ON THE CONIVEAU FILTRATION ON ALGEBRAIC $K$-THEORY
OF SINGULAR SCHEMES

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Abstract. We construct two functorial filtrations on the algebraic $K$-theory of schemes of finite type over a field $k$ that may admit arbitrary singularities and may be non-reduced, one called the coniveau filtration, and the other called the motivic coniveau filtration. Restricting to the subcategory of smooth $k$-schemes, our coniveau filtration coincides with the classical coniveau (also known as the topological) filtration on algebraic $K$-theory of D. Quillen, whereas our motivic coniveau filtration coincides with the homotopy coniveau filtration for algebraic $K$-theory of M. Levine.

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1. Introduction

The objective of this article is to propose two functorial filtrations on higher algebraic $K$-theory of schemes of finite type over an arbitrary field, where the schemes can be arbitrarily singular or non-reduced, extending two types of decreasing filtrations on the algebraic $K$-theory of smooth schemes.

We know that (SGA VI [7]) when $Y$ is a smooth irreducible quasi-projective $k$-scheme, there is the natural isomorphism $K_0(Y) \cong G_0(Y)$ from the Grothendieck group of locally free sheaves of finite type to that of coherent sheaves. The key idea behind it is given by projective resolutions of finite lengths of coherent sheaves backed by the Auslander-Buchsbaum theorem [6]. Since the group $G_0(Y)$ has the natural coniveau (also called topological, or codimension) filtration, the isomorphism induces a filtration on $K_0(Y)$.

We have the cycle class map

$$\text{CH}^q(Y) \to \text{gr}^q_{\text{top}} K_0(Y)$$

from the Chow group of codimension $q$-cycles to the associated graded of $K_0(Y)$ with respect to the filtration, and it is an isomorphism after tensoring with $\mathbb{Q}$. This result is often referred to as (a part of) the Grothendieck-Riemann-Roch theorem.

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It was further extended to higher Chow groups and higher algebraic $K$-theory of smooth $k$-schemes by S. Bloch combining [9] and [10], or M. Levine [24].

When one tries to generalize the result to a singular $Y$, unfortunately a few things in the above go wrong. For instance, the Chow groups $\text{CH}^q(Y)$ in (1.0.1) are not contravariant functorial in $Y$, while $K_0(Y)$ is. The homomorphism $K_0(Y) \to G_0(Y)$ is no longer an isomorphism so that the coniveau filtration on $G_0(Y)$ does not induce a natural filtration on $K_0(Y)$. Given these circumstances, so far an analogue of the Grothendieck-Riemann-Roch theorem for $K$-theory of singular schemes has not been available.

This is where this article aims to make a contribution by building some stepping stones. In this article, we define two functorial decreasing filtrations on the algebraic $K$-theory on the category of $k$-schemes of finite type.

The following first main theorem of the article is on the first filtration (see (5.3.3), Definition 5.3.4, Proposition 5.4.2, Theorem 5.6.16):

**Theorem 1.0.2.** Let $n \geq 0$ be an integer. For a $k$-scheme $Y$ of finite type with possibly arbitrary singularities, there exists a tower in the homotopy category of spectra $\hat{G}_Y^{\bullet,0} \to K(Y)$, which induces a decreasing filtration $F_{\text{cv}}^nK_n(Y)$ on $K_n(Y)$, such that

1. the tower and the filtration are functorial in $Y$, and
2. if $Y$ is smooth and equidimensional, the tower coincides with Quillen’s classical coniveau tower (recalled in (5.4.1)) up to weak-equivalence and the filtration coincides with the classical coniveau filtration on $K_n(Y)$.

We will call $F_{\text{cv}}^n$, the coniveau filtration on $K_n(Y)$.

Theorem 1.0.2 implies in particular that the pull-back $g^* : K_i(Y_2) \to K_i(Y_1)$ induced by a map between smooth $k$-schemes $g : Y_1 \to Y_2$ respects the classical coniveau filtration. This is already known, e.g. see H. Gillet [15, Theorem 83, Lemma 84, p.283], which is based on the technique of deformation to the normal cone and $\mathbb{A}^1$-invariance of the algebraic $K$-theory of regular schemes. However, the proof given in this article offers a new argument even for the classical smooth case, not directly relying on the $\mathbb{A}^1$-invariance.

In a future work, we plan to investigate the “Brown-Gersten-Quillen spectral sequence” determined by the tower in Theorem 1.0.2 and its potential application for a Bloch-Quillen-type formula for the cycle groups $\text{CH}^q(Y,0)$ defined by the first author [32].

The coniveau filtration, as in the above, is known to be unsuitable for constructing a motivic analogue of the Atiyah-Hirzebruch spectral sequence ([5]) even on smooth schemes. To resolve this issue on smooth schemes, different filtrations were considered by Friedlander-Suslin [13] and M. Levine [26], where the latter filtration is called the homotopy coniveau filtration.

The second filtration we define in this paper, is aimed to be an analogue of such filtrations on the category of schemes of finite type. This is a cubical enrichment of the above coniveau tower in a sense (see (5.3.2), Definition 5.3.4, Proposition 5.5.12, Theorem 5.6.16):

**Theorem 1.0.3.** Let $n \geq 0$ be an integer. For a $k$-scheme $Y$ of finite type with possibly arbitrary singularities, there exists a tower in the homotopy category of spectra $\hat{G}_Y^{\bullet,0} \to K(Y)$, which induces a decreasing filtration $F_{\text{m cv}}^nK_n(Y)$ on $K_n(Y)$, such that

1. the tower and the filtration are functorial in $Y$, and
2. if $Y$ is smooth and equidimensional, the tower coincides with Levine’s homotopy coniveau tower [26 §2.1], up to weak-equivalence.
We will call $F^\bullet_{\text{mot cvn}}$ the motivic coniveau filtration on $K_n(Y)$.

In particular, the spectral sequence associated to the tower $G^\bullet_Y$ restricts to the motivic Atiyah-Hirzebruch spectral sequence, when $Y$ is smooth and equidimensional.

For a general scheme of finite type, we plan to study the spectral sequence determined by the tower in Theorem 1.0.3 in a future work. We expect there may be natural isomorphisms from the $E_2$-terms to the new higher Chow groups $\operatorname{CH}^q(Y,n)$ of [32].

A basic idea in our constructions of the filtrations of Theorems 1.0.2 and 1.0.3 is to use formal neighborhoods $\hat{X}$ of $Y$ associated to a closed immersion $Y \hookrightarrow X$ into an equidimensional smooth $k$-scheme, if such an immersion exists. This idea was used in Grothendieck’s studies of étale fundamental groups in SGA 1 [20] as well as Hartshorne’s studies of the algebraic de Rham cohomology of singular varieties in [22]. This idea is also a cornerstone in [31], [32] on studies of the motivic cohomology of singular schemes.

To illustrate part of the strategy of the construction, for the sake of discussion, for a while suppose $Y$ is affine so that it does admit such a closed immersion $Y \hookrightarrow X$. From the regularity of $X$, we deduce that $\hat{X}$ is a regular formal scheme (Lemma 2.3.1). For such $\hat{X}$, the natural functor $D_{\text{perf}}(\hat{X}) \rightarrow D_{\text{coh}}(\hat{X})$ from the perfect complexes to pseudo-coherent complexes is an equivalence (Lemma 2.2.1). To have a higher structure, we propose to replace $\hat{X}$ by the co-cubical formal scheme $\hat{X} \times \Box^\bullet$, where $\Box := \mathbb{P}^1 \setminus \{0, \infty\}$, bringing in some flavors of cubical higher Chow cycles on formal schemes of [31] and [32]. Each $\hat{X} \times \Box^n$ is still regular (see [16] Theorem 8.10, p.39) or [34] Théorème 7, p.398).

For integers $q \geq 0$, we can consider the triangulated subcategory $D^q_{\text{coh}}(\hat{X},n) \subset D_{\text{coh}}(\hat{X} \times \Box^n)$ generated by the coherent sheaves whose associated cycles are “higher Chow cycles over $\hat{X}$” in $\mathbb{Z}^{\geq q}(\hat{X},n)$ (see Definition 2.4.3), i.e. they have the codimension $\geq q$ on $\hat{X} \times \Box^n$, intersect properly with all faces $\hat{X} \times F$ for faces $F \subset \Box^n$ as well as their “special fibers” $\hat{X}_{\text{red}} \times F$, defined by the ideals of definition, being cycles over formal schemes. The associated Waldhausen $K$-spaces ([37] and [39])

$$\cdots \rightarrow G^q(\hat{X},n) \rightarrow \cdots \rightarrow G^0(\hat{X},n) \rightarrow G(\hat{X} \times \Box^n)$$

define a tower of the cubical spaces $(\underline{n} \mapsto G^q(\hat{X},n))$ over $q \geq 0$, where spaces here mean spectra. The geometric realizations $G^q(\hat{X}) := |\underline{n} \mapsto G^q(\hat{X},n)|$ give a tower of spaces

$$\cdots \rightarrow G^q(\hat{X}) \rightarrow \cdots \rightarrow G^0(\hat{X}) \rightarrow |\underline{n} \mapsto G(\hat{X} \times \Box^n)|,$$

where the last space is weak-equivalent to $G(\hat{X})$.

Taking the images of their higher homotopy groups, we have a decreasing filtration $F^\bullet$ on $G_n(\hat{X}) = \pi_n G(\hat{X})$ for $n \geq 0$. This induces a filtration on $K_n(\hat{X})$ via the isomorphism $K_n(\hat{X}) \overset{\sim}{\rightarrow} G_n(\hat{X})$.

As the first attempt, consider the induced filtration on $K_n(Y)$ by taking the images under the natural map $K_n(\hat{X}) \rightarrow K_n(Y)$. More precisely, let

$$(1.0.4) \quad \left\{ \begin{array}{ll}
F^q_K(Y) := K_n(Y), & \text{for } q \leq 0, \\
F^q_K(Y) := \text{Im}(F^q_K(\hat{X}) \rightarrow K_n(Y)), & \text{for } q \geq 1.
\end{array} \right.$$ 

There are some apparent problems in the filtration of $\{1.0.4\}$ on $K_n(Y)$. Firstly, in general a finite type $k$-scheme $Y$ may not admit a global closed immersion $Y \hookrightarrow X$ into an equidimensional smooth $k$-scheme. Secondly, even if one has it, the filtration described in $\{1.0.4\}$ depends on the embedding. An additional subtle problem is
that the homotopy cofiber of $G^{p+1} (\hat{X}) \to G^q (\hat{X})$ fails to distinguish $Y$ and $Y_{\text{red}}$, because both of them have the same formal neighborhood $\hat{X}$.

We get around these difficulties by generalizing the Čech hypercohomology machine of R. Thomason [36 §1] via the structured Čech covers, called “systems of local embeddings” $U = \{(U_i, X_i)\}_{i \in A}$ (see Definition 5.1.1). This is a minor variant of the one first used by R. Hartshorne [22] in his studies of the algebraic de Rham cohomology of singular varieties. This $U$ consists of a (quasi-)affine open cover $|U| = \{U_i\}_{i \in A}$ of $Y$ and closed immersions $\iota_i: X_i \hookrightarrow X_i$ into equidimensional smooth $k$-schemes. For each $I = (i_0, \cdots, i_p) \in \Lambda^{p+1}$, $p \geq 0$, let $U_I := U_{i_0} \cap \cdots \cap U_{i_p}$ and $X_I := X_{i_0} \times \cdots \times X_{i_p}$. Consider the diagonal closed immersion $U_I \hookrightarrow X_I$, and the completion $\hat{X}_I$ of $X_I$ along $U_I$.

The spaces $G^q (\hat{X}_I)$ over all $I \in \Lambda^{p+1}$ and $p \geq 0$ still do not tell the difference between $Y$ and $Y_{\text{red}}$. Here, we devise a $K$-theoretic analogue of the mod equivalence for cycles developed in [31] and [32]. Namely, we construct (see §1) a space $S^q (D_{\hat{X}_I}, U_I, n)$, where $D_{\hat{X}_I} = \hat{X}_I \coprod_Y \hat{X}_I$ with two morphisms

\[(1.0.5) \quad S^q (D_{\hat{X}_I}, U_I, n) \cong G^q (\hat{X}_I, n),\]

arising from the derived Milnor patching of S. Landsburg [23]. Then take the homotopy coequalizer of (1.0.5) to define the space $G^q (\hat{X}_I, U_I, n)$, and take the geometric realization $G^q (\hat{X}_I, U_I) := |U| \mapsto G^q (\hat{X}_I, U_I, n)$ of the cubical object. It can be seen as “the $K$-space of $\hat{X}_I$ mod $U_I$, with the coniveau $\geq q$.”

Using the spaces $G^q (\hat{X}_I, U_I)$ over all $I \in \Lambda^{p+1}$ and $p \geq 0$, we want to apply a Čech-like machine. This requires a technical tool that is similar in spirit to moving lemmas for algebraic cycles, thus we also call it a moving lemma in this paper. The important case is the following, stated in Proposition 3.2.5 and Corollary 3.3.3.

**Proposition 1.0.6.** Let $Y$ be a quasi-projective $k$-scheme. Suppose we have closed immersions $Y \hookrightarrow X_1 \hookrightarrow X_2$, where $X_1$ and $X_2$ are equidimensional smooth $k$-schemes. Let $\hat{X}_1$ be the completion of $X_1$ along $Y$.

Let $D^q_{\text{coh.}(\hat{X}_1)} (\hat{X}_2, n)$ be the triangulated subcategory of $D^q_{\text{coh.}(\hat{X}_2, n)}$ generated by coherent sheaves whose associated cycles intersect properly with $\hat{X}_1$ (see Definition 4.2.4).

Then the inclusion

\[D^q_{\text{coh.}(\hat{X}_1)} (\hat{X}_2, n) \hookrightarrow D^q_{\text{coh.}(\hat{X}_2, n)}\]

is essentially surjective, and the induced map

\[G^q_{(\hat{X}_1)} (\hat{X}_2, n) \to G^q_{(\hat{X}_2, n)}\]

of their Waldhausen $K$-spaces is a weak-equivalence.

One important consequence of Proposition 1.0.6 is the following, stated in Theorems 3.5.7 and 4.3.7.

