Anisotropic inflation with coupled $p$—forms

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Abstract. We study the cosmology in the presence of arbitrary couplings between $p$-forms in 4-dimensional space-time for a general action respecting gauge symmetry and parity invariance. The interaction between 0-form (scalar field $\phi$) and 3-form fields gives rise to an effective potential $V_{\text{eff}}(\phi)$ for the former after integrating out the contribution of the latter. We explore the dynamics of inflation on an anisotropic cosmological background for a coupled system of 0-, 1-, and 2-forms. In the absence of interactions between 1- and 2-forms, we derive conditions under which the anisotropic shear endowed with nearly constant energy densities of 1- and 2-forms survives during slow-roll inflation for an arbitrary scalar potential $V_{\text{eff}}(\phi)$. If 1- and 2-forms are coupled to each other, we show the existence of a new class of anisotropic inflationary solutions in which the energy density of 2-form is sustained by that of 1-form through their interactions. Our general analytic formulas for the anisotropic shear are also confirmed by the numerical analysis for a concrete inflaton potential.
1 Introduction

The observational evidence for inflation and dark energy suggests that there may be additional degrees of freedom (DOFs) beyond those appearing in standard models of particle physics and general relativity. The simplest candidate for such new DOFs is a scalar field \( \phi \) (0-form), which can be compatible with the homogeneous and isotropic cosmological background. Indeed, the scalar field slowly evolving along a nearly flat potential \( V(\phi) \) drives the cosmic acceleration [1–7]. The mechanism of generating scalar and tensor perturbations during single-field inflation [8–13] is overall consistent with the observed scalar spectral index and the tensor-to-scalar ratio of Cosmic Microwave Background (CMB) temperature anisotropies [14–16].

On the other hand, it is known that there are some anomalies in the CMB data such as the hemispherical asymmetry between the North and South ecliptic hemispheres, the mutual alignment of lowest multipole moments, and the dipole modulation of very large-scale CMB signals [16, 17]. This may be attributed to the violation of the isotropic cosmological evolution. For a scalar field, the breaking of isotropy of space-time is limited apart from specific cases [18]. The vector field \( A_\mu \) (i.e., 1-form field) or the 2-form field \( B_{\mu\nu} \) can be the natural sources for generating anisotropies relevant to CMB anomalies.

If we apply the 1-form field to inflation, there is a no-hair theorem stating that, in the presence of an effective cosmological constant, the anisotropic shear generated by the 1-form dilutes exponentially fast, so, an isotropic de Sitter background is quickly obtained [19, 20]. Nevertheless, in the presence of the coupling \(-f_1(\phi)F_{\mu\nu}F^{\mu\nu}/4\), where \( f_1(\phi) \) is a function of the scalar field \( \phi \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength of the 1-form \( A_\mu \), it is possible to sustain the anisotropic hair on the quasi de Sitter background where the Hubble expansion rate slowly varies in time [21]. The power spectrum of curvature perturbations is modified by the broken rotational invariance, whose effect can be quantified by an anisotropic parameter.
The non-linear estimator \( f_{NL} \) of scalar non-Gaussianities can be as large as the order of 10 in the squeezed limit [24]. Several aspects of the coupling \( -f_1(\phi)F_{\mu\nu}F^{\mu\nu}/4 \) have been discussed in the literature, see Refs. [21, 23–42].

The interaction between the scalar \( \phi \) and the 2-form field strength \( H_{\mu\nu\lambda} \) of the form 
\[-f_2(\phi)H_{\mu\nu\lambda}H^{\mu\nu\lambda}/12 \]
can also lead to anisotropic inflation for a suitable choice of the coupling \( f_2(\phi) \). In Ref. [43], it was shown that the power spectrum of curvature perturbations is of the prolate-type anisotropy with \( g_* > 0 \), in contrast to the oblate-type anisotropy for the scalar-vector coupling. Moreover the non-linear estimator \( f_{NL} \) vanishes in the squeezed limit, while it can be of order 10 in the equilateral and enfolded limits as consistent with the recent CMB data [16]. Hence the couplings of 1-form and 2-form fields with the scalar field \( \phi \) can be observationally distinguished from each other [44] (see also Ref. [45]).

The 3-form field is also relevant to the dynamics of cosmic acceleration [46–57]. The coupling between the scalar \( \phi \) and the 3-form field generates the effective potential \( V_{\text{eff}}(\phi) \) for the scalar [48, 49]. This mechanism was used for the realization of chaotic inflation by the mixing between an axion and the 3-form [50]. In a more general setup with multiple scalar and axions, the interactions with 3-forms were studied at length in Refs. [58–60] (see also Ref. [61]). A crucial property exploited in those references rests on the fact that the 3-form field strength \( F_{\mu\nu\lambda\rho} \) is proportional to the volume element in 4-dimensional space-time, so it can be regarded as an effective cosmological constant term. In this regard, such couplings are not only relevant for inflation but also for the dynamics of dark energy.

Recently, three of the present authors [62] performed a systematic construction of gauge-invariant theories with coupled p-forms on a D-dimensional background. The construction was restricted to up to first-order derivatives of fields in the action, so the theories do not contain, for instance, the second-order derivatives typical from the Galileon class of interactions [63–66]. Indeed, even if we include the second-order derivatives of p-form fields in the action, there is a no-go theorem stating that Galilean interactions for odd p-forms preserving gauge invariance and Lorentz symmetry are forbidden\(^1\) [74, 75]. In Ref. [62], the authors derived the equations of motion on the homogenous and isotropic background for coupled p-form theories and applied them to the dynamics of dark energy in the presence of interactions between 0- and 3-forms.

If 1- and 2-forms are present besides 0- and 3-forms, we need to consider the Bianchi-type cosmological background to discuss the fate of anisotropies [76]. In this paper, we study the dynamics of anisotropic inflation for the most general parity-invariant coupled p-form theories with first-order derivatives of fields in the action. For the system in which the scalar field \( \phi \) is independently coupled to 1- or 2-form fields, the anisotropic hair can survive during slow-roll inflation. If 1- and 2-form fields coexist, it was shown in Ref. [77] that anisotropic inflation supported by their couplings with the scalar can occur for the exponential potential of \( \phi \). This case corresponds to anisotropic power-law inflation [78, 79], so there is no exit to the subsequent reheating stage. Instead, we will perform a more general analysis without specifying the scalar potential and derive conditions under which the anisotropic shear can be supported by both 1- and 2-form fields.

Our study is general enough to cover the interaction between 1- and 2-form fields (as advocated in Refs. [80–86]). Indeed, we show the existence of new anisotropic inflationary solutions along which the 2-form energy density is supported by the coupling with the 1-form. We obtain analytic formulas of the anisotropic shear to the Hubble expansion rate for general

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\(^1\)If we break the \( U(1) \) gauge symmetry, it is possible to construct massive vector-tensor theories with Galileon-type derivative interactions [67–73].
inflaton potentials. The survival of anisotropic hair will be also numerically confirmed for a concrete inflaton potential.

This paper is organized as follows. In section 2 we briefly review the basics of $p$-forms by following the general results of Ref. [62]. In section 3 we study the uncoupled system between 1- and 2-forms and find general conditions under which the anisotropic shear survives during inflation. In section 4 we take into account the interaction between 1- and 2-forms and obtain a new class of anisotropic inflationary solutions. Section 5 is devoted to conclusions.

Throughout the paper, we use the Lorentzian metric $g_{\mu\nu}$ with the sign convention $(-,+,+,+)$. Greek indices $\alpha, \beta, \gamma \cdots$ denote space-time coordinates, while latin indices $i, j, k, \cdots$ represent spatial coordinates.

2 Coupled $p$-forms and anisotropic cosmological background

The general gauge-invariant action of coupled $p$-forms was derived in Ref. [62] by restricting the derivatives of fields up to first order. In Sec. 2.1, we briefly review such a system and show how the 3-form coupled to the scalar field $\phi$ generates the effective potential for $\phi$. In Sec. 2.2, we derive the field equations of motion on the anisotropic cosmological background for parity- invariant coupled $p$-form theories.

2.1 Action of coupled $p$-forms

In the 4-dimensional space-time, the $p$-forms $A_{(p)}$, where $p = 1, 2, 3$, are defined, respectively, by

$$A_{(1)} = A_{(1)}^{\mu_1} dx^{\mu_1}, \quad A_{(2)} = \frac{1}{2} A_{(2)}^{\mu_1 \mu_2} dx^{\mu_1} \wedge dx^{\mu_2}, \quad A_{(3)} = \frac{1}{6} A_{(3)}^{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3},$$

(2.1)

where $A_{(2)}^{\mu_1 \mu_2}$ and $A_{(3)}^{\mu_1 \mu_2 \mu_3}$ are totally anti-symmetric tensors, and $\wedge$ represents the wedge product. Associated with them, we define the field strengths,

$$F_{(1)}^{\mu_1 \mu_2} = 2 \delta_{\mu_1}^{[\mu_1} A_{(1)\mu_2]}, \quad F_{(2)}^{\mu_1 \mu_2 \mu_3} = 3 \delta_{\mu_1}^{[\mu_1} A_{(2)\mu_2 \mu_3]}, \quad F_{(3)}^{\mu_1 \mu_2 \mu_3 \mu_4} = 4 \delta_{\mu_1}^{[\mu_1} A_{(3)\mu_2 \mu_3 \mu_4]},$$

(2.2)

and the Hodge duals,

$$\tilde{F}_{(1)}^{\mu_1 \mu_2} = \frac{1}{2} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} F_{(1)}^{\mu_3 \mu_4}, \quad \tilde{F}_{(2)}^{\mu_1} = \frac{1}{6} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} F_{(2)}^{\mu_2 \mu_3 \mu_4}, \quad \tilde{F}_{(3)} = \frac{1}{24} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} F_{(3)}^{(\mu_1 \mu_2 \mu_3 \mu_4)},$$

(2.3)

where $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} = \sqrt{-g} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$, $g$ is the determinant of metric tensor $g_{\mu\nu}$, and $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$ is the Levi-Civita symbol with the convention $\epsilon_{0123} = 1$.

