Eighth-order Derivative-Free Family of Iterative Methods for Nonlinear Equations

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ABSTRACT. In this note, we present an eighth-order derivative-free family of iterative methods for nonlinear equations. The proposed family shows optimal eight-order of convergence in the sense of the Kung and Traub conjecture [1] and is based on the Steffensen derivative approximation used in the Newton-method. As a final step, having in mind computational purposes, a derivative-free polynomial base interpolation is used in order to get optimal order of convergence with only four functional evaluations. Numerical experiments and few issues are discussed at the end of this note.

KEYWORDS. Non-linear equations; Steffensen’s method; polynomial interpolation; iterative methods.

1. Introduction

Let \( f : D \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a sufficiently differentiable function of single variable in some neighborhood \( D \) of \( \alpha \), where \( \alpha \) is a simple root \( (f'(\alpha) \neq 0) \) of nonlinear algebraic equation \( f(x) = 0 \). The well-known Newton method is defined by the iteration

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},
\]

(1)

which shows a second-order convergence. One can easily get Steffensen’s approximation for first order derivative as

\[
\begin{align*}
  f(x_n - \kappa f(x_n)) & \approx f(x_n) - \kappa f(x_n) f'(x_n), \\
  \kappa f(x_n) f'(x_n) & \approx f(x_n) - f(x_n - \kappa f(x_n)), \\
  f'(x_n) & \approx \kappa \frac{f(x_n) - f(x_n - \kappa f(x_n))}{f(x_n)}. 
\end{align*}
\]

(2)

If we substitute the derivative approximation (2) in (1), we obtain Steffensen’s second order accurate derivative-free iterative method for nonlinear equations [2].

\[
\begin{align*}
  w_n &= x_n - \kappa f(x_n), \\
  x_{n+1} &= x_n - \kappa \frac{f(x_n)^2}{f(x_n) - f(w_n)}. 
\end{align*}
\]

(3)
In 2012, an optimal eighth-order iterative method [3] was proposed by Y. Khan et al. as follows:

\[
\begin{align*}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    z_n &= y_n - G \left( \frac{f(y_n)}{f'(x_n)} \right), \\
    x_{n+1} &= z_n - \mu + \nu q_n \frac{f(z_n)}{f'(x_n)},
\end{align*}
\]

(4)

where \( \mu \neq 0, \nu \in \mathbb{R}, q_n = f(z_n)/f(x_n) \), \( G(t) \) is a real-valued function with \( G(0) = 1, G'(0) = 2, G''(0) < \infty \), and

\[
\begin{align*}
    f'(z_n) &\approx K - C(y_n - z_n) - D(y_n - z_n)^2, \\
    H &= \frac{x_n - y_n}{f(x_n) - f(y_n)}, \\
    K &= \frac{x_n - y_n}{y_n - z_n}, \\
    D &= \frac{f(x_n) - H}{(x_n - y_n)(x_n - z_n)} - \frac{H - K}{(x_n - z_n)^2}, \\
    C &= \frac{H - K}{x_n - z_n} - D(x_n + y_n - 2z_n).
\end{align*}
\]

(5)

In the original draft of paper [3] the expression for \( C \) has typo-mistake, which is corrected here. Actually (5) polynomial interpolation approximation for \( f'(z_n) \) is given in [4]. Clearly (4) iterative scheme is not derivative free.

The main contribution in this paper is to use the idea of iterative scheme (4) by introducing Steffensen’s derivative approximation for \( f'(x_n) \) and then finally construct derivative-free approximation for \( f'(z_n) \) without reducing order of convergence.

