Nonlinear dynamics and parameter control for metamaterial plate with negative Poisson’s ratio

S T Zhu\(^1\), J Li\(^1\), †, J Zhou\(^2\) and T T Quan\(^1,3\)

\(^1\)College of Applied Sciences, Beijing University of Technology, Beijing, P. R. China
\(^2\)State Key Laboratory of New Ceramics and Fine Processing, School of Materials Science and Engineering, Tsinghua University, Beijing, P. R. China
\(^3\)School of Science, Tianjin Chengjian University, Tianjin, P. R. China

†Corresponding autor. E-mail: leejing@bjut.edu.cn

Abstract. The metamaterial with negative Poisson’s ratio is widely used due to its special mechanical and physical properties. Based on the theory of periodic solution and bifurcation of nonlinear dynamics, we mainly focus on the nonlinear vibration behaviors and parameter control of a simply supported concave hexagonal composite sandwich plate with negative Poisson’s ratio in auxetic honeycombs subjected to in-plane and transverse excitation. The Melnikov function is improved by introducing the curvilinear coordinate frame and Poincaré map to detect the existence and number of the periodic solutions. The effects of the forcing excitation coefficient on nonlinear dynamics as well as the parameter control conditions are presented. Numerical method is performed to obtain the phase portraits of the number and corresponding positions of multiple periodic orbits.

1. Introduction

Metamaterials are man-made structures with special properties that may not be readily available from natural materials [1, 2]. The typical metamaterials are generally associated with four elastic constants, the Young’s modulus, shear modulus, bulk modulus and Poisson’s ratio. Poisson’s ratio, by definition, is the negative ratio of transverse to axial strain. As is well known, most natural materials have a positive Poisson’s ratio. However, the metamaterials with negative Poisson’s ratio, also called ‘auxetic’ materials [3], can exhibit an unconventional behavior, expanding laterally when stretched and contracting laterally when compressed [4, 5]. This kind of materials may have multiple potential applications for automotive, defense and aerospace industries due to their advantages of high energy absorption, fracture toughness, indentation resistance and so on [6].

Auxetic materials, as potential new materials, have attracted great interest. During the last two decades, a variety of materials and structures of negative Poisson’s ratio have been discovered [1, 4-6, 9] and references therein).

As is well known, making the cell of a conventional hexagonal honeycomb re-entrant produces a negative Poisson’s ratio, which is described as auxetic honeycomb. When auxetic materials are used as load bearing or energy absorption structures, there exist the geometric nonlinearity and shear
deformation which can lead to nonlinear oscillations of the structures. Thus, the nonlinear dynamic behaviors for the structures, especially the periodic solutions [10], bifurcation [10, 11] and chaos [12, 13] etc., becomes important and necessary. Hou et al. [14] demonstrated the potential use of auxetic and graded conventional-auxetic structures under flatwise compression and edgewise loading. Imbalzano et al. [15] conducted the numerical investigations of the dynamic responses and energy absorbing capabilities of auxetic composite panels and equivalent monolithic steel plates. Yang et al. [16] investigated the bifurcation and chaos behavior of a sandwich plate with viscoelastic soft core in supersonic flow by considering the in-plane periodic loading. Guo et al. [17] investigated the free vibration of graphene nanoplatelet reinforced laminated composite quadrilateral plates using the element-free IMLS-Ritz method. Duc et al. [18] applied the analytical solution to investigate the nonlinear dynamic response and vibration of sandwich auxetic composite cylindrical panels. Li, Quan and Zhang [10] performed the curvilinear coordinate transformation to study the bifurcation and number of subharmonic solutions of 4 dimensional non-autonomous slow-fast systems.

In this paper, we focus on the nonlinear vibration behaviors and parameter control of a simply supported concave hexagonal composite metamaterial sandwich plate with negative Poisson’s ratio in auxetic honeycombs subjected to its in-plane and transverse excitation. The nonlinear motion equation of the model is derived by the method of multiple scales. The bifurcation of multiple periodic solutions occurs under certain conditions. Furthermore, the numbers and relative positions of the periodic solutions can be clearly found from the numerical results.

