CREDAL NETS UNDER EPISTEMIC IRRELEVANCE

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ABSTRACT. We present a new approach to credal nets, which are graphical models that generalise Bayesian nets to imprecise probability. Instead of applying the commonly used notion of strong independence, we replace it by the weaker notion of epistemic irrelevance. We show how assessments of epistemic irrelevance allow us to construct a global model out of given local uncertainty models and mention some useful properties. The main results and proofs are presented using the language of sets of desirable gambles, which provides a very general and expressive way of representing imprecise probability models.

1. INTRODUCTION

This paper is under construction. At the current stage, it only aims to present some essential ideas and theorems. We intend to extend this preliminary work to a full size paper in the near future.

2. SETS OF DESIRABLE GAMBLES

Consider a variable $X$ taking values in some non-empty and finite set $\mathcal{X}$. Knowledge about the possible values this variable may assume can be modelled in various ways: probability mass functions, credal sets and coherent lower previsions are only a few of the many options. We choose to use a different approach, being a set of desirable gambles. We will model a subject’s beliefs regarding the value of a variable $X$ by means of his behaviour: which gambles (or bets) on the unknown value of $X$ would our subject be inclined to participate in?

Although they are not as well known as other (imprecise) probability models, sets of desirable gambles have a series of advantages. To begin with, sets of desirable gambles are more expressive then both credal sets and lower previsions. For example, sets of desirable gambles are easily able to model such things as conditioning on events with probability zero, which is something other imprecise probability models cannot do. Secondly, sets of desirable gambles have the advantage of being operational, meaning that there is a practical way of constructing a model that represents the subject’s beliefs. In the case of sets of desirable gambles this can be done by offering the subject certain gambles and asking him whether or not he wants to participate. And finally, it tends to be much easier to construct proofs in the language of coherent sets of desirable gambles than it is to do so in other languages. We will give a brief survey of the basics of sets of desirable gambles and refer to Refs. [1, 2, 3] for more details and further discussion.

2.1. Desirable gambles. A gamble $f$ is a real-valued map on $\mathcal{X}$ which is interpreted as an uncertain reward. If the value of the variable $X$ turns out to be $x$, the (possibly negative) reward is $f(x)$. A non-zero gamble is called desirable if we accept the transaction in which (i) the actual value $x$ of the variable is determined, and (ii) we receive the reward $f(x)$. The zero gamble is not considered to be desirable, mainly because we want desirability to represent a strict preference to the zero gamble.

We will model a subject’s beliefs regarding the possible values $\mathcal{X}$ that a variable $X$ can assume by means of a set $\mathcal{D}$ of desirable gambles, which will be a subset of the set $\mathcal{G}(\mathcal{X})$ of all gambles on $\mathcal{X}$. For any two gambles $f$ and $g$ in $\mathcal{G}(\mathcal{X})$, we say that $f \geq g$ if $f(x) \geq g(x)$ for all $x$ in $\mathcal{X}$ and $f > g$ if $f \geq g$ and $f \neq g$. We use $\mathcal{G}(\mathcal{X})_{>0}$ to denote the set of all gambles
For all \( f \), \( f_1 \), \( f_2 \) \( \in \mathcal{G}(\mathcal{X}) \) and all real \( \lambda > 0 \):

D1. if \( f \leq 0 \) then \( f \notin \mathcal{D} \)

D2. if \( f > 0 \) then \( f \in \mathcal{D} \)

D3. if \( f \in \mathcal{D} \) then \( \lambda f \in \mathcal{D} \) [scaling]

D4. if \( f_1, f_2 \in \mathcal{D} \) then \( f_1 + f_2 \in \mathcal{D} \) [combination]

Requirements D3 and D4 make \( \mathcal{D} \) a convex cone: \( \text{posi}(\mathcal{D}) = \mathcal{D} \), where we have used the positive hull operator posi which generates the set of finite strictly positive linear combinations of elements of its argument set:

\[
\text{posi}(\mathcal{D}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k : f_k \in \mathcal{D}, \lambda_k \in \mathbb{R}_{>0}, n \in \mathbb{N}_0 \right\}.
\]

Here \( \mathbb{R}_{>0} \) is the set of all strictly positive real numbers, and \( \mathbb{N}_0 \) the set of all natural numbers (positive integers).

2.3. Natural extension. In practice, a set of desirable gambles will usually be elicited by presenting an expert a number of gambles and asking him whether or not he finds them desirable. This results in a (finite) assessment of desirable gambles \( \mathcal{A} \subseteq \mathcal{G}(\mathcal{X}) \) and the question raises whether this can be extended to a coherent set. It is shown in Ref. [5] that if the assessment \( \mathcal{A} \) can be extended to a coherent set of desirable gambles, the smallest (most conservative) such coherent set is given by \( \mathcal{E}(\mathcal{A}) := \text{posi}(\mathcal{A} \cup \mathcal{G}(\mathcal{X})_{>0}) \) and we then call \( \mathcal{E}(\mathcal{A}) \) the natural extension of \( \mathcal{A} \).

