Moduli of continuity and average decay of Fourier transforms: two-sided estimates

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Abstract. We study inequalities between general integral moduli of continuity of a function and the tail integral of its Fourier transform. We obtain, in particular, a refinement of a result due to D. B. H. Cline [2] (Theorem 1.1 below). We note that our approach does not use a regularly varying comparison function as in [2]. A corollary of Theorem 1.1 deals with the equivalence of the two-sided estimates on the modulus of continuity on one hand, and on the tail of the Fourier transform, on the other (Corollary 1.5). This corollary is applied in the proof of the violation of the so-called entropic area law for a critical system of free fermions in [4, 5].

1. Introduction and statement of the main results

A result of this paper (Corollary 1.5 below) is applied in the proof of the violation of the so-called entropic area law for a critical system of free fermions, see [4] (6) et seq., [4] Section on Fractal Boundaries] and [5] Lemma 2.10. Corollary 1.5 follows from more general results of this paper (Theorems 1.1, 1.4) which are of independent interest.

It is well-known that the behavior of a modulus of continuity $\omega[f](h)$ of a function $f$ for $|h|$ small is related to the behavior of the Fourier transform $\hat{f}(\xi)$ of $f$ for $|\xi|$ large (precise definitions are given in (1.3) et seq. below), see e.g. [10] Proposition 5.3.4, [12] Theorem 85. The main object of our study are inequalities between general averaged moduli of continuity (m.c.) of $L^p$ functions (defined in (1.5) below) and tails of their Fourier transforms (F.t.). In [2] several results relevant for our purposes were obtained. Theorem 1.1 below gives a lower estimate for a general $L^p$ m.c., $1 \leq p \leq 2$, in terms of the modified tail integral of the F.t. improving one of the results in [2] (as in [2], we distinguish between the true and the modified F.t. tail integral, as defined in (1.5) and (1.6) below). Corollary 1.2 gives a two-sided
estimate for the m.c. in terms of the modified tail integral of the F.t. in the case 
$p = 2$. In applications it might be desirable to use the true F.t. tail instead of the 
modified tail that arises naturally in the mentioned inequalities. Theorem 1.4 gives 
the best possible power-scale description of the relationship between the true and 
the modified F.t. tails (see Remark 2.3).

Before stating our results we need to introduce some notation and recall two 
results in [2]. Let $d \in \mathbb{N}$ and denote by $\| \cdot \|_{p,\mathbb{R}^d}$ the standard norm in $L^p(\mathbb{R}^d)$, 
$1 \leq p \leq \infty$. The Fourier transform $\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx$, $\xi \in \mathbb{R}^d$, of a function 
$f \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq 2$ is defined in the standard way (see e.g. [3, Section IV.3]). In the case $2 < p \leq \infty$ we consider only the functions $f \in L^p(\mathbb{R}^d)$ whose 
transforms belong to $L^{p'}(\mathbb{R}^d)$, $p' := p/(p-1)$. Introduce the difference operator of 
order $m \in \mathbb{N}$ acting on functions with domain $\mathbb{R}^d$ by [11, Section 3.3]

\begin{equation}
(1.1) \quad \Delta_y^m f(x) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + ky), \quad x, y \in \mathbb{R}^d,
\end{equation}

where $\binom{m}{k}$ denotes the binomial coefficient. Note that the Fourier transform of 
(1.1) in $x$ equals $\left( \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} e^{ik\xi} \right) \hat{f}(\xi) = (e^{iy \cdot \xi} - 1)^m \hat{f}(\xi)$, and hence the functions

\begin{equation}
(1.2) \quad \Delta_y^m f(x), \quad (2i \sin(y \cdot \xi/2))^{m} e^{imy \cdot \xi/2} \hat{f}(\xi)
\end{equation}

form a Fourier pair for $m \in \mathbb{N}$. Now we define $\Delta_y^m f(x)$ for any $m > 0$, $m \not\in \mathbb{N}$, as the inverse Fourier transform of the second function in (1.2). Let $S^{d-1}$ denote the unit sphere in $\mathbb{R}^d$ and let $dS$ denote the standard measure on $S^{d-1}$. (All the results 
below with obvious modifications hold if one replaces the standard measure on $S^{d-1}$, with a measure $G$ invariant under orthogonal transformations and supported in the unit ball in $\mathbb{R}^d$ as in [2].) For $1 \leq p \leq \infty$ set

\begin{equation}
\delta_{p,m}[f](y) := \| \Delta_y^m f \|_{p,\mathbb{R}^d}, \quad y \in \mathbb{R}^d,
\end{equation}

and define a general averaged (integral) m.c. of $f$ as follows. For any $h > 0$ in the case $1 \leq q < \infty$ set

\begin{equation}
(1.3) \quad \omega_{p,m,q}[f](h) := \| \delta_{p,m}[f](hy) \|_{q,S^{d-1}} = \left( \int_{S^{d-1}} \| \Delta_y^m f \|_{p,\mathbb{R}^d}^q \, dS_y \right)^{1/q}
\end{equation}

(where in the case $d = 1$ the integral should be interpreted as a sum over $y \in S^0 = \{ \pm 1 \}$), and in the case $q = \infty$ set

\begin{equation}
\omega_{p,m,\infty}[f](h) := \sup_{|y| \leq h} \delta_{p,m}[f](y) = \sup_{|y| \leq h} \| \Delta_y^m f \|_{p,\mathbb{R}^d}.
\end{equation}

The Hölder inequality implies that for any $1 \leq q_1 \leq q_2 \leq \infty$ there exists $c = c(p, q_1, q_2, d) < \infty$ so that

\begin{equation}
(1.4) \quad \omega_{p,m,q_1}[f](h) \leq c \omega_{p,m,q_2}[f](h), \quad h > 0.
\end{equation}

Define the tail integral of the F.t. for $1 \leq p' < \infty$ and for $p' = \infty$, respectively, by

\begin{equation}
(1.5) \quad \psi_{p'}[\hat{f}](t) := \left( \int_{|\xi| \geq t} |\hat{f}(\xi)|^{p'} \, d\xi \right)^{1/p'}, \quad \psi_{\infty}[\hat{f}](t) := \sup_{|\xi| \geq t} |\hat{f}(\xi)|, \quad t > 0.
\end{equation}

Motivated by the results in [2], we wish to compare the m.c. $\omega_{p,m,q}[f](1/t)$ and the F.t. tail $\psi_{p}[\hat{f}](t)$, as $t \to \infty$. It will be clear from our Theorem 1.1 below that the natural choice of $q$ for the purpose of such a comparison is $q = p'$ (see also the
discussion preceding Theorem [1.3]. Note next that it is possible that \( \psi_p(\hat{f})(t) \) is rapidly decreasing, or simply zero, for large \( t \) (take e.g., \( \hat{f} \in C_0^\infty(\mathbb{R}^d) \)), whereas the modulus of continuity related to the \( m \)th finite difference of \( \hat{f} \) vanishes generally speaking at the rate \( 1/t^m \) only. This motivated the author in [2] to introduce the following modified F.t. tails: For \( 1 \leq p' < \infty \)

\[
\psi_{p',m}[\hat{f}](t) := \left( \int_{\mathbb{R}^d} \min\left(1, \frac{|\xi|}{t} \right)^{mp'} |\hat{f}(\xi)|^{p'} \, d\xi \right)^{1/p'}
\]

(1.6)

\[\left( t^{-mp'} \int_{|\xi| \leq t} |\xi|^{mp'} |\hat{f}(\xi)|^{p'} \, d\xi + (\psi_{p'} [\hat{f}](t))^{p'} \right)^{1/p'}, \quad t > 0,
\]

and for \( p' = \infty \)

\[
\psi_{\infty,m}[\hat{f}](t) := \sup_{\xi \in \mathbb{R}^d} \left( \min\left(1, \frac{|\xi|}{t} \right)^{m} |\hat{f}(\xi)| \right), \quad t > 0.
\]

It might not be immediately obvious why these tails are useful, we give the reason for that in Theorem [1.3] below. Note that for any \( 1 \leq p' \leq \infty \), \( \psi_{p',m}[\hat{f}](t) \) is nonincreasing as \( t \) grows, because so is the function \( \min(1, r/t) \) for any fixed \( r > 0 \).

