KIRCHHOFF’S EQUATIONS OF MOTION VIA A CONstrained ZAKHAROV SYSTEM

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Abstract. The Kirchhoff problem for a neutrally buoyant rigid body dynamically interacting with an ideal fluid is considered. Instead of the standard Kirchhoff equations, equations of motion in which the pressure terms appear explicitly are considered. These equations are shown to satisfy a Hamiltonian constraint formalism, with the pressure playing the role of the Lagrange multiplier. The constraint is imposed on the shape of a compact fluid surface whose dynamics is governed by the canonical variables introduced by Zakharov for a free-surface. It is also shown that the assumption of neutral buoyancy can be relaxed.

1. Introduction. Kirchhoff’s equations of motion [14, 21] govern the most basic form of a dynamically coupled fluid-solid system. In such a system, a solid—typically assumed to be neutrally buoyant—moves solely under the influence of the instantaneous fluid stresses acting on its surface and is not constrained in any other manner. The fluid flow, in turn, is simultaneously influenced by the motion of the body. The fluid flow in Kirchhoff’s equations is incompressible, homogeneous, irrotational and inviscid, and the solid is rigid. Moreover, there is no external boundary. Several extensions to Kirchhoff’s equations have been formulated over the years—the addition of discrete singular vortices [15, 27, 26, 28, 24, 5], non-coincident centers of gravity and buoyancy [18], and deformable bodies [11, 12].

Kirchhoff’s equations are typically written in the variables \( \mathcal{L} \) and \( \mathfrak{A} \) which are the linear and angular momenta of the body+fluid system referred to a body-fixed frame:

\[
\left( \frac{d}{dt} + \mathbf{\Omega} \times \right) \mathcal{L} = 0,
\]

\[
\left( \frac{d}{dt} + \mathbf{\Omega} \times \right) \mathfrak{A} + \mathbf{V} \times \mathcal{L} = 0
\]

In the above, \( \mathbf{V} \) and \( \mathbf{\Omega} \) are the velocity of the center of mass and angular velocity, respectively, of the rigid body, both again referred to the body-fixed frame. The above equations are of course a consequence of the conservation of the system (body+fluid) momenta in a spatially fixed frame,
These conservation laws, in turn, are a consequence of the SE(3) symmetries of the system. A derivation of (1) and (2) may be obtained, for example, from the Appendix sections of references [27] and [28], where a more general problem was considered involving the addition of discrete singular vortices. Setting the vortex strengths to zero in these references recovers the basic Kirchhoff problem.

From a dynamical systems perspective, interest in Kirchhoff’s equations mainly stems from the fact that the associated Neumann problem for the velocity potential function \( \Phi \) of the fluid flow is solvable in terms of the rigid body velocities for any given body shape. \( \Phi \) is expressible in body-fixed coordinates as

\[
\Phi = V(t) \cdot \phi(t) + \Omega(t) \cdot \xi(t),
\]

where \( \phi \) and \( \xi \) (time-invariant vector fields in the body-fixed frame) are the Kirchhoff potentials. Each of the three components of \( \phi \) and \( \xi \) satisfy a Neumann problem in which the body moves with unit translational/rotational velocity along each of the three coordinate directions. These potentials are completely fixed by the body shape. As a consequence, the Kirchhoff system is easily shown to be a closed 6-dimensional system of ODE similar to the system governing the motion of a free rigid body, the sole difference being that the rigid body mass tensor is modified to include the so-called added mass terms which capture the fluid dynamic effects. The system can then, by routine means, shown to be a Hamiltonian system governed by the Lie-Poisson bracket on \( \mathfrak{se}(3)^* \), the dual of the Lie algebra of SE(3) [3]. Nonlinear stability, chaos, integrability and geometric theories for Hamiltonian systems have all found applications to this system [16, 2, 23, 22].

One advantage of Kirchhoff’s formulation, apart from finite dimensionality, is that it avoids dealing with the pressure field on the surface of the body, by using momentum conservation laws. This paper deals with the equations in which pressure appears explicitly. An alternative way of writing (3) is in the ‘\( F = ma \)’ form, and this introduces the incompressible pressure field \( p \) explicitly,

\[
\begin{align*}
\frac{dL_s}{dt} &= -\int \Sigma pn \nu, \\
\frac{dA_s}{dt} &= -\int \Sigma p(l \times n) \nu,
\end{align*}
\]

where \( L_s, A_s \) are the spatial linear and angular momentum, respectively, of the body alone and \( n \) is the unit normal on \( \Sigma \) pointing into the fluid. However, these equations must be supplemented by appropriate fluid equations. The incompressible pressure field has no evolution equation but it does appear in Bernoulli’s equation.

Assuming the fluid to have unit density, the following system of equations is thus considered

\[
\begin{align*}
\frac{\partial \Sigma}{\partial t} &= \nabla \Phi \cdot n, \\
\left( \frac{\partial \Phi}{\partial t} \right) &= -\frac{1}{2} \left| \nabla \Phi \right|^2 \bigg|_{\Sigma} + p_{\infty},
\end{align*}
\]
\[
\frac{d\mathbf{q}}{dt} = V_s,
\]
\[
\frac{d(\mathbf{L}_s, \mathbf{A}_s)}{dt} = \left( -\int_{\Sigma} \rho m \nu, -\int_{\Sigma} \rho (\mathbf{l} \times \mathbf{n}) \nu \right),
\]
where \( \Sigma \) denotes the fluid boundary which coincides with the body boundary. A precise definition of \( \Sigma \) is given in the next section. The second equation is recognized as Bernoulli’s equation applied on the body surface and \( p_\infty \) is the constant pressure far field (where the fluid is assumed to be at rest). The equations do not form a closed system due to the absence of an evolution equation for \( p \). However, substituting Bernoulli’s equation in the rigid body momentum equations gets rid of \( p \) altogether. Proceeding along this line, it is shown later in this paper, explicitly by calculations, how the evolution equations for \( \Phi, \mathbf{L}_s \) and \( \mathbf{A}_s \) can be combined to give (3) and are therefore equivalent to Kirchhoff’s equations in the basic Kirchhoff problem. The pressure distribution can then be obtained, \textit{a posteriori}, from Bernoulli’s equation corresponding to a body trajectory \( g(t) \), and the associated \( \Phi(t) \), obtained from Kirchhoff’s equations. In this sense, the pressure is akin to the objects obtained in geometric mechanics via the process of reconstruction.

The system (6) is more useful, and perhaps even essential, in extensions of the basic Kirchhoff problem in which momentum conservation no longer holds, such as, for example, in the presence of an external boundary—solid or fluid—which breaks the SE(3) symmetry. It should be noted of course that in such extensions, other equations may have to be appended to system (6) [7]. Even in the case of an externally unbounded domain, system (6) has an advantage since it is known that there are issues with the convergence of the fluid momentum integral [25]. The association of Kirchhoff’s equations with (fluid+solid) momentum conservation is thus not immediately obvious but is obtained after some careful analysis of far-field terms [27, 28]. System (6), on the other hand, avoids dealing with momentum convergence issues in unbounded domains. Moreover, in applications, the knowledge of pressure distribution on the surface of the body is of great importance. Thrust and drag, structural integrity of the body under the pressure loading, and vortex shedding off the body—which typically occurs at moderate and high Reynolds numbers—are only a few of the issues whose study requires a good knowledge of the body surface pressure distribution.

The main objective of this paper is to present a Hamiltonian constraint formalism for system (6). In particular, it is shown that these equations for the basic Kirchhoff problem are governed by the sum of a rigid body bracket and a Zakharov bracket with a suitably imposed rigidity constraint. The pressure field on the surface of the rigid body is obtained as a Lagrange multiplier. The Zakharov bracket, it may be recalled, is a canonical Poisson bracket introduced in [29] for the free surface problem. It is also shown that the gravity term, in the case where the fluid and the rigid body have different (but constant) densities, can be accommodated in this framework. In this case, one obtains Kirchhoff’s equation with a non-zero ‘gravity minus buoyancy’ force term on the right and Bernoulli’s equation with the gravity term included. Historically, the idea of pressure as a Lagrange multiplier term in body+fluid systems may be traced back to the unpublished PhD thesis of Paul Ehrenfest [9].

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\( ^1 \) The author thanks Jair Koiller for alerting him to this rare work.
The ‘free’ surface that is considered here is a compact fluid surface, lacking surface tension, on which the constraint (to be defined later) is imposed. The canonical variables for the (unconstrained) fluid surface are the Zakharov variables $(\Sigma, \phi)$, as in Definition 2.1. In Zakharov’s original formulation, the variables $(\eta, \phi)$ are used where, it may be recalled, $\eta$ is the height (anti-parallel to the gravity direction) of a non-compact free surface with respect to some datum. There is no canonical vertical height variable here, one uses $\Sigma$ instead which is the image of the embedding in $\mathbb{R}^3$ of a reference configuration of the compact fluid surface. In Benjamin’s Hamiltonian formulation of irrotational bubble dynamics [4], the variable $\Sigma$ is essentially the same but is defined using parameters and coordinates. The Hamiltonian structure in [4] is the same as in Zakharov’s problem but is discussed without reference to Poisson brackets. Another paper relevant to this work is the one by Lewis, Marsden, Montgomery and Ratiu [19], where the focus is on a free surface, compact or non-compact, whose dynamics is coupled with the vorticity dynamics in the interior of the fluid. The Poisson bracket structure in [19], which is presented in great detail in the framework of Hamiltonian systems with symmetry, is a generalization of the Zakharov bracket to account for vorticity dynamics in the interior.

The main difference between this paper and [29, 4, 19] stems from the imposition of the rigid body constraint on the compact fluid surface which, unconstrained, is assumed to be governed by Zakharov dynamics. The rigid body bracket is then introduced and the body+fluid system, governed by the sum of the Zakharov and rigid brackets, is studied in the framework of Hamiltonian constraint formalism. For this an appropriate constraint problem and the associated geometric set-up, introducing, in particular, the notion of a normal displacement vector field, is first discussed (next section). The main result of the paper is then presented in Theorem 4.1, showing how all this leads to Kirchhoff’s equations.

