Conformal maps in higher dimensions and derived geometry

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Abstract

By Liouville’s theorem, in dimensions 3 or more conformal transformations form a finite-dimensional group, an apparent drastic departure from the 2-dimensional case. We propose a derived enhancement of the conformal Lie algebra which is an infinite-dimensional dg-Lie algebra incorporating not only symmetries but also deformations of the conformal structure. Our approach is based on (derived) deformation theory of the ambitwistor space of complex null-geodesics.

Introduction

The classical theorem of Liouville (1850) says that the behavior of conformal maps in dimensions \( \geq 3 \), as compared to dimension 2, is drastically different. While in 2 dimensions, holomorphic functions give an infinite-dimensional supply of local conformal maps, any conformal map between connected domains \( U, V \subset \mathbb{R}^n \), \( n \geq 3 \), comes from a global Möbius transformation of the conformal sphere \( S^n \), i.e., from an element of the finite-dimensional group \( O(n+1,1) \). This apparent discontinuity can be rather puzzling.

The goal of this note is to propose a resolution to this apparent puzzle by using the point of view of derived geometry, i.e., of the homological, derived category-style approach to algebraic and differential geometry [5, 11, 17, 24]. More precisely, we recover the missing infinite-dimensional part of the conformal group (it is technically easier to start with the Lie algebra) in a different cohomological degree. For \( n = 2 \) this degree is 0, and we observe infinite-dimensionality at the naive classical level.

For an \( n \)-dimensional complex analytic conformal manifold \( M \) of appropriate type we introduce a differential graded (dg-) Lie algebra \( Rconf(M) \) which is of infinite-dimensional nature regardless of \( n \). Its cohomology includes:

- \( H^0 = conf(M) \), the usual Lie algebra of conformal Killing vector fields (locally, infinite-dimensional for \( n = 2 \), finite-dimensional for \( n > 2 \)).
• $H^1$ being the space of infinitesimal deformations of the conformal structure (locally, zero for $n = 2$, infinite-dimensional for $n > 2$).

Thus, for $M = \mathbb{C}^n$ with flat metric, the total size of $H^\bullet Rconf(M)$ varies “continuously” with $n$. To illustrate this point, we identify $H^\bullet Rconf(\mathbb{C}^n)$ as a representation of $SO(n, \mathbb{C})$ in Theorem 5.3.

Our approach is based on the analogy with behavior of holomorphic functions on $\mathbb{C}^n$ vs. $\mathbb{C}^n \setminus \{0\}$. While passing from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$ increases the supply of holomorphic functions, passing from $\mathbb{C}^n$ to $\mathbb{C}^n \setminus \{0\}$, $n \geq 2$, does not (Hartogs’ theorem). But if we look at the total cohomology $H^\bullet(\mathbb{C}^n \setminus \{0\}, \mathcal{O})$ of the sheaf of holomorphic functions, we find the missing singular parts in the cohomological degree $n - 1$.

To relate Liouville’s theorem with Hartogs-type phenomena, we use the ambitwistor approach to holomorphic conformal geometry [16, 19, 20]. The principal object there is $L(M)$, the space of complex null-geodesics in a holomorphic conformal manifold $M$, with its natural contact structure. Its holomorphic contact geometry completely encodes the holomorphic conformal geometry of $M$. In particular, the space of (holomorphic) conformal Killing fields on $M$ is found as the space of sections

\[(0.1)\quad conf(M) = H^0(L(M), \mathcal{K}),\]

where $\mathcal{K}$ is the sheaf of holomorphic contact vector fields on $L(M)$. Already in the local case ($M$ is a small geodesically convex neighborhood of a point $x_0$), $L(M)$ is a complex manifold which (for $n \geq 3$) is not compact but has compact directions: it contains $(n - 2)$-dimensional complex projective quadrics $L_x$ formed by null-geodesics passing through various points $x \in M$. For manifolds of this type, coherent sheaves such as $\mathcal{K}$, can have finite-dimensional $H^0$ but infinite-dimensional higher cohomology. We define

\[Rconf(M) = R\Gamma(L(M), \mathcal{K}),\]

the dg-Lie algebra of derived global sections of the sheaf of Lie algebras $\mathcal{K}$.

There can be other approaches to defining $Rconf(M)$, for example, in the $C^\infty$, rather than holomorphic case. The holomorphic approach adopted here provides a natural way to arrive at the idea of a derived extension of the conformal algebra. It also leads to a conceptually transparent proof of (the infinitesimal, holomorphic version of) Liouville’s theorem, in the form of finite-dimensionality of the $H^0$-space in (0.1).

Since for $n \geq 3$, the infinite-dimensionality of the dg-Lie algebra $Rconf(M)$ is situated in the odd cohomological degree 1, one can integrate it, in a purely algebraic way, cf. [3, 18], to a derived group $RConf(M)$. The classical truncation’ of $RConf(M)$ is $Conf(M)$, the usual Lie group of (holomorphic) conformal diffeomorphisms, and the whole $RConf(M)$ can be seen as an infinite-dimensional formal derived thickening of $Conf(M)$. Such derived groups, and their analogs for conformal superspaces [4, 19] should be of importance for the study of (super)conformal quantum field theories in dimensions $n \geq 3$. See [22] for a somewhat different recent appearance of derived geometry constructions in that context.
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1 Theorems of Hartogs and Liouville

A. The classical theorems. The classical theorem of Hartogs says:

Theorem 1.1. For \( n \geq 2 \), every holomorphic function on \( \mathbb{C}^n \setminus \{0\} \) extends holomorphically to \( \mathbb{C}^n \).

In the algebro-geometric version, over the base field \( \mathbb{C} \) and over the Zariski topology, we consider the affine space \( \mathbb{A}^n \) (i.e., “\( \mathbb{C}^n \) ‘considered as an algebraic variety’”). The corresponding statement is that for \( n \geq 2 \)

\[
H^0(\mathbb{A}^n \setminus \{0\}, \mathcal{O}) = H^0(\mathbb{A}^n, \mathcal{O}) = \mathbb{C}[z_1, \ldots, z_n],
\]

(no increase of the ring of regular functions), while for \( n = 1 \) we have a manifest increase:

\[
\mathbb{C}[z, z^{-1}] = H^0(\mathbb{A}^1 \setminus \{0\}, \mathcal{O}) \supsetneq H^0(\mathbb{A}^1, \mathcal{O}) = \mathbb{C}[z].
\]

with the new part being \( z^{-1}\mathbb{C}[z^{-1}] \) (space of polar parts of functions with poles at 0).

This phenomenon looks like discontinuity: something seemingly disappears as we pass to higher dimensions.

There is an even more classical result in geometry where the situation changes drastically in passing to higher dimensions: the Liouville theorem. We consider the flat Euclidean space \( \mathbb{R}^n \) and look at conformal transformations between domains \( U, V \) in \( \mathbb{R}^n \). If \( n = 2 \), then locally, we have an infinite-dimensional supply of such transformations, as any holomorphic function on a domain in \( \mathbb{C} = \mathbb{R}^2 \) is conformal. However, the Liouville theorem says:

Theorem 1.4. Let \( n \geq 3 \), let \( U, V \) be connected open domains in \( \mathbb{R}^n \) and \( \varphi : U \to V \) be a conformal diffeomorphism. Then \( \varphi \) extends to a Moebius-type conformal map (composition of rigid motions, dilations and inversions) defined on the entire \( \mathbb{R}^n \) with the possible exception of one point, and belonging to the standard conformal group \( O(n+1,1) \).

