Stability of Landau-Ginzburg branes

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Abstract

We evaluate the ideas of II-stability at the Landau-Ginzburg point in moduli space of compact Calabi-Yau manifolds, using matrix factorizations to B-model the topological D-brane category. The standard requirement of unitarity at the IR fixed point is argued to lead to a notion of “R-stability” for matrix factorizations of quasi-homogeneous LG potentials. The D0-brane on the quintic at the Landau-Ginzburg point is not obviously unstable. Aiming to relate R-stability to a moduli space problem, we then study the action of the gauge group of similarity transformations on matrix factorizations. We define a naive moment map-like flow on the gauge orbits and use it to study boundary flows in several examples. Gauge transformations of non-zero degree play an interesting role for brane-antibrane annihilation. We also give a careful exposition of the grading of the Landau-Ginzburg category of B-branes, and prove an index theorem for matrix factorizations.

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1 Introduction

The main purpose of this work is to develop a stability condition, to be called “R-stability”, on the triangulated category of matrix factorizations describing D-branes at the Landau-Ginzburg point $p_{LG}$ in the Kähler moduli space $\mathcal{M}_k$ of a compact Calabi-Yau manifold $X$. The proposal is motivated by physical considerations similar to the ones leading to the notion of $\Pi$-stability on the derived category of coherent sheaves $\mathcal{D}(X)$, which describes the variation of the spectrum of B-type BPS branes over $\mathcal{M}_k$. In fact, the notion of R-stability can be thought of as the specialization of $\Pi$-stability to $p_{LG}$. It is expected, however, that R-stability should be intrinsic to the Landau-Ginzburg model and does in principle not depend on knowledge of the stable spectrum elsewhere in $\mathcal{M}_k$.

In this paper, section 2 is a brief review of the relevant aspects of $\Pi$-stability that we want to abstract to the Landau-Ginzburg model. Section 3 contains the basic definitions related to matrix factorizations. Section 4 explains how quasi-homogeneous matrix factorizations can be, first $\mathbb{Q}$-, then $\mathbb{Z}$-graded. Section 5 is a somewhat independent unit concerned with the RR charges of matrix factorizations in string theory and an index theorem. Section 6 gives a preliminary definition of R-stability and partial answers to the difficulties in relating it to the action of the gauge group on matrix factorizations. This general discussion is then applied in section 7 and the proposal shown to work well in several relevant examples. Section 8 gives a summary.

2 Review of $\Pi$-stability

$\Pi$-stability was introduced in [1, 2], and further sharpened and tested in [3, 4]. It was subsequently abstracted into a precise mathematical definition of stability condition on triangulated categories in [5]. We refer to these works for the categorical aspects of $\Pi$-stability, as well as to Aspinwall’s review [6] for more extensive background material. Instead, we begin with a slightly personal review of the worldsheet origin of $\Pi$-stability, following Douglas [2].

The basic physical intuition is quite simple. Consider a fixed 2-dimensional conformally invariant string worldsheet quantum field theory $\mathcal{C}$ defining a closed string background. By definition, a D-brane in this background is a conformally invariant boundary condition, $\mathcal{B}$, for $\mathcal{C}$. A popular way to define $\mathcal{C}$’s and $\mathcal{B}$’s is as IR fixed points of bulk or boundary RG flows, induced by turning on a relevant operator $\mathcal{O}$ in a known
bulk or boundary theory. Such a UV description is “stable” if it flows to a theory in the infrared which is “acceptable” in the sense of, e.g., having the right central charge, being unitary, etc.. Finding necessary and/or sufficient stability conditions on $\mathcal{O}$ is in general a very hard question.

A situation in which more can be said is when one requires bulk and boundary theories to preserve $\mathcal{N} = 2$ supersymmetry with a non-anomalous $U(1)$ R-symmetry, so that the chiral algebra underlying $\mathcal{C}$ and $\mathcal{B}$ will contain the $\mathcal{N} = 2$ superconformal algebra. A necessary condition on acceptable $\mathcal{B}$’s is that the R-charges of all open string NS chiral primary operators satisfy the unitarity constraint

\begin{equation}
0 \leq q \leq \hat{c},
\end{equation}

where $\hat{c}$ is the central charge of the superconformal algebra. Often, $\hat{c}$ and $q$ can be determined in the UV and the equation (2.1) therefore provides a stability condition in the above sense.

The ideas of $\Pi$-stability in fact go further than (2.1). Assume that $\hat{c}$ and the R-charges of $\mathcal{C}$ are all integral. The chiral algebra of $\mathcal{C}$ then contains, in addition to the $\mathcal{N} = 2$ superconformal algebra, the (square of the) spectral flow operator $\mathcal{S}$. One can then contemplate imposing a boundary condition of the form

\begin{equation}
\mathcal{S}_L = e^{i\pi \varphi} \mathcal{S}_R,
\end{equation}

involving an arbitrary phase, $\varphi$. Standard conformal field theory arguments\footnote{Using the doubling trick, one transports $\mathcal{S}^2$ around an open string vertex operator inserted at the boundary of the worldsheet, and notes that the total monodromy of $\mathcal{S}^2$, which evaluates to the difference of phases, measures the $U(1)$ charge of the operator up to an integer.} then show that the R-charges of an open string spanning between two branes $\mathcal{B}$ and $\mathcal{B}'$ (with phase $\varphi$ and $\varphi'$) satisfy

\begin{equation}
q = \varphi' - \varphi \mod \mathbb{Z}.
\end{equation}

If we bosonize the left- and right-moving $U(1)$ currents in terms of two canonically normalized chiral bosons $\phi_L$ and $\phi_R$, the spectral flow operators are $\mathcal{S}_{L,R} = e^{i\frac{\pi}{\sqrt{\hat{c}}} \phi_{L,R}}$, and we can visualize the boundary condition as Dirichlet or Neumann boundary condition on the compact boson $\phi = \phi_L \pm \phi_R$ with radius $\sqrt{\hat{c}}$. (The sign depending on which side (A or B) of the mirror one chooses to present the conformal field theory.) Of course, the equation $e^{i\frac{\pi}{\sqrt{\hat{c}}} \phi_L} = e^{i\pi \varphi} e^{i\frac{\pi}{\sqrt{\hat{c}}} \phi_R}$ leaves a $\hat{c}$-fold ambiguity on the
position/Wilson line of the boundary condition on $\phi$. In such a picture \cite{2}, the strings stretching between the different images correspond to the different values of $q$ in (2.3).

We emphasize that, in conformal field theory, $\varphi$ is defined as a real number modulo even integers. We should also like to stress that $\varphi$ is, in general, independent of the phases appearing in the boundary condition on the $\mathcal{N} = 2$ currents, as in, $G_L^\pm = e^{\pm i\alpha} G_R^\pm$, $G_L^\mp = e^{\pm i\beta} G_R^\mp$ for A- and B-type, respectively. (See, e.g., \cite{8} for a BCFT discussion of this.) $\varphi$ determines which $\mathcal{N} = 1$ spacetime supersymmetry is preserved by the brane, and can be different for different branes. On the other hand, the phases appearing in the boundary condition on the $\mathcal{N} = 2$ currents determine which $\mathcal{N} = 1$ worldsheet supersymmetry is preserved. This is a gauge symmetry and has to be the same for all branes.

Now recall that an $\mathcal{N} = 2$ field theory (conformal or not) with a conserved $U(1)$ R-current can be twisted to a topological theory. As anticipated in \cite{9}, and by now well appreciated in the physics literature, the set of branes in the topological theory together with open strings between them carries the algebraic structure of a “triangulated category” (plus more). Two important pieces of structure are, firstly, the so-called “distinguished triangles”, such as

\begin{align}
\begin{diagram}
1234567890
1&2&3&4&5&6&7&8&9&10
\end{diagram}
\end{align}

which expresses the fact that the topological brane $B$ can be obtained as a topological bound state of the two branes $B_1$ and $B_2$ by condensing the “topological tachyon” $T$ on the base of the triangle. Secondly, a triangulated category has a so-called shift functor, which in physics terms sends a brane $B$ to a copy of its antibrane $B[1].$

In relating the physical to the topological theory, one chooses a lift of the phase $\varphi$ to a real number called “grade” and identifies the ghost number $n$ of open strings as the integer appearing in (2.3), i.e.,

$$n = q + \varphi - \varphi'. \quad (2.5)$$

Consequently, for every physical brane $B$ there are in fact an infinite number of topological branes $B[m]$ whose grade differs by an integers. Shifting the grade shifts the ghost number by integers, and hence modifies the topological theory. On the other hand, the topological theory is unaware of the unitarity constraint (2.1). In particular,
the topological theory is independent of changes of \( q \) and \( \varphi \). This decoupling of the topological theory from the variation of \( q \) with (part of) the moduli is one of the central ideas underlying \( \Pi \)-stability.

\( \Pi \)-stability, then, is designed to decide when the bound state formation described in the topological theory by triangles such as (2.4) is stable in the physical theory, and thereby provides a picture of the spectrum of BPS branes in some given closed string background. Let us, for concreteness, focus on the case of B-type D-branes on a Calabi-Yau manifold \( X \). In that case, the topological branes are objects of the derived category of coherent sheaves, \( \mathbf{D}(X) \), of the algebraic variety underlying \( X \). \( \mathbf{D}(X) \) depends only on the complex structure of \( X \), and is independent of the Kähler moduli. Within this category of topological branes, the set of stable branes, conjectured to flow to BPS branes in the physical theory, varies over the stringy Kähler moduli space \( \mathcal{M}_k \) of \( X \). Essentially, one follows the continuous variation of the phases \( \varphi \), and hence of \( U(1) \) R-charges of open strings, over \( \mathcal{M}_k \). Charges leaving/entering the unitarity bound (2.1) signal loss/gain of stable branes, with decay and bound state formation described by the triangles (2.4).

For the details of this construction, consistency with monodromies in \( \mathcal{M}_k \), and a lot of examples, see ref. [6]. One peculiar aspect of the story is that \( \Pi \)-stability really only describes the changes of the BPS spectrum as one moves around in \( \mathcal{M}_k \). To determine the spectrum at any given point \( p \) of \( \mathcal{M}(X) \), one has to know the spectrum at some distinguished point \( p_0 \) and then follow it to \( p \) using \( \Pi \)-stability.

One natural choice for basepoint is large volume, \( p_{LV} \), in the compactification of \( \mathcal{M}_k \). At \( p_{LV} \), \( \Pi \)-stability reduces to \( \mu \)-stability for the Abelian category of coherent sheaves on \( X \) [1]. Although \( \mu \)-stability does not extend over an open neighborhood of \( p_{LV} \) (and hence does not allow determining the complete BPS spectrum there), it is at present the only useful handle on the spectrum elsewhere in \( \mathcal{M}_k \).

Another special point, which one expects exists when \( X \) is a non-compact Calabi-Yau manifold, is the so-called “orbifold point” \( p_O \) in \( \mathcal{M} \). Even if \( X \) is not the resolution of an actual orbifold singularity, one may define \( p_O \) as a point in \( \mathcal{M}_k \) at which the phases of all branes are aligned. In such a situation, determining the BPS spectrum is a problem of solving F- and D-flatness conditions in a supersymmetric quiver gauge theory, as argued in [1]. More rigorously, Aspinwall shows in [10] that in an open neighborhood of such an orbifold point, \( \Pi \)-stability reduces to \( \theta \)-stability for the Abelian category of quiver representations in the sense of King [11].
Such a point at which all phases align is expected not to exist in the moduli space of a generic compact Calabi-Yau model. The closest one can get seems to be the Landau-Ginzburg point, which resembles ordinary orbifolds in the appearance of a discrete quantum symmetry, but with the important difference that not all phases of branes are aligned. The purpose of the present paper, pursuing a suggestion made in [12, 13], is to investigate the ideas underlying II-stability at the Landau-Ginzburg orbifold point in the Kähler moduli space of compact Calabi-Yau manifolds, using the recently introduced description of the topological category using matrix factorizations.

We now turn to explaining various (old and new) aspects of matrix factorizations, and pick up the stability discussion in section 6.

