identity morphism on the sub curve $\tilde{X}_0$.

**Definition 2.13.** (1) Two Gieseker vector bundles $(\mathcal{X}_T^{\text{mod}}, E_T)$ and $(\mathcal{X}_T^{\text{mod}}, E_T')$ are called equivalent if there exists an isomorphism $\sigma_T : \mathcal{X}_T^{\text{mod}} \to \mathcal{X}_T^{\text{mod}}$ such that $\sigma_T$ commutes with the projection map $\pi_T$ and $\sigma_T^* E_T'$ is isomorphic to $E_T$ as vector bundles over $\mathcal{X}_T^{\text{mod}}$.

(2) Two Gieseker–Higgs bundles $(\mathcal{X}_T^{\text{mod}}, E_T, \phi_T)$ and $(\mathcal{X}_T^{\text{mod}}, E_T', \phi_T')$ are called equivalent if there exists an isomorphism $\sigma_T : \mathcal{X}_T^{\text{mod}} \to \mathcal{X}_T^{\text{mod}}$ such that $\sigma_T$ commutes with the projection map $\pi_T$ and $(\sigma_T^* E_T', \sigma_T^* \phi_T')$ is isomorphic to $(E_T, \phi_T)$ as Higgs bundles over $\mathcal{X}_T^{\text{mod}}$.

**Definition 2.14.** [3, Definition 3.4, 3.6, 3.8]

(1) Functor of Gieseker curves $F_{GC,S}$: We define the functor of Gieseker curves

$$F_{GC,S} : \text{Sch}/S \to \text{Sets}$$
\( T \mapsto \begin{cases} \text{Isomorphism classes of} \\ \text{modifications } \mathcal{X}_T^{\text{mod}} \to \mathcal{X}_T \end{cases} \) \quad (2.8)

(2) Functor of Gieseker vector bundles \( F_{GVB,S} \): We define the functor of Gieseker vector bundles

\[
F_{GVB,S} : \text{Sch}/S \to \text{Sets}
\]

\[
T \mapsto \begin{cases} \text{Gieseker-equivalent classes of families of Gieseker} \\ \text{vector bundles i.e., pairs } (\mathcal{X}_T^{\text{mod}}, \mathcal{E}_T), \text{where } \mathcal{X}_T^{\text{mod}} \text{ is a family of Gieseker curves and } \mathcal{E}_T \text{ a family of} \\ \text{Gieseker vector bundles on } \mathcal{X}_T^{\text{mod}} \end{cases}
\] \quad (2.9)

(3) Functor of Gieseker-Higgs bundles \( F_{GHB,S} \): We define the functor of Gieseker-Higgs bundles

\[
F_{GHB,S} : \text{Sch}/S \to \text{Sets}
\]

\[
T \mapsto \begin{cases} \text{Gieseker-equivalent classes of families of Gieseker-Higgs bundles i.e., triples} \\ (\mathcal{X}_T^{\text{mod}}, \mathcal{E}_T, \phi_T : \mathcal{E}_T \to \mathcal{E}_T \otimes \omega_{\mathcal{X}_T^{\text{mod}}/T}), \text{where } \mathcal{X}_T^{\text{mod}} \text{ is a family of Gieseker curves} \\ \text{and } \mathcal{E}_T \text{ a family of Gieseker vector bundles on } \mathcal{X}_T^{\text{mod}} \text{ and } \phi_T : \mathcal{E}_T \to \mathcal{E}_T \otimes \omega_{\mathcal{X}_T^{\text{mod}}/T} \\ \text{any } \mathcal{O}_{\mathcal{X}_T^{\text{mod}}} - \text{module homomorphism} \end{cases}
\] \quad (2.10)

Let us denote by \( F_{GVB,S}^{st} \) and \( F_{GHB,S}^{st} \) the open subfunctors of stable Gieseker vector bundles and stable Gieseker-Higgs bundles, respectively. Now we recall few results from [3] and [31], which are necessary for further discussion.

(1) [31, Theorem 2] Assume \((n, d) = 1\). The functor of stable Gieseker vector bundles \( F_{GVB,S}^{st} \) is represented by a scheme \( \mathcal{M}_{GVB,S} \) which is projective and flat over \( S \). Let us denote the closed fibre by \( \mathcal{M}_{GVB} \). The variety \( \mathcal{M}_{GVB,S} \) is regular as a scheme over \( k \), and the closed fibre \( \mathcal{M}_{GVB} \) is a normal crossing divisor.

(2) [3, Theorem 1.1] Assume \((n, d) = 1\). The functor of stable Gieseker-Higgs bundles \( F_{GHB,S}^{st} \) is represented by a scheme \( \mathcal{M}_{GHB,S} \) which is quasi-projective and flat over \( S \). Let us denote the closed fibre by \( \mathcal{M}_{GHB} \). The variety \( \mathcal{M}_{GHB,S} \) is regular as a scheme over \( k \), and the closed fibre \( \mathcal{M}_{GHB} \) is a normal crossing divisor. Moreover, there is a Hitchin map \( h_S : \mathcal{M}_{GHB,S} \to B_S \) to an affine space over \( S \). Moreover, the map \( h \) is proper.

2.14.1. **Construction of the moduli Gieseker vector bundles and Gieseker-Higgs bundles.** Let us briefly recall the constructions of moduli of Gieseker vector bundles (Gieseker-Higgs bundles) from [31, Section 3] ([3, Section 5.3]).

Let us choose a relatively ample line bundle \( \mathcal{O}_{\mathcal{X}/S}(1) \) for the family of curves \( \mathcal{X}/S \). The set of all flat families of stable torsion-free sheaves (Higgs pairs) of degree \( d \) and rank \( n \) over \( \mathcal{X} \) forms a bounded
family. Therefore we can choose a large integer $m$ such that given any family of stable torsion-free Higgs pairs $(\mathcal{F}_s, \varphi_s)$, the sheaf $\mathcal{F}_s \otimes \mathcal{O}_{\mathcal{X}_s}(m)$ is generated by global sections and $H^1(\mathcal{X}_s, \mathcal{F}_s \otimes \mathcal{O}_{\mathcal{X}_s}(m)) = 0$ for every geometric point $s \in S$. Set $N := H^0(\mathcal{X}_s, \mathcal{F}_s \otimes \mathcal{O}_{\mathcal{X}_s}(m))$ for any geometric point $s \in S$. We denote by $\text{Grass}(N, n)$ the Grassmannian of $n$ dimensional quotient vector spaces of $\mathbb{C}^N$.

**Definition 2.15.** Let $\mathcal{G}_S : \text{Sch}/S \to \text{Sets}$ be the functor defined as follows:

$$\mathcal{G}_S(T) = \{(\Delta_T, V_T)\},$$

where

$$\Delta_T \subset \mathcal{X}_S \times T \times \text{Grass}(N, n)$$

is a closed subscheme and $V_T$ is a vector bundle on $\Delta_T$ such that

1. the projection $j : \Delta_T \to T \times \text{Grass}(N, n)$ is a closed immersion,
2. the projection $\Delta_T \to \mathcal{X}_S \times T$ is a modification,
3. the projection $p_T : \Delta_T \to T$ is a flat family of Gieseker curves,
4. Let $V$ be the tautological quotient bundle of rank $n$ on $\text{Grass}(N, n)$ and $V_T$ its pullback to $T \times \text{Grass}(N, n)$. Then

$$V_T := j^*(V_T)$$

be such that $V_T$ is a Gieseker vector bundle on the modification $\Delta_T$ of rank $n$ and degree $d' := N + n(g-1)$.

5. for each $t \in T$, the quotient $\mathcal{O}_{\Delta_t}^N \to V_t$ induces an isomorphism

$$H^0(\Delta_t, \mathcal{O}_{\Delta_t}^N) \cong H^0(\Delta_t, V_t)$$

and $H^1(\Delta_t, V_t) = 0$.

We denote by $P$ the Hilbert polynomial of the closed subscheme $\Delta_s$ of $\mathcal{X}_S \times \text{Grass}(N, n)$ for any geometric point $s \in S$ with respect to the polarisation $\mathcal{O}_{\mathcal{X}_s}(1) \boxtimes \mathcal{O}_{\text{Grass}(N, n)}(1)$, where $\mathcal{O}_{\text{Grass}(N, n)}(1)$ is the line bundle $\text{det } V$.

**Remark 2.16.** It is shown in [31, Proposition 8] that the functor $\mathcal{G}_S$ is represented by a $PGL(N)$-invariant open subscheme $\mathcal{Y}_S$ of the Hilbert scheme $\mathcal{H}_S := \text{Hilb}^P(\mathcal{X} \times \text{Grass}(N, n))$. Moreover, the subfunctor $\mathcal{G}_S^{st}$ of stable Gieseker vector bundles is represented by an open subscheme $\mathcal{Y}_S^{st}$ of $\mathcal{Y}_S$. The moduli of Gieseker vector bundles

$$\mathcal{M}_{GVB, S} := \mathcal{Y}_S^{st} \parallel PGL(N)$$

is the GIT quotient. Moreover, the action of $PGL(N)$ is free; therefore $\mathcal{Y}_S^{st} \to \mathcal{M}_{GVB, S}$ is a principal $PGL(N)$-bundle.
Remark 2.17. Let $\Delta_{\mathcal{Y}_S}$ be the universal object defining the functor $\mathcal{Y}^s_{S,st}$. By definition, we have the following closed immersion

$$\Delta_{\mathcal{Y}^s_{S,st}} \to \mathcal{Y}^s_{S,st} \times \text{Grass}(N,n)$$

(2.16)

More precisely, $\Delta_{\mathcal{Y}^s_{S,st}} = \{(y,x) \in \mathcal{Y}^s_{S,st} \times \text{Grass}(N,n) \mid y \in \mathcal{Y}^s_{S,st} \& x \in \Delta_y\}$. Here $\Delta_y$ denotes the fiber of the morphism $\Delta_{\mathcal{Y}^s_{S,st}} \to \mathcal{Y}^s_{S,st}$ over the point $y \in \mathcal{Y}^s_{S,st}$. Using this description, it is clear that the action of $\text{PGL}(N)$ on $\mathcal{Y}^s_{S,st} \times \text{Grass}(N,n)$ restricts to an action on the subscheme $\Delta_{\mathcal{Y}^s_{S,st}}$ such that the morphism is equivariant under the action of $\text{PGL}(N)$. Since the action of $\text{PGL}(N)$ is free on $\mathcal{Y}^s_{S,st}$ the action is also free on $\Delta_{\mathcal{Y}^s_{S,st}}$.

Definition 2.18. We define a functor

$$\mathcal{Y}^H_S : \text{Sch}/\mathcal{Y}_S \to \text{Groups}$$

which maps

$$T \to H^0(T,(p_T)_* (\mathcal{O}_T \otimes \omega_{\Delta_T/T})),$$

where $p_T : \Delta_T := \Delta_{\mathcal{Y}_{st}} \times \mathcal{Y}_S \to T$ is the projection, and $\omega_{\Delta_T/T}$ denotes the relative dualising sheaf of the family of curves $p_T$.

Remark 2.19. Since $\mathcal{Y}_S$ is a reduced scheme the functor $\mathcal{Y}^H_S$ is representable i.e., there exists a linear $\mathcal{Y}_S$-scheme $\mathcal{Y}^H_{S,st}$ which represents it. For a $S$-scheme $T$, a point in $\mathcal{Y}^H_{S}(T)$ is given by $(V_T, \phi_T)$, where

1. $V_T \in \mathcal{Y}_S(T)$, and
2. $(V_T, \phi_T)$ is a Gieseker–Higgs bundle.

The subfunctor $\mathcal{Y}^H_{S,st}$ of stable Gieseker–Higgs bundles is represented by an open subscheme $\mathcal{Y}^H_{S,st}$ of $\mathcal{Y}^H_S$. The moduli of Gieseker-Higgs bundles

$$\mathcal{M}_{\text{GHB},S} := \mathcal{Y}^H_{S,st} \parallel \text{PGL}(N)$$

(2.18)

is the GIT quotient. As before, the action of $\text{PGL}(N)$ is free and therefore $\mathcal{Y}^H_{S,st} \to \mathcal{M}_{\text{GHB},S}$ is a principal $\text{PGL}(N)$-bundle. Let us pullback the universal curve $\Delta_{\mathcal{Y}_S}$ via the morphism $\mathcal{Y}^H_{S,st} \to \mathcal{Y}^s_{S,st}$ and denote it by $\Delta_{\mathcal{Y}^H_S}$. The action of $\text{PGL}(N)$ lifts to an action on $\Delta_{\mathcal{Y}^H_S}$. The morphism $\Delta_{\mathcal{Y}^H_S} \to \mathcal{Y}^H_{S,st}$ is equivariant under the action of $\text{PGL}(N)$.

2.20. On Poisson structures on schemes. Let $X$ be a scheme over $\mathbb{C}$. For any positive integer $k$, we write $\mathcal{X}^k_X := (\Omega_X^k)^\vee$, the dual of the $\mathcal{O}_X$-module $\Omega_X^k$. This is the $\mathcal{O}_X$-module of alternating $k$-multilinear forms on $\Omega_X$. The natural map $\wedge^k T_X \to \wedge^k \mathcal{X}^k_X$ is an isomorphism for $k = 1$ but need not be isomorphism in the higher degrees. We refer to the sections of $\mathcal{X}^k_X$ as $k$-derivations.
**Definition 2.21.** [17, Definition 1] A Poisson scheme is a pair \((X, \sigma)\), where \(X\) is a scheme and \(\sigma \in H^0(X, \mathcal{X}^2_X)\) is a 2-derivation such that the \(\mathbb{C}\)-bilinear morphism

\[
\{\cdot, \cdot\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X, \quad (g, h) \mapsto \sigma(dg \wedge dh)
\] (2.19)

defines a Lie algebra structure on \(\mathcal{O}_X\). This Lie bracket is the Poisson bracket.

Using the Hom-Tensor duality we get, \(\text{Hom}(\Omega_X, T_X) \cong H^0(X, \mathcal{X}^2_X)\). Therefore, the 2-derivation \(\sigma\) induces an \(\mathcal{O}_X\)-linear map (the anchormap) \(\sigma^\flat : \Omega^1_X \to T_X\) defined by

\[
\sigma^\flat(a)(\beta) = \sigma(a \wedge \beta)
\] (2.20)

for all \(a, \beta \in \Omega^1_X\). We say that \((X, \sigma)\) is a smooth Poisson scheme if the underlying scheme \(X\) is smooth.

**Definition 2.22.** [17, Definition 2] Let \((X, \sigma)\) and \((Y, \eta)\) be Poisson schemes with corresponding brackets \(\{\cdot, \cdot\}_X\) and \(\{\cdot, \cdot\}_Y\). A morphism \(f : X \to Y\) is a Poisson morphism if it preserves the Poisson brackets, i.e., the pull-back morphism \(f^* : \mathcal{O}_Y \to f_\ast \mathcal{O}_X\) satisfies

\[
f^*\{g, h\}_Y = \{f^* g, f^* h\}_X
\] (2.21)

for all \(g, h \in \mathcal{O}_Y\). Equivalently, \(f\) is a Poisson morphism if the following diagram is commutative.

\[
\begin{array}{ccc}
\Omega_X & \xrightarrow{(df)^*} & f^* \Omega_Y \\
\downarrow \sigma_X^\flat & & \downarrow f^* \sigma_Y^\flat \\
T_X & \xrightarrow{df} & f^* T_Y
\end{array}
\] (2.22)

**Definition 2.23.** Let \((X, \sigma_X)\) be a Poisson scheme. We say the Poisson scheme \((Y, \sigma_Y)\) is a Poisson subscheme of \((X, \sigma_X)\) if \(Y\) is a subscheme of \(X\) and the embedding \(i : Y \to X\) is a Poisson morphism.

Here we recall, from [17, section 3], few examples of natural Poisson subschemes of a Poisson scheme \((X, \sigma)\).

**Example 1.** An open embedding is a Poisson subscheme in a unique way. A closed subscheme \(Y\) of \(X\) admits the structure of a Poisson subscheme if and only if \([I_Y, \mathcal{O}_X] \subset I_Y\). Note that the condition is necessary and sufficient for \(\{\cdot, \cdot\}\) to descent to a Poisson bracket on \(\mathcal{O}_Y = \mathcal{O}_X/I_Y\). In this case, the induced Poisson structure on \(Y\) is unique. We denote it by \(\sigma|_Y\) [17, Proposition 2].

**Example 2.** The irreducible components of \(X\) are Poisson subvarieties. Similarly, the singular locus of \(X\) is a Poisson subscheme [17, Lemma 3].

**Definition 2.24.** [17, Definition 5] Let \((X, \sigma)\) be a Poisson scheme. The degeneracy loci \(D_{2k}(\sigma)\) of \(\sigma\) is the locus where the morphism \(\sigma^{\flat} : \Omega^1_X \to T_X\) has rank at most \(2k\). It is the closed subscheme whose ideal sheaf
is the image of the morphism
\[ \Omega_{X}^{2k+1} \stackrel{\sigma^{k+1}}{\longrightarrow} \mathcal{O}_{X} \]  

(2.23)

where
\[ \sigma^{k+1} = \sigma \wedge \cdots \wedge \sigma \in H^{0}(X, \mathcal{O}_{X}^{2k+2}). \] 

(2.24)

Example 3. From [17, Proposition 6], it follows that for \(0 \leq 2k \leq \dim X\), the degeneracy loci \(D_{2k}(\sigma)\) are Poisson subschemes of \(X\). Notice that \(D_{2k}(\sigma) \setminus D_{2k-2}(\sigma)\) is a subscheme of \(X\) consisting of points where the rank of the morphism \(\sigma^{\flat}\) is exactly equal to 2\(k\). From [17, Lemma 5], it follows that, If \((X, \sigma)\) is a Poisson scheme, and \(Y\) is a Poisson subscheme, then \(D_{2k}(\sigma) \cap Y = D_{2k}(\sigma|_{Y})\). In particular, it implies that if \(Y\) is a Poisson subscheme of \(X\) and \(y\) is any point of \(Y\), then the Poisson rank of \(\sigma\) at \(y\) is the same as the Poisson rank of \(\sigma|_{Y}\) at \(y\). We get a natural stratification of \(X\) by closed Poisson subschemes
\[ D_{\dim X}(\sigma) := X \supset D_{\dim X-2}(\sigma) \supset \cdots \supset D_{2k}(\sigma) \supset D_{2k-2}(\sigma) \supset \cdots \supset D_{0}(\sigma). \] 

(2.25)

We refer to it as the stratification by Poisson ranks.

2.25. On log-symplectic and relative log-symplectic structure.

Definition 2.26. Let \(S\) be a discrete valuation ring and \(f : Y_{S} \rightarrow S\) be a scheme over \(S\). Let us denote the closed fibre by \(Y\). We call \(Y_{S}\) a flat degeneration over \(S\) if it satisfies the following conditions

1. \(Y_{S}\) is regular as a scheme over \(k\),
2. the generic fibre of \(f : Y_{S} \rightarrow S\) is smooth, and
3. \(Y\) is a normal crossing divisor in \(Y_{S}\).

Let \(t\) be a uniformising parameter of \(S\). Then the divisor \(Y\) is the vanishing locus of the function \(t \circ f\) on \(Y_{S}\).

Definition 2.27. [37, Definition 1.1, 1.2] The sheaf of differentials on \(Y_{S}\) with logarithmic poles along \(Y\) is defined by
\[ \Omega_{Y_{S}}(\log Y) := \{ \text{meromorphic forms } \omega \text{ on } Y_{S} \mid t \cdot \omega \text{ and } t \cdot d\omega \text{ are both regular differential forms} \} \] 

(2.26)

Since any two uniformising parameter of \(S\) differs by an unit, the definition does not depend on the choice of the uniformising parameter. By a local calculation [12, Properties 2.2, (c)] it follows that in our case \(\Omega_{Y_{S}}(\log Y)\) is a locally free sheaf. We call it the log-cotangent bundle. We call the dual of this vector bundle the log-tangent bundle and denote it by \(T_{Y_{S}}(-\log Y)\).

A similar local calculation also shows that \(\text{Coker}(f^{*}\Omega_{S}(\eta_{0}) \rightarrow \Omega_{Y_{S}}(\log Y))\) is a vector bundle on \(Y_{S}\). We call it the relative log-cotangent bundle and denote it by \(\Omega_{Y_{S}/S}(\log Y)\). We call the dual vector bundle the relative log-tangent bundle and denote it by \(T_{Y_{S}/S}(-\log Y)\). We call the restriction of the vector bundle
From [13, Theorem 3.2], we have the following inclusion of sheaves

\[ \Omega_Y(\log \partial Y) \hookrightarrow q_* \Omega_X(\log \partial X) \]  \tag{2.27}

It induces the following inclusion.

\[ q^* \Omega_Y(\log \partial Y) \hookrightarrow \Omega_X(\log \partial X) \]  \tag{2.28}

The support of the cokernel is \( \partial X \). Let \( \partial^2 X := \text{singular locus of } \partial X \). We claim that the morphism is an isomorphism outside \( X \setminus \partial^2 X \).

Assuming the claim, we see that the morphism of two vector bundles is isomorphic outside co-dimension 2. Therefore the map must be an isomorphism.

The proof of the claim follows from the description [13, Equation 3.1.1, (3.1.2)]'.

Definition 2.29. A relative log-symplectic form on \( Y_S \) is a relative non-degenerate 2-form \( \omega_S \in H^0(Y_S, \Omega^2_{Y_S/S}(\log Y)) \) such that \( \omega \) is non-degenerate over \( T_{Y_S/S}(\log Y) \) and \( d\omega_S = 0 \), where \( d \) is the relative exterior derivative.

