Sharp $p$-Poincaré inequalities under measure contraction property

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Abstract

We obtain sharp estimate on $p$-spectral gaps, or equivalently optimal constant in $p$-Poincaré inequalities, for metric measure spaces satisfying measure contraction property. We also prove the rigidity for the sharp $p$-spectral gap.

Keywords: $p$-Poincaré inequality, $p$-spectral gap, $p$-Obata theorem, curvature-dimension condition, measure contraction property, metric measure space.

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1 Introduction

Sharp estimates on spectral gap for $p$-Laplacian, or equivalently, the optimal constant in $p$-Poincaré inequalities is a classical problem in comparison geometry. It addresses the following basic problem. Given a family $\mathcal{F} := \{(X_{\alpha}, d_{\alpha}, m_{\alpha}) : \alpha \in \mathcal{A}\}$

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of metric measure spaces, the corresponding optimal constant \( \lambda_{F} \) in \( p \)-Poincaré inequalities is defined by

\[
\lambda_{F} := \inf_{\alpha \in \mathcal{A}} \left\{ \frac{\int_{X_{\alpha}} |\nabla d_{\alpha} f|^{p} \, d\alpha}{\int_{X_{\alpha}} |f|^{p} \, d\alpha} : f \in \text{Lip} \cap L^{p}, \int_{X_{\alpha}} f |f|^{p-2} \, d\alpha = 0, f \neq 0 \right\},
\]

(1.1)

where the local Lipschitz constant \( |\nabla d_{\alpha} f| : X_{\alpha} \mapsto \mathbb{R} \) is defined by

\[
|\nabla d_{\alpha} f|(x) := \lim_{y \to x} \frac{|f(y) - f(x)|}{d_{\alpha}(y, x)}.
\]

One of the most studied families of metric measure spaces is Riemannian manifolds with lower Ricci curvature bound \( K \in \mathbb{R} \), upper dimension bound \( N > 0 \) and diameter bound \( D > 0 \). In this case, \( \lambda_{F} \) is the minimum of all first positive eigenvalues of the \( p \)-Laplacian (assuming Neumann boundary conditions if the boundary is not empty). Based on a refined gradient comparison technique and a careful analysis of the underlying model spaces, sharp estimate on the first eigenvalue of the \( p \)-Laplacian was finally obtained by Valtorta and Naber in \([22, 26]\).

Another important family is weighted Riemannian manifolds (called smooth metric measure spaces) satisfying \( \text{BE}(K, N) \) curvature-dimension condition à la Bakry-Émery \([5, 6]\). More generally, thanks to the development of optimal transport theory, it was realized that Bakry-Émery’s condition in smooth setting can be equivalently characterized by convexity of an entropy functional along \( L^{2} \)-Wasserstein geodesics (c.f. \([14]\) and \([27]\)). In this direction, metric measure spaces satisfying \( \text{CD}(K, N) \) condition was introduced by Lott-Villani \([20]\) and Sturm \([24, 25]\). This class of metric measure spaces with synthetic lower Ricci curvature bound and upper dimension bound includes the previous smooth examples, and is closed in the measured Gromov-Hausdorff topology. Recently, using measure decomposition technique on Riemannian manifolds developed by Klartag \([19]\) (and by Cavalletti-Mondino \([10]\) on metric measure spaces), sharp \( p \)-Poincaré inequalities under the \( \text{BE}(K, N) \) condition and the \( \text{CD}(K, N) \) condition have been obtained by E. Calderon in his Ph.D thesis \([9]\).

In addition, Measure Contraction Property \( \text{MCP}(K, N) \) was introduced independently by Ohta \([23]\) and Sturm \([25]\) as a weaker variant of \( \text{CD}(K, N) \) condition. The family \( \text{MCP}(K, N) \) is strictly larger than \( \text{CD}(K, N) \). It was discovered by Juillet \([18]\) that the \( n \)-th Heisenberg group equipped with the left-invariant measure, which is the simplest sub-Riemannian space, does not satisfy any \( \text{CD}(K, N) \) condition but do satisfy \( \text{MCP}(0, N) \) for \( N \geq 2n + 3 \). More recently, interpolation inequalities à la Cordero-Erausquin–McCann–Schmuckenschläger \([14]\) were obtained, under suitable modifications, by Barilari and Rizzi \([8]\) in the ideal sub-Riemannian setting, Badreddine and Rifford \([4]\) for Lipschitz Carnot group, and by Balogh, Kristály and Sipos \([7]\) for the Heisenberg group. As a consequence, more and more examples of spaces verifying \( \text{MCP} \) but not \( \text{CD} \) have been found, e.g. the generalized H-type groups and the Grushin plane (for more details, see \([8]\)).

In \([17]\), the author and E. Milman proved a sharp Poincaré inequality for subsets of (essentially non-branching) \( \text{MCP}(K, N) \) metric measure spaces, whose diameter is bounded from above by \( D \). The current paper is a subsequent work of \([17]\). We will
study the general $p$-poincaré inequality within the class of spaces verifying measure contraction property. Thanks to measure decomposition theorem (c.f. Theorem 3.5 [12]), it suffices to study the corresponding eigenvalue problems on one-dimensional model spaces introduced by E. Milman [21]. In particular, we identify a family of one-dimensional MCP($K, N$)-densities with diameter $D$, not verifying CD($K, N$), achieving the optimal constant $\lambda_{p, K, N, D}$.

