Reducibility of Euler integrals and multiintegrals

E.I. Ganzha∗
Department of Mathematics,
Krasnoyarsk State Pedagogical University,
ul. Lebedevoi, 89,
660060 Krasnoyarsk
e-mail: eiganzha@mail.ru

Abstract

We discuss the notion of reduction of a special type of explicit solutions which generalize the solutions appearing in the classical Laplace cascade method of integration of hyperbolic equations of the second order in the plane. We give algorithms of reduction and prove that different natural precise definitions of reduction are equivalent.

Keywords: cascade integration method, integrable systems, Euler integrals.

1 Introduction

Classical methods of integration of linear partial differential equations and systems of such equations (cf. [12, 13, 15]) were considerably generalized in the last decades in [3, 5, 11, 19]. New applications of these methods were found in two-dimensional spectral theory [6] and the theory of stochastic systems [14]. The following expressions appear as examples of exact solutions in the methods developed in the aforementioned publications:

\[ f(x, y) = a_0(x, y) \varphi(x) + a_1(x, y) \varphi'(x) + \ldots + a_n(x, y) \varphi^{(n)}(x) + b_0(x, y) \psi(y) + \ldots + b_m(x, y) \psi^{(m)}(y) \]  

(1.1)

with arbitrary functions \( \varphi(x) \), \( \psi(y) \) and fixed coefficients \( a_i(x, y), b_j(x, y) \). First examples of such solutions were found already very early by Euler [8]. For systems [3, 5, 11, 19] and higher-order equations [16, 17, 18] solutions may appear as sums of several expression of the form (1.1). Already Darboux and other authors of the classical period had noted that the expressions of the form (1.1) sometimes may be simplified (i.e. the orders \( n \) and \( m \) can be made lower) if one introduces new arbitrary

∗The author was supported by the Russian Foundation for Basic Research under the grant No 06-01-00814 and a grant 09-09-1/NSh from Krasnoyarsk State Pedagogical University.
functions, for example as $\vartheta(x) = r_0(x)\varphi(x) + \ldots + r_k(x)\varphi^{(k)}(x)$. No general theory of simplification (reduction) of the expression of type (1.1) is currently available. In this paper we give algorithms of reduction and prove that different natural precise definitions of reduction are equivalent. For simplicity we separate the parts of the expression (1.1) containing $\varphi(x)$ and $\psi(y)$, and give the following definition:

**Definition 1.1.** Euler integral in the plane is an expression of the form

$$I = a_0(x, y)\varphi(x) + a_1(x, y)\varphi'(x) + \ldots + a_n(x, y)\varphi^{(n)}(x), \quad (1.2)$$

where $\varphi(x)$ is an arbitrary function of the variable $x$ and $a_i(x, y)$ are given functions of the two variables $(x, y)$.

Below we assume that $a_i(x, y)$ belong to some constructive differential field of functions in the plane. In order to guarantee correctness of our algorithms and results we need to require that we can constructively decide if a given element of this field is zero; we also obviously need at least existence of derivatives of the coefficients $a_i(x, y)$ up to the order needed in the computations. All functions should have a common domain of definition—some open subset of $\mathbb{R}^2$. The simplest practically important example of such a field is the field of rational functions $\mathbb{Q}(x, y)$.

Hereafter we will often call Euler integrals (1.2) simply “integrals” and write them as $I = L(x, y)\varphi(x)$, where

$$L(x, y) = a_0(x, y) + a_1(x, y)D + \ldots + a_n(x, y)D^n, \quad D = \frac{d}{dx},$$

is a linear ordinary differential operator (LODO). All differential operators in this paper will be linear ordinary differential operators in $x$, i.e. they will include the derivatives $D^s = d^s/dx^s$ only. Dependence of the coefficients of an operator on the both variables $x$ and $y$ or on $x$ alone will be explicitly shown after the sign of an operator. Everywhere below we will denote linear ordinary differential operators in $x$ with upper-case Latin letters and all given functions with lower-case Latin letters; lower-case Greek letters will denote arbitrary functions on $x$.

**Definition 1.2.** Euler multiintegral in the plane is an expression of the form

$$J = a_{10}(x, y)\varphi_1(x) + a_{11}(x, y)\varphi_1'(x) + \ldots + a_{1n_1}(x, y)\varphi_1^{(n_1)}(x) +$$

$$+ a_{20}(x, y)\varphi_2(x) + a_{21}(x, y)\varphi_2'(x) + \ldots + a_{2n_2}(x, y)\varphi_2^{(n_2)}(x) +$$

$$\ldots$$

$$+ a_{k0}(x, y)\varphi_k(x) + a_{k1}(x, y)\varphi_k'(x) + \ldots + a_{kn_k}(x, y)\varphi_k^{(n_k)}(x) =$$

$$= L_1(x, y)\varphi_1(x) + \ldots + L_k(x, y)\varphi_k(x), \quad (1.3)$$

where $\varphi_1(x), \ldots, \varphi_k(x)$ are arbitrary functions of the variable $x$ alone and $a_{ij}(x, y)$ are given functions of the two variables $(x, y)$, $1 \leq i \leq k$, $1 \leq j \leq n_i$.

For the sake of brevity we will denote the operator row $(L_1(x, y), \ldots, L_k(x, y))$ as $\bar{L}(x, y)$, and the column of functions $(\varphi_1(x), \ldots, \varphi_k(x))^t$ as $\hat{\varphi}(x)$:

$$J = \bar{L}(x, y)\hat{\varphi}(x). \quad (1.4)$$
In Petrén’s thesis \cite{17} the following linear partial differential operators in the
plain were considered:

\[
P(x, y) = \sum_{i=0}^{p-1} a_i(x, y)D_y^iD_x + \sum_{i=0}^{p-1} b_i(x, y)D_y^i = A(x, y)D_x + B(x, y); \quad a_{p-1} \neq 0.
\]  

(1.5)

Now we will show that any Euler integral

\[
I = L(x, y)\varphi(x)
\]

(1.6)
can be an (incomplete) solution of some equation of Petrén type \(P(x, y)I = 0\).

Suppose we have an integral \(I\) generated by an operator \(L(x, y) = l_n(x, y)D_x^n + l_{n-1}(x, y)D_x^{n-1} + \ldots + l_0(x, y)\). We have to find an operator \(P(x, y)\) of the form (1.5), such that

\[
(P(x, y)L(x, y))\varphi(x) \equiv 0
\]

(1.7)

for any \(\varphi(x)\). Since

\[
P(x, y)L(x, y) = A(x, y)D_xL(x, y) + B(x, y)L(x, y) =
\]

\[
= A(x, y)L(x, y)D_x + A(x, y)[D_x, L(x, y)] + B(x, y)L(x, y) =
\]

\[
= A(x, y)\left[l_n(x, y)D_x^{n+1} + l_{n-1}(x, y)D_x^n + \ldots + l_0(x, y)D_x\right] +
\]

\[
+ A(x, y)\left[(l_n)'_x(x, y)D_x^n + (l_{n-1})'_x(x, y)D_x^{n-1} + \ldots + (l_0)'_x(x, y)\right] +
\]

\[
+ B(x, y)\left[l_n(x, y)D_x^n + l_{n-1}(x, y)D_x^{n-1} + \ldots + l_0(x, y)\right]
\]

and \(\varphi(x)\) is an arbitrary function, then (1.7) is equivalent to the system

\[
\begin{align*}
A(x, y)l_n(x, y) &= 0, \\
A(x, y)\left[l_n(x, y) + (l_{n-1})'_x(x, y)\right] + B(x, y)l_n(x, y) &= 0, \\
& \quad \ldots \\
A(x, y)\left[l_0(x, y) + (l_1)'_x(x, y)\right] + B(x, y)l_1(x, y) &= 0, \\
A(x, y)(l_0)'(x, y) + B(x, y)l_0(x, y) &= 0,
\end{align*}
\]

(1.8)

where \(l_i(x, y)\) are the given coefficients of the operator \(L(x, y)\). Since (1.8) is a system with \((n+2)\) linear homogeneous algebraic equations for \(2p\) unknown coefficients \(b_0, b_1, \ldots, b_{p-1}, a_0, \ldots, a_{p-1}\), one easily concludes that (1.8) always has a nontrivial solution if \(p > \frac{n+2}{2}\) and consequently every integral of the form (1.6) is a solution (certainly an incomplete solution in general) of some equation of Petrén type (1.5) of sufficiently high degree.

Analogously for multiintegrals (1.3), it is possible to find some equation of Petrén type satisfied by a given multiintegral. In this case the coefficients of the equation (1.3) satisfy a set of systems of the form (1.8) (a system (1.8) is formed for every operator \(L_i(x, y)\)) with the total number of linear homogeneous algebraic equations equal to \(N = \sum_{i=1}^k n_i + 2k\), where \(n_i\) is the order of the operator \(L_i(x, y)\). So if \(2p > N\) there is a nontrivial solution of such a system for the coefficients \(a_i, b_i\) of an equation of Petrén type.
The linear case considered in this paper can be a basis for a further study of exact solutions of nonlinear PDEs integrable by Darboux method: as we know (see [2, 9, 15]) Darboux integrability is equivalent to integrability of the corresponding linearized equation by the Laplace cascade method.

