The quest for a conifold conformal order

Alex Buchel

Department of Physics and Astronomy, University of Western Ontario,
London, Ontario N6A 5B7, Canada
Perimeter Institute for Theoretical Physics,
Waterloo, Ontario N2J 2W9, Canada

E-mail: abuchel@uwo.ca

ABSTRACT: The holographic duality between cascading gauge theory and type IIB supergravity on warped deformed conifold with fluxes reveals exotic thermal phases with nonzero expectation values of certain operators, persistent to high temperatures. These phases, in the limit of vanishing the strong coupling scale of the cascading gauge theory, would realize thermal ordered conformal phases in $\mathbb{R}^{3,1}$ relativistic QFT. We find that the dual Klebanov-Strassler/Klebanov-Tseytlin black branes in this limit are outside the regime of the supergravity approximation, rendering the construction of such conformal ordered states unreliable. While we have been able to construct conformal order in phenomenologically deformed effective theory of type IIB supergravity reduced on warped deformed conifold with fluxes, the removal of the deformation parameter causes the destruction of the thermal conformal ordered phases. Once again, we find that the holographic models with the conformal ordered phases are in the String Theory swampland.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Gauge-Gravity Correspondence

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1 Introduction

Conformal order stands for exotic thermal phases of conformal field theories (CFTs), characterized with nonzero one-point correlation function(s) of certain operator(s) [1–10]. For a CFT_{d+1} in Minkowski space-time R^{d,1} the existence of the ordered phases implies that there are at least two distinct thermal phases:

\[ \frac{\mathcal{F}}{T^{d+1}} = -C \times \begin{cases} 1, & T^{-\Delta_i} \langle O_{\Delta_i} \rangle = 0, \\ \kappa, & T^{-\Delta_i} \langle O_{\Delta_i} \rangle = \gamma_i \neq 0, \end{cases} \tag{1.1} \]

where \( \mathcal{F} \) is the free energy density, \( T \) is the temperature, \( C \) is a positive constant proportional to the central charge of the theory, and \( \{O_{\Delta_i}\} \) is the set of the order parameters with the conformal dimension spectrum \( \{\Delta_i\} \). The parameters \( \kappa \) and \( \{\gamma_i\} \) characterizing the thermodynamics of the ordered phase are necessarily constants. Conformal order can realize spontaneous breaking of discrete [1–3] or continuous [10] global symmetries; but it does not have to be the case: in the model we discussed in [9], the conformal order parameter is not associated with spontaneous breaking of any global symmetry.\(^1\)

From (1.1), note that when \( \kappa > 1 \) (\( \kappa < 1 \)), the symmetry broken phase dominates (is subdominant) both in the canonical and the microcanonical ensembles. Irrespectively of the value, provided \( \kappa > 0 \), the symmetry broken phase is thermodynamically stable. It is difficult to compute directly in a CFT the values \( \{\kappa, \gamma_i\} \), thus establishing the presence and the (in)stability of the ordered phase. Rather, the authors of [1, 6–8] established the

\(^1\)This is strictly true in the context of the effective Kaluza-Klein reduced holographic models. The nonvanishing scalar expectation value would signal the spontaneous breaking of the global symmetry of the compactification manifold upon uplift to 10d supergravity. Unfortunately, no model with a thermal conformal order has been constructed in top-down holography yet.
instability of the disordered thermal phases in discussed CFTs. The condensation of the identified unstable mode then leads to \( \langle O_{\Delta} \rangle \neq 0 \) for the new equilibrium thermal state — the conformal order.

Conformal order states are very interesting in the context of holography [11, 12], as they imply the existence of the dual black branes in a Poincare patch of asymptotically AdS\(_{d+2}\) bulk geometry that violate the no-hair theorem. In [9] we proved a theorem that the disordered conformal thermal states are always stable in dual holographic models of Einstein gravity with multiple scalars. Thus, the mechanism for the conformal order presented in [1] is not viable in these holographic models.

The first holographic model of the conformal order was discovered (though not appreciated in this context prior to the QFT construction [1]) purely by accident in [2]. The general framework for constructing holographic conformal order, the phenomenologically deformed effective theory, was presented in [5] — we now review those arguments as they will be utilized in this paper.\(^2\) Consider a top-down holographic model, dual to\(^3\) a CFT\(_d\) with a single operator \( O_{\Delta} \) of a dimension \( \Delta \). The five-dimensional gravitational bulk effective action takes the form

\[
S_5 = \frac{c}{2\pi^2 L^3} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \left\{ R - \frac{1}{2} (\nabla \phi)^2 - \mathcal{P}[\phi] \right\},
\]

where \( c \) is the central charge of the boundary CFT, \( \phi \) is the gravitational bulk scalar dual to the order parameter \( O_{\Delta} \), and the scalar potential \( \mathcal{P}[\phi] \) is

\[
\mathcal{P}[\phi] = -\frac{12}{L^2} + \frac{\Delta(\Delta - 4)}{2L^2} \phi^2 + \mathcal{O}(\phi^3).
\]

The \( \mathcal{O}(\phi^0) \) term in (1.3) is a negative cosmological constant, setting the radius of the asymptotically AdS\(_5\) bulk geometry to \( L \); the mass term,

\[
L^2 m^2 = \Delta(\Delta - 4),
\]

represents the standard encoding of the dimension of the order parameter \( O_{\Delta} \) [13, 14]. In what follows we set \( L = 1 \). It is vital that in real holographic models (contrary to the phenomenological toys), the full scalar potential \( \mathcal{P}[\phi] \) is nonlinear. Unfortunately, holography is not understood at the level where given a boundary CFT, with a spectrum of gauge invariant operators, we can engineer/compute the scalar potential. In specific holographic examples, like the \( \mathcal{N} = 2^* \) correspondence [15–17], the cascading gauge theory duality [18, 19] or the Maldacena-Nunez model [20], the scalar potential is computed from the realization of the duality correspondence in type IIB supergravity. The construction of the holographic conformal order proposed in [5] relies on (and is applicable to) models where the leading nonlinear correction is unbounded from below\(^4\) along certain directions.

\(^2\)All the known constructions of the holographic conformal order can be understood within this framework.

\(^3\)Extensions to AdS\(_{d+2}/\text{CFT}_{d+1}\) models with multiple order parameters \( O_{\Delta} \), is trivial — such models will be studied in section 3.

\(^4\)This feature is ubiquitous in top-down holographic examples.
on the scalar manifold. To be specific, we assume that

$$P[\phi] = -12 + \frac{\Delta(\Delta - 4)}{2} \phi^2 - \frac{s^2}{2} \phi^n + \mathcal{O}(\phi^{n+1}),$$  (1.5)

for some constant $s$ and an integer $n \geq 3$. Note that $\mathcal{O}(\phi^n)$-truncated scalar potential is unbounded as $\phi \to +\infty$. Of course, the full scalar potential $P[\phi]$ might be bounded, but this is not important for the perturbative construction of the thermal conformal order in the phenomenologically deformed model. The phenomenologically deformed model is defined as a holographic correspondence where the top-down scalar potential $P[\phi]$ is deformed as

$$P[\phi] \rightarrow P^b[\phi] \equiv -12 + \frac{\Delta(\Delta - 4)}{2} \phi^2 + b \left( P[\phi] + 12 - \frac{\Delta(\Delta - 4)}{2} \phi^2 \right),$$  (1.6)

with the constant deformation parameter $b$ being positive. The claim of [5], see also appendix A, is that the thermal conformal order always exists in the limit $b \to +\infty$, when the thermal ordered phase is holographically realized as AdS-Schwarzschild black brane, with a perturbatively small “scalar hair”

$$\phi \propto \left(\frac{1}{b}\right)^{\frac{1}{n-2}} \iff T^{-\Delta}\langle \mathcal{O}_\Delta \rangle = \gamma \propto \left(\frac{1}{b}\right)^{\frac{1}{n-2}}. \quad (1.7)$$

The existence of the thermal conformal order in real holography then boils to the question whether this perturbative constructions survives as $b$ decreases from $+\infty$ to 1, since

$$\lim_{b \to 1_+} P^b[\phi] = P[\phi]. \quad (1.8)$$

In this paper we continue the quest for constructing the thermal conformal order in String Theory holography. Our focus is on top-down holographic dualities between regular and fractional D3-branes on a conifold, the simplest non-compact Calabi-Yau threefold [21], and $\mathcal{N} = 1$ supersymmetric gauge theories - the Klebanov-Witten (KW) [22] and the Klebanov-Strassler (KS) [18] models. There are two reasons for this choice:

- Analysis of the phenomenological model [2] revealed for the first time the exotic thermal phases in a holographic system, which are associated with the spontaneous breaking of a discrete symmetry, and persist to arbitrary high temperature. As $T \to \infty$, the fact that the model of [2] was non-conformal becomes irrelevant since for any fixed mass scale $m/T \to 0$. In this way one can obtain a holographic thermal conformal order, as emphasized in [3]. Holographic duality between type IIB supergravity on warped deformed conifold with fluxes and the KS cascading gauge theory also reveals the exotic thermal phase [23] — the deconfined phase with spontaneously broken $U(1)_R$ chiral symmetry, that exists only above certain critical temperature $T_{\chiSB}$. Like the model in [2], the cascading gauge theory is non-conformal, and has a strong coupling scale $\Lambda$. It is natural to explore this exotic phase in the limit $\Lambda/T \to 0$, and potentially obtain a top-down holographic model of the thermal conformal order. We discuss this in section 2.
• The limit of $\Lambda/T \to 0$ in the above example effectively removes the fractional D3-branes from the holographic model. On the gravity side one ends with type IIB supergravity on warped deformed conifold with the self-dual five-form flux. The corresponding boundary gauge theory is $\mathcal{N} = 1$ superconformal KW model. The gravitational bulk scalars encode the gauge invariant operators (potential conformal order parameters). As we review in section 3, the resulting holographic model (when the conifold is not deformed) is a universal example of AdS$_5$/CFT$_4$ duality on warped and squashed Sasaki-Einstein manifolds [24]. Thus, analysis of the conformal order on the conifold will cover all such cases, see sections 3.1 and 3.2. In the case of the deformed conifold, we will have potentially the first example of the holographic conformal order with spontaneously broken continuous symmetry, see section 3.3.

