A short note on Godbersen’s Conjecture

S. Artstein-Avidan

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For compact convex sets $K_1, \ldots, K_m \subset \mathbb{R}^n$, and non-negative real numbers $\lambda_1, \ldots, \lambda_m$, a classical result of Minkowski states that the volume of $\sum \lambda_i K_i$ is a homogeneous polynomial of degree $n$ in $\lambda_i$,

$$\text{Vol} \left( \sum_{i=1}^{m} \lambda_i K_i \right) = \sum_{i_1, \ldots, i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}).$$ (1)

The coefficient $V(K_{i_1}, \ldots, K_{i_n})$, which depends solely on $K_{i_1}, \ldots, K_{i_n}$, is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$. The mixed volume is a non-negative, translation invariant function, monotone with respect to set inclusion, invariant under permutations of its arguments, and positively homogeneous in each argument. For $K$ and $L$ compact and convex, we denote $V(K[j], L[n-j])$ the mixed volume of $j$ copies of $K$ and $(n-j)$ copies of $L$. One has $V(K[n]) = \text{Vol}(K)$. By Alexandrov’s inequality, $V(K[j], -K[n-j]) \geq \text{Vol}(K)$, with equality if and only of $K = x_0 - K$ for some $x_0$, that is, some translation of $K$ is centrally symmetric. For further information on mixed volumes and their properties, see Section §5.1 of [8].

Recently, in the paper [2] we have shown that for any $\lambda \in [0, 1]$ and for any convex body $K$ one has that

$$\lambda^j (1-\lambda)^{n-j} V(K[j], -K[n-j]) \leq \text{Vol}(K).$$

In particular, picking $\lambda = \frac{j}{n}$, we get that

$$V(K[j], -K[n-j]) \leq \frac{n^n}{j^j (n-j)^{n-j}} \text{Vol}(K) \sim \binom{n}{j} \sqrt{2\pi j \frac{(n-j)}{n}}.$$ (2)

The conjecture for the tight upper bound $\binom{n}{j}$, which is what ones get for a body which is an affine image of the simplex, was suggested in 1938 by Godbersen [4] (and independently by Hajnal and Makai Jr. [5]).

**Conjecture 1** (Godbersen’s conjecture). For any convex body $K \subset \mathbb{R}^n$ and any $1 \leq j \leq n-1$,

$$V(K[j], -K[n-j]) \leq \binom{n}{j} \text{Vol}(K),$$ (2)

with equality attained only for simplices.

We mention that Godbersen [4] proved the conjecture for certain classes of convex bodies, in particular for those of constant width. We also mention that the conjecture holds for $j = 1, n-1$ by the inclusion $K \subset n(-K)$ for bodies $K$ with center of mass at the origin, and inclusion which is tight for the simplex, see Schneider [9]. The bound from [2] quoted above seems to be the currently smallest known upper bound for general $j$.

In this short note we improve the aforementioned inequality and show

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Theorem 2. For any convex body $K \subset \mathbb{R}^n$ and for any $\lambda \in [0, 1]$ one has
\[
\sum_{j=0}^{n} \lambda^j (1 - \lambda)^{n-j} V(K[j], -K[n-j]) \leq \text{Vol}(K).
\]

The proof of the inequality will go via the consideration of two bodies, $C \subset \mathbb{R}^{n+1}$ and $T \subset \mathbb{R}^{2n+1}$. Both were used in the paper of Rogers and Shephard [7].

We shall show by imitating the methods of [7] that

Lemma 3. Given a convex body $K \subset \mathbb{R}^n$ define $C \subset \mathbb{R} \times \mathbb{R}^n$ by
\[
C = \text{conv} \left( \{0\} \times (1 - \lambda)K \cup \{1\} \times -\lambda K \right).
\]

Then we have
\[
\text{Vol}(C) \leq \frac{\text{Vol}(K)}{n+1}.
\]

With this lemma in hand, we may prove our main claim by a simple computation

Proof of Theorem 2
\[
\text{Vol}(C) = \int_0^1 \text{Vol}((1 - \eta)(1 - \lambda)K - \eta \lambda K) d\eta
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} (1 - \lambda)^{n-j} \lambda^j V(K[j], -K[n-j]) \int_0^1 (1 - \eta)^{n-j} \eta^j d\eta
\]
\[
= \frac{1}{n+1} \sum_{j=0}^{n} (1 - \lambda)^{n-j} \lambda^j V(K[j], -K[n-j]).
\]

