ARITHMETIC VOLUMES OF UNITARY SHIMURA CURVES

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To Steve Kudla, on the occasion of his 70th birthday.

Abstract. We compute the arithmetic volumes of integral models of unitary Shimura curves. This establishes the base case of an inductive argument to compute the arithmetic volumes of unitary Shimura varieties of higher dimension, to appear in subsequent work of Bruinier and the author.

1. INTRODUCTION

We compute the arithmetic volume of the integral model of a Shimura curve of type GU(1, 1). This calculation provides the base case of an inductive argument to compute the arithmetic volumes of unitary Shimura varieties of higher dimension [BH].

1.1. Arithmetic volumes. Suppose $\mathcal{X}$ is an arithmetic surface, by which we mean a regular and separated Deligne-Mumford stack of dimension 2, flat and of finite type over $\mathbb{Z}$. The arithmetic Picard group $\widehat{\text{Pic}}(\mathcal{X})$ is the group of isomorphism classes of hermitian line bundles

$$\widehat{\Omega} = (\Omega, \cdot \cdot \cdot)$$

on $\mathcal{X}$. If $\mathcal{X}$ is proper over $\mathbb{Z}$ we may identify $\widehat{\text{Pic}}(\mathcal{X}) \cong \widehat{\mathcal{H}}^1(\mathcal{X})$, where the right hand side is the codimension one arithmetic Chow group of Gillet-Soulé [Sou92] (extended to Deligne-Mumford stacks in [KRY06]), and use the Arakelov intersection pairing

$$\langle \cdot, \cdot \rangle : \widehat{\mathcal{H}}^1(\mathcal{X}) \times \widehat{\mathcal{H}}^1(\mathcal{X}) \to \mathbb{R}$$

from (2.5.13) of [KRY06] to define the arithmetic volume $\widehat{\text{vol}}(\widehat{\Omega}) = \langle \widehat{\Omega}, \widehat{\Omega} \rangle$.

Of particular interest is the metrized Hodge bundle

$$\widehat{\omega}^\text{Hdg}_{A/\mathcal{X}} \in \widehat{\text{Pic}}(\mathcal{X})$$

associated to an abelian scheme $\pi : A \to \mathcal{X}$ of relative dimension $d$. The underlying line bundle is $\omega^\text{Hdg}_{A/\mathcal{X}} = \pi_* \Omega^d_{A/\mathcal{X}}$, where $\Omega^d_{A/\mathcal{X}} = \wedge^d_{\mathcal{O}_A} \Omega^1_{A/\mathcal{X}}$ is the

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determinant of the (locally free of rank $d$) sheaf of Kähler differentials on $A$, and the hermitian metric is defined by the relation

$$\|s_x\|^2 = \left| \frac{1}{(2\pi i)^d} \int_{A_x(\mathbb{C})} s_x \wedge \overline{s}_x \right|$$

for any vector $s_x \in H^0(A_x, \Omega^d_{A_x/\mathbb{C}})$ in the fiber at $x \in X(\mathbb{C})$.

1.2. **Unitary Shimura curves.** Let $k$ be a quadratic imaginary field of odd discriminant $\text{disc}(k)$, and let $W$ be a $k$-hermitian space of signature $(1,1)$ that contains a self-dual $\mathcal{O}_k$-lattice. For any prime $p \mid \text{disc}(k)$, abbreviate

$$p^\circ = \begin{cases} p & \text{if } W \otimes \mathbb{Q} / \mathbb{Q}_p \text{ is isotropic} \\ -p & \text{otherwise.} \end{cases}$$

To this data one can associate an arithmetic surface

$$X_W \to \text{Spec}(\mathbb{Z}),$$

defined as a moduli space of principally polarized abelian surfaces with $\mathcal{O}_k$-actions (satisfying some additional constraints that need not concern us at the moment). We show that there is a decomposition

$$X_W \cong \bigsqcup_{L \in \mathcal{L}_W} \mathcal{C}_L$$

into connected components indexed by the set $\mathcal{L}_W$ of isometry classes of self-dual $\mathcal{O}_k$-lattices $L \subset W$, and that

$$|\mathcal{L}_W| = \frac{|\text{CL}(k)|}{2^{o(k)-1}}$$

where $\text{CL}(k)$ is the ideal class group of $k$, and $o(k)$ is the number of prime divisors of $\text{disc}(k)$; see Propositions 3.2.1 and 4.4.1. Each $\mathcal{C}_L \to \text{Spec}(\mathbb{Z})$ has geometrically connected fibers, and is proper if $W$ is anisotropic.

**Theorem A.** Fix a connected component $\mathcal{C}_L \subset X_W$, and let $A \to \mathcal{C}_L$ be the restriction of the universal abelian surface. If $W$ is anisotropic then

$$\hat{\text{vol}}(\omega^\text{Hdg}_{A/\mathcal{C}_L}) = -\deg_{\mathcal{C}}(\omega^\text{Hdg}_{A/\mathcal{C}_L}) \left( 1 + \frac{2\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \sum_{p \mid \text{disc}(k)} \frac{1 - p^\circ}{1 + p^\circ} \cdot \log(p) \right),$$

where $\zeta(s)$ is the Riemann zeta function, and

$$\deg_{\mathcal{C}}(\omega^\text{Hdg}_{A/\mathcal{C}_L}) = \frac{1}{12 \cdot |\mathcal{O}_k^\times|} \prod_{p \mid \text{disc}(k)} (1 + p^\circ)$$

is the degree of the Hodge bundle restricted to the complex orbifold $\mathcal{C}_L(\mathbb{C})$.

Theorem A is stated in the text as Theorem 5.1.1. When $W$ is isotropic there is a similar statement, but one must use the Burgos-Kramer-Kuhn [BGKK07] extension of the Gillet-Soulé theory on a compactification of $\mathcal{C}_L$. See Theorem 5.3.2.
1.3. **Outline of the proof.** We will prove Theorem A by reducing it to the analogous result for compact quaternionic Shimura curves proved by Kudla-Rapoport-Yang [KRY06], and for modular curves proved by Bost and Kühn [Ku01].

Suppose $\mathcal{O}_B \subset B$ is a maximal order in an indefinite quaternion algebra over $\mathbb{Q}$, and $N$ is a squarefree integer prime to $\text{disc}(B)$. In §2 we recall the quaternionic Shimura curve $X_B(N)$ parametrizing triples $(A_0, A_1, f)$ consisting of abelian surfaces $A_0$ and $A_1$ with $\mathcal{O}_B$-actions, together with an $\mathcal{O}_B$-linear isogeny $f : A_0 \to A_1$ of degree $N^2$. To each level $N$ Eichler order $R \subset \mathcal{O}_B$ we associate an abelian surface with $R$-action

$$A_R \to X_B(N)$$

in such a way that the universal $f : A_0 \to A_1$ factors as a composition

$$A_0 \to A_R \to A_1$$

of $R$-linear isogenies of degree $N$. We call $A_R$ the **intermediate abelian surface** determined by the Eichler order $R$. The main result of §2 is Theorem 2.3.1 which computes the arithmetic volume of the metrized Hodge bundle of $A_R$ by comparing it to that of $A_0$, which is known by the works cited above.

In §3 we explain how the hermitian space $W$ of signature $(1, 1)$ determines an indefinite quaternion algebra $B$ with an embedding $k \to B$, and how the self-dual $\mathcal{O}_k$-lattice $L \subset W$ determines an Eichler order $R \subset B$ of level

$$N = -\text{disc}(k)/\text{disc}(B).$$

In §4 we construct a finite étale surjection $q_L : X_B(N) \to \mathcal{C}_L$ of degree $|\mathcal{O}_k^\times|$ in such a way that the pullback of the universal $A \to \mathcal{C}_L$ is precisely the intermediate abelian surface $A_R$ of the Eichler order determined by $L$.

As explained at the beginning of §5 Theorem A follows from the above constructions. The remainder of §5 is devoted to extending that theorem to the case of isotropic $W$.

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2. **Quaternionic Shimura curves**

Over a quaternionic Shimura curve with level structure there is a universal isogeny $f : A_0 \to A_1$ of abelian surfaces, each equipped with an action of a fixed maximal order $\mathcal{O}_B$ in an indefinite quaternion algebra. The main result of this section is the computation of the arithmetic volume of the metrized Hodge bundle of a third abelian surface, lying between $A_0$ and $A_1$, and depending on a choice of Eichler order in $\mathcal{O}_B$. 


2.1. The moduli problem. By a quaternion algebra we mean a central simple \( \mathbb{Q} \)-algebra of dimension 4. Any quaternion algebra \( B \) admits a \( \mathbb{Q} \)-basis \( \{1, i, j, ij\} \) with \( i^2 = a, j^2 = b, \) and \( ij = -ji \), for some \( a, b \in \mathbb{Q}^\times \). We write

\[
B \cong \left( \frac{a, b}{\mathbb{Q}} \right)
\]

to indicate the existence of such a basis. For any place \( p \ll \infty \) of \( \mathbb{Q} \) the local invariant is defined by

\[
\text{inv}_p(B) = (a, b)_p,
\]

where the right hand side is the Hilbert symbol. Two quaternion algebras are isomorphic if and only if their local invariants agree at all places. As usual, \( B \) is indefinite if \( \text{inv}_\infty(B) = 1 \).

Fix an indefinite quaternion algebra \( B \) and a maximal order \( \mathcal{O}_B \subset B \).

The maximal order is unique up to \( B^\times \)-conjugacy, and so the particular choice is unimportant. Let \( b \mapsto \overline{b} \) denote the main involution on \( B \), and denote by \( \text{Trd}(b) = b + \overline{b} \) and \( \text{Nrd}(b) = b\overline{b} \) the reduced trace and reduced norm.

**Definition 2.1.1.** An \( \mathcal{O}_B \)-abelian surface over a \( \mathbb{Z} \)-scheme \( S \) is an abelian scheme \( A \to S \) of relative dimension two, together with a ring homomorphism \( \mathcal{O}_B \to \text{End}(A) \) satisfying Drinfeld’s determinant condition: every \( b \in \mathcal{O}_B \) acts on the locally free \( \mathcal{O}_S \)-module \( \text{Lie}(A) \) with characteristic polynomial

\[
x^2 - (b + \overline{b})x + b\overline{b} \in \mathbb{Z}[x],
\]

viewed as a polynomial in \( \mathcal{O}_S[x] \).

**Remark 2.1.2.** The determinant condition stated here is different in appearance from Drinfeld’s special condition, as defined in Remark III.3.3 of [BC91]. The equivalence of the two conditions can be easily proved using Proposition 2.1.3 of [Har15].

Fix a squarefree integer \( N > 0 \) prime to \( \text{disc}(B) \). Let \( \mathcal{X}_B(N) \) be the Deligne-Mumford stack whose functor of points assigns to a \( \mathbb{Z} \)-scheme \( S \) the groupoid of tuples \( (A_0, A_1, f) \) in which \( A_0 \) and \( A_1 \) are \( \mathcal{O}_B \)-abelian surfaces over \( S \), and \( f : A_0 \to A_1 \) is an \( \mathcal{O}_B \)-linear isogeny of degree \( N^2 \) whose kernel is contained in the \( N \)-torsion subgroup scheme \( A_0[N] \subset A_0 \).

It is known that \( \mathcal{X}_B(N) \) is an arithmetic surface, in the sense of §111 with geometrically connected fibers. It is smooth outside characteristics dividing \( N\text{disc}(B) \). If \( B \) is a division algebra then \( \mathcal{X}_B(N) \) is proper over \( \mathbb{Z} \). For all of this, see [BC91] and [Buz97].

**Definition 2.1.3.** As in the classical theory of modular curves, the Atkin-Lehner involution

\[
\tau : \mathcal{X}_B(N) \to \mathcal{X}_B(N)
\]

is the automorphism sending \( f : A_0 \to A_1 \) to the isogeny \( f^\vee : A_1 \to A_0 \) characterized by \( f^\vee \circ f = [N] \) and \( f \circ f^\vee = [N] \).
From now on we let \( f : A_0 \to A_1 \) denote the universal object over \( \mathcal{X}_B(N) \), and fix a level \( N \) Eichler order \( R \subset \mathcal{O}_B \). The finite flat group scheme \( \ker(f) \to \mathcal{X}_B(N) \) has order \( N^2 \), and carries an action of (2.1.1) \[ \mathcal{O}_B/N\mathcal{O}_B \cong M_2(\mathbb{Z}/N\mathbb{Z}) \].