The Lagrangian of interacting $p$-forms consistent with Abelian gauge invariance was derived in Ref. [62] by restricting the field derivatives up to first order. Besides $p$-forms, we take into account a canonical scalar field $\phi$ with the potential $V(\phi)$. For the gravity sector we consider the Einstein-Hilbert Lagrangian $M_{pl}^2 R/2$, where $M_{pl}$ is the reduced Planck mass and $R$ is the Ricci scalar. We assume that the scalar and form fields are minimally coupled to gravity. Then, the total action of such a system is given by

$$S = \int d^4x \sqrt{-g} \left( \frac{M_{pl}^2}{2} R + \mathcal{L}_\phi + \mathcal{L}_p \right),$$

(2.4)
where
\[ L_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \] (2.5)
\[ L_p = L_M + L_T. \] (2.6)

The Lagrangian \( L_p \), which arises from \( p \)-forms, consists of two contributions:
\[ L_M = -\frac{1}{2} \sum_{p=1}^{3} \frac{f_p(\phi)}{(p+1)!} F_{(p)}^2, \quad L_T = -g(\phi) F_{(1)} \wedge F_{(1)} + \sum_{p=0}^{3} h_p(\phi) A_{(p)} \wedge F_{(3-p)}, \] (2.7)
where \( f_p(\phi), g(\phi), \) and \( h_p(\phi) \) are \( \phi \)-dependent functions, and
\[ F_{(p)}^2 = F_{(p)\mu_1 \mu_2 ... \mu_{p+1}} F_{(p)\mu_1 \mu_2 ... \mu_{p+1}}. \] (2.8)

In the sum in \( L_T \) we extended the notation \( (2.1) \) to include the scalar field as a 0-form, that is, \( A_{(0)} = \phi \) and \( F_{(0)\mu} = \partial_\mu \phi \). The Lagrangian \( L_M \) corresponds to dynamical, quadratic, Maxwell-like terms, whereas \( L_T \) represents topological terms. The contribution \( g(\phi) F_{(1)} \wedge F_{(1)} \) to \( L_T \) is the Chern-Pontryagin term (or \( \theta \)-term), which affects the background dynamics when it is coupled to the scalar field through the function \( g(\phi) \). The other contributions to \( L_T \) are the so called \( BF \)-terms [80–86]. In total, we have four terms of this type:
\[ A_{(0)} \wedge F_{(3)} \propto \phi \tilde{F}_{(3)}, \quad A_{(1)} \wedge F_{(2)} \propto A_{(1)\mu} \tilde{F}_{(2)}^\mu, \]
\[ A_{(2)} \wedge F_{(1)} \propto A_{(2)\mu_1 \mu_2} \tilde{F}_{(1)}^{\mu_1 \mu_2}, \quad A_{(3)} \wedge F_{(0)} \propto \tilde{A}_{(3)\mu} \partial^\mu \phi. \] (2.9)

After integration by parts, however, it can be seen that \( A_{(0)} \wedge F_{(3)} \) and \( A_{(1)} \wedge F_{(2)} \) are equivalent to \( A_{(3)} \wedge F_{(0)} \) and \( A_{(2)} \wedge F_{(1)} \), respectively. In the following, we will keep the terms \( A_{(0)} \wedge F_{(3)} \) and \( A_{(2)} \wedge F_{(1)} \). After redefining the coupling functions, the complete \( p \)-form Lagrangian is expressed as
\[ L_p = -\frac{1}{2} \sum_{p=1}^{3} \frac{f_p(\phi)}{(p+1)!} F_{(p)}^2 - \frac{1}{4} g_1(\phi) \tilde{F}_{(1)\mu_1 \mu_2} \tilde{F}_{(1)}^{\mu_1 \mu_2} - \frac{1}{2} g_2(\phi) \tilde{A}_{(2)\mu_1 \mu_2} \tilde{F}_{(1)}^{\mu_1 \mu_2} - g_3(\phi) \tilde{F}_{(3)}. \] (2.10)

We note that further couplings between 1-, 2-, and 3- forms are already included in the previous Lagrangian. For instance, we have
\[ F_{(3)\mu_1 \mu_2 \mu_3} \tilde{F}_{(2)}^{\mu_1} \tilde{F}_{(2)\mu_2 \mu_3} \propto \epsilon^{\sigma \mu_1 \mu_2 \mu_3} \tilde{F}_{(2)\sigma} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2 \mu_3} \propto F_{(3)}^{\sigma \mu_1 \mu_2 \mu_3} \tilde{F}_{(2)\sigma} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2 \mu_3} = 0, \]
\[ F_{(2)\mu_1 \mu_2} \tilde{F}_{(1)}^{\mu_1} \tilde{F}_{(2)\mu_2} \propto \epsilon^{\sigma \mu_1 \mu_2} \tilde{F}_{(2)\sigma} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2} = 0, \] (2.11)

Other possible combinations like
\[ F_{(2)\mu_1 \mu_2} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2} \propto \epsilon^{\sigma \mu_1 \mu_2} \tilde{F}_{(2)\sigma} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2} = 0, \]
\[ F_{(2)\mu_1 \mu_2} \tilde{F}_{(1)}^{\mu_1} \tilde{F}_{(2)\mu_2} \propto \epsilon^{\sigma \mu_1 \mu_2} \tilde{F}_{(2)\sigma} \tilde{F}_{(2)\mu_1} \tilde{F}_{(2)\mu_2} = 0, \] (2.12)
are null due to contractions between antisymmetric and symmetric indices. Moreover, couplings involving the kinetic term \( X = -\partial_\mu \phi \partial^\mu \phi/2 \) of the scalar, such as \( f(\phi, X) F_{(p)}^2 \) or
\[ \partial_{\mu} \phi F^{\mu\alpha} \partial_{\alpha} \phi F_{\nu}^{\nu}, \] 

suffer from Hamiltonian instability and non-hyperbolicity of the equations of motion \[87\], so, we will not consider those terms either.

Now, we will explicitly show that, after integrating out the 3-form field from the action, it can be absorbed into the scalar potential. To see this, we isolate the terms in \( L_\phi + L_p \) corresponding to the scalar field and the 3-form, as

\[ L_{\phi A(3)} = - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - \frac{f_3(\phi)}{48} F^{2}_{(3)} - \frac{g_3(\phi)}{24} \frac{\epsilon^{\mu\nu\sigma\rho}}{\sqrt{-g}} F_{(3)\mu\nu\sigma\rho}. \]  

(2.13)

Varying the corresponding action with respect to \( \phi \) and \( A_{(3)} \) respectively, it follows that

\[ \Box \phi - V_\phi - \frac{f_3(\phi)}{48} F^{2}_{(3)} - \frac{g_3(\phi)}{24} \frac{\epsilon^{\mu\nu\sigma\rho}}{\sqrt{-g}} F_{(3)\mu\nu\sigma\rho} = 0, \]  

(2.14)

\[ \nabla^\mu \left[ f_3 F_{(3)\mu\nu\sigma\rho} + g_3 \sqrt{-g} \epsilon^{\mu\nu\sigma\rho} \right] = 0, \]  

(2.15)

where the notation \( V_\phi = \frac{dV}{d\phi} \) is used. We exploit the fact that \( F_{(3)\mu\nu\sigma\rho} \) is proportional to the volume element \( \sqrt{-g} \), i.e.,

\[ F_{(3)\mu\nu\sigma\rho} = X(x^\mu) \sqrt{-g} \epsilon^{\mu\nu\sigma\rho}, \]  

(2.16)

where \( X(x^\mu) \) is a scalar function. Substituting Eq. (2.16) into (2.15), we obtain

\[ X(x^\mu) = \frac{c - g_3(\phi)}{f_3(\phi)}, \]  

(2.17)

where \( c \) is an integration constant. Plugging this solution into Eq. (2.13) and using the properties \( F^{2}_{(3)} = -4! X^2 \) and \( \epsilon^{\mu\nu\sigma\rho} F_{(3)\mu\nu\sigma\rho}/\sqrt{-g} = -4! X \), the Lagrangian \( L_{\phi A(3)} \) reduces to

\[ L_{\phi A(3)} = - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) + \frac{c^2 - g_3(\phi)^2}{2 f_3(\phi)}. \]  

(2.18)

Varying the corresponding action with respect to \( \phi \) and comparing the resulting equation of motion with Eq. (2.14), we find that the constant \( c \) is fixed to zero. Then, the coupled system of scalar and 3-forms is equivalent to the Lagrangian,

\[ L_{\phi A(3)} = - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V_{\text{eff}}(\phi), \]  

(2.19)

where \( V_{\text{eff}}(\phi) \) is the effective potential given by

\[ V_{\text{eff}}(\phi) \equiv V(\phi) + \frac{g_3(\phi)^2}{2 f_3(\phi)}. \]  

(2.20)

Thus, in the presence of coupling functions \( f_3(\phi) \) and \( g_3(\phi) \), the 3-form induces the potential \( g_3(\phi)^2/[2 f_3(\phi)] \) for the scalar field.

