A note on the action-angle variables for the rational Calogero–Moser system

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Abstract

A relationship between the action-angle variables and the canonical transformation relating the rational Calogero–Moser system to the free one is discussed.
The aim of this note is to answer the question of S. Ruijsenaars [1] concerning the relationship between the action-angle variables [2] for the rational Calogero-Moser model [3] and the equivalence of the latter to free particle systems described explicitly with the help of $sl(2, \mathbb{R})$ dynamical symmetry in [4].

We begin by recalling the construction of the canonical transformation in [4]. This construction is based on the observation that many features of the rational Calogero-Moser model with the Hamiltonian

$$H_{CM} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j)^2},$$

where $p_i, q_i$ are the canonical variables, and $g$ is a coupling constant, can be explained in terms of the dynamical $sl(2, \mathbb{R})$ symmetry. Consider the following four functions on the phase space:

$$T_+ = \frac{1}{\omega}H_{CM}, \quad T_- \equiv \omega \sum_{i=1}^{N} \frac{q_i^2}{2}, \quad T_0 \equiv \frac{1}{2} \sum_{i=1}^{N} q_i p_i, \quad \bar{T}_+ = \frac{1}{\omega} \sum_{i=1}^{N} \frac{p_i^2}{2},$$

where $\omega \neq 0$ is a parameter. One easily checks that each of the sets $\{T_+, T_-, T_0\}$ and $\{\bar{T}_+, T_-, T_0\}$ spans the $sl(2, \mathbb{R})$ Lie algebra with respect to the Poisson brackets, i.e.,

$$\{T_0, T_\pm\} = \pm T_\pm, \quad \{T_-, T_+\} = 2T_0,$$

and

$$\{T_0, \bar{T}_+\} = \bar{T}_+, \quad \{T_0, T_-\} = -T_-, \quad \{T_-, \bar{T}_+\} = 2T_0.$$

These $sl(2, \mathbb{R})$ algebras act on the phase space in the standard way by means of the Poisson brackets. The action can be integrated to the symplectic action of the $SL(2, \mathbb{R})$ group. In the construction of the transformation from the Calogero-Moser system to free particles an important role is played by the following one-parameter family of canonical transformations

$$q_k \to e^{i\lambda T_1} \ast q_k \equiv \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \{T_1, \ldots, \{T_1, q_k\} \ldots\},$$

$$p_k \to e^{i\lambda T_1} \ast p_k \equiv \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \{T_1, \ldots, \{T_1, p_k\} \ldots\}, \quad (1)$$
where \( T_1 = \frac{i}{2}(T_+ + T_-) \). Since \( T_1 = \frac{i}{2\omega} H_C \), where

\[
H_C = \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} \right) + \frac{g}{2} \sum_{i,j=1}^{N} \left( \frac{1}{(q_i - q_j)^2} \right)
\]

is the Hamiltonian of the Calogero model, the transformation (1) can be viewed as the time evolution generated by the Calogero Hamiltonian \( H_C \), with the time \( t = \lambda / 2\omega \).

On the other hand the transformation (2) is simply a rotation in the space spanned by \( T_0, T_\pm \) about the axis \( T_1 \) by an angle \( \lambda \). Thus for \( \lambda = \pi \) (i.e., \( t = \pi / 2\omega \)) we have \( H_{CM} = \omega T_+ \rightarrow \omega e^{i\pi T_1} \ast T_+ = \omega T_- \). Next we can make a rotation in the space spanned by \( T_0, T_-, \tilde{T}_+ \) about the axis \( \tilde{T}_1 = \frac{i}{2}(\tilde{T}_+ + T_-) \) through the angle \(-\pi\). In particular, this will rotate \( T_- \) to \( \tilde{T}_+ \). Since the latter is proportional to the Hamiltonian of the free theory \( H_0 = \sum_{i=1}^{N} \frac{\tilde{p}_i^2}{2} \), the canonical transformation obtained by the combination of two rotations transforms the rational Calogero-Moser model to the free particle theory, i.e.,

\[
H_{CM} \rightarrow e^{-i\pi \tilde{T}_1} \ast (e^{i\pi T_1} \ast H_{CM}) = H_0.
\]

