Soft Wilson lines in soft-collinear effective theory

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Abstract

The effects of the soft gluon emission in hard scattering processes at the phase boundary are resummed in the soft-collinear effective theory (SCET). In SCET, the soft gluon emission is decoupled from the energetic collinear part, and is obtained by the vacuum expectation value of the soft Wilson-line operator. The form of the soft Wilson lines is universal in deep inelastic scattering, in the Drell-Yan process, in the jet production from $e^+e^-$ collisions, and in the $\gamma^*\gamma^* \rightarrow \pi^0$ process, but its analytic structure is slightly different in each process. The anomalous dimensions of the soft Wilson-line operators for these processes are computed along the light-like path at leading order in SCET and to first order in $\alpha_s$, and the renormalization group behavior of the soft Wilson lines is discussed.

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I. INTRODUCTION

The factorization theorem [1] has been one of the most important issues in QCD. It states that scattering cross sections or amplitudes can be written as a product or a convolution of the hard, the collinear, and the soft parts, and each part depends on a single scale. The factorization of the hard, collinear, and soft parts in various processes was studied previously in the full theory, considering the appropriate kinematic regions in which the particles involved can be either collinear or soft [2, 3, 4, 5, 6]. It has been a complicated task to disentangle the contributions from all the possible kinematic regions, but the soft-collinear effective theory (SCET) [8, 9, 10, 11] can be helpful in understanding this difficult procedure since SCET is formulated in such a way that the collinear interactions and the soft interactions are decoupled to all orders in $\alpha_s$ from the beginning.

SCET successfully describes the factorization properties in $B$ decays such as $B \rightarrow D\pi$ [12], $B \rightarrow \gamma\nu\bar{\nu}$ [13], nonleptonic $B$ decays [14, 15, 16] and $B \rightarrow K^{*}\gamma$ [17]. And it can be applied to other high-energy processes to probe the factorization properties [18]. In order to show the utility of SCET in high-energy processes, we will consider deep inelastic scattering, the Drell-Yan process, jet production from $e^+e^-$ collisions, and the $\pi-\gamma$ form factor with two virtual photons, where all the hadrons in these processes are light and energetic so that we can apply SCET. The treatment of these high-energy processes with SCET has been extensively discussed in Ref. [19].

In SCET the factorization occurs in each process, in which the scattering cross section or the form factor can be written as a convolution of the hard part (the Wilson coefficients from matching the full theory onto SCET$_1$), the collinear part (the jet functions from matching SCET$_1$ onto SCET$_{II}$), and the soft part (the matrix elements of the remaining operators in SCET$_{II}$). The matrix elements between hadronic states are nonperturbative and cannot be computed from first principles, but their scaling behavior can be obtained using perturbation theory. In high-energy scattering processes, the matrix elements of gauge-invariant operators are usually parameterized as either the parton distribution functions for hadrons in the initial states, or the fragmentation functions for the final-state hadrons [20].

In this paper, we are mainly interested in the nonperturbative part from the soft interactions in SCET. The effect of soft interactions in the soft part does not usually appear at leading order in SCET since it cancels in many processes. However, near the boundary of
the phase space, the cancellation is incomplete and the nonperturbative effect of soft gluon emission can be important. In this case, the conventional operator product expansion breaks down, but still we can consider gauge-invariant nonlocal operators which are connected by an operator with a string of gauge fields. The endpoint behavior in deep inelastic scattering using SCET was first discussed in Ref. [21], focusing on the scaling behavior of the matrix elements of the collinear operators. The nonperturbative effects of the soft gluon emission in $e^+e^-$ collisions were first considered in terms of SCET in Refs. [22, 23].

Near the phase boundary, the final operators, of which the matrix elements between hadronic states describe nonperturbative effects, consist of collinear fields, which are connected by soft Wilson lines. In SCET, since the soft interactions are decoupled from the collinear sector, we can decompose the operators in terms of the collinear operators and the soft Wilson lines and we can consider the matrix elements of these two kinds of operators. Though the matrix element of the soft Wilson lines cannot be computed in perturbation theory, the renormalization group behavior of the soft Wilson-line operator can be derived.

There has been such consideration in the full theory. For example, in Ref. [6], the authors considered the vacuum expectation value of a soft Wilson line which is defined as

$$ W(C) \equiv \frac{1}{N} \langle 0 | P \exp \left( ig \oint_C dz_{\mu} A^\mu(z) \right) | 0 \rangle. $$  

(1)

In Ref. [4], the vacuum average of a soft Wilson loop was analyzed, which is given by

$$ W_T(C) = \frac{1}{N} \langle 0 | \text{tr} P \exp \left( ig \oint_C dz_{\mu} A^\mu(z) \right) | 0 \rangle. $$  

(2)

Here the integration path $C$ is determined by the kinematics of the processes, $T$ orders gauge fields $A^a_{\mu}(z)$ in time, and $P$ orders the generators $T^a$ of the $SU(N)$ gauge group along the path $C$. The difference between $W(C)$ and $W_T(C)$ is that the gluon fields in $W(C)$ are ordered along the path $C$, but not according to time. Therefore on different parts of the path $C$, the gluon fields are time or anti-time ordered. The minute difference in the definitions of $W_T(C)$ and $W(C)$ affects the analytic structure of the soft Wilson lines.

The main theme of the paper is to study the analytic structure of the Wilson lines, especially the soft Wilson lines in SCET near the boundary of the phase space. We show that the soft Wilson lines appear universally in deep inelastic scattering, in the Drell-Yan process and in the jet production from $e^+e^-$ collisions, in which the Wilson lines appear in the matrix elements squared or the discontinuity of the forward scattering amplitude. They
also appear in the $\pi$-$\gamma$ form factor, in which the soft Wilson line appears in an amplitude. After we identify the analytic structure of the soft Wilson lines, we compute the anomalous dimensions of the soft Wilson lines in SCET to see the renormalization group behavior.

The conventional approach [1] for this analysis is to separate the region of momentum and to extract the contribution from the collinear region and the soft region, but the advantage of SCET comes from the fact that this separation is performed automatically when we perform the two-step matching [24] with the two effective theories called SCET$_I$ and SCET$_II$. Near the boundary of the phase space, the intermediate states can have momentum of order $p_X^2 \sim Q^2(1-x)$, where $Q$ is the large momentum scale with $x \sim 1$, but still $p_X^2 \gg \Lambda_{\text{QCD}}^2$. First, we integrate the degrees of freedom of order $p^2 \sim Q^2$ from the full theory and match onto the intermediate effective theory, SCET$_I$. Here the ultrasoft (usoft) particles with momentum of order $\Lambda \sim \Lambda_{\text{QCD}}$ can interact with collinear particles. Then we integrate out the degrees of freedom of order $p^2 \sim Q^2(1-x)$ to go down to SCET$_II$. In SCET$_II$, the soft particles and the collinear particles are decoupled, and the effects of soft gluon emission in SCET can be studied without regard to the collinear sector. The soft Wilson line operator in SCET, which we will elaborate in detail, takes the form

$$K(\eta) = \frac{1}{N} \text{tr} \left( \bar{S} i S_\delta (\eta + in \cdot \partial) S \bar{S} \right), \quad (3)$$

where the Wilson lines $S$ and $\bar{S}$ are the Fourier transforms of

$$S(z) = \exp \left[ ig \int ds n \cdot A_s (ns + z) \right], \quad \bar{S}(z) = \exp \left[ ig \int ds \bar{n} \cdot A_s (\bar{n}s + z) \right], \quad (4)$$

and here we suggest how to prescribe the path ordering, which is determined by the processes under consideration. We also study the renormalization properties of the soft Wilson-line operators for a path $C$ lying on the light cone in SCET after we separate the soft interactions in SCET$_II$. This is in contrast to Eqs. (1) and (2) since the paths are partially lying on the light-cone. In our approach, we put all the energetic collinear particles on the light-cone and consider the soft Wilson lines on these light cones at leading order in SCET.

The structure of the paper is as follows: In Sec. II we study in detail the analytic structure of the soft Wilson lines in SCET. In Sec. III we explain how the operator $K(\eta)$ for the soft gluon emission appears in deep inelastic scattering, in the Drell-Yan process, in the jet production from $e^+e^-$ scattering, and in the $\pi$-$\gamma$ form factor with two virtual photons. In Sec. IV we compute the radiative correction for $K(\eta)$ at one loop and derive
the renormalization group equation. We also show that the result is consistent with the result obtained by Korchemsky et al. [4, 5, 6]. In the final section, we present conclusions and compare the conventional approach with the approach in SCET.

II. ANALYTIC STRUCTURE OF THE WILSON LINES

The analytic structure of the soft Wilson lines is important in computing radiative corrections because the position of the poles is determined by the appropriate $i\epsilon$ prescription. In SCET, the soft Wilson line is obtained by factorizing the usoft interactions in SCET\textsubscript{I}. One way of deriving the Wilson line is to attach usoft gluons to collinear fields and use the eikonal approximation. There are four possible cases and they are shown in Fig. 1.

In Fig. 1(a), soft gluons are attached to an incoming quark from $-\infty$ to $x$ or an outgoing antiquark from $x$ to $-\infty$, and the Wilson line is given by

$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perms}} \frac{(-g)^m}{m!} \frac{n \cdot A_s^{a_n} \cdots n \cdot A_s^{a_1}}{n \cdot \left(\sum_{i=1}^{n} k_i + i\epsilon\right) \cdots \left(n \cdot k_1 + i\epsilon\right)} T_{a_n} \cdots T_{a_1},$$

(5)

which is written as

$$Y = \sum_{\text{perm}} \exp \left[ \frac{1}{n \cdot \mathcal{P} + i\epsilon} (-g n \cdot A_s) \right],$$

(6)

where $\mathcal{P}^\mu$ is the momentum operator. Here $A_s^\mu$ denotes the usoft gluon in SCET\textsubscript{I}. In the literature, this is commonly denoted as $A_{us}^\mu$, and after we go down to SCET\textsubscript{II}, we relabel them as the soft gluon $A_s^\mu$. However, we will use $A_s^\mu$ for the usoft (soft) gluon in SCET\textsubscript{I}.

![Figure 1](attachment:image.png)

FIG. 1: Attachment of soft gluons to (a) an incoming quark, (b) an outgoing quark, (c) an outgoing antiquark, and (d) an incoming antiquark.
(SCET\textsubscript{ll}) for simplicity. The exponentiated form $Y$ is related to the Fourier transform $Y(x)$ of the path-ordered exponential

$$Y(x) = P \exp \left( ig \int_{-\infty}^{x} ds n \cdot A_{s}(ns) \right),$$

where the path ordering $P$ means that the fields are ordered in such a way that the gauge fields closer (farther) to the point $x$ are moved to the left (right).

Note that the Feynman “$i\epsilon$” prescription enforces the path ordering. This can be seen if we consider the Fourier transform of the exponent in Eq. (6). $n \cdot A_{s}(x)$ depends on the coordinate $x$, but since the soft gluons are attached to the collinear particle moving in the $n^\mu$ direction, $n \cdot A_{s}(x)$ depends only on $\vec{x} = \vec{\pi} \cdot x/2$ at leading order in $\Lambda$. It means that the Fourier transform of $n \cdot A_{s}(x)$ depends only on $n \cdot q$. Therefore we can consider the one-dimensional Fourier transform

$$n \cdot A_{s}(x) = \frac{1}{2\pi} \int dn \cdot q e^{-i n \cdot q \bar{x}} n \cdot A_{s}(n \cdot q), \quad n \cdot A_{s}(n \cdot q) = \int d\bar{x} e^{i n \cdot q \bar{x}} n \cdot A_{s}(\bar{x}),$$

since the remaining components of the Fourier transform yield only the delta functions. The Fourier transform of the exponent in Eq. (6) can be written as

$$-\frac{g}{2\pi} \int dn \cdot q e^{-i n \cdot \bar{x}} \frac{e^{-i n \cdot q \bar{x}} n \cdot A_{s}(n \cdot q)}{n \cdot q + i\epsilon} n \cdot A_{s}(n \cdot q) = -\frac{g}{2\pi} \int_{-\infty}^{\infty} dy \int dn \cdot q \frac{e^{i n \cdot q (\bar{y} - \bar{x})}}{n \cdot q + i\epsilon} n \cdot A_{s}(\bar{y}).$$

