REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH REEB PARALLEL RICCI TENSOR IN GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. There are several kinds of classification problems for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Among them, Suh classified Hopf hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel Ricci tensor in Levi-Civita connection. In this paper, we introduce a new notion of generalized Tanaka-Webster Reeb parallel Ricci tensor for $M$ in $G_2(\mathbb{C}^{m+2})$. By using such parallel conditions, we give complete classifications of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Introduction

In this paper, let $M$ represent a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and $S$ denote the Ricci tensor of $M$. Hereafter unless otherwise stated, we consider that $X, Y, Z$ are any tangent vector fields on $M$. Let $W$ be any tangent vector field on the distribution $\mathfrak{h} = \{X \in TM | X \perp \xi\}$. $k$ stands for a non-zero constant real number.

The classification of real hypersurfaces in Hermitian symmetric space is one of interesting parts in the field of differential geometry. Among them, we introduce a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is a kind of Hermitian symmetric space of compact irreducible type with rank 2. Remarkably, the manifolds are equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ satisfying $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$) where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$. When $m = 1$, $G_2(\mathbb{C}^4)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\text{Spin}(6) \simeq \text{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann Manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper we assume $m$ is not less than 3. (see [2]).

Let $N$ be a local unit normal vector field of $M$. Since $G_2(\mathbb{C}^{m+2})$ has the Kähler structure $J$, we may define a Reeb vector field $\xi = -JN$ and a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$. The Reeb vector field $\xi$ is said to be a Hopf if it is

\textbf{1} 2010 Mathematics Subject Classification : Primary 53C40; Secondary 53C15.

\textbf{2} Key words : Real hypersurfaces; complex two-plane Grassmannians; Hopf hypersurface; generalized Tanaka-Webster connection; Ricci tensor; Reeb parallel.

* This work was supported by Grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea. The first author by Grant Proj. No. NRF-2012-R1A1A3002031, the second by Grant Proj. No. NRF-2012-R1A2A2A01043023. And the third author supported by NRF Grant funded by the Korean Government (NRF-2013-Fostering Core Leaders of Future Basic Science Program).
invariant under the shape operator $A$ of $M$. The 1-dimensional foliation of $M$ by the integral curves of $\xi$ is said to be a Hopf foliation of $M$. We say that $M$ is a Hopf hypersurface if and if the Hopf foliation of $M$ is totally geodesic. By the formulas in [9] Section 2, it can be easily seen that $\xi$ is Hopf if and only if $M$ is Hopf.

From the quaternionic Kähler structure $\mathfrak{F}$ of $G_2(\mathbb{C}^{m+2})$, there naturally exists almost contact $3$-structure vector field $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Put $Q^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, which is a 3-dimensional distribution in a tangent vector space $T_x M$ of $M$ at $x \in M$. In addition, $Q$ stands for the orthogonal complement of $Q^\perp$ in $T_x M$. It becomes the quaternionic maximal subbundle of $T_x M$. Thus the tangent space of $M$ consists of the direct sum of $Q$ and $Q^\perp$ as follows: $T_x M = Q \oplus Q^\perp$.

For two distributions $[\xi]$ and $Q^\perp$ defined above, we may consider two natural invariant geometric properties under the shape operator $A$ of $M$, that is, $A[\xi] \subset [\xi]$ and $AQ^\perp \subset Q^\perp$. By using the result of Alekseevskii [1], Berndt and Suh [2] have classified all real hypersurfaces with two natural invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

**Theorem A.** Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $Q^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$,

or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

In the case (A), we say $M$ is of Type (A). Similarly in the case (B) we say $M$ is of Type (B). Using Theorem A, geometers have given characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometric quantities, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. Actually, Lee and Suh [9] gave a characterization for a real hypersurface of Type (B) as follows:

**Theorem B.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\xi$ belongs to the distribution $Q$ if and only if $M$ is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$. In other words, $M$ is locally congruent to a real hypersurface of Type (B).

In particular, there are various well-known results with respect to $S$ on Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. From such a point of view, Suh [17] gave a characterization of a model space of Type (A) in $G_2(\mathbb{C}^{m+2})$ under the condition $S\phi = \phi S$ where $\phi$ denotes the structure tensor field of $M$. In [18] and [19], he also considered the parallelism of Ricci tensor with respect to the Levi-Civita connection and gave, respectively,

**Theorem C.** [19] Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with non-vanishing geodesic Reeb flow. If the Ricci tensor is Reeb parallel, $\nabla_\xi S = 0$. Then $M$ is locally congruent to one of the following:

(i) a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \neq \frac{\pi}{4\sqrt{2}}$,

or

(ii) a tube over a totally geodesic $\mathbb{H}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius $r$ such that $\cot^2(2r) = \frac{1}{2m-1}$ and $\xi$-parallel eigenspaces $T_{\text{cot}}r$ and $T_{\text{tan}}r$.

Motivated by these works, we define the notion of Reeb parallel Ricci tensor with respect to the generalized Tanaka-Webster connection for a real hypersurface $M$
in $G_2(\mathbb{C}^{m+2})$. In order to do this, we first define the generalized Tanaka-Webster connection $\widehat{\nabla}^{(k)}$ on $M$ given by

$$\widehat{\nabla}^{(k)}_\xi Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,$$

where $k$ is a non-zero real number (see [3], [4], [5]). Hereafter, unless otherwise stated, a GTW connection means a generalized Tanaka-Webster connection. In addition, we put

$$F_X^{(k)} Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$ 

Then the operator $F_X^{(k)}$ becomes a skew-symmetric (1,1) type tensor, that is, $g(F_X^{(k)} Y, Z) = -g(Y, F_X^{(k)} Z)$ for any tangent vector fields $X, Y, Z$ on $M$ and said to be Tanaka-Webster (or $k$-th-Cho) operator with respect to $X$.

