Renormalization and finiteness of
topological $BF$ theories *

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Abstract

We show that the $BF$ theory in any space-time dimension, when quantized in a certain linear covariant gauge, possesses a vector supersymmetry. The generator of the latter together with those of the BRS transformations and of the translations form the basis of a superalgebra of the Wess-Zumino type. We give a general classification of all possible anomalies and invariant counterterms. Their absence, which amounts to ultraviolet finiteness, follows from purely algebraic arguments in the lower-dimensional cases.

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1 Introduction

Original inspiration for topological gauge field theory (TGFT) came from mathematics. For instance, the topology of four-dimensional manifolds as studied by Donaldson could be described through an action principle by the topological Yang-Mills theory of Witten. Another celebrated example, in three dimensions, is the description of invariants of knots in terms of the topological Chern-Simons theory.

The topological Yang-Mills theory and the Chern-Simons theory are prototypes of two distinct classes of TGFT’s which are sometimes classified as being “of Witten type”, or respectively “of Schwarz type” (see for instance [5]). We shall only be concerned with the latter in the following.

Schwarz type TGFT’s are characterized by the fact that their classical gauge fixed action can be written as the sum of a gauge-invariant term and a BRS-exact term. Besides the Chern-Simons theory there exists another TGFT of Schwarz type, namely the topological $BF$ theory. The latter represents a natural generalization of the Chern-Simons theory since it can be defined in arbitrary dimensions, whereas a Chern-Simons action exists only in odd-dimensional space-times. Moreover, the Lagrangean of the $BF$ theory contains the quadratic terms needed for defining a quantum theory, whereas a Chern-Simons action has this feature only in three dimensions.

The topological $BF$ model describes the coupling of an antisymmetric tensor field to a Yang-Mills potential (cf. [5] for references), hence the name “antisymmetric tensor field theory” which is sometimes used in the literature.

The aim of the present paper is to present a systematic study of the perturbative renormalization of the $BF$ theory in arbitrary space-time dimensions.

Before starting with our program we feel it necessary to motivate the recourse to perturbation theory and to compare its relevance to the one of non-perturbative approaches to TGFT.

Non-perturbative investigations are sensitive to the topology of the manifold on which the TGFT is defined. As an example, let us consider the quantization law for the Chern-Simons coupling constant $k$ [6]. This is a non-perturbative effect that is due to the existence of topologically non-trivial finite gauge transformations. In turn, the quantization law enforces the vanishing of the corresponding $\beta$-function $\beta_k$, which lead to the conjecture that the theory should be finite.

When dealing with perturbation theory, one sees only infinitesimal gauge transformations, i.e., gauge transformations which are connected to the identity. In the example above, the quantization law cannot be inferred perturbatively. The
β-function of the coupling constant has to be dealt with order by order. This β-
function has been shown to be zero perturbatively [7, 8].

Hence, perturbation theory is necessary for a rigorous discussion of results ob-
tained from non-perturbative considerations. In our example, a perturbative analy-
sis is necessary for clarifying the issue of ultraviolet finiteness. On a more abstract
and fundamental level, it is only in the perturbative regime that the existence of the
theory may a priori be guaranteed by the theorems of renormalization theory [9].

There is a further motivation for undertaking a perturbative study. Although
local observables do not exist in a TGFT, such objects may live on the boundary,
if any, of the manifold on which the theory is defined [4, 10]. Perturbative consider-
erations are then expected to lead to a better understanding of the structure of the
algebra of such local observables [11, 12].

The ultraviolet finiteness was studied and established for particular gauge fixings
in three-dimensional Chern-Simons theory [7, 8, 13, 14] as well as in two-, three-
and four-dimensional BF theory [15, 16, 17]. In all these cases (except [7]) the
main ingredient of the proof was the presence of a supersymmetric structure [18].
Furthermore, in most cases the validity of a ghost equation [14] controlling the
couplings of some Faddeev-Popov fields proved being an important feature.

We treat in a similar way the BF topological theory. Our paper is organized as
follows. In sect. 2 we recall the main features of the topological BF theory [5, 19, 20].
We choose a linear gauge fixing condition, in contrast with [19] where a non-linear
covariant gauge is used. We write the corresponding Slavnov identity, which ex-
presses the (off-shell nilpotent) BRS invariance. The supersymmetric structure of
the BF models is worked out in sect. 3. We show that beyond the BRS transforma-
tions there exists a vector valued generator whose anticommutation with BRS
yields the translation generator. For a specific value of the gauge fixing parameters
this supersymmetry turns out to be an invariance of the theory up to a trivial linear
breaking. The ghost equation is derived in sect. 4. The functional operators gen-
erating all the symmetries obey a closed algebra displayed in sect. 5. The problem
of the renormalization is treated in sect. 6 by cohomological methods. We deter-
mine all anomaly candidates and all possible invariant counterterms. By anomaly
candidates we mean all the possible obstructions to the renormalization program,
whereas “all possible invariant counterterms” stands for the ambiguities arising in
the renormalization procedure, usually also referred to as “ultraviolet infinities”.
The cases of space-time dimensions four to seven are discussed in more details in
sect. 7.

Let us remark that, the theory being topological, its physical content should be
independent of the metric of the manifold in which it is defined. For simplicity we
shall however restrict ourselves to the case of flat \( \mathbb{R}^D \) space-times. An extension of
our results to arbitrary manifolds – at least to manifolds such that a renormalized
perturbation theory can be defined—may be performed by following the renormalization theory arguments of Ref. [13] (see also [21] for a formal discussion based on the path integral).

2 BRS invariance and gauge fixing

The BF system in \((d = n + 2)\)-dimensional space-time is defined at the classical level by the gauge invariant action

\[
\Sigma_{\text{inv}} = \frac{1}{2n!} \int d^{n+2}x \, \text{Tr} \, \epsilon_{\mu_1 \cdots \mu_{n+2}} B_{\mu_1 \cdots \mu_n} F_{\mu_{n+1} \mu_{n+2}} ,
\]

(2.1)

where \(B_{\mu_1 \cdots \mu_n}\) is an antisymmetric tensor of rank \(n\) and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\) is the Yang-Mills strength. All the fields \(\varphi\) live in the adjoint representation of the gauge group \(G\), which we assume to be a simple Lie group. We use the conventions \([T_a, T_b] = i f_{abc} T_c\), \(\text{Tr} T_a T_b = \delta_{ab}\), the \(T_a\)'s being a basis of Lie \(G\) in the fundamental representation. We also adopt the matrix notation \(\varphi = \varphi^a T_a\) for any field \(\varphi\). In terms of the (Lie algebra valued) differential forms \(B = (1/n!) B_{\mu_1 \cdots \mu_n} dx^{\mu_1} \cdots dx^{\mu_n}\) and \(A = A_\mu dx^\mu\), the invariant action reads

\[
\Sigma_{\text{inv}} = \int_{\mathbb{R}^{n+2}} \text{Tr} \, B \wedge F ,
\]

(2.2)

with \(F = dA + 1/2 [A, A]\). We shall omit the wedge symbol in the sequel. The field equations for \(A\), resp. \(B\), are

\[
F = 0 , \quad DB = 0 ,
\]

(2.3)

The action \(\Sigma_{\text{inv}}\) is invariant under two sets of gauge transformations, \(\delta_\omega\) and \(\delta_\psi\):

\[
\delta_\omega A = D\omega , \quad \delta_\omega B = [B, \omega] , \\
\delta_\psi A = 0 , \quad \delta_\psi B = D\psi ,
\]

(2.4)

with the covariant exterior derivative \(D\) given for any Lie algebra valued form \(\varphi\) by \(D\varphi = d\varphi + [A, \varphi]\). Moreover \(\omega = \omega^a T_a\) and \(\psi = (1/(n-1)!) \psi^a_{\mu_1 \cdots \mu_{n-1}} T_a dx^{\mu_1} \cdots dx^{\mu_{n-1}}\) are Lie \(G\) valued forms of degree 0, resp. \(n - 1\).

The Yang-Mills symmetry \(\delta_\omega\) can be gauge fixed in the usual way by using the Faddeev-Popov procedure. This is not the case for the symmetry \(\delta_\psi\) since it contains zero-modes. Indeed, if \(\psi = D\psi'\), where \(\psi'\) is an arbitrary \((n - 2)\)-form,
\[ \delta_\psi B = D D \psi' = [F, \psi'] \] which vanishes on-shell due to the equation of motion \( F = 0 \). \( \psi' = D \psi'' \), with \( \psi'' \) an arbitrary \((n - 3)\)-form, is then an on-shell zero-mode of the former. This procedure stops when \( \psi^{(k-1)} = D \psi^{(k)} \) and \( \psi^{(k)} \) is an arbitrary 0-form, \( \text{i.e.} \), there are \( k = n - 1 = d - 3 \) such stages. The symmetry \( \delta_\psi \) is said to be \( k \)-th stage on-shell reducible.

Fixing a gauge symmetry is more involved when the symmetry is reducible, as is \( \delta_\psi \). One possible way to do this is to follow the Batalin-Vilkovisky (BV) prescription \[22\]. Nevertheless, the recourse to this prescription can be avoided if one applies carefully the usual BRS renormalization prescription, \( \text{i.e.} \), if one properly fixes all the gauge freedom corresponding to the zero-modes of the symmetry.

