Tree decomposition and postoptimality analysis in discrete optimization

Oleg Shcherbina
Faculty of Mathematics,
University of Vienna
Nordbergstrasse 15, A-1090 Vienna,
Austria
oleg.shcherbina@univie.ac.at

Summary. Many real discrete optimization problems (DOPs) are NP-hard and contain a huge number of variables and/or constraints that make the models intractable for currently available solvers. Large DOPs can be solved due to their special structure using decomposition approaches. An important example of decomposition approaches is tree decomposition with local decomposition algorithms using the special block matrix structure of constraints which can exploit sparsity in the interaction graph of a discrete optimization problem. In this paper, discrete optimization problems with a tree structural graph are solved by local decomposition algorithms. Local decomposition algorithms generate a family of related DO problems which have the same structure but differ in the right-hand sides. Due to this fact, postoptimality techniques in DO are applied.

1 Introduction

Discrete optimization (DO) problems arise in various application areas such as planning, economical allocation, logistics, scheduling, computer aided design, and robotics. Application areas of discrete optimization models of OR also include supply chain design and management, network optimization, telecommunications, VLSI routing, manufacturing, transportation, scheduling, and finance. The tremendous attention that DO particularly has received in the literature gives some indication of its importance in many research areas. Unfortunately, most of the interesting problems are in the complexity class NP-hard and may require searching a tree of exponential size (if $P \neq NP$) in the worst case. Many real DO problems contain a huge number of variables and/or constraints that make the models intractable for currently available DO solvers.

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One of the promising approaches to cope with NP-hardness in solving DO problems is the construction of decomposition methods \[34\]. Decomposition techniques usually determine subproblems the solutions of which can be combined to create a solution of the initial DO problem. Usually, DO problems from applications have a special structure, and the matrices of constraints for large-scale problems have a lot of zero elements (sparse matrices). The nonzero elements of the matrix often fall into a limited number of blocks. The block form of many DO problems is usually caused by the weak connectedness of subsystems of real-world systems.

The search for graph structures appropriate for the application of dynamic programming caused a series of papers dedicated to tree decomposition research (\[35\], \[3\], \[6\], \[28\], \[14\], \[7\], \[9\], \[22\]). Tree decomposition methods aim to merge variables such that the meta-graph is a tree of meta-nodes. Tree decomposition and the related notion of a \textit{treewidth} (Robertson, Seymour \[35\]) play a very important role in algorithms, for many NP-complete problems on graphs that are otherwise intractable become polynomial time solvable when these graphs have a tree decomposition with restricted maximal size of cliques (or have a bounded treewidth \[7\], \[9\]).

Most of the works based on tree decomposition approach only present theoretical results \[27\], see the recent survey by Hicks et al. \[24\]. But only few papers on applications of this powerful tool in the area of DO exist \[29\], \[24\].

The algorithmic importance of the tree decomposition was caused by results of Courcelle \[12\] and Arnborg et al. \[2\] which showed that several NP-hard problems posed in monadic second-order logic can be solved in polynomial time using dynamic programming techniques on input graphs with bounded treewidth.

Tree decomposition based algorithms demonstrated their efficiency on solving frequency assignment problem Koster et al. \[29\], ring-routing problems \[10\], and traveling salesman problem \[11\].

Efficiency of tree decomposition based algorithms crucially depends on interaction graph structure of the DO problem, so that it has a time complexity \(O(n \cdot 2^{tw+1})\), where \(tw\) is the treewidth of the graph. If the interaction graph is rather sparse or, in other words, it has a relatively small treewidth, then complexity of the tree decomposition algorithm is reasonable.

Necessity of reduction of enumeration while solving problems corresponding to meta-nodes of the tree decomposition causes expediency and an urgency of development of tools that could help to cope with this difficulty.

In this paper, discrete optimization problems with a tree structural graph are solved by local decomposition algorithms that belong to dynamic programming paradigm. Local decomposition algorithm generates a family of related DO problems which have the same structure but differ in the right-hand sides. Due to this fact, postoptimality techniques in DO are applied.
2 Discrete optimization problems with constraints and their graph representations

Consider a DOP with constraints:

\[
\max_X f(X) = \max_X \sum_{k \in K} f_k(X^k),
\]
subject to

\[
A_{iS_i} X_{S_i} \leq b_i, \quad i \in M = \{1, 2, \ldots, m\},
\]
\[
x_j = 0, 1, \quad j \in N = \{1, \ldots, n\},
\]

where

\[X = \{x_1, \ldots, x_n\}\] is a set of discrete variables, functions \(f_i(X^i)\) are called components of the objective function and can be defined in tabular form, \(X^k \subseteq X, \quad k \in K = \{1, 2, \ldots, t\}\), \(t\) is the number of components of objective function, \(K\) is a set of indices of components;

\[S_i \subseteq \{1, 2, \ldots, n\}, \quad i \in M.\]

