Is scalar-tensor gravity consistent with polytropic stellar models?

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Abstract. We study the scalar field potential $V(\phi)$ in the scalar-tensor gravity with self-consistent polytropic stellar configurations. Without choosing a particular potential, we numerically derive the potential inside various stellar objects. We restrict the potential to conform to general relativity or to $f(R)$ gravity inside and require the solution to arrive at SdS vacuum at the surface. The studied objects are required to obtain observationally valid masses and radii corresponding to solar type stars, white dwarfs and neutron stars. We find that the resulting scalar-tensor potential $V(\phi)$ for the numerically derived polytrope that conforms to general relativity, in each object class, is highly dependent on the matter configuration as well as on the vacuum requirement at the boundary. As a result, every stellar configuration arrives at a potential $V(\phi)$ that is not consistent with the other stellar class potentials. Therefore, a general potential that conforms to all these polytropic stellar classes could not be found.

Keywords: modified gravity, stars

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1 Introduction

General relativity (GR), describes the local gravitational phenomena very well [1]. At large scales and in the early universe this description does not seem to be adequate anymore. From the observations of distant supernovae and cosmic microwave background [2–6] the expansion of the universe is interpreted to be accelerated at late times. To encompass the late time accelerated phase GR needs to be modified. The simplest modification allowed by the Einstein-Hilbert action is the cosmological constant model with cold dark matter, called the ΛCDM model [7, 8]. In this work the ΛCDM model is denoted as GR+Λ. While ΛCDM is very successful, this model has its shortcomings as well [9, 10] and many ways to explain the current accelerated phase have been suggested. Models generally either modify the content of the Einstein’s equations, by including a dark energy component [9, 10], or by modifying the gravity sector itself [11–14]. Because of the great success of GR in predicting the local observations with high accuracy, viable modifications must allow only configurations with small deviations from the general relativistic solutions.

A class of GR modifications that does not require any extra assumptions in addition to standard physics is the $f(R)$ theories of gravity. In $f(R)$ theories of gravity, an algebraic function of the Ricci scalar extends the gravitational Lagrangian density from GR [15–17]. Another way of obtaining adequate modifications to GR are so called scalar-tensor theories [18, 19]. These scalar-tensor theories arise naturally from e.g. higher dimensions, string theory or from non-commutative geometry [14]. In the Jordan frame formulation of the theory the scalar field is non-minimally coupled to the metric tensor but does not couple to the matter sector. The scalar field, with an adequate potential, can give rise to the observed exponential acceleration today with a slow-roll behavior [20, 21] and still conform to the local gravitational experiments via a chameleon mechanism [22, 23]. There are viable alternatives among the gravity modifications that do not possess instabilities [24–27], can account for the correct expansion history of the universe [28–30] and even produce inflation in some cases [31]. The local tests that GR passes must also be accounted for [27, 32]. Some $f(R)$ models can resemble GR so much that even the spacetime outside the $f(R)$-sun conforms to

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the observations [33]. This is, however, not a general feature of \( f(R) \) theories of which most are excluded by the light deflection experiments [34, 35]. It is also known that the Birkhoff’s theorem [36, 37] is broken for the general relativity modifications [38]. According to the Birkhoff’s theorem, the vacuum field equations of general relativity obtain the Schwarzschild solution around a spherically symmetric object.

In this work, we are looking for stellar solutions in the Jordan frame formulation of a scalar-tensor theory with GR+Λ behavior inside the star. We also demand the configurations to arrive at the Schwarzschild-de Sitter (SdS) vacuum at the boundary. The polytropic stellar object classes: solar type main sequence stars (SUN), white dwarfs (WD) and neutron stars (NS); are the matter configurations for which the scalar field potential \( V(\phi) \) was solved numerically. We find that a potential \( V(\phi) \) that will describe all the studied object classes cannot be found. The solution for the potential is found to be specific to each stellar class and highly dependent on the matter configuration inside the star and also on the boundary vacuum conditions.

The paper is organized as follows: in section 2 we formulate the theoretical framework for the gravitation. In section 3 we discuss the spherically symmetric spacetime and the objects classes. In section 4 we describe how to derive the potential \( V(\phi) \) from the field equations derived in 2 with the adequate boundary conditions and stellar configurations. Finally, we draw our conclusions in section 5.

2 The gravity formalism

We consider a scalar-tensor action that is of Brans-Dicke type [39] with a potential and no kinetic term in the Jordan frame. This theory could also be equivalently described in the Einstein frame if also the units of length, mass and time are scaled accordingly. For this numerical work, however, the Jordan frame is preferred [40, 41]. The considered scalar-tensor action [15] is

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \phi R - V(\phi) \right] + S_m, \tag{2.1}
\]

where \( \kappa \equiv 8\pi G \), the \( S_m \) term gives the matter contribution and the scalar field \( \phi \) interacts non-minimally with the gravitational field. We use \( c = \hbar = 1 \) in this work. This type of action is also equivalent to metric \( f(R) \) theories of gravity [15]. The configuration is required to be GR+Λ inside the matter configuration, to pass the local gravity tests [32, 34] for the solar model. The interaction with the matter is realized with the potential \( V(\phi) \) and through the coupling term \( \phi R \) in the used Lagrangian. The field equations are derived from the action by varying with respect to the metric \( g_{\mu\nu} \) and also with respect to the field \( \phi \), respectively:

\[
G_{\mu\nu} = \frac{\kappa}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right) \tag{2.2}
\]

\[
R = V'(\phi).
\]

