Decay of charged particles near naked singularities and super-Penrose process without fine-tuning

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We consider the Penrose process near the naked singularity in the Reissner-Nordström metric. Particle 0 falls from infinity and decays to two fragments at some point $r_0$. We show that the energy extraction due to this process can be indefinitely large in the limit $r_0 \to 0$. In doing so, the value of the particle charge can remain bounded, in contrast to the previously known examples of the Penrose process in the electric field with unbounded energy extraction. The effect persists even in the limit of the flat space-time. Unbounded energy gain is obtained in the standard (not collisional) Penrose process and does not require fine-tuning of particle parameters or characteristics of space-time.

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I. INTRODUCTION

The Penrose process (PP) \cite{1} is one of the remarkable physical effects typical of general relativity and other theories of gravity. Let us suppose that in a space-time there exists a region (called ergosphere) where the Killing energy $E$ measured at infinity can be negative. If a particle 0 within the ergosphere splits to two fragments 1 and 2, one of them can have $E_1 < 0$. Then, the conservation of energy entails that the second fragment has $E_2 > E_0$, so some energy gain occurs. This is just the PP. However, it has a quite modest efficiency in realistic astrophysics. Its efficiency increases if instead of decay of one particle the process consists in collision of different particles. Such a process with energy gain is called the

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collisional Penrose process (see Ref. [2] for review and references therein).

Especially, it is relevant in the context of another effect that is called the Bañados-Silk-West (BSW) one [3]. It happens if particles collide near the black hole. Then, under certain conditions the energy $E_{c.m.}$ in the center-of-mass frame becomes unbounded. Meanwhile, the gain in the energy $E$ remains bounded [4] - [6].

Originally, all these effects were found in the vicinity of rotating black holes. However, the similar effect should occur in the background of static charged black holes [7]. It was investigated further in detail [8] - [11]. Moreover, there exists a limit to flat space-time when the effect under discussion persists [12], [13]. It amounts to the possibility of indefinitely large energy gain when $E$ is formally unbounded (in the test particle approximation, when backreaction on a metric is neglected). Then, such a process is called the super-Penrose one (SPP). In contrast to collisions of neutral particles near rotating black holes where the SPP is forbidden (see, e.g. Ref. [14] and references therein), the SPP is indeed possible for the charged black holes, even for pure radial particle motion in the Reissner-Nordström (RN) background [15], [16]. (More general scenarios can include both the electric charge and rotation [17].)

In the present work, we show that there exists one more type of the SPP. It is realized in the background of naked charged singularities. The SPP near naked singularities was already the subject of study for rotating geometries with neutral particles. In [18], it was shown that the SPP is possible for the Kerr overspinning metric when particles move within the equatorial plane. In [19], this result was generalized to generic rotating axially symmetric space-times with naked singularities. In [20], high energy extraction in the Kerr overspinning geometry was investigated for particle collisions taking place on the off-equatorial planes. These papers share two common features: (i) the unbounded (or at least very high) energy extraction requires fine-tuning between the parameters of geometry, so it is on a threshold of forming a horizon (but the horizon does not form), (ii) the effect under discussion implies the collisional Penrose process.

We would like to stress that in the scenarios considered in the present work, both conditions are relaxed: the effect is achieved for quite generic geometries without special conditions for particles and, also, it happens for the decay typical of the standard Penrose process instead of the collisional one.

The process with charged particles in the RN background discussed in the present work
differs not only from its counterpart in the rotating background with neutral particles but also from its analogs near the RN naked singularity considered earlier [21], [22]. In aforementioned papers (i) collisions of shells were studied, (ii) the high energy process implies high \(E_{\text{c.m.}}\). Meanwhile, we show that (i) the effect under discussion is valid for test particles, (ii) it involves ultra-high energies \(E\), (iii) there are no collisions at all and the process represents a standard PP, not a collisional one.

There is one more interesting property inherent to the scenarios considered in the present paper. The examples of the electric SPP, known before, share the common feature. If one wants to gain large energy, the electric charge is also should be large. This concerns both the standard and collisional PP as well as the confined one [23]. However, the electric charge of elementary particles, atoms or nuclei cannot be arbitrary large [15], [24]. Also, there exist similar restrictions for macroscopic bodies [16]. Meanwhile, we demonstrate that for naked singularities not only PP but also SPP does exist for a finite value of a particle charge.

