Research Article

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The Jensen–Pólya program for various $L$-functions

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Abstract: Pólya proved in 1927 that the Riemann hypothesis is equivalent to the hyperbolicity of all of the Jensen polynomials of degree $d$ and shift $n$ for the Riemann Xi-function. Recently, Griffin, Ono, Rolen, and Zagier [M. Griffin, K. Ono, L. Rolen and D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, Proc. Natl. Acad. Sci. USA 116 (2019), no. 23, 11103–11110] proved that for each degree $d \geq 1$ all of the Jensen polynomials for the Riemann Xi-function are hyperbolic except for possibly finitely many $n$. Here we extend their work by showing that the same statement is true for suitable $L$-functions. This offers evidence for the generalized Riemann hypothesis.

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1 Introduction and statement of results

By extending notes of Jensen, Pólya [12] proved that the Riemann hypothesis (RH) is equivalent to the hyperbolicity of the Jensen polynomials for Riemann’s Xi-function. The Riemann Xi-function is the entire function that shifts the zeros of the Riemann zeta-function, $\zeta(s)$, from the line with real part $\frac{1}{2}$ to the real line. It is given by

$$
\Xi(z) := \frac{1}{2} \left( -z^2 - \frac{1}{4} \right) \pi^{-\frac{1}{2}} \Gamma\left( -\frac{iz}{2} + \frac{1}{2} \right) \Gamma\left( -\frac{iz}{2} + \frac{1}{2} \right),
$$

where $\Gamma(s)$ is the gamma function. We can consider a change of variable and define the coefficients $y(n)$ by the Taylor expansion of this new function:

$$
\Xi_1(x) = 8 \cdot \Xi(i \sqrt{x}) = \sum_{n=0}^{\infty} \frac{y(n)}{n!} \cdot x^n.
$$

(1.1)

Pólya originally proved that RH is equivalent to $\Xi_1$ having an infinite product expansion of the form

$$
\Xi_1(x) = ce^{\sigma x} \prod_{n=1}^{\infty} \left( 1 + \frac{x}{x_n} \right),
$$

where $c$ is a constant, $\sigma \geq 0$, $x_n \in \mathbb{R}^+$, and $\sum x_n^{-1} < \infty$. This condition can be encoded by the hyperbolicity of Jensen polynomials.

We say that a polynomial $f \in \mathbb{R}[X]$ is hyperbolic if all of its roots are real. Given a sequence $a : \mathbb{N} \rightarrow \mathbb{R}$ and positive integers $d$ and $n$, the associated Jensen polynomial of degree $d$ and shift $n$ is defined by

$$
J_{d,n}^a(X) := \sum_{j=0}^{d} \binom{d}{j} a(n+j)X^j.
$$

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RH is equivalent to the hyperbolicity of $f^{d,n}_y(X)$ for all $d$ and $n$ and where $y$ is given in equation (1.1) as the Taylor coefficients of $\Xi_1(x)$ (see [5, 6, 12]). The historical context of this approach to RH and a commentary on the results of [8] is given in [2]. Due to the difficulty of proving RH, research before [8] focused on establishing hyperbolicity for all shifts $n$ for small $d$. Work of Cşordas, Norfolk, and Varga [4] and Dimitrov and Lucas [6] shows that $f^{d,n}_y(X)$ is hyperbolic for all $n$ when $d \leq 3$. In [8], Griffin, Ono, Rolen, and Zagier prove that for any $d \geq 1$, the polynomial $f^{d,n}_y(X)$ is hyperbolic with at most finitely many exceptions $n$. They prove this by showing that for a fixed $d$,
$$\lim_{n \to \infty} f^{d,n}_y(a(n)X + \beta(n)) = H_d(X),$$
where $H_d(X)$ is the $d$-th Hermite polynomial and $a(n)$ and $\beta(n)$ are certain sequences. The Hermite polynomials are known to have real distinct roots, so $f^{d,n}_y(X)$ must also eventually have real distinct roots. In fact, Griffin, Ono, Rolen, and Zagier show that there is a family of sequences whose Jensen polynomials share the same property.

**Definition 1.1.** A real sequence $a : \mathbb{N} \to \mathbb{R}$ is Hermitian–Jensen if there exist sequences of positive real numbers $\{A(n)\}$ and $\{\delta(n)\}$ with $\delta(n)$ tending to zero, which satisfy
$$\log\left( \frac{a(n+j)}{a(n)} \right) = A(n)j - \delta(n)j^2 + o(\delta(n)^d) \quad \text{as} \quad n \to \infty$$
for some $d \geq 1$ and all $0 \leq j \leq d$.

**Remark.** Griffin, Ono, Rolen, and Zagier [8] give a more general statement about the asymptotic behavior needed for the Jensen polynomials of a sequence to converge to other families of polynomials.

In order to show that the Taylor coefficients of Riemann’s Xi-function are Hermitian–Jensen, an arbitrary precision asymptotic formula for the derivatives $\Xi^{(2n)}(0)$ was found in [8]. To extend the results of [8] we show that any good Dirichlet series is Hermitian–Jensen.

**Definition 1.2.** A Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ is good if the following conditions hold:

(i) $L(s)$ has a completed form $\Lambda(s)$, given in equation (2.1), that has an integral representation of the form
$$\Lambda(s) = N^\frac{1}{2} \int_0^\infty \left[ f(t) - f(\infty) \right] t^s dt,$$
where the function $f(t)$ has the form
$$f(t) = a(0) + \sum_{n \geq n_0 > 0} a(n)e^{-nt},$$
where $f(\infty) = a(0)$.

(ii) The function $f(t)$ satisfies
$$f\left( \frac{1}{N^t} \right) = e^{\frac{1}{2} \epsilon t} f(t),$$
where $\epsilon \in \{\pm 1\}$, which gives rise to an analytic continuation and a functional equation $\Lambda(s) = \epsilon \Lambda(k - s)$ for some $k \in \mathbb{Q}$.

(iii) The coefficients of $\Lambda(s)$ are real.

For a good Dirichlet series $L(s)$, we define
$$\Xi(z) = \Xi_L(z) := \begin{cases} (-z^2 - \frac{k^2}{4})\Lambda\left( \frac{k}{2} - iz \right) & \text{if } \Lambda(s) \text{ has a pole at } s = k, \\
\Lambda\left( \frac{k}{2} - iz \right) & \text{otherwise.} \end{cases}$$

From here on we will abuse notation and drop the dependence on $L$. If $\Lambda(s) = \Lambda(k - s)$, then we define
$$\Xi_1(x) := \Xi(i\sqrt{x}) =: \sum_{n \geq 0} \frac{\gamma(n)}{n!} x^n,$$
where

\[ y(n) = (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0). \]

If \( \Lambda(s) = -\Lambda(k - s) \), then we define

\[ \Xi_1(x) := \Xi(i \sqrt{x}) = \sum_{n \geq 0} \frac{y(n)}{n!} x^n, \]

where

\[ y(n) = i^{2n+1} \frac{n!}{(2n+1)!} \cdot \Xi^{(2n+1)}(0). \]

**Theorem 1.3.** Suppose that \( L(s) \) is a good Dirichlet series. Then \( f_y^{d,n}(X) \) is hyperbolic with at most finitely many exceptions for each fixed \( d \geq 1 \).

