Algebraic Structures of $N=(4,4)$ and $N=(8,8)$ SUSY Sigma Models on Lie groups and SUSY WZW Models

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Abstract
In this study, algebraic structures of the $N=(4,4)$ and $N=(8,8)$ SUSY two dimensional sigma models on Lie groups (in general) and the $N=(4,4)$ and $N=(8,8)$ SUSY WZW models (in special) are obtained. These algebraic structures are reduced to the Lie bialgebraic structures as for the $N=(2,2)$ case; with the difference that there is a one 2-cocycle for the $N=(4,4)$ case and two 2-cocycles for the $N=(8,8)$ case. Some examples are investigated.

1 Introduction
Supersymmetric two dimensional nonlinear sigma models have important role in theoretical and mathematical physics such as their numerous string applications. Let us have a short bibliography for this subject. The relation between these theories and geometry of the target spaces have been studied about thirty five years ago [1]. The biHermitean geometry of the target spaces of the $N=2$ extended supersymmetric sigma models was first realized in [2] (see also [3]). Then the extensions to more supersymmetries $N=4$ and $N=8$ have been investigated [4] (see also [5]). The sigma models with extended supersymmetry can only be defined on a restricted class of target manifolds, more supersymmetry implies more restriction on these geometries [5]. The extended supersymmetric sigma models on Lie group manifolds and also SUSY WZW models have been studied in [6]. The $N=2$ and $N=4$ extended superconformal field theories in two dimensions and also their correspondence with Manin triples have been investigated in [7] and [8]. Also there are some notes about $N=8$ superconformal field theory in [9]. The algebraic study of $N=(2,2)$ SUSY WZW models and also $N=(2,2)$ SUSY sigma models on Lie groups (algebraic biHermitian structures) have been studied in [9] and [10], respectively.

In this paper, we try to obtain the algebraic structures of $N=(4,4)$ and $N=(8,8)$ SUSY sigma models on Lie groups (in general) and the algebraic structures of SUSY WZW models (especially).

The outline of the paper is as follows: in section two, we review the $N=(2,2)$ SUSY sigma models on Lie groups and their algebraic biHermitian structures [10] as well as SUSY WZW models and their correspondence to Manin triples. Then in section three, we obtain the algebraic bihypercomplex structures for the $N=(4,4)$ SUSY sigma models on Lie groups and specially for the $N=(4,4)$ SUSY WZW models. We show their correspondence to Lie bialgebra with one 2-cocycle, at the end of this section we give an example. Finally in section four the algebraic structure of the $N=(8,8)$ two dimensional SUSY sigma models on Lie group is investigated and show that for the $N=(8,8)$ SUSY WZW models these algebraic structure is the Manin triples with two 2-cocycles, an example is given at the end of this section.

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2 \( N = (2, 2) \) SUSY sigma models on Lie groups and SUSY WZW models

In this section, for self containing of the paper we will review briefly the geometric description of the \( N = (2, 2) \) SUSY WZW and sigma-models on Lie groups [2]-[6] and their algebraic structures [9],[10]. We will use the \( N = (1, 1) \) action to the description of \( N = (2, 2) \) model; and impose extended supersymmetry on the superfields. With the knowledge that \( N \) supersymmetric sigma-models have \( N \) supersymmetric generator \( (Q_i) \) and \( N - 1 \) complex structures \( (J_i) \) on manifolds \( M \) such that for \( N = (p, q) \) SUSY sigma-models in two-dimension then we will have \( p \) right-handed generators \( (Q^+_{\pm}) \) and \( q \) left-handed generators \( (Q^-_{\pm}) \) respectively, then \( N = (1, 1) \) SUSY sigma model have one right-handed generators \( (Q^+_{\pm}) \) and one left-handed generators \( (Q^-_{\pm}) \) and the action on the manifold \( M \) is written as follows [2]:

\[
S = \int d^2 \sigma d^2 \theta D_\Phi \Phi \mu D_\mu \Phi (G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)),
\]

such that this action is invariant under the following supersymmetry transformation:

\[
\delta_1(\epsilon) \Phi^\mu = i(\epsilon^+ Q^+ + \epsilon^- Q^-) \Phi^\mu,
\]

where \( \Phi^\mu \) are \( N = 1 \) superfields; so that their bosonic parts are the coordinates of the manifold \( M \). Further more the bosonic parts of the \( G_{\mu\nu}(\Phi) \) and \( B_{\mu\nu}(\Phi) \) are metric and antisymmetric tensors on \( M \) respectively. Note that in the above relations \( Q^\pm \) and \( D^\pm \) are supersymmetry generators and superderivative, respectively and \( \epsilon^\pm \) are parameters of supersymmetry transformations. The above action has also invariant under the following extended supersymmetry transformation [2]:

