PROJECTIONS IN NORMED LINEAR SPACES AND SUFFICIENT ENLARGEMENTS

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Abstract. Definition. A symmetric with respect to 0 bounded closed convex set $A$ in a finite dimensional normed space $X$ is called a sufficient enlargement for $X$ (or of $B(X)$) if for arbitrary isometric embedding of $X$ into a Banach space $Y$ there exists a projection $P : Y \to X$ such that $P(B(Y)) \subset A$ (by $B$ we denote the unit ball).

The notion of sufficient enlargement is implicit in the paper: B. Grünbaum, Projection constants, Trans. Amer. Math. Soc. 95 (1960) 451–465. It was explicitly introduced by the author in: M.I. Ostrovskii, Generalization of projection constants: sufficient enlargements, Extracta Math., 11 (1996), 466–474.

The main purpose of the present paper is to continue investigation of sufficient enlargements started in the papers cited above. In particular the author investigate sufficient enlargements whose support functions are in some directions close to those of the unit ball of the space, sufficient enlargements of minimal volume, sufficient enlargements for euclidean spaces.

§1. Introduction

We denote the unit ball (sphere) of a normed linear space $X$ by $B(X)$ ($S(X)$).

Convention. We shall use the term ball for symmetric with respect to 0 bounded closed convex set with nonempty interior in a finite dimensional linear space.

Definition 1. A ball $A$ in a finite dimensional normed space $X$ is called a sufficient enlargement for $X$ (or of $B(X)$) if for arbitrary isometric embedding $X \subset Y$ ($Y$ is a Banach space) there exists a projection $P : Y \to X$ such that $P(B(Y)) \subset A$. A minimal sufficient enlargement is defined to be a sufficient enlargement no proper subset of which is a sufficient enlargement.

The notion of sufficient enlargement is implicit in B. Grünbaum’s paper [2], it was explicitly introduced by the present author in [5].

The notion of sufficient enlargement is of interest because it is a natural geometric notion, it characterizes possible shadows of symmetric convex body onto a subspace, whose intersection with the body is given.

1991 Mathematics Subject Classification. Primary 46B07, 52A21.

Key words and phrases. Banach space, projection.

The research was supported by an INTAS grant and by a grant of TÜBİTAK, the first version of the paper was prepared when the author was visiting the University of Michigan (Ann Arbor) and Odense University. The author would like to thank N.J. Nielsen and M.S. Ramanujan for their hospitality. The author is obliged to the referees for many useful remarks.

Typeset by AMSTeX
The main purpose of the present paper is to continue investigation of sufficient enlargements started in [5]. In §2 we investigate sufficient enlargements whose support functions are in some directions close to those of the unit ball of the space, §3 is devoted to sufficient enlargements for euclidean spaces.

We refer to [4] and [7] for background on Banach space theory and to [6] for background on the theory of convex bodies.

Some recalls. Let $X$ and $Y$ be finite dimensional normed spaces and $T : X \rightarrow Y$ be a linear operator. An $l_\infty$-factorization of $T$ is a pair of operators $u_1 : X \rightarrow l_\infty$ and $u_2 : l_\infty \rightarrow Y$ satisfying $T = u_2u_1$. The $L_\infty$-factorable norm of $T$ is defined to be the inf $\|u_1\|\|u_2\|$, where the inf is taken over all $l_\infty$-factorizations.

An absolute projection constant of a finite dimensional normed linear space $X$ is defined to be the smallest positive real number $\lambda(X)$ such that for every isometric embedding $X \subset Y$ there exists a continuous linear projection $P : Y \rightarrow X$ with $\|P\| \leq \lambda(X)$.

We shall use the following observations.

Proposition 1. [5]. Let $A$ be a ball in a finite dimensional normed linear space $X$. The space $X$ normed by the gauge functional of $A$ will be denoted by $X_A$.

The ball $A$ is a sufficient enlargement for $X$ if and only if the $L_\infty$-factorable norm of the natural identity mapping from $X$ to $X_A$ is $\leq 1$.

Proposition 2. [2]. A symmetric with respect to 0 parallelepiped containing $B(X)$ is a sufficient enlargement for $X$.

