Pion scattering and the skyrmion in $\chi PT$ with many pions

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Abstract
The large $N$ effective action of the non-linear sigma model based in the coset $O(N+1)/O(N)$ is obtained. The renormalization of this effective action requires the introduction of an infinite set of counter terms. However, there exist particular cases where, at some scale, only a finite number of non-zero coupling constants are present. This fact make possible a one parameter fit of the $I = J = 0$ low-energy pion scattering. The corresponding non-local effective action is used to study the properties of the skyrmion which is shown to be unstable in this approximation.

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1 Introduction

From the pioneering work by Weinberg [1] on phenomenological lagrangians, a lot of work has been devoted to this field. Specially remarkable was the work by Gasser and Leutwyler [2] where this technique was applied, at the one-loop level, to the description of the low-energy behavior of the Nambu-Goldstone bosons (pions) corresponding to the spontaneous breaking of the chiral symmetry of hadron interactions, i.e. the so-called Chiral Perturbation Theory ($\chi$PT). More recently, the phenomenological lagrangian approach has also been used in the context of the symmetry breaking sector of Standard Model and, in particular, it was applied to the scattering of the longitudinal components of the weak gauge bosons [3].

In spite of the great phenomenological success of $\chi$PT, some problems still must be solved. Typically $\chi$PT computations are done to the one-loop level. This fact puts strong limitations to the region of applicability of $\chi$PT which is restricted to the very low-energy regime since, at higher energies, conflicts with unitarity appear [4]. Several methods have been proposed in the literature to improve the unitarity behavior of $\chi$PT like the introduction of new fields corresponding to resonances [5], the Padé approximants [4], the inverse amplitude method [6] etc. More recently it was suggested that it is possible to define a sensible large $N$ approach to $\chi$PT ($N$ being the number of Nambu-Goldstone bosons) which is consistent in principle at any energy. In particular, it was shown in [7] how this approach can be used to compute and
renormalize the pion scattering amplitude and in [8] it was used to study the \( \gamma \gamma \rightarrow \pi^0 \pi^0 \) reaction. More recently, some of the results of [7] were reobtained in [9] by using a different method.

In this work we pursue this issue further as follows: First we show a functional computation of the effective action for pions in the model \( O(N+1)/O(N) \) to the order of \( 1/N \) which is non-local. From this effective action we obtain the elastic scattering amplitude for pions. We use this result to make a one parameter fit of the experimental data corresponding to the \( I = J = 0 \) channel. Then we obtain the mass functional for the skyrmion from the large \( N \) effective action. We arrive to the conclusion that the skyrmion is not stable in the many pion approximation considered here and we finish by discussing which could be the reasons for that and giving the main conclusions of our work.

\section{The effective action for pions in the 1/N expansion}

To define the large \( N \) limit of \( \chi \)PT we start from the two-flavor chiral symmetry group \( SU(2)_R \times SU(2)_L \). Then we use the equivalence of the coset spaces \( SU(2) \times SU(2)/SU(2) = O(4)/O(3) = S^3 \) to extend this symmetry pattern to \( O(N+1)/O(N) = S^N \) so that \( N \) can be understood as the number of Nambu-Goldstone bosons (NGB) or pions. The fields describing these NGB can be chosen as arbitrary coordinates on the coset manifold \( S^N \). The most general \( O(N+1) \) invariant lagrangian can be obtained as a derivative expansion which is covariant respect to both, the space-time and the
$S^N$ coordinates. The lowest order is the well known non-linear sigma model lagrangian given by:

$$\mathcal{L}_{NLSM} = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b$$  \hspace{1cm} (1)