\[
\delta_2(\epsilon) \Phi^\mu = \epsilon^+ D_+ \Phi \mu J^\mu_{\nu,\rho}(\Phi) + \epsilon^- D_\Phi \Phi ^\mu J^\mu_{\nu,\rho},
\]

where \( J^\mu_{\nu,\rho} \in TM \otimes T^* M \). The consequence of invariance of the action (1) under the above transformations are the following conditions on \( J^\mu_{\nu,\rho} [2] \):

\[
J^\mu_{\pm,\pm} = -\delta^\mu, \quad J^\mu_{\pm,\pm} G_{\mu\nu} = -G_{\mu\nu} J^\mu_{\pm,\pm}, \quad \nabla^\pm_{\rho} J^\mu_{\pm,\pm} = J^\mu_{\pm,\pm,\rho} + G^\pm_{\rho\sigma} J^\mu_{\pm,\pm} - \Gamma^\pm_{\rho\sigma} J^\mu_{\pm,\pm} = 0,
\]

where the extended connections \( \Gamma^\pm_{\rho\sigma} \) have the following forms:

\[
\Gamma^\pm_{\rho\sigma} = \Gamma^\rho_{\rho\sigma} \pm G^\mu_{\rho\sigma} H_{\rho\sigma\mu},
\]

such that

\[
H_{\rho\sigma\mu} = \frac{1}{2}(B_{\rho\sigma,\mu} + B_{\rho\sigma,\mu} + B_{\sigma,\mu,\rho}),
\]

and \( \Gamma^\rho_{\rho\sigma} \) are Christofel symbols.

In order to have a closed supersymmetry algebra we must have the integrability condition on the complex structures \( (J^\pm) \) as follows [2]:

\[
N^\rho_{\mu\nu}(J^\pm) = J^\gamma_{\rho\nu} \partial_{\gamma} J^\mu_{\pm,\pm} - J^\sigma_{\mu\nu} \partial_{\sigma} J^\rho_{\pm,\pm} = 0.
\]

In this manner the \( N = (2, 2) \) SUSY structure of the sigma model on \( M \) is equivalent to existence of the biHermitian complex structure \( (J^\pm) \) on \( M \) such that their covariant derivatives with respect to extended connection \( \Gamma^\pm_{\rho\sigma} \) are equal to zero (6). If \( M \) is a Lie group \( G \) then in the non-coordinate bases, we have:

\[
G_{\mu\nu} = L^A_{\mu} L^B_{\nu} G_{AB} = R^A_{\mu} R^B_{\nu} G_{AB}, \quad f_{AB}^C = L^C_{\nu} (L^A_{\mu} \partial_{\mu} L^B_{\nu} - L^B_{\mu} \partial_{\mu} L^A_{\nu}) = -R^A_{\nu} (R^B_{\mu} \partial_{\mu} R^C_{\nu} - R^C_{\nu} \partial_{\mu} R^B_{\nu}),
\]
where $G_{AB}$ is the ad-invariant nondegenerate metric and $H_{ABC}$ is antisymmetric tensor on the Lie algebra $\mathfrak{g}$ of the Lie group $G$. Note that $L^\mu_A(R^\nu_A)$ and $L^\mu_A(R^\rho_A)$ are components of left(right) invariant one-forms and their inverses on the Lie group $G$; $f_{ABC}$ are structure constants of the Lie algebra $\mathfrak{g}$ and $J_B^A$ is an algebraic map $J : \mathfrak{g} \to \mathfrak{g}$ or algebraic complex structure. Now, using the above relations and the following relations for the covariant derivative of the left invariant veilbin [11]:

\[
\nabla^\rho L^\eta_A = -\frac{1}{2}[f^{\rho(\eta)}_A + f^{\eta \rho}_A - T^{(\eta)}_A - T^{\eta \rho}_A - L^{\eta B} \nabla^\rho G_{BA} + L^{\rho B} \nabla^\eta G_{AB} + L^{(\eta A}_A L^B \nabla_A G^{AB}],
\]

then, we have the following algebraic relations, for the bi-Hermitian geometry of the $N = (2, 2)$ SUSY sigma models [10]:

\[
G_{\chi A} = (\chi A G)^t,
\]

\[
J_C^B J_B^A = -\delta_C^A,
\]

