SUPPORTS OF IRREDUCIBLE CHARACTERS OF $p$-GROUPS

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Abstract. If $\chi$ is an irreducible character of a finite group $G$ then the support of $\chi$, denoted $\text{Supp}(\chi)$, is the set of $g \in G$ with $\chi(g) \neq 0$. In this note, we study the supports of characters of certain types of $p$-groups. We show that if $\chi \in \text{Irr}(P)$ where $P$ is a metabelian $p$-group, then $\text{Supp}(\chi)$ contains at most $|P|/\chi(1)^2$ conjugacy classes, and we give a bound on the orders of elements in $\text{Supp}(\chi)$. We give more precise results for $p$-groups of nilpotence class at most 3 and groups of order dividing $p^9$.

1. Introduction

Let $\text{Irr}(G)$ denote the set of complex irreducible characters of a finite group $G$. If $\chi \in \text{Irr}(G)$ then the support of $\chi$ is the subset of $G$ on which $\chi$ does not vanish. We write $\text{Supp}(\chi)$ to denote the support of $\chi$. In this note, we present some results about the supports of irreducible characters of $p$-groups. (A $p$-group is a finite group whose order is a power of the prime $p$.)

If $\chi$ is an irreducible character of a $p$-group $P$, then $\chi$ is induced from a linear character of a subgroup $H \subseteq P$, where $|H| = |P|/\chi(1)$. (See Corollary 6.14 in [5].) Then every element of $\text{Supp}(\chi)$ has a conjugate that is contained in $H$, and it follows that $\text{Supp}(\chi)$ contains at most $|P|/\chi(1)$ conjugacy classes, and also that every element of $\text{Supp}(\chi)$ has order dividing $|P|/\chi(1)$.

We show that these bounds can be strengthened for some interesting classes of $p$-groups. Our first result concerns metabelian $p$-groups.

**Theorem A.** Let $P$ be a metabelian $p$-group and let $\chi \in \text{Irr}(P)$. Then

(1) $\text{Supp}(\chi)$ contains at most $|P|/\chi(1)^2$ conjugacy classes.

(2) If $p$ is odd, each element of $\text{Supp}(\chi)$ has order at most $|P|/\chi(1)^{3/2}$.

Theorem A(1) is false for $p$-groups in general. Using GAP ([3]), we found that the group $P$ with GAP identifier SmallGroup(512,2015) has two characters of degree 8 that are non-vanishing on 10 conjugacy classes of $P$. Since $10 \times 64 > 512$, this violates the bound of Theorem A(1). The group $P$ has derived length 3.

The exponent of 2 in Theorem A(1) cannot be improved, but in the case of Theorem A(2), we do not know the best value of the exponent. Theorems

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B(4) and C below show that its correct value is 2 in some cases. Since groups of nilpotence class at most 3 are metabelian, Theorem B can be seen as a strong version of Theorem A for such groups.

**Theorem B.** Let $P$ be a $p$-group of nilpotence class at most 3. Let $\chi \in \text{Irr}(P)$ and $x \in \text{Supp}(\chi)$. Let $h(x)$ be the size of the conjugacy class of $x$. Then

1. If $\chi$ is faithful, then $\frac{\chi(x)\sqrt{h(x)}}{\chi(1)}$ is a root of unity.
2. If $\chi$ is faithful, $\text{Supp}(\chi)$ contains exactly $|P|/\chi(1)^2$ conjugacy classes.
3. If $p$ is odd, then $x^m \in \text{Supp}(\chi)$ for all $m \in \mathbb{Z}$.
4. If $p$ is odd, then the order of $x$ divides $|P|/\chi(1)^2$.

We remark that in the situation of Theorem B, often $\text{Supp}(\chi) = \mathbb{Z}(\chi)$, where $\mathbb{Z}(\chi)/\ker(\chi)$ is the centre of $P/\ker(\chi)$. In other words, $\chi$ is often of central type when regarded as a character of $G/\ker(\chi)$. This always occurs when $P$ has nilpotence class 2, by for example Lemma 2.27(f) and Theorem 2.31 in [5]. (In general, groups whose irreducible characters $\chi$ all satisfy $\text{Supp}(\chi) = \mathbb{Z}(\chi)$ are called GVZ groups, and are studied in [8].) Theorem B is trivial for such characters, but of course, not all characters of groups of nilpotence class 3 have this property. For example, if $P$ is the dihedral group of order 16, $x \in P$ is an element of order 8 and $\chi \in \text{Irr}(P)$ is faithful then $x$ is not central but $\chi(x) = \pm\sqrt{2}$. Since $\chi(1) = 2$ and $\chi(x^2) = 0$, this example also shows that $p$ does need to be odd for parts (3) and (4) of Theorem B.

We mentioned that we do not know what value is the best possible for the exponent in Theorem A(2). Theorem B shows that the correct exponent is 2 for $p$-groups of nilpotence class at most 3, when $p$ is odd. Our last result shows that the same holds for all $p$-groups of order at most $p^9$, where $p$ is odd (we do not know what happens for $p$-groups of larger order).

