Lindeberg’s central limit theorems for martingale like sequences under sub-linear expectations*

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Abstract

The central limit theorem of martingales is the fundamental tool for studying the convergence of stochastic processes, especially stochastic integrals and differential equations. In this paper, the central limit theorem and functional central limit theorem are obtained for martingale like random variables under the sub-linear expectation. As applications, the Lindeberg central limit theorem is obtained for independent but not necessarily identically distributed random variables, and a new proof of the Lévy characterization of a G-Brownian motion without using stochastic calculus is given. For proving the results, Rosenthal’s inequality and the exponential inequality for the martingale like random variables are established.

Keywords: capacity; central limit theorem; functional central limit theorem; martingale difference; sub-linear expectation.

AMS 2010 subject classifications: 60F05, 62F17; secondary 60G48, 60H05.

1 Introduction and notations.

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, cf. Denis and Martini (2006), Gilboa (1987), Marinacci (1999), Peng (1997, 1999, 2007a, 2007c, 2008a) etc. Peng (2007a) introduced the notion of the sub-linear expectation. Under the sub-linear expectation, Peng (2007a, 2007b, 2007c, 2008a, 2008b, 2009) gave the notions of the G-normal distributions, G-Brownian motions, G-martingales, independence of random variables, identical distribution of random variables and so on, and developed the weak law of large numbers and central limit theorem for independent and identically

*Research supported by grants from the NSF of China (Grant No. 11731012), Ten Thousands Talents Plan of Zhejiang Province (Grant No. 2018R52042), the Fundamental Research Funds for the Central Universities and the 973 Program (No. 2015CB352302).
distributed (i.i.d.) random variables. Furthermore, Peng established the stochastic calculus with respect to the G-Brownian motion. As a result, Peng’s framework of nonlinear expectation gives a generalization of Kolmogorov’s probability theory. Recently, Bayraktar and Munk (2016) proved an $\alpha$-stable central limit theorem for independent and identically distributed random variables. This paper considers the general central limit theorem for random variables which are not necessarily i.i.d. under the sub-linear expectation. We establish a central limit theorem and a functional central limit theorem under the conditional Lindeberg condition for a kind of martingale-difference like random variables. As applications, the central limit theorem for independent but not necessary identically distributed under the popular Lindeberg’s condition is obtained. The tool for proving the central limit theorem is a promotion of Peng (2008b)’s and gives also a new normal approximation method for classical martingale differences instead of the characteristic function. For proving the functional central limit theorem, we also establish the Rosenthal’s inequalities for the martingale like random variables. As the central limit theorem of classical martingales which is the fundamental tool for studying the convergence of stochastic processes under the framework of the probability and linear expectation, especially stochastic integrals and differential equations (cf. Jacod and Shiryaev, 2003), the (functional) central limit theorem of martingale-difference like random variables under the sub-linear expectation will provide a way to study the weak convergence of stochastic integrals and difference equations with respect to the G-Brownian motion.

In the rest of this section, we state some notations about sub-linear expectations. The main results on the central limit theorem and functional central limit theorem are stated in Sections 2 and 3 with the proofs given the last section. In Section 4, we will establish the Rosenthal-type inequalities and an exponential inequality for the maximal sums of the martingale-difference like random variables. In Section 5, we consider the Lévy characterization of a G-Brownian motion in a general sub-linear expectation space. The Lévy characterization of a $G$-Brownian motion under G-expectation in a Wiener space is established by Xu and Zhang (2009, 2010) and extended by Lin (2013) by the method of the stochastic calculus. We will give an elementary proof without using stochastic calculus. We will find that the functional central limit theorem gives a new way to show the Lévy characterization.
We use the framework and notations of Peng (2008b). Let \((\Omega, \mathcal{F})\) be a given measurable space and let \(\mathcal{H}\) be a linear space of real functions defined on \((\Omega, \mathcal{F})\) such that if \(X_1, \ldots, X_n \in \mathcal{H}\) then \(\varphi(X_1, \ldots, X_n) \in \mathcal{H}\) for each \(\varphi \in C_{l,\text{Lip}}(\mathbb{R}_n)\), where \(C_{l,\text{Lip}}(\mathbb{R}_n)\) denotes the linear space of (local Lipschitz) functions \(\varphi\) satisfying

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}_n,
\]

for some \(C > 0, m \in \mathbb{N}\) depending on \(\varphi\).

\(\mathcal{H}\) is considered as a space of "random variables". In this case, we denote \(X \in \mathcal{H}\). We also denote the space of bounded Lipschitz functions and the space of bounded continuous functions on \(\mathbb{R}_n\) by \(C_{b,\text{Lip}}(\mathbb{R}_n)\) and \(C_b(\mathbb{R}_n)\), respectively.

**Definition 1.1** A sub-linear expectation \(\hat{E}\) on \(\mathcal{H}\) is a function \(\hat{E}: \mathcal{H} \to \mathbb{R}\) satisfying the following properties: for all \(X, Y \in \mathcal{H}\),

1. **Monotonicity:** If \(X \geq Y\) then \(\hat{E}[X] \geq \hat{E}[Y]\);
2. **Constant preserving:** \(\hat{E}[c] = c\);
3. **Sub-additivity:** \(\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]\) whenever \(\hat{E}[X] + \hat{E}[Y]\) is not of the form \(+\infty - \infty\) or \(-\infty + \infty\);
4. **Positive homogeneity:** \(\hat{E}[\lambda X] = \lambda \hat{E}[X], \ \lambda \geq 0\).

Here \(\mathbb{R} = [-\infty, \infty]\). The triple \((\Omega, \mathcal{H}, \hat{E})\) is called a sub-linear expectation space. Give a sub-linear expectation \(\hat{E}\), let us denote the conjugate expectation \(\hat{E}\) of \(\hat{E}\) by \(\hat{E}[X] := -\hat{E}[-X], \ \forall X \in \mathcal{H}\).

A sub-linear expectation \(\hat{E}\) is countably sub-additive, if

\[
\hat{E}\left(\sum_{i=1}^{\infty} X_i\right) \leq \sum_{i=1}^{\infty} \hat{E}[X_i], \text{ for all random variables } X_i \geq 0.
\]

If \(X\) is not in \(\mathcal{H}\), we define its sub-linear expectation by \(\hat{E}^*[X] = \inf\{\hat{E}[Y] : X \leq Y \in \mathcal{H}\}\). When there is no ambiguity, we also denote it by \(\hat{E}\). From the definition, it is easily shown that \(\hat{E}[X] \leq \hat{E}[X], \ \hat{E}[X + c] = \hat{E}[X] + c\) and \(\hat{E}[X - Y] \geq \hat{E}[X] - \hat{E}[Y]\) for all \(X, Y \in \mathcal{H}\) with \(\hat{E}[Y]\) being finite. Further, if \(\hat{E}[|X|]\) is finite, then \(\hat{E}[X]\) and \(\hat{E}[X]\) are both finite.
Definition 1.2 (Peng (2007a, 2008b))

(i) (Identical distribution) Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\mathbb{E}$, if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ we have $\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)|x=X]]$, whenever $\overline{\varphi}(x) := \mathbb{E}[|\varphi(x,Y)|] < \infty$ for all $x$ and $\mathbb{E}[|\overline{\varphi}(X)|] < \infty$.

Random variables $X_1, \ldots, X_n$ are said to be independent if for each $2 \leq k \leq n$, $X_k$ is independent to $(X_1, \ldots, X_{k-1})$. A sequence of random variables is said to be independent if for each $n$, $X_1, \ldots, X_n$ are independent.

Next, we introduce the capacities corresponding to the sub-linear expectation. We denote the pair $(\mathbb{V}, \mathbb{V})$ of capacities on $(\Omega, \mathcal{H}, \mathbb{E})$ by setting

$$\mathbb{V}(A) := \inf \{ \hat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H} \}, \quad \mathbb{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where $A^c$ is the complement set of $A$. Then it is obvious that $\mathbb{V}$ is sub-additive, i.e. $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathbb{V}$ and $\hat{\mathbb{E}}$ are not. However, we have

$$\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$$

due to the fact that $\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A \setminus B) \geq \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\hat{\mathbb{E}}[-X-Y] \geq \hat{\mathbb{E}}[-X] - \hat{\mathbb{E}}[Y]$.

The Choquet integrals/expecations of $(C_\mathbb{V}, \mathbb{V})$ are defined by

$$C_\mathbb{V}[X] = \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt$$

with $V$ being replaced by $\mathbb{V}$ and $\mathbb{V}$, respectively.

Finally, we recall the notations of G-normal distribution and G-Brownian motion which are introduced by Peng (2008b, 2010).
Definition 1.3 (G-normal random variable) For $0 \leq \sigma^2 \leq \sigma^2 < \infty$, a random variable $\xi$ in a sub-linear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ is called a normal $N(0, [\sigma^2, \sigma^2])$ distributed random variable (written as $\xi \sim N(0, [\sigma^2, \sigma^2])$ under $\tilde{\mathbb{E}}$), if for any $\varphi \in C_{1,lip}(\mathbb{R})$, the function $u(x,t) = \tilde{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)] \ (x \in \mathbb{R}, t \geq 0)$ is the unique viscosity solution of the following heat equation:

$$\partial_t u - G(\partial_{xx} u) = 0, \ u(0, x) = \varphi(x),$$

where $G(\alpha) = \frac{1}{2}(\sigma^2 \alpha^+ - \sigma^2 \alpha^-)$.

That $\xi$ is a normal distributed random variable is equivalent to that, if $\xi'$ is an independent copy of $\xi$, then

$$\tilde{\mathbb{E}}[\varphi(\alpha \xi + \beta \xi')] = \tilde{\mathbb{E}}[\varphi(\sqrt{\alpha^2 + \beta^2}X)], \ \forall \varphi \in C_{1,lip}(\mathbb{R}) \text{ and } \forall \alpha, \beta \geq 0,$n

(cf. Definition II.1.4 and Example II.1.13 of Peng (2010)).

Definition 1.4 (G-Brownian motion) A random process $(W_t)_{t \geq 0}$ in the sub-linear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ is called a G-Brownian motion (cf. Definition III.1.2 of Peng (2010)) if

(i) $W_0 = 0$;

(ii) For each $0 \leq t_1 \leq \cdots \leq t_d \leq t \leq s$,

$$\tilde{\mathbb{E}}[\varphi(W_{t_1}, \ldots, W_{t_d}, W_s - W_t)] = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[\varphi(x_1, \ldots, x_d, \sqrt{t - s})\xi] \mid x_1 = W_{t_1}, \ldots, x_d = W_{t_d}\right] \tag{1.1}$$

where $\xi \sim N(0, [\sigma^2, \sigma^2])$.

In some papers, for example, Xu and Zhang (2009, 2010), the test functions $\varphi$ are only required to be elements in $C_{b,lip}(\mathbb{R}_{d+1})$. It can be shown that if $\tilde{\mathbb{E}}[|W_t|^p] < \infty$ for all $p > 0$ and $t$, then that (1.1) holds for all $\varphi \in C_{b,lip}(\mathbb{R}_{d+1})$ is equivalent to that it holds for all $\varphi \in C_{1,lip}(\mathbb{R}_{d+1})$. Further, if the sub-linear expectation $\tilde{\mathbb{E}}$ is countably sub-additive, then
this two kinds of definitions are equivalent because, if $X$ is a random variable in $(\Omega, \mathcal{H}, \mathbb{E})$ such that
\[ \widehat{\mathbb{E}}[\varphi(X)] = \mathbb{E}[\varphi(\xi)], \quad \forall \varphi \in C_{b, Lip}(\mathbb{R}), \]
then $\mathbb{E}[|X|^p] < \infty$ for all $p > 0$. In fact, if $\xi \sim N(0, [\sigma^2, \pi^2])$ under $\mathbb{E}$, then (cf. Peng(2010, page 22))
\[ \mathbb{E}[|\xi|^p] = \sigma^p \int_{-\infty}^{\infty} |x|^p \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = c_0 \mathbb{E}^p, \quad \forall p \geq 1. \]
Now, for any $z > 0$, one can choose a function $\varphi \in C_{b, Lip}(\mathbb{R})$ such that $I\{x > z\} \leq \varphi(x) \leq I\{x > z - \epsilon\}$. From (1.2), it follows that
\[ \forall(|X| > z) \leq \widehat{\mathbb{E}}[\varphi(X)] = \mathbb{E}[\varphi(\xi)] \leq \mathbb{V}(\{|\xi| > z - \epsilon\}). \]
Hence
\[ \forall(|X| > z) \leq 2 \mathbb{V}(\{|x| \geq z/2\}) \leq \frac{2^p \mathbb{E}[|\xi|^{2p}]}{z^{2p}} = \frac{\sigma^{2p} c_{2p}}{z^{2p}}. \]
It follows that
\[ C_\forall(|X|^p) = \int_0^\infty \forall(|X|^p > z) dz \leq 1 + \int_1^\infty \frac{2^p \mathbb{E}[|\xi|^{2p}]}{z^{2p}} dz \leq 1 + \frac{\sigma^{2p} c_{2p}}{z^{2p}} < \infty, \quad \forall p \geq 2. \]
So, if $\mathbb{E}$ is countably sub-additive or $\mathbb{E}[|X|^p] = \lim_{c \to \infty} \mathbb{E}(|X|^c \land c)^p$, then $\mathbb{E}[|X|^p] \leq C_\forall(|X|^p) < \infty$ for all $p > 0$ by Lemma 3.9 of Zhang (2016).

Let $C_{[0,1]}$ be a function space of continuous real functions on $[0,1]$ equipped with the supremum norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$ and $C_b(C_{[0,1]})$ is the set of bounded continuous functions $h(x) : C_{[0,1]} \to \mathbb{R}$. As showed in Peng (2006, 2008a, 2010) and Denis, Hu, and Peng (2011), there is a sub-linear expectation space $(\widehat{\Omega}, \widehat{\mathcal{H}}, \widehat{\mathbb{E}})$ with $\widehat{\Omega} = C_{[0,1]}$ and $C_b(C_{[0,1]}) \subset \widehat{\mathcal{H}}$ such that $(\widehat{\mathcal{H}}, \widehat{\mathbb{E}}(\cdot \land \cdot))$ is a Banach space, and the canonical process $W(t)(\omega) = \omega_t(\omega \in \widehat{\Omega})$ is a G-Brownian motion. In the sequel of this paper, the G-normal random variables and G-Brownian motions are considered in $(\widehat{\Omega}, \widehat{\mathcal{H}}, \widehat{\mathbb{E}})$.

2 Lindebergs central limit theorem for independent random variables.

We write $\eta_n \xrightarrow{v} \eta$ if $\forall (|\eta_n - \eta| \geq \epsilon) \to 0$ for any $\epsilon > 0$, and write $\eta_n \xrightarrow{d} \eta$ if $\widehat{\mathbb{E}}[\varphi(\eta_n)] \to \widehat{\mathbb{E}}[\varphi(\eta)]$ holds for all bounded and continuous functions $\varphi$. In this section, we consider the
independent random variables \( \{X_{n,k}; k = 1, \ldots, k_n\} \). Denote \( \sigma^2_{n,k} = \hat{\mathbb{E}}[X_{n,k}^2], \sigma^2_{n,k} = \hat{\mathbb{E}}[X_{n,k}^2], \) \( B_n^2 = \sum_{k=1}^{k_n} \sigma^2_{n,k} \). We have the following Lindeberg’s central limit theorem.

**Theorem 2.1** Suppose that the Lindeberg condition is satisfied:

\[
\frac{1}{B_n^2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ (X_{n,k}^2 - \epsilon B_n^2)^+ \right] \to 0 \quad \forall \epsilon > 0, \tag{2.1}
\]

and further, there is a constant \( r \in [0, 1] \) such that

\[
\frac{\sum_{k=1}^{k_n} \left| r \sigma^2_{n,k} - \sigma^2_{n,k} \right|}{B_n^2} \to 0, \quad \text{also}, \tag{2.2}
\]

\[
\frac{\sum_{k=1}^{k_n} \left\{ \left| \hat{\mathbb{E}}[X_{n,k}] \right| + \left| \hat{\mathbb{E}}[X_{n,k}] \right| \right\}}{B_n} \to 0. \tag{2.3}
\]

Then for any bounded continuous function \( \varphi \),

\[
\lim_{n \to \infty} \hat{\mathbb{E}} \left[ \varphi \left( \frac{\sum_{k=1}^{k_n} X_{n,k}}{B_n} \right) \right] = \tilde{\mathbb{E}}[\varphi(\xi)], \tag{2.4}
\]

where \( \xi \sim N(0, [r, 1]) \) under \( \tilde{\mathbb{E}} \).

