MULTIVARIABLE $q$-RACAH POLYNOMIALS

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ABSTRACT. The Koornwinder-Macdonald multivariable generalization of the Askey-Wilson polynomials is studied for parameters satisfying a truncation condition such that the orthogonality measure becomes discrete with support on a finite grid. For this parameter regime the polynomials may be seen as a multivariable counterpart of the (one-variable) $q$-Racah polynomials. We present the discrete orthogonality measure, expressions for the normalization constants converting the polynomials into an orthonormal system (in terms of the normalization constant for the unit polynomial), and we discuss the limit $q \to 1$ leading to multivariable Racah type polynomials. Of special interest is the situation that $q$ lies on the unit circle; in that case it is found that there exists a natural parameter domain for which the discrete orthogonality measure (which is complex in general) becomes real-valued and positive. We investigate the properties of a finite-dimensional discrete integral transform for functions over the grid, whose kernel is determined by the multivariable $q$-Racah polynomials with parameters in this positivity domain.

1. Introduction

Some years ago, Koornwinder [K] extended a construction of Macdonald [M1] (see also [VK1]) to arrive at a multivariable generalization of a family of basic hypergeometric polynomials commonly known as the Askey-Wilson polynomials [AW2, GR]. The multivariable polynomials of interest depend rationally on a number of parameters and for parameter values in a certain domain they form an orthogonal system with respect to an explicitly given (positive) continuous weight function with support on a (real) $n$-dimensional torus (where $n$ denotes the number of variables). Recently, it was shown that the parameter domain for which the multivariable Askey-Wilson polynomials admit such an interpretation as orthogonal polynomials may be extended if one allows the corresponding orthogonality measure to have a partly continuous and partly discrete support [S]. (Thus further generalizing the corresponding situation in the case of one single variable, where the phenomenon of discrete masses emerging in the Askey-Wilson orthogonality measure was already known to occur [AW2].)

In the present paper we will demonstrate that for a different parameter regime satisfying a certain truncation condition, the multivariable Askey-Wilson polynomials can be reduced to a finite-dimensional orthogonal system with respect to a purely discrete weight function living on (i.e. supported on) a finite grid. The polynomials amount for these parameters to a multivariable generalization of the $q$-Racah polynomials [AW1, GR]. We will compute the normalization constants

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turning the polynomials into an orthonormal system with respect to the discrete orthogonality measure (in terms of the corresponding normalization constant for the unit polynomial), and also discuss the limit $q \to 1$ giving rise to multivariable Racah type polynomials.

In general we will not worry much about the positivity of the weight function and we will, in fact, for most of the time allow parameters to be such that the discrete orthogonality measure becomes complex. However, as will be outlined below in further detail, it is possible to restrict the parameters for $|q| = 1$ in a rather natural way to a subdomain such that the discrete orthogonality measure for the multivariable $q$-Racah polynomials becomes a genuine positive measure. For parameters in this positivity domain the restriction of the multivariable $q$-Racah polynomials to the grid points entails (after renormalizing) an orthonormal basis for the finite-dimensional Hilbert space consisting of all (complex) functions over the grid. From a functional-analytic point of view, the renormalized polynomials determine the kernel of a unitary finite-dimensional integral transformation for these grid functions.

The material is structured as follows. In Section 2 we first recall the definition of the multivariable Askey-Wilson polynomials for generic parameters as eigenfunctions of the Koornwinder-Macdonald second order analytic $q$-difference operator. Next, the discrete orthogonality measure for the polynomials is introduced in Section 3 and it is explained how the multivariable Askey-Wilson polynomials descend to a $q$-Racah type finite-dimensional orthogonal system when being restricted to the finite grid. Crucial in the orthogonality proof is the observation that the Koornwinder-Macdonald second order $q$-difference operator diagonalized by the polynomials is symmetric with respect to the discrete inner product (just as this observation turned out to be essential when dealing with a purely continuous or mixed continuous/discrete orthogonality measure \[ M1, K, S \]). The proof for the symmetry property of the $q$-difference operator (in the discrete context) is relegated to an appendix at the end of the paper (Appendix A). The orthonormalization constants are given in Section 4. Their computation, which is outlined in Appendix B, makes use of a recently introduced system of recurrence relations (or Pieri type formulas) for the multivariable Askey-Wilson polynomials \[ D2, D3 \]. In Section 5 we continue by discussing the transition from the basic hypergeometric level to the hypergeometric level ($q \to 1$). In this limit—which one might also interpret as a transition from trigonometric polynomials to rational counterparts obtained by sending the period of the trigonometric functions to infinity—our multivariable $q$-Racah type polynomials degenerate into multivariable Racah type polynomials. The paper is concluded in Section 6 with the characterization of a parameter domain for which the weights determining the orthogonality measure for the multivariable $q$-Racah polynomials become positive when $q$ lies on the unit circle, and a description of some properties of the finite-dimensional discrete integral transform for grid functions that is associated to the polynomials with parameters in this positivity domain.

\textit{Note.} Most objects of interest in this paper (such as the polynomials, weight functions, difference equations, normalization formulas, etc.) depend rationally on a number of parameters. Below it will always be assumed (unless explicitly stated otherwise) that the parameters are such that one stays away from singularities (this is the case generically), without repeatedly stressing this point each time.
We will be mainly concerned with the study of complex orthogonality properties of our polynomials. We shall say that a basis \( \{ p_i \} \) for a complex vector space \( \mathcal{H} \) endowed with a nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \) is orthogonal (with respect to \( \langle \cdot, \cdot \rangle \)) if \( \langle p_i, p_j \rangle = 0 \) for \( i \neq j \). Furthermore, an orthogonal basis of \( \mathcal{H} \) will be called orthonormal (with respect to \( \langle \cdot, \cdot \rangle \)) if in addition the normalization is such that the quantities \( \langle p_i, p_i \rangle \) are all equal to one.

2. Multivariable Askey-Wilson polynomials

In this section the definition of the multivariable Askey-Wilson polynomials as eigenfunctions of the second order Koornwinder-Macdonald \( q \)-difference operator is recalled. The approach followed here is very much in the spirit of Macdonald’s original treatment in [M1].

Let \( \mathcal{H} = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}] \) be the space of Laurent polynomials in the variables \( z_1, \ldots, z_n \). On this space the \( (BC) \) type Weyl group \( W = S_n \ltimes (\mathbb{Z}_2)^n \) (i.e. the semidirect product of the permutation group \( S_n \) and the \( n \)-fold product of the cyclic group \( \mathbb{Z}_2 \)) acts naturally by permutation and inversion \( (z_j \rightarrow z_j^{-1}) \) of the variables \( z_1, \ldots, z_n \). The subspace \( \mathcal{H}^W \) of \( \mathcal{H} \) consisting of the \( W \)-invariant (Laurent) polynomials is spanned by the symmetrized monomials

\[
m_\lambda(z) = \sum_{\mu \in W(\lambda)} z_1^{\mu_1} \cdots z_n^{\mu_n}, \quad \lambda \in \Lambda,
\]

where \( \Lambda(\cong W \setminus \mathbb{Z}^n) \) denotes the integral cone

\[
\Lambda = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}
\]

and the summation in (2.1) is meant over the orbit of \( \lambda \) under the action of the group \( W \) (which acts on vectors \( \lambda \in \mathbb{Z}^n \) by permuting and flipping the signs of the vector components \( \lambda_1, \ldots, \lambda_n \)). For future reference we will partially order the basis elements \( m_\lambda, \lambda \in \Lambda \), by defining for \( \mu, \lambda \in \Lambda \)

\[
\mu \leq \lambda \text{ iff } \sum_{1 \leq j \leq k} \mu_j \leq \sum_{1 \leq j \leq k} \lambda_j \text{ for } k = 1, \ldots, n
\]

(and \( \mu < \lambda \) if \( \mu \leq \lambda \) with \( \mu \neq \lambda \)).

In [K], Koornwinder introduced the following generalization of the second order \( (BC) \) type Macdonald \( q \)-difference operator

\[
D = \sum_{1 \leq j \leq n} \left( V_j(z) \left( T_{j,q} - 1 \right) + V_{-j}(z) \left( T_{j,q}^{-1} - 1 \right) \right),
\]

with

\[
V_{\varepsilon j}(z) = \frac{(1 - t_0 z_j)(1 - t_1 z_j)(1 - t_2 z_j)(1 - t_3 z_j)}{(1 - z_j^2)(1 - q z_j^2)} \times \prod_{1 \leq k \leq n, k \neq j} \frac{(1 - t z_j^\varepsilon z_k)(1 - t z_j^{\varepsilon -1} z_k^{-1})}{(1 - z_j^\varepsilon z_k)(1 - z_j^{\varepsilon -1} z_k^{-1})}, \quad \varepsilon = \pm 1,
\]

\[
(T_{j,q} f)(z) = f(z_1, \ldots, z_{j-1}, q z_j, z_{j+1}, \ldots, z_n),
\]
and showed that this operator is triangular with respect to the partially ordered basis of monomial symmetric functions:

\[(2.5) \quad D m_\lambda = \sum_{\mu \in \Lambda, \mu \leq \lambda} E_{\lambda,\mu} m_\mu \quad \text{with} \quad E_{\lambda,\mu} \in \mathbb{C}[q^{\pm 1}, t, t_0, t_1, t_2, t_3] \]

(i.e. the expansion coefficients (or matrix elements) \(E_{\lambda,\mu}\) depend polynomially on \(q^{\pm 1}, t\) and \(t_0, \ldots, t_3\)). The leading coefficient (or diagonal matrix element) \(E_{\lambda,\lambda}\) in (2.5) reads explicitly

\[(2.6) \quad E_{\lambda,\lambda} = \sum_{1 \leq j \leq n} \left( q^{-1} t_0 t_1 t_2 t_3 t^{2n-j-1}(q^\lambda - 1) + q^j (q^{-\lambda} - 1) \right). \]

The triangularity (2.5) of \(D\) implies that the eigenvalue problem for the \(q\)-difference operator in the space \(H^W\) is essentially finite-dimensional, because it can be reduced to the invariant subspaces of the form \(H^W_{\lambda} = \text{Span}\{m_\mu\}_{\mu \in \Lambda, \mu \leq \lambda}\) (with \(\lambda \in \Lambda\) (2.2)). The eigenvalues of \(D\) in \(H^W\) are given by the diagonal matrix elements \(E_{\lambda,\lambda}\) (2.6), \(\lambda \in \Lambda\). It is immediate from the explicit expression in (2.6) that \(E_{\mu,\mu} \neq E_{\lambda,\lambda}\) as a polynomial in the parameters \(q^{\pm 1}, t\) and \(t_0, \ldots, t_3\) if \(\mu \neq \lambda\). In other words, the eigenvalues are nondegenerate as (Laurent) polynomial functions of the parameters. The Koornwinder-Macdonald multivariable Askey-Wilson polynomials are now—by definition—the corresponding eigenfunctions \(p_\lambda, \lambda \in \Lambda\).

**Definition.** The **multivariable Askey-Wilson polynomial** associated with a (dominant weight) vector \(\lambda \in \Lambda\) (2.2) is the (unique) monic \(W\)-invariant Laurent polynomial of the form

\[(2.7a) \quad p_\lambda(z) = m_\lambda(z) + \sum_{\mu \in \Lambda, \mu < \lambda} c_{\lambda,\mu} m_\mu(z) \quad \text{with} \quad c_{\lambda,\mu} \in \mathbb{C}(q, t, t_0, t_1, t_2, t_3), \]

such that

\[(2.7b) \quad D p_\lambda = E_{\lambda,\lambda} p_\lambda. \]

Following Macdonald [M1] (see also [D2, D3, SK]) it is possible to write down a somewhat more constructive formula for the polynomial \(p_\lambda\) in terms of the monomial \(m_\lambda\), the operator \(D\) and the eigenvalues \(E_{\mu,\mu} (\mu \leq \lambda)\)

\[(2.8) \quad p_\lambda = \left( \prod_{\mu \in \Lambda, \mu < \lambda} \frac{D - E_{\mu,\mu}}{E_{\lambda,\lambda} - E_{\mu,\mu}} \right) m_\lambda. \]

(Notice that the r.h.s. is well-defined as a rational expression in the parameters in view of the fact that the denominators are nonzero as (Laurent) polynomials in the parameters.) The validity of this representation for the multivariable Askey-Wilson polynomials is easily verified by inferring that \(p_\lambda\) (2.8) satisfies the defining properties (2.7a) and (2.7b). That the r.h.s. of (2.8) is of the form in (2.7a) is clear from the triangularity of \(D\); that it also satisfies (2.7b) is immediate from the fact that the operator \(\prod_{\mu \in \Lambda, \mu < \lambda} (D - E_{\mu,\mu})\) annihilates the subspace \(H^W_{\lambda} = \text{Span}\{m_\mu\}_{\mu \in \Lambda, \mu \leq \lambda}\) as consequence of the Cayley-Hamilton theorem (and hence applying \((D - E_{\lambda,\lambda})\) to the r.h.s. of (2.8) yields zero, which is precisely (2.7b)).

In the one variable case, there is no \(t\)-dependence and the eigenvalue equation \(D p_\lambda = E_{\lambda,\lambda} p_\lambda\) amounts in that case to the second order \(q\)-difference equation for
the Askey-Wilson polynomials. Thus, the polynomials \( p_\lambda \) then reduce to (monic) Askey-Wilson polynomials \([\text{AW2, GR}]\)

\[
(2.9) \quad p_\lambda(z) = \frac{(t_0 t_1, t_0 t_2, t_0 t_3; q)_\lambda}{(t_0^3; (t_0 t_1 t_2 t_3 q^{\lambda-1}; q)_\lambda)} 4 \phi_3 \left( \begin{array}{c}
q^{-\lambda}, t_0 t_1 t_2 t_3 q^{\lambda-1}, t_0 z, t_0 z^{-1} \\
t_0 t_1, t_0 t_2, t_0 t_3
\end{array} ; q, q \right),
\]

with \( \lambda = 0, 1, 2, \ldots \) Here we have employed standard notation (see e.g. \([\text{AW2, GR}]\)) for the basic hypergeometric series

\[
s_1 \phi_s \left( \begin{array}{c} \frac{a_1, \ldots, a_s+1}{b_1, \ldots, b_s} : q, z \end{array} ; q, q \right) = \sum_{m=0}^\infty \frac{(a_1, \ldots, a_s+1)_m}{(b_1, \ldots, b_s)_m} \frac{z^m}{(q)_m}
\]

and the \( q \)-shifted factorials

\[
(a_1, \ldots, a_s; q)_m = (a_1; q)_m \cdots (a_s; q)_m, \quad (a; q)_m = (1 - a)(1 - aq) \cdots (1 - aq^{m-1})
\]

(with \( (a; q)_0 = 1 \)).

3. A Discrete orthogonality measure

It follows from \([3]\) that if the parameters satisfy the constraints

\[
(3.1) \quad 0 < q < 1, \quad -1 < t \leq 1, \quad |t_r| \leq 1, \quad r = 0, 1, 2, 3,
\]

with possible non-real parameters \( t_r \) occurring in complex conjugate pairs and pairwise products of the \( t_r \) not lying in the interval \([1, \infty)\), then the polynomials \( p_\lambda, \lambda \in \Lambda \) constitute an orthogonal system with respect to a continuous weight function \( \Delta_{\text{AW}}^{\lambda\mu}(x) \):

\[
(3.2) \quad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} p_\lambda(e^{ix}) p_\mu(e^{ix}) \Delta_{\text{AW}}^{\lambda\mu}(x) dx_1 \cdots dx_n = 0 \quad \text{if} \quad \lambda \neq \mu
\]

where

\[
(3.3) \quad \Delta_{\text{AW}}^{\lambda\mu}(x) = \prod_{1 \leq j, k \leq n} \frac{(e^{i(x_j + x_k)}; q)_\infty}{(e^{i(x_j + x_k)}; q)^{x_j + x_k}} \prod_{1 \leq j, k \leq n} \frac{(e^{2ix_j}; q)_\infty}{(e^{2ix_j}; q)^{x_j + x_k}}.
\]

Here we have used the notation \( e^{ix} \equiv (e^{ix_1}, \ldots, e^{ix_n}) \) and (cf. above)

\[
(a_1, \ldots, a_r; q)_\infty = (a_1; q)_\infty \cdots (a_r; q)_\infty, \quad (a; q)_\infty = \prod_{m=0}^\infty (1 - aq^m).
\]

It should be noted that at this point of our presentation it is not even obvious that the multivariable Askey-Wilson polynomials defined in the previous section actually exist for all parameter values in the above domain, since the domain might a priori contain parameter values for which the coefficients \( c_{\lambda, \mu} \) \( [2,73] \) have a singularity. However, that such singularities indeed do not occur is seen from an alternative characterization of the multivariable Askey-Wilson polynomial \( p_\lambda \) as the polynomial of the form

\[
(3.4a) \quad p_\lambda(z) = m_\lambda(z) + \sum_{\mu < \lambda, \mu \in \Lambda} c_{\lambda, \mu} m_\mu(z) \quad \text{with} \quad c_{\lambda, \mu} \in \mathbb{C},
\]
satisfying
\begin{equation}
\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} p_\lambda(e^{ix}) m_\mu(e^{ix}) \Delta^{AW}(x) \, dx_1 \cdots dx_n = 0 \quad \text{for} \quad \mu < \lambda.
\end{equation}

(In \cite{K} these two properties were in fact used to define the multivariable Askey-Wilson polynomials.) It is clear that the polynomials determined by the conditions (3.4a), (3.4b) are well-defined for all parameter values in the domain given by (3.1) and, furthermore, that they are continuous in $t$ and $t_0, \ldots, t_3$ for these parameter values. By showing (as was done in \cite{K}) that the difference operator $D$ (2.4) is symmetric with respect to the $L^2$ inner product with weight function $\Delta^{AW}$, i.e.,
\begin{equation}
\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (D^m_{\lambda})(e^{ix}) m_\mu(e^{ix}) \Delta^{AW}(x) \, dx_1 \cdots dx_n = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} m_\lambda(e^{ix}) (D^m_{\mu})(e^{ix}) \Delta^{AW}(x) \, dx_1 \cdots dx_n
\end{equation}
and combining this with the triangularity property in (2.3), one finds that the polynomial $p_\lambda(z)$ of the form (3.4a), (3.4b) satisfies the eigenvalue equation $D p_\lambda = E_{\lambda,\lambda} p_\lambda$, which shows that for the parameter domain of interest this alternative characterization leads us to the same polynomial $p_\lambda$ as in the previous section (and hence that the coefficients $c_{\lambda,\mu}$ are regular for these parameter values). The orthogonality (3.2) of the multivariable Askey-Wilson polynomials $p_\lambda, \lambda \in \Lambda$ with respect to the weight function $\Delta^{AW}$ (3.3) now follows, first for generic parameters in the above domain from the fact that they are eigenfunctions of a symmetric operator $D$ corresponding to different (real) eigenvalues, and then for all parameters in this domain by a continuity argument \cite{K}.