As a basic problem in metric geometry, the rigidity theorem helps us to understand more about the spaces under study. For the family of metric measure spaces satisfying RCD($K, N$) condition with $K > 0$, a space that reaches the equality in (1.1) must have maximal diameter $\pi \sqrt{\frac{N-1}{K}}$. By maximal diameter theorem this space is isomorphic to a spherical suspension (see [11] and references therein for details). For MCP($K, N$) spaces, the situation is very different. For $K > 0$, due to lack of monotonicity, we do not know whether a space that reaches the minimal spectrum has maximal diameter. For $K > 0$, by monotonicity (Proposition 3.6) and one-dimensional rigidity (Theorem 3.13) we can prove the rigidity Theorem 4.2.

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2 Prerequisites

Let $(X, d)$ be a complete metric space and $m$ be a locally finite Borel measure with full support. Denote by Geo($X, d$) the space of geodesics. We say that a set $\Gamma \subset Geo(X, d)$ is non-branching if for any $\gamma^1, \gamma^2 \in \Gamma$, it holds:

$$\exists t \in (0, 1) \text{ s.t. } \gamma^1_s = \gamma^2_s, \forall s \in [0, t] \Rightarrow \gamma^1_s = \gamma^2_s, \forall s \in [0, 1].$$

Let $(\mu_t)$ be a $L^2$-Wasserstein geodesic. Denote by OptGeo($\mu_0, \mu_1$) the space of all probability measures $\Pi \in P(Geo(X,d))$ such that $(e_t)_\sharp \Pi = \mu_t$ (c.f. Theorem 2.10 [1]) where $e_t$ denotes the evaluation map $e_t(\gamma) := \gamma_t$. We say that $(X, d, m)$ is essentially non-branching if for any $\mu_0, \mu_1 \ll m$, any $\Pi \in \text{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

It is clear that if $(X, d)$ is a smooth Riemannian manifold then any subset $\Gamma \subset Geo(X, d)$ is a set of non-branching geodesics, in particular any smooth Riemannian manifold is essentially non-branching. In addition, many sub-Riemannian spaces are also essentially non-branching, which follows from the existence and uniqueness of the optimal transport map on some ideal sub-Riemannian manifolds (c.f. [15]).

Given $K, N \in \mathbb{R}$, with $N > 1$, we set for $(t, \theta) \in [0, 1] \times \mathbb{R}^+$,

$$\sigma^{(t)}_{K, N}(\theta) := \begin{cases} 
\infty, & \text{if } K\theta^2 \geq (N-1)\pi^2, \\
\sin(\theta \sqrt{K/(N-1)}), & \text{if } 0 < K\theta^2 < (N-1)\pi^2, \\
t, & \text{if } K\theta^2 = 0, \\
\sinh(\theta \sqrt{-K/(N-1)}), & \text{if } K\theta^2 < 0.
\end{cases}$$
and 
\[ r^{(t)}_{K,N} := t^{\frac{1}{K}} \left( \sigma^{(t)}_{K,N-1} \right)^{1 - \frac{1}{K}}. \]

**Definition 2.1** (Measure Contraction Property MCP\((K, N)\)). We say that an essentially non-branching metric measure space \((X, d, m)\) satisfies measure contraction property MCP\((K, N)\) if for any point \(o \in X\) and Borel set \(A \subset X\) with \(0 < m(A) < \infty\) (and with \(A \subset B(o, (N - 1)/K \text{ if } K > 0)\)), there is \(\Pi \in \text{OptGeo}(\frac{1}{m(A)}m|_{A}, \delta_o)\) such that the following inequality holds for all \(t \in [0, 1]\)
\[ \frac{1}{m(A)} m \geq (e_t)_{\tau^{(1-t)}_{K,N}} (d(\gamma_0, \gamma_1))^{N} \Pi(d\gamma). \] (2.1)

**Theorem 2.2** (Localization for MCP\((K, N)\) spaces, Theorem 3.5 [12]). Let \((X, d, m)\) be an essentially non-branching metric measure space satisfying MCP\((K, N)\) condition for some \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). Then for any 1-Lipschitz function \(u\) on \(X\), the non-branching transport set \(T_u\) associated with \(u\) (roughly speaking, \(T_u\) coincides with \(\{|\nabla u| = 1\}\) up to \(m\)-measure zero set) admits a disjoint family of unparameterized geodesics \(\{X_q\}_{q \in Q}\) such that
\[ m(T_u \cup \bigcup X_q) = 0, \]
and
\[ m|_{T_u} = \int_{\Omega} m_q dq(q), \quad q(\Omega) = 1 \text{ and } m_q(X_q) = 1 \quad q \text{ - a.e. } q \in \Omega. \]
 Furthermore, for \(q\)-a.e. \(q \in Q\), \(m_q\) is a Radon measure with \(m_q \ll \mathcal{H}^1|_{X_q}\) and \((X_q, d, m_q)\) satisfies MCP\((K, N)\).

## 3 One dimensional models

### 3.1 One dimensional MCP densities

Let \(h \in L^1(\mathbb{R}^+, \mathcal{L}^1)\) be a non-negative Borel function. It is known (see e.g. Lemma 4.1 [17]) that \((\text{supp } h, |\cdot|, h\mathcal{L}^1)\) satisfies MCP\((K, N)\) condition if and only if \(h\) is a MCP\((K, N)\) density in the following sense
\[ h(tx_1 + (1-t)x_0) \geq \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)^{N-1}h(x_0) \] (3.1)
for all \(x_0, x_1 \in \text{supp } h\) and \(t \in [0, 1]\).