**Theorem 1.0.7.** Suppose we have a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{g} & Y_2,
\end{array}
\]

where $Y_1, Y_2$ are affine $k$-schemes of finite type and the vertical maps are closed immersions into equidimensional smooth $k$-schemes. Let $\hat{X}_i$ be the completion of
Lemma 5.2.2). Hence we have the homotopy colimit
\[ \hat{\mathcal{G}}(\hat{X}_2, n) \rightarrow \mathcal{G}(\hat{X}_1, n), \]
\[ \hat{f}^* : \mathcal{G}(\hat{X}_2, n) \rightarrow \mathcal{G}(\hat{X}_1, n), \]
in the homotopy category. In particular, after geometric realizations of the cubical spaces over \( n \geq 0 \), we have the induced morphisms in the homotopy category
\[ \hat{\mathcal{G}}(\hat{X}_2, Y_2, n) \rightarrow \mathcal{G}(\hat{X}_1, Y_1, n) \]
\[ \hat{f}^* : \mathcal{G}(\hat{X}_2, Y_2) \rightarrow \mathcal{G}(\hat{X}_1, Y_1). \]

Coming back to the generalizations of the Čech-machine, how does Theorem 1.0.7 help? For a \( k \)-scheme \( Y \) of finite type and a system \( \mathcal{U} \) of local embeddings for \( Y \), we construct a cosimplicial object in the homotopy category
\[ \hat{\mathcal{G}}^0_\mathcal{U} : \left\{ \prod_{\lambda \in \Lambda} \mathcal{G}(\hat{X}_\lambda, U_\lambda) \right\} \]
where the cofaces and the codegeneracies are not solid morphisms of spaces: they are defined with helps of Theorem 1.0.7 in the homotopy category in general. Taking the homotopy limit over \( \Delta \), we define a version of Čech hypercohomology space over the system \( \mathcal{U} \)
\[ \mathbb{H}(\mathcal{U}, \hat{\mathcal{G}}) := \text{holim}_\Delta \hat{\mathcal{G}}^n_\mathcal{U}. \]

For systems \( \mathcal{U} \) of local embeddings, we introduce a notion of refinements (see Definition 5.2.3). This notion is more flexible than the one considered by R. Hartshorne [22]. With helps of Theorem 1.0.7 again, we check that for each refinement system \( \mathcal{V} \) of \( \mathcal{U} \), there exists the induced morphism in the homotopy category (see Lemma 5.2.2)
\[ \mathbb{H}(\mathcal{U}, \hat{\mathcal{G}}^n) \rightarrow \mathbb{H}(\mathcal{V}, \hat{\mathcal{G}}^n), \]
while for any two systems \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) for \( Y \), we have a common refinement (see Lemma 5.2.2). Hence we have the homotopy colimit
\[ \hat{\mathcal{G}}^n_{\mathcal{U}_1} := \text{hocolim}_\mathcal{U} \mathbb{H}(\mathcal{U}, \hat{\mathcal{G}}^n). \]

On the other hand, for each \( \mathcal{U} \), the closed immersions \( U_i \hookrightarrow X_i \) induce the closed immersions \( U_1 \hookrightarrow \hat{X}_1 \). We deduce morphisms in the homotopy category
\[ \mathcal{G}(\hat{X}_1, U_1) \rightarrow \mathcal{G}(\hat{X}_1, U_1) \rightarrow \mathcal{K}(U_1), \]
which in turn induces a morphism in the homotopy category
\[ \mathbb{H}(\mathcal{U}, \hat{\mathcal{G}}^n) \rightarrow \mathbb{H}([\mathcal{U}], \mathcal{K}), \]
where the right hand side is the original Čech hypercohomology space of \( K \)-theory in the sense of R. Thomason [36 §1]. This is weak-equivalent to the \( K \)-space \( \mathcal{K}(Y) \) of \( Y \) (see Remark 5.1.12). Thus, after taking the homotopy colimits over \( \mathcal{U} \), we deduce a tower of morphisms in the homotopy category
\[ \cdots \rightarrow \hat{\mathcal{G}}^{-1} \rightarrow \hat{\mathcal{G}}^0 \rightarrow \cdots \rightarrow \hat{\mathcal{G}}^n \rightarrow \mathcal{K}(Y), \]
Taking the images of the higher homotopy groups, we finally obtain the desired motivic coniveau filtration on \( K_*(Y) \) of the main result, Theorem 1.0.3.

On the other hand, if we used \( \mathcal{G}(\hat{X}_1, U_1, 0) \) instead of \( \mathcal{G}(\hat{X}_1, U_1) \) in (1.0.8) and repeated the above, then we obtain the coniveau filtration on \( K_*(Y) \) of Theorem 1.0.2.

We remark that, in the article [32] concurrently developed by the first named author, there is a new cycle class group denoted by \( \text{CH}^0(Y, n) \) in bold face letters
for each $Y$, possibly arbitrarily singular. This is in general distinct from the higher Chow groups $\text{CH}^q(Y, n)$ of S. Bloch [9], while they coincide when $Y$ is smooth. This new cycle theory also utilizes a Čech machine via the systems of local embeddings and the regular formal neighborhoods $\hat{X}_I$ of $U_I$. We wonder whether we have a cycle class map

$$\text{CH}^q(Y, n) \rightarrow \text{gr}_m \text{conv} \mathcal{K}^n(Y)$$

(1.0.9)

and whether this map is an isomorphism up to tensoring with $\mathbb{Q}$. Should this hold, we may regard it as the generalization of the Grothendieck-Riemann-Roch theorem for singular schemes.

**Conventions.** In this paper, a given base field $k$ is arbitrary. All noetherian formal schemes are assumed to have finite Krull dimensions unless said otherwise. We let $\text{Sch}_k$ be the category separated $k$-schemes of finite type, maybe singular, even non-reduced.

## 2. SOME RECOLLECTIONS

In §2, we recall some basic notions of noetherian formal schemes, and discuss some materials needed for our studies of algebraic $K$-theory of noetherian formal schemes, such as perfect complexes and pseudo-coherent complexes. Most of the materials are from SGA VI [14] and EGA I [17], but one new observation (Proposition 2.1.8) on regularity of formal schemes is made.

### 2.1. Formal schemes

Recall the notion of adic rings. The basic references are EGA I [17] Ch 0, §7, p.60] and [17] §10, p.180).

#### 2.1.1. Adic rings.

Recall ([17] Ch 0, Définition (7.1.2), p.60]) that a topological ring $A$ is said to be linearly topologized if there is a fundamental system of neighborhoods of 0 in $A$ given by ideals. In a linearly topologized ring $A$, we say that an ideal $I \subset A$ is an ideal of definition if $I$ is open, and for each neighborhood $V$ of 0, there is an integer $n > 0$ such that $I^n \subset V$. If an ideal of definition does exist, then we say $A$ is preadmissible. An admissible ring $A$ is a preadmissible ring that is also separated and complete.

When $A$ is a noetherian admissible ring, there exists the largest ideal of definition $I_0 \subset A$, such that $A/I_0$ is reduced. See [17] Ch 0, Corollaires (7.1.6), (7.1.7), pp.61-62.

Recall ([17] Ch 0, Définition (7.1.9), p.62]) that a preadmissible ring $A$ is called preadic if there is an ideal $I$ of definition, and the powers $I^n$ for $n > 0$ form a fundamental system of neighborhoods of 0. It is called adic, if this preadic ring is separated and complete.

Let $A$ be an admissible ring. Let $J \subset A$ be an ideal contained in an ideal of definition. The topology given by the fundamental system $J^n$ for $n > 0$ of neighborhoods of 0 is called the $J$-preadic topology. In this case $A$ is separated and complete with respect to the $J$-preadic topology. (See [17] Ch 0, (7.2.3), Proposition (7.2.4), p.63].)

#### 2.1.2. Completed rings of fractions.

Recall from [17] Ch 0, (7.6.1), (7.6.5), pp.72-72 the following. Let $A$ be a linearly topologized ring with a fundamental system of neighborhoods of 0 given by ideals $\{I_\lambda\}$. Let $S \subset A$ be a multiplicative subset. Let $u_\lambda : A \rightarrow A_\lambda := A/I_\lambda$ be the natural map. If $I_\mu \subset I_\lambda$, let $u_{\lambda \mu} : A_\mu \rightarrow A_\lambda$ be the natural map. Let $S_\lambda := u_\lambda(S)$. 
The maps $u_{A_P}$ then canonically induce surjective homomorphisms $S^{-1}_\mu A_\mu \to S^{-1}_\lambda A_\lambda$, and the data form a projective system. We define
\[
A\{S^{-1}\} := \lim_{\longleftarrow} S^{-1}_\lambda A_\lambda,
\]
called the **completed ring of fractions** of $A$ with the denominators in $S$.

There is another version of localization from [17] Ch 0, (7.6.15), p.74], that we recall. Let $A$ be a linearly topologized ring and let $f \in A$. Let $S_f := \{1, f, f^2, \cdots\}$, which is multiplicative in $A$. Then we let
\[
(2.1.3) \quad A(f) := A\{S_f^{-1}\}.
\]

If $g \in A$ is another element, then we have a canonical continuous homomorphism $A(f) \to A\{f\}_{g}$. (See [17] Ch 0, (7.6.7), p.73.) So, when $S \subset A$ is a multiplicative subset, and $f$ runs over $S$, the system $A(f)$ gives a filtered inductive system of rings. We define
\[
A(S) := \lim_{\longleftarrow} A(f).
\]

Since we have a natural homomorphism $A(f) \to A\{S^{-1}\}$, this induces a natural flat ([17] Ch 0, Proposition (7.6.16), p.75]) homomorphism $A(S) \to A\{S^{-1}\}$. We recall the following:

**Proposition 2.1.4** ([17] Ch 0, Prop. (7.6.17), Cor. (7.6.18), p.75]). Let $A$ be an admissible ring and let $P \subset A$ be an open prime ideal. Let $S := A \setminus P$. Then

1. The rings $A(S)$ and $A\{S^{-1}\}$ are local rings, and the homomorphism $A(S) \to A\{S^{-1}\}$ is a flat local homomorphism.

2. The residue fields of both rings are canonically isomorphic to $\text{Frac}(A/P)$.

Furthermore, in case $A$ is a noetherian adic ring, then the above local rings are both noetherian, and the homomorphism $A(S) \to A\{S^{-1}\}$ is faithfully flat.

### 2.1.5. Affine formal schemes

Recall that ([17] Définition (10.1.2), p.181]), for an admissible ring $A$, the formal spectrum $\mathfrak{X} = \text{Spf}(A)$ of $A$ is the closed subspace of $\text{Spec}(A)_{\text{Zar}}$ given by the open prime ideals of $A$, together with the structure sheaf $O_X$ given by the projective limit of the sheaves $(\mathcal{A}/I_\lambda)_{|X}$ over the fundamental system of neighborhoods $\{I_\lambda\}$, so that $\mathfrak{X}$ is a ringed space. Here the tilde means the sheaves associated to the modules. We denote the underlying topological space by $|X|$. An affine formal scheme is a ringed space that is isomorphic to an affine formal spectrum as ringed spaces.

The rings $A(f)$ of (2.1.3) give basic open subsets for affine formal schemes: Let $A$ be an admissible ring and let $\mathfrak{X} = \text{Spf}(A)$. For $f \in A$, let $\mathfrak{D}(f) := D(f) \cap |X|$, where $D(f) = \{P \in \text{Spec}(A) \mid f \notin P\}$. Then [17] Proposition (10.1.4), p.181] shows that the induced ringed space $\mathfrak{D}(f), O_{\mathfrak{X}}|_{\mathfrak{D}(f)}$ is isomorphic to the affine formal spectrum $\text{Spf}(A(f))$.

**Definition 2.1.6** ([17] (10.1.5, p.182]). Let $A$ be an admissible ring and let $\mathfrak{X} = \text{Spf}(A)$. Let $x \in |X|$. Define the stalk $O_{\mathfrak{X}, x}$ at $x$ to be
\[
O_{\mathfrak{X}, x} := A(S_x) = \lim_{\longleftarrow} A(f),
\]
where $S_x := A \setminus P_x$ and $P_x \subset A$ is the prime ideal corresponding to the point $x$. Since $x \in |X|$, this $P$ is an open prime ideal of $A$, with respect to the topology of $A$. This stalk is indeed a local ring by Proposition 2.1.4(1).
2.1.7. Regularity for affine formal schemes. The following regularity result plays an important role in this article:

**Proposition 2.1.8.** Let \( A \) be a noetherian adic ring and \( \mathfrak{X} = \text{Spf}(A) \). Let \( x \in |\mathfrak{X}| \), which is a regular scheme point of \( \text{Spec}(A) \). Then the local ring \( \mathcal{O}_{\mathfrak{X},x} \) at \( x \) of the affine formal scheme is a regular local ring.

To prove it, we need a few results. First recall the flat descent of regularity:

**Lemma 2.1.9** (EGA IV \text{2} [19] Proposition (6.5.1)-(i), p.143]). Let \( R \to S \) be a flat local homomorphism of noetherian local rings. Under this assumption, if \( S \) is regular, then so is \( R \).

We also need the following:

**Lemma 2.1.10.** Let \( A \) be a preadic ring, \( S \subset A \) be a multiplicative subset, and \( I \subset A \) be an ideal of definition. Then there is a canonical isomorphism of rings

\[
A\{S^{-1}\} \cong \hat{S}^{-1}A,
\]

where the completion notation of the latter is the \( S^{-1}I \)-adic completion.

**Proof.** When \( \{I_\lambda\}_\lambda \) is a fundamental system of neighborhoods of 0, we have isomorphisms of rings

\[
A\{S^{-1}\} \cong \varprojlim S^{-1}A/S^{-1}I_\lambda \cong \varprojlim S^{-1}A/(S^{-1}I)^n = \hat{S}^{-1}A,
\]

which is also a homeomorphism. \( \square \)

**Proof of Proposition 2.1.8** The point \( x \in |\mathfrak{X}| \) gives a regular scheme point of \( \text{Spec}(A) \). Let \( P_x \) be the prime ideal corresponding to \( x \) (thus \( P_x \) is an open prime ideal in \( A \)), and let \( S_x := A \setminus P_x \). By Definition 2.1.6 we have \( \mathcal{O}_{\mathfrak{X},x} = A\{S_x^{-1}\} \).

By Proposition 2.1.4, we have a flat local homomorphism \( A\{S_x^{-1}\} \to A\{S_x^{-1}\} \) of noetherian local rings. So, by Lemma 2.1.9 it is enough to show that \( A\{S_x^{-1}\} \) is a regular local ring.

By the given assumption, \( S_x^{-1}A := A_x \) is a regular local ring, with the unique maximal ideal given by \( S_x^{-1}P_x \). Here, \( S_x^{-1}I \) gives an ideal of definition of \( S_x^{-1}A \), and it is contained in the maximal ideal \( S_x^{-1}P_x \).

