OPERATOR REALIZATION OF THE $SU(2)$ WZNW MODEL

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Decoupling the chiral dynamics in the canonical approach to the WZNW model requires an extended phase space that includes left and right monodromy variables \( M \) and \( \bar{M} \). Earlier work on the subject, which traced back the quantum group symmetry of the model to the Lie–Poisson symmetry of the chiral symplectic form, left some open questions:

– How to reconcile the necessity to set \( M\bar{M}^{-1} = 1 \) (in order to recover the monodromy invariance of the local 2D group valued field \( g = uu \)) with the fact the \( M \) and \( \bar{M} \) obey different exchange relations?

– What is the status of the quantum symmetry in the 2D theory in which the chiral fields \( u(x - t) \) and \( \bar{u}(x + t) \) commute?

– Is there a consistent operator formalism in the chiral (and the extended 2D) theory in the continuum limit?

We propose a constructive affirmative answer to these questions for \( G = SU(2) \) by presenting the quantum fields \( u \) and \( \bar{u} \) as sums of products of chiral vertex operators and \( q \)-Bose creation and annihilation operators.
1 Introduction and summary of earlier results

A key step in the treatment of the Wess–Zumino–Novikov–Witten (WZNW) model [1], both axiomatic [2–4] and Lagrangean [5–9], is the construction of chiral vertex operators (CVO) and conformal current algebra blocks. While the resulting 2-dimensional (2D) braid invariant correlation functions satisfy all Wightman axioms and can be used to reconstruct the quantum field operator formalism in the physical state space, this is not the case for the chiral theory. Trying to combine, following [10, 11], the CVO of a given weight into a quantum group tensor we have to abandon Hilbert space positivity. The alternative of using (weak) quasi Hopf algebras [12] requires giving up coassociativity.

Here we continue our study [9] of the canonical approach to the problem (which follows Gawędzki et al. [7]) specializing to the case $G = SU(2)$. We proceed to summarizing relevant background and earlier results which will serve as a starting point for the present paper.

The general group valued periodic solution $g(t, x + 2\pi) = g(t, x)$ of the WZNW equations of motion factorizes into a product of right and left movers’ factors,

$$g(t, x) = u(x - t)\bar{u}(x + t), \quad u, \bar{u} \in SU(2),$$

satisfying a weaker, twisted periodicity condition:

$$u(x + 2\pi) = u(x)M, \quad \bar{u}(\bar{x} + 2\pi) = \bar{M}^{-1}\bar{u}(\bar{x}). \quad (1.2)$$

The symplectic form of the 2D theory can be presented as a sum of two decoupled closed chiral 2–forms at the price of considering the monodromy matrices $M$ and $\bar{M}$ as independent of each other additional dynamical variables. One then derives [7] quadratic Poisson bracket relations for the components $M_\pm$ of the Gauss decomposition of $M$:

$$M = M_+ M_-^{-1}, \quad M_+ = e^{-i\Delta} \begin{pmatrix} d & b \\ 0 & d^{-1} \end{pmatrix}, \quad M_- = e^{i\Delta} \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \quad (1.3)$$

($\Delta = \frac{3}{4h}$ being the conformal dimension of $u$, see Eq.(1.13)) – and similarly for the bar variables. Using the tensor product notation of Faddeev et al. [13], $\hat{u} = u \otimes 1$, $\hat{\bar{u}} = 1 \otimes u$, one can write the quantized exchange relations in the form [7, 9]

$$\hat{\bar{u}}(x_2) \hat{u}(x_1) = \hat{u}(x_1) \hat{\bar{u}}(x_2) R(x_{12}), \quad x_{12} = x_1 - x_2 \quad (1.4)$$

where the quantum $R$–matrix is given by

$$R(x) = R^- \theta(x) + R^+ \theta(-x). \quad (1.5)$$
The step function $\theta$ on the infinite cover of the circle is assumed to have a periodic derivative:

$$2\pi \theta'(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{inx}, \quad \theta(x) + \theta(-x) = 1.$$

(1.6)

$R^\pm$ are $4 \times 4$ matrices solving (properly extended on the tensor cube of spaces) the Yang–Baxter equation

$$R_{12}^\varepsilon R_{13}^\pm R_{23}^\pm = R_{23}^\pm R_{13}^\pm R_{12}^\varepsilon, \quad \varepsilon = +, -.$$

(1.7)

They yield (upon multiplication with a permutation $P$, $P(x \otimes y) = y \otimes x$) a pair of braid operators with inverse eigenvalues:

$$\tilde{R}^\pm = R^\pm P = q^{\mp \frac{1}{2}} \Pi_3 - q^{\pm \frac{3}{2}} \Pi_1.$$

(1.8)

Here $\Pi_i$ ($i = 1, 3)$ are $i$–dimensional orthogonal projectors,

$$\Pi_i^2 = \Pi_i = \Pi_i^*, \quad \Pi_1 \Pi_3 = 0, \quad \Pi_1 + \Pi_3 = 1, \quad \text{tr} \Pi_i = i (= 1, 3).$$

(1.9)

If we introduce the $SL_q(2)$ invariant “$q$–skew symmetric” tensor

$$(\mathbf{E}_{\alpha\beta}) = \begin{pmatrix} 0 & -q^{1/2} \\ q^{1/2} & 0 \end{pmatrix}, \quad \bar{q} = q^{-1}$$

(1.10a)

and its inverse

$$(\mathbf{E}^{\alpha\beta}) = (-\mathbf{E}_{\alpha\beta}), \quad \mathbf{E}^{\alpha\sigma} \mathbf{E}_{\sigma\beta} = \delta^\alpha_\beta,$$

(1.10b)

then we can write

$$\Pi_1^{\alpha\beta} = -\frac{1}{[2]} \mathbf{E}^{\alpha\beta} \mathbf{E}_{\rho\sigma}, \quad (-\mathbf{E}^{\alpha\beta} \mathbf{E}_{\alpha\beta} = [2] := q + \bar{q})$$

