Permutations of cubical arrays

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Abstract. The structure constants of an algebra determine a cube called the cubical array associated
with the algebra. The permuted indices of the cubical array associated with a finite semifield generate new
division algebras. We do not not require that the algebra be finite and ask "Is it possible to choose a basis for
the algebra such any permutation of the indices of the structure constants leaves the algebra unchanged?"
What are the associated algebras? Author shows that the property "weakly quadratic" is invariant under
all permutations of the indices of the corresponding cubical array and presents two algebras for which the
cubical array is invariant under all permutations of the indices.

1. Introduction
The structure constants $\Gamma_{ijk}$ of an $n$-dimensional algebra $\mathfrak{A}$ over a field $F$ with basis
$\{e_0, e_1, e_2, \ldots, e_{n-1}\}$ can be defined by

$$e_ie_j = \sum_{k=0}^{n-1} \Gamma_{ijk}e_k$$

These structure constants $\Gamma_{ijk}$ determine a $n \times n \times n$ cube $A$ called the cubical array (Knuth
[1]) associated with the algebra $\mathfrak{A}$. Let $\sigma$ be a permutation of the set of indices, $\{i, j, k\}$. Then
$A^\sigma$ will represent the 3-cube $A$ with subscripts permuted by $\sigma$; i. e., the $k\sigma$th subscript of $A^\sigma$
is the $k$th subscript of $A$. If $A = (A_{ijk})$, then if $B = A^{(123)} = (B_{ijk})$ we have $B_{ijk} = A_{kij}$.

The algebra corresponding to $A^{(12)}$ is called the opposite algebra and is denoted by $\mathfrak{A}^{op}$. Pairing each algebra with its opposite gives

| $\mathfrak{A}$  | $\mathfrak{A}^{op}$ |
|--------------|------------------|
| $A$          | $A^{(12)}$       |
| $A^{(123)}$  | $A^{(123)(12)}$  |
| $A^{(132)}$  | $A^{(132)(12)}$  |

Table 1. The algebra and opposite.

If the algebra $\mathfrak{A}$ is associative, then so is the algebra $\mathfrak{A}^{op}$.

The group of permutations on a three element set is usually denoted by $S_3$ and is generated
by the permutations $(12)$ and $(13)$. The cubical array was introduced to study finite geometries;
if \( \pi \) is a finite projective plane coordinatized by the division ring \( S \) and \( A \) is the cubical array associated with \( S \), then the six permutations of the indices of \( A \) determine a series of at most six non-isotopic planes. The permutations (12) and (13) have the geometrical interpretations: the dual of the of the semifield is generated by the action of (12) on the associated cubical array and the transpose is generated by (13). The same series of permutation can be applied to the cubical array corresponding to any arbitrary division ring and will result in division rings.

Kantor [2] proves that the transposition and dualization of a commutative semifield is a sympletic semifield and, conversely, the dualization and transposition of a sympletic semifield is a commutative semifield. Ball and Brown [3] constructed potentially six non-isotopic semifields from a Cohen-Ganley pair of functions \((f, g)\). Ball, Ebert and Lavrauw [4] use the cubical array to compute the twelve isotopy classes of semifields associated with a semifield of rank 2 over its left nucleus. Ball and Lavrauw [5] examine the series of division rings arising from semifields two dimensional over a weak nucleus and Kantor [6] notes that the planes determined by the Hughes Kleinfeld semifields [7] have exactly one “image” under the action of \( S_3 \).

We seek algebraic characterizations of the sequence of algebras obtained by cubical permutations.

There are two approaches: one addresses properties of the algebra; the second specifies the non-zero elements of the array.

**Lemma 1.1.** Let the finite dimensional algebra \( \mathfrak{A} \) be the algebra direct sum of the algebras \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \). The cube \( \mathfrak{A}^3 \) determines a direct sum for all \( \sigma \in S_3 \).

**Proof.** Let \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) be bases for \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) respectively. Construct a basis for \( \mathfrak{A}_1 \oplus \mathfrak{A}_2 \) as \( \{e_i, e_j : e_i \in \mathfrak{B}_1, e_j \in \mathfrak{B}_2 \} \). Then \( \Gamma_{ijk} = 0 \) when \( i, j, k \) are not all contained in the same \( \mathfrak{B}_i, i = 1 \) or 2.

