Exact relativistic theory of geoid’s undulation.

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Abstract
Precise determination of geoid is one of the most important problem of physical geodesy. The present paper extends the Newtonian concept of the geoid to the realm of Einstein’s general relativity and derives an exact relativistic equation for the unperturbed geoid and level surfaces under assumption of axisymmetric distribution of background matter in the core and mantle of the Earth. We consider Earth’s crust as a small disturbance imposed on the background distribution of matter, and formulate the master equation for the anomalous gravity potential caused by this disturbance. We find out the gauge condition that drastically simplifies the master equation for the anomalous gravitational potential and reduces it to the form closely resembling the one in the Newtonian theory. The master equation gives access to the precise calculation of geoid’s undulation with the full account for relativistic effects not limited to the post-Newtonian approximation. The geoid undulation theory, given in the present paper, utilizes the geometric methods of the perturbation theory of curved spacetime manifolds and is fully covariant.

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Keywords: gravity, relativity, geodesy, geoid
PACS: 04.20.-q, 04.25.Nx, 91.10.-v, 91.10.By

1. Introduction
The geoid in the classical Newtonian theory of gravity is a surface on which the gravity potential is constant and to which the local direction of the gravity force is perpendicular everywhere [1]. The shape of the geoid depends critically on the law of distribution of mass density in the interior of the Earth. For the density distribution is fairly irregular, it is not possible to describe geoid’s surface analytically with a simple mathematical equation. Therefore, determination of the geoid is a complicated geodetic problem.

Geoid plays an important role in solving many geodetic problems and, in particular, in space geodesy and navigation where it is used to link the geodetic height, obtained from the global navigation satellite system (GNSS) measurements, and the normal height, determined by geodetic precise levelling. Furthermore, knowledge of geoid’s surface is crucially important in physical oceanography to determine the mean sea level and its spatial and/or temporal variations. Geoid also serves as a geodetic datum in geophysical applications.

Therefore, determination of the geoid with a high accuracy is a primary task of physical geodesy. Its precise calculation is usually carried out by combining a global geopotential model of gravity field with terrestrial gravity
anomalies measured in the region of interest and supplemented with the local/regional topographic information. The gravity anomalies allows us to find out the undulation of geoid’s surface that is measured with respect to a reference-ellipsoid of the World Geodetic System [2] established in 1984 (WGS84) and last revised in 2004. Geoid’s undulation is given in terms of height above the ellipsoid taken along the normal line to the ellipsoid’s surface (see http://earth-info.nga.mil/GandG/wgs84/ for more detail).

The equation of unperturbed geoid’s surface, $\bar{S}$, in the Newtonian gravity is defined by the condition of a constant gravity potential $\bar{W}_N$ in the geocentric frame rigidly rotating with a constant angular velocity $\Omega$,

$$\bar{W}_N(r, \theta) \equiv \bar{V}(r, \theta) + \frac{1}{2} \Omega^2 r^2 \sin^2 \theta = \text{const.}$$  \hspace{1cm} (1)

where $x^i = (x^1, x^2, x^3) = (r, \theta, \phi)$ are the spherical coordinates with the angle $\theta$ (co-latitude) measured from the rotational axis, and $\phi$ (longitude) measured in the equatorial plane. The quantity $\bar{V} = \bar{V}(r, \theta)$ in (1) is the unperturbed gravitational potential determined by the Poisson equation,

$$\Delta_N \bar{V}(r, \theta) = -4\pi G \bar{\rho}(r, \theta) , \hspace{1cm} (2)$$

where $\bar{\rho} = \bar{\rho}(r, \theta)$ is the axisymmetric volume mass density, $G$ is the universal gravitational constant,

$$\Delta_N \equiv \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi\phi} , \hspace{1cm} (3)$$

is the Laplace operator in the spherical coordinates, and the partial derivatives $\partial_i \equiv \partial / \partial x^i$, $\partial_{ij} \equiv \partial^2 / \partial x^i \partial x^j$ (the Roman indices $i, j = 1, 2, 3$).

The unperturbed geoid $\bar{W}_N$ defined in (1) is not used in geodetic datum. Instead, the gravity potential $\bar{U}_N = \bar{U}_N(r, \theta)$ of a reference (level) ellipsoid is used. It is defined by

$$\bar{U}_N(r, \theta) \equiv \bar{V}_N(r, \theta) + \frac{1}{2} \Omega^2 r^2 \sin^2 \theta , \hspace{1cm} (4)$$

where $\bar{V}_N$ is the gravitational potential of the reference-ellipsoid that satisfies the Laplace equation in the space exterior to the ellipsoid that contains the total mass of the Earth. Inside the mass distribution the gravitational potential of the ellipsoid satisfies the Poisson equation

$$\Delta_N \bar{V}_N(r, \theta) = -4\pi G \bar{\rho}_N(r, \theta) , \hspace{1cm} (5)$$

where the density $\bar{\rho}_N (r, \theta) \equiv \bar{\rho}(r, \theta)$ in the most general case of an inhomogeneous distribution of mass.

