A one-velocity model of suspension flow through a high-porous medium

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Abstract. A one-velocity model for slow flows of a suspension through a high-porous medium is proposed. The model involves the momentum equation (Brinkman equation) and two continuity equations for the suspension and suspended particles. The particle migration due to spatially varying frequency of particle-matrix collisions is heuristically accounted for. As other known diffusion models, the proposed one is not universal and is valid only for simple flow patterns (e.g., shear flows). A generalization of the proposed model for deep bed filtration is also given.

1. Introduction
In this paper we propose a model for slow (creeping) flows of a dilute suspension through a high-porous medium. Such a model can be applied in oil industry for the description of hydraulic fracturing or, for example, in modeling the flows in fibrous materials.

The paper is organized as follows. First we briefly outline a version of the known diffusion model for the flow of a suspension [1, 2] that accounts for shear-induced particle migration (with no porous medium present). Next we propose a heuristic modification of the model for the case of a flow through a high-porous material that accounts for the particle migration due to spatially varying frequency of particle-matrix collisions. Then we give the solution to the plane problem of suspension flow in a porous channel between two parallel impermeable walls as an example of the application of the model and discuss possible generalizations of the model to deep bed filtration.

2. A diffusion model for concentrated suspension flow
For the phenomenological description of concentrated suspension flows, a number of one-velocity diffusion models were proposed. Although the use of these models is restricted to the flows with simple flow geometry (for example, Poiseuille and Couette flows), they have the advantage of relative simplicity. Typically, the models are applied to the class of slow flows (i.e., the ones with small particle Reynolds numbers) of suspensions of neutrally buoyant particles.

As an example we will describe a closed model for the mean velocity of suspension \( \vec{v}(x, y, z, t) \), the pressure \( p(x, y, z, t) \), and (finite) volume fraction \( \phi(x, y, z, t) \) of particles, \( x, y, z \), and \( t \) being the spatial coordinates and the time.

The suspension is assumed to be a quasi-Newtonian fluid with the stress tensor

\[
p_{ij} = -p \delta_{ij} + 2\mu(\phi)e_{ij},
\]
where \( e_{ij} \) is the strain rate tensor, \( \mu(\phi) \) is the concentration-dependent apparent viscosity, for example

\[
\mu(\phi) = \mu(0) \left( 1 - \frac{\phi}{\phi_{\text{max}}} \right)^{-1.82},
\]

where \( \phi_{\text{max}} \) is the maximum volume fraction for close-packed arrangement of particles.

The closed system comprises the continuity equation for the suspension as a whole

\[
\text{div} \, \vec{v} = 0;
\]

the momentum equation for the suspension in slow-flow case

\[
\nabla_k p_{ik} + \rho F_i = 0,
\]

where \( \nabla_k \) is the spatial derivative with respect to \( k \)th coordinate, \( F_i \) is the \( i \)th component of the body force, and \( \rho \) is the density assumed to be constant; and the continuity equation for suspended particles

\[
\frac{d \phi}{dt} + \text{div} \left( \vec{N}_{\text{collis}} + \vec{N}_{\text{visc}} + \vec{N}_{\text{Brown}} \right) = 0.
\]

The last equation involves the diffusion term that accounts for spatially varying collision frequency

\[
\vec{N}_{\text{collis}} = -K_c \cdot a^2 \phi \cdot \text{grad} \left( \dot{\gamma} \phi \right),
\]

where \( \dot{\gamma} = \sqrt{2\epsilon_{ij} \epsilon_{ij}} \) is the local shear rate, and \( a \) is the suspended particles size; the flux due to spatially varying viscosity (i.e., particle migration in the direction of low viscosity)

\[
\vec{N}_{\text{visc}} = -K_{\mu} \cdot \dot{\gamma} \phi^2 a^2 \cdot \text{grad} \ln \mu(\phi);
\]

and Brownian diffusion term

\[
\vec{N}_{\text{Brown}} = -D \cdot \text{grad} \phi,
\]

\( K_c, K_{\mu}, \) and \( D \) being some constant parameters.

