CONFORMAL NETS ARE FACTORIZATION ALGEBRAS

ANDRÉ HENRIQUES

Abstract. We prove that conformal nets of finite index are an instance of the notion of a factorization algebra. This result is an ingredient in our proof that, for $G = SU(n)$, the Drinfel’d center of the category of positive energy representations of the based loop group is equivalent to the category of positive energy representations of the free loop group.

1. Introduction

In this note, we prove that conformal nets of finite index (Definitions 1.1 and 3.1 in [2]) form an instance of the notion of a factorization algebra. Our main result, Theorem [2] is a key ingredient in the proof, announced in [8], that the category of solitons of a finite index conformal net is a bicommutant category.

Our main theorem is an analog, within the coordinate-free setup of [2], of the additivity property of conformal nets. Let $\mathcal{A}$ be a conformal net on $S^1$, let $I \subset S^1$ be a closed interval, and let $\{ I_i \subset I \}$ be collection of closed intervals whose interiors cover that of $I$. Additivity is the statement that the von Neumann algebras $\mathcal{A}(I_i)$ then generate a dense subalgebra in $\mathcal{A}(I)$:

$$\bigcup I_i = \bar{I} \Rightarrow \bigvee \mathcal{A}(I_i) = \mathcal{A}(I). \quad \text{(additivity)}$$

The additivity property of chiral conformal nets was proven in [5]. If one takes finitely many intervals $I_i$ whose union is $I$, then the corresponding property is called strong additivity:

$$\bigcup I_i = I \Rightarrow \bigvee \mathcal{A}(I_i) = \mathcal{A}(I). \quad \text{(strong additivity)}$$

It is a result of Longo–Xu that chiral conformal nets of finite index satisfy strong additivity [12, §5].

Let now $I$ be an abstract interval, and $\{ I_i \subset I \}$ a finite collection of multi-intervals (a multi-interval is a finite disjoint union of intervals) satisfying $\bigcup I_i \times I_i = I \times I$. Equivalently, this is the requirement that for every pair of points $p, q \in I$ there exists an element of the cover that contains both $p$ and $q$. In Theorem [2] we prove that for every coordinate free conformal net $\mathcal{A}$ of finite index, not necessarily chiral, we have:

$$\bigcup I_i \times I_i = I \times I \Rightarrow \text{colim} \mathcal{A}(I_i) = \mathcal{A}(I). \quad \text{(factorization algebra)}$$

The colimit which appears in the right hand side, informally denoted colim $\mathcal{A}(I_i)$, is that of a diagram involving the algebras $\mathcal{A}(I_i)$ and $\mathcal{A}(I_i \cap I_j)$. (The colimit is defined by a universal property in the category of von Neumann algebras and normal $*$-homomorphisms.) That diagram is written out in the left hand side of

\[ A \text{a coordinate-free conformal net is called \textit{chiral} if the action of the rotation group on its vacuum sector has positive energy.}\]
i.e., the largest subset of \(V \cup \mathcal{I}\), of \([0, \infty)\)

### Remark

The category of von Neumann algebras and normal \(*\)-homomorphisms is cocomplete \([11, \text{Prop. 5.7}]\) (see also \([7, \S7]\)).

#### 2. Factorization algebras

Let \(\text{Man}^n\) be the category whose objects are \(n\)-dimensional manifolds and whose morphisms are embeddings. We equip it with the symmetric monoidal structure given by disjoint union. A collection of open subsets \(\{U_i \subset M\}\) of a manifold \(M\) is a Weiss cover if for every finite subset \(S \subset M\), there exists an index \(i\) such that \(S \subset U_i\). \([4, \text{Chapt. 6}]\). Equivalently, being a Weiss cover means that for every \(n \in \mathbb{N}\), the condition \(\bigcup U^n_i = M^n\) is satisfied. Let \(\mathcal{C}\) be a symmetric monoidal category.

**Definition** \([4, \text{Chapt. 6}]\). An \(n\)-dimensional \(\mathcal{C}\)-valued factorization algebra is a symmetric monoidal functor \(\mathcal{A} : \text{Man}^n \to \mathcal{C}\) which is a co-sheaf with respect to Weiss covers.

Here, being a co-sheaf with respect to Weiss covers means that, for every Weiss cover \(\{U_i \subset M\}\), the natural map

\[
\text{colim} \begin{pmatrix}
\mathcal{A}(U_1 \cap U_2) \\
\mathcal{A}(U_1 \cap U_3) \\
\mathcal{A}(U_2 \cap U_3) \\
\mathcal{A}(U_1 \cap U_4) \\
\mathcal{A}(U_2 \cap U_4) \\
\vdots
\end{pmatrix} \quad \to \quad \mathcal{A}(M)
\]

is an isomorphism. For later notational convenience, we abbreviate the left hand side of \((1)\) as \(\text{colim}\{\{\mathcal{A}(U_i)\}\} \Rightarrow \{\mathcal{A}(U_i)\}\).

In this paper, we are interested in 1-dimensional factorization algebras (or rather, a small variant of the notion of 1-dimensional factorization algebra) with values in the category of von Neumann algebras and normal \(*\)-homomorphisms.