We choose (2.1.1) in such a way that \( R/N\mathcal{O}_B \) is identified with the upper triangular matrices, and define orthogonal idempotents in (2.1.1) by \[ e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \]

One can easily check that \( C_R = e \cdot \ker(f) \) is characterized as the unique \( R \)-stable finite flat subgroup scheme of \( \ker(f) \) of order \( N \). In particular, while it depends on the Eichler order \( R \subset \mathcal{O}_B \), it does not depend on the isomorphism (2.1.1).

**Definition 2.1.4.** The quotient \( A_R = A_0/C_R \) is the intermediate abelian surface determined by the Eichler order \( R \). It is an abelian surface over \( \mathcal{X}_B(N) \) endowed with an action \( R \to \text{End}(A_R) \), and with \( R \)-linear isogenies \[ A_0 \xrightarrow{\phi} A_R \xrightarrow{\psi} A_1 \] each of degree \( N \).

The following lemma will be needed in the proof of Theorem 2.3.1.

**Lemma 2.1.5.** There is an isogeny \( \tau^* A_R \to A_R \) of degree prime to \( N \).

**Proof.** If we set \( C_R' = e' \cdot \ker(f) \), there is a chain of finite flat group schemes \[ 0 \subset C_R \subset C_R \oplus C_R' \subset eA_0[N] \oplus C_R' \subset A_0[N] \] of orders 1, \( N \), \( N^2 \), \( N^3 \), and \( N^4 \), respectively. Taking the quotients of \( A_0 \) by these subgroups yields a chain of degree \( N \) isogenies \[ A_0 \to A_R \to A_1 \to eA_0[N] \oplus C_R' \to A_0 \] whose composition is \([N]\). The composition of the first two arrows is \( f \), hence the composition of the second two arrows is \( f' \). Thus \[ \tau^* A_R \cong \frac{A_0}{eA_0[N] \oplus C_R'}, \] as both sides are characterized as the unique \( R \)-stable finite flat subgroup scheme of \( \ker(f') \) of order \( N \).

Recalling the fixed isomorphism (2.1.1), choose any lift \( \gamma \in \mathcal{O}_B \) of \[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \cong \mathcal{O}_B/N\mathcal{O}_B \] of reduced norm \( \gamma^r = NM \) with \( M \) an integer coprime to \( N \). The degree \((NM)^2\) endomorphism \( \gamma \in \text{End}(A_0) \) descends to a degree \( M^2 \) isogeny \[ \tau^* A_R \cong \frac{A_0}{eA_0[N] \oplus C_R'} \xrightarrow{\gamma} A_0/C_R \cong A_R. \qed \]
Remark 2.1.6. If $B \cong M_2(\mathbb{Q})$, we may choose an isomorphism $\mathcal{O}_B \cong M_2(\mathbb{Z})$ in such a way that

$$R \cong \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}.$$  

Such a choice determines an isomorphism from the moduli space of elliptic curves with $\Gamma_0(N)$-structure to $X_B(N)$, under which $f : A_0 \to A_1$ is identified with the square

$$(2.1.2) \quad \varphi \times \varphi : E_0 \times E_0 \to E_1 \times E_1,$$

of the universal cyclic $N$-isogeny elliptic curves. Under this identification,

$$C_R = \ker(\varphi) \times \{0\},$$

the intermediate abelian surface is $A_R = E_1 \times E_0$, and $\tau^* A_R = E_0 \times E_1$. The isogeny

$$E_0 \times E_1 = \tau^* A_R \rightarrow A_R = E_1 \times E_0,$$

determined by the choice of

$$\gamma = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} \in M_2(\mathbb{Z})$$

in the proof of Lemma 2.1.5 is the isomorphism $(x_0, x_1) \mapsto (x_1, x_0)$.

2.2. Comparison of hermitian line bundles. We continue with the notation of the previous subsection. Our goal is to compare the metrized Hodge bundle of the universal $A_0 \to X_B(N)$ with that of the intermediate abelian surface $A_R \to X_B(N)$ determined by an Eichler order $R \subset \mathcal{O}_B$ of level $N$.

By results of Katz-Mazur [KMS5], extended to quaternionic Shimura curves in [Buz97], the reduction of $\mathcal{X}_B(N)$ at a prime $p \mid N$ is a union

$$\mathcal{X}_B(N)_{F_p} = \mathcal{F}_p \cup \mathcal{V}_p$$

of two irreducible components, interchanged by the Atkin-Lehner involution of Definition (2.1.3). Each has a natural structure of reduced closed substack, and the fiber product

$$(2.2.1) \quad \mathcal{S}_p = \mathcal{F}_p \times_{\mathcal{X}_B(N)} \mathcal{V}_p$$

is the reduced locus of supersingular points.

One distinguishes between $\mathcal{F}_p$ and $\mathcal{V}_p$ as follows: after restricting to the two components, the universal isogeny $f : A_0 \to A_1$ admits factorizations

$$(2.2.2) \quad A_0/\mathcal{F}_p \xrightarrow{\text{Fr}} A_0^{(p)}/\mathcal{F}_p \to A_1/\mathcal{F}_p$$

and

$$(2.2.3) \quad A_0/\mathcal{V}_p \to A_1^{(p)}/\mathcal{V}_p \xrightarrow{\text{Ver}} A_1/\mathcal{V}_p,$$

where Fr and Ver are the usual Frobenius and Verschiebung morphisms, and the unlabelled arrows are isogenies of degree $(N/p)^2$. 

By the regularity of \(X_B(N)\), the vertical Weil divisors \(F_p\) and \(V_p\) are each defined locally by a single equation, and so determine line bundles \(O_pF_p\) and \(O_pV_p\) on \(X_B(N)\). Noting that these line bundles are canonically trivialized in the complex fiber by the constant function 1, we endow each with the constant metric \(\|1\| = \sqrt{p}\). The resulting hermitian line bundles are denoted \((2.2.4)\)

\[ \hat{F}_p, \hat{V}_p \in \hat{\text{Pic}}(X_B(N)). \]

As the tensor product \(O(F_p) \otimes O(V_p) \cong O(X_B(N))\) is trivialized by the global section \(p^{-1}\) of constant norm 1, we have the relation \((2.2.5)\)

\[ \hat{F}_p + \hat{V}_p = 0 \]

in the arithmetic Picard group.

**Proposition 2.2.1.** The metrized Hodge bundles of \(A_0\) and \(A_R\) satisfy

\[ \hat{\omega}_{A_0/X_B(N)}^{\text{Hdg}} = \hat{\omega}_{A_R/X_B(N)}^{\text{Hdg}} + \sum_{p | N} \hat{F}_p \in \hat{\text{Pic}}(X_B(N)). \]

**Proof.** The canonical isogeny \(\phi : A_0 \to A_R\) induces a morphism \((2.2.6)\)

\[ \phi^* : \omega_{A_R/X_B(N)}^{\text{Hdg}} \to \omega_{A_0/X_B(N)}^{\text{Hdg}} \]

of Hodge bundles. If we view this as a section \((2.2.7)\)

\[ \phi^* \in H^0(X_B(N), \omega_{A_0/X_B(N)}^{\text{Hdg}} \otimes (\omega_{A_R/X_B(N)}^{\text{Hdg}})^{-1}), \]

then unpacking the definition of the metric \((1.1.1)\) shows that \((2.2.8)\)

\[ \|\phi^*\| = \sqrt{\text{deg}(\phi)} = \sqrt{N}. \]

The morphism of vector bundles \(\phi : \text{Lie}(A_0) \to \text{Lie}(A_R)\) restricts to an isomorphism over \(X_B(N)/\mathbb{Z}[1/N]\), and hence so does \((2.2.6)\). From this it follows that

\[ \text{div}(\phi^*) = \sum_{p | N} a_p F_p + \sum_{p | N} b_p V_p \]

for nonnegative \(a_p, b_p \in \mathbb{Z}\). We will show that all \(a_p = 1\) and all \(b_p = 0\).

Suppose

\[ \text{Spec}(k) \to X_B(N)_{\overline{\mathbb{F}}_p} = X_B(N)_{\mathbb{F}_p} \setminus S_p \]

is a geometric point of the ordinary locus. For \(i \in \{0, 1\}\) the \(p\)-divisible group of \(A_{i,x}\) sits in a connected-étale sequence

\[ 0 \to H^0_{i,x} \to A_{i,x}[p^\infty] \to H^\text{et}_{i,x} \to 0, \]

and the connected part satisfies \((2.2.9)\)

\[ H^0_{i,x} \cong \mu_{p^{\infty}/k} \times \mu_{p^{\infty}/k}. \]

The isogeny \(f : A_0 \to A_1\) induces an isogeny of \(p\)-divisible groups \((2.2.10)\)

\[ f^0_x : H^0_{0,x} \to H^0_{1,x}. \]
If the geometric point \( x \) lies in \( V_p^{\text{ord}} = V_p \setminus S_p \), then (2.2.3) implies that the isogeny (2.2.10) factors as
\[
H^0_{0,x} \to H^{0,(p)}_{1,x} \xrightarrow{\text{Ver}} H^0_{1,x}.
\]
The first arrow is an isomorphism because the first arrow in (2.2.3) has degree prime to \( p \), and the second arrow is an isomorphism by (2.2.9). Hence (2.2.11) is an isomorphism. As \( \text{Lie}(H^0_{1,x}) \cong \text{Lie}(A_{1,x}) \), we deduce that \( f_x : A_{0,x} \to A_{1,x} \) induces an isomorphism \( \text{Lie}(A_{0,x}) \to \text{Lie}(A_{1,x}) \). This last map factors as
\[
\text{Lie}(A_{0,x}) \xrightarrow{\phi_x} \text{Lie}(A_{R,x}) \to \text{Lie}(A_{1,x}),
\]
and so the first arrow in the composition is also an isomorphism. It follows that the section (2.2.7) is nowhere vanishing on \( V_p^{\text{ord}} \), and so \( b_p = 0 \).

Suppose instead that the geometric point \( x \) lies in \( F_p^{\text{ord}} = F_p \setminus S_p \). In this case (2.2.2) implies that the isogeny (2.2.10) factors as
\[
H^0_{0,x} \xrightarrow{\text{Fr}} H^{0,(p)}_{0,x} \to H^0_{1,x}.
\]
The second arrow is an isomorphism because the second arrow in (2.2.2) has degree prime to \( p \), while (2.2.9) implies that the kernel of the first arrow is the \( p \)-torsion subgroup of \( H^0_{0,x} \). Thus the isomorphisms (2.2.9) may be chosen so that
(2.2.11) \[
\mu_{p^\infty/k} \times \mu_{p^\infty/k} \cong H^0_{0,x} \xrightarrow{f_0} H^0_{1,x} \cong \mu_{p^\infty/k} \times \mu_{p^\infty/k}
\]
is multiplication by \( p \). Again using \( \text{Lie}(H^0_{1,x}) \cong \text{Lie}(A_{1,x}) \), we now find that the isogeny \( f : A_0 \to A_1 \) induces the trivial map on Lie algebras. Thus (2.2.7) vanishes identically along \( F_p^{\text{ord}} \), and so \( a_p > 0 \).

To go further, let \( W = W(k) \) be the Witt ring. Taking the Serre-Tate canonical lifts of \( A_{0,x} \) and \( A_{1,x} \) to \( W \) provides us with a diagram
\[
\text{Spec}(k) \xrightarrow{x} \mathcal{X}_B(N) \xrightarrow{\tilde{x}} \text{Spec}(W).
\]
If we denote by \( H^0_{1,\tilde{x}} \) the connected part of the \( p \)-divisible group \( A_{1,\tilde{x}}[p^\infty] \) over \( W \), the Grothendieck-Messing deformation theory implies that the maps in (2.2.11) lift uniquely to
(2.2.12) \[
\mu_{p^\infty/W} \times \mu_{p^\infty/W} \cong H^0_{0,\tilde{x}} \xrightarrow{f_0} H^0_{1,\tilde{x}} \cong \mu_{p^\infty/W} \times \mu_{p^\infty/W},
\]
and this composition is still multiplication by \( p \). Taking Lie algebras, we find that
\[
W^2 \cong \text{Lie}(H_{0,\tilde{x}}) \cong \text{Lie}(A_{0,\tilde{x}}) \xrightarrow{f_{\tilde{x}}} \text{Lie}(A_{1,\tilde{x}}) \cong \text{Lie}(H_{0,\tilde{x}}) \cong W^2
\]
is multiplication by \( p \), and so each map in the factorization
\[
\text{Lie}(A_{0,\tilde{x}}) \xrightarrow{\phi_{\tilde{x}}} \text{Lie}(A_{R,\tilde{x}}) \xrightarrow{\psi} \text{Lie}(A_{0,\tilde{x}})
\]
of \( f_\overline{z} \) has cokernel isomorphic to \( W/pW \). It follows that the restriction of (2.2.6) to a morphism of rank one \( W \)-modules
\[
\phi^*: \omega_{A_{0,1}/W}^{\text{Hdg}} \to \omega_{A_{R,1}/W}^{\text{Hdg}}
\]
has cokernel isomorphic to \( W/pW \), and from this one deduces \( a_p = 1 \).

