Inflationary gravitational waves from unified spinor fields

2 Luis Santiago Ridao∗, 1,2 Marcos R. A. Arcodia†,
2 Jesús Martín Romero‡ and 1,2 Mauricio Bellini§

1 Departamento de Física, Facultad de Ciencias Exactas y Naturales,
Universidad Nacional de Mar del Plata,
Funes 3350, C.P. 7600, Mar del Plata, Argentina.

2 Instituto de Investigaciones Físicas de Mar del Plata (IFIMAR),
Consejo Nacional de Investigaciones Científicas y
Técnicas (CONICET), Mar del Plata, Argentina.

Abstract

Recently observations of Gravitational Waves (GW) generated by black-hole collisions have opened a new window to explore the universe in diverse scales. It is expected that in the following years be possible the detection of primordial gravitational waves. However this formalism is developed for weak gravitational waves, when the dynamics of the waves can be linearized. In this work we develop a non-perturbative formalism to describe GW using the Unified Spinor Fields (USF) theory. The tensor index is calculated and we obtain that this must be $n_T = 0$ in order to the + and $\times$ polarisations modes can take the same spectrum. This impose some restriction on the constant of self-interaction $\xi$ of the fermionic source. We obtain that the amplitude of the signals today must be $\sim 10^{-112}$ times more weak than the amplitude generated during inflation, which is of the order of $\Delta_{GW}|_{Infl} \simeq 10^{-11}$.

∗ E-mail address: santiagoridao@hotmail.com
† E-mail address: marcodia@mdp.edu.ar
‡ E-mail address: jesusromero@conicet.gov.ar
§ E-mail address: mbellini@mdp.edu.ar
I. INTRODUCTION AND MOTIVATION

Gravitational waves were predicted close to 100 years ago by A. Einstein in the framework of his theory of General Relativity (GR)\cite{1}, and revisited by him with N. Rosen twenty years later\cite{2}. In the last few years, Advanced LIGO has opened a new era in the modern astrophysical research\cite{3}, when detected gravitational waves (GW), coming from collisions of pairs of black-holes. It is expected that in the future, new detectors like LISA, can measure GW emitted during the early universe\cite{4}, in particular, when the universe suffered a quasi-exponential expansion during inflation. A general prediction of cosmological inflation\cite{5} is the generation of a stochastic background of primordial gravitational waves (GW)\cite{6}. Its detection would be of great importance for the understanding and corroborating of inflation during the early phase of the expansion of the universe\cite{7}. Under the standard model of cosmology plus the theory of inflation, it is very natural to predict the existence of the GW with respect to the background geometry of the universe. In the standard single-field, slow-roll inflationary scenario the tensor fluctuations of the metric are characterized by a nearly scale-invariant power-spectrum on super-Hubble scales. The amplitude of GW signal is described by the tensor-to-scalar ratio $r$, defined as the ratio between the tensor and the scalar power-spectrum amplitudes at a given wave-number $k = k_* \simeq 0.05 \text{Mpc}^{-1}$, assuming $r = -8n_T$, where $n_T$ is the tensor spectral index.

However, the existence of a background Riemannian geometry should be explained from a more fundamental basis. In a recent work\cite{8} some of the authors of this work have proposed an unified spinor field (USF) formalism in order to describe non-background relativistic systems which are quantum mechanical in nature. This is a non-perturbative theory of unification for quantised spinor fields on extended manifolds, taking into account the self-interactions of the spinor fields. Each component of spin $\hat{S}_\mu$, is defined as the momentum corresponding to the inner dimension $\hat{\Phi}^\mu$, such that one can define an universal bi-vectorial invariant: $\left\langle B \left\| \hat{S}_\mu \hat{\Phi} \right\| B \right\rangle = (2\pi n h) I_{4\times 4}$. The description of a non-perturbative description for GW in this formalism is a nontrivial issue. In this work we develop the dynamics for GW on an extended manifold characterized by the spinor fields components $\hat{\Psi}^\mu$. This theory is described by Sect. II. With the aim to illustrate the theory, we describe the dynamics of GW in a model of inflation. This is done in Sect. III. Finally, in Sect. IV we conclude with some final comments.
II. GRAVITATIONAL WAVES

