Thin-film equations with mildly singular potentials: an alternative solution to the contact-line paradox

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In the regime of lubrication approximation, we look at spreading phenomena under the action of mildly singular potentials of the form $P(h) \approx h^{1-m}$ as $h \to 0^+$ with $1 < m < 3$, modeling repulsion between liquid-gas interface and substrate. We assume zero slippage at the contact line. Based on formal analysis arguments, here we report that for any $m \in (1, 3)$ and any value of the speed (both positive and negative) there exists a two-parameter, hence generic, family of fronts (i.e., traveling-wave solutions with a contact line): they have finite energy, finite rate of dissipation, and microscopic contact angle equal to $\pi/2$, in agreement with energy minimizers. A one-parameter family of advancing “linear-log” fronts also exists, having a logarithmically corrected linear behaviour in the liquid bulk. These facts indicate that mildly singular potentials stand as an alternative solution to the contact-line paradox.

We also propose a selection criterion for the fronts, based on thermodynamically consistent contact-line conditions modeling friction at the contact line. So as contact-angle conditions do in the case of slippage models, this criterion selects a unique linear-log front for each positive speed. Numerical evidences suggest that, fixed the speed and the frictional coefficient, its shape depends on the spreading coefficient, with steeper fronts in partial wetting and a more prominent precursor region in dry complete wetting.

Key words: Drops, Contact lines, Thin films, Wetting and wicking, Lubrication theory

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1. Introduction

1.1. The model

In lubrication approximation (Greenspan 1978; Hocking 1983; Oron et al. 1997; Giacomelli & Otto 2003; Knüpfer & Masmoudi 2015), the spreading of a droplet is described by the thin-film equation, which in non-dimensional form reads as

$$ h_t + (hV)_x = 0, \quad V = \frac{m(h)}{h}(h_{xx} - Q'(h))_x \quad \text{on } \{ h > 0 \}. \quad (1.1) $$

The free energy of the system is given by

$$ E[h] = \int_{\{ h > 0 \}} \left( \frac{1}{2} h_x^2 + Q(h) \right) dx = \int_{\{ h > 0 \}} \left( 1 + \frac{1}{2} h_x^2 + (Q(h) - 1) \right) dx. \quad (1.2) $$

In lubrication theory, the term \((1 + \frac{1}{2} h_x^2)\) is the leading-order approximation of the liquid-gas surface energy \(\sqrt{1 + h_x^2}\). The summand 1 is incorporated in the definition of the potential \(Q\), which usually combines the effects of intermolecular, surface, and gravitational forces (de Gennes 1985); here we shall ignore the latter ones for simplicity:

$$ Q(h) = (P(h) + G(h) - S)\chi_{\{ h > 0 \}}, \quad \text{with } G \equiv 0 \text{ and } S \in \mathbb{R} \text{ in this manuscript.} \quad (1.3) $$

The function \(P\) is a inter-molecular potential, singular as \(h \to 0^+\) and decaying to zero as \(h \to +\infty\). The constant \(S\) (assumed to be small in lubrication theory, cf. Remark 1.2) is the non-dimensional spreading coefficient:

$$ S = \frac{\text{spreading coefficient}}{\gamma} = \frac{\gamma_{SG} - \gamma_{SL} - \gamma}{\gamma} = \frac{\gamma_{SG} - \gamma_{SL}}{\gamma} - 1, $$

where \(\gamma, \gamma_{SL}, \text{ and } \gamma_{SG}\) are liquid-gas, solid-liquid, and solid-gas tensions, respectively. There is, however, a caveat to be made at this point. In thermodynamic equilibrium of the solid with the surrounding vapor phase (the so-called “moist” case, which concerns for instance a surface which has been pre-exposed to vapor), \(\gamma_{SG}\) is usually denoted by \(\gamma_{SV}\), and its value can never exceed \(\gamma_{SL} + \gamma\). Indeed, otherwise the free energy of a solid/vapor interface could be lowered by inserting a liquid film in between: the equilibrium solid/vapor interface would then comprise such film, leading to \(\gamma_{SV} = \gamma_{SL} + \gamma\). Therefore, \(S \leq 0\) in the “moist” case. On the other hand, when the solid and the gaseous phases are not in thermodynamical equilibrium (the so-called “dry” case), there is no constraint on the sign of \(S\).

1.2. Statics

Let us briefly discuss the statics first, considering absolute minimizers \(h_{\min}\) of \(E\) in \(\mathbb{R}\) under the constraint of fixed mass: \(\int_{\mathbb{R}} h(x) dx = M > 0\).

When \(P \equiv 0\) and \(S < 0\), \(h_{\min}\) is an arc of parabola characterized by \(M\) and \(|S|\); in particular, its slope at the contact line \(\partial \{ h > 0 \}\) where liquid, gas and solid meet, is determined by \(|S|\):

$$ h_{\min} = \frac{3M}{4s^3} (s^2 - x^2)^+, \quad s^2 = \frac{3M}{2 \tan \theta_S}, \quad \theta_S := \arctan \sqrt{2|S|}, \quad S < 0. $$

When \(P \equiv 0\) and \(S \geq 0\), \(h_{\min}\) instead does not exist, and minimizing sequences converge to an unbounded film with zero thickness. Therefore it is common to define the static (or equilibrium) microscopic contact angle \(\theta_S\), and to name the two regimes, as follows:

$$ \theta_S := \begin{cases} 
\arctan \sqrt{2|S|} & \text{if } S < 0 \quad \text{(partial wetting)} \\
0 & \text{if } S \geq 0 \quad \text{(complete wetting)}.
\end{cases} \quad (1.4) $$
The picture changes when intermolecular potential are taken into account. Consider\(^1\)

\[
P(h) \sim \frac{A}{m-1} h^{1-m} \quad \text{as } h \to 0^+ \quad \text{with } A > 0 \text{ and } m > 1, \quad P(0) = P(+\infty) = 0.
\]

By a rescaling, we may assume without losing generality that \(A = 1\):

\[
P(h) \sim \frac{1}{m-1} h^{1-m} \quad \text{as } h \to 0^+ \quad \text{with } m > 1, \quad P(0) = P(+\infty) = 0. \quad (1.5)
\]

The potential \(P\) is short-range repulsive, in the sense that it penalizes short distances between the solid and the liquid-gas interface. It is easy to realize that \(h_{\text{min}}\) does not exist if \(m \geq 3\), in the sense that \(E[h] \equiv +\infty\) in \(L^1(\mathbb{R})\): in words, repulsion is so strong that a liquid film of positive thickness is interposed between solid and gas. If on the other hand the singularity is milder, \(1 < m < 3\), minimizers do exist, and the minimization problem has been rigorously addressed in Durastanti & Giacomelli (submitted) for general forms of \(P\) and including gravity. It turns out that minimizers have a microscopic contact angle \(\theta_S = \pi/2\). In agreement with the formal analysis of \((1.2)\) performed by de Gennes (1985), the microscopic gravity. It turns out that minimizers have a microscopic contact angle \(\theta_{\text{mac}} = \arctan \sqrt{2|S|}\) for \(M \gg 1\) if \(S < 0\), that is

\[
h_{\text{min}} \sim e \chi_{\{|x| \leq s\}}, \quad s = \frac{M}{2e} \quad \text{for } M \gg 1 \quad \text{if } S > 0,
\]

where \(e\) is the smallest absolute minimum point of \(Q(h)/h\), and a “droplet” if \(S < 0\), that is

\[
h_{\text{min}} \sim \frac{3M}{4s^3} \left(s^2 - x^2\right)_+, \quad s^2 = \frac{3M}{2 \tan \theta_{\text{mac}}}, \quad \theta_{\text{mac}} := \arctan \sqrt{2|S|} \quad \text{for } M \gg 1 \quad \text{if } S < 0,
\]

where \(\theta_{\text{mac}}\) denotes the macroscopic contact angle (see Fig. 1). When \(S = 0\), generally speaking, the shape is droplet-like, resp. pancake-like, if the potential is repulsive, resp., attractive, for large \(h\). In this respect, mildly singular potentials provide a more realistic picture with respect to both slippage models in complete wetting and strongly singular potentials: indeed, in both cases global minimizers with compact support do not exist, and pancakes are replaced by an unbounded film of zero, resp. positive, thickness.