Since \( S_x^{-1}A \) is a regular local ring, so is the completion \( \hat{S}_x^{-1}A \) of \( S_x^{-1}A \) by the maximal ideal \( S_x^{-1}P_x \) (see H. Matsumura [20] Theorem 19.5, p.157). On the other hand, the local homomorphism \( \hat{S}_x^{-1}A \to \hat{S}_x^{-1}A \) is flat (see EGA I [17] Corollaire (10.8.9), p.197), so by Lemma 2.1.9 the regularity of \( S_x^{-1}A \) implies the regularity of \( \hat{S}_x^{-1}A \). The latter ring is isomorphic to \( A\{S_x^{-1}\} \) by Lemma 2.1.10 so, \( A\{S_x^{-1}\} \) is a regular local ring. This completes the proof. \( \square \)

2.1.11. Formal schemes. We recall some basic definitions around noetherian formal schemes from EGA I [17] (10.4.2), p.185). A noetherian formal scheme is a locally ringed space such that its underlying topological space is quasi-compact, and each point has an open neighborhood that is an affine open formal spectrum given by a noetherian adic ring. All formal schemes \( \mathfrak{X} \) will be assumed to be noetherian, unless said otherwise.

For a noetherian formal scheme \( \mathfrak{X} \), the **dimension** of \( \mathfrak{X} \) is defined to be the supremum of the Krull dimensions of the local rings \( \mathcal{O}_{\mathfrak{X},x} \) over all \( x \in |\mathfrak{X}| \). We have \( \dim \mathfrak{X} \geq \dim |\mathfrak{X}| \), where the latter is the dimension of the noetherian topological space \( |\mathfrak{X}| \). This becomes an equality if \( \mathfrak{X} \) is a scheme, but otherwise it is a strict
inequality in general. Each open subset $U \subset |\mathcal{X}|$ gives an open formal subscheme $(U, \mathcal{O}_X|_U)$, which we often denote by $\mathcal{X}|_U$.

We say that $\mathcal{X}$ is equidimensional if the dimensions of $\mathcal{X}|_U$ are uniform over all nonempty affine open subsets $U \subset |\mathcal{X}|$. We say $\mathcal{X}$ is integral if every nonempty affine open formal subscheme $U \subset \mathcal{X}$ is given by $U = \text{Spf}(A)$ for some integral domain $A$.

A closed formal subscheme $\mathcal{Y}$ of $\mathcal{X}$ is a ringed space $((\mathcal{Y}), \mathcal{O}_\mathcal{Y})$ given by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$; namely, we have $\mathcal{O}_\mathcal{Y} = \mathcal{O}_X/\mathcal{I}$ and $|\mathcal{Y}| = \text{Supp}(\mathcal{O}_\mathcal{Y}) = |\mathcal{O}_\mathcal{Y}|$. We often informally write that $\mathcal{Y} \subset \mathcal{X}$ is a closed formal subscheme. (See EGA I [17, Définition I-(10.14.2), p.210])

When $\mathcal{X}$ is an equidimensional and $\mathfrak{Z} \subset \mathcal{X}$ is a closed nonempty formal subscheme, we define the codimension to be

$$\text{codim}_{\mathcal{X}} \mathfrak{Z} := \dim \mathcal{X} - \dim \mathfrak{Z}.$$ 

As in the case of schemes, when $\mathfrak{Z}_1, \mathfrak{Z}_2 \subset \mathcal{X}$ are two closed formal subschemes, with their ideal sheaves $\mathcal{I}_{\mathfrak{Z}_1}, \mathcal{I}_{\mathfrak{Z}_2} \subset \mathcal{O}_X$, respectively, the formal scheme theoretic intersection $\mathfrak{Z}_1 \cap \mathfrak{Z}_2$ is defined to be the closed formal subscheme of $\mathcal{X}$ associated to the sum $\mathcal{I}_{\mathfrak{Z}_1} + \mathcal{I}_{\mathfrak{Z}_2}$. In case the intersection $\mathfrak{Z}_1 \cap \mathfrak{Z}_2$ is either empty, or nonempty and its codimension in $\mathcal{X}$ is greater than or equal to the sum of the codimensions of $\mathfrak{Z}_1, \mathfrak{Z}_2$, we say that the intersection is proper, or that $\mathfrak{Z}_1$ intersects properly with $\mathfrak{Z}_2$.

We say $\mathcal{X}$ is regular, if for each point $x \in |\mathcal{X}|$, the local ring $\mathcal{O}_{\mathcal{X},x}$, as in Definition 2.1.4, is a regular local ring. The notion of regular formal scheme is important in this paper.

2.2. Perfect complexes and pseudo-coherent complexes. Recall the notion of perfect complexes on ringed topoi, especially on noetherian formal schemes. (See e.g. [11, §2.4, 2.5] or more generally SGA VI [7, Exposé IV, §2.5, pp.280-281] and [17, Exposé I, Corollaire 5.10, p.138]; here there are two things required to check by [17, Exposé IV, §2.5, pp.280-281]).

For noetherian formal schemes, the perfect complexes can be described more precisely as follows: Let $\mathcal{X}$ be a noetherian formal scheme. Let $\mathcal{A}(\mathcal{X})$ be the category of all $\mathcal{O}_X$-modules. Let $\mathcal{D}(\mathcal{X})$ be the derived category of $\mathcal{A}(\mathcal{X})$.

A complex $\mathcal{E} \in \mathcal{D}(\mathcal{X})$ is called a perfect complex if each point $x \in |\mathcal{X}|$ has an open neighborhood $U \subset |\mathcal{X}|$ (which gives the open formal subscheme $\mathcal{X}|_U$) and a bounded complex $\mathcal{F}$ of locally free finite type $\mathcal{O}_{\mathcal{X}|_U}$-modules together with an isomorphism $\mathcal{F} \cong \mathcal{E}|_U$ in $\mathcal{D}(\mathcal{X}|_U)$. We let $\mathcal{D}_{\text{perf}}(\mathcal{X})$ be the triangulated subcategory of $\mathcal{D}(\mathcal{X})$ of perfect complexes on $\mathcal{X}$.

A bit more generally, a complex $\mathcal{E}$ is called a pseudo-coherent complex if every point $x \in |\mathcal{X}|$ has an open neighborhood $U$ over which $\mathcal{E}|_U$ is quasi-isomorphic to a bounded above complex of locally free finite type $\mathcal{O}_{\mathcal{X}|_U}$-modules. We let $\mathcal{D}_{\text{coh}}(\mathcal{X})$ be the triangulated subcategory of $\mathcal{D}(\mathcal{X})$ of pseudo-coherent complexes on $\mathcal{X}$ with bounded cohomologies.

An important result we need is the following:

**Lemma 2.2.1.** Let $\mathcal{X}$ be a noetherian formal scheme such that for each point $x \in |\mathcal{X}|$, the stalk $\mathcal{O}_{\mathcal{X},x}$ is a regular local ring.

Then the functor $\mathcal{D}_{\text{perf}}(\mathcal{X}) \rightarrow \mathcal{D}_{\text{coh}}(\mathcal{X})$ from the perfect complexes to pseudo-coherent complexes with bounded cohomologies is an equivalence.

In particular, the natural induced homomorphism $K_i(\mathcal{X}) \rightarrow G_i(\mathcal{X})$ is an isomorphism.

**Proof.** This is an immediate consequence of SGA VI [7, Exposé IV, §2.5, pp.280-281] and [17, Exposé I, Corollaire 5.10, p.138]; here there are two things required to check by [17, Exposé IV, §2.5, pp.280-281].
First of all, for the requirement of having sufficiently many points in the sense of SGA IV \[2\] Exposé IV, Définition 6.4.1, p.389, this is satisfied by P. Deligne’s completeness theorem SGA IV \[3\] Exposé VI, Proposition (9.0), p.336 because all (formal) schemes have locally coherent underlying topological spaces. (A kind modern reference would be Fujiwara-Kato \[14\], §2.2-2.7.) Secondly, the finite Tor dimension requirement is automatic by Auslander-Buchsbaum \[6\] because \(O_{X,x}\) is a regular local ring.

The last follows from the constructions of the Waldhausen \(K\)-spaces of the derived categories (see Thomason-Trobaugh \[37\], Waldhausen \[39\]).

2.3. **Examples of embeddable schemes.** The technique of embedding a scheme \(Y \hookrightarrow X\) was used by Grothendieck at a few places in the literature, e.g. in SGA I \[20\], in SGA II \[21\], as well as in R. Hartshorne \[22\]. This is also widely used in this article. Here is one large class of examples:

**Lemma 2.3.1.** Let \(Y\) be a quasi-projective \(k\)-scheme. Then \(Y\) is embeddable, in that there is a closed immersion \(Y \hookrightarrow X\) into a regular scheme. Furthermore, we can find a closed immersion \(Y \hookrightarrow X\) into a regular noetherian scheme \(X\) that is smooth over \(k\).

If desired, one can choose \(X\) to be equidimensional.

Given an embedding \(Y \hookrightarrow X\), let \(\hat{X}\) be the completion of \(X\) along \(Y\). Then \(\hat{X}\) is a regular noetherian formal \(k\)-scheme. If \(X\) is equidimensional, then so is \(\hat{X}\).

**Proof.** That there is a closed immersion \(Y \hookrightarrow X\) for a regular scheme \(X\) that is smooth and quasi-projective over \(k\), is apparent because \(Y\) is quasi-projective over \(k\).

If \(X\) is not equidimensional, being quasi-compact and smooth over \(k\), it has at most finitely many smooth connected components of possibly various dimensions. We can then find a big enough projective space \(\mathbb{P}^N_k\) in which all connected components of \(X\) are closed subschemes of an open subset \(U\) of \(\mathbb{P}^N_k\). Such \(U\) is smooth over \(k\), and it gives a closed immersion \(Y \hookrightarrow X \hookrightarrow U\). Replacing \(X\) by \(U\), we may assume \(X\) is equidimensional.

By taking the completion, the formal scheme \(\hat{X}\) is an equidimensional (EGA IV \[19\] Corollaire (7.1.5), p.184]) noetherian formal \(k\)-scheme. For the regularity, we need to see that for each \(x \in |\hat{X}|\), the local ring \(O_{\hat{X},x}\) is regular. This follows by first covering \(\hat{X}\) by open affine formal subschemes, and then applying Proposition 2.1.8. \(\square\)

2.4. **On cycles and associated cycles.** We recall a few definitions and facts on cycles on affine formal schemes from \[31\].

**Definition 2.4.1.** Let \(\mathfrak{X} = \text{Spf}(A)\) be an equidimensional noetherian affine formal scheme of finite Krull dimension for an equidimensional ring \(A\).

(1) Let \(\mathfrak{z}_d(\mathfrak{X})\) be the free abelian group on the set of integral closed formal subschemes of \(\mathfrak{X}\) of dimension \(d\). This is equal to the classical group \(\mathfrak{z}_d(\text{Spec}(A))\) of \(d\)-dimensional cycles by EGA III \[18\] Corollaire (5.1.8), p.495.

This naive group \(\mathfrak{z}_d(\mathfrak{X})\) of cycles may contain some undesirable cycles that have poor behaviors with respect to ideals of definition of \(\mathfrak{X}\). So, we consider the following subgroup.

(2) Let \(\mathfrak{z}_d(\mathfrak{X}_{\text{red}})\) be the subgroup of \(\mathfrak{z}_d(\mathfrak{X})\) generated by the integral closed formal subschemes that intersect properly with the subscheme \(\mathfrak{X}_{\text{red}}\) defined by the largest ideal of definition of \(\mathfrak{X}\). The largest ideal of definition exists by EGA I \[17\] Proposition (10.5.4), p.187.
(3) If \( d_X \) is the dimension of \( X \), and \( 0 \leq q \leq d_X \), then we define the group of codimension \( q \) cycles by \( z^q(X) := z_{d_X-q}(X) \).

(4) Let \( X' \subset X \) be a fixed closed formal subscheme. The subgroup \( z^q_{X'}(X) \) of \( z^q(X) \) is generated by the integral closed formal subschemes in \( z^q(X) \) that intersect properly with \( X' \).

Since the naive group of cycles on a noetherian affine formal scheme \( X = \text{Spf}(A) \) is essentially the group of cycles on the scheme \( \text{Spec}(A) \), for a coherent sheaf \( F \) on \( X \), we have its associated cycle \([F] \in \zeta_*(X)\).

**Remark 2.4.2.** When \( X \) is not (quasi)-affine, we do not know whether we can define the associated cycle \([F] \) on \( X \). But this situation does not appear in this article. \( \square \)

Recall from [31], [32] the following version of higher Chow cycles on noetherian affine formal schemes. While [32] discusses more generally cycles on quasi-affine formal schemes, we recall only those relevant to this article.

**Definition 2.4.3.** Let \( X = \text{Spf}(A) \) be an equidimensional noetherian affine formal \( k \)-scheme of finite Krull dimension.

Let \( \square := \mathbb{P}^1 \setminus \{1\} \). Consider the \( k \)-rational points \( \{0, \infty\} \subset \square \). Let \( \square^n \) be the \( n \)-fold self fiber-product of \( \square \) over \( k \). A face \( F \subset \square^n \) is a closed subscheme defined by a finite set of equations of the form

\[ \{y_{i_1} = \epsilon_1, \ldots, y_{i_s} = \epsilon_s\} \]

where \( 1 \leq i_1 < \cdots < i_s \leq n \) is an increasing sequence of indices and \( \epsilon_j \in \{0, \infty\} \).

We allow the set to be empty, in which case \( F = \square^n \).

Consider the fiber product \( X \times_k \square^n \) in the category of formal schemes, which exists by EGA I [17, Proposition (10.7.3), p.193]. It is equidimensional by Greco-Salmon [16, Theorem 7.6-(b), p.35]. In what follows we will simply write \( X \times \square^n \).

Let \( \mathcal{I}_0 \subset \mathcal{O}_X \) be the largest ideal of definition of \( X \) (EGA I [17, Proposition (10.5.4), p.187]), and let \( X_{\text{red}} \) be the closed subscheme defined by the ideal.

For integers \( n, q \geq 0 \), let \( \zeta^q(X, n) \) be the free abelian group on the set of integral closed formal subschemes \( \mathfrak{F} \subset X \times \square^n \) of codimension \( q \), subject to the following conditions:

1. **(GP) (General position)** The cycle \( \mathfrak{F} \) intersects properly with \( X \times F \) for each face \( F \subset \square^n \).
2. **(SF) (Special fiber)** For each face \( F \subset \square^n \), we have

\[ \text{codim}_{X_{\text{red}} \times F}(\mathfrak{F} \cap (X_{\text{red}} \times F)) \geq q. \]

The cycles in \( \zeta^q(X, n) \) are called **admissible** for simplicity. We let

\[ z^q(X, n) := \frac{\zeta^q(\hat{X}, n)}{\zeta^q(\hat{X}, n)_{\text{deg}}}, \]

which is the group of non-degenerate cycles. \( \square \)

For the codimension 1 face maps \( \iota_i^* : X \times \square^{n-1} \to X \times \square^n \) given by \( \{y_i = \epsilon\} \), where \( \epsilon \in \{0, \infty\} \), we have seen in [31] (or [32]) that it induces the face map \( \iota_i^* : z^q(X, n) \to z^q(X, n-1) \). For \( \partial := \sum_{i=1}^n (-1)^i (\partial_{\infty} - \partial_{\iota_i^*}) \), one checks that \( \partial \circ \partial = 0 \) and it defines a complex \( (z^q(X, \bullet), \partial) \).