We summarized our results in section 4.

2 Exotic phases of the cascading gauge theory at high temperature

Thermodynamics of the Klebanov-Strassler cascading gauge theory [18] has by now a long history [23, 25–32]. We refer the reader to a recent comprehensive review [32], and focus here on the results only.

Cascading gauge theory is $\mathcal{N} = 1$ supersymmetric SU($N + M$) $\times$ SU($N$) gauge theory with pairs of chiral multiplets $A_k$ and $B_\ell$, $k, \ell = 1, 2$, in the bifundamental $(N + M, N)$ and $(N + M, N)$ representations, and the superpotential

$$W \propto \epsilon^{ij} \epsilon^{kl} \text{Tr} (A_i B_k A_j B_\ell) .$$

(2.1)

The theory is not conformal, and the gauge couplings $g_1$ and $g_2$, of the gauge group factors SU($N + M$) and SU($N$) correspondingly, run with the renormalization group scale $\hat{\mu}$,

$$\frac{d}{d \ln(\hat{\mu} / \Lambda)} \frac{8\pi^2}{g_1^2} = 3(N + M) - 2N(1 - \gamma) ,$$

$$\frac{d}{d \ln(\hat{\mu} / \Lambda)} \frac{8\pi^2}{g_2^2} = 3N - 2(N + M)(1 - \gamma) ,$$

(2.2)

where $\gamma$ is the anomalous dimension of operators $\text{Tr} A_i B_j$ and $\Lambda$ is the strong coupling scale of the cascading gauge theory. To leading order in $M/N$, $\gamma = -\frac{1}{2}$ [18, 22], so that

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M \ln \frac{\hat{\mu}}{\Lambda} \times \left(1 + \mathcal{O}(M/N)\right) ,$$

(2.3)

while the sum of the gauge couplings is constant along the RG flow

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} = \text{const} .$$

(2.4)

The thermal phase diagram of the theory is rich:
Figure 1. The left panel: the reduced free energy density $\hat{F}$ (2.7) of the deconfined chirally symmetric phase (the black curve), and the deconfined (ordered) phase with spontaneously broken chiral symmetry (the magenta curve) of the cascading gauge theory plasma. The vertical dashed red lines indicate $T_{\chi SB}$ (2.8). The ordered phase is exotic: it extends for $T > T_{\chi SB}$. The right panel: the Kretschmann scalar $K$ for the corresponding phases computed for the dual black branes at the horizon. We use $K_{\chi SB} = K(T = T_{\chi SB})$.

- At large temperatures, $T \gg \Lambda$, its thermal equation of state is that of a conformal theory with the effective temperature-dependent central charge [25], e.g, the free energy density $F$ takes the form

$$F \propto -c_{\text{eff}}(T) T^4, \quad c_{\text{eff}} \propto M^4 \ln^2 \frac{T}{\Lambda}.$$  

(2.5)

- At

$$T = T_c = 0.614(1)\Lambda,$$  

(2.6)

the theory undergoes the first-order confinement/deconfinement phase transition [29]; precisely at the transition point the free energy density vanishes,

$$\hat{F} \equiv \frac{2^6 \pi^4}{3^5 M^4} \frac{F}{\Lambda^4} \bigg|_{T=T_c} = 0,$$  

(2.7)

where the first equality introduces dimensionless quantity $\hat{F}$ we use to describe cascading gauge theory thermodynamics in the canonical ensemble.

- The next critical temperature is [31]

$$T_{\chi SB} = 0.541(9)\Lambda,$$  

(2.8)

represented by the vertical dashed red lines in figures 1 and 2. For $T > T_{\chi SB}$ the deconfined phase, represented by the solid black curves in figures 1 and 2, is perturbatively stable to $U(1)_R \rightarrow Z_2$ chiral symmetry breaking fluctuations. The deconfined states of the cascading gauge theory plasma unstable to chiral symmetry breaking fluctuations are represented by the dashed solid black curves in figures 1 and 2.
Figure 2. The left panel: the reduced free energy density \( \hat{F} \) (2.7) of the deconfined chirally symmetric phase with the positive specific heat (the black curves), and the deconfined chirally symmetric phase with the negative specific heat (the brown curve) of the cascading gauge theory plasma. The vertical dashed brown lines indicate \( T_u \) (2.11). The negative specific heat phase is exotic: it extends for \( T > T_u \) with ever increasing thermal expectation values of certain gauge invariant operators. The right panel: the Kretschmann scalar \( K \) for the corresponding phases computed for the dual black branes at the horizon. As in figure 1, we normalize the Kretschmann scalar to its value at \( T_{\chi SB} \).

- Similar to the model of [2], the deconfined phase with spontaneously broken chiral symmetry, the solid magenta curves in figure 1, is exotic: it exists for \( T > T_{\chi SB} \) and is realized holographically as the Klebanov-Strassler black brane [23]. If we could extend this phase for \( \frac{T}{\Lambda} \rightarrow \infty \), by analogy with [3], we would have realized the conformal order on the conifold. Alas, our numerics allowed the construction of this exotic phase in the range

\[
\frac{T}{\Lambda} = \frac{T_{\chi SB}}{\Lambda} \times \{1 \cdots 3.9466\}. \tag{2.9}
\]

There is a practical and a conceptual reason for this:

- from the practical perspective, certain normalizable components of the scalar fields near the boundary become too large for a reliable numerics;\(^5\)

- the conceptual reason causing the above growth is the following: as the temperature increases, the curvature of the dual Klebanov-Strassler black brane evaluated at the horizon grows, and the construction becomes unreliable in the supergravity approximation. Specifically, in the right panel of figure 1 we present the value of the Kretschmann scalar of the KS black brane horizon, for the exotic phase (the magenta curve) and the deconfined chirally symmetric phase (the black curve), relative to the value of the Kretschmann scalar \( K_{\chi SB} \) evaluated at \( T = T_{\chi SB} \). Over the range (2.9), the Kretschmann scalar corresponding to the exotic phase changes as

\[
K = K_{\chi SB} \times \{1 \cdots 8890\}. \tag{2.10}
\]

\(^5\)In the notations of [32], e.g., it is the coefficient \( f_{c,8,0} \) — see (A.55) there.
Thus, we conclude that there can not be a reliable conformal order on the warped deformed conifold with fluxes, arising from the $T \to \infty$ limit of the Klebanov-Strassler black branes — such black branes are outside the validity of the supergravity approximation.

• Lastly, there is a terminal temperature \[ T_u = 0.537(3)\Lambda, \] represented by the vertical dashed brown lines in figure 2, for the deconfined chirally symmetric states of the cascading gauge theory plasma — these states exist only for $T \geq T_u$. In the vicinity of $T_u$, there are two branches of states: the stable branch with respect to the energy density fluctuations (the black curves in figure 2), and the unstable branch (the brown curves in figure 2). On the former branch the specific heat is positive, while it is negative on the latter branch, explaining the (in)stability to the energy density fluctuations [33]. The unstable branch of the deconfined chirally symmetric states is exotic: it extends for $T > T_u$ with the ever increasing values of the thermal expectation values of $\mathcal{O}_{\Delta=\{4,6,8\}}$, however, much like in the case of the exotic phase with the chiral symmetry breaking, we fail to extend it as $T \to \infty$. We constructed the latter states for fairly narrow temperature range:

\[ \frac{T}{\Lambda} = \frac{T_u}{\Lambda} \times \left\{1 \cdots 1.0181\right\}. \]  

(2.12)

As the temperature of the exotic deconfined chirally symmetric phase increases, the curvature of the corresponding dual Klebanov-Tseytlin black brane evaluated at the horizon rapidly grows, see the right panel of figure 2. Over the range (2.12), the Kretschmann scalar corresponding to this exotic phase (the brown curve) changes as

\[ K = K_{\text{SB}} \times \left\{1.669 \cdots 73.1\right\}. \]  

(2.13)

On the contrary, the deconfined chirally symmetric phase with the positive specific heat (the black curves in figure 2) can be extended as $\frac{T}{\Lambda} \to \infty$ — however it is not exotic, as the thermal expectation values of $\mathcal{O}_{\Delta=\{4,6,8\}}$ operators vanish [29] in this limit, and we end up with the log-dressed conformal equation of state (2.5). Thus, we conclude that there can not be a reliable conformal order on the warped squashed conifold with fluxes, arising from the $T \to \infty$ limit of the Klebanov-Tseytlin black branes with the negative specific heat — such black branes are outside the validity of the supergravity approximation.

