RIORDAN MATRIX REPRESENTATIONS OF EULER’S CONSTANT $\gamma$ AND EULER’S NUMBER $e$

EDRAY HERBER GOINS AND ASAMOAH NKWANTA

This paper is dedicated to David Harold Blackwell (April 24, 1919 – July 8, 2010).

Abstract. We show that the Euler-Mascheroni constant $\gamma$ and Euler’s number $e$ can both be represented as a product of a Riordan matrix and certain row and column vectors.

1. Introduction

It was shown by Kenter [2] that the Euler-Mascheroni constant

$$\gamma = \lim_{n \to \infty} \left[ \left( \sum_{m=1}^{n} \frac{1}{m} \right) - \ln n \right] = 0.5772156649 \ldots$$

can be represented as a product of an infinite-dimensional row vector, the inverse of a lower triangular matrix, and an infinite-dimensional column vector:

$$\left( \begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \\ \frac{1}{2} & 1 & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \end{array} \right) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n} \\ \vdots \end{array} \right)^{-1} \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n+1} \\ \vdots \end{array} \right)$$

Kenter’s proof uses induction, definite integrals, convergence of power series, and Abel’s Theorem. In this paper, we recast this statement using the language of Riordan matrices. We exhibit another proof as well as a generalization. Our main result is the

2010 Mathematics Subject Classification. 05A15, 11B83, 20Cxx, 30E99, 15A23, 13F25, 13J05, 41A58.

Key words and phrases. Riordan matrices; Appell subgroup; generating functions; representation theory; complex residues.
Theorem 1. Consider sequences \( \{a_0, a_1, \ldots, a_n, \ldots\} \), \( \{b_0, b_1, \ldots, b_n, \ldots\} \) and \( \{c_0, c_1, \ldots, c_n, \ldots\} \) of complex numbers such that \( a_0, b_0, c_0 \neq 0 \); as well as an integer exponent \( d \). Assume that

1. the power series \( a(x) = \sum_n a_n x^n \), \( b(x) = \sum_n b_n x^n \), \( c(x) = \sum_n c_n x^n \), and \( b(x)^d \) are convergent in the interval \( |x| < 1 \); and
2. the following complex residue exists:

\[
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right] = \frac{1}{2\pi i} \oint_{|z|=1} \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \, dz.
\]

Then the matrix product

\[
\begin{pmatrix}
  b_0 \\
  b_1 \\
  b_2 & b_0 \\
  \vdots & \vdots & \ddots \\
  b_{n-1} & \cdots & b_0 & c_n \\
  \vdots & \vdots & \cdots & \cdots & \ddots
\end{pmatrix}
\]

is equal to the above residue.

The infinite-dimensional lower triangular matrix is an example of a Riordan matrix. Specifically, it is that Riordan matrix associated with the power series \( b(x)^d \). Kenter’s result follows by careful analysis of the power series \( 1.3 \)

\[
a(x) = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2} x + \frac{1}{3} x^2 + \cdots + \frac{1}{n+1} x^n + \cdots
\]

\[
b(x)^{-1} = -\frac{x}{\log(1-x)} = 1 - \frac{1}{2} x - \frac{1}{12} x^2 - \frac{1}{24} x^3 - \cdots - L_n x^n - \cdots
\]

\[
c(x) = \frac{a(x) - 1}{x} = \frac{1}{2} + \frac{1}{3} x + \frac{1}{4} x^2 + \cdots + \frac{1}{n+2} x^n + \cdots
\]

The coefficients \( L_n \) are sometimes called the “logarithmic numbers” or the “Gregory coefficients”; these are basically the Bernoulli numbers of the second kind up to a choice of sign. (Kenter employs the coefficients \( c_k = -L_k \).)
The idea of this paper is that we have the matrix product

\[
\begin{pmatrix}
1 & 1 \\
\frac{1}{n} & 1 \\
\vdots & \vdots \\
\frac{1}{n} & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
\frac{1}{n} & 1 \\
\vdots & \vdots \\
\frac{1}{n} & 1 \\
1 & 1 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
\frac{1}{n} \\
\vdots \\
\frac{1}{n} \\
1 \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
\frac{1}{12} \\
\vdots \\
\frac{1}{24} \\
\end{pmatrix},
\]

which is equivalent to the recursive identity \( \sum_{m=0}^{n-1} L_m/(n-m) = 0 \), that is valid whenever \( n = 2, 3, 4, \ldots \). The matrix product, and hence the recursive identity, can be derived from properties of Riordan matrices. Kenter’s result follows from the identity \( \sum_{m=1}^{\infty} L_m/m = \gamma \), which in turn follows from an identity involving a definite integral.