Theorem 2.1 will be a directly corollary of our Theorem 3.1 on the central limit theorem for martingale like sequence. The central limit theorem for independent and identically distributed random variables under the sub-linear expectation was obtained by Peng (2008b). Li and Shi (2010) generalized Peng’s result to a central limit theorem for independent random variables \( \{X_n; n \geq 1\} \) satisfying \( \hat{\mathbb{E}}[X_i] = \hat{\mathbb{E}}[X_i] = 0, \hat{\mathbb{E}}[|X_i|^3] \leq M < \infty, \) \( i = 1, 2, \ldots, \)

and

\[
\frac{1}{n} \sum_{i=1}^{n} \left| \hat{\mathbb{E}}[X_i^2] - \sigma^2 \right| \to 0, \quad \frac{1}{n} \sum_{i=1}^{n} \left| \hat{\mathbb{E}}[X_i^2] - \sigma^2 \right| \to 0.
\]

It is easily seen that the array \( \left\{ \frac{1}{\sqrt{n}} X_k; k = 1, \ldots, n \right\} \) satisfies the conditions (2.2) with \( r = \sigma^2 / \sigma^2 \), (2.3) and (2.1).

When \( \hat{\mathbb{E}} \) is a classical linear expectation, (2.2) is automatically satisfied with \( r = 1 \). It is easily seen that (2.2) implies

\[
\frac{\sum_{k=1}^{k_n} \sigma^2_{n,k}}{\sum_{k=1}^{k_n} \sigma^2_{n,k}} \to r. \tag{2.5}
\]

One may conjecture that (2.2) can be weakened to (2.5). The following example tells us that it is not the truth.
Example 2.1 Let $0 < \tau_1, \tau_2 < 1$, and $\{X_{n,k}; k = 1, \ldots, 2n\}$ be a sequence of independent normal random variables such that

$$X_{n,k} \overset{d}{\sim} N(0, [\tau_1, 1]), k = 1, \ldots, n \text{ and } X_{n,k} \overset{d}{\sim} N(0, [\tau_2, 1]), k = n + 1, \ldots, 2n.$$  

It is easily seen that $\{X_{n,k}; k = 1, \ldots, 2n\}$ satisfies the conditions (2.1), (2.3) and (2.5) with $r = (\tau_1 + \tau_2)/2$, and $B_n^2 = 2n$. It is obvious that

$$\sum_{k=1}^{2n} \frac{X_{n,k}}{\sqrt{n}} = \sum_{k=1}^{n} \frac{X_{n,k}}{\sqrt{n}} + \sum_{k=n+1}^{2n} \frac{X_{n,k}}{\sqrt{n}} \sim \xi + \eta,$$

where $\xi, \eta$ are independent normal random variables with $\xi \overset{d}{\sim} N(0, [\tau_1, 1]),$ $\eta \overset{d}{\sim} N(0, [\tau_2, 1])$. Song (2015) showed that $\xi + \eta$ is not $G$-normal distributed if $\tau_1 \neq \tau_2$, and hence (2.4) fails.

3 Central limit theorem for martingale like sequence.

In this section, we consider a general martingale. First, we recall the definition of the conditional expectation under the sub-linear expectation. Let $(\Omega, \mathcal{H}, \hat{E})$ be a sub-linear expectation space. We write $X \leq Y$ in $L_p$ if $\hat{E}[(|X - Y|)^p] = 0$, $X = Y$ in $L_p$ if both $X \leq Y$ and $Y \leq X$ holds in $L_p$.

Let $\mathcal{H}_{n,0} \subset \ldots \subset \mathcal{H}_{n,k_n}$ be subspaces of $\mathcal{H}$ such that

1. any constant $c \in \mathcal{H}_{n,k}$ and,
2. if $X_1, \ldots, X_d \in \mathcal{H}_{n,k}$, then $\varphi(X_1, \ldots, X_d) \in \mathcal{H}_{n,k}$ for any $\varphi \in C_{l,lip}(\mathbb{R}_d)$, $k = 0, \ldots, k_n$.

Denote $\mathcal{L}(\mathcal{H}) = \{X : \hat{E}[|X|] < \infty, X \in \mathcal{H}\}$. We consider a system of operators in $\mathcal{L}(\mathcal{H})$,

$$\hat{E}_{n,k} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_{n,k})$$

and denote $\hat{E}[X|\mathcal{H}_{n,k}] = \hat{E}_{n,k}[X]$, $\hat{E}[X|\mathcal{H}_{n,k}] = -\hat{E}_{n,k}[-X]$. $\hat{E}[X|\mathcal{H}_{n,k}]$ is called the conditional sub-linear expectation of $X$ given $\mathcal{H}_{n,k}$, $\hat{E}_{n,k}$ is called the conditional expectation operator. Suppose that the operators $\hat{E}_{n,k}$ satisfy the following properties: for all $X, Y \in \mathcal{L}(\mathcal{H})$,

(a) $\hat{E}_{n,k}[X+Y] = X + \hat{E}_{n,k}[Y]$ in $L_1$ if $X \in \mathcal{H}_{n,k}$, and $\hat{E}_{n,k}[XY] = X^+ \hat{E}_{n,k}[Y] + X^- \hat{E}_{n,k}[-Y]$ in $L_1$ if $X \in \mathcal{H}_{n,k}$ and $XY \in \mathcal{L}(\mathcal{H})$;

(b) $\hat{E}[\hat{E}_{n,k}[X]] = \hat{E}[X]$. 


It is easily seen that (a) implies that \( \hat{E}_{n,k}[c] = c \) in \( L_1 \) and \( \hat{E}_{n,k}[\lambda X] = \lambda \hat{E}_{n,k}[X] \) in \( L_1 \) if \( \lambda \geq 0 \). The definition of the conditional sub-linear expectation can be found in Peng (2010), Xu and Zhang (2009, 2010) with the operators satisfying (a), (b) and, \( \hat{E}_{n,k}[X] \leq \hat{E}_{n,k}[Y] \) if \( X \leq Y \), \( \hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y] \leq \hat{E}_{n,k}[X - Y] \), \( \hat{E}_{n,k} \left[ \left( \hat{E}_{n,l}[X] \right) \right] = \hat{E}_{n,l \wedge k}[X] \). It can be showed that these properties can be implied by (a) and (b) (c.f. Lemma 4.3).

Now, we assume that \( \{Z_{n,k}; k = 1, \ldots, k_n\} \) is an array of random variables such that \( Z_{n,k} \in \mathcal{H}_{n,k} \) and \( \hat{E}[Z_{n,k}^2] < \infty \), \( k = 1, \ldots, k_n \). The following is the central limit theorem.

**Theorem 3.1** Suppose that the operators \( \hat{E}_{n,k} \) satisfy (a) and (b). Assume that the following Lindeberg condition is satisfied:

\[
\sum_{k=1}^{k_n} \hat{E} \left[ \left( Z_{n,k}^2 - \varepsilon \right)^+ \right| \mathcal{H}_{n,k-1} \right] \xrightarrow{\mathcal{V}} 0 \quad \forall \varepsilon > 0, \tag{3.1}
\]

and further, there are constants \( \rho \geq 0 \) and \( r \in [0, 1] \) such that

\[
\sum_{k=1}^{k_n} \hat{E} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \xrightarrow{\mathcal{V}} \rho, \tag{3.2}
\]

\[
\sum_{k=1}^{k_n} [r \hat{E} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}] - \hat{E} [Z_{n,k}^2 | \mathcal{H}_{n,k-1}]] \xrightarrow{\mathcal{V}} 0, \tag{3.3}
\]

\[
\sum_{k=1}^{k_n} \{ |\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}]| + |\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}]| \} \xrightarrow{\mathcal{V}} 0. \tag{3.4}
\]

Then for any bounded continuous function \( \varphi \),

\[
\lim_{n \to \infty} \hat{E} \left[ \varphi \left( \sum_{k=1}^{k_n} Z_{n,k} \right) \right] = \hat{E}[\varphi(\sqrt{r}\xi)], \tag{3.5}
\]

i.e., \( \sum_{k=1}^{k_n} Z_{n,k} \xrightarrow{d} \sqrt{r}\xi \), where \( \xi \sim N(0, [r, 1]) \) under \( \hat{E} \).

**Remark 3.1** When \( \hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0 \) and \( \hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0 \), then \( \{Z_{n,k}; k = 1, \ldots, k_n\} \) is an array of symmetric martingale differences (c.f. Xu and Zhang (2009)). If \( \hat{E}[\cdot] = E_P[\cdot] \) is a classical linear expectation, then (3.3) is satisfied with \( r = 1 \), and the conclusion coincides with Corollary 3.1 of Hall and Heyde (1980).

The following is a direct corollary of Theorem 3.1.

**Corollary 3.1** Let \( \{\eta_n\} \) be a sequence of independent random variables on \((\Omega, \mathcal{H}, \hat{E})\) with \( \hat{E}[\eta_n] = \hat{E}[\eta_n^2] = 0, \hat{E}[\eta_n^2] =: \sigma_n^2 \to \sigma^2, \hat{E}[\eta_n^2] := \sigma_n^2 \to \sigma^2 \) and \( \sup_n \hat{E}[\eta_n^2] \to 0 \) as
$c \to \infty$. Suppose that $\{a_{n,i}; i = 1, \ldots, k_n\}$ is an array of real random variables in $\mathcal{H}$ with $a_{n,i}$ being a function of $\eta_1, \ldots, \eta_{i-1}$,

$$\max_i |a_{n,i}| \xrightarrow{V} 0 \quad \text{and} \quad \sum_{i=1}^{k_n} a_{n,i}^2 \xrightarrow{V} \rho,$$

where $\rho \geq 0$ is a constant. Then

$$\lim_{n \to \infty} \mathop{\hat{E}} \left[ \varphi \left( \sum_{i=1}^{k_n} a_{n,i} \eta_i \right) \right] = \mathop{\tilde{E}}[\varphi(\xi)],$$

(3.6)

for any bounded continuous function $\varphi$, where $\xi \sim N(0, [\rho \sigma^2, \rho \sigma^2])$ under $\mathop{\tilde{E}}$.

The following corollary is a central limit theorem for moving average processes which include the ARMA model.

**Corollary 3.2** Let $\{\eta_n\}$ be a sequence of independent and identically distributed random variables in $(\Omega, \mathcal{H}, \mathop{\hat{E}})$ with $\mathop{\hat{E}}[\eta_1] = \mathop{\hat{E}}[\eta_1^2] = \mathop{\hat{E}}[\eta_2^2] = \mathop{\hat{E}}[\eta_2^3] = \sigma^2$, $\{a_n; n \geq 0\}$ be a sequence of real numbers with $\sum_{i=0}^{\infty} |a_n| < \infty$. Let $X_k = \sum_{i=0}^{\infty} a_i \eta_i + k$. Then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow{d} N(0, [a^2 \sigma^2, a^2 \sigma^2]),$$

(3.7)

where $a = \sum_{j=0}^{\infty} a_j$.

**Proof.** Let $a_n = 0$ if $n < 0$. Then $X_k = \sum_{i=1}^{\infty} a_{i-k} \eta_i$ and

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k = \sum_{i=1}^{\infty} \left( \frac{\sum_{k=1}^{n} a_{i-k}}{\sqrt{n}} \right) \eta_i.$$

Let $a_{n,i} = \sum_{k=1}^{n} a_{i-k} \sqrt{n}$. Then $\max_i |a_{n,i}| \leq n^{-1/2} \sum_{i=1}^{\infty} |a_i| \to 0$ and $\sum_{i=1}^{\infty} a_{n,i}^2 \to a^2$. The result follows from Corollary 3.1. \square

Finally, we give the functional central limit theorems.

Let $D_{[0,1]}$ be the space of right continuous functions having finite left limits which is endowed with the Skorohod topology, $\tau_n(t)$ be a non-decreasing function in $D_{[0,1]}$ which takes integer values with $\tau_n(0) = 0$, $\tau_n(1) = k_n$. Define $S_{n,i} = \sum_{k=1}^{i} Z_{n,k}$,

$$W_n(t) = S_{n,\tau_n(t)},$$

(3.8)
Theorem 3.2 Suppose that the operators $\hat{E}_{n,k}$ satisfy (a) and (b). Assume that the conditions (3.1), (3.3) and (3.4) in Theorem 3.1 are satisfied. Further, there is a continuous non-decreasing non-random function $\rho(t)$ such that

$$
\sum_{k \leq \tau_n(t)} \hat{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \overset{\mathcal{V}}{\rightarrow} \rho(t), \ t \in [0,1].
$$

(3.9)

Then for any $0 = t_0 < \ldots < t_d \leq 1$,

$$
\left(W_n(t_1), \ldots, W_n(t_d)\right) \overset{d}{\rightarrow} \left(W(\rho(t_1)), \ldots, W(\rho(t_d))\right),
$$

(3.10)

and for any bounded continuous function $\varphi : D_{[0,1]} \rightarrow \mathbb{R}$,

$$
\lim_{n \rightarrow \infty} \hat{E}[\varphi(W_n)] = \tilde{E}[\varphi(W \circ \rho)],
$$

(3.11)

where $W$ is $G$-Brownian motion on $[0,1]$ with $W(1) \sim N(0,[r,1])$ under $\tilde{E}$, and $W \circ \rho(t) = W(\rho(t))$.

Because the proofs of Theorems 3.1 and 3.2 are a little long and need some preparation, we will give them in the last section.

4 Moment inequalities and exponential inequalities.

To prove the central limit theorems and functional central limit theorems, we need some inequalities on the sums of martingale-difference like random variables as basic tools. Before we give the inequalities, we state some properties of the sub-linear expectations $\hat{E}$ and $\hat{E}_{n,k}$. The first is Hölder’s inequality which is Proposition 16 of Denis, Hu, and Peng (2011).

Lemma 4.1 (Hölder’s inequality) Let $p,q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for two random variables $X,Y$ in $(\Omega, \mathcal{H}, \hat{E})$ we have

$$
\hat{E}[|XY|] \leq \left(\hat{E}[|X|^p]\right)^{\frac{1}{p}} \left(\hat{E}[|Y|^q]\right)^{\frac{1}{q}}
$$

whenever $\hat{E}[|X|^p] < \infty$, $\hat{E}[|Y|^q] < \infty$.

The next two lemmas are on the properties of the sub-linear expectation, the capacity and the operators $\hat{E}_{n,k}$. The proofs will be given in Appendix A. We write $X \leq Y$ in capacity $\mathcal{V}$ if $\mathcal{V}(X - Y \geq \epsilon) = 0$ for all $\epsilon > 0$, and $X = Y$ in capacity $\mathcal{V}$ if both $X \leq Y$ and $Y \leq X$ holds in $\mathcal{V}$.
Lemma 4.2 We have

(1) if $X \leq Y$ in $L_p$, then $X \leq Y$ in $\mathcal{V}$;

(2) if $X \leq Y$ in $\mathcal{V}$ and $\hat{E}[(X - Y)^+]^p < \infty$, then $X \leq Y$ in $L_q$ for $0 < q < p$;

(3) if $X \leq Y$ in $\mathcal{V}$, $f(x)$ is non-decreasing continuous function and $\mathcal{V}|y| \geq M \rightarrow 0$ as $M \rightarrow \infty$, then $f(X) \leq f(Y)$ in $\mathcal{V}$;

(4) if $p \geq 1$, $X, Y \geq 0$ in $L_p$, $X \leq Y$ in $L_p$, then $\hat{E}[X]^p \leq \hat{E}[Y]^p$;

(5) if $\hat{E}$ is countably additive, then $X \leq Y$ in $\mathcal{V}$ is equivalent to $X \leq Y$ in $L_p$ for any $p > 0$.