3 Effective theories and the conformal order on the conifold

Consistent truncation in the $\text{SU}(2) \times \text{SU}(2) \times \mathbb{Z}_2$ invariant sector of type IIB supergravity on warped deformed conifold with fluxes to a five dimensional manifold $\mathcal{M}_5$ was derived
in [31].\(^6\)

\[
S_5 \left[ \hat{g}_{\mu
u}, \Omega_{i=1\ldots 3}, \Phi, h_{i=1\ldots 3}, \{ P, \Omega_0 \} \right] = \frac{108}{16\pi G_5} \int_{M_5} \hat{\text{vol}}_{M_5} \Omega_1 \Omega_2^2 \Omega_3^2 \\
\times \left\{ R_{10} - \frac{1}{2} (\hat{\nabla} \Phi)^2 - \frac{1}{2} e^{-\Phi} \left( \frac{(h_1 - h_3)^2}{2 \Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^2} \left( \hat{\nabla} h_1 \right)^2 + \frac{1}{\Omega_2^2} \left( \hat{\nabla} h_3 \right)^2 \right) \right. \\
- \frac{1}{2} e^\Phi \left( \frac{2}{\Omega_1^2 \Omega_3^2} \left( \hat{\nabla} h_2 \right)^2 + \frac{1}{\Omega_1^2 \Omega_2^2} \left( h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^2 \Omega_3^2} h_2^2 \right) \\
- \left. \frac{1}{2 \Omega_1^2 \Omega_2^2 \Omega_3^2} \left( 4 \Omega_0 + h_2 (h_3 - h_1) \right) + \frac{1}{9} P h_1 \right)^2 \right\}. \quad (3.1)
\]

It is a functional of the a five-dimensional metric \( \hat{g}_{\mu\nu} \) on \( M_5 \),

\[
ds_5^2 = \hat{g}_{\mu\nu}(y)dy^\mu dy^\nu, \quad (3.2)
\]

scalars \( \Omega_{i=1\ldots 3} \) describing the warping and the deformation of the conifold, a dilaton \( \Phi \), scalars \( h_{i=1\ldots 3} \) parameterizing the 3-form fluxes, a constant parameter \( \Omega_0 \) (necessary to define the self-dual 5-form flux), and a topological parameter \( P \),

\[
\frac{2P}{9\alpha'} \equiv M \in \mathbb{Z}, \quad (3.3)
\]

related to the number of fractional D3 branes on the conifold. Finally, \( R_{10} \) is the Ricci scalar of the ten-dimensional type IIB metric, obtained from uplifting \((3.2)\),

\[
R_{10} = \hat{R}_5 + \left( \frac{1}{2 \Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4 \Omega_1^2 \Omega_3^2} - \frac{\Omega_2^2}{4 \Omega_1^2 \Omega_3^2} - \frac{\Omega_2^2}{ \Omega_1^2 \Omega_3^2} \right) - 2 \hat{\square} \ln \left( \Omega_1 \Omega_2^2 \Omega_3^2 \right) \\
- \left\{ \left( \hat{\nabla} \ln \Omega_1 \right)^2 + 2 \left( \hat{\nabla} \ln \Omega_2 \right)^2 + 2 \left( \hat{\nabla} \ln \Omega_3 \right)^2 + 2 \left( \hat{\nabla} \ln \left( \Omega_1 \Omega_2 \Omega_3 \right) \right)^2 \right\}, \quad (3.4)
\]

and \( \hat{R}_5 \) is the five-dimensional Ricci scalar of the metric \((3.2)\).

We find it convenient to rewrite the action \((3.1)\) in five-dimensional Einstein frame. The latter is achieved with the following rescaling

\[
\hat{g}_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad \Omega^{-3} \equiv 108 \Omega_1 \Omega_2^2 \Omega_3^2, \quad (3.5)
\]

leading to

\[
108 \sqrt{-\hat{g}} \Omega_1 \Omega_2^2 \Omega_3^2 \hat{R}_5 = \sqrt{-g} \left( R - 8 \hat{\square} \ln \Omega - 12 \left( \hat{\nabla} \ln \Omega \right)^2 \right). \quad (3.6)
\]

Further introducing

\[
\Omega_1 = \frac{1}{3} e^{f-4w}, \quad \Omega_2 = \frac{1}{\sqrt{6}} e^{f+w+\lambda}, \quad \Omega_3 = \frac{1}{\sqrt{6}} e^{f+w-\lambda}, \quad (3.7)
\]

\(^6\)See [32] for a recent comprehensive review.
the five-dimensional effective action becomes

\[
S_5 = \frac{1}{16\pi G_5} \int_{M_5} \text{vol}_{M_5} \left\{ R - \frac{40}{3} (\nabla f)^2 - 20 (\nabla w)^2 - 4 (\nabla \lambda)^2 - \frac{1}{2} (\nabla \Phi)^2 \\
- 18 e^{-4f-4w-\Phi} \left[ e^{4\lambda} (\nabla h_1)^2 + e^{-4\lambda} (\nabla h_3)^2 \right] - 36 e^{-4f-4w+\Phi} (\nabla h_2)^2 - P_{\text{flux}} - P_{\text{scalar}} \right\},
\]

(3.8)

where

\[
P_{\text{flux}} = 81 e^{-28f+4w-\Phi}(h_1 - h_3)^2 + 162 e^{-28f+4w+\Phi} \left[ e^{-4\lambda} \left( h_2 - \frac{1}{9} P \right)^2 + e^{4\lambda} h_2^2 \right] \\
+ 72 e^{-40f} \left( h_1 (P - 9h_2) + 9h_2 h_3 + 36\Omega_0 \right)^2 ,
\]

(3.9)

\[
P_{\text{scalar}} = 4 e^{-16f-12w} - 24 e^{-16f-2w} \cosh(2\lambda) - \frac{9}{2} e^{-16f+8w} \left( 1 - \cosh(4\lambda) \right).
\]

(3.10)

Consistent truncation of (3.8), i.e.,

\[
\lambda = 0, \quad h_1 = h_3 = \frac{1}{P} \left( K \frac{12}{12} - 36\Omega_0 \right) , \quad h_2 = \frac{P}{18},
\]

(3.11)

produces effective action of \( SU(2) \times SU(2) \times U(1) \) invariant sector of type IIB supergravity on warped squashed conifold with fluxes derived in [34].

The boundary holographic dual represented by (3.8) is the \( \mathcal{N} = 1 \) supersymmetric \( SU(N + M) \times SU(N) \) cascading gauge theory, often referred to as a Klebanov-Strassler gauge theory [18]. This theory is not conformal, and has a strong coupling scale \( \Lambda \),

\[
\Lambda^2 = \frac{\sqrt{2}}{P^2 g_s} e^{-\frac{K_0}{P^2 g_s}},
\]

(3.12)

where \( g_s \) is the asymptotic value of the string coupling constant,\(^7\) and \( K_0 \) is a parameter that can always be adjusted (using the scaling symmetries of the holographic radial coordinate) to a fixed positive value. The precise definition of \( K_0 \) can be found in [32, 35]. We are interested here in the conformal holographic model, thus we must take the limit \( \Lambda \to 0 \), which is equivalent to sending \( P \to 0 \). Finding the conformal order in the model with 7 scalars is a daunting task — so\(^8\) we will follow the framework developed in [5], and reviewed in appendix A.

As a first step, we truncate the general potential \( P_{\text{flux}} + P_{\text{scalar}} \) in (3.8) to scalars that enter nonlinearly, and have “unbounded” directions on the field space manifold, at least close to the origin. Notice that \( h_i \), encoding the 3-form fluxes on the conifold, enter quadratically and generate always nonnegative contribution to the scalar potential, i.e., \( P_{\text{flux}} \geq 0 \). Thus, in addition to \( P = 0 \), we set \( h_i \equiv 0 \). Then,

\[
P_{\text{flux}} = 72(36\Omega_0)^2 e^{-\frac{40f}{3}} = 8 e^{-\frac{40f}{3}},
\]

(3.13)

\(^7\) It can always be fixed to \( g_s = 1 \).

\(^8\) While it is conceivable that other general constructions of the conformal order might exist in the model, finding them without any guidances is a lost cause.
where in the second equality we set a constant parameter $\Omega_0 = \frac{1}{108}$ to ensure that the radius of the asymptotically AdS$_5$ geometry is $L = 1$.

