On the First $\text{aff}(1)$-Relative Cohomology of the Lie Algebra of Vector Fields and Differential Operators

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Abstract

Let $\text{Vect}(\mathbb{R}^1)$ be the Lie algebra of smooth vector fields on $\mathbb{R}^1$. In this paper, we classify $\text{aff}(1)$-invariant linear differential operators from $\text{Vect}(\mathbb{R}^1)$ to $\mathcal{D}_{\text{aff}}(\mathbb{R}^1)$ vanishing on $\text{aff}(1)$, where $\mathcal{D}_{\text{aff}} := \text{Hom}(\text{aff}(1), \text{Vect}(\mathbb{R}^1))$ is the space of bilinear differential operators acting on weighted densities. This result allows us to compute the first differential $\text{aff}(1)$-relative cohomology of $\text{Vect}(\mathbb{R}^1)$ with coefficients in $\mathcal{D}_{\text{aff}}$.

Keywords: Differential operators; Transvectants; Lie algebra; Cohomology

Introduction

Let $g$ be a Lie algebra and let $\mathcal{M}$ and $\mathcal{N}$ be two $g$-modules. It is well-known that nontrivial extensions of $g$-modules:

$$0 \to \mathcal{M} \to \mathcal{N} \to 0$$

are classified by the first cohomology group $H^1(g; \text{Hom}(\mathcal{M}, \mathcal{N}))$ [1]. Any 1-cocycle $c$ generates a new action on $\mathcal{M}$ as follows: for all $m \in \mathcal{M}$,

$$c \cdot m = [c, m] = [\text{ad}_c(m), m],$$

where $\text{ad}_c(m)$ is the Lie algebra of smooth vector fields on $\mathbb{R}^1$ endowed with the defined $\text{aff}(1)$-module structure.

The space of bilinear differential operators as a $\text{Vect}(\mathbb{R}^1)$-module

Consider bilinear differential operators that act on tensor densities:

$$(\varphi \otimes \psi)(f) = \varphi(f) \cdot \psi(f),$$

for all $f \in \mathbb{R}^1$. We denote $D_{\text{aff}}^{\text{aff}}$ the space of bilinear differential operators (2) endowed with the defined $\text{Vect}(\mathbb{R}^1)$-module structure (3).

Relative Cohomology

Let us first recall some fundamental concepts from cohomology theory [1]. Let $g$ be a Lie algebra acting on a vector space $V$ and let $h$ be a sub-algebra of $g$. (If $h$ is omitted it assumed to be $[0]$.) The space of $h$-relative $n$-cochains of $g$ with values in $V$ is the $g$-module $C^n(h; V) = \text{Hom}_h(A^n(g; h); V)$.

The coboundary operator $\delta: C^n(h; V) \to C^{n+1}(h; V)$ is a $g$-map satisfying $\delta \delta = 0$. The kernel of $\delta_{n-1}$, denoted $Z^n(h; V)$, is the space of $h$-relative $n$-cocycles, among them, the elements in the range of $\delta_{n-1}$ are called $h$-relative $n$-coboundaries. We denote $B^n(h; V)$ the space of $n$-coboundaries.

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By definition, the $n$th $h$-relative cohomology space is the quotient space
\[
H^n(g,h;\mathcal{V}) = \mathcal{Z}^n(g,h;\mathcal{V})/\mathcal{B}^n(g,h;\mathcal{V}).
\]
We will only need the formula of $\delta_i$ (which will be simply denoted $\delta$) in degrees 0 and 1: for $v \in \mathcal{C}(g,h;\mathcal{V}) = \mathcal{V}^n$, $\delta(v) = (-1)^{i+j}gv$, where $v^n = \{v \in \mathcal{V} | h.v = 0 \}$ for all $h \in h$,

and for $Y \in \mathcal{C}(g,h;\mathcal{V})$,

\[
\delta(Y) = Y - Y(Y(X)-Y([x,y])) \quad \text{for any } x,y \in g.
\]

### $\mathfrak{aff}(1)$-Invariant Differential Operators

The following steps to compute the relative cohomology has intensively been used in refs. [3,5-8]. First, we classify $\mathfrak{aff}(1)$-invariant differential operators, then we isolate among them those that are 1-cocycles. To do that, we need the following Lemma.

**Lemma 4.1**

Any 1-cocycle vanishing on the subalgebra $\mathfrak{aff}(1)$ of $\text{Vect}(\mathbb{R}^1)$ is $\mathfrak{aff}(1)$-invariant.