Hence, we shall perform here the perturbative quantization of the \( BF \) theory along the familiar lines of the BRS renormalization procedure. We will point out the parallelism to the BV approach.

### 2.1 BRS invariance

The first step in the BRS approach is to introduce the Faddeev-Popov (FP) ghosts and ghosts-for-ghosts. We shall use the notation \[19\] \( B = B_{n-1}^0 \), where the upper index denotes the ghost number (or FP charge) and the lower one is the form degree. The FP ghost \( \psi \) of the symmetry \( \delta_\psi \) will be named \( B_{n-2}^1 \) and its chain of ghosts-for-ghosts is \( \psi' = B_{n-3}^2 \), \( \psi'' = B_{n-4}^3 \), \( \cdots \) up to \( \psi^{(k)} = B_{n-k}^n \). The FP ghost for the Yang-Mills symmetry \( \delta_\omega \) is denoted by \( c \). The gauge fields and the system of ghosts form the so-called “geometrical” sector of the theory.

In the geometrical sector, the BRS transformations write

\[
\begin{align*}
\delta A &= D c, \\
\delta c &= c^2,
\end{align*}
\]

and

\[
\begin{align*}
\delta B_{n-g}^g &= DB_{n-g-1}^{g+1} + \left[ c, B_{n-g}^g \right], \\
\delta B_0^g &= \left[ c, B_0^g \right].
\end{align*}
\]

This BRS transformation is nilpotent only \textit{on-shell}, since:

\[
\delta^2 B_{n-g}^g = -DDB_{n-g-2}^{g+2} = -\left[ F, B_{n-g-2}^{g+2} \right].
\]

Besides the geometrical fields, one has to introduce FP antighosts and Lagrange multipliers (or Stuckelberg fields) in order to implement the gauge fixing. For the Yang-Mills symmetry they are denoted by \( \bar{c} \), respectively \( \pi \). For the reducible symmetry \( \delta_\psi \) one needs the sets of antighosts \( \bar{C}_{n-g-k}^\gamma(k) \) and Lagrange multipliers \( \Pi_{n-g-k}^{\gamma(k)+1} \).
for $1 \leq k \leq n$, $0 \leq g \leq n-k$ and with $\gamma(k) = g$ for $k$ even and $\gamma(k) = -g - 1$ for $k$ odd.

The BRS transformations act on the antighosts and on the multiplier fields as

$$
\begin{align*}
    s \bar{c} &= \pi, & s \pi &= 0, \\
    s \bar{C}^{\gamma(k)}_{n-g-k} &= \Pi^{\gamma(k)+1}_{n-g-k}, & s \Pi^{\gamma(k)+1}_{n-g-k} &= 0, & 1 \leq k \leq n, & 0 \leq g \leq n-k .
\end{align*}
$$

The ghosts and antighosts of the reducible symmetry form a pyramid which starts from the gauge field $B^0_n$ and ends when its base is made out of 0-forms:

$$
\begin{array}{cccc}
    & B^0_n & & \\
    & \bar{C}^1_{n-1} & B^1_{n-1} & \\
    \bar{C}^2_{n-2} & \bar{C}^2_{n-2} & B^2_{n-2} & \\
\bar{C}^3_{n-3} & \bar{C}^3_{n-3} & \bar{C}^3_{n-3} & B^3_{n-3} \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

The \-diagonals from left to right correspond to $g = 0$, $g = 1$, \ldots and the \-diagonals from right to left have $k = 0$, $k = 1$, \ldots. The diagonal with $k = 0$ contains the gauge field $B$ and its tower of ghosts. The multiplier fields form a smaller pyramid which corresponds to the one of the antighosts:

$$
\begin{array}{cccc}
    & \Pi^0_{n-1} & & \\
    & \Pi^1_{n-2} & \Pi^{-1}_{n-2} & \\
    \Pi^2_{n-3} & \Pi^2_{n-3} & \Pi^{-2}_{n-3} & \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

Similarly, for the Yang-Mills symmetry, we have the (trivial) pyramids:

$$
\begin{array}{ccc}
    A & & \\
    \bar{c} & c & \pi \\
\end{array}
$$

### 2.2 Slavnov identity

The purpose of this subsection is to extend the on-shell nilpotent BRS operator $s$ to an operator which is nilpotent off-shell. This is possible \footnote{As usual the upper index denotes the ghost number and the lower one is the form degree. We use the notation of \cite{19}.} by introducing into the action terms which are non-linear in the external sources which we shall need.
in order to define the renormalized BRS transformations. The off-shell nilpotent operator will be expressed functionally by a Slavnov identity, which will be taken as the starting point for the definition of the theory\(^3\).

Let us thus introduce external sources \(\gamma = \gamma_{n+1}, \tau = \gamma_{n+2}\) and \(b_{g+2}^{-1}\), \(0 \leq g \leq n\), which we couple to the composite BRS variations of \(A\), \(c\) and \(B_{n-g}\), respectively.

The BRS invariance of the total classical action \(\Sigma\) is expressed through the non-linear Slavnov identity

\[
S(\Sigma) \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \left\{ \frac{\delta \Sigma}{\delta \gamma} \frac{\delta A}{\delta \gamma} + \frac{\delta \Sigma}{\delta \tau} \frac{\delta c}{\delta \tau} + \sum_{g=0}^{n} \frac{\delta \Sigma}{\delta b_{g+2}^{-1}} \frac{\delta B_{n-g}^{g}}{\delta \gamma} + \frac{\delta \Sigma}{\delta \gamma} \frac{\delta A}{\delta \gamma} + \frac{\delta \Sigma}{\delta A} \frac{\delta A}{\delta \gamma} + \sum_{g=0}^{n} \sum_{k=1}^{n-k} \Pi \gamma_{g}(k+1) \frac{\delta \Sigma}{\delta C_{n-g-k}^{\gamma}(k)} \right\} = 0 .
\] (2.9)

The associated linearized Slavnov operator writes

\[
S_{\Sigma} \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \left\{ \sum_{g=0}^{n} \left( \frac{\delta \Sigma}{\delta b_{g+2}^{-1}} \frac{\delta A}{\delta b_{g+2}^{-1}} + \frac{\delta \Sigma}{\delta B_{n-g}^{g}} \frac{\delta B_{n-g}^{g}}{\delta b_{g+2}^{-1}} \right) + \frac{\delta \Sigma}{\delta \gamma} \frac{\delta A}{\delta \gamma} + \frac{\delta \Sigma}{\delta A} \frac{\delta A}{\delta \gamma} + \sum_{k=1}^{n-k} \Pi \gamma_{g}(k+1) \frac{\delta \Sigma}{\delta C_{n-g-k}^{\gamma}(k)} \right\} = 0 .
\] (2.10)

\(S_{\Sigma}\) is the off-shell nilpotent extension of the BRS operator \(s\) we were looking for. Indeed, \(S_{\Sigma}^{2} = 0\) if the Slavnov identity (2.9) is satisfied.

### 2.3 Gauge fixing

Within the functional formalism, we fix the gauge by imposing the following gauge conditions:

\[
\frac{\delta \Sigma}{\delta \pi} = d \ast A , \\
\frac{\delta \Sigma}{\delta \Pi_{n-g-1}^{g}} = (-1)^{n+1} d \ast B_{n-g}^{g} + (-1)^{n+1+g} \ast d \bar{C}_{n-g-2}^{g} + x_{1,g,n} \ast \Pi_{n-g-1}^{g} , \quad 0 \leq g \leq n-1 , \quad (k=1) ,
\] (2.11)

\[
\frac{\delta \Sigma}{\delta \Pi_{n-g-k}^{\gamma(k)+1}} = (-1)^{n+1} d \ast \bar{C}_{n-g-(k-1)}^{\gamma(k)+1} + (-1)^{n+k+g} \ast d \bar{C}_{n-g-(k+1)}^{\gamma(k+1)}
\]

\(^{3}\)This approach is similar to the one of Batalin and Vilkovisky [22], our Slavnov identity playing the role of their “master equation”. There is however an important difference: each BV “antifield” keeps in our approach a classical part which we identify with the external source coupled to the BRS variation of the corresponding field.
where the gauge parameters \( x_{k,g,n} \) are (for the moment free) numerical coefficients \(^4\).

