The Brownian Motion in an Ideal Quantum Gas

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ABSTRACT: A Brownian particle in an ideal quantum gas is considered. The mean square displacement (MSD) is derived. The Bose-Einstein or Fermi-Dirac distribution, other than the Maxwell-Boltzmann distribution, provides a different stochastic force compared with the classical Brownian motion. The MSD, which depends on the thermal wavelength and the density of medium particles, reflects the quantum effect on the Brownian particle explicitly. The result shows that the MSD in an ideal Bose gas is shorter than that in a Fermi gas. The behavior of the quantum Brownian particle recovers the classical Brownian particle as the temperature raises. At low temperatures, the quantum effect becomes obvious. For example, there is a random motion of the Brownian particle due to the fermionic exchange interaction even the temperature is near the absolute zero.

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1 Introduction

The Brownian motion is first observed by Robert Brown in 1827 and then explained by Einstein (1905), Smoluchowski (1905), and Langevin (1908) in the early 20th century [1]. The early theory of the Brownian motion not only provides an evidence for the atomistic hypothesis of matter [2], but also builds a bridge between the microscopic dynamics and the macroscopic observable phenomena [2].

The classical understanding of the Brownian motion is quite well established. However, there is an assumption in the early theory of the Brownian motion that the medium particle obeys the Maxwell-Boltzmann distribution.

The behavior of a Brownian particle in an ideal quantum gas draws some attentions because to study the motion of a Brownian particle in an quantum system is now within reach of experimental tests. For example, an electron in a black body radiation [3]. In such systems, the quantum exchange interaction, which always leads to real difficulty in mechanics and statistical mechanics [4–7], exists and causes the medium particle obeying the Bose-Einstein or Fermi-Dirac distribution. It is difficult to make exact or even detailed dynamical calculations [1, 8], since a different stochastic force is provided by the Bose-Einstein or Fermi-Dirac distribution. At high-temperature and low-density, the classical theory serves as good approximation.

In this paper, we give an explicit expression of the mean square displacement (MSD) of a Brownian particle in an ideal quantum gas using, e.g., the virial expansion. Comparison
with the classical Brownian motion, a correction for the MSD, which depends on the thermal wavelength and the density of medium particles, is deduced. The result shows that the MSD in an ideal Bose gas is shorter than that in a Fermi gas. The behavior of the quantum Brownian particle recovers the classical Brownian particle as the temperature raises. At low temperature, the quantum effect becomes obvious. For example, there is a random motion of the Brownian particle due to the fermionic exchange interaction even the temperature is near the absolute zero.

The early studies of the Brownian motion inspired many prominent developments in various areas such as physics, mathematics, financial markets, and biology. In physics, exact solutions of Brownian particles in different cases, such as in a constant field of force [1] and in a harmonically potential field [1], are given. The Brownian motion with a time dependent diffusion coefficient is studied in Ref. [9]. The boundary problem of Brownian motions is studied in Refs. [10, 11]. The anomalous diffusion process, frequently described by the scaled Brownian motion, is studied in Refs. [12, 13]. The Kramers-Klein equation considers the Brownian particle that is in an general field of force [1]. The generalized Langevin equations and the master equation for the quantum Brownian motion are studied in Refs. [14, 15]. In mathematics, the rigorous interpretation of Brownian motions based on concepts of random walks, martingales, and stochastic processes is given [1, 8, 16]. In financial markets, the theory of the Brownian motion is used to describe the movement of the price of stocks and options [1, 8, 17–20]. The application of the fractional Brownian motion, which is a generalized Brownian motion, in financial markets is studied in Refs. [21–23]. Moreover, the Brownian motion plays a central and fundamental role in studies of soft matter and biophysics [8], e.g., active Brownian motions, which can be used to describe the motion of swarms of animals in fluid, are studied in Refs. [8, 24–28].

Among many quantities, the MSD, which is measurable, describes the Brownian motion intuitively. There are studies focus on the MSD related problems. For examples, the relation between the MSD and the time interval can be generally written as $\langle x^2_t \rangle \sim t^\alpha$ [9]. One distinguishes the subdiffusion with $0 < \alpha < 1$ and the superdiffusion with $\alpha > 1$ [29–31]. The relation between the MSD and the time interval of the so called ultraslow Brownian motions is $\langle x^2_t \rangle \sim (\ln t)^\gamma$ [32].

