ON THE ALEXANDROFF-BORSUK PROBLEM

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Abstract. We investigate the classical Alexandroff-Borsuk problem in the category of non-triangulable manifolds: Given an $n$-dimensional compact non-triangulable manifold $M^n$ and $\varepsilon > 0$, does there exist an $\varepsilon$-map of $M^n$ onto an $n$-dimensional finite polyhedron which induces a homotopy equivalence?

1. Introduction

In 1928 Alexandroff [1] proved the following important theorem:

Alexandroff Theorem. Every $n$-dimensional compact metric space $X$ has the following properties:

- for every $\varepsilon > 0$, $X$ admits an $\varepsilon$-map $f : X^n \to P^n$ onto some $n$-dimensional finite polyhedron $P^n$; and
- for some $\mu > 0$, $X$ does not admit any $\mu$-map $g : X^n \to Q^k$ of $X^n$ onto a polyhedron $Q^k$ of dimension $k < n$.

Recall that for a metric space $X$ and $\varepsilon > 0$, a continuous map $f : X \to P$ is called an $\varepsilon$-map if the preimage $f^{-1}(p)$ of every point $p \in P$ has diameter $< \varepsilon$.

The condition of compactness in Alexandroff’s theorem above is essential since in 1953 Sitnikov [8, 25] constructed an example of a 2-dimensional subspace of $R^3$ which can be $\varepsilon$-mapped onto a 1-dimensional polyhedron for arbitrarily small $\varepsilon > 0$.

In 1954 Borsuk [2] asked whether every compact absolute neighborhood retract (ANR) is homotopy equivalent to a finite polyhedron. This difficult question was answered in the affirmative in 1977 by West [27].

It follows by Wall’s obstruction theory [26] that for every $n > 2$, every $n$-dimensional compact ANR is homotopy equivalent to an $n$-dimensional polyhedron, having the structure of a finite simplicial complex (cf. [27]).

The following natural problem has been opened for a very long time (compare with [17]):

Alexandroff-Borsuk ANR Problem. Given any compact $n$-dimensional absolute neighborhood retract $X^n$ and any $\varepsilon > 0$, does there exist an $\varepsilon$-map $f : X^n \to P^n$ of $X^n$ onto a finite $n$-dimensional polyhedron $P^n$ which is a homotopy equivalence?

In this paper we shall consider the Alexandroff-Borsuk problem for the category of non-triangulable manifolds. Since every topological $n$-manifold is a separable metric locally Euclidean space of dimension $n$ (cf. [19]), it is a locally contractible finite-dimensional space and therefore it is an ANR (cf. [3]).

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It follows by the West Theorem mentioned above that every compact manifold has the homotopy type of a finite polyhedron. This fact was first proved in 1969 by Kirby and Siebenmann [18]. So we have the following natural special case of the Alexandroff-Borsuk problem:

**Alexandroff-Borsuk Manifold Problem.** Given a compact \( n \)-dimensional manifold \( M^n \) and \( \varepsilon > 0 \), does there exist an \( \varepsilon \)-map of \( M^n \) onto a finite \( n \)-dimensional polyhedron \( P^n \) which is a homotopy equivalence?

Recall that for every \( n \geq 4 \), there exists a closed \( n \)-dimensional manifold which is not a polyhedron. Such manifolds were first constructed by Freedman [10] in dimension 4, by Galewski and Stern [12] in dimension 5, and by Manolescu [20] in dimensions \( n \geq 6 \).

The following is the main result of our paper:

**Main Theorem.** The Alexandroff-Borsuk Manifold Problem has an affirmative solution for the non-triangulable closed manifolds of Freedman, Galewski and Stern, and Manolescu.

2. Preliminaries

We shall work with the categories of separable metric spaces, CW complexes and continuous maps. In these categories all classical definitions of dimension coincide: \( \text{dim } X = \text{ind } X = \text{Ind } X \) (cf. [8]).

We list some definitions and theorems which we shall need in the sequel:

**Theorem 2.1.** (Cellular Approximation Theorem [28, p. 77]). Let \( (X, A) \) and \( (Y, B) \) be relative CW complexes and let \( f : (X, A) \rightarrow (Y, B) \) be a continuous map. Then \( f \) is homotopic (rel \( A \)) to a cellular map.

Recall that a map \( f : X \rightarrow Y \) between CW complexes \( X \) and \( Y \) is said to be cellular if \( f(X_n) \subset Y_n \) for every \( n \). The Simplicial Approximation Theorem is a special case of the Cellular Approximation Theorem, cf. [28, p. 76] for details.

