Collisional relaxation in diffuse clouds of trapped bosons

G. M. Kavoulakis, C. J. Pethick, and H. Smith

1 NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark,
2 Ørsted Laboratory, H. C. Ørsted Institute, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

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The damping of collective modes and the relaxation of temperature anisotropies in a trapped Bose gas is determined at temperatures above the Bose-Einstein condensation temperature in the collisionless regime. We demonstrate for both cases how the effects of collisions may be treated perturbatively and calculate relaxation rates based on a variational principle. Our results are compared with experiment and with previous theoretical calculations.

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I. INTRODUCTION

In recent years a number of experiments have been carried out to study relaxation processes in trapped Bose gases. Such experiments may be divided into two broad categories, those on collective oscillations and those on temperature anisotropies. Relaxation rates of collective modes have been measured in rubidium vapors by Jin et al. [1,2], and in sodium ones by the MIT group [3,4]. The relaxation of temperature anisotropies has been investigated by Monroe et al. [5] and Arndt et al. [6] in cesium, Newbury et al. [7] and Myatt [8] in rubidium, and Davis et al. [9] in sodium.

In the experiments on the relaxation of collective modes [1-4], the atomic cloud is excited at the frequency of the normal mode of interest by means of a weak external perturbation. The amplitude of the oscillation is then measured in the absence of the perturbation, and from that the damping rate of the mode is extracted. In the experiments on temperature relaxation, the atomic cloud is either prepared in a nonequilibrium state with different effective temperatures for motion along and perpendicular to the symmetry axis of the trap [5-8], or it is driven out of equilibrium by displacing the trap center [9].

Most experiments on relaxation have been carried out in the collisionless regime, where the collision rate is small compared with the frequencies of particle motion in the trap and also, in the case of collective oscillations, the oscillation frequency. In addition the mean free path is large compared to the size of the cloud. In this paper we focus on the collisionless limit, at temperatures above the condensation temperature, $T_c$, but our results also apply to the condensed phase at temperatures close to the transition temperature, where the thermal excitations constitute the dominant component of the gas. A brief account of some of our results has been given previously [10].

Relaxation of temperature anisotropies in trapped clouds of alkali-atom vapors has been studied theoretically both analytically [11-13] and numerically [14-16]. Newbury et al. [14] and Myatt [15] used the Boltzmann equation with an ansatz for the form of the non-equilibrium distribution function to obtain an analytic expression for the temperature relaxation rate and solved the problem numerically for large temperature anisotropies. Monroe et al. [11] performed a Monte-Carlo simulation of temperature relaxation for atoms with an energy-independent scattering cross section, and Arndt et al. [12] made a similar calculation for resonant scattering [13]. Finally Wu and Foot solved the Boltzmann equation numerically and extracted the rate of temperature relaxation [16].

The present paper is based on the semi-classical Boltzmann equation, from which we derive an eigenvalue equation for the relaxation rate. Treating the collisions perturbatively, we obtain general expressions for the damping rate of the collective modes and the relaxation rate for temperature anisotropies. We include effects of quantum degeneracy and explore the role of energy dependence of the scattering cross section.

The paper is organized as follows: In Sec. II we develop the general formalism. We then examine in Sec. III the damping of modes with frequencies equal to twice the characteristic frequencies of the trap and calculate the damping rate for a general, energy-dependent cross section. The case of an energy-independent scattering cross section is studied in detail, and analytical results are obtained in the limit of Boltzmann statistics for a simple trial function. We perform calculations with improved trial functions, and find that the corrections are small, of the order of 1%. Our results are then compared with experiment.

In Sec. IV we examine the relaxation of temperature anisotropies and show that this problem is essentially the same as that of the damping of collective modes. Again we perform calculations with a simple trial function and with a more general one. The improved trial function yields a reduction of the relaxation rate by about 7% for the two cases which we examine. We then compare the results with data from experiment and from numerical simulations. Section V is a brief conclusion.
II. GENERAL FORMALISM

When the energy of a particle is much greater than the spacing between energy levels the motion may be described semiclassically. In a trap, energy spacings are of order the oscillator quantum, \( \hbar \omega_i \), where \( \omega_i \) is the frequency associated with motion in the \( i \) direction. Even when \( T \) is as low as the transition temperature, the ratio \( k_BT/\hbar \omega_i = [N/\zeta(3)]^{1/3} \) is very much greater that unity, where \( N \) is the total number of atoms in the trap, \( \zeta(\alpha) \) is the Riemann zeta function, and \( \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3} \), and therefore the level spacing is small provided the anisotropy of the trap is not extreme. Also we may neglect the influence of the mean field due to interactions between atoms. This gives rise to a potential of order \( n(r)U_0 \) where \( n(r) \) is the number density and \( U_0 = 4 \pi \hbar^2 a/m \) is the effective two-particle interaction with \( a \) being the scattering length for atom-atom collisions, and \( m \) being the atomic mass. Even when \( T = T_c \), the energy \( nU_0 \) is typically only about \% of \( k_BT \), and the molecular field may therefore be neglected in the equation of motion of the particles.

Since the motion can be described semiclassically the state of the system may be specified in terms of a particle distribution function \( f(p, r, t) \), where \( p \) is the momentum vector, \( r \) is the position vector, and \( t \) is the time. This satisfies the Boltzmann equation,

\[
\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f - \nabla V \cdot \nabla f = -I, \tag{1}
\]

where \( V \) is the external potential, and \( I \) is the collision integral, which is a functional of \( f(r, p, t) \). We write the distribution function as \( f = f^0 + \delta f \), where \( f^0 \) is the equilibrium distribution function and \( \delta f \) is the deviation from equilibrium, and linearizing Eq. (1) we find

\[
\frac{\partial \delta f}{\partial t} + \frac{p}{m} \cdot \nabla \delta f - \nabla V \cdot \nabla p \delta f = -I[\Phi], \tag{2}
\]

where \( \Phi \) is defined by \( \delta f = f^0(1 + f^0)\Phi \).

We now consider collisions. The energy levels of a single particle moving in a harmonic trap are discrete, and for a spherically symmetric trap there is strong shell structure. For the anisotropic traps used in experiments, the shell structure is less pronounced, and the density of states will be made more uniform by interatomic interactions. The ratio \( nU_0/\hbar \omega_i \) is of the order of unity for typical experimental conditions. Finally collisions will smear the density of states. As a consequence, we expect that under current experimental conditions the collision integral will be well approximated by the expression for a bulk system. The linearized collision integral is thus

\[
I[\Phi] = \int d\tau_1 \int d\sigma |v - v_1| (\Phi + \Phi_1 - \Phi - \Phi_1), \tag{3}
\]

where the momenta of the incoming particles in a collision are denoted by \( p \) and \( p_1 \), and those of the outgoing ones by \( p' \) and \( p'_1 \), and \( d\sigma = dp/(2\pi \hbar)^3 \). In the above expression we have introduced the differential cross section \( d\sigma \), which in general depends on the relative velocity \( u = |v - v_1| \) of the two incoming particles, as well as on the angle between the relative velocities of the colliding particles before and after the collision. Since the atoms obey Bose-Einstein statistics, their distribution function in equilibrium, \( f^0(p, r) \), is

\[
f^0(p, r) = \frac{1}{e^{(p^2/2m \pi V - \mu)/k_BT} - 1}, \tag{4}
\]

where \( \mu \) is the chemical potential. We shall consider harmonic traps where the potential has the form

\[
V = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{5}
\]

