Physical States in Topological Phase of N=2 Supersymmetric WZNW Models

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Abstract. N=2 supersymmetric WZNW models associated with finite-dimensional Manin triples (p, p+, p-), where p± are Borel subalgebras of any simple Lie algebra is considered. Physical states in topological phase of these models are computed. They ring stucture and N=2 WZNW representatives are constructed.

1. Introduction The extended superconformal field theories in two dimensions have recently attracted attention for two reasons. First, in connection with compactification of strings with space-time supersymmetry [1-4]. Second, as was noted in [5] the so-called twisted N=2 models introduced in [6-7] describe topological matter which can interact with topological gravity. From the point of view of classification of conformal theories, N=2 superconformal theories are in a sense the simplest type to classify, and a subset of them, supersymmetric Landau- Ginzburg theories, is related to catastrophe theory [8]. From the point of view of topological characterization of the theory, they have a ring of operators (chiral primary fields) which is believed to basically characterize them.
In [9-11] WZNW models were studied, which allow extended supersymmetry, and conditions were formulated that the Lie group must satisfy so that its WZNW model would have extended supersymmetry. In particular, in [11] a correspondence was established between N=2,4 super conformal Kac-Moody algebras and finite-dimensional Manin triples.

A generators of Super-Virasoro algebras arising in N=2,4 WZNW models are given by appropriate supersymmetric extension of Sugavara construction. In [12-16] were studied representations and modular properties of characters of these algebras. In the present paper we consider the "topological phase" of N=2 WZNW model which are associated with Manin triples of the form \( (p, p_+, p_-) \), where \( p_+, p_- \) are Borel subalgebras of any simple Lie algebra and \( p = p_+ \oplus p_- \) as a vector space. Here the "topological phase" means the topological theory which is obtained from N=2 WZNW model with help of twisting procedure [6-7]. We compute a physical states in topological phase. The ring structure of these states will be investigated in the next paper.

The paper is arranged as follows. In section 2 we briefly describe Sugavara construction following [10]. In section 3 we formulate the problem of describing a physical states in topological phase as a current algebra cohomology problem. Section 4 is devoted to the solution of this problem. In section 5 we give in an explicit form a representatives of physical states and show that they are a highest weights of N=2 Virasoro algebra. In section 6 we compute the ring structure of physical states.

2. N=2 WZNW models and finite-dimensional Manin triples

Following [11] we give the N=2 modification of Sugavara construction. Let \( p \) be finite-dimensional Lie algebra, endowed with invariant nondegenerate scalar product \( <,> \), and \( p_+, p_- \) be isotropic with respect to this scalar product Lie subalgebras of \( p \), such that \( p = p_+ \oplus p_- \) as vector space. Then the triple \( (p, p_+, p_-) \) is called Manin triple. Let's fixed any finite-dimensional Manin triple and orthonormal basis \( E^a, E_a, a = 0, ..., \frac{dim p}{2} - 1 \) in \( p \) such that \( f^{ab}_{c}, \ f^{ab}_{c} \) are the structure constants of subalgebras \( p_+, p_- \). Let \( J^a(z) \),
$J_a(z)$ be a generators of the affine Kac-Moody algebra $\hat{L}_p$, which are correspondes to the fixed basis $E^a$, $E_a$ and $J^a(z)$ currents generate $L_{p+}$ subalgebra, $J_a(z)$ currents generate $L_{p-}$ subalgebra. Let $\psi^a(z), \psi_a(z)$, $a = 1, \ldots, \frac{d_{\text{imp}}}{2} - 1$ be a free fermion currents which has a singular OPE with respect to the scalar product $\langle , \rangle$. The two spin $\frac{3}{2}$ currents:

$$G^+ = \alpha (J^a \psi_a + \beta^+ f^{ab}_c \psi_a \psi_b \psi^c)$$ (1)

$$G^- = \alpha (J_a \psi^a + \beta^- f_{ab}^c \psi^a \psi_b \psi_c)$$ (2)

where $\alpha$, $\beta^+$, $\beta^-$ are normalization constants, generate $N=2$ Virasoro superalgebra.