In order for describe tensor metric fluctuations that describes an wave dynamics on the extended manifold (with respecto to the Riemannian one), defined by the connection

\[ \hat{\Gamma}_{\beta\gamma}^{\alpha} = \left\{ \begin{array}{c} \alpha \\
\beta \\
\gamma \end{array} \right\} + \hat{\Psi}^{\alpha} g_{\beta\gamma}. \] (1)

In the connections (1), it is clear that

\[ \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} = \hat{\Psi}^{\alpha} g_{\beta\gamma}. \] (2)

Here, \( \hat{\Psi}^{\alpha} \) are the components of an operator spinor field which could describe bosons or fermions, depending on the algebra complied by them, and \( g_{\beta\gamma} \) are the components of the covariant tensor metric. It is expected that the expectation value of the quantum displacement of the extended manifold with respect to the classical Riemannian background, which is described by the Levi-Civita symbols in (1), to be null:

\[ \left\langle B \left| \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} \right| B \right\rangle = 0. \]

A. DYNAMICS OF SPINOR FIELDS REVISITED

The variation of the extended Ricci tensor \( \delta R_{\beta\gamma}^{\alpha} = \delta R_{\beta\gamma} \): \( \delta R_{\beta\gamma} = \left( \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} \right)_{||\gamma} - \left( \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} \right)_{||\alpha}, \)

\[ \delta \hat{R}_{\beta\gamma} = \nabla_{\gamma} \hat{\Psi}_{\beta} - 3 \left( 1 - \frac{\xi^2}{3} \right) g_{\beta\gamma} \left( \hat{\Psi}^{\nu} \hat{\Psi}_{\nu} \right) 
- \ g_{\beta\gamma} \left( \nabla_{\nu} \hat{\Psi}_{\nu} \right) + \left( 1 - \frac{\xi^2}{3} \right) \hat{\Psi}_{\beta} \hat{\Psi}_{\gamma} = \hat{U}_{\beta\gamma} + \hat{V}_{\beta\gamma}, \] (3)

where \( \hat{U}_{\beta\gamma} \), and \( \hat{V}_{\beta\gamma} \) are the symmetric and antisymmetric parts of \( \delta \hat{R}_{\beta\gamma} \):[8]:

\[ \hat{U}_{\beta\gamma} = \frac{1}{2} \left( \nabla_{\beta} \hat{\Psi}_{\gamma} + \nabla_{\gamma} \hat{\Psi}_{\beta} \right) - g_{\beta\gamma} \left( \nabla_{\nu} \hat{\Psi}_{\nu} \right) 
- 3 \left( 1 - \frac{\xi^2}{3} \right) g_{\beta\gamma} \left( \hat{\Psi}_{\beta} \hat{\Psi}_{\nu} \right) + 3 \left( 1 - \frac{\xi^2}{3} \right) \left\{ \hat{\Psi}_{\beta}, \hat{\Psi}_{\gamma} \right\} , \]

\[ \hat{V}_{\beta\gamma} = -\frac{1}{2} \left( \nabla_{\beta} \hat{\Psi}_{\gamma} - \nabla_{\gamma} \hat{\Psi}_{\beta} \right) + \frac{3}{2} \left( 1 - \frac{\xi^2}{3} \right) \left[ \hat{\Psi}_{\beta}, \hat{\Psi}_{\gamma} \right]. \]

1 where \( \left( \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} \right)_{||\gamma} \) denotes the covariant derivative of \( \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} \), given by [2], on the extended manifold, with self-interactions included.
The antisymmetric tensor $\hat{\delta} R_{\alpha \beta \gamma} \equiv \hat{\Sigma}_{\beta \gamma}$ is
\[
\hat{\Sigma}_{\beta \gamma} = \left( \nabla_\beta \hat{\Psi}_\gamma - \nabla_\gamma \hat{\Psi}_\beta \right) - (1 + \xi^2) \left[ \hat{\Psi}_\beta, \hat{\Psi}_\gamma \right].
\] (4)

Now we can introduce the varied Einstein tensor on the extended manifold $\hat{\Sigma}_{\beta \gamma} = \hat{G}_{\beta \gamma} = \hat{U}_{\beta \gamma} - \frac{1}{2} g_{\beta \gamma} \hat{U}$, where $\hat{U} = g^{\alpha \beta} \hat{U}_{\alpha \beta}$:
\[
\hat{\delta} G_{\beta \gamma} = \frac{1}{2} \left( \nabla_\beta \hat{\Psi}_\gamma + \nabla_\gamma \hat{\Psi}_\beta \right) + \frac{1}{2} g_{\beta \gamma} \left[ \left( 1 - \frac{\xi^2}{3} \right) \left( \hat{\Psi}^\alpha \hat{\Psi}_\alpha \right) \right.
\[
+ \left( \nabla_\nu \hat{\Psi}_\nu \right) \right] + \frac{1}{2} \left( 1 - \frac{\xi^2}{3} \right) \left\{ \hat{\Psi}_\beta, \hat{\Psi}_\gamma \right\}.
\] (5)

