Modeling the quantum evolution of the universe through classical matter

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We show that the quantum behavior of the radiation-dominated flat Friedmann-Robertson-Walker (FRW) universe can be reproduced by classical cosmology given that the universe is filled with an exotic matter. In the case of a fluid, we find an implicit equation of state (EoS) which asymptotically approaches \( p = \rho/3 \), showing consistency with the classical model. In the case of two non-interacting scalar fields, where one of them is of the phantom type, we find their potential energy. These scalar fields behaves asymptotically as radiation.

PACS numbers: 04.20.Dw, 04.60.Ds, 04.40.Nr

I. INTRODUCTION

Classical singularities are always present in cosmological models [1]. This is a burden of general relativity and there is a general agreement that a full quantum theory of gravitation will solve this problem, either indicating how to deal with the singularities or excluding them at all. There are several pieces of evidences suggesting that quantum mechanics will be able to exclude the classical singularities on cosmological models. It was shown in [2, 3] that the quantum evolution of the FRW universe filled with a perfect fluid with EoS \( p = w\rho \) is non-singular and matches the classical evolution when \( t \to \infty \). In this context we have shown that the energy conditions for matter are quantum mechanically violated [3], a necessary requirement for the exclusion of singularities.

Here we consider another possibility, a classical exotic matter which, through Einstein equations, reproduces the quantum behavior of the universe. For simplicity, we will consider the quantum flat FRW universe filled with radiation. We then show that in the classical limit, the matter content which fills the universe tends to become radiation with EoS \( p = \rho/3 \). This indicates that it is possible to get important results from quantum cosmology in the classical scenario and still recover the main features of classical cosmology as time flows. We will consider two kinds of matter. A perfect fluid, for which we will find an implicit EoS and two non-interacting scalar fields, one of them is of the phantom type, for which we will be looking for their potential energy.

Non-linear EoS have already been considered in previous works. For instance, in [3] the general quadratic EoS \( p = p_0 + \alpha \rho + \beta \rho^2 \) was studied and it was shown that with the right choice of the parameters \( p_0, \alpha \) and \( \beta \) this exotic fluid can model dark matter. In [4], it was shown that such EoS fits a realistic model where a decelerating universe turns into an accelerating one. More general EoS were studied in [5, 6] and again it was demonstrated that cosmic acceleration can be modeled by exotic fluids. In particular, in [4], it was proved that the early time inflation and late time acceleration can be unified by such fluids. An example of implicit EoS can be found in [9]. By considering an implicit EoS depending also on the Hubble parameter \( H = \dot{a}/a \), a universe with one big bang and one big rip singularity was found and it was shown that the phantom divide \( w = -1 \) is crossed thanks to the effect of the inhomogeneous term \( H \) in the EoS.

A universe filled with multiple (canonical and/or phantom) scalar fields was studied in [10]. There, they reconstructed the entire evolution of the universe, including early inflation and late-time acceleration, using these fields. It is also possible to reconstruct the evolution of the universe through modified theories, for example \( F(R) \) gravity. This can be seen in [11].

In this paper we search for an exotic matter in a non-singular universe. Since the energy conditions are violated due to the exclusion of singularities, cosmic acceleration comes as a bonus. However, the expression for the scale factor is algebraic in time, so this cosmic acceleration is not enough to generate inflation.

II. QUANTUM REPRESENTATIVE OF THE SCALE FACTOR

The action for general relativity with a perfect fluid in Schutz formalism [12, 13] is given by (in units where \( 16\pi G = 1 \))

\[
S_G = \int_M d^4x \sqrt{-g} R + 2 \int_{\partial M} d^3x \sqrt{h} h_{ab} K^{ab} + \int_M d^4x \sqrt{-g} p,
\]

where \( h_{ab} \) is the induced metric over the boundary \( \partial M \) of the four-dimensional manifold \( M \), \( K^{ab} \) is the second fundamental form of the hypersurface \( \partial M \) and \( p \) is the fluid pressure, which is linked to the energy density by the EoS \( p = w\rho \).