**Definition 3.1.** Given \(K \in \mathbb{R}, N > 1\). Denote by \(D_{K,N}\) the Bonnet–Meyers diameter upper-bound:
\[ D_{K,N} := \begin{cases} \frac{\pi}{\sqrt{K/(N-1)}} & \text{if } K > 0 \ \
+\infty & \text{otherwise}. \end{cases} \] (3.2)
For any \(D > 0\), we define \(\mathcal{F}_{K,N,D}\) as the collection of MCP\((K, N)\) densities \(h \in L^1(\mathbb{R}^+, \mathcal{L}^1)\) with \(\text{supp } h = [0, D \wedge D_{K,N}]\).
For $\kappa \in \mathbb{R}$, we define the function $s_\kappa : [0, +\infty) \mapsto \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta), & \text{if } \kappa > 0, \\ \theta, & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0. \end{cases}$$

It can be seen that (3.1) is equivalent to

$$\left( \frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left( \frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)} \right)^{N-1},$$

for all $[x_0, x_1] \subset [a, b] \subset \text{supp} \, h$.

Furthermore, we have the following characterization.

**Lemma 3.2.** Given $D \leq D_{K,N}$, a density $h$ is in $\mathcal{F}_{K,N,D}$ if and only if

$$\left( \frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left( \frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)} \right)^{N-1} \forall \ 0 \leq x_0 \leq x_1 \leq D.$$  

(3.4)

Furthermore, $h \in \mathcal{F}_{K,N,D}$ if and only if $\ln h$ is $\mathcal{L}^1$-a.e. differentiable and

$$-h(x) \cot_{K,N,D}(D-x) \leq h(x) \cot_{K,N,D}(x), \ \mathcal{L}^1 \text{- a.e. } x \in [0, D]$$

where the function $\cot_{K,N,D} : [0, D] \mapsto \mathbb{R}$ is defined by

$$\cot_{K,N,D}(x) := \begin{cases} \sqrt{K(N-1)} \cot(\sqrt{K/(N-1)} x), & \text{if } K > 0, \\ (N-1)/x, & \text{if } K = 0, \\ \sqrt{-K(N-1)} \coth(\sqrt{K/(N-1)} x), & \text{if } K < 0. \end{cases}$$

**Proof.** It can be checked that the function

$$a \mapsto \frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}$$

is non-decreasing on $[0, x_0]$, and the function

$$b \mapsto \frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}$$

is non-decreasing on $[x_1, D]$. Thus, (3.4) follows from (3.3).

Furthermore, for any $h \in \mathcal{F}_{K,N,D}$, it can be seen that (3.4) holds if and only if

$$x \mapsto \frac{(s_{K/(N-1)}(D-x))^{N-1}}{h(x)} \text{ is non-increasing},$$

(3.5)

and

$$x \mapsto \frac{(s_{K/(N-1)}(x))^{N-1}}{h(x)} \text{ is non-decreasing.}$$

(3.6)

From (3.4) we can see that $\ln h$ is locally Lipschitz, so $\ln h$ is differentiable almost everywhere. So, by (3.5) and (3.6) we know (3.4) is equivalent to

$$\left( \ln s_{K/(N-1)}^{N-1}(D-\cdot) \right)' \leq (\ln h)' \leq \left( \ln s_{K/(N-1)}^{N-1} \right)' \mathcal{L}^1 \text{- a.e. on } [0, D]$$

which is the thesis. \hfill \Box

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Notice that the function
\[ [0, D] \ni x \mapsto \frac{s_{K/(N-1)}(D - x)}{s_{K/(N-1)}(x)} \]
is decreasing. By Lemma 3.2 (or (3.5) and (3.6) ) we immediately obtain the following rigidity result.

**Lemma 3.3** (One dimensional rigidity). Denote \( h^1_{K,N,D} = \left( \frac{s_{K/(N-1)}(x)}{N-1} \right)^{N-1} \big|_{[0,D]} \) and \( h^2_{K,N,D} = \left( \frac{s_{K/(N-1)}(D - x)}{N-1} \right)^{N-1} \big|_{[0,D]} \). Then we have \( h^1_{K,N,D}, h^2_{K,N,D} \in \mathcal{F}_{K,N,D} \). Furthermore, \( h^1_{K,N,D} \) is the unique \( \mathcal{F}_{K,N,D} \) density (up to multiplicative constants) satisfying
\[ h'(x) = h(x) \cot_{K,N,D}(x) \]
and \( h^2_{K,N,D} \) is the unique \( \mathcal{F}_{K,N,D} \) density satisfying
\[ h'(x) = -h(x) \cot_{K,N,D}(D - x). \]

### 3.2 One dimensional p-Poincaré inequalities

**Definition 3.4.** For \( p \in (1, \infty) \) and \( h \in \mathcal{F}_{K,N,D} \), the p-spectral gap associated with \( h \) is defined by
\[ \lambda^{p,h} := \inf \left\{ \int |u'|^p h \, dx \middle| u \in \text{Lip} \cap L^p, \int u|u|^{p-2} h \, dx = 0, u \neq 0 \right\}. \tag{3.7} \]

**Definition 3.5.** Let \( K \in \mathbb{R}, D > 0 \) and \( N > 1 \). The optimal constant \( \lambda^{p}_{K,N,D} \) is defined as the infimum of all p-spectral gaps associated with admissible densities, i.e. \( \lambda^{p}_{K,N,D} \) is given by
\[ \lambda^{p}_{K,N,D} := \inf_{h \in \mathcal{F}_{K,N,D}^c} \lambda^{p,h}. \]

