Robust Iterative Interference Alignment for Cellular Networks with Limited Feedback

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Abstract

In theory coordinated multi-point transmission (CoMP) promises vast gains in spectral efficiency. But industrial field trials show rather disappointing throughput gains, whereby the major limiting factor is proper sharing of channel state information. Many recent papers consider this so-called limited feedback problem in the context of CoMP. Usually taking the assumptions: 1) infinite SNR regime, 2) no user selection and 3) ideal link adaptation; rendering the analysis too optimistic. In this paper we make a step forward towards a more realistic assessment of the limited feedback problem by introducing an improved metric for the performance evaluation which better captures the throughput degradation. We find the relevant scaling laws (lower and upper bounds) and show that they are different from existing ones. Moreover, we provide a robust iterative interference alignment algorithm and corresponding feedback strategies achieving the obtained scaling laws. The main idea is that instead of sending the complete channel matrix each user fixes a receive filter and feeds back a quantized version of the effective channel. Finally we underline our findings with simulations for the proposed system.

I. INTRODUCTION

Coordinated processing (or so-called coordinated multi-point transmission (CoMP)) of signals by multiple network nodes is a key design element in LTE-A (and beyond 4G) cellular networks: CoMP algorithms can range from: 1) joint transmission (fully coherent with message sharing), 2) coordinated beamforming (without message sharing), to 3) interference coordination (by exchanging e.g. simple interference indicators). A classical summary of coordination techniques in multi-cell MIMO cooperative networks can be found in [1], [2]. A prominent coordinated beamforming technique is interference alignment (IA) [3] which essentially aligns the signal space so that multiple interferer appear as a single one.

In theory coherent transmission from multiple base stations to multiple users promises vast gains in spectral efficiency [1], [2]. But, industrial field trials show rather disappointing throughput gains, whereby the major limiting factor is proper sharing of channel state information (CSI) and other overhead among cells [4]. Many papers consider the so-called limited feedback problem. For example, [5] and [6] considered multiuser MIMO systems and network MIMO systems, respectively. Reference [7], [8] considered IA for the interference channel. Recently, [9] considered IA for the interfering MAC. All with a focus on the infinite SNR regime carrying out a system degrees of freedom (DoF) analysis.

However, even though analytic treatment of the limited feedback problem has made significant
progress in the past, the DoF approach cannot really account for the throughput degradation experienced in practice. The main reasons are: 1) The infinite SNR regime where achieving DoF is optimal is considered. In this regime interference mitigation instead of signal enhancement is the primary goal. 2) No user selection is considered, i.e., it is assumed that the optimal scheduling decision is known. 3) Ideal link adaptation is assumed. Altogether, this renders the performance analysis too optimistic and motivates extended analysis of the limited feedback problem.

Now, the question is: How can we get reliable estimates of the performance degradation due to limited feedback. In this paper we take a step forward towards a more realistic answer to this question. Our approach is universal in the sense that we do not consider a specific transmit strategy. By considering the interfering broadcast channel [10] our results also hold for the interference channel and the broadcast channel, which are special cases of the interfering broadcast channel. In particular we:

- introduce an improved metric for the performance evaluation which better captures the throughput degradation due to limited feedback in practice. The metric is defined per user instead of sum rate.
- calculate the rate degradation for any scheduling decision, any beamforming strategy, and any SNR regime which is a useful performance benchmark for the design of systems.
- derive a lower bound on the throughput degradation for IA; replacing the too optimistic scaling laws for the number of feedback bits in the conventional analysis. We prove that the feedback scaling is $2^{-\frac{\mu}{2(n_t-1)}}$ instead of $2^{-\frac{\mu}{n_t-1}}$ in most of the previous work.
- introduce a robust iterative IA algorithm with user selection which achieves the optimal scaling under any SNR regime. The main idea is that instead of sending the complete channel matrix each user fixes a receive filter and feeds back a quantized version of the effective channel.
- show that the proposed distributed approach is favorable over centralized approaches in terms of performance, convergence speed and computational complexity.

We like to disclose that a summary of the results was presented in the workshop paper [11]. In contrast to [11] the paper at hand includes all proofs in detail. Moreover, we develop new approaches like a partial reverse of Jensen’s inequality (Lemma 4).

**Notation:** The inner product of $x \in \mathbb{C}^N$ and $y \in \mathbb{C}^N$ is $\langle x, y \rangle = x^H y$, where $x^H$ is the conjugate transpose of vector $x$. The vector $p$-norm is defined as $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$. The unit
sphere in $\mathbb{C}^N$ is defined as $S^{N-1}$. The expected value of a random variable $X$ is $\mathbb{E}[X]$. 

II. System Setup

A. System Model

Consider the downlink of a cellular network with $K$ base stations, each equipped with $n_t$ transmit antennas, and $U$ user equipments, each equipped with $n_r$ receive antennas. Throughout the paper we consider an arbitrary but fixed spectral resource element. On this resource element the channel between base station $b$ and user $m$ is modeled by the matrix $H_{m,b} \in \mathbb{C}^{n_r \times n_t}$ which is constant over one transmission frame and distributed complex Gaussian with zero mean and unit variance. In each transmission frame (time index omitted) all base stations $b = 1, \ldots, K$ select disjoint subsets of users $S_b \subseteq U = \{1, \ldots, U\}$ and transmit the signal $x_b \in \mathbb{C}^{n_t}$. The signal received by user $m \in S_b$ is given by

$$y_m = \sum_{l=1}^{K} \langle u_m, H_{m,l}x_l \rangle + \langle u_m, n_m \rangle,$$

where $u_m \in S^{n_r-1}$ is the receive filter and $n_m \sim \mathcal{CN}(0, I)$ is additive white Gaussian noise. The set of all scheduled users is defined as the set $S := S_1 \cup S_2 \cup \ldots \cup S_K$ and the beamforming vectors are given by the function

$$\pi : S \rightarrow S^{n_t-1}. \quad (2)$$

Assume that the complex information symbols can be modeled as complex Gaussian with zero mean and unit variance, $d_m \sim \mathcal{CN}(0, 1)$, then the signal transmitted by base station $b$

$$x_b = \sqrt{\frac{P}{|S_b|}} \sum_{m \in S_b} \pi(m)d_m, \quad (3)$$

fulfills the average power constrained $\mathbb{E}[\|x_b\|^2] = P$, for all $b$. We assume that each base station distributes its available power $P_b = P$ equally among all users $m \in S_b$.

Throughout the paper we make the assumption that all users $m \in U$ have perfect knowledge of their own channels $H_{m,l}$, for $l = 1, 2, \ldots, K$, and we assume no delay in reporting the CSI, the process of scheduling and the transmission.
B. Scheduling and Feedback Model

A scheduling decision consists of two steps: i) selection of users \( S_b \subseteq U \) and ii) computation of beamforming vectors \( \pi : S \rightarrow \mathbb{S}^{n_t-1} \). For a given scheduling decision \((\pi, S)\) the achievable system sum rate is given by

\[
R(\pi, S; H) = \sum_{l=1}^{K} \sum_{m \in S_l} r_m(\pi, S; H),
\]

where \( H = \{H_{m,l} : l = 1, 2, \ldots, K; m = 1, 2, \ldots, U\} \) is the list of all channels. The achievable rate of user \( m \in S_b \) is given by the Shannon rate

\[
r_m(\pi, S; H) = \max_{u \in \mathbb{S}^{n_t-1}} \log \left( 1 + \frac{p \|u, H_{m,b} \pi(m)\|^2}{1 + \sum_{l=1}^{K} \sum_{k \in S_l, k \neq m} \frac{p}{|S_l|} \|u, H_{m,l} \pi(k)\|^2} \right),
\]

where the receive filters can be optimized independently by each user; the receive filter of user \( m \) will be denoted by \( u_m \). In the sequel, we assume that the base stations aim at maximizing the system sum-rate. Thus, if all base stations have knowledge of all channels \( H \), the optimal scheduling decision \((\pi_H, S_H)\) is the solution to the optimization problem

\[
\max_{S \subseteq U} \max_{\pi : S \rightarrow \mathbb{S}^{n_t-1}} R(\pi, S; H).
\]

Therefore, the optimal system sum rate is \( R(\pi_H, S_H; H) \).

