Distinguished solutions for discontinuous signature change with weak junction conditions

Marcus Kriele*

5th January 2022

Abstract

We consider discontinuous signature change with the weak junction condition favoured by Ellis et. al. [4]. We impose certain regularity conditions and investigate the space of solutions (considered as one-parameter families of three-dimensional Riemannian manifolds) for dust and scalar field models.

PACS numbers: 04.20.Cv

1 Introduction

Hartle and Hawking [6] have suggested that our universe should be described by a signature changing rather than a Lorentzian manifold. Their motivation arose in connection with a Lorentzian path integral approach to quantum gravity, and through the hope that in such a setting it may be possible to calculate initial conditions for our Lorentzian universe. Unfortunately, path integrals in quantum gravity are mathematically ill defined. On the other hand, one can (as a semi-classical limit) consider a purely classical theory of signature change. One of the first questions to arises would be the determination of the space of solutions to Einstein’s equations in the presence of signature change.

The qualitative assumption of (classical) signature change may constrain the initial data, and therefore give rise to predictions which can be compared with observations of our universe. As far as I can see, this is the only way to test theories of signature change. Put another way, apart from purely theoretical motivations, such predictions are the only way to justify (classical) signature change.

There are several implementations of classical signature change. People distinguish between weak and strong junction conditions with smooth or discontinuous signature change. Each of these 4 flavours represent different conditions one may want to place on the transition from Riemannian to Lorentzian signature (for a short discussion, see [4]). In this paper we are concerned with discontinuous signature and weak junction conditions. This amounts to solving Einstein’s equation separately in the Riemannian and the Lorentzian region and then

*Technische Universität Berlin, Fachbereich Mathematik, Sekr. MA 8-3, Straße des 17. Juni 136, 10623 Berlin, GERMANY, 
Email: kriele@sfb288.math.tu-berlin.de, 
Telephone: +49 30 314 25776, 
Fax: +49 30 314 21577
matching these two solutions. This is different from the strong junction condition, i.e., from viewing Einstein’s equation distributionally and taking into account the non-differentiability of the metric at the hypersurface of signature change.

Discontinuous signature change with weak junction conditions recovers (in the analytic category with certain $C^2$-conditions on the metric) the whole generality of the usual Lorentzian formulation. Since we are situated in the Lorentzian region it follows that (classically) weak signature change does not give any physical prediction which can be used to differentiate it from the traditional, purely Lorentzian approach. It seems that the only way to arrive at predictions from the proposal of weak signature change is to impose natural additional conditions. It further seems that the only way to do so is to demand regularity conditions at the hypersurface of signature change on the metric, the energy density, and the principal pressures.

In this paper we will investigate a variety of natural conditions. In section 2 we will give a precise definition for “discontinuous signature change with weak junction conditions” and state our notation. In section 3 we investigate the effect of strong regularity conditions both for dust and scalar field spacetimes. It is shown that Einstein’s Equations reduce to a highly constrained system of ordinary differential equations. While the space of solutions shrinks considerably there exist non-trivial solutions. This will be shown in section 4 where pressureless dust is discussed more systematically. In this chapter we also investigate the effect of (rather weak) regularity conditions on the metric and different continuity assumption on the energy density. In section 5 we discuss the physical relevance of our results.

2 Discontinuous signature change and the weak junction condition

Different authors give slightly different definitions of discontinuous signature change. We will therefore first state our definitions of a “discontinuously signature type changing spacetime $(M, g)$” (see also [12]). We assume that there exists a hypersurface $D$ such that $M \setminus D$ consists of two connected components, $M^+$ and $M^-$. $(M^+, g)$ is a Riemannian and $(M^-, g)$ a Lorentzian manifold. Moreover, we will demand that both, $(M^+, g)$ and $(M^-, g)$ admit extensions beyond $D$. (But we do not demand a priori that the extension of $(M^-, g)$ is in any sense restricted by $(M^+, g)$ or vice versa). For points in the Lorentzian region $M^-$ we choose as the fourth coordinate, $t^-$, the Lorentzian distance from $D$. Then in $M^-$ the metric has the form

$$-(dt^-)^2 + g^-_{ij}(t^-, x^1, \ldots, x^{m-1})dx^i dx^j.$$  

Analogously, we take the negative of the Riemannian distance from $D$ as the fourth coordinate $t^+$ in the Riemannian region $M^+$. In $M^+$ the metric is then given by

$$(dt^+)^2 + g^+_{ij}(t^+, x^1, \ldots, x^{m-1})dx^i dx^j.$$  

There are several ways in which to join both regions at $D$. However, it is most natural to demand that $g_{ij}$ and $g^+_{ij}$ can be joined smoothly. Let $t(x) := t^-(x)$ for $x \in M^-$ and $t(x) := t^+(x)$ for $x \in M^+$. We demand that $t$ is a coordinate function on $M$. Thus we finally obtain

$$g = -\eta dt^2 + g_{ij}(t, x^1, \ldots, x^{m-1})dx^i dx^j,$$

1 later in the paper we will impose different regularity conditions — cf. Notation and Conventions below.
where $\eta(x) = 1$, $g_{ij}(x) = g_{ij}^-(x)$ for $x \in M^-$ and $\eta(x) = -1$, $g_{ij}(x) = g_{ij}^+(x)$ for $x \in M^+$.

Set $\mathcal{D}_t := \{x \in M | t(x) = t\}$. Then $(\mathcal{D}_t, g_{ij}(t, x^1, \ldots, x^{m-1}))$ can be viewed as a 1-parameter family of Riemannian three-dimensional manifolds.