Note also that in the case \( \hat{f} \in C_0^\infty(\mathbb{R}^d) \), \( \hat{f} \neq 0 \), there exist \( \tilde{c}_1 = \tilde{c}_1(\hat{f}) > 0 \), \( \tilde{c}_2 = \tilde{c}_2(\hat{f}) > 0 \) such that

\[
\tilde{c}_1 t^{-m} \leq \psi_{p',m}[\hat{f}](t) \leq \tilde{c}_2 t^{-m}, \quad t \to \infty,
\]

because \( \psi_{p'}[\hat{f}](t) = 0 \) for large \( t \).

We are ready to state the two results from [2] mentioned above. First [2 p. 512], for any \( d \in \mathbb{N} \) in the case \( 2 \leq p \leq \infty \), there exists \( c_3 = c_3(p, d, m) \) such that for for all functions in the set \( \{ f : f \in L^p(\mathbb{R}^d) \text{ and } \hat{f} \in L^{p'}(\mathbb{R}^d) \} \),

\[
\omega_{p,m,\infty}[f](1/t) \leq c_3 \psi_{p',m}[\hat{f}](t), \quad t > 0.
\]

(1.8)

Note that in the case \( p = 2 \), (1.8) holds for all \( f \in L^2(\mathbb{R}^d) \). (The mentioned formula in [2] involves in fact \( \psi_{p',m}[\hat{f}](2t) \). The formula (1.8) is then true since \( \psi_{p',m}[\hat{f}](t) \) is nonincreasing, and (1.8) suffices for our purposes.) Secondly [2 (9)], for any \( d \in \mathbb{N} \) in the case \( 1 \leq p \leq 2 \), for any \( a > 1 \) there exists \( c_4 = c_4(p,q,d,m,a) \) such that for any \( 1 \leq q \leq \infty \) and for all \( f \in L^p(\mathbb{R}^d) \)

\[
\psi_p[\hat{f}](t) \leq c_4 \sum_{k=1}^\infty a^m \omega_{p,m,q}[f](1/(a^k t)), \quad t > 0.
\]

(1.9)

It would be preferable to have instead of (1.9) a formula which does not involve an infinite sum, i.e., of the type (1.8). It turns out that such a result holds for \( d \geq 2 \) and if \( q \) is large enough, at least \( q = p' \).

**Theorem 1.1.** For any \( d \geq 2 \) and \( 1 \leq p \leq 2 \) there exists \( c = c(p,d,m) \) such that for all \( f \in L^p(\mathbb{R}^d) \)

\[
\psi_{p',m}[\hat{f}](t) \leq c \omega_{p,m,p'}[f](1/t), \quad t > 0.
\]

(1.10)

It is explained in Remark [2,2] below why (1.10) (and even its analog with \( \psi_{p',m}[\hat{f}](t) \) in the left-hand side, cf. (1.15) below) fails for \( d = 1 \). In view of (1.4), one can replace \( p' \) in the right-hand side in (1.10) with any \( q \geq p' \) (and a different \( c \)). It would be interesting to know if one could replace \( p' \) in the right-hand side of (1.10) with \( 1 \leq q < p' \). Note also that for \( q \geq p' \), our Theorem 1.1 implies readily all
the statements of [2] Theorem 2, and is slightly more general since no comparison function \( s \) as in [2] is required.

In the case \( p = 2 \) the following is an immediate consequence of (1.8), (1.10) and (1.4).

**Corollary 1.2.** For \( d \geq 2 \) and \( p = 2 \) there exist \( c_1, c_2 > 0 \) that depend on \( d \) and \( m > 0 \) only, such that for all \( f \in L^2(\mathbb{R}^d) \) and \( t > 0 \)
\[
(1.11) \quad c_1 \omega_{2,m,\infty}[f](1/t) \leq \psi_{2,m}[f](t) \leq c_2 \omega_{2,m,2}[f](1/t)
\]
\[
\leq c_2c(2, 2, \infty, d) \omega_{2,m,\infty}[f](1/t).
\]

The estimates (1.8), (1.10), (1.11) show that the modified F.t. tail \( \psi_{p',m} \) is more appropriate than the true F.t. tail \( \psi_{p'} \) to be compared with the m.c. \( \omega_{p,m,p'} \). The upper estimate in (1.11) is a Jackson-type inequality, see e.g. [8] Section I.8.

It follows from (1.11) that for \( p = 2 \) an inequality in the direction opposite to (1.4) holds: In the case \( d \geq 2 \) for all \( f \in L^2(\mathbb{R}^d) \)
\[
(1.12) \quad \omega_{2,m,\infty}[f](1/t) \leq \tilde{c}(2, \infty, 2) \omega_{2,m,2}[f](1/t), \quad t > 0,
\]
and so for any \( m > 0 \) all the moduli \( \omega_{2,m,q}, 2 \leq q \leq \infty \), are equivalent. It would be interesting to find a direct proof of (1.12).

We now explain why in the case \( d \geq 2 \) and \( 1 \leq p' < \infty \) the modified F.t. \( \psi_{p',m} \) defined in (1.6) is a natural quantity to consider. For \( d \geq 2 \) and any \( 0 < \alpha < \infty \) define
\[
(1.13) \quad G_\alpha(|w|) := 2^\alpha \int_{\mathbb{R}^{d-1}} (1 - \cos(y \cdot w))^\alpha dS_y, \quad w \in \mathbb{R}^d.
\]

Recalling that the functions in (1.2) form a Fourier pair and using the Hausdorff–Young inequality (see the proof of Theorem 1.1 below and the proof of (1.3) in [2]) we can compare for \( y \in S^{d-1} \) fixed and \( t > 0 \), the \( L^p \) norm of \( \Delta_n^\alpha f, \delta_{p,m}[f](y) \), and the \( L^{p'} \) norm of the function \( (2i \sin(y \cdot \xi/(2t)))^m e^{i\nu y \cdot \xi/2} \hat{f}(\xi) \) (in the case \( p > 2 \) we assume in addition as before that \( \hat{f} \in L^{p'}(\mathbb{R}^d) \)). Raising both quantities to the power \( q = p' \) (this explains why the choice \( q = p' \) is natural) we can compare the quantities \( \omega_{p,m,p'}[f](t) \) and
\[
\int_{\mathbb{R}^d} 2^{mp'/2} \left( \int_{\mathbb{R}^{d-1}} (1 - \cos y \cdot \xi/t)^{mp'/2} dS_y \right) |\hat{f}(\xi)|^{p'} d\xi.
\]