(1.11a)

$$\Pi_3^{\alpha\beta} = \delta^\alpha_\rho \delta^\beta_\sigma + \frac{1}{[2]} \mathbf{E}^{\alpha\beta} \mathbf{E}_{\rho\sigma}.$$

(1.11b)

The condition that the eigenvalues of $\tilde{R}^-$ coincide with those of the braid matrices (computed from the conformal and $SU(2)$ invariant 3–point functions) fixes the value of $q$; in particular, the triply degenerate eigenvalue is

$$q^{1/2} = \exp \left\{ i\pi \left( \Delta_1 - 2\Delta_{1/2} \right) \right\} = \exp \left( \frac{\pi i}{2h} \right), \quad h = k + 2$$

(1.12)

where $k$ is the Kac-Moody level, the conformal dimension $\Delta_I$ for a CVO of isospin $I$ (at height $h$) being

$$\Delta_I = \frac{1}{h} I(I + 1).$$

(1.13)
The $R$–matrix so obtained yields the correct $k \to \infty$ limit in terms of classical Poisson brackets.

The exchange relations involving the triangular factors $M_\pm$ of the monodromy matrix (1.3) are also expressed in terms of $R^\pm$:

$$\frac{1}{2} M_\pm \frac{2}{2} (x) = \frac{2}{2} (x) R^\pm \frac{1}{2} M_\pm,$$  \hspace{1cm} (1.14)

$$R^\varepsilon \frac{1}{2} M_\pm \frac{2}{2} M_\pm = \frac{2}{2} M_\pm R^\varepsilon, \hspace{1cm} \varepsilon = +, -, \hspace{1cm} (1.15a)$$

$$R^\pm \frac{1}{2} M_\pm \frac{2}{2} M_\mp = \frac{2}{2} M_\mp R^\pm. \hspace{1cm} (1.15b)$$

The symmetry of (1.4) and (1.14) under (local) left shifts of $u(x)$ is generated by a periodic chiral current $j(x) = j^a(x)\sigma_a \in su_2$ such that

$$\begin{align*}
[ \frac{1}{2} u(x_1), \frac{2}{2} j(x_2) ] & = C \frac{1}{2} u(x_1)\delta(x_{12}), \hspace{1cm} (1.16) \\
[ \frac{1}{2} j(x_1), \frac{2}{2} j(x_2) ] & = [C, \frac{1}{2} j(x_1)]\delta(x_{12}) + ikC\delta'(x_{12}). \hspace{1cm} (1.17)
\end{align*}$$

Here $C$ is the Casimir invariant

$$C = \frac{1}{2} \frac{1}{2} \sigma_a \sigma_a = P - \frac{1}{2} 1. \hspace{1cm} (1.18)$$

Classically the current $j$ is expressed in terms of the group valued field $u$ as $j = -ik u'u^{-1}$.

The quantum version of this relation is the operator Knizhnik–Zamolodchikov equation [2, 3]:

$$- ihu'(x) = : j(x)u(x) :, \hspace{1cm} (1.19)$$

where the normal product in the right–hand side is defined in terms of the frequency parts of $j$:

$$: j(x)u(x) : = \sigma_a \left\{ j^a_{(+)}(x)u(x) + u(x)j^a_{(-)}(x) \right\},$$

$$j^a_{(-)}(x) = \sum_{n=0}^{\infty} J_n e^{inx}, \hspace{1cm} j^a_{(+)}(x) = \sum_{n=-\infty}^{-1} J_n e^{inx}, \hspace{1cm} (1.20)$$

$$j(x) = j_{(+)}(x) + j_{(-)}(x), \hspace{1cm} j_{(-)}(x)|0\rangle = 0 = \langle 0|j_{(+)}(x).$$

Similar, although not identical, relations are derived [7, 9] for the left mover (bar) sector:

$$\frac{2}{2} \bar{u}(\bar{x}_2) \frac{1}{2} \bar{u}(\bar{x}_1) = \bar{R}(\bar{x}_{21}) \frac{1}{2} \bar{u}(\bar{x}_1) \frac{2}{2} \bar{u}(\bar{x}_2), \hspace{1cm} (1.21a)$$

where

$$\bar{R}(\bar{x}) = \bar{R}^- \theta(\bar{x}) + \bar{R}^+ \theta(-\bar{x}). \hspace{1cm} (1.21b)$$
and $\bar{R}^\pm$ are related to $R^\pm$ by

$$\bar{R}^\pm = P R^\pm P, \quad \bar{R}^\pm R^\mp = 1; \quad (1.21c)$$

$$\frac{1}{M_\pm} \bar{u} (\bar{x}) = R^\pm \frac{2}{\bar{\bar{u}}} (\bar{x}) \frac{1}{M_\pm}; \quad (1.22)$$

$$\frac{1}{M_\pm} M_\mp R^\varepsilon = R^\varepsilon \frac{1}{M_\pm}; \quad \varepsilon = +, - \quad (1.23a)$$

$$\frac{1}{M_\pm} M_\pm R^\mp = R^\mp \frac{1}{M_\pm}; \quad (1.23b)$$

$$\left[ \frac{1}{j} (\bar{x}_1), \frac{2}{\bar{\bar{u}}} (\bar{x}_2) \right] = \delta (\bar{x}_12) \frac{2}{\bar{\bar{u}}} (\bar{x}_2) C. \quad (1.24)$$

