THE $\Pi^1_1$ LÖWENHEIM-SKOLEM-TARSKI PROPERTY OF STATIONARY LOGIC

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ABSTRACT. Fuchino-Maschio-Sakai proved that the Löwenheim-Skolem-Tarski (LST) property of Stationary Logic is equivalent to the Diagonal Reflection Principle on internally club sets (DRP$_{IC}$) introduced in [4]. We prove that the restriction of the LST property to (downward) reflection of $\Pi^1_1$ formulas, which we call the $\Pi^1_1$-LST property, is equivalent to the internal version of DRP from [2]. Combined with results from [2], this shows that the $\Pi^1_1$-LST Property for Stationary Logic is strictly weaker than the full LST Property for Stationary Logic, though if CH holds they are equivalent.

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1. INTRODUCTION

Stationary Logic is a relatively well-behaved fragment of Second Order Logic introduced by Shelah [12], and first investigated in detail by Barwise et al [1]. Stationary Logic augments first order logic by introducing a new second order quantifier $\text{stat}$; we typically interpret “$\text{stat} Z \phi(Z,\ldots)$” to mean that there are stationarily many countable $Z$ such that $\phi(Z,\ldots)$ holds. The quantifier $aa$ stands for “almost all” or “for club many”; so

$$aaZ \phi(Z,\ldots)$$

is an abbreviation for

$$\neg \text{stat} Z \phi(Z,\ldots).$$

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$^1$Other interpretations, e.g. for uncountable $Z$, or for filters other than the club filter, are often considered too.
Section 2 provides more details.

By \textit{structure} we will always mean a first order structure in a countable signature. The question of whether every structure has a “small” elementary substructure in Stationary Logic was raised already in [1]. One cannot hope to always get countable elementary substructures; e.g. if $\kappa$ is regular and uncountable, then $(\kappa, \in)$ satisfies “$\in$ is a linear order and

$$aaZ \exists x x \text{ is an upper bound of } Z$$

but no countable linear order can satisfy that sentence. In a footnote in [1], it was observed that even the statement

“Every structure has an elementary (w.r.t. Stationary Logic) substructure of size $\leq \omega_1$”

(LST)

carries large cardinal consistency strength.\textsuperscript{2} The quoted statement above is now typically called the \textit{Löwenheim-Skolem-Tarski} (LST) property of Stationary Logic.\textsuperscript{3}

Fuchino et al. recently proved that LST is equivalent to a version of the Diagonal Reflection Principle introduced in Cox [4]:

\textbf{Theorem 1.1} (Fuchino-Maschio-Sakai [7]). \textit{LST is equivalent to the Diagonal Reflection Principle on internally club sets (DRP}\textsubscript{IC}\textit{).}

The purpose of the present note is to prove the following variant of Theorem 1.1 involving $\Pi^1_1$ formulas in Stationary Logic (defined in Section 2 below) and the principle DRP\textsubscript{internal} from [2]:

\textbf{Theorem 1.2.} The $\Pi^1_1$-LST property of Stationary Logic (see Definition 2.2) is equivalent to the principle DRP\textsubscript{internal}.

Cox [2] proved that DRP\textsubscript{IC} is strictly stronger than DRP\textsubscript{internal}. This was obtained by forcing over a model of a strong forcing axiom in a way that preserved DRP\textsubscript{internal} while killing DRP\textsubscript{IC} (in fact killing RP\textsubscript{IC}; the argument owed much to Krueger [10]). Furthermore, if CH holds, then DRP\textsubscript{IC} is equivalent to DRP\textsubscript{internal}. Combining those results with Theorem 1.2 immediately yields:

\textbf{Corollary 1.3.} The LST property of Stationary Logic is \textbf{strictly} stronger than the $\Pi^1_1$-LST property of Stationary Logic.

\textit{However, if the Continuum Hypothesis holds, they are equivalent.}\textsuperscript{4}

\textsuperscript{2}See Definition 2.2 for precisely what is meant by “elementary substructure” in this context.

\textsuperscript{3}The weaker assertion that every consistent theory (in Stationary Logic) has a model of size $\omega_1$, on the other hand, is a theorem of ZFC, as proven in [1].