3. **A moving lemma for some sheaves on formal schemes**

The central goal in [33] is to prove a moving lemma in Proposition 3.2.5 on the level of triangulated subcategories of certain pseudo-coherent complexes over the regular formal schemes of the form \( \hat{X} \times \square^n \), and to deduce a general pull-back in Theorem 3.5.7. They will be fundamental tools in §4 and §5.
3.1. A tower for formal schemes.

**Definition 3.1.1.** Let $X$ be an equidimensional noetherian formal scheme of finite Krull dimension. We say that a coherent $O_X$-module $F$ on $X$ is of codimension $q$ if there is a finite affine open cover $U$ of $|X|$ such that for each $U \in \mathcal{U}$, we have $[F|_U] \in z^q(|X|_U)$ for the cycle group in Definition 2.4.1 (2). (3).

**Lemma 3.1.2.** The notion of codimension in Definition 3.1.1 is independent of the choice of the cover.

*Proof.* We may assume $|X|$ is connected.

When $F = 0$, there is nothing to prove. So, suppose $F \neq 0$ in what follows.

Let $U$ and $V$ be two finite affine open covers of $|X|$ for which $F$ is of codimension $q$ and $q'$ with respect to $U$ and $V$, respectively, in the sense of Definition 3.1.1. Since $F \neq 0$, there is some $U_0 \in \mathcal{U}$ such that $F|_{U_0} \neq 0$.

Toward contradiction, suppose $q \neq q'$. For any $V \in \mathcal{V}$ such that $U_0 \cap V \neq \emptyset$, we are given that

$$[F|_{U_0}] \in z^q(|X|_{U_0}), \text{ and } [F|_V] \in z^{q'}(|X|_V).$$

Then by restriction, we have $[F|_{U_0 \cap V}] \in z^q(|X|_{U_0 \cap V})$ and $[F|_{U_0 \cap V}] \in z^{q'}(|X|_{U_0 \cap V})$ at the same time. Since $q \neq q'$, this implies that $[F|_{U_0 \cap V}] = 0$, which is possible only when $F|_{U_0 \cap V} = 0$.

Since the above holds for all $V \in \mathcal{V}$ such that $U_0 \cap V \neq \emptyset$, and $V$ is an open cover of $|X|$, this holds only when $F|_{U_0} = 0$. This is a contradiction. \(\square\)

**Definition 3.1.3.** Let $X$ be an equidimensional noetherian formal scheme of finite Krull dimension.

Let $D^b_{\text{coh}}(X) \subset D_{\text{coh}}(X)$ be the triangulated subcategory generated by coherent $O_X$-modules $F$ of codimension $\geq q$ in the sense of Definition 3.1.1. Over all $q \geq 0$, they form a decreasing sequence of triangulated subcategories of $D_{\text{coh}}(X)$. \(\square\)

**Remark 3.1.4.** Here the inclusion $D^b_{\text{coh}}(X) \hookrightarrow D_{\text{coh}}(X)$ may not be essentially surjective in general: the category $D^b_{\text{coh}}(X)$ of Definition 3.1.3 is generated by coherent sheaves whose associated cycles have the proper intersection condition with $X_{\text{red}}$, while the category $D_{\text{coh}}(X)$ has no such a requirement.

However, in the special case when $X_{\text{red}} = X$, clearly $D^b_{\text{coh}}(X) = D_{\text{coh}}(X)$. This situation occurs when $X = \hat{X}$, where $Y$ is a smooth equidimensional scheme and $X = Y$ in Lemma 2.3.1. In particular, $X = \hat{X} = Y = X$ in this case. \(\square\)

We need the following higher level version of Definition 3.1.3. This is analogous to the cubical version of higher Chow cycles on schemes (see [9]) and such cycles on formal schemes from [31], [32], recalled in Definition 2.4.13.

**Definition 3.1.5.** Let $X$ be an equidimensional noetherian formal $k$-scheme of finite Krull dimension.

We say that a coherent $O_{X \times \Delta^n}$-module $F$ on $X \times \Delta^n$ is admissible of codimension $\geq q$ if there is a finite affine open cover $U$ of $|X|$ such that for each $U \in \mathcal{U}$, we have $[F|_{X_\Delta^n}] \in z^{2q}(|X|_U, n)$ for the higher cycle group in Definition 2.4.13. Using an argument similar to that in Lemma 3.1.2 one checks that this codimension is independent of $U$.

Let $D^b_{\text{coh}}(X, n) \subset D_{\text{coh}}(X \times \Delta^n)$ be the full triangulated subcategory generated by coherent $O_{X \times \Delta^n}$-modules $F$ that are admissible of codimension $\geq q$ in the above sense. \(\square\)

**Remark 3.1.6.** When $n = 0$, for all $q \geq 0$, we have the equalities $D^b_{\text{coh}}(X, 0) = D^b_{\text{coh}}(X)$ of the categories in Definitions 3.1.3 and 3.1.4 in this case. \(\square\)
Definition 3.1.11. For a noetherian equidimensional formal scheme \( X \) which induces the tower of the cubical spaces
\[
\cdots \rightarrow G = G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \cdots
\]
(3.1.9)
\[
\text{Waldhausen K
\text{§ (see Thomason-Trobaugh [37, n □ 3.1.7).}
\]
Remark 3.1.7. For the closed immersion of a codimension 1 face \( \iota^n_i : F = \square^{n-1} \hookrightarrow \square^n \) given by the equation \( \{y_i = \epsilon\} \) for \( 1 \leq i \leq n \) and \( \epsilon \in \{0, \infty\} \), we have the induced derived functor
\[
L(\iota_i^n)^* : D^b_{\text{coh}}(X, n) \rightarrow D^b_{\text{coh}}(X, n-1).
\]
Note that we have the free resolution
\[
0 \rightarrow \mathcal{O}_{X \times \square^n} \otimes_{\mathcal{O}_{X \times \square^n}} \mathcal{O}_{X \times F} \rightarrow 0.
\]
Using it together with the proper intersection with faces, one checks immediately that for each coherent sheaf \( F \in D^b_{\text{coh}}(X, n) \), we have \( \text{Tor}_j^\mathcal{O}_{X \times \square^n}(\mathcal{O}_{X \times F}, F) = 0 \) for \( j > 0 \). Hence the derived functor \( L(\iota_i^n)^* \) at \( F \) is in fact given just by the usual pull-back \( (\iota_i^n)^* \) of \( F \).

Together with the apparent degeneracy functors, the above face functors define the cubical triangulated category
\[
D_{\text{coh}}^b(X, \bullet) := (\mathfrak{m} \mapsto D_{\text{coh}}^b(X, n)).
\]
Over \( q \geq 0 \), they give morphisms of cubical triangulated categories
\[
D_{\text{coh}}^b(X, \bullet) \hookrightarrow D_{\text{coh}}^b(X, \bullet) \hookrightarrow \cdots \hookrightarrow D_{\text{coh}}^b(X, 0) \hookrightarrow D_{\text{coh}}(X \times \square^\bullet).
\]
For each \( n \geq 0 \), we have the Waldhausen \( K \)-spaces
(3.1.8)
\[
G^q(X, n) := \mathcal{K}(D_{\text{coh}}^b(X, n))
\]
(see Thomason-Trobaugh [37 §1] and Waldhausen [39]), that form the tower of the Waldhausen \( K \)-spaces
(3.1.9)
\[
\cdots \rightarrow G^{q+1}(X, n) \rightarrow G^q(X, n) \rightarrow \cdots \rightarrow G^0(X, n) \rightarrow G(X \times \square^n),
\]
which induces the tower of the cubical spaces
(3.1.10)
\[
\cdots \rightarrow G^{q+1}(X, \bullet) \rightarrow G^q(X, \bullet) \rightarrow \cdots \rightarrow G^0(X, \bullet) \rightarrow G(X \times \square^\bullet).
\]
Definition 3.1.11. For a noetherian equidimensional formal scheme \( X \), we define
\[
\mathcal{G}(X) := \mathcal{K}(D_{\text{coh}}(X)).
\]
For \( q \geq 0 \), define
\[
\mathcal{G}^q(X) := \bigoplus_{\mathfrak{m}} \mathcal{G}^q(X, n)|_{\mathfrak{m}},
\]
\[
\mathcal{G}^\square(X) := \bigoplus_{\mathfrak{m}} \mathcal{G}(X \times \square^n)|_{\mathfrak{m}},
\]
the geometric realizations (see Remark 3.1.12 below) of the cubical spaces.

Thus we deduce the tower of spaces
\[
\cdots \rightarrow \mathcal{G}^{q+1}(X) \rightarrow \mathcal{G}^q(X) \rightarrow \cdots \rightarrow \mathcal{G}^0(X) \rightarrow \mathcal{G}^\square(X).
\]

Remark 3.1.12. We briefly mention that the geometric realization of a cubical space is defined as follows: recall that a cubical space (for us, a space is a spectrum) is a functor \( \text{Cube}^{\text{op}} \rightarrow \text{Spt} \). The category of cubical spaces is denoted by \( \text{Cube}^{\text{opSpt}} \).

Consider the 1-simplex \( \Delta^1 \), and for \( n \geq 1 \), let \( (\Delta^1)^n := \Delta^1 \times \cdots \times \Delta^1 \). When \( n = 0 \), let \( (\Delta^1)^0 \) be the zero simplex \( \Delta^0 \). For the simplicial set \( (\Delta^1)^n \), consider the infinite suspension \( n \mapsto C_n := \Sigma(\Delta^1)^n \). This gives a co-cubical spectrum denoted by
\[
C_\cdot : \text{Cube} \rightarrow \text{Spt}, \ n \mapsto C_n.
\]
This induces the enriched Yoneda embedding
\[
h : \text{Cube} \rightarrow \text{Cube}^{\text{opSpt}}
\]
\[
n \mapsto (h_n : \text{Cube}^{\text{op}} \rightarrow \text{Spt})
\]
where \( h_n(m) = \text{Hom}_{\text{Spt}}(C_m, C_n) \).
The geometric realization $|a|$ of a cubical space $a$ is then defined as the object evaluated with the left Kan extension $|-|$ of $C_-$ via $h$

\[
\begin{array}{ccc}
\text{Cube} & \xrightarrow{C_-} & \text{Spt} \\
\downarrow{h} & & \downarrow{|-|} \\
\text{Cube}^\oplus\text{Spt}. & & \\
\end{array}
\]

More specifically,

$$|a| = \text{colim}_{h_n, i^0} \Sigma^\infty(\Delta^1)^n_+,$$

which is the colimit over the comma category.

**Lemma 3.1.13.** For each $n \geq 0$, the morphism $G(\hat{X}) \to G(\hat{X} \times \Box^n)$ induced by the projection $\hat{X} \times \Box^n \to \hat{X}$ is a weak-equivalence.

In particular, the diagonal morphism $G(\hat{X}) \to G(\hat{X})$ to the geometric realization is also a weak-equivalence.

**Proof.** Under the localization theorem for $\mathcal{G}$-theory, it is enough to consider the affine case $\hat{X} = \text{Spf}(A)$. Here, the projection corresponds to the injective homomorphism $A \xhookrightarrow{} A[y_1, \cdots, y_n]$ into the restricted formal power series ring. Here, the induced map of the $K$-spaces is a weak-equivalence by D. Quillen [33, Theorem 7 of § 6, p. 120].

---

3.2. A moving lemma: the statement. Let $Y$ be a quasi-projective scheme. In § 3.2 we consider the special circumstance when there is a sequence of closed immersions $Y \xhookrightarrow{} X_1 \xhookrightarrow{} X_2$, where $X_1, X_2$ are equidimensional smooth $k$-schemes.

They induce closed immersions $Y \xhookrightarrow{} \hat{X}_1 \xhookrightarrow{} \hat{X}_2$ of formal schemes, where $\hat{X}_i$ is the completion of $X_i$ along $Y$.

The following Definition 3.2.1 is similar to Definition 3.1.1, but with an additional condition of proper intersections with respect to $\hat{X}_1$ (cf. Definition 2.4.1-(4)):

**Definition 3.2.1.** Let

(3.2.2) $$D^q_{\text{coh},(\hat{X}_1)}(\hat{X}_2, n) \subset D^q_{\text{coh}}(\hat{X}_2, n)$$

be the triangulated subcategory generated by all coherent $\mathcal{O}_{\hat{X}_2 \times \Box^n}$-modules $\mathcal{F}$ of codimension $\geq q$ intersecting properly with $\hat{X}_1 \times F$ for each face $F \subset \Box^n$. More precisely, there is an affine open cover $\mathcal{U}$ of $|\hat{X}_2| = |Y|$ such that for each $U \in \mathcal{U}$, we have $[\mathcal{F} \upharpoonright |\hat{X}_2|_{|U|}] \in \mathcal{D}^q_{\{X_1|_{|U|}\}}(\hat{X}_2|_{|U|}, n)$. As we did in Lemma 3.1.2 one checks that this notion is independent of the choice of $\mathcal{U}$.

Note that for any $\mathcal{F} \in D^q_{\text{coh},(\hat{X}_1)}(\hat{X}_2, n)$, we have the pull-back $\mathcal{L}\tau^* \mathcal{F} \in D^q_{\text{coh}}(\hat{X}_1, n)$ via the closed immersion $\tau: \hat{X}_1 \hookrightarrow \hat{X}_2$ (cf. [32, Proposition 5.1.1]), and it induces the “Gysin pull-back” functor

(3.2.3) $$\mathcal{L}\tau^*: D^q_{\text{coh},(\hat{X}_1)}(\hat{X}_2, n) \to D^q_{\text{coh}}(\hat{X}_1, n).$$

This (3.2.3) gives the induced morphism of the Waldhausen $K$-spaces

(3.2.4) $$\tau^*: \mathcal{K}(D^q_{\text{coh},(\hat{X}_1)}(\hat{X}_2, n)) \to \mathcal{K}(D^q_{\text{coh}}(\hat{X}_1, n)).$$

To show that $\tau^*$ in (3.2.4) is also defined on $\mathcal{K}(D^q_{\text{coh}}(\hat{X}_2, n))$ in the homotopy category, we want to check whether the inclusion functor $\mathcal{L}\tau^*$ induces a weak-equivalence of the respective $K$-spaces. The following is a key, that we would like to consider as a moving lemma:
Proposition 3.2.5. The inclusion functor \((3.2.5)\) is essentially surjective.

The following \(3.3\) and \(3.4\) are devoted to the proof of Proposition \(3.2.5\).

3.3. A moving lemma: the affine and complete intersection case. We continue to follow the notations and the assumptions of \(3.2\). Since \(X_1\) and \(X_2\) are smooth, the closed immersion \(X_1 \hookrightarrow X_2\) is an l.c.i. morphism.

