ON LOCAL CONVEXITY OF NONLINEAR MAPPINGS BETWEEN BANACH SPACES

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ABSTRACT. We find conditions for a smooth nonlinear map \( f : U \rightarrow V \) between open subsets of Hilbert or Banach spaces to be locally convex in the sense that for some \( c \) and each positive \( \varepsilon < c \) the image \( f(B_{\varepsilon}(x)) \) of each \( \varepsilon \)-ball \( B_{\varepsilon}(x) \subset U \) is convex. We give a lower bound on \( c \) via the second order Lipschitz constant \( \text{Lip}_2(f) \), the Lipschitz-open constant \( \text{Lip}_o(f) \) of \( f \), and the 2-convexity number \( \text{conv}_2(X) \) of the Banach space \( X \).

INTRODUCTION

The local convexity of nonlinear mappings of Banach spaces is important in many branches of applied mathematics \([1, 2, 13, 18, 20, 22]\), in particular, in the theory of nonlinear differential-operator equations, optimization and control theory etc. Locally convex maps appear naturally in various problems of Fixed Point Theory \([1, 2, 13, 18, 19, 20, 21]\), in particular, in the theory of nonlinear differential-operator equations, optimization and control theory etc. Locally convex maps appear naturally in various problems of Fixed Point Theory \([7, 8, 9]\) and Nonlinear Analysis \([12, 16, 17, 22]\).

Let \( X, Y \) be Banach spaces. A map \( f : U \rightarrow Y \) defined on an open subset \( U \subset X \) is called \textit{locally convex} at a point \( x \in U \) if there is a positive constant \( c > 0 \) such that for each positive \( \varepsilon \leq c \) and each point \( x \in U \) with \( B_{\varepsilon}(x) \subset U \) the image \( f(B_{\varepsilon}(x)) \) is convex. Here \( B_{\varepsilon}(x) = \{ y \in X : \| x - y \| < \varepsilon \} \) stands for the open \( \varepsilon \)-ball centered at \( x \). The local convexity of \( f \) at \( x \) can be expressed via the \textit{local convexity radius}

\[
\lcr_x(f) = \sup \{ c \in [0, +\infty) : \forall \varepsilon \leq c \forall x \in U \text{ with } B_{\varepsilon}(x) \subset U \text{ the set } f(B_{\varepsilon}(x)) \text{ is convex} \}.
\]

It follows that \( f \) is locally convex at \( x \in U \) if and only if \( \lcr_x(f) > 0 \).

A map \( f : U \rightarrow Y \) is defined to be

- \textit{locally convex} if \( f \) is locally convex at each point \( x \in U \);
- \textit{uniformly locally convex} if its local convexity radius \( \lcr(f) = \inf_{x \in U} \lcr_x(f) \) is not equal to zero.

For example, if a homeomorphism \( f : U \rightarrow V \) between open subsets \( U \subset X, V \subset Y \) with \( f(0) = 0 \in U \) is norm convex in the sense that

\[
\left\| f\left( \frac{x + x'}{2} \right) \right\| \leq \frac{1}{2} \left( \| f(x) \| + \| f(x') \| \right) \quad \text{for all } x, x' \in f(U),
\]

then the inverse map \( f^{-1} \) is locally convex at the point \( y = 0 \). In particular, if \( Y \) is a Banach lattice with the order \( \leq \) and a homeomorphism \( f : U \rightarrow V \) is Jensen convex, i.e.

\[
f\left( \frac{x + x'}{2} \right) \leq \frac{1}{2} \left( f(x) + f(x') \right)
\]

for all \( x, x' \in U \), then the inverse map \( f^{-1} \) is locally convex at the point \( y = 0 \).

In this paper we find some conditions on a map \( f : U \rightarrow Y \) guaranteeing that \( f \) uniformly locally convex, and give a lower bound on the local convexity radius \( \lcr(f) \) of \( f \). This bound depends on the second order Lipschitz constant \( \text{Lip}_2(f) \) of \( f \), the Lipschitz-open constant \( \text{Lip}_o(f) \) of \( f \), and the 2-convexity number \( \text{conv}_2(X) \) of the Banach space \( X \).

