FINITE DIMENSIONAL QUOTIENTS OF COMMUTATIVE OPERATOR ALGEBRAS

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ABSTRACT. It is shown that the matrix normed structure of a non-unital operator algebra determines that of its unitization. This makes the study of certain unital operator algebras much easier and provides several interesting counterexamples.

Every two-dimensional, unital operator algebra is completely isometrically isomorphic to an algebra of 2 × 2-matrices, and every contractive homomorphism between two such algebras is completely contractive. This is used to define analogues for commutative, unital operator algebras of the Carathéodory distance and the Carathéodory-Reiffen metric on complex manifolds.

There exists an isometric, completely contractive map between three-dimensional \( \mathcal{Q} \)-algebras that is not completely isometric. Moreover, for every strongly pseudoconvex domain \( \mathcal{M} \), the algebra \( \mathcal{H}^\infty(\mathcal{M}) \) has a contractive representation by \( 3 \times 3 \)-matrices that is not completely contractive.

Recently, Arveson introduced the \( d \)-shift as a model for \( d \)-contractions. Completely isometric representations of quotients of the operator algebra generated by the \( d \)-shift are computed explicitly. For \( d = 1 \), this gives a version of Nevanlinna-Pick theory. It happens that every quotient of finite dimension \( r \) has a completely isometric representation by \( r \times r \)-matrices. Finally, the class of operator algebras with this property is investigated.

1. INTRODUCTION

An operator algebra \( A \) is just a subalgebra of \( \mathcal{B}(\mathcal{H}) \), the bounded operators on a Hilbert space \( \mathcal{H} \). The operator norm on \( \mathcal{B}(\mathcal{H}) \) gives rise to a norm on \( A \). Moreover, \( A \otimes M_n \subset \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) in a natural way, where \( M_n \) denotes the algebra of \( n \times n \)-matrices with the usual \( C^* \)-norm. Thus every operator algebra comes with natural norms on all tensor products \( A \otimes M_n \). The main interest of this article lies on this additional structure.

It is the framework for the model theory of (commuting) operators on Hilbert space. The starting point of model theory was the Szőkefalvi-Nagy dilation theorem [27], [30], which asserts that for any contraction \( T \in \mathcal{B}(\mathcal{H}) \), there is an essentially unique unitary operator \( U \) on a Hilbert space \( \mathcal{D} \) containing \( \mathcal{H} \) such that \( T^n = P_{\mathcal{H}} U^n P_{\mathcal{H}} \) for all \( n \in \mathbb{N} \). Here \( P_{\mathcal{H}} \) denotes the projection onto the subspace \( \mathcal{H} \). The unitary \( U \) is called a (power) dilation of \( T \). This allows to apply the rich theory of unitary operators to the study of contractions.

Until recently, attempts at a generalization of this result to (multi)operators, i.e. \( d \)-tuples of commuting operators \( T = (T_1, \ldots, T_d) \) on a common Hilbert space, have been rather unsuccessful. It was soon discovered by Andô [3] that two commuting contractions still have a unitary dilation, which, however, is no longer unique. But Parrot [22] gave a counterexample of three commuting contractions that do not have a unitary dilation. At the same time, Arveson [5], [6] found necessary and sufficient conditions for the existence of dilations in terms of the matrix normed structure described above. An accessible account of these classical results is Paulsen’s monograph [23].

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In [7], finally an interesting model theory for multi-operators is developed. A d-contraction is a multi-operator $T = (T_1, \ldots, T_d)$ such that
\begin{equation}
\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2
\end{equation}
for all $\xi_1, \ldots, \xi_n \in \mathcal{H}$. An equivalent condition is that the $d \times d$-matrix
\begin{equation}
\begin{pmatrix}
T_1 & T_2 & \ldots & T_d \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{equation}
is a contraction. Hence the innocent-looking condition (1) already involves the matrix normed structure on $\mathbb{B}(\mathcal{H}) \otimes M_d$. Indeed, the norm on the operator algebra generated by $T$, in general, does not contain enough information to determine whether $T$ is a d-contraction.

A particular d-contraction is the d-shift $S = (S_1, \ldots, S_d)$, acting on the Hilbert space $H^2_d$, which will be described in greater detail below. The main result of [7] is that every d-contraction has an essentially unique dilation with additional properties to a multi-operator of the form $n \cdot S \oplus Z$, where $n \cdot S$ stands for the direct sum of $n$ copies of $S$ acting on $(H^2_d)^n$ and $Z$ is a normal multi-operator with spectrum contained in the boundary of the standard Euclidean unit ball $D_d \subset \mathbb{C}^d$. Moreover, every operator that has such a dilation is a d-contraction. The normal part $Z$ is often missing, e.g. if the matrix in (2) has norm strictly less than 1.

For $d = 1$, the 1-shift is just the usual unilateral shift, and the above dilation is the von Neumann-Wold decomposition of an isometry. This is almost as good as a unitary dilation, and actually what is needed in several applications of dilation theory, e.g. [2]. For $d > 1$, however, the d-shift is no longer subnormal\footnote{A subnormal (multi)operator is the restriction of a normal (multi)operator to an invariant subspace.}. This is the reason why the model theory for d-contractions was discovered so late.

Let $\Omega$ be a compact space. A uniform algebra on $\Omega$ is a closed unital subalgebra of $C(\Omega)$ that separates the points of $\Omega$. More generally, a function algebra on $\Omega$ is a subalgebra of $C(\Omega)$. A function algebra $F$ comes with a natural matrix normed structure, viewing elements of $F \otimes M_n$ as functions $f$ from $\Omega$ to $M_n$ with norm $\|f\|_\infty = \sup_{\omega \in \Omega} \|f(\omega)\|$. It follows easily from spectral theory that every function algebra $F$ on $\Omega$ is also a function algebra on its spectrum $\text{Spec}(F)$. Thus the space $\Omega$ is not very important. The operator algebra generated by a subnormal operator is always (completely isometric to) a function algebra.

Function algebras are interesting in their own right because they arise in complex analysis. A typical example of a uniform algebra is the algebra $H^\infty(\Omega)$ of bounded holomorphic functions on a complex manifold $\mathfrak{M}$. However, its matrix normed structure has not been of great use in complex analysis so far. Some results in complex analysis, e.g. Lempert’s theorem, can be proved quite naturally using dilation theory (see [2]), but there are also more elementary proofs [19], [20] of those results.

If $A$ is a unital operator algebra and $I \subset A$ is an ideal, there is a natural matrix normed structure on the quotient algebra $Q = A/I$. Somewhat surprisingly, the resulting object is again an abstract operator algebra, i.e. it can be represented completely isometrically on a Hilbert space $\mathcal{H}$ [9]. A representation $\rho: Q \to \mathbb{B}(\mathcal{H})$ is called completely isometric if all the maps
\[\rho_{(n)} = \rho \otimes \text{id}_{M_n}: Q \otimes M_n \to \mathbb{B}(\mathcal{H}) \otimes M_n \cong \mathbb{B}(\mathcal{H} \otimes \mathbb{C}^n)\]
are isometric. The proof due to Blecher, Ruan, and Sinclair uses an axiomatic characterization of unital operator algebras and is not constructive. Indeed, completely isometric representations of quotients are often quite hard to find.

A commutative operator algebra has a rich ideal structure and a lot of finite dimensional quotients. It might be expected that these quotients are simpler than the original object. This is true in the sense that all information contained in the quotient is already contained in the original object. But in fact, taking quotients instead often brings the hidden complexity of an operator algebra to the surface.

Function algebras appear to be rather simple objects, and this is certainly true, say, from the point of view of spectral theory. But $Q$-algebras, i.e. quotients of function algebras, are among the most complicated operator algebras. Let $F$ be a function algebra on $\Omega$ and let $I$ be the ideal

$$I = I(\omega_1, \ldots, \omega_n) = \{ f \in F \mid f(\omega_1) = \cdots = f(\omega_n) = 0 \}$$

(3)

with distinct points $\omega_1, \ldots, \omega_d \in \Omega$. Then an element of $Q = F/I$ is determined by its function values at the points $\omega_1, \ldots, \omega_d$. Write $[f]$ for the projection of $f \in F$ in $Q$. The norm of an element in $F \otimes M_n$ can, in principle, be computed from its range. The norm on $Q$, however, depends on the existence of a solution of an interpolation problem with prescribed range.

Solving interpolation problems with prescribed range is one of the most difficult problems in complex analysis. If there are just two points, i.e. $d = 1$, this amounts to computing the Carathéodory distance of $\omega_1$ and $\omega_2$, but this is possible only in very few special cases. For $d > 3$, quotients of the disk algebra $\mathcal{P}(\mathbb{T})$ can be computed explicitly. It is also possible to compute the norm on quotients of $\mathcal{P}(\mathbb{T}^2)$ [1], but there seems to be no theory for other examples. Thus quotients of function algebras usually cannot be computed. Moreover, in those cases where they can, they tend to have as few completely contractive representations as possible (Theorem 6.5 and Theorem 6.8).

The most basic requirement on a model theory is a simple criterion which operators can be modeled. Formally, this can be translated into a criterion to decide whether a given representation of a certain operator algebra (generated by the model multi-operator) is completely contractive. Thus taking quotients is very relevant to model theory: The (completely) contractive representations of the quotient $A/I$ are precisely the (completely) contractive representations of $A$ whose kernel contains $I$. Thus a criterion to decide whether a representation of $A$ is completely contractive automatically applies to representations of $A/I$.

In the opinion of the author, a good model theory should actually have the stronger property that completely isometric representations of all quotients can be computed explicitly. This criterion is not met by function algebras (with the exception of the disk algebra). However, completely isometric representations of the operator algebra $\text{Shift}_d$ generated by the $d$-shift $S$ can be computed explicitly. The last part of this article is concerned with the theory of the quotients of $\text{Shift}_d$.

The completely isometric representations of quotients of $\text{Shift}_d$ can be easily written down explicitly, but the proof that they are indeed completely isometric is formal and not constructive. It is based on the fact that the quotient is again an abstract operator algebra.

It turns out that every quotient of $\text{Shift}_d$ of finite dimension $r$ has a completely isometric representation by $r \times r$-matrices. A unital, commutative operator algebra with this property is said to have minimal quotient complexity. The author conjectures that this property already essentially characterizes the quotients of $\text{Shift}_d$. More precisely, the conjecture is that a finite dimensional, indecomposable operator algebra of minimal quotient complexity is either a quotient of $\text{Shift}_d$ of the transpose of such a quotient.
The structure of this article is as follows:

Section 1.1 contains general terminology used in this article.

In Section 2, the matrix normed structure on the unitization of an operator algebra $A$ is shown to be determined by the matrix normed structure of $A$ and not to depend on the choice of a completely isometric representation. Hence for many purposes a unital operator algebra can be replaced by a 1-codimensional ideal $I$. This is particularly useful if the multiplication on the 1-codimensional ideal $I$ is the zero map. This means that $I$ essentially is just a linear space of operators on a Hilbert space. Such a space with its matrix normed structure is called an operator (vector) space, and every operator space $V$, endowed with the zero multiplication, occurs as a maximal ideal of a unital abstract operator algebra $V^+$. This abstract operator algebra is uniquely determined and called the trivial unitization of $V$. A linear map $\rho: V_1 \to V_2$ can be extended uniquely to a unital homomorphism $\rho^+: V_1^+ \to V_2^+$. Then $\|\rho\|_{(n)} = \|\rho^+\|_{(n)}$, and thus $\|\rho^+\|_{cb} = \|\rho\|_{cb}$. Here $\|\rho\|_{(n)} = \|\rho_{(n)}\|$ and $\|\rho\|_{cb} = \sup_{n \in \mathbb{N}} \|\rho_{(n)}\|$.

Consequently, the representation theory of a trivial unitization is precisely as well-behaved, or pathological, as the linear representation theory of the underlying operator space. This is the basis for several counterexamples.

Finally, the unitization technique yields a slight refinement of the theorem of Smith that a $d$-contractive linear mapping into $M_d$ is automatically completely contractive. If $A$ is a unital, commutative operator algebra, then a $d-1$-contractive, unital homomorphism $A \to M_d$ is completely contractive. This generalizes Agler’s discovery that every contractive, unital homomorphism $A \to M_2$ is completely contractive.

In Section 3, two-dimensional, unital operator algebras are studied. The unitization technique reduces the classification of these operator algebras to that of one-dimensional operator algebras, which is rather trivial. Every two-dimensional, unital operator algebra has a completely isometric representation by $2 \times 2$-matrices. The norms and complete norms of all algebraic isomorphisms between two-dimensional operator algebras are computed. It turns out that, for any such automorphism, $\|\rho\| = \|\rho\|_{cb}$. By taking quotients, this immediately generalizes to unital homomorphisms between unital operator algebras which have rank two as linear maps. This generalizes results previously known about representations of function algebras by $2 \times 2$-matrices ([2], [28], [10], [12], [19]).

In Section 4, the classification of two-dimensional operator algebras is used to define analogues of the Carathéodory pseudodistance and the Carathéodory-Reiffen pseudometric [16] for (commutative) unital operator algebras $A$. Since Carathéodory has not much to do with these objects, they are called quotient distance and quotient metric. The quotient distance is a distance on the spectrum of $A$ that is defined precisely as in the case of complex manifolds. Essentially, it describes the equivalence class of the two-dimensional, unital operator algebra $A/I(\omega_1, \omega_2)$. The tangent space of $A$ consists of pairs $(\omega; d)$, where $\omega \in \text{Spec}(A)$ and $d$ is a derivation of $A$ at $\omega$, and the quotient metric is just the norm of this linear functional. These definitions also make sense for noncommutative operator algebras. But since elements of the form $[f, g]$ are in the kernel of all characters and derivations at some point of the spectrum, the spectrum and the tangent space ignore any noncommutativity of $A$.

For every model theory, there should be an explicit criterion which $2 \times 2$-matrices can be modeled. Hence the quotient distance and metric can be computed explicitly for the operator algebras involved.
An application of the quotient distance is a simple criterion when a finite dimensional, commutative, unital operator algebra is \textit{decomposable}, i.e. when it decomposes into a non-trivial orthogonal sum of two ideals: This happens if and only if some two-dimensional quotient is isometric to $C(\{0, 1\})$, i.e. iff some two-dimensional quotient can be decomposed orthogonally.

In Section 5, the usual transposition operation on $M_n$ is defined for abstract operator algebras. This operation is isometric, but usually not completely isometric. However, transposition is completely isometric for Q-algebras and thus serves as a simple criterion to show that an operator algebra is not a Q-algebra. Moreover, transposition provides the easiest examples of homomorphisms between two-dimensional, non-unital operator algebras that are not completely isometric. The transpose of the operator algebra $\text{Shift}_d$ is interesting for theoretical purposes because it models the adjoints of $d$-contractions and has similar formal properties as $\text{Shift}_d$.

If $A$ is a unital operator algebra and $\omega \in \text{Spec}(A)$, define $I(\omega)$ as in (3) and $I(\omega)^2$ as the closure of $I(\omega) \cdot I(\omega)$. Then define $A(\omega) = A/I(\omega)^2$ and the cotangent space $T^*_\omega A = I(\omega)/I(\omega)^2$ of $A$ at $\omega$. The tangent space with the quotient metric is its normed dual. Of course, $A(\omega)$ is the trivial unitization of $T^*_\omega A$. Section 6 contains several counterexamples of badly behaved cotangent spaces of function algebras.

For certain function algebras, the tangent and cotangent spaces were already introduced by Paulsen in [24]. He was interested in determining when it happens that every contractive representation of $\mathcal{R}(\mathfrak{M})$ is completely contractive, where $\mathcal{R}(\mathfrak{M})$ denotes the algebra of rational functions without singularities in $\mathfrak{M}$, considered as a subalgebra of $C(\mathfrak{M})$. If $\mathfrak{M}$ is a balanced domain, it is easy to show that the cotangent space of $\mathcal{R}(\mathfrak{M})$ at zero is completely equivalent to $\text{MIN}(V)$, where $V$ is the normed space whose unit ball is the polar of $\mathfrak{M}$ and $\text{MIN}(V)$ denotes the minimal $L^\infty$-matricially normed structure on $V$. Thus it is very rare that every contractive linear representation of $T^*_0 \mathcal{R}(\mathfrak{M})$ is completely contractive: This happens iff $\text{MIN}(V) = \text{MAX}(V)$. Paulsen shows in [24] that this cannot hold for $\dim V \geq 5$.

Thus $\mathcal{R}(\mathfrak{M})$ has a contractive representation that is not completely contractive whenever $\dim \mathfrak{M} \geq 5$.

In Section 6, Paulsen’s negative result is extended to bounded, strongly pseudoconvex domains: If $\mathfrak{M}$ is a bounded, strongly pseudoconvex domain in $C^d$, $d \neq 1$, then $H^\infty(\mathfrak{M})$ has a contractive representation by $3 \times 3$-matrices that is not 2-contractive. This uses that the matrix normed structure on the cotangent space can be computed approximately near the boundary and approaches $\text{MIN}(\ell_2^d)$. Moreover, the result of Lempert ([17]) that there exist domains not biholomorphic to $\mathbb{D}_2$ but with tangent space at some point isometric to $\ell_2^d$ provides an example of an isometric, completely contractive homomorphism between three-dimensional Q-algebras that is not 2-isometric. Using quite different techniques, such a homomorphism has recently been obtained by Paulsen in [25].

This adds to the evidence that Q-algebras are rather complicated operator algebras. In Section 7, two reasons will be given why the operator algebraic viewpoint should not be expected to give deep results in complex analysis. First, the matrix normed structure on Q-algebras does not distinguish between certain nice and pathological objects. Secondly, there is more structure on a Q-algebra than the matrix normed structure: A Q-algebra $Q$ comes with natural norms on $Q \otimes V$ for all normed spaces $V$. The relevance of this structure for interpolation theory is discussed in [21]. From the point of view of interpolation theory, the restriction to the spaces $M_n$ is artificial.

Section 8 starts with a brief account of Arveson’s model theory for $d$-contractions. There are two other approaches to Arveson’s Hilbert space $H^2_d$ with interesting
consequences. Both use the explicit form \( u_x(z) = (1 - \langle z, x \rangle)^{-1} \), \( x, z \in \text{Ball}(\ell_2^d) \), of the reproducing kernel for the Hilbert space \( H_d^2 \). That this is a reproducing kernel means essentially that \( (f, u_x) = f(x) \) for all \( f \in H_d^2 \), and this computation is done by Arveson. But he does not explore the connections this opens to other areas of mathematics.

Up to a constant, \( u_x(z) \) is the \( 1/(d + 1) \)st power of the Bergman kernel for the ball, so that \( H_d^2 \) is a “twisted Bergman space”. These spaces have been studied by harmonic analysts because they carry a natural projective representation of the automorphism group \( \text{PSU}(d, 1) \) of the ball. Especially, there is a natural projective representation of \( \text{PSU}(d, 1) \) on \( H_d^2 \). Consequently, the closed operator algebra \( \text{Shift}_d \) generated by the \( d \)-shift retains all symmetries of the ball, i.e. \( f \mapsto f \circ a \) is a completely isometric isomorphism of \( \text{Shift}_d \) for all \( a \in \text{Aut}(\mathbb{D}_d) \). This is the main consequence of the twisted Bergman space picture of \( H_d^2 \). Moreover, some less important results of Arveson are special cases of results about general twisted Bergman spaces on symmetric domains. In [8], Bagchi and Misra compute the spectrum of \( \text{Shift}_d \) and show that the \( d \)-shift is not subnormal. However, they fail to single out \( H_d^2 \) as a case of special interest because their main goal is to determine when the analogue of the \( d \)-shift on a twisted Bergman space is subnormal and when the closed unit ball is a (complete) spectral set for it.

The reproducing kernel of \( H_d^2 \) is also related to the Fantappiè transform for \( \mathbb{C} \)-convex domains. For the special case of \( \mathbb{D}_d \), the Fantappiè transform is a linear bijection \( \mathcal{F}: \mathcal{O}(\mathbb{D}_d)' \to \mathcal{O}(\mathbb{D}_d) \) and thus induces a canonical bilinear form on \( \mathcal{O}(\mathbb{D}_d) \subset \mathcal{O}(\mathbb{D}_d) \). In order to make this form sesquilinear, some complex conjugate signs must be added. The resulting conjugate Fantappiè transform \( \mathcal{F} \) is given by \( (\mathcal{F}(l))(x) = l(u_x) \) for \( l \in \mathcal{O}(\mathbb{D}_d)' \) and is a bijection onto the space of coanalytic functions on a neighborhood of \( \mathbb{D}_d \). Thus \( \langle f, g \rangle = \langle \mathcal{F}^{-1}(\mathcal{F}(f)) \rangle(f) \) defines a sesquilinear form on \( \mathcal{O}(\mathbb{D}_d) \). This is nothing but the inner product on \( H_d^2 \). Thus if \( l \in \mathcal{O}(\mathbb{D}_d)' \), then inner products in \( H_d^2 \) involving \( \mathcal{F}(l) \) can be computed easily by \( \langle f, l(f) \rangle = \langle f, \mathcal{F}(l) \rangle \).

In Section 9, completely isometric representations of quotients of the algebra \( \text{Shift}_d \) generated by Arveson’s \( d \)-shift are computed explicitly. It turns out that, up to a self-adjoint part coming from the boundary \( \partial \mathbb{D}_d \), the representation of \( \text{Shift}_d/1 \) on \( H_d^2 \oplus 1 \cdot H_d^2 \) is completely isometric. The proof that this representation is indeed completely isometric starts with any completely isometric representation and shows that it is a quotient of several copies of \( H_d^2 \oplus H_d^2 \). Thus it makes essential use of the fact that quotients of operator algebras are again abstract operator algebras. Therefore, the proof is not as elementary as it may seem at first glance. Another disadvantage of the proof is that it is not constructive: Given \( F \in (\text{Shift}_d/1)_{(n)} \), it does not produce a representative \( F \in (\text{Shift}_d)_{(n)} \) with equal norm.

A special case of particular interest is if the ideal is of the form \( I(x_1, \ldots, x_m) \) for distinct points \( x_1, \ldots, x_m \in \mathbb{D}_d \). There exists \( F \in (\text{Shift}_d)_{(n)} \) with prescribed values \( F(x_j) = y_j \) and positive, invertible real part if and only if the block matrix \( P(F) \) with entries

\[
P(F)_{ij} = \frac{y_i + y_j^*}{1 - \langle x_i, x_j \rangle} \in \mathbb{M}_n
\]

is positive definite and invertible. A similar criterion allows to check whether \( \|F\|_{(n)} < 1 \). These formulas are direct generalizations of the existence criteria of Nevanlinna-Pick theory, which is the special case \( d = 1 \) of the above. However, an important difference between \( d = 1 \) and \( d > 1 \) is that transposition is no longer completely isometric for \( d > 1 \). If there exists \( F \in (\text{Shift}_d)_{(n)} \) with \( F(x_j) = y_j \) and \( \|F\|_{(n)} < 1 \), there need not exist \( F \in (\text{Shift}_d)_{(n)} \) with \( F(x_j) = y_j^* \) and \( \|F\|_{(n)} < 1 \).

The computation of two-dimensional quotients of \( \text{Shift}_d \) yields the quotient distance and metric for \( \text{Shift}_d \). The result turns out to be the usual Carathéodory
distance and metric for $D_d$. This computation is greatly facilitated by the invariance of $\text{Shift}_d$ under automorphisms of the ball.

An important property of quotients of $\text{Shift}_d$ is that they have completely isometric representations by $r \times r$-matrices if they have dimension $r$. Unital, commutative operator algebras with this property are said to have minimal quotient complexity. This is a very rare property. Indeed, the author conjectures that the only finite dimensional operator algebras with this property are orthogonal direct sums of quotients of $\text{Shift}_d$ and $\text{Shift}_d'$.

As a first step towards proving this conjecture, the case of trivial unitizations is studied. We obtain that cotangent spaces of closed operator algebras of minimal complexity are completely isometric to $\mathbb{B}(C(K))$ or $\mathbb{B}(H, \mathbb{C})$ for some Hilbert space $H$ with the obvious matrix normed structure. Especially, the only possibilities for finite dimensional cotangent spaces are $T_0\text{Shift}_d$ and $T_0\text{Shift}_d'$ with $d \in \mathbb{N}$. In particular, if a function algebra $F$ has minimal complexity, its cotangent spaces can have dimension at most 1. This excludes algebras like $\mathcal{P}(\mathcal{M})$, $\mathcal{R}(\mathcal{M})$, etc., if $\dim \mathcal{M} \geq 2$.

1.1. Notation. If $K$ is a Hilbert space and $\mathcal{K} \subset K$, then $\mathcal{K} \ominus K$ denotes the orthogonal complement of $K$ in $\mathcal{K}$. If $L: V_1 \to V_2$ is a linear map, $\text{Ker} L$ and $\text{Ran} L$ denote its kernel and range, respectively. For $p \in [1, \infty]$, $n \in \mathbb{N}$, let $\ell^p_n$ be the $n$-dimensional $\ell^p$-space. Let $M_n$ be the $n \times n$-matrices with the usual $C^*$-norm and let $M_{n,m}$ be the $n \times m$-matrices, normed as operators from $\ell^2_n$ to $\ell^2_m$.

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $\partial \mathbb{D}$ its boundary, the unit circle. If $V$ is a normed space, let $\text{Ball}(V)$ be its open unit ball and $\overline{\text{Ball}}(V)$ its closed unit ball. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

If $\Omega$ is a compact space, $C(\Omega)$ stands for the $C^*$-algebra of $\mathbb{C}$-valued continuous functions on $\Omega$. If $\mathcal{M}$ is a complex manifold, $H^\infty(\mathcal{M})$ denotes the algebra of bounded holomorphic functions on $\mathcal{M}$ and $\mathcal{O}(\mathcal{M}, \mathcal{N})$ the algebra of holomorphic maps from $\mathcal{M}$ to $\mathcal{N}$. As usual, $\mathcal{O}(\mathcal{M}) = \mathcal{O}(\mathcal{M}, \mathbb{C})$. If $K \subset \mathbb{C}^n$ is a compact set, let $\mathcal{O}(K)$ be the algebra of functions holomorphic in a neighborhood of $K$ and $\overline{\mathcal{O}}(K)$ its closure in $\mathcal{C}(K)$. Similar conventions apply to the algebras $\mathcal{P}(K)$ of polynomials and $\mathcal{R}(K)$ of rational functions without singularities in $K$. View them as unital subalgebras of $\mathcal{C}(K)$ and write $\overline{\mathcal{P}}(K)$ and $\overline{\mathcal{R}}(K)$ for their closures.