Very recently, it was shown that the parameter domain for which the multivariable Askey-Wilson polynomials admit an interpretation as orthogonal polynomials may be further extended. Specifically, it follows from \cite{S} that for $t = q^m$, with $m$ an arbitrary nonnegative integer, one can remove the constraints $|t_r| \leq 1$ in (3.1) (while still keeping the other restrictions on these parameters though) to end up with an orthogonality relation for the polynomials $p_\lambda, \lambda \in \Lambda$ consisting of a continuous part of the form (3.2) and additional mixed continuous, mixed discrete parts.

The main purpose of the present section is to demonstrate that for parameters satisfying a truncation condition the multivariable Askey-Wilson polynomials give rise to a finite-dimensional orthogonal system with a purely discrete weight function living on a finite grid.

First it is needed to introduce some notation. Let $N$ be an arbitrary nonnegative integer and let $\Lambda_N$ be the alcove of dominant weight vectors of the form
\begin{equation}
\Lambda_N = \{ \lambda \in \Lambda \mid \lambda \leq N\omega \} \quad \text{with} \quad \omega \equiv e_1 + \cdots + e_n.
\end{equation}
(Here and below $e_j$ represents the $j$th unit vector in the standard basis of $\mathbb{R}^n$.) We will show that for parameters subject to a truncation condition of the type
\begin{equation}
t_a t_b t_n^{n-1} = q^{-N},
\end{equation}
with $a, b \in \{0, 1, 2, 3\}$ (fixed) and $b \neq a$, the multivariable Askey-Wilson polynomials $p_\lambda, \lambda \in \Lambda_N$ form a finite-dimensional orthogonal system with respect to a
(generally complex) discrete measure with support on the grid points
\[ \tau q^\nu = (\tau_1 q^{\nu_1}, \ldots, \tau_n q^{\nu_n}), \quad \nu \in \Lambda_N \]
where
\[ \tau_j = t^{n-j}t_a, \quad j = 1, \ldots, n. \]
Specifically, the (complex) orthogonality relation becomes for these parameters (i.e. satisfying condition (3.6))
\[ \sum_{\nu \in \Lambda_N} p_\lambda(\tau q^\nu) p_\mu(\tau q^\nu) \Delta^q_R(\nu) = 0 \quad \text{for} \quad \lambda \neq \mu \quad (\lambda, \mu \in \Lambda_N), \]
where the weights \( \Delta^q_R(\nu), \nu \in \Lambda_N \) are given explicitly by
\[ \Delta^q_R(\nu) = \frac{1}{C^q_R(\nu) C^q_R(\nu)}, \]
with
\[ C^q_R(\nu) = c_0(\nu) \prod_{1 \leq j < k \leq n} \left( \frac{(\tau_j \tau_k; q)_{\nu_j+\nu_k} (\tau_j \tau_k^{-1}; q)_{\nu_j-\nu_k}}{(\tau_j \tau_k; q)_{\nu_j} (\tau_j \tau_k^{-1}; q)_{\nu_j-\nu_k}} \right) \]
\[ \times \prod_{1 \leq j \leq n} \frac{(\tau_j^2; q)_{2\nu_j}}{\prod_{0 \leq r \leq 3} (t_r \tau_j; q)_{\nu_j}}, \]
\[ C^q_R(\nu) = c_0(\nu) \prod_{1 \leq j < k \leq n} \left( \frac{(t^{-1} q^2 \tau_j \tau_k; q)_{\nu_j+\nu_k} (t^{-1} q^2 \tau_j \tau_k^{-1}; q)_{\nu_j-\nu_k}}{(q^2 \tau_j \tau_k; q)_{\nu_j+\nu_k} (q^2 \tau_j \tau_k^{-1}; q)_{\nu_j-\nu_k}} \right) \]
\[ \times \prod_{1 \leq j \leq n} \frac{\prod_{0 \leq r \leq 3} (t_r^{-1} q \tau_j; q)_{\nu_j}}{(q \tau_j^2; q)_{2\nu_j}}, \]
and
\[ c_0(\nu) = \prod_{1 \leq j \leq n} \left( t^{n-j}(t_0 t_1 t_2 t_3 q^{-1})^{1/2} \right)^{\nu_j}. \]

Here we have introduced discretized functions \( C^q_R(\nu) \) that are reminiscent (upon dualization) of the c-functions of Harish-Chandra in harmonic analysis (see e.g. [He, HS]).

Lemma 3.1. a. For \( \nu \in \Lambda \) (3.3) the functions \( C^q_R(\nu) \) (3.11), (3.12) are well-defined and nonzero as meromorphic expressions in the parameters \( t, t_0, \ldots, t_3 \) and \( q \).

b. For \( \nu \in \Lambda_N \) (3.5) the functions \( C^q_R(\nu) \) (3.11), (3.12) are well-defined and nonzero as meromorphic expressions in the parameters \( t, t_0, \ldots, t_3 \) and \( q \) subject to the truncation condition (3.6).

Here and below when stating that an expression is meromorphic or rational in the parameters \( t, t_0, \ldots, t_3 \) and \( q \) subject to the truncation condition (3.6), it is meant that after elimination of \( t_b \) (or \( t_a \)) with the aid of the relation \( t_a t_b = t^{1-n}q^{-N} \), the resulting expression is meromorphic/rational in the remaining parameters \( t_a \) (or \( t_b \), \( t_c, t_d, t \) and \( q \) (where \( t_c \) and \( t_d \) denote the two parameters complementing \( t_a \) and \( t_b \) in \( \{t_0, t_1, t_2, t_3\} \) such that \( \{t_a, t_b, t_c, t_d\} = \{t_0, t_1, t_2, t_3\} \)).

The proof of this lemma is immediate from inspection of the explicit expressions for \( C^q_R(\nu) \) given above (no factor in the numerator or the denominator of \( C^q_R(\nu) \)
becomes identical to zero). Observe also that the functions $C_{\pm}^R(\nu)$ are rational in the parameters except for the square roots appearing in the common factor $c_0(\nu)$ (3.13). In the discretized weight function $\Delta^R(\nu)$ (3.14) these square roots can be collected (there is an even number of them) and rationality becomes restored.

Notice also that for parameters subject to the truncation condition $C^R_{\pm}(\nu)$ (3.11) would be infinite if $\nu \notin \Lambda \backslash \Lambda_N$ (i.e., when $\nu$ lies in the cone $\Lambda$ (2.2) but outside the alcove $\Lambda_N$ (3.3)). In that case we have that $\nu > N$ and hence the factor $(t_0t_1; q)_\nu = (t_0t_0^{1-L}; q)_\nu$ in the denominator of $C^R_{\pm}(\nu)$ (3.11) is identical to zero when the truncation condition (3.6) holds. As a result, the weight function $\Delta^R(\nu)$ (3.10) vanishes on $\Lambda \backslash \Lambda_N$ for parameters subject to the truncation condition.

The orthogonality relation (3.9) should be read as a set of identities for the multivariable Askey-Wilson polynomials $p_{\lambda}(z)$, $\lambda \in \Lambda_N$ with parameters subject to the truncation condition (3.6). It is clear from Macdonald’s representation in (2.8) and the explicit formula for the eigenvalues $E_{\lambda, \nu}$ (2.6) that the polynomials $p_{\lambda}(z)$ with this restriction on the parameters are well-defined as rational expressions in the parameters except for the square roots appearing in the common factor of the monic eigenfunctions of the difference operator $D$ (2.4). Here instead of treating each parameter subject to a condition of the type $t_0t_0 = q^{-N}$ (in the l.h.s. of (3.9) make sense as rational expressions in the parameters $q, t$ and $t_0, t_1, t_2, t_3$ subject to condition (3.6).

In the case of one single variable ($n = 1$) the orthogonality relation (3.9) reduces to the well-known orthogonality relation

$$\sum_{0 \leq \nu \leq N} p_{\lambda}(t_0q^\nu)p_{\mu}(t_0q^\nu)\Delta^R(\nu) = 0$$

for $\lambda \neq \mu$ ($\lambda, \mu \in \{0, \ldots, N\}$),

with

$$\Delta^R(\nu) = \frac{(1 - t_0^2q^{2\nu})}{(t_0t_1t_2t_3q^{-1})(1 - t_0^2)(t_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}qt_0; q)_\nu}$$

for the (monic) Askey-Wilson polynomials given by $p_{\lambda}(z)$ (2.9) with parameters subject to a condition of the type $t_0t_0 = q^{-N}$ ($b \neq a$) [AW1].

In the discrete case, with parameters subject to the truncation condition, these one-variable polynomials are usually referred to in the literature as $q$-Racah polynomials rather than Askey-Wilson polynomials.

In the above formulas (and throughout the paper) $t_a$ and $t_b$ denote two fixed, distinct, but otherwise arbitrary parameters from the set $\{t_0, t_1, t_2, t_3\}$. Since the monic (multivariable) Askey-Wilson polynomials $p_{\lambda}(z)$ are symmetric with respect to permutations of the parameters $t_0, \ldots, t_3$ (this is immediate from their definition as monic eigenfunctions of the difference operator $D$ (2.4)), the actual choices of $t_a$ and $t_b$ are all equivalent up to permutation of the parameters $t_0, t_1, t_2, t_3$. Notice, however, that the choice of $t_a$ does influence the position of the grid points $t_0q^\nu$ (2.7).

In the case of one single variable one usually works with a different normalization of the $q$-Racah polynomials that breaks the permutation symmetry and designates a preferred role to the parameter $t_0$. (Instead of employing the monic polynomials of (2.9) one then omits the constant factor in front of the terminating $q\phi_3$ series.) It is in that case custom to take $t_a$ to be $t_0$ (thus fixing the grid points to $t_0q^{\nu}$, $\nu = 0, \ldots, N$). This then leaves three possible choices for the other parameter $t_b$ entering the truncation condition $t_0t_b = q^{-N}$, which leads one to distinguish the three cases $t_0t_1 = q^{-N}$, $t_0t_2 = q^{-N}$ and $t_0t_3 = q^{-N}$. Here instead of treating each
of these cases separately we prefer not to make such distinctions and have tried rather to emphasize the symmetry in the parameters \( t_0, \ldots, t_3 \) with our notation.

In order to prove the discrete orthogonality relations (3.9) and analyze their properties in more detail we will need some further notation. Specifically, let us introduce a bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \) on the finite-dimensional subspace (of \( H^N \))

\[
\mathcal{H}_{qR}^N = \text{Span}\{m_{\lambda}\}_{\lambda \in \Lambda_N}
\]

determined by

\[
(f, g)_{qR}^N = \sum_{\nu \in \Lambda_N} f(\tau_q\nu)g(\tau_q\nu)\Delta_{qR}(\nu) \quad (f, g \in \mathcal{H}_{qR}^N).
\]

The orthogonality of the multivariable Askey-Wilson (or \( q \)-Racah) polynomials with respect to the discrete weight function \( \Delta_{qR} \) can now be phrased in terms of the following theorem.

**Theorem 3.2.** For parameters \( q, t \) and \( t_0, \ldots, t_3 \) subject to the truncation condition

\[
t_at^n-1 = q^{-N} \quad b \neq a
\]

(where \( N \) denotes an arbitrary nonnegative integer and \( a, b \) are two fixed, distinct but otherwise arbitrary numbers from the set \( \{0, 1, 2, 3\} \)), the multivariable Askey-Wilson polynomials \( p_{\lambda}(z) \), \( \lambda \in \Lambda_N \) (3.5) satisfy the orthogonality relation

\[
\langle p_{\lambda}, p_{\mu} \rangle_{qR}^N = 0 \quad \text{for} \quad \lambda \neq \mu \quad (\lambda, \mu \in \Lambda_N),
\]

with the bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \) being defined by (3.17).

The proof of the orthogonality theorem operates along the same lines as the orthogonality proof in [K] for the continuous case (cf. the beginning of this section) and is based on the statement that (for parameters satisfying the truncation condition) the difference operator \( D \) (2.4) is symmetric with respect to the bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \).

**Proposition 3.3.** For parameters in accordance with the assumptions in Theorem 3.2, the difference operator \( D \) (2.4) is symmetric with respect to the bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \) (3.17), i.e.

\[
\langle Df, g \rangle_{qR}^N = \langle f, Dg \rangle_{qR}^N \quad (f, g \in H_{qR}^N).
\]

The proof of this proposition has, in turn, been relegated to Appendix A at the end of the paper. The orthogonality (3.18) now follows from the fact that the polynomials \( p_{\lambda}, \lambda \in \Lambda_N \) are eigenfunctions of a symmetric operator \( D \) corresponding to eigenvalues \( E_{\lambda, \lambda} \) (2.4), \( \lambda \in \Lambda_N \) that are nondegenerate (i.e. distinct for different \( \lambda \)) as rational functions in the parameters \( q, t \) and \( t_0, \ldots, t_3 \) subject to condition (3.6).

An important property of the bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \) is that for generic parameters (subject to the truncation condition) it is nondegenerate on the space \( \mathcal{H}_{qR}^N \) (3.16) (i.e., if for a certain \( f \in \mathcal{H}^R \) one has that \( \langle f, g \rangle_{qR}^N = 0 \) for all \( g \in \mathcal{H}_{qR}^N \), then \( f \) must be zero).

**Proposition 3.4.** The bilinear form \( \langle \cdot, \cdot \rangle_{qR}^N \) (3.17) is nondegenerate on the space \( \mathcal{H}_{qR}^N \) (3.16) for generic parameters \( q, t \) and \( t_0, \ldots, t_3 \) (subject to condition (3.6)).
The proof of Proposition 3.4 readily follows after recalling that the weights \( \Delta^q R(\nu), \nu \in \Lambda_N \) are nonzero as rational expressions in the parameters subject to the truncation condition (no weight becomes identical to zero) and combining this with the following lemma.

**Lemma 3.5.** For generic \( q, t \) and \( t_a \) every function in the space \( H^q R_N \) is uniquely characterized by its values on the grid points \( \tau q^\lambda, \lambda \in \Lambda_N \).

To prove Lemma 3.5 one uses that the determinant of the matrix

\[
M = [m_{\mu}(\tau q^\nu)]_{\mu, \nu \in \Lambda_N}
\]

(where the columns and rows are ordered by means of some (any) total order of the weight vectors in the cone \( \Lambda \) extending the partial order (2.3)) is nonzero as a Laurent polynomial in \( q \) for generic \( t \) and \( t_a \) (this follows from the dominant behavior of the determinant for \( q \to +\infty \), see [D2, Lemma 4]). Hence, if \( \sum_{\mu \in \Lambda_N} c_{\mu} m_{\mu}(\tau q^\nu) = 0 \) for all \( \nu \in \Lambda_N \), then the coefficients \( c_{\mu} \) must all be zero.

Since the dimension of the space of all functions over the grid (3.7) is equal to the number of points in the grid, which is for generic \( q, t \) and \( t_a \) equal to the number of points in the alcove \( \Lambda_N \) (3.5) and hence identical to the dimension of the space \( H^q R_N \) (3.16), Lemma 3.5 actually says—when read in an appropriate way—that for generic \( q, t \) and \( t_a \) any complex function defined on the grid points \( \tau q^\lambda, \lambda \in \Lambda_N \) may be identified with the restriction of a function in the space \( H^q R_N \) to the grid. In other words, for such generic \( q, t \) and \( t_a \) every function on the grid can be approximated exactly by (the restriction to the grid of) a unique function in \( H^q R_N \).

**Proposition 3.6.** For generic \( q, t \) and \( t_a \) any complex function over the grid with points \( \tau q^\nu, \nu \in \Lambda_N \) can be represented exactly by the restriction to the grid points of a unique function in the space \( H^q R_N \).

In principle there is no real reason why the definition of the bilinear form \( \langle \cdot, \cdot \rangle^q R_N \) should be confined to the subspace \( H^q R_N \) and could not, e.g., be extended to the whole space \( H^W \) (which consist of all \( W \)-invariant Laurent polynomials in \( n \) variables). In fact, this is why the condition \( f, g \in H^q R_N \) in (3.17) has been put between parenthesis. It turns out that Theorem 3.2 and Proposition 3.3 remain valid (with the given proofs applying verbatim) if one allows \( \lambda \) and \( \mu \) to be any dominant weight vector from the cone \( \Lambda \) and \( f \) and \( g \) to be any function in the space \( H^W \), respectively. However, since such a generalization of Theorem 3.2 would imply that \( p_\lambda \) is orthogonal to \( H^q R_N \) if \( \lambda \) is not in \( \Lambda_N \), it follows from combination of Proposition 3.4 and Proposition 3.6 that in that case \( p_\lambda \) must actually be zero on the grid.

**Proposition 3.7.** For parameters subject to the truncation condition (3.6) one has that

\[
p_\lambda(\tau q^\nu) = 0 \quad \text{for } \nu \in \Lambda_N
\]

if \( \lambda \in \Lambda \setminus \Lambda_N \).

The upshot of Proposition 3.7 is that the multivariable Askey-Wilson polynomials \( p_\lambda(z), \lambda \in \Lambda \) with parameters subject to the truncation condition (3.6) descend to a finite-dimensional orthogonal system spanned by \( p_\lambda(z), \lambda \in \Lambda_N \) when being restricted to the grid points \( \tau q^\nu, \nu \in \Lambda_N \). Furthermore, the results of this section
can now be alternatively summarized in the following way: for generic \(q\) and parameters subject to the truncation condition, the restriction of the multivariable Askey-Wilson polynomials \(p_\lambda(z)\), \(\lambda \in \Lambda_N\) to the grid points \(\tau q\nu\), \(\nu \in \Lambda_N\) yields an orthogonal basis for the (finite-dimensional) space of functions over this grid endowed with the bilinear form \(\sum_{\nu \in \Lambda_N} f(\tau q\nu)g(\tau q\nu)\Delta^{qR}(\nu)\).

**Remark 3.1.** It is convenient to think of the discrete weight function determined by the weights \(\Delta^{qR}(\nu)\) (3.10) as arising from a continuous weight function \(\Delta^{qR}(z)\) restricted to the grid points \(\tau q\nu\) (3.7) (such that \(\Delta^{qR}(\tau q\nu) = \Delta^{qR}(\nu)\)). In principle such a function \(\Delta^{qR}(z)\) of course exists and is a priori highly non-unique. For \(0 < |q| < 1\) a possibility would be to take

\[
\Delta^{qR}(z) = D_0 \prod_{1 \leq j \leq n} (t^{2(n-j)}t_0t_1t_2t_3q^{-1})^{-\log(z_j)/\log(q)} \\
\times \left(1 - z_jz_k \right)(1 - z_jz_k^{-1}) \left( \frac{(t^{-1}qz_jz_k,t_0^{-1}qz_jz_k^{-1};q)_\infty}{(tz_jz_k,t_0z_jz_k^{-1};q)_\infty} \right) \\
\times \left(1 - z_j \right) \left( \frac{(t_0^{-1}qz_j,t_1^{-1}qz_j,t_2^{-1}qz_j,t_3^{-1}qz_j;q)_\infty}{(t_0z_j,t_1z_j,t_2z_j,t_3z_j;q)_\infty} \right),
\]

(3.22)

where \(D_0\) denotes a normalization constant determined by the requirement that evaluation of \(\Delta^{qR}(z)\) in \(z = \tau \equiv (\tau_1, \ldots, \tau_j, \ldots, \tau_n)\) 

\text{i.e.}, \(z = \tau q\nu\) (3.7) with \(\nu = 0\) should yield the value one (since \(\Delta^{qR}(0) = 1\)).