2. Metamaterial plate system and averaged equation

In this section, we focus on the mechanical model of a simply supported concave hexagonal composite metamaterial sandwich plate with length $a$, width $b$, core thickness $h_c$, and total thickness $h$. This plate is subjected to its in-plane excitation and transverse excitation. The section model of the plate and the unit cell of concave hexagonal honeycomb core are shown in figure 1, where $l_1$, $l_2$ represent the length of the inclined and horizontal cell rib, $t$ is the uniform thickness of cell rib, $\theta$ is the inclined angle. The honeycomb is auxetic if $\theta$ is positive and conventional if $\theta$ is negative.

![Figure 1. The section model of the plate and the unit cell of core.](image)

The non-dimensional governing equations of transverse motion of the plate is as follows [12].

\[
\begin{align*}
\ddot{x} + \omega_1^2 x + \epsilon \alpha_1 \mu \dot{x} + \epsilon \alpha_2 x \cos \Omega_{1} t + \epsilon \alpha_3 x^3 + \epsilon \alpha_4 y^3 + \epsilon \alpha_5 x y^2 + \epsilon \alpha_6 x^2 y &= \epsilon f_1 \cos \Omega_{1} t \\
\ddot{y} + \omega_2^2 y + \epsilon \beta_1 \mu \dot{y} + \epsilon \beta_2 y \cos \Omega_{1} t + \epsilon \beta_3 y^3 + \epsilon \beta_4 y x^2 + \epsilon \beta_6 y x^2 y &= \epsilon f_2 \cos \Omega_{1} t
\end{align*}
\]  

(1-1) 

(1-2)

where $\epsilon$ is a small parameter, $\omega_1$ and $\omega_2$ are the first and second order natural frequency of the corresponding linear system respectively, $\mu$ is damping coefficient, $\alpha_i, \beta_i \ (i=1, \cdots, 6)$ are non-dimensional coefficients, and $f_1, f_2$ are forcing excitations, $\Omega_1$ and $\Omega_2$ are the frequencies of transverse and in-plane excitation respectively. We focused on the case of 1:1 internal resonance and primary parametric resonance. In this resonant case, $\omega_1^2 = \Omega_1^2 + \epsilon \sigma_1$, $\omega_2^2 = \Omega_2^2 + \epsilon \sigma_2$, where $\sigma_1$ and $\sigma_2$ are two detuning parameters, and we assume that $\Omega_1 = \Omega_2 = 1$. By the methods of multiple scales, the averaged equation is obtained as follows.

2
\[
\dot{x}_i = (a_i - a_2)x_2 + X_i(x_1, x_2, x_3, x_4) \\
\dot{x}_2 = -(a_1 + 5a_2)x_1 + X_2(x_1, x_2, x_3, x_4) \\
\dot{x}_3 = (b_1 - b_2)x_4 + X_3(x_1, x_2, x_3, x_4) \\
\dot{x}_4 = -(b_1 + 5b_2)x_3 + X_4(x_1, x_2, x_3, x_4)
\]

where \( a_i = (\sigma_i^2 + \mu^2\alpha_i^2)/8 \), \( a_2 = \alpha_2^2/24 \), \( b_1 = (\sigma_i^2 + \mu^2\beta_i^2)/8 \), \( b_2 = \beta_2^2/24 \). \( X_j(j = 1, 2, 3, 4) \) are polynomials in variables of \( x_i(i = 1, 2, 3, 4) \), and their coefficients \( a_i, b_i(i \geq 3, i \in \mathbb{N}) \), are uniquely determined by the coefficients of equation (1).

3. Nonlinear dynamic analysis and parameter control

3.1. Transformations for the system

For convenience, we introduce the following rescaling transformation

\[
a_i \to \varepsilon a_i, \quad b_i \to \varepsilon b_i, \quad i \geq 2, \quad i \in \mathbb{N}
\]

where \( \varepsilon \) is a sufficiently small parameter. Then system (2) can be rewritten as

\[
\dot{x} = Ax + \varepsilon F(x)
\]