\textsuperscript{4}One doesn’t actually need the full continuum hypothesis for this equivalence to hold, but rather a variant of Shelah’s Approachability Property, namely that the class of internally stationary sets is the same (mod NS) as the class of internally club sets. See Cox [2] for more details.
Figure 1. An arrow indicates an implication, an arrow with an X indicates a non-implication.

Martin’s Maximum $\xrightarrow{\mathcal{X}}$ DRP_{internal} $\leftarrow$ Stationary Logic has the $\Pi_1^1$ LST property

Stationary Logic has the LST property

(\textit{these 4 statements are equivalent if CH holds})

We note that while the technical strengthening MM$^{++}$ of Martin’s Maximum implies DRP_{IC} (see [4]), recent work of Cox-Sakai [6] shows that Martin’s Maximum alone does not imply even the weakest version of DRP. Figure 1 summarizes the relevant implications and non-implications discussed in this introduction.

Section 2 covers the relevant preliminaries, and Section 3 proves Theorem 1.2. Section 4 ends with some concluding remarks.

2. Preliminaries

Recall that $S \subseteq [A]^\omega$ is stationary if it meets every closed, unbounded subset of $[A]^\omega$ (in the sense of Jech [9]). By Kueker [11] this is equivalent to requiring that for every $f : [A]^<\omega \to A$ there is an element of $S$ that is closed under $f$.

In what follows, we will use uppercase letters to denote second order variables/parameters, and lowercase letters to denote first order variables/parameters. We will also use some standard abbreviations; e.g. if our language includes the $\in$ symbol, $v$ is a first order variable, and $Z$ is a second order variable, “$v = Z$” is short for

$$\forall x \ x \in v \iff Z(x).$$

Given a structure $\mathfrak{A} = (A, \ldots)$ (which we always assume to have a countable signature), the satisfaction relation in Stationary Logic is defined recursively by:

$$\mathfrak{A} \models \text{stat}_Z \phi(Z, U_1, \ldots, U_\ell, p_1, \ldots, p_k)$$

$$\iff$$

$$\{ Z \in [A]^\omega : \mathfrak{A} \models \phi(Z, U_1, \ldots, U_\ell, p_1, \ldots, p_k) \} \text{ is stationary in } [A]^\omega.$$
We define a hierarchy of formulas in Stationary Logic that mimics the usual hierarchy in Second Order Logic. Since
\[ \text{aa}Z \phi(Z, \ldots) \]
roughly translates as
\[ \exists C \ C \text{ is club and } \forall Z \in C \phi(Z, \ldots), \]
the \text{aa} quantifier will correspond to the existential second order quantifier when constructing the hierarchy. Similarly, since
\[ \text{stat}Z \phi(Z, \ldots) \]
roughly translates as
\[ \forall C \ C \text{ is club } \Rightarrow \exists Z \in C \phi(Z, \ldots), \]
the \text{stat} quantifier will correspond to the universal second-order quantifier.

**Definition 2.1.** A formula in Stationary Logic without second order quantifiers will be denoted by \( \Sigma^1_0 \) or \( \Pi^1_0 \). For \( n > 0 \), a formula of the form
\[ \text{stat}Z_1 \ldots \text{stat}Z_k \phi(Z_1, \ldots, Z_k, \ldots) \]
where \( \phi \) is \( \Sigma^{1}_{n-1} \) will be called a \( \Pi^1_n \) formula, and a formula of the form
\[ \text{aa}Z_1 \ldots \text{aa}Z_k \psi(Z_1, \ldots, Z_k, \ldots) \]
where \( \psi \) is \( \Pi^{1}_{n-1} \) will be called a \( \Sigma^1_n \) formula.

For example, if \( \phi(Z_0, Z_1, v_1, \ldots, v_\ell) \) has no \text{stat} or \text{aa} quantifiers, then
\[ \text{stat}Z_0 \ \text{aa}Z_1 \phi(Z_0, Z_1, v_1, \ldots, v_\ell) \]
is a \( \Pi^1_2 \) formula.