2. Setup. In this section the constraint problem is defined, but first the geometric formalism required for this is introduced.

Let $D_0$ be a reference domain of the fluid, extending to infinity, with boundary $\Sigma_0$ of the following geometry. Embed a rigid body boundary shape $\partial B$—a smooth, compact, boundaryless two-dimensional manifold—in $\mathbb{R}^3$, via the inclusion $i : \partial B \to \mathbb{R}^3$, and let $\partial B_0 := i(\partial B)$. Obtain $\Sigma_0$ using a normal displacement vector field $\mathcal{N}_0 : \partial B_0 \to \mathbb{R}^3$. A normal displacement vector field $\mathcal{N}_0$ consists of elements of $(\partial B_0)^\perp$, the normal bundle of $\partial B_0$, viewed as vector fields in $\mathbb{R}^3$ with base points $x_b \in \partial B_0 \subset \mathbb{R}^3$. It is represented as

$$\mathcal{N}_0(x_b) = a_0 n_b(x_b), \quad x_b \in \partial B_0, \quad a_0 : \partial B_0 \to \mathbb{R}. \quad (7)$$

Here, $n_b$ is the unit normal vector field on $\partial B_0$. Equivalently, one can think of $\mathcal{N}_0$ as the ‘flow map’ of the vector field, rather than as the vector field itself, taking points $x_b \in \partial B_0$ to points $x_f \in \Sigma_0$. A schematic sketch is shown in Figure 1. It should be noted, while viewing this figure, that the body is not physically present in the fluid.

Now consider images $\Sigma$ of smooth maps $\Sigma_m : \partial B_0 \to \mathbb{R}^3$ with the following property. Each $\Sigma_m$ is the composition of two maps, the restriction of $\Phi_g$, the action of $\text{SE}(3)$ on $\mathbb{R}^3$, followed by a normal displacement $\mathcal{N}$ defined analogously to $\mathcal{N}_0$, i.e. $\Sigma_m \equiv \mathcal{N} \circ \Phi_g \circ a_0$, is defined as

$$\Sigma_m(x_b) = a n_b(\Phi_g(x_b)), \quad x_b \in \partial B_0, \quad g \in \text{SE}(3), \quad a : \partial B_0 \to \mathbb{R}, \quad (8)$$
where \( \partial B_g := \Phi_g(\partial B_0) \). To avoid notational clutter, the unit normal vector field on \( \partial B_g \) is still denoted by \( n_b \). A schematic sketch is shown in Figure 2. It is also useful to view \( \Sigma_m \) as a map of the reference configuration \( \Sigma_0 \),

\[
\Sigma_m(x_f) = N \circ \Phi_g \circ N_0^{-1}(x_f),
\]

where \( x_f \) is the image of the map defined in (7).

**Remark 1.** In the above set-up \( \partial B_0 \) is viewed as a rigid body ‘skeleton’ for a given \( \Sigma_0 \). The existence of \( \partial B_0 \), for a given \( \Sigma_0 \), is implicitly assumed, and is easier to justify if the elements of the normal displacement vector field \( N_0 \) are, in turn, assumed to be small. The smallness assumption does not affect the constraint formalism, discussed later in the paper, for which it is actually sufficient that these normal elements be small. These assumptions continue to be in place in a time interval in which \( \partial B_0 \) and \( \Sigma_0 \) evolve according to free rigid body dynamics and Zakharov dynamics, respectively, as depicted in Figure 2. For the objectives of this paper, it is also sufficient that this time interval be infinitesimally small, since the system equations considered are instantaneous evolution equations.

**Tangent and cotangent bundles.** It will be assumed that the set of all \( \Sigma \) has a manifold structure and denote this manifold by \( M_\Sigma \) and its points by \( m \). Tangent vectors \( Y \in T_m M_\Sigma \) are identified with vector fields \( v_\Sigma \) based on \( \Sigma \). Further, 1-forms

![Figure 1. Sketch showing the manner in which the internal fluid boundary \( \Sigma_0 \) is obtained by normally extending the shape of a rigid body, in the reference configuration. The arrows represent elements of the normal vector field \( N_0 \) of equation (7). Representative points on \( \partial B_0 \) and \( \Sigma_0 \), labeled \( x_b \) and \( x_f \), respectively, are also shown.](image-url)
$\Sigma_m$, based on $\Sigma$, where the flat operator $^b$ is with respect to the standard Euclidean metric on $\mathbb{R}^3$, can be formally identified with dual elements $\beta \in T^*_mM_\Sigma$ via the non-degenerate $L^2$-pairing

$$\langle \beta, Y \rangle := \int_{\Sigma} u^\flat_\Sigma (v_\Sigma) \, \nu. \quad (10)$$

Of particular interest in $T^*_mM_\Sigma$ are normal 1-forms. Since the normal spaces of $\Sigma$ have dimension 1, these can be written as $bn^\flat$, $b : \Sigma \to \mathbb{R}$, where $n$ is the unit normal vector field based on $\Sigma$ [19]. Since, for a given $\Sigma$ and $b$, a real-valued function on $\Sigma_0$ can be defined uniquely by pull-back as

$$\phi(x_f) := b(\Sigma_m(x_f)), \quad x_f \in \Sigma_0, \quad (11)$$

it follows that such a function can also be identified uniquely with a normal 1-form (note that $\Sigma_m$, as given by (9), is invertible). By taking $b = \Phi|_\Sigma$, the function $\phi$ can also be identified with a harmonic function $\Phi$ in the fluid domain $D_g$ (with appropriate decay conditions at infinity) via existence and uniqueness of solutions to the Dirichlet problem. Recall, that the incompressible and irrotational fluid velocity fields $v_q$ are gradients of harmonic functions $\Phi : D_g \to \mathbb{R}$. For the standard metric on $\mathbb{R}^3$, $v_q$ and $\Phi$ obey the familiar relations:

$$\nabla \cdot v_q = 0, \quad v_q = \nabla \Phi,$$
$$\nabla \times v_q = 0, \quad \nabla^2 \Phi = 0. \quad (12)$$

Figure 2. Schematic sketch showing the action of the maps $\Sigma_m$ and $\Phi_g$.
Definition 2.1. The Zakharov variables are the pairs \((\Sigma, \phi)\), where \(\Sigma\) is the image of the map in (8) or, equivalently, the map in (9), and \(\phi\) is the pull-back function defined in (11) with \(b\) replaced by \(\Phi\mid_\Sigma\).

3. Variational derivatives and Poisson brackets. Consider functions \(\bar{F} : T^* M_\Sigma \to \mathbb{R}\) of the form

\[
\bar{F}(m, \beta) := \int_\Sigma \bar{f} \left( u^b_\Sigma (p_f) \right) \nu, \quad p_f \in \Sigma,
\]

where \(\bar{f}\) is a real-valued function on the space of 1-forms based on \(\Sigma\). Define variational derivatives using the pairing (10), in the usual manner [20]. For any variation \(\delta m \in T_m M_\Sigma\),

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \bar{F}(m + \epsilon \delta m, \beta) - \bar{F}(m, \beta) \right] =: \left\langle \frac{\delta \bar{F}}{\delta m}, \delta m \right\rangle \equiv \int_\Sigma \left( \frac{\delta \bar{f}}{\delta m} \right) \left( \delta \Sigma \right) \nu,
\]

where \(\delta \Sigma\) is the vector field based on \(\Sigma\) corresponding to \(\delta m\) and \(\delta \bar{f}/\delta \Sigma\) is identified with a 1-form based on \(\Sigma\). Similarly, for any variation \(\delta \beta \in T_{(T^* M_\Sigma)}^\ast\),

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \bar{F}(m, \beta + \epsilon \delta \beta) - \bar{F}(m, \beta) \right] =: \left\langle \frac{\delta \bar{F}}{\delta \beta}, \delta \beta \right\rangle \equiv \int_\Sigma \left( \frac{\delta \bar{f}}{\delta \beta} \right) \left( \delta u^b_\Sigma \right) \nu,
\]

where \(\delta u^b_\Sigma\) is the 1-form based on \(\Sigma\) corresponding to \(\delta \beta\) and \(\delta \bar{f}/\delta u^b_\Sigma\) is identified with a vector field based on \(\Sigma\).

Remark 2. It is well-known in variational calculus that definition (14) requires, in addition, parallel transport relative to the choice of a connection in \(T^* M_\Sigma\). This ensures that \(\beta \in T_{m + \epsilon \delta m}^\ast M_\Sigma\) of \(\bar{F}(m + \epsilon \delta m, \beta)\) is the ‘same’ as \(\beta \in T_m M_\Sigma\) of \(\bar{F}(m, \beta)\). In this paper, as will be seen below, such a choice of connection is implicitly made by pulling back covector elements to the reference configuration.

Proposition 1. Consider functions \(F_Z : T^* M_\Sigma \to \mathbb{R}\) of the form

\[
F_Z(m, \beta) := \int_\Sigma f \left( b n^b(p_f) \right) \nu, \quad p_f \in \Sigma,
\]

where, as defined earlier, \(b n^b\) is a normal 1-form based on \(\Sigma\), viewed as the normal component of \(u^b_\Sigma\), with \(b : \Sigma \to \mathbb{R}\). The canonical Poisson bracket on \(T^* M_\Sigma\) applied to a pair of such functions \(F_Z, G_Z\) is

\[
\{F_Z, G_Z\}_{T^* M_\Sigma} = \int_\Sigma \left( - \frac{\delta f}{\delta (\delta \Sigma)n} \left( \frac{\delta g}{\delta b} \right) + \frac{\delta g}{\delta (\delta \Sigma)n} \left( \frac{\delta f}{\delta b} \right) \right) \nu,
\]

where \((\delta \Sigma)n\) denotes the normal component of the variational vector field \(\delta \Sigma\).