Theorem 2.30. There is a natural relative log-symplectic structure on the relative log-cotangent bundle \( \Omega_{Y_S/S}(\log Y) \).

Proof. Let \( \tilde{f} : \Omega_{Y_S/S}(\log Y) \to Y_S \) denote the projection map. The vector bundle \( \tilde{f}^* \Omega_{Y_S/S}(\log Y) \cong \Omega_{Y_S/S}(\log Y) \times_{Y_S} \Omega_{Y_S/S}(\log Y) \) has a diagonal section \( \lambda : \Omega_{Y_S/S}(\log Y) \to \Omega_{Y_S/S}(\log Y) \times_{Y_S} \Omega_{Y_S/S}(\log Y) \). But \( \tilde{f}^* \Omega_{Y_S/S}(\log Y) \) is a sub-bundle of the log cotangent bundle of \( \Omega_{Y_S/S}(\log Y) \), where the polar divisor of \( \Omega_{Y_S/S}(\log Y) \) is the inverse image \( \tilde{f}^{-1}(Y) \). Therefore we have a logarithmic 1-form \( \lambda \) over \( \Omega_{Y_S/S}(\log Y) \). We now define a two form \( \omega := -d\lambda \) by taking the exterior derivative. It is clearly a closed two form. By a local calculation, it follows that \( \omega \) is non-degenerate on \( T_{Y_S/S}(\log Y) \).

Definition 2.31. A log-symplectic form on \( Y \) is a 2-form \( \omega \in H^0(Y, \Omega^2_Y(\log \partial Y)) \) such that \( \omega \) is non-degenerate over \( T_Y(\log \partial Y) \) and \( d\omega = 0 \).

Inverting \( \omega \), we obtain a Poisson bivector

\[ \sigma \in H^0(Y, \mathfrak{X}^2_Y(-\log \partial Y)) \]  \tag{2.29}
Remark 2.32. Given a log symplectic form $\omega$ on $Y$ the pullback $\tilde{\omega} := q^* \omega$ is a log symplectic form on the normal crossing divisor $(X, \partial X)$.

Let us now discuss a prototype example of a variety with a log-symplectic form.

Example 4. Consider the smooth variety $C^k$ with coordinates $y_1, \ldots, y_k$. Consider the normal crossing divisor given by the equation $y_1 \cdots y_k = 0$. Then the coordinates of the log-cotangent bundle are $\{y_1, \ldots, y_k, p_1, \ldots, p_k\}$, where $p_j := y_j \partial_{y_j}$, for every $j = 1, \ldots, k$. Notice that the log cotangent bundle is a smooth variety isomorphic to $C^{2k}$ which has a natural normal crossing divisor given by the equation $y_1 \cdots y_k = 0$. There is a tautological logarithmic one form (Liouville 1-form) on the log-cotangent bundle which is given by

$$\lambda := \sum_{j=1}^k p_j \cdot \frac{dy_j}{y_j}$$

(2.30)

The exterior derivative

$$\omega := -d\lambda = \sum_{j=1}^k dp_j \wedge \frac{dy_j}{y_j}$$

(2.31)

is a logarithmic symplectic form. The corresponding Poisson bivector is

$$\sigma := \sum_{j=1}^k y_j \partial_{y_j} \wedge \partial_{p_j}$$

(2.32)

Now, for any skew-symmetric matrix $(B_{ij}) \in \mathbb{C}^{k \times k}$, consider the 2-form

$$B = \sum_{1 \leq i < j \leq k} B_{ij} \frac{dy_i}{y_i} \wedge \frac{dy_j}{y_j}$$

(2.33)

We can define a new 2-form

$$\omega' := \omega + B$$

(2.34)

The form $\omega'$ is again a log-symplectic form with the same polar divisor as $\omega$. The corresponding Poisson bivector

$$\sigma' := \sigma + \sum_{1 \leq i < j \leq k} B_{ij} \partial_{p_i} \wedge \partial_{p_j}$$

(2.35)

2.3.3. On the symplectic structure on the moduli of Higgs bundles on a curve. Let $X$ be a smooth projective curve. Let $\mathcal{M}_{VB}(\mathcal{M}_{HB})$ denote the moduli space of stable vector bundles (Higgs bundles) of rank $n$ and degree $d$, $(n, d) = 1$. In this subsection, we recall few results from [9] and [10] about the symplectic structure on $\mathcal{M}_{HB}$. The following results will be used in the subsequent sections of this paper.

2.3.3.1. Symplectic form on a cotangent bundle. Let $Z$ be a smooth variety. Let us denote by $f$ the projection map $\Omega_Z \to Z$. We have a natural morphism

$$\Omega_Z \xrightarrow{\Delta} \Omega_Z \times_Z \Omega_Z \cong f^* \Omega_Z \to \Omega_{\Omega^*_Z}$$

(2.36)
where $\Delta$ denotes the diagonal map. The above section induces a $1$–form on the cotangent bundle $\Omega_Z$, known as the tautological $1$–form or the Liouville $1$–form. We denote it by $\lambda$.

The negative of the exterior derivative i.e., $-d\lambda$ is a symplectic form on $\Omega_Z$. We denote it by $\omega$. Let $(z, w)$ be an element of $\Omega_Z$ over a point $z$. Let $v \in T_{\Omega_Z,(z,w)}$. Then by definition (2.36) we have

$$\lambda(v) = w(df(v)).$$

(2.37)

and for two elements $v_1, v_2 \in T_{\Omega_Z,(z,w)}$,

$$\omega(v_1, v_2) = -d\lambda(v_1, v_2).$$

(2.38)

2.33.2. The tangent space and cotangent space of $\mathcal{M}_{\mathcal{VB}}$ and $\mathcal{M}_{\mathcal{HB}}$. The tangent space of $\mathcal{M}_{\mathcal{VB}}$ at a point $\mathcal{E}$ is naturally isomorphic to the space of first-order infinitesimal deformations of the vector bundle $\mathcal{E}$. It is well-known that the latter space is isomorphic to $H^1(X, \mathcal{E} \otimes \Omega_X)$. The cotangent space of $\mathcal{M}_{\mathcal{VB}}$ at a point $\mathcal{E}$ is isomorphic to $H^1(X, \mathcal{E} \otimes \Omega_X) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega_X)$. It follows that $\Omega_{\mathcal{M}_{\mathcal{VB}}}$ is an open subset of $\mathcal{M}_{\mathcal{HB}}$ whose complement has codimension $2$. Therefore the natural Liouville form $\lambda$ and the symplectic form $\omega$ on $\Omega_{\mathcal{M}_{\mathcal{VB}}}$ extends over $\mathcal{M}_{\mathcal{HB}}$. We will now describe the forms $\lambda$ and $\omega$ on the tangent space of $\mathcal{M}_{\mathcal{HB}}$.

Given a Higgs bundle $(\mathcal{E}, \phi)$, we denote by $\mathcal{C}_*$ the following complex.

$$0 \to \mathcal{E} \otimes \Omega_X \xrightarrow{[\phi, \star]} \mathcal{E} \otimes \Omega_X \to 0,$$

(2.39)

where $[\star, \phi]$ is the morphism of $\mathcal{O}_X$-modules which maps $s \mapsto (s \otimes 1) \circ \phi - \phi \circ s$.

We denote by $\mathcal{C}_*$ the dual complex.

$$0 \to \mathcal{E} \otimes \Omega_X \xrightarrow{[\phi, \star]} \mathcal{E} \otimes \Omega_X \to 0,$$

(2.40)

where $[\phi, \star] := -[\star, \phi]$.

Let $X = \bigcup_{i \in \Lambda} U_i$ be an open cover such that $\mathcal{E}$ and $\Omega_X$ are trivial over $U_i$ for each $i \in \Lambda$. For any $i, j, k \in \Lambda$ (all distinct), we set $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. Let $\{A_{ij}\}_{i, j \in \Lambda}$ denote the transition functions of $\mathcal{E}$ with respect to the open cover $\{U_i\}_{i \in \Lambda}$. Then we have $A_{ij}A_{jk} = A_{ik}$. Let $\phi_i \in \Gamma(U_i, \mathcal{E} \otimes \Omega_X)$ denote the Higgs fields on each open sets in the open cover which glue to give the Higgs field $\phi$. In other words, $A_{ij} \circ \phi_j \circ A_{ij}^{-1} = \phi_i$ for all $i, j \in \Lambda$.

**Proposition 2.34.**

1. The tangent space of $\mathcal{M}_{\mathcal{HB}}$ at $(\mathcal{E}, \phi)$ is isomorphic to $H^1(X, \mathcal{C}_*)$.

2. The cotangent space is isomorphic to $H^1(X, \mathcal{C}_*)^\vee$.

**Proof.** By Kodaira-Spencer theory, the tangent space of $\mathcal{M}_{\mathcal{HB}}$ at a point $(\mathcal{E}, \phi)$ is the space of infinitesimal deformations of the Higgs bundle. The elements of this space can be expressed as pairs $(s_{ij}, t_i) \in \Gamma(U_{ij}, \mathcal{E} \otimes \Omega_X) \times \Gamma(U_i, \mathcal{E} \otimes \Omega_X)$ such that

1. $s_{ij}A_{jk} + A_{ij}s_{jk} = t_i$,
\[(2)\ t_i A_{ij} - A_{ij} t_j = s_{ij} \phi_j - \phi_i s_{ij}.\]

Therefore from [9, proof of theorem 2.3] and [10, Proposition 3.1.2], it follows that \((\{s_{ij}\}, \{t_i\})\) defines an element of \(\mathbb{H}^1(\mathcal{E}_*)\) and the space of infinitesimal first-order deformations of the Higgs bundle is isomorphic to \(\mathbb{H}^1(\mathcal{E}_*)\).

Using duality of hypercohomologies, one can show that the cotangent space of \(\mathcal{M}_{HB}\) is isomorphic to \(\mathbb{H}^1(X, \mathcal{E}_*^{\vee}(\mathcal{E}, \phi))\) whose elements can be expressed as pairs \((s_{ij}, t_i)\) satisfying the following condition:

1. \(s_{ij} A_{jk} + A_{ij} s_{jk} = s_{ik}\) as elements of \(\Gamma(U_{ijk}, \mathcal{E} \otimes \Omega_X)\),
2. \(t_i A_{ij} - A_{ij} t_j = - s_{ij} \phi_j + \phi_i s_{ij}\) as elements of \(\Gamma(U_{ij}, \mathcal{E} \otimes \Omega_X)\).

\[\square\]

**Remark 2.35.** By a different interpretation, as in [9, proof of Theorem 2.3] and [10, Proposition 3.1.2], the elements of the tangent space \(\mathbb{H}^1(\mathcal{E}_*)\) can be equivalently described as pairs of co-cycles \((s_{ij}, t_i)\) satisfying the following condition:

1. \(s_{ij} + s_{jk} = s_{ik}\),
2. \(t_i - t_j = s_{ij} \phi - \phi s_{ij}\).

Now, consider the short exact sequence of complexes:

\[0 \to \mathcal{E} \otimes \Omega_X[-1] \to \mathcal{E} \to \mathcal{E} \otimes \Omega_X \to 0\]  
(2.41)

The following is the long exact sequence of hypercohomologies of the above short exact sequence.

\[0 \to \mathbb{H}^0(\mathcal{E}_*) \to H^0(\mathcal{E} \otimes \Omega_X) \to H^0(\mathcal{E} \otimes \Omega_X) \to \mathbb{H}^1(\mathcal{E}_*) \to H^1(\mathcal{E} \otimes \Omega_X) \to H^1(\mathcal{E} \otimes \Omega_X) \to \mathbb{H}^2(\mathcal{E}_*) \to 0\]  
(2.42)

There is a natural forgetful morphism from the functor of Higgs bundles on \(X\) to the functor of the vector bundles on \(X\). Let us denote the morphism by \(f\). In the long exact sequence (2.42), the map

\[\mathbb{H}^1(\mathcal{E}_*) \to H^1(\mathcal{E} \otimes \Omega_X)\]

is the differential \(df\) of the forgetful map \(f\).

**2.35.1. Description of the symplectic potential.** Consider the morphism of vector spaces

\[\lambda : \mathbb{H}^1(\mathcal{E}_*) \to \mathbb{C}\]

given by

\[v \to \phi(df(v)).\]

where \(df\) is the morphism (2.43).
If \((s_{ij}, t_i)\) represents the tangent vector \(v\) (Proposition 2.34 and remark 2.35), then

\[
\phi(df(v)) = \text{Trace}(\phi \circ s_{ij}) \tag{2.45}
\]

Notice that \(\{\text{Trace}(\phi \circ s_{ij})\}_{i \in \Lambda}\) is an 1-cocycle of \(\Omega_X\) and hence an element of \(H^1(X, \Omega_X) \cong \mathbb{C}\). The equality on the right in (2.45) can be seen using the Trace paring on the sheaf \(\mathcal{E}nd\). By definition (2.37), \(\lambda\) is the extension of the Liouville 1-form \(\Omega_{db}\).

### 2.35.2. Description of the symplectic form

Consider the morphism of complexes \(\mathcal{E}^{\vee} \to \mathcal{E}\).

\[
\begin{array}{ccc}
\mathcal{E} nd \mathcal{E} & \xrightarrow{1} & \mathcal{E} nd \mathcal{E} \\
\downarrow{[\phi, \ast]} & & \downarrow{[\phi, \ast]} \\
\mathcal{E} nd \mathcal{E} \otimes \Omega_X & \xrightarrow{\text{Trace}} & \mathcal{E} nd \mathcal{E} \otimes \Omega_X
\end{array} \tag{2.46}
\]

It induces a skew-symmetric pairing

\[
H^1(\mathcal{E}^{\vee}) \to H^1(\mathcal{E}) \tag{2.47}
\]

From the diagram 5.7, it follows that the above morphism can be expressed as

\[
(s_{ij}, t_i) \mapsto (s_{ij}, -t_i) \tag{2.48}
\]

in terms of the co-cycle descriptions (Proposition 2.34 and remark 2.35). It induces a bilinear skew-symmetric pairing

\[
\omega: H^1(\mathcal{E}) \times H^1(\mathcal{E}) \to \mathbb{C} \tag{2.49}
\]

Alternatively, the pairing can be described as the following composition

\[
H^1(\mathcal{E}) \times H^1(\mathcal{E}) \to H^2(\mathcal{E} \otimes \mathcal{E}) \xrightarrow{\text{Trace}} H^2(\Omega_X[-1]) \cong H^1(X, \Omega_X) \cong \mathbb{C} \tag{2.50}
\]

given by

\[
((s_{ij}, t_i), (s'_{ij}, t'_i)) \mapsto s_{ij} \otimes t'_i - t_i \otimes s'_{ij} \mapsto \text{Trace}(s_{ij} \circ t'_i - t_i \circ s'_{ij}) \tag{2.51}
\]

Therefore, in terms of co-cycle, the pairing \(\omega\) is given by

\[
\omega((s_{ij}, t_i), (s'_{ij}, t'_i)) = \text{Trace}(s_{ij} \circ t'_i - t_i \circ s'_{ij}). \tag{2.52}
\]

Using this description, one can show that

\[
\omega = -d\lambda \tag{2.53}
\]

For further detail, we refer to [9] and [10].

**Remark 2.36.** The results discussed in this subsection hold for the moduli of Higgs bundles (or more generally Hitchin pairs) on any curve (not necessarily smooth). This is because the arguments in the proof...
of [9, Theorem 2.3] does not require the curve to be smooth. In §6, we will show that the moduli of Higgs bundles on a fixed Gieseker curve has a natural symplectic potential and a symplectic form. Moreover, the symplectic potential and the corresponding symplectic form can be described in terms of the co-cycles similarly as in the case of a smooth curve.

3. Functorial log structures on the moduli spaces

This section aims to define two natural logarithmic structures on the moduli space $\mathcal{M}_{GHBS}$ and to show that they are isomorphic. We refer to [22, 23] for basic definitions and results on log-geometry.

3.1. Existence of an universal family. To begin with, we show that there is a universal family over the moduli spaces $\mathcal{M}_{GVBS}$ and $\mathcal{M}_{GHBS}$. The proof is an easy adaptation of [38, proof of Theorem 3.2.1].

Proposition 3.2. Let $\mathcal{X} \to S$ be a family of curves with a section $\sigma$, as in 2.7.1. There exists a universal family of Gieseker vector bundles (Higgs bundles) on the moduli space $\mathcal{M}_{GVBS}$ ($\mathcal{M}_{GHBS}$). The varieties $\mathcal{M}_{GVBS}$ and $\mathcal{M}_{GHBS}$ are fine moduli spaces.

Proof. First let us recall that the moduli space $\mathcal{M}_{GVBS}$ is a GIT quotient of $Y_S$ by the action of $PGL(N)$ (remark 2.16). There is a universal curve $\Delta_S \subset Y_S \times Grass(N, n)$ (remark 2.17). More precisely,

$$\Delta_S = \{(h, x) \mid h \in Y_S, x \in \Delta_h\}. \quad (3.1)$$

For an element $h \in Y_S$, let $\Delta_h$ denote the fibre of $\Delta_S \to Y_S$ over the point $h$. It is the image of the morphism $X \to Grass(N, n)$ corresponding to the element $h$. From this description of the universal curve, $\Delta_S$ it follows that it is stable under the action of $PGL(N)$. Consider the $PGL(N)$-equivariant polarisation

$$\mathcal{O}_{Y_S}(s) \otimes \mathcal{O}_{Grass(N, n)}(t) \quad (3.2)$$

over $Y_S \times Grass(N, n)$, where $\mathcal{O}_{Y_S}(1)$ and $\mathcal{O}_{Grass(N, n)}(1)$ are the natural polarizations on $Y_S$ (remark 2.16) and $Grass(N, n)$, respectively and $s/t$ is sufficiently large. We denote by $\mathcal{O}_{\Delta_S}(s/t)$ the restriction of this polarisation to $\Delta_S$. Using this polarisation, we construct the GIT quotient $\Delta^{ss}_S \parallel PGL(N) \to Y^{st}_S \parallel PGL(N)$. Because of the assumption $g.c.d(\text{rank}, \text{deg}) = 1$, we have stable=semistable on $Y_S$. Since $s/t$ is sufficiently large, therefore on $\Delta_S$ we also have stable = semi-stable. In fact, the pre-image of $Y^{st}_S$ under the morphism $\Delta_S \to Y_S$ is precisely the set of semistable points in $\Delta_S$. We denote by $\mathcal{X}^{univ}_S$ the GIT quotient $\Delta^{st}_S \parallel PGL(N)$ and refer to it as the universal curve over $\mathcal{M}_{GVBS}$. Since the action of $PGL(N)$ is free on $Y^{st}_S$, from the description of the universal curve $\Delta_S$ it follows that the action of $PGL(N)$ is also free on $\Delta^{st}_S$. Therefore we have the following cartesian square

$$\begin{array}{ccc}
\Delta^{st}_S & \longrightarrow & Y^{st}_S \\
\downarrow & & \downarrow \\
\mathcal{X}^{univ}_S & \longrightarrow & \mathcal{M}_{GVBS}
\end{array} \quad (3.3)$$
where the vertical morphisms are $PGL(N)$-principal bundles.

Now let us discuss the descent of the universal vector bundle. Notice that there exists a universal bundle $U$ over $\Delta_{S}^{st}$, which is the pullback of the universal bundle over $\text{Grass}(N, n)$. Let us choose a line bundle over $\mathcal{X}$ such that the restriction of the line bundle on each fibre of $\mathcal{X} \to S$ is of degree one, e.g. the section $\sigma$ gives such a line bundle. Let us denote this line bundle by $\mathcal{O}_{\mathcal{X}/S}(1)$. With a choice of such a line bundle of relative degree one, the rest of the proof follows from similar arguments from [32, Lemma 5.11].

The relative moduli of Gieseker-Higgs bundles is a GIT quotient of $\mathcal{Y}_{S}^{H}$ by the action of $PGL(N)$ (remark 2.19). Over $\mathcal{Y}_{S}^{H}$, we can pull back the family of curves $\Delta_{S}$ and the universal vector bundle $U$ by the forgetful morphism $\mathcal{Y}_{S}^{H} \to \mathcal{Y}_{S}$. Let us denote the curve by $\Delta_{S}^{H}$ and the vector bundle by $U'$. By similar arguments, we can show that the curve and the vector bundle descend to the moduli space $\mathcal{M}_{GHB,S}$. Notice that we have a tautological section $\phi$ of the vector bundle $\Gamma(\Delta_{S}^{H}, \mathcal{E}nd U' \otimes p_{X}^{*} \omega_{X/S})$, where $p_{X}$ denotes the composite morphism $\Delta_{S} \to \mathcal{X} \times_{S} \mathcal{Y}_{S} \to \mathcal{X}$. Also, $\phi$ is $PGL(N)$-equivariant. Therefore it descends to the GIT quotient.

Remark 3.3. The functors $F_{GVB,S}^{st}$ and $F_{GHB,S}^{st}$ are represented by the varieties $\mathcal{M}_{GVB,S}$ and $\mathcal{M}_{GHB,S}$. We have natural forgetful morphisms

$$F_{GHB,S}^{st} \twoheadrightarrow F_{GVB,S}^{st} \twoheadrightarrow F_{GC,S}. \quad (3.4)$$

The natural transformations $F_{GHB,S} \to F_{GC,S}$ and $F_{GVB,S} \to F_{GC,S}$ are formally smooth. ([31, Appendix: Local theory, I] and [3, Proposition 5.12]).