We shall consider further a linear objective function (5):

\[
F(x_1, \ldots, x_n) = F(X) = C_N X_N = \sum_{j=1}^{n} c_j x_j \rightarrow \max
\]

**Definition 1.** [5]. Variables \(x \in X\) and \(y \in Y\) interact in DOP with constraints if they both appear either in the same component of objective function, or in the same constraint (in other words, if variables are both either in a set \(X^k\), or in a set \(X_{S_i}\)).

Graph representation of a DOP structure may be done with various detailization. Structural graph of a DOP defines which variables are in which constraints.

An interaction graph [5] represents a structure of the DOP in a natural way.

**Definition 2.** [5]. The interaction graph of a DOP is an undirected graph \(G = (X, E)\), such that

1. Vertices \(X\) of \(G\) correspond to variables of DOP;
2. Two vertices of \(G\) are adjacent iff corresponding variables interact.

Further, we shall use the notion of vertices that correspond one-to-one to variables.

**Definition 3.** The set of variables interacting with a variable \(x \in X\), is denoted by \(Nb(x)\) and called neighborhood of the variable \(x\). For corresponding vertices a neighborhood of a vertex \(v\) is a set of vertices of interaction graph that are linked by edges with \(v\). Denote the latter neighborhood as \(Nb_G(v)\).
Let $S$ be a vertex set of the graph. Introduce the following notions:

1. Neighborhood of a set $S \subseteq V$, $Nb(S) = \bigcup_{v \in S} Nb(v) - S$.
2. Closed neighborhood of a set $S \subseteq V$, $Nb[S] = Nb(S) \cup S$.
3. If $S = \{j_1, \ldots, j_q\}$ then $X_S = \{x_{j_1}, \ldots, x_{j_q}\}$.

**Example 1.**

\[
\begin{align*}
2x_1 + 3x_2 + x_3 + 5x_4 + 4x_5 + 6x_6 + x_7 & \rightarrow \text{max} \\
3x_1 + 4x_2 + x_3 & \leq 6, \quad (C_1) \\
2x_2 + 3x_3 + 3x_4 & \leq 5, \quad (C_2) \\
2x_2 + 3x_5 & \leq 4, \quad (C_3) \\
2x_3 + 3x_6 + 2x_7 & \leq 5, \quad (C_4) \\
x_j = 0, 1, \ j = 1, \ldots, 7.
\end{align*}
\]

![Interaction graph for example.](image)

We need following notions. A **clique** is a set of vertices that induce a complete subgraph of $G$, and a **maximal clique** is a clique which is not a subset of any other clique. A clique $C$ in graph $G$ is **maximal**, if $C$ is not a subset of any other clique in $G$. A **spanning tree** of a graph is a tree that includes every vertex in the graph. The **clique graph** of $G$, is the intersection graph of the family of maximal cliques of $G$. 
Dynamic programming is a very powerful algorithmic framework in which an optimization problem is solved by identifying a set of subproblems and solving them one-by-one, smallest first, storing solutions and using the stored solutions to small problems to find solutions of larger ones (recursively defining value of optimal solution), until the whole set of them are solved. Dynamic programming computes recurrences efficiently by storing partial results in tables.

A tree is a recursive data structure because each child of a node in the tree has a tree of its descendants. Due to this fact, many of the important algorithms to access and manipulate trees are easily expressed using recursion and particularly dynamic programming which computes recurrences efficiently by storing partial results in tables and hence can be effectively used for solving optimization problems on trees. Dynamic programming works best on objects which are linearly ordered, e.g. the left-to-right order of leaves in a tree. Dynamic programming starts at the leaves of the tree and proceeds from smaller to larger subproblems (corresponding to subtrees) that is to say, bottom-up in the rooted tree.

One reason why many optimization problems that are hard on general graphs are easy on trees is that trees do not contain cycles.

Above mentioned facts and an observation that many optimization problems which are hard to solve on general graphs are easy on trees makes the detection tree structures in a graph a very promising task. A powerful tool of algorithmic graph theory like tree decomposition \cite{Robertson} can help detect trees and obtain the treewidth, a measure of the "tree-likeness" of the graph.