In the absence of collisions the motions of particles in the three coordinate directions decouple, and particles execute simple harmonic motion for each of the coordinates. Let us consider motion in the \( z \) direction. In the absence of collisions \( I[\Phi] = 0 \) the general solution of the Boltzmann equation having time dependence \( e^{-n\omega_z t} \) has the form

\[
\delta f = (p_z + im\omega_z z)^n h(K_j), \tag{6}
\]

where \( h \) is an arbitrary function, and \( K_j \) are constants of the motion in the absence of collisions. So, neglecting collisions, for \( \delta f(\mathbf{r}, \mathbf{p}, t) = h(K_j)(p_z + im\omega_z z)^n e^{-n\omega_z t} \), one has

\[
\frac{\partial \delta f}{\partial t} + n\omega_z \delta f = 0. \tag{7}
\]

We now consider the effect of collisions. Provided the collision rate is small compared with the oscillator frequency, collisions will not mix modes of the system in the absence of collisions with different values of \( n \). Consequently we expect the solution to be of the form

\[
\delta f(\mathbf{p}, \mathbf{r}, t) = h(K_j)(p_z + im\omega_z z)^n e^{-n\omega_z t - \Gamma t}. \tag{8}
\]

This problem therefore amounts to doing degenerate perturbation theory as in quantum mechanics, because we consider mixing of modes which have the same frequency in the absence of the perturbation, in this case collisions. Collisions will generally couple particle motions in different directions, but we begin by assuming that the collision rate is small compared with \( |\omega_z - \omega_x| \) and \( |\omega_z - \omega_y| \), in which case this coupling may be neglected. It is important in traps with axial symmetry for modes in which particles move perpendicular to the symmetry axis, and we shall consider its effects later.

Substituting Eq. (3) into the Boltzmann equation one finds

\[
\frac{\partial \delta f}{\partial t} + n\omega_z \delta f = -\Gamma \delta f = -I[\Phi]. \tag{9}
\]
the eigenvalues of the collision integral are never negative, then for any trial function \( \Phi \) one finds \( 12 \) that because of
integration over both coordinate and momentum space. The above integration is over momentum space alone, in
terms.

It is worth pointing out the connection between
Eq. \( 13 \) and the usual variational expression for transport coefficients that applies in the static limit. A transport coefficient such as the shear viscosity is determined by an inhomogeneous Boltzmann equation of the form

\[
f^0(1 + f^0)X = I[\Xi],
\]

and the transport coefficient is proportional to the quantity \( \langle \Xi^* X f^0(1 + f^0) \rangle_p \), where

\[
\langle \Xi^* X f^0(1 + f^0) \rangle_p = \int d\tau f^0(1 + f^0)\Xi^*(p)X(p).
\]

The above integration is over momentum space alone, in contrast to the previous case [Eq. \( 13 \)], which involves integration over both coordinate and momentum space. Then for any trial function \( \Psi \) one finds \( 13 \) that because the eigenvalues of the collision integral are never negative,

\[
\langle \Xi^* X f^0(1 + f^0) \rangle_p^1 \leq \frac{\langle \Psi^* I[\Psi] \rangle_p}{\langle \Psi^* X f^0(1 + f^0) \rangle_p^2}.
\]

Comparing Eqs. \( 11 \) and \( 14 \), we see that both are given by the same sort of variational expression and involve the collision integral. The differences are in the normalization condition which appears in the denominators of the two expressions, and in the spatial integrations in Eq. \( 14 \).

III. DAMPING OF MODES

A. The damping rate

The inequality \( 11 \) may be used to calculate the damping rate of any mode in which particle motion in the \( z \) direction is perturbed, but let us focus on a trap with uniaxial symmetry, \( \omega_x = \omega_y \), and on modes with frequency \( 2\omega_z \), which are among those of experimental interest. In the same spirit as one adopts in calculating transport coefficients, we begin by making the simplest choice, Eq. \( 8 \) with \( n = 2 \), and with \( h \) put equal to unity. Thus as our initial trial function \( \Phi \) we use

\[
\Phi = (p_z + im\omega_z)^2.
\]

In Sec. III C we examine more general trial functions. The function \( \Phi \) contains terms of the form \( z^2, zp_z, \) and \( p_z \). It is clear from Eq. \( 8 \) that the collision integral gives zero when operating on the first two of these terms because collisions conserve particle number and momentum; it also gives zero when operating on \( p_z^2 \) because of the conservation of the kinetic energy. Therefore we may write

\[
\langle (p_z - im\omega_z)^2 I[(p_z + im\omega_z)^2] \rangle =
\]

\[
= \langle p_z^2 I[p_z^2] \rangle = \langle (p_z^2 - p_z^2/3)I[p_z^2 - p_z^2/3] \rangle.
\]

The second of these forms follows as a consequence of kinetic energy conservation, which allows us to substract any multiple of \( p_z^2 \) from \( I[p_z^2] \). The choice made above introduces the function \( p_z^2 - p_z^2/3 \), which is orthogonal to the collision invariants, since \( p_z^2 - p_z^2/3 = (2/3)p_z^2P_2(\cos \theta) \), where \( P_l(x) \) is the Legendre polynomial of order \( l \), and \( \theta \) is the polar angle of the momentum vector. Substitution of Eq. \( 16 \) into Eq. \( 11 \) leads to the result

\[
\langle \Xi^* X f^0(1 + f^0) \rangle_p \leq \frac{\langle \Psi^* I[\Psi] \rangle_p}{\langle \Psi^* X f^0(1 + f^0) \rangle_p^2}.
\]

Comparing Eqs. \( 11 \) and \( 14 \), we see that both are given by the same sort of variational expression and involve the collision integral. The differences are in the normalization condition which appears in the denominators of the two expressions, and in the spatial integrations in Eq. \( 14 \).
The factor of 1/6 comes from the integrals \(<(p^2_0 - p^2_1/3)f^0(1 + f^0)/<(p^2_1 + (m\omega z)^2)f^0(1 + f^0)\rangle\) and reflects the fact that collisions damp directly only the part of the distribution function varying as \(p^2_1 - p^2_0/3\), while all parts of the distribution contribute to the total energy associated with the oscillation. In the classical limit Eq. (20) takes the simple form

\[
\Gamma_0 \leq \frac{1}{6} \int dr \frac{n(r)}{n_{\text{var}}} \left\langle \frac{1}{\tau_{\text{var}}} \right\rangle \int dr \frac{n(r)}{\left\langle \frac{1}{\tau_{\text{var}}} \right\rangle}. \tag{21}
\]

We return to Eq. (17) now, starting with the numerator on the right side of the inequality. This quantity includes integration over real space as well as momentum space, and we write it as

\[
\left\langle (p^2_0 - p^2_1/3)f^0(1 + f^0) \right\rangle = \int dr \left\langle I_c(z(r)) \right\rangle, \tag{22}
\]

where \(z(r) = (0)e^{-V/k_BT}\) is the fugacity, with \(e^{-V/k_BT}\) being its value at the center of the cloud. \(I_c(z(r))\) is given by

\[
I_c(z(r)) = \int dr \int d\tau \int d\sigma |v - v_1| \times \Phi^*(\Phi + \Phi_1 - \Phi' - \Phi'_1)f^0f^0(1 + f^0')(1 + f^0)(1 + f^0'/1 + f^0'). \tag{23}
\]

In the above expression unprimed quantities refer to incoming particles and primed ones to outgoing particles. Using the symmetry of the collision integral under interchange of incoming and outgoing particle momenta, we may write Eq. (23) as

\[
I_c(z(r)) = \frac{1}{4} \int dr \int d\tau \int d\sigma |v - v_1| \times |\Phi + \Phi_1 - \Phi' - \Phi'_1|^2 f^0f^0(1 + f^0')(1 + f^0)(1 + f^0'/1 + f^0'). \tag{24}
\]