\[
J_C^A G_{AB} J_B^D = G_{CD},
\]

\[
H_{EFG} = J_E^A J_F^C H_{ACG} + J_G^A J_E^C H_{ACP} + J_F^A J_G^C H_{ACE},
\]

\[
(\chi_A + \chi A G) J = [(\chi_A G + H_A) J]^t,
\]

where $(\chi_A)_{BC} = -f_{ABC}$ are the matrices in the adjoint representation and we have $(H_A)_{BC} = H_{ABC}$ for the matrices $H_A$. Note that relation (15) represents the ad-invariance of the Lie algebra metric $G_{AB}$. One can use relation (15)-(19) as a definition of algebraic bi-Hermitian structure on Lie algebra [10]; and calculate and also classify such structures on the Lie algebras [10]. For the $N = (2, 2)$ SUSY WZW models we have $H_{ABC} = f_{ABC}$; then (19) automatically satisfy and from (16) we obtain the determinant of $J^2$ is $(-1)^n$, i.e the dimension of the Lie algebra $\mathfrak{g}$ ($n$) must be even and $J^B_A$ has eigenvalues $\pm i$. If we choose a basis $T_A = (T_a, T_\bar{a})$ for the Lie algebra $\mathfrak{g}$ we will have [9]:

\[
J = \begin{pmatrix}
  0 & 0 \\
  i \delta_b^a & 0 \\
  0 & -i \delta_b^a
\end{pmatrix},
\]

where this form of $J$ is satisfying in (18). In this basis according to (17) we must have the following form for $G_{AB}$:

\[
G = \begin{pmatrix}
  0 & g \\
  g^t & 0
\end{pmatrix},
\]

where $g$ is a $\frac{n}{2} \times \frac{n}{2}$ symmetric matrix. According to (18), we have $f_{abc} = 0$ and $f_{\bar{a} \bar{b} \bar{c}} = 0$, this means that $f_{\bar{a} \bar{b}} = f_{\bar{b} \bar{a}} = 0$ i.e $T_a$ and $T_\bar{a}$ form Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ such that $(\mathfrak{g}_+, \mathfrak{g}_-)$ is a Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple [9]. The relation between Manin triples and $N = 2$ superconformal models (from the algebraic OPE point of view) was first pointed out in [7]. Also the relation of $N = (2, 2)$ WZW models and Manin triple (from the action point of view) was pointed in [9]. In [10] we have obtained all algebraic bi-Hermitian structures related to four dimensional real Lie algebra. Let us consider a simple example for $N = (2, 2)$ SUSY WZW models correspond to the following non-Abelian four dimensional Manin triple $A_{4,8}$ [10]:

\[
[T_2, T_4] = T_2, \quad [T_3, T_4] = -T_3, \quad [T_2, T_3] = T_1,
\]

\[
[T_2, T_4] = T_2, \quad [T_3, T_4] = -T_3, \quad [T_2, T_3] = T_1,
\]

\[
G = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & -1 & 0 \\
  0 & -1 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}, \quad J = \begin{pmatrix}
  -i & 0 & 0 & 0 \\
  0 & i & 0 & 0 \\
  0 & 0 & -i & 0 \\
  0 & 0 & 0 & i
\end{pmatrix}.
\]
3 \(N=(4,4)\) SUSY WZW and sigma models on Lie groups

As mentioned above, the correspondence between \(N = 2\) and also \(N = 4\) and \(N = 8\) superconformal Kac-Moody algebra and Manin triples has been investigated in [7] and the Manin triples construction of \(N = 4\) superconformal field theories has also investigated in [8], but up to now the algebraic structures of the \(N = (4,4)\) and \(N = (8,8)\) SUSY sigma models on Lie groups and also the \(N = (4,4)\) and \(N = (8,8)\) SUSY WZW models and their relations to Manin triples (from the action point of view) are not studied explicitly. Here, in this section we consider \(N = (4,4)\) case and in the next section the \(N = (8,8)\) case.