Because all unperturbed functions depend only on $r$ and $\theta$, the unperturbed geoid is an axisymmetric body. In the most general case, equation (1) does not define a surface of the ellipsoid of revolution. Only in case of a uniform mass density, $\bar{\rho} = \text{const.}$, the geoid and the ellipsoid of revolution coincide [3, section 5.2]. Equation (1) also defines the level surface of a constant density and pressure. Because of the previous remark, the level surfaces do not coincide with the ellipsoid of revolution for inhomogeneous distribution of density. In most of the practical applications, however, the difference between the axisymmetric figure defined by (1) and the ellipsoid of revolution is ignored [4, Section 4.2.1]. In applying general relativity to calculation of the geoid surface, it may become important to distinguish the axisymmetric figure of the geoid defined in (1) from the ellipsoid of revolution. This problem, however, is lying beyond the scope of the present paper.

Earth’s crust is a thin surface layer having irregular mass density, $\mu = \mu(r, \theta, \phi)$, deviating significantly from the axisymmetric distribution. Therefore, the physical surface, $S$, of the geoid is perturbed and deviates from the surface $\bar{S}$ of the unperturbed (axisymmetric) figure defined by (1). We denote $W_N \equiv W_N(r, \theta, \phi)$ - the perturbed gravity potential, and $V = V(r, \theta, \phi)$ - the perturbed gravitational potential that is determined by

$$\Delta_N V(r, \theta, \phi) = -4\pi G \rho(r, \theta, \phi) , \hspace{1cm} (6)$$

with the perturbed mass density

$$\rho(r, \theta, \phi) = \bar{\rho}(r, \theta) + \mu(r, \theta, \phi) . \hspace{1cm} (7)$$
We shall call the difference
\[ T_N(r, \theta, \phi) \equiv W_N(r, \theta, \phi) - \bar{W}_N(r, \theta, \phi), \]
the anomalous (Newtonian) gravity potential, and both functionals, \( W_N \) and \( \bar{W}_N \), are calculated at the same point of space under assumption that the angular velocity \( \Omega \) remains unperturbed. The anomalous potential obeys the Poisson equation
\[ \Delta_N T_N(r, \theta, \phi) = -4\pi G \mu(r, \theta, \phi), \]
inside the mass distribution, and the Laplace equation
\[ \Delta_N T_N(r, \theta, \phi) = 0, \]
outside the mass.

It is important to emphasize that the anomalous gravity potential in classical Newtonian geodesy is defined differently from (8). Namely, the anomalous potential, \( T_e \), is defined as the difference between the real gravity potential \( W_N \) and the normal gravity potential \( U = U(r, \theta) \) of the reference-ellipsoid,
\[ T_e(r, \theta, \phi) \equiv W_N(r, \theta, \phi) - \bar{U}_N(r, \theta), \]
Because the unperturbed density \( \bar{\rho} \) defining \( \bar{V} \) and the density \( \bar{\rho}_e \) defining \( \bar{V}_e \) do not coincide, the differential Poisson equation for \( T_e \) inside the mass distribution contains besides the density \( \mu \) of the crust also the difference \( \bar{\rho} - \bar{\rho}_c \). Hence, inside the mass \( T_N \neq T_e \). Nonetheless, outside the mass both, \( T_N \) and \( T_e \), satisfy the same Laplace equation (10), and can be equated one to another up to a harmonic function.

Molodensky [5, 6] reformulated (10) into an integral equation
\[ 2\pi T_N + \sum \int \frac{T_N}{\ell} n(\partial) \ln (\ell T_N) d\Sigma = 0, \]
where \( \ell = |x - x'| \) denotes the distance between the source point, \( x' \), taken on the Earth’s surface \( \Sigma \) and the field point, \( x \), while \( d\Sigma \) is the surface element of integration at point \( x' \), and \( n(\partial) \) is the (outward) unit normal to \( \Sigma \) at \( x' \). The physical surface \( \Sigma \) of the Earth is known from the Global Navigation Satellite System (GNSS) measurements [1]. Thus, the only remaining unknown in (12) is the external gravity potential, \( T_N \). It can be found by solving either (10) or (12) by employing the gravity disturbances of \( T_N(\Sigma) \) taken on \( \Sigma \) as boundary values. As soon as \( T_N \) is known everywhere in space, the geoid’s undulation (height \( h \)) can be found from the Bruns equation [1]
\[ h = \frac{T_N(S)}{g_N}, \]
where the anomalous potential \( T_N(S) \) refers to the perturbed geoid, and \( g_N \) is normal gravity on the reference-ellipsoid.

Producing a precise global map of the geoid’s undulation has proven to be a challenge. The important discoveries in the classic (Stokesian) theory of geoid computation was made by Vaníček and cowokers [7] whose precise geoid solution enables millimetre-to-centimetre accuracy in geoid computation. The precision of geoid’s computation on the global scale has been further improved with the advent of gradiometric satellites like GRACE (http://www.csr.utexas.edu/grace) and GOSE (http://www.esa.int/Our_Activities/Observing_the_Earth/GOCE). It will continue to improve as new geodetic data will be accumulated.

