Quantum dwell times

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We put forward several inherently quantum characteristics of the dwell time, and propose an operational method to detect them. The quantum dwell time is pointed out to be a conserved quantity, totally bypassing Pauli’s theorem. Furthermore, the quantum dwell time in a region for one dimensional motion is doubly degenerate. In presence of a potential barrier, the dwell time becomes bounded, unlike the classical quantity. By using off-resonance coupling to a laser we propose an operational method to measure the absorption by a complex potential, and thereby the average time spent by an incoming atom in the laser region.

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Time has traditionally been a sore subject in quantum mechanics. Since the early days of the theory, the inclusion of time observables in the usual formalism has proved problematic (for a general presentation of several aspects of time in quantum mechanics, see [1]). The advent of single atom manipulation and fantastic cooling techniques, however, requires further thinking on the quantum aspects of measurements of time. In this letter we present an analysis of the dwell time observable, in which we shall prove that some hitherto unexplored properties of the self-adjoint operator are intrinsically quantum mechanical in origin. We analyze the possibility that these aspects could be experimentally observed in two different setups, and realize that indeed current techniques will allow us the exploration of quantum properties of dwell time.

The concept of a “dwell time” for a stationary regime was first clearly made distinct from “traversal time”, “delay time” and “reflection time” by Büttiker [2]. Without explicit mention of operators, similar quantities had been presented in the literature previously as components of a “delay time”, obtained by suitable substraction of the corresponding object for the free case, and a limiting procedure, adequate for scattering processes [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Much of the discussion of the dwell time has concerned stationary states (see however [4, 5, 8, 10, 11, 14, 15]). At any rate, the following (dwell time) operator was at least implicitly, and often explicitly [5, 6], part of those definitions:

\[
\hat{t}_D = \int_{-\infty}^{+\infty} d\tau e^{i\hat{H}\tau/\hbar} \chi_D(\hat{x}) e^{-i\hat{H}\tau/\hbar}.
\]  

(1)

In this expression, \(\chi_D(\hat{x})\) is the projector on the region of interest, in which we desire to understand and compute the time a quantum particle spends. The main emphasis in the literature has lied on the fact that the average value of this observable over any given state involves contributions from both reflection and transmission situations, as well as interference terms. Some striking facts about this operator seem however to be missing from current discussions.

In particular, the operator \(\hat{t}_D\) is prima facie symmetric; but it can further be proved that it is in fact essentially self-adjoint in the cases we shall be considering. For free particles moving in one dimension, the normalizability requirement of the image of the operator determines the (initial) domain to be that of square summable wave functions which, in momentum representation, fulfill \(\psi(p) = o(\sqrt{p})\). Symmetry imposes no further constraint on the domain, which is dense, and the deficiency indices are computed to vanish. For the case of a purely scattering spectrum, this proof is translated without further ado. Additionally, notice that \(\hat{t}_D\) is a positive definite operator.

Over the common domains of definition, it is immediate to observe, although not generally known, that the Hamiltonian commutes with the dwell time operator. This should come as no surprise even to those attuned to the general meaning of Pauli’s theorem, which is concerned with a covariant time operator, i.e., an observation of an instant, since here we deal with an interval observable. This is in fact the reason for the Wigner-Smith delay time [3, 4, 5] to be defined in terms of scattering data. As a consequence, it will be of interest to diagonalize the dwell time operator in the eigenspaces of the Hamiltonian. For one dimensional motion under a Hamiltonian with no bound states, this leads generically to two different eigenvalues for the dwell time operator in each energy
eigenspace. Thus we are faced with the following logical outcome: if we were to design a measuring procedure for the dwell time observable that indeed coincides with the dwell time operator presented above, the probability density for measured dwell times would be generically bimodal for particles with small energies (in fact, the requirement that the characteristic action, given by the characteristic momentum of the particle multiplied by the length of the region in which we measure the permanence of the particle, has to be much smaller than \( \hbar \)). In an interval of length \( l \), the dwell time eigenvalues for a free particle in one dimension with momentum \( p \) are

\[
t_{\pm}(p) = \frac{2m}{\hbar} \left( \frac{\hbar}{|p|l} \right) \left( 1 \pm \frac{\hbar}{ml} \sin \frac{q}{\hbar} \right).
\]

It should be observed that \( t_+(p) \) grows without bound when \( p \to 0 \).