After some algebra, we obtain the antisymmetric tensors $\hat{\Sigma}_{\beta \gamma}$ and $\hat{\mathcal{M}}_{\beta \gamma}$ such that the symmetric tensor $\hat{\delta} G_{\beta \gamma}$, with the antisymmetric ones $\hat{\mathcal{N}}_{\beta \gamma}$ and $\hat{\mathcal{M}}_{\beta \gamma}$. The spinor fields, must be conserved on the extended Weylian manifold:
\[
\left( \hat{\delta} G^\beta_\gamma \right)_{\| \gamma} = 0, \quad \left( \hat{\mathcal{M}}^\beta_\gamma \right)_{\| \gamma} = 0, \quad \left( \hat{\mathcal{N}}^\beta_\gamma \right)_{\| \gamma} = 0.
\] (8)

Taking into account the gauge-transformations: $\hat{\delta} G_{\alpha \beta} = \hat{\delta} G_{\alpha \beta} - g_{\alpha \beta} \hat{\Lambda}$, we obtain that
\[
\hat{\Lambda} = -\frac{3}{4} \left[ \nabla_\alpha \hat{\Psi}^\alpha + \left( 1 - \frac{\xi^2}{3} \right) \hat{\Psi}^\alpha \hat{\Psi}_\alpha \right].
\] (9)

B. Gravitational waves from USF

In order to describe GW, we shall propose the existence of a 2-range quantum operator $\hat{h}_{\mu \nu}$ such that
\[
\frac{1}{2} \Box \hat{h}_{\mu \nu} = -\hat{\delta} R_{\mu \nu},
\] (10)

where, as can be demonstrated, the variation on the extended manifold (11), of the Ricci tensor is
\[
\hat{\delta} R_{\mu \nu} = -\frac{1}{2} \left( \nabla_\mu \hat{\Psi}_\nu + \nabla_\nu \hat{\Psi}_\mu \right) + \frac{1}{2} (\xi^2 - 3) \left\{ \hat{\Psi}_\mu, \hat{\Psi}_\nu \right\} + \frac{2}{3} g_{\mu \nu} \hat{\Lambda},
\] (11)

where $\hat{\Lambda}$ is given by
\[
\hat{\Lambda} = -\frac{3}{4} \left[ \nabla_\alpha \hat{\Psi}^\alpha + \frac{1}{2} \left( 1 - \frac{\xi^2}{3} \right) g^{\alpha \beta} \left\{ \hat{\Psi}_\alpha, \hat{\Psi}_\beta \right\} \right].
\] (12)
and takes into account the source of the gravitational waves. The expectation values of $\hat{\delta}R_{\mu\nu}$ and $\hat{\Lambda}$ (on the Riemannian background), are

$$
\langle B \left| \hat{\delta}R_{\mu\nu} \right| B \rangle = - \left\langle B \left| \frac{1}{2} (\xi^2 - 3) \left\{ \hat{\Psi}_\mu, \hat{\Psi}_\nu \right\} \right| B \right\rangle + \frac{2}{3} \delta g_{\mu\nu} \left\langle B \left| \hat{\Lambda} \right| B \right\rangle,
$$

(13)

$$
\langle B \left| \hat{\Lambda} \right| B \rangle = - \frac{3}{8} \left\langle B \left| \left( 1 - \frac{\xi^2}{3} \right) g^{\alpha\beta} \left\{ \hat{\Psi}_\alpha, \hat{\Psi}_\beta \right\} \right| B \right\rangle,
$$

(14)

where we have used the fact that $\langle B \left| \nabla_\alpha \hat{\Psi}_\alpha \right| B \rangle = 0$. Because can be demonstrated that $\langle B \left| \hat{\delta}R \right| B \rangle = 4 \left\langle B \left| \hat{\Lambda} \right| B \right\rangle$, the wave equation is

$$
\langle B \left| \Box \delta h_{\mu\nu} \right| B \rangle = - 2\kappa \left\langle B \left| \delta T_{\mu\nu} \right| B \right\rangle,
$$

