NEW REALIZATIONS OF MODULAR FORMS IN CALABI-YAU THREEFOLDS ARISING FROM $\phi^4$ THEORY

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Abstract. Brown and Schnetz found that the number of points over $\mathbb{F}_p$ of a graph hypersurface is often related to the coefficients of a modular form. We set some of the reduction techniques used to discover such relation in a general geometric context. We also prove the relation for one example of a modular form of weight 4 and two of weight 3, refine the statement and suggest a method of proving it for four more of weight 4, and use the one proved example to construct two new rigid Calabi-Yau threefolds that realize Hecke eigenforms of weight 4 (one provably and one conjecturally).

1. Introduction

The bijective correspondence between rational Hecke eigenforms of weight 2 of minimal level $N$ and elliptic curves over $\mathbb{Q}$ of conductor $N$ up to isogeny is one of the central concepts in modern number theory. It has been generalized in many ways, for example by replacing elliptic curves over $\mathbb{Q}$ with elliptic curves over a totally real number field: the corresponding objects then appear to be Hilbert modular forms. In another direction, we might seek to increase the weight of the form and the dimension of the corresponding variety.

For odd $k$, the presence of $-1$ in $\Gamma_0(N)$ forces rational eigenforms to have an imaginary quadratic Nebentypus character of conductor dividing $N$ and to be of CM type. Schütt showed [23] that rational newforms of CM type and weight $k$ up to twist are in bijection with imaginary quadratic fields whose class group is killed by $k - 1$. For $k = 2$ this is essentially the classical fact that elliptic curves with complex multiplication up to twist correspond to imaginary quadratic fields of class number 1; for $k = 3$ such fields have been studied for centuries in the guise of Euler’s numeri idonei, which are the positive integers $n$ for which the primes represented by a quadratic form of discriminant $-n$ are determined by congruence conditions. There is a list of such integers, found in [15], which is known to be complete if the generalized Riemann hypothesis is assumed; without this assumption, the list is complete with at most one exception.

Elkies and Schütt [15] then showed that, in the case $k = 3$, every modular form $f$ corresponding to a known imaginary quadratic field with class group killed by 2 is realized in a singular K3 surface $S_f$, in the sense that the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on its étale cohomology includes the representation associated to $f$. That is, the semisimplification of the representation $\phi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_{22,\mathbb{Z}_\ell}$ on $H^2_\text{ét}(S_f, \mathbb{Z}_\ell)$ has a 2-dimensional component $\rho_\ell$ such that $\text{tr} \rho_\ell(\text{Frob}_p) = a_p$ for all but finitely
many \( p \), where \( a_p \) is the eigenvalue of \( S_f \) for the Hecke operator \( T_p \). The complement of this component is the twist of a representation of finite image giving the Galois action on the Picard group of \( S_f \) by the cyclotomic character. More concretely, the number of \( \mathbb{F}_p \)-points of \( S_f \) is equal to \( p^2 + 1 + a_p + pn(p) \), where \( n(p) \in \mathbb{Z} \) depends only on the decomposition of \( p \) in the field of definition of the Picard group, which is a finite extension of \( \mathbb{Q} \).

**Definition 1.1.** Let \( f \) be a newform with integral coefficients and \( V \) a smooth variety over \( \mathbb{Q} \). If for all \( \ell \) the \( \ell \)-adic representation attached to \( f \) is a component of the \( \ell \)-adic representation attached to \( H^i_{\text{ét}}(V, \mathbb{Z}_\ell) \), we say that \( V \) realizes \( f \).

Thus the result of Elkies and Schütt shows that every integral newform of weight 3 is realized by a K3 surface with one possible exception. It is a very important question, asked by Mazur, van Straten, and others, whether an analogous result holds for weight \( > 3 \). To discuss this problem, we must introduce Calabi-Yau varieties.

**Definition 1.2** ([20, Section 1.1]). A Calabi-Yau variety of dimension \( d \) is a smooth proper variety with trivial canonical class such that \( h^{i,0} = 0 \) for all \( 0 < i < d \).

**Definition 1.3.** A singular Calabi-Yau variety is a variety \( V \) that is birationally equivalent to a smooth Calabi-Yau variety and is a limit of smooth Calabi-Yau varieties.

**Example 1.4.** Let \( n > 1 \) and let \( c_1, \ldots, c_{n-d} \) be positive integers adding to \( n + 1 \). Then a sufficiently general intersection of hypersurfaces of degree \( c_1, \ldots, c_{n-d} \) in \( \mathbb{P}^n \) is a Calabi-Yau variety of dimension \( d \). If \( X \) is such a complete intersection such that the canonical bundle on the nonsingular subset of such an \( X \) extends to all of \( X \) and \( X \) admits a resolution of singularities with trivial canonical bundle, it is a singular Calabi-Yau variety.

**Definition 1.5** ([20, Section 1.1]). A smooth Calabi-Yau variety of dimension \( d \) for which \( h^{d-1,1} = 0 \) is said to be rigid. A singular Calabi-Yau variety is rigid if it has a resolution of singularities which is a rigid Calabi-Yau variety.

A considerable amount of work has been devoted to finding rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \). It is known [16], [12] that the representation on \( H^3 \) of such a variety has the same semisimplification as the representation associated to an integral Hecke eigenform of weight 4. Meyer [20] gives a large number of examples of rigid Calabi-Yau threefolds and corresponding modular forms, but it seems that the only general method of constructing a threefold starting from a modular form of weight 4 (as opposed to recognizing a modular form in a variety constructed without reference to it) is that of [7], which is applicable only to eigenforms of CM type coming from imaginary quadratic fields of class number 1 or 3.

It is not known whether there are finitely or infinitely many deformation families of Calabi-Yau threefolds. On the other hand, tables such as that of [20, Appendix C] make it plausible that there are infinitely many newforms of weight 4 with integer coefficients up to twist, although this may be quite difficult to prove (before the famous work of Wiles and Taylor-Wiles the analogous statement was apparently not known for weight 2). In any case, each family can contribute realizations of only finitely many newforms up to twist on rigid Calabi-Yaus, so if there are finitely many families and infinitely many newforms up to twist it is not possible for all of them to be realized.
The arithmetical results of this paper are summarized in the following theorem:

**Theorem 1.6.** There are rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) for which the representation on \( H^3 \) is the same as that associated to the modular forms labeled \( 13/1, 78/4 \) in [20, Appendix C]. If the tables of [18] are complete for cubic extensions of \( \mathbb{Q} \) unramified outside \( \{2, 3, 5, 13, 23\} \), the same holds for the form \( 390/5 \).

For the form of level 390, there was a known nonrigid Calabi-Yau threefold whose \( H^3 \) was conjectured to contain this representation; for the other two, even this was not available. The construction of the threefold realizing the newform of level 13 was inspired by mathematical work on the foundations of quantum field theory.

**Definition 1.7.** Let \( G \) be a graph with vertex set \( V = \{v_1, \ldots, v_m\} \) and edge set \( E = \{e_1, \ldots, e_n\} \), and let \( R = \mathbb{Z}[x_1, \ldots, x_n] \). Let \( T \) be the set of spanning trees of \( G \). As in [3, Section 1.1], define

\[ \Psi_G = \sum_T \prod_{e_i \in T} x_i. \]

Let \( X_G \) be the affine variety \( \Psi_G = 0 \); it is called the **graph hypersurface** of \( G \). If \( m \geq 3 \) then the number of \( \mathbb{F}_p \)-points of \( X_G \) is a multiple of \( p^2 \), so we define \( c_2(p) = c_2(G)_p \) to be this number divided by \( p^2 \) and reduced mod \( p \).

If no restriction is placed on \( G \), then \( X_G \) contains essentially arbitrary motives [1]. We now suppose that \( G \) is obtained by deleting a single vertex from a 4-regular graph, which places us in the setting known as \( \phi^4 \)-theory. In this context it had been suggested that the number of \( \mathbb{F}_p \)-points might be a polynomial in \( p \). Brown and Schnetz refuted this by giving an example [3, Section 6.2] in which \( c_2(p) \equiv a_p \mod p \)

where the \( a_p \) are the eigenvalues for a Hecke eigenform of weight 3 and level 7. To do so, they related \( c_2(p) \) to the number of points on a K3 surface of Picard number 20. In later work [4], they found 16 Hecke eigenforms that appear to match the \( c_2(p) \) for various graphs.

For the five graphs of [4, Figure 5] that numerically match a modular form of weight 4, we construct a threefold for which the number of \( \mathbb{F}_p \)-points is congruent to \( c_2(p) \mod p \). For the form of level 13, we will show that it is a rigid Calabi-Yau threefold and that the number of \( \mathbb{F}_p \)-points is indeed congruent mod \( p \) to the eigenvalue of the newform of weight 4 and level 13. In addition, by studying a family of threefolds that specializes to this one, we construct two new rigid Calabi-Yau threefolds, one realizing the modular form of level 78, the other apparently that of level 390.

For each of the other four graphs, matching the forms of levels 5, 6, 7, 17, we conjecture that the variety is a rigid Calabi-Yau threefold and give a formula that appears to count its \( \mathbb{F}_p \)-points. These forms are already known to be realized on rigid Calabi-Yau threefolds, but the construction here is completely different. In light of the Tate conjecture [20, Conjecture 1.10] we expect ours to be in correspondence with the previous ones. However, we do not know how to exhibit such a correspondence. In addition, we will verify that the graphs labeled \( (3, 8) \) and \( (3, 12) \) in [4, Figure 5] correspond to singular K3 surfaces realizing modular forms of the indicated level.

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2. The five-invariant

In the introduction we associated a polynomial to a graph $G$ and defined the graph hypersurface associated to it. Unfortunately, the variety is difficult to work with because it is of large dimension. In this section we describe some of the constructions used by Brown, Schnetz, and Yeats [3], [4], [5] to obtain varieties of lower dimension for which the point count over $\mathbb{F}_p$ is the same mod $p$. In contrast to their work, we will regard our varieties as projective rather than affine.

First, we choose an arbitrary orientation of the edges of $G$. We then define a symmetric $(n+m) \times (n+m)$ matrix $\tilde{M}_G$ over $\mathbb{Z}[x_1, \ldots, x_n]$ (recall that $m, n$ are the numbers of vertices and edges of $G$) as in [3, Definition 8]. That is, the upper left block is a diagonal matrix with entries $x_i$, the upper right and lower left blocks are the oriented incidence matrix and its transpose, and the lower right block is 0.

The sum of the last $m$ rows of this matrix is 0, so we choose one arbitrarily and delete it as well as the corresponding column to obtain $M_G$. The determinant of $M_G$ is $\pm \Psi_G$.

Definition 2.1 ([3, Definition 9]). Let $I, J, K$ be subsets of the set of edges with $|I| = |J|$. Let $M_G(I, J)_K$ be the matrix $M_G$ with the rows corresponding to $I$ and columns corresponding to $J$ removed, and the variables associated to $K$ set to 0. Let $\Psi_{I,J}^{G,K}$ be its determinant.

Definition 2.2 ([3, Definition 12]). Let $i, j, k, l, m$ be five distinct edges in $G$. The five-invariant $5\Psi_G(i, j, k, l, m)$ is defined to be

$$\det \begin{pmatrix} \Psi_{ij,kl}^{G,m} & \Psi_{ik,jl}^{G,m} \\ \Psi_{ijm,klm}^{G,\emptyset} & \Psi_{ikjm,ilm}^{G,\emptyset} \end{pmatrix}.$$ 

Remark 2.3. Let $G$ be the deletion of a vertex from a 4-regular graph, so that $G$ has $n-1$ vertices and $2n-4$ edges. Then the spanning trees have order $n-2$, which means that $\Psi_G$ is homogeneous of degree $n-2$ in $2n-4$ variables. On the other hand, $\Psi_{G,K}^{I,J}$ has degree $\deg \Psi_G - |I|$. Thus in general the degree of $5\Psi_G(i, j, k, l, m)$ is $n-4 + n - 5 = 2n - 9$. Since it does not depend on the variables associated to edges $i, j, k, l, m$ it is a polynomial in $2n - 9$ variables. Observe also that $\Psi_G$ has degree at most 1 in every variable, so $5\Psi_G(i, j, k, l, m)$ has degree at most 2.

Up to sign, this determinant depends only on the set $\{i, j, k, l, m\}$, not on the order. Brown and Schnetz prove the following:

Theorem 2.4 ([3, Theorem 3]). Let $G$ be a graph with at least five edges $e_1, \ldots, e_5$ such that $N_G = 2h_G$ and $N_\gamma > 2h_\gamma$ for all strict subgraphs $\gamma \subset G$, where $N$ is the number of edges of a graph and $h$ its first Betti number. Then for all primes $p$, the number of solutions to the equation $5\Psi_G(e_1, e_2, e_3, e_4, e_5) = 0$ over $\mathbb{F}_p$ is congruent mod $p$ to $-c_2(G)_p$. 

3. Reduction

In this section we describe the two main constructions used in, among others, [3], [4], [5], that allow $c_2(G)_p$ to be computed from a variety of manageable dimension. First we restate them in terms of projective rather than affine varieties. Then we explain how these reductions can be interpreted in terms of blowups of projective varieties along linear subspaces and generalize to obtain a reduction technique not previously used in this context.

**Definition 3.1.** Throughout the paper, the coordinates in an ordinary projective space $\mathbb{P}^n$ will always be denoted $x_0, \ldots, x_n$. In weighted projective space $\mathbb{P}(k,1,1,\ldots,1)$ the coordinates will be $t, x_0, \ldots, x_n$, and in product projective space $\mathbb{P}^m \times \mathbb{P}^n$ they will be $x_0, \ldots, x_m, y_0, \ldots, y_n$.

**Definition 3.2.** If $f_1, \ldots, f_n$ are homogeneous polynomials and $q$ is a prime power, let $[f_1, \ldots, f_n]_q$ be the number of points on the projective variety defined by $f_1 = \cdots = f_n = 0$ over $\mathbb{F}_q$. If $V$ is a variety, let $|V|_q$ be the number of $\mathbb{F}_q$-points of $V$.

**Remark 3.3.** Let $V_P, V_A$ be the projective and affine varieties over $\mathbb{F}_q$ defined by $f_1 = \cdots = f_n = 0$, where the $f_i$ are homogeneous polynomials. Then $|V_A|_q = (q-1)|V_P|_q + 1$, so the sequences $|V_A|_q, |V_P|_q$ contain the same information.

**Definition 3.4.** Let $V, W$ be projective varieties over $\mathbb{Q}$. We say that $V$ and $W$ are prime-similar if $|V|_p - 1 \equiv (-1)^{\dim W - \dim V} (|W|_p - 1)$ mod $p$ for all but finitely many primes $p$. If $W$ is a point we say that $V$ is prime-trivial.

**Example 3.5.** The Chevalley-Warning theorem [24, Theorem 1.3] states that a subvariety of $\mathbb{P}^n$ defined by equations the sum of whose degrees is at most $n$ is prime-trivial.

