Abstract. A criterion for the validity of the Riemann hypothesis reduced the problem to the search for a certain estimate, relative to a hermitian form associated by means of the Weyl symbolic calculus of operators to a distribution in the plane of an arithmetic nature. One can reduce the question further to an algebraic question, which leads to a proof of the conjecture.

1. Reminders

This proof of the Riemann hypothesis is a robust one, in the sense given to the word by computer scientists. To wit, errors (and we certainly made repeated ones in the algebraic part, which resulted in our having submitted a too large number of versions of this arXiv preprint: we apologize to our readers) do not lead to a collapse of the attempt, only to the necessity of making local corrections. We hope, and believe, that the present version is the definitive one. In Remarks 2.1, 3.1 and 3.2, we have pointed towards aspects of this proof which seem to us especially satisfactory. In the first, what substitutes for the current view (not truly founded) that R.H. might depend on a duality between zeros of zeta and primes is clearly established, and used. The second remark explains in which way the proof succeeded in finessing the too hard criterion based on estimates of cumulative sums of Möbius indicators. Finally, in the third remark, we explain the appearance of the critical line $\Re \rho = \frac{1}{2}$ as the one providing the best possible agreement between the demands of the algebraic part of the proof and the analytic properties of zeta at infinity on vertical lines.

The two ingredients of the present proof are the Weyl symbolic calculus of operators [5] and the theory of Eisenstein distributions, a chapter in the theory of modular distributions. The novel developments start with Lemma 2.1: all that precedes consists of reminders of results published at
least 4 years ago, and amply verified on many occasions.

We normalize the Weyl symbolic calculus as follows. Given \( S \in \mathcal{S}'(\mathbb{R}^2) \), the operator with symbol \( S \) is the linear operator \( \Psi(S) \) from \( \mathcal{S}(\mathbb{R}) \) to \( \mathcal{S}'(\mathbb{R}) \) weakly defined by the equation

\[
(\Psi(S)u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} S \left( \frac{x + y}{2}, \xi \right) e^{i\pi(x-y)\xi} u(y) \, dy \, d\xi,
\]

or \( (v \mid \Psi(S)u) = \langle S, W(v, u) \rangle \) with

\[
W(v, u)(x, \xi) = \int_{-\infty}^{\infty} \overline{v}(x+t) u(x-t) e^{2i\pi t \xi} \, dt.
\]

The Weyl calculus has been for more than half a century, under the name of “pseudodifferential analysis”, one of the main tools in the study of partial differential equations: but the methods used in the present context do not intersect the ones experienced there. In [3], the symbolic calculus \( \Psi \) was denoted as \( \text{Op}_2 \) and connected by a pair of rescalings [3, (2.1.10)] to the calculus \( \text{Op} \) used in most of the book. We shall quote results of [3] in their \( \Psi \)-version, rather than \( \text{Op} \)-version: it would be a trivial if lengthy job to make the transformations explicit, or we may rely on the fact that we have remade all calculations in [4].

With \( a(r) = \prod_{p \mid r} (1 - p) \) for \( r = 1, 2, \ldots \), consider the distribution

\[
\Psi_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k),
\]

where \( (j, k, N) = \text{g.c.d.}(j, k, N) \). Setting \( 2i\pi E = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} \), and \( (t^{2i\pi E} S)(x, \xi) = t S(tx, t\xi) \) for \( t > 0 \), one has \([3, (3.1.9)] \) or \([4, (4.11)] \) \( \Psi_N = \prod_{p \mid N} (1 - p^{-2i\pi E}) \text{Dir} \), where \( \text{Dir}(x, \xi) \) is the Dirac comb \( \sum_{j, k \in \mathbb{Z}} \delta(x - j) \delta(\xi - k) \). The distribution \( \Psi_N \) depends only on the “squarefree version” of \( N \).

We consider also the distribution \( \Psi_\infty \) defined as the limit as \( N \to \infty \) along a sequence of integers such that any given squarefree integer eventually divides \( N \), of the distribution \( \Psi_N' \) obtained from \( \Psi_N \) by dropping the term such that \( j = k = 0 \): this amounts to replacing \( a((j, k, N)) \) by \( a((j, k)) \) in the remaining terms. The prime 2 was recognized in [3] as a minor plague. This is the main reason for our having decided to use the version \( \Psi \) of the Weyl calculus: doing so makes it possible to make use of squarefree odd integers only. If taking \( N \) odd, we shall obtain in the limit
a distribution to be denoted as $\mathcal{T}_2$.

The distributions $\mathcal{T}_N$ and $\mathcal{T}_\infty$ are automorphic, i.e., invariant under the linear changes of coordinates in $\mathbb{R}^2$ with matrices in $SL(2, \mathbb{Z})$. Automorphic distributions homogeneous of some degree refine modular forms of the non-holomorphic type, and [2] was devoted to an exposition of the theory obtained. A special case consists of Eisenstein distributions: if $\nu \in \mathbb{C}$, $\text{Re}\, \nu > 1$, the Eisenstein distribution $\mathcal{E}_{-\nu}$ is the modular distribution homogeneous of degree $-1 + \nu$ defined by the equation [2, p.11], valid for every $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathcal{E}_{-\nu}, h \rangle = \sum_{|j| + |k| \neq 0} \int_0^\infty t^{-\nu} h(jt, kt) \, dt \quad (1.4)$$

One has the decomposition [3, p.19], valid in the weak sense in $\mathcal{S}'(\mathbb{R}^2)$,

$$\mathcal{T}_\infty = \frac{1}{2i\pi} \int_{\text{Re}\, \nu = c} \frac{\mathcal{E}_{-\nu}}{\zeta(\nu)} \, d\nu, \quad c > 1 \quad (1.5)$$

and, inserting an extra factor $(1 - 2^{-\nu})^{-1}$, one would obtain a similar decomposition for the distribution $\mathcal{T}_\infty$.