Let us recall the geometric meaning of the group \( O(n+1,1) \) in this case. For this, and for further analysis, it is convenient to work in the complex analytic situation.

B. The complex setting. Let \( M \) be a complex manifold of dimension \( n \). We can speak about holomorphic Riemannian metrics on \( M \). Such a metric is a holomorphic section of the vector bundle \( S^2(T^*_M) \) which is non-degenerate at each point. In holomorphic coordinates it is gives as a symmetric matrix \( g(z) = \|g_{ij}(z)\| \) of holomorphic functions.
A (holomorphic) conformal metric on $M$ is, naively, a “Riemannian metric defined up to a scalar”. This means that locally we have a representative $g_{ij}(z)$ which is a holomorphic Riemannian metric, with the understanding that we identify

$$g_{ij}(z) \sim \lambda(z)g_{ij}(z), \quad \lambda \in \mathcal{O}_M^*,$$

i.e., that $\lambda(z)g_{ij}(z)$ represents the same conformal metric as $g_{ij}(z)$.

If we fix a point $z \in M$, then a complete invariant of a “non-degenerate symmetric form on $T_zM$ defined modulo scalars”, is the null-cone $C_zM \subset T_zM$. Thus, globally:

**Definition 1.5.** (a) A holomorphic conformal metric on a complex manifold $M$ is a holomorphic family of non-degenerate quadratic cones $C_z \subset T_zM$. A complex conformal manifold is a complex manifold equipped with a holomorphic conformal metric.

(b) Let $(M, (C_z))$ and $(M', (C'_z))$ are two $n$-dimensional complex conformal manifolds. A conformal mapping $\varphi : M \to M'$ is a biholomorphic map whose differential takes null-cones to null-cones: $d_z\varphi(C_z) = C_{\varphi(z)}$.

An alternative definition would be that a conformal metric is given by a holomorphic line bundle $\Lambda$ on $M$ an $\Lambda$-valued scalar product $g \in \text{Hom}(S^2T_M, \Lambda)$ on $T_M$. It is easily seen to be equivalent to the above.

**Example 1.6 (Conformal quadric).** (a) Let $Q \subset \mathbb{P}^{n+1} = \mathbb{P}(\mathbb{C}^{n+2})$ be a non-degenerate quadric hypersurface. It has a canonical conformal structure defined as follows. For $z \in Q$ let $\mathcal{C}_zQ \subset \mathbb{P}^{n+1}$ be the projective tangent space to $Q$ at $z$. It is an algebraic variety isomorphic to $\mathbb{P}^n$, and we have a canonical identification of the (usual) tangent spaces $T_z\mathcal{C}_zQ = T_zQ$. The intersection $\mathcal{C}_z = Q \cap \mathcal{C}_zQ$ (inside $\mathbb{P}^{n+1}$) is a quadratic hypersurface in $\mathcal{C}_zQ$ with one singular point, namely $z$. It is called the projective tangent cone to $Q$. The (intrinsic) tangent cone to this intersection is a nondegenerate quadratic cone $C_z$ in $T_z\mathcal{C}_zQ$, i.e., in $T_zQ$. This defines the conformal structure on $Q$.

(b) Thus any automorphism of $\mathbb{P}^{n+1}$ preserving $Q$ defines a conformal mapping $Q \to Q$. Such automorphisms form the group $O(n + 2, \mathbb{C})$.

(c) If we fix one point $\infty \in Q$, then $Q \cap \overline{C}_\infty Q$ is isomorphic to $\mathbb{A}^n$ as an algebraic variety so to $\mathbb{C}^n$ as a complex manifold. The induced conformal structure on $\mathbb{C}^n$ is the standard flat conformal structure. Thus any $g \in O(n + 2, \mathbb{C})$ defines a conformal mapping from $\mathbb{C}^n$ (minus, possibly, a quadratic cone hypersurface) to itself.

(d) If we want to restrict to real points, then $Q$ gives the $n$-sphere $S^n$, with its standard conformal structure. The real group $O(n + 1, 1)$ acts by conformal mappings of $S^n$ to itself. Each projective tangent cone $\overline{C}_zS^n$ gives just the point $z$, and $S^n - \{z\}$ is identified with $\mathbb{R}^n$ via the stereographic projection. In this way any $g \in O(n + 1, 1)$ defines a conformal map of $\mathbb{R}^n$ (minus, possibly, a single point) to itself.

The holomorphic version of Liouville’s theorem can be formulated as follows.
Theorem 1.7. Let $n \geq 3$, let $U, V \subset Q$ be open domains and $\varphi : U \to V$ be a holomorphic conformal mapping. Then $\varphi$ extends to an isomorphism of algebraic varieties $\Phi : Q \to Q$.

Let us concentrate on the Lie algebra (infinitesimal) version. Let $M$ be a complex manifold with conformal structure. A conformal Killing field on $M$ is a holomorphic vector field preserving the conformal structure. We denote by $\text{conf}(M)$ the Lie algebra formed by conformal Killing fields. The Lie algebra version of the Liouville theorem is:

Theorem 1.8. Let $Q \subset \mathbb{P}^{n+1}$ be the $n$-dimensional quadric.

(a) For any $n \geq 1$ we have $\text{conf}(Q) = \mathfrak{so}(n+2, \mathbb{C})$.

(b) Let $n \geq 3$. For any connected open domain $U \subset Q$ the restriction map

$$\mathfrak{so}(n+2, \mathbb{C}) = \text{conf}(Q) \longrightarrow \text{conf}(U)$$

is an isomorphism.

As we shall see, this fact is a manifestation of the same phenomenon as the Hartogs theorem and can be overcome in a similar way.

2 Overcoming Hartogs

A way to recover the missing polar parts in Hartogs’ theorem is by using the full cohomology $H^\bullet(\mathbb{A}^n \setminus \{0\}, \mathcal{O})$, not just $H^0$. We have the following elementary fact.

Proposition 2.1. For any $n \geq 2$ we have

$$H^i(\mathbb{A}^n \setminus \{0\}, \mathcal{O}) \approx \begin{cases} \mathbb{C}[z_1, \ldots, z_n], & \text{if } i = 0, \\ z_1^{-1} \cdots z_n^{-1} \cdot \mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}], & \text{if } i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Here the new space, formed by the polar parts

$$H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathcal{O}) = H^0_{(0)}(\mathbb{A}^n, \mathcal{O}) = z_1^{-1} \cdots z_n^{-1} \cdot \mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}]$$

appears in cohomological degree $n-1$ and is therefore invisible if we remain in the classical (non-homological) framework. It can also be seen as the $n$th cohomology with support at 0. In fact, the second identification in (2.2) holds for any $n \geq 1$. Thus passing to the cohomology restores the continuity.