2. Construction of derivative-free family

First we construct an interpolation polynomial approximation for \( f'(z_n) \). Suppose we have \( f(x_n), f(w_n) \) (defined in (3)), \( f(y_n) \) and \( f(z_n) \), One could construct a three-degree polynomial as follows:

\[
\begin{align*}
    p(\phi) &= f(y_n) + r_1(\phi - y_n) + r_2(\phi - y_n)^2 + r_3(\phi - y_n)^3, \\
    p'(\phi) &= r_1 + 2r_2(\phi - y_n) + 3r_3(\phi - y_n)^2.
\end{align*}
\]

(6)

By using four functional values, we get the following system of equations:

\[
\begin{align*}
    v_1 &= r_1 a + r_2 a^2 + r_3 a^3, \\
    v_2 &= r_1 b + r_2 b^2 + r_3 b^3, \\
    v_3 &= r_1 c + r_2 c^2 + r_3 c^3,
\end{align*}
\]

(7)

where

\[
\begin{align*}
    v_1 &= f(x_n) - f(y_n), \\
    v_2 &= f(z_n) - f(y_n), \\
    v_3 &= f(w_n) - f(y_n), \\
    a &= x_n - y_n, \\
    b &= z_n - y_n, \\
    c &= w_n - y_n.
\end{align*}
\]

(8)
After solving (7) for \( r_1, r_2 \) and \( r_3 \) and substituting them in (6) implies the following approximation for \( f'(z_n) \):

\[
f'(z_n) \approx \psi_n = \frac{b(b-c)}{(a-b)(a-c)}\frac{v_1}{a} + \frac{-3b^2+2bc+2ab-ac}{(a-b)(a-c)}\frac{v_2}{b} + \frac{b(b-a)}{(a-c)(b-c)}\frac{v_3}{c}.
\]

We consider the following family of iterative methods for nonlinear equations:

\[
\begin{align*}
  w_n &= x_n - \kappa f(x_n), \\
  y_n &= x_n - \kappa \frac{f(x_n)}{f(w_n)}, \\
  z_n &= y_n - \kappa \frac{f(x_n)}{f(w_n)} G(t_1, t_2), \\
  x_{n+1} &= z_n - \frac{f(y_n)}{f(z_n)} H(s_1, s_2),
\end{align*}
\]

where \( t_1 = \frac{f(y_n)}{f(x_n)}, \ t_2 = \frac{f(y_n)}{f(w_n)}, \ s_1 = \frac{f(z_n)}{f(x_n)}, \ s_2 = \frac{f(z_n)}{f(z_n)} \) and \( \kappa(\neq 0) \in \mathbb{R}. \)

3. Convergence analysis

We state the following theorem about the order of convergence of the family described in (10).

**Theorem 3.1.** Let \( f : D \subseteq \mathbb{R} \to \mathbb{R} \) be a sufficiently differentiable function, and \( \alpha \in D \) is a simple root of \( f(x) = 0 \), for an open interval \( D \). If \( x_0 \) is chosen sufficiently close to \( \alpha \), then the iterative scheme given in (10) converges to \( \alpha \). If \( G \) and \( H \) satisfy

\[
G(0,0) = 1, \quad \frac{\partial G}{\partial t_1}(0,0) = 1, \quad \frac{\partial G}{\partial t_2}(0,0) = 1, \quad H(0,0) = 1, \quad \frac{\partial H}{\partial s_1}(0,0) = 0, \quad \frac{\partial H}{\partial s_2}(0,0) = 0,
\]

and

\[
\frac{\partial^2 G}{\partial t_1^2}, \quad \frac{\partial^2 G}{\partial t_1 \partial t_2}, \quad \frac{\partial^2 G}{\partial t_2^2}, \quad \frac{\partial^2 H}{\partial s_1^2}, \quad \frac{\partial^2 H}{\partial s_1 \partial s_2}, \quad \frac{\partial^2 H}{\partial s_2^2}
\]

are bounded at \((0,0)\) then the iterative scheme (10) shows an order of convergence at least equal to eight.