3.4. Log structures on $\mathcal{M}_{GVB,S}$ and $\mathcal{M}_{GHB,S}$. Any normal crossing divisor of a smooth variety induces a natural log structure on the variety. Let us consider the logarithmic structure on the discrete valuation ring $S$ induced by the closed point $\eta_{0}$. For any log scheme $S$, we denote by $\text{Log}_{S}$ the algebraic stack classifying fine log-structures on schemes over the log scheme $S$ ([33, section 4]). We have the following two natural log structures on $\mathcal{M}_{GVB,S}$.

(1) $(\mathcal{M}_{GVB,S}, \mathcal{X}_{S}^{\text{univ}})$: the curve $\mathcal{X}_{S}^{\text{univ}} \to \mathcal{M}_{GVB,S}$ is a prestable curve (remark 2.9). So it induces a log-structure on $\mathcal{X}_{S}^{\text{univ}}$ and $\mathcal{M}_{GVB,S}$ such that the projection morphism is a morphism of logarithmic schemes [21, Global construction]. By [33, section 4], this log structure induces a morphism $f_{\text{Cur}} : \mathcal{M}_{GVB,S} \to \text{Log}_{S}$.

(2) $(\mathcal{M}_{GVB,S}, \mathcal{N}_{GVB})$: the normal crossing divisor $\mathcal{N}_{GVB} \subset \mathcal{M}_{GVB,S}$ induces a log structure on $\mathcal{M}_{GVB,S}$.

Similarly, this also induces a morphism $f_{\text{Div}} : \mathcal{M}_{GHB,S} \to \text{Log}_{S}$.

Similarly, we have two log-structures on $\mathcal{M}_{GHB,S}$.

Remark 3.5. Notice that for the first log-structure to exist we need the universal curve, the existence of which is ensured by the assumption that the curve $\mathcal{X} \to S$ has a section $\sigma$, as in 2.7.1 and Proposition 3.2. From now on in this section we will work with this assumption.
Proposition 3.6. The two log-structures on \( \mathcal{M}_{GV,S} (\mathcal{M}_{GB,H,S}) \) are isomorphic.

Proof. Let \((X_r, S)\) be any \(\text{Spec} k\)-valued point of \(\mathcal{M}_{GV,S}\), where \(X_r\) is a Gieseker curve with a chain of rational curves of length \(r\) and \(S\) is a Gieseker vector bundle on \(X_r\). Let us denote by \(A\) the Henselian local ring of \(\mathcal{M}_{GV,S}\) at the point \((X_r, S)\). Denote the maximal ideal by \(m_A\). We have the following diagram

\[
\begin{array}{ccc}
X_r & \longrightarrow & \mathcal{X}_A^\text{univ} \\
\downarrow & & \downarrow \\
\text{Spec } A/m_A & \longrightarrow & \mathcal{M}_{GV,S}
\end{array}
\]  

Both the squares are cartesian. Let \(D\) be the closed subscheme of \(\mathcal{X}_S^\text{univ}\) defined by the first Fitting ideal \(\text{Fit}^1(\Omega_{\mathcal{X}_S^\text{univ}/\text{Spec } S})\).

We claim the following:

1. \(V(\text{Fit}^1(\Omega_{\mathcal{X}_A^\text{univ}/\text{Spec } A})) = D_A = \bigcup_{j=1}^{r+1} D_{A,j}\), where \(D_A = D \times_{\mathcal{M}_{GV,S}} \text{Spec } A\) and \(D_{A,j}\) are the connected components of \(D_A\).

2. around \(D_1\),

\[
R_i := \mathcal{O}_{\mathcal{X}_A^\text{univ}, D_i}^h = A[x,y]/x y - t_i, \text{ for some } t_i \in m_A.
\]

3. Set \(A_0 := A \otimes_{m_S} \mathcal{O}_S\), where \(m_S\) denotes the maximal ideal of \(\mathcal{O}_S\). Then \(A_0 = A/(t_1 \cdots t_{r+1})\), i.e., in the Henselian local ring, the normal crossing divisor is the vanishing locus of \((t_1 \cdots t_{r+1})\).

The first claim follows from the functoriality of the construction of the Fitting ideals. Second claim follows from the definition of a family of pre-stable curves. To prove the third claim note that the map

\[
A \rightarrow \frac{A[x,y]}{x y - t_i}
\]

is not smooth only over \(V(t_i)\). Therefore the map \(\mathcal{X}_A^\text{univ} \rightarrow \text{Spec } A\) is not smooth exactly over \(V(t_1 \cdots t_{r+1})\).

Let us denote the point \((X_r, S)\) of \(\mathcal{M}_{GV,B,S}\) by \(p\). For simplicity, let us denote \(\mathcal{M}_{GV,B,S}\) by \(\mathcal{M}\). So \(A = \mathcal{O}_{\mathcal{M}, p}\).

We have the following inclusions

\[
\mathcal{O}_{S, x} \rightarrow \mathcal{O}_{\mathcal{M}, p} \rightarrow \mathcal{O}_{\mathcal{M}, p}^h \rightarrow \hat{\mathcal{O}}_{\mathcal{M}, p}
\]  

For our purpose, we can assume that \(S = \text{Spec } k[[t]]\). The functor of Artin rings \(F_{GC,S}\) has a versal deformation space given by \(W := \text{Spec } k[[z_1, \ldots, z_{r+1}]]\) and a versal family of Gieseker curves \(B\) over \(W\) ([14, Lemma 4.2] and [31, "Appendix: Local Theory ", II. (a),(b), (c),(d)]). There is a morphism \(W \rightarrow S\) given by \(t \mapsto z_1 \cdots z_{r+1}\) and the fiber over \(t = 0\) is the versal space for the absolute functor \(F_{GC}\). From the construction of \(B\) and \(W\), it follows that the fiber over \(t = 0\) is precisely the locus in \(W\) over which the morphism \(B \rightarrow W\) is not smooth.

Now the restriction of the universal modification \(\mathcal{X}_S^\text{univ}\) is a modification on \(\text{Spec } \hat{\mathcal{O}}_{\mathcal{M}, p}\). By the versality property, there exists a formally smooth morphism \(\nu : \text{Spec } \hat{\mathcal{O}}_{\mathcal{M}, p} \rightarrow W\) such that \(\mathcal{X}_S^\text{univ} \cong \nu^* B\). It follows
from [14, Proposition 4.5] that $t \cdot \hat{O}_{\mathcal{M}, p} = (t_1 \cdots t_{r+1}) \hat{O}_{\mathcal{M}, p}$. Since $\mathcal{O}_{\mathcal{M}, p}^h \to \hat{O}_{\mathcal{M}, p}$ is faithfully flat, therefore $t \cdot \hat{O}_{\mathcal{M}, p} = (t_1 \cdots t_{r+1}) \mathcal{O}_{\mathcal{M}, p}$. This proves the claim (3). Since the local equations of the divisor coincide with the local equation of the nodes of the universal curve, the two log structures are isomorphic. The proof for $\mathcal{M}_{GB,S}$ is similar.

\[\square\]

Remark 3.7. Let us consider the logarithmic structure on the discrete valuation ring $S$ induced by the closed point $\eta_0$. Since the morphisms $\mathcal{M}_{GV,B} \to S$ and $\mathcal{M}_{GB,S} \to S$ are semistable degenerations; therefore, they are log-smooth morphisms. In other words, the induced morphisms $\mathcal{M}_{GV,B} \to \text{Log}_S$ and $\mathcal{M}_{GB,S} \to \text{Log}_S$ are smooth morphisms of algebraic stacks [34, 3.7 (Verification of (1.1 (iii)))].

4. Relative Log-tangent space

4.0.1. Relative Log-tangent space. In this subsection we want to compute the relative tangent space and relative log-tangent space of $\mathcal{M}_{GV,B} \to S$ and $\mathcal{M}_{GB,S} \to S$ using log-deformation theory [22],[33].

Remark 4.1. For this purpose, it is enough to concentrate on the special fibres $\mathcal{M}_{GV} \to \text{Spec} \, k$ and $\mathcal{M}_{GB} \to \text{Spec} \, k$ instead of the relative case over $S$. Therefore, we see that to compute the relative log-tangent space we can replace $S$ by a suitable etale neighbourhood of the closed point of $S$ as in 2.7.1.

Let $\pi_r : X_r \to X_0$ be a $\text{Spec} \, k$-valued point of $\mathcal{F}_{GC}$. From [21, "Global Construction", Proposition 2.1], it follows that there are canonical induced log-structures on $X_r$ and $X_0$ and as well as on $\text{Spec} \, k$ such that the arrows $X_r \to \text{Spec} \, k$ and $X_0 \to \text{Spec} \, k$ are log-smooth. It is straightforward to check that the log structure on $\text{Spec} \, k$ induced by the curve $X_r \to \text{Spec} \, k$ is the same as the log-structure induced by the following pre-log structure.

$$\oplus_{i=1}^{r+1} \mathbb{N} \to k \quad (4.1)$$

which sends

$$e_i \mapsto 0,$$

where $e_i$ is the $i$-th basis element of $\oplus_{i=1}^{r+1} \mathbb{N}$.

Lemma 4.2. Let $\mathcal{X}$ be a family of Gieseker curves over $\text{Spec} \, k[e]$ such that the fiber over the closed subscheme $\text{Spec} \, k$ is the Gieseker curve $\pi_r : X_r \to X_0$. If the induced log structure ([21, Proposition 2.1]) on $\text{Spec} \, k[e]$ is isomorphic to the pull back of the log structure of $\text{Spec} \, k$, defined above in (4.1) under the natural projection map $\text{Spec} \, k[e] \to \text{Spec} \, k$, then the deformation is trivial i.e., $\mathcal{X} \equiv X_r \times \text{Spec} \, k[e]$.

Proof. The vanishing locus of the first Fitting ideal of the relative cotangent sheaf of the morphism $\mathcal{X} \to \text{Spec} \, k[e]$ has $r + 1$ components. Let us denote them by $D_1, \ldots, D_{r+1}$. Let us denote by $U_i$ the complement
of the node $D_i$ and by $U^+_i$ the complement of the closed subset $\bigsqcup_{j \neq i} D_j$. Then $(U^+_i, U^-_i)$ is a Zariski-open covering of $\mathcal{X}$. Etale locally, around $D_i$ we have

$$\mathcal{O}_{U^+_i} \cong \frac{k[x_i, y_i, \epsilon]}{x_i y_i - \lambda_i \epsilon}. \quad (4.2)$$

We attach a log structure defined by a pre-log structure $\mathbb{N}^2 \to \frac{k[x_i, y_i, \epsilon]}{x_i y_i - \lambda_i \epsilon}$ which sends $(1, 0)$ to $x_i$ and $(0, 1)$ to $y_i$. On $U^-_i$ we consider the log structure defined by the pre-log structure $\mathbb{N} \to \mathcal{O}_{U^-_i}$ which maps $1 \to \lambda_i \epsilon$; these log structures can be glued along the intersection by using the diagonal homomorphism $\mathbb{N} \to \mathbb{N}^2$.

We denote the resulting log structure by $\mathcal{M}_i$. The induced log structure on $\text{Spec } k[\epsilon]$ is the log structure defined by the pre-log structure $\alpha_i : \mathbb{N} \to k[\epsilon]$ given by $1 \to \lambda_i \epsilon$. Let us denote the log structure by $\mathcal{L}_i$. Then we see that $\mathcal{L}_i \cong \mathbb{N} \otimes_{\alpha_i^{-1}(k[\epsilon]^*)} k[\epsilon]^* \cong \mathbb{N} \oplus k[\epsilon]^*$.

Finally the induced log structure on $\text{Spec } k[\epsilon]$ is the amalgumated sum

$$\mathcal{L}_{k[\epsilon]} := \mathcal{L}_1 \otimes_{k[\epsilon]} \cdots \otimes_{k[\epsilon]} \mathcal{L}_{r+1}. \quad (4.3)$$

It is isomorphic to the log structure associated with the pre-log structure

$$\oplus_{i=1}^{r+1} \mathbb{N} \to k[\epsilon] \quad (4.4)$$

given by

$$e_i := (0, \ldots, \underbrace{1}_{i\text{-th position}}, \ldots, 0) \mapsto \lambda_i \epsilon$$

Therefore it is isomorphic to the pull back of the log structure on $\text{Spec } k$, defined above in eq.4.1 under the natural projection map $\text{Spec } k[\epsilon] \to \text{Spec } k$ if and only if $\lambda_i = 0$ for all $i = 1, \ldots, r + 1$.

The space of infinitesimal deformations of the nodal curve $X_r$ is isomorphic to $\text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r})$. Using Local-to-global spectral sequence [1, eq. 1.2, page 169], we get

$$0 \to H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \to \text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r}) \to H^0(X_r, \mathcal{O}_{X_r}) \times \text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r}) \cong \bigoplus_{i=1}^{r+1} \text{Ext}^1(\Omega_{X_r}, \mathcal{O}_{X_r}) \to 0 \quad (4.5)$$

Since $\lambda_i = 0$ for all $i = 1, \ldots, r + 1$, from (4.2), it follows that the infinitesimal deformation $\mathcal{X}$ is an element of $H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r}))$. Now from [14, Corollary 4.4] we have the following inclusion

$$H^1(X_r, \mathcal{H}om(\Omega_{X_r}, \mathcal{O}_{X_r})) \hookrightarrow H^1(X_0, \mathcal{H}om(\Omega_{X_0}, \mathcal{O}_{X_0})). \quad (4.6)$$

Moreover, the image of any 1-cocycle/infinitesimal first order deformation $\mathcal{X}'$ of $X_r$ under this inclusion is the obstruction to extending the map $X_r \to X_0$ to a map $\mathcal{X}' \to X_0 \times \text{Spec } k[\epsilon]$. But since the deformation $\mathcal{X}$, by definition comes with a morphism to $X_0 \times \text{Spec } k[\epsilon]$, therefore the image under the inclusion is 0. Therefore $\mathcal{X} \cong X_r \times \text{Spec } k[\epsilon]$. \quad \square
Remark 4.3. [23, Example 2.5,(2)] Since our base field $k$ is algebraically closed, there is a bijection between the following two sets:

$$
\begin{align*}
\text{Isomorphism classes of integral} & \quad \text{log-structures on Spec } k \\
\text{Isomorphism classes of integral monoids} & \quad \text{having no invertible elements other than } 0
\end{align*}
$$

(4.7)

given by the following:

1. Given an integral monoid $P$ such that $P^* = \{0\}$, the corresponding log structure is $k^* \oplus P$ with

$$
k^* \oplus P \twoheadrightarrow k
$$

$$(\lambda, p) \mapsto
\begin{cases}
\lambda & \text{if } p = 0, \\
0 & \text{otherwise}
\end{cases}
$$

2. Given an integral log-structure $\alpha : \mathcal{P} \to k$, the corresponding integral monoid is $P := \mathcal{P}/\alpha^{-1}(k^*)$.

Consider the zero map

$$
P \to k
$$

(4.9)

Then the associated log structure $P \oplus k^* \cong \mathcal{P}$.

Lemma 4.4. Let $Q_0$ be an integral monoid with 0 as its only unit. Let $\mathcal{X} \to \text{Spec } k[e]$ be a family such that the fiber over $\text{Spec } k$ is isomorphic to $X_r$. The log structure on $\text{Spec } k[e]$ induced by the family $\mathcal{X}$ is isomorphic to the log structure associated with the pre-log structure

$$
\beta : Q_0 \to k[e]
$$

(4.10)

$$
q \mapsto
\begin{cases}
1 & \text{if } q = 0, \\
0 & \text{otherwise}
\end{cases}
$$

if and only if $Q_0 \cong \mathbb{N}^{r+1}$ and the pre-log structure

$$
\beta : \mathbb{N}^{r+1} \to k[e]
$$

(4.11)

is given by

$$
e_i \mapsto 0 \quad \text{for all } i = 1, \ldots, r + 1
$$

Proof. We leave it to the reader. □
Consider the following morphism of pre-log structures

\[ e_i \longrightarrow 0 \text{ for all } i = 1, \cdots, r + 1 \]

\[ e_1 + \ldots + e_{r+1} \quad \begin{array}{c}
\longrightarrow \quad \mathbb{N}^{r+1} \quad \longrightarrow \quad k[e] \\
\uparrow\quad \uparrow \quad \uparrow \\
e \quad \mathbb{N} \quad \longrightarrow \quad k \\
\end{array} \quad (4.12) \]

\[ \quad e \longrightarrow 0 \]

Let us denote the log structure induced on \( \text{Spec } k[e] \) by \( M \). We denote the log scheme \((\text{Spec } k[e], M)\) by \( X \), the log scheme \((\text{Spec } k, \mathbb{N})\) by \( Y \) and the log morphism above by \( f : X \to Y \).

**Lemma 4.5.** The group of the automorphisms \( \phi \) of the log structure \( M \) on \( \text{Spec } k[e] \) satisfying the following conditions

1. the automorphism of the underlying scheme \( \text{Spec } k[e] \) is identity,
2. the restriction \( \phi_0 \) of \( \phi \) on the closed log-subscheme \((\text{Spec } k, j^* M)\) is the identity morphism, where \( j \) is the closed immersion \( \text{Spec } k \hookrightarrow \text{Spec } k[e] \),
3. the automorphism \( \phi \) commutes with the log morphism \( f \)

has the structure of a \( k \)-vector space of dimension \( r \). We denote this group by \( \text{Aut}^{inf}_Y (M) \).

**Proof.** Notice that the log structure \( M \) is isomorphic to \( \mathbb{N}^{r+1} \oplus k[e]^* \), where the monoid product is given by addition in the first component and multiplication in the second component. The monoid morphism \( M \to k[e] \) is given by \((0, ae + b) \mapsto (0, ae + b) \) and \((e_i, 1) \mapsto 0 \) for all \( i = 1, \cdots, r + 1 \). Similarly, the log structure on \( \text{Spec } k \) associated to the prelog structure \( \mathbb{N}^{r+1} k \) is isomorphic to \( \mathbb{N} \oplus k^* \) and the monoid morphism is the morphism \( \mathbb{N} \oplus k^* \to k \) which sends \((0, \lambda) \to \lambda \) and \((e, 1) \to 0 \).

Since the restriction \( \phi_0 \) of the automorphism

\[ \mathbb{N}^{r+1} \oplus k[e]^* \xrightarrow{\phi_0} \mathbb{N}^{r+1} \oplus k[e]^* \]

to the reduced log subscheme \((\text{Spec } k, \mathbb{N}^{r+1} \oplus k^*)\) is the identity morphism. Therefore \( \phi_0(e_i, 1) = (e_i, 1) \) for \( i = 1, \cdots, r + 1 \). Therefore the first factor of \( \phi(e_i, a + be) \) also must be \( e_i \) for every \( i = 1, \cdots, r + 1 \).

Since \( \phi \) is a monoid isomorphism therefore \( \phi((0, 1)) = (0, 1) \), because \((0, 1)\) is the identity element in this monoid. Therefore \( \phi(0, a + be) = a + be \).

Now notice that the monoid \( \mathbb{N}^{r+1} \oplus (k[e])^* \) is generated by the elements of the following forms \( (e_i, 1) \mid i = 1, \cdots, r + 1 \) and \( (0, a + be) \mid a + be \in k[e]^* \). Therefore the images of these generators under \( \phi \) determine the automorphism \( \phi \). Since the first factor of \( \phi(e_i, 1) \) must be \( e_i \), therefore \( \phi(e_i, 1) = (e_i, c_i \cdot (1 + \lambda_i e)) \) for some
$c_i \in k^*$ and $\lambda_i \in k$. But since $\phi_0$ is the identity morphism therefore we see that $c_i = 1$ for all $i = 1, \ldots, r + 1$. Therefore $\phi(e_i, 1) = (e_i, 1 + \lambda_i e)$, for some $\lambda_i \in k$.

Since the isomorphism $\phi$ commutes with the log morphism $f$, we must have $\phi(e_1 + \cdots + e_{r+1}, 1) = (e_1 + \cdots + e_{r+1}, 1)$. Since $\phi$ is automorphism of a monoid we have $\phi(e_1 + \cdots + e_{r+1}, 1) = (e_1 + \cdots + e_{r+1}, 1 + \lambda_1 e) \cdots (1 + \lambda_{r+1} e) = (e_1 + \cdots + e_{r+1}, 1 + (\lambda_1 + \cdots + \lambda_{r+1}) e)$. Therefore the sum $\lambda_1 + \cdots + \lambda_{r+1} = 0$.

Therefore the group of such automorphism is isomorphic to the underlying additive group of the vector space $k^r$. If $\rho \in Aut^f_Y(M)$ such that $\phi(e_i, 1) = (e_i, 1 + \lambda_i e)$ and $\lambda \in k$ any scalar then we define $(\lambda \ast \phi)(e_i, 1) := (e_i, 1 + \lambda \cdot \lambda_i e)$ for all $i = 1, \ldots, r + 1$. Therefore the group has the structure of a vector space and is isomorphic to $k^r$.

The following two lemmas follow from [14, Lemma 4.6, 4.8]. Nevertheless, we include slightly different proofs, which are more suitable for our purpose.

**Lemma 4.6.**

(1) Let $E$ be a stable Gieseker vector bundle on the curve $X_r$ and $\psi \in Aut_{X_0}(X_r)$. Then $E \neq \psi^* E$.

(2) Let $(E, \phi)$ be a stable Gieseker-Higgs bundle on the curve $X_r$ and $\psi \in Aut_{X_0}(X_r)$. Then $E \neq \psi^* E$.

