INTEGRAL EXTENSIONS AND THE $a$-INVARIANT

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ABSTRACT. In this note we compare the $a$-invariant of a homogeneous algebra $B$ to the $a$-invariant of a subalgebra $A$. In particular we show that if $A \subset B$ is a finite homogeneous inclusion of standard graded domains over an algebraically closed field with $A$ normal and $B$ of minimal multiplicity then $A$ has minimal multiplicity. In some sense these results are algebraic generalizations of Hurwitz type theorems.

1. INTRODUCTION

The purpose of this note is to study the behavior of the $a$-invariant under homogeneous inclusions of positively graded algebras. Thus, let $k$ be a field and $A$ a positively graded Noetherian $k$-algebra of dimension $d$ and with homogeneous maximal ideal $m$. The $a$-invariant $a(A)$ is the top degree of the local cohomology module $H^d_m(A)$ or, equivalently, the negative of the initial degree of the graded canonical module $\omega_A$. If $A$ is standard graded, then the $a$-invariant is related to the Castelnuovo-Mumford regularity, via the inequality $a(A) + d \leq \text{reg} A$, which is an equality in case $A$ is Cohen-Macaulay.

We will consider homogeneous inclusions $A \subset B$ of positively graded algebras over a perfect field $k$. Our goal is to compare the $a$-invariants of $A$ and of $B$. Most notably, we wish to bound the $a$-invariant or the regularity of $A$ by means of the corresponding invariants of $B$. We now describe what our results say in the special case where the extension is integral and $B$ is a domain. If char $k = 0$ and $A$ is normal, then the trace map shows that $A$ is a direct summand of $B$, hence $a(A) \leq a(B)$. This bound still holds if the extension of quotient fields is separable, but the estimates change sharply in the inseparable case. In the next theorem we summarize what we prove, though not in the most general form.

**Theorem 1.1.** Let $k$ be a perfect field and $A \subset B$ a homogeneous integral extension of positively graded Noetherian $k$-domains. Assume in addition that $A$ is regular in codimension one.

1. If the field extension Quot$(A) \subset$ Quot$(B)$ is separable, then $a(A) \leq a(B)$.
2. If char $k = p > 0$, $d = 2$, and $A$ is standard graded, then $\lfloor \frac{a(A)}{p} \rfloor \leq a(B)$, where $p^e$ is the inseparable degree of the field extension; equality holds if $B$ is regular in codimension one and the field extension is purely inseparable.
(3) If \( \text{char } k = p > 0 \) and \( A, B \) are standard graded, then \( \left\lfloor \frac{a(A)+d-2}{p^e} \right\rfloor \leq \text{reg } B - 2 \), where \( p^e \) is the inseparable degree of the field extension.

Notice that the bound in item (3), the inseparable case, is considerably weaker than the estimate in (1), the separable case, but it is nevertheless sharp according to (2). No such bounds can be expected if \( A \) is not regular in codimension one, regardless of characteristic. For instance, let \( R = k[x,y] \) be a polynomial ring over any field and let \( B \) be the \( n \)th Veronese subring of \( R \), i.e., the \( k \)-subalgebra of \( R \) generated by all monomials of degree \( n \). Consider the subring \( A = k[x^n, x^{n-1}y, y^n] \) of \( B \). After regrading, \( A \subset B \) is a homogeneous integral extension of standard graded \( k \)-algebras. One has \( a(B) = -1 \), whereas \( a(A) = n - 3 \), which can be arbitrarily large. The first equality holds because \( B \) is a Veronese subring, whereas the second one follows from the fact that \( A \) is a hypersurface ring and has multiplicity \( n \).