- In the absence of 3-form fluxes the dilaton becomes an exact modulus, and we set
  \[ \Phi \equiv 0. \] (3.14)

- We now arrive at the effective action we use to analyze conformal order on the warped deformed conifold:
  \[ S_5 = \frac{c_{KW}}{2\pi^2} \int_{M_5} \text{vol}_{M_5} \left\{ R - \frac{40}{3} (\nabla f)^2 - 20 (\nabla w)^2 - 4 (\nabla \lambda)^2 - \mathcal{P}[f, w, \lambda] \right\}, \] (3.15)
  where
  \[ c_{KW} = \frac{27N^2}{64}, \] (3.16)
is the central charge of $\mathcal{N} = 1$ superconformal SU$(N) \times$SU$(N)$ Klebanov-Witten model [22], and
  \[ \mathcal{P} = 4e^{-\frac{16}{3}f-12w} - 24e^{-\frac{16}{3}f-2w} \cosh(2\lambda) - \frac{9}{2}e^{-\frac{16}{3}f+8w} \left( 1 - \cosh(4\lambda) \right) + 8e^{-\frac{16}{3}f}. \] (3.17)
  From (3.15) we obtain the following equations of motion
  \[ 0 = \Box f - \frac{3}{80} \frac{\partial \mathcal{P}}{\partial f}, \] (3.18)
  \[ 0 = \Box w - \frac{1}{40} \frac{\partial \mathcal{P}}{\partial w}, \] (3.19)
  \[ 0 = \Box \lambda - \frac{1}{8} \frac{\partial \mathcal{P}}{\partial \lambda}, \] (3.20)
  \[ R_{\mu\nu} = \frac{40}{3} \partial_\mu f \partial_\nu f + 20 \partial_\mu w \partial_\nu w + 4 \partial_\mu \lambda \partial_\nu \lambda + \frac{1}{3} g_{\mu\nu} \mathcal{P}. \] (3.21)

In the rest of this section we consider three conformal models, obtained by consistent truncations of the effective action $S_5 = S_5[f, w, \lambda]$ (3.15):

- **Model I:**
  \[ S_{5I}^I[f] \equiv S_5 \biggl|_{w=\lambda=0}; \] (3.22)

- **Model II:**
  \[ S_{5II}^I[f, w] \equiv S_5 \biggl|_{\lambda=0}; \] (3.23)

- **Model III:**
  \[ S_{5III}^I[f, w, \lambda] \equiv S_5. \] (3.24)
3.1 Conformal order in Model I

Conformal Model I realizes holographic dual to the KW gauge theory with a single dimension $\Delta = 8$ order parameter $O_8$, represented by the bulk scalar $f$. Since this scalar represents the warping of the $T^{1,1}$ base of the singular conifold [21], the model is universal in the sense that any holographic duality between a CFT$_4$ and a type IIB supergravity on AdS$_5 \times Y_5$, where $Y_5$ is a five-dimensional Sasaki-Einstein manifold, can be truncated to our Model I [24]. In the latter case, the scalar $f$ represents the breathing mode of $Y_5$.

We follow appendix A to construct conformal order in Model I. From (3.15) we find

$$\mathcal{P}_I[f] = \mathcal{P}[f, 0, 0] = -12 + \frac{1280}{3} f^2 - \frac{71680}{27} f^3 + \mathcal{O}(f^4), \quad (3.25)$$

i.e., the leading nonlinear term is $n = 3$ (see (1.5)) and the unbounded direction is along $f \to +\infty$. The $b$-deformed scalar potential takes the form, compare with (1.6),

$$\mathcal{P}_I^b[f] = -12 + \frac{1280}{3} f^2 + b \left( -20 e^{-\frac{16}{5} f} + 8 e^{-\frac{40}{5} f} + 12 - \frac{1280}{3} f^2 \right). \quad (3.26)$$

Using the radial coordinate as in (A.10), and introducing

$$c_2(x) = \frac{g(x)}{(2x - x^2)^{1/4}}, \quad (3.27)$$

we obtain the following equations of motion ($' \equiv \frac{d}{dx}$)

$$0 = f'' + \frac{f'}{x - 1} + \left( -\frac{1}{2} (f')^2 + \frac{9 h^2}{20} - \frac{9 (x^2 - 2 x + 2) h}{40 x (1-x)(2-x)} + \frac{9}{80 x^2 (2-x)^2} \right) \frac{\partial \mathcal{P}_I}{\partial f}, \quad (3.28)$$

$$0 = h' - 4 h^2 + \frac{3 x^2 - 6 x + 4}{x(1-x)(2-x)} \frac{h}{9} \left( f' \right)^2, \quad (3.29)$$

$$0 = g' - h \ g. \quad (3.30)$$

Eqs. (3.28)–(3.29) are solved subject to the boundary conditions:

- as $x \to 0_+$, i.e., at the AdS$_5$ boundary

$$f = f_2 x^2 + \mathcal{O}(x^3), \quad h = -\frac{32}{9} f_2^2 x^3 + \mathcal{O}(x^4), \quad g = A \left( 1 - \frac{8}{9} f_2^2 x^4 + \mathcal{O}(x^5) \right), \quad (3.32)$$

- as $y \equiv (1-x) \to 0_+$, i.e., at the horizon

$$f = f_h^b + \mathcal{O}(y^2), \quad h = h_1^b y + \mathcal{O}(y^3), \quad g = A \left( g_h^b + \mathcal{O}(y^2) \right). \quad (3.33)$$

Note that in total we have four parameters $\{f_2, f_h^b, h_1^b, g_h^b\}$, along with an arbitrary constant $A$. They determine the thermal conformal order of Model I, specifically,

$$s = \frac{2 \pi \chi_{KW}}{\pi} A^3 \left( g_0^b \right)^3, \quad \mathcal{F} = -\frac{s T}{4}, \quad T^{-8} \langle O_8 \rangle = f_2, \quad T^2 = \frac{A^2 \left( g_0^b \right)^2}{9 \pi^2 (1 - 2 h_1^b)} \left( (320 (f_h^b)^2 - 9)(b - 1) + 3 b \left( 5 e^{-\frac{16}{5} f_h^b} - 2 e^{-\frac{40}{5} f_h^b} \right) \right), \quad (3.34)$$

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Figure 3. Thermal conformal order parameter $\gamma_8 \equiv T^{-8}\langle O_8 \rangle$ of the deformed Model I diverges as $b \to b_{\text{crit},I} + 0 > 1$, represented by the dashed vertical red line.

for the entropy density $s$, the free energy density $F$, the temperature $T$, and the thermal order parameter $O_8$. Thus, see (1.1),

$$\kappa_I = 27 \left[ \frac{320(f_0^2 - 9)(b - 1) + 3b \left( 5e^{-\frac{4}{3}f_0} - 2e^{-\frac{4}{3}f_0} \right)}{1 - 2h_1} \right]^{-3/2}, \quad \gamma_{I,8} = f_2,$$

where the subscript $I$ refers to Model I.

In the limit $b \to +\infty$ we find, see appendix A,

$$f = \frac{9x^2(2-x)^2}{28(1 + (1-x)^3)} \frac{1}{b} + \mathcal{O}(b^{-2}), \quad h = \mathcal{O}(b^{-2}), \quad g = 1 + \mathcal{O}(b^{-2}).$$

Finite $b$ results are obtained solving (3.28)–(3.30), subject to the asymptotics (3.32) and (3.33). In figure 3 we plot the expectation value of the order parameter, see (1.1) and (3.34), (3.35). In figure 4 we plot the coefficient $\kappa$ of the thermal order equation of state in Model I as a function of the deformation parameter $b$, see (1.1) and (3.35). Notice that the order parameter diverges as $b \to b_{\text{crit},I}$,

$$b_{\text{crit},I} = 1.8370(4),$$

from above. Since $b_{\text{crit},I} > 1$, there is no conformal order in universal top-down holographic Model I, representing a CFT dual to type IIB supergravity with the self-dual five-form flux on a Sasaki-Einstein manifold with a breathing mode, dual to an order parameter of dimension $\Delta = 8$. Furthermore, since $\kappa_I(b) < 1$, the conformal order in deformed Model I is subdominant in canonical and microcanonical ensembles. Following [4] we expect that this conformal order is perturbatively unstable.

### 3.2 Conformal order in Model II

Conformal Model II realizes holographic dual to the KW gauge theory with a pair of order parameters: a dimension $\Delta = 8$ operator $O_8$ and a dimension $\Delta = 6$ operator $O_6$, represented by the bulk scalars $f$ and $w$ correspondingly. Since these scalars represent the
The vertical dashed red line denotes $b = b_{\text{crit},I}$. Warping of the $T^{1,1}$ base, along with the squashing of the $U(1)$ fibration of the Kähler-Einstein base of $T^{1,1}$, this model is universal as well: any holographic duality between a CFT$_4$ and a type IIB supergravity on $\text{AdS}_5 \times \tilde{Y}_5$, where $\tilde{Y}_5$ is a five-dimensional squashed Sasaki-Einstein manifold, can be truncated to our Model II [24]. In the latter case, the scalars $f$ and $w$ represent the breathing and the squashing modes of $\tilde{Y}_5$.