**Remark 3.2.** It is instructive to view the grid points (3.7) on which the discrete masses of the orthogonality measure determined by \(\Delta^{qR}\) are positioned as being of the form \(q^{e+n}, \nu \in \Lambda_N\), where \(\rho\) denotes an \(n\)-dimensional vector that is related to \(\tau\) (3.3) by a logarithm

\[
\rho = (\rho_1, \ldots, \rho_n) \quad \text{with} \quad \rho_j = \log(\tau_j)/\log(q), \quad j = 1, \ldots, n.
\]

Thus, up to an exponential scaling \((z \rightarrow q^z)\) the grid consists of the points in the alcove \(\Lambda_N\) (3.3) translated over the vector \(\rho\).

## 4. Orthonormalization

In this section we will present the normalization constants turning the multivariable Askey-Wilson polynomials \(p_\lambda(z)\), \(\lambda \in \Lambda_N\) (3.4) —with parameters subject to the truncation condition (3.6)—into an orthonormal basis for the space \(H^{qR}_N\) (3.10). To this end it is needed to compute the quantities \(\langle p_\lambda, p_\lambda \rangle^{qR}_N\), \(\lambda \in \Lambda_N\). The evaluation of these sums pivots on a previously introduced system of Pieri type recurrence formulas for the multivariable Askey-Wilson polynomials (D2, D3). At the time of their introduction the same formulas were also used to verify the explicit expressions (conjectured by Macdonald) for the squared norm of \(p_\lambda(z)\) with respect to Koornwinder’s inner product (3.2). In both cases (i.e. the discrete and the continuous case) the mechanism leading to the solution of the orthonormalization problem is very similar. Hence, we will refrain from providing full details here and refer the reader to Appendix B where the main ingredients of the recipe are outlined with an emphasis on those points at which the discrete case differs from the continuous case.

We would like to add that exactly the same approach may also serve to obtain the orthonormalization constants (in terms of the norm of the unit polynomial) for...
the parameter regime with the mixed continuous/discrete orthogonality measure considered in \cite{3}. The answer will in that case be formally identical to that in the case of Koornwinder’s purely continuous orthogonality measure $\Delta^{AW}$ \cite{3}, except that the parameter domain now gets extended in the way indicated at the beginning of the previous section.

In order to describe the evaluation formula for $(p_\lambda, p_\lambda)^{qr}_N$, it is convenient to introduce dual parameters $\hat{t}_r$, $r = 0, \ldots, 3$ that are related to the parameters $t_r$, $r = 0, \ldots, 3$ in the following way

$$
\begin{align*}
\hat{t}_a &= (t_at_bt_cq^{-1})^{1/2}, \\
\hat{t}_b &= (t_at_b^{-1}q^{-1})^{1/2}, \\
\hat{t}_c &= (t_at_b^{-1}t_cq^{-1})^{1/2}, \\
\hat{t}_d &= (t_at_b^{-1}t_c^{-1}t_dq)^{1/2},
\end{align*}
$$

where $c$ and $d$ denote the two indices that complement the indices $a$ and $b$ entering the truncation condition (3.6) such that $\{a, b, c, d\} = \{0, 1, 2, 3\}$ (cf. also the comment just after Lemma \cite{3}). It is worthwhile noticing that the parameter transformation defining the dual parameters in \cite{4} is an involution (the duals of $t_r$ bring us back to $t_r$, $r = 0, \ldots, 3$) and, furthermore, that the dual parameters $\hat{t}_r$ satisfy the truncation condition (3.6) if the parameters $t_r$ do so (because $\hat{t}_at_b = t_at_b$).

We now form discrete Harish-Chandra-like $c$-functions $\hat{C}_c^{qr}$ that are dual to the $c$-functions $C_c^{qr}$ \cite{3, 12} (i.e., $\hat{C}_c^{qr}$ is obtained from $C_c^{qr}$ by replacing the parameters $t_0, \ldots, t_3$ by the dual parameters $\hat{t}_0, \ldots, \hat{t}_3$)

$$
\begin{align*}
\hat{C}_c^{qr}(\lambda) &= \hat{c}_0(\lambda) \prod_{1 \leq j < k \leq n} \frac{(\hat{\tau}_j \hat{\tau}_k; q)_\lambda + \lambda_k}{(\hat{t}\tau_j \hat{\tau}_k; q)_\lambda + \lambda_k} \frac{(\hat{\tau}_j \hat{\tau}_k^{-1}; q)_\lambda - \lambda_k}{(\hat{t}\tau_j \hat{\tau}_k^{-1}; q)_\lambda - \lambda_k} \times \prod_{1 \leq j \leq n} \left(\prod_{0 \leq r < s \leq 3}(t_r \tau_j; q)_{\lambda_j}\right) \\
\hat{C}_c^{qr}(\lambda) &= \hat{c}_0(\lambda) \prod_{1 \leq j < k \leq n} \frac{(t^{-1}q\tau_j \hat{\tau}_k; q)_{\lambda_j} + \lambda_k}{(q\tau_j \hat{\tau}_k; q)_{\lambda_j} + \lambda_k} \frac{(t^{-1}q\tau_j \hat{\tau}_k^{-1}; q)_{\lambda_j} - \lambda_k}{(q\tau_j \hat{\tau}_k^{-1}; q)_{\lambda_j} - \lambda_k} \times \prod_{1 \leq j \leq n} \left(\prod_{0 \leq r < s \leq 3}(t_r^{-1}q\tau_j; q)_{\lambda_j}\right)
\end{align*}
$$

with

$$
\hat{c}_0(\lambda) = \prod_{1 \leq j \leq n} \left(t^{n-j}(t_0 \hat{t}_1 \hat{t}_2 q^{-1})^{1/2}\right)^{\lambda_j}
$$

and (cf. \cite{3, 12})

$$
\hat{\tau}_j = t^{n-j} \hat{t}_a = t^n \hat{t}_a t_1 t_2 t_3 q^{-1/2}, \quad j = 1, \ldots, n.
$$

It is important to convince oneself that, despite the appearances of square roots in the definitions of $t_r$ \cite{3} and $\tau_j$ \cite{3, 4}, the functions $\hat{C}_c^{qr}(\lambda)$ \cite{12} (including the common factor $\hat{c}_0$ \cite{14}) are rational in the parameters $t, t_0, \ldots, t_3$ and $q$. The point is that in the above expressions the quantities $t_r$ and $\tau_j$ always occur in rational combinations of the form $\hat{\tau}_j \hat{\tau}_k^{\pm 1}, \hat{\tau}_j \hat{\tau}_k^{\pm 2}$ and that $(t_at_b q^{-1})^{1/2} = (t_at_b t_c \hat{\tau}_d q^{-1})^{1/2} = t_a$. In this sense the square roots in $\hat{t}_r$ \cite{11} have merely a formal meaning and these dual parameters were introduced mostly for notational
convenience and to emphasize the duality between the expressions for the above c-functions $C^R_{\pm}$ and the c-functions $C^R_q$ appearing in the previous section.

Our main object of interest in this section will be a function $N^q_R(\lambda)$ on the cone $\Lambda$ (2.2) defined by

\begin{equation}
N^q_R(\lambda) = \frac{C^q_R(\lambda)}{C^q_q(\lambda)}.
\end{equation}

It is clear from (the dual version of) Lemma 3.1 (part a.) and the above comments that $C^q_R(\lambda)$ and hence $N^q_R(\lambda)$ are well-defined nonzero rational expressions in the parameters $t$, $t_0$, $t_1$, $t_3$ and $q$ for all $\lambda \in \Lambda$. Furthermore, since $t_0 \cdot t_1 = t_3$, the truncation condition (3.6) reads the same in the dual parameters $t_r$ as in the original parameters $t_r$. Thus, we may apply again (the dual version) of Lemma 3.1 (part b.) to conclude that for $\lambda \in \Lambda_N$ these expressions are well-defined and nonzero as rational expressions in the parameters $t$, $t_0$, $t_1$, $t_3$ and $q$ subject to the truncation condition (3.6).

The following theorem provides a summation formula expressing $\langle p_\lambda, p_\lambda \rangle^q_R$ in terms of $(1,1)^q_R$ (which corresponds to $\lambda = 0$).

**Theorem 4.1.** For parameters subject to the truncation condition (3.6) one has that

\begin{equation}
\langle p_\lambda, p_\lambda \rangle^q_R = N^q_R(\lambda) (1,1)^q_R \quad (\lambda \in \Lambda_N),
\end{equation}

with $N^q_R(\lambda)$ given by (4.6) and

\begin{equation}
(1,1)^q_R = \sum_{\nu \in \Lambda_N} \Delta^q(\nu)
\end{equation}

(where $\langle \cdot, \cdot \rangle^q_R$ and $\Delta^q(\nu)$ are defined by (3.17) and (3.10), respectively).

For an outline of the proof and further details regarding the Pieri type recurrence formulas lying at the basis of this proof the reader is referred to Appendix B.

It is clear that the summation formula (4.7) should be thought of as a set of identities for the multivariable Askey-Wilson polynomials with parameters subject to the truncation condition (3.6), which complements the orthogonality identities in (3.18). For $n = 1$ the formula reduces to the known (cf. [AW1, GR]) summation formula

\begin{equation}
\sum_{0 \leq \nu \leq N} \left(p_\lambda(t, q^\nu)\right)^2 \Delta^q(\nu) = \frac{(q^2; q^2)_\lambda \prod_{0 \leq r < s \leq 3} (t_r, t_s; q^\lambda-1)}{(t_0, t_1, t_2, t_3, q^{\lambda-1}, q, q^{\lambda})} (1,1)^q_R
\end{equation}

$(\lambda = 0, \ldots, N)$, with $\Delta^q(\nu)$ taken from (3.15) and

\begin{equation}
(1,1)^q_R = \sum_{0 \leq \nu \leq N} \Delta^q(\nu),
\end{equation}

for the monic $q$-Racah polynomial given by $p_\lambda(z)$ (2.9) with parameters subject to the condition $t_0 \cdot t_1 = q^{-N}$.

Although in principle the expression in (4.7) is in itself already explicit as a finite sum of explicitly given terms (this in contrast to the sum $\langle p_\lambda, p_\lambda \rangle^q_R$ with $\lambda \not= 0$, for which in general one does not know the terms in a very explicit form unless $n = 1$), it would be interesting to also evaluate the sum $(1,1)^q_R$ in terms of a product formula. For $n=1$ this was done by Askey and Wilson in [AW1] by means
of a \(q\)-Dougall summation formula for a very-well-poised (terminating) \(6\phi_5\) series (see also \(\text{GR}\)), entailing

\[
\sum_{0 \leq \nu \leq N} \Delta^q^{1R}(\nu) = \frac{(t_0 t_1 t_2 t_3, t_a^{-1} t_b; q)_N}{(t_b t_c, t_b t_d; q)_N} = \frac{(t_a^2 q, t_c^{-1} t_d^{-1} q; q)_N}{(t_a t_c^{-1} q, t_a t_d^{-1} q; q)_N}
\]

(where \(\Delta^q^{1R}(\nu)\) is given by (3.15) and \(t_a t_b = q^{-N}\)). For arbitrary number of variables summing \((1,1)^{1R}_N\) amounts to the evaluation of a finite Selberg type \(q\)-Jackson integral (related to the root system \(BC_n\)). Similar (but infinite) Selberg type \(q\)-Jackson integrals appear in the recent works of Aomoto and Ito \(\text{AI}\). More specifically, the \(q\)-Jackson integral corresponding to \((1,1)^{1R}_N\) constitutes a finite analogue of an infinite \(q\)-Jackson Selberg integral related to the root system \(BC_n\) that belongs to the same class as those studied in \(\text{AI}\). The connection with the \(q\)-Jackson integrals considered by Aomoto and Ito becomes particularly transparent after recalling (see Remark 3.1) that for \(0 < |q| < 1\) the weights \(\Delta^q^{1R}(\nu)\) may be thought of as the restriction of a function \(\Delta^q(z)\) of the form in (3.22) to the grid points \(\tau q^\nu, \nu \in \Lambda_N\).

We see from Theorem 4.1 that the quantities \(\langle p_\lambda, p_\nu \rangle_N^{1R}\), \(\lambda \in \Lambda_N\) are nonzero rational expressions in the parameters \(t, t_0, \ldots, t_3\) and \(q\) (subject the truncation condition (3.11)) because the factors \(\Lambda^q^{1R}(\nu)\) are nonzero. (Clearly the common factor \((1,1)_N^{1R}\) in (1.7) is nonzero because otherwise the bilinear form \(\langle \cdot, \cdot \rangle_N^{1R}\) would vanish identically on the space \(\mathcal{H}^{1R}\).) This checks with Proposition 3.4, which stated that for generic parameters (subject to the truncation condition) the bilinear form \(\langle \cdot, \cdot \rangle_N^{1R}\) is nondegenerate on the space \(\mathcal{H}^{1R}_N\).

The orthonormalization constants are given by square roots of \(\langle p_\lambda, p_\nu \rangle_N^{1R}\), \(\lambda \in \Lambda_N\). After division by these constants the basis \(p_\lambda, \lambda \in \Lambda_N\) becomes an orthonormal basis for the space \(\mathcal{H}^{1R}_N\) with respect to the bilinear form \(\langle \cdot, \cdot \rangle_N^{1R}\). The fact that (generically) the orthogonal basis \(p_\lambda, \lambda \in \Lambda_N\) can be turned into an orthonormal basis for \(\mathcal{H}^{1R}_N\) (or, equivalently, that the quantities \(\langle p_\lambda, p_\lambda \rangle^{1R}_N, \lambda \in \Lambda_N\) are nonzero) implies that (generically) no linear dependences arise between the functions \(p_\lambda, \lambda \in \Lambda_N\) when being restricted to the grid \(\tau q^\nu, \nu \in \Lambda_N\). This is of course precisely what we already saw in Lemma 3.2 and what—because of dimensional considerations—implied that (generically) any function over the grid could be represented exactly by a unique function in \(\mathcal{H}^{1R}_N\) (Proposition 3.6).

We also saw already in Section 3 that for parameters subject to the truncation condition the weights \(\Delta^q^{1R}(\nu)\) (3.10) vanish when \(\nu \in \Lambda \setminus \Lambda_N\), because then the \(c\)-function \(C_{1R}(\nu)\) (3.11) becomes infinite. Since the truncation condition (3.6) reads the same in the dual parameters \(\hat{t}_a\) (4.1) as in the original parameters \(t\) (recall \(\hat{t}_a \hat{t}_b = t_a t_b\)), the \(c\)-function \(C^{1R}(\lambda)\) (4.2) also becomes infinite for parameters subject to the truncation condition \(\lambda \in \Lambda \setminus \Lambda_N\). Hence, the sum \(\langle p_\lambda, p_\lambda \rangle^{1R}_N\) (4.7) vanishes if the parameters satisfy the truncation condition (3.6) when \(\lambda \in \Lambda \setminus \Lambda_N\). The vanishing of the sum \(\langle p_\lambda, p_\lambda \rangle^{1R}_N\) in this situation should of course not come as a surprise in view of the previously noted fact that the polynomials in question are actually zero on the grid points \(\tau q^\nu, \nu \in \Lambda_N\) for these parameters (Proposition 3.7).

It is possible to capture the orthogonality relations (3.15) together with the orthonormalization formulas (1.7) in terms of a purely linear-algebraic formulation. In this language Theorem 3.2 and Theorem 4.1 boil down to the property of the
matrix

\[(4.12) \quad [K_{\mu,\nu}]_{\mu,\nu \in \Lambda_N} \quad \text{with} \quad K_{\mu,\nu} \equiv P_{\mu}(\tau q^\mu) \frac{(\hat{\Delta}^qR(\mu))^{1/2} (\Delta^qR(\nu))^{1/2}}{(1,1)^{R\mu}_N^{1/2}}\]

being an orthogonal matrix for generic parameters subject to the truncation condition \((\ref{3.6})\). Here we have used the renormalized polynomial \(P_{\mu}(z) \equiv C^qR_{+}(\mu)p_{\mu}(z)\) (cf. Appendix 3) and the weights for the ‘Plancherel’ measure (cf. (3.10))

\[\hat{\Delta}^qR(\mu) = \frac{1}{C^qR_{+}(\mu)C^qR_{-}(\mu)}\]

and, we have again (cf. (4.24)) ordered the rows and columns of the matrix according to an arbitrary total extension of the partial order in \((\ref{2.3})\). The orthogonality (and thus invertibility) of the matrix \(K = [K_{\mu,\nu}]_{\mu,\nu \in \Lambda_N}\) in \((4.12)\) checks with the previously noted invertibility of the matrix \(M\) given by \((3.20)\) because—as should be clear from our discussion—one has that \(K = LMD\) with \(L\) triangular and \(D\) diagonal and both \(L\) and \(D\) (generically) nonsingular (as the matrix elements on their respective diagonals are nonzero as meromorphic functions in the parameters subject to the truncation condition).

**Remark 4.1.** The normalization of the polynomials \(P_{\mu}(z) = \hat{C}^qR_{+}(\mu)p_{\mu}(z)\) is such that \(P_{\mu}(\tau) = 1\) (see Appendix 3). Moreover, the renormalized multivariable Askey-Wilson polynomials \(P_{\mu}(z)\) satisfy Macdonald’s duality property (see also Appendix 3) stating that

\[(4.13) \quad P_{\mu}(\tau q^\mu) = \hat{P}_{\mu}(\hat{\tau} q^\mu),\]

where \(\hat{P}_{\mu}(z) = C^qR_{+}(\nu)\hat{p}_{\mu}(z)\) denotes the renormalized multivariable Askey-Wilson polynomial dual to \(P_{\nu}(z) = C^qR_{+}(\nu)p_{\nu}(z)\), i.e., with the parameters \(t_r\) being replaced by the dual parameters \(\hat{t}_r\) \((\ref{4.1})\). (For \(\nu = 0\) the equality in \((4.13)\) reduces to the evaluation formula \(P_{\mu}(\tau) = 1\).) The duality relation \(4.13\) for the polynomials gives rise to a similar duality property for the matrix \([K_{\mu,\nu}]_{\mu,\nu \in \Lambda_N}\) \((\ref{4.12})\), viz., transposition of \([K_{\mu,\nu}]_{\mu,\nu \in \Lambda_N}\) leads one to the matrix \([\hat{K}_{\mu,\nu}]_{\mu,\nu \in \Lambda_N}\) in which the parameters \(t_r\) are replaced by the dual parameters \(\hat{t}_r\). Here we also used that the orthogonality of the matrices fixes their normalization uniquely. In particular one sees that as a consequence the sum \((1,1)^{R\mu}_N\) is invariant with respect to the transformation \(t_r \rightarrow \hat{t}_r\) (for parameters satisfying the truncation condition \((3.3)\)).