The Eisenstein distribution $\mathcal{E}_{-\nu}$ extends as a meromorphic function of $\nu$ in the complex plane, with simple poles at $\nu = \pm 1$ which will not concern us. Since $(2i\pi \mathcal{E}) \mathcal{E}_{-\nu} = \nu \mathcal{E}_{-\nu}$, one has if R.H. does hold, for every $\varepsilon > 0$, the bound $Q^{2i\varepsilon} \mathcal{T}_\infty = O(Q^{\frac{1}{2} + \varepsilon})$ in the topology of $\mathcal{S}'(\mathbb{R}^2)$. The converse is true, but one can, and one must, do much better than that, combining the question with the Weyl symbolic calculus. A rephrasing of [3, Lemma 2.2.5], reproved as Lemma 3.4 in [4], starts as follows. Let $\nu \in \mathbb{C}$, $\nu \neq \pm 1$, and let $v, u \in \mathcal{C}_\infty(\mathbb{R})$ be two functions, the interiors of the supports of which are disjoint and both contained in $[a, b]$ with $a \geq 0$ and $b^2 - a^2 < 8$. One has

$$\langle v \mid \Psi(\mathcal{E}_{-\nu}) u \rangle = 2 \int_1^\infty t^{-1} \mathcal{E}_{-\nu}(t + t^{-1}) u(t - t^{-1}) \, dt \quad (1.6)$$

It follows that if $\nu \neq \pm 1$ and $\beta > 2$ are given, one can find a pair $v, u$ of $\mathcal{C}_\infty$ functions with disjoint supports contained in $[0, \beta]$ such that $(v \mid \Psi(\mathcal{E}_{-\nu}) u) \neq 0$: assuming $\beta < 2^\frac{3}{2}$ and taking $u$ supported in $[0, \sqrt{\beta^2 - 4}]$ and $v$ in $[2, \beta]$, this is easy to ascertain.

In [3, Prop. 3.4.2], we proved the following (necessary and) sufficient condition for the Riemann hypothesis to hold. That, for some $\beta > 2$ and every function $w \in \mathcal{C}_\infty$ supported in $[0, \beta]$, there should exist for every
ε > 0 a constant C > 0 such that
\[ |(w | Ψ(Q^{2iπE}\mathcal{F}_∞) w)| \leq C Q^{1+ε} \] (1.7)
for every squarefree integer Q. The trick consists in using the function, holomorphic for Re s > \frac{3}{2},
\[ F(s) = \sum_{Q \text{squarefree}} Q^{-s} (w | Ψ(Q^{2iπE}\mathcal{F}_∞) w) = \frac{1}{2iπ} \int_{\text{Re } ν = c} h(ν) f(s - ν) dν, \] (1.8)
where c > 1 and
\[ h(ν) = (ζ(ν))^{-1} (w | Ψ(E_ν) w), \quad f(s - ν) = \sum_{Q \text{sqf}} Q^{-s+ν} = \frac{ζ(s - ν)}{ζ(2(s - ν))}. \] (1.9)
Assuming that ρ₀ is a “bad” zero of zeta, choosing c such that 1 < c < \frac{1}{2} + Re ρ₀ and changing the line Re ν = c to a contour γ enclosing ρ₀ but no other zero, such that c - \frac{1}{2} < Re ν ≤ c for ν ∈ γ, one obtains from the theorem of residues, with the help of the special case of (1.6) for which v = u = w, that F(s) is singular at s = 1 + ρ₀, a contradiction. Taking a full benefit of this identity, one obtains the generalization of the criterion (1.7), reexamined as Theorem 5.2 in [4], in which the pair w, w is replaced by a pair v, u of functions both supported in [0, β] but with disjoint supports.

Then, we observe (this is a consequence of (1.2) together with the fact that W(v, u)(x, ξ) = 0 unless 0 < x < β if v and u are both supported in [0, β]) that one has \( (v | Ψ(Q^{2iπE}\mathcal{F}_∞) u) = (v | Ψ(Q^{2iπE}\mathcal{F}_N) u) \) under this support condition if N is a squarefree multiple of Q divisible by all primes < βQ. Finally, using \( \mathcal{F}_∞ \) to drop the prime 2, one obtains the following criterion for R.H. That, for some β > 2 and every pair v, u of \( C^∞ \) functions supported in [0, β] with disjoint supports, there should exist for every ε > 0 a constant C > 0 with the following property: that, for every squarefree odd integer Q, one should be able to find a squarefree odd integer \( N = RQ \) divisible by all odd primes < βQ such that
\[ |(v | Ψ(Q^{2iπE}\mathcal{F}_N) u)| \leq C Q^{\frac{1}{2}+ε}. \] (1.10)

The benefit is that this expression is amenable to a fully algebraic treatment, as shown in [3, Prop. 4.1.3] (and reconsidered in detail in [4] with the new normalization of the Weyl calculus). Indeed, introducing the
linear space $E[2N^2]$ of complex-valued functions on $\mathbb{Z}/(2N^2)\mathbb{Z}$ and, for every function $u \in \mathcal{S}(\mathbb{R})$, the function $\theta_N u \in E[2N^2]$ defined by the equation

$$(\theta_N u)(n) = \sum_{n_1 \in \mathbb{Z}} u\left(\frac{n_1}{N}\right), \quad n \text{ mod } 2N^2,$$  \hspace{1cm} (1.11) $$

one obtains an identity

$$(v \mid \Psi (Q^{2i\pi E} \mathcal{I}_N) u) = \sum_{m,n \text{ mod } 2N^2} c_{R,Q}(m,n) \overline{\theta_N v(m)} (\theta_N u)(n).$$  \hspace{1cm} (1.12) $$

The coefficients $c_{R,Q}(m,n)$ are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure.

If, under the isomorphism $\mathbb{Z}/(2N^2)\mathbb{Z} \sim \mathbb{Z}/R^2\mathbb{Z} \times \mathbb{Z}/(2Q^2)\mathbb{Z}$, $n$ identifies with a pair $(n', n'')$, let us denote as $\widetilde{n}$ the class that identifies with the pair $(n', -n'')$. Set, if $u \in \mathcal{S}(\mathbb{R})$ and $n \in \mathbb{Z}/(2N^2)\mathbb{Z}$, $(\Lambda_{R,Q} \theta_N u)(n) = (\theta_N u)(\widetilde{n})$. There exists a transformation $\Lambda^\sharp_{R,Q}$ of $\mathcal{S}(\mathbb{R})$ such that, for any $u \in \mathcal{S}(\mathbb{R})$, the transfer formula $\Lambda_{R,Q} \theta_N u = \theta_N \Lambda^\sharp_{R,Q} u$ should hold. One has then the identity [3, Theor. 4.2.2] and [3, Cor. 4.2.7]

$$(v \mid \Psi (Q^{2i\pi E} \mathcal{I}_N) u) = \mu(Q) \left(v \mid \Psi (\mathcal{I}_N) \Lambda^\sharp_{R,Q} u\right)$$  \hspace{1cm} (1.13) $$

involving the Möbius indicator $\mu$. The reflection $\Lambda^\sharp_{R,Q}$ is given explicitly [3, Prop. 4.2.3] by the formula

$$\left(\Lambda^\sharp_{R,Q} w\right)(x) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} w\left(x + \frac{2R\tau}{Q}\right) \exp\left(2i\pi \sigma(Nx + R^2\tau)\right) Q^{-1/2}.$$  \hspace{1cm} (1.14) $$

We have given three different proofs of the identity (1.13): the first has already been cited, and a second proof was given in [3, p.62-64]. A third, shorter, verification was given in [4, Theorem 8.2].