The strong junction condition would imply that $\partial_t g_{ij} = 0$ at $t = 0$. The weak junction condition only implies that the family $g_{ij}$ is a $C^1$-1-parameter family of Riemannian metrics.

**Notation and Conventions:** Expressions intrinsic to $(\mathcal{D}_t, g_{ij}dx^i dx^j)$ carry a superscript $(\mathcal{D}_t)$ (e.g. $(\mathcal{D}_t)g_{ij} = g_{ij}dx^i dx^j$ denotes the induced metric on the hypersurface $\mathcal{D}_t$). All indices run from 1 to $m-1$, where $m$ is the dimension of spacetime. We will employ Einstein’s summation convention. The curvature scalar is denoted by $s$. ‘tr’ is always understood as the trace with respect to the $(m-1)$-dimensional Riemannian metric $g_{ij}$. For a bilinear form $A$ defined on $\mathcal{D}_t$ we write $|A|$ for $\sqrt{g^{ij}g^{kl}A_{ik}A_{jl}}$. Differentiation with respect to $t$ is sometimes denoted by a dot. The word ‘smooth’ is always understood as a synonym for ‘$C^\infty$’.

We will always assume that $(\mathcal{D}_t)g_{ij}$ are analytic functions of $(x^1, \ldots, x^{m-1})$. In all papers we are aware of, this assumption has (implicitly) been made. It seems necessary because we are considering a system of partial differential equations which changes its type. *One cannot expect to find non-analytic solutions in the smooth category.* In particular, for smooth metrics the initial value problem would not be well posed in the Riemannian region (this is already apparent for the massless wave equation — the corresponding initial value problem has no solutions for non-analytical data). On the other hand, such problems are well posed in the analytic category. In [13] the authors have shown that the Einstein equation for smoothly signature changing spacetimes is also well posed in the analytic category.

We will make different assumptions on the dependence of $(\mathcal{D}_t)g$ on $t$. In most papers in the field it is assumed that $t \mapsto (\mathcal{D}_t)g(x^1, \ldots, x^{m-1})$ is $C^2$. We will make this assumption in theorems 4.3, 4.5. However, we will also investigate the strongest possible regularity condition, i.e. that $t \mapsto (\mathcal{D}_t)g(x^1, \ldots, x^{m-1})$ is a real analytic map (cf. theorems 3.2, 4.1, 4.2, 4.4).

### 3 The Einstein equation

We will study the Einstein equation for pressureless dust, where the world lines of dust particles intersect $\mathcal{D}$ orthogonally, and a single, non-interacting scalar field. Assume that $g$ satisfies

$$\text{Ric} - \frac{s}{2} g + \Lambda g = 8\pi (T_{\text{scalar}} + T_{\text{dust}}), \quad (3.1)$$

where

$$T_{\text{scalar}} = 8\pi \left( d\phi \otimes d\phi - \frac{1}{2} (g(\text{grad}(\phi), \text{grad}(\phi)) + V(\phi)) g \right) \quad (3.2)$$

and

$$T_{\text{dust}} = \epsilon dt \otimes dt. \quad (3.3)$$

Let $(M, g)$ be a spacetime with discontinuously signature changing metric $g$ which satisfies Equation (3.1) in $M^- \cup M^+$. Then $(M, g)$ is called a *spacetime with an adapted dust-scalar...*
field model. If \( \phi = 0 \) then we call \((M, g)\) a spacetime with an adapted dust model. Clearly, vacuum solutions are special cases.

The equation of motion, \( \text{div}(T_{\text{scalar}} + T_{\text{dust}}) = 0 \), implies

\[
0 = 8\pi \left( \Delta \phi - \frac{1}{2} V'(\phi) \right) d\phi + (\epsilon \text{div}(\partial_t) + \epsilon \text{e}(\partial_t)) \eta dt + \epsilon g(\nabla \partial_t \partial_t, \cdot).
\]

The last summand vanishes since \( \partial_t \) is a geodesic vector field. Assume the genericity condition that \( \phi \) is not constant in any open subset of \( D \). Applying the left hand side of Equation (3.4) to any \( y \in TD_t \) with \( d\phi(y) \neq 0 \) we obtain the wave equation

\[
0 = \Delta \phi - \frac{1}{2} V'(\phi).
\]

Inserting Equation (3.5) again into Equation (3.4) we get the dynamical equation for \( \epsilon \),

\[
\partial_t \epsilon = -\frac{1}{2} \epsilon g^{ij} \partial_t g_{ij}.
\]

Observe that these equations have been derived only in \( M \setminus D \). Again, one has to make a choice how to join these physical fields across \( D \). We will consider different conditions for \( \phi \) and \( \epsilon \).

**Lemma 3.1** Let \((M, g)\) be a discontinuously signature changing spacetime with an adapted dust-scalar field model. If \((D_t)g, \phi\) are \( C^{k+2} \) with respect to \((t, x^1, \ldots, x^{m-1})\) then at \( D \) we have

\[
0 = (\partial_t)^l \left( (D_t)\text{Ric}_{ij} - \left( \frac{1}{2} (D_t)s - \Lambda + 2\pi g^{kl} \partial_{x^k} \phi \partial_{x^l} \phi + 2\pi V(\phi) \right) g_{ij} - 4\pi \partial_{x^i} \phi \partial_{x^j} \phi \right)
\]

\[
0 = (\partial_t)^l \left( (D_t)\Delta \phi - \frac{1}{2} V'(\phi) \right)
\]

for all \( 0 \leq l \leq k \).