The left and right sectors are completely decoupled, their dynamical variables commute between each other:

$$\left[ \frac{1}{1} u (x), \frac{2}{\bar{\bar{u}}} (x') \right] = 0 = \left[ \frac{1}{M_\varepsilon}, \frac{2}{M_\varepsilon} \right] = \left[ \frac{1}{\bar{u} (x)}, \frac{2}{\bar{\bar{M}}_\varepsilon} \right] = \left[ \frac{1}{M_\varepsilon}, \frac{2}{\bar{\bar{u}} (\bar{x})} \right]. \quad (1.25)$$

It is instructive to verify that the above exchange relations imply the local commutativity of $g$ (1.1):

$$[g(t_1, x_1), g(t_2, x_2)] = 0 \quad \text{for} \quad (x_{12} - t_{12})(x_{12} + t_{12}) > 0. \quad (1.26)$$

There appears to be a price for the decoupling of the left and right dynamics. There is a difference between the monodromy exchange relations (1.15) and (1.23) (which can be traced back to a sign difference between the corresponding classical Poisson brackets – see [9]). Hence, one cannot identify the dynamical variables $M$ and $\bar{M}$. The question arises: can we then recover the monodromy invariance (i.e., the single valuedness) of the 2D field (1.1)? A related problem emerges in trying to make precise the quantum group invariance of $g$ (see Sec.2 below). The clue to the solution of these problems lies in the realization that the left–right extended WZNW model has features of a gauge quantum field theory. The physical Hilbert space of the 2D theory is a proper subquotient of the tensor product of state spaces of the chiral theories. The monodromy invariance of the product (1.1) is recovered in a weak sense – as an equation for matrix elements between physical states. The quantum group properties of the 2D theory are examined in similar terms. We start by factorizing the dependence of the chiral field $u(x) = (u_\beta^\alpha(x))$ on its quantum group index $\beta$ and use the ensuing $q$–Bose creation and annihilation operators to express the monodromy matrices $M_{\pm}$. 

5
2 Operator realization of the chiral exchange relations

2A Quantum group symmetry. Basic building blocks: the $U_q$ oscillators

The Lie–Poisson symmetry of the quadratic Poisson brackets of $u(x)$ and $M_{\pm}$ [6, 7] gives rise to a “quantum symmetry” under $GL_q(2)$ transformation

$$u(x) \to u(x)T, \quad M \to T^{-1}MT$$

where the $2 \times 2$ matrix $T$ has non–commuting elements characterized by the exchange relations [13]

$$\frac{1}{2} T^\epsilon R^\epsilon = R^\epsilon \frac{1}{2} T^\epsilon, \quad \epsilon = +, -$$

or

$$T^\beta T^\alpha = q T^\alpha T^\beta, \quad [T^1_2, T^2_1] = 0, \quad [T^1_2, T^2_2] = (\bar{q} - q) T^1_2 T^2_1. \quad (2.2b)$$

$GL_q(2)$ has for generic $q$ a 1–dimensional centre generated by the $q$–determinant:

$$\det_q T = T^1_1 T^2_2 - \bar{q} T^1_2 T^2_1 = T^2_2 T^1_1 - q T^1_2 T^2_1;$$

this allows to define the factor algebra

$$SL_q(2) = \{T \in GL_q(2); \ \det_q T = 1\}.$$ (2.3b)

It follows from (2.2) that the tensor product of quantum group matrices commutes with the braid operators (1.8):

$$[\hat{R}^\pm, \frac{1}{2} T^\epsilon] = 0 \Leftrightarrow \left[ \Pi_i, \frac{1}{2} T^\epsilon \right] = 0, \quad i = 1, 3.$$ (2.4)

The entries of $T$ can be viewed as linear functionals on the quantum universal enveloping algebra (QUEA) $U_q = U_q(s\ell_2)$ [13].

An elementary realization of the exchange relations (1.4) is given by two (conjugate) pairs of $SL_q(2)$ covariant oscillators $a^\pm_\alpha$ satisfying

$$a^\alpha_\alpha a^\beta_\beta = q^{1/2} a^\alpha_\beta a^\beta_\alpha (\hat{R}^\pm)^{\rho\sigma}_{\alpha\beta}$$

$$a^-_\alpha a^+_\beta = q^{1/2} a^+_\beta a^-_\alpha (\hat{R}^+)^{\rho\sigma}_{\alpha\beta} + \bar{q}^N \mathcal{E}_{\alpha\beta}.$$ (2.5b)
where $\mathcal{E}_{\alpha \beta}$ is the $SL_q(2)$ invariant tensor (1.10),

$$\mathcal{E}_{\rho \sigma} T_{\alpha}^{\rho} T_{\beta}^{\sigma} = \mathcal{E}_{\alpha \beta} \left( -q^{\pm} \mathcal{E}_{\rho \sigma} (\hat{R}^{\pm})_{\alpha \beta}^{\rho \sigma} \right). \tag{2.6}$$

We shall interpret $a_\alpha^+$ and $a_\alpha^-$ as creation and annihilation operators setting

$$a_\alpha^- |0\rangle = 0 = \langle 0 | a_\beta^+ \quad \Rightarrow \quad a_\alpha^- a_\beta^+ |0\rangle = \mathcal{E}_{\alpha \beta} |0\rangle. \tag{2.7}$$

The $SL_q(2)$ invariant combinations of $a_\alpha^\pm$ are expressed in terms of the number operator $N$ determined (mod $2h$ for $q^h = -1$) by

$$q^N a_\alpha^\pm = a_\alpha^\pm q^{N \pm 1}, \quad (q^N - 1) |0\rangle = 0; \tag{2.8}$$

we have

$$a_\alpha^+ \mathcal{E}_{\alpha \beta} a_\beta^- = [N] := \frac{q^N - \bar{q}^N}{q - \bar{q}}, \quad a_\alpha^- \mathcal{E}_{\alpha \beta} a_\beta^+ = -[N + 2], \tag{2.9a}$$

$$a_\alpha^+ \mathcal{E}_{\alpha \beta} a_\beta^+ = 0 \Leftrightarrow a_\alpha^\pm a_\beta^\pm = qa_\alpha^\pm a_\beta^\pm. \tag{2.9b}$$