The energy momentum tensor that describes the matter content is of standard perfect fluid form \( T = T^\mu_\mu = -\rho + 3p \) [42] and we use the polytropic equations of state to describe the adiabatic processes inside the star. The matter term in (2.2) needs to obey the equation of continuity \( D_\mu T^\mu_\nu = 0 \), so this equation is used when the polytropic profile is calculated.
The only non-trivial component of the equation of continuity for a spherically symmetric system is
\[ p' = \frac{B'}{2B}(\rho + p). \] (2.3)

Here comma stands for the radial derivative, \( \equiv \frac{d}{dr} \). If the scalar-tensor gravity is to resemble general relativity at small scales, it should obtain the Schwarzschild solution outside a spherically symmetric object or the Schwarzschild-de Sitter vacuum outside the GR+\( \Lambda \) star. Since the potential \( V(r) \) and the field \( \phi(r) \) are compared to polytropes with observational properties, we study the field equations in the Jordan frame throughout this work for numerical convenience.

In the chameleon theories of gravity [22, 23], the effective potential is chosen such that the second derivative of the potential is dependent on the local energy density. In this way the fifth force is evaded with a sufficiently high mass of the field in dense environments. In this work no fixed potential is selected, but the potential is required to stay close to the GR+\( \Lambda \) solution inside the objects and to arrive at the SdS vacuum solution. As a result the potential turns out to be very dependent on the matter density and the boundary conditions. Irrespective of whether the screening condition [43] is fulfilled or not, a general potential \( V(\phi) \) should be able to describe all the observed objects. With the above conditions, every studied object class occupies an unique range for both \( V(r,\phi(r)) \), namely \( V(\rho,\phi(r))_{\text{NS}} \leq V(r,\phi(r))_{\text{WD}} \leq V(r,\phi(r))_{\text{SUN}} \). Therefore, all the studied objects (SUN, WD, NS) cannot be described by one potential \( V(\phi) \).

2.1 Competence with the GR+\( \Lambda \) and \( f(R) \) models

We demand the studied object to be comparable to an already established general relativistic configurations. We, therefore, select for the scalar-tensor examination only the configurations that are similar enough to the general relativistic polytropes inside the star. We study all the configurations separately with the GR+\( \Lambda \) and the scalar-tensor field equations, integrating first from the center outwards with general relativistic field equations and then from the fixed SdS boundary inwards with the scalar-tensor field equations. We also, for comparison, study configurations with \( f(R) \) gravity interiors. The gravitational interaction that fixes the interior of the configurations is derived from the action
\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m. \] (2.4)

Now the Einstein-Hilbert action that gives the GR+\( \Lambda \) model is obtained with \( f(R) = R - 2\Lambda \). To numerically derive the polytropic profiles (\( \rho(r), B(r), A(r) \)) inside the configuration we use the temporal field equation from
\[ F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F(R) = 8\pi G T_{\mu\nu}. \] (2.5)

(with \( \mu = \nu = 0 \)). The scalar degree of freedom is denoted here as \( F(R) \equiv f'(R) \). After obtaining \( \rho(r), B(r) \) and \( A(r) \), the scalar tensor potential \( V(r) \) and field \( \phi(r) \) are numerically integrated inwards starting from the surface. Metric \( f(R) \) theories, with an algebraic function \( f(R) \) that replaces the Ricci scalar \( R \) in the Einstein-Hilbert Lagrangian, can be represented with the scalar-tensor gravity of (2.1) [15, 18, 19]. In this work we, therefore, constructed the potential \( V(\phi) \) and the field \( \phi \) also by demanding that the polytropes’ energy density and the metric follow the field equations for two \( f(R) \) models. We in particular studied
the chameleon $f(R)$ gravity models of Hu and Sawicki (HS) and Starobinsky (St), that can produce the correct cosmological dynamics and have the correct weak field limit. The $f(R)$ function for the Hu-Sawicki model \[44\] is

$$f(R)_{\text{HS}} = R - m^2 \frac{c_1 \left( \frac{R}{m^2} \right)^\alpha}{c_2 \left( \frac{R}{m^2} \right)^\beta + 1},$$

where $\alpha > 0$, $c_1$ and $c_2$ are dimensionless parameters and $m^2$ is the mass scale of the vacuum today. The Starobinsky model \[45\] is

$$f(R)_{\text{St}} = R + \lambda R \Lambda \left( \left( 1 + \frac{R^2}{R \Lambda^2} \right)^{-\beta} - 1 \right),$$

where $\beta, \lambda > 0$ and $R \Lambda$ is the vacuum scale. The field equations are highly non-linear, and for the functions (2.6) and (2.7), therefore, only solar type polytropes and white dwarfs are solvable with the current code. We also separately solve the Tolman-Oppenheimer-Volkov (TOV) equations \[48\] for all the selected objects to make sure the objects don’t deviate much from GR \[46\]. All the chosen polytropes (also the ones derived with the $f(R)_{\text{HS,St}}$ field equations) separately fulfill both (GR+$\Lambda/f(R)$ and TOV) requirements.