There are different approaches to build a quantum analog of the Brownian motion [33–36]. For examples, the method of the path integral is used to study the quantum Brownian motion [35]. The approach of a quantum analog or quantum generalization of the Langevin equation and the master equation, e.g., the quantum master equation [3] and the quantum Langevin equation [37] is used to build a quantum Brownian motion. Among them, the method of quantum dynamical semigroups [38] is prominent. They point it out that the quantum equation should be casted into the Lindblad form [38, 39]. A completely positive master equation describing quantum dissipation for a Brownian particle is derived in Ref. [39].

This paper is organized as follows. In Sec. 2, for the sake of completeness, we derive the brownian motion from the perspective of the particle distribution in an ideal Boltzmann gas. In Sec. 3, we derive the MSD of a Brownian particle in an ideal quantum gas. High-temperature and low-temperature expansions are given. The $d$-dimensional case is
considered. The conclusion and outlook are given in Sec. 4. Some details of the calculation is given in the Appendix.

2 A Brownian particle in an ideal classical gas: the Brownian motion

In this section, we consider a Brownian particle in an ideal classical gas. For the sake of completeness, we derive, in detail, the Brownian motion from the perspective of the particle distribution.

A brief review on the Langevin equation. For a Brownian particle with mass \( M \), the dynamic equation is given by Paul Langevin

\[
\frac{dv}{dt} = -\frac{\gamma}{M} v dt + \frac{1}{M} F_t dt,
\]

\[
\frac{dx}{dt} = v dt,
\]

(2.1)
(2.2)

where \( \gamma = 6\pi \eta r \) with \( \eta \) the viscous coefficient and \( r \) the radius of the medium particles. \( F_t \) is the stochastic force generated by numerous collisions of the medium particle. It is reasonable to make the assumption that \( F_t \) is isotropic, i.e.,

\[
\langle F_t \rangle = 0.
\]

(2.3)

If the collision of the medium particle is uncorrelated; that is, for \( t \neq s \), \( F_t \) is independent of \( F_s \):

\[
\langle F_s F_t \rangle \propto \delta (s - t),
\]

(2.4)

then, the solution of Eqs. (2.1) and (2.2) is

\[
v_t = v_0 \exp \left( -\frac{\gamma}{M} t \right) + \frac{1}{M} \int_0^t \exp \left[ -\frac{\gamma}{M} (t - s) \right] F_s ds,
\]

(2.5)

\[
x_t = x_0 + M \gamma v_0 \left[ 1 - \exp \left( -\frac{\gamma}{M} t \right) \right] + \frac{1}{\gamma} \int_0^t \left\{ 1 - \exp \left[ -\frac{\gamma}{M} (t - s) \right] \right\} F_s ds,
\]

(2.6)

where \( x_0 \) and \( v_0 \) are the initial position and velocity.

The stochastic force determined by the Maxwell-Boltzmann distribution and the MSD. In an ideal classical gas, the gas particle obeys the Maxwell-Boltzmann distribution [41]. The number of particles possessing energy within \( \varepsilon \) to \( \varepsilon + d\varepsilon \), denoted by \( \tilde{a}_\varepsilon \), is proportional to \( e^{-\beta \varepsilon} \) [41], i.e.,

\[
\tilde{a}_\varepsilon = \omega_\varepsilon e^{-\beta \varepsilon} d\varepsilon,
\]

(2.7)

where \( \omega_\varepsilon \) is the degeneracy of the energy \( \varepsilon \) and \( \beta = (kT)^{-1} \) with \( k \) the Boltzmann constant \( T \) the temperature [41]. A collision of the medium particle with energy \( \varepsilon \) gives a force of magnitude proportional to \( \sqrt{2m}\varepsilon \), which is the momentum of the particle. We have

\[
F' = \rho \sqrt{2m} \varepsilon,
\]

(2.8)

where \( \rho \) is a coefficient and \( m \) is the mass of the medium particle. Thus, the probability of the Brownian particle subjected to a stochastic force with magnitude within \( F \) to \( F + dF \) is
\[ P(F) dF = \sqrt{\frac{\beta}{2\pi m \rho^2}} \exp \left( -\frac{F^2 \beta}{2m \rho^2} \right) dF. \] (2.9)

