Off-diagonal Bethe Ansatz for the $D_3^{(1)}$ model

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Abstract

The exact solutions of the $D_3^{(1)}$ model (or the $so(6)$ quantum spin chain) with either periodic or general integrable open boundary conditions are obtained by using the off-diagonal Bethe Ansatz. From the fusion, the complete operator product identities are obtained, which are sufficient to enable us to determine spectrum of the system. Eigenvalues of the fused transfer matrices are constructed by the $T-Q$ relations for the periodic case and by the inhomogeneous $T-Q$ one for the non-diagonal boundary reflection case. The present method can be generalized to deal with the $D_n^{(1)}$ model directly.

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1 Introduction

Quantum integrable models play an important role in the condensed matter, cold atoms, theoretical and mathematical physics \[1, 2\]. Recently, the models with general integrable non-diagonal boundary reflections attract many attentions. Due to the $U(1)$-symmetry broken, the traditional Bethe Ansatz does not work. People developed many interesting methods, such as q-Onsager algebra \[3, 4, 5, 6\], separation of variables \[7, 8, 9\], modified algebraic Bethe Ansatz \[10, 11, 12, 13\], off-diagonal Bethe Ansatz (ODBA) \[14, 15\] and others \[16\].

The exact solution of integrable models associated with high rank Lie algebra is a very interesting issue. The case that the models with periodic or diagonal boundary reflection, where the $U(1)$-charge is conserved, has been studied extensively. For example, Reshetikhin derived the energy spectrum of the periodic quantum integrable models associated with $B_n$, $C_n$, $D_n$ and other Lie algebras by using the analytic Bethe Ansatz \[17, 18\]. Martins and Ramos studied this kind of models with periodic boundary condition by using the algebraic Bethe Ansatz \[19\]. Li, Shi and Yue investigated the open boundary cases where the reflection matrices only have the diagonal elements \[20, 21\].

The ODBA is a universal method to treat the quantum integrable models with or without $U(1)$-symmetry, even for the high rank cases. The nested ODBA was proposed to study the spin chain associated with $A_n$ Lie algebra with generic non-diagonal boundary reflections \[22, 23, 24\]. The ODBA also has been applied to the models associated with $A_2^{(2)}$ \[25\], $B_2$ \[26\] and $C_2$ \[27\] Lie algebras.

In this paper, we develop a nested ODBA method to approach the quantum integrable $D_3^{(1)}$ (simplest case of $D_n^{(1)}$) model with either periodic or non-diagonal open boundary conditions. We have obtained the complete operator product identities based on the fusion \[28, 29, 30, 31\] and the eigenvalues of the system based on the (inhomogeneous) $T - Q$ relations.

The paper is organized as follows. In section 2, we review the $R$-matrix associated with the $D_3^{(1)}$ model, which is the starting point. In section 3, we construct the $D_3^{(1)}$ model with periodic boundary condition. The Hamiltonian, closed functional relations among the eigenvalues of the fused transfer matrices, and the spectrum of the system are obtained with fusion techniques. In section 4, we study the $D_3^{(1)}$ model with an off-diagonal open boundary reflection. By constructing some operator product identities, we derive the exact eigenvalues...
of the transfer matrix in terms of an inhomogeneous $T - Q$ relation. Section 5 is attributed to concluding remarks.

## 2 The $R$-matrix

Throughout, $V$ denotes a six-dimensional linear space which endows the fundamental vectorial representation of $so(6)$ (or the $D_3$) algebra, and let $\{|i\}, i = 1, 2, \cdots, 6\}$ be an orthogonal basis of it. We adopt the standard notations: for any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as identity on the other factor spaces; For $B \in \text{End}(V \otimes V)$, $B_{ij}$ is an embedding operator of $B$ in the tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

The $R$-matrix $R(u) \in \text{End}(V \otimes V)$ of the $D_3^{(1)}$ model is

$$R_{ij}^{vv} = \begin{pmatrix}
\begin{matrix}
& a & b & b & b & g \\
& & b & a & b & d \\
& & & b & a & e \\
& & & & b & f \\
& & & & & g
\end{matrix}
\end{pmatrix}, \quad (2.1)$$
where the matrix elements are

\[ a = (u + 1)(u + 2), \quad b = u(u + 2), \quad f = 2, \]

\[ d = -u, \quad e = u(u + 1), \quad g = u + 2, \]

and \( u \) is the spectral parameter.

The \( R \)-matrix (2.1) satisfies the properties

\[
\text{regularity} : \quad R_{12}^{vv}(0) = \rho_1(0) \frac{1}{2} \mathcal{P}_{12},
\]

\[
\text{unitarity} : \quad R_{12}^{vv}(u) R_{21}^{vv}(-u) = \rho_1(u) = a(u) a(-u),
\]

\[
\text{crossing symmetry} : \quad R_{12}^{vv}(u) = V_1 \{ R_{12}^{vv}(-u - 2) \}^t_2 V_1 = V_2 \{ R_{12}^{vv}(-u - 2) \}^t_1 V_2, \quad (2.2)
\]

where \( \mathcal{P}_{12} \) is the permutation operator with the matrix elements

\[
\mathcal{P}_{12}^{ij} = \delta_{il} \delta_{jk},
\]

\( t_i \) denotes the transposition in the \( i \)-th space, and \( R_{21} = \mathcal{P}_{12} R_{12} \mathcal{P}_{12} \), and the crossing-matrix \( V \) is

\[
V = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}, \quad V^2 = \text{id}. \quad (2.3)
\]

Combining the crossing-symmetry and the unitarity of the \( R \)-matrix, one can derive the crossing-unitarity relation

\[
\text{crossing unitarity} : \quad R_{12}^{vv}(u)^t_1 R_{21}^{vv}(-u - 4)^t_1 = \rho_1(u + 2). \quad (2.4)
\]

The \( R \)-matrix (2.1) satisfies the Yang-Baxter equation

\[
R_{12}^{vv}(u - v) R_{13}^{vv}(u) R_{23}^{vv}(v) = R_{23}^{vv}(v) R_{13}^{vv}(u) R_{12}^{vv}(u - v). \quad (2.5)
\]

### 3 \( D_3^{(1)} \) model with periodic boundary condition

For the periodic boundary condition, we introduce the monodromy matrix

\[
T_0^{v}(u) = R_{01}^{vv}(u - \theta_1) R_{02}^{vv}(u - \theta_2) \cdots R_{0N}^{vv}(u - \theta_N), \quad (3.1)
\]

where the index 0 indicates the auxiliary space and the other tensor space \( V^\otimes N \) is the physical or quantum space, \( N \) is the number of sites and \( \{ \theta_j \} \) are the inhomogeneous parameters.

The monodromy matrix satisfies the Yang-Baxter relation

\[
R_{12}^{vv}(u - v) T_0^{v}(u) T_0^{v}(v) = T_0^{v}(v) T_0^{v}(u) R_{12}^{vv}(u - v). \quad (3.2)
\]
The transfer matrix is given by the trace of monodromy matrix in the auxiliary space

\[ t^{(p)}(u) = tr_0 T_0^v(u). \]  \hspace{1cm} (3.3)

From the Yang-Baxter relation (3.2), one can prove that the transfer matrices with different spectral parameters commute with each other, \( [t^{(p)}(u), t^{(p)}(v)] = 0 \). Therefore, \( t^{(p)}(u) \) serves as the generating function of the conserved quantities of the system. The Hamiltonian is given in terms of the transfer matrix (3.3) as

\[ H_p = \frac{\partial \ln t^{(p)}(u)}{\partial u} \bigg|_{u=0,\{\theta_j\}=0} = \sum_{k=1}^{N} H_{k,k+1}, \]  \hspace{1cm} (3.4)

where

\[ H_{k,k+1} = \mathcal{P}_{k,k+1} \frac{\partial}{\partial u} R_{k,k+1}(u) \bigg|_{u=0}, \]  \hspace{1cm} (3.5)

and the periodic boundary condition reads

\[ H_{N,N+1} = H_{N1}. \]  \hspace{1cm} (3.6)

### 3.1 The fusion

At some special points, the \( R \)-matrix (2.1) degenerates into the projector operators which enables us to do the fusion. For example, at the point of \( u = -2 \), we have

\[ R_{12}^{vv}(-2) = P_{12}^{vv(1)} S_{12}^{(1)}, \]  \hspace{1cm} (3.7)

where \( S_{12}^{(1)} \) is a constant matrix \( \in \text{End}(V \otimes V) \) omitted here, \( P_{12}^{vv(1)} \) is the one-dimensional projector

\[ P_{12}^{vv(1)} = |\psi_0\rangle \langle \psi_0|, \quad P_{21}^{vv(1)} = P_{12}^{vv(1)}, \]  \hspace{1cm} (3.8)

and the basis vector reads

\[ |\psi_0\rangle = \frac{1}{\sqrt{6}}(|16\rangle + |25\rangle + |34\rangle + |43\rangle + |52\rangle + |61\rangle). \]