We can perform the integration over $n \cdot q$ in the complex plane. Because the pole is in the lower half plane, the integration over $n \cdot q$ becomes $-2\pi i \theta(\bar{y} - \bar{x})$. And Eq. (9) is given by

$$ig \int_{-\infty}^{x} d\bar{y} n \cdot A_{s}(\bar{y}).$$

In order to see how the path ordering is specified, let us consider the Fourier transform of Eq. (6) at order $g^2$, which is given by

$$\frac{1}{n \cdot P + i\epsilon} e^{g n \cdot A_{s}} - \frac{1}{n \cdot P + i\epsilon} e^{g n \cdot A_{s}} \int \frac{dn \cdot q_{1} dn \cdot q_{2}}{(2\pi)^2} e^{-i(n_{1} + q_{2})\bar{x}} \frac{g n \cdot A_{s}(n \cdot q_{1}) g n \cdot A_{s}(n \cdot q_{2})}{(n \cdot (q_{1} + q_{2}) + i\epsilon)(n \cdot (q_{1} + q_{2}))}$$

$$= \int d\bar{y} d\bar{z} \int \frac{dn \cdot q_{1} dn \cdot q_{2}}{(2\pi)^2} e^{-i(n_{1} + q_{2})(\bar{y} - \bar{z})} \frac{g n \cdot A_{s}(\bar{y}) g n \cdot A_{s}(\bar{z})}{(n \cdot (q_{1} + q_{2}) + i\epsilon)(n \cdot (q_{1} + q_{2}))}$$

$$= \int d\bar{y} d\bar{z} \frac{g n \cdot A_{s}(\bar{y}) g n \cdot A_{s}(\bar{z})}{(2\pi)^2} \int \frac{dn \cdot q_{1}}{n \cdot q_{1} + i\epsilon} \int \frac{dn \cdot q_{2}}{n \cdot (q_{1} + q_{2}) + i\epsilon} e^{i(n_{1} + q_{2})(\bar{y} - \bar{z})}$$

$$= \int d\bar{y} d\bar{z} g n \cdot A_{s}(\bar{y}) g n \cdot A_{s}(\bar{z}) \int \frac{dn \cdot q_{1}}{2\pi} e^{i(n_{1} + q_{1})(\bar{y} - \bar{x})} (-i) e^{-i(n_{1} + q_{1})(\bar{y} - \bar{z})} \theta(\bar{y} - \bar{x}) \theta(\bar{z} - \bar{y}).$$
\[ (-i)^2 \int d\bar{\tau} \bar{\tau} g_n \cdot A_s(\bar{\tau}) g_n \cdot A_s(\bar{\tau}) \theta(\bar{\tau} - \tau) \theta(\tau - \bar{\tau}) \]
\[ = (-i)^2 \int_{-\infty}^{\infty} d\tau g_n \cdot A_s(\tau) \int_{-\infty}^{\infty} d\tau' g_n \cdot A_s(\tau') = \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' P[g_n \cdot A_s(\tau) g_n \cdot A_s(\tau')]. \]

In Eq. (11), the integrations over \( n \cdot q_1 \) and \( n \cdot q_2 \) are performed in the complex plane, and the final relation with the path ordering is obtained by changing the region of integration.

Similarly, for the \( m \)-th order term, we have the relation
\[ \left[ \frac{1}{n \cdot \mathcal{P} + i \epsilon} g_n \cdot A_s \right]^m \to \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d\tau_1 \cdots d\tau_m P\left[ g_n \cdot A_s(\tau_1) \cdots g_n \cdot A_s(\tau_m) \right], \]
which shows that Eq. (10) indeed specifies the path-ordered exponential in Eq. (7).

We can find an equivalent expression for \( Y \) if we replace the incoming gluons by the outgoing gluons in Fig. 1 (a). Then the relative sign of the quark momentum \( p \) and the gluon momenta \( k_i \) becomes opposite and \( Y \) can be written as
\[ Y = \sum_{\text{perm}} \exp\left[ -g_n \cdot A_s \frac{1}{-n \cdot \mathcal{P} + i \epsilon} \right], \]
but the corresponding Fourier transform is given by the same form in Eq. (7).

We can proceed in a similar way for other diagrams. Fig. 1 (b) describes an outgoing quark from \( x \) to \( \infty \), or an incoming antiquark from \( \infty \) to \( x \) emitting soft gluons, and the soft Wilson line, and its Fourier transform are written as
\[ \tilde{Y}^\dagger = \sum_{\text{perm}} \exp\left[ -g_n \cdot A_s \frac{1}{n \cdot \mathcal{P} + i \epsilon} \right], \quad \tilde{Y}^\dagger(x) = \mathcal{P} \exp\left( ig \int_x^{\infty} ds n \cdot A_s(ns) \right). \]

Our notation is such that the Wilson lines with a tilde propagate between \( x \) and \( \infty \), and the Wilson lines without a tilde propagate between \( -\infty \) and \( x \). Fig. 1 (c) is the case in which an outgoing antiquark is moving from \( x \) to \( \infty \), or an incoming quark from \( \infty \) to \( x \), emitting soft gluons, and the corresponding Wilson line is given by
\[ \tilde{Y} = \sum_{\text{perm}} \exp\left[ -g_n \cdot A_s \frac{1}{n \cdot \mathcal{P} - i \epsilon} \right], \quad \tilde{Y}(x) = \mathcal{P} \exp\left( -ig \int_x^{\infty} ds n \cdot A_s(ns) \right), \]
where \( \mathcal{P} \) means anti-path ordering. Fig. 1 (d) describes an incoming antiquark from \( -\infty \) to \( x \), or an outgoing quark from \( x \) to \( -\infty \) emitting soft gluons, and the Wilson line is given as
\[ Y^\dagger = \sum_{\text{perm}} \exp\left[ -g_n \cdot A_s \frac{1}{n \cdot \mathcal{P} + i \epsilon} \right], \quad Y^\dagger(x) = \mathcal{P} \exp\left( -ig \int_x^{\infty} ds n \cdot A_s(ns) \right). \]

From now on, we will omit the summation over all the possible permutations in \( Y, Y^\dagger, \tilde{Y} \) and \( \tilde{Y}^\dagger \). These results are summarized in Table I. And we explicitly construct the Wilson
TABLE I: Summary of the soft Wilson lines shown in Fig. 1

| type | Wilson line | Fourier transform |
|------|-------------|-------------------|
| (a): | $Y = \exp\left[\frac{1}{n \cdot \mathcal{P} + i \epsilon}(-g_n \cdot A_s)\right]$ | $Y(x) = \mathcal{P} \exp\left[ig \int_{-\infty}^{x} ds_n \cdot A_s(ns)\right]$ |
| (b): | $\tilde{Y}^\dagger = \exp\left[-g_n \cdot A_s\frac{1}{n \cdot \mathcal{P} + i \epsilon}\right]$ | $\tilde{Y}^\dagger(x) = \mathcal{P} \exp\left[ig \int_{x}^{\infty} ds_n \cdot A_s(ns)\right]$ |
| (c): | $\tilde{Y} = \exp\left[1 \frac{1}{n \cdot \mathcal{P} - i \epsilon}(-g_n \cdot A_s)\right]$ | $\tilde{Y}(x) = \mathcal{P} \exp\left[-ig \int_{\infty}^{x} ds_n \cdot A_s(ns)\right]$ |
| (d): | $\tilde{Y}^\dagger = \exp\left[-g_n \cdot A_s\frac{1}{n \cdot \mathcal{P} - i \epsilon}\right]$ | $\tilde{Y}^\dagger(x) = \mathcal{P} \exp\left[-ig \int_{-\infty}^{x} ds_n \cdot A_s(ns)\right]$ |

lines to satisfy the unitarity $Y^\dagger(x)Y(x) = \tilde{Y}^\dagger(x)\tilde{Y}(x) = 1$ and it is also true for $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$. With our notation, $YY^\dagger$ does not have to be 1 unless we fix the gauge as $n \cdot A_s = 0$, but in this case, $Y, \tilde{Y}$ become formally 1.

The factorization of the usoft interaction in SCET$_1$ can be achieved by redefining the collinear fields as

$$\xi \rightarrow Y(\tilde{Y})\xi, \quad \bar{\xi} \rightarrow \bar{\xi}Y^\dagger(\tilde{Y}^\dagger), \quad \chi \rightarrow \bar{\mathcal{Y}}(\tilde{\mathcal{Y}})\chi, \quad \bar{\chi} \rightarrow \bar{\mathcal{Y}}Y^\dagger(\tilde{\mathcal{Y}}^\dagger),$$

where the choice of the Wilson lines depends on the physical processes under consideration. In order to choose appropriate soft Wilson lines, we should consider whether the collinear particle, which emits soft gluons, is a particle or an antiparticle, and where it is directed.

The possible soft Wilson lines listed in Table I are the building blocks to construct soft Wilson lines in physical processes. The $i\epsilon$ prescription is irrelevant if we consider only tree-level processes, but it is critical when the radiative corrections are investigated to see the scaling behavior of the soft Wilson lines.

III. DERIVATION OF THE SOFT WILSON OPERATORS IN SCET

Now that the analytic structure of the soft Wilson lines is completely known, we can investigate which paths the soft Wilson lines take in various physical processes. We consider deep inelastic scattering, the Drell-Yan process, the jet production in $e^+e^-$ collisions, and the $\pi$-$\gamma$ form factor near the phase boundary, in which the same type of the soft Wilson line appears, but with a different analytic structure. In all these processes, we put all the collinear particles on the light cone, namely either in the $n^\mu$ or $\bar{n}^\mu$ direction and consider
the leading contributions in SCET. The effect of the soft gluon emission can be expressed as the matrix elements of the soft Wilson-line operator along the specific path which is determined by the kinematics. By applying the two-step matching in SCET near the phase space boundaries, we derive the soft Wilson-line operator in SCET\textsubscript{II}. We choose the collinear particles in each process in such a way that the resultant soft Wilson lines have the same form in Eq. (3).

We consider the electromagnetic current $j^\mu = \overline{\psi} \gamma^\mu \psi$ for these processes, which can be transparently extended to other currents. The first step is to express the current in SCET\textsubscript{I} by matching the full theory onto SCET\textsubscript{I}. The fermion field $\psi$ is replaced by the corresponding collinear fermions $\xi (\chi)$ in the $n^\mu (\pi^\mu)$ direction with the appropriate Wilson coefficient. Before we go down to SCET\textsubscript{II} below the scale $p_\perp^2 \sim Q^2 (1 - x)$, we factorize the usoft interactions by redefining the collinear fields. The form of the usoft Wilson lines depends on the physical processes under consideration. After the redefinition, we scale down to SCET\textsubscript{II} and extract the contribution of the soft gluon emission.

A. Deep inelastic scattering

In deep inelastic scattering, we choose the Breit frame in which the proton moves in the $\pi^\mu$ direction and the final state consists of a jet in the $n^\mu$ direction. The momentum transfer to the hadronic system is $Q^2 = -q^2$ which is the large scale, and $q^\mu = (\pi \cdot q, q_\perp, n \cdot q) = (Q, 0, -Q)$. The Bjorken variable $x$ is defined as $x = Q^2 / (2p \cdot q) \approx -n \cdot q / n \cdot p \sim Q / n \cdot p$, where $p^\mu$ is the momentum of the proton. We consider the region with $x \sim 1$, where almost all the momentum of the proton is carried by a parton undergoing a hard process. In this case, the final-state particles have invariant mass $p_X^2 = (p + q)^2 \sim Q^2 (1 - x)$. The momenta $p^\mu$, and $p_X^\mu$ in the Breit frame scale as

$$p^\mu = (\pi \cdot p, p_\perp, n \cdot p) \sim \left( \frac{\Lambda^2}{Q}, \Lambda, \frac{Q}{x} \right), \quad p_X^\mu \sim (Q, \Lambda, Q \frac{1 - x}{x}). \quad (18)$$

We integrate out the degrees of freedom of order $Q^2$ to obtain SCET\textsubscript{I}, and subsequently we integrate out the degrees of freedom of order $Q^2 (1 - x)$ to go down to SCET\textsubscript{II}.

The hadronic tensor in the full theory is defined as

$$W^{\mu\nu}(p, q) = \frac{1}{\pi} \text{Im} T^{\mu\nu}(p, q), \quad (19)$$
and \( T^{\mu\nu} \) is the spin-averaged matrix element of the forward Compton scattering amplitude which is given by

\[
T^{\mu\nu}(p, q) = \langle p | \hat{T}^{\mu\nu}(q) | p \rangle_{\text{spin av.}}, \quad \hat{T}^{\mu\nu} = i \int d^4z e^{i q \cdot z} T[j^\mu(z) j^\nu(0)],
\]

where \( j^\mu(x) = \bar{\psi} \gamma^\mu \psi(x) \) is the electromagnetic current for electroproduction. If the virtuality of the intermediate states is of order \( Q^2 \), we can perform the conventional operator product expansion in which the product of the current operators is expanded in terms of local operators in powers of \( 1/Q \) \cite{19}. However, when the virtuality of the final-state particles is of order \( p_X^2 \approx Q^2(1 - x) \sim Q \Lambda \), the operator product expansion breaks down. Still, we can apply SCET and express \( \hat{T}^{\mu\nu} \) in terms of the bilocal operators \cite{21}. In SCET, the momenta of the external collinear particles scale as

\[
p^\mu_n = (\vec{n} \cdot p_n, p_n, n \cdot p_n) \sim (\Lambda^2/Q, \Lambda, Q),
\]

\[
p^\mu_n = (\vec{n} \cdot p_n, p_n, n \cdot p_n) \sim (Q, \Lambda, 1 - x/Q).
\]

The time-ordered product of \( \hat{T}^{\mu\nu} \) is schematically shown in Fig. 2. A quark \( \chi \) from \(-\infty\) is annihilated at point 0, which is described by the current \( \bar{\xi} \gamma^\mu \gamma^\nu W \xi \). Here \( W \) and \( \overline{W} \) are the collinear Wilson lines which are defined as

\[
W = \sum_{\text{perm}} \exp \left[ \frac{1}{\mathcal{P}} (-g \vec{n} \cdot A_n) \right], \quad \overline{W} = \sum_{\text{perm}} \exp \left[ \frac{1}{\mathcal{P}} (g n \cdot A_n) \right],
\]

where \( \mathcal{P}^\mu \) is the label momentum operator. On the other hand, at point \( z \), a quark \( \chi \) is produced moving to \( \infty \), which is described by the current \( \bar{\chi} W \gamma^\mu W \dagger \xi \). The useft Wilson lines associated with the collinear field \( \chi \) are uniquely determined by the behavior of the external states. However, the prescription of the useft Wilson line associated with the quark

![FIG. 2: Forward Compton scattering amplitude in deep inelastic scattering. At the spacetime point 0, a quark \( \chi \) from \(-\infty\) is annihilated and a quark \( \xi \) is produced and moves to \( z \). At \( z \), a quark \( \xi \) is annihilated, and a quark \( \chi \) is produced and moves to \( \infty \).](image-url)

\( 10 \)
FIG. 3: The description of the usoft Wilson lines in deep inelastic scattering. The usoft Wilson lines with \( \xi \) (a) going to \(-\infty\), (b) going to \(\infty\), (c) the resultant usoft Wilson lines from (a) and (b).