Related to this connection, the Ricci tensor $S$ is said to be generalized Tanaka-Webster Reeb parallel (in short, GTW-parallel) if the covariant derivative in GTW connection $\nabla^{(k)}$ of $S$ along $\xi$ is vanishing, that is, $(\nabla^{(k)}_\xi S)Y = 0$. From this, we naturally see that this notion is weaker than generalized Tanaka-Webster parallel (shortly, GTW-parallel) Ricci tensor, that is, $(\nabla^{(k)}_\xi S)Y = 0$. Recently, Pérez and Suh [14] proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with GTW-parallel Ricci tensor. From such a viewpoint, we assert:

**Theorem 1.** Let $M$ be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\alpha = g(A\xi, \xi) \neq 2k$. The Ricci tensor $S$ of $M$ is GTW-Reeb parallel if and only if $M$ is locally congruent to one of the following:

(i) a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r$ such that $r \neq \frac{1}{2\sqrt{2}} \cot^{-1}(\frac{k}{\sqrt{2}})$, or

(ii) a tube over a totally geodesic $\mathbb{H}^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius $r$ such that $r = \frac{1}{2} \cot^{-1}(\frac{k}{4(2n-1)})$.

For the case $\alpha = 2k$, the Reeb vector field $\xi$ of Hopf hypersurface $M$ with GTW-Reeb parallel Ricci tensor belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. So, for the case $\xi \in \mathcal{Q}^\perp$, we obtain that the trace $\theta$ of the shape operator $A$ is constant along $\xi$, that is, $\theta \xi = 0$. In addition for the case $\xi \in \mathcal{Q}$ we have the following:

**Corollary 1.** Let $M$ be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with GTW-parallel Ricci tensor for $\alpha = 2k$. If $\xi$ belongs to the distribution $\mathcal{Q}$, then $M$ is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$ with radius $r$ such that $r = \frac{1}{2} \tan^{-1}(\frac{\sqrt{2}}{2n-1})$.

On the other hand, we consider the notion of GTW-Reeb parallel Ricci tensor on $\mathfrak{h}$, that is, $(\nabla^{(k)}_\xi S)W = 0$ for any $W \in \mathfrak{h}$. Then by virtue of Theorem C for the case $\alpha = 2k$, we assert the following:

**Theorem 2.** Let $M$ be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\alpha = 2k$. The Ricci tensor of $M$ satisfies the Reeb parallelism on $\mathfrak{h}$ in both GTW and Levi-Civita connections, that is, $(\nabla^{(k)}_\xi S)W = 0$ and $(\nabla^{(k)}_\xi S)W = 0$ for any $W \in \mathfrak{h}$ if and only if $M$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r$ such that $r = \frac{1}{2} \cot^{-1}(\frac{\sqrt{2}}{\sqrt{2}})$. 

Moreover, as a generalization of the assumption \( \hat{\nabla}_\xi^{(k)} S = 0 = \nabla_\xi S \) on \( \mathfrak{h} \) in Theorem 2, we want to consider that \( \hat{\nabla}_\xi^{(k)} S = \nabla_\xi S \), that is, the Reeb parallel Ricci tensor in GTW connection coincides with the Reeb parallel Ricci tensor in Levi-Civita connection. This condition has a geometric meaning such that \( S \) commutes with the Tanaka-Webster operator \( F_\xi \), that is, \( S \cdot F_\xi = F_\xi \cdot S \). This meaning gives any eigenspaces of \( S \) are invariant by the Tanaka-Webster operator \( F_\xi \). From such a point of view, we have the following:

**Theorem 3.** Let \( M \) be a Hopf hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). Then \( \hat{\nabla}_\xi^{(k)} S = \nabla_\xi S \) if and only if \( M \) is locally congruent to an open part of a tube around a totally geodesic \( G_2(\mathbb{C}^{m+1}) \) in \( G_2(\mathbb{C}^{m+2}) \).

But for the case where the derivative of the Ricci tensor in GTW connection is equal to the derivative in Levi-Civita connection, that is, \( \hat{\nabla}_X^{(k)} S = \nabla_X S \) for any \( X \in T M \), we assert the following:

**Corollary 2.** There does not exist any Hopf hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), satisfying \( (\nabla_X^{(k)} S) Y = (\nabla_X S) Y \) for arbitrary tangent vector fields \( X \) and \( Y \) on \( M \).

Obviously, we know that the condition \( \hat{\nabla}_X^{(k)} S = \nabla_X S \) has a geometric meaning that any eigenspaces of \( S \) are invariant by the Tanaka-Webster operator \( F_X \). Recently, Pérez and Suh [15] investigated the Levi-Civita and GTW covariant derivatives for the shape operator or the structure Jacobi operator of real hypersurfaces in complex projective space \( \mathbb{C}P^m \). Moreover, in [6] Jeong, Lee and Suh gave a characterization of Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with \( \hat{\nabla}_X^{(k)} A = \nabla A \).

In this paper, we refer [1], [2], [7], [9], [16] and [17] for Riemannian geometric structures of \( G_2(\mathbb{C}^{m+2}) \) and its geometric quantities, respectively. In order to get our results, in sections 1 we will give the fundamental formulas related to the Reeb parallel Ricci tensor. In section 2 we want to give a complete proof of Theorem 1 for \( \alpha = g(AX, \xi) \neq 2k \). In section 3 we will consider the case \( \alpha = 2k \) and give a proof of Corollary 1 and Theorem 2. Finally, in section 4 we will give a complete proof of Theorem 3 and Corollary 2.