From the Yang-Baxter equation (2.5), we obtain the following fusion identities

\[ P_{21}^{vv(1)} R_{13}^{vv}(u) R_{23}^{vv}(u-2) P_{21}^{vv(1)} = a(u) e(u-2) P_{21}^{vv(1)}, \]
\[ P_{12}^{vv(1)} R_{31}^{vv}(u) R_{32}^{vv}(u-2) P_{12}^{vv(1)} = a(u) e(u-2) P_{12}^{vv(1)}. \]  \hspace{1cm} (3.9)
At the point of \( u = -1 \), we have
\[
R_{12}^{\nu}(1) = P_{12}^{\nu(16)} S_{12}^{(16)},
\]
where \( S_{12}^{(16)} \) is a constant matrix omitted here and \( P_{12}^{\nu(16)} \) is a 16-dimensional projector
\[
P_{12}^{\nu(16)} = \sum_{i=1}^{16} |\phi_i^{(16)}\rangle \langle \phi_i^{(16)}|,
\]
with the basis vectors:
\[
|\phi_1^{(16)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |\phi_2^{(16)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle), \quad |\phi_3^{(16)}\rangle = \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle),
\]
\[
|\phi_4^{(16)}\rangle = \frac{1}{\sqrt{2}}(|15\rangle - |51\rangle), \quad |\phi_5^{(16)}\rangle = \frac{1}{\sqrt{2}}(|16\rangle - |61\rangle), \quad |\phi_6^{(16)}\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle),
\]
\[
|\phi_7^{(16)}\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle), \quad |\phi_8^{(16)}\rangle = \frac{1}{\sqrt{2}}(|25\rangle - |52\rangle), \quad |\phi_9^{(16)}\rangle = \frac{1}{\sqrt{2}}(|26\rangle - |62\rangle),
\]
\[
|\phi_{10}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|34\rangle - |43\rangle), \quad |\phi_{11}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|35\rangle - |53\rangle), \quad |\phi_{12}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|36\rangle - |63\rangle),
\]
\[
|\phi_{13}^{(16)}\rangle = \frac{1}{\sqrt{6}}(|34\rangle + |43\rangle + |25\rangle + |52\rangle + |16\rangle + |61\rangle), \quad |\phi_{14}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|45\rangle - |54\rangle),
\]
\[
|\phi_{15}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|46\rangle - |64\rangle), \quad |\phi_{16}^{(16)}\rangle = \frac{1}{\sqrt{2}}(|56\rangle - |65\rangle).
\]

From the Yang-Baxter equation (2.5) and the 16-dimensional projector \( P_{12}^{\nu(16)} \), we obtain the fusion identities
\[
P_{21}^{\nu(16)} R_{13}^{\nu(16)} R_{23}^{\nu(16)} (u - 1) P_{21}^{\nu(16)} = (u - 1)(u + 2) S_{12}^{(16)} R_{13}^{s, v} (u - \frac{1}{2}) R_{23}^{s, v} (u - \frac{1}{2}) S_{12}^{-1}^{(16)},
\]
\[
P_{12}^{\nu(16)} R_{31}^{\nu(16)} R_{32}^{\nu(16)} (u - 1) P_{12}^{\nu(16)} = (u - 1)(u + 2) S_{12}^{(16)} R_{31}^{s, v} (u - \frac{1}{2}) R_{32}^{s, v} (u - \frac{1}{2}) S_{12}^{-1}^{(16)}.
\]

(3.11)

In Eq.(3.11), we take the fusion in the auxiliary spaces and obtain a 16-dimensional auxiliary space \( V_{(12)} \). We can show that the resulting 16-dimensional auxiliary space is exact a direct tensor-product of two 4-dimensional auxiliary spaces \( V^{(s_+)} \) and \( V^{(s_-)} \), which endows the (anti) spinor representation of \( so(6) \) respectively. The matrix \( S_{12}^{(16)} \) is defined in the direct
The product space $V^{(s_+)} \otimes V^{(s_-)}$ with the matrix form

$$S_{1'2'} = \begin{pmatrix} 1 & -1 & -1 & 1 & b_2 & -b_2 & -b_2 & b_2 \\ -1 & 1 & -b_2 & b_2 & b_2 \\ b_2 & b_2 & 1 & b_2 & b_2 \\ b_2 & b_2 & -b_2 & b_2 & b_2 \\ -a_2 & a_2 & -a_2 & a_2 & -1 & -1 \end{pmatrix},$$

$$\bar{S}_{1'2'} = \begin{pmatrix} 1 & -1 & -1 & 1 & b_2 & -b_2 & -b_2 & b_2 \\ -1 & 1 & -b_2 & b_2 & b_2 \\ b_2 & b_2 & 1 & b_2 & b_2 \\ b_2 & b_2 & -b_2 & b_2 & b_2 \\ a_2 & -a_2 & -a_2 & a_2 & -1 & -1 \end{pmatrix},$$

where

$$a_2 = \frac{\sqrt{3}}{2}, \quad b_2 = \frac{1}{2}.$$  

The relation between $S_{1'2'}$ and $\bar{S}_{1'2'}$ is provided by (4.19). In Eq.(3.11), the matrices $R^{s_+v}_{1'2'}(u)$ are defined in the $V^{(s_\pm)} \otimes V$ space and take the forms of

$$R^{s_+v}_{1'2'} = \begin{pmatrix} a_1 & a_1 & a_1 & b_1 & b_1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & a_1 & b_1 & b_1 & a_1 & b_1 & b_1 & 1 & -1 \\ -1 & 1 & b_1 & a_1 & b_1 & a_1 & b_1 & b_1 & -1 & 1 \\ -1 & 1 & b_1 & a_1 & b_1 & a_1 & b_1 & b_1 & 1 & -1 \\ -1 & 1 & b_1 & a_1 & b_1 & a_1 & b_1 & b_1 & 1 & -1 \end{pmatrix}. $$
where the matrix elements are

\[ a_1 = u + \frac{3}{2}, \quad b_1 = u + \frac{1}{2}. \]

The matrices \( R_{1/2}^{s,v}(u) \) have the properties

\[
\begin{align*}
\text{unitarity:} & \quad R_{1/2}^{s,v}(u)R_{21}^{s,v}(-u) = \rho_s(u) = a_1(u)a_1(-u), \\
\text{crossing unitarity:} & \quad R_{1/2}^{s,v}(u)^{t,v}R_{21}^{s,v}(-u - 4)^{t,v} = \rho_s(u + 2),
\end{align*}
\]

(3.12)

and satisfy the Yang-Baxter equations

\[
R_{1/2}^{s,v}(u_1 - u_2)R_{1/3}^{s,v}(u_1 - u_3)R_{23}^{s,v}(u_2 - u_3) = R_{23}^{s,v}(u_2 - u_3)R_{1/3}^{s,v}(u_1 - u_3)R_{1/2}^{s,v}(u_1 - u_2). \quad (3.13)
\]

Now, we consider the degenerate case of matrix \( R_{1/2}^{s,v}(u) \). At the point of \( u = -\frac{3}{2} \), we have

\[
R_{1/2}^{s,v}(-\frac{3}{2}) = P_{1/2}^{(+)} S_{1/2}^{(+)},
\]

(3.14)

where \( S_{1/2}^{(+)} \) is a constant matrix omitted here and \( P_{1/2}^{(+)} \) is a 4-dimensional projector

\[
P_{1/2}^{(+)} = \sum_{i=1}^{4} |\phi_i^{(+)}\rangle \langle \phi_i^{(+)}|,
\]

with the basis vectors

\[
|\phi_1^{(+)}\rangle = \frac{1}{\sqrt{3}}(|14\rangle + |22\rangle + |31\rangle), \quad |\phi_2^{(+)}\rangle = \frac{1}{\sqrt{3}}(|15\rangle - |23\rangle + |41\rangle),
\]
\[ |\phi_3^{(+)} \rangle = \frac{1}{\sqrt{3}} (|16 \rangle - |33 \rangle - |42 \rangle), \quad |\phi_4^{(+)} \rangle = \frac{1}{\sqrt{3}} (|26 \rangle - |35 \rangle + |44 \rangle). \]

Taking the fusion by using the projector \( P_{1/2}^{(+)} \), we obtain
\[
P_{1/2}^{(+)} R_{23}^{vv}(u) R_{13}^{s_v}(u - \frac{3}{2}) P_{1/2}^{(+)} = (u - 1)(u + 2) R_{(12)3}^{s_v}(u - \frac{1}{2}),
\]
\[
P_{21}^{(+)} R_{32}^{vv}(u) R_{31}^{s_v}(u - \frac{3}{2}) P_{21}^{(+)} = (u - 1)(u + 2) R_{3(12)}^{s_v}(u - \frac{1}{2}). \quad (3.15)
\]

We note that the dimension of the fused auxiliary space \( V_{(1/2)} \) is 4.