2.4. Maximal sets of desirable gambles. A coherent set \( \mathcal{D} \) of desirable gambles on \( \mathcal{X} \) is called maximal if it is not strictly included in any other coherent set of desirable gambles on \( \mathcal{X} \). In other words, if adding any gamble \( f \) to \( \mathcal{D} \) makes sure we can no longer extend the set \( \mathcal{D} \cup \{ f \} \) to a set that is still coherent. We will denote maximal sets of desirable gambles as \( \mathcal{M} \) instead of using the general notation \( \mathcal{D} \).

These maximal sets of desirable gambles have a number of useful properties. For example, a coherent set \( \mathcal{D} \) of desirable gambles on \( \mathcal{X} \) is always the intersection of all the maximal coherent sets \( \mathcal{M} \) of desirable gambles on \( \mathcal{X} \) that include \( \mathcal{D} \); see Ref. [5]. In other words, \( f \in \mathcal{D} \) if and only if \( f \in \mathcal{M} \) for every \( \mathcal{M} \supseteq \mathcal{D} \). As a consequence, if a gamble \( f \in \mathcal{G}(\mathcal{X}) \) is not an element of \( \mathcal{D} \), there is at least one maximal set \( \mathcal{M} \supseteq \mathcal{D} \) for which \( f \notin \mathcal{M} \). Another useful property that holds for every maximal set \( \mathcal{M} \) is that for all gambles \( f \neq 0 \) in \( \mathcal{G}(\mathcal{X}) \), either \( f \) or \( -f \) is an element of \( \mathcal{M} \); see Ref. [2].

3. Credal nets under epistemic irrelevance

3.1. Directed acyclic graphs. A directed acyclic graph (DAG) is a graphical model that is well known for its use in bayesian networks. It consists of a finite set of nodes (vertices), which are joint together into a network by a set of directed edges, each edge connecting one node with another. Since this directed graph is assumed to be acyclic, it is not possible to follow a sequence of edges from node to node and end up back at the same node you started from.

We will call \( G \) the set of nodes \( s \) associated with a given DAG. For two nodes \( s \) and \( t \), if there is a directed edge from \( s \) to \( t \), we say that \( s \) is a parent of \( t \) and \( t \) is a child of \( s \). Note that a single node can have multiple parents and multiple children. For any node \( s \), its set
of parents is denoted by \( P(s) \) and its set of children by \( C(s) \). If a node \( s \) has no parents, we use the convention \( m(s) = \emptyset \) and we call it a root node. The set of all root nodes is denoted as \( G^r := \{ s \in G : P(s) = \emptyset \} \). If \( C(s) = \emptyset \), then we call \( s \) a leaf, or terminal node. We denote by \( G^o := \{ s \in G : C(s) \neq \emptyset \} \) the set of all non-terminal nodes.

For nodes \( s \) and \( t \), we write \( s \sqsubseteq t \) if \( s \) precedes \( t \), i.e., if there is a directed segment (sequence of directed edges) in the graph from \( s \) to \( t \). If \( s \sqsubseteq t \) and \( s \neq t \), we say that \( s \sqsubset t \).

For any node \( s \), we denote its set of descendants by \( D(s) := \{ t \in G : s \sqsubset t \} \) and its set of non-parent non-descendants is given by \( N(s) := G \setminus (P(s) \cup \{ s \} \cup D(s)) \).

3.2. Variables and gambles on them. With each node \( s \) of the tree, there is associated a variable \( X_s \) assuming values in a non-empty finite set \( \mathcal{X}_s \). We denote by \( \mathcal{G}(\mathcal{X}_s) \) the set of all gambles on \( \mathcal{X}_s \). We extend this notation to more complicated situations as follows. If \( S \) is any subset of \( G \), then we denote by \( X_S \) the tuple of variables whose components are the \( X_s \) for all \( s \in S \). This new joint variable assumes values in the finite set \( \mathcal{X}_S := \times_{s \in S} \mathcal{X}_s \) and the corresponding set of gambles is denoted by \( \mathcal{G}(\mathcal{X}_S) \). When \( S = \emptyset \), we let \( \mathcal{X}_\emptyset \) be a singleton. The corresponding variable \( X_\emptyset \) can then only assume this single value, so there is no uncertainty about it. \( \mathcal{G}(\mathcal{X}_\emptyset) \) can then be identified with the set \( \mathbb{R} \) of real numbers.

Generic elements of \( \mathcal{X}_S \) are denoted by \( x_s \) or \( z_s \) and similarly for \( x_S \) and \( z_S \) in \( \mathcal{X}_S \). Also, if we mention a tuple \( z_S \), then for any \( t \in S \), the corresponding element in the tuple will be denoted by \( z_t \). We assume all variables in the network to be logically independent, meaning that the variable \( X_S \) may assume all values in \( \mathcal{X}_S \), for all \( \emptyset \subset S \subset G \).