1. GTW-Reeb parallel Ricci tensor

From [13], the Ricci tensor \( S \) of a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), is given by

\[
SX = \sum_{i=1}^{4m-1} R(X, e_i) e_i = (4m + 7)X - 3\eta(X)\xi + hAX - A^2 X + \sum_{\nu=1}^{3} \{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X - \eta(\phi_\nu X)\phi_\nu \xi - \eta(X)\eta_\nu(\xi)\nu\}
\]

where \( h \) denotes the trace of the shape operator \( A \), that is, \( h = \text{Tr}A \).
And we also have
\[(\nabla_X S)Y = -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX\]
\[-3\sum_{\nu=1}^{3} \left\{ g(\phi_{\nu}AX, Y)\xi_{\nu} + \eta_{\nu}(Y)\phi_{\nu}AX \right\} \]
\[(3.2)\]
\[+ 3\sum_{\nu=1}^{3} \left\{ 2g(\phi AX, \xi_{\nu})\phi_{\nu}Y + g(AX, \phi_{\nu}\phi Y)\phi_{\nu}\xi \right. \]
\[-\eta(Y)g(AX, \xi_{\nu})\phi_{\nu}\xi + \eta_{\nu}(\phi Y)g(AX, \xi)\xi_{\nu} - \eta_{\nu}(\phi Y)\phi_{\nu}AX \]
\[-\eta(Y)g(\phi AX, \xi_{\nu})\xi_{\nu} - \eta(Y)g(\phi_{\nu}AX, \xi)\xi_{\nu} \}
\[+ (Xh)AY + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y. \]
Substituting \(X = \xi\) into (3.2) and using the condition that \(M\) is Hopf, that is, \(A\xi = \alpha\xi\), we get
\[(3.3)\]
\[(\nabla_\xi S)Y = -4\alpha \sum_{\nu=1}^{3} \left\{ g(\phi_{\nu}\xi, Y)\xi_{\nu} + \eta_{\nu}(Y)\phi_{\nu}\xi \right\} + (\xi h)AY \]
\[+ h(\nabla_\xi A)Y - (\nabla_\xi A)AY - A(\nabla_\xi A)Y. \]

In this section we assume that \(M\) is a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2})\) with GTW-Reeb parallel Ricci tensor, that is, \(S\) satisfies:
\[(3.1)\]
\[(\nabla_\xi^{(k)} S)X = 0. \]

By the definition of GTW connection \(\nabla_\xi^{(k)}\), the covariant derivative of \(S\) with respect to the GTW connection along \(\xi\) becomes
\[(3.4)\]
\[(\nabla_\xi^{(k)} S)X = \nabla_\xi^{(k)}(SX) - S(\nabla_\xi^{(k)} X) \]
\[= \nabla_\xi(SX) + g(\phi A\xi, SX)\xi - \eta(SX)\phi A\xi - k\eta(\xi)\phi SX \]
\[\quad - S(\nabla_\xi X) - g(\phi A\xi, X)S\xi + \eta(X)S\phi A\xi + k\eta(\xi)S\phi X \]
\[= (\nabla_\xi S)X - k\phi SX + kS\phi X. \]
Thus the condition \((C-1)\) is equivalent to
\[(3.5)\]
\[(\nabla_\xi S)X = k\phi SX - kS\phi X, \]
it yields
\[4(k - \alpha) \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\phi X)\xi_{\nu} - \eta_{\nu}(X)\phi_{\nu}\xi \right\} \]
\[(3.6)\]
\[= (\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X - kh\phi AX \]
\[+ k\phi A^2X + khA\phi X - kA^2\phi X \]
from \(3.1\), \(3.2\) and \([8, \text{Section 2}]\).

Using these equations, we prove that \(\xi\) belongs to either \(Q\) or \(Q^\perp\), where \(M\) is a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2})\) with GTW-Reeb parallel Ricci tensor.

**Lemma 1.1.** Let \(M\) be a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\). If \(M\) has GTW-Reeb parallel Ricci tensor, then \(\xi\) belongs to either \(Q\) or \(Q^\perp\).
From the Codazzi equation \([8, \text{Section 2}]\) and differentiating \((4.1)\) a real hypersurface of Type (B) is a Hopf hypersurface with GTW-Reeb parallel Ricci tensor, the equation \((3.6)\) becomes

\[ M \]

Using the equation \([8, \text{Lemma 2.1}]\) and the previous one, we get \(\alpha\) with GTW-Reeb parallel Ricci tensor and \(\xi\) belongs to \(Q\). Therefore from this, \((4.1)\) can be written as

\[ 4(k - \alpha)\eta_1(\xi)\phi_1\xi = \alpha(\xi\eta)\xi - h(\xi\alpha)\xi - 2\alpha(\xi\alpha)\xi, \]

where we have used \((\nabla_\xi A)\xi = (\xi\alpha)\xi\) and \((\nabla_\xi A)A\xi = \alpha(\xi\alpha)\xi\).

Taking the inner product of \((3.7)\) with \(\phi_1\xi\), we have

\[ 4(k - \alpha)\eta_1(\xi)\eta^2(X_0) = 0, \]

because of \(\eta^2(X_0) + \eta^2(\xi_1) = 1.\) From this, we have the following three cases.

**Case 1:** \(\alpha = k.\)

For this case, we see that \(\alpha\) becomes a non-zero real number. Using the equation in \([2, \text{Lemma 1}]\), we assert that \(\xi\) belongs to either \(Q\) or \(Q^\perp.\)

**Case 2:** \(\eta(\xi_1) = 0.\)

By the notation \([**]\), we see that \(\xi\) belongs to \(Q.\)

**Case 3:** \(\eta(X_0) = 0.\)

This case implies that \(\xi\) belongs to \(Q^\perp\) from \([**]\).

Accordingly, summing up these cases, the proof of our Lemma is completed. \(\square\)

### 2. Proof of Theorem 1

In this section, let \(M\) be a Hopf hypersurface, \(\alpha \neq 2k\), in \(G_2(\mathbb{C}^{m+2})\) with GTW-Reeb parallel Ricci tensor. Then by Lemma \([L.1]\) we shall divide our consideration in two cases depending on \(\xi\) belongs to either \(Q^\perp\) or \(Q\), respectively.