Furthermore, we can generally define the expected dwell time of a stationary state for a particle in one spatial dimension as its expected value in the two dimensional Hilbert space with the associated energy; for instance, for a free particle of momentum \( p \) the expected value would be \( \langle \tau_D \rangle = \frac{2m^2}{\hbar^2} \left( \frac{\hbar}{|p|l} \right) \), which actually coincides with the classical quantity.

As pointed out above, the eigenvalues of the dwell time operator are not bound from above for the free particle case. That is not the situation however when the particle faces a barrier. In this case both series of eigenvalues are bounded, which entails that there is a maximum dwell time for a quantum particle in a barrier, in sharp contrast to the classical situation, where the dwell time for particles whose energies allows them to overcome the barrier can be made as big as desired by smoothly diminishing the energy. The preceding statements can be proved as follows. First, observe that the matrix elements of the projector of the region of interest in the scattering basis \( |p^+\rangle \) can be easily computed if it encompasses completely the interaction region, since

\[
\langle p^+ | \chi_D(x) | q^+ \rangle = \delta(p-q) - \langle p^+ | \chi_D(x) | q^- \rangle,
\]

where \( \chi_D(x) \) is the complementary projector, and the explicit forms of the scattering eigenstates in the position representation outside the interaction region are known in terms of the scattering amplitudes. The possibly dangerous delta terms vanish in the end result thanks to unitarity, which in fact underpins the whole construction, and the explicit form of the eigenvalues of the dwell time operator can be obtained in terms of the transmission and reflection amplitudes. For the specific case of a square barrier of height \( V_0 \), the eigenvalues are given by

\[
t_{\pm}(p) = 2ml|p| \frac{1 \pm (\hbar/ql) \sin(ql/\hbar)}{p^2 + q^2 \pm 2mV_0 \cos(ql/\hbar)},
\]

where now \( q^2 = p^2 - 2mV_0 \). The maxima and minima of \( t_+ \) and \( t_- \) interleave, and both functions tend to the classical value for high incident momenta. In the same way as above, one can define the expected dwell time for a stationary state with momentum \( p \), which is given by \( \langle \tau_D \rangle = (t_+(p) + t_-(p))/2 \). This average dwell time is in fact what has been usually termed “dwell time” in the literature \[3\,8\,10\]. The decomposition of an average dwell time for a generic state in terms of a symmetric and an antisymmetric part was suggested by Nussenzveig \[3\], and it holds true for parity invariant hermitian Hamiltonians.

It now behooves us to propose an operational model to test the prediction that dwell times will present, for a number of particle states, a bimodal distribution. The first one that comes to mind is one based on fluorescence, in analogy to a time of arrival detection through fluorescence \[14\,15\,12\]: consider that the region of interest is illuminated by a laser on resonance with an internal transition of the particle, perpendicular to the initial motion of the particle; set this laser and the initial state so as to minimize motion transversal to the classical path of the particle; then measure the number of resonance photons emitted by the particle for each run of the experiment. It could be expected that indeed the distribution of numbers of emitted photons would be a proxy for the distribution of dwell times, and, therefore, that for some regime this distribution of emitted photons would also be bimodal. However, the regime of interest in which the bimodal distribution would be observed is actually inadequate for analysis with this operational model, since reflection and detection delays will be present due to the laser \[14\], leading to poor total signal and poor signal discrimination. In particular, for the bimodality to be observed, the characteristic interval between modes (\( \hbar/E \), where \( E \) is the particle’s energy) should be greater than the characteristic interval between successive emission of fluorescence photons \((2/\gamma + \gamma/\Omega^2)\), where \( \gamma \) is the decay constant, i.e. Einstein’s coefficient, and \( \Omega \) Rabi’s frequency); on the other hand, if reflection is to be avoided, the particle energy should be much bigger than the characteristic Rabi energy \((E \gg \hbar/\Omega)\). These conditions cannot be simultaneously met, making this procedure inadequate for revealing the bimodality of dwell times.