3 Matrix factorizations

Let $W \in \mathbb{R} = \mathbb{C}[x_1, \ldots, x_r]$ be a polynomial. To keep things simple, we will assume throughout that $W$ has an isolated critical point at the origin $x_i = 0$. A matrix factorization (of dimension $N$) of $W$ is a pair of square matrices $f, g \in \text{Mat}(N \times N, \mathbb{R})$ with polynomial entries satisfying

$$fg = gf = W \cdot \text{id}_{N \times N}. \quad (3.1)$$

A matrix factorization is called reduced if all entries of $f$ and $g$ have no constant term, i.e., $f(0) = g(0) = 0$.

Matrix factorizations $(f, g)$ and $(f', g')$ are called equivalent if they are related by a similarity transformation

$$U_1 f = f' U_2 \quad U_2 g = g' U_1 \quad (3.2)$$

where $U_1, U_2 \in \text{GL}(N, \mathbb{R})$ are invertible matrices with polynomial entries.

3.1 Maximal Cohen-Macaulay modules

Matrix factorizations originated in Eisenbud’s work [14] in the context of so-called maximal Cohen-Macaulay modules over local rings of hypersurface singularities. See [15,16] for some background. An example of such a ring is given by $\tilde{R}_m = R_m/(W)$, where $R_m = \mathbb{C}[[x_1, \ldots, x_r]]$ is the complete local ring of power series, with maximal ideal $m = (x_1, \ldots, x_r)$, and $W$ is a polynomial, as above. If $(f, g)$ is a matrix factorization
of $W$, consider the $\mathcal{R}_m$-module $M = \text{Coker} f$ with the $\mathcal{R}_m$-free resolution

$$0 \longrightarrow G \xrightarrow{f} F \longrightarrow M \longrightarrow 0, \quad (3.3)$$

where $F \cong G \cong (\mathcal{R}_m)^N$ are rank $N$ free modules. Since multiplication by $W$ on (3.3) is homotopic to zero, $M$ descends to a $\tilde{\mathcal{R}}_m$-module, with the infinite free resolution

$$\cdots \longrightarrow \tilde{G} \xrightarrow{f} \tilde{F} \xrightarrow{g} \tilde{G} \xrightarrow{f} \tilde{F} \longrightarrow M \longrightarrow 0. \quad (3.4)$$

with $\tilde{F} \cong \tilde{G} \cong (\tilde{\mathcal{R}}_m)^N$.

The resolution (3.3) being of length one, which is the codimension of a hypersurface, makes $M$ into a so-called maximal Cohen-Macaulay module (MCM) over $\tilde{\mathcal{R}}_m$ (see [15, 16] for the definitions). Eisenbud’s theorem [14] essentially says that all MCMs over hypersurface rings come from matrix factorizations.

The category of Cohen-Macaulay modules [15] will be denoted by $\text{MCM}(W)$. Objects of $\text{MCM}(W)$ are matrix factorizations of $W$ and morphism are morphisms of Cohen-Macaulay modules. In other words, a morphism from $(f, g)$ to $(f', g')$ in $\text{MCM}(W)$ is a pair of $N' \times N$-dimensional matrices $a, b$, with polynomial entries, satisfying

$$bg = g'a \quad af = f'b, \quad (3.5)$$

so that the diagram

$$\begin{array}{ccc}
F & \xrightarrow{g} & G \\
\downarrow a & & \downarrow b \\
F' & \xrightarrow{g'} & G'
\end{array} \xrightarrow{f} \begin{array}{ccc}
F & \xrightarrow{f} & F \\
\downarrow a & & \downarrow a \\
F' & \xrightarrow{f'} & F'
\end{array}$$

(3.6)

commutes. We will make no direct use of the category $\text{MCM}(W)$, but have included its definition here since it might play a role in a precise formulation of $R$-stability.

### 3.2 Triangulated category

A different category’s construction based on matrix factorization was observed by Kontsevich [17]. The construction starts from triples $(M, \sigma, Q)$, where $M$ is a free $\mathcal{R} = \mathbb{C}[x_1, \ldots, x_r]$-module with a $\mathbb{Z}_2$-grading $\sigma$, and $Q$ is an odd ($\sigma Q + Q \sigma = 0$) endomorphism of $M$ satisfying

$$Q^2 = W \cdot \text{id}_M. \quad (3.7)$$
Decomposing $M = M_0 \oplus M_1$ into homogeneous components, with equal rank $N$, $Q$ can be represented as the matrix

$$Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}, \quad (3.8)$$

making the relation of $3.7$ to $3.1$ obvious. The grading is then given by the matrix

$$\sigma = \begin{pmatrix} \text{id}_{N \times N} & 0 \\ 0 & -\text{id}_{N \times N} \end{pmatrix}. \quad (3.9)$$

Let us denote by $\text{DG}(W)$ the category which has such triples as objects and as morphisms the (even) morphisms of free modules (forgetting the $Q$’s). The gauge transformations in $\text{DG}(W)$ are the even automorphisms of $M$ as an $R$-module, $GL^+(2N, R)$, acting as

$$GL^+(2N, R) \ni U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : Q \mapsto UQU^{-1}, \quad (3.10)$$

with $U_1, U_2 \in GL(N, R)$, as in $3.2$.

The point of the construction $[30, 19]$ is that the category $\text{DG}(W)$ has the structure of a differential graded category. This means that morphism spaces $\text{Hom}_R(M, M')$ are equipped with an odd differential $D$ acting as a supercommutator

$$D\Phi = Q'\Phi - \sigma'\Phi \sigma \Phi Q$$

$$= \begin{pmatrix} 0 & f' \\ g' & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \quad (3.11)$$

on morphisms

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3.12)$$

in $\text{DG}(W)$. One easily checks that $D^2 = 0$ by the super-Jacobi identity. By a general construction $[18, 19]$, one can then associate a triangulated category, $\text{MF}(W)$ to $\text{DG}(W)$, which has the same objects as $\text{DG}(W)$ (i.e., triples $(M, \sigma, Q)$), but in which morphisms are given by the $\mathbb{Z}_2$-graded cohomology of $Q$. Thus $\text{Hom}_{\text{MF}(W)}(M, M') = H^*(D) = \text{Ker}D/\text{Im}D$. We shall usually embezzle $M$ and $\sigma$, and simply write $\text{Hom}_{\text{MF}(W)}(Q, Q') = H^0(Q, Q') \oplus H^1(Q, Q')$ for the morphisms in $\text{MF}(W)$. We also write $H^0(Q), H^1(Q)$ for the morphisms from $Q$ to itself.

For future reference, let us spell out a few triangulated constructions in the language of matrix factorizations. Firstly, the shift functor $[1]$ is nothing but the reversal of the
$\mathbb{Z}_2$-grading $\sigma \to -\sigma$, or, equivalently, the exchange of $f$ and $g$, i.e.,

$$Q[1] = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} [1] = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix},$$

(3.13)

with $M$ and $\sigma$ fixed. This operation obviously exchanges $H^0$ with $H^1$. Secondly, given two matrix factorizations $Q_1$ and $Q_2$ and an odd morphism $T \in H^1(Q_1, Q_2)$, we obtain a third factorization simply as

$$Q = \begin{pmatrix} Q_1 & 0 \\ T & Q_2 \end{pmatrix}$$

(3.14)

fitting into the triangle

$$Q,$$

(3.15)

$$\begin{array}{c}
Q_2 \\
\downarrow S_2 \\
T \\
\downarrow S_1 \\
Q_1
\end{array}$$

where

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(3.16)

The construction (3.14) is referred to as the “cone” over the map $T[1] \in H^0(Q_1, Q_2[1])$.

Let us also note explicitly that the construction of $\text{MF}(W)$ implies in particular that we identify matrix factorizations which differ by the direct addition of the trivial factorization $f = 1, g = W$,

$$Q \equiv Q \oplus \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix} \equiv Q \oplus \begin{pmatrix} 0 & W \\ 1 & 0 \end{pmatrix}$$

(3.17)

This identification occurs because adding the trivial factorization does not affect the cohomology of $D$ between $Q$ and any other factorization $Q'$.

### 3.3 Relation to $\mathcal{N} = 2$ Landau-Ginzburg model

Orbifoldized $\mathcal{N} = 2$ Landau-Ginzburg models [20] are known [21, 22] to describe the small-volume continuation of Calabi-Yau sigma models, see [23] for the background. (LG also describe, in particular, the mirrors of CY sigma models, as well as the mirrors of toric Fano and non-compact Calabi-Yau manifolds, but this will not be important here. We will stay on the B-side throughout.)
In the bulk, LG models are characterized by the worldsheet superpotential $W$, such as the polynomial we have been studying in this section. The $x_i$ are $\mathcal{N} = 2$ chiral field variables, whose interaction is described by $W$. The kinetic term for the $x_i$ is described by a Kähler potential $K(x_i, \bar{x}_i)$, and is usually ignored in the discussion of LG models because it does not affect topological quantities such as the chiral ring. What is more, in the quasi-homogeneous case, it is actually conjectured that there is a Kähler potential, uniquely determined by the superpotential, such that the associated model is conformal. This Kähler potential can be reached by RG flow along which $W$ is unchanged by non-renormalization theorems.

When adding boundaries to the worldsheet of an $\mathcal{N} = 2$ LG model, the supersymmetry variation of the superpotential exhibits a peculiar boundary term, whose non-vanishing is known as the Warner problem [24–28]. Following a proposal of Kontsevich, it was shown in [30, 31, 29] that matrix factorizations of $W$ provide a solution of the Warner problem. More precisely, it was argued there that the category of topological B-branes in a Landau-Ginzburg model is equivalent to the category $\text{MF}(W)$ we have described in the previous subsection. The extent to which $\text{MF}(W)$ also describes “physical” branes in the untwisted Landau-Ginzburg model will be the subject of the present paper.

In the LG application, the space $M$ is the Chan-Paton space of a (target space-time filling) DDbar-system, with equal number of branes and antibranes, and $f$ and $g$ describe a tachyon configuration. The shift functor [1] is nothing but the exchange of branes with antibranes. The matrix $Q$ is part of the BRST charge, and the matrix factorization equation $Q^2 = W$ is the condition that the tachyon configuration be BRST invariant (or preserve $\mathcal{N} = 2$ supersymmetry in the untwisted model). Open string states between two such brane systems are given by the cohomology of $D$, i.e., are elements of $H^*(Q, Q')$. ($H^0$ being referred to as bosonic, and $H^1$ as fermionic.) The cone (3.14) describes the formation of a “topological bound sate” between two such configurations. Finally, (3.17) simply corresponds to the addition of a brane-antibrane pair which is canceled by an identical tachyon.

For other recent work on matrix factorizations in their relation to D-branes in Landau-Ginzburg models, see [32–39, 41, 40, 42, 43, 12].
4 Graded matrix factorizations

By construction so far, our D-brane category $\text{MF}(W)$ is $\mathbb{Z}_2$-graded. In particular, the shift functor squares to the identity. On the other hand, the prime example of a triangulated category, namely the derived category of coherent sheaves on an algebraic variety $\text{D}(X)$, is $\mathbb{Z}$-graded (and has shifts by arbitrary integers). As pointed out in [12], there is a simple way to improve $\text{MF}(W)$ to a $\mathbb{Z}$-graded category in the special case that $W$ is quasi-homogeneous.

4.1 R-symmetry

$W$ being quasi-homogeneous is the condition that there exists an assignment of degrees to the variables $x_i$ such that $W$ has definite degree. In the physical model, this grading is worldsheet R-charge, and $W$ having R-charge 2 is the conventional normalization. Thus, we assume that there exist R-charges $q_i \in \mathbb{Q}$ such that

$$W(e^{i\lambda q_i} x_i) = e^{2i\lambda} W(x_i) \quad \text{for all } \lambda \in \mathbb{R} \quad (4.1)$$

One can think of R-charge as a $U(1)$ (or $\mathbb{C}^\times$) action on the space of polynomials with respect to which $W$ is equivariant. The $U(1)$ action closes for $\lambda = \pi H$, where $H$ is the smallest integer such that $Hq_i \in 2\mathbb{Z}$ for all $i$.