We will frequently use the simplifying device of identifying a gamble \( f_S \) on \( \mathcal{X}_S \) with its cylindrical extension to \( \mathcal{X}_U \), where \( S \subset U \subset G \). This is the gamble \( f_U \) on \( \mathcal{X}_U \) defined by \( f_U(x_U) := f_S(x_S) \) for all \( x_U \in \mathcal{X}_U \). To give an example, if \( \mathcal{X}_U \subset \mathcal{G}(\mathcal{X}_G) \), this trick allows us to consider \( \mathcal{X} \cap \mathcal{G}(\mathcal{X}_G) \) as the set of those gambles in \( \mathcal{X} \) that depend only on the variable \( X_S \).

As another example, this device allows us to identify the gambles \( \mathbb{I}_{[x_S]} \) and \( \mathbb{I}_{[x_S]} \times \mathcal{X}_{G \setminus S} \), and therefore also the events \( \{ x_S \} \) and \( \{ x_S \} \times \mathcal{X}_{G \setminus S} \). More generally, for any event \( A \subset \mathcal{X}_S \), we can identify the gambles \( \mathbb{I}_A \) and \( \mathbb{I}_{A \times \mathcal{X}_{G \setminus S}} \) and therefore also the events \( A \) and \( A \times \mathcal{X}_{G \setminus S} \).

3.3. Modelling our beliefs about the network. Throughout the paper, we consider sets of desirable gambles as models for a subject’s beliefs about the values that certain variables in the network may assume. One of the main contributions of this paper, further on in Section 4, will be to show how to construct a joint model for our network, being a coherent set \( \mathcal{D}_G \) of desirable gambles on \( \mathcal{X}_G \).

From such a joint model, one can derive both conditional and marginal models. Let us start by explaining how to condition the global model \( \mathcal{D}_G \). Consider a subset \( f \) of \( G \) and assume we want to update the model \( \mathcal{D}_G \) with the information that \( X_f = x_f \). This leads to the following updated set of desirable gambles:

\[
\mathcal{D}_G | x_f := \{ f \in \mathcal{G}(\mathcal{X}_G) : \mathbb{I}_{[x_f]} f \in \mathcal{D}_G \},
\]

which represents our subject’s beliefs about the value of the variable \( X_{G \setminus f} \), conditional on the observation that \( X_f \) assumes the value \( x_f \). This definition is very intuitive, since \( \mathbb{I}_{[x_f]} f \) is the unique gamble that is called off (is equal to zero) if \( X_f \neq x_f \) and equal to \( f \) if \( X_f = x_f \). Notice that since \( \mathbb{I}_{[x_f]} = 1 \), the special case of conditioning on the certain variable \( X_0 \) does not yield any problems. As wanted, it amounts to not conditioning at all.

Marginalisation is also very intuitive in the language of sets of desirable gambles. Suppose we want to derive a marginal model for our subject’s beliefs about the variable \( X_O \), where \( O \) is some subset of \( G \). This can be done by using the set of desirable gambles that belong to \( \mathcal{D}_G \) but only depend on the variable \( X_O \):

\[
\text{marg}_O(\mathcal{D}_G) := \{ f \in \mathcal{G}(\mathcal{X}_O) : f \in \mathcal{D}_G \}.
\]

Now let \( I \) and \( O \) be disjoint subsets of \( G \) and let \( x_I \) be any element of \( \mathcal{X}_I \). By sequentially applying the process of conditioning and marginalisation we can obtain conditional
marginal models for our subject’s beliefs about the value of the variable $X_D$, conditional on the observation that $X_I$ assumes the value $x_I$:

$$\text{marg}_D(\mathcal{D}_G | x_I) = \{ f \in \mathcal{G}(\mathcal{X}_D) : 1_{\{x_I\}} f \in \mathcal{D}_G \}.$$  \hspace{1cm} (2)

Since coherence is trivially preserved under both conditioning and marginalisation, we find that if the joint model $\mathcal{D}_G$ is coherent, all the derived models will also be coherent.

Conditional and/or marginal models do not necessarily have to be derived from a joint model, they can instead also be given as separate models on their own. In that case we will generally denote them as $\mathcal{D}_{O|x_I}$. The special case of an unconditional marginal model is sometimes denoted as $\mathcal{D}_0$ but we will also use the general notation above by letting $I = \emptyset$ in the general notation above.

3.4. **Epistemic irrelevance.** We now have the necessary tools to introduce one of the most important concepts for this paper, that of epistemic irrelevance. We describe the case of conditional irrelevance, as we will show that the unconditional version of epistemic irrelevance can easily be recovered as a special case.

Consider three disjoint subsets $C$, $I$, and $O$ of $G$. When a subject judges $X_I$ to be *epistemically irrelevant* to $X_O$ conditional on $X_C$, he assumes that if he knows the value of $X_C$, then learning in addition which value $X_I$ assumes in $\mathcal{X}_I$ will not affect his beliefs about $X_O$. More formally, assume that a subject has for every $x_C \in \mathcal{X}_C$ a coherent set of desirable gambles $\mathcal{D}_{O|x_C}$ over $\mathcal{X}_O$. If he assesses $X_I$ to be epistemically irrelevant to $X_O$ conditional on $X_C$, this implies that he can infer from these models $\mathcal{D}_{O|x_C}$ the following additional conditional models $\mathcal{D}_{O|x_C,I}$ on $\mathcal{D}_O$:

$$\mathcal{D}_{O|x_C,I} = \mathcal{D}_{O|x_C} \quad \text{for all } x_{C,I} \in \mathcal{X}_{C,I}.$$  

By now, it should be clear that it suffices for the unconditional case, in the discussion above, to let $C = \emptyset$. This makes sure the variable $X_C$ has only one possible value, so conditioning on that variable amounts to not conditioning at all.