Proof. The functional derivative of \(F_Z\) with respect to \(b n^b\) is taken keeping \(\Sigma\) fixed, and so

\[
\delta (b n^b) = (\delta b) n^b.
\]

It follows from (15) that

\[
\left\langle \frac{\delta F_Z}{\delta \beta}, \delta \beta \right\rangle = \int_\Sigma (\delta b) n^b \left( \frac{\delta f}{\delta \beta} \right) \nu,
\]
where $\delta f/\delta b$ is identified with a normal vector field based on $\Sigma$. The canonical Poisson bracket on $T^*M$ is written as

$$\{F_Z, G_Z\}_{T^*M} := -\left\langle \frac{\delta F_Z}{\delta m}, \frac{\delta G_Z}{\delta \beta} \right\rangle + \left\langle \frac{\delta F_Z}{\delta \Sigma}, \frac{\delta G_Z}{\delta \Sigma} \right\rangle$$

Since the tangential components of $\delta f/\delta \Sigma$, $\delta g/\delta \Sigma$ are annihilated. In turn, using (14), $(\delta f/\delta \Sigma)$, $(\delta g/\delta \Sigma)$ are determined by just the normal components, and are unaffected by the tangential components, of the variational vector field $\delta \Sigma$. This proves relation (16). The relation further shows that the set of functions $F_Z$ forms a Poisson subalgebra. The four properties of Poisson brackets can then be shown to be satisfied.

Referring to equation (11) and the accompanying discussion, make the identification $b \equiv \Phi|_{\Sigma}$, and write the function $F_Z$ as

$$F_Z(m, \beta) = \int_{\Sigma} f(\phi(x_f)) \, \nu, \quad x_f \in \Sigma_0.$$  

It can be seen that the bracket (16) remains the same for such an identification since, keeping $\Sigma$ fixed, $\delta \phi(x_f) = \delta b(p_f)$, implying that $\delta f/\delta \phi = \delta f/\delta b$. However, $\delta f/(\delta \Sigma)n$ is evaluated differently. Total variations in $\phi$, as per (11), are given by

$$\delta \phi = \delta \Phi|_{\Sigma} \circ \Sigma_m + \Phi \circ (\Sigma_m + \delta \Sigma_m) - \Phi \circ \Sigma_m,$$

$$= \delta \Phi|_{\Sigma} \circ \Sigma_m + (\nabla \Phi)|_{\Sigma} \cdot (\delta \Sigma).$$  

In evaluating $\delta f/(\delta \Sigma)n$, keeping $\phi$ fixed, the induced variation in $\Phi|_{\Sigma}$,  

$$\delta \Phi|_{\Sigma} = - (\nabla \Phi)|_{\Sigma} \cdot (\delta \Sigma)$$

should be taken into account. Equivalently, the Hamiltonian vector field in the $(\Sigma, \Phi|_{\Sigma})$ variables corresponding to (16) can be first obtained and then transformed to the Zakharov variables $(\Sigma, \phi)$ using (11).

**Definition 3.1.** The bracket (16), with $b$ replaced by $\phi$, will be referred to as the Zakharov bracket.

**Remark 3.** The reason for writing the canonical bracket in Proposition 1 as the negative of the conventionally accepted form for canonical brackets is related to the choice of the inward-pointing normal field on $\Sigma$. Equivalently, the conventional form of the bracket can be used but with the canonical momentum identified with negative $\phi$. With the choice of an outward-pointing normal field, these switches in signs are unnecessary.

**The unconstrained phase space and Hamilton’s equations.** Let $Q := M_\Sigma \times \text{SE}(3)$ denote the configuration space of the unconstrained fluid+rigid body system. Then the phase space of the unconstrained problem, which is the cotangent bundle of $Q$ is given by

$$T^*Q = T^* (M_\Sigma \times \text{SE}(3)).$$
The rigid body variables \((g, P_S) \in T^* \text{SE}(3)\) are the standard ones, \(g \in \text{SE}(3)\) denoting the body’s position and orientation, and \(P_S \equiv (L_s, A_s) \in T^*_g \text{SE}(3)\) the body’s spatial momenta.

Equip \(T^* Q\) with the sum Poisson bracket,
\[
\{\cdot,\cdot\}_{T^* Q} := \{\cdot,\cdot\}_{T^* M_{\Sigma}} + \{\cdot,\cdot\}_{T^* \text{SE}(3)}
\]
where \(\{\cdot,\cdot\}_{T^* M_{\Sigma}}\) is the Zakharov bracket of Definition 3.1 and \(\{\cdot,\cdot\}_{T^* \text{SE}(3)}\) is the canonical bracket on the rigid body phase space. Hamilton’s equations, obtained from
\[
\frac{dF}{dt} = \{F, H\}_{T^* Q}, \quad F, H : T^* Q \to \mathbb{R},
\]
take the following form for the uncoupled system
\[
\begin{align*}
\frac{\partial \Sigma}{\partial t} &= -\frac{\delta H}{\delta \phi}, \\
\frac{\partial \phi}{\partial t} &= \frac{\delta H}{\delta \Sigma}, \\
\frac{dg}{dt} &= \frac{\delta H}{\delta P_S}, \\
\frac{dP_S}{dt} &= -\frac{\delta H}{\delta g},
\end{align*}
\]
where
\[
H(m, \beta, g, P_S) = -\frac{1}{2} \int_{\Sigma} \Phi \nabla \Phi \cdot n \nu + \frac{1}{2} \left\langle \langle P_S, M^{-1} \cdot P_S \rangle \right\rangle_{\mathbb{R}^6}
\]
is the sum of the fluid and rigid body kinetic energies, respectively. Note that the kinetic energy of the (unit density) fluid in \(D_g\) is written as a surface integral for incompressible and irrotational flows whose velocity fields satisfy relations (12). The negative sign is due to the choice of the inward pointing unit normal field on \(\Sigma\). One checks that this term is in the form of the functions defined by Proposition 1 and, as per previous discussions, \(\beta\) is identified with the function \(\phi\) defined by (11). The second term is the kinetic energy of the rigid body with \(\langle\langle \cdot,\cdot\rangle\rangle_{\mathbb{R}^6}\) denoting the metric pairing on \(\mathbb{R}^3 \times \mathbb{R}^3\) after making the usual identification of \(\mathfrak{se}(3)^*\) with \(\mathbb{R}^3 \times \mathbb{R}^3\).

The Hamiltonian \(H\) is obtained from a Legendre transformation of the Lagrangian \(\mathcal{L} : TQ \to \mathbb{R}\) given by
\[
\mathcal{L}(m, \dot{m}, g, \dot{g}) = -\frac{1}{2} \int_{\Sigma} \Phi \nabla \Phi \cdot n \nu + \frac{1}{2} \left\langle \langle \mathfrak{V}_S, M \cdot \mathfrak{V}_S \rangle \right\rangle_{\mathbb{R}^6},
\]
where \(\dot{m} \in T_m M_{\Sigma}\) is identified with \(\partial \Sigma/\partial t\) and \(\mathfrak{V}_S\) denotes the rigid body’s spatial velocities. It can be shown that the Euler-Lagrange equations result in the same equations of motion as (19).

4. The constrained phase space and equations of motion. Thus far no interaction with the fluid boundary \(\Sigma\) and the rigid body \(B\) has been considered. Only the geometric shape of the rigid boundary boundary has been utilized. The fluid boundary and the free rigid body have been assumed to evolve independently of each other (on possibly different physical domains) in their respective phase spaces \(T^* M_{\Sigma}\) and \(T^* \text{SE}(3)\) and maintaining their respective kinetic energies constant.

To make the fluid and the rigid body ‘talk to each other’, one has to impose the constraint that the shape of \(\Sigma\) coincide with the shape of \(\partial B_g\), and the conservation of ‘fluid+solid’ kinetic energy has to be considered.
The constraint is imposed in the following manner. Going back to equations (8), it is obvious to each \( q \equiv (g, \Sigma) \in Q \) there exists a unique normal displacement vector field \( \mathcal{N} \) given by \( an_b(\Phi_g(x_b)) \), which can be viewed as the image of a map \( \varphi \) given, at each \( q \), by

\[
\varphi(q) = an_b.
\] (22)

**Definition 4.1.** Denote by \( V \) the vector space of all sections of the normal bundle \( (\partial B_g)^\perp \), equivalently the space of all normal vector fields based on \( \partial B_g \).

Following the fiber bundle picture of constraints and Lagrange multipliers described by Marsden and Ratiu [20], consider now a vector bundle \( \pi: E \rightarrow Q \), with fiber \( V := \pi^{-1}(q) \). The map \( \varphi \) of (22) therefore defines a unique section of \( (\partial B_g)^\perp \), at each \( q \), and the map itself, \( \varphi: Q \rightarrow E \), can be viewed as a section of \( \pi: E \rightarrow Q \).

The dual space \( V^* \) consists of elements that can be identified with normal 1-forms based on \( \partial B_g \). This space can be defined using \( L^2 \)-pairings similar to (10), with the integration carried out over \( \partial B_g \).

**Definition 4.2.** The configuration space of the constrained system is \( \varphi^{-1}(0) \subset Q \).

Here, 0 stands for the zero section of \( (\partial B_g)^\perp \).

4.1. **The case of a neutrally buoyant rigid body.** The Hamiltonian constraint formalism for a neutrally buoyant body is considered first. Later on, it will be shown that formalism is easily extended to the case where the body is not neutrally buoyant.

Consider the product space, also a cotangent bundle,

\[ T^*E \equiv T^*M_\Sigma \times T^*SE(3) \times V^* \times V \equiv T^*(M_\Sigma \times SE(3) \times V^*) \]

As usual, \( V^* \) is regarded as a cotangent bundle \( T^*V^* \) after making the tangent bundle identification \( TV^* \equiv V^* \times V^* \) and the duality of \( V^* \) and \( V \).

