RESTRICTIONS OF BROWNIAN MOTION

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Abstract. Let \( \{B(t) : 0 \leq t \leq 1\} \) be a linear Brownian motion and let \( \dim \) denote the Hausdorff dimension. Let \( \alpha > \frac{1}{2} \) and \( 1 \leq \beta \leq 2 \). We prove that, almost surely, there exists no set \( A \subset [0, 1] \) such that \( \dim A > \frac{1}{2} \) and \( B: A \to \mathbb{R} \) is \( \alpha \)-Hölder continuous. The proof is an application of Kaufman's dimension doubling theorem. As a corollary of the above theorem, we show that, almost surely, there exists no set \( A \subset [0, 1] \) such that \( \dim A > \frac{\beta}{2} \) and \( B: A \to \mathbb{R} \) has finite \( \beta \)-variation. The zero set of \( B \) and a deterministic construction witness that the above theorems give the optimal dimensions.

1. Introduction

This paper was motivated by questions of Kahane and Katznelson [5] about restrictions of Hölder continuous functions. For related restriction theorems for non-random functions see also papers of Máté [2] and Elekes [9].

We examine how large a set can be, on which linear Brownian motion is \( \alpha \)-Hölder continuous for some \( \alpha > \frac{1}{2} \) or has finite \( \beta \)-variation for some \( 1 \leq \beta \leq 2 \). The main goal of the paper is to prove the following two theorems.

**Theorem 1.1.** Let \( \{B(t) : 0 \leq t \leq 1\} \) be a linear Brownian motion and assume that \( \alpha > \frac{1}{2} \). Then, almost surely, there exists no set \( A \subset [0, 1] \) with \( \dim A > \frac{1}{2} \) such that \( B: A \to \mathbb{R} \) is \( \alpha \)-Hölder continuous.

**Theorem 1.2.** Let \( \{B(t) : 0 \leq t \leq 1\} \) be a linear Brownian motion and assume that \( 1 \leq \beta \leq 2 \). Then, almost surely, there exists no set \( A \subset [0, 1] \) with \( \dim A > \frac{\beta}{2} \) such that \( B: A \to \mathbb{R} \) has finite \( \beta \)-variation.

Clearly, the above theorems hold simultaneously for a countable dense set of parameters \( \alpha, \beta \), thus simultaneously for all \( \alpha, \beta \). Let \( Z \) be the zero set of a linear Brownian motion \( B \). Then, almost surely, \( \dim Z = \frac{1}{2} \) and \( B|_Z \) is \( \alpha \)-Hölder continuous for all \( \alpha > \frac{1}{2} \), so Theorem [1.1] gives the optimal dimension. We prove also that Theorem [1.2] is best possible, see Theorem [4.3]

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2. Preliminaries

The diameter of a metric space $X$ is denoted by $\text{diam} X$. For all $s \geq 0$ the $s$-dimensional Hausdorff measure of $X$ is defined as

$$\mathcal{H}^s(X) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(X),$$

where

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^\infty (\text{diam} X_i)^s : X \subset \bigcup_{i=1}^\infty X_i, \forall i \text{ diam } X_i \leq \delta \right\}.$$ 

The Hausdorff dimension of $X$ is defined as

$$\dim X = \inf \{ s \geq 0 : \mathcal{H}^s(X) < \infty \}. $$

Let $A \subset \mathbb{R}$ and $\alpha, \beta > 0$. A function $f : A \to \mathbb{R}$ is called $\alpha$-Hölder continuous if there exists a constant $c \in (0, \infty)$ such that $|f(x) - f(y)| \leq c|x-y|^\alpha$ for all $x, y \in A$. The $\beta$-variation of a function $f : A \to \mathbb{R}$ is defined as

$$\text{Var}^\beta(f) = \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^\beta : x_0 < \cdots < x_n, x_i \in A, n \in \mathbb{N}^+ \right\}. $$

Fact 2.1. If $f : A \to \mathbb{R}$ is $\alpha$-Hölder continuous then $\dim f(A) \leq \frac{1}{\alpha} \dim A$.

3. Hölder Restrictions

The goal of this section is to prove Theorem 1.1. First we need some preparation.