2 Euler integrals

In order to give the most general precise definition of reducibility of a Euler integral (1.2) we introduce the following natural definition:

Definition 2.1. The set of all functions $Z(I)$ of the two variables $(x, y)$ which will be obtained after substitution of arbitrary (smooth) functions $\varphi(x)$ into (1.2) will be called the function stock generated by some Euler integral $I$ of the form (1.2).

Any such function stock is an infinite-dimensional linear space; its elements may be obviously added and multiplied with constants.

Definition 2.2. For any Euler integral (1.2) we will call the order $n$ of the leading derivative of the arbitrary function $\varphi(x)$ the order of the integral.

Definition 2.3. We will call a Euler integral (1.2) order-reducible if there exists another integral $I_1 = b_0(x,y)\psi(x) + \ldots + b_m(x,y)\psi^{(m)}(x)$ of smaller order $m$, $m < n$, generating the same function stock as the given integral $I$.

Note that we do not assume any relation between the functions $\varphi(x)$ and $\psi(x)$ in $I$ and $I_1$. As we prove below, if an integral is order-reducible then there exists a differential relation between $\varphi(x)$ and $\psi(x)$.

Definition 2.4. We call a Euler integral (1.2) operator-reducible if the corresponding operator $L(x,y)$ may be represented as a composition of operators

$$L(x,y) = M(x,y)R(x),$$

(2.1)

where $M(x,y)$ is another operator of lower order and the coefficients of the operator $R(x)$ depend (as shown) only on $x$.

In this case the function stocks generated by the integrals $I = L(x,y)\varphi(x)$ and $I_1 = M(x,y)\psi(x)$ obviously coincide and there exists the differential relation $\psi(x) = R(x)\varphi(x)$.

Definition 2.5. We will say that a Euler integral (1.2) has no kernel (is kernel-irreducible), if $I = L(x,y)\varphi(x)$ identically vanishes as a function of the two variables only for $\varphi(x) \equiv 0$. In the opposite case we will call the integral $I$ kernel-reducible.

Below we will need the following constructive way of order reduction for any Euler integral.
Algorithm RI

The input data of the algorithm is a Euler integral $I(x, y)$. The output of the algorithm is a representation of this integral in the form (2.1) with an operator $M(x, y)$ of minimal possible order or a proof that the given integral is operator-irreducible.

Let us take two new formal variables $y_1$ and $y_2$. Consider $L(x, y_1)$ and $L(x, y_2)$ as two different ordinary operators in $D_y$ with coefficients depending on the parameters $y_1$ and $y_2$ respectively. One can find the right greatest common divisor of these operators $rGCD(L(x, y_1), L(x, y_2))$ using the well-known non-commutative version of the Euclidean algorithm (cf. for example [1]), performing divisions of these linear ordinary differential operators with remainders:

$$
L(x, y_1) = A_0(x, y_1, y_2)L(x, y_2) + B_1(x, y_1, y_2),
$$
$$
L(x, y_2) = A_1(x, y_1, y_2)B_1(x, y_1, y_2) + B_2(x, y_1, y_2),
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
B_{k-1}(x, y_1, y_2) = A_k(x, y_1, y_2)B_k(x, y_1, y_2).
$$

(2.2)

So we have $rGCD(L(x, y_1), L(x, y_2)) = B_k(x, y_1, y_2)$.

Case 1: $\operatorname{ord} B_k(x, y_1, y_2) = 0$. Then the operators $L(x, y_1)$ and $L(x, y_2)$ have no nontrivial right greatest common divisor and our algorithm terminates. The integral $\mathcal{I} = L(x, y)\varphi(x)$ is operator-irreducible.

Case 2: $\operatorname{ord} B_k(x, y_1, y_2) = \operatorname{ord} L(x, y_1) = \operatorname{ord} L(x, y_2) = n$. This means that the Euclidean algorithm has terminated on the first division step and $L(x, y_1) = a_0(x, y_1, y_2)L(x, y_2)$, $\operatorname{ord} a_0 = 0$. Consequently $L(x, y) = a_0(x, y, y_0)L(x, y_0)$, where $y_0$ is some fixed generic point (such that $a_n(x, y_0) \neq 0$ and $a_0(x, y, y_0)$ is correctly defined), i.e. $L(x, y)$ is a LODO with coefficients depending on $x$ only, multiplied on the left with a function in the two variables $(x, y)$. Thus we have obtained a representation (2.1) with $\operatorname{ord} M(x, y) = 0$ and $R(x) = L(x, y_0)$. As a result we conclude that the integral $\mathcal{I}$ is operator-reducible to another integral $\mathcal{I}_1 = a_0(x, y, y_0)\psi(x)$ of order zero.

Case 3: $\operatorname{ord} B_k(x, y_1, y_2) = m > 0 < \operatorname{ord} L(x, y)$. Then, if $B_k(x, y_1, y_2) = B_k(x)$, i.e. does not depend on $y_1, y_2$, $R(x) = B_k(x)$ so we again obtain the required reduction of the integral. In the opposite case we introduce a new formal variable $y_3$ and return to the stage (2.2) of our algorithm RI applying the Euclidean algorithm to the operators $L(x, y_3)$ and $B_k(x, y_1, y_2)$. Again some operator $B_r^{(1)}(x, y_1, y_2, y_3) = rGCD(L(x, y_1), L(x, y_2), L(x, y_3))$ will be found, and in the Cases 1 and 2 (when $\operatorname{ord} B_r^{(1)} = 0$ or $\operatorname{ord} B_r^{(1)} = \operatorname{ord} B_k$) the algorithm terminates; in the Case 3 we return to the stage (2.2) adding a new formal variable $y_4$ and finding $B_r^{(2)}(x, y_1, y_2, y_3, y_4) = rGCD(L(x, y_1), \ldots, L(x, y_4))$. Since on every cycle for the Case 3 the order of $B_r^{(k)}$ decreases at least by one, after a finite number of steps the algorithm terminates. As a result the operators $M(x, y)$ and $R(x)$ appearing in the representation (2.1) will be found; obviously the order of $R(x)$ will be maximal and the integral $\mathcal{I}_1 = M(x, y)\psi(x)$ will be operator-irreducible.

Note that for an operator-irreducible integral $\mathcal{I} = L(x, y)\varphi(x)$ in the process of the work of the algorithm one can find points $y_1, y_2, \ldots, y_s$, $s \leq n+1$, and operators
$P_1(x), \ldots, P_s(x)$ such that

$$P_1(x)L(x, y_1) + \ldots + P_s(x)L(x, y_s) = 1. \tag{2.3}$$

In fact one can take generic $y_i$, i.e. some points which don’t coincide with the poles of the coefficients of all operators appearing in the Euclidean algorithm and don’t make their leading coefficients identically zero. The operators $P_i(x)$ are found using the standard reverse substitution in the equations (2.2).

One should note that in the general case the number of the points $y_i$ can not be made smaller than $n + 1$. Below we construct an example of such second-order integral $\mathcal{I} = L(x, y)\varphi(x)$ that $\forall y_1, y_2$, $rGCD(L(x, y_1), L(x, y_2)) = B_k(x, y_1, y_2)$, ord $B_k(x, y_1, y_2) = 1$, but there exist points $y_1, y_2, y_3$, such that $rGCD(L(x, y_1), L(x, y_2), L(x, y_3)) = B^{(1)}(x, y_1, y_2, y_3) = 1$.

To this end let us consider the following three-dimensional linear space of second degree polynomials:

$$V = \{p(x) = \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2 | \alpha, \beta, \gamma \in \mathbb{R}\}.$$

Introduce the following two-dimensional subspaces $W_y$ depending on a parameter $y$ which are spanned by the basis

$$z_1 = y \cdot 1 + (y + 1) \cdot x, \quad z_2 = y \cdot x + (y + 1) \cdot x^2, \quad W_y = \langle z_1, z_2 \rangle.$$

As one can readily check, for any fixed $y$ the functions $z_1, z_2$ are linearly independent and for every two distinct values of the parameter $y_1 \neq y_2$, $W_{y_1}$ and $W_{y_2}$ have a one-dimensional intersection. On the other hand $W_{y_1} \cap W_{y_2} \cap W_{y_3} = \{0\}$. This allows us to construct the following LODO

$$L(x, y)\varphi(x) = \begin{vmatrix} \varphi & \varphi' & \varphi'' \\ z_1 & z'_1 & z''_1 \\ z_2 & z'_2 & z''_2 \end{vmatrix}^{-1} z_1 z'_1 z''_1 =$$

$$= \begin{bmatrix} D^2 - \frac{2(y + 1)}{xy + x + y}D + \frac{2(y + 1)^2}{(xy + x + y)^2} \end{bmatrix} \varphi(x),$$

with $rGCD(L(x, y_1), L(x, y_2)) = B_1(x, y_1, y_2) = (y_1 - y_2)((xy_1 + x + y_1)(xy_2 + x + y_2)D - (2xy_1y_2 + 2xy_1 + 2xy_2 + x + 2y_1y_2 + y_1 + y_2)$, but $rGCD(L(x, y_1), L(x, y_2), L(x, y_3)) = 1$. So one concludes that in our algorithm RI for the constructed operator $L(x, y)$ one can not find two points $y_1, y_2$ and operators $P_1(x), P_2(x)$, such that $P_1(x)L(x, y_1) + P_2(x)L(x, y_2) = 1$ although the integral $\mathcal{I} = L(x, y)\varphi(x)$ is operator-irreducible. On the other hand for three generic points $y_1, y_2, y_3$ the equality (2.3) holds. One can easily construct in the same way an example on an integral of order $n$ for which one has to choose at least $n + 1$ points $y_i$ in order to get (2.3).