Lemma 4.3 Suppose that the operators $\hat{E}_{n,k}$ satisfy (a) and (b). For $X, Y \in \mathcal{L}(\mathcal{H})$, we have

(c) if $X \leq Y$ in $L_1$, then $\hat{E}_{n,k}[X] \leq \hat{E}_{n,k}[Y]$ in $L_1$;

(d) $\hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y] \leq \hat{E}_{n,k}[X - Y] \leq \hat{E}_{n,k}[|X - Y|]$ in $L_1$;

(e) $\hat{E}_{n,k}\left[\hat{E}_{n,l}[X]\right] = \hat{E}_{n,l\wedge k}[X]$ in $L_1$;

(f) if $|X| \leq M$ in $L_p$ for all $p \geq 1$, then $|\hat{E}_{n,k}[X]| \leq M$ in $L_p$ for all $p \geq 1$.

For the martingale-difference like random variables, we have the following theorem on the Rosenthal-type inequalities.

Theorem 4.1 Set $S_0 = 0$, $S_k = \sum_{i=1}^k Z_{n,i}$. Suppose that $\{Z_{n,i}\}$ are a set of bounded random variables. Then,

$$\hat{E}\left[\max_{k \leq k_n}(S_{k_n} - S_k)^2\right] \leq \hat{E}\left[\sum_{k=1}^{k_n} \hat{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]\right]$$

(4.1)

when $\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}] \leq 0$, $k = 1, \ldots, k_n$, and in general,

$$\hat{E}\left[\max_{k \leq k_n} |S_k|^2\right] \leq 256 \left\{ \hat{E}\left[\sum_{k=1}^{k_n} \hat{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]\right] + \hat{E}\left[\sum_{k=1}^{k_n} \left( (\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right)^2 \right]\right\}$$

(4.2)
Moreover, for $p \geq 2$ there is a constant $C_p$ such that

\[
\mathbb{E} \left[ \max_{k \leq k_n} |S_k|^p \right] \leq C_p \left\{ \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}|^p \mathcal{H}_{n,k-1} \right] + \mathbb{E} \left[ \left( \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}|^2 \mathcal{H}_{n,k} \right)^{p/2} \right] \right. \\
\left. + \mathbb{E} \left[ \sum_{k=1}^{k_n} \left( (\mathbb{E} |Z_{n,k}| \mathcal{H}_{n,k})^+ + (\mathbb{E} |Z_{n,k}| \mathcal{H}_{n,k})^- \right)^p \right] \right\}. 
\]  

(4.3)

Proof. Let $Q_k = \max \{ Z_{n,k}, Z_{n,k} + Z_{n,k-1}, \ldots, Z_{n,k} + \ldots Z_{n,1} \}$, $M_k = \max_{i \leq k} |S_i|$. Then, $Q_k = Z_{n,k} + Q_{k-1}^+$, $Q_k^2 = Z_{n,k}^2 + 2Z_{n,k}Q_{k-1}^+ + (Q_{k-1}^+)^2$, $|Q_k| \leq 2M_k$. It follows that

\[
\left( \max_{k \leq k_n} (S_{k_n} - S_k) \right)^2 = (Q_{k_n}^+)^2 \leq \sum_{k=1}^{k_n} Z_{n,k}^2 + 2 \sum_{k=1}^{k_n} Z_{n,k}Q_{k-1}^+ \\
\leq \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}^2 \mathcal{H}_{n,k-1}\right) + \sum_{k=1}^{k_n} (Z_{n,k}^2 - \mathbb{E} |Z_{n,k}^2 \mathcal{H}_{n,k-1}\right) \\
+ 2 \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right|^+ + 2 \sum_{k=1}^{k_n} (Z_{n,k} - \mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right) Q_{k-1}^+ \\
\leq \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}^2 \mathcal{H}_{n,k-1}\right) + 4 \sum_{k=1}^{k_n} (\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right)^+ M_k \\
+ \sum_{k=1}^{k_n} (Z_{n,k} - \mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right) + 2 \sum_{k=1}^{k_n} (Z_{n,k} - \mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right) Q_{k-1}^+.
\]

By the fact that $Z_{n,i}$s are bounded, Lemma 4.3 (f) and Hölder’s inequality, the random variables considered above and in the sequel have finite moments of any order. So, the properties of the conditional expectation operator can be applied freely. The sub-linear expectations of the last two sums above are non-positive, and the sub-linear expectation of the second sum is also zero when $\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k} \right| \leq 0$, $k = 1, \ldots, k_n$. Taking the sub-linear expectation yields (4.1). By considering $\{-Z_{n,k}\}$, for $\max_{k \leq k_n} (-S_{k_n} + S_k)$ we have a similar estimate. Note $M_k \leq 2 \max_{k \leq k_n} |S_n - S_k|$. It follows that

\[
\mathbb{E} \left[ M_{k_n}^2 \right] \leq 8 \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}^2 \mathcal{H}_{n,k-1}\right] \\
+ 16 \mathbb{E} \left[ \sum_{k=1}^{k_n} \left\{ (\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right)^+ + (\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right)^- \right] M_k \\
\leq 8 \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E} |Z_{n,k}^2 \mathcal{H}_{n,k-1}\right] + \frac{1}{2} \mathbb{E} \left[ M_{k_n}^2 \right] \\
+ 128 \mathbb{E} \left[ \sum_{k=1}^{k_n} \left\{ (\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right)^+ + (\mathbb{E} |Z_{n,k} | \mathcal{H}_{n,k-1}\right)^- \right]^2.
\]
where the last inequality is due to $ab \leq \frac{a^2 + b^2}{2}$.

For (4.3), we apply the elementary inequality

$$|x + y|^p \leq 2^p p^2 |x|^p + |y|^p + px|y|^{p-1} \text{sgn}y + 2^p p^2 x^2 |y|^{p-2}, \quad p \geq 2,$$

and yields

$$|Q_k|^p \leq 2^p p^2 |Z_{n,k}|^p + |Q_{k-1}|^p + pZ_{n,k}(Q_{k-1}^+)^{p-1} + 2^p p^2 Z_{n,k}^2 (Q_{k-1}^+)^{p-2}.$$

It follows that

$$\left( \max_{k \leq k_n} (S_{k_n} - S_k) \right)^p \leq |Q_{k_n}|^p$$

$$\leq 2^p p^2 \sum_{k=1}^{k_n} |Z_{n,k}|^p + p \sum_{k=1}^{k_n} Z_{n,k}(Q_{k-1}^+)^{p-1} + 2^p p^2 \sum_{k=1}^{k_n} Z_{n,k}^2 (Q_{k-1}^+)^{p-2}$$

$$\leq 2^p p^2 \sum_{k=1}^{k_n} \mathbb{E}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] + p \sum_{k=1}^{k_n} (\mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ (Q_{k-1}^+)^{p-1}$$

$$+ 2^p p^2 \sum_{k=1}^{k_n} \mathbb{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] (Q_{k-1}^+)^{p-2} + 2^p p^2 \sum_{k=1}^{k_n} (|Z_{n,k}|^p - \mathbb{E}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}])$$

$$+ p \sum_{k=1}^{k_n} (Z_{n,k} - \mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}]) (Q_{k-1}^+)^{p-1}$$

$$+ 2^p p^2 \sum_{k=1}^{k_n} (Z_{n,k}^2 - \mathbb{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}]) (Q_{k-1}^+)^{p-2}.$$

The sub-linear expectations of the last three sums are non-positive. Note $Q_k \leq 2M_{k_n}$ and for $(\max_{k \leq k_n} (-S_{k_n} + S_k))^p$ we have a similar estimate. It follows that

$$\mathbb{E} \left[ M_{k_n}^p \right] \leq C_p \left\{ \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right] + \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] M_{k_n}^{p-2} \right]$$

$$+ \mathbb{E} \left[ \sum_{k=1}^{k_n} \left\{ (\mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right\} M_{k_n}^{p-1} \right] \right\}$$

$$\leq C_p \left\{ \mathbb{E} \left[ \sum_{k=1}^{k_n} \mathbb{E}[|Z_{n,k}|^p | \mathcal{H}_{n,k-1}] \right] + \mathbb{E} \left[ \left( \sum_{k=1}^{k_n} \mathbb{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] \right)^{p/2} \right]$$

$$+ \mathbb{E} \left[ \left( \sum_{k=1}^{k_n} \left\{ (\mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^+ + (\mathbb{E}[Z_{n,k} | \mathcal{H}_{n,k-1}])^- \right\} \right)^p \right] \right\} + \frac{1}{2} \mathbb{E}[M_{k_n}^p],$$

where the last inequality is due to $ab \leq \frac{2}{p} |a|^{p/2} + (1 - \frac{2}{p}) |b|^{p/(p-2)}$ and $ab \leq \frac{1}{p} |a|^p + (1 - \frac{1}{p}) |b|^{p/(p-1)}$. The proof is completed. □

The next one gives the exponential inequality of the martingale like sequences.
Theorem 4.2 Suppose that the operators $\hat{E}_{n,k}$ satisfy (a) and (b), $\{Z_{n,k}; k = 1, \ldots, k_n\}$ is an array of random variables such that $Z_{n,k} \in \mathcal{H}_{n,k}$ and $\hat{E}[Z_{n,k}^2] < \infty$, $k = 1, \ldots, k_n$. Assume that $\hat{E}[Z_{n,k}|\mathcal{H}_{n,k-1}] \leq 0$ in $L_1$, $k = 1, \ldots, k_n$. Then for all $x, y, A > 0$

$$\mathcal{V} \left( \max_{m \leq k_n} \sum_{k=1}^{m} Z_{n,k} \geq x \right) \leq \mathcal{V} \left( \max_{k \leq k_n} Z_{n,k} \geq y \text{ or } \sum_{k=1}^{k_n} \hat{E}[Z_{n,k}^2|\mathcal{H}_{n,k-1}] \geq A \right) + \exp \left\{ - \frac{x^2}{2(xy + A)} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy}{A} \right) \right) \right\}. \quad (4.4)$$

Proof. Let $X_k = Z_{n,k} \wedge y$. Then $Z_{n,k} - X_k = (Z_{n,k} - y)^+ \geq 0$. Denote $\sigma^2_{n,k} = \hat{E}[Z_{n,k}^2|\mathcal{H}_{n,k-1}]$, $\delta_k = \sum_{i=1}^{k} \sigma^2_{n,i}$, $k = 1, \ldots, k_n$. Let $f(x)$ be a function with bounded derivative such that $I\{x \leq A\} \leq f(x) \leq I\{x \leq A + \epsilon\}$. Let $Y_k = X_k f(\delta_k)$, $T_k = \sum_{i=1}^{k} Y_k$. Then $\hat{E}[Y_k|\mathcal{H}_{n,k-1}] \leq f(\delta_k) \hat{E}[Z_{n,k}|\mathcal{H}_{n,k-1}] \leq 0$ in $L_1$, $\hat{E}[Y_k^2|\mathcal{H}_{n,k-1}] \leq f^2(\delta_k) \hat{E}[Z_{n,k}^2|\mathcal{H}_{n,k-1}] = f^2(\delta_k) \sigma^2_{n,k}$ in $L_1$. Denote $\delta^*_k = \sum_{i=1}^{k} f^2(\delta_k) \sigma^2_{n,k}$. It follows that for any $x, y, A > 0$,

$$\mathcal{V} \left( \max_{m \leq k_n} \sum_{k=1}^{m} Z_{n,k} \geq x \right) \leq \mathcal{V} \left( \max_{k \leq k_n} Z_{n,k} \geq y \text{ or } \delta_{k_n} > A \right) + \mathcal{V} \left( \max_{k \leq k_n} T_k \geq x \right).$$

For any $t > 0$, by noting $Y_k \leq y$, $0 \leq f^2(\delta_k) \sigma^2_{n,k} \leq \delta^*_k \leq A + \epsilon$, and

$$e^{ty} = 1 + ty_k + \frac{e^{ty} - 1 - ty}{y^2} Y_k^2 \leq 1 + ty_k + \frac{e^{ty} - 1 - ty}{y^2} Y_k^2,$$

we have

$$\exp \left\{ - \frac{e^{ty} - 1 - ty}{y^2} f^2(\delta_k) \sigma^2_{n,k} \right\} \hat{E} [e^{ty_k}|\mathcal{H}_{n,k-1}] \leq \exp \left\{ - \frac{e^{ty} - 1 - ty}{y^2} f^2(\delta_k) \sigma^2_{n,k} \right\} \left\{ 1 + \frac{e^{ty} - 1 - ty}{y^2} \hat{E}[Y_k^2|\mathcal{H}_{n,k-1}] \right\} \leq 1 \text{ in } L_1.$$

Write

$$U_0 = 1, \quad U_k = \exp \left\{ - \frac{e^{ty} - 1 - ty}{y^2} \delta^*_k \right\} e^{ty_k}, \quad k = 1, \ldots, k_n.$$ Then

$$\hat{E} [U_k|\mathcal{H}_{n,k-1}] \leq U_{k-1} \text{ in } L_1, \quad k = 1, \ldots, k_n. \quad (4.5)$$

Next, we show that for any $\alpha > 0$,

$$\mathcal{V} \left( \max_{k \leq k_n} U_k \geq \alpha \right) \leq \frac{\hat{E}[U_0]}{\alpha}. \quad (4.6)$$
For given $\beta \in (0, \alpha)$, let $f(x)$ be a continuous function with bounded derivation such that $I\{x \leq \alpha - \beta\} \leq f(x) \leq I\{x \leq \alpha\}$. Define $f_0 = 1$, $f_k = f(U_1) \cdots f(U_k)$. Then $f_k \in \mathcal{H}_k$, $0 \leq f_k \leq 1$ and

$$f_0 U_0 + \sum_{k=1}^n f_{k-1}(U_k - U_{k-1}) = f_n U_n + \sum_{k=1}^n f_{k-1}(1 - f(U_k)) U_k \geq f_n U_n + \sum_{k=1}^n f_{k-1}(1 - f(U_k)) (\alpha - \beta) = (\alpha - \beta)(1 - f_n) + f_n U_n \geq (\alpha - \beta) I\{\max_k U_k \geq \alpha\}.$$

By (4.5),

$$\hat{E} \left[ f_{k-1}(U_k - U_{k-1}) \right] = \hat{E} \left[ \hat{E} \left[ f_{k-1}(U_k - U_{k-1}) \mid \mathcal{H}_{k-1} \right] \right] = \hat{E} \left[ f_{k-1}(U_k \mid \mathcal{H}_{k-1}) - U_{k-1} \right] \leq 0.$$

It follows that

$$(\alpha - \beta) \mathbb{V} \left( \max_{k \leq k_n} U_k \geq \alpha \right) \leq \hat{E}[f_0 U_0] = \hat{E}[U_0].$$

(4.6) is proved. Now, note $\delta_k^* \leq A + \epsilon$. We have for any $t > 0$,

$$\exp \left\{ t \max_{k \leq k_n} T_k \right\} \leq \max_{k \leq k_n} U_k \exp \left\{ \frac{e^{ty} - 1 - ty}{y^2}(A + \epsilon) \right\}.$$

Hence by (4.6),

$$\mathbb{V} \left( \max_{k \leq k_n} T_k \geq x \right) \leq \mathbb{V} \left( \max_{k \leq k_n} U_k \geq \exp \left\{ tx - \frac{e^{ty} - 1 - ty}{y^2}(A + \epsilon) \right\} \right) \leq \exp \left\{ -tx + \frac{e^{ty} - 1 - ty}{y^2}(A + \epsilon) \right\}.$$

Choosing $t = \frac{1}{y} \ln \left( 1 + \frac{xy}{A + \epsilon} \right)$ yields

$$\mathbb{V} \left( \max_{k \leq k_n} T_k \geq x \right) \leq \exp \left\{ \frac{x}{y} - \frac{x}{y} \left( \frac{A + \epsilon}{xy} + 1 \right) \ln \left( 1 + \frac{xy}{A + \epsilon} \right) \right\}.$$ 

Applying the elementary inequality

$$\ln(1 + t) \geq \frac{t}{1 + t} + \frac{t^2}{2(1 + t)^2} \left( 1 + \frac{2}{3} \ln(1 + t) \right)$$

yields

$$\left( \frac{A + \epsilon}{xy} + 1 \right) \ln \left( 1 + \frac{xy}{A + \epsilon} \right) \geq 1 + \frac{xy}{2(xy + A + \epsilon)} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy}{A + \epsilon} \right) \right).$$

(4.4) is proved by letting $\epsilon \to 0$. □
5 Lévy’s characterization of a G-Brownian motion.