Definition 3.1. A function $g : [0, 1] \to \mathbb{R}^2$ is called dimension doubling if

$$\dim g(A) = 2 \dim A \quad \text{for all } A \subset [0, 1].$$

The following result is Kaufman’s dimension doubling theorem.

Theorem 3.2 (Kaufman, [7]). The two-dimensional Brownian motion is almost surely dimension doubling.

The following theorem follows from [4] Lemma 2 together with the fact that the closed range of the stable subordinator with parameter $\frac{1}{2}$ coincides with the zero set of a linear Brownian motion. For a more direct reference see [6].

Theorem 3.3. Let $A \subset [0, 1]$ be a compact set with $\dim A > \frac{1}{2}$ and let $\mathcal{Z}$ be the zero set of a linear Brownian motion. Then $\dim (A \cap \mathcal{Z}) > 0$ with positive probability.

Lemma 3.4 (Key Lemma). Let $\{W(t) : 0 \leq t \leq 1\}$ be a linear Brownian motion. Assume that $\alpha > \frac{1}{2}$ and $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $(f, W)$ is almost surely dimension doubling. Then there is no set $A \subset [0, 1]$ such that $\dim A > \frac{1}{2}$ and $f$ is $\alpha$-Hölder continuous on $A$.

Proof. Assume to the contrary that there is a set $A \subset [0, 1]$ such that $\dim A > \frac{1}{2}$ and $f$ is $\alpha$-Hölder continuous on $A$. As $f$ is still $\alpha$-Hölder continuous on the closure of $A$, we may assume that $A$ itself is closed. Let $\mathcal{Z}$ be the zero set of $W$, then Theorem 3.3 implies that $\dim (A \cap \mathcal{Z}) > 0$ with positive probability. Then the $\alpha$-Hölder continuity of $f|_A$ and Fact 2.1 imply that, with positive probability,

$$\dim (f, W)(A \cap \mathcal{Z}) = \dim (f(A \cap \mathcal{Z}) \times \{0\}) = \dim f(A \cap \mathcal{Z}) \leq \frac{1}{\alpha} \dim (A \cap \mathcal{Z}) < 2 \dim (A \cap \mathcal{Z}),$$

which contradicts the fact that $(f, W)$ is almost surely dimension doubling. \qed
Proof of Theorem 1.1. Let \( \{W(t) : 0 \leq t \leq 1\} \) be a linear Brownian motion which is independent of \( B \). By Kaufman’s dimension doubling theorem \((B, W)\) is dimension doubling with probability one, thus applying Lemma 3.4 for an almost sure path of \( B \) finishes the proof. \( \square \)

4. Restrictions of bounded variation

We need the following lemma, which may be obtained by a slight modification of [11, Lemma 4.1]. For the reader’s convenience we outline the proof.

Lemma 4.1. Let \( \alpha, \beta > 0 \). Assume that \( A \subset [0, 1] \) and the function \( f : A \to \mathbb{R} \) has finite \( \beta \)-variation. Then there are sets \( A_n \subset A \) such that for any \( n \in \mathbb{N}^+ \)

\[
\text{dim} \left( A \setminus \bigcup_{n=1}^{\infty} A_n \right) \leq \alpha \beta.
\]

Proof. For all \( n \in \mathbb{N}^+ \) let

\[
A_n = \{ x \in A : |f(x + t) - f(x)| \leq 2t^\alpha \text{ for all } t \in [0, 1/n] \cap (A - x) \}.
\]

As \( A \) is bounded, \( f|_{A_n} \) is \( \alpha \)-Hölder continuous for all \( n \in \mathbb{N}^+ \). Let

\[
D = \left\{ x \in A : \limsup_{t \to 0^+} |f(x + t) - f(x)|t^{-\alpha} > 1 \right\}.
\]

Clearly \( A \setminus \bigcup_{n=1}^{\infty} A_n \subset D \), so it is enough to prove that \( \text{dim} D \leq \alpha \beta \). Let us fix \( \delta > 0 \) arbitrarily. Then for all \( x \in D \) there is a \( 0 < t_x < \delta \) such that

\[
|f(x + t_x) - f(x)| \geq t_x^\beta.
\]