Our proofs are, in some sense, algebraic generalizations of arguments that can be found in [6, Chap. IV, Sec. 2], for instance. To prove part (1) of Theorem 1.1 we produce an embedding of graded canonical modules \( \omega_A \subset \omega_B \). Part (3) is deduced from item (2) using an induction argument, and (2) in turn follows once we have shown that \( \omega_A^{(p^e)} \subset \omega_B^{p^e} \). Here \( B^{p^e} \) denotes the \( p^e \)th Frobenius power of \( B \) (in other words, \( B^{p^e} \) is the subring \( \{b^{p^e} \mid b \in B \} \) of \( B \)) and \( A^{(p^e)} \) the \( p^e \)th Veronese subring \( A \). Quite generally, if \( C \) is a positively graded Noetherian algebra over a field, then the \( n \)th Veronese subring \( C^{(n)} \) is defined as the graded subalgebra \( \oplus C_{ni} \) of \( C \). The formation of Veronese subrings commutes with taking local cohomology, hence \( a(C^{(p^e)}) = p^e \left\lfloor \frac{a(C)}{p^e} \right\rfloor \). On the other hand, when \( C \) is a domain and has characteristic \( p > 0 \), then \( a(C^{p^e}) = p^e a(C) \). Thus, the inequality of Theorem 1.1(2) follows once we have established the embedding \( \omega_A^{(p^e)} \subset \omega_B^{p^e} \).

The origin of this note was a problem that came up in our earlier work [9], namely the question of whether minimal multiplicity descends under integral extensions. Recall that a Noetherian standard graded domain \( A \) over an algebraically closed field is said to have minimal multiplicity if its multiplicity has the smallest possible value, namely the embedding codimension of \( A \) plus one. Minimal multiplicity is equivalent to the bound \( \text{reg } A \leq 1 \), which in turn holds if and only if \( A \) is Cohen-Macaulay with \( a(A) \leq 1 - d \), where \( d = \text{dim } A \). For \( d = 2 \) the inequality \( a(A) \leq 1 - d \) means that \( \omega_A|_0 \neq 0 \). On the other hand, the vector space dimension of \( \omega_A|_0 \) is the genus of \( C = \text{Proj } A \), at least when \( C \) is nonsingular. Thus, the homogeneous coordinate ring of a nonsingular curve in projective space has minimal multiplicity if and only if the genus of the curve is zero. In this setting, the descent of minimal multiplicity follows from the classical Hurwitz Formula that describes the change in genus under finite separable morphisms of curves. Our results apply to any homogeneous integral extension \( A \subset B \) of standard graded Cohen-Macaulay domains over an algebraically closed field as long as \( A \) is normal, and they show, in particular, that if \( B \) has minimal multiplicity, then \( A \) does also. Again, this implication no longer holds without the normality assumption on \( A \). It would be interesting to generalize the descent of minimal multiplicity from the graded to the local case.
2. The Separable Case

In this section we compare the $a$-invariants of $A \subset B$ when suitable separability assumptions are in place.

Let $k$ be an infinite perfect field and $A$ a positively graded, reduced, Noetherian, equidimensional $k$-algebra of dimension $d$. Then there is a natural homogeneous map

$$c_A : \wedge^d \Omega_A/k \to \omega_A,$$

called the canonical class, that is an isomorphism off the singular locus of $A$ (see [3, 8, 10]); here $\Omega$ denotes the module of differentials.

We will often use the following fact from dimension theory.

**Remark 2.1.** If $k$ is a field and $A \subset B$ is an inclusion of finitely generated $k$-algebras, then $\dim A \leq \dim B$. To prove this one reduces to the case when $A$ and $B$ are domains; then one uses the fact that the Krull dimension equals the transcendence degree over $k$.

Recall that if $A$ is a Noetherian ring of dimension $d < \infty$, then $A_{unm}$ stands for the ring $A/\cap Q_i$, where the $Q_i$ are the primary components of $(0)$ of dimension $d$.

**Theorem 2.2.** Let $k$ be a perfect field and $A \subset B$ a homogeneous inclusion of positively graded Noetherian $k$-algebras of dimension $d$. Assume every minimal prime $p_i$ of $A$ of dimension $d$ is the contraction of a minimal prime $q_i$ of $B$ so that the extension of quotient fields $\text{Quot}(A/p_i) \subset \text{Quot}(B/q_i)$ is separable. Further suppose that $\dim B/(\cap q_i + JB) \leq d - 2$ for some $A$-ideal $J$ with $\text{Sing}(A_{unm}) \subset V(JA_{unm})$. Then there is a homogeneous embedding of canonical modules