We want to find the most general classical action \( \Sigma \) obeying the Slavnov identity (2.9) and the gauge conditions (2.11). One easily sees that the compatibility of these two requirements implies the antighost equations

\[
\left( \frac{\delta}{\delta \bar{c}} - d * \frac{\delta}{\delta \bar{\gamma}} \right) \Sigma = 0 ,
\]

\[
\left( \frac{\delta}{\delta C_{n-g-1}} - d * \frac{\delta}{\delta b_{g+1}} \right) \Sigma = (-1)^g * d \Pi_{n-g-2}^{g+1} , \quad 0 \leq g \leq n-1 , \quad (k=1) , \tag{2.12}
\]

\[
\frac{\delta}{\delta C_{n-g-k}} \Sigma = d * \Pi_{n-g-(k-1)}^{(k-1)+1} + (-1)^{k+1} * d \Pi_{n-g-(k+1)}^{(k+1)+1} ,
\]

\[
0 \leq g \leq n-k , \quad 2 \leq k \leq n .
\]

Let us first write down the general solution of the gauge conditions (2.11) and of the antighost equations (2.12); we get

\[
\Sigma(A, c, B, \bar{c}, \pi, \bar{C}, \Pi, \gamma, \tau, b) = \hat{\Sigma}(A, c, B, \hat{\gamma}, \tau, \hat{b}) + \Sigma_{\Pi}(A, B, \bar{c}, \pi, \bar{C}, \Pi) , \tag{2.13}
\]

where

\[
\Sigma_{\Pi} = \int_{\mathbb{R}^n} \text{Tr} \left\{ -d\pi * A - \sum_{g=0}^{n-1} d\Pi_{n-g-1}^{g+1} * B_{n-g}^{g} + \sum_{k=1}^{n-1} \sum_{g=0}^{n-k} \left[ -d\Pi_{n-g-k}^{(k)+1} * \bar{C}_{n-g-k+1}^{(k-1)+1} + (-1)^{n+1} d\bar{C}_{n-g-k}^{(k)} * \Pi_{n-g-k+1}^{(k-1)+1} \right] \right\} . \tag{2.14}
\]

\(^4\text{Integrability of the equations has been enforced.}\)

\(^5\text{The symbol * denotes the Hodge duality, defined for any } p\text{-form } \omega \text{ by}\)

\[
*\omega = \frac{1}{(d - p)!} (\omega)_{\beta_1 \beta_2 \cdots \beta_{d-p}} \, dx^{\beta_1} dx^{\beta_2} \cdots dx^{\beta_{d-p}} ,
\]

where

\[
(\omega)_{\beta_1 \beta_2 \cdots \beta_{d-p}} = \frac{1}{p!} \varepsilon_{\beta_1 \beta_2 \cdots \beta_{d-p} \alpha_1 \cdots \alpha_p} \omega^{\alpha_1 \cdots \alpha_p} .
\]
The truncated action \( \hat{\Sigma} \) is the general solution of the homogenous gauge conditions and of the homogenous antighost equations. It is hence independent of the multiplier fields and depends on the antighosts only through the combinations
\[
\hat{\gamma} = \gamma - *d\bar{c},
\]
\[
\hat{b}_{g+2}^{-g-1} = b_{g+2}^{-g-1} + (-1)^{g+1} *d\bar{C}_{n-g-1}, \quad 0 \leq g \leq n-1, \ (k=1),
\]
\[
\hat{b}_{n+2}^{-n-1} = b_{n+2}^{-n-1} \ (g=n, \ k=1).
\]

The part of the action which depends on the multipliers completely fixes the gauge freedom. Apart from the terms quadratic in the multiplier fields it is a gauge-non-covariant version of the gauge fixing of Ref. [19]. The coefficients \( x_{k,g,n} \) of the quadratic terms are still free and will remain so after imposing the Slavnov identity. It is only the requirement of supersymmetry that will enforce a definite value for these coefficients, see sect. 3.

Before writing the Slavnov identity for the truncated action \( \hat{\Sigma} \), let us remark that there exists a very compact notation\(^6\) for the fields appearing as arguments of \( \hat{\Sigma} \). These can be arranged within two “field ladders”
\[
\hat{\phi}^{(1)} = \{ \hat{\phi}_p^{(1)}, \ p=0,\cdots,n+2 \} = \{ c, A, \hat{b}_{g+2}^{-g-1} (g=0,\cdots,n) \},
\]
\[
\hat{\phi}^{(2)} = \{ \hat{\phi}_p^{(2)}, \ p=0,\cdots,n+2 \} = \{ B_{n-g}^g (g=n,\cdots,0), \hat{\gamma}, \tau \}.
\]

Each of these ladders contains forms of degrees ranging from 0 up to the maximal degree \( n+2 \); we shall call such ladders “complete”. In these ladder variables, and after use of the gauge conditions and of the antighost equations, the Slavnov identity\(^7\) takes the form
\[
S(\Sigma) = \frac{1}{2} B_{\Sigma} \hat{\Sigma} = 0, \tag{2.17}
\]
where we define, for any functional \( \gamma \), the \( \gamma \)-dependent functional linear Slavnov operator \( B_\gamma \) to be
\[
B_\gamma \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \sum_{p=0}^{n+2} \left( \frac{\delta \gamma}{\delta \hat{\phi}^{(2)}_{n+2-p}} \frac{\delta}{\delta \hat{\phi}^{(1)}_p} + \frac{\delta \gamma}{\delta \hat{\phi}^{(1)}_{n+2-p}} \frac{\delta}{\delta \hat{\phi}^{(2)}_p} \right).
\]

The general solution of the Slavnov identity (2.17) reads, up to trivial field renormalizations,
\[
\hat{\Sigma} = \int_{\mathbb{R}^{n+2}} \text{Tr} \left( \sum_{p=0}^{n+1} \hat{\phi}_p^{(1)} d\hat{\phi}_{n+1-p} + \frac{1}{2} \sum_{p=0}^{n+2} \sum_{q=0}^{n+2-p} \left[ \hat{\gamma}^{(1)}_p, \hat{\gamma}^{(1)}_q \right] \hat{\phi}_{n+2-p-q}^{(2)} \right) \tag{2.19}
\]

\(^6\)An equivalent notation is found in Ref. [20].

\(^7\)One recognizes here the “master equation” of Batalin and Vilkovisky [22]. The fields (2.15) play the role of the BV “antifields”, but they possess here a classical part which has no analogue in the BV formalism and which consists of our external sources \( \gamma, b \) and \( \tau \) (for \( \tau \) the shift (2.15) is trivial: \( \hat{\tau} = \tau \)).
The cubic part contains terms quadratic in the external sources, and hence terms quadratic in the antighosts. The need for such non-linear terms was stressed at the beginning of subsect. 2.2.

In the case where the functional $\gamma$ in $B_\gamma$ (2.18) is the truncated action $\hat{\Sigma}$ given in (2.19), then $B_\gamma$ is nilpotent off-shell, and we shall write

$$B\hat{\Sigma} \equiv B, \quad B^2 = 0.$$  \tag{2.20}$$

The off-shell nilpotent BRS transformations of the components of the ladders (2.16) are now given by

$$B\hat{\varphi}^{(1)}_p = d\hat{\varphi}^{(1)}_{p-1} + \frac{1}{2} \sum_{q=0}^{n+2} \left[ \hat{\varphi}^{(1)}_q, \hat{\varphi}^{(1)}_{p-q} \right], \quad p=0,\ldots,n+2,$$

$$B\hat{\varphi}^{(2)}_p = d\hat{\varphi}^{(2)}_{p-1} + \sum_{q=0}^{n+2} \left[ \hat{\varphi}^{(1)}_q, \hat{\varphi}^{(2)}_{p-q} \right], \quad p=0,\ldots,n+2. \tag{2.21}$$

3 Supersymmetry

In analogy to what is known for the case of the Chern-Simons theory in three dimensions [18, 8, 5], the topological $BF$ theory exhibits a supersymmetric structure which, for the time being, has only been established in dimensions four and less [17, 16, 15]. In the present section, we shall give a systematic treatment of this supersymmetry in higher dimensions, at the classical level. The problems faced by its renormalization are addressed in sect. 6.

Let the following transformations of the ladder components (2.16) be

$$\delta^S((\xi))\hat{\varphi}^{(A)}_p = \begin{cases} -i\xi \hat{\varphi}^{(A)}_{p+1}, & p=0,\ldots,n+1, \\ 0, & p=n+2, \end{cases} \quad A = 1, 2, \tag{3.1}$$

where $i\xi$ is the inner derivative along a (constant) vector field $\xi^\mu$. In the sector of the antighosts and multipliers, for the Yang-Mills symmetry, let

$$\delta^S((\xi))\bar{c} = 0, \quad \delta^S((\xi))\bar{\pi} = \mathcal{L}\xi\bar{c}, \tag{3.2}$$

and, for the reducible symmetry, with $1 \leq k \leq n$, $0 \leq g \leq n-k$, let

$$\delta^S((\xi))\bar{C}^{g(k)}_{n-g-k} = \begin{cases} (-1)^{n+(k+1)/2} g(\xi) \bar{C}^{-(g-2)}_{n-g-k-1}, & (k \text{ odd}), \\ (-1)^{k/2+1} i\xi \bar{C}_{n-g-k+1}^g, & (k \text{ even}), \end{cases} \tag{3.3}$$

$$\delta^S((\xi))\Pi^{g(k)+1}_{n-g-k} = \begin{cases} (-1)^{n+(k+1)/2} g(\xi) \Pi^{-(g-1)}_{n-g-k-1} + \mathcal{L}\xi \bar{C}_{n-g-k}^g, & (k \text{ odd}), \\ (-1)^{k/2+1} i\xi \Pi^g_{n-g-k+1} + \mathcal{L}\xi \bar{C}_{n-g-k}^g, & (k \text{ even}). \end{cases}$$
$\mathcal{L}_\xi = [i_\xi, d]$ is the Lie derivative along $\xi^\mu$. $g(\xi) = g_{\mu\nu} \xi^\mu d\xi^\nu$ is a 1-form of ghost number 1, the vector field $\xi^\mu$ carrying an odd grading. $g_{\mu\nu}$ is a flat, Minkowskian or Euclidean metric in $(n+2)$-spacetime.