**Definition 2.2.** We say that the LST property holds for Stationary Logic iff for every structure \( \mathfrak{A} = (A, \ldots) \) there exists a \( W \subseteq A \) of size \( \leq \omega_1 \) such that for all formulas \( \phi \) in Stationary Logic with no free occurrences of second order variables, and all first order parameters \( p_1, \ldots, p_k \in W \),
\[ \mathfrak{A} \models \phi[p] \text{ if and only if } \mathfrak{A}|W \models \phi[p]. \]

We say that the \( \Pi^1_\downarrow \) LST property holds for Stationary Logic iff for every structure \( \mathfrak{A} = (A, \ldots) \) there exists a \( W \subseteq A \) of size \( \leq \omega_1 \) such that for all \( \Pi^1_n \) formulas \( \phi \) in Stationary Logic with no free occurrences of second order variables, and all first order parameters \( p_1, \ldots, p_k \in W \),
\[ \text{if } \mathfrak{A} \models \phi[p], \text{ then } \mathfrak{A}|W \models \phi[p]. \]

**Remark 2.3.** Note that in the definition of the \( \Pi^1_\downarrow \) LST property, we only require that \( \Pi^1_n \) formulas reflect downward. If there is always an \( \omega_1 \) sized substructure that reflects \( \Pi^1_n \) formulas both upward and downward, then the full LST property holds. This issue is discussed further in Section 4.

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5Recall we always assume countable signature, though for everything discussed in this paper an \( \omega_1 \)-sized signature would still be fine.
We consider variants of the **Diagonal Reflection Principle** introduced in Cox [4] and [2]. We use the following definition, which by Cox-Fuchs [5] is equivalent to the definitions from [4] and [2]:

**Definition 2.4.** \(\text{DRP}_{\text{internal}}\) asserts that for every sufficiently large regular \(\theta\), there are stationarily many \(W \in \wp(\omega_2(H_\theta))\) such that:

- \(|W| = \omega_1 \subset W\); and
- Whenever \(A \in W\) is uncountable and \(S \in W\) is a stationary subset of \([A]^{\omega}\), the set \(S \cap W \cap [W \cap A]^{\omega}\) is stationary in \([W \cap A]^{\omega}\).

The “internal” part of the definition refers to the fact that we require that \(S \cap W \cap [W \cap A]^{\omega}\) is stationary, not merely that \(S \cap [W \cap A]^{\omega}\) is stationary. Definition 2.4 is simply the diagonal version of an internal variant of WRP introduced in Fuchino-Usuba [8] (see Cox [2] for a discussion).

### 3. Proof of Theorem 1.2

We prove a slightly stronger variant of Theorem 1.2. The proof below is strongly influenced by Fuchino et al [7].

**Theorem 3.1.** The following are equivalent:

1. \(\text{DRP}_{\text{internal}}\).
2. For every structure \(\mathfrak{A} = (A, \ldots)\), there is a \(W \subseteq A\) of size at most \(\omega_1\) such that for every finite list \(p_1, \ldots, p_k \subseteq W \cap A\) and every formula \(\phi\) without 2nd order quantifiers,

\[
\left( \mathfrak{A} \models \text{stat}Z \phi[Z, \vec{p}] \right) \implies \left( \mathfrak{A}|W \models \text{stat}Z \phi[Z, \vec{p}] \right).
\]

3. The \(\Pi^1_1\)-LST property holds of Stationary Logic (as in Definition 2.2); 
4. For every structure \(\mathfrak{A} = (A, \ldots)\), there is a \(W \subseteq A\) of size at most \(\omega_1\) such that for every formula \(\psi\) in 2nd order prenex form with no free occurrences of second order variables, and every finite list \(p_1, \ldots, p_k\) of elements of \(W\), if

\[
\mathfrak{A} \models \psi[\vec{p}]
\]

then, letting \(\hat{\psi}\) be the formula obtained from \(\psi\) by changing all \(\forall\) quantifiers to \(\text{stat}\) quantifiers,

\[
\mathfrak{A}|W \models \hat{\psi}[\vec{p}].
\]

Before proving the theorem, we remark that in parts 2, 3, and 4 of Theorem 3.1 we only mentioned first order parameters from \(W \cap A\). If the structure \(\mathfrak{A}\) is sufficiently rich then it often makes sense to also speak of second-order parameters that are elements of \(W\). But in general (e.g. when \(\mathfrak{A}\) is a group) it is more natural to only speak of first order parameters from \(W \cap A\).
Proof. (of Theorem 3.1): (4) trivially implies (3), since if $\psi$ is represented as a prenex $\Pi^1_1$ formula, then $\hat{\psi} = \psi$ (because there are no $aa$ quantifiers in the original formula at all). Similarly, (3) trivially implies (2) because if $\phi$ has no second order quantifiers,

$$\text{statZ } \phi$$

is obviously a $\Pi^1_1$ formula.