The easiest way to establish Proposition 2.1 is by using the Čech complex associated to the covering of $\mathbb{A}^n \setminus \{0\}$ by the affine open sets $U_i = \{z_i \neq 0\}$, $i = 1, \ldots, n$. The space $\Gamma(U_{i_1, \ldots, i_p}, \mathcal{O})$ of regular functions on each $p$-fold intersection is realized inside the Laurent polynomial ring $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, and the $(n-1)$th cohomology will appear as the span of the Laurent monomials which will not appear in any of the $\Gamma(U_{i_1, \ldots, i_p}, \mathcal{O})$. This approach is
equivalent to the classical computation of the cohomology of the sheaves $\mathcal{O}(d)$ on $\mathbb{P}^{n-1}$ due to Serre, see [10].

If we consider the complex manifold $\mathbb{C}^n$ instead of the algebraic variety $\mathbb{A}^n$ and the sheaf $\mathcal{O}_{\text{hol}}$ of holomorphic functions, we have a statement similar to Proposition 2.1, but with

$$H^{n-1}(\mathbb{C}^n \setminus \{0\}, \mathcal{O}_{\text{hol}}) = H^n_{\{0\}}(\mathbb{C}^n, \mathcal{O}_{\text{hol}}) = z_1^{-1} \cdots z_n^{-1} \cdot \mathbb{C}[\{z_1^{-1}, \ldots, z_n^{-1}\}]_{\text{ent}}$$

being the space of Taylor series representing entire functions. This space is known as the space of holomorphic hyperfunctions on $\mathbb{C}^n$ with support at 0, see [23].

For any sheaf $\mathcal{F}$ of $\mathbb{C}$-vector spaces on a topological space $X$ we denote by $R\Gamma(X, \mathcal{F})$ the derived functor of sections of $\mathcal{F}$, i.e., “the” complex of $\mathbb{C}$-vector spaces whose cohomology is $H^\bullet(X, \mathcal{F})$. Such a complex is defined uniquely up to unique isomorphism in the derived category. If $\mathcal{F}$ has some additional algebraic structure (commutative algebra, Lie algebra etc.), then it is well known that $R\Gamma(X, \mathcal{F})$ can be defined in such a way as to inherit this structure.

**Examples 2.3.** (a) Let $X$ be a complex manifold and $\mathcal{F} = \mathcal{O}_X$ be the sheaf of holomorphic functions. It is a sheaf of commutative algebras. The Dolbeault complex $\Omega^0\bullet(X, \overline{\partial})$ is a model for $R\Gamma(X, \mathcal{O}_X)$ which has the structure of a commutative dg-algebra.

The sheaf $\mathcal{F} = T_X$ of holomorphic vector fields on $X$ is a sheaf of Lie algebras. The Dolbeault complex $(\Omega^0\bullet(X, T_X), \overline{\partial})$ is a dg-Lie algebra model for $R\Gamma(X, T_X)$, with the Lie structure given by the Schouten bracket.

(b) Let $X$ be an algebraic variety with Zariski topology and $\mathcal{F} = \mathcal{O}_X$ be the sheaf of regular functions. A commutative dg-aglebra model for $R\Gamma(X, \mathcal{O}_X)$ can be obtained as the goobal relative de Rham complex $\Gamma(J, \mathcal{O}^\bullet_{J/X})$ where $J \to X$ is a Jouianolou torsor, i.e., an affine variety which is made into a Zariski oially trivial bundle over $X$ with fibers being affine spaces and transition functions being affine transformations. See [2] for a general discussion and [6] for a concrete example with $X = \mathbb{A}^n \setminus \{0\}$.

(c) If $X$ is any topological space and $\mathcal{F}$ is any sheaf of commutative (resp. Lie, etc.) $\mathbb{C}$-algebras, then the Čech model for $R\Gamma(X, \mathcal{F})$ produces a cosimplicial commutative (resp. Lie, etc.) $\mathbb{C}$-algebra. There is a general procedure of converting a cosimplicial algebra of any given type into a dg-algebra of the same type using the Thom-Sullivan construction involving polynomial differential forms on simplices. It provides the most general way to make $R\Gamma(X, \mathcal{F})$ to inherit the algebra structure present on $\mathcal{F}$. We refer to [11] §5.2 for details.

Thus the correct $n$-dimensional replacement of the algebra $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials is the commutative dg-algebra

$$\mathfrak{A}_{[n]} = R\Gamma(\mathbb{A}^n \setminus \{0\}, \mathcal{O})$$

defined as in Example 2.3(b). In particular, tensoring $\mathfrak{A}_{[n]}$ with a finite-dimensional reductive Lie algebra $\mathfrak{g}$ leads to interesting higher-dimensional derived generalizations of Kac-Moody algebras [6, 8].
3 Ambitwistor description of conformal metrics

We want to show that Liouville’s theorem, at least in its complex, infinitesimal form (1.8),
can be seen as a Hartogs-type phenomenon and therefore can be “overcome” by introducing
cohomological degrees of freedom. For this, we recall the main points of the ambitwistor
approach [16, 19, 20] to holomorphic conformal metrics (in any dimension, in particular
without assuming self-duality in dimension 4).

A. The space of null-geodesics. Let \( (M, g) \) be an \( n \)-dimensional complex manifold
with a holomorphic Riemannian metric. We can then speak about null-geodesics in \( M \)
which are parametrized holomorphic curves \( \gamma : U \to M, U \subset \mathbb{C} \) open, satisfying the complex
version of the geodesic equation and such that \( \gamma'(t) \) is isotropic everywhere. The elementary
but fundamental fact is, see [16] §II.2:

**Proposition 3.1.** For two conformally equivalent metrics \( g(z) \) and \( \lambda(z)g(z) \), the null-
geodesics are the same up to a re-parametrization.

Put differently, let \( QT M \subset \mathbb{P}(TM) \) be the quadric bundle formed by the null-directions
in the tangent spaces \( T_x M, x \in M \). It is a complex manifold of dimension \( 2n - 2 \). The “complex geodesic flow” for \( g(z) \) is the 1-dimensional complex foliation \( \mathcal{L} \) on \( QT M \) whose
leaves are the tangent lifts of null-geodesics for \( g(z) \). Note that \( QT M \) depends only on the
conformal class of \( g(z) \). Proposition 3.1 says that so does \( \mathcal{L} \).

Let now \( (M, (C_x)) \) be a holomorphic conformal manifold of dimension \( n \). We then have
the quadric bundle \( QT M \subset \mathbb{P}(TM) \) with fibers \( Q(T_x M) = \mathbb{P}(C_x) \subset \mathbb{P}(T_x M) \). By the above, \( QT M \) carries a canonical 1-dimensional holomorphic foliation \( \mathcal{L} \) whose leaves, are, locally,
the lifts of complex null-geodesics for any holomorphic metric representing \( (C_x) \).

The space of null-geodesics \( L = L(M) \) is defined as the space of leaves of the foliation \( \mathcal{L} \).
In the sequel we assume that this space of leaves exists, i.e., intuitively, the global behavior of
complex null-geodesics in not too wild. More precisely, following [16], we make the following

**Definition 3.2.** A holomorphic conformal manifold \( (M, (C_x)) \) of dimension \( n \) is called civilized, if:

1. There is a Hausdorff complex manifold \( L \) of dimension \( 2n - 3 \) and a holomorphic
   submersion \( \rho : QT M \to L \) whose fibers are precisely the leaves of \( \mathcal{L} \), with the property:

2. The restriction of \( \rho \) to any quadric \( Q(T_x M), x \in M \), is a holomorphic embedding (that
   is, no complex null-geodesic passes through the same point twice).