**Hamiltonian constraint formalism.** According to the theory of Lagrange multipliers for manifolds [20], the Euler-Lagrange equations for the constrained system are obtained from the Lagrangian

\[
L_c(m, \dot{m}, g, \dot{g}, \alpha) := L - \int_{\partial B_g} \lambda(\varphi(q)) \mathring{\nu}
\]

where \( L \) is given by (21) and \( \alpha \in V^* \) is identified with

\[
\lambda \equiv f_\lambda n^b_b, \quad f_\lambda: \partial B_g \rightarrow \mathbb{R},
\] (23)

and plays the role of the Lagrange multiplier. The term \( \int_{\partial B_g} \lambda(\varphi(q)) \mathring{\nu} \) is a functional of the Lagrange multiplier. The Hamiltonian function for the constrained system, obtained from the Legendre transformation of \( L_c \), is

\[
H_c(q, \mu, \alpha) = H(q, \mu) + \int_{\partial B_g} \lambda(\varphi(q)) \mathring{\nu},
\] (24)

where \( H(q, \mu) \) is given by (20). \( H_c \) is viewed as a function on \( T^*E \), with \( q \equiv (m, g) \), \( \mu \equiv (\beta, P_S) \). Note that, as per the theory [20], since \( L_c \) is independent of \( \dot{\alpha} \), the canonical momentum conjugate to \( \alpha \) is constrained to be zero. The constrained system therefore lies on a submanifold \( C \subset T^*E \). An investigation of the existence
of a Poisson structure on $C$, which might require a more detailed implementation of the Dirac theory of constraints, is not considered in this paper.

The constrained equations of motion. The constrained equations of motion on the Hamiltonian side are given, following Dirac [8], by

$$
\frac{\partial \Sigma}{\partial t} = \frac{\delta H}{\delta \phi},
\frac{\partial \phi}{\partial t} = \frac{\delta H}{(\delta \Sigma)^n} + \lambda \circ \frac{\delta \varphi(q)}{(\delta \Sigma)^n},
\frac{dg}{dt} = \frac{\delta H}{\delta g},
\frac{dP_S}{dt} = -\frac{\delta H}{\delta P} - \int_{\Sigma} \lambda \circ \frac{\delta \varphi(q)}{\delta g} \nu,
$$

(25)

where $\circ$ denotes the coupling between the Lagrange multiplier and the variational derivative term. These terms are defined as

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{\partial B_g} \lambda (\varphi(q + \epsilon(\delta \Sigma)n) - \varphi(q)|_{\varphi(q)=0} \bar{\nu} \right] =: \int_{\Sigma} \lambda \circ \frac{\delta \varphi(q)}{(\delta \Sigma)^n} ((\delta \Sigma)n) \nu,
$$

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{\partial B_g} \lambda (\varphi(q + \epsilon(\delta \Sigma)n) - \varphi(q)|_{\varphi(q)=0} \bar{\nu} \right] =: \left\langle \left\langle \int_{\Sigma} \lambda \circ \frac{\delta \varphi(q)}{\delta g} \nu, \delta g \right\rangle \right\rangle_{\bar{g}}.
$$

Here, $\delta g \equiv (V_\delta, \Omega_\delta) \in T_g \text{SE}(3)$ and can be identified with an element of the Lie algebra $\mathfrak{se}(3)$ via left invariant vector fields on SE(3) [20]. Note that since $\varphi$ is independent of the $\mu$ variables, the $\partial \Sigma/\partial t, dg/dt$ equations are unchanged.

The main result of the paper is now stated.

**Theorem 4.3.** The constrained equations of motion (25) are equivalent to system (6) which, in turn, can be combined to give system (3).

**Proof.** In the first part of the proof it will be shown that system (25) gives system (6). The terms corresponding to $\delta H/(\delta \Sigma)n, \delta H/\delta \phi$ in (25) are available in the literature, although computed in different settings and using different approaches [29, 4, 7]. In particular, Zakharov [29], in the free surface problem, uses the height variable $\eta$ instead of $\Sigma$. The same terms can be obtained in the geometric setting of this paper. Details are presented in the Appendix section.

**The $\lambda_g \circ \frac{\delta \varphi_g(q)}{(\delta \Sigma)n}$ term.** Going back to the definition of $\varphi(q)$, equation (22), a normal variation $\epsilon(\delta \Sigma)n$ in the fluid boundary will change the $\varphi(q)$ field to $\varphi_g(q) + \epsilon(\delta \Sigma)n \cdot \nu_b$. On imposing the constraint, $n = n_b, \partial B_g = \Sigma$, and therefore

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{\partial B_g} \lambda (\varphi_g(q + \epsilon(\delta \Sigma)n) - \varphi(q)|_{\varphi(q)=0} \bar{\nu} \right] = \int_{\Sigma} \lambda ((\delta \Sigma)n) \nu,
$$

$$
\Rightarrow \lambda \circ \frac{\delta \varphi(q)}{(\delta \Sigma)n} = \lambda.
$$

The Lagrange multiplier $\lambda$ is identified with the dynamic pressure field $p$, i.e. the pressure field minus the hydrostatic component, by choosing

$$
f_\lambda = -(p - p_\infty)
$$

(26)
where \( p_\infty \) denotes the constant pressure value at infinity.

**The \( \int_\Sigma \lambda \circ \frac{\delta \varphi(q)}{\delta g} \) \( \nu \) term.** Computing this term involves elementary geometry again but is somewhat more involved than the previous term. The result is proved in \( \mathbb{R}^2 \) but essentially the same arguments apply in \( \mathbb{R}^3 \). Referring to Figure 4, identify a point \( p_f \) on \( \Sigma \). Since any non-zero \( \varphi(q) \) depends only on the relative orientations of \( \partial B_g \) and the fluid boundary, assume that the body is fixed and the fluid boundary \( \Sigma \) is acted upon in a rigid manner by \( -\epsilon(\delta g)_{\mathbb{R}^2} \). Here, \( \delta g \) is identified with an element of the Lie algebra \( se(2) \) and \( (\delta g)_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2 \) is the corresponding infinitesimal generator of the Lie algebra action. To make the figure more viewable \( \partial B_g \) is not shown, but it should be kept in mind that when the constraint is imposed \( \partial B_g \) coincides with \( \Sigma \). Under this action, the displacement of \( p_f \) is given by the vector 

\[
-\epsilon(\delta g)_{\mathbb{R}^2}(p_f)
\]

and the corresponding normal vector is \((-\epsilon(\delta g)_{\mathbb{R}^2} \cdot n)(p_f)\). However, this may not be an element of \( \delta \varphi(q) \) since (i) there is no guarantee that the tip of this vector lies on \( \Sigma - \epsilon(\delta g)_{\mathbb{R}^2} \) and (ii) it is a vector along \( n \) based on \( p_f \), and not along \( n_b \) based on \( p_b(\in \partial B_g) \). When the constraint \( \varphi(q) = 0 \) is imposed, however, (ii) is easily dealt with since then \( p_b, n_b \) coincide with \( p_f, n \), respectively. And so the more important objection is (i).

![Figure 3](https://via.placeholder.com/150)

**Figure 3.** Schematic sketch for the computation of the \( \int_\Sigma \lambda \circ \frac{\delta \varphi(q)}{\delta g} \) \( \nu \) term.

The following argument, using elementary geometry, suffices to show that the length by which \((-\epsilon(\delta g)_{\mathbb{R}^2} \cdot n)(p_f)\) fails to be an element of \( \delta \varphi(q) \) is an \( O(\epsilon^2) \) term as \( \epsilon \to 0 \). Draw two line segments perpendicular to each other intersecting at a
point \(o\), one being in the normal direction from \(p_f\) and the other, perpendicular to this segment, originating from \(p'_f\). Consider the right triangle right triangle with vertices \(p_f, p'_f\) and \(o\). The side \(p_f o\) has length \(|\epsilon(\delta g)_{\mathbb{R}^2} \cdot n| (p_f)\) and the side \(p'_f o\) has length \(|\epsilon(\delta g)_{\mathbb{R}^2} \times n| (p_f)\).

Parametrize the segment \(p'_f o\) by \(s\) such that \(s = 0\) corresponds to \(p'_f\) and \(s = |\epsilon(\delta g)_{\mathbb{R}^2} \times n| (p_f)\) to the point \(o\). Consider now the portion, curve \(C\), of \(\Sigma\) that goes from \(p'_f\) to the point where it is intersected by \(p_f o\). Call this point \(o'\). The objective is to show that the length of the segment \(o'o\) is \(O(\epsilon^2)\). Denote by \(\theta(s)\) the angle between the tangent to \(C\) at any point and the direction of \(p'_f o\) at \(s\). The length of segment \(o'o\) is given by

\[
\Delta l = \int_0^{\epsilon(\delta g)_{\mathbb{R}^2} \times n} \tan \theta(s) \, ds.
\]

Now, \(\tan \theta(0) = O(\epsilon)\), since \(\theta\) at \(p'_f\) is simply the angle by which the tangent vector at \(p_f\) has rotated. Now, if it was true that \(\tan \theta(s) = O(\epsilon)\) \(\forall s\), the proof is complete. However, this is not always true. There could be portions of curve \(C\) where \(\tan \theta(s)\) is really large i.e. \(O(1)\). But since \(C\) is smooth, there exists a curve neighborhood of \(p'_f\), i.e. a portion of \(C\) of arclength \(l\) containing \(p'_f\), at all points of which \(\tan \theta(s) = O(\tan(\theta(0))) = O(\epsilon)\). Note that \(l\) is independent of \(\epsilon\). As \(\epsilon \to 0\), \(C\) will get contained in this neighborhood for some \(\epsilon < \epsilon_0\). It follows that \(\Delta l = O(\epsilon^2)\).

