RIGOROUS DERIVATION OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS FROM THE TWO-FLUID NAVIER-STOKES-MAXWELL EQUATIONS

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Abstract. In this paper, we rigorously derive the compressible one-fluid Navier-Stokes equation from the scaled compressible two-fluid Navier-Stokes-Maxwell equations locally in time under the assumption that the initial data are well prepared. We justify the singular limit by proving the uniform decay of the error system, which is obtained by elaborate energy estimates.

1. Introduction

Besse-Degond-Deluzet [1] derived the scaled two-fluid Euler-Maxwell equation. Based on this work, assuming that the electrons and ions density are equal to \( n \), which implies \( \text{div} E = 0 \), and that the electron-neutral and ion-neutral collision frequencies \( \nu_i, \nu_e \) are ignored, we have

\[
\partial_t n + \text{div}(n u_i) = 0, \\
\tau \varepsilon (\partial_t (n u_e) + \text{div}(n u_e \otimes u_e)) - \mu \Delta u_e - (\mu + \lambda) \nabla \text{div} u_e + \eta \nabla P_e(n) \\
= -\kappa^{-1} n (E + u_e \times B) - \frac{\kappa_{ei} \beta}{\kappa^2} K n^2 (u_e - u_i), \\
\tau (\partial_t (n u_i) + \text{div}(n u_i \otimes u_i)) - \mu \Delta u_i - (\mu + \lambda) \nabla \text{div} u_i + \eta \nabla P_i(n) \\
= \kappa^{-1} n (E + u_i \times B) - \frac{\kappa_{ei} \beta}{\kappa^2} K n^2 (u_i - u_e), \\
\alpha \partial_t E - \nabla \times B = -\beta j, \quad \text{div} E = 0, \\
\partial_t B + \nabla \times E = 0, \quad \text{div} B = 0, \\
\kappa j = n(u_i - u_e).
\]

Where \( u_i, u_e \) denote the velocities, \( P_i, P_e \) stand for the pressures, \( E, B \) are the electric and magnetic field, \( j \) is the current density. The constants \( \mu \) and \( \lambda \) are the viscosity coefficients of the flow satisfying \( \mu > 0 \) and \( 2\mu + 3\lambda > 0 \). \( \varepsilon \) is the mass ratio of electron to ion, \( \tau \) is the mean time between ion-neutral collisions, \( \eta \) is the measure of the thermal energy, \( \kappa \) denotes the number of electron-neutral (or ion-neutral) collisions relative to a rotation period of an electron in the unit of \( B \) field, \( \kappa_{ei} \) stands for measure of the strength of electron-ion collisions. \( \beta \) measures the relative strength of the induced \( B \) field to the unit \( B \) field, \( \alpha \) denotes the squared reciprocal of light speed. The rate constant \( K \) evaluate collisions between electrons and ions. Equation (1.1a) is the mass balance law and (1.1b)–(1.1c) are momentum balance laws, while (1.1d)–(1.1e) are Maxwell equations.

Noted that using the fact that \( \text{div} j = 0 \) deduced by (1.1d) and (1.1f) we infer that

\[
\partial_t n + \text{div}(n u_e) = 0.
\]
Setting
\[ \frac{1}{\varepsilon} P_e = P_i = P, \quad u = u_i + \varepsilon u_e, \quad j = \frac{j}{n}, \]
(1.1) can be recast as
\[
\begin{align*}
\partial_t n + \frac{1}{1+\varepsilon} \text{div}(nu) &= 0, \\
\partial_t (nu) + \frac{1}{1+\varepsilon} \left( \text{div} \left( (nu \otimes u) + \varepsilon \kappa^2 \text{div} \left( n\tilde{j} \otimes \tilde{j} \right) \right) \right) &= -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \frac{1+\varepsilon}{\varepsilon} \Delta \nabla P(n) = \frac{\kappa}{\varepsilon} n\tilde{j} \times B, \\
\kappa \partial_t (n\tilde{j}) + \frac{\kappa}{1+\varepsilon} \kappa^2 \text{div} \left( n\tilde{j} \otimes \tilde{j} \right) &= \frac{\kappa}{1+\varepsilon} nE + \frac{\kappa}{1+\varepsilon} nu \times B + \frac{\kappa}{1+\varepsilon} \kappa^2 \nabla \text{div} \tilde{j} \quad \text{(1.3)}
\end{align*}
\]
where the second equation is obtained by the summation of \(\frac{1}{\varepsilon}(1.1b)\) and \(\frac{1}{\varepsilon}(1.1c)\), while the third one is obtained by the summation of \(-\frac{\varepsilon}{\kappa}(1.1b)\) and \(\frac{1}{\varepsilon}(1.1c)\). Inserting assumptions \(\beta = \alpha^2, \quad \alpha = \kappa^2\) and scalings \(B \to \kappa^2 B, \quad E \to \kappa E\) into (1.3), we obtain
\[
\begin{align*}
\partial_t n + \frac{1}{1+\varepsilon} \text{div}(nu) &= 0, \\
\partial_t (nu) + \frac{1}{1+\varepsilon} \left( \text{div} \left( (nu \otimes u) + \varepsilon \kappa^2 \text{div} \left( n\tilde{j} \otimes \tilde{j} \right) \right) \right) &= -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \frac{1+\varepsilon}{\varepsilon} \Delta \nabla P(n) = \frac{\kappa}{\varepsilon} n\tilde{j} \times B, \\
\kappa \partial_t (n\tilde{j}) + \frac{\kappa}{1+\varepsilon} \kappa^2 \text{div} \left( n\tilde{j} \otimes \tilde{j} \right) &= \frac{\kappa}{1+\varepsilon} nE + \frac{\kappa}{1+\varepsilon} nu \times B + \frac{\kappa}{1+\varepsilon} \kappa^2 \nabla \text{div} \tilde{j} \quad \text{(1.4)}
\end{align*}
\]
Extensive research has been done on two-fluid flows. The local existence of classical solution of (1.1) can be deduced by [10, 24, 29]. For more results about the well-posedness of the symmetrizable hyperbolic equations, see [9, 16]. In the case of the scaled Debye length), and they obtained the incompressible Euler equation by asymptotic expansions. A same result can be extended to two-fluid Euler-Maxwell system since the energy estimates for Euler equations and Maxwell equations can be carried out separately. For the non-relativistic limit, Yang-Wang [28] obtained the two-fluid compressible Euler-Poisson system while for the combined non-relativistic and quasi-neutral limit, Li-Peng-Xi [15] obtained the one-fluid compressible Euler equations. For more results on asymptotic limits with small parameters of two-fluid flows, we refer to [3, 6, 11, 27] and the references therein. For the asymptotic limit problem with boundary effects, see [7, 13] and the references therein.

For the one-fluid flows, Peng-Wang [19] considered the combined non-relativistic and quasi-neutral limit of the Euler-Maxwell system in the scaling case if the small parameters \(\gamma\) and \(\varepsilon\) are equal (where \(\gamma\) is the inversely proportional to the light speed and \(\varepsilon\) stands for the scaled Debye length), and they obtained the incompressible Euler equation by asymptotic expansions. As an improved work, Peng-Wang [20] rigorously proved the e-MHD
equations as the quasi-neutral limit of compressible one-fluid Euler-Maxwell equations through an elaborate nonlinear energy method. Noted that this result can be extended to the combined non-relativistic and quasi-neutral limit without any relation between $\gamma$ and $\varepsilon$. Later, Jiang-Li \[5\] verified the compressible MHD limit of the electromagnetic fluid system. Yang-Wang \[29\] obtained the incompressible Navier-Stokes equations as the combined non-relativistic and quasi-neutral limit of compressible Navier-Stokes-Maxwell equations. For more results on asymptotic limits with small parameters for one-fluid flows, see \[2,18\]. For the asymptotic limit problem in a bounded domain, see \[3,11,21,23,25\] and the references therein.

