FROM CM-FINITE TO CM-FREE

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Abstract. The aim of this paper is twofold. On one hand, we prove a slight generalization of the stability for Gorenstein categories in [SWSW] and [Huang]; and show that the relative Auslander algebra of a CM-finite algebra is CM-free. On the other hand, we describe the bounded derived category, and the Gorenstein defect category introduced in [BJO], via Gorenstein-projective objects; and we show that the Gorenstein defect category of a CM-finite algebra is triangle-equivalent to the singularity category of its relative Auslander algebra.

1. Introduction

1.1. M. Auslander and M. Bridger [AB] introduced the modules of G-dimension zero over two-sided noetherian rings. E. E. Enochs and O. M. G. Jenda [EJ1] generalized this concept and introduced Gorenstein-projective modules over any ring. This class of modules enjoys pleasant stable properties, becomes a main ingredient in the relative homological algebra, and widely used in the representation theory of algebras and algebraic geometry. See e.g. [AM], [AR], [BGS], [Buch], [EJ2], [Hap2], [Hol], [Kn]. Recent studies show that they are important not only to Gorenstein algebras, but also to non-Gorenstein ones (see e.g. [B2], [BK], [BR], [C], [CPST], [Rin], [T], [Y]).

1.2. Throughout A is an abelian category with enough projective objects unless stated otherwise, P(A) is the full subcategory of projective objects, and GP(A) is the full subcategory of Gorenstein-projective objects. Replacing P(A) by a full additive subcategory C, one similarly define a category GC(C), and G2(C) := G(G(C)). S. Sather-Wagstaff, T. Sharif and D. White proved that if C is self-orthogonal, then G2(C) = G(C); and they proposed the question: Whether G2(C) = G(C) holds for an arbitrary C? See [SWSW, 4.10, 5.8]. Recently Z. Y. Huang ([Huang, 4.1]) answered this question affirmatively. This shows that G(C), in particular GP(A), has a strong stability. We give a slight generalization of this stability by a different method. For details please see 2.1 and Theorem 3.1.

1.3. Throughout Λ-mod is the category of finitely generated left modules of Artin algebra Λ, P(Λ) the full subcategory of projective Λ-modules, and GP(Λ) the full subcategory of Gorenstein-projective Λ-modules. Clearly P(Λ) ⊆ GP(Λ). If GP(Λ) = P(Λ), then Λ

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is called CM-free. Algebras of finite global dimension are CM-free ([EJ2, 10.2.3]), and there are also many CM-free algebras of infinite global dimension ([C]). If $GP(\Lambda)$ has only finitely many isoclasses of indecomposable objects, then $\Lambda$ is called CM-finite. In this case, let $E_1, \cdots, E_n$ be all the pairwise non-isomorphic indecomposable Gorenstein-projective modules, and $\textbf{Aus}(\Lambda)$ the relative Auslander algebra $\text{End}_{\Lambda}(\bigoplus_{1 \leq i \leq n} E_i)^{op}$. CM-finiteness and CM-freeness have been recently studied for examples in [B2], [B3], [C], [CY], [GZ], [LZ], [Rin]. In fact, CM finiteness characterizes the simple hypersurface singularity ([BGS], [CPST], [Kn]).

A CM-finite algebra $\Lambda$ is Gorenstein if and only if $\text{gl.dim} \textbf{Aus}(\Lambda) < \infty$ ([LZ]; [B3]). Thus in this case $\textbf{Aus}(\Lambda)$ is CM-free. A CM-finite Gorenstein algebra $\Lambda$ which is not CM-free has a non-zero singularity category, but the singularity category of $\textbf{Aus}(\Lambda)$ becomes zero. So, this is a kind of categorical resolution of singularities ([Kuz]). However, by the recent work of C. M. Ringel [Rin] there are many CM-finite algebras which are non-Gorenstein and not CM-free (see 2.3). So it is natural to study $\textbf{Aus}(\Lambda)$ also for such algebras. We prove that for an arbitrary CM-finite algebra $\Lambda$, $\textbf{Aus}(\Lambda)$ is always CM-free (Theorem 4.5). It is a Gorenstein version of Auslander’s theorem ([ARS, p.215]). For those CM-finite algebras which are non-Gorenstein and not CM-free, Theorem 4.5 means a categorical resolution of Gorenstein singularities.

Put $\Omega$ to be the class of CM-finite algebras, and $\Theta$ the class of CM-free algebras. Theorem 4.5 implies that there is a surjective map $\textbf{Aus} : \Omega \rightarrow \Theta$; moreover, it sends CM-finite Gorenstein algebras to algebras of finite global dimension, and sends CM-finite non-Gorenstein algebras to CM-free non-Gorenstein algebras. Graphically, we have

1.4. An important feature is that $GP(\mathcal{A})$ is a Frobenius category ([Ke]) with relative projective-injective objects being projective objects ([B2]), and hence the stable category $GP(\mathcal{A})$ of $GP(\mathcal{A})$ modulo $\mathcal{P}(\mathcal{A})$ is triangulated (see D. Happel [Hap1, p.16]).

The singularity category $D_{sg}^b(\mathcal{A})$ of $\mathcal{A}$ is defined as the Verdier quotient $D^b(\mathcal{A})/K^b(\mathcal{P}(\mathcal{A}))$ (R.-O. Buchweitz [Buch], D. Orlov [O]), where $D^b(\mathcal{A})$ is the bounded derived category of
\( \mathcal{A} \), and \( K^b(\mathcal{P}(\mathcal{A})) \) is the thick subcategory consisting of bounded complexes of projective objects. It measures how far \( \mathcal{A} \) is from smoothness. Buchweitz’s Theorem ([Buch, 4.4.1]; also [Hap2, 4.6]) says that there is a triangle-embedding \( F : \mathcal{GP}(\mathcal{A}) \rightarrow D^b_{sg}(\mathcal{A}) \), and if each object of \( \mathcal{A} \) is of finite Gorenstein-projective dimension, then \( F \) is an equivalence (it is stated for \( R\text{-mod} \), but it holds also for \( \mathcal{A} \)). The converse is also true. See A. Beligiannis [B1, 6.9(8)], [BJO], and S. J. Zhu [Zhu].

Following P. A. Bergh, D. A. Jorgensen and S. Oppermann [BJO], the Gorenstein defect category of \( \mathcal{A} \) is defined as the Verdier quotient \( D^b_{\text{defect}}(\mathcal{A}) := D^b_{sg}(\mathcal{A})/\text{Im} F \). It measures how far \( \mathcal{A} \) is from Gorensteinness since \( D^b_{\text{defect}}(\mathcal{A}) = 0 \) if and only if each object of \( \mathcal{A} \) is of finite Gorenstein-projective dimension.

Another aim of this paper is to describe \( D^b_{\text{defect}}(\mathcal{A}) \).

1.5. Note that \( D^b(\mathcal{A}) \) can be interpreted as \( K^{-b}(\mathcal{P}(\mathcal{A})) \). This makes \( D^b(\mathcal{A}) \) more accessible. If \( \mathcal{A} \) is CM-contravariantly finite, we can describe \( D^b(\mathcal{A}) \) also via Gorenstein-projective objects, i.e., there is a triangle-equivalence

\[
D^b(\mathcal{A}) \cong K^{-, gp}(\mathcal{GP}(\mathcal{A}))/K^{b, ac}(\mathcal{GP}(\mathcal{A})),
\]

where \( K^{-, gp}(\mathcal{GP}(\mathcal{A})) \) is introduced in [GZ]. For details please see 2.2 and Theorem 5.1.

1.6. By introducing the category \( K_{-}^{-, b}(\mathcal{P}(\mathcal{A})) \), we describe \( D^b_{\text{defect}}(\mathcal{A}) \) as

\[
D^b_{\text{defect}}(\mathcal{A}) \cong K^{-, b}(\mathcal{P}(\mathcal{A}))/K_{-}^{-, b}(\mathcal{P}(\mathcal{A}));
\]

and if \( \mathcal{A} \) is CM-contravariantly finite, then we also have

\[
D^b_{\text{defect}}(\mathcal{A}) \cong K^{-, gp}(\mathcal{GP}(\mathcal{A}))/K^{b}(\mathcal{GP}(\mathcal{A})).
\]

For details please see Theorem 6.8. This implies that in this condition \( D^b_{\text{defect}}(\mathcal{A}) \) is completely controlled by \( \mathcal{GP}(\mathcal{A}) \). As an application, if \( \Lambda \) is CM-finite, then we get a triangle-equivalence (Corollary 6.10)

\[
D^b_{\text{defect}}(\Lambda) \cong D^b_{sg}(\mathcal{Aus}(\Lambda)).
\]

Thus the Gorenstein defect category of a CM-finite algebra can be reduced to the singularity category of a CM-free algebra.

1.7. Suppose that \( \mathcal{A} \) is CM-contravariantly finite with \( \mathcal{GP}(\mathcal{A}) \cong \mathcal{GP}(\mathcal{A}') \). One can not expect that \( \mathcal{A}' \) is also CM-contravariantly finite. However, \( D^b(\mathcal{A}') \) and \( D^b_{\text{defect}}(\mathcal{A}') \) share the same descriptions via \( \mathcal{GP}(\mathcal{A}') \). So \( D^b_{\text{defect}}(\mathcal{A}) \cong D^b_{\text{defect}}(\mathcal{A}') \). Please see Theorem 7.1.

1.8. The paper is organized as follows. In Section 2 we give necessary preliminaries and notations. In Section 3 we prove a version of the stability of \( \mathcal{G}(\mathcal{C}) \). Sections 4 is devoted to Theorem 4.5. Sections 5 and 6 are to describe \( D^b(\mathcal{A}) \) and \( D^b_{\text{defect}}(\mathcal{A}) \) via Gorenstein-projective objects. Section 7 is to prove Theorem 7.1.
2. Preliminaries and notations

2.1. Category $\mathcal{G}(\mathcal{C})$ and Gorenstein-projective objects. Let $\mathcal{C}$ be a full additive subcategory of an abelian category $\mathcal{A}$ (not necessarily has enough projective objects) which is closed under isomorphisms. A complex $X^\bullet$ over $\mathcal{A}$ is $\mathcal{C}$-exact (resp. $\mathcal{C}$-coexact) if $\text{Hom}_{\mathcal{A}}(C, X^\bullet)$ is exact (resp. $\text{Hom}_{\mathcal{A}}(X^\bullet, C)$ is exact) for each $C \in \mathcal{C}$. A $\mathcal{C}$-exact and $\mathcal{C}$-coexact complex is said to be $\mathcal{C}$-biexact.

Let $L$ be an object of $\mathcal{A}$. An exact complex $X^\bullet = (X^i, d^i)$ over $\mathcal{A}$ is a complete $\mathcal{C}$-resolution of $L$, if each $X^i \in \mathcal{C}$ and $X^\bullet$ is $\mathcal{C}$-biexact, such that $L \cong \text{Im} d^0$. Define $\mathcal{G}(\mathcal{C})$ to be the full subcategory of $\mathcal{A}$ consisting of the objects which admit complete $\mathcal{C}$-resolutions. Define $\mathcal{G}^2(\mathcal{C}) := \mathcal{G}(\mathcal{G}(\mathcal{C}))$ ([SWSW, 4.1]). Clearly $\mathcal{C} \subseteq \mathcal{G}(\mathcal{C}) \subseteq \mathcal{G}^2(\mathcal{C})$.