General relativistic corrections to the Newtonian theory of geoid can reach the magnitude of a centimetre [8, 9]. Though this number looks small but it is within the range of modern geodetic techniques which now include, besides conventional sensors, also atomic clocks [10, 11, 12] that allows us to measure the potential difference of gravitational field instead of deducing it from the measurement of its gradient. The rate of clocks is fully defined by the metric tensor of relativistic theory of gravity. Therefore, taking into account relativistic corrections in the determination of geoid’s undulation is getting practically important. Furthermore, there is a growing demand among geodetic community for merging the science of geodesy with a modern theoretical description of space, time and gravity - the Einstein general relativity. This requires working out an exact theory of relativistic geodesy. This paper contributes to the success of this goal and extends the Newtonian theory of the geoid and its undulation into the realm of general relativity.
The paper is organized as follows. Section 2 defines the unperturbed axisymmetric spacetime manifold and derives Einstein’s equations for the unperturbed metric tensor. Section 3 gives three definitions of the relativistic geoid and proves that they are equivalent. Section 4 discusses the anomalous gravitational potential leading to the geoid undulation. Section 5 derives the master equation for the anomalous gravity potential. Finally, section 6 yields the relativistic Bruns equation for the geoid undulation. Notations are explained in the main text as they appear.

We use the system of units adopted in theoretical physics with the speed of light \( c = 1 \). Though this system of units is not used in geodesy, it makes sense since relativistic equations look more elegant while \( c \) can be easily restored in the equations from the dimensional analysis.

2. Background spacetime manifold

Formulation of the relativistic theory of geoid begins from the unperturbed case of a uniformly rotating Earth under assumption that the tidal forces from the external bodies of the solar system are neglected and Earth’s matter has stationary, axisymmetric distribution. Mathematical description of the unperturbed (background) spacetime manifold corresponding to this configuration is described the most conveniently in the spherical coordinates \( x^a = (\chi^0, x^1, x^2, x^3) = (t, r, \theta, \phi) \) whose spatial axes rotates rigidly with a constant angular velocity, \( \Omega \), counter-clockwise. Naturally, the background spacetime is stationary and axisymmetric with the metric \( \bar{g}_{ab} \) defined as follows \[13\]

\[
d\bar{s}^2 = \bar{g}_{ab}dx^a dx^b = -\left[N^2 - (\Omega - \Theta)B^2 r^2 \sin^2 \theta \right] dt^2 + 2(\Omega - \Theta)B^2 r^2 \sin^2 \theta dtd\phi + A^2 \left(dr^2 + r^2 d\theta^2 \right) + B^2 r^2 \sin^2 \theta d\phi^2 ,
\]

where \( N = N(r, \theta), A = A(r, \theta), B = B(r, \theta), \Theta = \Theta(r, \theta) \) are functions of only two coordinates, \( r \) and \( \theta \), and the Greek indices here and everywhere else take values 0, 1, 2, 3. The metric, \( \bar{g}_{ab}, \) and its inverse, \( \bar{g}^{ab}, \) are used for rising and lowering the Greek (spacetime) indices.

We notice that the stationary, axisymmetric metric (14) possesses two Killing vectors corresponding to translations along time, \( x^0 = t, \) and azimuthal, \( x^3 = \phi, \) coordinates. In the Newtonian limit functions \( A = B = 1, \Theta = 0, \) and \( N = 1 - 2\bar{V}, \) where \( \bar{V} \) is the Newtonian gravitational potential defined by equation (2). General relativity predicts the values of these functions being different from their Newtonian limits. In particular, function \( \Theta \) represents a new type of gravitational field not being present in the Newtonian theory – the gravitomagnetic field – that arises in general relativity due to the rotation of the Earth [14]. It is very weak but can be presently measured with satellite laser ranging technique [15] and/or by means of a spinning gyroscope flying around the Earth in a drag-free satellite [16].

Let us assume that the core and mantle of the Earth are made of an ideal fluid rotating stationary with the angular velocity \( \bar{\Omega} \) which means that the fluid is at rest with respect to the spatial coordinates. Realistic interior of the Earth is not, of course, a perfect fluid since the mantle has a fairly large viscosity [17]. This would be important for relativity due to the rotation of the Earth [14]. It is very weak but can be easily restored in the equations from the dimensional analysis.

Four-velocity of the fluid, \( \bar{u}^a = dx^a/d\tau, \) where \( \tau \) is the proper time taken along a world line of the fluid element, \( d\tau^2 = -d\bar{s}^2. \) For the fluid at rest in the rotating coordinates, its four-velocity has the following components, \( \bar{u}^a = (\bar{u}^0, \bar{u}^r, \bar{u}^\theta, \bar{u}^\phi) = (\bar{u}^0, 0, 0, 0) \) where

\[
\bar{u}^0 = \frac{1}{\sqrt{N^2 - (\Omega - \Theta)^2 B^2 r^2 \sin^2 \theta}} .
\]

World lines of the fluid elements form a rotating and accelerating congruence without divergence. Indeed, the covariant derivative of the four-velocity of the fluid [19, Appendix]

\[
\bar{u}_{a;\beta} = \bar{u}_{a;\beta} + \bar{\sigma}_{a;\beta} + \frac{1}{3} \bar{\partial} \bar{h}_{a;\beta} - \bar{\partial} \bar{u}_{a;\beta} ,
\]

where here, and everywhere else, the vertical bar denotes a covariant derivative on the background manifold with the metric (14),