\( \xi \) is subtle because the quark \( \xi \) is a particle in the intermediate state in the forward Compton scattering amplitude.

Physically, as shown in Fig. 2 usoft gluons are emitted from the collinear quarks, and the usoft gluons in the \( n^\mu \) direction are emitted from the quark \( \xi \) between 0 and \( z \). In order to achieve this requirement, there are two possibilities which are shown in Fig. 3 (a) and (b), in which the soft Wilson lines are represented in spacetime. The lines with arrows represent the soft Wilson lines produced by the collinear particles moving in the specified direction. In Fig. 3 (a), a quark \( \xi \) is produced at point 0, moves to \(-\infty\). And it moves back from \(-\infty\) to point \( z \) and is annihilated. A second possibility is shown in Fig. 3 (b) in which a quark \( \xi \) is produced at point 0, moves to \( \infty \). And it moves from \( \infty \) back to \( z \) to be annihilated. In the region where two soft Wilson lines overlap [\((-\infty, 0)\) in Fig. 3 (a), and \((z, \infty)\) in Fig. 3 (b)], the soft Wilson lines cancel and the corresponding Wilson lines in both cases are equivalent to the soft Wilson lines in Fig. 3 (c). Therefore for the collinear fermions in the intermediate state, we can factorize the usoft interactions using either \( Y \) or \( \tilde{Y} \), and it does not affect the physical properties such as the radiative corrections. As we will see later in computing the radiative corrections, when the separation between 0 and \( z \) is lightlike in the \( n^\mu \) direction, the radiative corrections depend only on the analytic structure of the soft Wilson lines in the \( \overline{m}^\mu \) direction, and independent of the analytic structure of the soft Wilson lines in the \( n^\mu \) direction. The above statement holds true when \( n^\mu \) and \( \overline{m}^\mu \) are switched.

From the prescription described above, we factorize the usoft interactions by redefining the collinear fields as

\[
\begin{align*}
\xi W^\gamma \overline{W}^\dagger \chi : & \quad \xi \rightarrow \xi Y^\dagger(\tilde{Y}^\dagger), \quad A_n^\mu \rightarrow Y(\tilde{Y}) A_n^\mu Y^\dagger(\tilde{Y}^\dagger), \quad \chi \rightarrow \overline{Y} \chi, \quad A_n^\mu \rightarrow \overline{Y} A_n^\mu (\overline{Y}^\dagger), \\
\chi W^\gamma \overline{W}^\dagger \xi : & \quad \chi \rightarrow \chi \tilde{Y}^\dagger, \quad A_n^\mu \rightarrow \tilde{Y} A_n^\mu \tilde{Y}^\dagger, \quad \xi \rightarrow Y(\tilde{Y}) \xi, \quad A_n^\mu \rightarrow Y(\tilde{Y}) A_n^\mu Y^\dagger(\tilde{Y}^\dagger). \quad (23)
\end{align*}
\]
For the collinear particles in the \( n^\mu \) direction, the soft Wilson lines without (with) a parenthesis corresponds to the prescription described by Fig. 3 (a) [Fig. 3 (b)]. From now on, we describe the soft Wilson lines shown in Fig. 3 (a). The current operator in SCET \( _1 \) after the usoft factorization is written as

\[
  j^\mu(z) = C(Q) \left[ e^{i(\pi p_n n \cdot z/2 - n \cdot p_n n \cdot z/2)} \chi W Y^\dagger \gamma^\mu Y W^\dagger \chi(z) + e^{i(-\pi p_n n \cdot z/2 + n \cdot p_n n \cdot z/2)} \chi W Y^\dagger \gamma^\mu Y W^\dagger \chi(z) \right],
\]

where the exponential factors are the label momenta of order \( Q \). The Wilson coefficient \( C(Q) \) is obtained from the matching of the full theory onto SCET \( _1 \), and is given by

\[
  C(Q) = 1 + \frac{\alpha_s(Q) C_F}{4\pi} \left( -8 + \frac{\pi^2}{6} \right). \tag{25}
\]

The time-ordered product of the two currents in SCET \( _1 \) is given by

\[
  \hat{T}^{\mu\nu}_1 = i \int d^4 z e^{i q \cdot z} T[j^\mu(z) j^\nu(0)] = i C^2(Q) \int d^4 z e^{i(n_q + n_{p_h}) \cdot z/2} e^{i(\pi q - \pi p_n) n \cdot z/2} \chi W Y^\dagger \gamma^\mu Y W^\dagger \chi(0)
\]

\[
  \times T[\chi W Y^\dagger \gamma^\mu Y W^\dagger \chi(z) \chi W Y^\dagger \gamma^\mu Y W^\dagger \chi(0)]. \tag{26}
\]

Here we use the fact that, for label momenta \( p, p' \), and residual momenta \( k, k' \),

\[
  \int d^4 z e^{i(p - p' + k - k')z} = \delta_{p, p'} \int d^4 z e^{i(k - k')z}. \tag{27}
\]

The second exponential factor in the second line of Eq. (26) is converted to \( \delta_{n_q, n_{p_h}} = \delta_{Q, n_p} \), but the first exponential factor needs a careful treatment. Since \( n \cdot q + n \cdot p_h = -Q + Q/x = -Q + (Q + Q(1 - x))/x \), the label momentum of order \( Q \) turns into a Kronecker delta, and there is a slight mismatch in the large momenta in such a way that the difference of the two large momenta produces a small momentum of order \( Q(1 - x) \) for \( x \sim 1 \) and we explicitly keep it in Eq. (26).

Since there are no collinear particles in the \( n^\mu \) direction in the proton, we can write

\[
  \langle 0 | T[W^\dagger \xi](z)[\bar{\xi} W](0) | 0 \rangle \equiv i \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot z} J_P(k) \theta(k), \tag{28}
\]

which defines the jet function \( J_P(k) \). Here the label \( P \) is the sum of the label momenta carried by the collinear fields. This is obtained by integrating out the degrees of freedom with \( p_X^2 \sim Q^2(1 - x) \).
In SCET II after integrating out the degrees of freedom of order $Q^2 (1 - x)$ and renaming the usoft Wilson line $Y$ as the soft Wilson line $S$, the time-ordered product in SCET II is written as

$$
\hat{T}^{\mu \nu} = - \int d\omega C^2(\omega) \int \frac{d^4 k}{(2\pi)^4} \int d^4 z e^{i (Q(1-x) - n \cdot k)\pi z / 2} e^{-i k_{\perp} \cdot z_\perp - i n \cdot k n \cdot z / 2} 
\times J_P(n \cdot k) T \left[ \left( \chi \bar{W} S^\dagger \gamma^\mu S \right)(z) \delta(\omega - P_+) \frac{i}{2} \left( S^\dagger \gamma^\nu \bar{S}W^\dagger \chi \right)(0) \right],
$$

(29)

where $P_+ = n \cdot P + n \cdot P^\dagger$. Using the fact that

$$
\int \frac{d^4 k d^4 z}{(2\pi)^4} e^{-i k_{\perp} \cdot z_\perp - i n \cdot k n \cdot z / 2} = \frac{1}{2\pi} \int dn \cdot k d^4 z \delta^2 (z_\perp) \delta (n \cdot z / 2) = \frac{1}{4\pi} \int dn \cdot k d\pi \cdot z,
$$

(30)

Eq. (29) can be written as

$$
\hat{T}^{\mu \nu} = - \int d\omega C^2(\omega) \int \frac{dn \cdot k d\pi \cdot z}{4\pi} e^{i (Q(1-x) - n \cdot k)\pi z / 2} J_P(n \cdot k) 
\times T \left[ \left( \chi \bar{W} S^\dagger \gamma^\mu S \right)(\pi \cdot z) \delta(\omega - P_+) \frac{i}{2} \left( S^\dagger \gamma^\nu \bar{S}W^\dagger \chi \right)(0) \right]
= - \int d\omega C^2(\omega) \int \frac{dn \cdot k d\pi \cdot z}{4\pi} \int d\eta e^{i (Q(1-x) - n \cdot k - \eta)\pi z / 2} J_P(n \cdot k) 
\times T \left[ \left( \chi \bar{W} S^\dagger \gamma^\mu S \right)(0) \delta(\omega - P_+) \frac{i}{2} \delta(\eta + n \cdot i\partial) \left( S^\dagger \gamma^\nu \bar{W}^\dagger \chi \right)(0) \right]
= - \int d\omega C^2(\omega) \int dn \cdot k \int d\eta J_P(n \cdot k) \delta(Q(1-x) - n \cdot k - \eta) 
\times T \left[ \left( \chi \bar{W} \right)_{\alpha} \delta(\omega - P_+) \gamma^\mu \frac{i}{2} \gamma^\nu \left( \bar{W}^\dagger \chi \right)_\beta \left( S^\dagger S \delta(\eta + n \cdot i\partial) S^\dagger \bar{S} \right)_{\alpha \beta} \right]
\rightarrow - \int d\omega C^2(\omega) T \left[ \left( \chi \bar{W} \right) \delta(\omega - P_+) \gamma^\mu \frac{i}{2} \gamma^\nu \left( \bar{W}^\dagger \chi \right)(0) 
\times \int dn \cdot k J_P(n \cdot k) K(Q(1-x) - n \cdot k) \right],
$$

(31)

where $K(\eta)$ is the soft Wilson-line operator which is defined as

$$
K(\eta) = \frac{1}{N} \text{tr} \left( \bar{S}^\dagger S \delta(\eta + n \cdot i\partial) S^\dagger \bar{S} \right)
= \frac{1}{N} \text{tr} \exp \left[ -g \bar{\pi} \cdot A_s \frac{1}{\pi \cdot P^\dagger + i\epsilon} \right] \exp \left[ \frac{1}{n \cdot P + i\epsilon} (-g n \cdot A_s) \right] \delta(\eta + n \cdot i\partial)
\times \exp \left[ -g n \cdot A_s \frac{1}{n \cdot P^\dagger - i\epsilon} \right] \exp \left[ \frac{1}{n \cdot P - i\epsilon} (-g \bar{\pi} \cdot A_s) \right],
$$

(32)

while $K(\eta)$ becomes

$$
K(\eta) = \frac{1}{N} \text{tr} \left( \bar{S}^\dagger \tilde{S} \delta(\eta + n \cdot i\partial) \bar{S}^\dagger \bar{S} \right)
= \frac{1}{N} \text{tr} \exp \left[ -g \bar{\pi} \cdot A_s \frac{1}{\pi \cdot P^\dagger + i\epsilon} \right] \exp \left[ \frac{1}{n \cdot P - i\epsilon} (-g n \cdot A_s) \right] \delta(\eta + n \cdot i\partial)
\times \exp \left[ -g n \cdot A_s \frac{1}{n \cdot P^\dagger + i\epsilon} \right] \exp \left[ \frac{1}{n \cdot P + i\epsilon} (-g \bar{\pi} \cdot A_s) \right],
$$

(33)
if we use the prescription described by Fig. 3 (b).

In Eq. (31), the third equality is obtained because the soft particles are decoupled from
the collinear particles and $\alpha, \beta$ are color indices. And in the final result we take the color
average of the soft Wilson-line operator, represented by the trace since the soft Wilson line
will be sandwiched between the vacuum states. Care must be taken in separating the soft
part from the collinear part in Eq. (31) since the operator
\[ \delta(\eta + n \cdot i\partial) \text{ acts on } S^\dagger S W^\dagger \chi. \]
When $n \cdot i\partial$ acts on $W^\dagger \chi$, it produces the momentum of order $\Lambda$, that is,
$n \cdot i\partial W^\dagger \chi \sim \Lambda W^\dagger \chi$ since the largest momentum of order $Q$ is already extracted as the label momentum and the
remaining momentum can be of order $\Lambda$. However, due to the reparameterization invariance
\[ [25, 26],\]
we can transform away the momentum component of order $\Lambda$ in the collinear fields
$W^\dagger \chi$. Therefore the derivative $n \cdot i\partial$ applied to the collinear fields can be neglected. As a
result, the time-ordered product $\hat{T}^{\mu\nu}$ is given as the convolution of the Wilson coefficient
$C(\omega)$, the collinear operator, the jet function which can be obtained by matching between
SCET$_I$ and SCET$_II$, and finally the soft Wilson-line operator.