When considering matrix factorizations of $W$, it is natural to require that this $U(1)$ action can be extended to $(M, \sigma, Q)$. This condition that the boundary interactions preserve the $U(1)$ R-symmetry is a necessary condition for the existence of a conformal IR fixed point. We will call such matrix factorizations quasi-homogeneous. For compatibility with (3.7), we must require that $Q$ has R-charge 1. We will, at first, assume that this $U(1)$ acts on $M$ as an $\mathcal{R} = \mathbb{C}[x_1, \ldots, x_r]$-module (instead of as a $\mathbb{C}$-vectorspace). We will, however, assume that the action is even, i.e., commutes with $\sigma$. Explicitly, we assume that there exists a map

$$\rho : \mathbb{R} \to GL^+(2N, \mathcal{R}) = GL(N, \mathcal{R}) \times GL(N, \mathcal{R}), \quad (4.2)$$

such that

$$\rho(0, x_i) = \rho(\pi H, x_i) = \text{id}_{2N \times 2N} \quad (4.3)$$

$$\rho(\lambda, x_i) Q(e^{i\lambda q_i} x_i) = e^{i\lambda} Q(x_i) \rho(\lambda, x_i). \quad (4.4)$$
Note that this implies the slightly non-standard group law
\[ \rho(\lambda, x_i)\rho(\lambda', e^{i\lambda q_i} x_i) = \rho(\lambda + \lambda', x_i). \] (4.5)

In (4.2), \( GL(N, \mathbb{R}) \) is the group of invertible \( N \times N \) matrices with polynomial entries. Under gauge transformations, \( Q(x_i) \to U(x_i)Q(x_i)U(x_i)^{-1} \) with \( U \in GL^+(2N, \mathbb{R}) \), \( \rho \) transforms as
\[ \rho^U(\lambda, x_i) = U(x_i)\rho(\lambda, x_i)U(e^{i\lambda q_i} x_i)^{-1}. \] (4.6)

Note that if we can find a gauge transformation such that \( \rho^U \) is diagonal, then by (4.3), \( \rho^U \) must be independent of the \( x_i \). Hence \( \rho^U \) is an ordinary \( U(1) \) representation on \( M \) as a \( \mathbb{C} \)-vectorspace. On general grounds, one expects that one can always find such gauge transformation that makes \( \rho \) diagonal. We will assume that this is true. But, as will become clear later, we do not want to exclude altogether gauge transformations of nonzero degree which might make \( \rho \) non-diagonal (and \( x_i \)-dependent).

### 4.2 Gradability is a topological condition

Consider the vector field generating the \( U(1) \) action
\[ R(\lambda, x_i) = -i\partial_{\lambda}\rho(\lambda, x_i)\rho(\lambda, x_i)^{-1}. \] (4.7)

In general, this will depend on \( \lambda \), but it is easy to see that \( R(\lambda, x_i) \) is actually determined for all \( \lambda \) by (4.5) and \( R(0, x_i) \). At \( \lambda = 0 \), the condition (4.4) becomes
\[ EQ + [R, Q] = Q, \] (4.8)

where
\[ E = \sum_i q_i x_i \frac{\partial}{\partial x_i} \] (4.9)
is the “Euler vector field”. Note that \( W \) being quasi-homogeneous means \( EW = 2W \), and therefore, if \( Q^2 = W \),
\[ \{Q, EQ - Q\} = 0, \] (4.10)
where \( \{\cdot, \cdot\} \) is the anticommutator. In other words, \( EQ - Q \) defines a class in \( H^1(Q) \). The existence of \( R \) is the statement that this class is trivial.

The quasi-homogeneity condition on matrix factorizations is therefore a topological condition that is roughly analogous, by mirror symmetry, to the vanishing of the Maslov class of Lagrangian cycles. Recall [44] that the vanishing of the Maslov class ensures
that Floer cohomology can be $\mathbb{Z}$-graded. Here, requiring (4.8) will immediately only give a $\mathbb{Q}$-grading, which commutes with the $\mathbb{Z}_2$ grading. In subsection 4.6, we will combine the two gradings into a single $\mathbb{Z}$-grading.

Actually, requiring the infinitesimal version (4.8) is somewhat weaker than the integrated version (4.3), (4.4), because it does not guarantee that $R$ generates a compact $U(1)$ action. Equivalently, we might not be able to diagonalize $R$ by a gauge transformation. Deferring a discussion of this point to subsection 4.4, let us assume that $R$ is diagonalized (and hence its entries are in $\mathbb{Q}$). We then obtain an induced (diagonalizable) $U(1)$-action on the morphism spaces in the dg-category $\text{DG}(W)$. $\mathbb{Q}$-homogeneous elements of $\text{Hom}_{\text{DG}(W)}(Q, Q')$ satisfy

$$E\Phi + R'\Phi - \Phi R = q\Phi \Phi$$

(4.11)

By (4.8), this descends to a $\mathbb{Q}$-grading of $D$-cohomology, and hence, of $\text{MF}(W)$. To avoid confusion, we will use $\mathcal{H}^\bullet(Q, Q') = \bigoplus_{q\in\mathbb{Q}} \mathcal{H}^q(Q, Q')$ to denote this $\mathbb{Q}$-graded cohomology, and also use the split

$$\mathcal{H}^q(Q, Q') = \mathcal{H}^{q,0}(Q, Q') \oplus \mathcal{H}^{q,1}(Q, Q')$$

(4.12)

into $\mathbb{Z}_2$ even and odd pieces.

### 4.3 Serre duality

If the boundary tachyon configuration described by $Q$ and $Q'$ flows to a conformal theory in the IR, one expects the spectrum of Ramond ground states to be charge conjugation symmetric. As usual [7], by spectral flow, this means for the chiral primaries, which are given by $D$-cohomology

$$H^\bullet(Q, Q') = H^{\bullet+r}(Q', Q)$$

$$\mathcal{H}^q(Q, Q') = \mathcal{H}^{\tilde{c}-q}(Q', Q)$$

(4.13)

for the $\mathbb{Z}_2$ and $\mathbb{Q}$-graded cohomologies, respectively. Here $\tilde{c} = \sum_{i=1}^r (1-q_i)$ is the central charge of the bulk CFT associated with $W$. In mathematical terms, (4.13) expresses “Serre duality” for the category $\text{MF}(W)$, with trivial Serre functor given purely by a shift in rational degree by $\tilde{c}$, and reversal of $\mathbb{Z}_2$ degree if the number of variables is odd.

Serre duality is equivalent to non-degeneracy of the boundary topological metric, which was computed in [32, 39]. If $\Phi \in \text{Hom}_{\text{MF}(W)}(Q, Q')$ and $\Psi \in \text{Hom}_{\text{MF}(W)}(Q', Q)$,
this Serre pairing is given by

$$\langle \Psi \Phi \rangle = \oint \text{Str}_M [ (\partial Q)^r \Psi \Phi ] \frac{\partial_1 W \cdots \partial_r W}{\partial_1 W \cdots \partial_r W}$$  \hspace{1cm} (4.14)$$

where the integral is a multi-dimensional residue. It is easy to see that this pairing has $Q$-degree $\hat{c}$, i.e., $\langle \Psi \Phi \rangle = 0$ unless $q_\Psi + q_\Phi = \hat{c}$. It also has $\mathbb{Z}_2$ grading given by $r$, the number of variables in the model. Thus, proving non-degeneracy of (4.14) is equivalent to (4.13). It would be interesting to show this.

4.4 Ambiguities of $R$

As we have mentioned, the condition (4.8) does not guarantee that $R$ generates a compact $U(1)$ action that closes for $\lambda = \pi H$. On the other hand, it determines $R$ only up to an even matrix that commutes with $Q$, i.e., a representative of $H^0(Q)$. I am not aware of any example in which (4.8) has a solution, but no solution which does not generate a compact $U(1)$ action, or which is not diagonalizable.

For example, if all entries of $Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$ are in fact homogeneous polynomials, then one expects that (4.8) generically has a solution $R = \text{diag}(R_1, \ldots, R_{2N})$ which is diagonal. Indeed, denoting polynomial degree by $\deg$, equation (4.8) becomes

$$R_j - R_{k+N} = 1 - \deg(f_{jk})$$
$$R_{k+N} - R_j = 1 - \deg(g_{kj}) \quad \text{for } j, k = 1, \ldots, N,$$  \hspace{1cm} (4.15)

which is a system of $2N^2$ equations for $2N$ unknowns. The non-trivial relations on the left hand side of (4.15) are given by permutations $\pi \in \Sigma_N$ on $N$ indices,

$$\sum_j (R_j - R_{\pi(j)+N})$$  \hspace{1cm} (4.16)

being independent of $\pi$. On the right hand side, these relations become

$$N - \sum_j \deg(f_{j\pi(j)}) .$$  \hspace{1cm} (4.17)

On the other hand, on taking determinant of (3.1), we see that

$$\det(f) \det(g) = W^N,$$  \hspace{1cm} (4.18)

which, assuming that $W$ is irreducible, implies $\det(f) = W^k$ for some $0 \leq k \leq N$. Since $\sum \deg(f_{j\pi(j)})$ is the degree of a summand of $\det(f)$, we see that if there are no
exceptional cancellations, (4.17) is independent of \( \pi \). Similarly, \( gf = W \) generically implies \( \text{deg}(f_{jk}) + \text{deg}(g_{kj}) = 2 \).

Thus if all entries of \( Q \) are homogeneous, we expect that there is a diagonal solution of (4.8) (this is true in all examples I have studied). It is easy to see that the converse is also true: If \( R \) is diagonal, then all entries of \( Q \) must be homogeneous. (But there are factorizations that are not quasi-homogeneous, see subsection 7.6.)

We will generally assume that there is a solution of (4.8) that is diagonalizable, keeping in mind that this assumption can conceivably fail at singular loci in the moduli space of matrix factorizations. Let us then analyze the ambiguities of \( R \).

The proposal for fixing the ambiguity of \( R \) is motivated by the examples of section 7 and the general considerations of section 6. The essential idea is that \( R \) defines a character on the gauge group of similarity transformations. Infinitesimally, such gauge transformations are given by even endomorphisms of \( M \) as a \( \mathcal{R} \)-module, \( i.e. \), block-diagonal matrices \( V \in \text{Mat}^+(2N \times 2N, \mathcal{R}) \), with \( \text{Tr} V \in \mathbb{C} \). They act on \( Q \) by \( \delta Q = [V, Q] \), and the character induced by \( R \) is given by

\[
\chi_R(V) = \text{Tr}_M(RV) .
\]

The condition we would like to impose on \( R \) is that this character be trivial on the part of the gauge group acting trivially,

\[
\text{Tr}(RV) = 0 \quad \text{whenever} \quad [V, Q] = 0 .
\]

Note that under such infinitesimal gauge transformations, \( R \) transforms according to

\[
\delta R = -EV - [R, V]
\]

which leaves (4.8) invariant to first order. By all we have said, it might then seem natural to fix a diagonal \( R \) and restrict to gauge transformation of degree 0, \( i.e. \), those which satisfy \( EV + [R, V] = 0 \) leave \( R \) invariant. As we will see in section 7, however, this would be too restrictive, as we would not be able to describe brane-antibrane annihilation.