At the functional level, the invariance under the transformations $\delta_S^{(\xi)}$ given by eqs. (3.1) to (3.3) is implemented by the Ward operator

$$\mathcal{W}^{S}_{(\xi)} \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \sum_{\text{all fields } \varphi} \delta_S^{(\xi)} \varphi \frac{\delta}{\delta \varphi},$$

(3.4)

The transformations generated by the operator $\mathcal{W}^{S}_{(\xi)}$ form, together with the off-shell nilpotent linear Slavnov operator (2.10) an algebra which closes off-shell on the space-time translations:

$$[S_\Sigma, \mathcal{W}^{S}_{(\xi)}] = \mathcal{W}^{T}_{(\xi)} , \quad S_\Sigma^2 = 0 , \quad [\mathcal{W}^{S}_{(\xi)}, \mathcal{W}^{S}_{(\xi')}] = 0 ;$$

(3.5)

we denote by $\mathcal{W}^{T}_{(\xi)}$ the Ward operator for translations along a (constant) vector field $\xi^\mu$:

$$\mathcal{W}^{T}_{(\xi)} \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \sum_{\text{all fields } \varphi} \mathcal{L}_\xi \varphi \frac{\delta}{\delta \varphi}.$$  

(3.6)

Due to the algebraic structure (3.5), we call “supersymmetry transformations” the rules (3.1) to (3.3). The bosonic degrees of freedom are just the field components with even grading and the fermionic ones are those with odd grading. There is, in all dimensions, an equal number of bosonic components and of fermionic ones.

The Ward identity of this supersymmetry contains at the classical level a breaking which, being linear in the quantum fields, does not get renormalized. This is analogous to what has been proven for the Chern-Simons theory [8]. We get

$$\mathcal{W}^{S}_{(\xi)} \Sigma = \Delta^S_{(\xi)} \text{ class },$$

(3.7)

where the classical breaking $\Delta^S_{(\xi)} \text{ class }$ is given by

$$\Delta^S_{(\xi)} \text{ class } = \int_{\mathbb{R}^{n+2}} \text{Tr} \left\{ - \sum_{g=0}^n \frac{b_{-g+1}}{b_{g+2}} \mathcal{L}_\xi B_{n-g}^g + (-1)^n \gamma \mathcal{L}_\xi A + (-1)^n \tau \mathcal{L}_\xi c \\
+ d * i_\xi (\beta_1 b_2^{-1} \pi + \gamma \Pi_{n-1}^0) \right\},$$

(3.8)

provided the classical action $\Sigma$ is just the expression given by (2.13), (2.14) and (2.19), with fixed coefficients

$$x_{k,g,n} = (-1)^{n+g+(k+1)/2}. \quad \text{Note that the grading of some of the fields depends on the space-time dimension.}$$

10
On the truncated action \( \hat{\Sigma} \) (2.19), the Ward identity of supersymmetry becomes

\[
\hat{W}_S^\xi \hat{\Sigma} = \hat{\Delta}_S^\xi, \quad \text{(3.10)}
\]

where the Ward operator writes, in the ladder notation (2.16),

\[
\hat{W}_S^\xi(\xi) \equiv \int_{\mathbb{R}^{n+2}} \text{Tr} \left[ \sum_{A=1,2} \sum_{p=0,\ldots,n+1} -i\xi \phi^{(A)}_{p+1} \frac{\delta}{\delta \phi^{(A)}_p} \right], \quad \text{(3.11)}
\]

and the classical breaking is replaced by

\[
\hat{\Delta}_S^\xi = \int_{\mathbb{R}^{n+2}} \text{Tr} \left( -\sum_{p=0}^{n+2} \phi_{n+2-p}^{(2)} \mathcal{L}_\xi \phi^{(1)}_p \right). \quad \text{(3.12)}
\]

4 The ghost equation

It is known that, in gauge theories with a Landau gauge fixing, the dependence of the radiative corrections on the Faddeev-Popov ghosts is constrained by identities called the ghost equations [14]. Since in the present case the gauge fixing of the field \( A \) is precisely of the Landau type, one may expect such an identity to hold. Inspection of the classical action (given in (2.13), (2.14) and (2.19)) shows that this is indeed the case for the ghost zero-form \( B_0 \). This ghost equation reads, at the classical level:

\[
\mathcal{G} \Sigma \equiv \int_{\mathbb{R}^{n+2}} \left( \frac{\delta \Sigma}{\delta B_0} + \left[ \frac{\delta \Sigma}{\delta \pi}, \mathcal{C}_{-n} \right] \right) = \Delta^\mathcal{G}_{\text{class}}, \quad \text{(4.1)}
\]

where the right-hand-side

\[
\Delta^\mathcal{G}_{\text{class}} \equiv \int_{\mathbb{R}^{n+2}} \left( \frac{1}{2} \sum_{p=2}^{n} \left[ b^{-p+1}_p, b^{-p+1}_{n+2-p} \right] + \left[ b^{-n}_n, A \right] + \left[ b^{-n-1}_n, c \right] \right.
\]

\[
+ \sum_{p=2}^{n} (-1)^{n+p+1} \left[ b^{-p+1}_p, \star d \mathcal{C}_{n+p-1} \right] \right), \quad \text{(4.2)}
\]

is linear in the quantum fields, hence not subject to renormalization.

One can check that, for any functional \( \gamma \),

\[
\mathcal{G} \mathcal{S}(\gamma) + \mathcal{S}_\gamma (\mathcal{G} \gamma - \Delta^\mathcal{G}_{\text{class}}) = \mathcal{F} \gamma - \Delta^\mathcal{F}_{\text{class}}, \quad \text{(4.3)}
\]

with

\[
\mathcal{F} \equiv \int_{\mathbb{R}^{n+2}} \left( \left[ c, \frac{\delta}{\delta B_0} \right] + \left[ A, \frac{\delta}{\delta B_{n+1}} \right] + \sum_{p=2}^{n} \left[ b^{-p+1}_p \frac{\delta}{\delta B_{n+p}} \right] \right.
\]

\[
+ \left[ b^{-n}_n \frac{\delta}{\delta \gamma} \right] + \left[ b^{-n-1}_n \frac{\delta}{\delta \tau} \right] + \sum_{p=2}^{n} \left[ \star d \mathcal{C}_{n+p-1} \frac{\delta}{\delta B_{n+p}} \right] \right)
\]

\[
+ (-1)^{n+1} \left[ \mathcal{C}_{-n} \frac{\delta}{\delta \mathcal{C}} \right] + (-1)^{n+1} \left[ \Pi_{-n+1} \frac{\delta}{\delta \Pi} \right] \right), \quad \text{(4.4)}
\]
and

\[ \Delta^F_{\text{class}} \equiv \sum_{q=2}^{n} (-1)^{n+p+1} \left[ b_p^{-p+1} \right] . \] (4.5)

For \( \gamma = \Sigma \), solution of the Slavnov identity and of the ghost equation, \( (4.3) \) implies the "associated ghost equation"

\[ \mathcal{F} \Sigma = \Delta^F_{\text{class}} , \] (4.6)

whith again a right-hand-side which is linear in the quantum fields.

In term of the truncated action \( (2.15) \) and the ladder variables \( (2.16) \) the ghost equation and its associated equation take the much simpler forms

\[ \hat{G} \hat{\Sigma} \equiv \int_{\mathbb{R}^{n+2}} \frac{\delta \hat{\Sigma}}{\delta B_0} = \frac{1}{2} \int_{\mathbb{R}^{n+2}} \sum_{p=0}^{n+2} \left[ \hat{\phi}_p^{(1)}, \hat{\phi}_p^{(1)} \right] \equiv \Delta^g_{\text{class}} , \] (4.7)

\[ \hat{F} \hat{\Sigma} \equiv \int_{\mathbb{R}^{n+2}} \sum_{p=0}^{n+2} \left[ \hat{\phi}_p^{(1)}, \frac{\delta \hat{\Sigma}}{\delta \phi_p^{(2)}} \right] = 0 . \] (4.8)

5 Functional algebra

The Slavnov operator \( \mathcal{S} \) \( (2.9) \) and its linearized form \( \mathcal{S}_F \) (see \( (2.10) \)), the Ward identity operators\(^9\) for supersymmetry \( \mathcal{W}^S_{(\xi)} \) \( (3.4) \) and for translations \( \mathcal{W}^T_{(\varepsilon)} \) \( (3.6) \), as well as the ghost equation operator \( \mathcal{G} \) \( (4.1) \) and its associated operator \( \mathcal{F} \) \( (4.4) \) obey the non-linear algebra

\[ \mathcal{S}_F \mathcal{S}(F) = 0 , \]
\[ \mathcal{W}^S_{(\xi)} \mathcal{S}(F) - \mathcal{S}_F \left( \mathcal{W}^S_{(\xi)} F - \Delta^S_{\text{class}} \right) = \mathcal{W}^T_{(\varepsilon)} F , \]
\[ \mathcal{W}^T_{(\varepsilon)} \mathcal{S}(F) - \mathcal{S}_F \mathcal{W}^T_{(\varepsilon)} F = 0 , \] (5.1)
\[ \mathcal{G} \mathcal{S}(F) - (-1)^n \mathcal{S}_F \left( \mathcal{G} F - \Delta^g_{\text{class}} \right) = \mathcal{F} F - \Delta^F_{\text{class}} , \]
\[ \mathcal{F} \mathcal{S}(F) + (-1)^n \mathcal{S}_F \left( \mathcal{F} F - \Delta^F_{\text{class}} \right) = 0 , \]

valid for any functional \( F \), and the (anti)-commutation relations\(^10\)

---

\(^9\)The infinitesimal supersymmetry parameters \( \xi^\mu \) are anticommuting numbers, the translation parameters \( \varepsilon^\mu \) are commuting.