Let $\mathcal{A}$ be an abelian category with enough projective objects. Taking $\mathcal{C}$ to be $\mathcal{P}(\mathcal{A})$, we get the notion of a complete projective resolution. Let $\mathcal{G}\mathcal{P}(\mathcal{A})$ denote the category $\mathcal{G}(\mathcal{P}(\mathcal{A}))$, whose objects are called Gorenstein-projective objects ([AB], [EJ1]).

2.2. CM-contravariantly finite abelian categories. Let $\mathcal{A}$ be an abelian category (not necessarily has enough projective objects), $\mathcal{C}$ a full subcategory of $\mathcal{A}$, and $X \in \mathcal{A}$. A morphism $f : C \to X$ with $C \in \mathcal{C}$ is a right $\mathcal{C}$-approximation of $X$, if $\text{Hom}_{\mathcal{A}}(C', f) : \text{Hom}_{\mathcal{A}}(C', C) \to \text{Hom}_{\mathcal{A}}(C', X)$ is surjective for each $C' \in \mathcal{C}$. If each object $X \in \mathcal{A}$ admits a right $\mathcal{C}$-approximation, then $\mathcal{C}$ is said to be contravariantly finite in $\mathcal{A}$ ([AR]).

Let $\mathcal{A}$ be an abelian category with enough projective objects. For short, we say that $\mathcal{A}$ is CM-contravariantly finite, if $\mathcal{G}\mathcal{P}(\mathcal{A})$ is contravariantly finite in $\mathcal{A}$. If each object of $\mathcal{A}$ has a finite Gorenstein-projective dimension, then $\mathcal{A}$ is CM-contravariantly finite ([EJ2, 11.5.1], or H.Holm [Hol, 2.10]. We stress that the proof in [Hol, 2.10] is stated for module category over ring, but holds also for an abelian category with enough projective objects).

An Artin algebra $\Lambda$ is CM-contravariantly finite, if $\Lambda$-mod is CM-contravariantly finite. Recall that $\Lambda$ is a Gorenstein algebra, if the injective dimension of $\Lambda$ is finite and the injective dimension of $\Lambda$ is finite (in this case the both are same, see [I]). Note that $\Lambda$ is a Gorenstein algebra if and only if each $\Lambda$-module has finite Gorenstein-projective dimension ([Hos]). Thus a Gorenstein algebra is CM-contravariantly finite. Also, clearly a CM-finite algebra is CM-contravariantly finite. By [B2, Theorem 8.2(ix)] a virtually Gorenstein algebra is CM-contravariantly finite. On the other hand, there exists an Artin algebra which is not CM-contravariantly finite (see [BK], [Y]. Also [T]).

2.3. CM-finite algebras which are non-Gorenstein and not CM-free. There are many CM-finite algebras which are non-Gorenstein and not CM-free. For examples, by [Rin, Proposition 5] the Nakayama algebras with admissible sequences $(6, 6, 5), (8, 8, 8, 7), (10, 10, 9, 10, 9)$, are such examples.

2.4. Triangulated categories. Among the other conditions, we emphasize that a triangulated subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C}$ is a full subcategory and is closed.
under isomorphisms of $\mathcal{C}$. See A. Neeman [N]. For a triangle functor $F : \mathcal{A} \to \mathcal{B}$ between triangulated categories, let $\text{Im}F$ denote the full subcategory of $\mathcal{B}$ consisting of the objects which are isomorphic to $F(X)$ with $X \in \mathcal{A}$. If $F$ is full, then $\text{Im}F$ is a triangulated subcategory of $\mathcal{B}$.

**Lemma 2.1.** (J. L. Verdier [V], Corollary 4-3) Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are triangulated subcategories of triangulated category $\mathcal{C}$, and $\mathcal{D}_1$ is a subcategory of $\mathcal{D}_2$. Then $\mathcal{D}_2/\mathcal{D}_1$ is a triangulated subcategory of $\mathcal{C}/\mathcal{D}_1$, and there is a triangle-equivalence $(\mathcal{C}/\mathcal{D}_1)/(\mathcal{D}_2/\mathcal{D}_1) \cong \mathcal{C}/\mathcal{D}_2$.

**Lemma 2.2.** (J. Rickard [Ric], p. 446, line 1) A full triangle functor which sends non-zero objects to non-zero objects is faithful.

### 2.5. Notations and convention.

For convenience, we list some categories mainly used in Section 5 - Section 7.

- $K^b(\mathcal{A}) = \text{the homotopy category of bounded complexes over } \mathcal{A}$.
- $K^{b,ac}(\mathcal{A}) = \text{the homotopy category of bounded exact complexes over } \mathcal{A}$.
- $K^{-}(\mathcal{A}) = \text{the homotopy category of upper bounded complexes over } \mathcal{A}$.
- $K^{-,ac}(\mathcal{A}) = \text{the homotopy category of upper bounded exact complexes over } \mathcal{A}$.
- $K^b(\mathcal{P}(\mathcal{A})) = \text{the homotopy category of bounded complexes over } \mathcal{P}(\mathcal{A})$.
- $K^{-}(\mathcal{P}(\mathcal{A})) = \text{the homotopy category of upper bounded complexes over } \mathcal{P}(\mathcal{A})$.
- $K^{-,b}(\mathcal{P}(\mathcal{A})) = \text{the homotopy category of upper bounded complexes over } \mathcal{P}(\mathcal{A})$, with only finitely many non-zero cohomologies.
- $K^b(\mathcal{GP}(\mathcal{A})) = \text{the homotopy category of bounded complexes over } \mathcal{GP}(\mathcal{A})$.
- $K^{b,ac}(\mathcal{GP}(\mathcal{A})) = \text{the homotopy category of bounded exact complexes over } \mathcal{GP}(\mathcal{A})$. Here “exact” means exact as a complex over $\mathcal{A}$.
- $K^{-}(\mathcal{GP}(\mathcal{A})) = \text{the homotopy category of upper bounded complexes over } \mathcal{GP}(\mathcal{A})$.
- $\mathcal{D}^b(\mathcal{A}) = \text{the derived category of bounded complexes over } \mathcal{A}$, i.e., the Verdier quotient $K^b(\mathcal{A})/K^{b,ac}(\mathcal{A})$.
- $\langle \mathcal{GP}(\mathcal{A}) \rangle = \text{the triangulated subcategory of } \mathcal{D}^b(\mathcal{A}) \text{ generated by } \mathcal{GP}(\mathcal{A})$, i.e., the smallest triangulated subcategory of $\mathcal{D}^b(\mathcal{A})$ containing $\mathcal{GP}(\mathcal{A})$.
- $\mathcal{D}^-(\mathcal{A}) = \text{the derived category of upper bounded complexes over } \mathcal{A}$, i.e., the Verdier quotient $K^{-}(\mathcal{A})/K^{-,ac}(\mathcal{A})$.
- $\mathcal{GP}(\mathcal{A}) = \text{the stable category of } \mathcal{GP}(\mathcal{A}) \text{ modulo } \mathcal{P}(\mathcal{A})$.

**Convention.** Let $\mathcal{C}$ be a triangulated category and $\mathcal{D}$ a full additive subcategory. We say that $\mathcal{D}$ is a triangulated subcategory of $\mathcal{C}$, if the isomorphism closure $\bar{\mathcal{D}}$ of $\mathcal{D}$ is a triangulated subcategory of $\mathcal{C}$. In this case, we do not distinguish between $\mathcal{D}$ and $\bar{\mathcal{D}}$ (if no substantial difficulties occur): for example, the Verdier quotient $\mathcal{C}/\mathcal{D}$ always means $\mathcal{C}/\bar{\mathcal{D}}$. This convention occurs in many places in this paper.
2.6. **Singularity categories.** The Verdier quotient \( D^b(A) / K^b(\mathcal{P}(A)) \) is called the singularity category of \( A \) and denoted by \( D^b_{sg}(A) \) (see \([O]\); or the stabilized derived category in \([Buch]\)). Then \( D^b_{sg}(A) = 0 \) if and only if each object of \( A \) has finite projective dimension. By the canonical triangle-equivalence \( \rho : K^{-,b}(\mathcal{P}(A)) \longrightarrow D^b(A) \), there is a triangle-equivalence \( D^b_{sg}(A) \cong K^{-,b}(\mathcal{P}(A))/K^b(\mathcal{P}(A)). \)

2.7. **Buchweitz Theorem and Gorenstein defect categories.** Consider the composition of the embedding \( \mathcal{G\mathcal{P}(A)} \hookrightarrow D^b(\mathcal{A}) \) and the localization functor \( D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A})) \). It induces a functor 
\[
F : \mathcal{G\mathcal{P}(A)} \longrightarrow D^b_{sg}(\mathcal{A}) := D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A}))
\] (2.1)
which sends a Gorenstein-projective object \( G \) to the stalk complex of \( G \) at degree 0. By the triangle-equivalence \( \rho : K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A})) \cong D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A})) \) one gets a functor 
\[
\overline{F} : \mathcal{G\mathcal{P}(A)} \longrightarrow K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A}))
\] (2.2)
such that the diagram
\[
\begin{array}{ccc}
\mathcal{G\mathcal{P}(A)} & \xrightarrow{F} & K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A})) \\
\downarrow & & \downarrow \pi \\
\mathcal{G\mathcal{P}(A)} & \xrightarrow{\overline{F}} & D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A}))
\end{array}
\] (2.3)
commutes. For a Gorenstein-projective object \( G \), let \((P^\bullet, d)\) be a complete projective resolution such that \( \text{Im} d^0 \cong G \). Then 
\[
\overline{F}(G) = P^\bullet_{\leq 0} \in K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A})),
\]
where \( P^\bullet_{\leq 0} \) is the brutal truncation \( \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0 \) of \( P^\bullet \).

