THE DIXMIER PROBLEM, LAMPLIGHTERS AND BURNSIDE GROUPS

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ABSTRACT. J. Dixmier asked in 1950 whether every non-amenable group admits uniformly bounded representations that cannot be unitarisable. We provide such representations upon passing to extensions by abelian groups. This gives a new characterisation of amenability. Furthermore, we deduce that certain Burnside groups are non-unitarisable, answering a question raised by G. Pisier.

1. Introduction

A group $G$ is said to be unitarisable if every uniformly bounded representation $\pi$ of $G$ on a Hilbert space $H$ is unitarisable, i.e. there is an invertible operator $S$ on $H$ such that $S\pi(\cdot)S^{-1}$ is a unitary representation. Dixmier [Dix50] proved that all amenable groups are unitarisable and asked whether unitarisability characterises amenability. Since unitarisability passes to subgroups and non-commutative free groups are not unitarisable, every group containing a non-commutative free group is non-unitarisable. For these facts and more background, we refer to Pisier [Pis01, Pis05].

Recently, a criterion was discovered [EMxx] that lead to examples without free subgroups (see [Osixx, EMxx]). We shall improve a strategy proposed in [Mon06] in order to apply ergodic methods to the problem.

Let $G$ and $A$ be groups. Recall that the associated (restricted) wreath product, or lamplighter group, is the group

$$A \wr G = \bigoplus_G A \rtimes G,$$

wherein $\bigoplus_G A$ is the restricted product indexed by $G$ upon which $G$ acts by permutation. We shall be interested in the case where $A$ and hence also $\bigoplus_G A$ is abelian.

Theorem 1. For any group $G$, the following assertions are equivalent.

(i) The group $G$ is amenable.

(ii) The wreath product $A \wr G$ is unitarisable for all abelian groups $A$.

(iii) The wreath product $A \wr G$ is unitarisable for some infinite abelian group $A$.

The above theorem leads to a partial answer to a question of G. Pisier, namely whether free Burnside groups are unitarisable (see e.g. [Pis05]).

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1Shakespeare, Richard III, 1:1 (we quote from the 1623 First Folio).
Theorem 2. Let \( m, n, p \) be integers with \( m, n \geq 2, p \geq 665 \) and \( n, p \) odd. Then the free Burnside group \( B(m, np) \) of exponent \( np \) with \( m \) generators is non-unitarisable.

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2. Proofs

Let \( G \) be a group and \((\pi, \mathcal{H})\) be a unitary representation of \( G \). We write \( \mathcal{L}(\mathcal{H}) \) for the algebra of bounded operators of \( \mathcal{H} \). A map \( D: G \to \mathcal{L}(\mathcal{H}) \) is called a derivation if it satisfies the Leibniz rule \( D(gh) = D(g)\pi(h) + \pi(g)D(h) \), or equivalently if the map \( \pi_D \) defined by

\[
\pi_D(g) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})
\]

is a group homomorphism. In that case, \( \pi_D \) is a uniformly bounded representation if and only if \( D \) is a bounded derivation. Moreover, \( \pi_D \) is unitarisable if and only if \( D \) is inner, i.e. there is \( T \in \mathcal{L}(\mathcal{H}) \) such that \( D(g) = \pi(g)T - T\pi(g) \). (See Lemma 4.5 in [Pis01] for a proof of this fact.) To set up a cohomological framework for studying this problem, we will view \( \mathcal{L}(\mathcal{H}) \) as a coefficient \( G \)-module whose \( G \)-action is given by the conjugation \( g \cdot T = \pi(g)T\pi(g)^* \). Then, the space of bounded derivations modulo inner derivations is canonically isomorphic to the first bounded cohomology \( H^1_b(G, \mathcal{L}(\mathcal{H})) \). Hence, to prove non-unitarisability of \( G \), it suffices to produce a unitary \( G \)-representation \((\pi, \mathcal{H})\) for which \( H^1_b(G, \mathcal{L}(\mathcal{H})) \neq 0 \).

We now undertake the proof of Theorem 2. It suffices to show that if \( A \) is infinite abelian and \( G \) is non-amenable, then the wreath product \( H = A \wr G \) is non-unitarisable.