The Carathéodory* pseudodistance on a complex manifold $\mathcal{N}$ is defined by

\[ c^*_\mathcal{N}(\omega_1, \omega_2) = \sup\{|f(\omega_2)| \mid f \in \mathcal{O}(\mathcal{N}, \mathbb{D}), \ f(\omega_1) = 0\} \]

for $\omega_1, \omega_2 \in \mathcal{N}$. The Carathéodory-Reiffen pseudometric on $\mathcal{N}$ is defined by

\[ \gamma_{\mathcal{N}}(\omega, d) = \sup\{|d(f)| \mid f \in \mathcal{O}(\mathcal{N}, \mathbb{D}), \ f(\omega) = 0\} \]

for $(\omega, d) \in T\mathcal{N}$, i.e. $d: C^\infty(\mathcal{N}) \to \mathbb{C}$ is a derivation at $\omega \in \mathcal{N}$ ($C^\infty(\mathcal{N})$ is the algebra of smooth functions $\mathcal{N} \to \mathbb{C}$). The Carathéodory pseudodistance $c_{\mathcal{N}}$ (without *) is related to $c^*_\mathcal{N}$ by $c^*_\mathcal{N} = \tanh c_{\mathcal{N}}$. If $\mathcal{N} = \mathbb{D}$, these definitions yield the Möbius distance

\[ m(\lambda_1, \lambda_2) = c^*_\mathbb{D}(\lambda_1, \lambda_2) = \left| \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \right| , \]

the Poincaré distance $p = c_\mathbb{D}$, and the Poincaré metric

\[ \gamma(\lambda, l) = \frac{|l|}{1 - |\lambda|^2} , \]

where $(\lambda, l)$ stands for the derivation $f \mapsto l f'(\lambda)$. See [16] for these distances and metrics.
If $V \subset B(\mathcal{H})$ is an operator space, let

$$V_{(n)} = V \otimes M_n \subset B(\mathcal{H} \otimes \mathbb{C}^n)$$

Write $\|\cdot\|_{(n)}$ for the norm on $V_{(n)}$ coming from this representation of $V_{(n)}$.

Let $\phi: V_1 \to V_2$ be a linear map between operator spaces. Then $\phi$ induces linear maps $\phi_{(n)} = \phi \otimes \text{id}_{\mathbb{C}^n}: V_1 \otimes M_n \to V_2 \otimes M_n$. $\phi$ is called $n$-contractive if $\phi_{(n)}$ is contractive, and completely contractive if all the maps $\phi_{(n)}$, $n \in \mathbb{N}$, are contractive. Similarly, $\phi$ is called completely isometric if all the maps $\phi_{(n)}$ are isometric, and $n$-isometric if $\phi_{(n)}$ is isometric. Let

$$\|\phi\|_{cb} = \sup_{n \in \mathbb{N}} \|\phi_{(n)}\|$$

and call $\phi$ completely bounded if $\|\phi\|_{cb} < \infty$. An isometric map is not required to be surjective. A surjective, completely isometric map is called a complete equivalence. If both $V_1$ and $V_2$ are operator algebras, usually only homomorphisms are considered, and a complete equivalence of operator algebras is a completely isometric isomorphism. As a matter of convention, an abstract operator algebra $A$ is called unital only if it has a unit $e$ with $\|e\| = 1$ because if $\|e\| \neq 1$ it can have no “unital” completely isometric representations.

A matrix normed space or algebra satisfying certain axioms [9] is called an $L^\infty$-matricially normed space or an $L^\infty$-matricially normed algebra. The point of these axioms is that a matrix normed space has a completely isometric (linear) representation on a Hilbert space if and only if it is an $L^\infty$-matricially normed space, and a unital matrix normed algebra (with unit of norm 1!) has a completely isometric (unital, multiplicative) representation on a Hilbert space if and only if it is an $L^\infty$-matricially normed algebra. Thus we prefer to call them abstract operator spaces and abstract operator algebras. A not necessarily unital $L^\infty$-matricially normed algebra is only called an abstract operator algebra if it has a completely isometric representation on a Hilbert space.

There is a natural way to define the (orthogonal) direct sum of operator spaces $V_j \subset \mathcal{H}_j$, $j = 1, 2$, as $V_1 \oplus V_2 \subset B(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Thus

$$\|(v_1, v_2)\|_{(n)} = \max\{\|v_1\|_{(n)}, \|v_2\|_{(n)}\}$$

for $v_j \in V_j \otimes M_n$, $j = 1, 2$. Equation (6) shows that the resulting operator space does not depend on the chosen completely isometric representations of $V_j$, $j = 1, 2$. All this remains true for operator algebras.

Another natural construction is the quotient operator space structure. If $V_1$ is an abstract operator space and $V_2$ is a closed subspace, define a matrix normed structure on the quotient space $V_1/V_2$ by identifying

$$(V_1/V_2) \otimes M_n \cong (V_1 \otimes M_n)/(V_2 \otimes M_n)$$

and taking the quotient norm on the latter space. This gives an abstract operator space [9] because the result satisfies the axioms for an $L^\infty$-matricially normed space. Moreover, if $A$ is a unital $L^\infty$-matricially normed algebra (with unit of norm 1), and $I \subset A$ is a proper ideal, then $A/I$ is again a unital $L^\infty$-matricially normed algebra and thus an abstract operator algebra. This is the only place where the formalism of $L^\infty$-matricially normed algebras is needed in this article.

The natural projection $\pi: V_1 \to V_1/V_2$ is a complete quotient map, i.e. every map $\pi_{(n)}$ is a quotient map. This implies that if $\rho: V_1/V_2 \to V_3$ is any linear map, then $\|\rho_{(n)}\| = \|\rho \circ \pi_{(n)}\|$ for all $n \in \mathbb{N}$ and in particular $\|\rho\|_{cb} = \|\rho \circ \pi\|_{cb}$.

Further references for matricially normed spaces include [9], [11], [24], and [23].
2. Unitization of operator algebras

Let \( A \subset \mathcal{B}(\mathcal{H}) \) be an operator algebra with \( \text{id}_{\mathcal{H}} \not\in A \). The goal of this section is to express the matrix normed structure of its unitization \( A^+ = A \oplus \mathbb{C} \cdot \text{id}_{\mathcal{H}} \) in terms of the matrix normed structure of \( A \). Thus it is independent of the chosen representation. The basic idea is that the open unit balls of \( M_n \) are symmetric domains and especially have a transitive automorphism group. This easily implies the results of this section for uniform algebras and \( \mathcal{Q} \)-algebras (without any further explicit computations!). However, in order to get statements for arbitrary operator algebras, some calculations are necessary.

**Lemma 2.1.** Every 1-dimensional, unital abstract operator algebra is completely equivalent to \( \mathbb{C} \) with its usual operator algebra structure.

**Proof.** Let \( A \) be a 1-dimensional, unital abstract operator algebra and let \( \rho : A \to \mathcal{B}(\mathcal{H}) \) be any unital, completely isometric representation. If \( 1_A \) is the identity element of \( A \), then \( \rho(1_A) = \text{id}_{\mathcal{H}} \) and this determines \( \rho \). Now the lemma follows from the identity \( \| S \otimes T \| = \| S \| \cdot \| T \| \).

Define \( \mathcal{C}(X) = (1 - X)/(1 + X) \) for all \( X \in \mathcal{B}(\mathcal{H}) \) such that \( 1 + X \) is invertible (here \( 1 = \text{id}_{\mathcal{H}} \)). This differs from the usual Cayley transform that maps a self-adjoint operator to a unitary only by some factors of \( i \) and is indeed an analogue of the Cayley transform for skew-adjoint operators. Let \( \mathfrak{B} = \text{Ball}(\mathcal{B}(\mathcal{H})) \) and \( \mathfrak{B}^+ = \{ X \in \mathcal{B}(\mathcal{H}) \mid \text{Re} \, X \text{ positive and invertible} \} \).

**Lemma 2.2.** \( \mathcal{C}(X) \) is well-defined for \( X \in \mathfrak{B} \cup \mathfrak{B}^+ \). \( \mathcal{C} \) maps \( \mathfrak{B} \) bijectively onto \( \mathfrak{B}^+ \) and is its own inverse, i.e. \( \mathcal{C} \circ \mathcal{C}(X) = X \) for \( X \in \mathfrak{B} \cup \mathfrak{B}^+ \). More generally, \( 1 + \mathcal{C}(X) \) is invertible whenever \( \mathcal{C}(X) \) is defined and then \( \mathcal{C} \circ \mathcal{C}(X) = X \).

**Proof.** It is easy to see that elements of \( 1 + \mathfrak{B} \) and \( 1 + \mathfrak{B}^+ \) are invertible. The identity \( \mathcal{C} \circ \mathcal{C}(X) = X \) is easy to check, whenever the left side is well-defined. Thus it only remains to show \( \mathcal{C}(\mathfrak{B}) \subset \mathfrak{B}^+ \) and \( \mathcal{C}(\mathfrak{B}^+) \subset \mathfrak{B} \).

\[
2 \text{Re} \mathcal{C}(X) = (1 + X)^{-1}((1 - X)(1 + X^*) + (1 + X)(1 - X^*)) (1 + X^*)^{-1}
\]

\[
= 2(1 + X)^{-1}(1 - XX^*)(1 + X^*)^{-1}
\]

shows that \( \text{Re} \, \mathcal{C}(X) \) is positive and invertible for \( X \in \mathfrak{B} \), i.e. \( \mathcal{C}(\mathfrak{B}) \subset \mathfrak{B}^+ \). A similar calculation shows that for \( X \in \mathfrak{B}^+ \),

\[
1 - \mathcal{C}(X) \mathcal{C}(X)^* = 4(1 + X)^{-1} \text{Re}(X)(1 + X^*)^{-1}
\]

is positive and invertible, so that \( \mathcal{C}(X) \in \mathfrak{B} \).

**Theorem 2.3.** Let \( A \) be a closed, unital abstract operator algebra and \( X \in A_{(n)} \). Then the following are equivalent:

(i) \( X \) is of the form \( \mathcal{C}(Y) \) for some \( Y \in \text{Ball}(A_{(n)}) \);
(ii) \( 1 + X \) is invertible in \( A_{(n)} \) and \( \mathcal{C}(X) \in \text{Ball}(A_{(n)}) \);
(iii) \( \rho_{(n)}(X) \) has invertible and positive real part for all \( n \)-contractive, unital representations \( \rho : A \to \mathcal{B}(\mathcal{H}) \);
(iv) \( \rho_{(n)}(X) \) has invertible and positive real part for some \( n \)-isometric, unital representation \( \rho : A \to \mathcal{B}(\mathcal{H}) \).

**Definition 2.1.** The set of elements satisfying one of these equivalent conditions is called the (positive) cone \( \text{Cone}(A_{(n)}) \) of \( A_{(n)} \).

\footnote{The algebra \( A \) may well be unital, i.e. have a unit of norm 1, so that the notation \( A^+ \) is preferable to \( \tilde{A} \).}
Proof. Replacing $A$ by $A_{(n)}$, if necessary, it can be assumed that $n = 1$, i.e. $X \in A$. Since $A$ is closed, all elements of $1 + \text{Ball}(A)$ are invertible in $A$, so that $\mathcal{C}(Y)$ is defined and lies in $A$ for all $Y \in \text{Ball}(A)$. Thus the equivalence of (i) and (ii) follows as in Lemma 2.2. Moreover, (iii) trivially implies (iv).

If $\rho$ is some contractive ($n = 1$) representation and $X = \mathcal{C}(Y)$ with $Y \in \text{Ball}(A)$, then $\rho(X) = \mathcal{C}(\rho(Y))$ because $\rho$ is a homomorphism, and this lies in $B_+\mathcal{B}_+$ by Lemma 2.2. Hence (i) implies (iii), so that it remains to show that (iv) implies (ii).

Therefore, take any isometric unital representation $\rho: A \to \mathcal{B}(\mathcal{H})$ and $X \in A$ such that $\rho(X)$ has positive and invertible real part. It is not difficult to see that $A \in \mathcal{B}(\mathcal{H})$ is invertible if its real part is positive and invertible. Thus $\rho(X) + \lambda \text{id}_\mathcal{H}$ is invertible in $\mathcal{B}(\mathcal{H})$ for all $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \geq 0$. Therefore, the spectrum of $1 + \rho(X)$ is a compact subset of $\{\lambda \in \mathbb{C} \mid \text{Re}\lambda > 1\}$. The function $\lambda \mapsto 1/\lambda$ can be approximated uniformly on a neighborhood of this compact set by polynomials. Thus the inverse of $1 + \rho(X)$ lies in $\rho(A)$, so that $1 + X$ is invertible in $A$ and not just in $\mathcal{B}(\mathcal{H})$. Hence $\mathcal{C}(X)$ is a well-defined element of $A$. Moreover, $\rho(\mathcal{C}(X)) = \mathcal{C}(\rho(X))$. Since $\|\mathcal{C}(\rho(X))\| < 1$ by Lemma 2.2 and $\rho$ is isometric, $\|\mathcal{C}(X)\| < 1$. Thus (iv) implies (ii).

Due to this correspondence, the matrix normed structure of a closed unital operator algebra $A \subset \mathcal{B}(\mathcal{H})$ can equally well be described by its positive cone. This is especially advantageous if $\mathcal{H}$ does not come with an orthonormal basis but just with a frame. A set of vectors $(\xi_j)_{j \in J}$ in a Hilbert space $\mathcal{H}$ is a frame if there exist numbers $A, B \in (0, \infty)$ such that, for all $\eta \in \mathcal{H}$,

$$A\|\eta\|^2 \leq \sum_{j \in J} |\langle \xi_j, \eta \rangle|^2 \leq B\|\eta\|^2.$$ 

Thus a frame need not be linearly independent. With every frame one can associate a bounded linear map $S: \mathcal{H} \to \ell^2(J)$ mapping $\eta$ to $(\langle \eta, \xi_j \rangle)_{j \in J}$. The assumptions guarantee that $S$ is bounded and that $S^*S$ is invertible.

**Proposition 2.4.** Let $\mathcal{H}$ be a Hilbert space, $(\xi_j)_{j \in J}$ a frame, and $T \in \mathcal{B}(\mathcal{H})$. Then $T$ has positive real part iff $S^*TS \in \mathcal{B}(\ell^2(J))$ has positive real part. This happens iff the matrix $\tilde{T} \in \mathcal{B}(\ell^2(J))$ with entries

$$\tilde{T}_{ij} = \langle T\xi_j, \xi_i \rangle + \langle \xi_j, T\xi_i \rangle$$

is positive definite.

If $S$ is invertible, then $T$ has positive and invertible real part iff $S^*TS$ has positive and invertible real part.

Proof. If $T$ has positive real part, then so has $STS^* \in \mathcal{B}(\ell^2(J))$ because

$$2\text{Re}(STS^*) = STS^* + S^*TS^* = 2S\text{Re}(T)S^*.$$ 

Conversely, if $STS^*$ has positive real part then so has $S^*STS^*S \in \mathcal{B}(\mathcal{H})$ and hence $T$ because $S^*S$ is invertible. If $S$ is invertible then $STS^*$ has invertible real part iff $T$ has by (7).

Let $(e_j)_{j \in J}$ be the canonical orthonormal basis of $\ell^2(J)$. Then $S^*e_j = \xi_j$ and thus

$$\langle 2\text{Re}(STS^*)e_j, e_i \rangle = \langle 2S\text{Re}(T)S^*e_j, e_i \rangle = \langle 2\text{Re}(T)e_j, \xi_i \rangle = \langle T\xi_j, \xi_i \rangle + \langle \xi_j, T\xi_i \rangle.$$ 

Hence the matrix $\tilde{T}$ comes from the operator $2\text{Re}(STS^*)$. 

The situation of Proposition 2.4 will occur in Section 9: The Hilbert spaces on which quotients of $\text{Shift}_J$ are represented do not come with natural orthonormal bases, but linearly independent frames are rather easy to obtain.
It is easy to write down automorphisms of $\text{Cone}(A_{(n)})$: If $A \in M_n$ is invertible and $B \in M_n$ satisfies $\text{Re} B = 0$, then $\Phi_{A,B}: X \mapsto AXA^* + B$ defines a bijection from $\text{Cone}(A_{(n)})$ onto itself, with inverse $X \mapsto A^{-1}X(A^{-1})^* - A^{-1}B(A^{-1})^*$. It is easy to see that these maps really map $\text{Cone}(A_{(n)})$ into itself using the characterization (iv) of Theorem 2.3. Consequently, the map $C \circ \Phi_{A,B} \circ C$ gives a bijection $\text{Ball}(A_{(n)}) \rightarrow \text{Ball}(A_{(n)})$.

**Theorem 2.5.** Let $A \subset \mathbb{B}(\mathcal{H})$ be a closed operator algebra not containing $\text{id}_{\mathcal{H}}$. Then the matrix normed structure of $A^+$ is uniquely determined by the matrix normed structure of $A$: The open unit ball of $(A^+_{(n)})$ consists precisely of the elements $\Psi(X)$ where $X \in \text{Ball}(A_{(n)})$ and $\Psi = \mathcal{C} \circ \Phi_{A,B} \circ \mathcal{C}$ for $A,B \in M_n$, $A$ invertible and $\text{Re} B = 0$.

If $\rho: A \rightarrow \mathbb{B}(\mathcal{H})$ is a completely isometric representation and $\text{id}_{\mathcal{H}} \not\in \rho(A)$, then the unitization $\rho^+: A^+ \rightarrow \mathbb{B}(\mathcal{H})$, defined by $\rho^+(A) = \rho$, $\rho^+(1_{A^+}) = \text{id}_{\mathcal{H}}$, is also completely isometric.

More generally, let $\rho: A \rightarrow \mathbb{B}(\mathcal{H})$ be a representation with $\text{id}_{\mathcal{H}} \not\in \rho(A)$. Then $\rho^+$ is $n$-contractive if and only if $\rho$ is $n$-contractive. The map $(\rho^+)^{(n)}$ is a quotient map onto its image if and only if $\rho(n)$ is a quotient map onto its image.

**Proof.** All elements $\Psi(X)$ with $X \in \text{Ball}(A_{(n)})$ and $\Psi$ as above lie in $\text{Ball}((A^+_{(n)})$. Conversely, let $X$ in $\text{Ball}((A^+_{(n)})$. The map $A^+ \rightarrow A^+ / A$ is a completely contractive homomorphism. Since $A^+$ is unital, so is $A^+ / A$, so that $A^+ / A$ is completely equivalent to $\mathbb{C}$ with the usual matrix normed structure by Lemma 2.1. This yields a completely contractive unital homomorphism $\omega: A^+ \rightarrow \mathbb{C}$. Thus $X_0 = \omega_n(X)$ lies in $\text{Ball}(M_n)$.

For suitable $A,B \in M_n$, the corresponding $\Psi = \mathcal{C} \circ \Phi_{A,B} \circ \mathcal{C}: \text{Ball}(M_n) \rightarrow \text{Ball}(M_n)$ satisfies $\Psi(0) = X_0$. Let $\Psi$ also denote the map $\mathcal{C} \circ \Phi_{A,B} \circ \mathcal{C}: \text{Ball}((A^+)^{(n)}) \rightarrow \text{Ball}((A^+)^{(n)})$.

Then $\omega(n) \circ \Psi^{-1} = \Psi^{-1} \circ \omega(n)$ and $\omega(n)(\Psi^{-1}(X)) = 0$, i.e. $\Psi^{-1}(X) \in A_{(n)}$. Since also $||\Psi^{-1}(X)|| < 1$ and $X = \Psi(\Psi^{-1}(X))$, all elements of $\text{Ball}((A^+)^{(n)})$ are of the form $\Psi(Y)$ for some $Y \in \text{Ball}(A_{(n)})$.

The definition of $\Psi(X)$ is purely algebraic and does not use the matrix normed structure of $A^+$, but only that certain elements of $A^+$, namely those of the form $1 + \Phi_{A,B} \circ \mathcal{C}(X)$ with appropriate $X,A,B$, are invertible. Thus any completely isometric representation of $A$ whose image does not contain the identity, yields the same unit ball for $(A^+)^{(n)}$. Hence the unital extension of such a completely isometric representation of $A$ is completely isometric.

The map $\rho(n)$ is contractive iff it maps $\text{Ball}(A_{(n)})$ into $\text{Ball}(\mathbb{B}(\mathcal{H}^n \otimes \mathbb{C}^n))$, and similarly for $\rho^+_n$. Thus contractiveness of $\rho^+_n$ trivially implies contractiveness of $\rho(n)$. The converse follows because $\rho^+_n(\Psi(X)) = \Psi(\rho(n)(X))$ for all $X \in \text{Ball}(A_{(n)})$ and the right side lies in $\text{Ball}(\mathbb{B}(\mathcal{H}^n \otimes \mathbb{C}^n))$ if $||\rho(n)(X)|| < 1$.

The statement about quotient maps follows in the same way, using that $\rho(n)$ is a quotient map onto its image if it maps $\text{Ball}(A_{(n)})$ onto $\text{Ball}(\mathbb{B}(\mathcal{H}^n \otimes \mathbb{C}^n)) \cap \rho(n)(A_{(n)})$ and the same statement for $\rho^+_n$.

For the special case of quotients of the function algebra $\mathbb{H}^\infty(\mathbb{D})$, this corresponds to well-known facts of Nevanlinna-Pick theory. Theorem 2.3 means that interpolation problems from the unit disk to $M_n$ with values in $\text{Ball}(M_n)$ and $\text{Cone}(M_n)$ are equivalent. Theorem 2.5 means that one can always restrict to the case where the origin is mapped to $0 \in \text{Ball}(M_n)$ or $1 \in \text{Cone}(M_n)$.

If $A \subset \mathbb{B}(\mathcal{H})$ does contain $\text{id}_{\mathcal{H}}$, consider $A \subset \mathbb{B}(\mathcal{H} \oplus \mathbb{C})$ in the obvious way and define $A^+ \subset \mathbb{B}(\mathcal{H} \oplus \mathbb{C})$. Then Theorem 2.5 shows that $A^+$ is a well-defined abstract
operator algebra if \( \mathcal{A} \) is an abstract operator algebra (it does not matter whether \( \mathcal{A} \) is closed), i.e. it does not depend on the choice of a completely isometric representation. Every homomorphism \( \rho: \mathcal{A}_1 \to \mathcal{A}_2 \) can be extended uniquely to a unital homomorphism \( \rho^+: \mathcal{A}_1^+ \to \mathcal{A}_2^+ \).

**Corollary 2.6.** \( \mathcal{A} \to \mathcal{A}^+ \) is a functor from the category of abstract operator algebras with completely contractive homomorphisms as morphisms to the category of unital abstract operator algebras with unital, completely contractive homomorphisms as morphisms. Furthermore, it maps complete quotient maps to complete quotient maps and complete isometries to complete isometries. Thus \( (\mathcal{A}/I)^+ \cong \mathcal{A}^+/I \).

Let \( \mathcal{V} \) be an abstract operator space, furnish it with the zero multiplication. This yields an abstract operator algebra: Let \( \rho: \mathcal{V} \to \mathcal{B}(\mathcal{H}) \) be any completely isometric linear representation, define \( \rho_2: \mathcal{V} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) by putting

\[
\rho_2(x) = \begin{pmatrix} 0 & \rho(x) \\ 0 & 0 \end{pmatrix}.
\]

Then \( \rho_2(x)\rho_2(y) = 0 \) for all \( x, y \in \mathcal{V} \), i.e. \( \rho_2 \) is multiplicative. Now let \( \mathcal{V}^+ \) be the unitization of \( \mathcal{V} \), viewed as an abstract operator algebra with zero multiplication. \( \mathcal{V}^+ \) is called the **trivial unitization of \( \mathcal{V} \)**. Here trivial does not mean that the resulting objects cannot be very complicated but that \( \mathcal{V} \) is endowed with the trivial multiplication.

It is easy to see that every unital homomorphism \( \sigma: \mathcal{V}_1^+ \to \mathcal{V}_2^+ \) is of the form \( \sigma = \rho^+ \) for some \( \rho: \mathcal{V}_1 \to \mathcal{V}_2 \), namely \( \rho = \sigma|_{\mathcal{V}_1} \). \( \mathcal{V} \) is the unique maximal ideal of \( \mathcal{V}^+ \), and the multiplication on \( \mathcal{V} \) is trivial. Conversely, if \( \mathcal{A} \) is an abstract operator algebra with a 1-codimensional ideal \( \mathcal{V} \) such that the multiplication on \( \mathcal{V} \) is trivial, then \( \mathcal{A} = \mathcal{V}^+ \). Especially, the image of a unital homomorphism \( \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) again has these properties.

**Theorem 2.7.** Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be abstract operator spaces, let \( \rho: \mathcal{V}_1 \to \mathcal{V}_2 \) be a linear map, and let \( \rho^+: \mathcal{V}_1^+ \to \mathcal{V}_2^+ \) be its unitization. Then, for all \( n \in \mathbb{N} \),

\[
\| \rho^+(\alpha) \| = \max\{1, \| \rho \|_{(\alpha)} \}.
\]

**Proof.** The inequality “\( \geq \)” is trivial. To prove “\( \leq \)” we assume \( M = \| \rho(\alpha) \| < \infty \). If \( M \leq 1 \), the assertion follows from Theorem 2.6. Thus assume \( M > 1 \) and let \( m: \mathcal{V}_1 \to \mathcal{V}_1 \) be the map \( T \mapsto MT \). Then \( \rho = \rho \circ m^{-1} \circ m \), and \( \| (\rho \circ m^{-1})(\alpha) \| = 1 \). Hence \( \| (\rho^+ \circ (m^{-1})^+)(\alpha) \| \leq 1 \), so that it remains to prove \( \| m^+ \|_{cb} \leq M \).

Therefore, choose a representation of \( \mathcal{V}_1 \) on some Hilbert space \( \mathcal{H} \) and view \( \mathcal{V}_1^+ \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) as above. Let \( S = \text{diag}(M^{1/2}, M^{-1/2}) \) be a diagonal matrix with respect to this decomposition, i.e. \( S = M^{1/2}\text{id}_{\mathcal{H}} \) on the first copy of \( \mathcal{H} \) and \( S = M^{-1/2}\text{id}_{\mathcal{H}} \) on the second copy. Then \( m^+(T) = STS^{-1} \) for all \( T \in \mathcal{V}_1^+ \) because both sides of this equation are unitary maps that coincide on \( \mathcal{V}_1 \). Thus \( \| m^+ \|_{cb} \leq \| S \| \cdot \| S^{-1} \| = M \) as desired. \( \square \)

Any linear representation \( \rho: \mathcal{V} \to \mathcal{B}(\mathcal{H}) \) yields a unital, multiplicative representation \( \rho^+: \mathcal{V}^+ \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) with the same boundedness properties. Thus the representation theory of \( \mathcal{V}^+ \) is precisely as well-behaved (or pathological) as the linear representation theory of \( \mathcal{V} \).

There seems to be no analogue of Theorem 2.7 for unitizations of homomorphisms between operator algebras with non-trivial multiplication. Already for the two-dimensional examples, the estimate \( \| \rho^+ \| \leq \max\{1, \| \rho \|_{cb} \} \) does not hold. Instead, (9) implies

\[
\| \rho^+ \| = \| \rho^+ \|_{cb} \leq \max\{1, 2\| \rho \| \}.
\]

But this probably is a special feature of the two-dimensional case.
Theorem 2.8. Let $A$ be a unital, commutative operator algebra, $d \in \mathbb{N}$. Then any $d-1$-contractive unital homomorphism $\rho: A \to M_d$ is completely contractive.

Proof. Let $B = \rho(A)$, let $J$ be a maximal ideal in $B$, and $1 = \rho^{-1}(J)$. Then $A = 1^+$, $B = J^+$, and $\rho = (\rho|_1)^+$. By Theorem 2.5, it suffices to show that $\rho|_1$ is completely contractive.