In our computations we will select the standard coordinates $\pi_a = \pi^a$ where $g_{ab}(\pi) = \delta_{ab} + \pi_a \pi_b / (N F^2 - \pi^2)$ being $N F^2$ the squared sphere radius and $\pi^2 = \sum_{a=1}^{N} \pi^a \pi_a$. The above lagrangian gives rise not only to the kinetic term but also to interacting terms with arbitrary even number of pions. These two-derivative interactions produce some difficulties in the quantum theory which have been largely discussed in the literature [10]. From the path integral point of view, the proper quantization prescription is obtained by defining the path integral measure as $[d\pi \sqrt{g}]$ which is $O(N + 1)$ invariant provided $g$ is the $S^N$ metric determinant. This factor gives an extra contribution to the non-linear sigma model lagrangian proportional to $\delta^4(0)$. As it is well known, one can deal with this new term by using dimensional regularization [11]. In this case one work in a $n = 4 - \epsilon$ dimensional space-time so that the regularized classical lagrangian still is $O(N + 1)$ invariant and, in addition, the measure term can be ignored as a consequence of the rule $\delta^n(0) = 0$ or equivalently $\int d^n k = 0$. In the following we will work always in this regularization scheme and hence we will neglect completely the measure term (for a recent discussion about regularization methods in $\chi$PT see [12]).

In order to derive the regularized large $N$ effective action from the above non-linear sigma model we will extend a technique that has already been used in the case of the linear theory [13]. First we introduce an auxiliary vector field $B^a$ (in contrast with the $\lambda \Phi^4$ case where an auxiliary scalar field
has to be introduced). Then we consider the new modified lagrangian for the non-linear sigma model:

\[ \tilde{\mathcal{L}}_{\text{NLSM}} = \frac{1}{2} \partial_{\mu} \pi_a \partial^{\mu} \pi^a - \frac{1}{2} N F^2 B_{\mu} B^{\mu} + B_{\mu} \frac{\pi_a \partial_{\mu} \pi^a}{\sqrt{1 - \pi^2/N F^2}} \]  \hspace{1cm} (2)

As there is not any kinetic term for the \( B \) field in this lagrangian and it is quadratic in this field, a simple gaussian integration makes possible to integrate out this auxiliary field in such a way that the original lagrangian is recovered so that, the pion Green functions obtained from the lagrangians in eq.1 and eq.2 are the same. However, the new lagrangian simplifies the computation of the \( N \) factors for any given Feynman diagram. This is a consequence of the fact that the lagrangian in eq.2 does not produce pion self-interactions but \( B \) mediated pion interactions. The \( B \) propagator is just a constant of the order \( 1/N \) and we have a \( 2\pi B \) vertex of the order one, a \( 4\pi B \) vertex of the order \( 1/N \), a \( 6\pi B \) vertex of the order \( 1/N^2 \) etc.

Therefore, the most general pion Green function can be obtained by attaching the external pions in all possible ways to the \( B \) legs of the most general \( B \) Green function (see Fig. 1.a for an example). The most general connected \( B \) Green function (represented in Fig.1 by a black blob) will contain loops both of pions and \( B \)'s and in general can be written in terms of \( B \) Green functions containing only pions loops (represented in Fig.1 by a dashed blob) attached to \( B \) lines forming loops (see Fig.1b). In addition, it is not difficult to realize that the dashed \( B \) functions are as much of the order of \( N \) (see Fig.1c). Therefore, taking into account that the \( B \) propagators are of the order of \( 1/N \) we conclude that, to leading order in the \( 1/N \) expansion,
black $B$ Green functions are equal to the dashed $B$ Green functions or, in other words, to leading order we do not have to consider diagrams with $B$ loops (see Fig.1d).

Thus in the large $N$ limit we just need to compute the generating functional for the 1PI $B$ Green functions and then couple the pions to the external $B$ legs at the tree level. Moreover, we are interested only in the two point and the four point pion Green functions which are of the order of $1$ and $1/N$ respectively. This means that we need to consider only one point and two point $B$ functions. In this case it is not difficult to see that the only vertex that must be considered is $2\pi B$ since the others necessarily produce pion loops with only one vertex. However this kind of loop vanishes because our pions are massless and we are using dimensional regularization. In principle, there could be also a contribution of the order of $1/N$ to the two-pion function coming from the one leg $B$ diagram with one $B$ loop inside. Nevertheless, this diagram is proportional to the external $B$ momentum so that it does not contribute to the two-pion function.