For \(n = 1\) this property of \((1,1)^{R\mu}_N\) boils down to the equality

\[(4.14) \quad \frac{(t^2_d q, t^{-1}_d q; q)_N}{(t c_d^{-1} q, t a_d^{-1} q; q)_N} = \frac{(t^2_d q, t^{-1}_c q; q)_N}{(t c^{-1} q, t a^{-1} q; q)_N},\]

which is not difficult to check directly using \((3.6), (4.1)\) and the transformation property for \(q\)-shifted factorials

\[(a_1 q^{-N}; q)_N / (a_2 q^{-N}; q)_N = (a_3 / a_2)^N (a_1^{-1} q; q)_N / (a_2^{-1} q; q)_N.\]

In the special case that the parameters satisfy the additional constraint \(t_a q = t_b q = t_c q = t_d q\) one has that \(t_r = t_r\) \((r = 0, \ldots, 3)\). Hence, we are then in a self-dual situation \(\hat{P}_{\mu}(z) = P_{\mu}(z)\) and the matrix \([K_{\mu,\nu}]_{\mu,\nu \in \Lambda_N}\) \((\ref{4.12})\)—in addition to being orthogonal—now also becomes symmetric.
5. Transition to Racah type polynomials

We will now study the transition from the basic hypergeometric level to the hypergeometric level. To this end we substitute the variables

\[ z_j = q^{x_j}, \quad j = 1, \ldots, n \]

and in addition perform a reparametrization of the form

\[ t = q^g, \quad t_r = q^{g_r}, \quad r = 0, \ldots, 3. \]

After these substitutions and division by a constant factor \((1 - q^2)\), the \(q\)-difference operator \(D (2.4)\) passes for \(q \to 1\) over into a second order difference operator given by

\[ \tilde{D} = \sum_{1 \leq j \leq n} \left( \tilde{V}_j(x)(T_j - 1) + \tilde{V}_{-j}(x)(T_j^{-1} - 1) \right) \]

where

\[ \tilde{V}_{\varepsilon j}(x) = \prod_{0 \leq r < 3} (g_r + \varepsilon x_j) \prod_{1 \leq k \leq n, k \neq j} \left( \frac{g + \varepsilon x_j + x_k}{\varepsilon x_j + x_k} \right) \left( \frac{g + \varepsilon x_j - x_k}{\varepsilon x_j - x_k} \right), \quad \varepsilon = \pm 1 \]

and with the action of the operators \(T_j, j = 1, \ldots, n\) being of the form

\[ (T_j f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n). \]

It turns out (see [D3]) that the difference operator \(\tilde{D} (5.3)\) is triangular with respect to the (partially ordered) basis of symmetrized monomials for the space \(\mathbb{C} S_n [x_1^2, \ldots, x_n^2]\) (consisting of the permutation-invariant and even polynomials in the variables \(x_1, \ldots, x_n\)). Specifically, one has that

\[ \tilde{D} \tilde{m}_\lambda = \sum_{\mu \in \Lambda, \mu \leq \lambda} \tilde{E}_{\lambda, \mu} \tilde{m}_\mu \quad \text{with} \quad \tilde{E}_{\lambda, \mu} \in \mathbb{C}[g, g_0, g_1, g_2, g_3] \]

where

\[ \tilde{m}_\lambda(x) = \sum_{\mu \in S_n(\lambda)} x_1^{2\mu_1} \cdots x_n^{2\mu_n}, \quad \lambda \in \Lambda. \]

Here the summation in \(\tilde{D} \tilde{m}_\lambda = \sum_{\mu \in \Lambda, \mu \leq \lambda} \tilde{E}_{\lambda, \mu} \tilde{m}_\mu\) is meant over the orbit of \(\lambda \in \Lambda\) under the action of the permutation group \(S_n\) (which permutes the vector components \(\lambda_1, \ldots, \lambda_n\)) and the partial order of the cone \(\Lambda\) is taken to be the same as before (see \(\ref{2.2}\)). The diagonal matrix elements \(\tilde{E}_{\lambda, \lambda}\) in \(\tilde{D} \tilde{m}_\lambda\) (which can be obtained for \(q \to 1\) from \(E_{\lambda, \lambda}\) after substitution of \(\ref{5.2}\)) and division by \((1 - q^2)\) read explicitly

\[ \tilde{E}_{\lambda, \lambda} = \sum_{1 \leq j \leq n} \left( (\lambda_j + \hat{\rho}_j)^2 - \hat{\rho}_j^2 \right), \quad \lambda \in \Lambda \]

with

\[ \hat{\rho}_j = (n - j)g + (g_0 + g_1 + g_2 + g_3 - 1)/2, \quad j = 1, \ldots, n. \]

The triangularity of the difference operator again reduces the corresponding eigenvalue problem in the space of the permutation-invariant and even polynomials to an in essence finite-dimensional problem. Although the eigenvalues \(\tilde{E}_{\lambda, \lambda}\) are no longer nondegenerate, it still remains true that \(\tilde{E}_{\lambda, \lambda} \neq \tilde{E}_{\mu, \mu}\) as polynomial expression in
the parameters $q, g_0, \ldots, g_3$, if $\lambda \neq \mu$ and $\lambda, \mu$ are comparable with respect to the partial order (2.3). Fortunately, this is already sufficient to single out the eigenfunctions uniquely by means of conditions analogous to those entering the definition of the multivariable Askey-Wilson polynomials $p_\lambda$ in Section 2. It turns out (see [D3]) that in the present case we are in fact dealing with a multivariable analogue of the Wilson polynomials [W].

Definition. The multivariable Wilson polynomial associated with a (dominant weight) vector $\lambda \in \Lambda$ (2.2) is the (unique) monic permutation invariant and even polynomial of the form

\[ \tilde{p}_\lambda(x) = \tilde{m}_\lambda(x) + \sum_{\mu \in \Lambda, \mu < \lambda} c_{\lambda,\mu} \tilde{m}_\mu(x) \quad \text{with} \quad c_{\lambda,\mu} \in \mathbb{C}(g, g_0, g_1, g_2, g_3), \]

such that

\[ \tilde{D} \tilde{p}_\lambda = \tilde{E}_{\lambda,\lambda} \tilde{p}_\lambda. \]

We can again represent these polynomials in terms of a formula of the type (2.8):

\[ \tilde{p}_\lambda = \left( \prod_{\mu \in \Lambda, \mu < \lambda} \frac{\tilde{D} - \tilde{E}_{\mu,\mu}}{\tilde{E}_{\lambda,\lambda} - \tilde{E}_{\mu,\mu}} \right) \tilde{m}_\lambda. \]

Furthermore, one may always replace the monomial basis in the r.h.s. of such a formula by any other basis related to it via a unitriangular transformation (this is immediate from the comments following (2.8) that proved the validity of this type of representations for the polynomials). In [D3] it was demonstrated, by performing a suitable unitriangular transformation of the basis elements, that formula (2.8) for the multivariable Askey-Wilson polynomials tends to formula (5.9) for the multivariable Wilson polynomials in the limit $q \to 1$ after substitution of (5.1), (5.2) and division by a constant factor $(1 - q)^{2|\lambda|}$. Here we have used the (standard) notation

\[ |\lambda| \equiv \lambda_1 + \cdots + \lambda_n. \]

(The point of the unitriangular transformation is that the basis elements (2.1) all collapse into constant functions for $q \to 1$ after the substitution (5.1); by temporarily passing to a basis of $\mathcal{H}^W$ with elements of the form

\[ \sum_{\mu \in S_n(\lambda)} (z_1 + z_1^{-1} - 2)^{\mu_1} \cdots (z_n + z_n^{-1} - 2)^{\mu_n}, \quad \lambda \in \Lambda, \]

it is seen that a nontrivial limit is obtained after one divides out the constant factor $(1 - q)^{2|\lambda|}$.)

The upshot is that there exists the following limiting relation between the multivariable Askey-Wilson polynomials $p_\lambda$ of Section 2 and their Wilson type counterparts $\tilde{p}_\lambda$ of the present section D3.

Proposition 5.1. For Askey-Wilson parameters given by (5.2), one has that

\[ \tilde{p}_\lambda(x) = \lim_{q \to 1} (1 - q)^{-2|\lambda|} p_\lambda(q^x) \quad \lambda \in \Lambda \]

(where $q^x \equiv (q^{x_1}, \ldots, q^{x_n})$).

In [D3] the orthogonality properties of polynomials $\tilde{p}_\lambda(x)$ were investigated with respect to a continuous Wilson type weight function $\Delta^W$. For parameters satisfying

\[ g \geq 0, \quad \text{Re}(g_r) > 0 \quad (r = 0, 1, 2, 3), \]

the (standard) notation

\[ |\lambda| \equiv \lambda_1 + \cdots + \lambda_n. \]
with possible non-real parameters $g_r$ occurring in complex conjugate pairs, the relevant orthogonality relations read

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{p}_\lambda(ix) \hat{p}_\mu(ix) \Delta^W(x) \, dx_1 \cdots dx_n = 0 \quad \text{if} \quad \lambda \neq \mu
\]

where

\[
\Delta^W(x) = \prod_{1 \leq j < k \leq n} \frac{\Gamma(g + i(\varepsilon_1 x_j + \varepsilon_2 x_k))}{\Gamma(i(\varepsilon_1 x_j + \varepsilon_2 x_k))} \times \prod_{1 \leq j \leq n} \frac{\Gamma(g_0 + i\varepsilon x_j) \Gamma(g_1 + i\varepsilon x_j) \Gamma(g_2 + i\varepsilon x_j) \Gamma(g_3 + i\varepsilon x_j)}{\Gamma(2i\varepsilon x_j)}
\]

(with $\Gamma(\cdot)$ denoting the gamma function).

We will now apply the limiting relation of Proposition 5.1 to the results of Section 3 and 4 to infer that for generic parameters subject to the truncations condition

\[
(n-1) g + g_a + g_b + N = 0
\]

(with $N$ a nonnegative integer and $a, b \in \{0, 1, 2, 3\}$ such that $a \neq b$) the polynomials $\tilde{p}_\lambda(x)$, $\lambda \in \Lambda_N \{\lambda \}$ constitute an orthogonal basis for the finite-dimensional space

\[
H_N^R \equiv \text{Span}\{\tilde{p}_\lambda\}_\lambda \in \Lambda_N
\]

edowed with a nondegenerate bilinear form determined by

\[
\langle f, g \rangle_N^R = \sum_{\nu \in \Lambda_N} f(\rho + \nu) g(\rho + \nu) \Delta^R(\nu) \quad (f, g \in H_N^R).
\]

Here the vector $\rho$ is of the form (c.f. also the vector $\rho$ in Remark 3.2 with $\tau$ given by (3.3) and parameters taken from (5.2))

\[
\rho = (\rho_1, \ldots, \rho_n) \quad \text{with} \quad \rho_j = (n-j)g + g_a, \quad j = 1, \ldots, n
\]

and the weights are given by

\[
\Delta^R(\nu) = \frac{1}{C_+^R(\nu) C_-^R(\nu)},
\]

with

\[
C_+^R(\nu) = \prod_{1 \leq j < k \leq n} \left( \frac{(\rho_j + \rho_k)_{\nu_j+\nu_k}}{(g + \rho_j + \rho_k)_{\nu_j+\nu_k}} \frac{(\rho_j - \rho_k)_{\nu_j-\nu_k}}{(g + \rho_j - \rho_k)_{\nu_j-\nu_k}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{(2\rho_j)_{2\nu_j}}{\Pi_{0 \leq r < 3}(g_r + \rho_j)_{\nu_j}} \right),
\]

\[
C_-^R(\nu) = \prod_{1 \leq j < k \leq n} \left( \frac{(1-g + \rho_j + \rho_k)_{\nu_j+\nu_k}}{(1+\rho_j + \rho_k)_{\nu_j+\nu_k}} \frac{(1-g + \rho_j - \rho_k)_{\nu_j-\nu_k}}{(1+\rho_j - \rho_k)_{\nu_j-\nu_k}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{\Pi_{0 \leq r < 3}(1-g_r + \rho_j)_{\nu_j}}{(1+2\rho_j)_{2\nu_j}} \right),
\]

where we have used Pochhammer symbols defined by

\[
(a_1, \ldots, a_s)_m = (a_1)_m \cdots (a_s)_m,
\]

\[
(a)_m = a(a+1) \cdots (a+m-1)
\]
(with \(a_0 = 1\)).

In order to describe the corresponding orthonormalization constants we again need dual parameters

\[
(5.22) \quad \begin{pmatrix} \hat{g}_a \\ \hat{g}_b \\ \hat{g}_c \\ \hat{g}_d \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} g_a \\ g_b \\ g_c \\ g_d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]

\(\{a, b, c, d\} = \{0, 1, 2, 3\}\) and a function \(\mathcal{N}^R(\lambda)\) on the cone \(\Lambda(2.2)\)

\[
(5.23) \quad \mathcal{N}^R(\lambda) = \frac{\mathcal{C}_{\pm}^R(\lambda)}{\mathcal{C}_{\pm}^R(\lambda)}
\]

that is governed by \(c\)-functions dual to \(C_{\pm}^R(5.20), (5.22)\)

\[
(5.24) \quad \hat{C}_{\pm}(\lambda) = \prod_{1 \leq j < k \leq n} \left( \frac{(\hat{\rho}_j + \hat{\rho}_k)_{\lambda_j + \lambda_k}}{(g + \hat{\rho}_j + \hat{\rho}_k)_{\lambda_j + \lambda_k}} \right) \frac{(\hat{\rho}_j - \hat{\rho}_k)_{\lambda_j - \lambda_k}}{(g + \hat{\rho}_j - \hat{\rho}_k)_{\lambda_j - \lambda_k}} \\
\times \prod_{1 \leq j \leq n} \left( \frac{2\hat{\rho}_j}{\prod_{0 \leq r < s \leq n}(g_r + \hat{\rho}_j)_{\lambda_j}} \right),
\]

\[
(5.25) \quad \hat{\mathcal{C}}^R(\lambda) = \prod_{1 \leq j < k \leq n} \left( \frac{(1 - g + \hat{\rho}_j + \hat{\rho}_k)_{\lambda_j + \lambda_k}}{(1 + \hat{\rho}_j + \hat{\rho}_k)_{\lambda_j + \lambda_k}} \right) \frac{(1 - g + \hat{\rho}_j - \hat{\rho}_k)_{\lambda_j - \lambda_k}}{(1 + \hat{\rho}_j - \hat{\rho}_k)_{\lambda_j - \lambda_k}} \\
\times \prod_{1 \leq j \leq n} \left( \frac{\prod_{0 \leq r < s}(1 - \hat{g}_r + \hat{\rho}_j)_{\lambda_j}}{(1 + 2\hat{\rho}_j)_{2\lambda_j}} \right).
\]

(Recall that the components of the vector \(\hat{\rho}\) are given by \(5.7\), so \(\hat{\rho}_j = (n-j)g + \hat{g}_a\).

The following theorem, which describes the orthogonality properties of the polynomials \(\hat{p}_\lambda, \lambda \in \Lambda_N\) with respect to the bilinear form \(\langle \cdot, \cdot \rangle_N\), is an immediate consequence of the application of Proposition 5.1 to Theorem 3.2 and Theorem 4.1.

**Theorem 5.2.** For parameters subject to the truncation condition \(5.13\) one has that

\[
(5.26) \quad \langle \hat{p}_\lambda, \hat{p}_\mu \rangle_N^R = 0 \quad \text{for} \quad \lambda \neq \mu \quad (\lambda, \mu \in \Lambda_N)
\]

and that

\[
(5.27) \quad \langle \hat{p}_\lambda, \hat{p}_\lambda \rangle_N^R = \mathcal{N}^R(\lambda) \langle 1, 1 \rangle_N^R \quad (\lambda \in \Lambda_N)
\]

with \(\mathcal{N}^R(\lambda)\) given by \(5.23\) and

\[
(5.28) \quad \langle 1, 1 \rangle_N^R = \sum_{\nu \in \Lambda_N} \Delta^R(\nu)
\]

(where \(\langle \cdot, \cdot \rangle_N^R\) and \(\Delta^R(\nu)\) are defined by \(5.17\) and \(5.19\), respectively).

(The formulas \(5.26\) and \(5.27\) should again be interpreted as equalities between expressions that are rational in parameters subject to the truncation condition.)

To verify the theorem it suffices to infer that for \(q \to 1\) and parameters given by \(5.2\) the \(c\)-functions \(C_{\pm}^R(\nu) (3.1)\) and \(C_{\pm}^R(\lambda) (4.2)\) (multiplied by a factor \((1-q)^{2|\nu|}\) and \((1-q)^{2|\lambda|}\), respectively) converge to \(C_{\pm}^R(\nu) (5.21)\) and \(C_{\pm}^R(\lambda) (5.24)\); and that, similarly, the \(c\)-functions \(C^R(\nu) (3.12)\) and \(C^R(\lambda) (4.3)\) (divided by a factor \((1-q)^{2|\nu|}\) and \((1-q)^{2|\lambda|}\), respectively) tend to \(C^R(\nu) (5.21)\) and \(C^R(\lambda) (5.25)\).
in this limit. To this end one simply uses that the (renormalized) \( q \)-shifted factorial \( (a; q)_m/(1 - q)^m \) converges to the Pochhammer symbol \( (a)_m \) when \( q \) tends to one. Furthermore, for the parameters (5.2) the truncation condition (5.3) amounts to the condition \((n - 1)g + g_0 + N = 0 \pmod{2\pi i / \log(q)}\), which entails (5.15) in the limit \( q \to 1 \).

It is important to again convince oneself that the formulas (5.24) and (5.27) are indeed well-defined as rational expressions in the parameters \( g \) and \( g_0, \ldots, g_3 \) subject to the truncation condition (5.13) (i.e., no denominator becomes identical to zero) and, furthermore, that the r.h.s. of (5.27) is nonzero as a rational expression in these parameters. We thus have that the bilinear form \( \langle \cdot, \cdot \rangle_R^N \) is orthogonal for generic parameters subject to the truncation condition (5.15) and that any function defined on the grid points \( \rho + \lambda, \lambda \in \Lambda_N \) can be represented exactly by the restriction to the grid of a unique function in the space \( \mathcal{H}_N^R \). The restriction to the grid points of the orthonormalized basis \( \langle \hat{p}_\lambda, \hat{p}_\mu \rangle^R_N \) yields an orthonormal basis for the space of functions over the grid \( \rho + \Lambda_N \) endowed with the nondegenerate bilinear form \( \sum_{\nu \in \Lambda_N} f(\rho + \nu)\Delta(\rho + \nu) \Delta^R(\nu) \).