In \(3.3\) we prove Corollary \(3.3.10\) which is Proposition \(3.2.5\) in the special case when \(Y\) is a connected affine \(k\)-scheme of finite type, and \(X_1 \hookrightarrow X_2\) is a complete intersection. Using this and an argument resembling Zariski descent, in \(3.4\) we prove Proposition \(3.2.5\) in general.

Let \(\hat{X}_1 = \text{Spf}(A)\) for a regular noetherian \(k\)-domain \(A\) complete with respect to an ideal \(J \subset A\), such that for \(B := A/J\), we have \(Y = \text{Spec}(B)\). When \(r\) is the codimension of \(\hat{X}_1 \hookrightarrow \hat{X}_2\), we have \(\hat{X}_2 = \text{Spf}(A[[t]])\) for a set \(t := \{t_1, \cdots, t_r\}\) of indeterminates. If \(r = 0\), there is nothing to consider, so we suppose \(r \geq 1\).

A basic intuition is to use a version of translation in the formal power series setting. We remark that for cycles, an analogous situation is studied in \(32\ §5\). In what follows, some of the lemmas are very similar to some in \(ibid\). We give arguments independently for self-containedness.

When we have rings \(R_1 \subset R_2\) and a subset \(S \subset R_1\), we use the notational conventions that \((S)_{R_2}\) is the ideal of \(R_2\) generated by \(S\).

Definition 3.3.1. We introduce the following notations.

1. Let \(A_0 := A\) and let \(A_i := A[[t_1, \cdots, t_i]]\) for \(1 \leq i \leq r\). They give a sequence of inclusions

\[ A \subset A_1 \subset A_2 \subset \cdots \subset A_r. \]

2. Consider the ideal \(J_i := (J, t_1, \cdots, t_i)_{A_i} \subset A_i\) generated by \(J, t_1, \cdots, t_i\).

By convention \(J_0 := J\). The ring \(A_i\) is complete with respect to the ideal \(J_i\), so we form the formal scheme \(\hat{X}_{1,i} := \text{Spf}(A_i)\). Here, \(\hat{X}_{1,0} = \hat{X}_1\) and \(\hat{X}_{1,r} = \hat{X}_2\).

Regarding \(A_i\) as the quotient \(A_{i+1}/(t_{i+1})\), we have closed immersions

\[ \hat{X}_1 \subset \hat{X}_{1,1} \subset \cdots \subset \hat{X}_{1,r-1} \subset \hat{X}_2. \]

3. Consider the ideal \((J_i) \subset A_{i+1}\), generated by \(J_i\) in the bigger ring \(A_{i+1}\).

4. Define the set \(V_1 = (J_0)\), and the sets \(V_{i+1} := V_i \times (J_i)\) for \(1 \leq i \leq r - 1\), where \(\times\) is the Cartesian product of sets.

5. For each \(\xi = (c_1, \cdots, c_r) \in \bigvee_r\), we consider the automorphism of \(A[[t]]\) given by the translations \(t_i \mapsto t_i + c_i\) for all \(1 \leq i \leq r\). Note that \(c_i \in (J_i-1)\), and it may not necessarily be in \(J_i-1\).

Since \(A[[t]]\) is complete with respect to the ideal \(J_r := (J, (t))\), the above gives the induced automorphism of \(\hat{X}_2 = \text{Spf}(A[[t]])\)

\[ \psi_\xi : \hat{X}_2 \to \hat{X}_2. \]

This automorphism in turn induces the automorphism

\[ \psi_\xi : \hat{X}_2 \times \Box^n \to \hat{X}_2 \times \Box^n. \]

We will use the above notations repeatedly until the end of the proof of Proposition \(3.2.5\) \(\square\)

The automorphism \(\psi_\xi\) induces pull-backs of cycles on \(\hat{X}_2 \times \Box^n\). We first observe the following (cf. \(32\ §5\)):

Lemma 3.3.2. Let \(\xi \in \bigvee_r\). Let \(\zeta \in z^g(\hat{X}_2, n)\). Then we have \(\psi_\xi^*(\zeta) \in z^g(\hat{X}_2, n)\).
Proof. We may assume that $\mathfrak{J}$ is integral. We check the conditions (GP) and (SF) for $\psi^*_{\mathfrak{J}}(3)$.

Let $F \subset \mathbb{A}^n$ be a face. Since $\psi_{\mathfrak{J}}$ is an isomorphism, it preserves dimensions. Since we have

$$\psi^*_{\mathfrak{J}}(3) \cap (\hat{X}_2 \times F) = \psi^*_{\mathfrak{J}}(3 \cap (\hat{X}_2 \times F)),$$

we immediately deduce the condition (GP) for $\psi^*_{\mathfrak{J}}(3)$ from that of $\mathfrak{J}$.

The condition (SF) for $\psi^*_{\mathfrak{J}}(3)$ holds immediately, because all of $t_i, t_i + c_i, c_i$ belong to the largest ideal of definition.

For a given coherent sheaf $F$ on $\hat{X}_2 \times \mathbb{A}^n$ of codimension $\geq q$ whose associated cycle $[F]$ belongs to $z \geq q(\hat{X}_2, n)$, we want to know whether we can have the additional proper intersection property with $\hat{X}_1 \times F$ over the faces $F \subset \mathbb{A}^n$, for the translated sheaf $\psi_{\mathfrak{J}}^* F$ for a suitable choice of $\mathfrak{J} \in V_r$. The cardinality of $J$ matters in doing so. The trivial case is:

**Lemma 3.3.3.** Suppose $|J| < \infty$. Then

(1) $J = 0$ and

(2) $Y = \hat{X}_1 = X_1$ and $Y$ is smooth over $k$.

**Proof.** We have $Y = \text{Spec}(B)$ and $\hat{X}_1 = \text{Spf}(A)$ with $A/J = B$. Write $X_1 = \text{Spec}(A_0)$ for a smooth $k$-algebra $A_0$. Let $I \subset A_0$ be the ideal such that $A_0/I = B$. By definition, we have $A = \lim_{\leftarrow m} A_0/I^m$, $J = I$, and $A/J = A_0/I = B$.

Since $J$ is finite, the descending chain of ideals

$$\cdots \subset J^3 \subset J^2 \subset J$$

is stationary, so that there is some $N \geq 1$ such that $J^N = J^{N+1} = \cdots$.

Since $A$ is $J$-adically complete, we have

$$A = \lim_{\leftarrow m} A/J^m = A/J^N.$$ 

This implies that $J^N = 0$. But $Y$ is connected so that $A$ is a regular $k$-domain. In particular, it has no nonzero nilpotent element. Hence $J = 0$, proving (1).

That $J = 0$ implies $0 = \hat{I}$. Hence $I = 0$ and $A = A_0 = B$. This means that $Y = \hat{X}_1 = X_1$. Because $X_1$ is smooth over $k$ and $Y = X_1$, the scheme $Y$ is smooth over $k$, proving (2). □

When $J$ is infinite, the technical key is Lemma 3.3.9 below. The argument is somewhat involved. Analogous ideas are also to be used in [72, §5], in a slightly different situation.

Let’s begin with the following basic result from commutative algebra (see e.g. Atiyah-MacDonald [1] Proposition 1.11-i), p.8) needed in its proof.

**Lemma 3.3.4** (Prime avoidance). Let $A$ be a commutative ring with unity. Let $J \subset A$ be an ideal, and let $I_1, \ldots, I_N \subset A$ be prime ideals such that $J \nsubseteq I_i$ for all $1 \leq i \leq N$. Then $J \nsubseteq \bigcup_{i=1}^N I_i$.

The following, which is used in Subcases 1-2 and 1-3 of the proof of Lemma 3.3.3 needs the prime avoidance:

**Lemma 3.3.5.** Let $A$ be an integral domain. For a subset $S \subset A[[t]]$, let $(S) \subset A[[t]]$ be the ideal generated by $S$. Suppose that for an ideal $J \subset A$, the ideal $(J) \subset A[[t]]$ is proper.

Let $I_1, \ldots, I_N \subset A[[t]]$ be prime ideals such that $(J, t) \nsubseteq I_i$.

Suppose $t \in I_{i_1}$ for some indices $1 \leq i_1 < \cdots < i_n \leq N$. Then we have
(1) \( J \not\subset \bigcup_{j=1}^{u} I_{j} \).

(2) \(|J \setminus (J) \cap (\bigcup_{j=1}^{u} I_{j})| = \infty. \)

(3) For each \( c \in (J) \setminus (J) \cap (\bigcup_{j=1}^{u} I_{j}) \), we have \( t + c \not\in I_{j} \) for all \( 1 \leq j \leq u. \)

Proof. (1) For each \( 1 \leq j \leq u \), we are given that \( t \in I_{j}. \) If \( J \subset I_{j} \), then we would have \( (J, t) \subset I_{j}, \) contradicting the given assumption that \( (J, t) \not\subset I_{j}. \) Thus we deduce that \( J \not\subset I_{j}. \) By the prime avoidance (Lemma \ref{lemma:prime-avoidance}), we deduce that \( J \not\subset \bigcup_{j=1}^{u} I_{j} \), proving (1).

(2) Since the set is nonempty by (1), choose any \( c \in (J) \setminus (J) \cap (\bigcup_{j=1}^{u} I_{j}) \). This is nonzero. Since \( c \in (J) \) and \( (J) \) is an ideal, we have \( \{c, c^{2}, \cdots\} \subset (J). \)

We claim that the powers \( c^{1}, c^{2}, \cdots \), are all distinct. Indeed, suppose \( c^{p} = c^{p'} \) for some distinct positive integers \( p < p'. \) This gives \( c^{p}(1 - c^{p-p'}) = 0 \) in \( A[[t]]. \) Since \( A[[t]] \) is an integral domain and \( c \neq 0 \), we deduce that \( 1 - c^{p-p'} = 0. \) In particular \( c \) is a unit in \( A[[t]]. \) However, we are given that \( J \) is a proper ideal of \( A[[t]] \) so that \( J \) contains no unit, a contradiction. In particular, the set \( \{c, c^{2}, c^{3}, \cdots\} \) is infinite.

If any one of \( c^{p} \) is \( \in \bigcup_{j=1}^{u} I_{j} \) for some \( p \geq 1 \), then \( c^{p} \in I_{j} \) for some \( j. \) But \( I_{j} \) is a prime ideal so that it implies \( c \in I_{j}, \) which contradicts our choice of \( c. \) Hence the infinite set \( \{c, c^{2}, \cdots\} \) is disjoint from \( \bigcup_{j=1}^{u} I_{j}, \) i.e. it is contained in the set \( (J) \setminus (\bigcup_{j=1}^{u} I_{j}). \) This proves (2).

(3) Toward contradiction, suppose \( t + c \in I_{j} \) for some \( 1 \leq j \leq u. \) Since \( t \in I_{j}, \) by our given assumption, we have \( c = (t + c) - t \in I_{j}. \) But, this contradicts our choice of \( c \) that \( c \not\in I_{j}. \) Hence \( t + c \not\in I_{j} \) for all \( 1 \leq j \leq u, \) proving (3). \qed

The following is to be used in Subcase 1-3 of the proof of Lemma \ref{lemma:prime-avoidance}.

**Lemma 3.3.6.** Let \( A \) be an integral domain complete with respect to an ideal \( J \subset A. \) Let \( I \subset A[[t]] \) be a prime ideal such that \( t \not\in I. \) Let \( c \in (J) \) be any nonzero member.

(1) If \( c \in I, \) then \( t + c^{p} \not\in I \) for all integers \( p \geq 1. \)

(2) If \( c \not\in I, \) then there is at most one integer \( p \geq 1 \) such that \( t + c^{p} \in I. \)

**Proof.** (1) Suppose \( c \in I. \) Then for all \( p \geq 1, \) we have \( c^{p} \in I. \) If \( t + c^{p} \in I, \) then \( t = (t + c^{p}) - c^{p} \in I, \) which contradicts that \( t \not\in I. \) Hence we must have \( t + c^{p} \not\in I \) for all \( p \geq 1, \) proving (1).

(2) Now suppose \( c \not\in I. \) Toward contradiction, suppose for two positive integers \( p < p', \) we have \( t + c^{p}, t + c^{p'} \in I. \) This implies that

\[
(t + c^{p}) - (t + c^{p'}) = c^{p}(1 - c^{p-p'}) \in I.
\]

Since \( I \) is a prime ideal such that \( c \not\in I, \) we have \( 1 - c^{a} \in I, \) where \( a := p' - p. \)

Since \( A \) is complete with respect to \( J, \) the ring \( A[[t]] \) is complete with respect to \( (J, t). \) Since \( c \in (J) \subset (J, t), \) we have \( 1 + c^{a} + c^{2a} + \cdots \in A[[t]]. \) Thus multiplying it to the element \( 1 - c^{a} \) of the ideal \( I, \) we deduce that

\[
(1 + c^{a} + c^{2a} + \cdots)(1 - c^{a}) = 1 \in I,
\]

which is a contradiction because \( I \subset A[[t]] \) is a prime ideal, thus proper. This proves (2). \qed

**Lemma 3.3.7.** Let \( A \) and \( J \) be as the above. Suppose \( |J| = \infty. \) Consider the ring \( A[[t]] \) for a variable \( t. \) Let \( (J) \subset A[[t]] \) be the ideal generated by \( J \) in \( A[[t]]. \) This is proper.

Let \( I_{1}, \cdots, I_{N} \subset A[[t]] \) be prime ideals such that \( (J, t) \not\subset I_{i} \) for all \( 1 \leq i \leq N. \)

Then there exists some \( c \in (J) \) such that \( t + c \not\in I_{i} \) for all \( 1 \leq i \leq N. \)
Proof. There are a few cases to consider. The easiest case is:

**Case 1:** Suppose \( t \not\in I_i \) for all \( 1 \leq i \leq N \).

In this case, we may take \( c = 0 \) to deduce the desired conclusion that \( t+c = t \not\in I_i \) for all \( 1 \leq i \leq N \).

Here is the opposite case:

**Case 2:** Suppose \( t \in I_i \) for all \( 1 \leq i \leq N \).

In this case, applying Lemma 3.3.2, we can find some \( c \in (J) \setminus (\bigcup_{i=1}^{N} I_i) \) such that \( t+c \not\in I_i \) for all \( 1 \leq i \leq N \). This answers the lemma in this case.

Here is the “mixed” case:

**Case 3:** Suppose that \( t \in I_i \) for some indices \( i \), while we have \( t \not\in I_i \) for some other indices \( i' \).

After relabeling them, if necessary, we may assume that there is some positive integer \( 1 \leq s < N \) such that we have

\[
t \not\in I_i, \quad \text{for } 1 \leq i \leq s, \quad \text{and } t \in I_i, \quad \text{for } s + 1 \leq i \leq N.
\]

We apply Lemma 3.3.5 to the ideals \( I_{s+1}, \ldots, I_N \). Let \( \tilde{J} := (J) \setminus (\bigcup_{s+1}^{N} I_i) \).