$$\omega_A \hookrightarrow \omega_B.$$

In particular,

$$a(A) \leq a(B)$$

and, if $A$ and $B$ are homogeneous and $A$ is Cohen-Macaulay,

$$\text{reg } A \leq a(B) + d \leq \text{reg } B.$$

**Proof.** Notice that $A/p_i \hookrightarrow B/q_i$, thus $\dim B/q_i = d$ by Remark 2.1. Since $\dim B/(\cap q_i + JB) < \dim B/q_i$ it follows that $J$ cannot be contained in any $q_i$, hence is not in any $p_i$. Therefore $A_{unm}$ is reduced and $A_{unm} = A/\cap p_i$. Notice that $\omega_A = \omega_{A_{unm}}$ and $\omega_{B/\cap q_i} \hookrightarrow \omega_B$. Thus we may replace $A \subset B$ by $A_{unm} = A/\cap p_i \subset B/\cap q_i$ to assume that $A$ and $B$ are reduced and equidimensional with minimal primes $\{p_i\}$ and $\{q_i\}$, respectively. Furthermore one has $\text{Quot}(A) = \times K_i \subset \text{Quot}(B) = \times L_i$, where $K_i = \text{Quot}(A/p_i) \subset L_i = \text{Quot}(B/q_i)$ are separable algebraic field extensions.

Let $J'$ be the Jacobian ideal of the $k$-algebra $B$. By our assumption on $J$ and since $B$ is reduced, we have that $\dim B/JJ' \leq d - 1$. Thus $\dim A/(JJ' \cap A) \leq \dim B/JJ' \leq d - 1$ where the first inequality obtains by Remark 2.1. It follows that the homogeneous ideal $JJ' \cap A$ cannot be in any minimal
prime of $B$ because any such prime contracts to a prime of $A$ of dimension $d$. Hence by prime avoidance there exists a homogeneous element $f \in (J J') \cap A$ that is a non zerodivisor on $B$. Notice that $A_f$ and $B_f$ are regular rings. Hence $(c_A)_f$ is an isomorphism, which gives a commutative diagram

$$
\xymatrix{ \wedge^d \Omega_{A/k} \ar[r]^{c_A} \ar[d]^{\text{nat}} & \Omega_A \ar[d]^{\text{nat}} \\
(\wedge^d \Omega_{A/k})_f \ar[r]^{(c_A)_f^{-1}} & (\Omega_A)_f }
$$

where the right most map is an embedding since $f$ is $A$-regular. We also have a corresponding diagram for $B$.

Now there is a commutative diagram of natural homogeneous linear maps,

$$
\xymatrix{ \wedge^d \Omega_{A/k} \ar[r]^{e} \ar[d]^{c_A} & B \otimes \wedge^d \Omega_{A/k} \cong \wedge^d (B \otimes_A \Omega_{A/k}) \ar[d]^{\mu} \\
\omega_A \ar[r]^{\varphi} \ar[d]^{\text{nat}} & (\wedge^d \Omega_{A/k})_f \ar[d]^{\text{nat}} \\
(\omega_A)_f \cong (\wedge^d \Omega_{A/k})_f & (\wedge^d \Omega_{B/k})_f \cong (\omega_B)_f }
$$

where $\psi = (\mu e)_f$ and $\varphi$ is the restriction of $\psi$.

We show that $\psi$ is injective. Indeed, $A_f$ is a regular ring and therefore the $A_f$-module $(\wedge^d \Omega_{A/k})_f \cong \wedge^d \Omega_{A_f/k}$ is projective, hence torsionfree. Thus it suffices to show that $K \otimes_{A_f} \psi$ is injective for $K = \text{Quot}(A_f) = \text{Quot}(A)$. Since $\text{Quot}(A) = \times K_i \subset L = \text{Quot}(B) = \times L_i$ with $K_i \subset L_i$ separable algebraic field extensions, we have natural isomorphisms $L_i \otimes_K \Omega_{K_i/k} \cong \Omega_{L_i/k}$ and therefore $L \otimes_K \Omega_{K/k} \cong \Omega_{L/k}$. It follows that $L \otimes \wedge^d \Omega_{K/k} \cong \wedge^d (L \otimes_K \Omega_{K/k}) \cong \wedge^d \Omega_{L/k}$. Clearly $\wedge^d \Omega_{K/k}$ embeds into $L \otimes_K \wedge^d \Omega_{K/k}$ because $L$ is a faithfully flat $K$-module, for instance. In summary, we obtain the commutative diagram

$$
\xymatrix{ \wedge^d \Omega_{K/k} \ar[r]^{K \otimes \psi} \ar[d] & \wedge^d \Omega_{K \otimes A B/k} \\
L \otimes_K \wedge^d \Omega_{K/k} \sim \ar[r] & \wedge^d \Omega_{L/k}, }
$$

which shows that $K \otimes \psi$ is injective. Hence indeed, $\psi$ is injective.