One important comment is in order about the relation between the forward scattering
amplitude and the scattering cross sections. In our approach in which we consider the
forward scattering amplitude, we take the discontinuity of the hadronic tensor $T^{\mu\nu}$ in the
final step, which is given by

\[ 2 \text{Im } T^{\mu\nu} = \sum_f \int d\Pi_f \langle p|J^\mu(-q)|f\rangle \langle f|J^\nu(q)|p\rangle_{\text{spin av.}}. \]

\[ = \sum_f \int d\Pi_f \langle p|T[\chi W Y Y^\dagger \gamma^\mu \tilde{Y} W^\dagger \xi] |f\rangle \langle f|T[\xi W Y Y^\dagger \gamma^\mu \tilde{Y} W^\dagger \chi] |p\rangle_{\text{spin av.}}, \quad (34) \]

where the first matrix element $\langle p|T[\chi W Y Y^\dagger \gamma^\mu \tilde{Y} W^\dagger \xi] |f\rangle$ is the hermitian conjugate of
$\langle f|T[\xi W Y Y^\dagger \gamma^\mu \tilde{Y} W^\dagger \chi] |p\rangle$, in which the time ordering becomes the anti-time ordering. We may develop a calculational technique to compute the radiative corrections for the matrix
elements squared \[3, 20], but it is more convenient to use the optical theorem. That is, we
consider the time-ordered products $\hat{T}^{\mu\nu}$, and we can compute the radiative corrections using
the conventional Feynman rules. At the end we take the discontinuity of $\hat{T}^{\mu\nu}$.

This approach is also advantageous if we use SCET. In SCET, the hadronic tensor is
described by the gauge-invariant operators, whose matrix elements can be attributed to the
nonperturbative effects. We first match the current between the full theory and SCET$_I$ near
$\mu^2 = Q^2$, then we integrate out the degrees of freedom of order $\mu^2 \sim Q^2(1 - x)$ to obtain
nonlocal, but gauge-invariant operators in SCETII. In this procedure, the jet function $J_P(k)$ is obtained to a desired order in $\alpha_s$, but all the radiative corrections for $J_P(k)$ come from collinear interactions, not from usoft interactions, otherwise the radiative corrections cannot be integrated out. Therefore when we take the imaginary part of $T^\mu\nu$, the contribution only comes from the discontinuity of the jet function $J_P(k)$. Then we consider the contributions from soft particles in SCETII, which are decoupled from the collinear sector. The soft radiative corrections do not have discontinuity at least at one loop order. To summarize, the strategy for computing radiative soft interactions is to compute the radiative corrections in the left-hand side of Eq. (34), in which all the operators are time-ordered, instead of computing the right-hand side of Eq. (34). Finally we take its discontinuity to obtain the hadronic tensor.

This distinction has also been realized in the consideration using the full theory. For example, in Ref. [4], the radiative correction is computed using the time-ordered products. On the other hand, in Ref. [5], it is computed using the matrix element squared and in obtaining the final result, the authors carefully distinguished the analytic structure of the soft Wilson lines by taking a hermitian conjugate from the results in Ref. [4] for the part on the opposite side of the physical cut. We would like to stress that both approaches are equivalent and the same results should be obtained. However, we choose the approach with the forward scattering amplitude in which only time-ordered products appear and we can compute radiative corrections using the Feynman rules derived from the form $K(\eta)$.

B. Drell-Yan processes

In Drell-Yan processes $p\bar{p} \rightarrow \ell\bar{\ell} + X$, lepton pairs, preferably muon pairs, are produced in proton-antiproton scattering. This process can be regarded as $p\bar{p} \rightarrow \gamma^* X$, where $\gamma^*$ is a virtual photon which eventually produces a lepton pair. Let the momenta of the proton (antiproton) be $p$ ($\bar{p}$) in the $n^\mu$ ($n'^\mu$) direction. The kinematic variables in this process are given by

\[
\begin{align*}
    s &= (p + \bar{p})^2 = n \cdot p\bar{p}, \\
    Q^2 &= q^2, \\
    \tau &= \frac{Q^2}{s}, \\
    x &= \frac{Q^2}{2p \cdot q} = \frac{Q^2}{n \cdot p\bar{p} \cdot q} = \frac{n \cdot q}{n \cdot p}, \\
    \tau &= \frac{Q^2}{2p \cdot q} = \frac{n \cdot q}{n \cdot p}. 
\end{align*}
\]
where \( q^\mu = (\pi \cdot q, q_\perp, n \cdot q) = (Q, 0, Q) \) is the momentum of the virtual photon. The momentum \( p_X = p + \overline{p} - q \) of the final hadronic system satisfies the relation

\[
p_X^2 = s + Q^2 - 2p \cdot q - 2\overline{p} \cdot q = Q^2 \left( 1 + \frac{1}{\tau} - \frac{1}{x} - \frac{1}{\overline{x}} \right).
\]

(36)

If one of the partons approaches the endpoint, say, \( x \sim 1 \), \((\overline{x} \text{ away from the endpoint})\)
\(\tau = x\overline{x} \approx x\), and \( p_X^2 \approx Q^2(1 - x)(1 - \overline{x})/x\). The momenta scale as

\[
p_\mu = p_\mu^\prime = (\pi \cdot p, p_\perp, n \cdot p) \sim \left( \frac{\Lambda^2}{Q}, \Lambda, \frac{Q}{x} \right),
\]

\[
\overline{p}_\mu = (\pi \cdot \overline{p}, \overline{p}_\perp, n \cdot \overline{p}) \sim \left( \frac{Q}{x}, \Lambda, \frac{\Lambda^2}{Q} \right),
\]

\[
p_X^\mu \sim \frac{1 - x}{x} Q, \Lambda, (1 - x)Q.
\]

(37)

Therefore near the endpoint \( x \sim 1 \), the final-state hadrons move in the \( n^\mu \) direction, that is, in the direction of the antiproton, and we can apply SCET to integrate out the degrees of freedom of order \( Q^2 \) and \( Q^2(1 - x) \) successively.

The hadronic tensor \( W^{\mu\nu} \) is defined in close analogy to deep inelastic scattering as

\[
W^{\mu\nu} = \frac{1}{4} \sum_{\text{spins}} \sum_X (2\pi)^4 \delta^4(p + \overline{p} - q - p_X) \langle p\overline{p}|j^\mu(0)|X\rangle \langle X|j^\nu(0)|p\overline{p}\rangle
\]

\[
= \frac{1}{4} \sum_{\text{spins}} \int d^4ze^{-iq\cdot z} \langle p\overline{p}|j^\mu(z)j^\nu(0)|p\overline{p}\rangle.
\]

(38)

As in the case of deep inelastic scattering, instead of computing Eq. (38), we consider the forward scattering amplitude which corresponds to the matrix element of the time-ordered product of the two currents, which is given as

\[
\mathcal{W} = \int d^4ze^{-iq\cdot z} T[j^\mu(z)j^\nu(0)].
\]

(39)

The electromagnetic current in the Drell-Yan process before the usoft factorization is also given as

\[
j^\mu(z) = C(Q) \left[ e^{-i(\pi \cdot p_n z/2 + n \cdot p_n \pi \cdot z/2)} \xi W \gamma^\mu \overline{W} \chi(z)
\]

\[
+ e^{i(\pi \cdot p_n z/2 + n \cdot p_n \pi \cdot z/2)} \overline{\chi} W \gamma^\mu W^\dagger \xi(z) \right],
\]

(40)

but the redefinition of the collinear fields to factorize the usoft interactions becomes different compared to deep inelastic scattering. In order to compute the forward scattering amplitude in the Drell-Yan process, the current \( \xi W \gamma^\mu \overline{W}^\dagger \chi(z) \) describes a quark \( \chi \) and an antiquark \( \xi \) coming from \(-\infty\), which are annihilated at \( z \). The second current in Eq. (40) describes
FIG. 4: (a) The usoft Wilson lines in the Drell-Yan process at lowest order. (b) the configuration of the usoft Wilson lines for the Drell-Yan process at higher orders in $\alpha_s$, which is equal to that of deep inelastic scattering, hence giving the same radiative corrections.

a quark $\chi$ and an antiquark $\xi$, which are created at $z$ and move to $\infty$. Since the collinear particles which determine the structure of the usoft Wilson lines are in the initial and the final states, the prescription is specified without ambiguity, and the usoft Wilson lines are described in Fig. 4 (a). Fig. 4 (b) will be explained at the end of this subsection.

With the prescription shown in Fig. 4 (a), the field redefinition becomes

\[ \xi W \gamma^\mu Y \dagger \chi : \xi \rightarrow \xi Y^\dagger, \quad A_\mu \rightarrow Y A_\mu Y^\dagger, \quad \chi \rightarrow \chi Y, \quad A_\mu \rightarrow YA_\mu Y^\dagger. \]  

Therefore the current operator after the redefinition becomes

\[ j^\mu(z) = C(Q) \left[ e^{-i(p_n \cdot n - z)/2} e^{i(p_n \cdot n - q) n \cdot z/2} \bar{\xi} W Y Y^\dagger \gamma^\mu Y Y^\dagger \chi(z) + e^{i(p_n \cdot n - z)/2} e^{i(p_n \cdot n - q) n \cdot z/2} \bar{\chi} W Y Y^\dagger \gamma^\mu Y Y^\dagger \xi(z) \right]. \]  

The time-ordered product of the two currents $W$ can be written as

\[ W = C(Q)^2 \int d^4 z e^{i(p_n \cdot n - q) n \cdot z/2} e^{i(p_n \cdot n - q) n \cdot z/2} \]  

\[ \times T \left[ \bar{\xi} W Y Y^\dagger \gamma^\mu Y Y^\dagger \xi(z) \bar{\xi} W Y Y^\dagger \gamma^\mu Y Y^\dagger \chi(0) \right], \]  

where $p_n^\mu = \bar{p}^\mu - p_X^\mu \sim (Q, \Lambda, Q(1 - x)/x)$ is the net collinear momentum in the $n^\mu$ direction.

In order to obtain the scattering cross section, we evaluate the matrix element of $W$ between the states with a proton and an antiproton, which corresponds to the forward scattering amplitude. And we take the discontinuity of this amplitude. However there is a delicate problem on how to extract the soft Wilson lines. First at zeroth order in $\alpha_s$, there are no final-state hadrons ($p_X = 0$). There is no intermediate scale which separates SCET$_1$ and
SCET\textsubscript{II}, and there is no requirement that the separation between 0 and \( z \) be lightlike. From now on, we will only consider the Drell-Yan process with some hadrons in the final state.

The time-ordered product \( \mathcal{W} \) now includes the collinear Lagrangian \( \mathcal{L}_c \), and it can be written at nontrivial lowest order in SCET\textsubscript{I} as

\[
\mathcal{W} = C(Q)^2 \int d^4z \int d^4y e^{i(n \cdot p_n - n \cdot q)\overline{\mathcal{P}}z/2} e^{i(n \cdot p_n - \overline{\mathcal{P}}q)n \cdot z/2} 
\times T\left[\mathcal{W}^\dagger \gamma^\mu \mathcal{W}(z)\mathcal{W}^\dagger \gamma^\mu \mathcal{W}(0), i\mathcal{L}_c(y)\right],
\]

(44)

and we can include additional \( i\mathcal{L}_c \) at higher orders. We contract the collinear fields in the \( n^\mu \) directions such that the resultant operators contains two collinear quarks or gluons in the \( n^\mu \) direction. The Drell-Yan processes from Eq. (44) at order \( \alpha_s \) are schematically described in Fig. 5. In SCET\textsubscript{II}, we take the matrix elements of these operators after Fierz transformation to parameterize as parton distribution functions in the proton and the antiproton. Note that we need the gluon operators including \( A_{n\bot} \) to compute the contribution from the gluon distribution functions in the antiproton, described in Fig. 5 (b) because only the operators with \( A_{n\bot} \) contribute.

The form of the collinear operators is complicated, but there are two important points in constructing the soft Wilson lines. First, when we contract the collinear particles, the jet functions are obtained. For example, when we contract collinear fermions in the \( n^\mu \) direction, we obtain

\[
\langle 0|T[\mathcal{W}^\dagger \xi(z)\mathcal{W}(0)]|0\rangle \equiv i \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot z/2} J_P(k),
\]

(45)

and similarly for the collinear gluons. All these jet functions depend only on \( n \cdot k \). Secondly, we have freedom to choose the directions of the soft Wilson lines associated with the collinear fermions in the intermediate states, as explained in deep inelastic scattering and we use this freedom to construct soft Wilson lines.