In an ideal classical gas, there is no inter-particle interactions among medium particles, thus, for \( t \neq s \), the force \( F_s \) and \( F_t \) are independent. Substituting Eq. (2.9) into Eq. (2.4) gives

\[ \langle F_s F_t \rangle = \frac{m \rho^2}{\beta} \delta(s - t). \] (2.10)

By using Eqs (2.6), (2.9), and (2.10), a direct calculation of the MSD gives

\[ \langle (x_t - x_0)^2 \rangle = \frac{M^2}{\gamma^2} \left( v_0^2 - \frac{1}{2m\gamma} \frac{m \rho^2}{\beta} \right) \left[ 1 - \exp \left( -\frac{\gamma}{M} t \right) \right]^2 \]
\[ + \frac{1}{\gamma^2} \frac{m \rho^2}{\beta} \left( t - M \frac{1}{\gamma} \left[ 1 - \exp \left( -\frac{\gamma}{M} t \right) \right] \right), \] (2.11)

For a large-scale time, \( t \gg 1 \), Eq. (2.11) recovers

\[ \langle x_t^2 \rangle = \frac{kT}{3\pi \eta r} t, \] (2.12)

where \( \rho = \sqrt{12\pi \eta r / m} \) and \( x_0 \) is chosen to be 0 without lose of generality. Eq. (2.12) is the famous Einstein’s long-time result of the MSD. The motion of a brownian particle in an ideal classical gas is the Brownian motion.

3 The MSD of a Brownian particle in an ideal quantum gas

In this section, we give the MSD of a Brownian particle in an ideal quantum gas. High-temperature and low-temperature expansions explain the quantum effect intuitively.

3.1 The stochastic force determined by the Bose-Einstein or Fermi-Dirac distribution

In an ideal quantum gas, the gas particle obeys Bose-Einstein or Fermi-Dirac distribution other than the Maxwell-Boltzmann distribution. The stochastic force is different from that in an ideal classical gas. In this section, we discuss the properties of the stochastic force in an ideal quantum gas.

In an ideal quantum gas, the number of particles possessing energy within \( \varepsilon \) to \( \varepsilon + d\varepsilon \), denoted by \( a_\varepsilon \), is

\[ a_\varepsilon = \frac{\omega_\varepsilon}{\exp(\beta \varepsilon + \alpha) + g} d\varepsilon, \] (3.1)

where \( \alpha \) is defined by \( z = e^{-\alpha} \) with \( z \) the fugacity [41]. For Bose cases, \( g = -1 \), and for Fermi cases, \( g = 1 \). Thus, the probability of the Brownian particle subjected to a stochastic force with magnitude within \( F \) to \( F + dF \) is

\[ p(F) dF = \sqrt{\frac{\beta}{2\rho^2 m \pi}} \frac{1}{h_{1/2}(z)} \frac{1}{\exp[\beta F^2/(2\rho^2 m)] + \alpha} + g} dF, \] (3.2)
where we \( h_\nu (x) \) equals Bose-Einstein integral \( g_\nu (x) \) in Bose cases \([41]\) and Fermi-Dirac integral \( f_\nu (x) \) \([41]\) in Fermi cases.

In an ideal quantum gas, the stochastic force is also isotropic, that is, Eq. (2.3) holds. However, the collision, due to the overlapping of the wave package, can be correlated; that is, \( \langle F_s F_t \rangle \) is no longer a delta function but a function of \( s-t \) with a peak at \( s=t \). However, as the ratio of the thermal wavelength and the average distance between the medium particles decreases, \( \langle F_s F_t \rangle \), can be well approximated by a delta function:

\[
\langle F_s F_t \rangle \sim \frac{m \rho^2}{\beta \hbar^{3/2}} \frac{h_3/2(z)}{h_{1/2}(z)} \delta (s-t),
\]

(3.3)

for \( n\lambda \ll 1 \), where \( \lambda = \hbar / \sqrt{2\pi mkT} \) is the thermal wavelength and \( n \) is the density of the medium particle.

In this paper, we consider the case that the ratio of the thermal wavelength and the average distance between the medium particles is small.