Because of the Galilean invariance of the collision process it is convenient to use the center-of-mass momentum \(\mathbf{P}\) and the relative momentum coordinates \(\mathbf{p}_r\) and \(\mathbf{p}'_r\) instead of the momenta of the incoming and outgoing particles \(\mathbf{p}_1, \mathbf{p}_{1}', \mathbf{p}_r, \mathbf{p}'_r\). We introduce \(\Delta[\Phi]\) by the definition

\[
\Delta[\Phi] = \Phi + \Phi_1 - \Phi' - \Phi'_1, \tag{25}
\]

and for \(\Phi = p^2 - p^2/3\) one finds

\[
\Delta[p^2 - p^2/3] = 2(p^2_z - p^2'_z)^2 - \frac{1}{3} (p^2_z - p^2'_z)^2 \tag{26}
\]

\[
= \frac{4}{3} p^2_r \left[ P_2(\cos \theta_r) - P_2(\cos \theta'_r) \right], \tag{27}
\]

where \(\theta_r\) and \(\theta'_r\) are the polar angles of the vectors \(\mathbf{p}_r\) and \(\mathbf{p}'_r\) with respect to the \(z\) axis. We write the element of cross section \(d\sigma\) in Eq. (25) as \((d\sigma/d\Omega)d\Omega\), where \(d\Omega\) is the element of solid angle for the vector \(\mathbf{p}_r\). Since we are considering scattering of indistinguishable bosons, the final states corresponding to \(\mathbf{p}_r\) and \(-\mathbf{p}_r\) are identical, and we perform the summation over final states by summing over all \(\Omega\) and then dividing by 2 to avoid double counting. Using the addition theorem for Legendre polynomials and their orthogonality property, we find from Eq. (27) that the numerator of Eq. (17) is

\[
\left\langle (p^2_0 - p^2_1/3)f^0(1 + f^0) \right\rangle = \frac{16}{45m} \int dr \int d\tau \int d\tau_1 \times \frac{1}{2} \int d\Omega [1 - P_2(\cos \chi)] \frac{d\sigma}{d\Omega} p^2_r f^0f^0(1 + f^0')(1 + f^0), \tag{28}
\]

where \(\chi\) is the angle between \(\mathbf{p}_r\) and \(\mathbf{p}'_r\). The product of the Bose factors in Eq. (25) depends on the angles between \(\mathbf{P}\) and the vectors \(\mathbf{p}_r\) and \(\mathbf{p}'_r\), which implies that it is impossible to factorize the collision integral, except for the case of Boltzmann statistics. In order to perform the angular integrations we express \(\sin^2 \chi\) in terms of \(\theta_r\) and \(\theta'_r\), using the addition formula

\[
\cos \chi = \cos \theta_r \cos \theta'_r + \sin \theta_r \sin \theta'_r \cos(\phi_r - \phi'_r), \tag{29}
\]

where \(\phi_r\) and \(\phi'_r\) are the azimuthal angles of the vectors \(\mathbf{p}_r\) and \(\mathbf{p}'_r\). Equation (29) then gives

\[
\int_0^{2\pi} d(\phi_r - \phi'_r) \sin^2 \chi = \pi(1 + y^2 + y'^2 - 3y^2y'^2), \tag{30}
\]

with \(y = \cos \theta_r\) and \(y' = \cos \theta'_r\). The differential cross section \(d\sigma/d\Omega\) depends in general on \(y\) and \(y'\) and in principle we may consider any form of \(d\sigma/d\Omega\). At the low energies of interest it is, however, a good approximation to assume that the scattering is purely s wave. When the cross section is energy-independent, which is a very good approximation for rubidium and sodium atoms, \(d\sigma/d\Omega = 4a^2\). The factor of 4 reflects the fact that for identical bosons the scattering amplitude is twice that for distinguishable particles. For cesium atoms there is a bound state close to zero energy and the scattering is resonant. If the resonant state were at zero energy, the s wave phase shift \(\delta_0\) would be \(\pi\) (modulo \(2\pi\)) and therefore the cross section at low energy would be given by the unitarity limit for s waves, \(d\sigma/d\Omega = 4/k^2\). More generally, when the resonant state is close to, but not exactly at zero energy, one may take the leading term in the effective range expansion and write \(\cos \delta_0 = ka\), since \(d\sigma/d\Omega = (4/k^2)\sin^2 \delta_0\), one then finds

\[
\frac{d\sigma}{d\Omega} = \frac{4a^2}{1 + (ka)^2}, \tag{31}
\]

where \(k = p_r/h\) is the wavevector corresponding to \(\mathbf{p}_r\). The differential cross section thus depends on the energy of the colliding particles. For s wave scattering the
integral of the $P_2$ term in Eq. (28) vanishes, and the 
intergrant on the right side in this equation involves the 
total momentum-dependent cross section $\sigma(u)$, which is 
defined as 
\[
\sigma(u) = \frac{1}{2} \int d\Omega \frac{d\sigma}{d\Omega},
\] (32)
where for the relative velocity of the incoming particles, 
$u, u = 2p_r/m$. For a differential cross section of the form 
of Eq. (31) the total cross section is 
\[
\sigma(u) = \frac{8\pi a^2}{1 + (ka)^2},
\] (33)

We turn now to the denominator of Eq. (17) and define 
$D$ as 
\[
D = \int dx \int d\tau \left[ p_x^2 + (m\omega_x z)^2 \right] f^0(1 + f^0) 
= 2 \int dx \int d\tau \left[ p_x^2 + p_y^2 (m\omega_x z)^2 \right] f^0(1 + f^0),
\] (34)
where the latter result follows from the fact that $f^0$ is 
a function of $p_x^2 + (m\omega_x z)^2$, which is symmetric under 
interchange of $p_x$ and $m\omega_x z$. The integral in Eq. (34) is 
over a six-dimensional hyperspace with coordinates $p$ and 
$m\omega_x$, and the “angular” and radial integrals decouple. 
With the use of the equation $f^0(1 + f^0) = -k_B T \partial f^0/\partial E$, 
with $E$ being the particle energy, one finds 
\[
D = \frac{8}{15} \int dx \int d\tau p^2 f^0(1 + f^0) 
= \frac{8}{3} m k_B T \int dx \int d\tau p^2 f^0 
= \frac{215/2\pi}{3} (m k_B T)^{7/2} \int dx \bar{D}(x),
\] (35)

where 
\[
\bar{D}(x) = \int_0^\infty \frac{x^4 dx}{z^{-1}(x)e^{x\bar{z}} - 1} 
= \frac{3\sqrt{\bar{z}}}{8} g_{5/2}[z(x)],
\] (36)
with $g_n(z) = \sum_{l=1}^{\infty} z^l/l^n$ being the familiar Bose integral. 

From Eqs. (28), (32), (33) and (35) we obtain 
\[
\Gamma_0 \leq \frac{4}{15 m^2 k_B T} \int dx \int d\tau \int d\tau_1 
\times p_{x}^2 \sigma(u) f_0 f_0'(1 + f_0')(1 + f_0') \int dx \int d\tau p^2 f^0.
\] (37)

Our final variational inequality (35) can be written in the form 
\[
\Gamma_0 \leq \frac{1}{10\sqrt{2}} \left( \frac{m k_B T}{2\pi h^2} \right)^{3/2} \sigma_0 \bar{v} 
\times \int \bar{I}_c z(x) \int dx \bar{D}(x),
\] (38)
which is one of our main results. Using Eq. (36) we perform 
the spatial integration in the denominator of Eq. (38) and find 
\[
\Gamma_0 \leq \frac{1}{30\pi^{7/2}\sqrt{2}} \left( \frac{m\omega}{h} \right)^3 \sigma_0 \bar{v} \int \bar{I}_c \int dx \bar{D}(x).
\] (39)

In the above equations $\sigma_0 = 8\pi a^2$ is the scattering cross 
section at zero energy, and 
\[
\bar{v} = (8k_B T/\pi m)^{1/2}
\] (40)
is the mean thermal velocity. The quantity $\bar{I}_c$ is propor-
tional to $I_c$ and it is given by 
\[
\bar{I}_c = \int_0^\infty dx_r \int_0^\infty dx_0 \int_0^1 dy \int_0^1 dy' x_0^3 x_r^7 \sigma(x_r,t) 
\times F(x_r, x_0, y, y') (1 + y^2 + y'^2 - 3y^2 y'^2).
\] (41)