![Diagram](a) ![Diagram](b)

**FIG. 5:** The Drell-Yan processes at order \( \alpha_s \) in which a valence quark comes from the proton and (a) an antiquark, (b) a gluon from the antiproton.
Let us take a specific example shown in Fig. 5 (a). We can write the leading collinear Lagrangian as \( L_c(y) = \xi g A_n \cdot \Gamma \xi(y) \), where \( \Gamma \) is a Dirac structure. We can make it explicitly gauge invariant, but this form is enough to consider the soft Wilson lines. The usoft Wilson lines for the external states \( \chi(z) \) and \( \chi(0) \) are fixed as \( \chi(z) \tilde{Y}^\dagger \) and \( Y \chi(0) \). The intermediate states are obtained by contracting \( \xi(y), \tilde{\xi}(0), A_n(y), A_n(z) \). We are free to choose \( Y \) or \( \tilde{Y} \) for the intermediate states, but once we specify the soft Wilson line for one of the contracted fields, the usoft Wilson line for the remaining contracted field should be the same because the contracted fields should start from the same point. This was also illustrated in Fig. 3 (a) and (b) for the fermion in the intermediate state. If we choose the usoft Wilson line for \( \xi(z) \) as \( \tilde{Y} \xi(z) \), it fixes all the remaining usoft Wilson lines. Satisfying all these requirements, \( L_c \) becomes

\[
[\bar{\xi} \tilde{Y}^\dagger(y) [\tilde{Y} g A_n \cdot \Gamma \tilde{Y}^\dagger(y)] \bar{\xi} \chi(y)].
\]

(46)

The whole set of operators in Eq. (44) is written as

\[
(\bar{\chi} W \tilde{Y}^\dagger(z)) \gamma^\mu \left( \tilde{Y} W^\dagger \tilde{Y}^\dagger(z) \right) (\bar{\xi} g A_n \cdot \Gamma) \bar{\xi}(y) (\xi g A_n \cdot \Gamma) \gamma^\mu (\tilde{Y} W^\dagger \chi(0)),
\]

(47)

where the braces represent the contraction of the corresponding fields to produce the jet functions. After contracting the fields, if we express the color structure only, it becomes

\[
[(\bar{\chi} W(z))_\alpha \gamma^\mu (\tilde{Y} W^\dagger(z))_\alpha^\beta (T_a)_{\beta \tau} (\xi(z))_\tau ] \cdot [(\bar{\xi}(y))_\rho (T_a)_{\rho \sigma} \gamma^\mu (\tilde{Y} W^\dagger \chi(0))_\sigma \bar{\xi}(y)].
\]

(48)

Since the collinear quarks \( \xi, \tilde{\xi} \) and \( \chi, \bar{\chi} \) should form a color singlet to produce the parton distribution functions, we project out the color-singlet components for the usoft part. We can pull out the usoft part since it is decoupled from the collinear part, and it is written as

\[
\frac{\delta_{\alpha \nu} \delta_{\tau \rho}}{N} (\bar{Y} \tilde{Y}(\pi \cdot z))_\alpha^\beta (T_a)_{\beta \tau} (T_a)_{\rho \sigma} \bar{\xi}(y) \chi(0) = \frac{C_F}{N} \frac{1}{N} \text{tr} (\bar{Y} \tilde{Y}(\pi \cdot z) Y W^\dagger \chi(0)),
\]

(49)

which is described by Fig. 4 (b) with the lightlike separation between 0 and \( z \) because the jet functions depend only on \( n \cdot k \) and all the spacetime points are put on the light cone. For Fig. 5 (b) where gluons in the antiproton contribute, the same configuration of the soft Wilson lines is obtained. Interestingly enough, Fig. 4 (b) is the prescription for deep inelastic scattering, and we expect that the radiative corrections for deep inelastic scattering and the Drell-Yan process are the same. If we choose \( Y \xi(z), \tilde{Y} \) is replaced by \( Y \) in Eq. (49), and it yields the same configuration for the usoft Wilson lines, but corresponds to different a physical process.
We will not bother to write down $W$, but there is certainly a contribution from the soft Wilson lines of the form

$$\int dn \cdot k \tilde{J}_P(n \cdot k) \langle 0 | T \left[ K \left( Q(1 - x) - n \cdot k \right) \right] | 0 \rangle,$$

in the expression of $W$, where $\tilde{J}_P$ is a collection of jet functions. And the soft Wilson line operator $K(\eta)$ is given by

$$K(\eta) = \frac{1}{N} \text{tr} \tilde{S}^\dagger \tilde{S} \delta(\eta + n \cdot i \partial) \tilde{S}^\dagger \tilde{S} = \frac{1}{N} \text{tr} \exp \left[ -g \pi \cdot A_s + \frac{1}{n \cdot P + i\epsilon} (gn \cdot A_s) \right] \exp \left[ -g \pi \cdot A_s + \frac{1}{n \cdot P + i\epsilon} (gn \cdot A_s) \right].$$

C. Jet production in $e^+e^-$ scattering

The soft Wilson lines for the jet production in $e^+e^- \rightarrow Z \rightarrow q\bar{q}$ were first derived in Ref. [22, 23]. Our approach is equivalent, but different in the fact that we consider the time-ordered products of the hadronic electromagnetic current to evaluate the forward scattering amplitude. We choose the frame such that a quark (an antiquark) is produced in the $n_\mu$ direction, and the momentum for the final-state particles is given by $q_\mu = (n \cdot q, q_\perp, n \cdot q) = (Q, 0, Q)$. We consider the case in which a collinear jet in the $n_\mu$ direction is initiated by the antiquark. If the energy of the antiquark is close to its maximum, $x = n \cdot \bar{p}/Q \sim 1$, there are hadrons in the $n_\mu$ direction including the quark jet. The momentum $p_{\eta}^\mu$ of the collinear antiquark jet and the momentum $p_X^\mu$ of the hadrons in the $n_\mu$ direction scale as

$$p_{\eta}^\mu \sim \left( \frac{\Lambda^2}{xQ}, \Lambda, xQ \right), \quad p_X^\mu \sim (Q, \Lambda, (1 - x)Q).$$

Here we can apply the two-step matching of SCET, that is, we first integrate out the degrees of freedom of order $Q^2$, and go down to SCET II by integrating out the degrees of freedom of order $Q^2(1 - x)$.

The hadronic electromagnetic current is given by

$$j^\mu(z) = C(Q) \left[ e^{i(\pi \cdot p_n z/2 + n \cdot p_n \pi^\mu z/2)} \xi W^\gamma W^\dagger \chi(z) + e^{-i(\pi \cdot p_n z/2 + n \cdot p_n \pi^\mu z/2)} \chi W^\gamma W^\dagger \xi(z) \right].$$

In $e^+e^-$ collisions, the forward scattering amplitude corresponds to the photon polarization, in which a photon decays into collinear particles ($\xi$ and $\chi$ at leading order) and they turn
FIG. 6: (a) The description of the usoft Wilson lines in $e^+e^-$ collisions. (b) the resultant configuration of the usoft Wilson lines.

into a photon again. In this case, since both collinear particles are in the intermediate states, there can arise ambiguities as in the case of deep inelastic scattering. However, we can think of a collinear jet in the $\pi^\mu$ direction and the remainder becomes hadrons including the collinear jet in the $n^\mu$ direction. Then we can specify the direction of the antiquark $\chi$, and the description of the usoft Wilson lines is shown in Fig. 6. Of course, we can let $\xi$ go to $-\infty$, but as in deep inelastic scattering, that prescription also gives the usoft Wilson lines in Fig. 6 (b). And we will describe the process using the prescription in Fig. 6 (a).

The collinear fields are redefined as

$$
\xi W^\mu W^\dagger \chi: \xi \rightarrow \tilde{\chi} \tilde{Y}^\dagger, \ A_n^\mu \rightarrow \tilde{Y} A_n^\mu \tilde{Y}^\dagger, \ \chi \rightarrow \tilde{\chi} \chi, \ A_n^\mu \rightarrow \tilde{Y} A_n^\mu \tilde{Y}^\dagger,
\chi W^\mu W^\dagger \xi: \chi \rightarrow \tilde{\chi} W^\dagger, \ A_n^\mu \rightarrow \tilde{Y} A_n^\mu \tilde{Y}^\dagger, \ \xi \rightarrow \tilde{\chi} \chi, \ A_n^\mu \rightarrow \tilde{Y} A_n^\mu \tilde{Y}^\dagger,
$$

(54)

and the current is given by

$$
\begin{align*}
\mathcal{J}^\mu (z) &= C(Q) \left[ e^{i(\bar{n} \cdot p_n n \cdot z/2 + n \cdot p_n \bar{n} \cdot z/2)} \xi W^\mu W^\dagger \chi(z) \\
&\quad + e^{-i(\bar{n} \cdot p_n n \cdot z/2 + n \cdot p_n \bar{n} \cdot z/2)} \tilde{\chi} W^\dagger \gamma^\mu \bar{Y} W^\dagger \xi(z) \right].
\end{align*}
$$

(55)

Now we define the hadronic tensor $\Pi^{\mu\nu}$ as

$$
\Pi^{\mu\nu} = i \int d^4 z e^{i q \cdot z} T \left[ j^\mu (z) j^\nu (0) \right],
$$

(56)

which can be written as

$$
\begin{align*}
\Pi^{\mu\nu} &= i C^2(Q) \int d^4 z e^{i(\bar{n} \cdot q - n \cdot p_n) n \cdot z/2} e^{i(n \cdot q - n \cdot p_n) \bar{n} \cdot z/2} T \left[ \xi W^\dagger \gamma^\mu \bar{Y} W^\dagger \chi(z) \xi W^\dagger \gamma^\nu \bar{Y} W^\dagger \chi(0) \right] \\
&= i C^2(Q) \int d^4 z e^{i n \cdot q - n \cdot p_n \bar{n} \cdot z/2} T \left[ \tilde{\chi} W^\dagger \gamma^\mu \bar{Y} W^\dagger \chi(z) \tilde{\chi} W^\dagger \gamma^\nu \bar{Y} W^\dagger \chi(0) \right].
\end{align*}
$$

(57)
where the first exponential turns into $\delta_{Q,n_p}$, $n \cdot (q - p) = (1 - x)Q$, and the perpendicular part is omitted. Since there are no collinear particles in the final state, we can write

$$\langle 0 | T[W^+_{\bar{z}}(z)\bar{\xi}W(0)] | 0 \rangle \equiv \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} e^{-ikz} J_P(k),$$  \hspace{1cm} (58)$$

where this equation defines the jet function $J_P(k)$ which depends only on $n \cdot k$. We go down to SCET$_H$ and relabel the usoft Wilson line $Y, \bar{Y}$ as $S, \bar{S}$ respectively and $\Pi^{\mu \nu}$ is written as

$$\Pi^{\mu \nu} = -C^2(Q) \int \frac{d^4k}{(2\pi)^4} \int d^4ze^{i(Q(1-x) - n \cdot k)\bar{\pi}z/2} e^{-ikz} e^{-\bar{\pi}k_+n^\perp z/2} J_P(n \cdot k)$$

\hspace{1cm} \times T[\bar{W}S^\dagger \gamma^\mu S(z) \frac{\bar{W}}{2} S^\dagger \gamma^\nu \bar{W}^\dagger \chi(0)] \hspace{1cm} (59)$$

\begin{align*}
&= -C^2(Q) \int \frac{d\tau \cdot zd\cdot k}{4\pi} e^{i(Q(1-x) - n \cdot k)\bar{\pi}z/2} J_P(n \cdot k) T[\bar{W}S^\dagger \gamma^\mu S(z) \frac{\bar{W}}{2} S^\dagger \gamma^\nu \bar{W}^\dagger \chi(0)] \\
&= C^2(Q)g_{\perp}^{\mu \nu} T[\bar{W}S^\dagger \gamma^\mu S(z) \frac{\bar{W}}{2} S^\dagger \gamma^\nu \bar{W}^\dagger \chi(0)] \\
&\times T[\frac{1}{N} \text{tr}(\bar{S}^\dagger \bar{S} \delta(\eta + n \cdot i\partial) \bar{S}^\dagger S)] \\
&= g_{\perp}^{\mu \nu} \int d\omega C^2(\omega) T[\bar{W}S^\dagger \gamma^\mu S(z) \frac{\bar{W}}{2} S^\dagger \gamma^\nu \bar{W}^\dagger \chi(0)] \\
&\times T[\frac{1}{N} \text{tr}(\bar{S}^\dagger \bar{S} \delta(\eta + n \cdot i\partial) \bar{S}^\dagger S)] \\
&= g_{\perp}^{\mu \nu} \int d\omega C^2(\omega) T[\bar{W}S^\dagger \gamma^\mu S(z) \frac{\bar{W}}{2} S^\dagger \gamma^\nu \bar{W}^\dagger \chi(0)] \\
&\times T[\frac{1}{N} \text{tr}(\bar{S}^\dagger \bar{S} \delta(\eta + n \cdot i\partial) \bar{S}^\dagger S)],
\end{align*}$$

where $g_{\perp}^{\mu \nu} = g^{\mu \nu} - (n^\mu \bar{\pi}^\nu + \bar{\pi}^\mu n^\nu)/2$, and we use the fact that the soft part is decoupled from the collinear part and only the color-singlet component is projected out in the final result.