We can and shall assume that \( A \) and \( G \) are countable. Indeed, since amenability is preserved under direct limits, \( G \) contains some countable non-amenable group \( G_0 \). Further, \( A \) contains an infinite countable \( G_0 \)-invariant subgroup \( A_0 \) and \( A_0 \wr G_0 \) is a subgroup of \( A \wr G \). Thus our claim follows since unitarisability passes to subgroups.

Let \( F \) be a countable non-commutative free group. The proof relies on the following two facts. (1) \( H^1_b(F, \mathcal{L}(\ell_2 F)) \neq 0 \), see the proof of Theorem 2.7* in [Pis01]. (2) Every non-amenable countable group admits a free type \( II_1 \) action whose orbits contain the orbits of a free \( F \)-action \((\mathcal{L}(\mathcal{K}), \mathcal{L}(\ell_2 F))\), as described below. The strategy of the proof is to induce \( H^1_b(F, \mathcal{L}(\ell_2 F)) \) through this “randomembding” in the sense of [Mon06].

We henceforth consider a non-amenable countable group \( G \) and the corresponding Bernoulli shift action on the compact metrisable product space \( X = [0, 1]^G \) endowed with the product of the Lebesgue measures. Gaboriau and Lyons prove in [GLxx] that the resulting equivalence relation \( \mathcal{R} \subseteq X \times X \) contains the equivalence relation of some free measure-preserving \( F \)-action upon \( X \). In particular, we have commuting \( G \)- and \( F \)-actions on \( \mathcal{R} \) given by the action on the first, respectively the second coordinate. These actions preserve the \( \sigma \)-finite measure on \( \mathcal{R} \) provided by integrating over \( X \) the counting measure on orbits. Each of these actions admits a fundamental domain; let \( Y \subseteq \mathcal{R} \) be a fundamental domain for \( F \). We may now forget the orbit.
equivalence relation and view $\mathcal{R}$ just as a standard measure space with a measure-preserving $G \times F$-action such that $G$ admits a fundamental domain $X$ of finite measure and $F$ admits a fundamental domain $Y$. We identify $\mathcal{R}$ with $Y \times F$ in such a way that $t^{-1}y \in \mathcal{R}$ corresponds to $(y, t) \in Y \times F$. Then, $s \in F$ acts on $Y \times F$ by $s(y, t) = (y, ts^{-1})$ and $g \in G$ acts by $g(y, t) = (g \cdot y, \alpha(g, y)t)$, where $g \cdot y \in Y$ is the (essentially) unique element in $Fg y \cap Y \subset \mathcal{R}$ and $\alpha(g, y) \in F$ is the (essentially) unique element such that $\alpha(g, y)gy = g \cdot y$. It follows that $\alpha$ satisfies the cocycle relation $\alpha(gh, y) = \alpha(g, h \cdot y)\alpha(h, y)$.

We now consider any countable infinite abelian group $A$. We claim that $A$ has a representation into the unitaries of the von Neumann algebra $L^\infty(Y)$ whose image generates $L^\infty(Y)$ as a von Neumann algebra. By construction, $Y$ is a standard Borel space with a $\sigma$-finite non-atomic measure. Furthermore, as far as the present claim is concerned, we may temporarily assume this measure finite since only its measure class is of relevance. Since $A$ is countably infinite, its Pontryagin dual $\hat{A}$ (for $A$ endowed with the discrete topology) is a non-discrete compact metrisable group. In other words, we have reduced to the case where we may assume that $Y$ is $\hat{A}$ endowed with a Haar measure. Fourier transform establishes an isomorphism between $L^\infty(\hat{A})$ and the group von Neumann algebra $L(A) \subseteq \mathcal{L}(\ell_2 A)$, which is by definition generated by the unitary regular representation of $\hat{A}$; this proves the claim.

Returning to the main argument, we view $A$ in the unitary group of $L^\infty(Y) \cong L^\infty(Y) \otimes C_1 F \subset L^\infty(\mathcal{R})$. Since $A$ and $gAg^{-1} \subset L^\infty(Y)$ commute, this gives rise to a unitary representation of $\mathcal{L}(\mathcal{R})$ on $H = A \triangleleft G$ on $L^2(\mathcal{R})$. We will prove that $H^b_1(H, \mathcal{L}(L^2(\mathcal{R}))) \neq 0$.