![Diagram](image)

Figure 1: Global minimizers under mildly singular potentials \((1 < m < 3)\): pancake \((S > 0, \text{left})\) or droplet \((S < 0, \text{right})\).

1.3. Dynamics

Let us now turn to the dynamics. A simple formal computation shows that smooth, positive and, say, periodic solutions to \((1.1)\) (e.g. modelling a liquid film) satisfy

\[
\frac{d}{dt} E[h] = -\int_T \frac{h^2}{m(h)} V^2 \, dx, \quad E[h] = \int_T \left(\frac{1}{2} h_x^2 + Q(h)\right) \, dx. \quad (1.6)
\]

For droplets, however, the energy balance is more subtle, as boundary terms at \(\partial \{h > 0\}\) appear in \((1.6)\). Here we assume for simplicity that \(\{h > 0\}\) is and remains connected for all times (i.e. we exclude coalescence or splitting of droplets):

\[
\{h > 0\} = \{(t, x) \in \mathbb{R} \times \mathbb{R} : t > 0, x \in (s_-(t), s_+(t))\},
\]

As \(h \to h_0\), we write: \(f(h) \sim g(h)\) when \(f(h)/g(h) \to 1\); \(f(h) \approx g(h)\) when \(C > 0\) exists such that \(f(h)/g(h) \to C\); \(f(h) = O(g(h))\) when \(f(h)/g(h)\) remains bounded; \(f(h) = o(g(h))\) when \(f(h)/g(h) \to 0\).
$s_\pm(t)$ denoting the contact lines. Since $s_\pm(t)$ are unknown and (1.1) is of fourth order, three conditions are needed for well-posedness. Two of them are obvious:

$$h|_{x=s_\pm(t)} = 0 \quad \text{and} \quad \dot{s}_\pm(t) = V|_{x=s_\pm(t)}. \quad (1.7)$$

The first one defines the contact lines $s_\pm(t)$, while the second one is a kinematic condition guaranteeing no mass flux through $s_\pm$. The third condition, the so-called contact-line condition, is yet debated (to a certain extent inevitably, due to the variety of material properties and configurations, which may involve surface roughness and hysteretic effects; see e.g. Feldman & Kim (2018), Alberti & DeSimone (2005, 2011), as well as the reviews cited below). The most common one amounts to prescribing a constant microscopic contact angle equal to the static one:

$$|h_x|_{x=s_\pm(t)} = \tan \theta_S. \quad (1.8)$$

Of particular interest to us is a relatively recent proposal by Ren & E (2007) (see also Ren et al. (2010), Ren & E (2011)), based on consistency with the second law of thermodynamics. In lubrication approximation (Chiricotto & Giacomelli 2011, 2013), when $P \equiv 0$ and $S = -\frac{1}{2} \tan^2 \theta_S$ (the moist case, cf. (1.4)), its simplest form reads as follows:

$$\pm \left( h_x^2 - (\tan \theta_S)^2 \right)|_{x=s_\pm(t)} = \mu \left\{ \begin{array}{ll}
\dot{s} & \text{if } \theta_S > 0 \\
\max\{0, \dot{s}\} & \text{if } \theta_S = 0
\end{array} \right. \quad \text{(partial wetting)} \quad \text{(complete wetting)} \quad (1.9)$$

with $\mu > 0$ (more precisely one should write $(h_x^2 - 2\dot{h}_x h_{xx})$ in place of $h_x^2$, but it is expected that the second summand always vanishes at $x = s_\pm(t)$). Indeed, under (1.9), the energy balance reads as

$$\frac{d}{dt} \int_{s_\pm(t)} \left( \frac{1}{2} h_x^2 - S \right) = -\mu (|\dot{s}_+|^2 + |\dot{s}_-|^2) - \int_{s_\pm(t)} m(h) h_{xxx}^2 \quad (1.10)$$

(see Chiricotto & Giacomelli (2017)), which shows that (1.9) is consistent with the second law and accounts for frictional forces at the contact line, with $\mu > 0$ a friction coefficient. When $\mu = 0$ (null contact-line friction), (1.9) coincides with (1.8).

### 1.4. The contact-line paradox

When the no-slip condition is assumed at the liquid-solid interface, the mobility $m$ is given by

$$m(h) = h^3. \quad (1.11)$$

When $P \equiv 0$, it is well known that the no-slip condition yields to the so-called contact-line paradox (Huh & Scriven 1971; Dussan V. & Davis 1974): “not even Herakles could sink a solid”, that is, the droplet’s support remains fixed for all times (to our knowledge, however, a rigorous proof of it is yet missing). This paradox manifests itself with the non-existence of advancing traveling-wave solutions to (1.1), whereas receding ones are unphysical, in that the rate of energy dissipation in (1.6) is unbounded as $h \to 0^+$ (see Remark 3.1 below).

The contact-line paradox motivated the introduction of quite a few enrichments of the basic model, the most common ones being slip conditions, intermolecular potentials, and non-newtonian rheologies: we refer to the discussions in Oron et al. (1997); de Gennes (1985); Bonn et al. (2009); Snoeijer & Andreotti (2013) for the former two ones, and also to Flitton & King (2004); King (2001); Weidner & Schwartz (1994); Ansini & Giacomelli (2002, 2004) for the latter. All of them introduce at least one “microscopic” lengthscale into the model. For instance, the introduction of slip conditions modifies the mobility as
\( m(h) = h^3 + \lambda^{3-n}h^n \), where \( \lambda > 0 \) represents a slippage length-scale (its inverse can be taken as a measure of the friction between liquid and solid) and \( n \in (0, 3) \) depends on the slippage law (\( n = 2 \) for the classical Navier’s slip condition). Therefore, when \( P \equiv 0 \) the thin-film equation with slippage reads as

\[
h_t + ((h^3 + \lambda^{3-n}h^n)h_{xxx})_x = 0 \quad \text{on}\{h > 0\}.
\]

Besides slippage, a common and computationally very convenient circumvention of the contact-line paradox is the use of intermolecular potentials with a strong singularity:

\[ P(0) = +\infty \text{ and/or } m \geq 3 \text{ in (1.5)} \]

(see e.g. Eggers (2005), Pismen & Eggers (2008), as well as the recent review by Witelski (2020)). Indeed, as we mentioned already, the strength of the singularity forces the liquid to fully cover the solid with a film of positive (though small) thickness, so that the paradox itself becomes invisible.