With this in mind, for \( d \geq 2 \) and any \( \hat{f} \in L^{p'}(\mathbb{R}^d) \), \( 1 \leq p < \infty \), we introduce the **Bessel tail** of the Fourier transform
\[
\Psi_{p',m}[\hat{f}](t) := \left( \int_{\mathbb{R}^d} G_{mp'/2}(|\hat{\xi}|/t)|\hat{f}(\xi)|^{p'} d\xi \right)^{1/p'}
\]
and \( \Psi_{\infty,m}[\hat{f}](t) := \psi_{\infty,m}[\hat{f}](t) \). From the above discussion, \( \omega_{p,m,p'}[f](t) \) can be compared with the Bessel tail \( \Psi_{p',m}[\hat{f}](t) \) as in (1.8), (1.10). The relevance of the modified F.t. \( \psi_{p',m} \) is now apparent from the following

**Theorem 1.3.** For any \( d \geq 2 \) and any \( 0 < \alpha < \infty \), there exist \( C_1, C_2 > 0 \) that depend only on \( d \) and \( \alpha \) so that
\[
(1.14) \quad C_1(\min(1, v))^{2\alpha} \leq G_\alpha(v) \leq C_2(\min(1, v))^{2\alpha}, \quad v \geq 0,
\]
and hence for some \( \tilde{C}_1, \tilde{C}_2 > 0 \) that depend on \( 1 \leq p < \infty, \ m > 0, \ d \) only,
\[
\tilde{C}_1 \Psi_{p',m}[\hat{f}](t) \leq \psi_{p',m}[\hat{f}](t) \leq \tilde{C}_2 \Psi_{p',m}[\hat{f}](t)
\]
for all \( \hat{f} \in L^{p'}(\mathbb{R}^d) \) and all \( t > 0 \).

The relation (1.14) is illustrated for \( \alpha = 1 \), \( C_1 = \pi/3 \), \( C_2 = 6\pi \) in Fig. 1 (note that \( G_1(v) = 4\pi(1 - \sin(v)/v) \)).

We now describe the relationship between the true and the modified F.t. tails, \( \psi_{p'} \) and \( \psi_{p',m} \), respectively. From (1.6) it is clear that for any \( 1 \leq p' \leq \infty \), \( m > 0 \), and \( \hat{f} \in L^{p'}(\mathbb{R}^d) \)

\[
\psi_{p'}[\hat{f}](t) \leq \psi_{p',m}[\hat{f}](t), \quad t > 0.
\]

The following statement gives a converse to (1.15) that is optimal on the power scale (see Remark 2.3 below).

**Theorem 1.4.** Let \( d \in \mathbb{N} \), \( m > 0 \), \( \alpha > 0 \), \( 1 \leq p' \leq \infty \), \( g \in L^{p'}(\mathbb{R}^d) \). All the constants below depend on \( g \) and are strictly positive and finite.

1. Let \( 1 \leq p' < \infty \).
   (i) If \( \psi_{p'}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \to \infty \), then as \( t \to \infty \),

   \[
   \psi_{p',m}[g](t) \leq b_2(g) \cdot \begin{cases} t^{-\alpha}, & 0 < \alpha < m \\ t^{-\alpha} \log t^{1/p'}, & \alpha = m \\ t^{-m}, & \alpha > m. \end{cases}
   \]

   (ii) If \( c_1(g) \cdot t^{-\alpha} \leq \psi_{p',m}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \to \infty \), then as \( t \to \infty \),

   \[
   \psi_{p'}[g](t) \geq b_1(g) \cdot \begin{cases} t^{-\alpha}, & 0 < \alpha < m \\ 0, & \alpha \geq m. \end{cases}
   \]

2. Let \( p' = \infty \).
   (i) If \( \psi_{\infty}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \to \infty \), then as \( t \to \infty \),

   \[
   \psi_{\infty,m}[g](t) \leq b_2(g) \cdot \begin{cases} t^{-\alpha}, & 0 < \alpha < m \\ t^{-m}, & \alpha \geq m. \end{cases}
   \]
(ii) If $c_1(g) \cdot t^{-\alpha} \leq \psi_{\infty,m}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \to \infty$, then as $t \to \infty$,

$$
\psi_{\infty}[g](t) \geq b_1(g) \cdot \begin{cases} 
\ t^{-\alpha}, & 0 < \alpha < m \\
0, & \alpha \geq m.
\end{cases}
$$

We state finally a result that was applied in a study of the scaling of entanglement entropy for a certain physical system in [4][5].

**Corollary 1.5.** Let $d \in \mathbb{N}$. Assume that $f \in L^2(\mathbb{R}^d)$. Then for some $c_1 = c_1(f) > 0$, $c_2 = c_2(f) > 0$ and some $\gamma = \gamma(f) \in (0, 2)$, $f$ satisfies

$$
c_1 \epsilon^\gamma \leq \int_{\mathbb{R}^{d-1}} \|f(\cdot + cy) - f(\cdot)\|_{L^2,\mathbb{R}^d}^2 \ dS_y \leq c_2 \epsilon^\gamma, \quad 0 \leq \epsilon \leq 1,
$$

if and only if there exist $b_1 = b_1(f) > 0$, $b_2 = b_2(f) > 0$ such that

$$
b_1 t^{-\gamma} \leq \int_{|\xi| \geq t} |\hat{f}(\xi)|^2 \ d\xi \leq b_2 t^{-\gamma}, \quad t \geq 1.
$$

Note that Corollary 1.5 is true in all dimensions: In the proof of Corollary 1.5 below, we consider the cases $d \geq 2$ and $d = 1$ separately. In the former case we employ the general results stated above. In the case $d = 1$ we give a direct proof using in particular the ideas in the proofs of [1] Lemma 2.10, Lemma 4.2. The main reason why, in the case $d = 1$, Corollary 1.5 is true despite the fact that Theorem 1.1 fails, is because of the explicit (power-type) form of the estimates in Corollary 1.5.

The equivalence of the upper estimates in (1.16) and in (1.17) is well-known: it follows e.g. from [9] Lemma 3.3.1, and also from the results obtained in [2]. Note that [9] Lemma 3.3.1 deals with a Besov space $B_{2,\infty}^s(\mathbb{R}^d)$, $s > 0$, $s \notin \mathbb{N}$ (the case $s = \gamma/2 \in (0, 1)$ is relevant for the upper estimates in Corollary 1.5). In [1], (1.17) is derived in the case $\gamma = 1$ from a more restrictive pointwise condition

$$
c_1 |y|\gamma \leq (\delta_{2.1}[f](y))^2 \leq c_2 |y|\gamma, \quad |y| \leq 1,
$$

which in general can not be reversed because $\delta_{2.1}[f](y)$ can have singular directions.

A simple example of a function that satisfies (1.16) with $\gamma = 1$ is the characteristic function of a compact set with $C^1$ boundary. For any $0 < \gamma < 1$ there exists a compact set whose characteristic function satisfies (1.16), see [5] Lemma 2.9.

It turns out that Corollary 1.5 fails for $\gamma = 2$, see Remark 2.3. Note that for $\gamma > 2$ the condition (1.16) is not satisfied for any $f \in C^\infty_0(\mathbb{R}^d)$, $f \neq 0$, because it involves the finite difference of order 1.

Theorems 1.1, 1.3, 1.4 and Corollary 1.5 are proved in Section 2.