The 1-cocycle condition of $Y$ reads:

\[
X(Y(Y(X))-Y(Y([X,Y]))) = 0,
\]

where $X,Y \in \text{Vect}(\mathbb{R}^1)$. Thus, if $Y(X)=0$ for all $X \in \mathfrak{aff}(1)$, eqn. (4) becomes

\[
Y([X,Y]) = X(Y(Y(X))-Y(Y([X,Y])) = 0,
\]

expressing the $\mathfrak{aff}(1)$-invariance property of $Y$.

As our 1-cocycles vanish on $\mathfrak{aff}(1)$, we will investigate $\mathfrak{aff}(1)$-invariant linear differential operators that vanish on $\mathfrak{aff}(1)$.

**Proposition 4.2:** There exist $\mathfrak{aff}(1)$-invariant bilinear differential operators $J^{i\times} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ given by:

\[
J^{i\times}(gdx^i, \phi dx^\mu) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $k \in \mathbb{N}$ and the coefficients $\gamma_{ij}$ are constants.

**Proof.** Any differential operator $J^{i\times} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ is of the form

\[
J^{i\times}(gdx^i, \phi dx^\mu) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $\gamma_{ij}$ are constants. The $\mathfrak{aff}(1)$-invariant property of the operators $J^{i\times}$ with respect to the vector field $X = \frac{d}{dx}$ implies that $\gamma_{ij} = 0$. On the other hand, the invariant property with respect to the vector fields $X = \frac{d}{dx}$ implies that $\gamma_{ij} = 0$. Therefore, the space of solutions is $\frac{1}{2}(k-1)$-dimensional, spanned by

\[
Y_{0,0,0}, Y_{1,0,0}, Y_{2,1,0}, Y_{3,2,0}, Y_{4,3,0}, Y_{5,4,0}, \ldots
\]

where $Y_{r,\lambda,\mu}$ are functions. The $\mathfrak{aff}(1)$-invariant property of the operators $J^{i\times}$ reads as follows.

**Proposition 4.3:** There exist $\mathfrak{aff}(1)$-invariant differential operators $K^{i\times \phi} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ that vanish on $\mathfrak{aff}(1)$ given by:

\[
K^{i\times \phi}(X, \phi, \psi) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $i+j+k=0$ and the coefficients $\gamma_{ij}$ are constants. Moreover, the space of solutions is $\frac{1}{2}(k-1)$-dimensional, for all $\lambda$ and $\mu$.

**Proof of Proposition 4.3 and 4.4:** We are going to prove Proposition 4.3 and 4.4 simultaneously. Any differential operator $K^{i\times \phi} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ is of the form

\[
K^{i\times \phi}(X, \phi, \psi) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $\gamma_{ij}$ are functions. The $\mathfrak{aff}(1)$-invariant property of the operators $K^{i\times \phi}$ reads as follows.

\[
L^\gamma K^{i\times \phi} - K^{i\times \phi} L^\gamma \psi = K^{i\times \phi} L^\gamma \psi.
\]

The invariant property with respect to the vector field $X = \frac{d}{dx}$ implies that $\gamma_{ij} = 0$. On the other hand, the invariant property with respect to the vector fields $X = \frac{d}{dx}$ implies that $\gamma_{ij} = 0$. Therefore, the space of solutions is $\frac{1}{2}(k-1)$-dimensional, spanned by

\[
Y_{0,0,0}, Y_{1,0,0}, Y_{2,1,0}, Y_{3,2,0}, Y_{4,3,0}, Y_{5,4,0}, \ldots
\]

Now, the proof of Proposition 4.4 follows as above by putting $r=1$.

In this case, the space of solutions is $\frac{1}{2}(k-1)$-dimensional, spanned by

\[
Y_{2,0,0}, Y_{3,0,0}, Y_{4,0,0}, \ldots
\]

Therefore, the corresponding operator can be expressed as (5).

**Proposition 4.3:** There exist $\mathfrak{aff}(1)$-invariant differential operators $K^{i\times \phi} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ given by:

\[
K^{i\times \phi}(X, \phi, \psi) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $i+j+k=0$ and the coefficients $\gamma_{ij}$ are constants.

If $r, \lambda, \mu$ are generic, then the space of solutions is $\frac{1}{2}(k-1)(k+2)$-dimensional.