Keeping in mind that we are considering a gauge-invariant theory, the coupling \( g_2(\phi) \) in Eq. (2.10) is constrained to be constant. We will use the notation:

\[ g_2(\phi) = m_v = \text{constant}. \]  

(2.21)

Additionally, in this work, we focus on parity-conserving theories invariant under the transformation \( P: \vec{x} \rightarrow -\vec{x} \). Hence we set the coupling \( g_1(\phi) \) to 0 in the following. Despite
the presence of the Levi-Civita symbol in $A_{(2)\mu_1\mu_2}F_{(1)}^{\mu_1\mu_2}$, this can be absorbed as a parity conserving longitudinal mass term for the 1-form $A_{(1)\mu}$ [62, 83–85] \(^2\). Then, the Lagrangian (2.10) preserving gauge symmetry and parity invariance yields

$$\mathcal{L}_p = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} \sum_{p=1}^{2} \frac{f_p(\phi)}{(p+1)!} F_{(p)}^2 - \frac{1}{2} m_v A_{(2)\mu_1\mu_2} \tilde{F}_{(1)}^{\mu_1\mu_2}.$$  

(2.22)

For the later convenience, we will use the following notations for the $p$-forms and their respective field strengths:

$$A_{(1)\mu} \equiv A_{\mu}, \quad A_{(2)\mu_1\mu_2} \equiv B_{\mu_1\mu_2}, \quad F_{(1)\mu_1\mu_2} \equiv F_{\mu_1\mu_2}, \quad F_{(2)\mu_1\mu_2\mu_3} \equiv H_{\mu_1\mu_2\mu_3},$$

(2.23)

and

$$F^2 \equiv F_{\mu_1\mu_2} F_{\mu_1\mu_2}, \quad H^2 \equiv H_{\mu_1\mu_2\mu_3} H_{\mu_1\mu_2\mu_3}, \quad B \tilde{F} \equiv B_{\mu_1\mu_2} \tilde{F}_{\mu_1\mu_2}.$$  

(2.24)

Then, the total action (2.4) reduces to

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} R - \frac{1}{4} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} f_1(\phi) F^2 - \frac{1}{12} f_2(\phi) H^2 - \frac{1}{2} m_v B \tilde{F} \right].$$

(2.25)

Varying the action (2.25) with respect to $g_{\alpha\beta}$ on general curved backgrounds, it follows that

$$M_{pl}^2 \left( R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \right) = T_{\alpha\beta},$$

(2.26)

where $R_{\alpha\beta}$ is the Ricci tensor, and $T_{\alpha\beta}$ is the energy-momentum tensor defined by

$$T_{\alpha\beta} = \partial_{\alpha} \phi \partial_{\beta} \phi - \frac{1}{2} g_{\alpha\beta} \partial_{\sigma} \partial^{\sigma} \phi - g_{\alpha\beta} V_{\text{eff}}(\phi) + f_1(\phi) \left( F_{\beta\gamma} F_{\alpha}^{\gamma} - \frac{1}{4} g_{\alpha\beta} F^2 \right) + f_2(\phi) \left( \frac{1}{2} H_{\beta}^{\gamma\delta} H_{\alpha}^{\gamma\delta} - \frac{1}{12} g_{\alpha\beta} H^2 \right).$$

(2.27)

The topological $B \tilde{F}$ term does not contribute to the energy-momentum tensor as they are independent of the metric. The equations of motion for $\phi$, 1-form, and 2-form following from the action (2.25) are given, respectively, by

$$\Box \phi - V_{\text{eff, } \phi} - \frac{f_1(\phi)}{4} F^2 - \frac{f_2(\phi)}{12} H^2 = 0,$$

(2.28)

$$\nabla^\mu \left[ f_1(\phi) F_{\mu\nu} + m_v \tilde{B}_{\mu\nu} \right] = 0,$$

(2.29)

$$\nabla^\mu \left[ f_2(\phi) H_{\mu\nu\alpha} \right] - m_v \tilde{F}_{\mu\nu\alpha} = 0,$$

(2.30)

where $\tilde{B}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta}/2$. These equations are complemented with the Bianchi identities,

$$\nabla_{\mu} \tilde{F}^{\mu\nu} = 0, \quad \nabla_{\mu} \tilde{H}^\mu = 0,$$

(2.31)

together with the gauge choice.

\(^2\)The other way to see this is to use the equation of motion for the 2-form field, which implies $F_{(1)\mu\nu} = -1/(3g) \nabla^{[\mu} f_2 F_{(2)}^{\nu]}$. Replacing $F_{(1)\mu\nu}$ back into the Lagrangian, it can be seen that there are two Levi-Civita symbols in the $A_{(1)} \wedge F_{(2)}$ term, and hence the Lagrangian is parity-invariant on shell.
2.2 Background equations on the anisotropic cosmological background

We derive the dynamical equations of motion for the reduced action (2.25) on the anisotropic cosmological background. We consider the 1-form field $A_{\mu}$ aligned in the $x$ direction, such that $A_{\mu} = (0, v_{A}(t), 0, 0)$, where $v_{A}$ depends on the cosmic time $t$. For the 2-form field $B_{\mu\nu}$, we choose the configuration orthogonal to the 1-form field, such that $B_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = 2v_{B}(t)dy \wedge dz$, where $v_{B}$ depends on $t$. Since there is a rotational symmetry in the $(y, z)$ plane, we can take the line element in the following form,

$$ds^{2} = -N(t)^{2}dt^{2} + e^{2a(t)} \left[ e^{-4\sigma(t)}dx^{2} + e^{2\sigma(t)}(dy^{2} + dz^{2}) \right],$$

where $N(t)$ is a lapse function, $a \equiv e^{a(t)}$ is an isotropic scale factor, and $\sigma(t)$ is a spatial shear. The non-vanishing components $F_{\mu\nu}$ and $B_{\mu\nu}$ are given by $F_{01} = -F_{10} = \dot{v}_{A}$ and $B_{23} = -B_{32} = v_{B}$, where a dot represents a derivative with respect to $t$. Then, the action (2.25) yields

$$S = \int d^{4}x \left[ \frac{3M_{pl}^{2}3\lambda}{N} (\dot{\sigma}^{2} - \dot{\alpha}^{2}) + e^{3\lambda} \dot{\phi}^{2} \right. - N\varepsilon^{3\lambda}V_{\text{eff}}(\phi) + \frac{f_{1}(\phi)}{2N}e^{\alpha + 4\sigma}\dot{v}_{A}^{2}$$

$$\left. \quad + \frac{f_{2}(\phi)}{2N}e^{-\alpha - 4\sigma}\dot{v}_{B}^{2} + m_{v}\dot{v}_{A}v_{B} \right].$$

Varying the action (2.33) with respect to $v_{A}$ and $v_{B}$, respectively, and setting $N = 1$ at the end, it follows that

$$\frac{d}{dt} \left[ \frac{f_{1}(\phi)e^{\alpha + 4\sigma}}{f_{1}} \dot{v}_{A} + m_{v}v_{B} \right] = 0, \quad \Rightarrow \quad f_{1}(\phi)e^{\alpha + 4\sigma}\dot{v}_{A} + m_{v}v_{B} = p_{A},$$

$$\frac{d}{dt} \left[ \frac{f_{2}(\phi)e^{-\alpha - 4\sigma}}{f_{2}} \dot{v}_{B} - m_{v}v_{A} \right] = 0, \quad \Rightarrow \quad f_{2}(\phi)e^{-\alpha - 4\sigma}\dot{v}_{B} - m_{v}v_{A} = p_{B},$$

where $p_{A}$ and $p_{B}$ are constants. Expanding the time derivatives of Eqs. (2.34) and (2.35), we find

$$\ddot{v}_{A} + \left( \frac{f_{1}}{f_{1}} \dot{\phi} + \dot{\alpha} + 4\dot{\sigma} \right) \dot{v}_{A} + \frac{m_{v}^{2}}{f_{1}f_{2}} \left( v_{A} + \frac{p_{B}}{m_{v}} \right) = 0,$$

$$\ddot{v}_{B} + \left( \frac{f_{2}}{f_{2}} \dot{\phi} - \dot{\alpha} - 4\dot{\sigma} \right) \dot{v}_{B} + \frac{m_{v}^{2}}{f_{1}f_{2}} \left( v_{B} - \frac{p_{A}}{m_{v}} \right) = 0.$$

This means that both 1- and 2-form fields acquire the effective mass term $m_{v}/\sqrt{f_{1}f_{2}}$ through their interactions.