Following \([BJO]\), the Gorenstein defect category of \( \mathcal{A} \) is defined as the Verdier quotient 
\[
D^b_{\text{defect}}(\mathcal{A}) := D^b_{sg}(\mathcal{A}) / \text{Im} F = (D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A}))) / \text{Im} F.
\]
For an Artin algebra \( \Lambda \), the Gorenstein defect category \( D^b_{\text{defect}}(\Lambda) \) of \( \Lambda \) is defined to be \( D^b_{\text{defect}}(\Lambda\text{-mod}) \). Then \( D^b_{\text{defect}}(\Lambda) = 0 \) if and only if \( \Lambda \) is Gorenstein. By (2.3) the diagram
\[
\begin{array}{ccc}
\text{Im} \overline{F} & \longrightarrow & K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A})) \\
\downarrow & & \downarrow \pi \\
\text{Im} F & \longrightarrow & D^b(\mathcal{A}) / K^b(\mathcal{P}(\mathcal{A}))
\end{array}
\] (2.4)
commutes, where the horizontal functors are embeddings, and the vertical functors are triangle-equivalences. It follows that there is a triangle-equivalence 
\[
D^b_{\text{defect}}(\mathcal{A}) \cong (K^{-,b}(\mathcal{P}(\mathcal{A}))/K^b(\mathcal{P}(\mathcal{A}))) / \text{Im} \overline{F}.
\] (2.5)
3. Stability of Gorenstein categories

Throughout $\mathcal{A}$ is an abelian category (not necessarily has enough projective objects), and $\mathcal{C}$ is a full additive subcategory of $\mathcal{A}$, which is closed under isomorphisms. By [SWSW, 4.10] and [Huang, 4.1] $\mathcal{G}^2(\mathcal{C}) = \mathcal{G}(\mathcal{C})$. This shows that the category $\mathcal{G}(\mathcal{C})$ has a strong stability. By using a different method, we have a little generalization of this stability.

**Theorem 3.1.** Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ a full additive subcategory of $\mathcal{A}$ which is closed under isomorphisms. Let $X^\bullet = (X^i, d^i)$ be an exact complex with $X^i \in \mathcal{G}(\mathcal{C})$ for all $i \in \mathbb{Z}$. If $X^\bullet$ is $\mathcal{C}$-biexact, then $\text{Im} d^i \in \mathcal{G}(\mathcal{C})$ for all $i \in \mathbb{Z}$.

If $X^\bullet$ in Theorem 3.1 is required to be $\mathcal{G}(\mathcal{C})$-biexact, then it is exactly $\mathcal{G}^2(\mathcal{C}) = \mathcal{G}(\mathcal{C})$.

3.1. We need the following fact.

**Lemma 3.2.** Suppose there is a commutative diagram in $\mathcal{A}$ with exact rows $\delta$ and $\eta$:

$$
\begin{array}{ccc}
\delta: & 0 & \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \\
& \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\beta} \\
\eta: & 0 & \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \rightarrow 0.
\end{array}
$$

Denote by $\Delta$ the corresponding short exact sequence

$$
0 \rightarrow X \xrightarrow{(\alpha f)} Y \oplus X' \xrightarrow{(\beta, f')} Y' \rightarrow 0.
$$

Then we have

(i) $\delta$ is $\mathcal{C}$-exact if and only if both $\eta$ and $\Delta$ are $\mathcal{C}$-exact.

(ii) $\delta$ is $\mathcal{C}$-coexact if and only if both $\eta$ and $\Delta$ are $\mathcal{C}$-coexact.

**Proof.** (i) We need to prove that $\text{Coker}(\text{Hom}_\mathcal{A}(C, g)) = 0$ if and only if $\text{Coker}(\text{Hom}_\mathcal{A}(C, g')) = 0$ and $\text{Coker}(\text{Hom}_\mathcal{A}(C, (\beta, f'))) = 0$, for each $C \in \mathcal{C}$. This can be seen from the diagram chasing. However, for simplicity we use an argument from triangulated category.

Regard $\delta$, $\eta$ and $\Delta$ as complexes in $K^b(\mathcal{A})$, and the above commutative diagram as a morphism $h$ from $\delta$ to $\eta$. Then the mapping cone $\text{Cone}(h)$ is the complex

$$
0 \rightarrow X \xrightarrow{(\alpha f)} Y \oplus X' \xrightarrow{(\beta, f')} Z \oplus Y' \xrightarrow{(1, g')} Z \rightarrow 0.
$$

Then there is a homotopy equivalence $\Delta[1] \cong \text{Cone}(h)$, see the commutative diagram.
Thus we have a distinguished triangle \( \delta \to \eta \to \Delta[1] \to \delta[1] \) in \( K^b(A) \). Applying the cohomological functor \( \text{Hom}_{K^b(A)}(C, -) \) to this distinguished triangle we get the following long exact sequence of abelian groups

\[
\cdots \to \text{Hom}_{K^b(A)}(C, \eta[1]) \to \text{Hom}_{K^b(A)}(C, \Delta[2]) \to \text{Hom}_{K^b(A)}(C, \delta[2]) \to \text{Hom}_{K^b(A)}(C, \eta[2]) \to \text{Hom}_{K^b(A)}(C, \Delta[3]) \to \cdots.
\]

Using the formula \( \text{Hom}_{K^b(A)}(C, \eta[n]) = H^n \text{Hom}_A(C, \eta) \), the above exact sequence read is

\[
\cdots \to H^1 \text{Hom}_A(C, \eta) \to H^2 \text{Hom}_A(C, \Delta) \to H^2 \text{Hom}_A(C, \delta) \to H^2 \text{Hom}_A(C, \eta) \to H^3 \text{Hom}_A(C, \Delta) \to \cdots.
\]

That is we have the exact sequence

\[
0 \to \text{Coker}(\text{Hom}_A(C, (\beta, f'))) \to \text{Coker}(\text{Hom}_A(C, g)) \to \text{Coker}(\text{Hom}_A(C, g')) \to 0.
\]

This prove (i).

(ii) can be similarly proved. ■

3.2. We also need the following technical lemma.

**Lemma 3.3.** Let \( \delta : 0 \to X_1 \xrightarrow{f} U \xrightarrow{g} X_2 \to 0 \) be an exact sequence with \( U \in \mathcal{G}(C) \) such that \( \delta \) is \( C \)-biexact. Let \( \eta : 0 \to X_1 \xrightarrow{u} Y \xrightarrow{v} V \to 0 \) be an exact sequence with \( V \in \mathcal{G}(C) \) such that \( \eta \) is \( C \)-coexact. Then we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & X_1 & \xrightarrow{f} & U & \xrightarrow{g} & X_2 & \rightarrow & 0 \\
& & \downarrow{u} & & \downarrow{u'} & & \downarrow{u'} & & \\
0 & \rightarrow & Y & \rightarrow & C & \rightarrow & Z & \rightarrow & 0
\end{array}
\]

such that

(i) \( \delta' \) is exact with \( C \in \mathcal{C} \), and \( \delta' \) is \( C \)-biexact;

(ii) \( u' \) is a monomorphism with \( \text{Coker} u' \in \mathcal{G}(C) \), and \( \text{Hom}_A(u', C) \) is surjective.
Proof. Step 1. Since by assumption $U, V \in \mathcal{G}(\mathcal{C})$, by definition there exists two exact sequences $\epsilon_1 : 0 \to U \xrightarrow{a} C_1 \xrightarrow{b} L_1 \to 0$ and $\epsilon_2 : 0 \to V \xrightarrow{c} C_2 \xrightarrow{d} L_2 \to 0$ with $C_1, C_2 \in \mathcal{C}, L_1, L_2 \in \mathcal{G}(\mathcal{C})$, such that both $\epsilon_1$ and $\epsilon_2$ are $\mathcal{C}$-biexact. Since by assumption $\text{Hom}_A(\eta, C_1)$ is exact, by considering $af \in \text{Hom}_A(X_1, C_1)$ we see that there exists morphisms $e$ and $e'$ such that the following diagram commutes:

$$
\begin{array}{c}
\eta : & 0 & \to & X_1 & \xrightarrow{u} & Y & \xrightarrow{v} & V & \to & 0 \\
& & \downarrow f & & \downarrow e & & \downarrow e' & & \\
& 0 & \to & U & \xrightarrow{a} & C_1 & \xrightarrow{b} & L_1 & \to & 0.
\end{array}
$$

Put $\alpha = (\begin{smallmatrix} c \\ e' \end{smallmatrix}) : Y \to C_2 \oplus C_1$, $i = (\begin{smallmatrix} 0 \\ a \end{smallmatrix}) : U \to C_2 \oplus C_1$, $\pi = (\begin{smallmatrix} -1 & 0 \\ 0 & b \end{smallmatrix}) : C_2 \oplus C_1 \to C_2 \oplus L_1$, $x = (\begin{smallmatrix} -c \\ e' \end{smallmatrix}) : V \to C_2 \oplus L_1$. Then there exist morphisms $\beta, u', v', y$ such that the following diagram commutes:

$$
\begin{array}{c}
\eta & \Xi & \eta' \\
\delta : & 0 & \to & X_1 & \xrightarrow{f} & U & \xrightarrow{g} & X_2 & \to & 0 \\
& \downarrow u & & \downarrow f & & \downarrow g & & \downarrow u' & & \downarrow v' & & \downarrow \Xi \\
\delta' : & 0 & \to & Y & \xrightarrow{\alpha} & C_2 \oplus C_1 & \xrightarrow{\beta} & Z & \to & 0 \\
& \downarrow v & & \downarrow \alpha & & \downarrow \beta & & \downarrow \pi & & \downarrow \Xi & & \downarrow \Xi \\
\Delta : & 0 & \to & V & \xrightarrow{x} & C_2 \oplus L_1 & \xrightarrow{y} & L & \to & 0 \\
& \downarrow & & \downarrow x & & \downarrow y & & \downarrow & & \downarrow & & \downarrow \\
& 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0.
\end{array}
$$

where $Z = \text{Coker} \alpha$ and $L = \text{Coker} x$. Since $c$ is a monomorphism, so is $x$. Also, the middle column $\Xi$ is exact since $\epsilon_1$ is exact. Applying Snake Lemma to the left two columns we know that $\alpha$ is a monomorphism and that the right column $\eta'$ is exact. In particular, $u'$ is a monomorphism.

We will prove that the upper two rows of $(\ast)$ are what we need.

Step 2. Write $y = (l, m)$. Observe that $\Delta$ is exact means that there is the following commutative diagram with exact rows (note that $L$ is the push-out of $e'$ and $c$):

$$
\begin{array}{c}
\epsilon_2 : & 0 & \to & V & \xrightarrow{c} & C_2 & \xrightarrow{d} & L_2 & \to & 0 \\
& \downarrow e' & & \downarrow c & & \downarrow d & & \downarrow l & & \downarrow \\
\zeta : & 0 & \to & L_1 & \xrightarrow{m} & L & \xrightarrow{n} & L_2 & \to & 0
\end{array}
$$
Since $\epsilon_2$ is $C$-biexact, it follows from Lemma 3.2 that both $\zeta$ and $\Delta$ are $C$-biexact. Since $L_1, L_2 \in G(C)$, and $\zeta$ is $C$-biexact, it follows from Proposition 4.4 of [SWSW] that $L \in G(C)$, thus $\text{Coker } u' = L \in G(C)$.

Step 3. $\forall \ C \in C$, applying $\text{Hom}_A(C, -)$ to the right two columns of $(\ast)$, we get the following commutative diagram with exact rows

$$
\begin{array}{ccc}
\text{Hom}_A(C, U) & \rightarrow & \text{Hom}_A(C, C_2 \oplus C_1) \\
\downarrow \text{Hom}_A(C, g) & & \downarrow \text{Hom}_A(C, \beta) \\
\text{Hom}_A(C, X_2) & \rightarrow & \text{Hom}_A(C, Z) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_A(C, L).
\end{array}
$$

(Note that $\text{Hom}_A(C, \pi)$ is surjective since $\text{Hom}_A(C, \epsilon_1)$ is exact.) Also $\text{Hom}_A(C, y)$ and $\text{Hom}_A(C, g)$ are surjective, by Snake Lemma $\text{Hom}_A(C, \beta)$ is surjective. Thus $\delta'$ is $C$-exact.