2. Proofs and concluding remarks

**Proof of Theorem 1.1.** First consider the case $p = 1$, $p' = \infty$ (in this case the proof below goes through for all $d \in \mathbb{N}$). We have to prove that for some $c = c(p, d, m)$, $\psi_{\infty,m}[\hat{f}](t) \leq c \omega_{1,m,\infty}[f](1/t)$, $t > 0$. Recall that $\|\hat{f}\|_{L^\infty,\mathbb{R}^d} \leq \|f\|_{L^1,\mathbb{R}^d}$. Since the functions (1.2) form a Fourier pair we have

$$
\omega_{1,m,\infty}[f](1/t) = \sup_{|y| \leq 1/t} \delta_{1,m}(y) \geq \sup_{|y| \leq 1/t} \sup_{\xi \in \mathbb{R}^d} \left( |\hat{f}(\xi)| \sup_{|z| \leq 1} |\sin(z \cdot \xi/(2t))|^{\gamma} \right)\ 
$$

$$
(2.1) \quad = 2^m \sup_{\xi \in \mathbb{R}^d} \left( |\hat{f}(\xi)| \sup_{|z| \leq 1} |\sin(z \cdot \xi/(2t))|^{\gamma} \right).
$$
It is an easy exercise to prove that for any $\xi \in \mathbb{R}^d$
\[
\sup_{|z| \leq 1} |\sin(z \cdot \xi/(2t))| = \begin{cases} 
1, & |\xi| \geq |\pi| \\
\sin(|\xi|/(2t)), & |\xi| < |\pi| 
\end{cases} \geq \frac{1}{4} \min(1, |\xi|/t).
\]
This together with (2.1) and (1.6) proves the result.

Consider now the case $d \geq 2$, $1 < p \leq 2$, $p' < \infty$. By the Hausdorff–Young inequality $\|\hat{f}\|_{p',\mathbb{R}^d} \leq (2\pi)^{d/p'} \|f\|_{p,\mathbb{R}^d}$, $1 < p \leq 2$, and using the fact that (1.2) is a Fourier pair we get as in [2, (4.7)]
\[
(\delta_{p,m}[f](y/t))^{p'} \geq (2\pi)^{-d} \int_{\mathbb{R}^d} |2i \sin(y \cdot \xi/(2t))|^{mp'} |\hat{f}(\xi)|^{p'} d\xi
\]
\[
= (2\pi)^{-d} 2^{mp'/2} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^{p'} \left(1 - \cos \frac{y \cdot \xi}{t}\right)^{mp'/2} d\xi.
\]
Integrating over $y$ and recalling (1.13) we obtain
\[
(\omega_{p,m,p'}[f](1/t))^{p'} = \int_{S_{d-1}} (\delta_{p,m}[f](y/t))^{p'} dS_y
\]
\[
\geq (2\pi)^{-d} \int_{\mathbb{R}^d} G_{mp'/2}(|\xi|/t) |\hat{f}(\xi)|^{p'} d\xi.
\]
Now we need the following elementary result, the proof is given after the end of the present proof.

**Lemma 2.1.** Let $d \geq 2$ and fix any $0 < \alpha < \infty$. The function $G_{\alpha}(v)$, $v \geq 0$, defined in (1.13) satisfies the following: For any $v_0 > 0$ there exist $C_{\alpha}(v_0), c_{\alpha}(v_0) > 0$ such that
\[
G_{\alpha}(v) \geq C_{\alpha}(v_0) > 0, \quad v \geq v_0
\]
and
\[
G_{\alpha}(v) \geq c_{\alpha}(v_0) v^{2\alpha}, \quad 0 \leq v \leq v_0.
\]

Now we write the integral in (2.3) as a sum of $\int_{|\xi| \leq t}$ and $\int_{|\xi| \geq t}$, apply Lemma 2.1 with $\alpha = mp'/2$, $v_0 = 1$, and set $c_1(p, d, m) := c(m, d) \cdot \min(c_{\alpha}(1), C_{\alpha}(1))$ to obtain (2.6)
\[
(\omega_{p,m,p'}[f](1/t))^{p'} \geq c_1(p, d, m) \left( \int_{|\xi| \leq t} (|\xi|/t)^{mp'} |\hat{f}(\xi)|^{p'} d\xi + \int_{|\xi| \geq t} |\hat{f}(\xi)|^{p'} d\xi\right)
\]
\[
= c_1(p, d, m) (\psi_{p',m}[\hat{f}](t))^{p'}
\]
which finishes the proof of Theorem 1.1.

**Remark 2.2.** We show now that (1.10) is not true in the case $d = 1$ and $p = 2$. More precisely, we show that for $d = 1$ there is no $c \in (0, +\infty)$ so that for all $f \in L^2(\mathbb{R})$
\[
\psi_2[\hat{f}](t) \leq c \omega_{2,1,2}[f](1/t), \quad t > 0.
\]
By (1.2) with $m = 1$ using the Parseval formula we find
\[
(\omega_{2,1,2}[f](1/t))^2 = \frac{4}{\pi} \int_{\mathbb{R}} \sin^2 \left(\frac{\xi}{2t}\right) |\hat{f}(\xi)|^2 d\xi.
\]
For $\xi \in \mathbb{R}$, $t, \hat{c} > 0$, introduce the notation
\[
H(\xi, t, \hat{c}) = \sin^2 \left( \frac{\xi}{2t} \right) - \hat{c} \text{sgn} \left( \frac{\xi}{t} - 1 \right)
\]
where $\text{sgn} \ a = 1, 0, -1$ for $a > 0$, $a = 0$, $a < 0$, respectively, and $(a)_+ = \max(0, a)$. If (2.7) were true for some $c \in (0, +\infty)$ then there would exist $\hat{c} \in (0, +\infty)$ such that
\[
\int_{\mathbb{R}} H(\xi, t, \hat{c}) |\hat{f}(\xi)|^2 \, d\xi \geq 0 \quad \text{for all } f \in L^2(\mathbb{R}), \quad t > 0.
\]
But
\[
\{ |\hat{f}|^2 : f \in L^2(\mathbb{R}) \} = \{ |f|^2 : f \in L^2(\mathbb{R}) \} = \{ g : g \in L^1(\mathbb{R}) \text{ and } g \geq 0 \text{ almost everywhere (a.e.)} \}.
\]
We have arrived at a contradiction: If
\[
\int_{\mathbb{R}} H(\xi, t, \hat{c}) g(\xi) \, d\xi \geq 0 \quad \text{for all } g \in L^1(\mathbb{R}), \quad g \geq 0 \text{ a.e., } t > 0,
\]
then we must have $H(\xi, t, \hat{c}) \geq 0$ for a.e. $\xi$ and all $t > 0$, which is clearly false for any choice of $\hat{c} > 0$. This proves the result.

We note that the basic reason for the inapplicability of Theorem 1.1 to the case $d = 1$ is that for any $v_0 > 0$ it is not possible to insert a constant function between the graph of $\sin^2 v$ and the real axis on the interval $[v_0, +\infty)$, cf. Fig. 1 (In the case $d = 1$, $S^0 = \{ \pm 1 \}$ and so the function $G_\alpha$ in (1.13) would be given by $2^{2\alpha+1}(\sin^2(w/2))^{\alpha}$, $w \in \mathbb{R}$, and there is no helpful averaging over $S^{d-1}$.)