At the point of \( u = -\frac{3}{2} \), we also have
\[
R_{1/2}^{s_v}(-\frac{3}{2}) = P_{1/2}^{(-)} S_{1/2}^{(-)}, \quad (3.16)
\]
where \( P_{1/2}^{(-)} \) is a 4-dimensional projector
\[
P_{1/2}^{(-)} = \sum_{i=1}^{4} |\phi_i^{(-)} \rangle \langle \phi_i^{(-)}|,
\]
with the basis vectors
\[
|\phi_1^{(-)} \rangle = \frac{1}{\sqrt{3}} (|13 \rangle + |22 \rangle + |31 \rangle), \quad |\phi_2^{(-)} \rangle = \frac{1}{\sqrt{3}} (|15 \rangle - |24 \rangle + |41 \rangle),
\]
\[
|\phi_3^{(-)} \rangle = \frac{1}{\sqrt{3}} (|16 \rangle - |34 \rangle - |42 \rangle), \quad |\phi_4^{(-)} \rangle = \frac{1}{\sqrt{3}} (|26 \rangle - |35 \rangle + |43 \rangle),
\]
and the \( S_{1/2}^{(-)} \) is a constant matrix omitted here. Taking the fusion by using the projector \( P_{1/2}^{(-)} \), we obtain
\[
P_{1/2}^{(-)} R_{23}^{vv}(u) R_{13}^{s_v}(u - \frac{3}{2}) P_{1/2}^{(-)} = (u - 1)(u + 2) R_{(12)3}^{s_v}(u - \frac{1}{2}),
\]
\[
P_{21}^{(-)} R_{32}^{vv}(u) R_{31}^{s_v}(u - \frac{3}{2}) P_{21}^{(-)} = (u - 1)(u + 2) R_{3(12)}^{s_v}(u - \frac{1}{2}). \quad (3.17)
\]

### 3.2 The operator product identities

From the fused \( R \)-matrices \( R_{ij}^{s_{ij}} \), we can define the fused monodromy matrices
\[
T_{ij}^{(\pm)}(u) = R_{ij}^{s_{ij}}(u - \theta_1) R_{ij}^{s_{ij}}(u - \theta_2) \cdots R_{ij}^{s_{ij}}(u - \theta_N), \quad (3.18)
\]
which satisfy the Yang-Baxter relations
\[
R_{ij}^{s_{ij}}(u - v) T_{0}(u) T_{ij}^{(\pm)}(v) = T_{ij}^{(\pm)}(v) T_{0}(u) R_{ij}^{s_{ij}}(u - v). \quad (3.19)
\]
Taking the partial trace in the auxiliary space, we obtain the fused transfer matrices
\[ t_{\pm}^{(p)}(u) = \text{tr}_{\gamma'} T_{\gamma'}^{\pm}(u). \quad (3.20) \]

From the Yang-Baxter relations (3.2) and (3.19) at certain points and using the properties of projectors, we obtain
\[
T_1(\theta_j) T_2(\theta_j - 2) = P_{21}^{\text{ev}(1)} T_1(\theta_j) T_2(\theta_j - 2),
\]
\[
T_1(\theta_j) T_2(\theta_j - 1) = P_{21}^{\text{ev}(16)} T_1(\theta_j) T_2(\theta_j - 1),
\]
\[
T_2(\theta_j) T_{1}^{+}(\theta_j - \frac{3}{2}) = P_{12}^{(+)} T_2(\theta_j) T_{1}^{+}(\theta_j - \frac{3}{2}),
\]
\[
T_2(\theta_j) T_{1}^{-}(\theta_j - \frac{3}{2}) = P_{12}^{(-)} T_2(\theta_j) T_{1}^{-}(\theta_j - \frac{3}{2}). \quad (3.21)
\]

By using the fusion identities (3.9), (3.11), (3.15) and (3.17), we have following fusion identities
\[
P_{21}^{\text{ev}(1)} T_1(u) T_2(u - 2) P_{21}^{\text{ev}(1)} = P_{21}^{\text{ev}(1)} \prod_{i=1}^{N} a(u - \theta_i) e(u - \theta_i - 2) \times \text{id},
\]
\[
P_{21}^{\text{ev}(16)} T_1(u) T_2(u - 1) P_{21}^{\text{ev}(16)} = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) S_{12}^{+} T_{12}^{+}(u - \frac{1}{2}) T_{21}^{-}(u - \frac{1}{2}) S_{12}^{-1},
\]
\[
P_{12}^{(+)} T_2(u) T_{12}^{+}(u - \frac{3}{2}) P_{12}^{(+)} = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) T_{12}^{+}(u - \frac{1}{2}),
\]
\[
P_{12}^{(-)} T_2(u) T_{12}^{-}(u - \frac{3}{2}) P_{12}^{(-)} = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) T_{12}^{-}(u - \frac{1}{2}). \quad (3.22)
\]

where \( \tilde{\rho}_0(u) = (u - 1)(u + 2). \)

Taking the partial trace in the auxiliary spaces and using the relations (3.21) and (3.22), we obtain following operator product identities
\[
t^{(p)}(\theta_j) t^{(p)}(\theta_j - 2) = \prod_{i=1}^{N} a(\theta_j - \theta_i) e(\theta_j - \theta_i - 2) \times \text{id},
\]
\[
t^{(p)}(\theta_j) t^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}(\theta_j - \frac{1}{2}) t^{(p)}(\theta_j - \frac{1}{2}),
\]

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\[ t^{(p)}(\theta_j) t^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}(\theta_j - \frac{1}{2}), \]
\[ t^{(p)}(\theta_j) t^{(p)}_-(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}_+(\theta_j - \frac{1}{2}). \] (3.23)

From the definitions, the asymptotic behaviors of the transfer matrices can be calculated directly as
\[ t^{(p)}(u)|_{u \to \pm \infty} = 6u^{2N} \times \text{id} + \cdots, \]
\[ t^{(p)}_{\pm}(u)|_{u \to \pm \infty} = 4u^{N} \times \text{id} + \cdots. \] (3.24)

Let us denote the eigenvalues of the transfer matrices \( t^{(p)}(u) \) and \( t^{(p)}_{\pm}(u) \) as \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_{\pm}(u) \), respectively. We note that the eigenvalues \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_{\pm}(u) \) are the polynomials of \( u \) with degrees \( 2N \) and \( N \), respectively. Thus we need \( 4N + 3 \) conditions to determine the polynomials \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_{\pm}(u) \).

From the operator product identities (3.23), we have the functional relations among the eigenvalues
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 2) = \prod_{i=1}^{N} a(\theta_j - \theta_i)e(\theta_j - \theta_i - 2), \]
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}_+(\theta_j - \frac{1}{2}) \Lambda^{(p)}_{-}(\theta_j - \frac{1}{2}), \]
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}_{\pm}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}_{\pm}(\theta_j - \frac{1}{2}), \]
\[ \Lambda^{(p)}(\theta_j) \Lambda^{(p)}_{-}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}_{-}(\theta_j - \frac{1}{2}). \] (3.25)

And the corresponding asymptotic behaviors read
\[ \Lambda^{(p)}(u)|_{u \to \pm \infty} = 6u^{2N} + \cdots, \]
\[ \Lambda^{(p)}_{\pm}(u)|_{u \to \pm \infty} = 4u^{N} + \cdots. \] (3.26)

Then we arrive at that \( 4N \) functional relations (3.25) together with 3 asymptotic behaviors (3.26) give us sufficient conditions to determine the eigenvalues.
3.3 The $T - Q$ relations

Let us introduce some functions

\[
Z_1^{(p)}(u) = \prod_{j=1}^{N} a(u - \theta_j) \frac{Q_p^{(1)}(u - 1)}{Q_p^{(1)}(u)},
\]

\[
Z_2^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_p^{(1)}(u + 1)Q_p^{(2)}(u - 1)Q_p^{(3)}(u - 1)}{Q_p^{(1)}(u)Q_p^{(2)}(u)Q_p^{(3)}(u)},
\]

\[
Z_3^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_p^{(2)}(u - 1)Q_p^{(3)}(u + 1)}{Q_p^{(2)}(u)Q_p^{(3)}(u)},
\]

\[
Z_4^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_p^{(2)}(u + 1)Q_p^{(3)}(u - 1)}{Q_p^{(2)}(u)Q_p^{(3)}(u)},
\]

\[
Z_5^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_p^{(1)}(u)Q_p^{(2)}(u + 1)Q_p^{(3)}(u + 1)}{Q_p^{(1)}(u + 1)Q_p^{(2)}(u)Q_p^{(3)}(u)},
\]

\[
Z_6^{(p)}(u) = \prod_{j=1}^{N} e(u - \theta_j) \frac{Q_p^{(1)}(u + 2)}{Q_p^{(1)}(u + 1)},
\]

\[
Q_p^{(1)}(u) = \prod_{k=1}^{L_1} (u - \mu_k^{(1)} + \frac{1}{2}), \quad Q_p^{(2)}(u) = \prod_{k=1}^{L_2} (u - \mu_k^{(2)} + 1),
\]

\[
Q_p^{(3)}(u) = \prod_{k=1}^{L_3} (u - \mu_k^{(3)} + 1).
\]