**Theorem C.** Let $P$ be a $p$-group, where $p$ is odd, and let $\chi \in \text{Irr}(P)$. Suppose there exists $x \in \text{Supp}(\chi)$ such that the order of $x$ does not divide $|P|/\chi(1)^2$. Then $\chi(1)$ is divisible by $p^4$ and $|P|$ is divisible by $p^{10}$.

The proofs of Theorems A, B and C are in Sections 2, 3 and 4 respectively. As far as we know, these results have not been reported before, but similar questions have been considered in [2] and [7], for example. In particular, we draw attention to Theorem C in [7], which looks at the situation from the opposite point of view, giving a lower bound for the number of conjugacy classes on which an irreducible character of an arbitrary $p$-group vanishes.

## 2. Metabelian $p$-groups

In this section, we prove Theorem A. We need the following well-known fact about metabelian groups.

**Lemma 2.1.** Let $G$ be a finite metabelian group and let $\chi \in \text{Irr}(G)$. Then $G$ has a normal subgroup $K$ with $G/K$ abelian, such that $\chi = \lambda^G$ for some linear character $\lambda \in \text{Irr}(K)$.
Proof. This follows from Corollary 2.6 in [5], taking the subgroup $N$ in that corollary to be the derived subgroup of $G$. \qed

We begin with the proof of Theorem A(1). If $A$ and $H$ are groups with $H$ acting on $A$ (we always mean an action via automorphisms of $A$), then for $\lambda \in \text{Irr}(A)$, we define the trace of $\lambda$ with respect to $H$, denoted $\text{tr}_H(\lambda)$, by

$$\text{tr}_H(\lambda) = \sum_{h \in H} \lambda^h.$$ 

Equivalently, $\text{tr}_H(\lambda) = (\lambda^{AH})_A$, where $AH$ is the semidirect product of $A$ and $H$ corresponding to the given action and $\lambda^{AH}$ is the induced (not necessarily irreducible) character. Also define

$$\sigma(A, \lambda, H) = \sum_{x \in A \text{ with } \text{tr}_H(\lambda)(x) \neq 0} |C_H(x)|.$$ 

We prove Theorem A(1) via the following result:

**Proposition 2.2.** Suppose $A$ and $H$ are abelian $p$-groups for some prime $p$, with $H$ acting on $A$. Let $\lambda \in \text{Irr}(A)$. Then $\sigma(A, \lambda, H) \leq |A||C_H(\lambda)|$.

Our proof of Proposition 2.2 requires two lemmas.

**Lemma 2.3.** Suppose $A$ and $H$ are $p$-groups with $A$ abelian and $H$ acting on $A$. Let $\lambda \in \text{Irr}(A)$ be a character of $A$. Let $B$ be a $H$-invariant subgroup of $A$ with $|A : B| = p$. Then one of the following cases holds:

1. $C_H(\lambda_B) = C_H(\lambda)$.
2. $|C_H(\lambda_B)| = p|C_H(\lambda)|$ and $\text{tr}_H(\lambda)(x) = 0$ for all $x \in A \setminus B$.

**Proof.** Let $\text{Irr}(A|\lambda_B)$ denote the set of irreducible characters of $A$ that lie over $\lambda_B \in \text{Irr}(B)$. Then $C_H(\lambda_B)$ acts on $\text{Irr}(A|\lambda_B)$. If the action fixes $\lambda$ then case (1) holds. If not, then since $|C_H(\lambda_B) : C_H(\lambda)| \leq |\text{Irr}(A|\lambda_B)| = p$ and $H$ is a $p$-group, $|C_H(\lambda_B) : C_H(\lambda)| = p$. This is case (2), but we must also show that $\text{tr}_H(\lambda)$ vanishes outside of $B$ in this case. Fix $g \in C_H(\lambda_B)$ with $g \notin C_H(\lambda)$, so $\lambda^g = \lambda\mu$ for some $\mu \in \text{Irr}(A/B)$ with $\mu \neq 1$. Since $H$ centralizes $A/B$ and therefore fixes $\mu$, we have

$$\text{tr}_H(\lambda) = \sum_{h \in H} \lambda^h = \sum_{h \in H} \lambda^{gh} = \sum_{h \in H} (\lambda\mu)^h = \sum_{h \in H} \lambda^h\mu = (\text{tr}_H(\lambda))\mu.$$ 

Hence if $\text{tr}_H(\lambda)(x) \neq 0$ then $x \in \ker(\mu) = B$. \qed

**Lemma 2.4.** Suppose $A$ and $H$ are abelian $p$-groups with $H$ acting on $A$. Let $\lambda \in \text{Irr}(A)$ be a character of $A$, and suppose $B$ is a $H$-invariant subgroup of $A$ with $|A : B| = p$. Then

$$\sigma(A, \lambda, H) \leq \sigma(B, \lambda_B, H) + (|A| - |B||C_H(\lambda)|).$$

**Proof.** The proof is by induction on $|A|$. First, assume that $B$ is the unique $H$-invariant subgroup of $A$ of index $p$. If $x \in A$, then since $H$ is abelian,
Let this be \( B \), by induction, \( A \) with \( 1 | p \) and the result follows since \( C \) the result follows. Otherwise, case (1) of Lemma 2.3 applies to \( p \).