Somewhat surprisingly, the case of mildly singular potentials, i.e. \( 1 < m < 3 \), received much lesser attention. For \( 1 < m < 2 \) and \( m(h) = h^3 \), Bertozzi & Pugh (1994) observed that advancing fronts (i.e., traveling wave solutions with a contact line) exist with null microscopic contact angle (\( \theta_S = 0 \) in (1.8)). However, as we shall see, these fronts are non-generic and, most importantly, their rate of dissipation is unbounded near the contact line (as well as their energy if \( m \geq 3/2 \); see (TW0) in §3.1. In fact, based on the formal analysis of traveling wave solutions, in the next sections we will argue that for \( 1 < m < 3 \) and \( m(h) = h^3 \), generic fronts (both advancing and receding) have finite energy, finite rate of dissipation, and microscopic contact angle equal to \( \pi/2 \), in agreement with energy minimizers (cf. (A) below). In this sense, mildly singular potentials stand as an alternative solution to the contact-line paradox. Among them, advancing “linear-log” fronts also exists, having a logarithmically corrected linear behaviour in the liquid bulk (cf. (B) below). We will also propose a selection criterion for the fronts, based on a class of thermodynamically consistent contact-line conditions modelling friction at the contact line (cf. (C) below).

**Remark 1.1.** For the sake of simplicity, here we will only discuss the most relevant case of zero slippage, \( m(h) = h^3 \). However, the qualitative parts of (A)-(C) continue to hold for more general mobilities, e.g. of the form \( m(h) = h^3 + \lambda^{3-n}h^n \).

**Remark 1.2.** One may question whether a microscopic contact angle equal to \( \pi/2 \) is consistent with the regime of lubrication approximation. In this respect, we should mention that the rigorous derivation of the thin-film equation in Giacomelli & Otto (2003) only requires global smallness conditions on the ratio between vertical and horizontal length-scales, in form of relations between mass, energy, and second moments (the result is proved for Darcy’s flow in complete wetting, but it is plausible that similar conclusions may be drawn as well for Stokes flow and partial wetting). In fact, the validity of lubrication theory under global (hence weak) assumptions is most evident when looking at the statics: for instance, in partial wetting with \( P \equiv 0 \), it has been shown that

\[
\frac{1}{\varepsilon^2} \int_{\{h > 0\}} \left((1 + \varepsilon^2 h_x^2)^{1/2} - 1 - \varepsilon^2 S\right)dx \xrightarrow{\varepsilon \to 0} E[h] = \int_{\{h > 0\}} \left(\frac{1}{2} h_x^2 - S\right)dx
\]

under the sole assumptions that \( h \geq 0 \) has finite energy, mass, and second moment, cf. Giacomelli & Otto (2001, (3) in Proposition 1).
2. Results and open questions

2.1. Traveling waves

The first observation we wish to report is that mildly singular potentials generically solve the contact-line paradox. To this aim, in Section 3 we discuss traveling-wave solutions (or, equivalently, fronts) to (1.1)-(1.7) with finite energy and finite rate of bulk dissipation. These are solutions of the form

\[ h(t,x) = H(y), \quad y = x + Ut > 0, \quad U \in \mathbb{R} \setminus \{0\} \]

with

\[ H(0) = 0, \quad \text{supp } H = [0, +\infty), \quad (2.1) \]

(capitalizing on translation invariance, we have assumed without losing generality that the contact line is at \( y = 0 \), finite energy at the contact line in the sense that

\[ \int_0^1 \left( \frac{1}{2} (H_y)^2 + Q(H) \right) dy < +\infty, \quad (2.2) \]

and finite rate of bulk dissipation at the contact line in the sense that

\[ \int_0^1 H^3 \left( (H_y - Q'(H)) y \right)^2 dy = \int_0^1 U^2 H^{-1} dy < +\infty. \quad (2.3) \]

Consistently with (1.5), we assume that

\[ P''(h) = mh^{-m-1}(1 + o(1)) \quad \text{as } h \to 0^+, \quad 1 < m < 3, \quad P(0) = P(+\infty) = 0. \quad (2.4) \]

The formal argument in §3, supported by numerical evidences in §5.1, indicates that both advancing and receding fronts exist for any speed even in presence of a no-slip condition. We now summarize the outcome of the analysis.

2.1.1. Quadratically growing fronts

(A) Quadratic traveling-waves. Assume (1.3), (1.11), and (2.4). For any \( U \in \mathbb{R} \setminus \{0\} \), (1.1) has a two parameter \((c_1, c_2) \in \mathbb{R}\) family of traveling-wave solutions \( H(y) \) satisfying (2.1), (2.2), and (2.3). They have quadratic growth as \( y \to +\infty \) and satisfy \( H_y(0) = +\infty \) in the sense that

\[ \frac{1}{2} (H_y(y(H)))^2 = P(H) \left( 1 + c_1 H^{m-1} + o(H^{m-1}) \right) \quad \text{as } H \to 0, \quad (2.5) \]

where \( o(H^{m-1}) \) is a function determined by \( m, U, P, c_1, \) and \( c_2 \).

At leading order, (2.5) implies that

\[ H(y) = \left( \frac{(m+1)^2}{2(m-1)} \right)^{\frac{1}{m-1}} y^{\frac{2}{m-1}} (1 + o(1)) \quad \text{as } y \to 0^+, \quad (2.6) \]

which coincides with the behavior of the minima \( h_{\min} \) of \( E \). Of course, the non-generic one-parameter family of advancing zero-contact angle solutions analysed by Bertozzi & Pugh (1994) also exists; however, their rate of bulk dissipation is unbounded (as well as their energy if \( m \geq 3/2 \)); see (TW_0-b) in §3.1.

2.1.2. Linear-log fronts

Starting from the work of Voinov (1976), various formal asymptotic arguments have been developed for wetting phenomena. In this framework, advancing fronts \( (U > 0) \) are usually matched to a macroscopic profile (see e.g. Greenspan (1978); Hocking (1983); Cox (1986); Ehrhard & Davis (1991); Haley & Miksis (1991); Hocking (1992); Bertsch et al. (2000);
Eggers & Stone (2004); Eggers (2005); Pismen & Eggers (2008); Chiricotto & Giacomelli (2013)). Such matching (parts of which were made rigourous in Giacomelli & Otto (2002); Giacomelli et al. (2016); Delgadino & Mellet (2021)) requires to select those traveling waves which, instead of a quadratic one, display a linear (though logarithmically corrected) growth for $y \gg 1$. The formal argument in §3, supported by the numerical evidence in §5.1, indicates that these linear-log fronts exist also for mildly singular potentials:

(B) **Linear-log traveling-waves.** Assume (1.3), (1.11), and (2.4). For any $U > 0$, (1.1) has a one-parameter family of traveling-wave solutions $H$ satisfying (2.1), (2.2), and (2.3). They satisfy $H_y(0) = +\infty$, in the sense that (2.5) holds, and have linear-log behavior as $y \to +\infty$, in the sense that

$$H_y^3(y(H)) = 3U \left( \log H - \frac{1}{3} \log \log H + a + O \left( \frac{\log \log H}{\log H} \right) \right) \quad \text{as } H \to +\infty \quad (2.7)$$

(a $\in \mathbb{R}$).

The picture in (A) and (B) obviously contrasts the well-known case $P \equiv 0$, since in that case traveling waves with bounded dissipation do not exist at all for $n = 3$ (Remark 3.1).

On the other hand, the picture in (A) and (B) is analogous to the case $P \equiv 0$ with positive slippage (cf. (1.12)): also there, a two parameter family of quadratic fronts and a one-parameter family of linear-log fronts exist for all $U \in \mathbb{R}$, resp, $U > 0$ (Boatto et al. (1993); see also Remark 3.1). However, in that case a single linear-log front (and a one-parameter family of quadratic fronts) can be identified by imposing a contact-angle condition of the form (1.8) or, more generally, (1.9): in other words, the microscopic contact angle $H_y(0)$ may be taken as one of the parameters spanning the fronts. This additional condition on the contact angle is also necessary for uniqueness of generic solutions to (1.12) (Giacomelli et al. 2008, 2014; Knüpfer 2011, 2015; Knüpfer & Masmoudi 2013, 2015; Gnann 2015; Gnann & Petrache 2018).