3.5. **Local uncertainty models.** We now add *local uncertainty models* to each of the nodes $s$ in our network. These local models are assumed to be given beforehand and will be used further on in Section 4 as basic building blocks to construct a joint model for a given network.

If $s$ is not a root node of the network, i.e. has a non-empty set of parents $P(s)$, then we have a conditional local model for every instantiation of its parents. For each $x_{P(s)} \in \mathcal{X}_{P(s)}$, we have a conditional coherent set $\mathcal{D}_{s|x_{P(s)}}$ of desirable gambles on $\mathcal{X}_s$. It represents our subject’s beliefs about the variable $X_s$ conditional on the information that its parents $X_{P(s)}$ assume the value $x_{P(s)}$.

If $s$ is one of the root nodes, i.e. has no parents, then our subject’s local beliefs about the variable $X_s$ are represented by an unconditional local model. It should be a coherent set of desirable gambles and will be denoted by $\mathcal{D}_s$. As was explained in Section 3.3, we can also use the common generic notation $\mathcal{D}_{s|x_{P(s)}}$ in this unconditional case, since for a root node $s$, its set of parents $P(s)$ is equal to the empty set $\emptyset$.

3.6. **The interpretation of the graphical model.** In classical Bayesian nets, the graphical structure is taken to represent the following assessments: for any node $s$, conditional on its parent variables, the associated variable is independent of its non-parent non-descendant variables.

When generalising this interpretation to imprecise graphical networks, the classical notion of independence gets replaced by a more general, imprecise notion of independence that is usually chosen to be strong independence. In this paper we will not do so, we choose to use the weaker, assymetric notion of epistemic irrelevance instead, which was introduced earlier on in Section 3.4. In the special case of precise uncertainty models, both epistemic
irrelevance and strong independence will reduce to the usual classical notion of independence and the corresponding interpretations of the graphical network are equivalent with the one used in a classical Bayesian network.

In the present context, we assume that the graphical structure of the network embodies the following conditional irrelevance assessments, turning the network into a credal net under epistemic irrelevance. Consider any node $s$ in the network, its set of parents $P(s)$ and its set of non-parent non-descendants $N(s)$. Then conditional on its parent variables $X_{P(s)}$, the non-parent non-descendant variables $X_{N(s)}$ are assumed to be epistemically irrelevant to the variable $X_s$ associated with the node $s$.

For a coherent set of desirable gambles $\mathcal{D}_G$ that describes our subject’s global beliefs about all the variables in the network, this interpretation has the following consequences. It can easily be seen from Sections 3.3 and 3.4 that it implies for all $s \in G$ and all subsets $I$ of $N(s)$ that

$$\text{marg}_s(\mathcal{D}_G|X_{P(s)}):=\text{marg}_s(\mathcal{D}_G|X_{P(s)\cup I}) \text{ for all } x_{P(s)\cup I} \in \mathcal{X}_{P(s)\cup I}. \quad (3)$$

4. Constructing the Most Conservative Joint

Let us now show how to construct a global model for the variables in the network, and argue that it is the most conservative coherent model that extends the local models and expresses all conditional irrelevancies encoded in the network. But before we do so, let us provide some motivation. Suppose we have a global set of desirable gambles $\mathcal{D}_G$, how do we express that such a model is compatible with the assessments encoded in the network?

4.1. Defining properties of the joint. We will require our joint model to satisfy the following four properties. First of all, we require that our global model extends the local ones. This means that the local models derived from the global one should be equal to the given local models:

**G1.** For each node $s$ in $G$, $\text{marg}_s(\mathcal{D}_G|X_{P(s)}) = \mathcal{D}_{s|X_{P(s)}}$ for all $x_{P(s)} \in \mathcal{X}_{P(s)}$.

The second requirement is that our model reflects all epistemic irrelevancies encoded in the graphical structure of the network:

**G2.** $\mathcal{D}_G$ satisfies all equalities that are imposed by Eq. (3). In these equalities, the right hand side can be replaced by $\mathcal{D}_{s|X_{P(s)}}$ due to requirement G1.

The third requirement is that our model satisfies the rationality requirement of coherence:

**G3.** $\mathcal{D}_G$ is coherent (satisfies requirements D1–D4).

Since requirements G1–G3 do not uniquely determine a global model, there is also a final requirement, which guarantees that all inferences we make on the basis of our global models are as conservative as possible, and are therefore based on no other considerations than what is encoded in the tree:

**G4.** $\mathcal{D}_G$ is the smallest set of desirable gambles on $\mathcal{D}_G$ satisfying requirements G1–G3: it is a subset of any other set that satisfies them.

We will now show how to construct the unique global model $\mathcal{D}_G$ that satisfies all off the four requirements G1–G4 that were given above.