The term (1) has a simple physical meaning: a particle which experiences a higher frequency of collisions from one direction than from the opposing direction will migrate in the direction of the lower collision frequency [1], the frequency being proportional to the magnitude of the local shear rate \( \dot{\gamma} \).

3. A model of suspension flow through a high-porous medium

The main result of the paper is a modification of the above-described model to the case of low-concentration suspension flow through a high-porous medium.

The most important difference between the models is that for the flows in a porous medium the term \( \vec{N}_{\text{collis}} \) accounts for particle-porous matrix collisions rather than particle-particle collisions, the flux being determined by the velocities of the particles relative the porous matrix (\( N_{\text{collis}} \sim \text{grad} (\phi \cdot |\vec{u}|) \), where \( \vec{u} \) is the Darcy velocity) rather than by the relative velocities of adjacent particles, i.e., by the shear rate (\( N_{\text{collis}} \sim \text{grad} (\dot{\gamma} \phi) \)).

So, we assume the flux \( \vec{N}_{\text{collis}} \) to be of the form

\[
\vec{N}_{\text{collis}} = -K \cdot \phi \cdot d \cdot \text{grad} (\phi \cdot |\vec{u}|),
\]

where \( d \) is a characterisitc length scale of the internal porous space, \( K \) is a dimensionless constant parameter. In case of small changes in concentration (\( \phi \approx \phi_0 \)) we may use a simplified expression

\[
\vec{N}_{\text{collis}} \approx -K \cdot \phi_0 \cdot d \cdot \text{grad} (\phi \cdot |\vec{u}|).
\]
As to other fluxes, the Brownian diffusion term retains its form
\[ \vec{N}_{\text{Brown}} = -D \cdot \text{grad} \phi, \]
and the influence of the viscosity gradients for the case of a dilute suspension is negligible:
\[ \vec{N}_{\text{visc}} \approx 0. \]

Thus the closed system of equations for the unknown values of \( p, \vec{u}, \) and \( \phi \) involves the mass conservation equation for the suspension as a whole
\[ \text{div} \vec{u} = 0; \quad (2) \]
the momentum equation for the flow through a high-porous medium (Brinkman equation)
\[ 0 = -\nabla p - \frac{\mu(\phi)}{k} \vec{u} + \mu(\phi) \cdot \nabla \vec{u}, \]
where \( k \) is the permeability of the matrix (in writing the last term, \( \mu(\phi) \) is supposed to be a slightly varying function); and the mass balance equation for the suspended particles
\[ m \frac{\partial \phi}{\partial t} + \text{div} \left( \vec{N}_{\text{mean}} + \vec{N}_{\text{collis}} + \vec{N}_{\text{Brown}} \right) = 0, \]
where \( m \) is the porosity \( (m \approx 1) \), and \( \vec{N}_{\text{mean}} = \phi \vec{u} \).

Note that equation (2) is an exact one if diffusion effects are negligibly small [3], while in general (2) should be considered as an approximation.

As a set of boundary condition at an impermeable boundary, we can pose the Navier slip condition for the Darcy velocity and the condition of zero flux of the vector \( \vec{N}_{\text{mean}} + \vec{N}_{\text{collis}} + \vec{N}_{\text{Brown}} \) across the boundary. It is significant that the former boundary condition is not generally connected with the presence of a molecular slip or with the superhydrophobic nature of the internal surface of the porous matrix [4].

4. Shear flow in a porous channel
As an example, we will consider a steady plane-parallel pressure-driven flow of a suspension in a high-porous channel between two parallel impermeable walls placed \( 2h \) apart.

We seek the solution of the form \( u = u(y), \phi = \phi(y), \) where \( y \) is the transverse coordinate (the origin being at the channel centerline), and assume, for the sake of simplicity, the viscosity to be constant, \( \mu = \text{const} \), and the variations of concentration to be small, \( \phi \approx \phi_0 = \text{const} \).

The spatial Darcy velocity distribution is completely determined from the boundary value problem
\[ 0 = i - \frac{\mu}{k} u + \mu u', \quad u(0) = 0, \quad \beta u'(h) = -u(h), \]
where \( i \) is the absolute value of the pressure gradient, and \( \beta \) is the slip length in the Navier slip boundary condition.