First of all, if we assume \(\xi \in Q\), then a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2}), m \geq 3\), with GTW-Reeb parallel Ricci tensor and \(\alpha = q(A\xi, \xi) \neq 2k\) is locally congruent to a real hypersurface of Type (B) by virtue of Theorem B given in the introduction.

Next let us consider the case, \(\xi \in Q^\perp.\) Accordingly, we may put \(\xi = \xi_1.\) Since \(M\) is a Hopf hypersurface with GTW-Reeb parallel Ricci tensor, the equation \((3.6)\) becomes

\[ (\xi h)AX + h(\nabla_\xi A)X - (\nabla_\xi A)AX - A(\nabla_\xi A)X = k(h\phi AX - \phi A^2X - hA\phi X + A^2\phi X). \]

From the Codazzi equation \([8, \text{Section 2}]\) and differentiating \(A\xi = \alpha\xi,\) we obtain

\[ (\nabla_\xi A)X = (\nabla_X A)\xi + \phi X + \phi_1X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \]

\[ = (X\alpha)\xi + \alpha\phi AX - A\phi AX + \phi X + \phi_1X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \]

Using the equation \([8, \text{Lemma 2.1}]\) and the previous one, we get

\[ (\nabla_\xi A)X = \frac{\alpha}{2}\phi AX - \frac{\alpha}{2}A\phi X + (\xi\alpha)\eta(X)\xi. \]

Therefore from this, \((4.1)\) can be written as

\[ (\xi h)AX + \tilde{\kappa}h\phi AX - \tilde{\kappa}hA\phi X + (h - 2\alpha)(\xi\alpha)\eta(X)\xi - \tilde{\kappa}\phi A^2X + \tilde{\kappa}A^2\phi X = 0, \]

where \(\tilde{\kappa} = \left(\frac{\alpha}{2} - k\right).\)
Since $\tilde{\kappa} \neq 0$ is equivalent to the given condition $\alpha \neq 2k$, (4.2) yields

$$\frac{(\xi h)}{\tilde{\kappa}} AX + h\phi AX - hA\phi X + \frac{(h - 2\alpha)}{\tilde{\kappa}} (\xi \alpha) \eta(X) \xi - \phi A^2 X + A^2 \phi X = 0.$$  

Now we consider the case $\xi h = 0$. Then (4.3) can be reduced to

$$h\phi AX - hA\phi X + \frac{(h - 2\alpha)}{\tilde{\kappa}} (\xi \alpha) \eta(X) \xi - \phi A^2 X + A^2 \phi X = 0.$$  

Taking the inner product of (4.4) with $\xi$, we have

$$\frac{(h - 2\alpha)}{\tilde{\kappa}} (\xi \alpha) \eta(X) \xi = 0.$$  

Thus (4.4) becomes

$$h\phi AX - \phi A^2 X - hA\phi X + A^2 \phi X = 0.$$  

On the other hand, from the equation (3.1) we calculate

$$S\phi X - \phi SX = hA\phi X - A^2 \phi X - h\phi AX + \phi A^2 X,$$

then by (4.5) it follows that $S\phi X = \phi SX$ for any tangent vector field $X$ on $M$. Hence, by Suh [17] we assert that $M$ satisfying our assumptions must be a model space of Type $(A)$.

We now assume $\xi h \neq 0$. Putting $\sigma = \frac{(\xi h)}{\tilde{\kappa}}(\neq 0)$ and $\tau = \frac{(h - 2\alpha)}{\tilde{\kappa}}(\xi \alpha)$, the equation (4.3) becomes

$$\sigma AX + h\phi AX - hA\phi X + \tau \eta(X) \xi - \phi A^2 X + A^2 \phi X = 0.$$  

Applying $\phi$ to (4.6) and replacing $X$ by $\phi X$ in (4.6), respectively, we get the following two equations:

$$\sigma \phi AX - hAX + h\alpha \eta(X) \xi - h\phi A\phi X + A^2 X - \alpha^2 \eta(X) \xi + \phi A^2 \phi X = 0$$

and

$$\sigma A\phi X + h\phi A\phi X + hAX - h\alpha \eta(X) \xi - \phi A^2 \phi X - A^2 X + \alpha^2 \eta(X) \xi.$$

Summing up the above two equations, we obtain $\phi A + A\phi = 0$. Thus from this, the equation (4.6) implies

$$\sigma AX + 2h\phi AX + \tau \eta(X) \xi = 0.$$  

Let us $X_h$ be the orthogonal projection of $X$ onto the distribution $h = \{X \in TM| X \perp \xi\}$. Inserting this into the previous equation yields

$$\sigma AX_h + 2h\phi AX_h = 0.$$  

In addition, applying $\phi$ to this equation, it follows

$$\sigma \phi AX_h - 2hAX_h = 0.$$  

Thus we obtain

$$\begin{pmatrix} \sigma & 2h \\ -2h & \sigma \end{pmatrix} \begin{pmatrix} AX_h \\ \phi AX_h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

The determinant of the square matrix of order 2, that is, $\sigma^2 + 4h^2 \geq \sigma^2 \neq 0$, so we get $AX_h = 0$ for any $X_h \in h$. Substituting $X_h$ as $\xi_2$ and $\xi_3$, it implies $A\xi_2 = 0$ and $A\xi_3 = 0$, respectively. Hence, we can assert that the distribution $Q^\perp$ is invariant under the shape operator, that is, $M$ is a $Q^\perp$-invariant real hypersurface. Thus by virtue of Theorem A, we conclude that $M$ with our assumptions must be a model space of Type $(A)$. 


Summing up these discussions, we conclude that if a Hopf hypersurface $M$ in complex two-plane Grassmannians $G_{2}(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying (C-1) and $\alpha \neq 2k$, then $M$ is of Type (A) or (B).