\(^10\)Trivial commutation relations involving the translation operator are not written.
\[
\left[ \mathcal{W}_s^{(\xi)}, \mathcal{W}_s^{(\xi')} \right] = 0 , \\
\left[ \mathcal{G}, \mathcal{W}_s^{(\xi)} \right] = \left[ \mathcal{F}, \mathcal{W}_s^{(\xi)} \right] = 0 , \tag{5.2}
\]
\[
\left[ \mathcal{G}^a(x), \mathcal{G}^b(y) \right] = \left[ \mathcal{F}^a(x), \mathcal{G}^b(y) \right] = \left[ \mathcal{F}^a(x), \mathcal{F}^b(y) \right] = 0 .
\]

For a functional \( \hat{F} \) independent of the Lagrange multiplier fields and depending on the antighosts only through the shifted external fields (2.15) – like the truncated action \( \hat{\Sigma} \) defined in (2.13) – the non-linear algebra (5.1) becomes

\[
\mathcal{B} \hat{F} = 0 , \\
\frac{1}{2} \hat{W}^{S}_s(\xi) \mathcal{B} \hat{F} - \mathcal{B} \hat{F} \left( \hat{W}^{S}_s(\xi) - \hat{\Delta}^{S}_{\xi} \right) = \mathcal{W}^T_s(\xi) \hat{F} , \\
\frac{1}{2} \hat{W}^T_s(\xi) \mathcal{B} \hat{F} - \mathcal{B} \hat{F} \hat{W}^T_s(\xi) \hat{F} = 0 , \\
\frac{1}{2} \hat{G} \mathcal{B} \hat{F} - (-1)^n \mathcal{B} \left( \hat{G} \hat{F} - \hat{\Delta}^{G}_{\text{class}} \right) = \hat{F} \hat{F} - \hat{\Delta}^{F}_{\text{class}} , \\
\frac{1}{2} \hat{F} \mathcal{B} \hat{F} + (-1)^n \mathcal{B} \hat{F} \hat{F} = 0 . 
\]

\( \mathcal{B} \), was defined in (2.18), \( \hat{W}^{S}_s(\xi) \), \( \hat{\Delta}^{S}_{\xi} \) in (3.11), (3.12), and \( \hat{G} \), \( \hat{F} \), \( \hat{\Delta}^{G}_{\text{class}} \) in (4.7), (4.8). The operators \( \hat{W}^{S}_s(\xi), \hat{G} \) and \( \hat{F} \) obey the same linear algebra (5.2) as the unhatted ones.

Furthermore if this functional \( \hat{F} \) is the classical truncated action \( \hat{\Sigma} \) (2.13), (2.19), solution of the Slavnov identity (2.17) (and more generally if \( \hat{F} \) obeys the latter Slavnov identity), then to (5.1) there corresponds the linear algebra

\[
\mathcal{B}^2 = 0 , \\
\left[ \hat{W}^{S}_s(\xi), \mathcal{B} \right] = \mathcal{W}^T_s(\xi) , \quad \left[ \mathcal{W}^T_s(\xi), \mathcal{B} \right] = 0 , \tag{5.4}
\]
\[
\left[ \hat{G}, \mathcal{B} \right] = \hat{F} , \quad \left[ \hat{F}, \mathcal{B} \right] = 0 ,
\]
with \( \mathcal{B} \) defined by (2.18), (2.20).

## 6 Renormalization

### 6.1 Statement of the problem

Our aim is now to perform the perturbative quantization of the classical theory defined in the preceding sections. This means that we must:
1) construct a vertex functional $\Gamma = \Sigma + O(\hbar)$ satisfying to all orders of perturbation theory all the functional identities, in particular the Ward identities, which we have shown to hold for the classical action $\Sigma$. This is the problem of the anomalies;

2) look for the possible counterterms which one can freely add at each order to the action without spoiling the functional identities. This is the problem of the invariant counterterms.

Let us begin with the problem of the anomalies and let us denote by $\Delta_{\text{BRS}}$, $\Delta_S(\xi)$, $\Delta^\varphi$ and $\Delta^F$ the possible radiative breakings of the Slavnov identity (2.9), of the supersymmetry Ward identity (3.7), of the ghost equation (4.1) and of its associated equation (4.6). From the quantum action principle [24] the breakings are local insertions of dimensions constrained by power-counting. Their ghost numbers are fixed by ghost number conservation.

The renormalization scheme is assumed to preserve Poincaré invariance. The renormalizability of the gauge fixing conditions (2.11) and of the antighost equations (2.12) is trivial; we therefore assume these to hold exactly. As a consequence, the vertex functional $\Gamma$ is the sum of a term $\hat{\Gamma}$ depending only on the ladder fields (2.16), and of the explicit term $\Sigma_\Pi$ depending on the Lagrange multiplier and antighost fields which appears in the classical action (2.13). We have:

$$\Gamma(A, c, B, \bar{c}, \pi, \bar{C}, \Pi, \gamma, b) = \hat{\Gamma}(\hat{\phi}(1), \hat{\phi}(2)) + \Sigma_\Pi(A, B, \bar{c}, \pi, \bar{C}, \Pi) .$$

(6.1)

As a consequence of the gauge conditions and of the antighost equations the radiative breakings depend only on the ladder fields. Due to the algebra (5.3) (with $\hat{F} = \hat{\Gamma}$) and (5.2) the breakings obey, at the lowest order $N$ in $\hbar$ at which they are supposed to appear, the consistency conditions:

$$B\Delta_{\text{BRS}} = 0 , \quad \hat{\mathcal{V}}_\xi^{\text{S}}\Delta_{\text{BRS}} - B\Delta_{\text{S}}(\xi) = 0 ,$$

$$\hat{\mathcal{G}}\Delta_{\text{BRS}} + (-1)^{n+1}B\Delta^\varphi = \Delta^F , \quad \hat{\mathcal{F}}\Delta_{\text{BRS}} + (-1)^nB\Delta^F = 0 ,$$

$$\hat{\mathcal{V}}_\xi^{\text{S}}\Delta_{\text{S}}(\xi') - \hat{\mathcal{V}}_{(\xi')}^{\text{S}}\Delta_{\text{S}}(\xi) = 0 , \quad \hat{\mathcal{G}}\Delta_{\text{S}}(\xi) - \hat{\mathcal{V}}_{(\xi)}^{\text{S}}\Delta^\varphi = 0 ,$$

$$\hat{\mathcal{F}}\Delta_{\text{S}}(\xi) - \hat{\mathcal{V}}_{(\xi)}^{\text{S}}\Delta^F = 0 , \quad \hat{\mathcal{G}}_a\Delta_a^\varphi + (-1)^{n+1}\hat{\mathcal{G}}_b\Delta_b^\varphi = 0 ,$$

$$\hat{\mathcal{G}}_a\Delta_a^F - \hat{\mathcal{F}}_b\Delta_b^F = 0 , \quad \hat{\mathcal{F}}_a\Delta_a^F + (-1)^n\hat{\mathcal{F}}_b\Delta_b^F = 0 .$$

(6.2)

Anomalies are non-trivial solutions of these consistency conditions, i.e., solutions which cannot be written as:

$$\Delta_{\text{BRS}} = B\Delta , \quad \Delta_{\text{S}}(\xi) = \hat{\mathcal{V}}_{(\xi)}^{\text{S}}\Delta , \quad \Delta^\varphi = \hat{\mathcal{G}}\Delta , \quad \Delta^F = \hat{\mathcal{F}}\Delta .$$

(6.3)

\[11\]Poincaré invariance of the renormalization scheme being assumed, there is no radiative breaking of translation invariance. Hence the corresponding consistency conditions reduce to the condition of translation invariance of the other breakings.
\( \Delta \) being some local functional of the ladder fields \((2.10)\). In the anomaly-free situation, adding \(-\Delta\) to the action as a counterterm ensures the validity of the functional identities at the order \(N\) considered.

Let us now come to the invariant counterterms. Here the problem consists in finding the general solution \(\Delta^{\text{inv}}\) – a local functional with the dimensions of the classical action and of ghost number zero – of the equations

\[
\mathcal{B}\Delta^{\text{inv}} = 0, \quad \hat{W}^S_{(\xi)}\Delta^{\text{inv}} = 0, \quad \hat{G}\Delta^{\text{inv}} = 0, \quad \hat{F}\Delta^{\text{inv}} = 0.
\]

This yields all the possible local counterterms which one may recursively add to the action without spoiling the functional identities.

### 6.2 Cohomology

We have to solve the consistency conditions \((6.2)\) and the invariance conditions \((6.4)\) in the space of translation invariant local functionals. Since the translation operator \(\mathcal{W}^T_{(\varepsilon)}\) \((3.6)\) belongs to the algebra of functional operators of sect. \(3\) in a non-trivial way – it appears in a right-hand side – it will turn out to be convenient to keep it among the set of functional operators.