3 Static spherically symmetric solutions

We consider here static, spherically symmetric bodies embedded in background a SdS vacuum with $R_0 = 12 H_0^2$ \[47\]. We use the general spherically symmetric line element ((8.1.4) in \[48\])

$$ds^2 = B(r) dt^2 - A(r) dr^2 - r^2 (d\Theta^2 - \sin^2 \Theta d\Phi^2),$$

(3.1)

Where $A(r)$ and $B(r)$ are free functions that are numerically derived inside the stellar body from the GR+$\Lambda$ field equations with the polytropic EOS. The angle coordinates $\Theta$ and $\Phi$ in (3.1) do not enter the used equation ansatz at all. The polytropes are fixed to the SdS vacuum at the surface \[49\] to make sure the initial values $V_s, \phi_s, \phi'_s$ correspond to stellar objects with a cosmological vacuum. The scalar-tensor field equations are integrated from the surface to the center with the scalar tensor field equations (2.2) starting from the Schwarzschild-de Sitter metric with the boundary values

$$B(r_s) = c^2 \left( 1 - \frac{2GM}{c^2 r_s} - \frac{\Lambda}{3} r_s^2 \right),$$

$$A(r_s) = \left( 1 - \frac{2GM}{c^2 r_s} - \frac{\Lambda}{3} r_s^2 \right)^{-1}$$

(3.2)

at the boundary. Here the mass $M$ and the stellar radius $r_s$ are obtained first with the GR+$\Lambda$ field equations.

The scalar-tensor field equations (2.2) can be analytically solved in the de Sitter vacuum

$$G_{11} = -3A(r) H_0^2, \quad T_{11} = 0$$

and the first field equation boils down to

$$V(\phi) = - \left( \frac{B'}{BA} + \frac{4}{Ar} \right) \phi' + 6H_0^2 \phi$$

(3.3)
at the surface with $R = 4\Lambda \equiv 12H_0^2$ [47]. The boundary conditions, $\phi_s(r_s)$ and $\phi'_s(r_s)$, for all the stars can be straightforwardly solved from the equation (3.3) with $V(r_s) = 2\Lambda = 6H_0^2$ already at the surface. The Schwarzschild-de Sitter vacuum is obtained from (3.3) with $\phi = 1$ and $\phi' = 0$. In the numerical analysis the boundary value for the potential is, however, obtained with the field values deviating from these a little i.e. with $\phi \approx 1$ and $\phi' \approx 0$. The potential outside the body stays near the vacuum value, but the field and its derivative develop. The field behavior outside is dependent on the stellar boundary conditions and the field value will be a monotonically decreasing function of the radius. With identical initial $\phi(r_s)$ at the boundary, physically valid configurations can be found, but the potential magnitude $V(r)$ inside will still be dictated by the matter configuration and will range different values for each object class.

3.1 Polytropic stellar configurations

Polytropic model describes adiabatic processes. Stars are often modelled with the polytropic model because it naturally results in a monotonically decreasing density profile with a well defined boundary [50]. The polytropic equation of state (EOS) is

$$p(r) = K\rho(r)\gamma$$

(3.4)

Here $K$ is a constant, and $\gamma$ is related to the polytropic index $n$ by $\gamma = n/(1+n)$. This restricts the perfect fluid matter to form spherical objects that are most dense at the core. All the configurations we studied here are tested to be regular at the center [49], having physical central densities $\rho_c$ and obtaining masses and radii that conform to observations. We have considered only objects that are also numerically equivalent to their Tolman-Oppenheimer-Volkov counterparts [48] by separately solving the TOV equation for the same stellar configurations and by comparing the solutions.

The parameter space $(K, \rho_c, \gamma)$, yielding the physically and observationally acceptable object classes is extremely tight for compact configurations. However, one can find objects that represent relativistic and non-relativistic white dwarfs as well as neutron stars that also follow the $R(r) \sim \kappa\rho(r)$ behavior inside the body. One representative from each class was chosen for plotting the scalar field potential $V(\phi)$ in figure 2. We use the Eddington polytropic model of [33] with $n = 3$ to produce a representative of the stellar object class. The two other studied classes of compact stars are polytropic white dwarfs, and polytropic neutron stars. The polytropes’ matter density is parametrized as in the general relativistic Lane-Emden case [48] with $\rho(r) = \rho_c\theta^n(r)$ for the numerical work ($\theta$ is the scaled density parameter). The Lane-Emden equation itself is not used in this work, only the above parametrization.