In this section, we give a Lévy characterization of a G-Brownian motion as an application of Theorem 3.2. Let \( \{ H_t; t \geq 0 \} \) be a non-decreasing family of subspaces of \( \mathcal{H} \) such that (1) a constant \( c \in H_t \) and, (2) if \( X_1, \ldots, X_d \in H_t \), then \( \varphi(X_1, \ldots, X_d) \in H_t \) for any \( \varphi \in \mathcal{C}_{l,lip} \).

We consider a system of operators in \( \mathcal{L}(\mathcal{H}) \),

\[
\hat{E}_t : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_t)
\]

and denote \( \hat{E}_t[X|H_t] = \hat{E}_t[X], \hat{E}[X|H_t] = -\hat{E}_t[-X] \). Suppose that the operators \( \hat{E}_t \) satisfy the following properties: for all \( X, Y \in \mathcal{L}(\mathcal{H}) \),

(i) \( \hat{E}_t[X + Y] = X + \hat{E}_t[Y] \) in \( L_1 \) if \( X \in \mathcal{H}_t \), and \( \hat{E}_t[XY] = X + \hat{E}_t[Y] + X - \hat{E}_t[-Y] \) in \( L_1 \) if \( X \in \mathcal{H}_t \) and \( XY \in \mathcal{L}(\mathcal{H}) \);

(ii) \( \hat{E}[\hat{E}_t[X]] = \hat{E}[X] \).

Example 5.1 Let \( W_t \) be a G-Brownian motion in a sub-linear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \), and

\[
\mathcal{H} = \{ X = \varphi(W_{t_1}, \ldots, W_{t_d}) : 0 \leq t_1 \leq \cdots \leq t_d, \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}_d), d \geq 1 \}, \quad \mathcal{H}_t = \{ X = \varphi(W_{t_1}, \ldots, W_{t_d}) : 0 \leq t_1 \leq \cdots \leq t_d \leq t, \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}_d), d \geq 1 \}.
\]

For \( X = \varphi(W_{t_1}, \cdots, W_{t_d}) \in \mathcal{H}_t \), assume \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{i+1} \leq \cdots \leq t_d \), and define

\[
\hat{E}_t[X] = \hat{E}[\varphi(w_{t_1}, \cdots, w_{t_i}, W_{t_{i+1}} - W_t + w_t, \cdots, W_{t_d} - W_t + w_t)] \big|_{w_{t_1} = W_{t_1}, \cdots, w_{t_i} = W_{t_i}, w_t = W_t}.
\]

Then, in the sub-linear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \), the family \( \{ \mathcal{H}_t, \hat{E}_t \}_{t \geq 0} \) satisfies the properties (i)-(iii).

Definition 5.1 A process \( M_t \) is called a martingale, if \( M_t \in \mathcal{L}(\mathcal{H}) \), \( M_t \in \mathcal{H}_t \) and

\[
\hat{E}[M_t|\mathcal{H}_s] = M_s, \quad s \leq t.
\]

Denote

\[
\omega_T(M, \delta) = \sup_{|t-s| < \delta, t, s \in [0, T]} |M(t) - M(s)|
\]
and
\[
W_T(M, \delta) = \sup_{t_i} \hat{E} \left[ \max_{1 \leq i \leq n} |M(t_i) - M(t_{i-1})| \wedge 1 \right],
\]
where the supremum \( \sup_{t_i} \) is taken over all \( t_i \)s with
\[
0 = t_0 < t_1 < \cdots < t_n = T, \quad \delta/2 < t_i - t_{i-1} < \delta, \quad i = 1, \cdots, n.
\]

The following theorem gives a Lévy characterization of a G-Brownian motion.

**Theorem 5.1** Let \( M_t \) be a random process in \((\Omega, \mathcal{H}, \mathcal{H}_t, \hat{E})\) with \( M_0 = 0 \), for all \( p > 0 \) and \( t \geq 0 \), \( CV(|M_t|^p) < \infty \) \( \implies \hat{E}[|M_t|^p] < \infty \). (5.1)

Suppose that \( M_t \) satisfies

(I) both \( M_t \) and \( -M_t \) are martingales;

(II) for a constant \( \sigma^2 > 0 \), \( M_t^2 - \sigma^2 t \) is a martingale;

(III) for a constant \( 0 < \sigma^2 \leq \sigma_0^2 \), \( -(M_t^2 - \sigma^2 t) \) is a martingale;

(IV) for any \( T > 0 \), \( \lim_{\delta \to 0} W_T(M, \delta) = 0 \).

Then, \( M_t \) satisfies Property (ii) as in Definition 1.4 with \( M_1 \sim N(0, [\sigma^2, \sigma_0^2]) \).

**Remark 5.1** The assumption (I) implies that \( \hat{E}[M_t - M_s|\mathcal{H}_s] = \hat{E}[M_t - M_s|\mathcal{H}_s] = 0 \) for all \( t > s \). Also, under the assumptions (I), the assumption (II) is equivalent to that \( \hat{E}[(M_t - M_s)^2|\mathcal{H}_s] = \sigma^2(t - s) \) for all \( t > s \), (III) is equivalent to that \( \hat{E}[(M_t - M_s)^2|\mathcal{H}_s] = \sigma_0^2(t - s) \) for all \( t > s \).

The assumption of (IV) means that \( M_t \) is continuous. Note \( W_T(M, \delta) \leq \epsilon + V(w_T(M, \delta) > \epsilon) \). It is satisfied if

(IV') for any \( T, \epsilon > 0 \), \( \lim_{\delta \to 0} V(w_T(M, \delta) > \epsilon) = 0 \).

The condition (IV') means that \( M_t \) is continuous in capacity \( V \) uniformly in \( t \) on each finite interval. Also, \( W_T(M, \delta) \leq \sup_{t_i} \left( \sum_i \hat{E} \left[ |M(t_i) - M(t_{i-1})|^{2+\alpha} \right] \right)^{1/(2+\alpha)} \). (IV) is also satisfied if
(IV)′′ there is a constant $\alpha > 0$ such that for any $t > s > 0$, $\hat{E}[|M_t - M_s|^{2+\alpha}] = o(t-s)$ as $t-s \to 0$.

**Remark 5.2** The Lévy characterization of a G-Brownian motion is first established under G-expectation in a Wiener space by Xu and Zhang (2009, 2010) by using the stochastic calculus. We will give an elementary proof by using the functional central limit theorem.

**Remark 5.3** If $\hat{E}$ is countably sub-additive, then the condition (5.1) is automatically satisfied. The G-expectation space considered in Xu and Zhang (2009, 2010) is complete and so the sub-linear expectation is countably additive, and (5.1) is satisfied.

In Xu and Zhang (2009, 2010), the operators $\hat{E}_t$ are also supposed to have the following assumptions:

(iii) if $X \leq Y$, then $\hat{E}_t[X] \leq \hat{E}_t[Y]$;

(iv) $\hat{E}_t[X] - \hat{E}_t[Y] \leq \hat{E}_t[X - Y]$;

(v) $\hat{E}_t\left[\left(\hat{E}_s[X]\right)\right] = \hat{E}_{t \wedge s}[X]$.

As in Lemma 4.3, (iii), (iv), and (v) holds in $L_1$ if the operators satisfy (i) and (ii).

For proving Theorem 5.1 we need a more lemma.

**Lemma 5.1** Suppose that the operators $\hat{E}_t$ satisfy (i)-(ii), $M_t$ is a martingale in $(\Omega, \mathcal{H}, \mathcal{H}_t, \hat{E})$ such that (IV) in Theorem 5.1 is satisfied and $\hat{E}[(M_t - M_s)^2|\mathcal{H}_s] \leq (t-s)\sigma^2$ for all $t > s \geq 0$, where $\sigma$ is a positive constant. Then,

$$\forall (M_t - M_s \geq x) \leq \exp\left\{ -\frac{x^2}{2(t-s)\sigma^2} \right\}, \text{ for all } t > s \geq 0, x \geq 0. \quad (5.2)$$

In particular, for any $p > 0$, $C_{\mathcal{Y}}\left(\left[(M_t - M_s)^+\right]^p\right) \leq c_p(t-s)^{p/2}\sigma^p$.

**Proof.** Let $s = t_0 < t_1 < \ldots < t_k = t$ be a partition of $[s, t]$ with $\delta/2 < t_i - t_{i-1} < \delta$. Note $\hat{E}[M_{t_i} - M_{t_{i-1}}|\mathcal{H}_{t_{i-1}}] = 0$ and $\hat{E}[(M_{t_i} - M_{t_{i-1}})^2|\mathcal{H}_{t_{i-1}}] \leq (t_i - t_{i-1})\sigma^2$. So,
\[
\sum_{i=1}^{k} \hat{E}[ (M_{t_i} - M_{t_{i-1}})^2 | \mathcal{H}_{t_{i-1}} ] \leq (t-s) \sigma^2. \]
By Theorem 4.2, for \(0 < y < 1\) and \(x > 0\),
\[
\mathbb{V} (M_t - M_s \geq x) \\
\leq \mathbb{V} \left( \max_i (M_{t_i} - M_{t_{i-1}}) \geq y \right) \\
+ \exp \left\{ -\frac{x^2}{2(xy + (t-s)\sigma^2)} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy}{(t-s)\sigma^2} \right) \right) \right\} \\
\leq \frac{W_T(M, \delta)}{y} + \exp \left\{ -\frac{x^2}{2(xy + (t-s)\sigma^2)} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy}{(t-s)\sigma^2} \right) \right) \right\}.
\]
By letting \(\delta \to 0\) and then \(y \to 0\), we conclude (5.2). Finally, for \(p > 0\),
\[
C_V \left( |M_t - M_s|^p \right) \leq \int_0^\infty \mathbb{V} \left( M_t - M_s \geq x^{1/p} \right) \, dx \\
\leq (t-s)^{p/2} \sigma^p \int_0^\infty \exp \left\{ -\frac{x^{2/p}}{2} \right\} \, dx \leq c_p(t-s)^{p/2} \sigma^p. \quad \square
\]

**Proof of Theorem 5.1.** Suppose that (I)-(IV) are satisfied. Note that both \(M_t\) and \(-M_t\) are martingales, and \(\hat{E}[ (M_t - M_s)^2 | \mathcal{H}_{t_{i-1}} ] = (t-s) \sigma^2\). By Lemma 5.1,
\[
C_V \left( |M_t - M_s|^p \right) \leq c_p(t-s)^{p/2} \sigma^p.
\]
By the assumption (5.1), \(\hat{E}[ |M_t - M_s|^p ] < \infty\) for any \(p > 0\) and \(t, s\). Let \(W_t\) be a G-Brownian motion in a sub-linear expectation \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})\) with \(W_1 \sim N(0, [\mathcal{F}, \mathcal{H}]^2)\). It is sufficient to show that for any \(0 < t_1 < \ldots < t_d\) and \(\varphi \in C_{b,\text{Lip}}(\mathbb{R}_d)\)
\[
\hat{E} [ \varphi(M_{t_1}, \ldots, M_{t_d}) ] = \tilde{E} [ \varphi(W_{t_1}, \ldots, W_{t_d}) ]. \quad (5.3)
\]
Actually, by noting \(\hat{E}[ |M_t|^p ] < \infty\) for any \(p > 0\), we can extend \(\varphi\) from \(C_{b,\text{Lip}}(\mathbb{R}_d)\) to \(C_{t,\text{Lip}}(\mathbb{R}_d)\) by an elementary argument.

Now, without loss of generality, we assume \(0 < t_1 < \ldots < t_d \leq 1\). Note \(\hat{E} \left[ (|M_t - M_s|^3 - c^3)^+ \right] \leq \hat{E} \left[ |M_t - M_s|^4 \right] / c \to 0\) as \(c \to \infty\). Then \(\hat{E}[ |M_t - M_s|^3 ] \leq C_V (|M_t - M_s|^3) = o(t-s)\) as \(t-s \to 0\). Let
\[
k_n = 2^n, \quad Z_{n,k} = M_{k/2^n} - M_{(k-1)/2^n}, \quad \mathcal{H}_{n,k} = \mathcal{H}_{k/2^n}, \quad k = 1, \ldots, k_n,
\]
and \(\tau_n(t) = [t 2^n]\). Then \(\hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}] = \hat{E}[Z_{n,k} | \mathcal{H}_{n,k-1}] = 0\),
\[
\hat{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] = \frac{\sigma^2}{2^n}, \quad \hat{E}[Z_{n,k}^2 | \mathcal{H}_{n,k-1}] = \frac{\sigma^2}{2^n}.
\]
Hence it is easily seen that the sequence \( \{ Z_{n,k}, \mathcal{H}_{n,k} \} \) satisfy the conditions (3.3), (3.4) and (3.9) with \( \rho(t) = t^2, \ r = \sigma^2 / \sigma^2 \). Further,
\[
\sum_{k=1}^{k_n} \hat{\mathbb{E}} [ |Z_{n,k}|^3 ] = \sum_{k=1}^{2^n} o\left( \frac{1}{2^n} \right) \to 0.
\]
So, the Lindeberg condition (3.1) is satisfied. Let \( W_n(\cdot) \) be defined as in (3.8). By Theorem 3.2, \( (W_n(t_1), \cdots, W_n(t_d)) \overset{d}{\to} (W_{t_1}, \cdots, W_{t_d}) \). On the other hand,
\[
|W_n(t) - M_t| = |M_t - M_{[2nt]/2^n}| \overset{\mathcal{L}}{\to} 0.
\]
So, (5.3) holds for all \( \varphi \in C_{b,Lip}(\mathbb{R}_d) \). The proof is now completed. \( \square \)

6 Proofs of the central limit theorems for martingales.

6.1 Proof of the central limit theorem

We give the proof of Theorem 3.1. By (3.1), there exists a sequence of positive numbers \( 1/2 > \epsilon_n \searrow 0 \) such that
\[
\epsilon_n^2 \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ (Z_{n,k}^2 - \epsilon_n^2)^+ | \mathcal{H}_{n,k-1} \right] \overset{\mathcal{L}}{\to} 0.
\]
Let \( Z_{n,k}^* = (-2\epsilon_n) \vee Z_{n,k} \wedge (2\epsilon_n) \). Then
\[
\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ (Z_{n,k} - Z_{n,k}^*)^2 | \mathcal{H}_{n,k-1} \right] \leq \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ (Z_{n,k}^2 - \epsilon_n^2)^+ | \mathcal{H}_{n,k-1} \right] \overset{\mathcal{L}}{\to} 0
\]
and
\[
\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ |Z_{n,k} - Z_{n,k}^*| | \mathcal{H}_{n,k-1} \right] \leq \epsilon_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[ (Z_{n,k}^2 - \epsilon_n^2)^+ | \mathcal{H}_{n,k-1} \right] \overset{\mathcal{L}}{\to} 0.
\]
Hence, \( \{ Z_{n,k}^*; k = 1, \ldots, k_n \} \) satisfy the conditions (3.2)-(3.4). Further, let \( h_k = \epsilon_n^{-2} \sum_{i=1}^{k} \hat{\mathbb{E}} (Z_{n,k}^2 - \epsilon_n^2)^+ | \mathcal{H}_{n,k-1} \) and \( f \) be a bounded Lipschitz function such that \( I\{ x \leq \epsilon \} \leq f(x) \leq I\{ x \leq x \}

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Then,
\[
\forall (Z_{n,k} \not= Z^*_{n,k} \text{ for some } k)
\]
\[
= \forall \left( \max_{k \leq k_n} |Z_{n,k}| \geq 2\epsilon_n \right) \leq \forall \left( \sum_{k=1}^{k_n} \left[ 1 \wedge (Z^2_{n,k} - \epsilon_n^2)^+ \right] \geq \epsilon_n^2 \right)
\]
\[
\leq \forall \left( \sum_{k=1}^{k_n} \left[ 1 \wedge (Z^2_{n,k} - \epsilon_n^2)^+ \right] \geq \epsilon_n^2, h_{k_n} \leq \epsilon \right) + \forall (h_{k_n} \geq \epsilon)
\]
\[
= \forall \left( \sum_{k=1}^{k_n} \left[ 1 \wedge (Z^2_{n,k} - \epsilon_n^2)^+ \right] f(h_k) \geq \epsilon_n^2, h_{k_n} \leq \epsilon \right) + \forall (h_{k_n} \geq \epsilon)
\]
\[
\leq \mathbb{E} \left[ \epsilon_n^2 \sum_{k=1}^{k_n} \left[ 1 \wedge (Z^2_{n,k} - \epsilon_n^2)^+ \right] f(h_k) \right] + \forall (h_{k_n} \geq \epsilon)
\]
\[
\leq \mathbb{E} \left[ \epsilon_n^2 \sum_{k=1}^{k_n} f(h_k) \mathbb{E} \left[ \left[ 1 \wedge (Z^2_{n,k} - \epsilon_n^2)^+ \right] |\mathcal{H}_{n,k-1} \right] \right] + \forall (h_{k_n} \geq \epsilon)
\]
\[
\leq 2\epsilon + \forall (h_{k_n} \geq \epsilon) \to 0 \text{ as } n \to \infty \text{ and then } \epsilon \to 0.
\]

It follows that for any bounded function \(\varphi\),
\[
\mathbb{E} \left[ |\varphi(\sum_{k=1}^{k_n} Z_{n,k}) - \varphi(\sum_{k=1}^{k_n} Z^*_{n,k})| \right] \leq 2 \sup_x |\varphi(x)| \forall (Z_{n,k} \not= Z^*_{n,k} \text{ for some } k) \to 0.
\]

So, without loss of generality we can assume that there is a positive sequence \(1 \geq \epsilon_n \searrow 0\) such that \(|Z_{n,k}| \leq \epsilon_n, k = 1, \ldots, k_n\).

