Green’s function analysis of a triplet superconductor - ferromagnet - triplet superconductor Josephson junction

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Abstract. We present a theoretical analysis of a triplet superconductor - ferromagnet - triplet superconductor (TFT) junction. We describe in detail the construction of the quasiclassical scattering Green’s function and provide a derivation of the general expression for the Josephson current between two triplet superconductors. Specializing to the TFT junction, we compare and contrast the dependence of the Andreev bound state energies and the current upon the scattering at the ferromagnetic barrier, for the two cases of aligned and mis-aligned d-vectors in the triplet superconductors. We also discuss the unconventional temperature-dependence of the current through the junction.

1. Introduction
The discovery of the high-$T_c$ superconductors has sparked intensive research into the construction and properties of tunneling junctions involving superconductors with unconventional pairing symmetries [1]. More recently, attention has been directed towards the inclusion of superconductors with spin-triplet pairing in such devices [2, 3, 4, 5, 6, 7, 8, 9], stimulated by the confirmation of triplet superconductivity in Sr$_2$RuO$_4$ [10]. Because of the triplet state of the Cooper pairs in such systems, the interplay of triplet superconductivity with magnetic phases in a Josephson junction is of special interest [3, 6, 7, 8, 9].

It is well known that bound states form at the surfaces of conventional superconductors or at their interfaces with other materials [11]. This is of particular relevance to theory of ballistic transport through a Josephson junction, as the surface states overlap for sufficiently thin barriers. The tunneling through these so-called Andreev bound states dominates the low-temperature transport properties of the junction [12]. The quasiclassical Green’s function method provides a convenient formalism within which the Andreev bound states and the tunneling current may be calculated [13, 14, 15]. This method has been particularly useful in the study of junctions involving unconventional superconductors [1].

In this paper we use the quasiclassical Green’s function method to investigate the formation of Andreev bound states and the tunneling current through them in a triplet superconductor - ferromagnet - triplet superconductor (TFT) junction [6, 9]. In Sec. 2 we introduce the TFT junction and derive the Green’s function of the junction. We consider the dependence of the bound states and the current on the barrier parameters and the alignment of the d-vectors of each superconductor in Sec. 3. We also show that the current through the junction can reverse direction as a function of temperature.
2. Theory

In this section we outline the theoretical description of the TFT junction. In Sec. 2.1 we present the Hamiltonian model, which is followed in Sec. 2.2 by a pedagogical introduction to the theory of quasiclassical Green’s functions. We then show in Sec. 2.3 how to obtain the Andreev bound states and the tunneling current through the junction from the Green’s function.

![Figure 1. Schematic diagram of the TFT Josephson junction studied in this paper.](image)

2.1. Hamiltonian

The TFT junction that we study is shown schematically in Fig. (1). We assume that the barrier lies along the plane \( z = 0 \); since the system is translationally invariant along the \( x \)- and \( y \)-axes, the system is effectively a 1D problem. The TFT junction is therefore described by the Hamiltonian 

\[
H(z) = \int dz H(z) = \int dz \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} - \mu + U_P(z) \right] \hat{\psi}_\sigma(z) + \sum \sigma \hat{\psi}_\sigma^\dagger(z) \hat{\sigma}_{\alpha\beta} \hat{\psi}_\beta(z)
\]

where the Hamiltonian density is defined

\[
H(z) = \int dz H(z)
\]

\( \hat{\psi}_\sigma(z) \) and \( \hat{\psi}_\sigma^\dagger(z) \) are respectively the fermionic creation and annihilation operators for a particle with spin \( \sigma \) at \( z \). The first term on the RHS of Eq. (1) describes the kinetic energy of the quasiparticles and the charge scattering at the boundary by the potential \( U_P(z) \). The second term gives the scattering by the barrier moment, where \( \hat{\sigma} \) is the vector of Pauli matrices and the moment is parameterized as \( M(z) = (M_\perp \cos(\alpha), M_\perp \sin(\alpha), M_\parallel) \). The last line represents the triplet pairing in the bulk superconductors, specialized for \( p_\perp \)-wave symmetry. The \( \mathbf{d} \)-vectors of the triplet superconductors are assumed to lie within the spin \( x-y \) plane, i.e. \( \mathbf{d}_\nu = (\cos(\theta_\nu), \sin(\theta_\nu), 0) \). The subscript \( \nu \) refers to the side of the junction: for \( z < 0 \) we have \( \nu = L \), whereas for \( z > 0 \) we have \( \nu = R \). Without loss of generality, we take \( \Delta L \) to be oriented along the \( x \)-axis (i.e. \( \theta_L = 0 \)), while \( \Delta R \) is inclined to this by the angle \( \theta_R = \theta \). The magnitude of the superconducting gap is \( \Delta_0 \), and \( \varphi_\nu = 0(\varphi) \) for \( \nu = L(R) \). We assume that \( \Delta_0 \) is uniform within each superconducting slab; although the order parameter should be suppressed near the barrier by the proximity effect, it recovers its bulk value on a length scale much shorter than the decay length of the Andreev bound states.