For the asymptotic convergence of \[(1.1)\] mentioned in \[1\], there are few theoretical justifications. In this paper, we derive the compressible Navier-Stokes equation from \[(1.1)\] rigorously as $\kappa \to 0$, which means that the density of the plasma is large, and $\varepsilon \to 0$.

Before proceeding, let us first introduce the notations and lemmas used throughout this paper. We denote by $H^l(\mathbb{R}^3)$ the standard Sobolev’s space in the whole space $\mathbb{R}^3$, and shorted by $\| \cdot \|_l$ the norm of the Banach space $H^l(\mathbb{R}^3)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial^\alpha = \partial^{\alpha_1} \partial^{\alpha_2} \partial^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $A \leq B$ means that $A \leq C B$ for a constant $C > 0$.

The following basic Moser-type calculus inequalities \[11,12\] will be used frequently in the proof of the main result.

**Lemma 1.1.** For any nonnegative multi-index $\alpha$ with $|\alpha| \leq s$ and $f, g \in H^s$, it holds
\[
\| \partial_x^\alpha (fg) \|_{L^2} \leq C_s \left( \| f \|_{L^\infty} \| D_x^s g \|_{L^2} + \| g \|_{L^\infty} \| D_x^s f \|_{L^2} \right),
\]
and
\[
\| \partial_x^\alpha (fg) - f \partial_x^\alpha g \|_{L^2} \leq C_s \left( \| D_x^s f \|_{L^\infty} \| D_x^{s-1} g \|_{L^2} + \| g \|_{L^\infty} \| D_x^s f \|_{L^2} \right).
\]

Next, we will provide a formal asymptotic analysis of \[(1.4)\], which will shed light on its links with the compressible Navier-Stokes equation. In order to emphasize the unknowns depending on the singular perturbation parameter $\kappa$, we rewrite the system \[(1.4)\] as
\[
\begin{aligned}
\partial_t n^n + \frac{1}{1+\varepsilon} \text{div}(n^n u^n) &= 0, \\
\partial_t (n^n u^n) + \frac{1}{1+\varepsilon} \left( \text{div} (n^n u^n \otimes u^n) + \varepsilon n^n \nabla \text{div} (n^n j^n \otimes j^n) \right) &= -\mu \Delta u^n - \mu(\mu + \lambda) \nabla \text{div} u^n + \frac{\mu(1+\varepsilon) n^n}{\tau} \Delta j^n \times B^n, \\
k \partial_t (n^n j^n) + \frac{1}{1+\varepsilon} n^n \nabla \text{div} (n^n j^n \otimes j^n) &= \frac{1}{1+\varepsilon} n^n (\nabla \text{div} (n^n u^n \otimes j^n) + \nabla (n^n j^n \otimes u^n)) - \mu(\mu + \lambda) \nabla \text{div} j^n \\
&= \frac{1}{1+\varepsilon} n^n E^n \nabla + \frac{1}{1+\varepsilon} n^n \nabla u^n \times B^n + \frac{\varepsilon}{\tau} \nabla \kappa (n^n j^n \times j^n) - \frac{1}{1+\varepsilon} n^n \kappa \nabla (n^n j^n), \\
k \partial_t E^n - \nabla \times B^n &= -k^2 n^n j^n, \quad \text{div} E^n = 0, \\
k \partial_t B^n + \nabla \times E^n &= 0, \quad \text{div} B^n = 0,
\end{aligned}
\]
which equipped with the initial data
\[
(n^n, u^n, j^n, E^n, B^n) \big|_{t=0} = \left( n^n_0(x), u^n_0(x), j^n_0(x), E^n_0(x), B^n_0(x) \right).
\]
Setting $\kappa \to 0$, we formally get the one-fluid compressible Navier-Stokes system:
\[
\begin{aligned}
\partial_t n^0 + \frac{1}{1+\varepsilon} \text{div}(n^0 u^0) &= 0, \\
\partial_t (n^0 u^0) + \frac{1}{1+\varepsilon} \text{div}(n^0 u^0 \otimes u^0) &= -\mu \Delta u^0 - \mu(\mu + \lambda) \nabla \text{div} u^0 + \frac{\mu(1+\varepsilon) n^n}{\tau} \Delta j^n \times B^n = 0,
\end{aligned}
\]
with the initial data
\[
(n^0, u^0) \big|_{t=0} = \left( n^0_0(x), u^0_0(x) \right).
\]
Then letting $\varepsilon \to 0$, we derive the standard compressible Navier-Stokes equation formally. In this paper, we only prove the case of $\kappa \to 0$ rigorously. For $\varepsilon \to 0$, it is similar and much simpler. For solvability of $(n^0, u^0)$, we have the following proposition (see \[24\).
Theorem 1.2. Let 
\[ n_0^0, u_0^0 \in H^s(\mathbb{R}^3), \quad 0 < \hat{n}_0 \leq \inf_{x \in \mathbb{R}^3} n_0^0(x) \leq \sup_{x \in \mathbb{R}^3} n_0^0(x) \leq \hat{n}_0 < \infty, \]
for some positive constants \( \hat{n}_0, \hat{n}_0 \). Then there exist constants \( T^* \in (0, \infty) \) and \( \hat{n}_0, \hat{n}_0 > 0 \) such that the initial value problem \((1.7)-(1.8)\) has a unique solution 
\[ n^0 \in C^k([0, T^*], H^{s-k} \text{ or } H^{s-2k}(\mathbb{R}^3)), \quad u^0 \in C^k([0, T^*], H^{s-k}(\mathbb{R}^3)), \quad k = 0, 1, \]
\[ 0 < \hat{n} \leq \inf n^0(x, t) \leq \sup n^0(x, t) \leq \hat{n} < \infty. \]

Now, we are ready to state our main result of this paper.

Theorem 1.2. Let \( l > 2 + \frac{3}{2} \) and \( P(\cdot) \) be a smooth function on \((0, \infty)\) with \( P'(\cdot) > 0 \). Suppose \((n_0^0, u_0^0)\) be the unique classical solution to the equation \((1.7)-(1.8)\) on \([0, T^*]\) given in Proposition 1.1, where \( T^* \in (0, \infty) \) be the maximal existence time of \((n_0^0, u_0^0)\). Assume the initial data \((n_0(n), u_0(n), \tilde{\jmath}_0^0, E_0^0, B_0^0)\) satisfy 
\[ (n_0(n), u_0(n), \tilde{\jmath}_0^0, E_0^0, B_0^0) \in H^l(\mathbb{R}^3), \quad \text{div}B_0^0 = 0, \text{div}E_0^0 = 0, \]
and 
\[ \| (n_0^0 - n_0^0, u_0^0 - u_0^0, \tilde{\jmath}_0^0, E_0^0, B_0^0) \|_l \leq C_0 \kappa \quad (1.9) \]
for some constant \( C_0 > 0 \) independent of \( \kappa \). Then for any \( T_0 \in (0, T^*) \), there exist positive constants \( \kappa_0(T_0) \) and \( \hat{\mathcal{C}}(T_0) \), such that the system \((1.5)-(1.6)\) has unique classical solution \((n^\kappa, u^\kappa, \tilde{\jmath}_0^\kappa, E^\kappa, B^\kappa)\) on \([0, T_0]\) with 
\[ \| (n^\kappa - n_0^0, u^\kappa - u_0^0, \tilde{\jmath}_0^\kappa, E^\kappa, B^\kappa) (\cdot, t) \|_{H^l} \leq \hat{\mathcal{C}} \kappa \quad (1.10) \]
for any \( \kappa \in (0, \kappa_0] \) and \( t \in [0, T_0] \).

This paper is organized as follows. In Section 2, we derive the error system and introduce the local existence of the unique classical solution. In sections 3.1 and 3.2, we derive a uniform decay estimate with respect to \( \kappa \) of the error system \((1.7)-(1.8)\) by employing the energy method. In the last section, based on Lemma 3.2, we deduce Theorem 1.2 by the bootstrap principle.