The relations (3.25) and (3.26) enable us to parameterize the eigenvalues of the transfer matrices in terms of some homogeneous $T - Q$ relations as

\[
\Lambda^{(p)}(u) = Z_1^{(p)}(u) + Z_2^{(p)}(u) + Z_3^{(p)}(u) + Z_4^{(p)}(u) + Z_5^{(p)}(u) + Z_6^{(p)}(u),
\]

\[
\Lambda_+^{(p)}(u) = \prod_{i=1}^{N} a_1(u - \theta_i) \left[ \frac{Q_p^{(2)}(u - \frac{3}{2})}{Q_p^{(2)}(u - \frac{1}{2})} + \frac{Q_p^{(1)}(u - \frac{1}{2})Q_p^{(2)}(u + \frac{1}{2})}{Q_p^{(1)}(u + \frac{1}{2})Q_p^{(2)}(u - \frac{1}{2})} \right] + \prod_{i=1}^{N} b_1(u - \theta_i) \left[ \frac{Q_p^{(3)}(u + \frac{3}{2})}{Q_p^{(3)}(u + \frac{1}{2})} + \frac{Q_p^{(1)}(u + \frac{1}{2})Q_p^{(3)}(u - \frac{1}{2})}{Q_p^{(1)}(u - \frac{1}{2})Q_p^{(3)}(u + \frac{1}{2})} \right],
\]

\[
\Lambda_-^{(p)}(u) = \prod_{i=1}^{N} a_1(u - \theta_i) \left[ \frac{Q_p^{(3)}(u - \frac{3}{2})}{Q_p^{(3)}(u - \frac{1}{2})} + \frac{Q_p^{(1)}(u - \frac{1}{2})Q_p^{(3)}(u + \frac{1}{2})}{Q_p^{(1)}(u + \frac{1}{2})Q_p^{(3)}(u - \frac{1}{2})} \right] + \prod_{i=1}^{N} b_1(u - \theta_i) \left[ \frac{Q_p^{(2)}(u + \frac{3}{2})}{Q_p^{(2)}(u + \frac{1}{2})} + \frac{Q_p^{(1)}(u + \frac{1}{2})Q_p^{(2)}(u - \frac{1}{2})}{Q_p^{(1)}(u - \frac{1}{2})Q_p^{(2)}(u + \frac{1}{2})} \right].
\]
The regularities of the eigenvalues $\Lambda^{(p)}(u)$ and $\Lambda^{(p)}(u)\pm u$ lead to that the Bethe roots $\{\mu_k^{(m)}\}$ should satisfy the Bethe ansatz equations (BAEs):

$$Q_p(1)\frac{Q_p(1)(\mu_k^{(1)} - \frac{3}{2})Q_p(2)(\mu_k^{(1)} - \frac{1}{2})}{Q_p(2)(\mu_k^{(1)} + \frac{1}{2})Q_p(2)(\mu_k^{(1)} - \frac{3}{2})} = -\prod_{j=1}^{N} \frac{\mu_k^{(1)} - \frac{1}{2} - \theta_j}{\mu_k^{(1)} + \frac{1}{2} - \theta_j},$$

$$Q_k^{(1)}(\mu_l^{(2)} - 2) = -1, \quad l = 1, \cdots, L_1,$n

$$Q_k^{(1)}(\mu_l^{(2)} - 1)Q_k^{(2)}(\mu_l^{(2)}) = -1, \quad l = 1, \cdots, L_2,$n

$$Q_k^{(1)}(\mu_l^{(3)} - 2) = -1, \quad l = 1, \cdots, L_3.$$  \hspace{1cm} (3.29)

We note that the BAEs obtained from the regularity of $\Lambda^{(p)}(u)$ are the same as those obtained from the regularity of $\Lambda^{(p)}_\pm(u)$. It is easy to check that $\Lambda^{(p)}(u)$ and $\Lambda^{(p)}_\pm(u)$ satisfy the functional relations (3.25) and the asymptotic behaviors (3.26). Therefore, we conclude that $\Lambda^{(p)}(u)$ and $\Lambda^{(p)}_\pm(u)$ are the eigenvalues of the transfer matrices $t^{(p)}(u)$ and $t^{(p)}_\pm(u)$, respectively. It is easy to check that the homogeneous $T-Q$ relations (3.28) and associated BAEs (3.29) coincide with those obtained previously by others methods [17, 18, 19].

Then the eigenvalues of the Hamiltonian (3.4) can be obtained by the $\Lambda^{(p)}(u)$ as

$$E_p = \frac{\partial \ln \Lambda^{(p)}(u)}{\partial u} |_{u=0, \{\theta_j\}=0}. \hspace{1cm} (3.30)$$

4 \hspace{1cm} $D_3^{(1)}$ model with non-diagonal open boundary condition

In this section, we consider the system with general integrable open boundary condition. Let us introduce a pair of reflection matrices $K^v(u)$ and $\bar{K}^v(u)$. The former satisfies the reflection equation (RE)

$$R_{12}^v(u - v)K_1^v(u)R_{21}^v(u + v)K_2^v(v) = K_2^v(v)R_{12}^v(u + v)K_1^v(u)R_{21}^v(u - v), \hspace{1cm} (4.1)$$

and the latter satisfies the dual RE

$$\bar{R}_{12}^v(-u + v)\bar{K}_1^v(u)\bar{R}_{21}^v(-u - v - 4)\bar{K}_2^v(v) = \bar{K}_2^v(v)\bar{R}_{12}^v(-u - v - 4)\bar{K}_1^v(u)\bar{R}_{21}^v(-u + v). \hspace{1cm} (4.2)$$
For the open case, instead of the “row-to-row” monodromy matrix $T\hat{\nu}(u)$ given by (3.1), one needs consider the “double-row” monodromy matrix
\[
\hat{T}\nu(u) = R\nu_N(u + \theta_N) \cdots R\nu_2(u + \theta_2) R\nu_1(u + \theta_1),
\] (4.3)
which satisfies the Yang-Baxter relation
\[
R\nu_1(u - v)\hat{T}\nu_1(u)\hat{T}\nu_2(v) = \hat{T}\nu_2(v)\hat{T}\nu_1(u)R\nu_1(u - v).
\] (4.4)

The transfer matrix $t(u)$ is defined as
\[
t(u) = tr_0 \{ \bar{K}\nu_0(u) T\nu_0(u) K\nu_0(u) \hat{T}\nu_0(u) \}.
\] (4.5)

From the Yang-Baxter relation, reflection equation and its dual, one can prove that the transfer matrices with different spectral parameters commute with each other, $[t(u), t(v)] = 0$. Therefore, $t(u)$ serves as the generating function of all the conserved quantities of the system. The Hamiltonian is constructed by taking the derivative of the logarithm of the transfer matrix
\[
H = \frac{\partial \ln t(u)}{\partial u} \bigg|_{u=0, \{\theta_j\}=0} = \sum_{k=1}^{N-1} H_{kk+1} + \frac{K\nu_1(0)'}{2\xi} + \frac{tr_0 \{ \bar{K}\nu_0(0) H_N \}}{tr_0 K\nu_0(0)} + \text{constant},
\] (4.6)
where $H_{kk+1}$ is given by (3.3) and $K\nu(0) = \xi \times \text{id}$.

### 4.1 Reflection matrix

In this paper, we consider the general integrable open boundary condition where the reflection matrix has the non-diagonal elements which breaks the $U(1)$-symmetry of the system. The general reflection matrix for $D_n^{(1)}$ vertex model with its R-matrix being trigonometric has been obtained by Lima-Santos and Malara [34, 35]. Here we chose the reflection matrix $K\nu^-(u)$, which is the solution of reflection equation (4.1), as
\[
K\nu(u) = \begin{pmatrix}
K_{11}^\nu(u) & 0 & 0 & 0 & 0 & 0 \\
0 & K_{22}^\nu(u) & 0 & K_{24}^\nu(u) & 0 & 0 \\
0 & 0 & K_{33}^\nu(u) & 0 & K_{35}^\nu(u) & 0 \\
0 & K_{42}^\nu(u) & 0 & K_{44}^\nu(u) & 0 & 0 \\
0 & 0 & K_{53}^\nu(u) & 0 & K_{55}^\nu(u) & 0 \\
0 & 0 & 0 & 0 & 0 & K_{66}^\nu(u)
\end{pmatrix},
\] (4.7)
where the non-vanishing matrix elements are

\[
\begin{align*}
K_{11}^v(u) &= \frac{(2 - c_2 - 2c_2u)(2 + c_2 + 2c_2u)}{2}, \\
K_{22}^v(u) &= \frac{(2 - c_2 - 2cu)(2 + c_2 + 2c_2u)}{2}, \\
K_{24}^v(u) &= -c_1u(2 + c_2 + 2c_2u), \\
K_{33}^v(u) &= \frac{(2 + c_2 - 2cu)(2 - c_2 - 2c_2u)}{2}, \\
K_{35}^v(u) &= c_1u(2 - c_2 - 2c_2u), \\
K_{42}^v(u) &= -c_3u(2 + c_2 + 2c_2u), \\
K_{44}^v(u) &= \frac{(2 - c_2 + 2cu)(2 + c_2 + 2c_2u)}{2}, \\
K_{53}^v(u) &= -c_3u(-2 + c_2 + 2c_2u), \\
K_{55}^v(u) &= \frac{(2 + c_2 + 2cu)(2 - c_2 - 2c_2u)}{2}, \\
K_{66}^v(u) &= \frac{(2 - c_2 - 2cu)(2 + c_2 + 2c_2u)}{2}.
\end{align*}
\tag{4.8}
\]

Here \(c_1, c_2\) and \(c_3\) are free boundary parameters, where \(c\) is expressed in terms of them as

\[c^2 = c_2^2 - c_1c_3.\]

The Eq.(4.7) can also be obtained at some special free parameters by taking rational limit to the general reflection matrix in \[34, 35\].

The dual reflection matrix \(\bar{K}^v(u)\) is also a non-diagonal one and given by

\[
\bar{K}^v(u) = K^v(-u - 2)|_{(c_1, c_2, c_3) \rightarrow (c'_1, c'_2, c'_3)},
\tag{4.9}
\]

where \(c'_1, c'_2\) and \(c'_3\) are the boundary parameters at the other side. For a generic choice of the three boundary parameters \(\{c_i, c'_i\}_{i = 1, 2, 3}\), it is easily to check that \([K^v(u), \bar{K}^v(v)] \neq 0\). This implies that the matrices \(K^v(u)\) and \(\bar{K}^v(u)\) cannot be diagonalized simultaneously.