\[
\bar{h}_{a;\beta} \equiv \bar{g}_{a;\beta} + \bar{u}_{a;\beta} ,
\]
which represents a tensor of projection onto the 3-dimensional space being orthogonal to \( \ddot{u}^\alpha , \ddot{\alpha}^\alpha \equiv \ddot{u}^\alpha \ddot{u}_\alpha \) is a four-acceleration, \( \ddot{\theta} \equiv \ddot{u}^\alpha \ddot{u}_\alpha \) - divergence of the congruence (which should not be confused with the coordinate \( \theta \)), and \( \ddot{\alpha}_{\alpha \beta} \) and \( \ddot{\alpha}_{\alpha \beta} \) are the tensors of a shear and a rotation of the congruence:

\[
\ddot{\alpha}_{\alpha \beta} = \frac{1}{2} \left( \ddot{u}_\alpha \ddot{u}_\beta + \ddot{u}_\beta \ddot{u}_\alpha \right) - \frac{1}{3} \ddot{h}_{\alpha \beta} , \\
(18)
\]

\[
\ddot{\alpha}_{\alpha \beta} = \frac{1}{2} \left( \ddot{u}_\alpha \ddot{u}_\beta - \ddot{u}_\beta \ddot{u}_\alpha \right) . \\
(19)
\]

In case of a rigidly rotating axisymmetric configuration we have \( \ddot{\alpha}_{\alpha \beta} = \ddot{\theta} = 0 \) but \( \ddot{\alpha}_\alpha \neq 0 \) because the fluid particles do not move along geodesics, and \( \ddot{\alpha}_{\alpha \beta} \neq 0 \) because the fluid is rotating. We notice that the spatial components, \( \ddot{h}_{ij} \), of the projector form a 3-dimensional spatial metric on the hypersurface orthogonal to the four-velocity. This metric is used to measure the proper (physical) distances \([18, 20]\) in space.

The energy-momentum tensor of the fluid

\[
\bar{T}^{\alpha \beta} = \varepsilon \ddot{u}^\alpha \ddot{u}^\beta + \bar{p} \left( g^{\alpha \beta} + \ddot{u}^\alpha \ddot{u}^\beta \right) , \\
(20)
\]

where \( \varepsilon \) and \( \bar{p} \) are the background energy density and pressure of Earth’s matter. The energy density contains the specific internal energy \( \bar{\Pi} \) through the thermodynamic relation, \( \varepsilon = \bar{p} (1 + \bar{\Pi}) \), where \( \bar{p} \) is the background (unperturbed) value of the mass density of matter. The specific internal energy related to \( \bar{p} \) and \( \bar{\rho} \) by the first law of thermodynamics

\[
d\bar{\Pi} + \bar{p} d \left( \frac{1}{\bar{\rho}} \right) = 0 . \\
(21)
\]

Einstein’s field equations are

\[
\bar{R}_{\alpha \beta} = 8\pi G \left( \bar{T}_{\alpha \beta} - \frac{1}{2} \bar{g}_{\alpha \beta} \bar{T} \right) , \\
(22)
\]

where \( \bar{T} \equiv \\bar{T}_\alpha^a = \bar{g}^{\alpha \beta} \bar{T}_{\alpha \beta} \), \( \bar{R}_{\alpha \beta} \) is the Ricci tensor formed from the metric tensor \((14)\), its first and second derivatives \([8, \text{Section 3.7}]\). Substituting the metric \((14)\) and \((20)\) to \((22)\) yields differential equations for the yet unknown functions entering the metric tensor

\[
\left( \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta + \frac{1}{r \tan \theta} \partial_\phi \right) \nu = -8\pi A^2 \left[ \bar{T}_0^0 + (\Omega - \bar{\Omega}) \bar{T}_\phi^\phi - \frac{1}{2} \bar{T} \right] + \frac{B^2 r^2 \sin^2 \theta}{2N^2} \partial \bar{\Sigma} \partial \Sigma - \partial \nu \partial (\nu + \ln B) , \\
(23)
\]

\[
\left( \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta - \frac{1}{r \sin^2 \theta} \partial_\phi \right) \nu \theta = \frac{16\pi N^2 A^2 \bar{T}_\phi^\phi}{B^2 r^2 \sin^2 \theta} - \partial \bar{\Sigma} \partial \nu (\nu - 3 \ln B) r \sin \theta , \\
(24)
\]

\[
\left( \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta - \frac{1}{r \sin^2 \theta} \partial_\phi \right) \nu \bar{\Omega} r \sin \theta = 8\pi N A^2 \left( \bar{T}_r^r + \bar{T}_\phi^\phi \right) r \sin \theta , \\
(25)
\]

\[
\left( \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta - \frac{1}{r \sin^2 \theta} \partial_\phi \right) \nu \bar{\nu} = 8\pi A^2 \left[ \bar{T}_\phi^\phi - (\Omega - \bar{\Omega}) \bar{T}_\phi^\phi \right] + \frac{3B^2 r^2 \sin^2 \theta}{4N^2} \partial \bar{\Sigma} \partial \Sigma - \partial \nu \partial \nu , \\
(26)
\]

where \( \nu \equiv \ln N \), and we have used the following abbreviation \([13]\) for the product of two arbitrary functions, \( u \) and \( w \),

\[
\partial u \partial w \equiv (\partial u / \partial r) \partial_r w + \frac{1}{r^2} (\partial u / \partial \theta) \partial_\theta w , \\
(27)
\]