2. Derivation of the error system and local existence

Let \((n^\kappa, u^\kappa, \tilde{\jmath}_0^\kappa, E^\kappa, B^\kappa)\) be the solution to system \((1.5)-(1.6)\) and \((n_0^0, u_0^0)\) be the solution to the system \((1.7)-(1.8)\) given in Proposition 1.1. Set 
\[ N^\kappa = n^\kappa - n_0^0, \quad U^\kappa = u^\kappa - u_0^0. \]
For \( \rho > 0 \), define 
\[ h(\rho) = \int_1^\rho \frac{P'(s)}{s} ds. \]
Combining the equations \((1.5)-(1.8)\), we obtain the error system:
\[ \partial_t N^\kappa + \frac{1}{1 + \varepsilon} \left( \text{div} \left( (N^\kappa + n_0^0) U^\kappa \right) + \text{div} \left( N^\kappa u_0^0 \right) \right) = 0, \quad (2.1) \]
\[ \partial_t U^\kappa + \frac{\varepsilon}{1 + \varepsilon} \tilde{\jmath}_0^\kappa \cdot \nabla \tilde{\jmath}_0^\kappa + \frac{1}{1 + \varepsilon} \left( \left( U^\kappa + u_0^0 \right) \cdot \nabla U^\kappa + U^\kappa \cdot \nabla u_0^0 \right) \]
\[ - \frac{\mu}{N^\kappa + n_0^0} \Delta U^\kappa - \frac{\mu + \lambda}{N^\kappa + n_0^0} \nabla \text{div} U^\kappa + \frac{\eta(1 + \varepsilon)}{\tau} \nabla \left( h(N^\kappa + n_0^0) - h(n_0^0) \right) \]
\[ = \left( \frac{1}{N^\kappa + n_0^0} - \frac{1}{n_0^0} \right) \left( \mu \Delta u_0^0 + (\mu + \lambda) \nabla \text{div} u_0^0 \right) + \frac{\kappa^2}{\tau} \tilde{\jmath}_0^\kappa \times B^\kappa, \quad (2.2) \]
\[ \kappa \partial_t \tilde{j}^\kappa + \frac{\kappa}{1 + \varepsilon} \left( (U^\kappa + u^0) \cdot \nabla \tilde{j}^\kappa + \tilde{j}^\kappa \cdot \nabla (U^\kappa + u^0) \right) + \frac{\varepsilon - 1}{\varepsilon + 1} \kappa^2 \tilde{j}^\kappa \cdot \nabla \tilde{j}^\kappa = 0, \]

\[ \frac{\mu}{N^\kappa + n^0} \Delta \tilde{j}^\kappa - (\mu + \lambda) \frac{\kappa}{N^\kappa + n^0} \nabla \text{div} \tilde{j}^\kappa = \frac{1 + \varepsilon}{\tau \varepsilon} E^\kappa + \frac{\kappa}{\tau \varepsilon} (U^\kappa + u^0) \times B^\kappa + \frac{\varepsilon - 1 + \kappa^2 \tilde{j}^\kappa \times B^\kappa}{\tau \varepsilon} - \frac{\varepsilon + 1}{\tau \varepsilon} \kappa \varepsilon (N^\kappa + n^0) \tilde{j}^\kappa, \quad (2.3) \]

\[ \frac{\partial_t E^\kappa - 1}{\kappa} \nabla \times B^\kappa = -\kappa (N^\kappa + n^0) \tilde{j}^\kappa, \quad \text{div} E^\kappa = 0, \quad (2.4) \]

\[ \frac{\partial_t B^\kappa + \frac{1}{\kappa} \nabla \times E^\kappa = 0, \quad \text{div} B^\kappa = 0, \quad (2.5) \]

\[ (N^\kappa, U^\kappa, \tilde{j}^\kappa, E^\kappa, B^\kappa)_{|t=0} = (n^\kappa_0 - n^0_0, u^\kappa_0 - u^0_0, \tilde{j}^\kappa_0, E^\kappa_0, B^\kappa_0) =: (N^\kappa_0, U^\kappa_0, \tilde{j}^\kappa_0, E^\kappa_0, B^\kappa_0). \quad (2.6) \]

Let

\[ W_1^\kappa = \begin{pmatrix} U^\kappa \\ \kappa \tilde{j}^\kappa \end{pmatrix}, \quad W_2^\kappa = \begin{pmatrix} N^\kappa \\ E^\kappa \\ B^\kappa \end{pmatrix}, \quad W^\kappa = \begin{pmatrix} W_1^\kappa \\ W_2^\kappa \end{pmatrix}, \]

\[ A^\kappa = \begin{pmatrix} \frac{1}{2 \tau \varepsilon} (u^\kappa + U^\kappa) & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\kappa} B^\kappa & 0 \end{pmatrix}, \]

\[ A^\kappa_{ij} = \begin{pmatrix} \frac{\mu}{N^\kappa + n^0} e_{i}^{T} e_{j} I_{3 \times 3} + \frac{\mu + \lambda}{N^\kappa + n^0} e_{i} e_{j}^{T} \\ 0 \end{pmatrix}, \]

where

\[ B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \]

Then the error system (2.1)–(2.3) can be recast as

\[ \begin{align*}
\partial_t W_1^\kappa &= \sum_{i,j=1}^{3} A^\kappa_{ij} \partial_{x_i x_j} W_1^\kappa + F \left( \partial^3 W^\kappa, \partial^3 n^0, \partial^3 u^0 \right), \\
\partial_t W_2^\kappa &= \sum_{i=1}^{3} A^\kappa_i \partial_{x_i} W_2^\kappa + G \left( W_2^\kappa, \partial^3 W_1^\kappa, \partial^3 n^0, \partial^3 u^0 \right), \\
W^\kappa_{|t=0} &= W^\kappa_0. \quad (2.7)
\end{align*} \]

for some functions \( F \) and \( G \), and the multi-index \( \alpha \) and \( \beta \) satisfy \( |\alpha| \leq 2, |\beta| \leq 1 \). Noted that the first group of equations is in the form of parabolic system with respect to \( W_1^\kappa \) while the second group of equations is symmetric hyperbolic system with respect to \( W_2^\kappa \). Thus, by the theory in [24], we have the following result for initial value problem (2.7) (or (2.1)–(2.6)).

**Proposition 2.1.** Let \( l > 2 + \frac{3}{2} \) and assume that \((n^0, u^0)\) satisfy the conditions in Proposition 1.1. Moreover, suppose that the initial data satisfy

\[ (N^\kappa_0, U^\kappa_0, \tilde{j}^\kappa_0, E^\kappa_0, B^\kappa_0) \in H^{l} (\mathbb{R}^3), \quad \inf_{x \in \mathbb{R}^3} \| N^\kappa_0 \| > 0, \quad \| N^\kappa_0 \| \leq \delta \]

for some small positive constant \( \delta \). Then there exist positive constants \( T_k (0 < T_k < \infty) \) and \( M \) (\( M \) only depends on \( \delta \)) such that the initial data problem (2.7) has a unique classical solution \((N^\kappa, U^\kappa, \tilde{j}^\kappa, E^\kappa, B^\kappa)\) satisfying \( \| N^\kappa(t) \|_l \leq M_\delta \), \( \inf_{x \in \mathbb{R}^3} N^\kappa > 0 \) and

\[ (N^\kappa, E^\kappa, B^\kappa) \in C^{k} ([0, T_k], H^{l-k} (\mathbb{R}^3)), \quad (U^\kappa, \tilde{j}^\kappa) \in C^{k} ([0, T_k], H^{l-2k} (\mathbb{R}^3)), \quad k = 0, 1. \]
We will show that for any $T_0 < T^*$, there exists $\kappa_0 > 0$, such that the existence time $T_\kappa > T_0$ for any $0 < \kappa < \kappa_0$. For this purpose, one should establish the uniform estimates on $(N^\kappa, U^\kappa, \kappa j^\kappa, E^\kappa, B^\kappa)$.