We have now shown that (2.2.7) has divisor \( \sum_{p \mid N} \mathcal{F}_p \) and norm (2.2.8), and the claim follows. \( \square \)

2.3. Arithmetic volumes. We continue to let \( R \subset O_B \) be an Eichler order of squarefree level \( N \), prime to \( \text{disc}(B) \), and assume that \( B \) is a division algebra. Hence \( X_B(N) \) is proper over \( \mathbb{Z} \).

We now combine results of Kudla-Rapoport-Yang on the metrized Hodge bundle of \( A_0 \to X_B^1 \) with Proposition 2.2.1 to obtain results on the Hodge bundle of the intermediate abelian surface of Definition 2.1.4.

**Theorem 2.3.1.** The Hodge bundle of \( A_R \to X_B(N) \) has geometric degree
\[
\deg_C(\omega_{A_R/X_B(N)}^{\text{Hdg}}) = \frac{1}{12} \prod_{p \mid \text{disc}(B)} (1 - p) \prod_{p \mid N} (1 + p)
\]
and arithmetic volume
\[
\hat{\text{vol}}(\omega_{A_R/X_B(N)}^{\text{Hdg}}) = -\deg_C(\omega_{A_R/X_B(N)}^{\text{Hdg}}) \times \left( 1 + \frac{2 \zeta'(-1)}{\zeta(-1)} + \sum_{p \mid \text{disc}(B)} \frac{1 + p}{1 - p} \cdot \log(p) \frac{2}{2} + \sum_{p \mid N} \frac{1 - p}{1 + p} \cdot \log(p) \frac{2}{2} \right).
\]

**Proof.** The metrized Hodge bundle of the universal \( A_0 \to X_B(N) \) has geometric degree
\[
(2.3.1) \quad \deg_C(\omega_{A_0/X_B(N)}^{\text{Hdg}}) = \frac{1}{12} \prod_{p \mid \text{disc}(B)} (1 - p) \prod_{p \mid N} (1 + p)
\]
and arithmetic volume
\[
(2.3.2) \quad \hat{\text{vol}}(\omega_{A_0/X_B(N)}^{\text{Hdg}}) = -\deg_C(\omega_{A_0/X_B(N)}^{\text{Hdg}}) \left( 1 + \frac{2 \zeta'(-1)}{\zeta(-1)} + \sum_{p \mid \text{disc}(B)} \frac{1 + p}{1 - p} \cdot \log(p) \frac{2}{2} \right).
\]

When \( N = 1 \) this is Corollary 7.11.2 of [KRY06], adjusted to account for the fact that our metrized Hodge bundle differs from the \( \omega \) appearing there by a rescaling of the metric; see especially (3.3.4) of loc. cit.. The forgetful morphism
\[
X_B(N) \to X_B^1
\]
is finite flat of degree \( \prod_{p \mid N} (1 + p) \) by Theorem 4.7 of [Buz97], and the formulas at level \( N \) follow from those at level 1 (for the arithmetic volume use the projection formula of Section III.3.1 of [Sou92]).
It is immediate from Proposition 2.2.1 that the metrized Hodge bundles of $A_R$ and $A_0$ are isomorphic in the complex fiber, and so the first claim of the theorem follows from (2.3.1).

Fix a prime $p | N$, and recall that the Atkin-Lehner involution $\tau$ of Definition (2.1.3) interchanges $F_p$ and $V_p$. Using this and (2.2.5) we find that

$$\tau^* \hat{\omega}_p = \hat{\omega}_p = -\hat{\omega}_p$$

in the arithmetic Chow group, and so the invariance of $\hat{\omega}$ under $\tau^*$ implies

(2.3.3) $$\hat{\deg}(\hat{\omega}_p \cdot \hat{\omega}^{Hdg}_{A_R/X_B(N)}) = \hat{\deg}((\hat{\omega}_p \cdot \tau^* \hat{\omega}^{Hdg}_{A_R/X_B(N)}).$$

On the other hand, any isogeny as in Lemma 2.1.5 induces a morphism

$$\hat{\omega}^{Hdg}_{A_R/X_B(N)} \rightarrow \tau^* \hat{\omega}^{Hdg}_{A_R/X_B(N)}$$

that restricts to an isomorphism over the mod $p$ fiber of $X_B(N)$, and in particular over $F_p$. In general, we have

$$\hat{\deg}(\hat{\omega}_p \cdot \hat{\Omega}) = \hat{\deg}(\hat{\Omega} |_{F_p}) \cdot \log(p)$$

for any hermitian line bundle $\hat{\Omega}$ on $X_B(N)$, and so the existence of such an isomorphism implies

(2.3.4) $$\hat{\deg}(\hat{\omega}_p \cdot \hat{\omega}^{Hdg}_{A_R/X_B(N)}) = \hat{\deg}(\hat{\omega}_p \cdot \tau^* \hat{\omega}^{Hdg}_{A_R/X_B(N)}).$$

Combining (2.3.3) and (2.3.4) shows

(2.3.5) $$\hat{\deg}(\hat{\omega}_p \cdot \hat{\omega}^{Hdg}_{A_R/X_B(N)}) = 0.$$

It follows from (2.2.5) that

$$-\hat{\omega}_p \cdot \hat{\omega}_p = \hat{\omega}_p = \hat{\omega}_p = (S_p, 0) \in \CH^2(X_B(N)),$$

where $(S_p, 0)$ is the reduced supersingular locus (2.2.1) endowed with the trivial Green current. Hence

$$-\hat{\vol}(\hat{\omega}_p) = \sum_{x \in S_p(S_p^{ab})} \frac{\log(p)}{|\text{Aut}(x)|}$$

(2.3.6) $$= \frac{\log(p)}{24} \left( \frac{p-1}{p+1} \right) \prod_{\ell | \text{disc}(B)} (\ell - 1) \prod_{\ell | N} (\ell + 1),$$

where the second equality is the Eicher-Deuring mass formula, extended to quaternionic Shimura curves by Yu [Yu08].

Combining Proposition 2.2.1 with (2.3.5) shows that

$$\hat{\vol}(\hat{\omega}^{Hdg}_{A_0/X_B(N)}) = \hat{\vol}(\hat{\omega}^{Hdg}_{A_R/X_B(N)}) + \sum_{p | N} \hat{\vol}(\hat{\omega}_p),$$

and combining this with (2.3.2) and (2.3.6) proves the second claim of the theorem. $\square$
3. HERMITIAN SPACES AND LATTICES

Let \( \mathbf{k} \) be a quadratic imaginary field of discriminant \( \text{disc}(\mathbf{k}) \). Eventually we will specialize to the case where the discriminant is odd.

3.1. Quaternion embeddings. Suppose \( W \) is a hermitian space over \( \mathbf{k} \), which we always assume to be finite dimensional and non-degenerate, and denote by \( \langle -, - \rangle \) the hermitian form. At each place \( p \leq \infty \) we define the local invariant

\[
\text{inv}_p(W) = (\text{det}(W), \text{disc}(\mathbf{k}))_p \in \{\pm 1\},
\]

where the right hand side is the Hilbert symbol, and \( \text{det}(W) \in \mathbb{Q}^\times/\text{Nm}(\mathbf{k}^\times) \) is the determinant of \( \langle x_i, x_j \rangle \) for any \( \mathbf{k} \)-basis of \( x_1, \ldots, x_d \in W \). Two \( \mathbf{k} \)-hermitian spaces of the same dimension are isomorphic if and only if they have the same signature and the same local invariants.

**Definition 3.1.1.** A quaternion embedding of \( \mathbf{k} \) is a quaternion algebra \( B \) together with an embedding of \( \mathbb{Q} \)-algebras \( \mathbf{k} \hookrightarrow B \).

**Remark 3.1.2.** The isomorphism class of a quaternion embedding \( \mathbf{k} \to B \) is completely determined by the isomorphism class of the underlying quaternion algebra \( B \), as the Noether-Skolem theorem implies that any two embeddings \( \mathbf{k} \to B \) are \( B^\times \)-conjugate.

Suppose \( \mathbf{k} \to B \) is an indefinite quaternion embedding. The main involution on \( B \) restricts to complex conjugation on \( \mathbf{k} \), so the notation \( \overline{a} \) for \( a \in \mathcal{O}_k \) is unambiguous. Up to \( \mathbf{k}^\times \)-scaling, there exists a unique \( j \in B^\times \) satisfying \( aj = j\overline{a} \) for all \( a \in \mathbf{k} \). Any such element satisfies \( j^2 \in \mathbb{Q}^\times \), and determines a decomposition

\[
(3.1.1) \quad B = \mathbf{k} \oplus \mathbf{k}j.
\]

Fix a \( \delta \in \mathbf{k}^\times \) with \( \delta^2 = \text{disc}(\mathbf{k}) \). The positive involution on \( B \) defined by

\[
(3.1.2) \quad b^\dagger = \delta \overline{b} \delta^{-1}
\]

restricts to complex conjugation on \( \mathbf{k} \), and to the identity on \( \mathbf{k}j \).

**Lemma 3.1.3.** Up to \( \mathbb{Q}^\times \)-scaling,

\[
(3.1.3) \quad \lambda(x, y) = \text{Trd}(\delta^{-1} x \overline{y})
\]

is the unique symplectic form \( \lambda : B \times B \to \mathbb{Q} \) satisfying

\[
(3.1.4) \quad \lambda(bx, y) = \lambda(x, b^\dagger y).
\]

**Proof.** The proof that (3.1.3) satisfies (3.1.4) is elementary, and left to the reader. For the uniqueness claim, any symplectic form \( \lambda \) on \( B \) satisfying (3.1.4) also satisfies, for all \( x, y \in \mathbf{k} \subset B \),

\[
\lambda(xj, y) = \lambda(\overline{y} xj, 1) = -\lambda(1, \overline{y} xj) = -\lambda(1, xjy) = -\lambda(xj, y).
\]
Thus $\lambda(xj, y) = 0$, and the two summands in (3.1.1) satisfy $\lambda(kj, k) = 0$. This implies that $\lambda$ is determined by the restrictions $\lambda|_k$ and $\lambda|_{kj}$, which are related by

$$\lambda|_{kj}(jx, jy) = j^2 \cdot \lambda|_k(x, y).$$

This shows that $\lambda$ is determined by $\lambda|_k$, which, being a symplectic form on a 2-dimensional space, is unique up to scaling.

It follows from (3.1.4) that $B$ admits a unique $k$-hermitian form $\langle -,- \rangle$ satisfying

$$\lambda(x, y) = \text{Tr}_{k/\mathbb{Q}}(\delta^{-1}x, y).$$

This form is given by the explicit formula $\langle x, y \rangle = \pi(x \mathfrak{y})$, where $\pi : B \to k$ is projection to the first factor in (3.1.1).

**Proposition 3.1.4.** There is a bijection

$$\{\text{iso. classes of indefinite quaternion embeddings of } k\}$$

$$\cong \{\text{iso. classes of hermitian spaces over } k \text{ of signature } (1,1)\}$$

sending $k \to B$ to the hermitian space whose underlying $k$-vector space is $W = B$, endowed with the hermitian form of (3.1.5). If $B$ and $W$ are related in this way, then

$$(3.1.6) \quad \text{inv}_p(B) = (-1, \text{disc}(k))_p \cdot \text{inv}_p(W)$$

for all places $p \leq \infty$, and there is an isomorphism

$$(3.1.7) \quad (k^\times \times B^\times)/\mathbb{Q}^\times \cong \text{GU}(W)$$

that restricts to $\ker(\text{Nrd} : B^\times \to \mathbb{Q}^\times) \cong \text{SU}(W)$.

**Proof.** Most of this can be found in §5 of [Gro04], so we only add a few comments. An easy calculation shows that the hermitian space $W = B$ has determinant

$$\det(W) = -j^2 \in \mathbb{Q}^\times/\text{Nm}(k^\times),$$

and so the $\mathbb{Q}$-basis $\{1, \delta, j, \delta j\} \subset B$ exhibits an isomorphism

$$B \cong \left(\frac{\text{disc}(k), -\det(W)}{\mathbb{Q}}\right).$$

The relation (3.1.6) is clear from this. This also shows how to reconstruct the quaternion embedding from the abstract hermitian space $W$, providing the inverse to the function in the statement of the proposition.