Following the method developed in \[32\] and using the crossing-symmetry (2.2) of the \(R\)-matrix, the explicit expressions (4.7) and (4.9) of the reflection matrices, we find that the transfer matrix (4.5) possesses the crossing symmetry

\[
t(-u - 2) = t(u).
\tag{4.10}
\]
4.2 Fusion of the reflection matrix

In order to seek the relations that the transfer matrix \([4.5]\) satisfies, we first consider the fusion of the reflection matrix. The fusion of the one-dimensional projector \(P_{12}^{(1)}\) gives

\[
P_{12}^{(1)} K_2^v(u) R_{12}^{uv}(2u - 2) K_2^v(u - 2) P_{21}^{(1)} = (u - 2)(u - \frac{3}{2}) h(u) h(-u) P_{12}^{(1)},
\]

\[
P_{21}^{(1)} \tilde{K}_1^v(u - 2) R_{21}^{uv}(-2u - 2) \tilde{K}_2^v(u) P_{21}^{(1)} = (u + 2)(u + \frac{3}{2}) \tilde{h}(u) \tilde{h}(-u) P_{21}^{(1)},
\]

(4.11)

where

\[
h(u) = (2 - c_2 - 2c_2 u)(2 + c_2 + 2c_2 u), \quad \tilde{h}(u) = (2 - c_2' - 2c_2' u)(2 + c_2' + 2c_2'u).
\]

(4.12)

The fusion of the 16-dimensional projector \(P_{12}^{(1)}\) gives

\[
P_{12}^{(1)} K_1^v(u) R_{12}^{uv}(2u - 1) K_2^v(u - 1) P_{12}^{(16)} = 2(u - 1) h(u) S_{12}^{v'} K_{1'}^{s'} (u - \frac{1}{2}) R_{2}^{s_2 s_2'} (2u - 1) K_{2'}^{s_2'} (u - \frac{1}{2}) S_{12}^{-1},
\]

\[
P_{21}^{(1)} \tilde{K}_1^v(u - 1) R_{21}^{uv}(-2u - 3) \tilde{K}_2^v(u) P_{21}^{(16)} = -2(u + 2) \tilde{h}(u) S_{12}^{v'} \tilde{K}_{1'}^{s'} (u - \frac{1}{2}) R_{12}^{s_2 s_2'} (-2u - 3) \tilde{K}_{2'}^{s_2'} (u - \frac{1}{2}) S_{12}^{-1},
\]

(4.13)

where

\[
R_{12}^{s_2 s_2'} = \begin{pmatrix}
  a_3 & a_3 & a_3 & a_3 \\
  a_3 & 1 & -1 & 1 \\
  b_3 & 1 & -1 & 1 \\
  1 & 1 & b_3 & a_3 \\
  a_3 & 1 & -1 & 1 \\
  b_3 & 1 & b_3 & a_3 \\
  1 & -1 & 1 & b_3 \\
  a_3 & 1 & a_3 & a_3
\end{pmatrix},
\]

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The dual reflection matrices \( \bar{\mathbf{K}}^{s\pm}(u) \) which satisfy the dual REs

\[
K^{s\pm}(u) = \begin{pmatrix}
1 - cu & c_1 u & 0 & 0 \\
c_3 u & 1 + cu & 0 & 0 \\
0 & 0 & 1 + c_2 u & 0 \\
0 & 0 & 0 & 1 - c_2 u
\end{pmatrix}, \quad (4.14)
\]

\[
K^{s-}(u) = \begin{pmatrix}
1 + c_2 u & 0 & 0 & 0 \\
0 & 1 - c_2 u & 0 & 0 \\
0 & 0 & 1 - cu & c_1 u \\
0 & 0 & c_3 u & 1 + cu
\end{pmatrix}, \quad (4.15)
\]

and

\[
a_3 = u + 2, \quad b_3 = u + 1.
\]

The matrix \( R_{12}^{s+,s-}(u) \) has properties

unitarity : \( R_{12}^{s+,s-}(u)R_{21}^{s-,s+}(-u) = (2 + u)(2 - u) \),

crossing unitarity : \( R_{12}^{s+,s-}(u)R_{21}^{s-,s+}(-u - 4)^{t'v'} = \rho_{ss}(u) = -(u + 1)(u + 3) \), \( (4.16) \)

and satisfies the Yang-Baxter equations

\[
R_{12}^{s+,s-}(u_1 - u_2)R_{13}^{s-,s+}(u_1 - u_3)R_{23}^{s-,s+}(u_2 - u_3) = R_{23}^{s-,s+}(u_2 - u_3)R_{13}^{s-,s+}(u_1 - u_3)R_{12}^{s+,s-}(u_1 - u_2). \quad (4.17)
\]

The matrices \( K^{s\pm}(u) \) satisfy the reflection equations

\[
R_{12}^{s\pm}(u - v)K_{12}^{s\pm}(u)R_{21}^{s\pm}(u + v)K_{21}^{s\pm}(v) = K_{21}^{s\pm}(v)R_{12}^{s\pm}(u + v)K_{12}^{s\pm}(u)R_{21}^{s\pm}(u - v). \quad (4.18)
\]

The \( \bar{S}_{12} \) can be obtained by

\[
\bar{S}_{12} = \frac{1}{2} S_{12} R_{21}^{s-,s+}(0). \quad (4.19)
\]

The dual reflection matrices \( \bar{K}^{s\pm}(u) \) are constructed as

\[
\bar{K}^{s\pm}(u) = K^{s\pm}(-u - 2)|_{(c,c_1,c_2,c_3)\rightarrow(c',c'_1,c'_2,c'_3)}, \quad (4.20)
\]

which satisfy the dual REs

\[
R_{12}^{s\pm}(u - v)\bar{K}_{12}^{s\pm}(u)\bar{R}_{21}^{s\pm}(-u - v - 4)\bar{K}_{21}^{s\pm}(v)
\]

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By using the fusion identities (3.9), (3.11), (3.15) and (3.17), we also obtain the fusion of the 4-dimensional projector \( P_{v_{1/2}}^{(+)} \) gives

\[
P_{v_{1/2}}^{(+)} \tilde{K}_2^v(u) \tilde{R}_{v_{1/2}}^{s^+} (2u - \frac{3}{2}) K_1^{s^+} (u - \frac{3}{2}) P_{v_{1/2}}^{(+)} = (u - \frac{3}{2}) \bar{h}(u) K_{(1/2)}^{s^+} (u - \frac{1}{2}),
\]

\[
P_{2v_{1/2}}^{(+)} \tilde{K}_{v_{1/2}}^{s^+} (u - \frac{3}{2}) \tilde{R}_{2v_{1/2}}^{s^+} (-2u - \frac{5}{2}) \tilde{K}_2^v(u) P_{v_{1/2}}^{(+)} = -(u + 2) \bar{h}(u) \tilde{K}_{(1/2)}^{s^+} (u - \frac{1}{2}).
\]

4.3 Operators product relations

Define the fused monodromy matrices \( \hat{T}_{0^0}^{\pm}(u) \) by using the fused \( R \)-matrix \( R^{u s^+} \) as

\[
\hat{T}_{0^0}^{\pm}(u) = R_{N0^0}^{u s^+}(u + \theta_N) \cdots R_{20^0}^{u s^+}(u + \theta_2) R_{10^0}^{u s^+}(u + \theta_1),
\]

which satisfy the Yang-Baxter relation

\[
P_{00^0}^{u s^+} (u - v) \hat{T}_0(u) \hat{T}_{0^0}^{\pm}(v) = \hat{T}_{0^0}^{\pm}(v) \hat{T}_0(u) P_{00^0}^{u s^+}(u - v).
\]

From the Yang-Baxter relations (4.4) and (4.24) at certain points and using the properties of projectors, we obtain

\[
\hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j - 2) = P_{12}^{(1)}(1) \hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j - 2),
\]

\[
\hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j - 1) = P_{12}^{(16)}(16) \hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j - 1),
\]

\[
\hat{T}_2(-\theta_j) \hat{T}_1^{(1)}(-\theta_j - \frac{3}{2}) = P_{21'}^{(+)}(2) \hat{T}_2(-\theta_j) \hat{T}_1^{(1')}(-\theta_j - \frac{3}{2}),
\]

\[
\hat{T}_2(-\theta_j) \hat{T}_1^{(-)}(-\theta_j - \frac{3}{2}) = P_{21'}^{(-)}(1) \hat{T}_2(-\theta_j) \hat{T}_1^{(-)}(-\theta_j - \frac{3}{2}).
\]

