Diameter rigidity for Kähler manifolds with positive bisectional curvature

Ved Datar¹ · Harish Seshadri¹

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Abstract
We prove that a Kähler manifold with positive bisectional curvature and maximal diameter is isometric to complex projective space with the Fubini-Study metric.

1 Introduction
Let \((M, \omega)\) be a Kähler manifold. The bisectional curvature of \(\omega\) along real unit tangent vectors \(X, Y\) is defined to be

\[
BK(X, Y) = Rm(X, JX, JY, Y),
\]

where \(Rm\) denotes the Riemann curvature tensor of the Riemannian metric associated to \(\omega\). In this note we will be concerned with Kähler manifolds \((M, \omega)\) satisfying

\[
BK \geq 1,
\]

i.e., \(BK(X, Y) \geq 1\) for all real unit tangent vectors \(X, Y\).

A diameter comparison theorem was established for compact Kähler \(n\)-manifolds satisfying (1) in [5]. The comparison space here is the complex projective space \(\mathbb{C}P^n\)

¹ Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

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Harish Seshadri
harish@iisc.ac.in

Ved Datar
vvdatar@iisc.ac.in

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endowed with the Fubini-Study metric $\omega_{\mathbb{C}P^n}$, normalized so that

$$\int_{\mathbb{C}P^n} \omega_{\mathbb{C}P^n}^n = (2\pi)^n,$$

equivalently $\text{Ric} = (n + 1)\omega_{\mathbb{C}P^n}$.

**Theorem 1** (Li-Wang [5]) If $(M^n, \omega)$ is a compact Kähler manifold satisfying $\text{BK} \geq 1$, then

$$\text{diam}(M) \leq \text{diam}(\mathbb{C}P^n, \omega_{\mathbb{C}P^n}) = \frac{\pi}{\sqrt{2}}.$$

**Remark** In [5], the diameter bound is stated to be $\pi/2$. This is due to a different normalization for the Hermitian extension of the Riemannian metric.

The main result of this note is a characterization of the case of equality in Theorem 1:

**Theorem 2** Let $(M^n, \omega)$ be a compact Kähler manifold satisfying $\text{BK} \geq 1$. If

$$\text{diam}(M, \omega) = \text{diam}(\mathbb{C}P^n, \omega_{\mathbb{C}P^n}),$$

then $(M, \omega)$ is isometric to $(\mathbb{C}P^n, \omega_{\mathbb{C}P^n})$.

The diameter bound in Theorem 1 is analogous to the classical Bonnet-Myers diameter bound for compact Riemannian manifolds with positive Ricci curvature. However, one cannot relax the curvature assumption to a positive Ricci lower bound in the Kähler case, as pointed out in [6]: endow $\mathbb{C}P^1$ with the round metric of curvature $\frac{1}{n+1}$ and consider the product metric on the $n$-fold product

$$M = \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1.$$ The Ricci curvature of $M$ satisfies $\text{Ric} = (n + 1)\omega$, but

$$\text{diam}(M) = \sqrt{\frac{n}{n + 1}} \pi > \frac{\pi}{\sqrt{2}},$$

if $n \geq 2$.

In the Riemannian case, the equality case of the Bonnet-Myers diameter bound is the well-known maximal diameter theorem of Cheng. Theorem 2 can be regarded as the Kähler analogue of Cheng’s theorem.

Theorem 2 has been established under additional assumptions in [6,11]. In [6], the authors construct a totally geodesic $\mathbb{C}P^1$ with sectional curvature 2 and use this to show that rigidity holds if $\int_M \omega^n > \pi^n$. In [11], the authors assume that there are compact connected complex submanifolds $P$ and $Q$ in $M$ with $\text{dim}(P) + \text{dim}(Q) = n - 1$ and $d(P, Q) = \frac{\pi}{\sqrt{2}}$. An eigenvalue comparison theorem is then employed to show rigidity.
Our strategy for proving Theorem 2 is to establish a monotonicity formula for a function arising from Lelong numbers of positive currents on \( \mathbb{C}P^n \). In [7], the \( \partial \bar{\partial} \)-comparison theorem of [11] is reformulated as asserting the positivity of a certain \((1, 1)\)-current and this is the current we work with.