Proof of Lemma 2.1. Recall that $d \geq 2$. If $d \geq 3$ then integrating over the $d - 2$ angles in (1.13) as in [3] (II.3.4.2) we obtain for $v \geq 0$
\[
(2.9) \quad G_\alpha(v) = |S^{d-2}| \int_0^\pi (1 - \cos(v \cos \theta))^\alpha \sin^{d-2} \theta \, d\theta.
\]
If $d = 2$ then (2.9) and the subsequent formulae hold true with the convention $|S^0| = 2$.

We show first there is $M_\alpha > 0$ such that $G_\alpha(v) \geq M_\alpha > 0$ for sufficiently large $v$. Indeed, in the case $\alpha \geq 1$ by the Hölder inequality there is $\tilde{C}_\alpha > 0$ so that
\[
(2.10) \quad \int_0^\pi ((1 - \cos(v \cos \theta))^{(d-2)/\alpha} \theta)^\alpha \, d\theta 
\geq \tilde{C}_\alpha \left( \int_0^\pi (1 - \cos(v \cos \theta)) \sin^{d-2} \theta \, d\theta \right)^\alpha 
\geq \tilde{C}_\alpha \left( \int_0^\pi (1 - \cos(v \cos \theta)) \sin^{d-2} \theta \, d\theta \right)^\alpha
\]
where the second inequality follows from
\[
(2.11) \quad w^\beta \geq w, \quad 0 < \beta \leq 1, \quad 0 \leq w \leq 1
\]
with $w = \sin^{d-2} \theta$ and $\beta = 1/\alpha$. In the case $0 < \alpha < 1$, in view of (2.11) with $w = (1 - \cos(v \cos \theta))/2$ and $\beta = \alpha$ we obtain
\[
(2.12) \quad \int_0^\pi (1 - \cos(v \cos \theta))^\alpha \sin^{d-2} \theta \, d\theta 
\geq 2^{\alpha-1} \int_0^\pi (1 - \cos(v \cos \theta)) \sin^{d-2} \theta \, d\theta.
\]
Note that
\begin{equation}
\left| S^{d-2} \right| \int_0^\pi \left( 1 - \cos(v \cos \theta) \right) \sin^{d-2} \theta \, d\theta = \int_{S^{d-1}} \left( 1 - \cos \left( (v, 0, \ldots, 0) \cdot y \right) \right) \, dS_y
= \left| S^{d-1} \right| \left( 1 - 2^s \Gamma(s+1) v^{-s} J_s(v) \right)
\end{equation}
where $s = (d-2)/2$, $J_s$ is the Bessel function, and we have used [3] (II.3.4.2] and [7] (8.411.4]. By [7] (8.451], $v^{-s} J_s(v) = O(v^{-s-1/2}) \to 0$, as $v \to \infty$ for $d \geq 2$. Hence the right-hand side of (2.13), and also of (2.10) and (2.12), tends to a strictly positive limit, as $v \to \infty$. Therefore for any fixed $0 < \alpha < \infty$, there exist $M_\alpha > 0$ and $v_1(\alpha)$ so that $G_\alpha(v) \geq M_\alpha > 0$ for $v \geq v_1(\alpha)$.

But $G_\alpha(v)$ does not have zeros other than $v = 0$. Since $G_\alpha$ is continuous, it is for any $v_0 > 0$ bounded away from zero on the compact $[v_0, v_1(\alpha)]$. This proves (2.4).

Let us now prove (2.5) for $v_0 > 0$ small enough. We can rewrite (2.9) as
\begin{equation}
G_\alpha(v) = 2^\alpha \left| S^{d-2} \right| \int_0^\pi \sin^{2\alpha} (v \cos \theta) \sin^{d-2} \theta \, d\theta.
\end{equation}
Using the elementary estimate
\begin{equation}
\sin x \geq \frac{2}{\pi} x, \quad 0 \leq x \leq \frac{\pi}{2},
\end{equation}
we conclude that
\begin{equation}
G_\alpha(v) \geq v^{2\alpha} \cdot c(\alpha, d) \int_0^\pi \cos^{2\alpha} \theta \sin^{d-2} \theta \, d\theta, \quad 0 \leq v \leq \frac{\pi}{2},
\end{equation}
which proves (2.5) for any $0 < v_0 \leq \pi/2$. But now if we take any $v_0 > \pi/2$ then using (2.5) for $0 \leq v \leq \pi/2$ and the fact that $G_\alpha(v)$ is bounded away from zero for $\pi/2 \leq v \leq v_0$ by (2.4) we can always find $c_\alpha(v_0) > 0$ small enough so that (2.5) holds for $0 \leq v \leq v_0$. The proof of Lemma 2.1 is complete.

\textbf{Proof of Theorem 1.3} Recall that $d \geq 2$. The lower estimate in (1.14) follows immediately from Lemma 2.1 with $v_0 = 1$. As for the upper estimate, we note first that by the definition (1.13), the function $G_\alpha(v)$ is bounded above for $v \geq 1$. Next, using the estimate $\sin x \leq x$, $0 \leq x \leq 1$, in place of (2.15), we derive from (2.14) the upper estimate
\begin{equation}
G_\alpha(v) \leq v^{2\alpha} \cdot \tilde{c}(\alpha, d) \int_0^\pi \cos^{2\alpha} \theta \sin^{d-2} \theta \, d\theta, \quad 0 \leq v \leq 1.
\end{equation}
This proves the upper estimate in (1.14). The proof of Theorem 1.3 is complete.

\textbf{Proof of Theorem 1.4} Recall that $d \in \mathbb{N}$, $0 < \alpha < \infty$, $m > 0$.

1. Consider first the case $1 \leq p' < \infty$.
   (i) Assume $\psi_p^r[g](t) \leq c_1(g) \cdot t^{-\alpha}$, $t \geq 1$. By (1.6)
\begin{equation}
(\psi_{p,m}^r[g](t))^{p'} = t^{-mp'} \int_{|\xi| \leq t} |\xi|^{mp'} |g(\xi)|^{p'} \, d\xi + (\psi_{p,m}^r[g](t))^{p'}.
\end{equation}
Then the following gives the result for the case 1(i), all values of \( \alpha \):

\[
t^{-mp'} \int_{|\xi| \leq t} |\xi|^{mp'} |g(\xi)|^{p'} d\xi \\
\leq t^{-mp'} \left( \int_{|\xi| \leq 1} |g(\xi)|^{p'} d\xi + \sum_{k=0}^{|\log_2 t|} \int_{2^k \leq |\xi| \leq 2^{k+1}} |\xi|^{mp'} |g(\xi)|^{p'} d\xi \right) \\
\leq t^{-mp'} \left( \text{const}(g) + c_1(g) \sum_{k=0}^{|\log_2 t|} (2^{(m-\alpha)p'} k) \right)
\]

where \([\cdot]\) denotes the integer part of a real number. It is explained in Remark 2.3 below why the order in \( t \) cannot be improved in the case 1(i) and in all other cases.