An interval is an oriented 1-manifold diffeomorphic to \([0, 1]\). A multi-interval is a finite disjoint union of intervals. Let \(\text{INT}^*\) be the category whose objects are multi-intervals and whose morphisms are orientation preserving embeddings, and let \(\text{INT} \subset \text{INT}^*\) be its full subcategory of intervals. Let \(\mathcal{V}\) denote the category of von Neumann algebras and normal \(*\)-homomorphisms, equipped with the symmetric monoidal structure given by spatial tensor product. By the split property \([2, \text{Def. 1.1}]\), a conformal net can be viewed as a symmetric monoidal functor \(\mathcal{A} : \text{INT}^* \to \mathcal{V}\).

We introduce a variant of the notion of Weiss cover that accounts for the fact that morphisms in \(\text{INT}^*\) are not open but rather closed inclusions. Given a topological space \(X\), a Weiss c-cover is a family of closed subsets \(\{V_i \subset X\}\) that satisfies \(\bigcup V^n_i = X^n\) for every \(n \in \mathbb{N}\). Here, \(V_i\) denotes the relative interior of \(V_i\) inside \(X\), i.e., the largest subset of \(V_i\) which is open in \(X\). (For example, the relative interior of \([0, 1]\) inside \([0, 2]\) is the half-open interval \([0, 1]\).)

Throughout this paper, all conformal nets are assumed irreducible, i.e., all the algebras \(\mathcal{A}(I)\) are assumed to be factors (we work with conformal nets in the sense
of [2, Def. 1.1]). The following statement expresses the idea that conformal nets are factorization algebras:

**Theorem 1.** Let \( A : \text{INT}^* \to \text{VN} \) be a conformal net of finite index. Then \( A \) is a co-sheaf with respect to Weiss \( c \)-covers. Namely, for every multi-interval \( I \), and every Weiss \( c \)-cover of \( I \) by multi-intervals \( I_i \subset I \), the natural map

\[
q : \text{colim} \{ A(I_i \cap I_j) \} \to \{ A(I_i) \} \to A(I)
\]

is an isomorphism.

**Remark.** Here, it is crucial to use covers by closed multi-intervals. For a chiral conformal net \( A \) on \( S^1 \), the functor that sends an open multi-interval \( U \) to the algebra \( A(\bar{U}) \) is never a factorisation algebra, unless \( A \) is trivial (\( \bar{U} \) denotes the closure of \( U \) in \( S^1 \)), because the map

\[
A := \text{colim}_{\varepsilon \to 0} A([\varepsilon, 1 - \varepsilon]) \to A([0, 1])
\]

is not an isomorphism. Let \( I_\varepsilon \subset S^1 \) be the image of \( [\varepsilon, 1 - \varepsilon] \) under the exponential map \( t \mapsto e^{2\pi i t} : \mathbb{R} \to S^1 \). The obvious isomorphisms \( A([\varepsilon, 1 - \varepsilon]) \to A(I_\varepsilon) \) followed by the standard actions of \( A(I_\varepsilon) \) on the vacuum Hilbert space \( H_0 \) yield an irreducible representation of \( A \) on \( H_0 \). However, \( A([0, 1]) \) is a \( III_1 \) factor, which admits no irreducible representations [6, Thm. 2.13].

We can sharpen the above result a little bit. Given a compact topological space \( X \), a collection \( \{ V_i \subset X \}_{i \in I} \) of closed subsets is called a 2-cover if there exists a finite subset \( I' \subset I \) such that \( \bigcup_{i \in I'} V_i = X \). Any Weiss \( c \)-cover is a 2-cover, and Theorem 1 is a formal consequence of the following stronger result:

**Theorem 1'.** Let \( A \) be a conformal net of finite index, let \( I \) be a multi-interval, and let \( \{ I_i \subset I \} \) be a 2-cover by multi-intervals. Assume furthermore that there exists an element of the 2-cover that contains \( \partial I \). Then the natural map

\[
q : \text{colim} \{ A(I_i \cap I_j) \} \to \{ A(I_i) \} \to A(I)
\]

is an isomorphism.

When \( I \) is connected, the statement of Theorem 1 simplifies:

**Theorem 2.** Let \( A \) be a conformal net of finite index, let \( I \) be an interval, and let \( \{ I_i \subset I \} \) be a 2-cover by multi-intervals. Then the natural map

\[
q : \text{colim} \{ A(I_i \cap I_j) \} \to \{ A(I_i) \} \to A(I)
\]

is an isomorphism.