The required result follows by summing over \( x \).

Proof of Proposition 2.2. We proceed by induction on \( |A| \). This completes the proof.

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\( C_A(C_H(x)) \) is a \( H \)-invariant subgroup of \( A \) containing \( x \). As \( A \) and \( H \) are \( p \)-groups, every proper \( H \)-invariant subgroup of \( A \) is contained in a \( H \)-invariant subgroup of index \( p \). Hence in this case \( C_H(x) = C_H(A) \) for \( x \in A \backslash B \). Then

\[
\sigma(A, \lambda, H) \leq \sigma(B, \lambda_B, H) + (|A| - |B|)|C_H(A)|,
\]

and the result follows since \( C_H(A) \subseteq C_H(\lambda) \).

We may therefore choose another \( H \)-invariant subgroup of index \( p \) in \( A \). Let this be \( B_1 \) and set \( C = B \cap B_1 \). Then \( H \) acts trivially on \( A/C \), so the \( p+1 \) proper subgroups of \( A \) strictly containing \( C \) are all \( H \)-invariant. Two of these are \( B \) and \( B_1 \), and we label the others as \( B_2, \ldots, B_p \). Then \( A \backslash B \) is the disjoint union of the sets \( B_1 \backslash C, \ldots, B_p \backslash C \). Hence

\[
\sigma(A, \lambda, H) - \sigma(B, \lambda_B, H) = \sum_{i=1}^{p} (\sigma(B_i, \lambda_{B_i}, H) - \sigma(C, \lambda_C, H)).
\]

We first consider this equation assuming case (1) of Lemma 2.3 holds for \( B_i \) for each \( i \) with \( 1 \leq i \leq p \). Then \( C_H(\lambda_{B_i}) = C_H(\lambda) \) for \( 1 \leq i \leq p \). Since \( |B_i : C| = p \) for each \( i \), induction gives

\[
\sigma(B_i, \lambda_{B_i}, H) - \sigma(C, \lambda_C, H) \leq (|B_i| - |C|)|C_H(\lambda_{B_i})| = (|B_i| - |C|)|C_H(\lambda)|.
\]

The required result follows by summing over \( i \) with \( 1 \leq i \leq p \).

Now we may assume that case (2) of Lemma 2.3 holds for \( B_i \) for some \( i \) with \( 1 \leq i \leq p \). Then \( \text{tr}_H(\lambda) \) vanishes off \( B_i \), so for \( j \), with \( 1 \leq j \leq p \) and \( j \neq i \), \( \sigma(B_j, \lambda_{B_j}, H) = \sigma(C, \lambda_C, H) \). By induction, \( \sigma(B_i, \lambda_{B_i}, H) - \sigma(C, \lambda_C, H) \leq (|B_i| - |C|)|C_H(\lambda_{B_i})| \), and by Lemma 2.3 \( |C_H(\lambda_{B_i})| = p|C_H(\lambda)| \).

Hence

\[
\sigma(A, \lambda, H) - \sigma(B, \lambda_B, H) \leq p(|B_i| - |C|)|C_H(\lambda)| = (|A| - |B|)|C_H(\lambda)|.
\]

This completes the proof.

Proof of Theorem A(1). We write \( k(\text{Supp}(\chi)) \) to denote the number of conjugacy classes in \( \text{Supp}(\chi) \). We must prove that \( k(\text{Supp}(\chi)) \leq |P|/\chi(1)^2 \). Let \( K = \ker(\chi) \), so that \( \chi \) is the lift to \( P \) of a faithful character \( \chi_0 \in \text{Irr}(P/K) \). Clearly \( k(\text{Supp}(\chi)) \leq |K|k(\text{Supp}(\chi_0)) \), so it suffices to prove that \( k(\text{Supp}(\chi_0)) \leq |P/K|/\chi_0(1)^2 \). Hence, we can assume \( \chi = \chi_0 \) is faithful.
By Lemma 2.1, \( \chi = \lambda^p \) for some \( A \triangleleft P \) with \( P/A \) abelian and some linear character \( \lambda \in \text{Irr}(A) \). Since \( \chi \) is faithful, \( A \) is abelian. Let \( H = P/A \) with its natural action by conjugation on \( A \), and let \( \lambda \) be a constituent of \( \chi_A \). Then \( \chi_A = \text{tr}_H(\lambda) \) and the support of \( \chi \) is contained in \( A \). Proposition 2.2 applies and \( C_H(\lambda) = 1 \) since \( \chi = \lambda^p \) is irreducible. Hence \( \sigma(A, \lambda, H) \leq |A| \).