Thus, using Lemma 3 we have that
\[
\sum_{j=0}^{n} (1 - \lambda)^{n-j} \lambda^j V(K[j], -K[n-j]) \leq \text{Vol}(K).
\]

Before turning to the proof of Lemma 3 let us state a few consequences of Theorem 2. First, integration with respect to the parameter $\lambda$ yields

Corollary 4. For any convex body $K \subset \mathbb{R}^n$
\[
\frac{1}{n+1} \sum_{j=0}^{n} \frac{V(K[j], -K[n-j])}{\binom{n}{j}} \leq \text{Vol}(K),
\]

which can be rewritten as
\[
\frac{1}{n-1} \sum_{j=1}^{n-1} \frac{V(K[j], -K[n-j])}{\binom{n}{j}} \leq \text{Vol}(K).
\]

So, on average the Godbersen conjecture is true. Of course, the fact that it holds true on average was known before, but with a different kind of average. Indeed, the Rogers-Shephard inequality for the difference body, which is
\[
\text{Vol}(K - K) \leq \binom{2n}{n} \text{Vol}(K)
\]
(see for example [8] or [3]) can be rewritten as
\[ \frac{1}{(2n)^n} \sum_{j=0}^{n} \binom{n}{j} V(K[j], -K[n-j]) \leq \text{Vol}(K). \]

However, our new average, in Corollary 4 is a uniform one, so we know for instance that the median of the sequence \( (\binom{n}{j})^{-1}V(K[j], -K[n-j]) \) is less than two, so that at least for one half of the indices \( j = 1, 2, \ldots, n - 1 \), the mixed volumes satisfy Godbersen’s conjecture up to factor 2. More generally, apply Markov’s inequality for the uniform measure on \( \{1, \ldots, n - 1\} \) to get

**Corollary 5.** Let \( K \subset \mathbb{R}^n \) be a convex body with Vol\( (K) = 1 \). For at least \( k \) of the indices \( j = 1, 2, \ldots, n - 1 \) it holds that
\[ V(K[j], -K[n-j]) \leq \frac{n-1}{n-k} \binom{n}{j}. \]

We mention that the inequality of Theorem 2 can be reformulated, for \( K \) with Vol\( (K) = 1 \), say,
\[ \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} V(K[j], -K[n-j]) \leq \text{Vol}(K). \]

So that by taking \( \lambda = 0, 1 \) we see, once again, that \( V(K, -K[n-1]) \leq V(K[n-1], -K) \leq n \).

A key ingredient in the proof of Lemma 3 is Rogers-Shephard inequality for sections and projections from [7], which states that

**Lemma 6** (Rogers and Shephard). Let \( T \subset \mathbb{R}^m \) be a convex body, let \( E \subset \mathbb{R}^m \) be a subspace of dimension \( j \). Then
\[ \text{Vol}(P_{E^\perp} T)\text{Vol}(T \cap E) \leq \binom{m}{j} \text{Vol}(T), \]
where \( P_{E^\perp} \) denotes the projection operator onto \( E^\perp \).

We turn to the proof of Lemma 3 regarding the volume of \( C \).

**Proof of Lemma 3.** We borrow directly the method of [7]. Let \( K_1, K_2 \subset \mathbb{R}^n \) be convex bodies, we shall consider \( T \subset \mathbb{R}^{2n+1} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \) defined by
\[ T = \text{conv}\{(0,0,y) : y \in K_2 \} \cup \{(1, x, -x) : x \in K_1\}. \]

Written out in coordinates this is simply
\[ T = \{ (\theta, \theta x, -\theta x + (1-\theta)y) : x \in K_1, y \in K_2 \} \]
\[ = \{ (\theta, w, z) : w \in \theta K_1, z \in (1-\theta)K_2 \}. \]

The volume of \( T \) is thus, by simple integration, equal to
\[ \text{Vol}(T) = \text{Vol}(K_1)\text{Vol}(K_2) \int_0^1 \theta^n (1-\theta)^n d\theta = \frac{n!n!}{(2n+1)!} \text{Vol}(K_1)\text{Vol}(K_2). \]