**Remark.** This offers evidence for the generalized Riemann Hypothesis (GRH).

**Remark.** Notice that in order to study the Jensen polynomials associated to \( y(n) \) we must understand the derivatives of \( \Xi(z) \) at \( z = 0 \), or equivalently the derivatives of \( \Lambda(s) \) at \( s = \frac{1}{2} \). In order to prove Theorem 1.3 we will prove an asymptotic formula with arbitrary precision for these derivatives.

**Remark.** All good \( L \)-series satisfy the Gaussian Unitary Ensemble (GUE) random matrix prediction in derivative aspect. Dyson, Montgomery, and Odlyzko [7, 9, 11] conjectured that the non-trivial zeros of the Riemann zeta function and other suitable \( L \)-functions are distributed like the eigenvalues of random Hermitian matrices. These eigenvalues and the roots \( H_d(X) \) of the suitably normalized Hermite polynomials, as \( d \to \infty \), both satisfy Wigner’s Semicircular Law [1, Chapter 3]. The roots of \( f_y^{d,n}(X) \), as \( d \to \infty \), approximate the zeros of \( \Lambda(\frac{1}{2} - iz) \) (see [12]), so these roots are also expected to satisfy Wigner’s Semicircular Law. The derivatives of the completed \( L \)-function are also predicted to satisfy GUE and higher derivatives correspond to \( L \)-functions in degree aspect. For a good \( L \)-function the \( f_y^{d,n}(X) \) converge to the Hermite polynomials which satisfy GUE in degree aspect. This is what is meant by the statement that good \( L \)-functions satisfy GUE in derivative aspect.

The following corollaries give some examples of Hermite–Jensen Dirichlet series.

**Corollary 1.4.** Dirichlet \( L \)-functions for real primitive self-dual characters are good.

**Corollary 1.5.** If \( f \in S_{2k}^{\text{new}}(\Gamma_0(N)) \) is a weight \( 2k \) modular newform on \( \Gamma_0(N) \), then the modular \( L \)-function associated to \( f \) is good.

**Corollary 1.6.** The Dedekind zeta-function for a number field is good.

In each of these cases we prove an arbitrary precision asymptotic formula for the derivatives of the completed \( L \)-series at its central value. We do this to show that these \( L \)-series are Hermite–Jensen, but these results are also of independent interest. This paper is organized in the following way: Section 2 will include the asymptotic formulas for the derivatives of the \( L \)-functions mentioned above. In Section 3, we will prove Theorem 1.3. In Section 4, we will prove the three corollaries.

## 2 Asymptotics for \( \Xi^{(n)}(0) \)

Let \( L(s) = \sum_{n \geq 1} a(n) n^{-s} \) be a good Dirichlet series. We thus know that \( L(s) \) has a completed form

\[ \Lambda(s) = N^\frac{s}{2} \prod_{j=1}^{J} \Gamma_R(s) \prod_{m=1}^{M} \Gamma_C(s) \cdot L(s) = N^\frac{s}{2} \int_0^\infty [f(t) - f(\infty)] t^s \frac{dt}{t}, \quad (2.1) \]

where \( \Gamma_R(s) := \Gamma(\frac{s}{2}) \), \( \Gamma_C(s) := 2(2\pi)^{-s} \Gamma(s) \), and \( \Gamma(s) := \int_0^\infty e^{-t \Gamma(s)} dt \) is the usual gamma function. Because
of the transformation properties of \( f(t) \), we split the integral at \( \frac{1}{\sqrt{N}} \) to arrive at

\[
\Lambda(s) = \frac{(cs - s + k)f(\infty)}{s(s - k)} + \int_{\frac{1}{\sqrt{N}}}^{\infty} (f(t) - f(\infty))(cN^{\frac{1}{2}s}t^s + N^{\frac{1}{2}s}t^s) \frac{dt}{t}.
\]

(2.2)

We have the following expression for the derivatives of \( \Lambda(s) \):

\[
\Lambda^{(n)}(s) = \frac{(-1)^{n+1}(k - s)^{n+1} - cs^{n+1})f(\infty)n!}{s^{n+1}(k - s)^{n+1}} + \int_{\frac{1}{\sqrt{N}}}^{\infty} (f(t) - f(\infty))(N^{\frac{1}{2}s}t^s + (-1)^n cN^{\frac{1}{2}s}t^s) \left( \frac{1}{2}\log(N) + \log(t) \right)^n \frac{dt}{t}.
\]

At \( s = \frac{k}{2} \) and \( z = 0 \) we have

\[
\Lambda^{(n)} \left( \frac{k}{2} \right) = \frac{2^{n+1}(f(\infty)n!((-1)^{n+1} - e) + F(n)}{k^{n+1}}
\]

and

\[
\Xi^{(n)}(0) = \begin{cases} 
\frac{(-i)^n 8(\pi)^2 F(n - 2) - k^2 F(n)}{n!} & \text{if } \Lambda(s) \text{ has a pole at } s = k, \\
(-i)^n F(n) & \text{otherwise,}
\end{cases}
\]

where

\[
F(n) = \frac{1}{2^n} \int_{\frac{1}{\sqrt{N}}}^{\infty} (f(t) - f(\infty))(N^{\frac{1}{2}s}t^s - (-1)^n cN^{\frac{1}{2}s}t^s) \left( 1 + (-1)^n e(\log(N) + 2 \log(t)) \right)^n \frac{dt}{t}
\]

(2.3)

for all \( n \geq 0 \). Notice that from (2.2) it is clear that \( \Lambda(s) \) having a pole at \( s = k \) corresponds directly with \( f(\infty) \neq 0 \). The large asymptotics of \( \Lambda^{(n)} \left( \frac{k}{2} \right) \) and \( \Xi^{(n)}(0) \) are obtained from the following theorem.