The action of $k^\times \times B^\times$ on $W = B$ given by $(\alpha, b) \cdot w = \alpha wb^{-1}$ defines the isomorphism (3.1.7).

**Remark 3.1.5.** It follows from (3.1.6) that $\text{inv}_p(B) = 1$ if and only if $W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isotropic.
Proposition 3.1.6. Suppose $k \to B$ is an indefinite quaternion embedding, and let $W = B$ be the signature $(1,1)$ hermitian space of Proposition 3.1.4. For a 2-dimensional $\mathbb{C}$-subspace $F^0 W \subset W_\mathbb{C} = W \otimes \mathbb{Q} \mathbb{C}$, the following are equivalent:

1. $F^0 W$ is stable under left multiplication by $B$,
2. $F^0 W$ is stable under $k$, totally isotropic with respect to $(3.1.3)$, and free of rank one over $k \otimes \mathbb{Q} \mathbb{C}$.

Proof. First suppose $F^0 W$ satisfies condition (1). We may choose an isomorphism $B \otimes \mathbb{Q} \mathbb{C} \cong M_2(\mathbb{C})$ identifying $k \otimes \mathbb{Q} \mathbb{C}$ with the subalgebra of diagonal matrices. This isomorphism makes $F^0 W$ into a left module over the $\mathbb{C}$-algebra $M_2(\mathbb{C})$, whose standard representation is the unique left module of complex dimension 2. The standard representation is free of rank one over the subalgebra of diagonal matrices, hence so is $F^0 W$. To see that $F^0 W$ is totally isotropic under the symplectic form $(3.1.3)$, note that if we set $k' = \mathbb{Q}[j] \subset B$, the argument above shows that $F^0 W$ is free of rank one over $k' \otimes \mathbb{Q} \mathbb{C}$. If we choose a generator $x_0 \in F^0 W$ then, recalling that $j^\dagger = j$, we find

$$
\lambda(jx_0, x_0) = \lambda(x_0, j^\dagger x_0) = \lambda(x_0, jx_0) = -\lambda(jx_0, x_0).
$$

It follows that $\lambda$ is identically 0 on $\text{Span}_\mathbb{C}\{x_0, jx_0\} = F^0 W$.

Now suppose $F^0 W$ satisfies condition (2), and fix a $k \otimes \mathbb{Q} \mathbb{C}$-module generator $x_0 \in F^0 W$. For any $\alpha \in k$ we have

$$
\lambda(\alpha x_0, jx_0) = \lambda(j\alpha x_0, x_0) = \lambda(\overline{\alpha}jx_0, x_0) = \lambda(jx_0, \alpha x_0) = -\lambda(\alpha x_0, jx_0).
$$

It follows that $\lambda(F^0 W, jx_0) = 0$ and, as $F^0 W$ is a maximal isotropic subspace, that $jx_0 \in F^0 W$. Thus $F^0 W$ is stable under left multiplication by $j$, and hence under all of $B = k \otimes kj$. \hfill $\square$

3.2. Self-dual lattices. We wish to parametrize the set

$$
\mathcal{L}_{(1,1)} = \left\{ \text{isometry classes of self-dual hermitian } \mathcal{O}_k\text{-modules of signature (1,1)} \right\}
$$

where hermitian $\mathcal{O}_k$-module means a projective $\mathcal{O}_k$-module $L$ of finite rank endowed with an $\mathcal{O}_k$-valued hermitian form $\langle -, - \rangle$, and self-dual means that $L$ is equal to its dual lattice

$$
L^* = \{ x \in L \otimes \mathbb{Q} : \langle x, L \rangle \subset \mathcal{O}_k \}.
$$

There is a natural partition of $\mathcal{L}_{(1,1)}$, obtained by declaring $L, L' \in \mathcal{L}_{(1,1)}$ to be equivalent if $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$ as $k$-hermitian spaces. We express this as a decomposition

$$
(3.2.1) \quad \mathcal{L}_{(1,1)} = \bigsqcup_W \mathcal{L}_W
$$
where the disjoint union is over all \( k \)-hermitian spaces \( W \) of signature \((1, 1)\) that admit a self-dual \( \mathcal{O}_k \)-lattice, and
\[
\mathcal{L}_W = U(W) \setminus \{ \text{self-dual } \mathcal{O}_k \text{-lattices in } W \}.
\]

Let \( \text{CL}(k) \) be the ideal class group of \( k \). The principal genus \( \text{CL}_0(k) \subset \text{CL}(k) \) is characterized either as the subgroup of squares, or by
\[
\text{CL}_0(k) \cong k^\times \setminus \{ \text{fractional ideals } a \subset k : \text{Nm}(a) \in \text{Nm}(k^\times) \}.
\]
More adelically, if we let \( k^1 \) denote the kernel of the norm map \( k^\times \to \mathbb{Q}^\times \), and similarly for \( \hat{k}^1 \), then
\[
\text{CL}_0(k) \cong k^1 \backslash \hat{k}^1 / \hat{O}_k^1.
\]

Using this last isomorphism and the isomorphism \( k^\times / \mathbb{Q}^\times \cong k^1 \) of Hilbert’s Theorem 90, one can prove the classical formula (originating in Gauss’s genus theory of binary quadratic forms)
\[
|\text{CL}_0(k)| = \frac{|\text{CL}(k)|}{2^{o(k)-1}},
\]
where \( o(k) \) is the number of prime divisors of \( \text{disc}(k) \).

**Proposition 3.2.1.** Assume that \( \text{disc}(k) \) is odd. The Steinitz class map
\[
\text{St} : \mathcal{L}_{(1,1)} \to \text{CL}(k),
\]
which sends \( L \in \mathcal{L}_{(1,1)} \) to the rank 1 projective \( \mathcal{O}_k \)-module \( \text{St}(L) = \bigwedge^2 \mathcal{O}_k L \), is bijective. Under this bijection, the image of each subset \((3.2.2)\) is a coset of the principal genus, and in fact
\[
\text{St}(\mathcal{L}_W) = a\text{CL}_0(k)
\]
for any fractional ideal \( a \) with \( \text{Nm}(a) = -\det(W) \) in \( \mathbb{Q}^\times / \text{Nm}(k^\times) \).

**Proof.** As in Lemma 2.11(i) of [KR14], there are \( 2^{o(k)-1} \) distinct hermitian spaces \( W \) that contribute to the disjoint union \((3.2.1)\). Fix one. By a theorem of Jacobowitz [Jac62], any two self-dual lattices in \( W \) are isometric everywhere locally. Fixing one such lattice \( L \subset W \), we obtain a bijection
\[
U(W) \setminus U(W_{\hat{k}}) / U(L_{\hat{k}}) \xrightarrow{g \mapsto gL} \mathcal{L}_W.
\]

Using strong approximation for the (simply connected) group \( \text{SU}(W) \), one sees that the function
\[
U(W) \setminus U(W_{\hat{k}}) / U(L_{\hat{k}}) \xrightarrow{\det} \hat{k}^1 \backslash \hat{O}_k^1 \cong \text{CL}_0(k)
\]
is also bijective, and so
\[
|\mathcal{L}_W| = |\text{CL}_0(k)|.
\]

The above discussion, \((3.2.1)\), and \((3.2.3)\) show that \( |\mathcal{L}_{(1,1)}| = |\text{CL}(k)| \), and so to prove the bijectivity of \((3.2.4)\) it suffices to construct a section to the function. This can be done by sending a fractional ideal \( a \subset k \) to
\[1\text{and our assumption that } \text{disc}(k) \text{ odd} \]
the self-dual hermitian \( O_k \)-module \( L = O_k \oplus a \), where the first factor in the orthogonal direct sum is endowed with the hermitian form \( \langle x, y \rangle = x \overline{y} \), and the second with \( \langle x, y \rangle = -x \overline{y}/\Nm(a) \).

For any \( L \in \mathcal{L}_{(1,1)} \) and any fractional ideal \( \mathfrak{c} \subset k \), we endow the \( O_k \)-module \( L^\prime = \mathfrak{c} \otimes_{O_k} L \) with the hermitian form
\[
\langle c_1 \otimes x_1, c_2 \otimes x_2 \rangle = \frac{c_1 c_2^\prime}{\Nm(c)} \cdot \langle x_1, x_2 \rangle.
\]

This defines an action of \( \text{Cl}(k) \) on \( \mathcal{L}_{(1,1)} \), satisfying \( \text{St}(L^\prime) = c^2 \text{St}(L) \). The determinants of \( L \otimes \mathbb{Q} \) and \( L^\prime \otimes \mathbb{Q} \) are equal in \( \mathbb{Q}^\times/\Nm(k^\times) \), and so comparison of local invariants shows they are isomorphic as \( k \)-hermitian spaces. This proves that each subset \( \mathcal{L}_W \subset \mathcal{L}_{(1,1)} \) is stable under the action.

If we now use (3.2.4) to identify \( \mathcal{L}_{(1,1)} = \text{Cl}(k) \) as \( \text{Cl}(k) \)-sets, but with the action on the right hand side given by \( \mathfrak{c} \ast a = \mathfrak{c}^2 a \), then each \( \mathcal{L}_W \) is a union of orbits. The orbits are exactly the cosets of the principal genus, and the counting formula (3.2.5) shows that each \( \mathcal{L}_W \) is a single coset. The final claim, specifying which coset it is, follows from the explicit section to (3.2.4) constructed above.

\[\square\]

Remark 3.2.2. As a corollary of the proof, every \( L \in \mathcal{L}_{(1,1)} \) can be split as an orthogonal direct sum \( L = O_k \oplus a \). This is false if we drop the assumption that \( \text{Disc}(k) \) is odd.

3.3. Construction of Eichler orders. Fix a \( k \)-hermitian space \( W \) of signature \((1,1)\). Let \( k \to B \) be the indefinite quaternion embedding corresponding to \( W \) under the bijection of Proposition 3.1.4.

Proposition 3.3.1. Assume that \( \text{Disc}(k) \) is odd, and suppose \( L \subset W \) is a self-dual \( O_k \)-lattice. There exists an Eichler order \( R \subset B \) of level \((3.3.1)\)
\[N = -\text{Disc}(k)/\text{Disc}(B)\]
(in particular, this is an integer) containing \( O_k \) such that:

1. The positive involution (3.1.2) satisfies \( R^\dagger = R \).
2. Under the symplectic form (3.1.3), \( R \) is a self-dual \( \mathbb{Z} \)-lattice.
3. Under the \( k \)-hermitian form of (3.1.5), \( R \) is a self-dual \( O_k \)-lattice isometric to \( L \).

Proof. Write \( B = k \oplus kj \) be as in (3.1.1), and fix a fractional ideal \( \mathfrak{a} \subset k \) representing the Steinitz class \( \text{St}(L) \). Combining (3.1.8) with the final claim of Proposition 3.2.1 shows that
\[\Nm(\mathfrak{a}) \cdot j^2 \in \Nm(k^\times),\]
and so we may rescale \( \mathfrak{a} \) by an element of \( k^\times \) to assume that \( \Nm(\mathfrak{a}) \cdot j^2 = 1 \). This guarantees that
\[R = O_k \oplus \mathfrak{a} j\]
is an order in \( B \), and direct calculation shows that it has reduced discriminant \(-\text{Disc}(k)\). Note that the reduced discriminant is squarefree, by the
assumption that \( \text{disc}(k) \) is odd. The equality \( R^\dagger = R \) is clear, as \( \dagger \) restricts to complex conjugation on \( k \), and to the identity on \( kj \).

The reduced discriminant of an order is a multiple of the reduced discriminant of any order that contains it, and the reduced discriminant of any maximal order is \( \text{disc}(B) \). Hence (3.3.1) is an integer.

We claim that any order \( R \subset B \) of squarefree reduced discriminant \( N \text{disc}(B) \) is an Eichler order (necessarily of level \( N \)). This can be checked locally. At a prime \( p \) \( \nmid N \), \( R_p \) is maximal, and there is nothing to check. At a prime \( p \) \( \mid N \), we may identify \( B_p \) in such a way that \( R_p \subset M_2(\Z_p) \) with index \( p \). It follows that \( J(A) \cdot F^2_p \subset F^2_p \) is an \( A \)-stable line, and so \( A \) is conjugate to the algebra of upper triangular matrices. From this one can see that the isomorphism \( B_p \cong M_2(\Q_p) \) may be chosen so that

\[
R_p = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} : a, b, c, d \in \Z_p \right\} \subset M_2(\Z_p),
\]

which is an Eichler order of level \( N \).