As in the previous section, we consider the \(N = (1,1)\) SUSY sigma model action (1) where invariant under transformation (2). Now we will consider for \(N = (4,4)\) case the invariance of that action under the following SUSY transformations [2,3] (instead of (3)) and also \(N = (4,4)\) SUSY sigma model must be have four right-handed generators \((Q_{+r})\) and four left-handed generators \((Q_{-r})\) and three complex structures \((J_{\pm r})\):

\[
\delta_{2r}(\epsilon) \Phi^{\mu} = c_{r}^{\mu} D_{+} \Phi^{\nu} J_{+ r \nu}^{\mu}(\Phi) + c_{r}^{- \mu} D_{-} \Phi^{\nu} J_{- r \nu}^{\mu}(\Phi), \quad r = 1, 2, 3,
\]

such that the constrains on the complex structures are followed as [5]:

\[
J_{\pm r \nu} \lambda J_{\pm s \lambda} = - \delta_{r s}^{\mu},
\]

\[
J_{\pm r \nu} \lambda J_{\pm s \lambda} = J_{\pm t \nu} \lambda, \quad r \neq s \neq t = 1, 2, 3,
\]

\[
J_{\pm r \nu} \mu G_{\mu \nu} = - G_{\mu \nu} J_{\pm r \nu}^{\mu},
\]

\[
\nabla_{\mu}^{(\pm)} J_{\pm r \nu} = \partial_{\mu} J_{\pm r \nu}^{\mu} + \Gamma_{\mu \sigma}^{\pm \nu} J_{\pm r \nu}^{\sigma} - \Gamma_{\mu \nu}^{\pm \sigma} J_{\pm r \nu}^{\sigma} = 0,
\]

where the closed characteristic of the algebra of SUSY transformations (i.e \([\delta_{r}^{\pm}(\epsilon_{r}), \delta_{s}^{\pm}(\epsilon_{s})]\), and \([\delta_{r}^{+}(\epsilon_{r}), \delta_{s}^{+}(\epsilon_{s})]\) have been consequences the following relations [3]:

\[
J_{\pm r} \lambda_{[\mu} \partial_{\nu} J_{\pm s} \gamma_{\nu]} - J_{\pm s} \lambda_{[\mu} \partial_{\nu} J_{\pm r} \gamma_{\nu]} = 0,
\]

\[
J_{\pm r \nu} \lambda J_{\pm s \mu} + J_{\pm s \nu} \lambda J_{\pm r \mu} = 0, \quad r \neq s,
\]

\[
J_{\pm r} \gamma_{\nu} \partial_{\mu} J_{\pm s} \lambda_{\mu} + J_{\pm s} \gamma_{\nu} \partial_{\mu} J_{\pm r} \lambda_{\mu} + J_{\pm s} \lambda_{[\mu} \partial_{\nu} J_{\pm r} \gamma_{\nu]} = 0,
\]

such that these are Nijenhuis-concomitant [12] for complex structures \(J_{\pm r}\). When the background is a Lie group \(G\) then in non-coordinate bases ((10)-(13)) the geometrical relations (25)-(31) have the following algebraic forms:

\[
J_{C} C^{B} J_{F} B^{A} = - \delta_{C} A^{A}, \quad J_{C} B^{B} J_{F} A^{A} = J_{C} A^{A}, \quad J_{C} A^{B} J_{F} D^{D} = G_{C} D^{D},
\]

\[
H_{EFG} = J_{E} A^{A} J_{F} C^{B} H_{ACG} + J_{C} A^{A} J_{E} F^{B} H_{ACF} + J_{F} A^{A} J_{C} G^{C} H_{ACE},
\]

\[
(J_{A} + \chi_{A} G) J_{r} = [(\chi_{A} G + H_{A}) J_{r}],
\]

\[
J_{sD} B^{B} J_{r} B^{A} + J_{rD} B^{B} J_{s} B^{A} = 0, \quad r \neq s,
\]

\[
J_{B} A^{A} B^{B} J_{C} B^{C} A^{A} + J_{C} A^{A} B^{B} J_{B} A^{A} C^{C} J_{s} A^{A} + J_{s} A^{A} B^{B} J_{C} A^{A} J_{s} B^{B} C^{C} + J_{C} A^{A} B^{B} J_{B} A^{A} J_{s} B^{B} C^{C} + J_{s} A^{A} B^{B} J_{B} A^{A} J_{s} A^{A} C^{C} = 0,
\]

In this way, relation (32)-(38) define the algebraic bihypercomplex structures\(^{2}\) on the Lie algebra \(g\), such that we have three algebraic complex structures \(J_{r}\) (r = 1, 2, 3) where by use of (33) only two of them are independent i.e we have two algebraic independent complex structures (e.g \(J_{1}\) and \(J_{2}\)). As for the \(N = (2,2)\) case for the