### 3.2 The MSD

For \( n\lambda \ll 1 \), by using Eqs. (3.2), (2.3), and (2.6), a direct calculation of MSD gives

\[
\langle x_t^2 \rangle = \frac{M^2}{\gamma^2} \left\{ v_0^2 - \frac{1}{2m\gamma} \frac{m \rho^2 h_{3/2}(z)}{\beta h_{1/2}(z)} \right\} \left[ 1 - \exp \left( -\frac{\gamma M t}{M} \right) \right]^2
+ \frac{1}{\gamma^2} \frac{m \rho^2 h_{3/2}(z)}{\beta h_{1/2}(z)} \left\{ t - \frac{M}{\gamma} \left[ 1 - \exp \left( -\frac{\gamma M t}{M} \right) \right] \right\}. \tag{3.4}
\]

For a large-scale time, \( t \gg 1 \), Eq. (3.4) recovers

\[
\langle x_t^2 \rangle \approx \frac{kT}{3\pi\eta r \hbar^{3/2}} \frac{h_{3/2}(z)}{h_{1/2}(z)}. \tag{3.5}
\]

where \( h_{3/2}(z)/h_{1/2}(z) \) is a correction for the MSD due to the Bose-Einstein or Fermi-Dirac distribution, a result of the quantum exchange interaction among gases particles.

### 3.3 High-temperature and low-temperature expansions

In order to compare with Eq. (2.12) intuitively, we give high-temperature and low-temperature expansions of Eq. (3.5) by using the state equation of ideal Bose or Fermi gases \([40, 41]\)

\[
p = \frac{kT}{\lambda} h_{3/2}(z), \tag{3.6}
\]
\[
\frac{N}{V} = \frac{1}{\lambda} h_{1/2}(z). \tag{3.7}
\]

**The high-temperature expansion.** At high temperatures, the virial expansion of Eqs. (3.6) and (3.7) directly gives \([40, 41]\)

\[
\frac{pv}{N} \sim kT \left[ 1 + ga_1(T)n\lambda + \ldots \right], \tag{3.8}
\]

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where $a_1 (T)$ is the first virial coefficient [40]. For a 1-dimensional ideal Bose or Fermi gas, $a_1 (T) = 0.353553$ [41]. Substituting Eqs. (3.6) and (3.7) into Eq. (3.8) gives

$$\frac{h_{3/2} (z)}{h_{1/2} (z)} \sim [1 + ga_1 (T) n \lambda + ...]. \quad (3.9)$$

Substituting Eq. (3.9) into Eq. (3.5) gives the MSD at high temperatures:

$$\langle x_t^2 \rangle = \frac{kT}{3\pi \eta} t [1 + ga_1 (T) n \lambda + ...]. \quad (3.10)$$

The result, Eq. (3.10), shows that the MSD in an ideal Bose gas is shorter than that in a Fermi gas. Since $\lambda$ decreases as $T$ raises, the behavior of the quantum Brownian particle returns the classical Brownian particle as the temperature raises.

The low-temperature expansion for Fermi cases. At low temperatures, for Fermi cases, $g = -1$,

$$\frac{h_{3/2} (z)}{h_{1/2} (z)} = \frac{f_{3/2} (z)}{f_{1/2} (z)} \quad (3.11)$$

The expansion of the Fermi-Dirac integral at large $z$ gives [41]

$$f_\nu (\xi) = \frac{\xi^\nu}{\Gamma (1 + \nu)} \left\{ 1 + 2\nu \sum_{j=1,3,5,...} \left[ (\nu - 1) \ldots (\nu - j) \left( 1 - 2^{-j} \right) \frac{\zeta (j+1)}{\xi^j+1} \right] \right\}. \quad (3.12)$$

Keeping only the first two terms in Eq. (3.12) gives

$$f_\nu (z) = \frac{(\ln z)^\nu}{\Gamma (1 + \nu)} + 2\nu (\nu - 1) \frac{\zeta (2)}{2 (\ln z)^2}. \quad (3.13)$$

Substituting Eq. (3.13) into Eqs. (3.6) and (3.7) gives

$$p = \frac{kT (\ln z)^{3/2}}{\lambda \Gamma (5/2)} \left[ 1 + \frac{3}{4} \frac{\zeta (2)}{(\ln z)^2} \right], \quad (3.14)$$

$$\frac{N}{V} = \frac{1}{\lambda \Gamma (5/2)} \left[ 1 - \frac{1}{4} \frac{\zeta (2)}{(\ln z)^2} \right]. \quad (3.15)$$