The self-duality of \( R \) under the hermitian form of (3.1.5) is a direct calculation. It is isometric to \( L \) because both have Steinitz class \( a \), which determines the isometry class by Proposition 3.2.1. Self-duality under the symplectic form follows from self-duality under the hermitian form and the relation (3.1.5). \( \square \)

4. Unitary Shimura curves

Let \( k \) be a quadratic imaginary field of odd discriminant \( \text{disc}(k) \).

4.1. The moduli problem. As with quaternionic Shimura curves, we will define unitary Shimura curves as moduli spaces of abelian surfaces with additional structure.

**Definition 4.1.1.** An \( \mathcal{O}_k \)-abelian surface of signature \((1,1)\) over a scheme \( S \) is an abelian scheme \( A \to S \) of relative dimension two, together with a ring homomorphism \( \mathcal{O}_k \to \text{End}(A) \) such that every \( a \in \mathcal{O}_k \) acts on \( \text{Lie}(A) \) with characteristic polynomial

\[
x^2 - (a + \overline{a})x + a\overline{a} \in \Z[x],
\]

viewed as an element of \( \mathcal{O}_S[x] \).

Let \( \mathcal{X}_{(1,1)} \) be the Deligne-Mumford stack whose functor of points assigns to a \( \Z \)-scheme \( S \) the groupoid of pairs \((A, \lambda)\) in which \( A \) is an \( \mathcal{O}_k \)-abelian surface of signature \((1,1)\) over \( S \), and \( \lambda : A \to A^\vee \) is a principal polarization such that

\[
\overline{a} = \lambda^{-1} \circ a^\vee \circ \lambda \in \text{End}(A)
\]
for every $a \in \mathcal{O}_k$; in other words, the Rosati involution on $\text{End}(A)$ restricts to complex conjugation on $\mathcal{O}_k$.

It is known that $\mathcal{X}_{(1,1)}$ is a regular Deligne-Mumford stack of dimension 2, flat and of finite type over $\mathbb{Z}$, and smooth over $\mathbb{Z}/(\text{disc}(k))$. Note that regularity and flatness use the local model calculations of Pappas [Pap00], which requires our standing hypothesis that $\text{disc}(k)$ is odd.

By Proposition 2.12(i) of [KR14] and its proof, there is a decomposition $\mathcal{X}_{p_1, q_1} = \bigsqcup\mathcal{X}_W$ into open and closed substacks in which, recalling Proposition 3.2.1, the disjoint union is over the 2-torsion isometry classes of $k$-hermitian spaces $W$ of signature $(p_1, q_1)$ that contain a self-dual $\mathcal{O}_k$-lattice. The substack $\mathcal{X}_W$ is characterized as follows: for any geometric point $x : \overline{\mathbb{Q}} \to \mathcal{X}_W$, and any prime $\ell \not\equiv \text{char}(x)$, there exist a $k_\ell$-linear isomorphism $\mathcal{T}_\ell(A_x) \otimes \mathbb{Q}_\ell \cong W \otimes \mathbb{Q}_\ell$ and a $\mathbb{Q}_\ell$-linear isomorphism $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$ that identify the $\ell$-adic Weil pairing on the left (induced by the principal polarization on $A_x$) with the symplectic form $\langle s, t \rangle \mapsto \text{Tr}_{k/\mathbb{Q}}(\delta^{-1}s, t)$ on the right. Here $\langle - , - \rangle$ is the hermitian form on $W$, and $\delta \in k^\times$ is any element with $\delta^2 = \text{disc}(k)$.

**Remark 4.1.2.** The stacks $\mathcal{X}_W$ are typically disconnected. The above decomposition of $\mathcal{X}_{(1,1)}$ will be refined in Theorem 5.1.1 below.

### 4.2. A morphism of Shimura curves

Fix an indefinite quaternion embedding $k \hookrightarrow B$, let $W = B$ be the signature $(1, 1)$ hermitian space of Proposition 3.1.4 and suppose $L \subset W$ is a self-dual $\mathcal{O}_k$-lattice. Fix a level $N$ Eichler order $R \subset B$ as in Proposition 3.3.1 and a maximal order $\mathcal{O}_B$ containing it. After replacing $L$ by an isometric lattice in $W$, we may assume $L = R$.

In this subsection we will construct a morphism

\[
q_L : \mathcal{X}_B(N)/\mathbb{Q} \to \mathcal{X}_{W/\mathbb{Q}}
\]

of $\mathbb{Q}$-stacks by realizing the source and target as canonical models of Shimura varieties. We emphasize that the map depends on the isometry class of $L \subset W$. Indeed, the domain is connected but the codomain usually is not. As $L$ varies, the maps will take values in different connected components.

Define a chain of $\mathbb{Z}$-lattices

\[
\Lambda_0 \subset \Lambda_R \subset \Lambda_1
\]

in $B$ as follows. First set $\Lambda_1 = \mathcal{O}_B$ and $\Lambda_R = R$. The subgroup

\[
\Lambda_R/N\Lambda_1 \subset \Lambda_1/N\Lambda_1
\]

contains a unique index $N$ subgroup stable under the left action of $\mathcal{O}_B$. Namely, identify $\Lambda_R/N\Lambda_1$ with the subgroup of upper triangular matrices
in
\[ \Lambda_1/\Lambda_1 \cong \mathcal{O}_B/\mathcal{O}_B \cong M_2(\mathbb{Z}/\mathbb{Z}), \]
and then take the subgroup of matrices whose first column is 0. Thus there
is a unique left \( \mathcal{O}_B \)-stable sublattice \( \Lambda_0 \subset \Lambda_R \) of index \( N \) that contains \( N\Lambda_1 \).

Denote by \( \mathcal{H} \) the hermitian symmetric domain whose points \( z \in \mathcal{H} \) pa-
rametrize weight \(-1\) Hodge structures
\[ W_{\mathbb{C}} = W_z^{0,-1} \oplus W_z^{-1,0} \]
whose filtrations \( F_z^0 W = W_z^{0,-1} \) satisfy the equivalent conditions of Proposition 3.1.6. One can easily verify that \( \mathcal{H} \) is isomorphic to the union of the upper and lower complex half-planes.

The reductive group \( G_0 = B^\times \) acts on \( W = B \) by right multiplication
\[ g \cdot w = wg^{-1}, \]
and this induces a transitive action of \( G_0(\mathbb{R}) \) on \( \mathcal{H} \). If we define a compact open subgroup \( K_0 = \hat{R}^\times \) of \( G_0(\Lambda_f) \), there is an isomorphism
\[ G_0(\mathbb{Q}) \backslash (\mathcal{H} \times G_0(\Lambda_f)/K_0) \cong \mathcal{X}_B(N)(\mathbb{C}) \]
sending the double coset of a pair \( (z, g) \) to the isogeny \( f : A_0 \to A_1 \) defined
by the \( \mathcal{O}_B \)-abelian surfaces
\[ (4.2.3) \quad A_i(\mathbb{C}) = F_z^0 W \backslash W_{\mathbb{C}}/g \cdot \Lambda_i \]
and the inclusion \( g \cdot \Lambda_0 \subset g \cdot \Lambda_1 \).

The group \( G = GU(W) \) of unitary similitudes of \( W \) also acts transitively
on \( \mathcal{H} \), and the pair \( (G, \mathcal{H}) \) is a Shimura datum with reflex field \( \mathbb{Q} \). If we let \( K \subset G(\Lambda_f) \) be the stabilizer of the lattice \( L = \Lambda_R \), there is an isomorphism\(^2\)
of complex orbifolds
\[ G(\mathbb{Q}) \backslash (\mathcal{H} \times G(\Lambda_f)/K) \cong \mathcal{X}_W(\mathbb{C}) \]
sending the double coset of a pair \( (z, g) \) to the \( \mathcal{O}_k \)-abelian surface
\[ (4.2.4) \quad A_R(\mathbb{C}) = F_z^0 W \backslash W_{\mathbb{C}}/g \cdot \Lambda_R \]
of signature \((1,1)\). To polarize this abelian surface, let \( \nu(g) \in \hat{\mathbb{Q}}^\times \) be the similitude factor of \( g \), and write
\[ \nu(g) = r \cdot u \]
with \( r \in \hat{\mathbb{Q}}^\times \) and \( u \in \hat{\mathbb{Z}}^\times \). By Proposition 3.3.1, the \( \mathbb{Z} \)-lattice \( g \cdot \Lambda_R \) is self-
dual under the rescaled symplectic form \( r^{-1} \lambda \), which defines (plus or minus) a principal polarization.

Proposition 3.1.4 (and its proof) imply that \( G_0 \subset G \) as subgroups of
\( \text{GL}_k(W) \), and the pair \( (G_0, \mathcal{H}) \) is again a Shimura datum with reflex field \( \mathbb{Q} \). It is clear from the constructions that \( K_0 \subset K \), as right multiplication
by \( R \) stabilizes \( \Lambda_R = R \).

\(^2\)The surjectivity requires the theorem of Jacobowitz cited in the proof of Proposition 3.2.1. Jacobowitz’s result, which requires our assumption that \( \text{disc}(k) \) is odd, guarantees
that \( G(\Lambda_f) \) acts transitively on set of self-dual \( \mathcal{O}_k \)-lattices in \( W \).
Proposition 4.2.1. The morphism \((4.2.1)\) induced by the morphism of Shimura data \((G_0, \mathcal{H}) \to (G, \mathcal{H})\) is finite étale of degree \(|\mathcal{O}_k^\times|\) over its image (as we have noted already, it need not be surjective).

Proof. Fix a connected component \(\mathcal{H}^+ \subset \mathcal{H}\). The domain of the orbifold morphism
\[
q_L : \mathcal{X}_B(N)(\mathbb{C}) \to \mathcal{X}_W(\mathbb{C})
\]
is connected, and if we replace the codomain by the image, the morphism takes the form
\[
q_L : R^1 \backslash \mathcal{H}^+ \to U(L) \backslash \mathcal{H}^+,
\]
where \(R^1 \subset R^\times\) is the kernel of the reduced norm \(R^\times \to \{\pm 1\}\). The exactness of
\[
1 \to SU(L) \to U(L) \xrightarrow{\det} \mathcal{O}_k^\times \to 1
\]
follows easily from Remark 3.2.2 while the final claim of Proposition 3.1.4 identifies \(R^1 \cong SU(L)\). Thus \(\deg(q_L) = |U(L) : R^1| = |\mathcal{O}_k^\times|\). \(\square\)

4.3. Extension to integral models. The goal of this subsection is to show that the morphism of \(\mathbb{Q}\)-stacks \((4.2.1)\) admits a unique extension to a morphism \(q_L : \mathcal{X}_B(N) \to \mathcal{X}_W\) of integral models.

As in Definition 2.1.4, the Eichler order \(R \subset B\) determines a factorization
\[
A_0 \to A_R \to A_1
\]
of the universal degree \(N^2\) isogeny \(f : A_0 \to A_1\) over \(\mathcal{X}_B(N)\), in which the intermediate abelian surface comes with an action \(R \subset \text{End}(A_R)\). Restricting this action to \(\mathcal{O}_k \subset R\) makes
\[
A_R \to \mathcal{X}_B(N)
\]
into an \(\mathcal{O}_k\)-abelian surface of signature \((1, 1)\). Indeed, as \(\mathcal{X}_B(N)\) is flat over \(\mathbb{Z}\), it is enough to verify the signature condition over the generic fiber, where the natural morphism \(\text{Lie}(A_0) \to \text{Lie}(A_R)\) restricts to an isomorphism. Thus the signature condition on \(\text{Lie}(A_R)\) follows from Drinfeld’s determinant condition imposed on \(\text{Lie}(A_0)\) in \(\S\).

Proposition 4.3.1. The intermediate abelian surface admits a unique principal polarization \(\lambda_R : A_R \to A_R^\times\) such that the induced Rosati involution restricts to the positive involution \((3.1.2)\) on \(R \subset \text{End}(A_R)\).