**Proof.** Proof of (1) Suppose that there exists a Gieseker vector bundle $E$ on $X_r$ and $\psi \in Aut_{X_0}(X_r)$ such that $\psi^* E \cong E$. Then there exists an automorphism $\tilde{\psi} : E \to E$ such that the following diagram commutes

$$
\begin{array}{ccc}
E & \xrightarrow{\psi} & E \\
\downarrow & & \downarrow \\
X_r & \xrightarrow{\psi} & X_r
\end{array}
$$

Notice that $X_r = \tilde{X}_0 \cup R$, where $\tilde{X}_0 \to X_0$ is the normalization, $R$ is the rational chain of length $r$ and $\tilde{X}_0 \cap R = \{p_1, p_{r+1}\}$. Consider the push-forward $\pi_* E \xrightarrow{\pi_* \tilde{\psi}} \pi_* E$. Since $\pi_* E$ is a stable torsion-free sheaf the morphism $\pi_* \tilde{\psi} = \lambda \cdot \text{Identity}$, where $\lambda$ is a non-zero scalar.

Restricting the above diagram on $R$ we get

$$
\begin{array}{ccc}
E|_R & \xrightarrow{\tilde{\psi}} & E|_R \\
\downarrow & & \downarrow \\
R & \xrightarrow{\psi} & R
\end{array}
$$

The restriction of $\tilde{\psi}$ at the two points are $\lambda \cdot \text{Identity}$. Now notice that the vector bundle $E|_R$ is globally generated. Given a global section $\sigma \in \Gamma(R, E|_R)$, we get a new section $\psi^{-1} \circ \sigma \circ \psi$. Notice that $(\psi^{-1} \circ \sigma \circ \psi - \lambda \cdot \sigma)(p_i) = 0$ for $i = 1$ and $i = r + 1$. Since $E|_R$ is strictly standard (definition 2.10), we conclude that $\psi^{-1} \circ \sigma \circ \psi = \lambda \cdot \sigma$. Therefore the induced morphism $H^0(R, E|_R) \to H^0(R, E|_R)$ is multiplication by $\lambda^{-1}$. Since $E|_R$ is globally generated therefore the morphism $\tilde{\psi} = \lambda^{-1} \cdot \text{Identity}$.

This is not possible because $E|_R$ is strictly standard. To see this first notice that we can decompose $E|_R \cong L_1 \oplus \cdots \oplus L_n$, such that $L_i|_{R_i} = \mathcal{O}(1)$ and $L_i|_{R_j} = \mathcal{O}$ for all $i \neq j$. For $i \neq j$, the induced morphism $L_i \xrightarrow{\psi} L_j$
cannot be multiplication by a scalar because there is no $\psi$-equivariant homomorphism from $\mathcal{O}(1) \to \mathcal{O}$. Therefore the induced morphism $L_i \to L_j$ is 0 for any $i \neq j$. Therefore $\tilde{\psi}(L_i) \cong L_i$ for all $i = 1, \ldots, n$.

Since $\psi$ is a nontrivial automorphism of the curve $X_r$, it is nontrivial on at least one rational curve in the chain $R$. Without loss of generality, let us assume that $\psi|_{R_i} \neq 1$. Moreover, let $\psi|_{R_i}$ is given by the multiplication by a scalar $\mu \notin \{1, -1\}$ i.e., $\psi([x : y]) = [\mu \cdot x : \frac{1}{\mu} y]$. Notice that the morphism $L_i \to L_i$ given by the multiplication by a scalar $\lambda$ cannot be $\psi$-equivariant for any scalar $\lambda$. To see this let us focus on the rational curve $R_i$. We have $L_i|_{R_i} \cong \mathcal{O}(1)$ and therefore $L_i^*|_{R_i} \cong \mathcal{O}(-1)$. We have the following commutative square

$$
\begin{array}{ccc}
L^* & \xrightarrow{\tilde{\psi}^*} & L^* \\
\downarrow & & \downarrow \\
R_i & \xrightarrow{\psi^{-1}} & R_i
\end{array}
$$

(4.15)

Notice that the map $\tilde{\psi}^*$ is multiplication by the scalar $\lambda$.

The total space of $\mathcal{O}(-1)$ is the following subvariety of $R_i \times \mathbb{C}^2$

$$\{(\gamma \cdot [x : y], (\gamma \cdot x, \gamma \cdot y)) \mid \gamma \in \mathbb{C}\}$$

(4.16)

Consider the diagram

$$
\begin{array}{ccc}
((x \cdot y), (\gamma \cdot x, \gamma \cdot y)) & \longrightarrow & ([\frac{1}{\mu} \cdot x : \mu \cdot y], (\lambda \cdot \gamma \cdot x, \lambda \cdot \gamma \cdot y)), \forall \gamma \in \mathbb{C} \\
\downarrow & & \downarrow \\
\mathcal{O}(-1) & \longrightarrow & \mathcal{O}(-1) \\
\downarrow & & \downarrow \\
R_i & \longrightarrow & R_i
\end{array}
$$

(4.17)

But if $\mathcal{O}(-1)$ has to be equivariant then $\lambda \cdot \gamma = \frac{1}{\mu} \cdot \nu$ and $\lambda \cdot \gamma = \mu \cdot \nu$ for some $\nu \in \mathbb{C}$. But this implies that $\mu^2 \cdot \nu = \nu$. Since $\lambda \neq 0$ the scalar $\nu \neq 0$. Therefore $\mu^2 = 1$ i.e., $\mu \in \{1, -1\}$. But the multiplications by $\pm 1$ induce the identity morphism on $R_i$, which is a contradiction.

**proof of (2)** The proof of the second statement follows similarly using the fact that induced torsion-free Higgs pair $(\pi_* \mathcal{E}, \pi_* \phi)$ is stable and therefore the automorphism $\pi_* \tilde{\psi} = \lambda \cdot \text{Identity}$, where $\lambda$ is a non-zero scalar. □

The vector space $H^0(X_r, T_{X_r})$ parametrises the automorphisms of the variety $X_r \times \text{Spec } k[e]$, which commute with the projection to $\text{Spec } k[e]$ and whose restriction on the closed fiber is the identity morphism. Let us denote $\text{Spec } k[e]$ by $\mathbb{D}$. Now notice given an infinitesimal automorphism $X_r \times \mathbb{D} \xrightarrow{\psi} X_r \times \mathbb{D}$ and a vector bundle $\mathcal{E}_\mathbb{D}$ over $X_r \times \mathbb{D}$ such that the restriction to the closed fiber is a Gieseker vector bundle $\mathcal{E}$, we can
pull back the vector bundle $\mathcal{E}_D$ by the morphism $\psi$. We define the following action

$$H^0(X_r, T_{X_r}) \times H^1(X_r, \mathcal{E} \wedge nd \mathcal{E}) \to H^1(X_r, \mathcal{E} \wedge nd \mathcal{E}) \quad (4.18)$$

given by $(\psi, \mathcal{E}_D) \mapsto \psi^* \mathcal{E}_D$.

Similarly, given an infinitesimal automorphism $X_r \times \text{Spec } k[e] \to X_r \times \text{Spec } k[e]$ and a Higgs bundle $(\mathcal{E}_D, \phi_D)$ over $X_r \times \text{Spec } k[e]$ such that the restriction to the closed fiber is the Higgs bundle $(\mathcal{E}, \phi)$, we can pullback the Higgs field $\phi_D$ by the morphism $\psi$. We define the following action

$$H^0(X_r, T_{X_r}) \times \mathbb{H}^1(X_r, \mathcal{E}_* \mathcal{C}_*) \to \mathbb{H}^1(X_r, \mathcal{E}_* \mathcal{C}_*) \quad (4.19)$$

given by $(\psi, \mathcal{E}_D, \phi_D) \mapsto (\psi^* \mathcal{E}_D, \psi^* \phi_D)$.

**Lemma 4.7.**

1. Let $\mathcal{E}$ be a stable Gieseker vector bundle on $X_r$. The action of the group $H^0(X_r, T_{X_r})$ of infinitesimal automorphisms of $X_r$ on the space $H^1(X_r, \mathcal{E} \wedge nd \mathcal{E})$ of all first order infinitesimal deformations of the vector bundle $\mathcal{E}$ is free.

2. Let $(\mathcal{E}, \phi)$ be a stable Gieseker-Higgs bundle on $X_r$. The action of the group $H^0(X_r, T_{X_r})$ of infinitesimal automorphisms of $X_r$ on the space $\mathbb{H}^1(X_r, \mathcal{E}_* \mathcal{C}_*)$ of all first order infinitesimal deformations of the Higgs bundle $(\mathcal{E}, \phi)$ is free.

**Proof.** proof of (1) Notice that if there exists $\psi \in H^0(X_r, T_{X_r})$ and $\mathcal{E}_D \in H^1(X_r, \mathcal{E} \wedge nd \mathcal{E})$ such that there exists an isomorphism $\tilde{\psi} : \mathcal{E}_D \to \psi^* \mathcal{E}_D$, then we have a following cartesian square

$$\begin{array}{ccc}
\mathcal{E}_D & \xrightarrow{\tilde{\psi}} & \mathcal{E}_D \\
\downarrow & & \downarrow \\
X_r \times \text{Spec } k[e] & \xrightarrow{\psi} & X_r \times \text{Spec } k[e]
\end{array} \quad (4.20)$$

In other words, $\mathcal{E}_D$ is a $\psi$-equivariant bundle on $X_r \times \text{Spec } k[e]$. Let us denote by $\pi$ the morphism $X_r \to X_0$. Then we have an induced automorphism $X_0 \times \text{Spec } k[e] \xrightarrow{(\pi \times \mathbb{1})_*} X_0 \times \text{Spec } k[e]$ such that it commutes with the projection to Spec $k[e]$ and the induced automorphism on the special fiber is the identity. But since $X_0$ is a stable curve therefore $(\pi \times \mathbb{1}), \tilde{\psi}$ = Identity. We also have an induced automorphism of the torsion-free sheaf $(\pi \times \mathbb{1})_* \tilde{\psi} : (\pi \times \mathbb{1})_* \mathcal{E}_D \to (\pi \times \mathbb{1})_* \mathcal{E}_D$ such that induced morphism on the closed fiber $\pi_* \mathcal{E} \to \pi_* \mathcal{E}$ is the identity. Since the morphism $(\pi \times \mathbb{1})_* \tilde{\psi}$ is an $\mathcal{O}_{X_0}[e]$ module homomorphism and it is the Identity morphism modulo $\epsilon$, therefore the morphism is multiplication(on the left) by $I + \epsilon \Psi_0$, where $\Psi_0 : \pi_* \mathcal{E} \to \pi_* \mathcal{E}$ an $\mathcal{O}_{X_0}$ module homomorphism. Therefore if $\sigma_1 + \epsilon \sigma_2$ is a local section of $(\pi \times \mathbb{1})_* \mathcal{E}_D$ then $(\pi \times \mathbb{1})_* \tilde{\psi}(\sigma_1 + \epsilon \sigma_2) = \sigma_1 + \epsilon \sigma_2 + \epsilon \Psi_0(\sigma_1)$. Since the torsion free sheaf $\pi_* \mathcal{E}$ is stable therefore the morphism $\Psi_0$ must be multiplication by some scalar $\lambda$. Therefore on $\tilde{X}_0 \times \text{Spec } k[e]$ also the restriction of $\tilde{\psi}$ is given by

$$\tilde{\psi}(\sigma_1 + \epsilon \sigma_2) = \sigma_1 + \epsilon \cdot \sigma_2 + \epsilon \lambda \sigma_1 \quad (4.21)$$
By restricting the morphism $\tilde{\psi}$ over $R[c] := R \times \text{Spec } k[c]$ we get

$$ (\mathcal{E}_R)|_{R[c]} \xrightarrow{\tilde{\psi}} (\mathcal{E}_D)|_{R[c]} \quad (4.22) $$

But $\tilde{\psi}|_{p, r \times \text{Spec } k[c]}(\sigma_1 + \epsilon \sigma_2) = \sigma_1 + \epsilon \sigma_2 + \epsilon \lambda \sigma_1$ for $i = 1, r + 1$. Since the morphism $\tilde{\psi}$ is $\psi$-equivariant $\mathcal{O}_R[c]$-module homomorphism and is the identity morphism modulo $\epsilon$, therefore $\tilde{\psi}$ is multiplication (on the left) by $1 + \epsilon \psi$, where $\Psi : \mathcal{E}|_R \to \mathcal{E}|_R$ is $\mathcal{O}_R$-module. Therefore $\tilde{\psi}(\sigma_1 + \epsilon \sigma_2) = \sigma_1 + \epsilon \sigma_2 + \epsilon \Psi(\sigma_1)$. Now notice that at the two extremal points $p_1$ and $p_{r+1}$, the morphism $\Psi$ is multiplication by the scalar $\lambda$. Since the vector bundle $\mathcal{E}|_R$ is a strictly standard vector bundle, therefore $\Psi = \lambda \cdot I$.

But this is not possible unless the infinitesimal automorphism $\psi$ is trivial. To see this, notice that $\psi$ is given by

$$ \mathcal{O}_R[c] \to \mathcal{O}_R[c] \quad (4.23) $$

which maps $f + \epsilon \cdot g \to f + \epsilon (g + X_{\psi}(df))$, where $X_{\psi}$ is the vector field on $R$ corresponding to the infinitesimal automorphism $\psi$. Over a $\psi$-equivariant trivialization $U[c]$ of $\mathcal{E}$, where $U$ is an open subset of $R$, we have

$$ (\mathcal{O}_U)_{\mathcal{E}^2}[c] \xrightarrow{\tilde{\psi}} (\mathcal{O}_U)_{\mathcal{E}^2}[c] \quad (4.24) $$

such that $\tilde{\psi}(f_1 + \epsilon g_1, f_2 + \epsilon g_2) = \tilde{\psi}(f_1, f_2) + \epsilon (g_1, g_2) = \tilde{\psi}(f_1 \cdot (1, 0)) + \tilde{\psi}(f_2 \cdot (1, 0)) + \tilde{\psi}(g_1 \cdot (0, 1)) + \tilde{\psi}(g_2 \cdot (0, 1)) = \psi(f_1) \cdot (1, 0) + \psi(f_2) \cdot (1, 0) + \psi(g_1) \cdot (0, 1) + \psi(g_2) \cdot (0, 1) = (f_1 + \epsilon X_{\psi}(f_1))(1, 0) + (f_2 + \epsilon X_{\psi}(f_2))(0, 1) + (g_1)(0, 1) + (g_2)(0, 1) = (f_1 + \epsilon (g_1 + X_{\psi}(df_1)), f_2 + \epsilon (g_2 + X_{\psi}(df_2))).$ But then we must have $X_{\psi}(df_1) = \lambda$ and $X_{\psi}(df_2) = \lambda$ for all local functions $f_1$ and $f_2$, which is only possible when $\lambda = 0$ and the vector field $X_{\psi}$ is trivial, i.e., the infinitesimal automorphism $\psi$ is trivial.

proof of (2) The proof of the second statement follows similarly using the fact that the induced infinitesimal torsion free Higgs pair $(\pi_*, \mathcal{E}, \pi_D \ast \phi)$ is stable and therefore the $\Psi_0 = \lambda \cdot \text{Identity},$ where $\lambda$ is a scalar.

\begin{proof}
\end{proof}

Remark 4.8. From the above lemma it follows that if $0 \neq \psi \in H^0(X_r, T_X)$ and $\mathcal{E}_D$ is the trivial infinitesimal deformation of $\mathcal{E}$ over $X_r \times \text{Spec } k[c]$, then $\psi^* \mathcal{E}_D \not\cong \mathcal{E}_D$. Therefore we conclude that $H^0(X_r, T_X)$ is a subspace of $\mathcal{H}^1(X_r, \mathcal{E} \times \text{nd } \mathcal{E})$. Similarly, we can show that $H^0(X_r, T_X)$ is a subspace of $\mathcal{H}^1(\mathcal{E}_*)$.

Proposition 4.9. \hspace{1em} (1) The relative tangent space of $f_{\text{Cur}} : \mathcal{M}_{\text{GVB}} \to \mathcal{L}_{\text{Log}(\text{Spec } k[s])}$ at a point $(\pi_r : X_r \to X_0, \mathcal{E})$ is isomorphic to $H^1(X_r, \mathcal{E} \times \text{nd } \mathcal{E})$.

(2) The relative tangent space of $f_{\text{Cur}} : \mathcal{M}_{\text{GHB}} \to \mathcal{L}_{\text{Log}(\text{Spec } k[n])}$ at a point $(\pi_r : X_r \to X_0, \mathcal{E}, \phi : \mathcal{E} \to \mathcal{E} \otimes \pi_r^* \omega_{X_0})$ is isomorphic to $\mathcal{H}^1(\mathcal{E}_*)$, where $\mathcal{E}_*$ is the complex

$$ 0 \to \mathcal{E} \otimes \mathcal{E} \xrightarrow{[-\phi, \ast]} \mathcal{E} \otimes \mathcal{E} \otimes \pi_r^* \omega_{X_0} \to 0, \quad (4.25) $$

and the map $[\phi, \ast](s) = \phi \circ s - (\mathcal{I} \otimes s) \circ \phi$.  

\begin{proof}
\end{proof}
Proof. Since \( \mathcal{M}_{GVB} \to (\text{Spec } k, \overline{\mathbb{N}}) \) is a base-change of the log smooth morphism \( \mathcal{M}_{GVB,S} \to S \), it is also a log-smooth. Therefore the morphism \( f_{Cur} : \mathcal{M}_{GVB} \to \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})} \) is smooth. By definition \([25, \text{Definition } 17.14.2]\), the relative tangent space is the fiber product (for the notations see 1.6)

\[
\begin{array}{c}
\mathcal{M}_{GVB}(k[e]) \\ \downarrow \\
T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k)} \end{array} - \begin{array}{c}
\mathcal{M}_{GVB}(k[e]) \times_{T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}}(k)} \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k) \\ \downarrow \\
\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k)
\end{array}
\]

(4.26)

The vertical morphism on the left is the natural projection, and the horizontal morphism below is the null-section \([25, 17.11.4]\). The fibre product in the diagram is a vector space.

It is well-known that the isomorphism classes of first-order infinitesimal deformations of a vector bundle \( \mathcal{E} \) over a projective curve \( C \) are parametrised by the vector space \( H^1(C, \mathcal{E}nd\mathcal{E}) \). Therefore we have the following.

\[
\begin{aligned}
\{ \text{Isomorphism classes of stable Gieseker vector bundles}
\} \\
\{ \mathcal{E} \text{ over } X_r \times \text{Spec } k[e] \text{ such that the}
\} \\
\text{restriction over } X_r \text{ is } \mathcal{E}
\end{aligned}
\]  
\[
\cong H^1(X_r, \mathcal{E}nd\mathcal{E}) \tag{4.27}
\]

and from lemma 4.2 it follows that we have a surjective morphism of vector spaces

\[
H^1(X_r, \mathcal{E}nd\mathcal{E}) \to \mathcal{M}_{GVB}(k[e]) \times_{T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}}(k)} \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k). \tag{4.28}
\]

From remark 4.3 and lemma 4.4, it follows that the elements which lies in the image of the morphism \( \mathcal{M}_{GVB}(k[e]) \times_{T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}}(k)} \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k) \to \mathcal{M}_{GVB}(k[e]) \) are the Gieseker-equivalent classes (definition 2.13) of families of stable Gieseker vector bundles \( (\mathcal{X}^{mod}, \mathcal{E}) \) over \( \text{Spec } k[e] \) such that the induced logarithmic structure is on \( \text{Spec } k[e] \) is isomorphic to the pull back of the log structure of \( \text{Spec } k \), defined above in (4.1), under the natural projection map \( \text{Spec } k[e] \to \text{Spec } k \). Therefore from lemma 4.2, it follows that the image of the morphism \( \mathcal{M}_{GVB}(k[e]) \times_{T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}}(k)} \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k) \to \mathcal{M}_{GVB}(k[e]) \) is isomorphic to the quotient vector space

\[
\frac{H^1(X_r, \mathcal{E}nd\mathcal{E})}{H^0(X_r, T_{X_r})}. \tag{4.29}
\]

In lemma 4.7, we have shown that the action of \( H^0(X_r, T_{X_r}) \) on \( H^1(X_r, \mathcal{E}nd\mathcal{E}) \) is free and in 4.8, we have remarked that \( H^0(X_r, T_{X_r}) \) is a vector subspace of \( H^1(X_r, \mathcal{E}nd\mathcal{E}) \). Also, notice that the Gieseker equivalence on the infinitesimal families of Gieseker vector bundles is precisely the equivalence induced by the action of \( H^0(X_r, T_{X_r}) \) on \( H^1(X_r, \mathcal{E}nd\mathcal{E}) \).

It follows from the definition of the fibre product of algebraic stacks that the fibre of the surjective morphism of vector spaces

\[
\mathcal{M}_{GVB}(k[e]) \times_{T_{\mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}}(k)} \mathcal{L}og_{(\text{Spec } k, \overline{\mathbb{N}})}(k) \to \frac{H^1(X_r, \mathcal{E}nd\mathcal{E})}{H^0(X_r, T_{X_r})}
\]
is isomorphic to the vector space $\text{Aut}^\text{Inf}_Y(M)$ which is isomorphic to $k^r$ (by lemma 4.5). Also notice that $H^0(X_r, T_{X_r}) \cong k^r$. Since the morphism 4.28 is a surjective morphism between two vector spaces of the same dimension, it has to be an isomorphism. Therefore the relative log tangent space is isomorphic to $H^1(X_r, \mathcal{E} \otimes \pi^*_r \omega_{X_0})$.