And so to leading order, the length of segment \(p_f o'\) is also \(|\epsilon(\delta g)_{\mathbb{R}^2} \cdot n| (p_f)\). In general, without imposing the constraint, \(p_f o'\) cannot be identified with \(\varphi(q)\) since any element of \(\delta \varphi(q)(\varphi(q) + \delta \varphi(q)) - \varphi(q)\) must be normal to \(\partial B_g\). In fact, the element of \(\delta \varphi(q)\) originating at \(p_b \in \partial B_g\) is a different (directed) segment. But, as discussed above, once the constraint \(\varphi(q) = 0\) is imposed, \(n = n_b, \Sigma = \partial B_g\), and these segments coincide to finally give

\[
\delta \varphi(q) = - (\epsilon(\delta g)_{\mathbb{R}^2} \cdot n_b) n_b.
\]

In the general case of \(\mathbb{R}^3\), the term is

\[
\delta \varphi(q) = - (\epsilon(\delta g)_{\mathbb{R}^3} \cdot n_b) n_b
\]

and the corresponding variation in the Lagrange multiplier term can be written as

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{\partial B_g} \lambda (\varphi(q) + \epsilon \delta g) - \varphi(q) |_{\varphi(q) = 0} \right]
\]

\[
= \int_{\Sigma} -\lambda ((\delta g)_{\mathbb{R}^3} \cdot n) \nu
\]

\[
= \int_{\Sigma} -f_\lambda (\delta g)_{\mathbb{R}^3} \cdot n \nu,
\]

\[
= \left\langle \left( \int_{\Sigma} -f_\lambda n \nu, \int_{\Sigma} -f_\lambda (l \times n) \nu \right), (\delta g) \right\rangle_{\mathbb{R}^6},
\]

since \((\delta g)_{\mathbb{R}^3} \equiv V_\delta + \Omega_\delta \times l\).

Importing terms from the Appendix and \(f_\lambda\) from (26), system (25) is finally obtained as

\[
\frac{\partial \Sigma}{\partial t} = \nabla \Phi \cdot n,
\]
\[
\frac{\partial \phi}{\partial t} = \left( -\frac{1}{2} \nabla \Phi \cdot \nabla \Phi + (\nabla \Phi \cdot n)^2 - (p - p_\infty) \right)_{\Sigma(t)}, \quad (28)
\]

\[
\frac{dq}{dt} = V_S, \quad (29)
\]

\[
\frac{dP_S}{dt} = \left( \int_\Sigma -pn \nu, \int_\Sigma -p(l \times n) \nu \right). \quad (30)
\]

Note that the integrals involving \( p_\infty \) are zero due to the underlying assumption that the body is homogeneous and the origin (for \( l \)) is at the center of mass of the body. Moreover, since

\[
\frac{\partial \phi}{\partial t} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Sigma}{\partial t} \cdot \frac{\partial \Phi}{\partial n}, \quad (31)
\]

equation (28) is equivalent to the unsteady Bernoulli equation on \( \Sigma \), in system (6), with \( p_\infty \) playing the role of the integration ‘constant’ (which, it may be recalled, is, at most, a function of time alone). This completes the first part of the proof.

In the second part, using vector calculus in \( \mathbb{R}^3 \), it will be shown that the evolution equations for \( \phi \) and \( P_S \) can be combined to give (3). Since this part is an exercise in classical fluid mechanics, similar derivations are available in the literature such as, for example, in Landweber and Yih [17].

Use (31) to write (28) in the familiar Bernoulli equation form, applied to points on \( \Sigma \),

\[
p|_{\Sigma(t)} - p_\infty = \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi \right)_{|_{\Sigma(t)}}. \]

From (30) obtain

\[
\frac{dP_S}{dt} = \left( \int_\Sigma \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi \right) n \nu, \int_\Sigma \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi \right) (l \times n) \nu \right). \]

Examine the first component on the right. Since the flow is everywhere free of vorticity, one can apply a familiar vector identity for bounded domains (see, for example, (3) on p.47 of Saffman [25]) to write

\[
\int_{\Sigma \cup C_R} \frac{1}{2} \nabla \Phi \cdot \nabla \Phi n \nu = \int_{\Sigma \cup \tilde{D}_g} \nabla \Phi (\nabla \Phi \cdot n) \nu,
\]

where \( C_R \) is a large fixed imaginary external boundary enclosing the rigid body for all times. Without loss of generality, \( C_R \) can be taken to be a circle of radius \( R \) centered on the rigid body (and with an inward pointing normal). Denote the fluid domain bounded by \( \Sigma \cup C_R \) by \( \tilde{D}_g \). Since \( P_S \equiv (L_s, A_s) \), obtain

\[
\frac{dL_s}{dt} = \int_{\Sigma \cup C_R} \frac{\partial \Phi}{\partial t} n \nu + \int_{\Sigma \cup C_R} \nabla \Phi (\nabla \Phi \cdot n) \nu.
\]

Now,

\[
\frac{d}{dt} \int_{\Sigma \cup C_R} \phi \, n \nu = -\frac{d}{dt} \int_{\tilde{D}_g} \nabla \Phi \, \mu,
\]

\[
= -\int_{\tilde{D}_g} \frac{D(\nabla \Phi)}{Dt} \, \mu = -\int_{\tilde{D}_g} \left( \frac{\partial (\nabla \Phi)}{\partial t} + \nabla \Phi \cdot \nabla (\nabla \Phi) \right) \mu,
\]

\[
= \int_{\Sigma \cup C_R} \frac{\partial \Phi}{\partial t} n \nu + \int_{\Sigma \cup C_R} \nabla \Phi (\nabla \Phi \cdot n) \nu.
\]
Since \( C_R \) is invariant in time, only the integral over \( \Sigma \) remains for the first terms on the left and right. Moreover, \( \Phi = O(1/R^2) \), for large \( R \), so that the last integral on the right over \( C_R \) is \( O(1/R^4) \). In the limit \( R \to \infty, \tilde{D}_g \to D_g \), this integral is zero and one finally obtains
\[
\Rightarrow \frac{d}{dt} \left( L_s - \int \Phi n \nu \right) = 0. \tag{32}
\]

Similarly,
\[
\frac{dA_s}{dt} = \int \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi \right) (l \times n) \nu.
\]

Again, using standard vector identities it can be shown that
\[
\int_{\Sigma \cup C_R} \frac{1}{2} (\nabla \Phi \cdot \nabla \Phi) (l \times n) \nu = \int_{\tilde{D}_g} \nabla \Phi \cdot \nabla (\nabla \Phi) \times l \mu = -\int_{\Sigma \cup C_R} (\nabla \Phi \times l) \nabla \Phi \cdot n \nu.
\]

Now,
\[
\frac{d}{dt} \int_{\Sigma \cup C_R} \Phi (n \times l) \nu = -\frac{d}{dt} \int_{\tilde{D}_g} \nabla \times (\Phi l) \mu,
\]
\[
= -\frac{d}{dt} \int_{\tilde{D}_g} \nabla \Phi \times l \mu,
\]
\[
= -\int_{\tilde{D}_g} \frac{D(\nabla \Phi)}{Dt} \times l \mu - \int_{\tilde{D}_g} \nabla \Phi \times \frac{Dl}{Dt} \mu
\]
\[
= -\int_{\tilde{D}_g} \left( \frac{\partial (\nabla \Phi)}{\partial t} + \nabla \Phi \cdot \nabla (\nabla \Phi) \right) \times l \mu,
\]
\[
= \int_{\Sigma \cup C_R} \frac{\partial \Phi}{\partial t} (n \times l) \nu - \int_{\Sigma \cup C_R} \frac{1}{2} (\nabla \Phi \cdot \nabla \Phi) (l \times n) \nu.
\]

Canceling terms using the fact that \( C_R \) is invariant in time and the far-field decay rate of \( \Phi \) gives in the limit \( R \to \infty \) and \( \tilde{D}_g \to D_g \),
\[
\Rightarrow \frac{d}{dt} \left( A_s - \int \Phi l \times n \nu \right) = 0. \tag{33}
\]

Equations (32) and (33) are the same as system (3) when \( L_s \) and \( A_s \) are defined to be the quantities within the parentheses.

Decomposing \( \Phi \) using the Kirchhoff potentials (4) allows one to write the integral terms as a linear combination of the body velocities, where the coefficients—body surface integrals of the Kirchhoff potentials—are dependent on the body geometry alone and are identified with the added mass terms. Transferring to a body-fixed frame transforms (32) and (33) into Kirchhoff’s equations in their usual form. \( \square \)

**Remark 4.** Equation (27) is the analog of the evolution equation for \( \eta \) in the free-surface problem. Indeed when applied to the free surface problem, one easily obtains the corresponding evolution equation for \( \eta \). Write \( \Sigma(x,y) = \eta(x,y)\hat{k} \). In such a case, since (27) gives the evolution of \( \Sigma \) in normal directions,
\[
\frac{\partial \Sigma}{\partial t} = \frac{\partial \eta}{\partial t} \hat{k} \cdot n
\]
\[
= \frac{\partial \eta}{\partial t} \hat{k} \cdot \frac{\nabla \eta}{|\nabla \eta|}.
\]
where \( \hat{\eta}(x, y, z) := z - \eta(x, y) \), and the free surface is identified with the zero-level set of \( \hat{\eta} \). It follows that (27) becomes

\[
\frac{\partial \eta}{\partial t} \frac{1}{\sqrt{1 + (\partial \eta/\partial x)^2 + (\partial \eta/\partial y)^2}} = \nabla \Phi \cdot \frac{-(\partial \eta/\partial x)\hat{i} - (\partial \eta/\partial y)\hat{j} + \hat{k}}{\sqrt{1 + (\partial \eta/\partial x)^2 + (\partial \eta/\partial y)^2}}
\]

\[
\Rightarrow \frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial x} \frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial y} \frac{\partial \eta}{\partial y}
\]

which is exactly the equation for \( \eta \) obtained by Zakharov [29].

4.2. The case of a rigid body with positive or negative buoyancy. In the previous subsection, the neutrally buoyant assumption was made and therefore the hydrostatic pressure and the gravity body force terms do not appear in the equations. In this section, the Hamiltonian formalism of the previous section is extended to include the case where the rigid body has a different density \( \rho_b \) than the fluid density \( \rho_f \). This extension however is quite independent of the constraint formalism.