This concludes the list of necessary reminders: the proof of the Riemann hypothesis that follows is self-contained.
Lemma 2.1. Let $N$ be a squarefree odd integer, and let $v, u$ be two functions in $\mathcal{S}(\mathbb{R})$. One has for every squarefree odd integer $N$ the identity
\[
(v \, | \, \Psi(\mathcal{S}_N) \, u) = 2 \sum_{T \mid N} \mu(T) \sum_{j,k \in \mathbb{Z}} \mathbf{1}(Tj + \frac{k}{T}) \, u \left( T(j - \frac{k}{T}) \right) .
\] (2.1)

Proof. Together with the operator $2i\pi \mathcal{E}$, let us introduce the operator $2i\pi \mathcal{E}^\# = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}$ when the coordinates $(r, s)$ are used on $\mathbb{R}^2$. One has if $\mathcal{F}^{-1}_2$ denotes the inverse Fourier transformation with respect to the second variable $\mathcal{F}^{-1}_2[(2i\pi \mathcal{E}) \mathcal{S}] = (2i\pi \mathcal{E}^\#) \mathcal{F}^{-1}_2 \mathcal{S}$ for every tempered distribution $\mathcal{S}$. From the earlier given relation between $\mathcal{S}_N$ and the Dirac comb, and Poisson’s formula, one obtains
\[
\mathcal{F}^{-1}_2 \mathcal{S}_N = \prod_{p \mid N} \left( 1 - p^{-2i\pi \mathcal{E}^\#} \right) \mathcal{F}^{-1}_2 \text{Dir} = \sum_{T \mid N} \mu(T) T^{-2i\pi \mathcal{E}^\#} \text{Dir},
\] (2.2)
explicitly
\[
(\mathcal{F}^{-1}_2 \mathcal{S}_N) (r, s) = \sum_{T \mid N} \mu(T) \sum_{j,k \in \mathbb{Z}} \delta \left( \frac{T}{T} - j \right) \delta (Ts - k).
\] (2.3)

The integral kernel of the operator $\Psi(\mathcal{S}_N)$ is
\[
K(x, y) = \left( \mathcal{F}^{-1}_2 \mathcal{S}_N \right) \left( \frac{x + y}{2}, \frac{x - y}{2} \right)
= 2 \sum_{T \mid N} \mu(T) \sum_{j,k \in \mathbb{Z}} \delta \left( x - Tj - \frac{k}{T} \right) \delta \left( y - T(j + \frac{k}{T}) \right).
\] (2.4)

The equation (2.1) follows.

Remark 2.1. A popular way to conceive of R.H. would lie in a duality between discrete measures on the line, one of which would be carried in some spectral sense by the zeros of zeta and the other by the set of primes. Actually, replacing (2.1) by its limit as $N \not\to \infty$, we have obtained an identity close to it. But it is a two-dimensional version of it: also, primes have to be replaced by squarefree integers and the duality is the one between the symbol and the integral kernel of the same operator. This being said, we are still far, at this point, from having completed the algebraic-arithmetic part.
Lemma 2.2. Let $Q$ be a squarefree odd positive integer and let $\beta > 0$ be given. There exists $R > 0$, with $N = RQ$ squarefree odd divisible by all odd primes $< \beta Q$, such that $R \equiv 1 \mod 2Q^2$.

Proof. Choose $R_1$ positive, odd and squarefree, relatively prime to $Q$, divisible by all odd primes $< \beta Q$ relatively prime to $Q$, and $R_1$ such that $R_1 \equiv 1 \mod 2Q^2$ and $R_1 \equiv 1 \mod R_1$. Choose (Dirichlet’s theorem) a prime $r$ such that $r \equiv 1 \mod R_1$ and $r \equiv R_1 \mod 2Q^2$. The number $R = R_1r$ satisfies the desired condition. \hfill \Box

Proposition 2.3. Let $N = RQ$ be a squarefree odd integer such that $R \equiv 1 \mod 2Q^2$: given $T_1|R$, let $S$ be the integer $S = \frac{R}{T_1}$. Given $v, u \in S(\mathbb{R})$, one has the identity

$$
(v \mid \Psi(Q^{2i\pi\xi}u) = 2 \sum_{Q_1Q_2=Q} \mu(Q_1) \sum_{T_1|R} \mu(T_1) \sum_{j,k \in \mathbb{Z}} v(T_1Q_2 j + \frac{k}{T_1Q_2}) u(T_1Q_2 j - \frac{k}{T_1Q_2} + \frac{2R\omega}{Q_2}),
$$

(2.5)

where $\omega$ is the integer characterized by the conditions $0 \leq \omega < Q_1Q_2$ and $\omega \equiv Sk - T_1Q_2 j \mod Q_1Q_2$.

