Quark states near a threshold

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(MIT-CTP-2572, hep-ph/9610395 September 1996)

Abstract

We reduce the problem of many-channel hadron scattering at non-relativistic energies to calculations on the scale of a few fermis. Having thus disentangled kinematics from interior quark dynamics, we study their interplay when a quark state occurs near a hadronic threshold. Characteristic parameters, such as the observed peak width, the decay width, and the shape of a cross-section itself are highly affected by the threshold. A general pole-form expression for the $S$-matrix in an arbitrary background is given, and the pole structure of $S$ is examined. We show that at a hadronic threshold two poles in $S$ are generally important. We also classify the $S$-matrix pole structure considering an example where nonsingular coupled channels are closed at the threshold. The framework of our paper is the $P$-matrix formalism, which is reviewed and extended for use together with conventional methods of computing quark-gluon dynamics. Results and applications are illustrated for the doubly strange two-baryon system, the detailed analysis of which we postpone till our forthcoming paper.

Submitted to: Nuclear Physics B

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement #DF-FC02-94ER40818.
I. INTRODUCTION AND SUMMARY

A resonance shape can be dramatically distorted if one of its decay channels has a threshold within the resonance width. A tiny variation of coupling strength may lead to a wide spectrum of physical phenomena such as a slightly bound or a virtual state, a “shoulder”, or a resonance. All these effects are of kinematic origin. We will show that the underlying quark-gluon dynamics can be isolated and quantitatively estimated in a smooth way which is unaffected by such kinematic cataclysms.

There is little doubt that far from threshold singularities narrow and dramatic effects in scattering amplitudes are to be identified with quasi-stable states of QCD. Little sophistication is required to connect the $\rho(770)$ with $\bar{u}u - \bar{d}d$ or the $\phi(1020)$ with $\bar{s}s$. However, great care must be used when attempting to assign a fundamental QCD interpretation to broad effects like most of those seen in meson-meson scattering above 1 GeV or to striking effects like the $f_0(980)$ and $a_0(980)$ that lie near thresholds (in this case $K\bar{K}$). Identification many objects of great interest — exotics, hybrids, glueballs, quasi-molecular states, etc. — require us to consistently relate low energy scattering to microscopic quark-gluon dynamics.

We study hadron-hadron scattering at small kinetic energy, where non-relativistic methods suffice. This is an old problem, but there is no general agreement on how to associate quark-gluon “states” with effects seen in low energy scattering. One of the most popular phenomenological tools is the $K$-matrix parameterization and its pole analysis. The $K$-matrix emerges naturally in the study of dynamics that occurs at distances much smaller than the de Broglie wavelength of the scattering system. For a single channel close enough to threshold, the conditions for a $K$-matrix analysis might seem to be met. However, in the real world hadron-hadron systems with small relative momentum are often strongly coupled to other open or closed channels, where the relative momentum is large in absolute magnitude compared to the intrinsic sizes of hadrons. In this case results obtained from solving microscopic quark dynamics must not be directly associated with the $K$-matrix. We hope to make this clear in the course of our paper. In place of the $K$-matrix, we will argue that the $P$-matrix formalism is more suitable for this purpose.

This paper consists of two general divisions. Section 2 concerns with microdynamics on the hadron-size scale and the $P$-matrix formalism. The following Section 3 deals with observable objects such as $S$-matrix and cross-sections. Many of our results are outlined in the rest of the introduction below.

In Section 2A we review and extend the $P$-matrix formalism. $P$ is defined, similar to $K$, as an algebraic transform of the $S$-matrix but it involves an additional parameter $b$:

$$P(\varepsilon, b) = i\sqrt{k} \frac{e^{ikb}S(\varepsilon)e^{ikb} + 1}{e^{ikb}S(\varepsilon)e^{ikb} - 1} \sqrt{k},$$

where we consider a multichannel $s$-wave with the total energy $\varepsilon$. If the interaction for $r>b$ is absent or simple enough to be described with a potential, the $P$-matrix
generalizes the logarithmic derivative of the wave function at $r=b$ (see Section 2A for details). Then $P$ is fully determined by the dynamics in the inner domain $r<b$.

The poles of $P(\varepsilon, b)$ play an important role. Their positions and residues are shown to be related to the spectrum of the hadron-hadron system confined in a spherical cavity with a radius $R$ depending on $b$. Such boundary conditions are used in the bag model and could be simulated on a lattice.

We will vary the parameter $b$ changing the size of the cavity $R(b)$ and tracing the evolution of the $P$-poles. The smaller $R$, the more simple the quark dynamics inside the cavity. But when $R$ becomes equal or less than the confinement radius, the connection between the spectrum of the physical system and $P$-poles gets more and more complicated. In practice one has to stop at some $R_0(b_0)$. It was proposed in Ref. [2] that at this $R_0$ the quark system in the cavity may be treated as a single bag and its eigenstates can be calculated in perturbative QCD with current quark masses. This assumption reflects the idea that the bag interior is a phase built up on the perturbative vacuum. Alternatively, it could be a phase in which chiral symmetry is spontaneously broken, yielding constituent quarks with renormalized couplings and pion-like excitations. Finally one might also attempt to exploit lattice methods.