**Theorem 2.1.** The function \( F(n) \) defined in (2.3) is given to all orders in \( n \) by the asymptotic expansion

\[
F(n) \sim \frac{a(n_0)\sqrt{2\pi N^{\frac{3}{2}}(1 - (-1)^n e)}}{2^n} \frac{L^{n+1}}{\sqrt{(1 + \frac{k}{2})n - (\frac{k}{2} - 1)L^2}} \times e^{\frac{k}{2}(L - \log(N)) - \frac{n}{2} - \frac{1}{2} + \left( 1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots \right)} \quad (n \to \infty),
\]

(2.4)

where

\[
L = L(n) = 2 \log \left( \frac{n\sqrt{N}}{\log(n\sqrt{N})} \right)
\]

is the unique positive solution of the equation

\[
n = \frac{1}{2} \left( \pi n_0 e^{-\frac{1}{2} \log(N)} - \frac{k}{2} + 1 \right) L
\]

and each coefficient \( b_k \) belongs to \( \mathbb{Q}(L) \), the first value being

\[
b_1 = \frac{2(3L^4 + 189L^3 + 542L^2 + 744L + 496)}{3(L + 2)^3}.
\]

**Proof of Theorem 2.1.** We approximate the integrand in (2.3) by the function

\[
g(t) = a(n_0)N^{\frac{1}{2}}(1 - (-1)^n e)t^{\frac{1}{2} - 1}(\log(N) + 2 \log(t))^n,
\]

which comes from the first term in the expansion of \( f(t) - f(\infty) \). From now on we let \( n \) be fixed and will omit it from our notations. We have that

\[
t \frac{d}{dt} (\log(g(t))) = \frac{2n}{\log(N) + 2 \log(t)} - \pi n_0 t + \frac{k}{2} - 1,
\]
so \( g(t) \) assumes its unique maximum at \( t = a \), where \( a \) is the solution in \((\frac{1}{\sqrt{N}}, \infty)\) of
\[
  n = \frac{1}{2} \left( \pi n_0 a - \frac{k}{2} + 1 \right) (\log(N) + 2 \log(a)).
\]

For convenience, we define \( L = \log(N) + 2 \log(a) \), so we have
\[
  n = \frac{1}{2} \left( \pi n_0 e^{\frac{\log(N)}{2}} - \frac{k}{2} + 1 \right) L.
\]

We can then use Lambert’s \( W \) function to find an asymptotic expansion for \( L \). Lambert’s \( W \) function is defined as the solution to \( z = W(z)e^{W(z)} \). It has the nice property that \( Y = Xe^X \) if and only if \( X = W(Y) \). If we restrict to the real line so that \( W \) is real-valued, then we have that the principal branch (defined for \( x \geq -\frac{1}{e} \)) has a Taylor series around 0 given by
\[
  W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n.
\]

For large \( x \), we have the asymptotic expansion (see [3])
\[
  W(x) = \ln(x) - \ln(\ln(x)) + \frac{\ln(\ln(x))(\ln(\ln(x)) - 2)}{\ln^2(x)} + O\left( \left( \frac{\ln(\ln(x))}{\ln(x)} \right)^3 \right).
\]

Therefore, we have
\[
  L \approx 2 \log\left( \frac{n \sqrt{N}}{\log(n \sqrt{N})} \right).
\]

We now follow [8] and apply the saddle point method. The Taylor expansion of \( g(t) \) around \( t = a \) is given by
\[
  \frac{g((1 + \lambda)a)}{g(a)} = \left( 1 + \frac{2 \log(1 + \lambda)}{\log(N) + 2 \log(a)} \right)^n (1 + \lambda)^{\frac{k-1}{2}} e^{-\pi n_0 a} = e^{\frac{C\lambda^2}{2}} (1 + A_3 \lambda^3 + A_4 \lambda^4 + \cdots),
\]
where \( C = n(\frac{2}{2} + \epsilon^2) + \frac{k}{8} \frac{1}{4}, \)
\[
  \epsilon = \frac{1}{\log(N) + 2 \log(a)} = L^{-1},
\]
and the \( A_i \) are polynomials of degree \( \lfloor \frac{i}{2} \rfloor \) in \( n \) with coefficients in \( \mathbb{Q}[\epsilon] \). This expansion was found by expanding \( \log(g((1 + \lambda)a)) - \log(g(a)) \) in \( \lambda \). The linear term vanishes by the choice of \( a \) and the quadratic term is \( -\frac{C\lambda^2}{2} \).

The coefficients of the higher powers of \( \lambda \) are all linear expressions in \( n \) with coefficients in \( \mathbb{Q}[\epsilon] \). Exponentiating this expansion gives our expression for \( g((1 + \lambda)a)/g(a) \). The important behavior is that the dominant term of each \( A_i \) comes primarily from the exponential of the cubic term of the logarithmic expansion. The first few \( A_i \) are
\[
  A_3 = 2n\left( \frac{\epsilon}{3} + \epsilon^2 + \frac{4\epsilon^3}{3} \right) + \frac{k}{6} - \frac{1}{3},
\]
\[
  A_4 = -n\left( \frac{\epsilon}{2} + \frac{11\epsilon^2}{6} + 4\epsilon^3 + 4\epsilon^4 \right) - \frac{k}{8} + \frac{1}{4},
\]
\[
  A_5 = n\left( \frac{2\epsilon}{5} + \frac{5\epsilon^2}{3} + \frac{14\epsilon^3}{3} + 8\epsilon^4 + \frac{32\epsilon^5}{5} \right) + \frac{k}{10} - \frac{1}{5},
\]
\[
  A_6 = n^2\left( \frac{2\epsilon^2}{9} + \frac{4\epsilon^3}{3} + \frac{34\epsilon^4}{9} + \frac{16\epsilon^5}{3} + \frac{32\epsilon^6}{9} \right) + \frac{k^2 - 7k + 10}{36}
\]
\[
  + n\left( \frac{(10k - 50)\epsilon}{90} + \frac{(30k - 197)\epsilon^2}{90} + \frac{(40k - 530)\epsilon^3}{90} - \frac{34\epsilon^4}{3} - 16\epsilon^5 - \frac{32\epsilon^6}{3} \right).
\]

We plug in \( t = (1 + \lambda)a \) to arrive at the asymptotic expansion
\[
  \frac{1}{2^n} \int_{1/\sqrt{N}}^{\infty} g(t) \, dt = \frac{ag(a)}{2^n} \int_{-1 + \frac{1}{a^{\epsilon^2}}}^{\infty} e^{\frac{C\lambda^2}{2}} (1 + A_3 \lambda^3 + A_4 \lambda^4 + \cdots) \, d\lambda
\]
\[
  = \frac{ag(a)}{2^n} \sqrt{\frac{2\pi}{C}} \left( 1 + \frac{3A_4}{C^2} + \frac{15A_6}{C^3} + \cdots + \frac{(2i - 1)!!A_{2i}}{C^{i+1}} + \cdots \right).
\]
This expression and the one in Theorem 2.1 are interpreted as asymptotic expansions. These series do not converge for a fixed \( n \), but we can truncate the approximation at \( O(n^{-\delta}) \) for some \( A > 0 \), and as \( n \to +\infty \), this approximation becomes true to the given precision. We substitute the formulas for \( C_1 \) and \( A_i \) in terms of \( n \) in order to obtain the statement in the theorem. We also replace \( F(n) \) by the integral over \( g(t) \) with only the \( A_{2j} \) with \( i \leq 3k \) contributing to \( b_j \). The same asymptotic formula will hold with this replacement because the ratio of \( g(t) \) and the integrand of \( F(n) \) is equal to \( 1 + O(n^{-K}) \) for any \( K > 0 \) for \( t \) near \( a \). 

\[ \square \]