Proposition 3. [2]. Convex combination of sufficient enlargements for $X$ is a sufficient enlargement for $X$.

§2. Sufficient enlargements which are in some directions close to the balls

We start with an investigation of a sufficient enlargement which is contained in a homothetic image of a circumscribed parallelepiped with the coefficient of homothety close to 1 (and of course greater than 1). Next result gives a condition under which such enlargement contains a non-trivial homothetic image of the parallelepiped.

Theorem 1. Let $X$ be an $n$-dimensional normed space. Let $\{f_i\}_{i=1}^n \subset S(X^*)$ be a basis of $X^*$ and let vectors $x_i \in S(X)$ be such that $f_i(x_i) = 1$ and for some $c_2 > 0$ and each $f \in B(X^*)$ there exists at most one element $i$ in the set $\{1, \ldots , n\}$ for which $|f(x_i)| \geq 1 - c_2$.

Let $A$ be a sufficient enlargement for $X$ such that for some $c_1 \geq 0$ it is contained in the parallelepiped $\{x : |f_i(x)| \leq 1 + c_1, i \in \{1, \ldots , n\}\}$

Let $c_3 = 1 - \frac{2c_2}{c_1}$. Suppose $c_3 > 0$. Then $A$ contains the parallelepiped $Q := \{x : |f_i(x)| \leq c_3, i \in \{1, \ldots , n\}\}$.

Proof. Let $\{f_i\}_{i=n+1}^\infty \subset S(X^*)$ be such that $\forall x \in X \ (\|x\| = \sup\{|f_i(x)| : i \in \mathbb{N}\})$. Then the operator $E : X \rightarrow l_\infty$ defined by $Ex := \{f_i(x)\}_{i=1}^\infty$ is an isometric embedding. Let $P : l_\infty \rightarrow E(X)$ be a projection for which $P(B(l_\infty)) \subset E(A)$.

The condition of the theorem imply that there exists a partition of $\mathbb{N}$ into subsets $F_1, \ldots , F_n$ such that for $i \in F_j$ we have $f_j(x_k) < 1 - c_2$ for $k \neq j$.

Let us show that $P(B(l_\infty))$ contains $E(Q)$. Observe that the first $n$ coordinate functionals on $l_\infty$ are norm-preserving extensions of functionals $f_iE^{-1} : E(X) \rightarrow \mathbb{R}$.
Therefore in order to prove that \( A \supset Q \) it is sufficient to prove that for every collection \( \{ \theta_i \}_{i=1}^n \), \( \theta_1 = \pm 1 \) there exists a vector \( z_0 \in B(l_\infty) \) and real numbers \( b_1, \ldots, b_n \geq c_3 \) such that

\[
Pz_0 = (\theta_1 b_1, \theta_2 b_2, \ldots, \theta_n b_n, b_{n+1}, b_{n+2}, \ldots)
\]

for some \( b_{n+1}, b_{n+2}, \ldots \in \mathbb{R} \).

We introduce \( z_0 \) as the sequence \( \{ d_k \}_{k=1}^\infty \), where \( d_k = \theta_j f_i(x_i) \) if \( k \in F_j \). In particular, \( d_1 = \theta_1, \ldots, d_n = \theta_n \). Let us show that \( Pz_0 \) satisfies the requirement above. Let

\[
Pz_0 = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots).
\]

Suppose that for some \( m \in \{1, \ldots, n\} \) we have \( \alpha_m \notin [\theta_m c_3, \theta_m \infty) \). Let us consider the family of vectors

\[y_\delta = (1 + \delta)\theta_m E(x_m) - \delta z_0, \quad (\delta > 0).\]

When \( \delta > 0 \) is small enough, then \( y_\delta \in B(l_\infty) \). More precisely, by the conditions of the theorem it happens at least when \( (1 - c_2)(1 + \delta) + \delta \leq 1 \), that is, when \( \delta \leq \frac{c_2}{2 - c_2} \).