In Section 2B we illustrate the $P$-matrix calculation taking the bag model as an example. The issue of flavor symmetry is addressed. We show that $P_{ij}(\varepsilon, b)$ reflects this symmetry provided the cavity is sufficiently small. Then the flavor projections of the quark-bag states onto a two-particle state determine the corresponding projections of the $P$-pole residues. The latter, in turn, yield the partial decay width of the states, as it is shown in Section 3.

In Section 3A we explore the relation between the $P$-matrix and the $S$-matrix, concentrating on $S$ poles, their widths and their channel couplings. We will find a pole form equation for the $S_{ij}(\varepsilon)$ and thus obtain the formulas for the position, width, and the decay amplitudes of observed resonances or virtual states. All this will be done in the presence of an arbitrary phenomenological background $S_0(\varepsilon)$, and the unitarity of $S$ will be preserved.

At a threshold the momentum $k(\varepsilon)$ becomes singular. But unless $b$ is too big, the $P$-matrix does not “feel” the threshold and has a smooth behavior considered in Secs. 2A and 2B. This enables us to separate cusp effects from inner dynamics (in a way similar to $K$-matrix analysis). In the Section 3B we discuss the analytical structure of a many-channel $S$-matrix when a pole in $P(\varepsilon)$ occurs near some threshold. We find that the $P$-pole gives rise to two nearby poles in $S$, and track the movement of these poles on the many-sheeted energy plane, where they appear as a bound, quasi-bound states, and a resonance as the coupling strength decreases. We note the drastic energy dependence of hadronic shift and width at a threshold. One of its important consequences is a substantial difference between a quasi-bound state decay width and the corresponding observed resonance width (peak width).
An application of our methods can be found in a forthcoming paper where we consider the low energy production and scattering of two baryons with the total strangeness minus two. We will investigate the possibility for the 6-quark $H$-dibaryon to be unstable with respect to strong decay.

We would like to emphasize that we prefer this formalism to the $K$-matrix parameterization because:

- it is simply connected with dynamical calculations. One can, in principle, find the $P$-matrix solving a boundary value problem on a microscopic scale;
- $P$ obtained this way does not need correction for the coupling with open channels (hadronic shift).

$K$ matrix may also be considered for a many-channel system. Nevertheless, the connection between quark dynamics and $K$-pole structure or symmetries is not straightforward, unless at the given energy the wavelength is large ($kb << 1$) in all of the coupled channels. This never can be realized if some of the strongly coupled channels have different threshold energies. Even for one-channel $NN$ scattering this holds only in $10\ MeV$ energy interval while

- the $P$-matrix formalism gives simple dynamical interpretation over the whole nonrelativistic range ($\sim 10^3\ MeV$).

$P$ also inherits the main advantages of the $K$-matrix:

- it provides a parameterization of $S$ supporting its unitarity;
- it is insensitive to threshold singularities.

The formal definition of $P$ by eq. (1) resembles the definition of $K$. In fact,

$$K(\varepsilon) = P^{-1}(\varepsilon, b=0)$$

which implies that

- many analytical results in the well investigated $K$-matrix formalism are directly generalized to $P$-matrix.

\footnote{The choice of $b_0 \simeq 10^{-2}\ MeV^{-1}$ is made according to Ref. \footnote{2}; see Sec. 2B for details.}
II. THE P-MATRIX

This method of analyzing two-body reactions was proposed by Jaffe and Low in 1979 in order to test the spectroscopic predictions of quark models especially as they relate to exotic (e.g. multi-quark) states. The method was initially developed in the context of the bag model, where quarks are confined by a scalar vacuum pressure. However, it applies to any model in which quark and gluon eigenstates are studied without considering their coupling to decay channels. First we briefly review their formalism. We also present new arguments that give a further insight into the connection between low-energy scattering and quark model speculations. In the next subsection we quantitatively estimate the parameters of the $P$-matrix from the quark-bag model.

A. Formalism

At low kinetic energies hadron-hadron scattering may be described by nonrelativistic kinematics. Restricting our attention to $S=L=0$, we factor out the center-of-mass motion and consider the wave-function of the $n$-channel two-hadron system in the relative coordinate $r$. For a given value of a spatial parameter $b$, a definite energy $\varepsilon$, and $r$ greater than the interaction radius, the most general form of the wave function is

\[
\psi_i(r_i) = \sum_{j=1}^{n} \left\{ \cos[k_i(r_i - b)] \delta_{ij} + \sin[k_i(r_i - b)] \frac{P_{ij}}{k_i} \right\} A_j, \tag{3}
\]

where $i = 1, \ldots, n$ labels the channel and the $\{A_j\}$ are some amplitudes.