In (A) and (B), the microscopic contact angle can not be a selection criterion, since all fronts have $H_y(0) = +\infty$. This points to the necessity of a different criterion which, for instance, singles out a unique linear-log front. We will propose a robust choice of it in the next Section.

### 2.2. Thermodynamically consistent contact-line conditions

The second observation we wish to report is that thermodynamically consistent contact-line conditions exist, which replace the contact-angle conditions (1.8) and (1.9) for the thin-film equation with $P \equiv 0$. More precisely, in Section 4 we show that generic solutions to (1.1) satisfy

$$\frac{d}{dt} E[h(t)] = \pm \delta(t) \left( Q(h) - \frac{1}{2} h_x^2 \right) \big|_{x=s_\pm(t)} - \int_{s_- (t)}^{s_+ (t)} h^3(h_{xx} - Q'(h)) \frac{x}{h} dx,$$

provided they behave as traveling waves near $s_\pm(t)$ (here we convene that $\pm a_\pm = a_+ - a_-$. Imposing dissipativity of the system selects a class of contact-line conditions (see (4.7)). The simplest of such conditions, in analogy with (1.9), has the form

$$\left( \frac{1}{2} h_x^2 - Q(h) \right) \big|_{x=s_\pm(t)} = \pm \mu \delta(t). \quad (2.8)$$
With it, the energy balance is analogous to the one in (1.10):

\[
\frac{d}{dt} \int_{s_{+}(t)}^{s_{-}(t)} \left( \frac{1}{2} h_x^2 + Q(h) \right) = -\mu (|\dot{s}_+|^2 + |\dot{s}_-|^2) - \int_{s_{-}(t)}^{s_{+}(t)} m(h)(h_{xx} - Q'(h))^2, \quad (2.9)
\]

which encodes a quadratic dissipation of kinetic energy through frictional forces acting at the contact line. Note that the left-hand side of (2.8) is zero for the global minimizers \( h_{\text{min}} \) discussed in §1.2 (see Durastanti & Giacomelli (submitted, Theorem 4.5)). Note also that the left-hand side of (2.8) is well-defined on traveling waves. Indeed, though both summands are unbounded as \( x \to s_{\pm}(t) \), their difference is not: the left-hand side of (2.8) depends on one of the free parameters which appear in (A), since

\[
\frac{1}{2} H^2 - Q(H) \overset{(1.3),(2.4),(2.5)}{\sim} \frac{c_1}{m-1} + S \quad \text{as } H \to 0. \quad (2.10)
\]

On traveling waves, since \( s_{-}(t) = -Ut \), the contact-line condition (2.8) reads as

\[
\Theta[H] := \lim_{y \to 0^+} \left( \frac{1}{2} H_y^2(y) - P(H(y)) \right) = \mu U - S. \quad (2.11)
\]

Therefore, we expect that:

(C) **Selection criterion.** Assume (1.3), (1.11), and (2.4). Let \( \mu \geq 0 \).

1. For any \( U \in \mathbb{R} \setminus \{0\} \), (1.1) has a one-parameter family of quadratic traveling-wave solutions \( H \) (see (A)) such that (2.11) holds;
2. For any \( U > 0 \), (1.1) has a unique linear-log traveling-wave solution \( H_T \) (see (B)) such that (2.11) holds.

Numerical evidences in §5.2 and §5.3.1 indicate that the linear-log fronts in (C2) have a prominent precursor region ahead of the “macroscopic” contact line when \( \Theta[H_T] \ll -1 \), whereas they are steep for \( \Theta[H_T] \gg 1 \) (Fig. 2). Therefore, if (2.11) is adopted as contact-line conditions, we expect that \( H_T \) matches the following intuitive properties:

- for fixed \( \mu \) and \( S \), a greater speed \( U \) yields steeper profiles of \( H_T \);
- for fixed \( U \) and \( S \), a greater contact-line friction \( \mu \) yields steeper profiles of \( H_T \);
- for fixed \( \mu \) and \( U \), a larger spreading coefficient \( S \) yields gentler profiles of \( H_T \).

![Figure 2: Typical linear-log fronts \( H_T \) depending on \( \Theta[H_T] \). Under the contact-line condition (2.11), for given speed \( U > 0 \) and contact-line frictional coefficient \( \mu > 0 \), these are typical fronts for large and positive (left), resp. large and negative (right), values of the spreading coefficient \( S \) (“dry” complete wetting, resp. partial wetting).](image-url)
2.3. Open questions

The above three observations issue quite a few challenges.

- Though (A) and (B) are strongly supported in §3, resp. §5.1, by formal arguments, resp. numerical evidences, their rigorous validation is highly desirable. Once the asymptotics in (A) and (B) have been proved, we expect that (C) will follow as a byproduct.

- For relatively large values of $S$, it would be interesting to quantify height and length of the precursor region which appears to exist ahead of the “macroscopic” contact line.

- It would be very useful to have numerical simulations and/or matched asymptotic studies available for generic solutions to the PDE (1.1) with the contact-line conditions (1.7) and (2.8), for potentials $P$ of the form (2.4). Of particular interest would be the detection of scaling laws —such as the logarithmic correction to Tanner’s law (Tanner 1979) in complete wetting and the Voinov-Cox-Hocking relation in partial wetting—, an estimate of the rate of convergence to equilibria, and an insight on the evolution of the precursor region for relatively large values of $S$.

- Based on (A)-(C), for potentials $P$ of the form (2.4), we conjecture that for any non-negative $h_0 \in H^1(\mathbb{R})$ such that $E[h_0] < +\infty$ there exists a unique solution to (1.1) with the contact-line conditions (1.7) and (2.8). A difficult but extremely interesting task would be to develop a well-posedness theory at least when $h_0$ is a perturbation of a traveling wave, in the spirit of (Giacomelli et al. 2008, 2014; Knüpfer 2011, 2015; Knüpfer & Masmoudi 2013, 2015; Gnann 2015; Gnann & Petrache 2018).

- Interesting, though of seemingly lesser impact, would also be to face the above challenges for more general mobilities, e.g. of the form $m(h) = h^3 + \lambda^{3-n} h^n$ (cf. Remark 1.1).

3. Traveling-wave solutions

We look for solutions to (1.1)-(1.7) of the form

$$h(t,x) = H(y), \quad y = x + Ut, \quad U \in \mathbb{R} \setminus \{0\}.$$

Plugging this Ansatz into (1.1) and using (1.3) yields, in case of (1.11),

$$UH_y + (H^3(H_{yy} - P'(H))_y)_y = 0.$$

An integration using (1.7) then gives

$$U + H^2(H_{yy} - P'(H))_y = 0 \quad (3.1)$$

with

$$H(0) = 0 \quad (3.2a)$$

and

$$\text{supp } H = [0, +\infty) \quad \text{and} \quad H(+\infty) = +\infty. \quad (3.2b)$$

It will be useful to compare the features of (3.1) with parallel ones on the thin-film equation with $Q \equiv 0$. Its traveling wave solutions (if they exist) solve

$$U + H^2 H_{yyy} = 0 \quad \text{for zero slippage}, \quad U + (H^2 + H^{n-1})H_{yy} = 0 \quad \text{for positive slippage} \quad (3.3)$$

with the same boundary conditions, where by rescaling $H$ we have assumed without losing generality that $\lambda = 1$ in (3.3). To simplify the exposition, we shall limit our discussion on (3.3) to the case $n \in (\frac{3}{2}, 3)$.
3.1. Asymptotics near the contact line