As another consequence of our main result, we can also show that Euler’s number

\[
e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = 2.7182818284\ldots
\]
can be represented as a product of an infinite-dimensional row vector, a lower triangular matrix, and an infinite-dimensional column vector.

**Corollary 2.** For any integers \( p, q, \) and \( d \) with \( pq > 1 \), the number

\[
\frac{pq}{pq - 1} \sqrt{e^d} = \lim_{n \to \infty} \left[ \frac{pq}{pq - 1} \left( 1 + \frac{1}{pn} \right)^{dn} \right]
\]
is equal to the matrix product

\[
\begin{pmatrix}
1 & 1 \\
\frac{1}{n} & 1 \\
\vdots & \vdots \\
\frac{1}{n} & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{1}{2} \\
\vdots \\
\frac{1}{2} \\
1 \\
\end{pmatrix}^d
= \begin{pmatrix}
1 \\
\frac{1}{q} \\
\vdots \\
\frac{1}{q^2} \\
\frac{1}{q^n} \\
\end{pmatrix}.
\]

In the process of proving these generalizations, we present a representation theoretic view of Riordan matrices. That is, we consider the matrices as representations \( \pi : G \to GL(V) \) of a certain group \( G \) – namely, the Riordan group – acting on an infinite-dimensional vector space \( V \) – namely, the collection of those formal power series \( h(x) \) in \( \mathbb{C}[x] \) where \( h(0) = 0 \).
2. Introduction to Riordan Matrices

We wish to list several key results in the theory of Riordan matrices. To do so, we recast this theory using techniques from representation theory very much in the spirit of Bacher [1]. Our ultimate goal in this section is to explain how Riordan matrices are connected to a permutation representation \( \pi : G \to GL(V) \) of a certain group \( G \) acting on an infinite dimensional vector space \( V \). Some of the notation in the sequel will differ from standard notation such as that given by Shapiro et al. [4] and Sprugnoli [5, 6], but we will explain the connection.

2.1. Group Actions. Before developing the representation theoretic view, we give the definition of a Riordan matrix and few related useful properties.

Let \( k \) be a field; it is customary to set \( k = \mathbb{C} \) as the set of complex numbers, but, in practice, \( k = \mathbb{Q} \) is the set of rational numbers. Set \( k[[x]] \) as the collection of formal power series in an indeterminate \( x \); we will view this as a \( k \)-vector space with countable basis \( \{1, x, x^2, \ldots, x^n, \ldots\} \). For most of this article, we will not be concerned with regions of convergence for these series.

There are three binary operations \( k[[x]] \times k[[x]] \to k[[x]] \) which will be of importance to us, namely multiplication \( \bullet \), composition \( \circ \), and addition \( + \). Explicitly, if we write

\[
(f \bullet g)(x) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} f_m g_{n-m} \right) x^n
\]

(2.1) \[ f(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} g_n x^n \]

then we have the formal power series

\[
(f \circ g)(x) = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} f_m \left( \sum_{n_1 + \cdots + n_m = n} g_{n_1} \cdots g_{n_m} \right) \right] x^n
\]

(2.2) \[
(f + g)(x) = \sum_{n=0}^{\infty} \left[ f_n + g_n \right] x^n
\]

There are three subsets of the vector space \( k[[x]] \) which will be of interest to us in the sequel.

Proposition 3. Define the subsets

\[
H = \left\{ f(x) \in k[[x]] \mid f(0) \neq 0 \right\}
\]

\[
K = \left\{ g(x) \in k[[x]] \mid g(0) = 0 \text{ yet } g'(0) \neq 0 \right\}
\]

\[
V = \left\{ h(x) \in k[[x]] \mid h(0) = 0 \right\}
\]
(i) $H$ is a group under multiplication $\bullet$, $K$ is a group under composition $\circ$, and $V$ is a group under addition $\circ$. In particular, $V$ is a $k$-vector space with countable basis $\{x, x^2, \ldots, x^n, \ldots\}$.

(ii) The map $\varphi : K \to \text{Aut}(H)$ which sends $g(x) \in K$ to the automorphism $\varphi_g : f(x) \mapsto (f \circ \overline{g})(x)$ is a group homomorphism, where $\overline{g}(x)$ is the compositional inverse of $g(x)$. In particular, $G = H \rtimes \varphi K$ is a group under the binary operation $\star : G \times G \to G$ defined by

$$(f_1, g_1) \star (f_2, g_2) = (f_1 \bullet \varphi_{g_1}(f_2), g_1 \circ g_2).$$

(iii) The map $\star : G \times V \to V$ defined by $(f, g) \star h = f \bullet (h \circ \overline{g})$ is a group action of $G$ on $V$.