Similarly, if $(X_r, \mathcal{E}, \phi : \mathcal{E} \to \mathcal{E} \otimes \pi^*_r \omega_{X_0})$ is a Gieseker-Higgs bundle then the relative tangent space of $\mathcal{M}_{GHB} \to \mathcal{M}_{GHB,S}(\mathcal{E})$ at the point $(X_r, \mathcal{E}, \phi : \mathcal{E} \to \mathcal{E} \otimes \pi^*_r \omega_{X_0})$ is isomorphic to

$$\left\{ \text{Isomorphism classes of Higgs bundles } (\mathcal{E}, \phi) \text{ over } X_r \times \text{Spec} \, k[\epsilon] \text{ such that the restriction over } X_r \text{ is } (\mathcal{E}, \phi) \right\} \cong \mathcal{H}^1(\mathcal{E}_s), \text{(subsection 2.33 and remark 2.36)}$$

where $\mathcal{E}_s$ is the complex (4.25).

\begin{proof}

Theorem 4.10. Let $(X_r, \mathcal{E}, \phi)$ be any Gieseker-Higgs bundle. Then

$$T\mathcal{M}_{GVB,S}(\mathcal{E}) \cong H^1(X_r, \mathcal{E} \otimes \pi^*_r \omega_{X_0}), T\mathcal{M}_{GHB,S}(\mathcal{E}) \cong H^1(\mathcal{E}_s).$$

\end{proof}

\begin{remark}

It is easy to see that although the universal bundle $U$ on $\Delta^st$ may not, in general, descend to the quotient $X_S$, the vector bundle $\mathcal{E} \otimes nd U$ descends to $X_S$. It is because the $GL(N)$– action at a stable bundle has stabilizer $\mathbb{C}^*$ and the action of the stabilizer on the bundle $\mathcal{E} \otimes nd U$ is trivial and therefore the bundle $\mathcal{E} \otimes nd U$ is in fact $PGL(N)$ equivariant. Therefore, the vector bundle $\mathcal{E} \otimes nd U$ descends. We denote it by $\mathcal{E} \otimes nd \mathcal{E}$. Similarly, the vector bundle $R^1 \pi_* \mathcal{E} \otimes nd U$ (and $R^1 \pi_* \mathcal{E}_s$) also descends to $\mathcal{M}_{GVB,S}$ (and $\mathcal{M}_{GHB,S}$).

5. Relative Log symplectic structure on $\mathcal{M}_{GHB,S}$

In this section, we will show that there is a relative log symplectic structure on $\mathcal{M}_{GHB,S} \to S$ and also describe it functorially.

Consider the following composite morphism.

$$\wedge^2 \Omega_{\mathcal{M}_{GHB,S}}(\log \mathcal{M}_{GHB}) \longrightarrow \mathcal{M}_{GHB,S} \longrightarrow S$$

Now over the generic point $\eta$ of $S$, the first projection map has a natural section which corresponds to the symplectic form on moduli of Higgs bundles over the generic curve ([18, Section 8],[9, Section 4], subsection 2.33 and remark 2.36). Let us denote the generic fibre of $\mathcal{M}_{GHB,S} \to S$ by $\mathcal{M}_{HB}$. It is an open subset of $\mathcal{M}_{GHB,S}$.

Consider the relative logarithmic cotangent bundle.
\[\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}}) \longrightarrow \mathcal{M}_{\text{GVB},S} \longrightarrow S\]  \hspace{1cm} (5.2)

From theorem 4.31, it follows that for any stable Gieseker vector bundle \((X_r, \mathcal{E})\)
\[\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})(X_r, \mathcal{E}) \cong H^1(X_r, \mathcal{E} \otimes \pi_r^* \omega_{X_0})\]  \hspace{1cm} (5.3)

5.0.1. Few properties of the map \(\pi_r\). Let us remind here that \(\pi_r\) is the projection morphism \(X_r \rightarrow X_0\) and recall the following facts from [31, Proposition 3]

1. \(\pi_r^* \omega_{X_0} \cong \omega_{X_r}\),
2. \(R^i(\pi_r)_* \Omega_{X_r} = 0\), for all \(i > 0\),
3. \((\pi_r)_* \Omega_{X_r} \cong \Omega_{X_0}\).

Now, using Serre duality for nodal curves and the properties above, we see that
\[H^1(X_r, \mathcal{E} \otimes \pi_r^* \omega_{X_0}) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \pi_r^* \omega_{X_0})\]  \hspace{1cm} (5.4)

Therefore we have a morphism \(\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}}) \rightarrow \mathcal{M}_{\text{GHB},S}\), which is clearly injective. The objects of \(\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})\) are precisely those Gieseker-Higgs bundles whose underlying Gieseker vector bundle is stable. By the openness of stability of Gieseker vector bundles it follows that \(\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}}) \rightarrow \mathcal{M}_{\text{GHB},S}\) is an open immersion.

There is a natural relative log-symplectic structure \(\omega\) on \(\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})\) (theorem 2.30). Now consider the union of the two open subvarieties \(\mathcal{M}_{\text{HB}} \cup \Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})\). The symplectic structure on \(\mathcal{M}_{\text{HB}}\) and the relative log-symplectic structure on \(\Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})\) agree on the intersection \(\mathcal{M}_{\text{HB}} \cap \Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}}) = \Omega_{\mathcal{M}_{\text{HB}}}\). Therefore there is a relative log-symplectic structure on the union of the two open subsets \(\mathcal{M}_{\text{HB}} \cup \Omega_{\mathcal{M}_{\text{GVB},S}}(\log \mathcal{M}_{\text{GVB}})\). The following lemma shows that \(\mathcal{M}_{\text{HB}}\) is a dense open subset of \(\mathcal{M}_{\text{GHB},S}\).

Lemma 5.1. Let \((X_r, \mathcal{E}, \phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \pi_r^* \omega_{X_0})\) be any stable Gieseker-Higgs bundle. There exists a family of stable Gieseker-Higgs bundles \((\mathcal{E}_S^{\text{mod}}, \phi_S : \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \omega_S)\) over a complete discrete valuation ring \(S\), whose generic fiber is a stable Higgs bundle over the smooth curve \(X_{\eta}\) and the special fiber is \((X_r, \mathcal{E}, \phi)\).

Proof. The proof follows from the openness of a flat morphism and the fact that the relative moduli space \(\mathcal{M}_{\text{GHB},S} \rightarrow \mathcal{S}\) is flat over \(S\) [3, Theorem 1.1]. \(\Box\)

Theorem 5.2. There is a relative logarithmic-symplectic form on \(\mathcal{M}_{\text{GHB},S} \rightarrow \mathcal{S}\), which is the classical symplectic form on the generic fibre and is a log-symplectic form on the special fibre.

Proof. Given a Gieseker-Higgs bundle \((X_r, \mathcal{E}, \phi)\) consider the following complex \(\mathcal{E}_*\):
\[0 \rightarrow \mathcal{E} \otimes \phi^* \mathcal{E} \rightarrow \mathcal{E} \otimes \pi_r^* \omega_{X_0} \rightarrow 0\]  \hspace{1cm} (5.5)
where \([\phi, \bullet](s) = \phi \circ s - (1 \otimes s) \circ \phi\). The log-cotangent space of the moduli \(\mathcal{M}_{GHB}\) at \((X_r, E, \phi)\) is isomorphic to \(H^1(\mathcal{E}_*)\) (subsection 2.33 and remark 2.36). The dual of this complex (let us denote by \(\mathcal{E}_*')\) is

\[
0 \to \mathcal{E}ndE \xrightarrow{[-\phi, \bullet]} \mathcal{E}ndE \otimes \pi^*_r \omega_{X_0} \to 0 \tag{5.6}
\]

We have the following isomorphism of complexes:

\[
\begin{array}{ccc}
\mathcal{E}ndE & \xrightarrow{1} & \mathcal{E}ndE \\
\downarrow{[-\phi, \bullet]} & & \downarrow{[-\phi, \bullet]} \\
\mathcal{E}ndE \otimes \pi^*_r \omega_{X_0} & \xrightarrow{-10^1} & \mathcal{E}ndE \otimes \pi^*_r \omega_{X_0}
\end{array} \tag{5.7}
\]

This induces an isomorphism

\[(\sigma')^\flat : H^1(\mathcal{E}_*') \to H^1(\mathcal{E}_*) \tag{5.8}\]

Therefore, we have an isomorphism

\[(\sigma')^\flat : \Omega_{\mathcal{M}_{GHBS}/S} (\log \mathcal{M}_{GHB})(X_r, E, \phi) \cong T_{\mathcal{M}_{GHBS}/S} (-\log \mathcal{M}_{GHB})(X_r, E, \phi) \tag{5.9}\]

In other words, this gives a non-degenerate relative logarithmic two-form. We denote it by \(\omega'\). From the choice of the sign in the diagram 5.7, it follows that \(\omega'\) is skew symmetric. Therefore, this gives a section \(\omega' : \mathcal{M}_{GHBS} \to \Lambda^2 \Omega_{\mathcal{M}_{GHBS}/S} (\log \mathcal{M}_{GHB})\). Following theorem 2.30, we have another section \(\omega\) over \(\mathcal{M}_{HB} \cup \mathcal{M}_{GVBS}(\log \mathcal{M}_{GVB})\), and from [9, Section 4], [10, Theorem 4.5.1.], and [18, Section 8] it follows that these two sections are the same on the open subset \(\mathcal{M}_{HB}\) (see also subsection 2.33 and remark 2.36). Therefore these two 2-forms are the same and hence \(\omega'\) is the extension of the relative log-symplectic form \(\omega\) discussed in theorem 2.30.

\[\square\]

**Remark 5.3.** Suppose that the special fiber of the surface \(\mathcal{X} \to S\) is a reducible curve of the form \(C_1 \cup C_2\), where \(C_1\) and \(C_2\) are smooth curves transversally intersecting at a point \(C_1 \cap C_2\). We can similarly construct the moduli of Gieseker-Higgs bundles in this case. If we concentrate on the case \((\chi, r) = 1\), we can ensure that all the semistable objects are stable for a generic choice of polarisation. As a result, the moduli is a variety with normal crossing singularity (see [3, remark 9.3] and [4] for details). It follows similarly that there is a relative log-symplectic form on this degeneration. A particular case interesting for many computations is when the rank is 2, and \(\chi\) is odd. For a generic choice of polarisation, one can show that the moduli of stable torsion-free Hitchin pairs on the nodal curve \(C_1 \cup C_2\) consists of \((\mathcal{F}, \phi)\), where \(\mathcal{F}\) has local types either \(\mathcal{O} \oplus \mathcal{O}\) or \(\mathcal{O} \oplus m\). Since we can avoid the local type \(m \oplus m\), the moduli of stable torsion-free Higgs pair coincides with the moduli of Gieseker-Higgs bundles ([3, remark 9.3] and [4]). We will show in §7 that in this special case the special fibre \(\mathcal{M}_{GHB}\) is union of two smooth log-symplectic manifolds transversally intersecting along a divisor.
6. Moduli space of Higgs bundles on a fixed Gieseker curve

In this section, we discuss about the moduli of Higgs bundles of rank \( n \) and Euler-characteristic \( \chi \) on a fixed Gieseker curve \( X_r \) such that \( g.c.d(n, \chi) = 1 \). In [26], Kiem and Li introduced and studied semi-stability of vector bundles on a fixed Gieseker curve with respect to a special polarisation on the curve whose degree on every rational component is sufficiently smaller (in ratio) than the degree on the curve \( X_0 \). We will refer to this notion of semistability as the \( \epsilon \)-semistability. Moreover, they show that \( \epsilon \)-semistable vector bundles are quasi-Gieseker vector bundles, i.e., they are standard bundles \((6.5)\) and the push-forward is a torsion-free sheaf on the nodal curve \( X_0 \). Later in [40], Sun introduced the notion of \( 0 \)-semi-stability of vector bundles on a fixed Gieseker curve and showed that when the Euler characteristic of the bundle is positive and co-prime to the rank, the two notions, namely \( \epsilon \)-semistability and \( 0 \)-semistability coincide. In this section, we adapt these notions for the Higgs bundles on a fixed Gieseker curve.

To motivate this, we first notice that the moduli of Gieseker-Higgs bundles \( \mathcal{M}_{GHB} \) has the following Whitney stratification given by the successive singular locus.

\[
\mathcal{M}_{GHB} \supset \partial^1 \mathcal{M}_{GHB} \supset \cdots \supset \partial^n \mathcal{M}_{GHB} \supset \cdots
\] (6.1)

The purpose of the discussion in this section is to show that each stratum is a torus quotient of the moduli of \( \epsilon \)-semistable Higgs bundles on a fixed Gieseker curve.

Let \( 0 \leq \epsilon < 1 \) be an arbitrarily small non-negative number. Since the purpose is to describe the stratum, as mentioned above, it is safe to assume that \( \chi(\mathcal{E}) > 0 \) (see also remark \( 6.10 \)).

**Definition 6.1.** A Higgs bundle \((X_r, \mathcal{E}, \phi)\) is called \( \epsilon \)-semistable (\( \epsilon \)-stable) if for all \( \phi \)-invariant subsheaf \( \mathcal{F} \) of \( \mathcal{E} \) we have

\[
\chi(\mathcal{F}) \leq (\leq) \frac{\chi(\mathcal{E})}{n} r_k(\mathcal{F}),
\] (6.2)

where \( r_k(\mathcal{F}) := (1 - r) \text{rank } \mathcal{F}|_{X_0} + \epsilon \sum_{i=1}^r (\text{rank } \mathcal{F}|_{R_i}) \)

**Definition 6.2.** A Higgs bundle \((X_r, \mathcal{E}, \phi)\) is called \( 0 \)-semistable if for all \( \phi \)-invariant subsheaf \( \mathcal{F} \) of \( \mathcal{E} \) we have

\[
\chi(\mathcal{F}) \leq \frac{\chi(\mathcal{E})}{n} r_k(\mathcal{F}),
\] (6.3)

and it is called \( 0 \)-stable if it is \( 0 \)-semistable and

\[
\chi(\mathcal{F}) \leq \frac{\chi(\mathcal{E})}{n} r_k(\mathcal{F}), \quad \text{when } r_k(\mathcal{F}) \neq 0.
\] (6.4)

The proof of the following two lemmas are elementary; we leave it to the reader.

**Lemma 6.3.** \((X_r, \mathcal{E}, \phi)\) is \( \epsilon \)-stable if and only if it is \( 0 \)-stable.

**Lemma 6.4.** Assume \((\chi(\mathcal{E}), n) = 1\). Then
Lemma 6.5. If \((X_r, \mathcal{E}, \phi)\) is \(\epsilon\)-stable then \(\mathcal{E}\) satisfies the following two properties.

1. \(\mathcal{E}\) is a standard vector bundle i.e.,
   \[
   \mathcal{E}|_{R[r]} \cong \mathcal{O}^{a_1} \oplus \mathcal{O}(1)^{a_2} \quad \text{for all } i = 1, \ldots, r, \text{ and}
   \]
   \[
   (6.5)
   \]

2. \(\pi_{r*} \mathcal{E}\) is a torsion-free sheaf.

**Proof.** For each \(i \in \{1, \ldots, r\}\), we denote by \(x_i^+, x_i^-\) the two marked points on the rational curve \(R[r]_i\). We denote the Zariski-closure of the curve \(X_r \setminus R[r]_i\) by \(\tilde{X}_i\). We denote the two extremal points on \(\tilde{X}_0\) by \(x^+\) and \(x^-\).

Suppose \(\exists\) an integer \(i\) such that \(\mathcal{E}|_{R[r]_i}\) has a negative degree line sub bundle as a direct summand. Let \(m\) and \(a\) be the largest positive integers such that \(\mathcal{O}_{R[r]_i}(-m)^{a}\) is a direct summand of \(\mathcal{E}|_{R[r]_i}\). Let \(K\) be the kernel of the surjection \(\mathcal{E} \twoheadrightarrow \mathcal{O}_{R[r]_i}(-m)^{a}\). It follows that \(K\) is a \(\phi\)-invariant subsheaf of \(\mathcal{E}\). We have

\[
\chi(K) = \chi(\mathcal{E}) - \chi(\mathcal{O}_{R}[(-m)^a]) = \chi(\mathcal{E}) + a(m-1).
\]

Since rank \(K|_{\tilde{X}_0} = n\), we have \(\chi(\mathcal{E}) + a(m-1) < \chi(\mathcal{E})\) which implies \(a(m-1) < 0\). This is a contradiction. Therefore \(m \leq 0\).

Suppose \(\exists\) an integer \(i\) such that \(\mathcal{E}|_{R[r]_i}\) has a positive degree line sub bundle as a direct summand. Let \(m\) and \(a\) be the largest positive integers such that \(\mathcal{O}_{R[r]_i}(m)^{a}\) is a direct summand of \(\mathcal{E}|_{R[r]_i}\). We have \(\mathcal{E}|_{R[r]_i} = \mathcal{O}_{R[r]_i}(m)^{a} \oplus M\), where \(M\) is a vector bundle on \(R[r]_i\). We have a short exact sequence

\[
0 \to \mathcal{E}|_{\tilde{X}_i}(-x_i^+ - x_i^-) \to \mathcal{E} \to \mathcal{E}|_{R[r]_i} \to 0
\]

From the above short exact sequence it follows that \((\mathcal{O}_{R[r]_i}(m)^{a}) \oplus \mathcal{O}_{R[r]_i}(-x_i^+ - x_i^-)\) is a subsheaf of \(\mathcal{E}\). Notice that \((\mathcal{O}_{R[r]_i}(m)^{a}) \oplus \mathcal{O}_{R[r]_i}(-x_i^+ - x_i^-) \cong \mathcal{O}_{R[r]_i}(m-2)^{a}\). It follows that \(\mathcal{O}_{R[r]_i}(m-2)^{a}\) is a \(\phi\)-invariant subsheaf of \(\mathcal{E}\). The \(\epsilon\)-stability implies

\[
\chi(\mathcal{O}_{R[r]_i}(m-2)^{a}) = a(m-1) \leq 0 \implies m \leq 1.
\]

Therefore \(\mathcal{E}\) is a standard vector bundle.

The sheaf \(\pi_{r*} \mathcal{E}\) is torsion free if and only if \(H^0(\mathcal{E}|_{R[r]}(-x^+ - x^-)) = 0\). Suppose \(H^0(\mathcal{E}|_{R[r]}(-x^+ - x^-)) \neq 0\). Consider the sub-bundle \(F\) of \(\mathcal{E}|_{R[r]}(-x^+ - x^-)\) generated by \(H^0(\mathcal{E}|_{R[r]}(-x^+ - x^-))\). It is a \(\phi\)-invariant subsheaf of \(\mathcal{E}\). Since it is generically generated by global sections, \(\chi(F) \geq 1\). But \(\epsilon\)-stability implies \(\chi(F) \leq 0\), which is a contradiction. Therefore, \(H^0(\mathcal{E}|_{R[r]}(-x^+ - x^-)) = 0\).

\[\square\]

Remark 6.6. If a Higgs bundle \((\mathcal{E}, \phi)\) on \(X_r\) satisfies the condition (1) and (2) in Lemma 6.5, we call it a quasi-Gieseker-Higgs bundle.
Definition 6.7. A generalised parabolic Higgs bundle (GPH) on \( \tilde{X}_0 \) is a triple \((E, \phi, F(E))\), where \( E \) is a vector bundle, \( \phi : E \to E \otimes \omega_{\tilde{X}_0}(x^+ + x^-) \) is a homomorphism and \( F(E) \subseteq E_{x^+} \oplus E_{x^-} \) is any sub-vector space such that \((q_* \phi)(q_*(F(E)) \subseteq q_*(F(E)) \otimes \omega_{X_0}\), where \( q : \tilde{X}_0 \to X_0 \) is the normalisation morphism.

Given a GPH \((E, \phi, F(E))\) we have the following torsion-free sheaf

\[
\mathcal{T} := \text{Kernel } (q_* E \to \frac{E_{x^+} \oplus E_{x^-}}{F(E)})
\]

(6.8)

Since \((q_* \phi)(q_*(F(E)) \subseteq q_*(F(E)) \otimes \omega_{X_0}\) the morphism \( \phi \) induces a homomorphism \( \phi_0 : \mathcal{T} \to \mathcal{T} \otimes \omega_{X_0} \).

Proposition 6.8. The GPH \((E, \phi, F(E))\) is semistable (stable) if and only if the induced torsion-free Higgs pair \((\mathcal{T}, \phi_0)\) is semi-stable (stable).

Proof. The proof is similar to [7, Proposition 4.2]. \( \square \)

Proposition 6.9. If a Higgs bundle \((X_r, \mathcal{E}, \phi)\) is \( \epsilon \)-stable, then \(((\pi_r)_* \mathcal{E}, (\pi_r)_* \phi)\) is stable torsion-free Higgs pair. If \( \mathcal{E}|_R \) is positive and \(((\pi_r)_* \mathcal{E}, (\pi_r)_* \phi)\) stable then \((X_r, \mathcal{E}, \phi)\) is \( \epsilon \)-stable.

Proof. Set \( \tilde{E} := \mathcal{E}|_{\tilde{X}_0} \) and \( \tilde{F} := \mathcal{E}|_{R[r]} \). Since \((X_r, \mathcal{E}, \phi)\) is \( \epsilon \)-stable therefore from Lemma 6.5 it follows that \( H^0(\tilde{F}(-x^+ - x^-)) = 0 \). Therefore the first map in the following sequence is injective

\[
H^0(R[r], \tilde{F}) \xrightarrow{s_\mathcal{E}(s(x^+), s(x^-))} \tilde{F}_{x^+} \oplus \tilde{F}_{x^-} \oplus \theta_1 \oplus \theta_2 \cdot \tilde{E}_{x^+} \oplus \tilde{E}_{x^-},
\]

(6.9)

where \( \theta_1 : \tilde{F}_{x^+} \to \tilde{E}_{x^+} \) and \( \theta_2 : \tilde{F}_{x^-} \to \tilde{E}_{x^-} \) are the gluing isomorphisms. Since \( H^0(R[r], \tilde{F}) \) is \( \phi \) invariant therefore we get a GPH \((\tilde{E}, (\theta_1 \oplus \theta_2)(H^0(R[r], \tilde{F})), \tilde{\phi})\) whose induced torsion-free Higgs pair is \(((\pi_r)_* \mathcal{E}, (\pi_r)_* \phi)\).