The example of the Manin triple which will be important for us in the following is based on any simple Lie algebra $g$ with the scalar product $\langle , \rangle$ and its Cartan decomposition $g = n_- \oplus h \oplus n_+$, $b_+ = h \oplus n_+$, $b_- = h \oplus n_-$. Consider the Lie algebra

$$p = g \oplus \tilde{h}$$ (3)

where $\tilde{h}$ is the copy of the Cartan subalgebra $h$ and the Lie algebra structure on $p$ is defined by

$$[g, \tilde{h}] = 0$$ (4)

On $p$ we define the invariant scalar product

$$\langle (X_1, H_1), (X_2, H_2) \rangle = (X_1, X_2) - (H_1, H_2)$$ (5)

If we set

$$p_+ = \{(X, H) \in p | X \in b_+, H = X_h\}$$ (6)

$$p_- = \{(X, H) \in p | X \in b_-, H = -X_h\}$$ (7)

where $X_h$ is the projection of $X$ on the Cartan subalgebra $h$, then we have $p = p_+ \oplus p_-$ and $p_+$, $p_-$ are isotropic subalgebras of $p$, which are isomorphic to a Borel subalgebras $b_+$, $b_-$. Remark that there is the direct generalization of this construction, which is based on any
pair of parabolic sabalgebras $p_{\pm}$ such that $b_{\pm} \subset p_{\pm}$. The explicit N=2 Sugavara construction we give in the simplest case

$$g = sl(2, C)$$

(8)

For this case let $J^a, J_0, \psi^a, \psi_0, a = 0, 1$ be the bosonic and fermionic currents with the OPE

$$J^0(z)J^1(0) = z^{-1}J^1(0) + o(z)$$
$$J_0(z)J_1(0) = z^{-1}J_1(0) + o(z)$$
$$J^0(z)J^0(0) = -z^{-2} + o(z)$$
$$J_0(z)J_0(0) = -z^{-2} + o(z)$$
$$J^0(z)J_1(0) = -z^{-1}J_1(0) + o(z)$$
$$J^1(z)J_0(0) = z^{-1}J^1(0) + o(z)$$
$$J^0(z)J_0(0) = -z^{-2}(k + 1) + o(z)$$
$$J^1(z)J_1(0) = -z^{-2}k + z^{-1}(J_0 - J^0)(0) + o(z)$$
$$\psi^a(z)\psi_0(0) = z^{-1}i\delta^a_0 + o(z)$$

(9)

If we introduce currents $H = J_0 - J^0, P = J_0 + J^0$ instead of $J_0, J^0$, then the OPE (9) is nothing less than OPE of $\hat{L}sl(2)$ current algebra with level $k$ and OPE of $\hat{L}\tilde{h}$ current algebra with level $k+2$.

Generators of N=2 Virasoro superalgebra with central charge $c = \frac{4}{k+2}$ is given by

$$T = -\frac{i}{2}(\partial\psi^a\psi^a - \psi^a\partial\psi^a) - \frac{1}{2(k+2)}(J_aJ^a + J^aJ_a)$$

$$G^- = -\sqrt{\frac{2}{k+2}}(\psi^aJ_a + i\psi^0\psi^1\psi^1)$$

$$G^+ = -\sqrt{\frac{2}{k+2}}(\psi^aJ_a + i\psi^0\psi^1\psi^1)$$

$$K = \psi^0\psi^0 + \frac{k}{k+2}\psi^1\psi^1 - \frac{i}{k+2}(J_0 - J^0)$$

(10)