Stability considerations for the scalar-tensor polytropic configurations have not been examined in this work. However we list here some works done with general relativistic polytropes. Here, we present some aspects to take into account when considering the stability of static polytropes also with modified gravity. All static polytropic solutions in GR are considered stable in [51] if the solution is regular at the center and the density falls of rapidly after the boundary region. Also note that due to the higher order nature of the solutions for higher order gravity theories, the boundary matching at the surface is not discontinuous and SdS can be reached naturally. The same matching conditions at the boundary (i.e. $r_{s, inside} = r_{s, outside}$ and $\phi(r_{s, inside}) = \phi(r_{s, outside})$) were also studied in the scalar-tensor gravity in [52]. Furthermore, for static spherically symmetric solutions in scalar-tensor gravity, the presence of a non-negative effective potential implies the absence of unstable modes for linear perturbations in the scalar field [53]. Kosambi-Cartan-Chern and Lyapunov stability
properties of Lane-Emden equations have been studied in [54]. With both these methods the general relativistic polytropic index is stable only for the values $\gamma \in [1.2, 1.313708]$. This does not, however, include the solar Eddington polytrope with $\gamma_{\text{SUN}} = 4/3$ nor the non-relativistic white dwarfs with $\gamma_{\text{WDnr}} = 5/3$ although these polytropic equations of state are widely used for modelling polytropic stellar type stars and white dwarfs.

Polytropic stellar models are not realistic, but extremely useful for their simplicity and fairly good resemblance to more accurate models in the stellar case. Also, polytropic stars arise naturally in general relativity [48, 50] and provide a good approximation for the observed systems. We are using GR+Λ equations inside these objects. In this work we discuss observationally and physically acceptable self-consistent objects in the studied scalar-tensor gravitation. The configurations are demanded to verify the following requirements:

i) The studied polytropes were chosen from the object class such that they obtain masses and radii within observed ranges. Also, the central densities need to be of the right order w.r.t. solar standard model [55] and Harrison-Wheeler EOS [42]. The parameter values are discussed in the following subsections.

ii) The parameters $\rho(r), A(r)$ and $B(r)$ are first derived with the GR+Λ or $f(R)_{\text{HS,St}}$ field equations, which selects a scalar-tensor models with a potential $V(\phi)$ that conforms to general relativity inside the object [46].

iii) We demand the spacetime to conform to SdS at the boundary by numerically solving (3.3) with $V = 6H_0^2$ and the SdS metric parameters $B(r_s)$ and $A(r_s)$ in (3.2), that were obtained with the mass $M$ and the radius $r_s$ of a GR+Λ polytrope.

3.1.1 Solar and white dwarf polytropes

For the stellar type stars, we use the Eddington model that was also studies in context of Hu-Sawicki $f(R)$ gravity in [33]. Eddington model gives a fair approximation for the standard solar model density profile [55] with $n = 3$ and half the standard model central density. We studied two classes of WDs with different polytropic indices, $n = 1.5$ for non-relativistic and $n = 3$ for relativistic white dwarfs. The observational mass and radius ranges for the WDs were referenced from observations; see e.g. [56]. We accepted as valid values for the mass $\in [0.4, 0.7]M_\odot$ and for the radius $\in [8000, 11000]$ km. The coefficient for the polytropic equation of state for non-relativistic degenerate gas is $n = 1.5$ and for highly relativistic degenerate gas $n = 3$. Observational masses and radii for the white dwarfs were found only for the physical central densities that conform to the Harrison-Wakano-Wheeler stellar models ([42] p.625).

3.1.2 Polytropic neutron stars

There are many rival models for the structure of a neutron star and there is no favorite equation of state to be used. Many modern models build the neutron star from multiple polytropic layers as well as separate crust or core with different equation of state [57–59]. In this work we will use two layers of polytropes of which the first describes the core area $\in [r_0, r_c]$ (about 10% of the radius of the NS) and the second layer extends from the core to the surface $r_s$. The observational mass and radius ranges we use conform to [59, 60] As valid mass range we accepted the observational values $\in [1.4, 1.7]M_\odot$ and the corresponding radii estimates (that are in accordance with the studies [57, 58, 61]) to lie in the range $[9, 13]$ km. For the HS and St field equations, neutron stars do not solve even for the singly-polytropic object with the current code.
4 Numerical work

One observationally valid object (with particular $K, \rho_c$ and $\gamma$) was chosen to represent each stellar class. Inside the star, the GR+Λ field equations were solved starting from a smooth center to obtain $\rho(r), A(r)$ and $B(r)$. The scalar-tensor potential $V(r)$ and the field $\phi(r)$ inside the body were obtained with the equations (4.2) inside, starting from the boundary with the metric initial conditions set to SdS (3.2) and $\phi_s \equiv \phi(r_s), \phi'_s \equiv \phi'(r_s)$ as the field boundary conditions. The potential at the boundary is set to the vacuum value $V(r_s) = 2\Lambda = 6H_0^2$ already at the surface. These are required to fulfill the boundary condition (3.3).

Also the field equations corresponding to $f(R)$ gravities (2.6) and (2.7) were used to obtain the comparison $f(R)$ configurations. These configurations are only shortly discussed in this text and the focus is on the GR+Λ matter configurations. To constrain the parameters in the scalar-tensor field equations (2.2), we bind together the metric parameters, pressure and the matter density inside the configurations by demanding the GR+Λ field equations (from (2.4) with $f(R) = R - 2\Lambda$), the continuity equation (2.3) and the polytropic equation of state $p = p(\rho)$ to hold for each object class.