We compute

\[
\sum_{x \in A, \chi(x) \neq 0} \frac{|C_H(x)|}{|H|} = \frac{\sigma(A, \lambda, H)}{|H|} \leq \frac{|A|}{|H|} = \frac{|P|}{|H|^2} = \frac{|P|}{\chi(1)^2}.
\]

This completes the proof of Theorem A(1). \( \square \)

For the proof of Theorem A(2), we need some properties of the fields generated by the values of characters of \( p \)-groups. Denote by \( \mathbb{Q}(\chi) \) the field generated by the values of a character \( \chi \). Also, let \( \zeta_n = e^{2\pi i/n} \), and let \( \text{tr}_{K/L} \) be the relative trace from \( K \) to \( L \) where \( K \supseteq L \) are fields (no confusion should arise with the different operator \( \text{tr}_H \) used above).

**Lemma 2.5.** Let \( p \) be an odd prime and let \( n \geq 1 \). Let \( K \) be a field. Then

1. \( \mathbb{Q}(\zeta_p) \subseteq K \subseteq \mathbb{Q}(\zeta_p^*) \) if and only if \( K = \mathbb{Q}(\zeta_p^*) \) for some \( r \), \( 1 \leq r \leq n \).
2. If \( r > s \geq 1 \) and \( \zeta \) is any primitive \((p^r)^{\text{th}}\) root of unity, then

\[
\text{tr}_{\mathbb{Q}(\zeta_p^*)/\mathbb{Q}(\zeta_p^*)}(\zeta) = 0.
\]

**Proof.** The fields \( \mathbb{Q}(\zeta_p^*) \) for \( 1 \leq r \leq n \) clearly lie between \( \mathbb{Q}(\zeta_p) \) and \( \mathbb{Q}(\zeta_p^*) \). On the other hand, the Galois group of \( \mathbb{Q}(\zeta_p^*)/\mathbb{Q}(\zeta_p) \) is the subgroup of \((\mathbb{Z}/p^n\mathbb{Z})^\times\) consisting of classes that are congruent to \( 1 \) mod \( p \). As \( p \) is odd it is cyclic, generated for example by the class of \( 1+p \). Since a cyclic \( p \)-group has at most one subgroup of each order, it follows that there are no other intermediate fields. For part (2), since \( r > s \), we have \( \zeta^{p^r} \neq 1 \). Therefore

\[
\text{tr}_{\mathbb{Q}(\zeta_p^*)/\mathbb{Q}(\zeta_p^*)}(\zeta) = \sum_{n \equiv 1 \text{ mod } p^s} \sum_{j=0}^{p^r-1} \zeta^{j+p^r} = \zeta^{p^r-1} = 0,
\]

as required. \( \square \)

**Lemma 2.6.** Let \( P \) be a \( p \)-group, where \( p \) is odd. Let \( \chi \in \text{Irr}(P) \). Then \( \mathbb{Q}(\chi) = \mathbb{Q}(\zeta_p^r) \), where \( p^r \leq |P|/\chi(1)^2 \).

**Proof.** We can assume that \( \chi \neq 1 \). Then \( \mathbb{Q}(\zeta_p) \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}(\zeta_p^r) \), where \( p^r = |P| \). By Lemma 2.5, \( \mathbb{Q}(\chi) = \mathbb{Q}(\zeta_p^r) \) for some \( r \leq n \). Then \( \chi \) has \( |\mathbb{Q}(\chi) : \mathbb{Q}| = p^{r-1}(p-1) \) Galois conjugates, so \( p^{r-1}(p-1)\chi(1)^2 \leq |P| - 1 \). Since \( |P|/\chi(1)^2 \) is a power of \( p \), it follows that \( p^r \leq |P|/\chi(1)^2 \) as required. \( \square \)

**Lemma 2.7.** Let \( P \) be a \( p \)-group, where \( p \) is odd. Suppose \( \chi \in \text{Irr}(P) \) is faithful. Let \( x \in P \) and suppose \( \chi(x) \neq 0 \). Let \( C = \langle x \rangle \). If the order of \( x \) does not divide \( |P|/\chi(1)^2 \) then \( C \cap C^g = 1 \) for some \( g \in P \).
Proof. Let \( p^e \) be the order of \( x \). By Lemma 2.5(1) and Lemma 2.6, \( \mathbb{Q}(\chi) = \mathbb{Q}(\zeta_{p^e}) \) where \( p^f \leq |P|/\chi(1)^2 \). We can write
\[
|\mathbb{Q}(\zeta_{p^e}) : \mathbb{Q}(\zeta_{p^f})| = \text{tr}_{\mathbb{Q}(\zeta_{p^e})/\mathbb{Q}(\zeta_{p^f})}(\chi(x)).
\]
By hypothesis, \( e > f \) and \( \chi(x) \) is not zero. In view of Lemma 2.5(2), \( \chi(x) \) is not a sum of primitive \( (p^e) \)th roots of unity. Hence, some \( \lambda \in \text{Irr}(C) \) lying under \( \chi \) is not faithful. Therefore, if \( D \) is the unique subgroup of \( C \) of order \( p \), we have \([\chi, 1]_D > 0\), so \( D \) cannot be normal in \( P \) since \( \chi \) is faithful. Since \( C \) is cyclic, there must be \( g \in P \) such that \( C \cap C^g \) does not contain \( D \). Then \( C \cap C^g = 1 \), as required.