Step 4. $\forall \ C \in C$, applying $\text{Hom}_A(-, C)$ to $(\ast)$, we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
(\eta', C) & (\Xi, C) & (\eta, C) \\
\downarrow & & \downarrow \\
0 & 0 & 0 \\
\downarrow & & \downarrow \\
(\Delta, C) : & (L, C) & (V, C) \\
\downarrow & & \downarrow \\
0 & (g, C) & (V, C) \\
(\delta', C) : & (Z, C) & (Y, C) \\
\downarrow & & \downarrow \\
0 & (\beta, C) & (Y, C) \\
(\delta, C) : & (X_2, C) & (X_1, C) \\
\downarrow & & \downarrow \\
0 & (u', C) & (X_1, C) \\
\end{array}
$$

(Note that the exactness of $\text{Hom}_A(\delta, C)$ and $\text{Hom}_A(\eta, C)$ follows from assumptions; the exactness of $\text{Hom}_A(\Xi, C)$ follows from the exactness of $\text{Hom}_A(\epsilon_1, C)$; and the exactness of $\text{Hom}_A(\Delta, C)$ follows from Step 2). Applying Snake Lemma to the right two columns we see that both $\text{Hom}_A(u', C)$ and $\text{Hom}_A(\alpha, C)$ are surjective. This completes the proof. ■

3.3. Proof of Theorem 3.1. Without loss of generality, we only prove $\text{Im } d^0 \in G(C)$.

For each $i$, decompose $d^i$ as $X^i \xrightarrow{a_i} \text{Im } d^i \xrightarrow{b_i} X^{i+1}$. We claim that there exists an exact sequence $C^+ : 0 \rightarrow \text{Im } d^0 \xrightarrow{a^0} C^1 \xrightarrow{\gamma^1} C^2 \xrightarrow{\gamma^2} \cdots$ with $C^i \in C$ for all $i \geq 1$, such that $C^+$ is $C$-biexact.
In fact, applying Lemma 3.3 to the exact sequence $0 \to \text{Im}d^0 \xrightarrow{\partial^0} X^1 \xrightarrow{a^1} \text{Im}d^1 \to 0$ and the exact sequence $0 \to \text{Im}d^0 \xrightarrow{\cong} \text{Im}d^0 \to 0 \to 0$, we get the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & \text{Im}d^0 & \xrightarrow{b^0} & X^1 & \xrightarrow{a^1} & \text{Im}d^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Im}d^0 & \xrightarrow{v^0} & C^1 & \xrightarrow{w^1} & L^1 & \to & 0 \\
\end{array}
$$

such that $C^1 \in \mathcal{C}$, $w^1$ is a monomorphism with $\text{Coker} w^1 \in \mathcal{G}(\mathcal{C})$, and that $\xi_1$ is $\mathcal{C}$-biexact, and $\text{Hom}_A(w^1, \mathcal{C})$ is surjective. Now applying Lemma 3.3 to the exact sequence $0 \to \text{Im}d^1 \xrightarrow{b^1} X^2 \xrightarrow{a^2} \text{Im}d^2 \to 0$ and the exact sequence $0 \to \text{Im}d^1 \xrightarrow{w^1} L^1 \to \text{Coker} w^1 \to 0$, we get an exact sequence $\xi_2 : 0 \to L^1 \xrightarrow{v^1} C_2 \xrightarrow{u^2} L_3 \to 0$ and a monomorphism $w^2 : \text{Im}d^2 \hookrightarrow L^2$. Continuing this process and putting $\xi_1, \xi_2, \cdots$ together we finally obtain an exact sequence $C^+ \to C^0 \to \text{Im}d^0$, and this proves $\text{Im}d^0 \in \mathcal{G}(\mathcal{C})$. 

3.4. Applying Theorem 3.1 to $R$-mod, the category of finitely generated left $R$-modules, we get

**Corollary 3.4.** Let $R$ be a left noetherian ring. Let $X^\bullet = (X^i, d^i)$ be an exact complex with $X^i \in \mathcal{GP}(R)$ for all $i$. If $\text{Hom}_A(X^\bullet, R)$ is still exact, then $\text{Im}d^i \in \mathcal{GP}(R)$ for all $i$.

4. Relative Auslander algebras of CM-finite algebras

Throughout this section $\Lambda$ is an Artin algebra. All modules are finitely generated.

4.1. Let $E$ be a Gorenstein-projective generator, i.e., $E$ is a Gorenstein-projective $\Lambda$-module such that $\Lambda \in \text{add}E$, where $\text{add}E$ is the smallest full additive subcategory of $\Lambda$-mod containing $E$ and closed under direct summands. For short, we say that a complex $X^\bullet$ is $E$-biexact, if it is $E$-biexact. Denote $\mathcal{G}(\text{add}E)$ by $\mathcal{G}(E)$, i.e.,

$$
\mathcal{G}(E) = \{X \in \Lambda\text{-mod} \mid \exists \text{ an exact complex } X^\bullet = (X^i, d^i) \text{ with all } X^i \in \text{add}E, \text{ such that } X^\bullet \text{ is } E\text{-biexact, and } X \cong \text{Im}d^0\}.
$$

By Corollary 3.4(ii) we have

**Corollary 4.1.** Let $E$ be a Gorenstein-projective generator. Then $\mathcal{G}(E) \subseteq \mathcal{GP}(\Lambda)$.
4.2. Recall the Yoneda philosophy. Given a \( \Lambda \)-module \( E \), the functor \( \text{Hom}_\Lambda(E, -) : \Lambda\text{-mod} \to \Gamma\text{-mod} \) induces an equivalence between \( \text{add } E \) and \( \mathcal{P}(\Gamma) \), where \( \Gamma = (\text{End}_\Lambda E)^{\text{op}} \).

Also, for each \( E' \in \text{add } E \) and each \( X \in \Lambda\text{-mod} \) there is an isomorphism

\[
\text{Hom}_\Lambda(E', X) \to \text{Hom}_\Gamma(\text{Hom}_\Lambda(E, E'), \text{Hom}_\Lambda(E, X))
\]
given by \( f \mapsto \text{Hom}_\Lambda(E, f) \), \( \forall f \in \text{Hom}_\Lambda(E', X) \) (cf. [ARS], p.33).

Now, if \( E \) is a generator of \( \Lambda\text{-mod} \) (i.e., \( \Lambda \in \text{add } E \)), then we can say more.

**Lemma 4.2.** Let \( E \) be a generator of \( \Lambda\text{-mod} \), and \( \Gamma = (\text{End}_\Lambda E)^{\text{op}} \). Then the functor \( \text{Hom}_\Lambda(E, -) : \Lambda\text{-mod} \to \Gamma\text{-mod} \) is fully faithful.

**Proof.** Since \( E \) is a generator, for any \( \Lambda \)-module \( X \) there is a surjective map \( E^m \to X \)
for some positive integer \( m \). This implies that \( \text{Hom}_\Lambda(E, -) \) is faithful. Let \( X, Y \in \Lambda\text{-mod} \), \( f : \text{Hom}_\Lambda(E, X) \to \text{Hom}_\Lambda(E, Y) \) be a \( \Gamma \)-map. By taking right adjoint \( E \)-approximations, we get exact sequences \( T_1 \xrightarrow{u} T_0 \xrightarrow{\pi} X \to 0 \) and \( T'_1 \xrightarrow{u'} T'_0 \xrightarrow{\pi'} Y \to 0 \) with \( T_0, T_1, T'_0, T'_1 \in \text{add } E \) (note that since \( E \) is a generator, it follows that \( \pi \) and \( \pi' \) are surjective). Applying \( \text{Hom}_\Lambda(E, -) \) we have the following diagram with exact rows

\[
\begin{array}{ccccccc}
\text{Hom}_\Lambda(E, T_1) & \xrightarrow{\text{Hom}_\Lambda(E, u)} & \text{Hom}_\Lambda(E, T_0) & \xrightarrow{\text{Hom}_\Lambda(E, \pi)} & \text{Hom}_\Lambda(E, X) & \to & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\text{Hom}_\Lambda(E, T'_1) & \xrightarrow{\text{Hom}_\Lambda(E, u')} & \text{Hom}_\Lambda(E, T'_0) & \xrightarrow{\text{Hom}_\Lambda(E, \pi')} & \text{Hom}_\Lambda(E, Y) & \to & 0.
\end{array}
\]

Since the two rows are respectively parts of projective resolutions of \( \text{Hom}_\Lambda(E, X) \) and \( \text{Hom}_\Lambda(E, Y) \), \( f \) induces \( f_1 \) and \( f_0 \) such that the above diagram commutes. Thus \( f_i = \text{Hom}_\Lambda(E, f'_i) \) for some \( f'_i \in \text{Hom}_\Lambda(T_i, T'_i), i = 0, 1 \). So we get the following diagram

\[
\begin{array}{ccccccc}
T_1 & \xrightarrow{u} & T_0 & \xrightarrow{\pi} & X & \to & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f' & & \\
T'_1 & \xrightarrow{u'} & T'_0 & \xrightarrow{\pi'} & Y & \to & 0
\end{array}
\]

with commutative left square. So there exists \( f' \in \text{Hom}_\Lambda(X, Y) \) such that the above diagram commutes. Thus \( f \text{Hom}_\Lambda(E, \pi) = \text{Hom}_\Lambda(E, f') \text{Hom}_\Lambda(E, \pi) \) and hence \( f = \text{Hom}_\Lambda(E, f') \). This proves that \( \text{Hom}_\Lambda(E, -) \) is full. ✷

4.3. The following result shows that, after taking the opposite algebra of endomorphism algebra of a Gorenstein-projective generator, the category of the Gorenstein-projective modules can not be “enlarged”.

**Theorem 4.3.** Let \( \Lambda \) be an Artin algebra and \( E \) a generator, and \( \Gamma = (\text{End}_\Lambda E)^{\text{op}} \). Then \( \text{Hom}_\Lambda(E, -) \) induces an equivalence between \( \mathcal{G}(E) \) and \( \mathcal{GP}(\Gamma) \).

In particular, if \( E \) is a Gorenstein-projective generator, then \( \mathcal{GP}(\Gamma) \) is equivalent to a full subcategory of \( \mathcal{GP}(\Lambda) \).
Then Aus indecomposable Gorenstein-projective Θ bras, and
Let Theorem 4.5. Any Artin algebra is Morita equivalent to a basic Artin algebra.

4.4. Recall that an Artin algebra is basic dimension.

Γ = (End GP

Corollary 4.4. Let

E

there is a complete projective Γ-resolution Im(HomΛ(E, d0)). However,

since HomΛ(E, X) is exact and E is a generator, it follows that Im(HomΛ(E, d0)) = HomΛ(E, Im d0). That is, HomΛ(E, L) ∈ GP(Γ).