We wish to find a property that is invariant under the action of \( S_3 \) shared by a large class of algebras. Such a property is a generalization of the quadratic algebras.

**2. Weakly quadratic algebras**

A **quadratic algebra** \( \mathfrak{A} \) over a field \( F \) of characteristic not 2, is a unital algebra in which every \( x \in \mathfrak{A} \) satisfies the condition \( x^2 \in \text{span} \{1, x\} \). A generalization of the quadratic algebras are the **weakly quadratic algebra**.

A finite dimensional algebra \( \mathfrak{A} \) with a basis \( \{e_0, e_1, \ldots, e_{n-1}\} \), such that \( e_0 \) is idempotent, is called **weakly quadratic** (relative to that basis) if for any \( x \in \mathfrak{A} \) we have \( x^2 \in \text{span}\{e_0, x\} \) and

(i) \( \Gamma_{0ik} = a \delta_{ik} \), \( i \neq 1 \) for some \( a \in F \)

(ii) \( \Gamma_{i0k} = b \delta_{ik} \), \( i \neq 1 \) for some \( b \in F \)

(iii) \( \Gamma_{iik} = c \delta_{k0} \), \( c \in F \)

(iv) \( \Gamma_{ijk} = -\Gamma_{jik} = \Gamma_{jki} \), where no two of \( i, j, k \) are the same and none is equal to 0

**Example 2.1.** Table 3 gives the multiplication constants for a weakly quadratic algebra.

Table 2. A weakly quadratic algebra.

|   | I | J | K |
|---|---|---|---|
| e | e | aI | aJ | aK |
| I | bI | ce | K | -J |
| J | bJ | -K | ce | I |
| K | bK | J | -I | ce |
If \( a = b = 1 \) and \( c = -1 \) the table defines the real quaternion division ring.

**Theorem 2.2.** If the algebra \( \mathfrak{A} \) is weakly quadratic then so is \( \mathfrak{A}^\sigma \) for all \( \sigma \in S_3 \).

**Proof.** Clearly if \( \mathfrak{A} \) is weakly quadratic then so is \( \mathfrak{A}^{(12)} \). The structure constants \( \Phi_{ijk} \) of \( \mathfrak{A}^{(13)} \) are given in table 3.

**Table 3.** Structure constants for \( A \) and \( A^{(13)} \).

| \( \Gamma \) | \( \Phi \) |
|-------------|-------------|
| \( \Gamma_{0jk} = a \delta_{jk}, \ j \neq 1 \) for \( a \in F \) | \( \Phi_{kj0} = a \delta_{jk}, \ j \neq 1 \) for \( A \in F \) |
| \( \Gamma_{i0k} = b \delta_{ik}, \ i \neq 1 \) for \( b \in F \) | \( \Phi_{k0i} = b \delta_{ik}, \ i \neq 1 \) for \( b \in F \) |
| \( \Gamma_{ik} = c \delta_{ik}, \ c \in F \) | \( \Phi_{ki} = c \delta_{ik}, \ c \in F \) |
| \( \Gamma_{ijk} = -\Gamma_{jki} \) | \( \Phi_{kji} = -\Phi_{kij} \) |
| \( \Gamma_{ii} = \delta_{ii} \) | \( \Phi_{iij} = \delta_{iij} \) |

**Example 2.3.** The Jordan algebra \( J = F1 \oplus \mathfrak{B} \) of the symmetric bilinear form \( q \) on \( \mathfrak{B} \) is a Type D Jordan algebra (Jacobson [8]). Albert [9] showed any Type D algebra has a basis \( \{e_0, e_1, \ldots, e_{n-1}\} \) such that

\[
e_0 e_i = e_i, \quad e_i e_j = \alpha_i \delta_{ij} e_0
\]

If \( \alpha_i = 1 \) non-zero structure constants are

\[
\Gamma_{ii} = \Gamma_{kii} = \Gamma_{ikii} = \delta_{k0}
\]

We note that the associated cubical array is invariant under all \( \sigma \in S_3 \).