The matrix $P_{ij}$ generalizes the logarithmic derivative of $\psi(r)$ for the case of many channels. Comparing eq. (3) to the usual $S$-matrix parameterization of the scattering wave function, we find that $P$ and $S$-matrix are simply related as

\[
S = e^{-ikb} \frac{1}{\sqrt{k}} P \frac{1}{\sqrt{k}} + i \frac{1}{\sqrt{k}} P \frac{1}{\sqrt{k}} - i e^{-ikb}. \tag{4}
\]

Resolving this equation with respect to $P$ one gets eq. (4). For a unitary $S$ the matrix $P$ is hermitian. If the interaction is time reversal invariant, then $P$ is also real. $P$ depends on $b$ according to the equation

\[
\frac{\partial P}{\partial b} = -P^2 - k^2. \tag{5}
\]

For the present we treat $b$ as a free parameter. Suppose for a moment that the value of $b$ is large enough so that there is no interaction for $r \geq b$. If for the energy $\varepsilon = \varepsilon_p(b)$ and some choice of the amplitudes $A_j$ in eq. (3)
FIG. 1. (a) A two-hadron system is confined in a spherical cavity with a radius $R$. Suppose this system is in an eigenstate of a definite energy $\varepsilon_n$. (b) Then the wave function of the centers of its 3q-subsystems (solid line) strictly vanishes at the cavity boundary. At the same energy $\varepsilon_n$, the wave function of unconstrained two-hadron motion (dashed line) vanishes when the relative hadron-hadron separation equals $b = 2R - 2R_h$.

\[ \psi_i(b) = 0 \quad \forall i = 1, \ldots, n \]  

(6)

then the $P$-matrix has a pole at $\varepsilon_p(b)$. As shown in Ref. 2 its residue can be factorized:

\[ P_{ij}(\varepsilon) = P_{ij}(\varepsilon) + \xi_i \frac{r}{\varepsilon - \varepsilon_p(b)} \xi_j^T. \]  

(7)

We have chosen the vector $\xi$ to be normalized: $\sum_{i=1}^{n} \xi_i^2 = 1$.

Now let us explore the connection between the poles of the $P$-matrix and the quark-bag calculations. Remember that for now $b$ is taken to be larger than the range of the strong forces. In this case the $P$-matrix poles ("primitives") $\varepsilon_p(b)$ occur at the eigenenergies of the two hadron system with relative wave function constrained to vanish at $r = b$. We claim that these are just the eigenenergies of the multi-quark system that has the quantum numbers of the two-hadron system and is confined in a spherical cavity with a radius $R(b)$. The radius $R$ is approximately half of $b$. In fact, if two hadrons are placed in a hard sphere, the wave function of their relative motion $\psi(r)$ vanishes at

\[ b = 2R - 2R_h , \]  

(8)

were $R_h$ plays the role of the hadron radius, as shown in the Fig. 1.

\footnote{Note that we distinguish between the $P$-poles $\varepsilon_p$ and the eigenenergies $\varepsilon_n$ of a physical system.}
We see that for a large value of $b$ there is one-to-one correspondence between the $P$-matrix poles and the eigenenergies of a physical system which is put into a hard-wall cavity. Now let us make $b$ smaller. The $P$-matrix, as defined by eq. (1), will preserve a pole structure (see eq. (7)) but the parameters $\varepsilon_p$, $r$, and $\xi$ will change with $b$. If $b$ goes to $b'$ the related $P$-pole shifts to $\varepsilon'_p$, satisfying the equation

$$\varepsilon'_p = \varepsilon_p - \xi^T \frac{r}{P(\varepsilon'_p) + k'_p \cot k'_p \Delta b} \xi ,$$  \hspace{1cm} (9)

and

$$\Delta b = b' - b .$$  \hspace{1cm} (10)

It was noted by M. Soldate [5] that decreasing the cavity radius, $R$, imposes additional constraints on the system inside and, therefore, causes the eigenenergies of its states $\varepsilon_n(R)$ to grow. In accordance with this, one can show from eq. (9) that for $b' < b$

$$\varepsilon'_p > \varepsilon_p$$  \hspace{1cm} (11)

if the matrix $\partial P(\varepsilon)/\partial \varepsilon$ is negative semidefinite, in particular if $P(\varepsilon)$ is a constant.

The residue of the $P$-matrix $\xi r \xi^T$ varies with $b$ as

$$\xi' r' \xi'^T = R (\varepsilon'_p, \Delta b) \xi r \xi^T R^T(\varepsilon'_p, \Delta b)$$  \hspace{1cm} (12)

with

$$R (\varepsilon, \Delta b) \equiv \frac{k}{\sin(k \Delta b)} \frac{1}{P(\varepsilon) + k \cot (k \Delta b)} .$$  \hspace{1cm} (13)

We obtained this using the $b$-independence of the $S$-matrix in eq. (4) and the following identity:

$$\frac{1}{A + \xi a \xi^T} = \frac{1}{A} - \frac{1}{A} \xi \frac{a}{1 + \xi^T A^{-1} a} \xi^T \frac{1}{A} ,$$  \hspace{1cm} (14)

where $A$ is a nonsingular matrix, $\xi$ is a unit vector, and $a$ is a constant.