2.2. Quasiclassical Green’s Functions

The quasiclassical theory is based upon the use of scattering wavefunctions to construct the retarded Green’s function of the junction [1, 13, 14, 15]. In general there are eight distinct such wavefunctions: \( \Psi_{1\sigma}, \Psi_{2\sigma}, \Psi_{3\sigma}, \Psi_{4\sigma} \). These wavefunctions describe

- \( \Psi_{1\sigma} \): scattering of a spin-\( \sigma \) electron-like quasiparticle incident from the left

...
\( \Psi_{2\sigma} \): scattering of a spin-\( \sigma \) hole-like quasiparticle incident from the left
\( \Psi_{3\sigma} \): scattering of a spin-\( \sigma \) electron-like quasiparticle incident from the right
\( \Psi_{4\sigma} \): scattering of a spin-\( \sigma \) hole-like quasiparticle incident from the right

The wavefunctions contain contributions from the incident and the scattered waves. For example, the wavefunction describing the scattering of a spin-\( \uparrow \) electron-like quasiparticle incident from the left is

\[
\Psi_{1\uparrow}(z) = \{a_{1\uparrow} \psi_{Lh-\uparrow} + a_{1\uparrow} \psi_{Lh-\downarrow} + b_{1\uparrow} \psi_{Le-\uparrow} + b_{1\uparrow} \psi_{Le-\downarrow}\} \Theta(-z) \\
+ \{c_{1\uparrow} \psi_{Re+\uparrow} + c_{1\uparrow} \psi_{Re+\downarrow} + d_{1\uparrow} \psi_{Rh+\uparrow} + d_{1\uparrow} \psi_{Rh+\downarrow}\} \Theta(z)
\]

(2)

The wavefunctions \( \psi_{\nu,\mu,\pm,\sigma} \) describe a \( \mu = e, h \)-like quasiparticle with spin-\( \sigma \) traveling in the \pm z-direction in the \( \nu = L, R \)-superconductor. The coefficients of the \( \psi_{\nu,\mu,\pm,\sigma} \) are the reflection and transmission coefficients, which are determined by solving the Schrödinger equation. The first line describes the wavefunction in the superconductor to the left of the boundary: the incident right-moving electron-like quasiparticle \( \psi_{Le-\uparrow} \) is reflected as a left-moving hole-like quasiparticle \( \psi_{Lh-\downarrow} \) or a left-moving electron-like quasiparticle \( \psi_{Le-\downarrow} \). On the right-hand side of the barrier [bottom line of Eq. (2)] we have the transmitted wavefunction: right-moving electron-like quasiparticles \( \psi_{Re-\downarrow} \) and right-moving hole-like quasiparticles \( \psi_{Rh-\downarrow} \). We illustrate this situation schematically in Fig. (2). When \( M_{\perp} \neq 0 \) there is spin-flipping at the barrier, and so we allow reflected and transmitted quasiparticles of any spin orientation.

We have explicit expressions for the wavefunctions \( \psi_{\nu,\mu,\pm,\sigma} \):

\[
\psi_{\nu,\mu,\pm,\sigma} = \Phi_{\nu,\mu,\pm,\sigma} e^{\frac{i\mu q}{k_F} z}
\]

(3)

where \( \mu = +1(-1) \) for \( \mu = e(h) \), \( q = \frac{m}{h k_F} \Omega \), and \( \Omega = \sqrt{E^2 - k_F^2 \Delta^2} \) where \( E \) is the energy, \( m \) is the effective mass, and \( k_F \) is the Fermi wavevector. The basis vectors are defined

\[
\Phi_{\nu,e,\pm,\uparrow} = (u, v, 0, 0)^T
\]

(4)

\[
\Phi_{\nu,h,\pm,\uparrow} = (v, e^{-i(k_F \cdot \theta)} u, 0, 0)^T
\]