**Proof.** Let the error at step \( n \) be denoted by \( e_n = x_n - \alpha \) and let us define \( c_1 = f'(\alpha) \) and \( c_k = \frac{1}{k!} f^{(k)}(\alpha), \ k = 2, 3, \ldots \). If we expand \( f \) around the root \( \alpha \) and express it in terms of powers of error \( e_n \), we obtain

\[
f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9),
\]

\[
f(w_n) = -c_1 (1 + \kappa c_1) e_n + c_2 \left(-3\kappa c_1 + 1 + \kappa^2 c_1^2\right) e_n^2 - c_1 (4\kappa c_1 c_3 + 2\kappa^2 c_1 c_2 c_3 + 2\kappa c_3^2 c_3^2 - c_2 - 3\kappa c_1 c_2^2 + c_3 c_1^2 c_1^2) e_n^3 + c_1 (-5\kappa c_1 c_4 - 5\kappa c_2 c_1 c_3 + 8\kappa^2 c_1 c_2 c_3 + \kappa^2 c_1 c_2^2 + 3\kappa c_2 c_3 c_3 c_3 + c_4 + 6\kappa c_2 c_1^2 - 4\kappa^2 c_1^3 + c_4 c_1^4) e_n^4 + \cdots + O(e_n^5),
\]

\[
y_n - \alpha = -c_2 (-1 + \kappa c_1) e_n^2 + (2c_3 - 3\kappa c_1 c_3 + c_3 c_2^2 c_2 + 2\kappa c_2 c_1 - 2\kappa^2 c_1^2 c_1^2) e_n^3 + (3c_4 + 4\kappa c_1 c_3 c_3 - 6\kappa c_1 c_4 + 4\kappa c_2 c_1^2 - c_3 c_1^3 c_1 - 7\kappa^2 c_1 c_2 c_3 - 7\kappa^2 c_1 c_2^2 + 5\kappa c_1 c_3^2 + 2\kappa c_2 c_3 c_3 c_3 + 3\kappa^2 c_1^3 c_3 + 4\kappa^3 c_1^3 c_3 + \cdots + O(e_n^5),
\]

\[
f(y_n) = -c_1 c_2 c_3 (-1 + \kappa c_1) e_n^2 - c_1 (-2c_3 + 3\kappa c_1 c_3 - c_3 c_2^2 c_2 - 2\kappa^2 c_1 c_2 c_2 + 2\kappa^2 c_1^2 c_1^2) e_n^3 - c_1 (-3c_4 - 10c_3 c_2 c_1 c_3 + 6\kappa c_1 c_4 - 4\kappa^2 c_1 c_2 c_1^2 + c_4 c_1^4 c_1 + 7\kappa^2 c_1 c_2 c_3 + c_7 c_1 c_3^2 - 2c_2 c_3 c_3 c_3 + 5c^2 c_1^2 c_1^2) e_n^4 + \cdots + O(e_n^5),
\]

\[
f(y_n) - f(x_n) = c_2 e_n + (2c_3 - 3\kappa c_1 c_3 + c_3 c_2^2 c_2 + 2\kappa^2 c_1 c_2 c_1^2 - 3\kappa^2 c_1^2 c_1^2) e_n^2 + \cdots + O(e_n^2),
\]

\[
f(y_n) - f(w_n) = c_2 e_n + (-\kappa c_1 c_3 + 2\kappa^2 c_1 c_1 + 2c_4 + 3c_2^2) e_n^2 + \cdots + O(e_n^2).
\]
By using (12), (13), (15), (16), (17), we find

\[ H - \alpha + 32 = (20 - \alpha + (27 + 3)2 - 2c_4 - 78c_5c_7c_9 - 3\kappa c_5^2 c_7^2 + 5c_5 c_7 c_9 - 6c_5^2 A_3 c_7 c_9 + 10c_1 c_4 + 10c_2 A_3 + 10A_2 c_4) e_n + \cdots + O(e_n^5), \]  