3. Physical observables in topological phase of N=2 supersymmetric WZNW models and semi-infinite cohomology of current algebras.
Topological phase of N=2 supersymmetric WZNW model has the stress tensor \( T = T + \frac{i}{2} \partial K, \ T = T \pm \frac{i}{2} \partial \bar{K} \). In general there are two inequivalent models, depending on the choice of sign in the last formula. In this paper we will consider only the model with the plus sign. The model with opposite sign is believed to be equivalent by a duality that is related to the ”mirror manifold” phenomenon [17,18]. After above-described the stress tensor modification the operator

\[
Q = G^-(0) + \bar{G}^-(0)
\]  

where \( G^-(0) \) and \( \bar{G}^-(0) \) are zero-modes of the holomorphic and anti holomorphic spin-3/2 currents, can be considered as the BRST operator. The physical states are defined by the condition that they are Q-closed and equivalent up to Q-exact states:

\[
Qx = 0 \sim x + Qy
\]  

For brevity we will consider only holomorphic part of topological phase (the total physical space is the tensor product holomorphic and anti-holomorphic parts). That is rather then (12) we consider

\[
G^-(0)x = 0 \sim x + G^-(0)y
\]  

where \( x, y \) are any states from the holomorphic sector of N=2 supersymmetric model. For simplicity we consider conditions (13) in the case (3),(8).

Let \( L_{(l,k)} \) be irreducible \( \hat{L}s\hat{I}(2) \) module with integral highest weight \((l,k)\) and highest vector \( v_{(l,k)} \). Let \( F_{p,k+2} \) be irreducible representation of the Heisenberg algebra with generators \( P(n), n \in \mathbb{Z} \) and vacuum vector \( u_{p,k+2} \) annihilated by \( P(n), n \geq 1 \) and

\[
P(0)u_{(p,k+2)} = \frac{p}{\sqrt{2(k+2)}} u_{(p,k+2)}
\]  

Let \( Cl \) be irreducible representation of Clifford algebra with generators \( \psi^a(n), \psi_a(n), a = 0, 1, n \in \mathbb{Z} \) and vacuum vector \( \omega \) annihilated by \( \psi^a, n \geq 1 \) and \( \psi_a(n), n \geq 0 \). Consider the space

\[
A_{(l,p,k)} = L_{(l,k)} \otimes F_{(p,k+2)} \otimes Cl
\]
In view of commutatin relations of N=2 Virasoro superalgebra and Sugavara construction (10) the zero mode $G^{-}(0)$ endow $A_{(l,p,k)}$ with the structure of semi-infinite cochain complex

$$A_{(l,p,k)} = \oplus_{n \in \mathbb{Z}} A_{2+n}^{2+n}$$  \hspace{1cm} (16)

with the differential $d = G^{-}(0)$. Let $H^{\infty}_{\bullet,\bullet}(d, L_{(l,k)} \otimes F_{(p,k+2)})$ be semi-infinite cohomology of complex $A_{(l,p,k)}$. It is clear that all physical states coming from $A_{(l,p,k)}$ compose its cohomology.

The above-described considerations of the simplest case (3),(8) can be easily extended to more general Manin triples (3), (6), (7).

4. Computation of the cohomology groups.

For spin-3/2 current $G^{-}(z)$ there is a decomposition

$$G^{-} = G^{-}_0 + G^{-}_1$$

$$G^{-}_0 = -\sqrt{\frac{2}{k+2}} \psi^0 (J_0 + \psi^1 \psi_1)$$

$$G^{-}_0 = -\sqrt{\frac{2}{k+2}} \psi^1 J_1$$  \hspace{1cm} (17)

and relations

$$G^{-}_0(z)G^{-}_0(w) = o(z - w)$$

$$G^{-}_1(z)G^{-}_1(w) = o(z - w)$$

$$G^{-}_0(z)G^{-}_1(w) = o(z - w)$$  \hspace{1cm} (18)