## 2 Lelong numbers and a monotonicity formula on \( \mathbb{C}P^n \)

Let \( M \) be a Kähler manifold. In what follows, we frequently use the real operator

\[
d^c = \frac{\sqrt{-1}}{2\pi} (\bar{\partial} - \partial).
\]

Note that

\[
\text{dd}^c = \frac{1}{\pi} \sqrt{-1} \partial \bar{\partial}.
\]

If \( T \) is a non-negative current on a \( M \) such that

\[
T = \text{dd}^c \varphi,
\]

in a neighbourhood of a point \( q \in M \), then the Lelong number of \( T \) at \( q \) is defined as

\[
\nu(T, q) := \lim_{r \to 0^+} \sup_{B_{\mathbb{C}^n}(0, r)} \frac{\varphi(z)}{\log r},
\]

where \( z \) is a holomorphic coordinate in a neighbourhood of \( q \) such that \( z(q) = 0 \). It is not difficult to see (for instance using the maximum principle) that the quotient on the right is increasing in \( r \), and hence the limit \( \nu(T, q) \) exists and is moreover non-negative and independent of the choice of holomorphic coordinates. Note that the normalization is chosen so that if \( V \) is a smooth hypersurface with defining function \( f \), and \([V]\) denotes the current of integration along \( V \), then by the Poincaré-Lelong equation, \([V] = \text{dd}^c \log |f|\), and so \( \nu([V], q) = 1 \) for any point \( q \in V \).

The following proposition is well known (cf. [2, pg. 164–165]), but since the proof of our main theorem has a precise dependence on the constants involved, we provide a proof for the convenience of the reader.

**Proposition 3** Suppose \( T = \text{dd}^c \varphi \) as above in a neighbourhood of \( q \) with holomorphic coordinates \( z = (z^1, \ldots, z^n) \) such that \( z(q) = 0 \). We then have

\[
\nu(T, q) = \lim_{r \to 0^+} \frac{1}{\pi^{n-1}r^{2n-2}} \int_{B_{\mathbb{C}^n}(0, r)} T \wedge \omega_{\mathbb{C}^n}^{n-1},
\]

where \( B_{\mathbb{C}^n}(0, r) \) is the ball of radius \( r \) around the origin with respect to the Euclidean metric \( \omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \).
Note that quantity on the right above is increasing in $r$ (cf. [4, pg. 390]), and hence the limit, in particular, exists.

**Proof** First suppose that $\varphi$ is smooth. We let

$$v(dd^c \varphi, 0, t) := \frac{1}{\pi^{n-1} t^{2n-2}} \int_{B_{C^n}(0,t)} dd^c \varphi \wedge \omega_{C^n}^{n-1},$$

$$\mu_t(\varphi) := \frac{1}{\sigma_{2n-1}} \int_{\mathbb{S}^{2n-1}} \varphi(t, \theta) d\sigma(\theta),$$

where $\sigma_{2n-1} = 2\pi^n/(n-1)!$ is the volume of the unit sphere in $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, and $d\sigma$ is the standard Riemannian measure on $\mathbb{S}^{2n-1}$. Let $\mathbb{S}_{t}^{2n-1}$ be the sphere of radius $t$ centred at the origin, $d\sigma_t$ the Riemannian measure on it and let $\partial\varphi/\partial \nu$ be the normal derivative of $\varphi$. Differentiating in $t$,

$$\frac{d\mu_t(\varphi)}{dt} = \frac{1}{\sigma_{2n-1}} \int_{\mathbb{S}^{2n-1}} \frac{\partial \varphi}{\partial t}(t, \theta) d\sigma(\theta)$$

$$= \frac{1}{\sigma_{2n-1} t^{2n-1}} \int_{\mathbb{S}^{2n-1}_t} \frac{\partial \varphi}{\partial \nu} d\sigma_t$$

$$= \frac{2}{\sigma_{2n-1} t^{2n-1}} \int_{B_{C^n}(0,t)} \Delta_{\partial} \varphi \omega_{C^n}^{n-1}$$

$$= \frac{2}{\sigma_{2n-1} t^{2n-1}} \int_{B_{C^n}(0,t)} \Delta_{\partial} \varphi \wedge \omega_{C^n}^{n-1}$$