Finally, \( H(s_1, s_2) \) has Taylor’s expansion (by neglecting the linear terms)

\[ H \left( \frac{f(z_n)}{f(x_n)}, \frac{f(z_n)}{f(w_n)} \right) = 1 + B_1 \left( \frac{f(z_n)}{f(x_n)} \right)^2 + B_2 \left( \frac{f(z_n)}{f(w_n)} \right)^2 + B_3 \left( \frac{f(z_n)}{f(x_n)} \right) \left( \frac{f(z_n)}{f(w_n)} \right) + O(s_1^3, s_2^3). \]  

From (20) and (21), we deduced the following error equation which leads to the desired result

\[ e_n + c_2 = (-6c_2^2 c_3^2 c_4 + 10A_1 c_2^2 - 10A_2 c_2^2 - 10c_2^2 A_3 + 6c_2^2 c_4^2 + 31c_2^2 c_4 c_7 c_9 - 4c_2 c_4^2 c_7 c_9) e_n + \cdots + O(e_n^5). \]
+160\kappa^3 c_1^2 c_2^5 - 130\kappa^3 c_1^3 c_2^3 - 100\kappa^3 c_1 c_2 c_3 + 56\kappa^4 c_1^4 c_2^2 - 32A_6 c_2^5 c_3^2 + 30A_1 c_2^5 c_3 c_1 + 14A_2 c_2^5 c_3^3 c_1^3 \\
+46c_2^5 A_3 c_3^2 c_1 - 62c_2 A_2 c_3^2 c_1 + 16c_2 A_4 c_3^4 c_1 + 40c_2 A_3 c_3 c_1 + 108A_2 c_2^5 c_3^3 c_1^3 - 102A_2 c_2^5 c_3^2 c_1^3 \\
- 62A_2 c_2^5 c_3^4 c_1^2 + 50A_2 c_2^5 c_3^5 c_1^2 + A_1 c_2^5 c_3^2 c_1 + A_1 c_2^5 c_3^3 c_1^2 + 6c_2 A_2^3 c_3^3 c_1^2 - 4c_2 A_2^3 c_3^4 c_1 + 2c_2 A_3 c_3^5 c_1^2 + c_2 A_4 c_3^8 c_1^2 + 15A_2 A_3 c_3^4 c_1^2 \\
- 20A_2 A_3 c_3^5 c_1^2 - 15A_2 A_3 c_3^6 c_1^2 - 6A_2 c_3^8 c_1 - 6A_2 c_3^9 c_1^2 + A_2 c_3^10 c_1^3 - 2A_2 A_1 c_3^8 c_1^3 - 8A_1 c_3^9 c_1^3 c_1 \\
+ 12A_1 c_2 A_4 c_3^2 c_1^2 + 6A_1 c_2 A_3 c_3^5 c_1^2 - 8A_1 c_2^5 A_1 c_2 A_3 c_3^7 c_1 + 2A_1 c_2^5 c_3 c_1^2 A_2 - 2A_1 c_2^5 c_3^3 c_1^3 A_3 \\
+ 10c_2 A_3 c_3^4 c_1 A_2 - 20c_2^3 A_3 c_3^5 c_1^2 A_2 + 20c_2^3 A_3 c_3^6 c_1^2 A_2 - 2A_2 c_3^8 c_1 A_2 - 2A_2 c_3^9 c_1^2 <c_1 A_2 \\
+ 2A_1 c_2^5 A_3 + 2A_1 c_2^5 A_2 + 2A_2 c_2^3 A_3 + c_2 c_3^5 c_1^2 O(e_2^n). \\
\Box

It is clear that the considered family of numerical schemes requires four functional evaluations and attains optimal convergence order eight according to Kung and Traub conjecture which can be stated as follows [1]: If \( n \) is the total number of functional evaluations per iteration, then the optimal convergence order of the associated numerical procedure is \( 2^n-1 \).

4. Numerical results

**Definition 4.1.** The computational order of convergence [5], can be approximated by

\[
COC \approx \frac{\ln |x_{n+1} - \alpha|}{\ln |x_n - \alpha|},
\]

where \( x_{n+1} \), \( x_n \), and \( x_{n+1} \) are successive iterations closer to the root \( \alpha \) of \( f(x) = 0 \).