Using dynamic programming techniques on tree decompositions of bounded treewidth, hard optimization graph problems can often be solved in polynomial time \cite{Robertson}.

The notions of treewidth and tree decomposition were introduced by Robertson & Seymour in their seminal paper \cite{Robertson} on graph minors. The best known complexity bounds are given by the treewidth \( tw \) (Robertson, Seymour \cite{Robertson}) of an interaction graph associated with a DOP. This parameter is related to some topological properties of the interaction graph. It leads to a time complexity of \( O(n \cdot 2^{tw+1}) \). Tree decomposition based methods aim to merge variables so that the resulting meta-graph is a tree of meta-nodes. A more detailed introduction to tree decompositions is given in \cite{Robertson} and in surveys \cite{Robertson}, \cite{Robertson}, \cite{Robertson}. Most of the works based on tree decomposition approach only present theoretical results \cite{Robertson}.

**Definition 4.** Let \( G = (V,E) \) be a graph. A **tree decomposition** of \( G \) is a pair \((T;X)\) with \( T = (I;F) \) a tree and \( X = \{X_I \mid I \in I\} \) a family of subsets of \( V \), one for each node of \( T \), such that

\[ i \cup_{i \in I} X_i = V, \]
(ii) for every edge \((u, v) \in V\) there is an \(i \in I\) with \(u \in X_i\), \(v \in X_i\),

(iii) (running intersection property) for all \(i, j, l \in I\), if \(i < j < l\), then \(X_i \cap X_l \subseteq X_j\).

**Remark 1.** To construct a tree-decomposition we merge the vertices of \(G\) together to form meta-nodes ("supernodes" [1] or "bags" [42]); each meta-node is a subset of the vertices of \(G\) and we connect these meta-nodes to form a rooted tree \(T\). The meta-nodes do not have to be disjoint, i.e., there might be nodes of the graph contained in more than one meta-node. This grouping and connecting has to be done in such a way that – for all edges \(e\) of the graph: there exists a meta-node containing both endpoints of edge \(e\), and – for all vertices \(v\) of the graph: all meta-nodes containing \(v\) together with the edges between those meta-nodes in \(T\) form a connected subtree of \(T\).

The **width** of a tree-decomposition is the number of graph-vertices of the largest meta-node minus 1.

As finding an optimal tree-decomposition is \(NP\)-hard, approximate optimal tree decompositions using triangulation of a given graph are often exploited. Given a triangulated (or chordal) graph, the set of its maximal cliques corresponds to the family of subsets associated with a tree-decomposition (so called **clique tree** [6]). When a tree-decomposition is exploited, usually one considers approximations of optimal triangulations by clique trees [26]. Hence, the time complexity is then \(O(n \cdot 2^{w^++1})\) with \(w^+ + 1\) the size of the largest cluster \((w + 1 \leq w^+ + 1 \leq n)\). The space complexity is \(O(n \cdot s \cdot 2^s)\) with \(s\) the size of the largest minimal separator [27].

**Definition 5.** A **clique tree** of \(G\) is a tree \(T = (K, E)\) whose vertex set is the set of maximal cliques of \(G\) such that each of the induced subgraphs \(T[K_v]\) is connected.

**Definition 6.** A graph is **chordal** (triangulated, perfect elimination graph, rigid circuit [15], monotone transitive [36]) if every cycle of length \(> 3\) has a chord (i.e., an edge joining two nonconsecutive vertices of a cycle).

All induced subgraphs of a chordal graph are also chordal. If \(G\) is a chordal graph, then any clique tree of \(G\) is also a tree decomposition of \(G\). However, the converse is not necessarily true.

**Theorem 1.** [17] Let \(G = (V, E)\) be an undirected graph, and let \(K\) be the set of maximal cliques of \(G\), with \(K_v\) the set of all maximal cliques that contain vertex \(v\) of \(G\). The following statements are equivalent:

(i) \(G\) is chordal.

(ii) \(G\) is the intersection graph of a family of subtrees of a tree.

(iii) There exists a tree \(T = (K, E)\) whose vertex set is the set of maximal cliques of \(G\) such that each of the induced subgraphs \(T[K_v]\) is connected.
Clique graphs are not very useful on general graphs, since these can contain \( n!/(k!(n-k)!) \) different cliques of size \( k \). It follows from this that clique graphs of general graphs can be exponentially large, and it is no surprise that finding the maximal clique for general graphs is hard. Chordal graphs on the other hand have limitations that make clique graphs useful.