In the following we assume that the base stations collect quantized CSI through a rate-constrained feedback channel. For fixed receive filters \( u_k \) each user \( k \in U \) quantizes and feeds back the effective channels \( \hat{h}_{k,l} := (H_{k,l})^H u_k \) to all base stations \( l = 1, \ldots, K \). In particular user \( k \in U \) uses random vector quantization (RVQ) on the normalized effective channels

\[
\hat{h}_{k,l} := \frac{\hat{h}_{k,l}}{\|\hat{h}_{k,l}\|_2}, \quad \forall l \in [1, K].
\]

The normalized effective channels are quantized using a random codebook \( \mathcal{V}_k \subseteq \mathbb{S}^{n_t-1} \), with \( 2^B \) isotropically distributed elements. Each user uses an independent copy of the random codebook which ensures that the feedback messages from different users are linearly independent, almost surely. Each user \( k \) feeds back the indices of the elements

\[
\nu_{k,l} := \arg \min_{v \in \mathcal{V}_k} \left( 1 - |\langle h_{k,l}, v \rangle|^2 \right), \quad \forall l \in [1, K],
\]

\footnote{For simplicity of notation, we assume that with \( S \) also the information about the cardinality of the partial sets \( S_1, \ldots, S_K \) is delivered.}
to all base stations. Here, \(1 - |\langle h_{k,l}, v \rangle|^2\) is the squared chordal distance and \(\min_{v \in V_k} (1 - |\langle h_{k,l}, v \rangle|^2)\) is the quantization error. Later on in Section IV-C we will show that the chordal distance is a reasonable and robust quantization metric for the considered systems. Equivalently, the quantization problem can be formulated on the complex Grassmann manifold \(G(n_t, 1)\) which is the set of all one dimensional subspaces of \(\mathbb{C}^{n_t}\) (see e.g. [12] for further details). To simplify our analysis we assume that the channel norm \(\mu_{k,l} := \|\hat{h}_{k,l}\|_2\) is perfectly known to all base stations.

After receiving the feedback messages from all users \(k \in \mathcal{U}\), each base station \(l = 1, \ldots, K\), has knowledge of the quantized effective channels

\[
V := \{\hat{v}_{k,l} = \mu_{k,l}v_{k,l} : k \in \mathcal{U}, l \in [1, K]\}.
\]

Based on quantized CSI \(V\), the scheduling decision \((\pi_V, S_V)\) is found by solving the problem

\[
\max_{S \subseteq \mathcal{U}} \max_{\pi : S \to \mathbb{S}^{n_t-1}} R(\pi, S; V),
\]

instead of problem (6).

Remark 1. In general the scheduling decisions with quantized and perfect CSI are not equal, \((\pi_V, S_V) \neq (\pi_H, S_H)\). Therefore, the achievable sum rate with quantized CSI is smaller or equal the achievable sum rate with perfect CSI, \(R(\pi_V, S_V; H) \leq R(\pi_H, S_H; H)\).

The core of this paper is to explore the performance degradation under the scheduling decisions based on quantized CSI \(V\) (Section III) under suitable algorithms for the optimization problems (6) and (10) which we discuss in Section IV.

III. RATE LOSS GAP ANALYSIS

A. Known Results

In the literature usually the rate gap \(r_m(\pi_H, \mathcal{U}, H) - r_m(\pi_V, \mathcal{U}, H)\) is analyzed. That is, the set of active users is fixed and perfect link adaptation is assumed. Moreover, most papers consider a specific system setup (e.g. the broadcast channel or the \(K\)-user interference channel) and a specific beamforming strategy (e.g. zero forcing beamforming or IA). Based on these assumptions, the influence of quantized CSI on the sum rates or user rates is analyzed. Let us shortly summarize some of the results.
In [5], [13] a broadcast channel with $|\mathcal{U}| = n_t$ single antenna users and zero forcing beamforming is assumed. Let us denote $\pi_{ZF,H}$ and $\pi_{ZF,V}$ as the zero forcing beamforming solutions with perfect and quantized CSI, respectively. According to [13], limited feedback with $B$ feedback bits per user incurs a throughput loss relative to zero forcing with perfect CSI bounded by

$$\mathbb{E}[r_m(\pi_{ZF,H}, \mathcal{U}, H) - r_m(\pi_{ZF,V}, \mathcal{U}, H)] < \log(1 + P2^{-\frac{B}{n_t-1}}).$$

(11)

This result has been recently generalized in [6] for network MIMO.

For IA, a limited feedback scheme for the $K$-user interference channel is proposed in [14]. Quantization is based on Grassmannian representation of the channel matrices. The throughput loss due to the channel quantization scales like

$$\mathbb{E}[r_m(\pi_{IA,H}, \mathcal{U}, H) - r_m(\pi_{IA,V}, \mathcal{U}, H)] < \log \left(1 + P2^{-\frac{B}{Ng+1}}\right),$$

(12)

where $Ng = 2n_r((K-1)n_t - n_r)$ is the real dimension of the Grassmannian manifold. In [15] similar results have been obtained for a system using OFDM. An in depth treatment of the scaling law analysis in Grassmannian manifolds can be found in [16].

In [9] a cellular system with two base stations and four users was considered and it was shown that the rate gap scales exactly like (11). As we will see this result can not be generalized to systems with more than two cells.

In the following we show that the scaling law (11) is to optimistic, if we consider more general systems and a slightly different but more realistic metric. Moreover we will see that the scaling laws (12) can be significantly improved if we use a different feedback and IA strategy.

### B. An Improved Metric

In this paper we assume that the CSI is used by the base stations to perform (i) beamforming, (ii) scheduling, and (iii) link adaptation. If the base stations have only quantized CSI, each of these tasks causes a rate loss compared to the performance with perfect CSI. Therefore, we define the following per user performance metric

$$\Delta r_m(\pi_H, \pi_V) = \max\{r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H), r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; V)\}$$

$$= r_m(\pi_H, S_H; H) - \min\{r_m(\pi_V, S_V; H), r_m(\pi_V, S_V; V)\}.$$  

(13)

Because of the per user formulation $\Delta r_m(\pi_H, \pi_V)$ is not necessarily positive for all $H$. The rate loss gap $\Delta r_m(\pi_H, \pi_V)$ captures the following effects:
1) If $r_m(\pi_V, S_V; H) > r_m(\pi_V, S_V; V)$ the rate gap is $r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; V)$. Thus, the rate gap captures the rate loss due to beamforming, scheduling and link adaptation based on quantized CSI, since it is assumed that the base station transmits with a rate $r_m(\pi_V, S_V; V)$.

2) If $r_m(\pi_V, S_V; H) < r_m(\pi_V, S_V; V)$, the rate gap is $r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H)$ and describes the rate loss due to beamforming and scheduling based on quantized CSI. We do not consider link adaptation, because even if allocation of a rate $r_m(\pi_V, S_V; V) > r_m(\pi_V, S_V; H)$ causes an outage event with high probability in practice mechanisms like automatic repeat requests are used to handle such events.

As we will see, $\Delta r_m(\pi_H, \pi_V)$ is strong enough to address some of the drawbacks of the conventional analysis (summarized in Subsection III-A) and leads to indeed different results.

In the remainder, we will derive lower and upper bounds on $\Delta r_m(\pi_H, \pi_V)$ for symmetric systems which are defined as follows.

**Definition 1.** In a symmetric system the random channels $H_{m,l}$ are independent and identically distributed for all $m \in \mathcal{U}$ and all $l = 1, \ldots, K$. Further, the distribution of the effective channels $(H_{m,l})^H u_m$, for all $l = 1, \ldots, K$ and each user $m \in \mathcal{U}$ given some fixed arbitrary receive filter $u_m$, is the same and isotropic.

The following lemma sets the basis for our analysis; it allows us to bound the rate gap $\Delta r_m(\pi_H, \pi_V)$ in terms of the same scheduling decisions.

**Lemma 1.** Let $S_H$ and $\pi_H : S_H \rightarrow \mathbb{S}^{n-1}$ denote the optimal user selection and the optimal beamforming vectors under perfect CSI $H$ according to (6). Similarly, let $S_V$ and $\pi_V : S_V \rightarrow \mathbb{S}^{n-1}$ be the optimal user selection and the optimal beamforming vectors under quantized CSI $V$ according to (10). Assume a symmetric system and fix some arbitrary user $m \in \mathcal{U}$, then the expected rate gap $\mathbb{E} [\Delta r_m(\pi_H, \pi_V)]$ is bounded by

$$\mathbb{E} \left[ \max \{ r_m(\pi_V, S_V; H) - r_m(\pi_V, S_V; V), 0 \} \right] \leq \mathbb{E} [\Delta r_m(\pi_H, \pi_V)] \leq 3 \mathbb{E} \left[ \max_{S \subseteq \mathcal{U}} \max_{\pi : S \rightarrow \mathbb{S}^{n-1}} |r_m(\pi, S; H) - r_m(\pi, S; V)| \right].$$