The super-Hamiltonian for this action in a flat FRW universe given by the metric

\[
ds^2 = -N^2(t) dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)
\]

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and filled with radiation, is given by

\[ H = \frac{p_a^2}{24} - p_T \approx 0, \quad (3) \]

where \( p_a = -12 \frac{\partial}{\partial a} a/N \) is the momentum conjugated to the scale factor \( a(t) \) and \( p_T \) is the momentum conjugated to the dynamical degree of freedom of the fluid \[13\]. As we can see in Eq. \[3\], this super-Hamiltonian is a constraint. Following Dirac algorithm \[13\] for quantization of constrained Hamiltonian systems, we make the substitutions

\[ p_a \to -i \frac{\partial}{\partial a}; \quad p_T \to -i \frac{\partial}{\partial T} \quad (4) \]

and demand that the super-Hamiltonian operator annihilate the wave function of the universe. The result is

\[ \frac{\partial^2 \Psi}{\partial a^2} + 24i \frac{\partial \Psi}{\partial T} = 0, \quad (5) \]

where \( t = -T \) is the time coordinate in the conformal-time gauge \( N(t) = a(t) \) \[2\]. This is the so called Wheeler-DeWitt equation of the universe \[10\]. Note that in this case we do not have operator ambiguities.

The scale factor is defined on the half-line \((0, \infty)\), therefore with the usual scalar product

\[ \langle \Psi | \Phi \rangle = \int_0^\infty \Psi(a,t)^* \Phi(a,t) \, da, \quad (6) \]

a boundary condition in \( a = 0 \) is necessary in order to ensure self-adjointness of Eq. \[3\]. Lemos found in \[2\] that all possible boundary conditions are given by

\[ \Psi'(0,t) = \beta \Psi(0,t), \quad \beta \in \mathbb{R}. \quad (7) \]

For the sake of simplicity, let us consider only two possibilities \( \beta = 0 \) and \( \beta = \infty \), corresponding to Neumann and Dirichlet boundary conditions, respectively. Eq. \[3\] can be easily solved by finding its propagator \[2\] or separating variables \[3\]. Then the expected value of the scale factor can be readily computed from the wave function \( \Psi(a,t) \) through the formula

\[ \langle a \rangle (t) = \frac{\langle \Psi | a | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \quad (8) \]

The result is \[2,3\]

\[ a(t) = a_0 \sqrt{\gamma^2 + t^2}; \quad (9) \]

where \( a_0 \) and \( \gamma \) are positive constants, \( \gamma \) being related to the initial wave packet. Note that when \( t \to \infty \) we have that \( a(t) \sim t \). In the cosmic gauge, \( \tau \) is given by \( a(t) dt = d\tau \). Therefore \( a(\tau) \sim \tau^{1/2} \), as expected from the classical model \[17\].

## III. EXOTIC EQUATION OF STATE

The four-velocity of a perfect fluid in the comoving coordinates in the conformal gauge \( N(t) = a(t) \) is given by

\[ U_\mu = (a(t), 0, 0, 0), \quad (10) \]

in such a way that the energy-momentum tensor of the fluid

\[ T_{\mu\nu} = (p + \rho) U_\mu U_\nu + \rho g_{\mu\nu} \quad (11) \]

is given by

\[ T_{\mu\nu} = a^2(t) \text{diag}(\rho, p, p, p). \quad (12) \]

Einstein field equations (in units where \( 16\pi G = 1 \)) become

\[ \begin{cases} G_{00} = \frac{3}{\sqrt{2}} a^2(t) = \frac{1}{2} a^2(t) \rho \\
G_{i\bar{i}} = \frac{1}{2} a^2(t) \rho \\
G_{ii} = \frac{2}{\sqrt{2}} a^2(t) \rho \end{cases}, \quad (13) \]

Substituting Eq. \[9\] into the above equations leads to

\[ \rho = \frac{6t^2}{a_0^2(\gamma^2 + t^2)^2}, \quad (14a) \]

\[ p = \frac{\rho}{3} - \frac{4\gamma^2}{a_0^2(\gamma^2 + t^2)^2} = \frac{2}{a_0^2(\gamma^2 + t^2)^2} \quad (14b) \]

Prior to find the EoS of this fluid, let us study the expressions for \( \rho \) and \( p \) as functions of time.

In Fig. \[1\]a) we see that \( \rho(t) \) does not start with a singularity. This is expected since the universe with the scale factor \[9\] is non-singular. Then \( \rho(t) \) reaches a maximum \( \rho_{\text{max}} = 8/(9a_0^2\gamma) \) at \( t = \gamma/\sqrt{2} \), thereafter it starts to decay to 0 as \( t \to \infty \) (as the universe expands). Note that \( \rho(t) \) is not a one-to-one function, so if we wish to invert \( t \) as a function of \( \rho \), we must do this in two branches (more on this later). We know that the energy conditions, along with some reasonable restrictions on the causal structure of the spacetime, lead to singularities in cosmological models. Therefore, for a non-singular universe, we expect the violation of the energy conditions. This can be verified in Fig. \[1\]b), where we analyzed the strong energy condition, which states that \( \rho + 3p \geq 0 \). For small \( t \), we have \( \rho + 3p \leq 0 \), but we recover the classical inequality as time flows. Finally, we see in Fig. \[1\]c) that the classical expression \( p = \rho/3 \) is recovered as \( t \to \infty \). This is not a surprise since \( a(t) \sim t \) in this limit.