Define \( I_x = [x - t_x, x + t_x] \) for all \( x \in D \). By Besicovitch’s covering theorem (see [10, Thm. 2.7]) there is a number \( p \in \mathbb{N}^+ \) not depending on \( \delta \) and countable sets \( S_i \subset D \) \( (i \in \{1, \ldots, p\}) \) such that

\[
D \subset \bigcup_{i=1}^{p} \bigcup_{x \in S_i} I_x \text{ and } I_x \cap I_y = \emptyset \text{ for all } x, y \in S_i, \ x \neq y.
\]

Applying (4.1) and (4.2) implies that for all \( i \in \{1, \ldots, p\} \)

\[
\sum_{x \in S_i} |I_x|^\alpha \beta = 2^\alpha \sum_{x \in S_i} t_x^\beta \leq 2^\alpha \beta \sum_{x \in S_i} |f(x + t_x) - f(x)|^\beta \leq 2^\alpha \beta \text{ Var}^\beta (f).
\]

Equations (4.2) and (4.3) imply that

\[
\mathcal{H}^\alpha \beta (D) \leq \sum_{i=1}^{p} \sum_{x \in S_i} |I_x|^\alpha \beta \leq p2^\alpha \beta \text{ Var}^\beta (f).
\]

As \( \text{Var}^\beta (f) < \infty \) and \( \delta > 0 \) was arbitrary, we obtain that \( \mathcal{H}^\alpha \beta (D) < \infty \). Hence \( \text{dim} D \leq \alpha \beta \), and the proof is complete. \( \square \)

Proof of Theorem 1.2. Assume to the contrary that for some \( \varepsilon > 0 \), with positive probability, there is a set \( A \subset [0, 1] \) such that \( \text{dim} A \geq \beta/2 + 2\varepsilon \) and \( B|_A \) has finite \( \beta \)-variation. Let \( \alpha = 1/2 + \varepsilon/\beta > 1/2 \). Applying Lemma 4.1 we obtain that there are sets \( A_n \subset A \) such that \( f|_{A_n} \) is \( \alpha \)-Hölder continuous for every \( n \in \mathbb{N}^+ \) and

\[
\text{dim} \left( A \setminus \bigcup_{n=1}^{\infty} A_n \right) \leq \alpha \beta = \frac{\beta}{2} + \varepsilon.
\]
As \( \alpha > 1/2 \) and \( f|_{A_n} \) is \( \alpha \)-Hölder continuous, Theorem 1.1 implies that almost surely \( \dim A_n \leq 1/2 \) for all \( n \in \mathbb{N}^+ \), therefore (1.4) and the countable stability of the Hausdorff dimension yield that \( \dim A \leq \beta/2 + \varepsilon \). This is a contradiction, and the proof is complete.

The following two theorems imply that Theorem 1.2 is sharp for all \( \beta \).

**Theorem 4.2** (Lévy’s modulus of continuity, [8]). For the linear Brownian motion \( \{B(t) : 0 \leq t \leq 1\} \), almost surely,

\[
\limsup_{h \to 0^+} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h)-B(t)|}{\sqrt{2h \log(1/h)}} = 1.
\]

**Theorem 4.3.** Let \( 1 \leq \beta \leq 2 \) be fixed. Then there is a compact set \( A \subset [0,1] \) such that \( \dim A = \frac{\beta}{2} \) and if \( f : [0,1] \to \mathbb{R} \) is a function such that for all \( x, y \in [0,1] \)

\[
|f(x) - f(y)| \leq c|x-y|^\beta \log \frac{1}{|x-y|}
\]

with some \( c \in (0, \infty) \), then \( f|_A \) has finite \( \beta \)-variation.

**Proof.** Let \( \beta \in [1,2] \) be fixed, first we construct \( A \). For all \( n \in \mathbb{N} \) let

\[
\gamma_n = 2^{-2n/\beta}(n+1)^{-6}.
\]

As \( \beta \leq 2 \), for all \( n \in \mathbb{N} \) we have

\[
\gamma_{n+1} < \frac{\gamma_n}{2}.
\]

For all \( n \in \mathbb{N} \) and \( \{i_1, \ldots, i_n\} \in \{0,1\}^n \) we define intervals \( I_{i_1 \ldots i_n} \subset [0,1] \) by induction. We use the convention \( \{0,1\}^0 = \{\emptyset\} \) and let \( I_\emptyset = [0,1] \). If the interval \( I_{i_1 \ldots i_n} = [u,v] \) is already defined then let

\[
I_{i_1 \ldots i_n, 0} = [u, u + \gamma_{n+1}] \quad \text{and} \quad I_{i_1 \ldots i_n, 1} = [v - \gamma_{n+1}, v].
\]