Hereafter, let us check whether $S$ of a model space of Type (A) (or of Type (B)) satisfies the Reeb parallelism with respect to $\nabla^{(k)}$ by [2, Proposition 3] (or [2, Proposition 2], respectively).

Let us denote by $M_{A}$ a model space of Type (A). From now on, using the equations (3.1), (3.2) and [2, Proposition 3], let us check whether or not $S$ satisfies (3.0) which is equivalent to our condition (C-1) for each eigenspace $T_{\alpha}$, $T_{\beta}$, $T_{\lambda}$, and $T_{\mu}$ on $T_{x}M_{A}$, $x \in M_{A}$. In order to do, we find one equation related to $S$ from (3.0) using the property of $M_{A}$, $\xi = \xi_{1}$ as follows.

$$
(\nabla^{(k)}_{\xi}S)X = -h(\nabla_{\xi}A)X + (\nabla_{\xi}A)AX + A(\nabla_{\xi}A)X + kh\phi AX
- k\phi A^{2}X - khA\phi X + kA^{2}\phi X,
$$

(4.7)

since $h = \alpha + 2\beta + 2(m - 2)\lambda$ is a constant.

**Case A-1:** $X = \xi (= \xi_{1}) \in T_{\alpha}$.

Since $(\nabla_{\xi}A)\xi = 0$, we see that $(\nabla^{(k)}_{\xi}S)\xi = 0$ from the equation (4.7). It means that the Ricci tensor $S$ becomes GTW Reeb parallel on $T_{\alpha}$.

**Case A-2:** $X \in T_{\beta} = \text{Span}\{\xi_{2}, \xi_{3}\}$.

For $\xi_{\mu} \in T_{\beta}$, $\mu = 2, 3$ we have

$$(\nabla_{\xi}A)\xi_{\mu} = \beta(\nabla_{\xi}\xi_{\mu}) - A(\nabla_{\xi}\xi_{\mu})$$

$$= \beta q_{\mu+2}(\xi)\xi_{\mu+1} - q_{\mu+2}(\xi)\xi_{\mu+1} - \alpha \beta \phi_{\mu}\xi$$

$$- q_{\mu+2}(\xi)A\xi_{\mu+1} + q_{\mu+1}(\xi)A\xi_{\mu+2} - \alpha A\phi_{\mu}\xi,$$

which follows that $(\nabla_{\xi}A)\xi_{2} = 0$ and $(\nabla_{\xi}A)\xi_{3} = 0$. Therefore, from the equation (4.7) we obtain, respectively,

$$(\nabla^{(k)}_{\xi}S)\xi_{2} = kh\phi A\xi_{2} - k\phi A^{2}\xi_{2} - khA\phi \xi_{2} + kA^{2}\phi \xi_{2}$$

$$= (-kh\beta + k\beta^{2} + k\beta^{2})\xi_{3} = 0,$$

and $(\nabla^{(k)}_{\xi}S)\xi_{3} = 0$ by similar methods. So, we assert that the Ricci tensor $S$ of $M_{A}$ is Reeb parallel on $T_{\beta}$.

By the structure of a tangent vector space $T_{x}M_{A}$ at $x \in M_{A}$, we see that the distribution $Q$ is composed of two eigenspaces $T_{\lambda}$ and $T_{\mu}$. On this distribution $Q = T_{\lambda} \oplus T_{\mu}$ we obtain

$$
(\nabla_{\xi}A)X = \alpha \phi AX - A\phi AX + \phi X + \phi_{1}X
$$

(4.8)

by virtue of the Codazzi equation [3, Section 2]. Using this equation we consider the following two cases.

**Case A-3:** $X \in T_{\lambda} = \{X \mid X \in Q, JY = J_{1}Y\}$.

We naturally see that if $X \in T_{\lambda}$, then $\phi X = \phi_{1}X$. Moreover, the vector $\phi X$ also belong to the eigenspace $T_{\lambda}$ for any $X \in T_{\lambda}$, that is, $\phi T_{\lambda} \subset T_{\lambda}$. From these and (4.8), we obtain

$$(\nabla_{\xi}A)X = (\alpha \lambda - \lambda^{2} + 2)\phi X, \text{ for } X \in T_{\lambda}.$$
From (4.7) and together with these facts, we obtain
\[
(\nabla^{(k)}_{\xi} S)X = (\alpha \lambda - \lambda^2 + 2)(2\alpha - h)\phi X,
\]
which implies that \( S \) must be Reeb parallel for \( \nabla^{(k)}_{\xi} \) on \( T_\lambda \), since \( \alpha \lambda - \lambda^2 + 2 = 0 \).

**Case A-4:** \( X \in T_\mu = \{ X \mid X \in Q, JY = -J_1Y \} \).

If \( X \in T_\mu \), then \( \phi X = -\phi_1X, \phi T_\mu \subset T_\mu \) and \( \mu = 0 \). So, from (4.8), we obtain \( (\nabla_{\xi}A)X = 0 \), moreover \( (\nabla^{(k)}_{\xi} S)X = 0 \) for any \( X \in T_\mu \).

Summing up all cases mentioned above, we can assert that \( S \) of real hypersurfaces \( M_A \) of Type (A) in \( G_2(\mathbb{C}^{m+2}) \) is GTW Reeb parallel.