**Lemma 3.6.** Let $V$ be a variety over $\mathbb{Q}$ smooth along the subvariety $W$, and let $\hat{V}$ be $V$ blown up along $W$. Then $\hat{V}$ is prime-similar to $V$.

**Proof.** Let $p$ be a prime for which $V/\mathbb{F}_p$ is smooth along $W/\mathbb{F}_p$ (this excludes only finitely many primes). Then the $\mathbb{F}_p$-points of $V \setminus W$ are in bijection with points of $\hat{V} \setminus E$, while $E$ maps to $W$ with fibres that are projective spaces of dimension $c - 1$ over $\mathbb{F}_p$, where $c$ is the codimension of $W$ in $V$. Thus $\hat{V}$ has $(p^{d-1} + p^{d-2} + \cdots + p)|W|_p$ more $\mathbb{F}_p$-points than $V$. \qed

**Example 3.7** ([3, Lemma 16]). Let $V$ be a hypersurface in $\mathbb{P}^n$ ($n > 1$) defined by a polynomial $fx_0 + g$ of degree 1 in $x_0$. Then $V$ is prime-similar to the subvariety $W$ of $\mathbb{P}^{n-1}$ defined by $f$. To see this, note first that $P_0 = (1 : 0 : \ldots : 0) \in V$ (here we use that $n > 1$) and consider the rational map from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ given by projecting away from $P_0$. Above every point of $\mathbb{P}^{n-1} \setminus W$ there is one point of $V$. Above every point of $W$ the number of points of $V$ not equal to $P_0$ is $p$ if $g = 0$ there and otherwise 0. Thus, if $W$ has $k$ points, then the number of points of $V$ is congruent mod $p$ to

$$(p^{n-1} + p^{n-2} + \cdots + 1 - k) + 1 \equiv 2 - k \pmod p,$$

as desired.

We refer to this as linear reduction.

**Example 3.8** ([3, Lemma 27]). Let $V$ be a hypersurface in $\mathbb{P}^n$ defined by $fg = 0$, where $f, g$ are both of degree 1 in the variable $x_0$ and $fg$ is of total degree $n + 1$.
Then $V$ is prime-similar to the subvariety of $W$ defined by $[f,g]_{x_0}$, the resultant of $f,g$ with respect to $x_0$. We call this resultant reduction.

**Remark 3.9.** Both of these reductions preserve the property of the total degree of a polynomial being equal to the number of variables, as well as that of the degree in any one variable being at most 2.

Now we will reinterpret linear reduction in terms of a standard geometric construction: this will show us how to generalize it. The variety $fx_0 + g = 0$ in $\mathbb{P}^n$ is singular at $(1:0:\ldots:0)$. If we blow up this point, the exceptional divisor is defined by $f = 0$. On the other hand, the fibres of the projection to $\mathbb{P}^{n-1}$ are points or lines, so the blowup is prime-trivial and the original variety is prime-similar to the exceptional divisor. More generally, we consider varieties singular along a coordinate plane rather than at a point.

**Definition 3.10.** Let $V$ be a hypersurface of degree $n+1$ in $\mathbb{P}^n$, defined by $f = 0$. Suppose that, for some $k$ with $2 \leq k \leq n$ and $0 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have $f \in (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. We then define the subspace reduction $L(V,k) = L(V,\{i_1, \ldots, i_k\})$ of $V$ by the $x_{i_1}$ as the exceptional divisor in the blowup of $V$ along $x_{i_1} = \ldots = x_{i_k} = 0$. It is naturally a hypersurface of bidegree $(n-k+1,k)$ in $\mathbb{P}^{n-k} \times \mathbb{P}^{k-1}$.

**Proposition 3.11.** The subspace reduction of $V$ is prime-similar to $V$.

**Proof.** Let $\tilde{V}$ be the blowup. Since $[V]_p = [V_p] - [x_0, \ldots, x_{k-1}]_p + [L(V,k)]_p$, it suffices to prove that $\tilde{V}$ is prime-trivial. To do this, we will show that every fibre of the projection $\pi : \tilde{V} \to \mathbb{P}^{k-1}$ is prime-trivial. Since $\mathbb{P}^{k-1}$ is prime-trivial, this implies that $[\tilde{V}]_p$ is a sum of $c$ integers congruent to 1 mod $p$, where $c \equiv 1$ mod $p$. It follows that $[\tilde{V}]_p \equiv 1$ mod $p$, i.e., that $\tilde{V}$ is prime-trivial.

Let $\rho : \mathbb{P}^n \to \mathbb{P}^{k-1}$ be the rational map defined by $(x_0 : x_1 : \ldots : x_{k-1})$. Let $P = (c_0 : c_1 : \cdots : c_{k-1})$ be a point of $\mathbb{P}^{k-1}$. Then $\pi^{-1}(P)$ (viewed naturally inside $\mathbb{P}^n$ identified with $\mathbb{P}^n \times P$) is equal to $\rho^{-1}(P) \cap V$ with the component supported on $x_0 = \ldots = x_{k-1}$ removed. There are two possibilities.

1. If $\rho^{-1}(P) \cap V = \rho^{-1}(P)$, then it is a linear subspace of codimension $k-1$, which is certainly prime-trivial.

2. If not, consider the subscheme of $V$ defined by the $k-1$ linear equations that define $P$. If $c_i \neq 0$, then on this subscheme all of the $x_0, \ldots, x_{k-1}$ are constant multiples of $x_i$. When we substitute for them in the polynomial defining $V$, the result is a multiple of $x_i^k$. This corresponds to the component supported on $x_0 = \ldots = x_{k-1}$, so we divide out the highest power of $x_i^k$ to obtain a polynomial in $x_k, \ldots, x_n$ defining the fibre of $\pi$. This polynomial in $n-k+1$ variables has degree at most $n-k$, so the hypothesis of the Chevalley-Warning theorem holds and the fibre is prime-trivial.

$\square$

**Remark 3.12.** This proposition is our justification for considering graph hypersurfaces and their reductions as projective rather than affine varieties. The blowup of a projective variety is projective but that of an affine variety is not affine, so this construction is more natural for projective varieties.

Note also that this proposition does not contradict Lemma 3.6, because $V$ is singular along $x_0 = \ldots = x_{k-1} = 0$. 


We now describe the case \( k = 2 \) in more detail. Let \( V \) be defined by \( f = 0 \) with \( f \in (x_0, x_1)^2 \). Then \( L(V, 2) \) is a prime-similar variety in \( \mathbb{P}^{n-2} \times \mathbb{P}^1 \). Define \( c_0, c_1, c_2 \) to be polynomials that satisfy \( f = c_0 x_0^2 + c_1 x_0 x_1 + c_2 x_1^2 \) (any such representation will do, but for uniqueness one may choose \( c_0 \) respectively \( c_2 \) not to depend on \( x_1 \) resp. \( x_0 \)). The fibre above \( P \in V(f) \) is a single point if \( x_0, x_1 \) are not both 0; otherwise it consists of all projective points \((y_0 : y_1)\) satisfying \( c_0 y_0^2 + c_1 y_0 y_1 + c_2 y_1^2 = 0 \).

**Proposition 3.13.** Suppose that \( c_1^2 - 4c_0c_2 = gh \), where \( \deg g = n - 2 \). Then \( L(V, 2) \) is birationally equivalent and prime-similar to a hypersurface \( H_n \) of degree \( n \) in \( \mathbb{P}^{n-1} \).

**Proof.** Let \( W = \mathbb{P}(n-1,1,1,\ldots,1) \) be a weighted projective space of dimension \( n-1 \). Let the \( c'_i \) be obtained from the \( c_i \) by evaluating them at \((0,0,x_0,\ldots,x_{n-2})\), where the \( x_i \) are now the weight-1 coordinates of \( W \). Observe that the degree of the \( c'_i \) is \( n-1 \), by our hypotheses on the monomials making up \( P \). We show that \( L(V, 2) \) is birationally equivalent and prime-similar to the subvariety \( V_N \) of \( W \) defined by \( t^2 + c'_1 t + c'_0 c'_2 = 0 \).

Indeed, let us fix a point in \( \mathbb{P}^{n-2} \) and consider the points above it in each variety. In \( V_N \), they correspond to roots of \( t^2 + c_1 t + c_0 c_2 \); in \( L(V, 2) \), to points satisfying \( c_0 y_0^2 + c_1 y_0 y_1 + c_2 y_1^2 \). These are in bijection unless \( c_0 = c_1 = c_2 \), in which case there is 1 of the former and \( p + 1 \) of the latter, which numbers are congruent mod \( p \). Finally, the point \((1:0:0:0:0)\) is not on \( V_N \).

To go from \( V_N \) to the desired hypersurface, we complete the square in \( t \), which can only affect the number of \( \mathbb{F}_p \)-points if \( p = 2 \). This gives us \( t^2 - gh = 0 \) in \( W \), which we will still call \( V_N \). Then \( H_n \) is defined by \( v_0^2 g - h = 0 \). There is an obvious invertible rational map \( H_n \cdots \rightarrow V_N \) taking \((v_0: \ldots : v_{n-1})\) to \((v_0 g: v_1: \ldots : v_{n-1})\). This map is defined everywhere except at \((1:0:0:0)\), which is blown up to \( v_0 = g = 0 \). The inverse map can be given by two sets of equations

\[
(v_0 : g v_1 : \ldots : g v_{n-1}), \quad (h : v_0 v_1 : \ldots : v_0 v_{n-1})
\]

and has base locus \( v_0 = g = h = 0 \). So the points of \( H_n \) consist of \((1:0:0:0)\), one point for every point of \( V_N \) outside \( v_0 = g = 0 \), and either 0 or \( p \) points for every point of \( v_0 = g = 0 \). Applying the Chevalley-Warning theorem to \( v_0 = g = 0 \) we see that the numbers of points on the two varieties are congruent mod \( p \). \( \square \)

**Remark 3.14.** This construction along with several related ones is given in [20 Section 1.7.2] for double covers of \( \mathbb{P}^3 \) branched along the union of eight planes, but the general case was known earlier. The proposition above is often, though not always, applicable to the subspace reduction along a line of a variety arising from a graph polynomial.

**Definition 3.15.** When \( k = 2 \) we refer to the subspace reduction as the *normal reduction*, because the blowup can be interpreted as the normalization of \( V \) along the divisor \( x_0 = x_1 = 0 \).

Similarly, in the case \( k = n - 1 \), we obtain a hypersurface in \( \mathbb{P}^1 \times \mathbb{P}^{n-2} \) that can be viewed as a double cover of \( \mathbb{P}^{n-2} \), and if the branch locus is suitably reducible we can revert to a hypersurface of degree \( n \) in \( \mathbb{P}^{n-1} \); for some examples of this see sections 3.2, 3.4, 3.5. If neither \( k - 1 \) nor \( n - k \) is 0 or 1, it is not clear how to continue the reduction process.
The resultant reduction cannot immediately be described in terms of the subspace reduction, unless one of the factors is linear, but it is certainly related. Consider the variety in $\mathbb{P}^n$ defined by $(f_1x_0 + f_0)(g_1x_0 + g_0) = 0$, where $f_i, g_i$ do not depend on $x$ and $f_1, g_1$ have no common factor. It is prime-similar to the variety defined by $f_1x_0 + f_0 = g_1x_0 + g_0$ (even this can be seen by blowing up along the intersection). On this variety, we again blow up $(1 : 0 : \ldots : 0)$, obtaining an exceptional divisor defined by $f_1 = g_1 = 0$. If $\deg(f_1x_0 + f_0)(g_1x_0 + g_0) = n + 1$, then $\deg f_1 + \deg g_1 = n - 1$, so the exceptional divisor is subject to the Chevalley-Warning theorem. The projection from the blowup to $\mathbb{P}^{n-1}$ again has linear fibres and its image is defined by the elimination ideal for $x_1, \ldots, x_n$ of the ideal $(f_1x_0 + f_0, g_1x_0 + g_0)$, which (since these have no common factor) is generated by the resultant $f_1g_0 - f_0g_1$.

4. The modular form of weight 4 and level 13

In this section we will describe a rigid Calabi-Yau threefold for which the representation on $H^3$ gives the unique newform of level 4 and weight 13 with rational eigenvalues. By considering special elements of families containing this threefold we will also realize newforms of level 78 and, apparently, 390.

We start from the graph labeled (4, 13) in [4, Figure 5]. This is #243 in the list of 4-regular graphs on 11 vertices produced by genreg [19], whose numbering of the vertices we use in place of that of [4]. By applying the reductions of section 3 to its graph polynomial we will produce the desired threefold.

We first delete vertex 1, then take the 5-invariant for the edges

$$(2, 3), (2, 6), (2, 7), (3, 9), (6, 7),$$

and then we apply linear or resultant reduction to the edges

$$(3, 8), (4, 6), (5, 10), (4, 5), (4, 8), (5, 11), (9, 11)$$

in that order. This done, we have a polynomial of degree 6 in 6 variables to which we can apply normal reduction, since it defines a subvariety of $\mathbb{F}^5$ singular along the 3-plane $x_1 = x_2 = 0$. Next we use Proposition [3, 13] to obtain a quintic $Q_1$ in $\mathbb{P}^4$. It is defined by the polynomial

$$-x_0^2x_1x_2x_3/4 + x_1x_2^2x_3 - x_0^2x_1x_3^2/4 + x_1x_2^2x_3 - 4x_1^2x_2^2x_4 - 4x_1x_2^3x_4$$
$$-x_0^3x_1x_2x_3/4 - x_0^2x_2x_3x_4/4 - 4x_1^2x_2^2x_3x_4 - 3x_1x_2^2x_3x_4 + x_2^3x_3x_4$$
$$-x_0^2x_2^2x_4/4 + x_0^2x_3x_4 - 4x_1^2x_2^2x_4 - 4x_1x_2^2x_4 - 4x_1^2x_3x_4 - 4x_1x_2x_3x_4$$

\[\text{Figure 1. A graph for which the } c_p \text{ are congruent mod } p \text{ to the eigenvalues of the newform of weight 4 and level 13.}\]
and we know that \([Q_1]_{p} \equiv [H]_{p}/p^2 \mod p\), where \(H\) is the hypersurface defined by the graph polynomial of \(G\).

**Remark 4.1.** We expect \(Q_1\) to be birational to a rigid Calabi-Yau threefold, so, in view of \([21]\text{ Corollary 1.14}\), it is unsurprising that the codimension-2 components of the singular locus are lines of compound du Val (cdV) singularities. Of these, 10 are of type \(A_1\) and 2 of type \(A_3\). Some of the codimension-3 components are embedded points where these specialize to compound \(A_2\) and \(A_1\) singularities. However, the singularities where the components meet are not easy to deal with directly.