Denote \(S_0 = 0, \delta_0 = 0, S_k = \sum_{i=1}^{k} Z_{n,i}, a^2_{n,k} = \mathbb{E}[Z^2_{n,k}|\mathcal{H}_{n,k-1}], \delta_k = \sum_{i=1}^{k} a^2_{n,i}, k = 1, \ldots, k_n\). Let \(f(x)\) be a function with bounded derivative such that \(I \{x \leq \rho + \epsilon/2\} \leq f(x) \leq I \{x \leq \rho + \epsilon\}\. Let \(Z^*_{n,k} = Z_{n,k} f(\delta_k)\. Then \(\{Z^*_{n,k}, k = 1, \ldots, k_n\}\) satisfy the conditions (3.2)-(3.4), and
\[
\sum_{k=1}^{k_n} \mathbb{E}[(Z^*_{n,k})^2|\mathcal{H}_{n,k-1}] = \delta^*_k, \quad \text{(6.1)}
\]
where \(\delta^*_k = \sum_{k=1}^{k_n} f(\delta_k) \mathbb{E}[Z^2_{n,k}|\mathcal{H}_{n,k-1}] \leq \rho + \epsilon\). The above equalities hold in \(L_1\) by the Property (a) of the operators \(\mathbb{E}_{n,k}\) and then hold in any \(L_q\) by Lemma 4.2 (2) since \(\delta^*_k\) is bounded in \(L_q\) by Lemma 4.3 (f). Further,
\[
\{Z_{n,k} \not= Z^*_{n,k} \text{ for some } k\} \subset \left\{ \sum_{k=1}^{k_n} a^2_{n,k} > \rho + \epsilon/2 \right\}.
\]

So, without loss of generality we can further assume that \(\delta_k = \sum_{k=1}^{k_n} \mathbb{E}[Z^2_{n,k}|\mathcal{H}_{n,k-1}] \leq \rho + \epsilon\) in \(L_1\). Similarly, we can assume \(\chi_{k_n} =: \sum_{k=1}^{k_n} \left\{ \mathbb{E}[Z_{n,k}|\mathcal{H}_{n,k-1}] + |\mathbb{E}[Z_{n,k}|\mathcal{H}_{n,k-1}]| \right\} < \epsilon < 1\)

2 ε. Then,
in $L_1$. The property (f) in Lemma 4.3 implies that all random variables considered above and in the sequel are bounded in $L_p$ for all $p > 0$.

Now, by Theorem 4.1,

$$
\hat{\mathbb{E}} \left[ \max_{k \leq k_n} \left( \sum_{i=1}^{k_n} Z_{n,i} \right)^2 \right] \leq 256 \hat{\mathbb{E}} [\delta_{k_n}] + 256 \hat{\mathbb{E}} \left[ \chi_{k_n}^2 \right]. \quad (6.2)
$$

If $\rho = 0$, then $\delta_{k_n} \xrightarrow{v} 0$. Note $\chi_{k_n} \xrightarrow{v} 0$. So, $\hat{\mathbb{E}} \left[ \left( \sum_{i=1}^{k_n} Z_{n,i} \right)^2 \right] \to 0$, and then the result is obvious. In the sequel, we suppose $\rho \neq 0$. Let $\varphi$ be a bounded continuous function with bounded derivation. Without loss of generality, we assume $|\varphi(x)| \leq 1$. We want to show that

$$
\hat{\mathbb{E}}[\varphi(S_{k_n})] \to \hat{\mathbb{E}}[\varphi(\sqrt{\rho} \xi)]. \quad (6.3)
$$

In the classical probability space, the above convergence is usually shown by verifying the convergence of the related characteristic functions (cf. Hall and Heyde (1980), p. 60-63; Pollard (1984), p. 171-174). As shown by Hu and Li (2014), the characteristic function cannot determine the distribution of random variables in the sub-linear expectation space. Peng (2007a, 2008b) developed a method to show the above convergence for independent random variables. Here we promote Peng’s argument such that it is also valid for martingale differences which give also a new normal approximation method for classical martingale differences instead of the characteristic function.

Now, for a small but fixed $h > 0$, let $V(t, x)$ be the unique viscosity solution of the following equation,

$$
\partial_t V + G(\partial^2_{xx} V) = 0, \quad (t, x) \in [0, \rho + h] \times \mathbb{R}, \quad V|_{t=\rho+h} = \varphi(x), \quad (6.4)
$$

where $G(\alpha) = \frac{1}{2}(\alpha^+ - r\alpha^-)$. Then by the interior regularity of $V$,

$$
\|V\|_{C^{1+\alpha/2, 2+\alpha}([0, \rho+h] \times \mathbb{R})} < \infty, \quad \text{for some } \alpha \in (0, 1). \quad (6.5)
$$

According to the definition of $G$-normal distribution, we have $V(t, x) = \hat{\mathbb{E}}[\varphi(x+\sqrt{\rho} + h - \sqrt{\rho} \xi)]$ where $\xi \sim N(0, [r, 1])$ under $\hat{\mathbb{E}}$. In particular,

$$
V(h, 0) = \hat{\mathbb{E}}[\varphi(\sqrt{\rho} \xi)], \quad V(\rho + h, x) = \varphi(x).
$$

It is obvious that, if $\varphi(\cdot)$ is a global Lipschitz function, i.e., $|\varphi(x) - \varphi(y)| \leq C|x - y|$, then $|V(t, x) - V(t, y)| \leq C|x - y|$ and

$$
|V(t, x) - V(s, x)| \leq C\hat{\mathbb{E}}[|\xi|] \left| \sqrt{\rho + h - t} - \sqrt{\rho + h - s} \right| \leq C\hat{\mathbb{E}}[|\xi|] |t - s|^{1/2}.
$$
So, \(|\partial_x V(t, x)| \leq C, |\partial_t V(t, x)| \leq C\hat{E}|\xi|/\sqrt{\rho + h - t}, |V(\rho + h, x) - V(\rho, x)| \leq C\hat{E}|\xi|\sqrt{h}\)

and \(|V(h, 0) - V(0, 0)| \leq C\hat{E}|\xi|\sqrt{h}\). Following the proof of Lemma 5.4 of Peng (2008b), it is sufficient to show that

\[
\lim_{n \to \infty} \hat{E}[V(\rho, S_{kn})] = V(0, 0). \tag{6.6}
\]

As we have shown, we can assume that \(\delta_{kn} \leq \rho + h/4 =: h_0 < 2\rho\) in \(L_1\). It is obvious that \(|V(t, x)| \leq 1\), and

\[
\hat{E} \left[ \left| V(\rho, S_{kn}) - V(\delta_{kn} \wedge h_0, S_{kn}) \right| \right] \leq C \hat{E} \left[ |\delta_{kn} \wedge h_0 - \rho|^{1/2} \right] \to 0.
\]

Hence, it is sufficient to show that

\[
\lim_{n \to \infty} \hat{E} [V(\delta_{kn} \wedge h_0, S_{kn})] = V(0, 0). \tag{6.7}
\]

Let \(\tilde{\delta}_i = \delta_i \wedge h_0\). Then \(\tilde{\delta}_{i+1} - \tilde{\delta}_i \leq a_{n,i+1}^2, |\tilde{\delta}_i| \leq h_0 = \rho + h/4\). It follows that

\[
|\partial_x V(\tilde{\delta}_i, S_i)| \leq C, \quad |\partial_t V(\tilde{\delta}_i, S_i)| \leq C/\sqrt{h} \leq C.
\]

Also, by the fact that \(\partial_{xx} V\) is uniformly \(\alpha\)-Hölder continuous in \(x\) and \(\alpha/2\)-Hölder continuous in \(t\) on \([0, \rho + h/2] \times R\), it follows that

\[
\left| \partial_{xx}^2 V(\tilde{\delta}_i, S_i) \right| \leq \left| \partial_{xx}^2 V(0, 0) \right| + C|\tilde{\delta}_i|^{\alpha/2} + C|S_i|^{\alpha} \leq C + C|S_i|^{\alpha}.
\]

Now, applying the Taylor’s expansion yields

\[
V(\tilde{\delta}_{kn}, S_{kn}) - V(0, 0)
\]

\[
= \sum_{i=0}^{k_n-1} \left\{ [V(\tilde{\delta}_{i+1}, S_{i+1}) - V(\tilde{\delta}_i, S_{i+1})] + [V(\tilde{\delta}_i, S_{i+1}) - V(\tilde{\delta}_i, S_i)] \right\} =: \sum_{i=0}^{k_n-1} \{ I_i^n + J_i^n \},
\]

with

\[
J_i^n = \partial_t V(\tilde{\delta}_i, S_i)(\tilde{\delta}_{i+1} - \tilde{\delta}_i) + \frac{1}{2} \partial_{xx}^2 V(\tilde{\delta}_i, S_i)Z_{n,i+1}^2 + \partial_x V(\tilde{\delta}_i, S_i)Z_{n,i+1} \]

\[
= \left\{ a_{n,i+1}^2 \partial_t V(\tilde{\delta}_i, S_i) + \frac{1}{2} \partial_{xx}^2 V(\tilde{\delta}_i, S_i)Z_{n,i+1}^2 - \frac{1}{2} (\partial_{xx}^2 V(\tilde{\delta}_i, S_i)) - (ra_{n,i+1}^2 - \tilde{\alpha}[Z_{n,i+1}^2|H_{n,i}]) \right\}
\]

\[
+ \left\{ \partial_x V(\tilde{\delta}_i, S_i)Z_{n,i+1} \right\} + \left\{ \frac{1}{2} (\partial_{xx}^2 V(\tilde{\delta}_i, S_i)) - (ra_{n,i+1}^2 - \tilde{\alpha}[Z_{n,i+1}^2|H_{n,i}]) \right\}
\]

\[
+ \left\{ \partial_t V(\tilde{\delta}_i, S_i)(\tilde{\delta}_{i+1} - \tilde{\delta}_i - a_{n,i+1}^2) \right\}
\]

\[
=: J_{n,1}^i + J_{n,2}^i + J_{n,3}^i + J_{n,4}^i.
\]
and 

\[ I_n^i = (\delta_{i+1} - \delta_i) \left[ (\partial_i V(\tilde{\delta}_i + \gamma(\delta_{i+1} - \delta_i), S_{i+1}) - \partial_i V(\tilde{\delta}_i, S_{i+1})) \\
+ (\partial_i V(\tilde{\delta}_i, S_{i+1}) - \partial_i V(\tilde{\delta}_i, S_i)) \right] \\
+ \frac{1}{2} \left[ \partial_{xx}^2 V(\tilde{\delta}_i, S_i + \beta Z_{n,i+1}) - \partial_{xx}^2 V(\tilde{\delta}_i, S_i) \right] Z_{n,i+1}^2, \]

where \( \gamma \) and \( \beta \) are between 0 and 1. Thus

\[
\begin{align*}
\left| \mathbb{E}[V(\tilde{\delta}_{k_n}, S_{k_n})] - V(0, 0) - \mathbb{E}\left[ \sum_{i=0}^{k_n-1} (J_{n,1}^i + J_{n,2}^i) \right] \right| & \leq \mathbb{E}\left[ |V(\tilde{\delta}_{k_n}, S_{k_n}) - V(0, 0) - \sum_{i=0}^{k_n-1} (J_{n,1}^i + J_{n,2}^i)| \right] \\
& \leq \mathbb{E}\left[ \sum_{i=0}^{k_n-1} (|I_n^i| + |J_{n,3}^i| + |J_{n,4}^i|) \right].
\end{align*}
\]

For \( J_{n,1}^i \), it follows that

\[ \mathbb{E} [J_{n,1}^i | \mathcal{H}_{n,i}] = [\partial_i V(\tilde{\delta}_i, S_i) + G(\partial_{xx}^2 V(\tilde{\delta}_i, S_i))] a_{n,i+1}^2 = 0 \text{ in } L_1. \]

It follows that

\[ \mathbb{E}\left[ \sum_{i=0}^{k_n-1} J_{n,1}^i \right] = \mathbb{E}\left[ \sum_{i=0}^{k_n-2} J_{n,1}^i + \mathbb{E}\left[ J_{n,1}^{k_n-1} | \mathcal{H}_{n,k_n-1} \right] \right] = \mathbb{E}\left[ \sum_{i=0}^{k_n-2} J_{n,1}^i \right] = \ldots = 0. \tag{6.9} \]

For \( J_{n,2}^i \), we denote \( \tilde{J}_{n,2}^i = |\partial_x V(\tilde{\delta}_i, S_i)| \left( |\mathbb{E}[Z_{n,i+1} | \mathcal{H}_{n,i}]| + |\mathbb{E}[Z_{n,i+1} | \mathcal{H}_{n,i}]| \right) \). Then

\[
\begin{align*}
\mathbb{E}[J_{n,2}^i - \tilde{J}_{n,2}^i | \mathcal{H}_{n,i}] &= \mathbb{E}[J_{n,2}^i | \mathcal{H}_{n,i}] - \tilde{J}_{n,2}^i \\
& \leq (\partial_x V(\tilde{\delta}_i, S_i))^+ \mathbb{E}[Z_{n,i+1} | \mathcal{H}_{n,i}] - (\partial_x V(\tilde{\delta}_i, S_i))^+ \mathbb{E}[Z_{n,i+1} | \mathcal{H}_{n,i}] - \tilde{J}_{n,2}^i \leq 0 \text{ in } L_1.
\end{align*}
\]

Similarly \( \mathbb{E}[-J_{n,2}^i - \tilde{J}_{n,2}^i | \mathcal{H}_{n,i}] \leq 0 \text{ in } L_1. \) It follows that

\[
\begin{align*}
\mathbb{E}\left[ \sum_{i=0}^{k_n-1} (\pm J_{n,2}^i - \tilde{J}_{n,2}^i) \right] &= \mathbb{E}\left[ \sum_{i=0}^{k_n-2} (\pm J_{n,2}^i - \tilde{J}_{n,2}^i) + \mathbb{E}\left[ \pm J_{n,1}^{k_n-1} - \tilde{J}_{n,2}^{k_n-1} | \mathcal{H}_{n,k_n-1} \right] \right] \\
& \leq \mathbb{E}\left[ \sum_{i=0}^{k_n-2} (\pm J_{n,2}^i - \tilde{J}_{n,2}^i) \right] \leq \ldots \leq 0. \tag{6.10}
\end{align*}
\]