In the above expression we have introduced $\sigma(x_r,t)$ which 
is given by the ratio $\sigma(u)/\sigma_0$ and it is a function of $u = p_r/m$. 
We have also defined $x_r = p_r/(m k_B T)^{1/2}$ and 
$t = T/T_0$, with $k_B T_0 = h^2/ma^2$, which implies that $u = x_r t^{1/2}$. Also $x_0$ is given by $x_0 = P/(4m k_B T)^{1/2}$. Finally 
we have introduced the product $F = f^0 f_0'(1 + f_0')/(1 + f_0')$, 
which may be written in the useful form 
\[
F = \frac{2e^{-z^2} (x_0^2 + z^2)}{(1 - e^{-z^2} + x_0^2)(1 - e^{-z^2} + y_0^2)(1 - e^{-z^2} + y_0^2)},
\] (42)
where $x_i = p_i/(2m k_B T)^{1/2}$ with $p_i$ being the momentum of 
particle $i$. The indices $i = 1$ and 2 refer to incoming 
particles, and $i = 3$ and 4 to outgoing ones. The variables 
$x_i$ are explicitly given in terms of $x_r, x_0, y, y'$ by 
\[
x_1 = \frac{1}{2} x_0^2 + 2x_0 x_r y + x^2_r, 
x_2 = \frac{1}{2} (x_0^2 - 2x_0 x_r y + x^2_r), 
x_3 = \frac{1}{2} (x_0^2 + 2x_0 x_r y' + x^2_r), 
x_4 = \frac{1}{2} (x_0^2 - 2x_0 x_r y' + x^2_r).
\]

Taking the classical limit of Eq. (38), we find that the 
integrand in the numerator is proportional to $z^2(r)$ and 
therefore to the square of the density, as expected. 
The integrand in the spatial integration in the denominator 
is linear in the density in this limit. In addition, in the 
classical limit the integrations over the center of mass 
and the relative momenta decouple, and Eq. (38) may be 
written in the form 
\[
\Gamma_0 \leq \frac{\sqrt{\pi}}{30\pi^{7/2}} \sigma_0 \bar{v} \int dr \int dx_0 \int dr \int dr \int dr \int dr 
\times \int_0^\infty dx_r x_0^3 e^{-z^2} \sigma(x_r,t) \int_0^\infty dx x^4 e^{-x^2},
\] (43)
where $n_{c1}(x) = n_{c1}(0)e^{-V/k_B T}$ is the density in the classical 
limit, with $n_{c1}(0)$ being the density of the atoms at 
the center of the cloud, where it is a maximum.
To express Eq. (43) in a more physical form it is convenient to define an average density weighted by the particle number, or

\[ n_{av} = \int n_{cl}^2(r) \, dr / \int n_{cl}(r) \, dr = n_{cl}(0)/2^{3/2}, \]  

(44)

where the last equality holds for a harmonic potential. The result Eq. (43) may be written in a form that makes it physically more clear,

\[ \Gamma_0 \leq \frac{1}{12} \sigma(u) u \int dr n_{cl}^2(r) / \int dr n_{cl}(r) \leq \frac{1}{24\sqrt{2}} n_{cl}(0) \sigma(u) u, \]  

(45)

where

\[ \frac{\sigma(u)}{u} = \frac{\int_0^\infty du u^7 \exp(-mu^2/4k_BT)\sigma(u)}{\int_0^\infty du u^6 \exp(-mu^2/4k_BT)}. \]  

(46)

Using Eq. (33) we find that classically for energy-independent s-wave scattering,

\[ \Gamma_0 \leq \frac{1}{15} n_{cl}(0) \sigma_0 \bar{v} \equiv \Gamma_{0,cl}. \]  

(47)

We turn now to the effects of quantum degeneracy. For energy-independent scattering and for \( T \gg T_c \) one may derive an analytical expression for the damping, as shown in Appendix A, and the result is

\[ \Gamma_0 \leq \frac{1}{15} n_{cl}(0) \sigma_0 \bar{v} \left[ 1 + \frac{3}{16} \zeta(3) \left( \frac{T_c}{T} \right)^3 \right], \]  

(48)

where \( n_{cl}(0) = N\bar{\omega}^3/m/2\pi k_BT \) \( 3^{3/2} \). The coefficient \( 3\zeta(3)/16 \) is approximately equal to 0.23. The dashed line in Fig. 1 shows the result of the expansion given by Eq. (38), as a function of \( T_c/T \). Under other conditions the integrals in Eq. (38) must be calculated numerically. The relaxation rate can be written as

\[ \frac{\Gamma_0}{\Gamma_{0,cl}} = G(T_c/T). \]  

(49)

The solid line in Fig. 1 shows the function \( G(T_c/T) \) for s wave, energy-independent scattering. The value of \( \Gamma_0 \) at \( T_c \) is \( \approx 1.66 \Gamma_{0,cl} \). The assumption of an energy-independent cross section is a rather good approximation for Rb and Na atoms, but later we shall consider the case of an energy-dependent cross section, which is relevant for Cs.

**B. Traps with degenerate modes**

Up to now we have studied oscillations along one of the axes of the cloud, taken to be the \( z \) axis, and we have assumed that the mode is non-degenerate: \( \omega_z \neq \omega_x \), and \( \omega_z \neq \omega_y \). There are experimental situations, however, where the damping of oscillations perpendicular to the axis of the trap has been measured. Since \( \omega_x = \omega_y \), collisions couple the motion in the \( x \) and \( y \) directions. Because the collision integral is invariant under rotations of the spatial coordinates about the \( z \) direction, the appropriate solutions of the Boltzmann equation may be characterized by the “magnetic” quantum number, \( n_z \), specifying the symmetry under rotations about the \( z \) axis. From the trial functions \((p_x + im \omega_x x)^2\) and \((p_y + im \omega_y y)^2\) for motion in the \( x \) and \( y \) directions one can construct two sorts of trial function

\[ \Phi(m_z = 0) = (p_x + im \omega_x x)^2 + (p_y + im \omega_y y)^2, \]  

(50)

and

\[ \Phi(|m_z| = 2) = (p_x + im \omega_x x)^2 - (p_y + im \omega_y y)^2. \]  

(51)

For \( m_z = 0 \) the only part of this trial function that contributes to the collision integral is \( p_x^2 + p_y^2 \), and since \( p_x^2 + p_y^2 = p^2 - p_z^2 \), the collision integral gives the same result as in the case studied in Sect. III A. On the other hand with this choice of trial function the denominator is equal to \( 2D \), where \( D \) is given by Eq. (34), because the energy in the mode is twice as large. This is easily seen
by calculating $|\Phi(m_z = 0)|^2$ and performing the angular integrations, which yield

$$\int dr \int d\tau |\Phi(m_z = 0)|^2 f^0(1 + f^0) =$$

$$= 2 \int dr \int d\tau \left[ p_z^2 + (m \omega_z z)^2 \right] f^0(1 + f^0) =$$

$$= \frac{16}{15} \int dr \int d\tau p^4 f^0(1 + f^0), \quad (52)$$

since the integral of the cross terms vanishes. The denominator in the present problem is thus twice as large as the one obtained in Eq. (55), and one finds

$$\Gamma(m_z = 0) = \frac{1}{2} \Gamma_0. \quad (53)$$

This result is independent of the degree of quantum degeneracy.