Therefore the composition $\varphi$ is a homogeneous embedding of graded $A$-modules. It remains to show that $\text{imp} \subset \omega_B$. However, since $\text{Sing}(A) \subset V(J)$, some power $I$ of $J$ annihilates the cokernel of the canonical class $c_A$. Thus $I \cdot \text{imp} \subset \omega_B$ by the commutativity of the above diagram. On the other hand, since $\dim B/I \leq d - 2$, we have $\text{depth}_{B \omega_B} \geq 2$. This implies that $\text{imp} \subset \omega_B$. \qed
Remark 2.3. The assumption on $J$ obtains if $A$ is regular in codimension one and $B$ is integral over $A$. If $A$ is equidimensional, then the ideal $J$ can be taken to be the Jacobian ideal of $A$, that is, the $d$-th Fitting ideal of $\Omega_{A/k}$.

Remark 2.4. If $A \subset B$ is an integral extension, one can also obtain Theorem 2.2 using the trace map $B \rightarrow A$ and dualizing this map into $\omega_A$.

The assumption that $\dim A = \dim B$ in Theorem 2.2 can be removed if $B$ is standard graded.

Remark 2.5. Let $k$ be a perfect field and $A \subset B$ a homogeneous inclusion of positively graded Noetherian $k$-algebras. Assume that $B$ is standard graded. Further suppose every minimal prime $p_i$ of $A$ of maximal dimension is the contraction of a minimal prime $q_i$ of maximal dimension so that the extension of quotient fields $\text{Quot}(A/p_i) \subset \text{Quot}(B/q_i)$ is separable. In addition assume that $\dim B/(\cap q_i + JB) \leq \dim B - 2$ for some $A$-ideal $J$ with $\text{Sing}(A^{unm}) \subset V(JA^{unm})$. Then there is a homogeneous embedding of canonical modules

$$\omega_A(\dim A) \rightarrow \omega_B(\dim B).$$

In particular,

$$a(A) + \dim A \leq a(B) + \dim B$$

and, if $A$ is standard graded and Cohen-Macaulay, then

$$\text{reg} A \leq a(B) + \dim B \leq \text{reg} B.$$

Proof. We may suppose that $k$ is infinite and as in the proof of Theorem 2.2 we reduce to the case where $A$ and $B$ are reduced and equidimensional with minimal primes $\{p_i\}$ and $\{q_i\}$, respectively. Write $\delta = \dim B - \dim A$ and let $x_1, \ldots, x_\delta$ be general linear forms in $B$. One has $\delta = \dim B/q_i - \dim A/p_i = \text{trdeg}_{A/p_i} B/q_i$ for each of the finitely many minimal primes $q_i$. Thus the images of $x_1, \ldots, x_\delta$ in $B/q_i$ forms a separating transcendence basis of $\text{Quot}(B/q_i)$ over $\text{Quot}(A/p_i)$. Since $B$ is reduced, this also shows that the linear forms $x_1, \ldots, x_\delta$ are algebraically independent over $A$. Therefore $\omega_A = \omega_{A[x_1, \ldots, x_\delta]}(\delta)$. Now, replacing $A$ by the graded polynomial ring $A[x_1, \ldots, x_\delta]$ we may assume that $d = \dim A = \dim B$. Notice that the separability assumption is preserved by the definition of a separating transcendence basis. The assertion now follows from Theorem 2.2. □

3. The Two Dimensional Case

In this section we allow for inseparable extensions, but we require the dimension of the rings to be 2.

Theorem 3.1. Let $k$ be a perfect field of positive characteristic $p$ and $A \subset B$ a homogeneous inclusion of positively graded Noetherian $k$-algebras of dimension two. Assume $A$ is a domain containing a nonzero linear form. Further suppose that $\dim B/(q + JB) \leq 0$ for some $A$-ideal $J$ with $\text{Sing}(A) \subset V(J)$ and some minimal prime $q$ of $B$ such that $q \cap A = 0$. 