We use $\overline{g}(x)$ to denote the compositional inverse $g^{-1}(x)$ so that we will not confuse this with the multiplicative inverse $g(x)^{-1}$. Later, we will show that $G$ is isomorphic to the Riordan group $R$. Moreover, we will show that $H$, a normal subgroup of $G$, is isomorphic to the Appell subgroup of $R$. The motivation of this result is to use the action of $G$ on $V$ to write down a permutation representation $\pi : G \to GL(V)$, then use the canonical basis $\{x, x^2, \ldots, x^n, \ldots\}$ of $V$ to list infinite-dimensional matrices.

Proof. We show (i) to fix some notation to be used in the sequel. Since $(f \bullet g)(0) = f(0) g(0) \neq 0$ for any $f(x), g(x) \in H$, we see that $\bullet : H \times H \to H$ is an associative binary operation. The identity is the constant power series $e(x) = 1$, and the inverse of $f(x)$ is its reciprocal, seen to be a power series by expressing as a formal geometric series:

$$\frac{1}{f(x)} = \frac{1}{f(0)} \cdot \frac{1}{1 - \sum_{n=0}^{\infty} (-f_n/f_0) x^n}
= \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \sum_{n_1 + \cdots + n_m = n-m} (-1)^m \frac{f_{n_1+1} \cdots f_{n_m+1}}{f_0^{m+1}} \right] x^n.$$

(2.3)

Since $(f \circ g)(0) = f(g(0)) = f(0) = 0$ and $(f \circ g)'(0) = f'(g(0)) g'(0) = f'(0) g'(0) \neq 0$ for any $f(x), g(x) \in K$, we see that $\circ : K \times K \to K$ is an associative binary operation. The identity is the power series $\text{id}(x) = x$, and the inverse of $g(x)$ is its compositional inverse $\overline{g}(x) = \sum_n \overline{g}_n x^n$ having the implicitly defined coefficients

$$\overline{g}_0 = 0,$$
$$\overline{g}_1 = \frac{1}{g_1},$$

(2.4)

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n_1 + \cdots + n_m = n} g_{n_1} \cdots g_{n_m} = 0 \quad \text{for } n = 2, 3, \ldots.$$

Since $(f + g)(0) = f(0) + g(0) = 0$ for any $f(x), g(x) \in V$, we see that $+ : V \times V \to V$ is an associative binary operation. The identity is the
constant power series \( a(x) = 0 \), and the inverse of \( h(x) \) is the negation \( -h(x) \), seen to be a power series with \( (-h)(0) = -h(0) = 0 \).

Now we show (ii). Since \( (f \circ \overline{g})(0) = f(\overline{g(0)}) = f(0) \neq 0 \) for any \( f(x) \in H \) and \( g(x) \in K \), we see that \( \varphi : K \to \text{Aut}(H) \) is well-defined. Given \( g(x), h(x) \in K\) we have \( \varphi_g \circ \varphi_h = \varphi_{g \circ h} \) because for all \( f(x) \in H \) we have

\[
\left( \varphi_g \circ \varphi_h \right)[f(x)] = \varphi_g \left( (f \circ \overline{h})(x) \right) \\
= \left( f \circ \overline{h} \circ \overline{g} \right)(x) = \left( f \circ \overline{g \circ h} \right)(x) \\
= \varphi_{g \circ h}[f(x)].
\]

Hence \( \varphi : K \to \text{Aut}(H) \) is indeed a group homomorphism. The semi-direct product \( G = H \rtimes_{\varphi} K \) consists of pairs \((f(x), g(x))\) with \( f(x) \in H \) and \( g(x) \in K \), where the binary operation \( \ast : G \times G \to G \) is defined by

\[
(f_1(x), g_1(x)) \ast (f_2(x), g_2(x)) = (f_1(x) f_2(\overline{g_1(x)}), g_1(g_2(x))).
\]

Finally, we show (iii). The map \( \ast : G \times V \to V \) is defined as the formal identity

\[
(f(x), g(x)) \ast h(x) = f(x) h(\overline{g(x)}).
\]