Let’s introduce bigrading on $A_{(l,p,k)}$ putting

$$\text{deg}(J^a_n) = \text{deg}(J_a(n)) = (0,0)$$

$$\text{deg}(\psi^0_n) = -\text{deg}(\psi_0(n)) = (-1,0)$$

$$\text{deg}(\psi^1_n) = -\text{deg}(\psi_1(n)) = (0,-1)$$  \hspace{1cm} (19)

In view of (17),(18),(19) zero modes $G^{-}_0$, $G^{-}_1$ endowes $A_{(l,p,k)}$ with the structure of double complex

$$A^{n,m}_{(l,p,k)} = \oplus_{n,m \in \mathbb{Z}} A^{n,m}_{(l,p,k)}$$  \hspace{1cm} (20)

with differentials $d_0 = G^{-}_0(0)$, $d_1 = G^{-}_1(0)$, $\text{deg}(d_0) = (-1,0)$, $\text{deg}(d_1) = (0,-1)$. There is the spectral sequence

$$\{ E^{n,m}_r, \delta_r, r = 1, 2, \ldots \}$$  \hspace{1cm} (21)
associated with double complex (20), such that
\[ E_{1}^{n,m} = H_{-}^{\infty} + m(d_{1}, A_{(l,p,k)}^{*}) \]
\[ E_{2}^{n,m} = H_{-}^{\infty} + n(d_{0}, E_{1}^{*}) \]
and the limit term \( E_{\infty} \) of this spectral sequence gives the cohomology of the complex (16) with respect to \( d \).

In our case as we will show the spectral sequence (21) degenerates in the first term, that is the differential \( \delta_{r} \) in \( E_{r}^{n,m} \) has to be identically zero for \( r > 1 \) and \( E_{2}^{n,m} = E_{\infty}^{m} \). Therefore according to our rules, first of all we have to compute \( E_{1}^{n,m} \).

Let \( C_{l_{0}}, Cl_{1} \) be irreducible representations of Clifford algebras with generators \( \psi_{0}(n), \psi_{0}(n) \) and with generators \( \psi^{1}(n), \psi_{1}(n) \), correspondingly. It is obvious that
\[ A_{(l,p,k)}^{*} = A_{0}^{*} \otimes A_{1}^{*} \]
\[ A_{0}^{*} = F_{(p,k+2)} \otimes C_{l_{0}} \]
\[ A_{1}^{*} = L_{(l,k)} \otimes C_{l_{1}} \]

Because \( A_{1}^{*} \) is the complex with differential \( d_{1} \) which act by identity on \( A_{0}^{*} \) we have:
\[ E_{1}^{n,m} = A_{0}^{*} \otimes H_{-}^{\infty} + m(d_{1}, A_{1}^{*}) \]

Let’s denote
\[ I_{0} = J_{0} + \psi_{1}^{1} \psi_{1} \]
\[ I^{0} = J^{0} + \psi_{1}^{1} \psi_{1}^{1} \]
\[ I_{-} = I_{0} - I^{0} \]
\[ I_{+} = I_{0} + I^{0} \]

One has:
\[ [d_{1}, I_{0}(n)] = [d_{1}, I^{0}(n)] = 0 \]

From (23), (25), (26) follows that \( E_{1}^{n,m} \) is a representation of loop algebra \( L\hat{h}_{-} \oplus L\hat{h}_{+} \) generated by currents \( I_{-}(z), I_{+}(z) \). The cohomology \( H_{-}^{\infty} + m(d_{1}, A_{1}^{*}) \) is given by the semi-infinite Borel-Weil-Bott theorem:
\[ H_{-}^{\infty} + m(d_{1}, A_{1}^{*}) = F_{(w_{m}, l, k+2)} \]
where

\[ w_m = \begin{cases} 
  \underbrace{w_1 w_0 \cdots w_{(0,or1)}}_m & m > 0 \\
  \underbrace{w_0 w_1 \cdots w_{(0,or1)}}_{-m} & m < 0 
\end{cases} \quad (28) \]

and \( w_0, w_1 \) are simple reflection in affine Weil group of \( Lsl(2, C) \). This theorem was proved in [18].