Assume (1.3), (1.11), and (2.4). We will argue that:

(TW$_0$) Locally for $y \ll 1$, for any $U \in \mathbb{R} \setminus \{0\}$:

(a) there exists a two-parameter family of generic solutions $H$ to (3.1)-(3.2a) satisfying (2.5); in particular, $H_y(0) = +\infty$, their energy is finite in the sense that (2.2) holds, and their rate of bulk dissipation is finite, in the sense that (2.3) holds;

(b) if $U > 0$ and $m \in (1, 2)$, a one-parameter family of non-generic solutions to (3.1)-(3.2a) satisfying

\[ \frac{1}{2} H_y^2(y(H)) = \psi_p(H) + c_1 \psi_o(H) \sim \psi_p(H) \quad \text{as} \quad H \to 0^+, \quad c_1 \in \mathbb{R}, \quad (3.4) \]

also exist, where $\psi_o$ and $\psi_p$ are functions depending on $m$ and $U$, resp. $m$, $U$, and $P$; they satisfy

\[ \psi_o(H) \sim H^{3m-1} \exp \left( -\frac{m \sqrt{m}}{U |p|} H^\beta \right) \quad \text{and} \quad \psi_p(H) \sim \frac{U^2}{2m^2} H^{2(m-1)} \quad \text{as} \quad H \to 0^+ \]

with $\beta = \frac{3}{2}(1-m)$; hence $H_y(0) = 0$, their rate of bulk dissipation is unbounded in the sense that (2.3) does not hold, and their energy is unbounded if $m \geq 3/2$ in the sense that (2.2) does not hold.

In order to motivate (TW$_0$), it is convenient to let

\[ \psi(H) = \frac{1}{2} H_y^2(y(H)), \quad (3.5) \]

so that (3.1) reads as

\[ H^2 \psi''(H) = -\frac{U}{\sqrt{2\psi(H)}} + H^2 P''(H). \quad (3.6) \]

At leading order as $H \to 0$, a simple asymptotic expansion using (2.4) shows that two cases occur:

(a) $\psi(H) \sim \frac{1}{m-1} H^{1-m}$;

(b) $\psi(H) \sim \frac{U^2}{2m^2} H^{2(m-1)}$, $U > 0$. \quad (3.7)

Case (a), resp. (b), may be read off from (3.6) by neglecting the first term on the right-hand side, resp. the left-hand side. In terms of $H(y)$, as $y \to 0^+$ the two cases translate into

(a) $H(y) \sim \left( \frac{m+1}{2(m-1)} \right)^{1/m} y$;

(b) $H(y) \sim \left( \frac{2-m}{m} U y \right)^{1-m}$, $U > 0$, $m < 2$.

Note the constraint $m < 2$ in (b): if $m \geq 2$, $H$ diverges as $y \to 0^+$, hence it is not admissible. It is clear that solutions in (a) have finite rate of bulk dissipation since $\frac{2}{m+1} < 1$, whilst those in (b) don’t since $\frac{1}{2-m} > 1$ for $m < 2$ (cf. (2.3)). In addition, solutions in (a) have finite energy, whilst those in (b) have not if $m \geq 3/2$:

\[ \int_0^1 \left( \frac{1}{2} H_y^2 + Q(H) \right) dy \overset{(a), (2.4)}{\approx} \int_0^1 y^{\frac{2(1-m)}{m+1}} dy \overset{m \leq 3}{\approx} +\infty, \]

\[ \int_0^1 Q(H) dy \overset{(b), (2.4)}{\approx} \int_0^1 y^{\frac{(1-m)}{2-m}} dy \overset{m \geq 3/2}{\approx} +\infty. \]

We now distinguish the two cases.

(a). We define the function $\nu(H)$ as

\[ \psi(H) = P(H)(1 + \nu(H)), \quad \nu(0) \overset{(2.4), (3.7)}{=} 0. \]
It follows from (3.6) that
\[ L(v(H)) = -\frac{U}{\sqrt{2}P^3(H)} (1 + v(H))^{-\frac{1}{2}}, \] (3.8)
where
\[ L(v(H)) = H^2 \left( v''(H) + 2 \frac{P''(H)}{P(H)} v'(H) + \frac{P''(H)}{P(H)} v(H) \right). \]

The linearization of (3.8) around \( v = 0 \) is given by
\[ H^2 v''(H) - 2(m-1)Hv'(H) + m(m-1)v(H) \]
\[ = -\frac{U}{\sqrt{2}P^3(H)} - 2 \left( \frac{H P''(H)}{P(H)} + (m-1) \right) Hv'(H) + \left( m(m-1) - \frac{H^2 P''(H)}{P(H)} \right) v(H). \]

In view of (2.4), this equation has a two-parameter family of solutions of the form
\[ v(H) = c_1 H^{m-1} + c_2 H^m + v_p(H), \quad c_1, c_2 \in \mathbb{R}. \]

Generically, \( v_p(H) = o(H^{m-1}) \) as \( H \to 0^+ \) (if \( c_1 = 0 \), its regularity improves) is a function determined by \( U, P, m, c_1, \) and \( c_2 \). In terms of \( H \), this translates into (2.5); the additional degree of freedom coming from invariance under translation \( y \mapsto y - y_0 \) is spent to match the condition \( H(0) = 0 \). Therefore (TW0-a) holds.

(b). Arguing as above, a linearization around \( \psi(H) = \frac{U^2}{2m^2} H^{2(m-1)} \) yields
\[ \psi''(H) = A^2 H^{1-3m} \psi(H) - \frac{m}{2} H^{-m-1} (1 + o(1)), \quad A = \sqrt{\frac{m^3}{U^2}}, \quad \text{as } H \to 0^+ \]
(in this case it is more convenient to write the linearization directly on \( \psi \)). The change of variables
\[ \psi(H) = H^\frac{1}{2} \hat{\psi}(\eta), \quad \eta = BH^\beta, \quad \beta = \frac{3}{2}(1 - m) < 0, \quad B = \frac{A}{|\beta|}, \]
leads to a non-homogeneous modified Bessel equation,
\[ \eta^2 \hat{\psi}''(\eta) + \eta \hat{\psi}'(\eta) - \left( \eta^2 + (2\beta)^{-2} \right) \hat{\psi}(\eta) = -\frac{m}{2\beta^2} \left( \frac{|\beta|}{\eta} \right)^{\frac{2m-1}{3m-1}} (1 + o(1)) \quad \text{as } \eta \to +\infty. \]
The homogeneous solutions are spanned by modified Bessel functions (Abramowitz & Stegun 1992): \( \hat{\psi}(\eta) = c_1 K(\eta) + c_2 I(\eta) \), where \( K = K_{(2|\beta|^{-1})-1} \) and \( I = I_{(2|\beta|^{-1})-1} \). Since \( I \) is unbounded as \( \eta \to +\infty \), the condition \( \hat{\psi}(0) = 0 \) implies that \( c_2 = 0 \). Simple computations thus show that
\[ \hat{\psi}(\eta) = c_1 K(\eta) + \hat{\psi}_p(\eta) \]
with \( c_1 \in \mathbb{R} \), where the particular solution \( \hat{\psi}_p \) is given, at leading order as \( H \to 0^+ \), by
\[ \hat{\psi}_p(\eta) = \frac{m}{2\beta^2} B^{\frac{1-2m}{3(m-1)}} K(\eta) \int (\eta')^{\frac{2m-1}{3m-1}-1} I(\eta') \mathrm{d}\eta' - I(\eta) \int (\eta')^{\frac{2m-1}{3m-1}-1} K(\eta') \mathrm{d}\eta' \]
(here we used that the Wronskian \( K I' - IK' = \eta^{-1} \)). A simple asymptotic expansion, using \( I(\eta) \sim \frac{1}{\sqrt{2\pi}} \eta^{-1/2} e^{-\eta} \) and \( K(\eta) \sim \frac{\sqrt{\pi}}{\sqrt{2}} \eta^{-1/2} e^{-\eta} \) as \( \eta \to +\infty \), shows that
\[ \hat{\psi}_p(\eta) \sim \frac{m}{2\beta^2 B^2} \left( \frac{\eta}{B} \right)^{\frac{5-4m}{3m-1}} \quad \text{as } \eta \to +\infty. \]

Returning to original variables and using invariance under translation \( y \mapsto y - y_0 \) to match the condition \( H(0) = 0 \), we obtain (TW0-b).