3. **Proofs**

In this section, we present the proofs of the above theorems. We first prove Theorem 2. We then prove Theorem 1 by a slight variation of the argument. Theorem 1' is then a formal consequence of Theorem 1. We begin with some lemmas. We first note that, when working with multi-intervals, a 2-cover induces a cover in the usual sense:

**Lemma 3.** Let \( I \) be a multi-interval and let \( \{ I_i \subset I \}_{i \in I} \) be a 2-cover by multi-intervals. Then \( \bigcup_i I_i = I \).
Proof. By definition, \( \bigcup_{I \in \mathcal{I}} I^2 = I^2 \) for some finite subset \( I' \subset \mathcal{I} \). Given a point \( p \in \hat{I} \), pick sequences \( (x_n) \) and \( (y_n) \) in \( I \) converging to \( p \) and satisfying \( x_n < p < y_n \). For every \( n \), there exists an index \( i \in I' \) such that \( x_n \) and \( y_n \) are both in \( I_i \). The set \( I' \) being finite, there exists an \( I_i \) that contains infinitely many \( x_n \)'s and \( y_n \)'s. Since \( I_i \) is a multi-interval, it contains \( p \) in its interior. \( \square \)

The next lemma is technical in nature. It is a generalisation of [2, Lem. 1.9]. Let \( \mathcal{A} \) be a conformal net (not necessarily of finite index) and let \( I \) be a multi-interval:

Lemma 4. Let \( \mathcal{I} = \{I_i \subset I\} \) be a collection of multi-intervals satisfying \( \bigcup \hat{I}_i = \hat{I} \). Let \( \varphi \in \text{Diff}(I) \) be a diffeomorphism in the connected component of the identity, and let \( \hat{I} := \varphi(I_0) \) for some \( I_0 \in \mathcal{I} \). Let \( H \) be a Hilbert space equipped with actions \( \rho_i : \mathcal{A}(I_i) \to B(H) \) satisfying

\[
\begin{align*}
(1) & \quad \rho_i|_{\mathcal{A}(I_i \cap I_j)} = \rho_j|_{\mathcal{A}(I_i \cap I_j)} : \mathcal{A}(I_i \cap I_j) \to B(H). \\
(2) & \quad \text{For every } I_j, I_k \in \mathcal{I} \text{ and every intervals } J \subset I_j, K \subset I_k \text{ with disjoint interiors, the algebras } \rho_j(\mathcal{A}(J)) \text{ and } \rho_k(\mathcal{A}(K)) \text{ commute.}
\end{align*}
\]

Then the actions \( \rho_i|_{\mathcal{A}(I_i \cap I_j)} \) of \( \mathcal{A}(I_i \cap I_j) \) on \( H \) extend (uniquely) to an action \( \hat{\rho} : \mathcal{A}(\hat{I}) \to B(H) \).

Proof. We write \( \rho_0 \) for the action of \( \mathcal{A}(I_0) \) on \( H \). We may assume without loss of generality that \( \varphi \) fixes a neighbourhood of \( \partial I \). Provided that is the case, we can write \( \varphi \) as a product of diffeomorphisms \( \varphi = \varphi_1 \circ \ldots \circ \varphi_n \) with \( \text{supp}(\varphi_s) \subset \hat{I}_s \) for some \( \hat{I}_s \in \mathcal{I} \). Let \( u_s \in \mathcal{A}(I_s) \) be unitaries s.t. \( \text{Ad}(u_s) = \mathcal{A}(\varphi_s) \) [2, Def. 1.1(iv)]. Identifying the elements \( u_s \) with their images in \( B(H) \), we set

\[
\hat{\rho}(a) := u_1 \ldots u_n \rho_0(\mathcal{A}(\varphi^{-1})(a)) u_n^* \ldots u_1^*.
\]

For every \( I_\ell \in \mathcal{I} \) and every sufficiently small interval \( K \subset \hat{I} \cap I_\ell \), we will show that

\[
\hat{\rho}|_{\mathcal{A}(K)} = \rho_0|_{\mathcal{A}(K)}.
\]

Here, ‘sufficiently small’ means that the intervals \( K_s := \varphi_s^{-1}(\ldots(\varphi_1^{-1}(K))) \) are contained in \( \hat{I}_s \) for some \( \hat{I}_s \in \mathcal{I} \), and that for every \( s' \leq n \) either \( K_{s'} \subset \hat{I}_{s'} \), or \( K_s \cap \text{supp}(\varphi_{s'}) = \emptyset \).

For every \( s \leq n \), we claim that

\[
(3) \quad u_1 \ldots u_s \rho_{k_s}(\mathcal{A}(\varphi_{s-1} \circ \ldots \circ \varphi_1^{-1})(a)) u_s^* \ldots u_1^* = \rho_0(a) \quad \forall a \in \mathcal{A}(K).
\]

Equation (2) is the special case \( s = n \). We prove (3) by induction on \( s \). The base case \( (s = 0, k_0 = \ell) \) is trivial. The induction step reduces to the equation

\[
\rho_{k_s}(\mathcal{A}(\varphi^{-1})(b)) = u_s^* \rho_{k_{s-1}}(b) u_s,
\]

with \( b = \mathcal{A}(\varphi_{s-1} \circ \ldots \circ \varphi_1^{-1})(a) \). Recall that \( b \in \mathcal{A}(K_{s-1}) \), \( u_s \in \mathcal{A}(I_s) \) and that, by assumption, either \( K_{s-1} \subset I_s \), or \( K_{s-1} \cap \text{supp}(\varphi_s) = \emptyset \). In the first case, we have

\[
u_s^* \rho_{k_{s-1}}(b) u_s = u_s^* \rho_{k_s}(b) u_s = \rho_{k_s}(u_s^* b u_s) = \rho_{k_s}(\mathcal{A}(\varphi^{-1})(b)) = \rho_{k_s}(\mathcal{A}(\varphi^{-1})(b)).
\]