The $SL_q(2)$ invariance of the exchange relations (2.5) is equivalent to their $U_q$ invariance [14]. Introducing the raising and lowering Chevalley generators $E$ and $F$ such that

$$[E, F] = [H], \quad q^H E = Eq^{H+2}, \quad q^H F = Fq^{H-2} \tag{2.10a}$$

and defining their coproduct by

$$\Delta(E) = E \otimes q^H + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + \bar{q}^H \otimes F, \quad \Delta(q^H) = q^H \otimes q^H, \tag{2.10b}$$

we verify that the relations (2.5) are invariant under the following $U_q$ transformation law:

$$q^H a_1^\pm = a_1^\pm q^{H+1}, \quad q^H a_2^\pm = a_2^\pm q^{H-1} \tag{2.11a}$$

$$[E, a_1^\pm] = 0 = Fa_2^\pm - qa_2^\pm F, \quad [E, a_2^\pm] = a_1^\pm q^H, \quad Fa_1^\pm - qa_1^\pm F = a_2^\pm. \tag{2.11b}$$

The $U_q(gl_2)$ Cartan subalgebra generated by $q^H$ and $q^N$ involves the individual number operators $N_\alpha, \alpha = 1, 2$, satisfying

$$N_1 + N_2 = N, \quad N_1 - N_2 = H \tag{2.12a}$$

and consequently, in view of (2.8), (2.11a),

$$q^N a_\beta^+ = a_\beta^+ q^{N_\alpha + \delta_{\alpha \beta}}, \quad [N_\alpha, a_1^+ a_2^-] = 0 = [N_\alpha, a_2^+ a_1^-]. \tag{2.12b}$$
The Fock space of the q-oscillator algebra (with an \( U_q \) invariant vacuum vector satisfying (2.7)) possesses two \( U_q \) invariant forms: a hermitean (sesquilinear) one, \( \langle \ , \rangle \) , and a bilinear one, \( \langle \ , \rangle \). There are, accordingly, two antiinvolutions (i.e. involutive algebra antihomomorphisms) that extend the known ones for \( U_q \) – cf. [15]. One can define – for any \( O \) in the oscillator algebra – an antilinear hermitean conjugation \( O \to O^* \) for which
\[
E^* = F \ , \ F^* = E \ , \ (q^H)^* = \bar{q}^H
\] (2.13a)
(and \( \Delta(X^*) = \Delta(X)^* \) for \( (X_1 \otimes X_2)^* = X_2^* \otimes X_1^* \), \( \forall X, X_1, X_2 \in U_q \) such that the following counterpart of the familiar relation between (undeformed) creation and annihilation, and number operators holds:
\[
a_\alpha^+(a_\alpha^+)^* = [N_\alpha] \ , \ (a_\alpha^+)^*a_\alpha^+ = [N_\alpha + 1] \ , \ \alpha = 1, 2 \ ,
\] (2.13b)
and a linear transposition \( O \to \, ^tO \), satisfying
\[
^tE = Fq^H \ , \ ^tF = \bar{q}^H E \ , \ ^t(q^H) = q^H
\] (2.14a)
(so that \( \Delta(^tX) = \Delta(X)^* \) for \( ^t(X_1 \otimes X_2) = ^tX_1 \otimes ^tX_2 \), and
\[
\sum_{\alpha=1}^2 a_\alpha^+ ^t(a_\alpha^+) = [N] \ , \ \sum_{\alpha=1}^2 a_\alpha^- ^t(a_\alpha^+) = 0 \ .
\] (2.14b)

It is an easy exercise to verify that (2.13b) is satisfied if we set
\[
(a_1^+)^* = q^\frac{1}{2} \quad N_2 a_2^- = :a_1 \ , \ (a_2^+)^* = -q^\frac{1}{2} N_1 a_1^- = :a_2 \ , \ a_\alpha^* = a_\alpha^+ \ , \ (q^{N_\alpha})^* = \bar{q}^{N_\alpha} \ ,
\] (2.15)
while (2.14b) will take place if
\[
^t(a_\alpha^+) = \mathcal{E}^{\alpha\beta} a_\beta^- = :a_\alpha^+ \ , \ ^t(a_\alpha^-) = -\mathcal{E}^{\alpha\beta} a_\beta^+ , \ ^t(q^{N_\alpha}) = q^{N_\alpha}
\] (2.16a)
so that
\[
a^1 = q^{-N_2} a_1 \ , \ a^2 = q^{N_1} a_2 \ .
\] (2.16b)

The (infinite dimensional) Fock space \( \tilde{\mathcal{F}} \) is spanned by the vectors
\[
\Phi_{n_1n_2} = (a_1^+)^{n_1} (a_2^+)^{n_2} |0\rangle \ , \ n_1, n_2 \in \mathbb{Z}
\] (2.17a)
that form an orthogonal basis with respect to both forms:
\[
\langle \Phi_{n_1n_2}, \Phi_{m_1m_2} \rangle = \langle 0| a_2^{n_2} a_1^{n_1} (a_1^+)^{m_1} (a_2^+)^{m_2} |0\rangle =
\]
\[
= \delta_{n_1m_1} \delta_{n_2m_2} [n_1]![n_2]!
\] (2.17b)
\[
\langle \Phi_{n_1n_2}, \Phi_{m_1m_2} \rangle = \langle 0| (a_2^{n_2} a_1^{n_1}) (a_1^+)^{m_1} (a_2^+)^{m_2} |0\rangle =
\]
\[
= \delta_{n_1m_1} \delta_{n_2m_2} \bar{q}^{n_1n_2} [n_1]![n_2]! \ .
\] (2.17c)
The sesquilinear form (2.17b) is real but not positive definite (for \( q \bar{q} = 1, q \neq 1 \)). For \( q^{1/2} \) given by (1.12) \( (q^h = -1) \) it is semidefinite: \( \tilde{F} \) contains an infinite dimensional subspace \( \tilde{F}^{(0)} \) of null (zero norm) vectors spanned by \( |n_1, n_2) \) with \( \max(n_1, n_2) \geq h \). Setting

