THE SPECTRAL NORM OF GAUSSIAN MATRICES WITH CORRELATED ENTRIES

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Abstract. We give a non-asymptotic bound on the spectral norm of a $d \times d$ matrix $X$ with centered jointly Gaussian entries in terms of the covariance matrix of the entries. In some cases, this estimate is sharp and removes the $\sqrt{\log d}$ factor in the noncommutative Khintchine inequality. This paper is superseded by https://arxiv.org/abs/2108.06312

1. Introduction

Let $X$ be a $d \times d$ centered random matrix with (correlated) jointly Gaussian entries. We aim to provide an estimate for the expected spectral norm of $E\|X\|$ in terms of the $d^2 \times d^2$ covariance matrix $E(X \otimes X)$ of the Gaussian entries. This problem is settled by the noncommutative Khintchine inequality \cite{6, 12, 14} up to a $\sqrt{\log d}$ factor, namely,

\begin{equation}
E\|X\|^2 + E\|XX^*\|^2 \lesssim E\|X\| \lesssim \sqrt{\log d} \left( E\|X^*X\|^2 + E\|XX^*\|^2 \right),
\end{equation}

where $\lesssim$ denotes smaller or equal up to multiplicative dimension-free constant.

The $\sqrt{\log d}$ factor on the right hand side of (1.1) is, in general, required: if $X$ is diagonal with i.i.d. standard Gaussian diagonal entries, then $E\|X\| \sim \sqrt{\log d}$ and $E\|X^*X\| = E\|XX^*\| = 1$. By contrast, if the $d^2$ entries of $X$ are i.i.d. standard Gaussian random variables, then $E\|X\| \sim \sqrt{d}$ and $E\|X^*X\| = E\|XX^*\| = d$ so in this case, the $\sqrt{\log d}$ factor can be removed. More generally, if the entries of $X$ are independent and the variances of the entries are homogeneous enough, then the $\sqrt{\log d}$ factor can be removed \cite{3, 9, 11}.

Estimates for the spectral norm of random matrices are a central tool in both pure and applied mathematics, we point the interested reader to the monograph \cite{16} and references therein for applications. We note also that the extra dimensional factor often propagates to the applications resulting in suboptimal bounds.

The extent to which the $\sqrt{\log d}$ factor can be removed in (1.1), in general, is mostly unknown. A notable result in this direction, whose insights we build on, is the work of Tropp \cite{18} which introduces a quantity $w(X)$, for a self-adjoint Gaussian matrix $X$, and shows that

$$E\|X\| \lesssim \sqrt{\log d} \|E(X^2)\|^2 + \sqrt{\log d} \cdot w(X)$$

for all (correlated) self-adjoint Gaussian matrices $X$. When all the $d^2$ entries of $X$ are i.i.d. standard Gaussian, this estimate improves (1.1) but is still not sharp because of the $\sqrt{\log d}$ factor. Moreover, in general, computing $w(X)$ directly appears to be challenging.

The following is the main result of this paper.

Theorem 1.1. Let $X$ be a $d \times d$ random matrix with jointly Gaussian entries and $E X = 0$, then

$$E\|X\| \lesssim \|E(X^*X)\|^{\frac{1}{2}} + \|E(XX^*)\|^{\frac{1}{2}} + \|E(X \otimes X)\|^{\frac{1}{2}},$$

for all $\epsilon > 0$; here $\lesssim_\epsilon$ means less or equal up to a dimension-free multiplicative constant depending on $\epsilon$. 

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Note that $\mathbb{E}(X \otimes X)$ is a linear transformation on the $d^2$ dimensional inner product space $M_d(\mathbb{R})$ of $d \times d$ real matrices with $\langle A, B \rangle = \text{Tr}(AB^*)$ for $A, B \in M_d(\mathbb{R})$.

Before presenting a range of guiding examples and discussing the sharpness of this inequality, we state a “user-friendly” version of it. One can see that the first statement of Theorem 1.2 is equivalent to Theorem 1.1 by taking the $A_1, \ldots, A_n$ in Theorem 1.2 being certain appropriately scaled eigenvectors of $\mathbb{E}(X \otimes X)$ in Theorem 1.1. Moreover, when all entries of $A_1, \ldots, A_n$ are nonnegative, the $d^ε$ factor can be replaced by $(\log d)^2$.

**Theorem 1.2.** Let $g_1, \ldots, g_n$ be i.i.d. standard Gaussian random variables and $A_1, \ldots, A_n \in M_d(\mathbb{R})$ satisfy $\text{Tr}(A_kA_k^*) = 0$ for all $k \neq k_2$ in $[n]$. Then

$$\mathbb{E} \left\| \sum_{k=1}^{n} g_k A_k \right\| \lesssim \epsilon \left\| \sum_{k=1}^{n} A_k^* A_k \right\|^{\frac{1}{2}} + \left\| \sum_{k=1}^{n} A_k A_k^* \right\|^{\frac{1}{2}} + d^\epsilon \max_{k \in [n]} \left\| A_k \right\|_F,$$

for all $\epsilon > 0$. If moreover, all entries of $A_1, \ldots, A_n$ are nonnegative, then

$$\mathbb{E} \left\| \sum_{k=1}^{n} g_k A_k \right\| \lesssim \left\| \sum_{k=1}^{n} A_k^* A_k \right\|^{\frac{1}{2}} + \left\| \sum_{k=1}^{n} A_k A_k^* \right\|^{\frac{1}{2}} + (\log d)^2 \max_{k \in [n]} \left\| A_k \right\|_F.$$

While Theorem 1.2 is the one we use in the guiding examples, it is worth formulating an inequality for Gaussian series without the orthogonality condition; the following follows immediately from Theorem 1.1 by noticing that the Gaussian series is a Gaussian matrix.

**Corollary 1.3.** Let $g_1, \ldots, g_n$ be i.i.d. standard Gaussian random variables and $H_1, \ldots, H_n \in M_d(\mathbb{R})$. Then

$$\mathbb{E} \left\| \sum_{k=1}^{n} g_k H_k \right\| \lesssim \epsilon \left\| \sum_{k=1}^{n} H_k^* H_k \right\|^{\frac{1}{2}} + \left\| \sum_{k=1}^{n} H_k H_k^* \right\|^{\frac{1}{2}} + d^\epsilon \sup_{B \in M_d(\mathbb{R}) \atop \left\| B \right\|_F \leq 1} \left( \sum_{k=1}^{n} \langle H_k, B \rangle^2 \right)^{\frac{1}{2}}.$$

**Remark 1.4.** While outside the scope of this paper, we note that (i) Corollary 1.3 can be used to obtain non-asymptotic bounds on the expected spectral norm of sums of independent random matrices via the techniques described in 17, and (ii) it is, in general, possible to obtain tail bounds on the spectral norm of random matrices via a control on the expected spectral norm and a scalar concentration inequality.

**Remark 1.5.** Theorem 1.1 is not, in general, sharp. We expect that the $d^ε$ factor is not needed and could be replaced by a $\sqrt{\log d}$ factor, but were not able to prove it. Furthermore, the term $\| \mathbb{E}(X \otimes X) \|$ does not appear to be the correct quantity in general. In particular, there are situations in which it is even weaker than the noncommutative Khintchine inequality 1.1: namely, for $n = 1$, we have $X = gA$ and $\| \mathbb{E}X \| \sim \| A \|$ while $\| \mathbb{E}(X \otimes X) \| = \| A \|_F^2$, which can be a factor of $d$ larger than $\| A \|^2$. Nevertheless, as we will see in the next section, Theorem 1.1 captures the sharp behavior of the expected norm of a Gaussian matrix with correlated entries in several scenarios.

1.1. **A Conjecture involving a weak variance parameter.** It has been conjectured, first implicitly in 16, and then more explicitly in 2, 18, 8 that the correct parameter commanding the existence or not of the logarithmic factor in noncommutative Khintchine is the weak variance: for a $d \times d$ centered random matrix $X$ with jointly Gaussian entries and $\mathbb{E}X = 0$, let

$$\sigma_0(X) = \sup_{v, w \in \mathbb{R}^d \atop \| v \|_2 = \| w \|_2 = 1} (\mathbb{E}(Xv, w))^2.$$
This parameter can be viewed as the injective norm of $\mathbb{E}X \otimes X$ when viewed as a fourth order tensor. It is also worth noting that this is the parameter governing fluctuations per Gaussian elimination

$$\mathbb{P}(\|X\| - \mathbb{E}\|X\| \geq t) \leq 2e^{-t^2/(2\sigma_s(X)^2)}.$$  

Intuitively, in the language of Corollary 1.3 and the particular case of self-adjoint matrices, the cancellations responsible for the removal of the $\sqrt{\log d}$ factor appear to be due to non-commutativity of the matrices $H_k$’s.

**Conjecture 1.6.** Let $X$ be a $d \times d$ centered random matrix with jointly Gaussian entries and, then

$$\mathbb{E}\|X\| \leq \|\mathbb{E}(X^*X)\|^{\frac{1}{2}} + \|\mathbb{E}(XX^*)\|^{\frac{1}{2}} + \sqrt{\log d} \sup_{v, w \in \mathbb{R}^d} (\mathbb{E}(Xv, w))^2 \frac{1}{2}.$$  

We note that $\sigma_s(X) \leq \|\mathbb{E}(X \otimes X)\|^{\frac{1}{2}}$, since

$$\|\mathbb{E}(X \otimes X)\| = \sup_{B \in M_d(\mathbb{R})} \mathbb{E}(X, B) = \sup_{B \in M_d(\mathbb{R})} \|\mathbb{E}\text{Tr}(XB^*)\|^2.$$  

Also, the Cauchy-Schwarz inequality implies that $\sigma_s(X) \leq \|\mathbb{E}(X^*X)\|^{\frac{1}{2}}$.

Conjecture 1.6 has been verified in the case of independent entries 3. It is worth mentioning that when the matrix is very inhomogenous even the term $\sqrt{\log d} \sigma_s(X)$ may not be necessary 11. There are two ways in which Theorem 1.1 is weaker than Conjecture 1.6: (i) the dimensional factor is $d^e$ as opposed to $\sqrt{\log d}$; in the examples to be described, this limits the regimes in which our inequality is sharp; and (ii) the quantity $\|\mathbb{E}(X \otimes X)\|$ can in general be larger than $\sigma_s(X)$; it is worth mentioning however that the quantity $\sigma_s(X)$ in Conjecture 1.6 appears to be difficult to compute, whereas $\|\mathbb{E}(X \otimes X)\|$ can be viewed as an easily computable (sometimes sharp) upper bound, at least in several cases in the next section. In Remark 2.3 we highlight an interesting regime in which these two quantities are different and Conjecture 1.6 would imply a stronger result.

**Notation.** Throughout this paper, if $T$ is a matrix or a linear transformation on an inner product space, $\|T\|$ denotes the spectral norm of $T$. The trace and the Frobenius norm of $T$ are denoted by $\text{Tr}T$ and $\|T\|_F = \sqrt{\text{Tr}(T^*T)}$, respectively. For $a, b > 0$, we write $a \lesssim b$ when $a \leq Cb$ for some universal constant $C > 0$; we write $a \lesssim_c b$ when $a \leq Cib$ for some constant $C_i > 0$ that depends only on $c$; we write $a \sim b$ when $a \lesssim b$ and $b \lesssim a$; we write $a \sim_c b$ when $a \lesssim_c b$ and $b \lesssim_c a$. For $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$. For $d \in \mathbb{N}$, $(e_1, \ldots, e_d)$ is the canonical basis for $\mathbb{R}^d$.

2. Guiding Examples and Applications

2.1. Gaussian on a subspace. Consider the inner product space $M_d(\mathbb{R})$ of $d \times d$ real matrices with $\langle A, B \rangle = \text{Tr}(AB^*)$. Suppose that $\mathcal{M}$ is a subspace of $M_d(\mathbb{R})$ and $X$ is a standard Gaussian on $\mathcal{M}$, i.e., $X = \sum_{k=1}^{\dim \mathcal{M}} g_k A_k$, where $g_1, \ldots, g_{\dim \mathcal{M}}$ are i.i.d. standard Gaussian random variables and $(A_1, \ldots, A_{\dim \mathcal{M}})$ is any orthonormal basis for $\mathcal{M}$. (The distribution of $X$ is independent of the choice of the orthonormal basis.) When $\dim \mathcal{M} = d$, the $\sqrt{\log d}$ factor in (1.1) cannot always be removed, e.g., when $\mathcal{M}$ is the subspace of diagonal matrices. When $\dim \mathcal{M} = d^2$, we have $\mathcal{M} = M_d(\mathbb{R})$ so all the $d^2$ entries of $X$ are i.i.d. standard Gaussian and the $\sqrt{\log d}$ factor can be removed.