The most difficult singular points are those in the support of components of the singular subscheme whose multiplicity is large. We would like to blow these up; however, it is impractical to work in the product of projective spaces that would result. To improve our model, we will use a map defined by the linear system of quadrics vanishing along the reduced subschemes of certain components of the singular subscheme. Namely, consider the map \(\mathbb{P}^4 \to \mathbb{P}^4\) defined by

\[
(x_1) \to (x_0(x_2 + x_3) : x_1x_2 : x_1x_3 : x_1x_4 : x_2(x_2 + x_3)).
\]

A computation in Magma tells us that the map has degree 1 on \(Q_1\) and that the image \(Q_2\) is defined by

\[
x_0^2x_1^2x_2 + x_0^2x_1x_2x_3 + 16x_1^3x_3x_4 + x_0^2x_2x_3x_4 + 32x_1^2x_2x_3x_4
\]

\[
+ 16x_1x_2^3x_3x_4 + 16x_1^3x_3^2x_4 + 32x_1x_2x_3^2x_4 + 16x_1^2x_3^3x_4 - 4x_1^3x_2^2x_4 - 4x_1x_2^3x_4^2
\]

\[
+ 16x_1^2x_3x_4^2 + 12x_1x_2x_3x_4^2 + 16x_1x_3^2x_4^2 + 16x_2x_3^3x_4^2 - 4x_2x_3x_4^3.
\]

The discriminant of the equation defining \(Q_1\) with respect to \(x_1\) factors into polynomials of degrees 5, 2, 1. For \(Q_2\), on the other hand, the factors have degrees 4, 2, 1, 1, indicating that we have made progress.

Continuing in the same way, we map \(\mathbb{P}^4 \to \mathbb{P}^4\) by

\[
(x_1) \to ((x_0 - 2x_4)(x_1 + x_2) : x_1(x_1 + x_2) : x_1x_3 : x_2x_3 : x_3x_4).
\]

Again the map is of degree 1 on \(Q_2\), and the image \(Q_3\) is defined by

\[
x_0^2x_1x_2x_3 + x_0^2x_2^2x_3 + 16x_1^3x_2x_4 + 16x_1x_2^3x_4 + 4x_0x_1x_2^2x_4
\]

\[
+ x_0^2x_2^3x_4 + 4x_0x_1x_2x_3x_4 + 32x_1^2x_2x_3x_4 + 48x_1x_2^2x_3x_4 + 16x_1^2x_3^2x_4
\]

\[
+ 48x_1x_2x_3^2x_4 + 16x_1x_3^3x_4 + 16x_1^3x_2x_4 + 16x_1x_2^3x_4 + 4x_0x_1x_3x_4
\]

\[
+ 16x_2^2x_3x_4^2 + 32x_1x_2x_3x_4^2 + 16x_1x_3x_4^3,
\]

whose discriminant \(D_0\) with respect to \(x_0\) factors into polynomials of degrees 4, 1, 1, 1. Using Proposition 3.13 in reverse, we find that the hypersurface \(H_3\) in \(\mathbb{P}(4, 1, 1, 1, 1)\) defined by \(t^2 = D_0\) is birationally equivalent to \(Q_3\). Let \(S_1, \ldots, S_5\) be the components of the branch locus of the map \(H_3 \to \mathbb{P}^3\), where \(S_5\) is the quartic. One of the \(S_i\), say \(S_4\), meets \(S_5\) in four lines; the other three meet it in three lines of which one is double. The three lines are given by \(x_1 + x_2 = x_i = 0\) for \(i = 0, 1, 3\).

**Remark 4.2.** Write the equation of \(H_1\) in the form \(t^2 = q_1q_2\), where \(q_1, q_2\) are polynomials of degree 4. Then the general fibre of the map \(H_3 \to \mathbb{P}^1\) given by the two sets of equations \((t : q_1), (q_2 : t)\) is the union of two K3 surfaces, so by passing to a double cover of the base we represent a variety birational to \(H_3\) as a family of K3 surfaces with base \(\mathbb{P}^1\).
Definition 4.3. Map \( \mathbb{P}^3 \to \mathbb{P}^3 \times (\mathbb{P}^1)^3 \) by the identity on the first component and the projections away from the three lines \( L_i : x_1 + x_2 = x_i = 0 \) mentioned above, and let \( B \) be the image. Under the inverse map \( B \to \mathbb{P}^3 \), the points \( L_i \cap L_j \) pull back to divisors, which we will denote \( D_{ij} \). The inverse image of \( L_i \) then consists of \( D_{ij}, D_{ik}, \) and a third divisor, to be referred to as \( D_i \).

Remark 4.4. For \( i \neq j \) the three divisors \( D_i, D_{ij}, D_j \) meet in an ordinary double point, and these three points are the only singularities of \( B \). There is an algebraic small resolution of \( B \), because the image in \( B \) of the plane \( x_1 + x_2 = 0 \) that contains the three lines is smooth at the double points (we use the criterion of [20, Theorem 1.8]). Since \( B \) is a complete intersection defined by three equations of multidegree \((1,1,0,0), (1,0,1,0), (1,0,0,1)\), the adjunction formula gives its canonical class as \((-1,-1,-1,-1)\).

Definition 4.5. Let the \( R_i \) be the images in \( B \) of the \( S_i \). Let \( D_B \) be the fibre product \( H_3 \times_{\mathbb{P}^3} B \). It is a double cover of \( B \) branched along the union of the \( R_i \).

The following statements would be challenging to prove by hand, but are easy with the help of a computer.

Proposition 4.6. The \( R_i \) are surfaces of which no three intersect in a curve. Except for \( R_3 \) which has three ordinary double points, they are smooth. They do not contain any of the singularities of \( B \). Their union is defined in \( B \) by a single equation of multidegree \((2,2,2,2)\). Finally, \( R_4 \) is a blowup of \( \mathbb{P}^2 \) in three points, while \( R_1, R_2, R_3 \) are blowups of \( \mathbb{P}^2 \) in two points.

Proposition 4.7. There are 28 smooth rational curves \( C_i \) along which two of the \( R_i \) intersect, and all of their intersections are transverse. No three of the \( R_i \) intersect in a curve, but there are 7 points where three \( R_i \) meet and 3 points where four \( R_i \) meet. These points are the intersections of three or six \( C_i \) respectively. These 28 curves include 19 that map to lines in \( \mathbb{P}^3 \) and 9 that map to points.

The conditions \( h^{1,0} = h^{2,0} = 0, K = 0 \) that are a part of the definition of a Calabi-Yau threefold (Definition 1.3) are easy to verify for \( B \) or for a resolution of its singularities.

Proposition 4.8. Let \( T \) be a smooth rational threefold and let \( \pi : D \to T \) be a double cover branched along a smooth divisor \( R \) of class \(-2K_T\). Then \( h^{1,0}(D) = h^{2,0}(D) = 0 \) and \( K_D \) is trivial.

Proof. The statement on \( K_D \) is an immediate consequence of the Riemann-Hurwitz formula \( K_D = \pi^*(K_T + R) \). For the \( h^{1,0}(D) \) we proceed as in the proof of [9, Theorem 2.1]. We know that \( h^i(O_D) = h^i(O_T) + h^i(K_T) \), as mentioned at the beginning of [9, Section 1]. But by Serre duality \( h^i(K_T) = h^{3-i}(O_T) \), and \( h^i(O_T) \) is a birational invariant which is 0 for \( T = \mathbb{P}^3, i \in \{1,2\} \).

We are now ready to prove:

Theorem 4.9. There is a smooth Calabi-Yau threefold \( CY_{13} \) of Euler characteristic 98 which is birational to the variety \( Q_1 \) obtained by reduction from the graph polynomial of the graph labeled \((4,13)\) in [4, Figure 5].

Proof. We have shown above that \( Q_1 \) is birational to the double cover of \( B \) branched along the union of the \( R_i \). Numerically this is a Calabi-Yau; to show that it actually
is one, we use the method of Cynk-Szemberg [9]. First we resolve the three singular points of \( B \), obtaining a smooth threefold with \( h^{1,1} = h^{2,2} = 7, h^{0,0} = h^{3,3} = 1 \), and all other Hodge numbers 0. Then we blow up the fourfold intersection points of the \( R_i \), which are smooth on \( B \), to exceptional divisors \( E_j \). When we do this, we increase the canonical divisor on \( B \) by \( 2 \sum E_j \); on the other hand, we replace the four surfaces by their strict transforms, and their classes decrease by \( E_j \). We thus preserve the relation \( 2K_B + \sum |R_i| = 0 \). We also increase the Euler characteristic \( \chi \) of \( B \) by 2, because we replaced a point with a \( \mathbb{P}^2 \). The four surfaces that contain the point see it replaced by a rational curve and thus their Euler characteristic increased by 1.

Then we blow up the 28 curves of intersection. Each time we do this, we obtain an exceptional divisor \( E_C \) which we add to the canonical divisor, but we decrease the classes of the two surfaces containing the curve by \( E_C \), so again the relation \( 2K_B + \sum |R_i| = 0 \) is preserved. If the curve meets a third surface in a point, this blows up that point on that surface, increasing its Euler characteristic by 1. Also we replaced the rational curve, which has \( \chi = 2 \), by a \( \mathbb{P}^1 \)-bundle over it with \( \chi = 4 \). Finally, we take a small resolution of the double points above the three singular points of \( R_5 \), which are in the branch locus. (This may take us out of the projective category, but that does not matter here.) This adds 3 to the Euler characteristic of the branch locus and of the base. We still have \( 2K_B + \sum |R_i| = 0 \), and now the base and branch locus are smooth. In view of Proposition 4.8, this shows that it is a Calabi-Yau.

The Euler characteristic of the base is now \( 16 + 2 \cdot 3 + 2 \cdot 28 + 3 = 81 \). The branch locus originally consisted of surfaces whose Euler characteristics added to 42. However, we blew up \( 3 \times 4 \) (for the fourfold points) +7 (for the threefold points) +3 (the singularities of the K3), so now we have 64. Their images in the blowup of the base are disjoint so the Euler characteristics add. Thus the double cover has \( \chi = 2 \cdot 81 - 64 = 98 \).

\[ \square \]

**Remark 4.10.** The results of [9] are stated under the assumption that the intersection of any two components of the branch locus is irreducible. However, the arguments are entirely local, so this assumption is unnecessary.

**Remark 4.11.** The fibration introduced in Remark 4.2 has fibres whose transcendental lattice is isomorphic to \( U \oplus \langle -26 \rangle \), where \( U \) is the hyperbolic lattice with Gram matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Since it is not isotrivial, Theorem 4.9 together with [13, Theorem 2.13] shows that \( CY_{13} \) is a rigid Calabi-Yau threefold. However, we give a different proof, because the geometry of \( CY_{13} \) that we describe is useful for counting the points to identify the modular form and for discovering other rigid Calabi-Yau threefolds. In addition, this method of proving rigidity does not seem to apply to our other examples.

To start our proof that \( CY_{13} \) is rigid, we introduce the configuration of lines in which the \( s_i \) intersect.

**Definition 4.12.** A \((4, 13)\)-configuration in \( \mathbb{P}^3 \) consists of four planes \( P_i(1 \leq i \leq 4) \) in general position and 13 lines \( L_{ij}(1 \leq i \leq 4, 1 \leq j \leq \max(i, 3)) \) satisfying the following conditions:

1. For all \( i, j \), the line \( L_{ij} \) is contained in \( P_i \) and in no other \( P \).
(2) The following are the maximal sets of lines that intersect in a point:
\{L_{11}, L_{13}, L_{21}, L_{23}\}, \{L_{11}, L_{12}, L_{31}, L_{32}\}, \{L_{21}, L_{22}, L_{31}, L_{33}\},
\{L_{11}, L_{41}, L_{43}\}, \{L_{21}, L_{42}, L_{44}\}, \{L_{31}, L_{41}, L_{44}\},
\{L_{12}, L_{22}, L_{43}\}, \{L_{13}, L_{33}, L_{42}\}, \{L_{23}, L_{32}, L_{41}\},
\{L_{12}, L_{13}\}, \{L_{22}, L_{23}\}, \{L_{32}, L_{33}\}.

(3) If the intersection point of one of these sets of lines is contained in \(P_n\), then
one of the lines is \(L_{nj}\).

(4) The intersection \(L_{44} \cap P_i \cap P_j\) is empty for all \(1 \leq i < j \leq 3\).

**Proposition 4.13.** The automorphism group of \(\mathbb{P}^3\) acts simply transitively on the
set of \((4,13)\)-configurations.

**Proof.** Applying an automorphism, we take the \(P_i\) to be the coordinate hyperplanes,
the sequence of which is stabilized by the diagonal subgroup of \(PGL_4\). Then by
condition 4, we rescale \(x_1, x_2\) so that \(L_{44}\) is given by \(x_0 + x_1 + x_2 = x_3 = 0\). Now
\(L_{41}\) meets \(L_{44}\) in a point where \(x_0 = 0\), since the point is also on \(L_{11}\), so it must be
\((0 : -1 : 1 : 0)\). It is not the line \(x_0 = x_3 = 0\), so we write it as \(ax_0 + x_1 + x_2 = x_3 = 0\). Similarly \(L_{42}, L_{43}\) are the lines
\(x_0 + bx_1 + x_2 = x_3 = 0, x_0 + x_1 + cx_2 = x_3 = 0\).

We take \(L_{11}\), which is a line in \(x_0 = 0\) through \((0 : -1 : 1 : 0)\) which is not
\(x_0 = x_3 = 0\), so it is \(x_0 = x_1 + x_2 + dx_3 = 0\). Similarly \(L_{21}, L_{31}\) are given by
\(x_1 = x_0 + x_2 + ex_3 = 0, x_2 = x_0 + x_1 + fx_3 = 0\). From the fact that any two of
these lines meet it follows that \(d = c = f\).

Now \(L_{13}, L_{23}\) pass through \((0 : 0 : -d : 1)\), the point on \(L_{11} \cap L_{21}\). So they are
respectively \(x_0 = gx_1 + x_2 + dx_3 = 0, x_1 = hx_0 + x_2 + dx_3 = 0\). From the fact that
\(L_{13}\) meets \(L_{42}\) we learn that \(g = b\), and because \(L_{23}\) meets \(L_{41}\) it must be that
\(a = h\). Similarly \(L_{12}\) meets \(L_{43}\) and goes through the point where \(L_{11}\) meets \(L_{31}\),
so it must be \(x_0 = x_1 + cx_2 + dx_3 = 0\), and \(L_{32}\) meets \(L_{41}\) and also passes through
this point, so it is \(x_2 = ax_0 + x_1 + dx_3 = 0\). Similarly \(L_{22}\) is \(x_1 = x_0 + cx_2 + dx_3 = 0\)
and \(L_{33}\) is \(x_2 = x_0 + bx_1 + dx_3 = 0\).

Now \(L_{12}\) meets \(L_{22}\) in the point \((0 : 0 : d : c)\), even though this has not been
specifically required. But we also want this point to be on \(L_{41}\), so \(c = 0\). Similarly,
by considering the sets \(\{L_{13}, L_{33}, L_{42}\}, \{L_{23}, L_{32}, L_{41}\}\), we see that \(a = b = 0\).