It is enough to show that the GPH is stable. The rest of the proof follows from the observation that for any \( \phi \)-invariant sub-sheaf \( \tilde{E}' \subset \tilde{E} \) (on \( \tilde{X}_0 \)), the sub-sheaf \( E' \) of \( \mathcal{E} \) constructed in [40, Proposition 1.6] is \( \phi \)-invariant. The converse also follows from similar arguments. \( \square \)

Let \( \chi \) be a positive integer such that \( (\chi, n) = 1 \) and \( \epsilon \) be a sufficiently small positive number. Let us denote by \( \mathcal{M}_{\text{HB}, X_r}^{\chi, n, \epsilon} \) the moduli space of \( \epsilon \)-stable Higgs bundles \((\mathcal{E}, \phi)\) on the curve \( X_r \) of rank \( n \) and \( \chi(\mathcal{E}) = \chi \). For the construction of the moduli space we refer to [39] and [8, Theorem B.12]. Notice that if \((\mathcal{E}, \phi) \in \mathcal{M}_{\text{HB}, X_r}^{\chi, n, \epsilon}\), then from Lemma 6.5 it follows that it is a quasi-Gieseker-Higgs bundle (6.6). Since \((\chi, n) = 1\), one can show that the moduli space \( \mathcal{M}_{\text{HB}, X_r}^{\chi, n, \epsilon} \) is a fine moduli space. The proof of the existence of a universal family is similar to the proof of proposition 3.2. Let us fix a universal family \((\mathcal{E}^{\text{univ}}, \phi^{\text{univ}})\) over \( X_r \times \mathcal{M}_{\text{HB}, X_r}^{\chi, n, \epsilon} \). For every \( i = 1, \ldots, r \), let us choose a smooth point \( s_i \) of \( R[r],i \). Consider the map

\[
\mathcal{M}_{\text{HB}, X_r}^{\chi, n, \epsilon} \to \prod_{i=1}^{r} \{0,1,\ldots,r\}
\]

(6.10)

given by \([[(\mathcal{E}, \phi)] \to (dim H^0(\mathcal{E}|_{R[r]_i} \otimes \mathcal{O}_{R[r]}(-s_1)), \cdots, dim H^0(\mathcal{E}|_{R[r]_i} \otimes \mathcal{O}_{R[r]}(-s_r))\]

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We recall that we had fixed a positive integer \( \epsilon \).

**Remark 6.10.**

Notice that if \( E|_{R[r_i]} \cong \mathcal{O}_{R[r_i]}(1)^{ab} \), then \( \dim H^0(E|_{R[r_i]} \otimes \mathcal{O}_{R[r_i]}(-s_i)) = b \). Since the codomain is discrete, we see that the inverse image of every element of the codomain is a disjoint union of some connected components of \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \).

### 6.9.1. Action of Aut\((X_r/X_0)\) on the moduli space \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \)

Let \((\mathcal{E}^{\text{univ}}, \phi^{\text{univ}})\) be a universal family over \( X_r \times \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \). Given any \( \gamma \in \text{Aut}(X_r/X_0) \) \((2.7)\), consider the pullback family \((\gamma^* \mathcal{E}^{\text{univ}}, \gamma^* \phi^{\text{univ}})\) over \( X_r \times \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \). Notice that given any Gieseker-Higgs bundle \((\mathcal{E}, \phi)\) the pullback \((\gamma^* \mathcal{E}, \gamma^* \phi)\) induces the same torsion-free Higgs pair i.e., \((\pi_r)_* \mathcal{E} \cong (\pi_r)_*(\gamma^* \mathcal{E})\) and \((\pi_r)_* \phi = \pi_r*(\gamma^* \phi)\). Therefore \( \chi(\mathcal{E}) = \chi(\gamma^* \mathcal{E}) = \chi(\mathcal{F}) \). Moreover, \((\mathcal{E}, \phi)\) is \( \epsilon \)-stable if and only if \((\gamma^* \mathcal{E}, \gamma^* \phi)\) is \( \epsilon \)-stable. Therefore, we see that \( \gamma^* (\mathcal{E}^{\text{univ}}, \phi^{\text{univ}}) \) is also a family of \( \epsilon \)-stable Gieseker-Higgs bundles.

In other words, we have an action of \( \text{Aut}(X_r/X_0) \) on the moduli space \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \).

Given any \( a_* := (a_1, \ldots, a_r) \in \prod_{i=1}^r \{0, 1, \ldots, r\} \), let us denote by \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, a_*} \) the inverse image of \( a_* \) by the map \((6.10)\). Clearly the action of \( \text{Aut}(X_r/X_0) \) on \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, a_*} \) induces an action on \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon} \). We call a tuple \( a_* := (a_1, \ldots, a_r) \) admissible if \( a_i \geq 1 \) for every \( i = 1, \ldots, r \). From lemma 4.6, it follows that if \( a_* \) is admissible, the action of \( \text{Aut}(X_r/X_0) \) on \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, a_*} \) is free. We define

\[
\mathcal{M}_{\mathcal{V}, X_r}^{\chi, n, \epsilon, \text{ad}} := \bigcup_{a_* \text{ admissible}} \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, a_*}
\]

and

\[
\mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, \text{ad}} := \bigcup_{a_* \text{ admissible}} \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, a_*}
\]

**Remark 6.10.** We recall that we had fixed a positive integer \( n \) which denotes the rank and an integer \( d \) which denotes the degree satisfying \( \gcd(d, n, d) = 1 \). We denote by \( \mathcal{M}_{\mathcal{G}, \mathcal{H}} \) the moduli of Gieseker-Higgs bundles of rank \( n \) and degree \( d \) on the nodal curve \( X_0 \). Notice that for a Gieseker-Higgs bundle \((\mathcal{E}, \phi) \in \mathcal{M}_{\mathcal{G}, \mathcal{H}}\), we have \( \chi(\mathcal{E}) = \chi(\pi_r* \mathcal{E}) = d + n(1 - g) \). So if \( d < n(g - 1) \), we see that \( \chi(\mathcal{E}) \) is not positive. But if we choose a smooth point \( x \in X_0 \) and a positive integer \( N \) such that \( d + n \cdot N > n(g - 1) \), then tensoring every Gieseker-Higgs bundle with \( \mathcal{O}(N \cdot x) \) we get an isomorphism from the moduli space of Gieseker-Higgs bundles of rank \( n \) and degree \( d \) to the moduli of Gieseker-Higgs bundles of rank \( n \) and degree \( d + n \cdot N \).

So we can safely assume that \( \chi(\mathcal{E}) > 0 \). Therefore, we have a morphism \( \mathcal{M}_{\mathcal{H}, X_r}^{\chi, n, \epsilon, \text{ad}} \rightarrow \mathcal{M}_{\mathcal{G}, \mathcal{H}} \), where

1. \( \chi = d + n(1 - g) \), if \( d + n(1 - g) > 0 \)
2. \( \chi = d + N \cdot n + n(1 - g) \), for some sufficiently large positive integer \( N \), if \( d + n(1 - g) \geq 0 \).

**Remark 6.11.** Let

\[
\mathcal{M}_{\mathcal{G}, \mathcal{H}} \supset \partial^1 \mathcal{M}_{\mathcal{G}, \mathcal{H}} \supset \partial^2 \mathcal{M}_{\mathcal{G}, \mathcal{H}} \supset \cdots
\]
be the stratification of $\mathcal{M}_{GHB}$ given by the successive singular locus. We will show in Proposition 7.1 that $\partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB}$ is the locally closed subvariety of $\mathcal{M}_{GHB}$ consisting of the stable Gieseker-Higgs bundles $(X_\eta, \mathcal{E}, \phi)$. Therefore, the image of the morphism $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r} \to \mathcal{M}_{GHB}$ is precisely $\partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB}$.

In fact, $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r} \to \partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB}$ is a principal $\text{Aut}(X_r/X_0)$-bundle. Let us denote the morphism $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r} \to \mathcal{M}_{GHB}$ by $f_r$. Using the map (6.10), we see that the stratum

$$\partial^r \mathcal{M}_{GVB} \setminus \partial^{r+1} \mathcal{M}_{GVB} = \bigcup_{a, \text{ admissible}} \mathcal{M}^{a,\alpha}_{GVB}$$

(6.14)

and

$$\partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB} = \bigcup_{a, \text{ admissible}} \mathcal{M}^{a,\alpha}_{GHB}$$

(6.15)

where $\mathcal{M}^{a,\alpha}_{GVB}$ and $\mathcal{M}^{a,\alpha}_{GHB}$ are defined as in subsubsection (6.9.1) using the map (6.10). Moreover, $f_r^{-1}(\mathcal{M}^{a,\alpha}_{GVB}) = \mathcal{M}^{\chi,n,e,ad}_{V_B,X_r}$ and $f_r^{-1}(\mathcal{M}^{a,\alpha}_{GHB}) = \mathcal{M}^{\chi,n,e,ad}_{H_B,X_r}$.

6.11.1. **Symplectic structure on $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r}$**

**Lemma 6.12.** (1) The tangent space of the moduli space $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r}$ at a point $(\mathcal{E}, \phi)$ is naturally isomorphic to $\mathcal{H}^1(X_r, \mathcal{E}_* \mathcal{E}(\mathcal{E}, \phi))$, where $\mathcal{E}_* \mathcal{E}(\mathcal{E}, \phi)$ is the following complex:

$$\mathcal{E} \text{nd} \mathcal{E} \xrightarrow{[\cdot, \phi]} \mathcal{E} \text{nd} \mathcal{E} \otimes \omega_{X_r, s}$$

(6.16)

where $[s, \phi](s) = s \circ \phi - \phi \circ s$.

(2) The cotangent space of the moduli space $\mathcal{M}^{\chi,n,e,ad}_{H_B,X_r}$ at a point $(\mathcal{E}, \phi)$ is naturally isomorphic to $\mathcal{H}^1(X, \mathcal{E}_* \mathcal{E}^\vee(\mathcal{E}, \phi))$, where $\mathcal{E}_* \mathcal{E}^\vee(\mathcal{E}, \phi)$ is the following complex:

$$\mathcal{E} \text{nd} \mathcal{E} \xrightarrow{[\phi, \cdot]} \mathcal{E} \text{nd} \mathcal{E} \otimes \omega_{X_r, s}$$

(6.17)

where $[\phi, s](s) = -s \circ \phi + \phi \circ s$.

**Proof.** The proof follows from subsection 2.33 and remark 2.36. \(\square\)

Let $S$ be a complete discrete valuation ring. Let us fix $X_r \to S$, a flat family of projective curves such that the generic fibre $X_{r, \eta}$ is a smooth curve of genus $g$, the closed fibre is the nodal curve $X_r$ and the total space $X_r$ is regular over Spec $\mathbb{C}$. Again, the existence of such a family follows from [26, Theorem B.2 and Corollary B.3, Appendix B]. Let us denote by $\omega_{X_r/S}$ the relative dualising sheaf.

We can choose a line bundle $\mathcal{O}_{X_r/S}(1)$, which has the following property

$$\text{deg} \mathcal{O}_{X_r/S}(1)|_{\tilde{X}_0} = b_0 \quad \text{and} \quad \text{deg} \mathcal{O}_{X_r/S}(1)|_{R_i} = b \quad \text{for} \quad i = 1, \ldots, r, \quad \text{then} \quad b_0 \neq 0, b \neq 0 \quad \text{and} \quad \frac{b}{b_0} = \frac{e}{1 - re}.$$  

(6.18)
To construct such a line bundle, we first choose a line bundle $\mathcal{O}_{X_r}(1)$ on $X_r$ satisfying (6.18). Then a line bundle $\mathcal{O}_{X_r/S}(1)$ can be constructed using a standard spreading-out argument (may be after replacing $S$ by an étale neighbourhood of the closed point of $S$).

**Remark 6.13.** By Simpson’s method [39, Theorem 4.7], one can construct a relative moduli of Higgs bundles over the family of curves. To construct a total space and the GIT quotient, one must choose a relatively ample line bundle over the family $X_r/S$. We choose the line bundle $\mathcal{O}_{X_r/S}(1)$ (6.18) for this purpose. Then one can easily see from the definition that the relative moduli of Higgs bundle constructed using GIT with respect to this line bundle parametrises families of $\epsilon$-semistable Higgs bundles over the family $X_r/S$.

**Proposition 6.14.** There exists a family
\[
\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad} \to S
\]
of moduli of $\epsilon$-semistable Higgs bundles along the fibers of $X_r/S$ with Euler characteristic $\chi$. Moreover, the morphism $\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad} \to S$ is smooth.

**Proof.** We refer to [39, Theorem 4.7] for the construction of the family. The space of first order infinitesimal deformations of a Higgs bundle $(\mathcal{E}, \phi)$ is isomorphic to $\mathbb{H}^1(\mathcal{E}, (\mathcal{E}, \phi))$ and the space of obstructions to extend the Higgs bundle over a small thickening is isomorphic to $\mathbb{H}^2(\mathcal{E}, (\mathcal{E}, \phi))$. Since $\epsilon$-stability implies $0$-stability, from [3, Proposition 5.3], we have $\dim \mathbb{H}^2(\mathcal{E}, (\mathcal{E}, \phi)) = 1$. Therefore the dimension of the relative tangent space i.e., $\dim \mathbb{H}^1(\mathcal{E}, \phi)$ is constant and hence the morphism $\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad} \to S$ is smooth. \hfill $\square$

**Theorem 6.15.** There is a natural $\text{Aut}(X_r/X_0)$-equivariant symplectic form on $\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad}$.

**Proof.** As before (5.7 and 5.9), the following morphism of complexes induces a bi-linear pairing on the tangent space.
\[
\mathcal{C}^\circ (\mathcal{E}, \phi) \to \mathcal{C}_* (\mathcal{E}, \phi)
\]

Skew-symmetry and non-degeneracy of the above pairing follow from the description of the morphism of complexes. The closed-ness of the corresponding 2-form follows from the fact that $\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad}$ is the closed fibre of the smooth family $\mathcal{M}_{\text{HB},X_r}^{X,n,e,ad} \to S$ and the fact that the above pairing is closed on the generic fibre.

Let $t \in \text{Aut}(X_r/X_0)$ be an automorphism $t : X_r \to X_r$. Then we have a commutative diagram of complexes
\[
\begin{array}{ccc}
\mathcal{C}^\circ (\mathcal{E}, \phi) & \to & \mathcal{C}_* (\mathcal{E}, \phi) \\
\downarrow t^* & & \downarrow t^* \\
\mathcal{C}^\circ (t^* \mathcal{E}, t^* \phi) & \to & \mathcal{C}_* (t^* \mathcal{E}, t^* \phi)
\end{array}
\]

The commutativity follows from the fact that $t^*[\phi, s] = [t^* \phi, t^* s]$.
It induces the following commutative diagram of hypercohomologies

\[
\begin{array}{ccc}
\mathbb{H}^1(\mathcal{E}_\ast'(\mathcal{E}, \phi)) & \longrightarrow & \mathbb{H}^1(\mathcal{E}_\ast(\mathcal{E}, \phi)) \\
\downarrow f^\ast & & \downarrow f^\ast \\
\mathbb{H}^1(\mathcal{E}_\ast'(t^\ast \mathcal{E}, t^\ast \phi)) & \longrightarrow & \mathbb{H}^1(\mathcal{E}_\ast(t^\ast \mathcal{E}, t^\ast \phi))
\end{array}
\]  

(6.22)

Therefore, the symplectic form on \( \mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad} \) is \( Aut(X_r/X_0) \)-equivariant. \[\square\]

**Corollary 6.16.** The morphism \( f_r : \mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad} \to \mathcal{M}_{\text{GHB}} \) (remark 6.11) is a Poisson morphism.

**Proof.** Using the descriptions (Theorem 4.10 and Lemma 6.12) of the vector bundles in the following equations, we see that

\[
f_r^\ast \Omega_{\mathcal{M}_{\text{GHB}}} (\log \partial \mathcal{M}_{\text{GHB}}) \equiv \Omega_{\mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad}}, \quad \text{and} \quad f_r^\ast T_{\mathcal{M}_{\text{GHB}}} (\log \partial \mathcal{M}_{\text{GHB}}) \equiv T_{\mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad}}
\]  

(6.23)

The explicit descriptions given in (5.7), (5.9) and Theorem 6.15) of the morphisms \( f_r^\ast \Omega_{\mathcal{M}_{\text{GHB}}} \to f_r^\ast T_{\mathcal{M}_{\text{GHB}}} \) and \( \Omega_{\mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad}} \to T_{\mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad}} \) induced by the Poisson bivectors on \( \mathcal{M}_{\text{GHB}} \) and \( \mathcal{M}_{\text{HB}, X_r}^{X,n,e,ad} \), respectively clearly match at every point. Therefore the corollary follows. \[\square\]

### 7. Stratification of \( \mathcal{M}_{\text{GHB}} \) by Poisson ranks

Let us recall that we had chosen a degeneration of a smooth projective curve i.e., a family of curves \( \mathcal{X} \) over a discrete valuation ring \( S \) (2.7.1). Then one can construct a family of varieties \( \mathcal{M}_{\text{GHB}, S} \) over \( S \) such that the fibre over the generic point is the moduli of Higgs bundles over the generic curve and the fibre over the closed point is the moduli of Gieseker-Higgs bundles on the nodal curve. Moreover, the closed fibre is a normal-crossing divisor in \( \mathcal{M}_{\text{GHB}, S} \). It has a natural stratification given by its successive singular loci

\[
\mathcal{M}_{\text{GHB}} \supset \partial^1 \mathcal{M}_{\text{GHB}} \supset \cdots \supset \partial^n \mathcal{M}_{\text{GHB}} \supset \partial^{n+1} \mathcal{M}_{\text{GHB}} := \emptyset
\]  

(7.1)

By [5, Lemma 3.1], the stratification has the following description.

for every \( 0 \leq r \leq n \), \( \partial^r \mathcal{M}_{\text{GHB}} = \{ x \in \mathcal{M}_{\text{GHB}} \mid \text{cardinality of the set } q^{-1}(x) \geq r + 1 \} \),

where \( q \) denotes the normalisation \( \widetilde{\mathcal{M}}_{\text{GHB}} \to \mathcal{M}_{\text{GHB}} \).

**Proposition 7.1.**

1. For every integer \( 0 \leq r \leq n \), \( \partial^r \mathcal{M}_{\text{GHB}} \) is a closed Poisson sub-variety \( \mathcal{M}_{\text{GHB}} \). The closed points of \( \partial^r \mathcal{M}_{\text{GHB}} \) correspond to the equivalence classes of stable Gieseker-Higgs bundles \( (X_k, \mathcal{E}, \phi) \), where \( n \geq k \geq r \).

2. The \( r \)-th stratum \( \partial^r \mathcal{M}_{\text{GHB}} := \partial^r \mathcal{M}_{\text{GHB}} \setminus \partial^{r+1} \mathcal{M}_{\text{GHB}} \) is a smooth locally-closed Poisson sub-scheme of \( \mathcal{M}_{\text{GHB}} \).
(3) the most singular locus $\partial^n \mathcal{M}_{GHB}$ is a smooth Poisson variety of dimension $2n^2(g - 1) + 2 - n$, whose closed points correspond to the equivalence classes of stable Gieseker-Higgs bundles $(X_n, \mathcal{E}, \phi)$ of rank $n$ and degree $d$.

Proof. From [35, Corollary 2.4], it follows that for every $r$, the variety $\partial^i \mathcal{M}_{GHB}$ is a closed Poisson subvariety of $\mathcal{M}_{GHB}$ of dimension $2(n^2(g - 1) + 1) - r$. In particular, $\partial^r, o \mathcal{M}_{GHB}$ is a smooth locally-closed Poisson subvariety.

There is a universal curve $\mathcal{X}_{uni v}$ over $\mathcal{M}_{GHB}$, which is the restriction of the universal curve $\mathcal{X}_{uni v}$. We define

$$D := V(Fitt^1 \Omega_{\mathcal{X}_{uni v}} \mathcal{M}_{GHB}),$$

where $Fitt^1 \Omega_{\mathcal{X}_{uni v}} \mathcal{M}_{GHB}$ denotes the first Fitting ideal of $\Omega_{\mathcal{X}_{uni v}} \mathcal{M}_{GHB}$.

Claim: $D$ is the normalisation of $\mathcal{M}_{GHB}$.

Proof: Since the fibres of $D \to \mathcal{M}_{GHB}$ are the singular locus of the morphism $\mathcal{X}_{uni v} \to \mathcal{M}_{GHB}$, we see that the earlier map is finite, birational and surjective. So if we show that $D$ is smooth, then it follows that it is the normalisation of $\mathcal{M}_{GHB}$.