Proof of Theorem A(2). Let \( C \) be the cyclic group generated by \( x \). We have to prove that \( |C| \leq |P|/\chi(1)^{3/2} \). If \( K = \ker(\chi) > 1 \) then by induction on \( |P|, |CK/K| \leq |P/K|/\chi(1)^{3/2} \), and the result follows. Hence, we may assume \( \chi \) is faithful. By Lemma 2.7, either \( |C| \) divides \( |P|/\chi(1)^2 \) or there exists \( g \in P \) with \( C \cap C^g = 1 \). We may assume the latter holds. By Lemma 2.1, there is a normal subgroup \( A < P \) and a linear character \( \lambda \in \text{Irr}(A) \) with \( \lambda^P = \chi \). Since \( \chi(x) \neq 0, C \subseteq A \). Then as \( C \cap C^g = 1, |C|^2 \leq |A| = |P|/\chi(1) \). Since \( \chi(1)^2 \leq |P| \), it follows that \( |C|^2 \leq |P|^2/\chi(1)^3 \), as required.

3. Groups of nilpotence class 3

In this section, we will prove Theorem B. The proof of parts (3) and (4) depends on an important theorem of I.M. Isaacs [4]. Recall that if \( G \) is a group with a normal subgroup \( N \), then \( \chi \in \text{Irr}(G) \) and \( \varphi \in \text{Irr}(N) \) are said to be fully ramified with respect to \( G/N \) if \( \chi_N = e \varphi \) with \( e^2 = |G : N| \).

Following [4], we call a set \((G, K, L, \theta, \varphi)\) of groups and characters a character five if \( K \) and \( L \) are normal subgroups of the finite group \( G \) with \( L \subseteq K \), and \( \theta \in \text{Irr}(K) \) and \( \varphi \in \text{Irr}(L) \) are \( G \)-invariant characters and are fully ramified with respect to \( K/L \). The following only includes what we need for the proof of Theorem B; see [4] for the full statement.

Theorem 3.1 (I.M. Isaacs. See Theorem 9.1 in [4]). Let \((G, K, L, \theta, \varphi)\) be a character five. Suppose \( K/L \) is abelian and \(|K : L|\) is odd. Then there is a subgroup \( U \subseteq G \), and a (reducible, in general) character \( \Psi \) of \( G \), such that

1. \( KU = G \) and \( K \cap U = L \).
2. \( \Psi(1)^2 = |K/L| \) and \( \Psi \) has no zeros on \( G \).
3. For each \( \chi \in \text{Irr}(G/\theta) \) there exists \( \xi \in \text{Irr}(U/\varphi) \) such that \( \chi_U = \Psi_U \xi \) and \( \xi^G = \overline{\Psi}_\chi \).

The next lemma is essentially Problem 3.12 in [5].

Lemma 3.2. Let \( G \) be a finite group and let \( \chi \in \text{Irr}(G) \). Suppose \( N \) is a normal subgroup of \( G \) such that \( \chi_N \) is irreducible. Then for \( x \in G \), the following formula holds:
\[
|\chi(x)|^2 = \frac{\chi(1)}{|N|} \sum_{g \in N} \chi([x, g]).
\]
Proof. Fix \( x \in G \) and let \( H = N(x) \). Clearly \( \chi_H \in \text{Irr}(H) \). Let \( \rho \) be a representation of \( CH \) affording \( \chi_H \) and let \( C_x \in Z(CH) \) be the sum over the conjugacy class \( x^H = x^N \). Then \( \rho(C_x) = \omega(x)I \), where \( I \) is the identity matrix and \( \omega(x) = |x^N|\chi(x)/\chi(1) \). Hence \( \rho(x^{-1}C_x) = \omega(x)\rho(x^{-1}) \). Taking the matrix trace of both sides of this equation gives \( \omega(x)\chi(x^{-1}) = \sum_{y \in x^N} \chi(y^{-1}x) \), which is equivalent to the required result.

We also need the following, which will provide the implication \( (3) \implies (4) \) in the proof of Theorem B.

**Lemma 3.3.** Let \( P \) be a \( p \)-group, where \( p \) is any prime. Let \( \chi \in \text{Irr}(P) \) and let \( H \) be a subgroup of \( P \). Suppose \( \chi \) is not zero at any element of \( H \). Then \( |H| \) divides \( |P|/\chi(1)^2 \).

