Scaling laws at the critical point

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There are two independent critical exponents that describe the behavior of systems near their critical point. However, at the critical point only the exponent $\eta$, which describes the decay of the correlation function, is usually discussed. We emphasize that there is a second independent exponent $\eta'$ that describes the decay of the fourth-order correlation function. The exponent $\eta'$ is related to the exponents determining the behavior of thermodynamic functions near criticality via a fluctuation-response equation for the specific heat. We also discuss a scaling law for $\eta'$.

It is well known that the critical behavior of a system near its continuous phase transition can be completely described by only two independent exponents $\alpha$ This description can be expressed by the scaling behavior of the singular part of the free energy density

$$f_{\text{sing}} = |t|^{2-\alpha} f_\pm (h/|t|^{\Delta}), \quad (t \to 0^\pm)$$

(1)

where $t \equiv (T - T_c)/T_c$ is the reduced temperature, $T_c$ is the critical point temperature, and $h$ is the external field that is coupled to the order parameter. The scaling functions $f_\pm (x)$ depend on $t$ and $h$ only in the combination $x = h/|t|^{\Delta}$. [xx note change of notation xx] Because the singular contributions to various thermodynamic quantities are found from various derivatives of the free energy in Eq. 1, there are only two independent exponents, say $\alpha$ and $\Delta$, that characterize the critical behavior of the system and thus specify its universality class. For example, the dependence on $t$ in Eq. (1) leads to a power-law singularity for the specific heat, $C_{\text{sing}} \sim -\partial^2 f/\partial t^2$ when $h = 0$:

$$C_{\text{sing}} = |t|^{-\alpha} c_\pm (h/|t|^{\Delta}) = c_\pm (0) |t|^{-\alpha}, \quad (h = 0).$$

(2)

At the critical point the thermodynamic derivatives are not useful for characterizing the critical behavior as they usually take on infinite values. [xx most of the derivatives are infinite. don’t understand this sentence xx] The only well known exponent that is defined at the critical point is the anomalous dimension exponent $\eta$, which describes the decay of the order parameter correlation function:

$$G_m(r) \equiv \langle \delta m(r) \delta m(0) \rangle = \langle m(r) m(0) \rangle - \langle m \rangle^2 \sim 1/r^{d-2+\eta},$$

(3)

where $\delta m(r) = m(r) - \langle m \rangle$ is the order parameter density fluctuation from the mean at point $r$. The angular braces $\langle \cdots \rangle$ denote an ensemble average, and $d$ is the spatial dimension. Note that $G_m(r)$ is a two-point correlation function. Although we will use a notation appropriate to magnetic systems, the following discussion also is applicable to other systems that have a continuous phase transition, such as fluids at their critical point.

The search for a second independent exponent leads to the fourth-order correlation function in terms of the order parameter density with an exponent defined by $\Delta$

$$G_E(r) \equiv \langle H(r) H(0) \rangle - \langle H \rangle^2 \sim 1/r^{d-2+\eta'},$$

(4)

where $H(r)$ is the local Hamiltonian or the interaction energy density. Note that the local interaction energy density $H(r) \sim m(r)^2$ is quadratic in the order parameter density $m(r)$.

The exponents $\eta'$ and $\eta$ are related to the usual exponents via a fluctuation-response equation for the order parameter susceptibility

$$\chi = \beta \int d^d r G_m(r) \sim \int d^d r \frac{d^d r'}{r^{d-2+\eta}} \sim \xi^{2-\eta},$$

(5)

and the energy version of Eq. 5 which relates the thermal susceptibility or the specific heat:

$$C/k = \beta^2 \int d^d r G_E(r) \sim \int d^d r \frac{d^d r'}{r^{d-2+\eta'}} \sim \xi^{2-\eta'}.$$

(6)

The correlation length $\xi$ characterizes the exponential decay of the correlations, $\beta = 1/kT$, and $k$ is Boltzmann’s constant. Equation 6 yields the well-known Fisher scaling identity $\Delta$

$$\gamma = (2 - \eta) \nu,$$

(7)
where $\gamma$ is the linear response susceptibility exponent defined by $\chi \sim |t|^{-\gamma}$, and $\nu$ is the correlation length exponent, $\xi \sim |t|^{-\nu}$. Equation (3) yields an energy version of Fisher scaling:

$$\alpha = (2 - \eta')\nu. \quad (8)$$

The correlation length exponent $\nu$ appearing in Eqs. (4) and (5) must be the same because of the hyperscaling hypothesis, which states that close to criticality the correlation length $\xi$ is the only relevant length scale, and thus $\xi$ is solely responsible for the singular contributions to the thermodynamic quantities. Another important outcome of the hyperscaling hypothesis is the Josephson scaling identity, $2 - \alpha = d\nu$, which provides another relation between $\alpha$ and $\nu$ and thus gives a constraint on Eq. (5). If we substitute $\alpha = 2 - d\nu$ into Eq. (5), we obtain another scaling law involving $\eta'$:

$$\eta' = d + 2 - \frac{2}{\nu}. \quad (9)$$

We see that the critical exponent $\eta'$ is determined by the correlation length exponent $\nu$ and the spatial dimension $d$.