We use the geometric system of units in which fundamental constants \(G = c = 1\).

II. BASIC EQUATIONS

We consider the Reissner-Nordstr"om metric

\[
ds^2 = -dt^2 f + \frac{dr^2}{f} r^2 d\omega^2,
\]

where \(d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\),

\[
f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.
\]

Here \(M\) and \(Q\) are the mass and electric charge, respectively. We take \(Q > 0\). We assume that \(M < Q\), so there is a naked singularity at \(r = 0\) in this space-time.

Now, we consider motion of test particles. We restrict ourselves by pure radial motion. This is sufficient to demonstrate that the effect under discussion does exist. Then, equations of motion read

\[
m\dot{t} = \frac{X}{f}, \tag{3}
\]

\[
X = E - q\varphi, \tag{4}
\]

\[
m\dot{r} = \sigma P, \quad P = m\sqrt{U}, \quad U = \frac{X^2}{m^2} - f, \tag{5}
\]
\( \sigma = \pm 1 \) depending on the direction of motion, dot denotes differentiation with respect to the proper time \( \tau \). Here, \( q \) is the particle’s electric charge, \( m \) being its mass, \( E \) the energy. The forward-in-time condition \( \dot{t} > 0 \) entails

\[
X > 0. \tag{6}
\]

The electric Coulomb potential

\[
\varphi = \frac{Q}{r}. \tag{7}
\]

Hereafter, we use notations \( \varepsilon = \frac{E}{m}, \; \tilde{q} = \frac{q}{m} \). Then,

\[
U(r) = (\varepsilon - \tilde{q}Q)^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right). \tag{8}
\]

Then, the possible turning points \( r_t \) can be found from the condition \( P = 0 \), whence

\[
r_t^\pm = \frac{1}{\varepsilon^2 - 1}(\varepsilon \tilde{q}Q - M \pm \sqrt{C}), \tag{9}
\]

\[
C = (M - \varepsilon \tilde{q}Q)^2 + (1 - \tilde{q}^2)Q^2(\varepsilon^2 - 1). \tag{10}
\]

### III. SCENARIO OF DECAY

In what follows, we will be mainly interested in the situation when decay occurs near the singularity, so the point of decay \( r_0 \to 0 \). Then, for fixed \( q_0 \) and \( E_0 \), the forward-in-time condition (6) requires \( q_0 < 0 \). Assuming also that particle moves from infinity, so \( \varepsilon_0 > 1 \), we see from (9) that no more than one turning point \( r_t^+ \) can exist, provided \( |\tilde{q}_0| < 1 \).

Let particle 0 with \( \varepsilon > 1 \) fall from infinity. In some point \( r_0 \) it decays to two new fragments 1 and 2. We assume the conservation of the energy and electric charge in the point of decay, so

\[
E_0 = E_1 + E_2, \tag{11}
\]

\[
q_0 = q_1 + q_2. \tag{12}
\]

The necessary condition that makes decay possible is

\[
m_0 \geq m_1 + m_2. \tag{13}
\]

For given characteristics \( E_0, m_0, q_0 \) of particle 0, one can solve (11), (12) with (5) taken into account. We can take advantage of already obtained results - see eqs. (19) - (25) of
Ref. [25]. Only minimum changes are required: (i) instead of indices 3, 4 we use here 1,2, (ii) the quantity \( X \) is defined according to (5) instead of eq. (5) of [25].

Then,

\[
X_1 = \frac{1}{2m_0^2} \left( X_0 \Delta_+ + P_0 \delta \sqrt{D} \right),
\]

\[
X_2 = \frac{1}{2m_0^2} \left( X_0 \Delta_- - P_0 \delta \sqrt{D} \right),
\]

\[
P_1 = \left| \frac{P_0 \Delta_+ + \delta X_0 \sqrt{D}}{2m_0^2} \right|, \tag{16}
\]

\[
P_2 = \left| \frac{P_0 \Delta_- - \delta X_0 \sqrt{D}}{2m_0^2} \right|, \tag{17}
\]

where \( \delta = \pm 1, \)

\[
\Delta_\pm = m_0^2 \pm (m_1^2 - m_2^2), \tag{18}
\]

\[
D = \Delta_+^2 - 4m_0^2 m_1^2 = \Delta_-^2 - 4m_0^2 m_2^2. \tag{19}
\]

It follows from (13) that \( D \geq 0 \). The equality holds if \( m_0 = m_1 + m_2 \) only.