By abstract algebra, there exists a vector $x \in \mathbb{C}^d$ that is annihilated by all elements of $J$, because $J$ is a non-unital, commutative operator algebra. Apply the same reasoning to the algebra $J^*$ of all adjoints of elements of $J$. This yields $y \in \mathbb{C}^d$ that is annihilated by all adjoints $T^*$ of elements $T \in J$.

Thus elements of $J$ can be viewed as operators from $\mathbb{C}^d \ominus x$ to $\mathbb{C}^d \ominus y$. This yields a completely isometric linear representation of $J$ by $(d-1) \times (d-1)$-matrices. In general, this representation fails to be a homomorphism, but this does not matter. By a result of Smith (see [23]), every linear representation $\phi: V \to M_n$ of an operator space $V$ satisfies $\|\phi\|_{cb} = \|\phi(n)\|$. Application of Smith’s theorem to $\rho|_1$, with $n = d-1$, yields the assertion. \qed

The following obvious corollary will be strengthened in the next section.

Corollary 2.9. Let $A$ be a unital, commutative operator algebra. Then any contractive unital homomorphism $\rho: A \to M_2$ is completely contractive.

3. Two-dimensional unital operator algebras

In this section, two-dimensional unital operator algebras are classified up to complete equivalence. This is the smallest non-trivial dimension by Lemma 2.1. Most results will be false for two-dimensional non-unital operator algebras and for operator algebras of dimension at least 3. This makes the study of two-dimensional unital operator algebras even more important because it is the only case with a satisfactory theory. Theorem 2.5 reduces the classification of two-dimensional unital operator algebras $A$ to that of 1-dimensional operator algebras, which is rather trivial.

Lemma 3.1. Every 1-dimensional abstract operator space is completely equivalent to $C$ with its usual matrix normed structure.

Proof. Let $V$ be a 1-dimensional abstract operator space, choose a completely isometric linear representation $V \subset B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Choose $v \in V$ with $\|v\| = 1$. Then every element of $V(n)$ can be written as $v \otimes T$ for some $T \in M_n$. Since $\rho$ is completely isometric,

$$\|v \otimes T\| = \|\rho(v) \otimes T\| = \|\rho(v)\| \cdot \|T\| = \|T\|.$$ 

\qed

For $c \in [0, 1]$, let

$$T_c = \begin{pmatrix} 0 & \sqrt{1-c^2} \\ 0 & c \end{pmatrix}.$$ 

Clearly, $\|T_c\| = 1$. Since $T_c^2 = cT_c$, $C \cdot T_c$ is an operator algebra. Let $Q_c = \text{lin}\{1, T_c\}$ be the unital operator algebra generated by $T_c$.

Theorem 3.2. Let $A$ be a 1-dimensional abstract operator algebra. Then there exist unique $c \in [0, 1]$ and $T \in A$ such that $T \mapsto T_c$ defines a completely isometric representation of $A$.
Proof. There exist unique $T \in A$ and $c \in [0, \infty]$ with $\|T\| = 1$ and $T^2 = cT$: The first condition determines $T$ uniquely up to scalar multiplication by some $\lambda \in \partial D$. This can be used to make the constant $c$ real and nonnegative. In addition, $c = \|cT\| = \|T^2\| \leq \|T\|^2 = 1$, so that $c \in [0, 1]$. Using Lemma 3.1, it is easy to see that $T \mapsto T_c$ defines a completely isometric representation of $A$. 

**Theorem 3.3.** Let $A$ be a two-dimensional unital operator algebra. Then $A$ is completely equivalent to $Q_c$ for a unique $c \in [0, 1]$ and thus has a completely isometric, unital representation by $2 \times 2$-matrices. Indeed, the algebras $Q_c$ are not even isometrically isomorphic for different values of $c$.

Proof. Let $A$ be a 2-dimensional, unital abstract operator algebra. Then $A$ is necessarily commutative and thus contains a maximal ideal $I$. Thus $A = I^{+}$. By Theorem 3.2, there is a completely isometric representation $I \to M_2$ whose image does not contain the identity. The unitization of this representation gives a unital, completely isometric representation of $A$ by $2 \times 2$-matrices.

It will follow immediately from Theorem 3.4 that the algebras $Q_c$ for different values of $c$ are not isometrically isomorphic.

The following theorem lists all (algebraic) isomorphisms, i.e. bijective algebra homomorphisms, between the algebras $Q_c$, $c \in [0, 1]$. Such an isomorphism is necessarily unital and hence determined by the image of the generator $T_c$.

**Theorem 3.4.** (i) The automorphisms of $Q_0$ are the unital maps $m_\lambda : Q_0 \to Q_0$ defined by $T_0 \mapsto \lambda T_0$ for $\lambda \in \mathbb{C}^{+}$. There are no isomorphisms $Q_0 \to Q_c$ (or in the opposite direction) for $c \in (0, 1]$. The only non-identical automorphism of $Q_c$ for $c \neq 0$ is $\theta_c : Q_c \to Q_c$ given by

$$T_c \mapsto c - T_c = \begin{pmatrix} c & -\sqrt{1-c^2} \\ 0 & 0 \end{pmatrix}.$$ 

For $c, c' \in (0, 1]$, there are two isomorphisms $Q_c \to Q_{c'}$, namely $i_{c,c'}$ defined by $i_{c,c'}(T_c) = (c/c')T_{c'}$ and $\theta_c \circ i_{c,c'} = i_{c,c'} \circ \theta_{c'}$.

(ii) These isomorphisms are contractive if and only if they are completely contractive and isometric if and only if they are completely isometric.

(iii) The automorphism $m_\lambda$ is (completely) contractive iff $|\lambda| \leq 1$ and (completely) isometric iff $|\lambda| = 1$. More generally,

$$\|m_\lambda\| = \|m_\lambda\|_{cb} = \max\{1, |\lambda|\}. \hspace{1cm} (8)$$

(iv) The automorphism $\theta_c$ is completely isometric for all $c \in (0, 1]$. The isomorphism $i_{c,c'}$ is (completely) contractive iff $c \leq c'$ and isometric only for $c = c'$. More generally,

$$\|i_{c,c'}\| = \|i_{c,c'}\|_{cb} = \max\{1, h(c')/h(c)\}, \hspace{1cm} (9)$$

where

$$h(c) = c^{-1}(1 + \sqrt{1-c^2}).$$

Proof. (i) A unital map $\ell : Q_c \to Q_{c'}$ is an isomorphism iff $\ell(T_c)^2 = c\ell(T_c)$. This easily implies that the maps $m_\lambda$, $\theta_c$, and $i_{c,c'}$ are isomorphism, and that there are no other possibilities.

(ii) Theorem 2.5 and Lemma 3.1 imply that an isomorphism $\ell : Q_c \to Q_{c'}$ is (completely) contractive iff its restriction to the linear span of $T_c$ is (completely) contractive iff $\|\ell(T_c)\| \leq 1$, and that it is (completely) isometric iff $\|\ell(T_c)\| = 1$.

(iii) The restriction of $m_\lambda$ to the linear span of $T_0$ is just scalar multiplication by $\lambda$ and thus has norm and complete norm $|\lambda|$. Hence the claim is a special case of Theorem 2.7.
(iv) Since \( \|\theta_c(T_c)\| = 1 \) and \( \|\iota_{c,c'}(T_c)\| = c/c' \), the first part of the assertion follows from the proof of (ii). It remains to prove (9).

For \( c \in (0, 1) \), let \( T_c = -1 + 2c^{-1}T_c \). The matrix \( T_c \) has the eigenvalues \( \pm 1 \) and thus is a more “symmetric” generator for \( Q_c \). Moreover, \( \iota_{c,c'}(T_c) = T_{c'} \).

An elementary calculation shows \( \|T_c\| = h(c) \). This is, of course, where the function \( h \) comes from. Therefore,

\[
\|\iota_{c,c'}\|_{cb} \geq \|\iota_{c,c'}\| \geq \max\{1, h(c')/h(c)\},
\]

so that it remains to prove \( \|\iota_{c,c'}\|_{cb} \leq \max\{1, h(c')/h(c)\} \). This estimate is true for \( c \leq c' \) because then \( \iota_{c,c'} \) is completely contractive.

Thus it only remains to show \( \|\iota_{c,c'}\|_{cb} \leq h(c')/h(c) \) if \( 0 < c' < c \leq 1 \). This follows if there is an invertible \( S \in M_2 \) with \( \iota_{c,c'}(T) = S T S^{-1} \) for all \( T \in Q_c \) and \( \|S\| \cdot \|S^{-1}\| = h(c')/h(c) \). Let

\[
S_\gamma = \begin{pmatrix} 1 & \sqrt{1 - \gamma^2} \\ 0 & \gamma \end{pmatrix}
\]

for \( \gamma \in (0, 1] \) and \( S = S_\gamma S^{-1}. \) It is easy to check \( S^{-1}_\gamma T_c S_\gamma = \gamma T_1 \), so that \( ST_c S^{-1} = (c/c')T_{c'} \). Thus \( ST S^{-1} = \iota_{c,c'}(T) \) for all \( T \in Q_c \).

The computation of \( \|S\| \cdot \|S^{-1}\| \) is quite elementary but tedious, so that the details are left to the reader. A main step is to compute

\[
\left\| \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}^{-1} \right\| = \frac{1}{2|x|} (1 + |x|^2 + |y|^2 + \sqrt{(1 + |x|^2 + |y|^2)^2 - 4|x|^2})
\]

for \( x \in \mathbb{C}^*, y \in \mathbb{C} \). This can be applied to the matrix \( S \). Using the rules

\[
\sqrt{2 - c^2 - c'^2 - 2\sqrt{1 - c^2} \sqrt{1 - c'^2}} = \sqrt{1 - c'^2} - \sqrt{1 - c^2}
\]

and

\[
h(c)^{-1} = c^{-1}(1 - \sqrt{1 - c^2}),
\]

the result can be transformed into \( h(c')/h(c) \).

See also [12] and [19] for different looking but equivalent versions of (8) and (9).

The discovery that \( \|\rho\| = \|\rho\|_{cb} \) for homomorphisms between the algebras \( Q_c \) goes back to Holbrook [14].

Corollary 3.5. Every two-dimensional unital operator algebra is completely equivalent to a quotient of the disk algebra \( \mathcal{P}(\mathbb{D}) \).

Proof. By Theorem 3.3, it suffices to show that each of the algebras \( Q_c \) is completely equivalent to a quotient of the disk algebra. The quotient algebra \( \mathcal{P}(\mathbb{D})/l(0)^2 \) is two-dimensional and algebraically not isomorphic to \( Q_c \) for \( c \neq 0 \). Hence \( Q_0 \cong \mathcal{P}(\mathbb{D})/l(0)^2 \) by Theorem 3.3.

Let \( c \in (0, 1) \). The spectrum of \( Q_c \) consists of precisely two points, called 0 and \( c \), respectively, such that \( T_c(0) = 0 \), \( T_c(c) = c \). Clearly,

\[
\sup\{|T(c)| : T \in Q_c, |T| \leq 1, T(0) = 0\} = c.
\]

It is elementary that

\[
\sup\{|f(c)| : f \in \mathcal{P}(\mathbb{D}), f(0) = 0\} = c.
\]

By definition of the quotient norm, \( \mathcal{P}(\mathbb{D}) \) can be replaced by \( \mathcal{P}(\mathbb{D})/l(0,c) \) here. Comparison with (10) shows \( \mathcal{P}(\mathbb{D})/l(0,c) \not\cong Q_{c'} \) for \( c' \neq c \). Thus, by Theorem 3.3, \( \mathcal{P}(\mathbb{D})/l(0,c) \cong Q_c \).
The easier parts of Theorem 3.4 can be obtained from this realization of the algebras $Q_c$ as $Q$-algebras. $\theta_c[f] = [f \circ U_c]$, where $U_c \in Aut(D)$ is characterized by $U_c(c) = 0$ and $U_c(0) = c$. The map $f \mapsto f \circ U_c$ is an automorphism of $P(D)$ and obviously completely isometric. By definition of the quotient norm, this passes down to $Q_c$. Similarly, the complete contractiveness of $m_1$ and $\iota_1$, for appropriate $\lambda, c, c'$, can be deduced by lifting these maps to endomorphisms $f \mapsto f \circ g$ of $P(D)$, where $g(z) = \lambda z$. Moreover, the fact that every contractive homomorphism $Q_c \rightarrow \square$ is completely contractive follows from the Szőkefalvi-Nagy dilation theorem [30].

**Corollary 3.6.** Let $A_1$ and $A_2$ be unital operator algebras and let $\rho: A_1 \rightarrow A_2$ be a unital homomorphism, which has rank at most 2 as a linear map (for example, $\dim A_1 = 2$ or $\dim A_2 = 2$). If $\rho$ is contractive, then it is completely contractive. More generally, $\|\rho\| = \|\rho|_{cb}$.

**Proof.** Replace $A_1$ by the unital operator algebra $A_1/\ker \rho$ and replace $A_2$ by the unital operator algebra $\text{Ran} \rho$. This does not change the norm or complete norm of $\rho$. Thus both $A_1$ and $A_2$ may be assumed to have dimension at most 2.

The one-dimensional case follows immediately from Lemma 2.1. The two-dimensional case follows from Theorem 3.3 and Theorem 3.4.

A main tool in Agler’s proof of Lempert’s theorem [2] is that for $\mathcal{M} \subset \mathbb{C}^n$, contractive representations of $H^\infty(\mathcal{M})$ by $2 \times 2$-matrices are always completely contractive. This result about representations by $2 \times 2$-matrices has since then been more and more generalized [28], [10], [19]. The following should be the most general form possible.

**Corollary 3.7.** Let $A$ be a commutative, unital operator algebra. Then every unital, contractive representation of $A$ by $2 \times 2$-matrices is completely contractive. More generally, if $\rho: A \rightarrow M_2$ is a unital homomorphism, then $\|\rho\|_{cb} = \|\rho\|$.

**Proof.** It is not difficult to see that every commutative subalgebra $A_2 \subset M_2$ has dimension at most 2: Otherwise $A_2 \cap A_2^*$ has codimension at most 2 and thus contains an element that is not a multiple of 1. Hence $A_2$ contains all diagonal matrices in a particular basis. But only diagonal matrices commute with all diagonal matrices in a particular basis, so that $\dim A_2 = 2$ contrary to assumption. Since every representation of a commutative operator algebra has commutative range, Corollary 3.6 can be applied.

4. The Quotient Distance and Metric for Unital Operator Algebras

**Example 4.1.** Let $\mathcal{M}$ be a complex manifold and either $m_1, m_2 \in \mathcal{M}$ or $(m; X) \in T\mathcal{M}$. Let $U = H^\infty(\mathcal{M})$ and either $l = l(m_1, m_2)$ or

$$l = l(m; X) = \{f \in U \mid f(m) = 0, Df(m; X) = 0\}.$$  

Assume that the codimension of $l$ is two.

In the first case, the quotient algebra $U/l$ is completely equivalent to $Q_c$ with $c = c_0^*(m_1, m_2) \neq 0$ the Carathéodory* pseudodistance of $m_1$ and $m_2$. In the second case, the quotient algebra $U/l$ is completely equivalent to $Q_0$.

**Proof.** $U/l$ must be completely isometrically isomorphic to some $Q_c$ with $c \in [0, 1]$. In the first case, there are two distinct characters on $U/l$ because $\dim U/l = 2$. This excludes the possibility $c = 0$. Moreover, comparing (10) with the definition of the Carathéodory* pseudodistance shows that $c = c_0^*(m_1, m_2)$.

In the second case, there is one character and a non-trivial derivation on $U/l$. Thus $U/l \not\cong Q_c$ for any $c \in (0, 1]$, forcing $U/l \cong Q_0$. ∎
For a unital operator algebra $A$, let $\text{Spec}(A)$ be the set of all nonzero, continuous homomorphisms $A \to \mathbb{C}$. Endowed with the weak topology from $\text{Spec}(A) \subseteq \text{Ball}(A')$, this becomes a compact Hausdorff space. If $f \in A$, $\omega \in \text{Spec}(A)$, write $f(\omega)$ for $\omega(f)$. If $\omega \in \text{Spec}(A)$, a linear functional $d: A \to \mathbb{C}$ with $d(fg) = f(\omega)d(g) + d(f)g(\omega)$ for all $f, g \in A$ is called a derivation at $\omega$. Write $T_\omega A$ for the set of all derivations at $\omega$ and $T A = \bigsqcup_{\omega \in \text{Spec}(A)} T_\omega A$. Obviously, $T_\omega A$ is a complex vector space, called the tangent space of $A$ at $\omega$. $T A$ is called the tangent space of $A$.

These definitions can be made for not necessarily commutative operator algebras. However, it is easy to check that $[f, g](\omega) = 0$ and $d[f, g] = 0$ for all $f, g \in A$, $\omega \in \text{Spec}(A)$, $d \in T_\omega A$, and $[f, g] = fg - gf$. Let $[A, A]$ be the ideal generated by all commutators. Then all elements of the spectrum and all derivations at some point of the spectrum annihilate $[A, A]$. Hence they factor through the commutative operator algebra $A/[A, A]$. All the constructions in this section will ignore any noncommutativity of $A$ in this way.

**Definition 4.1.** Let $A$ be a unital operator algebra and $\omega_1, \omega_2 \in \text{Spec}(A)$. Define $c_A^*(\omega_1, \omega_2) = \sup\{|f(\omega_2)| \mid f \in \text{Ball}(A) \text{ and } f(\omega_1) = 0\}$ and let $c_A = \text{arctanh} \circ c_A^*$ with the convention $\text{arctanh}(1) = \infty$. The functions $c_A^*$ and $c_A$ are called the quotient* distance and quotient distance respectively.

If $(\omega, d) \in TA$, let $\gamma_A(\omega, d) = \sup\{|d(f)| \mid f \in \text{Ball}(A) \text{ and } f(\omega) = 0\}$. $\gamma_A$ is called the quotient metric for $A$.

Clearly, $c_A^*(\omega_1, \omega_2) \leq 1$ for all $\omega_1, \omega_2 \in \text{Spec}(A)$, so that $c_A$ is a well-defined function from $\text{Spec}(A)^2$ to $[0, \infty]$. We write $c_A^{(x)}$ if an assertion holds both for $c_A$ and for $c_{A}^x$.

A distance on a set $X$ is a symmetric function $d: X \times X \to [0, \infty]$ satisfying the triangle inequality and $d(x, y) = 0$ iff $x = y$. Thus infinite distances are allowed. This is necessary because it can easily happen that $c_A^*(\omega_1, \omega_2) = 1$ and thus $c_A(\omega_1, \omega_2) = \infty$. However, it will be shown below that $c_A^*$ and $c_A$ are distances on $\text{Spec}(A)$ in the above sense.

A first justification for the names “quotient distance” and “quotient metric” is that they behave well with respect to taking quotients:

**Lemma 4.1.** Let $A$ be a unital operator algebra, $1 \subset A$ an ideal, and $\omega_1, \omega_2 \in \text{Spec}(A/I)$. View $\omega_1, \omega_2 \in \text{Spec}(A)$ by putting $f(\omega_j) = |f(\omega_j)|$. Then $c_A^{(I)}(\omega_1, \omega_2) = c_A^*(\omega_1, \omega_2)$.

Similarly, if $(\omega, d) \in T(\omega/A/I)$, then $\gamma_A^{(I)}(\omega, d) = \gamma_A(\omega, d)$.

**Proof.** Trivial.

**Theorem 4.2.** Let $A$ be a unital operator algebra, $\omega_1, \omega_2 \in \text{Spec}(A)$, $\omega_1 \neq \omega_2$. Define $A(\omega_1, \omega_2) = A/[\omega_1, \omega_2]$. This is again a unital operator algebra and completely equivalent to $Q_\check{c}$, for $c = c_A^*(\omega_1, \omega_2)$.

There exists $f \in \text{Ball}(A)$ with $f(\omega_j) = \lambda_j$, $j = 1, 2$, if and only if $\lambda_1, \lambda_2 \in \mathbb{D}$ and $m(\lambda_1, \lambda_2) < c_A^*(\omega_1, \omega_2)$, where $m$ denotes the Möbius distance. Thus $c_A^*(\omega_1, \omega_2) = \sup\{m(f(\omega_1), f(\omega_2)) \mid f \in \text{Ball}(A)\}$.

In particular, $c_A^*$ and $c_A$ are distances on $\text{Spec}(A)$.
Proof. By Theorem 3.3, $A(\omega_1, \omega_2) \cong Q_c$ for some $c \in [0, 1]$. Clearly, $c = 0$ is impossible and $c = c_A^*(\omega_1, \omega_2)$ follows immediately from (10). For the special case $A = P(\mathbb{D})$, the second assertion is just the classical Schwarz-Pick lemma, and it is well-known that $m = c^*_P$ and $p = c_P$ are distances on $\mathbb{D}$. Lemma 4.1 and the definition of the quotient norm yield the second assertion for the algebras $Q_c$, since they are quotients of $P(\mathbb{D})$. The general case follows from this in the same way. (It is not difficult to give a direct proof using von Neumann’s inequality, paralleling the argument in [20] for uniform algebras.)

If $f \in \text{Ball}(A)$ then $f(\omega) \in \mathbb{D}$ for each $\omega \in \text{Spec}(A)$. Hence
$$f^*m = m \circ (f, f) : \text{Spec}(A) \times \text{Spec}(A) \to \mathbb{R}_+$$

is a well-defined pseudodistance on $\text{Spec}(A)$ since $m$ is a distance on $\mathbb{D}$. Clearly, the supremum of a family of pseudodistances is again a pseudodistance. It is trivial that $c^*_A(\omega_1, \omega_2) = 0$ implies $\omega_1 = \omega_2$. Thus $c^*_A$ is a distance. Replacing $m$ by $p$, the same argument yields that $c_A$ is a distance.

If $(\omega, d) \in T_\omega A$ with $d \neq 0$, define $l(\omega, d) = \{f \in A \mid f(\omega) = d(f) = 0\}$ and $A(\omega, d) = A/l(\omega, d)$. Clearly, $A(\omega, d) = A(\omega, \lambda d)$ for all $\lambda \in \mathbb{C}^*$.

Theorem 4.3. Let $(\omega, d) \in T_\omega A$, $d \neq 0$. Then there is a complete equivalence $\phi : A(\omega, d) \to Q_0$. There exists $f \in \text{Ball}(A)$ with $f(\omega) = \lambda$, $d(f) = l$ iff $\lambda \in \mathbb{D}$, $l \in \mathbb{C}$, and $\gamma(\lambda, l) < \gamma_A(\omega, d)$. Thus $\gamma_A(\omega, d) = \|d\|$, and this is a norm on $T_\omega A$.

Proof. By Theorem 3.3, there exists a complete equivalence $\phi : A(\omega, d) \to Q_0$. Thus $d$ induces some derivation $d \circ \phi^{-1}$ on $Q_0$. By Lemma 4.1, it suffices to prove the remaining claims for the special case $A = Q_0$. This case can further be translated to $P(\mathbb{D})$, where everything follows from the Schwarz-Pick lemma and $\gamma(\lambda, l) \leq \gamma(0, l) = \|l\|$ for all $\lambda \in \mathbb{D}$, $l \in \mathbb{C}$.

Corollary 4.4. Let $f \in A$, $\omega \in \text{Spec}(A)$, with $f(\omega) = \|f\|$. Then $d(f) = 0$ for all $d \in T_\omega A$ and $c^*_A(\omega, \omega_2) = 1$ for all $\omega_2 \in \text{Spec}(A)$ with $f(\omega_2) \neq f(\omega)$.

Proof. Apply Theorem 4.2 and 4.3 to $(\|f\| + \epsilon)^{-1}f$ for $\epsilon > 0$.

Example 4.2. Let $A = C(\Omega)$, $\Omega$ a compact Hausdorff space. Then $\text{Spec}(A)$ with the weak topology is homeomorphic to $\Omega$. However, it is easy to see that $c^*_A(\omega_1, \omega_2) = 1$ for all $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$: there exists $f \in A$ with $\|f\| = 1 = f(\omega_1)$, $f(\omega_2) = 0$. Moreover, Corollary 4.1 implies that $T_\omega A = \{0\}$ for all $\omega \in \Omega$. Thus the topology on $\Omega$ defined by $c^*_A$ is always the discrete topology, which is different from the usual topology unless $\Omega$ is finite.

Thus the quotient distance and metric do not say anything interesting about commutative $C^*$-algebras. In a sense, they measure how much a commutative operator algebra deviates from being self-adjoint. For function algebras they measure whether there is a relation (in form of inequalities) between the function values at different points.

The following theorem is the analogue of the holomorphic contractiveness of the classical Carathéodory (*) pseudodistance and the Carathéodory-Reiffen pseudometric.

Theorem 4.5. Let $\rho : A_1 \to A_2$ be a contractive, unital homomorphism. Then $\rho$ induces natural maps $\rho^* : \text{Spec}(A_2) \to \text{Spec}(A_1)$ and $D\rho^*(\omega) : T_\omega A_2 \to T_{\rho^*(\omega)} A_1$ for all $\omega \in \text{Spec}(A_2)$. This yields a map $D\rho^* : TA_2 \to TA_1$ over $\rho^*$. These maps are contractions for the quotient(*) distance and the quotient metric, respectively, i.e.

$$c^*_A(\rho^*(\omega_1), \rho^*(\omega_2)) \leq c^*_A(\omega_1, \omega_2),$$

$$\gamma_A((\rho^* \omega; D\rho^*(\omega; d)) \leq \gamma_A(\omega; d).$$
Proof. Define \( \rho^*(\omega) = \omega \circ \rho \) and \( D\rho^*(\omega; d) = (\omega \circ \rho \circ d \circ \rho) \). All the algebraic properties are easy to check. To get the contractiveness of the maps, note that, for \( \omega \in \text{Spec}(A_2) \), \( \rho^{-1}(l(\omega)) = l(\rho^*(\omega)) \) and that the restriction \( \rho \colon l(\rho^*\omega) \to l(\omega) \) is still contractive. Hence, for any linear functional \( l \) on \( A_2 \),

\[
\sup\{l(\rho(f)) \mid f \in \text{Ball}(l(\rho^*\omega))\} \leq \sup\{l(f) \mid f \in \text{Ball}(l(\omega))\}.
\]

Now the result follows by specializing to derivations and the linear functionals \( f \mapsto f(\omega_2) \). \( \square \)

**Proposition 4.6.** The topology on \( \text{Spec}(A) \) generated by the distance \( c_A^*(\cdot) \) is finer than the weak topology.

**Proof.** Assume that the net \( (\omega_j) \) in \( \text{Spec}(A) \) converges towards some \( \omega_\infty \in \text{Spec}(A) \) in the distance topology. This implies \( m(f(\omega_j), f(\omega_\infty)) \to 0 \) for all \( f \in \text{Ball}(A) \).