3 Proof of Theorem 1.3

Our goal is to show that \( \{\gamma(n)\} \) satisfies the growth conditions of Definition 1.1. Recall from Section 1 that

\[ \gamma(n) = \begin{cases} (-1)^n \frac{n!}{(2n)!} \Xi^{(2n)}(0) & \text{if } c = 1, \\ \frac{2^{n+1} n!}{(2n+1)!} \Xi^{(2n+1)}(0) & \text{if } c = -1, \end{cases} \]

where \( \gamma(n) \) are the Taylor coefficients of \( \Xi_1(x) \). Therefore, if we define

\[ \hat{F}(n) = \frac{a(n_0)\sqrt{2\pi N^e}((1 + (-1)^n)c)L^{m+1}}{2^{n+1}n!(1 + \frac{k}{2} - 1)L^2}e^{\frac{k}{2}(L\log(N))^{-\frac{1}{2}}-\frac{1}{2}+1\left(1 + \frac{b_1}{n}\right)} \]  

(3.1)

and

\[ \Xi^{(n)}(0) = \begin{cases} (-1)^n (\frac{n}{2})! \hat{F}(n - 2) & \text{if } \Lambda(s) \text{ has a pole}, \\ (-1)^n \hat{F}(n) & \text{otherwise} \end{cases} \]

we have the relation

\[ \tilde{\gamma}(n) = \begin{cases} (\frac{n!}{(2n-1)!})\hat{F}(2n-2) & \text{if } \Lambda(s) \text{ has a pole and } c = 1, \\ (\frac{n!}{(2n-1)!})\hat{F}(2n-1) & \text{if } \Lambda(s) \text{ has a pole and } c = -1, \\ (\frac{n!}{(2n)!})\hat{F}(2n) & \text{if } \Lambda(s) \text{ does not have a pole and } c = 1, \\ (\frac{n!}{(2n+1)!})\hat{F}(2n+1) & \text{if } \Lambda(s) \text{ does not have a pole and } c = -1. \end{cases} \]

(3.2)

We will show that

\[ \gamma(n) = \frac{n!}{m!} \hat{F}(m) \cdot (1 + O\left(\frac{1}{n^{2-\varepsilon}}\right)) \]

for \( m = 2n - 2, 2n - 1, 2n, \) or \( 2n + 1 \) form a Hermite–Jensen sequence. Recall that

\[ b_1 = \frac{2(31L^4 + 189L^3 + 542L^2 + 744L + 496)}{3(L + 2)^3}. \]

Using Stirling’s approximation \( r! = \sqrt{2\pi r^e} \cdot \left(1 + \frac{1}{12r}\right) \cdot \left(1 + O(1/r^2)\right) \) and (2.4), we have

\[ \gamma(n) = \frac{a(n_0)N^e((1 + (-1)^n)c)e^m - n^{m+\frac{1}{2}}(1 + \frac{1}{12m})L(m)^m}{2^m m^{m+\frac{1}{2}}(1 + \frac{1}{12m})C(m)^m} \times \exp\left(\frac{k}{4}(L(m) - \log(N)) - \frac{2m}{L(m)} - k \frac{2}{m} + 1\right) \left(1 + \frac{b_1(m)}{m}\right) \cdot \left(1 + O\left(\frac{1}{n^{2-\varepsilon}}\right)\right). \]

(3.3)

Recall that \( L(m) \) and \( b_1(m) \) are given in Theorem 2.1 and \( C(m) = m(L(m))^{-1} + L(m)^{-2} + \frac{k}{(L(m))^{-1}} \). Now, \( L(m) \) can be viewed as a holomorphic and non-vanishing function for \( \Re(m) > 0 \), so we have a Taylor expansion in \( j \) for the ratio \( L(m + 2j)/L(m) \) given by

\[ \mathcal{L}(j; m) := \frac{L(m + 2j)}{L(m)} = 1 + \sum_{r \geq 1} \frac{\ell_r(m)}{r!} \cdot j^r. \]
which converges when \(|j| < \frac{\pi}{2}\), so we will assume this throughout the proof. If we let \(J = \frac{4m}{\ell}\) for some \(-1 < \lambda < 1\), then the asymptotic
\[
L(m) = \log\left(\frac{\sqrt{Nm}}{\log(\sqrt{Nm})}\right)
\]
gives the limit
\[
\lim_{m \to \infty} \frac{L(j; m)}{L(m)} = 1.
\]
This implies that \(\ell_j(m) = o((\frac{2}{m})^j)\). If we expand
\[
m + 2j = \frac{L(m) \cdot L(j; m)}{2}\left(\pi n_0 e^{\frac{L(m) L(j; m) - \omega(0)}{2}} - \frac{k}{2} + 1\right)
\]
in \(j\), then we find
\[
\ell_1(m) = \frac{8}{4m(L/2 + 1) + L^2(k/2 - 1)} = \frac{2}{C \cdot L^2} \quad \text{and} \quad \ell_2(m) = \frac{-(L/2 + 2)(m + KL/4 - L/2)}{C^3 \cdot L^5},
\]
where \(L = L(m)\) and \(C = C(m)\). We will also define
\[
\mathcal{C}(j; m) := \frac{C(m + 2j)}{C(m)} = 1 + \sum_{r \geq 1} \frac{c_r(m)}{r!} \cdot j^r
\]
and
\[
\mathcal{B}(j; m) := \frac{1 + \frac{b_1(m + 2j)}{m^2j}}{1 + \frac{b_1(m)}{m}} = 1 + \sum_{r \geq 1} \frac{\beta_r(m)}{r!} \cdot j^r.
\]
We have the limits
\[
\lim_{m \to \infty} \mathcal{C}(j; m) = 1 + \lambda \quad \text{and} \quad \lim_{m \to \infty} \frac{m}{2} (\mathcal{B}(j; m) - 1) = 0,
\]
which imply \(c_r(m) = o((\frac{2}{m})^r)\) and \(\beta_r(m) = o((\frac{2}{m})^{r+1})\). Using the expansion for \(\mathcal{L}(j; m)\) and the expression for \(\ell_1(m)\), we can find that
\[
c_1(m) = \frac{L + 2}{C \cdot L^2} - mL + 4)
\]
If we define \(R_j(j; m) := \frac{\gamma(n + j)}{\gamma(n)}\), then after using (3.3) and some manipulations, we have
\[
R_j(j; m) = e^{\frac{1}{nL(m)}2j} \frac{\gamma(n + j)}{2j} \cdot \frac{(m!)^{n+j/2}}{2^{n+j/2} j^{n+j/2}} \cdot \frac{1}{\sqrt{\mathcal{C}(j; m)}} \cdot \frac{L(m)^{n+2j}}{(m)L(m)} \cdot \mathcal{L}(j; m) \cdot \mathcal{B}(j; m) \cdot \left(1 + O\left(\frac{1}{n^{2-\epsilon}}\right)\right).
\]
By equation (3.2), for \(j\) fixed and as \(m \to \infty\) (and thus \(n \to \infty\)), we have
\[
\frac{\gamma(n + j)}{\gamma(n)} = R_j(j; m) \cdot \left(1 + O\left(\frac{1}{n^{2-\epsilon}}\right)\right).
\]
Notice that the first factor in \(R_j(j; m)\) is the \(j\)-th power of \(e^{\frac{1}{nL(m)}2j}\). This factor will essentially be \(e^{A(n)}\). We will now look at the expansion
\[
\log R_j(j; m) = \sum_{r \geq 1} g_r(m) j^r.
\]
If we again let \(J = \frac{4m}{\ell}\) for \(-1 < \lambda < 1\), then we have
\[
\lim_{m \to \infty} \frac{2(\log R_j(j; m) - J \log(\frac{nL}{2m})) - f)}{m} = -(\lambda + 1) \log(\lambda + 1),
\]
which tells us that \(g_r(m) = O((\frac{m}{2})^{1-r})\). We can use our previous expansions and the formula for \(R_j(j; m)\) to
find