By (1.4) and the construction for all \( n \in \mathbb{N} \) and \( (i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \{0,1\}^n \)

(i) \( \text{diam } I_{i_1 \ldots i_n} = \gamma_n \),

(ii) \( I_{i_1 \ldots i_n} \cap I_{j_1 \ldots j_n} = \emptyset \) if \( (i_1, \ldots, i_n) \neq (j_1, \ldots, j_n) \),

(iii) \( I_{i_1 \ldots i_n, 0} \subset I_{i_1 \ldots i_n} \).

Let us define

\[
A = \bigcap_{n=0}^\infty \bigcup_{(i_1, \ldots, i_n) \in \{0,1\}^n} I_{i_1 \ldots i_n}.
\]

Let \( f : [0,1] \to \mathbb{R} \) be a function satisfying (1.5). Now we prove that \( \text{Var}^\beta(f|_A) < \infty \). Inequality (1.5), \( f \), the definition of \( \gamma_n \) and \( \beta \geq 1 \) imply that for all \( n \in \mathbb{N} \) and \( (i_1, \ldots, i_n) \in \{0,1\}^n \) we have

\[
(\text{diam } f(I_{i_1 \ldots i_n}))^\beta \leq (c_\beta n^{1/2} \log \gamma_n)^\beta \leq c_\beta 2^{-n}(n+1)^{-2},
\]

where \( c_\beta \in (0, \infty) \) is a constant depending on \( c \) and \( \beta \) only. For all \( x, y \in A \) let \( n(x,y) \) be the maximal number \( n \) such that \( x, y \in I_{i_1 \ldots i_n} \) for some \( (i_1, \ldots, i_n) \in \{0,1\}^n \). If \( \{x_k\}_{k=0}^\infty \) is a monotone sequence in \( A \) and \( n \in \mathbb{N} \) then the number of \( i \in \{1, \ldots, k\} \) such that \( n(x_{i-1}, x_i) = n \) is at most \( 2^n \). Therefore (4.7) implies that

\[
\text{Var}^\beta(f|_A) \leq \sum_{n=0}^\infty 2^n (c_\beta 2^{-n}(n+1)^{-2}) = \sum_{n=1}^\infty c_\beta n^{-2} < \infty.
\]
Finally, we prove \( \dim A = \beta/2 \). Then \( \dim A \leq \beta/2 \) is obvious, for the lower bound let \( \mu \) be the Borel measure on \( A \) such that for all \( n \in \mathbb{N} \) and \( (i_1, \ldots, i_n) \in \{0, 1\}^n \)
\[
\mu(I_{i_1 \ldots i_n}) = 2^{-n}.
\] (4.8)

Note that by (ii) and (iii) the measure \( \mu \) is well-defined on the algebra generated by the sets \( A \cap I_{i_1 \ldots i_n} \), so it can be uniquely extended to the generated \( \sigma \)-algebra by Carathéodory’s extension theorem [3, Thm. A, p. 54.]. Let us fix \( \varepsilon > 0 \), then clearly there is an \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have
\[
\gamma_n^{\beta/2 - \varepsilon} \geq 2^{-n}.
\] (4.9)

Let \( C \subset A \) be any set with \( \operatorname{diam} C \leq \gamma_N \). We can choose \( n \geq N \) such that
\[
\gamma_{n+1} < \operatorname{diam} C \leq \gamma_n.
\] (4.10)

Property (i) implies that \( C \) can intersect at most two \( n \)th level intervals \( I_{i_1 \ldots i_n} \), therefore (4.8), (4.9), and (4.10) yield that
\[
\mu(C) \leq 2^{-n+1} \leq 4^{\gamma_n^{\beta/2 - \varepsilon}} \leq 4(\operatorname{diam} C)^{\beta/2 - \varepsilon}.
\]

Hence the mass distribution principle [11, Thm. 4.19] implies that \( \dim A \geq \beta/2 - \varepsilon \). As \( \varepsilon > 0 \) was arbitrary, we obtain that \( \dim A \geq \beta/2 \). The proof is complete. \( \square \)

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