In this paper, we show that, for any $\epsilon > 0$, when $\dim \mathcal{M} \geq d^{1+\epsilon}$, the $\sqrt{\log d}$ factor can still be removed. Thus, there is a “phase transition” where the $\sqrt{\log d}$ factor cannot always be
removed for \( \dim M = d \), but can be removed for \( \dim M \geq d^{1+\epsilon} \). Intuitively, this is because when all matrices in \( M \) are self-adjoint, it is possible that all matrices in \( M \) commute if \( \dim M = d \), but it is impossible that all matrices in \( M \) commute when \( \dim M > d \). As \( \dim M \) gets larger, the matrices in \( M \) are “more noncommuting.”

**Corollary 2.1.** If \( X \) is a standard Gaussian on a subspace \( M \) of \( M_d(\mathbb{R}) \) and \( \dim M \geq d^{1+\epsilon} \) with \( \epsilon > 0 \), then

\[
\mathbb{E}[\|X\| \sim \|\mathbb{E}(X^*X)\|^\frac{1}{2} + \|\mathbb{E}(XX^*)\|^\frac{1}{2}].
\]

**Proof.** Since \( X \) is a standard Gaussian on \( M \), the expected Frobenius norm \( \mathbb{E}\|X\|_F = \dim M \) and the covariance \( \mathbb{E}(X \otimes X) \) is the orthogonal projection from \( M_d(\mathbb{R}) \) onto \( M \). So the spectral norm \( \mathbb{E}(X \otimes X) \| = 1 \). So \( \mathbb{E}(X^*X) \geq \frac{d}{n} \mathbb{E}(X \otimes X) \|_F = \frac{d}{n} \mathbb{E}(X \otimes X) = \frac{1}{d} \mathbb{E}(X \otimes X) \|_F = \frac{1}{d} \dim M \geq d^\epsilon \). Thus, \( d^\epsilon \mathbb{E}(X \otimes X) \|_F \leq \|X^*X\|^\frac{1}{2}. \) The result follows from Theorem 1.1. \( \square \)

We expect the sharp condition to be \( \dim M \gtrsim d \log d \), but were not able to prove it.

### 2.2. Independent blocks

In Theorem 1.2 if we let \( A_1, \ldots, A_d \) be \( A_{i,j} = b_{i,j} e_i e_j^T \in M_d(\mathbb{R}) \) for \( i, j \in [d] \), where \( b_{i,j} > 0 \) for \( i, j \in [d] \), then the second statement of Theorem 1.2 gives

\[
\mathbb{E} \left( \sum_{i,j \in [n]} g_{i,j} b_{i,j} e_i e_j^T \right) \leq \max_{j \in [d]} \left( \frac{d}{1+\epsilon} \right) \mathbb{E} \left( \sum_{i \in [d]} |b_{i,j}|^2 \right) + (\log d)^2 \max_{i,j \in [n]} |b_{i,j}|,
\]

where \( (g_{i,j})_{i,j \in [d]} \) are i.i.d. standard Gaussian random variables. This recovers a weaker version of a result by the first author and van Handel [3], who prove the estimate with the \( (\log d)^2 \) factor being replaced by \( \sqrt{\log d} \), which is in fact, the optimal factor.

A block version of this example better illuminates the difference between the weak variance and the quantity our inequality uses. We note this is different from the model of random lifts of graphs [15, 16, 17].

**Corollary 2.2.** For each \( i, j \in [d] \), let \( B_{i,j} \) be an \( r \times r \) matrix and \( g_{i,j} \) be independent standard Gaussian random variables. Consider the following \( dr \times dr \) matrix

\[
X = \begin{bmatrix}
g_{1,1} B_{1,1} & \cdots & g_{1,d} B_{1,d} \\
\vdots & \ddots & \vdots \\
g_{d,1} B_{d,1} & \cdots & g_{d,d} B_{d,d}
\end{bmatrix},
\]

and \( \gamma = \max_{j \in [d]} \left( \sum_{i=1}^d B_{i,j}^* B_{i,j} \right)^\frac{1}{2} + \max_{i \in [d]} \left( \sum_{j=1}^d B_{i,j} B_{i,j}^* \right)^\frac{1}{2} \). Then

\[
\gamma \lesssim \mathbb{E}\|X\| \lesssim \gamma + (dr)^\epsilon \max_{i,j \in [d]} \| B_{i,j} \|_F.
\]

If moreover, all entries of every \( B_{i,j} \) are nonnegative, then

\[
\gamma \lesssim \mathbb{E}\|X\| \lesssim \gamma + (\log(dr))^2 \max_{i,j \in [d]} \| B_{i,j} \|_F.
\]

**Proof.** This follows from Theorem 1.2 by taking \( A_1, \ldots, A_d \in M_{dr}(\mathbb{R}) \) to be \( A_{i,j} \in M_{dr}(\mathbb{R}) \) being the matrix with the \((i, j)\)-block being \( B_{i,j} \) and the other blocks being 0, where \( i, j \in [d] \). \( \square \)
Remark 2.3. We note that if Conjecture [1,6] is true, then

\[ \mathbb{E}\|X\| \lesssim \gamma + \sqrt{\log(dr)} \max_{i,j \in [d]} \|B_{i,j}\|, \]

where \( \|B_{i,j}\|_F \) is replaced by \( \|B_{i,j}\| \).

2.3. Independent rows.

Corollary 2.4. Suppose that \( X \) is a \( d_1 \times d_2 \) random matrix with independent rows and for \( i \in [d_1] \), the \( i \)th row of \( X \) is a centered Gaussian random vector with covariance matrix \( B_i \in M_{d_2}(\mathbb{R}) \). Then

\[ \left\| \sum_{i=1}^{d_1} B_i \right\|^{\frac{1}{2}} + \max_{i \in [d_1]} \left\| \text{Tr}(B_i) \right\|^{\frac{1}{2}} \lesssim \mathbb{E}\|X\| \lesssim \epsilon \left( \sum_{i=1}^{d_1} B_i \right)^{\frac{1}{2}} + \max_{i \in [d_1]} \left\| \text{Tr}(B_i) \right\|^{\frac{1}{2}} + \max(d_1', d_2') \max_{i \in [d_1]} \|B_i\|^{\frac{1}{2}}, \]

for all \( \epsilon > 0 \).

Proof. Write \( X = \sum_{i=1}^{d_1} e_i x_i^T \) where each \( x_i \) is a centered Gaussian random vector with covariance matrix \( B_i \in M_{d_2}(\mathbb{R}) \), and \( x_1, \ldots, x_{d_1} \) are independent. Thus, each \( x_i \) can be written as \( x_i = \sum_{j=1}^{d_2} g_{i,j} \sqrt{\lambda_{i,j}} v_{i,j}^T \), where \( \lambda_{i,1}, \ldots, \lambda_{i,d_2} \) are the eigenvalues of \( B_i \) and \( (v_{i,1}, \ldots, v_{i,d_2}) \) is an orthonormal basis for \( \mathbb{R}^{d_2} \) consisting of the corresponding eigenvectors. Moreover, the \( (g_{i,j})_{i \in [d_1], j \in [d_2]} \) are i.i.d. standard Gaussian random variables.

We have \( X = \sum_{i=1}^{d_1} \sum_{j \in [d_2]} A_{i,j}^* a_{i,j} \). Let \( A_{i,j} = \sqrt{\lambda_{i,j}} e_i v_{i,j}^T \) for all \( i \in [d_1], j \in [d_2] \). Note that \( \text{Tr}(A_{i_1,j_1}^* A_{i_2,j_2}) = 0 \) whenever \( (i_1,j_1) \neq (i_2,j_2) \). Thus, \( X = \sum_{i=1}^{d_1} \sum_{j \in [d_2]} g_{i,j} A_{i,j} \).

\[ \sum_{i \in [d_1], j \in [d_2]} A_{i,j}^* A_{i,j} = \sum_{i \in [d_1], j \in [d_2]} \lambda_{i,j} e_i e_i^T = \sum_{j \in [d_2]} \left( \sum_{i=1}^{d_1} \lambda_{i,j} \right) e_i e_i^T = \sum_{i=1}^{d_1} \text{Tr}(B_i) e_i e_i^T, \]

by Theorem 1.2 and adding some zero rows/columns to each \( A_{i,j} \) so that they become square matrices, the right hand side of the result follows. The left hand side is simply

\[ \| \sum_{i \in [d_1], j \in [d_2]} A_{i,j}^* A_{i,j} \|^{\frac{1}{2}} + \| \sum_{i \in [d_1], j \in [d_2]} A_{i,j} A_{i,j}^* \|^{\frac{1}{2}}. \]

Remark 2.5. In Corollary 2.4 if \( \text{Tr}(B_i) \geq \max(d_1', d_2') \) for all \( i \in [d_1] \), or if each \( B_i \) appears in \( B_1, \ldots, B_{d_1} \) at least \( \max(d_1', d_2') \) times, then we obtain

\[ \mathbb{E}\|X\| \sim \epsilon \left( \sum_{i=1}^{d_1} B_i \right)^{\frac{1}{2}} + \max_{i \in [d_1]} \left\| \text{Tr}(B_i) \right\|^{\frac{1}{2}}, \]

and so since \( (\mathbb{E}\|X\|^{2})^{\frac{1}{2}} \leq \mathbb{E}\|X\| \) (by a Gaussian version of Kahane’s inequality [10] or by concentration of \( \|X\| \)),

\[ \mathbb{E}(\|X\|^{2}) \sim \epsilon \left( \sum_{i=1}^{d_1} B_i \right) + \max_{i \in [d_1]} \text{Tr}(B_i). \]
2.4. Sample covariance.

**Corollary 2.6.** Suppose that \( \mu \) is a probability measure on \( \{ B \in M_{d_2}(\mathbb{R}) \mid B \text{ is positive semidefinite} \} \).

Let \( z_1, \ldots, z_{d_1} \) be i.i.d. random vectors in \( \mathbb{R}^{d_2} \) chosen according to \( \int N(0, B) \, d\mu(B) \), i.e.,

\[
\mathbb{P}(z_1 \in S) = \frac{1}{\sqrt{\det(2\pi B)}} \int_{S} \frac{1}{\sqrt{\det(2\pi B)}} \, d\mu(B)
\]

for all measurable \( S \subset \mathbb{R}^{d_2} \), where \( g \) is a standard Gaussian on \( \mathbb{R}^{d_2} \). Let \( Y = \sum_{i=1}^{d_1} z_i z_i^T \in M_{d_2}(\mathbb{R}) \). If \( \text{Tr}(B) \geq \max(d_1^2, d_2^2) \|B\| \) \( \mu \)-a.s., then

\[
\mathbb{E}\|Y\| \sim \epsilon \int B \, d\mu(B) \mathbb{E} \max_{i \in [d_1]} \text{Tr}(B_i),
\]

where \( B_1, \ldots, B_{d_1} \) in \( M_{d_2}(\mathbb{R}) \) are i.i.d. chosen according to \( \mu \).