Hence

\[ \mathbb{E}\left[ \pm \sum_{i=0}^{k_n-1} J_{n,2}^i \right] \leq \mathbb{E}\left[ \sum_{i=0}^{k_n-1} (\pm J_{n,2}^i - \tilde{J}_{n,2}^i) \right] + \mathbb{E}\left[ \sum_{i=0}^{k_n-1} \tilde{J}_{n,2}^i \right] \leq \mathbb{E}\left[ \sum_{i=0}^{k_n-1} J_{n,2}^i \right]. \tag{6.11} \]

Note \( |\partial_x V(\tilde{\delta}_i, S_i)| \leq C, \chi_{k_n} \xrightarrow{\mathcal{J}} 0 \text{ and } \chi_{k_n} \leq 1 \text{ in any } L_p. \) Combining (6.9) and (6.11) yields that

\[ \left| \mathbb{E}\left[ \sum_{i=0}^{k_n-1} (J_{n,1}^i + J_{n,2}^i) \right] \right| \leq \mathbb{E}\left[ \sum_{i=0}^{k_n-1} J_{n,2}^i \right] \leq C \mathbb{E}[\chi_{k_n}] \to 0. \]
For $J_{n,3}^i$, it is easily seen that

$$\sum_{i=0}^{k_n-1} |J_{n,3}^i| \leq C(1 + \max_{i \leq k_n} |S_i|^\alpha) \sum_{i=1}^{k_n} |r a_{n,i}^2 - \hat{E}[Z_{n,i}^2 | \mathcal{F}_{n-1}]|.$$ (6.12)

Write $\beta_{k_n} = \sum_{i=1}^{k_n} |r a_{n,i}^2 - \hat{E}[Z_{n,i}^2 | \mathcal{F}_{n-1}]|$. Note that

$$\beta_{k_n} \xrightarrow{p} 0 \text{ and } \beta_{k_n} \leq 2\delta_{k_n} \leq 2h_0 \text{ in any } L_p$$

and $\hat{E}[\max_{i \leq k_n} |S_i|^2] \leq 256\{\hat{E}[\delta_{k_n}] + \hat{E}[\chi_{k_n}^2]\} \leq 256(h_0 + 1)$ by (6.2). So

$$\hat{E}\left[\sum_{i=0}^{k_n-1} |J_{n,3}^i|\right] \leq C(\hat{E}[1 + \max_{i \leq k_n} |S_i|^\alpha])^{1/2} (\hat{E}[\beta_{k_n}^2])^{1/2} \to 0.$$

For $J_{n,4}^i$, note that $|\delta_{i+1} - \delta_i - a_{n,i+1}^2| \leq a_{n,i+1}^2$, and $\delta_{i+1} - \delta_i + a_{n,i+1}^2 = \delta_{i+1} - \delta_i - a_{n,i+1}^2 = 0$ when $\delta_{k_n} \leq h_0$. It follows that

$$\hat{E}\left[\sum_{i=0}^{k_n-1} |J_{n,4}^i|\right] \leq C\hat{E}\left[\delta_{k_n} I\{\delta_{k_n} > h_0\}\right] \leq C \left(\hat{E}[\delta_{k_n}^2]\right)^{1/2} (\hat{E}(\delta_{k_n} > h_0))^{1/2} = 0.$$

For $I_n^i$, note both $\partial_t V$ and $\partial_{xx} V$ are uniformly $\alpha$-Hölder continuous in $x$ and $\alpha/2$-Hölder continuous in $t$ on $[0, \rho + h/2] \times R$. Without loss of generality, we assume $\alpha < \tau$. Also, $\delta_{i+1} - \delta_i \leq a_{n,i+1}$. We then have

$$|I_n^i| \leq C |a_{n,i+1}|^{2+\alpha} + C a_{n,i+1}^2 |Z_{n,i+1}|^{\alpha} + |Z_{n,i+1}|^{2+\alpha}$$

$$\leq C \epsilon_n^{\alpha} a_{n,i+1}^2 + C \epsilon_n^\alpha Z_{n,i+1} = C \epsilon_n^{\alpha} a_{n,i+1}^2 + C \epsilon_n^{\alpha} (Z_{n,i+1}^2 - a_{n,i+1}^2)$$

in any $L_q$ by Lemma 4.2. And so,

$$\sum_{i=0}^{k_n-1} |I_n^i| \leq 2C \epsilon_n^{\alpha} + C \epsilon_n^{\alpha} \sum_{i=1}^{k_n} (Z_{n,i}^2 - a_{n,i}^2) \text{ in } L_1,$$ (6.13)

by noting $\sum_{i=1}^{k_n} a_{n,i}^2 \leq 2\rho$ in $L_1$, where the sub-linear expectation under $\hat{E}$ of the last term is zero. It follows that

$$\hat{E}\left[\sum_{i=0}^{k_n-1} |I_n^i|\right] \leq 2C \epsilon_n^{\alpha} \to 0.$$ (6.7)

(6.7) is proved. Hence, (6.3) holds for any bounded function $\varphi$ with bounded derivative.

If $\varphi$ is a bounded and uniformly continuous function, we define a function $\varphi_\delta$ as a convolution of $\varphi$ and the density of a normal distribution $N(0, \delta)$, i.e.,

$$\varphi_\delta = \varphi * \psi_\delta, \text{ with } \psi_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} \exp\left\{-\frac{x^2}{2\delta}\right\},$$
where $\varphi * \psi_\delta$ denotes the convolution of $\varphi$ and $\psi_\delta$. Then $|\varphi'(x)| \leq \sup_x |\varphi(x)|\delta^{-1/2}$ and
$\sup_x |\varphi_\delta(x) - \varphi(x)| \to 0$ as $\delta \to 0$. Hence, (6.3) holds for any bounded and uniformly continuous function $\varphi$.

Now, for a bounded continuous function $\varphi$ and a given number $N > 1$, we define $\varphi_1(x) = \varphi((-N) \vee (x \wedge N))$. Then, $\varphi_1$ is a bounded and uniformly continuous function, and $|\varphi(x) - \varphi_1(x)| \leq CI\{|x| > N\}$. And so,

$$
\sup_n \mathbb{E} \left[ |\varphi(\sum_{k=1}^{k_n} Z_{n,k}) - \varphi_1(\sum_{k=1}^{k_n} Z_{n,k})| \right] \leq CN^2 \sup_n \mathbb{E} \left[ (\sum_{k=1}^{k_n} Z_{n,k})^2 \right] \leq CN^{-2} \sup_n \left( \mathbb{E} \left[ \delta_{k_n} \right] + \mathbb{E} \left[ \chi_{k_n}^2 \right] \right)
$$

$$
\leq 3CN^{-2} \to 0 \quad \text{as } N \to \infty
$$

by (6.2). The proof of Theorem 3.1 is now completed. □

### 6.2 Proof of the functional central limit theorem

For proving the functional central limit theorem, we need a more lemma.

**Lemma 6.1** Suppose that the operators $\hat{\mathbb{E}}_{n,k}$ satisfy (a) and (b), $X_n \in \mathcal{H}_{n,k'_n} \subset \mathcal{H}$ is a $d_1$-dimensional random vector, and $Y_n \in \mathcal{H}$ is a $d_2$-dimensional random vector. Write $\mathcal{H}_n = \mathcal{H}_{n,k'_n}$. Assume that $X_n \overset{d}{\to} \bar{X}$, and for any bounded Lipschitz function $\varphi(x, y) : \mathbb{R}_{d_1} \otimes \mathbb{R}_{d_2} \to \mathbb{R}$,

$$
\hat{\mathbb{E}} \left[ \left| \mathbb{E}[\varphi(X_n, Y_n) | \mathcal{H}_n] - \mathbb{E}[\varphi(x, Y)] \right| \right] \to 0, \quad \forall x,
$$

(6.14)

where $X, Y$ are two random vectors in a sub-linear expectation space $(\Omega, \mathcal{H}, \tilde{\mathbb{E}})$ with $\tilde{\mathbb{V}}(\|X\| > \lambda) \to 0$ and $\tilde{\mathbb{V}}(\|Y\| > \lambda) \to 0$ as $\lambda \to \infty$. Then

$$
(X_n, Y_n) \overset{d}{\to} (\bar{X}, \bar{Y}),
$$

(6.15)

where $\bar{Y}$ is independent to $\bar{X}$, $\bar{X} \overset{d}{=} X$ and $\bar{Y} \overset{d}{=} Y$.

**Proof.** Suppose $\varphi(x, y) : \mathbb{R}_{d_1} \otimes \mathbb{R}_{d_2} \to \mathbb{R}$ is a bounded continuous function. We want to show that

$$
\hat{\mathbb{E}}[\varphi(X_n, Y_n)] \to \mathbb{E}[\varphi(\bar{X}, \bar{Y})].
$$

(6.16)
First we assume that \( \varphi(x, y) \) is a bounded Lipschitz function. Without loss of generality, we assume \( 0 \leq \varphi(x, y) \leq 1 \) and \( |\varphi(x_1, y_1) - \varphi(x_2, y_2)| \leq \|x_1 - x_2\| + \|y_1 - y_2\| \). Let 
\[
g_n(x) = \hat{E}[\varphi(x, Y_n)|\mathcal{H}_n] \quad \text{and} \quad g(x) = \hat{E}[\varphi(x, \tilde{Y})].
\]
Then
\[
|g(x_1) - g(x_2)| \leq \hat{E}[|\varphi(x_1, \tilde{Y}) - \varphi(x_2, \tilde{Y})|] \leq \|x_1 - x_2\|
\]
and
\[
\left| \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n] - g_n(x) \right| \leq \hat{E}[|\varphi(X_n, Y_n) - \varphi(x, Y_n)||\mathcal{H}_n] \leq \|X_n - x\| \quad \text{in } L_1,
\]
by Lemma 4.3. We use an argument of Hu, Li and Liu (2018) (c.f. Proposition 3.4) to approximate the function \( \varphi(x, y) \). For fixed \( N \geq 1 \), denote \( B_N(0) = \{x : \|x\| \leq N\} \). By partition of unity theorem, there exist \( h_i \in C_{b,\text{lip}}(\mathbb{R}^d) \), \( i = 1, \cdots, k_N \), such that \( 0 \leq h_i(x) \leq 1 \), \( I_{B_N(0)} \leq \sum_{i=1}^{k_N} h_i(x) \leq 1 \), and the diameter of support \( \lambda(supp(h_i)) \leq 1/N \). Choose \( x_i \) such that \( h_i(x_i) > 0 \). Then
\[
\left| \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n] - \sum_{i=1}^{k_N} h_i(X_n)g_n(x_i) \right|
\]
\[
\leq \sum_{i=1}^{k_N} h_i(X_n) \left| \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n] - g_n(x_i) \right| + (1 - \sum_{i=1}^{k_N} h_i(X_n)) \left| \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n] \right|
\]
\[
\leq \sum_{i=1}^{k_N} h_i(X_n)\|X_n - x_i\| + (1 - \sum_{i=1}^{k_N} h_i(X_n)) \leq \frac{1}{N} + (1 - \sum_{i=1}^{k_N} h_i(X_n)) \quad \text{in } L_1.
\]
It follows that
\[
\left| \hat{E}[\varphi(X_n, Y_n)] - \hat{E}\left[ \sum_{i=1}^{k_N} h_i(X_n)g_n(x_i) \right] \right|
\]
\[
= \hat{E}\left[ \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n]\right] - \hat{E}\left[ \sum_{i=1}^{k_N} h_i(X_n)g_n(x_i) \right]
\]
\[
\leq \hat{E}\left[ \hat{E}[\varphi(X_n, Y_n)|\mathcal{H}_n] - \sum_{i=1}^{k_N} h_i(X_n)g_n(x_i) \right]
\]
\[
\leq \frac{1}{N} + \hat{E}\left[ 1 - \sum_{i=1}^{k_N} h_i(X_n) \right].
\]
Similarly,
\[
\left| \hat{E}[\varphi(\tilde{X}, \tilde{Y})] - \hat{E}\left[ \sum_{i=1}^{k_N} h_i(\tilde{X})g(x_i) \right] \right| = \left| \hat{E}[g(\tilde{X})] - \hat{E}\left[ \sum_{i=1}^{k_N} h_i(\tilde{X})g(x_i) \right] \right|
\]
\[
\leq \hat{E}\left[ |g(\tilde{X}) - \sum_{i=1}^{k_N} h_i(\tilde{X})g(x_i)| \right] \leq \frac{1}{N} + \hat{E}\left[ 1 - \sum_{i=1}^{k_N} h_i(\tilde{X}) \right].
\]
On the other hand, we have
\[
\left| \hat{E}\left[ \sum_{i=1}^{k_N} h_i(\mathbf{X}_n)g_n(x_i) \right] - \hat{E}\left[ \sum_{i=1}^{k_N} h_i(\mathbf{X})g(x_i) \right] \right| \\
\leq \sum_{i=1}^{k_N} \hat{E}\left[ |g_n(x_i) - g(x_i)| \right] \text{ as } n \to \infty,
\]
by (6.14), and
\[
\hat{E}\left[ \sum_{i=1}^{k_N} h_i(\mathbf{X}_n)g(x_i) \right] \to \hat{E}\left[ \sum_{i=1}^{k_N} h_i(\mathbf{X})g(x_i) \right],
\]
\[
\hat{E}[1 - \sum_{i=1}^{k_N} h_i(\mathbf{X}_n)] \to \hat{E}[1 - \sum_{i=1}^{k_N} h_i(\mathbf{X})]
\]
as \(n \to \infty\), by the fact that \(\mathbf{X}_n \overset{d}{\to} \mathbf{X}\). Combining the above arguments yields
\[
\limsup_{n \to \infty} \left| \hat{E}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] - \hat{E}[\varphi(\mathbf{X}, \mathbf{Y})] \right| \\
\leq \frac{2}{N} + 2\hat{E}[1 - \sum_{i=1}^{k_N} h_i(\mathbf{X})] \leq \frac{2}{N} + 2\bar{\nu}(\|\mathbf{X}\| > N) \to 0 \text{ as } N \to \infty.
\]
Hence (6.16) is proved for any bounded Lipschitz function \(\varphi\). For a bounded and uniformly continuous function \(\varphi\), we define
\[
\varphi_\delta = \varphi \ast \psi_\delta \text{ with } \psi_\delta(x, y) = \frac{1}{(2\pi \delta)^{(d_1 + d_2)/2}} \exp \left\{ -\frac{\sum_{i=1}^{d_1} x_i^2 + \sum_{j=1}^{d_2} y_j^2}{2\delta} \right\}.
\]
Then \(\varphi_\delta\) is a bounded Lipschitz function with \(\sup_{x,y} |\varphi_\delta(x, y) - \varphi(x, y)| \to 0 \text{ as } \delta \to 0\).
Hence, (6.16) holds for any bounded and uniformly continuous function \(\varphi\). Finally, let \(\varphi(x, y)\) be a bounded continuous function with \(|\varphi(x, y)| \leq M\). Let \(\lambda > 0\). For \(x = (x_1, \ldots, x_d)\), denote \(x_\lambda = ((-\lambda) \lor (x_1 \land \lambda))\), \ldots, \((-\lambda) \lor (x_{d_1} \land \lambda)\) and define \(y_\lambda\) similarly. Let \(\varphi_\lambda(x, y) = \varphi(x_\lambda, y_\lambda)\). Then \(\varphi_\lambda\) is a bounded uniformly continuous function with
\[
|\varphi_\lambda(x, y) - \varphi(x, y)| \leq 2MI\{\|x\| > \lambda\} + 2MI\{\|y\| > \lambda\}.
\]
It follows that
\[
\limsup_{n \to \infty} \left| \hat{E}[\varphi(\mathbf{X}_n, \mathbf{Y}_n)] - \hat{E}[\varphi(\mathbf{X}, \mathbf{Y})] \right| \\
\leq \limsup_{n \to \infty} \left| \hat{E}[\varphi_\lambda(\mathbf{X}_n, \mathbf{Y}_n)] - \hat{E}[\varphi_\lambda(\mathbf{X}, \mathbf{Y})] \right| \\
+ 2M \limsup_{n \to \infty} \left\{ \bar{\nu}(\|\mathbf{X}_n\| > \lambda) + \bar{\nu}(\|\mathbf{Y}_n\| > \lambda) \right\} \\
+ 2M \left\{ \bar{\nu}(\|\mathbf{X}\| > \lambda) + \bar{\nu}(\|\mathbf{Y}\| > \lambda) \right\} \\
\leq 4M\left\{ \bar{\nu}(\|\mathbf{X}\| > \lambda/2) + \bar{\nu}(\|\mathbf{Y}\| > \lambda/2) \right\} \to 0 \text{ as } \lambda \to \infty.
\]
The proof is completed. □

**Remark 6.1** In the original proofs of Lemma 6.1 and Theorem 3.2, we need an additional assumption on the operators \( \hat{E}_{n,k} \) as follows.