(ii) Assume \( c_1(g) \cdot t^{-\alpha} \leq \psi_{p',m}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \geq 1 \ (p' < \infty) \). The example \( g \in C_0^\infty(\mathbb{R}^d), \ g \neq 0 \), shows that in the case \( \alpha \geq m \) we can only claim the trivial bound \( \psi_{p'}[g](t) \geq 0, \ t \to \infty \). Let now \( 0 < \alpha < m \). We use the idea in the proof of [1, Lemma 4.2]. Let \( 0 < B \leq 1 \) be a number to be chosen later. By the definition (2.16)

\[
(\psi_{p',m}[g](t))^{p'} = \left( \int_{|\xi| \leq Bt} + \int_{|\xi| \geq Bt} \right) \min^{mp'}(1, |\xi|/t) |g(\xi)|^{p'} d\xi.
\]

We have

\[
\int_{|\xi| \leq Bt} \min^{mp'}(1, |\xi|/t) |g(\xi)|^{p'} d\xi = t^{-mp'} \int_{|\xi| \leq Bt} |\xi|^{mp'} |g(\xi)|^{p'} d\xi \\
\leq t^{-mp'} \left( \int_{|\xi| \leq B} |\xi|^{mp'} |g(\xi)|^{p'} d\xi + \sum_{l=1}^{|\log_2 t|+1} \int_{2^{l-1}B \leq |\xi| \leq 2^l B} |\xi|^{mp'} |g(\xi)|^{p'} d\xi \right) \\
\leq t^{-mp'} \left( \|g\|^{p'}_{p',\mathbb{R}^d} + \sum_{l=1}^{|\log_2 t|+1} (2^l B)^{mp'} \cdot c_2^{p'}(g) \cdot (2^{l-1}B)^{-\alpha p'} \right) \\
\leq t^{-mp'} \cdot \text{const}(g) \cdot t^{(m-\alpha)p'} \cdot B^{(m-\alpha)p'}, \ t \to \infty.
\]

Therefore choosing \( 0 < B \leq 1 \) small enough (recall that \( m - \alpha > 0 \)) we obtain that

\[
(2.18) \quad \int_{|\xi| \leq Bt} \min^{mp'}(1, |\xi|/t) |g(\xi)|^{p'} d\xi \leq \frac{c_2^{p'}(g)}{2} \cdot t^{-\alpha p'}, \ t \to \infty.
\]

Since \( \psi_{m,p'}[g](t) \geq c_1^{p'}(g) \cdot t^{-\alpha p'}, \ t \to \infty \), (2.18) together with (2.17) gives

\[
\int_{|\xi| \geq Bt} \min^{mp'}(1, |\xi|/t) |g(\xi)|^{p'} d\xi \geq \frac{c_1^{p'}(g)}{2} \cdot t^{-\alpha p'}, \ t \to \infty.
\]

Setting \( s = Bt \) we obtain

\[
(2.19) \quad \int_{|\xi| \geq s} \min^{mp'}(1, B|\xi|/s) |g(\xi)|^{p'} d\xi \geq c_1(g) B^{\alpha p'} \cdot s^{-\alpha p'}, \ s \to \infty.
\]

Noting that \( 1 \geq \min(1, B|\xi|/s) \) we conclude from (2.19) that

\[
\psi_{p'}^{p'}[g](t) = \int_{|\xi| \geq s} |g(\xi)|^{p'} d\xi \geq c_1(g) B^{\alpha p'} \cdot s^{-\alpha}, \ s \to \infty.
\]

2. The case \( p' = \infty \).
(i) Assume \( \psi_\infty[g](t) \leq c_1(g) \cdot t^{-\alpha}, \ t \geq 1. \) Note that

\[
(2.20) \quad \psi_{\infty,m}[g](t) = \max \left( t^{-m} \sup_{|\xi| \leq t} |\xi|^m |g(\xi)|, \sup_{|\xi| \geq t} |g(\xi)| \right).
\]

Clearly

\[
(2.21) \quad |g(\xi)| \leq \sup_{|\eta| \geq |\xi|} |g(\eta)| \leq c_1(g) \cdot |\xi|^{-\alpha}, \quad |\xi| \geq 1.
\]

Therefore

\[
\sup_{|\xi| \leq t} |\xi|^m |g(\xi)| = \max \left( \sup_{|\xi| \leq 1} |\xi|^m |g(\xi)|, \sup_{1 \leq |\xi| \leq t} |\xi|^m |g(\xi)| \right)
\]

\[
\leq \max \left( c_1(g), c_1(g) \cdot \sup_{1 \leq |\xi| \leq t} |\xi|^m |g(\xi)| \right) = \begin{cases} C(g), & 0 < \alpha < m \\ C_1(g) \cdot t^{-m-\alpha}, & \alpha \geq m. \end{cases}
\]

This together with (2.20) proves the case 2(i).

(ii) Let \( c_1(g) \cdot t^{-\alpha} \leq \psi_{\infty,m}[g](t) \leq c_2(g) \cdot t^{-\alpha}, \ t \to \infty. \) Again if \( \alpha \geq m \) then the example of \( g \in C_0^\infty(\mathbb{R}^d), \ g \neq 0, \) shows that generally speaking only the trivial bound \( \psi_\infty[g](t) \geq 0 \) holds for large \( t. \) Let now \( 0 < \alpha < m. \) Let \( 0 < B \leq 1 \) be a number to be chosen later. We have

\[
\psi_{\infty,m}[g](t) = \sup_{\xi \in \mathbb{R}^d} \min \left( 1, |\xi|/t \right) |g(\xi)|
\]

\[
= \max \left( \sup_{|\xi| \leq Bt} \min \left( 1, |\xi|/t \right) |g(\xi)|, \sup_{|\xi| \geq Bt} \min \left( 1, |\xi|/t \right) |g(\xi)| \right).
\]

Now

\[
\sup_{|\xi| \leq Bt} \min \left( 1, |\xi|/t \right) |g(\xi)| = t^{-m} \sup_{|\xi| \leq Bt} |\xi|^m |g(\xi)|
\]

\[
\leq t^{-m} \max \left( \sup_{0 \leq |\xi| \leq B} |\xi|^m |g(\xi)|, \max_{l=1, \ldots, \lfloor \log_2 t \rfloor + 1} \sup_{2^{l-1} B \leq |\xi| \leq 2^l B} |\xi|^m |g(\xi)| \right)
\]

\[
\leq t^{-m} \max \left( B^m \|g\|_{\infty, \mathbb{R}^d}, \max_{l=1, \ldots, \lfloor \log_2 t \rfloor + 1} 2^{lm} B^m \cdot c_2(g) \cdot (2^{l-1} B)^{-\alpha} \right)
\]

\[
\leq \text{const}(g) \cdot B^{m-\alpha} \cdot t^{-\alpha}, \quad t \to \infty.
\]

Choosing \( B \) small enough (note that \( m - \alpha > 0 \)) we obtain

\[
c_1(g) \cdot t^{-\alpha} \leq \psi_{\infty,m}[g](t) \leq \max \left( c_1(g) \cdot t^{-\alpha}, \sup_{|\xi| \geq Bt} \min \left( 1, |\xi|/t \right) |g(\xi)| \right), \quad t \to \infty,
\]

which implies that \( \sup_{|\xi| \geq Bt} \min \left( 1, |\xi|/t \right) |g(\xi)| \geq c_1(g) \cdot t^{-\alpha}. \) Replacing \( s = Bt \) we obtain

\[
(2.22) \quad \sup_{|\xi| \geq s} \min \left( 1, B|\xi|/s \right) |g(\xi)| \geq c_1(g) B^\alpha \cdot s^{-\alpha}, \quad s \to \infty.
\]

Noting that \( 1 \geq \min(1, B|\xi|/s) \) we conclude from (2.22) that

\[
\psi_\infty[g](t) = \sup_{|\xi| \geq s} |g(\xi)| \geq c_1(g) B^\alpha \cdot s^{-\alpha}, \quad s \to \infty.
\]