By using the fusion identities (3.9), (3.11), (3.15) and (3.17), we also obtain the fusion identities

\[
P_{12}^{(1)}(1) \hat{T}_1(u) \hat{T}_2(u - 2) P_{12}^{(1)}(1) = P_{12}^{(1)}(1) \displaystyle \prod_{i=1}^{N} a(u + \theta_i) e(u + \theta_i - 2) \times \text{id},
\]

\[
P_{12}^{(16)}(1) \hat{T}_1(u) \hat{T}_2(u - 1) P_{12}^{(16)}(16) = \displaystyle \prod_{i=1}^{N} \tilde{\rho}_0(u + \theta_i) S_{1v_{1/2}} \tilde{T}_1^{(1)}(u - \frac{1}{2}) \tilde{T}_{1'}^{(1')}(-u - \frac{1}{2}) S_{1v_{1/2}}^{-1},
\]

\[
P_{21'}^{(+)}(2) \hat{T}_2(u) \hat{T}_1^{(2)}(u - \frac{3}{2}) P_{21'}^{(+)}(2) = \displaystyle \prod_{i=1}^{N} \tilde{\rho}_0(u + \theta_i) \hat{T}_1^{(1)}(u - \frac{1}{2}).
\]
\[ P_{2i}^\pm(u) \hat{T}_2(u) \hat{T}_1^\pm(u - \frac{3}{2}) P_{2\bar{i}}^\pm = \prod_{i=1}^{N} \bar{\rho}_0(u + \theta_i) \hat{T}_{\mp(v/2)}^\pm(u - \frac{1}{2}). \]  

(4.26)

The fused transfer matrices are defined as

\[ t_\pm(u) = tr_{0'} \{ \bar{K}_{\theta}^{\pm\pm}(u) T_{\theta}^\pm(u) K_{\theta}^{\pm\pm}(u) \hat{T}_{\theta}^\pm(u) \}. \]  

(4.27)

Direct calculation shows

\[ t(u)t(u + \Delta) = [\rho_1(2u + \Delta + 2)]^{-1} tr_{12} \{ \bar{K}_2^{u\pm}(u + \Delta) R_{12}^{uv}(-2u - 4 - \Delta) \]
\[ \times \bar{K}_1^u(u) T_1(u) T_2(u + \Delta) K_2^u(u + \Delta) \hat{T}_1(u) \hat{T}_2(u + \Delta) \}, \]  

(4.28)

\[ t(u)t_\pm(u + \Delta) = [\rho_3(2u + \Delta + 2)]^{-1} tr_{12} \{ \bar{K}_2^{s\pm}(u + \Delta) R_{12}^{s\pm}(-2u - 4 - \Delta) \]
\[ \times \bar{K}_1^u(u) T_1(u) T_2(u + \Delta) K_2^{s\pm}(u + \Delta) \hat{T}_1(u) \hat{T}_2(u + \Delta) \}, \]  

(4.29)

\[ t_+(u)t_-(u + \Delta) = [\rho_{ss}(2u + \Delta)]^{-1} tr_{12} \{ \bar{K}_2^{s-}(u + \Delta) R_{12}^{s+s-}(-2u - 4 - \Delta) \]
\[ \times \bar{K}_1^{s+}(u) T_1^+(u) T_2^-(u + \Delta) K_2^{s+}(u + \Delta) \hat{T}_1^+(u) \hat{T}_2^-(u + \Delta) \}. \]  

(4.30)

Here \( \Delta \) choose \(-2, -1, -\frac{3}{2}, 0 \) for Eq. (4.28), Eq. (4.28), Eq. (4.29) and Eq. (4.30), respectively.

With the help of relations (3.21), (3.22), (4.11), (4.13), (4.22), (4.25), (4.26) and considering Eqs. (4.28)-(4.30) at certain points, we arrive at

\[ t(\pm \theta_j) t(\pm \theta_j - 2) = \frac{1}{2^4} \frac{(\pm \theta_j - 2)(\pm \theta_j + 2)(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})}{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})} h(\pm \theta_j) h(\mp \theta_j) \bar{h}(\pm \theta_j) \]
\[ \times \bar{h}(\mp \theta_j) \prod_{i=1}^{N} a(\pm \theta_j - \theta_i) e(\pm \theta_j - \theta_i - 2) a(\pm \theta_j + \theta_i) e(\pm \theta_j + \theta_i - 2) \times id, \]

\[ t(\pm \theta_j) t(\pm \theta_j - 1) = \frac{(\pm \theta_j - 1)(\pm \theta_j + 2)}{(\pm \theta_j - 1)(\pm \theta_j + 1)} \prod_{i=1}^{N} \bar{\rho}_0(\pm \theta_j - \theta_i) \bar{\rho}_0(\pm \theta_j + \theta_i) \]
\[ \times h(\pm \theta_j) \bar{h}(\pm \theta_j) t_+(\pm \theta_j - \frac{1}{2}) \times t_-(\pm \theta_j - \frac{1}{2}), \]

\[ t(\pm \theta_j) t_+(\pm \theta_j - \frac{3}{2}) = \frac{1}{2^2} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + 2)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + 1)} \prod_{i=1}^{N} \bar{\rho}_0(\pm \theta_j - \theta_i) \bar{\rho}_0(\pm \theta_j + \theta_i) \]
\[ \times h(\pm \theta_j) \bar{h}(\pm \theta_j) t_-(\pm \theta_j - \frac{1}{2}), \]

\[ t(\pm \theta_j) t_-(\pm \theta_j - \frac{3}{2}) = \frac{1}{2^2} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + 2)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + 1)} \prod_{i=1}^{N} \bar{\rho}_0(\pm \theta_j - \theta_i) \bar{\rho}_0(\pm \theta_j + \theta_i) \]
\[ \times h(\pm \theta_j) \bar{h}(\pm \theta_j) t_+(\pm \theta_j - \frac{1}{2}). \]  

(4.31)
Meanwhile, from the definitions we also know the values of transfer matrices at some special points

\[ t(0) = \frac{3}{2} (2 - c_2)(2 + c_2)(2 - c'_2)(2 + c'_2) \prod_{l=1}^{N} \rho_l(-\theta_l), \]

\[ t(-\frac{1}{2}) = 6 \prod_{l=1}^{N} \rho_l(-\theta_l)t_{\pm}(-1), \]

\[ t_{+}(0) = t_{-}(0) = 4 \prod_{l=1}^{N} \rho_l(-\theta_l). \tag{4.32} \]

The corresponding asymptotic behaviors read

\[ t(u)|_{u \to \pm \infty} = -\frac{8c_2c'_2((c'_2)^2c^2 + c'^2c'^2 - c_2^2(c'_2)^2 - c^2(c'_2)^2 - 2cc'_3c'^2 - c_2c'_3c'^2)}{c_3c'_3} u^{4N+4} \times \text{id} \]

\[ t_{+}(u)|_{u \to \pm \infty} = \frac{(c'_3 - c'_2c - c'_2c - c'_2c)(c'_3c_3 - c'_2c + c'_2c_3 + c'_2c_3)}{c_3c'_3} u^{2N+2} \times \text{id} + \cdots, \]

\[ t_{-}(u)|_{u \to \pm \infty} = \frac{(c'_3 - c'_2c - c'_2c - c'_2c)(c'_3c_3 - c'_2c + c'_2c_3 + c'_2c_3)}{c_3c'_3} u^{2N+2} \times \text{id} + \cdots. \tag{4.33} \]

Denote the eigenvalues of the fused transfer matrices \( t(u) \) and \( t_{\pm}(u) \) as \( \Lambda(u) \) and \( \Lambda_{\pm}(u) \), respectively. From Eq. (4.31), we obtain the functional relations among the eigenvalues of the transfer matrices

\[ \Lambda(\pm \theta_j) \Lambda(\pm \theta_j - 2) = \frac{1}{2^4} \frac{\big((\pm \theta_j - 2)(\pm \theta_j + 2)(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})\big)}{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})} h(\pm \theta_j) h(\mp \theta_j) \tilde{h}(\pm \theta_j) \]

\[ \times \tilde{h}(\mp \theta_j) \prod_{i=1}^{N} a(\pm \theta_j - \theta_i)e(\pm \theta_j - \theta_i - 2)a(\pm \theta_j + \theta_i)e(\pm \theta_j + \theta_i - 2), \]

\[ \Lambda(\pm \theta_j) \Lambda(\pm \theta_j - 1) = \frac{(\pm \theta_j - 1)(\pm \theta_j + 2)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{3}{2})} \prod_{i=1}^{N} \tilde{\rho}_0(\pm \theta_j - \theta_i)\tilde{\rho}_0(\pm \theta_j + \theta_i) \]

\[ \times h(\pm \theta_j) \tilde{h}(\pm \theta_j) \Lambda_{\pm}(\pm \theta_j - \frac{1}{2}), \]

\[ \Lambda(\pm \theta_j) \Lambda_{\pm}(\pm \theta_j - \frac{3}{2}) = \frac{1}{2^2} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + 2)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + 1)} \prod_{i=1}^{N} \tilde{\rho}_0(\pm \theta_j - \theta_i)\tilde{\rho}_0(\pm \theta_j + \theta_i) \]

\[ \times h(\pm \theta_j) \tilde{h}(\pm \theta_j) \Lambda_{\pm}(\pm \theta_j - \frac{1}{2}), \]

\[ \Lambda(\pm \theta_j) \Lambda_{\pm}(\pm \theta_j - \frac{3}{2}) = \frac{1}{2^2} \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + 2)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + 1)} \prod_{i=1}^{N} \tilde{\rho}_0(\pm \theta_j - \theta_i)\tilde{\rho}_0(\pm \theta_j + \theta_i) \]