Varying the action (2.33) with respect to $N, \alpha, \sigma$, and $\phi$, we obtain

$$3M_{pl}^{2} \left( \ddot{\alpha}^{2} - \ddot{\sigma}^{2} \right) = \frac{1}{2} \dot{\phi}^{2} + V_{\text{eff}}(\phi) + \rho_{A} + \rho_{B},$$

$$M_{pl}^{2} \left( \ddot{\alpha} + 3\dot{\alpha}^{2} \right) = V_{\text{eff}}(\phi) + \frac{1}{3}\rho_{A} + \frac{2}{3}\rho_{B},$$

$$M_{pl}^{2} \left( \ddot{\sigma} + 3\dot{\sigma}^{2} \right) = \frac{2}{3}\rho_{A} - \frac{2}{3}\rho_{B},$$

$$\ddot{\phi} + 3\dot{\phi}^{2} + V_{\text{eff},\phi} - \frac{f_{1}}{f_{1}} \rho_{A} - \frac{f_{2}}{f_{2}} \rho_{B} = 0.$$
where $\rho_A$ and $\rho_B$ are the energy densities of 1- and 2-forms defined, respectively, by

$$\rho_A = \frac{f_1(\phi)}{2} e^{-2\alpha - 4\sigma} \dot{\phi}^2, \quad \rho_B = \frac{f_2(\phi)}{2} e^{-4\alpha - 4\sigma} \dot{\phi}^2.$$  \hfill (2.42)

On using Eqs. (2.34)-(2.35) and (2.36)-(2.37), (2.42), the energy densities $\rho_A$ and $\rho_B$ obey

$$\dot{\rho}_A = -4\rho_A \left( \dot{\alpha} + \dot{\sigma} + \frac{f_1}{4f_2} \right) - 2m_v \sqrt{\frac{p_{APB}}{f_1f_2}}, \quad \hfill (2.43)$$

$$\dot{\rho}_B = -2\rho_B \left( \dot{\alpha} - 2\dot{\sigma} + \frac{f_2}{2f_2} \right) + 2m_v \sqrt{\frac{p_{APB}}{f_1f_2}}, \quad \hfill (2.44)$$

where we assumed that $\dot{\nu}_A > 0$ and $\dot{\nu}_B > 0$ without loss of generality. In subsequent sections, we exploit the above background equations of motion to study the dynamics of anisotropic inflation.

3 Anisotropic inflation for uncoupled 1-form and 2-form fields

It is known that anisotropic inflation can be realized by the couplings of the scalar field $\phi$ with 1-form [21] or 2-form [43] fields. We study the dynamics of anisotropic inflation driven by both 1- and 2-form fields without specifying the form of the effective scalar potential $V_{\text{eff}}(\phi)$. In this section, we consider the system in which the 1- and 2-form fields are uncoupled to each other, i.e.,

$$m_v = 0.$$  \hfill (3.1)

In this case, we obtain $\dot{\nu}_A = p_A f_1^{-1}(\phi) e^{-\alpha - 4\sigma}$ and $\dot{\nu}_B = p_B f_2^{-1}(\phi) e^{\alpha + 4\sigma}$ from Eqs. (2.34) and (2.35). Then, the 1- and 2-form energy densities in Eq. (2.42) reduce to

$$\rho_A = \frac{p_A^2}{2f_1(\phi)} e^{-4\alpha - 4\sigma}, \quad \rho_B = \frac{p_B^2}{2f_2(\phi)} e^{-2\alpha + 4\sigma}.$$  \hfill (3.2)

Provided that $|\sigma| \ll \alpha$, the couplings $f_1(\phi) \propto e^{-4\alpha}$ and $f_2(\phi) \propto e^{-2\alpha}$ lead to nearly constant values of $\rho_A$ and $\rho_B$, respectively. During inflation in which the scalar field $\phi$ evolves slowly along the nearly flat potential $V_{\text{eff}}(\phi)$, we can resort to the slow-roll approximations $\dot{\phi}^2/2 \ll V_{\text{eff}}(\phi)$ and $|\dot{\phi}| \ll |3\ddot{\phi}|$. Ignoring also the contributions of 1- and 2-form fields to Eqs. (2.38) and (2.41), we obtain $3M^2_{\text{pl}} \dot{\alpha}^2 \simeq V_{\text{eff}}(\phi)$ and $3\dot{\alpha} \dot{\phi} \simeq -V_{\text{eff},\phi}$, so that $d(\alpha/\dot{\phi}) \simeq -V_{\text{eff}}/(M^2_{\text{pl}} V_{\text{eff},\phi})$. Then, the critical couplings $f_1(\phi) \propto e^{-4\alpha}$ and $f_2(\phi) \propto e^{-2\alpha}$ correspond, respectively, to

$$f_1(\phi) = e^{\int \frac{V_{\text{eff}}}{M^2_{\text{pl}} V_{\text{eff},\phi}} d\phi}, \quad f_2(\phi) = e^{2\int \frac{V_{\text{eff}}}{M^2_{\text{pl}} V_{\text{eff},\phi}} d\phi}. \quad \hfill (3.3)$$

In Refs. [21, 43, 44], it was shown that these couplings separately give rise to anisotropic inflation with the non-vanishing shear $\Sigma \equiv \dot{\sigma}$ satisfying

$$\frac{\Sigma}{H} \simeq \frac{\epsilon}{12(\alpha + \alpha_0)}, \quad \text{for } p_A \neq 0, \quad p_B = 0,$$  \hfill (3.4)

$$\frac{\Sigma}{H} \simeq -\frac{\epsilon}{3(\alpha + \alpha_0)}, \quad \text{for } p_A = 0, \quad p_B \neq 0.$$  \hfill (3.5)
where $\alpha_0$ is a constant, and
\[
H \equiv \dot{\alpha}, \quad \epsilon \equiv -\frac{\dot{H}}{H^2}.
\] (3.6)

It is also possible to sustain the anisotropic hair for the functions given by
\[
f_1(\phi) = e^{4c_1 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi}, \quad f_2(\phi) = e^{2c_2 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi},
\] (3.7)
where $c_1$ and $c_2$ are constants. In the following, we study the dynamics of anisotropic inflation for the system in which the couplings of the form (3.7) coexist. The similar analysis was carried out in Ref. [77] for the exponential potential $V_{\text{eff}}(\phi) = V_0 e^{\lambda \phi/M_{\text{pl}}}$, in which case the couplings (3.7) correspond to $f_1(\phi) = e^{4c_1 \phi/(\lambda M_{\text{pl}})}$ and $f_2(\phi) = e^{2c_2 \phi/(\lambda M_{\text{pl}})}$. In this case there is no exit from the inflationary stage, so we will perform a more general treatment without assuming the form of $V_{\text{eff}}(\phi)$.

If the contributions of 1- and 2-form fields to Eq. (2.41) are negligibly small, there is the approximation relation $d\alpha/d\phi \simeq -V_{\text{eff}}/(M_{\text{pl}}^2 V_{\text{eff},\phi})$. In this regime, the couplings evolve as $f_1(\phi) \propto e^{-4c_1 \alpha}$ and $f_2(\phi) \propto e^{-2c_2 \alpha}$, so that $\rho_A \propto e^{4(c_1-1)\alpha-4\sigma}$ and $\rho_B \propto e^{2(c_2-1)\alpha+4\sigma}$. Provided that $|\sigma| \ll \alpha$, $\rho_A$ and $\rho_B$ increase for $c_1 > 1$ and $c_2 > 1$, respectively. Then, the energy densities of 1- and 2-form fields should start to contribute to the background cosmological dynamics.

In the following, we perform a more refined treatment for the dynamics of anisotropic inflation without ignoring the couplings of $\phi$ with 1- and 2-form fields in Eq. (2.41). Applying the slow-roll approximation $|\dot{\phi}| \ll |3\ddot{\phi}|$ to Eq. (2.41), it follows that
\[
\frac{d\phi}{d\alpha} \simeq -\frac{M_{\text{pl}}^2 V_{\text{eff},\phi}}{V_{\text{eff}}} \left(1 - \frac{c_1}{\epsilon_V V_{\text{eff}}} e^{-4\alpha-4\sigma-4c_1 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi} - \frac{c_2}{2 \epsilon_V V_{\text{eff}}} e^{-2\alpha+4\sigma-2c_2 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi}\right),
\] (3.8)
where
\[
\epsilon_V \equiv \frac{M_{\text{pl}}^2}{2} \left(\frac{V_{\text{eff},\phi}}{V_{\text{eff}}}\right)^2.
\] (3.9)

The evolution of inflaton $\phi$ slows down by the existence of couplings $f_1(\phi)$ and $f_2(\phi)$. Then, the solutions eventually enter the stage in which either $\rho_A$ or $\rho_B$ approaches a constant value smaller than $V_{\text{eff}}$. To understand this behavior, we first write Eq. (3.8) in the form
\[
\frac{d\alpha}{d\phi} + \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} = \left(\frac{c_1}{\epsilon_V V_{\text{eff}}} e^{-4\alpha-4\sigma-4c_1 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi} + \frac{c_2}{2 \epsilon_V V_{\text{eff}}} e^{-2\alpha+4\sigma-2c_2 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi}\right) \frac{d\alpha}{d\phi}.
\] (3.10)

Introducing the quantity
\[
x \equiv e^{2c_1 \alpha + 2c_1 \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi},
\] (3.11)
Eq. (3.10) can be further expressed as
\[
\frac{dx}{d\phi} = \frac{c_1}{\epsilon_V V_{\text{eff}}} \left[2c_1 p_A^2 e^{4(c_1-1)\alpha-4\sigma} x^{-1} + c_2 p_B^2 e^{2(c_2-1)\alpha+4\sigma} x^{-1} - c_2 / c_1\right] \frac{d\alpha}{d\phi}.
\] (3.12)

In what follows, we integrate Eq. (3.12) under the approximation that the quantity $\epsilon_V V_{\text{eff}}$ is constant. We also neglect the time dependence of $\sigma$ under the condition that $|\sigma| \ll \alpha$. 