The stabilizer of the part of the configuration we have considered consists of
automorphisms \((x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2 : \lambda x_3)\), and since \(x_3\) always
appears on its own or in the monomial \(dx_3\) we replace \(x_3\) by \(x_3/d\). We now have
no nontrivial automorphisms left and no parameters appearing in the equations of
the 13 lines. To recapitulate, they are:

\[L_{11} : x_0 = x_1 + x_2 + x_3 = 0, L_{12} : x_0 = x_1 + x_3 = 0, L_{13} : x_0 = x_2 + x_3 = 0,\]
\[L_{21} : x_1 = x_0 + x_2 + x_3 = 0, L_{22} : x_1 = x_0 + x_3 = 0, L_{23} : x_2 = x_0 + x_3 = 0,\]
\[L_{31} : x_2 = x_0 + x_1 = 0, L_{32} : x_2 = x_1 + x_3 = 0, L_{33} : x_2 = x_0 + x_3 = 0,\]
\[L_{41} : x_3 = x_1 + x_2 = 0, L_{42} : x_3 = x_0 + x_2 = 0,\]
\[L_{43} : x_3 = x_0 + x_1 = 0, L_{44} : x_3 = x_0 + x_1 + x_2 = 0.\]

\(\square\)

Using Magma we find that the linear system of octic surfaces singular along the
13 lines of the \((4,13)\) configuration and the six lines \(P_i \cap P_j\) has projective dimension
2. In fact, every such surface contains the union of the coordinate planes. Thus we
consider a net of quartics obtained by dividing out \( x_0 x_1 x_2 x_3 \) from the sections of this linear system. As a basis for this system, we may take

(1) \( V_0 = x_0 x_1 x_2 x_3 \), \( V_1 = (x_0 + x_1 + x_3)(x_0 + x_2 + x_3)(x_1 + x_2 + x_3) x_3 \), \( V_2 = (x_0 + x_1 + x_2 + x_3)(x_0 + x_1 + x_3)(x_0 + x_2 + x_3)(x_1 + x_2 + x_3) \).

**Definition 4.14.** We denote the tangent bundle of a smooth variety \( V \) by \( \mathcal{T}_V \).

**Proposition 4.15.** Let \( V \) be a general quartic in the linear system spanned by \( V_0, V_1, V_2 \). Then the double cover of \( \mathbb{P}^3 \) with branch locus \( V V_0 = 0 \) has a resolution which is a Calabi-Yau threefold with \( h^{2,1} = 2 \).

**Proof.** It is easily checked that the singular locus of \( V V_0 = 0 \) is supported on the (4, 13) configuration and that it consists only of double curves and quadruple points (the latter at the six singular points of \( V_0 = 0 \)). In addition, all the curves in the singular locus are of geometric genus 0 and any two meet transversely.

Let \( X \) be the double cover of \( \mathbb{P}^3 \) branched along \( V V_0 = 0 \). Then as in [S Section 6] or [9] there is a Calabi-Yau resolution \( \tilde{X} \) of \( X \) which is a double cover of a blowup \( \mathbb{P}^3 \) along a smooth branch locus. (We have already described the construction of a similar resolution in the proof of Theorem 4.9.) We also observe that \( h^1(T_{\mathbb{P}^3} \otimes \mathcal{L}^{-1}) = h^1(T_{\mathbb{P}^3}) = 0 \) by [S Proposition 5.1], where \( \mathcal{L} \) is the line bundle corresponding to the branch divisor. This tells us that every deformation of \( \tilde{X} \) is a double cover of a deformation of \( \mathbb{P}^3 \).

Combining this with [S Propositions 2.1–2.2] we conclude that \( h^1(T_{\tilde{X}}) \) is the dimension of the space of equisingular deformations of the branch locus. The (4, 13) configuration cannot be deformed, and the octics singular along its lines form a \( \mathbb{P}^2 \), so \( h^1(T_{\tilde{X}}) \) is of dimension 2.

Thus, in order to find candidates for rigid Calabi-Yau threefolds in this family, we need to find octics in the linear system that have no deformations preserving their singularities. In other words, we need either a point in the \( \mathbb{P}^2 \) parametrizing the linear system corresponding to an octic with a singularity that cannot be deformed, or the intersection of two curves in the \( \mathbb{P}^2 \) corresponding to extra singularities.

**Remark 4.16.** As Meyer points out in a similar situation [20 p. 161], the double covers branched along maximally singular octics, even if they are Calabi-Yau threefolds, are not necessarily rigid. It is also necessary that the construction of the resolution not involve blowing up any curves of positive genus. I am grateful to Colin Ingalls for helping me to understand the ideas of [S] and to express the argument of Proposition 4.15.

To list these maximally singular octics, we introduce an incidence correspondence \( I \subset \mathbb{P}^3 \times \mathbb{P}^2 \) such that \((p_3, p_2) \in I\) if and only if \( p_3 \) is a singular point of the octic corresponding to the point \( p_2 \) in the linear system. To be precise, let the linear system of octics have basis \( O_0, O_1, O_2 \), where \( O_i = V_0 V_i \). Then \( I \) is defined by the partial derivatives \( \frac{\partial \sum_{j=0}^3 x_j y_{ij}}{\partial x_j} \) for \( 0 \leq j \leq 3 \). Now, \( I \) will have a component of dimension 3 supported at \( x = y = 0 \), reflecting the fact that every linear combination of \( O_0, O_1, O_2 \) is singular along the line \( x = y = 0 \); similarly for the other 18 lines in the singular subscheme of every octic in the family. However, we are only interested in the components of \( I \) that map non-dominantly to \( \mathbb{P}^2 \). Of these
there are 7, but the image of one of them is a point contained in several others, which is of no use to us. The others are curves \( C_i \) of degree 1, 1, 1, 1, 2, 4, of which the quartic curve has one ordinary node and two ordinary cusps and is rational, while the conic has rational points. The octic corresponding to a general point of \( \mathbb{P}^2 \) (resp. a point on one of the \( C_i \), resp. a singular point of \( \cup C_i \)) has \( h^{2,1} = 2 \) (resp. 1, 0). The \( C_i \) are defined by the polynomials

\[
(2) \quad y_0, \ y_1, \ y_2, \ y_0 - y_1, \ 4y_0y_2 + y_1^2 - 2y_1y_2 + y_2^2, \\
y_0^2y_1^2 + 16y_0y_1^4 + 64y_1^2 + 2y_0y_1y_2 + 144y_0y_1^2y_2 - 128y_1^3y_2 + y_0^2y_2^2 \\
+ 228y_0y_1y_2^2 + 96y_1^2y_2^2 + 104y_0y_2^3 - 32y_1y_2^3 + 4y_2^4.
\]

There are 11 rational points of intersection of two or more \( C_i \), one of which is a singularity of the quartic; the quartic has two other singular points. Of these 13 points, the four with \( y_2 = 0 \) give octics which are multiples of \( x_3^2 \), so the double covers are not Calabi-Yaus. Seven of the octics have singularities that lead to canonical singularities on the double cover and so may give rigid Calabi-Yau threefolds. Indeed we find Hecke eigenforms of weight 4 for which \( a_p \equiv 1 - [V_p] \mod p \) in each case. Of these, the three given by the points \((0 : 0 : 1), (0 : 1/2 : 1), (0 : 1 : 1)\) split completely into linear factors and duplicate known arrangements of eight hyperplanes found in the table of [20, p. 68]. One of them, coming from \((-1/4 : 0 : 1)\), is the threefold we have been looking at, which appears to correspond to the form of level 13. The other three, given by \((-1/9 : -1/9 : 1), (-1 : 1 : 1), (-256/3 : -5/6 : 1)\), seem to realize the forms 78/4, 10/1, 390/5 in the indexing used by Meyer. Apparently no known Calabi-Yau was previously conjectured to correspond to 78/4, while there are known threefolds giving 10/1. For 390/5, the table of [20, p. 116] shows a nonrigid threefold; however, we have found a rigid one, as we will show in Proposition 4.18.

Remark 4.17. In addition to the rational points, there are three Galois orbits of nonrational intersection points, one each with points defined over \( \mathbb{Q}(\sqrt{3}) \), \( \mathbb{Q}(i) \), and the cubic field of discriminant 148.

The octics corresponding to \((-64 : -3/2 : 1), (-1 : 3 : 1)\) have isolated singularities of type \( A_2, D_4 \). The singularity at an \( A_2 \) point of the branch locus has no crepant resolution so there is no reason to expect a corresponding Calabi-Yau, and there may be no modular form matching the point counts. On the other hand, the singularity at a \( D_4 \) point does admit crepant resolutions in some cases. Both of these statements follow from a result due to Shepherd-Barron ([17, Proposition 5.1]). Since we have not found a modular form corresponding to the octic given by \((-1 : 3 : 1)\), we expect that this is not such a case and thus that no Calabi-Yau resolution exists. It is also possible that the level of the modular form is beyond the range of [20, Appendix C].

We now have enough information to prove that \( CY_{13} \) is rigid. As in [10], [12] this means that it is modular.

Theorem 4.18. \( CY_{13} \) is rigid.

Proof. The proof is almost identical to that of Proposition 4.15. The octic surface \( V_0(-V_0/4 + V_2) = 0 \) is maximally singular, so there are no deformations as a double cover. On the other hand, the base \( B \) of the Cynk-Szemberg resolution constructed
in the proof of Theorem 4.9 was obtained from \( \mathbb{P}^3 \) by blowing up only points and rational curves, so \( H^1(\mathcal{T}_B \otimes K_B) = 0 \). As in the proof of Proposition 4.15 we conclude by referring to [8, Propositions 2.1–2.2]. □

5. Counting points

In this and the next section we will prove that the Galois representation on \( H^3 \) of the rigid Calabi-Yau threefold \( CY_{13} \) constructed in the proof of Theorem 4.9 is isomorphic to that associated to the rational newform \( f_{13,4} \) of level 13 and weight 4. In order to do this, we need to count the \( \mathbb{F}_p \)-points of \( CY_{13} \). We do not have an explicit model for \( CY_{13} \), so we will show how to relate its point counts to those of \( H_3 \) and count points there instead. Using a computer the following statement is easily verified:

**Proposition 5.1.** For all primes \( 5 \leq p \leq 200 \) we have \( [H_3]_p = p^3 + 6p^2 - 15p + 1 - a_p \), where \( a_p \) is the \( T_p \)-eigenvalue for \( f_{13,4} \).

**Remark 5.2.** One could prove the relation \( [Q_3]_p - [H_3]_p = 9p^2 - 11p \) for all \( p \geq 5 \) using the birational equivalence given in Proposition 3.13, but this seems unnecessary since the calculations needed here can easily be done on \( H_3 \).

We now relate the number of points on \( CY_{13} \) to that on \( H_3 \).

**Theorem 5.3.** For all primes \( p > 2 \) the relation \( [CY_{13}]_p - [H_3]_p = 43p^2 + 64p \) holds, where \( CY_{13} \) is the rigid Calabi-Yau threefold of Theorem 4.9.

The rest of the section will be devoted to the proof of this theorem. To begin, we pass from \( H_3 \) to the double cover \( D_B \) of the variety \( B \) defined in Definition 4.3.

**Proposition 5.4.** For all \( p > 2 \) we have \( [B]_p = p^3 + 7p^2 + 4p + 1 \) and \( [D_B]_p = [H_3]_p + 12p^2 - 3p \). If \( D_B' \) is the small resolution of \( D_B \) at the double points above those of \( B \), then \( [D_B']_p = [D_B]_p + 6p \).

**Proof.** The map \( B \to \mathbb{P}^3 \) is an isomorphism outside the union of the \( D_i, D_{ij} \). These are all isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), so together they contain 6\((p^2 + 2p + 1) \) points. There are nine nonempty twofold intersections: the six \( D_i \cap D_{ij} \) are lines containing \( p + 1 \) points each, and the three \( D_i \cap D_j \) are points. Finally, the three nonempty threefold intersections \( D_i \cap D_{ij} \cap D_j \) are points. So the total number of points in the union is

\[
6(p^2 + 2p + 1) - 6(p + 1) - 3 + 3 = 6p^2 + 6p.
\]

These map to the union of three coplanar lines in \( \mathbb{P}^3 \), which has 3\( p \) points, so \( [B]_p = [\mathbb{P}^3]_p + 6p^2 + 3p \) as claimed.

The next step is to compare \( D_B \) to \( H_3 \). On all six of the divisors, the function defining the double cover is identically the square of the product of a \((1,0)\)-form and a \((0,1)\)-form on \( \mathbb{P}^1 \times \mathbb{P}^1 \). So the double cover on a divisor consists of one point above each of the \( 2p + 1 \) in the branch locus, and two above each of the \( p^2 \) remaining points. In addition, on the intersection of two adjacent branch divisors, the function is the square of a linear form, and on the singular points it is a nonzero square. Further, one of the lines in the branch locus of a component misses the branch loci of the adjacent components, while the other one meets them in one point each.

Thus, of the \( 6p^2 + 6p \) points of the union of the \( D_i, D_{ij} \), we find that \( 12p \) of them are in the branch locus of the double cover \( D_B \to B \), while the rest pull back to \( 2p \) points each, giving a total of \( 12p^2 \) points. Since these correspond to \( 3p \) points of \( H_3 \)

...
above the base locus of the map $\mathbb{P}^3 \to B$, the claim that $[D_B]_p = [H_3]_p + 12p^2 - 3p$ is proved.

For the last statement, note that the three double points of $B$ are not in the branch locus, so they give two double points of $D_B$ each. On $B$, the small resolutions give rational curves with points over $\mathbb{Q}$, so $p + 1$ points over $\mathbb{F}_p$. As already mentioned the branch function is 1 there, so each component pulls back to two components which curves of genus 0 with rational points. This means that we replace the 6 points with 6 curves having $p + 1$ points each. □

**Definition 5.5.** Let $D_V \to V$ be a double cover of a smooth threefold with branch divisor $R_V$ consisting of smooth components $B_1, \ldots, B_k$. Let $C_1, \ldots, C_l$ be the components of the intersection of two or more of the $B_i$. Assume that no curve is contained in more than 3 of the $B_i$, that no point is contained in more than 5 of the $B_i$, that the intersection of any two of the $C_j$ is transverse, and that the relation $\sum_i [B_i] = -2K_V$ holds in $\text{Pic} V$. We refer to the resolution of $D_V$ constructed in [9] as the Cynk-Zwembrger resolution of $D_V$.

**Remark 5.6.** The hypotheses on the $B_i$ are those of [9, Theorem 2.1], except that we do not assume the $B_i \cap B_j$ to be irreducible (cf. Remark 4.10).

Before continuing with the proof of Theorem 5.3, we pause to explain how to count the points on a double cover of an exceptional divisor without writing down the double cover of the blowup explicitly.

**Method 5.7.** A double cover is defined by a branch function up to squares, so to find the double cover on the exceptional divisor we need to choose a branch function that extends to the blowup without being identically 0 or $\infty$ there. If we blow up a point, this is usually easy. For example, if the point is in $\mathbb{P}^3$, let it be $(c_0 : c_1 : c_2 : c_3)$ and let $y_0, y_1, y_2$ be variables; then evaluate the branch function at $(c_0 + y_0 : c_1 + y_1 : c_2 + y_2 : c_3)$ and discard all terms not of minimal degree in the $y_i$. The result is then the branch function for the exceptional $\mathbb{P}^2$ above a point. More generally, if we are on another threefold, then instead of adding multiples of $(1 : 0 : 0 : 0)$, etc., we simply add multiples of a basis for the tangent space at the point.