We follow appendix A to construct conformal order in Model II. From (3.15) we find

$$P_{II}[f, w] = P[f, w, 0] = -12 + \frac{1280}{3} f^2 + 240 w^2 + \mathcal{O}((f, w)^4),$$

i.e., the leading nonlinear terms are $n = 3$ (see (1.5)) and the unbounded direction is along $\{f, w\} \to +\infty$. The $b$-deformed scalar potential takes the form,

$$P_{II}^b[f, w] = -12 + \left(\frac{1280}{3} f^2 + 240 w^2\right) + b \left(P[f, w, 0] + 12 - \frac{1280}{3} f^2 - 240 w^2\right).$$

Repeating the analysis as in section 3.1, in the limit $b \to +\infty$, we find

$$f = \frac{1}{b} f_1 + \mathcal{O}(b^{-2}), \quad w = \frac{1}{b} w_1 + \mathcal{O}(b^{-2}), \quad h = \mathcal{O}(b^{-2}), \quad g = 1 + \mathcal{O}(b^{-2}),$$

where

$$0 = f_1'' + \frac{f_1'}{x - 1} + \frac{4}{3} \frac{56 f_1^2 + 9 s w_1^2 - 6 f_1}{x^2(x - 2)^2},$$

$$0 = w_1'' + \frac{w_1'}{x - 1} + \frac{w_1 (16 s f_1 + 21 w_1 - 3)}{x^2(x - 2)^2}.$$  

Notice that we modified the leading nonlinear interactions in (3.38) as

$$\left(-\frac{71680}{27} f^3 - 1280 f w^2 - 1120 w^3\right) \to \left(-\frac{71680}{27} f^3 - 1280 s f w^2 - 1120 w^3\right),$$

where a constant parameter $s$ dials the strength of the leading nonlinear coupling between $f_1$ and $w_1$ scalars. Ultimately, we need to set $s = 1$, however, it is convenient to start at
Figure 5. Coefficients \( \{ f_{1,4}, f_{1,0}^h \} \), see (3.44) and (3.45), of the perturbative conformal order in \( b \to +\infty \) deformation of Model II, with a parameter \( s \) characterizing the cubic coupling between the scalars \( f \) and \( g \), see (3.43). The black dot indicates \( s = 0 \) results reported in appendix A. There are two branches of the conformal order for \( s < s_{\text{max}} \), represented by the vertical dashed red lines. Since \( s_{\text{max}} < 1 \), there is no perturbative conformal order in Model II.

Figure 6. Coefficients \( \{ w_{1,3}, w_{1,0}^h \} \), see (3.44) and (3.45), of the perturbative conformal order in \( b \to +\infty \) deformation of Model II, with a parameter \( s \) characterizing the cubic coupling between the scalars \( f \) and \( g \), see (3.43). The black dot indicates \( s = 0 \) results reported in appendix A. There are two branches of the conformal order for \( s < s_{\text{max}} \), represented by the vertical dashed red lines. Since \( s_{\text{max}} < 1 \), there is no perturbative conformal order in Model II.

\( s = 0 \), so that the scalars \( f_1 \) and \( w_1 \) are decoupled and we can use the results of appendix A, and then increase \( s \to 1 \). Eqs. (3.41)–(3.42) are solved subject to the boundary conditions:

- as \( x \to 0_+ \), i.e., at the AdS\(_5 \) boundary
  \[
  f_1 = f_{1,4} x^2 + (f_{1,4} - \frac{3}{4} s w_{1,3}^2) x^3 + \mathcal{O}(x^4), \quad w_1 = w_{1,3} x^{3/2} + \frac{3}{4} w_{1,3} x^{5/2} + \mathcal{O}(x^3), \quad (3.44)
  \]

- as \( y \equiv (1 - x) \to 0_+ \), i.e., at the horizon
  \[
  f_1 = f_{1,0}^h + \mathcal{O}(y^2), \quad w_1 = w_{1,0}^h + \mathcal{O}(y^2). \quad (3.45)
  \]

In figures 5 and 6 we present results for \( \{ f_{1,4}, w_{1,3}, f_{1,0}^h, w_{1,0}^h \} \) as a function of \( s \). The black
dots represent $s = 0$ results\(^9\) from appendix A for order parameters of dimensions $\Delta = 8$ and $\Delta = 6$. For any $0 < s < s_{\text{max}}$,

\[
s_{\text{max}} = 0.3956(7),
\]

represented by the vertical red dashed lines, we find two branched of the perturbative conformal order in deformed Model II. However, since $s_{\text{max}} < 1$, we can not construct even a perturbative conformal order in $b \to +\infty$ deformation of Model II — recall that the latter requires $s = 1$. We conclude that there is no conformal order (at least within the deformation framework of [5]) in universal top-down holographic Model II, representing a CFT dual to type IIB supergravity with the self-dual five-form flux on a squashed Sasaki-Einstein manifold with a breathing and a squashing modes, dual to the order parameters of dimensions $\Delta = 8$ and $\Delta = 6$ correspondingly.

### 3.3 Conformal order in Model III

Conformal Model III realizes holographic dual to the KW gauge theory with a triplet of order parameters: a dimension $\Delta = 8$ operator $O_8$, a dimension $\Delta = 6$ operator $O_6$ and a dimension $\Delta = 3$ operator $O_3$, represented by the bulk scalars $f$, $w$ and $\lambda$ correspondingly. A noticeable different of Model III from the models discussed in sections 3.1 and 3.2 is that the former ones represent holographic duals to the boundary CFTs with an unbroken $U(1)_R$ symmetry; while $\langle O_3 \rangle \neq 0$ would spontaneously break this continuous symmetry as $U(1) \to \mathbb{Z}_2$.

We follow appendix A to construct conformal order in Model III. From (3.15) we find

\[
\mathcal{P}_{III}[f, w, \lambda] = \mathcal{P} = -12 + \left( \frac{1280}{3} f^2 + 240 w^2 - 12 \lambda^2 \right) + \left( - \frac{71680}{27} f^3 - 1280 f w^2 - 1120 w^3 + 64 \lambda^2 (f + 6 w) \right) + \mathcal{O}(\{f, w, \lambda\}^4),
\]

i.e., the leading nonlinear terms are $n = 3$ (see (1.5)). The relevant unbounded directions of the truncated potential are less obvious to identify: as in Model II, one would expect for $\{f, w\}$ scalars to become large-positive to drive their effective mass (A.19) below the BF bound; while they would be expected to become large-negative for the effective mass of $\lambda$ to dip below its effective BF bound near the horizon. As we shortly demonstrate, conformal order in deformed Model III exists when both scalars $\{f, w\}$ are negative at the horizon. The $b$-deformed scalar potential takes the form,

\[
\mathcal{P}_{III}^b[f, w, \lambda] = -12 + \left( \frac{1280}{3} f^2 + 240 w^2 - 12 \lambda^2 \right) + b \left( \mathcal{P} + 12 - \frac{1280}{3} f^2 - 240 w^2 + 12 \lambda^2 \right).
\]

Using the radial coordinate as in (A.10), and introducing

\[
c_2(x) = \frac{g(x)}{(2x - x^2)^{1/4}},
\]

\(^9\)With appropriate rescaling due to different normalization of the scalar kinetic terms in (3.15) and (A.1).
we obtain the following equations of motion \((' \equiv \frac{d}{dx})\)

\[
0 = f'' + \frac{f'}{x - 1} + \left( -\frac{1}{2} (f')^2 \right) - \frac{3}{4} (w')^2 - \frac{3}{2} \left( \lambda' \right)^2 + \frac{9h^2}{20} - \frac{9(x^2 - 2x + 2)h}{40x(1 - x)(2 - x)} \\
+ \frac{9}{80x^2(2 - x)^2} \left( \frac{\partial P_{III}}{\partial f} \right),
\]

\[
0 = w'' + \frac{w'}{x - 1} + \left( -\frac{1}{3} (f')^2 \right) - \frac{1}{2} (w')^2 - \frac{1}{10} \left( \lambda' \right)^2 + \frac{3h^2}{10} - \frac{3(x^2 - 2x + 2)h}{20x(1 - x)(2 - x)} \\
+ \frac{3}{40x^2(2 - x)^2} \left( \frac{\partial P_{III}}{\partial w} \right),
\]

\[
0 = \lambda'' + \frac{\lambda'}{x - 1} + \left( -\frac{5}{3} (f')^2 \right) - \frac{5}{2} (w')^2 - \frac{1}{2} \left( \lambda' \right)^2 + \frac{3h^2}{2} - \frac{3(x^2 - 2x + 2)h}{4x(1 - x)(2 - x)} \\
+ \frac{3}{8x^2(2 - x)^2} \left( \frac{\partial P_{III}}{\partial \lambda} \right),
\]

\[
0 = h' - 4h^2 + \frac{(3x^2 - 6x + 4)h}{x(1 - x)(2 - x)} + \frac{40}{9} (f')^2 + \frac{20}{3} (w')^2 + \frac{4}{3} \left( \lambda' \right)^2,
\]

\[
0 = g' - h g.
\]