Let \( \Gamma_0 \) be Heisenberg superalgebra with generators \( I^0(n), I_0(n), \psi^0(n), \psi_0(n) \). As a consequence of the semi-infinite Borel-Weil-Bott theorem and (24) one gets:

\[ E_{n,m}^{*} = F_{(w_m * l,k+2)} \otimes F_{(p,k+2)} \otimes Cl_0 \]

and \( E_{n,m}^{*} \) is the representation of \( \Gamma_0 \) with the vacuum vector \( v_{(l_m,p)} \) annihilated by \( \psi^0(n), I^0(n), I_0(n) \), with \( n > 0 \) and \( \psi_0(n) \) with \( n \geq 0 \), and zero modes act as following:

\[ I_-(0) v_{(l_m,p)} = l_m v_{(l_m,p)} \]
\[ I_+(0) v_{(l_m,p)} = p v_{(l_m,p)} \]

(30)

, where \( l_m = w_m * l \). Now we have to compute the second term \( E_{2,n,m}^{*} \) of spectral sequence (21). There are relations:

\[ [d_0, \psi^0(n)]_+ = 0 \quad [d_0, I^0(n)] = -\sqrt{2(k+2)n} \psi^0(n) \]

(31)

\[ [d_0, \psi^0(n)]_+ = -i \sqrt{\frac{2}{(k+2)}} I_0(n) \quad [d_0, I_0(n)] = 0 \]

(32)

The relations in (32) means that \( \psi_0(0) \) is the contracting homotopy operator for \( d_0 \). Using this fact it is not difficult to show that:

\[ E_{2,n,m}^{*} = E_{2,n,m}^{*,rel} \oplus \psi^0(0) \otimes E_{2,n,m}^{*,rel} \]

(33)

where \( E_{2,n,m}^{*,rel} \) is \( d_0 \)-cohomology of the relative complex \( E_{1,n,m}^{*,rel} \):

\[ E_{1,n,m}^{*,rel} = \{ c \in E_{1,n,m}^{*,rel} \mid \psi_0(0) c = I_0(0) c = 0 \} \]

(34)

The cohomology \( E_{2,n,m}^{*,rel} \) is very simple because they are cohomology of Heisenberg algebra generated by \( I_0(z) \). We have:

\[ E_{2,n,m}^{*,rel} = \begin{cases} 
  0 & n \neq 0 \\
  C & \text{and generated by } v_{(l_m,p)} \text{ if} 
\end{cases} \]

(35)
\[ I_0(0)v_{(l,m,p)} = 0 \] (36)

From this result follows that the spectral sequence (21) degenerates at the first term \( E_2^{m,m} = E_\infty^{n,m} \) and one gets the following:

\[
H_{rel}^{\infty+**}(d, L_{(l,k)} \otimes F_{(p,k+2)}) = \\
\psi^0(0) \otimes H_{rel}^{\infty+**}(d, L_{(l,k)} \otimes F_{(p,k+2)})
\]

\[ (37) \]

\[
H_{rel}^{\infty+n}(d, L_{(l,k)} \otimes F_{(p,k+2)}) = \delta_{n,m}\delta_{(l,m+p+1)}C
\]

\[ (38) \]

where \( H_{rel}^{\infty+**}(d, L_{(l,k)} \otimes F_{(p,k+2)}) \) denote \( d \)-cohomology of the relative complex

\[ A_{rel}^* = \{ c \in A_{(l,p,k)} | \psi^0(0)c = I_0(0)c = 0 \} \] (39)

and delta-function \( \delta_{(l,m+p+1)} \) code the condition (36).

The generalization of the last formulas for Manin triples (3), (6), (7) is straightforward because in [19] semi-infinite Borel-Weil-Bott theorem proved for general case.