where \( x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \), \( F(x) = (F_1, F_2, F_3, F_4)^T \) is the vector-valued polynomials in variables of \( x_i(i = 1, 2, 3, 4) \). \( A = \delta_{ij}^i(j)(a_i + \delta_{ij}^i(a_j) + \delta_{ij}^i(b_i) + \delta_{ij}^i(b_j) - b_i) \) and \( \delta_{ij}^i(M) \) denotes an \( m \times n \) block matrix with the \( (i, j) \)-th block \( M \), a smaller matrix, and all other blocks are zero matrices [19]. When \( \varepsilon = 0 \), system (3) degenerates to two Hamilton systems on plane \((x_1, x_2)\) and \((x_3, x_4)\) with Hamiltonian \( H(x) = (H_1, H_2)^T \), where

\[
H_1(x_1, x_2) = \frac{1}{2}a_2(x_1^2 + x_2^2), \quad H_2(x_1, x_4) = \frac{1}{2}b_2(x_3^2 + x_4^2)
\]

Then there exists an open interval \( J \subset \mathbb{R}^2 \), \( h = (h_1, h_2)^T \in J \), such that each system has a family of periodic orbits

\[
\Gamma_h = \{x_h \mid H_1(x_1, x_2) = h_1\}, \quad \Gamma_h = \{x_h \mid H_2(x_3, x_4) = h_2\}
\]

Suppose that the family of periodic obits \( \Gamma_h \) and \( \Gamma_h \) can be expressed as

\[
x_1 = \sqrt{m} \cos(a_1 t), \quad x_2 = \sqrt{m} \sin(a_1 t), \quad x_3 = \sqrt{n} \cos(b_1(t + t_0)), \quad x_4 = \sqrt{n} \sin(b_1(t + t_0))
\]

where \( m = 2h_1/a_1, n = 2h_2/b_1 \). The periods of the orbits are \( T_1(h_1) = 2\pi/a_1 \), \( T_2(h_2) = 2\pi/b_1 \) respectively.

3.2. Melnikov function

Introducing curvilinear coordinates in the neighbourhood of \( \Gamma_h \times \Gamma_h \), and establishing a curvilinear coordinate frame. Furthermore, we define a global cross section \( \Sigma \) in the phase space, and construct the \( k \) th iteration of Poincaré map \( P^k : \Sigma \to \Sigma \). The fixed point of \( P^k \), which corresponds to the periodic solution of system (3), can be obtained by calculating the zero point of the improved Melnikov function \( M = (M_1, M_2, M_3)^T \), where

\[
M_1 = \int_0^{2\pi} a_i(x_iF_1 + x_2F_2) dt, \quad M_2 = \int_0^{2\pi} b_i(x_3F_1 + x_2F_2) dt
\]
\[ M_3 = \int_0^{2\pi} \frac{(x_4 F_3 - x_3 F_2)h_2 - (x_2 F_1 - x_1 F_4)h_2}{2h_2} \, dt \]  

Assuming that \( b_1 = 2a_2 = 2, \ b_2 = 0 \), then the period \( T_1(h_1) = 2T_2(h_2) = 2\pi \). we have

\[ M_1 = 4\pi h_1 (2a_4 h_2 + 2a_6 h_1 - a_3 \sqrt{h_2} \sin(2t_0)) = 0, \ M_2 = 4\pi h_2 (4b_1 h_1 + b_6 h_2) = 0 \]

\[ M_3 = \pi (2a_4 \cos(2t_0)) \sqrt{h_2} + (b_3 - a_2 - 2a_1) h_2 + (b_7 + 2b_3 - 4a_4) h_1 + 2b_2 - 4a_3 = 0 \]  

(7)

For convenience, denote \( a_j = d_{i,j}, b_j = d_{2,j}, \alpha_j = \gamma_{1,k}, \beta_j = \gamma_{2,k} \ (j, k \in \mathbb{N}^+) \). Then the unknown coefficients of equation (7) are expressed as follows.

\[ d_{i,3} = \frac{1}{4} \gamma_{1,5} (2\sigma_i - \sigma_{3,i}), \ d_{i,4} = \frac{1}{4} \mu \gamma_{1,5} \gamma_{3,i+1}, \ d_{i,5} = \frac{3}{4} \gamma_{1,3} \sigma_i, \ d_{i,6} = \frac{3}{4} \mu \gamma_{1,3} \gamma_{1,3} \]

\[ d_{i,7} = \frac{1}{2} \gamma_{1,5} \sigma_{3,i+1}, \ d_{i,8} = \frac{1}{8i} (\gamma_{1,5} \sigma_{3,i+1} + (2i-1) \gamma_{1,6} \sigma_{3,i+1}) \]