Remark 3.1. At leading order for \( y \ll 1 \), the slippage model (3.3) reads as \( U + H^{n-1} H_{yyy} = 0 \) with \( n \leq 3 \). In that case:
for

The asymptotic expansion yielding

is

In both cases,

and are unbounded as

Giacomelli 2011; Giacomelli

unphysical, in the sense their rate of bulk dissipation density is not integrable near

\( \theta \)

any

of solutions with

\( H_y(0) = \theta \) (Boatto et al. 1993; Buckingham et al. 2002; Chiricotto & Giacomelli 2011; Giacomelli et al. 2016).

Remark 3.2. We note for further reference that the traveling waves in \((TW_0-a)\) satisfy

\[
H_{yy} - P'(H) \overset{(3.5)}{=} \psi'(H) - P'(H) \overset{(2.4),(2.5)}{=} o(H^{-1}) \quad \text{as } H \to 0^+.
\]  

3.2. Asymptotics in the liquid bulk

We now look at the behavior of generic solutions to (3.1) which are defined up to

\( y \to +\infty \)

and are unbounded as \( y \to +\infty \). We assume (1.3), (1.11), and

\[
P''(h) = Bp h^{-1-p}(1 + o(1)) \quad \text{as } h \to +\infty, \quad p > 1, \quad B \in \mathbb{R}.
\]  

(3.10)

The cases \( B > 0 \), resp. \( B < 0 \), correspond to long-range repulsive, resp. attractive, potentials. We will argue that:

\((TW_\infty)\) Consider (3.1) with (3.2b).

(Q) For any \( U \in \mathbb{R} \setminus \{0\} \) there exists a generic, three-parameter family of quadratically growing solutions:

\[
H(y) = a(y-y_0)^2 + b(y-y_0) + O(y^{-\gamma}) \quad \text{as } y \to +\infty, \quad \gamma = \min\{1, 2p-2\}, \quad (3.11)
\]

with \( a > 0 \) and \( b, y_0 \in \mathbb{R} \).

(L) For any \( U > 0 \) there exists a non-generic, two-parameter family of linear-log solutions to (3.1)-(3.2b):

\[
H^3_y(y(H)) = 3U \left( \log H - \frac{1}{3} \log \log H + a + O\left( \frac{\log \log H}{\log H} \right) \right) \quad \text{as } H \to +\infty \quad (3.12)
\]

with \( a \in \mathbb{R} \). In particular,

\[
H^3_y(y) = 3U \left( \log \left( (3U)^{1/3}(y-y_0) \right) + a + o(1) \right) \quad \text{as } y \to +\infty, \quad y_0 \in \mathbb{R}. \quad (3.13)
\]

In both cases, \( H \) is convex for \( y \gg 1 \).

To motivate \((TW_\infty)\), in view of (3.10) we rewrite (3.1) as

\[
H_{yyy} = -UH^{-2} + pBH^{-p-1}H_y(1 + r(H)), \quad r(H) = o(1) \quad \text{as } H \to +\infty \quad (3.14)
\]

The asymptotic expansion yielding (Q) is straightforward. For (L), let \( U > 0 \). The equation for

\[
u(H) = \frac{1}{3U}H^3_y(y(H))
\]

is

\[
u'' = \frac{(u')^2}{3u} - \frac{1}{H^2} + 3pBH^{-p-1}(3U)^{-2/3}u^{1/3}(1 + r(H)).
\]
Following Giacomelli et al. (2016), we exploit the homogeneity of the \((B = 0)\)-part of the equation letting

\[
s = \log H, \quad r(s) = o(1) \quad \text{as } s \to +\infty
\]

which yields

\[
\frac{d^2u}{ds^2} = \frac{du}{ds} + \frac{1}{3u} \left( \frac{du}{ds} \right)^2 - 1 + f(s, u), \quad f(s, u) = 3pB(3U)^{-2/3}e^{s(1-p)}u^{1/3}(1+r(s)). \tag{3.15}
\]

This equation has been analysed in Giacomelli et al. (2016, Section 4 and 5), with a slippage-type perturbation (namely, \(f(s, u) = (1 + e^{(3-n)s})^{-1}\)) whose specific form is however immaterial as long as \(f(s) = O(s^{-2} \log s)\). Their analysis shows that (3.15) with \(f = 0\) has a one-parameter family of solutions such that \(u(s)/s \to 1\) and \(u'(s) \to 1\) as \(s \to +\infty\), with an asymptotic expansion of the form

\[
u(s) = \left( s - \frac{1}{3} \log s + a + O(s^{-1} \log s) \right) \quad \text{for } s \gg 1, \quad a \in \mathbb{R}.
\]

Since \(f(s, u) \approx e^{s(1-p)}u^{1/3}\) and \(p > 1\), it is apparent that \(f\) produces only an exponentially small perturbation: thus solutions to (3.15) have the same behavior, which yields (3.12). Recalling the translation invariance of (3.14), (3.12) yields (3.13).

**Remark 3.3.** For later reference, we mention that when \(p \geq 2\) and \(UB < 0\) there exists another non-generic family of solutions, which however are decaying as \(y \to +\infty\):

\[
H(y) = \left( \frac{pB}{(2-p)U} \frac{1}{(y-y_0)} \right)^{\frac{1}{p-2}} (1 + o(1)) \quad \text{as } y \to +\infty, \quad p > 2, \quad UB < 0,
\]

\[
H(y) = e^{\frac{LU}{2\pi} (y-y_0)} (1 + o(1)) \quad \text{as } y \to +\infty, \quad p = 2, \quad UB < 0.
\]

**Remark 3.4.** The picture of \((\text{TW}_\infty)\) is identical to the case \(P \equiv 0\), (3.3), which at leading order for \(y \gg 1\) reads as \(U + H^2 H_{yyy} = 0\). Indeed, also for this equation there exist a generic three-parameter family of quadratic solutions satisfying (3.11) (though with a different remainder) and a non-generic two-parameter family of linear-log solutions satisfying (3.13)-(3.12).

### 3.3. Global behavior

For the global picture, one has to make sure that local solutions are global. This is not always the case, in the sense that (3.1) also have generic solutions with compact support, a feature which is common to the slippage model (3.3) (see §5.1). However, we have strong numerical evidence that any of both the three-parameter family \((\text{TW}_\infty-\text{Q})\) and the two-parameter family \((\text{TW}_\infty-\text{L})\) touches down to \(H = 0\) at some point \(y_0 \in \mathbb{R}\). Thus, capitalizing on translation invariance, the observations made in the previous two subsections yield (A) and (B). Indeed, out of the three parameters in \((\text{TW}_\infty-\text{Q})\), one is used to match \(H(0) = 0\), which together with \((\text{TW}_0)\) yields (A); analogously, out of the two parameters in \((\text{TW}_\infty-\text{L})\), one is used to match \(H(0) = 0\), which together with \((\text{TW}_0)\) yields (B).