Note that in the third line we have the $\bar{\partial}$-Laplacian $\Delta_{\partial}$, and hence the factor of 2 on application of Green’s formula. Integrating the above equality from $r$ to 1, we obtain the so-called Jensen-Lelong formula (cf. [2, pg. 163]):

$$\mu_1(\varphi) - \mu_r(\varphi) = \int_r^1 v(dd^c \varphi, 0, t) \frac{dt}{t}.$$

By regularization, the above equality also holds for a general, possibly non-smooth, plurisubharmonic function $\varphi$. Changing variables $s = \log t$ and dividing by $\log r$ we have

$$\frac{\mu_r(\varphi)}{\log r} = \frac{\mu_1(\varphi)}{\log r} - \frac{1}{\log r} \int_{\log r}^0 v(dd^c \varphi, 0, e^s) ds.$$
and letting \( r \to 0^+ \) we obtain

\[
\lim_{r \to 0^+} \nu(T, 0, r) = \lim_{r \to 0^+} \frac{\mu_r(\varphi)}{\log r}.
\]

Next proceeding as in [2, pg. 165], by Harnack inequality and maximum principle, we have that

\[
\lim_{r \to 0^+} \mu_r(\varphi) \log r = \lim_{r \to 0^+} \sup_{z \in \partial B_{C^n}(0, r)} \varphi(z) \log r = \lim_{r \to 0^+} \sup_{z \in B_{C^n}(0, r)} \varphi(z) \log r.
\]

We require the following modification, which as far as we can tell, seems to be new.

**Proposition 4** Let \( T \) be a non-negative current on \( \mathbb{C}P^n \) in a Kähler class, and \( q \in \mathbb{C}P^n \). Then

\[
\Theta(T, q, r) := \frac{1}{(2\pi)^{n-1} \sin^{2n-2}(r/\sqrt{2})} \int_{B_{C^n}(q, r)} T \wedge \omega_{C^n}^{n-1}
\]

is increasing in \( r \). Here \( B_{C^n}(q, r) \) is the ball of radius \( r \) with respect to \( \omega_{C^n} \). Moreover, we also have that

\[
\lim_{r \to 0^+} \Theta(T, q, r) = \nu(T, q).
\]

Note that the factor in the denominator is precisely the volume of a ball of radius \( r \) in \( \mathbb{C}P^{n-1} \) with respect to the Fubini-Study metric \( \omega_{C^n} \) up to a factor of \((n-1)!\).

**Proof** Let us first assume that \( T \) is a smooth \((1, 1)\) Kähler form. We use homogenous coordinates \([\xi_0 : \xi_1 : \cdots : \xi_n] \) on \( \mathbb{C}P^n \) with \( q = [1 : 0 : \cdots : 0] \), and the usual in-homogenous coordinates \( Z_i = \frac{\xi_i}{\xi_0} \) on \( \xi_0 \neq 0 \). Then

\[
\omega = \sqrt{-1} \partial \bar{\partial} \log |\xi|^2 = \sqrt{-1} \partial \bar{\partial} \log(1 + |Z|^2).
\]

We then compute

\[
\Theta(T, q, r) = \frac{1}{2^{n-1} \sin^{2n-2}(r/\sqrt{2})} \int_{B_{C^n}(q, r)} T \wedge (dd^c \log |\xi|^2)^{n-1}
\]

\[
= \frac{1}{2^{n-1} \sin^{2n-2}(r/\sqrt{2})} \int_{\partial B_{C^n}(q, r)} T \wedge d^c \log(1 + |Z|^2)
\]

\[
\wedge (dd^c \log(1 + |Z|^2))^{n-2}.
\]