4.2. Constructing the joint. Let us start by looking at a single given marginal model $\mathcal{D}_{s|X_{P(s)}}$, and investigate some of its implications for the joint model $\mathcal{D}_G$. Consider any node $s$ in the network and fix values $x_{P(s)}$ for its parents. For the local model $\mathcal{D}_{s|X_{P(s)}}$, we now introduce a corresponding (non-coherent) set $\mathcal{A}^{irr}_{s|X_{P(s)}}$ of desirable gambles on $\mathcal{D}_G$:

$$\mathcal{A}^{irr}_{s|X_{P(s)}}:=\{f|_{X_{P(s)\cup N(s)}} : x_{N(s)} \in \mathcal{X}_{N(s)}, f \in \mathcal{D}_{s|X_{P(s)}}\}. \quad (4)$$
We now propose the following expression for the joint model $\mathcal{D}_{G}$, and this results in a set $S_{G}^{irr}$ of desirable gambles on $\mathcal{X}_{G}$ that will become essential further on for our construction of the joint model $\mathcal{D}_{G}$. The importance of this set $S_{G}^{irr}$ is already manifested by the following proposition, which is proven in Appendix A (as are all other important results of this paper).

**Proposition 1.** Consider any $s \in G$ and $x_{Pi(s)} \in \mathcal{X}_{Pi(s)}$. Then the set $S_{x_{Pi(s)}}^{irr}$ will be a subset of any joint model $\mathcal{D}_{G}$ satisfying requirements G1 and G2. A consequence, their union $S_{G}^{irr}$ will also be a subset of any joint model $\mathcal{D}_{G}$ satisfying requirements G1 and G2 and thus a subset of the unique joint model that satisfies all four requirements G1–G4.

We now propose the following expression for the joint model $\mathcal{D}_{G}$, describing our subject’s beliefs about the variables in the network and satisfying all four requirements G1–G4:

$$ \mathcal{D}_{G} := \text{posi}(S_{G}^{irr}) \tag{6} $$

Since our eventual joint model $\mathcal{D}_{G}$ should be coherent (satisfy requirement G3), and thus in particular should be a convex cone (satisfy properties D3 and D4), we know that posi($\mathcal{D}_{G}$) should be equal to $\mathcal{D}_{G}$. It is therefore very intuitive to consider the set given above, since posi(posi($\mathcal{D}$)) = posi($\mathcal{D}$) for any set of desirable gambles $\mathcal{D}$. On the other hand, it is not obvious that this set is indeed the unique joint model $\mathcal{D}_{G}$ satisfying all four requirements G1–G4. Therefore, the next part of this paper consists of three propositions that will lead to the main theorem, which states that the joint model $\mathcal{D}_{G}$ does satisfy all four requirements G1–G4. We start by showing that it contains all positive gambles.

**Proposition 2.** $\mathcal{G}(\mathcal{D}_{G})_{>0}$ is a subset of posi($S_{G}^{irr}$). As a consequence, we have that posi($S_{G}^{irr}$) = posi($S_{G}^{irr} \cup \mathcal{G}(\mathcal{D}_{G})_{>0}$) = $\mathcal{G}(S_{G}^{irr})$

This proposition serves as a first step towards the following coherence result, which proves that our joint model satisfies requirement G3.

**Proposition 3.** posi($S_{G}^{irr}$) is a coherent set of desirable gambles on $\mathcal{X}_{G}$.

The proof is given in Appendix A, but it contains an interesting result that deserves to be pointed out. The crucial step of the proof hinges on the assumption that if the local models of our network were precise probability mass functions, we would be able to construct a joint probability mass function that satisfies all irrelevancies (in that case independencies) that are encoded in our network. Since the precise version of a credal tree under epistemic irrelevance is a classical Bayesian network, this assumption is indeed true. However, what is nice about this approach is that it can easily be extended to credal networks with irrelevance assumptions that differ from the ones we use, as long as the assumption above is satisfied. This enables us to use existing coherence results for precise networks to prove their counterparts for credal networks.

We now turn to an important proposition that will be essential to prove that our joint model extends the local models and expresses all conditional irrelevancies encoded in the network (satisfies requirements G1 and G2).

**Proposition 4.** Consider any $s \in G$ and any subset $I$ of its non-parent non-descendants $N(s)$. If we fix a value $x_{Pi(s),i} \in \mathcal{X}_{Pi(s),i}$, then it holds for every $f \in \mathcal{G}(\mathcal{X}_{s})$ that

$$ 1_{(x_{Pi(s),i})}f \in \text{posi}(S_{G}^{irr}) \iff f \in \mathcal{D}_{s|x_{Pi(s)}}. $$
We now have all necessary tools to formulate our most important result. It is the main contribution of this paper and provides a justification for the joint model $\mathcal{F}_G$ that was proposed by Eq. (6).

**Theorem 5.** Consider any credal network under epistemic irrelevance with given conditional marginal models $\mathcal{F}_G$, then $\mathcal{F}_G = \text{posi}(\mathcal{A}^{irr}_G)$ is the unique set of desirable gambles on $\mathcal{G}$ that satisfies all four requirements G1–G4.