To fix the ambiguity, and impose (4.20), one may proceed as follows. Start with a reference solution \( R_0 \), assumed to be diagonal. The ambiguities of (4.8), which are

\footnote{It is also reminiscent of the “\( a \)-maximization” procedure used to find the R-charge of \( \mathcal{N} = 1 \) superconformal gauge theories in four dimensions [45].}
parametrized by even cycles of $Q$, can be decomposed according to the degree with respect to $R_0$,

$$C^0(Q) = \oplus_q \mathcal{C}^{q,0}(Q)$$

(4.22)

where

$$\mathcal{C}^{q,0} = \{V \in \text{Mat}(2N \times 2N, \mathcal{R}); [Q, V] = 0, [\sigma, V] = 0, EV + [R_0, V] = qV\}. \ (4.23)$$

To see what can happen if we modify $R_0$ by an element of $C^0(Q)$, it is instructive to consider the following example. Let

$$R_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ (4.24)$$

with $a$ and $b$ rational. Then (we are neglecting $Q$ and $\sigma$ in this discussion—$R_0$ and $V$ could be submatrices in a larger problem),

$$V = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \ (4.25)$$

is of (total) degree $q(V) = \deg(x) + a - b$ with respect to $R_0$. Clearly, $R_0$ generates a compact $U(1)$ by $\rho_0(\lambda) = e^{i\lambda R_0}$, but $R_0 + V$ not necessarily so. Indeed, it is easy to see that the solution of (4.5) generated by $R_0 + V$ at $\lambda = 0$ is

$$\rho(\lambda, x) = \begin{pmatrix} e^{i\lambda a} & x^{\frac{e^{i\lambda(a + \deg(x))} - e^{i\lambda b}}{a + \deg(x) - b}} \\ 0 & e^{i\lambda b} \end{pmatrix}, \ (4.26)$$

which for $b = a + \deg(x)$ goes over into

$$\rho(\lambda, x) = \begin{pmatrix} e^{i\lambda a} & i\lambda x e^{i\lambda b} \\ 0 & e^{i\lambda b} \end{pmatrix}. \ (4.27)$$

We see that as long as $q(V) = \deg(x) + a - b \neq 0$, $R = R_0 + V$ generates a compact $U(1)$ and can be diagonalized. This fails when $q(V) = 0$.

Thus, by rediagonalizing $R$ if necessary, we can neglect the modifications of $R_0$ by elements of $\mathcal{C}^{q,0}(Q)$ for $q \neq 0$. And clearly, $\chi_{R_0}$ vanishes automatically on $V \in \mathcal{C}^{q,0}(Q)$, because such $V$ doesn’t have diagonal entries.

What about the ambiguities parametrized by $\mathcal{C}^{0,0}(Q)$? As we have just seen, we cannot add $V$’s without diagonal entries. Among those with diagonal entries, we
choose a maximal commuting subalgebra, with basis \( \{ V_i \}_{i=1,...,s} \), and impose (4.20) on the ansatz
\[
R = R_0 + \sum_{i=1}^{s} a_i V_i .
\] (4.28)

In all examples I have studied, this procedure leads to an unambiguously determined \( R \) which is diagonalizable and satisfies (4.20).

### 4.5 Cone construction

We next show that the grading we just introduced is compatible with the triangulated structure. In particular, we show that the cone (3.14) over a map \( T[1] \in H^0(Q_1, Q_2[1]) \) between two quasi-homogeneous matrix factorizations is again quasi-homogeneous.

Indeed,
\[
EQ - Q = \begin{pmatrix}
EQ_1 - Q_1 & 0 \\
ET - T & EQ_2 - Q_2
\end{pmatrix}
\]
\[
= \begin{pmatrix}
[Q_1, R_1] & 0 \\
(qT - 1)T - R_2 T + TR_1 & [Q_2, R_2]
\end{pmatrix}
\]
\[
= \begin{pmatrix}
Q, (R_1 + (qT - 1)id_1 & 0 \\
0 & R_2
\end{pmatrix}
\] (4.29)

is exact. More properly, we could choose
\[
R = \begin{pmatrix}
R_1 + (qT - 1)^{N_2/N_1 + N_2} id_1 & 0 \\
0 & R_2 - (qT - 1)N_2 id_2
\end{pmatrix}
\] (4.30)

so as to satisfy \( \text{Tr}(R) = 0 \) as well as \( EQ - Q = [Q, R] \). But in the generic case, \( C^0(Q) \) will contain more elements than just the identity so that we will not satisfy (4.20) in general.

### 4.6 Orbifolding and the phase of matrix factorizations

Recall that the Calabi-Yau/Landau-Ginzburg correspondence relates Calabi-Yau manifolds given as complete intersections in toric varieties to Landau-Ginzburg orbifold models [22]. In the simplest case, the Calabi-Yau is a hypersurface \( X \) given as the vanishing locus of a polynomial \( P \) of total degree \( H \) in weighted projective space \( \mathbb{P}_{w_1,\ldots,w_r}^{r-1} \) such that \( \sum_{i=1}^{r} w_i = H \). Such an \( X \) corresponds, via CY/LG correspondence, to the Landau-Ginzburg orbifold model with superpotential \( W = P \) and orbifold group
\( \Gamma = \mathbb{Z}_H \), with central charge \( \hat{c} = r - 2 \). The \( x_i \) have R-charge \( q_i = 2w_i/H \) and \( \Gamma \) is generated by \( x_i \mapsto \omega_i x_i \) with \( \omega_i = e^{i\pi q_i} \). The R-charges of the invariant part of the bulk chiral ring \( J = (\mathcal{R}/\partial W)_{\Gamma} \) are then all even integers.

Now let \( Q \) be a quasi-homogeneous matrix factorization of \( W \) with R-matrix \( R \) uniquely determined as in subsection (4.4), and diagonal. Obviously, we would like to extend the \( \Gamma \) action to \( Q \), and we require that it commutes with the rational and the \( \mathbb{Z}_2 \)-grading. In other words, we are looking for a representation of \( \Gamma \) on (the associated \( \mathbb{Z}_2 \)-graded \( \mathcal{R} \)-module of CP factors) \( M \) such that

\[
\gamma Q(\omega_i x_i) \gamma^{-1} = Q(x_i) .
\]

It is easy to see that such a representation must be related to \( R \) in a simple manner. Indeed, we see that

\[
\tilde{\gamma} = \sigma e^{-i\pi R} \gamma
\]

commutes with \( Q \). It is no restriction to assume that it is diagonal. If \( Q \) is reduced (i.e., contains not entries with a constant term), then all diagonal degree 0 elements of \( C^0(Q) \) are actually non-trivial in \( S^{0,0}(Q) \). We conclude that if \( Q \) is reduced and irreducible (i.e., \( S^{0,0}(Q) \) is one-dimensional), then \( \tilde{\gamma} \) is a multiple of the identity, \( \tilde{\gamma} = e^{i\pi \phi} \). In other words, we find

\[
\gamma = \sigma e^{i\pi R} e^{-i\pi \phi}
\]

Imposing \( \gamma^H = 1 \) fixes \( \phi \in \mathbb{R} \mod 2/H \). Lifting to \( \phi \in \mathbb{R} \mod 2 \) gives \( H \) different equivariant factorizations for each factorization of \( W \). (To be sure, if \( H \) is even, these correspond to \( H/2 \) branes together with their antibranes.)

A \( \Gamma \)-action on the objects induces an action on the morphism spaces \( \mathcal{R}^*(Q, Q') \), and we can project onto invariant morphisms by requiring

\[
\gamma' \Phi(\omega_i x_i) \gamma^{-1} = \Phi
\]

By combining the definitions, it is easy to see that invariant morphisms satisfy the condition

\[
e^{i\pi q_{\Phi}} (-1)^{\Phi} e^{\pi i (\phi - \phi')} = 1
\]

In other words, \( q_{\Phi} = \phi' - \phi + n \), where \( n \) has the same parity as \( \Phi \). This constraint on the \( U(1) \) charges is the same as (2.3), and leads to the identification of \( \phi \) as the phase of the matrix factorization.
Let us then define the category $\mathcal{MF}(W)$, in which objects are quasi-homogeneous matrix factorizations $Q$ together with a lift of the phase $\varphi$ to a real “grade”, and morphism spaces are
\[
\text{Hom}^n((Q, \varphi), (Q', \varphi')) = \mathcal{S}_{n+\varphi'-\varphi}(Q, Q')
\] (4.36)
This is the promised $\mathbb{Z}$-graded category of matrix factorizations. Note in particular that the shift functor, which, because of (4.32), must be accompanied by $\varphi \to \varphi + 1$, does not square to the identity in $\mathcal{MF}(W)$.

### 4.7 Conjecture
The general decoupling statements of [2], the result that B-branes are described at large volume by the derived category of coherent sheaves, together with the assumption that all topological B-branes of the Landau-Ginzburg model have a description using matrix factorizations, naturally lead to the statement that—in appropriate cases—there should be an equivalence of categories
\[
\mathcal{MF}(W) \cong \mathcal{D}(X),
\] (4.37)
where $\mathcal{MF}(W)$ is the category of quasi-homogeneous $\Gamma$-equivariant matrix factorizations of $W$ and $\mathcal{D}(X)$ is the derived category of coherent sheaves on the Calabi-Yau manifold $X$ related to $W/\Gamma$ by Witten’s gauged linear sigma model construction [22]. Cases in which one expects such a correspondence include those GLSM’s in which both large volume and Landau-Ginzburg points exist and are unique, such as the quintic in $\mathbb{P}^4$, or hypersurfaces in weighted projective spaces.

I hope that such a correspondence appears well-motivated from the physics point of view. It has essentially already been stated by Ashok, Dell’Aquila and Diaconescu in [35] (for the quintic case and without the homogeneity condition). I should, however, add that the correspondence is somewhat different from existing (and mathematically proven!) equivalences between categories of matrix factorizations and other structures. Besides Eisenbud’s canonical correspondence [14] with maximal Cohen-Macaulay modules, there is also an equivalence between matrix factorizations and a so-called “triangulated category of singularities” which was proven by Orlov [19]. Moreover, there is the classical correspondence of Grothendieck and Serre between graded modules over graded rings and vector bundles over the associated projective variety. This correspondence was exploited by Laza, Pfister and Popescu in [46] for the case of the elliptic...
curve. If (4.37) is true, it is likely that the equivalence favored by physics is different from those just mentioned. For the elliptic curve, for instance, the methods of [46] on the one hand and [47,42] on the other hand yield quite different bundles corresponding to some given matrix factorization.

4.8 Landau-Ginzburg monodromy

We can make one further check that our conjecture makes sense. The Landau-Ginzburg point $p_{LG}$ is an orbifold point in the Kähler moduli space $\mathcal{M}_k$. The action of the monodromy around $p_{LG}$ acts on matrix factorizations in $\mathfrak{M}(W)$ simply by rotating the choice of lift of $\varphi$ in (4.33),

$$\varphi \to \varphi + 2/H.$$ (4.38)

As a consequence, the $H$-th power of the Landau-Ginzburg monodromy operator acts by $\varphi \to \varphi + 2$. This does nothing on the physical brane associated with $Q$, but is a shift by 2 in the triangulated category. This solves a problem posed in [6], in which the 5-th power of the Landau-Ginzburg monodromy on the quintic Calabi-Yau was computed and found to correspond to a shift by 2 on the derived category. We can simply confirm this result using matrix factorizations, and in fact extend it to all Calabi-Yau manifolds with a Landau-Ginzburg description.

5 RR charges and index theorem

If matrix factorizations represent D-branes in string theory, they must carry Ramond-Ramond (RR) charge. This charge takes value in the dual of the appropriate space $\mathcal{H}_{RR}^B$ of closed string RR ground states. Because of the boundary condition on the worldsheet $U(1)$ current, B-branes couple to those RR ground states with opposite left and right-moving R-charge, $q_L = -q_R$. The purpose of this section is to determine these RR charges of matrix factorizations. We first describe $\mathcal{H}_{RR}^B$.

The space of Ramond-Ramond ground states in Landau-Ginzburg orbifolds and their left-right R-charges was computed in [20]. For simplicity, we will restrict here to a cyclic orbifold group $\Gamma = \mathbb{Z}_H$, as well as to integer central charge (we mostly have in mind, of course, $\hat{c} = 3$). The generalization of at least some of the formulas to the more general case should be obvious. In general, those RR ground states with $q_L \neq q_R$
arise purely from the twisted sector, and if \( c \) is integer, the \( \mathbb{Z}_H \) projection on twisted sectors implies that the RR ground states have \( q_L \equiv q_R \equiv \hat{c}/2 \mod \mathbb{Z} \).