Furthermore, this transformation sends the standard integrals of motion of the Calogero-Moser model \( \frac{1}{m} \text{Tr}(L^m) \), \( m = 1, \ldots, N \), where \( L \) is the Lax matrix,

\[
L_{jk} = \delta_{jk}p_k + (1 - \delta_{jk}) \frac{ig}{q_j - q_k}, \quad (2)
\]

to their free counterparts (obtained by setting \( g = 0 \)). The same applies to the functions \( \text{Tr}(Q L^m) \), \( m = 1, \ldots, N \), with \( Q = \text{diag}(q_1, q_2, \ldots, q_N) \).

The Ruijsenaars construction of the action-angle variables for the Calogero-Moser model can be most clearly explained in terms of the Hamiltonian reduction [5]. We now briefly recall how the reduction procedure can be applied to the Calogero-Moser model [5]. One starts with the space of pairs \((A, B)\) of \( N \times N \) hermitian matrices. This space is equipped with the symplectic form

\[
\Omega = \text{Tr}(dB \wedge dA). \quad (3)
\]

The action of the unitary group \( U(N) \),

\[
U \in U(N) : \quad (A, B) \rightarrow (UAU^\dagger, UBU^\dagger), \quad (4)
\]
preserves the form $\Omega$ in (3) and thus is a symplectic action. The reduced phase space is obtained with the help of the momentum map equation

$$i[A, B] = g(I - \nu^\dagger \otimes \nu), \quad \nu = (1, 1, \ldots, 1).$$

Using the symplectic action of the group $U(N)$ in equation (4), one can fix a gauge in which $A = \text{diag}(q_1, q_2, \ldots, q_N)$. In this gauge $B$ is the Lax matrix in equation (2), and $\Omega$ takes the standard form $\Omega = \sum_{i=1}^N dp_i \wedge dq_i$. Thus we conclude that the Calogero-Moser model can be obtained by the Hamiltonian reduction of a simple dynamical system in $\Gamma$ defined by the Hamiltonian $H = \frac{1}{2} \text{Tr} B^2$.

On the other hand the symplectic transformation $A \mapsto \tilde{A} = B$, $B \mapsto \tilde{B} = -A$ preserves the momentum map. Following Ruijsenaars we can fix a gauge in which $\tilde{A} = B$ is diagonal, i.e., $\tilde{A} = \text{diag}(I_1, I_2, \ldots, I_N)$. In this gauge the Hamiltonian $H = \frac{1}{2} \text{Tr} B^2$ is $H = \frac{1}{2} \sum_{i=1}^N I_i^2$. Clearly, the variables $I_1, \ldots, I_N$ are constants of motion and together with the diagonal elements $-\phi_1, \ldots, -\phi_N$ of $\tilde{B} = -A$ in this gauge, form the complete set of canonical variables. Thus we conclude that $(\phi_i, I_i)$ are the action-angle variables for the matrix model.

It is now not difficult to relate this construction of action-angle variables to that of the canonical map [4] recalled at the beginning of this note. The action of the $\mathfrak{sl}(2, \mathbb{R})$ symmetry on the reduced phase space can be lifted to $\Gamma$. Using the explicit form of the Poisson brackets induced by the canonical form $\Omega$ (3), $\{A_{ij}, B_{kl}\} = \delta_{il} \delta_{jk}$ one easily verifies that the functions

$$t_+ = \frac{1}{2\omega} \text{Tr} B^2, \quad t_- = \frac{\omega}{2} \text{Tr} A^2, \quad t_0 = \frac{1}{2} \text{Tr} AB,$$

generate the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, i.e., $\{t_0, t_{\pm}\} = \pm t_{\pm}, \{t_-, t_+\} = 2t_0$. The relationship between the actions of $\mathfrak{sl}(2, \mathbb{R})$ on the unreduced and reduced phase spaces can be
summarised in the following commutative diagram:

Again using the explicit form of the Poisson brackets one finds that $t_0, t_{\pm}$ act linearly on $A, B$. This means that for any fixed $k, l$, $(A_{kl}, B_{kl})$ is an $sl(2, \mathbb{R})$ doublet. Therefore a general $sl(2, \mathbb{R})$ transformation of $\Gamma$ can be represented as

$$\left( \begin{array}{c} A \\ \frac{1}{\omega} B \end{array} \right) \rightarrow \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left( \begin{array}{c} A \\ \frac{1}{\omega} B \end{array} \right), \quad \alpha\delta - \beta\gamma = 1. \quad (5)$$

The transformation (5) is a lift of the $sl(2, \mathbb{R})$ action on the reduced phase space. Thus, in particular, the lift of the canonical transformation induced by $e^{i\pi T_1}$ (cf. equation (4)), must be of the form (4). We have

$$\left( \begin{array}{c} A \\ \frac{1}{\omega} B \end{array} \right) \rightarrow e^{i\pi T_1} \left( \begin{array}{c} A \\ \frac{1}{\omega} B \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} A \\ \frac{1}{\omega} B \end{array} \right) = \left( \begin{array}{c} \frac{1}{\omega} B \\ -A \end{array} \right).$$

This shows that the Ruijsenaars procedure corresponds to the lifting of the construction of the canonical mapping of the Calogero-Moser system to free particles in [4]. One has to keep in mind, however, that the diagonal elements of $B$ are viewed as momentum variables in the Ruijsenaars approach while in [4] they are proportional to the position variables. This explains the need for the additional transformation $e^{-i\pi T_1}$ which exchanges the momentum and position variables (and kills the factor $\omega$).

The reasoning presented above explains also in a straightforward way why the functions $\text{Tr} L^n = \text{Tr} B^n$ and $\text{Tr}(Q L^n) = \text{Tr}(A B^n)$ are transformed to their free counterparts, while it is no longer the case for $\text{Tr}(Q^m L^n)$, $m \geq 2$. The point is that the $\text{Tr} B^n$ and $\text{Tr}(A B^n)$ depend only on the eigenvalues of $B$ and diagonal elements of $A$ in the gauge in which $B$ is diagonal, while the $\text{Tr}(Q^m L^n)$, $m \geq 2$ depend on non-diagonal elements of $A$ too.
One can quantise the matrix theory on unreduced phase space $\Gamma$. Since the action of $\mathfrak{sl}(2,\mathbb{R})$ is linear, it can easily be implemented on the quantum level too. Then one can use the quantum Hamiltonian reduction \[8\] and carry the Ruijsenaars procedure over to the quantum case (for a different approach see \[9\]). At this point the main advantage of the procedure producing the symplectic map in \[4\] is that it can be immediately quantised.

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References

[1] S.N.M. Ruijsenaars, private communication during the Needs ’99 conference.

[2] S.N.M. Ruijsenaars, Commun. Math. Phys. 115 (1988), 127.

[3] F. Calogero, J. Math. Phys. 12 (1971), 419;
    F. Calogero, G. Marchioro, J. Math. Phys. 15 (1974), 1425;
    J. Moser, Adv. Math. 16 (1975) 1.

[4] T. Brzeziński, C. Gonera and P. Maślanka, Phys. Lett. A 254 (1999), 185.

[5] D. Kazhdan, B. Kostant and S. Sternberg, Commun. Pure Appl. Math. 31 (1978), 481.

[6] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 71 (1981), 313; A. Gorsky, Theor. Math. Phys. 103 (1995), 681.

[7] J. Moser [in:] Dynamical Systems, Progress in Mathematics No. 8, Brikhaüser, Boston, 1980, p. 233.

[8] A.P. Polychronakos, Phys. Lett. B 266 (1991), 29;
    A. Gorsky and N. Nekrasov, Nucl. Phys. B 414 (1994), 213; P. Etingof, J. Math. Phys. 36 (1995), 2636.
[9] S.N.M. Ruijsenaars, Commun. Math. Phys. 110 (1987), 191.