In the second case, the elements \( \rho_{k_{s-1}}(b) \) and \( u_s \) commute:

\[
u_s^* \rho_{k_{s-1}}(b) u_s = \rho_{k_{s-1}}(b) = \rho_{k_s}(\mathcal{A}(\varphi^{-1})(b)),
\]

where the last equality holds since \( b \in \mathcal{A}(K_{s-1}) \) and \( \varphi_s \) acts trivially on \( K_{s-1} \). This finishes the proof of (3) and hence of (2). Finally, by strong additivity (which is one of the axioms in [2, Def. 1.1]), it follows from (2) that \( \hat{\rho}(a) = \rho_0(a) \) for every \( a \in \mathcal{A}(\hat{I} \cap I_\ell) \). \( \square \)
We now establish some assumptions under which the hypotheses of Lemma 3 are satisfied:

**Lemma 5.** Let \( A \) be a multi-interval, and let \( I = \{ I_i \subset I \} \) be a 2-cover. Let \( \rho_i : A(I_i) \to B(H) \) be actions satisfying \( \rho_i|_{A(I_i \cap I_j)} = \rho_j|_{A(I_i \cap I_j)} \). Then, for every \( I_j, I_k \in \mathcal{I} \) and every intervals \( J \subset I_j, K \subset I_k \) with disjoint interiors, we have

\[
[\rho_j(A(J)), \rho_k(A(K))] = 0.
\]

**Proof.** We assume without loss of generality that the 2-cover is finite. The finite set \( S := \bigcup_{I_i \in \mathcal{I}} \partial I_i \) decomposes \( J \) and \( K \) into a finitely many intervals: \( J = J_1 \cup \ldots \cup J_n \) and \( K = K_1 \cup \ldots \cup K_m \). For each pair \( J_r, K_s \) of above intervals, we will argue that

\( (4) \quad [\rho_j(A(J_r)), \rho_k(A(K_s))] = 0. \)

Pick interior points \( x \in J_r \) and \( y \in K_s \). Since \( \mathcal{I} \) is a 2-cover, there exists an \( i \in \mathcal{I} \) such that \( \{x, y\} \subset I_i \). It follows that \( J_r \cup K_s \subset I_i \). The actions of \( A(J_r) \) and \( A(K_s) \) on \( H \) factor through that of \( A(I_i) \), so equation (4) follows.

Equation (4) being true for every pair \( J_r, K_s \) as above, by strong additivity, it follows that the algebras \( \rho_j(A(J)) = \bigvee_r \rho_j(A(J_r)) \) and \( \rho_k(A(K)) = \bigvee_s \rho_k(A(K_s)) \) commute with each other. \( \square \)

The following lemma contains the main argument of the proof of Theorem 2.

**Lemma 6.** Let \( A \) be a conformal net of finite index:

**Lemma 6.** Let \( I \) be an interval and let \( \mathcal{I} = \{ I_i \subset I \} \) be a 2-cover. Let \( H \) be a Hilbert space equipped with actions \( \rho_i : A(I_i) \to B(H) \) satisfying \( \rho_i|_{A(I_i \cap I_j)} = \rho_j|_{A(I_i \cap I_j)} \). Then those maps extend to an action of \( A(I) \).

**Proof.** We may assume, without loss of generality, that the 2-cover is closed under taking subsets: \( I_i \in \mathcal{I} \) and \( J \subset I_i, J \) a multi-interval \( \Rightarrow (J \in \mathcal{I}). \)

By Lemmas 4 and 5, we are in a situation to apply Lemma 3. The latter implies that for every interval \( J \subsetneq I \), the actions of \( A(I \cap J) \) extend (uniquely) to an action of \( A(J) \). We may therefore assume without loss of generality that \( I = [0,5] \), and that the 2-cover contains the multi-intervals \( [0,2] \cup [3,5] \) and \( [1,4] \) as elements.

Recall that \( L^2(-) \) is the unit for the operation \( \boxtimes \) of Connes fusion. We have \( H \cong L^2A([1,4]) \boxtimes A([1,4]), H \), both as \( A([1,4]) \)-modules and as \( A([0,1] \cup [4,5]) \)-modules. By [3 Cor. 2.9], the vacuum sector \( L^2A([1,4]) \) is isomorphic to

\[
L^2A([2,3]) \boxtimes A([2,3] \cup [2,3], L^2A([1,4]) \boxtimes A([1,4]) \) \]

as an \( A([1,4]) \)-\( A([1,4]) \)-bimodule (this is where we use the assumption that \( A \) has finite index). Here, \([2,3]\) denotes the interval \([2,3]\) equipped with the opposite orientation, and the algebra \( A([2,3] \cup [2,3], [2,3]) \) associated to the circle \([2,3] \cup [2,3], [2,3]) \) is described in [3 Prop. 1.25]. The action of \( A([2,3] \cup [2,3], [2,3]) \) on \( L^2A([1,4]) \boxtimes A([1,4]) \) comes from the left action of \( A([2,3], [2,3]) \) on the second copy of \( L^2A([1,4]) \) and the right action of \( A([2,3], [2,3]) \) on \( L^2A([1,4]) \) on the first copy of \( L^2A([1,4]) \).