The components of the energy-momentum tensor in \((23)-(26)\) are \( \bar{T}_0^0 = -\varepsilon, \bar{T}_r^r = \bar{T}_\phi^\phi = \bar{T}_0^\theta = \bar{P}, \) and

\[
\bar{T}_\phi^\phi = \frac{(\varepsilon + \bar{p}) (\Omega - \bar{\Omega}) B^2 r^2 \sin^2 \theta}{N^2 - (\Omega - \bar{\Omega})^2 B^2 r^2 \sin^2 \theta} , \\
(28)
\]

and all other components of \( \bar{T}^{\alpha \beta} \) are nil. After solving \((23)-(26)\) we get a complete description of the background spacetime manifold in terms of functions \( A, B, N, \bar{\Omega} \) entering the metric tensor \((14)\).
3. Relativistic geoid

Pioneering study of relativistic geodesy including the geoid definition have been conducted by Bjerhammar [21]. The Newtonian concept of the geoid was extended to the post-Newtonian approximation of general relativity in [22, 23]. More recent discussion of the post-Newtonian gravimetry and geodesy is given in [8, 9]. In this section we make a next step and introduce an exact concept of the relativistic geoid in general relativity that is not limited to the post-Newtonian approximation.

General relativity offers three definitions of the relativistic geoid [22, 23] which are proven to be identical.

**Definition 1.** The relativistic \( u \)-geoid represents a two-dimensional surface at any point of which the rate of the proper time, \( r \), of an ideal clock carried out by a static observer with the fixed coordinates \( r, \theta, \phi \), is constant. The \( u \)-geoid is defined by the function

\[
\tilde{W}(r, \theta) = -\left( \frac{d\tau}{dt} \right)_{r,\theta,\phi \text{ fixed}} = \text{const. ,}
\]

where we have put a minus sign in front of the derivative in order to match the signs in the relativistic and Newtonian definitions of the geoid. This condition, applied to the metric (14) yields equation of the surface of the \( u \)-geoid in the form \( \tilde{W} = -1/\tilde{u}^0 = \text{const.} \) which is formally similar to (1) up to a constant. Picking up the value of \( \tilde{u}^0 \) from (15), the equation of the \( u \)-geoid reads

\[
\tilde{W}(r, \theta) \equiv -\sqrt{N^2 - (\Omega - \Phi)^2 B^2 r^2 \sin^2 \theta} = \text{const.}
\]

In the Newtonian approximation \( N(r, \theta) = 1 - 2V(r, \theta), B(r, \theta) = 1 \) and \( \Phi(r, \theta) = 0 \). Expanding the root square in (30) into the post-Newtonian series yields \( \tilde{W}(r, \theta) = \tilde{W}_N(r, \theta) + \text{(relativistic terms)} + \text{const}, \) that matches with the Newtonian definition of the geoid (1) and the post-Newtonian definitions of the geoid given in [22, 23].

Differential equation for \( \tilde{W} \), defining the \( u \)-geoid, is derived from the Landau-Raychaudhuri equation [24, p. 84] which, in case of the rigidly rotating axisymmetric configuration, takes on the following form [25, Problem 14.10]

\[
\tilde{h}_{\alpha\beta} \tilde{a}_{\alpha\beta} = \tilde{h}_{0\beta} \tilde{u}^0 \tilde{u}^\beta - \tilde{a}_\alpha \tilde{a}^\alpha - 2 \tilde{\omega}^2 ,
\]

where \( \tilde{\omega}^2 \equiv (1/2)\tilde{\omega}_{0\alpha} \tilde{\omega}^{0\beta} \), and we notice that in the Newtonian approximation \( \tilde{\omega}^2 = \Omega^2 \). Stationary axisymmetric spacetime admits two Killing vectors, \( \xi^\alpha = \partial_t \) and \( \chi^\alpha = \partial_\phi \), associated with the translations along \( t \) and \( \phi \) coordinates respectively [26]. Existence of the Killing vectors allows us to represent the four-acceleration of the fluid congruence in the form of a gradient taken from the time component of the four-velocity, \( \tilde{a}_\alpha = -\partial_\alpha (\ln \tilde{u}^0) \), where \( \tilde{u}^0 = (-\tilde{g}_{00})^{-1/2} = (-\tilde{g}_{0\alpha})^{-1/2} \) is interpreted as a scalar [25, Problem 10.14]. Accounting for the above-given identity, \( \tilde{W} = -1/\tilde{u}^0 \), it yields

\[
\tilde{a}_\alpha = \partial_\alpha (\ln \tilde{W}) .
\]

After replacing (32) in (31) one gets a non-linear equation for the potential \( \tilde{W} \),

\[
\Delta \tilde{W} + 2 \left( \tilde{\omega}^2 + \tilde{a}_\alpha \tilde{a}^\alpha \right) \tilde{W} = \frac{8\pi G}{W} \left( \tilde{F}_{00} + \frac{1}{2} \tilde{F} \right) ,
\]

where both \( \tilde{a}_\alpha \) and \( \tilde{\omega}^2 \) are functions of \( \tilde{W} \), and

\[
\Delta f \equiv \tilde{h}^{\alpha\beta} (\partial_\alpha f) (\partial_\beta f) ,
\]

is the covariant form of the Laplace operator of the spatial metric \( \tilde{h}_{\alpha\beta} = \tilde{g}_{\alpha\beta} + \tilde{a}_\alpha \tilde{a}_\beta \) for an arbitrary scalar function \( f \equiv f(x^\alpha) \).