The whole set of functional operators can be incorporated into one single operator

\[
\delta = \mathcal{B} + \hat{W}^S_{(\xi)} + \mathcal{W}^T_{(\varepsilon)} + \text{Tr} \left( u\hat{G} + v\hat{F} \right) + \xi^\mu \frac{\partial}{\partial \varepsilon^\mu} + \text{Tr} \left( u \frac{\partial}{\partial v} \right),
\]

where the “global ghosts” \(\xi^\mu, \varepsilon^\mu, u^a\) and \(v^a\) are the infinitesimal parameters of the supersymmetry transformations, of the translations, of the ghost equation and of its associated equation \((4.6)\). Their ghost numbers are 2, 1, \(n\), and \(n-1\), respectively, so that \(\delta\) has ghost number one. Their gradation is equal to the parity of their ghost numbers: for \(\xi\) and \(\varepsilon\) it is opposite to the natural gradation used previously. Their own transformation laws are given by the last two terms in \((6.5)\); this together with the algebra \((5.2), (5.4)\) makes \(\delta\) a coboundary operator \([13]\):

\[
\delta^2 = 0.
\]

In this cohomological setting the problems of the anomalies and of the invariant counterterms, as described above, reduce to one single cohomology problem. Indeed, both the consistency conditions \((6.2)\) and the invariance conditions \((6.4)\) can be written as

\[
\delta\Delta^G_{(n+2)} = 0, \quad G=0,1,
\]

where \(\Delta^G_{(n+2)}\) belongs to the space of integrated local functionals of ghost number \(G\) of the ladder fields \(\hat{\phi}^{(1)}\) and \(\hat{\phi}^{(2)}\) \((2.10)\).
1) The possible anomalies, obeying the consistency conditions (6.2), are given by the non-trivial solutions of (6.7) with ghost number \( G = 1 \) and which are homogeneous of degree one in the infinitesimal parameters.

2) The invariant counterterms, obeying the conditions (6.4), are obtained as the general solution of (6.7) with ghost number \( G = 0 \), independent of the infinitesimal parameters \( \xi^\mu, \cdots, v^a \). The non-trivial solutions correspond to the renormalization of the physical parameters of the theory.

We have thus to solve the cohomology of the coboundary operator \( \delta \), i.e., to look for the equivalence classes modulo–\( \delta \) of solutions of the equation (6.7). “Modulo–\( \delta \)” means up to a “trivial” term of the form \( \delta \hat{\Delta}^{G^{-1}} \) with \( \hat{\Delta}^{G^{-1}} \) taken in the same space of functionals as \( \hat{\Delta}^G \).

In order to solve this cohomology problem it is useful to introduce the filtering operator
\[
F = \xi^\mu \frac{\partial}{\partial \xi^\mu} + \varepsilon^\mu \frac{\partial}{\partial \varepsilon^\mu} + \text{Tr} \left( u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right).
\]

The insertion \( \Delta_G^{(n+2)} \) and the operator \( \delta \) can be expanded according to the eigenvalues \( f \) of the filter \( F \), i.e., according to the degree in the global ghosts. The expansion of \( \delta \) reads
\[
\delta = \delta_0 + \delta_1,
\]
with:
\[
\delta_0 = B + \xi^\mu \frac{\partial}{\partial \xi^\mu} + \text{Tr} \left( u \frac{\partial}{\partial u} \right),
\]
\[
\delta_1 = \hat{\mathcal{W}}_{\xi} + \hat{\mathcal{W}}_{\varepsilon} + \text{Tr} \left( u \hat{\mathcal{G}} + v \hat{\mathcal{F}} \right).
\]

One has
\[
\delta_0^2 = \{ \delta_0, \delta_1 \} = \delta_1^2 = 0. \quad (6.10)
\]

The reason for performing this filtration is that the cohomology of \( \delta \) is isomorphic to a subspace of the cohomology of \( \delta_0 \) (see [23, 24]). Let us hence begin by solving
\[
\delta_0 \Delta_G^{(n+2)} = 0. \quad (6.11)
\]

We first observe that the global ghosts \( \varepsilon, \xi = \delta_0 \varepsilon, \nu \) and \( u = \delta_0 v \) form two \( \delta_0 \)-doublets. It follows [23, 27] that the \( \delta_0 \)-cohomology does not depend on them. Accordingly \( \Delta_G^{(n+2)} \) will be assumed to depend only on the field ladders (2.16) \( \hat{\phi}^{(A)}, \) \( A = 1, 2. \)

Let us write
\[
\Delta_G^{(n+2)} = \int_{\mathbb{R}^{n+2}} \chi_G^{n+2}, \quad (6.12)
\]
where the upper index indicates as usual the ghost number and the lower one the form degree. The condition (6.11) implies that the \( \delta_0 \)-variation of the integrand \( \chi_G^{n+2} \) is a total derivative \( d\chi_G^{n+2} \). Then, nilpotency of \( \delta_0 \) together with the triviality [28, 27].
of the cohomology of the exterior derivative $d$ in the space of local functionals, imply
the following set of descent equations:

$$
\begin{align*}
\delta_0 \chi_{n+2+G-g}^g &= d \chi_{n+1+G-g}^{g+1}, & g &= G, \ldots, G+n+1, \\
\delta_0 \chi_{n+2+G} &= 0,
\end{align*}
$$

(6.13)

which relate $\chi_{n+2}^g$ to a zero-form of ghost number $n + 2 + G$. The most general
expression for this zero-form $\chi_{0}^{n+2+G}$ is a polynomial in the ladder fields $\phi_0^{(A)}$.

Let us first consider the case of space-time dimensions greater or equal to 5, i.e.,
$n \geq 3$. The general solution $\chi_{0}^{n+2+G}$ of the last equation (6.13) (for $G = 0$ or 1) is
an arbitrary linear superposition of the monomials 28, 27

$$
\chi_0^{n+2+G} = \prod_{k=1}^{K} \text{Tr} c^{n_k} \left| \sum_k n_k = n+2+G, \quad n_k \text{ odd} \right.,
$$

(6.14)

and of the monomial (for $G = 0$)

$$
\chi_0''^{n+2} = \Tr (c^2 B_0^n) = -\delta_0 \Tr (cB_0^n),
$$

(6.15)

For $n = 2$ the general solution is a superposition of:

$$
\chi_0^5 = \Tr c^5 \quad (G = 1),
\chi_0''^4 = \Tr (c^2 B_0^4) = -\delta_0 \Tr (cB_0^4) \quad (G = 0),
$$

(6.16)

(6.17)

and

$$
\chi_0'''^4 = \Tr (B_0^4)^2 \quad (G = 0).
$$

(6.18)

Finally for $n = 1$ one finds the solutions (there is none for $G = 1$):

$$
\chi_0^3 = \Tr c^3 \quad (G = 0),
\chi_0''^3 = \Tr (c^2 B_0^3) = -\delta_0 \Tr (cB_0^3) \quad (G = 0),
$$

(6.19)

(6.20)

and

$$
\chi_0'''^3 = \Tr (B_0^3)^3 \quad (G = 0).
$$

(6.21)

Climbing the ladder (6.13) from $\chi_{0}^{n+2+G}$ up to $\chi_{n+2}^G$ can be achieved through the
repeated application of the “ladder climbing operator” 17

$$
\nabla = dx^\mu \hat{\mathcal{W}}_\mu^S,
$$

(6.22)

where $\hat{\mathcal{W}}_\mu^S$ is the supersymmetry generator defined by 12

$$
\hat{\mathcal{W}}_\mu^S = \xi^\mu \hat{\mathcal{W}}_\mu^S.
$$

(6.23)

12The supersymmetry Ward operator $\hat{\mathcal{W}}_\mu^S$ (6.11) defines a linear map, i.e., a one-form, from
the vectors $\xi$ to the space of the functional differential operators. This is the intrinsic definition of
the functional operator valued one-form $\nabla$. 17
The action of $\nabla$ on the ladder fields is given by:

$$\nabla \tilde{\phi}^{(A)}_p = (p + 1) \tilde{\phi}^{(A)}_{p+1}, \quad p=0,\ldots,n+1,$$

(6.24)

It is a derivation, which commutes with the exterior derivative $d$. The commutation rules of $\nabla$ with the other operators are:

$$[\delta_0, \nabla] = d,$$

$$[\hat{W}_S^{(\xi)}, \nabla] = [W^{T_\epsilon}, \nabla] = [\hat{G}, \nabla] = [\hat{F}, \nabla] = 0,$$

(6.25)

from which it obviously follows that

$$[\delta, \nabla] = d.$$

(6.26)

These commutation rules are simple consequences of the algebra (5.2), which also implies the anticommutation rules

$$\{\hat{W}_\mu^S, \hat{W}_\nu^S\} = 0,$$

(6.27)

from which follows the identity

$$\nabla^{n+2} \hat{W}_S^{(\xi)} = 0.$$

(6.28)

One easily checks now that, for a given zero-form $\chi^{n+2+G}_0$, which is a solution of the last descent equation (6.13), the forms

$$\chi^{n+2+G-p}_p = \frac{1}{p!} \nabla^p \chi^{n+2+G}_0, \quad p=0,\ldots,n+2,$$

(6.29)

do indeed solve the whole set of descent equations (6.13).\(^{13}\)

We remark that, due to the first commutation rule (6.25), the solutions constructed from (6.15), (6.17) and (6.20) yield trivial cocycles, i.e., the corresponding insertions (6.12) are $\delta_0$-variations. On the other hand the solutions derived from the zero-forms (6.14), (6.16), (6.18), (6.19) and (6.21) lead to non-trivial $\delta_0$-cocycles.