**Proof:** The set of partial differential equations for the components \( g_{ij} \) is analogous to [13, Equation 5.4] and is given by

\[
\partial_t \partial_t g_{ij} = \eta \left( A_{ij} - \frac{1}{m-2} g^{kl} A_{kl} g_{ij} \right),
\]

where

\[
A_{ij} = -2 \left( (D_t)\text{Ric}_{ij} - \left( \frac{1}{2} (D_t)s - \Lambda \right) g_{ij} \right) +
\]

\[
+ \eta \left( g^{kl} \partial_t g_{ik} \partial_t g_{jl} - \frac{1}{2} g^{kl} \partial_t g^{kl} g_{ij} + \frac{1}{4} \left( (g^{kl} \partial_t g_{kl})^2 - 3 g^{kl} g^{np} \partial_t g_{kn} \partial_t g_{lp} \right) g_{ij} \right)
\]

\[
+ 8\pi \left( \partial_{x^i} \phi \partial_{x^j} \phi - \frac{1}{2} \left( -\eta (\partial_t \phi)^2 + g^{kl} \partial_t \phi \partial_{x^l} \phi + V(\phi) \right) g_{ij} \right).
\]

\footnote{There is a misprint in the quoted set of equations: \( m - 1 \) should be replaced by \( m - 2 \).}
This set of equations has the form

\[ \partial_t \partial_t g_{ij} = \eta \left( B_{ij} - \frac{1}{m - 2} g^{kl} B_{kl} g_{ij} \right) + C_{ij}, \]

where both \( B_{ij} \) and \( C_{ij} \) depend analytically on \( g_{ij}, \partial_t g_{ij}, \partial_x^k g_{ij}, \partial_x^k \partial_x^l g_{ij}, \phi, \partial_t \phi, \partial_x^k \phi. \) Thus it follows that \( (\partial_t)^l \left( B_{ij} - \frac{1}{m - 2} g^{kl} B_{kl} g_{ij} \right) = 0 \) (\( l \leq k \)) for any \( C^{k+2} \)-solution. Taking the trace we obtain \( (\partial_t)^l g_{ij} B_{kl} g_{ij} = 0 \) and therefore also \( (\partial_t)^l B_{ij} = 0 \) which proves the first assertion of the lemma.

Equation (3.5) can be written as \( 0 = (D_t) \Delta \phi - \frac{1}{2} V' (\phi) - \eta \left( \partial_t \partial_t \phi + \frac{1}{2} g^{kl} \partial_t g_{kl} \partial_t \phi \right). \) Thus the second assertion follows by an analogous argument.

**Theorem 3.2** Let \((M, g)\) be a discontinuously signature changing spacetime with an adapted dust-scalar field model. If \((D_t) g, \phi\) are real analytic with respect to \((t, x^1, \ldots, x^{m-1})\), then the dynamical part of the Einstein equation reduces to the system of ordinary differential equations

\[
\begin{align*}
\partial_t \partial_t g_{ij} &= C_{ij} - \frac{1}{m - 2} g^{kl} C_{kl} g_{ij}, \\
\partial_t \partial_t \phi &= -\frac{1}{2} g^{ij} \partial_t g_{ij} \partial_t \phi \\
\partial_t \epsilon &= -\frac{1}{2} g^{kl} \partial_t g_{kl},
\end{align*}
\]

where \( C_{ij} = g^{kl} \partial_t g_{ik} \partial_t g_{jl} - \frac{1}{2} g^{kl} \partial_t g_{kl} \partial_t g_{ij} + \frac{1}{4} \left( \left( g^{kl} \partial_t g_{kl} \right)^2 - 3 g^{kl} g^{np} \partial_t g_{kn} \partial_t g_{lp} \right) g_{ij} + 4 \pi (\partial_t \phi)^2. \)

In addition, in each surface \( D_t \) the intrinsic equations

\[
\begin{align*}
0 &= (D_t) \Delta \phi - \frac{1}{2} V' (\phi) \\
0 &= (D_t) \text{Ric}_{ij} - \left( \frac{1}{2} (D_t) s - \Lambda + 2 g^{kl} \partial_x^k \phi \partial_x^l \phi + 2 V (\phi) \right) g_{ij} - 4 \pi \partial_x^i \phi \partial_x^j \phi
\end{align*}
\]

are satisfied.

**Proof:** Since \( g_{ij}, \phi \) are analytic lemma \([3.1]\) implies \( B_{ij} = 0 \) and \( 0 = (D_t) \Delta \phi - \frac{1}{2} V' (\phi) \) at all hypersurfaces \( D_t \). The theorem follows by inserting these equations into the Einstein equations for \( g_{ij} \) and Equation (3.5).

We see that the system is strongly over-determined and it should not come as a surprise that there exist only very few solutions.

## 4 Pure, pressureless dust \( \phi = 0 \)

In order to arrive at concrete results, we will consider one of the simplest matter models, pure, pressureless dust. The only matter quantity is the energy density \( \epsilon. \) Our main physical assumption will be a continuity assumption on \( \epsilon. \) There are two possibilities which seem to be especially natural. One may assume that \( \epsilon \) is a continuous function. This is carried out in
section [4.1]. However, in the definition of energy, time enters in a fundamental way. Hence it seems also plausible to expect that the change of signature is reflected in the energy density by a change of sign. In subsection [4.2] we will therefore investigate the assumption that $\eta\epsilon$ is continuous.

We will consider the two extreme regularity conditions on the map $t \mapsto (\mathcal{D}_t)g(x^1, \ldots, x^{m-1})$, namely that $(\mathcal{D}_t)g$ is an analytic function of $t$ and that it is merely a $C^2-$function of $t$.