Thus, the $B$ 1PI generating functional (or $B$ effective action) that we have to compute can be written as:

$$i\Gamma_{eff}[B] = i \int d^n x \left( -\frac{1}{2} N F^2 B^2 \right) + \log \int [d\pi] \exp i \tilde{S}[\pi, B]$$

where:

$$\tilde{S}[\pi, B] = \int d^n x \left( -\frac{1}{2} \pi_a \Box \pi^a + B_\mu \pi_a \partial^\mu \pi^a \right)$$

As the functional integral is quadratic in the pion field it can be computed
exactly by the use of standard methods. We introduce the operator:

\[
K_{ab}(x, y) = \left. \frac{-\delta^2 \tilde{S}}{\delta \pi^a(x) \delta \pi^b(y)} \right|_{\pi = \bar{\pi}}
\]

\[
= \delta_{ab} \Box_x \delta(x - y) + \delta_{ab} \partial^\mu B^\mu(x) \delta(x - y)
\]

\[
= \Box_{ab}(x, y) + \Delta_{ab}(x, y)
\]

where \(\bar{\pi}\) is the solution of the equation of motion \((\partial_\mu \partial^\mu + \partial_\mu B^\mu)\bar{\pi} = 0\) and therefore we can write:

\[
\Gamma_{eff}[B] = \int d^n x \left( -\frac{1}{2} NF^2 B^2 \right) + \tilde{S}[\bar{\pi}, B] + \frac{i}{2} Tr \log K
\]

As usual the trace of the log \(K\) operator is expanded as:

\[
Tr \log K = Tr \log \Box + Tr \log (1 + \Box^{-1} \Delta)
\]

\[
= Tr \log \Box + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} Tr (\Box^{-1} \Delta)^k
\]

As it was mentioned before, we only need to compute the first two terms of this sum corresponding to the one and two points \(B\) functions respectively. In addition, the one point function vanishes since it is proportional to a tadpole integral. Therefore, the only contribution to the \(B\) effective action is:

\[
Tr (\Box^{-1} \Delta)^2 = N \int d^n x d^n y B_\mu(x) B_\nu(y) \int \frac{d^n k}{(2\pi)^n} k^\mu k^\nu e^{ik(x-y)} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(k + q)^2 q^2}
\]

so that, by neglecting the no \(B\)-leg contribution and terms with more than two legs we find:

\[
\Gamma_{eff}[B] = -\frac{1}{2} \int d^n x d^n y B_\mu(x) \Gamma_{\mu\nu}(x, y) B_\nu(y)
\]

where:

\[
\Gamma_{\mu\nu}(x, y) = NF^2 g_{\mu\nu} \delta(x-y) - \frac{N}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(x-y)} k_\mu k_\nu}{(4\pi)^2} (N_\epsilon + 2 + \log \frac{\mu^2}{-k^2})
\]
and as usual $N_\epsilon = 2/\epsilon + \log 4\pi - \gamma$ and $\mu$ is an arbitrary dimensional scale.

Next we add to this action the coupling with the external pions $B_\mu \pi_a \partial^\mu \pi^a$:

$$\Gamma_{eff}[\pi, B] = \int d^4 x B_\mu \pi_a \partial^\mu \pi^a + \Gamma_{eff}[B]$$

(11)

Now we use this action to obtain the $B$ field as a functional of the $\pi$ field through the corresponding equation of motion:

$$\frac{\delta \Gamma_{eff}}{\delta B^\mu} = \pi_a \partial^\mu \pi^a - \int d^4 y \Gamma^{\mu\nu}(x, y) B_\nu(y) = 0$$

(12)

the solution to this equation can be written as:

$$\tilde{B}_\mu(x) = \frac{1}{2} \int d^4 y \Gamma^{-1}_{\mu\nu}(x, y) \partial^\nu \pi^2(y)$$

(13)

The effective action for pions can be written as:

$$S_{eff}[\pi] = \int d^4 x \left( \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi^a + \pi_a \partial^\mu \pi^a B_\mu - \frac{1}{2} \int d^4 y B_\mu(x) \Gamma^{\mu\nu}(x, y) B_\nu(y) \right)$$