\[ \times h(\pm \theta_j) \tilde{h}(\pm \theta_j) \Lambda_{\pm}(\pm \theta_j - \frac{1}{2}). \tag{4.34} \]
Eqs. (4.32) and (4.33) give rise to the relations

\[ \Lambda(0) = \frac{3}{2}(2 - c_2)(2 + c_2)(2 - c'_2)(2 + c'_2) \prod_{l=1}^{N} \rho_l(-\theta_l), \]

\[ \Lambda(-\frac{1}{2}) = 6 \prod_{l=1}^{N} \rho_s(-\theta_l) \Lambda_{\pm}(-1), \]

\[ \Lambda_{\pm}(0) = \Lambda_{\mp}(0) = 4 \prod_{l=1}^{N} \rho_s(-\theta_l), \quad (4.35) \]

and

\[ \Lambda(u)|_{u \to \pm \infty} = -\frac{8c_2c'_2((c'_3)^2c^2 + c'^2c_3 - c'_3(c'_3)^2 - c^2_3(c'_3)^2 - 2cc'_3c'_3 - c_2c_3c'_3)}{c_3c'_3} u^{-4N+4} \]

\[ + \cdots, \]

\[ \Lambda_{\mp}(u)|_{u \to \pm \infty} = \frac{(c'c_3 - c'_3c - c'_2c_3 - c'_2c_2)(c'c_3 - c'_3c + c'_2c_3 + c'_2c_2)}{c_3c'_3} u^{-2N+2} + \cdots, \]

\[ \Lambda_{\pm}(u)|_{u \to \pm \infty} = \frac{(c'c_3 - c'_3c - c'_2c_3 - c'_2c_2)(c'c_3 - c'_3c + c'_2c_3 + c'_2c_2)}{c_3c'_3} u^{-2N+2} + \cdots \quad (4.36) \]

From the definitions, we know that the eigenvalues \( \Lambda(u) \) and \( \Lambda_{\pm}(u) \) are the polynomials of \( u \) with degrees \( 4N + 4 \) and \( 2N + 2 \), respectively. Meanwhile, \( \Lambda(u) \) and \( \Lambda_{\pm}(u) \) enjoy the crossing symmetries

\[ \Lambda(-u - 2) = \Lambda(u), \quad \Lambda_{\pm}(-u - 2) = \Lambda_{\mp}(u). \quad (4.37) \]

Therefore, in order to determine the explicit expressions of the polynomials, we need \( 4N + 7 \) conditions, which are all listed by Eqs. (4.34)–(4.36).

### 4.4 Inhomogeneous T-Q relations

Let us introduce some functions

\[ Z_1(u) = 2^2 \frac{(u + 2)(u + \frac{3}{2})}{(u + 1)(u + \frac{1}{2})} \prod_{j=1}^{N} a(u - \theta_j)a(u + \theta_j) \]

\[ \times h_1(u + \frac{1}{2})h_1(u - \frac{1}{2}) \tilde{h}_1(u + \frac{1}{2}) \tilde{h}_1(u - \frac{1}{2}) \frac{Q^{(1)}(u - 1)}{Q^{(1)}(u)}, \]

\[ Z_2(u) = 2^2 \frac{u(u + 2)(u + \frac{3}{2})}{(u + 1)(u + 1)(u + \frac{1}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j) \]

\[ \times h_1(u + \frac{1}{2})h_2(u + \frac{3}{2}) \tilde{h}_1(u + \frac{1}{2}) \tilde{h}_2(u + \frac{3}{2}) \frac{Q^{(1)}(u + 1)Q^{(2)}(u - 1)Q^{(3)}(u - 1)}{Q^{(1)}(u)Q^{(2)}(u)Q^{(3)}(u)}, \]

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where

\[Z_3(u) = 2^2 \frac{u(u+2)}{(u+1)(u+1)} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j)\]

\[\times h_1(u + \frac{1}{2})h_2(u + \frac{3}{2}) \tilde{h}_1(u + \frac{1}{2}) \tilde{h}_2(u + \frac{3}{2}) \frac{Q^{(2)}(u-1)Q^{(3)}(u+1)}{Q^{(2)}(u)Q^{(3)}(u)},\]

\[Z_4(u) = 2^2 \frac{u(u+2)}{(u+1)(u+1)(u+\frac{3}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j)\]

\[\times h_1(u + \frac{1}{2})h_2(u + \frac{3}{2}) \tilde{h}_1(u + \frac{1}{2}) \tilde{h}_2(u + \frac{3}{2}) \frac{Q^{(1)}(u)Q^{(2)}(u+1)Q^{(3)}(u+1)}{Q^{(1)}(u+1)Q^{(2)}(u)Q^{(3)}(u)},\]

\[Z_6(u) = 2^2 \frac{u(u+\frac{1}{2})}{(u+1)(u+\frac{3}{2})} \prod_{j=1}^{N} e(u - \theta_j)e(u + \theta_j)\]

\[\times h_2(u + \frac{5}{2})h_2(u + \frac{3}{2}) \tilde{h}_2(u + \frac{5}{2}) \tilde{h}_2(u + \frac{3}{2}) \frac{Q^{(1)}(u+2)}{Q^{(1)}(u+1)},\]

\[f_1(u) = 2^2x \frac{u(u+2)(u+\frac{3}{2})}{u+1} \prod_{j=1}^{N} a(u - \theta_j)a(u + \theta_j)(u - \theta_j)(u + \theta_j)\]

\[\times h_1(u + \frac{1}{2})\tilde{h}_1(u + \frac{1}{2}) \frac{Q^{(2)}(u-1)Q^{(3)}(u-1)}{Q^{(1)}(u)},\]

\[f_2(u) = 2^2x \frac{u(u+2)(u+\frac{1}{2})}{u+1} \prod_{j=1}^{N} a(u - \theta_j)a(u + \theta_j)(u - \theta_j)(u + \theta_j)\]

\[\times h_2(u + \frac{3}{2})\tilde{h}_2(u + \frac{3}{2}) \frac{Q^{(2)}(u+1)Q^{(3)}(u+1)}{Q^{(1)}(u+1)},\]

where

\[Q^{(1)}(u) = \prod_{k=1}^{L_1}(u - \mu_k^{(1)} + \frac{1}{2})(u + \mu_k^{(1)} + \frac{1}{2}),\]

\[Q^{(2)}(u) = \prod_{k=1}^{L_2}(u - \mu_k^{(2)} + 1)(u + \mu_k^{(2)} + 1),\]

\[Q^{(3)}(u) = \prod_{k=1}^{L_3}(u - \mu_k^{(3)} + 1)(u + \mu_k^{(3)} + 1),\]

\[h_1(u) = 1 + c_2u, \quad h_2(u) = 1 - c_2u, \quad \tilde{h}_1(u) = 1 - c_2u, \quad \tilde{h}_2(u) = 1 + c_2u.\]
The constraints (4.34)-(4.36) enables us to parameterize the eigenvalues of the transfer matrices $\Lambda(u)$ and $\Lambda_\pm(u)$ in terms of the inhomogeneous $T - Q$ relations

$$
\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + Z_5(u) + Z_6(u) + f_1(u) + f_2(u),
$$

$$
\Lambda_+(u) = \prod_{i=1}^{N} a_1(u - \theta_i) a_1(u + \theta_i) h_1(u) \tilde{h}_1(u) \times \left[ \frac{u + 2 Q^{(2)}(u - \frac{3}{2})}{u + \frac{1}{2} Q^{(2)}(u - \frac{1}{2})} + \frac{u(u + 2)}{(u + 1)(u + \frac{1}{2})} Q^{(1)}(u - \frac{1}{2}) Q^{(2)}(u + \frac{1}{2}) \right] + \prod_{i=1}^{N} b_1(u - \theta_i) b_1(u + \theta_i) h_2(u + 2) \tilde{h}_2(u + 2) \times \left[ \frac{u Q^{(3)}(u + \frac{3}{2})}{u + \frac{1}{2} Q^{(3)}(u + \frac{1}{2})} + \frac{u(u + 2)}{(u + 1)(u + \frac{3}{2})} Q^{(1)}(u + \frac{1}{2}) Q^{(3)}(u - \frac{1}{2}) \right] + xu(u + 2) \prod_{i=1}^{N} a_1(u - \theta_i) a_1(u + \theta_i) b_1(u - \theta_i) b_1(u + \theta_i) \frac{Q^{(2)}(u + \frac{1}{2}) Q^{(3)}(u - \frac{1}{2})}{Q^{(1)}(u + \frac{1}{2})}.
$$

$$
\Lambda_-(u) = \prod_{i=1}^{N} a_1(u - \theta_i) a_1(u + \theta_i) h_1(u) \tilde{h}_1(u) \times \left[ \frac{u + 2 Q^{(3)}(u - \frac{3}{2})}{u + \frac{1}{2} Q^{(3)}(u - \frac{1}{2})} + \frac{u(u + 2)}{(u + 1)(u + \frac{1}{2})} Q^{(1)}(u - \frac{1}{2}) Q^{(3)}(u + \frac{1}{2}) \right] + \prod_{i=1}^{N} b_1(u - \theta_i) b_1(u + \theta_i) h_2(u + 2) \tilde{h}_2(u + 2) \times \left[ \frac{u Q^{(2)}(u + \frac{3}{2})}{u + \frac{1}{2} Q^{(2)}(u + \frac{1}{2})} + \frac{u(u + 2)}{(u + 1)(u + \frac{3}{2})} Q^{(1)}(u + \frac{1}{2}) Q^{(2)}(u - \frac{1}{2}) \right] + xu(u + 2) \prod_{i=1}^{N} a_1(u - \theta_i) a_1(u + \theta_i) b_1(u - \theta_i) b_1(u + \theta_i) \frac{Q^{(2)}(u - \frac{1}{2}) Q^{(3)}(u + \frac{1}{2})}{Q^{(1)}(u + \frac{1}{2})}.
$$