Finally, as degeneration of Proposition 3.7 we arrive at a similar statement for the Proposition 5.3.

For parameters subject to the truncation condition (5.15) one has that

\[
\hat{p}_\lambda(\rho + \nu) = 0 \quad \text{for} \quad \nu \in \Lambda_N
\]

if \( \lambda \in \Lambda \setminus \Lambda_N \).

The corresponding linear-algebraic formulation of Theorem 5.2 states that the matrix (cf. (1.13))

\[
[K_{\mu, \nu}]_{\mu, \nu} \equiv \hat{K}_{\mu, \nu} \equiv \hat{P}_\mu(\tau q^\nu) \left( \frac{\Delta^R(\mu)}{(1, 1)_\mu^R} \right)^{1/2} \left( \frac{\Delta^R(\nu)}{(1, 1)_\nu^R} \right)^{1/2}
\]

is orthogonal for generic parameters \( g, g_0, \ldots, g_3 \) subject to the truncation condition (5.15). Here \( \hat{P}_\mu \) is the renormalized polynomial \( \hat{P}_\mu(x) \equiv C^R_\mu(\mu) \hat{p}_\mu(x) \) and \( \hat{\Delta}^R \) denotes the weight function for the discrete ‘Plancherel’ measure \( \hat{\Delta}^R(\mu) = 1/(\hat{C}^R_\mu(\mu) \hat{C}^R_\nu(\mu)) \) dual to the weight function \( \Delta^R \) (5.13).

As a side remark we mention that the orthogonality of \([K_{\mu, \nu}]_{\mu, \nu} \in \Lambda_N \) implies that the matrix

\[
\hat{M} \equiv \hat{m}_{\mu}(\rho + \nu)\]

has a determinant that does not vanish as a polynomial in the parameters \( g \) and \( g_0 \). This is because \( \hat{M} \) is related to \( \hat{K} \equiv \hat{K}_{\mu, \nu} \) (5.30) by \( \hat{K} = \hat{L}\hat{M}\hat{D} \), where \( \hat{L} \) is triangular and \( \hat{D} \) is diagonal with both matrices having elements on the diagonal that do not vanish as meromorphic functions in the parameters \( g \) and \( g_0, \ldots, g_3 \). It is interesting to observe that the invertibility of the matrix \( \hat{M} \) does not seem so easily established directly (i.e. without using \( \hat{K} \)), as was the case when dealing with its \( q \)-version \( \hat{M} \equiv m_{\mu}(\tau q^\nu)\) (3.20) just after Lemma 3.3. (The problem is of course that here in the degenerate version we have lost the possibility to play with the parameter \( q \).) Notice, however, that in the special case of only one single variable the matrix \( \hat{M} \) (5.31) becomes a Vandermonde matrix.
[(g_\alpha + \nu)^2]_{0 \leq \mu, \nu \leq N}$, from which it immediately follows that the determinant is nonzero for generic parameter values.

The difference equation (5.32) tells us that for $n = 1$ the polynomials $\hat{p}_\lambda$ reduce to the monic Wilson polynomials

\begin{equation}
\hat{p}_\lambda(x) = \frac{(g_0 + g_1, g_0 + g_2, g_0 + g_3)_\lambda}{(g_0 + g_1 + g_2 + g_3 + \lambda - 1)_\lambda} \times
\end{equation}

\[ _4F_3 \left( -\lambda, g_0 + g_1 + g_2 + g_3 + \lambda - 1, g_0 + x, g_0 - x ; 1 \right),\]

where we have used standard notation for the hypergeometric series (see e.g. [AW2, GR]).

(The explicit hypergeometric representation (5.32) for $\hat{p}_\lambda$ in the case of one variable also follows from Proposition 5.1 and the corresponding basic hypergeometric formula for $p_\lambda$ in (2.3).) The identities in Theorem 5.2 amount in this special case to the discrete orthogonality relations for the monic Wilson polynomials (5.32) subject to the parameter condition $g_a + g_b + N = 0$, subject to the parameter condition $g_a + g_b + N = 0$, and

\[ \sum_{0 \leq \nu \leq N} \hat{p}_\lambda(g_a + \nu) \hat{p}_\mu(g_a + \nu) \Delta^R(\nu) = 0 \quad \text{for} \quad \lambda \neq \mu \]

($\lambda, \mu \in \{0, \ldots, N\}$) and

\[ \sum_{0 \leq \nu \leq N} \hat{p}_\lambda(g_a + \nu) \hat{p}_\lambda(g_a + \nu) \Delta^R(\nu) = \]

\[ \frac{\lambda! \prod_{0 \leq r < s \leq 3}(g_r + g_s) \lambda}{(g_0 + g_1 + g_2 + g_3 + \lambda - 1) \lambda (g_0 + g_1 + g_2 + g_3 + \lambda - 1) \lambda} (1, 1)_N^R \]

($\lambda \in \{0, \ldots, N\}$), where

\[ (1, 1)_N^R = \sum_{0 \leq \nu \leq N} \Delta^R(\nu), \]

\[ \Delta^R(\nu) = \left( 1 + \frac{\nu}{g_a} \right) \frac{(g_0 + g_a, g_1 + g_a, g_2 + g_a, g_3 + g_a)_\nu}{(1 - g_0 + g_a, 1 - g_1 + g_a, 1 - g_2 + g_a, 1 - g_3 + g_a)_\nu}. \]

In the finite-dimensional case with the truncation condition $g_a + g_b + N = 0$ and discrete orthogonality properties, the one-variable polynomials $\hat{p}_\lambda$ in (5.32) are usually referred to as Racah polynomials rather than Wilson polynomials. The normalization factor $(1, 1)_N^R$ for the one-variable Racah polynomials can be evaluated in product form by means of a summation formula for a very-well-poised (terminating) $_5F_4$ series due to Dougall, which entails [AW] (see also [GR] for the Dougall $_5F_4$ summation formula)

\begin{equation}
\sum_{0 \leq \nu \leq N} \Delta^R(\nu) = \frac{(1 + 2g_a, 1 - g_c - g_d)_N}{(1 + g_a - g_c, 1 + g_a - g_d)_N}. \tag{5.33}
\end{equation}

Remark 5.1. The duality relations for the (renormalized) multivariable Askey-Wilson polynomials in Remark 4.1 give in the limit $q \to 1$ rise to analogous duality relations for the (renormalized) multivariable Wilson polynomials $\hat{P}_\mu(x) = C^R_+(\mu)\hat{p}_\mu(x)$.
\[ \hat{P}_\nu(\rho + \nu) = \hat{P}_\nu(\rho + \mu), \]

where \( \hat{P}_\nu(x) = C_\nu^{\rho}(x) \) is the dual of \( \tilde{P}_\nu(x) = \tilde{C}_\nu^{\rho}(x) \) with the parameters \( \gamma_r \) being replaced by the dual parameters \( \hat{\gamma}_r \) \((5.23)\). (For \( \nu = 0 \) the duality relation \((5.34)\) reduces to the evaluation formula \( \tilde{P}_\mu(\rho) = 1 \) characterizing the normalization of the polynomials \( \tilde{P}_\mu(z) \).) Just as in the \( q \)-case, the duality properties for the polynomials are again inherited by the matrix \( [\tilde{K}_{\mu,\nu}]_{\mu,\nu} \in \Lambda_N \) in \((5.30)\). We now have (for parameters satisfying the truncation condition \((5.15)\)) that transposition of the matrix \( [\tilde{K}_{\mu,\nu}]_{\mu,\nu} \in \Lambda_N \) amounts to the parameter transformation \( \gamma_r \to \hat{\gamma}_r \) and, in particular, that the sum \( (1,1)_N^{\lambda} \) is invariant with respect to such a transformation of the parameters. For parameters satisfying the additional constraint \( g_a - g_b - g_c - g_d = -1 \) we have that \( \hat{\gamma}_r = \gamma_r \). Hence, in that case we are again in a self-dual situation (i.e. \( \hat{P}_\nu(x) = \tilde{P}_\nu(x) \)) and the matrix \( [\tilde{K}_{\mu,\nu}]_{\mu,\nu} \in \Lambda_N \) now also becomes symmetric (in addition to being orthogonal).

**Remark 5.2.** It turns out that our multivariable Racah polynomials (i.e., the multivariable polynomials \( \tilde{p}_\lambda(x) \) with parameters subject to the truncation condition \((5.15)\)) are not the first generalization of Wilson’s one-variable Racah polynomials to the case of several variables. Already several years ago Gustafson reported on a finite system of multivariable orthogonal polynomials with discrete orthogonality measure tied to the so-called multiplicity-free Racah coefficients for the group \( U(m+1) \) \([54]\). In the rank one situation \((m = 1)\), the orthogonal polynomials in question can be reduced to the Racah polynomials of \([55]\). It is not clear (at least not to us), though an interesting question, whether also for higher rank Gustafson’s multivariable Racah polynomials may be linked with the multivariable Racah polynomials of the present paper. From the explicit expressions for the weight functions it seems that both approaches generalize the one-variable Racah polynomials to several variables along very different directions. However, at present we are not able to rule out completely the possibility that there might not be some transformation connecting the two approaches. Should there indeed exist such a connection (which, however, would seem more likely after trading the group \( U(m+1) \) for \( Sp(m) \) say), then this would imply an interesting link between Gustafson’s group-theoretical program and a degenerate case of the Macdonald theory.

**Remark 5.3.** The reader may find it illuminating to view the multivariable \( q \)-Racah polynomials \( p_\lambda \) in Sections \( \S3 \) and \( \S4 \) as a trigonometric version of the multivariable Racah polynomials \( \tilde{p}_\lambda \) in the present section. If we substitute Askey-Wilson parameters in accordance with \((5.2)\) and set \( q = e^{i\alpha} \), then we may rewrite the weight function \( \Delta_{\text{R}}^{\lambda} \) \((3.10)\) as

\[ \Delta_{\text{R}}^{\lambda}(\nu) = \frac{1}{C_\nu^{\lambda}(\nu) C_\nu^{\lambda}(\nu)}, \]
with
\[ C_{\pm}^{qR}(\nu) = (-4)^{-|\nu|} \prod_{1 \leq j < k \leq n} \left( \frac{(\rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}}{(g + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}} \frac{(\rho_j - \rho_k : \sin \alpha)_{\nu_j - \nu_k}}{(g + \rho_j - \rho_k : \sin \alpha)_{\nu_j - \nu_k}} \right) \]
\[ \times \prod_{1 \leq j \leq n} \left( \frac{(2\rho_j : \sin \alpha)_{2\nu_j}}{\prod_{0 \leq r \leq 3}(g_r + \rho_j : \sin \alpha)_{\nu_j}} \right), \]
and
\[ C_{-}^{qR}(\nu) = (-4)^{|\nu|} \prod_{1 \leq j < k \leq n} \left( \frac{(1 - g + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}}{(1 + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}} \right) \]
\[ \times \prod_{1 \leq j \leq n} \left( \frac{\prod_{0 \leq r \leq 3}(1 - g_r + \rho_j : \sin \alpha)_{\nu_j}}{(1 + 2\rho_j : \sin \alpha)_{2\nu_j}} \right), \]

(where the components of the vector \( \rho \) are given by (5.18)). In the above formula we have used ‘trigonometric Pochhammer symbols’ defined by
\[ (a : \sin \alpha)_m = \sin \alpha(a) \sin \alpha(a+1) \cdots \sin \alpha(a+m-1) \]
with \((a : \sin \alpha)_0 \equiv 1\) and \(\sin \alpha(\xi) \equiv \sin(\alpha \xi/2)\). Furthermore, we also arrive at corresponding trigonometric expressions for \(\hat{\Delta}^{qR}(\lambda) = 1/(\hat{C}_+^{qR}(\lambda)\hat{C}_-^{qR}(\lambda))\) and \(\hat{N}^{qR}(\lambda) = \hat{C}_+^{qR}(\lambda)/\hat{C}_-^{qR}(\lambda)\), which are governed by dual \(c\)-functions \(\hat{C}_+^{qR}\) obtained by replacing the parameters \(g_r\) by \(\tilde{g}_r\) (5.22) and the vector \(\rho\) by \(\tilde{\rho}\) (5.7).

It is manifest from these representations that \(\Delta^{qR}, \hat{\Delta}^{qR}\) and \(\hat{N}^{qR}\) can be interpreted as trigonometric versions of \(\Delta^R\) (5.19), \(\hat{\Delta}^R\) (see (5.30)) and \(\hat{N}^R\) (5.23), respectively. The transition \(q \to 1\) corresponds to the limit \(\alpha \to 0\) in which the renormalized trigonometric Pochhammer symbols \((2/\alpha)^m(a : \sin \alpha)_m\) go over in the ordinary Pochhammer symbols \((a)_m\). As far as the polynomials are concerned, we see that \(p_\lambda(e^{i\alpha x})\) becomes a trigonometric polynomial in the variables \(x_1, \ldots, x_n\) with period \(2\pi/\alpha\). Proposition 5.4 describes the rational limit in which the period of the trigonometric functions tends to infinity (cf. (5.13))
\[ \tilde{p}_\lambda(x) = \lim_{\alpha \to 0} (i\alpha)^{-2|\lambda|} p_\lambda(e^{i\alpha x}). \]

Notice also that the modified monomial basis elements in (5.11) are in the trigonometric notation of the form \((-4)^{|\lambda|}(\sin \alpha(x))\) and converge, after division by the constants \((i\alpha)^{2|\lambda|}\), to \(\tilde{m}_\lambda(x)\) for \(\alpha \to 0\). Finally, in the trigonometric coordinates the grid points on which the discrete orthogonality measure \(\Delta^{qR}\) for the polynomials \(p_\lambda(e^{i\alpha x})\) is supported become of the form \(\rho + \nu, \nu \in \Lambda_N\). This means, in particular, that in these coordinates the grid points do not move when performing the limit \(q \to 1\) (or equivalently \(\alpha \to 0\)).

### 6. Positivity Domain for \(|q| = 1\) and the \(q\)-Racah Transform

In the preceding sections we have viewed the polynomials and other objects of interest (such as the discrete weight function and the difference equation) as rational expressions in the parameters. As a consequence, we arrived at their properties for generic values of the parameters. It is clear, however, that the above generic picture does not hold for all values of the parameters. A most drastic way in which the generic picture breaks down occurs when the cardinality of the grid \(\tau q^\nu, \nu \in \Lambda_N\)
becomes less than the dimension of the space $\mathcal{H}_N^{qR}$ (3.10) (or, equivalently, when it becomes less than the number of points in the alcove $\Lambda_N$ (3.9)). This may for instance happen when $q^M = 1$ for $M \in \{1, \ldots, N\}$. The special case $M = 1$ (so $q = 1$) corresponds of course precisely to what we already analyzed in more detail in the previous section by means of a limit transition. In this section, however, rather than to present any in-depth analysis of the case that $q$ is such a root of unity (cf. Remark 5.3), we will focus on a parameter domain with $|q| = 1$ for which the generic picture sketched in the preceding sections does apply for all parameter values in the domain and, moreover, for which the weight function $\Delta^{qR}(\nu)$ becomes real-valued and positive when $\nu$ lies in the alcove $\Lambda_N$.

To describe the positivity domain for $|q| = 1$ it is convenient to perform the following trigonometric substitution of the parameters (cf. Remark 5.3)

$$q = e^{ia}, \quad t = e^{iaq}, \quad t_a = e^{iaq_{sa}}, \quad t_b = -e^{iaq_{sb}}, \quad t_c = e^{ia(q_{sa}+1/2)}, \quad t_d = -e^{ia(q_{sa}+1/2)}.$$  

The weights $\Delta^{qR}$ (3.10) can then be rewritten with the aid of the trigonometric Pochhammer symbols (cf. Remark 5.3)

$$\begin{align*}
(a_1, \ldots, a_s : \sin \alpha)_m &= (a_1 : \sin \alpha)_m \cdots (a_s : \sin \alpha)_m \\
(a : \sin \alpha)_m &= \sin(a)(\sin(a + 1) \cdots \sin(a + m - 1)) \\
(a_1, \ldots, a_s : \cos \alpha)_m &= (a_1 : \cos \alpha)_m \cdots (a_s : \cos \alpha)_m \\
(a : \cos \alpha)_m &= \cos(a)(\cos(a + 1) \cdots \cos(a + m - 1))
\end{align*}$$

where $(a : \sin \alpha)_0 = (a : \sin \alpha)_1 = 1$ and $\sin(\alpha \xi/2), \cos(\alpha \xi/2)$. Specifically, we have

$$\Delta^{qR}(\nu) = \frac{1}{C_+^{qR}(\nu) C_-^{qR}(\nu)},$$

with

$$C_+^{qR}(\nu) = \prod_{1 \leq j < k \leq n} \left( \frac{(\rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}}{(g + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}} \frac{(\rho_j - \rho_k : \sin \alpha)_{\nu_j - \nu_k}}{(g + \rho_j - \rho_k : \sin \alpha)_{\nu_j - \nu_k}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{(\rho_j, 1/2 + \rho_j : \sin \alpha)_{\nu_j}}{(g_a + \rho_j, g_c + 1/2 + \rho_j : \sin \alpha)_{\nu_j}} \right),$$

$$C_-^{qR}(\nu) = \prod_{1 \leq j \leq n} \left( \frac{(1 - g + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}}{(1 + \rho_j + \rho_k : \sin \alpha)_{\nu_j + \nu_k}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{(1 - g + \rho_j, 1/2 - g_a + \rho_j : \sin \alpha)_{\nu_j}}{(1 + \rho_j, 1/2 + \rho_j : \sin \alpha)_{\nu_j}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{(1 - g + \rho_j, 1/2 - g_a + \rho_j : \sin \alpha)_{\nu_j}}{(1 + \rho_j, 1/2 + \rho_j : \sin \alpha)_{\nu_j}} \right) \times \prod_{1 \leq j \leq n} \left( \frac{(1 - g + \rho_j, 1/2 - g_a + \rho_j : \sin \alpha)_{\nu_j}}{(1 + \rho_j, 1/2 + \rho_j : \sin \alpha)_{\nu_j}} \right)$$

and

$$\rho_j = (n - j)g + g_a.$$  

The corresponding dual objects $\hat{C}_+^{qR}$ and $\hat{C}_-^{qR}$ are again obtained by replacing the parameters $g_r$ by dual parameters $\hat{g}_r$ with

$$\begin{pmatrix}
\hat{g}_a \\
\hat{g}_b \\
\hat{g}_c \\
\hat{g}_d
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
g_a \\
g_b \\
g_c \\
g_d
\end{pmatrix}.$$
and the vector $\rho$ with $\hat{\rho}_j = (n - j)g + \hat{g}_a$.