(14)

by computing the inverse of the $\Gamma_{\mu\nu}(x, y)$ operator appearing in eq.10 we finally obtain:

$$S_{eff}[\pi] = \int d^4 x \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi^a$$

$$+ \frac{1}{8NF^2} \int d^4x d^4y \pi^2(x) \pi^2(y) \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{q^2}{1 - \frac{q^2}{2(4\pi)^2F^2}(N_\epsilon + 2 + \log \frac{\mu^2}{q^2})} + O(1/N^2)$$

(15)

Here the denominator must be understood as the corresponding formal geometric series.
3 Renormalization of the effective action

Once the regularized effective action for pions has been obtained in the large \( N \) limit we have to deal with the renormalization of this result. As the kinetic term of this effective action is just the classical one, no renormalization of the pion wave function is needed i.e. \( Z_\pi = 1 \). However, in the effective action we find four pion interaction terms which are divergent and proportional to arbitrary high even powers of the momenta. This of course reveals the well known fact that the non-linear sigma model is not renormalizable in the standard sense. Nevertheless, we can absorb all the divergences by adding to the original lagrangian in eq.1 an infinite set of new counter terms that can be chosen for example as:

\[
\mathcal{L}^{ct} = \sum_{k=1}^{\infty} \frac{g_k}{8N} \pi^2 \left( -\frac{\Box}{F^2} \right)^{k+1} \pi^2 + O(1/N^2) \tag{16}
\]

Apart from these, one could also in principle introduce many other terms. However, here we will adopt the philosophy of introducing only a minimal set of terms just to absorb the divergences appearing in the non-linear sigma model to leading order in the \( 1/N \) expansion.

Now, following similar steps as in [7], it is straightforward to obtain the renormalized pion effective action:

\[
S_{\text{eff}}[\pi] = \int d^4x \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi^a + \frac{1}{8NF^2} \int d^4x d^4y \pi^2(x) \pi^2(y) \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{q^2G^R(q^2;\mu)}{1 - \frac{q^2}{2(4\pi)^2 F^2} G^R(q^2;\mu) \log \frac{\mu^2}{-q^2}} + O(1/N^2)
\]

where \( G^R(q^2;\mu) \) is the generating function for the renormalized coupling.
constants $g_k(\mu)$ at the scale $\mu$:

$$G^R(q^2; \mu) = \sum_{k=0}^{\infty} g_k(\mu)(\frac{q^2}{F^2})^k$$  \hspace{1cm} (18)

where by definition $g_0(\mu) = 1$ for every $\mu$. Since $Z_\pi = 1$, the renormalization group equation followed by the pion effective action is:

$$\left(\frac{\partial}{\partial \log \mu} + \sum_{k=0}^{\infty} \beta_k \frac{\partial}{\partial g_k(\mu)}\right)S_{\text{eff}}[\pi] = 0$$  \hspace{1cm} (19)

where $\beta_k = \frac{dg_k(\mu)}{d \log \mu}$. From this equation we can obtain very easily the renormalization group equation for $G^R(q^2; \mu)$ which can be integrated to give the evolution equation:

$$G^R(q^2; \mu) = \frac{G^R(q^2; \mu_0)}{1 + \frac{q^2}{2(4\pi)^2 F^2} G^R(q^2; \mu_0) \log(\frac{\mu^2}{\mu_0^2})}$$  \hspace{1cm} (20)

By using this equation and eq.18 it is very easy to get the evolution equations for all the renormalized coupling constants $g_k(\mu)$. Again because $Z_\pi = 1$ and the fact that pions continue being massless we can read very easily the elastic pion scattering amplitude from eq.17 recovering the result of [7] (one missing factor 2 in this reference has been corrected here):

$$T_{abcd}(s, t, u) = A(s)\delta_{ab}\delta_{cd} + A(t)\delta_{ac}\delta_{bd} + A(u)\delta_{ad}\delta_{bc}$$  \hspace{1cm} (21)

where:

$$A(s) = \frac{s}{NF^2} \frac{G^R(s; \mu)}{1 - \frac{s}{2(4\pi)^2 F^2} G^R(s; \mu) \log(\frac{\mu^2}{s})} + O(1/N^2)$$  \hspace{1cm} (22)