**Definition 2.2.** A manifold \( M \) is said to be almost-smooth if it admits a smooth structure in the complement \( M \setminus \{p\} \) of any point \( p \in M \).

**Theorem 2.3.** (Cairns-Whitehead [4, 5, 16, 29]). Every smooth manifold can be given a simplicial structure.

Freedman has proved the following important theorem (cf. [10, Corollary 1.6]):

**Theorem 2.4.** There is a closed connected almost-smooth 4-manifold \( |E_8| \) with the intersection matrix \( E_8 \).

Freedman also established the following surprising fact:

**Theorem 2.5.** Either \( |E_8| \) is the first example in any dimension of a manifold which is not homeomorphic to a polyhedron, or the 3-dimensional Poincaré conjecture is false.

Since Perel’man has proved the Poincaré conjecture (cf. e.g. [21]), it follows that the 4-manifold \( |E_8| \) is not homeomorphic to any polyhedron. However, the complement of any point in \( |E_8| \) can be given a polyhedral structure, since it admits a smooth structure on the complement of any point (cf. [11, Theorem on p. 116]) and by the Cairns-Whitehead Theorem 2.3 every smooth manifold is triangulable.
We briefly recall the Kirby-Siebenmann and the Galewski-Stern obstructions and some of their theorems (cf. [13, 22]).

**Definition 2.6.** (cf. [22, pp. xii-xiii, 78]). \(BTOP\) is the classifying space for stable topological bundles and \(\chi\) is some element of the 4-dimensional cohomology group \(H^4(BTOP; \mathbb{Z}_2)\) which is called the universal Kirby-Siebenmann class.

The construction of the CW complex \(BTOP\) and the element \(\chi\) is given e.g. in [22, pp. xii-xiii, 78].

**Definition 2.7.** (cf. e.g. [9, pp. 403-404]). Let \(M^n\) be a topological manifold. Then the topological tangent bundle \(\tau^n\) of \(M^n\) is a neighborhood \(U\) of the diagonal \(\Delta \subset M^n \times M^n\) such that the projection \(p_1: U \to M^n\) is a topological \(\mathbb{R}^n\) bundle. Here, \(p_1(x, y) = x\).

Let \(M\) be a topological manifold, and let \(f: M \to BTOP\) classify the stable tangent bundle of \(M\). The class \(f^4(\chi) \in H^4(M; \mathbb{Z}_2)\) is a well-defined invariant of \(M\) since \(f\) is unique up to homotopy. The Kirby-Siebenmann class \(\Delta(M)\) of \(M\) is by definition the element \(f^4(\chi)\).

If \(U \subset M\), \(i: U \to M\) is the inclusion, and \(f_M, f_U\) classify the corresponding stable tangent bundles then \(f_M \circ i \simeq f_U\) and we have \(i^4(\Delta(M)) = \Delta(U)\).

**Theorem 2.8.** (cf. e.g. [22, p. 78]). Let \(M\) be a topological manifold. If \(M\) admits a PL structure, in particular if it admits a smooth structure, then \(\Delta(M) = 0\). For \(\dim M \geq 5\) the converse also holds: if \(\Delta(M) = 0\), then \(M\) admits a PL structure.

**Theorem 2.9.** (cf. [7, 13, 23]). Let \(M^n\) be a topological manifold, \(n \geq 5\). Then the Galewski-Stern obstruction for a manifold \(M^n\) to having a simplicial triangulation is the image

\[
\beta(\Delta(M^n)) \in H^5(M^n; \ker \mu)
\]

of the Kirby-Siebenmann class \(\Delta(M^n)\) by the Bockstein homomorphism \(\beta\) for the exact sequence of coefficients:

\[
0 \to \ker \mu \to \theta^H_3 \xrightarrow{\mu} \mathbb{Z}_2 \to 0.
\]

The homomorphism \(\mu: \theta^H_3 \to \mathbb{Z}_2\) is the Rokhlin invariant homomorphism of the abelian group \(\theta^H_3\) of homology cobordism classes of oriented PL homology 3-spheres with the operation of connected sum, cf. e.g. [7, 23]. For the purposes of our paper it suffices to know that \(\ker \mu\) is some nontrivial abelian group.

**Theorem 2.10.** (cf. [15, for \(n = 3\)], [9, for \(n = 4\)], and [6, for \(n \geq 5\)]). Let \(\alpha\) be an open cover of an \(n\)-manifold \(M\). Then there exists an open cover \(\beta\) of \(M\) such that any proper \(\beta\)-map \(g: M \to N\) onto any \(n\)-manifold \(N\) is homotopic through \(\alpha\)-maps to a homeomorphism.