\(^{1}\)The Nijenhuis concomitant of \(J_{r}\) and \(J_{s}\) has the following form [12]:

\[
N(1, J)_{\mu \nu} = [I^{r} \mu \partial_{\nu} J^{\gamma \nu} - [\mu \leftrightarrow \nu] - (I^{r} \gamma_{\nu} \partial_{\mu} J^{\gamma \nu} - (\mu \leftrightarrow \nu)) + (I \leftrightarrow J)
\]

\(^{2}\)Similar to the name of bihypercomplex geometry [13].
where with the following 2-cocycle:

\[ \delta \gamma = \delta \gamma _{-} \otimes \delta \gamma _{+} \]

Using the following form for the 2-cochain:

\[ (J_2)_A^B = R_A^B = \begin{pmatrix} R_a^b & R_a^b \\ R_b^a & R_b^a \end{pmatrix}, \]

and

\[ (J_2)_A^B = R_A^B = \begin{pmatrix} R_a^b & R_a^b \\ R_b^a & R_b^a \end{pmatrix}, \]

where we have the basis \( T_A = \{ T_a, T_a \} \) for the Lie algebra \( g \). Then from (37) one can obtain \( R_a^b = R_b^a = 0 \), and from (34) we obtain that \( R^F = -R \), then from (32) we see that dimension of \( J_2 \) must be \( 4n \) where \( n \) is an integer number. So the dimension of Lie algebra \( g \) must be \( 4n \). Note that from (35) as for \( N = (2, 2) \) case we see that \( g = g_+ \oplus g_- \) where \( g_+ \) and \( g_- \) are Lie subalgebras with basis \( T_1 = \{ T_a, T_a \} \) and \( T_1' = \{ T_a', T_a' \} \), \( a, \bar{a}, 1, ..., n \), such that the basis for \( g \) are now \( T_A = \{ T_1, T_1' \} \) and they form a Lie bialgebra. Now from (38) we have:

\[ f_{AB}^C R_{CD}^E + f_{DA}^C R_{DB}^E - f_{DB}^C R_{DA}^E = 0. \]

This means that we have a 2-cocycle. To show this we consider the definition of coboundary operator \( \delta \) on an \( i \)-cochain \( \gamma \) on the Lie algebra \( g \) with values in the space \( M \) as follows [13]:

\[ \delta \gamma (T_0, T_1, ..., T_i) = \Sigma_{j=0}^i T_j \otimes (\gamma (T_0, ..., \hat{T}_j, ..., T_i)) + \Sigma_{j<k} (-1)^{j+k} \gamma ([T_j, T_k], T_0, ..., \hat{T}_j, ..., \hat{T}_k, ..., T_i), \]

\[ \forall T_A \in g. \]

The 2-cochain \( \gamma \) is 2-cocycle when \( \delta \gamma = 0 \). Now for the case that \( M = \mathbb{C} \) we have:

\[ -\delta \gamma (T_0, T_1, T_2) + T_0 \otimes (\gamma (T_0, T_2)) + T_1 \otimes (\gamma (T_0, T_2)) + T_2 \otimes (\gamma (T_0, T_1)) \]

\[ -\gamma ([T_0, T_1], T_2) + \gamma ([T_0, T_2], T_1) - \gamma ([T_1, T_2], T_0) = 0. \]

Using the following form for the 2-cochain:

\[ \gamma (T_A, T_B) = (R_{AB})^{\Gamma \Lambda} T_\Gamma \otimes T_\Lambda + (R_{AB})^{\Gamma \Lambda} T_\Gamma \otimes T_\Lambda + (R_{AB})^{\Gamma \Lambda} T_\Gamma \otimes T_\Lambda + (R_{AB})^{\Gamma \Lambda} T_\Gamma \otimes T_\Lambda, \]

in (43) after some calculation one can obtain (41). In this way the algebraic structure of \( N = (4, 4) \) WZW models is also Lie bialgebra as for the \( N = (2, 2) \) WZW models with this difference that for the \( N = (4, 4) \) case, we have Lie bialgebra with a 2-cocycle, such that the independence algebraic complex structures \( (J_1, J_2) \) are anticommutate (37).