Note that at low energies this $A(s)$ function behaves as $s/NF^2$ and this fact is important for two reasons. First, we realize that it is compatible with the low energy theorems which state that it must go in this limit as $s/f_\pi^2$ (in
the chiral limit that we are considering here). Second we can use this low-energy theorems to set the constant $F$ in terms of the pion decay constant $f_\pi \simeq 96\text{MeV}$ as $f_\pi^2 = NF^2$ with $N = 3$.

In addition to this scattering amplitude, there is another important object that can be obtained from the effective action in eq.17. For static pion field configurations $\pi_a(\vec{x})$ the energy density is given by minus the effective lagrangian i.e. $-\mathcal{L}_{\text{eff}}(\vec{x})$ which can be read off from eq.17. Therefore, the mass functional or, equivalently, the total energy of the configuration can be written as:

$$M[\pi] = \int d\vec{x} \left( \frac{1}{2} \partial_i \pi_a \partial^i \pi^a + \frac{1}{8NF^2} \pi^2 \left[ \frac{1}{1 - \frac{1}{2(N\pi)^2}G^R(\nabla;\mu)\log(\frac{\mu^2}{\pi^2})\nabla} \right] \right)$$

where the meaning of this non-local mass functional is given by the appropriate Fourier transform.

In order to use eq.22 or eq.23 to do phenomenology we need to define the generating function $G^R(q^2;\mu)$ or, in other words, the infinite number of renormalized couplings constants $g_k(\mu)$. A typical example is the linear sigma model. In this case, the appropriate choice is:

$$g_k(\mu) = \left( \frac{1}{2\lambda(\mu)} \right)^k$$

$$G^R(s;\mu) = \frac{1}{1 - \frac{s}{2\lambda(\mu)F^2}}$$

where $\lambda(\mu)$ is the renormalized coupling constant of the linear sigma model computed in the large $N$ limit [14]. With this definition, eq.22 and eq.20 provide the corresponding well known scattering amplitude and evolution equation for $\lambda$. In addition, the amplitude in eq.22 has a pole in the second
Riemann sheet that has to be understood as the physical scalar resonance appearing in the linear sigma model i.e. the Higgs resonance. (see [14] and [15] for a discussion on this point).

However, the $O(N + 1)$ linear sigma model is not the theory that reproduces the hadron interactions at low energies. In principle, there are two possible approaches for the determination of the renormalized coupling constants appearing in the pion effective action of eq.17. From the theoretical point of view one could try to compute these parameters from a more fundamental theory of strong interactions like QCD. From the phenomenological point of view, one can try to fit these constants from the experiment, for example the low-energy pion scattering data. As the theoretical computation seems to be quite involved, it is worth to make the phenomenological determination of the couplings first in order to test the quality of the whole $N$ approach to this case.

4 Fitting the elastic pion scattering

Using eq.22 to fit the experimental pion scattering data requires to face the fact of having an infinite number of parameters i.e. the scale $\mu$ and the values of the renormalized coupling constants $g_k(\mu)$ at the scale $\mu$. However one can solve this problem in a easy way just by considering only particular cases where all the coupling constants but a finite set $g_1, g_2 \ldots g_r$ vanish at some scale $\mu$. These models are defined just by a finite number of parameters ($\mu$ and the $r$ coupling constants renormalized at this scale) and therefore can be used to fit the experimental data. In particular, one can consider the
extremal case of having all the renormalized couplings equal to zero at some scale \( \mu \) i.e. \( g_k(\mu) = 0 \) for all \( k > 0 \). The model so obtained has only one parameter (the \( \mu \) scale) and it can be considered in some sense as the nonlinear sigma model renormalized at the leading order of the \( 1/N \) expansion.