(15)

where $\kappa = 8\pi G/c^4$. The variation of the stress tensor $\delta T_{\mu\nu}$ with respect to the background, is

$$
\delta \hat{T}_{\mu\nu} = 2 \frac{\delta \hat{\cal L}}{\delta g_{\mu\nu}} - g_{\mu\nu} \hat{\cal L},
$$

(16)

such that the Lagrangian density is given by $\hat{\cal L} = \frac{2}{3\kappa} \hat{\Lambda}$. Therefore, the stress tensor must be written as

$$
\kappa \delta \hat{T}_{\alpha\beta} = - \frac{1}{2} \left( \nabla_\alpha \hat{\Psi}_\beta + \nabla_\beta \hat{\Psi}_\alpha \right) - \frac{1}{2} g_{\alpha\beta} \nabla_\nu \hat{\Psi}^\nu - \frac{1}{4} g_{\alpha\beta} \left( 1 - \frac{\xi^2}{3} \right) g^{\mu\nu} \left\{ \hat{\Psi}_\mu, \hat{\Psi}_\nu \right\}
$$

$$
+ \frac{1}{2} (\xi^2 - 3) \left\{ \hat{\Psi}_\alpha, \hat{\Psi}_\beta \right\}.
$$

(17)

Therefore, the equation for gravitational waves is

$$
\Box \delta h_{\alpha\beta} = - 2\kappa \delta \hat{T}_{\alpha\beta},
$$

(18)

where $\delta \hat{T}_{\alpha\beta}$ given by (20). We are interested in the study of gravitational waves produced during inflation. In that case we must to adopt a co-moving frame with $\hat{U}^0 \left| B \right\rangle = \mathbb{I}_{4\times4} \left| B \right\rangle$ and $\hat{U}^j \left| B \right\rangle = 0$. The components $\delta \hat{h}_{0j}$ in (18) describes the evolution of electromagnetic waves produced by photons, but the components $\delta \hat{h}_{ij}$ describes gravitational waves produced by gravitons. The TT (Transverse-Traceless) components of these waves: $\delta \hat{h}_{ij} = \delta \hat{h}_{ij} - \frac{1}{2} g_{ij} \delta h$, have a dynamics governed by the equations

$$
\Box \delta h_{ij} = - 2\kappa \left\langle B \left| \delta \hat{S}_{ij} \right| B \right\rangle,
$$

(19)

where $\delta \hat{S}_{\alpha\beta} = \delta \hat{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \delta \hat{T}$, is

$$
- 2\kappa \delta \hat{S}_{\alpha\beta} = \left( \nabla_\alpha \hat{\Psi}_\beta + \nabla_\beta \hat{\Psi}_\alpha \right) - 2 g_{\alpha\beta} \left[ \nabla_\nu \hat{\Psi}^\nu + \frac{3}{2} \left( 1 - \frac{\xi^2}{3} \right) g^{\mu\nu} \left\{ \hat{\Psi}_\mu, \hat{\Psi}_\nu \right\} \right]
$$

$$
- (\xi^2 - 3) \left\{ \hat{\Psi}_\alpha, \hat{\Psi}_\beta \right\}.
$$

(20)
Finally, the expectation values of the source terms in the wave equations, are
\[ -2\kappa \langle B \left| \hat{S}_{ij} \right| B \rangle = -3g_{ij} \left( 1 - \frac{\xi^2}{3} \right) g^{\mu\nu} \langle B \left| \{ \hat{\Psi}_\mu, \hat{\Psi}_\nu \} \right| B \rangle - (\xi^2 - 3) \langle B \left| \{ \hat{\Psi}_i, \hat{\Psi}_j \} \right| B \rangle, \]  
(21)

which is due exclusively by a fermionic contribution, because fermions comply with the quantization algebra\[8\]
\[ \langle B \left| \{ \hat{\Psi}_\mu(x, \phi), \hat{\Psi}_\nu(x', \phi') \} \right| B \rangle = \frac{s^2}{\hbar^2} g_{\mu\nu} \mathbb{I}_{4\times4} \sqrt{\frac{\eta}{g}} \delta^{(4)} (x - x') \delta^{(4)} (\phi - \phi'). \]  
(22)