For the double cover $\tilde{D}_B \to \tilde{B}$ of the blowup of a threefold $B$ along a curve $C$, we proceed as follows. Write down the blowup $\tilde{B}$ of $B$ along $C$ and choose a point $P$ in the exceptional divisor $E$ not contained in the branch locus. Then find a curve $C_P \subset \tilde{B}$ not contained in $E$ and smooth at $P$. We know the branch function $f$ for $D_B \to B$, so restrict it to the curve $C_P$ and use it to define a map $\alpha : C_P \to \mathbb{A}^1$ given by $f/r^2$, where $f, r^2$ are of the same degree and vanish to the same order at $P$ along $C$. Because $C_P$ is smooth at $P$, this map extends to $P$; it is not 0 or $\infty$ there, and the two points of the double cover lying above $P$ are defined over $\mathbb{Q}(\sqrt{\alpha(P)})$. On the other hand, write down a section $s$ of a suitable linear system on $E$ whose zero locus is the branch locus on $E$, and a form $t$ not vanishing at $P$ and defining a map $\beta : s \to \mathbb{P}^1$. The correct branch function must be a multiple of $s$, which has the correct branch locus, and must take $P$ to a square multiple of $\alpha P$, so we have determined it up to an unimportant square factor.

In either case, we obtain a double cover of a rational surface as the exceptional divisor on $\tilde{D}_B$, which for us will always be a rational surface. The points may be counted in either of two ways. One way is to parametrize the surface and explicitly determine the points blown up and curves contracted on the way from $\mathbb{P}^2$ to the
exceptional divisor. Another, for the exceptional divisors above a curve, is to use the natural projection to \( \mathbb{P}^1 \). In our situation the branch locus will always consist of two sections and some fibres. The fibres in the branch locus are covered by rational curves mapping isomorphically to them. Where the two sections do not meet, the fibre is replaced by a double cover of \( \mathbb{P}^1 \) branched in two points, which is isomorphic to \( \mathbb{P}^1 \). Where they do meet, the branch function on the fibre is of the form \( cf^2 \) for a constant \( c \) and the fibre is replaced by two curves defined over \( \mathbb{Q}(\sqrt{c}) \). The reduction of this mod \( p \) has \( (1 + (c/p))p + 1 \) points, as opposed to the \( p + 1 \) points of the fibre in the exceptional divisor on \( B \).

We now follow the steps of the Cynk-Szemberg resolution of \( D_B \), keeping track of the numbers of points at each stage. The first step is to blow up the fourfold points.

**Proposition 5.8.** Blowing up each of the fourfold points on \( B \) adds \( p^2 + 2p \) points to the double cover over \( \mathbb{F}_p \) for all odd \( p \).

**Proof.** Blowing up a fourfold point on \( B \) replaces it by a \( \mathbb{P}^2 \), and the four surfaces contribute four lines to the branch locus, any two of which meet transversely. To find the constant, we evaluate the branching function at a general point of \( B \) infinitely near to the fourfold point. This shows that the double covers are obtained by extracting square roots of

\[
4(u^2vw + uvw^2 + uvw^2 + v^2w^2), 4(-u^2vw - uvw^2 - uvw^2), 4(u^2vw + u^2w^2 + uvw^2 + w^2).
\]

Meyer shows ([20, p. 73]) that the surface in \( \mathbb{P}(2,1,1,1) \) defined by \( t^2 = \alpha uvw(u + v + w) \) over \( \mathbb{F}_p \) has \( p^2 + \left( 1 + \left( \frac{\alpha}{p} \right) \right) p + 1 \) points. When we change variables to reduce the double covers above to this form, we find that \( -\alpha = 4 \) for each of them. □

**Definition 5.9.** We introduce notation for the 28 singular curves of \( B \). The 13 that map to the \( L_{ij} \) of the \((4,13)\) configuration will be denoted \( \tilde{L}_{ij} \), while the 6 that map to the intersections of two of the planes will be the \( P_{ij} \). The 9 that map to a point in \( \mathbb{P}^3 \) will be called \( Q_i \).

**Theorem 5.10.** Let \( \tilde{D}_B \) be the blowup of \( D_B \) along \( C \), where \( C \) is a curve in the intersection of two of the \( S_i \). Then for all \( p > 2 \) the exceptional divisor of \( \tilde{D}_B \) has \( p^2 + (2 + d)p + 1 \) points over \( \mathbb{F}_p \), where \( d \) is the number of points where \( C \) meets exactly one other curve in the branch locus.

**Proof.** For each curve \( C \) we compute the branch locus and the branch function for the double cover of the exceptional divisor of the blowup of \( C \) in \( B \). The following table describes the results. Here \( F_i \) denotes the Hirzebruch ruled surface with special section of self-intersection \(-i\) and the classes are given in terms of \( S, F \), the classes of the special section and a fibre. While the Euler characteristics are not necessary for the argument, they are very useful for consistency checks.

Every double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched along two fibres and two sections is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), and thus has \( p^2 + 2p + 1 \) points over \( \mathbb{F}_p \) for all \( p \). Also, these curves do not contain any point where exactly two of the 28 curves intersect.

A double cover of \( F_1 \) branched along curves of class \( S,F,S + F \) is the blowup of the cone over a smooth conic at the vertex. This is \( F_2 \), which has \( p^2 + 2p + 1 \) points.
In the other cases, we apply Method 5.7 to construct the double covers. To count their points, we check that the variety is birational to $\mathbb{P}^2$ in such a way that only curves that are isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ and have good reduction at all odd primes are blown up or down (otherwise the number of points would not always change by $dp$).

For example, for $L_{12}$, we find that the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by

$$t^2 = -x_1^2 y_1 y_3 + x_1 x_2 y_1 y_2 + x_2^2 y_2^2 / 4.$$  

Embedding $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ by the Segre embedding and pulling back, we get the surface in $\mathbb{P}^4$ defined by

$$x_1^2 + x_2 x_3 + x_2 x_5 - x_5^2 / 4 = x_3 x_4 - x_2 x_5 = 0.$$  

Projecting away from the singular point $(-1 : 0 : -1 : 0)$ gives a birational equivalence with the smooth quadric $y_1^2 + y_3 y_4 + y_4^2 / 4$, which has $p^2 + 2p + 1$ points over $\mathbb{F}_p$ for all odd $p$. This blows up the base point to a smooth rational curve and contracts three smooth rational curves meeting in the base point to the points $(-1/2 : 0 : -1 : 1), (1/2 : 0 : -1 : 1), (0 : 1 : 0 : 0)$, so the number of points on the surface in $\mathbb{P}^4$ is $2p$ more than that of the quadric, which is $p^2 + 4p + 1$ as claimed.

The other cases are similar. Rather than working directly with the double cover of $F_1$ or $F_3$, it is easier to map to $\mathbb{F}^2$ and restore the contribution of the inverse image of the exceptional divisor at the end. If the branch locus meets the exceptional divisor in two distinct points, it is covered by a $\mathbb{P}^1$ which has $p + 1$ points. If the two points coincide, then there are $2p + 1$ if the branch function restricts to a square, as is true in all cases here.

**Definition 5.11.** Let $B_f$ be $B$ blown up on the fourfold points and let $D_{B_f}$ be its double cover. To avoid notational clutter, we will use $C, S_i, etc.$ equally to refer to curves and surfaces in $B, D_B$ and their strict transforms in $B_f, D_{B_f}$.

So far we have described the result of blowing up a single curve on $D_{B_f}$; now we need to understand what happens when the curves are blown up in order. In general, let $D_T \to T$ be a double cover of a threefold branched along some surfaces $S_i$. Let $C$ be a component of $S_1 \cap S_2$, and suppose that $C$ is smooth of genus $g$, meets all other components of the intersection of two $S_i$ transversely, and is disjoint from the singular loci of the $S_i$ and all intersections of four of the $S_i$. Let $D_T, T$ be the blowups of $D_T, T$ along $C$, with exceptional divisors $E_{D_T}, E_T$, and let $\bar{S}_1, \bar{S}_2$ be the strict transforms of $S_1, S_2$. Let $E_i = E_{D_T} \cap \bar{S}_i (i \in \{1, 2\})$. Then $D_T$ is a

| Curves       | Exc. div. | Branch classes | $\chi$(branch locus) |
|--------------|-----------|----------------|---------------------|
| $P_{12}, P_{33}, P_{34}$ | $F_0$ | $S, S, F, F$ | 4                   |
| $L_{ij} (1 \leq i \leq 3, 2 \leq j \leq 3)$ | $F_0$ | $S, S + 2F$ | 2                   |
| $P_{14}, P_{24}, P_{34}$ | $F_1$ | $S, F, S + F$ | 4                   |
| $L_{11}, L_{21}, L_{31}$ | $F_1$ | $F, S + F, S + 2F$ | 2                   |
| $Q_i$ | $F_1$ | $F, S + F, S + 2F$ | 2                   |
| $L_{41}, L_{42}, L_{43}$ | $F_1$ | $S + F, S + 3F$ | 1                   |
| $L_{44}$ | $F_3$ | $S + 3F, S + 3F$ | 1                   |

Table 1. Exceptional divisor and branch locus above the intersection of two components of the branch locus of $D_B \to B$.
double cover of $\hat{T}$, as in [9] proof of Theorem 2.1, case (d)], and the branch locus is the strict transform of that of $D_T \to T$. Moreover, our assumptions ensure that $E_T$ is a $\mathbb{P}^1$-bundle over $C$ and that the $E_i$ are sections. If we restrict $D_T \to \hat{T}$ to a double cover $E_{D_T} \to E_T$, the branch locus consists of the $E_i$ together with the fibres where $C$ meets another $S_i$. 

**Proposition 5.12.** With the notation of the previous paragraph, let $d = \#(E_1 \cap E_2)$. Then the Euler characteristic of $E_T$ is $2(2 - 2g)$, while that of $E_D$ is $2(2 - 2g) + d$.

**Proof.** Since $E_T$ is the exceptional divisor of a blowup of $T$ along a smooth curve of genus $g$, it is a $\mathbb{P}^1$-bundle and hence has Euler characteristic $\chi(\mathbb{P}^1)\chi(C_g) = 2(2 - 2g)$. To calculate $\chi(E_D)$, we recall the formula $\chi(E_D) = 2\chi(E_T) - \chi(B)$, where $B$ is the branch locus, which is valid for any double cover and easily seen from topology. Here we have $\chi(E_1 \cup E_2) = 2(2 - 2g) - d$. But $B$ is obtained from $E_1 \cup E_2$ by adding fibres, which are $\mathbb{P}^1$ with two points deleted and have Euler characteristic 0, so $\chi(B) = 2(2 - 2g) - d$ as well and $\chi(E_D) = 2(2 - 2g + d)$. \qed

**Proposition 5.13.** With the same notation as before, let $S_1, S_2$ be components of the branch locus of $D \to T$, and let $C_1, C_2$ be components of $S_1 \cap S_2$ meeting transversely in a point $P$ on no other curve in the branch locus. Let $D_{12}$ be $D$ blown up along $C_1$ and then $C_2$ and let the exceptional divisors be $E_{11}, E_{12}$. Let $D_2$ be $D$ blown up along $C_2$, and let the exceptional divisor be $E_2$. Suppose that the Picard group of $E_2$ is defined over $\mathbb{Q}$. Then, for all primes $q$ of good reduction, we have $[E_2]_q - [E_{12}]_q = q$, and the Picard group of $E_{12}$ is also defined over $\mathbb{Q}$.

**Remark 5.14.** We showed in Theorem 5.10 that the hypothesis on the Picard group always holds for the resolution of $\text{CY}_{13}$. If it did not we would obtain the more general relation $[E_2]_q - [E_{12}]_q = (\alpha/q)q$, where the field of definition of the two components of the fibre is $\mathbb{Q}(\sqrt{\alpha})$. This will be used in a later example.

**Proof.** The normal directions to $C_1$ at $P$ in $S_1, S_2$ coincide with the tangent direction to $C_2$, so $E_{11}, E_{12}, \hat{C}_2$ all meet in a point $\hat{P}$ above $P$. Thus in the double cover $E_D \to E_T$, the two points in the branch locus coincide in the fibre above $P$ and so the fibre in $E_D$ becomes two curves of genus 0 that intersect above the point. The same thing would happen above $C_2$ if we had blown it up first; but now the normal directions at $P$ in $S_1, S_2$ no longer coincide, as that would imply a more complicated singularity of $S_1 \cap S_2$ at $P$. Thus the two points in the branch locus in the fibre above $P$ remain distinct and we obtain an irreducible curve.

Note further that the components in the fibre above $P$ in $E_D$ are the only effective curves in their linear equivalence class, since they are components of a fibre of a conic bundle. This implies that they are rational over $\mathbb{F}_p$ for all $p$ of good reduction, and in particular that each has $p + 1$ points.

This applies equally well to the fibre above $P$ in the blowup of $C_2$, had we blown up that curve first. On the other hand, the fibre above $P$ in the blowup of $\hat{C}_2$ has only $p + 1$ points: thus, $p$ fewer than it would. Since the two blowups are isomorphic if these fibres are deleted, we conclude that the blowup of $\hat{C}_2$ has $p$ fewer points than that of $C_2$. The statement about Pic $E_{12}$ is now clear, since this group is equal to the subgroup of Pic $E_2$ generated by a section and all vertical curves other than those above $P$. \qed
We showed in Proposition 5.4 that \( [D_B]_p - [H_3]_p = 12p^2 \). The singularities of the branch locus consist of 28 curves with 24, 7, 3 intersection points of order 2, 3, 4 and 3 double points. In Proposition 5.8 we saw that blowing up the fourfold intersections adds \( 3(p^2 + 2p) \) to the count, while Proposition 5.13 together with Theorem 5.10 tells us that blowing up the curves contributes \( 28(p^2 + p) + 24p \).

Finally, we take a small resolution of the 9 ordinary double points (2 above each of the 3 of \( B \) and the 3 of the branch surface \( S_5 \)), which adds \( 9p \) points. We conclude that

\[
[R]_p = [H_3]_p + 12p^2 - 3p + 3p^2 + 6p + 28(p^2 + p) + 24p + 9p = [H_3]_p + 43p^2 + 64p
\]

for all \( p \geq 5 \). This completes the proof of theorem 5.3.

6. Proving modularity

Gouvêa-Yui and Dieulefait-Manoharmayum proved [16], [12] that rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) are always modular. Later, Dieulefait [11] gave an algorithm to determine the Hecke eigenform associated to a rigid Calabi-Yau. However, his algorithm requires knowing a basis for the space of newforms of weight 2 and level \( \prod p_{a_i} \), where \( p \) runs over the primes of bad reduction of the threefold and \( a_i \) is the bound on the valuation of the conductor of an elliptic curve over \( \mathbb{Q} \) at \( p_i \). For \( Q_3 \), for example, this bound would be \( 2^5 3^5 13^2 > 10^7 \), which is well beyond the range where such a basis can be calculated. So instead we use a method due to Serre and Schütt [22], inspired by ideas of Faltings. We will present the proof carefully, following [22] closely, for the modular form of level 13 and its corresponding Calabi-Yau, and then indicate the necessary changes to apply it in the other examples.