Of course, it could happen that the underlying theory (say QCD) were not of this type. However this would not be a serious problem at very low energies because, for natural values of the renormalized coupling constants and the renormalization scale \( \mu \), only the very first terms of the generating function \( G^R(q^2;\mu) \) series in eq.18 contribute to the effective action and the scattering amplitude. However, at higher energies the effect of the infinite set of couplings constants, when different from zero, could be important. For example, they can conspire to give rise to resonances which appear as poles in the second Riemann sheet of the partial waves amplitudes as it happens in the linear sigma model. Of course, in this case the value of the couplings can also be obtained from the underlying theory or by making extra assumptions on the behavior of the amplitudes.

For comparison with the experiment we need also to compute the phase shifts for the different channels. These are obtained from the corresponding partial waves amplitudes. The proper generalization of the standard \( SU(2) \) isospin projections to the \( O(N) \) case is given by [9]:

\[
T_0 = NA(s) + A(t) + A(u)
\]
\[
T_1 = A(t) - A(u)
\]
\[
T_2 = A(t) + A(u)
\]
and the partial waves are given by:

\[ a_{IJ} = \frac{1}{64\pi} \int_{-1}^{1} T_I(s, \cos\theta) P_J(\cos\theta) d(\cos\theta) \]  

(26)

Then, the \( a_{00} \) amplitude is of the order of one but, for instance, the \( a_{11} \) and the \( a_{20} \) amplitudes are of the order of \( 1/N \). Thus at the leading order of the large \( N \) expansion \( a_{00}(s) = NA(s)/32\pi + O(1/N) \), \( a_{11}(s) = 0 + O(1/N) \) and \( a_{20}(s) = 0 + O(1/N) \) (remind that \( A(s) \) is of the order of \( 1/N \)). Now it is easy to show that in this approximation the partial waves have the proper cut structure and also are elastic unitary i.e. \( \text{Im} a_{00} = |a_{00}|^2 + O(1/N) \) just above the unitarity cut as they must be.

In fig. 2 we show the result of our fit of the \( I = J = 0 \) phase shift channel for elastic pion scattering. This fit has been done with only one parameter, or in other words, by using the very particular model where all the renormalized coupling constants are zero at some scale \( \mu \). As discussed above, the only parameter in this specially simple case is the scale \( \mu \). The fit in fig. 2 corresponds to a value of \( \mu^2 = 600000 \text{MeV}^2 (\mu \simeq 775 \text{MeV}) \). For other channels like \( I = J = 1 \) or \( I = 2, J = 0 \) the prediction of the leading order of the large \( N \) expansion is that there is no interaction. This could appear as a rather poor prediction but it is not the case. By looking at the experimental data one can realize that the phase shifts in these channels grow very slowly with energy, for instance, at 500 \( \text{MeV} \) of center of mass energy we have \( \delta_{11} \simeq 5 \) degrees and \( \delta_{20} \simeq -7 \) degrees whilst \( \delta_{00} \simeq 38 \) degrees at the same energy. The conclusion is that for energies below, let say, 500 \( \text{MeV} \) our simple one parameter fit gives rise to errors in the phase shifts which are lesser than 10 per cent (refered to the \( I = J = 0 \) channel) for the three channels.
considered. Of course, for larger energies things become completely different particularly in the $I = J = 1$ channel where the onset of the $\rho$ resonance makes this channel to be strongly interacting. One is then tempted to make an interpretation of the scale $\mu$ as some kind of cutoff signaling the range of applicability of our approach and, in fact, this interpretation was discussed in [7]. From this point of view, it is quite interesting to realize how close is the $\mu$ fitted value ($\mu \simeq 775\,\text{MeV}$) to the $\rho$ mass. In some way, by fitting the $I = J = 0$ channel, the model is telling us where new things can appear in other channels like the $I = J = 1$ (the $\rho$ resonance) and thus setting the limits of applicability of the model. However, we would like to stress that our renormalization method is completely consistent and our results are formally valid at any energy independently of the goodness of the fits we can obtain with them.