(14)
Proof: First we need to show that \( \mathbb{E}[r_m(\pi_H, S_H; V)] \leq \mathbb{E}[r_m(\pi_V, S_V; V)] \) which does not trivially follow from the sum rate maximization (10). To see this assume that \( \mathbb{E}[r_m(\pi_H, S_H; V)] > \mathbb{E}[r_m(\pi_V, S_V; V)] \) for some user \( m \). Since \( \mathbb{E}[r_m(\pi_H, S_H; V)] = \mathbb{E}[r_l(\pi_H, S_H; V)] \) and \( \mathbb{E}[r_m(\pi_V, S_V; V)] = \mathbb{E}[r_l(\pi_V, S_V; V)] \) when \( m \neq l \), it follows from the symmetry of the system that \( \mathbb{E}[r_m(\pi_H, S_H; V)] > \mathbb{E}[r_m(\pi_V, S_V; V)] \) for all \( m \in \mathcal{U} \). Hence, we have

\[
\mathbb{E} \left[ \sum_{m \in \mathcal{U}} r_m(\pi_V, S_V; V) \right] < \mathbb{E} \left[ \sum_{m \in \mathcal{U}} r_m(\pi_H, S_H; V) \right],
\]

which contradicts with the definition of \( S_V \) and \( \pi_V \) given in (10). Therefore,

\[
\mathbb{E}[r_m(\pi_H, S_H; V)] \leq \mathbb{E}[r_m(\pi_V, S_V; V)],
\]

must hold for all \( m \in \mathcal{U} \). In a similar manner we can show that

\[
\mathbb{E}[r_m(\pi_V, S_V; H)] \leq \mathbb{E}[r_m(\pi_H, S_H; H)]
\]

(17) holds for all \( m \in \mathcal{U} \). Inequalities (16) and (17) state that in expectation the sum-rate optimal scheduling decision maximizes also the individual per user rates. To prove the upper bound we write \( \mathbb{E}[\Delta r_m(\pi_H, \pi_V)] \) as

\[
\mathbb{E}[\Delta r_m(\pi_H, \pi_V)] = \mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H) + \max\{r_m(\pi_V, S_V; H) - r_m(\pi_V, S_V; V), 0\}].
\]

(18)

The first term can be bounded by

\[
\mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H)]
\]

\[
= \mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; V) + r_m(\pi_V, S_V; V) - r_m(\pi_V, S_V; H)]
\]

\[
\leq \mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_H, S_H; V) + r_m(\pi_V, S_V; V) - r_m(\pi_V, S_V; H)]
\]

\[
\leq 2\mathbb{E}\left[ \max_{S \subseteq \mathcal{U}} \max_{\pi_m \rightarrow \mathcal{S}_{m-1}} |r_m(\pi, S; H) - r_m(\pi, S; V)| \right].
\]

(19) holds since we simply added a \( 0 = -r_m(\pi_V, S_V; V) + r_m(\pi_V, S_V; V) \). The first inequality (20) holds according to (16). Further by (17) we have \( \mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H)] \geq 0 \). Since

\[
\mathbb{E}[\max\{r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; V), 0\}] \leq \mathbb{E}\left[ \max_{S \subseteq \mathcal{U}} \max_{\pi_m \rightarrow \mathcal{S}_{m-1}} |r_m(\pi, S; H) - r_m(\pi, S; V)| \right].
\]

(21)
is also true the upper bounds follows. Since according to (17)
\[
\mathbb{E}[r_m(\pi_H, S_H; H) - r_m(\pi_V, S_V; H)] \geq 0,
\]
the lower bound follows by setting the first term in (13) equal to 0.

We offer some brief remarks.

**Remark 2.** Lemma 1 is tight if \( H = V \). On the other hand, if \( H \) and \( V \) are not related the bound can be arbitrary loose. However, since we assume that \( V \) is a good approximation of \( H \) the bounds in Lemma 1 can be assumed to be reasonably tight.

**Remark 3.** Even though we assumed achievable rates \( \log(1 + x) \), it is possible to consider extensions of this lemma which incorporate more general utility functions. In addition, the assumptions on the channel distribution may be relaxed here but they are required in the subsequent theorems.

**C. Main Result**

The main result in this subsection is an upper bound on the expected rate gap \( \mathbb{E}[\Delta r(\pi_H, \pi_V)] \) defined in (13). In contrast to previous results (summarized in Subsection III-A), the following theorem holds for any receive and transmit strategy, incorporates user selection and is valid for any SNR regime.

**Theorem 1.** Assume a symmetric system with limited feedback. Let the transmit beamformer \( \pi \) and the user selection \( S \) be arbitrary but fixed. If each user \( m \in \mathcal{U} \) uses RVQ (8) with \( B \) bits per base station feedback link,

\[
\mathbb{E}[|r_m(\pi, S; H) - r_m(\pi, S; V)|] 
\leq 2 \log \left( 1 + \frac{P}{2} \left[ K (K - 1) \mathbb{E}[\mu^{4}_{m,1}] 2^{-\frac{B}{\pi_1}} + 2^{-\frac{B}{\pi_1}} \left( \mathbb{E}[\mu^{8}_{m,1}] + K \mathbb{E}[\mu^{2}_{m,1}] \right) \right] \right),
\]

(23) holds for any SNR \( P \).

The proof is presented in Appendix A. Theorem 1 holds regardless of the transmit strategy, i.e., for any beamforming and user selection strategy. None of the known results presented in Section III-A hold with such generality.
Remark 4. In contrast to the known results (11) and (12) we obtain a different scaling for the RVQ scheme (8). It is significantly better than the result (12) for IA with Grassmannian feedback, but requires two times more feedback bits than the result (11) for the broadcast channel with zero forcing.

Remark 5. Assuming the worst-case decision for each transmit beamformer (given fixed receive filters), the upper bound (53) in the proof of Theorem 1 is tight. Therefore, also the upper bound in Lemma 1 is tight. Hence, using Lemma 3 it follows that the minimum chordal distance is a robust feedback metric for the considered systems.

In the next section we present an interference alignment algorithm and derive a corollary which adapts Theorem 1 to this algorithm.

IV. INTERFERENCE ALIGNMENT WITH QUANTIZED CSI AND USER SELECTION

Without requiring further constraints, scheduling problem (6) and (10) are NP-hard [17]. Therefore, good sub-optimal solutions are required. In this section we present an algorithm that efficiently solves the scheduling problem by alternating the optimization of receive filters and transmit beamformers. As we will see, the algorithm is robust to CSI quantization and keeps the feedback overhead low.

A. Cellular Interference Alignment

The IA algorithm presented below uses concepts of spatial IA and user selection; it aims on finding beamforming vectors $\pi_{IA}$ and user sets $S_1, \ldots, S_K$ such that the following conditions hold

$$\left|\langle u_m, H_{m,b}\pi_{IA}(m) \rangle \right|^2 \geq c_0, \quad \forall b = 1, 2, \ldots, K \text{ and } m \in S_b$$

$$\left|\langle u_m, H_{m,l}\pi_{IA}(k) \rangle \right|^2 = 0, \quad \forall b, l = 1, 2, \ldots, K \text{ and } k \in S_l, m \in S_b \text{ with } k \neq m$$

where $c_0 > 0$ is a positive constant. Condition (24) states that for each active user the desired effective channels are non zero and condition (25) states that all interfering channels are zero. Typically, IA is analyzed using the concept of DoF [18] which are defined as

$$d = \lim_{P \to \infty} \frac{R(\pi, S; H)}{\log(P)} + o(\log(P)).$$

Intuitively, the DoF can be seen as the number of interference-free parallel data streams that can be transmitted simultaneously in a network. The DoFs for symmetric cellular networks with
spatial interference alignment have been analyzed in [19], where we have shown that condition (24) and (25) can be fulfilled, almost surely, if

$$n_t \geq \frac{SK + 1}{2},$$

(27)

with $n_r = n_t$, $S = |S_b| = |S_l|$, for all $l, b = 1, 2, \ldots, K$, and a single data stream per user. Hence, spatial IA is feasible, almost surely, if the number of active users per base station is bounded by

$$|S_b| \leq \frac{1}{K} (2n_t - 1), b = 1, \ldots, K.$$  

(28)

**B. Minimum Interference Algorithm**

The algorithm presented here aims on minimizing interference and may achieves interference alignment. In the sequel we will call this algorithm minimum interference algorithm. The minimum interference algorithm with user selection is summarized in Algorithm 1. To ensure that all interference can be canceled, the maximum number of active users is selected according to the feasibility condition (28). At the beginning of each transmission frame the active users are selected according to some metric, e.g., maximum fairness, maximum channel gain or other requirements, possibly defined by higher layers. Having determined the set of active users the alternating optimization of receive filters and beamformers is performed.

Many algorithms that achieve interference alignment or other related objectives have been proposed in the literature, e.g., [20], [10], [21], [19]. To our best knowledge we are the first to propose that the quantized CSI is given by the quantized effective channel $V$ defined in (2).

Even if we only consider single stream transmissions, extensions to multi-stream transmissions are straightforward. Multi-stream transmission requires that each user feeds back the effective channel for all streams that it wants to transmit. Based on this additional information additional streams can be treated like additional users.

1) **Receive filter optimization based on perfect CSI:** At the beginning of each transmission frame, orthogonal common pilots are transmitted, so that, all users $k$ can measure the channel matrices $H_{k,b}$, for all $b$. Common pilots are necessary to compute the effective channels at the terminals and must be retransmitted in intervals depending on the coherence time of the channel.