As for the \( N = (2, 2) \) case we consider the non-Abelian four dimensional Manin triple \( \mathbf{A}_{4,8} \). Now in this case \( (N = (4, 4)) \) we have the following forms for the metric \( G \) and complex structures \( J_1 \) and \( J_2 \):

\[ G = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

\[ J_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \]

with the following 2-cocycle:

\[ R = \begin{pmatrix} 0 & i \bar{I} \\ i \bar{I} & 0 \end{pmatrix}, \]

where \( I \) is \( 2 \times 2 \) unit matrix.
4 N=(8,8) WZW and sigma models on Lie groups

Now, as for the N = (4,4) case we consider the action (1) again; such that this action is a invariant under SUSY transformation (2) as well as under the following second SUSY transformations [3]:

\[
\delta_{2r}(\epsilon) \Phi^\mu = \epsilon^*_r D_+ \Phi^\nu J_{+\nu}^\mu + \epsilon^r_- D_- \Phi^\nu J_{-\nu}^\mu, \quad r = 1, \ldots, 7
\]

(47)

where for these transformations we have fourteen \( J_{\mp\nu} \) geometric complex structures. As for the \( N = (4,4) \) case from the invariance of the action (1) under transformation (47) and also closed characteristic of the algebra of transformations one can obtain again relations similar to (25)-(31) with \( (r = 1, \ldots, 7) \) [3] and also the same algebraic relations (32)-(38). For this case from (34) we have the following relations among algebraic complex structures

\[
J_1 J_2 J_3 J_4 J_5 J_6 = J_7, \quad J_2 J_3 J_4 J_5 = J_6, \quad J_3 J_4 = J_5, \quad J_1 J_2 = J_3,
\]

(48)

therefore only three of them (e.g. \( J_1, J_2 \) and \( J_3 \)) are independent. As for the \( N = (4,4) \) case for \( N = (8,8) \) SUSY WZW we obtain the following forms for the complex structures \( J_1, J_2 \) and \( J_3 \) and also for \( G \):

\[
J_1 = \begin{pmatrix} i \delta^a_b & 0 \\ 0 & -i \delta^a_b \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g \\ g^t & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & R_{1b}^a \\ R_{1b}^a & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & R_{2b}^a \\ R_{2b}^a & 0 \end{pmatrix},
\]

(49)

where in this case from (32) we conclude that the dimension of the algebra \( g \) must be \( 8n \) with \( n \) is an integer and relation (35) for \( J_2 \) reduce the Lie bialgebra structures with Lie subalgebras \( g_+ \) and \( g_- \) with dimension \( 4n \). In this case relation (38) reduce to the following relations:

\[
f_{AB}^D R_{1D}^C + f_{DA}^C R_{1B}^D - f_{BD}^C R_{1A}^D = 0, \tag{50}
\]

\[
f_{AB}^D R_{2D}^C + f_{DA}^C R_{2B}^D - f_{BD}^C R_{2A}^D = 0, \tag{51}
\]

i.e the algebraic structures of \( N = (8,8) \) WZW models are Lie bialgebras with two 2-cocycles and three algebraic complex structure \( J_1, J_2 \) and \( J_3 \) where anticommutate under (37).

As an example, consider a four dimensional complex Lie algebra \( L_9 \) with the following commutations relations [5]:

\[
[T_1, T_2] = T_2, \quad [T_3, T_4] = T_4.
\]

(52)

one of the dual Lie algebra for the above Lie algebra is \( \tilde{L}_9 \) that satisfy in the following mixed Jacobi identities:

\[
f_{mk}^i \tilde{f}^{jm} t - f_{ml}^i \tilde{f}^{jm} k - f_{mk}^j \tilde{f}^{im} t + f_{ml}^j \tilde{f}^{im} k = f_{kl}^m \tilde{f}^{ij} m.
\]

(53)

with following commutation relations:

\[
[T_1, T_2] = T_2, \quad [T_3, T_4] = T_4.
\]

(54)

Now, for this 8 dimensional Lie algebra \( g \) we have obtained the following algebraic complex structures \( J_1, J_2 \) and \( J_3 \) and metric \( G \):

\[
J_1 = \begin{pmatrix} -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{pmatrix},
\]

(55)
with the following two 2-cocycles:

\[ R_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]  

(56)

5 conclusion

We have obtained the algebraic structures of the \( N = (4, 4) \) and \( N = (8, 8) \) SUSY two dimensional sigma models on Lie groups (in general) and the \( N = (4, 4) \) and \( N = (8, 8) \) SUSY WZW models (in special). We have shown that as for the \( N = (2, 2) \) case these structures correspond to the Lie bialgebra structures with one 2-cocycle for the \( N = (4, 4) \) and two 2-cocycles for the \( N = (8, 8) \) case. As an open problem in the forthcoming work one can use the relations of algebraic structures for \( N = (4, 4) \) and \( N = (8, 8) \) ((32)-(38)) to obtain and classify all these structures on low dimensional Lie algebra as for the \( N = (2, 2) \) case [10].

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