**Proof.** By assumption, \( z_1, \ldots, z_{d_1} \) are chosen as follows: first, choose i.i.d. \( B_1, \ldots, B_{d_1} \) in \( M_{d_2}(\mathbb{R}) \) according to \( \mu \) and then for each \( i \in [d_1] \), take \( z_i = B_i^{1/2} g_i \), where \( g_1, \ldots, g_{d_1} \) are i.i.d. standard Gaussian random variables. Let \( X \) be the \( d_1 \times d_2 \) matrix with the \( i \)-th row of \( X \) being \( z_i \) for every \( i \in [d_1] \). Note that \( Y = X^*X \). Since \( \text{Tr}(B) \geq \max(d_1^2, d_2^2) \|B\| \) \( \mu \)-a.s., by Corollary 2.4 and the remark after Corollary 2.4, conditioning on \( B_1, \ldots, B_{d_1} \), we have

\[
\mathbb{E}(\|X\|^2 \mid B_1, \ldots, B_{d_1}) \sim \epsilon \sum_{i=1}^{d_1} B_i + \max_{i \in [d_1]} \text{Tr}(B_i).
\]

Thus, since \( Y = X^*X \),

\[
\mathbb{E}\|Y\| \sim \epsilon \mathbb{E} \left( \sum_{i=1}^{d_1} B_i \right) + \mathbb{E} \max_{i \in [d_1]} \text{Tr}(B_i).
\]

By \cite{L7} Theorem 5.1(1),

\[
\mathbb{E} \left( \sum_{i=1}^{d_1} B_i \right) \lesssim \mathbb{E} \left( \sum_{i=1}^{d_1} \epsilon B_i \right) + (\log d_2) \mathbb{E} \max_{i \in [d_1]} \|B_i\|.
\]

But by assumption, \( \text{Tr}(B) \geq d_2^2 \|B\| \) \( \mu \)-a.s. Therefore,

\[
\mathbb{E}\|Y\| \sim \epsilon \sum_{i=1}^{d_1} \mathbb{E} B_i + \mathbb{E} \max_{i \in [d_1]} \text{Tr}(B_i).
\]

Since \( \mathbb{E} B_i = \int B \, d\mu(B) \) for all \( i \in [d_1] \), the result follows. \( \square \)

**Remark 2.7.** If the assumption \( \text{Tr}(B) \geq \max(d_1^2, d_2^2) \|B\| \) \( \mu \)-a.s. is removed, Corollary 2.6 may fail. For example, take \( d_1 = d_2 \) and \( \mu \) to be the uniform probability measure over the subset \( \{e_1 e_1^T, \ldots, e_{d_2} e_{d_2}^T\} \) of \( M_{d_1}(\mathbb{R}) \). Then \( d_1 \| \int B \, d\mu(B) \| + \mathbb{E} \max_{i \in [d_1]} \text{Tr}(B_i) \sim 1 \), while \( \mathbb{E}\|Y\| \geq \mathbb{E} \max_{i \in [d_1]} \|z_i\|^2 \sim \log d_1 \).

2.5. Glued entries.

**Corollary 2.8.** Suppose that \( \{S_1, \ldots, S_n\} \) is a partition of \([d] \times [d]\) such that \( |S_1| = \ldots = |S_n| \leq \frac{d}{(\log d)^4} \). Let \( g_1, \ldots, g_n \) be i.i.d. standard Gaussian random variables. Consider the random matrix \( X \) in \( M_d(\mathbb{R}) \) defined by \( X_{i,j} = g_k \) for all \( (i, j) \in S_k \) and \( k \in [n] \). Then

\[
\mathbb{E}\|X\| \sim \left( \sum_{k=1}^{n} A_k^* A_k \right)^{1/2} + \left( \sum_{k=1}^{n} A_k A_k^* \right)^{1/2},
\]

where for \( k \in [n] \), the matrix \( A_k \in M_d(\mathbb{R}) \) is defined by \( (A_k)_{i,j} = \begin{cases} 1, & (i, j) \in S_k \\ 0, & \text{ Otherwise} \end{cases} \).
Proof. Observe that \( X = \sum_{k=1}^{n} g_k A_k \) and that \( \text{Tr}(A_{k_1} A_{k_2}^*) = 0 \) for all \( k_1 \neq k_2 \). Thus, by Theorem 1.2

\[
\mathbb{E} \| X \| \lesssim \left\| \sum_{k=1}^{n} A_k^* A_k \right\|^{\frac{1}{2}} + \left\| \sum_{k=1}^{n} A_k A_k^* \right\|^{\frac{1}{2}} + (\log d)^2 \max_{k \in [n]} \| A_k \|_F.
\]

Since \( \| A_k \|_F^2 = \text{Tr}(A_k^* A_k) = |S_k| \) for all \( k \in [n] \),

\[
\left\| \sum_{k=1}^{n} A_k^* A_k \right\| \geq \frac{1}{d} \text{Tr} \left( \sum_{k=1}^{n} A_k^* A_k \right) = \frac{n}{d} |S_k|.
\]

Thus, if \( \frac{\sqrt{n}}{d} \geq (\log d)^2 \), then \( \| \sum_{k=1}^{n} A_k^* A_k \|^{\frac{1}{2}} \geq (\log d)^2 \max_{k \in [n]} \| A_k \|_F \) and the result follows. To show that \( \frac{\sqrt{n}}{d} \geq (\log d)^2 \), note that \( n |S_1| = \sum_{k=1}^{n} |S_k| = d^2 \) so \( \frac{n}{d} = \frac{d}{|S_1|} \geq (\log d)^4 \) by assumption.

**Remark 2.9.** When \( |S_1| = 1 \), this result recovers the classical estimate for the spectral norm of a standard Gaussian matrix. When \( |S_1| = d \), this result could fail. For example, take \( S_k = \{(i, j) \in [d] \times [d] | i - j \equiv k \mod d\} \) for \( k \in [d] \). Then \( A_k = A_1^k \), for all \( k \in [d] \), and \( X = \sum_{k=1}^{d} g_k A_k \) is a random circulant matrix. We have \( \| \sum_{k=1}^{d} g_k A_k \| = \| \sum_{k=1}^{d} g_k A_1^k \| = \sup_{w \in \mathbb{C}} | \sum_{k=1}^{d} g_k u^k | \) has expected value \( \sim \sqrt{d \log d} \). On the other hand, since \( A_k \) is a unitary for all \( k \in [d] \), we have \( \| \sum_{k=1}^{d} A_k^* A_k \|^{\frac{1}{2}} = \| \sum_{k=1}^{d} A_1^k A_1^k \|^{\frac{1}{2}} = \sqrt{d} \). Or if \( X \) is a random self-adjoint Toeplitz matrix where in each row, the entries are i.i.d. standard Gaussian entries, then the \( \sqrt{\log d} \) factor is also needed in this case, though \( |S_1|, \ldots, |S_d| \) are all different [13].

A particularly interesting case is when, for some \( r > 0 \), the partition \( \{S_1, \ldots, S_n\} \) of \([d] \times [d]\) satisfies, for all \( k \in [n] \),

1. \( |S_k| = r \);
2. \( S_k \) has at most one entry in each row of \([d] \times [d]\);
3. \( S_k \) has at most one entry in each column of \([d] \times [d]\).

For \( k \in [n] \), consider the matrix \( A_k \in M_d(\mathbb{R}) \) defined by \( (A_k)_{i,j} = \begin{cases} 1, & (i,j) \in S_k \\ 0, & \text{Otherwise} \end{cases} \). We have

\[
\left\| \sum_{k=1}^{n} A_k^* A_k \right\|^{\frac{1}{2}} = \left\| \sum_{k=1}^{n} A_k A_k^* \right\|^{\frac{1}{2}} = \sqrt{d}.
\]

Indeed, \( A_k^* A_k \) and \( A_k A_k^* \) are diagonal matrices for all \( k \in [n] \). For every \( r \in [d] \), their \( r \)th diagonal entries are

\[
\langle A_k^* A_k e_r, e_r \rangle = \| A_k e_r \|_2^2 = \begin{cases} 1, & S_k \text{ has one entry in the } r \text{th column} \\ 0, & \text{Otherwise} \end{cases}
\]

and

\[
\langle A_k A_k^* e_r, e_r \rangle = \| A_k^* e_r \|_2^2 = \begin{cases} 1, & S_k \text{ has one entry in the } r \text{th row} \\ 0, & \text{Otherwise} \end{cases}
\]

Since each row/column has \( d \) entries and each entry belongs to exactly one \( S_k \) (by assumption that \( \{S_1, \ldots, S_n\} \) is a partition), it follows that \( \sum_{k=1}^{n} \langle A_k^* A_k e_r, e_r \rangle = \sum_{k=1}^{n} \langle A_k A_k^* e_r, e_r \rangle = d \) for every \( r \in [d] \). So \( \| \sum_{k=1}^{n} A_k^* A_k \| = \| \sum_{k=1}^{n} A_k A_k^* \| = d \).
In this case, if \( r \leq \frac{d}{(\log d)^r} \), Corollary 2.5 implies that

(2.1) \[ \mathbb{E} \left\| \sum_{k=1}^{n} g_k A_k \right\| \sim \sqrt{d}, \]

where \( g_1, \ldots, g_n \) are i.i.d. standard Gaussian random variables. Again, we expect this to hold for \( r \leq \frac{d}{(\log d)^d} \) but were not able to prove it.

\section{Proof of the main theorem}

\subsection{Some estimations}

The first step is to prove Lemma 3.5, which is a result about real (random) matrices. However, it uses Lemma 3.4, which is over the complex, in an essential way. So the first two lemmas, which are needed to prove Lemma 3.5, involve both real and complex matrices. Let \( M_d(\mathbb{C}) \) be the space of all \( d \times d \) complex matrices.

**Lemma 3.1.** If \( \{B_1, \ldots, B_{d^2}\} \) is an orthonormal basis for \( M_d(\mathbb{R}) \), i.e., \( \text{Tr}(B_{k_1} B_{k_2}^*) = \begin{cases} 1, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases} \), then \( \sum_{k=1}^{d^2} B_k^{*} L B_k = \text{Tr}(L) I \) for all \( L \in M_d(\mathbb{C}) \).

**Proof.** Without loss of generality, \( L \in M_d(\mathbb{R}) \). Let \( g_1, \ldots, g_{d^2} \) be i.i.d. standard Gaussian random variables. Since

\[ \sum_{k=1}^{d^2} B_k^{*} L B_k = \mathbb{E} \left( \sum_{k=1}^{d^2} g_k B_k \right)^* \left( \sum_{k=1}^{d^2} g_k B_k \right) \]

and \( \sum_{k=1}^{d^2} g_k B_k \) is independent of the choice of the orthonormal basis \( \{B_1, \ldots, B_{d^2}\} \), the matrix \( \sum_{k=1}^{d^2} B_k^{*} L B_k \) is independent of the choice of the orthonormal basis \( \{B_1, \ldots, B_{d^2}\} \). We may take \( \{B_1, \ldots, B_{d^2}\} = \{e_i e_j^T | i, j \in [d]\} \). We have

\[ \sum_{k=1}^{d^2} B_k^{*} L B_k = \sum_{i=1}^{d} \sum_{j=1}^{d} e_i e_j^T L e_j = \left( \sum_{j=1}^{d} e_j^T L e_j \right) \sum_{i=1}^{d} e_i e_i^T = \text{Tr}(L) I. \]

\hfill \Box

**Lemma 3.2.** Suppose that \( Q_1, \ldots, Q_5 \in M_d(\mathbb{C}) \) are unitary, \( Y \in M_d(\mathbb{R}) \) is self-adjoint, \( A_1, \ldots, A_n \in M_d(\mathbb{R}) \) are self-adjoint matrices and \( \text{Tr}(A_{k_1} A_{k_2}) = 0 \) for all \( k_1 \neq k_2 \) in \([n]\). Then

\[ \left| \sum_{k_1, k_2 \in [n]} \text{Tr}(Q_1 Y^2 Q_2 A_{k_1} Q_3 A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2}) \right| \leq \left( \max_{k \in [n]} \| A_k \|_F \right)^2 \left\| \sum_{k=1}^{n} A_k^2 \right\| \text{Tr}(Y^2). \]

**Proof.** Without loss of generality, assume that \( A_k \neq 0 \) for all \( k \in [n] \). Let \( \beta = \max_{k \in [n]} \| A_k \|_F \). For each \( k \in [n] \), let \( \lambda_k = \| A_k \|_F \) and write \( A_k = \lambda_k B_k \). Then \( B_1, \ldots, B_n \) are orthonormal in \( M_d(\mathbb{R}) \). Extend \( B_1, \ldots, B_n \) to an orthonormal basis \( \{B_1, \ldots, B_{d^2}\} \) for \( M_d(\mathbb{R}) \). Note that \( B_{n+1}, \ldots, B_{d^2} \) are not necessarily self-adjoint. For a matrix \( D \in M_d(\mathbb{C}) \), define \( |D|^2 = D^* D \). We have

\[ \left| \sum_{k_1, k_2 \in [n]} \text{Tr}(Q_1 Y^2 Q_2 A_{k_1} Q_3 A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2}) \right| \]
\[
\begin{aligned}
&= \left| \sum_{k_1=1}^{n} \text{Tr} \left( (Y Q_2 A_{k_1} Q_3) \sum_{k_2=1}^{n} A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right| \\
&\leq \sum_{k_1=1}^{n} \left[ \text{Tr}(Y Q_2 A_{k_1}^2 Q_2 Y)^{\frac{1}{2}} \left( \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right) \right]^{\frac{1}{2}} \\
&\leq \left( \sum_{k_1=1}^{n} \text{Tr}(Y Q_2 A_{k_1}^2 Q_2 Y) \right)^{\frac{1}{2}} \left( \sum_{k_1=1}^{n} \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right) \left( \sum_{k_1=1}^{n} \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{k_1=1}^{n} \text{Tr}(Y Q_2 A_{k_1}^2 Q_2 Y) \right)^{\frac{1}{2}} \left( \sum_{k_1=1}^{n} \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 A_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{k_1=1}^{n} \text{Tr}(Y Q_2 A_{k_1}^2 Q_2 Y) \right)^{\frac{1}{2}} \left( \sum_{k_1=1}^{n} \beta^2 \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 B_{k_1} Q_5 A_{k_2} Q_1 Y \right) \right)^{\frac{1}{2}},
\end{aligned}
\]

where we use the cyclic property of the trace in the first equality, we use Cauchy-Schwarz inequality in the first and second inequalities, and we use the fact that \( A_k = \lambda_k B_k \) with \( 0 \leq \lambda_k \leq \beta \) and extend the sum over \( k_1 \) to \( 1, \ldots, d^2 \) in the last inequality.