(5)

\[
\Phi_{\nu,e,\pm,\downarrow} = (0, 0, u, e^{i(k_F \cdot \theta)} v)^T
\]

(6)

\[
\Phi_{\nu,h,\pm,\downarrow} = (0, 0, v, e^{-i(k_F \cdot \theta)} u)^T
\]

(7)

where

\[
u = \sqrt{\frac{E + \Omega}{2E}}, \quad u = \sqrt{\frac{E - \Omega}{2E}}.
\]

(8)

The 4\times4 matrix retarded Green’s function is constructed from a linear combination of outer products of the wavefunctions \( \Psi_{\nu,\sigma} \) with their conjugates \( \overline{\Psi}_{\nu,\sigma} \). We have the general expression for \( z < z' < 0 \)

\[
G^r(z, z'; E) = \sum_{\sigma, \sigma'} \left\{ \alpha_{1,\sigma,\sigma'} \overline{\Psi}_{3,\sigma}(z) \overline{\Psi}_{1,\sigma'}^T(z') + \alpha_{2,\sigma,\sigma'} \overline{\Psi}_{3,\sigma}(z) \overline{\Psi}_{2,\sigma'}^T(z') + \alpha_{3,\sigma,\sigma'} \overline{\Psi}_{4,\sigma}(z) \overline{\Psi}_{1,\sigma'}^T(z') \\
+ \alpha_{4,\sigma,\sigma'} \overline{\Psi}_{4,\sigma}(z) \overline{\Psi}_{2,\sigma'}^T(z') \right\}
\]

(9)

and for \( z' < z < 0 \)
\[ G^r(z, z'; E) = \sum_{\sigma, \sigma'} \left\{ \beta_{1, \sigma, \sigma'} \Psi_{1, \sigma}(z) \bar{\Psi}_{3, \sigma'}^T(z') + \beta_{2, \sigma, \sigma'} \Psi_{1, \sigma}(z) \bar{\Psi}_{4, \sigma'}^T(z') + \beta_{3, \sigma, \sigma'} \Psi_{2, \sigma}(z) \bar{\Psi}_{3, \sigma'}^T(z') \\
+ \beta_{4, \sigma, \sigma'} \Psi_{2, \sigma}(z) \bar{\Psi}_{4, \sigma'}^T(z') \right\} \] (10)

Figure 2. Schematic representation of the wavefunction \( \Psi_{1\uparrow} \) showing the reflected and transmitted trajectories with their weighting coefficients. Solid lines indicate electron-like quasiparticles (elq) while broken lines show hole-like quasiparticles (hlq). The details of the scattering region (i.e. the “barrier”) are not important, with all scattering information contained in the coefficients.

In Eq. (9) and Eq. (10) the wavefunction \( \bar{\Psi}_{a, \sigma} \) is obtained from \( \Psi_{a, \sigma} \) by making the replacements \( \phi_{\nu} \rightarrow -\phi_{\nu}, \theta_{\nu} \rightarrow -\theta_{\nu} \) and \( \Delta_{\nu} \rightarrow -\Delta_{\nu} \). The \( c \)-number coefficients \( \alpha_{n, \sigma, \sigma'} \) and \( \beta_{n, \sigma, \sigma'} \) are deduced from two conditions: continuity of the Green’s function

\[ G^r(z, z + 0^+; E) = G^r(z, z - 0^+; E) \] (11)

and the discontinuity of the first derivative of the Green’s function

\[ \frac{\partial}{\partial z} G^r(z, z'; E)|_{z=0^+} - \frac{\partial}{\partial z} G^r(z, z'; E)|_{z=-0^+} = \frac{2m}{\hbar^2} \begin{pmatrix} r_3 & 0 \\ 0 & r_3 \end{pmatrix} \] (12)

The last condition arises from integrating the usual equation of motion obeyed by the Green’s function [16]. Suppressing details of the lengthy computation for the Green’s function, we obtain the Green’s function for \( z' < z < 0 \)

\[ G^r(z, z'; E) = -\frac{im}{\hbar^2 k_F} \frac{E}{\Omega} \left\{ e^{i(k_F+q)(z-z')} \Phi_{L, e, -\sigma} \bar{\Phi}_{L, e, +\sigma}^T + e^{-i(k_F-q)(z-z')} \Phi_{L, h, +\sigma} \bar{\Phi}_{L, h, -\sigma}^T + \Phi_{L, e, -\sigma} e^{i(k_F+q)(z+z')} \Phi_{L, e, +\sigma} + a_1 \Phi_{L, h, +\sigma} e^{i(k_F-q)(z+z')} \Phi_{L, h, -\sigma}^T + a_2 \Phi_{L, h, +\sigma} e^{-i(k_F+q)(z-z')} \Phi_{L, h, -\sigma}^T \right\} \] (13)
The expression for $z < z' < 0$ is of the same general form, however only Eq. (13) is necessary to calculate the transport properties of the junction.