Since \( [(f, g) \ast h](0) = f(0) h(\overline{g(0)}) = f(0) h(0) = 0 \), we see that the map \( \ast : G \times V \to V \) is well-defined. As the identity element of \( G \) is \( (e(x), \text{id}(x)) = (1, x) \), we see that \( (e(x), \text{id}(x)) \ast h(x) = h(x) \) so that it acts trivially on \( V \). Given two elements \((f_1, g_1), (f_2, g_2) \in G\) and \( h(x) \in V \), we have the identity

\[
(f_1(x), g_1(x)) \ast \left[ (f_2(x), g_2(x)) \ast h(x) \right] \\
= (f_1(x), g_1(x)) \ast \left[ f_2(x) h(\overline{g_2(x)}) \right] \\
= f_1(x) f_2(\overline{g_1(x)}) h(\overline{g_2 \circ g_1(x)}) \\
= f_1(x) f_2(\overline{g_1(x)}) h(\overline{g_1 \circ g_2(x)}) \\
= \left( f_1(x) f_2(\overline{g_1(x)}), g_1(g_2(x)) \right) \ast h(x) \\
= \left[ (f_1(x), g_1(x)) \ast (f_2(x), g_2(x)) \right] \ast h(x).
\]
Similarly, given two elements \( h_1(x), h_2(x) \in V \) and \((f, g) \in G\), we have the identity
\[
(f(x), g(x)) \ast [h_1(x) + h_2(x)] = f(x) \left[ h_1(\overline{g}(x)) + h_2(\overline{g}(x)) \right]
\]
\[
= (f(x), g(x)) \ast h_1(x) + (f(x), g(x)) \ast h_2(x).
\]
Hence \( \ast : G \times V \to V \) is indeed a group action. \( \square \)

2.2. Riordan Matrices. Recall that the set
\[
V = \left\{ h(x) \in k[x] \mid h(0) = 0 \right\}
\]
is a \( k \)-vector space with countable basis \( \{ x, x^2, \ldots, x^n, \ldots \} \). Since the semi-direct product \( G = H \rtimes \varphi K \) acts on \( V \), we have a “permutation” representation \( \pi : G \to GL(V) \). Explicitly, this representation is defined on the basis elements of \( V \) via the formal identity
\[
(f(x), g(x)) \ast x^m = f(x) \left[ \overline{g}(x) \right]^m
\]
\[
= \sum_{n=1}^{\infty} l_{n,m} x^n \quad \text{for } m = 1, 2, 3, \ldots
\]
(Recall that \( \overline{g}(x) \) is the compositional inverse of \( g(x) \).) The matrix with respect to the basis \( \{ x, x^2, \ldots, x^n, \ldots \} \) is given by the lower triangular matrix
\[
\pi \begin{pmatrix} f(x), g(x) \end{pmatrix} = \begin{pmatrix} l_{1,1} & & & \\ l_{2,1} & l_{2,2} & & \\ l_{3,1} & l_{3,2} & l_{3,3} & \\ \vdots & \vdots & \vdots & \ddots \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & l_{n,n} \\ \vdots & \vdots & \vdots & \cdots & \ddots \end{pmatrix}.
\]
Recall that \( g(0) = 0 \) yet \( f(0), g'(0) \neq 0 \). The following result explains the main multiplicative property of these matrices.

**Theorem 4.** Continue notation as above.

(i) \( \pi : G \to GL(V) \) is a group homomorphism. That is,
\[
\pi \left( f_1(x), g_1(x) \right) \pi \left( f_2(x), g_2(x) \right) = \pi \left( f_1(x) f_2(\overline{g_1}(x)), g_1(g_2(x)) \right).
\]
(ii) For a generating function \( t(x) = t_0 + t_1 x + t_2 x^2 + \cdots \) with \( t_0 \neq 0 \),
\[
\pi\left( f(x), g(x) \right) \pi\left( t(x), \text{id}(x) \right) = \left( \sum_{p=1}^{m} l_{n,p} t_{p-m} \right)_{n,m \geq 1}.
\]

Such matrices \( \pi(f, g) \) are called the Riordan matrices associated to the pair \( (f, g) \). The collection \( R \) of Riordan matrices is a group which is isomorphic to \( G = H \rtimes \varphi K \); this is the Riordan group. The collection of matrices \( \pi(f, \text{id}) \) is a group which is isomorphic to \( H \); this normal subgroup is the Appell subgroup of \( R \).

Proof. We show (i). In the proof of Proposition 3, we found that for each \( h(x) \in V \) we have the following formal identity involving power series as elements of \( k[[x]] \):
\[
\left( f_1(x), g_1(x) \right) * \left[ \left( f_2(x), g_2(x) \right) * h(x) \right] = \left[ \left( f_1(x), g_1(x) \right) * \left( f_2(x), g_2(x) \right) \right] * h(x) = \left( f_1(x) f_2(g_1(x)), g_1(g_2(x)) \right) * h(x).
\]
(2.13)

In particular, this holds for the basis elements \( h(x) = x^n \), so the result follows.