Since \( m \) is a distance on \( \mathbb{D} \), the net \( (f(\omega_j)) \) converges to \( f(\omega_\infty) \), even for \( f \in A \) with \( \|f\| \) arbitrary. Hence \( \omega_j \to \omega_\infty \) in the weak topology. \( \square \)

**Example 4.3.** Let \( \mathcal{M} \) be a complex manifold, \( A = H^\infty(\mathcal{M}) \). There is a canonical map \( \mathcal{M} \to \text{Spec}(A) \). The quotient\(^{(*)}\) distance on \( \text{Spec}(A) \) pulls back to a pseudodistance on \( \mathcal{M} \) (which may fail to separate points if \( H^\infty(\mathcal{M}) \) does not separate the points of \( \mathcal{M} \)). This is nothing but the classical Carathéodory\(^{(*)}\) pseudodistance. Similarly, there is a canonical map \( \mathcal{T}\mathcal{M} \to \mathcal{T}A \) by restricting a derivation on \( C^\infty(\mathcal{M}) \) to \( H^\infty(\mathcal{M}) \). Under this map, the quotient metric on \( \text{Spec}(A) \) pulls back to the classical Carathéodory-Reiffen metric on \( \mathcal{M} \).

Complex analysts may be surprised by Proposition 4.6 because the \( c \)-topology, which is the topology on \( \mathcal{M} \) defined by the distance \( c^{(*)}_{\mathcal{M}} \) is always weaker than the usual topology on \( \mathcal{M} \) as a complex manifold, and may be strictly weaker (even if \( \mathcal{M} \to \text{Spec}(A) \) is injective) \((16)\).

On \( H^\infty(\mathcal{M}) \), there is also the topology of locally uniform convergence, which is weaker than the norm topology and has the nice property that the closed unit ball of \( H^\infty(\mathcal{M}) \) is compact in this topology. This implies that the function \( c_{\mathcal{M}}^* \) is continuous with respect to the manifold topology on \( \mathcal{M} \times \mathcal{M} \). Thus the \( c \)-topology is weaker than the manifold topology. However, the mapping \( \mathcal{M} \to \text{Spec}(A) \) may fail to be a homeomorphism onto its image. In fact, by Proposition 4.6 this must happen if the \( c \)-topology on \( \mathcal{M} \) is strictly weaker than the manifold topology.

The cases \( c_A(\omega_1, \omega_2) = \infty \) and \( c_A(\omega_1, \omega_2) < \infty \) are qualitatively different. Call \( \omega_1, \omega_2 \in \text{Spec}(A) \) \( A \)-related if \( c_A(\omega_1, \omega_2) < \infty \) and write \( \omega_1 \sim \omega_2 \). Since \( c_A \) satisfies the triangle inequality, this is an equivalence relation. Call \( \text{Spec}(A) \) \( A \)-connected if all elements of \( \text{Spec}(A) \) are \( A \)-related. Conversely, if \( \omega_1 \sim \omega_2 \) implies \( \omega_1 = \omega_2 \), then \( \text{Spec}(A) \) is called totally \( A \)-disconnected.

By definition, \( \text{Spec}(A) \) is \( A \)-connected if and only if there are no two-dimensional quotients that are equivalent to \( \mathbb{Q}_1 \). \( \mathbb{Q}_1 \) is the only two-dimensional unital operator algebra that can be written as an orthogonal direct sum of two one-dimensional operator algebras. Call an operator algebra decomposable if it is an orthogonal direct sum of two nonzero operator algebras, and indecomposable otherwise. Indecomposability is closely linked with \( A \)-connectedness of the spectrum.

Let \( A \) be a finite dimensional commutative operator algebra. Then \( \text{Spec}(A) \) is a finite set, and both the weak and the \( c \)-topology are discrete. First, recall some general algebraic facts about finite dimensional algebras.

Let \( \omega \in \text{Spec}(A) \), and let \( I = l(\omega) \subset A \) be the corresponding maximal ideal. The sequence of ideals \( I \supset I^2 \supset \cdots \) becomes constant, i.e. \( I^k = I^{k+1} \) for some \( k \in \mathbb{N} \). Write \( I^\infty = I^k \). These ideals are still coprime, i.e. \( l(\omega_1)^\infty + l(\omega_2)^\infty = A \)
for \( \omega_1, \omega_2 \in \text{Spec}(A) \), \( \omega_1 \neq \omega_2 \). Moreover, \( \bigcap_{\omega \in \text{Spec}(A)} I(\omega)^\infty = \{0\} \). The Chinese remainder theorem yields a direct sum decomposition

\[
A \cong \bigoplus_{\omega \in \text{Spec}(A)} A/I(\omega)^\infty \quad \text{(algebraically)}.
\]

Given a completely isometric representation of \( A \) on \( \mathcal{H} \), this corresponds to a decomposition of \( \mathcal{H} \) into generalized eigenspaces

\[
\mathcal{H}(\omega) = \{ \xi \in \mathcal{H} \mid (f - f(\omega))^k \xi = 0 \text{ for some } k \in \mathbb{N}, \text{ all } f \in A \}
\]

\[
= \bigcup_{k \in \mathbb{N}} \bigcap_{f \in A} \ker(f - f(\omega))^k = \bigcap_{k \in \mathbb{N}} \bigcup_{f \in A} \ker(f - f(\omega))^k.
\]

(11)

The subspaces \( \mathcal{H}(\omega) \) are \( A \)-invariant, and \( I(\omega)^\infty \) acts on \( \mathcal{H}(\omega) \) by zero. Moreover, in a suitable orthonormal basis, the action of all \( f \in A \) on \( \mathcal{H}(\omega) \) is (jointly) upper triangular with entries \( f(\omega) \) in the diagonal. In most cases, the spaces \( \mathcal{H}(\omega) \) will not be orthogonal. But always \( \sum \mathcal{H}(\omega) = \mathcal{H} \).

Now decompose \( \text{Spec}(A) \) into classes of related elements \( C_1, \ldots, C_m \), and let \( \bigcap_{\omega \in C_j} I(\omega)^\infty = \bigcup_{k \in \mathbb{N}} \bigcap_{f \in A} \ker(f - f(\omega))^k \). Since \( \bigcup C_k = \text{Spec}(A) \), the Chinese remainder theorem yields that, algebraically, \( A \cong \bigoplus_{k=1}^m A/I(C_k) \). However, in this case the decomposition is orthogonal:

**Theorem 4.7.** Let \( A \) be a finite dimensional, unital, commutative operator algebra and let \( C_1, \ldots, C_m \) be the \( \sim \)-equivalence classes of \( \text{Spec}(A) \). Then the canonical isomorphism \( \phi: A \to \bigoplus_{k=1}^m A/I(C_k) \) is completely isometric, where the right side has the orthogonal direct sum matrix normed structure. Furthermore, the operator algebras \( A/I(C_k) \) are indecomposable.

In fact, if the commutative operator algebra \( B \) has a \( B \)-connected spectrum, then \( B \) is not even isometrically isomorphic to a non-trivial direct sum, where non-trivial means that both summands are nonzero.

**Proof.** The quotient maps \( A \to A/I(C_k) \) for \( k = 1, \ldots, m \) induce a completely contractive map \( \phi: A \to \bigoplus_{k=1}^m A/I(C_k) \) by definition of the orthogonal direct sum of operator algebras. Moreover, \( \phi \) is an isomorphism by abstract algebra. The point is to show that \( \phi^{-1} \) is completely contractive. Choose a completely isometric representation \( \rho: A \to \mathcal{B}(\mathcal{H}) \) on some Hilbert space \( \mathcal{H} \). Let \( \omega \in \text{Spec}(A) \), and let \( \mathcal{H}(\omega) \) be as in (11). Choose an orthonormal basis of \( \mathcal{H}(\omega) \) making the \( A \)-action upper triangular.

Let \( \omega_j, j = 1, 2 \), lie in different \( \sim \)-equivalence classes. Since \( A \) is finite dimensional, its unit ball is compact. Thus by Theorem 4.2 there is \( f \in A \) with \( \|f\| \leq 1 \), \( f(\omega_1) = 1 \), and \( f(\omega_2) = -1 \). For \( j = 1, 2 \), the restriction of \( \rho(f) \) to \( \mathcal{H}(\omega_j) \) must be of the form

\[
\begin{pmatrix}
 f(\omega_j) & * & * & * & \cdots \\
 0 & f(\omega_j) & * & * & \cdots \\
 0 & 0 & f(\omega_j) & * & \cdots \\
 0 & 0 & 0 & f(\omega_j) & \cdots \\
 0 & 0 & 0 & 0 & f(\omega_j)
\end{pmatrix}.
\]

Since \( \|\rho(f)\| = \|f\| = 1 \) and \( |f(\omega_j)| = 1 \), this is only possible if \( \rho(f)|_{\mathcal{H}(\omega_j)} \) is diagonal for \( j = 1, 2 \). Pick unit vectors \( x_j \in \mathcal{H}(\omega_j) \) for \( j = 1, 2 \) and let \( \lambda = \langle x_1, x_2 \rangle \).

The goal is to show \( \lambda = 0 \). Since \( \rho(f)x_1 = x_1 \) and \( \rho(f)x_2 = -x_2 \),

\[
|a_1|^2 + |a_2|^2 - 2 \mathsf{Re}(a_1a_2\lambda) = \|a_1x_1 - a_2x_2\|^2 = \|\rho(f)(a_1x_1 + a_2x_2)\|^2 \leq \|f\|^2 \|a_1x_1 + a_2x_2\|^2 = |a_1|^2 + |a_2|^2 + 2 \mathsf{Re}(a_1a_2\lambda)
\]
for all $a_1, a_2 \in \mathbb{C}$. Thus $\text{Re}(a_1 a_2 \lambda) \geq 0$ for all $a_1, a_2 \in \mathbb{C}$. This is only possible if $\lambda = 0$, so that $\mathcal{H}(\omega_1) \perp \mathcal{H}(\omega_2)$. Thus the subspaces $\mathcal{H}_j = \sum_{\omega \in C_j} \mathcal{H}(\omega)$ are orthogonal to each other. Clearly, $\bigoplus \mathcal{H}_j = \mathcal{H}$.

Define $\rho_j : A \rightarrow \mathbb{B}(\mathcal{H}_j)$ by $\rho_j(f) = \hat{P}_j \rho(f)|_{\mathcal{H}_j}$. Then $\rho_j|_{\mathcal{H}_j} = 0$, so that $\rho_j$ induces a completely contractive representation $\hat{\rho}_j : A/\mathcal{H}_j \rightarrow \mathbb{B}(\mathcal{H}_j)$. Together, these mappings induce a completely contractive representation

$$\bigoplus \hat{\rho}_j : \bigoplus A/\mathcal{H}_j \rightarrow \mathbb{B} \left( \bigoplus \mathcal{H}_j \right) = \mathbb{B}(\mathcal{H}).$$

Clearly, $(\bigoplus \hat{\rho}_j) \circ \phi = \rho$. Since $\rho$ is completely isometric and both $\bigoplus \hat{\rho}_j$ and $\phi$ are completely contractive, $\phi$ must be completely isometric. This proves the first claim.

For the second claim, it suffices to show that if $\mathcal{B}$ is isometrically isomorphic to the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ for some nonzero operator algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, then $\text{Spec}(\mathcal{B})$ is not $\mathcal{B}$-connected. $\mathcal{B}_1$ and $\mathcal{B}_2$ must be ideals of $\mathcal{B}$ and $\mathcal{B}/\mathcal{B}_1 \cong \mathcal{B}_2$, $\mathcal{B}/\mathcal{B}_2 \cong \mathcal{B}_1$ isometrically. Choose $\omega_j \in \text{Spec}(\mathcal{B})$ such that $I(\omega_j) \supset \mathcal{B}_j$. Write $1 \in \mathcal{B}$ as a sum $e_1 + e_2$, $e_j \in \mathcal{B}_j$, $j = 1, 2$. Then

$$1 = \|1\| = \|e_1 + e_2\| = \max\{\|e_1\|, \|e_2\|\} = \|e_1 - e_2\|.$$

Moreover,

$$\omega_1(e_1 - e_2) = -\omega_1(1) + 2\omega_1(e_1) = -1,$$

$$\omega_2(e_1 - e_2) = \omega_2(1) - 2\omega_2(e_2) = 1.$$

Together with $\|e_1 - e_2\| = 1$ this implies $c_0^*(\omega_1, \omega_2) = 1$, so that $\text{Spec}(\mathcal{B})$ is not $\mathcal{B}$-connected. $\square$

It is easy to find examples of indecomposable operator algebras whose spectrum is not $A$-connected, for example $C(\Omega)$ for a connected space $\Omega$. However, this only occurs for infinite dimensional operator algebras, where there are also topological obstructions to decomposability.

If $\text{Spec}(A)$ is totally $A$-disconnected it cannot be concluded that $A \cong C(\text{Spec}(A))$ because of examples like $\mathbb{Q}_0$. Such operator algebras can be distinguished from the self-adjoint case by their tangent space.

Let $A$ be a unital operator algebra, $\omega \in \text{Spec}(A)$, and $d \in T_\omega A$. Then $d|_{I(\omega)^2} = 0$, so that $d|_{I(\omega)}$ determines a continuous linear functional $\delta$ on $I(\omega)/I(\omega)^2$. Moreover, $d$ is uniquely determined by $\delta$ because $d(1) = 0$. Conversely, if $\delta : I(\omega)/I(\omega)^2 \rightarrow \mathbb{C}$ is a linear functional, then $d(f) = \delta([f - f(\omega)])$ defines a derivation at $\omega$. Hence $T_\omega A$ is the dual space of $I(\omega)/I(\omega)^2$ (compare this with Exercise 2.12 of [29]).

**Definition 4.2.** Let $A$ be a unital operator algebra, $\omega \in \text{Spec}(A)$. The **cotangent space of $A$ at $\omega$** is the abstract operator algebra

$$T^*_\omega A = I(\omega)/I(\omega)^2.$$

$A(\omega) = A/I(\omega)^2$ is the trivial unitization of $T^*_\omega A$.

Let $\varsigma : T_\omega A \rightarrow (T^*_\omega A)'$ be the bijection constructed above. One of the consequences of Theorem 4.3 is that, for any $d \in T_\omega A$,

$$\|d\| = \|d|_{I(\omega)^2}\| = \gamma_A(\omega, d) = \|\varsigma(d)\|.$$

Thus $T_\omega A$ with the quotient metric is the normed dual of $T^*_\omega A$.

For the special case of the function algebras $\mathcal{R}(K)$, $K \subset \mathbb{C}$ compact, the tangent and cotangent spaces at points $k \in K$ were already introduced by Paulsen in [24]. He also endowed $T_\omega A$ with a $L^1$-matricially normed structure making it the standard operator space dual of $T^*_\omega \mathcal{R}(K)$.

A commutative, unital operator algebra $A$ is said to have **zero tangent space** if $T_\omega A = \{0\}$ for all $\omega \in A$. Equivalently, $T^*_\omega A = \{0\}$ for all $\omega \in \text{Spec} A$, i.e.
Theorem 4.10. Let $A$ be a unital, commutative operator algebra of dimension $n \in \mathbb{N}$. Then $A$ is completely equivalent to $C(\{1, \ldots, n\})$ if and only if $A$ has zero tangent space and Spec$(A)$ is totally $A$-disconnected.

Proof. Of course, $C(\{1, \ldots, n\})$ has zero tangent space and totally $A$-disconnected spectrum. Assume conversely that $A$ has zero tangent space and totally $A$-disconnected spectrum. By Theorem 4.7, $A \cong \bigoplus A/\|\omega\|^{\infty}$, where the sum runs over all $\omega \in$ Spec$(A)$. But $T_{\omega}A$ means $\|\omega\|^2 = \|\omega\|$ and thus $\|\omega\|^\infty = \|\omega\|$, so that $A/\|\omega\|^\infty \cong \mathbb{C}$. Thus $A$ is an orthogonal direct sum of several copies of $\mathbb{C}$, which is completely equivalent to $C(\{1, \ldots, n\})$. \hfill $\Box$

Remark 4.9. There are commutative, unital operator algebras $A_1$ and $A_2$ that have zero tangent space and isometric spectra (Spec$(A)$, $e_\lambda'$), but which are not isometric. Examples are appropriate quotients of $\mathcal{P}(\mathbb{D}_q)$ and Shift$_d$. Thus a lot of information is lost by looking only at the tangent space and the spectrum with the quotient distance.

4.1. The quotient distance and metric for tensor products. Let $A_1$ and $A_2$ be unital operator algebras. There is no unique way to turn their algebraic tensor product into an operator algebra. The most natural choices are the spatial and the maximal tensor product [26]. If $A_j \subset \mathcal{B}(\mathcal{H}_j)$, then the spatial tensor product structure comes from the natural representation $A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The maximal tensor product structure has the maximal matrix norms for which the embedding $A_1 \to A_1 \otimes A_2$ and $A_2 \to A_1 \otimes A_2$ are still completely contractive. Thus a representation of the maximal tensor product is completely contractive if its restrictions to $A_1 \otimes \{\id\}$ and $\{\id\} \otimes A_2$ are completely contractive. Their is no such criterion for a representation of the spatial tensor product to be completely contractive. However, the maximal tensor product, like most universal objects, does not come with an interesting completely isometric representation.

Both tensor product structures are functorial for completely contractive maps in the sense that if $\rho_j: A_j \to B_j$, $j = 1, 2$, are completely contractive maps, then $\rho_1 \otimes \rho_2$ is also completely contractive. This is trivial for the maximal tensor product, but not for the spatial tensor product (the corresponding statement for contractive representations is false because otherwise every contractive map would be completely contractive). See [23] for a proof.

If $\omega_j \in$ Spec$(A_j)$, $j = 1, 2$, then $(\omega_1, \omega_2): f \otimes g \mapsto f(\omega_1)g(\omega_2)$ defines a character of $A_1 \otimes A_2$, which is continuous both for the spatial and the maximal tensor product structure. Moreover, every character of $A_1 \otimes A_2$ is of this form. If $d_j \in T_{\omega_j}A_j$, $j = 1, 2$, then
\[
(d_1, d_2): f \otimes g \mapsto f(\omega_1)d_2(g) + d_1(f)g(\omega_2)
\]
is a continuous derivation at $(\omega_1, \omega_2)$. Moreover, every derivation at $(\omega_1, \omega_2)$ is of this form.

The spatial tensor product of uniform algebras $U_j \subset C(\Omega_j)$, $j = 1, 2$, is a unital function algebra on $\Omega_1 \times \Omega_2$. In this special case, there is the following formula for the quotient distance and metric of the spatial tensor product:

Theorem 4.10 ([20]). Let $U_1$ and $U_2$ be uniform algebras or, more generally, unital function algebras. If $\omega_{j,k} \in$ Spec$(U_k)$, $j, k \in \{1, 2\}$, then
\[
e(s)_{U_1 \otimes U_2}(\omega_{1,1}, \omega_{1,2}, (\omega_{2,1}, \omega_{2,2})) = \max\{e_{U_1}^{(s)}(\omega_{1,1}, \omega_{2,1}), e_{U_2}^{(s)}(\omega_{1,2}, \omega_{2,2})\}.
\]
If $d_j \in T_{\omega_j}U_j$, then
\[
\gamma_{U_1 \otimes U_2}(d_1, d_2) = \max\{\gamma_{U_1}(d_1), \gamma_{U_2}(d_2)\}.
\]
The inclusion $A_1 \to A_1 \otimes A_2$ is completely contractive for any reasonable tensor product structure. Hence Theorem 4.5 shows easily that the inequality “$\geq$” holds in (12) and (13) also for more general operator algebras and any reasonable tensor product structure. Moreover, the estimate “$\leq$” is trivial for the maximal tensor product structure. Hence the analogues of (12) and (13) hold for the maximal tensor product of unital operator algebras. However, this is not a generalization of Theorem 4.10 because the spatial tensor product of uniform algebras considered there is usually different from the maximal tensor product.

E.g., the maximal tensor product of $P(\mathbb{B}^2)$ and $P(\mathbb{B})$ is the universal operator algebra for three commuting contractions and thus different from $P(\mathbb{B}^3)$.

However, the situation is more complicated for the spatial tensor product.

**Theorem 4.11.** For $c, d \in (0, 1]$, let

$$\phi(c, d) = c_{Q_c \otimes Q_d}^2((0, 0), (c, d)).$$

If $A_1$ and $A_2$ are any unital operator algebras and $\omega_{j,k} \in \text{Spec}(A_k)$, then

$$\max\{c_{A_1}^2(\omega_{1,1}, \omega_{2,1}), c_{A_2}^2(\omega_{1,2}, \omega_{2,2})\}$$

(14)

$$\leq c_{A_1 \otimes A_2}^2((\omega_{1,1}, \omega_{1,2}), (\omega_{2,1}, \omega_{2,2}))$$

$$\leq \phi(c_{A_1}^2(\omega_{1,1}, \omega_{2,1}), c_{A_2}^2(\omega_{1,2}, \omega_{2,2})).$$

**Proof.** The lower bound in (14) follows from the above discussion. It remains to prove the upper bound. Let $c = c_{A_1}^2(\omega_{1,1}, \omega_{2,1})$ and $d = c_{A_2}^2(\omega_{1,2}, \omega_{2,2})$. There are completely isometric isomorphisms $A_1/I(\omega_{1,1}, \omega_{2,1}) \to Q_c$ and $A_2/I(\omega_{1,2}, \omega_{2,2}) \to Q_d$. The induced map $A_1 \otimes A_2 \to Q_c \otimes Q_d$ is completely contractive. Hence the induced map $\text{Spec}(Q_c \otimes Q_d) \to \text{Spec}(A_1 \otimes A_2)$ is a contraction for the quotient distance. This is the assertion.

Hence, with respect to estimating the quotient distance for spatial tensor products, the operator algebras $Q_c$ are the worst case. Unfortunately, the computation of $\phi(c, d)$ is more complicated than one might expect.

**Problem 4.1.** Compute the function $\phi$ of Theorem 4.11.

Compressing the standard representation of $Q_c \otimes Q_d \subset B(\mathbb{C}^4)$ to the subspace spanned by the joint eigenvectors corresponding to $(0, 0)$ and $(c, d)$ yields a completely contractive representation of $Q_c \otimes Q_d$ by $2 \times 2$-matrices. This yields an upper bound for $\phi(c, d)$, which can be computed from the angle between the two joint eigenvectors. The result is

$$\phi(c, d) \leq \sqrt{c^2 + d^2 - c^2 d^2}.$$ 

This is better than the estimate $\phi(c, d) \leq (c + d)/(1 - cd)$ which follows because the quotient distance is a distance and from the addition theorem for tanh.

The computations are much cleaner for the quotient metric:

**Theorem 4.12.** Let $A = Q_0 \otimes Q_0 \subset M_4$ be the spatial tensor product. Then $A$ has a unique 1-dimensional ideal $I$ spanned by $T_0 \otimes T_0$. The quotient $A/I$ is the trivial unitization of a two-dimensional operator space $B$ with the matrix normed structure

$$\|([T_0 \otimes 1] \otimes A + [1 \otimes T_0] \otimes B)\|_{(n)} = \max\{\|AA^* + BB^*\|^{1/2}, \|A^*A + B^*B\|^{1/2}\}$$

for $A, B \in M_n$.

**Proof.** The algebraic assertions are all easy. If $A, B, C \in M_n$, then

$$M = T_0 \otimes 1 \otimes A + 1 \otimes T_0 \otimes B + T_0 \otimes T_0 \otimes C \in Q_0 \otimes Q_0 \otimes M_n$$
is represented by the block matrix
\[
\begin{pmatrix}
0 & A & B & C \\
0 & 0 & 0 & B \\
0 & 0 & 0 & A
\end{pmatrix}.
\]

There exist unitary matrices \( U, V \in M_{2n} \) with
\[
U \begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} (A^*A + B^*B)^{1/2} \\ 0 \end{pmatrix}, \quad (A \cdot B)V = 0 \begin{pmatrix} (AA^* + BB^*)^{1/2} \end{pmatrix}.
\]

Hence the matrix \( M \) is unitarily equivalent to the matrix
\[
\begin{pmatrix}
0 & 0 & (AA^* + BB^*)^{1/2} & C \\
0 & 0 & 0 & (A^*A + B^*B)^{1/2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Clearly, the norm of this is minimal for \( C = 0 \) and has the value that occurs in (15).

**Corollary 4.13.** For \( j = 1, 2 \), let \( A_j \) be a unital operator algebra, \( \omega_j \in A_j \), and \( d_j \in T_{\omega_j} A_j \). Then, if \( A_1 \otimes A_2 \) is the spatial tensor product,
\[
\max \{ \gamma_{A_1}(d_1), \gamma_{A_2}(d_2) \} \leq \gamma_{A_1 \otimes A_2}(d_1, d_2) \leq (\gamma_{A_1}(d_1)^2 + \gamma_{A_2}(d_2)^2)^{1/2}.
\]

**Proof.** If \( A_1 = A_2 = Q_0 \), this follows at once from Theorem 4.12 because the norm on \( Q_0 \otimes Q_0 / (T_0 \otimes T_0) \) is of the form
\[
\| \lambda \cdot T_0 \otimes 1 + \mu \cdot 1 \otimes T_0 \| = (|\lambda|^2 + |\mu|^2)^{1/2}.
\]

For general \( A_1, A_2 \), the estimate follows as in the proof of Theorem 4.11.

5. **Transposition**

Let \( \mathcal{H} \) be a Hilbert space and let \( U : \mathcal{H} \rightarrow \mathcal{H} \) be an anti-unitary operator. Then \( T \mapsto UT^*U^{-1} \) defines a linear isometry \( U : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) called a transposition. Let \( V \subset \mathcal{B}(\mathcal{H}) \) be an operator space. Then \( V^t = U(V) \) is another operator space and \( \omega^t : V \rightarrow V^t \) is an isometric representation of \( V \), which usually fails to be completely contractive.

View \( V^t \) as an abstract operator algebra. A priori, \( V^t \) depends on the choice of a completely isometric representation of \( V \) and on the anti-unitary \( U \). But the matrix normed structure of \( V^t \) turns out to be independent of these choices and can be defined intrinsically as follows:

**Definition 5.1.** Let \( A \) be an abstract operator algebra. Define the transposed algebra \( A^t \) as follows: Algebraically, \( A^t \) is the opposite algebra, i.e. has the same vector space structure, but the order of multiplication is reversed to \( \hat{x} \circ \hat{y} = \hat{y} \hat{x} \). Here \( \hat{\cdot} \) is used to signify that \( x, y \) are viewed as elements of \( A^t \). However, the \( M_n \)-bimodule structure of \( A_{(n)} \) is as usual: \( S \circ \hat{x} \cdot T = \hat{S} \hat{x} T \) for \( x \in A_{(n)}, S, T \in M_n \).

If \( x = \sum_{j=1}^N a_j \otimes T_j, a_j \in A, T_j \in M_n \), define \( x^t = \sum_{j=1}^N a_j \otimes T_j^t \), where \( T_j^t \) is the transpose of \( T_j \). Writing \( \omega^t : M_n \rightarrow M_n \) for the transpose operation, this is just \( id_A \otimes \omega^t : A_{(n)} \rightarrow A_{(n)} \) and thus well-defined. The norm \( \|x^t\|_{(n)} \) on \( A_{(n)}^t \) is now defined by \( \|x^t\|_{(n)} = \|x^t\|_{(n)} \).