The solutions (14) - (19) are classified according to 4 quantities \( (\sigma_2, h_2, h_1, \delta) \). The corresponding allowed combinations are listed in eq. (30) of [25]. (A reader should bear in mind that the role of particle 3 in [25] is played now by particle 2). Here,

\[
h_1 = sgnH_1, \quad H_1 = \Delta_+ \sqrt{f - 2m_1 X_0}, \tag{20}
\]

\[
h_2 = sgnH_2, \quad H_2 = \Delta_- \sqrt{f - 2m_2 X_0}. \tag{21}
\]

We are interested in the situation when particle 2 escapes. If it moves after decay immediately to infinity, \( \sigma_2 = +1 \). We will consider this type of scenario first. (Afterwards, we will also discuss an alternative scenario when particle 2 bounces back from the potential barrier.)

Then, according to [25], there are only two possibilities: 1(++, +, +, −) and 2(++ +, −, −). Thus \( \delta = -1 \) and we should also have

\[
H_2 > 0 \tag{22}
\]

while \( H_1 \) can have any sign.
IV. ENERGY EXTRACTION

The Penrose process implies that particle 1 moves toward the center with $E_1 < 0$, whereas particle 2 escapes to infinity with $E_2 > 0$. In doing so, $\sigma_1 = -1$ and $\sigma_2 = +1$. Our goal is to obtain the energy extraction as large as possible. According to (1),

$$E_2 = X_2 + \frac{q_2 Q}{r_0}. \quad (23)$$

Taking into account (6), we see that if $q_2 > 0$ and $r_0 \to 0$, the energy $E_2$ is unbounded. Is it possible to achieve this goal in our scenario?

When $r_0 \to 0$, condition (6) requires $q_0 < 0$. Thus we should have

$$q_0 < 0, q_2 > 0. \quad (24)$$

Further, we should consider two different cases depending on whether or not there is a turning point for particle 0.

A. No turning point

This case is realized if $|\tilde{q}_0| > 1$ since both roots (9) become negative. Then, we can take the limit $r_0 \to 0$ directly. In this limit, it follows from (4), (5), (7) that

$$X_0 \approx \frac{|q_0| Q}{r_0}, \quad (25)$$

$$P_0 \approx \frac{Q}{r_0} \sqrt{\frac{q_0^2}{r_0^2} - m_0^2}. \quad (26)$$

Now, using (15), (23), (25) and (26), one obtains in the main approximation

$$E_2 \approx \frac{Q}{r_0} [q_2 + \frac{1}{2m_0^2} (|q_0| \Delta - \sqrt{q_0^2 - m_0^2 \sqrt{D}})] \quad (27)$$

and $E_2 \to \infty$ when $r_0 \to 0$.

Now,

$$H_2 \approx \frac{Q}{r_0} (\Delta - 2m_2 |q_0|). \quad (28)$$

Eq. (22) is valid, if

$$|q_0| < \frac{\Delta}{2m_2}. \quad (29)$$

It is consistent with $|\tilde{q}_0| > 1$. 
This is not the end of story. We must check that particle 2 does escape, so it does not have a new turning point for all \( r > r_0, U > 0 \). We should verify that \( U_2(r) > U_2(r_0) \) for any \( r > r_0 \). Using (30), it is easy to find that

\[
U_2(r) - U_2(r_0) \approx \frac{(r - r_0)}{rr_0^2} B,
\]

\[
B = Q^2[1 + \bar{q}_2^2 + \frac{r_0(1 - \bar{q}_2^2)}{r} + \frac{q_2}{m_2^2m_0^2}(|q_0| \Delta_+ + \sqrt{\bar{q}_0^2 - m_0^2\sqrt{D}})] - 2Mr_0.
\]