Now, it is well known fact that

\[
\cos^2 \frac{d_{C^n}(q, Z)}{\sqrt{2}} = \frac{|\xi_0|^2}{|\xi|^2} = \frac{1}{1 + |Z|^2}.
\]
For instance exploiting the $U(n)$ symmetry one needs to check this only for $\mathbb{CP}^1$ which can be done easily. We then have that for any $Z \in \partial B_{\mathbb{C}P^n}(q, r)$,

$$d^c \log(1 + |Z|^2) = \frac{|Z|^2}{1 + |Z|^2} d^c \log |Z|^2 = \sin^2 \left( \frac{r}{\sqrt{2}} \right) d^c \log |Z|^2.$$  

Putting this back in the formula above we have that

$$\Theta(T, q, r) = \frac{1}{2n-1} \int_{\partial B_{\mathbb{C}P^n}(q, r)} T \wedge d^c \log |Z|^2 \wedge (dd^c \log |Z|^2)^{n-2}.$$  

(3)

So if $r_1 < r_2$, then integrating by parts we have

$$\Theta(T, q, r_2) - \Theta(T, q, r_1) = \frac{1}{2n-1} \int_{A_{\mathbb{C}P^n}(q, r_1, r_2)} T \wedge (dd^c \log |Z|^2)^{n-1},$$

where $A_{\mathbb{C}P^n}(q, r_1, r_2) = B_{\mathbb{C}P^n}(q, r_2) \setminus B_{\mathbb{C}P^n}(q, r_1)$. Now if $\mu : \mathbb{CP}^n \to \mathbb{CP}^{n-1}$ is the projection from $q$ to $[\xi_0 = 0]$, then we have

$$\Theta(T, q, r_2) - \Theta(T, q, r_1) = \frac{1}{(2\pi)^{n-1}} \int_{A_{\mathbb{C}P^n}(q, r_1, r_2)} T \wedge (\mu^* \omega_{\mathbb{CP}^{n-1}})^{n-1} \geq 0.$$  

This proves the monotonicity for smooth currents. For a general positive current $T$ we can proceed by regularization. In fact in our case we can first let $r_1 < r_2 < R < \pi / \sqrt{2}$. Then $B(q, R)$ is contained in Euclidean ball (of radius $\tan R$) with respect to the inhomogenous coordinates. We can then use the standard convolution to find sequence of smooth non-negative forms $T_j$ converging weakly to $T$. Then since $r_1 < r_2 < R$,

$$\Theta(T, q, r_2) - \Theta(T, q, r_1) = \lim_{j \to \infty} \left( \Theta(T_j, q, r_2) - \Theta(T_j, q, r_1) \right) \geq 0.$$  

If $r_2 = \pi / \sqrt{2}$, then the result follows by the monotonic convergence.

Next, to compute the limit, we again first work with smooth Kahler forms. If $T$ is smooth then in formula (3), we observe that

$$d^c \log |Z|^2 = \frac{d^c |Z|^2}{|Z|^2} = \frac{d^c |Z|^2}{\tan^2(r / \sqrt{2})},$$

where notice that $d(q, Z) = r$ implies that

$$|Z|^2 = \tan^2 \left( \frac{r}{\sqrt{2}} \right).$$

Then we have

$$\Theta(T, q, r) = \frac{1}{2n-1} \int_{\partial B_{\mathbb{C}P^n}(q, r)} T \wedge d^c \log |Z|^2 \wedge (dd^c \log |Z|^2)^{n-2}$$

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\[
\begin{align*}
&= \frac{1}{2^{n-1} \tan^{2n-2}(r/\sqrt{2})} \int_{B_{\mathbb{C}^n}(q, r)} T \wedge d^c |Z|^2 \wedge (dd^c |Z|^2)^{n-2} \\
&= \frac{1}{2^{n-1} \tan^{2n-2}(r/\sqrt{2})} \int_{B_{\mathbb{C}^n}(q, r)} T \wedge (dd^c |Z|^2)^{n-1} \\
&= \frac{1}{\pi^{n-1} r^{2n-2}} \int_{B_{\mathbb{C}^n}(0, t)} T \wedge \omega_{\mathbb{C}^n}^{n-1},
\end{align*}
\]

where we integrated by parts in the third line and set \( t = \tan(r/\sqrt{2}) \), and noted that in terms of the \( Z \)-coordinates \( B_{\mathbb{C}^n}(q, r) = B_{\mathbb{C}^n}(0, t) \). Once again by regularization, as above, the above formula holds for general possibly non-smooth currents. Letting \( t \to 0^+ \) and applying Proposition 3 we obtain (2). \( \square \)