Proof. Standard arguments (see Chapters 6 and 7 of [Hid04], for example), show that the functor that assigns to a scheme \(S\) the groupoid
\[
\mathcal{X}_B^I(N)(S) = \left\{ \begin{array}{c} \text{morphisms } S \to \mathcal{X}_B(N) \text{ together with a} \\ \text{principal polarization } A_{R/S} \to A_{R/S}^\times \\ \text{whose Rosati involution restricts to } \dagger \text{ on } R \end{array} \right\}
\]
is represented by a proper and unramified morphism of Deligne-Mumford stacks
\[
g : \mathcal{X}_B^I(N) \to \mathcal{X}_B(N).
\]
The morphism $g$ is injective on geometric points of any characteristic. Indeed, given a geometric point $x \to X_B(N)$, choose a prime $\ell \neq \text{char}(x)$. A lift of $x$ to $X^\dagger_B(N)$ is determined by a principal polarization $\lambda_x : A_{R,x} \to A^\vee_{R,x}$ whose induced $\ell$-adic Weil pairing

$$\lambda_x : \text{Ta}_\ell(A_{R,x}) \times \text{Ta}_\ell(A_{R,x}) \to \mathbb{Z}_\ell$$

satisfies $\lambda_x(r,s,t) = \lambda_x(s,r^t)$ for all $r \in R$. As $\text{Ta}_\ell(A_{R,x}) \otimes \mathbb{Q}_\ell$ is free of rank one over $B_\ell$, it follows from Lemma 3.1.3 (or rather, its $\ell$-adic analogue) that any two such polarizations span the same $\mathbb{Q}_\ell$-line in $\text{Hom}(A_{R,x}, A^\vee_{R,x}) \otimes \mathbb{Q}_\ell$, from which it follows that they are equal.

The morphism $g$ is surjective on complex points. Indeed, at any complex point of $X_B(N)$ the universal $f : A_0 \to A_1$ has the form (4.2.3) for some $p \in H^\dagger$, and the intermediate abelian surface takes the form (4.2.4) for the same pair. We have already explained how to construct a polarization of the desired type on (4.2.3).

The morphism $g$ is quasi-finite and proper, so finite by [Sta18, Tag 02OG]. Being finite, unramified, and injective on geometric points, it becomes a closed immersion after pullback to an étale cover of $X_B(N)$, by [Sta18, Tag 04HG]. By [Sta18, Tag 02L6] it was already a closed immersion before pullback to the étale cover.

At this point we know that $g$ is a closed immersion, bijective on complex points, whose codomain is integral and flat of finite type over $\mathbb{Z}$. Étale locally, such a morphism has the form $\text{Spec}(D/I) \to \text{Spec}(D)$, where $D$ is an integral domain flat and of finite type over $\mathbb{Z}$ and $I \subset D$ is an ideal contained in the kernel of any ring homomorphism $D \to \mathbb{C}$. But $D$ admits an injective homomorphism to $\mathbb{C}$, and so $I = 0$. It follows that $g$ is an isomorphism. □

**Proposition 4.3.2.** There is a unique morphism

$$q_L : X_B(N) \to X_W$$

extending the morphism (4.2.1) already constructed in the generic fiber. It is relatively representable and finite étale of degree $|O_k^\times|$ over its image, and the pullback via $q_L$ of the universal $O_k$-abelian surface $A \to X_W$ is isomorphic to the intermediate abelian surface $A_R \to X_B(N)$ of Definition 2.1.4.

**Proof.** By endowing the intermediate abelian surface $A_R \to X_B(N)$ with its action of $O_k \subset R$ and the polarization of Proposition 4.3.1 we obtain a morphism

$$q_L : X_B(N) \to X_{(1,1)}.$$

It is clear from the explicit constructions of 4.2 that this extends the morphism (4.2.1) already constructed in the complex fiber. The flatness of the integral models over $\mathbb{Z}$ implies both that $q_L$ takes values in $X_W$ (as this is
true in the generic fiber) and that the extension of (4.2.1) to integral models is unique.

It remains to prove that $q_L$ is relatively representable and finite étale. To do so, factor $q_L$ as a composition

$\mathcal{X}_B(N) \to \mathcal{X}'_W \to \mathcal{X}_W$, 

where the Deligne-Mumford stack in the middle is defined as follows: for any scheme $S$, an object of the groupoid $\mathcal{X}'_W(S)$ is a principally polarized $\mathcal{O}_k$-abelian surface $(A, \lambda) \in \mathcal{X}_W(S)$ together with an extension of the $\mathcal{O}_k$-action to an action $R \subset \text{End}(A)$ such that the Rosati involution induced by $\lambda$ restricts to $\dagger$ on $R$. Standard arguments (again, see Chapters 6 and 7 of [Hid04]) show that the forgetful morphism $\mathcal{X}'_W \to \mathcal{X}_W$ is relatively representable, finite, and unramified.

Of course, the first arrow in (4.3.1) sends $f : A_0 \to A_1$ to its intermediate abelian surface, endowed with the polarization of Proposition 4.3.1.

Lemma 4.3.3. The first arrow in (4.3.1) is a closed immersion. In particular, it is relatively representable, finite, and unramified.

Proof. Set $Z_N = \prod_{p | N} \mathbb{Z}_p$, fix an isomorphism $\mathcal{O}_B \otimes \mathbb{Z} Z_N \cong M_2(\mathbb{Z}_N)$ identifying

$$R \otimes \mathbb{Z} Z_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_N \right\},$$

and define elements of $R \otimes \mathbb{Z} Z_N$ by

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}.$$

Suppose we have a scheme $S$ and object $f : A_0 \to A_1$ of $\mathcal{X}_B(N)(S)$. If we define finite flat $R$-stable subgroup schemes

(4.3.2) $P = e' \ker(w : A_R[N] \to A_R[N]), \quad Q = eA_R[N] \oplus P$

of $A_R$ then, recalling the subgroup scheme $C_R = e \ker(f) \subset A_0$ used in Definition 2.1.4, we have

$$P = \ker(f)/C_R \subset A_0/C_R = A_R,$$

and the induced morphism

$$A_1 \cong A_0/\ker(f) \cong A_R/P \to A_R/Q$$

has the same kernel as the dual isogeny $f^\vee : A_1 \to A_0$ of Definition 2.1.3. Indeed, this is easily verified if we are in the situation $\mathcal{O}_B \cong M_2(\mathbb{Z})$ of Remark 2.1.6, so that $f : A_0 \to A_1$ has the form (2.1.2); the general case is no different, as one can check the claims after replacing $f : A_0 \to A_1$ with the induced morphisms of $p$-divisible groups for all $p \mid N$, which again have the form (2.1.2).
The above paragraph shows that we can recover $f : A_0 \to A_1$ from $A_R$ (with its $R$-action), as the dual isogeny to $A_R/P \to A_R/Q$. In other words, for any scheme $S$ the first arrow in (4.3.1) defines a fully faithful functor

$$\mathcal{X}_B(N)(S) \to \mathcal{X}_W'(S),$$

whose inverse over the essential image is explicitly known. We must show the essential image is defined by the inclusion of a closed substack of $\mathcal{X}_W'$.

Consider the universal object $A \to \mathcal{X}_W'$. There are finite flat $R$-stable subgroup schemes $P \subset Q$ of $A$ defined exactly as in (4.3.2), but with $A_R$ replaced by $A$. Imposing the conditions

$$\text{rank}(P) = N \quad \text{and} \quad \text{rank}(Q) = N^3$$

cuts out an open and closed substack of $\mathcal{X}_W'$, and imposing the condition that the actions of $R$ on

$$A_1 = A/P \quad \text{and} \quad A_0 = A/Q$$

extend to $\mathcal{O}_B$-actions satisfying the determinant condition of Definition 2.1.1 cuts out a closed substack. Thus there is a maximal closed substack over which all of these conditions are satisfied.

From what was said above, the first arrow in (4.3.1) factors through this closed substack, and over this closed substack the arrow has an inverse, defined by dualizing the $\mathcal{O}_B$-linear isogeny $A_1 \to A_0$ determined by the inclusion $P \subset Q$. □

We can now complete the proof of Proposition 4.3.2. The morphism $q_L : \mathcal{X}_B(N) \to \mathcal{X}_W$ is relatively representably, finite, and unramified, as each of the arrows in the composition (4.3.1) has these properties. In particular, by [Sta18, Tag 04HG], the induced maps on étale local rings are surjective. As the source and target of $q_L$ are regular of dimension 2, their étale local rings are integral domains of the same dimension, and Krull’s Hauptidealsatz implies that any surjection between them is an isomorphisms. Hence $q_L$ induces isomorphisms on étale local rings, so is étale. As the source and target of $q_L$ are flat over $\mathbb{Z}$, the calculation of its degree can be done in characteristic 0. This was done in Proposition 4.2.1. □

4.4. Connected components and level structure. Recall from (3.2.2) that $\mathcal{Z}_W$ denotes the set of isometry classes of self-dual $\mathcal{O}_k$-lattices in $W$. It follows from the discussion of (4.2) that

$$\mathcal{X}_W(\mathbb{C}) = \bigsqcup_{L \in \mathcal{Z}_W} U(L) \backslash \mathcal{H}^+, \tag{4.4.1}$$

where each $U(L) \backslash \mathcal{H}^+$ is the orbifold quotient of a connected domain $\mathcal{H}^+$ (isomorphic to the complex upper half-plane) by the action of a discrete group. Compare with Proposition 3.1 of [KR13].
**Proposition 4.4.1.** There is a unique decomposition
\[ X_W = \bigsqcup_{L \in \mathcal{L}_W} C_L \]
into a disjoint union of open and closed substacks, such that taking complex points recovers (4.4.1). For each \( L \in \mathcal{L}_W \) the morphism \( C_L \to \text{Spec}(\mathbb{Z}) \) is flat with geometrically connected fibers, and the morphism \( q_L \) of Proposition \( \text{(4.3.2)} \) has image \( C_L \).

**Proof.** For each \( L \in \mathcal{L}_W \) we have constructed in Proposition \( \text{(4.3.2)} \) a morphism \( q_L : \mathcal{X}_B(N) \to X_W \). As this morphism is finite étale, its image \( C_L \subset X_W \) is both open and closed. As \( \mathcal{X}_B(N) \) has geometrically connected fibers, so does \( C_L \). The morphism \( q_L \) was constructed so that its image in the complex fiber is the component \( U^+ \) of (4.4.1), and so we have found a connected component satisfying \( C_L(\mathbb{C}) \cong U(L) \backslash \mathcal{H}^+ \). The existence and uniqueness of the decomposition follow from this and the flatness of \( X_W \) over \( \mathbb{Z} \).

\[ \square \]

Fix a self-dual \( \mathcal{O}_k \)-lattice \( L \subset W \). The hermitian form on \( W \) determines an alternating \( \mathbb{Q} \)-bilinear form \( \lambda : W \times W \to \mathbb{Q} \), as in (3.1.5), under which \( L \) is again self-dual. Hence \( \lambda \) induces a perfect alternating form
\[ \lambda_m : L/mL \times L/mL \to \mathbb{Z}/m\mathbb{Z} \]
for any integer \( m \geq 1 \).

Let \( A \to C_L \) be the restriction of the universal polarized \( \mathcal{O}_k \)-abelian surface over \( X_W \) to the connected component determined by \( L \). Denote by
\[ C_L(m) \to \text{Spec}(\mathbb{Z}[1/m]) \]
the Deligne-Mumford stack whose functor of points assigns to a \( \mathbb{Z}[1/m] \)-scheme \( S \) the groupoid whose objects consist of a morphism \( S \to C_L \) together with \( \mathcal{O}_k \)-linear isomorphisms of group schemes
\[ A_S[m] \cong L/mL, \quad \mu_m \cong \mathbb{Z}/m\mathbb{Z} \]
identifying the Weil pairing on \( A_S[m] \) with the pairing (4.4.2).

Let \( f : A_0 \to A_1 \) be the universal object over \( \mathcal{X}_B(N) \). For any \( m \) prime to \( N \), we denote by
\[ \mathcal{X}_B(N,m) \to \text{Spec}(\mathbb{Z}[1/m]) \]
the Deligne-Mumford stack whose functor of points assigns to a \( \mathbb{Z}[1/m] \)-scheme \( S \) the groupoid whose objects consist of a morphism \( S \to \mathcal{X}_B(N) \) together with an \( \mathcal{O}_B \)-linear isomorphism of group schemes
\[ A_{0/S}[m] \cong \mathcal{O}_B/m\mathcal{O}_B. \]
Proposition 4.4.2. For any $m \geq 1$ prime to $N$ there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_B(N, m) & \rightarrow & \mathcal{C}_L(m) \\
\downarrow & & \downarrow \pi_m \\
\mathcal{X}_B(N)_{/\mathbb{Z}[1/m]} & \rightarrow & \mathcal{C}_L/\mathbb{Z}[1/m]
\end{array}
\]

in which all arrows are finite étale and relatively representable. If $m \geq 3$, the top horizontal arrow is an isomorphism of quasi-projective schemes.

Proof. This amounts to keeping track of étale level structure in the proof of Proposition 4.3.2, and we leave everything except for the final claim as an exercise for the reader.