During the receive filter optimization all transmit beamformers $\pi$ are fixed. In the first iteration the beamforming vectors are set to $\pi(k) = 1/\sqrt{n_t}(1, 1, \ldots, 1)^T$. Based on dedicated (precoded)
Algorithm 1 minimum interference algorithm with user selection

**Begin of transmission frame:**

Transmit common pilots to all users and make an estimate $H_{m,b}$, with $b \in [1, K]$ and $m \in U$.

For $b \in [1, K]$ select $S_b \subseteq U$ according to (28) by central control.

Set $\pi(k) = 1/\sqrt{n_t}(1, 1, \ldots, 1)^T$ for all $k \in \{S_1, S_2, \ldots, S_K\}$.

repeat

Transmit dedicated pilots.

for $b = 1, 2, \ldots, K$ do

Compute receive filter matrix $u_k$, for all $k \in S_b$, according to (29).

Quantize and feed back the effective channels $\hat{v}_{k,l}$, for all $k \in S_b$, $l = 1, \ldots, K$, according to (9).

end for

for $b = 1, 2, \ldots, K$ do

Compute beamforming vectors $\pi(k)$, for all $k \in S_b$, according to (33).

end for

until termination condition is satisfied (e.g. maximum number of iterations, minimum residual interference, ...).

**End of transmission frame**

pilots each user $k \in S_b$ measures the effective channels $H_{k,l}\pi(m), l = 1, \ldots, K, l \neq b, m \in S_l$.

Based on the measured channels, the receive filter of user $k \in S_b$ is given by

$$u_k = \nu_{\min}(\Theta_{b,k}),$$

(29)

where $\nu_{\min}(X)$ is defined as the eigenvector corresponding to the smallest eigenvalue of the Hermitian matrix $X$. The out of cell interference covariance matrix $\Theta_{b,k}$ is defined as

$$\Theta_{b,k} = \sum_{l=1}^{K} \sum_{m \in S_l, l \neq b} H_{k,l}\pi(m)(H_{k,l}\pi(m))^H.$$  

(30)

In the receive filter optimization no intra-cell interference is considered. Intra-cell interference is considered in the beamformer optimization only. This approach ensures that the intra-cell interference gets aligned with the out-of-cell interference.
2) **Transmit beamformer optimization based on quantized CSI:** The transmit beamformer optimization is performed at the base stations and is based on quantized CSI \( V \). The transmit beamformers are computed in two steps. First, the transmit subspace which causes minimum out-of-cell interference is determined. Second, the intra-cell interference is canceled by a zero forcing step. Consider the reciprocal network. The reciprocal precoded channel from user \( k \) in cell \( b \) to base station \( l \) is given by \( \tilde{v}_{k,l} = \hat{v}_{k,l} \). For base station \( b \) the transmit subspace which causes minimum out-of-cell interference is given by

\[
\Pi_b = \nu_{\text{min}}^{S_b}(\tilde{\Theta}_b) \in \mathbb{C}^{n_t \times |S_b|},
\]

where \( \nu_{\text{min}}^{N}(X) \) is defined as the eigenvectors corresponding to the \( N \) smallest magnitude eigenvalues of the Hermitian matrix \( X \). The interference covariance matrix \( \tilde{\Theta}_b \) is defined as

\[
\tilde{\Theta}_b = \sum_{l=1}^{K} \sum_{m \in S_l} \tilde{v}_{m,b} (\tilde{v}_{m,b})^H.
\]

Finally, the intra-cell interference is canceled by an additional zero forcing step. The zero forcing beamformer \( w_{m,b} \in \mathbb{C}^{|S_b|} \) for user \( m \) in cell \( b \) is chosen from the null space of the effective channels \( \nu_{k,b}^H \Pi_b \), with \( k \in S_b \) and \( k \neq m \), such that, the transmit beamformer for user \( m \in S_b \) is given by

\[
\pi(m) = \Pi_b w_{m,b}.
\]

3) **Convergence of the residual interference:** The convergence of the residual interference using the minimum interference algorithm can be proved in a similar manner as the proof of convergence in [20] for the \( K \)-user interference channel. Key observations are the following. First, the out-of-cell interference is monotonically decreased when computing the receive filter (29). Second, the transmit beamformer computations (33) decreases the out-of-cell interference and nulls all intra-cell interference.

Note that even if the residual interference converges to a local minimum, it is not guaranteed that the algorithm converges to a unique solution.

**C. Rate Loss Gap Analysis**

Together with Lemma \( \square \) we have the following corollary which tailors Theorem \( \square \) to the minimum interference algorithm.
Corollary 1. Under the assumptions of Theorem 7 in any iteration of Algorithm 3 the average rate loss per user is upper bounded by

\[ \mathbb{E}[\Delta r_m(\pi_H, \pi_V)] \leq 6 \log \left( \frac{P}{2} \left[ K^2 n_r^2 2^{-\frac{B}{2(n_r+1)}} + (n_r + K) 2^{-\frac{B}{2(n_r+1)}} \right] \right) \leq 3P \left( K^2 n_r^2 2^{-\frac{B}{2(n_r+1)}} + (n_r + K) 2^{-\frac{B}{2(n_r+1)}} \right). \] (34)

Proof: Using Lemma 1 we have

\[ \mathbb{E}[\Delta r_m(\pi_H, \pi_V)] \leq 3 \mathbb{E} \left[ \max_{\mathcal{S} \subseteq \mathcal{U}} \max_{\pi : \mathcal{S} \rightarrow \mathcal{S}^{n_t-1}} |r_m(\pi, \mathcal{S}; H) - r_m(\pi, \mathcal{S}; V)| \right]. \] (35)

Now, we can use Theorem 1 and obtain

\[ \mathbb{E}[\Delta r_m(\pi_H, \pi_V)] \leq 6 \log \left( 1 + \frac{P}{2} \left[ K (K - 1) \mathbb{E} \left[ \mu_{m,1}^4 \right] 2^{-\frac{B}{2(n_r+1)}} + 2^{-\frac{B}{2(n_r+1)}} \left( \mathbb{E}^{\frac{1}{2}} \left[ \mu_{m,1}^8 \right] + K \mathbb{E}^{\frac{1}{2}} \left[ \mu_{m,1}^2 \right] \right) \right] \right). \] (36)

When optimizing the beamformers using Algorithm 1 all receive filters are fixed. Thus, we can compute the expected values

\[ \mathbb{E} \left[ \mu_{m,1}^2 \right] = \mathbb{E} \left[ \| (H_{m,1})^H u_m \|^2 \right] = \mathbb{E} \left[ (u_m)^H H_{m,1} (H_{m,1})^H u_m \right] = 1 \]
\[ \mathbb{E} \left[ \mu_{m,1}^4 \right] \leq \mathbb{E} \left[ \left( \sum_{i=1}^{n_t} \| h_i \|^2 \right)^2 \right] \leq n_r^2 \]
\[ \mathbb{E}^{\frac{1}{4}} \left[ \mu_{m,1}^8 \right] \leq n_r, \]

where \( h_i \) is the \( i \)th column of \( H_{m,1} \) and we used the Cauchy-Schwarz inequality and the fact that \( n_t \sum_{i=1}^{n_r} \| h_i \|^2 \) is chi-squared distributed with \( n_t n_r \) degrees of freedom.

Remark 6. The upper bound in Lemma 1 also holds (up to a constant) if we consider perfect link adaptation. Therefore, Corollary 1 is also true (up to a constant) if we consider perfect link adaptation.

The following theorem shows that the scaling \( 2^{-\frac{B}{2(n_r+1)}} \) can not be improved if we consider IA with RVQ, as described in Section II-B, and link adaptation.
Theorem 2. Under the assumptions of Theorem 1 for sufficiently high SNR the average rate loss is bounded from below by

$$\mathbb{E} [\Delta r_m(\pi_H, \pi_V)] \geq \max_{c_1 > 0} \frac{1}{c_1} \left( 1 - \frac{n_t^2 (n_t + 1)}{4c_1(n_t - 1) 2^{-n_t}} \right)$$

$$\mathbb{E} \left[ \log \left( 1 + \frac{c_1 \mathbb{E} [\mu_{m,b}^2]}{|S_b| (1 + \mathbb{E} [\mu_{m,b}^2])} \sqrt{\frac{4 \cdot 2^{-n_t n_t - 1}}{n_t^2 (n_t + 1)} 2^{\frac{-B}{2(n_t - 1)}}} \right) \right]$$

$$- \log \left( 1 + PK \sqrt{\mathbb{E} [\mu_{m,b}^4]} 2^{\frac{-B}{n_t - 1}} \right)$$

(37)

where \(|S_b|\) satisfies the IA feasibility condition (28), for all \(b = 1, \ldots, K\), with equality. In particular, for some \(c_2 > 0\)

$$\mathbb{E} [\Delta r_m(\pi_H, \pi_V)] \geq c_2 \log \left( 1 + \frac{P \mathbb{E} [\mu_{m,b}^2]}{|S_b| (1 + \mathbb{E} [\mu_{m,b}^2])} 2^{\frac{-B}{2(n_t - 1)}} \right) + o \left( 2^{\frac{-B}{(n_t - 1)}} \right)$$

(38)

holds.