**Theorem 2.6.** The three definitions 2.4, 2.2 and 2.3 of reducibility of Euler integrals are equivalent.
Proof. If an integral $\mathcal{I}$ is operator-reducible, i.e. if $\mathcal{I} = L(x, y)\varphi(x) = M(x, y)R(x)\varphi(x)$ with some non-trivial $R(x)$ then the integral $\mathcal{I}$ has a nontrivial kernel which at least contains the space of solutions of the equation $R(x)\varphi(x) = 0$. Hence $\mathcal{I}$ is kernel-reducible.

If an integral $\mathcal{I}$ is kernel-reducible, that is there exists a non-zero function $\varphi(x)$ such that $\mathcal{I} = L(x, y)\varphi(x) \equiv 0$ then obviously $L(x, y)$ is representable in the form

$$L(x, y) = L_1(x, y) \left( \frac{d}{dx} - \frac{\varphi_1'}{\varphi} \right),$$

so the integral $\mathcal{I}$ is operator-reducible.

It is obvious that operator-reducible integrals are order-reducible. In order to prove the converse we need the following Lemma:

**Lemma 2.7.** Let two Euler integrals $\mathcal{I} = L(x, y)\varphi(x)$ and $\mathcal{I}_1 = M(x, y)\psi(x)$ of the form (1.2) are given, and $\mathcal{I}_1$ is operator-irreducible. Let the following inclusion for the function stocks generated by the integrals hold: $Z(\mathcal{I}) \subseteq Z(\mathcal{I}_1)$. Then the order of $\mathcal{I}$ is greater or equal to the order of $\mathcal{I}_1$, the integral $\mathcal{I}$ can be operator-reduced to the integral $\mathcal{I}_1$ (i.e. $L(x, y) = M(x, y)R(x)$ for some operator $R(x)$), and the function stocks generated by the integrals $\mathcal{I}$ and $\mathcal{I}_1$ coincide.

Proof. Applying the algorithm RI to the integral $\mathcal{I}_1 = M(x, y)\psi(x)$ we find points $y_1, y_2, \ldots, y_s$ and operators $P_1(x), \ldots, P_s(x)$, such that (2.3) holds for $M(x, y)$, since $\mathcal{I}_1$ is already operator-irreducible. Substituting the points $y_i, i = 1, \ldots, s$, into the operator $L(x, y)$ which generates the integral $\mathcal{I}$, we obtain $s$ LODO $L(x, y_i)$ with coefficients depending on $x$.

For any function $\varphi(x)$ there exists a function $\psi(x)$, such that

$$L(x, y)\varphi(x) = M(x, y)\psi(x),$$

since $Z(\mathcal{I}) \subseteq Z(\mathcal{I}_1)$. Hence

$$L(x, y_i)\varphi(x) = M(x, y_i)\psi(x) \quad (2.4)$$

for $i = 1, \ldots, s$. Multiplying the equalities (2.4) with the corresponding $P_i(x)$ and adding we obtain

$$\sum_{i=1}^{s} P_i(x)L(x, y_i)\varphi(x) = \sum_{i=1}^{s} P_i(x)M(x, y_i)\psi(x),$$

or in view of (2.3) for the operator $M(x, y)$, we obtain

$$\sum_{i=1}^{s} P_i(x)L(x, y_i)\varphi(x) = \psi(x).$$

This means that $\psi(x)$ is differentially expressible in terms of $\varphi(x)$: $\psi(x) = R(x)\varphi(x)$, $R(x) = \sum_i P_i(x)L(x, y_i)$, so $L(x, y)\varphi(x) = M(x, y)R(x)\varphi(x)$, i.e. $\mathcal{I}$ is operator-reducible to $\mathcal{I}_1$. Since for every function $\psi(x)$ there exists a solution $\varphi(x)$ of
an ordinary differential equation $\psi(x) = R(x)\varphi(x)$, we have $M(x,y)\psi(x) = M(x,y)R(x)\varphi(x) = L(x,y)\varphi(x)$ for any $\psi(x)$ and the corresponding $\varphi(x)$. Consequently $Z(I) = Z(I_1)$. □

The end of the proof of Theorem 2.6

Let an integral $I = L(x,y)\varphi(x)$ be order-reducible to some $I_1 = L_1(x,y)\psi(x)$. We must show that $I$ is order-reducible. One can suppose $I_1$ to be already order-irreducible since in the opposite case one can use the algorithm $RI$ to find an operator-irreducible $I_2 = M(x,y)\xi(x)$ with the same function stock. Since the function stocks generated by $I_1$ and $I_2$ coincide, we can apply Lemma 2.7 to the integrals $I$ and $I_2$ and obtain that $L(x,y) = M(x,y)R(x)$, i.e. the integral $I$ is order-reducible.

In view of the obtained result we will call integrals simply “reducible” or “irreducible” without specifying the precise meanings given in the definitions 2.3, 2.4 and 2.5.

**Corollary 2.8.** The kernel of a Euler integral $I = L(x,y)\varphi(x)$ coincides with the kernel of the operator $R(x)$, obtained in the algorithm $RI$.

*Proof.* From (2.1) one can easily see that $\text{Ker } R(x) \subseteq \text{Ker } I$. Let $\varphi(x) \in \text{Ker } I$. Then $0 = L(x,y)\varphi(x) = M(x,y)R(x)\varphi(x) = M(x,y)\psi(x)$, $\psi = R\varphi$. From this we conclude that $\psi(x) = 0$, since $M(x,y)$ is irreducible. Consequently $\varphi(x) \in \text{Ker } R(x)$. □

We are summing up the results about the structure of reducible integrals obtained so far in the following Proposition:

**Proposition 2.9.** Among Euler integrals generating the same function stocks there is a unique (up to gauge transformations $L_0 \mapsto L_0(x,y)g(x)$) irreducible integral $I_0 = L_0(x,y)\psi(x)$. All other integrals with the same function stock have the following form: $I = L_0(x,y)R(x)\varphi(x)$, where $R(x)$ is an arbitrary LODO with coefficients depending on $x$ only.

*Proof.* Let $Z(I_1) = Z(I_2)$. Using the algorithm $RI$ if necessary we may assume that $I_1$ and $I_2$ are irreducible. Applying Lemma 2.7 we see that $L_1(x,y) = L_2(x,y)R_1(x)$, $L_2(x,y) = L_1(x,y)R_2(x)$, so $R_1(x) = g(x)$ must be invertible in the ring of LODO so it must be a zero-order operator. If $I$ is an arbitrary (reducible) integral and $I_0$ is irreducible, we can use Lemma 2.7 again obtaining the required identity $L(x,y) = L_0(x,y)R(x)$. □

**Theorem 2.10.** Let $I$ and $I_1$ be Euler integrals of the form (1.2). Then the function stocks generated by these integrals either coincide or have a finite-dimensional intersection.