5 The skyrmion in the large $N$ limit

As it was mentioned above, the second phenomenological application of the large $N$ expansion that will be considered here is the description of the nucleon in the context of the so-called skyrmion model. As it is well known, the skyrmion [21] is a topological soliton of the $SU(2)_L \times SU(2)_R/ SU(2)_{L+R}$ chiral model. Finite energy and static pion field configurations are continuous maps from the compactified space $S^3$ to the coset space $S^3$. These applications can be classified in homotopically non-equivalent classes since $\pi_3(S^3) = Z$. Therefore every map can be labeled by an integer number. In the skyrmion model of the nucleon this topological number is interpreted as
the baryonic number of the field. The skyrmion model has proved to give a qualitative description of many of the nucleon properties [22] including (in an highly non-trivial way) the spin [23]. The standard ansatz for the skyrmion field is \( U(\vec{x}) = \exp(\sqrt{N} f(\vec{r}) \hat{x}_a \sigma_a) \) with \( f(0) = \pi \) and \( f(\infty) = 0 \) being the right boundary conditions for the chiral angle \( f(\vec{r}) \) in order to have baryonic number equal to one. In this description of the skyrmion the chiral field is taken to be an element of \( SU(2) \) which is equivalent to our coset space \( S^3 \). In terms of the coordinates used here, the skyrmion ansatz is given by \( \pi^a(\vec{x}) = \sqrt{NF} \sin f(\vec{r}) \hat{x}^a \). In order to obtain the mass of the skyrmion in the approximation considered in this work we have to introduce this ansatz in the mass functional in eq.22 with the renormalized coupling constants obtained from the fit of the elastic pion scattering described in the previous section. Then, it is not very difficult to find:

\[
M[f] = NF^22\pi \int_0^\infty drr^2\left(2\frac{\sin^2 f(r)}{r^2} + \cos^2 f(r)f'(r)^2\right) + NF^2 \int_0^\infty drr \sin^2 f(r) \int_0^\infty dr' \sin^2 f(r') \int_0^\infty dk \sin(kr) \sin(kr') \frac{k^2G^R(-k^2; \mu)}{1 + \frac{k^2}{2(2\pi)^2}G^R(-k^2; \mu) \log \frac{\mu^2}{k^2}}
\]

Now the mass functional depends only on the chiral angle \( f(\vec{r}) \). By minimizing this functional we can find in principle the mass and the shape of the skyrmion in this large \( N \) approximation. In order to make this task easier we can use some reasonable parametrization of the chiral angle such as the one by Atiyah and Manton [24]:

\[
f_R(r) = \pi(1 - (1 + \frac{R^2}{r^2})^{-1/2})
\]
where the parameter $R$ is a measure of the size of the Skyrmion. In fig.3 we show the results obtained when this parametrization is inserted in eq.26 with the $G^{R}(k^2; \mu)$ obtained in the previous section i.e. we follow the philosophy of [25] where the authors use the skyrmion model to predict the mass of the proton from the pion scattering data. The first and obvious conclusion from fig.3 is that, at least for this parametrization, the mass functional has not any minimum but the trivial one at $R = 0$ so that the skyrmion tends to shrink to zero size in this approximation. In fact we have checked that this is also the case when other parametrizations of the chiral angle are used.

Therefore, everything seems to indicate that in the large $N$ limit considered here, the skyrmion is not stable i.e. we are in a similar situation to the one found when one works with the two derivatives term in eq.1. This is a bit disappointing since one of the most appealing properties of the large $N$ approximation is that it includes the effect of higher derivative terms and the corresponding higher loops contributions. This fact improves the unitarity behavior of the amplitudes and one could think that it could also improve the skyrmion description. However, this is not the case, at least in the leading order considered here. A simple way to understand why this is not so is the following. In the standard approach to the skyrmion one typically adds to the two derivatives term in the lagrangian one four derivative term called the skyrme term. This term produce an stable skyrmion in such a way that the contribution to the total energy of the skyrmion coming from the two terms is exactly the same. It is also possible to add another four derivative term called the non-skyrme term but its contribution can turn the skyrmion un-
stable. One-loop effects can also be included in the mass functional [26]. One possible interpretation of the origin of the skyrmion and the non-skyrmion terms is to understand them as the low energy remnant of the tail of one vector \((I = J = 1)\) and one scalar \((I = J = 0)\) resonance respectively. In fact it is not difficult to relate the skyrme and the non-skyrme parameters with the masses of the corresponding resonances (see for instance [4]). Then, the consequence of this picture is that the existence of one vector resonance like the \(\rho\) produces an stable skyrmion (and in fact it is possible to relate the \(\rho\) and the nucleon masses) but one scalar resonance do not produce an stable skyrmion by itself and even it can make unstable the skyrmion when the vector resonance is also present.