In derivation of (33) we have used the fact that all time derivatives vanishes due to the stationary character of the problem, and the shear \( \tilde{\sigma}_{0\alpha} = 0 \). In particular, it means that the covariant wave operator, \( \Box f \equiv f^{\alpha\beta}_{;\alpha\beta} = \tilde{g}^{\alpha\beta} f_{;\alpha\beta} \), from any scalar function \( f \) is reduced to the following form,

\[
\Box f = \Delta f + \tilde{u}^0 \tilde{a}_\alpha f ,
\]

because \( \tilde{h}^{\alpha\beta} \tilde{a}_{\alpha\beta} = 0, \tilde{h}^{\alpha\beta} \tilde{a}_\beta = 0 \) and we have used the stationary character of the metric to transform \( \tilde{u}^0 \tilde{u}^{0\beta} f_{;\alpha\beta} \).
**Definition 2.** The relativistic $a$-geoid represents a two-dimensional surface at any point of which the direction of a plumb line measured by a static observer is orthogonal to the tangent plane of geoid’s surface (30).

In order to derive equation of $a$-geoid, we notice that the direction of the plumb line is given by the four-vector of gravity, $\bar{g}^\alpha = -\bar{a}^\alpha$. We consider an arbitrary displacement, $dx^\alpha_\perp \equiv \bar{h}^\alpha_{\beta\gamma} dx^\beta$, on the spatial hypersurface orthogonal to $\bar{u}^\alpha$, and make a scalar product of it with the direction of a plumb line. It gives,

$$dx^\alpha_\perp \bar{g}_\alpha = dx^\alpha \bar{g}_\alpha = -dx^\alpha \bar{a}_\alpha = -d \ln \bar{W}.$$  \hspace{1cm} (36)

From the definition of the $a$-geoid the left side of (36) must vanish due to the condition of the 3-dimensional orthogonality of the two vectors, $dx^\alpha_\perp$ and $\bar{g}_\alpha$. Therefore, it makes $d \ln \bar{W} = 0$ which means the constancy of $\bar{W}$ on the 3-dimensional surface of the $a$-geoid. Thus, equation of the $a$-geoid is exactly the same as that for the $u$-geoid.

**Definition 3.** The relativistic $p$-geoid represents a two-dimensional level surface of a constant pressure of the rigidly rotating perfect fluid.

Relativistic Euler’s equation for the perfect fluid is [25, Problem 14.3]

$$(\bar{\varepsilon} + \bar{p}) \bar{u}_\alpha = -\partial_\alpha \bar{p} - \bar{u}_\alpha \bar{u}_\beta \partial_\beta \bar{p}.$$  \hspace{1cm} (37)

A second term in the right side of this equation vanishes because pressure, $\bar{p}$, does not depend on time $u^\alpha \partial_\alpha \bar{p} = u^0 \partial_0 \bar{p} = 0$. Contracting (37) with an infinitesimal vector of displacement, $dx^\alpha$, yields

$$d \bar{p} = - (\bar{\varepsilon} + \bar{p}) d \ln \bar{W}.$$  \hspace{1cm} (38)

But the right side of (38) vanishes on the geoid which means that pressure $\bar{p} = \text{const}$. on the geoid surface. It can be shown [25, Problem 16.18] that the density, $\bar{\rho}$ and the specific internal energy, $\bar{\Pi}$, are also constant on the level surfaces.

4. **The gravity anomaly potential**

In real physical situation the surface of the geoid, modelled by a rigidly rotating perfect fluid, is disturbed by the presence of a thin, irregular surface layer – Earth’s crust. The crust can be thought as floating on the mantle and rotating on average with the same angular velocity $\Omega$ as the Earth’s mantle. The presence of the crust violates the axial symmetry of the problem and causes perturbations both in the background matter of Earth’s interior and in the spacetime metric. Depending on physical situation, the perturbation can bring about a time variability of the undergoing geophysical processes but in this paper we shall not dwell upon this problem and consider merely the time-independent perturbations. The perturbed physical metric, $g_{\alpha\beta} = g_{\alpha\beta}(r, \theta, \phi)$, now depends on all three spatial coordinates, and can be formally split into an algebraic sum of the background metric and a perturbation,

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \kappa_{\alpha\beta},$$  \hspace{1cm} (39)

where the metric perturbation $\kappa_{\alpha\beta} \equiv \kappa_{\alpha\beta}(r, \theta, \phi)$ is induced by the presence of the crust, and we assume that it does not depend on time.

We define the gravity anomaly potential $T \equiv T(\nabla, \theta, \phi)$ as the difference between the perturbed gravity potential, $W \equiv W(r, \theta, \phi)$, and its unperturbed counterpart,

$$T(r, \theta, \phi) = W(r, \theta, \phi) - \bar{W}(r, \theta),$$  \hspace{1cm} (40)

where the gravity potential $W$ defines the equation of the perturbed relativistic $u$-geoid

$$W(r, \theta, \phi) = - \left( \frac{dr}{dt} \right)_{r, \theta, \phi \text{ fixed}} = \text{const}.$$  \hspace{1cm} (41)
and $\tau$ is the proper time measured by static observers in spacetime with the metric (39),

$$d\tau^2 = -(g_{00} + \xi_{00}) \, dt^2.$$  \hfill (42)

Equation (40) represents a relativistic generalization of the Newtonian anomalous potential $T_N$ defined above in (8). We notice that it would be more consistent with the current geodetic practice to define the anomalous potential as a perturbation of the relativistic reference-ellipsoid, like in equation (11). Unfortunately, there exists a serious conceptual problem in relativity with such a definition because it is not clear at the time being if the relativistic gravitational field of such an ellipsoid is compatible with the Einstein field equations at all. The answer to this question seems to be not trivial and will be published somewhere else. The present paper operates with equation (40) which refers to the relativistic unperturbed geoid $\bar{W}$ that is well-defined in the sense of the existence as a solution of Einstein’s equations.