Now we show (ii). For a generating function \( t(x) = t_0 + t_1 x + t_2 x^2 + \cdots \), we have the product
\[
\left( t(x), \text{id}(x) \right) * x^m = t(x) x^m = \sum_{n=1}^{\infty} t_{n-m} x^n;
\]
(2.14)

so that matrices in the Appell subgroup are in the form
\[
\pi\left( t(x), \text{id}(x) \right) = \begin{pmatrix}
  t_0 & t_0 & \cdots \\
  t_1 & t_0 & \cdots \\
  t_2 & t_1 & t_0 \\
  \vdots & \vdots & \ddots \\
  t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \\
  \vdots & \vdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]
(2.15)

This gives the matrix product
\[
\pi\left( f(x), g(x) \right) \pi\left( t(x), x \right) = \left( l_{n,p} \right)_{n,p \geq 1} \left( t_{p-m} \right)_{p,m \geq 1}
\]
\[
= \left( \sum_{p=1}^{m} l_{n,p} t_{p-m} \right)_{n,m \geq 1}
\]
(2.16)
so the result follows. □

2.3. **Examples.** Let $k = \mathbb{Q}$. Using elementary Calculus, we find the power series expansions

\[
-\frac{\ln (1 - x)}{x} = 1 + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \frac{1}{5} x^4 + \frac{1}{6} x^5 + \cdots ,
\]

\[
\frac{x}{\ln (1 - x)} = 1 - \frac{1}{2} x - \frac{1}{12} x^2 - \frac{1}{24} x^3 - \frac{19}{720} x^4 - \frac{3}{160} x^5 + \cdots ;
\]

which are valid on whenever $|x| < 1$. Hence the formal power series

\[
f(x) = 1 + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \cdots + \frac{1}{n + 1} x^n + \cdots
\]

is an element of $H$, and has multiplicative inverse

\[
\frac{1}{f(x)} = 1 - \frac{1}{2} x - \frac{1}{12} x^2 - \frac{1}{24} x^3 - \frac{19}{720} x^4 - \frac{3}{160} x^5 + \cdots .
\]

We have the product

\[
(f(x), \text{id}(x)) * x^m = f(x) x^m = \sum_{n=1}^{\infty} \frac{1}{n - m + 1} x^n
\]

which yields the matrix

\[
\pi(f, \text{id}) = \begin{pmatrix}
1 \\
\frac{1}{2} & 1 \\
\frac{1}{3} & \frac{1}{2} & 1 \\
\vdots & \vdots & \vdots & \ddots \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\]

Similarly, we have the product

\[
\left( \frac{1}{f(x)}, \text{id}(x) \right) * x^m = \frac{1}{f(x)} x^m
\]

\[
= x^m - \frac{1}{2} x^{m+1} - \frac{1}{12} x^{m+2} - \frac{1}{24} x^{m+3} - \frac{19}{720} x^{m+4} + \cdots .
\]
Since we may use Theorem 4 to conclude that \( \pi(f, \text{id})^{-1} = \pi(1/f, \text{id}) \), we find the identity

\[
\begin{pmatrix}
1 \\
\frac{1}{2} & 1 \\
\frac{1}{3} & \frac{1}{2} & 1 \\
\vdots & \vdots & \vdots & \ddots \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}^{-1}
= 
\begin{pmatrix}
1 \\
-\frac{1}{2} & 1 \\
-\frac{1}{12} & -\frac{1}{2} & 1 \\
-\frac{1}{24} & -\frac{1}{12} & -\frac{1}{2} & 1 \\
-\frac{1}{720} & -\frac{1}{24} & -\frac{1}{12} & -\frac{1}{2} & 1 \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}.
\]

These matrices are elements of the Appell subgroup of \( \mathbf{R} \).

2.4. Relation with Standard Notation. Standard references for Riordan matrices are Shapiro et al. [4] and Sprugnoli [5], [6]. The notation \( \pi(f, g) \) employed above is not the typical one, so we explain the connection. Consider sequences \( \{G_0, G_1, G_2, \ldots, G_n, \ldots\} \) and \( \{F_1, F_2, F_3, \ldots, F_n, \ldots\} \) of complex numbers \( k = \mathbb{C} \), where \( G_0, F_1 \neq 0 \). Upon associating generating functions \( G(x) = G_0 + G_1 x + G_2 x^2 + \cdots \) and \( F(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots \) to these sequences respectively, the standard notation for a Riordan matrix is that infinite-dimensional matrix given by

\[
L = \begin{bmatrix} G(x), & F(x) \end{bmatrix} = \pi(G(x), \overline{F}(x)) = \begin{pmatrix} l_{n,m} \end{pmatrix}_{n,m \geq 1}
\]

in terms of the compositional inverse \( \overline{F}(x) \) of \( F(x) \). Indeed, the entry \( l_{n,m} \) in the \( n \)th row and \( m \)th column satisfies the relation

\[
G(x) \begin{bmatrix} F(x) \end{bmatrix}^m = \sum_{n=1}^{\infty} l_{n,m} x^n \quad \text{for } m = 1, 2, 3, \ldots
\]

as formal power series in \( \mathbb{C}[x] \). Equivalently, a Riordan matrix \( L \) can be defined by a pair \( (G(x), F(x)) \) of generating functions.