Now let us consider our problem for a model space of Type (B), which will be denoted by \( M_B \). In order to do this, let us calculate the fundamental equation related to the covariant derivative of \( S \) of \( M_B \) along the direction of \( \xi \) in GTW connection. On \( T_xM_B, x \in M_B \), since \( \xi \in Q \) and \( h = \text{Tr}(A) = \alpha + (4n-1)\beta \) is a constant, the equation (3.6) is reduced to
\[
(\nabla^{(k)}_{\xi} S)X = 4k - \alpha \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\phi X)\xi_{\nu} - \eta_{\nu}(X)\phi_{\nu} \right\}
\]
\[
- h(\nabla_{\xi}A)X + (\nabla_{\xi}A)AX + A(\nabla_{\xi}A)X
\]
\[
+ kh\phi AX - k\phi A^2X - khA\phi X + kA^2\phi X.
\]

Moreover, by the equation of Codazzi and [2] Proposition 2] we obtain that for any \( X \in T_xM_B \)
\[
(\nabla_{\xi}A)X = \alpha \phi AX - A\phi AX + \phi X - \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(X)\phi_{\nu} \xi + 3g(\phi_{\nu} \xi, X)\xi_{\nu} \right\}
\]
\[
(4.9)
\]
\[
= \begin{cases}
0 & \text{if } X \in T_\alpha \\
\alpha\beta\phi\xi_\ell & \text{if } X \in T_\beta = \text{Span}\{\xi_\ell \mid \ell = 1, 2, 3\} \\
-4\xi_\ell & \text{if } X \in T_\gamma = \text{Span}\{\phi\xi_\ell \mid \ell = 1, 2, 3\} \\
(\alpha \lambda + 2)\phi X & \text{if } X \in T_\lambda \\
(\alpha \mu + 2)\phi X & \text{if } X \in T_\mu.
\end{cases}
\]

From these two equations, it follows that
\[
(\nabla^{(k)}_{\xi} S)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
(\alpha - k)(4 - h\beta + \beta^2)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\
(4(\alpha - k) + (h - \beta)(4 + k\beta))\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\
(h - \beta)(k\mu - k\lambda - \alpha \lambda - 2)\phi X & \text{if } X \in T_\lambda \\
(h - \beta)(k\mu - k\lambda - \alpha \mu - 2)\phi X & \text{if } X \in T_\mu.
\end{cases}
\]
\[
(4.10)
\]

So, we see that \( M_B \) has Reeb parallel GTW-Ricci tensor, when \( \alpha \) and \( h \) satisfies the conditions \( \alpha = k \) and \( h - \beta = 0 \), which means \( r = \frac{1}{2} \cot^{-1}\left( \frac{-k}{4(2n-1)} \right) \). Moreover, this radius \( r \) satisfies our condition \( \alpha \neq 2k \).

Hence summing up these considerations, we give a complete proof of our Theorem 1 in the introduction. \( \Box \)
3. Proofs of Corollary 1 and Theorem 2

In section 2 we obtained the classification of Hopf hypersurfaces $M$ with GTW-Reeb parallel Ricci tensor and $\alpha \neq 2k$. Thus in present section we will consider the case $\alpha = 2k$ related to the GTW-Reeb parallelism of Ricci tensor of a Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

Now let us prove Corollary 1 in the introduction.

Our condition $\alpha = 2k$ means that $\alpha$ is constant. From this we assert that $\xi$ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. For $\xi \in \mathcal{Q}$, it is a well-known fact that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ must be a model space $M_B$ of Type (B) (see [9]). On the other hand, from (4.10) and $\alpha = 2k$, the GTW covariant derivative of Ricci tensor $S$ of $M_B$ along the direction of $\xi$ is given

$$
(\tilde{\nabla}^{(k)}_\xi S)X = \begin{cases} 
0 & \text{if } X = \xi \in T_\alpha \\
(4k - h\beta + \beta^2)\phi\ell & \text{if } X = \phi\ell \in T_{\beta} \\
(4k + (h - \beta)(4 + k\beta))\ell & \text{if } X = \ell \in T_{\gamma} \\
-(h - \beta)(k\beta + 2)\phi X & \text{if } X \in T_\lambda \\
-(h - \beta)(k\beta + 2)\phi X & \text{if } X \in T_{\mu}.
\end{cases}
$$

Actually, since $\alpha = 2k$, we naturally have $k\beta + 2 = 0$. It follows that $S$ is GTW Reeb parallel on $T_\lambda$ and $T_{\mu}$. In order to be the GTW-Reeb parallel Ricci tensor on the other eigenspaces $T_{\beta}$ and $T_{\gamma}$, we should have the following two equations,

$$(4 - h\beta + \beta^2) = 0$$

and

$$4k + (h - \beta)(4 + k\beta) = 0.$$ 

Combining these two equations, we have $2k + h - \beta = 0$. Since $h = \alpha + 3\beta + (4n - 4)(\lambda + \mu) = \alpha + (4n - 1)\beta$ and $\alpha = 2k$, it follows that $\beta = -2n - 1$. By virtue of [2] Proposition 2, $\alpha = -2\tan(2r)$ and $\beta = 2\cot(2r)$ where $r \in (0, \pi/4)$, we obtain $\tan(2r) = \sqrt{2n - 1}$. From such assertions, we conclude that a model space of Type (B) has GTW-Reeb parallel Ricci tensor for special radius $r$ such that $r = \frac{1}{2}\tan^{-1}(\sqrt{2n - 1})$, which gives us a complete proof of Corollary 1.

□

On the other hand, for the case $\xi \in \mathcal{Q}^\perp$, the equation (1.2) becomes

$$(\xi h)AX = 0$$

under the assumption of $\alpha = 2k$. For the case $\xi h \neq 0$, it follows that $AX = 0$. If $X = \xi$, then $\alpha = 0$, which gives a contradiction. From this, we assert the following for the case $\xi \in \mathcal{Q}^\perp$:

**Remark.** Let $M$ is a Hopf hypersurface, that is, $A\xi = \alpha\xi$ where $\alpha = 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with GTW-Reeb parallel Ricci tensor, $\tilde{\nabla}^{(k)}_\xi S = 0$. If $\xi \in \mathcal{Q}^\perp$, then we only get the result that the trace $h$ of the shape operator $A$ is constant along the direction of $\xi$, that is, $\xi h = 0$.