Obviously, if \( r_0 \to 0, B > 0 \) for any \( r > r_0 \). Thus,

\[
U_2(r) > U_2(r_0) > 0
\]

and there are no additional turning points, so particle 2 escapes to infinity freely.

\[\text{B. Decay in the turning point}\]

Let us suppose now that the turning point for particle 0 does exist. As we try to obtain the maximum possible \( E_2 \), it makes sense to choose (for given values of other parameters) the minimum possible value of \( r_0 \). To this end, we put \( r_0 = r_t^{(-)} \). We assumed (as is explained above) that \( q_0 < 0 \). Then, according to eqs. (9), (10) the existence of a turning point requires \( |\bar{q}_0| \leq 1 \). In this point we have \( P_0 = 0 \) by definition, so it follows from (14) - (17) that

\[
X_1 = \frac{X_0\Delta_+}{2m_0^2},
\]

\[
X_2 = \frac{X_0\Delta_-}{2m_0^2},
\]

\[
P_1 = P_2 = \frac{X_0}{2m_0^2}\sqrt{D},
\]

where now

\[
X_0 = E_0 + \frac{|q_0| Q}{r_0},
\]

\[
E_2 = \frac{E_0\Delta_-}{2m_0^2} + \frac{Q}{r_0}(q_2 + |q_0| \frac{\Delta_-}{2m_0^2}).
\]

In doing so, \( X_0 > 0 \) for any \( r_0 \) due to \( q_0 < 0 \), so condition (6) holds. As we want to minimize \( r_0 \), we choose

\[
|\bar{q}_0| = 1 - \beta, \beta \ll 1.
\]
Then, it follows from (9) that
\[
r_0 \approx \frac{Q^2 \beta}{M + \varepsilon Q}.
\] (39)

can be made as small as one likes. As a result, we have from (37) that
\[
E_2 \approx \frac{(M + \varepsilon Q)}{Q \beta} (q_2 + |q_0| \frac{\Delta - 2m_0^2}{2m_0^2}).
\] (40)

When \( \beta \to 0 \), \( E_2 \to \infty \), so the SPP does exist.

Eq. (32) is valid in the case under consideration as well. It is worth noting that if \( q = 0 \), the expression (9) coincides with eq. (13) of \[21\]. However, we saw that in both versions of the scenario under consideration (with a turning point or without it) it is essential for the SPP that \( q_0 \neq 0 \). Thus, this process is possible for charged particles and is absent for neutral ones.

V. ALTERNATIVE TYPE OF SCENARIO

For completeness, we must consider the case when particle 2 after decay moves in the same direction as particle 1, so \( r \) continues to decrease. However, immediately after decay particle 2 bounces back from the potential barrier. This means that \( r_0 \) is the turning point for particle 2, so \( P_2 = 0 \) and, therefore,
\[
X_2 = m_2 \sqrt{f}.
\] (41)

When \( r_0 \to 0 \),
\[
X_2 \approx m_2 \frac{Q}{r_0}.
\] (42)

According to (17), we must take \( \delta = +1 \) and
\[
P_0 \Delta_+ = X_0 \sqrt{D}.
\] (43)

It is easy to check that this is equivalent to \( H_2 = 0 \) in \[21\], so
\[
\Delta_+ \sqrt{f} = 2m_2 X_0.
\] (44)

If \( q_0 < 0 \), there is no turning point for particle 0. In the limit \( r_0 \to 0 \), \( X_0 \approx \frac{|q_0| Q}{r_0} \), \( \sqrt{f} \approx \frac{Q}{r_0} \) and we obtain from (44)
\[
|q_0| \approx \frac{\Delta_+}{2m_2}.
\] (45)
If \( q_0 > 0 \), choosing \( r_0 \) to be a turning point for particle 0 as well, we have \( X_0 = m_0 \sqrt{f} \), so it follows from (44) that \( \Delta_- = 2m_0m_2 \), whence \( m_0 = m_1 + m_2 \). In both cases, according to (42),

\[
E_2 \approx \frac{Q(m_2 + q_2)}{r_0}.
\]  

(46)

Here, we should take \( q_2 > -m_2 \). Thus the SPP does exist in this case also.