For the purpose of comparison between newly developed family and other derivative-free methods, a list of derivative-free methods for nonlinear equations is presented here.

4.1. The Kung-Traub eighth-order derivative-free method (K-T). The Kung-Traub eighth-order derivative-free method is discussed in [1, 6], and also considered in [7] is given as

\[
\begin{align*}
\phi_1 &= \left( 1 - \frac{f(y_n)}{f(w_n)} \right)^{-1}, \\
\phi_2 &= 1 + \frac{f(y_n)}{f(w_n)} + \left( \frac{f(y_n)}{f(w_n)} \right)^2, \\
\phi_3 &= f \left[ x_n, w_n \right],
\end{align*}
\]

4.2. R. Thukral \( M_1, M_2, M_3 \) methods. In 2011, R. Thukral [7] presented three variants of his proposed eighth-order three-point derivative-free method. Three members of the family called by author namely, \( M_1, M_2, \) and \( M_3 \) are listed as

\[
\phi_1 = \left( 1 - \frac{f(y_n)}{f(w_n)} \right)^{-1}, \\
\phi_2 = 1 + \frac{f(y_n)}{f(w_n)} + \left( \frac{f(y_n)}{f(w_n)} \right)^2, \\
\phi_3 = f \left[ x_n, w_n \right],
\]
and

\[
\begin{aligned}
    w_n &= x_n + \beta f(x_n), \\
    y_n &= x_n - \left( \frac{\beta f(x_n)^2}{f(w_n) - f(x_n)} \right), \\
    z_n &= y_n - \phi_k \left( \frac{f(y_n)}{f(x_n, y_n)} \right), \\
    x_{n+1} &= z_n - \left( 1 - \frac{f(z_n)}{f(w_n)} \right)^{-1} \left( 1 - \frac{f(y_n)^2}{f(w_n)^2 f(x_n)} \right) \left( \frac{f(x_n, y_n) f(z_n)}{f(y_n, x_n) f(x_n, z_n)} \right).
\end{aligned}
\] (29)

| Functions | Roots |
|-----------|-------|
| \( f_1(x) = \exp(x) \sin(x) + \ln(1 + x^2) \) | \( \alpha = 0 \) |
| \( f_2(x) = x^{15} + x^4 + 4x^2 - 15 \) | \( \alpha = 1.148538... \) |
| \( f_3(x) = (x - 2)(x^{10} + x + 1) \exp(-x - 1) \) | \( \alpha = 2 \) |
| \( f_4(x) = \exp(-x^2 + x + 2) - \cos(x + 1) + x^3 + 1 \) | \( \alpha = -1 \) |
| \( f_5(x) = (x + 1) \exp(\sin(x)) - x^2 \exp(\cos(x)) - 1 \) | \( \alpha = 0 \) |
| \( f_6(x) = \sin(x)^2 - x^2 + 1 \) | \( \alpha = 1.40449165... \) |
| \( f_7(x) = 10 \exp(-x^2) - 1 \) | \( \alpha = 1.517427... \) |
| \( f_8(x) = (x^2 - 1)^{-1} - 1 \) | \( \alpha = 1.414214... \) |
| \( f_9(x) = \ln(x^2 + x + 2) - x + 1 \) | \( \alpha = 4.15259074... \) |
| \( f_{10}(x) = \cos(x)^2 - x/5 \) | \( \alpha = 1.08598268... \) |
| \( f_{11}(x) = \sin(x) - \frac{x}{2} \) | \( \alpha = 0 \) |
| \( f_{12}(x) = x^{10} - 2x^3 - x + 1 \) | \( \alpha = 0.59144593... \) |
| \( f_{13}(x) = \exp(\sin(x)) - x + 1 \) | \( \alpha = 2.63066415... \) |

