Quasi-partial sums of the generalized Bernardi integral of certain analytic functions

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Abstract.

In this short note we extend a result of Jahangiri and Farahmand [5] concerning functions of bounded turning to a more general class of functions.

1. Introduction

Let \( C \) be the complex plane. Denote by \( A \) the class of functions:

\[
f(z) = z + a_2 z^2 + \cdots
\]

which are analytic in the unit disk \( E = \{ z : |z| < 1 \} \).

In [5] Jahangiri and Farahmand studied the partial sums of the Liberal integral of the class \( B(\beta) \), which consists of functions in \( A \) satisfying \( \text{Re} f'(z) > \beta \), \( 0 \leq \beta < 1 \). Functions in \( B(\beta) \) are called functions of bounded turning. It is known that functions of bounded turning are generally univalent and close-to-convex in the unit disk. In particular they proved that the \( m \)th partial sums

\[
F_m(z) = z + \sum_{k=2}^{m} \frac{2}{k+1} a_k z^k
\]

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of the Libera integral
\[ F(z) = \int_0^z f(t) dt \] (1.3)
is also of bounded turning. Their result was stated as:

**Theorem A**

If \( \frac{1}{4} \leq \beta < 1 \) and \( f \in B(\beta) \), then \( F_m \in B \left( \frac{4\beta - 1}{3} \right) \).

Earlier, Li and Owa [6] have proved that if \( f \in A \) is univalent in \( E \), then the partial sums \( F_m(z) \) is starlike in the subdisk \( |z| < \frac{3}{8} \), the number \( \frac{3}{8} \) being the best possible.

The result of Jahangiri and Farahmand [5] naturally leads to inquistion about a more general class of functions (including \( B(\beta) \) as a special case), which was introduced in [7] by Opoola, and has been studied extensively in [2]. This is the class \( T_\alpha^n(\beta) \) consisting of functions \( f \in A \) which satisfy the inequality:

\[ \Re \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta \] (1.4)

where \( \alpha > 0 \) is real, \( 0 \leq \beta < 1 \), \( D^n (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}) \) is the Salagean derivative operator defined as:

\[ D^n f(z) = D[D^{n-1} f(z)] = z[D^{n-1} f(z)]' \] (1.5)

with \( D^0 f(z) = f(z) \) and powers in (1.4) meaning principal values only. Observe that the geometric condition (1.4) slightly modifies the one given originally in [7] (see [2]). Observe also that the class \( B(\beta) \) corresponds to \( n = \alpha = 1 \).

In a recent work we considered the generalized Bernardi integral operator given by:

\[ F(z)^\alpha = \frac{\alpha + c}{z^\alpha} \int_0^z t^{c-1} f(t)^\alpha dt, \quad \alpha + c > 0 \] (1.6)

and sharpened and extended many earlier results concerning closure, under the integral, of several classes of functions. In the present paper we define a concept of quasi-partial sums and follow a method of Jahangiri and Farahmand [5] to extend their result (Theorem A) to the class \( T_\alpha^n(\beta) \).

As we noted in [1], the binomial expansion of (1.1) gives

\[ f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \] (1.7)

where \( a_k(\alpha) \) is a polynomial depending on the coefficients of \( f(z) \) and the index \( \alpha \). Hence

\[ F(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} \frac{\alpha + c}{\alpha + k + c - 1} a_k(\alpha) z^{\alpha+k-1} \] (1.8)
and we define the \( m \)th quasi-partial sums of the integral (1.6) as follows

\[
F_m(z)^\alpha = z^\alpha + \sum_{k=2}^{m} \frac{\alpha + c}{\alpha + k + c - 1} a_k(\alpha) z^{\alpha+k-1}
\]  

(1.9)

In the next section we state the preliminary results.

2.0 Preliminary Results

We will require the following lemmas.

Lemma 2.1\([3]\)

Let \( \theta \) be a real number and \( l \) a positive integer. If \(-1 < \gamma \leq A\), then \( \frac{1}{1+\gamma} + \sum_{k=1}^{l} \frac{\cos k\theta}{k+\gamma} \geq 0 \). The constant \( A = 4.5678018 \cdots \) is the best possible.

Lemma 2.2

For \( z \in E \) and \(-1 < \gamma \leq A = 4.5678018 \cdots \), \( \text{Re} \left( \sum_{k=1}^{l} z^k \right) \geq -\frac{1}{1+\gamma} \).