5. **Conclusions**

This paper has presented a new approach to credal nets. We replaced the commonly used notion of strong independence with the weaker notion of epistemic irrelevance and expressed both our local models and the eventual joint model in the language of sets of desirable gambles. This has lead to an intuitive, easy expression for a joint model, that is proven to be the most conservative coherent model that extends the local models and expresses all conditional irrelevancies encoded in the network.

**References**

[1] De Cooman, G., Quaeghebeur, E.: Exchangeability and sets of desirable gambles. International Journal of Approximate Reasoning (2010), in print. Special issue in honour of Henry E. Kyburg, Jr.
[2] Couso, I., Moral, S.: Sets of desirable gambles: conditioning, representation, and precise probabilities. International Journal of Approximate Reasoning 52(7), 1034–1055 (2011)
[3] Walley, P.: Towards a unified theory of imprecise probability. International Journal of Approximate Reasoning 24, 125–148 (2000)
[4] Gert de Cooman and Enrique Miranda Independent Natural Extension for Sets of Desirable Gambles In *ISPTA 2011: Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*, pages 169–178, Innsbruck, 2011.
[5] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

**Appendix A. Proofs of Important Results**

In this Appendix, we give proofs for Propositions 1, 2, 3 and 4 and Theorem 5.

**Proof of Proposition 1.** The second part of this proposition is trivial and we thus only need to prove the first part. To do so, consider any $s \in G$ and $x_{P(s) \cup N(s)} \in \mathcal{F}_{P(s) \cup N(s)}$. As a consequence of requirements G1 and G2, we see that $\text{marg}_{s}(\mathcal{F}_G | x_{P(s) \cup N(s)})$ should be equal to the given local model $\mathcal{F}_{s|x_{P(s)}}$. If we now apply Eq. (2), it follows immediately that $\text{I}_{(x_{P(s) \cup N(s)})} f$ is an element of $\mathcal{F}_G$, thereby completing the proof.

**Proof of Proposition 2.** The essential step is to see that for any $x_G \in \mathcal{G}_G$, the indicator function $\text{I}_{\{x_G\}}$ is an element of $\mathcal{A}^{irr}_G$. To prove this, pick an arbitrary leaf $s \in G$. This is possible because a DAG with a finite amount of nodes always has at least one leaf. Since $s$ is a leaf, it has no descendants and we therefore have that $G = s \cup P(s) \cup N(s)$. Due to the coherence of the local models, and in particular property D2, the indicator function $\text{I}_{\{s\}}$ is an element of $\mathcal{F}_{s|x_{P(s)}}$. We can now apply Eqs. (4) and (5) to see that $\text{I}_{\{s\}} = \text{I}_{\{s,x_{P(s) \cup N(s)}\}}$ is an element of $\mathcal{A}^{irr}_G$.

Since every $f > 0$ is a finite strictly positive linear combination of the indicator functions that were constructed above, it follows that $\text{posi}(\mathcal{A}^{irr}_G)$ does indeed contain all positive gambles in $\mathcal{F}(\mathcal{G})_{>0}$. As a consequence, we have that $\text{posi}(\mathcal{A}^{irr}_G) = \text{posi}(\mathcal{A}^{irr}_G) \cup \mathcal{F}(\mathcal{G})_{>0}$ and because $\text{posi}(\mathcal{G}) = \text{posi}(\mathcal{G})$ for any set of desirable gambles $\mathcal{G}$, we find that $\text{posi}(\mathcal{A}^{irr}_G) = \text{posi}(\mathcal{A}^{irr}_G) \cup \mathcal{F}(\mathcal{G})_{>0}$. The right hand side of this equality is trivially equal to $\text{posi}(\mathcal{A}^{irr}_G) \cup \mathcal{F}(\mathcal{G})_{>0} = \mathcal{F}^{irr}_G$, thereby completing the proof.

Our proof of Proposition 3 uses the following convenient version of the separating hyperplane theorem. It is proven in Ref. [4, Lemma 2] and repeated here to make the paper more self-contained.
Lemma 6. Consider any finite subset \( \mathcal{A} \) of \( \mathcal{G}(\mathcal{X}) \). Then \( 0 \notin \mathcal{B}(\mathcal{A}) := \text{posi}(\mathcal{A} \cup \mathcal{G}(\mathcal{X})_{>0}) \) if and only if there is some probability mass function \( p \) such that \( \sum_{x \in \mathcal{X}} p(x) f(x) > 0 \) for all \( f \in \mathcal{A} \) and \( p(x) > 0 \) for all \( x \in \mathcal{X} \).

Proof of Proposition 3. Proving that \( \text{posi}(\mathcal{A}^{irr}_{G}) \) is coherent, means showing that it satisfies the properties D1–D4. Property D2 is a direct consequence of Proposition 2 and the properties D3 and D4 are trivial since \( \text{posi}(\mathcal{A}^{irr}_{G}) \) is a convex cone due to the use of the positivity operator. We thus only need to prove the first property, stating that any gamble \( f \in \mathcal{G}(\mathcal{X}_G) \) for which \( f \leq 0 \) can not be an element of \( \text{posi}(\mathcal{A}^{irr}_{G}) \).