Let us abbreviate \( A([a,b]) \) by \( A_{ab} \), \( A([a,b] \cup [c,d]) \) by \( A_{ab,cd} \), and \( A([a,b] \cup [a,b]) \) by \( A_{\geq b} \). We have:

\[
H \cong L^2A_{14} \boxtimes A_{14} H \quad \text{and} \quad L^2A_{14} \cong L^2A_{23} \boxtimes A_{13} \) (\( L^2A_{14} \boxtimes A_{12,34} L^2A_{14} \)).
Combining those two facts, one gets
\[ H \cong L^2A_{14} \boxtimes_A_{14} H \]
\[ \cong (L^2A_{23} \boxtimes_A_{23} (L^2A_{14} \boxtimes_A_{14} L^2A_{14})) \boxtimes_A_{14} H \]
\[ \cong L^2A_{23} \boxtimes_A_{23} (L^2A_{14} \boxtimes_A_{14} L^2A_{14} \boxtimes_A_{14} H) \]
\[ \cong L^2A_{23} \boxtimes_A_{23} (L^2A_{14} \boxtimes_A_{14} L^2A_{14} \boxtimes_A_{14} H). \] (5)

Using that \( H \cong L^2A_{02U35} \boxtimes_A_{02U35} H \) and the existence of a (non-canonical) isomorphism

\[ L^2A_{14} \boxtimes_A_{14} L^2A_{02U35} \cong L^2A_{02} \boxtimes_A_{14} L^2A_{14} \boxtimes_A_{35} L^2A_{35} \cong L^2A_{05} \]

which is compatible with the left actions of \( A_{14} \) and \( A_{01U45} \) and the right actions of \( A_{02U35} \) and \( A_{23} \) (\[2\] Cor. 1.33 and \[3\] Lem. A.4), we get the following sequence of isomorphisms of \( A_{14} \)- and \( A_{01U45} \)-modules:

\[ H \cong L^2A_{23} \boxtimes_A_{23} (L^2A_{14} \boxtimes_A_{14} L^2A_{14} \boxtimes_A_{14} H) \]
\[ \cong L^2A_{23} \boxtimes_A_{23} (L^2A_{14} \boxtimes_A_{14} L^2A_{02U35} \boxtimes_A_{02U35} H) \]
\[ \cong L^2A_{23} \boxtimes_A_{23} (L^2A_{05} \boxtimes_A_{02U35} H). \] (6)

The actions of \( A_{14} \) and of \( A_{01U45} \) on

\[ L^2A_{23} \boxtimes_A_{23} (L^2A_{05} \boxtimes_A_{02U35} H) \]

extend to an action of \( A_{05} \) because they both act on \( L^2A_{05} \). The actions of \( A_{14} \) and of \( A_{01U45} \) on \( H \) therefore also extend to an action of \( A_{05} \). \( \square \)

To help the reader digest the argument in the above proof, we include a graphical rendering of the isomorphisms which appear in (5) and (6):

With Lemma 6 in place, the proof of Theorem 2 is now easy:

Proof of Theorem 2. We first note that, by the strong additivity property of conformal nets \[2\] Def. 1.1, the map \( q \) has dense image. It is therefore surjective, as any morphism of von Neumann algebras whose image is dense is automatically surjective \[13\] Chapt. III, Prop. 3.12).

To show that \( q \) is injective, pick a faithful representation

\[ \pi : \text{colim} \left\{ A(I_i \cap I_j) \to A(I_i) \right\} \to B(H) \]

and let \( \rho_i := \pi|_{A(I_i)} \). By Lemma 3 this extends to an action \( \rho : A(I) \to B(H) \). As \( \pi \) is injective and \( \pi = \rho \circ q \), the map \( q \) is also injective. \( \square \)

The proof of Theorem 1 follows closely that of Theorem 2:

Proof of Theorem 1. Let \( I \) be a multi-interval, and let \( \mathcal{I} = \{ I_i \subset I \} \) be a 2-cover one of whose elements contains \( \partial I \). As in the proof of Theorem 2 it is enough to argue that if \( H \) is a Hilbert space equipped with actions \( \rho_i : A(I_i) \to B(H) \) satisfying \( \rho_i|_{A(I_i \cap I_j)} = \rho_j|_{A(I_i \cap I_j)} \), then those extend to an action of \( A(I) \).

Without loss of generality, we may assume that the 2-cover is closed under taking subsets: \( (I_i \in \mathcal{I} \text{ and } J \subset I_i) \Rightarrow (J \in \mathcal{I}) \). In particular, we may assume that the
2-cover admits an element which doesn’t intersect its boundary, and which has exactly one connected component in each connected component on $I$.