\[ g_1(m) = \log \left( \frac{nL^2}{4m^2} \right) + m \cdot \ell_1(m) \left( \frac{L+2}{L} \right) - \frac{k \cdot \ell_1(m) \cdot L}{4} - \frac{c_1(m)}{2} + O \left( \frac{1}{n^{2-\varepsilon}} \right), \]

\[ g_2(m) = -\frac{2}{m} + (4 \ell_1(m) + m \cdot \ell_2(m)) \left( \frac{L+2}{2L} \right) - m \cdot \ell_1(m)^2 \left( \frac{L+4}{2L} \right) + O \left( \frac{1}{n^{2-\varepsilon}} \right). \]

We can use the formulas for \( \ell_1(m) \) and \( \ell_2(m) \) to simplify these to

\[ g_1(m) = \log \left( \frac{nL^2}{4m^2} \right) + \frac{L - 2}{2C \cdot L^2} + \frac{m(L + 4)}{2C^2 \cdot L^4} + O \left( \frac{1}{n^{2-\varepsilon}} \right), \]

\[ g_2(m) = -\frac{2}{m} + \frac{4}{C \cdot L^2} + O \left( \frac{1}{n^{2-\varepsilon}} \right). \]

We now let

\[ A(n) = \log \left( \frac{nL^2}{4m^2} \right) + \frac{L - 2}{2C \cdot L^2} + \frac{m(L + 4)}{2C^2 \cdot L^4}, \]

\[ \delta(n) = -\frac{2}{m} - \frac{4}{C \cdot L^2}. \]

These functions satisfy the conditions of Definition 1.1 for the sequence \( \{y(n)\} \). The fact that \( \delta(n) \to 0 \) follows from the asymptotics given above, and the precision of \( O \left( \frac{1}{n^{2-\varepsilon}} \right) \) satisfies the necessary growth conditions given in Definition 1.1.

# 4 Proofs of corollaries

## 4.1 Dirichlet \( L \)-functions

### 4.1.1 Proof of Corollary 1.4

Let \( \chi \) be a Dirichlet character modulo \( N > 1 \). Then we define the Dirichlet \( L \)-function by

\[ L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s} \]

for \( \text{Re}(s) > 1 \). If we let \( \chi \) be the trivial character, then our \( L \)-function is the Riemann zeta function. This case was handled in [8]. Next, recall the twisted theta function

\[ \theta_\chi(z) = \chi(0) + 2 \sum_{n \geq 1} \chi(n)n^v e^{i\pi n^2 z}, \]

where \( v = 0 \) if \( \chi \) is even and \( v = 1 \) if \( \chi \) is odd. The twisted theta function satisfies the functional equation

\[ \theta_\chi(z) = \frac{\tau(\chi)}{i^v \sqrt{N}(-iNz)^{\frac{1}{2}+v}} \theta_\chi \left( \frac{1}{N^2 z} \right), \tag{4.1} \]

where \( \tau(\chi) \) is a Gauss sum and \( \chi \) is the dual character. We will focus on real primitive self-dual characters, so we have \( \chi = \bar{\chi} \) and \( \tau(\chi) = i^v \sqrt{N} \). Define the completed Dirichlet \( L \)-function by

\[ \Lambda(\chi, s) := \left( \frac{N}{\pi} \right)^{i\frac{v}{2}} \Gamma \left( \frac{s + v}{2} \right) L(\chi, s) = \frac{1}{2} N^{i\frac{v}{2}} \int_0^\infty \theta_\chi(iy) y^{i\frac{v}{2}} dy. \]

Using equation (4.1) and the fact that \( \chi \) is a real primitive self-dual character, we have the following functional equation:

\[ \Lambda(\chi, s) = \Lambda(\chi, 1 - s). \]

The completed Dirichlet \( L \)-function has the required integral representation, functional equation, and real coefficients, so it is good.
4.1.2 Derivatives at central values and Dirichlet–Jensen polynomials

We want to study the derivatives of $\Lambda(\chi, s)$, which are given by

$$\Lambda^{(n)}(\chi, s) = \frac{1}{2^{2n+1}} \int_{\mathbb{R}} \frac{\theta_\chi(iy)((Ny)^{\frac{1}{2}+\frac{s}{2}} + (-1)^n(Ny)^{-\frac{s}{2}+\frac{1}{2}})(\log(N^2) + 2 \log(y))^n}{y} \, dy.$$  

At the central value $s = \frac{1}{2}$, we have

$$\Lambda^{(n)}(\chi, \frac{1}{2}) = \frac{1}{2^{2n+1}} \int_{\mathbb{R}} \frac{\theta_\chi(iy)(Ny)^{\frac{1}{2}+\frac{s}{2}}(1 + (-1)^n)(\log(N^2) + 2 \log(y))^n}{y} \, dy.$$  

Because the Dirichlet $L$-functions fit into our framework, we have the following theorem which gives an arbitrary precision asymptotic formula for these derivatives.

**Theorem 4.1.** Assume the notation above. The large $n$ asymptotics for $\Lambda^{(n)}(\chi, \frac{1}{2})$ and $\Xi^{(n)}(\chi, 0)$ are given by all orders by the asymptotic expansion

$$F(n) \sim \frac{\sqrt{2\pi N^{\frac{1}{2}+\frac{s}{2}}(1 + (-1)^n)}}{2^{2n}} \frac{L^{n+1}}{\sqrt{4n(1 + \frac{s}{2}) - \left(\frac{3}{4} - \frac{v}{2}\right)L^2}} \cdot e^{\left(\frac{1}{2} + \frac{\nu(n)}{N} - \frac{1}{2} - \frac{1}{4}\right)f + \frac{1}{2} \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots\right)} \quad (n \to \infty),$$

where $L = L(n) \approx 2 \log\left(\frac{\log(N^2)}{\log(n)}\right)$ is the unique positive solution to

$$n = \frac{1}{2} \left(\pi e^{\frac{L - \log(N^2)}{2}} + \frac{3}{4} - \frac{v}{2}\right)L$$

and each coefficient $b_k$ belongs to $\mathbb{Q}(L)$, the first value being

$$b_1 = \frac{L^4 + 9L^3 + 31L^2 + 24L + 16}{24(L + 2)^3}.$$  

**Example 4.2.** Let $\chi_4$ be the odd Dirichlet character of modulus 4. Using the two-term approximation $\hat{F}(n)$ given in equation (3.1), we give some approximations $\hat{\gamma}_{\chi_4}(n)$ in Table 1.