*Proof.* We again can assume that the given integrals $I$ and $I_1$ are irreducible. Let $I = L(x,y)\varphi(x)$, $I_1 = L_1(x,y)\psi(x)$ and some function $f(x,y)$ is contained in the intersection of the corresponding function stocks. Then the following equality holds:

$$f(x,y) = L(x,y)\varphi(x) = L_1(x,y)\psi(x) \tag{2.5}$$
for some $\varphi(x)$, $\psi(x)$. Apply the algorithm $RI$ to the integral $\mathcal{I}$. In the process we obtain a finite number of points $y_1, y_2, \ldots, y_s$ and operators $P_i(x), \ldots, P_s(x)$, such that (2.3) holds. Substituting the points $y_i$ into (2.5), making the appropriate linear combination and using (2.3) we obtain

$$\varphi(x) = N(x)\psi(x), \tag{2.6}$$

where $N(x) = \sum_{i=1}^s P_i(x)L_1(x,y_i)$. Substituting (2.6) into (2.5) we get

$$L(x,y)N(x)\psi(x) = L_1(x,y)\psi(x) \tag{2.7}$$
or

$$\left(L(x,y)N(x) - L_1(x,y)\right)\psi(x) = 0. \tag{2.8}$$

If the operator $L(x,y)N(x) - L_1(x,y)$ vanishes identically then any function $\psi(x)$ satisfies (2.8), and, consequently, also (2.7). From this we conclude that the function stocks generated by $\mathcal{I}$ and $\mathcal{I}_1$, coincide (Lemma 2.7). If the operator $L(x,y)N(x) - L_1(x,y)$ does not vanish, we deduce from (2.8) that $\psi(x)$ belongs to the kernel of the integral, generated by the operator $F(x,y) = L(x,y)N(x) - L_1(x,y)$. Applying the algorithm $RI$ to $F(x,y)$ one obtains the decomposition $F(x,y) = F_1(x,y)S(x)$, where $F_1(x,y)$ is irreducible and $\psi(x)$ belongs to the finite-dimensional space of solutions of the ordinary differential equation $S(x)\psi(x) = 0$. From (2.5) we conclude that the intersection of the function stocks, generated by the integrals $\mathcal{I}$ and $\mathcal{I}_1$ is the image of this finite-dimensional space under the action of the operator $L_1(x,y)$. This concludes the proof. \hfill \Box

**Remark.** Obviously the proofs of the Proposition 2.9 and Theorem 2.10 imply that we have a simple algorithmic way to check if the function stocks generated by two given integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ coincide. For this we find the corresponding irreducible operators $L_1(x,y)$ and $L_2(x,y)$ with the same function stocks using the algorithm $RI$. If the orders of $L_1(x,y)$, $L_2(x,y)$ do not coincide then the function stocks $\mathcal{I}_1$ and $\mathcal{I}_2$ are different. If the orders of $L_1(x,y)$ and $L_2(x,y)$ coincide, we divide $L_1(x,y)$ by $L_2(x,y)$ on the left:

$$L_1(x,y) = L_2(x,y)g(x,y) + B(x,y), \quad \text{ord } B < \text{ord } L_1 = \text{ord } L_2.$$ 

The function stocks of $\mathcal{I}_1$ and $\mathcal{I}_2$ coincide if and only if $B(x,y) \equiv 0$, $g_y \equiv 0$. In order to find the dimension of the intersection $Z(\mathcal{I}_1) \cap Z(\mathcal{I}_2)$ we repeat the procedure described in the proof of Theorem 2.10 i.e. we form the operators in the left hand side of (2.8) and apply the algorithm $RI$ to split off the maximal right divisor $S(x)$ of the operator $F(x,y)$. The dimension of the intersection $Z(\mathcal{I}_1) \cap Z(\mathcal{I}_2)$ will be equal to the order of the operator $S(x)$.

### 3 Multiintegrals

In this Section we prove the equivalence of two possible definitions of reducibility of Euler multiintegrals (1.3) and give the algorithms of reduction and other necessary technical stuff.
Definition 3.1. The function stock $Z(\mathcal{J})$, generated by a multiintegral (1.3) is the set of all functions of two variables $(x, y)$ which is obtained after the substitution of arbitrary (smooth) functions $\varphi_1(x), \ldots, \varphi_k(x)$ into (1.3).

Definition 3.2. We will call a multiintegral (1.3) operator-reducible, if one can represent its operator row $(L_1(x, y), \ldots, L_k(x, y))$ as a composition of operator matrices

$$(L_1(x, y), \ldots, L_k(x, y)) = (M_1(x, y), \ldots, M_p(x, y)) \left( R_{11}(x) \ldots R_{1k}(x) \right) \cdots \left( \begin{array}{cc} \vdots & \vdots \\ R_{p1}(x) & \cdots & R_{pk}(x) \end{array} \right),$$

(3.1)

where $R_{ij}(x)$ are differential operators with coefficients depending on $x$ only, $M_i(x, y)$ are differential operators with coefficients depending on $(x, y)$; the matrix $(R_{ij}(x))$ is required to be non-invertible and $p \leq k$.

Below we will briefly represent the formula (3.1) as

$$\tilde{L}(x, y) = \tilde{M}(x, y)\tilde{R}(x).$$

(3.2)

Here and below $\tilde{R}(x)$ stands for a matrix of differential operators (not necessarily a square matrix). Invertibility of such a matrix is understood in the algebraic sense, i.e. as existence of another operator matrix $\tilde{S}(x)$, such that $\tilde{R}\tilde{S} = \tilde{S}\tilde{R} = (\delta_{ij})$.

Definition 3.3. We will call a multiintegral (1.3) kernel-irreducible (or we say that the multiintegral has no kernel) if $\mathcal{J} = L_1(x, y)\varphi_1(x) + \ldots + L_k(x, y)\varphi_k(x)$ vanishes as a function of two variables only for $\varphi_i(x) \equiv 0$, $i = 1, \ldots, k$. In the contrary case the multiintegral $\mathcal{J}$ will be called kernel-reducible.

Note that we do not find appropriate to introduce here an analogue of order-reducibility of multiintegrals. As we will see below, even irreducible multiintegrals with the same function stocks admit a representation (1.3) with operators $L_i(x, y)$ of arbitrary high order. This follows from the fact that for arbitrary multiintegral one can find representations (3.2) with invertible matrices $\hat{R}(x)$ of operators $R_{ij}(x)$ of arbitrary high order.

One can perform the following operations on the matrix $\hat{R}(x)$: transposition of two rows and addition to one of the row of another row multiplied on the left with an ordinary differential operator with coefficients depending only on $x$.

The following algorithm uses these operations for reduction of the matrix $\hat{R}(x)$ to a special convenient form.

Algorithm RM

First of all, note that the matrix $\hat{R}(x)$ in the representation (3.2) may be considered to be free from zero rows and zero columns.

If the first column has only one non-zero element, transpose the rows and put it into the first (uppermost) place, then start processing of the second column. If
there are several non-zero elements in the first column, we perform the division of one of them with another one with the remainder, for example

\[ R_{11}(x) = T(x)R_{21}(x) + Q(x). \]

Then, if one subtracts the second row of the matrix \( \hat{R}(x) \) multiplied on the left with the differential operator \( T(x) \) from the first row, one obtains an equivalent matrix, which has the operator \( Q(x) \) of lower order than the previous entry \( R_{11}(x) \).

It is obvious that such operations correspond to the operations of the Euclidean algorithm (2.2) and result in an equivalent matrix with \( R_{11}^{(1)}(x) = rGCD(R_{11}, R_{21}) \), and \( R_{21}^{(1)} \equiv 0 \). Consecutively applying this procedure to the other non-zero elements of the first column we obtain a matrix which has only one non-zero element \( R_{i1}^{(2)}(x) = rGCD(R_{11}, \ldots, R_{p1}) \) in the first column.

Next we proceed to the second column. If it has some of the entries \( R_{22}, R_{32}, \ldots, R_{p2} \) different from zero, we apply to them the same procedure as above and obtain \( R_{22}^{(2)}(x) = rGCD(R_{22}, \ldots, R_{p2}) \). If, contrarily, all of \( R_{22}, \ldots, R_{p2} \) vanish, we start processing of the elements \( R_{23}, R_{33}, \ldots, R_{p3} \) of the third column. Obviously this procedure is a direct non-commutative analogue of the standard Gauss elimination algorithm, so in a finite number of steps we obtain a matrix of the form

\[
\hat{R}_{st}(x) = \begin{pmatrix}
R_{11}(x) & \cdots & * & * & \cdots & * & \cdots & * & \cdots \\
0 & \cdots & 0 & R_{2q1}(x) & \cdots & * & * \\
0 & \cdots & 0 & 0 & \cdots & 0 & R_{3q1}(x) & \cdots \\
\vdots & \vdots & \vdots & 0 & \vdots & 0 & 0 & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]

which will be called below echelon matrix. In a more general case (which is of no interest to us) echelon matrices may have a few zero starting columns. First non-zero elements of each row \( R_{11}(x), R_{2q1}(x), R_{3q1}(x), \ldots \) will be called (as usual) pivots. The number of non-zero rows of the obtained echelon matrix is called the rank of the initial operator matrix. As proved in \[1, 4\], the rank is an invariant of a matrix and does not depend on the method of its reduction to an echelon form.

The procedure of reduction to the echelon form is equivalent to multiplication of the initial matrix \( \hat{R}(x) \) on the left with some invertible matrix \( \hat{T}(x) \). Therefore in the formula (3.2) we can substitute \( \hat{M}(x, y) \) with \( \hat{M}(x, y)\hat{T}^{-1}(x) \) and \( \hat{R}(x) \) with \( \hat{R}_{st}(x) = \hat{T}(x)\hat{R}(x) \); so we can assume below that the matrix \( \hat{R}(x) \) already is echelon matrix.