**Proposition 6.1.** For parameters satisfying the constraints

\[(6.5a)\quad \alpha > 0, \quad g \geq 0, \quad 0 \leq g_a, g_b < \frac{\pi}{\alpha}, \quad -g_a \leq g_c \leq g_a, \quad -g_b \leq g_d \leq g_b
\]

and the truncation condition

\[(6.5b)\quad (n - 1)g + g_a + g_b + N = \frac{\pi}{\alpha},
\]

one has that

\[0 < C_q^R(\nu) < \infty \quad \text{for } \nu \in \Lambda_N,
\]

\[0 < \hat{C}_q^R(\mu) < \infty \quad \text{for } \mu \in \Lambda_N.
\]

**Proof.** Notice that for $N = 0$ the proposition is valid trivially, because in that case $C_q^R, \hat{C}_q^R \equiv 1$ (and $\Lambda_N = \{0\}$). Let us from now on assume that $N$ is positive and, furthermore, let us also temporarily assume that the parameters $g$ and $g_a, g_b$ are nonzero. It is not very difficult to verify that the conditions (6.5a) and (6.5b) then imply that the arguments of the sinus functions in $C_q^R(\nu)$ (with $\nu \in \Lambda_N$) lie in the open interval $(0, \pi)$ and, similarly, that the arguments of the cosinus functions lie in the open interval $(-\pi/2, \pi/2)$. Hence, it follows that the c-functions $C_q^R(\nu), \nu \in \Lambda_N$ are positive and finite for these parameters. When one or more of the parameters $g, g_a, g_b$ (or $g_c, g_d$ for that matter) become zero, one may cancel the factors in the numerator/denominator carrying that parameter (with the value zero) against the corresponding term in the denominator/numerator (thus resulting in a trivial unit factor). By proceeding in this manner one readily infers that the positivity and finiteness of the c-functions is conserved also when one or more of the parameters $g$ and $g_a, g_b$ are allowed to become zero.

The dual statement that $\hat{C}_q^R(\mu)$ is positive for $\mu \in \Lambda_N$ is now immediate from the observation that the conditions (6.5a), (6.5b) are self-dual in the sense that they read the same in the parameters $g_a, g_b, g_c, g_d$ as in the dual parameters $\hat{g}_a, \hat{g}_b, \hat{g}_c, \hat{g}_d$. Specifically, the conditions (6.5a), (6.5b) (with $N > 0$) are equivalent to the conditions

\[(6.6a)\quad \alpha > 0, \quad g \geq 0, \quad 0 \leq \hat{g}_a, \hat{g}_b < \frac{\pi}{\alpha}, \quad -\hat{g}_a \leq \hat{g}_c \leq \hat{g}_a, \quad -\hat{g}_b \leq \hat{g}_d \leq \hat{g}_b
\]

and

\[(6.6b)\quad (n - 1)g + \hat{g}_a + \hat{g}_b + N = \frac{\pi}{\alpha}.
\]

That the conditions (6.5a) and (6.5b) imply the dual conditions (6.6a) and (6.6b) is seen with the aid of the definition of the dual parameters in (6.4); that both the conditions and the dual conditions are actually equivalent is then clear from the fact that the parameter transformation determining the dual parameters in (6.4) is an involution.

**Corollary 6.2.** For parameters subject to the conditions in Proposition 6.1, the weight functions $\Delta_q^R(\nu)$ and $\hat{\Delta}_q^R(\mu)$ are positive and finite when $\nu, \mu$ lie in the alcove $\Lambda_N$.

The positivity of Proposition 6.1 implies that $p_\lambda, \lambda \in \Lambda_N$ is well-defined not just generically but for all parameter values in the domain determined by the conditions (6.5a), (6.5b).
Proposition 6.3. The polynomials $p_{\lambda}, \lambda \in \Lambda_N$ are well-defined for all parameters \( (6.1) \) with values in the domain determined by the conditions (6.5a) and (6.5b) (i.e., the expansion coefficients $c_{\lambda, \mu}$ in (2.7a) are regular for these parameter values).

Proof. Let us first assume that the parameters are generic (complex say) but subject to the truncation condition (3.6). For arbitrary $\lambda \in \Lambda_N$, the bilinear form $\langle \cdot, \cdot \rangle_{qR}^N$ restricted to the subspace $\text{Span}\{m_{\mu} | \mu < \lambda\}$ is nondegenerate as a consequence of Theorem 3.2 and Theorem 4.1. Using the orthogonality of the polynomials and the nondegeneracy of the bilinear form it is seen that one can characterize the multivariable $q$-Racah polynomial corresponding to a dominant weight vector $\lambda \in \Lambda_N$ as the unique polynomial of the form

$$p_{\lambda}(z) = m_{\lambda}(z) + \sum_{\mu \in \Lambda_N, \mu < \lambda} c_{\lambda, \mu} m_{\mu}(z)$$

such that

$$\langle p_{\lambda}, m_{\mu} \rangle_{qR}^N = 0 \quad \text{for} \quad \mu < \lambda.$$ 

In other words, $p_{\lambda}$ consists of $m_{\lambda}$ minus its (unique) orthogonal projection with respect to the (nondegenerate) bilinear form $\langle \cdot, \cdot \rangle_{qR}^N$ onto $\text{Span}\{m_{\mu} | \mu < \lambda\}$. We thus have the following inductive Gram-Schmidt-like formula for the (orthogonal) polynomials $p_{\lambda}, \lambda \in \Lambda_N$

$$(6.7) \quad p_{\lambda}(z) = m_{\lambda}(z) - \sum_{\mu \in \Lambda_N, \mu < \lambda} \langle m_{\lambda}, p_{\mu} \rangle_{qR}^N \langle p_{\mu}, p_{\mu} \rangle_{qR}^N p_{\mu}(z).$$

(At this point it is helpful again to view Equation (6.7) as a rational identity in the parameters $q, t$ and $t_r$ subject to the truncation condition (3.6).) Using induction on the weight $\lambda$ one sees from this inductive formula and Theorem 4.1 that the polynomial $p_{\lambda}$ is regular at parameter values where the $c$-functions $C^{qR}, \hat{C}^{qR}$ are regular and nonzero. The proposition is then immediate from Proposition 6.1.

With the aid of Proposition 6.1 and 6.3 it is seen that the results of Section 3 and 4 hold for all parameter values in the domain determined by the conditions (6.5a) and (6.5b). At some points the positivity of the measure will enable us to even formulate a somewhat stronger version of the results stated there. For instance, for the parameters in the positivity domain (6.5a), (6.5b) the polynomial $p_{\lambda}(e^{i\alpha x})$ \((x \in \mathbb{R}^n)\) and thus also the matrix $[K_{\mu, \nu}]_{\mu, \nu \in \Lambda_N}$ (4.12) is real. Hence, in addition to being orthogonal the matrix $[K_{\mu, \nu}]_{\mu, \nu \in \Lambda_N}$ now also becomes unitary. (The real-valuedness of polynomials $p_{\lambda}(e^{i\alpha x})$ for parameters in the positivity domain (6.5a), (6.5b) follows e.g. from the inductive formula (6.7) together with the observation that the (even) monomials $m_{\lambda}(e^{i\alpha x})$ are real.) The fact that both polynomials and weight function are real for parameters in the positivity domain (6.5a), (6.5b) allows us to restrict the bilinear form $\langle \cdot, \cdot \rangle_{qR}^N$ (4.17) to a real form (by restricting to the real vector space spanned by $m_{\lambda}, \lambda \in \Lambda_N$), which in turn can be extended to a positive definite sesquilinear form on the complex vector space $H_{qR}^N$ (3.16) (in the standard way).

In the remainder of this section we will interpret the orthogonality and orthonormalization properties of the multivariable $q$-Racah polynomials with parameters in the positivity domain in terms of a finite-dimensional discrete integral transformation for grid functions. To this end we need to introduce some further notation. Let
$L^2(\rho + \Lambda_N, \Delta^{q_R})$ be the finite-dimensional Hilbert space of complex functions over the grid points $\rho + \nu, \nu \in \Lambda_N$ endowed with the standard inner product determined by the positive weights $\Delta^{q_R}(\nu), \nu \in \Lambda_N$

$$(f, g)_{\Delta} = \sum_{\nu \in \Lambda_N} f(\rho + \nu) g(\rho + \nu) \Delta^{q_R}(\nu).$$

Similarly, the space $L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R})$ denotes the corresponding dual Hilbert space consisting of the complex functions over the grid points $\hat{\rho} + \mu, \mu \in \Lambda_N$ endowed with the standard inner product $\langle \cdot, \cdot \rangle_{\hat{\Delta}}$ determined by the positive weights $\hat{\Delta}^{q_R}(\mu), \mu \in \Lambda_N$. We define the operator $\mathcal{K} : L^2(\rho + \Lambda_N, \Delta^{q_R}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R})$ as the map with kernel

$$(6.8) \quad \mathcal{K}(\hat{\rho} + \mu, \rho + \nu) = \frac{P_\mu(e^{i\alpha(\rho+\nu)}) \Delta^{q_R}(\nu)}{\sqrt{(1,1)^{q_R}_{\nu}}}$$

i.e., the operator $\mathcal{K}$ acts on a grid function $f : \rho + \Lambda_N \to \mathbb{C}$ as

$$(6.9) \quad (\mathcal{K}f)(\rho + \mu) = \sum_{\nu \in \Lambda_N} \mathcal{K}(\hat{\rho} + \mu, \rho + \nu)f(\rho + \nu)$$

(thus producing a function $(\mathcal{K}f)(\rho + \Lambda) \to \mathbb{C}$. Here we have employed the renormalized multivariable Askey-Wilson/$q$-Racah polynomials $P_\lambda(z) = \hat{C}^{q_R}_{+}(\lambda)p_\lambda(z)$ satisfying the normalization condition $P_\lambda(\tau) = 1$ (cf. Remark 4.1). We also define the dual map $\hat{\mathcal{K}} : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R}) \to L^2(\rho + \Lambda_N, \Delta^{q_R})$ determined by the kernel

$$(6.10) \quad \hat{\mathcal{K}}(\rho + \nu, \hat{\rho} + \mu) = \frac{\hat{P}_\nu(e^{i\alpha(\rho+\mu)}) \hat{\Delta}^{q_R}(\mu)}{\sqrt{(1,1)^{q_R}_{\mu}}}$$

($\hat{P}_\lambda(z) = C_+^{q_R}(\lambda)\hat{p}_\lambda(z)$) and acting on a function $\hat{f} : \hat{\rho} + \Lambda_N \to \mathbb{C}$ over the dual grid as

$$(6.11) \quad (\hat{\mathcal{K}}\hat{f})(\rho + \nu) = \sum_{\mu \in \Lambda_N} \hat{\mathcal{K}}(\rho + \nu, \hat{\rho} + \mu)\hat{f}(\hat{\rho} + \mu).$$

The following theorem describes a discrete integral transform—the ‘$q$-Racah transform’—between grid functions in $L^2(\rho + \Lambda_N, \Delta^{q_R})$ and $L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R})$ together with its inversion formula.

**Theorem 6.4.** For parameters subject to the conditions $(6.5a), (6.5b)$ the map $\mathcal{K} : L^2(\rho + \Lambda_N, \Delta^{q_R}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R})$ is an isometric isomorphism. The inverse of $\mathcal{K}$ is given by the map $\hat{\mathcal{K}} : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}^{q_R}) \to L^2(\rho + \Lambda_N, \Delta^{q_R})$.

**Proof.** Let $\mathcal{K} = [K_{\mu,\nu}]_{\mu,\nu\in\Lambda_N}$ be the matrix with elements $K_{\mu,\nu} = \mathcal{K}(\rho + \mu, \rho + \nu) = P_\mu(e^{i\alpha(\rho+\nu)})\Delta^{q_R}(\nu)/((1,1)^{q_R}_{\nu})^{1/2}$ and let $\Delta$ and $\hat{\Delta}$ be the diagonal matrices with the quantities $\Delta^{q_R}(\nu)$ and $\Delta^{q_R}(\mu)$ ($\nu, \mu \in \Lambda_N$) on the diagonal, respectively. (Here it is of course again assumed that the columns and rows are ordered by a total extension of the partial order $(2.3).$) We then have that

$$\mathcal{K} = \hat{\Delta}^{-1/2}K\Delta^{1/2},$$

where $K = [K_{\mu,\nu}]_{\mu,\nu\in\Lambda_N}$ denotes the matrix given in $(6.12)$. The unitarity of the (real orthogonal) matrix $K$ (for parameters in the positivity domain) and the
inversion formula $K^{-1} = \hat{K}$ imply for the matrix $K$ that
\begin{equation}
K^* \Delta K = \Delta \quad \text{and that} \quad K^{-1} = \hat{K}
\end{equation}
(where $\hat{K}$ and $K^*$ are the dual and the adjoint (transpose) of $K$). Clearly, Formula (6.13) boils down to a reformulation of the statement in the theorem. (The first property in (6.13) says that the matrix $K$ determines an isometry between the inner product spaces endowed with the positive sesquilinear forms associated to $\Delta$ and $\Delta$, respectively.)

It is instructive to view the $q$-Racah transform as a Fourier type transformation between the grid functions in $L^2(\rho + \Lambda_N, \Delta^{qR})$ and $L^2(\hat{\rho} + \Lambda_N, \Delta^{qR})$:
\begin{align}
\hat{f}(\hat{\rho} + \mu) &= (Kf)(\hat{\rho} + \mu) = \frac{\langle f, P_{\mu}\rangle_{\Delta}}{\sqrt{\langle 1, 1 \rangle_{\Delta}}} \\
\hat{f}(\rho + \nu) &= (K^{-1}\hat{f})(\rho + \nu) = \frac{\langle f, \hat{P}_{\nu}\rangle_{\Delta}}{\sqrt{\langle 1, 1 \rangle_{\Delta}}},
\end{align}
where $\langle \cdot, \cdot \rangle_{\Delta}$ is taken from (5.8) and $\langle \cdot, \cdot \rangle_{\Delta}$ denotes its dual version (and the functions $P_{\mu}$ and $\hat{P}_{\nu}$ stand for the restrictions to the grids of $P_{\mu}(e^{i\alpha x})$ and $\hat{P}_{\nu}(e^{i\alpha x})$, respectively). (Recall also that for parameters in the positivity domain (5.5a), (5.5b) one has that $\langle 1, 1 \rangle_{\Delta} = \langle 1, 1 \rangle_{\hat{\Delta}}$ as a consequence of Remark 4.1.)

We will next discuss the behavior of a discretization of the difference operator $D (2.4)$ with respect to the $q$-Racah transform. Let $D^{qR} : L^2(\rho + \Lambda_N, \Delta^{qR}) \longrightarrow L^2(\hat{\rho} + \Lambda_N, \Delta^{qR})$ be the discrete difference operator of the form
\begin{equation}
D^{qR} = \sum_{\nu + e_j \in \Lambda_N} V_{\nu-j}(\rho + \nu)(T_j - 1) + \sum_{\nu - e_j \in \Lambda_N} V_{\nu-j}(\rho + \nu)(T_j^{-1} - 1)
\end{equation}
where
\begin{equation}
V_{\pm j}(x) = w(\pm x_j) \prod_{1 \leq k \leq n \atop k \neq j} v(\pm x_j + x_k) v(\pm x_j - x_k),
\end{equation}
with
\begin{equation}
v(\xi) = \frac{\sin \frac{\alpha}{2} (g + \xi)}{\sin \frac{\alpha}{2} (g)} \quad \text{and} \quad \sin \frac{\alpha}{2},
\end{equation}
\begin{equation}
w(\xi) = \frac{\sin \frac{\alpha}{2} (g_a + \xi)}{\sin \frac{\alpha}{2} (g_a)} \cos \frac{\alpha}{2} (g_b + \xi) \sin \frac{\alpha}{2} (g_c + \frac{1}{2} + \xi) \cos \frac{\alpha}{2} (g_d + \frac{1}{2} + \xi)
\end{equation}
\begin{equation}
\frac{\sin \frac{\alpha}{2} (1/2 + \xi)}{\cos \frac{\alpha}{2} (1/2 + \xi)} \quad \text{and} \quad \cos \frac{\alpha}{2} (1/2 + \xi)
\end{equation}
and the action of the operators $T_j^{\pm 1}$ is given by
\begin{equation}
(T_j^{\pm 1}f)(\rho + \nu) = f(\rho + \nu \pm e_j).
\end{equation}

Notice that the conditions $\nu + e_j \in \Lambda_N$ and $\nu - e_j \in \Lambda_N$ in the summations of (6.13) guarantee that the function $(D^{qR}f)(\rho + \nu)$ for $\nu \in \Lambda_N$ indeed depends only on the values of $f$ on the grid points in $\rho + \Lambda_N$. Up to (multiplication by) an overall constant factor with value $t^{n-1}(t_d t_2 t_3 q^{-1})^{1/2} = e^{\alpha \alpha \alpha (n-1)} g + (g_a + g_b + g_c + g_d + \ldots + g_{d+2})/2$ the operator $D^{qR}$ (6.13) amounts to the restriction of the operator $D (2.4)$ to grid functions (cf. Remark 3.2) rewritten in trigonometric form (recall (6.1)). Such a restriction is well-defined because the coefficients of the difference operator $D (2.4)$ are regular on the grid points and they vanish (cf. Lemma A.2) when the
$q$-shift operators $T_{j,q}^{\pm}$ in \([2.4]\) shift the argument of a function out of the grid (which leads us to the above-mentioned restrictions on the sums in the discretized operator \((6.13)\)). Let us furthermore introduce the multiplication operator $E : L^2(\rho + \Lambda_N, \Delta_{\text{av}}) \to L^2(\rho + \Lambda_N, \Delta_{\text{av}})$ defined by

\[(6.17) \quad (Ef)(\rho + \nu) = E_\nu f(\rho + \nu)\]

with

\[E_\nu = 2 \sum_{1 \leq j \leq n} \left( \cos(\rho_j + \nu_j) - \cos(\alpha \rho_j) \right),\]

together with the corresponding dual operators $\hat{D}^{\text{av}} : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}})$ and $\hat{E} : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}})$ in which $g_\nu \to g_\nu$ and $\rho \to \hat{\rho}$. (The dual quantities $E_{\lambda,\lambda}$ are self-adjoint and the map $\hat{D}^{\text{av}}$ defined by $e^{\alpha i \nu_{\lambda,\lambda}}$ up to multiplication by the constant factor $t^{-1/2}$ relating $D^{\text{av}}$ \((6.14)\) and $D$ \((6.4)\).