**Definition 2.4** (Osborn [10]). If \( \mathfrak{A} \) is an \( F \) algebra and \( \eta \) an element of \( F \) not equal to \( \frac{1}{2} \). Let \( \mathfrak{A}^\ast \) denote the vector space \( \mathfrak{A} \) under a new multiplication \( \ast \) defined in terms of the old multiplication by

\[
a \ast b = \eta ab + (1 - \eta)ba
\]

The algebra \( \mathfrak{A}^\ast \) is said to be strongly quasi-equivalent to \( \mathfrak{A} \).

**Theorem 2.5.** If the algebra \( \mathfrak{A} \) is weakly quadratic then so is \( \mathfrak{A}^\ast \).

**Proof.** We must show that if the multiplication constants of the weakly quadratic algebra \( \mathfrak{A} \) satisfy the conditions 1 through 4 then the multiplication constants \( \Phi_{ijk} \) of \( \mathfrak{A}^\ast \) satisfy analogous conditions

\[
e_0 \ast e_i = \eta e_0 e_i + (1 - \eta)e_i e_0 = [\eta a + (1 - \eta)b]e_i, \quad \Phi_{0ii} = [\eta a + (1 - \eta)b]
\]

Similarly,

\[
\Phi_{i0i} = [\eta b + (1 - \eta)a]
\]
If \( i \neq 0 \), then \( e_i * e_i = e_i e_i = c e_0 \) and \( \Phi_{iik} = c \delta_{0k} \). If \( i = 0 \), then \( \Phi_{00k} = \delta_{0k} \) and
\[
e_i * e_j = \eta \Gamma_{ijk} e_k + (1 - \eta) \Gamma_{jik} e_k - (1 - \eta) \Gamma_{ijk} e_k = (2\eta - 1) \Gamma_{ijk} e_k, \quad \Phi_{ijk} = (2\eta - 1) \Gamma_{ijk}
\]
Similarly
\[
\Phi_{jik} = (2\eta - 1) \Gamma_{jik}
\]
From the above relations we have
\[
\Phi_{ijk} = (2\eta - 1) \Gamma_{ijk} = -(2\eta - 1) \Gamma_{ikj} = -\Phi_{ikj}
\]
Note that it is sufficient to prove that \( A^\sigma \) is quadratic for \( \sigma = (12) \) and \( \sigma = (13) \), since these two permutations generate \( S_3 \).

**Example 2.6.** Let \( A \) be the algebra of Example 3. Let \( \eta = 6 \) The multiplication in \( A^* \) is given in table 4.

### Table 4. A quasi-equivalent algebra.

|   |   |   | J | K |
|---|---|---|---|---|
| e | e | (6a - 5b) I | (6a - 5b) J | (6a - 5b) K |
| I | (6b - 5a) I | ce | 11K | -11J |
| J | (6b - 5a) J | -11K | ce | 11I |
| K | (6b - 5a) K | 11J | -11I | ce |

3. **Algebras invariant under \( S_3 \)**

Knuth [1] was interested in the cubical arrays associated with finite semifields, that is, finite division rings with unit element.

Oehmke [11] showed that any finite semifield \( S \) has a cyclic basis \( \{1, x, x^2, \ldots, x^n\} \) for some \( x \in S \). Using this basis and \( n \geq 2 \), the structure constants become \( \Gamma_{0ik} = \Gamma_{0ik} = \delta_{ik} \) and \( \Gamma_{11k} = \delta_{k2} \). If the cubical array is to be invariant under the action of \( S_3 \), we must have \( \Gamma_{110} = \delta_{11} \) and \( \Gamma_{110} = \delta_{02} \). Hence if the dimension is greater than or equal to 2, then cubical array is not invariant under \( S_3 \). If the dimension is one or two, the semifields is a finite field.

We examine three patterns of non-zero entries in the cubical array. The first has non-zero elements given by
\[
\Gamma_{iii} = \alpha_i \neq 0, \quad i = 1, \ldots, n
\]
If the algebra \( A \) is a finite sum of copies of the field \( F \), then \( A \) has a basis \( \{e_i : i = 0, 1, \ldots, n - 1\} \) of pairwise orthogonal idempotents.

We assume that the algebra \( A \) has a basis \( \{e_0, e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\} \).