| $n$   | $\hat{\gamma}_{\chi_4}(n)$ | $\gamma_{\chi_4}(n)$ | $\gamma_{\chi_4}(n)/\hat{\gamma}_{\chi_4}(n)$ |
|-------|-----------------------------|-----------------------|-----------------------------------------------|
| 10    | $8.6123842782 \times 10^{-14}$ | $8.5921206983 \times 10^{-14}$ | 0.997647158                                     |
| 100   | $1.0054943805 \times 10^{-174}$ | $1.0057957216 \times 10^{-174}$ | 0.9997361785                                    |
| 1000  | $1.7838444188 \times 10^{-2350}$ | $1.7838866878 \times 10^{-2350}$ | 0.9999763051                                    |
| 10000 | $1.7271165350 \times 10^{-30650}$ | $1.7271200653 \times 10^{-30650}$ | 0.99999979560                                  |
| 100000| $8.1291521235 \times 10^{-384416}$ | $8.1291531304 \times 10^{-384416}$ | 0.9999998761                                    |

**Table 1:** Approximations of $\hat{\gamma}_{\chi_4}$.  

In the previous section, we showed that the Dirichlet $L$-function $L(\chi, s)$ is good. Dirichlet $L$-functions have a pole at $s = 1$ precisely when $\chi$ is principal, so we define

$$\Xi(\chi, z) := \begin{cases} 
(-z^2 - \frac{1}{4})\Lambda\left(\frac{1}{2} - iz\right) & \text{if } \chi \text{ is principal}, \\
\Lambda\left(\chi, \frac{1}{2} - iz\right) & \text{else,}
\end{cases}$$

and

$$\Xi_1(\chi, x) := \Xi(\chi, i\sqrt{x}) = \sum_{n \geq 0} \frac{\gamma_{\chi}(n)}{n!} x^n.$$
where
\[ y_\chi(n) = (-1)^n \frac{n!}{(2n)!} \cdot \zeta(2n)(\chi, 0). \]

By Theorem 1.3 or by using the asymptotic expansion above, we know that if \( d \geq 1 \), then \( f_{y_\chi}^d(n) \) is hyperbolic with at most finitely many exceptions \( n \).

**Example 4.3.** To exemplify Corollary 1.4 we will again consider the odd Dirichlet character of modulus 4. Let \( L(n) \) be the unique solution to
\[ n = \frac{L(n)}{2} \left( ne^{\frac{(ln(n-\log(16)))}{2}} + \frac{1}{4} \right) \]
and set \( L = L(2n) \). Also set
\[ C = C(2n) = 2n \left( \frac{2}{L} + \frac{4}{L^2} \right) - \frac{1}{4}, \]
\[ A(n) = \log \left( \frac{L^2}{64n} \right) + \frac{2(L-2)}{CL^2} + \frac{16n(L+4)}{C^2L^4}, \]
\[ \delta(n) = \sqrt{\frac{1}{2n} - \frac{8}{CL^2}}. \]

Table 2 demonstrates for \( d = 2 \) and \( d = 3 \) that
\[ f_{y_\chi}^d(n)(X) = \frac{\delta(n)X - 1}{e^A(n)} \]
converges to \( H_d(X) \) as \( n \to \infty \). The polynomials have been normalized so that their leading coefficients are 1.

| \( n \) | \( \gamma_{y_\chi}^{2,n}(X) \) | \( \gamma_{y_\chi}^{3,n}(X) \) |
|---|---|---|
| 100 | \( X^2 + 0.3332X - 1.9985 \) | \( X^3 + 0.8306X^2 - 5.8678X - 1.3254 \) |
| 1000 | \( X^2 + 0.1136X - 1.9997 \) | \( X^3 + 0.2839X^2 - 5.9847X - 0.4414 \) |
| 10000 | \( X^2 + 0.0375X - 1.9999 \) | \( X^3 + 0.0936X^2 - 5.9984X - 0.1435 \) |
| 100000 | \( X^2 + 0.0012X - 1.9999 \) | \( X^3 + 0.0304X^2 - 5.9998X - 0.0444 \) |
| \( \infty \) | \( H_2(X) = X^2 - 2 \) | \( H_3(X) = X^3 - 6X \). |

Table 2: Degree 2 and 3 Jensen polynomials.

**4.2 Modular L-functions**

**4.2.1 Proof of Corollary 1.5**

Let \( f \in S_k(\Gamma_0(N)) \) be an even weight newform with real coefficients and write \( f(z) = \sum_{n \geq 1} a(n) e^{2\piinx} \). Assume that \( f \) is normalized so that \( a(1) = 1 \). We focus on newforms with trivial character. Define the \( L \)-function associated to \( f \) by
\[ L(f, s) := \sum_{n \geq 1} \frac{a(n)}{n^s} \]
for \( \text{Re}(s) > 1 + \frac{k}{2} \). Define the completed modular \( L \)-function by
\[ \Lambda(f, s) := N^\frac{z}{2} (2\pi)^{-s} \Gamma(s) L(f, s). \]

We have the transformation property
\[ f \left( \frac{i}{Ny} \right) = i^k e_f N^\frac{k}{2} y^k f(iy), \]
which gives rise to the functional equation
\[ \Lambda(f, s) = i^k e_f \Lambda(f, k - s), \]
where \( \epsilon_f \in \{\pm 1\} \) is the eigenvalue of \( f \) under the Atkin–Lehner involution. The completed modular \( L \)-function \( \Lambda(f, s) \) has the required integral representation, the modular properties of \( f(z) \) gives a functional equation, and the coefficients are real, so \( L(f, s) \) is good.

### 4.2.2 Derivatives at central values and modular Jensen polynomials

Similarly to the Dirichlet \( L \)-function case, the \( n \)-th derivative takes the form

\[
\Lambda^{(n)}(f, s) = \frac{1}{2^n} \int_{\frac{i}{\sqrt{N}}}^\infty f(iy)(N^{\frac{1}{2}}y^s + (-1)^n i^n \epsilon_f N^{\frac{k-1}{2}} y^{k-s})(\ln(N) + 2 \ln(y))^n \frac{dy}{y}.
\]

At the central value \( s = \frac{k}{2} \) we have

\[
\Lambda^{(n)}(f, \frac{k}{2}) = \frac{1}{2^n} \int_{\frac{i}{\sqrt{N}}}^\infty f(iy)N^{\frac{k}{2}}y^{\frac{k-1}{2}}(1 + (-1)^n i^n \epsilon_f)(\ln(N) + 2 \ln(y))^n dy.
\]

The following theorem gives an arbitrary precision asymptotic formula for these derivatives at central values.