The following theorem states that the $q$-Racah transform (i.e. the discrete integral transformation $K$) is the eigenfunction transformation that diagonalizes the operators $D^{\text{av}}$ and $\hat{D}^{\text{av}}$.

**Theorem 6.5.** For parameters in the positivity domain \((6.5a), (6.5b)\) the discrete difference operators $D^{\text{av}} : L^2(\rho + \Lambda_N, \Delta_{\text{av}}) \to L^2(\rho + \Lambda_N, \Delta_{\text{av}})$ and $\hat{D}^{\text{av}} : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}})$ are self-adjoint and the map $K : L^2(\rho + \Lambda_N, \Delta_{\text{av}}) \to L^2(\rho + \Lambda_N, \Delta_{\text{av}})$ constitutes the corresponding (unitary) eigenfunction transformation diagonalizing these operators

\[(6.17) \quad KD^{\text{av}} K^{-1} = \hat{E}, \quad K^{-1} \hat{D}^{\text{av}} K = E.\]

**Proof.** Clearly it is sufficient to prove only one of the diagonalization formulas in \((6.17)\) because the other will then automatically follow upon dualization. Since for parameters in the positivity domain the (real) functions $P_{\lambda}(e^{i \alpha (\rho + \nu)})$, $\lambda \in \Lambda_N$ form an orthogonal basis for the space $L^2(\rho + \Lambda_N, \Delta_{\text{av}})$, it is enough to show that

\[(6.18) \quad (KD^{\text{av}})P_{\lambda} = (\hat{E}K)P_{\lambda}, \quad \text{for all } \lambda \in \Lambda_N\]

(where $P_{\lambda}$ stands for the discretized trigonometric polynomial $P_{\lambda}(e^{i \alpha (\rho + \nu)})$). Equation \((6.18)\) is immediate from the discretized eigenvalue equation (cf. Remark 3.2)

\[D^{\text{av}} P_{\lambda} = \hat{E}_{\lambda} P_{\lambda}\]

and the observation that $(KP_{\lambda})(\hat{\rho} + \mu)$ is nonzero only in the point $\hat{\rho} + \lambda$ (i.e. for $\mu = \lambda$) in view of the orthogonality of the multivariable $q$-Racah polynomials. (We thus have that $KD^{\text{av}} P_{\lambda} = \hat{E}_{\lambda} \hat{K} P_{\lambda} = \hat{E} K P_{\lambda}$.) The self-adjointness of $D^{\text{av}}$ and $\hat{D}^{\text{av}}$ now follows from the fact that the discrete difference operators are unitarily equivalent to the real (for parameters in the positivity domain) multiplication operators $\hat{E}$ and $E$, respectively.

**Remark 6.1.** If the parameters satisfy the additional constraint

\[(6.19) \quad g_a - g_b - g_c - g_d = 0,\]

then $g_r = g_r$, $r = 0, \ldots, 3$ (see \((6.4)\)). Hence, we are then in a self-dual situation (cf. Remark 1.1) with both the Hilbert spaces $L^2(\rho + \Lambda_N, \Delta_{\text{av}})$ and $L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{\text{av}})$ coinciding and the map $K : L^2(\rho + \Lambda_N, \Delta_{\text{av}}) \to L^2(\rho + \Lambda_N, \Delta_{\text{av}})$ being an involution ($K^2 = \hat{K} K = Id$).
Remark 6.2. A positivity domain for a self-dual one-parameter subfamily of the one-variable $q$-Racah polynomials with $|q|=1$ similar to the domain considered in this section can be found in Section 3C2 of [127] together with a discussion of the corresponding finite-dimensional discrete integral transform.

Remark 6.3. The trigonometric polynomials $p_\lambda(e^{i\alpha x})$ are invariant with respect to permutations, sign flips $(x_j \rightarrow -x_j)$ and translations $(x_j \rightarrow x_j + 2\pi/\alpha)$ of the variables $x_1, \ldots, x_n$. A fundamental domain for $\mathbb{R}^n$ modulo the action of the discrete symmetry group generated by the permutations, sign flips and translations over a period $2\pi/\alpha$ is given by the (Weyl) alcove

\begin{equation}
\{ x \in \mathbb{R}^n | \pi/\alpha \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \}. \tag{6.20}\end{equation}

It is therefore natural to consider the trigonometric polynomials $p_\lambda(e^{i\alpha x})$, $\lambda \in \Lambda$ as polynomials over the alcove (6.20). In the trigonometric context the necessity to truncate the grid with points $\rho + \nu$, $\nu \in \Lambda$ on which the masses of the discrete orthogonality measures are concentrated arises as a natural consequence of the periodicity of the trigonometric functions (which demands that the support of the discrete orthogonality measure be finite). The conditions (6.5a), (6.5b) in Proposition 6.1 arrange things in such a manner that $\Delta^{qR}(\nu)$ (6.3) is positive for $\nu \in \Lambda_N$ and zero for $\nu \in \Lambda \setminus \Lambda_N$, and guarantee furthermore that the grid $\rho + \Lambda_N$ supporting the discrete orthogonality measure for the polynomials $p_\lambda(e^{i\alpha x})$, $\lambda \in \Lambda_N$ fits in the fundamental domain (6.20).

Remark 6.4. The second-order operator $D$ (2.4) sits in a commutative algebra generated by $n$-independent commuting analytic difference operators $D_1, D_2, \ldots, D_n$ of order $2, 4, \ldots, 2n$, respectively [127]. The first of these operators, viz. $D_1$, corresponds to the Koornwinder-Macdonald difference operator $D$ of Section 2. After restriction to functions on the grid $\rho + \Lambda_N$ the analytic difference operators go over in a family of (commuting) discrete difference operators $D^{qR}: L^2(\rho + \Lambda_N, \Delta^{qR}) \rightarrow L^2(\rho + \Lambda_N, \Delta^{qR})$ ($r = 1, \ldots, n$) given explicitly by (cf. Remark 3.2)

\begin{equation}
D^{qR}_r = \sum_{j \leq l \leq n} U_{e_j, e_l}(\rho + \nu) V_{\varepsilon J, J}(\rho + \nu) T_{\varepsilon J}, \quad r = 1, \ldots, n, \tag{6.21}\end{equation}

with $(T_{\varepsilon J} f)(\rho + \nu) = f(\rho + \nu + e_{\varepsilon J})$, $e_{\varepsilon J} = \sum_{j \in J} \varepsilon_j e_j$, and

\begin{align*}
V_{\varepsilon J, K}(x) &= \prod_{j \in J} w(\varepsilon_j x_j) \prod_{j, j' \in J, j < j'} v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + 1) \\
&\quad \times \prod_{\varepsilon_j x_j + x_k} v(\varepsilon_j x_j - x_k), \\
U_{K, \rho}(x) &= (-1)^{\rho} \sum_{L \subset K, |L| = \rho} \prod_{l \in L} w(\varepsilon_l x_l) \prod_{i, j \in L, i < j} v(\varepsilon_l x_l + \varepsilon_j x_j) v(-\varepsilon_l x_l - \varepsilon_j x_j - 1) \\
&\quad \times \prod_{\varepsilon_l x_l + x_k} v(\varepsilon_l x_l - x_k). \end{align*}
(where \( v \) and \( w \) are the same as in (5.15)). In the above expressions we have used the conventions that empty products are equal to one, and that \( U_{K, b} = 1 \) for \( p = 0 \). Notice that the coefficient functions \( V_{r, l, K}(x) \) and \( U_{K, p}(x) \) are regular for \( x \in \rho + \Lambda_N \) and that the condition \( e_{x, l} + \nu \in \Lambda_N \) in the summation again guarantees that \((D_{qR}^\rho f)(\rho + \nu)\) only depends on the values of \( f \) on the grid points in \( \rho + \Lambda_N \) when \( \nu \) lies in \( \Lambda_N \). (Hence the operator \( D_{qR}^\rho \) is well-defined as an operator in \( L^2(\rho + \Lambda_N, \Delta_{qR}) \).) After introducing also the corresponding multiplication operators \( E_r : L^2(\rho + \Lambda_N, \Delta_{qR}) \to L^2(\rho + \Lambda_N, \Delta_{qR}) \) given by

\[
(E_r f)(\rho + \nu) = E_{r, \nu} f(\rho + \nu)
\]

\((r = 1, \ldots, n)\) with

\[
E_{r, \nu} = 2^r \sum_{J \subset \{1, \ldots, n\}} (-1)^{|J|} \left( \prod_{j \in J} \cos(\alpha \rho_j + \nu_j) \right) \times \sum_{r \leq l_1 \leq \cdots \leq l_{|J|} \leq n} \cos(\alpha \rho_{l_1}) \cdots \cos(\alpha \rho_{l_{|J|}})
\]

(where the second sum in \( E_{r, \nu} \) should be read as 1 when \(|J| = r\)) and the associated dual difference operators \( \hat{D}_{qR}^\rho : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \) and dual multiplication operators \( \hat{E}_r : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \), we are in the position to formulate a generalization of Theorem 6.5 pertaining to these higher-order discrete difference operators.

**Theorem 6.6.** For parameters in the positivity domain (5.5a), (5.5b) the commuting discrete difference operators \( D_{qR}^\rho : L^2(\rho + \Lambda_N, \Delta_{qR}) \to L^2(\rho + \Lambda_N, \Delta_{qR}) \) and \( \hat{D}_{qR}^\rho : L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \) are self-adjoint and the map \( K : L^2(\rho + \Lambda_N, \Delta_{qR}) \to L^2(\hat{\rho} + \Lambda_N, \hat{\Delta}_{qR}) \) constitutes the (unitary) joint eigenfunction transformation simultaneously diagonalizing these operators

\[
KD_{qR}^\rho K^{-1} = \hat{E}_r, \quad K^{-1} \hat{D}_{qR}^\rho K = E_r
\]

\((r = 1, \ldots, n)\).

The proof of Theorem 6.6 runs along the same lines as that of Theorem 6.5 and hinges on the discrete difference equations in Remark 3.2 and the real-valuedness of the multiplication operators \( E_r \) and \( \hat{E}_r \) for parameters in the positivity domain (5.5a), (5.5b). For \( r = 1 \) Theorem 6.6 reduces to Theorem 6.5.

**Appendix A. Proof for the Symmetry of **\( D \)

In this appendix we prove Proposition 3.3 which stated that the \( q \)-difference operator \( D \) (2.4) is symmetric with respect to the bilinear form \( \langle \cdot, \cdot \rangle_{qR} \) (3.17), for parameters satisfying the truncation condition (3.6). This proposition was a key ingredient in our orthogonality proof for the multivariable \( q \)-Racah polynomials (i.e. the multivariable Askey-Wilson polynomials with parameters subject to the truncation condition) with respect to the bilinear form \( \langle \cdot, \cdot \rangle_{qR} \) (3.17) (Theorem 5.2).
Let us first recall the explicit form of the functions $V_{\pm j}(z)$ that determine the coefficients of the difference operator $D\ (2.4)$:

\begin{equation}
V_{\epsilon j}(z) = \frac{(1-t_0 z_j^\varepsilon)(1-t_1 z_j^\varepsilon)(1-t_2 z_j^\varepsilon)(1-t_3 z_j^\varepsilon)}{(1-z_j^\varepsilon)(1-q z_j^\varepsilon)} \times \prod_{i \leq k \leq n, k \neq j} \frac{(1-t z_j^\varepsilon z_k)(1-t z_j^\varepsilon z_k^{-1})}{(1-z_j^\varepsilon z_k)(1-z_j^\varepsilon z_k^{-1})}, \quad \varepsilon = \pm 1.
\end{equation}

The symmetry proof for the operator $D$ hinges on two lemmas. The first lemma describes a relation between the discrete weight function $\Delta_{qR}(\nu)$ \([3.7]\) and the coefficients $V_{\epsilon j}(z)$ of $D$ evaluated at the grid points $\tau q^{\nu}$ \([3.7]\).

**Lemma A.1.** Let us assume that $\nu$ and $\nu + \varepsilon e_j$ are in the dominant cone $\Lambda\ (2.2)$ (here $\varepsilon$ is $+1$ or $-1$). Then

\begin{equation}
\Delta_{qR}(\nu + \varepsilon e_j)V_{\epsilon -j}(\tau q^{\nu + \varepsilon e_j}) = \Delta_{qR}(\nu)V_{\epsilon j}(\tau q^{\nu})
\end{equation}

(where $\tau q^{\nu}$, $\Delta_{qR}(\nu)$ and $V_{\epsilon j}(z)$ are given by \([3.7]\), \([3.10]\) and \([A.1]\)).

**Proof.** The relation \([A.2]\) between $\Delta_{qR} = 1/(C^\nu_+ C^\nu_-)$ \([3.10]\) and $V_{\pm j}$ follows from the difference equations

\begin{equation}
\frac{C^R_+(\hat{\nu} - e_j)}{C^R_-(\hat{\nu})} = V_{+j}(\tau q^{\nu - e_j})f_j(\hat{\nu}) \quad \text{for} \quad \hat{\nu}, \hat{\nu} - e_j \in \Lambda
\end{equation}

and

\begin{equation}
\frac{C^R_-(-\hat{\nu} + e_j)}{C^R_+(\hat{\nu})} = V_{-j}(\tau q^{\nu + e_j})f_j(\hat{\nu} + e_j) \quad \text{for} \quad \hat{\nu}, \hat{\nu} + e_j \in \Lambda
\end{equation}

linking the $c$-functions $C^R_{\pm}$ to the coefficients $V_{\pm j}$. The factors $f_j$ in \([A.3]\) and \([A.4]\) represent certain intermediate products of the form

$$f_j(\hat{\nu}) \equiv t^{j-n}(t_0 t_1 t_2 t_3 q^{-1})^{-1/2} \prod_{1 \leq k < j} \frac{(1-t_j^{-1} t_k q^{\hat{\nu}_k - \hat{\nu}})(1-t_j^{-1} q^{\hat{\nu}_k - \hat{\nu}_k^{-1}})}{(1-t_j^{-1} t_k q^{\hat{\nu}_k - \hat{\nu}})(1-t_j^{-1} t_k^{-1} q^{\hat{\nu}_k - \hat{\nu}_k^{-1}})}$$

(where we have used the convention that an empty product is equal to one), which cancel each other in the final relation \([A.2]\). The verification of the difference equations \([A.3], [A.4]\) is straightforward using the explicit expressions for $C^R_+$, $C^R_-$ and $V_{\pm j}$ (in \([3.11], [3.13]\) and \([A.1]\)) and the elementary shift property $(a; q)_{l+1} = (1 - a q^l)(a; q)_l$ for the $q$-shifted factorial $(l = 0, 1, 2, 3 \ldots)$.

**Lemma A.2.** For $\nu \in \Lambda_N\ (3.3)$ and parameters subject to the truncation condition \([3.6]\), one has that

\begin{equation}
V_{\epsilon j}(\tau q^{\nu}) = 0 \quad \text{if} \quad \nu + \varepsilon e_j \not\in \Lambda_N \quad (\varepsilon = +1 \text{ or } -1).
\end{equation}

(Here $V_{\epsilon j}(z)$ and $\tau q^{\nu}$ are taken from \([A.1]\) and \([3.7]\)).

**Proof.** There are two situations that need to be distinguished: either $\nu + \varepsilon e_j$ is in $\Lambda\ (2.2)$ but outside the the alcove $\Lambda_N\ (3.5)$, or $\nu + \varepsilon e_j$ does not even lie in the cone $\Lambda\ (2.2)$.

The first situation occurs (only) when $\varepsilon = +1$ with $j = 1$ and $\nu_1 = N$. We then have that $V_{+1}(\tau q^{\nu}) = 0$ because in the numerator the factor $(1 - t_1 z_1) =$
\[(1 - t_0 \tau_1 q^{\nu_0}) = (1 - t^{-1} t_a t_b q^N)\] becomes identical to zero in view of the truncation condition (3.6).

The second situation can occur both when \(\varepsilon = +1\) or when \(\varepsilon = -1\). For \(\varepsilon = +1\) we have that \(\nu + \varepsilon e_j \not\in \Lambda\) iff \(j > 1\) and \(\nu_{j-1} = \nu_j\). Then \(V_{+j}(\tau q^{\nu}) = 0\) because in the numerator the factor \((1 - t z_j z_j^{-1}) = (1 - t \tau_j \tau_j^{-1} q^{\nu_j - \nu_{j-1}}) = (1 - t t^{-1})\) is identically zero. For \(\varepsilon = -1\) we have that \(\nu + \varepsilon e_j \not\in \Lambda\) iff either \(j = n\) and \(\nu_n = 0\) or if \(j < n\) and \(\nu_{j+1} = \nu_j\). In the former case \(V_{-n}(\tau q^{\nu}) = 0\) because in the numerator contains a factor \((1 - t a z_n^{-1}) = (1 - t a \tau_n^{-1} q^{-\nu_n}) = (1 - t a t^{-1}) = 0\), whereas in the latter case one has that \(V_{-j}(\tau q^{\nu}) = 0\) because in the numerator one has a factor \((1 - t z_j^{-1} z_{j+1}) = (1 - t \tau_j^{-1} \tau_{j+1} q^{-\nu_j + \nu_{j+1}}) = (1 - t t^{-1}) = 0\). \(\square\)

After these preliminaries we are finally set to prove the symmetry relation

\[(A.6) \quad \langle Df, g \rangle_{\mathcal{N}}^{\text{qR}} = \langle f, Dg \rangle_{\mathcal{N}}^{\text{qR}} \quad (f, g \in \mathcal{H}_{\mathcal{N}}^{\text{qR}})\]

for parameters subject to the truncation condition (3.6). Evidently this amounts to showing that (cf. the definition of \(\langle \cdot, \cdot \rangle_{\mathcal{N}}^{\text{qR}}\) in (3.16))

\[(A.7) \quad \sum_{\nu \in \Lambda_N} \sum_{1 \leq j \leq n, \varepsilon = \pm 1} V_{\varepsilon j}(\tau q^{\nu}) f(\tau q^{\nu + \varepsilon e_j}) g(\tau q^{\nu}) \Delta_{\text{qR}}(\nu) = \sum_{\nu \in \Lambda_N} \sum_{1 \leq j \leq n, \varepsilon = \pm 1} V_{-\varepsilon j}(\tau q^{\nu}) f(\tau q^{\nu - \varepsilon e_j}) g(\tau q^{\nu}) \Delta_{\text{qR}}(\nu).\]

(At both sides of this equation the coefficients \(V_{\pm j}\) are well-defined as rational expressions in the parameters subject to the truncation condition (3.6), i.e., no denominator becomes identical to zero.) Using Lemma (A.3) the sums at both sides of (A.7) can be restricted resulting in the equation

\[(A.8) \quad \sum_{1 \leq j \leq n, \varepsilon = \pm 1} \sum_{\nu \in \Lambda_N} V_{\varepsilon j}(\tau q^{\nu}) f(\tau q^{\nu + \varepsilon e_j}) g(\tau q^{\nu}) \Delta_{\text{qR}}(\nu) = \sum_{1 \leq j \leq n, \varepsilon = \pm 1} \sum_{\nu \in \Lambda_N} V_{-\varepsilon j}(\tau q^{\nu}) f(\tau q^{\nu - \varepsilon e_j}) g(\tau q^{\nu}) \Delta_{\text{qR}}(\nu).\]

To check the identity (A.8) (and thus proving (A.6)), one uses Lemma (A.4) to infer that for given \(j\) and \(\varepsilon\) each term on the l.h.s. coincides with a term on the r.h.s., where \(\nu\) and \(\nu\) are related by \(\tilde{\nu} = \nu + \varepsilon e_j\). Phrased in other words: substituting in (the terms corresponding to given \(j\) and \(\varepsilon\) at) the r.h.s. of (A.8) \(\tilde{\nu} = \nu + \varepsilon e_j\) and invoking of Lemma (A.1) results in (the corresponding terms at) the l.h.s. of (A.8), which completes the proof of Proposition (3.3).