For uniformity we denote the manifolds of Freedman, Galewski and Stern, and Manolescu by \(M^4_F, M^5_{GS}\) and \(M^5_{M+}^n\), respectively.
3. Proof of the Main Theorem

First consider the manifold $M^4_F$. According to Theorems 2.3 and 2.4, the complement $M^4_F \setminus \{p\}$ of a point $p \in M^4_F$ is a polyhedron. Let $\varepsilon > 0$ be any positive number. Let $U \subset M^4_F$ be an open topological ball in $M^4_F$ with center at $p$ and with diameter less than $\varepsilon$. Consider a triangulation $T$ of $M^4_F \setminus \{p\}$ and let $K$ be the polyhedron which is the union of all simplices of $T$ which intersect with $M^4_F \setminus U$ and let $L$ be the compactum which is the union of all the remaining simplices in $U$ and the point $p$.

Obviously, $L$ is a locally contractible 4-dimensional space and therefore by the Borsuk theorem [3] it is a compact ANR. By the above mentioned theorems of West and Wall it follows that $L$ is homotopy equivalent to a finite 4-dimensional polyhedron, call it $B$. We have a homotopy equivalence $f: L \to B$. Consider the closed star $st(f(p))$ of the point $f(p)$ in the finite polyhedron $B$, that is the union of all simplices of $B$ containing the point $f(p)$. This is a closed neighborhood of the point $p$. Consider the preimage of $st(f(p))$. We get a closed neighborhood of the point $p$ in $L$.

Consider the second barycentric subdivision of $st(f(p))$ and the closed star of the point $f(p)$ in it. Call it $st_2(f(p))$. Obviously,\n\[ f^{-1}(st_2(f(p))) \subset \text{Int} \ f^{-1}(st(f(p))).\]

Let $d$ be the minimal distance between the points of the boundary $\partial(f^{-1}(st(f(p))))$ and the points of compactum $f^{-1}(st_2(f(p)))$. Clearly, $d > 0$.

Consider small triangulation of $T$ such that the diameters of all of its simplices are less than $d$ and let $Q$ be the union of all simplices of this triangulation which intersect $f^{-1}(st_2(f(p)))$ and the point $p$. We get a relative CW complex $(L, Q)$ and the mapping\n\[ f: (L, Q) \to (B, st(f(p))).\]

By the Cellular Approximation Theorem 2.1, the map $f$ is homotopy equivalent to a map $g$ such that its restriction to $L \cap K$ is a simplicial map (in our case the relative CW complexes are obviously relative simplicial complexes).

Since $M^4_F$ and $L$ are ANR’s, the pair $(M^4_F, L)$ has the homotopy extension property with respect to any space [14, p. 120], [28]. Therefore $M^4_F$ is homotopy equivalent to the quotient space $M^4_F \cup g B$, cf. [28, p. 26, Corollary (5.12)].

The space $M^4_F \cup g B$ is the union of two finite polyhedra intersection of which is $g(L \cap K)$, a subpolyhedron of both of these polyhedra and therefore $M^4_F \cup g B$ is a finite polyhedron. Obviously, the quotient map of $M^4_F$ onto $M^4_F \cup g B$ is an $\varepsilon$-map.

Consider now the Galewski-Stern manifold $M^5_{GS}$. The obstruction to triangulability of any manifold $M^n$ for $n \geq 5$ is the obstruction
\[ \beta(\Delta)(M^n) \in H^5(M^n; \text{Ker} \ \mu) \]
(cf. Theorem 2.9). Manolescu [20] has showed that $\beta(\Delta)(M^5_{GS})$ is nontrivial and therefore the manifold $M^5_{GS}$ is non-triangulable.

However, for any connected closed 5-dimensional manifold,
\[ H^5(M^5 \setminus \{p\}; \text{Ker} \ \mu) = 0 \]
and therefore the Galewski-Stern obstruction $\beta(\Delta)(M^5_{GS} \setminus \{p\})$ is trivial and $M^5_{GS} \setminus \{p\}$ is an infinite polyhedron. Using the arguments similar to those used in the proof of the theorem for the Freedman manifold it follows that for every $\varepsilon > 0$ there
exists an $\varepsilon$-map of $M^5_{GS}$ onto a 5-dimensional polyhedron $P^5$ which is a homotopy equivalence.