We claim that HomΓ(E, E) is a Gorenstein-projective generator, then

More precisely, put Ω to be the class of pairwise non-isomorphic basic CM-finite algebras, and Θ the class of pairwise non-isomorphic basic CM-finite algebras. Then

(i) The map Aus : Ω → Θ is surjective.

(ii) The map Aus sends CM-finite Gorenstein algebras to algebras of finite global dimension.

(iii) The map Aus sends CM-finite non-Gorenstein algebras to CM-free non-Gorenstein algebras.
(iv) A CM-free Gorenstein algebra $\Lambda$ is of finite global dimension.

Proof. Write $\Gamma = \text{Aus}(\Lambda)$. By definition we have $\mathcal{G}P(\Lambda) = \text{add } E \subseteq \mathcal{G}(E)$. By Corollary 4.1 $\mathcal{G}(E) \subseteq \mathcal{G}P(\Lambda)$. Thus $\mathcal{G}(E) = \text{add } E$. By Theorem 4.3 we have $\mathcal{G}P(\Gamma) = \text{Hom}_{\Lambda}(E, \mathcal{G}(E)) = \text{Hom}_{\Lambda}(E, \text{add } E) = \mathcal{P}(\Gamma)$.

This means that $\Gamma$ is CM-free.

(i) The map $\text{Aus}$ sends a basic CM-free algebra to itself, and from this (i) follows.

(ii) Recall a well-known fact: for a CM-finite algebra $\Lambda$, $\Lambda$ is Gorenstein if and only if $\Gamma$ is of finite global dimension (see [LZ]). From this the assertion (ii) follows.

(iii) By the fact stated above, $\text{Aus}$ sends CM-finite non-Gorenstein algebra $\Lambda$ to a CM-free algebra $\Gamma$ of infinite global dimension. Note that $\Gamma$ can not be Gorenstein: otherwise, by (ii) $\text{Aus}(\Gamma) = \Gamma$ is of finite global dimension. This proves (iii).

(iv) By (ii) $\text{Aus}(\Lambda)$ is of finite global dimension. Since $\Lambda$ is Morita equivalent to $\text{Aus}(\Lambda)$, $\Lambda$ is of finite global dimension. ■

5. Description of bounded derived categories

Throughout this section $\mathcal{A}$ is an abelian category with enough projective objects unless stated otherwise. The bounded derived category $D^b(\mathcal{A})$ can be interpreted as $K^{b,ac}(\mathcal{P}(\mathcal{A}))$. The aim of this section is to describe $D^b(\mathcal{A})$ via $\mathcal{G}P(\mathcal{A})$, under the condition that $\mathcal{A}$ is CM-contravariantly finite.

5.1. To describe $D^b(\mathcal{A})$ via $\mathcal{G}P(\mathcal{A})$, we need the full additive subcategory $K^{-,gp}(\mathcal{G}P(\mathcal{A}))$ of $K^{-}(\mathcal{G}P(\mathcal{A})) ([GZ, 3.3])$, where

$$K^{-,gp}(\mathcal{G}P(\mathcal{A})) = \{ X^\bullet \in K^{-}(\mathcal{G}P(\mathcal{A})) \mid \text{there exists an integer } N \text{ such that } H^n \text{Hom}_{\mathcal{A}}(G, X^\bullet) = 0, \forall n \leq N, \forall G \in \mathcal{G}P(\mathcal{A}) \}.$$ 

As pointed in [GZ], $K^{-,gp}(\mathcal{G}P(\mathcal{A}))$ is a triangulated subcategory of $K^{-}(\mathcal{G}P(\mathcal{A}))$. Clearly, it is the Gorenstein version of $K^{-,b}(\mathcal{P}(\mathcal{A}))$.

Let $X^\bullet \in K^{-,gp}(\mathcal{G}P(\mathcal{A}))$. Since $\mathcal{A}$ has enough projective objects and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{G}P(\mathcal{A})$, we see that $H^n(X^\bullet) = 0$ for $n << 0$.

The main result of this section is as follows.

Theorem 5.1. Let $\mathcal{A}$ be a CM-contravariantly finite abelian category. Then there is a triangle-equivalence

$$D^b(\mathcal{A}) \cong K^{-,gp}(\mathcal{G}P(\mathcal{A}))/K^{b,ac}(\mathcal{G}P(\mathcal{A}))$$

which induces a triangle-equivalence $\langle \mathcal{G}P(\mathcal{A}) \rangle \cong K^b(\mathcal{G}P(\mathcal{A}))/K^{b,ac}(\mathcal{G}P(\mathcal{A}))$. 

5.2. To prove Theorem 5.1 we need some preparations. First, we have

**Lemma 5.2.** (i) $K^{b,ac} (\mathcal{GP}(A))$ is a triangulated subcategory of $K^{-,gp} (\mathcal{GP}(A))$.

(ii) Let $G^\bullet = (G^i, d^i) \in K^{-,gp} (\mathcal{GP}(A))$. If $G^\bullet$ is exact, then $G^\bullet \in K^{b,ac} (\mathcal{GP}(A))$.

**Proof.** (i) is clear. We prove (ii). Since $\mathcal{GP}(A)$ is closed under kernels of epimorphisms, $\text{Im} d^i \in \mathcal{GP}(A), \forall i \in \mathbb{Z}$. By definition there exists an integer $N$ such that $H^n \text{Hom}_A(G, G^\bullet) = 0, \forall n \leq N, \forall G \in \mathcal{GP}(A)$. In particular $H^n \text{Hom}_A(\text{Im} d^{n-1}, G^\bullet) = 0$. This implies that the induced epimorphism $d^{n-1}_1 : G^{n-1} \to \text{Im} d^{n-1}$ splits for $n \leq N$, and hence $G^\bullet \cong G'^\bullet \in K^{b,ac} (\mathcal{GP}(A))$, where $G'^\bullet$ is the complex $\cdots \to 0 \to \text{Im} d^{N-1} \leftarrow G^N \to G^{N+1} \to \cdots$. $lacksquare$

The following observation plays an important role in our argument.

**Lemma 5.3.** Assume that $A$ is CM-contravariantly finite abelian category, $P^\bullet \in K^{-,b} (\mathcal{P}(A))$. Then

(i) There exists a quasi-isomorphism $P^\bullet \to G^\bullet$ with $G^\bullet \in K^{-,gp} (\mathcal{GP}(A))$.

(ii) For chain maps $f_i^\bullet : P^\bullet \to G^\bullet_i$ with $G^\bullet_i \in K^{-,gp} (\mathcal{GP}(A)), \ 1 \leq i \leq m$, there exist a quasi-isomorphism $g^\bullet : P^\bullet \to G^\bullet$ and chain maps $h_i^\bullet : G^\bullet \to G^\bullet_i$ with $G^\bullet \in K^{-,gp} (\mathcal{GP}(A))$, such that $f_i^\bullet = h_i^\bullet g^\bullet, \ 1 \leq i \leq m$.

**Proof.** (i) Write $P^\bullet = (P^i, d^i)$. Let $N$ be an integer such that $H^n P^\bullet = 0$ for $n \leq N$. Since $\mathcal{GP}(A)$ is a contravariantly finite subcategory in $A$, we can take a right $\mathcal{GP}(A)$-approximation $G^{N-1} \to \text{Ker} d^N$ of Ker $d^N$, and then take right $\mathcal{GP}(A)$-approximations step by step

\[
\cdots \to G^{N-2} \xrightarrow{\partial^{N-2}} G^{N-1} \xrightarrow{} P^N \xrightarrow{} \text{Ker} d^{N-2} \xrightarrow{} \text{Ker} d^{N-1} \xrightarrow{} \text{Ker} d^N
\]

In this way we get a complex

$G^\bullet : \cdots \to G^{N-2} \to G^{N-1} \to P^N \to P^{N+1} \to \cdots$

in $K^{-,gp} (\mathcal{GP}(A))$ with $H^n \text{Hom}_A(G, G^\bullet) = 0, \forall n \leq N, \forall G \in \mathcal{GP}(A)$. Since $G^n$ with $n \leq N$ is constructed via right $\mathcal{GP}(A)$-approximations, it is clear that there exists an induced chain map

\[
P^\bullet = \cdots \to P^{N-2} \to P^{N-1} \to P^N \to P^{N+1} \to \cdots
\]

\[
G^\bullet = \cdots \to G^{N-2} \to G^{N-1} \to P^N \to P^{N+1} \to \cdots,
\]

which is a quasi-isomorphism, since $H^n P^\bullet = 0 = H^n G^\bullet$ for $n \leq N$. 
(ii) Let $N$ be an integer such that $H^n \text{Hom}_A(G, G^*) = 0 = H^n P^*$, $\forall \ n \leq N$, $\forall \ G \in \mathcal{GP}(A)$, $1 \leq i \leq m$. By the proof of (i) we have the following quasi-isomorphism

$$
P^* = \cdots \rightarrow P^{N-2} \rightarrow P^{N-1} \rightarrow P^N \rightarrow P^{N+1} \rightarrow \cdots
$$

with $G^* \in K^{-,gpb}(\mathcal{GP}(A))$. Since for each $i$ we have $H^n \text{Hom}_A(G, G^*) = 0, \forall \ n \leq N$, $\forall \ G \in \mathcal{GP}(A)$, by applying $\text{Hom}_A(G^{N-1}, -)$, $\text{Hom}_A(G^{N-2}, -)$, $\cdots$, to $G^*$ respectively, we obtain the following chain map $h_i^*$:

$$
G^* = \cdots \rightarrow G_i^{N-2} \rightarrow G_i^{N-1} \rightarrow P^{-N} \rightarrow P^{N+1} \rightarrow \cdots
$$

For $1 \leq i \leq m$, if $l \geq N$, then $f_i^l - h_i^l g^l = 0$. If $l = N - 1$, then $f_i^{N-1} - h_i^{N-1} g^{N-1}$ can factor through $G_i^{N-2} \rightarrow G_i^{N-1}$ since $H^{N-1} G_i^* = 0$. By induction we get the following null homotopy $f_i^* - h_i^* g^*$.

$$
P^* = \cdots \rightarrow P^{N-2} \rightarrow P^{N-1} \rightarrow P^N \rightarrow P^{N+1} \rightarrow \cdots
$$

This completes the proof.

5.3. **Proof of Theorem 5.1.** Let $\eta : K^{-,gpb}(\mathcal{GP}(A)) \rightarrow D^-(A)$ be the composition

$$
K^{-,gpb}(\mathcal{GP}(A)) \xrightarrow{\sigma} K^-(A) \xrightarrow{Q} D^-(A) = K^-(A)/K^{-,ac}(A)
$$

where $\sigma$ is the embedding and $Q$ is the localization functor.