Eqs. \((3.50)\)–\((3.53)\) are solved subject to the boundary conditions:

- as \(x \to 0_+\), i.e., at the AdS\(_5\) boundary

\[
f = -\frac{3}{25} b \lambda_1^2 x^{3/2} + f_4 \quad \text{or} \quad \mathcal{O}(x^{5/2}) , \quad w = \left( \frac{3}{10} b \lambda_1^2 \ln x + w_3 \right) x^{3/2} + \mathcal{O}(x^{5/2} \ln x), \]

\[
\lambda = x^{1/4} \left( \lambda_1 x^{1/2} + \frac{3}{8} \lambda_1 x^{3/2} + \mathcal{O}(x^2 \ln x) \right), \quad h = -\frac{3}{10} \lambda_1^2 x^{1/2} - \frac{3}{8} \lambda_1^2 x^{3/2} + \mathcal{O}(x^2 \ln^2 x) ,
\]

\[
g = A \left( 1 - \frac{1}{5} \lambda_1^2 x^{3/2} - \frac{3}{20} \lambda_1^2 x^{5/2} + \mathcal{O}(x^3 \ln^2 x) \right),
\]

- as \(y \equiv (1 - x) \to 0_+\), i.e., at the horizon

\[
f = f_0^h + \mathcal{O}(y^2), \quad w = w_0^h + \mathcal{O}(y^2) , \quad \lambda = \lambda_0^h + \mathcal{O}(y^2),
\]

\[
h = h_1^h y + \mathcal{O}(y^3), \quad g = A \left( g_0^h + \mathcal{O}(y^2) \right).
\]

Note that in total we have eight parameters \(\{f_4, f_0^h, w_3, w_0^h, \lambda_1, \lambda_0^h, h_1^h, g_0^h\}\), along with an arbitrary constant \(A\). They determine the thermal conformal order of Model III, speci-
fically, 
\[
T^{-8} \langle \mathcal{O}_8 \rangle = f_4, \quad T^{-6} \langle \mathcal{O}_6 \rangle = w_3, \quad T^{-3} \langle \mathcal{O}_3 \rangle = \lambda_1, 
\]
\[
T^2 = \frac{A^2 \langle g_0^h \rangle^2}{\pi^2 (1 - 2h_1^h)} \left[ \frac{320}{9} \left( f_0^h \right)^2 - \left( \lambda_0^h \right)^2 + 20 \left( w_0^h \right)^2 - 1 \right] (b - 1) 
+ \frac{2 e^{-\frac{16}{3} f_0^h - 2w_0^h}}{3} \cosh(2\lambda_0^h) + \frac{3}{8} e^{-\frac{16}{3} f_0^h + 8w_0^h} \left( 1 - \cosh(4\lambda_0^h) \right) 
- \frac{2}{3} e^{-\frac{4w_0^h - 1}{3} e^{-\frac{16}{3} f_0^h - 12w_0^h}} \right]^{-3/2}, 
\]
(3.57)
for the entropy density \( s \), the free energy density \( \mathcal{F} \), the temperature \( T \), and the thermal order parameters \( \{ \mathcal{O}_8, \mathcal{O}_6, \mathcal{O}_3 \} \). Thus, see (1.1),
\[
\kappa_{III} = \left[ 1 - 2h_1^h \right]^{3/2} \times \left[ \frac{320}{9} \left( f_0^h \right)^2 - \left( \lambda_0^h \right)^2 + 20 \left( w_0^h \right)^2 - 1 \right] (b - 1) 
+ \frac{2 e^{-\frac{16}{3} f_0^h - 2w_0^h}}{3} \cosh(2\lambda_0^h) + \frac{3}{8} e^{-\frac{16}{3} f_0^h + 8w_0^h} \left( 1 - \cosh(4\lambda_0^h) \right) 
- \frac{2}{3} e^{-\frac{4w_0^h - 1}{3} e^{-\frac{16}{3} f_0^h - 12w_0^h}} \right]^{-3/2}, \quad \gamma_{III,(8,6,3)} = \{ f_4, w_3, \lambda_1 \}, 
\]
(3.58)
where the subscript \( III \) refers to Model III.

In the limit \( b \to +\infty \) we find, see appendix A,
\[
\{ f, w, \lambda \} = \{ f_1, w_1, \ell_1 \} \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad h = \mathcal{O} \left( b^{-2} \right), \quad g = 1 + \mathcal{O} \left( b^{-2} \right), \quad \] (3.59)
with
\[
f_4 = 0.27(3) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad f_0^h = -0.008(6) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad w_3 = 0.16(5) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad w_0^h = -0.094(4) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad \lambda_1 = 0.95(2) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right), \quad \lambda_0^h = 0.59(3) \frac{1}{b} + \mathcal{O} \left( b^{-2} \right). 
\]
(3.60)
Finite \( b \) results are obtained solving (3.50)–(3.54), subject to the asymptotics (3.55) and (3.56). Notice that both \( f_0^h \) and \( w_0^h \) are negative as \( b \gg 1 \); as figure 7 shows they continue to stay negative for finite \( b \), driving the effective mass of \( \lambda \) at the Schwarzschild horizon below the effective BF bound,
\[
m_{eff,\lambda}^2 \bigg|_{\text{horizon}} = \Delta(\Delta - 4) + 16b(f + 6w) \bigg|_{\Delta=3} = -3 + 16b \left( f_0^h + 6w_0^h \right) 
= -12.16(1) + \mathcal{O} \left( b^{-1} \right), 
\]
(3.61)
and causing the condensation of all the scalars, see figure 8.

In figures 9–11 we plot the expectation values of the order parameters \( \gamma_{III,(8,6,3)} \), see (1.1) and (3.34), (3.58). In figure 12 we plot the coefficient \( \kappa \) of the thermal order
Figure 7. Conformal order in the deformed Model III arises due to negative values (3.56) of $f$ and $w$ scalars at the Schwarzschild horizon (blue curves), resulting in the effective mass of $\lambda$-scalar, see (3.61), below the effective BF bound. The green curves represent large-$b$ approximations to $\{f^b_0, w^b_0\}$, see (3.60).

Figure 8. Effective mass of the $\lambda$-scalar evaluated at the Schwarzschild horizon (blue curve), see (3.61). The red line represents AdS$_5$ BF bound.

equation of state in Model III as a function of the deformation parameter $b$, see (1.1) and (3.58). We obtained reliable numerical results for $b \gtrsim 1.03$; their extrapolation to smaller values of $b$ predicts that all the order parameters diverge as $b \to b_{\text{crit,III}},$

$$ b_{\text{crit,III}} = 0.99(7) \bigg|_{\text{extrapolation}}, $$

from above. We take closeness of $b_{\text{crit,III}}$ to 1 as the strong indication that the actual value of $b_{\text{crit,III}}$ is precisely 1. We thus conclude that while there is a conformal order in $b$-deformed Model III, it disappears once this model becomes a realization of the top-down holography, much like in top-down holographic models discussed in [5, 9]. Since $\kappa_{\text{III}}(b) < 1$, the conformal order in deformed Model III is subdominant in canonical and microcanonical ensembles. Following [4] we expect that this conformal order is perturbatively unstable.
Figure 9. Thermal conformal order parameter $\gamma_8 \equiv T^{-8}\langle O_8 \rangle$ of the deformed Model III diverges as $b \to b_{\text{crit}, III} + 0$, represented by the dashed vertical red line.

Figure 10. Thermal conformal order parameter $\gamma_6 \equiv T^{-6}\langle O_6 \rangle$ of the deformed Model III diverges as $b \to b_{\text{crit}, III} + 0$, represented by the dashed vertical red line.

Figure 11. Thermal conformal order parameter $\gamma_3 \equiv T^{-3}\langle O_3 \rangle$ of the deformed Model III diverges as $b \to b_{\text{crit}, III} + 0$, represented by the dashed vertical red line.
4 Conclusion

In this paper we searched for the holographic conformal order [3] on warped deformed conifold with fluxes [32] — representing the supergravity dual to Klebanov-Strassler cascading gauge theory [18]. We focused on two exotic phases [2] of the black branes on the conifold:

- the Klebanov-Strassler black branes [23], realizing the spontaneous breaking of the continuous $R$-symmetry;
- the branch of Klebanov-Tseytlin black branes with the negative specific heat [30].

The two classes of the black branes have an intrinsic scale, the holographic dual to the strong coupling scale $\Lambda$ of the boundary cascading gauge theory. Thus, to find the conformal order one needs to construct these black branes in the high temperature limit $T/\Lambda \to 0$. We showed that as one increases the temperature, these exotic black branes on the conifold become ever more stringy. As a result, the conformal order can not be reached within the controllable supergravity approximation.