Equation (42) can be recast to the following form

$$\left( \frac{d\tau}{dt} \right)^2 = -\bar{g}_{00} \left( 1 + \frac{\xi_{00}}{\bar{g}_{00}} \right) = \left( \frac{1}{\bar{u}^0} \right)^2 \left( 1 - (\bar{u}^0)^2 \xi_{00} \right) = \left( \frac{1}{\bar{u}^0} \right)^2 \left( 1 - \bar{u}^\alpha \bar{u}^\beta \xi_{0\beta} \right),$$  \hfill (43)

because the unperturbed four-velocity, $\bar{u}^\alpha$ has only a time component, $\bar{u}^0 \neq 0$. Accounting for the definition (29) of the unperturbed $u$-geoid and equations (40), (41), we get the anomalous gravity potential in the form,

$$T = \bar{W} \left( 1 - \sqrt{1 - \bar{u}^\alpha \bar{u}^\beta \xi_{0\beta}} \right).$$  \hfill (44)

Equation (44) is exact. For practical applications we linearise (44) by expanding its right side in the Taylor series and discarding the non-linear terms. It yields

$$T \approx \frac{1}{2} \bar{W} \bar{u}^\alpha \bar{u}^\beta \xi_{0\beta}. \hfill (45)$$

Our next task is to derive the differential equation for the anomalous gravity potential $T$.

5. The master equation for the anomalous gravity potential

To this end let us assume that the matter of the crust is described by the energy-momentum tensor

$$\Sigma^{\alpha\beta} = \epsilon \, u^\alpha u^\beta + p^{\alpha\beta},$$  \hfill (46)

where $u^\alpha$ is a four-velocity, $\epsilon$ – the energy density, and $p^{\alpha\beta}$ – the symmetric stress tensor of crust’s matter. The stress tensor is orthogonal to $u^\alpha$, that is $\xi_{0\beta} u^\beta = 0$. The energy density

$$\epsilon = \mu \left( 1 + \mathcal{P} \right),$$  \hfill (47)

where $\mu$ is the mass density of the crust - the same as in (7), and $\mathcal{P}$ is the internal (compression) energy.

A more convenient metric variable is

$$l_{\alpha\beta} \equiv -\xi_{\alpha\beta} + \frac{1}{2} \bar{g}_{\alpha\beta} \kappa,$$  \hfill (48)

where $\kappa \equiv \bar{g}^{\alpha\beta} \xi_{0\beta}$. The dynamic field theory of manifold perturbations leads to the following equation for $l_{\alpha\beta}$ [27, 28],

$$l_{\alpha\beta} l^\mu_{\beta} + \bar{g}_{\alpha\beta} A^\mu_{\beta} - 2 A^\mu_{\beta} - \bar{R}^\mu_{\beta} l_{\alpha\beta} = -2 \bar{R}_{\alpha\beta\gamma} b^\gamma + 2 F^m_{\alpha\beta} = 16\pi \Sigma_{\alpha\beta},$$  \hfill (49)

where $A^\mu \equiv p^{0\beta}_{\beta}$ is the gauge function, depending on the choice of the coordinates, $\bar{R}_{\alpha\beta\gamma} b^\gamma$ is the Riemann (curvature) tensor of the background manifold depending on the metric tensor $\bar{g}_{\alpha\beta}$, its first and second derivatives, $\bar{R}_{\alpha\beta\gamma} = \bar{g}^{\mu\nu} \bar{R}_{\alpha\beta\gamma\nu}$ - the Ricci tensor, and $F^m_{\alpha\beta}$ is the perturbation of the background matter induced by the presence of the crust [27, Eqs. 148-150].

In what follows, we focus on derivation of the master equation for the anomalous gravity potential $T$ for the case of empty space that is outside of the background mass distribution that is the most important in geodetic applications.
Derivation of the master equation for $\mathcal{T}$ inside the background mass distribution will be given somewhere else. To achieve our goal, let us introduce two scalars,

$$q = \bar{u}^\alpha \bar{u}_\alpha l_{ij} + \frac{1}{2}l,$$

$$p = \bar{R}^\alpha l_{ij},$$

where the scalar

$$l \equiv \bar{g}^{\alpha \beta} l_{\alpha \beta} = 2(p - q).$$

In terms of the scalar $q$ the anomalous gravity potential reads

$$\mathcal{T} = -\frac{1}{2} W q .$$

Taking the Laplace operator (34) from both sides of (53) yields

$$\Delta \mathcal{T} = \left( \frac{\Delta \bar{W}}{W} - 2\partial_{\alpha} \bar{a}^\alpha \right) \mathcal{T} + 2\bar{a}^\alpha \partial_{\alpha} \mathcal{T} - \frac{1}{2} W \Delta q ,$$

where $\Delta \bar{W}$ is given in (33), while $\Delta q$ is to be calculated from (49).