Since $K^{b,ac}(\mathcal{GP}(A))$ is a triangulated subcategory of $K^{-,gpb}(\mathcal{GP}(A))$, we have the Verdier quotient $K^{-,gpb}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A))$. Since $\eta(K^{b,ac}(\mathcal{GP}(A))) = 0$, by the universal property $\eta$ induces a unique triangle functor $\bar{\eta} : K^{-,gpb}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A)) \rightarrow D^-(A)$.

If $\mathcal{A}$ is CM-contravariantly finite, then by Lemma 5.3(i) we have $\text{Im} \bar{\eta} = K^{-,b}(\mathcal{P}(A)) \cong D^b(A)$. So we get a dense triangle functor from $K^{-,gpb}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A))$ to $D^b(A)$, again denoted by $\bar{\eta}$.

We prove that $\bar{\eta}$ is fully faithful. Suppose $G^* \in K^{-,gpb}(\mathcal{GP}(A))$ and $\bar{\eta}(G^*) = 0$. Then $G^*$ is exact. By Lemma 5.2 we have $G^* \in K^{b,ac}(\mathcal{GP}(A))$. That is $\bar{\eta}$ maps non-zero objects to non-zero objects. By Lemma 2.2 it suffices to prove that $\bar{\eta}$ is full.

Let $G_1^*, G_2^* \in K^{-,gpb}(\mathcal{GP}(A))$, and $\alpha^*/\sigma^*$ be a morphism in $\text{Hom}_{D^-(A)}(\bar{\eta}(G_1^*), \bar{\eta}(G_2^*)) = \text{Hom}_{D^-(A)}(G_1^*, G_2^*)$, where $\sigma^* : X^* \Rightarrow G_1^*$ is a quasi-isomorphism with $X^* \in K^-(A)$,
and \( \alpha^\bullet : X^\bullet \to G_2^\bullet \) is a morphism in \( K^-(A) \). Then there exists a quasi-isomorphism \( t^\bullet : P^\bullet \to X^\bullet \) with \( P^\bullet \in K^-(\mathcal{P}(A)) \). Since \( s^\bullet \) and \( t^\bullet \) are quasi-isomorphisms and \( G_1^\bullet \in K^{-,gpb}(\mathcal{GP}(A)) \), it follows that \( P^\bullet \in K^{-,b}(\mathcal{P}(A)) \). Thus we get the commutative diagram

\[
\begin{array}{ccc}
X^\bullet & \xrightarrow{\alpha^\bullet} & G_2^\bullet \\
\downarrow{s^\bullet} & & \downarrow{\alpha^\bullet} \\
G_1^\bullet & \xleftarrow{t^\bullet} & P^\bullet
\end{array}
\]

where the double arrowed morphisms mean quasi-isomorphisms. By Lemma 5.3(ii) we have the following commutative diagram

\[
\begin{array}{ccc}
G_1^\bullet & \xleftarrow{z^\bullet} & P^\bullet & \xrightarrow{\alpha^\bullet} & G_2^\bullet \\
\downarrow{t^\bullet} & & \downarrow{\alpha^\bullet} & & \downarrow{g^\bullet} \\
G^\bullet & \xleftarrow{l^\bullet} & G_2^\bullet & \xrightarrow{\beta^\bullet} & G^\bullet
\end{array}
\]

where \( G^\bullet \in K^{-,gpb}(\mathcal{GP}(A)) \), and \( g^\bullet : P^\bullet \Rightarrow G^\bullet \) is a quasi-isomorphism. Note that \( l^\bullet \) is also a quasi-isomorphism, hence the mapping cone \( \text{Cone}(l^\bullet) \) is exact. Since \( K^{-,gpb}(\mathcal{GP}(A)) \) is a triangulated subcategory of \( K^-(A) \), \( \text{Cone}(l^\bullet) \in K^{-,gpb}(\mathcal{GP}(A)) \). By Lemma 5.2(ii) \( \text{Cone}(l^\bullet) \in K^{b,ac}(\mathcal{GP}(A)) \). This proves \( \beta^\bullet/l^\bullet \in \text{Hom}_{K^{-,gpb}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A))}(G_1^\bullet, G_2^\bullet) \) and \( \alpha^\bullet/s^\bullet = \beta^\bullet/l^\bullet = \tau(\beta^\bullet/l^\bullet) \). This proves the first triangle-equivalence in Theorem 5.1.

By Lemma 2.1 we know that \( K^{b}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A)) \) is a triangulated subcategory of \( K^{-,gpb}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A)) \). Thus \( \tau(K^{b}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A))) \) is a triangulated subcategory of \( D^b(A) \). Obviously this image is generated by \( \mathcal{GP}(A) \), i.e.,

\[
\tau(K^{b}(\mathcal{GP}(A))/K^{b,ac}(\mathcal{GP}(A))) = \langle \mathcal{GP}(A) \rangle.
\]

So we get the second triangle-equivalence in Theorem 5.1.

6. Descriptions of Gorenstein defect categories

Throughout this section \( A \) is an abelian category with enough projective objects. The aim of this section is to describe \( D^{b,\text{defect}}(A) \).

6.1. A quick description is

Lemma 6.1. Let \( F : \mathcal{GP}(A) \to D^{b}(A) := D^b(A) / K^b(\mathcal{P}(A)) \) be the fully-faithful triangle functor defined in (2.1). Then

\[ \text{Im} F = \langle \mathcal{GP}(A) \rangle / K^b(\mathcal{P}(A)), \]

and hence we have a triangle-equivalence

\[ D^{b,\text{defect}}(A) \cong D^b(A) / \langle \mathcal{GP}(A) \rangle. \]
Proof. Since \(\mathcal{GP}(A)\) is the triangulated subcategory of \(D^b(A)\) generated by \(\mathcal{GP}(A)\), it follows that \(\mathcal{GP}(A)/K^b(\mathcal{P}(A))\) is the triangulated subcategory of \(D^b(A)/K^b(\mathcal{P}(A))\) generated by \(\mathcal{GP}(A)\), here view objects in \(\mathcal{GP}(A)\) as stalk complexes at degree 0. Since \(F\) is full, it follows that \(\text{Im}F\) is a triangulated subcategory of \(D^b(A)/K^b(\mathcal{P}(A))\) containing \(\mathcal{GP}(A)\). It follows that \(\text{Im}F \supseteq \mathcal{GP}(A)/K^b(\mathcal{P}(A))\).

On the other hand, by definition \(\text{Im}F \subseteq \mathcal{GP}(A)/K^b(\mathcal{P}(A))\). It follows that \(\text{Im}F = \mathcal{GP}(A)/K^b(\mathcal{P}(A))\), and hence by Lemma [2.1] we have a triangle-equivalence

\[
D^b_{\text{defect}}(A) = (D^b(A)/K^b(\mathcal{P}(A)))/(\mathcal{GP}(A)/K^b(\mathcal{P}(A))) \cong D^b(A)/\mathcal{GP}(A).
\]

By definition we have \(K^b(\mathcal{P}(A)) \subseteq \mathcal{GP}(A) \subseteq D^b(A)\).

Corollary 6.2. We have

(i) \(K^b(\mathcal{P}(A)) = \mathcal{GP}(A)\) if and only if \(A\) is CM-free, i.e., \(\mathcal{P}(A) = \mathcal{GP}(A)\).

(ii) \(\mathcal{GP}(A) = D^b(A)\) if and only if each object of \(A\) has finite Gorenstein-projective dimension.

Proof. (i) If \(\mathcal{P}(A) = \mathcal{GP}(A)\), then it is clear that \(K^b(\mathcal{P}(A)) = \mathcal{GP}(A)\). Conversely, assume that \(K^b(\mathcal{P}(A)) = \mathcal{GP}(A)\). Then one easily see that any Gorenstein-projective object \(G\) is of finite projective dimension, and hence \(G\) is projective ([EJ2, 10.2.3]).

(ii) By Lemma [6.1] \(\mathcal{GP}(A) = D^b(A)\) if and only if \(D^b_{\text{defect}}(A) = 0\), and if and only if each object of \(A\) has finite Gorenstein-projective dimension.

Remark 6.3. Although the generating process inside a triangulated category is clear (see e.g. R. Rouquier [Rou]), but the output of this generating process is not so clear. In other words, the description in Lemma [6.1] is rough in the sense that the shape of \(\mathcal{GP}(A)\) is vague. We will give an explicit description of \(D^b_{\text{defect}}(A)\).

6.2. Define \(K^b_{\mathcal{GP}}(\mathcal{P}(A))\) to be the full subcategory of \(K^b(\mathcal{P}(A))\) consisting of all the complexes \((P^\bullet, d) \in K^b(\mathcal{P}(A))\) such that there exists \(n_0 \in \mathbb{Z}\) with \(H^m(P^\bullet) = 0, \forall m \leq n_0\), and \(\text{Ker}d_{n_0}^A \in \mathcal{GP}(A)\).

In order to say that \(K^b_{\mathcal{GP}}(\mathcal{P}(A))\) is a thick triangulated subcategory of \(K^b(\mathcal{P}(A))\), first, we need to say that \(K^b_{\mathcal{GP}}(\mathcal{P}(A))\) is closed under isomorphisms of \(K^b(\mathcal{P}(A))\).

The following fact is well-known.

Lemma 6.4. Let \(A\) be the stable category of \(A\) modulo \(\mathcal{P}(A)\). Let \(X, Y \in A\). Then \(X \cong Y\) in \(A\) if and only if there exist projective objects \(P\) and \(Q\) such that \(X \oplus P \cong Y \oplus Q\) in \(A\).

The following fact is also well-known.

Lemma 6.5. Let \(f^\bullet : P^\bullet \to Q^\bullet\) be a null-homotopy with \(P^\bullet, Q^\bullet \in K^b(\mathcal{P}(A))\). Assume that \(H^m(P^\bullet) = 0 = H^m(Q^\bullet), \forall m \leq n_0\). Then the restriction \(f_{n_0} : \text{Ker}d_{n_0}^{P^\bullet} \to \text{Ker}d_{n_0}^{Q^\bullet}\) of \(f_{n_0}\) factors through a projective object.
Proof. For convenience we include a proof. Let $s : f^\bullet \sim 0$ be a homotopy. Put $K = \text{Kerd}_{P^\bullet}$ and $L = \text{Kerd}_{Q^\bullet}$.

Then we have $f_{n_0}^i = i'\overrightarrow{f_{n_0}}$ and $\overrightarrow{f_{n_0}}\pi = \pi' f_{n_0}^{-1}$. Since

$$i'\overrightarrow{f_{n_0}}\pi = d_{n_0}^{-1}f_{n_0}^{-1} = d_{n_0}^{-1}(s_{n_0}d_{n_0}^{-1} + d_{n_0}^{-2}s_{n_0}^{-1}) = d_{n_0}^{-1}s_{n_0}d_{n_0}^{-1} = i'\pi' s_{n_0}i\pi,$$

we have $\overrightarrow{f_{n_0}} = \pi's_{n_0}i$. Thus $\overrightarrow{f_{n_0}}$ factors through a projective object. ■

The following fact shows that $K^\leftarrow_{-b}(\mathcal{P}(A))$ is closed under isomorphisms of $K^\leftarrow_{-b}(\mathcal{P}(A))$.