Consider the \( l \)-th twisted sector, and divide the field variables of the LG model into two classes, according to whether \( l q_i \in 2\mathbb{Z} \) or \( l q_i \notin 2\mathbb{Z} \). Those fields, \( \{x^l_i\} \), with \( l q_i \notin 2\mathbb{Z} \) satisfy twisted boundary condition in this sector, and must be set to zero in the semi-classical analysis used to determine the RR ground states. The contribution of those fields to the R-charges is

\[
q^L_L = -q^L_R = \sum_{l q_i \notin 2\mathbb{Z}} \left( \frac{l q_i}{2} - \left\lfloor \frac{l q_i}{2} \right\rfloor - \frac{1}{2} \right)
\]  

(5.1)

On the other hand, those fields, \( \{x^u_i\}_{i=1,\ldots,r_l} \), with \( l q_i \in 2\mathbb{Z} \) satisfy untwisted boundary conditions in the \( l \)-th twisted sector. Their quantization leads to a spectrum of RR ground states which is that corresponding to the effective potential \( W_l(x^u_i) = W(x^u_i, x^l_i = 0) \). In particular, they contribute, \( q^u_L = q^u_R \), equal amounts to left and right charge.

What is important for us, the ground states with \( q_L = q^u_L + q^l_L = -q_R = q^u_R + q^l_R \) from the \( l \)-th twisted sector correspond precisely to the neutral ground states of the effective potential \( W_l(x^u_i) \) obtained by setting those fields with \( l q_i \notin 2\mathbb{Z} \) to zero. These ground states have \( q_L \equiv q_R \equiv \hat{c}/2 \mod \mathbb{Z} \) if the number, \( r_l \), of fields with \( l q_i \in 2\mathbb{Z} \) is even. A basis of these ground states can be labeled as \( |l; \alpha\rangle \), where \( l \) ranges between 0 and \( H - 1 \), and \( \alpha \) ranges over a basis \( \phi^u_\alpha = (x^u)^\alpha = \prod_{i=1}^{r_l} (x^u_i)^{\alpha_i} \) of the subspace, \( \mathcal{J}_l^0 \), of the untwisted chiral ring \( \mathcal{J}_l = \mathbb{C}[x^u_i] / \partial W_l \) with R-charge \( q^u_L = q^u_R = \sum_{l q_i \in 2\mathbb{Z}} \alpha_i q_i / 2 = \hat{c}^u / 2 \).

Here, \( \hat{c}^u = \sum_{l q_i \in 2\mathbb{Z}} (1 - q_i) \) is the central charge corresponding to \( W_l \). The states \( |l; \alpha\rangle \) can be thought of as being obtained by acting with \( \phi^l_\alpha \) on the unique state \( |l; 0\rangle \), which has R-charge \( -\hat{c}^u / 2 \).

Now by definition, the RR charge is the correlation function on the disk with the RR ground state inserted in the bulk. We propose that for a matrix factorization

\footnote{What we have called \( q_i \) is the sum of left- and right-moving charges of the variables \( x_i \) in the normalization of [20].}
$Q \in \mathfrak{M}(W)$, this is given by

$$
\begin{align*}
\text{ch}(Q) : H^B_{\text{RR}} &\to \mathbb{C} \\
\text{ch}(Q)(|l; \alpha\rangle) &\equiv \langle l; \alpha|Q\rangle_{\text{disk}} \\
&= \frac{1}{r_l!} \text{Res}_{W_l}(\phi_l^\alpha \text{Str}[\gamma^l(\partial Q_i)^{q_l}] ) \\
&= \frac{1}{r_l!} \oint \phi_l^\alpha \frac{\text{Str}[\gamma^l(\partial Q_i)^{q_l}]}{\partial_1 W_l \cdots \partial_{r_l} W_l}
\end{align*}
$$

where $\gamma$ is the representation of the generator of $\mathbb{Z}_H$ on the matrix factorization, and $\text{Str}(\cdot) = \text{Tr}_M(\sigma \cdot)$ is the supertrace over the $\mathbb{Z}_2$-graded module $M$. The residue is the same as the one appearing in the Serre pairing $\langle \gamma, \alpha \rangle$. It is normalized [48] such that the determinant of the Hessian of the superpotential $W_l$ has residue equal to the dimension of the chiral ring,

$$
\text{Res}_l(\det(\partial_i \partial_j W_l)) = \oint \frac{\det(\partial_i \partial_j W_l)}{\partial_1 W_l \cdots \partial_{r_l} W_l} = \dim \mathcal{J}_l = \mu_l = \prod_{q_i \in 2\mathbb{Z}} \frac{2 - q_i}{q_i}.
$$

Moreover, in (5.2), $Q_l(x_u^i, x_t^i = 0)$ is the restriction of $Q$ to the untwisted fields in the $l$-th sector. It satisfies $Q_l^2 = W_l$. Formula (5.2) makes sense since by (4.31),

$$
\gamma^l Q_l(x_u^i) = Q_l(x_u^i) \gamma^l,
$$

so $\gamma^l$ represents a cohomology class of the matrix factorization $Q_l$, and (5.2) computes the disk correlation function [32,39] of $\gamma^l \phi_i^\alpha$ in this model. Moreover, since $\text{Res}_l$ has $Q$ degree $\hat{c}^u$ and $\mathbb{Z}_2$ degree $r_l$, we see that (5.2) would vanish if $r_l$ were odd or if we tried to insert an element of $\mathcal{J}_l$ with charge not equal to $\hat{c}^u$.

In those twisted sectors with $r_l = 0$, i.e., $lq_i \notin 2\mathbb{Z}$ for all $i$, (5.2) reduces to

$$
\text{ch}(Q)(|l; 0\rangle) = \text{Str}\gamma^l.
$$

One can check that (5.2) gives the correct value for the RR charges in those cases where an alternate computation exists, namely minimal models and their tensor products. Also, we see immediately that $\text{ch}(Q[1]) = -\text{ch}(Q)$. The main evidence, however, that (5.2) is the correct expression for the RR charge is the index theorem for matrix factorizations, i.e., the fact that the Witten index for open strings between two matrix
factorization $Q$ and $Q'$ can be computed via $\text{ch}(Q)$ and $\text{ch}(Q')$ as

$$\text{Tr}(-1)^F = \sum_{n \in \mathbb{Z}} (-1)^n \dim \text{Hom}^n_{\text{DG}(W)}(Q, Q')$$

$$= \langle \text{ch}(Q'), \text{ch}(Q) \rangle$$

(5.6)

where $(-1)^F$ is the $\mathbb{Z}_2$ grading (fermion number) of matrix factorizations. The Chern pairing is given by

$$\langle \text{ch}(Q'), \text{ch}(Q) \rangle = \frac{1}{H} \sum_{l = 0}^{H-1} \sum_{\alpha, \beta} \text{ch}(Q')(|l; \alpha\rangle) \prod_{i \in \mathbb{Z}} \eta^\alpha_{l, \beta} \text{ch}(Q)(|l; \beta\rangle)^*$$

(5.7)

For fixed $l$, $\sum_{\alpha, \beta}$ is a sum over the chosen basis of $\mathcal{J}^0_l$ of elements of the chiral ring $\mathcal{J}_l$ with charge $\hat{c}_l/2$, and $\eta^\alpha_{l, \beta}$ is the inverse of the closed string topological metric in this sector,

$$\eta^l_{\alpha, \beta} = \text{Res}_t(\phi^\alpha_l \phi^\beta_l).$$

(5.8)

We will now prove (5.6) in the case that $r_l = 0$ in all twisted sectors. The index of interest is the equivariant index of the operator $D$ acting as in (3.11) on the complex given by the morphism space in $\text{DG}(W)$, i.e.,

$$\text{Tr}(-1)^F = \frac{1}{H} \sum_{l = 0}^{H-1} \text{Tr}(-1)^F \tilde{\gamma}_l,$$

(5.9)

where $\tilde{\gamma}$ is the action of the generator of $\mathbb{Z}_H$ on the cohomology spaces. We can regularize the computation of $\text{Tr}(-1)^F \tilde{\gamma} = \lim_{t \to 1} Z_l(t)$ by using the $\mathbb{Q}$-grading by $U(1)$ charge

$$Z_l(t) = \text{Tr}(-1)^F t^q \tilde{\gamma}_l.$$

(5.10)

(More precisely, we should use an appropriate covering of this $U(1)$ to make the charges integer.) By a standard argument, we can then replace the trace over the space of ground states by the trace over $\text{Hom}_{\text{DG}(W)}(Q, Q') = \text{Hom}_R(M, M')$, effectively reducing the computation to the setting $Q = Q' = 0$. We decompose

$$\text{Hom}_R(M, M') = \bigoplus_{j, k = 1}^{2N} \bigoplus_{\alpha} \mathcal{V}_{j, k, \alpha}$$

(5.11)

into one-dimensional pieces indexed by matrix entries $(j, k)$ and monomials $x^\alpha = \prod x_i^{\alpha_i}$ with multi-index $\alpha = (\alpha_1, \ldots, \alpha_r)$. Note that the combination of fermion number and $\mathbb{Z}_H$-action restricts on $\mathcal{V}_{j, k, \alpha}$ to

$$((-1)^F \tilde{\gamma})_{\mathcal{V}_{j, k, \alpha}} = \sigma_j^\gamma \gamma_j^\prime \prod_{i = 1}^{r} \omega_i^{\alpha_i} \sigma_k \gamma_k^{-1}$$

(5.12)
where \( \omega_i = e^{ \pi i q_i} \), and we are using that both \( \sigma \) and \( \gamma \) are diagonal matrices. Therefore,

\[
Z_l(t) = \sum_{j,k=1}^{2N} \sigma_j \gamma^l \left( \prod_{i=1}^r \omega_i^l \right) \sigma_k \gamma^{-l} \tag{5.13}
\]

\[
= \text{Str}(\gamma^l t^R) \frac{1}{\prod_i (1 - t^q \omega_i^l)} \text{Str}(\gamma^{-l} t^{-R}) .
\]

Since \( \text{Str}(\text{id}) = 0 \), and we are assuming that \( lq_i \notin \mathbb{Z} \) for all other \( l \) and \( i \), we can smoothly take \( t \to 1 \), and obtain

\[
\text{Tr}(-1)^F = \frac{1}{H} \sum_{l=1}^{H-1} \text{Str}(\gamma^l) \frac{1}{\prod_i (1 - \omega_i^l)} \text{Str}\gamma^{-l} , \tag{5.14}
\]

as was to be shown.

To establish (5.6) and (5.7) in general, one should combine the proof we just gave with the formula

\[
\frac{1}{(r_l)^2} \text{Res}_l (\text{Str}[(\partial Q_l^\alpha)^\wedge\eta_l]) \text{Str}[(\partial Q_l^\beta)^\wedge\eta_l]) \tag{5.15}
\]

\[
= \sum_{\alpha, \beta} \frac{1}{r_l} \text{Res}_l (\phi_l^\alpha \text{Str}[(\partial Q_l^\alpha)^\wedge\eta_l]) \eta_l^{\alpha\beta} \frac{1}{r_l} \text{Res}_l (\phi_l^\beta \text{Str}[(\partial Q_l^\beta)^\wedge\eta_l]) .
\]

This formula expresses the factorization rule for the topological annulus correlator [32] with no boundary insertions via two disk amplitudes and (the inverse of) the closed string topological metric \( \eta_{\alpha,\beta} \) (5.8) given by the sphere amplitude [48]. (Note that in (5.15), the sum over \( \alpha, \beta \) can be extended to the full chiral ring \( J_l \) because the disk correlators vanish outside of \( J_l^0 \).) In the general axioms of open-closed topological field theory [49,50], this factorization is known as the “Cardy condition.” By the same axioms, the annulus correlator (5.15) computes the open string Witten index \( \text{Tr}(-1)^F \) between the matrix factorizations \( Q_l, Q_l' \) in the untwisted Landau-Ginzburg model corresponding to \( W_l \). I have checked the equality of the two sides of (5.15), and that they compute the open string Witten index, in all examples I know, but I do not know a proof based directly on the residue formula.

We close this section with a few comments.

Firstly, we note that there is an obvious analogy between (5.6), (5.7) and the well-known Hirzebruch-Riemann-Roch theorem which computes the Witten index for open strings coupled to two vector bundles \( E \) and \( F \) on the Calabi-Yau manifold \( X \)

\[
\text{Tr}(-1)^F = \int \text{ch}(E^*) \text{ch}(F) \text{Td}(X) . \tag{5.16}
\]
Our formula is simply the small volume version of this. In particular, the factor \( \eta_{l}^{\alpha \beta} / \prod (1 - \omega_{l}) \) can be viewed as the analog of the Todd class of \( X \). From this perspective, the normalization in which the square-root of this factor is included in the charge might seem more natural.