**Proposition 4.4:** There exist $\mathfrak{aff}(1)$-invariant differential operators $K^{i\times \phi} : \mathcal{F}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_{i+j+k}$ that vanish on $\mathfrak{aff}(1)$ given by:

\[
K^{i\times \phi}(X, \phi, \psi) = \sum_{(i+j+k)} \gamma_{ij} \phi \psi dx^{i+j+k},
\]

where $i+j+k=0$ and the coefficients $\gamma_{ij}$ are constants.

If $r, \lambda, \mu$ are generic, then the space of solutions is $\frac{1}{2}(k-1)(k+2)$-dimensional.
The proof of Theorem 5.1 follows three steps:

1. Constructing a 1-cocycle on $Vect(\mathbb{R}^p)$.

2. Investigating the dimension of the space of operators that satisfy the cocycle condition.

3. Studying the triviality of the constructed 1-cocycle.

Proof of Theorem 5.1:

To construct a 1-cocycle on $Vect(\mathbb{R}^p)$, we define the following operators $L_{XB}$ for $X, Y \in Vect(\mathbb{R}^p)$:

$$L_{XB}(X, Y) = [X, Y] + \lambda(X, Y) + \mu(X, Y)$$

where $\lambda$ and $\mu$ are generic operators.

Now, we investigate the dimension of the space of operators that satisfy the 1-cocycle condition.

Proof of Lemma 5.2:

For $\lambda, \mu \in \mathbb{R}$, we define the following operators $L_{XB}$ for $X, Y \in Vect(\mathbb{R}^p)$:

$$L_{XB}(X, Y) = [X, Y] + \lambda(X, Y) + \mu(X, Y)$$

where $\lambda, \mu \geq 0$.

We also need the following Lemma.

Lemma 5.3:

Every 1-cocycle on $Vect(\mathbb{R}^p)$ with values in $D_{\lambda, \mu}$ is differentiable.

Proof [7]:

Now we are in position to prove Theorem 5.1. By Lemma 5.3, any 1-cocycle on $Vect(\mathbb{R}^p)$ should retain the following general form:

$$C(X, \phi, \psi) = \sum_{i, j, k} c_{ij, k} X^{(i)} \phi^{(j)} \psi^{(k)}$$

where $c_{ij, k}$ are constants.

The 1-cocycle condition reads as follows: for all $\phi \in \mathcal{F}_\lambda$ and all $\psi \in \mathcal{F}_\mu$ and for all $X \in Vect(\mathbb{R}^p)$, one has

$$c(\mathcal{X}, Y, \phi, \psi) - L_{XB}(Y, \phi, \psi) - L_{XB}^\mu(B(X, \phi, \psi)) = 0$$

The 1-cocycle condition can also be expressed as follows:

$$Y(X, \phi, \psi) = c_{\lambda, 0, 0} X^0 \phi \psi$$

By a direct computation, we can see that the 1-cocycle condition is always satisfied. Let us study the triviality of this 1-cocycle. A direct computation proves that

$$H_{\lambda, \mu} = 0$$
\[ L_x J^{+\pm} = \beta_{x;3,0} X^\nu \phi^\nu = - (\lambda \gamma_{1,0} + \mu \gamma_{0,1}) X^\nu \phi^\nu. \]

So, for \((\lambda, \mu) = (0,0)\), the coefficient \(c_{2;3,0}\) cannot be eliminated by adding a coboundary. Hence, the cohomology space is one-dimensional. While for \((\lambda, \mu) = (0,0)\), we can see that the coefficient \(c_{2,0} \) can be eliminated because \(\beta_{2,0} = 0\). Hence, the cohomology space is zero-dimensional.

**The case where \(v - \mu = -2\):** In this case, according to Proposition 4.4, the 1-cocycle (24) can be expressed as follows:

\[ Y(x, \phi^\nu) = c_{1,0} X^\nu \phi^\nu + c_{2,0} X^\nu \phi^\nu + c_{3,0} X^\nu \phi^\nu. \]

By a direct computation, we can see that the cohomology space is always satisfied. Let us study the triviality of this 1-cocycle. A direct computation proves that

\[ L_x J^{+\pm} = \beta_{x;3,0} X^\nu \phi^\nu + \beta_{y,3,0} X^\nu \phi^\nu + \beta_{z,3,0} \phi^\nu. \]

where

\[ \beta_{3,0} = - \lambda \gamma_{1,0} - \mu \gamma_{0,1} ; \beta_{y,3,0} = - (\lambda + 1) \gamma_{1,0} - \mu \gamma_{0,1} ; \beta_{z,3,0} = - \lambda \gamma_{0,1} - (\mu + 1) \gamma_{0,1}. \]