2.3. The Josephson Current

The Josephson current through the junction is derived by an equation of motion argument. We define the charge density operator

$$\hat{\rho}(z) = e \sum_{\sigma} \hat{\psi}^{\dagger}_{\sigma}(z) \hat{\psi}_{\sigma}(z)$$

(14)

Within the bulk superconductors $\hat{\rho}(z)$ obeys the continuity equation

$$\frac{\partial}{\partial t} \hat{\rho}(z) = -\frac{i e \hbar}{2m} \sum_{\sigma} \left\{ \frac{\partial^2}{\partial z^2} \hat{\psi}^{\dagger}_{\sigma}(z) \hat{\psi}_{\sigma}(z) - \frac{\partial^2}{\partial z^2} \hat{\psi}^{\dagger}_{\sigma}(z) \hat{\psi}_{\sigma}(z) \right\}$$

$$+ \frac{\epsilon \Delta_0}{\hbar} \sum_{\sigma} \left\{ e^{-i(\phi_{v} - \sigma \theta_{v})} \frac{\partial}{\partial z} \hat{\psi}_{\sigma}(z) \hat{\psi}^{\dagger}_{\sigma}(z) + e^{i(\phi_{v} - \sigma \theta_{v})} \frac{\partial}{\partial z} \hat{\psi}^{\dagger}_{\sigma}(z) \hat{\psi}_{\sigma}(z) \right\}$$

$$- \frac{\partial}{\partial z} \hat{I}(z) = - \frac{\partial}{\partial z} \left( \hat{I}_e(z) + \hat{I}_s(z) \right)$$

(15)

The last line defines the total current operator $\hat{I}(z)$. This can be decomposed into two distinct contributions: a term giving the usual electronic current operator

$$\hat{I}_e(z) = -\frac{i e \hbar}{2m} \sum_{\sigma} \left\{ \hat{\psi}^{\dagger}_{\sigma}(z) \frac{\partial}{\partial z} \hat{\psi}_{\sigma}(z) - \frac{\partial}{\partial z} \hat{\psi}^{\dagger}_{\sigma}(z) \hat{\psi}_{\sigma}(z) \right\}$$

(16)

and also a “source term”

$$\hat{I}_s(z) = \sum_{\sigma} \frac{\epsilon \Delta_0}{\hbar} \int_0^z dy' \left\{ e^{-i(\phi_{v} - \sigma \theta_{v})} \frac{\partial}{\partial y} \hat{\psi}_{\sigma}(y) \hat{\psi}^{\dagger}_{\sigma}(y) + e^{i(\phi_{v} - \sigma \theta_{v})} \frac{\partial}{\partial y} \hat{\psi}^{\dagger}_{\sigma}(y) \hat{\psi}_{\sigma}(y) \right\}$$

(17)

The thermal average of $\hat{I}_s(z)$ is vanishing if $\Delta_0 = \Delta_0(z)$ is self-consistently determined. In the case considered here, however, self-consistency is not satisfied and so this term gives a non-zero contribution to the current. The source term is interpreted to be the contribution to the current from Cooper pairs which are not included in $\hat{I}_s(z)$ [15].

To obtain the current flowing through the junction, we must perform the thermal average of the current operator $I_J(z) = \langle \hat{I}(z) \rangle$. The thermal average can be expressed in terms of the temperature matrix Green’s function, defined

$$G(z, \tau; z', \tau') = -\left\langle T_{\tau} \left[ \hat{\Psi}(z, \tau), \hat{\Psi}^{\dagger}(z', \tau') \right] \right\rangle$$

(18)

where

$$\hat{\Psi}(z, \tau) = \left( \hat{\psi}_1(z, \tau) \hat{\psi}_{1}^{\dagger}(z, \tau) \hat{\psi}_1(z, \tau) \hat{\psi}_{1}^{\dagger}(z, \tau) \right)^T$$