The equation ansatz that was used to derive the stellar interior consists of the $\mu = \nu = 0$-field equations (2.5), the trace equation
\[ F(R)R - 2f(R) + 3\Box F(R) = 8\pi G(\rho - 3p), \] (4.1)
the Ricci scalar as a function of the metric parameters $R[B(r), A(r)]$ and the continuity equation (2.3). The curvature scalar $R$ follows the energy density $\kappa \rho$ for all the selected WDs and SUN and the solution was checked to correspond to the general relativistic counterpart that was solved separately for the same polytropic parameters with the TOV equations [48]. The TOV equation was built as a separate code and used only for comparison to judge if the density $\rho(r)$ in the modified code obtains the general relativistic TOV $\rho_{TOV}(r)$ solution with high accuracy.

The polytropic coefficient $K$ was chosen such that mass and radius, that conform to the observations, are produced when the object is integrated from inside out with the GR+Λ field equations. A wide range of polytropic coefficients $K$ and central densities, $\rho_c \in [10^4, 10^{11}] \text{kg m}^{-3}$, in (3.4) were also scanned to be sure no other region of solutions for observationally and physically valid solutions exist. The domain of observationally and physically valid polytropic solutions shrinks essentially to a point in the parameter space.

The potential $V(r)$ and the field $\phi(r)$ were solved numerically from the contraction of the 11-component of the scalar-tensor field equations (2.2) with the metric (3.1)
\[ V(r) = 2\kappa p(r)(1 - \phi(r)) - \frac{4 + B'(r)r}{A(r)r} - \phi'(r), \]
\[ V'(r) = \phi'(r)R(r). \] (4.2)
Note that we have reparametrized the potential $V(\phi) \to V(r)$, thus the form of the second field equation is changed. The integration of $V(r)$ and $\phi(r)$ was started from the surface inwards with the initial conditions chosen by the SdS metric vacuum requirement (3.3).

The dimensionless field $\phi$ must be of order unity [15] to be consistent with the solar system experiments. The field naturally varies extremely little from unity with the SdS condition at the boundary. The used field initial values for these objects can be seen in the figure 2. Therefore, the effective parameter in this work is the deviation of the field $\phi = 1 + \delta \phi$. 

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Figure 1. (Color online) As an example a non-relativistic (green/dashed) and a relativistic (blue/solid curve) white dwarf, showing the typical field and potential behavior inside the configuration. The potential $V(\phi)$ is highly dependent on the matter density inside the object. The initial value at the surface was chosen to be the first $\phi_s'$ value that produces the SdS vacuum solution (3.3) with the potential value $V_s = 2\Lambda = 6H_0^2 = 3.2667 \times 10^{-35} \text{s}^{-1}$.

4.1 Results

The results presented here are qualitative in their nature although specific numerical values play a crucial role in the numerical analysis. In this article we have derived the scalar field potential $V(\phi)$ of the action (2.1) for polytropic stellar configurations that conform to general relativistic stars (solar, white dwarf and neutron star polytropes). As the main result, we consider the dependence of the numerically derived scalar field potential $V(\phi)$ on the matter configuration inside the stellar body in each case. This property does not allow to describe all the studied stellar objects with the same potential (that is with the same theory). We also find, that the resulting field inside the configuration $\delta \phi(r)$ is highly dependent on the initial value $\delta \phi_s$ at the surface. The initial condition $V_s \equiv V(r_s)$ was set to be the vacuum value $6H_0^2$ and the field values $\phi_s$ and $\phi'_s$ can be solved from (3.3). The sensitive boundary conditions for $\delta \phi_s$ range from $10^{-20}$ to $10^{-29}$ for the SUN and NS case respectively, see figure 2.

One can fix the higher order terms of the metric parameters \cite{15} at the boundary, therefore, the transition from the polytropic configuration to the outside solution at the surface $r_s$ is smooth. The potential and the field are monotonically increasing functions of the radius from the surface inwards and will produce $V(\phi)$ that is monotonically increasing in the positive field direction figure 1. The potential values in the plots for the examined stellar objects ranges from the surface values, $V_s$, outside up to a value depending on the density at the core. Because of this, a neutron star will always reach higher potential values than a white dwarf or a solar type main sequence star $V(\phi)_{NS} \leq V(\phi)_{WD} \leq V(\phi)_{SUN}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure1.png}
\caption{(Color online) As an example a non-relativistic (green/dashed) and a relativistic (blue/solid curve) white dwarf, showing the typical field and potential behavior inside the configuration. The potential $V(\phi)$ is highly dependent on the matter density inside the object. The initial value at the surface was chosen to be the first $\phi_s'$ value that produces the SdS vacuum solution (3.3) with the potential value $V_s = 2\Lambda = 6H_0^2 = 3.2667 \times 10^{-35} \text{s}^{-1}$.}
\end{figure}
Figure 2. Here the non-relativistic neutron star (red, leftmost), white dwarfs (blue and green/dashed, center) and a Sun like star (black/dotted, right) all have de Sitter vacuum as the surface initial condition (at $log_{10}(V(\phi_s)) = -34.485891$). The smaller figure is a zoom in to the WDs’ and the SUN’s region. All the objects show similar dependence with respect to the matter density. That is, all the potential curves are monotonically increasing functions of the radius, increasing toward the center.