The fugacity $z$ can be solved from Eq. (3.15):

$$\ln z \sim \frac{\epsilon_f}{kT} \left[ 1 + \frac{1}{2} \zeta (2) \left( \frac{kT}{\epsilon_f} \right)^2 \right], \quad (3.16)$$

where $\epsilon_f = \lambda^2 kT \left( \frac{1}{\lambda \Gamma (\frac{5}{2})} n \right)^2$ is the Fermi energy [41]. By substituting Eq. (3.13) into Eq. (3.11) with fugacity $z$ given by Eq. (3.16), we have

$$\frac{f_{3/2} (z)}{f_{1/2} (z)} = \frac{\Gamma (3/2) \epsilon_f}{\Gamma (5/2) kT} \left\{ 1 + \left[ \frac{\zeta (2)}{2} + \zeta (2) \right] \left( \frac{kT}{\epsilon_f} \right)^2 \right\}. \quad (3.17)$$
Substituting Eq. (3.17) into Eq. (3.5) gives the MSD of Fermi cases at low temperatures:
\[
\langle x_t^2 \rangle \sim \frac{1}{3\pi \eta r} \left[ \frac{1}{\Gamma (5/2)} \Gamma (3/2) \right] \left[ 1 + \frac{3}{2} \frac{\zeta (2)}{\epsilon_f} \left( \frac{kT}{\epsilon_f} \right)^2 \right].
\] (3.18)

The first term of Eq. (3.18) is independent of the temperature \(T\), which means that there is a random motion of the Brownian particle due to the fermionic exchange interaction even the temperature is near the absolute zero. It is a result of Pauli exclusion principle [41].

**The low-temperature expansion for Bose cases.** At low temperatures, for Bose cases, \(g = 1\),
\[
\frac{h_{1+d/2} \left( z \right)}{h_{d/2} \left( z \right)} = \frac{g_{1+d/2} \left( z \right)}{g_{d/2} \left( z \right)}.
\] (3.19)

Expanding \(g_{\nu} \left( z \right)\) around \(z = 1\) gives [41]
\[
g_{\nu} \left( z \right) = \frac{\Gamma \left( 1 - \nu \right)}{\left( -\ln z \right)^{1-\nu}} + \sum_{j=0}^{\infty} \frac{\left(-1\right)^j}{j!} \left( \nu - j \right) \left( -\ln z \right)^j.
\] (3.20)

Substituting Eq. (3.20) into Eqs. (3.6) and (3.7) gives
\[
p = \frac{1}{\lambda^d} \Gamma \left( - \frac{1}{2} \right) \left( -\ln z \right)^{1/2} + \zeta \left( \frac{3}{2} \right) - \zeta \left( \frac{1}{2} \right) \left( -\ln z \right) ,
\] (3.21)
\[
\frac{N}{V} = \frac{1}{\lambda^d} \Gamma \left( 1/2 \right) + \zeta \left( \frac{1}{2} \right) - \zeta \left( \frac{1}{2} \right) \left( -\ln z \right).
\] (3.22)

The fugacity can be solved from Eq. (3.22):
\[
\ln z = -\frac{\pi}{n^2 \lambda^2}.
\] (3.23)

By substituting Eq. (3.20) into Eq. (3.19) with fugacity \(z\) given by Eq. (3.23), we have
\[
\frac{g_{3/2} \left( z \right)}{g_{1/2} \left( z \right)} = \frac{\zeta \left( 3/2 \right)}{\sqrt{n^2 \lambda^2}} = \left( 2 + \frac{\zeta \left( 3/2 \right) \zeta \left( 1/2 \right)}{\pi} \right) \frac{n}{n^2 \lambda^2}.
\] (3.24)

Substituting Eq. (3.24) into Eq. (3.5) gives the MSD of Bose cases at low temperatures:
\[
\langle x_t^2 \rangle \sim \frac{kT}{3\pi \eta r^t} \left[ \frac{3}{2} \frac{1}{n \lambda} - \left( 2\pi + \zeta \left( 3/2 \right) \zeta \left( 1/2 \right) \right) \frac{1}{n^2 \lambda^2} \right].
\] (3.25)
\[
\sim \frac{1}{3\pi \eta r^t} \zeta \left( \frac{3}{2} \right) \frac{\sqrt{2\pi \eta m}}{\hbar} \frac{1}{n} \left( kT \right)^{3/2} t.
\] (3.26)

The MSD is proportional to \(T^{3/2}\) and is reversely proportional to the density of particle, which is also different from that of the Brownian motion.
3.4 The \( d \)-dimensional case

In this section, a similar procedure gives the MSD of a Brownian particle in a \( d \)-dimensional space. For the sake of clarity, we list the result. The detail of the calculation can be found in the Appendix.