| \((f_n(x), x_0)\) | L | K-T | M1 | M2 | M3 | P1 | P2 |
|-------------------|---|-----|----|----|----|----|----|
| \( f_1, 0.25 \)   | (L1) 6.38e-247 | 3.14e-136 | 1.69e-141 | 7.43e-142 | 1.69e-141 | 3.20e-113 | 8.98e-120 |
| \( f_2, 1.1 \)    | (L1) 1.2376e-652 | 3.72e-62 | 3.44e-62 | 3.44e-62 | 3.44e-62 | 0.26e-7 | 2.67e-7 |
| \( f_3, 2.1 \)    | (L1) 1.057e-422 | 1.91e-60 | 1.49e-60 | 1.49e-60 | 1.49e-60 | 7.71e-8 | 7.56e-8 |
| \( f_4, -0.5 \)   | (L1) 2.952e-383 | 5.11e-362 | 1.92e-362 | 1.93e-362 | 1.92e-362 | 9.99e-367 | 8.78e-366 |
| \( f_5, 0.25 \)   | (L1) 2.336e-407 | 4.13e-328 | 6.52e-326 | 9.47e-326 | 6.52e-326 | 1.98e-322 | 2.56e-332 |
| \( f_6, 1.2 \)    | (L8) 1.719e-421 | 1.00e-327 | 4.58e-341 | 7.57e-344 | 4.58e-341 | 1.79e-381 | 1.72e-405 |
| \( f_7, 2 \)     | (L2) 7.264e-238 | 5.19e-88 | 1.24e-120 | 6.40e-124 | 1.24e-120 | 1.51e-187 | 6.79e-228 |
| \( f_8, 1.7 \)   | (L3) 1.429e-234 | 1.23e-113 | 1.74e-171 | 5.45e-188 | 1.74e-171 | 5.96e-211 | 4.84e-167 |
| \( f_9, 0.4 \)   | (L4) 2.504e-997 | 1.15e-928 | 4.52e-942 | 1.27e-965 | 4.52e-941 | 6.15e-904 | 4.11e-937 |
| \( f_{10}, 1.5 \)| (L5) 2.81e-305 | 7.19e-303 | 5.07e-284 | 1.84e-245 | 5.07e-285 | 4.91e-244 | 1.78e-275 |
| \( f_{11}, 0.25 \)| (L6) 2.35e-1143 | 3.65e-782 | 9.82e-823 | 1.00e-819 | 4.98e-823 | 1.00e-819 | 5.13e-812 |
| \( f_{12}, 0.25 \)| (L6) 7.86e-318 | 2.03e-256 | 5.65e-256 | 1.82e-254 | 5.65e-256 | 1.07e-264 | 6.31e-268 |
| \( f_{13}, 2.0 \)| (L7) 2.54e-436 | 2.63e-396 | 1.94e-378 | 5.1e-378 | 1.94e-378 | 8.70e-380 | 6.80e-379 |
where $k = 1, 2, 3$, $\beta \in \mathbb{R}^+$, $\phi_k$ are listed in (26)-(28). (29) is called $M1$, $M2$ and $M3$ for $\phi_1$, $\phi_2$ and $\phi_3$, respectively.

### 4.3. Petkovic et al. type methods.

In [7], author developed Petkovic type 1 (P1) and type 2 (P2) derivative-free methods for the comparison of numerical efficiency, (P1) and (P2) respectively, are written as

\[
\begin{align*}
   w_n &= x_n + \beta f(x_n), \\
   y_n &= x_n - \frac{\beta f(x_n)^2}{f(w_n) - f(x_n)}, \\
   z_n &= y_n - \left(1 + \frac{f(y_n)}{f(w_n)} + \frac{f(y_n)}{f(x_n)} \right) \frac{(w_n-x_n)f(y_n)}{f(w_n) - f(x_n)}, \\
   x_{n+1} &= z_n - \frac{1 - \frac{f(z_n)}{f(w_n)}}{1 - \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)^2} - 3 \left(\frac{f(y_n)}{f(w_n)}\right)^3} \frac{f[x_n, y_n, z_n]}{f[y_n, z_n, f[x_n, z_n]]},
\end{align*}
\]