### 4. Thermodynamically consistent contact-line conditions

In this Section we will identify a class of thermodynamically consistent contact-line conditions for (1.1), in the spirit of those proposed by Ren & E (2007); Ren et al. (2010); Ren & E (2011). To this aim, we will refine an argument introduced in Chiricotto & Giacomelli...
Using the convention which is integrable at finally, the integral on the right-hand side of (4.3) is finite: indeed, \[ h(t, x) \approx y^{\frac{2}{m+1}} \text{ as } y \to 0^+, \] (4.1a)
\[ h_x(t, x) \approx y^{\frac{1-m}{m+1}} \text{ as } y \to 0^+, \] (4.1b)
\[ V(t, x) = \dot{s}_x(t) (1 + o(1)), \text{ as } y \to 0^+, \] (4.1c)
\[ \frac{1}{2} h_x^2 - Q(h) \approx O(1) \text{ as } y \to 0^+, \] (4.1d)
\[ h_{xx} - Q'(h) \approx o(\varepsilon^{-1}) \text{ as } y \to 0^+. \] (4.1e)

Let \( \varepsilon > 0 \). Locally around \( x = s_\varepsilon(t) \), we may define \( s_\varepsilon^\varepsilon(t) \) by
\[ h(t, s_\varepsilon^\varepsilon(t)) := \varepsilon, \quad \text{hence} \quad (h_t + s_\varepsilon^\varepsilon h_x)|_{x = s_\varepsilon^\varepsilon} = 0. \] (4.2)

Using the convention \( \pm a_\pm|_{x = s_\varepsilon^\varepsilon} = a_+|_{x = s_\varepsilon^\varepsilon} - a_-|_{x = s_\varepsilon^\varepsilon} \) for the boundary terms, we compute:
\[
\frac{d}{dt} \int_{s_\varepsilon^\varepsilon} \left( \frac{1}{2} h_x^2 + Q(h) \right) = \pm \dot{s}_\varepsilon \left( \frac{1}{2} h_x^2 + Q(h) \right) |_{x = s_\varepsilon^\varepsilon} + \int_{s_\varepsilon^\varepsilon} (h_x h_t + Q'(h) h_t) dx \]
\[
= \pm \dot{s}_\varepsilon \left( \frac{1}{2} h_x^2 + Q(h) \right) |_{x = s_\varepsilon^\varepsilon} \pm (h_x h_t)|_{x = s_\varepsilon^\varepsilon} - \int_{s_\varepsilon^\varepsilon} (h_{xx} - Q'(h)) h_t dx \]
\[
\approx \pm B_1^{\pm} \pm B_2^{\pm} - \int_{s_\varepsilon^\varepsilon} i_3. \] (4.3)

We now notice three facts. Firstly, and crucially, the first boundary term remains bounded as \( \varepsilon \to 0 \): indeed,
\[ B_1^{\pm}(t) = [Q(h) - \frac{1}{2} h_x^2]|_{x = s_\varepsilon^\varepsilon(t)} \approx O(1) \text{ as } \varepsilon \to 0. \] (4.4)

In addition, the second boundary term vanishes as \( \varepsilon \to 0 \):
\[ B_2^{\pm}(t) = o(1) \text{ as } \varepsilon \to 0. \] (4.5)

Finally, the integral on the right-hand side of (4.3) is finite: indeed,
\[
i_3 = h^3 ((h_{xx} - Q'(h)) h_x)^2 = V^2 h^{-1} \approx \frac{2}{m+1} |s_\varepsilon^\varepsilon(t) - x|^{-\frac{2}{m+1}} \quad \text{as } x \to (s_\varepsilon^\varepsilon(t))^x, \]
which is integrable at \( x = s_\varepsilon^\varepsilon(t) \) since \( \frac{2}{m+1} < 1 \). Passing to the limit as \( \varepsilon \to 0 \), we obtain from (4.2)-(4.5) that
\[
\frac{d}{dt} E[h(t)] = \pm \dot{s}_\varepsilon(t) \left( Q(h) - \frac{1}{2} h_x^2 \right) |_{x = s_\varepsilon^\varepsilon(t)} - \int_{s_\varepsilon^\varepsilon(t)} h^3 ((h_{xx} - Q'(h)) h_x)^2. \] (4.6)

In order to be thermodinamically consistent, the contact-line condition has to be such that the free energy is dissipated along the flow: this leads to the following class of contact-line conditions:
\[
\left( \frac{1}{2} h_x^2 - Q(h) \right) |_{x = s_\varepsilon^\varepsilon(t)} = \pm f(\dot{s}_\varepsilon) \quad \text{with } f \text{ such that } f(\dot{s}_\varepsilon) \dot{s}_\varepsilon \geq 0 \text{ for all } \dot{s}_\varepsilon \in \mathbb{R}. \] (4.7)
The simplest choice of \( f \), i.e. the linear relation \( f(s) = \mu s \) (\( \mu \geq 0 \)), leads to (2.8):

\[
\left( \frac{1}{2} h_s^2 - Q(h) \right) \bigg|_{x=s_+(t)} = \pm \mu s_+(t).
\]

Substituting (4.8) into (4.6) we obtain (2.9).

5. Numerical observations

We now explore the main features of traveling-wave solutions to (1.1). We take as prototype example

\[ Q(h) = h^{-1} - S, \]

which corresponds to \( m = 2 \) in (2.4). Due to the homogeneity of \( Q' \), \( U \) may be scaled out of (3.1)-(3.2b): letting

\[ H = |U|^{-2/3} \hat{H}, \quad y = |U|^{-1} \hat{y} \]

and removing hats, in what follows we will consider without loss of generality solutions to (3.1)-(3.2b) with

\[ H^2(H_{yy} + H^{-2})_y = -\frac{U}{|U|}. \]

Note that the spreading coefficient \( S \) is immaterial in (5.2) (though it strongly affects droplets’ equilibria, see Durastanti & Giacomelli (submitted)). However, it enters the contact-line condition: for instance, after the rescaling in (5.1), (2.11) reads as

\[
\Theta[H] := \lim_{y \to 0} \left( \frac{1}{2} H_y^2(y) - H^{-1}(y) \right) = \frac{\mu U - S}{|U|^{2/3}}.
\]

Let us also notice that a change in sign of \( U \) is equivalent, in (5.2), to a change of sign of \( y \), and that the left-hand side of (5.3) is unaffected by the latter change: hence in place of (5.2) we may equivalently consider

\[ H^2(H_{yy} + H^{-2})_y = -1 \]

with the understanding that

- advancing fronts \((U > 0)\) correspond to solutions to (5.4) with \( H(0) = 0 \), \( \text{supp } H = [0, +\infty) \) and \( H(+\infty) = +\infty \);
- receding fronts \((U < 0)\) correspond to solutions to (5.4) with \( H(0) = 0 \), \( \text{supp } H = (-\infty, 0] \) and \( H(-\infty) = +\infty \).