**Example 5** (The “model” case) On \( \mathbb{C}P^n \) consider the current \( T = \sqrt{-1} \partial \overline{\partial} \log |\xi|^2 = 2\pi [\xi_n = 0] \), and \( q = [1 : 0 : \cdots : 0] \). We regard this as the model case for reasons given in Section 3. Then for any \( r > 0 \),

\[
\int_{B_{\mathbb{C}^n}(q, r)} T \wedge \omega_{\mathbb{C}^n}^{n-1} = 2\pi \int_{B_{\mathbb{C}^n}(q, r) \cap \{ \xi_n = 0 \}} \omega_{\mathbb{C}^n}^{n-1} \\
= 2\pi \int_{B_{\mathbb{C}^n-1(q, r)}} \omega_{\mathbb{C}^n-1}^{n-1} \\
= (2\pi)^n \sin^{2n-2} \left( \frac{r}{\sqrt{2}} \right),
\]

and so \( \Theta(T, q, r) = 2\pi \) and is independent of \( r \). Note that if we consider a modified

\[
\hat{\Theta}(T, q, r) := \frac{1}{(2\pi)^{n-1} r^{2n-2}} \int_{B_{\mathbb{C}^n}(q, r)} T \wedge \omega_{\mathbb{C}^n}^{n-1},
\]

where we have \( r^{2n-2} \) in the denominator as in the usual Euclidean case, then for \( T \) and \( q \) as above, we would have that

\[
\hat{\Theta}(T, q, r) = 2\pi \frac{\sin^{2n-2}(r/\sqrt{2})}{r^{2n-2}}.
\]

It is easy to see that this function is *decreasing* in \( r \). The increasing property of \( \Theta(T, q, r) \) is crucial for our proof of Theorem 2.

### 3 Proof of the Theorem

In [7], Lott introduces the following current:

\[
T_{\omega, p} := \omega + \sqrt{-1} \partial \overline{\partial} \psi_p, \quad \psi_p := \log \cos^2 \left( \frac{d_p}{\sqrt{2}} \right),
\]

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where $p$ is some fixed point in $M$ and $d_p$ is the distance function from $p$. Note that \textit{a priori}, $T_{\omega, p}$ is only defined (and also smooth) away from the cut-locus of $p$. Using the Hessian comparison theorem in [11], which holds away from the cut-locus, Lott observed that $T$ is in fact a global non-negative current if $\omega$ satisfies (1).

If $\omega = \omega_{\mathbb{C}P^n}$, and $p = [0:0: \cdots :1]$, then as observed before

$$\cos^2 \left( \frac{d_{\omega_{\mathbb{C}P^n}, p}}{\sqrt{2}} \right) = \frac{|\xi_n|^2}{|\xi|^2},$$

and so

$$T_{\omega_{\mathbb{C}P^n}, p} = \sqrt{-1} \partial \bar{\partial} \log |\xi_n|^2,$$

is precisely the current considered in Example 5 above.

\textbf{Proof of Theorem} First note that by the proof of the Frankel conjecture (cf. [8,10]), $M$ is biholomorphic to $\mathbb{C}P^n$. So from now on we set $M = \mathbb{C}P^n$. Let $p, q \in \mathbb{C}P^n$ such that $d_{\omega, p}(q) = \pi/\sqrt{2}$.