For a civilized \( M \) the manifold \( L = L(M) \) is defined uniquely up to a unique isomorphism.
In the sequel we will assume that \( M \) is civilized. Here are some examples.

**Examples 3.3.** (a) (Flat case, noncompact) \( M = \mathbb{C}^n \) with a flat conformal metric. In
this case \( L(\mathbb{C}^n) \) is a closed subvariety in the space of all straight lines in \( \mathbb{C}^n \). It is an
algebraic variety which, for \( n \geq 3 \), is not affine and not projective. More precisely, let \( Q^{n-2} \subset \mathbb{P}^{n-1} = \mathbb{P}(M) \) be the quadric formed by null-lines in \( M \) through 0. Any null-line in \( M \) can be seen, in a unique way, as a translation of a line passing through 0. This means that \( L(M) \) is the total space of the vector bundle \( l \mapsto M/l \) on \( Q \), i.e., of the bundle whose fiber over the point \([l]\) represented by a null-line \( l \) through 0, is \( M/l \). The bundle \( l \mapsto M/l \) is in fact defined on the entire \( \mathbb{P}(M) = \mathbb{P}^{n-1} \) and as such, is identified with \( T_{\mathbb{P}(M)}(-1) \), because of the “Euler sequence” [7]:

\[
0 \to O_{\mathbb{P}(M)}(-1) \to M \otimes O_{\mathbb{P}(M)} \to T_{\mathbb{P}(M)}(-1) \to 0.
\]

(We recall that \( O_{\mathbb{P}(M)}(-1) \) is the tautological line bundle, i.e., \( l \mapsto l \) in the above notation.) So we conclude that

\[
(3.4) \quad L(\mathbb{C}^n) = \text{Tot}(T_{\mathbb{P}^{n-1}}(-1)|_{Q^{n-2}}).
\]

(b) (Flat case, compact) \( M = Q^n \subset \mathbb{P}^{n+1} \) is the \( n \)-dimensional projective quadric. In this case \( L(Q^n) \) consists of all straight lines in \( \mathbb{P}^{n+1} \) which lie on \( Q^n \). It is a projective algebraic variety identified with \( G^{\text{is}}(2, \mathbb{C}^{n+2}) \), the Grassmannian of 2-dimensional subspaces in \( \mathbb{C}^{n+2} \) which are isotropic with respect to the quadratic form defining \( Q^n \).

(c) (Local case) We can always replace \( M \) by a sufficiently small neighborhood around a fixed point \( x_0 \) (small with respect to the curvature data of the metric near \( x_0 \)). Then the situation will be similar to the flat case (a), so we get a civilized manifold. See [16], §II.1.

For each \( x \in M \) we define

\[
(3.5) \quad L_x = \{ \gamma \in L \mid x \in \gamma \} \subset L
\]

to consist of null-geodesics that pass through \( x \). The condition (2) of Definition 3.2 implies that \( L_x \) is identified with the quadric \( Q(T_x M) \), i.e., is isomorphic to \( Q^{n-2} \). The conformal geometry of \( M \) is encoded by the system of subvarieties \( L_x \). That is, \( x \) and \( y \) are null-separated, if and only if \( L_x \cap L_y \neq \emptyset \). Further, comparison with the flat case identifies the normal bundle of each \( L_x \) in \( L \). That is, with respect to any identification \( L_x \simeq Q^{n-2} \) we have

\[
(3.6) \quad N_{L_x/L} \simeq T_{\mathbb{P}^{n-1}}(-1)|_{Q^{n-2}}.
\]

The bundle in the RHS of (3.6) is homogeneous (equivariant under automorphisms of \( Q^{n-2} \)), so one can write (3.6) without reference to a particular way of identifying \( L_x \) with \( Q^{n-2} \).

B. \( L(M) \) as a contact manifold. Let \( X \) be a complex manifold of odd dimension \( 2m + 1 \). We recall [1, 16, 21] that a (holomorphic) contact structure on \( X \) is a (holomorphic) vector subbundle \( \Theta \subset T_X \) of rank \( 2m \) which is maximally non-integrable in the following sense. Let \( \mathcal{N} = T_X/\Theta \) be the quotient line bundle. Then \( \Theta \) is given by the vanishing of the
tautological \( \kappa \)-valued contact form \( \theta : T_X \to \kappa \). A local trivialization of \( \kappa \) makes \( \theta \) into a usual holomorphic 1-form. The maximal non-integrability conditions means that

\[
\theta \wedge d\theta \wedge \cdots \wedge d\theta \neq 0 \quad \text{everywhere.}
\]

This condition is known to be independent on the way we represent \( \theta \) as a usual form by choosing a trivialization of \( \kappa \). More precisely, since \( \theta \) is, intrinsically, a 1-form with values in a line bundle, \( d\theta \) is not invariantly defined, but it is invariantly defined modulo (the wedge ideal generated by) \( \theta \). Therefore \( \theta \wedge (d\theta)^n \) is invariantly defined as a volume form with values in \( \kappa^{\otimes (m+1)} \), because it involves only \( d\theta \) modulo \( \theta \), as \( \theta \wedge \theta = 0 \). Since it is nowhere vanishing, we get a canonical identification of line bundles

\[
\kappa^{\otimes (m+1)} \simeq \omega_X^{\otimes (-1)},
\]

where \( \omega_X \) is the line bundle of volume forms.

Another consequence of the same remark is that \( d\theta \) is invariantly defined and non-degenerate on \( \text{Ker} (\theta) = \Theta \), that is, we have a non-degenerate skew-symmetric form

\[
d\theta : \Lambda^2 (\Theta) \longrightarrow \kappa.
\]

If now \( Z \subset X \) is a smooth hypersurface which is transversal to \( \Theta \) everywhere, then it carries the 1-dimensional \( \text{bicharacteristic foliation} \ \mathcal{B} \), with tangent spaces to the leaves being the 1-dimensional subspaces

\[
\mathcal{B}_z = \text{Ker}(d\theta|_{T_z Z \cap \Theta_z}) \subset T_z Z, \quad z \in Z.
\]

It is classical (the “contact reduction”) that the space of leaves of \( \mathcal{B} \) (the space of bicharacteristics), if it exists, is again a contact manifold, now of dimension \( 2m - 1 \).

Let now \( (M, (C_x)) \) be a holomorphic conformal manifold of dimension \( n \). Then \( T^* M \) is a symplectic manifold, so \( \mathbb{P}(T^* M) \) is a contact manifold \([1]\) which is identified with \( \mathbb{P}(TM) \) by the conformal structure. The hypersurface \( QT M \subset \mathbb{P}(TM) \) of null-directions is transversal to \( \Theta_{\mathbb{P}(TM)} \) and its bicharacteristic foliation \( \mathcal{B} \) is just the null-geodesic foliation \( \mathcal{L} \). This shows that the space \( L \) of null-geodesics carries a canonical contact structure \( \Theta = \Theta_L \).