For the first term,
\[
\sum_{k_1=1}^{n} \text{Tr}(Y Q_2 A_{k_1}^2 Q_2 Y) = \text{Tr} \left( Y Q_2 \left( \sum_{k_1=1}^{n} A_{k_1}^2 \right) Q_2 Y \right) \leq \left\| \sum_{k=1}^{n} A_k^2 \right\| \text{Tr}(Y^2).
\]

For the second term,
\[
\sum_{k_1=1}^{d^2} \beta^2 \text{Tr} \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 B_{k_1} Q_5 A_{k_2} Q_1 Y \right)^2 \\
= \beta^2 \sum_{k_1=1}^{d^2} \text{Tr} \left( \sum_{k_3=1}^{n} Y Q_1^* A_{k_1} Q_5^* B_{k_1} Q_4^* Q_1 A_{k_3} \right) \left( \sum_{k_2=1}^{n} A_{k_2} Q_4 B_{k_1} Q_5 A_{k_2} Q_1 Y \right)^2 \\
= \beta^2 \sum_{k_2, k_3 \in [n]} \text{Tr}(Q_1^* A_{k_1} A_{k_2} Q_4) \text{Tr}(Y Q_1^* A_{k_1} Q_5^* Q_5 A_{k_2} Q_1 Y) \\
= \beta^2 \sum_{k_2, k_3 \in [n]} \text{Tr}(A_{k_1} A_{k_2}) \text{Tr}(Y Q_1^* A_{k_1} A_{k_2} Q_1 Y) \\
= \beta^2 \sum_{k_1=1}^{n} \left\| A_k \right\|_F^2 \text{Tr}(Y Q_1^* A_k^2 Q_1 Y) \\
\leq \beta^4 \sum_{k_1=1}^{n} \text{Tr}(Y Q_1^* A_k^2 Q_1 Y) = \beta^4 \text{Tr} \left( Y Q_1^* \left( \sum_{k=1}^{n} A_k^2 \right) Q_1 Y \right) \leq \beta^4 \left\| \sum_{k=1}^{n} A_k^2 \right\| \text{Tr}(Y^2),
\]

where we expand the \( \ldots \) \( \|^2 \) in the first equality, rearrange the sums in the second equality, use Lemma \ref{lemma} in the third equality, use \( Q_1 Q_4^* = Q_5^* Q_5 = I \) in the fourth equality, use \( \text{Tr}(A_{k_1} A_{k_2}) = 0 \), for all \( k_1 \neq k_2 \), in the fifth equality and use \( \left\| A_k \right\|_F \leq \beta \) in the first inequality.
Therefore, the result follows. \qed

**Remark 3.3.** By modifying the proof of Lemma 3.2 slightly, one can see that if $A_1, \ldots, A_n \in M_d(\mathbb{R})$ are self-adjoint matrices and $\text{Tr}(A_k A_k) = 0$ for all $k_1 \neq k_2$ in $[n]$, then

$$\left| \sum_{k_1, k_2 \in [n]} \left( A_{k_1} Q_1 A_{k_2} Q_2 A_{k_1} Q_3 A_{k_2} v, v \right) \right| \leq \left( \max_{k \in [n]} \| A_k \|_F \right)^2 \left( \sum_{k=1}^n A_k^2 v, v \right),$$

for all $v \in \mathbb{R}^d$ and unitary $Q_1, Q_2, Q_3 \in M_d(\mathbb{C})$. Thus, in this case, the quantity $w(\sum_{k=1}^n g_k A_k)$, introduced in [IS], satisfies

$$\sup_{Q_1, Q_2, Q_3} \left\| \sum_{k_1, k_2 \in [n]} A_{k_1} Q_1 A_{k_2} Q_2 A_{k_1} Q_3 A_{k_2} \right\| \leq 2 \left( \max_{k \in [n]} \| A_k \|_F \right)^{\frac{1}{2}} \left\| \sum_{k=1}^n A_k^2 \right\|^{\frac{1}{2}}.$$

**Lemma 3.4 ([IS], Proposition 8.3).** Suppose that $F : (M_d(\mathbb{C}))^s \to \mathbb{C}$ is a multilinear function and $X_1, \ldots, X_s$ are random (not necessarily independent) self-adjoint matrices in $M_d(\mathbb{C})$ such that $\mathbb{E}\|X_i\|^{p_5} < \infty$ for all $i \in [s]$. Then

$$\| EF(X_1, \ldots, X_s) \| \leq \max_{j \in [s]} \mathbb{E} \max_{Q_1, \ldots, Q_s} |F(Q_1, \ldots, Q_{j-1}, Q_j X_j^s, Q_{j+1}, \ldots, Q_s)|,$$

where the second maximum is over all $d \times d$ (random) unitary matrices $Q_1, \ldots, Q_s$ in $M_d(\mathbb{C})$.

**Lemma 3.5.** Suppose that $A_1, \ldots, A_n \in M_d(\mathbb{R})$ are self-adjoint matrices and $\text{Tr}(A_k A_k) = 0$ for all $k_1 \neq k_2$ in $[n]$. Let $p_1 \leq \ldots \leq p_5$ in $\mathbb{N}$ with $p_5$ being even and let $X_1, \ldots, X_{p_5}$ be real random self-adjoint matrices such that $\mathbb{E}\|X_i\|^{p_5} < \infty$ for all $i \in [s]$. Then

$$\left| \mathbb{E} \sum_{k_1, k_2 \in [n]} \text{Tr} \left( \prod_{i=p_1+1}^{p_1} X_i A_{k_1} \prod_{i=p_2+1}^{p_2} X_i A_{k_2} \prod_{i=p_3+1}^{p_3} X_i A_{k_1} \prod_{i=p_4+1}^{p_4} X_i A_{k_2} \prod_{i=p_5+1}^{p_5} X_i \right) \right| \leq \left( \max_{k \in [n]} \| A_k \|_F \right)^2 \left\| \sum_{k=1}^n A_k^2 \right\| \max_{j \in [p_5]} \mathbb{E} \text{Tr}(X_j^{p_5}),$$

where empty products are the identity, e.g., $\prod_{i=p_1+1}^{p_1} X_i = I$.

**Proof.** Define $F : (M_d(\mathbb{C}))^{p_5} \to \mathbb{C}$ by

$$F(Y_1, \ldots, Y_{p_5}) = \sum_{k_1, k_2 \in [n]} \text{Tr} \left( \prod_{i=1}^{p_1} Y_i A_{k_1} \prod_{i=p_1+1}^{p_2} Y_i A_{k_2} \prod_{i=p_2+1}^{p_3} Y_i A_{k_1} \prod_{i=p_3+1}^{p_4} Y_i A_{k_2} \prod_{i=p_4+1}^{p_5} Y_i \right).$$

For all $j \in [p_5]$ and $d \times d$ unitary matrices $Q_1, \ldots, Q_{p_5}$, there exist $d \times d$ unitary matrices $Q'_1, \ldots, Q'_5$ such that

$$F(Q_1, \ldots, Q_{j-1}, Q_j X_j^{p_5}, Q_{j+1}, \ldots, Q_{p_5}) = \sum_{k_1, k_2 \in [n]} \text{Tr}(Q'_1 X_j^{p_5} Q'_2 A_{k_1} Q'_3 A_{k_2} Q'_4 A_{k_1} Q'_5 A_{k_2}),$$

by the cyclic property of the trace. So by Lemma 3.3,

$$\left| \mathbb{E}F(X_1, \ldots, X_{p_5}) \right| \leq \max_{j \in [p_5]} \mathbb{E} \max_{Q_1, \ldots, Q'_5} \left| \sum_{k_1, k_2 \in [n]} \text{Tr}(Q'_1 X_j^{p_5} Q'_2 A_{k_1} Q'_3 A_{k_2} Q'_4 A_{k_1} Q'_5 A_{k_2}) \right|.$$
where the second maximum is over all $d \times d$ (random) unitary matrices $Q_1', \ldots, Q_5'$. Thus, since $p_5$ is even, by Lemma 3.2

$$|EF(X_1, \ldots, X_{p_5})| \leq \left( \max_{k \in [n]} \| A_k \|_F \right)^2 \max_{j \in [p_5]} \| A_j^2 \| \| \text{ETr}(X_{p_5}^p) \|.$$ 

\[ \square \]

3.2. Tensor products. Suppose that $S$ is a finite set. If $\nu$ is a partition of $i, j \in S$, then $i \sim j$ means that $i$ and $j$ are in the same block of $\nu$. For partitions $\nu_1$ and $\nu_2$ of $S$, we write $\nu_1 \leq \nu_2$ if whenever $i \sim j$, we have $i \overset{\nu_1}{\sim} j$. For example, $\{\{1\}, \{2\}, \{3, 4\}\} \leq \{\{1, 2\}, \{3, 4\}\}$. For a partition $\nu$ of $S$, a subset $S_0$ of $S$ splits $\nu$ if whenever $i \sim j$ and $j \in S_0$, we have $i \in S_0$, or equivalently, $S_0$ is a union of blocks of $\nu$. For a function $f : S \to T$, where $T$ is a set, we write $f \sim \nu$ if whenever $i \sim j$ in $S$, we have $f(i) = f(j)$, or equivalently, $f$ is constant on each block of $\nu$.

A pair partition of $S$ is a partition where each block has exactly two elements. The set of all pair partitions of $S$ is denoted by $\mathbb{P}_2(S)$. Note that if $|S|$ is odd then $\mathbb{P}_2(S) = \emptyset$.

Suppose that $S$ is totally ordered. A partition $\nu$ of $S$ is noncrossing if whenever $i_1 < i_2 < i_3 < i_4$ in $S$ and $i_1 \sim i_3$ and $i_2 \sim i_4$, we have $i_1 \overset{\nu}{\sim} i_2 \sim i_3 \overset{\nu}{\sim} i_4$. The set of all noncrossing pair partitions of $S$ is denoted by $\text{NC}_2(S)$. A partition is crossing if it is not noncrossing. The set of all crossing pair partitions of $S$ is denoted by $\text{Cr}_2(S) = \mathbb{P}_2(S) \setminus \text{NC}_2(S)$.

In the following two lemmas, the tensor products are the usual multilinear tensor products for vector spaces.