Turning to the trial function with $|m_z| = 2$, Eq. (51), we see that it is a superposition of terms with $m_z = \pm 2$. Since the $m_z = +2$ and $m_z = -2$ modes are degenerate, this mode has the same damping. The only part of the trial function that contributes to the collision integral is $p_z^2 - p_y^2$. Making use of the expressions

$$p_x^2 - p_y^2 = (8\pi/15)^{1/2} p^2 (Y_2^2 + Y_2^{-2}) \quad (54)$$

and

$$p_x^2 + p_y^2 - 2p^2/3 = (2/3)^{1/2} (8\pi/15)^{1/2} p^2 Y_2^0, \quad (55)$$

where $Y_l^m$ are the spherical harmonics, we find for the damping rate $\Gamma(|m_z| = 2)$ of the $|m_z| = 2$ mode that

$$\Gamma_0(|m_z| = 2) = 3 \Gamma_0(m_z = 0) = \frac{3}{2} \Gamma_0, \quad (56)$$

since the denominator for $\Phi(|m_z| = 2)$ is the same as for $\Phi(m_z = 0)$. The above ratios are also independent of the quantum degeneracy.

### C. Improved estimates

One can improve the estimates of mode damping by using more general trial functions. We start with Eq. (42) which gives the most general form of $\delta f$. In the calculation presented in Sec. III A we assumed that the function $h(K)$ is 1, or $\Phi_0 = (p_z + i m \omega_z z)^2$. More general trial functions can be used, and one systematic method for generating these is to expand $h$ in terms of complete sets of functions of the variables $K$. For the problem under consideration the conserved quantities are the energies $E_i = p_i^2/2m + m \omega^2 r_i^2/2$ for each of the coordinate axes. For a trap with an axis of symmetry the angular momentum about this axis is conserved, but this will not be a relevant variable, since in the experiments there was no preferred direction of rotation about the axis. The number of possible functions of $E_i$ is vast, and we shall not perform a general search. Rather we shall look for improved functions that depend only on the total energy. Also for simplicity we consider the classical limit only. We expand $h$ in terms of a set of orthogonal polynomials, and a convenient choice is the Sonine polynomials introduced in the theory of transport coefficients in classical gases. Because of the form of the normalizing factors in the denominator in Eq. (43) the appropriate orthogonality condition for the polynomials is

$$\langle h_i h_j E^2 f^0 \rangle = \delta_{i,j}, \quad (57)$$

where the factor $E^2$ reflects the presence of the prefactor $p_x^2 + (m \omega_z z)^2$ in the trial function, and $\delta_{i,j}$ is the Kronecker delta function. The functions $h_i$ are thus proportional to the Sonine polynomials $S_j^1 \quad (58)$. We remark that the normalization we have chosen, which is convenient for our purposes, differs from the one used for Sonine polynomials. The simplest example is $h_0 = 1$, and the next one is $h_1 = \sqrt{5}(1 - E/5kT)$ and thus

$$\Phi_1 = \sqrt{5} (p_z + i m \omega_z z)^2 (1 - E/5k_B T). \quad (59)$$

As our improved trial function we take a linear combination of $h_0$ and $h_1$, which is equivalent to $c + E$, where $c$ is a constant. We write the improved trial function as

$$\Phi_{\text{imp}} = (1 - \alpha^2)^{1/2} \Phi_0 + \alpha \Phi_1, \quad (60)$$

where $\alpha$ is a variational parameter. We proceed by calculating the damping rate $\Gamma_{0,\text{imp}}$ corresponding to this trial function and minimizing it with respect to $\alpha$. In terms of the quantities

$$\gamma_{i,j} = \langle \Phi_i^* \delta_{ij} \rangle, \quad (61)$$

we obtain

$$\Gamma_{0,\text{imp}} = \frac{1}{D} \left[ (1 - \alpha^2) \gamma_{0,0} + 2\alpha (1 - \alpha^2)^{1/2} \gamma_{1,0} + \alpha^2 \gamma_{1,1} \right]. \quad (62)$$

For the calculation of the quantities $\gamma_{1,0}$ and $\gamma_{1,1}$, we notice that the only difficult part is the one that involves the combination $(p_z + i m \omega z) E/5k_B T$. One can easily see that due to the conservation laws, only the parts $(p_z^2/2m + V)p_z^2 + i \omega_z z p_z^2$ give non-vanishing terms, in addition to the terms we considered earlier with $\Phi_0$ only. We find that in terms of the coordinates $p_r, p_{r'}$, and $\mathbf{P}$, the combination of deviation functions $\Delta[\Phi_{\text{imp}}]$ is given by

$$\Delta[\Phi_{\text{imp}}] = 2 \left( p_{r,z}^2 - p_{r',z}^2 \right) \left[ (1 - \alpha^2)^{1/2} + \alpha \sqrt{5} \frac{V}{5k_B T} + C \frac{1}{m k_B T} \left( p^2 + \frac{p_r^2}{4} \right) \right].$$

$$\Delta[\Phi_{\text{imp}}] = \alpha \frac{2}{5} \sqrt{5} \frac{1}{k_B T} \left( \frac{p_z}{2m} + i \omega_z z \right) (p_{r,z} \mathbf{P} - p_{r',z} \mathbf{P}' \cdot \mathbf{P}) \quad (62)$$

For the calculation of the quantities $\gamma_{1,0}$ and $\gamma_{1,1}$, we notice that the only difficult part is the one that involves the combination $(p_z + i m \omega z) E/5k_B T$. One can easily see that due to the conservation laws, only the parts $(p_z^2/2m + V)p_z^2 + i \omega_z z p_z^2$ give non-vanishing terms, in addition to the terms we considered earlier with $\Phi_0$ only. We find that in terms of the coordinates $p_r, p_{r'}$, and $\mathbf{P}$, the combination of deviation functions $\Delta[\Phi_{\text{imp}}]$ is given by

$$\Delta[\Phi_{\text{imp}}] = 2 \left( p_{r,z}^2 - p_{r',z}^2 \right) \left[ (1 - \alpha^2)^{1/2} + \alpha \sqrt{5} \frac{V}{5k_B T} + \frac{1}{m k_B T} \left( p^2 + \frac{p_r^2}{4} \right) \right].$$

$$\Delta[\Phi_{\text{imp}}] = \alpha \frac{2}{5} \sqrt{5} \frac{1}{k_B T} \left( \frac{p_z}{2m} + i \omega_z z \right) (p_{r,z} \mathbf{P} - p_{r',z} \mathbf{P}' \cdot \mathbf{P}) \quad (62)$$
which reproduces Eq. (27) for $\alpha = 0$. We calculate the matrix elements and find that $\gamma_{1,0}/\gamma_{0,0} = -\sqrt{5}/20$, and $\gamma_{1,1}/\gamma_{0,0} = 199/80$. Identifying the ratio $\gamma_{0,0}/D$ as the quantity $\Gamma_0$ which we calculated earlier, we write Eq. (61) as

$$
\Gamma_{0,\text{imp}}(\alpha) = \Gamma_0 \left[ 1 - \frac{\alpha^2}{\gamma_{0,0}(\gamma_{1,1} - \gamma_{0,0})} \right].
$$

Minimizing the above expression with respect to $\alpha$, we find that $\alpha \approx 0.073$; the minimum is $\approx 0.992\Gamma_0$ and therefore the correction is of the order of 0.8%. Since $\alpha$ is very small, it is a good approximation to expand Eq. (63) in powers of $\alpha$ and write $\Gamma_{0,\text{imp}}$ in the form of a perturbative expansion as in quantum mechanics:

$$
\Gamma_{0,\text{imp}} = \Gamma_0 \left[ 1 - \frac{\gamma_{2,0}}{\gamma_{0,0}(\gamma_{1,1} - \gamma_{0,0})} \right].
$$

The correction term $1 - \gamma_{2,0}/\gamma_{0,0}(\gamma_{1,1} - \gamma_{0,0})$ is equal to $118/119 \approx 0.992$. Our final result is therefore

$$
\Gamma_{0,\text{imp}} = \frac{118}{1785}\nu_0(0)\sigma_0\bar{v}.
$$

This provides evidence for the result of the earlier calculations being very close to the actual value.