**Proposition 3.6.** Let \( K \in \mathbb{R}, D > 0 \) and \( N > 1 \). The function \( D \mapsto \lambda^{p}_{K,N,D} \) is non-increasing, and
\[ \lambda^{p}_{K,N,D} = \inf_{h \in \mathcal{F}_{K,N,D}^c \cup C^\infty} \lambda^{p,h}. \tag{3.8} \]

If \( K \leq 0 \), the map \( D \mapsto \lambda^{p}_{K,N,D} \) is strictly decreasing, and
\[ \lambda^{p}_{K,N,D} = \inf_{h \in \mathcal{F}_{K,N,D}^c \cup C^\infty} \lambda^{p,h}. \tag{3.9} \]

**Proof.** By Lemma 3.2 we know MCP densities are locally Lipschitz. Thus, using a standard mollifier we can approximate \( h \) uniformly by smooth MCP densities. Then by a simple approximation argument (see e.g. Proposition 4.8 [17]) we can prove
\[ \lambda^{p}_{K,N,D} = \inf_{h \in \mathcal{F}_{K,N,D}^c \cup C^\infty} \lambda^{p,h}. \]

Let \( h \in \mathcal{F}_{K,N,D}^c \) be a MCP density for some \( D' > 0 \), and \( u \) be an admissible function in (3.7). Then \( \tilde{h}(x) := h\left(\frac{D'}{D}x\right) \in \mathcal{F}_{K',N,D} \) with \( K' = \left(\frac{D'}{D}\right)^2 K \), and \( \tilde{u}(x) := u\left(\frac{D'}{D}x\right) \)
is also an admissible function. By computation, we have
\[
\int |\bar{u}'|^{p} \bar{h} \, dx = \int |u'|^{p} h \, dx = (D')^{p} \int |u'|^{p} h \, dx.
\]
Therefore, if \(K \leq 0\) and \(D' < D\), we have
\[
\inf_{h \in \mathcal{G}_{K,N,D}} \lambda^{p,h} \leq \inf_{h \in \mathcal{G}_{K',N,D'}} \lambda^{p,h} \leq \left( \frac{D'}{D} \right)^{p} \left( \inf_{h \in \mathcal{G}_{K,N,D'}} \lambda^{p,h} \right) < \inf_{h \in \mathcal{G}_{K,N,D'}} \lambda^{p,h}
\]
and so
\[
\lambda^{p}_{K,N,D} < \lambda^{p}_{K',N,D'}.
\]
Then we obtain (3.9). \(\square\)

Remark 3.7. The difference between the cases \(K \leq 0\) and \(K > 0\) was already observed in \([13]\) in the isoperimetric context and in \([17]\) in the 2-Poincaré context. It is known that the monotonicity property (3.9) is false when \(K > 0\).

In order to study the equation (3.18) in Theorem 3.10, we recall some basic facts about generalized trigonometric functions \(\sin_{p}\) and \(\cos_{p}\).

Definition 3.8. For \(p \in (1, +\infty)\), define \(\pi_{p}\) by
\[
\pi_{p} := \int_{-1}^{1} \frac{dt}{(1 - |t|^{p})^{\frac{1}{p}}} = \frac{2\pi}{p \sin(\pi/p)} > 0.
\]
The periodic \(C^{1}\) function \(\sin_{p} : \mathbb{R} \mapsto [-1, 1]\) is defined on \([-\pi/p, 3\pi/p/2]\) by:
\[
\begin{align*}
t &= \int_{0}^{\sin_{p}(t)} \frac{ds}{(1 - |s|^{p})^{\frac{1}{p}}} & \text{if } t \in [-\pi/p, \frac{\pi}{2}], \\
\sin_{p}(t) &= \sin_{p}(\pi/p - t) & \text{if } t \in [\frac{\pi}{2}, 3\pi/p].
\end{align*}
\]
It can be seen that \(\sin_{p}(0) = 0\) and \(\sin_{p}\) is strictly increasing on \([-\pi/p, \pi/p]\). Define \(\cos_{p}(t) = \frac{d}{dt} \sin_{p}(t)\), then we have the following generalized trigonometric identity
\[
|\sin_{p}(t)|^{p} + |\cos_{p}(t)|^{p} = 1.
\]

Definition 3.9. Let \(h_{K,N,D}^{i} : i = 1, 2\) be MCP(\(K, N\)) densities defined in Lemma 3.3. Define \(h_{K,N,D}^{i}\) by
\[
h_{K,N,D}(x) := \begin{cases} 
    h_{K,N,D}^{1}(x) & \text{if } x \in [\frac{D}{2}, D], \\
    h_{K,N,D}^{2}(x) & \text{if } x \in [0, \frac{D}{2}].
\end{cases}
\]
Define \(T_{K,N,D}\) by
\[
T_{K,N,D} := (\ln h_{K,N,D})' = \begin{cases} 
    \cot_{K,N,D}(x) & \text{if } x \in [\frac{D}{2}, D], \\
    -\cot_{K,N,D}(D - x) & \text{if } x \in [0, \frac{D}{2}].
\end{cases}
\]

By Lemma 3.2 we know \(h_{K,N,D}\) is a MCP(\(K, N\)) density. It can be seen that (c.f. Lemma 3.4 \([13]\)) \(h_{K,N,D}\) does not satisfy any forms of CD condition.
Theorem 3.10 (One dimensional $p$-spectral gap). Let $K \in \mathbb{R}$, $N > 1$, $D > 0$. Denote by $\hat{\lambda}_{K,N,D}^p$ the minimal $\lambda$ such that the following initial value problem has a solution:

\[
\begin{align*}
\varphi' &= \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T_{K,N,D} \cos_p^{p-1}(\varphi) \sin_p(\varphi), \\
\varphi(0) &= -\frac{\pi}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(D) = \frac{\pi}{2}.
\end{align*}
\] (3.11)

Then $\lambda^{n,h} \geq \hat{\lambda}_{K,N,D}^p$ for any $h \in \mathcal{F}_{K,N,D}$.