**Lemma 3.6.** Suppose that $V$ is a vector space over $\mathbb{R}$, $A_1, \ldots, A_n \in V$ and $g_1, \ldots, g_n$ are i.i.d. standard Gaussian random variables. Let $X = \sum_{k=1}^n g_k A_k$ and $X^{\otimes p} = \bigotimes_{i=1}^n X$. Then

$$\mathbb{E}(X^{\otimes p}) = \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f \sim \nu} A_{f(1)} \otimes \ldots \otimes A_{f(p)}.$$

**Proof.** If $p$ is odd, then both sides are 0 by symmetry of $X$ and $\mathbb{P}_2([p]) = \emptyset$. It is easy to see that the result holds for $p = 2$. For an even number $p \geq 4$, by Gaussian integration by parts, we have

$$\mathbb{E}(X^{\otimes p}) = \sum_{k=1}^n \mathbb{E}g_k A_k \otimes X^{\otimes p-1} = \sum_{k=1}^n A_k \otimes \mathbb{E}X^{\otimes (j-1)} \otimes A_k \otimes X^{\otimes (p-1-j)}),$$

where when $j = 1$ or $p - 1$, the term $X^{\otimes 0}$ is not present. So applying induction hypothesis to $\mathbb{E}(X^{\otimes (j-1)} \otimes X^{\otimes (p-1-j)})$, we obtain

$$\mathbb{E}(X^{\otimes p}) = \sum_{j=1}^{p-1} \sum_{k=1}^n A_k \otimes (A_{h(2)} \otimes \ldots \otimes A_{h(j)}) \otimes A_k \otimes (A_{h(j+2)} \otimes \ldots \otimes A_{h(p)}$$

$$= \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f \sim \nu} A_{f(1)} \otimes \ldots \otimes A_{f(p)},$$

via the identification $\nu = \sigma \cup \{\{1, j + 1\}\}$ and $f(i) = \begin{cases} k, & i = 1 \text{ or } j + 1 \\ h(i), & \text{otherwise} \end{cases}$.
**Lemma 3.7.** Suppose that $V$ is a vector space over $\mathbb{R}$, $A_1, \ldots, A_n \in V$ and $g_1, \ldots, g_n$ are i.i.d. standard Gaussian random variables. Let $\sigma$ be a partition of $[p]$. Then there exist random variables $X_1, \ldots, X_p$ taking values in $V$ such that each individual $X_i$ has the same distribution over $V$ as $\sum_{k=1}^n g_k A_k$ and

$$
\sum_{\nu \in \mathbb{P}_2([p])} \sum_{\nu \leq \sigma} A_{f(1)} \otimes \ldots \otimes A_{f(p)} = \mathbb{E}(X_1 \otimes \ldots \otimes X_p).
$$

**Proof.** Without loss of generality, by permuting the order of the tensor product, we may assume that $\sigma$ is an interval partition of $[p]$. Write $\sigma = \{B_1, \ldots, B_r\}$ in the ascending order. Each partition $\nu \in \mathbb{P}_2([p])$ with $\nu \leq \sigma$ corresponds to partitions $\nu_1 \in \mathbb{P}_2(B_1), \ldots, \nu_r \in \mathbb{P}_2(B_r)$, via the correspondence $\nu \mapsto (\nu|_{B_1}, \ldots, \nu|_{B_r})$. Thus,

$$
\sum_{\nu \in \mathbb{P}_2([p])} \sum_{\nu \leq \sigma} A_{f(1)} \otimes \ldots \otimes A_{f(p)}
$$

$$
= \sum_{\nu_1 \in \mathbb{P}_2(B_1)} \ldots \sum_{\nu_r \in \mathbb{P}_2(B_r)} \sum_{f_1 \sim \nu_1} \ldots \sum_{f_r \sim \nu_r} \left( \bigotimes_{i \in B_1} A_{f_1(i)} \right) \otimes \ldots \otimes \left( \bigotimes_{i \in B_r} A_{f_r(i)} \right)
$$

$$
= \left( \sum_{\nu_1 \in \mathbb{P}_2(B_1)} \ldots \sum_{\nu_r \in \mathbb{P}_2(B_r)} \bigotimes_{i \in B_1} A_{f_1(i)} \right) \otimes \ldots \otimes \left( \sum_{\nu_r \in \mathbb{P}_2(B_r)} \bigotimes_{i \in B_r} A_{f_r(i)} \right),
$$

where $\otimes_i \in B_j$ is the tensor product in the ascending order of $B_j$; for example, if $B_1 = \{1, 2, 3\}$ then $\otimes_i \in B_1 A_{f_1(i)} = A_{f_1(1)} \otimes A_{f_1(2)} \otimes A_{f_1(3)}$. Suppose that $g_{k,j}$, for $k \in [n]$ and $j \in [r]$, are i.i.d. Gaussian random variables. By Lemma 3.6,

$$
\mathbb{E} \left( \sum_{k=1}^n g_{k,j} A_k \right) \otimes \vert B_j \vert = \sum_{\nu_j \in \mathbb{P}_2(B_j)} \sum_{f_j \sim \nu_j} \bigotimes_{i \in B_j} A_{f_j(i)},
$$

for every $j \in [r]$. Therefore,

$$
\sum_{\nu \in \mathbb{P}_2([p])} \sum_{\nu \leq \sigma} A_{f(1)} \otimes \ldots \otimes A_{f(p)}
$$

$$
= \mathbb{E} \left( \sum_{k=1}^n g_{k,1} A_k \right) \otimes \vert B_1 \vert \ldots \otimes \mathbb{E} \left( \sum_{k=1}^n g_{k,r} A_k \right) \otimes \vert B_r \vert
$$

$$
= \mathbb{E} \begin{bmatrix}
\left( \sum_{k=1}^n g_{k,1} A_k \right) \otimes \vert B_1 \vert \\
\ldots \\
\left( \sum_{k=1}^n g_{k,r} A_k \right) \otimes \vert B_r \vert
\end{bmatrix},
$$

where the last equality follows from independence of the $g_{k,j}$. For each $j \in [r]$ and each $i \in B_j$, take $X_i = \sum_{k=1}^n g_{k,j} A_k$. (The $X_i$ is the same for all $i$ in the same block.) The result follows. □
3.3. **Proof of the second statement of Theorem 1.2.**

**Lemma 3.8** ([], Corollary 3). Suppose that $A_1, \ldots, A_n \in M_d(\mathbb{R})$ are self-adjoint matrices. Then

\[
\left| \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)}) \right| \leq \text{Tr} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}},
\]

for all even number $p \in \mathbb{N}$ and $\nu \in \mathbb{P}_2([p])$.

**Lemma 3.9.** Suppose that $g_1, \ldots, g_n$ are i.i.d. standard Gaussian random variables, $A_1, \ldots, A_n \in M_d(\mathbb{R})$ are self-adjoint with nonnegative entries and $\text{Tr}(A_{k_1}A_{k_2}) = 0$ for all $k_1 \neq k_2$ in $[n]$. Let $X = \sum_{k=1}^{n} g_k A_k$, where $g_1, \ldots, g_n$ are i.i.d. Gaussian random variables. Then

\[
\mathbb{E} \text{Tr}(X^p) \leq 2^p \text{Tr} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}} + p^4 \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left\| \sum_{k=1}^{n} A_k^2 \right\| \mathbb{E} \text{Tr}(X^{p-4}),
\]

for all even number $p \geq 4$.

**Proof.** By Lemma 3.8,

\[
\mathbb{E}(X^p) = \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} A_{f(1)} \cdots A_{f(p)},
\]

so

\[
(3.1) \quad \mathbb{E} \text{Tr}(X^p) = \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)})
\]

\[
= \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)})
\]

\[
= \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)}) + \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)})
\]

\[
\leq 2^p \text{Tr} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}} + \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)}),
\]

where the last inequality follows from Lemma 3.8 and the fact that there are at most $2^p$ noncrossing pair partitions of $[p]$. For every $\nu \in \mathbb{P}_2([p])$, there exist $i_1 < i_2 < i_3 < i_4$ in $[p]$ such that \{i_1, i_3\}, \{i_2, i_4\} $\in \nu$. So

\[
(3.2) \quad \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)}) \leq \sum_{i_1 < \ldots < i_4 \text{ in } [p]} \sum_{\nu \in \mathbb{P}_2([p])} \sum_{\{i_1, i_3\}, \{i_2, i_4\} \in \nu} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)}).
\]

Note that this is only an inequality since it involves some overcounting. We have also used the assumption that the entries of $A_1, \ldots, A_n$ are nonnegative. Fix $i_1 < i_2 < i_3 < i_4$ in $[p]$. We have

\[
\sum_{\nu \in \mathbb{P}_2([p])} \sum_{\{i_1, i_3\}, \{i_2, i_4\} \in \nu} \sum_{f:[p] \to [n]} \text{Tr}(A_{f(1)} \cdots A_{f(p)})
\]
\[
= \sum_{k_1, k_2 \in \{n\}} \sum_{\sigma \in \mathbb{P}_2([p]) \setminus \{i_1, \ldots, i_4\}} \sum_{f \sim \sigma} \text{Tr}(A_{f(1)} \cdots A_{f(i_1 - 1)} A_k A_{f(i_1 + 1)} \cdots A_{f(i_2 - 1)} A_{k_2})\\
A_{f(i_1 + 1)} \cdots A_{f(i_3 - 1)} A_{k_1} A_{f(i_3 + 1)} \cdots A_{f(i_4 - 1)} A_{k_2} A_{f(i_4 + 1)} \cdots A_{f(p)}),
\]
via the identification \(k_1 = f(i_1) = f(i_3), k_2 = f(i_2) = f(i_4)\) and \(\sigma = \nu|_{\{i_1, \ldots, i_4\}}\). Thus, by Lemma 3.6

\[
\sum_{\nu \in \mathbb{P}_2([p])} \sum_{f : [p] \rightarrow [n]} \sum_{f \sim \nu} \text{Tr}(A_{f(1)} \cdots A_{f(p)})
\]

By (3.1), the result follows. □

Proof of the second statement of Theorem 1.2 Without loss of generality, we may assume that \(A_1, \ldots, A_n\) are self-adjoint by replacing each \(A_k\) by the self-adjoint matrix \(\begin{bmatrix} 0 & A_k \\ A_k^* & 0 \end{bmatrix}\).

By Lemma 3.9 for all even number \(4 \leq p \leq \log d\),

\[
\mathbb{E} \text{Tr}(X^p) \leq d \cdot 2^p \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}} + (\log d)^4 \left( \max_{k \in \{n\}} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \mathbb{E} \text{Tr}(X^{p-4}).
\]

Let \(b_1 = 2\|\sum_{k=1}^{n} A_k^2\|^{\frac{1}{2}}\) and \(b_2 = (\log d)^4 (\max_{k \in \{n\}} \|A_k\|_F^2) \sum_{k=1}^{n} A_k^2\|\). For \(p \leq \log d\), let \(a_p = \mathbb{E} \text{Tr}(X^p)\). Then \(a_p \leq d (b_1^p + b_2^{p-4} + b_2^{p-8} + \ldots + b_2^{p-2} b_1^2 + b_2^p)\), so by Young’s inequality, \(a_p \leq d \left( b_1^p + (b_1^p + b_2^{p-4}) \right)^{\frac{1}{p}}\). Since \(\mathbb{E} \|X\| \leq \left( \mathbb{E} \text{Tr}(X^p) \right)^{\frac{1}{p}} = a_p^{\frac{1}{p}}\), taking \(p\) to be the largest number divisible by 4 and such that \(p \leq \log d\), we obtain

\[
\mathbb{E} \|X\| \leq b_1 + b_2^\frac{1}{2} \approx \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{1}{2}} + (\log d) (\max_{k \in \{n\}} \|A_k\|_F^2)^{\frac{1}{2}} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{1}{4}}.
\]

But \((\log d) (\max_{k \in \{n\}} \|A_k\|_F^2)^{\frac{1}{2}} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{1}{4}} \leq \|\sum_{k=1}^{n} A_k^2\|^{\frac{1}{4}} + (\log d)^2 \max_{k \in \{n\}} \|A_k\|_F\). Thus, the result follows.

\(\square\)

3.4. Proof of the first statement of Theorem 1.2 Recall the notation at the beginning of Section 3.2

Lemma 3.10. Assume that \(p \in \mathbb{N}\) is even. There exists \(\phi : \mathbb{C}_2([p]) \rightarrow \{\text{Partitions of } [p]\}\) such that

(1) \(\nu \leq \phi(\nu)\) for all \(\nu \in \mathbb{C}_2([p])\);
(2) whenever \(\nu, \tilde{\nu} \in \mathbb{C}_2([p])\) satisfy \(\tilde{\nu} \leq \phi(\nu)\), we have \(\phi(\nu) = \phi(\tilde{\nu})\);
(3) for every \(\sigma \in \mathfrak{R} \phi\), there exist \(i_1 < i_2 < i_3 < i_4\) in \([p]\) such that \(\{i_1, i_3\}, \{i_2, i_4\} \in \sigma\);
(4) \(\mathfrak{R} \phi\) has at most \(4p^2\) elements.
Proof. For \( \nu \in \text{Cr}_2([p]) \) and \( k \in [p] \), let
\[
S(\nu, k) = \{ j \in [p] \mid j \sim k \text{ for some } i \in [k] \}.
\]
Clearly \( S(\nu, k) \) splits \( \nu \) for all \( k \in [p] \) and \( \nu \in \text{Cr}_2([p]) \).