5. Representatives of physical states and ground states in Ramond sector.

Now we give in an explicit form a representatives of \( H_{rel}^{\infty+**}(d, L_{(l,k)} \otimes F_{(p,k+2)}) \).

Let us suppose that \( H_{rel}^{\infty+m} \) is nonzero. As is clear from calculations (29), (35) of spectral sequence the representative \( H_{rel}^{\infty+m} \) is vacuum vector \( v_{(l,m,p)} \) of Heisenberg superalgebra \( \Gamma_0 \). But in view of semi-infinite Borel-Weil-Bott theorem this representative is in one to one correspondence with vacuum vector out of \( H_{rel}^{\infty+m}(d_1, A_1^*) \). Therefore in order for the representative of \( H_{rel}^{\infty+m} \) to write out suffice it to write out the vacuum vector \( v_{l,m} \in H_{rel}^{\infty+m}(d_1, A_1^*) \) and tensor multiply its by fermion vacuum \( \omega_0 \) of \( C\ell_0 \) and by vacuum vector \( u_{(p,k+2)} \) of \( F_{(p,k+2)} \) , where momentum \( p \) is defined from (36).

For any ghost number \( m \) the vacuum vector of \( H_{rel}^{\infty+m}(d_1, A_1^*) \) can be obtained as follows. Let \( v_{(l,k)} \) be highest vector of \( L_{(l,k)} \) and \( \omega_1 \) be
vacuum vector of $C_{l1}$. The direct calculation shown that $v_{(l,k)} \otimes \omega_1$ is vacuum vector of $H^{\hat{\mathfrak{sl}}+0(d_1, A_1^*)}$. It is possible to prove that vacuum vector with nonzero ghost number $m$ is given by the action of the element $w_m$ out of affine Weil group of $L(\hat{\mathfrak{sl}}(2))$ on vacuum $v_{(l,k)} \otimes \omega_1$ ($w_m$ is determined in (28)).

In according to this procedure we can to write out the representative $H^{\hat{\mathfrak{sl}}+m}_{rel}$ for any ghost number $m$. Recall that $\omega$ is the vacuum out of $Cl$ and let $n$ be a nonnegative integer number. Then

$$X_{-2n} = \psi_1(-2n) \psi_1(-2n+1) \cdots \psi_1(-1) \omega \otimes (w_0 w_1)^n v_{(l,k)} \otimes u_{(-1-l-2n,k+2)}$$

$$X_{-2n-1} = \psi_1(-2n+1) \psi_1(-2n) \cdots \psi_1(-1) \omega \otimes w_1 (w_0 w_1)^n v_{(l,k)} \otimes u_{(-1-l-2n,k+2)}$$

$$X_{2n+1} = \psi_1(-2n) \psi_1(-2n+1) \cdots \psi_1(-1) \omega \otimes (w_0 w_1)^n w_0 v_{(l,k)} \otimes u_{(-1-l_{2n+1},k+2)}$$

$$X_{2n} = \psi_1(-2n) \psi_1(-2n+1) \cdots \psi_1(-1) \omega \otimes (w_0 w_1)^n v_{(l,k)} \otimes u_{(-1-l_{2n},k+2)}$$

(40)

be representatives with ghost numbers $-2n$, $-2n-1$, $2n+1$, $2n$ correspondingly. Using this explicit formulas and properties of the affine Weil group representation on the $L_{(l,k)}$ [20] it is possible to show that a representatives (40) are ground states in Ramound sector of the $N=2$ Virasoro superalgebra, that is they annihilated by positive modes of currents $K(z)$, $G^\pm(z)$, $T(z)$ and the action of zero modes is given by

$$G^\pm(0)X_m = 0$$

(41)

$$L(0)X_m = \frac{c}{24}X_m$$

(42)

$$K(0)X_m = (\frac{l}{k+2} - \frac{c}{6}) X_m \quad \text{if } m \text{ is even}$$

$$K(0)X_m = (\frac{k-l}{k+2} - \frac{c}{6}) X_m \quad \text{if } m \text{ is odd}$$

(43)

In order for a representatives of $H^{\hat{\mathfrak{sl}}+*}(d, L_{(l,k)} \otimes F_{(p,k+2)})$ to write out and compute its $U(1)$-charges suffice it to use a representatives
of $H_{rel}^{\infty+m}$ and formula (37). It’s easy to see that they are also ground states in Ramound sector.