(a') If \( X = (X_1, \ldots, X_d) \in \mathcal{H}_{n,k}, Z \in \mathcal{H} \) and \( \varphi(x, y) \) is a bounded Lipschitz function, then

\[
\hat{E}[\varphi(X, Z)] = \hat{E}\left[\hat{E}_{n,k}[\varphi(x, Z)] \mid x = X\right].
\]

We thank one of the referees mentioning us the Proposition 3.4 of Hu, Li and Liu (2018) which helps us to remove this condition, though we fail to verify this proposition when the point by point monotonicity of the conditional sub-linear expectation (c.f. Definition 3.1 (1) of Hu, Li and Liu (2018)) is replaced by the \( L_1 \)-monotonicity (c.f. Lemma 4.3 (c)).

**Proof of Theorem 3.2.** With the same argument as that at the beginning of the proof of Theorem 3.1, we can assume that

\[
\delta_{n} = \sum_{k=1}^{k_{n}} \mathbb{E}[Z_{n,k}^{2} \mid \mathcal{H}_{n,k-1}] \leq 2\rho(1) \text{ in } L_{1},
\]

\[
\chi_{n} = \sum_{k=1}^{k_{n}} \left\{ \mathbb{E}[Z_{n,k} \mid \mathcal{H}_{n,k-1}] + \mathbb{E}[Z_{n,k} \mid \mathcal{H}_{n,k-1}] \right\} < 1 \text{ in } L_{1} \text{ and } |Z_{n,k}| \leq \epsilon_{n}, k = 1, \ldots, k_{n},
\]

with a sequence \( 0 < \epsilon_{n} \to 0 \). Let \( 0 < t_{1} < t_{2} \leq 1 \). Consider \( \{Z_{n,k} = Z_{n,\tau_{n}(t_{1}) + k} ; k = 1, \ldots, k_{n}^{*}\} \), \( S_{i}^{*} = \sum_{k=1}^{i} Z_{n,\tau_{n}(t_{1}) + k}, k_{n}^{*} = \tau_{n}(t_{2}) - \tau_{n}(t_{1}) \). Then \( S_{k_{n}^{*}}^{*} = S_{n,\tau_{n}(t_{2})} - S_{n,\tau_{n}(t_{1})} = \sum_{k=1}^{k_{n}^{*}} Z_{n,\tau_{n}(t_{1}) + k} \),

\[
\sum_{k=1}^{k_{n}^{*}} \mathbb{E}\left[Z_{n,\tau_{n}(t_{1}) + k}^{2} \mid \mathcal{H}_{n,\tau_{n}(t_{1}) + k-1}\right] \frac{\nu}{\rho(t_{2}) - \rho(t_{1})}
\]

By Theorem 2.1,

\[
S_{n,\tau_{n}(t_{2})} - S_{n,\tau_{n}(t_{1})} \rightharpoonup W(\rho(t_{2})) - W(\rho(t_{1})).
\]

Further, for any a bounded Lipschitz function \( \varphi(u, x) \), let \( V^{u}(t, x) \) be the unique viscosity solution of the following equation,

\[
\partial_{t}V^{u} + G(\partial_{xx}V^{u}) = 0, \quad (t, x) \in [0, \varrho + h] \times \mathbb{R}, \quad V^{u}|_{t=\varrho+h} = \varphi(u, x),
\]

where \( \varrho = \rho(t_{2}) - \rho(t_{1}) \). With the same argument for showing (6.3), we can show that

\[
\hat{E}\left[\hat{E}\left[\varphi(u, S_{n,\tau_{n}(t_{2})} - S_{n,\tau_{n}(t_{1})}) \mid \mathcal{H}_{n,\tau_{n}(t_{1})}\right]\right] - \hat{E}\left[\varphi(u, W(\rho(t_{2})) - W(\rho(t_{1})))\right] \to 0.
\]

(6.17)
The only difference is that (6.8), (6.9) and (6.10) are needed to be replaced, respectively, by

\[
\mathbb{E} \left[ \mathbb{E} \left[ V^u(\delta_{k_n}^* \wedge h_0, S_{k_n}^* \mid \mathcal{H}_{n,\tau_n(t_1)}) - V^u(0, 0) - \mathbb{E} \left[ \sum_{i=0}^{k_n-1} (J_{n,1,*}^i + J_{n,2,*}^i) \mid \mathcal{H}_{n,\tau_n(t_1)}) \right] \right] \right] \\
\leq \mathbb{E} \left[ \left| V^u(\delta_{k_n}^* \wedge h_0, S_{k_n}^*) - V^u(0, 0) - \sum_{i=0}^{k_n-1} (J_{n,1,*}^i + J_{n,2,*}^i) \right| \right],
\]

\[
\mathbb{E} \left[ \sum_{i=0}^{k_n-1} J_{n,1,*}^i \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] = \mathbb{E} \left[ \sum_{i=0}^{k_n-1} J_{n,1,*}^i \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] + \mathbb{E} \left[ \sum_{i=0}^{k_n-2} J_{n,1,*}^i \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] = \ldots = 0 \text{ in } L_1,
\]

and

\[
\mathbb{E} \left[ \sum_{i=0}^{k_n-1} (\pm J_{n,2,*}^i - \tilde{J}_{n,2,*}^i) \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] \\
\mathbb{E} \left[ \sum_{i=0}^{k_n-2} (\pm J_{n,2,*}^i - \tilde{J}_{n,2,*}^i) \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] + \mathbb{E} \left[ \pm J_{n,1,*}^{k_n-1} - \tilde{J}_{n,2,*}^{k_n-1} \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] \\
\leq \mathbb{E} \left[ \sum_{i=0}^{k_n-2} (\pm J_{n,2,*}^i - \tilde{J}_{n,2,*}^i) \bigg| \mathcal{H}_{n,\tau_n(t_1)} \right] \leq \ldots \leq 0 \text{ in } L_1.
\]

where \(J_{n,1,*}^i, J_{n,2,*}^i\) and \(\tilde{J}_{n,2,*}^i\) are defined the same as \(J_{n,1,*}^i, J_{n,2,*}^i\) and \(\tilde{J}_{n,2,*}^i\) with \(Z_{n,k}\) taking the place of \(Z_{n,k}\). On the other hand, note \(S_{n,\tau_n(t_1)} \xrightarrow{d} W(\rho(t_1))\). Hence,

\[
\left( S_{n,\tau_n(t_1)}, S_{n,\tau_n(t_2)} - S_{n,\tau_n(t_1)} \right) \xrightarrow{d} \left( W(\rho(t_1)), W(\rho(t_2)) - W(\rho(t_1)) \right).
\]

by (6.17) and Lemma 6.1. By induction, for any \(0 = t_0 < \ldots < t_d \leq 1\),

\[
\left( S_{n,\tau_n(t_1)} - S_{n,\tau_n(t_0)}, \ldots, S_{n,\tau_n(t_d)} - S_{n,\tau_n(t_{d-1})} \right) \xrightarrow{d} \left( W(\rho(t_1)) - W(\rho(t_0)), \ldots, W(\rho(t_d)) - W(\rho(t_{d-1})) \right),
\]

which implies (3.10). So, we have shown the convergence of finite dimensional distributions of \(W_n\). By Theorem 9 of Peng (2010) on the tightness and the argument of Lin and Zhang (2017) or Zhang (2015), to show that (3.11) holds for bounded continuous function \(\varphi\), it is sufficient to show that for any \(\epsilon' > 0\),

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{V} \left( \omega_{\delta} (W_n) \geq 3\epsilon' \right) = 0,
\]

(6.18)
where \( \omega_\delta(x) = \sup_{|t-s|<\delta, t,s \in [0,1]} |x(t) - x(s)| \) (c.f. Proposition B.1 in Appendix B). Assume \( 0 < \delta < 1/10 \). Let \( 0 = t_0 < t_1 \ldots < t_K = 1 \) such that \( t_k - t_{k-1} = \delta \), and let \( t_{K+1} = t_{K+2} = 1 \). It is easily seen that

\[
\forall \left( w_\delta(W_n) \geq 3\epsilon' \right) \leq 2 \sum_{k=0}^{K-1} \forall \left( \max_{s \in [t_k,t_{k+2}]} |S_n,\tau_n(s) - S_n,\tau_n(t_k)| \geq \epsilon' \right).
\]

On the other hand, for \( t, \gamma > 0 \), by (4.3) we have

\[
\widehat{E} \left[ \max_{s \leq \gamma} |S_n,\tau_n(t+s) - S_n,\tau_n(t)|^4 \right] \\
\leq C \widehat{E} \left[ \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{E} \left[ Z_{n,k}^4 |_{\mathcal{H}_{n,k-1}} \right] + \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \left( \widehat{E} \left[ Z_{n,k}^2 |_{\mathcal{H}_{n,k-1}} \right] \right)^2 \right] \\
+ C \widehat{E} \left[ \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \left( |\widehat{E}[Z_{n,k}|\mathcal{H}_{n,k-1}]| + |\widehat{E}|Z_{n,k}|\mathcal{H}_{n,k-1}]| \right)^4 \right] \\
\leq C \widehat{E} \left[ \left( \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{E} \left[ Z_{n,k}^2 |_{\mathcal{H}_{n,k-1}} \right] \right)^2 \right] + C\epsilon_n^2 \cdot 2\rho + C\epsilon_n^2 \left[ \chi_{n,k} \right].
\]

The last two terms above will go to zero by (3.4). For considering the first term, we note

\[
2\rho(1) \geq \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{E} \left[ Z_{n,k}^2 |_{\mathcal{H}_{n,k-1}} \right] \overset{\forall}{\rightarrow} \rho(t + \gamma) - \rho(t).
\]

It follows that

\[
\widehat{E} \left[ \left( \sum_{k=\tau_n(t)+1}^{\tau_n(t+\gamma)} \widehat{E} \left[ Z_{n,k}^2 |_{\mathcal{H}_{n,k-1}} \right] \right)^2 \right] \rightarrow (\rho(t + \gamma) - \rho(t))^2.
\]

So, we conclude that

\[
\limsup_n \sum_{k=0}^{K-1} \forall \left( \max_{s \in [t_k,t_{k+2}]} |S_n,\tau_n(s) - S_n,\tau_n(t_k)| \geq \epsilon' \right) \\
\leq \limsup_n \sum_{k=0}^{K-1} \left( \frac{1}{\epsilon^4} \right)^4 \widehat{E} \left[ \max_{s \in [t_k,t_{k+2}]} |S_n,\tau_n(s) - S_n,\tau_n(t_k)|^4 \right] \\
\leq C \sum_{k=0}^{K-1} \left( \frac{1}{\epsilon^4} \right)^4 \rho(t_{k+2})^2 \leq C \rho(1) \left( \frac{1}{\epsilon^4} \right)^4 \sup_{|t-s| \leq 2\delta} |\rho(t) - \rho(s)| \rightarrow 0
\]

by taking \( \delta \rightarrow 0 \). Hence, (6.18) is verified. And the proof is completed. \( \square \)
Appendix

A The properties of the conditional expectations.

In this appendix, we give the proofs of Lemmas 4.2 and 4.3 on the properties of the conditional expectation.

Proof of Lemma 4.2. (1) is obvious. For (2), note that

\[
\hat{E}[(X - Y)^q] \leq \epsilon^q + \epsilon^q \mathbb{P}(X - Y \geq \epsilon) + \hat{E}[(X - Y - c)^q]
\]

and

\[
\hat{E}[(X - Y - c)^q] \leq \frac{\hat{E}[(X - Y)^p]}{c^{p-q}} \rightarrow 0 \text{ as } c \rightarrow \infty.
\]

The results follows.

For (3), let \( \epsilon > 0 \) and \( M > 0 \) be given. Let \( 0 < \delta < 1 \) such that \( |x - y| \leq \delta \) and \( |y| \leq M \) implies \( |f(x) - f(y)| \leq \epsilon \). Then,

\[
\mathbb{V}(f(X) - f(Y) \geq \epsilon) \leq \mathbb{V}(X - Y \geq \delta) + \mathbb{V}|Y| \geq M).
\]

The result follows.

For (4), note for \( y, x \geq 0 \), \( x^p - y^p \leq px^{p-1}(x - y) \). So,

\[
\hat{E}[X^p] - \hat{E}[Y^p] \leq p \hat{E}[X^{p-1}(X - Y)^+] \leq p(\hat{E}[X^p])^{1/q} \left( \hat{E}[(X - Y)^p] \right)^{1/p} = 0.
\]

For (5), note that the countable additivity of \( \hat{E} \) implies

\[
\hat{E}[(X - Y)^p] \leq \int_0^\infty \mathbb{V}((X - Y)^p > y) \, dy = \int_0^\infty \mathbb{V}(X - Y > y^{1/p}) \, dy
\]

(cf. Lemma 3.9 of Zhang (2016)). The result follows. \( \square \)

Proof of Lemma Lemma 4.3. (c) Let \( 0 \leq f \in \mathcal{H}_{n,k} \) be a bounded random variable. Then

\[
\hat{E}\left[f(\hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y])\right] = \hat{E}\left[\hat{E}_{n,k}[fX - \hat{E}_{n,k}[fY]]\right]
\]

\[
\leq \hat{E}\left[fX - \hat{E}_{n,k}[fY]\right] \leq \hat{E}\left[fX - fY + fY - \hat{E}_{n,k}[fY]\right]
\]

\[
\leq \hat{E}\left[f(X - Y)^+\right] + \hat{E}\left[fY - \hat{E}_{n,k}[fY]\right] \leq \hat{E}[fY - \hat{E}_{n,k}[fY]]
\]

\[
= \hat{E}\left[\hat{E}_{n,k}[fY - \hat{E}_{n,k}[fY]]\right] = \hat{E}\left[\hat{E}_{n,k}[fY] - \hat{E}_{n,k}[fY]\right] = 0,
\]

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which will implies \( \mathbb{E} \left[ \left( \hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y] \right)^+ \right] = 0 \). In fact, let \( Z = \hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y] \) and choose \( f \) to be a bounded Lipschitz function of \( Z \) such that \( I\{Z \geq 2\epsilon\} \leq f \leq I\{Z \geq \epsilon\} \). Then,

\[
\mathbb{E}[Z^+] \leq 2\epsilon + \mathbb{E}[fZ] \leq 2\epsilon.
\]

(d) The second inequality is due to (c). For the first one, let \( Z = \hat{E}_{n,k}[X] - \hat{E}_{n,k}[Y] - \hat{E}_{n,k}[X - Y] \). With the same argument as in (c), it is sufficient to show that \( \mathbb{E}[fZ] \leq 0 \) for any bounded \( 0 \leq f \in \mathcal{H}_{n,k} \). Now,

\[
\mathbb{E}[fZ] = \mathbb{E} \left[ \hat{E}_{n,k} \left[ fX - \hat{E}_{n,k}[fY] - \hat{E}_{n,k}[fX - fY] \right] \right] \\
= \mathbb{E} \left[ fX - \hat{E}_{n,k}[fY] - \hat{E}_{n,k}[fX - fY] \right] \\
= \mathbb{E} \left[ (fY - \hat{E}_{n,k}[fY]) + (fX - fY - \hat{E}_{n,k}[fX - fY]) \right] \\
\leq \mathbb{E} \left[ fY - \hat{E}_{n,k}[fY] \right] + \mathbb{E} \left[ fX - fY - \hat{E}_{n,k}[fX - fY] \right] = 0.
\]

(e) Suppose \( k < l \). Let \( Z = \hat{E}_{n,k} \left[ \left( \hat{E}_{n,l}[X] \right) \right] - \hat{E}_{n,k}[X] \) and \( f \geq 0 \) be a bounded random variable in \( \mathcal{H}_{n,k} \). Then,

\[
\mathbb{E}[fZ] = \mathbb{E} \left[ \hat{E}_{n,k} \left[ \left( \hat{E}_{n,l}[fX] \right) \right] - \hat{E}_{n,k}[fX] \right] \\
= \mathbb{E} \left[ \hat{E}_{n,k} \left[ fX - \hat{E}_{n,k}[fX] \right] \right] = \mathbb{E} \left[ fX - \hat{E}_{n,k}[fX] \right] = 0.
\]

which will imply \( \mathbb{E}[Z^+] = 0 \). On the other hand, note \( -Z \leq \hat{E}_{n,k} \left[ \left( \hat{E}_{n,l}[X] \right) \right] - \hat{E}_{n,k}[X] \) by Property (d). We have

\[
\mathbb{E}[f(-Z)] \leq \mathbb{E} \left[ \hat{E}_{n,k} \left[ fX - \hat{E}_{n,k}[fX] \right] \right] = \mathbb{E} \left[ fX - \hat{E}_{n,k}[fX] \right] = 0,
\]

which implies \( \mathbb{E}[(Z^+)^+] = 0 \). So, \( \mathbb{E}[|Z|] = 0 \).