Corollary 5 (Fundamental Theorem of the Riordan Group: [3], [4], [6]). Continue notation as above.

(i) The product of Riordan matrices is again a Riordan matrix. Explicitly, their product satisfies the relation

\[
\begin{bmatrix} G_1(x), & F_1(x) \end{bmatrix} \begin{bmatrix} G_2(x), & F_2(x) \end{bmatrix} = \begin{bmatrix} G_1(x) G_2(F_1(x)), & F_2(F_1(x)) \end{bmatrix}.
\]
(ii) For a generating function $T(x) = T_0 + T_1 x + T_2 x^2 + \cdots$ with $T_0 \neq 0$, we have the product

$$
[G(x), F(x)] [T(x), x] = \left( \sum_{p=1}^{m} \lambda_{n,p} T_{p-m} \right)_{n,m \geq 1}
$$

**Proof.** Statement (i) is shown in [4, Eq. 5] and [3, Proof of Thm. 2.1], but we give an alternate proof. Upon denoting $f_i(x) = G_i(x)$ and $g_i(x) = F_i(x)$ for $i = 1$ and 2, we find the matrix product

$$
\begin{align*}
[G_1(x), F_1(x)] [G_2(x), F_2(x)] &= \pi \left( f_1(x), g_1(x) \right) \pi \left( f_2(x), g_2(x) \right) \\
&= \pi \left( f_1(x) f_2(x), g_1(x) g_2(x) \right) \\
&= \left[ G_1(x) G_2(F_1(x)), F_1(F_1(x)) \right]
\end{align*}
$$

which follows directly from Theorem 4. Statement (ii) is also shown in [3], but it follows directly from Theorem 4 as well. \qed

3. **Proof of Kenter's Result and Generalizations**

3.1. **Main Result.** We now prove the following:

**Theorem 1** Consider sequences $\{a_0, a_1, \ldots, a_n, \ldots\}$, $\{b_0, b_1, \ldots, b_n, \ldots\}$ and $\{c_0, c_1, \ldots, c_n, \ldots\}$ of complex numbers such that $a_0, b_0, c_0 \neq 0$; as well as an integer exponent $d$. Assume that

(i) the power series $a(x) = \sum_n a_n x^n$, $b(x) = \sum_n b_n x^n$, $c(x) = \sum_n c_n x^n$, and $b(x)^d$ are convergent in the interval $|x| < 1$; and

(ii) the following complex residue exists:

$$
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right] = \frac{1}{2\pi i} \oint_{|z|=1} a(z) b(z^{-1})^d c(z^{-1}) \frac{dz}{z}.
$$

Then the matrix product

$$
\begin{pmatrix}
    b_0 \\
    b_1 & b_0 \\
    b_2 & b_1 & b_0 \\
    \vdots & \vdots & \vdots & \ddots \\
    b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \\
    \vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
$$

is equal to the above residue.
Proof. With the three power series $a(x) = \sum_n a_n x^n$, $b(x) = \sum_n b_n x^n$, and $c(x) = \sum_n c_n x^n$ convergent in the interval $|x| < 1$, consider the power series

$$f(x) = b(x)^d c(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{where } |x| < 1.$$  

As elements of the Appell subgroup of $\mathbb{R}$, we invoke Theorem 4 to see that we have the matrix product $\pi(f(x), x) = \pi(b(x), x)^d \pi(c(x), x)$. In particular, the first column is given by

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 b_0 \\ b_2 b_1 b_0 \\ \vdots \\ b_n b_{n-1} b_{n-2} \cdots b_0 \\ \vdots \end{pmatrix} \begin{pmatrix} d \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$  

Hence the matrix product

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & \cdots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 b_0 \\ b_2 b_1 b_0 \\ \vdots \\ b_n b_{n-1} b_{n-2} \cdots b_0 \\ \vdots \end{pmatrix} \begin{pmatrix} d \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is equal to the sum $\sum_n a_n f_n$. We wish to evaluate this sum using complex analysis.