For every \( \nu \in \text{Cr}_2([p]) \), let \( k_\nu \) be the largest \( k \in [p] \) for which \( \nu|_{S(\nu, k)} \) is noncrossing. Take
\[
\phi(\nu) = (\nu|_{S(\nu, k_\nu + 1)}) \cup \{ [p] \setminus S(\nu, k_\nu + 1) \},
\]
for \( \nu \in \text{Cr}_2([p]) \).

(1): Since \( S(\nu, k_\nu + 1) \) splits \( \nu \), we have \( \nu \leq \phi(\nu) \) for all \( \nu \in \text{Cr}_2([p]) \). This proves (1).

(2): Suppose that \( \nu, \nu \in \text{Cr}_2([p]) \) and \( \nu \leq \phi(\nu) \). Then
\[
\nu \leq (\nu|_{S(\nu, k_\nu + 1)}) \cup \{ [p] \setminus S(\nu, k_\nu + 1) \} \leq \{ S(\nu, k_\nu + 1), [p] \setminus S(\nu, k_\nu + 1) \}.
\]
Thus \( S(\nu, k_\nu + 1) \) splits \( \nu \). Taking restriction to \( S(\nu, k_\nu + 1) \) in the first inequality, we obtain
\[
\nu|_{S(\nu, k_\nu + 1)} \leq \nu|_{S(\nu, k_\nu + 1)}.
\]
Since \( S(\nu, k_\nu + 1) \) splits each of \( \nu \) and \( \nu \) and since each of \( \nu \) and \( \nu \) are pair partitions, the restrictions \( \nu|_{S(\nu, k_\nu + 1)} \) and \( \vee|_{S(\nu, k_\nu + 1)} \) are still pair partitions. Thus, the only way \( \nu|_{S(\nu, k_\nu + 1)} \leq \nu|_{S(\nu, k_\nu + 1)} \) can happen is when \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \).

So we have \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \).

To show that \( \phi(\nu) = \phi(\nu) \), it remains to show that \( S(\nu, k_\nu + 1) = S(\nu, k_\nu + 1) \). First we show that
\[
(3.3) \quad S(\nu, k) = S(\nu, k) \quad \text{for all } k \in [k_\nu + 1].
\]
Recall that we have proved that \( S(\nu, k_\nu + 1) \) splits \( \nu \) and \( \nu|_{S(\nu, k_\nu + 1)} \). We will use repeatedly use these in the next few paragraphs.

Fix \( k \in [k_\nu + 1] \). If \( j \in S(\nu, k) \), i.e., \( j \sim i \) for some \( i \in [k] \), then \( i \in [k_\nu + 1] \subset S(\nu, k_\nu + 1) \) so since \( S(\nu, k_\nu + 1) \) splits \( \nu \), it follows that \( j \in S(\nu, k_\nu + 1) \). Since \( i, j \in S(\nu, k_\nu + 1) \), \( j \sim i \) and \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \), we have \( j \sim i \). So \( j \in S(\nu, k) \). Thus, \( S(\nu, k) \subset S(\nu, k) \).

Conversely, if \( j \in S(\nu, k) \) then \( j \sim i \) for some \( i \in [k] \). Thus \( i \in [k_\nu + 1] \) so \( j \in S(\nu, k_\nu + 1) \) by definition of \( S(\nu, k_\nu + 1) \). Since \( i, j \in S(\nu, k_\nu + 1) \), \( j \sim i \) and \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \), we have \( j \sim i \). So \( j \in S(\nu, k) \). Thus, \( S(\nu, k) \subset S(\nu, k) \). This proves (3.3).

Since \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \), we have \( \nu|_{S(\nu, k)} = \nu|_{S(\nu, k)} \), for all \( k \in [k_\nu + 1] \), since \( S(\nu, k) \subset S(\nu, k_\nu + 1) \). So by (3.3), \( \nu|_{S(\nu, k)} = \nu|_{S(\nu, k)} \), for all \( k \in [k_\nu + 1] \), where the restriction on the left hand side becomes \( S(\nu, k) \). Thus, by definition of \( k_\nu \), we have that \( \nu|_{S(\nu, k_\nu)} = \nu|_{S(\nu, k_\nu)} \) is noncrossing and \( \nu|_{S(\nu, k_\nu + 1)} = \nu|_{S(\nu, k_\nu + 1)} \) is crossing. So by definition of \( k_\nu \), we have \( k_\nu = k_\nu \). So
\[
S(\nu, k_\nu + 1) = S(\nu, k_\nu + 1) = S(\nu, k_\nu + 1),
\]
by (3.3). This proves the remaining thing needed to obtain \( \phi(\nu) = \phi(\nu) \) as mentioned above.

(3): Let \( \nu \in \text{Cr}_2([p]) \). By definition of \( k_\nu \), the partition \( \nu|_{S(\nu, k_\nu + 1)} \) is crossing. Since \( S(\nu, k_\nu + 1) \) splits \( \nu \) and \( \nu \) is a pair partition, \( \nu|_{S(\nu, k_\nu + 1)} \) is still a pair partition. Thus, \( \nu|_{S(\nu, k_\nu + 1)} \) is a crossing pair partition. So there exist \( i_1 < i_2 < i_3 < i_4 \) in \( [p] \) such that \( \{ i_1, i_3 \}, \{ i_2, i_4 \} \in \nu|_{S(\nu, k_\nu + 1)} \). So \( \{ i_1, i_3 \}, \{ i_2, i_4 \} \in \phi(\nu) \). This proves (3).

(4): For every \( \nu \in \text{Cr}_2([p]) \),
\[
\phi(\nu) = (\nu|_{S(\nu, k_\nu)}) \cup (\nu|_{S(\nu, k_\nu + 1)} \setminus S(\nu, k_\nu)) \cup \{ [p] \setminus S(\nu, k_\nu + 1) \}.
\]
Since \( \nu \) is a pair partition, \( S(\nu, k + 1) \setminus S(\nu, k) \) has at most 2 elements for every \( k \in [p] \), namely, \( k + 1 \) and another one in the same \( \nu \)-block as \( k + 1 \).
Lemma 3.12. This follows from Lemma 3.10 by enumerating the range of 
Proof. 

(3)

\( q \) for every \( t \)

Assume that \( q \) are self-adjoint and \( \) for all even number \( p \).

There are at most 2 \( p \) possible noncrossing pair partitions \( \nu \).

Therefore, there are at most 2 \( p \) - \( 2p \) partitions of the form \( \nu \) for some \( \nu \in \text{Cr}_2([p]) \). □

Lemma 3.11. Assume that \( p \in \mathbb{N} \) is even. There exist partitions \( \nu_1, \ldots, \nu_q \) of \( [p] \) such that

(1) every \( \nu \in \text{Cr}_2([p]) \) is in exactly one of the sets

\( \{ \nu \in \text{Cr}_2([p]) | \nu \leq \nu_1 \}, \ldots, \{ \nu \in \text{Cr}_2([p]) | \nu \leq \nu_q \} \).

(2) for every \( t \in [q] \), there exist \( i_1 < i_2 < i_3 < i_4 \) in \( [p] \) such that \( \{ i_1, i_3 \}, \{ i_2, i_4 \} \in \nu_t \).

(3) \( q \leq 4p^2 \).

Proof. This follows from Lemma 3.10 by enumerating the range of \( \phi \) as \( \nu_1, \ldots, \nu_q \).

Lemma 3.12. Suppose that \( g_1, \ldots, g_n \) are i.i.d. standard Gaussian random variables, \( A_1, \ldots, A_n \in M_d(\mathbb{R}) \) are self-adjoint and \( \text{Tr}(A_{k_1}A_{k_2}) = 0 \) for all \( k_1 \neq k_2 \) in \([n]\). Let \( X = \sum_{k=1}^{n} g_k A_k \), where \( g_1, \ldots, g_n \) are i.i.d. Gaussian random variables. Then

\[
\mathbb{E}(X^p) = \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f: \nu \to [n]} A_{f(1)} \ldots A_{f(p)},
\]

so

\[
(3.4) \quad \mathbb{E}(X^p) \leq 2^p \text{Tr} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}} + 8^p \left( \max_{i \in [n]} \| A_i \|_F \right)^2 \sum_{k=1}^{n} A_k^2 \left\| \text{Tr}(X^{p-4}) \right\|,
\]

for all even number \( p \geq 4 \).

Proof. By Lemma 3.6

\[
\mathbb{E}(X^p) \leq \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)})
\]

\[
\leq \sum_{\nu \in \mathbb{P}_2([p])} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)}) + \sum_{\nu \in \text{Cr}_2([p])} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)})
\]

\[
\leq 2^p \text{Tr} \left( \sum_{k=1}^{n} A_k^2 \right)^{\frac{p}{2}} + \sum_{\nu \in \text{Cr}_2([p])} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)})
\]

where the last inequality follows from Lemma 3.8 and the fact that there at most 2 \( p \) noncrossing pair partitions of \([p]\). Let \( \nu_1, \ldots, \nu_q \) be obtained from Lemma 3.11 with \( q \leq 4p^2 \). Since every \( \nu \in \text{Cr}_2([p]) \) satisfies \( \nu \leq \nu_t \) for exactly one \( t \in [q] \),

\[
(3.5) \quad \sum_{\nu \in \text{Cr}_2([p])} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)}) = \sum_{t=1}^{q} \sum_{\nu \leq \nu_t} \sum_{f: \nu \to [n]} \text{Tr}(A_{f(1)} \ldots A_{f(p)}).
\]
Fix \( t \in [q] \). By the properties of \( \nu \) from Lemma 3.11 there exist \( i_1 < i_2 < i_3 < i_4 \) in \([p]\) such that \( \{i_1,i_3\}, \{i_2,i_4\} \in \nu \). For every \( \nu \in \mathcal{C}_2([p]) \) such that \( \nu \leq \nu \), since \( \{i_1,i_3\}, \{i_2,i_4\} \in \nu \) and \( \nu \) is a pair partition, we have \( \{i_1,i_3\}, \{i_2,i_4\} \in \nu \). Let \( \omega \) be the partition, we have \( \{i_1,i_3\}, \{i_2,i_4\} \in \nu \). We have

\[
\sum_{\nu \in \mathcal{C}_2([p])} \sum_{f:|p| \to [n]} \Tr(A_{f(1)} \cdots A_{f(p)})
\]

\[
= \sum_{k_1,k_2 \in [n]} \sum_{\sigma \in \mathcal{P}_2([p]) \setminus \{i_1,\ldots,i_4\}} \Tr(A_{f(1)} \cdots A_{f(i_1-1)} A_{k_1} A_{f(i_1+1)} \cdots A_{f(i_2-1)} A_{k_2} A_{f(i_2+1)} \cdots A_{f(i_3-1)} A_{k_1} A_{f(i_3+1)} \cdots A_{f(i_4-1)} A_{k_2} A_{f(i_4+1)} \cdots A_{f(p)}),
\]

via the identification \( k_1 = f(i_1) = f(i_3) \), \( k_2 = f(i_2) = f(i_4) \) and \( \sigma = \nu \). Thus, by Lemma 3.7,

(3.6) \[
\sum_{\nu \in \mathcal{C}_2([p])} \sum_{f:|p| \to [n]} \Tr(A_{f(1)} \cdots A_{f(p)})
\]

\[
= \sum_{k_1,k_2 \in [n]} \mathbb{E} \Tr \left( \prod_{i=1}^{i_1-1} X_{i_1} A_{k_1} \prod_{i=i_1+1}^{i_2-1} X_i A_{k_1} \prod_{i=i_2+1}^{i_3-1} X_i A_{k_2} \prod_{i=i_3+1}^{i_4-1} X_i A_{k_2} \prod_{i=i_4+1}^{p} X_i \right),
\]

for some random matrices \( X_1, \ldots, X_p \) (with \( X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4} \) skipped) in \( M_n(\mathbb{R}) \) such that each individual \( X_i \) has the same distribution as \( X = \sum_{k=1}^{n} g_k A_k \). By Lemma 3.5 the absolute value of the expression (3.6) is at most \( \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \mathbb{E} \Tr(X^{p-4}) = \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \mathbb{E} \Tr(X^{p-4}) \), since each \( X_j \) has the same distribution as \( X \). Thus, by (3.5),

\[
\sum_{\nu \in \mathcal{C}_2([p])} \sum_{f:|p| \to [n]} \Tr(A_{f(1)} \cdots A_{f(p)}) \leq q \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \mathbb{E} \Tr(X^{p-4}).
\]