**Lemma 1.** (Dirac [13]) A chordal graph \( G \) contains at most \( n \) maximal cliques.

Gavril [17] proved that every chordal graph can be represented by a clique tree limiting the number of edges to \( O(n) \). Indeed, from theorem 1 it follows that for the chordal graph \( G \) there exists a tree \( T = (K, E) \) whose vertex set is the set of maximal cliques of \( G \) such that each of the induced subgraphs \( T[K_v] \) is connected. From Lemma 4 and the fact that \( T \) is a tree, follows that a clique tree has at most \( n \) nodes and \( n - 1 \) edges.

**Theorem 2.** (Bernstein and Goodman [4]) Any maximum weight spanning tree of the clique graph of a chordal graph \( G \) is a clique tree of \( G \).

**Theorem 3.** (Ho and Lee [25]) Given a chordal graph \( G \) and a clique tree \( T \) of \( G \), a set of vertices \( S \) is a minimal separator of \( G \) iff \( S = C_i \cap C_j \) for an edge \((C_i, C_j)\) in \( T \).

**Corollary 1.** (Ho and Lee [25]) A chordal graph \( G \) has at most \( n - 1 \) minimal separators.

These results show how to build tree decompositions using elimination game algorithm which triangulates an initial interaction graph. For triangulated graphs it is rather simply to find maximal cliques and to build the clique tree.

### 2.2 Elimination game and tree decomposition

The process of interaction graph transformation known as *Elimination Game* was first introduced by Parter [33] as a graph analogy of Gaussian elimination. The input of the elimination game is a graph \( G \) and an ordering \( \alpha \) of \( G \) (i.e. \( \alpha(v) = i \) if \( v \) is \( i \)-th vertex in ordering \( \alpha \)). Elimination Game according to [23] consists in the following. At each step \( i \), the neighborhood of vertex \( x_i \) is turned into a clique, and \( x_i \) is deleted from the graph. This is referred to as eliminating vertex \( x_i \). The filled graph \( G^+_{\alpha} = (V, E^+) \) is obtained by adding to \( G \) all the edges added by the algorithm. This resulting graph \( G^+_{\alpha} \) is a triangulation of \( G \) (Fulkerson & Gross [16]), i.e., a chordal graph.

Different filled graphs result from processing the vertices of \( G \) in different orders. Thus in order to find a low fill-in, it is necessary to find a good order on the vertices of the given graph before running elimination game. Finding an ordering that results in the minimum fill-in is an \( NP \)-complete problem [43].

Elimination Game can also be implemented so that \( \alpha \) is generated during the
course of the algorithm. In this case, we can at each step $i$ choose a vertex $v$ of the elimination graph $G^{i-1}$ according to any desired criteria, and set $\alpha(v) = i$, to define an elimination ordering $\alpha$. One well known heuristic called Minimum Degree chooses a vertex $v$ of minimum degree in $G^{i-1}$ at each step $i$.

![Diagram](image)

**Fig. 2.** Elimination Game.

The procedure to solve an optimization problem with bounded treewidth involves two steps:

(i) computation of a (good) tree decomposition, and
(ii) application of an (dynamic programming) algorithm that solves instances of bounded treewidth in polynomial time.

To describe how tree-decompositions are used to solve problems with dynamic programming, let us assume we find a tree-decomposition of a graph $G$. Since this tree-decomposition is represented as a rooted tree $T$, the ancestor/descendant relation is well-defined. We can associate to each meta-node $X$ the subgraph of $G$ made up by the vertices in $X$ and all its descendant meta-nodes, and all the edges between those vertices. Starting at the leaves of the tree $T$, we
can compute information typically stored in a table, in a bottom-up manner for each bag until we reach the root. This information is sufficient to solve the subproblem for the corresponding subgraph. To compute the table for a meta-node of the tree-decomposition, we only need the information stored in the tables of the children (i.e. direct descendants) of this node. The problem for the entire graph can then be solved with the information stored in the table of the root of $T$.

3 Local decomposition algorithm for discrete optimization

During the study of complex objects it is not always possible (and expedient) to obtain (or to calculate) complete information about the object as a whole; therefore is of interest to obtain information about the object, examining it in parts, i.e., locally.

Yu.I. Zhuravlev [44] introduced and investigated local algorithms for calculating the information about properties of objects. The local algorithm can be described as follows. For a given set of sets $\{m\}$, $m = \{U_i\}$, $i = 1,2,\ldots,|m|$, and for each element $U \in m$ let us determine a neighborhood $S(U,m)$ in $m$; these neighborhoods should satisfy the following conditions:

- $U \in S(U,m)$;
- $S(U,m) \subseteq m$;
- If $U \in m_1$, $U \in m_2$, $S(U,m) \subseteq m_2 \subseteq m_1$ then $S(U,m_1) = S(U,m_2)$.