The MSD. The MSD for a Brownian particle in an ideal quantum gas in a \( d \)-dimensional space is

\[
\langle x_t^2 \rangle = \frac{kT d}{3\pi \eta r} \frac{h_{1+d/2}(z)}{h_{d/2}(z)}. \tag{3.27}
\]

The high-temperature expansion. At high temperatures, the MSD, Eq. (3.27), becomes

\[
\langle x_t^2 \rangle = \frac{kT d}{3\pi \eta r} t \left[ 1 + ga_1(T) n\lambda^d + \ldots \right], \tag{3.28}
\]

where \( a_1(T) = \frac{1}{2^1 \pi^1} \) and is the first virial coefficient of ideal Bose or Fermi gases in a \( d \)-dimensional space \([40, 41]\).

The low-temperature expansion for Fermi cases. At low temperatures, for Fermi cases, the MSD, Eq. (3.27), becomes

\[
\langle x_t^2 \rangle \sim \frac{d}{3\pi \eta r} \frac{\Gamma(1+d/2)}{\Gamma(2+d/2)} t \epsilon_f \left[ 1 + \left( \frac{d}{2} + 1 \right) \zeta(2) \left( \frac{kT}{\epsilon_f} \right)^2 \right], \tag{3.29}
\]

where \( \epsilon_f \) is the Fermi energy in a \( d \)-dimensional space and \( \epsilon_f = \lambda^2 kT \left( \frac{1}{4} \Gamma(1 + \frac{d}{2}) n \right)^{2/d} \) \([40, 41]\).

The low-temperature expansion for Bose cases with \( d = 2 \). The low-temperature expansion for a Bose gas at any dimension higher than 2 is not given in this section, because the Bose-Einstein condensation (BEC) occurs. Here, we only consider the 2-dimensional case. At low temperatures, for Bose cases, the MSD, Eq. (3.27), becomes

\[
\langle x_t^2 \rangle = \frac{2kT}{3\pi \eta r} t \left[ -\exp(-n\lambda^2) - \frac{\exp(-n\lambda^2)}{n\lambda^2} + \frac{\pi^2}{6n\lambda^2} \right]. \tag{3.30}
\]

4 Conclusions and outlooks

The difficulty in the calculation of the behavior of a Brownian particle in an ideal quantum gas directly comes from the stochastic force caused by the Bose-Einstein and Fermi-Dirac distribution other than the Maxwell-Boltzmann distribution. Comparison with the classical Brownian motion, on one hand, the distribution of the stochastic force is different; on the other hand, the collision, due to the overlapping of the wave package, could be correlated, that is, \( \langle F_s F_t \rangle \) is no longer a delta function but a function of \( s - t \) with a peak at \( s = t \). Thus, it is difficult to make exact or even detailed dynamical calculations \([1, 8]\).

In this paper, we consider the motion of a Brownian particle in an ideal quantum gas. We give an explicit expression of the MSD, which depends on the thermal wavelength and the density of medium particles. High-temperature and low-temperature expansions explain
the quantum effect intuitively. For examples, the MSD in an ideal Bose gas is shorter than that in a Ferm gas. There is a random motion of the Brownian particle due to the fermionic exchange interaction even the temperature is near the absolute zero.

The result in this work can be verified by experiment test.

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6 Appendix

In the appendix, we give the detail of the calculation of Eqs (3.27), (3.28), (3.29), and (3.30).