The Manolescu non-triangulable manifold $M^{5+n}_{\mathcal{M}}$ is the product of $M^5_{GS}$ with the torus $T^n$. Since $M^5_{GS}$ admits an $\varepsilon$-map onto $P^5$ which is a homotopy equivalence for an arbitrarily small $\varepsilon > 0$, it follows that obviously,

$$M^{5+n}_{\mathcal{M}} = M^5_{GS} \times T^n$$

also admits an $\varepsilon$-map which is a homotopy equivalence onto $P^5 \times T^n$ for arbitrarily small $\varepsilon > 0$.

4. Some complementary results and remarks

**Theorem 4.1.** There do not exist any non-triangulable almost-smooth manifolds $M^n$ for any $n > 4$.

**Proof.** Since $M^n$ is non-triangulable it does not have a PL structure and therefore the Kirby-Siebenmann obstruction

$$\Delta(M^n) \in H^4(M^n; \mathbb{Z}_2)$$

is nonzero, by Theorem 2.8. It follows from the exact sequence of the pair $(M^n, M^n \setminus \{p\})$:

$$0 = H^4(M^n, M^n \setminus \{p\}; \mathbb{Z}_2) \to H^4(M^n; \mathbb{Z}_2) \to H^4(M^n \setminus \{p\}; \mathbb{Z}_2)$$

that $\Delta(M^n \setminus \{p\})$ is nonzero for $n > 4$. Therefore $M^n \setminus \{p\}$ does not have a PL structure and is thus not smooth. □

**Theorem 4.2.** A non-triangulable connected manifold of dimension $n > 4$ has an infinite simplicial complex structure in the complement of a point if and only if $n = 5$.

**Proof.** The obstruction class $\beta(\Delta)(M \setminus \{p\})$ is obtained as the image of $\beta(\Delta)(M)$ by the restriction

$$H^5(M; \text{Ker } \mu) \to H^5(M \setminus \{p\}; \text{Ker } \mu).$$

When $n = 5$, we have

$$H^5(M \setminus \{p\}; \text{Ker } \mu) = 0$$

since $M$ is a connected manifold, hence the obstruction vanishes. When $n > 5$, we see that by the exact sequence of the pair $(M, M \setminus \{p\})$ we get

$$0 = H^5(M, M \setminus \{p\}; \text{Ker } \mu) \to H^5(M; \text{Ker } \mu) \to H^5(M \setminus \{p\}; \text{Ker } \mu).$$

The restriction is injective, so the image of $\beta(\Delta)(M)$ is non-zero and $M \setminus \{p\}$ is non-triangulable as simplicial complex, by Theorem 2.9. □

**Remark 4.3.** For non-triangulable manifolds there do not exist $\varepsilon$-maps onto a triangulable manifold for small enough $\varepsilon > 0$. This follows from deep results of Chapman and Ferry, Ferry and Weinberger, and Jakobsche (cf. Theorem 2.10).
5. Epilogue

The non-triangulable closed 4-manifolds of Freedman [10, 24], 5-manifolds of Galewski and Stern [7, 12], and n-manifolds of Manolescu for \( n \geq 6 \) [20, p. 148] have nice geometric descriptions. All of them are homotopy equivalent to polyhedra of the corresponding dimension. We have polyhedral homotopy representatives \( PH^4_F, PH^5_{GS}, PH^{5+n}_M \) of the above mentioned non-triangulable manifolds.

**Problem 5.1.** Find a geometric description of the polyhedra \( PH^4_F, PH^5_{GS} \) and \( PH^{5+n}_M \).

The Alexandroff-Borsuk problem is solved for the special class of non-triangulable manifolds and is still open for general non-triangulable manifolds. The following problems seem to be of interest:

**Problem 5.2.** Let \( M^n \) be any non-triangulable manifold. Does there exist any polyhedron \( P \) embeddable in \( M^n \), such that \( M^n \setminus P \) is also a polyhedron?

According to our Main Theorem, for every positive number \( \varepsilon \) and for the manifolds of Freedman, Galewski and Stern, and Manolescu there exist \( \varepsilon \)-maps of these manifolds onto some polyhedra \( PH^4_F, PH^5_{GS}, PH^{5+n}_M \), respectively. The following version of the Alexandroff-Borsuk Manifold Problem remains open:

**Problem 5.3.** Does there exist for every compact \( n \)-dimensional manifold \( M^n \), a finite \( n \)-dimensional polyhedron \( P^n \) such that for an arbitrarily small \( \varepsilon > 0 \) there exists an \( \varepsilon \)-map \( f: M^n \to P^n \) which is a homotopy equivalence?

The answer for the corresponding version of Alexandroff-Borsuk ANR Problem is negative, even for 1-dimensional compact absolute retracts (AR), i.e. for the dendrites.

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