By assumption, the power series $a(x)$, $b(x)$, and $c(x)$ are convergent in the interval $|x| < 1$. Hence, for each fixed real number $r$ satisfying $0 < r < 1$, the functions $a(z)$ and $f(z)$ are uniformly convergent inside a closed disk $|z| \leq r$. Hence we can interchange summation and integration to find the
integral around the boundary to be equal to

\[
\frac{1}{2\pi i} \oint_{|z|=r} a(z) b(z^*)^d c(z^*) \frac{dz}{z} = \frac{1}{2\pi i} \oint_{|z|=r} a(z) f(z^*) \frac{dz}{z}
\]

(3.4)

\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1} f_{n_2} r^{n_1+n_2} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i(n_1-n_2)\theta} d\theta
\]

\[
= \sum_{n=0}^{\infty} a_n f_n r^{2n}.
\]

Here \( z^* \) is the complex conjugate of \( z \). As \( r \to 1 \), the integral exists so by Cauchy's Residue Theorem it must be equal to

\[
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right]
\]

\[
= \lim_{r \to 1} \left[ \frac{1}{2\pi i} \oint_{|z|=1} a(z) b(z^{-1})^d c(z^{-1}) \frac{dz}{z} \right]
\]

(3.5)

\[
= \lim_{r \to 1} \sum_{n=0}^{\infty} a_n f_n r^{2n}
\]

\[
= \sum_{n=0}^{\infty} a_n f_n.
\]

The Theorem follows upon equating this with equation (3.3). \( \square \)

3.2. Applications. We explain how to use Theorem 1 in order to express Euler's number \( e = 2.7182818284\ldots \) in terms of Riordan matrices.

**Corollary 2.** For any integers \( p, q, \) and \( d \) with \( pq > 1 \), the number

\[
\frac{pq}{p q - 1} q^{\sqrt{\varepsilon d}} = \lim_{n \to \infty} \left[ \frac{pq}{p q - 1} \left( 1 + \frac{1}{pn} \right)^{dn} \right]
\]

is equal to the matrix product

\[
\begin{pmatrix}
1 & 1 & 1 & \\
\frac{1}{1!} & 1 & \\
\frac{1}{2!} & \frac{1}{1!} & 1 \\
\vdots & \vdots & \ddots \\
\frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & 1 \\
\vdots & \vdots & \cdots & \cdots & \ddots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
\frac{1}{q} \\
\frac{1}{q^2} \\
\vdots \\
\frac{1}{q^n} \\
\vdots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
\frac{1}{1!} \\
\frac{1}{2!} \\
\vdots \\
\frac{1}{n!} \\
\vdots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
\frac{1}{q} \\
\frac{1}{q^2} \\
\vdots \\
\frac{1}{q^n} \\
\vdots \\
\end{pmatrix}
\]
Proof. The coefficients of the matrices above correspond to the three power series

\[
\begin{align*}
a(x) &= \frac{1}{1-x/p} = \sum_{n=0}^{\infty} \frac{x^n}{p^n} \\
b(x) &= e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
c(x) &= \frac{1}{1-x/q} = \sum_{n=0}^{\infty} \frac{x^n}{q^n}
\end{align*}
\]

where \( |x| < 1 \).

For a complex number \( z \) with \( |z| < 1 \), we have the identity

\[
\frac{a(z) b(z^{-1})^d c(z^{-1})}{z} = \left[ \frac{1}{1-z/p} \right] \left[ e^{dz^{-1}} \right] \left[ \frac{1}{1-z^{-1}/q} \right] = \sum_{n=-\infty}^{\infty} \left[ \sum_{n_1-n_2-n_3=n+1} \frac{d^{n_2}}{p^{n_1} n_2! q^{n_3}} \right] z^n
\]

The residue corresponds to the coefficient of the \( z^{-1} \) term, so we consider the terms where \( n = -1 \):

\[
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right] = \sum_{n_1=n_2+n_3} \frac{d^{n_2}}{p^{n_1} n_2! q^{n_3}}
\]

\[
= \left[ \sum_{n_2=0}^{\infty} \frac{1}{n_2!} \left( \frac{d}{p} \right)^{n_2} \right] \left[ \sum_{n_3=0}^{\infty} \frac{1}{(pq)^{n_3}} \right]
\]

\[
= e^{d/p} \frac{pq}{pq-1}.
\]

The Corollary follows now from Theorem \([\mathbb{1}]\) \( \square \)

Kenter’s result is also an application of Theorem \([\mathbb{1}]\)