Pick \( c_0 \in \tilde{J} \). By Lemma 3.3.5, we have \( t + c_0^p \not\in I_i \) for \( s + 1 \leq i \leq N \) and \( p \geq 1 \).

On the other hand, by Lemma 3.3.6 there exists a finite subset \( B \subset \mathbb{N} \) of size \( |B| \leq s \) such that for all \( p \in \mathbb{N} \setminus B \), we have \( t + c_0^p \not\in I_i \) for all \( 1 \leq i \leq s \).

Thus for all \( p \in \mathbb{N} \setminus B \) and for all \( 1 \leq i \leq N \), we have proven that \( t + c_0^p \not\in I_i \).

This completes the proof of the lemma. \( \square \)

Lemma 3.3.8. Suppose \( |J| = \infty \).

Suppose we are given finitely many integral cycles \( \mathcal{Z}_i \in z^q(\tilde{X}_2, n_i) \) for some integers \( q_i, n_i \geq 0 \) over the indices \( 1 \leq i \leq N \).

Then there exists \( \mathcal{Z} = (c_1, \ldots, c_r) \in \mathbb{V}_r \) such that \( \mathcal{Z}^*\mathcal{Z}_i \in z^q(\tilde{X}_1, n_i) \) over all \( 1 \leq i \leq N \).

Proof. By Lemma 3.3.2, for any \( \mathcal{Z} \in \mathbb{V}_r \), we already have \( \mathcal{Z}^*\mathcal{Z}_i \in z^q(\tilde{X}_2, n_i) \). It remains to show that for a suitable \( \mathcal{Z} \in \mathbb{V}_r \), each cycle \( \mathcal{Z}^*\mathcal{Z}_i \) intersects \( \tilde{X}_1 \times F \) properly for all faces \( F \subset \square^s \).

We prove it by induction on the codimension \( r \) of \( \tilde{X}_1 \) in \( \tilde{X}_2 \).

**Step 1:** Consider the case when \( r = 1 \). Note that \( \tilde{X}_2 \times \square^s = \text{Spf}(A[[t_1]](y_1, \ldots, y_{n_1})) \).

Let \( I_i \subset A[[t_1]](y_1, \ldots, y_{n_1}) \) be the prime ideal of the integral cycle \( \mathcal{Z}_i \) over the indices \( 1 \leq i \leq N \).

Let \( F \subset \square^s \) be a face. For the intersection \( \mathcal{Z}_i \cap (\tilde{X}_2 \times F) \), its integral components are given by a finite collection of prime ideals \( I_{i,j} \subset A[[t_1]](y_1, \ldots, y_{n_1}) \). They all satisfy \( (J, t_1)_{A[[t_1]]} \not\subset I_{i,j} \) by the condition (SF) in definition 2.4.3.

To obtain proper intersections with \( \tilde{X}_1 \times F \) after a translation, we want some \( c \in (J) \) such that \( t_1 + c \not\in I_{i,j} \) for all \( i, j \) and \( F \) at the same time. This can be achieved if the contracted prime ideals

\[
I_{i,j,0} := A[[t_1]] \cap I_{i,j} \quad \text{in } A[[t_1]]
\]

over all \( i, j, F \) have the property that \( t_1 + c \not\in I_{i,j,0} \). Here we note that we still have \( (J, t_1)_{A[[t_1]]} \not\subset I_{i,j,0} \) by the condition (SF) in definition 2.4.3.

Since \( (I_{i,j,0})_{i,j,F} \) is a finite collection of prime ideals in \( A[[t_1]] \), where none of them contains \( (J, t_1)_{A[[t_1]]} \), we deduce from Lemma 3.3.7 that there does exist some \( c \in (J) \) such that \( t_1 + c \not\in I_{i,j,0} \) for all \( i, j, F \).
Thus for such $c \in (J)$, the translations $\psi^*_c(3_i)$ intersect properly with $\hat{X}_1 \times F$ for all $1 \leq i \leq N$ and all faces $F \subset \square^n$. This proves the lemma for $r = 1$.

Step 2: Now suppose $r \geq 2$. Suppose that the lemma holds when the codimension of $\hat{X}_1$ in $\hat{X}_2$ is $\leq r - 1$. The ring $A_{r-1} := A[[t_1, \cdots, t_{r-1}]]$ is complete with respect to $J_{r-1}$, and $\hat{X}_{1,r-1} := \text{Spf}(A_{r-1})$ is a closed formal subscheme of codimension $1$ in $\hat{X}_2$, given by the ideal generated by $t_r$.

For the given cycles $3_i$ for $1 \leq i \leq N$, by Step 1 applied to $\hat{X}_{1,r-1} \subset \hat{X}_2$, there exists some $c_r \in (J_{r-1})$ such that for all faces $F \subset \square^n$,

$$\psi^*_{t_r+c_r}(3_i) \quad \text{and} \quad \hat{X}_{1,r-1} \times F$$

intersect properly.

For $1 \leq i \leq N$, let $3'_i$ be the cycle associated to the intersection

$$\psi^*_{t_r+c_r}(3_i) \cap (\hat{X}_{1,r-1} \times \square^n).$$

Since $\hat{X}_1 \subset \hat{X}_{1,r-1}$ is of codimension $r - 1$, given by the ideal generated by $t_1, \cdots, t_{r-1}$, by the induction hypothesis we can find some $c_r' = (c_1, \cdots, c_{r-1}) \in \mathcal{V}_{r-1}$ such that for all faces $F \subset \square^n$,

$$\psi^*_c(3'_i) \quad \text{and} \quad \hat{X}_1 \times F$$

intersect properly.

Combining the above choices of $c_r' \in \mathcal{V}_{r-1}$ and $c_r \in (J_{r-1})$, let $c := (c_r', c_r) \in \mathcal{V}_r$.

By construction, $\psi^*_c(3_i)$ and $\hat{X}_1 \times F$ intersect properly for all faces $F \subset \square^n$ and all $1 \leq i \leq N$.

This completes the proof of the lemma. \hfill \Box

Lemma 3.3.9. Suppose $|J| = \infty$.

If $F$ is a coherent $\mathcal{O}_{\hat{X}_2 \times \square^n}$-module such that $[F]$ belongs to the group $z_{\geq q}(\hat{X}_2,n)$, then there exists $c = (c_1, \cdots, c_r) \in \mathcal{V}_r$ such that for the translated coherent sheaf $\psi^*_c(F)$, the associated cycle $[\psi^*_c(F)] \in z_{\geq q}(\hat{X}_2,n)$.

Proof. Write $[F] = \sum_{i=1}^N m_i 3_i$, for some integers $m_i > 0$ and finitely many distinct integral cycles $3_i \in z_{= q}(\hat{X}_2,n)$, for some integers $q_i \geq q$ and $n_1 = \cdots = n_N = n$

We apply Lemma 3.3.8 to those $3_i$. Indeed by this lemma, we have some $c \in \mathcal{V}_r$ such that $\psi^*_c(3_i) \in z_{\geq q}(\hat{X}_2,n)$ for all $1 \leq i \leq N$.

On the other hand, since $\psi_c$ is an isomorphism given by a translation, we have $[\psi^*_c(F)] = \sum_{i=1}^N m_i \psi^*_c(3_i)$, which is in $z_{\geq q}(\hat{X}_1) \hat{X}_2, n)$ as desired. \hfill \Box

Corollary 3.3.10. Let $Y$ be a connected affine $k$-scheme of finite type, with closed immersions $Y \hookrightarrow X_1 \hookrightarrow X_2$ into equidimensional smooth $k$-schemes $X_1, X_2$ such that the immersion $X_1 \hookrightarrow X_2$ is a complete intersection.

Then Proposition 3.2.3 holds in this case, i.e. the inclusion functor

$$D^q_{\text{coh},(X_1)}(\hat{X}_2, n) \hookrightarrow D^q_{\text{coh},(\hat{X}_2, n)}$$

is essentially surjective.

Proof. Write $Y = \text{Spec}(B)$, $\hat{X}_1 = \text{Spf}(A)$, and $\hat{X}_2 = \text{Spf}(A[[t]])$, where $B = A/J$ for an ideal $J \subset A$ as above. If $X_1 = X_2$, then there is nothing to prove. So, we suppose $X_1 \subset X_2$ and $r := \text{codim}_{X_2} X_1 \geq 1$.

Let $F$ be a coherent $\mathcal{O}_{\hat{X}_2 \times \square^n}$-module of codimension $\geq q$, such that its associated cycle $[F]$ belongs to $z_{\geq q}(\hat{X}_2,n)$.

Case 1: Suppose $|J| < \infty$. By Lemma 3.3.3 we have $J = 0$ and $Y = \hat{X}_1 = X_1$. \hfill \Box
Our requirement is that \([F] \in \mathcal{Z}_2^Y(\tilde{X}_2, n)\) so that each component already intersects properly with \(Y \times F\), which is \(\tilde{X}_1 \times F\), for all faces \(F \subset \Box^n\). Thus in this case the functor (3.3.11) is the identity and there is nothing to show.

**Case 2:** Now suppose \(|J| = \infty\). By Lemma (3.3.9) there exists \(\xi \in \mathcal{V}_r\) such that \(\psi_\xi^*F\) is a coherent \(\mathcal{O}_{\tilde{X}_2 \times \Box^n}\)-module that intersects \(\tilde{X}_1 \times F\) properly for all faces \(F \subset \Box^n\). But, \(\psi_\xi^*\) is an automorphism of \(\tilde{X}_2 \times \Box^n\) so that \(\psi_\xi^*F\) is isomorphic to \(F\) as \(\mathcal{O}_{\tilde{X}_2 \times \Box^n}\)-modules.

Thus after replacing \(F\) by \(\psi_\xi^*F\), which is in \(\mathcal{D}^\mathcal{q}_{\text{coh},\{\tilde{X}_1\}}(\tilde{X}_2, n)\), we see that the inclusion (3.3.11) is essentially surjective. This proves the corollary. □

The above Corollary (3.3.10) answers part of Proposition (3.2.5) when \(Y\) is connected affine and \(X_1 \hookrightarrow X_2\) is a complete intersection.

**3.4. A moving lemma: the general case.** The argument via translations given in (3.3.3) does not extend to a general quasi-projective \(Y\) and a general closed immersion \(X_1 \hookrightarrow X_2\) between smooth \(k\)-schemes. This may not be a complete intersection, but it is always a l.c.i., so a version of local-to-global Zariski descent-type gluing may offer a way-out.

In (3.4) we finish the proof of Proposition (3.2.5) using such an argument.

**Proof of Proposition (3.2.5)** If \(Y\) is not connected, then we can argue for each connected component separately. Hence we may assume \(Y\) is connected.

Since the morphism \(X_1 \hookrightarrow X_2\) is l.c.i., there is a finite affine open cover \(\mathcal{V} = \{V_1, \ldots, V_N\}\) of \(X_2\) such that \(X_1 \cap V_i \hookrightarrow X_2 \cap V_i\) is a complete intersection for each \(1 \leq i \leq N\).

Let \(U_i := Y \cap V_i\). We regard \(U_i\) as an open subset of \(Y\) as well as an open subscheme of \(Y\). The collection \(\mathcal{U} := \{U_1, \ldots, U_N\}\) gives an affine open cover of \(Y\). We may assume each \(U_i\) is connected.

Recall that \(\tilde{X}_\ell\) is a ringed space whose underlying topological space \(|\tilde{X}_\ell|\) is exactly equal to that of \(Y\). So, when \(U \subset Y\) is an open subset, recall \(\tilde{X}_\ell|_U\) means the open formal subscheme \((U, \mathcal{O}_{\tilde{X}_\ell}|_U)\). For \(\ell = 1, 2\) and \(1 \leq i \leq N\), note that \(\tilde{X}_\ell|_{U_i}\) is equal to the completion of \(X_\ell \cap V_i\) along \(U_i\).

By Corollary (3.3.10) applied to each \(1 \leq i \leq N\), we see that the inclusion functor

\[
\mathcal{D}^\mathcal{q}_{\text{coh},\{\tilde{X}_1|_{U_i}\}}(\tilde{X}_2|_{U_i}, n) \subset \mathcal{D}^\mathcal{q}_{\text{coh}}(\tilde{X}_2|_{U_i}, n)
\]

is essentially surjective. To prove Proposition (3.2.5) by induction, it is enough to show that if the proposition holds for two (not necessarily affine) open subsets of \(Y\) that cover \(Y\), then the proposition holds for \(Y\) as well.

So, suppose for two open sets \(V_1, V_2 \subset X_2\), with \(U_i = Y \cap V_i\), the proposition holds for \((U_\ell, \tilde{X}_1|_{U_\ell}, \tilde{X}_2|_{U_\ell})\), \(\ell = 1, 2\), and \(Y = U_1 \cup U_2\) so that \(\tilde{X}_2 = \tilde{X}_2|_{U_1 \cup U_2}\).

Let \(F\) be a coherent sheaf in \(\mathcal{D}^\mathcal{q}_{\text{coh}}(\tilde{X}_2, n)\). We prove that for some \(\mathcal{F}' \in \mathcal{D}^\mathcal{q}_{\text{coh},\{\tilde{X}_1\}}(\tilde{X}_2, n)\), there is a quasi-isomorphism between \(F\) and \(\mathcal{F}'\).

In what follows, for each open subset \(U \subset Y\), as a shorthand let us write \(\mathcal{F}|_U\) instead of the bulky notation \(\mathcal{F}|_{\tilde{X}_2|_{U \times \Box^n}}\), for notational simplicity.

By the given assumptions, for \(\ell = 1, 2\) there are quasi-isomorphisms

\[
\alpha_\ell : \mathcal{F}|_{U_\ell} \to \mathcal{M}_\ell
\]

for some pseudo-coherent complexes \(\mathcal{M}_\ell \in \mathcal{D}^\mathcal{q}_{\text{coh},\{\tilde{X}_1|_{U_\ell}\}}(\tilde{X}_2|_{U_\ell}, n)\). By Lemma (2.2.11) each \(\mathcal{M}_\ell\) is a perfect complex on \(\tilde{X}_2|_{U_\ell \times \Box^n}\). Let \(j_\ell\) (resp. \(j_{12}\)) denote the
open immersion \( \tilde{X}_2|_{U_t} \times \Box^n \hookrightarrow \tilde{X}_2 \times \Box^n \) (resp. \( \tilde{X}_2|_{U_t \cap U_2} \times \Box^n \hookrightarrow \tilde{X}_2 \times \Box^n \)). Since \( F \) is an \( O_{\tilde{X}_2 \times \Box^n} \)-module, we then have a short exact sequence of \( O_{\tilde{X}_2 \times \Box^n} \)-modules

\[
0 \to j_{12!} F|_{U_t \cap U_2} \to j_{11!} F|_{U_t} \oplus j_{21!} F|_{U_2} \to F \to 0.
\]

We can map it to the corresponding distinguished triangle in the derived category of \( O_{\tilde{X}_2 \times \Box^n} \)-modules.