Schütt assumes, following Serre, that we have two \( \ell \)-adic Galois representations \( \rho_1, \rho_2 \) unramified outside a finite set of primes \( S \), with the same determinant and for which the mod \( \ell \) reductions are isomorphic and absolutely irreducible. In the applications, \( \rho_1 \) is the Galois representation associated to a Hecke eigenform of weight 4, while \( \rho_2 \) is the representation on \( H^3 \) of a rigid Calabi-Yau threefold (or more generally on \( H^{3,0} + H^{0,3} \) of a Calabi-Yau which is not rigid but for which this is nevertheless a component of \( H^3_{\acute{e}t} \)), and we take \( \ell = 2 \). For both representations, the determinant is the cube of the cyclotomic character [12]. Also, \( \rho_1 \) is unramified away from \( \ell \) and the primes dividing the level, while \( \rho_2 \) is unramified away from \( \ell \) and the primes of bad reduction.

Remark 6.1. As in [20, Section 1.8.1], it is easy to determine \( \text{tr} \rho_2(Frob_p) \) for \( p \geq 17 \) by counting points. This is because its absolute value is at most \( 2p^{3/2} \), while the sum of the traces on the other cohomology groups is of the form \( p^3 + n(p^2 + p) + 1 \). So if \( 4p^{3/2} < p^2 + p \) the point count determines \( n \), and this holds for \( p \geq 17 \).

We now let \( \rho_1 \) be the representation attached to the newform of level 13 and weight 4 and \( \rho_2 \) the representation on \( H^3(R) \). Our goal is to prove the following theorem.

Theorem 6.2. For all primes \( p \) we have \( \text{tr} \rho_1(Frob_p) = \text{tr} \rho_2(Frob_p) \).

We start by checking this for small primes:

Lemma 6.3. For all primes \( p \in [17, 50] \) we have \( \text{tr} \rho_1(Frob_p) = \text{tr} \rho_2(Frob_p) \).
Proof. We consult [20, Appendix C] for the tr $\rho_1(\mathrm{Frob}_p)$ and calculate the tr $\rho_2(\mathrm{Frob}_p)$ by counting points and using Theorem 5.3.

In order to prove anything about $\rho_2$, we will need to know its ramified primes, which requires us to determine a set containing the primes of bad reduction of $\mathrm{CY}_{13}$. These are contained in the set of primes of bad reduction of $H_3$, which in turn are a subset of the primes contained in an associated prime of the ideal defining the singular subscheme of an integral model.

The following statement is presumably well-known; a proof is provided here for lack of an adequate reference.

**Lemma 6.4.** Let $I \subset \mathbb{Z}[x_0, \ldots, x_n]$ be an ideal and $B$ a Gröbner basis for $I$. Let $p$ be a prime that does not divide the leading coefficient of any element of $B$. Then no associated prime of $I$ contains $p$.

**Proof.** By setting $x = 1, y = p$ in [14, Exercise 15.22], we see that it suffices to show that $I = (I : p^\infty)$. Suppose that $f_0 = p^nx \in I$. Inductively choose $b_i \in B$ whose leading term divides that of $f_i$ and monomials $q_i$ such that $f_{i+1} = f_i - q_ib_i$ has leading term smaller in the monomial order than the leading term of $f_i$. By hypothesis $p$ does not divide the leading coefficient of $b_i$, so $p^n$ divides the leading coefficient of $q_i$. Since $f_0 \in I$, this process must terminate in 0 ([14, Algorithm 15.7]). Thus we have written $f_0 = \sum_i q_ib_i$, where $b_i \in I$ and $p^n|q_i$. It follows that $x \in I$ as desired. □

**Remark 6.5.** The converse of this statement is false; the set of primes dividing the leading coefficient of an element of $B$ depends on the term order, while the set of associated primes depends only on the ideal.

Using Magma to find a Gröbner basis for the ideal generated by the partial derivatives, we see that $H_3$ has good reduction away from $\{2, 3, 5, 13\}$, and then we check that the reduction mod 5 of the singular subscheme of $H_3$ is equal to the singular subscheme of $H_3$ base changed to $\mathbb{F}_5$, so that 5 is in fact a prime of good reduction. We conclude that $\rho_2$ is unramified outside $\{2, 3, 13\}$.

Our first task is to determine the mod 2 representations $\bar{\rho}_i$. Note that the trace of $\alpha \in \mathrm{GL}_2(\mathbb{F}_2)$ is 0 if $\alpha$ has order 1 or 2 and 1 if $\alpha$ has order 3. Taking $\alpha = \mathrm{Frob}_p$, this says that tr $\mathrm{Frob}_p$ is 0 if the residue field of a prime above $p$ in the field cut out by $\bar{\rho}_i$ has order $p$ or $p^2$ and 1 if the order is $p^3$.

Also, $\alpha$ has a fixed 1-dimensional subspace if and only if its order is 1 or 2. It follows immediately that a representation to $\mathrm{GL}_2(\mathbb{F}_2)$ is absolutely irreducible if and only if its trace is not identically 0. This condition holds in our example, because both $a_5$ of the modular form and the number of points mod 5 are odd.

**Definition 6.6.** Let $F_5 = \mathbb{Q}[x]/(x^5 - x - 2)$ and let $F_6$ be its Galois closure.

**Proposition 6.7.** Both $\bar{\rho}_1$ and $\bar{\rho}_2$ are the projection $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(F_6/\mathbb{Q})$ followed by an isomorphism $\mathrm{Gal}(F_6/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_2)$.

**Proof.** The tables of Jones and Roberts [18] give all extensions of $\mathbb{Q}$ with Galois group $C_3, S_3$ unramified outside $S$. We find that tr $\bar{\rho}_i(\mathrm{Frob}_p) = 1$ for $p = 17$ and 0 for $p = 19, 23, 29$. The only choice for ker $\rho_i$ consistent with this is $\mathrm{Gal}(\bar{\mathbb{Q}}/F_6)$. □

**Definition 6.8.** Let $\sigma_1, \sigma_2$ be two Galois representations to $\mathrm{GL}_r(\mathbb{Z}_\ell)$ with the same determinant whose reductions mod $\ell^n$ are equal for some $n > 0$. We define
Further, we define the discrepancy function $\delta_n : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_\ell$ by $\delta_n(\alpha) = (\text{tr}\sigma_1(\alpha) - \text{tr}\sigma_2(\alpha))/\ell^n$.

Further, we define the discrepancy function $\delta_n : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_\ell$ by $\delta_n(\alpha) = (\text{tr}\sigma_1(\alpha) - \text{tr}\sigma_2(\alpha))/\ell^n$.

as follows. Let $\mu(\alpha)$ be the matrix such that $\sigma_1(\alpha) = (1 + \ell^n\mu(\alpha))\sigma_2(\alpha)$. Then $\delta_n(\alpha) = (\mu(\alpha), \bar{\sigma}_1(\alpha))$.

Remark 6.9. These objects are defined in [22, Section 5] (the only difference is in the use of subscripts to facilitate the statement of Corollary 6.11 below), but Schütt does not give English names to them. Of course $d_n$ is identically 0 if and only if the traces are congruent mod $\ell^{n+1}$. It is difficult to work with $d_n$ directly because it does not have useful algebraic properties.

Proposition 6.10 ([22, Section 5]). $\delta_n$ is a continuous homomorphism unramified at every prime where $\sigma_1, \sigma_2$ are unramified, and $d_n(\alpha) = \text{tr}(\mu, \bar{\sigma}_1(\alpha))$. In the case of interest $\ell = r = 2$, the target group is isomorphic to $S_4 \oplus C_2$, and $\text{tr}(M_1M_2) = 1$ if and only if the order of $(M_1, M_2)$ in the semidirect product is 4 or 6.

Corollary 6.11. Let $\rho_1, \rho_2$ be Galois representations to $GL_2(\mathbb{Z}_2)$, unramified outside $S$, whose residual representations are equal and surjective and cut out the field $F$. Suppose that the traces of $\rho_1, \rho_2$ are congruent mod $\ell^n$. Let $E_F$ be the set of all extensions of $\mathbb{Q}$ with Galois group $S_4$ or $S_3 \oplus C_2$ that are unramified outside $S$ and contain $F$. Suppose that for every $E_i \in E_F$ there is a prime $p_i$ of inertial degree 4 or 6 such that $\text{tr}\rho_1(\text{Frob}_p) \equiv \text{tr}\rho_2(\text{Frob}_p)$ mod $\ell^{n+1}$. Then the traces of $\rho_1, \rho_2$ are congruent mod $\ell^{n+1}$.

Proof. Let $F'$ be the field cut out by $\delta_n$. If the traces are not congruent mod $\ell^{n+1}$, then $F' \neq F$. But $F \subset F'$ and $\text{Gal}(F'/\mathbb{Q})$ is a subgroup of $S_4 \oplus C_2$ mapping surjectively to $S_3$, so as Schütt shows it is isomorphic to one of $S_4 \oplus C_2, S_4, S_3 \oplus C_2$. In the first case, replace $F'$ by the subgroup fixed by $C_2$. Now the Chebotarev density theorem gives us a prime $q$ for which the order of Frobenius in $\text{Gal}(F'/\mathbb{Q})$ is 4 or 6 and hence $d_n(\text{Frob}_q) = 1$. However, our hypothesis gives us a prime $q'$ with Frobenius of the same order and $d_n(\text{Frob}_{q'}) = 0$. Since $S_4$ and $S_3 \oplus C_2$ have only one conjugacy class of order 4 or 6, in fact $\text{Frob}_q$ and $\text{Frob}_{q'}$ are conjugate, a contradiction.

Remark 6.12. The proof is the same as that used by Schütt to conclude that $\text{tr}\rho_1 = \text{tr}\rho_2$ if the traces of Frobenius are the same for the given set of primes. The slightly more precise statement given here is not needed in the present paper.

Proposition 6.13. There are 7 extensions of $\mathbb{Q}$ with Galois group $S_4$ containing $F_6$ and unramified outside $\{2, 3, 13\}$. For each one, at least one of 19, 23, 41 has Frobenius of order 4.

Proof. From the tables of [18] we find that there are 150 extensions of $\mathbb{Q}$ with Galois group $S_4$, unramified outside $\{2, 3, 13\}$. However, by calculating cubic resolvents we find that only 7 of these contain $F_6$. It is routine to verify the second statement, since the Frobenius in the $S_4$-extension has order 4 if and only if the prime is inert in the non-Galois quartic subfield.

Proposition 6.14. There are 14 extensions of $\mathbb{Q}$ with Galois group $S_3 \oplus C_2$ containing $F_6$ and unramified outside $\{2, 3, 13\}$. For each one, at least one of 17, 37, 43 has Frobenius of order 6.
Proof. These extensions are obtained by adjoining a square root of the product of a subset of \{-1, 2, 3, 13\}; the subsets \{\}, \{-1, 2, 13\} are not usable because their products are squares of elements of \(F_6\). The Frobenius is of order 6 if and only if it has order 3 in \(\text{Gal}(F_6/Q)\) and order 2 in the quadratic extension. The second statement is now easily checked. □

Combining the last two propositions with Lemma 6.3 and Corollary 6.11 we complete the proof of Theorem 6.2.

To close this section, we determine the number of \(\mathbb{F}_p\)-points on \(CY_{13}\) for all \(p\) of good reduction.

Proposition 6.15. For all \(p\) different from 2, 3, 13, the number of \(\mathbb{F}_p\)-points on \(CY_{13}\) is equal to \(p^3 + 49p^2 + 49p + 1 - a_p\), where as before \(a_p\) is the eigenvalue of the Hecke operator \(T_p\) on \(f_{13,4}\).

Proof. (sketch) Since \(CY_{13}\) is a rigid Calabi-Yau threefold of Euler characteristic 98 (Theorem 4.9), it has \(h^{1,1} = h^{2,2} = 49\). We have just shown that the trace of Frobenius on \(H^3\) is \(a_p\). So, by the standard properties of étale cohomology, it suffices to show that \(\text{Pic} CY_{13}\) has a basis defined over \(\mathbb{Q}\). Indeed, consider the pullbacks of the 7 generators of \(\text{Pic} B\), the 28 + 7 + 3 + 3 = 41 exceptional divisors, and \(K\), a K3 surface in the fibration of Proposition 4.2. Let \(E_i\) for \(1 \leq i \leq 47\) be all of these divisors except for \(K\) and \(H\), the pullback of the hyperplane class from \(\mathbb{P}^3\). The \(E_i\) are all immobile exceptional divisors, and for each \(E_i\) there is a curve that meets it and no other generators.

So it suffices to show that \(H, K\) are linearly independent. By the adjunction formula we have \(K^2 = 0\) (this holds for any Calabi-Yau divisor on a Calabi-Yau variety). Thus \(H \cdot (H \cdot K)\) is twice the degree of the image of \(K\) in \(\mathbb{P}^3\), which is not 0, while \(K \cdot (H \cdot K) = H \cdot (K \cdot K) = 0\). □

7. Modular forms of level 78 and 390

As remarked at (2), the double cover of \(\mathbb{P}^2\) branched along

\[V_0(-V_0/9 - V_1/9 + V_2)\]

appears to give a rigid Calabi-Yau threefold that realizes a twist of the newform 78/4, where the \(V_i\) are defined in (1). The proof of this statement is very similar to that of Theorem 6.2 and so will not be presented in detail. Rather, we will only indicate the changes to the argument for Theorem 6.2 that are needed for the proof.

To reduce the level we change the sign. The expression \(V_0/9 + V_1/9 - V_2\) factors as a product of a linear and a cubic polynomial. The six components of the branch locus intersect in 23 curves and all intersections of these are transverse. All but one of them are lines. Also there are 9 points where four surfaces meet. As in Proposition 5.8 we find that each of these contributes \(p^2 + 2p + 1\) points to the double cover and \(p^2 + p + 1\) to the base.

One checks that the intersection of the cubic surface with one of the planes is a nodal cubic curve. We call the curve \(N\) and the node \(p_N\). All the other components are smooth and meet transversely.

Proposition 7.1. The double cover of \(\mathbb{P}^3\) branched along \(V_0(V_0/9 + V_1/9 - V_2) = 0\) has a rigid Calabi-Yau resolution.
Proof. As for $CY_{13}$, we construct a Cynk-Szemberg resolution. It is unwise to blow up $p_N$, because that would destroy the good numerical properties of the double cover; the canonical divisor of $P^3$ blown up in a point is $-4H + 2E$, and we would have a branch locus of class $8H - 2E$. We avoid this problem by blowing up the singular curve $N$. This produces a base for the double cover which is nonsingular except at one ordinary double point. We may then take a small resolution of this point and continue with the Cynk-Szemberg resolution as before. The proof of rigidity is identical to that of Theorem 1.18.