Since \( q \leq 4^p p^2 \leq 8^p \), by (3.4), the result follows.

\[ \square \]

Proof of the first statement of Theorem 1.2 Without loss of generality, we may assume that \( A_1, \ldots, A_n \) are self-adjoint by replacing each \( A_k \) by the self-adjoint matrix \( \begin{bmatrix} 0 & A_k \\ A_k & 0 \end{bmatrix} \). Fix \( \epsilon > 0 \). By Lemma 3.12 for all even number \( 4 \leq p \leq \epsilon \log_8 d \),

\[
\mathbb{E} \Tr(X^p) \leq d \cdot 2^p \left( \sum_{k=1}^{n} A_k^2 \right)^\frac{p}{2} + d^p \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \mathbb{E} \Tr(X^{p-4}).
\]

Let \( b_1 = 2 \left( \sum_{k=1}^{n} A_k^2 \right)^\frac{p}{2} \) and \( b_2 = d^p \left( \max_{k \in [n]} \|A_k\|_F \right)^2 \left( \sum_{k=1}^{n} A_k^2 \right) \). For \( p \leq \epsilon \log_8 d \), let \( a_p = \mathbb{E} \Tr(X^p) \). Then \( a_p \leq d^{p} b_1 + 2b_2a_{p-4} \), for all even number \( 4 \leq p \leq \epsilon \log_8 d \), and \( a_0 = d \). Thus, for all \( p \leq \epsilon \log_8 d \) with \( p \) divisible by \( 4 \), we have

\[
a_p \leq d^{b_1} + (b_1 + b_2)^p + b_2 b_1^{p-8} + \ldots + b_2^{p-1} b_1 + b_2^p,
\]

so by Young’s inequality, \( a_p \leq d^{\left( \frac{p}{4} + 1 \right)}(b_1 + b_2^p) \). Since \( \mathbb{E} \|X\| \leq \left( \mathbb{E} \Tr(X^p) \right)^\frac{1}{p} = a_p^\frac{1}{p} \), taking \( p \) to be the largest number divisible by \( 4 \) and such that \( p \leq \epsilon \log_8 d \), we obtain

\[
\mathbb{E} \|X\| \lesssim b_1 + b_2^\frac{1}{p} \lesssim \left( \sum_{k=1}^{n} A_k^2 \right)^\frac{1}{p} + d^\frac{p}{4} \left( \max_{i \in [n]} \|A_i\|_F \right)^\frac{p}{4} \left( \sum_{k=1}^{n} A_k^2 \right)^\frac{1}{p}.
\]
But \( d \hat{r} (\max_{i \in [n]} \| A_i \|_F)^{\frac{1}{2}} \geq \sum_{k=1}^{n} \| A_k \|_F^{\frac{1}{2}} \leq \| \sum_{k=1}^{n} A_k \|_F^{\frac{1}{2}} + d \hat{r} \max_{i \in [n]} \| A_i \|_F \). Thus, the result follows.

4. Sample covariance matrix

**Theorem 4.1.** Suppose that \( X_1, \ldots, X_M \) are \( d \times d \) independent random matrices and for each \( r \in [M] \), the entries of \( X_r \) are jointly Gaussian entries and \( \mathbb{E} X_r = 0 \). Then

\[
\mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \mathbb{E}(X_r^* X_r)) \right\| \lesssim \epsilon \left( \sum_{r=1}^{M} \left( \| \mathbb{E}(X_r^* X_r) \|^2 + \| \mathbb{E}(X_r X_r^*) \|^2 + d^\epsilon \| \mathbb{E}(X_r \otimes X_r) \|^2 \right) \right)^{\frac{1}{2}},
\]

for all \( \epsilon > 0 \).

**Proof.** For each \( r \in [M] \), let \( \tilde{X}_r \) be an independent copy of \( X_r \) so that \( X_1, \ldots, X_M, \tilde{X}_1, \ldots, \tilde{X}_M \) are independent. Then

\[
(4.1) \quad \mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \mathbb{E}(X_r^* X_r)) \right\| \leq \mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \tilde{X}_r^* \tilde{X}_r) \right\|.
\]

For every \( r \in [M] \),

\[
X_r X_r^* - \tilde{X}_r \tilde{X}_r^* = \int_0^{\frac{\pi}{2}} \frac{d}{d\theta} ([\tilde{X}_r \cos \theta + X_r \sin \theta]^* (\tilde{X}_r \cos \theta + X_r \sin \theta)) d\theta
\]

\[
= \int_0^{\frac{\pi}{2}} (\tilde{X}_r \sin \theta + X_r \cos \theta)^* (\tilde{X}_r \cos \theta + X_r \sin \theta) d\theta
\]

\[
+ \int_0^{\frac{\pi}{2}} (\tilde{X}_r \cos \theta + X_r \sin \theta)^* (\tilde{X}_r \sin \theta + X_r \cos \theta) d\theta,
\]

and since \( X_r, \tilde{X}_r \) are independent and have the same centered Gaussian distribution on \( M_d(\mathbb{R}) \), the pair of random matrices \( (\tilde{X}_r, X_r) \) has the same distribution as \( (X_r, \tilde{X}_r) \) for every \( t \in [0, \frac{\pi}{2}] \). Thus,

\[
(4.2) \quad \mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \tilde{X}_r^* \tilde{X}_r) \right\| \leq \frac{\pi}{2} \mathbb{E} \left\| \sum_{r=1}^{M} X_r^* \tilde{X}_r \right\| + \frac{\pi}{2} \mathbb{E} \left\| \sum_{r=1}^{M} \tilde{X}_r^* X_r \right\| = \pi \mathbb{E} \left\| \sum_{r=1}^{M} \tilde{X}_r^* X_r \right\|.
\]

Fix deterministic \( D_1, \ldots, D_M \in M_d(\mathbb{R}) \). Then \( \sum_{r=1}^{M} D_r X_r \) has jointly Gaussian entries and mean 0. So by Theorem [1.1]

\[
\mathbb{E} \left\| \sum_{r=1}^{M} D_r X_r \right\|
\]

\[
\lesssim \epsilon \left\| \mathbb{E} \left( \sum_{r=1}^{M} D_r X_r \right) \right\| + \mathbb{E} \left( \sum_{r=1}^{M} D_r X_r \right) \left( \sum_{r=1}^{M} D_r X_r \right)^*
\]

\[
= \left\| \sum_{r=1}^{M} \mathbb{E}(X_r^* D_r^* D_r X_r) \right\| + \left\| \sum_{r=1}^{M} \mathbb{E}(D_r^* X_r^* X_r D_r) \right\| + d^\epsilon \left\| \sum_{r=1}^{M} \mathbb{E}(D_r X_r \otimes D_r X_r) \right\|
\]
\[ \leq 2 \left( \sum_{r=1}^{M} \| D_r \|^2 \mathbb{E} \| X_r \|^2 \right)^{\frac{1}{2}} + d^\epsilon \left( \sum_{r=1}^{M} \| \mathbb{E}[(D_r X_r) \otimes (D_r X_r)] \| \right)^{\frac{1}{2}} \]

But for every deterministic \( D \in M_d(\mathbb{R}) \) and every random matrix \( X \in M_d(\mathbb{R}) \) with \( \mathbb{E} \| X \|^2 < \infty \),

\[ \| \mathbb{E}[(DX) \otimes (DX)] \| = \sup_{B \in M_d(\mathbb{R})} \mathbb{E}[\text{Tr}(DXB^*)]^2 \]

\[ = \sup_{B \in M_d(\mathbb{R}) \atop \| B \|_F \leq 1} \mathbb{E}[\text{Tr}(XB^*)]^2 \]

\[ \leq \| D \|^2 \sup_{B \in M_d(\mathbb{R}) \atop \| B \|_F \leq 1} \mathbb{E}[\text{Tr}(XB^*)]^2 = \| D \|^2 \mathbb{E} \| X \otimes X \|. \]

Therefore,

\[ \mathbb{E} \left\| \sum_{r=1}^{M} X_r D_r \right\| \lesssim \epsilon \left( \sum_{r=1}^{M} \| D_r \|^2 \mathbb{E} \| X_r \|^2 \right)^{\frac{1}{2}} + d^\epsilon \left( \sum_{r=1}^{M} \| D_r \|^2 \mathbb{E} \| X_r \otimes X_r \| \right)^{\frac{1}{2}}. \]

for all deterministic \( D_1, \ldots, D_M \in M_d(\mathbb{R}) \). So

\[ \mathbb{E} \left\| \sum_{r=1}^{M} X_r \bar{X}_r \right\| \lesssim_{\epsilon} \left( \sum_{r=1}^{M} \mathbb{E} \| X_r \|^2 \mathbb{E} \| X_r \|^2 \right)^{\frac{1}{2}} + d^\epsilon \left( \sum_{r=1}^{M} \mathbb{E} \| \bar{X}_r \|^2 \mathbb{E} \| X_r \otimes X_r \| \right)^{\frac{1}{2}} \]

\[ = \left( \sum_{r=1}^{M} (\mathbb{E} \| X_r \|^2)^2 \right)^{\frac{1}{2}} + \left( \sum_{r=1}^{M} \mathbb{E} \| X_r \|^2 d^\epsilon \mathbb{E} \| X_r \otimes X_r \| \right)^{\frac{1}{2}} \]

\[ \leq \left( \sum_{r=1}^{M} (\mathbb{E} \| X_r \|^2)^2 \right)^{\frac{1}{2}} + \left( \sum_{r=1}^{M} (\mathbb{E} \| X_r \|^2)^2 + d^\epsilon \mathbb{E} \| X_r \otimes X_r \|^2 \right)^{\frac{1}{2}}. \]

By a Gaussian version of Kahane’s inequality \cite{10} or by concentration of \( \| X \| \), we have \( \mathbb{E} \| X \|^2 \sim (\mathbb{E} \| X \|)^2 \). Therefore,

\[ \mathbb{E} \left\| \sum_{r=1}^{M} X_r \bar{X}_r \right\| \lesssim_{\epsilon} \left( \sum_{r=1}^{M} ((\mathbb{E} \| X_r \|)^4 + d^\epsilon \mathbb{E} \| X_r \otimes X_r \|)^2 \right)^{\frac{1}{2}}. \]

So by \ref{4.1} and \ref{4.2},

\[ \mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \mathbb{E}(X_r^* X_r)) \right\| \lesssim_{\epsilon} \left( \sum_{r=1}^{M} ((\mathbb{E} \| X_r \|)^4 + d^\epsilon \mathbb{E} \| X_r \otimes X_r \|)^2 \right)^{\frac{1}{2}}. \]

Thus, by Theorem \ref{1.1} the result follows. \hfill \Box

**Corollary 4.2.** Suppose that \( \mu \) is a probability measure on \( \{ B \in M_d(\mathbb{R}) \mid B \text{ is positive semidefinite} \} \) and \( \text{Tr}(B) \geq \max(d_1^2, d_2^2)\| B \| \) \( \mu \)-a.s. Let \( M \in \mathbb{N} \). Let \( z_1, \ldots, z_{Md_1} \) be i.i.d. random vectors in \( \mathbb{R}^{d_1} \) chosen according to \( \int \mathcal{N}(0, B) d\mu(B) \), i.e., \( \mathbb{P}(z_1 \in \mathcal{S}) = \int \mathbb{P}(B \in \mathcal{S}) d\mu(B) \) for all measurable \( \mathcal{S} \subset \mathbb{R}^{d_1} \), where \( g \) is a standard Gaussian on \( \mathbb{R}^{d_2} \). Then

\[ \mathbb{E} \left\| \frac{1}{Md_1} \sum_{i=1}^{Md_1} z_i z_i^T - \int B d\mu(B) \right\|. \]
\[
\frac{1}{d_1 \sqrt{M}} \left( d_1 \left\| \int B \, d\mu(B) \right\| + \left( \mathbb{E} \max_{i \in [d_1]} \left[ \text{Tr}(B_i) \right]^2 \right)^{\frac{1}{2}} + \sqrt{d_1 \log d_2} \left\| \int B^2 \, d\mu(B) \right\|^{\frac{1}{2}} \right),
\]
where \( B_1, \ldots, B_{d_1} \) in \( M_{d_2}(\mathbb{R}) \) are i.i.d. chosen according to \( \mu \).