An operational method for determining the average dwell time does indeed exist, using absorbing potentials \[16\,17\,18\]. Consider the non hermitian Hamiltonian \( H_{V_I} \) obtained from adding to a scattering Hamiltonian a complex term of the form \(-iV_I \chi(\hat{z})\). The evolution under this Hamiltonian will result in absorption; nonetheless, it is possible to define transmission and reflection coefficients in the usual manner. Following the method described by Smith \[4\] to relate the delay time of a stationary state to the derivatives of transmission and reflection coefficients, but in this case with respect to \( V_I \), we obtain the average dwell time in the region of interest as \( \langle \tau_D(p) \rangle = \lim_{V_I \to 0} (\hbar/2) \partial_{V_I} A(p) \), where \( A(p) \) is the total absorption probability for incident momentum \( p \).

The equivalence of this quantity with \( \langle \tau_D(p) \rangle = \lim_{V_I \to 0} (\hbar/2) \partial_{V_I} A(p) \) can be readily checked. Since the absorption in a region can be tuned in a number of manners, we can check the reality of the quantum prediction at hand, differing from
the classical one; namely, that for all ingoing waves the quantum mechanical dwell time is bounded, unlike the classical one.

In particular, consider a two level system coupled in a spatial region to an off-resonance laser with large detuning, $\Delta \gg \gamma, \Omega$, where $\Delta$ is defined as the laser frequency minus the frequency of the atomic transition. The amplitude for the atomic ground state up to the first photon detection is governed then by the effective potential [19, 20],

$$V(x) = (V_R - iV_I)\chi(x) = \left(\frac{\hbar \Omega^2}{4\Delta} - i\frac{\hbar \gamma \Omega^3}{8\Delta^2}\right)\chi(x),$$

so that the average detection delay is now $4\Delta^2/\Omega^2 \gamma$. Whereas $\gamma$ is fixed for the atomic transition, both $\Omega$ and $\Delta$ may be controlled experimentally. The ratio $\Omega^2/\Delta$ can always be chosen so that the real part of $V$ remains constant. The remnant freedom can be used to set the value of the imaginary part. We desire that at most one fluorescence photon is emitted per atom, so that the fluorescence signal produced by an atomic ensemble will be proportional to the absorption probability $A$. This requirement can be met in the regime determined by $\langle \tau_D(p) \rangle \ll 4\Delta^2/\Omega^2 \gamma$; that is to say when the average time spent in the region is much smaller than the delay. After carrying out the adequate calibration to take into account the detector solid angle and efficiency, successive measurements of $A$ are carried out for different (automatically small) values of $V_I$. In this manner we can obtain an approximate value for the derivative of the absorption probability with respect to the imaginary part of the potential, and hence $\langle \tau_D(p) \rangle$. In fact, it is easier to compute $\tau_{\text{approx}} = hA/(2V_I)$, for small values of $V_I$. The limitation inherent to the procedure lies in the weak signal for small absorption, whereas strong absorption leads to larger errors when using the approximate expression $\tau_{\text{approx}}$. However, for absorption of the order of 0.2 the relative error in estimating $\langle \tau_D(p) \rangle$ with $\tau_{\text{approx}}$ is of the same order of magnitude. In fig. 1 we depict the exact average dwell time as a function of incident velocity for Cs atoms, as well as approximations given by $\tau_{\text{approx}}$ for several values of $V_I$.

To assess better the weak signal limitation of the procedure, in fig. 2 we depict both the relative error $(\langle \tau_D(p) \rangle - \tau_{\text{approx}})/\langle \tau_D(p) \rangle$ and the quotient between the average dwell time and the fluorescence delay time versus absorption, at the dwell time peak. The validity of the approximation is controlled by the smallness of this latter quotient.

In conclusion, we have signalled a number of hitherto unnoticed properties of the dwell time operator, which are intrinsically quantum mechanical, namely that it is a stationary observable and with double degeneracy in one-dimensional collisions. We have analyzed two possible operational approaches to unveil these properties, and we have detailed a procedure making use of absorption that could experimentally lead to measurement of atomic dwell times.

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