### D. Comparison with experiment

The MIT group has reported two measurements of the damping rate of oscillations along the axis of a cigar-shaped cloud. In the first one [1], $N = 5 \times 10^7$ atoms [19] were observed to oscillate at a frequency of $\approx 35$ Hz at $T = 2T_c$ and the damping time was measured to be about 80 ms. The frequencies of the trapping potential, which was axially symmetric, were $\nu_z \approx 19$ Hz along the axis of the trap, and $\nu_\perp \approx 250$ Hz perpendicular to it. Using these numbers we find that classically the central density is $\approx 2.9 \times 10^{13}$ cm$^{-3}$ and assuming that $a = 53$ Å for rubidium atoms [24], our calculations of the relaxation times Eqs. (49) and (61) imply that the $m_z = 0$ mode has a damping time of $\approx 88$ ms, and the $|m_z| = 2$ mode has a damping time of $\approx 29$ ms. The observed damping time was, however, $\approx 50$ ms for the $m_z = 0$ mode, and $\approx 100$ ms for the $|m_z| = 2$ mode, indicating the need for further work to resolve this discrepancy.

### IV. RELAXATION OF THERMAL ANISOTROPIES

#### A. Form of the trial function

We turn now to temperature relaxation. As we mentioned in the Introduction, in the experiments described in Refs. [1, 2] the energies associated with the motion of atoms in the various directions in the trap are unequal and to a first approximation these anisotropies can be characterized by ascribing different temperatures to the motions in different directions. This problem is essentially the same as the one we presented earlier. The most general form of the change in the particle distribution function $\delta f_T$ is in this case any function $p$ of the energies $E_i$ along the direction $i$,

$$
\delta f_T = p(E_x, E_y, E_z).
$$

It can be checked easily that this form of $\delta f_T$ satisfies the Boltzmann equation in the absence of collisions with a frequency equal to zero. If one changes the temperature associated with particle motion in the $z$ direction by an amount $\delta T$ and that associated with motion in the transverse direction by an amount $-\delta T/2$, conserving in this
way the average particle energy and the density up to first order in $\delta T$, the change in the particle distribution function $\delta f_T$ is thus

$$\delta f_T \propto f^0(1 + f^0) \left[ E_z - \frac{1}{2} (E_x + E_y) \right]$$

$$\propto 3 f^0(1 + f^0) \left[ E_z - \frac{E}{3} \right].$$

(67)

Since the results do not depend on the anisotropy of the trap, we assume that the external potential $V$ is isotropic with a frequency $\omega$. As our initial trial function we therefore take a function proportional to $E_z - E/3 = (p_z^2 - p^2/3)/2m + m\omega^2(z^2 - r^2/3)/2$,

$$\Phi_T = p^2 P_z(\theta_p) + (m\omega^2/2) P_z(\theta_r).$$

(68)

where $\theta_p$ is the polar angle of the momentum vector, and $\theta_r$ is the polar angle of the position vector. Later we shall consider improved trial functions.

B. Calculation of the thermal relaxation rate

Let us now calculate the thermal relaxation rate $\Gamma_T$. For the trial function $\Phi_T$ given by Eq. (68), the only part of it that gives a non-zero contribution when acted on by the collision integral is the term proportional to $p^2 \cos^2(\theta_p) = p_z^2$. Since in the previous problem of the damping of oscillation modes we assumed that $\Phi = (p_z + im\omega z)^2$, the quantity $\langle \Phi_T^* I[\Phi_T] \rangle$ for thermal relaxation is the same as that for the relaxation of oscillations, Eq. (28).

Turning to the denominator $D_T = \langle \Phi_T^2 f^0(1 + f^0) \rangle$ which appears in the thermal relaxation rate, the cross terms that arise from squaring $\Phi_T$ vanish because of the orthogonality of the Legendre polynomials in Eq. (68), and therefore

$$D_T = \frac{4}{45} \int dx \int d\tau \left[ p^4 + (m\omega r)^4 \right] f^0(1 + f^0)$$

$$= \frac{8}{45} \int dx \int d\tau p^4 f^0(1 + f^0).$$

(69)

Comparing the above equation with Eq. (35), $D_T = D/3$. Finally we find that

$$\Gamma_T \leq \frac{\langle \Phi_T^* I[\Phi_T] \rangle}{\langle \Phi_T^2 f^0(1 + f^0) \rangle} = 3 \Gamma_0,$$

(70)

or

$$\Gamma_T \leq \frac{1}{2} \frac{\tau_{\text{var}}^{-1}}{\int dx \langle (p_z^2 - p^2/3)^2 f^0(1 + f^0) \rangle_p}.$$

(71)

This result may be understood by noting that the decay rate for temperature anisotropies in a uniform gas is given by $\Gamma_T \leq \tau_{\text{var}}^{-1}$ in the corresponding approximation. In a trap there are two facts that must be taken into account. The first is that the system is inhomogeneous, and therefore the relaxation time must be averaged over space. This accounts for the ratio of integrals in Eq. (71). The second is that the total energy associated with a temperature anisotropy has contributions from the potential energy as well as from the kinetic energy. For an harmonic trap, the average potential and kinetic energies associated with motion along a given axis are equal and therefore the total energy for a given direction is twice the kinetic energy. Since collisions isotropize directly only the kinetic energy, the time to isotropize both the kinetic and potential energies is twice as large as that calculated neglecting the effect of the potential energy. This is the origin of the factor of 1/2 in Eq. (71).

Neglecting the effects of quantum degeneracy, Eq. (71) gives

$$\Gamma_T \leq \frac{1}{8\sqrt{2}} n_{\text{el}}(0) \sigma(0) v.$$

(72)

Equation (72) gives for an energy-independent cross section $\sigma(0) = \sigma_0$,

$$\Gamma_T \leq \frac{1}{5} n_{\text{el}}(0) \sigma_0 v.$$  

(73)

Considering the case of a more general form for the scattering cross section, Eq. (72) can be written in the form

$$\Gamma_T \leq \frac{1}{15} n_{\text{el}}(0) \sigma_0 \bar{\sigma} F(t).$$

(74)

As defined earlier, $t = T/T_0$, with $k_BT_0 = \hbar^2/ma^2$ being the characteristic energy that enters the scattering cross section and

$$F(t) = \int_0^\infty dx_r x_r^5 e^{-x_r^2} \bar{\sigma}.$$  

(75)

For the specific choice $\bar{\sigma} = (1 + x_r^2 t)^{-1}$, the above expression takes the form

$$F(t) = \int_0^\infty \frac{x_r^5 e^{-x_r^2}}{1 + x_r^2 t} dx_r.$$  

(76)

For $T_0 < T_0$ the scattering cross section is essentially energy independent, $\sigma(u) \approx \sigma_0$, whereas in the opposite limit the cross section is resonant, $\sigma(u) \approx 8\pi/k^2$. For Cs atoms, a lower bound on the magnitude of their scattering length has been determined to be $260 \pm 8 \text{nm}$, where $a_B$ is the Bohr radius; therefore $T_0 \leq 15 \mu$K. Figure 2 shows the result of Eq. (74) for Cs atoms, for a peak density $n_{\text{el}}(0) = 10^{10} \text{cm}^{-3}$. The $x$ axis shows the temperature in units of $T_0$ and the $y$ axis shows $\Gamma_T$ measured in units of

$$\Gamma_{T,0} = \frac{1}{15} n_{\text{el}}(0) \sigma_0 \bar{v}_0,$$

(77)

where $\bar{v}_0$ is the velocity $\bar{v}_0 = (8\pi k_BT_0/m)^{1/2}$.
FIG. 2. The thermal relaxation rate $\Gamma_T$ as a function of $T/T_0$, as given by the classical version of Eq. (74), for a cross section $\sigma(k) = 8\pi a^2/[1 + (ka)^2]$. $\Gamma_T$ is measured in units of $\Gamma_{T,0} = n_{cl}(0)\sigma_0\bar{v}/15$.