On the other hand the \( m \)-th coordinate of \( Py_\delta \) is equal to

\[(1 + \delta)\theta_m - \delta \alpha_m = \theta_m + \delta(\theta_m - \alpha_m).\]

So for \( 0 \leq \delta \leq c_2/(2 - c_2) \) we have \( |\theta_m + \delta(\theta_m - \alpha_m)| \leq 1 + c_1 \). Hence

\[1 + \frac{c_2}{2 - c_2}(1 - c_3) < 1 + c_1 \quad \text{or} \quad c_3 > 1 - \frac{2 - c_2}{c_2} c_1.\]

This contradicts the condition on \( c_3 \). \( \square \)

**Corollary.** Let \( X \) be an \( n \)-dimensional normed space and \( Q \) be a parallelepiped circumscribed about \( B(X) \). Suppose there exist points \( \{x_i\}_{i=1}^n \) on faces of \( Q \) (one point on the union of each pair of symmetric faces) such that \( x_i \in B(X) \) and for every pair \( (x_i, x_j) \), \( x_i \neq x_j \) and every \( f \in B(X^*) \) at least one of the numbers \( |f(x_i)| \) is less than \( 1 \). Then \( Q \) is a minimal sufficient enlargement for \( X \).

**Proof.** By Proposition 2 only minimality requires a proof. Let \( \{f_i\}_{i=1}^n \subset B(X^*) \) be such that \( Q = \{ x : |f_i(x)| \leq 1, \quad i \in \{1, \ldots, n\} \} \).

By compactness of \( B(X^*) \) there exists \( c_2 > 0 \) satisfying the condition of Theorem 1. Let \( A \subset Q \) be a sufficient enlargement for \( X \). Applying Theorem 1 with \( c_1 = 0 \) we get \( A \supset Q \). Hence the sufficient enlargement \( Q \) is minimal. \( \square \)

**Remark 1.** Condition \(|f(x_i)| \geq 1 - c_2 \) in Theorem 1 cannot be omitted. This statement can be derived e.g. from the following observation which is interesting itself: the proof of the M.Kadets–Snobar theorem as it is given in [3], (see, also [7], §15) shows the following. Let \( X \) be an \( n \)-dimensional normed linear space and \( E \subset B(X) \) be the ellipsoid of maximal volume in \( B(X) \). Then \( \sqrt{n} E \) is a sufficient enlargement for \( X \). In particular \( B(l_\infty^n) \) is a sufficient enlargement for \( B(l_1^n) \). Letting \( A = B(l_\infty^n) \) we get the statement.
Remark 2. The following example shows that there are no direct generalizations of Theorem 1 for non-trivially large values of $c_1$:

For arbitrary $h \in S(l_2^\infty)$ there exists a sufficient enlargement for $l_2^\infty$ which is contained in the intersection of $3B(l_\infty^\infty)$ and the set $\{x : |\langle h, x \rangle| \leq 1\}$.

In fact, let $P_h$ be a projection onto the hyperplane orthogonal to $h$ with minimal possible norm as an operator on $l_\infty^\infty$ and let $A = [-h, h] + P_h(B(l_\infty^\infty))$. It is easy to see that $A$ is a sufficient enlargement for $l_2^\infty$ satisfying all the requirements.

The next result shows that the condition of the Corollary is not necessary for $Q$ to be a minimal sufficient enlargement.

**Theorem 2.** There exist a two-dimensional normed linear space $X$ and functionals $f_1, f_2 \in B(X^*)$ such that the following conditions are satisfied:

1) There exists precisely one point $x_1 \in B(X)$ such that $f_1(x_1) = 1$ and precisely one point $x_2 \in B(X)$ such that $f_2(x_2) = 1$.

2) The parallelogram $C = \{x : |f_1(x)| \leq 1, |f_2(x)| \leq 1\}$ is a minimal sufficient enlargement.

3) There exist a linear functional $f_3 \in B(X^*)$ such that $|f_3(x_1)| = |f_3(x_2)| = 1$.

**Proof.** Consider the space whose unit ball is the euclidean disc intersected with the strip

$$\{(a_1, a_2) : |a_1 - a_2| \leq 1\}.$$ 

Let $x_1 = (1, 0)$, $x_2 = (0, 1)$ and let $f_1$ and $f_2$ be the coordinate functionals. It is clear that Condition 1 of the theorem is satisfied.