When \( a_6 b_6 - 4a_4 b_4 \neq 0 \), equation (7) becomes

\[ \sqrt{h_1} = 2ba \sin(2t_0), \ \sqrt{h_2} = 2a \sin(2t_0) \]

\[ 4(A^2 + B^2) \sin^4(2t_0) - 4(B^2 - AC) \sin^2(2t_0) + C^2 = 0 \]

(8-1)

in which \( A = 2a^2 c, B = 2a_6 a, C = 2b_2 - 4a_2, \) and

\[ a = \frac{a_6 b_4}{4a_6 b_4 - a_6 b_6}, \ b = - \frac{b_6}{2b_4}, \ c = b_3 - a_7 - 2a_3 - \frac{b_7}{2}(4a_3 - 2b_3 - b_7) \]

The number of solutions for \((t_0, h_1, h_2)\) in equation (8) is closely related to the number of periodic solutions of system (3).

### 3.3. Bifurcation and control of multiple periodic solutions

We choose a group of parameters as

\[ PC = (\mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = (0.2, 10, 2, 2, -1, 1, 1, 20, 1, 1, 0.5, -1, 1) \]

to investigate the influence of the forcing excitation coefficient \( f_i \) on the number and relative positions of the periodic orbits. Denote

\[ f_{11} = -\frac{8}{15} \sqrt{210 + 21\sqrt{221}}, \ f_{12} = \frac{8}{15} \sqrt{210 + 21\sqrt{221}} \]

Theorem 1 For the given parameter condition of \( PC \), and the assumption of \( b_1 = 2a_2 = 2, \ b_2 = 0 \),

(1) when \( f_i < f_{11} \) or \( f_i > f_{12} \), equation (8) has 4 sets of positive real solutions, then system (3) has 4 periodic solutions;

(2) when \( f_i = f_{11} \) or \( f_i = f_{12} \), equation (8) has 2 sets of positive real solutions, then system (3) has 2 periodic solutions;

(3) when \( f_{11} < f_i < f_{12} \), equation (8) has no real solution, then system (3) has no periodic solution.

Proof. For convenience, we define (8-2) as

\[ f(\sin^2(2t_0)) = 4(A^2 + B^2) \sin^4(2t_0) - 4(B^2 - AC) \sin^2(2t_0) + C^2 \]

(9)

The forcing excitation coefficient \( f_i \) is left as the only unknown parameter. The number of solutions for equation (9) is closely related to the value of the parameter \( f_i \).
If $A^2 + B^2 = 0$, then $f_0 = 0$ and $B^2 - AC = 0$, equation (9) becomes $f(\sin^2(2t_0)) = C^2$, there is no real solution for the equation. Therefore, system (3) has no periodic solution.

If $A^2 + B^2 \neq 0$ and $B^2 - AC = 0$, then $f_0 = \pm \frac{8}{15}\sqrt{210}$. Equation (9) becomes

$$f(\sin^2(2t_0)) = 4(A^2 + B^2)\sin^4(2t_0) + C^2$$

there is no real solution for the equation. Therefore, system (3) has no periodic solution.

If $A^2 + B^2 \neq 0$ and $B^2 - AC \neq 0$, equation (9) becomes a quadratic equation about $\sin^2(2t_0)$ with

$$\Delta = 16B^2(B^2 - C^2 - 2AC) .$$

The image of function $\Delta$ is shown in figure 2. Obviously, the equation $\Delta = 0$ has 3 solutions: $0$, $f_{11}$ and $f_{12}$, where

$$f_{11} = -\frac{8}{15}\sqrt{210} + 21\sqrt{221},$$

$$f_{12} = \frac{8}{15}\sqrt{210} + 21\sqrt{221} .$$

![Figure 2. The image of the function $\Delta$ respect to $f_1$.](image)

The following analysis is divided into 3 parts:

(1) When $f_1 < f_{11}$ or $f_1 > f_{12}$, we have $\Delta > 0$ and $B^2 - AC > 0$, then there are 2 positive solutions for $\sin^2(2t_0)$ in equation (9), i.e.,

$$\sin^2(2t_0) = \frac{4(B^2 - AC) \pm \sqrt{\Delta}}{8(A^2 + B^2)} ,$$

according to the relations between roots and coefficients. Therefore, equation (8) has 4 sets of solutions for $(t_0, h_1, h_2)$, and system (3) has 4 periodic solutions correspondingly.