Algorithm $A$ is completely determined by the set of predicates $P_1,\ldots,P_l$, by the partition of this set into a subset of basic predicates $P_1,\ldots,P_r$, and auxiliary predicates $P_{r+1},\ldots,P_l$, by the set of monotonic functions $\varphi_1,\ldots,\varphi_l$, $\varphi_i = \varphi_i(U,\alpha_1,\ldots,\alpha_l,S,m^*)$ and by the ordering algorithm $A_\pi$.

LA uses a dynamic programming paradigm and computes optimal partial solutions of subproblems that correspond to the blocks of the DOP.

Each step of the LA $A$ [38] consists of changing neighborhoods and replacing index $p$ with $p + 1$ (although it is possible to pass, also, from $S_p$ to $S_{p+\rho}$); for each fixed assignment of the variables of boundary ring the values of the variables of the corresponding neighborhood are stored, in this consists one of the important differences between LA $A$ and LA $A$: an information not only about the predicates, but also about the values of variables is memorized; this information is called as an indicator information.

Let us consider the LA $A_{BT}$ for the solution BT ILP problems [5], [2], [3], where the matrix $A$ has BT structure with the tree $D$ which contains of $k$ blocks.

Consider a vertex $r$ of the tree $D$ and introduce a tree $D_r$ which consists of the vertex $r$ and all its descendants.

Introduce the necessary notations:
• $S_r$ is a set of indices of variables which belong to block $B_r$;
• $S_{rr'}$ is a set of indices of variables which belong simultaneously to blocks $B_r$ and $B_{r'}$;
• if $S = \{j_1, \ldots, j_q\}$, then $X_S = \{x_{j_1}, \ldots, x_{j_q}\}$;
• $p_r$ is a vertex-ancestor for the node $r$;
• $J_r$ is a set of descendants of the node $r$.

Consider a LA [38] for solving DO problems with a tree structural graph, i.e., problems in which it is possible to find the set of the neighborhoods of different variables so that one variable can belong to two neighborhoods only and the graph of intersections of these neighborhoods is a tree. The LA solves this DO problem, moving bottom-up, i.e., from the neighborhoods corresponding to leaves of the tree, to the neighborhood corresponding to the root of the tree $T$. Let $B_1 = (\bar{S}_1, U_1), B_2 = (\bar{S}_2, U_2), \ldots, B_k = (\bar{S}_k, U_k)$ be a set of the blocks (neighborhoods) of some indices $j_1, \ldots, j_k$ of some variables, where $S_r, U_r$ are, respectively, the sets of the indices of variables and constraints for the $r$th neighborhood, $r = 1, \ldots, k$ and

$$\bigcup_{r=1}^{k} U_r = M = \{1, \ldots, m\},$$  

$$\bigcup_{r=1}^{k} S_r = N = \{1, \ldots, n\},$$

$$U_{r_1} \cap U_{r_2} = \emptyset, r_1 \neq r_2,$$

$$S_{r_1} \cap S_{r_2} \cap S_{r_3} = \emptyset$$ for any triple of different indices $r_1, r_2, r_3$.  

Consider a node $r$ of the tree $T$ and define a tree $T_r$ containing the node $r$ and its children.

Introduce notions:
• $\bar{S}_r$ is a set of indices of variables that are in the block $B_r$;
• $S_{rr'}$ is a set of indices of variables that are in the blocks $B_r$ and $B_{r'}$, i.e., $S_{rr'} = \bar{S}_r \cap \bar{S}_{r'}$;
• $p_r$ is a node-parent of the node $r$;
• $J_r$ is a set of children of the node $r$.

Then $X_{S_{rr'}}$ is a meta-variable consisting of variables common for blocks $B_{p_r}$ and $B_r$ (here $S_{p_r, r} = S_{p_r} \cap \bar{S}_r$).