**Proof.** Let \( P = N_0 \supset N_1 \ldots \supset N_n = 1 \) be a chief series of \( P \). For \( 1 \leq i \leq n \), let \( e_i = |\chi, \chi|_{N_i}/|\chi, \chi|_{N_{i-1}} \). The \( e_i \) are integers because \( |\chi, \chi|_{N_i} \) is a power of \( p \) for each \( i \) with \( 0 \leq i \leq n \), and \( |\chi, \chi|_{N_i} \geq |\chi, \chi|_{N_{i-1}} \). Since \( |\chi, \chi|_{N_i} \leq p|\chi, \chi|_{N_{i-1}} \), it follows that for each \( 1 \leq i \leq n \), either \( e_i = 1 \) or \( e_i = p \), and also clearly \( e_i = p \) if and only if \( \chi \) vanishes on \( N_{i-1}\setminus N_i \).

Now let \( f_i = |HN_{i-1} : HN_i| \). Then \( f_i = p \) if \( H \cap N_i = H \cap N_{i-1} \) and \( f_i = 1 \) otherwise. Since \( \chi \) does not vanish on \( H \), \( e_i = p \) only if \( f_i = p \). In particular, \( e_i \) divides \( f_i \) for each \( 1 \leq i \leq n \). Hence \( \chi(1)^2 = \prod_{i=1}^n e_i \) divides \( |P : H| = \prod_{i=1}^n f_i \), which is the required result. \( \square \)

**Proof of Theorem B.** We first show that \( (1) \) implies \( (2) \) and \( (3) \) implies \( (4) \).

Let \( k \) be the number of conjugacy classes in \( \text{Supp}(\chi) \) and let \( x_i \), for \( 1 \leq i \leq k \) be representatives of those classes. Let \( h_i \) be the size of the conjugacy class of \( x_i \). As \( \chi \) is faithful, assuming \( (1) \) gives \( h_i|\chi(x_i)|^2 = \chi(1)^2 \) for each \( i \). Hence \( |P| = \sum_i h_i|\chi(x_i)|^2 = k\chi(1)^2 \), so \( k = |P|/\chi(1)^2 \) which is \( (2) \). Also \( (4) \) follows from \( (3) \) and Lemma 3.3 applied to the cyclic group generated by \( x \).

To complete the proof of Theorem B, we need to prove parts \( (1) \) and \( (3) \).

We can clearly assume \( \chi \) is faithful. Since \( P \) has nilpotence class 3, it is metabelian, so by Lemma 2.1 we may choose a normal subgroup \( A \triangleleft P \) with \( P/A \) abelian and a linear character \( \lambda \in \text{Irr}(A) \) such that \( \lambda^G = \chi \). Since \( \chi \) is faithful, \( A \) is abelian. Note that \( x \in A \) since \( \chi \) vanishes off \( A \).

Let \( B = \langle x^P \rangle \) and \( X = C_P(\lambda B) \). Then \( X \supsetneq A \), so \( X \triangleleft P \) and hence also \( [B, X] \triangleleft P \). Also \( \lambda([B, X]) = 1 \) by definition of \( X \), and we conclude that \( [B, X] \subseteq \ker \chi \). Since \( \chi \) is faithful, this implies that \( X \) centralizes \( B \). Hence, writing \( H = P/X \), we can form the semidirect product \( P_0 = BH \) corresponding to the natural action of \( H \) on \( B \).

Let \( \chi_0 = (\lambda B)^{P_0} \). Since \( X = C_P(\lambda B) \) we see that \( \chi_0 \in \text{Irr}(P_0) \). Also \( (\chi_0)/\chi(1) = \chi_B/\chi(1) \); in particular, this shows that \( \chi_0 \) is faithful. Also, \( h(x) = |H| \) whether computed in \( P \) or \( P_0 \). Hence if parts \( (1) \) and \( (3) \) of Theorem B are true for \( P_0, \chi_0 \) and \( x \) (regarded as an element of \( P_0 \)), then those statements also hold for \( P, \chi_0 \) and \( x \). We can therefore work with \( P_0 \) and \( \chi_0 \) for the remainder of the proof. For ease of notation, we will write \( P_0 = P \) and \( \chi_0 = \chi \) from now on.
Set $N = [B, H]H$. Then $N < P$. Since $B = \langle x^H \rangle$, it follows that $P = BH = \langle x \rangle N$. Let $\varphi \in \text{Irr}(N)$ be a constituent of $\chi_N$ and let $J$ be the stabilizer of $\varphi$. Then $N \subseteq J < G$ and $\chi$ is induced from a character of $J$; since $\chi(x) \neq 0$ this implies $x \in J$ and so $J = P$. Therefore $\chi_P$ is homogeneous. Since $P/N$ is cyclic, it follows that $\chi_N$ is irreducible. We have $[N, N] = [B, H, H]$, so since $P$ has nilpotence class 3, $[N, N] \subseteq Z(P)$. Since $\chi_N \in \text{Irr}(N)$ is faithful, $\chi_N$ is fully ramified over $Z(P)$ and in particular, $\chi_N$ vanishes on $N \setminus Z(P)$ (see Theorem 3.1 in [5]). Then by Lemma 3.2,
\[
|\chi(x)|^2 = \frac{1}{|N|} \sum_{g \in Q} \chi([x, g]),
\]
where $Q$ is the pre-image in $N$ of $C_{N/Z(P)}(xZ(P))$. Let $\mu$ be the map which takes $g \in Q$ to $\chi([x, g])/\chi(1)$. It is clear that $\mu$ is a linear character of $Q$, and
\[
|\chi(x)|^2 = \frac{1}{|N|} |\mu, 1|_Q.
\]
Since $\chi(x) \neq 0$, we must have $\mu = 1$, so $|\chi(x)|^2 = \chi(1)^2/|N : Q|$. Since $\chi$ is faithful, $\mu = 1$ implies that $[x, g] = 1$ for all $g \in Q$, so $Q = C_N(x)$. Since $N = HC_N(x)$, we have $|N : Q| = |H : C_H(x)|$. We obtain
\[
|\chi(x)|^2 = \chi(1)^2/|H : C_H(x)| = \chi(1)^2/h(x).
\]
(We know that $h(x) = |H|$, but we do not need this here.)