\[
(a_n^\pm)^h = 0 ,
\]

one obtains a finite \( (h^2) \) dimensional Fock quotient space \( F_h = \tilde{F}/\tilde{F}^{(0)} \) on which the sesquilinear form (2.17b) is already positive definite. \( F_h \) splits into a direct sum of unitary irreducible representations of \( U_q \) with spins \( 0 \leq I \leq \frac{h-1}{2} \). Except for \( I = \frac{h-1}{2} \), all other spins appear twice – in a “standard” and a “shadow” representation differing by the eigenvalue of \( q^N \).

One can identify the \( U_q \) generators \( E, F \) and \( q^{\pm H} \) with

\[
E = a_1^* a_2 q^{N_1} , \quad F = \bar{q}^{N_1} a_2^* a_1 , \quad q^{\pm H} = q^{\pm (N_1-N_2)} .
\]

In verifying the consistency between (2.19) and (2.10a), (2.11) one uses (2.12b), (2.13b) and the definitions (2.5), (2.15) which imply

\[
a_2 a_1 = q a_1 a_2 , \quad a_1 a_2^* = q a_2^* a_1 ( \Rightarrow a_2^* a_1^* = q a_1^* a_2^* , \quad a_2 a_1^* = \bar{q} a_1^* a_2 ) .
\]

We note that (2.10a), (2.11) and (2.19) (implying \( EF = [N_1 [N_2+1], FE = [N_1+1] [N_2] \) yield the basic (anti) commutation relations of the Chevalley generators of the quantum superalgebra \( U_{q^{1/2}}(osp(1, 4)) \). We are, in fact, dealing with its \( (h^2 \) dimensional Fock space) singleton representation [16].

2B Factorized form of \( u(x) \). Monodromy matrices

The correlation functions of the chiral field \( u(x) = (u_{\beta}^\beta(x)) \) can be reconstructed if we factorize the dependence on the \( SU(2) \) index \( \alpha \) and the quantum group index \( \beta \) setting

\[
u_{\beta}^\beta(x) = u_+^\alpha(x,N) a_\beta^+ + a_\beta^- u_-^\alpha(x,N) ,
\]

with

\[
u_- |0) = 0 = (0|u_+ .
\]

It is assumed that \( u_\pm \) and \( a_\pm \) are only coupled through the number operator:

\[
a_\beta^- u_\beta^\alpha(x,N) = u_\beta^\alpha(x,N-\varepsilon) a_\beta^- .
\]

Inserting (2.21) into (1.4) and using (2.5) we “diagonalize” the exchange relations for the CVO \( u_\varepsilon^\alpha \). The result is particularly simple for an equal frequency pair:

\[
u_{\pm}^{\alpha_2}(x_2, N \pm 1) u_{\pm}^{\alpha_1}(x_1, N) = q_{\pm}^{x(x_{12})} u_{\pm}^{\alpha_1}(x_1, N \pm 1) u_{\pm}^{\alpha_2}(x_2, N)
\]

(2.23a)
where
\[\varepsilon(x) = \theta(x) - \theta(-x)\quad (2.23b)\]

For a product of opposite frequency \( u \)'s we introduce a (symmetric) 3–vector and a (skewsymmetric) scalar bilocal combination (obtained by applying the projectors \( \Pi_3 \) and \( \Pi_1 \) \((1.11)\), respectively):

\[
V^{\alpha_1\alpha_2}(x_1, x_2; N) = u^{\alpha_1}_+(x_1, N) u^{\alpha_2}_-(x_2, N) +
+ q u^{\alpha_1}_-(x_1, N + 1) u^{\alpha_2}_+(x_2, N + 1)\quad (2.24a)
\]

\[
S^{\alpha_1\alpha_2}(x_1, x_2; N) = u^{\alpha_1}_+(x_1, N) u^{\alpha_2}_-(x_2, N)[N] -
- u^{\alpha_1}_-(x_1, N + 1) u^{\alpha_2}_+(x_2, N + 1)[N + 2] \quad (2.24b)
\]

These are again just multiplied by a phase, under an exchange of the arguments:

\[
V^{\alpha_2\alpha_1}(x_2, x_1; N) = q^{\frac{1}{2}\varepsilon(x_{12})} V^{\alpha_1\alpha_2}(x_1, x_2; N) \quad (2.25a)
\]

\[
S^{\alpha_2\alpha_1}(x_2, x_1; N) = -q^{\frac{3}{2}\varepsilon(x_{12})} S^{\alpha_1\alpha_2}(x_1, x_2; N) \quad (2.25b)
\]

(In deriving \((2.25)\) we have used the properties \((2.5)\) of \( a^\pm \) which imply, in particular,
\((a^-_{\alpha_1} a^+_{{\alpha_2}} - q a^+_{{\alpha_2}} a^-_{\alpha_1}) \Pi_3 a^{\alpha_1\alpha_2} = 0.\)

The term “CVO” for \( u_\pm \) is justified by the fact that they diagonalize the monodromy:

\[
u^{\alpha}_\pm(x + 2\pi, N) = e^{-2\pi i L_0} u^{\alpha}_\pm(x, N) e^{2\pi i L_0} = q^{\mp \left(N + \frac{1}{2}\right)} u^{\alpha}_\pm(x, N) \quad (2.26)
\]