The proof can be found in Appendix B. Note that, the lower bound is bounded in \(P\). Therefore it can not be used for degrees of freedom analysis, where \(P\) is taken to infinity. But, Theorem 2 shows that the scaling \(2^{(-B/(2(n_t - 1)))}\) can not be improved for finite SNR \(P\).

V. Simulations

A. Baseline

As a baseline scheme we consider centralized IA which was proposed in [21] by the authors. The baseline scheme requires a central processing unit which has global (quantized) CSI. Each user \(m\) quantizes and feeds back the channel matrix \(H_{m,l}\), for all \(l = 1, \ldots, K\), to the central processing unit. The central processing unit computes the transmit beamformers in an iterative manner similar to Algorithm 1 proposed in Section IV. To quantize the channel matrices, we apply a scalar quantization or a vector quantization scheme.

1) Scalar Quantization: Each user maps each element of the channel matrix to an element of a scalar feedback codebook with \(2^{B_s}\) elements. Scalar quantization (SQ) leads to a feedback load of \(2Kn_t n_r B_s\) bits per user and per feedback message. As we will see, the feedback and control overhead is significantly larger than for the proposed distributed algorithm.
Fig. 1. Spectral efficiency over SNR. Convergence of the proposed minimum interference algorithm (Algorithm 1). Observation: The proposed algorithm converges quickly.

2) Vector Quantization: Vector quantization (VQ) is a popular quantization scheme for multi-antenna channels. As a baseline we consider the following scheme which was also used in [22] and [15]. Each user \( m \) quantizes the channel matrices \( H_{m,l} \), for all \( l = 1, \ldots, K \), by applying RVQ (see Subsection II-B) on the vector \( \text{vec}(H_{m,l}) \), where \( \text{vec}(X) \) stacks the columns of the matrix \( X \) one over the other.

B. Simulation Setup

In the simulations we consider a cellular network with \( K = 3 \) base stations and \( U = 9 \) users. Each node is equipped with \( n_t = n_r = 5 \) antennas. From [19] we know that IA is feasible if each base station \( b \) serves \( |S_b| = 3 \) users. For the minimum interference algorithm (Algorithm 1) each base station is assigned randomly to three users. Power allocation is assumed to be uniform. The channels are modeled as described in Section II.
C. Simulation Results

First, we investigate the convergence of the proposed Algorithm 1. Figure 1 depicts the residual interference of Algorithm 1. We observe that with quantized CSI the residual interference converges rapidly to its minimum of approximately 4.5 dB, 3.5 dB and 2.5 dB for 8, 10 and 12 bit per user per iteration, respectively. In contrast, with ideal CSI the residual interference keeps decreasing with the number of iterations. We conclude that with quantized CSI the number of iterations can be kept low (≈ 5 iterations) without loosing a significant part of the performance that can be achieved with more iterations. Hence, the algorithm seems to be well suited for practical applications where a small number of iterations is mandatory.

This observation is further supported by Figure 2 which depicts the spectral efficiency over the SNR for the minimum interference algorithm (Algorithm 1). Each user uses an independent random codebook with $2^{16}$ isotropic elements. Again, we observe that the proposed IA algorithm converges rapidly, i.e., going from 4 to 6 iterations the performance is increased only slightly. This is a remarkable results since a small number of iterations keeps the feedback load small.
In addition, Figure 2 shows the performance of the SQ baseline and VQ baseline schemes, defined above. Due to the centralized approach the feedback load of these schemes does not increase with the number of iterations. After 10 iterations the baseline schemes have converged. The SQ baseline is clearly outperformed by the minimum interference algorithm with 4 iterations and 192 bit feedback load per user and per transmission. That is, with the iterative minimum interference algorithm we require less than half the number feedback bits to outperform the SQ baseline. The VQ base line with a 15 bit random codebook (45 bit feedback per transmission per user) performs very poorly. To obtain a better performance significantly larger codebooks are required. However, computing the feedback decision for larger codebooks becomes quickly infeasible.

VI. CONCLUSION

We introduced an improved metric for the performance evaluation of the interfering broadcast channel. The improved metric captures the throughput degradation due to quantized channel state information by considering the beamformer offset and the link adaptation problem. We obtained the relevant scaling laws and showed that they are different from existing ones. Moreover, we provided an iterative IA algorithm and corresponding feedback strategies which achieve the derived scaling laws.

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**APPENDIX A**

**PROOF OF THEOREM 1**

Before proving Theorem 1 we prove two lemmas which are required in the proof.
Lemma 2. Let $x \in \mathbb{S}^{n_t-1}$ be an independent isotropic random vector and $\mathcal{V} = (v_1, ..., v_{2^B}) \subset \mathbb{S}^{n_t-1}$ be a collection of $2^B$ independent isotropic random vectors. If we define

$$Z := \min_{v \in \mathcal{V}} (1 - |\langle x, v \rangle|^2),$$

then for some $n \geq 1$ we have

$$\left( \frac{n_t - 1}{n_t} \frac{2^{B/n_t - 1}}{n} \right)^n \leq \mathbb{E}[Z^n] \leq \left( \frac{n_t - 1}{n_t - 1} \mathbb{E}[Z] \right)^n. \quad (39)$$

**Proof:** Since, $1 - |\langle x, v \rangle|^2$ is beta distributed with parameters $n_t - 1$ and $1$ [13], $Z$ is the minimum of $2^B$ beta($n_t - 1, 1$) distributed random variables. Therefore, $\mathbb{E}[Z] \geq \frac{n_t - 1}{n_t} 2^{-B/n_t}$ (see e.g. [13]) and by Jensen’s inequality we get the lower bound

$$\mathbb{E}[Z^n] \geq \left( \frac{n_t - 1}{n_t - 1} \mathbb{E}[Z] \right)^n = \left( \frac{n_t - 1}{n_t} \frac{2^{B/n_t - 1}}{n} \right)^n. \quad (40)$$

For the upper bound we use [23, Lemma 1] which states that $\Pr(Z > x) = (1 - x^{n_t-1})^{2^B}$. Thus, $\Pr(Z^n > x) = \Pr(Z > x^{1/n}) = \left(1 - x^{n_t-1} \frac{n_t}{n} \right)^{2^B}$, with $n \geq 1$, and therefore

$$\mathbb{E}[Z^n] \leq \left(2^{B/n_t - 1} \right)^n. \quad (41)$$

Using, $\mathbb{E}[Z] \geq \frac{n_t - 1}{n_t} 2^{-B/n_t}$ once more the second upper bound follows. \qed

Let us now define the following metric.

**Definition 2.**

$$\omega(x, y) := \max_{w \in \mathbb{S}^{n_t-1}} \left| |\langle x, w \rangle|^2 - |\langle y, w \rangle|^2 \right|^2$$

As we will see, this metric essentially dictates the rate loss gap in Theorem 1. We have the following lemma, which shows that this metric is equal to the chordal distance.

Lemma 3. Let $x \in \mathbb{S}^{n_t-1}$ and $y \in \mathbb{S}^{n_t-1}$ be unit norm vectors, then

$$\max_{w \in \mathbb{S}^{n_t-1}} \left| |\langle x, w \rangle|^2 - |\langle y, w \rangle|^2 \right|^2 = \sqrt{1 - |\langle x, y \rangle|^2}. \quad (43)$$

**Proof:** We have

$$\left| |\langle x, w \rangle|^2 - |\langle y, w \rangle|^2 \right|^2 = |w^H(xx^H - yy^H)w|.$$ 

Consider the matrix $A := xx^H - yy^H$. Since, rank $(\cdot)$ is a subadditive function we have that the matrix $A$ has maximum rank of two and, therefore, has only two non-zero eigenvalues $\lambda_1$
and $\lambda_2$. But the matrix $A$ is trace-less as well

$$\text{Tr}(A) = \lambda_1 + \lambda_2 = \text{Tr}(xx^H - yy^H) = \|x\|^2_2 - \|y\|^2_2 = 0.$$ (46)

Therefore, $\lambda_1 = -\lambda_2$ must hold. On the other hand, we get from the Frobenius norm $\|A\|_F^2 = \text{Tr}(A^HA)$ that

$$\text{Tr}(A^HA) = \lambda_1^2 + \lambda_2^2 = \|x\|^4_2 + \|y\|^4_2 - 2|\langle x, y \rangle|^2$$ (47)

Thus, using $\lambda_1 = -\lambda_2$ we get for the two non-zero eigenvalues

$$|\lambda_1| = |\lambda_2| = \sqrt{1 - |\langle x, y \rangle|^2},$$ (48)

which proves the claim.

We are now ready to prove Theorem I.