This completes the proof of Theorem 1.4. \( \square \)
Remark 2.3. Let us explain why the estimates in Theorem 1.4 have the best possible order in $t$. In view of (1.13), the estimates 1(i) and 2(i) for $0 < \alpha < m$ cannot be improved. The example of $g \in C_0^\infty(\mathbb{R}^d)$, $g \not\equiv 0$, in view of the lower estimate in (1.7) shows that estimates 1(i) for $\alpha > m$ and 1(ii), 2(i), 2(ii) for $\alpha \geq m$ cannot be improved. The example of $g \in L^p(\mathbb{R}^d)$, $1 \leq p' < \infty$, such that $g(\xi) = |\xi|^{-(d/p')-\alpha}$ for $|\xi| \geq 1$ and $g$ smooth for $|\xi| < 1$ shows that the estimate 1(ii) for $0 < \alpha < m$ cannot be improved (note that the mentioned $g$ satisfies $\psi_{p',m}[g](t) \leq c_1(g) \cdot t^{-\alpha}$, $t \to \infty$, and also $\psi_p[g](t) = (|\mathbb{S}^{d-1}|/(\alpha p'))^{1/p'} \cdot t^{-\alpha}$, $t \geq 1$). Choosing $g \in L^\infty(\mathbb{R}^d)$ such that $g(\xi) = |\xi|^{-\alpha}$ for $|\xi| \geq 1$ and $g$ smooth for $|\xi| < 1$ shows that the estimate 2(ii) for $0 < \alpha < m$ cannot be improved (for this $g$ we have $\psi_\infty[g](t) = t^{-\alpha}$, $t \geq 1$). Finally, the example $g \in L^p(\mathbb{R}^d)$, $1 \leq p' < \infty$, such that $g(\xi) = |\xi|^{-1/(p'\alpha)}$, $|\xi| \geq 1$, and $g$ smooth for $|\xi| < 1$ shows that 1(i) for $\alpha = m$ cannot be improved. Indeed for this $g$, $\psi_p[g](t) = (|\mathbb{S}^{d-1}|/(mp'))^{1/p'} \cdot t^{-m}$, $t \geq 1$, but $\psi_{p',m}[g](t) \geq |\mathbb{S}^{d-1}|/p' \cdot t^{-m}(\log t)^{1/p'}$, $t \geq 1$.

**Proof of Corollary 1.5.** The case $d \geq 2$. The result follows readily from Corollary 1.2 combined with Theorem 1.4 for $p = p' = 2$, $\alpha = \gamma/2$, $m = 1$, $g = \hat{f}$ with $\epsilon = 1/t$. Note that the quantity in the middle in (1.10) is $(\omega_{2,1,2}[f](\epsilon))^2$. The integral in (1.17) is the true tail integral of the F.T. of $f$, $\psi_2[\hat{f}](t)$. By Corollary 1.2 the exist $C_1, C_2 > 0$ that depend only on $\Delta$ so that for all $f \in L^2(\mathbb{R}^d)$

$$
C_1 \omega_{2,1,2}[f](1/t) \leq \psi_{2,1}[\hat{f}](t) \leq C_2 \omega_{2,1,2}[f](1/t), \quad t > 0.
$$

Set $\alpha = \gamma/2$. Note that since $0 < \gamma < 2$ we have $0 < \alpha = \gamma/2 < m = 1$, and the result follows from Theorem 1.4 cases 1 (i) and (ii), applied to the comparison function $\alpha^{-\alpha} = t^{-\gamma/2}$, $t \geq 1$.

The case $d = 1$. We prove first that (1.10) implies (1.17). This follows from a straightforward modification of the proof of [1] Lemma 4.2. We prefer to give the details for the convenience of the reader. Below, $c$ will denote a positive constant whose precise value may change from equation to equation and may depend on $f$ but which is independent of $\epsilon$ and $t$. By (2.8)

$$(\omega_{2,1,2}[f](\epsilon))^2 = \frac{4}{\pi} \int_\mathbb{R} \sin^2 \left( \frac{\epsilon \xi}{2} \right) |\hat{f}(\xi)|^2 d\xi \geq c \int_{\pi/2 \xi \leq |\xi| \leq 2\pi/(2\epsilon)} |\hat{f}(\xi)|^2 d\xi.$$

Setting $\xi = 2\epsilon/\pi$ and denoting $\xi$ by again, we find from the upper inequality in (1.10)

$$c_2 \cdot (\pi \epsilon/2)^\gamma \geq \omega_{2,1,2}[f](\pi \epsilon/2) \geq c \int_{1/\epsilon \leq |\xi| \leq 2/\epsilon} |\hat{f}(\xi)|^2 d\xi$$

which after setting $t = 1/\epsilon$ implies

$$
\int_{t \leq |\xi| \leq 2t} |\hat{f}(\xi)|^2 d\xi \leq c \cdot t^{-\gamma}, \quad t \geq 1.
$$

Using the latter estimate and representing

$$
\int_{|\xi| \geq 2^j} |\hat{f}(\xi)|^2 d\xi = \sum_{j=0}^{\infty} \int_{2^jt \leq |\xi| \leq 2^{j+1}t} |\hat{f}(\xi)|^2 d\xi,
$$
we obtain the lower inequality in (1.16). In order to prove the lower inequality in (1.17), we note first that (2.24) implies
\[
\int_{|\xi| \leq r} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \leq c_3 \cdot r^{2-\gamma}, \quad r \geq 1.
\]
Indeed using the upper inequality in (1.17) we obtain
\[
(2.25) \quad \int_{|\xi| \leq r} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \leq \int_{|\xi| \leq 1} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi + \sum_{j=0}^{[^\log_2 r]+1} \int_{2^j \leq |\xi| \leq 2^{j+1}} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \\
\leq \text{const} + c \sum_{j=0}^{[^\log_2 r]+1} (2^{j+1})^2 \cdot b_2 \cdot (2^j)^{-\gamma} \\
\leq c' \cdot r^{2-\gamma}
\]
(recall that $\gamma < 2$). Next, let $\alpha, \beta > 0$ be two numbers to be chosen later. From the lower inequality in (1.16) and (2.8) using $|\sin x| \leq x$ and $|\sin x| \leq 1$ for $x \geq 0$, we obtain
\[
c_1 \cdot \epsilon^\gamma \leq (\omega_{2,1,2}[f](\epsilon))^2 \\
= \frac{4}{\pi} \int_{\mathbb{R}} \sin^2 \left(\frac{\epsilon \xi}{2}\right) |\hat{f}(\xi)|^2 \, d\xi \\
\leq c \left( c^2 \int_{|\xi| \leq \alpha/\epsilon} |\hat{f}(\xi)|^2 \, d\xi + \int_{\alpha/\epsilon \leq |\xi| \leq \beta/\epsilon} |\hat{f}(\xi)|^2 \, d\xi + \int_{|\xi| \geq \beta/\epsilon} |\hat{f}(\xi)|^2 \, d\xi \right).
\]
Using (2.25) and the upper inequality in (1.17) we find
\[
\int_{\alpha/\epsilon \leq |\xi| \leq \beta/\epsilon} |\hat{f}(\xi)|^2 \, d\xi \geq c_1 \epsilon^{-1} \cdot \epsilon^\gamma - \beta^{-\gamma} b_2 \cdot \epsilon^\gamma - \alpha^\gamma c_3 \cdot \epsilon^\gamma
\]
which after choosing $\alpha > 0$ small enough and $\beta > 0$ large enough gives
\[
\int_{|\xi| \geq \alpha/\epsilon} |\hat{f}(\xi)|^2 \, d\xi \geq \int_{\alpha/\epsilon \leq |\xi| \leq \beta/\epsilon} |\hat{f}(\xi)|^2 \, d\xi \geq c \cdot \epsilon^\gamma, \quad 0 < \epsilon \leq 1.
\]
Setting $t = \alpha/\epsilon$ we prove the lower inequality in (1.17).