We now take the section of \( T \) by the \( n \) dimensional affine subspace
\[ E = \{(\theta_0, x, 0) : x \in \mathbb{R}^n \} \]
and project it onto the complement \( E^\perp \). We get for the section:
\[ T \cap E = \{ (\theta_0, x, 0) : x \in \theta_0 K_1 \cap (1-\theta_0)K_2 \} \]
and so $\text{Vol}_n(T \cap E) = \text{Vol}(\theta_0 K_1 \cap (1 - \theta_0)K_2)$. As for the projection, we get
\[
P_{E \perp T} = \{(\theta, 0, y) : \exists x \text{ with } (\theta, x, y) \in T}\]
\[
= \{(\theta, 0, y) : \theta K_1 \cap ((1 - \theta)K_2 - y)\}
\[
= \{(\theta, 0, y) : y \in (1 - \theta)K_2 - \theta K_1\}.
\]
Thus $\text{Vol}_n(P_{E \perp T}) = \text{Vol}((\theta, y) : y \in (1 - \theta)K_2 - \theta K_1)$ which is precisely a set of the type we considered before in $\mathbb{R}^{n+1}$. In fact, putting instead of $K_1$ the set $\lambda K$ and instead of $K_2$ the set $(1 - \lambda)K$ we get that $P_{E \perp T} = C$.

Staying with our original $K_1$ and $K_2$, and using the Rogers-Shephard Lemma bound for sections and projections, we see that
\[
\text{Vol}(P_{E \perp T}) \text{Vol}(T \cap E) \leq \binom{2n + 1}{n} \text{Vol}(T),
\]
which translates to the following inequality
\[
\text{Vol}(\text{conv}\{\{0\} \times K_2 \cup \{1\} \times (-K_1)\}) \leq \frac{1}{n + 1} \frac{\text{Vol}(K_1)\text{Vol}(K_2)}{\text{Vol}(\theta_0 K_1 \cap (1 - \theta_0)K_2)}.
\]

We mention that this exact same construction was preformed and analysed by Rogers and Shephard for the special choice $\theta_0 = 1/2$, which is optimal if $K_1 = K_2$.

For our special choice of $K_2 = (1 - \lambda)K$ and $K_1 = \lambda K$ we pick $\theta_0 = (1 - \lambda)$ so that the intersection in question is simply $\lambda(1 - \lambda)K$, which cancels out when we compute the volumes in the numerator. We end up with
\[
\text{Vol}(\text{conv}\{\{0\} \times (1 - \lambda)K \cup \{1\} \times (-\lambda K)\}) \leq \frac{1}{n + 1} \text{Vol}(K),
\]
which was the statement of the lemma.

Our next assertion is connected with the following conjecture regarding the unbalanced difference body
\[
D_\lambda K = (1 - \lambda)K + \lambda(-K).
\]

**Conjecture 7.** For any $\lambda \in (0, 1)$ one has
\[
\frac{\text{Vol}(D_\lambda K)}{\text{Vol}(K)} \leq \frac{\text{Vol}(D_\lambda \Delta)}{\text{Vol}(\Delta)}
\]
where $\Delta$ is an $n$-dimensional simplex.

Reformulating, Conjecture asks whether the following inequality holds
\[
\sum_{j=0}^{n} \binom{n}{j} \lambda^j (1 - \lambda)^{n-j} V_j \leq \sum_{j=0}^{n} \binom{n}{j}^2 \lambda^j (1 - \lambda)^{n-j},
\]  \hspace{1cm} \text{(3)}
where we have denoted $V_j = V(K[j], -K[n - j]) / \text{Vol}(K)$.

Clearly Conjecture follows from Godbersen’s conjecture. Conjecture holds for $\lambda = 1/2$ by the Rogers-Shephard difference body inequality, it holds for $\lambda = 0, 1$ as then both sides are 1, and it holds on average over $\lambda$ by Lemma (one should apply Lemma for the body $2K$ with $\lambda_0 = 1/2$). We rewrite two of the inequalities that we know on the sequence $V_j$:
\[
\sum_{j=0}^{n} \lambda^j (1 - \lambda)^{n-j} V_j \leq \sum_{j=0}^{n} \binom{n}{j} \lambda^j (1 - \lambda)^{n-j},
\]  \hspace{1cm} \text{(4)}
In all inequalities we may disregard the $0^{th}$ and $n^{th}$ terms as they are equal on both sides. We may take advantage of the fact that the $j^{th}$ and the $(n-j)^{th}$ terms are the same in each inequality, and sum only up to $(n/2)$ (but be careful, if $n$ is odd then each term appears twice, and if $n$ is even then the $(n/2)^{th}$ term appears only once).