All the objects in the figures 1, 2 in principle reach the SdS with $V = 6H_0^2$. The plots, however, will only show data down to a finite field value due to discrete numerical methods. This is the initial value at the boundary of the star. Also, there is an unique limiting initial value for the field, $\delta \phi_{sat}$, that saturates the field range to an object dependent minimum value. For smaller initial values than this $\delta \phi < \delta \phi_{sat}$ and $\phi' < \phi'_{sat}$ each object class obtains the saturated configuration. This will define a typical field range for each object class that is shown for all the studied classes on the figure 2. For the solar class, SUN, the initial condition (3.3) is fulfilled in the saturated case with $\delta \phi_{sat} = \mathcal{O}(10^{-20})$, $\phi'_{sat} = \mathcal{O}(10^{-27})$ with $V(r_s) = 6H_0^2$. In the WD and NS saturated cases the initial conditions are even tighter, $\delta \phi_{sat, WD} = \mathcal{O}(10^{-24})$ and $\delta \phi_{sat, NS} = \mathcal{O}(10^{-29})$ for the same vacuum initial value $V(r_s)$. Note here, that the initial values are bounded from above to be in accordance with (3.3), and that the allowed range for $\delta \phi$ allows the filed ranges to overlap. The effects of identical initial conditions are addressed later in this text.

We will consider the solar case, SUN, as a general relativistic polytrope that conforms to the Cassini results [34]. The initial condition $\delta \phi_s$ can be relaxed from the saturated case such that the WDs and the NS obtain solutions that lie within the same $\delta \phi$ range. However, in the case of identical initial conditions the field value ranges still correspond to the size of the object, so that the more compact objects will always obtain smaller ranges than the more extended cases. The potential values, being dependent on the density and not on the field initial value, will also in any case be higher for the denser objects. Due to these two properties of the system, a common function describing all the object classes $V(\phi)$ cannot
be found. Considering this, a potential that matches all the polytropic stellar configurations and also conforms to the observations could not be found, even with fine-tuning.

The polytropic profiles and the metric parameters calculated from the Hu-Sawicki and Starobinsky field equations yield solutions for the potential $V(r)$ and the field $\phi(r)$ that are similar to the GR+$\Lambda$ solutions. Almost identical solutions can be found for all the objects except the NSs that could not be solved for the $f(R)$ field equations. These models have also been studied in the case of a massive collapsing star in [62]. Inhomogeneous, polytropic-like energy density is required by the matching conditions at the boundary for the matter distribution in $f(R) = R + \alpha R^2$ gravity to be in accordance with GR (i.e. for the boundary to be matched to the Schwarzschild spacetime).

5 Conclusions and discussion

We considered configurations of general spherically symmetric, static spacetimes with adiabatic perfect fluid matter in the scalar-tensor gravity (2.2). These solutions were required to follow general relativity with $\Lambda$, according to the Einstein-Hilbert action, or $f(R)$ gravity with (2.6) and (2.7) inside the polytropic configurations. The field equations (2.2) were numerically solved to arrive at the scalar-tensor potential $V(\phi)$ inside the stellar object. We studied examples of polytropic stellar object classes; a solar type star, non-relativistic and relativistic white dwarfs and a neutron star with observationally acceptable parameters. We first solve the field equations numerically for the metric functions $A(r)$ and $B(r)$ and for the energy density $\rho(r)$ inside the polytropic object with (2.5) starting from a smooth center. Then obtain the field $\phi(r)$ and the potential $V(r)$ with (2.2) starting the iteration from the surface inwards by choosing the boundary conditions to accept the Schwarzschild-de Sitter solution with $V_S = 6H_0^2$.

As a result, we find that the potential of a scalar-tensor polytrope that conforms to general relativity is highly dependent on the matter configuration. Also, the possible field values are defined by the vacuum initial conditions down to a minimum value that is unique to the stellar object class. A potential $V(\phi)$ that would correspond to all the polytropic objects could not be found. The stability with respect to the polytropic equation of state in scalar-tensor gravity has not been studied yet, so it is possible that some of the objects described here are not stable.

This issue should be further studied with non-zero kinetic term in the action. This, however, will not change the situation for the f(R) gravities, that are dynamically equivalent to this particular action.

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References

[1] C.M. Will, The Confrontation between general relativity and experiment, Living Rev. Rel. 9 (2006) 3 [gr-qc/0510072] [esSPIRE].

[2] Supernova Search Team collaboration, A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. 116 (1998) 1009 [astro-ph/9805201] [esSPIRE].
[3] Supernova Cosmology Project collaboration, S. Perlmutter et al., Measurements of Omega and Lambda from 42 high redshift supernovae, Astrophys. J. 517 (1999) 565 [astro-ph/9812133] [inSPIRE].

[4] SNLS collaboration, P. Astier et al., The Supernova legacy survey: measurement of \( \Omega_M, \Omega_L \) and \( w \) from the first year data set, Astron. Astrophys. 447 (2006) 31 [astro-ph/0510447] [inSPIRE].

[5] WMAP collaboration, D.N. Spergel et al., First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: determination of cosmological parameters, Astrophys. J. Suppl. 148 (2003) 175 [astro-ph/0302209] [inSPIRE].

[6] WMAP collaboration, D.N. Spergel et al., Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology, Astrophys. J. Suppl. 170 (2007) 377 [astro-ph/0603449] [inSPIRE].