### 4.1 Continuous energy density $\epsilon$

**Theorem 4.1** Let $(M, g)$ be a 4-dimensional discontinuously signature changing spacetime with adapted dust model. If $(\mathcal{D}_t)g$ is real analytic and $\epsilon$ is continuous with respect to $(t, x^1, x^2, x^3)$ then $(M, g)$ has the following properties:

1. $(M, g)$ is a vacuum spacetime: $\epsilon = 0$;
2. The submanifolds $(\mathcal{D}_t, (\mathcal{D}_t)g)$ are flat;
3. The eigenspaces of $g^{ij}\dot{g}_{jk}$ are constant with respect to $t$ (here we are identifying the submanifolds $\mathcal{D}_t$ via projection along the $t$-coordinate);
4. The eigenvalues of $g^{ij}\dot{g}_{jk}$ are given by $c_i = \frac{2\hat{c}_i}{2+4\sum_{k=1}^3 c_k}$, where $\hat{c}_i(x^1, x^2, x^3) = c_i(0, x^1, x^2, x^3)$;
5. $c_1c_2 + c_2c_3 + c_3c_1 = 0$.

**Proof:** It follows from the additional intrinsic equations in theorem [3.2] that the hypersurfaces $\mathcal{D}_t$ must be Einstein manifolds: $(\mathcal{D}_t)\text{Ric}_{ij} = \left(\frac{1}{2}(\mathcal{D}_t)s - \Lambda\right)g_{ij}$. Taking the trace it follows immediately that $(\mathcal{D}_t)\text{Ric} = 2\Lambda(\mathcal{D}_t)g$. In particular, $(\mathcal{D}_t)\text{Ric}$ does not depend on $t$.

\[
8\pi\epsilon = G_{tt} + \Lambda(-\eta) = \frac{\eta}{2} (\mathcal{D}_t)s + \frac{1}{8} \left(\left(\text{tr}((\mathcal{D}_t)g)\right)^2 - \left|((\mathcal{D}_t)g)^\cdot\right|^2\right) - \eta\Lambda = 2\eta\Lambda + \frac{1}{8} \left(\left(\text{tr}((\mathcal{D}_t)g)\right)^2 - \left|((\mathcal{D}_t)g)^\cdot\right|^2\right)
\]

implies $\Lambda = 0$ by the continuity of $\epsilon$. But this means $(\mathcal{D}_t)\text{Ric} = 0$. Since for 3-dimensional manifolds the Ricci tensor already determines the Riemann tensor all surfaces $t = \text{const.}$ must be flat (hence [\[ii\] follows).

Consider a coordinate system of $\mathcal{D}$ such that at a given point $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ $g_{ij}$ and $\dot{g}_{ij}$ are simultaneously diagonal. (Such a coordinate system always exist since $g_{ij}$ is positive definite and $\dot{g}_{ij}$ is symmetric). Then $g_{ij}$ and $\dot{g}_{ij}$ are diagonal at $(t, \hat{x}^1, \hat{x}^2, \hat{x}^3)$ for all $t$. This follows from the uniqueness of solutions for ordinary differential equations and the fact that the ansatz

\[
g_{ij} = \begin{pmatrix} a_1(t) & 0 & 0 \\ 0 & a_2(t) & 0 \\ 0 & 0 & a_3(t) \end{pmatrix}, \quad \dot{g}_{ij} = \begin{pmatrix} b_1(t) & 0 & 0 \\ 0 & b_2(t) & 0 \\ 0 & 0 & b_3(t) \end{pmatrix}
\]

leads to a consistent system of differential equations. In fact, one obtains
\[ C_{ij} = \sum_{k=1}^{3} \frac{1}{a_k} b_i b_j \delta_{ik} \delta_{jk} - \frac{1}{2} \sum_{k=1}^{3} \frac{b_k}{a_k} b_i \delta_{ij} + \frac{1}{4} \left( \frac{3}{a_k} - 3 \sum_{k=1}^{3} \frac{b_k}{a_k} \right) a_i \delta_{ij} \]

(no summation over \( i, j \)) whence \( C_{ij} = \frac{1}{2} g^{kl} C_{kl} g_{ij} \) is diagonal. Thus we have

\[ \dot{b}_i = \frac{b_i}{a_i} - \frac{1}{2} \sum_{k=1}^{3} \frac{b_k}{a_k} b_i + \frac{1}{4} \left( \frac{3}{a_k} - 3 \sum_{k=1}^{3} \frac{b_k}{a_k} \right) a_i - \frac{1}{2} \left( \sum_{k=1}^{3} \frac{b_k}{a_k} \right)^2 - \frac{1}{2} \left( \sum_{k=1}^{3} \frac{b_k}{a_k} \right)^2 + \frac{3}{4} \left( \frac{3}{a_k} - 3 \sum_{k=1}^{3} \frac{b_k}{a_k} \right) a_i \]

\[ = \frac{b_i}{a_i} - \frac{1}{2} \sum_{k=1}^{3} \frac{b_k}{a_k} b_i + \frac{1}{8} \left( \frac{3}{a_k} - \sum_{k=1}^{3} \frac{b_k}{a_k} \right) a_i. \]

In particular, we have proved (iii). The eigenvalues of \( g^{ij} \delta_{jk} \) are given by \( c_i = \frac{b_i}{a_i} \). Expressing the system of differential equations with respect to \( c_i \) and using \( \dot{c}_i = \frac{b_i}{a_i} - (c_i)^2 \) we obtain

\[ \dot{c}_i = -\frac{1}{2} \sum_{k=1}^{3} c_k c_i + \frac{1}{8} \left( \frac{3}{a_k} - \sum_{k=1}^{3} (c_k)^2 \right). \]