3. Proof of the main result

For any $0 < T_1 < 1$ independent of $\kappa$, let $T = T^\kappa = \min \{T_1, T_\kappa \}$. The positive constant $C$ depends upon $C_0$ and $T_0$ with $C > C_0$. For the sake of convenience, we define

$$\| W^\kappa(t) \|^2 = \| (N^\kappa, U^\kappa, \kappa j^\kappa, E^\kappa, B^\kappa)(t) \|^2.$$

3.1. Zero-order estimates.

Lemma 3.1. Under the hypothesis in Theorem 1.2, for any $t \in (0, T)$ and sufficiently small $\kappa$, we have

$$\left( \| U^n \|^2 + \| \kappa \tilde{j} \|^2 + \| E^n \|^2 + \| G^n \|^2 \right)(t) + \int_0^t \int \left( h(n + s) - h(n) \right) dsdx$$

$$+ \int_0^t \mu \left( \| \nabla U^n \|^2 + \| \nabla (\kappa \tilde{j}) \|^2 \right)(\tau) + (\mu + \lambda) \left( \| \text{div} U^n \|^2 + \| \text{div} (\kappa \tilde{j}) \|^2 \right)(\tau)d\tau$$

$$\leq C \left( \| U^n \|^2 + \| \kappa \tilde{j} \|^2 + \| E^n \|^2 + \| G^n \|^2 \right) + \int_0^t \int \left( h(n + s) - h(n) \right) dsdx \quad (t = 0)$$

$$+ C \int_0^t \left( \| W^n \|^2 + \| W \|^2 \right)(\tau)d\tau. \quad (3.1)$$

Proof. Multiplying (2.2) by $(N^\kappa + n^0) U^k$, and integrating over the whole space, we obtain

$$\frac{1}{2} \frac{d}{dt} \int (N^\kappa + n^0) |U^k|^2 dx + \mu \| \nabla U^n \|^2 + (\mu + \lambda) \| \text{div} U^n \|^2$$

$$= \frac{\eta(1 + \epsilon)}{\tau} \left( h(N^\kappa + n^0) - h(n^0) \right) \text{div} \left( \left( N^\kappa + n^0 \right) U^k \right) + \frac{1}{2} \int \partial_k \left( N^\kappa + n^0 \right) |U^n|^2 dx$$

$$- \frac{1}{1 + \epsilon} \left( \left( U^n + u^0 \right) \cdot \nabla U^n + U^n \cdot \nabla u^0 \right) \left( N^\kappa + n^0 \right) U^k$$

$$- \frac{\epsilon}{1 + \epsilon} \kappa^2 \tilde{j} \cdot \nabla \tilde{j}, \left( N^\kappa + n^0 \right) U^k + \frac{\kappa^2}{\tau} \left( \tilde{j} \times B^k \right) \left( N^\kappa + n^0 \right) U^k$$

$$+ \int \left( \frac{1}{N^\kappa + n^0} - \frac{1}{n^0} \right) \left( \mu \Delta u^0 + (\mu + \lambda) \nabla \text{div} u^0 \right) \left( N^\kappa + n^0 \right) U^k dx. \quad (3.2)$$

Firstly, we deal with the second term on the right-hand side of (3.2). Noted that

$$\partial_k \left( N^\kappa + n^0 \right) + \frac{1}{1 + \epsilon} \text{div} \left( \left( N^\kappa + n^0 \right) (U^n + u^0) \right) = 0, \quad (3.3)$$

in view of the regularity of $(n^0, u^0)$, Hölder’s inequality and Sobolev’s imbedding theorem, we have

$$\frac{1}{2} \int \partial_k \left( N^\kappa + n^0 \right) |U^n|^2 dx = - \frac{1}{2(1 + \epsilon)} \int \text{div} \left( \left( N^\kappa + n^0 \right) (U^n + u^0) \right) |U^n|^2 dx$$

$$\lesssim (1 + \| N^\kappa \|_1 + \| U^n \|_{11} + \| N^\kappa \|_1 \| U^n \|_2) \| U^n \|^2. \quad (3.4)$$

For the last four terms on the right-hand side of (3.2), by using the regularity of $(n^0, u^0)$, Hölder’s inequality, Sobolev’s imbedding theorem and the bounds on $N^\kappa$ and $n^0$ stated in Proposition 1.1 and Proposition 2.1, we have

$$\left| - \frac{1}{1 + \epsilon} \left( \left( U^n + u^0 \right) \cdot \nabla U^n + U^n \cdot \nabla u^0 \right) \left( N^\kappa + n^0 \right) U^k \right| \lesssim (1 + \| N^\kappa \|_1 + \| U^n \|_{11}) \| U^n \|^2, \quad (3.5)$$

$$\left| - \frac{\epsilon}{1 + \epsilon} \kappa^2 \tilde{j} \cdot \nabla \tilde{j}, \left( N^\kappa + n^0 \right) U^k \right| \lesssim (1 + \| N^\kappa \|_1) \| U^n \|_{11} \| \kappa \tilde{j} \|^2 \quad (3.6)$$
Now, we deal with the first term on the right-hand side of (3.2). From equation (2.1), we deduce

\[
\frac{\eta(1 + \varepsilon)^2}{\tau} \int \left( h (N^\kappa + n^0) - h (n^0) \right) \left( - (1 + \varepsilon) \partial_t N^\kappa - \text{div} \left( N^\kappa u^0 \right) \right) dx
\]

and then integrating over the whole space, we have

\[
\frac{1}{2} \frac{d}{dt} \int (N^\kappa + n^0) |U^\kappa|^2 dx + \frac{\eta(1 + \varepsilon)^2}{\tau} \frac{d}{dt} \int_0^{N^\kappa} h (s + n^0) - h (n^0) ds dx + \mu \| \nabla U^\kappa \|^2 + (\mu + \lambda) \| \text{div} U^\kappa \|^2 \]

\[
\lesssim \| W^\kappa \|^2 + \| W^\kappa \|^2 + \| W^\kappa \|^2.
\]

Similarly, multiplying (2.3) by \((N^\kappa + n^0) \kappa \tilde{j}^k\), and then integrating over the whole space, we have

\[
\frac{1}{2} \frac{d}{dt} \int |\kappa \tilde{j}^k|^2 (N^\kappa + n^0) dx + \mu \| \nabla (\kappa \tilde{j}^k) \|^2 + (\mu + \lambda) \| \text{div} (\kappa \tilde{j}^k) \|^2
\]

and then integrating over the whole space, we have

\[
\frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) + \frac{\eta(1 + \varepsilon)^2}{\tau} \frac{d}{dt} \int_0^{N^\kappa} h (s + n^0) - h (n^0) ds dx + \mu \| \nabla U^\kappa \|^2 + (\mu + \lambda) \| \text{div} U^\kappa \|^2 \]

\[
\lesssim \| W^\kappa \|^2 + \| W^\kappa \|^2 + \| W^\kappa \|^2.
\]