Lemma 6.6. Let $f^\bullet : P^\bullet \rightarrow Q^\bullet$ be a homotopy equivalence in $K^\leftarrow_{-b}(\mathcal{P}(A))$ with $P^\bullet \in K^\leftarrow_{-b}(\mathcal{P}(A))$. Then $Q^\bullet \in K^\leftarrow_{-b}(\mathcal{P}(A))$.

Proof. Let $n_0 \in \mathbb{Z}$ such that $H^m(P^\bullet) = 0, \forall \ m \leq n_0$, and Ker$d_{n_0}^P \in \mathcal{G}\mathcal{P}(A)$. Then $H^m(Q^\bullet) = 0, \forall \ m \leq n_0$. Let $g^\bullet : Q^\bullet \rightarrow P^\bullet$ be the inverse of $f^\bullet$. Then we have the restriction $\overrightarrow{f_{n_0}} : \text{Kerd}_{Q^\bullet} \rightarrow \text{Kerd}_{P^\bullet}$ of $f_{n_0}$, and the restriction $g_{n_0} : \text{Kerd}_{P^\bullet} \rightarrow \text{Kerd}_{Q^\bullet}$ of $g^\bullet$. Since $g^\bullet f^\bullet - \text{Id}_{P^\bullet}$ and $f^\bullet g^\bullet - \text{Id}_{Q^\bullet}$ are null-homotopy, by Lemma 6.5 the induced morphism $\overrightarrow{g_{n_0}} f_{n_0}^{-1} - \text{Id}_{\text{Kerd}_{Q^\bullet}}$ factors through a projective object, and $\overrightarrow{f_{n_0}} g_{n_0}^{-1} - \text{Id}_{\text{Kerd}_{P^\bullet}}$ factors through a projective object. This means that Ker$d_{n_0}^P$ and Ker$d_{n_0}^Q$ are isomorphic in $\mathcal{A}$.

By Lemma 6.3 there exist projective objects $P$ and $Q$ such that there is an isomorphism Ker$d_{n_0}^P \oplus P \cong \text{Kerd}_{Q^\bullet} \oplus Q$ in $\mathcal{A}$. Since Ker$d_{n_0}^P \in \mathcal{G}\mathcal{P}(A)$ and $\mathcal{G}\mathcal{P}(A)$ is closed under direct summands, it follows that Ker$d_{n_0}^Q \in \mathcal{G}\mathcal{P}(A)$. This proves $Q^\bullet \in K^\leftarrow_{-b}(\mathcal{P}(A))$. ■

Proposition 6.7. $K^\leftarrow_{-b}(\mathcal{P}(A))$ is a thick triangulated subcategory of $K^\leftarrow_{-b}(\mathcal{P}(A))$.

Proof. It is clear that $K^\leftarrow_{-b}(\mathcal{P}(A))$ is an additive category. By Lemma 6.6 $K^\leftarrow_{-b}(\mathcal{P}(A))$ is a full subcategory of $K^\leftarrow_{-b}(\mathcal{P}(A))$ closed under isomorphisms. Since $\mathcal{G}\mathcal{P}(A)$ is closed under direct summands, it follows that $K^\leftarrow_{-b}(\mathcal{P}(A))$ is closed under direct summands. It is also clear that $K^\leftarrow_{-b}(\mathcal{P}(A))$ is closed under the shift functor $[1]$ and $[-1]$. Let $f^\bullet : P^\bullet \rightarrow Q^\bullet$ be a chain map with $P^\bullet, Q^\bullet \in K^\leftarrow_{-b}(\mathcal{P}(A))$. It remains to prove that the mapping Cone($f^\bullet$) $\in K^\leftarrow_{-b}(\mathcal{P}(A))$. 

\[ \text{\textbullet} \]
Since $P^\bullet, Q^\bullet \in K^-_G \cdot b(P(A))$, there exists an integer $n_0$ such that $H^m(P^\bullet) = 0 = H^m(Q^\bullet)$ for $m \leq n_0$, and $\text{Kerd}_{P^\bullet}^{n_0} \in G\mathcal{P}(A)$, $\text{Kerd}_{Q^\bullet}^{n_0} \in G\mathcal{P}(A)$. It follows that there are complete projective resolutions $X^\bullet$ and $Y^\bullet$ such that

$$\text{Kerd}_{X^\bullet}^{n_0} = \text{Kerd}_{P^\bullet}^{n_0}, \text{Kerd}_{Y^\bullet}^{n_0} = \text{Kerd}_{Q^\bullet}^{n_0}.$$ 

Construct the following two complexes

$$P^\bullet = \ldots \rightarrow P^{n_0-2} \xrightarrow{d_{P^\bullet}^{n_0-2}} P^{n_0-1} \xrightarrow{d_{P^\bullet}^{n_0-1}} X^{n_0} \xrightarrow{d_{P^\bullet}^{n_0}} X^{n_0+1} \xrightarrow{d_{X^\bullet}^{n_0+1}} X^{n_0+2} \rightarrow \ldots,$$

and

$$Q^\bullet = \ldots \rightarrow Q^{n_0-2} \xrightarrow{d_{Q^\bullet}^{n_0-2}} Q^{n_0-1} \xrightarrow{d_{Q^\bullet}^{n_0-1}} Y^{n_0} \xrightarrow{d_{Q^\bullet}^{n_0}} Y^{n_0+1} \xrightarrow{d_{Y^\bullet}^{n_0+1}} Y^{n_0+2} \rightarrow \ldots,$$

where $d_{P^\bullet}^{n_0} : P^{n_0-1} \rightarrow X^{n_0}$ is the composition of the canonical morphisms

$$P^{n_0-1} \rightarrow \text{Kerd}_{P^\bullet}^{n_0}, \text{Kerd}_{X^\bullet}^{n_0} \rightarrow X^{n_0},$$

and $d_{Q^\bullet}^{n_0} : Q^{n_0} \rightarrow Y^{n_0+1}$ is the composition of the canonical morphisms

$$Q^{n_0-1} \rightarrow \text{Kerd}_{Q^\bullet}^{n_0}, \text{Kerd}_{Y^\bullet}^{n_0} \rightarrow Y^{n_0}.$$ 

Now $f^\bullet : P^\bullet \rightarrow Q^\bullet$ induces a morphism $\text{Kerd}_{P^\bullet}^{n_0} \rightarrow \text{Kerd}_{Q^\bullet}^{n_0}$, namely a morphism $\text{Kerd}_{X^\bullet}^{n_0} \rightarrow \text{Kerd}_{Y^\bullet}^{n_0}$. Since projective objects are injective objects in the category $G\mathcal{P}(A)$, it follows that the $\text{Kerd}_{X^\bullet}^{n_0} \rightarrow \text{Kerd}_{Y^\bullet}^{n_0}$ induces morphisms $f^i : X^i \rightarrow Y^i$ for $i \geq n_0$, such that the diagram

$$P^\bullet : \quad \ldots \quad \xrightarrow{d_{P^\bullet}^{n_0-2}} P^{n_0-1} \xrightarrow{d_{P^\bullet}^{n_0-1}} X^{n_0} \xrightarrow{d_{P^\bullet}^{n_0}} X^{n_0+1} \xrightarrow{d_{X^\bullet}^{n_0+1}} \ldots$$

$$f^\bullet \quad \xrightarrow{f^{n_0-2}} \quad \xrightarrow{d_{P^\bullet}^{n_0-2}} \quad \xrightarrow{d_{P^\bullet}^{n_0-1}} \quad \xrightarrow{d_{X^\bullet}^{n_0+1}} \ldots$$

$$Q^\bullet : \quad \ldots \quad \xrightarrow{d_{Q^\bullet}^{n_0-2}} Q^{n_0-1} \xrightarrow{d_{Q^\bullet}^{n_0-1}} Y^{n_0} \xrightarrow{d_{Q^\bullet}^{n_0}} Y^{n_0+1} \xrightarrow{d_{Y^\bullet}^{n_0+1}} \ldots$$

commutes. By construction $P^\bullet$ and $Q^\bullet$ are complete projective resolutions. It is clear that $\text{Cone}(f^\bullet)$ is again a complete projective resolution, and in particular $\text{Kerd} d_{\text{Cone}(f^\bullet)}^{n_0-3}$ is a Gorenstein-projective object. However by construction we have

$$d_{\text{Cone}(f^\bullet)}^{m} = \begin{pmatrix} -d_{P^\bullet}^{m+1} & 0 \\ f_{m+1} & d_{Q^\bullet}^{m} \end{pmatrix} = d_{\text{Cone}(f^\bullet)}^{m}, \quad \forall \ m \leq n_0 - 3.$$ 

This proves that $\text{Cone}(f^\bullet) \in K^-_G \cdot b(P(A))$. □

6.3. We have the following description of $D^{b}_{\text{defect}}(A)$.

**Theorem 6.8.** Let $A$ be an abelian category with enough projective objects. Then

(i) There is a triangle-equivalence

$$D^{b}_{\text{defect}}(A) \cong K^-_G \cdot b(P(A)) / K^-_G \cdot b(P(A)).$$
(ii) If in addition $\mathcal{A}$ is CM-contravariantly finite, then there is a triangle-equivalence

$$D_{\text{defect}}^b(\mathcal{A}) \cong K^{-\text{gp}}(\mathcal{G}\mathcal{P}(\mathcal{A}))/K^b(\mathcal{G}\mathcal{P}(\mathcal{A})).$$

**Proof.** (i) We claim: the restriction of the canonical triangle-equivalence $\rho : K^{-b}(\mathcal{P}(\mathcal{A})) \to D^b(\mathcal{A})$ to $K_G^{-b}(\mathcal{P}(\mathcal{A}))$ gives rise to a triangle-equivalence $K_G^{-b}(\mathcal{P}(\mathcal{A})) \to \langle \mathcal{GP}(\mathcal{A}) \rangle$.

In fact, let $P^\bullet \in K_G^{-b}(\mathcal{P}(\mathcal{A}))$. By definition there is an integer $n_0 \in \mathbb{Z}$ such that $H^m(P^\bullet) = 0$, $\forall \ m \leq n_0$, and $\text{Ker} P^{n_0} \in \mathcal{GP}(\mathcal{A})$. Then there is a quasi-isomorphism $P^\bullet \to G^\bullet$, where $G^\bullet$ is the bounded complex

$$0 \to \text{Ker} P^{n_0} \hookrightarrow P^{n_0} \xrightarrow{d^{n_0}} P^{n_0+1} \to \cdots.$$ 

It follows that $\rho(P^\bullet) = P^\bullet \cong G^\bullet$ in $D^b(\mathcal{A})$. Since $P^\bullet$ is an upper-bounded complex of projective objects, it follows that $G$ is a bounded complex of Gorenstein-projective objects. Thus $G^\bullet \in \langle \mathcal{GP}(\mathcal{A}) \rangle$. So $\rho(K_G^{-b}(\mathcal{P}(\mathcal{A}))) \subseteq \langle \mathcal{GP}(\mathcal{A}) \rangle$.