By Lemmas 4 and 5 we are in a situation to apply Lemma 4. We can therefore assume, without loss of generality, that $I = \bigcup_{k=1}^{n} [0, 5]$, and that $\mathcal{I}$ contains the multi-intervals $\bigcup_{k=1}^{n} ([0, 2] \cup [3, 5])$ and $\bigcup_{k=1}^{n} [1, 4]$ as elements.

Following the structure of the proof of Lemma 6 we have isomorphisms:

$$H \cong L^2(A_{14})^{\otimes n} \boxtimes_{A_{14}^{\otimes n}} H$$

$$\cong (L^2(A_{23})^{\otimes n} \otimes A_{23}^{\otimes n}) \otimes (L^2(A_{14})^{\otimes n} \boxtimes A_{14}^{\otimes n}) \boxtimes A_{14}^{\otimes n} H$$

$$\cong L^2(A_{23})^{\otimes n} \otimes A_{23}^{\otimes n} (L^2(A_{14})^{\otimes n} \boxtimes A_{14}^{\otimes n} L^2(A_{14})^{\otimes n} \boxtimes A_{14}^{\otimes n} H)$$

$$\cong L^2(A_{23})^{\otimes n} \boxtimes A_{23}^{\otimes n} (L^2(A_{14})^{\otimes n} \otimes A_{14}^{\otimes n} H)$$

$$\cong L^2(A_{23})^{\otimes n} \otimes A_{23}^{\otimes n} (L^2(A_{14})^{\otimes n} \boxtimes A_{14}^{\otimes n} L^2(A_{02;35})^{\otimes n} \boxtimes A_{02;35}^{\otimes n} H)$$

$$\cong L^2(A_{23})^{\otimes n} \boxtimes A_{23}^{\otimes n} (L^2(A_{05})^{\otimes n} \otimes A_{02;35}^{\otimes n})$$

of $\mathcal{A}([0, 2] \cup [3, 5])$- and $\mathcal{A}([1, 4])$-modules.

The actions of $\mathcal{A}([0, 2] \cup [3, 5])$ and of $\mathcal{A}([1, 4])$ on the Hilbert space $L^2(\otimes_n A_{23}) \otimes A_{23}^{\otimes n} (L^2(\otimes_n A_{05}) \boxtimes A_{02;35}^{\otimes n} H)$ visibly extend to an action of the von Neumann algebra $\mathcal{A}([0, 5]) = \mathcal{A}(I)$. They therefore also extend to an action of $\mathcal{A}(I)$ on $H$. \qed

4. An application

In our recent preprint [1], we introduced higher categorical analogs of von Neumann algebras called bicommutant categories. A bicommutant category is a tensor category which is equivalent to its bicommutant inside $\Bim(R)$. (The latter is the category of all bimodules over a hyperfinite factor; it plays the role of the algebra of bounded operators on a Hilbert space.) A bicommutant category is also equipped with a higher categorical analog of a $*$-structure, called a bi-involutive structure [9 Def. 2.3].

In [8], we made the following announcement: for $G$ the group $SU(n)$ and for $k$ a positive integer, the category of positive energy representations of the based loop group of $G$ at level $k$ is a bicommutant category. Moreover, its Drinfel’d center is the category of positive energy representations of the free loop group of $G$:

$$Z(\Rep^k_{\text{ens}}(\Omega G)) = \Rep^k_{\text{ens}}(LG).$$

We then argued that this result provides good evidence for our claim that the tensor category of positive energy representation of the based loop group is the value of Chern-Simons theory on a point.

Remark 7. The tensor category of positive energy representations of $LG$, as defined using conformal nets (see, e.g., [8, 15]), has, to our knowledge, not been compared to the corresponding tensor category defined using affine Lie algebras (or vertex algebras, or quantum groups). The right hand side of (7) refers to the tensor category defined in [15].
Remark 8. We expect the relation (7) to hold true for every compact connected Lie group $G$ and every level $k \in H^1(BG, \mathbb{Z})$. It is conjectured by many people that all chiral WZW conformal nets have finite index (see [6] [14] [15] [1] §4.C [11] §8 for the definition of these conformal nets in various degrees of generality). The finite index property is known when $G = SU(n)$ [10] [15], and in a few other cases. Our proof of (7) relies crucially on the fact that the chiral WZW conformal nets associated to $G$ have finite index. However, our dependence on this result is the only place where we use that $G$ is the group $SU(n)$.

We can generalize (7) to arbitrary conformal nets of finite index. The analog of (7) relies crucially on the fact that the chiral WZW conformal nets have finite index. However, our dependence on this result is the only place where we use that $G$ is the group $SU(n)$.