(2) When $f_1 = f_{11}$ or $f_1 = f_{12}$, we have $\Delta = 0$ and $B^2 - AC > 0$, then there is only 1 positive solution for $\sin^2(2t_0)$ in equation (9), i.e.,

$$\sin^2(2t_0) = \frac{B^2 - AC}{2(A^2 + B^2)} .$$

Therefore, equation (8) has 2 sets of solutions for $(t_0, h_1, h_2)$, and system (3) has 2 periodic solutions correspondingly.

(3) When $f_{11} < f_1 < f_{12}$, $f_1 = \pm \frac{8}{15}\sqrt{210}$ and $f_1 \neq 0$, we have $\Delta < 0$, then there is no real solution for equation (9). Therefore, system (3) has no periodic solution.

The proof of theorem 1 is completed.

According to theorem 1, the periodic solutions of the system appear in pairs. When $f_1 < f_{11}$, system (3) has 4 periodic orbits. We divide them into two pairs: $\Gamma^+$ and $\Gamma^-$, which satisfy

$$\Gamma^+ = \left\{ x_h \big| h_1 = a^2 b \frac{4(B^2 - AC) + \sqrt{\Delta}}{4(A^2 + B^2)}, h_2 = a^2 \frac{4(B^2 - AC) + \sqrt{\Delta}}{2(A^2 + B^2)} \right\}$$

$$\Gamma^- = \left\{ x_h \big| h_1 = a^2 b \frac{4(B^2 - AC) - \sqrt{\Delta}}{4(A^2 + B^2)}, h_2 = a^2 \frac{4(B^2 - AC) - \sqrt{\Delta}}{2(A^2 + B^2)} \right\}$$

(10-1)  (10-2)
While $f_i$ crosses $f_{i_1}$ from $f_i < f_{i_1}$ to $f_i > f_{i_1}$, the two pairs of periodic orbits coincide, forming one pair of periodic orbits at the critical point $f_i = f_{i_1}$, which satisfies

$$\Gamma^0 = \left\{ x|h_i = a^2 + B^2 - AC, h_2 = 2a^2 + B^2 \right\}$$

and then disappear. As the value of parameter $f_i$ increases continually, a new pair of periodic orbits generates at $f_i = f_{i_2}$, and then split into two pairs of periodic orbits immediately while $f_i$ crosses $f_{i_2}$ from $f_i < f_{i_2}$ to $f_i > f_{i_2}$. Throughout the process, bifurcation of periodic solutions occurs twice, the corresponding bifurcation values are $f_i = f_{i_1}$ and $f_i = f_{i_2}$ respectively.

3.4. Numerical simulation

For a more visual understanding, the numerical simulation is performed to obtain the phase portraits of the periodic motions as well as the patterns and relative positions under certain conditions. We choose the perturbation parameter $\varepsilon = 0.001$ in the situations below.

1) $f_i = -13.5 < f_{i_1}$. There are 4 periodic solutions which can be divided into two pairs:

$\Gamma^+: \{ x|h_1 = 0.0686, h_2 = 0.0457 \}$, the pair of periodic orbits with bigger amplitude;

$\Gamma^-: \{ x|h_1 = 0.0126, h_2 = 0.0084 \}$, the pair of periodic orbits with smaller amplitude.

The patterns and relative positions of the periodic motions for system (3) are shown in figure 3. Figure 3 (a)-(c) represent the phase portraits of the periodic orbits projected on various planes, the projections of the two periodic orbits in same pair coincide. Figure 3 (d)-(f) represent the phase portraits of the periodic orbits projected into three dimensional spaces. Furthermore, we will clearly find that the moving patterns of a periodic orbit in $\Gamma^+$ is one-to-one correspondence with that in $\Gamma^-$.