The integration of the continuity equation for suspended particles with regard to the zero-flux boundary condition yields
\[ K d \cdot \phi_0 \cdot \phi u + D \phi = C = \text{const}. \]
As in [1], the constant \( C \) is then determined from the normalization condition
\[ \frac{1}{h} \int_0^h \phi(y) \, dy = \phi_{\text{mean}}, \]
where $\phi_{\text{mean}}$ is the average volume fraction, with the result that

$$\phi(y) = C \frac{K}{\phi_0} \cdot d \cdot u(y) + D.$$  

This simple solution predicts the increase in the volume fraction $\phi$ near the walls of the channel. The use of a more complicated model that accounts for particulate transport due to spatially varying viscosity gives qualitatively similar results.

For the channel flow, upon introducing the auxiliary variable

$$F(y) = \int_0^y \phi(y_1) \, dy_1,$$

the solution of the problem reduces to the integration of the boundary value problem

$$0 = i - \frac{\mu(\phi)}{k} u + (\mu_1(\phi) \cdot u_y)'y, \quad u'(0) = 0, \quad \beta u'(h) = -u(h),$$

$$K d \cdot \phi \cdot (\phi u)'y + D \phi y + K_\mu \cdot |u_y| \cdot \phi^2 |(\ln \mu(\phi))|^y = 0,$$

$$F'(y) = \phi(y), \quad F(0) = 0, \quad F(h) = h \phi_{\text{mean}}.$$  

As an example of numerical calculations, the dimensionless dependencies of $u$ and $\phi$ on the transverse coordinate $y$ for a model case $\mu(\phi) = \mu_1(\phi) = 1 + \phi$, $K_\mu = 1$, $K = 10$, $D = 1$, $\phi_{\text{mean}} = 0.1$, $d = 0.1$, $a = 0.05$, $k = 1$, $\beta = 1$ are given in figure 1 (non-dimensionalization with the use of $i$, $h$, and $\mu$ as scaling factors is not denoted for simplicity sake). For the parameters chosen, a slight increase in particle concentration in near-wall regions is observed.

![Figure 1. The plots of the Darcy velocity $u(y)$ and the volume fraction $\phi(y)$ with regard to the effect of spatially varying viscosity.](image-url)

5. Deep bed filtration in a high-porous medium

The proposed model may be straightforwardly generalized to account for the particle deposition onto the internal surface of a porous matrix or, on the contrary, to the case of particle entrainment by the flow.

The one-velocity version of deep bed filtration equations [5, 6] for the case under consideration includes the continuity equation for the suspended particles

$$\frac{\partial(m\phi)}{\partial t} + \text{div} \vec{N} = \frac{\partial m}{\partial t}, \quad \vec{N} = \vec{N}_{\text{mean}} + \vec{N}_{\text{collis}} + \vec{N}_{\text{Brown}};$$
the continuity equation for the suspension as a whole (at least, provided that the mean particle flux \( \vec{N}_{\text{mean}} = \phi \vec{u} \) is much greater than the diffusive ones)

\[
\text{div} \vec{u} = 0;
\]

the momentum equation

\[
0 = -\nabla p - \frac{\mu(\phi)}{k} \vec{u} + \mu(\phi) \cdot \Delta \vec{u};
\]

and the kinetic equation for the rate of particle deposition and/or entrainment

\[
\frac{\partial m}{\partial t} = G(\vec{N}, \phi, m)
\]

with a known function \( G \). These equations make up a closed system for determining the values of \( \phi, m, p, \) and \( \vec{u} \).

6. Conclusion
In this paper we proposed some versions of a new model for dilute particulate suspension flows through high-porous media. The model accounts for the particle migration due to particle-matrix interactions and is a modification of known diffusion models for concentrated suspension flows in the absence of porous media.

For simple flow patterns (shear flows in channels, etc.) the model is expected to predict qualitative changes in particle spatial distribution as opposed to standard models for low-porous materials (e.g., concentration increase in near-wall regions for the case of channel flows).

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