Theorem 8. For $n = 4, 5$ Conjecture holds.

Proof. For $n = 4$ We have that $V_0 = V_4 = 1$ and $V_1 = V_3$. We thus know that

$$8V_1 + 6V_2 \leq 32 + 36$$

and that for any $\lambda \in [0,1]$ we have

$$(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)V_1 + \lambda^2(1-\lambda)^2V_2 \leq 4(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3) + 6\lambda^2(1-\lambda)^2.$$  

We need to prove that

$$4(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)V_1 + 6\lambda^2(1-\lambda)^2V_2 \leq 16(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3) + 36\lambda^2(1-\lambda)^2.$$  

If we find $a, b \geq 0$ such that

$$(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)a + 8b = 4(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)$$

and

$$\lambda^2(1-\lambda)^2a + 6b = 6\lambda^2(1-\lambda)^2$$

then by summing the two inequalities with these coefficients, we shall get the needed inequality.

We thus should check whether the following system of equations has a non-negative solution in $a, b$:

$$
\begin{pmatrix}
\lambda^3(1-\lambda) + \lambda(1-\lambda)^3 & 8 \\
\lambda^2(1-\lambda)^2 & 6
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
4(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3) \\
6\lambda^2(1-\lambda)^2
\end{pmatrix}.
$$

The determinant of the matrix of coefficients is positive:

$$6(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3) - 8\lambda^2(1-\lambda)^2 =$$

$$2\lambda(1-\lambda)[3(\lambda^2 + (1-\lambda)^2) - 4\lambda(1-\lambda)] =$$

$$2\lambda(1-\lambda)[3(1-2\lambda)^2 + 2\lambda(1-\lambda)] \geq 0$$

We invert it to get, up to a positive multiple, that

$$
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
6 & -8 \\
-\lambda^2(1-\lambda)^2 & (\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)
\end{pmatrix}
\begin{pmatrix}
4(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3) \\
6\lambda^2(1-\lambda)^2
\end{pmatrix}
= 
\begin{pmatrix}
2(\lambda(1-\lambda))(1-2\lambda)^2 \\
2(\lambda^3(1-\lambda) + \lambda(1-\lambda)^3)\lambda^2(1-\lambda)^2
\end{pmatrix}.
$$

We see that indeed the resulting $a, b$ are non-negative.

For $n = 5$ we do the same, namely we have $V_0 = V_5 = 1$ and $V_1 = V_4$ and $V_2 = V_3$ so we just have two unknowns, for which we know that

$$5V_1 + 10V_2 \leq 25 + 100$$

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and that for any \( \lambda \in [0, 1] \) we have
\[
(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4)V_1+(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2)V_2 \leq 5(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4)+10(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2).
\]

We need to prove that
\[
5(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4)V_1+10(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2)V_2 \leq 25(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4)+100(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2).
\]

We are thus looking for a non-negative solution to the equation
\[
\begin{pmatrix}
(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4) & 5 \\
(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2) & 10
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
5(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4) \\
10(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2)
\end{pmatrix}.
\]

The determinant is positive since the left hand column is decreasing and the right hand column increasing. Up to a positive constant \( c \) we thus have
\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
= c \begin{pmatrix}
10 \\
-5
\end{pmatrix}
\begin{pmatrix}
(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4) \\
(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2)
\end{pmatrix}
\begin{pmatrix}
5(\lambda^4(1-\lambda)+\lambda(1-\lambda)^4) \\
10(\lambda^2(1-\lambda)^3+\lambda^3(1-\lambda)^2)
\end{pmatrix}.
\]

Multiplying we see that the solution is non-negative. (We use that \((\lambda^j(1-\lambda)^n-j+\lambda^{n-j}(1-\lambda)^j)\) is decreasing in \( j \in \{0, 1, \ldots, n/2\} \), an easy fact to check.)