Let us show that these are the only non-trivial solutions of (6.11). It is clear that the problem of solving the descent equation

$$\delta_0 \chi^g_{n+2+G-g} = d \chi^{g+1}_{n+1+G-g}, \quad g=G,\ldots,G+n+1,$$

(6.30)

for $\chi^g_{n+2+G-g}$ (a solution $\chi^{g+1}_{n+1+G-g}$ of the lower descent equation being given) is a problem of local $\delta_0$-cohomology, i.e., of solving the homogeneous equation

$$\delta_0 \omega = 0.$$

(6.31)

\(^{13}\)See [17] for a previous version of this construction.
The most convenient way to solve the latter equation is to introduce a new filtration, with the counting operator

$$N = \sum_{\text{all fields } \varphi} \int_{\mathbb{R}^{n+2}} \text{Tr} \frac{\delta}{\delta \varphi},$$

(6.32)
as filtering operator. According to this filtration $\delta_0$ splits into

$$\delta_0 = \delta_{0,0} + \delta_{0,1}, \quad \text{with} \quad \delta_{0,0}^2 = 0.$$  

(6.33)

The action of the coboundary operator $\delta_{0,0}$ on the two field ladders $\hat{\phi}^{(A)}$, $A = 1, 2$ reads:

$$\delta_{0,0} \hat{\phi}^{(A)} = 0 ,$$

$$\delta_{0,0} \hat{\phi}^{(A)}_p = d \hat{\phi}^{(A)}_{p-1} , \quad p=1, \ldots, n+2.$$  

(6.34)

It is shown in [29] that in the case of complete field ladders – as realized here – the local cohomology of $\delta_{0,0}$ depends only on the ladder components which are non-differentiated zero-forms. These zero-forms in the present case are $\hat{\phi}^{(1)}_0 = c$ and $\hat{\phi}^{(2)}_0 = \hat{B}_n^0$. It follows that the local $\delta_{0,0}$-cohomology is empty in the space of forms of degree $p > 0$. The local $\delta_0$-cohomology, being isomorphic to a subset of the local $\delta_{0,0}$-cohomology, is empty as well in this space. The consequence of this result is [30] that the general solution $\chi^g_{n+2+G-g}$ of the descent equations for a given non-trivial $\chi^g_0$ (see (6.14)) is unique modulo-$\delta_0$ and modulo-$d$:

$$\chi^g_{n+2+G-g} = \chi^g_{n+2+G-g} + \delta_0(\cdots) + d(\cdots).$$

(6.35)

Hence the whole $\delta_0$-cohomology for $G = 0, 1$ is given by all the linear superpositions of the $\delta_0$-cocycles

$$\Delta^G_{(n+2)} = \int_{\mathbb{R}^{n+2}} \frac{1}{(n+2)!} \nabla^{n+2} \prod_{k=1}^{K} \text{Tr} c^{n_k} \left| \sum_{k} n_k = n+2+G, \quad n_k \text{ odd} \right| , \quad n \geq 1 ,$$

(6.36)

and, in addition for $n = 2$ or $1$, of the $\delta_0$-cocycles

$$\Delta^0_{(4)} = \int_{\mathbb{R}^4} \frac{1}{4!} \nabla^4 \left( \text{Tr} \left( B_0^2 \right)^2 \right),$$

$$\Delta^0_{(3)} = \int_{\mathbb{R}^3} \frac{1}{3!} \nabla^3 \left( \text{Tr} \left( B_0^1 \right)^3 \right).$$

(6.37)

The cocycles (6.36) are solutions of the full cohomology condition (6.7). This follows indeed from the commutator (6.26) of the full coboundary operator $\delta$ with the climbing operator $\nabla$, from the identity (6.28) and from the fact that $\prod \text{Tr} c^{n_k}$ is annihilated by $\hat{\mathcal{G}}$ (1.7) and $\hat{\mathcal{F}}$ (1.8), and is independent of $\varepsilon$ and $v$. Moreover they cannot be $\delta$-coboundaries since they are independent of the global ghosts $\varepsilon$, $\xi$, $v$ and $u$.  

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The same argument applied to the cocycles (6.37) shows that these are not $\delta$-invariant because of their dependence on $B^0_n$, and thus must be rejected.

Hence (6.36) represents the whole $\delta$-cohomology for $G = 0, 1$ and $n \geq 1$.

6.3 Anomalies and invariant counterterms

1) The solutions (6.36) with ghost number $G = 1$ yield all the possible anomalies of the $(n + 2)$-dimensional $BF$ systems. Some examples are shown in the next section. The actual presence of anomalies depends on a case by case investigation of the group theoretical structure of the model and on explicit computations of the anomaly coefficients, possibly with the help of a non-renormalization theorem.

2) The cocycles (6.38) with $G = 0$ ghost number yield the non-trivial counterterms, i.e., those which generate the renormalization of physical coupling constants. Trivial $\delta$-invariant counterterms, if any, of the form

$$\Delta_{\text{trivial}}(\hat{\phi}^{(1)}, \hat{\phi}^{(2)}) = \delta \Delta^{-}(\hat{\phi}^{(1)}, \hat{\phi}^{(2)}) ,$$

(6.38)

where $\Delta^{-}$ is a local functional of ghost number $-1$, would correspond to the renormalization of parameters which can be compensated by field redefinitions.

Let us show that there is no such trivial counterterm. In order for a trivial counterterm to be independent of the global ghosts $\xi, \varepsilon, u$ and $v$, $\Delta^{-}$ has to be independent from them too and it has to fulfill the conditions

$$\hat{\mathcal{W}}_{\xi}^S \Delta^{-} = 0 ,$$

(6.39)

$$\hat{\mathcal{G}} \Delta^{-} = \hat{\mathcal{F}} \Delta^{-} = 0 .$$

(6.40)

It is shown in Appendix A that the general solution of the supersymmetry condition (6.39) reads

$$\Delta^{-} = \int_{\mathbb{R}^{n+2}} \nabla^{n+2} \Omega_{0}^{n+1} .$$

(6.41)

The zero-form $\Omega_{0}^{n+1}$ is a superposition of the monomials $\chi'^{n+1}_0$ (6.14) and of

$$\chi^{\prime\prime\prime}_0 = \text{Tr} (cB^0_n) .$$

(6.42)

$\chi'$, yielding a $\Delta^{-}$ which is $\delta$-invariant, leads to a vanishing counterterm, whereas $\chi^{\prime\prime\prime}$ leads to a $\Delta^{-}$ which does not obey (6.40) and thus must be discarded.

In conclusion the possible invariant counterterms are given by the non-trivial cocycles (6.36) with ghost number $G = 0$. Their actual occurrence has to be tested case by case as for the anomalies.
7 Discussion of the result: some examples

The purpose of this section is to discuss in details the algebraic structure of some $BF$ systems by using the general results on the cohomology of the operator $\delta$. As explicit models we will analyze the cases $d = 4, 5, 6$ and $7$.

7.1 Four-dimensional model

Let us begin with the case $n = 2$ which corresponds to a four dimensional $BF$ system $[17]$. On one hand the non-trivial cocycle of ghost number one is given by

$$\Delta^{1}_{(4)} = \frac{1}{5!} \int_{\mathbb{R}^4} \nabla^4 \text{Tr} \ c^5$$

$$= \int_{\mathbb{R}^4} \left( A^4 c + \hat{b}_4^{-1} c^4 + \hat{b}_3^{-2} (Ac^3 + cAc^2 + c^2 Ac + c^3 A) ight. \right.$$

$$\left. + \hat{b}_2^{-1} (A^2 c^2 + Ac^2 A + c^2 A^2 + AAc + cAcA + cA^2 c) ight.$$  \hspace{1cm} (7.1)

$$+ \hat{b}_2^{-1} \hat{b}_2^{-1} c^3 + \hat{b}_2^{-1} c \hat{b}_2^{-1} c^2 \right).$$

and has the quantum numbers of a Slavnov anomaly. Actually it is well known $[31]$ that the ghost polynomial $\text{Tr} \ c^5$ is related via descent equations to the non-abelian gauge anomaly (or Adler-Bardeen-Bell-Jackiw anomaly)

$$A^{(4)} = \frac{1}{2} \int_{\mathbb{R}^4} \text{Tr} \left( dc (AdA + dAA + A^3) \right).$$  \hspace{1cm} (7.2)

The expression (7.1) is in fact nothing but the gauge anomaly (7.2) (modulo a $\mathcal{B}$-coboundary) written in a way which is compatible with the Ward identity of supersymmetry (3.10). In this case however, the absence of anomalies is ensured by the fact that all the fields belong to the adjoint representation of the gauge group. Indeed this implies that the Feynman rules involve only the structure constants $f^{[abc]}$. This forbids the appearance of the totally symmetric tensor $d^{(abc)}$ which is present in the expression (7.1).

On the other hand there is no invariant counterterm. Thus the four dimensional model is finite to all orders of perturbation theory.