A simple interpolation formula for the ratio $\Gamma_T/\Gamma_{T,0}$ is given by

$$\frac{\Gamma_T}{\Gamma_{T,0}} = \frac{3\sqrt{2}}{[1 + (3t)^\nu]^{1/\nu}},$$

(78)

where $\nu \simeq 0.9$. For Cs atoms we find that $\Gamma_{T,0} \geq 0.2 \text{ s}^{-1}$, assuming that the magnitude of the scattering length is greater than, or equal to $260 \text{ a}_B$ [6] and for $n_{cl}(0) = 10^{10}$ cm$^{-3}$, which is the typical density in the experiments reported in Ref. [6] performed with Cs atoms.

Equation (72) for $\Gamma_T$ is useful for the case of Cs atoms; under the experimental conditions [6] it is a rather good approximation to assume that the scattering is resonant, with $\sigma = 8\pi/k^2$, where $k = m\bar{v}/\hbar$ is the relative momentum, since $T > T_0$. We then obtain the result in the classical regime

$$\Gamma_T \leq \frac{64}{15} \frac{n_{cl}(0)\hbar^2}{m^2\bar{v}}.$$  

(79)

It is instructive to define an effective cross section $\sigma_{\text{eff}}$ according to

$$\sigma_{\text{eff}} = \frac{\Gamma_T}{n_{av}\bar{v}}.$$  

(80)

From Eq. (74) we see that

$$\sigma_{\text{eff}} = 2\sqrt{2} \frac{\sigma_0 F(t)}{15}.$$  

(81)

Figure 3 shows $\sigma_{\text{eff}}$ given by Eq. (80) as function of $T/T_0$. For energy-independent scattering,

$$\sigma_{\text{eff}} = \frac{2\sqrt{2}}{5} \sigma_0 \equiv \sigma_{\text{eff},0},$$  

(82)

whereas for resonant scattering,

$$\sigma_{\text{eff}} = \frac{128\sqrt{2}}{15} \frac{\hbar^2}{(m\bar{v})^2}. $$  

(83)

Finally using Eq. (81) we plot in Fig. 4 the effective cross section $\sigma_{\text{eff}}$ for Cs atoms as a function of the temperature for four different values of the scattering length, $a = 100 \text{ a}_B, 200 \text{ a}_B, 300 \text{ a}_B$, and $400 \text{ a}_B$. For very small values of the temperature $\sigma_{\text{eff}} = \sigma_{\text{eff},0} = 0.142, 0.569, 1.278$ and $2.272 \times 10^{-11}$ cm$^2$, respectively.
that the matrix elements by analogy with Eq. (58). In this case we find that only given by Eq. (81), as a function of $a=100 a_B, 200 a_B, 300 a_B,$ and $400 a_B$ from bottom to top.

C. Improved estimates

Following the same method as in Sec. III C we calculate the corrections to the thermal relaxation rate $\Gamma_T$ using better trial functions. We choose the improved trial function to be

$$\Phi_{T,\text{imp}} = (1 - \alpha^2)^{1/2} \Phi_{T,0} + \alpha \Phi_{T,1}, \quad (84)$$

where $\alpha$ is a variational parameter. $\Phi_{T,0}$ is given by Eq. (58) and for $\Phi_{T,1}$ we take

$$\Phi_{T,1} = \sqrt{5} \left[(p_z^2 - p^2/3 + (\hbar \omega)^2 (z^2 - r^2/3)) \right] \times (1 - E/5k_B T) \quad (85)$$

by analogy with Eq. (58). In this case we find that only the parts proportional to $E p_z^2 - p^4/6m$ give non-vanishing terms in addition to the terms we considered earlier with $\Phi_{T,0}$ only, and thus

$$\Delta[\Phi_{T,\text{imp}}] = 3 \langle p_{r,z}^2 - p_{r,z}^{'2} \rangle \left[(1 - \alpha^2)^{1/2} + \alpha \sqrt{5} \right] \left. \frac{\alpha \sqrt{5}}{5} \frac{V}{k_B T} + \frac{\alpha \sqrt{5}}{10} \frac{1}{m k_B T} \left[p^2 + \frac{P^2}{4} \right] \right]$$

$$- \alpha 3 \sqrt{5} \frac{1}{10} \frac{1}{m k_B T} (P_z p_{r,z} \mathbf{P} \cdot \mathbf{p}_r - P_z p_{r,z}^{'} \mathbf{P} \cdot \mathbf{p}_r^{'}),$$

$$= \alpha 3 \sqrt{5} \frac{1}{10} \frac{1}{m k_B T} \left[(\mathbf{p} \cdot \mathbf{P})^2 - (\mathbf{p}^{'} \cdot \mathbf{P})^2 \right]. \quad (86)$$

For the case of energy-independent scattering, we find that the matrix elements $\gamma_{1,0}/\gamma_{0,0} = -\sqrt{5}/20,$ and $\gamma_{1,1}/\gamma_{0,0} = 277/240.$ Therefore for energy-independent scattering the thermal relaxation rate $\Gamma_{T,\text{imp}}$ corresponding to $\Phi_{T,\text{imp}}$ is given by

$$\Gamma_{T,\text{imp}}(\alpha) = \Gamma_T \left[1 - \frac{\sqrt{5}}{10} \alpha(1 - \alpha^2)^{1/2} + \frac{37}{240} \alpha^2 \right]. \quad (87)$$

The minimum is at $\alpha \approx 0.47$ and the value of the function is $\approx 0.941 \Gamma_T,$ i.e., the correction is $\approx 6\%.$ Therefore, our final answer for $\Gamma_{T,\text{imp}}$ for the case of energy-independent scattering is

$$\Gamma_{T,\text{imp}} \approx 0.189 n_{\text{el}}(0) \sigma_0 \bar{v}. \quad (88)$$

In a study of the problem of temperature relaxation, Myatt has used the Boltzmann equation to obtain an analytic expression for the temperature relaxation rate, and has solved the problem numerically for large temperature anisotropies. The coefficient 1/5 which appears in Eq. (73) is consistent with his results. In a numerical approach of the same problem, Monroe et al. have found that a Monte Carlo calculation gives for this coefficient a value of 0.185, which is remarkably close to our result, 0.189.

For the case of resonant scattering we find $\gamma_{1,0}/\gamma_{0,0} = \sqrt{5}/20,$ and $\gamma_{1,1}/\gamma_{0,0} = 53/48$ and thus the correction is $\approx 7\%$ in this case, or

$$\Gamma_{T,\text{imp}} \approx 4 \frac{n_{\text{el}}(0) \hbar^2}{m \bar{v}^2}. \quad (89)$$

As shown by Arndt et al. in Ref. (8), a Monte Carlo calculation for $\Gamma_T$ for the case of resonant scattering gives a value for this coefficient of $\approx 64/(2 \times 10.7) \approx 64/21 \approx 3,$ instead of the coefficient 4 that appears in Eq. (89).

D. Comparison with experiment

The experiment performed by Arndt et al. with Cs atoms shows that the collisional cross section is inversely proportional to the temperature in the range $5 - 60 \mu K.$ This indicates that $T_0 \lesssim 5 \mu K.$ Since these temperatures are larger than $T_0$ for Cs, according to our study the cross section should indeed vary as $T^{-1}.$ Our result given by Eq. (85) fits the data of Ref. (8) rather well.