If this simple picture were correct (as it seems to be) the \(\rho\) resonance would be the responsible of the stability of the nucleon in the skyrmion model (or at least of the pion cloud). Then it would be possible to understand why the skyrmion is not stable in the large \(N\) limit approximation considered in this paper. As we saw in the previous section this approach can describe well the elastic pion scattering in the \(I = J = 0\) channel but it does not reproduce the \(\rho\) resonance (yet in some way it predicts its existence) and therefore it does not contain the piece that seems to keep the skyrmion stable in other much more simple approach that do not include higher derivatives neither higher loops effects.
6 Conclusions

We have computed the effective action of the $O(N + 1)/O(N)$ non-linear sigma model to leading order on the $1/N$ expansion. This action is divergent but it can be properly renormalized by introducing an infinite set of counter terms in a systematically way. The energy dependence of the corresponding renomalized couplings can be obtained thus providing a very nice example of how $\chi PT$ can work beyond the one-loop approximation. With an appropriate choice of these renormalized couplings we can reproduce particular models with the same symmetry breaking pattern as, for example, the linear sigma model.

We can consider also models with a finite number of couplings which can be used to fit the elastic pion scattering experimental data since the partial waves amplitudes have a very good unitarity behavior. Then, with only one parameter it is possible to reproduce well the phase shifts for energies below 500$\text{MeV}$. This parameter can be also interpreted as some kind of cutoff setting the limits of the model and, in fact, the fitted value is basically the $\rho$ mass which in principle cannot be reproduced since, at leading order in the $1/N$ expansion, the $I = J = 1$ channel is not interacting.

We have also studied the mass functional of the skyrmion solutions and we arrived to the conclusion that the skyrmion solution is not stable at the leading order in the $1/N$ expansion. This fact is probably related with the above mentioned absence of interaction in the $I = J = 1$ since in other approaches the $\rho$ resonance stabilizes the skyrmion.

In any case, we consider the $1/N$ expansion of $\chi PT$ as a very interest-
ing alternative complementary to the standard one-loop computations. Of course, much more work is needed like including pion mass effects, going to the next to leading order in $1/N$, considering other ways to take the limit like the one based in the $SU(N)_L \times SU(N)_R/SU(N)_{L+R}$ non-linear sigma model and finally (and this was our original motivation) the applications to the description of the symmetry breaking sector of the Standard Model as a way to parametrize the future Large Hadron Collider (LHC) data. Work is in progress in all these directions.

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**Figure Captions**

**Figure 1:** a) A typical eight pion legs Feynman diagram where the three $B$ legs function appears (pion lines are narrow and $B$ lines are broader) b) The complete connected three $B$ legs function (black blob) in terms of the $B$ functions containing only pion loops (dashed blobs) c) Some diagrams contributing to the $B$ functions containing only pion loops d) Relation between the complete $B$ functions and the $B$ functions containing only pion loops at the leading order in the $1/N$ expansion.

**Figure 2:** Phase shifts of the elastic pion scattering for the $I = J = 0$ channel. The continuous line represents the one parameter fit described in the text. The experimental data corresponds to: △ ref.[16], ○ ref.[17], □ ref.[18], ◊ ref.[19], ▽ ref.[20].

**Figure 3:** Mass of the skyrmion in the Large $N$ limit for the Atiyah and Manton parametrization in terms of the $R$ parameter. Dotted line is the contribution of the kinetic term, dashed line is the interaction energy and the full line is the total mass of the skyrmion.
This figure "fig1-1.png" is available in "png" format from:

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