**VI. FLAT SPACE-TIME LIMIT**

Decay in the case of the flat space-time is of special interest. More precisely, we put \( M = Q = 0 \) in the metric thus neglecting the influence of the electromagnetic field on space-time. However, we take into account the electric charge in equations of motion. Again, let particle 0 decay in the point \( r_0 \). Then, eqs. (14), (15) are now valid with \( f = 1 \). Now we will show that the SPP is still possible for a finite value of \( |q_0| \) (the corresponding scenarios were overlooked in our previous paper [13]).

If the turning point \( P_0 = 0 \) exists, its coordinate is given by

\[
r_0 = \frac{q_0Q}{E_0 - m_0}.
\]  

(47)

Here, it is assumed that \( E_0 > m_0 \). Now there are two different cases.

**A. No turning point**

Let \( q_0 < 0 \). Then, the turning point for particle 0 is absent. We want particle 2 to escape to infinity, so the turning point for particle 2 should be absent as well, \( q_2 = -|q_2| < 0 \). Proceeding along the same lines as before, we obtain for \( r_0 \to 0 \)

\[
E_2 \approx \frac{Q |q_0|}{r_0} (\Delta_- + \sqrt{D}) - |q_2| \]

(48)

This expression differs from (27) since in (27) it was implied that \( f \sim \frac{Q^2}{r_0^2} \to \infty \), whereas now \( f = 1 \).

The positivity of \( E_2 \) for small \( r_0 \) requires

\[
|q_2| < \frac{|q_0|}{2m_0^2} (\Delta_- + \sqrt{D}).
\]  

(49)
Formally, (48) grows indefinitely when \( r_0 \to 0 \). Actually, as we neglected the corresponding term \( \frac{Q^2}{r^2} \) in the RN metric, there is an additional constraint on \( r_0 \). Restoring dimensionality, we have \( r_0 \gg \sqrt{GQc^2} \), so there is restriction \( E_2 \ll \frac{c^2}{\sqrt{G}} \left| |q_0| (\Delta_- + \sqrt{D}) - |q_2| \right| \) that is rather weak in the case \( G \to 0 \). Now,

\[ H_2 = \Delta_- - 2m_2(E_0 + \frac{|q_0|Q}{r_0}). \]  

(50)

It is seen that for a fixed \( |q_0| \) and \( r_0 \to 0 \) eq. (22) cannot be fulfilled. It means that the scenario under discussion cannot be realized, so particle 2 falls in the center along with particle 1 and does not escape. The situation changes if we take \( |q_0| = \alpha r_0 \), where \( \alpha = O(1) \). Then, we have from (50) that

\[ \alpha < Q^{-1}(\frac{\Delta_-}{2m_2} - E_0). \]  

(51)

B. Decay in the turning point

Now, \( P_0 = 0 \), so eqs. (33) - (36) apply. As a particle falls from infinity, \( E_0 > m_0 \). According to (47), we must also take \( q_0 > 0 \). For particle 2 we take \( q_2 < 0 \) to exclude a possible turning point after decay. As we want \( r_0 \to 0 \), it is seen from (47) that we must assume \( q_0 \to 0 \). Then, in eq. (37) the main contribution becomes negative, so the SPP is absent in this case.

C. Alternative scenario

Is the alternative scenario possible in the flat case? It implies that particle 2 bounces back from the turning point. Then, \( P_2 = 0 \), so

\[ X_2 = m_2. \]  

(52)

Now, one should put \( f = 1 \) in eq. (41) typical of a turning point. Then, putting there \( X_0 \approx \frac{|q_0|Q}{r_0} \), we obtain

\[ |q_0| \approx \frac{r_0 \Delta_-}{Q 2m_2}. \]  

(53)

Thus for a fixed \( q_0 \) we have

\[ E_2 = m_2 + \frac{q_2 Q}{r_0}. \]  

(54)
where we took into account (52). Then,

\[ E_2 \approx \frac{q_2 \Delta - 2m_2 |q_0| Q}{2m_2 |q_0| Q}, \quad (55) \]

where it is assumed \( q_2 > 0 \). If \( q_0 \to 0 \), \( E_2 \to \infty \), so the SPP occurs. As now \( X_2 \) is monotonically increasing function of \( r \), there are no other turning points and particle 2 escapes.