Remark A.1. Notice that in the restriction of the sums, i.e. in passing from (A.7) to (A.8), the truncation condition (3.6) became essential. Moreover, it was only after this restriction of the sums that the terms at both sides of the equation could be put into one-to-one correspondence. For generic parameters not satisfying any truncation condition the proof breaks down at this step because there will be nonzero terms of the form

\[V_{+1}(\tau q^{\nu}) f(\tau q^{\nu + e_1}) g(\tau q^{\nu}) \Delta_{\text{qR}}(\nu)\]
with \( \nu_1 = N \) in the l.h.s. of (A.7) that do not match with terms in the r.h.s. and, reversely, will there be nonzero terms of the form

\[
V_{\nu+1}(\tau q^\nu) f(\tau q^\nu) g(\tau q^{\nu+1}) \Delta^{\nu R}(\hat{\nu})
\]

with \( \hat{\nu}_1 = N \) that have no counterpart in the l.h.s. Therefore, in general equation (A.7) (and hence the symmetry of \( D \) with respect to \( \langle \cdot, \cdot \rangle_N^{\nu R} \)) will no longer hold if no truncation condition is assumed.

**Appendix B. Computation of orthonormalization constants using Pieri type formulas**

The key to the computation of the sum \( \langle p_\lambda, p_\lambda \rangle_N^{\nu R} \) is a system of Pieri type recurrence relations for the renormalized multivariable Askey-Wilson polynomials

\[(B.1)\quad P_\lambda(z) \equiv C^{\nu R}_+(\lambda) p_\lambda(z), \quad \lambda \in \Lambda\]

with \( C^{\nu R}_+(\lambda) \) given by (4.2). Basically, the recurrence relations in question provide explicit expansion formulas of the type \( E_r(z) P_\lambda(z) = \sum_c c_r P_c(z) \) for the products of the basis elements \( P_\lambda(z) \) with certain \( W \)-invariant polynomials \( E_1(z), \ldots, E_n(z) \) that form a set of generators for the algebra \( H^W \) of all \( W \)-invariant Laurent polynomials in the variables \( z_1, \ldots, z_n \). Specifically, we have (see \cite{D1, D2})

\[(B.2)\quad E_r(z; \tau) P_\lambda(z) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} \sum_{e_j = \pm 1, j \in J; \ e_{\lambda,j} + \lambda \in \Lambda} \hat{U}_{r,Jr-|J|}(\hat{\tau} q^\lambda) \hat{V}_{e_J, J, \tau}(\hat{\tau} q^\lambda) P_{\lambda+e_J}(z), \quad r = 1, \ldots, n,
\]

where

\[
E_r(z; \tau) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r} \left( (-1)^{r-|J|} \prod_{j \in J} (z_j + z_j^{-1}) \right. \\
\left. \times \sum_{r \leq l_1 \leq \cdots \leq l_{r-|J|} \leq n} (\tau_{l_1} + \tau_{l_1}^{\lambda}) \cdots (\tau_{l_{r-|J|}} + \tau_{l_{r-|J|}}^{-1}) \right)
\]

(with \( \tau, \hat{\tau} \) taken from (8.8), (4.3)) and the expansion coefficients are governed by

\[
\hat{V}_{e_J, K}(z) = \prod_{j \in J} \hat{w}(z_j), \quad \hat{U}_{K, \nu}(z) = (-1)^{\nu} \sum_{L \subset K, |L| = \nu} \prod_{i \in L} \hat{v}(z_i) \prod_{i,k \in L, i < k} \hat{v}(q z_i z_k) \hat{v}(z_i z_k),
\]

\[
\hat{v}(z) = t^{-1/2} \frac{(1 - t \zeta)}{1 - \zeta}, \quad \hat{w}(z) = (1 - t \hat{\zeta})^{-1/2} \prod_{0 \leq r \leq 3} (1 - \hat{t}^r \zeta)(1 - q \zeta^2).
\]

Here we have employed the notation

\[
e_{e_J} = \sum_{j \in J} e_j e_j
\]
and we also used the conventions that empty products are equal to one, that $\hat{U}_{K,p} \equiv 1$ for $p = 0$, and that the second sum in $E_r(z; \tau)$ is equal to one when $|J| = r$.

Even though formula (3.2) holds for generic parameters, it is not an entirely trivial matter to perform the reduction to the case of parameters satisfying the truncation condition (3.4). The problem is that for parameters subject to the truncation condition the c-function $\hat{C}_{\varphi}^\mathcal{R}(\lambda)$ (3.11) becomes infinite for $\lambda \in \Lambda \setminus \Lambda_N$. (Recall that we have a zero in the denominator from the factor $(\hat{t}_a\hat{t}_b; q)_{\lambda_1} = (t^{n-1}\hat{t}_a\hat{t}_b; q)_{\lambda_1} = (t^{n-1}t_a t_b; q)_{\lambda_1}$, which is zero for $\lambda_1 > N$ when the parameters satisfy the truncation relation $t^{n-1}t_a t_b = q^{-N}$.) Consequently, the renormalized polynomials $P_\lambda(z)$ (3.1) are no longer well-defined for such parameters when $\lambda$ lies outside the alcove $\Lambda_N$ (3.5).

Given the fact that even if we start in the l.h.s. with a polynomial associated to a weight $\lambda \in \Lambda_N$, we may end up in the r.h.s. with some polynomials corresponding to weights outside the alcove $\Lambda_N$, it is clear that in its present form the Pieri type recurrence formulas (3.4) do not make sense for all $\lambda \in \Lambda_N$ when the parameters satisfy the truncation condition (3.4).

**Lemma B.1.** For $\lambda \in \Lambda_N$ and $\lambda + e_{\varepsilon J} \in \Lambda \setminus \Lambda_N$, we have that $\hat{V}_{\varepsilon J, \varepsilon J}((\hat{t}_J q^\lambda))$ is the product of a factor of the form $(1 - \hat{t}_a \hat{t}_b t^{n-1} q^N)$ and an expression that is rational in the parameters subject to the truncation condition (3.4) (no denominator becomes zero after imposing (3.4)).

**Proof.** For $\lambda \in \Lambda_N$ one has that $\lambda + e_{\varepsilon J} \in \Lambda \setminus \Lambda_N$ iff $\lambda_1 = N$ and the index set $J \subset \{1, \ldots, n\}$ contains the number 1 with $\varepsilon_1 = +1$. Then $\hat{V}_{\varepsilon J, \varepsilon J}((\hat{t}_J q^\lambda))$ picks up a factor $(1 - \hat{t}_a \hat{t}_b t^{n-1} q^N)$ from the part $\hat{w}(\hat{t}_\varepsilon q^\lambda) = \hat{w}(\hat{t}_a t^{n-1} q^N)$.

Lemma 3.1 tells us that the factor $(1 - \hat{t}_a \hat{t}_b t^{n-1} q^N)^{-1}$ in $P_{\lambda + e_{\varepsilon J}}(z)$ when $\lambda + e_{\varepsilon J} \in \Lambda \setminus \Lambda_N$ (stemming from the denominator of the normalization factor $\hat{C}_{\varphi}^\mathcal{R}(\lambda + e_{\varepsilon J})$), is compensated in formula (3.2) by a corresponding factor $(1 - \hat{t}_a \hat{t}_b t^{n-1} q^N)$ in the numerator of $\hat{V}_{\varepsilon J, \varepsilon J}((\hat{t}_J q^\lambda))$. Moreover, by combining this observation with Proposition 3.7 it is seen that if we restrict the variable $z$ to the grid points $\tau q^\nu$, $\nu \in \Lambda_N$, then we end up with a recurrence formula of the form in (3.2) in which the sum in the r.h.s. gets restricted to the weights of the form $\lambda + e_{\varepsilon J}$ that lie inside the alcove $\Lambda_N$.

**Proposition B.2.** For parameters subject to the truncation condition (3.4) and $\lambda, \nu \in \Lambda_N$ one has that

\[
E_r(\tau q^\nu; \tau) P_\lambda(\tau q^\lambda) = \sum_{\substack{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq r \\ \varepsilon_j = \pm 1, j \in J; e_{\varepsilon J} + \lambda \in \Lambda_N}} \hat{U}_{J, r - |J|}\left((\hat{t}_J q^\lambda)\right) \hat{V}_{\varepsilon J, \varepsilon J}(\hat{t}_J q^\lambda) P_{\lambda + e_{\varepsilon J}}(\tau q^\nu),
\]

$r = 1, \ldots, n$.

After shrinking of the domain of the variables to the grid points in the Pieri formulas and implementation of the truncation condition, we are now ready to compute the the sums $\langle p_\lambda, p_\lambda \rangle_N^\mathcal{R}$, $\lambda \in \Lambda_N$ by means of the same method that also led to the norms of the polynomials in the continuous case (3.2). Replacing the products at both sides of the identity

(B.3) \[
\langle E_r P_\lambda, P_{\lambda + \omega_r} \rangle_N^\mathcal{R} = \langle P_\lambda, E_r P_{\lambda + \omega_r} \rangle_N^\mathcal{R}, \quad \omega_r = \varepsilon_1 + \cdots + \varepsilon_r
\]

by the corresponding r.h.s. of the Pieri formula in Proposition 3.2 and using the orthogonality of the polynomials $P_\mu$ with respect to the bracket $\langle \cdot, \cdot \rangle_N^\mathcal{R}$ leads us to
a relation between \((P_\lambda, P_\lambda)_N^{qR}\) and \((P_{\lambda+\omega_r}, P_{\lambda+\omega_r})_N^{qR}\) in terms of the coefficients of the Pieri formula

\[
(B.4) \quad \hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(\hat{T}q^\lambda)/(P_{\lambda+\omega_r}, P_{\lambda+\omega_r})_N^{qR} = \hat{V}_{\{-1,\ldots,-r\},\{r+1,\ldots,n\}}(\hat{T}q^{\lambda+\omega_r})/(P_\lambda, P_\lambda)_N^{qR}
\]

(recall \(\hat{U}_{K,p} = 1\) for \(p = 0\)). In the above manipulations we have assumed that \(\lambda\) and \(\hat{\lambda} + \omega_r\) (i.e. \(\lambda\) augmented by the fundamental weight vector \(\omega_r = e_1 + \cdots + e_r\)) lie in the cone \(\Lambda_N\) and furthermore that the parameters satisfy the truncation condition. To solve the recurrence relation \((B.4)\) for \((P_\lambda, P_\lambda)_N^{qR}\) we exploit the following connection between the \(c\)-functions \(\hat{C}_c^{qR}\) and the coefficients \(\hat{V}_{\{\pm 1,\ldots,\pm r\},\{r+1,\ldots,n\}}\)

\[
\frac{\hat{C}_c^{qR}(\hat{\lambda})}{C_+^{qR}(\lambda + \omega_r)} = \hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(\hat{T}q^\lambda) \quad \text{for} \quad \hat{\lambda}, \lambda + \omega_r \in \Lambda_N,
\]

\[
\frac{\hat{C}_c^{qR}(\hat{\lambda} + \omega_r)}{C_+^{qR}(\hat{\lambda})} = \hat{V}_{\{-1,\ldots,-r\},\{r+1,\ldots,n\}}(\hat{T}q^{\hat{\lambda} + \omega_r}) \quad \text{for} \quad \hat{\lambda}, \lambda + \omega_r \in \Lambda_N,
\]

which is not difficult to derive with the aid the elementary shift property \((a;q)_l = (1 - aq^l)/(a;q)_l\) for the \(q\)-shifted factorial \((l = 0, 1, 2, 3\ldots)\). (Notice also that for \(r = 1\) these two formulas amount to the (dual versions) of the formulas \((A.3), (A.4)\) in the proof of Lemma \((A.1)\) specialized to the case \(j = 1\).) Using these two relations we can eliminate the coefficient functions \(\hat{V}_{\{\pm 1,\ldots,\pm r\},\{r+1,\ldots,n\}}\) from \((B.4)\) entailing

\[
(B.5) \quad (P_\lambda, P_\lambda)_N^{qR}\hat{\Delta}^{qR}(\lambda) = (P_{\lambda+\omega_r}, P_{\lambda+\omega_r})_N^{qR}\hat{\Delta}^{qR}(\lambda + \omega_r)
\]

(where \(\hat{\Delta}^{qR} = 1/(\hat{C}_c^{qR}\hat{C}_c^{qR})\) again denotes the ‘Plancherel’ measure. Since the (fundamental weight) vectors \(\omega_1, \ldots, \omega_n\) positively generate the cone \(\Lambda\) \((2.2)\) it follows that the l.h.s. of \((B.5)\) does not depend on \(\lambda \in \Lambda_N\) and so we obtain by comparing with the evaluation in \(\lambda = 0\) (so \(P_\lambda = 1\)) that

\[
(B.6) \quad (P_\lambda, P_\lambda)_N = \frac{\langle 1, 1 \rangle_N^{qR}}{\hat{\Delta}^{qR}(\lambda)}, \quad \lambda \in \Lambda_N
\]

which reads in monic form

\[
(B.7) \quad (p_\lambda, p_\lambda)_N = \Lambda^{qR}(\lambda)\langle 1, 1 \rangle_N \quad \lambda \in \Lambda_N
\]

with \(\Lambda^{qR}(\lambda) = \hat{C}_c^{qR}(\lambda)/\hat{C}_c^{qR}(\hat{\lambda})\).

**Remark B.1.** The proof for the Pieri type recurrence formulas in \([D2]\) is complete only for parameters satisfying a self-duality condition of the type

\[
(B.8) \quad t_aq = t_bt_ct_d
\]

(where \(t_c\) and \(t_d\) denote the two parameters complementing \(t_a\) and \(t_b\) such that \(\{t_a, t_b, t_c, t_d\} = \{t_0, t_1, t_2, t_3\}\)). This condition on the parameters implies that \(t_r = t_r\) and thus that \(P_\lambda(z) = P_\lambda(z)\). As was pointed out in Section 7.2 of \([D2]\), however, the Pieri type recurrence formulas would immediately follow for general parameters without self-duality condition once one would succeed in proving that Macdonald’s conjectured evaluation formula stating that \(P_\lambda(\tau) = 1\) holds for such general parameters (cf. also \([D2]\) Theorem 3) for a proof of the evaluation formula in the self-dual case with parameters subject to the condition \((B.8)\)). At the ‘CRM Workshop on algebraic methods and \(q\)-special functions’ in Montreal, May 1996, we learned from Prof. Macdonald that he has managed to produce such proof for
the evaluation formula with general parameters using an extension of the Cherednik approach towards the Macdonald polynomials. (See [C1, C2, M2] for this approach, which is deeply connected with the representation theory of affine Hecke algebras.) Therefore, we have formulated all our results here without imposing the self-duality condition, even though our own direct proof (i.e. without using the representation theory of affine Hecke algebras) at present actually requires at one point, viz. the verification of the Pieri type recurrence formulas in [D2] (or equivalently the proof of the evaluation formula \( P_\lambda(\tau) = 1 \)), assuming that this additional condition be satisfied.

**Remark B.2.** Another consequence of the evaluation formula \( P_\lambda(\tau) = 1 \) is (see [D2]) the duality relation for the renormalized multivariable Askey-Wilson polynomials originally conjectured by Macdonald

\[
P_\mu(\tau q^\nu) = \hat{P}_\nu(\hat{\tau} q^\mu), \quad \mu, \nu \in \Lambda,
\]

where \( \hat{P}_\nu(z) = \hat{C}_{\nu}^{\text{gr}}(\nu)\hat{p}_\nu(z) \) denotes the renormalized multivariable Askey-Wilson polynomial dual to \( P_\nu(z) = C_{\nu}^{\text{gr}}(\nu)p_\nu(z) \), i.e. with the parameters \( t_r \) being replaced by the dual parameters \( \hat{t}_r \). Applying the duality relation to the restricted recurrence formulas of Proposition B.2 leads us (up to dualization) to a system of discrete difference equations for the multivariable Askey-Wilson/q-Racah polynomials with parameters subject to the truncation condition

\[
\sum_{J \subseteq \{1, \ldots, n\}, 0 \leq |J| \leq r, \varepsilon_j = \pm 1, j \in J, e_{\varepsilon_j} + \nu \in \Lambda_N} U_{J; r - |J|}(\tau q^\nu) V_{\varepsilon_j, J; r - |J|}(\tau q^\nu) P_\lambda(\tau q^{\nu + e_{\varepsilon_j}}) = E_{r}(\tau q^\lambda; \hat{\tau}) P_\lambda(q^\nu),
\]

\( r = 1, \ldots, n \) (where \( V_{\varepsilon_j, J; r - |J|} \) and \( U_{J; r - |J|} \) are the dual versions of \( \hat{V}_{\varepsilon_j, J; r} \) and \( \hat{U}_{J; r - |J|} \) with the parameters \( t_r \) replacing \( \hat{t}_r \)). This system of discrete difference equations is the restriction to the grid \( \tau q^\nu, \nu \in \Lambda_N \) of the system of analytic difference equations for the multivariable Askey-Wilson polynomials introduced in [D1] (see also [D2, D3]). For \( r = 1 \) the discrete difference equation in question boils down (after multiplication by a constant factor \( t_0 \cdots t_3 q^{-1/2} \)) to the restriction to the grid points of the second order \( q \)-difference equation in Section 3 (Eq. (2.7b)) with parameters subject to the truncation condition (3.6):

\[
\sum_{\varepsilon_j = \pm 1, \nu + e_{\varepsilon_j} \in \Lambda_N} V_{\varepsilon_j}(\tau q^\nu) \left( p_\lambda(\tau q^{\nu + e_{\varepsilon_j}}) - p_\lambda(\tau q^\nu) \right) = E_{\lambda, \lambda} p_\lambda(\tau q^\nu),
\]

where \( V_{\varepsilon_j} \) and \( E_{\lambda, \lambda} \) are taken from (2.4) and (2.6).

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