The Hamiltonian \( H_{c,g} \) in this case includes the potential energy due to gravity,

\[
H_{c,g} := H_g + \int_{\partial B_g} \lambda(\varphi(q)) \bar{\nu}
\]

\[
:= -\frac{1}{2} \rho_f \int_{\Sigma} \Phi \nabla \Phi \cdot n \nu - \rho_f \int_{D_g} \vec{g}_r \cdot z \hat{k} \mu - \rho_b \int_{B_g} \vec{g}_r \cdot z \hat{k} \mu
\]

\[
+ \int_{\partial B_g} \lambda(\varphi(q)) \bar{\nu}
\]

where \( \vec{g}_r \) is the constant gravity vector in the direction \( -\hat{k} \) and \( z \) is the height above or below a datum.

Using standard vector identities, rewrite the potential energy terms as

\[
- \rho_f \vec{g}_r \cdot \int_{D_g} \nabla \frac{z^2}{2} \mu - \rho_b \vec{g}_r \cdot \int_{B_g} \nabla \frac{z^2}{2} \mu,
\]

\[
= \rho_f \vec{g}_r \cdot \int_{\Sigma} \frac{z^2}{2} \ n \nu - \rho_b \vec{g}_r \cdot \int_{\partial B_g} \frac{z^2}{2} n_b \nu,
\]

recalling that \( n \) is inward pointing and \( n_b \) is outward pointing (the terms due to the boundary at infinity can be easily shown to cancel). Variations in \( \Sigma \) and \( g \) will thus cause variations in the above two integrals and make contributions to \( \delta H_g/(\delta \Sigma)_n \) and \( \delta H_g/\delta g \).

The variations \( \delta z \) and \( (\delta \Sigma)_n \) are related by

\[
\delta z \cdot n = (\delta \Sigma)_n,
\]

\[
\Rightarrow \delta z = \frac{(\delta \Sigma)_n}{k \cdot n}
\]

And so the variation in the first of the two integrals takes the form (modulo multiplying constants)

\[
\rho_f \vec{g}_r \cdot \int_{\Sigma} z \delta z \ n \nu = -\rho_f \vec{g}_r \cdot \int_{\Sigma} z (\delta \Sigma)_n \ n \nu
\]
Similarly, the variations $\delta z$ and $(\delta g)_{\mathbb{R}^3}$ are related by

$$\delta z \cdot n_b = (\delta g)_{\mathbb{R}^3} \cdot n_b,$$

$$\Rightarrow \delta z = \frac{(\delta g)_{\mathbb{R}^3} \cdot n_b}{k \cdot n_b}.$$  

The variation in the second of the two integrals therefore takes the form (modulo multiplying constants)

$$-\rho_b g_r \cdot \int_{\partial B_g} z \delta z \, n_b \, \nu = \rho_b g_r \int_{\partial B_g} z (\delta g)_{\mathbb{R}^3} \cdot n_b \, \nu,$$

$$= \rho_b g_r \left( \left( V_\delta, \Omega_\delta \right) \cdot \left( \int_{\partial B_g} z n_b \, \nu, \int_{\partial B_g} z (l \times n_b) \, \nu \right) \right),$$

$$= \rho_b g_r \left( \left( V_\delta, \Omega_\delta \right) \cdot \left( \int_{B_g} \nabla \cdot \mu, \int_{B_g} l \times \nabla \cdot \mu \right) \right),$$

$$= ((V_\delta, \Omega_\delta) \cdot (-\rho_b g_r V_B, 0)), $$

the second component being zero since $l$ is the position vector from center of mass.

The variational derivatives of $H_g$ and $H$ are therefore related as

$$\frac{\delta H_g}{(\delta \Sigma)_{n_b}} = \rho_f \frac{\delta H}{(\delta \Sigma)_{n_b}} - \rho_f g_r z,$$

$$\frac{\delta H_g}{\delta g} = \frac{\delta H}{\delta g} - (\rho_b g_r V_B, 0),$$

where $V_B$ is the volume of the rigid body.

To obtain the right form of equations (27), (28), (29) and (30) one has to further modify the Zakharov bracket as

$$\frac{1}{\rho_f} \int_{\Sigma} \left( -\frac{\delta f}{(\delta \Sigma)_{n_b}} \frac{\delta g}{\delta \phi} + \frac{\delta g}{(\delta \Sigma)_{n_b}} \left( \frac{\delta f}{\delta \phi} \right) \right) \nu.$$  

The constrained system of equations, for the case of positive or negative buoyancy are therefore obtained as

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{\rho_f} \frac{\delta H_g}{\delta \phi} = \nabla \Phi \cdot n,$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{\rho_f} \frac{\delta H_g}{\delta \phi} + \frac{\lambda}{\rho_f} \circ \frac{\delta \varphi(q)}{(\delta \Sigma)_{n_b}},$$

$$= \left( -\frac{1}{2} \nabla \Phi \cdot \nabla \Phi + \left( \nabla \Phi \cdot n \right)^2 - g_r z + \frac{f_\lambda}{\rho_f} \right)_{|_{\Sigma(t)}},$$

$$\frac{dg}{dt} = \frac{\delta H_g}{\delta P_S} = V_S,$$

$$\frac{dP_S}{dt} = -\frac{\delta H_g}{\delta g} - \int_{\Sigma} \lambda \circ \frac{\delta \varphi(q)}{\delta g} \nu$$

$$= \left( \int_{\Sigma} f_\lambda \nu + \rho_b g_r V_B, \int_{\Sigma} f_\lambda (l \times n) \nu \right),$$

with $f_\lambda$ again given by (26). In this case, the pressure field $p$ includes the hydrostatic contribution. Finally, as in the neutrally buoyant case, the $\phi$ and $P_S$ evolution
equations can be combined to give
\[
\frac{d}{dt}\left( L_s - \rho_f \int_\Sigma \Phi n \nu \right) = g_r \left( \rho_b - \rho_f \right) V_B, 
\tag{34}
\]
\[
\frac{d}{dt}\left( A_s - \rho_f \int_\Sigma \Phi l \times n \nu \right) = 0. 
\tag{35}
\]

5. **Summary and future directions.** The purpose of this paper is to present a Hamiltonian constraint framework for the system of equations (6)—and its extension to the case of non-zero buoyancy—which is an alternative way of writing the equations governing the problem of the coupled dynamics of a rigid body and an ideal fluid. In contrast with Kirchhoff’s equations for this problem, the pressure appears explicitly in these equations. A Hamiltonian constraint formalism is used, the constraint being imposed on the shape of a compact fluid surface evolving as per Zakharov dynamics, and the pressure is shown to be a Lagrange multiplier.

It would be interesting to see if this formalism allows one to make any connections between the dynamics of a rigid body in Kirchhoff’s problem—or of a deformable body with the constraints slightly relaxed—and the dynamics of free surfaces. The latter topic, as is well-known, has led to the development of models like the KdV [10, 13] and Camassa-Holm equations [6] which have been extensively studied for their integrability and their soliton (and peakon) solutions. Are there analogues of such solutions in Kirchhoff’s problem? Linearizing system (25) following Zakharov’s approach [29] may provide the answer to this question.

A few other intriguing issues present themselves. The associated Laplace equation in the free surface problem and in Kirchhoff’s problem have different boundary conditions—in the former they are Dirichlet but in the latter they are Neumann. Kirchhoff’s equations do not contain surface tension and it is not clear to the author how to include the surface tension term in Zakharov’s equations in this formalism. A more detailed implementation of Dirac’s theory of constraints may reveal if system (25) itself has a Poisson structure and, if yes, its relation to the Poisson structure of Kirchhoff’s equations.

Finally, as stated in the Introduction, equations in which the pressure appears explicitly are the appropriate way of formulating extensions of the basic Kirchhoff problem in which the SE(3) symmetry is broken and global momentum conservation cannot be invoked. The formulation suggested in this paper may prove useful in such problems.

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**Appendix.** In this appendix, calculations of the terms \( \delta H / (\delta \Sigma)_n, \delta H / \delta \phi, \delta H / \delta g \) and \( \delta H / \delta P_S \) in system (25) are presented. The following notation is introduced.

\[
\delta_A H := \lim_{\epsilon \to 0} \frac{\left[ \left( H(q + \epsilon \delta A, \mu) - H(q, \mu) \right) \right]}{\epsilon}
\]

OR

\[
\delta_A H := \lim_{\epsilon \to 0} \frac{\left[ \left( H(q, \mu + \epsilon \delta A) - H(q, \mu) \right) \right]}{\epsilon}
\]

where \( H \) is given by (20) and \( A \) denotes \( \Sigma, \phi, g \) or \( P_S \).

\[\text{Typically, in applications to water wave problems, the free surface problem also includes bottom or/and side fixed boundaries on which Neumann conditions apply [13].}\]
Variations in the rigid body kinetic energy term due to variations in $g$ and $P_S$ are standard results in variational mechanics. Making the identifications $T \in \mathbb{R}^6$, $T_g \in \mathbb{R}^6$, $T_g \in \mathbb{R}^6$.

\[
\delta_g H = \delta_g \frac{1}{2} \left\langle P_S, M^{-1} \cdot P_S \right\rangle_{\mathbb{R}^6} = 0,
\]

\[
\delta_P S H = \delta_P S \frac{1}{2} \left\langle P_S, M^{-1} \cdot P_S \right\rangle = \left\langle \delta P S, M^{-1} \cdot P S \right\rangle_{\mathbb{R}^6} = V_S.
\]

Next, variations in $\phi$ are easily shown to give

\[
\delta \phi := -\frac{1}{2} \int_{\Sigma} \delta \Phi \nabla \Phi \cdot n \nu - \frac{1}{2} \int_{\Sigma} \Phi \nabla (\delta \Phi) \cdot n \nu
\]

\[
= \int_{\Sigma} \delta \Phi (\nabla \Phi \cdot n) .
\]

Finally, variations in $\Sigma$ are considered.