**VII. DISCUSSION: COMPARISON OF DIFFERENT TYPES OF HIGH ENERGY PROCESSES**

It is instructive to compare the obtained results with those already existing in literature, both for the rotating and static charged case. Usually, the outcome of particle collisions depends strongly on several factors: (i) the parameters of geometry, (ii) parameters of particles, (iii) direction of motion. Let us remind a reader at first the situation for processes with neutral particles in the background of rotating black holes. If one wants to have unbounded \( E_{c.m.} \) for collision of particles falling from infinity, a black hole, typically, has to be extremal. This implies a special relation between the mass \( M \) and angular momentum \( a \) of a black hole: \( a = M \) (in corresponding units) for the Kerr metric. Also, one of particles should be fine-tuned (so-called critical) which implies a special relation between the parameters of a particle itself \[3\]. However, even with unbounded \( E_{c.m.} \), the energy \( E \) remains quite modest \[4\] - \[6\]. It can be significantly increased if a fine-tuned particle is outgoing \[26\], \[27\]. However, the energy \( E \) remains bounded anyway.

If collision occurs in the field of a naked singularity, there is no need in fine-tuning particle parameters but the geometry itself should be on the threshold of forming a horizon \[18\], \[19\], \[20\]. For example, for the Kerr metric \( a \) has to be only slightly bigger than \( M \). Alternatively, one can try to relax the requirement of having the extremal black hole and consider the Schnittman scenario \[26\] (collision between an infalling and outgoing particles) even for the nonextremal one. However, in this case, the input of energy should be very large from the very beginning, that depreciates the value of the process \[23\]. Meanwhile, in the current setup, neither geometry nor particles should be fine-tuned. Also, the relative direction of motion is irrelevant. Although details of the process look different for each separate scenario, the SPP turns out to be possible almost for all scenarios considered above.
One can also compare our problem with collisions of charged particles near the RN black hole. One should require one of infalling particles to be a critical as well. If both particles are usual (not fine-tuned), neither unbounded $E_{\text{c.m.}}$ or unbounded $E$ are possible \[^{15},^{16},^{17}\]. And, for the critical particle there is a bound on the electric charge $q < Ze$, where $e$ is the value of an elementary charge and $Z = 170$ comes from quantum electrodynamics. As the energy of a particle produced in collision turns out to be proportional to the charge $q$ \[^{15}\], this gives rise to the restriction of the efficiency of the PP. But this circumstance is irrelevant in the case under discussion since high energies can be obtained even for a modest electric charge.

And, the last but not the least, there is no necessity in arranging the collisional Penrose process for achieving the SPP. A standard decay of one particle is quite sufficient. Thus, a naked singularity provides us with the most favorable situation for gaining energy and accelerating particles to ultrahigh energies.

**VIII. CONCLUSIONS**

As is known, the electric charge of astrophysical objects is rather small. (Although it can, in principle, lead to observable effects \[^{29}\].) However, in some aspects the electrical charge can model what happens in more complicated realistic astrophysical systems with rotation. Meanwhile, the RN metric is much easier for analysis than, say, the Kerr metric describing the vacuum solution of the Einstein equations with rotation. In this sense, the results obtained in the present paper can be of some use for the analysis of more realistic processes.

They are also interesting on their own right. We found one more type of systems for which the SPP is possible. It exists in almost all scenarios considered above. In the configuration considered in the present paper (i) the SPP can occur without fine-tuning at all and (ii) to gain large $E$, there is no need to have large $q$. More precisely, fine-tuning is required for the subcase with the turning point \(^{38}\) only but is absent if such a point is missing. One more difference between the present version of SPP and that near a black hole consists in that now the original particle needs not be ultrarelativistic having any finite value $\varepsilon > 1$ (cf. Sec. IV C3 in \[^{24}\]). In this sense, the present version of the SPP is less restrictive than the previous one typical of the BSW effect. We would also like to stress that the effect under
discussion concerns indefinitely large Killing energy $E$, whereas $E_{\text{c.m.}}$ is irrelevant since we considered particle decay, not collision.

In combination with the previous results [7], [8], [12], [13], this means that any type of the Reissner-Nordström space-time (black hole, flat space-time as its limit, naked singularity) is pertinent to the SPP.

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