It is easy to see that the matrix \( \hat{R}(x) \) is invertible if and only if it is a square matrix and after its reduction to echelon form we obtain an upper-triangular matrix with non-zero diagonal elements which are invertible elements in the ring of differential operators, that is if they are zero-order operators (functions). The elements strictly above the main diagonal may be arbitrary differential operators.

Analogously to the algorithm \( \text{RI} \) we give below another algorithm which extracts the “maximal” right factor \( \hat{R}(x) \) from a given multiintegral, i.e. for a given operator row \( \hat{T}(x, y) \) a representation (3.2) will be found, with a row \( \hat{M}(x, y) \) defining an operator-irreducible multiintegral with the same function stock.
Algorithm RMI

The initial data of the algorithm is an operator row \((L_1(x, y), \ldots, L_k(x, y))\).

According to the decomposition (3.1) with an echelon matrix \(\hat{R}(x)\) we see that one has to find operators \(M_1(x, y), R_{ij}(x)\), such that

\[
L_1(x, y) = M_1(x, y)R_{11}(x), \quad (3.3)
\]
\[
L_2(x, y) = M_1(x, y)R_{12}(x) + M_2(x, y)R_{22}(x), \quad (3.4)
\]
\[
\ldots \ldots
given \hat{R}(x) = \sum_{i=1}^{s} P_i(x)M_1(x, y_i), \ldots, P_s(x)M_1(x, y_s) = 1. \quad (3.5)
\]

Now we should find \(R_{12}(x), M_2(x, y), R_{22}(x)\) such that the equality (3.4) holds. We obtain this decomposition in three stages.

**Stage A. Finding \(R_{22}(x)\).**

Substituting the found points \(y_i\) into (3.4) we obtain

\[
L_2(x, y_i) = M_1(x, y_i)R_{12}(x) + M_2(x, y_i)R_{22}(x).
\]

Multiplying these identities with the operators \(P_i(x)\) on the left and adding, we get

\[
\left[ \sum_{i=1}^{s} P_i(x)M_1(x, y_i) \right] R_{12}(x) + \left[ \sum_{i=1}^{s} P_i(x)M_2(x, y_i) \right] R_{22}(x) = \sum_{i=1}^{s} P_i(x)L_2(x, y_i),
\]
\[
(3.6)
\]
or, in view of (3.5),

\[
R_{12}(x) + \left[ \sum_{i=1}^{s} P_i(x)M_2(x, y_i) \right] R_{22}(x) = \sum_{i=1}^{s} P_i(x)L_2(x, y_i).
\]
\[
(3.7)
\]

Multiplying (3.7) on the left with \(M_1(x, y)\) and subtracting from (3.4), we arrive at:

\[
[M_2(x, y) - M_1(x, y) \sum_i P_i(x)M_2(x, y_i)] R_{22}(x) = L_2(x, y) - M_1(x, y) \sum_i P_i(x)L_2(x, y_i).
\]
\[
(3.8)
\]
The right hand side of this equality is some known operator \(N(x, y)\). Hence in order to find \(R_{22}(x)\) one has to apply the algorithm RI to \(N(x, y)\), obtaining the decomposition

\[
K(x, y)R_{22}(x) = N(x, y)
\]
\[
(3.9)
\]
with \(R_{22}(x)\) of maximal order; the obtained operator \(K(x, y)\) is irreducible.

**Stage B. Finding \(R_{12}(x)\).**

Rewrite (3.7) as

\[
R_{12}(x) + S(x)R_{22}(x) = T(x) \equiv \sum P_i(x)L_2(x, y_i),
\]
\[
(3.10)
\]
where the operators $T(x)$ and $R_{22}(x)$ are known. The operator $S(x)$ can be arbitrarily chosen (this corresponds to multiplication of the echelon matrix $\hat{R}(x)$ on the left with an invertible upper-triangular matrix) and define the corresponding $R_{12}(x)$.

**Stage C. Finding $M_2(x, y)$**

Recall that (3.8) was obtained from (3.7) and (3.4). Now we do the inverse operation: multiply (3.10) with $M_1(x, y)$ on the left and add (3.9) to it:

$$M_1(x, y)R_{12}(x) + [M_1(x, y)S(x) + K(x, y)] R_{22}(x) = L_2(x, y).$$

Thus we obtain the required equality (3.4) with $M_2(x, y) = M_1(x, y)S(x) + K(x, y)$.

Note that since the obtained $K(x, y)$ is irreducible we can find points $y_{2,1}, y_{2,2}, \ldots, y_{2,s_2}$ and operators $P_{2,1}(x), \ldots, P_{2,s_2}(x)$, such that

$$\sum_{i=1}^{s_2} P_{2,i}(x)K(x, y_{2,i}) = 1. \quad (3.11)$$

If the number of operators $L_i(x, y)$ in the row $\overline{L}(x, y)$ is greater than two, we can analogously find the required operators $M_j(x, y)$, $R_{ij}(x)$, $j \geq 3$. We demonstrate this below for $j = 3$.

In the equality

$$L_3(x, y) = M_1(x, y)R_{13}(x) + M_2(x, y)R_{23}(x) + M_3(x, y)R_{33}(x) \quad (3.12)$$

the operators $M_1(x, y)$ and $M_2(x, y)$ are known. Using the points $y_1, \ldots, y_s$ chosen above and (3.3) we arrive at the following analogues of (3.7) and (3.8):

$$K(x, y)R_{23}(x) + \left[ M_3(x, y) - M_1(x, y) \sum_i P_i(x)M_3(x, y_i) \right] R_{33}(x) = L_3(x, y) - M_1(x, y) \sum_i P_i(x)L_3(x, y_i). \quad (3.13)$$

Using now the points $y_{21}, \ldots, y_{2s_2}$ and the equality (3.11), we can cancel the term $K(x, y)R_{23}(x)$ and obtain an analogue of (3.9):

$$K_3(x, y)R_{33}(x) = N_3(x, y)$$

with the known operator $N_3(x, y)$. This allows us to find $R_{33}(x)$ using the algorithm RII. Repeating stage B we find $R_{13}(x), R_{23}(x)$ with some freedom, analogous to the freedom of choice of $S(x)$ in (3.10). Repeating the stage C one finds $M_3(x, y)$.

**Remark.** It may happen that the right hand side of (3.3) vanishes. Then we set $R_{22}(x) \equiv 0$ and find $R_{12}(x)$ from (3.7). Thus $M_2(x, y)$ is actually absent in (3.3) and we can set $R_{33}(x) \equiv 0$ on the following step in (3.12); in this case the pivot element becomes $R_{23}(x)$. Thus $R_{ij}(x)$ becomes a non-square echelon matrix. The number of operators $M_i(x, y)$ in this case will be smaller that the number of the initial operators $L_i(x, y)$, i.e. $p < k$.

As the result of the algorithm RMI we obtain an operator row $\overline{M}(x, y) = (M_1(x, y), \ldots, M_p(x, y))$ and an operator echelon matrix $\hat{R}(x)$ in the representation (3.1). It is not obvious that the complete multiintegral $J_1 = \overline{M}(x, y)\hat{\psi}(x)$
is operator-irreducible. This fact will be proved below (Corollary 3.9). Inclusion $Z(\mathcal{J}) \subseteq Z(\mathcal{J}_1)$ is obvious. The inverse inclusion follows from the simple fact that the echelon matrix $\hat{R}(x)$ obtained in the algorithm RMI has no zero rows so the system of linear ordinary differential equations $\hat{R}(x)\hat{\phi} = \hat{\psi}$ for the unknown $\hat{\phi}$ is solvable for any right hand side $\hat{\psi}$.

**Remark.** If the given multiintegral was operator-irreducible then the algorithm RMI will give the decomposition (3.1) with $k = p$ and a triangular invertible matrix $\hat{R}(x)$ as the output. The ambiguity of the choice of the operator $S(x)$ on stage B will correspond to the possibility to write a representation (3.1) with arbitrary triangular invertible matrix $\hat{R}(x)$ for any multiintegral $\mathcal{J}$. Note that the requirement of non-invertibility of the matrix $\hat{R}(x)$ without the requirement $p \leq k$ in the definition 3.2 is not sufficient: it is easy to give an example of decomposition (3.1) with $p > k$ (so with a non-invertible matrix $\hat{R}(x)$) for any multiintegral $\mathcal{J} = \mathcal{L}(x,y)\hat{\phi}(x)$.

**Proposition 3.4.** The multiintegral $(M_1(x,y), \ldots, M_p(x,y))$ obtained in the process of the work of the algorithm RMI is kernel-irreducible.