1. Banach spaces with modulus of convexity of power type 2

The \textit{modulus of convexity} of a Banach space \( X \) is the function \( \delta_X : [0, 2] \rightarrow [0, 1] \) assigning to each number \( t \geq 0 \) the real number

\[
\delta_X(t) = \inf \{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \| x - y \| \geq t \},
\]

\textbf{Key words and phrases.} Locally convex mapping, Hilbert and Banach spaces, modulus of convexity, modulus of smoothness, Lipschitz-open maps.
where \( S_X = \{ x \in X : \| x \| = 1 \} \) is the unit sphere of the Banach space \( X \). By [15, p.60], the modulus of convexity can be equivalently defined as

\[
\delta_X(t) = \inf \left\{ 1 - \frac{\| x + ty \|}{\| x \| + t\| y \|} : x, y \in B_X, \| x - y \| \geq t \right\},
\]

where \( B_X = \{ x \in X : \| x \| \leq 1 \} \) is the closed unit ball of \( X \).

Any Hilbert space \( E \) of dimension \( \dim(E) > 1 \) has modulus of convexity

\[
\frac{1}{8} t^2 \leq \delta_E(t) = 1 - \sqrt{1 - (t/2)^2} \leq \frac{1}{4} t^2.
\]

By [15, p63] or [10], \( \delta_X(t) \leq \delta_L(t) \leq \frac{1}{8} t^2 \) for each Banach space \( X \).

Following [15, p63], [3, p.154], we say that the Banach space \( X \) has modulus of convexity of power type \( p \) if there is a constant \( L > 0 \) such that \( \delta_X(t) \geq L \cdot t^p \) for all \( t \in [0, 2] \). It follows from \( L t^p \leq \delta_X(t) \leq \frac{1}{8} t^2 \) that \( p \geq 2 \). Hilbert spaces have modulus of convexity of power type 2. Many examples of Banach spaces with modulus of convexity of power type 2 can be found in [15, §1.e], [3, Ch.V], [3, §3], [14], and [11]. In particular, the class of Banach spaces with modulus of convexity of power type 2 includes the Banach spaces \( L_p \) for \( 1 < p \leq 2 \), and reflexive subspaces of the Banach space \( L_1 \). By [10], a Banach space \( X \) has modulus of convexity of power type 2 if and only if for any sequences \( (x_n)_{n \in \omega} \) and \( (y_n)_{n \in \omega} \) in \( X \) the convergence \( 2(\| x_n \|^2 + \| y_n \|^2) - \| x_n + y_n \|^2 \to 0 \) implies \( \| x_n - y_n \| \to 0 \).

For a Banach space \( X \) consider the constant

\[
\conv_2(X) = \inf \left\{ \frac{1}{\| x - y \|^2} : x, y \in B_X, \ x \neq y \right\} \geq 0
\]

called the 2-convexity number of \( X \) and observe that \( \conv_2(X) > 0 \) if and only if \( X \) has modulus of convexity of power type 2. It follows from [15, p.63] or [10] that

\[
0 \leq \conv_2(X) \leq \conv_2(\ell_2) = \frac{1}{8}
\]

for each Banach space \( X \).

2. Moduli of smoothness of maps of Banach spaces

In this section we recall known information [6, §2.7] on the moduli of smoothness \( \omega_n(f, t) \) of a function \( f : U \to Y \) defined on a subset \( U \subset X \) of a Banach space \( X \) with values in a Banach space \( Y \).

The \( n \)-th modulus of smoothness of \( f \) is defined as

\[
\omega_n(f, t) = \sup \{ \| \Delta_h^n(f, x) \| : h \in X, \ |h| \leq t, \ [x, x + nh] \subset U \}
\]

where

\[
\Delta_h^n(f, x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + kh)
\]

is the \( n \)-th difference of \( f \).

In particular,

\[
\omega_1(f, t) = \sup \{ \| f(x + h) - f(x) \| : \ |h| \leq t, \ [x, x + h] \subset U \}
\]

and

\[
\omega_2(f, t) = \sup \{ \| f(x + h) - 2f(x) + f(x - h) \| : \ |h| \leq t, \ [x - h, x + h] \subset U \}.
\]

Here \( [x, y] = \{ tx + (1 - t)y : t \in [0, 1] \} \) stands for the segment connecting two points \( x, y \in X \).