(f) Let \( Z = \hat{E}_{n,k}[X] \) and \( 0 \leq f \in \mathcal{H}_{n,k} \) be a bounded random variable with \( fZ^+ = 0 \) and \(|f| \leq 1\). Then \( Z \in \mathcal{L}(\mathcal{H}) \). We first show that \( f|Z|^p \in \mathcal{L}(\mathcal{H}) \) for any \( p \geq 1 \). It is obvious that \( f|Z| \in \mathcal{L}(\mathcal{H}) \). Assume that \( k \geq 1 \) is an integer, and \( f|Z|^p \in \mathcal{L}(\mathcal{H}) \) for \( p \leq k \). Let \( p' \geq k \), \( p' \leq p < p' + 1 \). Note

\[
0 \leq f|X||Z|^{p-1} \leq \frac{p' + 1 - p}{p'} |X|^{p' - p} \frac{p - 1}{p'} f|Z|^{p'}.
\]

Choosing \( p' = k \) yields \( X, f|Z|^{p-1}, f|X||Z|^{p-1} \in \mathcal{L}(\mathcal{H}) \). So by the Properties (a), (b) and (d),

\[
\mathbb{E}[f|Z|^p] = \mathbb{E} \left[ f|Z|^{p-1}(\hat{E}_{n,k}[X]) \right] = \mathbb{E} \left[ -\hat{E}_{n,k}[X| |Z|^{p-1}] \right] \\
\leq \mathbb{E} \left[ \hat{E}_{n,k}[|X| \cdot f|Z|^{p-1}] \right] = \mathbb{E}[f|X||Z|^{p-1}] < \infty
\]
if \( k \leq p < k+1 \). Choosing \( p' = k+1/2 \) and repeating the same argument yield \( \mathbb{E}[f|Z|p] < \infty \) if \( k+1/2 \leq p < k+3/2 \). So, \( f|Z|^p \in \mathcal{L}(\mathcal{H}) \) for \( k \leq p \leq k+1 \). By the induction, for any \( p \geq 1 \), \( \mathbb{E}[f|Z|^p] < \infty \) which will imply \( \mathbb{E}[(Z^-)^p] < \infty \). And similarly by choosing \( f \) such that \( fZ^- = 0 \), we will have \( \mathbb{E}[(Z^+)^p] < \infty \). So, we have \( \mathbb{E}[|Z|^p] < \infty \) for any \( p \geq 1 \).

Finally, by (c), \( |Z| \leq M \) in \( L_1 \). Hence, by Lemma 4.2 (2), the result follows. The proof is now completed. \( \square \)

### B Tightness.

**Proposition B.1** Let \( \{Z_{n,k}; k = 1, \ldots, k_n\} \) be an array of random variables with \( \mathbb{E}[|Z_{n,k}|] < \infty \), \( k = 1, \ldots, k_n \), and \( \tau_n(t) \) be a non-decreasing function in \( D_{[0,1]} \) which takes integer values with \( \tau_n(0) = 0, \tau_n(1) = k_n \). Define \( S_{n,i} = \sum_{k=1}^{i} Z_{n,k} \),

\[
W_n(t) = S_{n,\tau_n(t)}. \tag{B.1}
\]

Assume that for any \( \epsilon > 0 \),

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup V(w_{\delta}(W_n) \geq \epsilon) = 0, \tag{B.2}
\]

where \( \omega_{\delta}(x) = \sup_{|t-s| < \delta, t,s \in [0,1]} |x(t) - x(s)| \). Then \( \{W_n\} \) is tight in \( D_{[0,1]} \) endowed the Skorohod topology, i.e., for any \( \eta > 0 \), there exists a compact set \( K \) in \( D_{[0,1]} \) such that

\[
\sup_n V(W_n \not\in K) \leq \eta. \tag{B.3}
\]

Further, if (3.10) holds for any \( 0 < t_1 < \cdots, t_d \leq 1 \), then (3.11) holds.

**Proof.** The proof of the tightness is similar to that of the tightness of probability measures (c.f. Billingsley (1968)). The only difference we shall note is that \( V \) may be not countably additive and may be not continuous. For \( T_0 \subset [0,1] \), define

\[
w(x, T_0) = \sup_{t,s \in T_0} |x(t) - x(s)|,
\]

and

\[
w_{\delta}'(x) = \inf \max_{t_i, 1 \leq i \leq \nu} w\left(x, [t_{i-1}, t_i]\right),
\]

where the infimum extends over all sets \( \{t_i\} \) with

\[
0 = t_0 < t_1 < \cdots < t_{\nu-1} < t_\nu = 1, \min_{1 \leq i \leq \nu} (t_i - t_{i-1}) > \delta.
\]
Note \( w'_\delta(x) \leq w_{2\delta}(x) \),
\[
|x(t)| \leq |x(0)| + \sum_{i=1}^{k} |x(it/k) - x((i-1)t/k)| \leq |x(0)| + kw_{1/k}(x),
\]
and \( W_n(0) = 0 \). From (B.2) it follows that
\[
\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{V} \left( \sup_{t} |W_n(t)| > a \right) = 0 \tag{B.4}
\]
and
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{V} (w'_\delta(W_n) \geq \epsilon) = 0, \ \forall \epsilon > 0. \tag{B.5}
\]
For fixed \( n \), let \( 0 < t^n_1 < \cdots < t^n_{\nu-1} \leq 1 \) be the jump times of the step function \( \tau_n(t) \), \( t^n_0 = 0 \), \( t^n_{\nu} = 1 \). Then
\[
w(W_n, [t^n_{i-1}, t^n_i]) = 0, \quad i = 1, \cdots, \nu.
\]
Let \( \delta^n_0 = \min_{1 \leq i \leq \nu-1} (t^n_i - t^n_{i-1}) \) if \( t^n_{\nu-1} = 1 \), and \( = \min_{1 \leq i \leq \nu} (t^n_i - t^n_{i-1}) \) if \( t^n_{\nu-1} < 1 \). Then
\[
w'_\delta(W_n) = 0 \text{ when } \delta < \delta^n_0. \tag{B.6}
\]
On the other hand, it is obvious that
\[
\lim_{a \to \infty} \mathbb{V} \left( \sup_{t} |W_n(t)| > a \right) \leq \lim_{a \to \infty} \frac{\sum_{k=1}^{k_n} \hat{E} |Z_{n,k}|}{a} = 0.
\]
Hence, (B.4) and (B.5) imply that
\[
\lim_{a \to \infty} \sup_n \mathbb{V} \left( \sup_{t} |W_n(t)| > a \right) = 0 \tag{B.7}
\]
and
\[
\lim_{\delta \to 0} \sup_n \mathbb{V} (w'_\delta(W_n) \geq \epsilon) = 0, \ \forall \epsilon > 0. \tag{B.8}
\]
Now, for any \( \eta > 0 \) and a sequence \( 0 < \epsilon_k \to 0 \), choose \( a > 0 \) and \( 0 < \delta_k \to 0 \) such that
\[
\sup_n \mathbb{V} \left( \sup_{t} |W_n(t)| > a \right) < \eta/2 \quad \text{and} \quad \sup_n \mathbb{V} (w'_\delta(W_n) > \epsilon_k) < \eta/2^{k+1}.
\]
Now, let \( B_0 = \{ x \in D_{[0,1]} : \sup_t |x(t)| \leq a \} \), \( B_k = \{ x \in D_{[0,1]} : w'_\delta(x) \leq \epsilon_k \} \) and \( A = \bigcap_{k=0}^{\infty} B_k \).
Then \( \sup_{x \in A} \sup_t |x(t)| \leq a \) and \( \lim_{\delta \to 0} \sup_{x \in A} w'_\delta(x) = 0 \). By the Arzalá-Ascoli theorem, the closure of \( A \) is a compact set in \( D_{[0,1]} \). On the other hand, by noting (B.6),
\[
\{ W_n \notin d(A) \} \subset \{ \sup_{t} |W_n(t)| > a \} \bigcup_{k=1}^{\infty} \{ w'_\delta(W_n) > \epsilon_k \}
\]
\[
\subset \{ \sup_{t} |W_n(t)| > a \} \bigcup_{k : \delta_k \geq \delta^n_0} \{ w'_\delta(W_n) > \epsilon_k \}.
\]
By the (finite) sub-additivity of $\mathcal{V}$, it follows that
\[
\mathcal{V}(W_n \notin \text{cl}(A)) \leq \mathcal{V}\left(\sup_t |W_n(t)| > a\right) + \sum_{k: \delta_k \geq \delta_0} \mathcal{V}(w^*_k(W_n) > \epsilon_k) \tag{B.9}
\]
\[
< \frac{\eta}{2} + \sum_{k=1}^{\infty} \frac{\eta}{2^{k+1}} = \eta.
\]
The proof of the tightness (B.3) is completed.

Now, consider the G-Brownian motion $W$. In Zhang (2015), it is proved that
\[
\lim_{\delta \to 0} \mathcal{V}(w^{\delta}_1(W) \geq \epsilon) = 0 \quad \text{for any} \quad \epsilon > 0.
\]
Note that $\rho(\cdot)$ is a uniformly continuous function on $[0, 1]$. It follows that
\[
\lim_{\delta \to 0} \mathcal{V}(w^{\delta}_1(W \circ \rho) \geq \epsilon) = 0 \quad \text{for any} \quad \epsilon > 0.
\]
With the same argument as (B.9) one can show that for any $\eta > 0$, there exists a compact set $K$ in $D_{[0,1]}$ such that
\[
\mathcal{V}(W \circ \rho \notin K) < \eta.
\]

For $0 = t_0 < t_1 < t_2 \ldots < t_{d-1} < t_d = 1$, we define the projection $\pi_{t_1, \ldots, t_d}$ from $D_{[0,1]}$ to $\mathbb{R}^d$ by
\[
\pi_{t_1, \ldots, t_d} x = (x(t_1), \ldots, x(t_d)),
\]
and define a map $\Pi^{-1}_{t_1, \ldots, t_d}$ from $\mathbb{R}^d$ to $D_{[0,1]}$ by
\[
\Pi^{-1}_{t_1, \ldots, t_d}(x_1, \ldots, x_d) = \begin{cases} 0, & \text{if } t \in [0, t_1); x_k, & \text{if } t \in [t_k, t_{k+1}) (k = 1, \ldots, d); x_d, & \text{if } t = t_d. \end{cases}
\]
Then $\Pi^{-1}_{t_1, \ldots, t_d}$ is a continuous map. Denote $\tilde{\pi}_{t_1, \ldots, t_d} = \Pi^{-1}_{t_1, \ldots, t_d} \circ \pi_{t_1, \ldots, t_d}$. Let $\varphi \in C_b(D_{[0,1]})$. Then $\varphi(\tilde{\pi}_{t_1, \ldots, t_d} x) = \varphi \circ \Pi^{-1}_{t_1, \ldots, t_d}(x(t_1), \ldots, x(t_d))$ and $\varphi \circ \Pi^{-1}_{t_1, \ldots, t_d} \in C_b(\mathbb{R}^d)$. By (3.10) on the convergence of the finite-dimensional distributions of $W_n$, it follows that
\[
\lim_{n \to \infty} \mathbb{E}[\varphi(\tilde{\pi}_{t_1, \ldots, t_d} W_n)] = \lim_{n \to \infty} \mathbb{E}[\varphi \circ \Pi^{-1}_{t_1, \ldots, t_d}(W_n(t_1), \ldots, W_n(t_d))]
\]
\[
= \mathbb{E}[\varphi \circ \Pi^{-1}_{t_1, \ldots, t_d}(W(\rho(t_1)), \ldots, W(\rho(t_d)))]
\]
Now, suppose that $t_{i+1} - t_i < \delta$ for $i = 0, \ldots, d - 1$. Recall $\omega(\cdot) = \sup_{|t-s| < \delta} |x(t) - x(s)|$, and let $d_0(\cdot, \cdot)$ be the Skorohod distance in $D_{[0,1]}$ and $|x| = \sup_{0 \leq t \leq 1} |x(t)|$. It is easily seen
that \( d_0(\bar{\pi}_{t_1, \ldots, t_d}, x, x) \leq \|\bar{\pi}_{t_1, \ldots, t_d} x - x\| \leq \omega_\delta(x) \). Let \( \epsilon > 0 \) be given. Since \( \varphi \) is a continuous function, for each \( x \), there is an \( \epsilon_x > 0 \) such that

\[
|\varphi(x) - \varphi(y)| < \epsilon/2 \text{ whenever } d_0(x, y) < \epsilon_x.
\]

Let \( K \subset D_{[0,1]} \) be a compact set. Then it can be covered by a union of finite many of the sets \( \{ y : d_0(x, y) < \epsilon_x/2 \}, \ x \in K \). So, there is an \( \epsilon_K > 0 \) such that \( |\varphi(x) - \varphi(y)| < \epsilon \) whenever \( d_0(x, y) < \epsilon_K \) and \( x \in K \). Denote \( M = \sup_x |\varphi(x)| \). It follows that

\[
|\varphi(\bar{\pi}_{t_1, \ldots, t_d} x) - \varphi(x)| < \epsilon + 2M\{\omega_\delta(x) \geq \epsilon_K \} + 2M\{x \notin K \}.
\]

By the tightness of \( \{W_n\} \) and \( W \circ \rho \), respectively, we can choose \( K \) and \( \delta \) such that

\[
sup_n \vee (\omega_\delta(W_n) \geq \epsilon_K) + sup_n \vee (W_n \notin K) \leq \frac{\epsilon}{4M} \text{ and } \vee (\omega_\delta(W \circ \rho) \geq \epsilon_K) + \vee (W \circ \rho \notin K) \leq \frac{\epsilon}{4M}.
\]

Hence

\[
\begin{align*}
&\left| \mathbb{E}[\varphi(W_n)] - \mathbb{E}[\varphi(W \circ \rho)] \right| \\
\leq &\left| \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W_n)] - \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W \circ \rho)] \right| \\
&+ \left| \mathbb{E}[\varphi(W_n)] - \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W_n)] \right| + \left| \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W \circ \rho)] \right| \\
\leq &\left| \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W_n)] - \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W \circ \rho)] \right| \\
&+ 2\epsilon + 2M\vee (\omega_\delta(W_n) \geq \epsilon_K) + 2M\vee (W_n \notin K) \\
&+ 2M\vee (\omega_\delta(W \circ \rho) \geq \epsilon_K) + 2M\vee (W \circ \rho \notin K) \\
&\leq \left| \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W_n)] - \mathbb{E}[\varphi(\bar{\pi}_{t_1, \ldots, t_d} W \circ \rho)] \right| + 3\epsilon.
\end{align*}
\]

Letting \( n \to \infty \) and then \( \epsilon \to 0 \) completes the proof of (3.11). \( \Box \)

**Acknowledgements**

Special thanks go to the anonymous referees and the associate editor for their constructive comments, which led to a much improved version of this paper.

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