Since \( \Lambda = 0 \) Equation (4.1) implies \( 64 \pi \epsilon = \left( \sum_{k=1}^{3} c_k \right)^2 - \sum_{k=1}^{3} (c_k)^2 \). Taking the derivative we obtain

\[ 64 \pi \dot{\epsilon} = 2 \left( \sum_{k=1}^{3} c_k \sum_{l=1}^{3} \dot{c}_l - \sum_{k=1}^{3} c_k \dot{c}_k \right) \]

and inserting Equation (4.2) gives (with \( \alpha := \sum_{k=1}^{3} c_k, \beta^2 := \sum_{k=1}^{3} (c_k)^2 \))

\[ 64 \pi \dot{\epsilon} = 2 \left( \alpha \sum_{l=1}^{3} \left( -\frac{1}{2} \alpha + \frac{1}{8} (\alpha^2 - \beta^2) \right) - \sum_{k=1}^{3} c_k \left( -\frac{1}{2} \alpha + \frac{1}{8} (\alpha^2 - \beta^2) \right) \right) \]

\[ = -\alpha \left( \alpha^2 - \beta^2 \right) = -64 \pi \alpha \epsilon. \]

On the other hand, Equation (3.6) reads \( \dot{\epsilon} = -\frac{1}{2} \alpha \epsilon \). Hence we have \( \epsilon = 0 \) and (i) is proved. Assertion (v) is equivalent to \( \left( \sum_{k=1}^{3} c_k \right)^2 - \sum_{k=1}^{3} (c_k)^2 = 0 \) and therefore follows from \( \epsilon = 0 \).

Equation (4.2) simplifies to \( \dot{c}_i = -\frac{1}{2} \sum_{k=1}^{3} c_k c_i \). Thus \( \partial \left( \sum_{k=1}^{3} c_k \right) / \partial t = -\frac{1}{2} \left( \sum_{k=1}^{3} c_k \right)^2 \) which is easily integrated. Inserting the result into our differential equation for \( c_i \) we have a system of linear, uncoupled differential equations which immediately gives (iv).
The conditions in theorem 4.1 together with the usual constraint equations \((T_{ij} = 0 \text{ in our case})\) are necessary but need not be sufficient. On the other hand, there do exist non-flat vacuum solutions. In the next theorem we specialize to a case where one can solve the constraint equation easily.

**Theorem 4.2** Let \((\mathcal{D}, (\mathcal{D}g))\) be a 3-dimensional, flat manifold and assume that \(\hat{c}\) is an analytic, bilinear, symmetric tensor field such that there exists a Gaussian, orthonormal frame with respect to which \(\hat{c}\) is diagonal.

Then there exists a 4-dimensional, discontinuously signature changing vacuum spacetime \((M, g)\) such that

1. \((\mathcal{D}, (\mathcal{D}g))\) is the hypersurface of signature change,
2. \(\hat{c}(x^1, x^2, x^3) = \left((\mathcal{D}c)\right) (0, x^1, x^2, x^3)\)
3. \((\mathcal{D}c)\) is analytic

if and only if the eigenvalues \(\hat{c}_i\) satisfy \(\hat{c}_1 \hat{c}_2 + \hat{c}_2 \hat{c}_3 + \hat{c}_3 \hat{c}_1 = 0\) and either

(i) the \(\hat{c}\) are constant with respect to \((x^1, x^2, x^3)\) or
(ii) \(\hat{c}\) depends on only one variable (say, \(x^1\)) and the components \(\hat{c}_2, \hat{c}_3\) vanish identically.

Moreover, \((M, g)\) is flat if and only if two of the \(\hat{c}_i\) vanish identically.

**Proof:** In this proof we will dismiss Einstein’s summation convention if the repeated index is \(i\). Our assumptions imply that there exists coordinates such that at \(t = 0\) we have \(g_{ij} = \delta_{ij}\) and \(c_{ij} = \hat{c}_i \delta_{ij}\). The (usual) constraint equations takes the form

\[
T_{it} = \text{Ric}_{it} = \frac{1}{2} g^{ijk} (\partial_k \dot{g}_{ij} - \partial_i \dot{g}_{jk}) = 0
\]

which simplifies to \(\partial_i \hat{c}_i - \partial_j (\hat{c}_1 + \hat{c}_2 + \hat{c}_3) = 0\). Thus there exist positive functions \(f(x^2, x^3), g(x^1, x^3), h(x^1, x^2)\) and constants \(\alpha, \beta, \gamma = \pm 1\) such that \(\hat{c}_1(x^1, x^2, x^3) + \hat{c}_2(x^1, x^2, x^3) = \gamma (h(x^1, x^2))^2, \hat{c}_2(x^1, x^2, x^3) + \hat{c}_3(x^1, x^2, x^3) = \alpha (f(x^2, x^3))^2, \hat{c}_3(x^1, x^2, x^3) + \hat{c}_1(x^1, x^2, x^3) = \beta (g(x^1, x^3))^2\). Without loss of generality we can assume \(\alpha = \gamma\). Now we can express \(g(x^1, x^3)\) in terms of \(f(x^2, x^3), h(x^1, x^2)\) using theorem 4.1 (v). We obtain \(f = g = h = 0\) or \(\alpha = \beta\) and \(g(x^1, x^3) = f(x^2, x^3) + \delta h(x^1, x^2)\), where \(\delta = \pm 1\). Hence there exist functions \(u(x^1), v(x^2), w(x^3)\) such that \(f(x^2, x^3) = v(x^2) + w(x^3), g(x^1, x^3) = w(x^3) + \delta u(x^1), h(x^1, x^2) = u(x^1) - \delta v(x^2)\). With theorem 4.1 (iv) and \(c_i = (\ln g_{ij}) \delta_{ij}\) we obtain for the metric components

\[
g_{ij}(t, x^1, x^2, x^3) = \delta_{ij} \left(1 + \sum_{k=1}^{3} \hat{c}_k(x^1, x^2, x^3) t/2 \right)^{2\gamma/3}.
\]

Now it is straightforward to calculate the curvature expressions for our candidates of solution.