In the experiment of Monroe et al. the collisional cross section of Cs atoms was measured to be $\approx 1.5 \times 10^{-12} \text{ cm}^2,$ and it was independent of the temperature in the range between $30 - 250 \mu K.$ This value of the cross section is of the same order as the data in Ref. (8) for the highest temperatures. As Fig. 4 implies, our calculated cross sections for these temperatures are also of order $10^{-12} \text{ cm}^2.$

In the experiment of Davis et al. a quadrupole trap which gives a potential proportional to $(x^2 + y^2 + z^2)^{1/2}$ was used to confine sodium atoms. The atoms were observed to oscillate with an effective frequency of the order of kHz. A non thermal distribution was created by

FIG. 4. The effective cross section $\sigma_{\text{eff}}$ for Cs atoms given by Eq. (34) as a function of $T,$ for a cross section of the form $\sigma(k) = 8\pi a^2/[1 + (ka)^2].$ The four lines correspond to the following choices for the scattering length, $a = 100 a_B, 200 a_B, 300 a_B,$ and $400 a_B$ from bottom to top.
temporarily displacing the trap center along the $z$ axis. The time for relaxation to equilibrium was measured and found to be of the order of seconds, for a temperature of $200 \, \mu K$ and for densities in the center of the trap of order $10^{10} \, cm^{-3}$.

We cannot immediately apply our results to this experiment because the potential is not separable in cartesian coordinates. However Monte Carlo simulations show that motions are effectively separable (non ergodic) for periods of at least $10 \, s$ \[^{[9]}\]. To estimate the relaxation time for temperature anisotropies, we observe that the rate at which the particle distribution is isotropized in momentum space by collisions is given by the ratio of the integrals in Eq. (71). If the potential were separable in cartesian coordinates and were a homogeneous function of the coordinates of degree $\nu$, the potential energy associated with motion in any direction would be $\nu/2$ times the kinetic energy, and therefore the relaxation rate allowing for the potential energy would be

$$
\Gamma_{T,\nu} \leq \frac{1}{1 + \nu/2} \int \frac{dr}{f} \frac{\tau_{\nu,\text{var}}^{-1}(p_z^2 - p^2/3) f^{0}(1 + f^{0})}{f^{0}}. \quad (90)
$$

This result reduces to Eq. (11) for a harmonic potential ($\nu = 2$), and to the expression $\Gamma_{T} \leq \tau_{\nu,\text{var}}^{-1}$ for a uniform gas ($\nu = 0$). Because the mixing times due to the non-separability of the trapping potential are long, we would expect that for the quadrupole trap this result, with $\nu = 1$, should give a good estimate of the thermal relaxation rate. For classical statistics, energy-independent scattering and $\nu = 1$, Eq. (90) gives

$$
\Gamma_{T,1} \leq \frac{1}{15\sqrt{2}} n_{cl}(0) \sigma_0 \bar{v}. \quad (91)
$$

The spatial average that occurs in the present case is $\int n_{cl}^2(r) \, dr / \int n_{cl}(r) \, dr = n_{cl}(0)/2^3$, instead of $n_{cl}(0)/2^{3/2}$ for harmonic potentials.

The relaxation rate that was measured in Ref. \[^{[3]}\] can be written as

$$
\Gamma_{T,1} \approx \frac{1}{4.88} n_{cl}(0) \sigma_0 \bar{v}, \quad (92)
$$

if we assume that $a = 27.5 \, \text{Å}$ for sodium atoms \[^{[14]}\]. Comparing Eqs. (91) and (92) we see that there is a difference of a factor $\approx 4.3$. We regard the agreement as satisfactory because of the uncertainty in our estimate due to the unknown role of the lack of separability of the potential, and because of the difficulty of measuring particle densities experimentally \[^{[15]}\].

V. CONCLUDING REMARKS

In this paper we have calculated the damping of low vibrational modes and of thermal anisotropies using a Boltzmann equation approach. The main difference between the present problem and that of calculating the rate of relaxation processes in a homogeneous gas is that the distribution function has a more complicated spatial variation. This is because the motion is essentially collisionless, and consequently the distribution function is a function of both momentum and position, whereas for a homogeneous gas in the hydrodynamic limit the relevant distribution function is a function only of momentum, apart from a rather simple spatial dependence. We formulated the problem as a variational one, and found that for the simplest trial functions the damping times for vibrational modes and for thermal relaxation differ only by numerical factors. With the simplest variational functions, the most long-lived modes in thermal relaxation correspond to an anisotropy of the temperature, as Monroe et al. assumed in their calculation. This picture is not true in general, and an improved trial function for this mode led to a damping time $6\%$ longer than the simplest variational estimate, and in agreement with what was found in Monte Carlo simulations of the collisional dynamics for a constant cross section. For a resonant cross section there appears to be a significant discrepancy between our results and the Monte Carlo simulations of the collisional dynamics of Ref. \[^{[4]}\], and the reason for this is at present unclear. For damping of vibrational modes, the simplest trial function gives a very good description, and an improved one gave changes in the damping time of less than one per cent. If necessary, further improvement of our estimates is possible by employing more general trial functions. The existence of rather precise expressions for relaxation rates should prove a useful tool in analysis of experimental data.

Our calculations were carried out under the assumption that the semiclassical approximation is valid, not only for the streaming terms in the Boltzmann equation, but also for the collision term. For smaller numbers of particles, and/or larger trap frequencies the effects of the discreteness of the spectrum of single-particle energies in the trap will become important, and the situation will be closer to that encountered in studies of collective motion in finite nuclei \[^{[24]}\].

APPENDIX A: QUANTUM DEGENERACY CORRECTIONS

Including effects of quantum degeneracy to leading order in the fugacity $z(r)$, we find for energy-independent scattering, $\sigma(u) = \sigma_0$, that the collision integral in Eq. (38) gives

$$
\bar{I}_c[z(r)] = z^2 \sqrt{\pi} \left( 1 + z \frac{9}{16} \sqrt{\frac{3}{2}} \right). \quad (A1)
$$

Expanding the denominator of Eq. (38) we find

$$
\bar{D}(r) = z^2 \left( 1 + z \frac{1}{4\sqrt{2}} \right). \quad (A2)
$$
Combining Eqs. (A1) and (A2) we thus find that Eq. (38) takes the form

$$\Gamma_0 \leq \frac{4}{15\sqrt{2}} \left( \frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} \sigma_0 \bar{v} \times$$

$$\int z^2(r) \left( 1 + z(r) \frac{9}{16} \sqrt{\frac{3}{2}} \right) dr \times \int z(r) \left( 1 + \frac{z(r)}{4\sqrt{2}} \right) dr .$$

(A3)

Performing the spatial integrations in Eq. (A3), we find for a harmonic potential

$$\Gamma_0 \leq \frac{1}{15} \sigma_0 \bar{v} \left( \frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} z(0) \left[ 1 + \frac{5z(0)}{16} \right],$$

where $z(0)$ is the fugacity calculated at the center of the cloud, $r = 0$. We now calculate $z(0)$ as a function of the total number of particles $N = \sum_p \int dr f^0(r, p)$, which gives up to order $z^2(0)$

$$N \equiv \left( \frac{k_BT}{\hbar \omega} \right)^3 z(0) \left( 1 + \frac{z(0)}{8} \right).$$

(A5)

Solving the above equation in terms of $z(0)$ up to $N^2$ and using the expression for the critical temperature $k_BT_c = \hbar \omega [N/(\zeta(3))]^{1/3}$, we find

$$z(0) = \left( \frac{T_c}{T} \right)^3 \zeta(3) \left[ 1 - \frac{\zeta(3)}{8} \left( \frac{T_c}{T} \right)^3 \right].$$

(A6)

Inserting Eq. (A6) into Eq. (A3) we obtain

$$\Gamma_0 \leq \frac{1}{15} n_{cl}(0) \sigma_0 \bar{v} \left[ 1 + \frac{3}{16} \zeta(3) \left( \frac{T_c}{T} \right)^3 \right],$$

(A7)

where $n_{cl}(0) = N\bar{v}^3 [m/2\pi k_BT]^3/2$.

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