For an open immersion \( j \), since \( (j_!, j^*) \) is an adjoint pair (see e.g. SGA IV₁ [2 Exp. IV, 11.3.3] and SGA IV₂ [3 Exp. V, 1.3.1]), we have a morphism

\[
F(3.4.1) \implies \gamma
\]

of distinguished triangles in the derived category of \( O_{\tilde{X}_2 \times \Box^n} \)-modules, with \( \beta = (\iota, \alpha_2 \circ \alpha_1^{-1}|_{U_t \cap U_2}) \), where \( \iota \) the canonical map induced by the counit of the adjunction \((j_!, j^*)\), and

\[
F' := \text{Cone}(\beta).
\]

Notice that the morphism \( \gamma \) exists by the axiom \( \text{TR}_3 \) of triangulated categories (see A. Neeman [30]). Since \( F' \) restricted over \( U_\ell \) is just \( M_\ell \) for \( \ell = 1, 2 \), we see that \( F' \) is a perfect complex. By construction we see that \( F' \in \mathcal{D}_\text{coh}(\tilde{X}_2, n) \).

Since the first two vertical morphisms of (3.4.1) are quasi-isomorphisms, so is the morphism \( \gamma \). This shows that the inclusion functor (3.3.11) is essentially surjective, finishing the proof of the proposition.

Remark 3.4.2. Note that when \( F|_U \) is quasi-coherent, the sheaf \( j_! F|_U \) is not necessarily quasi-coherent in general. But this is not a problem here; we use implicitly the fact that the algebraic \( K \)-groups constructed from various kinds of derived categories are all equivalent. See Thomason-Trobaugh [37 Lemmas 3.5-3.7, p.313].

We also remark that the above construction of \( F' = \text{Cone}(\beta) \) is somewhat similar to the argument around [37] (3.20.4.2, p.326), where a perfect complex is obtained by gluing two perfect complexes defined on open subsets.

Corollary 3.4.3. Let \( Y \) be a quasi-projective \( k \)-scheme. Let \( Y \hookrightarrow X_1 \hookrightarrow \tilde{X}_2 \) be closed immersions, where \( X_1, X_2 \) are equidimensional smooth \( k \)-schemes. Let \( \tilde{X}_i \) be the completion of \( X_i \) along \( Y \).

Then the zigzag of functors

\[
\mathcal{D}_\text{coh}(\tilde{X}_2, n) \leftarrow \mathcal{D}_{\text{coh}(\tilde{X}_1)}(\tilde{X}_2, n) \leftarrow \mathcal{D}_{\text{coh}(\tilde{X}_1)}(\tilde{X}_1, n),
\]

induces morphisms of the induced Waldhausen \( K \)-spaces

\[
\mathcal{G}(\tilde{X}_2, n) \leftarrow \mathcal{G}(\tilde{X}_1, n) \leftarrow \mathcal{G}(\tilde{X}_1, n),
\]

where the first arrow \( i \) is a weak-equivalence. In particular, we have the Gysin morphisms of spaces in the homotopy category

\[
(3.4.5) \quad \left\{ \begin{array}{ll} \tau^- : \mathcal{G}(\tilde{X}_2, n) & \to \mathcal{G}(\tilde{X}_1, n), \\ \tau^+ : \mathcal{G}(\tilde{X}_2) & \to \mathcal{G}(\tilde{X}_1), \end{array} \right. \]

where \( \mathcal{G}(-) \) are as in Definition 3.3.17.

Proof. By Proposition 3.2.5, the inclusion functor \( i : \mathcal{D}_{\text{coh}(\tilde{X}_1)}(\tilde{X}_2, n) \hookrightarrow \mathcal{D}_{\text{coh}(\tilde{X}_1)}(\tilde{X}_2, n) \) in (3.2.2) is essentially surjective. In addition, the functors \( i, \tau^+ \) commute with the respective faces and degeneracies, see Remark 3.1.7.
Hence in the homotopy category, we obtain the morphisms (3.4.4) for each $n \geq 0$, and by [37] Theorems 1.9.1, 1.9.8, p.263, 271], $i$ induces a weak-equivalence of their respective $K$-spaces, i.e. the first arrow of (3.4.4) is a weak-equivalence. Thus we deduce the first morphism of (3.4.5).

The morphisms in (3.4.4) over $n \geq 0$ form morphisms of cubical spaces, and taking their respective geometric realizations, we deduce the second morphism of (3.4.5).

3.5. Some general pull-backs. In [3.5], we generalize Corollary 3.4.3 a bit. The extension we obtain is given as Theorem 3.5.7 below.

Note that we have:

**Lemma 3.5.1.** For $i = 1, 2$, let $Y_i$ be quasi-projective $k$-schemes. Choose any closed immersions $Y_i \hookrightarrow X_i$ into smooth $k$-schemes, and let $\hat{X}_i$ be the completion of $X_i$ along $Y_i$ for $i = 1, 2$. We also consider $Y_1 \times_k Y_2$ as a closed subscheme of $X_1 \times_k X_2$, and let $\hat{X}_1 \times_k \hat{X}_2$ be the completion of $X_1 \times_k X_2$ along $Y_1 \times Y_2$.

Then we have a natural isomorphism of formal schemes

\[
\hat{X}_1 \times_k \hat{X}_2 \simeq \hat{X}_1 \times_k X_2.
\]

**Proof.** The question is local, so we may assume $Y_i$ and $X_i$ are affine.

For $i = 1, 2$, write $Y_i = \Spec(B_i)$, $X_i = \Spec(A_i)$. We have natural surjections $A_i \to B_i$ and let $I_i$ be the kernels. Let $A_3 := A_1 \otimes_k A_2$, $B_3 := B_1 \otimes_k B_2$, so that $X_1 \times_k X_2 = \Spec(A_3)$ and $Y_1 \times_k Y_2 = \Spec(B_3)$.

We have

\[
B_3 = \frac{A_1}{I_1} \otimes_k \frac{A_2}{I_2} \simeq \frac{A_1 \otimes_k A_2}{I_1 \otimes_k A_2 + A_1 \otimes_k I_2},
\]

and let $I_3 := I_1 \otimes_k A_2 + A_1 \otimes_k I_2$. For $1 \leq i \leq 3$, we have $\hat{X}_i = \Spf(\hat{A}_i)$, where $\hat{A}_i = \lim_{\to} A_i$.

The product $\hat{X}_1 \times \hat{X}_2$ of the formal schemes is the formal spectrum of the completed tensor product $A_1 \hat{\otimes}_k A_2$. This is given also by

\[
\hat{A}_1 \hat{\otimes}_k \hat{A}_2 = \lim_{m,n} \frac{A_1}{I_1^m} \hat{\otimes}_k \frac{A_2}{I_2^n} \simeq \lim_{m,n} \frac{A_1 \otimes_k A_2}{I_1^m \otimes_k A_2 + A_1 \otimes_k I_2^n}.
\]

For integers $m, n \geq 1$, if $N \geq m + n$, by the binomial theorem

\[
I_3^N = (A_1 \otimes_k A_2 + A_1 \otimes_k I_2)^N \subset I_1^m \otimes_k A_2 + A_1 \otimes_k I_2^n.
\]

Conversely for an integer $N \geq 1$, we have

\[
I_3^N \otimes_k A_2 + A_1 \otimes_k I_2^N \subset (I_1 \otimes_k A_2 + A_1 \otimes_k I_2)^N = I_3^N.
\]

By (3.5.5) and (3.5.6), the two systems $\{I_1^m \otimes_k A_2 + A_1 \otimes_k I_2^N\}_{m,n \geq 1}$ and $\{I_3^N\}_{N \geq 1}$ define the same topology on $A_3$. Hence the ring $A_1 \hat{\otimes}_k A_2$ of (3.5.4) is equal to the ring $\lim_{\to} A_3/I_3^N$.

The following generalization of Corollary 3.4.3 is used repeatedly for the rest of the article:

**Theorem 3.5.7.** Let $g : Y_1 \to Y_2$ be a morphism of quasi-projective $k$-schemes. Let $Y_i \hookrightarrow X_i$ be closed immersion into equidimensional smooth $k$-schemes for $i = 1, 2$. Let $\hat{X}_i$ be the completion of $X_i$ along $Y_i$. Suppose there is a morphism $f : X_1 \to X_2$...
such that the diagram

\[(3.5.8)\]

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{g} & Y_2 
\end{array}
\]

commutes. Then for each \(q \geq 0, n \geq 0\), we have the induced morphisms in the homotopy category

\[(3.5.9)\]

\[
\begin{cases}
\hat{f}^*: G^q(\hat{X}_2, n) \to G^q(\hat{X}_1, n), \\
\hat{f}^*: G^q(\hat{X}_2) \to G^q(\hat{X}_1),
\end{cases}
\]

where \(G^q(-)\) are as in Definition 3.3.11.

Furthermore they form the commutative diagrams

\[
\begin{array}{ccc}
G^{q+1}(\hat{X}_2, n) & \xrightarrow{\hat{f}^*} & G^{q+1}(\hat{X}_1, n) \\
\downarrow & & \downarrow \\
G^q(\hat{X}_1, n) & \xrightarrow{\hat{f}^*} & G^q(\hat{X}_1, n),
\end{array}
\]

in the homotopy category, where the vertical arrows are as in (3.1.9)-(3.1.10).

Proof. From the given diagram (3.5.8), we deduce the commutative diagram

\[
\begin{array}{ccc}
\hat{X}_1 & \xrightarrow{\hat{f}} & \hat{X}_2 \\
\downarrow & & \downarrow \\
\hat{Y}_1 & \xrightarrow{\hat{g}} & \hat{Y}_2.
\end{array}
\]

The diagram (3.5.8) also induces the commutative diagram

\[
\begin{array}{ccc}
X_1 \xrightarrow{gr_f} X_1 \times X_2 & \xrightarrow{pr_2} & X_2 \\
\downarrow & & \downarrow \\
Y_1 \xrightarrow{gr_g} Y_1 \times Y_2 & \xrightarrow{pr_2} & Y_2.
\end{array}
\]

Let \(X_{12} := X_1 \times X_2\) and let \(\hat{X}_{12}\) be the completion of \(X_{12}\) along \(Y_1 \times Y_2\). This is equal to \(\hat{X}_1 \times \hat{X}_2\) by Lemma 3.5.1.

We also have the closed immersions \(Y_1 \hookrightarrow X_1 \xleftarrow{\eta} X_1 \times X_2 = X_{12}\). We let \(\hat{X}'_{12}\) be the completion of \(X_{12}\) along \(Y_1\). This is equal to the further completion \(\alpha: \hat{X}'_{12} \to \hat{X}_{12}\) of \(\hat{X}_{12}\) along \(Y_1\) as well. Thus we have the commutative diagram

\[(3.5.10)\]

Since the morphisms \(\alpha\) (see EGA I [17 Corollaire (10.8.9), p.197] or [29 Theorems 8.8 and 8.12, p.60-61]) and \(\hat{p}_2\) are flat, if a coherent sheaf is of codimension \(\geq q\), so are its flat pull-backs. Thus we have the associated pull-backs

\[(3.5.11)\]

\[
\begin{cases}
\hat{p}_2^*: D^q_{coh}(\hat{X}_2, n) \to D^q_{coh}(\hat{X}_{12}, n), \\
\alpha^*: D^q_{coh}(\hat{X}_{12}, n) \to D^q_{coh}(\hat{X}'_{12}, n),
\end{cases}
\]
thus applying the Waldhausen constructions, we have morphisms of the Waldhausen $K$-spaces

\[
\begin{align*}
\tilde{\mu}\ast_2 & : \mathcal{G}(\tilde{X}_2, n) \to \mathcal{G}(\tilde{X}_{12}, n), \\
\alpha\ast & : \mathcal{G}(\tilde{X}_{12}, n) \to \mathcal{G}(\tilde{X}_{12}, n).
\end{align*}
\]

(3.5.12)

On the other hand, the morphism $\tilde{\gamma}$ is obtained from the closed immersion $X_1 \hookrightarrow X_{12}$ so that by Corollary 3.4.3 we have the pull-back in the homotopy category

\[
\tilde{\gamma} : \mathcal{G}(\tilde{X}_{12}, n) \to \mathcal{G}(\tilde{X}_1, n).
\]

(3.5.13)

Composing all pull-backs in (3.5.12) and (3.5.13) in the correct order, we obtain the first desired pull-back of (3.5.9)

\[
\tilde{f}^\ast : \mathcal{G}(\tilde{X}_2, n) \to \mathcal{G}(\tilde{X}_1, n).
\]

This induces a morphism of cubical spaces in the homotopy category with the faces and degeneracies, see e.g. Remark 3.1.7. Thus taking the geometric realizations, we deduce the second morphism in (3.5.9) in the homotopy category.

The second part is apparent and we omit details.

The above construction can be composed:

**Lemma 3.5.14.** Let $g_1 : Y_1 \to Y_2$ and $g_2 : Y_2 \to Y_3$ be morphisms of quasi-projective $k$-schemes. Suppose we have closed immersions $Y_i \hookrightarrow X_i$ into equidimensional smooth $k$-schemes for $i = 1, 2, 3$, and there are morphisms $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_3$ that form a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{g_1} & Y_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{g_2} & Y_3,
\end{array}
\]

Let $\tilde{X}_i$ be the completion of $X_i$ along $Y_i$ for $i = 1, 2, 3$, and let

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{\tilde{f}_1} & \tilde{X}_2 \\
\downarrow & & \downarrow \\
\tilde{Y}_1 & \xrightarrow{\tilde{f}_2} & \tilde{Y}_2 \\
\downarrow & & \downarrow \\
\tilde{Y}_1 & \xrightarrow{\tilde{f}_3} & \tilde{Y}_3,
\end{array}
\]

be the induced commutative diagram.

Then we have

\[
\begin{align*}
(\tilde{f}_2 \circ \tilde{f}_1)^\ast & = \tilde{f}_3^\ast \circ \tilde{f}_2^\ast : \mathcal{G}(\tilde{Y}_3, n) \to \mathcal{G}(\tilde{Y}_1, n), \\
(f_2 \circ f_1)^\ast & = f_3^\ast \circ f_2^\ast : \mathcal{G}(\tilde{X}_3, n) \to \mathcal{G}(\tilde{X}_1, n),
\end{align*}
\]

(3.5.15)

in the homotopy category.