and

\[
\begin{align*}
   w_n &= x_n + \beta f(x_n), \\
   y_n &= x_n - \frac{\beta f(x_n)^2}{f(w_n) - f(x_n)}, \\
   z_n &= y_n - \left(1 + \frac{f(y_n)}{f(w_n)} + \frac{f(y_n)}{f(x_n)} \right) \frac{(w_n-x_n)f(y_n)}{f(w_n) - f(x_n)}, \\
   x_{n+1} &= z_n - \frac{1 - \frac{f(z_n)}{f(w_n)}}{1 - \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)^2} - 3 \left(\frac{f(y_n)}{f(w_n)}\right)^3} \frac{f[x_n, y_n, z_n]}{f[y_n, z_n, f[x_n, z_n]]},
\end{align*}
\]
4.4. Proposed family (L). We define the following weight functions:

\[
G_1(t_1, t_2) = \frac{1}{1 - (t_1 + t_2) + \omega(t_1 + t_2)^2}, \quad \omega \in \mathbb{R}, \quad (32)
\]

\[
G_2(t_1, t_2) = 1 + t_1 + t_2 + t_1^2 + 1.9t_2^2 + 4.4t_1t_2, \quad (33)
\]

\[
H_1(s_1, s_2) = 1, \quad (34)
\]

\[
H_2(s_1, s_2) = \frac{1}{1 + s_1s_2 + s_1^2 + s_2^2}, \quad (35)
\]

\[
H_3(s_1, s_2) = 1 + s_2^4 + s_2^6, \quad (36)
\]

\[
H_4(s_1, s_2) = 1 + s_1^2 + s_1^2 + 2s_1s_2, \quad (37)
\]

\[
H_5(s_1, s_2) = \frac{1}{1 - 20s_1s_2}, \quad (38)
\]

where \( t_1 \) and \( s_1 \) are defined in (10). Further we give names to methods for the purpose of simplicity as follows

\[
\begin{aligned}
L_1 &= (G_1, \ H_1, \ \omega = +0.01, \ \kappa = 0.01), \quad L_2 = (G_1, \ H_1, \ \omega = -0.022, \ \kappa = 0.01), \\
L_3 &= (G_1, \ H_1, \ \omega = -0.001, \ \kappa = 0.01), \quad L_4 = (G_2, \ H_1, \ \omega = +0.01, \ \kappa = 0.01), \\
L_5 &= (G_1, \ H_3, \ \omega = -0.01, \ \kappa = 0.01), \quad L_6 = (G_1, \ H_2, \ \omega = +0.01, \ \kappa = 0.01), \\
L_7 &= (G_1, \ H_4, \ \omega = +0.01, \ \kappa = 0.01), \quad L_8 = (G_1, \ H_5, \ \omega = +0.01, \ \kappa = 0.01).
\end{aligned}
\]

A set of thirteen nonlinear equations is used for numerical computations from [7], in Table 1. All the families in the numerical implementation are derivative-free and use four function evaluations to get the order of convergence eight. For all methods, 12 (TNFE) total number of function evaluations are used, and absolute error (\( |x_n - \alpha| \)) is displayed. Computational order of convergence is calculated according to (24) for the method. All numerical values for methods K-T, M1, M2, M3, P1, P2 are taken from [7].

5. Conclusion

In this note, we have presented a family of eighth-order derivative-free methods. The proper selection of weight functions showed a reasonable reduction in error as compared to other referenced derivative-free methods. It is obvious that constructed family has broad choice for the weight function in the third and fourth step of the method. The true essence of the family is hidden in the construction of interpolation polynomial for the approximation of \( f'(z) \) and weight functions make it more flexible to get higher performance and efficiency.

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