By standard manipulations, we find for $L(z) = \langle \hat{I}_e(z) \rangle$
Since the temperature Green’s function can be found from the retarded Green’s function by the analytic continuation $E \mapsto i\omega_n$, substituting Eq. (13) into Eq. (19) and performing the trace we obtain the electronic current for $z < 0$

$$I_e(z) = -\frac{e^2}{\hbar} \lim_{z' \to z} \frac{1}{\beta \hbar} \sum_n \left( \frac{\partial}{\partial z'} - \frac{\partial}{\partial z} \right) \text{Tr} \left\{ G(z, z'; i\omega_n) \right\} e^{i\omega_nt}$$

where $\tilde{\Delta}_n = \sqrt{\omega_n^2 + k^2_F \Delta_0^2}$ and $\tilde{q}_n = im\tilde{Q}_n / \hbar^2 k_F$.

We now examine the contribution from the source term. This may be expressed in terms of off-diagonal components of the matrix Green’s function:

$$I_a(z) = -\frac{e^2}{\hbar} \int_0^z dy' \lim_{y' \to y} \frac{1}{\beta \hbar} \sum_n \left\{ e^{-i(\phi_n - \theta_n)} \frac{\partial}{\partial y} G_{12}(y, y'; i\omega_n) + e^{i(\phi_n - \theta_n)} \frac{\partial}{\partial y} G_{21}(y, y'; i\omega_n) ight\} e^{i\omega_nt}$$

where

$$G_{12}(y, \tau; y', \tau') = -\left\langle T_\tau [\hat{\psi}_1(y, \tau), \hat{\psi}_1^\dagger(y', \tau')] \right\rangle$$

$$G_{21}(y, \tau; y', \tau') = -\left\langle T_\tau [\hat{\psi}_1^\dagger(y, \tau), \hat{\psi}_1(y', \tau')] \right\rangle$$

$$G_{34}(y, \tau; y', \tau') = -\left\langle T_\tau [\hat{\psi}_1(y, \tau), \hat{\psi}_1(y', \tau')] \right\rangle$$

$$G_{43}(y, \tau; y', \tau') = -\left\langle T_\tau [\hat{\psi}_1^\dagger(y, \tau), \hat{\psi}_1^\dagger(y', \tau')] \right\rangle$$

are the usual anomalous Green’s functions in a triplet superconductor. Inserting the expressions for these Green’s functions into Eq. (21) and performing the integration, we have

$$I_a(z) = -\frac{e^2}{\hbar} \lim_{y' \to y} \frac{1}{\beta \hbar} \sum_n \left\{ a_{11}(i\omega_n) + a_{11}(i\omega_n) - a_{21}(i\omega_n) - a_{21}(i\omega_n) \right\} (1 - e^{-2\tilde{q}_n z}) e^{i\omega_nt}$$

Adding together Eq. (20) and Eq. (26), we obtain the generalization of the Furusaki-Tsukuda formula for the Josephson current between triplet superconductors

$$I_J = \frac{e^2}{\hbar} \lim_{y' \to y} \frac{1}{\beta \hbar} \sum_n \left\{ a_{11}(i\omega_n) + a_{11}(i\omega_n) - a_{21}(i\omega_n) - a_{21}(i\omega_n) \right\} e^{i\omega_nt}$$

Unlike $\hat{I}_J(z) > \hat{I}_J(z)$, this expression is independent of $z$ as is required for a bulk current. The formula for $I_J$ has a straightforward physical interpretation as given in Ref. [15]. The coefficient $a_{1\sigma\sigma}$ describes the reflection of a spin-$\sigma$ electron-like quasiparticle as a spin-$\sigma$ hole-like quasiparticle [see Fig. (2)], whereas $a_{2\sigma\sigma}$ describes the reflection of a spin-$\sigma$ hole-like quasiparticle as a spin-$\sigma$ electron-like quasiparticle. These processes respectively correspond to the tunneling of a spin-2$\sigma$ Cooper pair from the left to the right superconductor, and the tunneling of a spin-2$\sigma$ Cooper pair from the right to the left superconductor. It is therefore natural that the difference $a_{1\sigma\sigma} - a_{2\sigma\sigma}$ appears in the expression for $I_J$.