Denote as $Z_{T_r}$ the following problem: for each assignment $X_{S_{rr'}}$ to find $X_{S_r}$ $X_{S_{rr'}}$, such that

$$h_{B_r}(X_{S_{rr'}}) = f_{D_r}(X_{S_{rr'}}) = \max_{X_{S_r}, X_{S_{rr'}}} \left\{ C_S X_{S_r} + \sum_{r' \in J_r} [f_{D_{r'}}(X_{S_{rr'}}) + C_{S_{rr'}} X_{S_{rr'}}] \right\} =$$
= \max_{X_{Sr}, X_{Sr'}} \left\{ C_{Sr} X_{Sr} + \sum_{r' \in J_r} \left[ h_{B_{r'}} (X_{S_r \cap S_{r'}}) + C_{Sr'} X_{Sr'} \right] \right\}

subject to

\begin{align*}
A_{Sr} X_{Sr} &\leq b_r - \sum_{r' \in J_r} A_{Sr' r} X_{Sr' r} - A_{Sp r} X_{Sp r}.
\end{align*}

Here \( f_T(X_{Sp r}) \) is an objective value of subproblem corresponding to the tree \( T_r \), \( f_{T'} (X_{Sr'}) = C_{T r} X_{T r} \). It is possible to assign this value to the root \( B_{r'} \) of the tree \( T_r \) and write: \( h_{B_{r'}} (X_{S_r \cap S_{r'}}) = f_{T'} (X_{Sr'}) \).

It is easy to see that if we fix a vector \( X_{Sp r} \), then the problem is decomposed into two problems: the first one corresponds to the tree \( T_r \); and the second one to \( T - T_r \). An application of LA \( \mathcal{A}_{BT} \) for solving DP problems with a tree structural graph is based on this property.

4 Solving of a concrete DOP with the finding of tree like structure and applying of local decomposition algorithm

Consider the DOP of example 1.

**Finding of the tree structural graph (tree decomposition)**

In Fig. 2 results of elimination game algorithm are shown. Since during elimination process new fill-in edges are not added, the elimination game process is equivalent to searching simplicial vertices and corresponding maximal cliques. In Fig. 3 these maximal cliques and links between them are shown. Local decomposition algorithm can be applied to this clique tree. Other possible way of finding of the clique tree is using of maximal spanning tree in the dual graph.

**Applying the local decomposition algorithm to DO problem**

Let us solve the subproblem corresponding to the block \( B_1 \). Since this block is adjacent to the block \( B_4 \), we have to solve DOP with variables \( X_{B_1 \cap B_4} \) for all possible assignments \( X_{B_1 \cap B_4} \). Thus, since \( X_{B_1 \cap B_4} = \{ x_3 \} \) and \( X_{B_1 \cap B_4} = \{ x_2 \} \), then induced subproblem has a form:

\( h_{B_1} (x_2) = \max_{x_3} \{ 4x_3 \} \)

subject to

\( 2x_2 + 3x_3 \leq 4, \quad x_j = 0, 1, \quad j \in \{ 2, 5 \} \)

Solution of the problem can be written in tabular form:
Fig. 3. Tree decomposition for the example 1.

Table 1. Calculation of $h_{B_1}(x_2)$

| $x_2$ | $h_{B_1}$ | $x^*_2(x_2)$ |
|-------|-----------|---------------|
| 0     | 4         | 1             |
| 1     | 0         | 0             |

Next subproblem corresponding to a leaf (or meta-node) $B_2$ of the clique tree is
\[ h_{B_2}(x_3) = \max_{x_6, x_7} \{6x_6 + x_7\} \]

subject to
\[ 2x_3 + 3x_6 + 2x_7 \leq 5, \quad x_j = 0, 1, \quad j \in \{3, 6, 7\} \]

Solution of this subproblem:

Table 2. Calculation of \( h_{B_2}(x_3) \)

| \( x_3 \) | \( x_6^*(x_3) \) | \( x_7^*(x_3) \) |
|---|---|---|
| 1 | 1 | 7 |
| 1 | 0 | 6 |

Subproblem corresponding to the block \( B_3 \) has the form:
\[ h_{B_3}(x_2, x_3) = \max_{x_4} \{ h_{B_2}(x_3) + 5x_4 \} \]

subject to
\[ 2x_2 + 3x_3 + 3x_4 \leq 5, \quad x_j = 0, 1, \quad j \in \{2, 3, 4\} \]

Table 3. Calculation of \( h_{B_3}(x_2, x_3) \)

| \( x_2 \) | \( x_3 \) | \( h_{B_3} \) | \( x_4^*(x_2, x_3) \) |
|---|---|---|---|
| 0 | 0 | 12 | 1 |
| 0 | 1 | 6 | 0 |
| 1 | 0 | 12 | 1 |
| 1 | 1 | 6 | 0 |

The last problem left to be solved is:
\[ h_{B_4} = \max_{x_1, x_2, x_3} \{ h_{B_1}(x_2) + h_{B_3}(x_2, x_3) + 2x_1 \} \]

subject to
\[ 3x_1 + 4x_2 + x_3 \leq 6, \quad x_j = 0, 1, \quad j \in \{1, 2, 3\} \]