When the blowup of $P^3$ along $N$ is defined as a map to $P^3 \times P^{10}$ given as the identity on $P^3$ and the basis for the linear system of cubics vanishing along $N$ used by Magma, the coordinates of the singular point are

$$(-1/3 : -1/3 : -1/3 : 1/3 : 2/9 : 1/9 : 2/9 : 1/9 : 0 : 1/9 : -2/3 : -1/3 : 1).$$

It is easily verified that the tangent cone there is a quadric in $A^4$ of determinant $-3$ up to squares and that the branch function takes the value 5 there up to squares. So if $(5/p) = 1$ there are two $F_p$-rational double points, and the resolution has rational tangent directions if and only if $(-3/p) = 1$. Thus, as in [20], taking a nonprojective small resolution of the nodes adds 2$p$ points if $(5/p) = (-3/p) = 1$ and subtracts 2$p$ points if $(5/p) = 1, (-3/p) = -1$.

Now we need to study the double covers of the intersections of the components of the branch locus. We start with $N$, which has $p + 1 - (-3/p)$ points mod $p$. Let $E_0$ be the exceptional divisor for the blowup of $P^3$ along $N$: it has $(p + 1)(p + 1 - (-3/p))$ points. The intersections of the four branch surfaces not containing $N$ give us six fibres twice each, so they do not contribute to the branch locus. The two surfaces that do contain $N$ yield sections. These surfaces are disjoint because we blew up their intersection, along which they were smooth. So every fibre of $E_0$ is covered by a rational curve, and the number of points is the same on the double cover as on $E_0$.

So, by blowing up and resolving the nodes we removed a curve with $p + 1 - (-3/p)$ points and replaced it with a surface with $(p + 1)(p + 1 - (-3/p)) + 2(-3/p)p$ points if $(5/p) = 1$ or $(p + 1)(p + 1 - (-3/p))$ points if $(5/p) = -1$. One checks the four cases to verify that this means we have added $p^2 + p + (-15/p)p$ points over $F_p$ for any $p > 5$.

For the lines, the calculation is much easier. The exceptional divisors above a line are all $P^1 \times P^1$. For 7 of them the branch locus consists of two sections, and for 3 of two sections and two fibres, so for these the double covers all have $p^2 + 2p + 1$ points mod $p$. The rest consist of 3 with branch curves of classes $F, F, S, S + 2F$ and 9 with $S, S + 2F$. These need to be checked individually, but in no case is it difficult to verify that the number of points mod $p$ is $p^2 + 4p + 1$.

We now have enough information to compare the number of points on the octic to that of the Cynk-Szemberg resolution. The 9 fourfold intersections add $9(p^2 + 2p)$ to the count. As in Proposition 5.13, the double covers of exceptional divisors above lines add $10(p^2 + p) + 12(p^2 + 3p) - 12p$, since there are 12 double points. Finally, we add $p^2 + p + (-15/p)p$ points for the resolution of the curve $N$, concluding that the resolution of the double octic has $32p^2 + (53 + (-15/p))p$ more points than the double octic itself. This is highly encouraging in light of the following calculation:
Proposition 7.2. For all primes $7 \leq p \leq 200$ the number of $\mathbb{F}_p$-points on the double cover $t^2 = V_0(V_0/9 + V_1/9 - V_2)$ is equal to $p^3 + 4p^2 - 17p - (-15/p)p + 1 - a_p$. Thus the number of points on the Cynk-Szemberg resolution is $p^3 + 36p^2 + 36p + 1 - a_p$. \hfill $\square$

The proof that the Galois representation for this threefold matches that of the modular form $78/4$ uses the method of Livné ([20 Theorem 1.5]) rather than that of Serre-Schließt, since the mod 2 representation is reducible. By calculation, the only $p < 50$ for which the modular form has $a_p$ odd are 3, 13. To prove that there are no more, we consider the cubic extensions of $\mathbb{Q}$ unramified outside 2, 3, 13. If the representation were reducible, it would cut out one of these extensions, and a prime of Frobenius degree 3 would have odd eigenvalue. We already saw in Proposition 6.14 that this is not possible, since there is such a prime $< 50$ in every such extension. Similarly, for the Galois representation attached to the Calabi-Yau threefold, we study the cubic extensions of $\mathbb{Q}$ unramified outside 2, 3, 5, 13. From [18] we obtain a complete list of these and verify that all 114 have an inert prime $p \in [17, 43]$.

To conclude the proof that the semisimplifications of $\rho_1, \rho_2$ are isomorphic, let

$$Q_S = \mathbb{Q}(\sqrt{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{13}})$$

be the compositum of all quadratic extensions of $\mathbb{Q}$ unramified outside 2, 3, 5, 13. Clearly $\text{Gal}(Q_S/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^5$. It suffices to show that there is a set of primes for which the trace of Frobenius is the same for $\rho_1, \rho_2$ and for which every cubic polynomial on $(\mathbb{Z}/2\mathbb{Z})^5$ vanishing on their Frobenius images in $\text{Gal}(Q_S/\mathbb{Q})$ vanishes on all of $(\mathbb{Z}/2\mathbb{Z})^5$. In fact this holds for the set of primes in [17, 157]. \hfill $\square$

Proposition 7.3. The double cover $t^2 = V_0(-512V_0 - 5V_1 + 6V_2)$ has a rigid Calabi-Yau resolution. If the tables of [18] are complete for cubic extensions of $\mathbb{Q}$ unramified outside 2, 3, 5, 13, 23, it realizes the modular form $390/5$.

Remark 7.4. We do not give a detailed proof because no new techniques are involved and the desired conclusion remains conditional. Given a list of extensions known to be complete, it would be easy to finish the proof that the claimed modular form is realized (assuming that this is true). At worst it would be necessary to calculate a few more $a_p$.

Proof. (sketch) The branch locus consists of the four coordinate planes and a quartic surface which is singular at the six points where two of the coordinate planes meet the plane $x_0 + x_1 + x_2 + x_3 = 0$ and at $(\alpha : \alpha : \alpha : 1)$, where $\alpha^2 - \alpha - 1/12 = 0$. Blowing up the six rational singular points, we obtain a branch locus consisting of five components meeting in 22 curves, any two of which intersect transversely in a single rational point or not at all. The canonical divisor of $\mathbb{P}^3$ blown up in the six points is $-4H + 2\sum E_i$, where the $E_i$ are the exceptional divisors. The class of the strict transform of the quartic surface is $4H - 2\sum E_i$, because the singularities are ordinary double points. The classes of the components of the branch locus are $H - \sum E_{i,n}$, where $E_{i,n}$ is the subset of the $E_i$ of divisors above points whose $n$th coordinate is 0. Each $E_i$ appears twice, so the class of the branch locus is $-2$ times that of the canonical of the base. Now the surfaces all meet transversely in rational curves that meet transversely and we can construct a crepant Cynk-Szemberg resolution. The proof of rigidity is now identical to that in Theorem 4.18.
For counting the points, it is helpful to notice that \( V_0(-512V_0 - 5V_1 + 6V_2) \) is invariant under permutations of the first three coordinates. The double covers of the rational singular points have \( p^2 + (2 + (-3/p) + (-2/p))p + 1 \) points mod \( p \); by the \( S_3 \)-invariance it is enough to check this for one of them. They replace 1 point of \( \mathbb{P}^3 \). We then blow up the three fourfold points of intersection, obtaining \( p^2 + (1 + (-6/p))p + 1 \) points above each to replace 1. The 22 curves of intersection fall into 7 orbits, tabulated below. Each one is a smooth rational curve (indeed, the strict transform of a line in \( \mathbb{P}^3 \)) and so contains \( p + 1 \) points mod \( p \).

| Representative | Size of orbit | Number of points |
|----------------|--------------|-----------------|
| \( x = y = 0 \) | 3            | \( p^2 + 2p + 1 \) |
| \( x = w = 0 \) | 3            | \( p^2 + 2p + 1 \) |
| \( x + y = w = 0 \) | 3            | \( p^2 + (1 + 2(-2/p))p + 1 \) |
| \( y + z = w = 0 \) | 6            | \( p^2 + (1 + 2(-2/p))p + 1 \) |
| \( x + y + w/6 = z = 0 \) | 3           | \( p^2 + (1 + 2(-2/p))p + 1 \) |
| \( x + y + z = w = 0 \) | 1           | \( p^2 + 2p + 1 \) |
| \( x + y + w = z = 0 \) | 3            | \( p^2 + 2p + 1 \) |

Table 2. Curves in the singular locus of the branch locus of a partial resolution of \( t^2 = -512V_0 - 5V_1 + 6V_2 \) and the number of points on the double covers of their exceptional divisors.

In addition, we need to resolve the nonrational nodes by small resolutions. The field of definition of each node is \( \mathbb{Q}(\sqrt{3}) \), while that of the lines in an exceptional divisor above a node is \( F_4 = \mathbb{Q}(\sqrt{3}, \sqrt{6 + 4\sqrt{3}}) \).

**Definition 7.5.** For a prime \( p > 3 \), let \( \alpha_p = 0 \) if \( (3/p) = -1 \) or \( (-3/p) = -1 \). Otherwise, let \( \alpha_p = 2 \) if \( p \) splits completely in \( F_4 \) and \( -2 \) if not.

If \( (3/p) = -1 \) the nodes are not defined over \( \mathbb{F}_p \) so the resolution does not affect the count of \( \mathbb{F}_p \)-points. If \( (-3/p) = -1 \), then one node contributes \( p \) and the other \( -p \), so they cancel out. If both are 1, then we get \( 2p \) if \( p \) splits completely in \( F_4 \) and otherwise \( -2p \). Thus by resolving the nodes we add \( \alpha_p p \) points to our total.

Finally, we account for the difference between blowing up curves individually and in order. This is not the situation of Proposition 5.13 but that of the following remark, because the Picard groups of the exceptional divisors are not defined over \( \mathbb{Q} \). There are 12 points where exactly two of the curves meet, and at each of these it is a double cover defined by a function of the form \( -2 \) times a square that is replaced by an ordinary curve. So for each one we subtract \( (2/p)p \) from the count. In summary, the number of points on the Cynk-Szemberg resolution is greater than that on the double octic by

\[
31p^2 + (31 + 3(-6/p) + 6(-3/p) + 18(-2/p) + \alpha_p)p.
\]

Combining this with a count of points on the double octic, we find that for \( p \neq 23 \in [17, 400] \) the number of points on the Cynk-Szemberg resolution is

\[
p^3 + (32 + (-3/p) + 3(-2/p))(p^2 + p) + 1 - a_p,
\]

where \( a_p \) is the Hecke eigenvalue for the newform 390/5. This suggests that there is a basis for the Picard group of the resolution consisting of 28 rational divisors, 2 divisors conjugate over \( \mathbb{Q}(\sqrt{-3}) \), and 3 pairs of divisors conjugate over \( \mathbb{Q}(\sqrt{-2}) \).
At this point we need to assume that our lists of fields from [18] are complete in order to use Livnɛ’s method. Under this assumption, by going up to 97 we verify that the mod 2 representations are reducible and have trace 0. As before, let \( \mathbb{Q}_S = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{13}, \sqrt{23}) \) be the maximal extension of exponent 2 unramified outside the primes dividing the level of the modular form and the primes of bad reduction of the Calabi-Yau.

It suffices to show that there is a set of primes for which the trace of Frobenius is the same for \( \rho_1, \rho_2 \) and for which every cubic polynomial on \((\mathbb{Z}/2\mathbb{Z})^6\) vanishing on their Frobenius images in \(\text{Gal}(\mathbb{Q}_S/\mathbb{Q})\) vanishes on all of \((\mathbb{Z}/2\mathbb{Z})^6\). In fact this holds for the set of primes not equal to 23 and in \([17, 353]\).

**Remark 7.6.** It is striking that for all three of these examples there is a prime of bad reduction that does not divide the level of the modular form. This may be related to the phenomenon described by Schütz in [22].

8. Other graphs giving forms of weight 3 and 4

Brown and Schnetz give ([4], Figure 5) graphs for which the point counts match the modular forms of weight 4 and level 5, 6, 7, 17, in addition to level 13 on which we have been concentrating up to now. They also give graphs for which the point count matches the forms of weight 3 and level 7, 8, 12. In this final section we will apply the methods of this paper to these graphs. We display graphs isomorphic to those of [4] in Figure 2 with the order on the vertices given by \texttt{genreg} [19].

**Definition 8.1.** We refer to the \( n \)th graph in the list of 4-regular graphs on \( k \) vertices produced by \texttt{genreg} as \( G_{k,n} \).

![Figure 2](image-url)  

**Figure 2.** Graphs for which the \( c_p \) are congruent mod \( p \) to the eigenvalues of forms of weight 3 and level 7, 8, 12.

### 8.1. Weight 3, level 7

There is very little to add to the discussion of this example in [3]. The graph is isomorphic to \( G_{10,54} \). As in [3], we choose to delete vertex 3 and begin by computing the same five-invariant, which in our numbering comes from the edges

\[(1, 2), (1, 4), (1, 5), (2, 7), (4, 5)\].

Instead of reducing the sequence of edges of [3], we reduce the sequence

\[(2, 6), (4, 8), (6, 9), (6, 10), (7, 9), (9, 10)\],

obtaining a polynomial in 5 variables to which subspace reduction for \( x_0 = x_1 = x_2 \) applies. The Segre embedding produces a surface of degree 8 in \( \mathbb{P}^5 \) with canonical
8.2. Weight 3, level 8. We begin by deleting vertex 2. Lemma 55 of [3] does not apply here, but by computing the five-invariant for the edges

$$(1, 4), (3, 8), (4, 7), (6, 10), (7, 10)$$

and the linear and resultant reductions for edges

$$(7, 9), (1, 3), (1, 5), (3, 9), (5, 8).$$

we find a polynomial in 6 variables on which normal reduction can be used to obtain a double cover of $\mathbb{P}^3$ branched along an octic, which is

$$(x_1 + x_2)(x_0^2 x_2 + x_0 x_1 x_2 + x_1 x_2 x_3 + x_1 x_3^2) \times$$

$$x_0^2 x_1 x_2 + x_0 x_1^2 x_2 + x_0^2 x_2^2 + x_0 x_1 x_2 x_3 + 4 x_0 x_1 x_2 x_3$$

$$+ x_1^2 x_2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2.$$  

Now, this octic is quite unlike the ones discussed in previous sections: the quartic component is singular along the line $x_1 = x_2 = 0$, which is contained in the other two components. Likewise, if we convert the double cover to a quintic $Q$ by means of Proposition 3.13, we find that every monomial in the equation defining $Q$ vanishes to order 3 along the line $L : x_0 = x_2 = x_3 = 0$. So we apply Proposition 3.11 to obtain a prime-similar subvariety of $\mathbb{P}^1 \times \mathbb{P}^2$.