A small variation of $b$ in eqs. (9,12) yields

$$\frac{\partial \varepsilon_p}{\partial b} = -r$$  \hspace{1cm} (15)

and

$$\frac{\partial}{\partial b} (\xi r \xi^T) = - \{ \xi r \xi^T, P \} \equiv - \xi r \xi^T P - P \xi r \xi^T .$$  \hspace{1cm} (16)
The last equation can be converted to

$$\mathcal{P}_{ij}(\varepsilon, b)\big|_{\varepsilon=\varepsilon_p} = -\frac{1}{2r} \frac{\partial r}{\partial b} \xi_i \xi_j - \xi_i \frac{\partial \xi_j}{\partial b} - \frac{\partial \xi_i}{\partial b} \xi_j + \mathcal{P}_{ij}(\varepsilon_p, b)$$

(17)

where the matrix $\mathcal{P}$ is orthogonal to the vector $\xi$:

$$\mathcal{P}_{ij} \xi_j = \xi_j \mathcal{P}_{ji} = 0.$$  

(18)

When the cavity radius reaches the size of a few fm we should treat the system inside as a single quark-bag rather than two hadrons, but due to the interaction outside the bag eq. (3) is no longer valid. Then it becomes difficult to relate the quark eigenenergies $\varepsilon_n$ to the position of the $P$ poles $\varepsilon_p$. Nevertheless, we might expect that there is a size of the cavity $R_0$ when the quark-gluon system inside is already simple enough for our theoretical tools, while $\varepsilon_n(R_0)$ and $\varepsilon_p(b_0)$ are still close to each other. In this case we can estimate the position of the $P$ poles $\varepsilon_p(b_0)$ and their orientation in the channel space $\xi_i(b_0)$ from quark-bag model calculations, and the eqs. (15,17) will provide us with the other ingredients of the $P$-matrix.

B. Calculation of $P$

To specify $P$ we require the pole positions ($\varepsilon_p$), the residues ($r^{(p)}$), the channel coupling vectors ($\xi^{(p)}$), and the nonsingular part ($\mathcal{P}_{ij}(\varepsilon)$). We treat each of them in sequence below.

The pole positions were already considered in the previous subsection. Let us remember that they were identified with the eigenenergies $\varepsilon_n$ of a quark-gluon system subject to confining boundary conditions at a sphere $R(b)$.

In determining the vectors $\xi^{(0)}$ it is important to take account of flavor symmetry. For example, one may consider $SU(3)$ when describing baryon octet scattering or $SU(2)$ for the $np$ system. For two scalar meson scattering $SU(3)$ is badly violated and $SU(2)$ isospin symmetry is more appropriate. If the flavor symmetry were exact, the mass of all hadrons belonging to one multiplet would be the same. The states of an interacting system confined by a cavity would also form flavor multiplets. The eigenenergies of the states in one multiplet would be equal, and the $P$-matrix would be $SU(n_f)$ symmetric, whatever the size of the cavity. We do not observe this in reality because of the difference in the current quark masses. Nevertheless, the smaller the cavity, the better the coupling vector reflects the flavor symmetry. Let us show this in specific examples.

Baryon-Baryon: Imagine two $\Lambda$-particles inside a macroscopic spherical cavity. To be specific, suppose they are in the ground energy state with $J = 0$ and assume that the fusion of the $\Lambda$’s into one $H$-dibaryon $\bar{\Lambda}$ is energetically forbidden, i.e. $M_H > 2M_\Lambda$. For the macroscopic cavity the lowest eigenenergy will be $\varepsilon_p \simeq 2M_\Lambda$, and the $\Lambda-\Lambda$ interaction is negligible. This $\Lambda\Lambda$ system belongs to the symmetrized
product of two \(SU(3)\) baryon octets that decomposes into the following irreducible representations:

\[
(8 \otimes 8)_{\text{sym}} = 27 \oplus 8 \oplus 1 .
\]  

(19)

However the \(\Lambda \Lambda\) state can not be attributed to any of those irreducible parts, so the coupling vector \(\xi_i\) in the \(P\) pole corresponding to this state is not \(SU(3)\) symmetric.

Now we gently contract the cavity so that the system remains in its ground state. When the cavity radius reaches the order of 1 fm., the scale of confinement starts to overcome the s-quark mass, and \(SU(3)\) symmetry gradually emerges. The \(\Lambda\)'s inside split into a “gas” of 6 strongly interacting quarks. Due to the color-magnetic interaction, the ground state of this system now does occur at the flavor singlet:\[^{[4]}\]

\[
|H\rangle = \frac{1}{\sqrt{5}} |BB\rangle + \frac{4}{\sqrt{5}} |8 \cdot 8\rangle
\]

(20)

where \(8 \cdot 8\) denotes two color octet baryons coupled to an overall singlet, and

\[
|BB\rangle = \frac{1}{\sqrt{8}} \left\{ |\Xi^- p\rangle - |\Xi^0 n\rangle + |p \Xi^-\rangle - |n \Xi^0\rangle + |\Sigma^- \Sigma^+\rangle + |\Sigma^+ \Sigma^-\rangle - |\Sigma^0 \Sigma^0\rangle + |\Lambda \Lambda\rangle \right\}
\]

(21)

is the flavor singlet state composed of two color singlet, flavor octet baryons.