**Corollary 6** (\([\mathbb{2}]\)). The Euler-Mascheroni constant

\[
\gamma = \lim_{n \to \infty} \left[ \left( \sum_{m=1}^{n} \frac{1}{m} \right) - \ln n \right] = 0.5772156649 \ldots
\]
is equal to the matrix product

\[
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \cdots \\
\frac{1}{3} & \frac{1}{2} & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \\
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\vdots \\
\vdots \\
\frac{1}{n+1} \\
\vdots \\
\end{pmatrix}
\]

\[
(1 \hspace{0.5em} \frac{1}{2} \hspace{0.5em} \frac{1}{3} \hspace{0.5em} \cdots \hspace{0.5em} \frac{1}{n} \hspace{0.5em} \cdots)
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \cdots \\
\frac{1}{3} & \frac{1}{2} & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \\
\frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \\
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\vdots \\
\vdots \\
\frac{1}{n+1} \\
\vdots \\
\end{pmatrix}
\]

Proof. The coefficients of the matrices above correspond to the three power series

\[
a(x) = -\frac{\log (1 - x)}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n+1}
\]

(3.9)

\[
b(x) = -\frac{\log (1 - x)}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n+1}
\]

(3.9)

\[
c(x) = \frac{a(x) - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n+2}
\]

We will choose the exponent \(d = -1\). We will express the reciprocal as the power series

\[
\frac{x}{\log (1 - x)} = -1 + \frac{1}{2} x + \frac{1}{12} x^2 + \frac{1}{24} x^3 + \frac{19}{720} x^4 + \frac{3}{160} x^5 + \cdots
\]

(3.10)

\[
= \sum_{n=0}^{\infty} L_n x^n
\]

which is also convergent in the interval \(|x| < 1\). (Recall that the coefficients \(L_n\) are sometimes called the “logarithmic numbers” or the “Gregory coefficients”.) For a complex number \(z\) with \(|z| < 1\), we have the identity

\[
a(z) b(z^{-1})^d c(z^{-1})
\]

(3.11)

\[
= -\frac{\log (1 - z)}{z} + \left[ -\frac{\log (1 - z)}{z} \right] \cdot \left[ \frac{z^{-1}}{\log (1 - z^{-1})} \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{z^n}{n+1} + \sum_{n=-\infty}^{\infty} \left[ \sum_{m=-n}^{\infty} \frac{L_m}{n+m+1} \right] z^n.
\]
The residue corresponds to the coefficient of the $z^{-1}$ term, so we consider the terms where $n = -1$:

$$
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1}) c(z^{-1})}{z} \right] = \sum_{m=1}^{\infty} \frac{L_m}{m} = \int_0^1 \left[ \sum_{m=1}^{\infty} L_m x^{m-1} \right] dx
$$

\begin{equation}
= \int_0^1 \left[ \frac{1}{x} + \frac{1}{\log(1-x)} \right] dx
\end{equation}

$$
= \gamma.
$$

The Corollary follows now from Theorem 1. \qed

We conclude with stating that Theorem 1 can also be used to show Riordan matrix representations for $\ln 2$ and $\pi^2/6$. Finding matrix representations of other constants, like $\sqrt{2}$, $\pi$, and the Golden Ratio $\phi$, are of interest.

4. ACKNOWLEDGMENTS

The authors would like to dedicate this work to the memory of David Harold Blackwell (April 24, 1919 – July 8, 2010). Both authors gave the recent annual Blackwell Lectures, organized by the National Association of Mathematicians (NAM) as part of the MAA MathFest. The first author gave his presentation during the summer of 2009, whereas the second gave his during the summer of 2010.

REFERENCES

[1] Roland Bacher. Sur le Groupe d’Interpolation. \textit{arXiv, math.CO}, Sept 2006.
[2] Frank K. Kenter. A Matrix Representation for Euler’s Constant, $\gamma$. \textit{Amer. Math. Monthly}, 106(5):452–454, 1999.
[3] Asamoah Nkwanta and Louis W. Shapiro. Pell Walks and Riordan Matrices. \textit{Fibonacci Quart.}, 43(2):170–180, 2005.
[4] Louis W. Shapiro, Seyoum Getu, Wen Jin Woan, and Leon C. Woodson. The Riordan Group. \textit{Discrete Appl. Math.}, 34(1-3):229–239, 1991.
[5] Renzo Sprugnoli. Riordan Arrays and Combinatorial Sums. \textit{Discrete Math.}, 132(1-3):267–290, 1994.
[6] Renzo Sprugnoli. Riordan Arrays and the Abel-Gould Identity. \textit{Discrete Math.}, 142(1-3):213–233, 1995.

Purdue University, Department of Mathematics, Mathematical Sciences Building, 150 North University Street, West Lafayette, IN 47907-2067
E-mail address: egoins@math.purdue.edu

Morgan State University, Department of Mathematics, 1700 East Cold Spring Lane, Baltimore, MD 21251
E-mail address: asamoah.nkwanta@morgan.edu