By solving the Schrödinger equation for the different wavefunctions $\Psi_{\sigma\sigma}$, we obtain explicit
expressions for the reflection coefficients. These are very lengthy, and so we only state their sum as it appears in Eq. (27):

\[
\alpha_{1\parallel}(i\omega_n) + \alpha_{1\parallel}(i\omega_n) - \alpha_{2\parallel}(i\omega_n) - \alpha_{2\parallel}(i\omega_n) =
\]

\[-8k_F^2\Delta_0\tilde{\Omega}_n \left(-4\omega_n^2g'Z\cos\phi\sin\theta + \left[-2\omega_n^2g^2\cos(2\alpha - \theta) + (k_F^2\Delta_0^2 + 2\omega_n^2g^2 + Z^2 + 1)\cos\theta + k_F^2\Delta_0^2\cos\phi\right]\sin\phi\right) / \Gamma
\]

(28)

where

\[
\Gamma = 8k_F^2\Delta_0^2 \left\{\frac{1}{D^2} + \frac{\omega_n^4}{k_F^2\Delta_0^4} + 4A\frac{\omega_n^2}{k_F^2\Delta_0^2} + 4B^2\right\}
\]

(29)

\[
D = \left[(1 + g^2 + g'^2 + Z^2)^2 + 4Z^2\right]^{-1/2}
\]

(30)

\[
A = \frac{1}{4} \left[(1 + 2g^2 + g'^2 + Z^2) + (1 + g^2 + Z^2)\cos\phi\cos\theta + g^2\cos(\theta - 2\alpha)\{\cos\theta - \cos\phi\} + 2g'Z\sin\theta\sin\phi\right]
\]

(31)

\[
B = \frac{1}{2}\cos\left(\phi + \frac{\theta}{2}\right)\cos\left(\phi - \frac{\theta}{2}\right)
\]

(32)

The dimensionless parameters \(g, g'\) and \(Z\) are respectively proportional to the magnitude of the magnetic moment transverse to the barrier \(M\), the magnetic moment normal to the barrier \(M_{||}\), and the strength of the charge scattering \(U_P\):

\[
g = \frac{mM_{\perp}}{\hbar^2k_F}, \quad g' = \frac{mM_{||}}{\hbar^2k_F}, \quad Z = \frac{mU_P}{\hbar^2k_F}
\]

(33)

We find it convenient to discuss the scattering at the barrier in terms of these variables.

The condition \(\Gamma = 0\) defines the Andreev bound state energies. We find two distinct solutions, with positive branch

\[
E_{a(b)} = k_F\Delta_0\sqrt{D} \left|\sqrt{DA - B} + (-\sqrt{DA + B})\right|
\]

(34)

The negative-energy states \(-E_{a(b)}\) are also solutions. The tunneling through these states dominates the current, reflected here by the fact that the Matsubara sum is equivalent to a sum of residues at these points. After standard manipulations, we then obtain the explicit expression for the current

\[
I_J = -e^2k_F^2\Delta_0^2D^2 \left\{\left[4g'Z\sin\theta\cos\phi + \left\{2g^2\cos(\theta - 2\alpha) - 2(1 + g'^2 + Z^2)\cos\theta\right\}\sin\phi\right]E_a^2
\]

+ \left[k_F^2\Delta_0^2\cos\theta + \cos\phi\sin\phi\right]\tan(\beta E_a/2)/[2E_a(E_a^2 - E_b^2)]
\]

\[-\left[4g'Z\sin\theta\cos\phi + \left\{2g^2\cos(\theta - 2\alpha) - 2(1 + g^2 + Z^2)\cos\theta\right\}\sin\phi\right]E_b^2
\]

+ \left[k_F^2\Delta_0^2\cos\theta + \cos\phi\sin\phi\right]\tan(\beta E_b/2)/[2E_b(E_a^2 - E_b^2)]\}
\]

(35)

The first two lines inside the braces give the contribution to the current \(I_j^a\) through the \(a\) states, while the last two lines give the contribution to the current \(I_j^b\) through the \(b\) states. Note the dependence of these contributions on the occupancy of the states.
Figure 3. $\phi$-dependence of (a) the Andreev bound states and (b) the corresponding Josephson current for $g = 1.0$ and $\alpha = \pi/4$. In (c) we show the current as a function of $\nu$, where $\nu$ is either $Z$ or $g'$. We take $\phi = 2\pi/5$ and $\alpha = \pi/4$. The dependence of the current on $\alpha$ is shown in (d), for $Z = g' = 0$. In all plots we have $\theta = 0$.