The regularities of the eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ lead to that the Bethe roots $\{\mu_k^{(m)}\}$ should satisfy the BAEs

$$
\left( \mu_k^{(1)} + \frac{1}{2} \right) h_1(\mu_k^{(1)} - 1) \tilde{h}_1(\mu_k^{(1)} - 1) \quad \frac{Q^{(1)}(\mu_k^{(1)} - \frac{3}{2})}{\prod_{j=1}^{N}(\mu_k^{(1)} - \frac{1}{2} - \theta_j)(\mu_k^{(1)} - \frac{1}{2} + \theta_j) Q^{(2)}(\mu_k^{(1)} - \frac{3}{2}) Q^{(3)}(\mu_k^{(1)} - \frac{3}{2})}
$$

$$
+ \left( \mu_k^{(1)} - \frac{1}{2} \right) h_2(\mu_k^{(1)} + 1) \tilde{h}_2(\mu_k^{(1)} + 1) \quad \frac{Q^{(1)}(\mu_k^{(1)} + \frac{1}{2})}{\prod_{j=1}^{N}(\mu_k^{(1)} + \frac{1}{2} - \theta_j)(\mu_k^{(1)} + \frac{1}{2} + \theta_j) Q^{(2)}(\mu_k^{(1)} + \frac{1}{2}) Q^{(3)}(\mu_k^{(1)} - \frac{1}{2})}
$$

$$
= -x \mu_k^{(1)} (\mu_k^{(1)} - \frac{1}{2}) (\mu_k^{(1)} + \frac{1}{2}), \quad k = 1, \ldots, L_1.
$$
\[
\frac{Q^{(1)}(\mu_l^{(2)})Q^{(2)}(\mu_l^{(2)} - 2)}{Q^{(1)}(\mu_l^{(2)} - 1)Q^{(2)}(\mu_l^{(2)})} = -\frac{\mu_l^{(2)} - \frac{1}{2}}{\mu_l^{(2)} + \frac{1}{2}}, \quad l = 1, \ldots, L_2,
\]
\[
\frac{Q^{(1)}(\mu_l^{(3)})Q^{(3)}(\mu_l^{(3)} - 2)}{Q^{(1)}(\mu_l^{(3)} - 1)Q^{(3)}(\mu_l^{(3)})} = -\frac{\mu_l^{(3)} - \frac{1}{2}}{\mu_l^{(3)} + \frac{1}{2}}, \quad l = 1, \ldots, L_3,
\]
(4.38)

where the numbers of Bethe roots should satisfy the constraint
\[L_1 = L_2 + L_3 + N,\]
(4.39)

and the parameter \( x \) is given by
\[x = \frac{(c'_3 c_3 - c'_2 c_2 - c'_3 c_2)(c'_2 c_3 - c'_3 c_3 + c'_2 c_3 + c'_3 c_2)}{c_3 c'_3} + 4c_2 c'_2.
\]
(4.40)

Again, the BAES obtained from the regularity of \( \Lambda(u) \) are the same as those obtained from the regularity of \( \Lambda_{\pm}(u) \). The function \( Q^{(m)}(u) \) has two zero points, and the BAES obtained from these two points are the same. It is easy to check that \( \Lambda(u) \) and \( \Lambda_{\pm}(u) \) satisfy the functional relations (4.34), the values at the special points (4.35) and the asymptotic behaviors (4.36). Therefore, we conclude that \( \Lambda(u) \) and \( \Lambda_{\pm}(u) \) are the eigenvalues of the transfer matrices \( t(u) \) and \( t_{\pm}(u) \), respectively.

Finally, the eigenvalue \( E \) of Hamiltonian (4.6) can be obtained by the \( \Lambda(u) \) as
\[E = \frac{\partial \ln \Lambda(u)}{\partial u}|_{u=0,\{\theta_j\}=0}.
\]
(4.41)

5 Discussion

In this paper, we study the exact solution of \( D^{(1)}_3 \) model, with various boundary conditions including the periodic one and the non-diagonal reflection one. By using the fusion technique, we obtain the complete operator product identities of the fused transfer matrices. Based on them and the asymptotic behaviors as well as the special values at certain points, we obtain the Bethe Ansatz solutions of the system. The method and the results in this paper could be generalized to the \( D^{(1)}_n \) case directly.

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Appendix A: Spinorial R-matrix

In this Appendix, we show that the vectorial $R$-matrix $R_{vv}^{u}(u)$ (2.1) and the fused ones $R_{v}^{u}(u)$ (see above (3.12)) can be obtained from the spinorial $R$-matrix of $D_{3}^{(1)}$ model [33] by using the fusion. The spinorial $R$-matrix $R_{1/2}^{ss}(u)$ of the $D_{3}^{(1)}$ model is the fundamental $R$-matrix of $su(4)$ one, and is a $16 \times 16$ matrix with the form

\[
R_{1/2}^{ss}(u) = \begin{pmatrix}
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1 \\
1 & u & 1 & 1
\end{pmatrix}.
\]

The spinorial $R$-matrix has following properties

\[
\text{unitarity} : \quad R_{1/2}^{ss}(u)R_{2}^{ss}(-u) = (1 + u)(1 - u),
\]

\[
\text{crossing – unitarity} : \quad R_{1/2}^{ss}(u)^{t\prime}R_{2}^{ss}(-u)^{t\prime} = -u(u + 4), \tag{A.1}
\]
and satisfies the Yang-Baxter equation

\[ R_{1'2'}^{ss}(u_1 - u_2)R_{1'2'}^{ss}(u_1 - u_3)R_{2'3'}^{ss}(u_2 - u_3) = R_{2'3'}^{ss}(u_2 - u_3)R_{1'3'}^{ss}(u_1 - u_3)R_{1'2'}^{ss}(u_1 - u_2). \] (A.2)

At the point of \( u = -1 \), we have

\[ R_{1'2'}^{ss}(-1) = P_{1'2'}^{(6)}S_{1'2'}^{(6)}, \] (A.3)

where \( P_{1'2'}^{(6)} \) is a 6-dimensional projector

\[ P_{1'2'}^{(6)} = \sum_{i=1}^{6} |\phi_i^{(6)}\rangle \langle \phi_i^{(6)}|, \] (A.4)

and the corresponding basis vectors are

\[ |\phi_1^{(6)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |\phi_2^{(6)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle), \quad |\phi_3^{(6)}\rangle = \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle), \]
\[ |\phi_4^{(6)}\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \quad |\phi_5^{(6)}\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle), \quad |\phi_6^{(6)}\rangle = \frac{1}{\sqrt{2}}(|34\rangle - |43\rangle). \]

The \( S_{1'2'}^{(6)} \) is a 6 × 6 constant matrix omitted here. The fusion of the 6-dimensional projector \( P_{1'2'}^{(6)} \) gives

\[ P_{1'2'}^{(6)}R_{1'2'}^{ss}(u + \frac{1}{2})R_{1'3'}^{ss}(u - \frac{1}{2})P_{2'3'}^{(6)} = (u - \frac{1}{2})R_{1'2'}^{ss}(u), \] (A.5)
\[ P_{1'2'}^{(6)}R_{2'3'}^{s+v}(u + \frac{1}{2})R_{1'3'}^{s+v}(u - \frac{1}{2})P_{2'3'}^{(6)} = R_{1'2'}^{v}(u). \] (A.6)

The dimension of the fused space \( V_{(1'2')} \) is 6. From Eq.(A.5), we obtain the fused \( R \)-matrix \( R^{s+v}(u) \). The \( R^{s-v}(u) \) can be obtained via Eq.(3.15). From Eq.(A.6), we obtain the vectorial \( R \)-matrix \( R^{v}(u) \) (2.1).

For the open case, the spinorial \( R \)-matrix \( R^{ss}(u) \) and the spinorial reflection matrix \( K^s(u) \) satisfy the reflection equation

\[ R_{1'2'}^{ss}(u - v)K^s_{1'}(u)R_{2'1'}^{ss}(u + v)K^s_{2'}(v) = K^s_{2'}(v)R_{1'2'}^{ss}(u + v)K^s_{1'}(u)R_{2'1'}^{ss}(u - v). \] (A.7)

One can check that matrix (4.14) is a solution of Eq.(A.7), \( K^s+(u) = K^s(u) \). By using Eq.(4.22), we arrive at \( K^s-(u) \) (4.15). The vectorial reflection matrix \( K^v(u) \) (1.8) is obtained from \( K^s+(u) \) by using the fusion with 6-dimensional projector \( P_{1'2'}^{(6)} \)

\[ P_{1'2'}^{(6)}K^s_{2'}(u + \frac{1}{2})R_{1'2'}^{ss}(2u)K^s_{1'}(u - \frac{1}{2})P_{2'1'}^{(6)} = (u - \frac{1}{2})K^v_{(1'2')}(u). \] (A.8)
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