In our case $C = \{(a_1, a_2) : |a_1| \leq 1, |a_2| \leq 1\}$.

It is clear that the functional $f_3(a_1, a_2) = a_1 - a_2$ satisfies Condition 3 of the theorem.

It remains to show, that $C$ is a minimal sufficient enlargement.

Let $\{f_i\}_{i=1}^\infty \subset S(X^*)$ be such that $(\forall x \in X) (|\|x\|| = \sup \{|f_i(x)| : i \in \mathbb{N}\})$, Then the operator $E : X \to l_\infty$ defined by $Ex := \{f_i(x)\}_{i=1}^\infty$ is an isometric embedding.

Now, if we suppose that $C$ is not a minimal sufficient enlargement, then there exists a projection $P : l_\infty \to E(X)$, such that the closure of its image is a proper part of $E(C)$. We show that this gives us a contradiction.

Consider the vectors

$$x_1(\varepsilon) := (\cos \varepsilon, \sin \varepsilon), \quad x_2(\varepsilon) := (\sin \varepsilon, \cos \varepsilon) \in B(X), \quad 0 < \varepsilon < \pi/4.$$ 

It is clear that for $0 < \varepsilon < \pi/4$ the following is true (the reader is advised to draw the picture): for each $f \in B(X^*)$ either

$$|f(x_1(\varepsilon))| \leq 1 - \tan \varepsilon \quad \text{or} \quad |f(x_2(\varepsilon))| \leq 1 - \tan \varepsilon.$$ 

Therefore there exists a partition $\mathbb{N} = A_1(\varepsilon) \cup A_2(\varepsilon)$ such that $|f_i(x_1(\varepsilon))| \leq 1 - \tan \varepsilon$ for $i \in A_2(\varepsilon)$ and $|f_i(x_2(\varepsilon))| \leq 1 - \tan \varepsilon$ for $i \in A_1(\varepsilon)$.

Now for $\theta = (\theta_1, \theta_2)$, where $\theta_1 = \pm 1$, $\theta_2 = \pm 1$, we define $z_\theta(\varepsilon) \in l_\infty$ as the vector, whose $i$-th coordinates coincide with the coordinates of $\theta_1 Ex_1(\varepsilon)$ for $i \in A_1(\varepsilon)$ and with the coordinates of $\theta_2 Ex_2(\varepsilon)$ for $i \in A_2(\varepsilon)$.

It is clear that $z \in B(l_\infty)$. Let

$$Pz_\theta(\varepsilon) = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in l_\infty.$$
Let us show that
\[ \theta_1 \alpha_1 \geq \cos \varepsilon - 2(1 - \cos \varepsilon) \varepsilon. \]  
(1)

\[ \theta_2 \alpha_2 \geq \cos \varepsilon - 2(1 - \cos \varepsilon) \varepsilon. \]  
(2)

Because \( \varepsilon > 0 \) and \( \theta = (\theta_1, \theta_2) = (\pm 1, \pm 1) \) are arbitrary (1) and (2) imply \( P(B(l_\infty)) \supset E(C) \), so we get a contradiction.

Suppose that either (1) or (2) is not satisfied. Without loss of generality, we may assume that (1) is not satisfied.

Consider the family of vectors
\[ y_\delta = (1 + \delta) \theta_1 E(x_1(\varepsilon)) - \delta z_\theta(\varepsilon) \in l_\infty \quad (\delta > 0). \]

From the definition of \( z_\theta(\varepsilon) \) it is easy to derive that
\[ ||y_\delta||_\infty \leq \max\{1, (1 + \delta)(1 - \tan \varepsilon) + \delta\}. \]

Hence if \( \delta \) is such that \( 2\delta/(1 + \delta) \leq \tan \varepsilon \), then \( ||y_\delta||_\infty \leq 1 \). In particular, \( \varepsilon > 0 \) and \( \theta = (\theta_1, \theta_2) = (\pm 1, \pm 1) \) are arbitrary (1) and (2) imply \( P(B(l_\infty)) \supset E(C) \), so we get a contradiction.