The representative constructing procedure may be easily extended for Manin triples (3), (6), (7).

6. The ring structure of physical states.

In the last section a correspondence between cohomology classes and ground states in Ramound sector was established. The spectral flow of N=2 Virasoro superalgebra [21] connects the Ramound (R) and Neveu- Schwarz (NS) sectors. Under this flow the ground states of R-sector flow to the chiral primary fields of ns-sector. U(1)-charges of chiral primary fields differs from U(1)-charges of ground states by $-\frac{c}{6}$ and from (43) we conclude that U(1)-charges of chiral primary correspondings to the representatives of $H_{rel}^{\infty+m}$ belongs to the segment of unitarity bound of N=2 Virasoro superalgebra [22-23]:

$$ g_{\frac{q}{2}} = 0 \quad 0 \leq q \leq \frac{c}{3} - 1 $$

(44)

where $q$ is U(1)-charge. U(1)-charges of chiral primary fields correspondings to the representatives of $\psi^0 \otimes H_{rel}^{\infty+m}$ belongs to the segment of unitarity bound:

$$ g_{\frac{q}{2}} = 0 \quad 1 \leq q \leq \frac{c}{3} $$

(45)

$$ g_{\frac{q}{2}} < 0 \quad f_{1,2} \geq 0 $$

The operator product of chiral primary fields modulo setting to zero the descendants of chiral primary fields defines the ring structure of of chiral primary fields and thus- the ring structure on the cohomology classes [21].

Let’s denote the conformal fields correspondings to the vectors $v_{(l,k)}$, $u_{(p,k+2)}$ by the same letters: $v_{(l,k)}(z)$, $u_{(p,k+2)}(z)$. Then the chiral primary fields correspondings to the representatives of the relative cohomology classes are given by:

$$ X_{-2n,t} = \partial^{2n-1}\psi^1 \partial^{2n-2}\psi^1 \cdots \psi^1 \otimes (J^1(-2n+1))^{k-l}(J^1(-2n+2))^{l-t} \cdots (J^1(0))^{k-l}v_{(l,k)} \otimes u_{(-l+2n(k+2),k+2)} $$
\[ X_{-2n-1,l} = \partial^{2n} \psi^1 \partial^{2n-1} \psi^1 \cdots \psi^1 \otimes (J^1(-2n))^l(J^1(-2n+1))^{k-l} \cdots (J^1(0))^l u_{(l+2+2n(k+2),k+2)} \]

\[ X_{2n+1,l} = \partial^{2n} \psi^1 \partial^{2n-1} \psi^1 \cdots \psi^1 \otimes (J^1(-2n-1))^{k-l}(J^1(-2n))^l \cdots (J^1(-1))^{k-l} v_{(l,k)} \otimes u_{(l-2n(k+2),k+2)} \]

The rest chiral primary are given by multiplication of the above on the chiral primary field \( \psi^0 \). The ring structure of the chiral primary fields is restricted by Ward identities and may be described as follows. Let's denote \( X_{-1,k} = \theta_1, \psi^0 = \theta_2, X_{0,1} = y, X_{2,0} = x, X_{-2,0} = x^{-1}, X_{0,0} = 1 \). Then the chiral ring is the associative supercommutative ring with unit generated by \( \theta_1, \theta_2, x, x^{-1}, y \) and relations:

\[ [\theta_1, \theta_2]_+ = \theta_1^2 = \theta_2^2 = 0 \]
\[ xx^{-1} = x^{-1}x = 1 \]
\[ y^{k+1} = 1 \]

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