The constants

\[
\text{Lip}_1(f) = \sup_{t > 0} \frac{\omega_1(f, t)}{t} \quad \text{and} \quad \text{Lip}_2(f) = \sup_{t > 0} \frac{\omega_2(f, t)}{t^2}
\]

are called the Lipschitz constant and the second order Lipschitz constant of \( f \), respectively.

A function \( f : U \to Y \) is called (second order) Lipschitz if its (second order) Lipschitz constant \( \text{Lip}_1(f) \) (resp. \( \text{Lip}_2(f) \)) is finite. The second order Lipschitz property of a weakly Gâteaux differentiable function \( f \) can be deduced from the Lipschitz property of its derivative \( f' \).
Let us recall [4, p.154] that a function $f : U \to Y$ is weakly Gâteaux differentiable at a point $x \in U$ if there is a bounded linear operator $f'_x : X \to Y$ (called the derivative of $f$ at $x$) such that for each $h \in X$ and each linear continuous functional $y^* \in Y^*$ we get

$$\lim_{t \to 0} \frac{y^*(f(x + th) - f(x))}{t} = y^* \circ f'_x(h).$$

If

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - f'_x(h)\|}{\|h\|} = 0,$$

then $f$ is Fréchet differentiable at $x$.

The derivative $f'_x$ belongs to the Banach space $L(X, Y)$ of all bounded linear operators from $X$ to $Y$, endowed with the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$.

The following two propositions are known and we present their short proofs for completeness.

**Proposition 2.1.** Let $X, Y$ be Banach spaces and $U \subset X$ be an open subset. A function $f : U \to Y$ is Lipschitz if $f$ is weakly Gâteaux differentiable at each point of $U$ and the derivative map $f' : U \to L(X, Y)$, $f' : x \mapsto f'_x$, is bounded. In this case $\text{Lip}_1(f) \leq \|f'\|_\infty = \sup_{x \in U} \|f'_x\|$.

**Proof.** Let $L = \|f'\|_\infty$. The inequality $\text{Lip}_1(f) \leq L = \|f'\|_\infty$ will follow as soon as we check that

$$\|f(x + h) - f(x)\| \leq L\|h\|$$

for any $x \in U$ and $h \in X$ with $[x, x + h] \subset U$. Using the Hahn-Banach Theorem, find a linear continuous functional $y^* \in Y^*$ with unit norm $\|y^*\| = 1$ such that $y^*(f(x + h) - f(x)) = \|f(x + h) - f(x)\|$. The weak Gâteaux differentiability of $f$ implies that the function

$$g : [0, 1] \to \mathbb{C}, \ g : t \mapsto y^*(f(x + th) - f(x))$$

is differentiable and $g'(t) = y^* \circ f'_{x + th}(h)$ for each $t \in [0, 1]$. Then

$$\|g'\|_\infty \leq \|y^*\| \cdot \|f'_{x + th}\| \cdot \|h\| \leq 1 \cdot \|f'\|_\infty \cdot \|h\| = L \cdot \|h\|$$

and

$$\|f(x + h) - f(x)\| = \|g(1) - g(0)\| = \left| \int_0^1 g'(t)dt \right| \leq \int_0^1 |g'(t)|dt \leq L\|h\| \int_0^1 dt = L\|h\|.$$

\[\square\]

**Proposition 2.2.** Let $X, Y$ be Banach spaces and $U \subset X$ be an open subset. Assume that a function $f : U \to Y$ is weakly Gâteaux differentiable at each point of $U$ and the derivative map $f' : U \to L(X, Y)$, $f' : x \mapsto f'_x$, is Lipschitz. Then

1. $f$ is Fréchet differentiable at each point of $U$;
2. $f$ is second order Lipschitz with $\text{Lip}_2(f) \leq \text{Lip}_1(f')$.