Therefore, the expectation value for the source term in the wave equation (19), is
\[ -2\kappa \langle B \left| \hat{S}_{ij} \right| B \rangle = 3g_{ij} \frac{s^2}{\hbar^2} \sqrt{\frac{\eta}{g}} (\xi^2 - 3) \mathbb{I}_{4\times4} \delta^{(4)} (x - x') \delta^{(4)} (\phi - \phi'). \]  
(23)

The factor \( \sqrt{\frac{\eta}{g}} \) takes into account the ratio between the determinants of tensor metrics in a Minkowski metric: \( \eta \), and in a generic metric: \( g \). Furthermore, the factor \( \frac{s^2}{\hbar^2} \) takes into account the spin of the fermions of the source.

III. GW DURING INFLATION

To explore the model we shall consider a de Sitter (inflationary) expansion, where the background spacetime is described by the line element
\[ dS^2 = a^2(\tau) \left[ d\tau^2 - \delta_{ij} dx^i dx^j \right], \]  
(24)

where \( \tau \), that runs from \(-\infty\) to zero, is the conformal time of the universe, which is considered as spatially flat, isotropic and homogeneous. If the expansion is governed by the inflaton field \( \varphi \), and it is non-minimally coupled to gravity, the universe can be described by the action
\[ \mathcal{I} = \int \sqrt{-g} \left[ \mathcal{R} \frac{2\kappa}{2\kappa} + \mathcal{L}_\varphi \right], \]  
(25)

where \( \mathcal{L}_\varphi = \frac{1}{2} (\varphi'\varphi')^2 - V(\varphi) \). In a de Sitter expansion the scale factor of the universe is \( a(\tau) = -\frac{1}{\pi \tau} \), and the scalar potential is a constant \( V(\varphi) = \frac{3H^2}{\kappa} \). Furthermore, the kinetic component of \( \mathcal{L}_\varphi \) is zero, so that \( \varphi(\tau) = \varphi_0 \).
Now, we must calculate the equation of motion for GW \([19]\), during inflation. The \(ij\)-components of \(\langle B \left| \hat{\delta} h_{ij} \right| B \rangle \equiv \tilde{h}_{ij}\), can be expanded as
\[
\tilde{h}_{ij} = \frac{1}{(2\pi)^3} \int d^4s \int d^4k \sum_{n=+,-,\times} (n)_{ij} \left[ C_{k,s}^{(n)} \right] \theta_k(\tau) e^{ikx^j} e^{\frac{i}{\hbar} \delta_{\alpha} \phi^\alpha} + \left[ C_{k,s}^{(n)} \right]^\dagger \theta_k^*(\tau) e^{-ikx^j} e^{-\frac{i}{\hbar} \delta_{\alpha} \phi^\alpha},
\]  
(26)

where \(+,\times\) denote the polarization states in the Transverse-Traceless (TT) gauge, defined by
\[
\tilde{h}_{0j} = 0, \quad \tilde{h} = 0.
\]  
(27)

For \((n)\epsilon_{00} = 0\), the polarization tensor is transverse: \(k^i (n)\epsilon_{ji} = 0\) to the propagation of the wave characterized by the wavenumber components \(k^i\). Accounting for \(h = 0\) and \((n)\epsilon_{ij} = (n)\epsilon_{ji}\), we can define\(^2\)
\[
(n)\epsilon_{11} = \frac{1}{2} \{\hat{\gamma}_1, \hat{\gamma}_1\}, \quad (n)\epsilon_{22} = -(n)\epsilon_{11}, \quad (n)\epsilon_{12} = \frac{1}{2} \{\hat{\gamma}_1, \hat{\gamma}_2\}, \quad (n)\epsilon_{21} = \frac{1}{2} \{\hat{\gamma}_2, \hat{\gamma}_1\}.
\]  
(28)

The creation and destruction operators comply
\[
\langle B \left| C_{k,s}^{(n)} \right\rangle C_{k',s'}^{(n')} \right| B \rangle = M_p^2 \delta_{mm'} \delta^{(4)}(\vec{k} - \vec{k}') \delta^{(4)}(\vec{s} - \vec{s}').
\]  
(29)

By imposing that
\[
\langle B \left| \sum_{n=1,2} (n)\epsilon_{ij} C_{k,s}^{(n)} \right| B \rangle = \langle B | \delta_{ij} M_p \theta_k^*(\tau') e^{-ikx^j} e^{-\frac{i}{\hbar} \delta_{\alpha} \phi^\alpha'},
\]  
(30)