Proof. Combining Lemma 2.1 and (1.14), one has

$$
(v \mid \Psi(Q^{2i\pi\xi}u) = \frac{2\mu(Q)}{Q^2} \sum_{T|N} \mu(T) \sum_{0 \leq \sigma, \tau < Q^2} \sum_{j,k \in \mathbb{Z}} v(Tj + \frac{k}{T}) u(Tj - \frac{k}{T} + \frac{2R\tau}{Q}) \exp \left( \frac{2i\pi\sigma}{Q^2} \left( N(Tj - \frac{k}{T}) + R^2\tau \right) \right).
$$

(2.6)

With $\lambda = Q^{-2} \left[ N(Tj - \frac{k}{T}) + R^2\tau \right]$, so that $Q^2\lambda \in \mathbb{Z}$ since $T|N$, one has

$$
\frac{1}{Q^2} \sum_{0 \leq \sigma < Q^2} e^{2i\pi\sigma\lambda} = \text{char}(\lambda \in \mathbb{Z}) = \text{char} \left( N(Tj - \frac{k}{T}) + R^2\tau \equiv 0 \mod Q^2 \right)
$$

(2.7)
Let $(T, Q) = Q_2$, and $T = T_1Q_2$, $Q = Q_1Q_2$. One has $(T_1, Q) = 1$, $T_1|R$ and $\frac{N}{T} = \frac{QR}{T_1Q_2} = \frac{Q_1R}{T_1}$. The condition $\lambda \in \mathbb{Z}$, or $NTj - \frac{Q_1Rk}{T_1} + R^2\tau \equiv 0 \mod Q^2$, implies $\tau \equiv 0 \mod Q_1$. Since $R \equiv 1 \mod 2Q^2$, the quotient $S = \frac{R}{T_1}$ is a representative of the inverse of the class of $T_1$ in $(\mathbb{Z}/(2Q^2\mathbb{Z})^\times$. Using repeatedly the fact that $R \equiv 1 \mod 2Q^2$, the condition $NTj - \frac{Q_1Rk}{T_1} + R^2\tau \equiv 0 \mod Q^2$, which expresses that $\lambda \in \mathbb{Z}$, can be rewritten as $\omega \equiv Sk - T_1Q_2^2j \mod Q_1Q_2^2$.

Regarding $Q_1, Q_2$ as fixed, we now rewrite the corresponding part of (2.6), taking as summation variables $T_1, j, k$. One has $T_1|\mu$ and $N\mu = QR T_1Q_2 = Q_1R T_1$, and $\nu_1 \equiv 0 \mod Q_2$, the quotient $S = \frac{R}{T_1}$ is a representative of the inverse of the class of $T_1$ in $(\mathbb{Z}/(2Q^2\mathbb{Z})^\times$. Using repeatedly the fact that $R \equiv 1 \mod 2Q^2$, the condition $NTj - \frac{Q_1Rk}{T_1} + R^2\tau \equiv 0 \mod Q^2$, which expresses that $\lambda \in \mathbb{Z}$, can be rewritten as $\omega \equiv Sk - T_1Q_2^2j \mod Q_1Q_2^2$.

Lemma 2.4. Assume that $Q \geq 3$, that $u$ is supported in $[-2, 2]$ and $v$ in $[2, 2\frac{1}{2}]$. The expression (2.5) can be simplified to

$$(v \mid \Psi \left( Q^{2\pi \xi} \Xi_N \right) u) = 2 \sum_{a=1,2,3} \sum_{T_1|R} u(T_1) \sum_{j,k \in \mathbb{Z}} \frac{1}{T_1Q} \left( T_1Qj + \frac{k}{T_1Q} \right) u \left( T_1Qj + \frac{k}{T_1Q} - \frac{2aQ}{T_1} \right).$$

(2.8)

One has $2^{-\frac{1}{2}}Q < T_1 < \left( 1 + 2^{\frac{1}{2}} \right) Q$.

Proof. Fixing $Q_1, Q_2$, one has for every nonzero term of the sum (2.5) the inequalities

$$(2 < x) = T_1Q_2j + \frac{k}{T_1Q_2} < 2\frac{1}{2}, \quad |y| = \left| T_1Q_2j - \frac{k}{T_1Q_2} + \frac{2R\omega}{Q_2} \right| < 2.$$ \hspace{1cm} (2.9)

If $\omega = 0$, the factor $Q_1$ of $Q$ does not appear in the general term, not even as an arithmetic constraint on $\omega$. Since, given a divisor $Q_2$ of $Q$ distinct from $Q$, one has $\sum \{ \mu(Q_1) : Q_1 \mid \frac{Q}{Q_2} \} = 0$, only the terms such that $Q_2 = Q$ remain in this case. Then, the inequalities (2.9) yield $0 < T_1Qj < 1 + 2^{\frac{1}{2}}$, which is impossible since $Q \geq 3$. Hence, $\omega \geq 1$.

Multiplying by $\frac{Q_2}{Q_1}$ the inequality obtained by taking the half-sum of the two equations (2.9) and using $R = ST_1$, one obtains $0 < Q_2^2j +
$S \omega < \frac{(1 + 2^{1/2}) Q_2}{T_1}$, from which it follows that $T_1 < (1 + 2^{1/2}) Q_2$. One has $\omega \equiv Sk \mod Q_2^2$ and

$$S(k - RT_1 \omega) = Sk - R^2 \omega \equiv Sk - \omega \equiv 0 \mod Q_2^2,$$  \quad (2.10)

so that $k - RT_1 \omega \equiv 0 \mod Q_2^2$. Noting that $\frac{k - RT_1 \omega}{T_1 Q_2} = \frac{x - y}{T_1 Q_2}$, one obtains that $0 < k - RT_1 \omega < 2^{1/2} T_1 Q_2 < (2 + 2^{1/2}) Q_2^2$. It follows on one hand that $k - RT_1 \omega = a Q_2^2$ with $a = 1, 2$ or $3$, on the other hand that

$$\frac{Q_2}{T_1} \leq \frac{a Q_2}{T_1 Q_2} = \frac{k - RT_1 \omega}{T_1 Q_2} < 2^{1/2},$$  \quad (2.11)

from which $T_1 > 2^{-1/2} Q_2$.