7.2 Five-dimensional model

The general expression (6.36) shows that the ghost polynomial $\text{Tr} \ c^5$ – which gave the Slavnov anomaly in four dimensions – determines also the cohomology for the
five-dimensional BF system (i.e., \( n = 3 \)), but now in the sector of ghost number \( G = 0 \). The relevant cocycle for this case is

\[
\Delta^0_{(5)} = \frac{1}{5!} \int_{\mathbb{R}^5} \nabla^5 \text{Tr} c^5 .
\]

(7.3)

This cocycle, whose explicit form is given by

\[
\Delta^0_{(5)} = \int_{\mathbb{R}^5} \text{Tr} \left( \frac{1}{5} A^5 + \hat{b}^{-1}_2 c + \hat{b}^{-1}_3 (Ac^2 + A^2 c + c^3 A) + \hat{b}^{-1}_3 (A^2 c + c^2 A) + \hat{b}^{-1}_2 (A^3 c + A^2 c + c^3 A) \right)
\]

(7.4)

has the quantum numbers of the five-dimensional BF action and corresponds to a possible counterterm. Moreover one should note that this term, like the cocycle (7.1), contains the totally symmetric tensor \( d^{abc} \) which cannot be generated in a model containing only fields in the adjoint representation. This means that the expression (7.4) can appear as a counterterm only if it has been included in the initial classical action. In this case one has to deal with a more general model which contains a generalized Chern-Simons term. It is not difficult in fact to show that the above expression coincides, modulo a \( \mathcal{B} \)-coboundary, with the Chern-Simons five-form [31]

\[
\mathcal{A}_{(5)} = \int_{\mathbb{R}^5} \text{Tr} \left( AdAdA + \frac{2}{3} A^3 dA + \frac{3}{5} A^5 \right) .
\]

(7.5)

In this sense, this BF system can be regarded as a higher dimensional generalization of the three dimensional Chern-Simons theory. It is apparent that the generalized Chern-Simons action can be likewise consistently included in any odd dimension.

But for the pure BF theory there is no possible counterterm. Since moreover no anomalies are allowed by the cohomology (no cocycles with \( G = 1 \)), the pure BF theory in five dimensions is finite.

### 7.3 Six- and seven-dimensional models

Let us close this section by considering the six- and the seven-dimensional (i.e., \( n = 4, 5 \)) models. The relevant ghost polynomials are given by \( \text{Tr} c^7 \) and \( \text{Tr} c^3 \text{Tr} c^5 \) which allow to define the cocycles:

\[
\Delta^1_{(6)} = \frac{1}{7!} \int_{\mathbb{R}^6} \nabla^6 \text{Tr} c^7 ,
\]

(7.6)
in six dimensions, and
\[
\Delta^0_{(7)} = \frac{1}{7!} \int_{\mathbb{R}^7} \nabla^7 \text{Tr} c^7,
\] (7.7)

in seven dimensions. The expression (7.6) corresponds to a possible Slavnov anomaly for the six-dimensional theory, while expressions (7.7) and (7.8) yield a generalized Chern-Simons counterterm and a possible Slavnov anomaly in the seven-dimensional case.

It is interesting to note that we cannot exclude the presence of the six-dimensional anomaly (7.6) or of the seven-dimensional counterterm (7.7) by using group theoretical arguments as in the four- and five-dimensional cases. One easily sees, indeed, that the expressions (7.6), (7.7) contain a totally symmetric tensor of rank four which could be generated by using the structure constants. A detailed analysis of the Feynman graphs which could contribute to these terms and the possibility of an Adler-Bardeen non-renormalization theorem is beyond the aim of this paper and will be reported on in a future work.

8 Conclusions

We have shown the existence of a supersymmetric structure generated by the BRS transformations and by a vector valued operator obeying, together with the translations, a Wess-Zumino-like superalgebra. In an appropriate linear gauge of the Landau type, the vector operator yields a symmetry of the theory which is broken, but only linearly.

We have given a complete classification of the possible BRS anomalies and invariant counterterms. It has turned out that none of them is present in the pure BF models in space-time dimensions four and five: these theories are finite. For arbitrary dimensions the analysis of our general algebraic results cannot exclude the presence of anomalies or of counterterms of the Chern-Simons type. Definite conclusions would require explicit knowledge of the numerical coefficients of the anomaly or of the counterterms, which one can obtain by evaluating the contributing Feynman diagrams.

The classification of the anomalies and of the invariant counterterms has been given by solving the BRS and supersymmetry cohomology, constrained by the ghost equation. The explicit construction of these objects has been performed by applying \( d \) times (\( d \) being the dimension of space-time) an operator \( \nabla \) to the zero-form ghost cocycles (7.14). This operator \( \nabla \), which is nothing else than the supersymmetry generator, has a well-defined geometrical meaning: mapping p-forms into \((p+1)\)-
forms, it naturally solves the descent equations owing to the fact that its commutator with the cohomology operator \( \delta \) yield the exterior derivative as shown by eq.(6.26).

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A Appendix: Supersymmetry invariants

We will show that the most general supersymmetric integrated local functional of the ladder fields (2.16) has the form (6.41). The result is stated in prop. A.2 below.

It is convenient to introduce a superspace formalism [32]. Superspace is the set of points 

\[ z = (x^\mu, \theta^\mu), \]

\( x \) being a point of \( \mathbb{R}^D \) and \( \{ \theta^\mu | \mu = 1, \ldots, D \} \) a set of \( D \) Grassmann coordinates. A superspace integration measure is provided by

\[ \int dz f(z) = \int d^D x d^D \theta f(z) = \int d^D x \frac{\partial}{\partial \theta^1} \cdots \frac{\partial}{\partial \theta^D} f(z). \quad (A.1) \]

**Definition A.1** Superfields are superspace local functionals \( \Psi(z) \) of the ladder fields \( \hat{\phi}^{(A)}_p \), transforming under supersymmetry (3.11) as

\[ \hat{W}^S_{(\xi)} \Psi(z) = \xi^\mu \frac{\partial}{\partial \theta^\mu} \Psi(z) \quad (A.2) \]

Superfields build up an algebra, a basis of which is provided by the superfields

\[ \Phi^{(A)}(z) = \sum_{p=0}^{D} \frac{1}{p!} \theta^{\mu_1} \cdots \theta^{\mu_p} \phi^{(A)}_{\mu_1 \cdots \mu_p}(x), \quad (A.3) \]

and their derivatives, where \( \phi^{(A)}_{\mu_1 \cdots \mu_p}(x) \) is a component of the \( p \)-form \( \hat{\phi}^{(A)}_p \). According to the transformation rules (3.11) \( \Phi^{(A)} \) as well as its \( x \)- and \( \theta \)-derivatives indeed transform as superfields.

**Proposition A.1** Every local functional \( \Delta \) of the ladder fields, invariant under supersymmetry, i.e., obeying

\[ \hat{W}^S_{(\xi)} \Delta = 0, \quad (A.4) \]

may be expressed as the superspace integral

\[ \Delta = \int dz \Psi(z) \quad (A.5) \]

of a composite superfield \( \Psi \) made out of the superfields \( \Phi^{(A)} \) and their derivatives.
\textbf{Proof}: The most general $\Delta$ (of degree $N$ in the fields) may be written as a multiple integral (cf. \cite{33}):

$$\Delta = \int dz_1 \cdots dz_N \Phi^{(A_1)}(z_1) \cdots \Phi^{(A_N)}(z_N) K_{A_1 \cdots A_N}(z_1, \ldots, z_N). \quad (A.6)$$

Due to the locality hypothesis and to translation invariance, the kernel $K$ is a linear combination of products of derivatives of Dirac distributions $\delta^D(x_i - x_1)$ with $\theta$-dependent coefficients. The invariance condition (A.4) is equivalent to the condition

$$\left( \sum_{i=1}^{N} \frac{\partial}{\partial \theta_1^i} \right) K = 0 \quad (A.7)$$

for the kernel. The latter condition implies that the kernel depends on the $\theta$'s only through their differences $\theta_i - \theta_1$. Then it is easy to see that its most general expression consists in a sum of terms of the form

$$K_{A_1 \cdots A_N}(z_1, \ldots, z_N) = \prod_{a=2}^{N} \frac{\partial}{\partial \theta_1^a} \cdots \frac{\partial}{\partial \theta_{a_{M_1}}} \frac{\partial}{\partial x_{a_{M_1}}} \cdots \frac{\partial}{\partial x_{a_{M_a}}} \delta(z_a - z_1), \quad (A.8)$$

where we have used the superspace delta distribution:

$$\int dz_2 \delta(z_1 - z_2) f(z_2) = f(z_1),$$

$$\delta(z_1 - z_2) = \left( \prod_{\mu=1}^{D} (\theta_1^\mu - \theta_2^\mu) \right) \delta^D(x_1 - x_2). \quad (A.9)$$

Introducing this result in (A.6) we conclude that $\Delta$ is a sum of terms of the form (A.5) with

$$\Psi(z) = \Phi^{(A_1)}(z) \prod_{a=2}^{N} \frac{\partial}{\partial \theta_1^a} \cdots \frac{\partial}{\partial \theta_{a_{M_1}}} \frac{\partial}{\partial x_{a_{M_1}}} \cdots \frac{\partial}{\partial x_{a_{M_a}}} \Phi^{A_a}(z) \quad (A.10)$$

is a superfield.

\textbf{Proposition A.2} The supersymmetric local functional of Prop. A.1 can be written as the space-time integral

$$\Delta = \int_{R^D} \nabla^D \Omega_0, \quad (A.11)$$

where $\Omega_0$ is a zero-form and $\nabla$ the ladder climbing operator defined in eqs (6.22), (6.23).

\textbf{Proof}: This follows from the result (A.3), the definition (A.1) of superspace integration and from the supersymmetry transformation law (A.2). The zero-form $\Omega_0$ is proportional to the $\theta = 0$ component of the superfield $\Psi(z)$. 

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