From such a point of view, we now only focus our attention to the Ricci Reeb parallelism in GTW connection on the distribution $\mathfrak{h} = \{X \in TM \mid X \perp \xi\}$, as given by the proof of Theorem 2.
As mentioned above in the proof of Corollary 1, we see that \( \xi \in \mathcal{Q} \) or \( \xi \in \mathcal{Q}^\perp \), because \( M \) is a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with \( \alpha = 2k \). Moreover, if \( \xi \in \mathcal{Q} \), then \( M \) must be a model space of Type \((B)\).

Now, let us consider the case \( \xi \in \mathcal{Q}^\perp \). Then by Suh [19] we have the following key lemma in the proof of Theorem 2.

**Lemma 3.1.** Let \( M \) be a Hopf hypersurface, that is, \( A\xi = \alpha \xi \) where \( \alpha = 2k \), in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). If \( M \) satisfies the following properties:

(i) the Reeb vector field \( \xi \) belongs to the distribution \( \mathcal{Q}^\perp \),

(ii) the Ricci tensor \( S \) is Reeb parallel with respect to both the Levi-Civita and GTW connections on \( \mathfrak{h} \), that is, \((\nabla^G_k)S)X = 0 \) and \((\nabla^S)X = 0 \) for any tangent vector field \( W \in \mathcal{H} \),

then \( M \) must be a model space of Type \((A)\) or Type \((B)\) in \( G_2(\mathbb{C}^{m+2}) \).

**Proof.** As investigated above, from the assumption of \( \alpha = 2k \) and the equation \((1.2)\) we have

\[ (\xi h)AW = 0 \]

for any tangent vector field \( W \in \mathcal{H} \).

From this, we see that the distribution \( \mathfrak{h} \) is totally geodesic, that is, \( AW = 0 \) for any \( W \in \mathfrak{h} \), if \((\xi h) \neq 0 \). So, we can assert that \( M \) is a \( \mathcal{Q}^\perp \)-invariant hypersurface in \( G_2(\mathbb{C}^{m+2}) \), that is, \( g(AQ, Q^\perp) = 0 \).

Next, we consider the case \((\xi h) = 0 \). From \((1.1)\) we get \( S\xi = (4m+ha-\alpha^2)\xi \). Differentiating this formula along the direction of \( \xi \) and using our assumptions, \( A\xi = \alpha \xi \), \((\xi h) = (\xi \alpha) = 0 \), it follows that \((\nabla^hS)\xi = 0 \). It implies that the Ricci tensor \( S \) becomes Reeb parallel. Then by virtue of the result given by Suh [19] we give a complete proof of our Lemma. \( \square \)

As a consequence, we assert that if \( M \) is a Hopf hypersurface, \( \alpha = 2k \), in \( G_2(\mathbb{C}^{m+2}) \) satisfying two Ricci Reeb parallelism defined by \((\nabla^hS)W = 0 \) and \((\nabla^hS)W = 0 \) for any \( W \in \mathfrak{h} \), then it must be either a real hypersurface of Type \((A)\) or Type \((B)\).

From now on, let us consider the converse problem. In other words, we now check whether the Ricci tensor \( S \) of model spaces \( M_A \) or \( M_B \) in \( G_2(\mathbb{C}^{m+2}) \) satisfies the conditions in Theorem 2 or not.

By [2] Proposition 3 and the checking for a model space \( M_A \) given in the introduction and section [2], respectively, we see that \( M_A \) is a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with the GTW-Reeb parallel Ricci tensor on \( \mathfrak{h} \subset TM_A \).

Now let us show that the Ricci tensor \( S \) of \( M_A \) is Reeb parallel in \( \nabla \) on \( \mathfrak{h} \), that is, \((\nabla^S)W = 0 \) for \( W \in \mathfrak{h} \subset TM_A \). By virtue of [2] Proposition 3], the equation \((5.3)\) can be written as

\[ (\nabla^S)Y = h(\nabla^A)Y - (\nabla^A)AY - A(\nabla^A)Y \]

\[ = h(\nabla^A)Y - \hat{\kappa}(\nabla^A)Y - A(\nabla^A)Y, \]

where \( AY = \hat{\kappa}Y \) for any \( W \in \mathfrak{h} \subset TM_A \). Moreover, from the equation of Codazzi, we obtain

\[ (\nabla^A)Y = (\nabla^A)\xi + \phi Y + \phi_1Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2 \]

\[ = \alpha \phi Y - A\phi Y + \phi Y + \phi_1Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2, \]
Reeb parallel on $T$ becomes ($M$

Summing up three cases above, ($Civita connection $∇$

Case A-2 : $Y \in T_β = \text{Span}\{ξ_2, ξ_3\}$

From (5.3), we obtain $(∇_ξA)ξ_2 = (β^2 - αβ - 2)ξ_3$, which implies $(∇_ξA)ξ_2 = 0$ since $β^2 - αβ - 2 = 0$. So, we see that $(∇_ξS)ξ_2 = 0$ by (5.2). Similarly, if we put $Y = ξ_3$ in (5.3), then $(∇_ξA)ξ_3 = -(β^2 - αβ - 2)ξ_2 = 0$, because $αβ = 2 \cot^2(\sqrt{2}r) - 2$. From this and (5.2), we see that $(∇_ξS)ξ_3 = 0$.

Case A-3 : $Y \in T_μ = \{Y \perp ξ_1, ξ_2, ξ_3 \mid φY = φ_1Y\}$

If $Y \in T_μ$, then $φY \perp T_μ$ and $φ_1Y \perp T_μ$. From these, the equation (5.3) becomes $(∇_ξA)Y = (αλ - λ^2 + 2)φY$. It follows $(∇_ξA)Y = 0$, since $αλ = 2 \tan^2(\sqrt{2}r) - 2$. Hence we see that the Ricci tensor $S$ of $M_λ$ becomes Reeb parallel on $T_β$, that is, $(∇_ξS)Y = 0$ for any $Y \in T_λ$.