**Proof.** Let $L = \text{Lip}_1(f')$. The Fréchet differentiability of $f$ at a point $x \in U$ will follow as soon as we check that

$$\|f(x + h) - f(x) - f'_x(h)\| \leq \frac{1}{2}L\|h\|^2$$

for each $h \in X$ with $[x, x + h] \subset U$. Using the Hahn-Banach Theorem, choose a linear continuous functional $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(f(x + h) - f(x) - f'_x(h)) = \|f(x + h) - f(x) - f'_x(h)\|$. The weak Gâteaux differentiability of $f$ implies that the function

$$g : [0, 1] \to \mathbb{C}, \ g : t \mapsto y^*(f(x + th) - tf'_x(h)),$$

is differentiable. Moreover, for each $t \in [0, 1]$ we get $g'(t) = y^* \circ f'_{x + th}(h) - y^* \circ f'_x(h)$ and

$$|g'(t)| = |y^*(f'_{x + th}(h) - f'_x(h))| \leq \|y^*\| \cdot \|f'_{x + th} - f'_x\| \cdot \|h\| \leq \|f'_{x + th} - f'_x\| \cdot \|h\| \leq \text{Lip}_1(f') \cdot \|h\| \leq tL\|h\|^2.$$

Then

$$\|f(x + h) - f(x) - f'_x(h)\| = |g(1) - g(0)| = \left| \int_0^1 g'(t)dt \right| \leq \int_0^1 |g'(t)|dt \leq \int_0^1 tL\|h\|^2 dt = \frac{1}{2}L\|h\|^2.$$
To see that $f$ is second order Lipschitz, observe that for each $h \in X$ with $[x - h, x + h] \subset U$ we get

$$
\|f(x + h) - 2f(x) + f(x - h)\| = \|f(x + h) - f(x) - f'(x)(h) + f(x - h) - f(x) - f'(x)(-h)\| \leq 
\leq \|f(x + h) - f(x)\| + \|f(x - h) - f(x)\| \leq 2L\|h\|^2 = L\|h\|^2,
$$

which implies that $\operatorname{Lip}_2(f) \leq L = \operatorname{Lip}_1(f')$. \hfill \Box

3. Lipschitz-open maps

Let $X, Y$ be Banach spaces. A map $f : U \to Y$ defined on an open subset $U \subset X$ is called Lipschitz-open if there is a positive constant $c$ such that for each $x \in X$ and $\varepsilon > 0$ with $B_\varepsilon(x) \subset U$ we get $B_c(f(x)) \subset f(B_\varepsilon(x))$. Observe that a map $f : U \to Y$ is Lipschitz-open if and only if its Lipschitz-open constant

$$
\operatorname{Lip}_o(f) = \sup \{ c \in [0, \infty) : \forall x \in U \forall \varepsilon > 0 \ B_\varepsilon(x) \subset U \Rightarrow B_c(f(x)) \subset f(B_\varepsilon(x)) \}
$$
is strictly positive.

A map $f : U \to Y$ is locally Lipschitz-open if each point $x \in U$ has an open neighborhood $W \subset U$ such that the restriction $f|W : W \to Y$ is Lipschitz-open.

Observe that a bijective map $f : X \to Y$ between Banach spaces is Lipschitz-open if and only if the inverse map $f^{-1} : Y \to X$ is Lipschitz. In this case $\operatorname{Lip}_o(f) = \operatorname{Lip}_1(f^{-1})$.

The following proposition can be derived from Theorem 15.5 of [4].

**Proposition 3.1.** Let $X, Y$ be Banach spaces. A map $f : U \to Y$ defined on an open subspace $U$ of $X$ is locally Lipschitz-open if

1. $f$ is weakly Gâteaux differentiable and the derivative $f'_x : X \to Y$ is surjective at each point $x \in U$;
2. the derivative $f' : U \to L(X, Y)$ is Lipschitz.

4. Main Results

**Theorem 4.1.** Let $X, Y$ be Banach spaces. A map $f : U \to Y$ defined on an open subspace $U \subset X$ is uniformly locally convex if

1. the Banach space $X$ has modulus of convexity of power type 2,
2. $f$ is second order Lipschitz;
3. $f$ is Lipschitz-open.