Applying the inequality of \((n^0, u^0)\), Hölder’s inequality and Sobolev’s imbedding theorem, the terms on the right-hand side of (3.11) can be estimated as:

\[
\left| - \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \| \kappa \tilde{j}^\kappa \|^2 \| (1 + \| N^\kappa \| + \| U^\kappa \| + \| N^\kappa \| + \| U^\kappa \|) \|
\]

\[
\left| - \frac{\eta(1 + \varepsilon)^2}{\tau} \frac{d}{dt} \int_0^{N^\kappa} h (s + n^0) - h (n^0) ds dx \right|
\]

\[
\lesssim \| \kappa \tilde{j}^\kappa \|^3 \| (1 + \| N^\kappa \|) \|
\]

\[
\left| \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \kappa \| \kappa \tilde{j}^\kappa \| \| B^\kappa \| \| (1 + \| N^\kappa \|) \|
\]

\[
\left| \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \kappa \| \kappa \tilde{j}^\kappa \| \| B^\kappa \| \| (1 + \| N^\kappa \|) \|
\]

\[
\left| \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \kappa \| \kappa \tilde{j}^\kappa \| \| B^\kappa \| \| (1 + \| N^\kappa \|) \|
\]

\[
\left| \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \kappa \| \kappa \tilde{j}^\kappa \| \| B^\kappa \| \| (1 + \| N^\kappa \|) \|
\]

\[
\left| \frac{\kappa^2}{1 + \varepsilon} \left( \left( (U^\kappa + u^0) \cdot \nabla j^\kappa + j^\kappa \cdot \nabla (U^\kappa + u^0), (N^\kappa + n^0) \tilde{j}^\kappa \right) \right) \right|
\]

\[
\lesssim \kappa \| \kappa \tilde{j}^\kappa \| \| B^\kappa \| \| (1 + \| N^\kappa \|) \|
\]
\[
\left| \frac{\varepsilon-1}{\varepsilon T} \kappa^3 \left( \bar{\mathbf{j}}^\kappa \times \mathbf{B}^\kappa, (N^\kappa + n^0) \bar{\mathbf{j}}^\kappa \right) \right| \lesssim \kappa \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|_2^2 \left\| B^\kappa \right\|_I (1 + \left\| N^\kappa \right\|_I),
\] (3.16)
\[
\left| \frac{\varepsilon+1}{\varepsilon T} \kappa, K \kappa^4 \left( (N^\kappa + n^0) \bar{\mathbf{j}}^\kappa, (N^\kappa + n^0) \bar{\mathbf{j}}^\kappa \right) \right| \lesssim \kappa^2 \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|_I^2 (1 + \left\| N^\kappa \right\|_I + \left\| N^\kappa \right\|_I^2),
\] (3.17)
\[
\frac{1}{2} \int \left| \kappa \bar{\mathbf{j}}^\kappa \right|^2 \partial_t \left( N^\kappa + n^0 \right) \, dx = \left| -\frac{1}{2(1+\varepsilon)} \int \left| \kappa \bar{\mathbf{j}}^\kappa \right|^2 \text{div} \left( (N^\kappa + n^0) \left( U^\kappa + u^0 \right) \right) \, dx \right| 
\lesssim \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|_I^2 (1 + \left\| N^\kappa \right\|_I + \left\| U^\kappa \right\|_I + \left\| N^\kappa \right\|_I \left\| U^\kappa \right\|_I),
\] (3.18)
where we have used (3.3) in the last equality. Substituting (3.12)–(3.15) into (3.11), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \int \left( N^\kappa + n^0 \right) \left| \kappa \bar{\mathbf{j}}^\kappa \right|^2 \, dx + \mu \left\| \nabla \left( \kappa \bar{\mathbf{j}}^\kappa \right) \right\|^2 + (\mu + \lambda) \left\| \text{div} \left( \kappa \bar{\mathbf{j}}^\kappa \right) \right\|^2 
\lesssim \left\| W^\kappa \right\|_I^2 + \left\| W^\kappa \right\|_I^3 + \left\| W^\kappa \right\|_I^4.
\] (3.19)

Then, multiplying (2.14), (2.5) by $E^\kappa$ and $G^\kappa$ respectively, and then integrating over the whole space, one has
\[
\frac{1}{2} \frac{d}{dt} \left( \left\| E^\kappa \right\|^2 + \left\| G^\kappa \right\|^2 \right) = -\left( \kappa \left( N^\kappa + n^0 \right) \bar{\mathbf{j}}^\kappa, E^\kappa \right) 
\lesssim \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|_I \left\| E^\kappa \right\|_I (1 + \left\| N^\kappa \right\|_I) 
\lesssim \left\| W^\kappa \right\|_I^2 + \left\| W^\kappa \right\|_I^3,
\] (3.20)
where we have used the regularity of $n^0$, Hölder’s inequality and Sobolev’s imbedding theorem.

Therefore, combining (3.10), (3.19) and (3.20), we have
\[
\frac{1}{2} \frac{d}{dt} \int \left( N^\kappa + n^0 \right) \left( \left| U^\kappa \right|^2 + \left| \kappa \bar{\mathbf{j}}^\kappa \right|^2 \right) \, dx + \frac{1}{2} \frac{d}{dt} \left( \left\| E^\kappa \right\|^2 + \left\| G^\kappa \right\|^2 \right) 
+ \frac{9(1+\varepsilon)}{\tau} \frac{d}{dt} \int_0^{N^\kappa} h \left( s + n^0 \right) - h \left( n_0 \right) \, ds \, dx + \mu \left\| \nabla U^\kappa \right\|^2 + (\mu + \lambda) \left\| \text{div} U^\kappa \right\|^2 
\lesssim \left\| W^\kappa \right\|_I^2 + \left\| W^\kappa \right\|_I^3 + \left\| W^\kappa \right\|_I^4.
\] (3.21)

By employing the fact that $N^\kappa + n^0 > \hat{n} > 0$, which is deduced by $\inf_{x \in \mathbb{R}^3} N^\kappa > 0$, then we conclude (3.11).

\[\square\]

3.2. Higher order estimates.

\textbf{Lemma 3.2.} Under the hypothesis in Theorem 1.2, for any $t \in (0, T)$ and sufficiently small $\kappa$, we have
\[
\left( \left\| U^\kappa \right\|^2 + \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|^2 + \left\| E^\kappa \right\|^2 + \left\| G^\kappa \right\|^2 \right) (t) + \int_0^t \left( \left\| U^\kappa \right\|^2_{l+1} + \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|^2_{l+1} \right) (\tau) \, d\tau 
\leq \left( \left\| U^\kappa \right\|^2 + \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|^2 + \left\| E^\kappa \right\|^2 + \left\| G^\kappa \right\|^2 \right) (t = 0) + \left( \sum_{1 \leq \alpha \leq l} \int \frac{h' \left( N^\kappa + n^0 \right)}{N^\kappa + n^0} \left\| \partial^\alpha U^\kappa \right\|^2 \, dx \right) (t = 0) 
\leq \left( \left\| U^\kappa \right\|^2 + \left\| \kappa \bar{\mathbf{j}}^\kappa \right\|^2 + \left\| E^\kappa \right\|^2 + \left\| G^\kappa \right\|^2 \right) (t = 0) + \left( \sum_{1 \leq \alpha \leq l} \int \frac{h' \left( N^\kappa + n^0 \right)}{N^\kappa + n^0} \left\| \partial^\alpha U^\kappa \right\|^2 \, dx \right) (t = 0) 
+ \left( \int_0^t \frac{6}{\tau} \left\| W^\kappa \right\|_I (\tau) \, d\tau \right) (3.22)
\]
Proof. Set $1 \leq \alpha \leq l$. Applying the operator $\partial^\alpha$ to (2.2), and taking the inner product of the resulting equation and $\partial^\alpha U^\kappa$, we obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial^\alpha U^\kappa\|^2 &= \mu \left( \partial^\alpha \left( \frac{\Delta U^\kappa}{N^\kappa + n^0} \right), \partial^\alpha U^\kappa \right) + (\mu + \lambda) \left( \partial^\alpha \left( \frac{\nabla \text{div} U^\kappa}{N^\kappa + n^0} \right), \partial^\alpha U^\kappa \right) \\
&- \frac{\eta(1 + \varepsilon)}{\tau} \left( \partial^\alpha \nabla \left( h(N^\kappa + n^0) - h(n^0) \right), \partial^\alpha U^\kappa \right) \\
&- \frac{1 + \varepsilon}{\tau} \left( \partial^\alpha \left( (U^\kappa + n^0) \cdot \nabla U^\kappa \right), \partial^\alpha U^\kappa \right) - \frac{1 + \varepsilon}{\tau} \left( \partial^\alpha \left( U^\kappa \cdot \nabla u^0 \right), \partial^\alpha U^\kappa \right) \\
&- \frac{\varepsilon}{\tau} \left( \partial^\alpha \left( \nabla \cdot \nabla \kappa \right), \partial^\alpha U^\kappa \right) \\
&\quad + \mu \left( \partial^\alpha \left( \frac{1}{N^\kappa + n^0} - \frac{1}{n^0} \right) \Delta u^0, \partial^\alpha U^\kappa \right) \\
&\quad + (\mu + \lambda) \left( \partial^\alpha \left( \frac{1}{N^\kappa + n^0} - \frac{1}{n^0} \right) \nabla \text{div} u^0, \partial^\alpha U^\kappa \right) \\
&\quad + \frac{\kappa^2}{\tau} \left( \partial^\alpha \left( \nabla \times \vec{B}^\kappa \right), \partial^\alpha U^\kappa \right) \\
&\quad = \sum_{i=1}^9 I^{(i)}.
\end{align*}