Proof: Fix the user selection $S$ and the transmit beamformers $\pi$. Denote the optimal receive filter as $u_m^*$ with respect to the collection $H$, we have $\hat{h}_{m,b} = (H_{m,b})^H u_m^*$, $\mu_{m,b} = \|\hat{h}_{m,b}\|_2$ and $h_{m,b} = \hat{h}_{m,b}/\mu_{m,b}$ for all $b = 1, \ldots, K$. Hence, the achievable rate of user $m \in S_b$ is

$$r_m(\pi, S; H) = \log \left(1 + \frac{P\mu_{m,b}^2|\langle h_{m,b}, \pi(m) \rangle|^2}{1 + \sum_{l=1}^K \sum_{k \in S_l \setminus \{m\}} \frac{P\mu_{m,l}^2}{|S_l|} |\langle h_{m,l}, \pi(k) \rangle|^2} \right).$$ (49)

Since, the base station does not know the channels $H$ it must use the imperfect CSI $V = \{\hat{v}_{m,b} = \mu_{m,b}v_{m,b} : m \in \mathcal{U}, b = 1, \ldots, K\}$. Based on $V$ the base station estimates the rates achievable by user $m$ as

$$r_m(\pi, S; V) = \log \left(1 + \frac{P\mu_{m,b}^2|\langle v_{m,b}, \pi(m) \rangle|^2}{1 + \sum_{l=1}^K \sum_{k \in S_l \setminus \{m\}} \frac{P\mu_{m,l}^2}{|S_l|} |\langle v_{m,l}, \pi(k) \rangle|^2} \right).$$ (50)

August 12, 2014 DRAFT
For the ease of presentation, we define the following variables (index $m$ omitted),

\[
\Phi_{b,k} := \frac{P \mu_{m,b}^2}{|S_b|} |\langle h_{m,b}, \pi(k) \rangle|^2 \quad (51)
\]

\[
\Psi_{b,k} := \frac{P \mu_{m,b}^2}{|S_b|} |\langle v_{m,b}, \pi(k) \rangle|^2 \quad (52)
\]

\[
\Delta_{b,k} := \Phi_{b,k} - \Psi_{b,k}
\]

\[
\Psi_{\Sigma} = \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k} \right)^{-1}
\]

\[
\Phi_{\Sigma} = \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l, k \neq m} \Phi_{l,k} \right)^{-1}
\]

Equation (51) and (52) can be interpreted as the effective received SNR of user $m$ through the channels $h_{m,b}$ and $v_{m,b}$, respectively. Further, define $\delta := |r_m(\pi, S; H) - r_m(\pi, S; V)|$, which can be bounded from above by

\[
\delta \leq \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} |r_m(\pi, S; H) - r_m(\pi, S; V)|
\]

\[
= \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} \left| \log \left( \frac{1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Phi_{l,k}}{1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k}} \right) + \log \left( \frac{1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k}}{1 + \sum_{l=1}^{K} \sum_{k \in S_l, k \neq m} \Phi_{l,k}} \right) \right|
\]

\[
= \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} \left| \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k} \cdot \Delta_{l,k} \right) + \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l, k \neq m} \Phi_{l,k} \cdot (-\Delta_{l,k}) \right) \right| \quad . (53)
\]

Using $\log (1 + a) + \log (1 + b) \leq 2 \log \left(1 + \frac{1}{2}(a+b)\right)$, with $a, b > -1$, yields

\[
\delta \leq \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} \left| \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k} \cdot \Delta_{l,k} \right) + \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_l} \Phi_{l,k} \cdot (-\Delta_{l,k}) + \sum_{l=1}^{K} \Phi_{l,k} \cdot \Delta_{l,m} \right) \right|
\]

\[
= 2 \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} \left| \log \left( 1 + \frac{1}{2} \left( \Psi_{\Sigma} - \Phi_{\Sigma} \right) \sum_{l=1}^{K} \sum_{k \in S_l} \Delta_{l,k} + \frac{1}{2} \sum_{l=1}^{K} \Phi_{l,k} \cdot \Delta_{l,m} \right) \right|
\]

\[
\leq 2 \max_{S \subseteq U} \max_{\pi: S \rightarrow S^{n_t-1}} \left| \log \left( 1 + \frac{1}{2} \left| \Psi_{\Sigma} - \Phi_{\Sigma} \right| \sum_{l=1}^{K} \sum_{k \in S_l} \Delta_{l,k} + \frac{1}{2} \sum_{l=1}^{K} \left| \Delta_{l,m} \right| \right) \right|
\]
Since,
\[ \Psi_\Sigma - \Phi_\Sigma = \frac{1}{\Psi_\Sigma^{-1} \Phi_\Sigma^{-1}} \left( \sum_{l=1}^{K} \sum_{k \in S_l} \Phi_{l,k} \sum_{l=1}^{K} \sum_{k \in S_l} \Psi_{l,k} \right) \leq \frac{1}{\Psi_\Sigma^{-1} \Phi_\Sigma^{-1}} \sum_{l=1}^{K} \sum_{k \in S_l} |\Delta_{l,k}| \]
holds, we have the result
\[ \delta \leq 2 \max_{S \subseteq U} \max_{\pi,\Sigma \to \Sigma^{n-1}} \log \left( 1 + \frac{1}{2} \left( \sum_{l=1}^{K} \sum_{k \in S_l} |\Delta_{l,k}| \right)^2 + \frac{1}{2} \sum_{l=1}^{K} \sum_{i \in S_l} |\Delta_{l,m}| \right) \]
\[ \leq 2 \log \left( 1 + \frac{1}{2} \sum_{l=1}^{K} \sum_{i \in S_l} \max_{\pi,\Sigma \to \Sigma^{n-1}} \max_{l',m} |\Delta_{l,i,k}| \right) \cdot \max_{\pi,\Sigma \to \Sigma^{n-1}} \max_{l,m} |\Delta_{l,m}| \).

Now, observe that
\[ \max_{\pi,\Sigma \to \Sigma^{n-1}} |\Delta_{l,k}| = \frac{P \mu_{m,l}^2}{|S_l|} \max_{\pi,\Sigma \to \Sigma^{n-1}} \left| \langle h_{m,l}, \pi(k) \rangle \right|^2 - \left| \langle v_{m,l}, \pi(k) \rangle \right|^2 \]
\[ = \frac{P \mu_{m,l}^2}{|S_l|} \max_{\pi \in \Sigma^{n-1}} \left| \langle h_{m,l}, x \rangle \right|^2 - \left| \langle v_{m,l}, x \rangle \right|^2 \]
\[ = \frac{P \mu_{m,l}^2}{|S_l|} \omega(h_{m,l}, v_{m,l}), \]
where \( \omega(\cdot, \cdot) \) was defined in (43) and the last term \( \frac{P \mu_{m,b}^2}{|S_b|} \omega(h_{m,b}, v_{m,b}) \) is actually independent of \( k \). Taking expectation and applying Jensen’s inequality we obtain
\[ \delta \leq 2 \log \left( 1 + \frac{P}{2} \sum_{l=1}^{K} \sum_{i \in S_l} \sum_{l',m} E \left[ \mu_{m,l_1}^2 \omega(h_{m,l_1}, v_{m,l_1}) \mu_{m,l_2}^2 \omega(h_{m,l_2}, v_{m,l_2}) \right] \]
\[ + \frac{P}{2} \sum_{l=1}^{K} E \left[ \mu_{m,l}^2 \omega(h_{m,l}, v_{m,l}) \right] \right). \quad (54) \]

Now, for \( l_1 \neq l_2 \) we have from the Cauchy-Schwarz inequality
\[ E \left[ \mu_{m,l_1}^2 \omega(h_{m,l_1}, v_{m,l_1}) \mu_{m,l_2}^2 \omega(h_{m,l_2}, v_{m,l_2}) \right] = E \left[ \mu_{m,l_1}^2 \omega(h_{m,l_1}, v_{m,l_1}) \right] E \left[ \mu_{m,l_2}^2 \omega(h_{m,l_2}, v_{m,l_2}) \right] \]
\[ \leq E^{\frac{1}{2}} \left[ \mu_{m,l_1}^4 \right] E^{\frac{1}{2}} \left[ \omega^2(h_{m,l_1}, v_{m,l_1}) \right] E^{\frac{1}{2}} \left[ \mu_{m,l_2}^4 \right] E^{\frac{1}{2}} \left[ \omega^2(h_{m,l_2}, v_{m,l_2}) \right] \quad (55) \]
and for \( l_1 = l_2 \) we have from the Cauchy-Schwarz inequality
\[ E \left[ \mu_{m,l_1}^4 \omega^2(h_{m,l_1}, v_{m,l_1}) \right] \leq E \left[ \mu_{m,l_1}^4 \omega(h_{m,l_1}, v_{m,l_1}) \right] \]
\[ \leq E^{\frac{1}{2}} \left[ \mu_{m,l_1}^8 \right] E^{\frac{1}{2}} \left[ \omega^2(h_{m,l_1}, v_{m,l_1}) \right]. \quad (56) \]
Using Lemma 3 and Lemma 2 we get
\[
E \left[ \omega^2 (h_{m,t}, v_{m,t}) \right] = E \left[ \min_{v \in V} \left( 1 - |\langle h_{k,t}, v \rangle|^2 \right) \right] < \frac{\theta}{2^{n-t}}. \tag{57}
\]
Such that, the claim follows by plugging (55), (56) and (57) in (54).