Secondly, we return to the split of the Ramond ground states into those from twisted sectors with \( r_{l} = 0 \) and those from twisted sectors with \( r_{l} \neq 0 \) and even. In the RCFT description of LG models as Gepner models [51], the ground states with \( r_{l} \neq 0 \) are not left-right symmetric in each individual \( \mathcal{N} = 2 \) minimal model.\(^4\) As a consequence, the BCFT constructions of boundary states in Gepner models [52] did not produce boundary states with charge under those RR ground states with \( r_{l} \neq 0 \), the only exception being related to the so-called fixed point resolution phenomenon discussed in [53,54] (see also [55,32]). On the other hand, it is easy to find matrix factorizations for which charges with \( r_{l} \neq 0 \) do not vanish. (The two-variable factorizations of subsection 7.4, when embedded in the appropriate Calabi-Yau model, provide useful examples.) It seems likely that matrix factorizations span the free part of K-theory that is expected from cohomology. What the Chern classes miss, of course, is the torsion part of the K-theory. Unorbifolded minimal models, for example, have K-theory that is purely torsion. One might expect that some of this will survive the orbifold procedure, conceivably in the twisted sectors with \( r_{l} \) odd. It would be interesting to determine the full K-theory of these Landau-Ginzburg orbifolds and compare with their geometric computation. This would be a zeroth order check of (4.37).

6 A stability condition

In mathematical models of D-branes similar to the one we are studying, such as Lagrangian submanifolds of symplectic manifolds, holomorphic vector bundles on complex manifolds, or representations of a quiver algebra, a stability condition is introduced with the purpose of identifying a subset of objects whose orbits under the group of appropriate automorphisms fit together into “nice” moduli spaces. Often, the stable orbits admit a distinct (unique) representative at the zero of a “moment map” associated with the stability condition (for instance, the special condition for Lagrangians or the hermitian Yang-Mills equation for the connection on the holomorphic vector

\(^4\)Geometrically, they correspond to non-toric blowups of \( X \). For this reason, most models that have been studied geometrically in any depth do not have such states.
bundle). (See, e.g., chapter 38 of [23] for a recount of these stories.)

In physics, the zeroes of the moment map are associated with the solution of the condition that the D-brane preserve supersymmetry in the uncompactified part of spacetime. Stability is the condition that such a supersymmetric configuration can be reached by boundary renormalization group (RG) flow on the string worldsheet. In the unstable (including semistable) case, the theory is expected to split at singular points along RG flow into the direct sum of several decoupled theories. The endpoint of the flow is the decomposition into the stable pieces.

For a general Landau-Ginzburg model (orbifolded or not, with arbitrary central charge), the interpretation involving spacetime supersymmetry is not necessarily available, and we will factor it out accordingly. What remains is the unitarity constraint \( (2.1) \) and the assertion that if this condition is satisfied, worldsheet RG flow should lead to a single unitary boundary CFT in the IR (i.e., a theory with a unique open string vacuum). This is a stability condition that can be imposed on the triangulated category of any quasi-homogeneous Landau-Ginzburg model.

If the model has a geometric interpretation, then in view of the expected equivalence \((4.37)\), this is a particular stability condition on \(D(X)\). It is distinguished by the fact that it arises only from data involving the unorbifolded model (or equivalently, the orbifolded model divided by the quantum symmetry). In the general framework of [5], the space of (numerical) stability conditions is locally modeled on the free part of the K-theory. As we have seen in section 5, most of the K-theory (all of it for an odd number of variables) appears during orbifolding. Therefore, the stability condition in the Landau-Ginzburg model should be more rigid than the ones on \(D(X)\).

### 6.1 A notion of stability

As we have reviewed in section 2, the basic idea underlying \(\Pi\)-stability is that open strings between physical branes should satisfy the unitarity constraint \(0 \leq q \leq \hat{c} \). It is hard, however, to impose such a constraint directly on individual objects to determine whether they are stable, essentially because this would involve an infinite number of checks, and moreover because a stable object does not only have strings satisfying \((2.1)\) ending on it. Physically [2], one should not try to impose the condition \((2.1)\) on configurations described as (topological) boundstates containing both branes and antibranes. One expects that in certain regimes [2, 4], or even at all points in the space of stability conditions [5], integrating out all canceling brane-antibrane pairs will
reduce the problem to a stability condition on an Abelian category, which involves only a finite number of checks. Still, these discussions leave open the question whether \( \Pi \)-stability is sufficient or just a self-consistent “bootstrap” condition. Our point of view is that the Landau-Ginzburg model should possess an intrinsic (rigid!) stability condition that does not depend on what is going on in the rest of the moduli space. It is this notion of stability that we are after.

In the Landau-Ginzburg context, “integrating out brane-antibrane pairs” simply corresponds to restricting to reduced matrix factorizations, \( i.e., \) those without scalar entries. It is not unreasonable to expect, therefore, that by going to reduced matrix factorizations, one obtains the Abelian category of interest for the discussion of [5]. This Abelian category could be simply related to the category of Cohen-Macaulay module of subsection 3.1. In any case, we now make the following tentative definition.

Let \( W(x_1, \ldots, x_r) \) be a quasi-homogeneous Landau-Ginzburg polynomial, \( EW = 2W \), where \( E = \sum q_i x_i \partial_i \). Let \( Q \) be a reduced quasi-homogeneous matrix factorization of \( W \). \( Q \) is called R-semistable if in all triangles

\[
\begin{array}{c}
\text{Q} \\
\downarrow s_2 \\
\text{Q}_2 \quad \text{R} \quad \text{T} \quad \text{Q}_1 \\
\uparrow s_1
\end{array}
\]

(6.1)
in which \( Q \) participates opposite to the fermionic morphism \( T \), we have

\[
q_T \leq 1 \iff q_{S_1} \geq 0 \iff q_{S_2} \geq 0 .
\]

(6.2)

\( Q \) is stable if the only triangles for which \( q_T = 1 \) are those with \( Q_1 \) or \( Q_2 \) equal to \( Q \) (and the other equivalent to 0).

Here \( q_T \) is defined by the condition

\[
ET + R_2 T - TR_1 = q_T T
\]

(6.3)

where \( R_1 \) and \( R_2 \) are the R-matrices of \( Q_1 \) and \( Q_2 \), respectively.

We can give one simple check that relates R-stability to a stability condition in the sense of Bridgeland [5]. Recall that in the orbifolded case (subsection 4.6), we have defined morphism between objects in \( \mathcal{M}_\Phi(W) \) by \( \text{Hom}^0(Q, Q') = \mathcal{H}^q = \varphi' - \varphi(Q, Q') \). Therefore, our condition (6.2) directly implies

\[
\varphi > \varphi' \Rightarrow \text{Hom}^0(Q, Q') = 0 ,
\]

(6.4)
which is one of the axioms of [5].

We also note that our formulation is similar to those of a stability condition for Lagrangian submanifolds proposed by Thomas [56] and further studied in [57].

6.2 A moment map problem?

The stability condition we have proposed is physically well-motivated. It only deserves its name, however, if it can be related to the moduli space problem for matrix factorizations. In other words, one would like to show that stable matrix factorizations have nicely behaved orbits under the group of gauge equivalences. As we have mentioned before, this group is the group of similarity transformations

\[ G \cong GL^+(2N, \mathcal{R}) \cong GL(N, \mathcal{R}) \times GL(N, \mathcal{R}) \]  

acting as in (3.2). Thus we have an algebraic group acting on a linear space with a constraint. This problem is quite similar to the one studied by King [11].

In [11], the general setup of geometric invariant theory (GIT) [58] is used to define moduli spaces for representations of finite-dimensional algebras, which can be equivalently described as the representations of quiver diagrams. Quivers arise naturally as world-volume theories for D-branes at singularities, and the theory of quivers has played in important role in the development of Π-stability [1, 10]. In the quiver case, the gauge group \( \mathcal{G} \) is the product of general linear groups acting on the vector spaces at each node of the quiver. King uses GIT to give a geometric description of the algebraic quotient of the representation space \( \mathcal{Y} \) with respect to a character \( \chi : \mathcal{G} \to \mathbb{C}^\times \) via “Mumford’s numerical criterion”: A representation \( y \in \mathcal{Y} \) is \( \chi \)-semistable iff \( \chi \) is trivial on the stabilizer of \( y \) and if every one-parameter subgroup \( g(\lambda) = e^{\lambda a} \) of \( \mathcal{G} \), for which \( \lim_{\lambda \to \infty} y \) exists, satisfies \( \langle d\chi, a \rangle \geq 0 \), where \( d\chi \) is the infinitesimal version of \( \chi \) evaluated on the generator \( a \) of \( g(\lambda) \).

Our stability condition is precisely equivalent to such a “numerical criterion”. To see this, note that all triangles in \( \text{MF}(W) \) are isomorphic to the standard cone (1.29), namely

\[ Q = Q_1 \oplus Q_2 + T, \quad R = R_1 \oplus R_2 + (q_1 - 1) \left[ \frac{N_2}{N_1 + N_2} S_1 - \frac{N_1}{N_1 + N_2} S_2 \right], \]  

where \( S_i = \text{id}_i \). Under the one-parameter group of gauge transformations generated by \( V = S_1 \), this cone transforms as

\[ Q_\lambda = e^{\lambda V} Q e^{-\lambda V} = Q_1 \oplus Q_2 + e^{-\lambda T} \xrightarrow{\lambda \to \infty} Q_1 \oplus Q_2. \]  

29
The limit $\lambda \to \infty$ simply splits the cone back into its constituents. The condition $q_T \leq 1$ is equivalent to

$$-\text{Tr}(RV) = -(q_t - 1) \frac{2N_2 N_1}{N_2 + N_1} \geq 0,$$

thus identifying $\text{Tr}(R \cdot \cdot)$ as the character of $G$ with respect to which we are defining stability. The condition (4.20) we are imposing to fix the ambiguities of $R$ is precisely the condition that $\text{Tr}(R \cdot \cdot)$ should vanish on the trivially acting gauge transformations. Similarly as in [11], we can then formulate a “numerical criterion” that a matrix factorization $Q$ is $R$-semistable if all one-parameter subgroups $e^{\lambda V}$ of the gauge group, for which the limit $\lim_{\lambda \to \infty} e^{\lambda V} Q e^{-\lambda V}$ exists, satisfy $\text{Tr}(RV) \leq 0$.

In [11], King then goes on to describing a symplectic quotient construction of the moduli space, which is the basis for the relation to quiver gauge theories.

We can at present see two difficulties in making such a relation in our situation more precise, both of which due to the fact that $G$ is not as simple a gauge group as the one acting on quiver representations. First of all, as a complex Lie group, $G$ is infinite-dimensional. This is similar to the situation with vector bundles or Lagrangian submanifolds, giving reason for hope. The second difficulty appears if, as might seem natural, we would restrict to the degree 0 gauge transformations, i.e., those generated by

$$g^0 = \{ V \in g; EV + [R, V] = 0 \}.$$  \hspace{1cm} (6.9)

The problem with $g^0$ is that it is non-reductive. Indeed, since both polynomial and total degree are preserved in matrix multiplication, $g^0$ has a maximal solvable subalgebra consisting of those matrices without constant term. In other words, we can decompose $g^0$ into its maximal reductive subalgebra consisting of those elements annihilated by $E$, and the nilpotent part. This non-reductiveness of $g^0$ makes it more difficult to apply the general results of GIT and to find a relation with a moment map problem. In any case, however, restricting to gauge transformations of degree 0 makes the description of brane antibrane annihilation somewhat unnatural, see subsection 7.2.