The soft Wilson line operator $K(\eta)$ in this case is given by

$$K(\eta) = \frac{1}{N} \text{tr}(\bar{S}^\dagger \bar{S} \delta(\eta + n \cdot i\partial) \bar{S}^\dagger S)$$

\hspace{1cm} \times \delta(\eta + n \cdot i\partial) \exp[-g\bar{\pi} \cdot A_s \frac{1}{n \cdot \bar{P} + i\epsilon}] \exp[\frac{1}{n \cdot \bar{P} - i\epsilon}(-g\bar{n} \cdot A_s)]$$

\hspace{1cm} \times \delta(\eta + n \cdot i\partial) \exp[-g\bar{n} \cdot A_s \frac{1}{n \cdot \bar{P} + i\epsilon}] \exp[\frac{1}{n \cdot \bar{P} - i\epsilon}(-g\bar{n} \cdot A_s)].$$  \hspace{1cm} (60)$$

D. Toy $\pi$-$\gamma$ form factor

If we consider the $\pi$-$\gamma$ form factor in the process $\gamma^*\gamma \rightarrow \pi^0$ in which $\gamma^*$ denotes a virtual photon, there is no soft Wilson lines of the form $K(\eta)$, as we will explain below. However, let us consider the fictitious process $\gamma^*(q_1)\gamma^*(q_2) \rightarrow \pi^0$, where both photons are virtual with $q_1^\mu + q_2^\mu = Q^\mu/2$. In this example, we only want to show that the soft Wilson lines in this process has a different analytic structure, and it appears in the matrix element. This may not be of physical relevance, and it is considered as a toy calculation.
The Feynman diagrams at lowest order for the form factor in $\gamma^* \gamma \to \gamma^0$ is shown in Fig. 7. The matrix element for this transition defines the $\pi^-\gamma$ form factor in the full theory \[19\]

\[
\langle \pi^0(p_{\pi}) | j_\mu(0) | \gamma(p_{\gamma}, \epsilon) \rangle = i e \epsilon^\nu \int d^4 z e^{-i p_{\gamma} \cdot z} \langle \pi^0(p_{\pi}) | T[j_\mu(0) j_\nu(z)] | 0 \rangle
\]  
\[= -i e F_{\pi\gamma}(Q^2) \epsilon_{\mu
u\alpha\beta} p_{\pi}^\alpha q_{\beta}, \tag{61}\]

where $\epsilon^\nu$ is the polarization vector for the real photon. The electromagnetic current is given by $j_\mu = \bar{\psi} \gamma^\mu \psi$. In the Breit frame the momentum of the virtual photon $q^\mu$, the momentum of the real photon $p_{\gamma}^\mu$, and the momentum of the pion $p_{\pi}^\mu$ can be written as

\[
q^\mu = \frac{Q}{2}(\bar{n}^\mu - n^\mu), \quad p_{\gamma}^\mu = E n^\mu \sim \frac{Q}{2} n^\mu, \quad p_{\pi}^\mu = E_{\pi} n^\mu. \tag{62}\]

We consider the case where the momentum transfer from the virtual photon is large $-q^2 = Q^2 \gg \Lambda_{\text{QCD}}^2$ where $q = p_{\pi} - p_{\gamma}$. If we write the momentum of the quark inside a pion as $p = x p_{\pi}$ and that of the antiquark as $p' = -(1-x)p_{\pi}$, where $x$ is the longitudinal momentum fraction of the pion, the virtualities of the intermediate states in Fig. 7 are given by

\[
p_{a}^2 = -x Q^2, \quad p_{b}^2 = -Q^2 (1-x), \tag{63}\]

with $E_{\pi} \sim Q/2$ at leading order in $\Lambda_{\text{QCD}}$.

Away from the endpoint region, $p_{a}^2 \sim p_{b}^2 \sim Q^2 \gg \Lambda_{\text{QCD}}^2$, and these states are integrated out to yield effective local operators in SCET$_I$. This is not true near the endpoint region. Near $x \sim 1$, $p_{a}^2$ is of order $Q^2$, but $p_{b}^2 \sim (1-x)Q^2$ is small, and the intermediate state in the second Feynman diagram of Fig. 7 cannot be integrated out in SCET$_I$. But we can still apply SCET and the two-step matching process is useful. First we integrate out the modes of order $p^2 \sim Q^2$ to go down to SCET$_I$. The intermediate collinear state in the first Feynman diagram of Fig. 7 is integrated out to produce local operators in SCET$_I$, in which the effect of the soft Wilson line does not appear since they cancel. Now we integrate out the modes of order $p^2 \sim (1-x)Q^2$ in the second diagram of Fig. 7 to go down to SCET$_{II}$. The problem here is that the momentum $p'_{\mu}$ of the antiquark becomes usoft near the endpoint.
FIG. 8: Feynman diagrams for the process $\gamma^* \gamma^* \rightarrow \pi^0$.

$x \sim 1$, and the effective current operator is an usoft-collinear current. The effect of the usoft interactions near the endpoint can be interesting on its own, but it is not appropriate in our context.

Instead, let us consider a hypothetical process $\gamma^* \gamma^* \rightarrow \pi^0$, which is shown in Fig. 8. The momenta of the virtual photons are given by

$$q_1^\mu = \frac{Q}{2}(y n^\mu - n^\mu), \quad q_2^\mu = \frac{Q}{2}((1 - y) n^\mu + n^\mu), \quad 0 \leq y \leq 1,$$

such that $q_1^\mu + q_2^\mu = Q n^\mu/2 = E_\pi n^\mu = p_\pi^\mu$. And we write the momentum of the quarks $p$ and $p'$ as $p = x p_\pi$, $p' = -(1 - x) p_\pi$. The momentum of the intermediate state is given by

$$p_a^\mu = \frac{Q}{2}(n^\mu + (x - y)n^\mu).$$

If $x - y \ll 1$ such that $p_a^2 = Q^2(x - y) \sim Q \Lambda$, the intermediate state becomes a collinear particle in the $n^\mu$ direction, and we can apply the two-step matching in this case.

Let us define the hadronic tensor

$$W_{\mu\nu} = i \int d^4z e^{-iq_1^\mu z} T \left[ j_\mu(z) j_\nu(0) \right],$$

where the current $j_\mu$ is given by

$$j_\mu(z) = C(Q) \left( e^{i(\vec{p}_n \cdot n - 2 - n \cdot p_\pi z/2)} \chi W \gamma_\mu \bar{W}^\dagger \chi + e^{-i(\vec{p}_n \cdot n - 2 - n \cdot p_\pi z/2)} \chi \bar{W} \gamma_\mu W^\dagger \chi \right),$$

where $p_n, p_\pi$ are the label momenta of the corresponding fields. Now we factorize the usoft interactions by redefining the collinear fields. Since the collinear field $\xi$ is the intermediate

FIG. 9: The description of the soft Wilson lines in the $\pi-\gamma$ form factor. (a) $\xi$ from $-\infty$, (b) $\xi$ to $\infty$, (c) the resultant soft Wilson lines from (a) and (b).
state, there are two possibilities to assign the usoft Wilson lines as shown in Fig. 8 (a), and (b), which are equivalent to the configuration in Fig. 8 (c). We follow the prescription given by Fig. 8 (a) and the collinear fields are redefined as

\[
\begin{align*}
\bar{\xi}W\gamma^\mu W^\dagger \chi : & \quad \bar{\xi} \to \bar{\xi} Y^\dagger, \quad A_\mu^\xi \to Y A_\mu^\xi Y^\dagger, \quad \chi \to \bar{\chi} \chi, \quad A_\mu^\chi \to \bar{\chi} A_\mu^\chi \bar{\chi}^\dagger, \\
\bar{\chi}W\gamma^\mu W^\dagger \xi : & \quad \bar{\chi} \to \bar{\chi} Y^\dagger, \quad A_\mu^\xi \to \bar{\xi} A_\mu^\xi \bar{\xi}^\dagger, \quad \xi \to \bar{\xi} \xi, \quad A_\mu^\xi \to \bar{\xi} A_\mu^\xi \bar{\xi}^\dagger.
\end{align*}
\]

(68)

The hadronic tensor can be written as

\[
W_{\mu\nu} = i \int d^4z e^{-i q \cdot z} T[j_\mu(z) j_\nu(0)] = iC^2(Q) \int d^4z e^{i(Q\cdot y)Q\cdot z/2} T[\bar{\xi}WY^\dagger \gamma_\mu Y W^\dagger \xi(z)\bar{\xi}Y^\dagger \gamma_\nu Y W^\dagger \chi(0)].
\]

(69)

Since there are no collinear particles in the \(n^\mu\) direction in the final state, we can write

\[
\langle 0| T[W^\dagger \xi(z)\bar{\xi}W(0)]|0\rangle = \frac{e^{ik\cdot z}}{2\pi^4} J_P(n \cdot k),
\]

(70)

where \(J_P(n \cdot k)\) is a function of \(n \cdot k\) only. We can write \(W_{\mu\nu}\) in SCETII as

\[
W_{\mu\nu} = -\frac{C^2(Q)}{4\pi} \int d\bar{\eta} \cdot zd\eta \cdot k d\eta e^{i(Q\cdot y - n \cdot \bar{\eta} - \eta \cdot n)\bar{\eta}} J_P(n \cdot k)
\]

\[
\times T[\bar{\xi}W\gamma_{\mu} S^\dagger 2 S \delta(\eta + n \cdot i \partial) S^\dagger S \eta \overline{S} W^\dagger \chi]
\]

\[
\to \frac{i}{4} \epsilon_{\mu\nu\alpha\beta} n^\alpha \bar{\eta}^\beta \int d\omega C^2(\omega) T[\bar{\xi}W\gamma_{\mu} S^\dagger 2 S \delta(\omega - P_+) \gamma_5 W^\dagger \chi]
\]

\[
\times \int d\eta \cdot k J_P(n \cdot k) T[K(Q(x - y) - n \cdot k)],
\]

(71)

where we extract the collinear operator proportional to \(\gamma_5\). Here the effect of the soft gluon emission is decoupled from the collinear sector, and it is described by the soft Wilson line as

\[
K(\eta) = \frac{1}{N} \text{tr } \bar{S}^\dagger S \delta(\eta + n \cdot i \partial) S^\dagger S
\]

\[
= \exp\left[-gn \cdot A_\mu \frac{1}{\bar{n} \cdot P_+ + i\epsilon}\right] \cdot \exp\left[\frac{1}{n \cdot P + i\epsilon}(-gn \cdot A_\mu)\right] \delta(\eta + n \cdot i \partial)
\]

\[
\times \exp\left[-gn \cdot A_\mu \frac{1}{\bar{n} \cdot P_+ - i\epsilon}\right] \cdot \exp\left[\frac{1}{n \cdot P - i\epsilon}(-gn \cdot A_\mu)\right].
\]

(72)

The forms of the soft Wilson lines for all the processes are summarized in Table II.

| process      | \(K(\eta)\)                                                                 | process      | \(K(\eta)\)                                                                 |
|--------------|-----------------------------------------------------------------------------|--------------|-----------------------------------------------------------------------------|
| DIS          | \(\frac{1}{N} \text{tr } \bar{\xi}W^\dagger S \delta(\eta + n \cdot i \partial) S^\dagger S\) | Drell-Yan    | \(\frac{1}{N} \text{tr } \bar{\xi}W^\dagger S \delta(\eta + n \cdot i \partial) S^\dagger S\) |
| \(e^+e^-\)  | \(\frac{1}{N} \text{tr } \bar{\xi}W^\dagger S \delta(\eta + n \cdot i \partial) S^\dagger S\) | \(\pi^-\gamma\) | \(\frac{1}{N} \text{tr } \bar{\xi}W^\dagger S \delta(\eta + n \cdot i \partial) S^\dagger S\) |
IV. RADIATIVE CORRECTIONS FOR THE SOFT WILSON LOOP

As we have seen in the previous section, the effects of the soft gluon emission expressed by the soft Wilson lines can be important near the boundary of the phase space. The matrix element of the soft Wilson line \( K(\eta) \) between the vacuum state describes a nonperturbative effect. Away from the boundary of the phase space, the soft Wilson lines cancel \( \text{[19]} \) or the radiative corrections can vanish \( \text{[21, 23]} \). In Refs. \( \text{[21, 23]} \), the authors considered the region where \( \eta \gg n \cdot i\partial \sim \Lambda \) and the delta function \( \delta(\eta + n \cdot q) \) can be expanded using the multipole expansion. However, we consider the region \( \eta \sim n \cdot i\partial \sim \Lambda \), where the multipole expansion is not useful. And if we include quantum corrections, the matrix elements and the radiative corrections can be nonzero.