The field extension Quot(A) ⊂ Quot(B/q) is algebraic; write $p^e$ for its inseparable degree. There is a homogeneous embedding of canonical modules

$$\omega_{A^{(p^e)}} \hookrightarrow \omega_{B^{p^e}}.$$ 

In particular,

$$\left\lfloor \frac{a(A)}{p^e} \right\rfloor \leq a(B)$$

and, if A is standard graded and Cohen-Macaulay,

$$\left\lfloor \frac{\text{reg} A - 2}{p^e} \right\rfloor \leq a(B).$$

Proof. As $A \hookrightarrow B/q$, Remark 2.1 gives that $\dim B/q \geq \dim A$, and therefore $\dim B/q = 2 = \dim A$. In particular, the field extension Quot(A) ⊂ Quot(B/q) is algebraic. Since furthermore $\omega_{B/q} \hookrightarrow \omega_B$ we may replace B by $B/q$ to assume that B is a domain. Write $K = \text{Quot}(A)$, $L = \text{Quot}(B)$, and let $K_{\text{sep}}$ be the separable closure of K in L. One has $L^{p^e} \subset K_{\text{sep}}$, which gives $K_{\text{sep}} = KL^{p^e}$ because the extension $KL^{p^e} \subset L$ is purely inseparable. Since $k$ is perfect and L has transcendence degree 2 over $k$, there exist elements $u, v$ in $L$ so that the extension $k(u, v) \subset L$ is separable algebraic. Therefore $L = k(u, v)L^{p^e} = L^{p^e}(u, v)$, where the last equality uses the perfection of $k$ again. We conclude that the extension $L^{p^e} \subset L$ has degree at most $p^{2e}$. Let $x$ be a nonzero linear form in $A$. Notice that the set $\{x_i \mid 0 \leq i \leq p^e - 1\}$ forms a basis of $K$ over Quot$(A^{(p^e)})$ and is linearly independent over the ring $B^{(p^e)}$, which in turn contains $A^{(p^e)}[B^{p^e}]$. Thus the purely inseparable field extensions Quot$(A^{(p^e)}) \subset K$ and Quot$(A^{(p^e)})L^{p^e} \subset KL^{p^e}$ have degrees $p^e$ and $\geq p^e$, respectively. We summarize our findings in the following diagram of field extensions and their respective degrees,

\[
\begin{align*}
K & \rightarrow_{\text{sep}} K_{\text{sep}} = KL^{p^e} \rightarrow_{p^{2e}} L \\
\text{Quot}(A^{(p^e)}) & \rightarrow_{p^e} \text{Quot}(A^{(p^e)})L^{p^e} \rightarrow_{\geq p^e} L^{p^e}
\end{align*}
\]

Calculating inseparable degrees in the square on the left shows that the extension Quot$(A^{(p^e)}) \subset \text{Quot}(A^{(p^e)})L^{p^e}$ is separable, and computing field degrees along the triangle on the right hand side we see that $L^{p^e} = \text{Quot}(A^{(p^e)})L^{p^e}$.

Since the extension $A^{(p^e)}[B^{p^e}] \subset B$ is integral we have that $J^{p^e}$ generates an ideal of dimension at most zero in $A^{(p^e)}[B^{p^e}]$. Now applying Theorem 2.2 to the inclusion $A^{(p^e)} \subset A^{(p^e)}[B^{p^e}]$ yields an embedding of canonical modules $\omega_{A^{(p^e)}} \hookrightarrow \omega_{A^{(p^e)}[B^{p^e}]}$. On the other hand, since $B^{p^e} \subset A^{(p^e)}[B^{p^e}]$ is a finite and birational extension we obtain an inclusion $\omega_{A^{(p^e)}[B^{p^e}]} \hookrightarrow \omega_{B^{p^e}}$.

Finally, we prove the inequality $\left\lfloor \frac{a(A)}{p^e} \right\rfloor \leq a(B)$. One has $a(A^{(p^e)}) = p^e \left\lfloor \frac{a(A)}{p^e} \right\rfloor$, since the local cohomology functors commute with the formation of Veronesse submodules. On the other hand, $a(B^{p^e}) = p^e a(B)$. 

$\blacksquare$
The next proposition shows that the estimate of Theorem 3.1 is sharp and that the stronger inequality of Theorem 2.2 is not valid without the separability assumption.