Let $p^n = |P|$ and $E = Q(\zeta_{p^n})$ and let $\mathcal{O}$ be the ring of integers of $E$. Let $a = \chi(x)$ and $p^r = \chi(1)^2/h(x)$. Then $a \in \mathcal{O}$ and we have shown so far that $a \tilde{a} = p^r$. In particular, $(a)(\tilde{a}) = (p)^r$ where the brackets denote ideals generated by elements of $\mathcal{O}$. However $(p)$ factorizes as $(p) = \pi^{p-1}(p-1)$, where $\pi = (1 - \zeta_{p^n})$ is the unique ramified prime of $\mathcal{O}$. (See for example page 9 in [9].) Since $(a)$ divides $(p)^r$, $(a)$ is a power of $\pi$ and as $\pi = \bar{\pi}$, also $(a) = (\tilde{a})$. Hence $a = \tilde{a}\omega$ for some $\omega \in \mathcal{O}$. We see that $|\omega| = 1$ for any automorphism $\sigma$ of $E$, so $\omega$ is a root of unity (see Lemma 1.6 in [9]). Hence $a^2 = p^r(a/\tilde{a}) = p^r\omega$ and so $a = p^{r/2}\zeta$ where $\zeta$ is a root of unity having $\zeta^2 = \omega$. This completes the proof of part (1).

For part (3), we assume that $p$ is odd. Let $\varphi$ be the unique irreducible constituent of $\chi_Z$. Then $(P, N, Z, \chi_N, \varphi)$ is a character five and Theorem 3.1 applies. By Theorem 3.1(1) and (3), there exists $U \subseteq P$ with $UN = P$, $U \cap N = Z$ and $\chi_U = \Psi_U\xi$ for some $\xi \in \text{Irr}(U)$. Since $\chi(1)^2 = |N/Z|$, Theorem 3.1(2) shows that $\xi(1) = 1$ and $\chi_U$ has no zeros. Since also $\chi = \Psi G$ by Theorem 3.1(3), we see that $\text{Supp}(\chi)$ is exactly the union of the conjugates of $U$. Part (3) follows, since if $x$ is contained in a conjugate of $U$ then so are its powers. This completes the proof of part (3), and the proof of Theorem B is complete.

4. Groups of small order

In this last section, we prove Theorem C.
Lemma 4.1. Let $N$ be a normal subgroup of the $p$-group $P$. Suppose $\chi \in \text{Irr}(P)$ and let $\varphi$ be an irreducible constituent of $\chi_N$. If $x \in \text{Supp}(\chi)$ then $x^m \in N$, where $m = \frac{|P:N|}{\chi(1)/\varphi(1)}$.

Proof. We use induction on $|P : N|$. The result is clear when $N = P$, so assume $N \neq P$ and let $M$ be a normal subgroup of $P$ with $N \subset M \subseteq P$ and $|M/N| = p$. Let $\theta$ be an irreducible constituent of $\chi_M$ that lies over $\varphi$. By induction, $x^r \in M$, where $r = \frac{|P:M|}{\chi(1)/\theta(1)}$. There are two possibilities; either $\theta_N = \varphi$ and $rp = m$, or $\theta = \varphi^M$ and $r = m$. In the first case, $x^r \in M$ immediately implies $x^m \in N$ as required. In the second case, let $J$ be the stabilizer of $\varphi \in G$. Then $J \cap M \subseteq N$. But $\chi(x) \neq 0$, so $x$ must be contained in some conjugate $J^g$ of $J$, and since also $J^g \cap M \subseteq N$, it follows that $\langle x \rangle \cap M \subseteq N$. Therefore $x^r = x^m \in N$, as required. \qed

Proof of Theorem C. We have a $p$-group $P$ with $\chi \in \text{Irr}(P)$ and $x \in \text{Supp}(\chi)$ such that the order of $x$ does not divide $|P|/\chi(1)^2$, and we have to show that $\chi(1) \geq p^4$ and that $|P| \geq p^{10}$. If $Z$ is the centre of $P$ then $\chi(1)^2$ divides $|P/Z|$, so $x \notin Z$, and in particular $\chi$ is not of central type and so $|P|/\chi(1)^2$ is at least $p^2$. We therefore need only prove that $\chi(1) \geq p^4$, since then $|P| \geq p^{8+2} = p^{10}$.