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When \( p_B = 0 \), the integrated solution to Eq. (3.12) is given by

\[
x^2 = \frac{c_1^2 p_A^2}{(c_1 - 1)\epsilon_V V_{\text{eff}}} e^{4(c_1 - 1)\alpha - 4\sigma} + \Sigma_A ,
\]

(3.13)

where \( \Sigma_A \) is a constant. For \( c_1 > 1 \), the first term on the right hand side of Eq. (3.13) exponentially increases during inflation. Hence the quantity \( x \) approaches the value

\[
x_A = \sqrt{\frac{c_1^2 p_A^2}{(c_1 - 1)\epsilon_V V_{\text{eff}}} e^{2(c_1 - 1)\alpha - 2\sigma}} .
\]

(3.14)

If \( p_A = 0 \), then the solution to Eq. (3.12) reads

\[
x^{c_2/c_1} = \frac{c_2^2 p_B^2}{2(c_2 - 1)\epsilon_V V_{\text{eff}}} e^{2(c_2 - 1)\alpha + 4\sigma} + \Sigma_B ,
\]

(3.15)

where \( \Sigma_B \) is a constant. For \( c_2 > 1 \), the variable \( x \) eventually approaches the value

\[
x_B = \left[ \frac{c_2^2 p_B^2}{2(c_2 - 1)\epsilon_V V_{\text{eff}}} e^{2(c_2 - 1)\alpha + 4\sigma} c_1/c_2 \right]^{c_1/c_2}. 
\]

(3.16)

Ignoring the \( \sigma \)-dependent terms in Eqs. (3.14) and (3.16), \( x_A \) grows faster than \( x_B \) for \( c_1 > c_2 \). Then, there are the three qualitatively different cases: (A) \( c_1 > c_2 > 1 \), (B) \( c_2 > c_1 > 1 \), and (C) \( c_1 = c_2 > 1 \). In the following, we study each case separately.

### 3.1 \( c_1 > c_2 > 1 \)

In this case, Eq. (3.12) is dominated by the term arising from the 1-form, so we can ignore the 2-form contribution to Eq. (3.8). From Eq. (3.14), we obtain

\[
e^{-4\alpha - 4\sigma - 4c_1} \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff}}} d\phi = \frac{(c_1 - 1)\epsilon_V V_{\text{eff}}}{c_1^2 p_A^2}.
\]

(3.17)

Then, the 1-form energy density reduces to

\[
\rho_A = \frac{c_1 - 1}{2c_1^2} \epsilon_V V_{\text{eff}},
\]

(3.18)

which is nearly constant during inflation. From Eq. (2.40), the ratio between the shear and the Hubble expansion rate approaches the value

\[
\frac{\Sigma}{H} \simeq \frac{2\rho_A}{3V_{\text{eff}}} = \frac{c_1 - 1}{3c_1^2} \epsilon_V ,
\]

(3.19)

where we used the slow-roll approximation \( 3M_{\text{pl}}^2 H^2 \simeq V_{\text{eff}} \). Substituting Eq. (3.17) into Eq. (3.8), we find

\[
\frac{d\phi}{d\alpha} = -\frac{M_{\text{pl}}^2 V_{\text{eff}}}{}.
\]

(3.20)

For \( c_1 > 1 \), the evolution of \( \phi \) slows down by the coupling with the 1-form field. From Eqs. (2.38) and (2.39), the slow-roll parameter \( \epsilon = -\dot{H}/H^2 \) can be estimated as

\[
\epsilon \simeq \frac{\dot{\phi}^2}{2M_{\text{pl}}^2 H^2} + \frac{2\rho_A}{3M_{\text{pl}}^2 H^2}.
\]

(3.21)
Since \( \Phi^2 / (2M_{\text{pl}}^2 H^2) = \epsilon V / c_1^2 \) and \( 2\rho_A / (3M_{\text{pl}}^2 H^2) = (c_1 - 1)\epsilon V / c_1^2 \) from Eqs. (3.18) and (3.20), we have

\[
\epsilon \approx \frac{\epsilon V}{c_1}, \tag{3.22}
\]

and hence \( \epsilon < \epsilon V \) for \( c_1 > 1 \). Then, Eq. (3.19) is expressed as

\[
\frac{\Sigma}{H} \approx \frac{c_1 - 1}{3c_1} \epsilon, \tag{3.23}
\]

which is positive and approximately constant during inflation. This is the region in which the anisotropic hair is generated by the coupling between \( \phi \) and the 1-form [21].

### 3.2 \( c_2 > c_1 > 1 \)

If \( c_2 > c_1 > 1 \), then the 2-form field gives the dominant contribution to Eq. (3.8). Since the solution (3.16) translates to

\[
e^{-2\alpha + 4\sigma - 2c_2} \int \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff},\phi}} d\phi = \frac{2(c_2 - 1)\epsilon V V_{\text{eff}}}{c_2^2 \rho_B}, \tag{3.24}
\]

the 2-form energy density is given by

\[
\rho_B = \frac{c_2 - 1}{c_2^2} \epsilon V V_{\text{eff}}. \tag{3.25}
\]

The shear divided by the Hubble expansion rate can be estimated as

\[
\frac{\Sigma}{H} \approx -\frac{2\rho_B}{3V_{\text{eff}}} = -\frac{2(c_2 - 1)}{3c_2^2} \epsilon V. \tag{3.26}
\]

Applying Eq. (3.24) to Eq. (3.8), we find

\[
\frac{d\phi}{d\alpha} = -\frac{M_{\text{pl}}^2 V_{\text{eff},\phi}}{V_{\text{eff}}} \frac{1}{c_2}, \tag{3.27}
\]

so that the coupling between \( \phi \) and the 2-form leads to the decrease of inflaton velocity for \( c_2 > 1 \). On using Eqs. (3.25) and (3.27), the slow-roll parameter \( \epsilon \) yields

\[
\epsilon \approx \frac{\dot{\phi}^2}{2M_{\text{pl}}^2 H^2} + \frac{\rho_B}{3M_{\text{pl}}^2 H^2} = \frac{\epsilon V}{c_2}, \tag{3.28}
\]

Then, from Eq. (3.26), it follows that

\[
\frac{\Sigma}{H} \approx -\frac{2(c_2 - 1)}{3c_2} \epsilon, \tag{3.29}
\]

which is approximately a negative constant during inflation. This result matches with that derived in Ref. [43] for the system in which the inflaton is coupled to the 2-form field alone.
3.3 \( c_1 = c_2 > 1 \)

If \( c_1 = c_2 > 1 \), then the differential Eq. (3.12) reduces to

\[
\frac{dx}{d\phi} = \frac{c_1^2}{\epsilon_V V_{\text{eff}}} \left[ 2P_A^2 e^{4(c_1 - 1)\alpha} x^{-1} + P_B^2 e^{2(c_1 - 1)\alpha} \right] \frac{d\alpha}{d\phi},
\]

(3.30)

where \( P_A = p_A e^{-2\sigma} \) and \( P_B = p_B e^{2\sigma} \). When we integrate Eq. (3.30) with respect to \( \alpha \), we deal with the term \( \epsilon_V V_{\text{eff}} \) and the \( \sigma \)-dependent terms as constants. Then, Eq. (3.30) admits the asymptotic solution of the form:

\[
x = C e^{2(c_1 - 1)\alpha},
\]

(3.31)

where the constant \( C \) obeys

\[
\frac{c_1}{\epsilon_V V_{\text{eff}}} \left( \frac{P_A^2}{C^2} + \frac{P_B^2}{2C} \right) = \frac{c_1 - 1}{c_1}.
\]

(3.32)

The solution (3.31) translates to

\[
e^{2\alpha + 2c_1} \int \frac{V_{\text{eff}}}{M_{\text{Pl}}^2 V_{\text{eff}}^2} d\phi = C.
\]

(3.33)

On using this relation and ignoring the \( \sigma \)-dependent terms in Eq. (2.42), the energy densities arising from 1- and 2-form fields are given, respectively, by

\[
\rho_A \simeq \frac{P_A^2}{2C^2}, \quad \rho_B \simeq \frac{P_B^2}{2C}.
\]

(3.34)

From Eq. (3.32), there is the particular relation

\[
2\rho_A + \rho_B = \frac{c_1 - 1}{c_1^2} \epsilon_V V_{\text{eff}}.
\]

(3.35)

Substituting Eq. (3.33) into Eq. (3.8) and using Eq. (3.32), we obtain

\[
\frac{d\phi}{d\alpha} = -\frac{M_{\text{Pl}}^2 V_{\text{eff}}^2}{\epsilon_V V_{\text{eff}}^2} \frac{1}{c_1},
\]

(3.36)

and hence \( \phi^2/(2M_{\text{Pl}}^2 H^2) = \epsilon_V/c_1^2 \). From Eqs. (2.38) and (2.39), the slow-roll parameter \( \epsilon \) yields

\[
\epsilon \simeq \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2 H^2} + \frac{2\rho_A + \rho_B}{3M_{\text{Pl}}^2 H^2} = \frac{\epsilon_V}{c_1},
\]

(3.37)

where we employed the relation (3.35) and the slow-roll approximation \( 3M_{\text{Pl}}^2 H^2 \simeq V_{\text{eff}} \).