It is clear how to define \( V^t \) for an abstract operator space.
Theorem 5.1. Let $V$ be an abstract operator space. Choose a completely isometric representation $\rho: V \to B(\mathcal{H})$ and an anti-unitary $U \in B(\mathcal{H})$. Then $T \mapsto U\rho(T)^*U^{-1}$, viewed as a representation of $V^t$, is a completely isometric linear representation. Moreover, $(V^t)^t$ is completely equivalent to $V$. The “identity map” $\iota: V \ni v \mapsto \hat{v} \in V^t$ is isometric. If it is $n$-contractive, it is necessarily  $n$-isometric.

If $V$ is a (unital) operator algebra to start with, then this representation is also multiplicative (and unital). Moreover, $(V^t)^t$ is completely equivalent to $V$, and $\iota$ is an anti-isomorphism.

Proof. Let $v_j \in V$, $T_j \in M_n$ for $j = 1, \ldots, N$. Let $V: \mathbb{C}^n \to \mathbb{C}^n$ be the standard anti-unitary operator given by $(x_1, \ldots, x_n) \mapsto (\overline{x_1}, \ldots, \overline{x_n})$. Then the usual transposition operation for matrices is given by $T^t = V T^* V^{-1}$. Moreover, $U \otimes V^{-1}$ is an anti-unitary operator on $\mathcal{H} \otimes \mathbb{C}^n$.

$$\left\| \sum_{j=1}^N U \rho(v_j)^* U^{-1} \otimes T_j \right\| = \left\| (U \otimes V^{-1}) \sum_{j=1}^N \rho(v_j)^* \otimes (T_j^*)^t (U \otimes V^{-1})^{-1} \right\|$$

$$= \left\| \sum_{j=1}^N \rho(v_j) \otimes T_j^t \right\| = \left\| \sum_{j=1}^N v_j \otimes T_j^t \right\|_{(n)}.$$

Thus the representation is completely isometric.

The algebraic assertions are all trivial, it only remains to show that if $\iota$ is $n$-contractive, it is also $n$-isometric. If this were false there would exist $X \in V_{(n)}$ with $\|X\|_{(n)} > \|\hat{X}\|_{(n)}$ and $\iota_{(n)}$ would be contractive. But then

$$\|X^t\|_{(n)} = \|\hat{X}\|_{(n)} < \|X\|_{(n)} = \|\hat{X}^t\|_{(n)},$$

counter to the assumption that $\iota_{(n)}$ is contractive.

Nevertheless, transposition is functorial for completely contractive maps as follows:

Lemma 5.2. Let $\rho: V_1 \to V_2$ be a linear map between abstract operator spaces. Then $\rho$ gives rise to a transposed linear map $\rho^t: V_1^t \to V_2^t$ mapping $\hat{v}$ to $\rho(\hat{v})$. This map satisfies $\|\rho\|_{(n)} = \|\rho^t\|_{(n)}$ for all $n \in \mathbb{N}$ and hence $\|\rho\|_{cb} = \|\rho^t\|_{cb}$. Moreover, if $\rho_{(n)}$ is a quotient map, then so is $\rho^t_{(n)}$, and if $\rho_{(n)}$ is isometric, then so is $\rho^t_{(n)}$. The same holds for (unital) homomorphisms between (unital) abstract operator algebras.

Proof. This is immediate from the abstract definition of the transposed operator algebra.

Corollary 5.3. Let $Q$ be a $Q$-algebra. Then $Q \to Q^t$ is completely isometric.

Proof. It is easy to prove this directly, but here we give an abstract nonsense proof using Lemma 5.2. Let $F \subset C(\Omega)$ be a function algebra and $I \subset F$ a closed ideal such that $Q = F/I$. Then $Q^t \cong F^t/I^t$ naturally. The transposition map $Q \to Q^t$ lifts to the transposition map $F \to F^t$. By definition of the quotient structure, it suffices to show that this lifted map is completely isometric. By Theorem 5.1, this map is isometric and it suffices to show that its inverse $F^t \to F \subset C(\Omega)$ is completely contractive. This is clear because any contractive map into $C(\Omega)$ is completely contractive.
6. Cotangent spaces and counterexamples

It is easy to classify those unital operator algebras that have a completely isometric representation by $3 \times 3$-matrices and are of the form $V^*$ for some two-dimensional operator space $V$.

Fix a basis $X,Y$ for $V$ and let $N = \text{Ran} \ X + \text{Ran} \ Y$. Then $X$ and $Y$ must vanish on $N$ because the multiplication on $V$ is trivial. $X,Y \neq 0$, implies $N \neq 0, N \neq \mathbb{C}^3$. Choose an orthonormal basis of $N$ and extend it to an orthonormal basis of $\mathbb{C}^3$.

First consider the case $\dim N = 2$. Then all elements of $V$ must be of the form

\[
\begin{pmatrix}
0 & 0 & * \\
0 & 0 & * \\
0 & 0 & 0
\end{pmatrix}.
\]

Since $\dim V = 2$, all such matrices occur, yielding the first example

\[
\text{Shift}_2(0) = \left\{ \begin{pmatrix} c & 0 & a \\ 0 & c & b \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.
\]

Shift$_2$ is defined in Section 8. It will turn out that the operator algebra above is indeed completely equivalent to $\text{Shift}_2(1)(0)^2$. For the time being, this name should be viewed just as a symbol.

If $\dim N = 1$, all elements of $V$ must be of the form

\[
\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Again, this determines $V$. After permuting the basis of $\mathbb{C}^3$, this is just the operator algebra $\text{Shift}_2(0)^t$ of all transposes of elements of $\text{Shift}_2(0)$.

We have just proved the following theorem, which will be essential in Section 10.

**Theorem 6.1.** Let $A \subset M_3$ be a trivial unitization of some operator space $V \subset A$. Then $A$ is unitarily equivalent to $\text{Shift}_2(0)$ or $\text{Shift}_2(0)^t$.

**Theorem 6.2.** The maximal ideals of both $\text{Shift}_2(0)$ and $\text{Shift}_2(0)^t$ are isometric to $\ell_2^2$. The transposition map $\iota : \text{Shift}_2(0) \rightarrow \text{Shift}_2(0)^t$ is an isometric isomorphism that is not 2-contractive, and whose inverse $\iota^{-1} : \text{Shift}_2(0)^t \rightarrow \text{Shift}_2(0)$ is not 2-contractive either.

**Proof.** Transposition always is an isometric anti-homomorphism by Theorem 5.1. Thus for commutative operator algebras it is a homomorphism. The map $\iota_{(n)}$ is given by

\[
\iota_{(n)} : \begin{pmatrix}
0 & 0 & A \\
0 & 0 & B \\
0 & 0 & 0
\end{pmatrix} \mapsto \begin{pmatrix}
0 & A & B \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The norm of the matrix on the left is $\|A^*A + B^*B\|$, whereas the matrix on the right has norm $\|AA^* + BB^*\|$. Especially, if $A,B \in M_1 \cong \mathbb{C}$, then both norms are $|A|^2 + |B|^2$, so that the maximal ideal of $\text{Shift}_2(0)$ is isometric to $\ell_2^2$. For $A = e_{11}, B = e_{12}$, the left and right side are 1 and 2, respectively, so that $\iota_{(2)}$ is not contractive. If $\iota_{(2)}^{-1}$ were contractive then so would be $\iota_{(2)}$ by Theorem 5.1. 

Thus for three-dimensional operator algebras, it is no longer true that every contractive (or even isometric) homomorphism is completely contractive. The proof shows that this also fails for two-dimensional operator algebras without unit.

**Corollary 6.3.** Neither $\text{Shift}_2(0)$ nor $\text{Shift}_2(0)^t$ is completely equivalent to a $Q$-algebra.
Proof. The transpose map \( \text{Shift}_2(0) \to \text{Shift}_2(0) \) is not completely isometric. But it is completely isometric for \( \mathcal{Q} \)-algebras by Corollary 5.3.

Thus no three-dimensional \( \mathcal{Q} \)-algebra with a unique maximal ideal can be represented completely isometrically by \( 3 \times 3 \)-matrices. The three-dimensional \( \mathcal{Q} \)-algebra in Example 6.3 has no finite dimensional isometric representation at all.

For any normed algebra \( \mathcal{A} \), define the corresponding maximal operator algebra structure \( \text{MAX}(\mathcal{A}) \) as follows: Let \( (\rho_j) \) be the class of all contractive representations of \( \mathcal{A} \) on Hilbert spaces. These induce representations \( (\rho_j)_{(n)} \) of \( \mathcal{A}(n) \). For \( X \in \mathcal{A}(n) \), let \( \|X\|_{(n)} = \sup \|\rho_j(X)\| \). It is easy to check that this yields an abstract operator algebra structure on \( \mathcal{A} \). The norm \( \|\cdot\|_{(1)} \) coincides with the given norm on \( \mathcal{A} \) if and only if \( \mathcal{A} \) is isometric to an operator algebra. Evidently, every contractive representation of \( \mathcal{A} \) with this maximal operator algebra structure is completely contractive.

The above construction works equally well for normed vector spaces. For these there also is a minimal operator space structure \( \text{MIN}(\mathcal{V}) \) given by the embedding \( \mathcal{V} \to C(\text{Ball}(\mathcal{V}')) \), where \( \mathcal{V}' \) is the dual space. Since every contractive map into \( C(\Omega) \) is completely contractive [23], every contractive map into \( \text{MIN}(\mathcal{V}) \) is completely contractive.

For any normed space \( \mathcal{V} \), define the map \( \iota = \text{id}_\mathcal{V}: \text{MIN}(\mathcal{V}) \to \text{MAX}(\mathcal{V}) \) and let \( \alpha(\mathcal{V}) = \|\iota\|_{\text{cb}} \). One of the main results of [24] is that
\[
\sqrt{\dim \mathcal{V}}/2 \leq \alpha(\mathcal{V}) \leq \dim \mathcal{V}.
\]
Especially, \( \alpha(\mathcal{V}) = \infty \) for \( \dim \mathcal{V} = \infty \), and \( \alpha(\mathcal{V}) > 1 \) for \( \dim \mathcal{V} \geq 5 \). Of course, \( \alpha(\mathcal{V}) = 1 \) if and only if \( \text{MIN}(\mathcal{V}) \) is completely equivalent to \( \text{MAX}(\mathcal{V}) \). The only known cases with \( \alpha(\mathcal{V}) = 1 \) are \( \ell^\infty_2 \cong \mathbb{C} \), \( \ell^\infty_2 \), and \( \ell^2_2 \), i.e. this is a very rare behavior.

Example 6.1. Paulsen shows in [24] that \( \alpha(\ell^2_2) \leq n/\sqrt{2} \). Especially, \( \alpha(\ell^2_2) \leq \sqrt{2} \). This bound is realized by the algebras \( \text{Shift}_2(0) \) and \( \text{Shift}_2(0)^t \). Both are isometric to \( \ell^2_2 \) and thus yield isometric representations \( \rho: \text{MIN}(\ell^2_2) \to \mathcal{M}_3 \). An easy calculation shows that \( \|\rho\|_{(2)} \geq \sqrt{2} \). Hence
\[
\|\rho(\ell^2_2)\| = \|\rho\|_{\text{cb}} = \alpha(\ell^2_2) = \sqrt{2}.
\]

Let \( \mathcal{M} \) be a complex manifold and \( m \in \mathcal{M} \). Let
\[
I_1 = \{ f \in H^\infty(\mathcal{M}) \mid f(m) = 0 \}, \quad I_2 = \{ f \in H^\infty(\mathcal{M}) \mid f(m) = 0 \text{ and } Df(m) = 0 \},
\]
and let \( \mathcal{M}(m) = H^\infty(\mathcal{M})/I_2, \mathcal{T}_m^* \mathcal{M} = I_1/I_2 \). Taking \( I_2 \) instead of \( I(m) = I_1/I_1 \) avoids the problem whether there exist derivations on \( H^\infty(\mathcal{M}) \) at \( m \) that do not come from a geometric tangent vector of \( \mathcal{M} \) at \( m \). If \( \mathcal{M} \) is sufficiently nice ("\( \gamma \)-hyperbolic at \( m \)) then \( \mathcal{T}_m^* \mathcal{M} \) as a vector space is the usual cotangent space. If \( \mathcal{M} \subset \mathbb{C}^n \) is a bounded domain, there are also the Q-algebras \( \mathcal{O}(\overline{\mathcal{M}})/\mathcal{I}(m)^2, \mathcal{R}(\overline{\mathcal{M}})/\mathcal{I}(m)^2, \text{ and } \mathcal{P}(\overline{\mathcal{M}})/\mathcal{I}(m)^2 \). They are always algebraically isomorphic to \( \mathcal{M}(m) = (\mathcal{T}_m^* \mathcal{M})^+ \) but they may have different matrix normed structures.

It is shown in [24] that cotangent spaces of absolutely convex domains\(^3\) at the origin are completely equivalent to a certain \( \text{MIN}(\mathcal{V}) \). For a proof, it is convenient to use the following formula for the Carathéodory-Reiffen metric on balanced domains\(^4\).

Lemma 6.4 ([16]). Let \( \mathcal{B} \subset \mathbb{C}^n \) be a balanced domain. Under the usual identification \( \mathcal{T}_0^* \mathcal{B} \cong \mathbb{C}^n \), the open unit ball for the Carathéodory-Reiffen metric on \( \mathcal{T}_0^* \mathcal{B} \) corresponds to the convex hull \( \text{co}(\mathcal{B}) \) of \( \mathcal{B} \).

---

\(^3\)A domain is absolutely convex if it is the unit ball of a normed space.

\(^4\)A domain \( \mathcal{B} \subset \mathbb{C}^n \) is balanced iff \( \lambda x \in \mathcal{B} \) whenever \( \lambda \in \mathbb{C} \) and \( x \in \mathcal{B} \).
Theorem 6.5. Let $\mathcal{B} \subset \mathbb{C}^n$ be a bounded, balanced domain, let $V$ be the normed space with $\text{co}(\mathcal{B})$ as its open unit ball, and let $V'$ be the corresponding dual space. Then

$$T'_0\mathcal{B} \cong T'_0\mathcal{O}(\mathcal{B}) \cong T'_0\mathcal{R}(\mathcal{B}) \cong T'_0\mathcal{P}(\mathcal{B}) \cong \text{MIN}(V').$$

Proof. The claim is that the unit ball of $T'_0\mathcal{O}(\mathcal{B})$ coincides with the unit ball of $\text{MIN}(V')/(\mathfrak{m})$ for $\mathfrak{m} \in H^\infty$. Let $f \in \text{Ball}(\text{MIN}(V'))$. According to the definition, the components of $f$ are to be viewed as functions on $\text{Ball}(V') \cong \text{Ball}(V)$. Thus $\|f\| < 1$ means that $f$ maps $\text{co}(\mathcal{B})$ to $\text{Ball}(\mathcal{M}_n)$. Especially, $f(\mathcal{B}) \subset \text{Ball}(\mathcal{M}_n)$ and of course $f(0) = 0$. Thus $f$, viewed as a (polynomial, rational, or holomorphic) map from $\mathcal{B}$ to $\text{Ball}(\mathcal{M}_n)$ defines an element of $T'_0\mathcal{B}(\mathfrak{m})$, etc., of norm at most 1. This yields a canonical completely contractive map from $\text{MIN}(V')$ to $T'_0 \mathcal{B}$.

Conversely, if $f \in \mathcal{O}(\mathcal{B}, \text{Ball}(\mathcal{M}_n))$, $f(0) = 0$, then $Df|_0$ is a linear map with the same derivative and function value at the origin. Since $Df|_0$ is a contraction for the respective Carathéodory-Reiffen metrics, Lemma 6.4 shows that $Df|_0$ maps $\text{co}(\mathcal{B})$ into $\text{Ball}(\mathcal{M}_n)$. Hence the image of the closed unit ball of $\text{MIN}(V')/(\mathfrak{m})$ contains the open unit ball of $T'_0\mathcal{B}$. This is true a fortiori for $T'_0\mathcal{O}(\mathcal{B})$, $T'_0\mathcal{R}(\mathcal{B})$, and $T'_0\mathcal{P}(\mathcal{B})$. $\square$

It is important that there is no version of Theorem 6.5 for arbitrary points of a domain. The Carathéodory-Reiffen metric usually can only be computed in highly symmetric situations as above, so that there is no general method to compute the normed structure of $T_m\mathfrak{M}$. Furthermore, $T_m\mathfrak{M}$ can fail to be of the form $\text{MIN}(V)$.

Corollary 6.6. Let $\mathcal{B} \subset \mathbb{C}^n$ be a bounded, balanced domain, $A$ any unital operator algebra, and $Q = \mathcal{B}(0)$. Then every unital, contractive homomorphism $A \to Q$ is completely contractive.

Proof. By Theorem 2.5, it suffices to check that the restriction of a unital homomorphism to the preimage of $T'_0\mathcal{B}$ is completely contractive. This follows from $T'_0\mathcal{B} \cong \text{MIN}(V')$. $\square$

Theorem 6.7 (Paulsen [24]). Let $\mathcal{B}$ be a bounded domain and let $V$ be the normed space with unit ball $\text{co}(\mathcal{B})$. If all contractive representations of $\mathcal{R}(\mathcal{B})$ are completely contractive, then $\alpha(V) = 1$.

Proof. The homomorphism $\iota : \text{MIN}(V') \to \text{MAX}(V')$ extends to a contractive homomorphism $\iota^+ : \text{MIN}(V')^+ \to \text{MAX}(V')^+$. Since $T'_0\mathcal{B}$ is completely isometric to $\text{MIN}(V')$ by Theorem 6.5, $\text{MIN}(V')^+$ is completely equivalent to $\mathcal{B}(0)$. If every contractive representation of $\mathcal{R}(\mathcal{B})$ is completely contractive, then every contractive representation of $\mathcal{B}(0)$ is completely contractive because $\mathcal{B}(0)$ is a quotient of $\mathcal{R}(\mathcal{B})$. Especially, $\iota^+$ is completely contractive, forcing $\iota$ to be completely contractive, i.e. $\alpha(V') = 1$. It is shown in [24] that $\alpha(V) = \alpha(V')$.

Example 6.2. There exists a strongly convex, bounded domain $\mathfrak{M} \subset \mathbb{C}^2$ with smooth boundary and $m \in \mathfrak{M}$ with $T_m\mathfrak{M}$ isometric but not 2-isometric to $T_0\mathbb{D}_2$. This yields an isometric, completely contractive isomorphism $\rho : \mathfrak{M}(m) \to \mathbb{D}_2(0)$ between $Q$-algebras that is not 2-isometric.

Proof. In [17], Lempert proves the existence of a strongly convex, bounded domain $\mathfrak{M} \subset \mathbb{C}^2$ with smooth boundary that is not biholomorphic to $\mathbb{D}_2$, but such that for some $m \in \mathfrak{M}$ the unit ball for the Carathéodory-Reiffen metric on $T_m\mathfrak{M}$ is $\mathbb{D}_2$. Dualizing an isometry $T' : \mathbb{D}_2 \to T_m\mathfrak{M}$ yields an isometry $T : T_m\mathfrak{M} \to \mathbb{D}_2$. $\square$
Let $\rho = T^+$. In order to prove the various properties of $\rho$, it suffices to check them for $T$ and to invoke Theorem 2.5.

By construction, $T$ is isometric. Since $T^*_0\mathbb{D}_2 \cong \text{MIN}(\ell_2^2)$ by Theorem 6.5, $T$ is completely contractive. There exist contractive linear maps $\iota: \ell_2^2 \to M_2$ and $\pi: M_2 \to \ell_2^2$ with $\pi \circ \iota = \text{id}_{\ell_2^2}$, i.e. $\ell_2^2$ is a linear retract of $M_2$. For example, put

$$\iota(x, y) = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}, \quad \pi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = (x, y).$$

Consider $[i] \in (T^*_0\mathbb{D}_2)(2) \cong \text{MIN}(\ell_2^2)(2)$, then $\|[i]\|_{(2)} \leq 1$. View $\pi$ as a holomorphic map $\text{Ball}(M_2) \to \mathbb{D}_2$.

If $T^{-1}$ were 2-contractive, then $\|[T^{-1}][i]\|_{(2)} \leq 1$, i.e. there would be $i \in H^\infty(\mathcal{M})(2)$ with $T(2)i = [i]$ and $\|[i]\|_{(2)} \leq 1$ (using that the unit ball of $H^\infty(\mathcal{M})$ is compact in the topology of locally uniform convergence). Thus $i(\mathcal{M}) \subset \text{Ball}(M_2)$. Since $i(0) = 0$ and $\text{Ball}(M_2)$ is convex, even $i(\mathcal{M}) \subset \text{Ball}(M_2)$ (use Proposition 1.6 of [20]). Put $\phi = \pi \circ i$.

The condition $T(2)i = \iota$ determines $i$ and hence $\phi$ up to first order at $m$. It is easy to check that $\phi(m) = 0$ and $D\phi(m) = (T')^{-1}$. This is an isometry for the Carathéodory-Reiffen metric at $m$, so that $\phi$ is biholomorphic by a theorem of Vigué (Proposition 8.7.2 of [16]). Thus $\mathcal{M}$ is biholomorphic to $\mathbb{D}_2$, contrary to assumption. Hence $T$ is not 2-contractive.

Examples of isometric but not completely isometric homomorphisms between Q-algebras have also been obtained by Paulsen in [25]. Thus, already for Q-algebras, the matrix normed structure on the cotangent space contains additional information besides the quotient metric. Notice that the unital Q-algebras involved are 3-dimensional and that they are realized as quotients of $H^\infty(\mathcal{M})$ for what are supposed to be very well-behaved domains in $\mathbb{C}^2$.

The next goal is to show that every strongly pseudoconvex domain $\mathcal{M} \subset \mathbb{C}^d$ has a contractive representation by $3 \times 3$-matrices that is not 2-contractive. The idea of the proof is that near a point in the boundary of $\mathcal{M}$, the cotangent spaces $T^*_m\mathcal{M}$ look more and more like $T^*_0\mathbb{D}_d$. This reduces the assertion to Example 6.1.

Let $\mathcal{M} \subset \mathbb{C}^d$ be a bounded, strongly pseudoconvex domain (with $C^2$-boundary). Then the Carathéodory-Reiffen metric can be computed approximately near the boundary of $\mathcal{M}$ (see [13], [18], or [16]). A consequence of this is that the unit ball of $T^*_m\mathcal{M}$ looks more and more like $\mathbb{D}_d$ for $m \to \partial\mathcal{M}$. To make this precise, define the distance of two (bicontinuous) Banach spaces $V_1$, $V_2$ to be

$$\text{dist}(V_1, V_2) = \log\left(\inf\{\|\rho\| \cdot \|\rho^{-1}\| \mid \rho: V_1 \to V_2 \text{ invertible}\}\right).$$

There is an obvious version of this distance for matrix normed spaces:

$$\text{dist}_{\infty}(V_1, V_2) = \log\left(\inf\{\|\rho\|_{cb} \cdot \|\rho^{-1}\|_{cb} \mid \rho: V_1 \to V_2 \text{ invertible}\}\right).$$

**Theorem 6.8.** Let $\mathcal{M} \subset \mathbb{C}^d$ be a bounded, strongly pseudoconvex domain. Then $\text{dist}_{\infty}(T^*_m\mathcal{M}, T^*_0\mathbb{D}_d) \to 0$ for $m \to \partial\mathcal{M}$.

**Proof.** The estimates given in [16] imply immediately that $\text{dist}(T^*_m\mathcal{M}, \ell_2^2) \to 0$ for $m \to \partial\mathcal{M}$. Clearly, $\text{dist}(V_1, V_2) = \text{dist}(V'_1, V'_2)$, so that $T^*_m\mathcal{M}$ can be replaced by $T^*_m\mathcal{M}$. To get the assertion for $\text{dist}_{\infty}$, it is convenient to use some intermediate results of Ma’s proof in [18].

**Theorem 6.9.** Let $\mathcal{M} \subset \mathbb{C}^d$ be a bounded, strongly pseudoconvex domain, choose $\epsilon > 0$. Then for all $m \in \mathcal{M}$ sufficiently near to $\partial\mathcal{M}$, there exist $\phi \in \mathcal{O}(\mathcal{M}, \mathbb{D}_d)$ and $\psi \in \mathcal{O}(\mathbb{D}_d, \mathcal{M})$ with $\phi(m) = 0$, $\psi(0) = m$, and $D(\phi \circ \psi)|_0 = (1 - \epsilon) \cdot \text{id}_{T^*_0\mathbb{D}_d}$. 


This follows from Theorem 3.3 and Lemma 3.4 of [18]. Ma’s Theorem 3.3 gives maps \( \phi, \psi \) as above but with \( |\det D(\phi \circ \psi)|_0 \geq 1 - \epsilon \) instead of \( D(\phi \circ \psi)|_0 = 1 - \epsilon \). By Lemma 3.4 of [18], this implies \( \|D(\phi \circ \psi)|_0 X\|_2 \geq (1 - \epsilon)n\|X\|_2 \), where \( \|\cdot\|_2 \) denotes the L2-norm. Hence the linear mapping that sends \( D(\phi \circ \psi)|_0 \) to \( (1 - \epsilon)X \) is contractive with respect to the L2-norms and thus maps \( D \) into \( Dd \). Composing \( \phi \) with this endomorphism of \( Dd \) yields Theorem 6.9.

The induced maps \( \phi^*: T^*_1 \rightarrow T^*_1 M \) and \( \psi^*: T^*_m M \rightarrow T^*_0 M \) are obviously completely contractive. Moreover, the inverse of \( \phi^* \) is \( \psi^* \) up to a factor of \( 1 - \epsilon \). Thus

\[
\exp(\text{dist}_\infty(T^*_0 M, T^*_m M)) \leq \|\phi^*\|_{cb}(1 - \epsilon)^{-1}\psi^* \leq (1 - \epsilon)^{-1}.
\]

Theorem 6.8 follows because this holds for all \( \epsilon > 0 \).

**Corollary 6.10.** Let \( d > 1 \) and let \( M \subset \mathbb{C}^d \) be a bounded, strongly pseudoconvex domain. Then \( H^\infty(M) \) has a contractive representation by \( 3 \times 3 \)-matrices that is not 2-contractive.

**Proof.** \( \text{MIN}(\ell^2_2) \) has a contractive linear representation \( \rho: \ell^2_2 \rightarrow M_3 \), where \( \ell^2_2 \) is some orthogonal projection and \( \ell^2_2 \rightarrow \text{Shift}_2(0) \) is the map of Example 6.1. Clearly, \( \|\rho\|_{(2)} = \sqrt{2} \). Choose \( \epsilon > 0 \) such that \( (1 - \epsilon)\sqrt{2} > 1 \).

According to Theorem 6.8, for \( m \) sufficiently near to some boundary point of \( M \), there exist completely contractive maps \( \phi^*: T^*_0 M \rightarrow T^*_m M, \psi^*: T^*_m M \rightarrow T^*_0 M \) satisfying \( \psi^* \circ \phi^* = (1 - \epsilon)\text{id} \). Since \( T^*_0 M \cong \text{MIN}(\ell^2_2) \), \( \rho \circ \psi^* \) is a contractive representation of \( T^*_m M \). It is not 2-contractive because \( \rho \circ (\psi^* \circ \phi^*) \) is not 2-contractive.