By a prism in \( \mathbb{R}^n \) we mean the Minkowski sum of a set \( A \) lying in an \( (n - 1) \)-dimensional hyperplane and a line segment that is not parallel to the hyperplane. The set \( A \) is called a basis of the prism.

It turns out that if a sufficient enlargement \( A \) for \( X \) is such that its boundary intersects \( S(X) \) in a smooth point, then \( A \) should contain a prism, which is also a sufficient enlargement, so the investigation of such enlargement can be in certain sense reduced to investigation of \( (n - 1) \)-dimensional sufficient enlargement.

**Theorem 3.** Let \( X \) be an \( n \)-dimensional normed space and let \( x_1 \in S(X) \) be a smooth point and \( h \in S(X^*) \) be its supporting functional. Let \( \{x_i\}_{i=2}^n \subset S(X) \) be such that \( \{x_i\}_{i=1}^n \) is a basis in \( X \) and \( h(x_i) = 0 \) for \( i \in \{2, \ldots, n\} \). Suppose that \( A \) is a sufficient enlargement for \( X \), which is contained in the set \( \{x \in X : |h(x)| \leq 1\} \).

Then there exists a symmetric with respect to 0 prism \( M \) with basis parallel to \( \text{lin}\{x_2, \ldots, x_n\} \) such that

(a) \( M \subset A; \)
(b) $M$ is a sufficient enlargement for $X$.

Proof. We consider the natural isometric embedding $E$ of $X$ into $C(S(X^*))$: every vector is mapped onto its restriction (as a function on $X^*$) to $S(X^*)$. We introduce the following notation: $C = C(S(X^*))$ and $B_C = B(C(S(X^*)))$.

Since $A$ is a sufficient enlargement for $X$, then there exists a projection $P : C \to \text{lin}\{Ex_i\}_{i=1}^{n}$, such that

$$P(B_C) \subset E(A).$$

Projection $P$ can be represented as $P(f) = \sum_{i=1}^{n} \mu_i(f)Ex_i$, where $\mu_i$ are measures on $S(X^*)$.

Inclusion (3) implies that $||\mu_i|| \leq 1$. Since $P$ is a projection we have $\mu_i(Ex_1) = \delta_{i,j} (i,j = 1, \ldots, n)$. In particular, $\mu_1(Ex_1) = 1$. Because $x_1$ is a smooth point, the function $|Ex_1| \in C$ attains its maximum only at $h$ and $-h$. Hence $\mu_1$ can be represented as $\mu_1 = b_1,1\delta_h + b_{1,1}\delta_{-h}$, where $\delta_h$ and $\delta_{-h}$ are Dirac measures, $b_{1,1} \geq 0$, $b_{2,1} \leq 0$ and $b_{1,1} - b_{2,1} = 1$.

Now, for $i = 2, \ldots, n$ we find representations

$$\mu_i = b_{1,i}\delta_h + b_{2,i}\delta_{-h} + \nu_i,$$

where $\nu_i$ don't have atoms in $h$ and $-h$. To unify the notation we set $\nu_1 = 0$.

We introduce new measures

$$\omega_i := (b_{1,i} - b_{2,i})\delta_h + \nu_i.$$

It is clear that $\omega_j(Ex_i) = \delta_{i,j} (i,j = 1, \ldots, n)$. Hence $Q(f) := \sum_{i=1}^{n} \omega_i(f)Ex_i$ is also a projection onto $\text{lin}\{Ex_i\}_{i=1}^{n}$.

Let us show that

$$Q(B_C) \subset \text{cl}(P(B_C)).$$

Let $f \in B_C$. Since $\nu_i$ don't have atoms in $\pm h$, then for every $\varepsilon > 0$ there exists a function $g \in B_C$ such that $g(-h) = -f(h), g(h) = f(h)$ and $|\nu_i(f) - \nu_i(g)| < \varepsilon$ for all $i \in \{1, \ldots, n\}$. This implies that

$$\forall i \in \{1, \ldots, n\} |\omega_i(f) - \mu_i(g)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary (4) follows. Hence $Q(B_C) \subset E(A)$. Now we shall show that $M := E^{-1}(\text{cl}(Q(B_C)))$ is the required prism.