(In choosing the sign of \( L_0 \) in the middle expression we have taken into account the fact that \( u \) depends on \( x - t \).) The true justification of the representation \((2.21)\) stems from the possibility to express \( M \), satisfying

\[
a^-_{\sigma} M^\sigma_\beta = a^-_{\beta} q^{N + \frac{1}{2}}, \quad a^+_{\sigma} M^\sigma_\beta = a^+_{\beta} \bar{q}^{N + \frac{1}{2}} = \bar{q}^{N + \frac{1}{2}} a^+_{\beta} \quad (2.27)
\]

and the exchange relations \((1.14), (1.15)\), in terms of \( a^\pm \):

\[
M^\sigma_\beta = (q^{\frac{3}{2}} - q^{\frac{1}{2}}) \mathcal{E}^\sigma_{\alpha\beta} a^+_{\alpha} a^-_\beta \bar{q}^{N + \frac{1}{2}} \delta^\sigma_\beta \quad (2.28)
\]

Eq.\((2.27)\) follows from the last equation \((2.9)\). To verify the exchange relations we work out the Gauss decomposition \((1.3)\) for \( M \) \((2.28)\) with the result

\[
d = q^{\frac{1}{2}H}, \quad b = (1 - q^2) q^{\frac{1}{2}H} F, \quad c = (q^2 - 1) E q^{\frac{1}{2}H} \quad (2.29)
\]

We note that due to the non–commutativity of \( b, c \) and \( d \) the inverse of \( M_\pm \) \((1.3)\) are

\[
M_+^{-1} = q^{\frac{3}{2}} \begin{pmatrix} d^{-1} & -\bar{q}b \\ 0 & d \end{pmatrix}, \quad M_-^{-1} = q^{\frac{3}{2}} \begin{pmatrix} d & 0 \\ -qc & d^{-1} \end{pmatrix} \quad (2.30)
\]
(for \( db = qbd \), \( cd = qdc \)). Eq. (1.14) is translated into

\[
(M_\pm)^\alpha_\beta a_\gamma^\epsilon = (R^{\pm})^\alpha_\sigma a_\rho^\sigma (M_\pm)^\beta_\sigma ,
\]

which, in view of (2.29), expresses the \( U_q \) transformation law of \( a^\epsilon \) (cf. (2.12)).

We note that the \( n \)-point correlation function of the chiral field (2.21) combines together all \( n \)-point conformal blocks (cf. [10]).

2C Factorization and monodromy for the bar sector

In the second factor, \( \bar{u} \), of the product (1.1) the role of the \( U_q \) and \( SU(2) \) indices is reversed and we can write

\[
\bar{u}_\gamma^\beta (\bar{x}) = \bar{u}_\gamma^\beta (\bar{x}, \bar{N}) \bar{a}_+^\beta + \bar{a}_-^\alpha \bar{u}_\gamma^\epsilon (\bar{x}, \bar{N}) .
\]

Here \( \bar{a}_\pm \) are new (independent) pairs of \( q \)-Bose oscillators,

\[
[a_\beta^\epsilon, \bar{a}_\epsilon^\beta] = 0,
\]

such that \( \bar{a}_\beta^\epsilon |0\rangle = 0 = \langle 0|\bar{a}_+^\beta \) and

\[
\bar{a}_+^\alpha \bar{a}_-^\beta = q^{\frac{1}{2}} (R^\pm)_{\rho\sigma}^\alpha \bar{a}_\rho^\sigma \bar{a}_\sigma^\beta , \quad (2.34a)
\]

\[
\bar{a}_-^\alpha \bar{a}_+^\beta = q^{\frac{1}{2}} (R^\pm)_{\rho\sigma}^\alpha \bar{a}_\rho^\sigma \bar{a}_\sigma^\beta - \bar{q}^\bar{N} E^{\alpha\beta} \quad (2.34b)
\]

where we have noted that, in view of (1.8), (1.21c),

\[
P \bar{R}^\pm = \bar{R}^\pm .
\]

The symmetry of the braid matrices \( \bar{R}^\pm \) and the relation \( (-E^{\alpha\beta}) = (E_{\alpha\beta}) \) (see (1.10b)) imply that \( \bar{a}_\pm^\beta \) satisfy exactly the same exchange relations as \( a_\pm^\beta \). We have, in particular,

\[
\langle 0|\bar{a}_-^\alpha \bar{a}_+^\beta |0\rangle = -E^{\alpha\beta} , \quad (2.36)
\]

\[
\bar{a}_+^\alpha E_{\alpha\beta} \bar{a}_-^\beta = -[\bar{N}] , \quad \bar{a}_-^\alpha E_{\alpha\beta} \bar{a}_+^\beta = [\bar{N} + 2] , \quad \bar{a}_+^\alpha E_{\alpha\beta} \bar{a}_+^\beta = 0 . \quad (2.37)
\]