**Proof.** Let $(\psi_1(x), \ldots, \psi_p(x))$ be a nonzero element of the kernel of the multiintegral $\mathcal{M} = (M_1(x,y), \ldots, M_p(x,y))$:

$$\mathcal{M} \psi = 0. \quad (3.14)$$

By construction of the operators $M_i(x,y)$ they satisfy

$$\mathcal{T} = \mathcal{M} \hat{R}. \quad (3.15)$$

First we carry out the proof of the Proposition for the number of elements in the operator rows $\mathcal{L}$ and $\mathcal{M}$ equal to 2. In this case (3.14) means that $M_i(x,y), \psi_i(x), i = 1, 2,$ satisfy the equation

$$0 = M_1(x,y)\psi_1 + M_2(x,y)\psi_2. \quad (3.16)$$

The operators $M_1(x,y), M_2(x,y)$ satisfy in turn

$$L_2(x,y) = M_1(x,y)R_{12}(x) + M_2(x,y)R_{22}(x), \quad (3.17)$$

where $R_{12}(x), R_{22}(x)$ are found in the algorithm RMI. Now we carry out with the equality (3.16) the same operations as we did in the algorithm RMI for the equation (3.17) in order to find $R_{12}(x), R_{22}(x)$, i.e. we take the same points $y_i$ and the same operators $P_i(x)$ which were found in the algorithm RMI, then we compose the same linear combination of the equations that in (3.17). We get

$$\psi_1(x) + \left[ \sum_i P_i(x)M_2(x,y_i) \right] \psi_2(x) = 0. \quad (3.18)$$

From (3.18) and (3.19) we have

$$\left[ M_2(x,y) - M_1(x,y) \sum_i P_i(x)M_2(x,y_i) \right] \psi_2(x) = 0 \quad (3.19)$$
(an analogue of (3.15) in the algorithm RMI). Thus \( \psi_2(x) \equiv 0 \), since the algorithm \( \text{RI} \) gave an irreducible integral \( K(x, y) \) in the left hand side of this equality.

Then from (3.16) we get \( M_1(x, y)\psi_1 \equiv 0 \). But \( M_1(x, y) \) is irreducible by construction so \( \psi_1(x) \equiv 0 \).

Thereby the Proposition is proved for the multiintegral \( \mathcal{J} = (M_1(x, y), M_2(x, y)) \).

In order to prove the Proposition for the general case \( \mathcal{J} = (M_1(x, y), \ldots, M_p(x, y)) \) we analogously consider the equality

\[
0 = M_1(x, y)\psi_1(x) + \ldots + M_p(x, y)\psi_p(x),
\]

(3.20)

where \((\psi_1(x), \ldots, \psi_p(x))\) is an element of the kernel of the multiintegral \( \mathcal{J} \) instead of the equality

\[
L_k(x, y) = M_1(x, y)R_{1k}(x) + \ldots + M_p(x, y)R_{pk}(x).
\]

(3.21)

Applying to (3.20) all operations described in the algorithm RMI one arrives at \( K_p(x, y)\psi_p(x) = 0 \), where \( K_p(x, y) \) is irreducible, which implies \( \psi_p(x) = 0 \). Performing the reverse run of the algorithm we find consecutively \( \psi_{p-1}(x) = 0, \ldots, \psi_1(x) = 0 \). \( \square \)

**Corollary 3.5.** The kernel of the multiintegral \( \mathcal{J} = (L_1(x, y), \ldots, L_k(x, y)) \) coincides with the kernel of the matrix \( \hat{R}(x) \) obtained in the algorithm RMI.

**Proof.** The algorithm gives us \( \overline{L}(x, y)\hat{\psi}(x) = \overline{M}(x, y)\hat{R}(x)\hat{\psi}(x) \), where \( \overline{M}(x, y) \) defines a kernel-irreducible multiintegral. If \( \hat{\psi}(x) \) belongs to the kernel of the matrix \( \hat{R}(x) \) then \( \hat{\psi}(x) \) obviously belongs to the kernel of the multiintegral \( \mathcal{J} \). Conversely if \( \hat{\psi}(x) \) belongs to the kernel of the multiintegral, i.e. \( \overline{L}(x, y)\hat{\psi}(x) = 0 \), then \( \overline{M}(x, y)\hat{\psi}(x) = 0 \), \( \hat{\psi} = \hat{R}(x)\hat{\psi}(x) \) and in virtue of the kernel-irreducibility of \( \overline{M}(x, y) \), \( \hat{\psi} = \hat{R}(x)\hat{\psi}(x) = 0 \), so \( \hat{\psi}(x) \) belongs to the kernel of the matrix \( \hat{R}(x) \). \( \square \)

**Proposition 3.6.** A multiintegral \( \mathcal{J} \) has an infinite-dimensional kernel if and only if the matrix \( \hat{R}(x) \) obtained in the algorithm RMI is non-square (the number of its columns is greater than the number of its rows).

**Proof.** Suppose that the matrix \( \hat{R}(x) \) in the decomposition (3.2) obtained by the algorithm RMI is a square matrix. By Corollary 3.5 the kernel of the multiintegral coincides with the kernel of \( \hat{R} \). The dimension of the space of columns \((\varphi_1(x), \ldots, \varphi_k(x))\) which are solutions of the matrix differential equation \( \hat{R}(x)\varphi(x) = 0 \) is finite. In fact, since \( \hat{R}(x) \) is a square echelon matrix without zero rows, we see that \( \varphi_k(x) \) satisfies the equation \( R_{kk}(x)\varphi_k(x) = 0 \) and consequently belongs to a finite-dimensional space. Substituting any of the found \( \varphi_k(x) \) into the previous equation \( R_{k-1,k-1}(x)\varphi_{k-1}(x) + R_{k-1,k}(x)\varphi_k(x) = 0 \), we find that \( \varphi_{k-1} \) also belongs to a finite-dimensional space. Then analogously we define all the other \( \varphi_{k-2}(x), \ldots, \varphi_1(x) \), each of them belong to a finite-dimensional space of solutions of some linear ordinary differential equation. Therefore the dimension of the kernel of the multiintegral is finite and equals the sum of the orders of the differential operators on the principal diagonal of the matrix \( \hat{R} \).
Lemma 3.7. Let an operator matrix \( \hat{R}(x) \) of the size \( k \times p \) with \( k \leq p \) be given. Then \( \hat{R}(x) \) is non-invertible if and only if it has a nontrivial kernel.

Proof. It is easy to see that nontriviality of the kernel implies that \( \hat{R}(x) \) is non-invertible.

Conversely, let \( \hat{R}(x) \) be non-invertible. Reduce \( \hat{R}(x) \) to echelon form. Multiplying the rows of the resulting \( \hat{R}(x) \) with appropriate non-zero functions we can without limitation of generality assume that its pivots of order zero are equal to 1.

Case A: let \( k = p \), i.e. the matrix \( \hat{R}(x) \) is square. Then among its pivots \( R_{11}(x), \ldots, R_{pp}(x) \) at least one is not equal to 1 (otherwise \( \hat{R}(x) \) is invertible). Take the first pivot different from 1. Let it be \( R_{mm}(x) \). The differential equation \( R_{mm}(x)\varphi(x) = 0 \) always has a non-zero solution \( \tilde{\varphi}(x) \). Consequently the kernel \( \tilde{\mathcal{R}}(x) \) contains a non-zero element \( \tilde{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_p(x))^t \), where

\[
\begin{align*}
\varphi_{m+1}(x) &= \ldots = \varphi_p(x) = 0, \\
\varphi_m(x) &= \tilde{\varphi}(x), \\
\varphi_{m-1}(x) &= -R_{m-1,m}(x)\varphi_m(x), \\
\varphi_{m-2}(x) &= -R_{m-2,m-1}(x)\varphi_{m-1}(x) - R_{m-2,m}(x)\varphi_m(x), \\
&\vdots \\
\varphi_1(x) &= -\sum_{j=2}^m R_{1,j}(x)\varphi_j(x).
\end{align*}
\]

case B: now we assume \( \hat{R}(x) \) to be non-square, i.e. \( k < p \). Take the first row of \( \hat{R}(x) \) with the pivot \( R_{mm} \) such that the next pivot is not diagonal (or take the last row). We have the entries \( R_{mm} \neq 0, R_{m,m+1}, \ldots, R_{m,m+r} \) of this row, such that \( R_{m+1,m+r+1} \neq 0 \) is the next pivot, \( r \geq 1 \) (or \( m = p \) so we take all elements of the last row). If the pivot \( R_{mm} \) is not equal to 1, we can easily find a nontrivial element of the kernel \( \tilde{\mathcal{R}}(x) \), using the argumentation of the case A. If we have \( r + 1 \geq 2 \) then we can find an infinite-dimensional kernel of \( \hat{R}(x) \) using the argumentation of the proof of Proposition 3.6.
Theorem 3.8. A multiintegral $\mathcal{J}$ is operator-reducible if and only if $\mathcal{J}$ is kernel-reducible.

Proof. Applying to $\mathcal{J} = \mathcal{T}(x,y)\hat{\varphi}(x)$ the algorithm RMI, we obtain the representation $\mathcal{T}(x,y) = \mathcal{M}(x,y)\hat{R}(x)$. If $\mathcal{J}$ is kernel-reducible, then according to Corollary 3.5 the matrix $\hat{R}(x)$ has a nontrivial kernel and is non-invertible. So $\mathcal{J}$ is operator-reducible.