The fact that $M$ is a sufficient enlargement follows by a standard argument from the fact that $C$ is an $L_\infty$-space (see [4]).

It remains to show that $E(M)$ is a prism with basis parallel to $\text{lin}\{Ex_2, \ldots, Ex_n\}$. We have

$$E(M) = \text{cl}\{f(h)Ex_1 + \sum_{i=2}^{n} (b_{1,i} - b_{2,i})f(h)Ex_i + \sum_{i=2}^{n} \nu_i(f)Ex_i : f \in B_C\}.$$  

It is clear that the closures of the sets

$$\Gamma_\alpha := \{\sum_{i=2}^{n} \nu_i(f)Ex_i : f \in B_C, f(h) = \alpha\}$$

don’t depend on $\alpha$. So $M$ is a prism of required form. The theorem is proved. □
§3. SUFFICIENT ENLARGEMENTS FOR EUCLIDEAN SPACES

Dealing with sufficient enlargements for $l^n_2$ it is useful to introduce the following definition.

**Definition 2.** A sufficient enlargement $A$ for $l^n_2$ is said to be small if

$$\int_{O(n)} T(A) d\mu(T) = \lambda(l^n_2) B(l^n_2),$$

where $\mu$ is the normalized Haar measure on the orthogonal group $O(n)$ and $\lambda(l^n_2)$ is the absolute projection constant.

**Remark.** Proposition 3 implies that $\int_{O(n)} T(A) d\mu(T) \supset \lambda(l^n_2) B(l^n_2)$ for arbitrary sufficient enlargement $A$. This explains the choice of the term “small”.

The following result supplies us with a wide and interesting class of small sufficient enlargements.

**Theorem 4.** Let $G$ be a finite subgroup of $O(n)$ such that each linear operator on $\mathbb{R}^n$ commuting with all elements of $G$ is a scalar multiple of the identity. Then for every $y \in S(l^n_2)$ the Minkowski sum of segments

$$A = \frac{n}{|G|} \sum_{g \in G} [-g(y), g(y)]$$

is a small sufficient enlargement for $l^n_2$.

**Proof.** First we prove

$$\forall x \in \mathbb{R}^n \ (x = \frac{n}{|G|} \sum_{g \in G} \langle x, g(y) \rangle g(y)). \quad (5)$$

Let us introduce a linear operator $T : l^n_2 \rightarrow l^n_2$ by the equality

$$Tx = \sum_{g \in G} \langle x, g(y) \rangle g(y). \quad (6)$$

Let us show that $hT = Th$ for each $h \in G$. In fact

$$hT(x) = \sum_{g \in G} \langle x, g(y) \rangle hg(y) = \sum_{g \in G} \langle h(x), hg(y) \rangle hg(y) = \sum_{g \in G} \langle h(x), g(y) \rangle g(y) = Th(x).$$

Hence $T = \lambda I$ for some $\lambda \in \mathbb{R}$.

The equality of traces in (6) shows that $\lambda n = |G|$. Hence $\lambda = \frac{|G|}{n}$. The assertion (5) follows.
Now, (5) implies that the identity operator on \( l^2 \) admits factorization \( I = T_2 T_1 \), where \( T_1 : l^2 \to l^2_C \) and \( T_2 : l^2_C \to l^2 \) are defined as follows

\[
T_1(x) = \{ (x, g(y)) \}_{g \in G} \quad \text{and} \quad T_2(\{a_g\}_{g \in G}) = \frac{n}{|G|} \sum_{g \in G} a_g g(y).
\]

It is clear that \( \|T_1\| = 1 \) and \( A = T_2(B(l^2_C)) \), therefore \( A \) is a sufficient enlargement (see Proposition 1).

The enlargement \( A \) is small by the following observation. A calculation of B.Grünbaum [2] shows that

\[
\forall z \in l^2 \int_{O(n)} T([-z, z]) d\mu(T) = \frac{||z|| \lambda(l^2)}{n} B(l^2).
\]

Therefore

\[
\int_{O(n)} T(A) d\mu(T) = \frac{n}{|G|} \sum_{g \in G} ||g(y)|| \lambda(l^2) \frac{B(l^2)}{n} = \lambda(l^2) B(l^2).
\]

\( \square \)

Remark 1. It is easy to find examples showing that for different \( y \in S(l^2) \) we get quite different sufficient enlargements.