For the final claim, $\mathcal{C}_L(m)$ is relatively representable and finite over the moduli stack of all principally polarized abelian surfaces with full level $m$ structure over $\mathbb{Z}[1/m]$-schemes. This moduli stack is a quasi-projective scheme when $m \geq 3$, so the same is true of both $\mathcal{C}_L(m)$ and $\mathcal{X}_B(N, m)$.

As the top horizontal arrow is finite étale, it suffices to show that it induces an isomorphism in the complex fiber, which can be done by examining the proof of Proposition 4.2.1. Indeed, every connected component of the complex fiber of $\mathcal{C}_L(m)$ has the form $\Gamma$ where $g \in \mathcal{G}$ acts trivially on $\mathcal{L}$.

The fiber of the top horizontal arrow over that component is exactly the same, but with $U(L)$ replaced by $R^1 \cong SU(L)$. The determinant $\det(g)$ of any $g \in \Gamma(m)$ is an element of $G \mathcal{K}$ satisfying $\det(g) \in 1 + mG \mathcal{O}_k$, and hence $\det(g) = 1$ by the proof of Theorem 5 in Chapter 21 of [Mum08]. Thus replacing $U(L)$ by $SU(L)$ does not change the group $\Gamma(m)$.

\[\square\]

5. Arithmetic volumes of unitary Shimura curves

Let $k$ be a quadratic imaginary field of odd discriminant $\text{disc}(k)$. Suppose $W$ is a $k$-hermitian space of signature $(1, 1)$ that contains a self-dual $\mathcal{O}_k$-lattice $L \subset W$, and let $\mathcal{C}_L \subset \mathcal{X}_W$ be the associated connected component, as in Proposition 4.3.1. Let $B$ be the indefinite quaternion algebra associated to $W$ by Proposition 3.1.4. As in (3.3.1), set

\[N = -\text{disc}(k)/\text{disc}(B)\]

5.1. The anisotropic case. Assume that $W$ is anisotropic, so that $B$ is a division algebra. For every prime $p \mid \text{disc}(k)$, define $p^\circ = \pm p$ as in (1.2.1). Equivalently, by Remark 3.1.5

\[p^\circ = \begin{cases} 
p & \text{if } p \nmid \text{disc}(B) \\
-p & \text{if } p \mid \text{disc}(B)\end{cases}\]
Theorem 5.1.1. The morphism $C_L \to \text{Spec}(\mathbb{Z})$ is proper, and the metrized Hodge bundle of the universal $O_k$-abelian surface $A \to C_L$ has geometric degree

$$\deg_C(\omega_{A/C_L}^{\text{Hdg}}) = \frac{1}{12 \cdot |O_k^*|} \prod_{p|\text{disc}(k)} (1 + p^\varphi)$$

and arithmetic volume

$$\widehat{\text{vol}}(\omega_{A/C_L}^{\text{Hdg}}) = \deg_C(\omega_{A/C_L}^{\text{Hdg}}) \left( 1 + \frac{2\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \sum_{p|\text{disc}(k)} \frac{1 - p^\varphi}{1 + p^\varphi} \cdot \log(p) \right).$$

In particular, these quantities depend only on the hermitian space $W$, and not on the connected component $C_L \subset X_W$.

Proof. We have constructed in Proposition 4.3.2 a finite étale surjection $q_L : X_B(N) \to C_L$ of degree $|O_k^*|$, under which the metrized Hodge bundle of $A \to C_L$ pulls back to the metrized Hodge bundle of the intermediate abelian surface $A_R \to X_B(N)$ determined by a level $N$ Eichler order $R \subset O_B$.

Remark 3.1.5 implies that $B$ is a division algebra, and so $X_B(N)$ is proper. The properness of $C_L$ follows. The projection formula for arithmetic intersections, as in Section III.3.1 of [Sou92], implies that

$$\widehat{\text{vol}}(\omega_{A/R/\mathcal{X}_B(N)}^{\text{Hdg}}) = |O_k^*| \cdot \widehat{\text{vol}}(\omega_{A/C_L}^{\text{Hdg}}),$$

and similarly for the geometric degree, and so the desired formulas follow from those of Theorem 2.3.1.

5.2. Pre-log singular Hermitian line bundles. Fix an integer $m \geq 1$. Suppose $C^*$ is a regular Deligne-Mumford stack, proper and flat over $\mathbb{Z}[1/m]$. Let $\partial C^* \subset C^*$ be a reduced effective divisor, flat over $\mathbb{Z}[1/m]$, whose generic fiber has normal crossings. Set $C = C^* \setminus \partial C^*$.

The 

arithmetic Picard group

$\hat{\text{Pic}}(C)$

is defined, as before, as the group of isomorphism classes of hermitian line bundles on $C$. The pre-log-singular arithmetic Picard group $\hat{\text{Pic}}(C^*, \mathcal{D}_{\text{pre}})$ is the group of isomorphism classes of line bundles on $\omega$ on $C^*$ equipped with a hermitian metric on $\omega|_C$ that is pre-log singular along the boundary $\partial C^*$ in the sense of Definition 1.20 of [BBGK07].

Proposition 5.2.1. The natural restriction map

$$\hat{\text{Pic}}(C^*, \mathcal{D}_{\text{pre}}) \to \hat{\text{Pic}}(C)$$

is injective, and is an isomorphism if $\partial C^* = \emptyset$.

Proof. The final claim is clear from the definitions, so we only need to prove the injectivity.

\[\text{In [BBGK07] this definition is made under the assumption that } C^* \text{ is a scheme, but there is no difficulty in extending the definition to Deligne-Mumford stacks.}\]
Suppose the pre-log singular hermitian line bundle \( \hat{\omega} \) on \( \mathcal{C}^* \) becomes trivial after restriction to \( \mathcal{C} \). A choice of trivializing section

\[
s \in H^0(\mathcal{C}, \omega|_\mathcal{C})
\]
determines a rational section of \( \omega \) such that \( \|s\| = 1 \) identically on \( \mathcal{C}(\mathbb{C}) \), and whose divisor is supported on the boundary \( \partial \mathcal{C}^* \). We are done if we can show that this divisor is trivial. As the boundary is flat over \( \mathbb{Z}[1/m] \) by hypothesis, this can be checked on the complex fiber.

We now view \( s \) as a meromorphic section of the holomorphic line bundle \( \omega \) on \( \mathcal{C}^*(\mathbb{C}) \). Locally for the orbifold topology, we may choose holomorphic coordinates \( z_1, \ldots, z_d \) near a point \( P \in \mathcal{C}^*(\mathbb{C}) \) so that \( \partial \mathcal{C}^*(\mathbb{C}) \) is defined by the equation \( z_1 \cdots z_r = 0 \), for some \( 0 \leq r \leq d \), and write

\[
s = z_1^{e_1} \cdots z_r^{e_r} s_0
\]
for a nowhere vanishing holomorphic local section \( s_0 \). By definition of a pre-log singular hermitian metric, the absolute value of

\[
\sum_{i=1}^r e_i \log |1/z_i| = \log \|s_0\|
\]
is bounded by the absolute value of \( c \prod_{i=1}^r (\log \log |1/z_i|)^\rho \) for positive constants \( c \) and \( \rho \) on some polydisc \( |z_1|, \ldots, |z_d| < \epsilon \). This can only happen if all \( e_i = 0 \), and hence \( s \) is holomorphic and nonvanishing on a neighborhood of \( P \in \mathcal{C}^*(\mathbb{C}) \).

5.3. The isotropic case. Now suppose that \( W \) is the unique isotropic \( k \)-hermitian space of signature \( (1,1) \). Thus \( N = -\text{disc}(k) \), and the associated quaternion algebra is \( B = M_2(\mathbb{Q}) \) by Remark 3.1.5.

The center of the unipotent radical of any proper parabolic subgroup of \( \text{GU}(W) \) is isomorphic to the additive group scheme \( \mathbb{G}_a \) over \( \mathbb{Q} \). This implies that the Shimura datum defining \( \mathcal{X}_W \) has a unique complete admissible rational polyhedral cone decomposition, and \[\text{PinS9}\] gives us a canonical toroidal compactification of the generic fiber of \( \mathcal{X}_W \). We want to extend this canonical compactification to the integral model. This can be done as in \[\text{Lan13}\] and \[\text{How15}\], but we will instead follow \[\text{MP19}\].

Let \( \mathcal{A} \to \text{Spec}(\mathbb{Z}) \) be the moduli stack of principally polarized abelian surfaces. It is a Deligne-Mumford stack smooth over \( \mathbb{Z} \) of relative dimension 3. Forgetting the \( \mathcal{O}_k \)-action defines a relatively representable, finite, and unramified morphism \( \mathcal{C}_L \to \mathcal{A} \).

The work of Faltings-Chai \[\text{FC90}\] gives us a family of smooth toroidal compactifications \( \mathcal{A} \to \mathcal{A} \), each of which depends on an auxiliary choice of combinatorial data. After making such a choice we define \( \mathcal{C}_L \) as the normalization of \( \mathcal{C}_L \to \mathcal{A} \). See \[\text{Sta18} \text{Tag 0BAK}\] for normalization.

We now add level structure as in \[\text{§4.4}\]. For any \( m \geq 1 \) define \( \mathcal{C}_L(m) \) as the normalization of \( \mathcal{C}_L(m) \to \mathcal{C}_L/\mathbb{Z}[1/m] \), or, equivalently, as the normalization of
\( \mathcal{C}_L(m) \rightarrow \bar{A}/\mathbb{Z}[1/m] \). In particular, there is a finite and relatively representable morphism
\[
\pi_m : \bar{\mathcal{C}}_L(m) \rightarrow \bar{\mathcal{C}}_L/\mathbb{Z}[1/m]
\]
extending the morphism \( \pi_m \) of \((4.4.4)\).

**Proposition 5.3.1.** The stack \( \bar{\mathcal{C}}_L(m) \) does not depend on the choice of Faltings-Chai compactification \( \bar{A} \) used in its construction, and satisfies the following properties.

1. If \( m \geq 3 \), it is a projective \( \mathbb{Z}[1/m] \)-scheme.
2. It is regular, and smooth over \( \mathbb{Z}[1/m] \) outside of finitely many points, all contained in the interior \( \mathcal{C}_L(m) \) and supported in characteristics dividing disc(\( k \)).
3. The boundary \( \partial \bar{\mathcal{C}}_L(m) = \bar{\mathcal{C}}_L(m) \setminus \mathcal{C}_L(m) \), with its reduced substack structure, is a Cartier divisor smooth over \( \mathbb{Z}[1/m] \).

**Proof.** The independence of \( \bar{\mathcal{C}}_L(m) \) on the choice of Faltings-Chai compactification follows from Remark 4.1.6 of \([\text{MP}19]\), together with the observation above that the Shimura datum defining \( \mathcal{X}_W \) admits a unique complete admissible rational polyhedral cone decomposition.

For any integer \( m \geq 1 \), denote by \( \mathcal{A}(m) \rightarrow \text{Spec}(\mathbb{Z}[1/m]) \) the moduli space of principally polarized abelian surfaces over \( \mathbb{Z}[1/m] \)-schemes, equipped with isomorphisms of group schemes as in \((4.4.3)\), compatible with the Weil pairing. As explained in Chapter V.5 of \([\text{FC}90]\), if \( m \geq 3 \) we may choose the compactification of \( \mathcal{A} \) in such a way that \( \mathcal{A}(m) \), defined as the normalization of \( \mathcal{A}(m) \rightarrow \bar{A}/\mathbb{Z}[1/m] \) is a projective \( \mathbb{Z}[1/m] \)-scheme. The stack \( \bar{\mathcal{C}}_L(m) \) can then be realized as the normalization of the natural map \( \mathcal{C}_L(m) \rightarrow \mathcal{A}(m) \), yielding a finite and relatively representable morphism \( \bar{\mathcal{C}}_L(m) \rightarrow \mathcal{A}(m) \) whose codomain is a projective scheme. Hence the domain is a projective scheme.