The detail of the calculate for the MSD, Eq. (3.27), of a Brownian particle in a d-dimensional space. The Langevin equation in d-dimensional is

$$M \frac{dv}{dt} = -\gamma v + F_t,$$  \hspace{1cm} (6.1)

$$\frac{dx}{dt} = v.$$  \hspace{1cm} (6.2)

In a d-dimensional space, the stochastic force \( F_t \) is isotropic:

$$\langle F \rangle = 0.$$  \hspace{1cm} (6.3)

For different time \( t \) and \( s \), \( F_s \) and \( F_t \) are almost independent when the ratio of the thermal wavelength and the average distance between the medium particles is small, that is,

$$\langle F_s \cdot F_t \rangle \sim \delta (s - t)$$  \hspace{1cm} (6.4)

holds for \( n\lambda^d \ll 1 \). The solution of Eqs. (6.1) and (6.2) is

$$v_t = v_0 e^{-\gamma t} + \frac{1}{M} \int_0^t \exp \left[ -\frac{\gamma}{M} (t - s) \right] F_s ds,$$  \hspace{1cm} (6.5)

$$x_t = x_0 + \frac{M}{\gamma} v_0 \left[ 1 - \exp \left( -\frac{\gamma}{M} t \right) \right] + \frac{1}{\gamma} \int_0^t \left[ 1 - \exp \left[ -\frac{\gamma}{M} (t - s) \right] \right] F_s ds.$$  \hspace{1cm} (6.6)

In the d-dimensional case, the number of particle possessing momentum within \( P \) to \( P + dP \), denoted by \( a(P) \), is [40, 41]

$$a(P) = \frac{1}{\exp \left[ \frac{\beta}{2} \left( p_1^2 + p_2^2 + ... p_d^2 \right) / (2m) + \alpha \right] + g}.$$  \hspace{1cm} (6.7)

The force given by a collision of a particle with momentum \( P \), \( F = \rho P \). Thus the probability of the stochastic force with magnitude within \( |F| \) to \( |F + dF| \) is

$$P(F) dF = \left( \frac{\beta}{2\pi m \rho^2} \right)^d \frac{1}{\hbar d/2(z)} \exp \left[ \frac{\beta}{2m \rho^2} \frac{1}{(2m \rho^2) + \alpha} \right] + g.$$  \hspace{1cm} (6.8)
Substituting Eq. (6.8) into Eq. (6.4) gives

\[
\langle \mathbf{F}_s \cdot \mathbf{F}_t \rangle \sim \frac{dm \rho^2 h_{1+d/2}(z)}{\beta h_{d/2}(z)} \delta(s-t). \quad (6.9)
\]

By using Eqs. (6.6), (6.8), and (6.9), a direct calculation of MSD gives

\[
\langle x_t^2 \rangle = M^2 \gamma^2 \left[ v_0^2 - \frac{d}{2m \gamma} \frac{m \rho^2 h_{1+d/2}(z)}{h_{d/2}(z)} \right] \left[ 1 - \exp \left( -\frac{\gamma M t}{M} \right) \right]^2 \nonumber \\
+ \frac{1}{\gamma^2} \frac{dm \rho^2 h_{1+d/2}(z)}{h_{d/2}(z)} \left[ t - \frac{M \gamma}{\gamma} \left[ 1 - \exp \left( -\frac{\gamma M t}{M} \right) \right] \right]. \quad (6.10)
\]

where \( x_0 \) is chosen to be the origin.

For a large-scale time, \( t \gg 1 \), Eq. (6.10) recovers Eq. (3.27).

**The high-temperature expansion.** For the \( d \)-dimensional case, the state equation of an ideal quantum gas is \([40, 41]\)

\[
p = \frac{kT}{\lambda^d} h_{1+d/2}(z), \quad (6.11)
\]

\[
\frac{N}{V} = \frac{1}{\lambda^d} h_{d/2}(z). \quad (6.12)
\]

The virial expansion \([40, 41]\) directly gives

\[
\frac{pV}{N} \sim kT \left[ 1 + ga_1(T) n \lambda^d + ... \right]. \quad (6.13)
\]

Substituting Eqs. (6.11) and (6.12) into Eq. (6.13) gives

\[
\frac{h_{1+d/2}(z)}{h_{d/2}(z)} \sim \left[ 1 + ga_1(T) n \lambda^d + ... \right]. \quad (6.14)
\]

Substituting Eq. (6.14) into Eq. (3.27) gives Eq. (3.28).