To achieve this goal, we contract (49) with $\bar{R}_{\alpha \beta}$ cancel out, $q_\alpha \equiv \partial_\alpha q$, $\bar{\Sigma} \equiv \bar{g}^{\alpha \beta} \bar{\Sigma}_{\alpha \beta}$, and we still have the gauge field $A^\alpha$ being unrestricted. Now, we make use of (55) for replacing $\Delta q$ in (54) along with the gauge field $A^\alpha$ defined by the following condition,

$$A_\alpha = -\bar{a}_\alpha q - p_\alpha,$$

where $p_\alpha \equiv \partial_\alpha p$. This gauge allows us to eliminate function $p$ completely, and to reduce equation (54) to a simple form of the Poisson equation

$$\Delta \mathcal{T} = 4\pi W \bar{\Sigma} ,$$

where the source in the right side is the trace, $\bar{\Sigma} \equiv \bar{g}^{\alpha \beta} \bar{\Sigma}_{\alpha \beta}$, of the energy-momentum tensor $\bar{\Sigma}_{\alpha \beta}$ of the surface layer (crust). In the Newtonian approximation the trace of the energy-momentum tensor is reduced to the negative value of the density of the Earth crust, $\bar{\Sigma} \approx -\mu$, and $W \approx 1$. Hence, equation (57) matches with its Newtonian counterpart (9). Outside the mass distribution the master equation for the anomalous gravity potential is reduced to the Laplace equation

$$\Delta \mathcal{T} = 0 .$$

Equations (57), (58) extend similar equations (9), (10) of the Newtonian gravity theory to the realm of general relativity. We emphasize that the Laplace operator in (57), (58) is taken in curved space with the spatial metric $\bar{h}_{ij}$. The explicit form of the Laplace operator in the spherical coordinates, $(r, \theta, \phi)$, is obtained from the general expression (34) which reads in this case

$$\Delta \equiv \frac{1}{\sqrt{h}} \partial_i \left( \sqrt{h} h^{ij} \partial_j \right),$$

where $\bar{h}^{ij} = \bar{g}^{ij}$, and $\bar{h} = \det[\bar{h}_{ij}] = r^4 A^4 B^2 N^2 \sin^2 \theta / W^4$. It brings (58) to the following form

$$\frac{\partial}{\partial r} \left( r^2 \frac{BN}{W^2} \frac{\partial \mathcal{T}}{\partial r} \right) + \frac{1}{\sin \theta \partial \theta} \left( \frac{BN}{W^2} \frac{\partial \mathcal{T}}{\partial \theta} \right) + \frac{A^2}{BN \sin^2 \theta \partial \phi^2} \frac{\partial^2 \mathcal{T}}{\partial \phi^2} = 0 ,$$

where $W$ is given in (30), and functions $A, B, N, \Theta$ are solutions of Einstein’s equations (23)-(26).
6. Geoid’s height

We introduce the relativistic geoid height, $H$, by making use of a relativistic generalization of Bruns’ formula (13). Let a point $Q$ on the level surface of the unperturbed geoid $W_0$ has coordinates $x_{0}^\alpha$, and the point $P$ on the surface of the perturbed geoid $W$ has coordinates $x_{P}^\alpha$. The height difference, $H$, between the two surfaces is defined as the absolute value of the integral taken along the direction $n_\alpha$ of the plumb line started at the point $Q$ and ended up at the point $P$,

$$H = \left| \int_{Q}^{P} n_\alpha \frac{dx^\alpha}{d\ell} \right|,$$  \hspace{1cm} (61)

where $n_\alpha = g_\alpha/|g_\alpha|$ is the unit (co)vector along the plumb line, $g_\alpha = \partial_\alpha W$ is the acceleration of gravity taken at the point of integration of the line integral (61), $|g_\alpha| = \sqrt{g_\alpha g_\alpha}$, and $\ell$ is the proper length defined by [18, 20]

$$d\ell^2 = \delta_{\alpha\beta}dx^\alpha dx^\beta.$$  \hspace{1cm} (62)

In case, when the height difference is small enough we can approximate the integral (61) as follows

$$H = \frac{1}{|g_\alpha|} \int_{Q}^{P} \partial_\alpha W dx^\alpha = \frac{W(P) - W(Q)}{|g_\alpha|},$$  \hspace{1cm} (63)

where $g_\alpha = g_\alpha(Q)$. Taking into account $W(Q) = \bar{W}$ and definition (40) of the anomalous gravity potential $\mathcal{T}$, we obtain from (63)

$$H = \frac{\mathcal{T}(P)}{|g_\alpha(Q)|},$$  \hspace{1cm} (64)

where the anomalous potential is measured at the point $P$ on the surface of $W$, and the acceleration of gravity is measured at point $Q$ on the surface of $\bar{W}$. Formula (64) yields the geoid undulation with respect to the unperturbed geoid that coincides with the surface of the mean sea level in accordance with the definition of $p$-geoid given above in section 3.

Acknowledgement

The present work has been supported by the Faculty Fellowship 2014 in the College of Arts and Science of the University of Missouri and the grant 14-27-00068 of the Russian Scientific Foundation.

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