**Example 6.3.** \( D^2(0) \) has no isometric representation on a finite dimensional Hilbert space.

**Proof.** It suffices to show this for \( T^*_n D^2 \), which is isometric to the dual \( \ell^2_2 \) of \( \ell^2_\infty \) by Theorem 6.5. Assume that \( \phi: \ell^2_2 \rightarrow M_n \) is a finite dimensional isometric representation. Let \( e_1, e_2 \) be the standard basis of \( \ell^2_2 \), so that \( \|a_1e_1 + a_2e_2\| = |a_1| + |a_2| \). Let \( T_1 = \phi(e_1), T_2 = \phi(e_2) \). Then \( \|T_1\| = \|T_2\| = 1 \). Moreover, \( \|T_1 + \lambda T_2\| = 2 \) for all \( \lambda \in \partial D \), \( x \in M_n \), with \( \|x\|_1 = 1 \) and \( \|T_1 + \lambda T_2\|_x \lambda = 2 \), because the unit ball of \( M_n \) is compact.

Since the \( T_j \) are contractions, \( \|T_1 x\| = 1 \) and \( \|T_2 x\| = 1 \). Moreover, \( \lambda T_2 x = T_1 x \) because otherwise \( \|T_1 + \lambda T_2\|_x \lambda < 2 \). Let \( K \subset C^n \) be the 1-eigenspace of \( T_1^* T_1 \). All the vectors \( x_\lambda \) lie in \( K \) because \( \|T_1 x\| = 1 \) and \( \|T_1\| = 1 \). Furthermore, \( x_\lambda \in K \) and \( \lambda T_2 x_\lambda = T_1 x_\lambda \) imply \( T_1^* T_2 x_\lambda = T_1^* T_1 x_\lambda = x_\lambda \). Thus \( x_\lambda \) is an eigenvalue of the operator \( T_1^* T_2 \) with eigenvalue \( \lambda \). However, \( T_1^* T_2 \in M_n \) has only finitely many eigenvalues. Thus \( x_\lambda \) cannot exist for all \( \lambda \in \partial D \), contradiction.

**7. Q-algebras in complex analysis**

Let \( F \subset C(\Omega) \) be a function algebra. If \( V \) is any normed vector space, then there is a natural norm on \( F \otimes V \subset C(\Omega) \otimes V \): View elements as functions from \( \Omega \) to \( V \) and take the supremum norm. This well-known tensor product norm can actually be defined for tensor products of arbitrary normed spaces, and it has the property that \( \|L_1 \otimes L_2\| \leq \|L_1\| \cdot \|L_2\| \) for any linear maps \( L_1: F_1 \rightarrow F_2, L_2: V_1 \rightarrow V_2 \). In particular, any contractive linear map \( L: F_1 \rightarrow F_2 \) induces contractions \( L_V = L \otimes \text{id}_V \).

If \( Q = F/1 \) for some closed ideal \( 1 \), then \( Q \otimes V \) carries a natural quotient norm from the identification \( (F/1) \otimes V \equiv (F \otimes V)/(1 \otimes V) \). The collection of these norms for all finite dimensional normed spaces is called the *ac-normed structure* of \( Q \) in [21]. It can be defined more generally for a quotient of a normed space by a closed ideal. Obviously, if a linear map \( F_2 \rightarrow F/1 \) can be lifted to a contractive linear map...
$F_2 \to F$, then it must be \textit{ac-contractive}, i.e. its tensor product with \textit{id}_V must be contractive for all normed spaces $V$. Conversely, such a contractive linear lifting exists for ac-contractive maps if the range is a dual Banach space and $I$ is weakly closed \cite{21}.

Although it looks formally quite similar to the matrix normed structure for operator algebras, this ac-normed structure is considerably finer. This can be seen most easily from the following example.

Let $\mathcal{B} \subset \mathbb{C}^d, \ d > 1$, be an absolutely convex domain. Define $U(\mathcal{B})$ to consist of all bounded, not necessarily continuous functions $f: \mathcal{B} \to \mathbb{C}$ such that $f|_{P \cap \mathcal{B}}$ is holomorphic for all 1-dimensional planes $P$ through the origin and $f(z) = a + l(z) + O(z^2)$ for $z \to 0$, with some $C$-linear functional $l$ and $a \in \mathbb{C}$. This is a function algebra on some compact Hausdorff space, e.g. the Stone-\v{C}ech compactification of $(\mathcal{B}, \text{discrete topology})$.

Clearly, $H^\infty(\mathcal{B}) \subset U(\mathcal{B})$. This induces a homomorphism $\theta: \mathcal{B}(0) \to U(0)$. Indeed, it is easy to check that this is an isomorphism, the inverse being

$$U(0) \ni [a + l(z) + O(z^2)] \mapsto [a + l(z)] \in \mathcal{B}(0).$$

**Proposition 7.1.** $\theta: \mathcal{B}(0) \to U(0)$ is a complete equivalence.

**Proof.** $\theta$ is obviously completely contractive because it can be lifted to the inclusion map $H^\infty(\mathcal{B}) \to U(\mathcal{B})$. To show that $\theta^{-1}$ is completely contractive, it suffices to check this for the restriction to $T_0U(\mathcal{B})$ by Theorem 2.5.

Therefore, take $f \in \text{Ball}(T_0U(\mathcal{B}))(n))$, i.e., $f: \mathcal{B} \to \text{Ball}(\mathcal{M}_n)$ and $f(0) = 0$. Write $f(z) = l(z) + O(z^2)$ for some linear functional $l$. An application of the Schwarz lemma to the restriction of $f$ to each 1-dimensional plane through the origin yields that $|l(z)| \leq 1$ for all $z \in \mathcal{B}$. Thus $l \in O(\mathcal{B}, \text{Ball}(\mathcal{M}_n))$ is a representative for $\theta^{-1}(f)$ in $T_0\mathcal{B}$, so that $\theta^{-1}$ is completely contractive.

Although the algebra $U(\mathcal{B})$ is quite pathological from the point of view of complex analysis, the Q-algebras $U(0)$ and $\mathcal{B}(0)$ cannot be distinguished by dilation theory. However, they can be distinguished by their ac-normed structure:

**Proposition 7.2.** $\theta$ is not ac-isometric for $\mathcal{B} = \mathbb{D}_2$.

**Proof.** Let $(\mathcal{M}, m), \mathcal{M} \subset \mathbb{C}^2$, be a pointed convex domain as in Example 6.2 and let $L: T_0\mathbb{D}_2 \to T_m\mathcal{M}$ be an isometry for the Carathéodory-Reifen metric. For each plane $P$ through 0 in $T_m\mathcal{M}$, choose an element $X \in P$ with $\gamma_{\mathcal{M}}(m; X) = 1$, and choose a complex geodesic $\phi_X \in O(\mathcal{D}, \mathcal{M})$ for $(m; X)$ according to Lempert’s Theorem \cite{16}. Via $L$, the plane $P$ corresponds to a plane in $T_0\mathbb{D}_2$, and $\mathbb{D}_2 \cap L^{-1}(P) = \mathbb{D}$. View the complex geodesic $\phi_X$ as a holomorphic map $L^{-1}(P) \cap \mathbb{D}_2 \to \mathcal{M}$ with derivative $L|_{L^{-1}(P)}$ at 0. These functions can be pieced together to a map $\phi: \mathbb{D}_2 \to \mathcal{M}$ whose components lie in $U(\mathbb{D}_2)$.

If $\theta$ were ac-isometric, there would be a contractive linear map $\hat{\theta}^{-1}: U(\mathbb{D}_2) \to H^\infty(\mathbb{D}_2)$ that lifts $\theta^{-1}$ by \cite{21}. Then $\hat{\phi} = \hat{\theta}^{-1}(\phi)$ is a function in $O(\mathbb{D}_2, \mathcal{M})$ because $\mathcal{M}$ is bounded and convex \cite{21}. Moreover, $\hat{\phi}(0) = \phi(0) = m$ and $D\hat{\phi}(0) = D\phi(0) = L$. Hence $\hat{\phi}$ is biholomorphic by Proposition 8.7.2 of \cite{16}, contradicting the choice of $\mathcal{M}$.

The proof Proposition 7.2 goes through for an absolutely convex domain $\mathcal{B}$ whenever there is a pointed, bounded, convex domain $(\mathcal{M}, m)$ not biholomorphic to $\mathcal{B}$ whose unit ball with respect to the quotient metric is $\mathcal{B}$. By \cite{17}, this is the case for all two-dimensional absolutely convex domains with smooth boundary.

The reason why the ac-normed structure contains more information than the matrix normed structure is Theorem 2.5: The vector spaces $T_0U(\mathcal{B})$ and $T_0\mathcal{B}$ are ac-isometric by an obvious generalization of the proof of Proposition 7.1. But this
does not imply that the unitizations are ac-isometric. From the point of view of complex analysis, this corresponds to the fact that the unit balls of $M_n$ have a transitive automorphism group, but general absolutely convex domains do not. Thus interpolation with values in arbitrary absolutely convex domains is more general than interpolation with values in the domains $\text{Ball}(M_n)$.

From the point of view of complex analysis, restricting attention to interpolation with values in the domains $\text{Ball}(M_n)$ does not seem very fruitful. On the one hand, this special case is too symmetric to be “generic”. It does not contain enough information to tackle more general interpolation problems whose range is not a symmetric domain. On the other hand, this special case is still too complicated for an interesting theory. This is exemplified by the counterexamples in Section 6.

8. Arveson’s model theory for $d$-contractions

In [7], Arveson develops a model theory for $d$-contractions, which are commuting $d$-tuples of operators $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{H}$ satisfying (1). He defines a particular $d$-contraction, the $d$-shift $S = (S_1, \ldots, S_d)$ acting on a Hilbert space $H_d^2$ which can be viewed as a variant of Bosonic Fock space, or as the closure of $\mathcal{P}(\overline{\mathbb{D}_d})$ under a somewhat strange norm. All elements of $H_d^2$ can be viewed as continuous functions on the closed Euclidean ball $\overline{\mathbb{D}_d}$ holomorphic on the interior, but not all such functions arise. Let $u_x(z) = (1 - \langle z, x \rangle)^{-1}$, where $\langle z, x \rangle = z_1 \overline{x_1} + \cdots + z_d \overline{x_d}$ for $z, x \in \mathbb{D}_d$. Then $u_x \in H_d^2$ for all $x \in \mathbb{D}_d$, and

$$\langle f, u_x \rangle = f(x)$$

for all $f \in H_d^2$. Especially,

$$\langle u_x, u_y \rangle = (1 - \langle y, x \rangle)^{-1}.$$  

Moreover, the vectors $\{u_x\}$ span a dense subset of $H_d^2$. As remarked by Arveson, the inner product on $H_d^2$ does not come from any measure on $\mathbb{C}^d$ and the $d$-shift is not subnormal.

The $d$-Toeplitz algebra $\text{Toeplitz}_d$ is the $C^*$-algebra generated by the $d$-shift. The closed subalgebra of $\text{Toeplitz}_d$ generated by $S$ is called $\text{Shift}_d$. Elements of $\text{Shift}_d$ are often viewed as functions $f$ on $\overline{\mathbb{D}_d}$. Then $M_f$ denotes the corresponding operator on $H_d^2$. It is shown in [7] that the $d$-shift has joint spectrum $\overline{\mathbb{D}_d}$. Hence the functional calculus yields $\mathcal{O}(\overline{\mathbb{D}_d}) \subset \text{Shift}_d$. Indeed, the following inclusions hold:

$$\mathcal{O}(\overline{\mathbb{D}_d}) \subset \text{Shift}_d \subset H_d^2 \subset \mathcal{O}(\mathbb{D}_d).$$

The $d$-shift is important because it is the “universal” $d$-contraction in the following sense:

**Theorem 8.1** [Arveson [7]]. Let $T = (T_1, \ldots, T_d)$ be a $d$-contraction on $\mathcal{H}$ and let $S$ be the $d$-shift. Then $S_j \mapsto T_j$ defines a unital, completely contractive representation of $\text{Shift}_d$ on $\mathcal{H}$. Conversely, if $\rho: \text{Shift}_d \to \mathcal{B}(\mathcal{H})$ is a unital, completely contractive representation, then $\rho(S) = (\rho(S_1), \ldots, \rho(S_d))$ is a $d$-contraction.

**Theorem 8.2** [Arveson [7]]. Let $d = 1, 2, \ldots$, let $T = (T_1, \ldots, T_d)$ be a $d$-contraction acting on a separable Hilbert space and let $S = (S_1, \ldots, S_d)$ be the $d$-shift. Then there is a triple $(n, Z, \mathcal{K})$ consisting of an integer $n = 0, 1, \ldots, \infty$, a spherical operator $Z$ and a full co-invariant subspace $\mathcal{K}$ for the operator $n \cdot S \oplus Z$ such that $T$ is unitarily equivalent to the compression of $n \cdot S \oplus Z$ to $\mathcal{K}$.

Here a spherical operator is a $d$-tuple $Z = (Z_1, \ldots, Z_d)$ of commuting normal operators acting on a common Hilbert space with joint spectrum $\partial \mathbb{D}_d$, i.e. $Z_1 Z_2^* + \cdots + Z_d Z_d^* = 1$. $n \cdot S$ denotes the direct sum of $n$ copies of $S$ acting on $(H_d^2)^n$. A subspace is called co-invariant if its orthogonal complement is invariant and full (for
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a collection of operators) if it generates the whole Hilbert space under the action of the C*-algebra generated by these operators. Another result of [7] is that this dilation is essentially unique. Of course, uniqueness can only hold if one restricts attention to dilations which involve compression to a full, co-invariant subspace.

It is elementary that every spherical operator is a d-contraction. Especially, the “coordinate functions” \( z_j \in C(\partial \mathbb{D}_d) \) form a spherical operator, and it is easy to check that \( S_j \rightarrow z_j \) defines a unital \(*\)-homomorphism \( \pi \) from Toeplitz\(_d\) onto \( C(\partial \mathbb{D}_d) \). Its kernel consists of the compact operators on \( H_d^2 \), so that Toeplitz\(_d\) as a C*-algebra is not different from more classical higher-dimensional Toeplitz algebras.

For \( d = 1 \), the 1-shift is just the unilateral shift and \( \text{Shift}_1 \cong \overline{O(\mathbb{D})} \) is contained in the well-known 1-Toeplitz algebra as usual.

The transposed algebra \( \text{Shift}_d^t = \text{Shift}_d \) is also of interest. Let \( S^t = (S^t_1, \ldots, S^t_d) \) be the multioperator \( (U S^t_1 U^{-1}, \ldots, U S^t_d U^{-1}) \) for any anti-unitary operator \( U : H_d^2 \rightarrow H_d^2 \). Then \( \text{Shift}_d^t \) is the algebra generated by \( S^t \). Call a multi-operator \( T = (T_1, \ldots, T_d) \) a transposed \( d \)-contraction if the matrix

\[
\begin{pmatrix}
T_1 & 0 & \cdots \\
T_2 & 0 & \cdots \\
\vdots & \vdots & \ddots \\
T_d & 0 & \cdots
\end{pmatrix}
\]

is a contraction, i.e. \( T_1^* T_1 + \cdots + T_d^* T_d \leq 1 \). Theorem 5.1 implies that \( T \) is a transposed \( d \)-contraction iff \( T^t \) is a \( d \)-contraction. The dilation theory of transposed \( d \)-contractions is quite similar to that of ordinary \( d \)-contractions:

**Theorem 8.3.** Let \( T = (T_1, \ldots, T_d) \) be a transposed \( d \)-contraction on \( \mathcal{H} \). Then \( S^t_j \rightarrow T_j \) defines a unital, completely contractive representation of \( \text{Shift}_d^t \) on \( \mathcal{H} \). Conversely, if \( \rho : \text{Shift}_d^t \rightarrow \mathcal{B}(\mathcal{H}) \) is a unital, completely contractive representation, then \( \rho(S^t_j) = (\rho(S^t_1), \ldots, \rho(S^t_d)) \) is a transposed \( d \)-contraction.

**Theorem 8.4.** Let \( d = 1, 2, \ldots \) and let \( T = (T_1, \ldots, T_d) \) be a transposed \( d \)-contraction acting on a separable Hilbert space. Then there is a triple \( (n, \mathbb{Z}, \mathcal{K}) \) consisting of an integer \( n = 0, 1, \ldots, \infty \), a spherical operator \( \mathbb{Z} \) and a full invariant subspace \( \mathcal{K} \) for the operator \( n \cdot S^t \oplus \mathbb{Z} \) such that \( T \) is unitarily equivalent to the compression of \( n \cdot S^t \oplus \mathbb{Z} \) to \( \mathcal{K} \).

**Proof of Theorem 8.3 and 8.4.** If \( T \) is a transposed \( d \)-contraction, then \( S^t_j \rightarrow T_j^t \) defines a completely contractive representation of \( \text{Shift}_d \) by Theorem 8.1. The transpose of this representation is given by \( S^t_j \mapsto T_j \). It is completely contractive by Lemma 5.2. The other half of Theorem 8.3 follows similarly.

Theorem 8.4 follows from Theorem 8.2 because the transposes of unitarily equivalent operators are again unitarily equivalent and because the transpose of a spherical operator is again a spherical operator. Notice that a co-invariant subspace for \( S \) is invariant for \( S^* \) and hence corresponds under the anti-unitary \( U \) to an invariant subspace for \( S^t \).

An important question that is not addressed in [7] is whether the symmetry group of the ball gives rise to symmetries of \( H_d^2 \) and thus of \( \text{Shift}_d \). Recall that \( \mathbb{D}_d \) is a homogeneous domain. Its symmetry group at the origin is just the unitary group \( U(d) \). It is elementary to check that the unitaries give rise to unitary operators in \( H_d^2 \). In fact, Arveson shows in [7] that every contraction in \( M_d \) gives rise to a contraction in \( H_d^2 \) and thus a completely contractive endomorphism of \( \text{Shift}_d \).

In order to see that automorphisms not fixing the origin also give rise to unitary operators in \( H_d^2 \), another characterization of \( H_d^2 \) as a “twisted Bergman space” is necessary.
8.1. $H^2_d$ as a twisted Bergman space. The Bergman kernel of the domain $\mathbb{D}_d$ is [16]

$$K_{\mathbb{D}_d}(z, w) = \frac{d^d}{\pi^d}(1 - \langle z, w \rangle)^{-(d+1)}.$$  

Hence, up to a constant factor, $u_z(w) = K_{\mathbb{D}_d}(w, z)^{1/(d+1)}$. Equation (16) means that $u_z(w)$ is a reproducing kernel for the Hilbert space $H^2_d$. A twisted Bergman space is a Hilbert space with reproducing kernel $K_{\mathbb{D}_d}$ for some $\lambda \in \mathbb{R}$. Such Hilbert spaces have been studied (also for other symmetric domains) by harmonic analysts, mainly for the reason that they carry a natural projective representation of the semi-simple Lie group $\text{Aut}(\mathbb{D}_d) = \text{PSU}(d, 1)$:

**Theorem 8.5.** Let $h \in \text{Aut}(\mathbb{D}_d)$ be an automorphism of $\mathbb{D}_d$. Let

$$\delta(z) = \left(\det Dh(z)\right)^{1/(d+1)},$$

where any holomorphic branch of the root is chosen and let $(Tf)(z) = \delta(z)f(h(z))$ for $f \in H^2_d$, $z \in \mathbb{D}_d$. Then $T$ defines a unitary operator $H^2_d \to H^2_d$. This gives rise to a projective representation of $\text{Aut}(\mathbb{D}_d)$ on $H^2_d$.

Moreover, $M_{\text{shift}} \circ T = T \circ M_T$ for all $f \in \text{Shift}_d$, so that $f \mapsto f \circ h$ is a completely isometric automorphism of $\text{Shift}_d$.

The proof is based on the behavior of the Bergman kernel under biholomorphic mappings ([16], Proposition 6.1.7), which implies

$$(\det Dh(z))^\lambda (\det Dh(w))^\lambda K_{\mathbb{D}_d}(h(z), h(w))^\lambda = K_{\mathbb{D}_d}(z, w)^\lambda$$

for all $\lambda \in \mathbb{R}$, $z, w \in \mathbb{D}_d$, and $h \in \text{Aut}(\mathbb{D}_d)$. See [4].

However, more general holomorphic mappings $f : \mathbb{D}_d \to \mathbb{D}_d$ do not give rise to completely contractive endomorphisms of $\text{Shift}_d$, although this is the case for linear contractions. The reason is that any $f \in \mathcal{O}(\mathbb{D}_d, \mathbb{D})$ can be viewed as an endomorphism of $\mathbb{D}_d$ mapping $z$ to $(f(z), 0, \ldots, 0)$. But if $f \notin \text{Shift}_d$ or $\|f\|_{\text{Shift}_d} > 1$, then this cannot produce a completely contractive endomorphism of $\text{Shift}_d$.

In the article [8] by Bagchi and Misra, some interesting results are proved for the analogue of the $d$-shift on twisted Bergman spaces over the symmetric domains $\text{Ball}(\mathbb{M}_{n,m})$. They determine when the generalized shift operator is bounded and they show that its joint spectrum is $\text{Ball}(\mathbb{M}_{n,m})$ whenever it is bounded. Their criterion implies that $S$ is bounded and that $\text{Spec \, Shift}_d = \mathbb{D}_d$. Moreover, they find necessary and sufficient criteria for the generalized shift operators to be subnormal. Their criterion implies that $S$ is not jointly subnormal and that the inner product on $H^2_d$ does not come from a measure on $\mathbb{C}^d$. However, they fail to notice the special role of the $d$-shift on $H^2_d$. The most important contribution of the twisted Bergman space picture of $H^2_d$ is the projective representation of $\text{Aut}(\mathbb{D}_d)$.

8.2. $H^2_d$ and the Fantappiè transform. The inner product on $H^2_d$ is also related to the Fantappiè transform, which is discussed here very briefly, following Hörmander [15]. Let $\mathfrak{M} \subset \mathbb{C}^d$ be a domain. Endow $\mathbb{C}^d$ with the usual bilinear form $(x, y) = x_1y_1 + \cdots + x_dy_d$ and define $\mathfrak{M}^1 = \{x \in \mathbb{C}^d \mid (x, y) \neq 1 \forall y \in \mathfrak{M}\}$. Notice that this is a compact set for open $\mathfrak{M}$ with $0 \in \mathfrak{M}$.

If $l : \mathcal{O}(\mathfrak{M}) \to \mathbb{C}$ is a linear functional continuous with respect to the topology of locally uniform convergence on $\mathfrak{M}$, its Fantappiè transform $\mathcal{F}l \in \mathcal{O}(\mathfrak{M}^1)$ is defined by $\mathcal{F}l(x) = l(\hat{u}_x)$, where $\hat{u}_x(y) = (1 - \langle y, x \rangle)^{-1}$ for $y \in \mathfrak{M}$. Actually, $\mathcal{F}l(x)$ is holomorphic on a neighborhood of $\mathfrak{M}^1$ because the support of $l$ is a compact subset of $\mathfrak{M}$ by continuity. The main theorem about the Fantappiè transform is that, if $\mathfrak{M}$ is $\mathbb{C}$-convex (especially if $\mathfrak{M}$ is convex), then $\mathcal{F}$ is a bijection between the

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\[\text{Not all } \lambda \text{ are admissible, i.e. give rise to a positive definite kernel.}\]
space of continuous linear functionals on $\mathcal{O}(\mathfrak{M})$ and the space $\mathcal{O}(\mathfrak{M}^1)$ of functions holomorphic in a neighborhood of $\overline{\mathfrak{M}}^1$. Since $\mathbb{D}^1_d = \mathbb{D}_d$ and $\mathcal{O}(\mathbb{D}_d) \subset \mathcal{O}(\mathbb{D}_d)$, this yields a bilinear form on $\mathcal{O}(\mathbb{D}_d) \supset \mathcal{R}(\mathbb{D}_d) \supset \mathcal{P}(\mathbb{D}_d)$ in the special case $\mathfrak{M} = \mathbb{D}_d$.

However, a sesquilinear form is necessary in order to get a Hilbert space. Therefore, replace the bilinear form $(\cdot, \cdot)$ by a sesquilinear form $(\cdot, \cdot)$ and apply the same reasoning. This yields a “conjugate Fantappiè” transform $\overline{\mathcal{T}}: I \mapsto I(u_x)$ where $u_x$ is defined as above, which maps the dual of $\mathcal{O}(\mathfrak{M})$ to the space of conjugate-holomorphic functions on a neighborhood of the conjugate $(\mathfrak{M}^1)^* = \{ \tau \mid z \in \mathfrak{M}^1 \}$ of $\mathfrak{M}^1$. For the ball, this yields a sesquilinear form $B(\cdot, \cdot): \mathcal{O}(\mathbb{D}_d) \times \mathcal{O}(\mathbb{D}_d) \to \mathbb{C}$ given by $B(f, g) = \overline{\mathcal{T}}^{-1}(\overline{\tau})(f)$.

For the simplest linear functionals, the point masses $\delta_x$ for $x \in \mathbb{D}_d$, we have $\overline{\mathcal{T}}(\delta_x)(z) = u_x(z) = u_{\tau_x}(z)$. Hence $B(f, u_x) = \delta_x(f) = f(x)$ as for the inner product on $H^2_d$. Thus $B$ coincides with the inner product on $H^2_d$.

9. Models for $d$-contractions with prescribed spectrum

In this section, completely isometric representations of quotients of $\text{Shift}_d$ are determined. For $d = 1$, this was already done by Arveson in [5].

**Theorem 9.1.** Let $I \subset \text{Shift}_d$ be a closed ideal. Let $H^2_d(I) = H^2_d \ominus I \cdot H^2_d \subset H^2_d$ and let $S(I) = (S_1(I), \ldots, S_d(I))$ be the compression of the $d$-shift $S$ to $H^2_d(I)$. Moreover, let $sa(I)$ (the self-adjoint part of $I$) be the quotient of $C(\partial \mathbb{D}_d)$ by the closed ideal generated by $\pi(I) \subset C(\partial \mathbb{D}_d)$, where $\pi$: Toeplitz$_d \to C(\partial \mathbb{D}_d)$.

Then $sa(I)$ is completely isometric to a commutative $C^*$-algebra. A completely isometric representation of $\text{Shift}_d/I$ can be obtained as a direct sum of a faithful $*$-representation of $sa(I)$ and of the representation of $\text{Shift}_d/I$ on $H^2_d$ generated by $\text{Shift}_d/I$.

**Proof.** Since any closed ideal in $C(\partial \mathbb{D}_d)$ is necessarily self-adjoint, $sa(I)$ is a commutative $C^*$-algebra.