Using the equation $\frac{R \omega}{Q_2} = \frac{k}{T_1 Q_2} - \frac{a Q_2}{T_1}$, one can write the argument of $u$ as

$$y = T_1 Q_2 j + k \frac{T_1}{T_1 Q_2} - 2 a Q_2 \frac{T_1}{T_1}.$$  \quad (2.12)

The number $\omega$ has disappeared from the general term, and so has $Q_1$, which only occurred before in the constraint on $\omega$. With the same argument as the one used in the case when $\omega = 0$, one sees that only the terms such that $Q_2 = Q$ remain.

\[\square\]

The sum (2.8) is finite, which means that the hermitian form $(v \mid \Psi (Q^{2 \pi i \xi} T_\infty) u)$ coincides, for $v$ and $u$ in $C^\infty(\mathbb{R})$ satisfying the support conditions in Lemma 2.4, with $\langle K, \Psi \otimes u \rangle$, where the integral kernel $K(x, y)$ is a certain linear combination of point masses $\delta(x - x_m) \delta(y - y_n)$ with $x_m, y_n \in \mathbb{Q}$. On the other hand, from the decomposition (1.5) and from (1.6), the hermitian form under consideration admits an integral kernel $K_1 \in \mathcal{S}'(\mathbb{R}^2)$ (cf. the first equation (2.4)), which must coincide with $K$ when tested on functions $\Psi \otimes u$ of the species indicated, hence also when tested on functions $w \in C^\infty(\mathbb{R}^2)$ with the associated support property. But, from (1.5) and (1.6), the hermitian form under study does not change if one multiplies the integral kernel $K_1(x, y)$ by any function $\gamma(x^2 - y^2 - 4)$ with $\gamma \in C^\infty(\mathbb{R})$ such that $\gamma(0) = 1$. This implies that, in the sum (2.8), we may keep only the terms such that, $x$ and $y$ denoting the arguments of $\Psi$ and $u$ there, one has $x = \sqrt{y^2 + 4}$. 
Lemma 2.5. The number $a$, as it occurs in (2.8), can always be taken to the value 1. Given $R, Q$ and $T_1$, there are unique values of $j, k$ corresponding to nonzero terms of (2.5).

Proof. We have already used, in the proof of Lemma 2.4, the equation

\[ \frac{x-y}{2} = \frac{k-RT_1 \omega}{T_1 Q} = \frac{aQ}{T_1}. \]

Since $x^2 - y^2 = 4$, one has $x + y = \frac{2T_1}{aQ}$, and

\[ x = \frac{T_1}{aQ} + \frac{aQ}{T_1}, \quad y = \frac{T_1}{aQ} - \frac{aQ}{T_1}. \]  \hfill (2.13)

Then,

\[ (T_1 Q)x = (T_1 Q)^2 j + k = (T_1 Q)^2 j + aQ^2 + RT_1 \omega \quad \text{and} \quad (T_1 Q)x = a^{-1} T_1^2 + aQ^2, \]

from which

\[ (T_1 Q)^2 j + RT_1 \omega = a^{-1} T_1^2 \quad \text{and} \quad Q^2 j + S \omega = a^{-1}. \]  \hfill (2.15)

That $a = 1$ follows.

Then, $S \omega \equiv 1 \mod Q^2$: since, as seen in the proof of Proposition 2.3, $\omega \equiv Sk \mod Q^2$, one has $S^2 k \equiv 1 \mod Q^2$ so that, given $T_1$, $S$ is known, $k$ is known mod $Q^2$ and so is $RT_1 \omega = k - Q^2$, finally so is $\omega$. Since $0 < \omega < Q^2$, there is when $T_1$ is known a unique possible value for $\omega$, next for $k$, ultimately for $j$.

To prove R.H., we must manage so that the number of possible values of $T_1$ will be a $O \left( Q^{\frac{1}{2} + \epsilon} \right)$.

\[ \Box \]

3. The Riemann hypothesis

With $y_0$ chosen so that $0 < y_0 < 2$ and $x_0 = \sqrt{4 + y_0^2}$, we shall use in place of a fixed pair of functions a pair $v_Q, u_Q$, where the support of $v_Q$ (resp. $u_Q$) concentrates towards $\{x_0\}$ (resp. $\{y_0\}$) as $Q \to \infty$. To do so, fixing a pair $v, u$ of $C^\infty$ even functions the supports of which contain 0, we define

\[ u_Q(y) = Q^{\frac{1}{4}} u \left( Q^{\frac{1}{4}} (y - y_0) \right), \]

\[ v_Q(x) = \text{char}(x > 2) \times Q^{\frac{1}{4}} v \left( Q^{\frac{1}{4}} \left( \sqrt{x^2 - 4} - y_0 \right) \right). \]  \hfill (3.1)

Note that the condition $x = \sqrt{4 + y^2}$ is equivalent to the fact that the arguments of $v$ and $u$ in this pair of equations coincide. If choosing for $v$ and
u functions supported in \([-2 - y_0, 2 - y_0]\), the support of \(u_Q\) is characterized by the condition \(\frac{-2 - y_0}{Q^{\frac{1}{2}}} \leq y - y_0 \leq \frac{2 - y_0}{Q^{\frac{1}{2}}}\) and the support of \(v_Q\) by the same condition, after we have substituted \(\sqrt{x^2 - 4}\) for \(y\).

In particular, when \(u_Q(y) \neq 0\), one has \(|y - y_0| < 4Q^{-\frac{1}{2}}\). Taking \(y = \sqrt{x^2 - 4}\), one has \(x - x_0 = (y - y_0) \frac{y_1}{\sqrt{4 + y_1^2}}\) for some \(y_1\), so that \(|x - x_0| < 4Q^{-\frac{1}{2}}\) as well.

**Lemma 3.1.** With \(v_Q\) and \(u_Q\) as defined in the beginning of this section, one has for some \(C > 0\) independent of \(Q\) the estimate

\[
\left| \left( v_Q \mid \Psi \left( Q^{2i\pi\varepsilon} \frac{x}{Q} \right) \right) u_Q \right| < CQ^{\frac{1}{2}}. \tag{3.2}
\]

**Proof.** We use Lemma 2.4 and Lemma 2.5, after \(v\) and \(u\) have been replaced by \(v_Q\) and \(u_Q\). We have seen in the beginning of the proof of Lemma 2.5 that \(\frac{x - y}{2} = \frac{Q}{T_1}\) for all nonzero terms of (2.5). Since \(x - y = x_0 - y_0 + O\left(Q^{-\frac{1}{2}}\right)\), one obtains \(\frac{Q}{T_1} = \frac{x_0 - y_0}{2} \left[ 1 + O\left(Q^{-\frac{1}{2}}\right) \right]\) and

\[
T_1 = \frac{2Q}{x_0 - y_0} + O\left(Q^\frac{1}{2}\right). \tag{3.3}
\]

The number of available \(T_1\)’s is thus \(O\left(Q^{\frac{1}{2}}\right)\), which implies Lemma 3.1 in view of Lemma 2.5.

\[\square\]

**Remark 3.1.** One of the most well-know criteria for R.H. is given as the validity of the estimate \(\sum_{T \leq Q} \mu(T) = O\left(Q^{1/2 + \varepsilon}\right)\). Instead of trying this approach, a notoriously too hard one, we have managed to create a situation in which only squarefree integers \(T\) differing from some multiple of \(Q\) by a \(O\left(Q^{\frac{1}{2}}\right)\) will occur.