Proof. Step 1. Firstly we will show the existence of $\hat{\lambda}_{K,N,D}^p$.

By Lemma 3.2 we know $T_{K,N,D} \in C^\infty((0, \frac{D}{2}) \cup (\frac{D}{2}, D))$ and $-\cot_{K,N,D}(D - \cdot) \leq T_{K,N,D} \leq \cot_{K,N,D}$. Denote $T = T_{K,N,D}$, and denote by $u = u^{T,\lambda}$ the (unique) solution of the following equation:

\[
\begin{align*}
\begin{cases}
(u'|u|^{p-2})' + Tu'|u|^{p-2} + \lambda u|u|^{p-2} &= 0, \\
u\left(\frac{D}{2}\right) &= 0.
\end{cases}
\] (3.12)

Next we will study the equation (3.12) using a version of the so-called Pf"{u}fer transformation. Define the functions $e = e^{T,\lambda}$ and $\varphi = \varphi^{T,\lambda}$ by:

\[
\alpha := \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}, \quad \alpha u = e \sin_p(\varphi), \quad u' = e \cos_p(\varphi).
\]

By Lemma 3.11 we know that $\varphi, e$ solve the following equation:

\[
\begin{align*}
\begin{cases}
\varphi' &= \alpha + \frac{1}{p-1} T|\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\frac{d}{dt} \ln e &= \frac{\alpha}{e} = -\frac{1}{p-1} T|\cos_p(\varphi)|^p.
\end{cases}
\] (3.13)

Consider the following initial valued problem on $(0, \frac{D}{2}) \cup (\frac{D}{2}, D)$.

\[
\begin{align*}
\begin{cases}
\varphi' &= \alpha + \frac{1}{p-1} T|\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi\left(\frac{D}{2}\right) &= 0.
\end{cases}
\] (3.14)

By Cauchy’s theorem we have the existence, uniqueness and continuous dependence on the parameters. Fix an $\epsilon \in (0, \frac{D}{2})$. We can find $\alpha = \alpha(\epsilon) > 0$, such that $\varphi'(x) > \frac{\pi}{pD-2\epsilon} > 0$ for all $x \in (\epsilon, \frac{D}{2})$. So there exists $a_\alpha \in [0, \frac{D}{2})$ such that $\varphi(a_\alpha) = -\frac{\pi}{2}$. Similarly, there is $b_\alpha \in (\frac{D}{2}, D]$ such that $\varphi(b_\alpha) = \frac{\pi}{2}$. Conversely, assume there is $\alpha > 0$ such that the following problem has a solution $\varphi$ for some $a_\alpha \in [0, \frac{D}{2})$ and $b_\alpha \in (\frac{D}{2}, D]$:

\[
\begin{align*}
\begin{cases}
\varphi' &= \alpha + \frac{1}{p-1} T|\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi(a_\alpha) &= -\frac{\pi}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(b_\alpha) = \frac{\pi}{2}.
\end{cases}
\] (3.15)

Then for any $\alpha' > \alpha$, the following problem also has a solution for some $a'_\alpha \in (a_\alpha, \frac{D}{2})$ and $b'_\alpha \in (\frac{D}{2}, b_\alpha)$:

\[
\begin{align*}
\begin{cases}
\varphi' &= \alpha' + \frac{1}{p-1} T|\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi(a'_\alpha) &= -\frac{\pi}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(b'_\alpha) = \frac{\pi}{2}.
\end{cases}
\] (3.16)
Therefore, by connectedness, there is a minimal \( \bar{\lambda} \geq 0 \) such that for any \( \lambda > \bar{\lambda} \), there exist \( \varphi = \varphi^{T,\lambda} \), \( 0 \leq a^\lambda < \frac{D}{2} \) and \( \frac{D}{2} < b^\lambda \leq D \) such that

\[
\begin{align*}
\varphi' &= \left( \frac{\lambda}{p-1} \right)^{\frac{1}{p}} + \frac{1}{p-1} \cos_{p}^{\varphi}(\varphi) \sin_{p}(\varphi), \\
\varphi(a^\lambda) &= -\frac{\pi}{2}, \varphi(D) = \frac{\pi}{2},
\end{align*}
\tag{3.17}
\]

By continuous dependence on the parameter \( \lambda \), we know (3.17) has a solution \( \varphi_{\infty} \) for \( \gamma = \tilde{\gamma} \), some \( a^\lambda \in [0, \frac{D}{2}) \) and \( b^\lambda \in (\frac{D}{2}, D] \). In particular, \( \lambda > 0 \).

Since \( T(x) = -T(D - x) \) on \([0, \frac{D}{2})\), by symmetry and minimality (or domain monotonicity) of \( \bar{\lambda} \), we have \( a^\lambda = 0 \) and \( b^\lambda = D \) (otherwise we can find a smaller \( \lambda \)). In particular, there is a minimal \( \lambda^p_{K,N,D} \) such that the initial value problem (3.11) has a solution \( \varphi^{T,K,N,D,\lambda^p_{K,N,D}} \).

**Step 2.** Given \( h \in \mathcal{F}_{K,N,D} \cap C^\infty \), we will show that \( \lambda^p_{K,N,D} \leq \lambda^{p,h} \).