Explicitly, \( \Theta \) can be defined as follows. Let \( \gamma \in L \) be a null-geodesic, considered as a 1-dimensional complex submanifold in \( M \). At any \( x \in \gamma \), the line \( T_x \gamma \subset T_x M \) is isotropic, so its orthogonal \( (T_x \gamma)^\perp \) is a hyperplane in \( T_x M \) containing \( T_x \gamma \). Now, the contact hyperplane \( \Theta_x \gamma \subset T_x \gamma L \) consists of infinitesimal displacements of \( \gamma \) which, for each \( x \in \gamma \), move \( x \) inside \( (T_x \gamma)^\perp \).

We further recall that an \( m \)-dimensional smooth submanifold \( W \) of a \( 2m + 1 \)-dimensional contact manifold \( (X, \Theta) \) is called \textit{Legendrian}, if the tangent spaces of \( W \) lie in \( \Theta \). In our case \( X = L \), it is clear from the above explicit definition of \( \Theta \) that any subvariety \( L_x \), see (3.5), is Legendrian. The main result of the ambitwistor description of conformal metrics \([15, 16, 20]\) can be summarized as follows.
Theorem 3.10. (a) Any local Legendrian deformation of any $L_x$ inside $L$ is of the form $L_y$ for some $y$.

(b) Let $M_1, M_2$ be two civilized holomorphic conformal manifolds of dimension $n$. Let $x_i \in M_i$, $i = 1, 2$, be two points. Holomorphic conformal diffeomorphisms $M_1 \to M_2$ taking $x_1 \to x_2$ are in bijection with holomorphic contact diffeomorphisms $L(M_1) \to L(M_2)$ taking $L_{x_1} \to L_{x_2}$. \hfill $\square$

Remark 3.11. In dimensions $\geq 4$ one can drop the reference to the contact structures thus reducing the problem to purely holomorphic geometry. That is, any holomorphic deformation of any $L_x$ is automatically Legendrian, and any holomorphic diffeomorphism $L(M_1) \to L(M_2)$ is automatically contact, see [16]. For special consideration of the case of dimension 3 see [14, 15]. Nevertheless, it seems natural to keep track of the contact structure in all dimensions, since it is the natural geometric structure present in the problem.

4 Overcoming Liouville

A. Contact Hamiltonians and conformal Killing fields. Let $(X, \Theta)$ be a holomorphic contact manifold of dimension $2m + 1$. A contact vector field on $X$ is a holomorphic vector field $\xi$ preserving the distribution $\Theta$. That is, if we trivialize $\mathcal{X} = TX/\Theta$ and view the contact form $\theta$ as a usual 1-form, then we should have $\operatorname{Lie}_\xi(\theta) = f \cdot \theta$ for some function $f$. It is well known [21] that such a $\xi$ is uniquely determined by the contact Hamiltonian

$$H = \theta(\xi) \in H^0(X, \mathcal{X}),$$

where we now view $\theta$ as a $\mathcal{X}$-valued 1-form. In this way the sheaf of contact vector fields is identified with the sheaf of holomorphic sections of the line bundle $\mathcal{X}$. The Lie algebra structure on contact vector fields translates to a canonical bi-differential operator (Poisson-Jacobi bracket) $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$.

We now specialize to $X = L(M)$ where $M$ is a civilized holomorphic conformal manifold of dimension $n$. Theorem 3.10(b) gives, as the infinitesimal version, the following.

Corollary 4.2. We have an identification of Lie algebras

$$\operatorname{conf}(M) \simeq H^0(L(M), \mathcal{X}). \quad \square$$

B. The derived conformal algebra. The above suggests the following definition.

Definition 4.3. Let $(M, (C_x))$ be a civilized holomorphic conformal manifold. The derived conformal algebra of $M$ is the dg-Lie algebra

$$R\operatorname{conf}(M) := R\Gamma(L(M), \mathcal{X}).$$

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In particular, the 1st cohomology of this dg-Lie algebra is \( H^1(L(M), \mathcal{X}) \) which is the space of infinitesimal deformations of \( L(M) \) as a contact manifold, i.e., by Theorem 3.10, of infinitesimal deformations of \( M \) as a conformal manifold. We now see that the infinite-dimensionality of the 2-dimensional conformal group does not “disappear” in dimensions \( \geq 3 \), but is transformed into the infinite-dimensionality of the moduli space of local conformal metrics. Indeed, symmetries and deformations are, from the point of view of derived geometry [11], always governed by the same algebraic structure: an appropriate differential graded Lie algebra.

5 The derived conformal algebra of the flat space

A. Statement of the result. We now analyze the cohomology of the derived conformal algebra of the \( n \)-dimensional flat space, \( n \geq 3 \). We will be interested in the algebraic skeleton of the problem, i.e., in dealing with polynomials rather than power series. Therefore we will work with the algebraic variety \( \mathbb{A}^n \) instead of the complex manifold \( \mathbb{C}^n \), and understand \( L(\mathbb{A}^n) \) as an algebraic variety as well. So we form the dg-Lie algebra

\[
R_{\text{conf}}(\mathbb{A}^n) = R\Gamma(L(\mathbb{A}^n), \mathcal{X}),
\]

considering \( \mathcal{X} \) as the sheaf of regular sections on the Zariski topology of \( L(\mathbb{A}^n) \). We will identify the cohomology if this dg-Lie algebra, i.e., \( H^\bullet(L(\mathbb{A}^n), \mathcal{X}) \) as a module over the orthogonal group \( SO(n, \mathbb{C}) \).

More precisely, we denote by \( M = \mathbb{C}^n \) the standard \( n \)-dimensional complex vector space and think of \( \mathbb{A}^n \) as “\( M \) considered an an algebraic variety”, i.e., as the spectrum of the algebra \( S^\bullet(M^*) \). Fixing a nondegenerate quadratic form \( q \in S^2(M^*) \), we get a flat metric on \( \mathbb{A}^n \) and the variety \( L(\mathbb{A}^n) \).

Recall the basics of representation theory of \( GL(n, \mathbb{C}) \), see [9]. Given a sequence of integers \( a = (a_1 \geq \cdots \geq a_n) \) (a dominant weight for \( GL(n) \)), we have the Schur functor \( \Sigma^a \) from the category of \( n \)-dimensional \( \mathbb{C} \)-vector spaces and their isomorphisms to the category of finite-dimensional \( \mathbb{C} \)-vector spaces, with \( \Sigma^a(V) \) being “the” space of irreducible representation of \( GL(V) \) with highest weight \( a \). If all \( a_i \geq 0 \), we think of \( a \) as a Young diagram with rows of lengths \( a_1, \cdots, a_n \). If \( a = (a_1, \cdots, a_p, 0, \cdots, 0) \), we write \( \Sigma^{a_1, \cdots, a_p} \) for \( \Sigma^a \), dropping the zeroes at the end. We also write \( 1^p = (1, \cdots, 1, 0, \cdots, 0) \), \( p \leq n \). Note the particular cases and properties:

\[
\Sigma^{d}(V) = \Sigma^{d,0,\cdots,0}(V) = S^d(V), \quad \Sigma^{1^p}(V) = \Lambda^p(V), \quad \Sigma^{a_1,\cdots,a_n}(V)^* \simeq \Sigma^{a_1,\cdots,a_n}(V^*) \simeq \Sigma^{-a_1,\cdots,-a_1}(V).
\]

Given two weights \( a = (a_1, \cdots, a_n) \) and \( b = (b_1, \cdots, b_n) \), the decomposition of the tensor product

\[
\Sigma^a(V) \otimes \Sigma^b(V) \simeq \bigoplus_c \Sigma^c(V)^{\otimes N^c_{ab}}
\]
into irreducibles is given by the Littlewood-Richardson rule, with $N_{ab}^c$, known as the Littlewood-Richardson coefficients. There are two important cases when $N_{ab}^c = 1$.