We can now find the exotic EoS of the fluid which will reproduce the quantum behavior of the universe. By Eqs. \[14a\] and \[14b\] we have

\[ \begin{cases} \frac{\alpha^2}{4} \rho = \frac{\rho}{(\gamma^2 + t^2)^2} \\
\frac{\alpha^2}{4} (\frac{\rho}{3} - p) = \frac{\gamma^2}{(\gamma^2 + t^2)^2}. \end{cases} \quad (15) \]
Eq. (14b) asserts that
\[
\frac{1}{(\gamma^2 + t^2)^2} = \frac{a_0^{4/3}}{4^{2/3}} \left( \frac{\rho}{\gamma} - p \right)^{2/3},
\]  
so that
\[
\frac{a_0^2}{4} (ρ - p) = \frac{a_0^{4/3}}{4^{2/3}} \left( \frac{\rho}{\gamma} - p \right)^{2/3}. 
\]  

Therefore, we have the implicit EoS \( f(ρ, p) = 0 \), where
\[
f(ρ, p) = \left( \frac{\rho}{\gamma} - p \right)^2 - \frac{a_0^2 γ^4}{4} (ρ - p)^3. 
\]  

Eq. (19) can be used to express \( p \) as a function of \( ρ \). Doing this inversion we come out with three branches, as can be seen in Fig. 2. Two of these branches (denoted by dashed lines) are smoothly connected and they represent the physical “trajectory” in the diagram \( ρ-p \). The third branch (the dotted line) is spurious. In fact, in Eq. (14a) we can invert \( t^2 \) as a function of \( ρ \) for \( 0 < ρ < ρ_{\text{max}} \). We find three different expressions for \( t^2 \) (since we are working with a third order polynomial in \( t^2 \)), two positives and one negative. The negative one is non-physical since it corresponds to an imaginary time. Because \( ρ(t) \) is not a one-to-one function, it must be inverted in two steps. Therefore, one of the positive expressions for \( t^2 \) corresponds to the phase where \( ρ \) grows up to \( ρ_{\text{max}} \) while the other one corresponds to the phase where \( ρ \) decays from \( ρ_{\text{max}} \) to 0. Finally, replacing the expressions of \( t^2 \) in Eq. (14b) we find the three branches of \( p(ρ) \), the one corresponding to the imaginary time being spurious.

Now we will show that the classical EoS \( p = ρ/3 \) is recovered when \( t \to \infty \), or similarly, \( ρ \to 0 \) and \( p \to 0 \) (see Fig. 2). To do this, we expand \( f(ρ, p) \) in Taylor series up to second order. The result is
\[
f(ρ, p) = 2 \left( \frac{ρ}{3} - p \right)^2 + O(3). 
\]  

FIG. 1: a) Letting \( a_0 = γ = 1 \), we see that the curve of \( ρ(t) \) starts at 0, it has a maximum \( ρ_{\text{max}} = 9/8 \) at \( t = 1/\sqrt{2} \) and it decays to 0 as \( t \to \infty \). b) Also, looking to the graphic of \( ρ(t) + 3p(t) \) for the same values of \( a_0 \) and \( γ \), we clearly see that the strong energy condition is violated for small values of \( t \). c) Finally, we consider the graphic of \( p(t)/ρ(t) \) for \( a_0 = γ = 1 \). In the classical limit \( (t \to \infty) \), \( p/ρ \to 1/3 \).

FIG. 2: The three branches of \( p = p(ρ) \) obtained from Eq. (19). Two of them (denoted by dashed lines) are connected smoothly and represent the physical trajectory of \( p \) as a function of \( ρ \) as time flows (see the arrows representing the direction of time). The other one is spurious. Inset we see that as \( t \to ∞ \), we recover the classical EoS \( p = ρ/3 \).
When $\rho \to 0$ and $p \to 0$ we have $f(\rho, p) = 0 \Rightarrow p = \rho/3$. This shows that when $t \to \infty$ the classical limit is recovered.

IV. SCALAR FIELDS

We can try to reproduce the quantum behavior of the universe by adding a scalar field instead of a perfect fluid. We choose to add a phantom and a standard scalar field. As we will see in a moment, with these two non-interacting fields we can fit the initial cosmic acceleration and still recover the classical behavior of the universe as $t \to \infty$.