By definition, a soliton is a Hilbert space equipped with compatible actions of the algebras $\mathcal{A}(I)$, where $I$ ranges over all subintervals of the standard circle whose interior does not contain the base point $1 \in S^1$. Equivalently, it ranges over all subintervals $I \subset S_{\text{cut}}^1$, where $S_{\text{cut}}^1$ is the manifold obtained from the standard circle by removing its base point and replacing it by two points:

$$S^1: \quad S_{\text{cut}}^1:$$

The monoidal structure on $\mathcal{T}_A$ is defined as follows. Let $H$ and $K$ be two solitons. Let $I_+$ be the upper half of $S^1$, and let $I_-$ be its lower half. Precomposing the left action of $\mathcal{A}(I_-)$ on $H$ by the map

$$\mathcal{A}(z \mapsto \bar{z} : I_+ \to I_-) : \mathcal{A}(I_+)^{\text{op}} \to \mathcal{A}(I_-)$$

yields a right action of $\mathcal{A}(I_+)$ on $H$. We let

$$H \boxtimes K := H \boxtimes_{\mathcal{A}(I_+)} K.$$ 

Here, $\mathcal{A}(I_+)$ acts on $K$ in the usual way, and acts on $H$ on the right via the action described above. The left actions of $\mathcal{A}(I_+)$ on $H$ and of $\mathcal{A}(I_-)$ on $K$ induce corresponding actions on $H \boxtimes K$. Given an interval $J \subset S^1$, $1 \notin J$, $-1 \in J$, then, by the same argument as in [2] Def. 1.31], the actions of $\mathcal{A}(J \cap I_+)$ and $\mathcal{A}(J \cap I_-)$ on $H \boxtimes K$ extend to an action of $\mathcal{A}(J)$. By Lemma 4, it follows that for every interval $J \subset S_{\text{cut}}^1$, the actions of $\mathcal{A}(J \cap I_+)$ and $\mathcal{A}(J \cap I_-)$ on $H \boxtimes K$ extend to an action

$$\rho_J : \mathcal{A}(J) \to B(H \boxtimes K).$$

All together, these actions equip $H \boxtimes K$ with the structure of a soliton.

Given a soliton $H$, with actions $\rho_I : \mathcal{A}(I) \to B(H)$ for $I \subset S_{\text{cut}}^1$, its conjugate $\overline{\mathcal{P}}$ is the complex conjugate Hilbert space equipped with the actions

$$\mathcal{A}(I) \overset{\mathcal{A}(z \mapsto \bar{z})}{\longrightarrow} \mathcal{A}(\bar{I})^{\text{op}} \overset{\mathcal{A}(\bar{I})}{\longrightarrow} B(\overline{\mathcal{P}}) = B(\overline{\mathcal{P}}).$$

Here, $\bar{I}$ denotes the image of $I \subset S^1$ under the complex conjugation map $S^1 \to S^1$. The conjugation on $\mathcal{T}_A$ squares to the identity and satisfies $H \boxtimes K \cong \overline{K} \boxtimes \overline{H}$.

**Definition** ([11] Def. 5.3). An object $\Omega$ of a tensor category $(\mathcal{T}, \otimes)$ is called absorbing if it is non-zero and satisfies

$$(X \neq 0) \Rightarrow (X \otimes \Omega \cong \Omega \cong \Omega \otimes X) \quad \forall X \in \mathcal{T}.$$
Remark. If $\mathcal{T}$ admits a conjugation, it is a little bit easier to check that an object is absorbing. $\Omega \in \mathcal{T}$ is absorbing if it is non-zero and if $X \otimes \Omega \cong \Omega$ for every $X \neq 0$; see the comments after [9, Def. 5.3] for a proof.

The next result, about the existence of absorbing objects, is a key ingredient in the proof, announced in [8], that the category of solitons of a conformal net with finite index is a bicommutant category, and that its Drinfel’d center is the category of representations of $A$. The proof relies on Lemma [5] (which is essentially equivalent to Theorem [2]).

Given a non-trivial conformal net $\mathcal{A}$, let

$$\Omega_\mathcal{A} := L^2(\mathcal{A}(S^1_{\text{cut}})) \in \mathcal{T}_{\mathcal{A}}.$$  

with actions of $\mathcal{A}(I)$ for $I \subseteq S^1_{\text{cut}}$ provided by the obvious inclusion $\mathcal{A}(I) \to \mathcal{A}(S^1_{\text{cut}})$ followed by the left action of $\mathcal{A}(S^1_{\text{cut}})$ on its $L^2$-space.

Alternatively, the soliton $\Omega_\mathcal{A}$ can be described as follows. Let its underlying Hilbert space be the vacuum Hilbert space $H_0$ of the conformal net $\mathcal{A}$. Given an interval $I = [e^{ia}, e^{ib}] \subseteq S^1$ with $0 \leq a < b \leq 2\pi$, let $\sqrt{I} := [e^{ia/2}, e^{ib/2}]$. The square root function induces an isomorphism $\mathcal{A}(I) \to \mathcal{A}(\sqrt{I})$. For $I \subseteq S^1$, $1 \notin I$, the action of $\mathcal{A}(I)$ on $H_0$ is the composite $\mathcal{A}(I) \to \mathcal{A}(\sqrt{I}) \to B(H_0)$ of the above isomorphism with the standard action of $\mathcal{A}(\sqrt{I})$ on $H_0$.