We define the ratio between 1- and 2-form energy densities, as

\[
r_{AB} \equiv \frac{\rho_A}{\rho_B} = \frac{P_A^2}{CP_B^2}.
\]

(3.38)

Solving Eq. (3.32) for \( C \) and using Eq. (3.37), we can write \( r_{AB} \) in the form,

\[
r_{AB} = \frac{4(c_1 - 1)\epsilon_V V_{\text{eff}} P_A^2}{c_1 P_B^4} \left[ 1 + \sqrt{1 + \frac{16(c_1 - 1)\epsilon_V V_{\text{eff}} P_A^2}{c_1 P_B^4}} \right]^{-1}.
\]

(3.39)
From Eqs. (3.35) and (3.38), the 1- and 2-form energy densities are expressed, respectively, as
\[
\rho_A = \frac{r_{AB}}{2r_{AB} + 1} \frac{c_1 - 1}{c_1} \epsilon V_{\text{eff}}, \quad \rho_B = \frac{1}{2r_{AB} + 1} \frac{c_1 - 1}{c_1} \epsilon V_{\text{eff}}. \tag{3.40}
\]
Then, from Eq. (2.40), the ratio between \(\Sigma\) and \(H\) can be estimated as
\[
\frac{\Sigma}{H} \approx \frac{2(r_{AB} - 1)}{2r_{AB} + 1} \frac{c_1 - 1}{3c_1} \epsilon. \tag{3.41}
\]
In the limit \(r_{AB} \to \infty\), we have \(\rho_A \to (c_1 - 1)\epsilon V_{\text{eff}}/(2c_1)\) and \(\rho_B \to 0\) from Eq. (3.40). In this case, the formula (3.41) reduces to \(\Sigma/H = (c_1 - 1)\epsilon/(3c_1)\), which is identical to Eq. (3.23).

In the limit \(r_{AB} \to 0\), we have \(\rho_A \to 0\), \(\rho_B \to (c_1 - 1)\epsilon V_{\text{eff}}/c_1\), and \(\Sigma/H = -2(c_1 - 1)\epsilon/(3c_1)\). This value of \(\Sigma/H\) is equivalent to Eq. (3.29) with \(c_1 = c_2\). For \(r_{AB} = 1\), i.e., \(\rho_A = \rho_B\), the anisotropic shear vanishes by the compensation of \(\rho_A\) and \(\rho_B\) in Eq. (2.40). In other cases, the anisotropic hair survives during inflation with \(\Sigma/H\) given by Eq. (3.41).

### 3.4 Numerical solutions

To confirm the accuracy of analytic solutions derived above, we perform numerical integrations for the quadratic potential
\[
V_{\text{eff}}(\phi) = \frac{1}{2} \mu^2 \phi^2, \tag{3.42}
\]
where \(\mu\) is a constant having a dimension of mass. We introduce the following dimensionless quantities:
\[
\hat{v}_A = \frac{v_A}{M_{\text{pl}}}, \quad \hat{v}_B = \frac{v_B}{M_{\text{pl}}}, \quad \hat{\rho}_A = \frac{\rho_A}{\mu^2 M_{\text{pl}}^2}, \quad \hat{\rho}_B = \frac{\rho_B}{\mu^2 M_{\text{pl}}^2}, \quad \hat{\phi} = \frac{\phi}{M_{\text{pl}}},
\]
\[
\hat{H} = \hat{\alpha}/\mu = \alpha', \quad \hat{m}_v = \frac{m_v}{\mu}, \tag{3.43}
\]
where a prime represents a derivative with respect to the dimensionless time \(\hat{t} = \mu t\).

For the couplings given by Eq. (3.7), we can express Eqs. (2.38)-(2.41) in the forms:
\[
\hat{H} = \sqrt{\sigma'^2 + \frac{1}{6} \hat{\phi}^2 + \frac{1}{6} \hat{\phi}^2 + \frac{1}{3} \hat{\rho}_A + \frac{1}{3} \hat{\rho}_B}, \tag{3.44}
\]
\[
\hat{H}' = -3\sigma'^2 - \frac{1}{2} \phi'^2 - \frac{2}{3} \hat{\rho}_A - \frac{2}{3} \hat{\rho}_B, \tag{3.45}
\]
\[
\sigma'' = -3\hat{H} \sigma' + \frac{2}{3} \hat{\rho}_A - \frac{2}{3} \hat{\rho}_B, \tag{3.46}
\]
\[
\hat{\phi}'' = -3\hat{H} \hat{\phi}' - \hat{\phi} + 2c_1 \hat{\phi} \hat{\rho}_A + c_2 \hat{\phi} \hat{\rho}_B. \tag{3.47}
\]

From Eqs. (2.43) and (2.44), the 1- and 2-form energy densities obey the differential equations:
\[
\hat{\rho}_A' = -4\hat{\rho}_A \left( \hat{H} + \sigma' + \frac{c_1}{2} \hat{\phi} \hat{\sigma}' \right) - 2\hat{m}_v \sqrt{\frac{\hat{\rho}_A \hat{\rho}_B}{f_1 f_2}}, \tag{3.48}
\]
\[
\hat{\rho}_B' = -2\hat{\rho}_B \left( \hat{H} - 2 \sigma' + \frac{c_2}{2} \hat{\phi} \hat{\phi}' \right) + 2\hat{m}_v \sqrt{\frac{\hat{\rho}_A \hat{\rho}_B}{f_1 f_2}}. \tag{3.49}
\]

For \(\hat{m}_v = 0\) the last terms on the right hand sides of Eqs. (3.48) and (3.49) vanish, so the anisotropic inflationary dynamics is known by integrating Eqs. (3.45)-(3.49) with Eq. (3.44) for given initial values of \(\hat{\phi}', \hat{\phi}, \sigma, \sigma', \hat{\rho}_A, \hat{\rho}_B\).
In the left panel of Fig. 1, we exemplify the evolution of $\epsilon$, $\hat{\rho}_A$, and $\hat{\rho}_B$ for $m_v = 0$ and $c_1 = c_2 = 2$. After the initial transient period, the Universe enters the stage in which both $\hat{\rho}_A$ and $\hat{\rho}_B$ are nearly constant. In this case, the ratio (3.38) is $r_{AB} = \hat{\rho}_A/\hat{\rho}_B = 4.89$ at $\alpha = 20$. Inflation ends around $\alpha = 80.4$ at which the slow-roll parameter $\epsilon$ exceeds 1. The 1- and 2-form energy densities start to decrease around the end of inflation. The case (a) in the right panel of Fig. 1 corresponds to the same model parameters and initial conditions as those in the left panel. We observe that the anisotropic shear survives during inflation with the slow increase of $\Sigma/H$. On using the values $r_{AB} = 4.89$ and $\epsilon = 8.22 \times 10^{-3}$ at $\alpha = 20$, the analytic formula (3.41) gives $\Sigma/H \simeq 0.120\epsilon = 9.89 \times 10^{-4}$, which exhibits good agreement with its numerical value.

The case (b) in the right panel of Fig. 1 corresponds to $r_{AB} = 1.77$ and $\epsilon = 8.27 \times 10^{-3}$ at $\alpha = 20$, so that $\Sigma/H \simeq 5.65 \times 10^{-2}\epsilon = 4.68 \times 10^{-4}$ from Eq. (3.41). In this case, anisotropic inflation occurs with the smaller ratio $\Sigma/H$ compared to case (a) by reflecting the fact that $r_{AB}$ is smaller. In case (c), the quantity $r_{AB}$ at $\alpha = 20$ is $r_{AB} = 1.13$, so this is quite close to the border value $r_{AB} = 1$ at which $\Sigma/H$ changes its sign. If the value of $r_{AB}$ is smaller than 1 before the solutions reach the regime with the nearly constant ratio $\Sigma/H$, we find that $\Sigma/H$ is negative during anisotropic inflation.

For $r_{AB}$ close to 0, our numerical simulations show that there exists the anisotropic inflationary period with the negative value $\Sigma/H \simeq -2(c_1 - 1)\epsilon/(3c_1)$. The accuracy of the analytic formula (3.41) is also numerically checked for arbitrary positive values of $r_{AB}$. Thus, for $c_1 = c_2 > 1$, we have confirmed that anisotropic inflation with the shear in the range

$$- \frac{2(c_1 - 1)}{3c_1} \epsilon < \frac{\Sigma}{H} < \frac{c_1 - 1}{3c_1} \epsilon,$$  

(3.50)
is indeed realized. There is a specific case around $r_{AB} = 1$ in which $\Sigma/H$ is vanishingly small due to the compensation of 1- and 2-form contributions to the shear.

In Fig. 1, we find that $\Sigma/H$ starts to decrease around the end of inflation. After inflation, the Universe enters the reheating stage in which the inflaton field oscillates around the potential minimum. The precise evolution of $\Sigma$ during reheating depends on how the scalar and form fields decay to radiation, but as long as $\rho_A$ and $\rho_B$ quickly decrease toward 0, Eq. (2.40) shows that $\Sigma$ decreases in proportion to $a^{-3}$.