2) $f_i = -12.5 < f_{i_1}$. There are also 4 periodic solutions which can be divided into two pairs:

$\Gamma^+: \{ x|h_1 = 0.0445, h_2 = 0.0297 \}$, the pair of periodic orbits with bigger amplitude;

$\Gamma^-: \{ x|h_1 = 0.0195, h_2 = 0.0130 \}$, the pair of periodic orbits with smaller amplitude.
The patterns and relative positions of the periodic motions for the system are shown in figure 4, where (a)-(c) represent the two dimensional projections, and (d)-(f) represent the three dimensional projections. Compared with situation (1), the amplitude of the periodic orbits decreased in $\Gamma^+$ while the amplitude increased in $\Gamma^-$. 

![Graphs showing periodic motions](image)

Figure 4. Patterns and relative positions of periodic motions when $f_i = -12.5$.

(3) $f_i = f_{11}$. There are 2 periodic solutions which satisfy $\Gamma^0: \{x \mid h_i = 0.0294, h_2 = 0.0196\}$. The patterns and relative positions of the periodic motions are shown in figure 5. Likewise, (a)-(c) represent the two dimensional projections, while (d)-(f) represent the three dimensional projections.

![Graphs showing periodic motions](image)

Figure 5. Patterns and relative positions of periodic motions when $f_i = f_{11}$. 


(4) \( f_i = -2 > f_{i1} \). There is no periodic solution for system (3). The phase portraits of two and three dimensional projections are shown in figure 6.

![Phase portraits](attachment:image.png)

**Figure 6.** Phase portraits of the orbits when \( f_i = -2 \).

From the above four cases, bifurcations of the phase portrait as \( f_i \) passing through \( f_{i1} \) can be easily found. This bifurcation can also be presented in \( (x_1, x_2, f_i) \)-space, see figure 7. The appearing \( f_i \)-family of periodic orbits forms two pairs of rotating surfaces which are symmetric about \( (x_1, x_2) \)-plane. In detail, when \( f_i < f_{i1} \), there are two pairs of periodic orbits. As the value of parameter \( f_i \) increases, the amplitude of the periodic orbits decreasing gradually in \( \Gamma^+ \) while the amplitude of the periodic orbits increasing in \( \Gamma^- \). The two pairs of periodic orbits coincide, forming one pair of periodic orbits \( \Gamma^0 \) at the bifurcation value \( f_i = f_{i1} \), and then disappear. Similarly, the bifurcation behaviors when \( f_i > 0 \) can also be described. The only difference is that the bifurcation direction is exactly the opposite with the case of \( f_i < 0 \), that is, generating a pair of periodic orbits at \( f_i = f_{i2} \) and then splitting into two pairs of periodic orbits immediately while \( f_i \) crosses \( f_{i2} \).

![Bifurcation](attachment:image.png)

**Figure 7.** Bifurcation of multiple periodic orbits in the phase-parameter space.
4. Conclusions
In this paper, we focus on the nonlinear vibration behaviours and parameter control of a simply supported concave hexagonal honeycomb sandwich plate subjected to its in-plane and transverse excitation. The honeycomb core is chosen as auxetic composite material with negative Poisson’s ratio which is promising in aircraft and aerospace engineering applications. Melnikov function, improved by the methods of curvilinear coordinate frame and Poincaré map, is presented to detect the existence and number of the periodic solutions. The forcing excitation coefficient \( f_1 \) is considered as a bifurcation parameter to investigate its influence on the number and relative positions of the periodic orbits. \( f_1 = f_{11} \) and \( f_1 = f_{12} \) are two key critical values to determine bifurcation of multiple periodic orbits. Numerical simulation is performed to obtain the phase portraits of the multiple periodic orbits under various conditions.

Periodic motion of averaged equation may lead to amplitude modulated periodic oscillations for the model under certain conditions. The large amplitude nonlinear oscillations of metamaterial plates will lead to the serious damages of structures. Results obtained in this paper will provide theoretical guidance to nonlinear vibration control and optimization design for metamaterials sandwich structures with negative Poisson’s ratio in auxetic honeycombs.

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