Summing up three cases above, $M_λ$ have Reeb parallel Ricci tensor in the Levi-Civita connection $∇$ on the distribution $h$.

On the other hand, let us check whether $M_B$ satisfies our conditions, $∇_ξS = 0$ and $∇_ξ^{(k)}S = 0$ on $h ⊂ TM_B$. Suppose that the Ricci tensor $S$ of $M_B$ is Reeb parallel, $(∇_ξS)X = 0$ for $X \in h$. From (5.3) and (5.4) we obtain

$$(∇_ξS)X = \begin{cases} 
-4α + hαβ - αβ^2 & \text{if } X = ξ_β \\
-4α + h - β & \text{if } X = φξ_β \\
(h - β)(αλ + 2)φX & \text{if } X \in T_γ \\
(h - β)(αμ + 2)φX & \text{if } X \in T_δ \end{cases}$$

Since the Ricci tensor $S$ is Reeb parallel on the eigenspace $T_γ$, we have $(h - β)(αλ + 2) = 0$. It implies that

(5.4) \hspace{1cm} (h - β) = 0,

because $(αλ + 2) ≠ 0$. On the other hand, for $T_γ$ we get $(α + h - β) = 0$, which means $α = 0$ from (5.4). It makes a contraction. Thus we assert that there does not exist $M_B$ satisfying the conditions in Theorem 2.

With such assertions we give a complete proof of Theorem 2 in the introduction.

4. Proofs of Theorem 3 and Corollary 2

First we want to give a proof of Theorem 3. Among the conditions in Theorem 2, we focus our attentions to the assumptions related to the Reeb parallelism of Ricci tensor $S$. Actually, we consider that on $h$ two covariant derivatives of $S$ in Levi-Civita and GTW connections are equal to zero, that is, $(∇_ξS)W = 0 = (∇_ξ^{(k)}S)W$ for any tangent vector field $W \in h = \{X ∈ TM \mid X ⊥ ξ\}$. So, in this section, we will
By straightforward calculation it is 
\[(\nabla_\xi S)X = (\hat{\nabla}_{\xi}^{(k)} S)X\]
for any tangent vector field \(X\) on \(M\). By virtue of the equation (3.3), the condition \((C-2)\) is equivalent to the \(S\phi = \phi S\). On the other hand, Suh proved in [17] that a Hopf hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), with commuting Ricci tensor is locally congruent a tube of radius \(r\) over a totally geodesic \(G_2(\mathbb{C}^{m+1})\) in \(G_2(\mathbb{C}^{m+2})\) Then we conclude that a Hopf hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), satisfying the condition \((C-2)\) if and only if \(M\) is of Type \((A)\), which gives us a complete proof of Theorem 3.

By Theorem 3, if a real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\) satisfies \(\nabla S = \hat{\nabla}_{Y}^{(k)} S\), then naturally \((C-2)\) holds on \(M\). So \(M\) is of Type \((A)\). Now let us check whether a model space \(M_A\) of Type \((A)\) satisfies our condition.

(C-3) 
\[(\hat{\nabla}_{X}^{(k)} S)Y = (\nabla_{X} S)Y\]
for any tangent vector fields \(X, Y \in T_x M_A\), \(x \in M_A\). In order to do this, we assume that the Ricci tensor \(S\) of \(M_A\) satisfies \((C-3)\). That is, we have

\[0 = (\hat{\nabla}_{X}^{(k)} S)Y - (\nabla_{X} S)Y\]
(6.1)
\[= g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY - g(\phi AX, Y)\xi + \eta(Y)S\phi AX + k\eta(X)\phi Y\]
for any \(X, Y \in T_x M_A\).

Since \(T_x M_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\gamma\), the equation (6.1) holds for \(X \in T_\beta\) and \(Y \in T_\alpha\). For the sake of convenience we put \(X = \xi_2 \in T_\beta\) and \(Y = \xi \in T_\alpha\). Since \(S\xi = \delta\xi\) and \(S\xi_3 = \sigma\xi_3\) where \(\delta = (4m + h\alpha - \alpha^2)\) and \(\sigma = (4m + 6 + h\beta - \beta^2)\), the equation (6.1) reduces to \(\beta(\delta - \sigma)\xi_3 = 0\). By [2] Proposition 3, since the principal curvature \(\beta = \sqrt{2}\cot(\sqrt{2}r)\) for \(r \in (0, \pi/\sqrt{8})\) is non-zero, it follows \((\delta - \sigma) = 0\). In other words, by [2] Proposition 3 we obtain

\[-(\delta - \sigma) = 6 - \alpha\beta + \beta^2 + (2m - 2)\beta\lambda - (2m - 2)\alpha\lambda\]
\[= 8 - 4(m - 1)\tan^2(\sqrt{2}r),\]
which gives us

(6.2)
\[\tan^2(\sqrt{2}r) = \frac{2}{m - 1}.\]
In addition, since (6.1) holds for \(X \in T_\lambda\) and \(Y = \xi\), we obtain

\[0 = (\hat{\nabla}_{X}^{(k)} S)\xi - (\nabla_{X} S)\xi = \lambda(\tau - \delta)\phi X,\]
where in the second equality we have used \(\phi X \in T_\lambda\) and \(S X = (4m + 6 + h\lambda - \lambda^2)X = \tau X\) for any \(X \in T_\lambda\). Because \(\lambda = -\sqrt{2}\tan(\sqrt{2}r)\) where \(r \in (0, \pi/\sqrt{8})\) is non-zero, we have also

\[\tau - \delta = 0.\]
By straightforward calculation it is

\[\tau - \delta = 6 + h\lambda - \lambda^2 - h\alpha + \alpha^2\]
\[= 4m - 4\cot^2(\sqrt{2}r) = 0.\]
From (6.2), it becomes \(2m + 2 = 0\), which gives us a contradiction. Accordingly, it completes our Corollary 2 given in the introduction.
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