\[
\delta \Sigma H = -\delta \Sigma \frac{1}{2} \int_{\Sigma} \Phi \nabla \Phi \cdot n \nu,
\]

\[
= -\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left[ \int_{\Sigma^+ + \epsilon(\delta \Sigma)_{\nu}} \left( (\Phi + \epsilon \delta \Phi) \nabla (\Phi + \epsilon \delta \Phi) \right) \cdot n \nu + \int_{\Sigma^+ + \epsilon(\delta \Sigma)_{\nu}} \Phi \nabla \Phi \cdot n \nu \right],
\]

\[
= -\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left[ \int_{\Sigma^+ + \epsilon(\delta \Sigma)_{\nu}} (\epsilon \delta \Phi \nabla \Phi \cdot n + \Phi \nabla (\epsilon \delta \Phi) \cdot n) \nu \right],
\]

where $V_S \subset \mathbb{R}^3$ is the infinitesimal volume, i.e. $O(\epsilon)$ volume, between $\Sigma$ and $\Sigma + \epsilon(\delta \Sigma)_{\nu}$—see Figure 4. In the above, Stokes theorem has been used to obtain the $V_S$ integral. And since this applies to oriented manifolds, the minus sign in the previous line accounts for the opposite orientations of $\Sigma$ and $\Sigma + \epsilon(\delta \Sigma)_{\nu}$ viewed as the boundaries of $V_S$. In addition, the $V_S$ integral will have opposite signs in regions $A$ and $B$ shown in Figure 4, due to the fact that the orientation of each surface switches in these two regions. Specifically, the positive sign is for region $B$ and the negative sign is for region $A$.

Using Stokes theorem again on the other boundary integral gives

\[
\int_{\Sigma + \epsilon(\delta \Sigma)_{\nu}} (\epsilon \delta \Phi \nabla \Phi \cdot n + \Phi \nabla (\epsilon \delta \Phi) \cdot n) \nu,
\]

\[
= \int_{\Sigma} (\epsilon \delta \Phi \nabla \Phi \cdot n + \Phi \nabla (\epsilon \delta \Phi) \cdot n) \nu + \int_{V_S} (\epsilon \nabla (\delta \Phi) \cdot \nabla \Phi + \nabla \Phi \cdot \nabla (\epsilon \delta \Phi)) \mu.
\]

Note that the boundary of $V_S$ is piecewise smooth because the old and new boundaries can intersect but these intersections have measure zero in the boundary and so Stokes’ theorem for piecewise smooth manifolds still applies [1]. The $V_S$ integral in this case can be ignored since it is $O(\epsilon^2)$, and so

\[
\delta \Sigma H = -\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left[ \pm \int_{V_S} \nabla \Phi \cdot \nabla \Phi \mu + \int_{\Sigma} (\epsilon \delta \Phi \nabla \Phi \cdot n + \Phi \nabla (\epsilon \delta \Phi) \cdot n) \nu \right].
\]
Figure 4. Schematic sketch showing the fluid surface $\Sigma$ and the perturbed fluid surface $\Sigma + \epsilon(\delta\Sigma)_n$. Elements of the unit normal vector field $n$ on $\Sigma$ and the perturbation vector field $\epsilon(\delta\Sigma)_n$ are also shown.

Returning to the $V_\Sigma$ integral, there exists a local set of orthogonal coordinates at each point of $V_\Sigma$ such that
\[
\mu = \nu_\epsilon \wedge dy.
\]
Such a set of orthogonal, and orientation-preserving, coordinates for $V_\Sigma$ can be constructed, for example, by drawing line segments normal to $\Sigma$ and ending on $\Sigma + \epsilon\delta\Sigma$, and surfaces orthogonal to these straight lines. In the above, $y$ represents direction perpendicular to $\Sigma$, and $\nu_\epsilon$ at any point is volume form on the coordinate surfaces. It is not hard to see that $\nu_\epsilon$ differs by $O(\epsilon)$ from the volume form $\nu$ on $\Sigma$ at the corresponding ‘base’ point and, for $\epsilon$ sufficiently small, every point in $V_\Sigma$ lies on just one of these line segments. Without loss of generality, positive $y$ is taken to point in the same direction as positive $n$.

With these estimates, the integral can be written as
\[
+ \int_{V_\Sigma} \nabla \Phi \cdot \nabla \Phi \mu = \int_0^{\epsilon(\delta\Sigma)_n} \int_\Sigma \nabla \Phi \cdot \nabla \Phi \nu \, dy + O(\epsilon^2)
\]
in region $B$ where $(\delta\Sigma)_n \cdot n > 0$, and becomes
\[
- \int_{V_\Sigma} \nabla \Phi \cdot \nabla \Phi \mu = - \int_{\epsilon(\delta\Sigma)_n}^0 \int_\Sigma \nabla \Phi \cdot \nabla \Phi \nu \, dy + O(\epsilon^2)
\]
in region $A$, where $(\delta\Sigma)_n \cdot n < 0$. And one obtains
\[
\pm \int_{V_\Sigma} \nabla \Phi \cdot \nabla \Phi \mu = \epsilon(\delta\Sigma)_n \cdot n \int_\Sigma \nabla \Phi \cdot \nabla \Phi \nu + O(\epsilon^2).
\]
Invoking (18), finally obtain
\[
\delta \Sigma H = - \left[ \frac{1}{2} \int_\Sigma \nabla \Phi \cdot \nabla \Phi (\delta\Sigma)_n \cdot n \, \nu + \int_\Sigma \delta \Phi \nabla \Phi \cdot n \, \nu \right],
\]
\[
= - \left[ \frac{1}{2} \int_\Sigma \nabla \Phi \cdot \nabla \Phi (\delta\Sigma)_n \cdot n \, \nu + \int_\Sigma - (\nabla \Phi \cdot (\delta\Sigma)_n) \nabla \Phi \cdot n \, \nu \right],
\]
\[
= \int_\Sigma \left( - \frac{1}{2} \nabla \Phi \cdot \nabla \Phi + (\nabla \Phi \cdot n)^2 \right) (\delta\Sigma)_n \cdot n \, \nu.
\]
To summarize, the variational derivative terms are obtained as
\[
\frac{\delta H}{\delta \phi} = -\nabla \Phi \cdot n, \\
\frac{\delta H}{\delta \Sigma}_n = -\frac{1}{2} \nabla \Phi \cdot \nabla \Phi + (\nabla \Phi \cdot n)^2, \\
\frac{\delta H}{\delta P_S} = V_S, \\
\frac{\delta H}{\delta g} = 0.
\]
Strictly speaking, as per the general definitions of the variational derivatives introduced earlier, the first derivative should be written as a normal vector field and the second as normal 1-form. However, in the final form of the equations of motion these qualifications are unnecessary.

REFERENCES

[1] R. Abraham, J. E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, volume 75 in series Applied Mathematical Sciences, 2nd edition, Springer-Verlag, New York, 1988.
[2] H. Aref and S. W. Jones, Chaotic motion of a solid through ideal fluid, *Phys. Fluids A*, 5 (1993), 3026–3028.
[3] V. I. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, volume 125 of series Applied Mathematical Sciences, Springer-Verlag, 1998.
[4] T. B. Benjamin, Hamiltonian theory for motions of bubbles in an infinite liquid, *J. Fluid Mech.*, 181 (1987), 349–379.
[5] A. V. Borisov, I. S. Mamaev and S. M. Ramodanov, Motion of a circular cylinder and n point vortices in a perfect fluid, *Reg. Chaotic Dyn.*, 8 (2003), 449–462.
[6] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (1993), 1661–1664.
[7] E. F. G. van Daalen, E. van Groesen and P. J. Zandbergen, A Hamiltonian formulation for nonlinear wave-body interactions, in *Proceedings of the Eighth International Workshop on Water Waves and Floating Bodies*, 23-26 May 1993, St John's, Newfoundland, Canada, 159–163. Available online from http://www.iwwwfb.org/Abstracts/iwwwfb08/iwwwfb08-41.pdf.
[8] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Second printing of the 1964 original. Belfer Graduate School of Science Monographs Series, 2. Belfer Graduate School of Science, New York; produced and distributed by Academic Press, Inc., New York, 1967.
[9] P. Ehrenfest, *Die Bewegung starrer Körper in Flüssigkeiten und die Mechanik von Hertz*, PhD Thesis, University of Vienna, 1904.
[10] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Classics in Mathematics, Springer, 2007.
[11] A. Galper and T. Miloh, Generalized Kirchhoff equations for a deformable body moving in a weakly non-uniform flow field, *Proc. Roy. Soc. Lond. A*, 446 (1994), 169–193.
[12] A. Galper and T. Miloh, Dynamic equations of motion for a rigid or deformable body in an arbitrary non-uniform potential flow field, *J. Fluid. Mech.*, 295 (1995), 91–120.
[13] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1997.
[14] G. Kirchhoff, Ueber die Bewegung eines Rotationskörpers in einer Flüssigkeit, *Journal für die reine und angewandte Mathematik (Crelle’s Journal)*, 1870 (1870), 237–262.
[15] J. Koiller, Note on coupled motions of vortices and rigid bodies, *Physics Letters A*, 120 (1987), 391–395.
[16] V. V. Kozlov and D. A. Oniščenko, Nonintegrability of Kirchhoff’s equations, *Soviet Math. Dokl.*, 26 (1982), 495–502.
[17] L. Landweber and C. S. Yih, Forces, moments, and added masses for Rankine bodies, *J. Fluid Mech.*, 1 (1956), 319–336.
[18] N. E. Leonard, Stability of a bottom-heavy underwater vehicle, *Automatica*, 33 (1997), 331–346.

[19] D. Lewis, J. Marsden, R. Montgomery and T. Ratiu, The Hamiltonian structure for dynamic free boundary problems, *Physica D*, 18 (1986), 391–404.

[20] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, volume 17 of series Texts in Applied Mathematics, 2nd edition, Springer-Verlag, 1999.

[21] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, 5th edition, Dover, New York, 1996.