The numerical results of Fig. 1 correspond to the case $c_1 = c_2 > 1$, but we also confirmed that, for $c_1 > c_2 > 1$ and $c_2 > c_1 > 1$, there are the anisotropic inflationary periods in which $\Sigma/H$ is given by Eqs. (3.23) and (3.29) respectively. These properties hold for other choices of the inflaton potential $V_{\text{eff}}(\phi)$ as well.

4 Anisotropic inflation for coupled 1-form and 2-form fields

In this section, we study whether anisotropic inflation in which both $\rho_A$ and $\rho_B$ are nearly constant can be realized by the presence of the non-vanishing coupling $m_v$ between 1- and 2-form fields.

For $m_v = 0$, we showed that this is possible for the functions (3.7) with $c_1 = c_2 > 1$. In the regime where the anisotropic hair is present, there is the particular relation (3.33) and hence $d\alpha/d\phi = -c_1 V_{\text{eff}}/(M_p^2 V_{\text{eff},\phi})$. Then, after ignoring the $\dot{\sigma}$ term, the first terms on the right hand sides of Eqs. (2.43) and (2.44) vanish. To estimate the effect of non-vanishing coupling $m_v$ on the solutions derived for $m_v = 0$ and $c_1 = c_2 > 1$, we write $\rho_A$ and $\rho_B$ in the forms $\rho_A = \bar{\rho}_A + \delta\rho_A(t)$ and $\rho_B = \bar{\rho}_B + \delta\rho_B(t)$, where $\bar{\rho}_A$ and $\bar{\rho}_B$ are constants. On using the relation (3.33) with $c_1 = c_2$, the couplings given by Eq. (3.7) evolve as $f_1 = \bar{f}_1 a^{-4}$ and $f_2 = \bar{f}_2 a^{-2}$, where $\bar{f}_1$ and $\bar{f}_2$ are constants. Then, the homogeneous perturbations $\delta\rho_A(t)$ and $\delta\rho_B(t)$ obey

$$
\begin{align*}
\delta\dot{\rho}_A &= -2m_v \sqrt{\frac{\bar{\rho}_A\bar{\rho}_B}{f_1f_2}} \dot{a}^3, \\
\delta\dot{\rho}_B &= 2m_v \sqrt{\frac{\bar{\rho}_A\bar{\rho}_B}{f_1f_2}} \dot{a}^3. 
\end{align*}
$$

(4.1)

Approximating the inflationary background as the de-Sitter expansion, i.e., $a = e^{Ht}$, the integrated solutions to Eq. (4.1) are

$$
\begin{align*}
\delta\rho_A &= -\frac{2m_v}{3H} \sqrt{\frac{\bar{\rho}_A\bar{\rho}_B}{f_1f_2}} e^{3Ht}, \\
\delta\rho_B &= \frac{2m_v}{3H} \sqrt{\frac{\bar{\rho}_A\bar{\rho}_B}{f_1f_2}} e^{3Ht},
\end{align*}
$$

(4.2)

where we dropped the integration constants. From Eq. (4.2), the non-vanishing coupling $m_v$ leads to the deviation from the solutions $\rho_A = \bar{\rho}_A$ and $\rho_B = \bar{\rho}_B$. In other words, the anisotropic inflationary solutions supported by uncoupled 1- and 2-form fields for $c_1 = c_2 > 1$ tend to disappear by taking into account their interactions.

Let us discuss the other choices of couplings $f_1$ and $f_2$ for the realization of solutions $\dot{\rho}_A \approx 0$ and $\dot{\rho}_B \approx 0$. For $c_1$ and $c_2$ of order unity, the terms $\dot{\alpha} + \dot{\sigma} + \dot{f}_1/(4f_1)$ and $\dot{\alpha} - 2\dot{\sigma} + \dot{f}_2/(2f_2)$ in Eqs. (2.43) and (2.44) are at most of order $H$. Then, the conditions $\dot{\rho}_A \approx 0$ and $\dot{\rho}_B \approx 0$ translate to $\rho_A/\rho_B \propto m_v^2/(H^2 f_1f_2)$ and $\rho_B/\rho_A \propto m_v^2/(H^2 f_1f_2)$, respectively. During inflation, the compatibility of these two conditions implies that $f_1f_2 = \text{constant}$, i.e.,

$$
c_2 = -2c_1.
$$

(4.3)
In this case, the mass term defined by
\[ \bar{m}_v = \frac{m_v}{\sqrt{f_1 f_2}}, \tag{4.4} \]
is constant.

### 4.1 Analytic solutions

We derive analytic solutions to the shear \( \Sigma = \dot{\sigma} \) during inflation for the constants \( c_1 \) and \( c_2 \) satisfying the condition \((4.3)\). Then, we can express Eqs. \((2.43)\) and \((2.44)\) in the following forms:

\[
\dot{\rho}_A = -4\rho_A \left( \dot{\alpha} + \dot{\sigma} + c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi \right) - 2\bar{m}_v \sqrt{\rho_A \rho_B}, \tag{4.5}
\]
\[
\dot{\rho}_B = -2\rho_B \left( \dot{\alpha} - 2\dot{\sigma} - 2c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi \right) + 2\bar{m}_v \sqrt{\rho_A \rho_B}. \tag{4.6}
\]

Imposing the conditions \( \dot{\rho}_A = 0 \) and \( \dot{\rho}_B = 0 \), we obtain the following relation
\[
\frac{\rho_B}{\rho_A} = \left( \frac{9H^2}{4\bar{m}_v^2} \right)^{2} \left[ \frac{1}{1 + \frac{4\bar{m}_v^2}{9H^2}} - 1 \right]. \tag{4.7}
\]

If \( \bar{m}_v^2/H^2 \ll 1 \), then Eq. \((4.7)\) reduces to
\[
\frac{\rho_B}{\rho_A} \approx \frac{\bar{m}_v^2}{9H^2} \ll 1. \tag{4.8}
\]

In another limit \( \bar{m}_v^2/H^2 \gg 1 \), the ratio \( \rho_B/\rho_A \) approaches 1. In the latter regime, we can set \( \rho_B \approx \rho_A \) in Eq. \((4.5)\), so that
\[
\dot{\rho}_A \approx -2\rho_A \left( 2H + 2\dot{\sigma} + 2c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi + \bar{m}_v \right). \tag{4.9}
\]

Since \( \bar{m}_v \) is much larger than \( H \), the exponential decrease of \( \rho_A \) occurs during inflation. Hence the anisotropic shear does not survive in the regime \( \bar{m}_v^2/H^2 \gg 1 \). As we will see below, this is not the case for \( \bar{m}_v^2 \) smaller than the order of \( H^2 \).

Let us explore the dynamics of anisotropic inflation in the regime \( \bar{m}_v^2/H^2 \ll 1 \). On using Eq. \((4.8)\), Eqs. \((4.5)\) and \((4.6)\) reduce, respectively, to

\[
\dot{\rho}_A \approx -4\rho_A \left( \dot{\alpha} + \dot{\sigma} + c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi + \frac{\bar{m}_v^2}{6H} \right), \tag{4.10}
\]
\[
\dot{\rho}_B \approx 4\rho_B \left( \dot{\alpha} + \dot{\sigma} + c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi \right). \tag{4.11}
\]