The Lagrangian for these two fields is given by

$$L = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_{st} \partial_\nu \phi_{st} - V(\phi_{st})$$
$$+ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_{ph} \partial_\nu \phi_{ph} - U(\phi_{ph}),$$

where $\phi_{st}$ corresponds to the standard scalar field while $\phi_{ph}$ corresponds to the phantom one.

The energy density and the pressure for these fields can be expressed as

$$\rho = \frac{\dot{\phi}_{st}^2}{2a^2} - \frac{\dot{\phi}_{ph}^2}{2a^2} + V(\phi_{st}) + U(\phi_{ph})$$
$$+ \frac{\dot{\phi}_{st}^2}{2a^2} - \frac{\dot{\phi}_{ph}^2}{2a^2} - V(\phi_{st}) - U(\phi_{ph}).$$

By adding and subtracting the above equations we have

$$\rho + p = \frac{\dot{\phi}_{st}^2}{a^2} - \frac{\dot{\phi}_{ph}^2}{a^2},$$
$$\rho - p = 2V(\phi_{st}) + 2U(\phi_{ph}).$$

Using Eqs. (14a) and (14b) leads to

$$\dot{\phi}_{st}^2 - \dot{\phi}_{ph}^2 = 4 \left[ \frac{2t^2 - \gamma^2}{(\gamma^2 + t^2)^2} \right],$$
$$V(\phi_{st}) + U(\phi_{ph}) = \frac{2}{a_0^2 \gamma^4 (\gamma^2 + t^2)^2}. \tag{24b}$$

We demand the standard scalar field and the phantom one to take care of the positive and negative parts of the right side of Eq. (24), respectively. In this way

$$\dot{\phi}_{st}^2 = \frac{8t^2}{(\gamma^2 + t^2)^2} \Rightarrow \phi_{st}(t) = \phi_{bst} + \sqrt{2} \ln \left(1 + t^2/\gamma^2\right)$$
$$\dot{\phi}_{ph}^2 = \frac{4\gamma^2}{(\gamma^2 + t^2)^2} \Rightarrow \phi_{ph}(t) = \phi_{ph} + 2 \arctan(t/\gamma).$$

Since

$$\gamma^2 + t^2 = \gamma^2 \exp\left(\frac{\phi_{st} - \phi_{bst}}{\sqrt{2}}\right),$$

it is easy to show that [by using Eq. (24b)]

$$V(\phi_{st}) + U(\phi_{ph}) = \frac{2}{a_0^2 \gamma^4} \exp\left[-\sqrt{2}(\phi_{st} - \phi_{bst})\right]. \tag{27}$$

Equivalently, we have

$$t = \gamma \tan \left(\frac{\phi_{ph} - \phi_{ph}}{2}\right),$$

so that

$$V(\phi_{st}) + U(\phi_{ph}) = \frac{2}{a_0^2 \gamma^4 \left[1 + \tan^2\left(\frac{\phi_{ph} - \phi_{ph}}{2}\right)\right]^2}. \tag{29}$$

Obviously the right hand sides of Eqs. (27) and (29) are equal. Therefore, both equations are satisfied if

$$V(\phi_{st}) = \alpha \frac{2}{a_0^2 \gamma^4} \exp\left[-\sqrt{2}(\phi_{st} - \phi_{bst})\right],$$
$$U(\phi_{ph}) = (1 - \alpha) \frac{2}{a_0^2 \gamma^4 \left[1 + \tan^2\left(\frac{\phi_{ph} - \phi_{ph}}{2}\right)\right]^2},$$

where $0 \leq \alpha \leq 1$.

It is also simple to check that in this case $p = \rho/3$ as $t \to \infty$.

V. FINAL REMARKS

We have found an exotic classical fluid which reproduces the quantum behavior of the universe. This fluid obeys an implicit equation of state. When we use this equation to find $p$ as a function of $\rho$ we find three different branches. Two of them correspond to the physical “trajectory” in the $\rho$-$p$ diagram. The third one is spurious. Moreover, this equation of state approaches $p = \rho/3$ as $t \to \infty$ (or $\rho$ and $p$ goes to zero as the universe expands). We can also reproduce the quantum behavior of the universe with the insertion of two non-interacting scalar fields, one of them being usual and the other of the phantom type. We are free to choose the fraction of potential energy for each field. In particular, we can set one of them to be free. In either case, we showed that it is possible to get important results from quantum cosmology and still recover the classical limit with the use of classical matter.

Acknowledgments

J.P.M.P. thanks Alberto Saa for enlightening conversations. This work was supported by FAPESP (Grant No. 2008/01310-5). One of the authors, Patricio S. Letelier, sadly passed away at the end of this work. J.P.M.P. acknowledges his partnership in this and other papers.
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