5.1. Generic solutions

Let us first describe generic solutions to (5.2). They can be obtained by noting that \( H \) is concave near \( H = 0 \) (see (2.6)) and convex (with either quadratic or linear-log behavior) near \( H = +\infty \). Therefore there exists a point \( y \) such that \( H_{yy} = 0 \). By translation invariance, we may fix the point to be \( y = 1 \). Shooting from \( y = 1 \) with the two parameters \( H(1) \) and \( H_y(1) \) produces a two-parameter family of solutions, which can then be translated in \( y \) to match \( H(0) = 0 \). Consider therefore

\[ 1 + H^2(H_{yy} + H^{-2})_y = 0, \quad H(1) = \alpha > 0, \quad H_y(1) = \beta \in \mathbb{R}, \quad H_{yy}(1) = 0. \]

For a fixed \( \alpha \), the generic picture is the following (cf. Fig. 3):

- advancing, quadratic traveling waves for \( \beta > \beta_0(\alpha) \);
- an advancing, linear-log traveling wave for \( \beta = \beta_0(\alpha) \);
- compactly supported solutions for \( \beta \in (\beta_1(\alpha), \beta_0(\alpha)) \);
- a separatrix for \( \beta = \beta_1(\alpha) \);
- receding (quadratic) traveling waves for \( \beta < \beta_1(\alpha) \).

For \( m \geq 2 \), as in (5.5), the separatrix is unbounded (cf. Remark 3.3), whereas it is compactly supported for \( 1 < m < 2 \). In Fig. 4 we also provide an analogous picture for the thin-film equation with slippage, (3.3)\(_2\), for \( n = 2 \).

![Figure 3: Generic solutions to (5.5) with \( \alpha = 1/4 \) (top) and \( \alpha = 1 \) (bottom), at two different scales.](image)

![Figure 4: Generic solutions to the thin-film equation with slippage (3.3) with \( \alpha = 1/4 \), at two different scales.](image)

5.2. Linear-log solutions

For each \( \alpha > 0 \), a unique linear-log solution \( H_T \) may be selected by finding the corresponding value of \( \beta_0(\alpha) \). The shapes reported in Fig. 5 show that linear-log solutions increase as \( \alpha = H_T|_{(H_T)_{\alpha=0}} \) increases, and that a prominent precursor region forms ahead of the “macroscopic contact line” for small values of \( \alpha \).
5.3. The contact-line condition

To support the choices of both $\Theta[H]$ as selection parameter and of (2.8) as free boundary condition, we now discuss the numerical values of $\Theta[H]$.

5.3.1. Linear-log solution and separatrix

In Fig. 6(A) we report numerical values of $\Theta[H_T]$, where $H_T$ are linear-log solutions. There, it is apparent that $\Theta[H_T]$ monotonically covers the whole real line as linear-log solutions are spanned: in particular, a unique linear-log front can be selected such that the contact-line condition (5.3) holds. Combining Fig. 6(A) with Fig. 5, it is also apparent that, as $\Theta[H_T]$ decreases, a more prominent precursor region forms ahead of the macroscopic contact line.

Thus, if (5.3) is assumed as a contact-line condition, the picture qualitatively matches the following intuitive expectations:

- for fixed $\mu$ and $S$, a greater speed $U$ yields steeper profiles of $H_T$;
- for fixed $U$ and $S$, a greater contact-line friction $\mu$ yields steeper profiles of $H_T$;
- for fixed $\mu$ and $U$, a larger spreading coefficient $S$ yields gentler profiles of $H_T$.

As discussed in Section 5.1, another separatrix, $H_S$, exists besides the linear-log solution: $H_S$ discriminates between receding and compactly supported solutions. It is apparent from Fig. 6(B) that $\Theta[H_S]$ increases with $\alpha$ and diverges to $-\infty$ as $\alpha \to 0^+$.

5.3.2. Generic advancing and receding fronts

When $H_y|_{H_{yy}=0} > (H_T)_y|_{(H_T)_{yy}=0}$, resp. $H_y|_{H_{yy}=0} < (H_S)_y|_{(H_S)_{yy}=0}$, traveling wave solutions are advancing, resp. receding, and have a quadratic profile for large $y$. In Fig. 7 we report numerical values of $\Theta[H]$ for such solutions. There, it is apparent that, for each value of $\alpha = H|_{H_{yy}=0}$, $\Theta[H]$ diverges to $+\infty$ as $|H_y||_{H_{yy}=0}$ does. Since $\Theta[H_T]$, resp. $\Theta[H_S]$, diverge to $-\infty$ as $H|_{H_{yy}=0} \to 0$, (Fig. 6), $\Theta[H]$ covers the whole real line as advancing, resp.
receding, traveling waves are spanned. There is, however, a lack of monotonicity of $\Theta[H]$ for fixed $\alpha$ for receding waves, a phenomenon which is analogous to the slippage case (3.3) when $\Theta[H]$ is replaced by the contact angle (cf. Fig. 4). Note that the minimum of $\Theta[H]$ appears to be on the left half-plane, hence the corresponding waves have negative derivative at their inflection point, thus they are monotonic. Therefore the branch of receding fronts emanating from $\Theta[H] = +\infty$ consists of monotone ones: this indicates that non-monotonic receding fronts (see Fig. 3) might be non-generic ones.

6. Conclusions

It has already been observed in Durastanti & Giacomelli (submitted) that, in thin-film models, mildly singular potentials of the form

$$P(h) \sim \frac{A}{m-1} h^{1-m} \quad \text{as} \quad h \to 0^+ \quad \text{with} \quad 1 < m < 3, \quad A > 0, \quad P(0) = P(+\infty) = 0,$$

always produce compactly supported minimizers (which may be droplet-shaped or pancake-shaped). In this respect, these potentials provide a more realistic picture with respect to both slippage in complete wetting and strongly singular potentials: indeed, in these other two cases global minimizers with compact support do not exist, and “pancakes” are replaced by infinite films of zero, resp. positive, thickness.

Here, based on formal arguments supported by numerical evidences, we have argued that mildly singular potentials generically solve the contact-line paradox, in the sense that

$$h_t + (h^3(h_{xx} - P'(h))_x)_x = 0$$

admits a two-parameter family of both advancing and receding traveling-wave solutions, all of them having microscopic contact angle equal to $\pi/2$ (in agreement with minimizers). A one-parameter family of advancing “linear-log” solutions exists as well, displaying a logarithmically corrected linear behavior in the bulk. This picture is analogous to that of the thin-film equation with, say, Navier slip $\lambda > 0$,

$$h_t + ((h^3 + \lambda h^2)h_{xxx})_x = 0.$$  

As is well-known, however, for the slippage model the microscopic contact angle may be used as a parameter to span traveling waves, thus selecting a unique advancing linear-log front. This is not the case for the mild-potential model, since the microscopic contact angle is always $\pi/2$. Here, we have also proposed a class of thermodynamically consistent contact-line conditions, which replaces contact-angle conditions and is capable to single out a unique
advancing linear-log front. The simplest among such conditions reads as

\[ \Theta[h(t)] := \left( \frac{1}{2} h_x^2 - P(h) \right)|_{x=s_\pm(t)} = \pm \mu \dot{s}_\pm(t) - S, \]

where \( \{h(t) > 0\} = (s_-(t), s_+(t)) \), \( \mu > 0 \) is a contact-line frictional coefficient, and \( S \) is the non-dimensional spreading coefficient. Under this condition, numerical evidences suggest that linear-log fronts are steep for \( \Theta[h] \gg 1 \), whereas they display a precursor region ahead of the macroscopic contact line for \( \Theta[h] \ll -1 \).

A number of challenging questions are issued, ranging from the rigorous validation of the above results for both traveling-wave and generic solutions, to a global formal asymptotic and/or numerical analysis of the evolution of droplets under mildly singular potentials and its characteristic scaling laws.

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