[7] V. Sahni and A.A. Starobinsky, The Case for a positive cosmological \( \Lambda \)-term, Int. J. Mod. Phys. D 9 (2000) 373 [astro-ph/9904398] [inSPIRE].

[8] M.S. Turner, The case for LambdaCDM, astro-ph/9703161 [inSPIRE].

[9] J. Frieman, M. Turner and D. Huterer, Dark Energy and the accelerating universe, Ann. Rev. Astron. Astrophys. 46 (2008) 385 [arXiv:0803.0982] [inSPIRE].

[10] M.J. Mortonson, D.H. Weinberg and M. White, Dark Energy: a short review, arXiv:1401.0046 [inSPIRE].

[11] S. Nojiri and S.D. Odintsov, Modified gravity as an alternative for \( \Lambda \)-CDM cosmology, J. Phys. A 40 (2007) 6725 [hep-th/0610164] [inSPIRE].

[12] P. Brax, C. van de Bruck, A.-C. Davis and D.J. Shaw, \( f(R) \) Gravity and Chameleon Theories, Phys. Rev. D 78 (2008) 104021 [arXiv:0806.3415] [inSPIRE].

[13] T.P. Sotiriou, V. Faraoni and S. Liberati, Theory of gravitation theories: a no-progress report, Int. J. Mod. Phys. D 17 (2008) 399 [arXiv:0707.2748] [inSPIRE].

[14] Y. Fujii and K. Maeda, The scalar-tensor theory of gravitation, Cambridge University Press, New York U.S.A. (2003).

[15] T.P. Sotiriou and V. Faraoni, \( f(R) \) theories of gravity, Rev. Mod. Phys. 82 (2010) 451 [arXiv:0805.1726] [inSPIRE].

[16] S. Nojiri and S.D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, eConf C 0602061 (2006) 06 [Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115 [hep-th/0601213] [inSPIRE].

[17] A. De Felice and S. Tsujikawa, \( f(R) \) theories, Living Rev. Rel. 13 (2010) 3 [arXiv:1002.4928] [inSPIRE].

[18] C.H. Brans, The Roots of scalar-tensor theory: an approximate history, gr-qc/0506063 [inSPIRE].

[19] V. Faraoni, Cosmology in Scalar-Tensor Gravity, Kluwer Academic, Dordrecht (2004).

[20] B. Ratra and P.J.E. Peebles, Cosmological Consequences of a Rolling Homogeneous Scalar Field, Phys. Rev. D 37 (1988) 3406 [inSPIRE].

[21] R.R. Caldwell, R. Dave and P.J. Steinhardt, Cosmological imprint of an energy component with general equation of state, Phys. Rev. Lett. 80 (1998) 1582 [astro-ph/9708069] [inSPIRE].

[22] J. Khoury and A. Weltman, Chameleon fields: Awaiting surprises for tests of gravity in space, Phys. Rev. Lett. 93 (2004) 171104 [astro-ph/0309300] [inSPIRE].
[23] D.F. Mota and D.J. Shaw, Evading equivalence principle violations, cosmological and other experimental constraints in scalar field theories with a strong coupling to matter, Phys. Rev. D 75 (2007) 063501 [hep-ph/0608078] [inSPIRE].

[24] S.A. Appleby, R.A. Battye and A.A. Starobinsky, Curing singularities in cosmological evolution of F(R) gravity, JCAP 06 (2010) 005 [arXiv:0909.1737] [inSPIRE].

[25] P. Zhang, Testing f(R) gravity against the large scale structure of the universe, Phys. Rev. D 75 (2007) 123504 [astro-ph/0610532] [inSPIRE].

[26] S.A. Appleby, R.A. Battye and A.A. Starobinsky, Curing singularities in cosmological evolution of F(R) gravity, JCAP 06 (2010) 005 [arXiv:0909.1737] [inSPIRE].

[27] V. Faraoni, Solar System experiments do not yet veto modified gravity models, Phys. Rev. D 74 (2006) 023529 [gr-qc/0607016] [inSPIRE].

[28] S. Nojiri and S.D. Odintsov, Modified f(R) gravity consistent with realistic cosmology: from matter dominated epoch to dark energy universe, Phys. Rev. D 74 (2006) 086005 [hep-th/0608008] [inSPIRE].

[29] S.A. Appleby and R.A. Battye, Do consistent F(R) models mimic General Relativity plus \Lambda?, Phys. Lett. B 654 (2007) 7 [arXiv:0705.3199] [inSPIRE].

[30] Y.-S. Song, W. Hu and I. Sawicki, The Large Scale Structure of f(R) Gravity, Phys. Rev. D 75 (2007) 044004 [astro-ph/0610532] [inSPIRE].

[31] S. Nojiri and S.D. Odintsov, Modified non-local-F(R) gravity as the key for the inflation and dark energy, Phys. Lett. B 659 (2008) 821 [arXiv:0708.0924] [inSPIRE].

[32] S.G. Turyshev, Experimental Tests of General Relativity, Ann. Rev. Nucl. Part. Sci. 58 (2008) 207 [arXiv:0806.1731] [inSPIRE].

[33] K. Henttunen and I. Vilja, Consistency of f(R) gravity models around solar polytropes, Phys. Lett. B 731 (2014) 110 [arXiv:1110.6711] [inSPIRE].