The question is local. So, let us concentrate around a point $p$ representing an equivalence class of Gieseker-Higgs bundle $(X_r, \mathcal{E}, \phi) \in \mathcal{M}_{GHB}$, as in the proof of Proposition 3.6. The Henselian local ring of $\mathcal{M}_{GHB}$ at $p$ is $A_0$ (see proposition 3.6), whose local components $\{D_i\}_{i=1}^{r+1}$ are given by

$$\text{Spec} A_{0, i} := \text{Spec} \frac{A_0}{(t_i)} \text{ for } i = 1, \ldots, r + 1.$$

Moreover, if $p_i$ denotes the $i$-th node of $X_r$, then

$$\mathcal{O}_{\mathcal{X}_{uni v}, p_i} \cong \frac{A_0[x, y]}{x y - t_i} \text{ for every } i = 1, \ldots, r + 1.$$

Therefore, using the description of $A_0$ ((3) in the proof of Proposition 3.6) we have

$$\mathcal{O}_{D_i, p_i} \cong A_{0, i}, \text{ and } D_i \text{ is smooth for every } i = 1, \ldots, r + 1.$$

This proves that the normalisation is isomorphic to the vanishing locus of the first Fitting ideal. □

Since $\tilde{\mathcal{M}}_{GHB}$ is the vanishing locus of the first Fitting ideal $Fitt^1 (\Omega_{\mathcal{X}_{uni v}} \mathcal{M}_{GHB})$, the fibre of the normalisation $\tilde{\mathcal{M}}_{GHB} \to \mathcal{M}_{GHB}$ over a point $(X_r, \mathcal{E}, \phi)$ is $\{(X_r, \mathcal{E}, \phi, x) | x \text{ is a node of the curve } X_r\}$. □

Since $\partial^r, o \mathcal{M}_{GHB}$ is a smooth locally closed Poisson sub-scheme, the Poisson bi-vector $\sigma$ induces a morphism $\sigma^b : \Omega^{\text{reg}, o} \mathcal{M}_{GHB} \to T^{\text{reg}, o} \mathcal{M}_{GHB}$. To compute the Poisson rank of $\sigma$ at a point of this stratum it is enough to compute the rank of the morphism $\sigma^b$ (see Example 3). Before computing the Poisson ranks, we need a preliminary lemma 7.2. Let us denote the torus $Aut(\mathcal{X}_r / X_0)$ by $A_r$, for convenience. Consider the principal $A_r$-bundle $\mathcal{M}_{HB, X_r} \to \partial^r, o \mathcal{M}_{GHB}$. 40
Let \( \tilde{\rho} \) denote an isomorphism class of a stable Gieseker-Higgs bundle \((X_k, E, \phi) \in \mathcal{M}_{HB,X_r}^{X,n,e,ad}\) and \( p \) denote the image of \( \tilde{\rho} \) i.e., the Gieseker-equivalent class of \((X_k, E, \phi)\). Then we have the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{\vartheta},\mathcal{G}_{\text{HGB},p} & \xrightarrow{u} & \Omega_{\vartheta},\mathcal{G}_{\text{HGB},p}^{L} & \xrightarrow{f} & \Omega_{A,e} & \equiv & H^0(X_r, T_{X_r}) & 0 \\
\downarrow{\sigma^p} & & \downarrow{\sigma^p} & & \downarrow{\sigma^p} & & \downarrow{\sigma^p} & & \downarrow{\sigma^p} & & \downarrow{\sigma^p} \\
0 & \longleftarrow & T_{\vartheta},\mathcal{G}_{\text{HGB},p} & \xleftarrow{v} & T_{\vartheta},\mathcal{G}_{\text{HGB},p}^{L} & \xleftarrow{i} & T_{A,e} & \equiv & H^0(X_r, T_{X_r}) & \longleftarrow 0
\end{array}
\]  

(7.3)

Notice that \( v \circ \sigma^p \circ u = \sigma^p \) and \( j \circ \sigma^p \circ i = B_r \).

**Lemma 7.2.** Let \((X_r, \mathcal{E}, \phi)\) be a stable Gieseker-Higgs bundle. Let \( X_r = \bigcup_{i \in \Lambda} U_i \) be an open cover of \( X_r \) such that the vector bundle \( \mathcal{E} \) and \( \omega_{X_r} \) are trivial over every \( U_i \). Let us denote the co-cycle (with respect to the cover \( \{U_i\} \)) of \( \mathcal{E} \) by \( \{A_{ij}\} \). Let \( \{\phi_i\}_{i \in \Lambda} \) denote the collection of Higgs fields on the cover \( \{U_i\}_{i \in \Lambda} \) which glue to give the global Higgs field \( \phi \). Then

1. the morphism \( i \) is given by \( i(\psi) = ([X\psi, \phi_i], [X\psi, A_{ij}]) \).
2. the morphism \( \omega^# \) is given by \( \omega^#([\alpha_i], [\eta_{ij}]) = ([\alpha_i], [\eta_{ij}]) \).
3. the composite \( j \circ \omega^# \circ i = 0 \).

**Proof.** proof of (1). Let \( \psi \in H^0(X_r, T_{X_r}) \). It is, by definition, an isomorphism

\[
\psi : X_r[e] \rightarrow X_r[e]
\]

which on the sheaf of rings can be described as follows.

\[
\psi^\#: \mathcal{O}_{X_r}[e] \rightarrow \mathcal{O}_{X_r}[e]
\]

(7.4)

given by

\[
f + e \cdot g \mapsto f + e \cdot (g + X\psi(df)),
\]

where \( X\psi \) is the vector field on \( X_r \) corresponding to the infinitesimal automorphism \( \psi \).

We denote by \( \mathcal{E}[e] \) the trivial deformation of \( \mathcal{E} \) over \( X_r[e] \). We want to write the co-cycle of \( \psi^* \mathcal{E}[e] \) in terms of the cocycle of \( \mathcal{E} \). The transition functions of \( \psi^* \mathcal{E}[e] \) are given by \( A_{ij} + eB_{ij} \) for some \( B_{ij} \) which fits into the following commutative diagram.

\[
\begin{array}{cccccc}
(\mathcal{O}_{U_j}[e])^\oplus_n & \longrightarrow & (\mathcal{O}_{U_j}[e])^\oplus_n & \xrightarrow{A_{ij} + e \cdot 0} & (\mathcal{O}_{U_j}[e])^\oplus_n \\
(\mathcal{O}_{U_j}[e])^\oplus_n & \longrightarrow & (\mathcal{O}_{U_j}[e])^\oplus_n & \xrightarrow{(1 + eX\psi)} & (\mathcal{O}_{U_j}[e])^\oplus_n
\end{array}
\]

(7.5)

where the map

\[
(\mathcal{O}_{U_j}[e])^\oplus_n \xrightarrow{(1 + eX\psi)} (\mathcal{O}_{U_j}[e])^\oplus_n
\]

(7.6)

is given by

\[
[(f_i + e g_i)]_{i=1}^n \mapsto [(1 + eX\psi)(f_i + e g_i)]_{i=1}^n.
\]
Since the above diagram commutes we have
\[
(A_{ij} + \epsilon B_{ij}) \circ (1 + \epsilon X_{\psi}) = (1 + \epsilon X_{\psi}) \circ A_{ij}
\]
\[
\implies B_{ij} = [X_{\psi}, A_{ij}]
\]

It can be easily checked that \( B_{ik} = A_{ij} B_{jk} + B_{ij} A_{jk} \) for any \( i, j, k \) and hence \( B_{ij} \) defines an element of \( H^1(\mathcal{E} \otimes \mathcal{E}) \).

Consider the Higgs field \( \phi + \epsilon \cdot 0 : \mathcal{E}[e] \to \mathcal{E}[e] \otimes \omega_X \) over \( X_r[e] \). It can be expressed as Higgs fields \( \{\phi_i\} \) over each \( U_i \) satisfying the following
\[
A_{ij} \phi_j A_{ij}^{-1} = \phi_i, \forall i, j. \tag{7.7}
\]

Similarly, the Higgs field \( \psi^* \phi \) can be expressed as \( \{\phi_i + \epsilon \phi'_i\} \) which fits into the following commutative diagram.
\[
\begin{array}{ccc}
\mathcal{E}[e]|_{U_i[e]} & \xrightarrow{\phi_i + \epsilon 0} & ((\mathcal{E} \otimes \omega_X)[e]|_{U_i[e]} \\
\downarrow (1 + \epsilon X_{\psi}) \circ 1 & & \downarrow (1 + \epsilon X_{\psi}) \circ 1 \\
(\psi^* \mathcal{E}[e])|_{U_i[e]} & \xrightarrow{\phi_i + \epsilon \phi'_i} & ((\psi^* \mathcal{E}[e]) \otimes \omega_X[e]|_{U_i[e]} \\
\end{array}
\tag{7.8}
\]

Since we have \( (\phi_i + \epsilon \phi'_i) \circ ((1 + \epsilon X_{\psi}) \otimes 1) = ((1 + \epsilon X_{\psi}) \otimes 1) \circ \phi_i \), it follows that \( \phi'_i = [X_{\psi}, \phi_i] \). It can be easily verified that \( \phi'_i A_{ij} - A_{ij} \phi'_j = B_{ij} \phi_j - \phi_i B_{ij} \) for all \( i, j \). Hence \( \{\phi'_i\}, \{B_{ij}\} \) defines an element of \( H^1(\mathcal{E}_*) \).

Therefore, we conclude that \( i(\psi) = \{[X_{\psi}, \phi_i]\}, \{[X_{\psi}, A_{ij}]\} \).

proof of (2). From the description of the morphism of complexes \( \mathcal{E}_*^\vee \to \mathcal{E}_* \), we see that \( \omega^\#(\{\alpha_j\}, \{\eta_{ij}\}) = \{-\alpha_j\}, \{\eta_{ij}\}\). 

proof of (3). We have \( \omega^\# \circ i(\psi) = ([X_{\psi}, \phi_j]), ([X_{\psi}, A_{ij}]) \). Since the map \( j \) is just the dual of the morphism \( i \), we have
\[
(j \circ \omega^\# \circ i)(\psi') = \text{Trace}([X_{\psi}, A_{ij}] \circ [X_{\psi}, \phi_j] - [X_{\psi}, \phi_i] \circ [X_{\psi}, A_{ij}]) \quad (\text{2.50, 2.52, and remark 2.36})
\]

The vector fields \( X_{\psi} \) and \( X_{\psi'} \) are elements of \( H^0(X_r, T_{X_r}) \). Since \( X_0 \) is a stable curve, the vector fields have support only along \( R[r] \), the chain of \( \mathbb{P}^1 \)'s. Moreover, it is not difficult to see that
\[
H^0(X_r, T_{X_r}) \cong \bigoplus_{i=1}^r H^0(R[r]_i, T_{R[r]_i}(x^+_i - x^-_i)), \quad (7.9)
\]
where \( x^+_i \) and \( x^-_i \) denote the two nodes of \( R[r]_i \). Therefore, we see that the vector fields \( X_{\psi} \) and \( X_{\psi'} \) have support on \( R[r] \) and they vanish at the nodes. Moreover, over any particular \( \mathbb{P}^1 \), say \( R[r]_i \), two such vector fields only differ by a scalar. In order to compute the term \( \text{Trace}([X_{\psi}, A_{ij}] \circ [X_{\psi}, \phi_j] - [X_{\psi}, \phi_i] \circ [X_{\psi}, A_{ij}] \) we will concentrate and compute it on every \( \mathbb{P}^1 \).
Let us concentrate on \( R[r]_k \) for any \( k \). Let us denote by \( x^+_k \) and \( x^-_k \) the two nodes on \( R[r]_k \). Then \( R[r]_k = (R[r]_k \setminus x^+_k) \cup (R[r]_k \setminus x^-_k) \) is an open cover. Let \( A \) denote the transition function \( R[r]_k \setminus \{x^+_k, x^-_k\} \rightarrow GL_n \), and \( \phi^+_k \) and \( \phi^-_k \) denote the Higgs fields on \( R[r]_k \setminus \{x^-\} \) and \( R[r]_k \setminus \{x^+\} \), respectively corresponding to the Higgs field \( \phi|_{R[r]_k} \). From (7.7), it follows that

\[
A \circ \phi^+_k = \phi^-_k \circ A. \tag{7.10}
\]

We recall that \( \Theta|_{R[r]_k} \equiv \Theta(1)^{a_k} \oplus \Theta^{b_k} \), for some positive integer \( a_k \) and some non-negative integer \( b_k \) such that \( a_k + b_k = n \). Therefore the matrix function \( A \) has the following form

\[
z \mapsto \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{7.11}
\]

where \( B = \frac{1}{2} \cdot I_{a_k} \) and \( D = I_{b_k} \). Similarly, \( \phi^+_k \) is also of the following form

\[
\begin{pmatrix} \phi^+_1 \\ \phi^+_3 \\ 0 \\ \phi^+_2 \end{pmatrix} \tag{7.12}
\]

We want to compute \( \text{Trace}([X_\psi, A] \circ [X_\psi, \phi^+_k] - [X_\psi, \phi^-_k] \circ [X_\psi, A]) \). It follows from the description above that

\[
\text{Trace}([X_\psi, A] \circ [X_\psi, \phi^+_k] - [X_\psi, \phi^-_k] \circ [X_\psi, A]) = \text{Trace}([X_\psi, B] \circ [X_\psi, \phi^+_1] - [X_\psi, \phi^-_1] \circ [X_\psi, B]) \tag{7.13}
\]

Now let \( \vec{f} \in \Gamma(R[r]_k \setminus \{x^+_k, x^-_k\}, \Theta^{|n|}_{R[r]_k}) \). Then

\[
[X_\psi, B](\vec{f}) = (X_\psi \circ B)(\vec{f}) = (B \circ X_\psi)(\vec{f}) = (X_\psi B)(\vec{f}) + B(X_\psi \vec{f}) - B(X_\psi \vec{f}) = (X_\psi B)(\vec{f}). \tag{7.14}
\]

Similarly,

\[
[X_\psi, \phi^+_1] = X_\psi \phi^+_1. \tag{7.15}
\]

Therefore,

\[
\text{Trace}(X_\psi B \circ [X_\psi, \phi^+_1] - [X_\psi, \phi^-_1] \circ [X_\psi, B]) = \text{Trace}(X_\psi B \circ (X_\psi \phi^+_1) - (X_\psi \phi^-_1) \circ (X_\psi, B)) = \frac{1}{\text{rank} B} \cdot \text{Trace}(X_\psi B) \cdot \text{Trace}(X_\psi \phi^+_1 - X_\psi \phi^-_1) \quad (\text{Since } X_\psi B \text{ is a diagonal matrix})\]

Now from (7.10), we have \( \phi^+_k = A^{-1} \circ \phi^-_k \circ A \). Therefore, \( \text{Trace}(\phi^+_k) = \text{Trace}(\phi^-_k) \) and \( \text{Trace}(\phi^+_1) = \text{Trace}(\phi^-_1) \). Since \( X_\psi(\text{Trace}(\phi^+_1)) = \text{Trace}(X_\psi \phi^+_1) \), we have

\[
\text{Trace}(X_\psi \phi^+_1) = \text{Trace}(X_\psi \phi^-_1). \tag{7.16}
\]
Therefore,

\[
\text{Trace}([X_\psi, B] \circ [X_\psi, \Phi^1] - [X_\psi, \Phi^1] \circ [X_\psi, B]) = \frac{1}{\text{rank } B} \cdot \text{Trace}(X_\psi B) \cdot \text{Trace}(X_\psi \Phi^1 - X_\psi \Phi^1) = 0
\]

Therefore, we conclude that \((j \circ \omega^\# \circ i)(\psi)(\psi') = 0\). \(\square\)

**Remark 7.3.** Notice that the statement (1) in lemma 7.2 gives an alternative proof of lemma 4.7.

**Theorem 7.4.** The stratification of the Poisson variety \(\mathcal{M}_{GHB}\) given by the successive degeneracy loci of the Poisson structure (2.25) is the same as the stratification given by the successive singular loci (7.1). Moreover, \(\partial^r \mathcal{M}_{GHB} \setminus \partial^{r+1} \mathcal{M}_{GHB}\) is a smooth Poisson subvariety of dimension \(2(n^2(g - 1) + 1) - r\) with constant Poisson rank \(2(n^2(g - 1) + 1) - 2r\). In particular, the most singular locus \(\partial^n \mathcal{M}_{GHB}\) is a smooth Poisson variety of dimension \(2(n^2(g - 1) + 1) - n\) with constant Poisson rank \(2(n^2(g - 1) + 1) - 2n\).

*Proof.* Follows from the fact that \(B_r = 0\) for all \(r = 0, \ldots, n\). \(\square\)

**Remark 7.5.** Let us denote by \(\mathcal{M}_{PV}^r\) the moduli of the parabolic vector bundles of rank \(n\) and degree \(d - n\) over \(\tilde{X}_0\) with full-flags at the two points \(x^+\) and \(x^-\) and semi-stable with respect to sufficiently small and generic choice of parabolic weights \(\epsilon_+\). In [5], it is shown that the most singular locus \(\partial^n \mathcal{M}_{GVB}\) is isomorphic to \(\mathcal{M}_{PV}^r\). Let us denote by \(\mathcal{M}_{PHB}^r\) the moduli of non-strongly parabolic-Higgs bundles over the curve \(\tilde{X}_0\) of rank \(n\) and degree \(d - n\) with full-flagged parabolic structures at the two pre-images \(x^+\) and \(x^-\) and with sufficiently small and generic choice of parabolic weights \(\epsilon_+\). One can show that the most singular locus \(\partial^n \mathcal{M}_{GHB}\) is isomorphic to the closed Poisson sub-scheme of \(\mathcal{M}_{PHB}^r\) consisting of non-strongly parabolic-Higgs bundles whose eigenvalues of the Higgs field at the two points \(x^+\) and \(x^-\) are the same. We will discuss this in a separate note.

**7.6. The induced Poisson structure on the normalisation of \(\mathcal{M}_{GHB}\).** Consider the normalization \(q : \widetilde{\mathcal{M}}_{GHB} \to \mathcal{M}_{GHB}\). It is a smooth variety with normal-crossing divisor \(q^{-1}(\partial \mathcal{M}_{GHB})\). The pullback of the log-symplectic form induces a log-symplectic structure on \(\widetilde{\mathcal{M}}_{GHB}\). The variety \(\widetilde{\mathcal{M}}_{GHB}\) has the following stratification.

\[
\widetilde{\mathcal{M}}_{GHB} \supset \partial \widetilde{\mathcal{M}}_{GHB} := q^{-1}(\partial \mathcal{M}_{GHB}) \supset \partial^2 \widetilde{\mathcal{M}}_{GHB} := q^{-1}(\partial^2 \mathcal{M}_{GHB}) \supset \cdots \tag{7.17}
\]

It is straightforward to check that \(\partial^{r+1} \widetilde{\mathcal{M}}_{GHB}\) is the singular locus of \(\partial^r \widetilde{\mathcal{M}}_{GHB}\) for every \(r \geq 1\). The sub-scheme \(\partial^{r,0} \mathcal{M}_{GHB}\) is locally an intersection of \(r\) connected components of \(\mathcal{M}_{GHB}\). Therefore, locally the inverse image of \(\partial^{r,0} \mathcal{M}_{GHB}\) is the disjoint union of \(r\) sub-varieties, each of which is isomorphic (locally) to \(\partial^{r,0} \mathcal{M}_{GHB}\). Therefore, we see that the Poisson rank at a point of the strata \(\partial^{r,0} \mathcal{M}_{GHB}\).
\( \partial^{r+1}\tilde{\mathcal{M}}_{GHB} \) is the same as the Poisson rank at the image of this point under the normalisation map. We can compute the Poisson rank at a point using the same diagram (7.3).

**Remark 7.7.** By lemma 7.2, it follows that the bi-residues/magnetic terms [30, Example 3.3] is 0 for every stratum of the normal crossing divisor \( \partial\tilde{\mathcal{M}}_{GHB} \). Therefore, by [30, proposition 3.6], it follows that the local normal form of the Poisson structure \( \mathcal{M}_{GHB} \) is stably equivalent to

\[
\omega = \sum_{j=1}^{k} dp_j \wedge \frac{dy_j}{y_j}, \text{ for some integer } k,
\]

as in the example 4.

8. Description of the symplectic foliation

We recall from §6 that the moduli space \( \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \) of \( \epsilon \) stable admissible Higgs bundles of rank \( n \) and Euler characteristic \( \chi \) on the curve \( X_r \) is a smooth variety. The torus \( A_r := Aut(X_r/X_0) \) acts freely on \( \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \). The quotient is isomorphic to \( \partial^{r,\partial}\mathcal{M}_{GHB} := \partial^{r,\partial}\mathcal{M}_{GHB} \setminus \partial^{r+1}\mathcal{M}_{GHB} \). From theorem 6.15, it follows that \( \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \) is a \( A_r \)-symplectic manifold. We will see that the action is Hamiltonian i.e., it has a momentum map. The Hamiltonian action of an algebraic group and the momentum map can be defined in the algebraic setting. We refer to [27], [28], and [41, Chapter II, §1, §2] for the details. Before describing the momentum map notice that \( T_{A_r,e} \equiv H^0(X_r,T_{X_r}) \equiv \oplus_{i=1}^r H^0(R[r],\mathcal{O}_{R[r]}) \).

**Theorem 8.1.** (1) The map

\[
\mu_r : \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \rightarrow (T_{A_r,e})^\vee
\]

declared by

\[
\mu_r(\mathcal{E},\phi)(X_\psi) = \lambda(i(X_\psi)) = Trace(\phi \circ i(X_\psi)), \text{ for } X_\psi \in H^0(X_r,T_{X_r})
\]

is a momentum map, where

(a) \( i : H^0(X_r,T_{X_r}) \hookrightarrow \mathcal{H}^1(\mathcal{E}_\ast) \) denotes the differential of the orbit map \( A_r \rightarrow \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \) at the point \( (\mathcal{E},\phi) \).

(b) \( \lambda \) denotes the symplectic potential on \( \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \) (2.35.1 and remark 2.36).

(2) \( \mu_r(\mathcal{E},\phi) = (Trace \phi|_{\mathcal{O}_{R[r]}(1)^{a_0}}, \ldots, Trace \phi|_{\mathcal{O}_{R[r]}(1)^{a_0}}) \), where \( \mathcal{E}|_{R[r]} \equiv \mathcal{O}_{R[r]}(1)^{a_0} \oplus \mathcal{O}_{R[r]}^{a_0} \) for every \( i = 1, \ldots, r \).