---

3These calculations have been performed using the software package ‘GRTensorII’ for Maple V release 3 [16].

From a purely computational point of view it is advantageous to replace the expression \(\sum_{k=1}^{3} \hat{c}_k(x^1, x^2, x^3)\) by a general function \(L(x^1, x^2, x^3)\).
Recall that by theorem 4.1 (ii) \((D_t)\text{Ric}\) must vanish for all \(t\). Thus at least two of the functions \(u, v, w\) are constant. We will assume that \(v\) and \(w\) are constant. But then we have

\[
\frac{\partial (D_t)\text{Ric}_{x_1x_2}^x}{\partial t}(0, x^1, x^2, x^3) = \frac{1}{2} \alpha \delta \frac{d^2 u}{dx^1}(x^1) \frac{d^2 u}{dx^1}(x^1),
\]

whence \(v = -w\) or \(d^2 u/(dx^1)^2(x^1) = 0\). If we assume \(d^2 u/(dx^1)^2(x^1) = 0\) then we obtain

\[
\frac{\partial (D_t)\text{Ric}_{x_3x_3}^x}{\partial t}(0, x^1, x^2, x^3) = \frac{1}{2} \alpha \delta (v + w) \left( \frac{d^2 u}{dx^1}(x^1) \right)^2.
\]

Thus either \(u, v, w\) are all constant or we have \(v = -w\) and \(u(x)\) arbitrary. Both cases lead to solutions of the vacuum equation \(\text{Ric} = 0\). This gives (i) and (ii). In the case \(v = -w\) the flat solution is recovered. In the other case the metric is flat if and only if \(v = \delta u\) or \(v = -w\) or \(w = -\delta u\). (This can be shown by explicitly calculating the Riemann tensor). By inserting these conditions into \(\hat{c}_i\) we conclude that \((M, g)\) is flat if and only if two of the \(\hat{c}_j\) vanish identically.

We will now consider solutions where \(\epsilon\) is continuous but where \(t \to (D_t)g_{ij}\) is only \(C^{2-}\), i.e. the one sided limits of the second \(t\)-derivative exist. These solutions are still restricted considerably:

**Theorem 4.3** Let \((M, g)\) be an \(m\)-dimensional, discontinuously signature changing spacetime with an adapted dust model. If \(t \to (D_t)g(x^1, \ldots, x^{m-1})\) is \(C^1\) and \(\epsilon\) is continuous with respect to \((t, x^1, \ldots, x^{m-1})\), then the surface of signature change \((D, (D_t)g)\) satisfies \((D_t)s = 2\Lambda\).

Conversely, assume that \((D, (D_t)g)\) is a 3-dimensional, analytic Riemannian manifold with \((D_t)s = 2\Lambda\). If there exists a real analytic, bilinear form \(\hat{c}\) on \(D\) such that \(g^{ik}(\nabla_k \hat{c}_{ij} - (D)\nabla_j \hat{c}_{ik}) = 0\), then there exists a discontinuously signature changing spacetime with an adapted dust model such that

(i) \((D_t)g\) is \(C^{2-}\) with respect to \(t\) and real analytic with respect to \((x^1, \ldots, x^{m-1})\),

(ii) at \(D\), \(\hat{g}_{ij}(0, x^1, x^2, x^3) = \hat{c}(x^1, x^2, x^3)\) holds,

(iii) \(\epsilon\) is continuous with respect to \((t, x^1, \ldots, x^{m-1})\),

**Proof:** The first part follows directly from

\[
8\pi\epsilon = \frac{\eta}{2}(D_t)s + \frac{1}{8} \left( \left( \text{tr}((D_t)g) \right)^2 - \left| (D_t)g \right|^2 \right) - \eta\Lambda.
\]

For the converse note that for both the Riemannian and the Lorentzian region each there exists a unique solution \(g_{ij}\) satisfying the system of equations (3.7), (3.8) with \(\phi = 0\) and
\( \hat{g}_{ij}(0,x^1,x^2,x^3) = \hat{c}(x^1,x^2,x^3) \). By assumption the constraints are initially satisfied and as in the proof of theorem 2 in [13] one sees that they are then also satisfied everywhere. Clearly, the solution is \( C^{2-} \) since it is \( C^1 \) and obtain by matching of analytic solutions.

Notice that \( (D)_s = 2\Lambda \) is an additional condition. If one replaces \( C^1 \) by \( C^2 \) in the above theorem then one obtains that \( (D),(D)_s \) is flat. The argument is almost identically but employs lemma 3.3 for \( l = 0 \). On the other hand, if one does not assume that \( \epsilon \) is continuous then there exist as much real analytic solutions as in a purely Lorentzian setting.

### 4.2 Continuity of the modified energy density \( \eta \epsilon \)

Since \( \epsilon = T_{tt} \) it is conceivable that one should not assume smoothness of \( \epsilon \) but smoothness of \( \hat{c} = \eta \epsilon \).

**Theorem 4.4** Let \( (M,g) \) be a 4-dimensional, discontinuously signature changing spacetime with an adapted dust model. If \( (D)g, \eta \epsilon \) are real analytic with respect to \( (t,x^1,x^2,x^3) \) and \( \epsilon \neq 0 \), then \( g \) is static and the surfaces \( t = \text{const} \) have constant curvature.