Next, we will handle all terms on the right-hand side of (3.23) one by one. For the term $I^{(1)}$, integrating by parts and using the fact that $N^\kappa + n^0 > \tilde{n} > 0$, we have
\begin{align*}
I^{(1)} &= \mu \left( \partial^\alpha \frac{\Delta U^\kappa}{N^\kappa + n^0}, \partial^\alpha U^\kappa \right) + \mu \left( \mathcal{H}^{(1)}_{U^\kappa}, \partial^\alpha U^\kappa \right) \\
&= -\mu \int \frac{1}{N^\kappa + n^0} \left| \partial^\alpha \nabla U^\kappa \right|^2 \, dx - \mu \left( \partial^\alpha \nabla U^\kappa, \partial^\alpha U^\kappa \nabla \left( \frac{1}{N^\kappa + n^0} \right) \right) + \mu \left( \mathcal{H}^{(1)}_{U^\kappa}, \partial^\alpha U^\kappa \right) \\
&\leq -\mu \int \frac{1}{N^\kappa + n^0} \left| \partial^\alpha \nabla U^\kappa \right|^2 \, dx + C \left\| U^\kappa \right\|_{l+1} \left\| U^\kappa \right\|_l (1 + \left\| N^\kappa \right\|_l) + \mu \left\| \mathcal{H}^{(1)}_{U^\kappa} \right\| \left\| U^\kappa \right\|_l,
\end{align*}
where
\begin{equation*}
\mathcal{H}^{(1)}_{U^\kappa} = \partial^\alpha \left( \frac{\Delta U^\kappa}{N^\kappa + n^0} \right) - \frac{\partial^\alpha \Delta U^\kappa}{N^\kappa + n^0}.
\end{equation*}
Applying the Moser-type inequalities in Lemma 1.1, one has
\begin{align*}
\left\| \mathcal{H}^{(1)}_{U^\kappa} \right\| &\lesssim D \left( \frac{1}{N^\kappa + n^0} \right) \left\| \partial^{l-1} \Delta U^\kappa \right\| + \| \Delta U^\kappa \|_\infty \left\| D^l \left( \frac{1}{N^\kappa + n^0} \right) \right\| \\
&\lesssim (1 + \left\| N^\kappa \right\|_l) \left\| U^\kappa \right\|_{l+1}.
\end{align*}
Thus, one deduces that
\begin{equation*}
I^{(1)} \leq -\mu \int \frac{1}{N^\kappa + n^0} \left| \partial^\alpha \nabla U^\kappa \right|^2 \, dx + C \left\| U^\kappa \right\|_{l+1} \left\| U^\kappa \right\|_l (1 + \left\| N^\kappa \right\|_l). \tag{3.24}
\end{equation*}

Similarly, for the term $I^{(2)}$, we have
\begin{equation*}
I^{(2)} \leq -(\mu + \lambda) \int \frac{1}{N^\kappa + n^0} \left| \partial^\alpha \text{div} U^\kappa \right|^2 \, dx + C \left\| U^\kappa \right\|_{l+1} \left\| U^\kappa \right\|_l (1 + \left\| N^\kappa \right\|_l). \tag{3.25}
\end{equation*}

Next, we will deal with the term $I^{(3)}$. Integrating by parts and applying Leibniz’s formula, we obtain
\begin{align*}
I^{(3)} &= \frac{\eta(1 + \varepsilon)}{\tau} \left( \partial^\alpha \left( h(N^\kappa + n^0) - h(n^0) \right), \partial^\alpha \text{div} U^\kappa \right) \\
&= \frac{\eta(1 + \varepsilon)}{\tau} \sum_{\beta \leq \alpha, |\beta| = 1} \left( \partial^{\alpha - \beta} \left( h'(N^\kappa + n^0) \partial^\beta N^\kappa + h'(N^\kappa + n^0) \partial^\beta N^\kappa \right), \partial^\alpha \text{div} U^\kappa \right) \\
&= \frac{\eta(1 + \varepsilon)}{\tau} \left( h'(N^\kappa + n^0) \partial^\alpha N^\kappa, \partial^\alpha \text{div} U^\kappa \right) + \frac{\eta(1 + \varepsilon)}{\tau} \left( \mathcal{H}^{(3)}_{U^\kappa}, \partial^\alpha \text{div} U^\kappa \right)
\end{align*}
\[ + \frac{\eta(1 + \varepsilon)}{\tau} \sum_{\beta \leq \alpha, \beta = 1} \left( \partial^{\alpha - \beta} \left( (h'(N^\kappa + n^0) - h'(n^0)) \partial^\beta n^0 \right) \partial^\alpha \text{div} U^\kappa \right) \]
\[ =: I^{(31)} + I^{(32)} + I^{(33)}, \quad (3.26) \]

where

\[ \mathcal{H}^{(31)}_{U} = \sum_{\beta \leq \alpha, \beta = 1} \partial^{\alpha - \beta} \left( h'(N^\kappa + n^0) \partial^\beta N^\kappa \right) - h'(N^\kappa + n^0) \partial^\alpha N^\kappa. \]

Applying the Moser-type inequalities in Lemma [21], we have

\[ \left\| \mathcal{H}^{(31)}_{U} \right\| \lesssim \left\| D h'(N^\kappa + n^0) \right\|_\infty \left\| D^{l-1} \partial^\beta N^\kappa \right\| + \left\| \partial^\beta N^\kappa \right\|_\infty \left\| D^{l} h'(N^\kappa + n^0) \right\| \]
\[ \lesssim (1 + \| N^\kappa \|_l) \| N^\kappa \|_l. \]

Therefore, for the term \( I^{(32)} \), one has

\[ I^{(32)} \lesssim \| \partial^\alpha \text{div} U^\kappa \| \left( \| W^\kappa \|_l + \| W^\kappa \|_l^2 \right) \]
\[ \quad (3.27) \]

For the term \( I^{(33)} \), integrating by parts yields that

\[ I^{(33)} \lesssim \| U^\kappa \|_l \| N^\kappa \|_l \lesssim \| W^\kappa \|_l^2 \]
\[ \quad (3.28) \]

For the term \( I^{(31)} \), using the equation [21], we divide it into three parts:

\[ I^{(31)} = \left( h'(N^\kappa + n^0) \partial^\alpha N^\kappa, \partial^\alpha \left( \frac{1}{N^\kappa + n^0} \left( (1 + \varepsilon) \partial h N^\kappa + U^\kappa \cdot \nabla (N^\kappa + n^0) + \text{div} (N^\kappa u^0) \right) \right) \right) \]
\[ \times \left( \frac{\eta(1 + \varepsilon)}{\tau} \right) \]
\[ =: I^{(31)} + I^{(312)} + I^{(313)}. \]