**Proposition 3.2.** Let $k$ be a perfect field of positive characteristic $p$ and $A \subset B$ a homogeneous integral extension of positively graded Noetherian $k$-domains of dimension two. Assume $A$ is normal and contains a nonzero linear form. Further suppose that the field extension $\text{Quot}(A) \subset \text{Quot}(B)$ is purely inseparable of degree $p^e$. Then $B^{p^e} \subset A^{(p^e)}$ is a finite birational extension. In particular, $B$ is normal if and only if $B^{p^e} = A^{(p^e)}$, in which case

$$\left\lfloor \frac{a(A)}{p^e} \right\rfloor = a(B).$$

**Proof.** Write $K = \text{Quot}(A)$ and $L = \text{Quot}(B)$. One has $B^{p^e} \subset L^{p^e} \subset K$ and therefore $B^{p^e} \subset K \cap B \subset A$ where the last inclusion obtains because $B$ is integral over $A$ and $A$ is normal. It follows that $B^{p^e} \subset A^{(p^e)}$. This extension is integral since $A^{(p^e)} \subset B$, and it is birational because the proof of Theorem 3.1 shows that $\text{Quot}(A^{(p^e)})L^{p^e} = L^{p^e}$, hence $\text{Quot}(A^{(p^e)}) \subset L^{p^e}$. \hfill $\square$

4. **The General Case**

In this section we continue our treatment of possibly inseparable extensions in positive characteristic. We allow the dimensions of the rings to be arbitrary, but we do not recover the full strength of the result in dimension 2. The $t$-th Fitting ideal of a finitely presented $C$-module $M$ is denoted by $\text{Fitt}^t M$.

**Theorem 4.1.** Let $k$ be a perfect field of positive characteristic $p$ and let $A \subset B$ a homogeneous integral extension of standard graded Noetherian $k$-algebras. Assume that $A$ is a domain, regular in codimension one, and write $p^e$ for the inseparable degree of the field extension $\text{Quot}(A) \subset \text{Quot}(B/q)$, where $q$ is some minimal prime of $B$ of maximal dimension. Then

$$\left\lfloor \frac{a(A) + \dim A - 2}{p^e} \right\rfloor \leq \text{reg} B - 2.$$

**Proof.** We may assume that $k$ is infinite. Write $d = \dim A = \dim B$. We prove the theorem by induction on $d$. If $d \leq 1$ then $A$ is regular, thus $\left\lfloor \frac{a(A) + \dim A - 2}{p^e} \right\rfloor = \left\lfloor \frac{-2}{p^e} \right\rfloor$. On the other hand, $\text{reg} B - 2 \geq -1$ unless $\text{reg} B = 0$, in which case $B = A$ and hence $p^e = 1$. For $d = 2$ the assertion follows from Theorem 3.1 because $a(A) + \dim A \leq \text{reg} A$ and $a(B) \leq \text{reg} B - 2$.

Now assume that $d \geq 3$. Let $\ell_1, \ldots, \ell_n$ be linear forms generating the homogeneous maximal ideal of $A$ and let $z_1, \ldots, z_n$ be indeterminates over $k$. In the ring $A \otimes_k k(z_1, \ldots, z_n)$ we consider the generic linear form $x = \sum z_i \ell_i$. Without changing the notation introduced in the theorem, we now replace $k$ by the purely transcendental extension field $k(z_1, \ldots, z_n)$. Write $m$ and $n$ for the homogeneous maximal ideals of $A$ and $B$, respectively. Notice that $\sqrt{m}B = n$. Consider the induced homogeneous module finite map of standard graded $k$-algebras

$$\overline{A} = (A/xA)/H^0_m(A/xA) \longrightarrow \overline{B} = (B/xB)/H^0_m(B/xB).$$
Both algebras have dimension \(d - 1\). Since \(x\) is a generic linear form, the proof of [71] Theorem shows that \(\overline{A}\) is a domain and is regular in codimension one. It follows that the above map is an embedding \(\overline{A} \hookrightarrow \overline{B}\). As \(d - 1 > 0\) and \(x\) is \(A\)-regular it follows that \(a(\overline{A}) = a(A/xA) \geq a(A) + 1\). On the other hand, the natural map \(B \rightarrow \overline{B}\) factors through the ring \(B' = B/H_m^0(B)\), and the linear form \(x\) is regular on the latter. As \(\overline{B} = (B'/xB'/H_m^0(B'/xB'))\) it follows that \(\text{reg} \overline{B} \leq \text{reg} B'/xB' = \text{reg} B' \leq \text{reg} B\). Hence the statement of the theorem follows from the induction hypothesis once we have shown that there exists a minimal prime \(\overline{q}\) of \(\overline{B}\) of maximal dimension so that the field extension \(\text{Quot}(\overline{A}) \subset \text{Quot}(\overline{B}/\overline{q})\) has inseparable degree at most \(p^r\).