Conversely, let $\mathcal{J}$ be operator-reducible. i.e. we have the corresponding decomposition $\mathcal{T}(x,y) = \mathcal{M}(x,y)\hat{R}(x)$, where $\hat{R}(x)$ is a non-invertible matrix, such that the number of its rows is smaller or equal than the number of its columns. According to Lemma 3.7 the operator matrix $\hat{R}(x)$ has a nontrivial kernel. So $\mathcal{J}$ is kernel-reducible.

From this Theorem and Proposition 3.4 we immediately get

Corollary 3.9. The obtained in the algorithm RMI multiintegral $\mathcal{J} = \mathcal{M}(x,y)\hat{\psi}(x)$ is operator-reducible.

Theorem 3.10. Let two multiintegrals $\mathcal{J}_1 = (M_1(x,y),\ldots,M_p(x,y))$, $\mathcal{J}_2 = (L_1(x,y),\ldots,L_k(x,y))$ be given, and $\mathcal{J}_1$ is operator-irreducible. Let the function stock generated by the multiintegral $\mathcal{J}_2$ be contained (as a subset) in the function stock generated by $\mathcal{J}_1$. Then

$$
(L_1(x,y),\ldots,L_k(x,y)) = (M_1(x,y),\ldots,M_p(x,y))\hat{R}(x)
$$

(3.22)

with some matrix $\hat{R}(x)$ of operators whose coefficients depend only on $x$. Moreover the rank of the matrix $\hat{R}(x)$ coincides with the number $p$ of its rows if and only if the function stocks generated by $\mathcal{J}_1$ and $\mathcal{J}_2$, coincide.

Proof. Since we have $Z(\mathcal{J}_2) \subseteq Z(\mathcal{J}_1)$, then for every column $\hat{\varphi}(x) = (\varphi_1(x),\ldots,\varphi_k(x))^t$ one can find another column $\hat{\psi}(x) = (\psi_1(x),\ldots,\psi_p(x))^t$, such that

$$
(M_1(x,y),\ldots,M_p(x,y))\hat{\psi} = (L_1(x,y),\ldots,L_k(x,y))\hat{\varphi}.
$$

(3.23)

Reproduce the procedures of the algorithm RMI for the equation (3.23) with the obvious modifications. Since $M_1(x,y)$ is irreducible, one can choose the values $y_1,\ldots,y_s$ and operators $P_1(x),\ldots,P_s(x)$ so that

$$
P_1(x)M_1(x,y_1) + \ldots + P_s(x)M_1(x,y_s) = 1.
$$

Therefore from (3.23) we can obtain

$$
\psi_1(x) + \sum_{i=1}^s P_i(x)M_2(x,y_i)\psi_2(x) + \ldots = \sum_{i=1}^s P_i(x)L_1(x,y_i)\varphi_1(x) + \ldots
$$

(3.24)

Applying the operator $M_1(x,y)$ to the both sides of this equation and subtracting the result from (3.23) we arrive at an analogue of the equality (3.13) of the algorithm RMI. Proceeding with the algorithm we finally obtain an equality of the form

$$
Q(x,y)\hat{\psi}_p(x) = \sum_{j=1}^k T_j(x)\varphi_j(x).
$$

(3.25)
Since by the assumption of the Theorem $J_1$ is operator-irreducible, the obtained operator $Q(x,y)$ is also operator-irreducible so choosing some appropriate points $\hat{y}_1$, $\ldots$, $\hat{y}_t$ and operators $\tilde{P}_1(x)$, $\ldots$, $\tilde{P}_t(x)$ one has

$$\tilde{P}_1(x)Q_1(x,\hat{y}_1)+\ldots+\tilde{P}_t(x)Q_1(x,\hat{y}_t) = 1.$$ 

Then from (3.25) we can easily deduce

$$\psi_p(x) = \sum_{j=1}^{k} R_{pj}(x)\varphi_j(x). \tag{3.26}$$

The found operators $R_{pj}(x)$ give us precisely the last row of the matrix $\hat{R}(x)$. Its previous rows can be readily found from the obtained in the process of the algorithm equalities of the form (3.23). Substituting (3.26) and the analogous expressions for $\psi_{p-1}(x)$, $\ldots$, $\psi_1(x)$ into (3.23) we obtain the required formula (3.22) since the functions $\varphi_i(x)$ were chosen to be arbitrary.

From (3.22) one concludes that the function stocks generated by the multiintegrals in question coincide iff the system of linear ordinary differential equations

$$\hat{R}(x)\hat{\varphi}(x) = \hat{\psi}(x) \tag{3.27}$$

is solvable for every column $\hat{\psi}(x)$. Reduce $\hat{R}$ to echelon form (this is equivalent to multiplication of the both sides of (3.27) on the left with some invertible operator matrix $\hat{S}(x)$). The system $\hat{S}(x)\hat{R}(x)\hat{\varphi}(x) = \hat{S}(x)\hat{\psi}(x)$ is solvable for any $\hat{\psi}(x)$ if and only if the echelon matrix $\hat{S}(x)\hat{R}(x)$ has no zero rows. This proves the last statement of the Theorem.

Note that contrary to the case of Euler integrals, a strict inclusion $Z(J_2) \subset Z(J_1)$ is possible for multiintegrals. The simplest case can be given by any multiintegral of the form $J_1 = (L_1(x,y), L_2(x,y))(\varphi_1(x), \varphi_2(x))^t$ where both $L_1(x,y)$, $L_2(x,y)$ generate irreducible integrals with different nonintersecting function stocks $Z(L_1\varphi_1)$, $Z(L_2\varphi_2)$. First of all we deduce that $J_1$ is kernel-irreducible and its function stock coincides with the direct sum $Z(L_1\varphi_1) \oplus Z(L_2\varphi_2)$. Take now $J_2 = L_1(x,y)\psi_1(x)$. We see that $Z(J_2) = Z(L_2\varphi_2)$ is strictly contained in $Z(J_1)$.

This example motivates the introduction of the following notion

**Definition 3.11.** If an operator-irreducible multiintegral $J = (L_1(x,y), \ldots, L_k(x,y))$ is given, then its submultiintegral is any multiintegral $\Bar{J}$ of the form

$$\Bar{J} = (\Bar{L}_1(x,y), \ldots, \Bar{L}_m(x,y))\hat{\varphi}(x) = (L_1(x,y), \ldots, L_k(x,y))\hat{N}(x)\hat{\varphi}(x),$$

where $\hat{N}(x)$ is an arbitrary matrix of LODO with coefficients depending on $x$. As obvious from the results proved above, this is equivalent to the requirement $Z(\Bar{J}) \subset Z(J)$.

**Remark.** As we can see from the proof of Theorem 3.10 there exists an algorithmic way to decide for any two given multiintegrals $J_1$ and $J_2$, wether the function
stocks generated by them coincide or are different, or whether we have a strict inclusion of one stock into another and what is the dimension of their intersection $Z(J_2) \cap Z(J_1)$.

To this end we find using the algorithm RMI the irreducible representations of the given multiintegrals $J_1 = (L_1(x,y), \ldots, L_k(x,y)) \hat{\varphi}(x)$, $J_2 = (M_1(x,y), \ldots, M_p(x,y)) \hat{\psi}(x)$ (with the same function stocks). Write the following formal equality

$$(M_1(x,y), \ldots, M_p(x,y)) \hat{\psi}(x) = (L_1(x,y), \ldots, L_k(x,y)) \hat{\varphi}(x),$$

where $\hat{\psi}(x) = (\psi_1(x), \ldots, \psi_p(x))^t$, $\hat{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_k(x))^t$. Applying the procedures of the algorithm RMI to the equality (3.28), as in the proof of Theorem 3.10, we find a matrix $\tilde{R}(x)$ such that

$$\hat{\psi}(x) = \tilde{R}(x)\hat{\varphi}(x).$$

From (3.28) and (3.29) one deduce that the intersection of the function stocks of $J_1$ and $J_2$ coincides with the set $\overline{M}(x,y)\tilde{R}(x)\hat{\varphi}(x)$, where $\hat{\varphi}(x)$ are solutions of the equations

$$\overline{M}(x,y)\tilde{R}(x)\hat{\varphi}(x) = \overline{L}(x,y)\hat{\varphi}(x).$$

Thus if we have the operator identity

$$\overline{L}(x,y) = \overline{M}(x,y)\tilde{R}(x),$$

then either $Z(J_2) \subseteq Z(J_1)$ so by definition $J_1$ is a proper submultiintegral of $J_2$ (in the case when the rank of $\tilde{R}$ is smaller than the number of its rows $p$), or $Z(J_2) = Z(J_1)$ (if the rank of $\tilde{R}$ equals the number of its rows $p$). If the equality (3.30) does not hold we form a new multiintegral $J_3 = \overline{K}(x,y)\xi(x)$ with $\overline{K}(x,y) = \overline{L}(x,y) - \overline{M}(x,y)\tilde{R}(x)$. Applying to $J_3$ the algorithm RMI, we obtain the representation

$$\overline{K}(x,y) = \overline{N}(x,y)\tilde{R}_1(x).$$

The intersection of the function stocks of $J_1$ and $J_2$ coincides with the set $\{\overline{L}(x,y)\xi(x) \mid \xi(x) \in \text{Ker}\tilde{R}_1(x)\}$ so by Proposition 3.6 it is finite-dimensional if and only if the matrix $\tilde{R}_1(x)$ is a square matrix. In the contrary case (when the number of its columns $k$ is greater than the number of its rows $p_1$) $Z(J_2)$ and $Z(J_1)$ has an infinite-dimensional intersection with the functional dimension equal to the difference $k - p_1$.