Firstly, we deal with the term \( I^{(31)} \). To this end, we rewrite \( I^{(31)} \) as

\[ I^{(31)} = -\frac{\eta(1 + \varepsilon)^2}{\tau} \left( h'(N^\kappa + n^0) \partial^\alpha N^\kappa, \frac{\partial^\alpha \partial h N^\kappa}{N^\kappa + n^0} \right) \]
\[ = -\frac{\eta(1 + \varepsilon)^2}{\tau} \frac{d}{dt} \left( \frac{1}{N^\kappa + n^0} \left( \| \partial^\beta N^\kappa \|_2 dx + \frac{\eta(1 + \varepsilon)^2}{2\tau} \int \partial_t \left( \frac{h'(N^\kappa + n^0)}{N^\kappa + n^0} \right) \| \partial^\alpha N^\kappa \|_2^2 dx \right) \]
\[ - \frac{\eta(1 + \varepsilon)^2}{\tau} \left( h'(N^\kappa + n^0) \partial^\alpha N^\kappa, \mathcal{H}^{(32)}_{U^\kappa} \right), \]
\[ \quad (3.30) \]

where

\[ \mathcal{H}^{(32)}_{U^\kappa} = \partial^\alpha \left( \frac{\partial_t N^\kappa}{N^\kappa + n^0} \right) - \frac{1}{N^\kappa + n^0} \partial^\alpha \partial h N^\kappa. \]

Combining the Moser-type inequalities in Lemma [21] and the equation [21], we have

\[ \left\| \mathcal{H}^{(32)}_{U^\kappa} \right\| \lesssim \left\| D \left( \frac{1}{N^\kappa + n^0} \right) \right\|_\infty \left\| D^{l-1} \partial_t N^\kappa \right\| + \left\| \partial_t N^\kappa \right\|_\infty \left\| D^{l} \left( \frac{1}{N^\kappa + n^0} \right) \right\| \]
\[ \lesssim (1 + \| N^\kappa \|_l) \left( \| U^\kappa \|_l + \| N^\kappa \|_l + \| U^\kappa \|_l \| N^\kappa \|_l \right) \]
\[ \lesssim \| W^\kappa \|_l + \| W^\kappa \|_l^2 + \| W^\kappa \|_l^2. \]

from which we can infer that the last term on the right-hand side of \( I^{(33)} \) can be controlled by

\[ C \left( \| W^\kappa \|_l^2 + \| W^\kappa \|_l^2 + \| W^\kappa \|_l^2 \right). \]

Then, we deal with the second term on the right-hand side of \( I^{(33)} \). By using the equation [33], the smoothness of \( h \) and the bounds on \( N^\kappa + n^0 \), we deduce

\[ \left\| \partial_t \left( \frac{h'(N^\kappa + n^0)}{N^\kappa + n^0} \right) \right\|_\infty \]
\[ \lesssim \left\| \text{div} \left( \left( (N^\kappa + n^0)(U^\kappa + u^0) \right) \right) \right\|_\infty \]
\[ \lesssim 1 + \left\| U^\kappa \right\|_l + \| N^\kappa \|_l + \| U^\kappa \|_l \| N^\kappa \|_l. \]
Thus, we obtain
\[
I^{(311)} \leq -\frac{\eta(1+\varepsilon)^2}{\tau} \frac{1}{2} \frac{d}{dt} \int \frac{h'(\|\nabla \psi^\alpha\|^2 + \|\nabla \psi^\alpha\|^4)}{\psi^\alpha \|\nabla \psi^\alpha\|^2} \frac{d}{dt} \int \frac{\partial^\alpha N^\alpha}{\psi^\alpha} \frac{\partial^\alpha N^\alpha \cdot \nabla (N^\alpha + n^0)}{\psi^\alpha} \frac{d}{dt} \int C \left( \|W^\kappa\|^2 + \|W^\kappa\|^4 \right). 
\tag{3.31}
\]

Next, we deal with the term \(I^{(312)}\). Integrating by parts, using the regularity of \((n^0, u^0)\), Sobolev’s imbedding theorem and the bounds on \(N^\alpha + n^0\), we have
\[
I^{(312)} = -\frac{\eta(1+\varepsilon)}{\tau} \left( h'(N^\alpha + n^0) \partial^\alpha N^\alpha \cdot \frac{U^\kappa}{N^\alpha + n^0} \cdot \nabla (N^\alpha + n^0) \right) 
- \frac{\eta(1+\varepsilon)}{\tau} \left( h'(N^\alpha + n^0) \partial^\alpha N^\alpha \cdot \frac{U^\kappa}{N^\alpha + n^0} \right) 
+ \frac{1}{\tau} \left( \partial^\alpha N^\alpha \partial^\alpha N^\alpha \partial \frac{h'(N^\alpha + n^0)U^\kappa}{N^\alpha + n^0} \right) 
\leq \|N^\kappa\|^2 \|U^\kappa\| + 2 \|N^\kappa\|^2 \|N^\kappa\|^2 \|U^\kappa\| + 2 \|N^\kappa\|^2 \|H_{U^\kappa}^{(33)}\|,
\]
where
\[
H_{U^\kappa}^{(33)} = \partial^\alpha \left( \frac{1}{N^\alpha + n^0} U^\kappa \cdot \nabla (N^\alpha + n^0) \right) - \frac{1}{N^\alpha + n^0} U^\kappa \cdot \nabla \partial^\alpha (N^\alpha + n^0).
\]

Furthermore, in view of Moser-type inequalities in Lemma \[13\], we have
\[
\|H_{U^\kappa}^{(33)}\| \lesssim \left( 1 + \|N^\kappa\| \right) \left( D \left( \frac{U^\kappa}{N^\alpha + n^0} \right) \right) + \left( 1 + \|N^\kappa\| \right) \left( D \left( \frac{U^\kappa}{N^\alpha + n^0} \right) \right) \lesssim \left( 1 + \|N^\kappa\| \right)^2 \|U^\kappa\|/I.
\]

Therefore, one deduces that
\[
I^{(312)} \lesssim \|W^\kappa\|^2 + \|W^\kappa\|^4 + \|W^\kappa\|^4.
\tag{3.32}
\]

Now, we deal with the term \(I^{(313)}\). Integrating by parts, utilizing the regularity of \((n^0, u^0)\), Sobolev’s imbedding theorem and the bounds on \(N^\alpha\) and \(n^0\), we have
\[
I^{(313)} = -\frac{\eta(1+\varepsilon)}{\tau} \left( h'(N^\alpha + n^0) \partial^\alpha N^\alpha \cdot \frac{u^0}{N^\alpha + n^0} \cdot \nabla N^\alpha \right) 
- \frac{\eta(1+\varepsilon)}{\tau} \left( h'(N^\alpha + n^0) \partial^\alpha N^\alpha \cdot \frac{u^0}{N^\alpha + n^0} \right) 
\leq \frac{\eta(1+\varepsilon)}{\tau} \left( \frac{1}{N^\alpha + n^0} \right) \left( \frac{u^0}{N^\alpha + n^0} \right) \left( \frac{N^\alpha}{N^\alpha + n^0} \right) 
\leq \frac{\eta(1+\varepsilon)}{\tau} \left( \frac{1}{N^\alpha + n^0} \right) \left( \frac{u^0}{N^\alpha + n^0} \right) \left( \frac{N^\alpha}{N^\alpha + n^0} \right) 
\leq \frac{\eta(1+\varepsilon)}{\tau} \left( \frac{1}{N^\alpha + n^0} \right) \left( \frac{u^0}{N^\alpha + n^0} \right) \left( \frac{N^\alpha}{N^\alpha + n^0} \right) 
\leq \frac{\eta(1+\varepsilon)}{\tau} \left( 1 + \|N^\kappa\| \right) \left( D \left( \frac{u^0}{N^\alpha + n^0} \right) \right) \left( D \left( \frac{u^0}{N^\alpha + n^0} \right) \right). 
\]