Setting aside these difficulties for the moment, we will naively follow the usual steps to write down a moment map-like flow equation on the gauge orbits. As we will see in section 7, this naive flow works quite well in a number of examples. Imitating [11], we introduce a metric on the space of matrix factorizations,

$$\langle Q, Q' \rangle = \sum_{\alpha} \text{Tr}(Q_{\alpha}^i Q'_{\alpha}^i),$$

(6.10)
where
\[ Q = \sum_{\alpha} Q_\alpha x^\alpha, \quad Q' = \sum_{\alpha} Q'_\alpha x^\alpha \] (6.11)
is the decomposition of \( Q \) and \( Q' \) into a sum over monomials \( x^\alpha = \prod_i x_i^{\alpha_i} \). In any given case, we restrict to a finite-dimensional subgroup of \( G \) and choose a basis of generators, \( \{V_i\} \). The flow equation then is
\[ \frac{dQ}{dt} = -\left( \langle Q, [V^i, Q] \rangle - \text{Tr} RV^i \right) [V^i, Q]. \] (6.12)

Note that by construction, (6.12) is indeed a moment map for the maximal reductive subgroup of the degree 0 gauge group.

Moreover, one can see that the flow (6.12) indeed reproduces the correct splitting (6.7) of the standard cone (4.29) in the case that \( q_T \geq 1 \). A simple calculation gives
\[ \frac{d\lambda}{dt} = \left( e^{-2\lambda} ||T||^2 + (q_T - 1) \beta \right) \] (6.13)
where \( \beta = 2N_2N_2/(N_2 + N_1) \). Evidently, this has a solution at finite \( \lambda \) if \( q_T < 1 \), whereas for \( q_T \geq 1 \), the flow drives us to \( \lambda \to \infty \). The form of eq. (6.13) is of course familiar in the context of solving D-flatness conditions in four-dimensional \( \mathcal{N} = 1 \) supersymmetric gauge theories.

7 Examples

We conclude the paper with several concrete examples of matrix factorizations and flows on their gauge orbits defined by (6.12). As alluded to before, one can view these flows as toy models for boundary flows in Landau-Ginzburg models. (Landau-Ginzburg descriptions of boundary flows have also recently been discussed in [59].)

7.1 Minimal models

Matrix factorizations of A-type minimal models with type 0A GSO projection, corresponding to the LG superpotential \( W = x^h \) were discussed detail in [31, 33, 36, 41]. They are given by
\[ Q_n = \begin{pmatrix} 0 & x^n \\ x^{h-n} & 0 \end{pmatrix} \quad \text{for } n = 1, \ldots, h-1. \] (7.1)
The R-matrix is
\[ R = \left( \begin{array}{cc} \frac{1}{2} - \frac{n}{\hbar} & 0 \\ 0 & -\frac{1}{2} + \frac{n}{\hbar} \end{array} \right). \] (7.2)

It is easy to see that there is only one non-trivial element of the degree 0 gauge algebra,
\[ V = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{array} \right). \] (7.3)

The orbit generated by \( V \) looks like
\[ Q_n(\lambda) = e^{\lambda V} Q_n e^{-\lambda V} = \left( \begin{array}{cc} 0 & e^{\lambda x^n} \\ e^{-\lambda x^{\hat{h}-n}} & 0 \end{array} \right), \] (7.4)

so that the flow \( \frac{d\lambda}{dt} \) becomes
\[ \frac{d\lambda}{dt} = -\left( \langle Q_n(\lambda), [V, Q_n(\lambda)] \rangle - \text{Tr} RV \right) 
= -\left( e^{2\text{Re}\lambda} - e^{-2\text{Re}\lambda} - \frac{1}{2} + \frac{n}{\hbar} \right). \] (7.5)

Obviously, this flow has just one stationary point, which is stable. This is as expected. Indeed, one can check explicitly that all open strings between different minimal model factorizations satisfy \( 0 \leq q \leq \hat{c} = 1 - \frac{2}{\hbar} < 1. \)

### 7.2 Brane-antibrane annihilation

We have been tempted several times in this paper to restrict attention to the gauge transformations of degree 0 only. In this subsection, we show that in fact, the description of brane-antibrane annihilation in the context of matrix factorizations requires the inclusion of gauge transformation of non-zero degree.

Let \((f, g)\) be a matrix factorization of \(W\) with R-matrix \(R = (R_+, R_-)\), and consider the cone over the identity \(\text{id} : (f, g) \to (f, g)\),

\[ Q_0 = \left( \begin{array}{cccc} 0 & 0 & f & 0 \\ 0 & 0 & 1 & g \\ g & 0 & 0 & 0 \\ -1 & f & 0 & 0 \end{array} \right). \] (7.6)

This is gauge equivalent to direct sums of the trivial factorization \(W = 1 \cdot W\) via the
gauge transformation

\[ U_\lambda = \begin{pmatrix} 1 & -\lambda f & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda g \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

(7.7)

Namely,

\[ Q_\lambda = U_\lambda Q_0 U_\lambda^{-1} = \begin{pmatrix} 0 & 0 & (1 - \lambda)f & (\lambda^2 - 2\lambda)W \\ 0 & 0 & 1 & (1 - \lambda)g \\ (1 - \lambda)g & (2\lambda - \lambda^2)W & 0 & 0 \\ -1 & (1 - \lambda)f & 0 & 0 \end{pmatrix} \]  

(7.8)

which for \( \lambda = 1 \) becomes

\[ Q_1 = \begin{pmatrix} 0 & 0 & 0 & -W' \\ 0 & 0 & 1 & 0 \\ 0 & W & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]  

(7.9)

What is the R-matrix associated with \( Q_0 \)? The cone construction of subsection 4.5 gives one possible solution

\[ R_{\text{cone}} = \begin{pmatrix} R_+ - \frac{1}{2} & 0 & 0 & 0 \\ 0 & R_- + \frac{1}{2} & 0 & 0 \\ 0 & 0 & R_- - \frac{1}{2} & 0 \\ 0 & 0 & 0 & R_+ + \frac{1}{2} \end{pmatrix} \]  

(7.10)

However, this R-matrix does not satisfy (4.20). By using the equivalence with \( Q_1 \), one finds that the generators of \( C^0(Q_0) \) with non-vanishing diagonal entries and degree 0 with respect to \( R_{\text{cone}} \) are

\[ V_i = \begin{pmatrix} e_i & -f_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -g^i & 0 \\ 0 & 0 & 0 & e_i \end{pmatrix} \]  

and

\[ V^i = \begin{pmatrix} 0 & f^i & 0 & 0 \\ 0 & e_i & 0 & 0 \\ 0 & 0 & e_i & g_i \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  

(7.11)

where \( e_i \) is the \( N \times N \) matrix with a 1 at the \( i \)-th position on the diagonal, and zeroes elsewhere, and \( f_i = e_i f \) and \( f^i = f e_i \) are the \( i \)-th row and column of \( f \), respectively.
Then the combination of $R^{\text{cone}}$, $V_i$ and $V^i$ satisfying (4.20) is

$$R_0 = \begin{pmatrix}
-\frac{1}{2} & R_+ f - f R_- & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & g R_+ - R_- g \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2} & -Ef + f & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & Eg - g \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}$$

(7.12)

Under the similarity transformation (7.8), $R_0$ transforms into the diagonal matrix $R_1 = \text{diag}(-1/2, 1/2, -1/2, 1/2)$. This $R_1$ is the R-matrix one would naturally assign to a sum of copies of the trivial branes described by $Q_1$.

Note that while the gauge transformation relating $Q_0$ and $Q_1$ has degree zero with respect to $R^{\text{cone}}$, it does not have definite degree with respect to $R_0$. We conclude that either we are forced to work with gauge transformations of non-zero degree or we should be using $R^{\text{cone}}$ as R-matrix for the cone. We cannot completely exclude the second possibility since (by definition) the factorization $(1, W)$ does not have any non-trivial morphism ending on it, so there are no R-charges to check. But the symmetric end-result, $R_1$, is good justification for the procedure we have proposed. And the moment map equation only makes sense if we use $R_0$. One can also check that the flow defined by (6.12) on the gauge orbit (7.8) flows to $\lambda = 1$.

### 7.3 Boundary flows in minimal models

Having argued for the general relevance of gauge transformations of non-zero degree, we now return to minimal models and study boundary flows associated with perturbations by a boundary condition changing operator. General aspects of such boundary flows in $\mathcal{N} = 2$ minimal models were discussed recently in [36] and in [41]. In particular, these works discuss the similarity transformations relating the different minimal model branes at the topological level, as well as the operators inducing these relations. Our flow equation (6.12) gives a handle on the complete flow in the physical theory.

We will consider as an example the starting point

$$Q_0 = \begin{pmatrix}
0 & 0 & x^2 & 0 \\
0 & 0 & -x & x^3 \\
x^{h-2} & 0 & 0 & 0 \\
x^{h-4} & x^{h-3} & 0 & 0
\end{pmatrix}$$

(7.13)
The R-matrix, obtained by methods as above is

\[
\begin{pmatrix}
\frac{1}{2} - \frac{4}{h} & -\frac{x}{h} & 0 & 0 \\
0 & \frac{1}{2} - \frac{1}{h} & 0 & 0 \\
0 & 0 & -\frac{1}{2} + \frac{1}{h} & -\frac{x^2}{h} \\
0 & 0 & 0 & -\frac{1}{2} + \frac{4}{h}
\end{pmatrix}
\] (7.14)

We have studied the flow induced by \((6.12)\) on the orbit of \(Q_0\) under the gauge transformations generated by

\[
V = \begin{pmatrix}
\lambda_1 & \lambda_3 x & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & -\lambda_2 & \lambda_4 x^2 \\
0 & 0 & 0 & -\lambda_1
\end{pmatrix}
\] (7.15)

and find that it does converge to the diagonal

\[
\begin{pmatrix}
0 & 0 & 0 & \alpha x^4 \\
0 & 0 & -\beta x & 0 \\
0 & -\beta^{-1} x^{h-1} & 0 & 0 \\
\alpha^{-1} x^{h-4} & 0 & 0 & 0
\end{pmatrix} \cong Q_1 \oplus Q_4,
\] (7.16)

where \(\alpha\) and \(\beta\) are the appropriate solutions of \((7.5)\). The R-matrix becomes \(\text{diag}(1/2 - 4/h, 1/2 - 1/h, -1/2 + 1/h, -1/2 + 4/h)\), which is certainly the correct result for this factorization.

We should note that the perturbation we have turned on in \((7.13)\) is not the most relevant between the two minimal model branes \(Q_2\) and \(Q_3\). We have chosen this one to illustrate that there are various possible flow patterns in \(\mathcal{N} = 2\) minimal models. In this simple case the range of possibilities is essentially governed by the K-theory, isomorphic to \(\mathbb{Z}_H\). One can create a free K-theory (and make the perturbation in \((7.13)\) the most relevant one) by considering an appropriate orbifold.

In any case, the end-result of the flow is consistent with the predictions made, for instance, in [36, 59].

### 7.4 D0-brane in quintic Gepner model

In [35], matrix factorizations were constructed which describe (at the topological level) D0-branes at the Landau-Ginzburg orbifold point in the Kähler moduli space of the
quintic Calabi-Yau. We here want to address the issue whether these factorizations can be stable by checking that the open strings stretched between this D0-brane and the rational tensor products of minimal model branes satisfy the unitarity bound. While this does of course not settle the question whether the D0-brane can become unstable far away from the large volume limit, it is certainly a non-trivial check.

The superpotential of interest is \( W = \sum_{i=1}^{5} x_i^5 \). The factorizations of [35] are tensor products of minimal model factorizations in three of the five minimal factors together with a non-factorisable factorization in the remaining two factors. Since taking tensor products simply adds \( U(1) \) charges, but does not affect the unitarity bound, it will suffice to consider this two-variable factorization. We can factorize

\[
x^5 - y^5 = (x - y)(x^4 + x^3 y + x^2 y^2 + xy^3 + y^4).
\]

(7.17)

One can see that the R-matrix associated with this factorization is \( \text{diag}(3/10, -3/10) \).