**Proposition 3.12.** Among multiintegrals generating the same function stock, there is a unique irreducible $\mathcal{I}_0 = \overline{L}_0(x,y)\hat{\psi}(x)$ (unique up to transformations $L_0 \mapsto L_0(x,y)\tilde{R}(x)$ with any invertible operator matrix $\tilde{R}(x)$). All other multiintegrals with the same function stock have the form $\mathcal{I} = \overline{L}_0(x,y)\hat{S}(x)\varphi(x)$, where $\hat{S}(x)$ is an arbitrary operator matrix with coefficients depending on $x$, the number of the rows of this matrix is equal to its rank.
The proof of this Theorem repeats the proof of Proposition 2.9 and uses Theorem 3.10 instead of Lemma 2.7.

Note that in the approach we used so far we did not allow operations with the arbitrary functions \( \varphi_i(x) \). The natural admissible operations with the function sets 

\[ \hat{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_k(x))^t \]

are:

- transposition of two functions \( \varphi_i \leftrightarrow \varphi_j \);
- multiplication of one function with a given nonzero multiplier \( g(x) \);
- substitution \( \varphi_j \mapsto \varphi_j + A(x)\varphi_i, i \neq j \), where \( A(x) \) is a LODO with coefficients depending only on \( x \).

These operations correspond in the representation (3.1) to:

- transposition of the corresponding operators \( L_i(x, y) \leftrightarrow L_j(x, y) \) of the operator row \( \hat{L}(x, y) \) and the corresponding columns of the matrix \( \hat{R}(x) \);
- multiplication of the corresponding operator \( L_i(x, y) \) and the \( i \)-th column of the matrix \( \hat{R}(x) \) on the right with \( g(x) \);
- substitution \( L_i \rightarrow L_i + L_j A(x) \) and addition to the \( i \)-th column of \( \hat{R}(x) \) of the \( j \)-th column multiplied on the right with \( A(x) \).

All these operations correspond to multiplication of the both sides of the equality (3.1) on the right with an invertible operator matrix. It was proved in [1] that carrying out simultaneously the aforementioned operations with both the columns and the rows of the matrix \( \hat{R}(x) \) lets us to reduce it to the form \( \hat{R}(x) \) where \( \tilde{R}_{ii}(x) = 1 \) for \( 1 \leq i \leq p - 1 \), \( \tilde{R}_{pp}(x) \) is a LODO, and all the other elements of the matrix \( \hat{R}(x) \) vanish.

From this we deduce that applying to the obtained in the algorithm RMI decomposition \( \hat{L} = \hat{M}\hat{R} \) the described above operations with the columns and the rows we get the following representation of any multiintegral:

\[
(\tilde{L}_1(x, y), \ldots, \tilde{L}_k(x, y)) = (\tilde{M}_1(x, y), \ldots, \tilde{M}_p(x, y))\hat{R}(x),
\]

that is \( \tilde{L}_1 = \tilde{M}_1, \ldots, \tilde{L}_{p-1} = \tilde{M}_{p-1}, \tilde{L}_p = \tilde{M}_p\tilde{R}_{pp} \) and, if \( k > p \), \( \tilde{L}_{p+1} = 0, \ldots, \tilde{L}_k = 0 \).

Thus the following Proposition is proved:

**Theorem 3.13.** Using the admissible operations with the set \( \hat{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_k(x))^t \) and dropping zero components, any multiintegral can be reduced to the form

\[ J = (\tilde{L}_1(x, y), \ldots, \tilde{L}_{p-1}(x, y), \tilde{M}_p(x, y)\hat{R}(x))\hat{\varphi}(x), \]

where the row \( (\tilde{L}_1, \ldots, \tilde{L}_{p-1}, \tilde{M}_p) \) gives an irreducible integral.
Acknowledgements

The author expresses her sincere gratitude to Prof. O.V.Kaptsov and Yu.V.Shanko for fruitful discussion of the results.

References

[1] Jacobson N. The theory of rings. AMS, 1943.

[2] Zhiber A.V., Sokolov V.V., Exactly integrable hyperbolic equations of Liouville type. Russian Math. Surveys. 2001. V. 56, No 1. p. 61–101.

[3] Zhiber V., Startsev S.Ya., Integrals, Solutions, and Existence Problems for Laplace Transformations of Linear Hyperbolic Systems. Mathematical Notes, 2003, V. 74, No. 5–6, p. 803–811.

[4] Lopatinskij Ya.B., Linear differential operators: Doctoral Thesis, 71 p. Baku, 1946. reprinted in: Ya. B. Lopatinskij. General theory of boundary problems. Kiev, 1984.

[5] Startsev S.Ya., Cascade method of Laplace integration for linear hyperbolic systems of equations. Mathematical Notes, 2008, V. 83, No. 1–2, P. 97–106.

[6] Taimanov I.A., Tsarev S.P., Two-dimensional Schrödinger operators with fast decaying potential and multidimensional $L_2$-kernel, Russian Math. Surveys, 2007, v. 62, No 3, p. 217–218.

[7] Tsarev S.P., On Darboux-Integrable Nonlinear Partial Differential Equations. Proceedings of the Steklov Institute of Mathematics, 1999, v. 225, p. 372–381.

[8] Euler L. Institutionum calculi integralis. V. III, Ac. Sc. Petropoli, 1770.

[9] Anderson I.M., Kamran N. The Variational Bicomplex for Second Order Scalar Partial Differential Equations in the Plane. Duke Math. J., 1997. V. 87. N 2. P. 265–319.

[10] Athorne C., Nimmo J.J.C. On the Moutard transformation for integrable partial differential equations. Inverse Problems, 1991, v. 7(6), p. 809–826.

[11] Athorne C. A $\mathbb{Z}^2 \times \mathbb{R}^3$ Toda system. Phys. Lett. A. 1995. v. 206, p. 162–166.

[12] Forsyth A.R. Theory of differential equations. Part IV, vol. VI. Cambridge, 1906.

[13] Darboux G. Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, T. 2. Gauthier-Villars, 1889.
[14] Ganzha E.I., Loginov V.M., Tsarev S.P. Exact solutions of hyperbolic systems of kinetic equations. Application to Verhulst model with random perturbation. Mathematics of Computation, 2008, v. 1, No 3, p. 459–472. e-print http://www.arxiv.org/math.AP/0612793

[15] Goursat É. Leçons sur l’intégration des équations aux dérivées partielles du second ordre a deux variables indépendants. T. 2. Paris: Hermann, 1898.

[16] Le Roux J. Extensions de la méthode de Laplace aux équations linéaires aux dérivées partielles d’ordre supérieur au second. Bull. Soc. Math. France. 1899. V. 27. P. 237–262. A digitized copy is obtainable from http://www.numdam.org/

[17] Petrén L. Extension de la méthode de Laplace aux équations $\sum_{i=0}^{n-1} A_{i1} \frac{\partial^{i+1} z}{\partial x_{i+1}} + \sum_{i=0}^{n} A_{0i} \frac{\partial^i z}{\partial y^i} = 0$. Lund Univ. Arsskrift. 1911. Bd. 7. Nr. 3. p. 1–166.

[18] Pisati, L. Sulla estensione del metodo di Laplace alle equazioni differenziali lineari di ordine qualunque con due variabili indipendenti. Rend. Circ. Matem. Palermo. 1905. t. 20. P. 344–374.

[19] Tsarev S.P. Generalized Laplace Transformations and Integration of Hyperbolic Systems of Linear Partial Differential Equations. Proc. ISSAC’2005 (July 24–27, 2005, Beijing, China) ACM Press. 2005. P. 325–331; also e-print cs.SC/0501030 at http://www.archiv.org/

[20] Tsarev S.P. On factorization and solution of multidimensional linear partial differential equations. in: ”COMPUTER ALGEBRA 2006. Latest Advances in Symbolic Algorithms”, Proc. Waterloo Workshop, Canada, 10–12 April 2006, World Scientific, 2007. p. 181-192. e-print http://www.archiv.org/cs.SC/0609075