Moreover, in this case $f$ has local convexity radius $lcr(f) \geq 8 \cdot \operatorname{Lip}_o(f) \cdot \operatorname{conv}_2(X)/\operatorname{Lip}_2(f) > 0$.

**Proof.** Given any point $x_0 \in U$ and a positive $\varepsilon \leq 8 \cdot \operatorname{Lip}_o(f) \cdot \operatorname{conv}_2(X)/\operatorname{Lip}_2(f)$ with $B_\varepsilon(x_0) \subset U$, we need to prove that the image $f(B_\varepsilon(x_0))$ is convex. Without loss of generality, $x_0 = 0$.

**Claim 4.2.** For any points $a, b \in f(B_\varepsilon(x_0))$ we get $(a + b)/2 \in f(B_\varepsilon(x_0))$.

**Proof.** Find two points $x, y \in B_\varepsilon(x_0) = B_\varepsilon(0)$ with $a = f(x)$ and $b = f(y)$, and consider the midpoint $z = (x + y)/2$. Observe that the points $x_\varepsilon = x/\varepsilon$, $y_\varepsilon = y/\varepsilon$, and $z_\varepsilon = z/\varepsilon$ have norms $\leq 1$.

The definition of the $2$-convexity number $\operatorname{conv}_2(X)$ guarantees that

$$
1 - \frac{1}{\varepsilon}\|z\| = 1 - \|z_\varepsilon\| \geq \operatorname{conv}_2(X) \cdot \|x_\varepsilon - y_\varepsilon\|^2 = \frac{1}{\varepsilon^2}\operatorname{conv}_2(X)\|x - y\|^2
$$

and thus

$$
\varepsilon - \|z\| \geq \frac{1}{\varepsilon}\operatorname{conv}_2(X)\|x - y\|^2.
$$

Then $B_\delta(z) \subset B_\varepsilon(x_0)$, where

$$
\delta = \frac{1}{\varepsilon}\operatorname{conv}_2(X)\|x - y\|^2 \geq \frac{\operatorname{Lip}_2(f)}{8\operatorname{Lip}_o(f) \cdot \operatorname{conv}_2(X)}\operatorname{conv}_2(X)\|x - y\|^2 = \frac{\operatorname{Lip}_2(f)}{8\operatorname{Lip}_o(f)}\|x - y\|^2
$$

and hence

$$
f(B_\varepsilon(x_0)) \supset f(B_\delta(z)) \supset B_{\operatorname{Lip}_o(f)\delta}(f(z)) = B_\eta(f(z))
$$
where $\eta = \operatorname{Lip}_o(f)\delta = \frac{1}{8}\operatorname{Lip}_2(f)\|x - y\|^2$. 
The definition of the constant Lip₂(f) implies that for \( h = z - x \), we get
\[
\| (a + b) / 2 - f(z) \| = \| (f(x) + f(y)) / 2 - f(z) \| = \frac{1}{2} \| f(z - h) - 2f(z) + f(z + h) \| \\
\leq \frac{1}{2} \text{Lip}_2(f) \| h \|^2 = \frac{1}{8} \text{Lip}_2(f) \| x - y \|^2 = \eta
\]
and hence \((a + b) / 2 \in B_\eta(f(z)) \subset f(B_\epsilon(x_0))\). □

Claim 4.2 implies that the closure \( \text{cl}(f(B_\epsilon(x_0))) \) is convex. The Lipschitz-openness of the map \( f \) implies that for any numbers \( \delta < \eta < \epsilon \) we get \( \text{cl}(f(B_\delta(x_0))) \subset f(B_\eta(x_0)) \). Then the open set \( f(B_\epsilon(x_0)) \) is convex, being the union
\[
f(B_\epsilon(x_0)) = f \left( \bigcup_{0<\delta<\epsilon} B_\delta(x_0) \right) = \bigcup_{0<\delta<\epsilon} \text{cl}(f(B_\delta(x_0)))
\]
of a linearly ordered chain of convex sets. □

Taking into account that each Hilbert space \( X \) has 2-convexity number \( \text{conv}_2(E) \geq \frac{1}{8} \), and applying Theorem 4.1, we get:

**Corollary 4.3.** Let \( Y \) be a Banach space and \( U \) be an open subspace of a Hilbert space \( X \). Each Lipschitz-open second order Lipschitz map \( f : U \to Y \) is uniformly locally convex and has local convexity radius \( \text{ler}(f) \geq \text{Lip}_o(f) / \text{Lip}_2(f) > 0 \).