Remark 2. Many different groups satisfying the condition of Theorem 4 are given by the representation theory of finite groups. In particular, every irreducible real representation of a finite group satisfies the condition (after a proper choice of an inner product on \( \mathbb{R}^n \)).

Small sufficient enlargements have the following nice property.

**Theorem 5.** Let \( A \) be a sufficient enlargement for \( l^{n+m}_2 = l^m_2 \oplus l^n_2 \) and suppose that the images \( A_1 \) and \( A_2 \) of \( A \) by the orthogonal projections onto \( l^2 \) and \( l^m \) are small sufficient enlargements for \( l^2 \) and \( l^m \). Then \( A = A_1 + A_2 \) (Minkowski sum).

**Proof.** We claim: if \( A_1 \) and \( A_2 \) are small sufficient enlargements for \( l^2 \) and \( l^m \), then \( A_1 + A_2 \subset l^{n+m}_2 \) is a small sufficient enlargement.

At the moment we do not need the fact that \( A_1 + A_2 \) is a sufficient enlargement, but because the proof is simple, we sketch it. By Proposition 1 the fact that \( A_1 \) is a sufficient enlargement for \( l^2 \) means that the \( L_\infty \)-factorable norm of the identical embedding of \( l^2_1 \) into \( \mathbb{R}^n \) normed by the gauge functional of \( A_1 \) is not greater than 1, the analogous assertion is valid for \( l^2_2 \) and \( A_2 \). Now, it is easy to see that the \( L_\infty \)-factorable norm of the identical embedding of \( l^2_2 \oplus_2 l^m_2 \) into \( \mathbb{R}^{n+m} \) normed by the gauge functional of \( A_1 + A_2 \) is \( \leq 1 \).

The fact that the sufficient enlargement \( A_1 + A_2 \) is small can be proved in the following way:

\[
\int_{O(n+m)} T(A_1 + A_2) d\mu(T) = \\
\int_{O(n+m)} T(\int_{O(n)} T_1(A_1) d\mu_1(T_1) + \int_{O(m)} T_2(A_2) d\mu_2(T_2)) d\mu(T) = \\
\int_{O(n)} T_1(A_1) d\mu_1(T_1) + \int_{O(m)} T_2(A_2) d\mu_2(T_2) = \\
\frac{n}{|G|} \sum_{g \in G} ||g(y)|| \lambda(l^2) \frac{B(l^2)}{n} = \lambda(l^2) B(l^2).
\]
(here $\mu_1$ and $\mu_2$ are normalized Haar measures on $O(n)$ and $O(m)$ respectively)

$$\int_{O(n+m)} T(\lambda(l_2^n)B(l_2^n) + \lambda(l_2^m)B(l_2^m))d\mu(T) =$$

$$\int_{O(n+m)} T(\int_{O(n)} T_1(Q_1)\,d\mu_1(T_1) + \int_{O(m)} T_2(Q_2)\,d\mu_2(T_2))d\mu(T) =$$

(here $Q_1$ and $Q_2$ are cubes circumscribed about $B(l_2^n)$ and $B(l_2^m)$ respectively)

$$\int_{O(n+m)} T(Q_1 + Q_2)\,d\mu(T) = \lambda(l_2^{n+m})B(l_2^{n+m})$$

(by B. Grünbaum’s result [2]).

Now the theorem follows from the following direct consequence of the remark after Definition 2: small sufficient enlargements are minimal, in particular, no proper subset of $A_1 + A_2$ is a sufficient enlargement for $l_2^{n+m}$. □

Let $X$ be a finite dimensional normed linear space. Denote by $M$ the set of all sufficient enlargements of minimal volume for $X$. Results of [1] (Theorem 6) imply the following result.

**Theorem 6.** The set $M$ contains a parallelepiped.

Easy examples (e.g. two-dimensional space whose ball is regular hexagon) show that $M$ may contain balls which are not parallelepipeds. But it turns out that for Euclidean spaces $M$ contains only parallelepipeds.