3. Results
In this section we present a selection of results for the Andreev bound state energies and the current through them, highlighting the dependence upon the orientation of the barrier magnetic moment. In Sec. 3.1 we examine the case of aligned $d$-vectors, while the effect of $d$-vector mis-alignment is discussed in Sec. 3.2. The temperature-dependence of the current is presented in Sec. 3.3. All results in Sec. 3.1 and Sec. 3.2 are for $T = 0$. A more complete discussion of the properties of the junction can be found in Ref. [6] and Ref. [9].

3.1. Aligned $d$-vectors
We first consider the case when the $d$-vectors of the two superconductors are aligned. The Andreev bound state energies and the associated Josephson current in this situation are shown in Fig. (3)(a) and Fig. (3)(b) respectively as a function of the phase $\phi$ for fixed $g = 1$. For $Z = g' = 0$ [solid black line], the bound states and the current are strongly dependent upon the alignment of the magnetic moment with the $x$-axis: for $\alpha \neq (2n + 1)\pi/2$, the Andreev bound states are non-degenerate except at $\phi = 2n\pi$, while the current is a continuous function of $\phi$. At $\alpha = (2n + 1)\pi/2$, the states are degenerate and have cosine dependence $E_{a,b} = k_F\Delta_0 \sqrt{D} \cos(\phi/2)$, while the current is given by $I_J = e k_F \Delta_0 \sqrt{D} \sgn(\cos(\phi/2)) \sin(\phi/2)/2$ and has jump discontinuities at $\phi = (2n + 1)\pi$ (not shown).

The effect of a finite $g'$ [dot-dash blue line] is mainly to reduce the amplitude of the Andreev bound state energies; there is very little change in the current. In contrast, including a finite potential barrier $Z$ when $g' = 0$ has a very significant effect upon the $\phi$-dependence of the Andreev states. For $\alpha \neq (2n + 1)\pi/2$ it lifts the degeneracy of the states at $\phi = 2n\pi$ and strongly modifies the $\phi$-dependence of $E_a$. In the vicinity of these points, interestingly, the Josephson current displays a moderate enhancement above the $Z = 0$ values for all $\phi$, as can be seen in Fig. (3)(b). For the case $\alpha = (2n + 1)\pi/2$ (not shown), the Andreev bound states $E_{a,b} \propto \cos(\phi/2)$, but the amplitude of the $a$ and the $b$ states...
is different. The jump discontinuities in the current remain and there is also a slight enhancement of $|I_J|$. The dependence the current on $g'$ and $Z$ is presented in Fig. (3)(c), where we plot $I_J$ at fixed $\phi = 2\pi/5$ as a function of $\nu = Z, g'$. Note that if $\nu = Z$, then $g' = 0$ and vice versa. When $g = 0$, the dependence of the Josephson current upon each parameter is identical. For $g \neq 0$, however, the dependence upon $g'$ is very different from that upon $Z$: in the former case, increasing $g'$ at fixed $g = 1.5$ we find a very weak maximum at $g' = 0.8$; as $Z$ is varied, however, there is a strong maximum at $Z \approx 1.8$, with a $\sim 40\%$ increase in the current over the $Z = 0$ value.

An important feature of our results is that the Andreev bound state energies and the current depend sensitively upon the orientation of the barrier magnetic moment. For example, as shown in Fig. (3)(d) for fixed $\phi$ the current displays a periodic dependence upon the angle $\alpha$ between the component of the magnetization in the $x$-$y$ plane and the $x$-axis. For $\phi$ close to $(2n+1)\pi$, the current is strongly peaked about $\alpha = (2m+1)\pi/2$, and there is the basis of a “switch” effect: small changes in $\alpha$ can produce very large changes in the current. The dependence of the current (and Andreev bound states) upon the orientation of $M$ is a significant point of difference with singlet superconductor-ferromagnet-singlet superconductor junctions, where only the magnitude of the moment is important [17].

3.2. Mis-aligned $d$-vectors
When the $d$-vectors are mis-aligned (i.e. $\theta \neq n\pi$) and $g \neq 0$, the Andreev bound states are non-degenerate for all $\phi$ [solid black line in Fig. (4)(a)]. The positive and negative branches of the $b$ state nevertheless have a level crossing at $\phi = (2n+1)\pi \pm \theta$. As at $T = 0$ only the negative energy branch contributes to the current, the discontinuous change in the $\phi$-dependence of $-|E_b|$ at these points results in jump discontinuities in the Josephson current through the junction [Fig. (4)(b)]. These jump discontinuities remain in the presence of potential scattering at the barrier or a component of the
magnetization along the $z$-axis. It is interesting to note that for $(2n-1)\pi + \theta < \phi < (2n+1)\pi + \theta$ the addition of the extra scattering potentials enhances the Josephson current, whereas for $(2n+1)\pi - \theta < \phi < (2n+1)\pi + \theta$ the extra terms at the barrier lead to a suppression of $I_J$.