The Riemann hypothesis would thus be proven if we could apply the criterion (1.7) (or, rather, its modification involving a pair \(v, u\)). But there is of course a price to pay for our having changed \(v\) and \(u\) to \(v_Q\) and \(u_Q\). The first lemma to follow prepares for the \(d\nu\)-integrability in integrals generalizing (1.8).
Lemma 3.2. One has for \( \nu \) on a line \( \text{Re} \nu = c > 0 \) and every \( \alpha \in ]1, 2[ \), for some \( C > 0 \), the estimate

\[
|\nu|^{\alpha} \left| \left( v_Q \left( \Psi \left( \mathcal{E}_{-\nu} \right) u_Q \right) \right) \right| \leq C Q^{\frac{\alpha}{2}}. \tag{3.4}
\]

Proof. One has for \( t > 1 \)

\[
u(t-t^{-1}) = Q^{\frac{\nu}{2}} u \left( Q^{\frac{\nu}{2}} \left( t-t^{-1} - y_0 \right) \right),
\]

\[
u(t+t^{-1}) = v_Q \left( \sqrt{4 + (t-t^{-1})^2} \right) = Q^{\frac{\nu}{2}} v \left( Q^{\frac{\nu}{2}} \left( t-t^{-1} - y_0 \right) \right). \tag{3.5}
\]

Then,

\[
\left( v_Q \left| \Psi \left( \mathcal{E}_{-\nu} \right) u_Q \right) \right) = 2 \int_1^{\infty} t^\nu v_Q \left( t+t^{-1} \right) u_Q \left( t-t^{-1} \right) \frac{dt}{t} \]

\[
= 2Q^{\frac{\nu}{2}} \int_1^{\infty} t^\nu t_Q \left( Q^{\frac{\nu}{2}} \left( t-t^{-1} - y_0 \right) \right) u \left( Q^{\frac{\nu}{2}} \left( t-t^{-1} - y_0 \right) \right) \frac{dt}{t} \]

\[
= 2Q^{\frac{\nu}{2}} \int_0^{\infty} \left( \frac{y + \sqrt{4 + y^2}}{2} \right)^\nu \left( Q^{\frac{\nu}{2}} \left( y - y_0 \right) \right) \frac{dy}{\sqrt{4 + y^2}}. \tag{3.6}
\]

This expression is bounded independently of \( Q \).

In the domain where the integrand is nonzero, \( t \) cannot approach 1 and, with \( y = t - t^{-1} \), \( \left| \frac{dy}{dt} \right|^{-1} \) is bounded. Using then the integrations by parts associated to the identities \( \nu t^\nu = t \frac{d}{dt} t^\nu \) and \( \nu^2 t^\nu = \left( t \frac{d}{dt} \right)^2 t^\nu \), we obtain

\[
|\nu| \left| \left( v_Q \left( \Psi \left( \mathcal{E}_{-\nu} \right) u_Q \right) \right) \right| \leq C Q^{\frac{\nu}{2}}, \quad |\nu|^2 \left| \left( v_Q \left( \Psi \left( \mathcal{E}_{-\nu} \right) u_Q \right) \right) \right| \leq C Q. \tag{3.7}
\]

To save \( |\nu|^\alpha \) for some \( \alpha > 1 \), we make use, with \( \frac{1}{p} = 2 - \alpha, \frac{1}{q} = \alpha - 1 \) and \( \xi = s^\frac{1}{p}, \eta = \left( Q^{-\frac{\nu}{2}} s^2 \right)^{\frac{1}{q}} \), of Young’s inequality

\[
Q^{-\frac{\nu}{2}} s^\alpha = \xi \eta \leq \frac{\xi^p}{p} + \frac{\eta^q}{q} = (2 - \alpha) s + (\alpha - 1) Q^{-\frac{\nu}{2}} s^2, \tag{3.8}
\]

from which

\[
|\nu|^\alpha \leq (2 - \alpha) Q^{\frac{\nu}{2}} |\nu| + (\alpha - 1) Q^{\frac{\nu}{2}} |\nu|^2. \tag{3.9}
\]

This gives (3.4). \( \square \)
Remark 3.2. Having rescaled \( v \) and \( u \) by the factor \( Q^{\frac{1}{2}} \) led to the optimal result. If taking a higher power of \( Q \), we would have obtained a better estimate of the number of possible choices for \( T_1 \), hence a better estimate of the hermitian form \( \langle v_Q \mid \Psi \left( Q^{2i\pi \xi \frac{\xi}{2}} \right) u_Q \rangle \); but we would have deteriorated the estimate (3.4).

Theorem 3.3. The Riemann zeta function has no zero \( \rho_0 \) such that \( \text{Re} \rho_0 > \frac{1}{2} \).

Proof. We start from the identities, in which \( c > 1 \),

\[
\left( v_Q \mid \Psi \left( \frac{\xi}{2} \right) \right) u_Q = \frac{1}{2i\pi} \int_{\text{Re} \nu=c} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} \left( v_Q \mid \Psi \left( \xi_{-\nu} \right) u_Q \right) d\nu,
\]

\[
\left( v_Q \mid \Psi \left( Q^{2i\pi \xi \frac{\xi}{2}} \right) u_Q \right) = \frac{1}{2i\pi} \int_{\text{Re} \nu=c} \frac{(1 - 2^{-\nu})^{-1} Q^{\nu}}{\zeta(\nu)} \left( v_Q \mid \Psi \left( \xi_{-\nu} \right) u_Q \right) d\nu,
\]

(3.10)
a consequence of (1.5).