**Examples 5.1.** (a) (Horizontal Young multiplication) $c = a + b$, i.e., $c_i = a_i + b_i$. If all $a_i, b_i \geq 0$, then $c$ is the Young diagram obtained by “adding” $a$ and $b$ in the horizontal direction (row by row). The resulting projection $\Sigma^a(V) \otimes \Sigma^b(V) \rightarrow \Sigma^{a+b}(V)$ is induced, via the Borel-Weil theorem, by tensor multiplication of line bundles on the flag variety.

(b) (Vertical Young multiplication). Dually, suppose that $a, b$ are nonnegative and $c$ is the Young diagram obtained by adding $a$ and $b$ in the vertical direction, column by column. Then $N_{ab}^c = 1$ as well. The resulting projection $y : \Sigma^a(V) \otimes \Sigma^b(V) \rightarrow \Sigma^c(V)$ can be called the vertical Young multiplication. For instance, if $a = 1^r$, $b = 1^s$, then $c = 1^{r+s}$ and we get the exterior multiplication. We will be particularly interested in the projection

$$ y_d : S^d(V) \otimes S^2(V) \rightarrow \Sigma^{d+2}(V), \quad d \geq 2. $$

We now specialize to $V = M^*$ where $(M, q)$ is as above and write $SO(n) = SO(n, \mathbb{C})$ for the group of automorphisms of $(M, q)$ with determinant 1. Note that $M \simeq M^*$ as an $SO(n)$-module. The projection (5.2) gives an $SO(n)$-equivariant map

$$ y_{d,q} : y(- \otimes q) : S^d(M^*) \rightarrow \Sigma^{d+2}(M^*). $$

**Theorem 5.3.** The dg-Lie algebra $R\text{Conf}(\mathbb{A}^n)$ has the following cohomology spaces:

- $H^0 = \Lambda^2(M^*) \oplus M^* \oplus M^* \oplus \mathbb{C} = \Lambda^2(M^* \oplus \mathbb{C}^2) = \mathfrak{so}(n + 2)$ (the usual conformal algebra).
- $H^1 = \bigoplus_{d \geq 2} \text{Coker}(y_{d,q})$, with each $y_{d,q}$, $d \geq 2$, being injective.
- $H^i = 0$ for $i \geq 2$.

**B. Moduli space interpretation.** We now explain why the space $H^1$ in Theorem 5.3 can be seen as the space of local deformations of the conformal class of the flat metric. For this we think of the components of a Riemannian metric $g_{ij}(z)$ as a formal Taylor series on $\mathbb{C}^n$ near 0 and view the symmetric algebras below as the spaces of polynomials dense in the spaces of power series.

The symmetric algebra $S^\ast(M^*)$ is (after completion) the space of formal germs of functions on $M = \mathbb{C}^n$ near 0. So the corresponding space of germs of the metric itself is the tensor product $S^2(M^*) \otimes S^\ast(V^*)$. As this is a linear space, we view it as the space of infinitesimal deformations of the flat metric. The Pieri formula [9] gives

$$ S^2 \otimes S^d \cong S^{d+2} \oplus \Sigma^{d+1,1} \oplus \Sigma^{d,2}. $$

Let us now quotient by changes of coordinates (understood infinitesimally, as vector fields). The space of vector fields (understood in the same sense as above) is $M \otimes S^\ast(M^*)$. We identify $M$ with $M^*$ as a $SO(n)$-module. Again, the Pieri formula gives

$$ M^* \otimes S^d(M^*) \cong S^{d+1}(M^*) \oplus \Sigma^{d,1}(M^*). $$
So the “moduli space” of metrics modulo coordinate changes has, as the tangent space at the trivial metric, the result of subtracting the contributions from (5.5) for all \( d \) from the contributions from (5.4) for all \( d \), which gives \( \bigoplus_{d \geq 2} \Sigma^{d,2}(M^*) \). For instance, the lowest summand here, \( \Sigma^{2,2}(M^*) \), is precisely the space of all possible values of the Riemann curvature tensor at the origin.

Further, let us look at the effect of passing to conformal classes, i.e., quotienting by scalar functions, on the tangent space to the moduli space. The space of functions is \( S^p V^q \). So taking the cokernel of the map

\[
y_q : S^{d,2}(V) \rightarrow \bigoplus_{d \geq 2} \Sigma^{d,2}(V)
\]

has the effect of passing to the tangent space of the moduli space of conformal classes.

**Remark 5.6.** Finally, it is instructive to compare the situation with the 2-dimensional case when we have an infinite-dimensional conformal algebra in homological degree 0. The difference is that for \( \dim(M) = 2 \) the map

\[
y_{d,q} : S^d(M^*) \rightarrow \Sigma^{d,2}(M^*) = S^{d-2}(M^*) \otimes \Lambda^2(M^*)^{\otimes^2}
\]

is surjective, not injective. The kernel of \( y_{d,q} \) has dimension 2, it is the space of traceless symmetric tensors in 2 variables. So in each degree we have two basis vectors contributing to the kernel. This matches the identification

\[
\text{conf}(\Lambda^2) = \mathbb{C}[z] \partial_z \oplus \mathbb{C}[\bar{z}] \partial_{\bar{z}}.
\]

### 6 Proof of Theorem 5.3.

**A. Identifying the bundle \( \kappa \).** We first identify the line bundle \( \kappa \), the target of the contact form, starting from the compact flat case. That is, let \( V = \mathbb{C}^{n+2} \) with a non-degenerate scalar product \( \langle -, - \rangle \) and let \( Q^n \subset \mathbb{P}(V) = \mathbb{P}^{n+1} \) be the quadric of null-directions. Then \( L(Q^n) \) is the isotropic Grassmannian \( G^\text{is}(2,V) \subset G(2,V) \). We denote by \( S \) the tautological rank 2 bundle on both \( G(2,V) \) and \( G^\text{is}(2,V) \) and put \( \mathcal{O}(1) = \Lambda^2(S^*) \).