Let $\rho: \text{Shift}_d/I \to \mathbb{B}(\mathcal{K})$ be any completely isometric representation. Thus $\rho|S = (\rho|S_1, \ldots, \rho|S_d)$ is a $d$-contraction by Theorem 8.1, so that Theorem 8.2 yields certain $(n, Z, \mathcal{K})$. Call the Hilbert space on which $Z$ acts $\mathcal{K}(Z, Z)$, and then $n \cdot S \otimes Z$ acts on $\mathcal{H} = (H^2_d)^n \otimes \mathcal{K}Z$. Let $\hat{\rho}: \text{Toeplitz}_d \to \mathbb{B}(\mathcal{H})$ be the corresponding $*$-representation given by $\hat{\rho}(S_i) = n \cdot S_i \otimes Z$. Since $\mathcal{K}$ is co-invariant for $n \cdot S \otimes Z$, its orthogonal complement $\mathcal{K}^\perp$ is $\hat{\rho}(\text{Shift}_d)$-invariant.

Let $f \in I$ and $\xi \in \mathcal{K}$. Write $\xi = \xi_0 + \xi^\perp$ with $\xi_0 \in \mathcal{K}$ and $\xi^\perp \in \mathcal{K}^\perp$. Then $\hat{\rho}(f)\xi^\perp \in \mathcal{K}^\perp$ because $\mathcal{K}^\perp$ is $\hat{\rho}(\text{Shift}_d)$-invariant. Moreover, the compression of $\hat{\rho}(f)$ to $\mathcal{K}$ is $\hat{\rho}(f) = \rho(0) = 0$, so that $\hat{\rho}(f)\xi_0 \in \mathcal{K}^\perp$ as well. Thus $\hat{\rho}(f)\xi \perp \mathcal{K}$.

Let $\mathcal{K}_2 = (\mathcal{K})^\perp = \{ \hat{\rho}(f)\xi \mid f \in I, \xi \in \mathcal{K} \}^\perp$, then $\mathcal{K} \subset \mathcal{K}_2$.

The representation $\rho$ is obtained from $\hat{\rho}$ by compressing to $\mathcal{K}$. Since $\mathcal{K} \subset \mathcal{K}_2$, first compressing to $\mathcal{K}_2$ and then to $\mathcal{K}$ does not change the result. If $f \in I$, then the compression $\tilde{\sigma}(f) = P_{\mathcal{K}_2} \hat{\rho}(f) P_{\mathcal{K}_2}$ is zero by construction. Hence $\tilde{\sigma}$ induces a completely contractive representation $\sigma: \text{Shift}_d/I \to \mathbb{B}(\mathcal{K}_2)$. Since the completely isometric representation $\rho$ is a compression of $\sigma$ to a subspace, $\sigma$ must be completely isometric as well.

By construction, $\mathcal{K}$ decomposes into a direct sum of $n$ copies of $H^2_d$ and a space $\mathcal{K}_Z$ on which Toeplitz$_d$ acts by the spherical operator $Z$. This yields a direct sum decomposition of $\mathcal{K}_2$ into $n$ copies of $H^2_d(I)$ and the part $\mathcal{K}_Z \ominus I(\mathcal{K}_Z) = \mathcal{K}_Z \ominus \{ f(Z)\xi \mid f \in I, \xi \in \mathcal{K}_Z \}$.

Since $Z$ is normal, $I(\mathcal{K}_Z)$ and therefore $\mathcal{K}_Z \ominus I(\mathcal{K}_Z)$ is invariant under the $C^*$-algebra generated by $Z$. Therefore, the compression of $Z$ to $\mathcal{K}_Z \ominus I(\mathcal{K}_Z)$ is still a normal (multi)operator with spectrum contained in $\partial \mathbb{D}_d$, i.e. a spherical operator. Thus the representation of $\text{Shift}_d$ on $\mathcal{K}_Z \ominus I(\mathcal{K}_Z)\mathcal{K}_Z$ extends to a $*$-representation of $C(\partial \mathbb{D}_d)$.
The kernel of this extension is a closed ideal of $C(\partial D_d)$. It must contain $I$, so that $Z$ comes from a $*$-representation of $sa(I)$.

Hence some completely isometric representation of $\text{Shift}_d/I$ can be obtained as a direct sum of $n$ copies of $S(I)$ and a $*$-representation of $sa(I)$. The representation of $sa(I)$ need not be faithful, but replacing it by a faithful representation can only increase norms and thus still gives a completely isometric representation. Moreover, replacing the $n$ copies of $S(I)$ with just one does not change matrix norms either. □

It is not always necessary to add a faithful representation of $sa(I)$. For example, if $I = \{0\}$, then $S(I) = S$ and $Z$ can be omitted, although $sa(I) = C(\partial D_d)$. Indeed, nothing really interesting happens in the boundary part coming from $Z$. In the finite dimensional case, it only contributes a direct sum of several copies of $C$ to the quotient algebra. To see this, the following lemma is necessary:

**Lemma 9.2.** Let $\omega \in \partial D_d$. Then $c_{\text{Shift}_d}(\omega, \omega) = 1$ for all $\omega_2 \in \overline{D_d} \setminus \{\omega\}$ and $T_\omega\text{Shift}_d = \{0\}$.

*Proof.* Since rotation by unitaries operates transitively on $\partial D_d$, we can assume without loss of generality that $\omega = (1, 0, \ldots, 0)$. Since $S$ is a $d$-contraction, $\|S_1\| \leq 1$. Moreover, $S_1(\omega) = 1$ and $S_1(\omega_2) \neq 1$ for any $\omega_2 \neq \omega$. Hence Corollary 4.4 implies $c_{\text{Shift}_d}(\omega, \omega) = 1$ for all $\omega_2 \in \overline{D_d} \setminus \{\omega\}$ and $\delta(S_1) = 0$ for any derivation $\delta$ of $\text{Shift}_d$ at $\omega$.

In order to get $T_\omega\text{Shift}_d = \{0\}$, it remains to show that, if $\delta$ is a derivation at $\omega$, then $\delta(S_j) = 0$ also for $j = 2, \ldots, d$. For then $\delta$ vanishes on the polynomial algebra which is dense in $\text{Shift}_d$. If $\delta$ were a non-zero derivation at $\omega$, then, without loss of generality, $\|\delta\| = 1$. Then the representation

$$\rho: f \mapsto \begin{pmatrix} f(\omega) & \delta(f) \\ 0 & f(\omega) \end{pmatrix}$$

is a completely contractive representation of $\text{Shift}_d$. Indeed, it is a completely isometric representation of the quotient $\text{Shift}_d/I\ker \rho$ by $2 \times 2$-matrices. Hence the matrix

$$\rho(S_1, S_2, \ldots, S_d) = \begin{pmatrix} 1 & \delta(S_1) & 0 & \delta(S_2) & \cdots & 0 & \delta(S_d) \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a contraction because $\rho(S)$ is a $d$-contraction. This implies $\delta(S_1) = \cdots = \delta(S_d) = 0$ as desired. Hence $T_\omega\text{Shift}_d = \{0\}$.

**Proposition 9.3.** Let $I \subset \text{Shift}_d$ be a closed, finite codimensional ideal. Then $Q = \text{Shift}_d/I$ is completely equivalent to an orthogonal direct sum $\text{Shift}_d/I \oplus C \oplus \cdots \oplus C$, where $I \subset \text{Shift}_d$ is a closed ideal with $sa(I) = \{0\}$ and the number of copies of $C$ occurring is the dimension of $sa(I)$.

*Proof.* The spectrum of $Q$ can be viewed as a subset of $\overline{D_d}$. By Lemma 9.2, a point in the boundary is $Q$-related only to itself and $I(\omega)^\infty = I(\omega)$ for $\omega \in \partial D_d$. Hence the assertion follows from Theorem 4.7. □

**Corollary 9.4.** Let $I \subset \text{Shift}_d$ be a closed ideal of finite codimension $r$. Then $\text{Shift}_d/I$ has a completely isometric representation by $r \times r$-matrices.

*Proof.* We may assume without loss of generality that $sa(I) = \{0\}$ by Proposition 9.3. By Theorem 9.1, $\text{Shift}_d/I$ can be represented completely isometrically on $H_2^d \oplus I H_2^d$. Since $\text{Shift}_d$ is dense in $H_2^d$, there is a map $\text{Shift}_d/I \to H_2^d \oplus I H_2^d$ whose image is dense. Since $\text{Shift}_d/I$ has finite dimension $r$, the image must be all of $H_2^d \oplus I H_2^d$ and this Hilbert space has dimension $r$. □
Theorem 9.5. Let $x_1, \ldots, x_m \in \mathbb{D}_d$ and let $y_1, \ldots, y_m \in \mathbb{M}_n$. Then there exists $F \in \text{Ball}(\text{Shift}_d)_{(n)}$ with $F(x_j) = y_j$ for all $j = 1, \ldots, m$ if and only if the block matrix $A \in \mathbb{M}_m \otimes \mathbb{M}_n$ with entries

$$ \frac{1 - y_i y_j^*}{1 - \langle x_i, x_j \rangle} \in \mathbb{M}_n $$

is positive definite and invertible. There exists $F \in \text{Cone}(\text{Shift}_d)_{(n)}$ with $F(x_j) = y_j$ for all $j = 1, \ldots, m$ if and only if the block matrix $B \in \mathbb{M}_m \otimes \mathbb{M}_n$ with entries

$$ \frac{y_i + y_j^*}{1 - \langle x_i, x_j \rangle} \in \mathbb{M}_n $$

is positive definite and invertible. Moreover, solutions $F$ can be chosen to have polynomial entries.

Proof. The theorem amounts to a computation of the matrix normed structure of the quotient $Q = \text{Shift}_d/\mathbb{L}(x_1, \ldots, x_n)$. By Theorem 9.1, $Q$ has a completely isometric representation on the Hilbert space $\mathcal{H} = H^2_d \otimes \mathbb{L}(x_1, \ldots, x_m)H^2_d$.

The first step is to write down a (non-orthogonal) basis of $\mathcal{H}$. We claim that the vectors $e_j = u_j$ for $j = 1, \ldots, m$ form such a basis. Viewing elements of $H^2_d$ as functions on $\mathbb{D}_d$, the relation $\langle f, e_j \rangle = f(x_j)$ for all $f \in H^2_d$ shows that the vectors $e_j$ are all orthogonal to the subspace $\mathbb{L}(x_1, \ldots, x_m)H^2_d$ and hence lie in $\mathcal{H}$. Moreover, they are linearly independent. Since $\dim \mathcal{H} = m$ by Proposition 9.4, they span $\mathcal{H}$.

Now $\text{Cone}(\text{Shift}_d/\mathbb{L}(x_1, \ldots, x_n))_{(n)}$ can be computed using Proposition 2.4. Let $[F] \in (\text{Shift}_d/\mathbb{L}(x_1, \ldots, x_n))_{(n)}$ be given by $[F](x_j) = y_j$ for $j = 1, \ldots, m$. Let $v_1, \ldots, v_n$ be the standard basis of $\mathbb{C}^n$. Then $\{e_i \otimes v_\mu\}$ is a frame for $\mathcal{H} \otimes \mathbb{C}^n$ and $\langle [F](e_j \otimes v_\mu), e_i \otimes v_\mu \rangle = \langle [F]_{\mu\nu}(x_i)u_j(x_i), (y_i)_{\mu\nu} \cdot (1 - \langle x_i, x_j \rangle)^{-1} \rangle$. This proves the correctness of the criterion for $[F] \in \text{Cone}(\cdot \cdot \cdot)$.

In order to compute the unit ball, however, the action of $[F]$ must be determined in an orthonormal basis. This will also give a new proof of the criterion for $[F] \in \text{Cone}(\cdot \cdot \cdot)$.

The inner products between the basis elements are given by the matrix $B$ with entries

$$ \beta_{i,j} = \langle e_j, e_i \rangle = (1 - \langle x_i, x_j \rangle)^{-1} $$

by (17). Since the inner product is positive definite, the matrix $B$ is positive and invertible. Hence the vectors

$$ \hat{e}_j = B^{-1/2} e_j = \sum_{k=1}^m (B^{-1/2})_{kj} e_k $$

are well-defined. It is easy to check that they form an orthonormal basis. Moreover, the operator $B: e_j \rightarrow \sum \beta_{kj} e_k$ still has the matrix $(\beta_{ij})$ in the basis $(\hat{e}_j)$ because $B$ and $B^{-1/2}$ commute.

For $f \in \text{Shift}_d$, the relation $\langle Mf e_i, e_j \rangle = f(x_j) \langle e_i, e_j \rangle$ shows that the action of $M_f^*$ is given by $M_f^* e_j = f(x_j) e_j$, i.e. the basis $e_j$ is a joint eigenbasis for these adjoints. Thus the action of the compression of $M_f^*$ to $\mathcal{H}$ is given by

$$ B^{1/2} \text{diag}(f(x_1), \ldots, f(x_m)) B^{-1/2} $$

in the basis $(\hat{e}_j)$. Hence the action of $\text{Shift}_d$ on $\mathcal{H}$ in the orthonormal basis $(\hat{e}_j)$ is given by

$$ f \mapsto B^{-1/2} \text{diag}(f(x_1), \ldots, f(x_m)) B^{1/2} $$

for $f \in \text{Shift}_d$.

$[F]$ is represented by $(B \otimes \text{id})^{-1/2} \text{diag}(y_1, \ldots, y_m)(B \otimes \text{id})^{1/2}$ on $\mathcal{H} \otimes \mathbb{C}^n$. This matrix has norm less than 1 iff

$$ 1 - (B \otimes \text{id})^{-1/2} \text{diag}(y_j)(B \otimes \text{id}) \text{diag}(y_j^*)(B \otimes \text{id})^{-1/2} $$
is positive and invertible. Since \( B \otimes \text{id} \) is invertible, this is equivalent to \( B \otimes \text{id} - \text{diag}(y_j)(B \otimes \text{id}) \text{diag}(y_j^*) \) being positive and invertible. This is just the matrix in the statement of the theorem, proving the first assertion.

The real part of \([F]\) is represented by the matrix
\[
(1/2) \cdot (B \otimes \text{id})^{-1/2} \text{diag}(y_j)(B \otimes \text{id})^{1/2} + (1/2) \cdot (B \otimes \text{id})^{1/2} \text{diag}(y_j^*)(B \otimes \text{id})^{-1/2}
\]
This is positive and invertible iff the matrix
\[
\text{diag}(y_j)(B \otimes \text{id}) + (B \otimes \text{id}) \text{diag}(y_j^*)
\]
is positive and invertible. Again, this is the matrix occurring in the statement of the theorem, proving again the second assertion.

**Theorem 9.6.** Let \( x_1, \ldots, x_m \in \mathbb{D}_d \) and let \( y_1, \ldots, y_m \in \mathbb{M}_n \). Then there exists \( F \in \text{Ball}(\text{Shift}_d^{(l)}(n)) \) with \( F(x_j) = y_j \) for all \( j = 1, \ldots, m \) if and only if the block matrix \( A \in \mathbb{M}_m \otimes \mathbb{M}_n \) with entries
\[
\frac{1 - y_i^* y_j}{1 - \langle x_j, x_i \rangle} \in \mathbb{M}_n
\]
is positive definite and invertible. There exists \( F \in \text{Cone}(\text{Shift}_d^{(l)}(n)) \) with \( F(x_j) = y_j \) for all \( j = 1, \ldots, m \) if and only if the block matrix \( B \in \mathbb{M}_m \otimes \mathbb{M}_n \) with entries
\[
\frac{y_i^* + y_j}{1 - \langle x_j, x_i \rangle} \in \mathbb{M}_n
\]
is positive definite and invertible. Moreover, solutions \( F \) can be chosen to have polynomial entries.

**Proof.** Since such an \( F \in (\text{Shift}_d^{(l)}(n)) \) exists iff there is \( F^t \in (\text{Shift}_d^{(l)}(n)) \) with \( F^t(x_j) = y_j^* \) for \( j = 1, \ldots, m \), this is an immediate consequence of Theorem 9.5.

For \( d = 1 \), Theorem 9.5 is equivalent to the existence part of Nevanlinna-Pick theory. However, the proof above is not constructive.

**Problem 9.1.** Find a constructive proof of Theorem 9.5. Is there a criterion for the existence of \( F \in \text{Shift}_d \otimes \mathbb{M}_n \) with \( \|F\|(n) \leq 1 \) and prescribed values in finitely many points of \( \mathbb{D}_d \)?

Since \( \text{Shift}_1 \cong \mathcal{P}(\mathbb{D}) \) is a uniform algebra, \( \text{Shift}_1^{(l)} \cong \text{Shift}_1 \). Thus quotients of \( \text{Shift}_1^{(l)} \) and \( \text{Shift}_1 \) are completely equivalent. This is not so clear, however, from the formulas in Theorem 9.5 and 9.6.

If \( \text{Shift}_d \) has non-trivial tangent space, the orthogonal complement of \( 1 \cdot H_d^2 \) can be computed most easily using the characterization of the inner product by the Fantappiè transform. There are additional conditions \( l(f) = 0 \), where the linear functional \( l \) is a differential operator \( l = \sum c_a \partial_a |a \) at some \( a \in \mathbb{D}_d \). Since \( l \) is a continuous linear functional on \( \mathcal{O}(\mathbb{D}_d) \), the Fantappiè transform yields
\[
(18) \quad l(f) = \overline{\mathcal{F}}^{-1}(\mathcal{F}(l))(f) = (f, \overline{\mathcal{F}(l)})_{H_d^2}.
\]
Hence the function \( z \mapsto (\mathcal{F}(l))(z) = l(u_z) \) lies in the orthogonal complement of \( 1 \). If \( l \) is some differential operator as above, this function can easily be computed. Appropriate differential operators \( l \) provide a basis of \( H_d^2 \otimes \mathbb{M}_d \). The inner products between these vectors can be computed using (18). More generally,
\[
(19) \quad (M_l 1_{(u_z)}, l_2(u_z)) = (M_l 1_{(u_z)}, (\overline{\mathcal{F}(l_2)})) = l_2 \left( f \cdot 1_{(u_z)} \right),
\]
for any continuous linear functionals \( l_1, l_2 \) on \( \mathcal{O}(\mathbb{D}_d) \) and \( f \in \text{Shift}_d \).

However, the method of the proof of Theorem 9.5 does not apply to this situation; there is no natural basis for \( H_d^2 \otimes \mathbb{M}_d \) in which the action of \( M_f \) can be computed easily. However, the following recipe still works. Let \( I \subset \text{Shift}_d \) be a closed ideal
of finite codimension \( r \) such that \( sa(I) = \{ 0 \} \). Let \( Q = \text{Shift}_d I \) and \( F \in Q(n) \). In order to determine whether \( F \in \text{Ball}(Q(n)) \), do the following:

1. Choose a basis \( l_1, \ldots, l_r \) for the vector space of differential operators annihilating \( I \) and \( g_1, \ldots, g_r \in \text{Shift}_d \) representing a basis of \( Q \). Write \( F \) as a matrix with entries

\[
F_{\mu\nu} = \sum_{k=1}^{r} F_{\mu\nu}^k [g_k].
\]

2. Compute \( \lambda_j = \mathcal{F}(l_j) \) for \( j = 1, \ldots, r \); these functions form a basis for \( H^2_d \otimes H^2_d \).

3. Using (19), compute the inner products

\[
\beta_{ij} = \langle \lambda_j, \lambda_i \rangle = l_i(\lambda_j),
\]

\[
\gamma_{kij} = \langle g_k \lambda_j, \lambda_i \rangle = l_i(g_k \lambda_j).
\]

4. Let \( M(F) \) be the block matrix with \( \mu, \nu \)th entry \( \left( \sum_k \gamma_{kij} F_{\mu\nu}^k \right)_{ij} \in M_r \) and let \( B = (\beta_{ij}) \).

5. Check whether the matrix \( B \otimes \text{id} - M(F)(B^{-1} \otimes \text{id})M(F)^* \) is positive definite and invertible. This happens iff \( F \in \text{Ball}(Q(n)) \).

In order to determine whether \( F \in \text{Cone}(Q(n)) \), it suffices to compute the matrix \( M(F) \), the matrix \( B \) is not necessary: \( F \in \text{Cone}(Q(n)) \) iff \( \text{Re} \, M(F) \) is positive and invertible.

The proof that the above algorithm works is left to the reader. It is also left to the reader to check that it gives the same answer in the special case of Theorem 9.5. It is essential to flip the indices \( i, j \) in the definition of \( \beta_{ij} \) and \( \gamma_{kij} \) in order to get the matrix normed structure right. This is because if the matrix \( A \) has entries \( A_{ij} \) in the orthonormal basis \( \{ E_j \} \), then \( \langle AE_i, E_j \rangle = A_{ji} \).

It is of special interest to compute the two-dimensional quotients of the multiplier algebra. First look at the quotient by \( I(x, y) \) with \( x, y \in \overline{D}_d \). If one of the points lies in the boundary, then \( c_{\text{Shift}_d}^2(x, y) = 1 \) by Lemma 9.2. If \( x, y \in D_d \), the quotient is represented by \( 2 \times 2 \)-matrices. The adjoints of the representing matrices have eigenvectors \( u_x \) and \( u_y \). The angle between these two vectors is

\[
\frac{|\langle u_x, u_y \rangle|}{|u_x| \cdot |u_y|} = \frac{\sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2}}{|1 - \langle x, y \rangle|}.
\]

Comparing this with the angle between the eigenvectors of the matrices \( T c^* \) of Section 3 shows that this number is \( \sqrt{1 - c_{\text{Shift}_d}^2(x, y)^2} \), so that

\[
c_{\text{Shift}_d}^2(x, y) = \left( 1 - \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y \rangle|^2} \right)^{1/2}.
\]

This coincides with the classical Carathéodory* distance for the unit ball, computed using \( H^\infty(\mathbb{D}_d) \) instead of \( \text{Shift}_d \) [16]. This is not too surprising because \( \text{Shift}_d \) is constructed to model \( d \)-contractions, and the quotient distance is constructed to describe two-dimensional quotients. Multi-operators that generate two-dimensional, unital operator algebras can be modeled just as well by \( H^\infty(\mathbb{D}_d) \). Only for higher dimensional quotients does \( H^\infty(\mathbb{D}_d) \) fail to give a satisfactory theory. Note also that the above distance is invariant under \( \text{Aut}(\mathbb{D}_d) \) because the classical Carathéodory* distance is invariant. This must happen because of Theorem 8.5.

Now let us compute the quotient metric for \( \text{Shift}_d \). For this purpose, it is convenient to use Theorem 8.5, which implies that all automorphisms of \( \mathbb{D}_d \) induce isometries for the quotient metric. It is easy to see that \( \text{T}_a \text{Shift}_d \equiv \text{T}_a \mathbb{D}_d \equiv \mathbb{C}^d \) for \( a \in \mathbb{D}_d \). Moreover, Lemma 9.2 yields \( \text{T}_a \text{Shift}_d = \{ 0 \} \) for \( a \in \partial \mathbb{D}_d \). Let \( X = \partial / \partial z_1 = \text{T}_0 \partial \mathbb{D}_d \), then every element of \( \text{T} \mathbb{D}_d \) is mapped to a multiple of \( X \) by some automorphism
of $\mathbb{D}_d$ (first map the base point to the origin and then rotate by a unitary). Hence the value of $\gamma_{\text{Shift}_d}(0; X)$ determines the quotient metric completely.

The orthogonal complement of $l(0; X)H^2_d$ is spanned by the functions $1, z_1 \in H^2_d$. Indeed, $\{1, z_1\}$ is an orthonormal basis of $H^2_d \ominus l(0; X)H^2_d$. Moreover, the compression of $M_{z_1}$ is the matrix $T_0$ of Section 3 in the basis $\{z_1, 1\}$. Thus the derivation $[f] \mapsto Df(0; X)$ corresponds to the derivation $T_0 \mapsto 1$ and therefore has norm 1. Consequently, $\gamma_{\text{Shift}_d}(0; X) = 1$. Lemma 6.4 gives the same result for the classical Carathéodory-Reiffen metric $\gamma_{H^\infty(\mathbb{D}_d)}(0; X)$. Since both $\gamma_{H^\infty(\mathbb{D}_d)}(a; X)$ and $\gamma_{\text{Shift}_d}(a; X)$ are invariant under automorphisms of $\mathbb{D}_d$, these functions coincide everywhere. Hence the formula in [16] for the classical Carathéodory-Reiffen metric on $\mathbb{D}_d$ can be copied:

$$\gamma_{\text{Shift}_d}(a; X) = \left( \frac{\|X\|^2}{1 - \|a\|^2} + \frac{|\langle a, X \rangle|^2}{(1 - \|a\|^2)^2} \right)^{1/2}.$$ 

Here $\|X\|$ stands for the norm of $X \in \ell^2_d$, not for the norm of the associated linear functional on $\text{Shift}_d$, of course. However, $T_0^* \text{Shift}_d$ and $T_0^* \mathbb{D}_d$ are no longer completely isometric for $d \geq 2$.

**Example 9.1.** The representation of $\text{Shift}_d(0)$ on $H^2_d \ominus l(0)^2 H^2_d$ is completely isometric by Theorem 9.1. Clearly, this orthogonal complement is spanned by the constant function 1 and the linear functions $z_1, \ldots, z_d$. Thus $T_0^* \text{Shift}_d$ is represented by the $d$-dimensional subspace

$$\begin{pmatrix}
0 & 0 & 0 & \cdots \\
* & 0 & 0 & \cdots \\
* & 0 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

of $M_{d+1}$. Especially, the algebra called $\text{Shift}_2(0)$ in Section 6 is indeed completely equivalent to $\text{Shift}_2/l(0)^2$.

The proof of Theorem 6.2 can easily be generalized to show that transposition $\text{Shift}_d(0) \to \text{Shift}_d(0)^t$ is not 2-isometric. Thus $\text{Shift}_d(0)$ is not a Q-algebra by Corollary 5.3. Therefore, $\text{Shift}_d$ cannot be a function algebra. Thus the $d$-shift cannot be subnormal.

**Theorem 9.7.** Let $\mathbb{Q}$ be a $d'$-dimensional quotient of $\text{Shift}_d$. Then $\mathbb{Q}$ is completely equivalent to quotients of $\text{Shift}_e$ for all $e \geq \min\{d, d' - 1\}$.

**Proof.** Viewing $(S_1, \ldots, S_d, 0)$ as a $d + 1$-contraction on $H^2_d$ gives a completely contractive unital homomorphism $i_1: \text{Shift}_{d+1} \to \text{Shift}_d$. Viewing $(S_1, \ldots, S_d)$ as a $d$-contraction on $H^2_{d+1}$ produces a completely contractive unital homomorphism $i_1^*: \text{Shift}_d \to \text{Shift}_{d+1}$. Clearly, $i_1 \circ i_1^* = \text{id}_{\text{Shift}_d}$. Thus $i_1^*$ is a complete isometry and $i_1$ is a complete quotient map. Consequently, every quotient of $\text{Shift}_d$ is also a quotient of $\text{Shift}_{d+1}$, so that it remains to show that $\mathbb{Q}$ is completely equivalent to a quotient of $\text{Shift}_{d'-1}$ if $d' \leq d < \infty$.

Since arbitrary finite dimensional quotients of $C(\partial \mathbb{D}_d)$ are also quotients of $\text{Shift}_1$, it can be assumed that the spectrum of $\mathbb{Q}$ is not contained in $\partial \mathbb{D}_d$. Here view $\text{Spec} \mathbb{Q} \subset \text{Spec}(\text{Shift}_d) \cong \mathbb{D}_d$ via the map induced by the quotient map $\rho: \text{Shift}_d \to \mathbb{Q}$. Using an automorphism of $\text{Shift}_d$ according to Theorem 8.5, it can be achieved that $0 \in \text{Spec} \mathbb{Q}$. Let $\delta = d - d' + 1$.