**Theorem 7.** If $A$ is a sufficient enlargement of minimal volume for $l_2^n$, then $A$ is a cube circumscribed about $B(l_2^n)$.

**Proof.** Let $A$ be a sufficient enlargement for $l_2^n$ and vol$A = 2^n$. We may assume without loss of generality (see Proposition 1) that $A$ is a zonoid. Therefore (see [6], p. 183), its support function can be represented in the form

$$h(A, x) = \int_{S^{n-1}} |\langle x, v \rangle|\,d\rho(v) \text{ for } x \in \mathbb{R}^n$$

with some even measure $\rho$ on $S^{n-1}$.

We denote by $D$ the set of all smooth points on the boundary of $A$. It is known (see [6], p. 73) that the complement of $D$ in the surface of $A$ has zero surface measure. Let $T : D \to S^{n-1}$ be the spherical image map (see [6], p. 78), that is: $T(d)$ is the unique outer unit normal vector of $A$ at $d$. Let $\mu$ be the measure on $S^{n-1}$ defined by

$$\mu(\Omega) = m_{n-1}(T^{-1}(\Omega)),$$

where $m_{n-1}$ is the surface area measure on the boundary of $A$.

It is clear that

$$\text{vol}A = \frac{1}{n} \int_{S^{n-1}} h(A, x)\,d\mu(x) = \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, v \rangle|\,d\rho(v)\,d\mu(x).$$
The \((n - 1)\)-dimensional volume of the orthogonal projection of \(A\) onto the hyperplane orthogonal to \(w \in S^{n-1}\) can be computed as

\[
\alpha(w) = \frac{1}{2} \int_{S^{n-1}} |\langle x, w \rangle| d\mu(x).
\]

We proceed by induction on the dimension. The case \(n = 1\) is trivial. Suppose that we have proved the result for \(n - 1\). Now, let \(A\) be a sufficient enlargement for \(l_2^n\) and \(\text{vol}A = 2^n\).

By Fubini theorem

\[
2^n = \text{vol}A = \frac{1}{n} \int_{S^{n-1}} 2\alpha(w) d\rho(w).
\]

Since \(A\) is a sufficient enlargement, it is easy to derive from (7) that \(\text{var}(\rho) \geq n\).

It is clear that an orthogonal projection of \(A\) onto an \((n - 1)\)-dimensional subspace is a sufficient enlargement for \(l_2^{n-1}\). It is clear also that every parallelepiped containing \(B(l_2^{n-1})\) has volume \(\geq 2^{n-1}\). Therefore by Theorem 6 \(\alpha(w) \geq 2^{n-1}\). It follows that almost everywhere (in the sense of \(\rho\)) \(\alpha(w) = 2^{n-1}\).

By induction hypothesis orthogonal projections in directions \(w\) for which \(\alpha(w) = 2^{n-1}\) are cubes. Let us choose one such direction, say \(w_1\), and let us denote by \(w_2, w_3, \ldots, w_n\) an orthonormal basis in the subspace orthogonal to \(w_1\) such that the orthogonal projection of \(A\) onto \(\text{lin}\{w_2, \ldots, w_n\}\) is

\[
[-w_2, w_2] + \cdots + [-w_n, w_n].
\]

In particular

\[
A \subset \{ x : |\langle x, w_2 \rangle| \leq 1 \}.
\]

By Theorem 3 \(A\) contains a prism \(M\) with the basis parallel to \(\text{lin}\{w_1, w_3, w_4, \ldots, w_n\}\) such that \(M\) is a sufficient enlargement for \(l_2^n\). Since \(A\) is a sufficient enlargement of minimal volume then \(M = A\). Let \(N = A \cap \text{lin}\{w_1, w_3, w_4, \ldots, w_n\}\). It is easy to see that \(N\) is a sufficient enlargement for \(l_2^{n-1}\) and \(\text{vol}_nA = 2\text{vol}_{n-1}N\). Hence \(\text{vol}_{n-1}N = 2^{n-1}\). By induction hypothesis \(N\) is a cube. Hence \(A\) is also a cube. \(\Box\)

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