APPENDIX B

PROOF OF THEOREM 2

The following lemma allows us to bound the expected value of certain concave functions from below. The lemma is a partial reverse of Jensen’s inequality when certain conditions on the moments are fulfilled; more precisely, if \( 1 - \sqrt{\frac{E[z^2]}{c_1 E[z]^2}} \geq c_3 \) holds, with \( c_1 > 0 \) and \( c_3 \neq 0 \) being constants. Note that this is exactly the case for the quantization error, as we will see in the proof of Theorem 2.

The following lemma is a partial reverse of Jensen’s inequality for super linear functions. The lemma will be useful since \( \log(1 + x) \) is a super linear function.

**Lemma 4.** If \( f : \mathbb{R} \to \mathbb{R} \) is superlinear, then for any constant \( c_1 > 0 \) and any random variable \( z > 0 \),
\[
E[f(z)] \geq \frac{f(c_1 E[z])}{c_1} \left( 1 - \sqrt{\frac{E[z^2]}{c_1 E[z]^2}} \right) \tag{58}
\]
holds.

**Proof:** By assumption \( f(z) \) is superlinear and therefore \( f(0) = 0 \). Thus, for any \( x \) and \( y \) and any \( t \in [0, 1] \), \( f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) \). Setting \( x = 0 \) and \( y = z^* \), \( f((1-t)z^*) \geq \frac{z^*}{2^t} (1-t)f(z^*) \). Hence, for any \( 0 \leq z \leq z^* \),
\[
\frac{f(z)}{z} \geq \frac{f(z^*)}{z^*} \tag{59}
\]
For any \( z^* > 0 \) we have
\[
\mathbb{E} [f(z)] \geq \mathbb{E} [f(z) \mathbb{I} \{ z \leq z^* \}]
\]
\[
= \mathbb{E} \left[ \frac{f(z)}{z} z \mathbb{I} \{ z \leq z^* \} \right]
\]
\[
\geq \frac{f(z^*)}{z^*} \mathbb{E} [z \mathbb{I} \{ z \leq z^* \}]
\]
\[
= \frac{f(z^*)}{z^*} \mathbb{E} [z (1 - \mathbb{I} \{ z > z^* \})]
\]
\[
= \frac{f(z^*)}{z^*} \left( \mathbb{E} [z] - \mathbb{E} [z \mathbb{I} \{ z > z^* \}] \right)
\]
\[
\geq \frac{f(z^*)}{z^*} \left( \mathbb{E} [z] - \sqrt{\mathbb{E} [z^2]} \sqrt{\mathbb{E} [\mathbb{I} \{ z > z^* \}]} \right)
\]
\[
= \frac{f(z^*)}{z^*} \mathbb{E} [z] \left( 1 - \sqrt{\frac{\mathbb{E} [z^2]}{\mathbb{E} [z]^2}} \sqrt{\mathbb{P}(z > z^*)} \right)
\]
\[
\geq \frac{f(z^*)}{z^*} \mathbb{E} [z] \left( 1 - \sqrt{\frac{\mathbb{E} [z^2]}{\mathbb{E} [z] z^*}} \right).
\] (60)

Inequality (60) follows from (59), (62) follows from the Cauchy-Schwarz inequality and (63) follows from Markov’s inequality. If we choose \( z^* = c_1 \mathbb{E} [z] \) the claim follows.

The following lemma is a modification of Lemma 3.

**Lemma 5.** Let \( x \in \mathbb{S}^{n_t-1} \) and \( y \in \mathbb{S}^{n_t-1} \) be unit norm vectors. If \( w \) is uniformly distributed on the unit sphere \( \mathbb{S}^{n_t-1} \) and \( m \geq 2 \), then
\[
\frac{4 \cdot 2^{-nt}}{n_t (n_t + 1)} \left( 1 - |\langle x, y \rangle|^2 \right)^{\frac{m}{2}} \leq \mathbb{E} \left[ \max \left\{ |\langle x, w \rangle|^2 - |\langle y, w \rangle|^2, 0 \right\}^m \right] \leq \left( 1 - |\langle x, y \rangle|^2 \right)^{\frac{m}{2}}
\] (64)
holds.

**Proof:** We have
\[
||\langle x, w \rangle|^2 - |\langle y, w \rangle|^2| = |w^H (xx^H - yy^H) w|.
\] (65)

Consider the eigen-decomposition of the Hermitian matrix \( A := xx^H - yy^H = QAQ^H \). Since, \( A \) has maximum rank of 2, it has at most two non-zero eigenvalues. Therefore, the diagonal matrix \( \Lambda \) can be written as \( \Lambda = \text{diag} (\lambda_1, \lambda_2, 0, \ldots, 0) \), with \( \lambda_1 \leq \lambda_2 \). Since, \( Q \) is a Hermitian matrix, with columns given by the eigenvectors of \( A \), and \( w \) is uniformly distributed on \( \mathbb{S}^{n_t-1} \), the following is true
\[
\mathbb{E} \left[ (w^H A w)^m \right] = \mathbb{E} \left[ (w^H \Lambda w)^m \right].
\] (66)
According to (46) \( \lambda_1 = -\lambda_2 \). Thus, we have
\[
\mathbb{E} \left[ \max \{ w^H A w, 0 \}^m \right] = \mathbb{E} \left[ \max \{ \lambda_1 |w_1|^2 - \lambda_1 |w_2|^2, 0 \}^m \right]
\geq \lambda_1^m \mathbb{E} \left[ \max \{ 2|w_1|^2 - 1, 0 \}^m \right]
= \lambda_1^m \mathbb{E} \left[ (2|w_1|^2 - 1)^m \right] \quad (67)
= \lambda_1^m \int_0^1 (2\varepsilon^2 - 1)^m d\mu(\varepsilon).
\]

Inequality (67) holds since \( \|w\|_2 = 1 \) and therefore \( |w_1|^2 + |w_2|^2 \leq 1 \). Equation (68) is true because \( 2|w_1|^2 - 1 \) is non-negative only for \( |w_1|^2 \geq 1/2 \). In (69) \( \mu(\varepsilon) \) is the Haar-measure of \( \{ w \in S^{n-1} : |w_1| \leq \varepsilon \} \). Now we apply a result by Rudin [24, page 15 equation (2)] for functions on the sphere \( S^{n-1} \) in one parameter. We have, for some function \( f : \mathbb{R} \to \mathbb{R} \) and a normalized measure \( \sigma(S^{n-1}) = 1 \),
\[
\int_{S^{n-1}} f(\sigma) d\sigma = \frac{n_t - 1}{\pi} \int_0^{2\pi} d\theta \int_0^1 (1 - r^2)^{n_t - 2} f(r) r \, dr
= 2(n_t - 1) \int_0^1 (1 - r^2)^{n_t - 2} f(r) r \, dr.
\quad (70)
\]
By setting \( f(r) := \lambda_1^m (2r^2 - 1)^m \chi_{[\sqrt{1/2},1]}(r) \), where \( \chi_I(x) \) is the characteristic function, the lower bound is proved as follows. Plugging \( f(r) \) in (70) and using (69) we have
\[
\mathbb{E} \left[ \max \{ w^H A w, 0 \}^m \right] \geq \lambda_1^m \int_0^1 \begin{array}{c}
\boxed{(2r^2 - 1)^m} \\
\text{d}\mu(r)
\end{array} 2(n_t - 1)(1 - r^2)^{n_t - 2} r \, dr
= \lambda_1^m (n_t - 1) \int_{1/2}^1 (2u - 1)^m (1 - u)^{n_t - 2} du \quad (71)
= \lambda_1^m (n_t - 1) \int_{1/2}^1 (2(u - 1) + 1)^m (1 - u)^{n_t - 2} du \quad (72)
= \lambda_1^m (n_t - 1) \int_0^{1/2} (1 - 2v)^m v^{n_t - 2} dv \quad (73)
= 4\lambda_1^m \frac{2^{n_t} - 1}{n_t(n_t + 1)}. \quad (74)
\]
Equation (71) is obtained by substituting \( u = r^2 \) and in (73) we substituted \( v = 1 - u \). Finally, (74) follows by solving the integral. Using (48) in the proof of Lemma 3 which states that the largest eigenvalue of \( A \) is \( \lambda_1 = \sqrt{1 - |\langle x, y \rangle|^2} \), the lower bound is obtained.
The upper bound follows, since \(0 \leq \max\{|w_1|^2 - |w_2|^2, 0\}^m \leq 1\) holds for any \(w \in \mathbb{S}^{n-1}\). Therefore, \(\mathbb{E}\left[\max\{w^H A w, 0\}^m\right] = \lambda_m^m \mathbb{E}\left[\max\{|w_1|^2 - |w_2|^2, 0\}^m\right] \leq 1\) together with (48) proves the upper bound.