To construct \(\overline{q}\) notice that the \(B\)-ideal \((q, xB) = m^\infty\). Let \(q'\) be a prime ideal of dimension \(d - 1\) containing the latter ideal and define \(\overline{q} = q'\overline{B}\). Clearly \(\overline{q}\) is a prime ideal of \(\overline{B}\) of maximal dimension and it contains the image of \(q\). Write \(K = \text{Quot}(A)\), \(L = \text{Quot}(B/q)\), let \(K_{\text{sep}}\) be the separable closure of \(K\) in \(L\), and define \(C = K_{\text{sep}} \cap (B/q)\). As \(L = K[B/q]\) one has \(K_{\text{sep}} = \text{Quot}(C)\). Thus \(B/q\) is a \(C\)-module of rank \(p^r\). Set \(q'' = (q'/q) \cap C\), which is a \((d - 1)\)-dimensional prime ideal of \(C\) containing \(x\). Consider the \(C\)-ideal \(J = \text{Fitt}^p_C(B/q)\). This ideal is not zero because the field extension \(\text{Quot}(A) = K \subset \text{Quot}(C) = K_{\text{sep}}\) is separable and because \(B/q\) has rank \(p^r\) as \(C\)-module. Hence there are at most finitely many \((d - 1)\)-dimensional prime ideals containing \(J\), and \(x\) cannot be contained in any of them, since \(x\) is generic for \(m\) and \(m\) generates an ideal of dimension \(0 < d - 1\) in \(C\). It follows that \(J\) cannot be in \(q''\), a \((d - 1)\)-dimensional prime containing \(x\). Therefore \((\Omega_{C/A})_{q''} = 0\) and \((B/q)_{q''}\) is a free \(C_{q''}\)-module of rank \(p^r\). Tensoring with the residue field \(k(q'')\) of \(q''\) we conclude that \(\Omega_{C/A} \otimes C k(q'') = 0\) and \(B/q \otimes C k(q'')\) is a \(k(q'')\)-vector space of dimension \(p^r\). Finally, write \(\overline{C} = C/q''\), which yields the inclusions of domains \(\overline{A} \subset \overline{C} \subset \overline{B}/\overline{q}\). Notice that \(\text{Quot}(\overline{C}) = k(q'')\).

As \(\Omega_{\overline{C}/\overline{A}} \otimes_{\overline{C}} k(q'')\) and \(\overline{B}/\overline{q} \otimes_{\overline{C}} k(q'')\) are epimorphic images of \(\Omega_{C/A} \otimes C k(q'')\) and \(B/q \otimes C k(q'')\), respectively, we conclude that \(\Omega_{\overline{C}/\overline{A}} \otimes_{\overline{C}} \text{Quot}(\overline{C}) = 0\) and \(\overline{B}/\overline{q} \otimes_{\overline{C}} \text{Quot}(\overline{C})\) is a \(\text{Quot}(\overline{C})\)-vector space of dimension at most \(p^r\). Therefore the field extension \(\text{Quot}(\overline{A}) \subset \text{Quot}(\overline{C})\) is separable and the extension \(\text{Quot}(\overline{C}) \subset \text{Quot}(\overline{B}/\overline{q})\) has degree at most \(p^r\). It follows that the inseparable degree of \(\text{Quot}(\overline{A}) \subset \text{Quot}(\overline{B}/\overline{q})\) is at most \(p^r\).

5. Applications

Now we come to the original goal of our work, which is showing that minimal multiplicity descends under integral extensions.