**Theorem 4.4.** Assume the notation above. Large \( n \) asymptotics for \( \Lambda^{(n)}(f, \frac{k}{2}) \) and \( \Xi^{(n)}(f, 0) \) is given to all orders by the asymptotic expansion

\[
F(n) \sim \sqrt{2\pi N^{\frac{k}{2}}(1 + (-1)^n i^n \epsilon_f)} \frac{L^{n+1}}{2^{n+1} \sqrt{(1 + \frac{k}{2})n - \frac{k}{2}L^2}} \times \epsilon^{\frac{1}{2}(L - \log(N)) - \frac{k-1}{2}} 1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots (n \to \infty),
\]

where

\[ L = L(n) = 2 \log \left( \frac{n \sqrt{N}}{\log(n \sqrt{N})} \right) \]

is the unique solution of the equation

\[ n = \frac{1}{2} \left( \pi e^{-\frac{L - \log(n)}{2}} - \frac{k}{2} + 1 \right) L \]

and each coefficient \( b_k \) belongs to \( \mathbb{Q}(L) \), the first value being

\[ b_1 = \frac{L^4 + 9L^3 + 32L^2 + 24L + 16}{24(L + 2)^3}. \]

We have shown that the modular \( L \)-function \( L(f, s) \) is good and does not have a pole, so we define

\[ \Xi(f, z) := \Lambda(f, \frac{k}{2} - iz). \]

Depending on the sign of the functional equation, we define the Taylor coefficients by

\[ \Xi_1(f, x) = \sum_{n \geq 0} \frac{y_f(n)}{n!} x^n = \begin{cases} \Xi(i \sqrt{x}) & \text{if } i^n \epsilon_f = 1, \\ \Xi(i \sqrt{x}) & \text{if } i^n \epsilon_f = -1, \end{cases} \]

where

\[ y_f(n) = \begin{cases} (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0) & \text{if } i^n \epsilon_f = 1, \\ \frac{i^{2n+1}}{2^{2n+1}} \frac{n!}{(2n + 1)!} \cdot \Xi^{(2n+1)}(0) & \text{if } i^n \epsilon_f = -1. \end{cases} \]

By Theorem 1.3 or from the asymptotic expansion above, we have that if \( d \geq 1 \), then \( J_y^{d,n}(X) \) is hyperbolic with at most finitely many exceptions \( n \).
4.3 Dedekind zeta-functions

4.3.1 Proof of Corollary 1.6

The Dedekind zeta-function case will require some setup and notation. We will mostly follow the notation in [10]. Let $K$ be a number field of degree $j$ and let $\mathcal{O}_K$ be its ring of integers. Denote the embeddings by $\sigma_1, \ldots, \sigma_r, \rho_1, \overline{\rho}_1, \ldots, \rho_{r_2}, \overline{\rho}_{r_2}$, where there are $r_1$ real embeddings and $r_2$ pairs of complex embeddings so that $r_1 + 2r_2 = j$. Denote the class group of $K$ by $\text{Cl}(K)$. Let $\mathcal{C} = \prod_{r} \mathbb{C}$ and $\mathcal{R} = \{z \in \mathbb{C} : z = \overline{z}\}$ be the Minkowski space of $K$, where $\mathcal{Z} = \overline{\mathcal{Z}} = (\mathbb{Z}_p)$ is the usual complex conjugation and $\tau$ runs over the $j$ embeddings. We define the trace and norm by

$$\text{Tr}(z) = \sum_{\tau} z_\tau, \quad N(z) = \prod_{\tau} z_\tau,$$

and have a Hermitian scalar product given by

$$\langle x, y \rangle = \sum_{\tau} x_\tau \overline{y}_\tau.$$

We will also require the spaces

$$\mathbb{R}_+ = [\prod_{r} \mathbb{R}]^+ = \{x \in \mathbb{R} : x_\tau = x\}$$

and

$$\mathbb{R}_+^* = [\prod_{r} \mathbb{R}_+]^+ = \{x \in \mathbb{R}_+ : x_\tau > 0 \text{ for all } \tau\}$$

in order to define the two homomorphisms

$$| \cdot | : \mathbb{R}_+^* \to \mathbb{R}_+^*, \quad x = (x_\tau) \mapsto |x| = (|x_\tau|),$$

$$\log : \mathbb{R}_+^* \to [\mathbb{R}_+]^+, \quad x = (x_\tau) \mapsto \log x = (\log x_\tau).$$

Let $p = \{\sigma, \overline{\sigma}\}$ denote a conjugacy class of embeddings (so $p$ has one or two elements depending on whether the embedding is real or complex) and observe that there is an isomorphism between $\mathbb{R}_+^*$ and $\prod_p \mathbb{R}_+^*$. We now have a Haar measure, which we denote by $\psi_p$, that corresponds to the product measure $\prod_p \psi_t$, where $\psi_t$ is the usual Haar measure on $\mathcal{R}_+$. We can now define a suitable generalization of the gamma function by

$$\Gamma_K(s) = 2^{(1-2s)r_2} \Gamma(s)^{r_1} \Gamma(2s)^{r_2} = \int_{\mathbb{R}_+^*} N(e^{-y})^{s} \frac{dy}{y}.$$

The Dedekind zeta-function for $K$ is given by

$$\zeta_K(s) = \sum_{a \in \mathcal{O}_K \text{ integral}} N(a)^{-s},$$

for $\text{Re}(s) > 1$, where $N(a) = [\mathcal{O}_K : a]$ is the norm of the ideal $a$. For each $B \in \text{Cl}(K)$ we define the partial zeta function by

$$\zeta_B(s) = \sum_{a \in B \text{ integral}} N(a)^{-s}.$$

We therefore have

$$\zeta_K(s) = \sum_{B \in \text{Cl}(K)} \zeta_B(s),$$

We define the completed partial Dedekind zeta-function by

$$\Lambda(B, s) = |d_K|^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma_K\left(\frac{s}{2}\right) \zeta_B(s) = \int_{\mathbb{R}_+^*} g(iy) N(y)^{\frac{s}{2}} \frac{dy}{y},$$
where $d_k$ is the discriminant of $K$ and $g$ is some theta function that we will not specify now. The image of the unit group $\mathcal{O}_K^*$ under the mapping $|\cdot| : \mathbb{R}^* \to \mathbb{R}_+^*$, which we will denote by $|\mathcal{O}_K^*|$, is contained in the norm-one hypersurface

$$S = \{ x \in \mathbb{R}_+^* : N(x) = 1 \}.$$  

We obtain a direct decomposition $\mathbb{R}_+^* = S \times \mathbb{R}_+^*$ by writing

$$y = xt^\frac{1}{h}, \quad x = \frac{y}{N(y)^{\frac{1}{h}}}, \quad t = N(y),$$

for any $y \in \mathbb{R}_+^*$. We will need to choose a fundamental domain $F$ for the action of the group

$$|\mathcal{O}_K^*|^2 = \{ |e|^2 : e \in \mathcal{O}_K^* \}$$

on $S$. The log map $\log : \mathbb{R}_+^* \to \mathbb{R}$, takes $S$ to the trace-zero space

$$H = \{ x \in \mathbb{R}_+ : \text{Tr}(x) = 0 \}$$

and by Dirichlet’s unit theorem the group $|\mathcal{O}_K^*|$ is taken to a complete lattice $G$ in $H$. We may choose $F$ to be the preimage of any fundamental mesh of the lattice $2G$. Now using this decomposition, we have that