The definition of the \( \bar{u} \) monodromy,

\[
\bar{M}^{-1} \bar{u}(\bar{x}) = \bar{u}(\bar{x} + 2\pi) = e^{2\pi i \bar{L}_0} \bar{u}(\bar{x})e^{-2\pi i \bar{L}_0} = \bar{a}_- \bar{q}^{\bar{N} + \frac{1}{2}} \bar{u}^-(\bar{x}, \bar{N}) + q^{\bar{N} + \frac{1}{2}} \bar{u}^+(\bar{x}, \bar{N}) \bar{a}_+ \quad (2.38)
\]
yields the expression
\[ (\bar{M}^{-1})^\beta_\gamma = (q^{1/2} - \bar{q}^{1/2})\bar{a}_+^\beta \bar{a}_-^\gamma \mathcal{E}_{\sigma \gamma} + \bar{q}^{N+1/2} \delta^\beta_\gamma \] (2.39)
which satisfies
\[ \bar{M}^{-1}a_+ = \bar{q}^{N+1/2}a_+ = \bar{a}_-\bar{q}^{N+1/2}, \quad \bar{M}^{-1}a_- = q^{N+1/2}a_- . \] (2.40)
The Borel components of \( \bar{M}^{-1} = \bar{M}_-\bar{M}_+^{-1} \) are
\[ \bar{M}_- = q^{1/4} \begin{pmatrix} q^{1/2} \bar{H} & 0 \\ (q^2 - 1)q^{1/2} \bar{F} & q^{1/2} \bar{H} \end{pmatrix}, \quad \bar{M}_+^{-1} = q^{1/4} \begin{pmatrix} q^{1/2} \bar{H} & -q(1 - q^2)\bar{E}\bar{q}^{1/2} \bar{H} \\ 0 & q^{1/2} \bar{H} \end{pmatrix} \] (2.41)
where the \( \bar{U}_q \) generators are given by
\[ E = \bar{a}^1\bar{a}^2q^{\bar{N}_1}, \quad F = \bar{q}^{\bar{N}_1}\bar{a}^2\bar{a}^1, \quad q^{\pm\bar{H}} = q^{\pm(\bar{N}_1-\bar{N}_2)} , (2.42)\]
while conjugation is defined by the bar counterpart of (2.15):
\[ \bar{a}_1 := q^{1/2+\bar{N}_2}a_2^1, \quad \bar{a}_2 := -q^{1/2+\bar{N}_1} a_2^1 . \] (2.43)
We recover the exchange relation (1.22) by noting the identity
\[ (\bar{M}_\pm)^\alpha_\beta \bar{a}_\varepsilon^\gamma = (R^\pm)^\alpha_\gamma_\sigma \bar{a}_\varepsilon^\sigma (\bar{M}_\pm)^\sigma_\beta , \quad \varepsilon = +, - . \] (2.44)
We shall impose again the relation
\[ (\bar{a}_\pm^\alpha)^h = 0 , \quad h = k + 2 , \] (2.45)
(cf. (2.18)) thus defining the (bar) \( h^2 \) dimensional Fock space \( \bar{F}_h \).
To sum up, we expressed the monodromy of both chiral sectors in terms of the corresponding QUEA generators. Thus the QUEA \( U_q \) and \( \bar{U}_q \) not only express the hidden symmetry of the (extended) WZNW model, they realize (the monodromy) part of the dynamical variables.

3 The 2D theory: weak monodromy and \( U_q \otimes \bar{U}_q \) invariance

We now address the question raised in the introduction: how are the expected properties of the local (observable) 2D field \( g \) (1.1) realized in the extended state space of the theory?
To answer this question we need to identify the physical space \( \mathcal{H} \). Let us denote by \( \widetilde{\mathcal{H}} \) the extended (tensor product) space generated from the vacuum by the action of \( u, \bar{u}, M_\pm \) and \( \bar{M}_\pm \). Concentrating on diagonal theories (which exist for all levels \( k \) and are the only ones present for odd \( k \) – see [17]) we consider the 2D field algebra \( \mathcal{A} = A_h(g(t, x)) \) generated by polynomials in the group valued field \( g = u\bar{u} \) and in the Lie algebra valued chiral currents \( j \) and \( \bar{j} \), and set \( \mathcal{H}' = \mathcal{A}|0\rangle \). Then the physical space is the subquotient \( \mathcal{H} = \mathcal{H}'/\mathcal{H}'' \) where \( \mathcal{H}'' \subset \mathcal{H}' \) is the maximal subspace orthogonal to all vectors in \( \mathcal{H}' \).

Note that the currents and the field are related since we can rewrite Eq.(1.19) in terms of \( g \):

\[
-\frac{i}{2}\hbar \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) g(t, x) = :j(x - t)\ g(t, x) : \quad \text{etc.} \tag{3.1}
\]

and the same is true for the commutation relation (1.16).

The algebra \( \mathcal{A} \) is reducible in \( \widetilde{\mathcal{H}} \). One can indeed verify using (1.14) and (1.22) that the operator matrices \( L_\pm \),

\[
(L_\pm)_\alpha^\beta = q^{\frac{\pm 3}{2}} (\bar{M}_\pm^{-1}M_\pm)_\alpha^\beta \tag{3.2}
\]

commute with \( g(t, x) \) (and – trivially – with the currents) and hence with \( \mathcal{A} \). The space \( \mathcal{H}' \) (and hence also the physical space \( \mathcal{H} \)) is an eigenspace of each of their matrix elements so that \( ((L_\pm)_\beta^\alpha - \delta_\beta^\alpha)\mathcal{H}' = 0 \). This becomes obvious if we compute the products (3.2),

\[
L_+ = \begin{pmatrix} D & (1 - q^2)B \\ 0 & D^{-1} \end{pmatrix}, \quad L_- = \begin{pmatrix} D^{-1} & 0 \\ (q^2 - 1)C & D \end{pmatrix}, \tag{3.3a}
\]

with

\[
D = q^{\frac{1}{2}(H - \bar{H})}, \quad B = q^{\frac{1}{2}(H + \bar{H})}F - \bar{E} q^{\frac{1}{2}(H - \bar{H} + 2)}, \quad C = E q^{\frac{1}{2}(H - \bar{H})} - q^{\frac{1}{2}(H + \bar{H} - 2)}\bar{F} \tag{3.3b}
\]

and act on the vacuum vector. Since

\[
[C, B] = [H - \bar{H}], \quad q^{H - \bar{H}}C = q^2Cq^{H - \bar{H}}, \quad q^{H - \bar{H}}B = q^{-2}Bq^{H - \bar{H}}, \tag{3.3c}
\]

comparison between (1.3), (2.29) and (3.3) suggests that \( CD^{-1}, DB \) and \( q^{\pm(H - \bar{H})} = D^{\mp 2} \) should be viewed as generators of the true \( U_q \) symmetry of the 2D theory.