Denote as \( S_{q_{\text{odd}}} \) the set of squarefree odd integers (taking only odd integers was necessitated by the algebraic part). Following the line of proof recalled in (1.8), (1.9), one must replace the function \( F(s) \) in (1.8) by the function, still denoted as \( F(s) \),

\[
F(s) = \sum_{Q \in S_{q_{\text{odd}}}} Q^{-s} \left( v_Q \mid \Psi \left( Q^{2i\pi \xi \frac{\xi}{2}} \right) u_Q \right),
\]

(3.11)
a function holomorphic for \( \text{Re} s > \frac{3}{2} \) in view of (3.2). One can piece this decomposition with (3.10) and with (3.6), rewritten as

\[
\left( v_Q \mid \Psi \left( \xi_{-\nu} \right) u_Q \right) = 2 \int_0^{\infty} \left( \frac{y + \sqrt{4 + y^2}}{2} \right)^\nu \left[ Q^{\frac{1}{2} + \frac{1}{2}(y-y_0)} \left( \frac{y}{y_0} \right) \frac{dy}{\sqrt{4 + y^2}} \right] (y - y_0).
\]

(3.12)
Depending on Lemma 3.2 to ensure the $d\nu$ summability and commute the $d\nu$-integration with the $Q$-summation, one obtains for Re $s$ large

$$F(s) = \frac{1}{i\pi} \int_{\text{Re } \nu = c} \frac{(1 - 2 - \nu)^{-1}}{\zeta(\nu)} \, d\nu \int_0^\infty \left( \frac{y + \sqrt{4 + y^2}}{2} \right)^\nu \sum_{Q \in Sq_{\text{odd}}} \left[ Q^{-s+\nu+\frac{1}{2} + \frac{1}{2}(y-y_0)\frac{dy}{y}} (\pi u) \right] (y - y_0) \frac{dy}{\sqrt{4 + y^2}}.$$  \hspace{1cm} (3.13)

Now, one has for Re $\theta > 1$

$$\sum_{Q \in Sq_{\text{odd}}} Q^{-\theta} = \prod_{p \neq 2} \left( 1 + p^{-\theta} \right) = \left( 1 + 2^{-\theta} \right)^{-1} \prod_{p} \frac{1 - p^{-2\theta}}{1 - p^{-\theta}} = \left( 1 + 2^{-\theta} \right)^{-1} \frac{\zeta(\theta)}{\zeta(2\theta)},$$  \hspace{1cm} (3.14)

so that

$$A = \sum_{Q \in Sq_{\text{odd}}} Q^{-s+\nu+\frac{1}{2} + \frac{1}{2}\xi \frac{d}{dx}}$$

$$= \left[ 1 + 2^{-s+\nu+\frac{1}{2} + \frac{1}{2}\xi \frac{d}{dx}} \right]^{-1} \frac{\zeta \left( s - \nu - \frac{1}{2} - \frac{1}{2}\xi \frac{d}{dx} \right)}{\zeta \left( 2(s - \nu) - 1 - \xi \frac{d}{dx} \right)}.$$  \hspace{1cm} (3.15)

The estimate (3.4) ensures the $d\nu$-summability and the identity (3.15) is obtained for Re $s > 1 + \frac{\alpha}{2}$; since $\alpha$ can be taken arbitrarily close to 1, this identity is valid for Re $s > \frac{3}{2}$.

The operator $A$ can be made explicit with the help of a decomposition of functions into generalized eigenfunctions of the self-adjoint operator $i \left( \frac{1}{2} + \xi \frac{d}{dx} \right)$ (a Mellin transformation), to wit

$$w = \frac{1}{i} \int_{\text{Re } \mu = -\frac{1}{2}} w_\mu \, d\mu, \quad w_\mu(\xi) = \frac{1}{2\pi} \int_0^\infty r^\mu w(r\xi) \, dr.$$  \hspace{1cm} (3.16)

One has $\left( \frac{1}{2} + \xi \frac{d}{dx} \right) w_\mu = \left( -\frac{1}{2} - \mu \right) w_\mu$ and, for $w \in \mathcal{S}(\mathbb{R})$,

$$(Aw)(\xi) = \frac{1}{i} \int_{\text{Re } \mu = -\frac{1}{2}} \left[ 1 + 2^{-s+\nu+\frac{1}{2}} \right]^{-1} \frac{\zeta \left( s - \nu + \frac{\xi}{2} \right)}{\zeta \left( 2(s - \nu) + \mu \right)} w_\mu(\xi) \, d\mu.$$  \hspace{1cm} (3.17)

We trade the variable $\nu$, on the line Re $\nu = c$, for the variable $\tau = \nu - \frac{\xi}{2}$ on the line Re $\tau = c + \frac{1}{4}$. Using (3.13), (3.15) and (3.17), one obtains with
\[ w = \pi u \]

\[ F(s) = \frac{1}{2i\pi} \int_{\text{Re}\tau = c + \frac{i}{4}} f(s - \tau) h(\tau) d\tau \quad (3.18) \]

with

\[ h(\tau) = \frac{1}{i} \int_{\text{Re}\mu = -\frac{1}{2}} \left( \frac{1 - 2^{-\tau - \frac{\mu}{2}}}{\zeta(\tau + \frac{\mu}{2})} \right)^{-1} \mu d\mu \]

\[ \int_{0}^{\infty} \left( y + \sqrt{4 + y^2} \right)^{\tau + \frac{\mu}{2}} w_\mu(y - y_0) \frac{dy}{\sqrt{4 + y^2}} \quad (3.19) \]

and

\[ f(s - \tau) = (1 + 2^{-s+\tau})^{-1} \frac{\zeta(s - \tau)}{\zeta(2(s - \tau))} \quad (3.20) \]

**Lemma 3.4.** One has

\[ h(\tau) = \sum_{j \geq 0} \sum_{k \geq 1} \text{Möb}(k) \int_{1}^{\infty} \left( \frac{2j}{t} \right)^{-\tau + \frac{\mu}{2}} w \left[ \left( \frac{2j}{t} \right)^{\frac{\mu}{2}} (t - t^{-1} - y_0) \right] dt, \quad (3.21) \]

where the Möbius indicator is here denoted as Möb to avoid confusion.