To see that (2) implies (1), assume (2) and suppose $\theta$ is a regular cardinal $\geq \omega_2$. We need to find a $W \prec (H_\theta, \in)$ such that $|W| = \omega_1 \subset W$ and for every $s \in W$ that is a stationary collection of countable sets,

$$s \cap W \cap [W \cap \bigcup s]^\omega \text{ is stationary.}$$

Consider $\mathfrak{A} = (H_\theta, \in)$. Let $W \subset H_\theta$ be as in the statement of (2). Fix any $s \in W$ that is a stationary collection of countable sets. Then

$$\mathfrak{A} \models \text{statZ } \exists p \exists \alpha \ (p = Z \cap \omega_1, \ \alpha < \omega_1, \ \text{and } \alpha \text{ is an upper bound of } p),$$

so by assumption on $W$, this statement is also satisfied by $\mathfrak{A}|W$ (note that the parameter $\omega_1$ is an element of $W$ because $\omega_1$ is first-order definable in $\mathfrak{A}$ and $W$ is at least first-order elementary in $\mathfrak{A}$). If $W \cap \omega_1$ were countable, say $W \cap \omega_1 = \delta < \omega_1$, it would follow that for stationarily many $Z \in W \cap [W]^{\omega}$, there is an $\alpha < W \cap \omega_1 = \delta$ such that $\alpha$ is an upper bound of $Z \cap \delta$. This would be a contradiction, since due to the countability of $\delta$, the set of $Z \in [W]^{\omega}$ such that $\delta \subseteq Z$ is a club.

Finally, to prove that (1) implies (4): fix a structure $\mathfrak{A} = (A, \ldots)$ and let $\theta$ be a sufficiently large regular cardinal with $\mathfrak{A} \in H_\theta$. By (1) there is a $W \prec (H_\theta, \in, \mathfrak{A})$ witnessing $\text{DRP}_{\text{internal}}$. We prove by induction on complexity of formulas $\psi$ in 2nd order prenex form that if $p_1, \ldots, p_k \in W \cap A$ and

$$\mathfrak{A} \models \psi[\vec{p}]$$

then, letting $\hat{\psi}$ be the result of replacing all $aa$ quantifiers with $\text{stat}$ quantifiers,

$$\mathfrak{A}|(W \cap A) \models \hat{\psi}[\vec{p}].$$
We actually need to inductively prove a slightly stronger statement: namely, that whenever $\psi$ is a 2nd order prenex formula, $p_1, \ldots, p_k \in W \cap A$, and $Z_1, \ldots, Z_\ell \in W \cap [A]^\omega$,

\[(1)\quad \mathfrak{A} \models \psi[Z, \vec{p}] \implies \mathfrak{A}[(W \cap A)] \models \hat{\psi}[Z, \vec{p}].\]

So suppose

\[(2)\quad \mathfrak{A} \models QZ \phi[Z, U_1, \ldots, U_k, p_1, \ldots, p_\ell]\]

where $Q$ is either the $aa$ or $stat$ quantifier, $U_1, \ldots, U_k$ are each elements of $W \cap [A]^\omega$, $p_1, \ldots, p_\ell \in W \cap A$, and the inductive hypothesis holds of the formula $\phi$.

Now regardless of whether $Q$ is the $aa$ or $stat$ quantifier,

\[\bar{QZ} \phi \equiv \text{stat}Z \hat{\phi}.\]

and by (2) (since the $aa$ quantifier is stronger than the $stat$ quantifier)

\[\mathfrak{A} \models \text{stat}Z \phi[Z, U_1, \ldots, U_k, p_1, \ldots, p_\ell].\]

Hence, by the definition of the stationary logic satisfaction relation,

\[s := \{Z \in [A]^\omega : \mathfrak{A} \models \phi[Z, \vec{U}, \vec{p}]\}\]

is stationary in $[A]^\omega$.

Note that since $\vec{U}, \vec{p}, \phi$, and $\mathfrak{A}$ are elements of $W$, it follows that $s \in W$. Since $W$ is internally diagonally reflecting,

\[s \cap W \cap [W \cap A]^\omega\]

is stationary in $[W \cap A]^\omega$.