Let us explore the interpretation of \(\xi_i\) as the bag state orientation in the channel space. Consider the parameter \(b\) in eq. (3) independently for each channel. If at the cavity boundary the interaction is negligible, the “partial residue” of the \(i\)-th channel will be

\[
r\xi^2_i = -\frac{\partial}{\partial b_i} \varepsilon_p \propto \left| \frac{\partial \psi_i}{\partial r_i} \right|^2 \biggr|_{r_i=b}
\]

(22)

(\(\psi\) is the normalized wave function of the confined system, obeying \(\psi|_{r=b}=0\)). Thus \(r\xi^2_i\) is associated with the “partial pressure” on the cavity walls. Eq. (22) shows that for a small \(b\) the residue \(\xi_i r \xi^T_j\) of the lowest \(P\)-matrix pole is almost a \(SU(3)\) singlet. As a first approximation we can take the vector \(\xi\) corresponding to the exact \(SU(3)\) symmetry (cf. eq. (21)):

\[
\xi_i = \pm \frac{1}{\sqrt{8}} , \quad i = (\Xi^- p, \Xi^0 n, p \Xi^-, n \Xi^0, \Sigma^- \Sigma^+, \Sigma^+ \Sigma^-, \Sigma^0 \Sigma^0, \Lambda \Lambda).
\]

(23)

[^{[4]}]: Modulo small \(SU(3)\) violation due to current quark masses.
Meson-Meson: The $f_0(980)$ resonance has the quantum numbers $I(J^{PC}) = 0(0^{++})$ and decays strongly into $\pi\pi$ and $K\bar{K}$. Because of the great difference between the $\pi$ and $K$ masses, it is not realistic to assume SU(3) symmetry even within the confinement radius. The SU(2) symmetric decomposition for $f_0$ reads:

$$|f_0\rangle = \alpha_K \sqrt{\frac{1}{4} \left\{ |K^- K^+\rangle - |\bar{K}^0 K^0\rangle + |K^+ K^-\rangle - |K^0 \bar{K}^0\rangle \right\} + \alpha_\pi \sqrt{\frac{1}{3} \left\{ |\pi^- \pi^+\rangle + |\pi^+ \pi^-\rangle - |\pi^0 \pi^0\rangle \right\} + \alpha_\eta |\eta \eta\rangle + \alpha_c |c\rangle}$$

where $|c\rangle$ stands for confined channels, e.g. glueball, and $\Sigma |\alpha_i|^2 = 1$. Therefore, the $P$-matrix for $\pi\pi$ scattering has a pole around 980 MeV, and its “orientation” in the channel space $\xi$ is given by the normalized projection of the decomposition eq. (24) onto the two-particle channels $\pi\pi$, $K\bar{K}$, and $\eta\eta$.

Without a deeper understanding of confinement, we are only able to provide crude estimate for the dynamical parameters $r$ and $P$. These will serve as a guide in the next sections. Ref. [2] contains rather visual reasoning concerning the residue $r$ that we paraphrase as follows. Let us consider the “partial pressure” $p_i$ on the cavity walls due to the $i$-th flavor component of the system:

$$4\pi R^2 p_i \equiv -\frac{\partial}{\partial R} \varepsilon p_i.$$

We suppose that there is a size of the cavity $R_0$ when $p_i$ can either be calculated perturbatively in the quark-bag model or attributed to the hadrons in the $P$-matrix approach. In the two-baryon example above the pressure exerted by the $\Lambda-\Lambda$ subsystem is

$$4\pi R^2 p_{\Lambda\Lambda} = -\frac{\partial}{\partial R} \varepsilon p_{\Lambda\Lambda}^2$$

with $\zeta_{\Lambda\Lambda}^2 = \frac{1}{40}$,

as calculated from the bag model (eqs. (20,21)). In the $P$-matrix formalism it is

$$4\pi R^2 p_{\Lambda\Lambda} = -\frac{\partial b}{\partial R} \frac{\partial}{\partial b_{\Lambda\Lambda}} \zeta_{\Lambda\Lambda}^2 = \frac{\partial b}{\partial R} r_{\Lambda\Lambda}^2$$

with $\zeta_{\Lambda\Lambda}^2 = \frac{1}{8}$,

see eqs. (22,23). That is generally we have:

$$r_{\xi_i^2} \simeq \left. \frac{\partial R}{\partial b} \frac{\partial}{\partial R} \zeta_{\Lambda\Lambda}^2 \right|_{R=R_0}.$$

The important result is that the “partial residue” $r_{\xi_i^2}$ is suppressed by the factor $\zeta_i^2/\xi_i^2 < 1$ with respect to the natural scale $1/R_0^2$.

To make the choice of $R_0$ and $b_0$, we follow the original paper [4]. In Ref. [2] Jaffe and Low employed the MIT bag-model where the bag with the mass $M$ had the radius...
\[ R_0 \simeq 5M^{1/3} \text{ GeV}^{-1} , \]  

for \( M \) in GeV. \( b_0 \) was obtained by matching the density of the free hadron-hadron wave function, vanishing as the relative hadron separation reached \( b_0 \), to the density of the free quarks inside the bag. In the case of two mesons this procedure yielded

\[ b_0 \simeq 1.4 \, R_0 . \]

We can say even less about the matrix \( \mathcal{P}(\epsilon) \). Definitely, it has the poles corresponding to the other bag states. Eqs. (16) or (17) suggest that in the interstitial region

\[ \mathcal{P}_{ij} \sim \frac{1}{b_0} . \]

In principle, all the information about \( P \)-matrix can be rigorously obtained from calculations involving only hadronic sizes. To this end one should solve the quark dynamics and parameterize the hadronic wave function according to eq.(3). The external interaction can be taken into account as described in Ref. [2]. In the absence of powerful methods applicable to scales of order 1 fm we have resorted to bag model phenomenology.