\(^2\) In this paper we shall consider the Weyl basis, which, in Minkowski spacetime, is:
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
\[
\gamma^2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix},
\]
such that \(\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}_{4\times4}\). Here, the Pauli matrices are
\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
The basis on the metric \([24]\) is \(\tilde{\gamma}^\mu = a^{-1} \gamma^\mu\). They comply with the Clifford algebra
\[
\tilde{\gamma}^\mu = \frac{1}{3!} (\tilde{\gamma}^\mu)^2 \epsilon_{\alpha \beta \gamma} \tilde{\gamma}^\alpha \tilde{\gamma}^\beta \tilde{\gamma}^\gamma, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu \nu} \mathbb{I}_{4\times4},
\]
where \(\mathbb{I} = \gamma^0\gamma^1\gamma^2\gamma^3, \mathbb{I}_{4\times4}\) is the identity matrix, \(\tilde{\gamma}^\alpha \tilde{\gamma}^\beta = \frac{1}{2} [\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta]\).
we obtain the following differential equation for the time dependent modes, for a de Sitter expansion:

\[ \theta''_k(\tau) - \frac{2}{\tau} \theta'_k(\tau) + k^2 \theta_k(\tau) = -\frac{6M_p^2}{\tau^2 H^2} \delta_{ij} \left( \frac{s^2}{\hbar^2} (\xi^2 - 3) \right) \theta_k(\tau), \]  

where the Hubble parameter in a de Sitter expansion is a constant \( H \), \( k^\mu = (\omega, k^j) \) and \( x^\mu = (\tau, x^j) \). The case \( i \neq j \) describes the modes with polarisation \( \times \). Notice that in this case the right hand of Eq. (31) cancels, so that the solution for the modes \( \times \) must be a particular one of the modes with \( + \)-polarisation. We must suppose that the spectrum of modes \( + \) and \( \times \) should be the same, with identical spectral indices. If we impose the normalization condition for the modes \( \theta_k(\tau) \):

\[ \theta_k(\tau) [\theta^*_k(\tau)]' - \theta^*_k(\tau) [\theta_k(\tau)]' = i \frac{4\pi H^2}{9M_p^2}, \]

we obtain the solution for the Eq. (31)

\[ \theta_k(\tau) = \frac{i\pi H^2}{3M_p} \left( \frac{\tau}{\tau_0} \right)^{3/2} \mathcal{H}^{(2)}_\nu(\zeta(\tau)), \]

where \( G = M_p^{-2} \) is the gravitational constant, \( M_p = 1.2 \times 10^{19} \text{ GeV} \) is the Planck mass, \( \mathcal{H}^{(2)}_\nu(\zeta(\tau)) \) is the second kind Hankel function with argument \( \zeta(\tau) = k \tau \), and parameter

\[ \nu = \frac{1}{2} \sqrt{9 + \left[ \frac{6(3 - \xi^2)M_p^2}{H^2} \right]}, \]

such that, due to the fact that the tensor index is given by \( n_T = 3 - 2\nu \), we obtain

\[ n_T = 3 - \sqrt{9 + 24 \frac{s^2}{\hbar^2} (3 - \xi^2) \frac{M_p^2}{H^2}}. \]

If we require that both polarisation spectrums (i.e., \( + \) and \( \times \)) to be the same, they must satisfy the condition

\[ \xi^2 = 3, \]

---

\( ^3 \) We use the Dirac functions representation on the background spacetime \[ \delta^{(4)}(x - x') = \frac{4\pi}{(2\pi)^4} \int_{-\infty}^{\infty} dk \theta_k(\tau) \theta^*_k(\tau') \int_{-\infty}^{\infty} dk' k'^2 e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}, \]

\[ \delta^{(4)}(\phi - \phi') = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^4s e^{i\vec{s} \cdot (\vec{\phi} - \vec{\phi}')}. \]
which implies that, if this supposition were correct, \( n_T \) would not depend on the coupling \( \xi \) of primordial fermion fields, or the spin \( s \) of the fermion sources on cosmological scales. As can be demonstrated, the amplitude of gravitational waves is

\[
\Delta_{GW} \simeq \frac{H^2}{2\pi^2 M_p^2},
\]  

which, during inflation would be very of the order (for \( H \simeq 10^{-5} M_p \))