**Lemma 6.1.** The line bundle \( \kappa_{G^\text{is}(2,V)} \) is identified with \( \mathcal{O}(1) \).

**Proof:** Let \( E \subset V \) be a 2-dimensional isotropic subspace and \( [E] \in G^\text{is}(2,V) \) be the corresponding point. Then it is standard that

\[
T_{[E]} G(2,V) \cong \text{Hom}(E, V/E).
\]

Inside this, \( T_{[E]} G^\text{is}(2,V) \) consists of linear maps \( f : E \rightarrow V/E \) such that

\[
\langle f(e), e \rangle = 0 \quad \text{for any } e \in E.
\]

(6.2)
(Since $E$ is isotropic, $\langle f(e), e \rangle$ is well defined.) This is a codimension 3 subspace in $\text{Hom}(E, V/E)$. Further, the contact hyperplane $\Theta_E \subset T[E]^\text{is}(2, V)$ is $\text{Hom}(E, E^\perp/E)$ (a codimension 4 subspace in $\text{Hom}(E, V/E)$), see the general discussion in §3B. Given $f$ satisfying (6.2), we have

$$
\langle f(e_1), e_2 \rangle = -\langle f(e_2), e_1 \rangle, \quad \text{for any } e_1, e_2 \in E.
$$

Therefore the expression $\langle f(e_1), e_2 \rangle$ is a linear map $\Lambda^2(E) \to \mathbb{C}$. Vanishing of this map means that $f : E \to E^\perp/E$, i.e., $f \in \Theta_E$. This gives an identification of vector spaces

$$
T[E]^\text{is}(2, V)/\Theta_E \simeq \Lambda^2(E^*),
$$

and so an identification of line bundles $\mathcal{K} \simeq \Lambda^2(S^*) = \mathcal{O}(1)$. \hfill \Box

We now pass from $L(Q^n)$ to the Zariski open part $L(\mathbb{A}^n)$ which is, by (3.4), the total space of an algebraic vector bundle whose projection we denote by $\pi$:

$$
L(\mathbb{A}^n) = \text{Tot}(T_{\mathbb{P}^{n-1}}(-1)|_{\mathbb{Q}^{n-2}}) \xrightarrow{\pi} Q^{n-2}.
$$

Let us write for short

$$
Q := Q^{n-2}, \quad G := (T_{\mathbb{P}^{n-1}}(-1))^* = \Omega_{\mathbb{P}^{n-1}}^1(1).
$$

In a more algebro-geometric language the identification of $L(\mathbb{A}^n)$ with the total space reads:

$$
L(\mathbb{A}^n) = \text{Spec} \bigoplus_{d \geq 0} S^d(G)|_Q.
$$

Lemma 6.1 implies that

$$
\mathcal{K}_{L(\mathbb{A}^n)} \simeq \pi^*\mathcal{O}_Q(1),
$$

and therefore

$$
H^i(L(\mathbb{A}^n), \mathcal{K}) = \bigoplus_d H^i(Q, S^d(G)(1)|_Q).
$$

\textbf{B. Cohomology on $\mathbb{P}(M)$ using Borel-Weil-Bott.} We invoke the short exact sequence of sheaves on $\mathbb{P}^{n-1} = \mathbb{P}(M)$

$$
0 \to S^d(G)(-1) \xrightarrow{\eta} S^d(G)(1) \to S^d(G)(1)|_Q \to 0
$$

and analyze first the cohomology of $S^d(G)(\pm 1)$ on $\mathbb{P}(M)$.

\textbf{Lemma 6.4. On $\mathbb{P}(M)$,}

(–1) \quad \text{The sheaf } S^d(G)(-1) \text{ has } H^1 = S^{d-1}(M^*) \text{ (understood as } 0 \text{ for } d = 0) \text{ and no other cohomology.}

(+) \quad - \quad \text{The sheaf } S^0(G)(1) \text{ has } H^0 = M^* \text{ and no other cohomology.}

- \quad \text{The sheaf } S^1(G)(1) \text{ has } H^0 = \Lambda^2(M^*) \text{ and no other cohomology.
Thus to find \( H_a \), a representation is impossible, i.e., if \( a \) has repetitions, then \( H_a \) has no cohomology.

Proof: We use the Borel-Weil-Bott theorem for flag varieties, see [19], Ch.1, §2 for a treatment convenient for us.

Let \( F = F(M) \) be the space of complete flags

\[
M_1 \subset M_2 \subset \cdots \subset M_n = M = \mathbb{C}^n, \quad \dim(M_i) = i,
\]

with the natural projection \( p : F \to \mathbb{P}(M) = \{M_1 \subset M\} \). We denote by \( M_i \) the tautological bundle on \( F \) of rank \( i \). To a weight \( a = (a_1, \cdots, a_n) \in \mathbb{Z}^n \) (not necessarily dominant) we associate the line bundle

\[
\mathcal{O}_F(a) = (M/M_{n-1})^{\otimes a_1} \otimes (M_{n-1}/M_{n-2})^{\otimes a_2} \otimes \cdots \otimes (M_2/M_1)^{\otimes a_n} \otimes M_1^{a_n}
\]

on \( F \). As mentioned in Example 3.3(a), \( G^* = T_{\mathbb{P}(M)}(-1) \) is the universal quotient bundle whose fiber at \( M_1 \subset M \) is \( M/M_1 \). This implies that

\[
S^d(G) = p_* \mathcal{O}_F(0, \cdots, 0, -d, 0).
\]

Indeed, taking the space of sections of the line bundle \((M_2/M_1)^{\otimes(-d)} = \mathcal{O}_{\mathbb{P}(M/M_1)}(d)\) on the projective space \( \mathbb{P}(M/M_1) \) or, equivalently, of the pullback of this line bundle to the full flag variety of \( M/M_1 \), gives \( S^d(M/M_1)^* \). This implies that for any \( b \in \mathbb{Z} \)

\[
S^d(G)(b) = p_* \mathcal{O}_F(0, \cdots, 0, -d, b),
\]

and so

\[
(6.5) \quad H^\bullet(\mathbb{P}(M), S^d(G)(b)) = H^\bullet(F, \mathcal{O}_F(0, \cdots, 0, -d, b)).
\]

We now recall the procedure of finding \( H^\bullet(F, \mathcal{O}_F(a)) \) for \( a \in \mathbb{Z}^n \) given by Bott’s theorem. That is, if \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \) is a dominant weight, \( w \in S_n \) is a permutation of length \( \ell(w) \) and \( \rho = (n, n-1, \cdots, 1) \), then

\[
H^i(F, \mathcal{O}_F(w(\lambda + \rho) - \rho)) = \begin{cases} 
\Sigma^\lambda(M), & \text{if } i = \ell(w), \\
0, & \text{otherwise}.
\end{cases}
\]

Thus to find \( H^\bullet(F, \mathcal{O}_F(a)) \) we need to represent \( a = w(\lambda + \rho) - \rho \) with \( \lambda \) dominant. If such a representation is impossible, i.e., if \( a + \rho \) has repetitions, then \( \mathcal{O}_F(a) \) has no cohomology.

After these preparations, let us establish part (+1) of Lemma 6.4. From (6.5) we see that we need to find \( H^\bullet(F, \mathcal{O}_F(a)) \), where

\[
a = (0, \cdots, 0, -d, 1), \quad \text{so} \quad a + \rho = (n, n-1, \cdots, 3, 2-d, 2).
\]
For $d = 0$ we have a repetition so no cohomology. For $d \geq 1$, a single elementary transposition (length 1) takes $a + \rho$ to $(n, n-1, \cdots, 3, 2, 2-d)$, then subtracting $\rho$ we get $(0, \cdots, 0, 1-d)$. So in this case the only non-trivial cohomology is

$$H^1(F, \mathcal{O}_F(a)) = \Sigma^{0, \cdots, 0, 1-d}(M) = S^{d-1}(M^*)$$

as claimed.