We claim that

$$\nu(T_{\omega, p}, q) = \nu(\omega + \pi dd^c \psi_{\omega, p}) \geq 2\pi.$$

To see this, we fix holomorphic coordinates $z := (z^1, \ldots, z^n)$ near $q$ with $z(q) = 0$. Then $C^{-1}|z(x)| \leq d(q, x) \leq C|z(x)|$ for some constant $C > 0$, and hence it is enough to show that

$$\lim_{\varepsilon \to 0^+} \sup_{B(q, \varepsilon)} \psi_{\omega, p} \geq 2\log \varepsilon,$$

since $\omega$ being smooth does not contribute to the Lelong number. It is more convenient to work with

$$\delta_p = \frac{\pi}{2} - \frac{d_p}{\sqrt{2}}.$$

Then $\psi_p = 2 \log \sin \delta_p$. Note that by the diameter upper bound we have $\delta_p(z) \geq 0$ for all $z$, and that $\delta_p$ is Lipshitz with constant $1/\sqrt{2}$. Then for any $x \in \mathbb{C}P^n$,

$$\delta_p(x) \leq \frac{1}{\sqrt{2}} d(q, x),$$

and so $\sup_{B(q, \varepsilon)} \psi_{\omega, p} \leq C + 2 \log \varepsilon$. But then

$$\frac{\sup_{B(q, \varepsilon)} \psi_{\omega, p}}{\log \varepsilon} \geq \frac{C}{\log \varepsilon} + 2 \xrightarrow{\varepsilon \to 0^+} 2.$$

But then by monotonicity, if $\omega \in c[\omega_{\mathbb{C}P^n}]$, putting $R = \pi/\sqrt{2}$, we have

\[\text{ Springer} \]
2\pi c = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{C}^n} T \wedge \omega_{\mathbb{C}^n}^{n-1} = \Theta(T_\omega, p, q, R) \geq \lim_{r \to 0^+} \Theta(T_\omega, p, q, r) \geq \nu(T_\omega, p, q) \geq 2\pi,$

and so $c \geq 1$. On the other hand note that the bisectional curvature lower bound gives

$\text{Ric}(\omega) \geq (n + 1)\omega,$

and so $c \leq 1$ since $[\text{Ric}(\omega)] = (n + 1)[\omega_{\mathbb{C}^n}]$, and hence $c = 1$. But then the lower bound on the Ricci curvature, and the $\sqrt{-1}\partial \bar{\partial}$-lemma imply that $\omega$ must be Kähler-Einstein and hence isometric to $\omega_{\mathbb{C}^n}$.

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References

1. Colding, T.H.: Shape of manifolds with positive Ricci curvature. Invent. Math. 124(1–3), 175–191 (1996)
2. Demailly, J.P.: Complex analytic and differential geometry. https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf
3. Demailly, J.P.: Pseudoconvex-Concave Duality and Regularization of Currents. Several Complex Variables (Berkeley, CA, 1995–1996), 233–271, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge (1999)
4. Griffiths, P., Harris, J.: Principles of Algebraic Geometry, Reprint of the 1978 Original. Wiley Classics Library, Wiley, New York (1994). (xvi+813 pp. ISBN: 0-471-05059-8)
5. Li, P., Wang, J.: Comparison theorem for Kähler manifolds and positivity of spectrum. J. Differ. Geom. 69(1), 43–74 (2005)
6. Liu, G., Yuan, Y.: Diameter rigidity for Kähler manifolds with positive bisectional curvature. Math. Z. 290(3–4), 1055–1061 (2018)
7. Lott, J.: Comparison geometry of holomorphic bisectional curvature for Kähler manifolds and limit spaces, to appear in Duke Math J. arXiv:2005.02906
8. Mori, S.P.: Projective manifolds with ample tangent bundles. Ann. Math. (2) 110(3), 593–606 (1979)
9. Ni, L.: A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature. J. Am. Math. Soc. 17(4), 909–946 (2004)
10. Siu, Y.T., Yau, S.T.: Compact Kähler manifolds of positive bisectional curvature. Invent. Math. 59(2), 189–204 (1980)
11. Tam, L.-F., Yu, C.: Some comparison theorems for Kähler manifolds. Manuscripta Math. 137(3–4), 483–495 (2012)
12. Zhang, K.: On the optimal volume upper bound for Kähler manifolds with positive Ricci curvature, (with an appendix by Liu, Y.). arXiv:2001.04169v2

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