In the second part of the paper we changed the strategy: instead of starting with the non-conformal exotic black branes and taking the high-temperature limit, we took the conformal limit in the effective action of type IIB supergravity on warped deformed conifold with fluxes [31]. The latter limit effectively removes the fractional D3 branes from the model. The resulting five-dimensional effective action has seven bulk scalars; but only three of these scalars have the nonlinear potential. The nonlinearity of the scalar potential appears to be crucial in holographic examples of the conformal order constructed so far [2, 3, 5, 9]. Taking this fact as a hint, we consistently truncated the “conformal” effective action to these three scalars. Within this action, we further identified two additional universal consistent truncations:

- type IIB supergravity on warped Sasaki-Einstein manifolds with the self-dual five-form flux [24, 30];
- type IIB supergravity on warped and squashed Sasaki-Einstein manifolds with the self-dual five-form flux [24, 30].

**Figure 12.** The coefficient $\kappa_{III}$, see (1.1), of the thermal order equation of state in the deformed Model III. The vertical dashed red line denotes $b = b_{\text{crit,III}}$. 
We showed that in both truncations one can construct the conformal order, provided one deforms the bulk scalar potential obtained from the Kaluza-Klein reduction on the Sasaki-Einstein manifold. However, before the deformation parameter is removed, the conformal order disappears. The same story repeats in the full effective action on the warped deformed conifold with the three bulk scalars — the conformal order exists, as long as the scalar potential is (arbitrarily small) deformed from the one obtained from the type IIB supergravity Kaluza-Klein reduction. Along with the previous results [5, 9] we believe this sends a powerful message: there is a ‘real’ holography and a ‘toy’ one; conformal order is possible only in the latter. This begs a question: how much one can trust phenomenological holographic models to predict String Theory Universe phenomena?

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A Perturbative holographic thermal conformal order

Consider thermal conformal order in the phenomenological model

$$ S_5 = \frac{c}{2\pi^2} \int_{M_5} \text{vol}_{M_5} \left\{ R - \frac{1}{2} (\nabla \phi)^2 - P^b[\phi] \right\}, \quad (A.1) $$

with the scalar potential $P^b[\phi]$ given by (1.6), i.e.,

$$ P^b = -12 + \frac{1}{2}\Delta(\Delta - 4)\phi^2 - b \sum_{k=n+3}^{\infty} p_k \phi^k, \quad (A.2) $$

where $p_n > 0$, and the following next nonzero coefficient $p_k$ is for $k = n + p \geq n + 1$. We assume $\Delta \geq 2$. To this end, we assume the black brane metric ansatz

$$ ds_5 = -c_1^2 dt^2 + c_2^2 dx^2 + c_3^2 dr^2, \quad c_i = c_i(r), \quad (A.3) $$

along with $\phi = \phi(r)$, all depending on the radial coordinate $r \in [r_0, +\infty)$. There is the smooth Schwarzschild horizon as $r \to r_0^+$, i.e.,

$$ \lim_{r \to r_0^+} c_1^2 = 0, \quad (A.4) $$

and asymptotic AdS$_5$ geometry, i.e., as $r \to \infty$,

$$ c_1^2 \to r^2, \quad c_2 \to r^2, \quad c_3^2 \to r^{-2}, \quad \phi \to \frac{\langle O_\Delta \rangle}{r^\Delta}, \quad (A.5) $$

where $\langle O_\Delta \rangle \neq 0$ is the order parameter, see (1.1).
From the effective action \((A.1)\) we derive the following equations of motion

\[
0 = c_1'' + c_1' \left[ \ln \frac{c_3^3}{c_3} \right]' + \frac{1}{3} c_1 c_3^2 P^b, \tag{A.6}
\]

\[
0 = c_2'' + c_2' \left[ \ln \frac{c_1^2 c_3^2}{c_3} \right]' + \frac{1}{3} c_2 c_3^2 P^b, \tag{A.7}
\]

\[
0 = (\phi')^2 - 12 \left[ \ln c_2 \right]' \left[ \ln (c_1 c_2) \right]' - 2 c_3^2 P^b, \tag{A.8}
\]

\[
0 = \phi'' + \phi' \left[ \ln c_3^3 \right]' - c_3^2 \frac{\partial P^b}{\partial \phi}, \tag{A.9}
\]

where \(\frac{d}{dx}\). Equations (A.6)–(A.9) can be systematically analyzed perturbatively as \(b \to +\infty\). Introducing a new radial coordinate \([36]\),

\[
1 - x = \frac{c_1}{c_2}, \quad x \in (0, 1), \tag{A.10}
\]

with \(x \to 1\) representing the regular Schwarzschild horizon \((A.4)\), and \(x \to 0\) the boundary asymptotes \((A.5)\), we find

\[
c_1 = (1 - x) c_2, \quad c_2 = \frac{\pi T}{(2x - x^2)^{1/4}} \left[ 1 + \mathcal{O} \left( \frac{1}{b} \right)^{\frac{\Delta - 2}{4}} \right],
\]

\[
c_3^2 \, dr^2 = \frac{dx^2}{4(2x - x^2)^2} \left[ 1 + \mathcal{O} \left( \frac{1}{b} \right)^{\frac{\Delta - 2}{4}} \right], \tag{A.11}
\]

\[
\phi = \left( \frac{1}{2p_n b} \right)^{\frac{1}{4}} \left[ \phi_1(x) + \mathcal{O} \left( \frac{1}{b} \right)^{\frac{\Delta - 2}{4}}, \left( \frac{1}{b} \right)^{\frac{\Delta - 2}{4}} \right],
\]

where \(T\) is the Hawking temperature of the horizon, and the leading as \(b \to +\infty\) scalar hair \(\phi_1\) satisfies

\[
0 = \phi_1'' + \frac{\phi_1'}{x - 1} + \phi_1 (n \phi_1 - 2 \Delta (\Delta - 4)) \frac{1}{8x^2(2 - x)^2}, \tag{A.12}
\]

subject to the boundary conditions:

- as \(x \to 0\), i.e., at the AdS\(_5\) boundary

\[
\phi_1 \to f_{1,0} \, x^{\Delta/4} \quad \implies \quad T^{-\Delta} \langle \mathcal{O}_\Delta \rangle \propto f_{1,0} \left( \frac{1}{b} \right)^{\frac{1}{4}}, \tag{A.13}
\]

- as \(x \to 1\), i.e., at the horizon

\[
\phi_1 \to f_{1,h}. \tag{A.14}
\]

Once the perturbative \((as \, b \to +\infty)\) solution \((A.11)\) is constructed, the finite-\(b\) conformal order can be analyzed numerically, incrementally decreasing the deformation parameter \(b\) \([3, 5, 9]\).

The reason why we expect the specific large-\(b\) scaling of the conformal order as in \((A.11)\) was given in \([5]\):
Recall the story of the holographic superconductor [37, 38]. A scalar field in asymptotically AdS geometry must have a mass above the (space-time dependent) Breitenlohner-Freedman (BF) bound to avoid the condensation. In the vicinity of the Schwarzschild horizon, the BF bound can be modified either by changing the effective dimensionality of the space-time (as in the extremal limit of a Reissner-Nordstrom black brane) [38], or via nonlinear scalar interactions, leading to large negative contribution of the effective mass [37]. Thus, it is possible for a scalar field to be above the BF bound close to the AdS boundary, and below the effective BF bound close to the horizon. This scenario triggers the condensation of the scalar - the black brane develops the scalar hair. Close to the transition point, the condensation can be studied in the probe approximation — neglecting the backreaction.

The conformal order realizes the black brane horizon scalarization mechanism of [37]. It becomes a probe approximation in the limit $b \to +\infty$. Indeed, with the scaling

$$\phi \propto \left(\frac{1}{b}\right)^{\frac{1}{n-2}}, \quad n \geq 3, \quad (A.15)$$

the scalar backreaction on the AdS$_5$-Schwarzschild geometry vanishes as in (A.11). Furthermore, an order $k \geq n$ monomial in the scalar potential (A.2) scales as

$$b p_k \phi^k \propto p_k \left(\frac{1}{b}\right)^{-\frac{k-n}{n-2}}, \quad (A.16)$$

while the mass term in the scalar potential (A.2), as well as the kinetic term, scale as

$$(\nabla \phi)^2 \sim \Delta (\Delta - 4) \phi^2 \propto \left(\frac{1}{b}\right)^{-\frac{2}{n-2}}. \quad (A.17)$$

Thus, any nonlinear term in the scalar potential with $k > n$ is subleading in the $b \to +\infty$ limit compare to the quadratic scalar terms in the effective action (A.1), while the leading nonlinear $k = n$ term scales precisely as the former.