Consider for the moment an arbitrary $Z \in s \cap W \cap [W \cap A]^\omega$. Then

\[\mathfrak{A} \models \phi[Z, \vec{U}, \vec{p}]\]

and it follows by the induction hypothesis (and that $Z, \vec{U}$, and $\vec{p}$ are each elements of $W$) that:

\[\mathfrak{A}[(W \cap A)] \models \hat{\phi}[Z, \vec{U}, \vec{p}].\]

Hence, we have shown that

\[s \cap W \cap [W \cap A]^\omega \subseteq \{Z \in [W \cap A]^\omega : \mathfrak{A}[(W \cap A)] \models \hat{\phi}[Z, \vec{U}, \vec{p}]\}\]

Since the set on the left side is stationary, the set on the right side is too. So by the definition of the satisfaction relation,

\[\mathfrak{A}[(W \cap A)] \models \text{stat}Z \hat{\phi}[Z, \vec{U}, \vec{p}].\]

This completes the proof of the (1) $\implies$ (4) direction. $\square$
4. Concluding remarks

We remark that it is straightforward to show, in ZFC alone, that:

**Lemma 4.1.** For every structure $\mathcal{A} = (A, \ldots)$ there exists a $W \subseteq A$ of size at most $\omega_1$ such that

$$\mathcal{A}|W \prec_{\Sigma^1_1} \mathcal{A}$$

(i.e. such that $\Sigma^1_1$ formulas satisfied by $\mathcal{A}$ are also satisfied by $\mathcal{A}|W$).

In fact, if $\theta$ is a regular cardinal such that $\mathcal{A} \in H_\theta$, and

$$W \prec_{\text{1st order}} (H_\theta, \in, \mathcal{A})$$

is such that $|W| = \omega_1$ and

(3) $W \cap [W \cap A]^\omega$ contains a club in $[W \cap A]^\omega$

(this always holds for stationarily many $W$, e.g. for those $W$ that are internally approachable), then

$$\mathcal{A}|(W \cap A) \prec_{\Sigma^1_1} \mathcal{A}.$$ 

We briefly sketch the proof of the lemma; more details, and other related results, can be found in Cox [3]. One proves by induction on complexity of formulas, making use of (3), that if $\phi$ is $\Sigma^1_1$, $p_1, \ldots, p_k \in W \cap A$, and $Z_1, \ldots, Z_\ell \in W \cap [A]^\omega$, then

if $\mathcal{A} \models \phi[Z,p]$, then $\mathcal{A}|(W \cap A) \models \phi[Z,p].$

This was basically part of the proof from Fuchino et al [7] that DRP$_{\text{IC}}$ implied the LST for Stationary Logic. See [3] for some other related ZFC theorems.

So by Lemma 4.1 one can always get an $\omega_1$ sized substructure that reflects all $\Sigma^1_1$ statements downward. And if DRP$_{\text{internal}}$ holds, one can also get an $\omega_1$ sized substructure that reflects all $\Pi^1_1$ statements downward. But it is consistent that both of these are true, yet no single $\omega_1$-sized substructure downward reflects all $\Pi^1_1$ and all $\Sigma^1_1$ statements. In particular, in any model where DRP$_{\text{internal}}$ holds and DRP$_{\text{IC}}$ fails, Theorem 1.2 tells us that there is a structure such that no $\omega_1$-sized substructure reflects all $\Pi^1_1$ and all $\Sigma^1_1$ statements (though there are structures that reflect one or the other).

Another way to view this phenomenon, in terms of DRP-like principles, is that DRP$_{\text{internal}}$ yields stationarily many $W \in \wp_{\omega_2}(H_\theta)$ such that the transitive collapse $H_W$ of $W$ is “correct about stationary sets”; i.e. whenever $s \in H_W$ and $H_W \models \text{“s is a stationary set of countable sets”}$, then $V$ believes this too. However, if $W$ is not internally club, it is possible (by [2]) that $H_W$ is correct about stationary sets, but is not correct about clubs; i.e. there can be a $c \in H_W$ such that $H_W \models \text{“c is a club of countable sets”}$, but $V$ does not believe this. If, on the other hand, $W$ witnesses DRP$_{\text{IC}}$, then $H_W$ is correct about both stationarity and clubness.
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