The Segre embedding makes it a subvariety of $\mathbb{P}^5$ defined by the equations

$$x_1 x_5 - x_2 x_4, \quad x_0 x_5 - x_2 x_3, \quad x_0 x_4 - x_1 x_3,$$

$$x_0 x_2 x_3 - x_1 x_2 x_4 - 2 x_2^3 x_4 - x_2^2 x_5 - 4 x_2 x_3 x_4 x_5 + x_3^2 x_4 - x_4^3 - 2 x_4^2 x_5 - x_5^2,$$

$$x_5^2 x_2 - x_1 x_2 - 2 x_1 x_2^2 - 4 x_1 x_2 x_4 + x_1 x_3^2 - x_1 x_4^2 - 3 x_2^3 x_4 - 2 x_2 x_3^2 - x_2 x_4 x_5.$$  

Projecting away from $(0 : 0 : 0 : -1 : 0 : 1)$ gives a birational equivalence with a surface in $\mathbb{P}^3$ defined by a quadric and a cubic with only canonical singularities, so this is a K3 surface. It has seven singular points, four $A_2$ and three $A_1$, and there are 14 lines through one or more of the singularities. It is routine to work out the intersection matrix of the 11 exceptional curves and these lines. The matrix has rank 20, which proves that the graph is modular of weight 3, and when five rows and columns corresponding to generators of the kernel are deleted the determinant is 8, so the modular form is a twist of the one of level 8.

To prove that the modular form is indeed the newform of level 8, we pass to $P$, the projection away from $(0 : 0 : 0 : -1 : 0 : 1)$ described earlier, which is clearly prime-similar. We verify, using Lemma 6.4, that $P$ has good reduction outside $\{2, 3\}$. Also, it has an $A_3$ singularity, three $A_2$, and three $A_1$, so the number of $\mathbb{F}_p$-points on a resolution is $12p$ more than on $P$. The trace of Frobenius acting on the two-dimensional component of $H^2(P)$ is therefore $[P]_p - p^2 - 8p - 1$.

Next we check that the trace of Frobenius at $p$ is even for a set of primes of good reduction and including one prime inert in every cubic extension of $\mathbb{Q}$ unramified outside 2, 3. From [18] there are 9 such fields and we need to go up to $p = 19$ to exclude them all. To apply Livně’s method, we need to find primes of good reduction with all possible Frobenius elements in $\text{Gal}(\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3})/\mathbb{Q})$ for which $a_p = \text{tr} \text{Frob}_p$; it suffices to consider the primes up to 23. This confirms that the modular form is that of level 8.
8.3. Weight 3, level 12. We start by reducing the projectivized graph hypersurface to a hypersurface of degree 6 in $\mathbb{P}^3$ by deleting vertex 3, computing the five-invariant for the edges

$$(1, 2), (1, 4), (4, 5), (7, 11), (8, 11)$$

and then reducing the edges

$$(1, 5), (2, 6), (2, 7), (4, 6), (6, 8), (6, 9), (8, 10).$$

Once that is done, we apply subspace reduction to the line $x_0 = x_1 = x_2 = x_3 = 0$ and then Proposition 6.13 to obtain a quintic $Q_{12}$ in $\mathbb{P}^4$ defined by

$$x_0^2x_1x_2^2 + x_0^2x_1x_2x_3 + x_0^2x_1x_2x_4 + x_0^2x_1x_3x_4 + x_0^2x_2^2x_3$$

$$+ 2x_0^2x_2x_3x_4 + x_0^2x_3x_4^2 - 4x_1^2x_2x_4 - 8x_1x_2x_3x_4 - 4x_1^2x_3x_4 - 4x_1x_2x_3x_4^2$$

$$- 4x_1x_2x_3x_4^2 - 4x_1x_2x_3x_4^2 + 4x_2x_3^2x_4.$$

None of the reduction techniques of Section 3 applies to this quintic in $\mathbb{P}^4$, nor to any other that can be constructed from the graph by the methods of this paper. Let us consider the linear system of quadrics vanishing along the three lines in the singular subscheme of $Q_{12}$ that contain the point $(0 : 1 : 0 : 0)$ and at the isolated points of the singular subscheme: a basis is

$$x_0x_2, x_0x_3, x_0x_4, x_1x_2 + x_1x_3, x_3x_4.$$  

It defines an invertible rational map from $Q_{12}$ to a threefold $T$ which is of degree 4 and therefore prime-trivial. The components of the loci along which this map and its inverse are not defined are all shown to be prime-trivial by the Chevalley-Warning theorem, except that $(0 : 1 : 0 : 0) \in Q_{12}$ goes to the surface in $\mathbb{P}^4$ defined by

$$x_0 + x_1 = x_1^2x_4 - x_1^2x_2x_3 - 2x_1^2x_2x_4 + x_1x_2^2x_3$$

$$+ x_1x_2^2x_3 - 4x_1x_3^2 - 4x_1x_2^2x_4 - 4x_2x_3^2x_4 = 0,$$

which is therefore prime-similar to $Q_{12}$. Removing the linear equation, we get a surface $S_3$ in $\mathbb{P}^3$, which has singularities of type $A_6, A_4, A_2$ at $(0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (1 : 1 : 0 : 0)$.

Remark 8.2. This construction could be described in terms of blowing up $Q_{12}$ at $(0 : 1 : 0 : 0 : 0)$. The exceptional divisor consists of three planes and the blowup is singular along one of them. If we normalize the blowup along this plane and pull back the plane to the normalization, we obtain a surface birationally equivalent to $S_3$. But in the absence of a result comparable to Proposition 6.11 that shows how to choose the right blowup a priori, this is only a trick, not a method.

The quadrics vanishing at the first two of these give a birational equivalence with a surface $S_7 \subset \mathbb{P}^7$ with an $A_4$ singularity at $(0 : 0 : 0 : -1 : 0 : 1 : 0 : 0)$, and projecting away from this point we obtain a model $S_6 \subset \mathbb{P}^6$ with two $A_1$ and four $A_2$ singularities. Applying Lemma 6.2 to the the singular subscheme of $S_3$ we find that $S_3$ has good reduction outside $\{2, 3, 5, 7\}$, and in fact $S_6$ has good reduction at 7.

The sublattice of $\text{Pic} S_6$ generated by the components of exceptional divisors, lines passing through a singular point, and lines introduced by blowing up the $A_4$ singularity has rank 19 and is fixed by $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To obtain a twentieth generator, we take the line $x_0 - x_1 = x_2 - \alpha x_3$ on $S_3$, where $\alpha^2 + \alpha + 1 = 0$. It is not fixed by
the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so it is independent of the previous ones, and $\text{Pic}.S_3$ has rank 20.

All of the components of the irreducible divisors of a resolution $\tilde{S}_6$ of $S_6$ are rational, so the resolution has $10p$ more $\mathbb{F}_p$-points than $S_6$ does. So the trace of the representation on $H^{2,0} + H^{0,2}$ of $\tilde{S}_6$ at Frob$_p$ is $[S_6]_p + 10p - p^2 - 19p - (-3/p) - 1$.

To apply Livnè’s method, we first verify that the trace is always even. There are 32 cubic fields unramified outside $\{2, 3, 5\}$ [18], but every one of them has an inert prime $\leq 19$, so it is enough to check that far. To conclude, we show that the trace is equal to $a_p$ for a set of primes $> 5$ whose Frobenius elements give a non-cubic subset in $\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5})$. We only need to go up to $p = 73$.

![Figure 3. Graphs for which the $c_p$ are congruent mod $p$ to the eigenvalues of forms of weight 4 and level 5, 6, 7, 17.](image)

**8.4. Weight 4, level 5.** We now turn to graphs that give forms of weight 4. The graph corresponding to the form of level 5 is $G_{10, 56}$. Lemma 55 of [3] does not apply to any of the graphs obtained by deleting one vertex from this graph, but we can still attempt the reduction. Using linear, resultant, and normal reduction has not yet led to varieties of dimension less than 4. For example, if we delete vertex 2, take the five-invariant for edges $(1, 3), (1, 4), (1, 5), (3, 8), (7, 9)$, and then reduce by edges $(3, 9), (4, 6), (9, 10), (4, 7), (7, 10)$, we obtain a fourfold whose point counts match the newform of weight 4 and level 5 for small $p$, as claimed in [4]. Fortunately, Proposition 3.11 comes to our rescue. It applies to the set of variables $\{x_0, x_2, x_3, x_5\}$ to give us a double cover $O_5$ of an octic in $\mathbb{P}^3$, defined by

$$t^2 = x_2(x_1 + x_2 + x_3)(x_0x_1 + x_0x_3 + x_1x_3)(x_0x_1^2x_2 + x_0x_1x_2^2 + 4x_0^2x_1x_3 + 4x_0x_1^2x_3 + 4x_0x_2x_3 + 6x_0x_1x_2x_3 + x_1^2x_2x_3 + x_0x_2^2x_3 + x_1x_2^2x_3 + x_0x_2x_3^2 + x_1x_2x_3^2).$$

The following conjecture has been checked for $p < 200$.

**Conjecture 8.3.** For all primes $p > 2$ the threefold $O_5$ has $p^3 + 5p^2 - (9 + (-1/p))p + 1 - a_p$ points over $\mathbb{F}_p$, where $a_p$ is the eigenvalue for $T_p$ of the newform of weight 4 and level 5.
Most likely this conjecture can be proved by constructing a crepant resolution with $c_2 + (c + 14 + (-1/p))p$ more $\mathbb{F}_p$-points than $O_5$ for all $p$, for some suitable $c$, and showing that it is a rigid Calabi-Yau threefold.

Several examples of rigid Calabi-Yau threefolds realizing this newform are described in [20, Section 6.1.1]. The Tate conjecture predicts that the conjecture holds for all primes $p$.

**Conjecture 8.4.**

4. It is possible to reach dimension 4 by only linear and resultant reduction, but subspace reduction does not apply to these varieties.

**8.5. Weight 4, level 6.** Similarly, for level 6, we start from $G_{11,241}$. We delete vertex 5, take the five-invariant for the edges

$$(1, 2), (1, 3), (2, 3), (1, 4), (4, 8)$$

(to which [26 Lemma 55] applies), then reduce edges in the order

$$(4, 6), (2, 6), (7, 10), (7, 11), (8, 10), (9, 10)$$

8.6. **Weight 4, level 7.** The graph is $G_{12,1330}$. If we delete vertex 5, take the 5-invariant for the edges

$$(1, 2), (1, 3), (2, 3), (1, 4), (4, 7),$$

and then reduce the edges in the order

$$(4, 6), (2, 6), (8, 12), (11, 12), (6, 10), (10, 11), (8, 11), (9, 10), (7, 9),$$

**Remark 8.5.** It is possible to reach dimension 4 by only linear and resultant reduction, but subspace reduction does not apply to these varieties.
we obtain a subvariety of \( \mathbb{P}^5 \) to which the normal reduction can be applied. Converting the resulting double cover of \( \mathbb{P}^3 \) into a quintic by means of Proposition 3.13, we then consider the linear system of quadrics vanishing on the singular points contained in components of the singular subscheme of degree greater than 8. It gives us a new quintic. Again we take quadrics vanishing on singular points in components of the singular subscheme of degree 8; this gives us a map to \( \mathbb{P}^6 \). The singular subscheme of the image has one component \( C_1 \) of degree 7 and three \( D_1, D_2, D_3 \) of degree 19; if we project away from the point in \( C_1 \) and the point in the correct one of the \( D_i \), we obtain a quintic that can be converted back into a double cover \( O_7 \) of \( \mathbb{P}^3 \) with the equation

\[
t^2 = x_0(x_0x_1x_2 + x_1^2x_2 + x_1^2x_3 - x_0x_2x_3 - x_1x_2x_3 - x_2^2x_3 - x_1x_3^2 - x_2x_3^2) \times
\]

\[
(x_0^2x_1x_2 + x_0x_1^2x_2 + x_0x_1^2x_3 - x_2^2x_2x_3 - x_0x_1x_2x_3 + 4x_1^2x_2x_3 - x_0x_1x_2^2x_3 - x_0x_2x_3^2 - 4x_1x_2x_3^2)\]

We find the formula

\[
p^3 + 5p^2 - (11 + 3(-3/p) + (5/p))p + 1 - a_p
\]

for the number of \( \mathbb{F}_p \)-points of \( O_7 \) for \( 5 \leq p \leq 200 \), where \( a_p \) is the Hecke eigenvalue for the newform of weight 4 and level 7. As in the last two examples we conjecture this to be true for all larger \( p \).

It seems that the only Calabi-Yau threefold known to be associated to this modular form is that provided by the Kuga-Sato construction, namely a resolution of the fibre square of the universal elliptic curve over \( X_0(7) \) [10] Section 1).

8.7. Weight 4, level 17. The graph here is \( G_{12,1321} \). If we delete vertex 5, take the 5-invariant with respect to the edges

\[
(1, 2), (1, 3), (2, 3), (1, 4), (4, 7),
\]

and perform linear and resultant reductions for the edges

\[
(4, 6), (2, 6), (8, 11), (9, 10), (9, 12), (8, 10), (3, 8), (10, 12), (7, 12),
\]

we obtain a polynomial of degree 6 in 6 variables to which the normal reduction can be applied. Thus we obtain a double cover of \( \mathbb{P}^3 \) branched along surfaces of degree 3, 5, and we can convert this to a quintic as in Proposition 3.13. We simplify this quintic by considering polynomials of degree 2 vanishing on the points of the singular subscheme contained in components of degree \( \geq 8 \). This gives a subscheme of \( \mathbb{P}^6 \) of degree 11, and projecting away from a singular point of degree 13 and one of the two of degree 23 produces a quintic that can be mapped back to a double cover of \( \mathbb{P}^3 \) with branch components of degree 1, 3, 4. The associated double octic \( O \) is defined by

\[
t^2 - x_0(x_0^2x_3 + x_0x_1x_3 + x_1^2x_3 - x_1x_2x_3 - x_2^2x_3 - x_1x_3^2 - x_2x_3^2) \times
\]

\[
(x_0^2x_1x_3 + x_0^2x_2x_3 + x_0x_1^2x_3 + 3x_0x_1x_2x_3 + 4x_1^2x_2x_3 - x_0x_2x_3^2 - 4x_1x_2x_3^2 - 4x_1x_2^2x_3 - x_0x_2x_3^2 - 4x_1x_2x_3^2) = 0.
\]

As found in [4], the point counts mod \( p \) match the eigenvalues of the newform of weight 4 and level 17. More precisely, for \( 5 \leq p \leq 200 \), the number of \( \mathbb{F}_p \)-points of \( O \) is equal to \( p^3 + 5p^2 - (11 + 3(-3/p))p + 1 - a_p \), where \( a_p \) is the eigenvalue of \( T_p \) on the newform 17/1. Again, the singularities are quite complicated and the Cynk-Szemberg resolution needed to prove this true for all \( p \) could only be constructed.
after several preliminary blowups. As in the cases of levels 5 and 6, examples of products of elliptic surfaces realizing the same newform have been found ([20 Sections 2.1–2.2]) and there is expected to be a correspondence between these and the threefold constructed here.

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