Table 4. Calculation of \( h_{B_4} \)

| \( h_{B_4} \) | \( x_1^* \) | \( x_2^* \) | \( x_3^* \) |
|---|---|---|---|
| 18 | 1 | 0 | 0 |

The maximal objective value is 18. To find the optimal values of the variables, it is necessary to do backward step of the dynamic programming procedure: from table 4 we have \( x_1^* = 1 \), \( x_2^* = 0 \), \( x_3^* = 0 \). From table 3 using the information \( x_2^* = 0 \), \( x_3^* = 0 \) we find \( x_4^* = 1 \). Considering table 2 we have for \( x_3^* = 0 \): \( x_6^* = 1 \), \( x_7^* = 1 \). From table 1 we find for \( x_5^* = 0 \): \( x_5^* = 1 \). The solution is \((1, 0, 0, 1, 1, 1, 1)\); maximal objective value is 18.
5 Postoptimal analysis and local algorithms

5.1 Postoptimal analysis in DO

Decomposition and sensitivity analysis in DO are closely related. Sensitivity analysis follows naturally from the duality theory. Decomposition methods consist of generating and solving families of related DO problems that have the same structure but differ as the values of coefficients. Sensitivity analysis allows using information obtained during solving one DO problem of the family of related DO problems in solving other problems of this family. Due to the lack of full-fledged duality theory in DO, sensitivity analysis for DO problems is not sufficiently developed [19], [31]. A number of useful tools of sensitivity analysis in DO are derived for integer programming in [19]. A technique of sensitivity analysis proposed in [37] computes a piecewise linear value function that provides a lower bound on the optimal value that results from changing the right-hand sides of constraints.

Recently, an interesting application of binary decision diagrams (BDD) (introduced earlier in computer science community) was proposed by HADZIC & HOOKER [21] for the purposes of postoptimal analysis in DO.

5.2 Postoptimal analysis in LA

LA systematically proceeds with so called parametric DO problems [5]:

Definition 7. A parametric DO problem is

$$\min_{X \in \mathcal{C}} \{ f(X) | X \in C_i, \ i = 1, \ldots, m; \ x_j \in \{0, 1\}, \ j = 1, \ldots, n \},$$

where $C_i$ is a set of feasible solutions of the constraint $i, \ i = 1, \ldots, m$.

Thus, an optimization problem is in parametric form when the objective function is optimized not over the entire set $X$, but only over a subset $X - P$, for all possible assignments of the variables of $P$.

Below we show that these parametric DOPs generated by LA lead to the possibility of exploiting postoptimality and sensitivity tools in the LA procedure. Consider DOP [3], (2), (3) with tree structural graph and use LA for its solving. Then for the block $B_r$ we have to solve a family of DOPs:

$$C_{S_r}X_{S_r} + [C_{S_{r-1},r}X_{S_{r-1},r} + C_{\sigma_{r-1}}X_{\sigma_{r-1}}] \rightarrow \max$$

s.t.

$$\sum_{j \in S_r} a_{ij}x_j \leq b_i - A^k_{S_{r-1},r}X_{S_{r-1},r} - A^k_{S_{r,r+1}}X_{S_{r,r+1}}, \ i \in U_r.$$

These DOPs should be solved for all binary assignments $X_{S_{r-1},r}$ and $X_{S_{r,r+1}}$. Denote

$$b_i (X_{S_{r-1},r} | X_{S_{r,r+1}}) = b_i - A^k_{S_{r-1},r}X_{S_{r-1},r} - A^k_{S_{r,r+1}}X_{S_{r,r+1}}.$$
It is clear, that it is better for each $X_{S_{r-1},r}$ and $X_{S_{r},r+1}$ to solve a problem:

$$C_{S_{r}}X_{S_{r}} \rightarrow \max$$

s.t.

$$\sum_{j \in S_{r}} a_{ij}x_{j} \leq b_{i} \left( X_{S_{r-1},r} | X_{S_{r},r+1} \right), \ i \in U_{r},$$

$$x_{j} = 0, 1, \ j \in S_{r}.$$  \hspace{1cm} (12)

It is possible to use information obtained during the solution of some DO problems of the family (10–12) for solving other problems of this family using postoptimality analysis (PA) [19], [20], [21], [37]. The more efficient the PA procedure is, the better LA will work.