[34] B. Bertotti, L. Iess and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, Nature 425 (2003) 374 [arXiv:0806.1731] [inSPIRE].

[35] V. Faraoni, Matter instability in modified gravity, Phys. Rev. D 74 (2006) 104017 [arXiv:0610734] [inSPIRE].

[36] G.D. Birkhoff, Relativity and Modern Physics, Harvard University Press, Cambridge (1923).

[37] S. Deser and J. Franklin, Schwarzschild and Birkhoff a la Weyl, Am. J. Phys. 73 (2005) 261 [gr-qc/0408067] [inSPIRE].

[38] V. Faraoni, The Jebsen-Birkhoff theorem in alternative gravity, Phys. Rev. D 81 (2010) 044002 [arXiv:1001.2287] [inSPIRE].

[39] C. Brans and R.H. Dicke, Mach’s principle and a relativistic theory of gravitation, Phys. Rev. 124 (1961) 925 [inSPIRE].

[40] V. Faraoni and S. Nadeau, The (pseudo)issue of the conformal frame revisited, Phys. Rev. D 75 (2007) 023501 [gr-qc/0612078] [inSPIRE].

[41] E.E. Flanagan, The conformal frame freedom in theories of gravitation, Class. Quant. Grav. 21 (2004) 3817 [gr-qc/0403063] [inSPIRE].

[42] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation, W.H.Freeman and Company, New York U.S.A. (1973).

[43] T.P. Waterhouse, An introduction to chameleon gravity, astro-ph/0611816 [inSPIRE].

[44] W. Hu and I. Sawicki, Models of f(R) cosmic acceleration that evade Solar-System tests, Phys. Rev. D 73 (2006) 064004 [arXiv:0705.1158] [inSPIRE].
[45] A.A. Starobinsky, *Disappearing cosmological constant in f(R) gravity*, JETP Lett. **86** (2007) 157 [arXiv:0706.2041] [INSPIRE].

[46] T. Multamaki and I. Vilja, *Static spherically symmetric perfect fluid solutions in f(R) theories of gravity*, Phys. Rev. **D 76** (2007) 064021 [astro-ph/0612775] [INSPIRE].

[47] J.P. Gazeau and M. Lachieze Rey, *Quantum Field Theory in de Sitter space: a survey of recent approaches*, PoS(IC2006)007 [hep-th/0610296] [INSPIRE].

[48] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons (1972).

[49] K. Henttunen, T. Multamaki and I. Vilja, *Stellar configurations in f(R) theories of gravity*, Phys. Rev. **D 77** (2008) 024040 [arXiv:0705.2683] [INSPIRE].

[50] S. Chandrasekhar, *An introduction to the study of stellar structure*, Dover Publications, New York U.S.A. (1958).

[51] C. Fronsdal, *Stability of polytropes*, Phys. Rev. **D 77** (2008) 104019 [arXiv:0705.0774] [INSPIRE].

[52] C. Barrabes and G.F. Bressange, *Singular hypersurfaces in scalar-tensor theories of gravity*, Class. Quant. Grav. **14** (1997) 805 [gr-qc/9701026] [INSPIRE].

[53] T. Harada, *Stability analysis of spherically symmetric star in scalar-tensor theories of gravity*, Prog. Theor. Phys. **98** (1997) 359 [gr-qc/9706014] [INSPIRE].

[54] C.G. Boehmer and T. Harko, *Nonlinear stability analysis of the Emden-Fowler equation*, J. Nonlin. Math. Phys. **17** (2010) 503 [arXiv:0902.1054] [INSPIRE].

[55] J.N. Bahcall, A.M. Serenelli and S. Basu, *New solar opacities, abundances, helioseismology and neutrino fluxes*, Astrophys. J. **621** (2005) L85 [astro-ph/0412440] [INSPIRE].

[56] J.L. Provencal, H.L. Shipman, E. Hog and P. Thejll, *Testing the white dwarf mass-radius relation with HIPPARCOS*, Astrophys. J. **494** (1998) 759 [INSPIRE].

[57] A.W. Steiner, J.M. Lattimer and E.F. Brown, *The neutron star mass-radius relation and the equation of state of dense matter*, Astrophys. J. **765** (2013) L5 [arXiv:1205.6871] [INSPIRE].

[58] H. Thomas, B. Roettgers and W. Weise, *How neutron stars constrain the nuclear equation of state*, arXiv:1307.4582 [INSPIRE].

[59] J.M. Lattimer, *The nuclear equation of state and neutron star masses*, Ann. Rev. Nucl. Part. Sci. **62** (2012) 485 [arXiv:1305.3510] [INSPIRE].

[60] A.K. Harding, *The Neutron Star Zoo*, Front. Phys. China **8** (2013) 679 [arXiv:1302.0869] [INSPIRE].

[61] F. Douchin and P. Haensel, *A unified equation of state of dense matter and neutron star structure*, Astron. Astrophys. **380** (2001) 151 [astro-ph/0111092] [INSPIRE].

[62] R. Goswami, A.M. Nzioki, S.D. Maharaj and S.G. Ghosh, *Collapsing spherical stars in f(R) gravity*, Phys. Rev. **D 90** (2014) 084011 [arXiv:1409.2371] [INSPIRE].

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