**The low-temperature expansion for Fermi cases.** For Fermi cases, \( g = -1 \),

\[
\frac{h_{1+d/2}(z)}{h_{d/2}(z)} = \frac{f_{1+d/2}(z)}{f_{d/2}(z)}. \quad (6.15)
\]

By the expansion of the Fermi-Dirac integral at large \( z \), we have \([41]\)

\[
f_{\nu} (e^z) = \frac{\zeta(\nu)}{\Gamma (1+\nu)} \left\{ 1 + 2\nu \sum_{j=1,3,5,...} \left[ (\nu - 1) \ldots (\nu - j) \left( 1 - 2^{-j} \right) \frac{\zeta(j+1)}{\xi_{j+1}} \right] \right\}. \quad (6.16)
\]

Keeping only the first two terms gives

\[
f_{\nu} (z) = \frac{(\ln z)^\nu}{\Gamma (1+\nu)} + 2\nu (\nu - 1) \frac{\zeta(2)}{2(\ln z)^2}. \quad (6.17)
\]
Substituting Eq. (6.17) into Eqs. (6.11) and (6.12) gives

\[
\frac{p}{kT} = \frac{1}{\lambda \Gamma(2 + d/2)} \left[ 1 + \frac{d}{2} \left( 1 + \frac{d}{2} \right) \frac{\zeta(2)}{(\ln z)^2} \right], \quad (6.18)
\]

\[
N = \frac{\Omega}{\lambda \Gamma(1 + d/2)} \left[ 1 + \frac{d}{2} \left( \frac{d}{2} - 1 \right) \frac{\zeta(2)}{(\ln z)^2} \right], \quad (6.19)
\]

where \( \Omega \) is the volume. The fugacity can be solved from Eq. (6.19):

\[
\ln z \sim \frac{\epsilon_f}{kT} \left[ 1 - \zeta(2) \left( \frac{d}{2} - 1 \right) \left( \frac{kT}{\epsilon_f} \right)^2 \right], \quad (6.20)
\]

where \( \epsilon_f = \lambda^2 kT \left[ \frac{1}{2} \Gamma(1 + \frac{d}{2}) \right]^{2/d} \) is the Fermi energy. By substituting Eq. (6.17) into Eq. (6.15) with fugacity \( z \) given by Eq. (6.20), we have

\[
\frac{f_{1+d/2}(z)}{f_d/2(z)} = \frac{\Gamma \left( 1 + \frac{d}{2} \right) \epsilon_f}{\Gamma \left( 2 + \frac{d}{2} \right) kT} \left\{ 1 + \left[ \frac{d\zeta(2)}{2} + \zeta(2) \right] \left( \frac{kT}{\epsilon_f} \right)^2 \right\}. \quad (6.21)
\]

Substituting Eq. (6.21) into Eq. (3.27) gives Eq. (3.29).

The low-temperature expansion for Bose cases in the 2-dimensional space. For Bose cases, \( g = 1 \),

\[
\frac{h_2(z)}{h_1(z)} = \frac{g_2(z)}{g_1(z)}, \quad (6.22)
\]

where \( d = 2 \). For \( d = 2 \),

\[
g_1(z) = - \ln (1 - z). \quad (6.23)
\]

Substituting Eq. (6.23) into Eq. (6.12) gives

\[
\frac{N}{V} = - \frac{1}{\lambda^2} \ln (1 - z). \quad (6.24)
\]

Then, the fugacity can be solved from Eq (6.24):

\[
z = 1 - \exp \left( -n\lambda^2 \right). \quad (6.25)
\]

Expanding \( g_2(z) \) around \( z = 1 \) gives

\[
g_2(z) = \frac{\pi^2}{6} - (1 - z) - \frac{(1 - z)^2}{2^2} - \frac{(1 - z)^3}{3^2} - \ldots + (1 - z) \ln (1 - z) + \frac{(1 - z)^2}{2} \ln (1 - z) + \frac{(1 - z)^3}{3} \ln (1 - z) + \ldots \quad (6.26)
\]

Substituting Eqs. (6.26) and (6.23) with fugacity given in Eq. (6.25) into Eq. (6.22) gives

\[
\frac{g_2(z)}{g_1(z)} = - \exp \left( -n\lambda^2 \right) - \frac{\exp \left( -n\lambda^2 \right)}{n\lambda^2} + \frac{\pi^2}{6n\lambda^2}. \quad (6.27)
\]

Substituting Eq. (6.27) into Eq. (3.27) gives Eq. (3.30).
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