[22] S. P. Novikov, Variational methods and periodic solutions of equations of Kirchhoff-type. II, *Funktional Anal. i Prilozhen.*, 15 (1981), 37–52, Available online from http://www.mi.ras.ru/ snovikov/70.pdf.

[23] S. P. Novikov and I. Shmel’ter, Periodic solutions of the Kirchhoff equations for the free motion of a rigid body in a fluid and the extended Lyusternik-Shnirel’man-Morse theory. I, *Funktional Anal. i Prilozhen.*, 15 (1981), 54–66, Available online from http://www.mi.ras.ru/ snovikov/69.pdf.

[24] B. N. Shashikanth, Poisson brackets for the dynamically interacting system of a 2D rigid boundary and N point vortices: The case of arbitrary smooth cylinder shapes, *Reg. Chaotic Dyn.*, 10 (2005), 1–14.

[25] B. N. Shashikanth, J. E. Marsden, J. W. Burdick and S. D. Kelly, The Hamiltonian structure of a 2-D rigid cylinder interacting dynamically with N point vortices, *Phys. Fluids*, 14 (2002), 1214–1227.

[26] B. N. Shashikanth, A. Sheshmani, S. D. Kelly and J. E. Marsden, Hamiltonian structure for a neutrally buoyant rigid body interacting with N vortex rings of arbitrary shape: the case of arbitrary smooth body shape, *Theoretical and Computational Fluid Dynamics*, 22 (2008), 37–64.

[27] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.*, 9 (1968), 190–194. Originally published in *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi,* 9 (1968), 86–94.

**Table of symbols.** Note that the symbols $g$ and $\mu$ have multiple usage in the text, the particular usage should be clear from the context.

| $D_0 \subset \mathbb{R}^3$ | domain of fluid in the reference configuration | |
| $\Sigma_0$ | boundary of $D_0$, a smooth, compact 2D manifold | |
| $D_g \subset \mathbb{R}^3$ | domain of fluid in the current configuration | |
| $\Sigma$ | boundary of $D_g$, a smooth, compact 2D manifold, and also image of map $\Sigma_m$ (below) | |
| $\partial B$ | a smooth, compact, boundaryless 2D manifold representing the boundary of a rigid body | |
| $\partial B_0$ | the boundary of the rigid body (embedded in $\mathbb{R}^3$) in the reference configuration | |
| $\partial B_g$ | the boundary of the rigid body (embedded in $\mathbb{R}^3$) in the current configuration, and also image of map $\Phi_g$ (below) restricted to $\partial B_0$ | |
| $\partial B_0^\perp$ | normal bundle of $\partial B_0$ | |
| $\partial B_g^\perp$ | normal bundle of $\partial B_g$ | |
| $N_0, N$ | normal displacement vector fields on $\partial B_0$ and $\partial B_g$, respectively | |
| $x_b, x_f$ | points on $\partial B_0$ and $\Sigma_0$, respectively | |
| Symbol | Definition |
|--------|------------|
| $p_0, p_f$ | points on $\partial B_g$ and $\Sigma$, respectively |
| $\Sigma_m$ | $\partial B_0 \rightarrow \mathbb{R}^3$ |
| $n, n^lat$ | inward-pointing unit normal vector field and corresponding normal 1-form field (w.r.t. $\mathbb{R}^3$ metric) on $\Sigma$ |
| $n_0, n_0^\flat$ | outward-pointing unit normal vector field and corresponding normal 1-form field (w.r.t. $\mathbb{R}^3$ metric) on $\partial B_g$ (and $\partial B_0$) |
| $\text{SE}(3)$ | special Euclidean group of rigid body translations and rotations in $\mathbb{R}^3$ |
| $g$ | (a) element of $\text{SE}(3)$, (b) a function appearing in the definition of $G_Z$ (see below) |
| $P_g \equiv (L_s, A_s)$ | element of $T^*_g \text{SE}(3)$ |
| $M$ | mass matrix of rigid body |
| $L_s$ | linear momentum of rigid body in spatially-fixed frame, |
| $A_s$ | angular momentum of rigid body in spatially-fixed frame, |
| $\mathfrak{L}_s$ | linear momentum of rigid body+fluid system in spatially-fixed frame |
| $\mathfrak{A}_s$ | angular momentum of rigid body+fluid system in spatially-fixed frame |
| $\mathfrak{L}$ | linear momentum of rigid body+fluid system in body-fixed frame |
| $\mathfrak{A}$ | angular momentum of rigid body+fluid system in body-fixed frame |
| $\Phi_g$ | $: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for each $g \in \text{SE}(3)$ |
| $(\delta g)_\mathbb{R}^2, (\delta g)_\mathbb{R}^3 \equiv V_\delta + \Omega_\delta \times l$, vector field in $\mathbb{R}^2, \mathbb{R}^3$ identified with | the infinitesimal generator of the $\text{SE}(3)$ action on $\mathbb{R}^2, \mathbb{R}^3$ |
| $l$ or $\mathbf{l}$ | position vector w. r. t. center of mass (centroid) of rigid body |
| $\mathbf{V}, \Omega$ | the rigid body’s translational and rotational velocities in a body-fixed frame |
| $\mathfrak{V}_g \equiv (V_g, \Omega_g)$ | the rigid body’s translational and rotational velocities in a spatially-fixed frame |
| $V_\delta, \Omega_\delta$ | vectors representing variations in $V_g, \Omega_g$, respectively |
| $a_0$ | $: \partial B_0 \rightarrow \mathbb{R}$ |
| $a$ | $: \partial B_g \rightarrow \mathbb{R}$ |
| $M_\Sigma$ | manifold representing space of all $\Sigma$ |
| $m$ | element of $M_\Sigma$ |
| $T_m M_\Sigma$ | tangent space of $M_\Sigma$ at point $m$ |
| $Y$ | element of $T_m M_\Sigma$ |
| $T^*_m M_\Sigma$ | cotangent space of $M_\Sigma$ at point $m$ |
\[ \beta_{\Sigma} \text{ element of } T^*_m M_{\Sigma} \]

\[ v_{\Sigma} \text{ vector field based on } \Sigma, \text{ identified with } Y \]

\[ u^1_{\Sigma} \text{ 1-form based on } \Sigma, \text{ identified with } \beta \]

\[ b : \Sigma \to \mathbb{R} \]

\[ \phi : \Sigma_0 \to \mathbb{R}, \text{ pull-back of } b \text{ under the map } \Sigma_m \]

\[ \Phi : D_g \to \mathbb{R}, \Phi \text{ harmonic, solution to a Neumann or a Dirichlet problem satisfying appropriate decay conditions at infinity} \]

\[ v_q \text{ incompressible and irrotational fluid velocity field in } D_g \]

\[ F, G : T^* M_{\Sigma} \to \mathbb{R}, \quad F = \int_{\Sigma} f \nu, \text{ where } f \text{ is a real-valued function on the space of 1-forms based on } \Sigma, \text{ similarly } G \]

\[ F_Z, G_Z : T^* M_{\Sigma} \to \mathbb{R}, \quad F_Z = \int_{\Sigma} f \nu, \text{ where } f \text{ is a real-valued function on the space of normal 1-forms based on } \Sigma, \text{ similarly } G_Z \]

\[ Q := M_{\Sigma} \times \mathbb{R} \]

\[ q \equiv (m, g), \text{ element of } Q \]

\[ \mu \equiv (\beta, P_3), \text{ element of } T^*_q Q, \text{ (b) measure of integration in } \mathbb{R}^3 \]

\[ (\delta \Sigma)_n \text{ variation of } \Sigma \text{ in normal directions} \]

\[ F : T^* Q \to \mathbb{R} \]

\[ (\cdot, \cdot) \text{ } L^2\text{-pairing between spaces of 1-forms and vector fields based on } \Sigma \text{ or } \partial B_g \]

\[ E, V \quad \pi : E \to Q \text{ vector bundle over } Q \text{ with fiber } V, \text{ the vector space of all sections of the normal bundle } \partial B_g^\perp \]

\[ V^* \text{ vector space dual to } V \text{ via the } L^2\text{-pairing, identified with the space of normal 1-forms based on } \partial B_g \]

\[ \varphi : Q \to E, \text{ a section of the bundle } \pi : E \to Q \]

\[ \varphi(q) \text{ a uniquely defined section of } (\partial B_g)^\perp \text{ at each } q \]

\[ \alpha \text{ element of } V^*, \text{ identified with the Lagrange multiplier} \]

\[ \lambda := f_\lambda u^1_{\hat{\lambda}}, \text{ normal 1-form based on } \partial B_g, \text{ identified with } \alpha \]

\[ f_\lambda : \partial B_g \to \mathbb{R} \]

\[ H : T^* Q \to \mathbb{R}, \text{ Hamiltonian function of the unconstrained problem} \]

\[ H_c : T^* E \to \mathbb{R}, \text{ Hamiltonian function of the constrained problem (neutrally buoyant case)} \]

\[ H_{c,q} : T^* E \to \mathbb{R}, \text{ Hamiltonian function of the constrained problem} \]

\[ (\text{positive or negative buoyancy case}) \]

\[ \rho_b \text{ uniform density of rigid body} \]
\begin{tabular}{|l|l|}
\hline
$\rho_f$ & uniform density of fluid \\
$\mathcal{V}_B$ & volume of rigid body \\
$p$ & fluid pressure field \\
$\nu$ & volume form on $\Sigma$ \\
$\nu_\epsilon$ & volume form on the perturbed fluid boundary $\Sigma + \epsilon(\delta \Sigma)_n$ \\
$\bar{\nu}$ & volume form on $\partial B_g$ \\
$\langle\langle , \rangle\rangle_{\mathbb{R}^6}$ & $\mathbb{R}^6$ pairing between elements in $T_g \text{SE}(3) \equiv \mathbb{R}^3 \times \mathbb{R}^3$ and elements in $T^*_g \text{SE}(3) \equiv \mathbb{R}^3 \times \mathbb{R}^3$ \\
\hline
\end{tabular}

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