(3) The coordinate functions of \( \mu_r \) are the Casimir functions of \( \mathcal{M}_{GHB}^{\alpha} \) (6.15). In particular, the variety \( \mu_r^{-1}(0)\cap \mathcal{M}_{GHB}^{\alpha} \) is a symplectic leaf of \( \mathcal{M}_{GHB}^{\alpha} \) containing \( \Omega_{\mathcal{M}_{GHB}^{\alpha}} \). Moreover, it consists of triples \( (X_r,\mathcal{E},\phi) \) such that the trace of \( \phi|_{\mathcal{O}_{R[r]}(1)^{a_0}} : \mathcal{O}_{R[r]}(1)^{a_0} \rightarrow \mathcal{O}_{R[r]}(1)^{a_0} \) is zero for all \( i = 1, \ldots, r \).

**Proof.** proof of (1). Similar to the case of a smooth curve, the symplectic form on \( \mathcal{M}_{X_{HB,Xr}}^{\chi,n,e,ad} \) is exact, i.e., there exists a 1-form \( \lambda \) such that the symplectic form is given by \( -d\lambda \). In the literature, such a form is called the symplectic potential. In our case, the symplectic potential is an extension of the Liouville 1-form, as in
the case of smooth curves. The symplectic potential form $\lambda$ can be described similarly as in 2.35.1 (remark 2.36).

It is well-known that any $G$-variety $Z$ equipped with an equivariant symplectic potential $\lambda$ has an equivariant momentum map.

$$\mu : Z \to g^\vee, \ g := \text{Lie algebra of } G$$ (8.3)
given by

$$\mu(z)(\xi) = \lambda(z)(\iota(\xi)),$$

where

1. $z \in Z$, and $\xi \in g$.
2. $\iota : g \to T_z Z$, denotes the differential of the orbit map at the point $z$.

Therefore, it is enough to check that the symplectic potential in our case is preserved by the action of the torus $A_r$. From 2.35.1, it follows that

$$\lambda(\mathcal{E}, \phi)([s_{ij}], [t_{ij}]) = \{\text{Trace}(\phi \circ s_{ij})\},$$ (8.4)

where

1. $(\mathcal{E}, \phi) \in \mathcal{M}^{x, n, e, ad}_{HB, X_r}$, and
2. $([s_{ij}], [t_{ij}]) \in H^1(\mathcal{E}_*)$, the tangent space of $\mathcal{M}^{x, n, e, ad}_{HB, X_r}$ at the point $(\mathcal{E}, \phi)$ (remark 2.35).

Therefore, we see that

$$(f^* \lambda)(\mathcal{E}, \phi)(s_{ij}, t_{ij}) = \text{Trace}(f^\# \circ \phi \circ (f^\#)^{-1} \circ f^\# \circ s_{ij} \circ (f^\#)^{-1})$$

$$= \text{Trace}(\phi \circ s_{ij})$$

$$= \lambda(\mathcal{E}, \phi)(s_{ij}, t_{ij}),$$

where $f \in A_r$ is an automorphism of $X_r$, and $f^\#$ denotes the induced morphism $f^* \mathcal{E} \to \mathcal{E}$. This completes the proof of (1).

Proof of (2) and (3). Since $\mu_r$ is $A_r$-invariant map, it descends to $\mathcal{M}^{\alpha}_{GHB}$.

Claim: The morphism $\mu_r$ is a smooth morphism.

Proof: To prove this, it is enough to show that the morphism $d\mu_r : T_{\mathcal{M}^{x, n, e, ad}_{HB, X_r}} \to (T_{A_r, e})^\vee$ is surjective at every point of $\mathcal{M}^{x, n, e, ad}_{HB, X_r}$. Let $(\mathcal{E}, \phi)$ be a point in $\mathcal{M}^{x, n, e, ad}_{HB, X_r}$. Recall that the tangent space of $\mathcal{M}^{x, n, e, ad}_{HB, X_r}$ is isomorphic to $H^1(\mathcal{E}_*)$. Since $\mu_r$ is a momentum map, the morphism

$$d\mu_r : H^1(\mathcal{E}_*) \to T_{A_r, e}^\vee$$ (8.5)
is the same as \( j \circ \omega \) (diagram 7.3). By lemma 4.7 and remark 4.8, the morphism \( j \) is surjective. Since \( \omega \) is an isomorphism, the morphism \( d\mu_r \) is also surjective. Hence, \( \mu_r \) is a smooth morphism.

Therefore, \( \mu_r^{-1}(0) \cap \mathcal{M}_{PHB}^n \) is a symplectic leaf of \( \mathcal{M}_{PHB}^n \). Notice that \( \mu_r \) is the quotient map in the following short exact sequence

\[
0 \to \pi^* \Omega_{\mathcal{E}_{GVB}} \to \Omega_{\mathcal{E}_{GVB},X_r} \to \Theta_{\mathcal{E}_{GVB},X_r} \otimes \Omega_{\mathcal{E}} \to 0
\]  

(8.6)

Therefore, it follows that \( \mu_r^{-1}(0) \cap \mathcal{M}_{PHB}^n \) contains \( \mathcal{M}_{PHB}^n \). Here, by abuse of notation, we denote both the sheaf and its total space by the same notation \( \Omega_{\mathcal{E}_{GVB}} \).

Now let \((\mathcal{E}, \phi)\) be a Gieseker-Higgs bundle on \( X_r \) and \( X_\psi \) be an element of \( H^0(R[r],\omega_{X_r}^n) \). Let us denote by \( x_i^+ \) and \( x_i^- \) the two nodes on \( R[r] \). Then \( R[r] = (R[r] \setminus x_i^+) \cup (R[r] \setminus x_i^-) \) is an open cover. Let \( A \) denote the transition function \( R[r] \setminus \{x_i^+, x_i^-\} \to GL_n \). We recall that \( \mathcal{E}|_{R[r]} \equiv \mathcal{E}(1)^{a_i} \oplus \mathcal{E}^{b_i} \), for some positive integer \( a_i \) and some non-negative integer \( b_i \) such that \( a_i + b_i = n \). Therefore the matrix function \( A \) has the following form

\[
A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},
\]  

(8.7)

where \( B = \frac{1}{z} \cdot I_{a_i} \) and \( D = I_{b_i} \). Similarly, \( \phi \) is also of the following form

\[
\begin{bmatrix} \phi_1 & \phi_3 \\ 0 & \phi_2 \end{bmatrix}.
\]  

(8.8)

We easily see that

\[
\text{Trace}(\phi \circ [X_\psi, A]) = \text{Trace}(\phi_1 \circ [X_\psi, B]).
\]  

(8.9)

Let \( \mathbf{f} \in \Gamma(R[r] \setminus \{x_i^+, x_i^-\}, \mathcal{E}_{R[r]}^n) \). Then

\[
[X_\psi, B](\mathbf{f}) = X_\psi(\frac{1}{z} \cdot \mathbf{f}) - \frac{1}{z} \cdot X_\psi(\mathbf{f}) = \mathbf{f} \cdot X_\psi(\frac{1}{z}).
\]  

(8.10)

Also, notice that \( X_\psi(z) \) is some scalar multiple of \( z \cdot \frac{d}{dz} \). Therefore,

\[
\text{Trace}(\phi_1 \circ [X_\psi, B]) = \text{Trace}(\phi_1 \circ (-\frac{1}{z} \cdot I)) = -\frac{1}{z} \cdot \text{Trace}(\phi_1).
\]  

(8.11)

Now using the identification \( \Omega_{R[r]}(x_i^+ + x_i^-) \equiv \mathcal{E}_{R[r]} \), we can identify \( \frac{1}{z} \cdot \text{Trace}(\phi_1) \) with \( \text{Trace}(\phi_1) \). Hence the \( i \)-th component of \( \mu_r(\mathcal{E}, \phi) \) is \( \text{Trace} \phi|_{\mathcal{E}_{R[r]}(1)^{a_i}} \).

\[\square\]

Remark 8.2. Let us denote by \( \mathcal{M}_{SPHB}^c \) the closed subscheme of \( \mathcal{M}_{PHB}^c \) (remark 7.5) consisting of parabolic Higgs bundles whose eigen-values of the Higgs field at the two points \( x^+ \) and \( x^- \) are all 0. It follows from proposition 8.1 that \( \mathcal{M}_{SPHB}^c \) is a symplectic leaf of the most singular locus \( \partial^n \mathcal{M}_{PHB}^n \).
Remark 8.3. When the nodal curve is reducible as in remark 5.3, the moduli space $\mathcal{M}_{GHB}$ is the union of two log-symplectic manifolds transversally intersecting along a smooth divisor ([3, §9, remark 9.3] and [4]). It follows from remark 7.5 that the divisor is isomorphic (as a Poisson scheme) to the moduli space of stable (with respect to parabolic weights determined by the polarisation on the moduli space [4, Lemma 3.4.2]) parabolic-Higgs bundles with the same eigenvalues (of the Higgs field) at the two pre-images of the node. It follows from proposition 8.1 that the moduli space of strongly-parabolic Higgs bundles is a symplectic leaf of the divisor.

9. Algebraically completely integrability

9.1. The Hitchin map and its general fibres. There is a Hitchin map on the moduli of Gieseker-Higgs bundles which is defined as follows.

$$h: \mathcal{M}_{GHB} \to B := \bigoplus_{i=1}^{n} H^0(X_0, \omega_X \otimes \pi^* \omega_{X_0})$$

(9.1)

given by

$$(X_r, \mathcal{E}, \phi: \mathcal{E} \to \mathcal{E} \otimes \pi^* \omega_X) \mapsto (\text{Trace} \phi, \ldots, (-1)^{i-1} \text{Trace} (\wedge^i \phi), \ldots, (-1)^{n-1} \text{Trace} (\wedge^n \phi))$$

Notice, using properties (5.0.1), it follows that

$$\bigoplus_{i=1}^{n} H^0(X_r, \omega_{X_r} \otimes i) \equiv \bigoplus_{i=1}^{n} H^0(X_0, \omega_{X_0} \otimes i).$$

(9.2)

It is shown in [3], that the Hitchin map $h$ is proper.

For a general element $\xi \in B$, [3, Section 7] constructs a spectral curve $X_\xi$, which is an irreducible vine curve, ramified outside the nodes, such that there is the following correspondence.

$$\begin{cases} \text{line bundles on the curve } X_\xi \text{ of degree } \delta \to \text{Gieseker-Higgs bundles on } X_0 \\ \text{of rank } n \text{ and degree } \delta - n(n-1)(g-1) \text{ with characteristic polynomial } \xi \end{cases}$$

(9.3)

Therefore the subvariety consisting of the objects on the right is isomorphic to the Picard $\text{Pic}^\delta_{X_\xi}$ of the vine curve $X_\xi$, which is a semiabelian variety. The full Hitchin-fiber $h^{-1}(\xi)$ is a compactification of this semi-abelian variety with normal crossing singularity. Moreover, the smooth locus of $h^{-1}(\xi)$ is precisely $\text{Pic}^\delta_{X_\xi}$. For the precise statements we refer to [3, Theorem 8.16((Quasi-abelianization))].

9.2. Completely integrability.
Definition 9.3. Let \( X \) be a variety with normal crossing singularity with a log-symplectic form i.e., a non-degenerate closed section of \( \wedge^2 \Omega_X(\log \mathcal{O}_X) \). An irreducible subvariety \( Y \subset X \) is co-isotropic (resp. Lagrangian) if it is generically a co-isotropic (resp. Lagrangian) subvariety of a symplectic leaf; i.e., \( Y \) is contained in the closure \( \overline{S} \) of a symplectic leaf \( S \subset X \) and the intersection \( Y \cap S \) is a co-isotropic (resp. Lagrangian) subvariety of \( S \).

Definition 9.4. A Poisson structure (may not be of uniform rank) on a variety \( X \) (possibly singular) is an algebraically completely Integrable system structure on \( h: X \to B \) if \( h \) is a Lagrangian fibration over the complement of some properly closed subvariety of \( B \).

From Theorem 7.4, it follows that \( \mathcal{M}_{GHB} \setminus \partial \mathcal{M}_{GHB} \) is the maximal symplectic leaf (dense open) of \( \mathcal{M}_{GHB} \). From the description of the fibre of the Hitchin map \( \mathcal{M}_{GHB} \setminus \partial \mathcal{M}_{GHB} \to B \) over a general point \( \xi \) is the Picard variety \( J_\xi \) of the vine curve \( X_\xi \). We will now show that the general fibre is a Lagrangian in \( \mathcal{M}_{GHB} \setminus \partial \mathcal{M}_{GHB} \) following the strategy of [29] and [26].

Lemma 9.5. Let \( f: X \to Y \) be a morphism between two projective nodal curves of degree \( d \) and unramified along the nodes. Then

1. we have a short exact sequence
   \[
   0 \to f^* \omega_Y \to \omega_X \to \omega_{X/Y} \to 0, \tag{9.4}
   \]
   where \( \omega_X \) and \( \omega_Y \) are the sheaf of differentials of the nodal curves with logarithmic poles along the nodes and \( \omega_{X/Y} \) denote the sheaf of relative logarithmic differentials.

2. \( \omega_{X/Y} \cong \mathcal{O}_R \), where \( R \) is the divisor \( R := \sum_{p \in X} \text{length} (\omega_{X/Y})_p \cdot p \).

3. \( 2g_X - 2 = d(2g_Y - 2) + \deg R \), where \( g_X \) and \( g_Y \) are the arithmetic genus of \( X \) and \( Y \), respectively.

Proof. Since \( f \) is unramified at the node by the functorial property of the sheaf of logarithmic differentials, we have the following inclusion of rank 1-locally free sheaves \( f^* \omega_Y \to \omega_X \). The locus, where the inclusion is not an isomorphism, is, by definition, the ramification divisor \( R \). Therefore \( \mathcal{O}_R \cong \omega_{X/Y} \). The statement (3) follows from 9.4. \( \square \)

The following lemma is a straightforward generalisation of [19, §5], [6, Remark 3.7] and [20, §4.3]. One can also find a proof in [11, Proposition 4.1].

Lemma 9.6. Let \( L \) be a line bundle on the vine curve \( X_\xi \) such that the push-forward of \( L \to L \otimes f^* \omega_{X_0} \) is a Gieseker-Higgs bundle \( \phi: \mathcal{E} \to \mathcal{E} \otimes \omega_{X_\xi} \), where the map \( L \to L \otimes f^* \omega_{X_0} \) is given by the multiplication by the canonical section of \( f^* \omega_{X_0} \) on \( X_\xi \). Suppose also that the spectral curve \( f: X_\xi \to X_0 \) is ramified along a divisor \( R \subset X_\xi \), which does not map to any of the nodes. Then we have the following exact sequence over \( X_\xi \)

\[
0 \to L(-R) \to f^* \mathcal{E} \to f^* \mathcal{E} \otimes \pi_\xi^* \omega_{X_0} \to L \otimes f^* \omega_{X_0} \to 0. \tag{9.5}
\]
Proposition 9.7. Let \( p := (X_0, \mathcal{E}, \phi) \) be a Gieseker-Higgs bundle in \( h^{-1}(\xi) \). Then we have the following short exact sequence

\[
0 \to H^1(f_* \mathcal{O}_{X_\xi}) \to H^1(\mathcal{E}_*) \to H^0(f_* \omega_{X_\xi}) \to 0 \tag{9.6}
\]

Moreover, \( T_{J_\xi, p} \cong H^1(f_* \mathcal{O}_{X_\xi}) \) and \( \mathcal{N}_{J_\xi} = \pi M_G H B \cong H^0(f_* \omega_{X_\xi}) \).

Proof. Tensoring the sequence (9.5) with \( L^{-1} \otimes \mathcal{O}(R) \) we get

\[
0 \to \mathcal{O}_{X_\xi} \to \pi^* \mathcal{E} \otimes L^{-1} \otimes \mathcal{O}(R) \to \pi^* \mathcal{E} \otimes \pi^* \omega_{X_0} \otimes L^{-1} \otimes \mathcal{O}(R) \to \pi^* \omega_{X_\xi} \otimes \mathcal{O}(R) \to 0 \tag{9.7}
\]

Notice that the morphism \( \pi : X_\xi \to X_0 \) is a finite cover. Using Riemann-Hurwitz formula, we see that the push forward (under \( \pi \)) of the above exact sequence is the same as

\[
0 \to \pi_* \mathcal{O}_{X_\xi} \to \mathcal{E} nd E \xrightarrow{[\cdot, \phi]} \mathcal{E} nd E \otimes \omega_{X_0} \to \pi_* \omega_{X_\xi} \to 0 \tag{9.8}
\]

Using this exact sequence, we can form the following short exact sequence of chains:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \pi_* \mathcal{O}_{X_\xi} \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{E} nd E \\
\downarrow & & \downarrow \\
0 & \to & Im ([\cdot, \phi] \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\tag{9.9}
\]

The last chain is quasi-isomorphic to the chain

\[
0 \to 0 \to \pi_* \omega_{X_\xi} \to 0 \tag{9.10}
\]

This follows from the following short exact sequence of complexes:

\[
\begin{array}{ccc}
0 & \to & Im ([\cdot, \phi]) \\
\downarrow & = & \downarrow \\
0 & \to & \mathcal{E} nd E \otimes \omega_{X_0} \\
\downarrow & & \downarrow \\
0 & \to & \pi_* \omega_{X_\xi} \\
\end{array}
\tag{9.11}
\]

Since the first vertical complex from the left is quasi-isomorphic to 0, the vertical complex in the middle and the first vertical complex from the right are quasi-isomorphic.

Now from 9.9 and using the long exact sequence of hypercohomology, we get equation 9.6. The tangent space \( T_{J_\xi, p} \cong H^1(X_\xi, \mathcal{O}_{X_\xi}) \cong H^1(X_0, \pi_* \mathcal{O}_{X_\xi}). \) Therefore, \( \mathcal{N}_{J_\xi} = \pi M_G H B \cong H^0(\pi_* \omega_{X_\xi}). \)

\[\square\]
Theorem 9.8. The generic fiber $h^{-1}(\xi)$ is Lagrangian in a symplectic leaf for the log-symplectic structure on $\mathcal{M}_{GHB}$. Therefore the Hitchin map $h : \mathcal{M}_{GHB} \to B := \oplus_{i=1}^n H^0(\mathcal{X}_i, \omega^{\xi_i}_{\mathcal{X}_i})$ is an algebraically completely integrable system (9.4).

Proof. By definition 9.3, it is enough to show that the isomorphism $\sigma^\ast : \mathbb{H}^1(\mathcal{E}^\vee) \to \mathbb{H}^1(\mathcal{E})$ maps $\mathcal{N}^\vee_{\mathcal{X}_i} \rightarrow \mathcal{M}_{GHB}$ to $T_p \mathcal{J}_i$. To see this, consider the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1(\pi_\ast \mathcal{O}_{\mathcal{X}_i}) & \longrightarrow & \mathbb{H}^1(\mathcal{E}) & \longrightarrow & H^0(\pi_\ast \omega_{\mathcal{X}_i}) & \longrightarrow & 0 \\
\sigma & \downarrow & \uparrow \sigma & & & & & & \\
0 & \longrightarrow & H^0(\pi_\ast \omega_{\mathcal{X}_i})^\ast & \longrightarrow & \mathbb{H}^1(\mathcal{E}^\vee) & \longrightarrow & H^1(\pi_\ast \mathcal{O}_{\mathcal{X}_i})^\ast & \longrightarrow & 0
\end{array}
$$

(9.12)

Since $\mathcal{N}^\vee_{\mathcal{X}_i} \rightarrow \mathcal{M}_{GHB}$ and $T_p \mathcal{J}_i \cong H^1(\pi_\ast \mathcal{O}_{\mathcal{X}_i})$ it is enough to show that $\sigma^\ast (H^0(\pi_\ast \omega_{\mathcal{X}_i})^\ast) \subseteq H^1(\pi_\ast \mathcal{O}_{\mathcal{X}_i})^\ast$.

Since the horizontal short exact sequences are exact, it is enough to show that the composite map

$$
H^0(\pi_\ast \omega_{\mathcal{X}_i})^\ast \to H^0(\pi_\ast \omega_{\mathcal{X}_i})
$$

(9.13)

is 0. Notice that $H^0(\pi_\ast \omega_{\mathcal{X}_i})^\ast \cong H^1(\pi_\ast \mathcal{O}_{\mathcal{X}_i})$ and also recall from the proof of proposition 9.7 that the complex

$$
\pi_\ast \mathcal{O}_{\mathcal{X}_i} \to 0
$$

(9.14)

is quasi-isomorphic to the complex (9.10). Now it is clear that the morphism (9.13) is the same as the composite of the morphisms induced on the $\mathbb{H}^1$’s of the following morphism of complexes.

$$
\begin{array}{cccccc}
\pi_\ast \mathcal{O}_{\mathcal{X}_i} & \longrightarrow & \mathcal{E} ndE & \longrightarrow & \mathcal{E} ndE & \longrightarrow & \text{Im}([\ast, \phi]) \\
\downarrow & & \downarrow [\ast, \phi] & & \downarrow [\ast, \phi] & & \\
0 & \longrightarrow & \mathcal{E} ndE \otimes \omega_{\mathcal{X}_0} & \longrightarrow & \mathcal{E} ndE \otimes \omega_{\mathcal{X}_0} & \longrightarrow & \mathcal{E} ndE \otimes \omega_{\mathcal{X}_0}
\end{array}
$$

(9.15)

Since the composition is 0, therefore the morphism $H^0(\pi_\ast \omega_{\mathcal{X}_i})^\ast \to H^0(\pi_\ast \omega_{\mathcal{X}_i})$ is also 0. □

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