To summarize, given (A.15), the holographic model (A.1) represents the probe approximation of the scalar $\phi$ on AdS$_5$-Schwarzschild geometry with the effective action:

$$S_{\text{scalar}} = \int_{\mathcal{M}_5 = \text{AdS}_5 - \text{Schwarzschild}} \text{vol}_{\mathcal{M}_5} \left\{ -\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m_{\text{eff}}^2 \phi^2 \right\} \propto \int_{\mathcal{M}_5 = \text{AdS}_5 - \text{Schwarzschild}} \text{vol}_{\mathcal{M}_5} \left\{ -\frac{1}{2} (\nabla \phi_1)^2 - \frac{1}{2} m_{\text{eff}}^2 \phi_1^2 \right\}, \quad (A.18)$$

where

$$m_{\text{eff}}^2 = \Delta (\Delta - 4) - 2 p_n b \phi^{n-2} = \Delta (\Delta - 4) - \phi_1^{n-2}, \quad (A.19)$$

and the scalar field $\phi_1$ satisfies (A.12). Notice from (A.19) that a large positive value of $\phi_1$ at the horizon, see (A.14), can make $m_{\text{eff}}^2$ sufficiently negative, and trigger the scalarization as in [37].
Figure 13. Normalizable coefficients, (A.13) and (A.14), of the bulk scalar field dual to the conformal order parameter $O_\Delta$ in the limit $b \to +\infty$. The red squares represent the analytical results obtained for $\Delta = 4$ and $\Delta = 8$ correspondingly.

We now return to the analysis of (A.12)–(A.14). We consider $n = 3$, as this case will be of relevance for the discussion in section 3, and consider integer values of $\Delta = \{2, \cdots, 10\}$. Numerical results for $\{f_{1,0}, f_{1,h}\}$ are presented in figure 13; for $\Delta = 4$ and $\Delta = 8$ (A.12)–(A.14) can be solved analytically:

$$
\Delta = 4: \quad \phi_1 = \frac{32}{3} x(2-x) \quad \implies \quad \{f_{1,0}, f_{1,h}\} = \left\{ \frac{64}{3}, \frac{32}{3} \right\},
$$

$$
\Delta = 8: \quad \phi_1 = 64 \frac{x^2(2-x)^2}{(1+(1-x)^2)^2} \quad \implies \quad \{f_{1,0}, f_{1,h}\} = \{64, 64\}. \quad (A.20)
$$

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References

[1] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici and M. Smolkin, Thermal Order in Conformal Theories, Phys. Rev. D 102 (2020) 065014 [arXiv:2005.03676] [insPIRE].

[2] A. Buchel and C. Pagnutti, Exotic Hairy Black Holes, Nucl. Phys. B 824 (2010) 85 [arXiv:0904.1716] [insPIRE].

[3] A. Buchel, Thermal order in holographic CFTs and no-hair theorem violation in black branes, Nucl. Phys. B 967 (2021) 115425 [arXiv:2005.07833] [insPIRE].

[4] A. Buchel, Fate of the conformal order, Phys. Rev. D 103 (2021) 026008 [arXiv:2011.11509] [insPIRE].

[5] A. Buchel, SUGRA/Strings like to be bald, Phys. Lett. B 814 (2021) 136111 [arXiv:2007.09420] [insPIRE].

$^{10}$Generalizations to other $n$ and $\Delta$ are straightforward.
[6] S. Chaudhuri, C. Choi and E. Rabinovici, Thermal order in large $N$ conformal gauge theories, *JHEP* 04 (2021) 203 [arXiv:2011.13981] [inSPIRE].

[7] N. Chai, A. Dymarsky and M. Smolkin, Model of Persistent Breaking of Discrete Symmetry, *Phys. Rev. Lett.* 128 (2022) 011601 [arXiv:2106.09723] [inSPIRE].

[8] S. Chaudhuri and E. Rabinovici, Symmetry breaking at high temperatures in large $N$ gauge theories, *JHEP* 08 (2021) 148 [arXiv:2106.11323] [inSPIRE].

[9] A. Buchel, Compactified holographic conformal order, *Nucl. Phys. B* 973 (2021) 115605 [arXiv:2107.05086] [inSPIRE].

[10] N. Chai, A. Dymarsky, M. Goykhman, R. Sinha and M. Smolkin, A model of persistent breaking of continuous symmetry, *SciPost Phys.* 12 (2022) 181 [arXiv:2111.02474] [inSPIRE].

[11] J.M. Maldacena, The Large $N$ limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998) 231 [hep-th/9711200] [inSPIRE].

[12] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, Large $N$ field theories, string theory and gravity, *Phys. Rept.* 323 (2000) 183 [hep-th/9905111] [inSPIRE].

[13] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge theory correlators from noncritical string theory, *Phys. Lett. B* 428 (1998) 105 [hep-th/9802109] [inSPIRE].

[14] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998) 253 [hep-th/9802150] [inSPIRE].

[15] K. Pilch and N.P. Warner, $N=2$ supersymmetric RG flows and the IIB dilaton, *Nucl. Phys. B* 594 (2001) 209 [hep-th/0004063] [inSPIRE].

[16] A. Buchel, A.W. Peet and J. Polchinski, Gauge dual and noncommutative extension of an $N=2$ supergravity solution, *Phys. Rev. D* 63 (2001) 044009 [hep-th/0008076] [inSPIRE].

[17] N.J. Evans, C.V. Johnson and M. Petrini, The Enhance and $N=2$ gauge theory: Gravity RG flows, *JHEP* 10 (2000) 022 [hep-th/0008081] [inSPIRE].

[18] I.R. Klebanov and M.J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities, *JHEP* 08 (2000) 052 [hep-th/0007191] [inSPIRE].

[19] C.P. Herzog, I.R. Klebanov and P. Ouyang, Remarks on the warped deformed conifold, in *Modern Trends in String Theory: 2nd Lisbon School on g Theory Superstrings*, Lisbon, Portugal (2001) [hep-th/0108101] [inSPIRE].

[20] J.M. Maldacena and C. Núñez, Towards the large $N$ limit of pure $N=1$ superYang-Mills, *Phys. Rev. Lett.* 86 (2001) 588 [hep-th/0008001] [inSPIRE].

[21] P. Candelas and X.C. de la Ossa, Comments on Conifolds, *Nucl. Phys. B* 342 (1990) 246 [inSPIRE].

[22] I.R. Klebanov and E. Witten, Superconformal field theory on three-branes at a Calabi-Yau singularity, *Nucl. Phys. B* 536 (1998) 199 [hep-th/9807080] [inSPIRE].

[23] A. Buchel, Klebanov-Strassler black hole, *JHEP* 01 (2019) 207 [arXiv:1809.08484] [inSPIRE].

[24] D. Cassani, G. Dall’Agata and A.F. Faedo, Type IIB supergravity on squashed Sasaki-Einstein manifolds, *JHEP* 05 (2010) 094 [arXiv:1003.4283] [inSPIRE].
[25] A. Buchel, *Finite temperature resolution of the Klebanov-Tseytlin singularity*, *Nucl. Phys. B* **600** (2001) 219 [hep-th/0011146] [insPIRE].

[26] A. Buchel, C.P. Herzog, I.R. Klebanov, L.A. Pando Zayas and A.A. Tseytlin, *Nonextremal gravity duals for fractional D-3 branes on the conifold*, *JHEP* **04** (2001) 033 [hep-th/0102105] [insPIRE].

[27] S.S. Gubser, C.P. Herzog, I.R. Klebanov and A.A. Tseytlin, *Restoration of chiral symmetry: A Supergravity perspective*, *JHEP* **05** (2001) 028 [hep-th/0102172] [insPIRE].

[28] O. Aharony, A. Buchel and A. Yarom, *Holographic renormalization of cascading gauge theories*, *Phys. Rev. D* **76** (2007) 086005 [arXiv:0706.1768] [insPIRE].

[29] O. Aharony, A. Buchel and P. Kerner, *The Black hole in the throat: Thermodynamics of strongly coupled cascading gauge theories*, *Phys. Rev. D* **72** (2005) 066003 [hep-th/0506002] [insPIRE].

[30] A. Buchel, *Hydrodynamics of the cascading plasma*, *Nucl. Phys. B* **820** (2009) 385 [arXiv:0903.3606] [insPIRE].

[31] A. Buchel, *Chiral symmetry breaking in cascading gauge theory plasma*, *Nucl. Phys. B* **847** (2011) 297 [arXiv:1012.2404] [insPIRE].

[32] A. Buchel, *A bestiary of black holes on the conifold with fluxes*, *JHEP* **06** (2021) 102 [arXiv:2103.15188] [insPIRE].

[33] A. Buchel, *A Holographic perspective on Gubser-Mitra conjecture*, *Nucl. Phys. B* **731** (2005) 109 [hep-th/0507275] [insPIRE].

[34] A. Buchel, *Transport properties of cascading gauge theories*, *Phys. Rev. D* **72** (2005) 106002 [hep-th/0509083] [insPIRE].

[35] I. Bena, A. Buchel and S. Lüst, *Throat destabilization for profit and for fun*, arXiv:1910.08094 [insPIRE].

[36] A. Buchel and J.T. Liu, *Thermodynamics of the \(N=2^*\) flow*, *JHEP* **11** (2003) 031 [hep-th/0305064] [insPIRE].

[37] S.S. Gubser, *Phase transitions near black hole horizons*, *Class. Quant. Grav.* **22** (2005) 5121 [hep-th/0505189] [insPIRE].

[38] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Holographic Superconductors*, *JHEP* **12** (2008) 015 [arXiv:0810.1563] [insPIRE].