Let us now establish part $(-1)$ of Lemma 6.4. We have

$$a = (0, \cdots, 0, -d, -1), \quad a + \rho = (n, n-1, \cdots, 3, 2-d, 0).$$

Now,

- If $d = 0$, then $a$ is dominant so we have only
  $$H^0(F, \mathcal{O}_F(a)) = \Sigma^{0, \cdots, 0, -1}(M) = M^*.$$

- If $d = 1$, then $a$ is still dominant, so we have only
  $$H^0(F, \mathcal{O}_F(a)) = \Sigma^{0, \cdots, 1, -1}(M) = \Lambda^2(M^*).$$

- If $d = 2$, we get $a + \rho = (\cdots, 3, 0, 0)$, a repetition so no cohomology.

- If $d \geq 3$, then $a + \rho = (\cdots, 3, 2-d, 0)$ which is ordered, by an elementary transposition, to $(\cdots, 3, 0, 2-d)$. Subtracting $\rho$, we get $(0, \cdots, 0, -2, 1-d)$, so the only cohomology is
  $$H^1(F, \mathcal{O}_F(a)) = \Sigma^{0, \cdots, 0, -2, 1-d}(M) = \Sigma^{d-1,2}(M^*).$$

Lemma 6.4 is proved.

\[\square\]

C. Cohomology on $Q$. We now finish the proof of Theorem 5.3. Let us display the cohomology (known from Lemma 6.4) of the first two sheaves $S^d(G)(\pm 1)$ in (6.3) in a table (Fig. 1), under these sheaves. Under the third sheaf, $S^d(G)(1)|_Q$, let us write the conclusion about its cohomology preceded by the sign “$\Rightarrow$”. We note that in the last row, the map $H^1(\mathbb{P}(M), S^d(G)(-1)) \to H^1(\mathbb{P}(M), S^d(G)(1))$ induced by multiplication with $q$, is proportional to $y_{d-1,q}$. This follows by invariance, by letting $q \in S^2(M^*)$ vary and using the fact (Example 5.1(b)) that

$$\dim \text{Hom}_{GL(M)}(S^{d-1}(M^*) \otimes S^2(M^*), \Sigma^{d-1,2}(M^*)) = 1.$$ 

The fact that the coefficient of proportionality is non-zero, is implied by the next lemma.

Lemma 6.6. The map

$$\tilde{q}: H^1(\mathbb{P}(M), S^d(G)(-1)) \to H^1(\mathbb{P}(M), S^d(G)(1))$$

induced by multiplication with $q$, is injective.
This lemma, together with the table in Fig. 1, establish Theorem 5.3.

\[ 0 \rightarrow S^d(G)(-1) \xrightarrow{\cdot \eta} S^d(G)(1) \rightarrow S^d(G)(1)|_Q \rightarrow 0 \]

\begin{align*}
  d = 0 & \quad H^\bullet = 0 & \quad H^0 = M^* & \quad \Rightarrow H^0 = M^* \\
  d = 1 & \quad H^1 = \mathbb{C} & \quad H^0 = \Lambda^2(M^*) & \quad \Rightarrow H^0 = \mathbb{C} \oplus \Lambda^2(M^*) \\
  d = 2 & \quad H^1 = M^* & \quad H^\bullet = 0 & \quad \Rightarrow H^0 = M^* \\
  d \geq 3 & \quad H^1 = S^{d-1}(M^*) \xrightarrow{\cdot \eta} H^1 = \Sigma^{d-1,2}(M^*) & \quad \Rightarrow H^1 = \operatorname{Coker}(y_{d-1,q}).
\end{align*}

Figure 1: Calculating cohomology on \( Q \subset \mathbb{P}(M) \).

Proof of Lemma 6.6: Let \( \varpi : F \rightarrow G(2, M) \) be the projection. We denote by \( M_2 \) the tautological rank 2 bundle on \( G(2, M) \). If \( E \subset M \) is a 2-dimensional subspace and \( [E] \in G(2, M) \) is the corresponding point, then \( \varpi^{-1}(E) = F(M/E) \times \mathbb{P}(E) \). By applying the Borel-Weil-Bott theorem to the fibers of \( \varpi \), we find that \( \gamma \) is identified with the morphism

\[ S^{d-1}(M^*) = H^0(G(2, M), S^{d-1}(M_2)^*) \rightarrow H^0(G(2, M), \Sigma^{d-1,2}(M^*)) = \Sigma^{d-1,2}(M^*) \]

induced by the morphism of vector bundles on \( G(2, M) \)

\[ y_{d-1,q}|_{M-2} : S^{d-1}(M_2)^* \rightarrow \Sigma^{d-1,2}(M_2)^* \]

which, on each fiber, i.e., on each \( E \subset M \) as above, is the morphism

\[ y_{d-1,q}|_E : S^{d-1}(E^*) \rightarrow \Sigma^{d-1,2}(E^*) \]

corresponding to the 2-dimensional space \( E \) and the quadratic form \( q|_E \). This morphism has been discussed in Remark 5.6, and its kernel is the subspace in \( S^{d-1}(E^*) \) formed by polynomials harmonic (traceless) with respect to \( q|_E \). So we are reduced to the following fact.

Lemma 6.7. Let \( M \) be a complex vector space of dimension \( \geq 3 \) and \( q \in S^2(M^*) \) be a non-degenerate quadratic form. If \( f \in S^{d-1}(M^*) \), \( d \geq 3 \), is such that for any 2-dimensional subspace \( E \subset M \), the restriction \( f|_E \) is harmonic with respect to \( q|_E \), then \( f = 0 \).

Proof: We can assume \( q \) to come from a positive definite quadratic form on a real form \( M_\mathbb{R} = \mathbb{R}^n \) of \( M \). Then it is enough to prove the lemma under the assumptions that \( f \) is
a real homogeneous polynomial of degree \(d - 1\) on \(\mathbb{R}^n\) and the restriction of \(f\) to any real subspace \(E\) is harmonic with respect to \(q|_E\). If \(E\) is a 2-dimensional real space with a positive definite quadratic form, then we can use Euclidean geometry in \(E\). In particular, a harmonic polynomial homogeneous of degree \(m\) is, in polar coordinates \((R, \phi)\) a linear combination of \(R^m \cos(m\phi)\) and \(R^m \sin(m\phi)\), and therefore it is invariant under Euclidean rotations by \(2\pi/m\) in \(E\). So our assumptions on \(f : \mathbb{R}^n \to \mathbb{R}\) imply that the restriction of \(f\) to any 2-plane \(E \subset \mathbb{R}^n\) is invariant under rotations by \(2\pi/(d - 1)\) in this plane. If \(d \geq 4\), this implies that \(f(x)\) depends only on the radius \(\|x\| = q(x)^{1/2}\), so by homogeneity \(f(x) = \text{const} \cdot \|x\|^{d-1}\), which contradicts the above trigonometric shape of \(f|_E\), so \(f = 0\).

In the remaining case \(d = 3\), our \(f\) is a quadratic form. The condition that \(f|_E\) is harmonic with respect to \(q|_E\) means, in the classical terminology, that \(f|_E\) and \(q|_E\) are “anharmonic”, i.e., that the sum of the two eigenvalues of \(f|_E\) with respect to \(q|_E\) is 0. Since \(q\) is assumed to be positive definite, this implies that each \(f|_E\) must always have signature \((+, -)\), if non-degenerate and must be zero, if degenerate. This is impossible unless \(f = 0\).

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