**Theorem 3.1.** \( A \) is an \( n \)-dimensional algebra whose set of nonzero \( \Gamma_{ijk} \)'s invariant under \( S_3 \) is the set \( \{\Gamma_{iii} = \alpha_i \neq 0 : i = 0, 1, \ldots, n - 1\} \) if and only if the algebra \( A \) is a direct sum of fields.

**Proof.** \( \Gamma_{iii} = \alpha_i \neq 0 \) says that for each element \( e_i \) of the basis, \( e_i^2 = \alpha_i e_i \neq 0 \). The set \( \{b_i = e_i/\alpha_i : i = 1, \ldots, n\} \) of pairwise orthogonal idempotent will also be a basis of \( A \) over \( F \). Relative to this new basis \( \Gamma_{ijk} = 0 \) if \( i \neq j \) for all \( k \), and in general, \( b_i b_j = \delta_{ij} \). Then \( A = \oplus_{i=0}^{n-1} F b_i \). \( \square \)
These algebras will all be associative, commutative, and have an identity element
\[ e = \hat{b}_0 + \hat{b}_2 + \cdots + \hat{b}_{n-1} \]

**Corollary 3.2.** If \( \mathfrak{A} \) is an \( m \)-dimensional algebra whose set of nonzero \( \Gamma_{ijk} \)s invariant under \( S_3 \) is the set \( \{ \Gamma_{ii} = \alpha_i \neq 0 : i = 0, \ldots, n \leq m \} \) then \( \mathfrak{A} \) is a direct sum of a finite direct sum of copies of the field \( F \) and a zero ring.

Our next collection of algebras will not be associative. Since there are usually many bases of a vector space, we choose one that is particularly useful.

Let \( \mathfrak{A} \) denote an unital algebra \( \mathfrak{A} \) with basis \( \{ e_0, e_1, e_2, \ldots, e_{n-1} \} \) over the field \( F \). Relative to this basis the cubical array has elements \( \Gamma_{0ii} = \Gamma_{i0i} = 1 \) for all \( i = 0, \ldots, n \). This time the non-zero elements are generated by the set

\[ \Gamma_{i0i} = 1, \quad i = 0, \ldots, n - 1 \]

with the axes permuted by \( S_3 \). The set of triples

\[ \{(0, i, i), (i, 0, i), (i, i, 0) : i = 0, \ldots, n - 1\} \]

is invariant under the action of any \( \sigma \in S_n \).

This condition and invariance under \( S_3 \) implies that the algebra is commutative, unital and, if \( n \geq 2 \), the algebra will not be associative since \((e_1, e_1, e_2) = e_2\). The multiplication for the algebra is given by

\[ e_0e_i = e_i = e_ie_0, \quad e_ie_j = \delta_{ij}, \quad i, j = 0, \ldots, n - 1 \]

In summary, we have

**Theorem 3.3.** The set of nonzero entries is \( \Gamma_{ii}, \Gamma_{jj}, \Gamma_{jjj} \), all equal to 1, if and only if the algebra is a Type D Jordan algebra with multiplication as given above.

By Jacobson [12], these algebras will be semisimple.

4. Conclusion

To summarize the results about the cubical arrays associated with finite dimensional algebra under actions of \( S_3 \):

(i) A finite division ring remains a finite division (not necessarily the same division ring).

(ii) A direct sum of algebras remains a direct sum (again, not necessarily the same direct sum).

(iii) A subclass of weakly quadratic algebras remains weakly quadratic.

(iv) The planes determined by the Hughes Kleinfeld semifields have exactly one "image" under the action of \( S_3 \).

(v) A finite direct sum of fields is invariant under the action of \( S_3 \).

(vi) A Type D Jordan algebra is invariant if and only if the multiplication satisfies

\[ e_0e_i = e_i = e_ie_0, \quad 1 \leq i \leq n \quad \text{and} \quad e_ie_j = \delta_{ij}e_0 \]

(vii) A quadratic algebra is invariant if and only if it is a Type D Jordan algebra with the multiplication described in 5.

There are additional patterns that can be analyzed, i.e., the pattern generated by \( \{(i, i, i), (i, i, 0)\} \). A slight modification of the above arguments show that the algebras need not be finite dimensional.
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