### III. CORRESPONDING S-MATRIX

Now we turn our attention to the quantities measured in actual scattering experiments, such as the \( S \)-matrix and singularities in cross-sections. In the Subsection 3A we express \( S \) and its singularities in terms of the \( P \)-matrix discussed earlier. Then we consider in detail threshold effects and their interference with \( P \)-poles.

#### A. General equations

In the previous section we argued that the poles of the \( P \)-matrix have fundamental significance. Taking \( P \) in the form

\[ P_{ij}(\epsilon) = \mathcal{P}_{ij}(\epsilon) + \xi_i \frac{r}{\varepsilon - \varepsilon_p} \xi_j^T , \]

one can easily reconstruct the corresponding \( S \)-matrix using eq. (4). In the denominator of eq. (6) one has to deal with the inversion of a matrix like \( A_{ij} + \xi_i a \xi_j^T \) and the identity (14) comes handy. After some calculations we obtain

\[ S_{ij} = \tilde{S}_{ij} - i\chi_i \frac{1}{\varepsilon - \varepsilon_r + \frac{r}{2} \chi_j^T} . \]
In this formula $\overline{S}$ is the background scattering produced by $\overline{P}(\varepsilon)$:

$$
\overline{S}(\varepsilon) = e^{-ikb} \frac{1}{\sqrt{k}} \overline{P} \frac{1}{\sqrt{k}} + i e^{-ikb} .
$$

(34)

The pole term in eq. (33) has diverse manifestations in cross-sections, that are discussed in the next subsection. As shorthand for them we will make free use of the word “resonance”. The “resonance” channel couplings $\chi_i$ are

$$
\chi_i(\varepsilon) = \sqrt{2} r e^{-ikb} \sqrt{k} \left( \frac{1}{P - ik} \right)_{ij} \xi_j .
$$

(35)

For the energy dependent “resonance” position and the width in the denominator of eq. (33) we have

$$
\varepsilon_r(\varepsilon) - i \frac{\gamma(\varepsilon)}{2} = \varepsilon_p - \xi^T \frac{r}{P - ik} \xi .
$$

(36)

The real and imaginary parts in eq. (36) can be easily separated. To this end we write the many-channel momentum matrix $k$ as

$$
k = q + i\kappa ,
$$

(37)

where $q$ and $\kappa$ are real and refer to the open and closed channels correspondingly. Recalling that for the strong interaction $\overline{P}$ is also real, we find

$$
\varepsilon_r(\varepsilon) = \varepsilon_p - \xi^T \frac{r}{P + \kappa} \left( \frac{1}{P + \kappa} q \right) \xi ,
$$

(38)

$$
\gamma(\varepsilon) = 2r \xi^T \frac{1}{\sqrt{1 + \left( \frac{1}{P + \kappa} q \right)^2}} \frac{1}{P + \kappa} q \frac{1}{P + \kappa} \frac{1}{\sqrt{1 + \left( \frac{1}{P + \kappa} q \right)^2}} \xi .
$$

(39)

These equations are valid for a nonsingular $\overline{P} + \kappa$ and an arbitrary $q$. One may write the total width in eq. (39) as a sum of partial width $\gamma_i$ over only the open channels:

$$
\gamma = \sum_{\text{open channels}} \gamma_i
$$

(40)

with the $i$-th partial width:

$$
\gamma_i = 2r q_i \left( \frac{1}{P + \kappa} \sqrt{1 + \left( \frac{1}{P + \kappa} q \right)^2} \xi \right)^2 .
$$

(41)
As expected, the $S$-matrix eq. (33) is unitary,

$$SS^\dagger = 1,$$

(42)

when $k$ is real. This is ensured by the following properties of the amplitudes $\chi_i$:

$$S_{ij}\chi_j^* = \chi_i,$$

(43)

and

$$\chi_i^\dagger\chi_i = \gamma.$$

(44)

If at some energy only the first $m < n$ channels are open, then only the upper-left $m \times m$ sub-matrix of $S_{ij}$ is unitary. Note that the unitarity of the $S$-matrix does not impose any additional restrictions on its background part $\overline{S}$, except that it be unitary by itself, or on the $P$-matrix poles and residues. As long as $P$ is hermitian, $S$-matrix unitarity is automatically taken care by eq. (4).

As we saw earlier, $P(\varepsilon, b_0)$ is completely determined by dynamics in the microscopic domain where the interaction is strong, and $P$ is not influenced by the region in configuration space where the system is represented by two freely moving hadrons. Thus, all kinematical effects are absorbed in eqs. (33-39). We proceed to study them next.

**B. Cusp analysis**

At threshold kinematics plays a key role. The analytical structure of the $S$-matrix at threshold is well known and conveniently described via the $K$-matrix parameterization. Since the $K$ matrix can be viewed as a special case of $P$ (see eq. (2)), the $P$-matrix formalism should provide similar results, with a different dynamical interpretation however. In this section we outline the analytical structure of the general equation eq. (33) at threshold and point out the specific features that arise from the dynamical estimates in the Sec. 2A.