**Proof.** We use (3.16) to compute \( w_\mu(y - y_0) = w_\mu(t - t^{-1} - y_0) \). Starting from (3.19) and using (3.16), one has

\[ h(\tau) = \frac{1}{2i\pi} \int_{0}^{\infty} dr \int_{\text{Re}\mu = -\frac{1}{2}} \left( \frac{1 - 2^{-\tau - \frac{\mu}{2}}}{\zeta(\tau + \frac{\mu}{2})} \right)^{-1} \mu \frac{r^\mu d\mu}{t^{\tau + \frac{\mu}{2} - 1}} \int_{1}^{\infty} w \left( r(t - t^{-1} - y_0) \right) dt. \quad (3.22) \]

We perform first the \( d\mu \)-integration. Moving back here to the variable \( \nu \), one has

\[ \frac{1}{2i\pi} \int_{\text{Re}\mu = -\frac{1}{2}} \left( \frac{1 - 2^{-\tau - \frac{\mu}{2}}}{\zeta(\tau + \frac{\mu}{2})} \right)^{-1} \mu \frac{r^\mu (r^2 t)^{\nu - 1} d\mu}{\zeta(\nu)} \int_{\text{Re}\nu = c} (\zeta(\nu))^{-1} (r^2 t)^{\nu - 1} d\nu. \quad (3.23) \]
Now, if \( x > 0 \) and \( c > 1 \), one has [3, (3.3.11)]

\[
\frac{1}{2i\pi} \int_{\Re \nu = c} \frac{x^{\nu-1}}{\zeta(\nu)} \, d\nu = \sum_{k \geq 1} \text{Möb}(k) \delta(x - k) \tag{3.24}
\]

and, expanding \((1 - 2^{-\nu})^{-1}\) as a series of powers of \(2^{-\nu}\),

\[
\frac{1}{2i\pi} \int_{\Re \nu = \epsilon} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} \, x^{\nu-1} d\nu = \sum_{j \geq 0} \sum_{k \geq 1} \text{Möb}(k) \delta\left( x - 2^j k \right). \tag{3.25}
\]

One obtains for the integral (3.23) the value

\[
2 \, r^{2-2\tau} \sum_{j \geq 0} \sum_{k \geq 1} \text{Möb}(k) \delta\left( r^2 t - 2^j k \right). \tag{3.26}
\]

Given any function \( f \in C^\infty([0, \infty[) \), one has for \( r > 0 \)

\[
\int_0^\infty f(r) \delta\left( r^2 t - 2^j k \right) \, dr = \frac{1}{2} \left( \frac{2^j k}{t} \right)^{-\frac{1}{2}} f \left( \frac{2^j k}{t} \right)^{\frac{1}{2}}. \tag{3.27}
\]

Using (3.22) and taking \( f(r) = r^{2-2\tau} w\left( r \left( t^{-1} - y_0 \right) \right) \), one obtains the lemma.

**Lemma 3.5.** Let \( \phi \) and \( \psi \) be two \( C^\infty \) functions on the line, with \( \phi \) compactly supported. The function defined for \( \Re \tau > 1 \) as the difference

\[
\sum_{j \geq 0} \sum_{k \geq 1} \text{Möb}(k) \left( 2^j k \right)^{-\tau + \frac{1}{2}} \int_{-\infty}^\infty \phi\left( \left( 2^j k \right)^{\frac{1}{2}} \theta \right) \psi(\theta) \, d\theta - \hat{\phi}(0) \psi(0) \frac{1 - 2^{-\tau}}{\zeta(\tau)} \tag{3.28}
\]

extends as an analytic function for \( \Re \tau > \frac{1}{2} \), bounded for \( \Re \tau \geq \frac{1}{2} + \epsilon \).

The function \( h(\tau) \) defined in (3.16) is the sum of \( c_1 \frac{1 - 2^{-\tau}}{\zeta(\tau)} \), where \( c_1 \) is some constant, and of a function which extends as an analytic function for \( \Re \tau > \frac{1}{2} \), bounded for \( \Re \tau \geq \frac{1}{2} + \epsilon \).
Proof. Writing $\psi(\theta) = \psi(0) + \theta \psi_1(\theta)$, one has
\[
\int_{-\infty}^{\infty} \phi \left( (2^j k) \frac{1}{2} \theta \right) \psi(\theta) \, d\theta
= \psi(0) \left( 2^j k \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(\theta) \, d\theta + \int_{-\infty}^{\infty} \phi \left( (2^j k) \frac{1}{2} \theta \right) \theta \psi_1(\theta) \, d\theta
= \psi(0) \hat{\phi}(0) \left( 2^j k \right)^{-\frac{1}{2}} + \left( 2^j k \right)^{-1} C(2^j k), \quad (3.29)
\]
where the collection of constants $C(2^j k)$ is bounded. The first part follows in view of the validity, for $\text{Re} \, \tau > 1$, of the equation
\[
\sum_{j \geq 0} \sum_{k \geq 1} \text{M"{o}b}(k) (2^j k)^{-\tau} = \frac{(1 - 2^{-\tau})^{-1}}{\zeta(\tau)}. \quad (3.30)
\]
In view of (3.21), the special case in which $\phi(\theta) = w(\theta)$ and $\psi(\theta) = t^{\tau - \frac{1}{2}} \theta^t (t - t^{-1} - y_0)$ would yield for the series of integrals under examination the value $f(\tau)$. It is not truly a special case of what precedes since $\psi$ depends also on $\tau$; however, nothing is changed in the proof, apart from replacing the constants $C(2^j k)$ by functions $C(2^j k, \tau)$ entire with respect to $\tau$ and bounded when $|\text{Re} \, \tau|$ is. Note that, when $\theta = 0$, one has $t = \frac{1}{2} \left( y_0 + \sqrt{4 + y_0^2} \right)$, which provides the value $\psi(0)$.

\square

**End of the proof of Theorem 3.3**

The end of the proof reproduces that of [3, Prop. 3.4.2] (revisited as Theorem 5.2 in [4]). Assume that $\rho_0$ is a zero of zeta such that $\text{Re} \, \rho_0 > \frac{1}{2}$, and perform the same change of contour as the one defined after (1.8), (1.9). Using (3.18), we must show that, for some choice of the pair $v, u$, the function $F(s)$ has a pole at $s = 1 + \rho_0$. Since the function $f = f(z)$ has a pole at $z = 1$, this would follow, as a consequence of the easy lemma [3, Lemma 3.4.1] (or [4, Lemma 5.1]) from the fact that the function $h(\nu)$ has a pole (of any order) at $\tau = \rho_0$. Paying attention to the main term on the right-hand side of (3.28), it suffices to choose for $v, u$ real non negative functions, not vanishing at 0.

\square
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