To verify the scaling equation (9) and to test its predictions, we first consider the $d = 2$ ferromagnetic Ising model in the absence of an external field for which exact analytic solutions are available. In particular, we are interested in the four-point correlation function of the form

$$w_4(r) = \langle S_1S_2S_rS_{r+1} \rangle - \langle S_1S_2 \rangle \langle S_rS_{r+1} \rangle \quad (10)$$

which also can be interpreted as a two-point energy correlation, with $S_1S_2$ characterizing the energy of a reference bond and $S_rS_{r+1}$ the energy of a bond at a distance $r$ from the reference bond, as can be seen in Fig. 1. The Ising spins take on the values, $S_r = \pm 1$. The fourth-order correlation function $w_4(r)$, when summed over $r$ for all distinct pairs of bonds is related to the specific heat in agreement with Eq. (6): $\sum_r w_4(r) = \partial^2 \partial^2 - 1 + \epsilon^2$, where $\epsilon = \langle S_1S_2 \rangle$ characterizes the mean energy per bond.

From Eq. (9) we see that $\eta' = 2$ for the $d = 2$ Ising model, where we have used the exactly known correlation length exponent $\nu = 1$. We note that when $\eta' = 2$ is substituted into the Eq. (6), we find a logarithmic divergence for the specific heat as we should. The asymptotic behavior of $w_4(r)$ has been calculated rigorously for the triangular and the square Ising lattice and has been found to be identical for the two lattices (apart from an amplitude factor of 1/2) and has the usual Ornstein-Zernick forms:

$$w_4(r) \sim \frac{e^{-2r/\xi}}{r^2}, \quad (r \to \infty). \quad (11)$$

We see that the predicted value $\eta' = 2$ agrees with results from exact analytic calculations.

For comparison the critical exponents relevant to our discussion for the $d = 2$ and $d = 3$ Ising models are summarized in Table 1. For $d = 3$ Eq. (4) predicts $\eta' = 1.822 (3)$, where we have used the numerical estimate, $\nu = 0.629 (1)$.

A direct measurement of the ratio $\alpha/\nu = 2 - \eta'$ can be obtained by a finite size scaling analysis of the specific heat obtained by Monte Carlo simulations. The slope of a log-log plot of $C$ versus $L$ provides an estimate for the ratio $\alpha/\nu$: $C \sim t^{-\alpha} \sim L^{\alpha/\nu}$. A reliable estimate of this ratio for $d = 3$ is $\alpha/\nu = 0.178 (5)$. Thus we have $\eta' = 2 - \alpha/\nu = 1.822 (5)$, which is in good agreement with the value predicted by Eq. (9).

In summary, the exponent $\eta'$ of the four-point correlation function provides a second independent exponent at the critical point. Thus $\{\eta', \eta\}$ form an independent exponent pair. The discussion presents a convenient framework for introducing the concept of fourth-order correlation functions, which are important in the study of viscous liquids and glasses.

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1. See, J. Tobochnik, “Resource letter CPPPT-1: Critical point phenomena and phase transitions,” Am. J. Phys. 69, 255–263 (2001).
2. K. Huang, Statistical Mechanics (Wiley, New York, 1987), 2nd ed., pp. 399–406.
3. M. Kardar, “Statistical mechanics of fields” (lecture notes), <ocw.mit.edu/OcwWeb/Physics/8-334Spring-2004/CourseHome/index.htm> Chap. 6.
4. M. E. Fisher, “Correlation functions and the critical region of simple fluids,” J. Math. Phys. 5, 944–962 (1964).
5. S. Davatolhagh, “The relation between the positional specific heat and the static relaxation length: Application to supercooled liquids,” J. Phys.: Condens. Matter 17, 1275–1285 (2005).
6. See also J. Cardy, Scaling and Renormalization in Statistical Physics (Cambridge University Press, Cambridge, 1996), where similar scaling properties are discussed in the context of conformal field-theory.
TABLE I: Comparison of various critical point exponents for the $d = 2$ and $d = 3$ Ising models.

| System                  | $\alpha$ | $\nu$ | $\eta$ | $\eta'^a$ | $\eta'^b$ |
|-------------------------|----------|-------|--------|-----------|-----------|
| 2D Ising model (exact)  | 0        | 1     | $1/4$  | 2         | 2         |
| 3D Ising model (approx) | 0.112(3) | 0.629 (1) | 0.037 (1) | 1.822 (3) | 1.822 (5) |

$^a$From Eq. (9)

$^b$From analytical/numerical calculations

![Schematic of four-point spin correlations in the $d = 2$ Ising model. The line segments represent bonds whose energies are determined by the product of nearest-neighbor spins at each end.](image)