Theorem 4.1 combined with Propositions 2.2 and 3.1 implies the following two corollaries.

**Corollary 4.4.** Let \( X, Y \) be Banach spaces. A map \( f : U \to Y \) defined on an open subspace \( U \subset X \) is uniformly locally convex if
\[
\begin{align*}
(1) & \text{ the Banach space } X \text{ has modulus of convexity of power type } 2, \\
(2) & f \text{ is weakly Gâteaux differentiable and the derivative } f' : U \to L(X, Y) \text{ is Lipschitz;} \\
(3) & f \text{ is Lipschitz-open.}
\end{align*}
\]

**Corollary 4.5.** Let \( X, Y \) be Banach spaces. A map \( f : U \to Y \) defined on an open subspace \( U \subset X \) is locally convex if
\[
\begin{align*}
(1) & \text{ the Banach space } X \text{ has modulus of convexity of power type } 2, \\
(2) & f \text{ is weakly Gâteaux differentiable and the derivative } f' : U \to L(X, Y) \text{ is Lipschitz;} \\
(3) & \text{ for each } x \in U \text{ the derivative } f'_x : X \to Y \text{ is surjective.}
\end{align*}
\]

5. **AN OPEN PROBLEM**

We do not know if the requirement on the convexity modulus of the Banach space \( X \) is essential in Theorem 4.1 and Corollaries 4.4, 4.5.

**Problem 5.1.** Assume that \( X \) is a Banach space such that any Lipschitz-open second order Lipschitz map \( f : U \to X \) defined on an open subset \( U \subset X \) is locally convex. Has \( X \) the modulus of convexity of power type 2? Is \( X \) (super)reflexive?

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References

[1] Augustynowicz A., Dzedzej Z., Gelman B.D. The solution set to BVP for some functional-differential inclusions, Set-Valued Analysis. 6 (1998) 257–263.

[2] Blackmore D., Prykarpatsky A.K. A solution set analysis of a nonlinear operator equation using a Leray Schauder type fixed point approach. Topology. 48 (2009) 182–185.

[3] Borwein J., Guirao A., Hájek P., Vanderwerff J. Uniformly convex functions on Banach spaces. Proc. Amer. Math. Soc. 137 (2009), no. 3, 1081–1091.

[4] Deimling K. Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.

[5] Deville R., Godefroy G., Zizler V. Smoothness and renormings in Banach spaces, Longman Scientific & Technical, Harlow, New York, 1993.

[6] DeVore R., Lorentz G. Constructive Approximation, Springer-Verlag, Berlin, 1993.

[7] Goebel K. Twierdzenia o punktach stałych. Wykłady. Wydawnictwo Uniwersytetu Marii-Curie Skłodowskiej, Lublin, 2005.

[8] Gorniewicz L. Topological fixed point theory of multivalued mappings. Kluwer, Dordrecht, 1999.

[9] Krasnosel’sky M.A., Zabreyko P.P. Geometric methods of nonlinear analysis. "Nauka" Publisher, Moscow, 1975 (in Russian).

[10] Lindenstrauss J., Tzafriri L. Classical Banach spaces. II. Function spaces, Springer-Verlag, Berlin-New York, 1979.

[11] Linke Y.E. Application of Michael’s theorem and its converse to sublinear operators. Mathematical Notes. 52, (1993) 1, 680-686

[12] Nirenberg L. Topics in Nonlinear Functional Analysis. AMS Publisher, 1974

[13] Samoilenko A.M., Prykarpats’kyi A.K., Samoilenko V.H. Lyapunov–Schmidt approach to studying homoclinic splitting in weakly perturbed Lagrangian and Hamiltonian systems. Ukr. Mat. Zh. 55 (2003), no.1, 82–92.

[14] Schwartz J.T. Nonlinear functional analysis. Gordonand Breach Science Publisher, NY, 1969.

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