In order to compute the anomalous dimension of the soft Wilson line, we consider the

\[
\nu, b \quad q \quad \mu, a
\]

- **DIS:**

\[
g^2 \frac{\delta_{ab}}{2N} \left[ \left( \frac{n^\mu \cdot n^\nu}{(n \cdot q + ie)(n \cdot q + ie)} + \frac{n^\mu \cdot n^\nu}{(n \cdot q - ie)(n \cdot q - ie)} \right) \delta(\eta) + \frac{1}{n^\mu \cdot n^\nu} \right]
\]

\[
\overset{\nu \leftrightarrow b, \mu \leftrightarrow \mu, q \leftrightarrow -q}
\]

- **Drell-Yan:**

\[
g^2 \frac{\delta_{ab}}{2N} \left[ \left( \frac{n^\mu \cdot n^\nu}{(n \cdot q + ie)(n \cdot q - ie)} + \frac{n^\mu \cdot n^\nu}{(n \cdot q + ie)(n \cdot q - ie)} \right) \delta(\eta) + \frac{1}{n^\mu \cdot n^\nu} \right]
\]

\[
\overset{\nu \leftrightarrow b, \mu \leftrightarrow \mu, q \leftrightarrow -q}
\]

- **\( e^+e^- \):**

\[
g^2 \frac{\delta_{ab}}{2N} \left[ \left( \frac{n^\mu \cdot n^\nu}{(n \cdot q - ie)(n \cdot q - ie)} + \frac{n^\mu \cdot n^\nu}{(n \cdot q - ie)(n \cdot q - ie)} \right) \delta(\eta) + \frac{1}{n^\mu \cdot n^\nu} \right]
\]

\[
\overset{\nu \leftrightarrow b, \mu \leftrightarrow \mu, q \leftrightarrow -q}
\]

- **\( \pi\gamma \):**

\[
g^2 \frac{\delta_{ab}}{2N} \left[ \left( \frac{n^\mu \cdot n^\nu}{(n \cdot q + ie)(n \cdot q - ie)} + \frac{n^\mu \cdot n^\nu}{(n \cdot q - ie)(n \cdot q - ie)} \right) \delta(\eta) + \frac{1}{n^\mu \cdot n^\nu} \right]
\]

\[
\overset{\nu \leftrightarrow b, \mu \leftrightarrow \mu, q \leftrightarrow -q}
\]


**TABLE III:** Feynman rules for the soft Wilson operator with two external gluons for deep inelastic scattering (DIS), the Drell-Yan process, the jet production from \( e^+e^- \) collisions, and the \( \pi\gamma \) form factor. Only the terms with one \( n \) and one \( \pi \) are shown. The arrows indicate the momentum flow.
radiative corrections induced from the operator $K(\eta)$, and the Feynman rules for the two-point vertex in various processes is given in Table III. The Feynman diagram for the radiative correction at one loop is shown in Fig. 10 and, in deep inelastic scattering, it is given by

$$I_{\text{DIS}}(\eta) = -ig^2 C_F \int \frac{d^D l}{(2\pi)^D} \left[ \frac{4\delta(\eta)}{l^2(\vec{\pi} \cdot l - \lambda_1)(n \cdot l - \lambda_2)} + \frac{\delta(\eta + n \cdot l)}{(l^2 + i0)(\vec{\pi} \cdot l - \lambda_1 + i0)} \left( - \frac{1}{n \cdot l + \lambda_1 - i0} + \frac{1}{n \cdot l + \lambda_1 + i0} \right) + \frac{\delta(\eta - n \cdot l)}{(l^2 + i0)(-\vec{\pi} \cdot l - \lambda_1 + i0)} \left( \frac{1}{n \cdot l + \lambda_2 - i0} + \frac{1}{n \cdot l + \lambda_2 + i0} \right) \right],$$

(73)

where $\lambda_1$, and $\lambda_2$ are the infrared cutoffs, and $i\epsilon$ is replaced by $i0$ to avoid confusion. Here the $i0$ prescription is omitted in the first integral including $\delta(\eta)$ because it does not affect the calculation. In order to extract the ultraviolet divergence as poles in $1/\epsilon$, we employ dimensional regularization with $D = 4 - 2\epsilon$, and we introduce infrared cutoffs to regulate infrared divergences. A rigorous method to introduce the infrared cutoff without violating the gauge invariance is discussed in Ref. [27], and it should be followed. But here we put simple infrared cutoffs in order to extract the ultraviolet divergence only and to see that there is no mixing of the ultraviolet and infrared divergences.

The integral proportional to $\delta(\eta)$ in Eq. (73) is given by

$$I^\eta_{\text{DIS}}(\eta) = -ig^2 C_F \delta(\eta) \int \frac{d^D l}{(2\pi)^D} \left( \frac{1}{l^2(\vec{\pi} \cdot l - \lambda_1)(n \cdot l - \lambda_2)} \right)$$

$$= -4ig^2 C_F \delta(\eta) 8 \int_0^\infty du \int_0^\infty dv \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 - (4uv + 2u\lambda_1 + 2v\lambda_2)^2}$$

$$= -\frac{\alpha_s}{\pi} \left( \frac{\lambda_1 \lambda_2}{\mu^2} \right)^{-\epsilon} \delta(\eta) \frac{1}{\epsilon^2},$$

(74)

where only the ultraviolet divergent term is kept.

For other integrals, we first integrate over $n \cdot l$ with the delta functions, and the remaining integral is computed using the contour integral in the complex $\vec{\pi} \cdot l$ plane, and we finally
integrate over \( l \perp \) using the dimensional regularization in \( D - 2(= 2 - 2\epsilon) \) dimensions. Since the integration of the delta functions \( \delta(\eta \pm n \cdot l) \) over \( n \cdot l \) produces a number, the \( i0 \) prescription does not affect the computation, while the position of the poles in the complex \( \vec{\Pi} \cdot l \) plane becomes important. Therefore the two integrands proportional to \( \delta(\eta + n \cdot l) \) and those proportional to \( \delta(\eta - n \cdot l) \), which differ only in the \( i0 \) prescription in the terms containing \( n \cdot l \), give the same result. That is, the analytic structure of the soft Wilson lines in the \( n^\mu \) directions does not alter the radiative corrections when the separation between \( 0 \) and \( z \) is lightlike in the \( n^\mu \) direction. As long as the analytic structure is the same for the soft Wilson lines in the \( \vec{\Pi}^\mu \) direction, the radiative corrections for the soft Wilson lines are the same though different specifications of the soft Wilson lines in the \( n^\mu \) direction may correspond to different physical processes. For this reason, the configuration of the soft Wilson lines in the Drell-Yan process shown in Fig. 4 (a) gives the same radiative corrections as the configuration give in Fig. 4 (b) as long as the separation is lightlike.

The integral with \( \delta(\eta + n \cdot l) \) is given by

\[
I_{\text{DIS}}^b(\eta) = -2ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{1}{\vec{\Pi} \cdot l - l_\perp^2 + i0} \frac{1}{\eta \cdot l - \lambda_1 + i0} \frac{1}{-\eta \cdot l + \lambda_2 - i0} \delta(\eta + n \cdot l)
\]

\[
= ig^2C_F \frac{1}{\eta + \lambda_2} \int \frac{d^{D-2}l_\perp d\vec{\Pi} \cdot l}{(2\pi)^D} \frac{1}{\eta \cdot l - \lambda_1 + i0} \frac{1}{\eta \cdot l - \lambda_2 - i0}.
\]

In order for this integral to have nonzero value, the poles in the complex \( \vec{\Pi} \cdot l \) plane should be on the opposite side of the real axis, hence \( \eta \) should be positive, and the integral is given by

\[
I_{\text{DIS}}^b(\eta) = \frac{\alpha_s C_F}{2\pi} \Gamma(\epsilon) \theta(\eta) \frac{(\eta \lambda_1/\mu^2)^{-\epsilon}}{\eta + \lambda_2}.
\]

Similarly, the integral with \( \delta(\eta - n \cdot l) \) is given by

\[
I_{\text{DIS}}^c(\eta) = \frac{\alpha_s C_F}{2\pi} \Gamma(\epsilon) \theta(\eta) \frac{(\eta \lambda_1/\mu^2)^{-\epsilon}}{\eta + \lambda_2},
\]

which is the same as \( I_{\text{DIS}}^b(\eta) \). In order to extract the ultraviolet divergence, we can write \( I_{\text{DIS}}^b(\eta) \) in the form

\[
I_{\text{DIS}}^b(\eta) = A\delta(\eta) + B\theta(\eta) \left( \frac{1}{\eta} \right)_+,
\]

since the singularity resides at \( \eta = 0 \) and the above equation defines the “+”-distribution which satisfies

\[
\int_{-\infty}^{\infty} d\eta \theta(\eta) \left( \frac{1}{\eta} \right)_+ = 0, \quad \int_{-\infty}^{\infty} d\eta \theta(\eta) f(\eta) \left( \frac{1}{\eta} \right)_+ = \int_{0}^{\infty} d\eta f(\eta) - f(0),
\]

28
where \( f(\eta) \) is a regular function at \( \eta = 0 \). Determining \( A \) and \( B \) by explicit calculation, we obtain

\[
I_{\text{DIS}}^{b,c}(\eta) = \frac{\alpha_s C_F}{2\pi} \left[ \left( \frac{\lambda_1 \lambda_2}{\mu^2} \right) - \frac{\epsilon}{\epsilon^2} \delta(\eta) + \frac{1}{\epsilon} \frac{\theta(\eta)}{\epsilon(\eta)} \right].
\]  
(80)

By adding all the contributions, the result is given by

\[
I_{\text{DIS}}(\eta) = I_{\text{DIS}}^a(\eta) + I_{\text{DIS}}^b(\eta) + I_{\text{DIS}}^c(\eta) = \frac{\alpha_s C_F 1}{\pi} \frac{\theta(\eta)}{\epsilon(\eta)}. \]  
(81)

The relation between the bare operator and the renormalized operator can be written, in general, as

\[
K_B(\eta) = \int d\eta' Z_{\text{DIS}}(\eta, \eta') K_R(\eta'),
\]  
(82)

where \( Z(\eta, \eta') \) is given by

\[
Z_{\text{DIS}}(\eta, \eta') = \delta(\eta - \eta') + \frac{\alpha_s C_F 1}{\pi} \frac{\theta(\eta - \eta')}{\epsilon(\eta - \eta')}.
\]  
(83)

Therefore the renormalization group equation for the renormalized operator is given by

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) K(\eta) = -\int d\eta' \gamma_{\text{DIS}}(\eta, \eta') K(\eta'),
\]  
(84)

where the anomalous dimension \( \gamma_{\text{DIS}}(\eta, \eta') \) is given by

\[
\gamma_{\text{DIS}}(\eta, \eta') = -2 \frac{\alpha_s C_F}{\pi} \frac{\theta(\eta - \eta')}{(\eta - \eta')}.
\]  
(85)

Now let us consider the radiative correction for the soft gluon emission in the Drell-Yan process. The Feynman diagram is also given by Fig. 10 but with the different Feynman rule in Table III for the Drell-Yan process. The radiative correction at one loop is given by

\[
I_{\text{DY}}(\eta) = -2ig^2 C_F \int \frac{d^Dl}{(2\pi)^D} \left[ \frac{2\delta(\eta)}{l^2(\vec{\eta} \cdot l - \lambda_1)(n \cdot l - \lambda_2)} \right.
\]  
\[
+ \frac{\delta(\eta + n \cdot l)}{(l^2 + i0)(\vec{\eta} \cdot l - \lambda_1 + i0)(-n \cdot l + \lambda_2 + i0)}
\]  
\[
+ \left. \frac{\delta(\eta - n \cdot l)}{(l^2 + i0)(\vec{\eta} \cdot l - \lambda_1 + i0)(n \cdot l + \lambda_2 + i0)} \right].
\]  
(86)

Compared to the case of deep inelastic scattering in Eq. (73), the only difference is the \( i0 \) prescription in the \( n \cdot l \) part in the denominators of the second and the third lines in Eq. (86). Since the integration over \( n \cdot l \) is governed by the delta functions \( \delta(\eta \pm n \cdot l) \), the \( i0 \) prescription does not have any effect on the result of the integration and the radiative
correction for the Drell-Yan process is the same as the case in deep inelastic scattering. And
the renormalization group equation for the soft Wilson line in Drell-Yan process is given by
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) W(\rho - i0) = - \left[ \Gamma_{\text{cusp}}(g) \left( \ln(\rho - i0) + \ln(-\rho + i0) \right) + \Gamma(g) \right] W(\rho - i0),
\] (88)
where \( \Gamma(g) \) is the integration constant, \( \rho = (v \cdot y)\mu \), and \( \Gamma_{\text{cusp}} \) is called the cusp anomalous
dimension. This is the renormalization group equation for the soft Wilson loop with
the initial quark with velocity \( v^\mu \). However, we can put the initial particle on the light cone,
say, in the \( \overline{p}^\mu \) direction. In this case, the integration constant becomes \( \Gamma(g) = 0 + O(\alpha_s^2) \).
At order \( \alpha_s \), the renormalization group equation for the soft Wilson loop becomes
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) W(\rho - i0) = - \left[ \Gamma_{\text{cusp}}(g) \left( \ln(\rho - i0) + \ln(-\rho + i0) \right) + \Gamma(g) \right] W(\rho - i0),
\] (89)
where \( \rho = (\overline{p} \cdot y)\mu \), and the \( i0 \) prescription comes from the Feynman prescription.

Instead of considering \( W \) as a function of \( \rho \), we will consider \( W \) as a function of \( \overline{p} \cdot y \),
\( W(\rho) = W(\overline{p} \cdot y, \mu, g) \) since its Fourier transform is what we have computed so far. Let us
define the Fourier transform of \( W(\overline{p} \cdot y) \) as
\[
W(\eta) = \int \frac{d\overline{p} \cdot y}{2\pi} e^{i\overline{p} \cdot y\eta} W(\overline{p} \cdot y - i0), \quad W(\overline{p} \cdot y - i0) = \int d\eta e^{-i\overline{p} \cdot y\eta} W(\eta),
\] (90)
and we take the Fourier transform of Eq. (89). It becomes
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) W(\eta)
= - \Gamma_{\text{cusp}} \int \frac{d\overline{p} \cdot y}{2\pi} e^{i\overline{p} \cdot y\eta} \left( \ln(\overline{p} \cdot y\mu - i0) + \ln(-\overline{p} \cdot y\mu + i0) \right) W(\overline{p} \cdot y\mu - i0)
= - \Gamma_{\text{cusp}} \int \frac{d\overline{p} \cdot y}{2\pi} e^{i\overline{p} \cdot y\eta} \left( \ln(\overline{p} \cdot y\mu - i0) + \ln(-\overline{p} \cdot y\mu + i0) \right) \int d\eta' e^{-i\overline{p} \cdot y\eta'} W(\eta')
= - \Gamma_{\text{cusp}} \int d\eta' W(\eta') \int \frac{d\overline{p} \cdot y}{2\pi} e^{i\overline{p} \cdot y(\eta - \eta')} \left( \ln(\overline{p} \cdot y\mu - i0) + \ln(-\overline{p} \cdot y\mu + i0) \right)
= - \Gamma_{\text{cusp}} \int d\eta' W(\eta - \eta') W(\eta'),
\] (91)
where $V(\eta - \eta')$ is given by

$$V(\eta - \eta') = \int \frac{d\tau \cdot y}{2\pi} e^{i\tau y(\eta - \eta')} \left( \ln(\tau \cdot y_{\mu} - i0) + \ln(-\tau \cdot y_{\mu} + i0) \right) = -2\frac{\theta(\eta - \eta')}{(\eta - \eta')_+}. \quad (92)$$

Since $\Gamma_{\text{cusp}} = \alpha_s C_F/\pi + O(\alpha_s^2)$, the anomalous dimensions in Eq. (91) and in Eq. (85) are the same.