Suppose the energy $\varepsilon_p$ of a $P$-pole lies at the vicinity of a threshold in some channel, which we denote by $i = 1$. The pole may correspond to $f_0(980)$ or $a_0(980)$ at $\bar{K}\bar{K}$ threshold (990 $MeV$); $\Lambda(1405)$ at $\bar{K}N$ (1430 $MeV$); $H$-dibaryon (2090–2240 $MeV$) at $\Lambda\Lambda$ (2230 $MeV$); even the deuteron or the $pp$ virtual state at the $pn$ or $pp$ thresholds correspondingly. At energies $\varepsilon_s$, satisfying the equation

$$\varepsilon_s = \varepsilon_r(\varepsilon_s) - i \frac{\gamma(\varepsilon_s)}{2} = \varepsilon_p - \xi^T \frac{r}{P(\varepsilon_s) - ik(\varepsilon_s)} \xi,$$

(45)

the denominator in eq. (33) vanishes, i.e. the $S$-matrix has a pole. Note that the right hand side of eq. (45) is a multivalued function because the momentum
FIG. 2. Finding $S$-matrix poles $\varepsilon_s$ as solutions of eq. (45) for real energies $\varepsilon_s$ just below the lowest threshold $\varepsilon_{th1}$. The solid parabola-like curve represents the right-hand side of eq. (45) as a function of $\varepsilon_s$. The intersections of the curve with the dashed diagonal line yield the solutions. (a) There are two real energy solutions. The upper one (black circle) corresponds to a bound state. The other one is located on the unphysical energy sheet. (b) Again, the $S$-matrix has two real poles with $\varepsilon_s < \varepsilon_{th1}$ but both of them lie on the unphysical sheet. Now the circle corresponds to a virtual state. (c) There are no real solutions. In this case the $S$-poles occur at complex energies, as shown in Figure 3.

\[ k(\varepsilon) = \text{diag} \left( \sqrt{2m_1 (\varepsilon - \varepsilon_{th1})}, \sqrt{2m_2 (\varepsilon - \varepsilon_{th2})}, \ldots, \sqrt{2m_n (\varepsilon - \varepsilon_{th n})} \right) \]  

(46)

has branch points at all threshold energies $\varepsilon_{th i}$. A pole in $S(\varepsilon)$ gives rise to an anomaly in the cross-section if only it is close enough to the physical region, where each $k_i$ is either real and positive (for the open channels) or $k_i = i\kappa_i$ with real and positive $\kappa_i$ (for the closed channels). Accordingly, we are interested in the solutions of eq. (45) in which the nonsingular momenta $k_i \neq 1$ are taken to be close to the positive real or upper imaginary semi-axis. Having specified the branches for $k_i$ with $i \neq 1$, we still in general have two sets of solutions corresponding to different $k_1$ branches.

Let us discuss the case when at our threshold nearby the $P$-pole, all the other channels $i \neq 1$ are closed. Consider the energies below the threshold,

\[ \varepsilon_s \leq \varepsilon_{th1} < \varepsilon_{th i \neq 1} , \]  

(47)

and choose the nonsingular momenta on the upper imaginary semi-axis:

\[ k_i = i\kappa_i \text{ with } \kappa_i > 0 , \text{ for } i \neq 1 . \]  

(48)

In Fig. 2 we plot the two remaining branches ($k_1 = \pm i\kappa_1$) of the right hand side of eq. (45) verses $\varepsilon_s$. The solutions of eq. (45) are given by the intersections of this curve.
FIG. 3. The $S$-matrix pole dynamics with the coupling strength decreasing. The complex plane of the channel momentum $k_1$ (left) and the energy plane (right) are shown. Note that while the pole marked by the circle goes from the upper $k_1$ half-plane down to the lower half-plane, it moves under the cut from the physical energy sheet onto the unphysical sheet.

and the dash diagonal line. If attraction in the system is strong enough, we have two solutions, one on each branch (Fig. 2(a)). The $S$-pole with $k_1=i\kappa_1$ (the upper branch in the figure) corresponds to a stable state. When the primitive energy $\epsilon_p$ goes up or its residue $r$ becomes weaker this pole moves onto the branch $k_1=-i\kappa_1$ (Fig. 2(b)), and the stable state turns into a virtual one. At last, when there is no intersection (Fig. 2(c)) the $S$-poles leave the imaginary $k_1$-axis. If one of them moves close to the physical sheet, it will appear as a resonance in the open scattering channel. The evolution of the $S$-matrix poles in the $k_1$ and $\epsilon$ complex planes is shown on the Fig. 3.

Due to interaction with the open channels the location of a pole in the $S$-matrix is shifted with respect to the $P$-pole energy $\epsilon_p$, as given by eq. (45). For a bound state this shift, $\epsilon_s-\epsilon_p$, is negative. In fact, if there is a bound state at the energy $E$ then the wave function of the system in this state vanishes at infinity. From the arguments of the Sec. 2A we conclude that there is a $P$-matrix pole approaching $E$ as $b\to\infty$. In that section we also showed that the primitive energy $\epsilon_p(b)$ is a monotonically decreasing function of $b$, therefore

$$\epsilon_p(b) < \epsilon_p(\infty) = E.$$  \hspace{1cm} (49)

Of course, $S$-matrix also has a pole at the bound state energy:

$$\epsilon_s = E.$$  \hspace{1cm} (50)

Combining the eqs. (49) and (50),

$$\epsilon_s - \epsilon_p < 0.$$  \hspace{1cm} (51)

This result as well as an equation similar to eq. (45) were already obtained in Ref. [7] for a simplified dynamical model.