First of all, by a standard variational argument we can see that \( \lambda^{p,h} \) is the smallest positive real number such that there exists a non-zero \( u \in W^{1,p}([0, D], h\mathcal{L}^1) \) solving the following equation (in weak sense):

\[
\Delta^h_p u = -\lambda u |u|^{p-2}
\tag{3.18}
\]

with Neumann boundary condition, where \( \Delta^h_p u \) is the weighted \( p \)-Laplacian on \(([0, D], |\cdot|, h\mathcal{L}^1)\):

\[
\Delta^h_p u = \Delta_p u + u'|u|^{p-2}(\log h)' = (u'|u|^{p-2})' + u'|u|^{p-2} \frac{h'}{h}.
\]

By regularity theory we know \( u \in C^{1,\alpha} \cap W^{1,p} \) for some \( \alpha > 0 \), and \( u \in C^{2,\alpha} \) if \( u' \neq 0 \). Conversely, for any \( u \) solving the Neumann problem (3.18), we have \( \int u|u|^{p-2} h \, dx = 0 \) and \( \int |u'|^p h \, dx = \lambda \int |u| h \, dx \).

Assume by contradiction that \( \lambda^{p,h} < \lambda^p_{K,N,D} \). From the monotonicity argument in **Step 1**, we can see that there is \( \lambda < \lambda^p_{K,N,D} \) such that the following equation has a (monotone) solution \( \varphi = \varphi^{K,h,\lambda} \):

\[
\begin{align*}
\varphi' &= \left( \frac{\lambda}{p-1} \right)^{\frac{1}{p}} + \frac{1}{p-1} k' \cos_{p}^{\varphi}(\varphi) \sin_{p}(\varphi), \\
\varphi(0) &= -\frac{\pi}{2}, \varphi(D) = \frac{\pi}{2},
\end{align*}
\tag{3.19}
\]

Without loss of generality (or by symmetry), we may assume there is \( a' \in [\frac{D}{2}, D] \) such that \( \varphi^{K,h,\lambda}(a') = 0 \). Suppose there is a point \( x_0 \in [a', D) \) such that \( \varphi^{K,h,\lambda}(x_0) = \varphi^{T,K,N,D,\lambda^p_{K,N,D}}(x_0) \). From Lemma 3.2 we know that \( \frac{k'}{n} \leq T_{K,N,D} \). So we know

\[
(\varphi^{K,h,\lambda})'(x_0) < (\varphi^{T,K,N,D,\lambda^p_{K,N,D}})'(x_0).
\]

Therefore,

\[
\varphi^{K,h,\lambda}(x) < \varphi^{T,K,N,D,\lambda^p_{K,N,D}}(x)
\]

for all \( x \in (a', D] \), which contradicts to the fact that \( \varphi^{K,h,\lambda}(\frac{D}{2}) = \varphi^{T,K,N,D,\lambda^p_{K,N,D}}(\frac{D}{2}) = \frac{\pi}{2} \). \( \square \)
The following formulas has been used in [22, 26]. We give a proof for completeness.

**Lemma 3.11.** Let $e, \varphi, T$ be functions defined in the proof of Theorem 3.10. Then we have

$$\begin{align*}
\varphi' = & \alpha + \frac{1}{p-1}T\cos_p(\varphi)\cos_p(\varphi)\sin_p(\varphi), \\
\frac{d}{dt} \ln e = & -\frac{1}{p-1}T\cos_p(\varphi)|^p.
\end{align*}$$

(3.20)

**Proof.** Firstly, we have

$$\begin{align*}
(u'|u|^{p-2})' = & (e \cos_p(\varphi)|e \cos_p(\varphi)|^{p-2})' \\
= & |e \cos_p(\varphi)|^{p-2}(e' \cos_p(\varphi) + e \sin_p(\varphi)\varphi') \\
& + e \cos_p(\varphi)(p-2)e \cos_p(\varphi)|e \cos_p(\varphi)|^{p-4}(e' \cos_p(\varphi) + e \sin_p(\varphi)\varphi') \\
= & |e \cos_p(\varphi)|^{p-2}(p-1)(e' \cos_p(\varphi) + e \sin_p(\varphi)\varphi').
\end{align*}$$

Combining with (3.12) we obtain

$$\begin{align*}
|e \cos_p(\varphi)|^{p-2}(e' \cos_p(\varphi) + e \sin_p(\varphi)\varphi') \sin_p(\varphi) \\
& + \frac{1}{p-1}Te \cos_p(\varphi)\sin_p(\varphi)|e \cos_p(\varphi)|^{p-2} + \frac{\lambda}{p-1}\alpha^{1-p}\epsilon^{p-1}|\sin_p(\varphi)|^p = 0.
\end{align*}$$

Differentiating the equation $\alpha u = e \sin_p(\varphi)$ and substituting $u'$ by $e \cos_p(\varphi)$, we get

$$\alpha e \cos_p(\varphi) = e' \sin_p(\varphi) + e \cos_p(\varphi)\varphi'.$$

Differentiating the identity $|\sin_p(t)|^p + |\cos_p(t)|^p = 1$ we also have

$$|\sin_p(t)|^{p-2}\sin_p(t) \cos_p(t) + |\cos_p(t)|^{p-2}\cos_p(t) \sin_p(t) = 0.$$ 

Therefore,

$$\begin{align*}
|e \cos_p(\varphi)|^{p-2}(e' \cos_p(\varphi) + e \sin_p(\varphi)\varphi') \sin_p(\varphi) \\
& + \frac{1}{p-1}Te \cos_p(\varphi)\sin_p(\varphi)|e \cos_p(\varphi)|^{p-2} + \frac{\lambda}{p-1}\alpha^{1-p}\epsilon^{p-1}|\sin_p(\varphi)|^p = 0.
\end{align*}$$

Combining the results above, we prove the lemma.

\[\square\]

Combining Proposition 3.6 and Theorem 3.10, we get the following corollary immediately.