On the other hand, by Proposition 6.7 $K_G^{-b}(\mathcal{P}(\mathcal{A}))$ is a triangulated subcategory of $K^{-b}(\mathcal{P}(\mathcal{A}))$. Since $\rho : K^{-b}(\mathcal{P}(\mathcal{A})) \to D^b(\mathcal{A})$ is full, it follows that $\rho(K_G^{-b}(\mathcal{P}(\mathcal{A})))$ is a triangulated subcategory of $D^b(\mathcal{A})$. Let $G \in \mathcal{GP}(\mathcal{A})$. Then there is a complete projective resolution $E^\bullet$ such that $\text{Ker} d_{E_i} \cong G$. Then the brutal truncation

$$E_{\leq 0}^\bullet = \cdots \to E^{-2} \to E^{-1} \xrightarrow{d_{E_{i-1}}} E^{i} \to 0$$

is in $K_G^{-b}(\mathcal{P}(\mathcal{A}))$, and $\rho(E_{\leq 0}^\bullet) \cong \text{Im} d_{E_1} = \text{Ker} d_{E_2} \cong G$. Thus $\rho(K_G^{-b}(\mathcal{P}(\mathcal{A})))$ is a triangulated subcategory of $D^b(\mathcal{A})$ containing $\mathcal{GP}(\mathcal{A})$. So $\langle \mathcal{GP}(\mathcal{A}) \rangle \subseteq \rho(K_G^{-b}(\mathcal{P}(\mathcal{A})))$. Thus $\rho(K_G^{-b}(\mathcal{P}(\mathcal{A}))) = \langle \mathcal{GP}(\mathcal{A}) \rangle$. This proves the claim.

By the claim the diagram

$$
\begin{array}{ccc}
K^b(\mathcal{P}(\mathcal{A})) & \longrightarrow & K_G^{-b}(\mathcal{P}(\mathcal{A})) \longrightarrow \mathcal{K}^-b(\mathcal{P}(\mathcal{A})) \\
\Downarrow & & \Downarrow \\
K^b(\mathcal{P}(\mathcal{A})) & \longrightarrow & \langle \mathcal{GP}(\mathcal{A}) \rangle \longrightarrow D^b(\mathcal{A})
\end{array}
$$

commutes, where the horizontal functors are embeddings, and the vertical functors are triangle-equivalences. This induces a triangle-equivalence

$$K^{-b}(\mathcal{P}(\mathcal{A}))/K_G^{-b}(\mathcal{P}(\mathcal{A})) \cong D^b(\mathcal{A})/\langle \mathcal{GP}(\mathcal{A}) \rangle = D_{\text{defect}}^b(\mathcal{A})$$

where the last equality follows from Lemma 6.1.

(ii) Assume that $\mathcal{A}$ is CM-contravariantly finite. By Theorem 5.1 and Lemma 2.1 we get a triangle-equivalence

$$K^{-\text{gp}}(\mathcal{GP}(\mathcal{A}))/K^b(\mathcal{GP}(\mathcal{A})) \to D^b(\mathcal{A})/\langle \mathcal{GP}(\mathcal{A}) \rangle.$$ 

By Lemma 6.1 we get a triangle-equivalence

$$D_{\text{defect}}^b(\mathcal{A}) \cong K^{-\text{gp}}(\mathcal{GP}(\mathcal{A}))/K^b(\mathcal{GP}(\mathcal{A})).$$
This completes the proof. ■

Remark 6.9. Let \( \overline{F} : \mathcal{GP}(A) \rightarrow K^{-,b}(\mathcal{P}(A))/K^{b}(\mathcal{P}(A)) \) be the fully-faithful triangle functor defined in (2.2). Then it is easy to see \( \text{Im} \overline{F} = K^{-,b}(\mathcal{P}(A))/K^{b}(\mathcal{P}(A)) \).

6.4. Note that a CM-finite algebra is CM-contravariantly finite. By Theorem 6.8 we get

Corollary 6.10. Let \( \Lambda \) be a CM-finite algebra. Then there is a triangle-equivalence

\[ D^{b}_{\text{defect}}(\Lambda) \cong D^{b}_{sg}(\text{Aus}(\Lambda)). \]

Proof. Put \( E \) to be the direct sum of all the pairwise non-isomorphic indecomposable Gorenstein-projective \( \Lambda \)-modules. Then \( \text{Hom}_{\Lambda}(E, -) : \mathcal{P}(\text{Aus}(\Lambda)) \cong \mathcal{GP}(\Lambda) \) as additive categories. This is extended to a triangle-equivalences

\[ K^{-,b}(\mathcal{P}(\text{Aus}(\Lambda)))/K^{b}(\mathcal{P}(\text{Aus}(\Lambda))) \cong K^{-,gp}(\mathcal{GP}(\Lambda))/K^{b}(\mathcal{GP}(\Lambda)), \]

and hence we get the triangle-equivalences

\[ D^{b}_{sg}(\text{Aus}(\Lambda)) = D^{b}(\text{Aus}(\Lambda))/K^{b}(\mathcal{P}(\text{Aus}(\Lambda))) \]
\[ \cong K^{-,b}(\mathcal{P}(\text{Aus}(\Lambda)))/K^{b}(\mathcal{P}(\text{Aus}(\Lambda))) \]
\[ \cong K^{-,gp}(\mathcal{GP}(\Lambda))/K^{b}(\mathcal{GP}(\Lambda)) \]
\[ \cong D^{b}_{\text{defect}}(\Lambda), \]

where the final triangle-equivalence follows from the second triangle-equivalence in Theorem 6.8. This completes the proof. ■

7. Final Remarks

In fact, we have the following more general result.

Theorem 7.1. Suppose that \( \mathcal{A} \) and \( \mathcal{A}' \) are abelian categories with enough projective objects such that \( \mathcal{GP}(\mathcal{A}) \cong \mathcal{GP}(\mathcal{A}') \) as categories, and that \( \mathcal{A} \) is CM-contravariantly finite. Then

(i) There is a triangle-equivalence

\[ D^{b}(\mathcal{A}') \cong K^{-,gp}(\mathcal{GP}(\mathcal{A}'))/K^{b,ac}(\mathcal{GP}(\mathcal{A}')); \]

(ii) There is a triangle-equivalence

\[ D^{b}_{\text{defect}}(\mathcal{A}') \cong K^{-,gp}(\mathcal{GP}(\mathcal{A}'))/K^{b}(\mathcal{GP}(\mathcal{A}')), \]

and hence there is a triangle-equivalence

\[ D^{b}_{\text{defect}}(\mathcal{A}) \cong D^{b}_{\text{defect}}(\mathcal{A}'). \]
Proof. Let $F : \mathcal{GP}(A) \to \mathcal{GP}(A')$ be an equivalence of categories, with quasi-inverse $F^{-1}$. Since an equivalence between additive categories is an additive functor, $F$ is an additive functor.

By construction we have $H^i(GP(A)) = 0$ for $i \geq 0$, so $H^i(GP(D)) = 0$ is an equivalence, we have a triangle-equivalence $A \to \mathcal{GP}(A)$ is a contravariantly finite subcategory in $N$. Consider complex $F^{-1}P^\bullet$. Since $\mathcal{GP}(A)$ is an additive functor, we can take a right $\mathcal{GP}(A)$-approximation $G^N \to \text{Ker}F^{-1}(d^N)$ of $\text{Ker}F^{-1}(d^N)$, and then take right $\mathcal{GP}(A)$-approximations step by step (as in the proof of Lemma 5.3) we get a complex

$$G^\bullet : \cdots \to G^{N-2} \xrightarrow{\partial^{N-2}} G^{N-1} \xrightarrow{\partial^{N-1}} F^{-1}(P^N) \xrightarrow{F^{-1}(d^N)} F^{-1}(P^{N+1}) \to \cdots.$$ 

By construction we have $H^n \text{Hom}_A(G, G^\bullet) = 0$ for $G \in \mathcal{GP}(A)$ and $n \leq N$. Since $F : \mathcal{GP}(A) \to \mathcal{GP}(A')$ is an equivalence, we have the complex isomorphism $\text{Hom}_A(G, G^\bullet) \cong \text{Hom}_{A'}(FG, FG^\bullet)$ for $G \in \mathcal{GP}(A)$. Thus $FG^\bullet \in K^{-,gpb}(\mathcal{GP}(A'))$.

Since $F^{-1}$ is a quasi-inverse of $F$, there is a complex $Q^\bullet$ and a complex isomorphism

$$Q^\bullet : \cdots \to F(G^{N-2}) \xrightarrow{d^N} F(G^{N-1}) \xrightarrow{P^N} F^{-1}(P^N) \xrightarrow{F^{-1}(d^N)} F^{-1}(P^{N+1}) \to \cdots$$

By construction we have $H^n \text{Hom}_A(G, G^\bullet) = 0$ for $n \leq N$. Thus we have a quasi-isomorphism

$$P^\bullet : \cdots \to P^{N-2} \xrightarrow{d^N} P^{N-1} \xrightarrow{P^N} P^{N+1} \to \cdots$$

This proves that Lemma 5.3(i) holds for $A'$. Hence Lemma 5.3(ii) also holds for $A'$.

Repeating the proof of Theorem 5.1, we can prove

$$D^b(A') \cong K^{-,gpb}(\mathcal{GP}(A'))/K^{\text{h,ac}}(\mathcal{GP}(A')).$$

This proves (i).

Repeating the proof of Theorem 6.8(ii) we have

$$D^b_{\text{defect}}(A') \cong K^{-,gpb}(\mathcal{GP}(A'))/K^b(\mathcal{GP}(A')).$$

Note that the equivalence $\mathcal{GP}(A') \cong \mathcal{GP}(A')$ extends to a triangle-equivalence

$$K^{-,gpb}(\mathcal{GP}(A'))/K^b(\mathcal{GP}(A')) \cong K^{-,gpb}(\mathcal{GP}(A'))/K^b(\mathcal{GP}(A')).$$

By Theorem 6.8(ii) we have a triangle-equivalence $D^b_{\text{defect}}(A) \cong K^{-,gpb}(\mathcal{GP}(A'))/K^b(\mathcal{GP}(A))$. Altogether we get a triangle-equivalence $D^b_{\text{defect}}(A) \cong D^b_{\text{defect}}(A')$. This proves (ii).
Corollary 7.2. Let $\Lambda$ and $\Lambda'$ be Artin algebras such that $\mathcal{GP}(\Lambda) \cong \mathcal{GP}(\Lambda')$ as categories. Then $\Lambda$ is Gorenstein if and only if $\Lambda'$ is Gorenstein.

Proof. If $\Lambda$ is Gorenstein, then $\mathcal{GP}(\Lambda)$ is contravariantly finite. By Theorem 7.1 we have

$$D^b_{\text{defect}}(\Lambda') \cong D^b_{\text{defect}}(\Lambda) = 0,$$

thus $\Lambda'$ is Gorenstein. $\blacksquare$

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