So consider any \( f \in \text{posi}(\mathcal{A}^{irr}_{G}) \) and assume \textit{ex absurdo} that \( f \leq 0 \). We will show that this leads to a contradiction. Since \( f \) is an element of \( \text{posi}(\mathcal{A}^{irr}_{G}) \), it follows from Eqs. (1), (4) and (5) that

\[
f = \sum_{x \in \mathcal{G}} \sum_{x_{P(s)}} \sum_{x_{N(s)}} \mathbb{I}_{\{x_P(s),N(s)\}} f_{x,P(s),N(s)},
\]

where every \( f_{x,P(s),N(s)} \) is an element of \( \mathcal{D}_s \cap \{0\} \) and at least one of them differs from zero. The only perhaps surprising fact about the equation above, is that it does not contain any (strictly positive) scaling factors \( \lambda_{x,P(s),N(s)} \). The reason why these factors can be omitted is that the local models \( \mathcal{D}_s \cap \{0\} \) are coherent and thus invariant under strictly positive linear scaling. Therefore, scaling a gamble \( f_{x,P(s),N(s)} \in \mathcal{D}_s \cap \{0\} \) with a strictly positive factor \( \lambda_{x,P(s),N(s)} \) will still yield a gamble in \( \mathcal{D}_s \cap \{0\} \).

Next, for every \( s \in \mathcal{G} \) and \( x_{P(s)} \in \mathcal{X}_{P(s)} \) we construct a finite subset of the local model \( \mathcal{D}_s \cap \{0\} \):

\[
\mathcal{A}^{f}_{S|x_{P(s)}} := \{ f_{x,P(s),N(s)} : x_{N(s)} \in \mathcal{X}_{N(s)} \text{ and } f_{x,P(s),N(s)} \neq 0 \}.
\]

Due to the coherence of \( \mathcal{D}_s \cap \{0\} \), we have that \( 0 \notin \mathcal{B}(\mathcal{A}^{f}_{S|x_{P(s)}}) \subseteq \mathcal{B}(\mathcal{D}_s \cap \{0\}) = \mathcal{D}_s \cap \{0\} \) and we can therefore apply Lemma 6. This gives us for every \( s \in \mathcal{G} \) and \( x_{P(s)} \in \mathcal{X}_{P(s)} \) a mass function \( p_s \) on \( \mathcal{X}_{P(s)} \) with expectation operator \( E_s(\cdot|x_{P(s)}) \) on \( \mathcal{X}_{P(s)} \) such that \( p_f(x_{P(s)}) > 0 \) for all \( x_{P(s)} \) and \( E_s(g|x_{P(s)}) > 0 \) for each \( g \in \mathcal{A}^{f}_{S|x_{P(s)}} \).

The trick is now to create a Bayesian network that has the conditional mass functions \( p_f(\cdot|x_{P(s)}) \) as its local models and has the same graphical structure as our credal net under epistemic irrelevance. If we let \( E_G \) be the expectation operator for this Bayesian net, we find that

\[
E_G(f) = \sum_{s \in \mathcal{G}} \sum_{x_{P(s)} \in \mathcal{X}_{P(s)}} \sum_{x_{N(s)} \in \mathcal{X}_{N(s)}} E_G(\mathbb{I}_{\{x_{P(s),N(s)}\}} f_{x,P(s),N(s)})
\]

\[
= \sum_{s \in \mathcal{G}} \sum_{x_{P(s)} \in \mathcal{X}_{P(s)}} \sum_{x_{N(s)} \in \mathcal{X}_{N(s)}} E_G(\mathbb{I}_{\{x_{P(s),N(s)}\}}) E_G(f_{x,P(s),N(s)} | x_{P(s)})
\]

\[
= \sum_{s \in \mathcal{G}} \sum_{x_{P(s)} \in \mathcal{X}_{P(s)}} \sum_{x_{N(s)} \in \mathcal{X}_{N(s)}} p_G(x_{P(s),N(s)}) E_G(f_{x,P(s),N(s)} | x_{P(s)})
\]

in which \( p_G \) is the global mass function of the Bayesian net. Since all the local probabilities \( p_s(\cdot|x_{P(s)}) \) are strictly positive, this is also true for the global ones and we find that \( p_G(x_{P(s),N(s)}) > 0 \). For the conditional expectations \( E_G(f_{x,P(s),N(s)} | x_{P(s)}) \) there are two possibilities. Either \( f_{x,P(s),N(s)} = 0 \), in which case \( E_G(f_{x,P(s),N(s)} | x_{P(s)}) = 0 \), either \( f_{x,P(s),N(s)} \in \mathcal{A}^{f}_{S|x_{P(s)}} \), in which case \( E_G(f_{x,P(s),N(s)} | x_{P(s)}) > 0 \). However, since at least one of the gambles \( f_{x,P(s),N(s)} \) in Eq. (7) has to differ from zero, it is not possible that \( E_G(f_{x,P(s),N(s)} | x_{P(s)}) = 0 \) for all gambles \( f_{x,P(s),N(s)} \) and we can conclude that \( E_G(f) > 0 \). If we now apply our assumption \textit{ex absurdo} that \( f \leq 0 \) and thus \( E_G(f) \leq 0 \), this leads to a contradiction and completes the proof. □
Proof of Proposition 4. The reverse implication is trivial due to the way \(\text{pos}(\mathcal{A}_G^{irr})\) is constructed; see Eqs. (1), (4) and (5). It therefore suffices to prove the direct implication. Consider any \(s \in G\), any subset \(I\) of its non-parent non-descendants \(N(s)\) and fix a value \(x_{P(s),I} \in \mathcal{I}_{P(s),I}\). We set out to prove for every \(f \in \mathcal{I}_s\) that \(f \notin \mathcal{I}_{s|x_{P(s),I}}\) implies \(\mathcal{I}_{s|x_{P(s),I}} f \notin \text{pos}(\mathcal{A}_G^{irr})\).