### 5.3 Postoptimal analysis for an implicit enumeration algorithm

We show how to use PA in LA using as an illustrative example, the simplest case of an implicit enumeration algorithm [18] that generates partial solutions and tries to fathom them using 3 tests.

The DO problem (10)–(12) with vector right hand side $b_{i} \left( X_{S_{r-1},r} | X_{S_{r},r+1} \right)$ will be denoted as $Z_{r} \left( X_{S_{r-1},r} | X_{S_{r},r+1} \right)$. Introduce a partial order over a set of DO problems $\{Z_{r} \left( X_{S_{r-1},r} | X_{S_{r},r+1} \right)\}$: DOP $Z'_{r} = Z_{r} \left( X'_{S_{r-1},r} | X'_{S_{r},r+1} \right)$ precedes DOP $Z''_{r} = Z_{r} \left( X''_{S_{r-1},r} | X''_{S_{r},r+1} \right)$ if $b_{i} \left( X'_{S_{r-1},r} | X'_{S_{r},r+1} \right) \leq b_{i} \left( X''_{S_{r-1},r} | X''_{S_{r},r+1} \right), \ i \in U_{r}$ or graphically

![Partial order of DOPs.](image)

Let us solve the problem $Z'_{r}$ by implicit enumeration. Let P be a partial solution (PS) fathomed by the tests of the implicit enumeration. There are 3 possible cases:
a) PS $P$ is fathomed by test 2, i.e., the best completion of $P$ is feasible. Then this completion is feasible for the problem $Z_r'$ too, so as

$$b_i \left( X_{S_{r-1},r} | X_{S_{r,r+1}} \right) \leq b_i \left( X_{S_{r-1},r} | X_{S_{r,r+1}} \right).$$

Thus, each incumbent of $Z_r'$ is a feasible solution of $Z_r''$, i.e., the objective function value of $Z_r''$ is higher (better) than the corresponding objective function value of $Z_r'$.

b) PS $P$ is fathomed by test 1 in the problem $Z_r'$, i.e., $\varpi \leq z^*$. Then $P$ is fathomed in $Z_r''$, too, as $\varpi \leq z^* \leq z''^*$.

c) PS $P$ is fathomed by test 3 in $Z_r$.

It is clear, that in cases a) and b) it is senseless to fathom PS $P$ in the problem $Z_r''$, because $P$ is automatically fathomed. Thus, while fathoming PS in $Z_r''$, it is interesting to study only those PS $P$ fathomed by test 3 in $Z_r$.

Let us consider a family of $2^{|S_{r-1,r}|+|S_{r,r+1}|}$ DOPs (10–12) with nonnegative coefficients (multidimensional knapsack problems). Using partial order of DOPs described above it is possible to order members of this family as it shown in Fig. 5. Consider a process of solving the family of DOPs $\{Z_r\}$. Let $P$ be a PS fathomed in $\{Z_r(11|1)\}$ (level 0 in Fig. 5). If PS $P$ is fathomed by tests 1, 2, we can exclude this PS $P$ from consideration (and fathoming) in other DOPs. If PS $P$ was fathomed by the test 3 in $\{Z_r(11|1)\}$, then we

![Fig. 5. Partial order of DOPs (multidimensional knapsack).](attachment:image.png)
pass to a DOP from the level 1, say, \( \{Z_r(01|1)\} \) and consider the PS \( P \) in this problem.

If the PS \( P \) is not fathomed by the test 3 in \( \{Z_r(01|1)\} \), then there exists one of the following cases:

- test 1 is true, then backtrack to \( \{Z_r(11|1)\} \) and pass to the next problem of level 1;
- test 2 is true, then also backtrack to \( \{Z_r(11|1)\} \) and pass to the next problem of level 1;
- both tests 1 and 2 are not true, then extend PS \( P : P' = (P, j_1) \) and try to fathom \( P' \).

If any of the tests 1, 2 are true, then do usual backtracking, i.e., PS \( P'' = (P, -j_1) \) is considered. If some extension of \( P : P''' = (P, j_1, -j_2, \ldots, j_f) \) is fathomed by the test 3, then we go to one of problem’s ancestors (say, \( \{Z_r(00|1)\} \)) and try to fathom by the tests of implicit enumeration.

**Conclusion**

Local decomposition algorithms combined with tree decomposition methods are a promising approach that enables solving sparse discrete optimization problems from applications. The performance of these algorithms can be improved with the aid of postoptimality analysis.

**A promising direction of future research** is the development for efficient schemes of postoptimality analysis embedded in local decomposition algorithms combined with tree decomposition methods.

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