Conversely, for each 3-dimensional, Riemannian constant curvature manifold \( (D),(D)_s \) there exists a static, discontinuously signature changing dust solution such that \( \eta \epsilon \) is analytic and \( (D),(D)_s \) is the hypersurface of signature change.

**Proof:** As in the proof of theorem 3.1 we obtain \( (D)_t \text{Ric} = 2\Lambda (D)_t g \) for all \( t \). Since \( \dim(D_t) = 3 \) the Ricci tensor of \( (D_t),(D)_t g \) completely determines the Riemann tensor. Hence the hypersurfaces \( D_t \) have constant curvature. We also have Equation (4.1). But instead of concluding \( \Lambda = 0 \) we infer \( \text{tr}((D)_t g) = \text{tr}((D)_t g) = 0 \) for \( t = 0 \). Moreover, by successive differentiation and analyticity of \( (D)_t g \) we obtain that \( \text{tr}((D)_t g) = \text{tr}((D)_t g) = 0 \) vanishes for all \( t \). It follows that \( 8\pi \eta \epsilon = 2\Lambda \) is constant. Since \( \epsilon \neq 0 \) theorem 3.2 implies \( \text{tr}((D)_t g) = 0 \) for all \( t \) and therefore \( |(D)_t g| = 0 \) for all \( t \). But this means that the components \( \hat{g}_{ij} \) vanish and hence that \( (M,g) \) is static.

For the converse it is sufficient to check that \( \partial_t g_{ij} = 0 \) and the condition that \( (D_t,g_{ij}) \) has constant curvature imply that the Einstein equation is satisfied for an appropriate cosmological constant \( \Lambda \). \( \blacksquare \)

For completeness, we give also the existence and uniqueness theorem for the case where \( t \mapsto (D)_t g(x^1,\ldots,x^{m-1}) \) is a \( C^{2-} \)-function.

**Theorem 4.5** Let \( (M,g) \) be an \( m \)-dimensional, discontinuously signature changing spacetime with an adapted dust model. If \( t \mapsto (D)_t g(x^1,\ldots,x^{m-1}) \) is \( C^1 \) and \( \eta \epsilon \) is continuous with respect to \( (t,x^1,\ldots,x^{m-1}) \), then at the hypersurface of signature change the second fundamental form \( \hat{c}(x^1,x^2,x^3) = \hat{g}_{ij}(0,x^1,x^2,x^3) \) satisfies \( \text{tr}(\hat{c})^2 - |\hat{c}|^2 = 0 \).

Conversely, assume that \( (D),(D)_s \) is a 3-dimensional, analytic Riemannian manifold. If there exists a real analytic, bilinear form \( \hat{c} \) on \( D \) such that \( g^{ik}(D)\nabla_k \hat{c}_{ij} - (D)\nabla_j \hat{c}_{ik} = 0 \) and \( (\text{tr}(\hat{c}))^2 - |\hat{c}|^2 = 0 \), then there exists a continuously signature changing spacetime with an adapted dust model such that

(i) \( (D)_t g \) is \( C^{2-} \) with respect to \( t \) and real analytic with respect to \( (x^1,\ldots,x^{m-1}) \),
\[ \hat{g}_{ij}(0, x^1, x^2, x^3) = \hat{c}(x^1, x^2, x^3) \] holds,

(iii) \( \eta \epsilon \) is continuous with respect to \( (t, x^1, \ldots, x^{m-1}) \),

**Proof:** The proof is analogous to the proof of theorem 4.3. \( \square \)

5 Conclusion

It is straightforward to define discontinuously signature changing spacetimes as smooth (or even analytic) objects: Writing \( g = -\eta dt^2 + g_{ij}(t, x^1, \ldots, x^{m-1})dx^i dx^j \) one can consider the one-parameter family of Riemannian metrics \( g_{ij} dx^i dx^j \). Weak junction conditions are implied by the requirement that this one-parameter family depends smoothly on the parameter \( t \) (many authors only demand \( C^1 \) which is also possible). Requiring the Einstein equation in both the Riemannian and the Lorentzian region is then a natural generalization of general relativity.

This theory can only give rise to explanations if additional assumptions are imposed. Our main assumption in this paper was that \( (M, g) \) is a dust spacetime with energy density \( \epsilon \) such that either \( \epsilon \) or \( \eta \epsilon \) is continuous. In theorems 4.1, 4.2 and 4.4 we have seen that only for very special solutions the 1-parameter family \( \{ (D_t)^\epsilon g_{ij} \} \) is analytic. If one assumes that \( t \mapsto (D_t)^\epsilon g_{ij}(x^1, \ldots, x^{m-1}) \) is \( C^2 \), the class of solutions is still more restricted than in the purely Lorentzian case.\[ \square \] It follows that the regularity conditions imposed by us effectively restrict the space of solutions but still leave room for non-trivial spacetimes. This result should be compared with analogous results for other implementations of signature change:

(i) In the case of smooth signature with strong junction condition the hypersurface of signature change must be totally geodesic. This accounts for half the initial conditions for Einstein’s equations. The existence of a totally geodesic hypersurface is a highly non-generic feature and has been used by Hayward to link smooth signature change to inflation. It should be noted that this interpretation is only possible if the strong energy condition is not valid near the hypersurface of signature change. In fact, by a slight modification of a singularity theorem of Hawking, spacetime would collapse if this energy condition was satisfied. This would clearly be in disagreement with observation.