It turns out that the gist of the matter is contained in the corresponding finite dimensional – oscillator algebra – problem. We therefore proceed to describe the subquotient \( \mathcal{F} = \mathcal{F}'/\mathcal{F}'' \) where \( \mathcal{F}' \) is the projection of \( \mathcal{H}' \) into the \( h^4 \) dimensional tensor product Fock space \( \mathcal{F}_h \otimes \bar{\mathcal{F}}_h \) and \( \mathcal{F}'' \) is the subspace of zero norm vectors in \( \mathcal{F}' \).
Proposition 3.1  The bilinear form $\langle \ , \ \rangle$ satisfying $\langle \Phi, O\psi \rangle = \langle O\Phi, \psi \rangle$ for any $O$ in the oscillator algebra (see (2.16), (2.17)) is positive semidefinite on the subspace $\mathcal{F}'$ spanned by vectors of the form

$$|n\rangle = (A^+)^n|0\rangle := (a^+\bar{a}_+)^n|0\rangle, \quad n = 0, 1, \ldots, 2h - 2.$$ 

One has

$$(N_\alpha - \bar{N}_\alpha)\mathcal{F}' = 0 = a^+\bar{a}_+\mathcal{F}' . \quad (3.4)$$

$\mathcal{F}'$ admits an $h$ dimensional subspace $\mathcal{F}''$ of zero norm vectors. The quotient space $\mathcal{F} = \mathcal{F}'/\mathcal{F}''$ is $h - 1$ dimensional.

**Proof**  To establish the second equation (3.4) we shall prove the identity $[a^+\bar{a}_+, A^+] = 0$.

Taking, to fix the ideas, the upper sign and applying (2.34b) and (2.5), (2.9) we indeed find

$$a^+\bar{a}_-A^+ = a^+_\alpha a^+_\beta (q^\frac{1}{2}(\hat{R}^+)^{\alpha\beta}_{\rho\sigma} \bar{a}_+^\rho \bar{a}_-^\sigma - q^\bar{N}\mathcal{E}^{\alpha\beta}) = A^+a^+\bar{a}_-. \quad (3.5)$$

The rest of the proof reduces to a computation of the norm square of the basis vectors in $\mathcal{F}'$. To do this we first establish the commutation relation

$$_{[A^-, A^+]} = [N + \bar{N} + 2] \quad (A^- := a^-\bar{a}_-) . \quad (3.6)$$

Indeed, a straightforward computation using (2.5b), (2.34b) gives

$$A^- A^+ = (q^\frac{1}{2} a^+_\alpha a^-_\beta (\hat{R}^+)_{\alpha\beta}^{\rho\sigma} + q^\bar{N}\mathcal{E}^{\alpha\beta}) (q^\frac{1}{2} (\hat{R}^+)_{\rho\sigma}^{\alpha\beta} \bar{a}_+^\rho \bar{a}_-^\sigma - q^\bar{N}\mathcal{E}^{\alpha\beta}) = A^+ A^- + q^2( q^\bar{N}[N] + q^\bar{N} \bar{N} + (q - \bar{q})[N][\bar{N}] ) + [2]q^{N+\bar{N}}$$

which yields (3.6). As a simple corollary we derive

$$A^- |n\rangle = |n| [n + 1] |n\rangle \quad (3.7)$$

and hence

$$\langle n, n \rangle = [n + 1]! [n]! \quad ( [n + 1]! = [n + 1][n]!, \quad [0]! = 1 ) . \quad (3.8)$$

It follows that this norm square is non–zero for $n \leq h - 2$ only. Proposition 3.1 follows.

We are now ready to answer the first question stated in the beginning.

Proposition 3.2  The field $g$ (1.1) is single valued (monodromy invariant) on the physical subquotient $\mathcal{H}$:

$$(g(t, x + 2\pi) - g(t, x))\mathcal{H} = 0 . \quad (3.9)$$
Proof Using the factorized expressions for \( u \) and \( \tilde{u} \) we can restate (3.9) as a finite dimensional equation

\[
(a^\varepsilon \ M \tilde{M}^{-1} \ \tilde{a}_\varepsilon' - a^\varepsilon \tilde{a}_\varepsilon') F = 0 .
\] (3.10)

Due to (2.27) and (2.40) we have

\[
a^\pm M \tilde{M}^{-1} \tilde{a}_\pm = a^\pm \tilde{a}_\pm q^{\pm(N-\tilde{N})}, \quad a^\pm M \tilde{M}^{-1} \tilde{a}_\mp = a^\pm \tilde{a}_\mp q^{\mp(N+\tilde{N}+2)}
\] (3.11)

which allows to prove (3.10) in view of (3.4).

We come finally to the meaning of \( SL_q(2) \) symmetry of the 2D theory.

Quantum group invariance of vacuum expectation values of products of \( g(t_i, x_i) \) follows from the observation that the \( 2n \)-point correlation function of \( g \) is given by a sum of products of manifestly \( SL_q(2) \) invariant conformal current algebra blocks with matrix elements of powers of \( A^+ \) and \( A^- \). The computation of the latter (which could be the only source of breaking the \( SL_q(2) \otimes SL_q(2) \) symmetry) only relies – as we saw – on the exchange relations (2.5) and (2.34) and on the commutativity between \( a^\varepsilon \) and \( \tilde{a}_\varepsilon' \), all of which are quantum group invariant.

To sum up: the methods of covariant (indefinite metric space) formulation of quantum gauge field theory apply to the (left–right) monodromy extended \( SU(2) \) WZNW model. They provide an understanding of monodromy and quantum group invariance in a weak sense – as equations valid when applied to the physical subquotient. Extension of these results to the \( SU(n) \) models, for which the \( R \)-matrices are known explicitly, appears to be straightforward [9]. Incorporation of nondiagonal models in such a canonical approach is still a challenge.

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