We want to compute the charges of open strings between this factorization and the tensor product of minimal model branes

\[
f = \begin{pmatrix} x & -y \\ -y^4 & x^4 \end{pmatrix}, \quad g = \begin{pmatrix} x^4 & y \\ y^4 & x \end{pmatrix}, \quad R = \text{diag}(3/5, -3/5, 0, 0).
\]

(7.18)

As computed in [35], there is one bosonic and one fermionic cohomology class between (7.17) and (7.18), represented by

\[
\Phi_0 = \begin{pmatrix} y^3 & 1 & 0 & 0 \\ 0 & 0 & y^3 & x^3 + x^2 y + xy^2 + y^3 \end{pmatrix},
\]

\[
\Phi_1 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ x^3 + x^2 y + xy^2 + y^3 & -1 & 0 & 0 \end{pmatrix},
\]

(7.19)

respectively. We easily find

\[
q(\Phi_0) = \frac{9}{10}, \quad q(\Phi_1) = \frac{3}{10}.
\]

(7.20)

satisfying the unitarity bound \( 0 \leq q \leq \hat{c} = \frac{6}{5} \). We have also checked the open strings between the D0-brane factorization and the other minimal model branes. They all satisfy the bound.

7.5 Decay of an unstable factorization

In this subsection, we give an example of a matrix factorization that is unstable and investigate to what extent our flow (6.12) can detect this without having to check the
charges of open strings. The superpotential is \( W = x^5 + y^5 \), and the factorization given by

\[
 f_{\text{unst}} = \begin{pmatrix} x & y & 0 \\ 0 & x^3 & y \\ y^3 & 0 & x \end{pmatrix} \quad g_{\text{unst}} = \begin{pmatrix} x^4 & -xy & y^2 \\ y^4 & x^2 & -xy \\ -x^3 y^3 & y^4 & x^4 \end{pmatrix}
\] (7.21)

\[ R = \text{diag}(7/10, -1/10, -1/10, 1/10, 1/10, -7/10) \]

A morphism between this factorization and the tensor product of minimal model branes (7.18) is given by

\[
 T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\] (7.22)

and one can easily check that this field has R-charge \(-\frac{1}{10}\), violating the unitarity bound. 
(Note that since \( T \) has scalar entries, it cannot be exact.) The cone over \( T \) is another copy of the tensor product of minimal branes, thus exhibiting (7.21) as an unstable bound state, obtained by “condensing” a field with \( q > 1 \) between two such objects.

Namely, \((f_{\text{unst}}, g_{\text{unst}})\) is stably equivalent to

\[
 F_{\text{unst}} = \begin{pmatrix} x & y & 0 & 0 \\ -y^4 & x^4 & 0 & 0 \\ 0 & -x^3 & x^4 & -y \\ y^3 & 0 & y^4 & x \end{pmatrix}, \quad \text{with corresponding } G_{\text{unst}}.
\] (7.23)

To be precise, we should note that \((F_{\text{unst}}, G_{\text{unst}})\) is really a cone over a brane and its own antibrane, but via a field that is not the identity. As a consequence, \((f_{\text{unst}}, g_{\text{unst}})\) has a unitarity violating field in the spectrum with itself. But since this field can easily be projected out by going to an appropriate orbifold, it should not be viewed as the cause of the instability.

In discussing the flow, it is useful to contrast the unstable factorization with a very similarly structured stable (with the same caveat as before) bound state of two minimal model tensor products, namely

\[
 F_{\text{stab}} = \begin{pmatrix} x & y^2 & 0 & 0 \\ -y^3 & x^4 & 0 & 0 \\ 0 & -x^3 & x^4 & -y^2 \\ y & 0 & y^3 & x \end{pmatrix}, \quad G_{\text{stab}},
\] (7.24)
which can be reduced to

\[
f_{\text{stab}} = \begin{pmatrix} x & y^2 & 0 \\ 0 & x^3 & y^2 \\ y & 0 & x \end{pmatrix}, \quad g_{\text{stab}} = \text{adj}(f_{\text{stab}}).
\] (7.25)

We have studied numerically the flow defined by (6.12) on the 12-parameter gauge orbit of \((F_{\text{unst}}, G_{\text{unst}})\) and \((F_{\text{stab}}, G_{\text{stab}})\) generated by

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & x\lambda_3 & y\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_7 & 0 & y\lambda_8 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_9 & x\lambda_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{12}
\end{pmatrix}
\] (7.26)

and

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & x\lambda_3 & y^2\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_7 & 0 & y^2\lambda_8 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_9 & x\lambda_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{12}
\end{pmatrix}
\] (7.27)

respectively. We find that starting from quite general initial conditions, (6.12) indeed drives \((F_{\text{unst}}, G_{\text{unst}})\) to the split into the direct sum of two copies of the tensor product brane. On the other hand \((F_{\text{stab}}, G_{\text{stab}})\) flows to the direct sum of \((f_{\text{stab}}, g_{\text{stab}})\) and a copy of the trivial factorization \((1, W)\).

It is worthwhile emphasizing that this statement does not hold for all initial conditions. One of the consequence of non-reductiveness is that the flow defined by (6.12) is not convex. Taking a second derivative on the right hand side does not produce something positive definite because \(\langle Q, [V, Q'] \rangle \neq \langle [V^\dagger, Q], Q' \rangle\) in general. If the flow is not convex, there is no guarantee that stationary points will be unique. In the present case, there does exist a stationary point for the flow close to \((f_{\text{unst}}, g_{\text{unst}} \oplus (1, W))\). But
this point is a saddle point of the flow, i.e., it is unstable in the sense of dynamical systems. One has to account for this possibility if one wants to make sense of (6.12) in general.

Another example of the same character as the one we have been discussing in this subsection arises in the series of models $W = x^h + y^3$, with factorizations given by

$$f = \begin{pmatrix} x^n & y & 0 \\ 0 & x^n & y \\ y & 0 & x^{h-2n} \end{pmatrix}$$

(7.28)

(and, as by now familiar, $g = \text{adj}(f)$). By considerations similar to those we have given above, one finds that the factorizations (7.28) are stable when $n < h/6$ and (apparently) stable otherwise. (In particular, for $h < 6$, where $W$ describes an $E$-type minimal model, these factorizations are all stable.)

### 7.6 A non-homogeneous factorization

Lest we leave the impression that all matrix factorizations of quasi-homogeneous polynomials are quasi-homogeneous, here is a counter-example.

The superpotential $W = x^3 + y^7$ is one of the simplest superpotentials that is not a simple singularity. In fact, it is unimodular. Torsion free rank one modules over local rings of unimodular singularities were classified in [60]. It is a simple exercise to determine the associated matrix factorizations. On the list for $W = x^3 + y^7$, one finds the following one-parameter family of factorizations.

$$f = \begin{pmatrix} x^2 - \lambda y^5 & xy \\ xy + \lambda^2 y^4 & -\lambda x + y^2 \end{pmatrix}$$

$$g = \begin{pmatrix} x - \frac{y^2}{\lambda} & \frac{xy}{\lambda} \\ \frac{xy}{x} + \lambda y^4 & -\frac{x^2}{x} + y^5 \end{pmatrix}$$

(7.29)

For $\lambda \neq 0$, this is stably equivalent to the following factorization

$$\tilde{f} = \begin{pmatrix} -xy^5 & \lambda xy^4 + y^6 & -x^2 \\ y^6 & x^2 - \lambda y^5 & xy \\ -x^2 - \lambda y^5 & yx + \lambda^2 y^4 & -\lambda x + y^2 \end{pmatrix}$$

$$\tilde{g} = \begin{pmatrix} \lambda & y & -x \\ y & x & 0 \\ -x & \lambda y^4 & y^5 \end{pmatrix}$$

(7.30)
which is non-reduced, but has a limit as $\lambda \to 0$. While $W$ is quasi-homogeneous with $q_x = 2/3$, $q_y = 2/7$, we see that

$$E f - f + R_+ f - f R_- = \frac{2}{21} \lambda \begin{pmatrix} -y^5 & 0 \\ 2\lambda y^4 & -x \end{pmatrix} = \frac{2}{21} \lambda \partial_\lambda f ,$$

(7.31)

where

$$R = \begin{pmatrix} R_+ & 0 \\ 0 & R_- \end{pmatrix} = \begin{pmatrix} -\frac{7}{42} & 0 & 0 & 0 \\ 0 & 0 & \frac{9}{42} & 0 \\ 0 & 0 & \frac{7}{42} & 0 \\ 0 & 0 & 0 & -\frac{9}{42} \end{pmatrix} .$$

(7.32)

Since this $\partial_\lambda f$ is the marginal deformation of the family (7.29), it is a non-trivial cohomology class (this can also be checked directly). As a consequence, the matrix factorization $(f, g)$ is not quasi-homogeneous.

It is interesting to ask for a geometric interpretation of this example. For example, one could embed $W = x^3 + y^7$ into the appropriate Calabi-Yau Landau-Ginzburg model, and try to identify a mirror Lagrangian cycle. Non-homogeneity of $(f, g)$ should be mirror to non-vanishing of the Maslov class. Of course, it is not clear to what extent Lagrangians with non-vanishing Maslov anomaly participate in mirror symmetry. The intriguing point is that the limit of (7.29) for $\lambda \to 0$ is actually quasi-homogeneous and therefore might have a good mirror. One way to avoid the paradox conclusion that the deformation of a non-anomalous Lagrangian is anomalous would be to show that the brane described by this factorization is never stable on the moduli space. (It is unstable in the Landau-Ginzburg model, as the examples in subsection 7.5.)

8 Summary

For convenience and definiteness, we shall here give a summary of the main ingredients that are proposed to enter into a stability condition for matrix factorizations.

As explained in section 2, the physical origin of stability conditions in string theory is the grading by worldsheet R-charge. In the context of matrix factorizations, which originate in local commutative algebra, it is also quite natural to consider the graded situation, so it would not seem that physics has much input to give. Before repeating the claim that it does, it is worthwhile to fix the convenient normalization of the grading: Physics suggests a normalization in which $W$ has charge 2, giving the field
variables fractional charge, whereas a more standard mathematical choice is to make all degrees integer.

With this in mind, we have associated to any matrix factorization

\[ Q^2 = W \]

of a Landau-Ginzburg potential \( W \), satisfying the anomaly-free condition that \( EQ - Q \) is cohomologically trivial, a matrix \( R \), defined by the conditions (4.8) and (4.20),

\[ EQ - Q = [Q, R] \quad \text{and} \quad \text{Tr}(RV) = 0 \quad \text{whenever} \ [V, Q] = 0, \]

where we have argued that the latter condition would fix \( R \) uniquely. Via (4.11), this induces a grading, \( q \), of the morphism spaces.

The choice of normalization of the grading is important because we intend to compare the grading of morphisms with another natural quantity that can be associated to a Landau-Ginzburg potential, namely the central charge

\[ \hat{c}. \]

A mathematical quantity that is closely related to \( \hat{c} \) is the so-called “singular index” that appears in singularity theory (see [61]), but it does not seem to have played a crucial role in the purely algebraic context so far.

The basic idea, motivated by \( \Pi \)-stability as we have explained, is to impose the \textit{unitarity constraint} (2.1)

\[ 0 \leq q \leq \hat{c} \]

as a stability condition on the category of topological D-branes.

The problem at this point, which is inherited from \( \Pi \)-stability, is that it is not \textit{a priori} clear exactly \textit{how} to impose this condition. For example, should it be imposed on all morphisms, or only on all morphisms involving stable objects? Or should one rather attempt to define the stable branes as a “maximal set” of objects satisfying this (and maybe some other) condition? Although the latter option would seem to depend on too many arbitrary choices, such ambiguities might not be unnecessary. The set of stable objects is expected to be unique only up to auto-equivalences of the topological category or monodromies in the moduli space [2,6].

In the mathematical approach of [5], the problem is circumvented by postulating the existence of abelian subcategories at each point in moduli space, on which a stability condition can be imposed in a more standard well-defined form.
We have argued here that there should be a way to identify uniquely a set of stable objects at the Landau-Ginzburg point, essentially because our definition of the grading does not depend on the rest of moduli space, and is hence insensitive to monodromies. A posteriori, this should also provide an abelian subcategory.

To gain further confidence that such an approach is possible, we have then proposed a relation to a moduli space problem via a “moment map-like” flow equation (6.12)

\[ \frac{dQ}{dt} = -\left(\langle Q, [V^i, Q]\rangle - \text{Tr}RV^i\right)[V^i, Q], \]

which is expected to provide the split of any given object into its stable constituents. We have implemented this flow in various relevant examples, with reasonable results.

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