Our equations (35), (36), or (45) involve the a-priori nontrivial matrix $(P-ik)^{-1}$. We want to show that many of its non-diagonal elements at the threshold are small

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and may be neglected. In fact, we mentioned in Sec. 2B that away from the other bag states (primitives) the characteristic scale for $\mathcal{P}$ is $1/b_0$, that is much less than the system mass. If for all $i \neq 1$ channels

$$|k_i| \gg |\mathcal{P}| \sim \frac{1}{b_0},$$

(52)
or equivalently

$$|\varepsilon - \varepsilon_{\text{th}}| \gg \frac{(1/b_0)^2}{2m_{i,\text{reduced}}} \simeq \begin{cases} 10 \text{ MeV} & \text{at two-baryon threshold}, \\ 30 \text{ MeV} & \text{at } \bar{K}K \text{ threshold}; \end{cases}$$

(53)
then $k$ dominates the matrix $P_0-ik$ in every $i \neq 1$ direction. Consequently,

$$\left(\frac{1}{P_0-ik}\right)_{ij} \ll \left\{ \left(\frac{1}{P_0-ik}\right)_{ii}, \left(\frac{1}{P_0-ik}\right)_{1i}\right\} \ll \left(\frac{1}{P_0-ik}\right)_{11},$$

(54)
where $i \neq j$ and $i, j \neq 1$.

With this remark in hand we can easily analyze the energy dependence of the effective “resonance” position $\varepsilon_r$ (eq.(38)), its width $\gamma$ (eq.(39)), the channel couplings $\chi_i$ (eq.(35)), and hence the cross-section itself. If at the first threshold the other $i \neq 1$ channels are closed and satisfy the condition eq.(53), then

$$\varepsilon_r \simeq \varepsilon_p - \sum_{i \neq 1} \left(\xi_i - \xi_1\frac{\mathcal{P}_{1i}}{\mathcal{P}_{11}}\right) \frac{r}{\kappa_i} \left(\xi_i - \frac{\mathcal{P}_{1i}}{\mathcal{P}_{11}}\xi_1\right) +$$

$$+ \sum_{i \neq 1} \xi_1\frac{\mathcal{P}_{1i}}{\mathcal{P}_{11}} \frac{r}{\kappa_i} \xi_1 - \xi_1\frac{r}{\mathcal{P}_{11} + z_1} \xi_1$$

(55)
with

$$z_1 = \begin{cases} \kappa_1, & \varepsilon < \varepsilon_{\text{th}1}; \\ q_1 \frac{1}{\mathcal{P}_{11}} q_1, & \varepsilon > \varepsilon_{\text{th}1}. \end{cases}$$

(56)
(we do not consider the degenerate case $\mathcal{P}_{11} \simeq 0$). The momentum $q_1$ or $ik_1$ in $z_1$ varies rapidly at the threshold. Together with the smallness of $\mathcal{P}_{11}$, it may introduce dramatic energy dependence in the last term of eq.(55), as demonstrated in the Fig. 4(a). The width $\gamma$, given by eqs.(40-41), also contains a rapidly changing factor $q_1$. The experimentally observed width of the resonance or the virtual state is determined by both $\gamma(\varepsilon)$ and $\varepsilon_r(\varepsilon)$. To show this, let us consider the resonance phase

$$e^{i\varphi_r} = \frac{(\varepsilon_r - \varepsilon) + i \frac{\gamma}{2}}{|(\varepsilon_r - \varepsilon) + i \frac{\gamma}{2}|}.$$
FIG. 4. The effective resonance position ($\varepsilon_r$), effective width ($\gamma$), and elastic cross section ($\sigma$) as functions of energy $\varepsilon$ for a two-channel model with a $P$-pole close to a threshold. All quantities are in $MeV$, and the threshold occurs at 990 $MeV$. The half-width of the peak in the cross-section is significantly narrowed with respect to $\gamma$.

Strong energy dependence of $\varepsilon_r(\varepsilon)$ may lead to rapid variation of $\varphi_r(\varepsilon)$ and substantial narrowing of the observed width. An example is presented in the Fig. 4, where two hypothetical particles with the masses 140 $MeV$ and 495 $MeV$ are coupled by a $P$-pole at 1040 $MeV$, which is 50 $MeV$ above the second threshold. The couplings $\xi_1$ and $\xi_2$ are taken to be equal. For this model the formal width $\gamma(\varepsilon)$ is no less than 70 $MeV$ at any energy, whereas the observed half-width of the corresponding resonance is as small as 30 $MeV$. Of course, this simple example does not pretend to describe a real world and any resemblance of the Fig. 4(c) to a known resonance is a mere coincidence.

IV. ACKNOWLEDGMENTS

We are grateful to B. Kerbikov for providing useful references and X. Yang for assistance with numerical work.
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