Proof. Fix \( B_1, \ldots, B_{M_{d_1}} \in M_{d_2}(\mathbb{R}) \). For each \( r \in [M] \), let \( X_r \) be a \( d_1 \times d_2 \) random matrix with independent rows such that for \( i \in [d_1] \), the \( i \)th row of \( X_r \) is a centered Gaussian random vector with covariance matrix \( B_i + (r-1)d_1 \in M_{d_2}(\mathbb{R}) \). Then using computations from the proof of Corollary 2.4 we have, by Theorem 4.3:

\[
\mathbb{E} \left\| \sum_{r=1}^{M} (X_r^* X_r - \mathbb{E}(X_r^* X_r)) \right\| \lesssim \epsilon \left[ \sum_{r=1}^{M} \left( \left\| \sum_{i=1}^{d_1} B_{i + (r-1)d_1} \right\|^2 + \max_{i \in [d_1]} \left[ \text{Tr}(B_{i + (r-1)d_1}) \right]^2 + \max_{i \in [d_1]} \left( d_1, d_2 \right) \max_{i \in [d_1]} \left\| B_{i + (r-1)d_1} \right\|^2 \right)^{\frac{1}{2}} \right].
\]

By assumption, \( z_1, \ldots, z_{M_{d_1}} \) are chosen as follows: first, choose i.i.d. \( B_1, \ldots, B_{M_{d_1}} \) in \( M_{d_2}(\mathbb{R}) \) according to \( \mu \) and then for each \( i \in [M_{d_1}] \), take \( z_i \sim \mathcal{N}(0, B_i) \) Thus, conditioning on \( B_1, \ldots, B_{M_{d_1}} \), we have

\[
\mathbb{E} \left( \left\| \sum_{i=1}^{M_{d_1}} (z_i z_i^T - B_i) \right\| \mid B_1, \ldots, B_{M_{d_1}} \right) \lesssim \epsilon \left( \mathbb{E} \left\| \sum_{i=1}^{d_1} B_i \right\|^2 + \mathbb{E} \max_{i \in [d_1]} \left[ \text{Tr}(B_i) \right]^2 \right)^{\frac{1}{2}}.
\]

Since by assumption, \( \text{Tr}(B) \geq \max(d_1, d_2) \| B \| \) a.s., So

\[
\mathbb{E} \left( \left\| \sum_{i=1}^{M_{d_1}} (z_i z_i^T - B_i) \right\| \right) \lesssim \epsilon \sqrt{M} \left( \mathbb{E} \left\| \sum_{i=1}^{d_1} B_i \right\|^2 + \mathbb{E} \max_{i \in [d_1]} \left[ \text{Tr}(B_i) \right]^2 \right)^{\frac{1}{2}}.
\]

By a modified version of [17] Theorem 5.1(1),

\[
\mathbb{E} \left( \sum_{i=1}^{d_1} B_i \right)^2 \lesssim \sum_{i=1}^{d_1} \mathbb{E} B_i^2 + (\log d_2)^2 \mathbb{E} \max_{i \in [d_1]} \left\| B_i \right\|^2 = d_2 \left( \int B \, d\mu(B) \right)^2 + (\log d_2)^2 \mathbb{E} \max_{i \in [d_1]} \left\| B_i \right\|^2.
\]

But \( \text{Tr}(B) \geq d_2 \| B \| \) a.s. Therefore,

\[
\mathbb{E} \left( \left\| \sum_{i=1}^{M_{d_1}} (z_i z_i^T - B_i) \right\| \right) \lesssim \epsilon \sqrt{M} \left( d_1 \left\| \int B \, d\mu(B) \right\| + \left( \mathbb{E} \max_{i \in [d_1]} \left[ \text{Tr}(B_i) \right]^2 \right)^{\frac{1}{2}} \right).
\]

But by [17] Theorem 5.1(2),

\[
\mathbb{E} \left( \left\| \sum_{i=1}^{M_{d_1}} (B_i - \mathbb{E} B_i) \right\| \right) \lesssim \sqrt{M_{d_1} \log d_2} \left\| \int B^2 \, d\mu(B) \right\|^{\frac{1}{2}} + (\log d_2) \left( \mathbb{E} \max_{i \in [M_{d_1}]} \left\| B_i - \mathbb{E} B_i \right\|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \sqrt{M_{d_1} \log d_2} \left\| \int B^2 \, d\mu(B) \right\|^{\frac{1}{2}} + (\log d_2) \left( M \mathbb{E} \max_{i \in [d_1]} \left\| B_i \right\|^2 \right)^{\frac{1}{2}}.
\]
\[ \sum_{i=1}^{Md_1} (z_i z_i^T - \mathbb{E} B_i) \leq \varepsilon \sqrt{M} \left( d_1 \left\| \int B d\mu(B) \right\| + \left( \mathbb{E} \max_{i \in [d_1]} [\text{Tr}(B_i)]^2 \right)^{\frac{1}{2}} + \sqrt{d_1 \log d_2} \left\| \int B^2 d\mu(B) \right\| \right). \]

Since \( \text{Tr}(B) \geq d_2 \| B \| \mu\text{-a.s.} \), Therefore,
\[ \mathbb{E} \left\| \sum_{i=1}^{Md_1} (z_i z_i^T - \mathbb{E} B_i) \right\| \leq \varepsilon \sqrt{M} \left( d_1 \left\| \int B d\mu(B) \right\| + \left( \mathbb{E} \max_{i \in [d_1]} [\text{Tr}(B_i)]^2 \right)^{\frac{1}{2}} + \sqrt{d_1 \log d_2} \left\| \int B^2 d\mu(B) \right\| \right). \]

Thus, the result follows.

**Corollary 4.3.** Suppose that \( \mu \) is a probability measure on \( \{ B \in M_{d_2}(\mathbb{R}) \mid B \text{ is positive semidefinite} \} \) and \( \text{Tr}(B) \leq t \) and \( \| B \| \leq \frac{t}{\max(d_1', d_2')} \) \( \mu\text{-a.s.} \), where \( t \in [d_2] \) is fixed. Let \( M \in \mathbb{N} \).

Let \( z_1, \ldots, z_{Md_1} \) be i.i.d. random vectors in \( \mathbb{R}^{d_2} \) chosen according to \( \int N(0, B) d\mu(B) \), i.e.,
\[ \mathbb{P}(z_1 \in S) = \mathbb{P}(B \frac{1}{2} g \in S) d\mu(B) \text{ for all measurable } S \subset \mathbb{R}^{d_2}, \text{ where } g \text{ is a standard Gaussian on } \mathbb{R}^{d_2}. \]

Then
\[ \mathbb{E} \left\| \frac{1}{Md_1} \sum_{i=1}^{Md_1} z_i z_i^T - \int B d\mu(B) \right\| \leq \frac{1}{\sqrt{M}} \left( \left\| \int B d\mu(B) \right\| + \frac{t}{d_1} + \sqrt{d_1 \log d_2} \left\| \int B^2 d\mu(B) \right\| \right). \]

**Proof.** By Corollary 4.2
\[ \mathbb{E} \left\| \frac{1}{Md_1} \sum_{i=1}^{Md_1} z_i z_i^T - \int B d\mu(B) \right\| \leq \frac{1}{\sqrt{M}} \left( \left\| \int B d\mu(B) \right\| + \frac{t}{d_1} + \sqrt{d_1 \log d_2} \left\| \int B^2 d\mu(B) \right\| \right). \]

Since
\[ \left\| \int B^2 d\mu(B) \right\| \leq \left\| \| B \| d\mu(B) \right\| \leq \frac{t}{\max(d_1', d_2')} \left\| \int B d\mu(B) \right\|, \]
we have
\[ \sqrt{d_1} \left\| \int B^2 d\mu(B) \right\| \leq \frac{t}{d_1} + \left\| \int B d\mu(B) \right\|. \]

Thus, the result follows.

**Remark 4.4.** Suppose that \( \mu \) is a probability measure on \( \{ B \in M_{d_2}(\mathbb{R}) \mid B \text{ is positive semidefinite} \} \) and \( \text{Tr}(B) = t \) and \( \| B \| \leq \frac{t}{d_2} \) \( \mu\text{-a.s.} \), where \( t \in [d_2] \) is fixed. Let \( 0 < \gamma < 1 \).

How many samples \( z_1, \ldots, z_m \) do we need so that
\[ \mathbb{E} \left\| \frac{1}{m} \sum_{i=1}^{m} z_i z_i^T - \int B d\mu(B) \right\| \leq \gamma \left\| \int B d\mu(B) \right\| ? \]

Since
\[ \mathbb{E} \left\| \frac{1}{m} \sum_{i=1}^{m} z_i z_i^T \right\| \geq \frac{1}{m} \mathbb{E} \| z_1 \|^2 = \frac{1}{m} \int \text{Tr}(B) d\mu(B) = \frac{t}{m}, \]
we need at least \( \frac{t}{2\| \int B d\mu(B) \|} \) samples. Let \( d_1 = \left\lceil \frac{t}{2\| \int B d\mu(B) \|} \right\rceil \). Note that \( d_1 \in [d_2] \). By Corollary 4.3, we have
\[ \mathbb{E} \left\| \frac{1}{Md_1} \sum_{i=1}^{Md_1} z_i z_i^T - \int B d\mu(B) \right\| \leq \frac{1}{\sqrt{M}} \left( \left\| \int B d\mu(B) \right\| + \frac{t}{d_1} \right) \leq \frac{3}{\sqrt{M}} \left\| \int B d\mu(B) \right\|. \]
Thus, the answer to the above question is between $d_1$ and $C_\epsilon \frac{\gamma^2}{\epsilon^2} d_1$ samples, where $C_\epsilon > 0$ is a constant that depends only on $\epsilon$.

Acknowledgement: The authors are grateful to Ramon van Handel and Joel Tropp for many insightful comments and discussions. The second author is supported by NSF DMS-1856221.

References

[1] J. Baik, Jinho and J. W. Silverstein, Eigenvalues of large sample covariance matrices of spiked population models, J. Multivariate Anal. 97 (2006), no. 6, 1382-1408
[2] A. S. Bandeira, Ten Lectures and Forty-Two Open Problems in the Mathematics of Data Science, Lecture Notes, 2015
[3] A. S. Bandeira and R. van Handel, Sharp nonasymptotic bounds on the norm of random matrices with independent entries, Ann. Probab. 44 (2016), no. 4, 2479-2506
[4] A. S. Bandeira and Y. Ding The Spectral Norm of Random Lifts of Matrices, Preprint.
[5] C. Bordenave and B. Collins, Eigenvalues of random lifts and polynomials of random permutation matrices, Annals of Mathematics, 190(3):811–875, 2019.
[6] A. Buchholz, Operator Khintchine inequality in non-commutative probability, Math. Ann. 319 (2001), no. 1, 1-16.
[7] X. Ding, F. Yang, Spiked separable covariance matrices and principal components, Ann. Statist. 49 (2) 1113-1138, April 2021.
[8] R. van Handel, Structured random matrices, Convexity and concentration, 107-156, IMA Vol. Math. Appl., 161, Springer, New York, 2017.
[9] R. van Handel, the spectral norm of Gaussian random matrices, Trans. Amer. Math. Soc. 369, 8161-8178 (2017).
[10] J.-P. Kahane, Sur les sommes vectorielles $\sum \pm u_n$, Comptes Rendus de l’Académie des Sciences (Paris) 259 (1964), 2577-2580.
[11] R. Latała, R. van Handel and P. Youssef, The dimension-free structure of nonhomogeneous random matrices, Invent. math. 214, 1031–1080 (2018)
[12] F. Lust-Piquard, Inégalités de Khintchine dans $C_p$ (1 < p < $\infty$), (French) C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 7, 289-292
[13] M. W. Meckes, On the spectral norm of a random Toeplitz matrix, Electron. Comm. Probab. 12 (2007), 315-325.
[14] R. I. Oliveira, Sums of random Hermitian matrices and an inequality by Rudelson, Electron. Commun. Probab. 15 (2010), 203-212.
[15] R. I. Oliveira, The spectrum of randomk-lifts of large graphs (with possibly large k), Journal of Combinatorics, 1(3):285-306, 2010
[16] J. A. Tropp, An introduction to matrix concentration inequalities, Foundations and Trends in Machine Learning, 2015.
[17] J. A. Tropp, The expected norm of a sum of independent random matrices: an elementary approach, High dimensional probability VII, 173-202, Progr. Probab., 71, Springer, 2016.
[18] J. A. Tropp, Second-order matrix concentration inequalities, Appl. Comput. Harmon. Anal. 44 (2018), no. 3, 700-736.

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