The equivalence between the above two descriptions of $\Omega_\mathcal{A}$ is provided by the linear map $L^2(\mathcal{A}(\sqrt{I})) : L^2(\mathcal{A}(S^1_{\text{cut}})) \to L^2(\mathcal{A}([e^{ia}, e^{ib}])) = H_0$.

Theorem 9. Let $\mathcal{A}$ be a non-trivial conformal net. Then $\Omega_\mathcal{A} \in \mathcal{T}_{\mathcal{A}}$ is characterized up to isomorphism by the following three properties:

(a) it is non-zero,

(b) $\Omega_\mathcal{A} \oplus \Omega_\mathcal{A} \cong \Omega_\mathcal{A}$, and

(c) the actions of $\mathcal{A}(I)$ for $I \subset S^1$, $1 \notin I$, factor through an action of $\mathcal{A}(S^1_{\text{cut}})$.

Moreover, if $\mathcal{A}$ has finite index, then $\Omega_\mathcal{A}$ is an absorbing object.

Proof. We first check that $\Omega_\mathcal{A}$ satisfies the above three properties. The first one is obvious. The third one holds by construction. The second property is a consequence of Lemma [10] below: since $\mathcal{A}(S^1_{\text{cut}})$ is an infinite factor, the Hilbert spaces $L^2(\mathcal{A}(S^1_{\text{cut}})) \oplus L^2(\mathcal{A}(S^1_{\text{cut}}))$ and $L^2(\mathcal{A}(S^1_{\text{cut}}))$ are isomorphic as left $\mathcal{A}(S^1_{\text{cut}})$-modules. It follows that $\Omega_\mathcal{A} \oplus \Omega_\mathcal{A} \cong \Omega_\mathcal{A}$.

Let $\Omega'$ be another soliton that satisfies the same three properties. Then $\Omega'$ is naturally an $\mathcal{A}(S^1_{\text{cut}})$-module. By the classification of modules over factors, there is a unique non-zero $\mathcal{A}(S^1_{\text{cut}})$-module (up to isomorphism) that satisfies $\Omega' \oplus \Omega' \cong \Omega'$. It follows that $\Omega' \cong \Omega_\mathcal{A}$.

Let us now assume that $\mathcal{A}$ has finite index. Given a non-zero soliton $X$, we need to show that $X \boxtimes \Omega_\mathcal{A} \cong \Omega_\mathcal{A}$. Equivalently, we need to show that $X \boxtimes \Omega_\mathcal{A}$ satisfies the three properties listed above. The first property, $X \boxtimes \Omega_\mathcal{A} \neq 0$, holds because fusing over a factor sends non-zero Hilbert spaces to non-zero Hilbert spaces (see, e.g., [1, Prop. 5.2]). The second property, $X \boxtimes \Omega_\mathcal{A} \boxplus X \boxtimes \Omega_\mathcal{A} \cong X \boxtimes \Omega_\mathcal{A}$, is an immediate consequence of the corresponding property of $\Omega_\mathcal{A}$. The third property is more tricky and its verification will occupy the rest of this proof.

Let $\mathcal{I}_0$ be the collection of all subintervals of $S^1$ whose interior does not contain the base point $1 \in S^1$. Equivalently, $\mathcal{I}_0$ is the collection of all subintervals $I \subset S^1_{\text{cut}}$. By definition, a soliton is a Hilbert space equipped with compatible actions of all
the algebras \( A(I) \) for \( I \in \mathcal{I}_0 \). Let \( I_1 := [e^{i0}, e^{i\pi/2}] \) and \( I_4 := [e^{i3\pi/2}, e^{i2\pi}] \) be the first and fourth quadrants of the standard circle, and let \( I_{14} := I_1 \cup I_4 \subset S^1_{\text{cut}} \) be their disjoint union (whereas \( I_1 \) and \( I_4 \) are not disjoint in \( S^1 \), these intervals are disjoint when viewed as subsets of \( S^1_{\text{cut}} \)). The collection \( \mathcal{I}_0 \) is not a 2-cover of \( S^1_{\text{cut}} \)

Now, given two intervals \( I_1 \) and \( I_4 \) as above.

**Proof.** The algebra \( A(I_1) \) is infinite dimensional as it contains infinitely many non-trivial commuting subalgebras.

Let \( I_0 \) be the upper half of the standard circle, so that the vacuum sector \( H_0 \) is \( L^2(A(I_0)) \). Assume by contradiction that the algebra \( A(I_0) \) is of type \( II_1 \). Then the von Neumann dimension of \( H_0 \) as an \( A(I_0) \)-module is equal to 1. By diffeomorphism covariance, for every interval \( I \subset S^1 \), the dimension of \( H_0 \) as an \( A(I) \)-module is also 1. Given two intervals \( I \subseteq J \subset S^1 \), we have

\[
\dim_{A(J)}(H_0) = [A(J) : A(I)] \cdot \dim_{A(J)}(H_0).
\]
It follows that $[A(J) : A(I)] = 1$. The inclusion $A(I) \to A(J)$ is therefore an isomorphism, a contradiction. \qed

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