In the jet production from $e^+e^-$ collisions, the radiative correction is given by

$$I_{\text{jet}}(\eta) = -2ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{2\delta(\eta)}{(l^2(\pi \cdot l - \lambda_1)(n \cdot l - \lambda_2))} \delta(\eta + n \cdot l)$$

$$+ \frac{\delta(\eta - n \cdot l)}{(l^2 + i0)(\pi \cdot l - \lambda_1 - i0)(n \cdot l + \lambda_2 + i0)}$$

$$+ \frac{\delta(\eta + n \cdot l)}{(l^2 + i0)(-\pi \cdot l - \lambda_1 - i0)(n \cdot l + \lambda_2 + i0)}.$$ \quad (93)

As in the previous calculations, the $i0$ prescription for the $n \cdot l$ part does not matter, but the $i0$ prescription for the $\pi \cdot l$ part has opposite signs compared to the cases in deep inelastic scattering and the Drell-Yan process. The integration of the term proportional to $\delta(\eta)$ is the same, and the integrals $I_{\text{jet}}^0$ and $I_{\text{jet}}^c$ with the delta functions $\delta(\eta \pm n \cdot l)$ are given by

$$I_{\text{jet}}^0 = \int \frac{d^Dl}{(2\pi)^D} \frac{1}{\pi \cdot l - l^2} + i0 \frac{1}{\pi \cdot l - \lambda_1 - i0} - \frac{1}{n \cdot l + \lambda_2 + i0} \delta(\eta + n \cdot l)$$

$$= \frac{\alpha_s C_F}{2\pi} \left[ (\ln \lambda_1 \lambda_2) \frac{1}{\mu^2} \delta(\eta) + \frac{1}{\epsilon} \frac{\theta(-\eta)}{(-\eta)_+} \right] = I_{\text{jet}}^c.$$ \quad (94)

Combining with the part proportional to $\delta(\eta)$, the $1/\epsilon^2$ pole cancels and the radiative correction at one loop in $e^+e^-$ collisions is given by

$$I_{\text{jet}} = \frac{\alpha_s C_F \theta(-\eta)}{\pi} \frac{\theta(-\eta)}{(-\eta)_+}. \quad (95)$$

The relation between the bare operator $K_B$ and the renormalized operator $K_R$ is written as

$$K_B(\eta) = \int d\eta' Z_{\text{jet}}(\eta, \eta') K_R(\eta'), \quad (96)$$

where $Z_{\text{jet}}(\eta, \eta')$ is given by

$$Z_{\text{jet}}(\eta, \eta') = \delta(\eta - \eta') + \frac{\alpha_s C_F \theta(\eta' - \eta)}{\pi} \frac{\theta(\eta' - \eta)}{(\eta' - \eta)_+}.$$ \quad (97)

And the renormalized operator satisfies the renormalization group equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) K(\eta) = - \int d\eta' \gamma_{\text{jet}}(\eta, \eta') K(\eta'), \quad (98)$$
where the anomalous dimension \( \gamma_{\text{jet}}(\eta, \eta') \) is given by

\[
\gamma_{\text{jet}}(\eta, \eta') = -2\alpha_s C_F \frac{\theta(\eta' - \eta)}{\pi (\eta' - \eta)_+}.
\]  

(99)

When we compare the kernel, or the anomalous dimension of deep inelastic scattering and the Drell-Yan process in Eq. (84), the anomalous dimension has the same form, but the argument changes sign.

Let us finally consider the radiative correction for the soft Wilson line in \( \pi-\gamma \) form factor. The radiative correction at one loop in given by

\[
I_\pi(\eta) = -ig^2 C_F \int \frac{d^Dl}{(2\pi)^D l^2 + i0} \left[ \frac{2\delta(\eta)}{\lambda_1 (\eta_1 - \lambda_1)(\eta_2 - \lambda_2)} \right]
\]  

(100)

\[+ \delta(\eta + n \cdot l) \left\{ \frac{1}{\lambda_1 (\eta_1 - \lambda_1)(n \cdot l - \lambda_2)} + \frac{1}{\lambda_2 (n \cdot l + \lambda_1)(n \cdot l + \lambda_2)} \right\} \]

\[+ \delta(\eta - n \cdot l) \left\{ \frac{1}{\lambda_2 (\eta_1 + \lambda_1)(n \cdot l + \lambda_2)} + \frac{1}{\lambda_1 (n \cdot l - \lambda_1)(n \cdot l - \lambda_2)} \right\} \]

In Eq. (100), the integral which contains \( \delta(\eta) \) is the same integral which appears previously and is given by

\[I_\pi(\eta) = ig^2 C_F \int \frac{d^Dl}{(2\pi)^D l^2 + i0} \left[ \frac{2\delta(\eta)}{\lambda_1 \lambda_2} \right] = -\alpha_s C_F \left( \frac{\lambda_1 \lambda_2}{\mu^2} \right)^{-\epsilon} \delta(\eta) \frac{1}{\epsilon^2} \]  

(101)

Using the same technique, the integrals \( I_b(\eta) \) and \( I_c(\eta) \) in the second and the third line of Eq. (100) can be evaluated and they are given as

\[I_b(\eta) = I_c(\eta) = \frac{\alpha_s C_F}{4\pi} \Gamma(\epsilon)(\eta_1 - \eta_2) \frac{\theta(\eta) - \theta(-\eta)}{\eta + \lambda_2}. \]  

(102)

By adding all the contributions, the radiative correction for the soft Wilson line in \( \pi-\gamma \) form factor is given by

\[I_\pi(\eta) = I_\pi^b(\eta) + I_\pi^c(\eta) + I_\pi^\eta(\eta) = \frac{\alpha_s C_F}{2\pi} \frac{1}{\epsilon} \frac{\theta(\eta) - \theta(-\eta)}{\eta}. \]  

(103)

The relation between the bare operator \( K_B(\eta) \) and the renormalized operator \( K_R(\eta) \) can be written as

\[K_B(\eta) = \int d\eta' Z_{\pi\gamma}(\eta, \eta') K_R(\eta'), \]  

(104)

where \( Z_{\pi\gamma}(\eta, \eta') \) is given by

\[Z_{\pi\gamma}(\eta, \eta') = \delta(\eta - \eta') + \alpha_s C_F \frac{1}{2\pi} \frac{\theta(\eta - \eta')}{(\eta - \eta')_+} + \frac{\theta(\eta' - \eta)}{(\eta' - \eta)_+}. \]  

(105)
The renormalized soft Wilson-line operator satisfies the renormalization group equation

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) K(\eta) = - \int d\eta' \gamma_{\pi\gamma}(\eta, \eta') K(\eta'),
$$

(106)

where the anomalous dimension \( \gamma_{\pi\gamma}(\eta, \eta') \) is given by

$$
\gamma_{\pi\gamma}(\eta, \eta') = - \frac{\alpha_s}{\pi} \frac{\theta(\eta - \eta')}{(\eta - \eta')_+} + \frac{\theta(\eta' - \eta)}{(\eta' - \eta)_+}.
$$

(107)

This result gives the Brodsky-Lepage kernel, and is also consistent with the result in Ref. [4]. The authors considered a soft Wilson loop with four cusps, so there is a difference of factor 2 in front of \( \Gamma_{\text{cusp}} \) in the renormalization group equation. If we perform the same analysis for the case of deep inelastic scattering, the result, in terms of \( W(\eta) \), can be written as

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) W(\eta, g) = - \Gamma_{\text{cusp}} \int d\eta' V(\eta - \eta') W(\eta', g)
$$

$$
= \frac{\alpha_s C_F}{\pi} \int d\eta' \left( \frac{\theta(\eta - \eta')}{(\eta - \eta')_+} + \frac{\theta(\eta' - \eta)}{(\eta' - \eta)_+} \right) W(\eta', g).
$$

(108)

This renormalization group equation is exactly the same as Eqs. (106) and (107).

V. CONCLUSION

We have considered the effect of the soft gluon emission in high-energy processes near the phase boundaries. The soft gluon emission can be expressed in terms of the soft Wilson-line operators. Since the collinear and energetic particles are decoupled from the soft interactions in SCETII, the matrix element of the soft Wilson lines is shown to be factorized. Therefore the decay rates or the scattering cross sections can be written as a convolution of the Wilson coefficients, the jet functions, the matrix elements of the collinear operators, and the matrix elements of the soft Wilson lines. The separation of long-distance physics and short-distance physics is complicated in the full theory, and SCET offers a simple tool to see this factorization clearly. And the scaling behavior of each factorized part can be computed in perturbation theory order by order. In this paper, we have focused on the scaling behavior of the soft Wilson lines.

The appearance of the soft Wilson line is universal when we consider the effect of the soft gluon emission near the phase boundary. But the analytic structure of the soft Wilson line is different in different processes, which appear as the \( i\epsilon \) prescription in the factorized
exponential, or the path- and anti path-ordering of the gauge fields in the soft Wilson lines.

At tree level, the difference of the analytic structure in the soft Wilson lines may not appear and the effect of the soft gluon emission is truly universal. If we include quantum corrections, the different analytic structure affects the scaling behavior of the soft Wilson lines and it has been explicitly shown in this paper by computing the anomalous dimensions of the soft Wilson lines in various processes. The anomalous dimensions for the soft Wilson line in deep inelastic scattering and in the Drell-Yan process are the same, while the anomalous dimension in the jet production from $e^+e^-$ collisions has the same form but the sign of the argument is opposite. In the $\pi-\gamma$ form factor, the anomalous dimension is proportional to $\theta(\eta - \eta')/(\eta - \eta')_+ + \theta(\eta' - \eta)/(\eta' - \eta)_+$, and this gives the Brodsky-Lepage kernel. This originates from the different analytic structure of the soft Wilson lines in the $n^\mu$ direction for the lightlike separation in the $n^\mu$ direction.

Another advantage in adopting SCET in considering the effect of the soft gluon emission is that it is possible to compute radiative corrections in momentum space expanding $S$, $\tilde{S}$, $\mathfrak{S}$, $\mathfrak{S}$ and their hermitian conjugates in powers of $g$, while previous approaches performed the radiative corrections mainly in coordinate space by expanding $S(x)$, $\tilde{S}(x)$, $\mathfrak{S}(x)$, $\mathfrak{S}(x)$ which are the Fourier transforms of $S$, $\tilde{S}$, $\mathfrak{S}$, $\mathfrak{S}$. In Ref. [28], the authors computed the cusp anomalous dimension in the momentum space. The study of the analytic structure on the soft Wilson lines can be extended to the collinear Wilson line and the processes in which the different analytic structure of the collinear Wilson lines may affect the scaling behavior are under investigation.

Finally, let us comment on the approaches using the forward scattering amplitudes and the matrix element squared, which should give the same physical results on the radiative corrections due to the optical theorem. The goal of SCET is to obtain the effective operators in SCET$_\Pi$ from the time-ordered products of the currents and take the matrix elements of the operators. The approach using the forward scattering amplitude fits this purpose. In the approach using the matrix elements squared, it is possible to compute radiative corrections, but the final result cannot be expressed in terms of operators. And the computation is more involved since we should consider the Feynman rules for the propagators with and without the cuts, and we should also take into account which particles are on which side of the cut. Though it is formidable, it has been done. In this approach also, we can draw the soft Wilson lines in spacetime. Due to the cancellation of the soft Wilson lines, the overall soft Wilson
line becomes a connected line, and it is easy to see which cusp angles contribute. This is seen, for example, in Refs. [5, 22]. However, care should be taken in obtaining the final result. In deep inelastic scattering in Ref. [5], there are contributions from two cusp angles, but the contribution of the cusp angle from the complex-conjugated amplitude should be changed to the cusp angle from the original amplitude by taking the hermitian conjugate because it is on the opposite side of the cut. This corresponds to taking the same cusp angle in the original part, not the cusp angle on the opposite side of the cut. This also happens in $e^+e^-$ collisions if we take the path of the soft Wilson line given in Ref. [22], and we have to consider choosing the correct cusp angle as in the case of the deep inelastic scattering. On the other hand, we can draw the soft Wilson lines in a similar way in the approach using the forward scattering amplitudes, but with straightforward rules. As we have explained, we can specify the configurations of the soft Wilson lines for a given physical process and compute the effects of the soft gluon emission in a straightforward manner and can see which cusp angles contribute.

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