**Corollary 3.12.** We have the following sharp $p$-spectral gap estimates for one dimensional models:

$$\lambda^p_{K,N,D} = \begin{cases} 
\hat{\lambda}^p_{K,N,D} & \text{if } K \leq 0 \\
\inf_{D' \in (0, \min(D,K,N))} \hat{\lambda}^p_{K,N,D'} & \text{if } K > 0
\end{cases}$$
Theorem 3.13 (One dimensional rigidity). Given $K \leq 0$, $N > 1$ and $D > 0$. If $\lambda^{p,h} = \hat{\lambda}^{p,h}_{K,N,D}$ for some $h \in F_{K,N,D}$. Then $h = h_{K,N,D}$ up to a multiplicative constant.

Proof. Assume $\lambda^{p,h} = \hat{\lambda}^{p,h}_{K,N,D}$ for some $h \in F_{K,N,D}$. Then there is $h_n \in F_{K,N,D} \cap C^\infty$ with $h_n \to h$ uniformly, and a decreasing sequence $(\lambda^{p,h_n})$ with $\lambda^{p,h_n} \to \hat{\lambda}^{p,h}_{K,N,D}$, such that $\varphi_n = \varphi_n^{h_n \lambda^{p,h_n}}$ solves the following equation:

\[
\begin{cases}
\varphi'_n = \left(\frac{\lambda^{p,h_n}}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} \frac{h'_n}{h_n} \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n), \\
\varphi_n(0) = -\frac{\pi}{2}, \varphi_n(D) = \frac{\pi}{2}.
\end{cases}
\] (3.21)

From Lemma 2.1 we know that $\{\varphi_n\}$ and $\{\varphi_n\}$ are uniformly bounded. By Arzelà-Ascoli theorem we may assume $\varphi_n \to \varphi_\infty$ uniformly for some Lipschitz function $\varphi_\infty$.

By minimality of $\hat{\lambda}^{p,h}_{K,N,D}$ and symmetry, we can see that $\lim_{n \to \infty} \varphi_n^{-1}(t)$ exists for any $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$\lim_{n \to \infty} \varphi_n^{-1} = \left(\varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}\right)^{-1}.$$ 

In fact, assume by contradiction that $\lim_{n \to \infty} \varphi_n^{-1}(t) \neq \left(\varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}\right)^{-1}(t)$ for some $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By symmetry we may assume there are $N_1 \in \mathbb{N}$ and $\delta > 0$, such that $\delta_n := \varphi_n^{-1}(t) - \varphi_n^{-1}(t) \geq \delta$ for all $n \geq N_1$. Define a MCP $(K, N)$ density $\tilde{h}_n$ by

$$\tilde{h}_n(x) := \begin{cases} h_n(x) & \text{if } x \in [0, \varphi_n^{-1}(t)], \\
\frac{h_n(\varphi_n^{-1}(t))}{h_{K,N,D}(\varphi_n^{-1}(t)+\delta_n)} h_{K,N,D}(x + \delta_n) & \text{if } x \in [\varphi_n^{-1}(t), D - \delta_n].
\end{cases}$$

Then $\tilde{\varphi}_n = \varphi_n^{h_n \lambda^{p,h_n}}$ satisfies $(\tilde{\varphi}_n)^{-1}(\frac{\pi}{2}) < D - \frac{\pi}{2}$ for $n$ large enough, which contradicts to Proposition 3.6 and the minimality of $\hat{\lambda}^{p,h}_{K,N,D}$.

In conclusion, $\varphi_\infty = \varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}$ and we have $\varphi_n \to \varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}$ uniformly.

Then we get

$$\frac{\pi_p}{2} = \varphi_n(\varphi_n^{-1}(0)) - \varphi_n(0)$$

$$= \lim_{n \to \infty} \int_0^{\varphi_n^{-1}(0)} \left(\frac{\lambda^{p,h_n}}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} \frac{h'_n}{h_n} \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n) \, dx$$

$$\leq \lim_{n \to \infty} \int_0^{\varphi_n^{-1}(0)} \left(\frac{\lambda^{p,h_n}}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T_{K,N,D} \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n) \, dx$$

$$= \int_0^{\frac{\pi_p}{2}} \left(\frac{\hat{\lambda}^{p,h}_{K,N,D}}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T_{K,N,D} \cos_p^{-1}(\varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}) \sin_p(\varphi^{T_{K,N,D},\hat{\lambda}^{p,h}_{K,N,D}}) \, dx$$

$$= \frac{\pi_p}{2}.$$
in $L^1([0, \frac{D}{2}], \cos_p^{p-1}(\varphi^{T_{K,N,D},\lambda_{K,N,D}}) \sin_p(\varphi^{T_{K,N,D},\lambda_{K,N,D}})) \mathcal{L}^1$). By symmetry, we can see that $(\ln h_n)' \to (\ln h_{K,N,D})'$ in $L^1([0, D], \mathcal{L}^1)$. Hence $h = h_{K,N,D}$ up to a multiplicative constant.

\[ \square \]

## 4 $p$-spectral gap

### 4.1 Sharp $p$-spectral gap estimates

Using standard localization argument (c.f. Theorem 1.1 [17], Theorem 4.4 [11]), we can prove the sharp $p$-Poincaré inequality with one dimensional results.