So, for \((\lambda, \mu) = (0,0)\), the cohomology space is one-dimensional, since only one of the coefficients \(c_{3,0} \) or \(c_{2,0} \) cannot be eliminated by adding a coboundary. While for \((\lambda, \mu) = (0,0)\), the coefficient \(c_{2,0} \) and \(c_{3,0} \) can be eliminated because \(\beta_{2,0} = 0\) and \(\beta_{3,0} \) are nonzero. Hence, the cohomology space is zero-dimensional.

**The case where \(v - \mu = 3\):** In this case, according to Proposition 4.4, the space of solutions is spanned by:

\[ c_{3,0} \phi^\nu \text{ or } c_{2,0} \phi^\nu. \]

Moreover, by formula (25), we readily obtain:

\[ -2c_{2,0}, (3 \lambda + 1) c_{2,0} - 2(\lambda + 1) c_{2,0} + \mu c_{2,0} - \mu c_{1,0} = 0, \]

\[ -2c_{2,0}, \lambda c_{2,0}, - c_{1,0}, c_{1,0} + (3 \mu + 1) c_{1,0} - (3 \mu + 1) c_{1,0} = 0, \]

\[ -c_{2,0}, c_{2,0} - 2c_{3,0}, c_{3,0} - \mu c_{1,0} - \mu c_{2,0} = 0. \]

Thus, we have just proved that the coefficients of every 1-cocycle is expressed in terms of \(c_{2,0} \text{ or } c_{3,0}\).

A direct computation confirms that, the coefficients of \(L_x J^{+\pm} \) are expressed in terms of:

\[ c_{2,0} = - (3 \lambda + 1) c_{2,0} - \mu c_{2,0} , \]

\[ \beta_{3,0} = - (3 \lambda + 1) c_{3,0} - \mu c_{3,0} , \]

\[ \beta_{2,0} = - (3 \lambda + 1) c_{2,0} - \mu c_{2,0}. \]

So, for \((\lambda, \mu) = (0,0)\), the cohomology space is one-dimensional, since only one of the coefficients \(c_{2,0} \) or \(c_{3,0} \) cannot be eliminated by adding a coboundary.

While for \((\lambda, \mu) = (0,0)\), the coefficient \(c_{2,0} \) and \(c_{3,0} \) can be eliminated because \(\beta_{2,0} = 0\) and \(\beta_{3,0} \) are nonzero. Hence, the cohomology space is zero-dimensional.

**The case where \(v - \mu = 4\):** In this case, according to Proposition 4.4, the space of solutions is spanned by:

\[ c_{2,0} \phi^\nu \text{ or } c_{3,0} \phi^\nu. \]

Moreover, by formula (25), we readily obtain:

\[ -2c_{2,0}, (3 \lambda + 2) c_{2,0} - 2(\lambda + 1) c_{2,0} + 2 \mu c_{2,0} - \mu c_{1,0} = 0, \]

\[ -2c_{2,0}, \lambda c_{2,0}, - c_{1,0}, c_{1,0} + (3 \mu + 1) c_{1,0} - (3 \mu + 1) c_{1,0} = 0, \]

\[ - c_{2,0}, c_{2,0} - 2c_{3,0}, c_{3,0} - \mu c_{1,0} - \mu c_{2,0} = 0. \]

Thus, we have just proved that the coefficients of every 1-cocycle is expressed in terms of \(c_{2,0} \text{ or } c_{3,0}\).

A direct computation confirms that, the coefficients of \(L_x J^{+\pm} \) are expressed in terms of:

\[ c_{2,0} = - (3 \lambda + 1) c_{2,0} - \mu c_{2,0} , \]

\[ \beta_{3,0} = - (3 \lambda + 1) c_{3,0} - \mu c_{3,0} , \]

\[ \beta_{2,0} = - (3 \lambda + 1) c_{2,0} - \mu c_{2,0}. \]

So, for \((\lambda, \mu) = (0,0)\), the cohomology space is one-dimensional, since only one of the coefficients \(c_{2,0} \) or \(c_{3,0} \) cannot be eliminated by adding a coboundary.
Thus, we have just proved that the coefficients of every 1-cocycle is expressed in terms of:

\[ c_{2,4,0}, c_{2,0,4}, c_{3,3,0}, c_{3,0,3}, c_{3,2,1}, c_{2,3,1}, c_{3,2,2}, c_{2,2,2}. \]