(ii) If one considers discontinuous signature change with strong junction conditions, then one obtains different answers according to the differentiability conditions one imposes on the spacelike metric components \( g_{ij} \). If one merely assumes that the \( g_{ij} \) are \( C^1 \) but not necessarily \( C^2 \) then one can recover the result from smooth signature change. If the \( g_{ij} \) are assumed to be analytic functions and the energy momentum tensor is assumed to be smooth then spacetime must be static. Under these conditions, the only solution to the vacuum equation is flat space.\[ \square \]

If the theory of discontinuous signature change with weak junction conditions is viable then our regular solutions should be of special interest. Unfortunately, assuming the regularity conditions of theorem 4.1 or theorem 4.4, the surviving solutions are static or vacuum and

\[ ^4 \text{Observe that there exist as many solutions of low differentiability (} C^{2-} \text{) with discontinuous energy density as in the purely Lorentzian case. One can just solve the Riemannian and the Lorentzian part for the same analytical initial conditions separately using the theorem of Cauchy Kowalewska.} \]
therefore do not seem to agree with observation. While this may be considered as a hint against signature change with weak junction conditions, it should be kept in mind that we have only considered a very simple macroscopic matter model. It would be interesting to learn whether more sophisticated matter models could lead to solutions which are physically more realistic. Theorem 3.2 shows that are also very restricted, which may be viewed as a preliminary result in this direction. Theorems 4.3 and 4.5 can be easily generalized to scalar field matter models.

It is interesting to observe that assuming vanishing cosmological constant, isotropy, a dust matter model, and continuous energy density, only \((k = 0)\)-Robertson-Walker-spacetimes are compatible with signature change under weak regularity conditions (theorem 4.3, see also [4]). Adopting signature change, we would therefore have a simple explanation of the flatness problem which partially motivated inflation. If (in this setting) the assumption of continuous energy density is replaced by the condition that \(\eta e\) is continuous, then the hypersurface of signature change must be totally geodesic. This would imply that the universe is collapsing in contradiction to experience.

The reader should also keep in mind that smooth and discontinuous signature changing dust models with strong junction conditions are ruled out by a very different mechanism. Pressureless dust satisfies the strong energy condition and therefore dust spacetimes should be collapsing in the Lorentzian region — in contradiction to observation. Still, in this context it may be worthwhile to study alternative matter models which violate the energy conditions near the hypersurface of signature change. Another way to save signature change with strong junction conditions would be to impose signature change in the future rather than in the past.

It seems that the weak junction conditions for continuous energy density corresponds best to observation if one assumes that matter satisfies the strong energy condition near the hypersurface of signature change. However, it should be remarked that this energy condition may not be justified, given the early stage of the universe’s evolution where signature change is supposed to occur.

Finally, it should be noted that there is still much controversy about which implementation of signature change should be considered ‘correct’. See [1, 7, 8, 12, 13, 1, 11, 5, 3, 10, 2], for instance. While in Hayward’s papers there have been used many harsh words with respect to weak junction conditions, I am not aware of any previous work which puts them to the physical test examining the consequences of the proposal.

Acknowledgement I would like to thank Franz Embacher and Tevian Dray for discussions about the weak junction conditions.

References

[1] M. Carfora and G. Ellis. The geometry of classical change of signature. Int. J. Mod. Phys., D4:175–188, 1995.

[2] T. Dray and C. Hellaby. Comparison of approaches to classical signature change. Phys. Review D, 52:7333–7339, 1995. Reply to [10].

[3] T. Dray and C. Hellaby. Comment on “smooth and discontinuous signature change in general relativity”. Gen. Rel. Grav., 1996. Comment on [12]. To be published.
[4] G. F. R. Ellis, A. Sumeruk, D. Coule, and C. Hellaby. Change of signature in classical relativity. *Class. Quantum Grav.*, 9:1535–1554, 1992.

[5] Franz Embacher. Actions for signature change. *Phys. Review D*, 51:6764–6777, 1995.

[6] J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Review D*, 28(12):2960–2975, 1983.

[7] S. A. Hayward. Signature change in general relativity. *Class. Quantum Grav.*, 9:1851–1862, 1992.

[8] S. A. Hayward. Junction conditions for signature change. *Preprint gr-qc/9303034*, 1993.

[9] S. A. Hayward. On cosmological isotropy, quantum cosmology and the Weyl curvature hypothesis. *Class. Quantum Grav.*, 10:L7–L11, 1993.

[10] S. A. Hayward. Comment on “failure of standard conservation laws at a classical change of signature”. *Phys. Review D*, 52:7331–7332, 1995. Cf. [11, 2].

[11] C. Hellaby and T. Dray. Failure of standard conservation laws at a classical change of signature. *Phys. Review D*, 49:5096–5104, 1994.

[12] M. Kossowski and M. Kriele. Smooth and discontinuous signature type change in general relativity. *Class. Quantum Grav.*, 10:2363–2371, 1993.

[13] M. Kossowski and M. Kriele. The Einstein equation for signature type changing space-times. *Proc. Roy. Soc. Lond. A*, 446:115–126, 1994.

[14] M. Kriele. Reply to: Comment on “smooth and discontinuous signature change in general relativity”. *Gen. Rel. Grav.*, 1996. Reply to [3]. To be published.

[15] M. Kriele and J. Martin. Black holes, cosmological singularities and change of signature. *Class. Quantum Grav.*, 12(2):503–511, 1995.

[16] P. Musgrave, D. Pollney, and K. Lake. GRTensorII. The program and its documentation can be found at astro.queensu.ca in the directory /pub/grtensor, 1994. Queen’s University, Kingston, Ontario, Canada.