We end this note with a simple geometric proof of the following inequality from [2] (which reappeared independently in [1])

**Theorem 9.** Let \( K, L \subset \mathbb{R}^n \) be convex bodies which include the origin. Then
\[
\text{Vol}(\text{conv}(K \cup -L)) \text{Vol}((K^\circ + L^\circ)^\circ) \leq \text{Vol}(K) \text{Vol}(L).
\]

We remark that this inequality can be thought of as a dual to the Milman-Pajor inequality [6] stating that when \( K \) and \( L \) have center of mass at the origin one has
\[
\text{Vol}(\text{conv}(K \cap -L)) \text{Vol}(K + L) \geq \text{Vol}(K) \text{Vol}(L).
\]

**Simple geometric proof of Theorem 2.** Consider two convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) and build the body in \( \mathbb{R}^{2n} \) which is
\[
C = \text{conv}(K \times \{0\} \cup \{0\} \times L)
\]

The volume of \( C \) is simply
\[
\text{Vol}(C) = \text{Vol}(K)\text{Vol}(L)\frac{1}{(2n)^n}.
\]

Let us look at the two orthogonal subspaces of \( \mathbb{R}^{2n} \) of dimension \( n \) given by \( E = \{(x, x) : x \in \mathbb{R}^n\} \) and \( E^\perp = \{(y, -y) : y \in \mathbb{R}^n\} \). First we compute \( C \cap E \):
\[
C \cap E = \{(x, x) : x = \lambda y, x = (1-\lambda)z, \lambda \in [0, 1], y \in K, z \in L\}.
\]

In other words,
\[
C \cap E = \{(x, x) : x \in \bigcup_{\lambda \in [0, 1]} (\lambda K \cap (1-\lambda)L)\} = \{(x, x) : x \in (K^\circ + L^\circ)^\circ\}.
\]

Next let us calculate the projection of \( C \) onto \( E^\perp \). Since \( C \) is a convex hull, we may project \( K \times \{0\} \) and \( \{0\} \times L \) onto \( E^\perp \) and then take a convex hull. In other words we are searching for all \((x, -x)\) such that there exists \((y, y)\) with \((x+y, -x+y)\) in \( K \times \{0\} \) or \( \{0\} \times L \). Clearly this means that \( y \) is either \( x \), in the first case, or \( -x \), in the second, which means we get
\[
P_{E^\perp} C = \text{conv}\{(x, -x) : 2x \in K \text{ or } -2x \in L\} = \{(x, -x) : x \in \text{conv}(K/2 \cup -L/2)\}.
\]

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In terms of volume we get that
\[ \text{Vol}_n(C \cap E) = \sqrt{2^n} \text{Vol}_n((K^\circ + L^\circ)^\circ) \]
and
\[ \text{Vol}_n(P_{E^\perp}(C)) = \sqrt{2^{-n}} \text{Vol}_n(\text{conv}(K \cup -L)) \]
and so their product is precisely the quantity in the right hand side of Theorem\[9\] and by the Rogers Shephard inequality for sections and projections, Lemma\[6\] we know that
\[ \text{Vol}_n(C \cap E)\text{Vol}_n(P_{E^\perp}(C)) \leq \text{Vol}_n(C) \left( \frac{2^n}{n} \right). \]
Plugging in the volume of \( C \), we get our inequality from Theorem\[9\].

**Remark 10.** Note that taking, for example, \( K = L \) in the last construction, but taking \( E_{\lambda} = \{(\lambda x, (1-\lambda)x) : x \in \mathbb{R}^n\} \), we get that
\[ C \cap E = \{(\lambda x, (1-\lambda)x) : x \in K\} \]
and
\[ P_{E_{\lambda}}C = \{( (1-\lambda)x, -\lambda x) : x \in \frac{1}{\lambda^2 + (1-\lambda)^2} \text{conv}((1-\lambda)K \cup \lambda K)\}. \]
In particular, the product of their volumes, which is simply
\[ \text{Vol}(\text{conv}((1-\lambda)K \cup \lambda K)) \text{Vol}(K) \]
is bounded by \( \left( \frac{2^n}{n} \right) \text{Vol}(C) \) which is itself \( \text{Vol}(K) \), giving yet another proof of the following inequality from \[3\], valid for a convex body \( K \) such that \( 0 \in K \)
\[ \text{conv}((1-\lambda)K \cup \lambda K) \leq \text{Vol}(K), \]
and more importantly a realization of all these sets as projections of a certain body.

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Shiri Artstein-Avidan School of Mathematical Science, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel.

Email address: shiri@post.tau.ac.il