From Eq. \((4.11)\), the condition \( \dot{\rho}_B = 0 \) holds for
\[
\dot{\alpha} + \dot{\sigma} + c_1 \frac{V_{\text{eff}}}{M_{\text{pl}}^2 V_{\text{eff,\phi}}} \phi = 0. \tag{4.12}
\]
Substituting Eq. (4.12) into Eq. (4.10), it follows that
\[ \dot{\rho}_A = -\frac{2\tilde{m}_V^2}{3H}\rho_A, \tag{4.13} \]
whose solution is
\[ \rho_A = \rho_{A0} \exp \left( -\int_0^\alpha \frac{2\tilde{m}_V^2}{3H^2} \, d\tilde{\alpha} \right), \tag{4.14} \]
where \( \rho_{A0} \) is an integration constant. The critical time \( t_c \) after which \( \rho_A \) is subject to the exponential suppression is identified by the moment at which the integral \( \int_0^\alpha \frac{2m_V^2}{3H^2} \, d\tilde{\alpha} \) reaches the order 1. For \( t < t_c \), \( \rho_A \) stays nearly constant with \( \rho_B \simeq \rho_A \tilde{m}_V^2/(9H^2) \). Then, from Eq. (2.40), it is possible to realize anisotropic inflation characterized by
\[ \Sigma_H \simeq \frac{2\rho_A}{3V_{\text{eff}}} \left( 1 - \frac{\tilde{m}_V^2}{9H^2} \right), \tag{4.15} \]
where we employed the slow-roll approximation \( 3M_p^2H^2 \simeq V_{\text{eff}} \) together with the condition \( \rho_A \ll V_{\text{eff}} \). Using the slow-roll approximation in Eq. (2.41), we obtain
\[ \frac{d\phi}{d\alpha} \simeq -\frac{M_p^2V_{\text{eff,}\phi}}{V_{\text{eff}}} + \frac{4c_1\rho_A}{V_{\text{eff,}\phi}} \left( 1 - \frac{\tilde{m}_V^2}{9H^2} \right), \tag{4.16} \]
From Eqs. (4.12), (4.15), and (4.16), the 1-form energy density can be expressed as
\[ \rho_A \simeq \frac{c_1 - 1}{2c_1^2} V_{\text{eff}} \left( 1 - \frac{\tilde{m}_V^2}{9H^2} \right)^{-1}, \tag{4.17} \]
where we ignored the slow-roll corrections to \( \rho_A \) higher than the linear order in \( \epsilon_V \). The positivity of \( \rho_A \) requires that \( c_1 > 1 \). In the limit \( \tilde{m}_V \to 0 \), the solution (4.17) recovers Eq. (3.18) derived for \( \tilde{m}_V = 0 \) and \( c_1 > c_2 > 1 \). Substituting Eq. (4.17) into Eq. (4.16), we have
\[ \frac{d\phi}{d\alpha} \simeq -\frac{M_p^2V_{\text{eff,}\phi}}{V_{\text{eff}}} \frac{1}{c_1}, \tag{4.18} \]
which is analogous to Eq. (3.20). From Eqs. (2.38) and (2.39), the slow-roll parameter \( \epsilon = -\dot{H}/H^2 \) is expressed as
\[ \epsilon \simeq \frac{\epsilon_V}{c_1} \left( 1 + \frac{c_1 - 1}{6c_1} \frac{\tilde{m}_V^2}{H^2} \right), \tag{4.19} \]
where we exploited Eq. (4.18) and picked up the leading-order term in the expansion of \( \tilde{m}_V^2/H^2 \). Applying Eqs. (4.17) and (4.19) to Eq. (4.15), we obtain
\[ \frac{\Sigma}{H} \simeq \frac{c_1 - 1}{3c_1^2} \epsilon_V \simeq \frac{c_1 - 1}{3c_1} \epsilon \left( 1 - \frac{c_1 - 1}{6c_1} \frac{\tilde{m}_V^2}{H^2} \right). \tag{4.20} \]
Hence the anisotropic shear can survive during inflation for the coupled system of 1- and 2-forms. This period ends after the exponential decrease of \( \rho_A \) characterized by Eq. (4.14) becomes significant at \( t > t_c \). For \( c_1 > 1 \), the 2-form energy density provides the negative contribution to Eq. (4.20), but the ratio \( \Sigma/H \) remains positive during anisotropic inflation due to the first approximate equality of Eq. (4.20).
we chose the specific values $c_2 = -2c_1 = -4$. The initial conditions are chosen to be $\phi' = 0, \phi = 13, \sigma' = 10^{-10}, \hat{\rho}_A = 10^{-5}$, and $\hat{\rho}_B = 10^{-10}$ at the onset of integration ($\alpha = 0$).

### 4.2 Numerical solutions

To confirm the analytic estimation given in Sec. 4.1, we consider the quadratic potential (3.42) and numerically integrate Eqs. (3.48)-(3.49) with Eqs. (3.44)-(3.47) for $\hat{m}_v \neq 0$. In the left panel of Fig. 2, we show one example for the evolution of $\hat{\rho}_A$ and $\hat{\rho}_B$ versus $\alpha = \ln a$ for $\hat{m}_v = \mu, c_1 = 2$, and $c_2 = -4$. This is the case in which the condition $\hat{m}_v^2 / H^2 \ll 1$ is satisfied at the onset of integration. As estimated by Eq. (4.14) in the regime $\int_0^\alpha 2\hat{m}_v^2 / (3H^2) d\tilde{\alpha} \ll 1$, there exists the anisotropic inflationary period in which $\rho_A$ stays nearly constant. Once the integral $\int_0^\alpha 2\hat{m}_v^2 / (3H^2) d\tilde{\alpha}$ reaches the order 1, the 1-form energy density $\hat{\rho}_A$ is subject to the exponential suppression. Soon after this decrease of $\hat{\rho}_A$, the inflationary period ends around $\alpha = 79.7$ in the numerical simulation of Fig. 2.

In the regime $\hat{m}_v^2 \ll H^2$, the 2-form energy density slowly grows as $\hat{\rho}_B \simeq \rho_A \hat{m}_v^2 / (9H^2)$ with the decrease of $H$. Eventually, $\hat{\rho}_B$ catches up with $\rho_A$ around the moment at which $\hat{m}_v^2 / (9H^2)$ exceeds the order 1. As estimated by Eq. (4.20), the right panel of Fig. 2 shows that there exists the period of anisotropic inflation in which the ratio $\Sigma / H$ stays nearly a constant. The energy density $\rho_A$ is the dominant source for the shear, but $\rho_B$ also contributes to $\Sigma$. In the numerical simulation of Fig. 2 we chose the specific values $c_2 = -2c_1 = -4$, but for the general coupling constants satisfying

$$c_2 = -2c_1 \quad \text{and} \quad c_1 > 1,$$

we numerically confirmed the existence of anisotropic hairs endowed with the coupled 1- and 2-form fields. As we discussed in Sec. 4.1, we require the condition $\hat{m}_v^2 / H^2 \ll 1$ to avoid the exponential suppression of $\hat{\rho}_A$ and hence $\hat{\rho}_B \simeq \rho_A \hat{m}_v^2 / (9H^2) \ll \rho_A$ during most stage of anisotropic inflation. Since $\rho_B$ cannot dominate over $\rho_A$ to keep the condition $\hat{\rho}_A \simeq 0$, the couplings satisfying $c_2 > 1$ and $c_2 = -2c_1$ do not sustain the anisotropic shear. We recall that, for $m_v = 0$, the anisotropic hair induced by both 1- and 2-form fields survives only for
The non-vanishing coupling $m_v$ allows the possibility for realizing new hairy solutions for the negative constant $c_2$ satisfying the condition (4.21).

5 Conclusions

We studied the dynamics of anisotropic inflation in gauge-invariant coupled $p$-form theories with parity invariance. The 3-form coupled to the scalar field $\phi$ generates the effective scalar potential after integrating out interacting Lagrangians from the action. As a result, the reduced action of coupled $p$-forms minimally coupled to gravity is of the form (2.25), which contains the interacting term $-m_v B\tilde{F}/2$ between 1- and 2-forms. In Sec. 2.2, we derived the equations of motion on the anisotropic background (2.32) to study the evolution of the cosmic shear during inflation.

If $m_v = 0$, it is known that the couplings $-f_1(\phi)F^2/4$ and $-f_2(\phi)H^2/12$ in the action (2.25) can separately sustain the anisotropic shear during slow-roll inflation for the couplings (3.7) [21, 43]. When these two couplings coexist, the presence of an anisotropic inflationary attractor was shown in Ref. [77] for the exponential potential $V_{\text{eff}}(\phi) = V_0 e^{\lambda \phi}$. Without specifying any inflaton potential, we derived the general analytic formulas of the shear $\Sigma$ to the Hubble expansion rate $H$ during inflation in the presence of two couplings mentioned above. As we showed in Sec. 3, there are three qualitatively different cases depending on the coupling constants $c_1$ and $c_2$: (A) $c_1 > c_2 > 1$, (B) $c_2 > c_1 > 1$, and (C) $c_1 = c_2 > 1$. The case (C) is particularly of interest, as both 1- and 2-forms contribute to the shear. As we see in the formula (3.41), there is a special case in which $\Sigma/H$ vanishes at $r_{AB} = 1$ due to the compensation of 1- and 2-form contributions to the shear. In Fig. 1, we confirmed that our analytic formulas are sufficiently accurate during the anisotropic inflationary period driven by the quadratic potential (3.42).

When $m_v \neq 0$, we explored the possibility for realizing anisotropic inflation supported by both 1- and 2-forms. In Sec. 4, we first showed that the anisotropic shear present for the couplings $c_1 = c_2 > 1$ with $m_v = 0$ tends to disappear by the non-vanishing coupling $m_v$ between 1- and 2-forms. However, for the couplings satisfying $c_2 = -2c_1$ and $c_1 > 1$, we found a new class of anisotropic inflationary solutions along which both $\rho_A$ and $\rho_B$ are approximately constant. In the regime $\bar{m}_v^2/H^2 \ll 1$, $\rho_B$ is sustained by $\rho_A$ according to the relation (4.8). As we observe in Eq. (4.14), $\rho_A$ is nearly constant by the time at which the integral $\int_0^\alpha 2\bar{m}_v^2/(3H^2) \, d\alpha$ reaches the order 1. We showed that the ratio $\Sigma/H$ is analytically given by Eq. (4.20) during slow-roll anisotropic inflation. Our analytic formulas were also numerically confirmed for the inflaton potential (3.42), see Fig. 2.

There are several issues we did not address in this paper. First, it will be of interest to estimate the effect of the inflationary anisotropic shear on the primordial power spectra of scalar and tensor perturbations as well as on the non-linear estimator $f_{NL}$ of primordial non-Gaussianities with/without the interactions between 1- and 2-forms. In particular, the anisotropy parameter $g_*$ in the scalar power spectrum would be subject to change by the partial compensation of 1- and 2-form contributions to the shear. They can be exploited to confront the models of anisotropic inflation with the observational data of CMB temperature anisotropies. Finally, the application of our coupled $p$-form theories to the late-time cosmology, in particular, to the dynamics of dark energy and associated fixed points with their stabilities will be also interesting.
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