We now derive (1.16) from (1.17). Using the upper inequality in (1.17) and employing (1.15) and Theorem 1.4 (recall that in our case $p = p' = 2$, $m = 1$, $\alpha = \gamma/2 \in (0,1)$)
\[
(\psi_2[f](t))^2 \leq (\psi_{2,1}[f](t))^2 \leq c \cdot t^{-\gamma}.
\]
Combining this with (1.8) (that holds for $d = 1$) and using (1.4) we obtain
\[
(\omega_{2,1,2}[f](1/t))^2 \leq c \cdot (\omega_{2,1,\infty}[f](1/t))^2 \leq c' \cdot (\psi_{2,1}[f](t))^2 \leq c'' \cdot t^{-\gamma}
\]
which proves the upper estimate in (1.16). It remains to prove the lower estimate in (1.16). Note that the two-sided estimate (1.17) implies that for $A > 1$ large enough
\[
\left( \int_{|\xi| \geq t} - \int_{|\xi| \geq At} \right) |\hat{f}(\xi)|^2 \, d\xi = \int_{t \leq |\xi| \leq At} |\hat{f}(\xi)|^2 \, d\xi \geq c \cdot t^{-\gamma}
\]
or after setting $\epsilon = \pi/(2t)$
\[
(2.26) \quad \int_{\pi/(2\epsilon) \leq |\xi| \leq A\pi/(2\epsilon)} |\hat{f}(\xi)|^2 \, d\xi \geq c \cdot \epsilon^\gamma, \quad 0 < \epsilon \leq 1.
\]
On the other hand again using (2.28)
\[
(\omega_{1,2}[f](\epsilon)) = \frac{4}{\pi} \int_{\mathbb{R}} \sin^2 \left(\frac{\xi}{2}\right) |\hat{f}(\xi)|^2 d\xi \geq c \int_{\pi/(2\epsilon) \leq |\xi| \leq 3\pi/(2\epsilon)} |\hat{f}(\xi)|^2 d\xi
\]
and replacing \(\epsilon\) with \(\epsilon/3^j\), \(j \in \mathbb{N}\), we find
\[
(\omega_{1,2}[f](3^{-j}\epsilon)) = c \int_{3^{j}\pi/(2\epsilon) \leq |\xi| \leq 3^{j+1}\pi/(2\epsilon)} |\hat{f}(\xi)|^2 d\xi, \quad j = 0, 1, 2, \ldots.
\]
Choosing \(N = [\log A]\) we then obtain
\[
\sum_{j=0}^{N} (\omega_{1,2}[f](3^{-j}\epsilon))^2 \geq c \cdot \sum_{j=0}^{N} \int_{3^{j}\pi/(2\epsilon) \leq |\xi| \leq 3^{j+1}\pi/(2\epsilon)} |\hat{f}(\xi)|^2 d\xi
\]
\[
\geq c \cdot \sum_{j=0}^{N} \int_{\pi/(2\epsilon) \leq |\xi| \leq A\pi/(2\epsilon)} |\hat{f}(\xi)|^2 d\xi
\]
\[
\geq c' \cdot \epsilon^\gamma, \quad 0 < \epsilon \leq 1,
\]
where we have used (2.20). Therefore
\[
\liminf_{\epsilon \to 0} \epsilon^{-\gamma} \sum_{j=0}^{N} (\omega_{1,2}[f](3^{-j}\epsilon))^2 \geq c > 0
\]
and hence as \(N \in \mathbb{N}\) is fixed, there exists at least one \(J \in \{1, \ldots, N\}\) such that
\[
\hat{c} := \liminf_{\epsilon \to 0} \epsilon^{-\gamma} (\omega_{1,2}[f](3^{-J}\epsilon))^2 > 0.
\]
Then
\[
(\omega_{1,2}[f](3^{-J}\epsilon))^2 \geq \hat{c} \cdot \epsilon^\gamma, \quad 0 < \epsilon \leq 1,
\]
and denoting \(3^{-J}\epsilon\) by \(\epsilon\) we finish the proof of the lower inequality in (1.16). \(\square\)

We note finally that the proof of Corollary 1.5 for \(d = 1\) can be modified to give an alternative proof of Corollary 1.6 also for all \(d \geq 2\) from scratch (in this connection, see an explanation of an argument from [1] given in the proof of [6] Lemma 3.4.1).

**Remark 2.4.** After the above general discussion it is not difficult to understand why Corollary 1.5 fails for \(\gamma = 2\). Recall that in (2.23), \(p = p' = 2\) and \(m = 1\). Let \(\hat{f} \in L^2(\mathbb{R}^d)\) be defined by \(\hat{f}(\xi) := \xi^{-(d/2) - 1}\) for \(|\xi| \geq 1\) and smooth for \(|\xi| < 1\). Let \(f \in L^2(\mathbb{R}^d)\) be the inverse F.t. of this \(\hat{f}\). It is easy to check that for some \(\hat{b}_1, \hat{b}_2 > 0\) that depend on \(\hat{f}\)
\[
\hat{b}_1 t^{-1}(\log t)^{1/2} \leq \psi_{2,1}[\hat{f}] (t) \leq \hat{b}_2 t^{-1}(\log t)^{1/2}, \quad t \geq 2
\]
whereas for certain \(\hat{c}_1, \hat{c}_2 > 0\) that depend on \(\hat{f}\)
\[
\hat{c}_1 t^{-1} \leq \psi_{2}[\hat{f}] (t) \leq \hat{c}_2 t^{-1}, \quad t \geq 2.
\]
Note that by (2.23) and (2.27) for \(c_1, c_2 > 0\) that depend on \(f\)
\[
c_1 t^{-1}(\log t)^{1/2} \leq \omega_{2,1,2}[f](1/t) \leq c_2 t^{-1}(\log t)^{1/2}, \quad t \geq 2.
\]
This shows that Corollary 1.5 fails for \(\gamma = 2\). It is only true in the case \(\gamma = 2\) that the upper estimate in (1.14) implies the upper estimate in (1.17) (simply because for any \(f \in L^2(\mathbb{R}^d)\) and all \(t > 0\), \(\psi_2[f](t) \leq \psi_{2,1}[\hat{f}] (t) \leq c(2, d, 1) \omega_{2,1,2}[f](1/t)\) in
view of (1.15) and (1.10)). The fact that the lower estimate in (1.17) need not hold is shown by considering the example of $\hat{f} \in C_0^\infty(\mathbb{R}^d)$, $\hat{f} \not\equiv 0$ (for which $\psi_2(\hat{f})(t) = 0$ identically for large $t$). Finally, the first example of this remark shows the upper estimate in (1.16) for $\gamma = 2$ need not follow even from a two-sided estimate in (1.17).

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