**Theorem 4.1** (The sharp $p$-spectral gap under MCP($K,N$)). Let $(X,d,m)$ be an essentially non-branching metric measure space satisfying MCP($K,N$) for some $K \in \mathbb{R}, N \in (1, \infty)$ and $\text{diam}(X) \leq D$. For any $p > 1$, define $\lambda^p_{K,d,m}$ as the optimal constant in $p$-Poincaré inequality on $(X,d,m)$:

$$
\lambda^p_{(X,d,m)} := \inf \left\{ \frac{\int \| \nabla f \|^p \, dm}{\int |f|^p \, dm} : f \in \text{Lip} \cap L^p, \int |f|^p \, dm = 0, f \neq 0 \right\}.
$$

Then we have the following sharp estimate

$$
\lambda^p_{(X,d,m)} \geq \lambda^p_{K,N,D} = \begin{cases} 
\hat{\lambda}^p_{K,N,D} & \text{if } K \leq 0 \\
\inf_{D' \in (0, \min(D,D_{K,N})]} \hat{\lambda}^p_{K,N,D'} & \text{if } K > 0.
\end{cases}
$$

**Proof.** Let $\bar{f} = f/\|f\|_p^2$ be a Lipschitz function with $\int \bar{f} = 0$. Let $\bar{f}^\pm$ denote the positive and the negative parts of $\bar{f}$ respectively. Then we have $\int \bar{f}^+ = - \int \bar{f}^-$. Consider the $L^1$-optimal transport problem from $\mu_0 := \bar{f}^+ m$ to $\mu_1 := \bar{f}^- m$. By Theorem 2.2, there exists a family of disjoint unparameterized geodesics $\{X_q\}_{q \in \Omega}$ of length at most $D$, such that

$$
m(X \setminus \cup X_q) = 0, \quad m = \int_\Omega m_q \, dq(q)
$$

where $m_q = h_q \mathcal{H}^1|_{X_q}$ for some $h_q \in \mathcal{F}_{K,N,D}$ with $D_q \leq D$, $m_q(X_q) = m(X)$ and

$$
\int \bar{f}h_q \, d\mathcal{H}^1|_{X_q} = 0
$$

for $q$-a.e. $q \in \Omega$.

Denote $f_q = \bar{f}|_{X_q}$. By definition we obtain

$$
\int |f_q|^p h_q \, d\mathcal{H}^1|_{X_q} \geq \lambda^{p,h_q} \int |f_q|^p h_q \, d\mathcal{H}^1|_{X_q} \geq \lambda^p_{K,N,D} \int |f_q|^p h_q \, d\mathcal{H}^1|_{X_q}.
$$

Notice that $|f_q'| \leq |\nabla f|$. Thus, we have

$$
\lambda^p_{K,N,D} \int |f|^p \, dm = \lambda^p_{K,N,D} \int \int_{X_q} |f_q|^p m_q \, dq(q) \leq \int \int_{X_q} |f_q'|^p m_q \, dq(q) = \int |\nabla f|^p \, dm.
$$

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4.2 Rigidity for $p$-spectral gap

In this part, we will study the rigidity for $p$-spectral gap under the measure contraction property. We adopt the notation $|Df|$ to denote the weak upper gradient of a Sobolev function $f$. We refer the readers to [2] and [16] for details about Sobolev space theory and calculus on metric measure spaces.

**Theorem 4.2** (Rigidity for $p$-spectral gap). Let $(X,d,m)$ be an essentially non-branching metric measure space satisfying MCP($K, N$) for some $K \leq 0, N \in (1, \infty)$ and $\text{diam}(X) \leq D$. Assume there is a non-zero Sobolev function $f \in W^{1,p}(X,d,m)$ with $\int f|f|^{p-2} \, dm = 0$ such that

$$\int |Df|^p \, dm - \hat{\lambda}_p^{K,N,D} \int |f|^p \, dm = 0.$$ 

Then $\text{diam}(X) = D$ and there are disjoint unparameterized geodesics $\{X_q\}_{q \in \Omega}$ of length $D$ such that $m(X \cup \bigcup X_q) = 0$. Moreover, $m$ has the following representation

$$m = \int_{\Omega} h_q \, d\mathcal{H}^1|_{X_q} \, dq(q),$$

where $h_q = T_{K,N,D}^p$ for $q$-a.e. $q \in \Omega$.

**Proof.** Similar to the proof of Theorem 4.1, we can find a measure decomposition associated with $\tilde{f} := f|f|^{p-2}$, such that

$$m(X \cup \bigcup X_q) = 0, \quad m = \int_{\Omega} m_q \, dq(q)$$

where $m_q = h_q \mathcal{H}^1|_{X_q}$ for some $h_q \in \mathcal{F}_{K,N,D'}$ with $D'_q \leq D$, $m_q(X_q) = m(X)$ and

$$\int \tilde{f} h_q \, d\mathcal{H}^1|_{X_q} = 0$$

for $q$-a.e. $q \in \Omega$.

By Theorem 7.3 [3] we know $f_q := f|_{X_q} \in W^{1,q}(X_q)$ and $|Df_q| \leq |Df|$. Then from the proof of Theorem 4.1 we can see that $\lambda_p^{\cdot, h_q} = \hat{\lambda}_p^{K,N,D}$ for $q$-a.e. $q \in \Omega$. By Proposition 3.6 we know that the function $D \mapsto \hat{\lambda}_p^{K,N,D}$ is strictly decreasing, so $D'_q = D$ and $\text{diam}(X) = D$. Finally, by Theorem 3.13 we know that $h_q = T_{K,N,D}^p$.

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