Now we are ready to prove Theorem 2.

**Proof:** Consider an arbitrary but fixed user \(m \in S_b\) where \(|S_b|\) is non-random and fulfills the feasibility condition (28), for all \(b = 1, \ldots, K\), with equality. Define an IA solution \(\pi_{IA}\) as \(|\langle v_{m,b}, \pi_{IA}(k)\rangle|^2 = 0\), for all \(b = 1, \ldots, K\) and \(k \in S \setminus \{m\}\). For sufficiently high SNR, \(R(\pi_{IA}, S, V)\) achieves the optimal capacity scaling (10). Thus, we can use Lemma 1 and get

\[
\mathbb{E}\left[\Delta r_m(\pi_H, \pi_V)\right] \geq \mathbb{E}\left[\max \left\{ r_m(\pi_{IA}, S; H) - r_m(\pi_{IA}, S; V), 0 \right\} \right],
\]

with

\[
r_m(\pi_{IA}, S; H) = \log \left(1 + \frac{P_{m,b}^2 |\langle h_{m,b}, \pi_{IA}(m)\rangle|^2}{1 + \sum_{l=1}^K \sum_{k \in S_l \setminus \{m\}} P_{l,m}^2 |\langle h_{m,l}, \pi_{IA}(k)\rangle|^2}\right),
\]

\[
r_m(\pi_{IA}, S; V) = \log \left(1 + \frac{P_{m,b}^2 |\langle v_{m,b}, \pi_{IA}(m)\rangle|^2}{|S_b|} \right).
\]

Similar to (51) and (52) we define the following variables (index \(m\) omitted)

\[
\Phi^*_{b,k} = \frac{P_{m,b}^2 |\langle h_{m,b}, \pi_{IA}(k)\rangle|^2}{|S_b|},
\]

\[
\Psi^*_{b,k} = \frac{P_{m,b}^2 |\langle v_{m,b}, \pi_{IA}(k)\rangle|^2}{|S_b|},
\]

\[
\Delta^*_{b,k} = \frac{|S_b|}{P_{m,b}^2} \max \left\{ \Phi^*_{b,k} - \Psi^*_{b,k}, 0 \right\},
\]

which can be interpreted as the effective receive SNR of user \(m\) for the IA solution. Using this notation the rate gap can be written in compact form,

\[
\Delta r_m(\pi_H, \pi_V) \geq \max \left\{ \log \left(1 + \sum_{l=1}^K \sum_{k \in S_l} \Phi^*_{l,k}\right) - \log \left(1 + \sum_{l=1}^K \sum_{k \in S_l \setminus \{m\}} \Phi^*_{l,k}\right) - \log \left(1 + \Psi^*_{b,m}\right), 0 \right\}
\]

and since \(\max\{a - b - c, 0\} \geq \max\{a - b, 0\} - c\) for \(c > 0\), the rate gap for user \(m\) is bounded.
from below by
\[
\Delta_{m}^{r}(\pi_{H}, \pi_{V}) \geq \log \left( 1 + \frac{\max \{ \Phi_{l,m}^{*} - \Psi_{b,m}^{*}, 0 \} }{1 + \Psi_{b,m}^{*}} \right) - \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_{l}} \Phi_{l,k}^{*} \right) \\
\geq \log \left( 1 + \frac{P\mu_{m,b}^{2}}{|S_{b}| (1 + P\mu_{m,b}^{2})} \Delta_{b,m}^{*} \right) - \log \left( 1 + \sum_{l=1}^{K} \sum_{k \in S_{l}} \Phi_{l,k}^{*} \right).
\]

(77)

To bound \( \mathbb{E}[\Delta_{m}^{r}(\pi_{H}, \pi_{V})] \) from below, we will derive a lower bound on the expected value of \( A \) and an upper bound on the expected value of \( B \).

We start with the upper bound for \( B \). Since, \( \pi_{IA} \) is an IA solution according to \( V \), we have
\[
|\langle h_{m,b}, \pi_{IA}(k) \rangle|^{2} \leq Z, \quad \text{for } k \neq m,
\]
where \( Z = \min_{v \in V} (1 - |\langle h_{k,l}, v \rangle|^{2}) \) is the quantization error (defined in Lemma 2) under RVQ. By Lemma 2 and the Cauchy-Schwarz inequality we have
\[
\mathbb{E}[\Phi_{l,k}^{*}] \leq P|S_{l}| \sqrt{\mathbb{E}[\mu_{m,b}^{4}] \sqrt{\mathbb{E}[Z^{2}]}} \leq P|S_{l}| \sqrt{\mathbb{E}[\mu_{m,b}^{4}] 2^{B_{m,b} - 1}}, \quad \forall k \neq m, l.
\]

(78)

Using Jensen’s inequality and \( |S_{l}|^{-1} \leq 1 \), for all \( l \), we obtain the upper bound
\[
\mathbb{E}[B] \leq \log \left( 1 + KP \sqrt{\mathbb{E}[\mu_{m,b}^{4}] 2^{B_{m,b} - 1}} \right).
\]

(79)

To lower bound \( A \) we define the positive random variable
\[
Y := (\Delta_{b,m}^{*})^{2} = \max \left\{ |\langle h_{m,b}, \pi_{IA}(k) \rangle|^{2} - |\langle v_{m,b}, \pi_{IA}(k) \rangle|^{2}, 0 \right\}^{2},
\]
where the mapping between \( Y \) and \( \Delta_{b,m}^{*} \) is bijective, since \( \Delta_{b,m}^{*} \) is positive per definition. Taking expectation conditioned on \( \mu_{m,b} \) and \( h_{m,b} \) (denoted \( \mathbb{E}[\cdot |\mu_{m,b}, h_{m,b}] := \mathbb{E}_{|\mu, h}[\cdot] \)) and using Lemma 4 with the concave function \( f(x) = \log (1 + \sqrt{x}) \) we get
\[
\mathbb{E}_{|\mu, h}[A] = \mathbb{E}_{|\mu, h}[\log \left( 1 + \frac{P\mu_{m,b}^{2}}{|S_{b}| (1 + P\mu_{m,b}^{2})} \Delta_{b,m}^{*} \right)] \\
= \mathbb{E}_{|\mu, h}[\log \left( 1 + \frac{P\mu_{m,b}^{2}}{|S_{b}| (1 + P\mu_{m,b}^{2})} \sqrt{Y} \right)] \\
\geq \frac{1}{c_{1}} \left( 1 - \sqrt{\frac{\mathbb{E}_{|\mu, h}[Y^{2}]}{c_{1} \mathbb{E}_{|\mu, h}[Y]^{2}}} \right) \log \left( 1 + c_{1} \frac{P\mu_{m,b}^{2}}{|S_{b}| (1 + P\mu_{m,b}^{2})} \sqrt{\mathbb{E}_{|\mu, h}[Y]} \right).
\]

(80)
It remains to compute the first and second moment of $Y$. Since, conditioned on $\mu_{m,b}$ and $h_{m,b}$ the beamformer $\pi_{IA}(m)$ is isotropic distributed, we have by Lemma 5 (first step, $n = 2$) and Lemma 2 (last step)

$$\mathbb{E}_{|\mu, h} (Y) = \mathbb{E}_{|\mu, h} \max \left\{ |\langle h_{m,b}, \pi_{IA}(m) \rangle|^2 - |\langle v_{m,b}, \pi_{IA}(m) \rangle|^2, 0 \right\}^2$$

$$\geq \frac{4 \cdot 2^{-nt}}{nt (nt + 1)} \mathbb{E}_{|\mu, h} \min_{\nu \in \nu} (1 - |\langle h_{m,b}, \nu \rangle|^2)$$

$$= \frac{4 \cdot 2^{-nt}}{nt (nt + 1)} \mathbb{E}_{|\mu, h} (Z)$$

$$\geq \frac{4 \cdot 2^{-nt}}{nt (nt + 1)} \frac{nt - 1}{nt} 2^{-n_{t-1}}.$$

Again by Lemma 5 (first step) and Lemma 2 (second step) we have

$$\mathbb{E}_{|\mu, h} (Y^2) \leq \mathbb{E}_{|\mu, h} \left[ Z^2 \right] \leq \left( \frac{nt}{nt - 1} \mathbb{E}_{|\mu, h} [Z] \right)^2. \quad (81)$$

Such that,

$$\mathbb{E}_{|\mu, h} [A] \geq \frac{1}{c_1} \left( 1 - \frac{n_t^2 (n_t + 1)}{4 \sqrt{c_1} (n_t - 1) 2^{-nt}} \right)$$

$$\log \left( 1 + \frac{c_1 P \mu_{m,b}^2}{|S_b| (1 + P \mu_{m,b}^2)} \sqrt{\frac{4 \cdot 2^{-nt} (nt - 1)}{n_t^2 (nt + 1)} 2^{-n_{t-1}}} \right). \quad (82)$$

Plugging (82) and (79) in (77) and taking expectation with respect to $\mu_{m,b}$ the claim follows.