$$\Lambda(B, s) = \int_0^\infty (f(a, t) - f(a, \infty))t^{\frac{s}{2}} \frac{dt}{t},$$

where $B$ is the class of $a^{-1}$ and

$$f(a, t) = \frac{1}{w_K} \int \theta(a, ixt^\frac{1}{h})d^*x.$$ 

In the above equation $w_K$ is the number of roots of unity in $K$, $d^*x$ is the appropriate Haar measure such that $d^*x \times \frac{dt}{t} = \frac{dy}{y}$, and the theta function is defined by

$$\theta(a, z) = \sum_{a \in \mathcal{O}_K^*} e^{\pi i d^{-1/2}(az, a)},$$

where $d_a = |N(a)|^2 d_K$ is the absolute value of the discriminant of $a$. By using the properties of the theta function, it is not difficult to show

$$f_F(a, \frac{1}{t}) = t^\frac{s}{2} f_{F^{-1}}((a\mathcal{O}_K)^{-1}, t)$$

and

$$f_F(a, \infty) = f(\infty) = \frac{2^{\frac{s}{2}} \pi^\frac{s}{2} \Gamma(\frac{s}{2})}{w_K} R(K),$$

where $F^{-1}$ is again a fundamental domain, $\mathcal{O}_K$ is the different ideal of $K$, and $R(K)$ is the regulator of $K$. Note that $(a\mathcal{O}_K)^{-1}$ is the dual lattice of $a$ and that $f(\infty)$ does not depend on the fundamental domain or ideal choice, so we will suppress notation whenever possible. We now define the completed Dedekind zeta-function by

$$\Lambda(K, s) = \sum_{B \in \text{Cl}(K)} \Lambda(B, s) = |d_K|^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta_K(s)$$

$$= \frac{2^{r_1+r_2} R(K) h_K}{s(s-1) w_K} + \sum_{i=1}^{h_K} \int_1^\infty (f(a_i, t) - f(\infty))(t^{\frac{s}{2}} + t^{\frac{s}{2}}) \frac{dt}{t},$$

where $h_K$ is the class number of $K$ and if $B_i$, $1 \leq i \leq h_K$, are ideal classes, then $B_i$ is the class of $a_i^{-1}$. This shows that we have the functional equation

$$\Lambda(K, s) = \Lambda(K, 1 - s).$$

The completed Dedekind zeta-function has suitable integral representation, functional equation, and real coefficients, so $\zeta_K(s)$ is good.
4.3.2 Derivatives at central values and Dedekind–Jensen polynomials

The $n$-th derivative of the completed Dedekind zeta-function has the form

$$
\Lambda^{(n)}(K, s) = \frac{2^{r_1 + r_2} R(K) h_K \cdot n!}{w_K} \frac{(s - 1)^{n+1} - s^{n+1}}{s^{n+1}(1 - s)^{n+1}} + \sum_{i=1}^{h_K} \frac{1}{2\pi i} \int_{\gamma_i} (f(a_i, t) - f(\infty)) (t^{1/2} + (1 - 1)^n t^{1/2}) \log^n(t) \frac{dt}{t}.
$$

At the central value $s = \frac{1}{2}$, we have

$$
\Lambda^{(n)}(K, 1/2) = \frac{2^{r_1 + r_2 + n+1} R(K) h_K ((-1)^{n+1} - 1) n!}{w_K} + \sum_{i=1}^{h_K} \frac{1}{2\pi i} \int_{\gamma_i} (f(a_i, t) - f(\infty)) t^{1/2} (1 + (-1)^n) \log^n(t) \frac{dt}{t}.
$$

In order to state the asymptotic expansion we need to find the first nonzero coefficient of each $f(a_i, t)$. Let $e$ be a unit with norm 1. Then the smallest nonzero exponent in $f(a_i, t)$ is given by

$$
m_{a_i} = \min \{ \langle ae, a \rangle : a \in a_i, \ a \neq 0 \}.
$$

Let

$$
M_{a_i} = \# \{ a \in a_i : \langle ae, a \rangle = m_{a_i} \}.
$$

Then the expansion of $f(a_i, t)$ begins as follows:

$$
f(a_i, t) = f(\infty) + \frac{2^{r_1 + r_2 - 1} R(K)}{w_K} M_{a_i} e^{-nm_{a_i}\left(\frac{i}{2\pi m_{a_i}}\right)^{1/2}} + \cdots.
$$

We will let

$$
C_i = \frac{2^{r_1 + r_2 - 1} R(K)}{w_K} M_{a_i} \quad \text{and} \quad F_i(n) = \frac{1}{2\pi i} \int_{\gamma_i} (f(a_i, t) - f(\infty)) t^{1/2} (1 + (-1)^n) \log^n(t) \frac{dt}{t}
$$

in order to simplify the next theorem.

**Theorem 4.5.** Assume the notation above. Then we have

$$
\Lambda^{(n)}(K, 1/2) = \frac{2^{r_1 + r_2 + n+1} R(K) h_K ((-1)^{n+1} - 1) n!}{w_K} + \sum_{i=1}^{h_K} F_i(n)
$$

and $F_i(n)$ is given to all orders by the asymptotic expansion

$$
F_i(n) \sim C_i \sqrt{2\pi (1 + (-1)^n)} \frac{L_i^{n+1}}{2^n} \left( \frac{L_i}{n + \frac{L_i}{T}} - \frac{3}{4} \right) \times e^{-n \frac{1}{4} \pi n \frac{L_i}{n} + \frac{b_{i,1}}{n} + \frac{b_{i,2}}{n^2} + \cdots} \quad (n \to \infty),
$$

where $L_i = L_i(n) \approx j \log \left( \frac{n}{\log(n)} \right)$ is the unique solution of the equation

$$
n = \left( \frac{m_{a_i} d_{a_i}^{3/2}}{j} \pi e^{\frac{L_i}{2} - 3/4} \right) L_i
$$

and each coefficient $b_{i,k}$ belongs to $\mathbb{Q}(L_i)$.

We have shown that $\zeta_K(s)$ is good, so we define

$$
\Xi(z) \coloneqq (-z^2 - \frac{1}{4}) \Lambda \left( K, \frac{1}{2} - iz \right)
$$

and

$$
\Xi_1(x) \coloneqq \Xi(i \sqrt{x}) = \sum_{n \geq 0} \frac{\gamma_K(n)}{n!} x^n,
$$

where the Taylor coefficients are given by

$$
\gamma_K(n) = (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0).
$$
The derivatives $\Xi^{(2n)}(0)$ are given by
\[
\Xi^{(2n)}(0) = (-1)^n \sum_{i=1}^{h_k} \frac{8^{(2n)} F_i(2n - 2) - F_i(2n)}{4},
\]
and so we can use the above asymptotic expansion or Theorem 1.3 to show that if $d \geq 1$, then $f_{\lambda_k}^{(d,n)}(X)$ is hyperbolic with at most finitely many exceptions $n$.

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