Let \(k\) be a field and \(A\) a quasi-standard graded \(k\)-algebra, by which we mean a positively graded Noetherian \(k\)-algebra integral over a subalgebra generated by linear forms. Write \(m\) for the homogeneous maximal ideal of \(A\). Assume that either \(A\) is Cohen-Macaulay or else \(k\) is algebraically closed and \(A\) is a domain. In this case \(e(A) \geq \text{edim} A - \dim A + 1\), [11] and [4], pg. 112]. If equality holds one says that \(A\) has minimal multiplicity. It is known that \(A\) has minimal multiplicity if and only if \(A\) is
Cohen-Macaulay and \( m^2 \subset J \) for some (every) ideal \( J \) generated by a linear system of parameters. In case \( A \) is standard graded the following conditions are equivalent as well [4, Introduction]:

- \( A \) has minimal multiplicity
- \( \text{reg} A \leq 1 \)
- \( A \) is Cohen-Macaulay and \( a(A) \leq 1 - \dim A \)
- \( A \) is Cohen-Macaulay and \( m^2 \subset J \) for some (every) ideal \( J \) generated by a linear system of parameters.

Integral extensions of such algebras are somewhat restricted. Thus let \( k \) be an algebraically closed field and \( A \) a standard graded \( k \)-domain of minimal multiplicity. The integral closure \( \overline{A} \) of \( A \) is a quasi-standard graded \( k \)-algebra. Since the extension \( A \subset \overline{A} \) is birational, one has \( e(\overline{A}) \leq e(A) \). Thus one concludes that

\[
edim \overline{A} \leq e(\overline{A}) + \dim \overline{A} - 1 \leq e(A) + \dim A - 1 = \text{edim} A.
\]

On the other hand, as \( A \) is standard graded, a homogeneous minimal generating set of the homogeneous maximal ideal of \( A \) extends to a homogeneous minimal generating set of the homogeneous maximal ideal of \( \overline{A} \). Thus the inequality \( \text{edim} \overline{A} \leq \text{edim} A \) implies that \( A = \overline{A} \). This recovers the well known fact that any standard graded domain over an algebraically closed field is normal, provided it has minimal multiplicity (much more is true, see for instance [5, 19.9]).

This discussion shows that in the next Corollary it is natural to assume that \( A \) is normal. This condition is approximated by the assumptions of Theorems 2.2, 3.1, or 4.1.

**Corollary 5.1.** In addition to the assumptions of either Theorems 2.2, 3.1, or 4.1, suppose that \( A \) and \( B \) are standard graded and \( A \) is Cohen-Macaulay. Further assume that \( B \) is Cohen-Macaulay or that \( k \) is algebraically closed and \( B \) is a domain. If \( B \) has minimal multiplicity so does \( A \).

**Proof.** The theorems show that \( \text{reg} A \leq 1 \) if \( \text{reg} B \leq 1 \). \( \square \)

In particular, we obtain the following statement that we use in [9].

**Corollary 5.2.** Let \( k \) be a perfect field and \( A \subset B \) homogeneous integral extensions of standard graded \( k \)-domains. Further assume that \( A \) is normal and Cohen-Macaulay. If \( B \) has minimal multiplicity, then so does \( A \).

**Proof.** We may assume that \( k \) is infinite. Let \( m \) be the homogeneous maximal ideal of \( A \). Let \( \underline{x} = x_1, \ldots, x_d \) be linear forms that are a system of parameters of \( A \) and let \( J \) be the \( A \)-ideal they
generate. Write $C = B/A$. To show that $\mathfrak{a}$ form a regular sequence on $C$ we need to prove that the first Koszul homology of $\mathfrak{a}$ with coefficients in $C$ vanishes or, equivalently, $\text{Tor}_1^A(C,A/J) = 0$. The vanishing of Tor is equivalent to the equality $JB \cap A = J$, as can be seen from the exact sequence

$$0 = \text{Tor}_1^A(B,A/J) \longrightarrow \text{Tor}_1^A(C,A/J) \longrightarrow A/J \longrightarrow B/JB \longrightarrow C/JC \longrightarrow 0.$$ 

Thus let $\alpha$ be a homogeneous element of $JB \cap A$. If $\alpha$ has degree one then $\alpha \in J$ because $B_0 = k = A_0$. If on the other hand, $\alpha$ has degree at least two, then $\alpha \in m^2 \subset J$, where the last inclusion holds because $A$ has minimal multiplicity.

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