Part of the assertion of Theorem 1 of \([\text{MP}19]\) is that the singularities of \( \bar{\mathcal{C}}_L(m) \) are no worse than those of \( \mathcal{C}_L(m) \). To spell this out in our simple setting, given a geometric point \( z \rightarrow \mathcal{C}_L(m) \) contained in the boundary, one can find a \( \mathbb{Z}[1/m] \)-scheme \( \mathcal{B}_z \) such that the inclusion \( \mathcal{C}_L(m) \rightarrow \bar{\mathcal{C}}_L(m) \) is, étale locally near \( z \), isomorphic to the torus embedding

\[
\mathbb{G}_{m/\mathcal{B}_z} \rightarrow \mathbb{A}^1_{\mathcal{B}_z}.
\]

The nonsmooth locus of \( \mathcal{C}_L(m) \) is finite over \( \mathbb{Z}[1/m] \), and supported in characteristics dividing disc(\( k \)). The finiteness allows us to choose the étale neighborhood of \( z \) small enough that it does not meet any nonsmooth points of the interior. This implies that \( \mathbb{G}_{m/\mathcal{B}_z} \) is smooth over \( \mathbb{Z}[1/m] \), which implies the smoothness of \( \mathcal{B}_z \), which implies the smoothness of \( \mathbb{A}^1_{\mathcal{B}_z} \), which implies the smoothness of \( \bar{\mathcal{C}}_L(m) \) near \( z \). It follows that \( \bar{\mathcal{C}}_L(m) \) is regular, its nonsmooth points are contained in the interior, and its boundary is a smooth Cartier divisor.

Pullback of hermitian line bundles defines a homomorphism
\[
\pi_m^* : \bar{\text{Pic}}(\mathcal{C}_L/m) \rightarrow \bar{\text{Pic}}(\mathcal{C}_L(m)).
\]
By Proposition 7.5 of [BGKK07], and recalling Proposition 5.2.1, this restricts to a homomorphism
\[ \pi^*_m : \widehat{\text{Pic}}(\mathcal{C}_L/\mathbb{Z}[1/m], \mathcal{D}_{\text{pre}}) \to \widehat{\text{Pic}}(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}). \]

Now suppose \( m \geq 3 \), so that \( \mathcal{C}_L(m) \) is a projective scheme, and recall the arithmetic Chern class map
\[ \widehat{\text{Pic}}(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}) \to \widehat{\text{CH}}^1(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}) \]
from (1.13) of [BBGK07]. The codomain here is the arithmetic Chow group with pre-log-log forms with respect to the boundary \( \partial \mathcal{C}_L(m) \), as in §1 of [BBGK07] and §7 of [BGKK07]. Given a pre-log-log hermitian line bundle \( \Omega \in \widehat{\text{Pic}}(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}) \), we may form the self-intersection
\[ \Omega \cdot \Omega \in \widehat{\text{CH}}^2(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}) \otimes \mathbb{Q} \]
of its arithmetic Chern class. If we set \( \mathbb{R}_m = \mathbb{R}/\sum_{p|m} \mathbb{Q}\log(p) \), then Remark 1.19 of [BBGK07] provides us with an arithmetic degree
\[ \widehat{\text{deg}} : \widehat{\text{CH}}^2(\mathcal{C}_L(m), \mathcal{D}_{\text{pre}}) \to \mathbb{R}_m, \]
and we define the arithmetic volume \( \widehat{\text{vol}}(\Omega) = \widehat{\text{deg}}(\Omega \cdot \Omega) \in \mathbb{R}_m \).

As in §6.3 of [BBGK07], any pre-log-log hermitian line bundle \( \Omega \in \widehat{\text{Pic}}(\mathcal{C}_L, \mathcal{D}_{\text{pre}}) \) has an arithmetic volume
\[ \widehat{\text{vol}}(\Omega) = \lim_{m \to 3} \frac{\widehat{\text{vol}}(\pi^*_m \Omega)}{\text{deg}(\pi^*_m)} \in \lim_{m \to 3} \mathbb{R}_m = \mathbb{R}, \]
where the limit is with respect to the natural maps \( \mathbb{R}_m \to \mathbb{R}_m \) for \( m' | m \).

**Theorem 5.3.2.** The metrized Hodge bundle of the universal \( \mathcal{O}_k \)-abelian surface \( A \to \mathcal{C}_L \) lies in the subgroup
\[ \widehat{\text{Pic}}(\mathcal{C}_L, \mathcal{D}_{\text{pre}}) \subset \widehat{\text{Pic}}(\mathcal{C}_L) \]
of Proposition 5.2.1. It has geometric degree
\[ \text{deg}_C(\omega^{\text{Hdg}}_{A/\mathcal{C}_L}) = \frac{1}{12 \cdot |\mathcal{O}_k^\times|} \prod_{p|\text{disc}(k)} (1 + p) \]
and arithmetic volume
\[ \widehat{\text{vol}}(\omega^{\text{Hdg}}_{A/\mathcal{C}_L}) = -\text{deg}_C(\omega^{\text{Hdg}}_{A/\mathcal{C}_L}) \left( 1 + \frac{2\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \sum_{p|\text{disc}(k)} \frac{1-p}{1+p} \cdot \log(p) \right). \]

In particular, these are independent of the connected component \( \mathcal{C}_L \subset \mathcal{X}_W \).
Proof. As in Remark 2.1.6, fix $O_B \cong M_2(\mathbb{Z})$, and identify the quaternionic Shimura curve $X_B(N)$ with the open modular curve of level $\Gamma_0(N)$. The universal isogeny of $O_B$-abelian surfaces takes the form

$$A_0 \cong E_0 \times E_1 \rightarrow E_1 \times E_1 \cong A_1$$

for elliptic curves $E_0, E_1 \rightarrow X_B(N)$. For any integer $m \geq 3$ relatively prime to $N$, the finite étale cover

$$X_B(N, m) \rightarrow X_B(N)/\mathbb{Z}[1/m]$$

of Proposition 4.4.2 classifies full level $m$ structures on $E_0$, and there is a canonical isomorphism

$$C_L(m) \cong X_B(N, m)$$

under which the universal $O_k$-abelian surface $A \rightarrow C_L(m)$ on the left is identified with the intermediate abelian surface $A_R = E_1 \times E_0$ of Remark 2.1.6.

It follows from calculations of Kühn and Kramer, see especially Remark 4.10 and Corollary 6.2 of [Ku01], that the metrized Hodge bundle of the universal elliptic curve $E_0 \rightarrow X_B(N, m)$ lies in the subgroup

$$\widetilde{\text{Pic}}(\tilde{X}_B(N, m), \mathcal{D}_{\text{pre}}) \subset \widetilde{\text{Pic}}(X_B(N, m)).$$

Here $X_B(N, m) \hookrightarrow \tilde{X}_B(N, m)$ is the Deligne-Rapoport compactification, constructed as a moduli space of generalized elliptic curves with level structure. Moreover,

$$\deg_C(\omega_{E_0/X_B(N, m)}^{\text{Hdg}}) = \frac{d(m)}{24}$$

and

$$\text{vol}(\omega_{E_0/X_B(N, m)}^{\text{Hdg}}) = \frac{d(m)}{2} \cdot \left( \frac{\zeta(-1)}{2} + \zeta'(-1) \right) \in \mathbb{R}_m$$

where

$$d(m) = \varphi(m) \cdot [\text{PSL}_2(\mathbb{Z}) : \Gamma_0(N) \cap \Gamma(m)]$$

is the degree of the forgetful morphism from $X_B(N, m)$ to the moduli stack of all elliptic curves.

We have constructed $\tilde{C}_L(m)$ as the normalization of a morphism $C_L(m) \rightarrow \mathcal{A}$ to a toroidal compactification of the Siegel 3-fold, and must compare this compactification of $C_L(m) \cong X_B(N, m)$ with the compactification appearing in (5.3.3).

**Lemma 5.3.3.** The isomorphism (5.3.2) extends (necessarily uniquely) to an isomorphism

$$\tilde{C}_L(m) \cong \tilde{X}_B(N, m).$$

---

4Our $d(m)$ is twice the integer $d$ appearing in Corollary 6.2 of [Ku01], because Kühn’s $\Gamma_0(N) \cap \Gamma(m)$ is a subgroup of $\text{PSL}_2(\mathbb{Z})$, not $\text{SL}_2(\mathbb{Z})$. 

---
Proof. It follows from the moduli interpretation of $\tilde{X}_B(N, m)$ that the elliptic curves $E_0$ and $E_1$ over $X_B(N, m)$ extend to smooth group schemes over $\tilde{X}_B(N, m)$ with toric degeneration along the boundary. In other words, they extend to semi-abelian schemes, and hence the same is true of the product $A_R = E_1 \times E_0$. Theorem 5.7(5) in Chapter IV of [FC90] therefore implies that the map

$$X_B(N, m) \to \tilde{A}$$

defined by the polarized abelian surface $A_R \to X_B(N, m)$ extends uniquely to a morphism

$$\tilde{X}_B(N, m) \to \tilde{A}.$$

By the universal property of normalization [Sta18, Tag 035I], there is a unique arrow $i$ making the diagram

$$\begin{array}{ccc}
\mathcal{C}_L(m) & \to & \tilde{X}_B(N, m) \\
i & & \downarrow \\
\mathcal{C}_L(m) & \to & \tilde{A}
\end{array}$$

commute. The diagonal arrow $i$ restricts to an isomorphism of generic fibers, because this restriction is a birational map between smooth proper curves. Moreover, the diagonal arrow is quasi-finite: the fiber over a closed point in the interior $X_B(N, m)$ consists of a single point, while the fiber over a closed point in characteristic $p$ of the boundary $\partial \tilde{X}_B(N, m)$ is contained in the mod $p$ fiber of $\partial \mathcal{C}_L(m)$, which is finite. Being proper and quasi-finite, $i$ is finite by [Sta18, Tag 02OG].

It follows that we may cover $\tilde{X}_B(N, m)$ by open affines $U = \text{Spec}(R)$ in such a way that $i^{-1}(U) = \text{Spec}(R')$ is affine, and $R \to R'$ is a finite morphism of normal domains inducing an isomorphism on fraction fields. Any such $R \to R'$ is an isomorphism, and hence $i$ is itself an isomorphism. □

The first isomorphism in (5.3.1) determines an isomorphism of hermitian line bundles

$$\hat{\omega}_{A_0/\mathcal{C}_L(m)}^{\text{Hdg}} \cong \hat{\omega}_{E_0/\mathcal{C}_L(m)}^{\text{Hdg}} \otimes \hat{\omega}_{E_0/\mathcal{C}_L(m)}^{\text{Hdg}},$$

while the commutativity of (4.4.4) implies

$$\deg(\pi_m) = \frac{|\mathcal{O}_k^\times| \cdot d(m)}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}.$$ 

Thus from the formulas of Kühn and Kramer cited above, one obtains formulas for the complex degree and arithmetic volume of the metrized Hodge bundle of $A_0 = E_0 \times E_0 \to X_B(N, m)$, exactly analogous to the formulas of Kudla-Rapoport-Yang cited in the proof of Theorem 2.3.1.

With these formulas in hand, the proofs of Theorem 2.3.1 and Theorem 5.1.1 extend to the compactification of $X_B(N, m) \cong \mathcal{C}_L(m)$ without change, and show that the metrized Hodge bundle of the universal $\mathcal{O}_k$-abelian surface
$A \to \mathcal{C}_L(m)$, which is identified with the metrized Hodge bundle of the intermediate abelian surface $A_R \to \mathcal{X}_B(N, m)$, lies in the subgroup

$$\hat{\text{Pic}}(\mathcal{C}_L(m), \mathcal{D}_{\text{pro}}) \subset \hat{\text{Pic}}(\mathcal{C}_L(m))$$

and satisfies

$$\deg_C(\omega_{A/\mathcal{C}_L(m)}^{\text{Hdg}}) = \frac{\deg(\pi_m)}{12} \cdot |O_k^\times| \cdot \prod_{p|N} (1 + p)$$

and

$$\hat{\text{vol}}(\omega_{A/\mathcal{C}_L(m)}^{\text{Hdg}}) = -\deg_C(\omega_{A/\mathcal{C}_L(m)}^{\text{Hdg}}) \cdot \left(1 + \frac{2\zeta'(-1)}{\zeta(-1)} + \sum_{p|N} \frac{1 - p \cdot \log(p)}{1 + p} \cdot \frac{2}{2}\right),$$

where the latter equality holds in $\mathbb{R}_m = \mathbb{R}/\sum_{p|m} \mathbb{Q} \log(p)$.

To complete the proof of Theorem 5.3.2, one upgrades this to an equality in $\mathbb{R}$ by choosing relatively prime integers $m_1, m_2 \geq 3$, both prime to $N$. The $\mathbb{Q}$-linear independence of $\{\log(p) : p \text{ prime}\}$ implies that

$$\mathbb{R} = \mathbb{R}_{m_1} \times \mathbb{R}_{m_1 m_2} \mathbb{R}_{m_2},$$

and so all of the stated properties of the metrized Hodge bundle of $A \to \mathcal{C}_L$ follow by combining the corresponding properties of $A \to \mathcal{C}_L(m_i)$ proved above. □

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