\[
\Delta_{GW}|_{Infl} \simeq 10^{-11},
\]

but the present day value should be (for \( H_0 \simeq 10^{-61} M_p \))

\[
\Delta_{GW}|_0 \simeq 10^{-123},
\]

where suffix 0 indicates the today’s value. It means that the present day value, \( \Delta_{GW}|_0 \), is \( 10^{-112} \) orders of magnitude smaller than of the value during inflation. Finally, it is important to notice that the \( \Delta_{GW}|_0 \)-decay is the same than of the cosmological constant \( \Lambda \) since the origin of the universe, and the inverse of the increasing of entropy since the big-bang, at the Planck epoch.

**IV. FINAL COMMENTS**

Following the unified spinor field theory recently introduced, we have studied the production of GW during inflation. This is an important issue that should be tested in the next years and could, if were confirmed, provide new and relevant information about the early stages of the inflationary expansion of the universe. Under the ansatz that the spectrums for the + and \( \times \) modes are the same, we have obtained the coupling value of the fermion source on cosmological scales. The scale invariant value, \( n_T = 0 \), here obtained, agrees very well with values obtained using slow-roll parameters \( \epsilon \) and \( \eta \) [10], and other models as bouncing cosmology [11, 12]. However, in our work, we have used direct calculations in order to obtain a scale invariant spectrum for both polarisations. In order to estimate the wavelengths of GW emitted during inflation, we must consider regions of size around \( 10^3 \) times (or biggest than) the size of the horizon at the end of inflation, when the universe suffered an expansion close to \( e^{60} \) times \( 1/H \), i.e.

\[
\lambda_{Inf} \geq 10^{-28} e^{60} 10^3 \simeq 10 \text{ cm}.
\]  

9
This means that GW should be with wavelengths larger than 10 cm. These wavelengths range would be detected by LISA (Laser Interferometer Space Antenna) in the future, when NASA/ESA launch three satellites into orbits to form an equilateral triangle with a distance of $5 \times 10^6$ kilometers between each spacecraft.

Acknowledgements

The authors acknowledge CONICET, Argentina (PIP 11220150100072CO) and UNMdP (EXA852/18), for financial support.

[1] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math.Phys.) 1918: 154-16 (1918).
[2] A. Einstein, N. Rosen, J. Franklin Inst. 223: 43 (1937).
[3] B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. Lett. 116: 061102 (2016);
    B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. Lett. 116: 241103 (2016);
    B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. X6: 041015 (2016);
    B. P. Abbott et al. (VIRGO, LIGO Scientific), Phys. Rev. Lett. 118: 221101 (2017);
    B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. Lett. 119: 141101 (2017);
    B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. Lett. 119: 161101 (2017).
[4] S.W. Hawking, Thomas Hertog, Neil Turok, Phys. Rev. D62: 063502 (2000);
    Michael S. Turner, Frank Wilczek, Phys. Rev. Lett. 65 3080 (1990)
    Jai-chan Hwang, Class. Quant. Grav. 15: 1401 (1998).
[5] A. A. Starobinsky, Phys. Lett. B91: (1980) 99;
    A. H. Guth, Phys. Rev. D23: (1981) 347;
    A. D. Linde, Phys. Lett. B129: (1983) 177.
[6] S. P. Gómez Martínez, J. E. Madriz Aguilar, M. Bellini, Phys. Lett. B649: 343 (2007);
    S. P. Gómez Martínez, L. P. da Silva, J. E. Madriz Aguilar, M. Bellini, Nuovo Cim. B122: 897 (2007).
[7] M. Ch. Guzzetti, N. Bartolo, M. Liguori, S. Matarrese, Riv. Nuovo Cim. 39: 399 (2016).
[8] M. R. A. Arcodía, M. Bellini, Towards unified spinor fields: confinement of gravitons on a dS background. E-print: arXiv 1703.01355.
[9] M. R. A. Arcodia, L. S. Ridao, M. Bellini, Astrophys. Space Sci. 361: 296 (2016).

[10] Planck Collab. 2015 Results XIII, Astron. & Astrophys. A13: 594 (2016).

[11] N. Pinto-Neto, A. Scardua, Phys. Rev. D95: 123522 (2017).

[12] J. I. Musmarra, M. Anabitarte, M. Bellini. Inflationary expansion of the universe with variable timescale. E-print: arXiv 1805.02565