The case \(f = 0\) is trivial because \(\mathcal{I}_{s|x_{P(s),I}} f = 0\) is then equal to zero, which can not be an element of \(\text{pos}(\mathcal{A}_G^{irr})\) due to its coherence; see Proposition 3. If \(f \neq 0\), we start by applying some of the properties of maximal coherent sets of desirable gambles that were introduced in Section 2.4. To do so, consider any \(\mathcal{M}_s^*\text{irr}\) \(\supseteq \mathcal{D}_{s|x_{P(s)}}\) for which \(f \notin \mathcal{M}_s^*\text{irr}\). Due to the second property and the fact that \(f \neq 0\), this in turn implies that \(- f \notin \mathcal{M}_s^*\text{irr}\).

We now denote by \(\mathcal{A}_G^{irr}\) the set that is obtained by Eq. (5) if we replace the local model \(\mathcal{D}_{s|x_{P(s)}}\) by the specific maximal superset \(\mathcal{M}_s^*\text{irr}\) that was introduced above. It should be clear that \(\mathcal{A}_G^{irr} \supseteq \mathcal{A}_G^{irr}\). Next, since \(- f \notin \mathcal{M}_s^*\text{irr}\), it follows from the construction of \(\mathcal{A}_G^{irr}\) that \(\mathcal{I}_{s|x_{P(s),I}} (- f) \in \mathcal{A}_G^{irr} \subseteq \text{pos}(\mathcal{A}_G^{irr})\). The proof can now be completed if we realise that \(\mathcal{I}_{s|x_{P(s),I}} f \notin \text{pos}(\mathcal{A}_G^{irr})\) because this would contradict with its coherence and notice that it implies that \(\mathcal{I}_{s|x_{P(s),I}} f \notin \text{pos}(\mathcal{A}_G^{irr})\) because \(\mathcal{A}_G^{irr} \supseteq \mathcal{A}_G^{irr}\).

Proof of Theorem 5. We start by proving that the joint model \(\mathcal{D}_G = \text{pos}(\mathcal{A}_G^{irr})\) satisfies requirements G1 and G2. To do so, consider any \(s \in G, I \subseteq N(s)\) and \(x_{P(s),I} \in \mathcal{I}_{P(s),I}\) and an arbitrary gamble \(h \in \mathcal{I}_s\). It can be seen from the following chain of equivalences that \(\text{marg}_s(\mathcal{D}_G|x_{P(s),I}) = \mathcal{D}_{s|x_{P(s)}}\).

\[
h \in \text{marg}_s(\mathcal{D}_G|x_{P(s),I}) \iff h \in \text{marg}_s(\text{pos}(\mathcal{A}_G^{irr}), x_{P(s),I}) \iff \mathcal{I}_{s|x_{P(s),I}} h \in \text{pos}(\mathcal{A}_G^{irr})
\]

The second equivalence is a direct application of Eq. 2 and the third one is due to Proposition 4. Requirement G1 is now proven by letting \(f = \emptyset\) and requirement G2 is fulfilled because \(\text{marg}_s(\mathcal{D}_G|x_{P(s),I}) = \mathcal{D}_{s|x_{P(s)}} = \text{marg}_s(\mathcal{D}_G|x_{P(s)})\). The next step is to show that the joint model \(\mathcal{D}_G = \text{pos}(\mathcal{A}_G^{irr})\) also satisfies requirements G3 and G4.

Requirement G3 demands that \(\mathcal{D}_G = \text{pos}(\mathcal{A}_G^{irr})\) is coherent, but since this is proven in Proposition 3, the only thing that is left to prove is requirement G4. This final requirement demands that \(\mathcal{D}_G = \text{pos}(\mathcal{A}_G^{irr})\) is included in any set of desirable gambles satisfying the requirements G1–G3. This is easy to prove since we know from Proposition 1 that \(\mathcal{A}_G^{irr}\) is a subset of any joint model satisfying all four requirements. It then follows from the coherence requirement G3 that \(\mathcal{D}_G = \text{pos}(\mathcal{A}_G^{irr})\) is a subset of all joint models satisfying G1–G3 and thus the unique smallest model that also satisfies requirement G4. 

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