Proof. Let \( z = re^{i\theta} \) where \( 0 \leq r < 1 \) and \( 0 < |\theta| \leq \pi \). Then by De Moivre’s law and the minimum principle for harmonic functions

\[
\text{Re} \left( \sum_{k=1}^{l} \frac{z^k}{k+\gamma} \right) = \sum_{k=1}^{l} \frac{r^k \cos k\theta}{k+\gamma} > \sum_{k=1}^{l} \frac{\cos k\theta}{k+\gamma}.
\]

Hence by Abel’s lemma \([8]\) and Lemma 2.1 above the conclusion follows. \( \square \)

Let \( P \) denote the class of functions of the form

\[
p(z) = 1 + c_1 z + \cdots
\]

(2.1)

normalized by \( p(0) = 1 \) and satisfy \( \text{Re} \ p(z) > 0 \) in \( E \). The next lemma concerns convolution of analytic functions with functions in \( P \). The convolution (or Hadamard product) of two power series \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) (written as \( f * g \)) is defined as \( (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \).

Lemma 2.3\([4]\)

Let \( p(z) \) be analytic in \( E \) and satisfy \( p(0) = 1 \) and \( \text{Re} \ p(z) > \frac{1}{2} \) in \( E \). For analytic function \( q(z) \) in \( E \), the convolution \( p * q \) takes values in the convex hull of the image of \( E \) under \( q(z) \).

3.0 Main Results

Theorem 3.1

Let \( f(z) \) given by (1.1) be in the class \( T_\alpha^n(\beta) \). Then

\[
\text{Re} \frac{D^n F_m(z)^\alpha}{\alpha^n z^\alpha} > 1 - \frac{2(1-\beta)(\alpha + c)}{\alpha + c + 1}, \quad \alpha + c \leq 4.5678018 \cdots
\]

(3.1)
Furthermore if \( \beta \geq \frac{1}{2} \frac{\alpha + c - 1}{\alpha + c} \), then \( F_m(z) \) belongs to some subclasses of the class \( T_n^\alpha(\beta) \).

**Proof.** From (1.7) and the condition (1.4) we have

\[
\text{Re} \left\{ 1 + \frac{1}{2(1 - \beta)} \sum_{k=2}^\infty \left( \frac{\alpha + k - 1}{\alpha} \right)^n a_k(\alpha) z^{k-1} \right\} > \frac{1}{2} \tag{3.2}
\]

Also from (1.9) we have

\[
\frac{D^n F_m(z)^\alpha}{\alpha^n z^\alpha} = 1 + \sum_{k=2}^\infty \left( \frac{\alpha + k - 1}{\alpha} \right)^n \frac{\alpha + c}{\alpha + c + k - 1} a_k(\alpha) z^{k-1} = p \ast q \tag{3.3}
\]

where

\[
p(z) = 1 + \frac{1}{2(1 - \beta)} \sum_{k=2}^\infty \left( \frac{\alpha + k - 1}{\alpha} \right)^n a_k(\alpha) z^{k-1}, \tag{3.4}
\]

\[
q(z) = 1 + 2(1 - \beta) \sum_{k=2}^\infty \frac{\alpha + c}{\alpha + c + k - 1} z^{k-1}. \tag{3.5}
\]

Thus by Lemma 2.3 and the condition (3.1), the geometric quantity \( \frac{D^n F_m(z)^\alpha}{\alpha^n z^\alpha} \) takes values in the convex hull of \( q(E) \). But Re

\[
\text{Re} q(z) = 1 + 2(1 - \beta)(\alpha + c) \text{Re} \left( \sum_{k=1}^\infty \frac{z^k}{\alpha + c + k} \right) \tag{3.6}
\]

We know from (1.6) that \( \alpha + c > 0 \). Now suppose \( \alpha + c \leq 4.5678018 \cdots \), then by taking \( l = m - 1 \) in Lemma 2.2, the real part of the series on the right of (3.6) is greater than \(- (\alpha + c + 1)^{-1}\) so that

\[
\text{Re} \frac{D^n F_m(z)^\alpha}{\alpha^n z^\alpha} = \text{Re} q(z) > 1 - \frac{2(1 - \beta)(\alpha + c)}{\alpha + c + 1}. \tag{3.7}
\]

Now observe that the real number \( 1 - \frac{2(1 - \beta)(\alpha + c)}{\alpha + c + 1} \) is nonnegative only for \( \beta \geq \frac{1}{2} \frac{\alpha + c - 1}{\alpha + c} \). Thus only in this case it is clear \( F_m(z) \) belongs to some subclasses of the class \( T_n^\alpha(\beta) \). This completes the proof. \( \Box \)

**Remark** For \( \alpha = 1 \), \( c = 0 \), the partial sums

\[
F_m(z) = z + \sum_{k=2}^m \frac{a_k}{k} z^k \tag{3.8}
\]

of the integral

\[
F(z) = \int_0^z t^{-1} f(t) dt \tag{3.9}
\]
for each \( f \in B_n(1) \), belongs to the class \( B_n(1) \) in general. More particularly, the partial sum (3.8) of the integral (3.9) of a function of bounded turning in the unit disk is also a function of bounded turning in the unit disk.

4.0 Conclusion

In the paper we defined a new concept of quasi-partial sums of the generalized Bernardi integral. We used the new concept to extend an earlier result of Jahangiri and Farahmand [5] concerning functions of bounded turning to a more general class of functions.

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