Let $C$ be the cyclic group generated by $x$ and write $|P| = p^a, \chi(1) = p^a, |C| = p^f$, so our assumption is that $e \geq n - 2a + 1$, yet $x \in \text{Supp}(\chi)$. We may assume that $\chi(1)$ is as small as possible subject to these conditions, and we must show that $a \geq 4$.

If $K$ is the kernel of $\chi$ then the image $xK$ of $x$ in $P/K$ has order at least equal to $p^e/|K|$ and lies in $\text{Supp}(\chi)$ where $\chi$ is regarded as a character of $P/K$, so $P/K, \chi$ and $xK$ constitute another example with the same value $a$. Hence, we can assume $\chi$ is faithful.

Since $\chi$ is not linear, $\chi$ is induced from a character $\varphi$ of some maximal subgroup $M$ of $P$. Since $\chi(x) \neq 0$, $x \in M$ and $x^g \in \text{Supp}(\varphi)$ for some $g \in P$. Since $\varphi(1) < \chi(1)$, it follows that $p^g$ divides $|M|/\varphi(1)^2 = p^{n-2a+1}$. On the other hand $e > n - 2a$, so in fact $e = n - 2a + 1$.

Since $\chi$ is faithful, Lemma 2.7 shows that there exists $h \in P$ such that $C \cap C^h = 1$. Hence $|C|^2 \leq |M|$, or $2e + 1 \leq n$. Also $n - 2a \geq 2$ because $\chi$ cannot be of central type, as remarked at the beginning of the proof. Hence $e = n - 2a + 1 \geq 3$ and $n \geq 7$. If $n = 7$ then $e$ is even, so $e \geq 4$ and $n < 2e + 1$, which is not the case. Hence $n \geq 8$.

To complete the proof, we need normal subgroups $X$ and $Y$ of $P$ with $|X| = p^3$, $|Y| = p^5$ and $X$ contained in the centre of $Y$. By a classical result, if an abelian $p$-group $A$ has order $p^m$ then the $p$-part of $|\text{Aut}(A)|$ divides $p^{m(m-1)/2}$. (This can be found in [1].) Let $A$ be any maximal abelian normal subgroup of $P$. Then $A = Cp(A)$ and so if $|A| = p^m$ then $|P| \leq p^{m(m+1)/2}$. Since $|P| > p^6$, $m \geq 4$. Choose $X$ to be any normal subgroup of $P$ contained
in $A$ and of order $p^3$. Now $|P : C_P(X)| \leq p^3$, so since $|P| \geq p^8$, $|C_P(X)| \geq p^5$.

Choose any $Y < P$ with $X \subseteq Y \subseteq C_P(X)$ and $|Y| = p^5$.

Let $\theta \in \text{Irr}(Y)$ be an irreducible constituent of $\chi_Y$ and let $\lambda$ be the unique irreducible constituent of $\theta_X$. Either $\theta$ is linear and $\theta_X = \lambda$ or $\theta(1) = p$ and $\theta$ and $\lambda$ are fully ramified with respect to $Y/X$.

If the first of these possibilities holds, Lemma 4.1 gives $D \subseteq Y$, where $D = \langle x^{P : Y / x(1)} \rangle$. Since $D \cap D^h = 1$, it follows that $|D|^2 \leq |Y|$ or $2(e - n + 5 + a) \leq 5$. Thus $e + a \leq n - 3$ and since $n - 2a + 1 = e$, we obtain $a \geq 4$, as required.

Assuming now the second possibility holds, let $J$ be the stabilizer of $\theta$. Then $(J, Y, X, \theta, \lambda)$ is a character five. Since $Y/X$ is abelian of odd order, by Theorem 3.1(1) and (3) there exists a subgroup $U$ such that $YU = J$, $Y \cap U = X$ and $\theta$ vanishes on elements of $J$ that are not conjugate to an element of $U$. Some conjugate of $x$ must be contained in $J$ since $\chi$ is induced from a character of $J$. In summary, some conjugate of $x$ lies in $U$. It follows that $C \cap Y \subseteq X$. Lemma 4.1 gives $E \subseteq Y$ where $E = \langle x^{P : Y / x(1)} \rangle$, so $E \subseteq X$. Since $E \cap E^h = 1$, $|E|^2 \leq |X|$ and so $2(e - 1 - n + 5 + a) \leq 3$, or $e + a \leq n - 3$. Again using $n - 2a + 1 = e$, it follows that $a \geq 4$. This completes the proof. 

\[ \square \]

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