Applying Moser-type inequalities in Lemma \[13\] again, we infer that
\[
\|H_{U^\kappa}^{(34)}\| \lesssim \left( 1 + \|N^\kappa\| \right) \left( D \left( \frac{u^0}{N^\alpha + n^0} \right) \right) \left( D \left( \frac{u^0}{N^\alpha + n^0} \right) \right) \left( D \left( \frac{u^0}{N^\alpha + n^0} \right) \right). 
\]
Employing the operator \( \partial^\alpha \) and Lemma 3.1, summing (3.31), (3.32) and (3.33), we obtain

\[
\sum_{\alpha = 0}^{\alpha \leq l} \| \partial^\alpha U^\kappa \|^2 \leq C \left( \| U^\kappa \|_{l+1}^2 + \| W^\kappa \|_{l+1}^2 \right),
\]

(3.33)

Therefore, one has

\[
I^{(313)} \lesssim \| W^\kappa \|^2_l + \| W^\kappa \|^3_l.
\]

(3.34)

Combining (3.31), (3.32) and (3.33), we obtain

\[
I^{(31)} \leq \frac{7}{2} \frac{\left( \| U^\kappa \|^2 + \| W^\kappa \|^2 + \| W^\kappa \|^3 \right)}{\| \partial^\alpha N^\kappa \|^2} \left( \| U^\kappa \|_{l+1} + \| W^\kappa \|_{l+1} \right).
\]

(3.35)

By the regularity of \((n^0, u^0)\) and Sobolev’s imbedding theorem, we get

\[
I^{(4)} \lesssim \| U^\kappa \|_{l+1} \| U^\kappa \|^2_l \lesssim \| U^\kappa \|_{l+1} \left( \| U^\kappa \|_{l+1} + \| W^\kappa \|_{l+1} \right).
\]

(3.36)

Similarly, we have

\[
I^{(5)} \lesssim \| U^\kappa \|^2_l,
\]

(3.37)

\[
I^{(9)} \lesssim \| U^\kappa \|^2_l \| B^\kappa \|_{l+1} \| U^\kappa \|_{l+1}.
\]

(3.38)

Integrating by parts, the regularity of \((n^0, u^0)\), Sobolev’s imbedding theorem and the bounds on \(N^\kappa\) and \(n^0\) imply that

\[
I^{(6)} \lesssim \| U^\kappa \|_{l+1} \| U^\kappa \|^2_l,
\]

(3.39)

\[
I^{(7)}, I^{(8)} \lesssim \| U^\kappa \|^3_l \| N^\kappa \|_{l+1}.
\]

(3.40)

Inserting (3.31), (3.32), (3.33), (3.34)–(3.40) into (3.26), we conclude that

\[
\frac{d}{dt} \left( \frac{\left( \| U^\kappa \|^2 + \| W^\kappa \|^2 + \| W^\kappa \|^3 \right)}{\| \partial^\alpha N^\kappa \|^2} \right) \leq \sigma_\alpha \| U^\kappa \|^2_{l+1} + C(\sigma_\alpha) \left( \| U^\kappa \|^2_l + \| W^\kappa \|^2_l + \| W^\kappa \|^3_l + \| W^\kappa \|^4_l + \| W^\kappa \|^5_l + \| W^\kappa \|^6_l \right),
\]

for some small constant \(\sigma_\alpha\), which will be determined later.

Applying a same argument on \((\partial^\alpha)\) and \((\partial^\alpha E^\kappa)\) and \((\partial^\alpha G^\kappa)\) respectively, we have

\[
\frac{d}{dt} \left( \frac{\left( \| U^\kappa \|^2 + \| W^\kappa \|^2 + \| W^\kappa \|^3 \right)}{\| \partial^\alpha N^\kappa \|^2} \right) \leq \sigma_\alpha \| U^\kappa \|^2_{l+1} + C(\sigma_\alpha) \left( \| U^\kappa \|^2_l + \| W^\kappa \|^2_l + \| W^\kappa \|^3_l + \| W^\kappa \|^4_l + \| W^\kappa \|^5_l + \| W^\kappa \|^6_l \right).
\]

(3.41)

Employing the operator \(\partial^\alpha\) to (3.7) and (3.8), and then taking the inner product of the resulting equations, \(\partial^\alpha E^\kappa\) and \(\partial^\alpha G^\kappa\) respectively, we have

\[
\frac{d}{dt} \left( \frac{\left( \| U^\kappa \|^2 + \| W^\kappa \|^2 \right)}{\| \partial^\alpha N^\kappa \|^2} \right) \leq \| W^\kappa \|^2_l + \| W^\kappa \|^3_l.
\]

(3.42)

Applying the bounds on \(N^\kappa + n^0\) stated in Proposition 1.1 and Proposition 2.1 combining (3.41), (3.32), (3.33) and Lemma 3.1 summing \(\alpha\) over \(1 \leq \alpha \leq l\), and choosing \(\sigma_\alpha\) small enough, we obtain the desired result of Lemma 3.2.
3.3. Proof of Theorem 1.2. Now, we are ready to prove Theorem 1.2.

Proof. As in [3], set \( \Gamma^\kappa(t) = ||W^\kappa(t)||^2 \). By Lemma 3.2 for any \( t \in (0, T) \) and sufficiently small \( \kappa \), we have

\[
\Gamma^\kappa(t) \leq \Gamma^\kappa(0) + C \int_0^t \Gamma^\kappa \left( 1 + \left( \Gamma^\kappa \right)^2 + \Gamma^\kappa + \left( \Gamma^\kappa \right)^2 \right) (\tau) d\tau.
\] (3.44)

Choose \( \kappa_1 > 0 \) and \( T_1 \in (0, 1) \) small enough such that

\[
1 + (3C)^2 \kappa_1 + 3C^2 \kappa_1^2 + (3C^2)^2 \kappa_1^3 + (3C)^2 \kappa_1^4 \leq 2, \quad 1 + 2CT_1 e^{2CT_1} \leq \frac{3}{2} \] (3.45)

Suppose that \( \Gamma^\kappa(t) \leq 3C \kappa^2 \). Noted that by using the assumption \( \Gamma^\kappa(0) \leq C_0 \kappa^2 \) in Theorem 1.2 and (3.44) and (3.45), for any \( \kappa \in (0, \kappa_1) \) and \( t \in (0, T_1] \), we have

\[
\Gamma^\kappa(t) \leq C \kappa^2 + C \int_0^t \Gamma^\kappa(\tau) \left( 1 + (3C)^2 \kappa + 3C \kappa^2 + (3C^2)^2 \kappa^3 + (3C^2)^2 \kappa^4 \right) d\tau
\leq C \kappa^2 + 2C \int_0^t \Gamma^\kappa(\tau) d\tau,
\] (3.46)

which combined with Gronwall’s inequality and \( \frac{3}{2} \) implies that

\[
\Gamma^\kappa(t) \leq C \kappa^2 \left( 1 + 2CT_1 e^{2CT_1} \right) \leq C \kappa^2 \left( 1 + 2CT_1 e^{2CT_1} \right) \leq \frac{3}{2} C \kappa^2.
\] (3.47)

By the bootstrap principle, we conclude that \( \Gamma^\kappa(t) \leq 3C \kappa^2 \) for any \( \kappa \in (0, \kappa_1) \) and \( t \in (0, T_1] \). Hence, by standard continuous induction method, for any \( T_0 < T^* \), there exists a positive constant \( \kappa_0(T_0) \), such that for any \( \kappa \in (0, \kappa_0) \), \( T_\kappa \geq T_0 \) and \( \Gamma^\kappa(t) \leq C \kappa^2 \) on \([0, T_0]\), for some constant \( \hat{C} \) depending only on \( T_0 \) and the initial data. Therefore, we complete the proof of Theorem 1.2. 

Acknowledgments

The research was supported by the National Natural Science Foundation of China (No. 11901066), the Natural Science Foundation of Chongqing (No. cstc2019jcyj-msxmX0167) and projects No. 2019CDYST0015, No. 2020CDJQY-A040 supported by the Fundamental Research Funds for the Central Universities.

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