For misaligned $d$-vectors and $g'$, $Z \neq 0$, the current vs phase relationship shows a remarkable feature: a finite current flows even when $\phi = 0$ [red dash and blue dot-dash lines in Fig. (4)(d)]. This exotic Josephson effect originates from a $\cos(\phi)$-dependence of $I_J$: examining Eq. (35), we see that this is only present if $g'Z \sin(\theta) \neq 0$. Understanding of this effect is best achieved when $g = 0$: because of the absence of spin-flip scattering at the barrier, spin is a good quantum number. We find that when $\theta \neq 0$, the effective phase difference seen in each spin channel is $spin-dependent$ and given by $\phi_\sigma = \phi - \sigma\theta$. This is clearly evidenced by the $Z = g = 0$ Andreev bound states in Fig. (4)(c), which are given by $E_a = k_\sigma \Delta_0 \cos(\phi_\sigma)/\sqrt{1 + (g')^2}$. Thus, at $\phi = 0$, the effective phase difference in each spin channel is $-\sigma\theta$. Although a finite current flows in each spin sector the net current flowing through the junction is vanishing, because the effective potential scattering at the barrier is equal in both spin channels. When we have $g', Z \neq 0$, however, the effective scattering potential is also spin-dependent and given by $Z - \sigma g'$. This more strongly suppress the current in the spin-\(\downarrow\) sector than the current in the spin-\(\uparrow\) sector, hence leading to a net current at $\phi = 0$. This effect persists in the presence of a finite $g$ [blue dot-dash lines in Fig. (4)(c) and (d)], but the interpretation is more complicated as the spin sectors are then not disconnected.

![Figure 5](image.png)

**Figure 5.** (a) Temperature dependence of the Josephson current for various values of $g$ with $\alpha = 0$ fixed. (b) Temperature dependence of the Josephson current for various values of $\alpha$ with $g = 2$ fixed. In both panels, we take $\phi = \pi/2$, $\theta = 0$ and $g' = Z = 0$.

### 3.3. Temperature Dependence

Assuming a BCS temperature dependence of $\Delta_0$, in Fig. (5) we plot $I_J$ as a function of $T/T_c$ for fixed $\phi = 0.5\pi$, $\theta = 0$ and $g' = Z = 0$. In panel (a) we see that the current reverses sign with increasing temperature for sufficiently large $g$, while panel (b) indicates that this effect is only encountered for small $\alpha$. The origin of the current reversal is linked to the structure of the Andreev bound states. In general we have $|E_a| > |E_b|$, and so as the temperature is raised from zero $I_J^a$ decreases more slowly than the $I_J^b$. If at zero temperature the currents have opposite sign and $|I_J^a| < |I_J^b|$, at some finite temperature the faster decrease of $|I_J^a|$ reduces it below $|I_J^b|$ and the total current hence reverses direction. Clearly, this is not possible when the Andreev bound states are degenerate, explaining the absence of the current reversal when $\alpha$ is too close to $(2n+1)\pi/2$. The origin of this behaviour is very different to the temperature-reversal of the current in a singlet-ferromagnet-singlet junction [18]: in our case, it is due to changes in the occupation of the Andreev energy states, whereas in Ref. [18] it stems from the temperature-dependence of the decay and oscillation lengths of the superconducting order parameter within the barrier.
4. Conclusion
In conclusion, we have studied an unconventional TFT Josephson junction using the quasiclassical Green’s function formalism. The dependence of the Andreev bound states and the current on the scattering potentials has been obtained for the two cases of aligned and mis-aligned d-vectors. The Josephson current has also been shown to display a temperature-dependent sign reversal. The orientation of the barrier magnetic moment plays a crucial role in controlling these results, a situation which is unique to triplet superconductors. The prospects for further work are promising, as the Green’s function method can be used to study the spin transport properties of the TFT junction: unlike junctions constructed from singlet superconductors, junctions involving triplet superconductors also allow for the appearance of spontaneous spin currents [19]. Like the charge current, we expect the spin current to have a rich dependence upon the junction parameters.

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