We write $N = \bigoplus_G A$. Since $N$ is amenable and $\mathcal{L}(L^2(\mathcal{R}))$ is a dual module, a weak-* averaging argument shows that there is a canonical isomorphism

$$H^b_0(H, \mathcal{L}(L^2(\mathcal{R}))) \cong H^b_0(G, \mathcal{L}(L^2(\mathcal{R})))^N$$

(see Corollary 7.5.10 in [Mon01]). With the identification $\mathcal{R} = Y \times F$, one has

$$\mathcal{L}(L^2(\mathcal{R}))^N = N \cap \mathcal{L}(L^2(\mathcal{R})) = L^\infty(Y) \otimes \mathcal{L}(\ell_2 F) \cong L^\infty(Y, \mathcal{L}(\ell_2 F))$$

(see Theorem IV.5.9 in [Tak02]). Keeping track of the $G$-representation, one sees that $g \in G$ acts on $L^\infty(Y, \mathcal{L}(\ell_2 F))$ by $(g \cdot f)(y) = \tau_{\alpha(g, g^{-1} \cdot y)}(f(g^{-1} \cdot y))$, where $\tau$ denotes the $F$-action on $\mathcal{L}(\ell_2 F)$. For ease of notation, we denote the coefficient $F$-module $\mathcal{L}(\ell_2 F)$ by $V$. Then, one further has a $G$-isomorphism

$$L^\infty(Y, V) \cong L^\infty(\mathcal{R}, V)^F,$$

where $f \in L^\infty(Y, V)$ corresponds to $\tilde{f} \in L^\infty(\mathcal{R}, V)^F$ defined by $\tilde{f}(y, t) = \tau^{-1}_t(f(y))$. Now, $F$ acts on $L^\infty(\mathcal{R}, V)$ by $(s \cdot F)(z) = \tau_s(F(s^{-1}z))$ and $G$ acts by $(g \cdot F)(z) = F(g^{-1}z)$.

Since both the $F$-action and the $G$-action on $\mathcal{R}$ admit a fundamental domain, Proposition 4.6 in [MS06] implies that

$$H^b_0(G, L^\infty(\mathcal{R}, V)^F) \cong H^b_0(F, L^\infty(\mathcal{R}, V)^G) \cong H^b_0(F, L^\infty(X, V)).$$

(See also Proposition 5.8 in [Mon06].) Since $X = \mathcal{R}/G$ has a finite $F$-invariant measure, the inclusion $V \hookrightarrow L^\infty(X, V)$ has a $G$-equivariant left inverse. It follows that the corresponding morphism

$$H^b_0(F, V) \longrightarrow H^b_0(F, L^\infty(X, V))$$
is an injection. Therefore, putting all identifications together, we conclude that there are injections
\[ H^*_b(F, \mathcal{L}(\ell_2 F)) \rightharpoonup H^*_b(H, \mathcal{L}(L^2(\mathcal{A}))) \]
in all degrees. Since \( H^*_b(F, \mathcal{L}(\ell_2 F)) \neq 0 \), this completes the proof. \( \square \)

Analysing the proof at the level of derivations, the above injection maps \( D: F \to \mathcal{L}(\ell_2 F) \) to \( \tilde{D}: H \to \mathcal{L}(L^2(Y, \ell_2 F)) \) defined by
\[ (\tilde{D}(ag)\xi)(y) = a(y)D(\alpha(g, g^{-1} \cdot y))\xi(g^{-1} \cdot y), \]
where \( a \in N \) is viewed as an element of \( L^\infty(Y) \), \( g \in G \) and \( \xi \in L^2(Y, \ell_2 F) \).

\textbf{Proof of Theorem 2.} By a theorem of Adyan [Ady82], the free Burnside group \( G = B(2, p) \) is non-amenable. Therefore, Theorem 1 implies that \( (\bigoplus_N \mathbb{Z}/n\mathbb{Z}) \wr G \) is non-unitarisable. Notice that this wreath product is a countable group of exponent \( np \). Therefore, by the universal property of free Burnside groups, it is a quotient of \( B(\aleph_0, np) \). In particular, the latter is non-unitarisable. It was shown by Širvanjan [Šir76] that \( B(\aleph_0, np) \) embeds into \( B(2, np) \) which is therefore also non-unitarisable. Finally, each \( B(m, np) \) surjects onto \( B(2, np) \) as long as \( m \geq 2 \), concluding the proof. \( \square \)

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