A direct computation confirms that, the coefficients of \( L_a J_1^{a\mu} \) are expressed in terms of:

\[ \beta_{1,1,1} = -(3\lambda + 2\mu)\gamma_{1,1,1}, \quad \beta_{1,2,1} = -(3\lambda + 1)\gamma_{1,2,1}, \quad \beta_{1,3,1} = -(3\lambda + 0)\gamma_{1,3,1}, \quad \beta_{1,4,1} = -(3\lambda - 2)\gamma_{1,4,1}. \]

In this case, according to Proposition 4.4 together with formulas (25), we check that the coefficients of every 1-cocycle are expressed in terms of:

\[ c_{1,0,0,0}, c_{1,0,1}, c_{2,0,2}, c_{2,0,4}, c_{2,0,6}, c_{3,0,3}, c_{3,0,5}, c_{3,0,7}, c_{2,0,8}. \]

A direct computation confirms that, the coefficients of \( L_a J_0^{a\mu} \) expressed in terms of:

\[ \beta_{1,0,0}, \beta_{1,0,1}, \beta_{1,0,2}, \beta_{1,0,3}, \beta_{1,0,4}, \beta_{1,0,5}, \beta_{1,0,6}, \beta_{1,0,7}, \beta_{1,0,8}. \]

So, in the same way as before, by Lemma 5.2, we can see, with the help of the maple, that the cohomology space is given as in (20).

**The case where \( v - \mu = 6 \):** In this case, according to Proposition 4.4 together with formulas (25), we check that the coefficients of every 1-cocycle are expressed in terms of:

\[ c_{1,1,0,1}, c_{1,2,0,1}, c_{1,3,1,1}, c_{2,0,2,0}, c_{2,0,3,0}, c_{3,0,3,0}, c_{3,0,4,0}, c_{3,0,5,0}, c_{2,0,6,0}. \]

A direct computation confirms that, the coefficients of \( L_a J_0^{a\mu} \) expressed in terms of:

\[ \beta_{1,1,0,1}, \beta_{1,2,0,1}, \beta_{1,3,1,1}, \beta_{1,4,2,1}, \beta_{1,5,3,1}, \beta_{1,6,4,1}, \beta_{1,7,5,1}, \beta_{1,8,6,1} \]

So, in the same way as before, by Lemma 5.2, we can see, with the help of the maple, that the cohomology space is given as in (21).

**The case where \( v - \mu = 8 \):** In this case, according to Proposition 4.4 together with formulas (25), we check that the coefficients of every 1-cocycle are expressed in terms of:

\[ c_{1,0,0,0}, c_{1,0,1}, c_{2,0,2}, c_{2,0,4}, c_{2,0,6}, c_{3,0,3}, c_{3,0,5}, c_{3,0,7}, c_{2,0,8}. \]

A direct computation confirms that, the coefficients of \( L_a J_0^{a\mu} \) expressed in terms of:

\[ \beta_{1,0,0}, \beta_{1,0,1}, \beta_{1,0,2}, \beta_{1,0,3}, \beta_{1,0,4}, \beta_{1,0,5}, \beta_{1,0,6}, \beta_{1,0,7}, \beta_{1,0,8}. \]

So, in the same way as before, by Lemma 5.2, we can see, with the help of the maple, that the cohomology space is given as in (22). This completes the proof.

**Conjecture 5.1**

For \( v - \mu \in \mathbb{N} + 12, \) \( \mu \) and \( \lambda \) are generic, one has

\[ H_{\text{aff}}^1(\text{Vect}(\mathbb{R}^p), \mathfrak{a}(1); D_{\lambda,\mu}) = 0. \]

**Conclusion**

In this paper, we classify \( \mathfrak{a}(1) \) -invariant linear differential operators from \( \text{Vect}(\mathbb{R}^p) \) to \( D_{\lambda,\mu} \) vanishing on \( \mathfrak{a}(1) \), where \( D_{\lambda,\mu} := \text{Homdiff}(\mathcal{F}_1, \mathcal{F}_2) \) is the space of bilinear differential operators acting on weighted densities. This result allows us to compute the first differential \( \mathfrak{a}(1) \)-relative cohomology of \( \text{Vect}(\mathbb{R}^p) \) with coefficients in \( D_{\lambda,\mu} \).

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