The space spanned by $1, \rho(S_1), \ldots, \rho(S_d)$ in $\mathbb{Q}$ has dimension at most $d'$, hence there must be at least $d - d'$ independent linear dependence relations among these elements. Since $0 \in \text{Spec} \mathbb{Q}$ and $S_j(0) = 0$ for all $j = 1, \ldots, d$, these relations cannot involve 1 with a nonzero coefficient. Hence there are $d - d'$ linearly independent functionals $l_j$ on $\mathbb{C}^d$ with $l_j(\rho(S_1), \ldots, \rho(S_d)) = 0$ for all $j = 1, \ldots, d - d'$. By a
unitary transformation (and thus a complete isometry of \( \text{Shift}_d \)) it can be achieved that the kernel of these linear functionals is spanned by \( e_d, e_{d-1}, \ldots, e_{d-d'} \).

Hence it can be assumed without loss of generality that \( \rho(S_{d-d'}) = \cdots = \rho(S_d) = 0 \), so that \( \rho = \rho \circ i^d_1 \circ i^d_2 \). Here, of course, \( i^d_1 = i_1 \circ \cdots \circ i_{d'} \) with \( d \) factors, so that

\[
i^d_1 \circ i^d_2(S_1, \ldots, S_d) = (S_1, \ldots, S_{d-1}, 0, \ldots, 0).
\]

Now define \( \rho_1 = \rho \circ i^d_1 : \text{Shift}_{d-1} \to Q \). By definition, this is completely contractive, and \( \rho = \rho \circ i^d_1 \circ i^d_2 = \rho_1 \circ i^d_1 \).

If \( l_1, l_2 \) are contractive linear maps and \( l_1 \circ l_2 \) is an isometry, then \( l_2 \) must be an isometry. The dual statement of this is that if \( l_1 \circ l_2 \) is a quotient map, then \( l_1 \) must be a quotient map. This dual statement can be proved either directly or by realizing that \( l : V_1 \to V_2 \) is a quotient map iff \( l^* : V_2^* \to V_1^* \) is an isometry. Of course, these statements remain true if “contractive”, “isometry”, and “quotient map” are replaced by “completely contractive”, “completely isometric”, and “complete quotient map” everywhere. Especially, this can be applied to \( \rho = \rho_1 \circ i^d_1 \) to obtain that \( \rho_1 \) is a complete quotient map.

**10. The Quotient Complexity of a Commutative Operator Algebra**

**Definition 10.1.** Let \( A \) be a unital, commutative operator algebra. Then \( A \) is said to have minimal (quotient) complexity if every \( r \)-dimensional quotient of \( A \) has a completely isometric representation by \( r \times r \)-matrices.

The idea behind the concept of quotient complexity is that a subalgebra of \( M_r \) for \( r' > r \) can have a more complicated matrix normed structure than an isomorphic subalgebra of \( M_r \). Moreover, an \( r \)-dimensional, algebraically semisimple, commutative operator algebra (i.e. a direct sum of \( r \) copies of \( \mathbb{C} \)) cannot be represented faithfully in \( M_{r-1} \). However, it is easy to find nilpotent commutative subalgebras of \( M_r \) of dimension greater than \( r \):

**Example 10.1.** Consider \( M_r \) as an operator space, then its trivial unitization is by definition a subalgebra of \( M_{r+1} \) of dimension \( r^2 + 1 \). However, it follows from Lemma 10.6 that the unitization of \( M_r \) does not have minimal quotient complexity.

Nevertheless, the term minimal seems appropriate, especially if the following conjecture should turn out to be true:

**Conjecture 10.1.** A unital, commutative subalgebra of \( M_r \) of dimension greater than \( r \) does not have minimal quotient complexity.

Theorem 10.5 proves this conjecture for trivial unitizations.

**Remark 10.2.** Let \( A \subset M_{n-1} \) be a unital, commutative operator algebra. Then \( A \) has a unital, completely isometric representation by \( n \times n \)-matrices.

**Proof.** It is trivial to find a non-unital completely isometric representation \( A \to M_n \). Restrict this to any maximal ideal and take the unitization of this representation. By Theorem 2.5, this defines a completely isometric, unital representation of \( A \).

**Proposition 10.3.** Let \( A_1 \) and \( A_2 \) be unital, commutative operator algebras. Then \( A_1 \oplus A_2 \) has minimal complexity if and only if both \( A_1 \) and \( A_2 \) have minimal complexity. Moreover, \( A_1 \) has minimal complexity if and only if \( A \) has minimal complexity.

**Proof.** Let \( e_j \in A_j \) be the identity elements. Any ideal \( l \subset A_1 \oplus A_2 \) is of the form \( l = l_1 \oplus l_2 \) with ideals \( l_j \subset A_j \). Hence \( (A_1 \oplus A_2)/l \cong (A_1/l_1) \oplus (A_2/l_2) \). If \( \rho_j : A_j/l_j \to M_{n_j} \), \( j = 1, 2 \), are completely isometric representations, then \( \rho_1 \oplus \rho_2 : (A_1 \oplus A_2)/l \to M_{n_1+n_2} \) is a completely isometric representation. Thus if \( A_1 \) and \( A_2 \) have minimal
complexity, so has their direct sum. The reverse implication is trivial because every quotient of $A_j$, $j = 1, 2$, is completely equivalent to a quotient of $A_1 \oplus A_2$.

By Lemma 5.2, every quotient of $A_1^\omega$ is the transpose of a quotient of $A$. Hence it has low-dimensional representations iff this is true for the corresponding quotient of $A$ by the concrete description of the transpose operation.

Thus it suffices to study indecomposable algebras of minimal complexity. Corollary 9.4 can be rephrased as follows: All quotients of $\text{Shift}_d$ have minimal quotient complexity. The proof uses only formal properties of the dilation theory for $d$-contractions. Hence in order to find other model theories with similar formal properties, it is interesting to know whether there are more algebras of minimal complexity.

The only finite dimensional, indecomposable algebras of minimal complexity known at the moment are the quotients of $\text{Shift}_d$ and $\text{Shift}_d^t$. It is quite conceivable that there are no other examples (Conjecture 10.12).

Other infinite dimensional operator algebras of minimal quotient complexity are easy to obtain. An obvious candidate is the injective limit of the algebras $\text{Shift}_d$ for $d \to \infty$. More generally, if $M$ is a set and $\Lambda$ is the net of finite dimensional subsets of $M$, we can associate to it an injective system $S \mapsto \text{Shift}_{\#S}$ for $S \in \Lambda$ with the obvious maps used already in the proof of Theorem 9.7. If $M$ is uncountable, then the corresponding injective limit will be an operator algebra of minimal quotient complexity that is not separable. Indeed, any finite dimensional quotient of this injective limit is also a quotient of some $\text{Shift}_d$.

**Example 10.2.** The 4-dimensional, unital operator algebra $Q_0 \otimes Q_0 \subset M_4$ does not have minimal quotient complexity: Its unique 2-dimensional cotangent space is invariant under transposition. Hence it cannot be completely isometric to $\text{Shift}_2(0)$ or $\text{Shift}_2(0)^t$.

**Remark 10.4.** The previous example shows that the spatial tensor product does not preserve quotient complexity. Another example is $\mathcal{P}(D^2) \cong \mathcal{P}(D) \otimes \mathcal{P}(D)$. This function algebra has a quotient with no finite dimensional completely isometric representations by Example 6.3, whereas $\mathcal{P}(D)$ has minimal quotient complexity because it is the algebra generated by the 1-shift. This example shows that the maximal tensor product does not preserve quotient complexity either.

However, if $A_j \subset M_{n_j}$, $j = 1, 2$, are $n_j$-dimensional subalgebras, then $A_1 \otimes A_2 \subset M_{n_1 \cdot n_2}$ is an $n_1 \cdot n_2$-dimensional subalgebra. Hence the reason for the above problem is that taking tensor products is not well-behaved with respect to quotients.

**Example 10.3.** I expect that the spatial tensor product $Q_c \otimes Q_d$ does not have minimal quotient complexity if $c, d < 1$. For $c = d = 0$, this is shown in Example 10.2. It can be shown that $Q_c \otimes Q_d$ is indecomposable and not a quotient of $\text{Shift}_1$ if $c, d < 1$. Hence the general case would follow from Conjecture 10.12. It is difficult to compute quotients of $Q_c \otimes Q_d$ directly, however.

The best structure theorem for algebras of minimal complexity that we can prove at the moment is the following:

**Theorem 10.5.** Let $A$ be a closed, commutative, unital operator algebra of minimal quotient complexity and let $\omega \in \text{Spec}(A)$. Then $T_\omega A$ is completely equivalent to $B(C, \mathcal{H})$ or $B(\mathcal{H}, C)$ for some Hilbert space $\mathcal{H}$. Especially, if $\dim_T A = d \in \mathbb{N}$, then $T_\omega A$ is completely equivalent to $T_0^t \text{Shift}_d$ or $T_0^t \text{Shift}_d^t$.

**Proof.** Let $\mathcal{H}$ be a Hilbert space and choose a unit vector $x \in \mathcal{H}$. This induces an isometric embedding $C \subset \mathcal{H}$ and a projection $\mathcal{H} \to C$, which turn $B(C, \mathcal{H})$ and $B(\mathcal{H}, C)$ into closed subspaces of $B(\mathcal{H})$. The resulting abstract operator space structure on $B(\mathcal{H}, C)$ and $B(C, \mathcal{H})$ does not depend on the choice of $x$, of course.
Since $\mathbb{B}(\mathbb{C}, \mathcal{H})$ and $\mathbb{B}(\mathcal{H}, \mathbb{C})$ are clearly isometric to a Hilbert space, the first step is to show that $T^*_\omega A$ is isometric to a Hilbert space.

**Lemma 10.6.** Assume that every 3-dimensional quotient of the unital, commutative operator algebra $A$ has an isometric representation on $\mathbb{C}^3$. Then, for any $\omega \in \text{Spec}(A)$, $T^*_\omega A$ is isometric to a pre-Hilbert space.

**Proof.** The only 3-dimensional subalgebras of $M_3$ that are trivial unitizations are $\text{Shift}_2(0)$ and $\text{Shift}_2(0)^t$, and both are isometric to $(\ell_2^3)^*$. By Theorem 2.5, there is a one-to-one correspondence between quotients of $A(\omega) = (T^*_\omega A)^+$ and quotients of $T^*_\omega A$. Hence any two-dimensional quotient of $T^*_\omega A$ is isometric to $\ell_2^2$.

The quotient map $T^*_\omega A \rightarrow \ell_2^2$ dualizes to an isometric embedding $\ell_2^2 \rightarrow (T^*_\omega A)' = T_{\omega} A$. Clearly, any two elements of $T_{\omega} A$ lie in the image of such a map. Thus the parallelogram identity holds in $T_{\omega} A$, because it only involves vectors in a two-dimensional subspace. Hence the norm on $T_{\omega} A$ comes from a pre-inner product. Another dualization yields that $T_{\omega} A$ is a pre-Hilbert space as well. □

The next step is to study subspaces of $M_{n,m}$ that are isometric to a Hilbert space.

**Theorem 10.7.** Let $n, m \geq 2$ and let $\mathcal{H} \subset M_{n,m}$ be a subspace of dimension $r$ that is isometric to a Hilbert space. Then $r \leq n + m - 2$.

**Remark 10.8.** The bound $n + m - 2$ in Theorem 10.7 probably is not optimal. The only candidates of Hilbert spaces contained in $M_{n,m}$ that immediately come to mind have dimensions $n$ and $m$, respectively, so a likely conjecture is that even $r \leq \max\{n, m\}$. However, this stronger estimate is more difficult and not relevant for our purposes.

**Proof.** Assume the contrary, then there exists a subspace $\mathcal{H} \subset M_{n,m}$ that is isometric to $\ell_2^r$ with $r = n + m - 1$. Since transposition $M_{n,m} \rightarrow M_{m,n}$ is isometric, we can assume without loss of generality that $m \leq n$.

The proof depends on the singular value decomposition of a matrix. Every $A \in M_{n,m}$ can be written as $A = U \text{ diag}(a_1, \ldots, a_m)V$, where $U \in M_{n,n}$ and $V \in M_{m,m}$ are unitary matrices and $\|A\| = a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. Here $\text{ diag}(a_1, \ldots, a_m)$ stands for the $n \times m$-matrix with $i,j$th entry $\delta_{ij}a_j$. The matrices $U$ and $V$ are usually not unique, but the “singular values” $a_1, \ldots, a_n$ are. Indeed, they are the eigenvalues of the matrix $(A^* A)^{1/2}$. This also shows that $a_k$ depends continuously on $A$. Let $\alpha_k : M_{n,m} \rightarrow \mathbb{R}_+$ be the map $A \mapsto a_k$.

For a normed space $V$, let $S(V) = \{v \in V \mid \|v\| = 1\}$. Every element of $S(\mathcal{H})$ has the singular value 1, possibly with multiplicity. Choose $A_0 \in S(\mathcal{H})$ for which this multiplicity is minimal.

Since left and right multiplication by unitary matrices induces an isometry of $M_{n,m}$, we can assume without loss of generality that $A_0$ is of the form

$$A_0 = \text{ diag}(1, \alpha_2(A_0), \ldots, \alpha_m(A_0)).$$

Let $N \cong M_{n-1,m-1}$ be the subspace of $M_{n,m}$ of matrices with zeroes in the first row and first column.

**Claim 10.9.** The natural map $p : \mathcal{H} \rightarrow M_{n,m}/N$ is a vector space isomorphism.

**Proof.** Since $\dim(M_{n,m}/N) = n + m - 1 = \dim \mathcal{H}$, it suffices to show $\mathcal{H} \cap N = \{0\}$. Assume the contrary and take $A \in \mathcal{H} \cap N \setminus \{0\}$. Let $A' \in M_{n-1,m-1}$ be the lower right block of $A$, i.e. forget the zero column and row of $A$. First consider the case $\alpha_2(A_0) < 1$. Then

$$\|A_0 + \lambda A\| = \max\{1, \|\text{ diag}(\alpha_2(A_0), \ldots, \alpha_m(A_0)) + \lambda A'\|\} = 1$$
for all scalars $\lambda$ in a neighborhood of 0. This, however, cannot happen in a Hilbert space. Now assume
\[ 1 = \alpha_2(A_0) = \cdots = \alpha_k(A_0) > \alpha_{k+1}(A_0) \]
for some $k \geq 2$. By construction of $A_0$, this means that the singular value 1 occurs $k$ times for all elements of $S(\mathcal{H})$. Since $A \in \mathcal{N}$, the matrix $A_0 + \lambda A$ still has the singular value 1 for all $\lambda \in \mathbb{C}$. Furthermore, $\alpha_{k+1}(A_0 + \lambda A) < 1$ and $\|A_0 + \lambda A\| \neq 1$ for sufficiently small $|\lambda|$. Hence, for small $|\lambda| > 0$, the matrix $(A_0 + \lambda A)/\|A_0 + \lambda\| \in S(\mathcal{H})$ has $k$ singular values that are at least $\|A_0 + \lambda\|^{-1}$. One of them is $\|A_0 + \lambda\|^{-1} < 1$, so that the singular value 1 occurs with multiplicity less than $k$. This contradicts the choice of $A_0$. Thus the assumption that $\mathcal{H} \cap N \neq \{0\}$ gives rise to a contradiction both if $\alpha_2(A_0) < 1$ and if $\alpha_2(A_0) = 1$. This proves the claim.

Indeed, the case $\alpha_2(A_0) = 1$ cannot occur at all:

**Claim 10.11.** $\alpha_2(A_0) < 1$.

**Proof.** Assume that $\alpha_2(A_0) = 1$. Since $\mathcal{H} \rightarrow M_{n,m}/N$ is an isomorphism, there exists $B \in \mathcal{H}$ of the form
\[
B = \begin{pmatrix}
0 & 1 & * & * & \ldots \\
1 & b & * & * & \ldots \\
* & * & * & * & \ldots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Since $\|A_0 + \lambda B\| \geq 1$ for all $\lambda \in \mathbb{C}$, the matrix $B$ must be orthogonal to $A_0$ in the Hilbert space $\mathcal{H}$. Therefore, $\|A_0 + \lambda B\|^2 = 1 + \text{const} \cdot |\lambda|^2$ for all $\lambda \in \mathbb{C}$. On the other hand,
\[
\|A_0 + \lambda B\|^2 \geq \left\| \begin{pmatrix} 1 & \lambda \\ \lambda & 1 + b\lambda \end{pmatrix} \right\|^2
\]
This can further be estimated below by
\[
\left\| \begin{pmatrix} 1 & \lambda \\ \lambda & 1 + b\lambda \end{pmatrix} \right\|^2 \geq \frac{1}{2} \left\| \begin{pmatrix} 1 & \lambda \\ \lambda & 1 + b\lambda \end{pmatrix} \left( \pm 1 \right) \right\|^2 = \frac{1}{2} \left( |\pm 1 + \lambda|^2 + |\pm \lambda + 1 + b\lambda|^2 \right) = 1 + \text{Re}(\pm 2\lambda + b\lambda) + O(\lambda^2)
\]
for $\lambda \rightarrow 0$. For a suitable choice of the sign, the $\lambda$-term does not vanish, so that $\|A_0 + \lambda B\|^2 \geq 1 + \text{Re}(\pm \lambda) + O(\lambda^2)$ for $\lambda \rightarrow 0$, with some $c \neq 0$. However, this contradicts $\|A_0 + \lambda B\|^2 = 1 + \text{const} \cdot |\lambda|^2$. Hence $\alpha_2(A_0) \neq 1$.

Now all the preliminary work is done, the following claim is the main part of the proof.

**Claim 10.11.** Any $A \in S(\mathcal{H})$ with $\alpha_2(A) < 1$ has rank at most 1.

**Proof.** Take any such $A$ and bring it into diagonal form $A = \text{diag}(1, a_2, \ldots, a_m)$ as above. Let $B \in S(\mathcal{H})$ with $B \perp A$ and entries $B = (b_{jk})$. Then $b_{11} = 0$ because otherwise $\|A + \lambda B\| \geq |1 + b_{11}|$ is not $1 + O(|\lambda|^2)$. Let
\[
M(\lambda) = (A + \lambda B)^*(A + \lambda B) - 1 - |\lambda|^2.
\]
Since $\|A + \lambda B\|^2 = 1 + |\lambda|^2$ for all $\lambda \in \mathbb{C}$, the matrix $M(\lambda)$ is not invertible for any $\lambda \in \mathbb{C}$. Hence $\lambda \mapsto \det(M(\lambda))$ must be the zero function.

The entries of $M(\lambda)$ are quadratic functions of $\lambda$ and $\overline{\lambda}$, so that $\det(M(\lambda))$ can be written as a polynomial in $\lambda$ and $\overline{\lambda}$. Since $M(\lambda)$ is Hermitian for all $\lambda \in \mathbb{C}$, the function $\det(M(\lambda))$ is real-valued. Thus $\det(M(\lambda)) = c + 2\text{Re}(d\lambda) + 2\text{Re}(e\lambda^2) + \cdots$.
f|\lambda|^2 + O(|\lambda|^3)$ with some constants $c,d,e,f \in \mathbb{C}$. All of them have to vanish. However, $c$, $d$, and $f$ are irrelevant for this proof.

In order to compute $e$, it suffices to look at the terms of the form $\text{const} \cdot \lambda^2$ in the expansion of $\det M(\lambda)$. Therefore, the summand $(\lambda B)^*(A + \lambda B) - |\lambda|^2$ in $M(\lambda)$ can be ignored. It remains to compute the second order term of $\det(A^*A - 1 + \lambda A^*B)$. Since the 1,1th entry of $A^*A - 1 + \lambda A^*B$ is zero, all nonzero terms in the determinant expansion involve an off-diagonal term in the first column and one off-diagonal term in the first row. Since $A^*A - 1$ is diagonal, terms involving more than two off-diagonal terms are $o(\lambda^2)$ and can be ignored. Finally, the result is $\det(A^*A - 1 + \lambda A^*B) = c\lambda^2 + o(\lambda^2)$ with

$$
eq - \sum_{k=2}^n \left( a_kb_kb_{1k} \prod_{2 \leq l \leq n, \ l \neq k} (a_l^2 - 1) \right) = \prod_{l=2}^n (a_l^2 - 1) \sum_{k=2}^n \frac{a_kb_kb_{1k}}{1 - a_k^2}.$$

Since $a_l^2 \neq 1$ for $l \geq 2$, the assumption $e = 0$ implies

$$(21) \quad \sum_{k=2}^n \frac{a_kb_kb_{1k}}{1 - a_k^2} = 0.$$

(21) holds for any $B \in S(\mathcal{H})$ with $B \perp A$. But then it must hold for all $B \in \mathcal{H}$. However, an application of Claim 10.9 shows that the constants $b_{k1}, b_{1k}$ can be prescribed arbitrarily. Thus $a_k = 0$ for all $k = 2, \ldots, n$. But that means that $A$ has rank 1.

Now the proof of Theorem 10.7 is almost finished. Pick any $A_0 \in S(\mathcal{H})$ with $\alpha_2(A) < 1$, then $A_0$ has rank 1. The same holds for all $A \in \mathcal{H}$ in a suitable neighborhood of $A_0$ because $\alpha_2$ is continuous. We can assume that $A_0 = \text{diag}(1,0,\ldots,0)$. By Claim 10.9, there is $B_1 \in \mathcal{H}$ with first row zero and first column $(0,1,0,\ldots,0)^t$. Moreover, there is $B_2 \in \mathcal{H}$ with first column zero and first row $(0,1,\ldots,0)$.

The only chance for $A_0 + \lambda B_1$ to have rank 1 is if $(B_1)_{jk} = 0$ for all $(j,k) \neq (2,1)$, and similarly for $B_2$. But then $A_0 + \lambda(B_1 + B_2)$ has rank 2 for all $\lambda \in \mathbb{C}^*$. This contradiction proves Theorem 10.7.

Now we continue the proof of Theorem 10.5. Assume first that $T_0^*A$ has finite dimension $r$. Let $\rho: A(\mathcal{H}) \to M_{r+1}$ be a completely isometric representation. Let $K$ be the intersection of the kernels of all $\rho(A), A \in T_0^*A$, and let $n = \dim K$. Since the multiplication on $T_0^*A$ is trivial, $\text{Ran} \rho(A) \subset K$ for all $A \in T_0^*A$.

By a unitary transformation, we can achieve that $K$ is spanned by the vectors $e_1, \ldots, e_n$. Then all elements of $\rho(T_0^*A)$ are of the form

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

Thus ignoring all but the upper right corner gives a (completely) isometric representation $T_0^*A \to M_{m,n}$ with $m = r+1-n$. By Lemma 10.6, the image is isometric to a Hilbert space of dimension $r = n + m - 1$. By Theorem 10.7, this is impossible unless $m = 1$ or $n = 1$. These two cases correspond to $T_0^*A = T_0^*\text{Shift}_r$ and $T_0^*A = T_0^*\text{Shift}_l$, respectively. This proves Theorem 10.5 in the finite dimensional case.

Now assume that $T_0^*A = B$ is infinite dimensional. Lemma 10.6 implies that $B$ is isometric to a Hilbert space $\mathcal{H}$ because $A$ and hence $B$ is closed. We have to show that $B$ is completely equivalent to $\mathbb{B}(\mathcal{H},\mathcal{H})$ or $\mathbb{B}(\mathcal{H},\mathbb{C}) = \mathbb{B}(\mathcal{H},\mathcal{H})^\perp$.

We already know that every $r$-dimensional quotient of $B$ is completely equivalent to $T_0^*\text{Shift}_r^l$ or $T_0^*\text{Shift}_r^r$. It is easy to see that $T_0^*\text{Shift}_r^r$ is not a quotient of $T_0^*\text{Shift}_r^r$ if $r > 1$, and similarly $T_0^*\text{Shift}_r^r$ cannot be a quotient of $T_0^*\text{Shift}_r^r$ for $r > 1$. Hence
either all quotients are of the form \( T_0^* \text{Shift}_r \), or of the form \( T_0^* \text{Shift}_r^t \), but both types cannot occur at the same time.

Replacing \( \mathcal{B} \) by \( \mathcal{B}' \) if necessary, we can achieve that every \( r \)-dimensional quotient of \( \mathcal{B} \) is completely isometric to \( T_0^* \text{Shift}_r \). Let \( \iota : \mathcal{B} \to \mathcal{H} \to \mathcal{B}(\mathbb{C}, \mathcal{H}) \) be the canonical isometry. We claim that \( \iota \) is completely isometric. This follows if the dual map \( \iota' : \mathcal{B}(\mathbb{C}, \mathcal{H})' \to \mathcal{B}' \) is a complete isometry. Here both dual spaces are equipped with their natural \( L^1 \)-matricially normed structure [11].

If \( X \in \mathcal{B}'(\iota) \), its entries span a finite dimensional subspace \( L \) of \( \mathcal{B}' \), which determines a finite dimensional quotient \( \mathcal{B}/L^+ \) of \( \mathcal{B} \). The map \( \mathcal{B}/L^+ \to \mathcal{B}(\mathbb{C}, \mathcal{H})/\iota(L^+) \) induced by \( \iota \) is completely isometric. Dualizing this shows that \( \iota'(\iota(L)) \) is completely isometric. This implies \( \| \iota'(\iota(X)) \|_{\iota(n)} = \| X \|_{\iota(n)} \), which is what we need.

Thus every finite dimensional operator algebra of minimal complexity that is a trivial unitization is a quotient of \( \text{Shift}_d \) or of \( \text{Shift}_d^t \). Notice that trivial unitizations are automatically indecomposable. In general, a direct sum of quotients of \( \text{Shift}_d \) need not be a quotient of \( \text{Shift}_d^t \) again. However, the following conjecture has a chance to be true:

**Conjecture 10.12.** Let \( A \) be an \( r \)-dimensional, indecomposable, (unital, commutative) operator algebra of minimal quotient complexity. Then there exists a closed, finite codimensional ideal \( I \subset \text{Shift}_{r-1} \) with \( \text{sa}(I) = \{ 0 \} \) such that \( A \cong \text{Shift}_{r-1}/I \) or \( A \cong (\text{Shift}_{r-1}/I)^t \). Moreover, if \( A \cong A^t \), then \( A \) is a quotient of \( \text{Shift}_d \).

The converse of this conjecture is easy. Hence it would provide us with a complete classification of finite dimensional operator algebras of minimal quotient complexity.

**Remark 10.13.** Every unital, commutative subalgebra of \( M_3 \) has minimal quotient complexity. Indeed, it can have at most dimension 3. Its non-trivial quotients have dimensions 1 and 2, so that the assertion follows from the classification of two-dimensional, unital operator algebras.

For subalgebras of \( M_3 \), I have verified Conjecture 10.12 by direct computations. But the proof is to messy to be included here. Moreover, new features arise in dimension 4 because no longer all 4-dimensional, unital, commutative subalgebras of \( M_4 \) have minimal quotient complexity.

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