**Proof.** Following the notational conventions in Theorem 3.5.7 and its proof, we define $\tilde{X}_{12}, \tilde{X}_{12}', \tilde{X}_{13}, \tilde{X}_{13}', \tilde{X}_{23}, \tilde{X}_{23}'$ similarly. They form part of the commutative diagram

\[
\begin{array}{ccccccccc}
\tilde{X}_1 & \to & \tilde{X}_{12} & \to & \tilde{X}_{12}' & \to & \tilde{X}_2 & \to & \tilde{X}_{23} & \to & \tilde{X}_{23}' & \to & \tilde{X}_3 \\
& & \downarrow & \uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & \tilde{X}_{13} & & & & \tilde{X}_{13}' & & & & & & \tilde{X}_{13}.
\end{array}
\]

This diagram and the construction in Theorem 3.5.7 imply (3.5.15). We omit details. □
4. The $K$-spaces under the mod equivalences

For a while, let $Y$ be an affine $k$-scheme of finite type. Let $Y \hookrightarrow X$ be a closed immersion into an equidimensional smooth $k$-scheme, and let $\hat{X}$ be the completion of $X$ along $Y$.

In [31] we construct a tower of “$K$-spaces modulo $Y$” in the homotopy category, using ideas inspired by the derived Milnor patching of S. Landsburg [23].

4.1. A pushout and derived Milnor patching. We recall a few basic results around the Milnor patching needed in this article. The readers may find some analogies in the constructions given in [32]. We would like to give self-contained treatments here as much as we could, and some basic lemmas may be repeated here. First recall from D. Ferrand [12, Théorème 5.4]:

Lemma 4.1.1. Let $X'$ be a scheme, $Y' \subset X'$ a closed subscheme, and $g : Y' \to Y$ is a finite morphism of schemes. Consider the pushout $X := X' \sqcup Y'/Y$ in the category of ringed spaces so that we have the co-Cartesian square

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow u \\
X' & \longrightarrow & X.
\end{array}
$$

Suppose that $X'$ and $Y$ satisfy the following:

(FA) Each finite subset of points is contained in an affine open subset.

Then $X$ is a scheme satisfying (FA), the diagram is Cartesian as well, and the morphism $f$ is finite, while $u$ is a closed immersion.

Lemma 4.1.2. Let $Y$ be an affine $k$-scheme of finite type. Let $Y \hookrightarrow X$ be a closed immersion into an equidimensional smooth $k$-scheme, and let $\hat{X}$ be the completion of $X$ along $Y$.

Then the push-out $D_{\hat{X}} = \hat{X} \coprod_{Y} \hat{X}$, taken as a locally ringed space, is a noetherian formal scheme.

Proof. Let $I$ be the ideal sheaf of the closed immersion $Y \hookrightarrow \hat{X}$. This is also an ideal of definition of the noetherian formal scheme $\hat{X}$ as well. For each $m \geq 1$, let $Y_m \subset \hat{X}$ be the closed subscheme defined by $I^m$.

Since $Y_m$ is affine, it satisfies the condition (FA) of Lemma 4.1.1. In particular, the pushout $D_m := Y_m \coprod Y_m$ exists as a noetherian affine $k$-scheme. Then we take $D_{\hat{X}} := \colim D_m$ in the category of noetherian formal schemes. We have $\hat{X} = \colim Y_m$, and there are closed immersions $j_i : \hat{X} \hookrightarrow D_{\hat{X}}$ for $i = 1, 2$. Since colimits commute among themselves, we have $D_{\hat{X}} = \hat{X} \coprod_{Y} \hat{X}$ as desired. □

We say that the above $D_{\hat{X}}$ is the double of $\hat{X}$ along $Y$. We have the following version of the derived Milnor patching of S. Landsburg [23]:

Lemma 4.1.3. Let $Y$ be an affine $k$-scheme of finite type. Let $Y \hookrightarrow X$ be a closed immersion into an equidimensional smooth $k$-scheme, and let $\hat{X}$ be the completion of $X$ along $Y$. Consider the double $D_{\hat{X}}$ of Lemma 4.1.2 and the associated co-Cartesian diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \hat{X} \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
\hat{X} & \longrightarrow & D_{\hat{X}}.
\end{array}
$$
Let \( F_1 \) and \( F_2 \) be perfect complexes on \( \hat{X} \) such that we have an isomorphism \( L_{\ast}^i F_1 \simeq L_{\ast}^i F_2 \). Then there exists a perfect complex \( \bar{F} \) on \( D_\hat{X} \) such that \( L_{\ast}^i \bar{F} \simeq F_i \) for \( i = 1, 2 \).

**Proof.** This essentially follows from the main theorem of [24]. We sketch the argument. Let \( T \) be the ideal sheaf of the closed immersion \( Y \hookrightarrow \hat{X} \). Let \( Y_m \hookrightarrow \hat{X} \) be the closed subscheme defined by \( T^m \) for integers \( m \geq 1 \). Let \( F_i,m \) be the derived pull-back of \( F_i \) to \( Y_m \), for \( i = 1, 2 \). We let \( i_{\ast,m} : Y \hookrightarrow Y_m \) be the closed immersion.

The given condition \( L_{\ast}^i F_1 \simeq L_{\ast}^i F_2 \) implies that \( L_{\ast}^i_{\ast,m} F_{1,m} \simeq L_{\ast}^i_{\ast,m} F_{2,m} \). Thus by the main theorem of [24], there is a perfect complex \( \bar{F}_m \) on \( D_m := Y_m \prod_m Y_m \) whose derived restrictions to \( Y_m \) are quasi-isomorphic to \( F_{i,m} \) for \( i = 1, 2 \). For each closed immersion \( i_{\ast,m} : D_m \hookrightarrow D_{m+1} \), we have \( i_{\ast,m}^* \bar{F}_{m+1} \simeq \bar{F}_m \). We then take \( \bar{F} := \mathbb{R}\operatorname{lim} \bar{F}_m \), which is a perfect complex on \( D_{\hat{X}} \) by [35] Lemma 0CQG (or B. Bhatt [8, Lemma 4.2]). One checks that this is a desired perfect complex. \( \square \)

### 4.2. A mod equivalent pair of sheaves.

For the cycles on formal schemes over \( \hat{X} \), in [32 §3], the notion of “mod \( Y \)-equivalence” was defined. There, it was first defined for a pair \( A_1, A_2 \) of coherent \( \mathcal{O}_{\hat{X} \times \Box^n} \)-algebras, and then we took the subgroup \( M^q(\hat{X}, Y, n) \) of the differences of the associated cycles \([A_1] - [A_2]\). The quotient \( z^q(\hat{X}, n)/M^q(\hat{X}, Y, n) \) defined the equivalence on cycles.

This process used the language of the derived algebraic geometry in terms of \( \infty \)-categories. For some basics of derived algebraic geometry, one can look at J. Lurie [25] and B. Toën [38], while for \( \infty \)-categories, we refer the reader to J. Lurie [27].

In this paper, we would like to try its \( K \)-theoretic counterpart. After some attempts, we realized that we should work with coherent *modules* unlike in the case of cycles, while we can proceed without the language of derived algebraic geometry, and just stick to the classical homological language of derived categories, together with the derived Milnor patching recalled in §4.1.

We start with the following (cf. [32 §3.2]):

**Definition 4.2.1.** Let \( Y \) be an affine \( k \)-scheme of finite type. Let \( Y \hookrightarrow X \) be a closed immersion into an equidimensional smooth \( k \)-scheme, and let \( \hat{X} \) be the completion of \( X \) along \( Y \). Recall we had the notion of admissible coherent \( \mathcal{O}_{\hat{X} \times \Box^n} \)-modules in Definition 3.1.4.

Two admissible coherent \( \mathcal{O}_{\hat{X} \times \Box^n} \)-modules (resp. of codimension \( \geq q \)) \( \mathcal{M}_1, \mathcal{M}_2 \) are said to be *mod \( Y \)-equivalent* if we have an isomorphism

\[
\mathcal{M}_1 \otimes_{\mathcal{O}_{\hat{X} \times \Box^n}} \mathcal{O}_{Y \times \Box^n} \simeq \mathcal{M}_2 \otimes_{\mathcal{O}_{\hat{X} \times \Box^n}} \mathcal{O}_{Y \times \Box^n}
\]

in the derived category of \( \mathcal{O}_{Y \times \Box^n} \)-modules. Let \( \mathcal{L}(\hat{X}, Y, n) \) (resp. \( \mathcal{L}^{\geq q}(\hat{X}, Y, n) \)) be the set of all pairs of *mod \( Y \)-equivalent* admissible coherent \( \mathcal{O}_{\hat{X} \times \Box^n} \)-modules (resp. of codimension \( \geq q \)). \( \square \)

There are two notable points here. The first is that \( \hat{X} \times \Box^n \) is a regular formal scheme, so that by Lemma 2.2.1, a coherent \( \mathcal{O}_{\hat{X} \times \Box^n} \)-module is a perfect complex on \( \hat{X} \times \Box^n \). The second point is that (4.2.2) is a quasi-isomorphism of the derived pull-backs to \( Y \times \Box^n \) of perfect complexes on \( \mathcal{O}_{\hat{X} \times \Box^n} \), so that the derived Milnor patching discussed in Lemma 1.1.3 applies. We state it as follows:

**Corollary 4.2.3.** Under the notations and the assumptions of Definition 4.2.1, there exists a perfect complex \( \bar{\mathcal{M}} \) on \( D_{\hat{X} \times \Box^n} \) whose derived pull-backs \( L_{j_1 \ast} \bar{\mathcal{M}} \) via the two closed immersions \( j_i : \hat{X} \times \Box^n \hookrightarrow D_{\hat{X} \times \Box^n} \) for \( i = 1, 2 \) are quasi-isomorphic to \( \mathcal{M}_i \) for \( i = 1, 2 \).
We collect all such \( \tilde{M} \):

**Definition 4.2.4.** Under the above notations and the assumptions:

1. Let \( \tilde{L}(D_{\hat{X}}, Y, n) \) (resp. \( \tilde{L}^{\geq q}(D_{\hat{X}}, Y, n) \)) be the set of perfect complexes \( \tilde{M} \) (resp. of codimension \( \geq q \)) on \( D_{\hat{X}} \times \square^n \) obtained in Corollary 4.2.3.

2. Let \( \mathcal{V}(D_{\hat{X}}, Y, n) \) (resp. \( \mathcal{V}^q(D_{\hat{X}}, Y, n) \)) be the triangulated subcategory of \( D_{\text{coh}}(D_{\hat{X}} \times \square^n) \) generated by the perfect complexes in \( \tilde{L}(D_{\hat{X}}, Y, n) \) (resp. \( \tilde{L}^{\geq q}(D_{\hat{X}}, Y, n) \)).

3. Let \( \mathcal{S}(D_{\hat{X}}, Y, n) \) (resp. \( \mathcal{S}^q(D_{\hat{X}}, Y, n) \)) be the Waldhausen \( K \)-space of the triangulated category \( \mathcal{V}(\hat{X}, Y, n) \) (resp. \( \mathcal{V}^q(\hat{X}, Y, n) \)).

The two closed immersions \( j_i : \hat{X} \times \square^n \hookrightarrow D_{\hat{X}} \times \square^n \) for \( i = 1, 2 \) induce the following commutative diagram of functors, where the pull-backs are the derived pull-backs of perfect complexes:

\[
\begin{align*}
\mathcal{V}^q(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{V}^q(D_{\hat{X}}, Y, n) \\
\mathcal{V}(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{V}(D_{\hat{X}}, Y, n) \\
\mathcal{S}^q(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{S}^q(D_{\hat{X}}, Y, n) \\
\mathcal{S}(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{S}(D_{\hat{X}}, Y, n)
\end{align*}
\]

This diagram in turn induces the commutative diagram of the spaces

\[
\begin{align*}
\mathcal{S}^q(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{S}^q(D_{\hat{X}}, Y, n) \\
\mathcal{S}(D_{\hat{X}}, Y, n) & \xrightarrow{j_1^*} \mathcal{S}(D_{\hat{X}}, Y, n)
\end{align*}
\]

**Definition 4.2.6.** Take the homotopy coequalizers of the horizontal maps of \( \mathcal{V}^q(D_{\hat{X}}, Y, n) \) to define

\[
\begin{align*}
\mathcal{G}^q(\hat{X}, Y, n) & := \text{hocolim} \left( \mathcal{S}^q(D_{\hat{X}}, Y, n) \xrightarrow{j_1^*} \mathcal{S}^q(D_{\hat{X}}, Y, n) \right), \\
\mathcal{G}(\hat{X}, Y, n) & := \text{hocolim} \left( \mathcal{S}(D_{\hat{X}}, Y, n) \xrightarrow{j_1^*} \mathcal{S}(D_{\hat{X}}, Y, n) \right).
\end{align*}
\]

We can regard them as the “Waldhausen \( K \)-spaces of \( \hat{X} \) mod \( Y \)” in a sense.

**Remark 4.2.7.** If \( Y \) is a smooth affine \( k \)-scheme of finite type, then we can take \( X = Y \) in Lemma 4.1.3.

In this case, we claim that we have \( \mathcal{G}^q(\hat{X}, Y, n) = \mathcal{G}^q(Y, n) \).

Indeed, in this case we get \( \hat{X} = Y \), so \( D_{\hat{X}} = \hat{X} \amalg_Y \hat{X} = Y \) and all the maps in the cartesian diagram in 4.1.3 are equal to the identity on \( Y \).

For two admissible coherent \( \mathcal{O}_{X \times \square^n} \)-modules \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), they are mod \( Y \)-equivalent in the sense of Definition 4.2.3 if and only if \( \mathcal{M}_1 \simeq \mathcal{M}_2 \) as \( \mathcal{O}_{X \times \square^n} \)-modules.

Hence we have \( \mathcal{V}^q(D_{\hat{X}}, Y, n) = \mathcal{D}_{\text{coh}}^q(Y, n) \). In particular, applying the Waldhausen \( K \)-spaces, we deduce that \( \mathcal{S}^q(D_{\hat{X}}, Y, n) = \mathcal{G}^q(Y, n) \).

Since \( j_1^* = j_2^* = \text{Id} \) in 4.2.3, we deduce that \( \mathcal{G}(\hat{X}, Y, n) = \mathcal{G}(Y, n) \), as desired. Similarly, we have \( \mathcal{G}(\hat{X}, Y, n) = \mathcal{G}(Y, n) \).

We will use this observation later when we compare our constructions of the filtrations on \( K \)-theory with the pre-existing ones in the literature when \( Y \) is smooth.

Coming back to Definition 4.2.6 by construction, we deduce the morphisms in the homotopy category

\[
\mathcal{G}^{q+1}(\hat{X}, Y, n) \rightarrow \mathcal{G}^q(\hat{X}, Y, n) \rightarrow \mathcal{G}(\hat{X}, Y, n).
\]
We also note that there are natural morphisms in the homotopy category
\[ G^q(\hat{X}, n) \to G^q(\hat{X}, n) \to g(\hat{X} \times □^n) \cong g(\hat{X}) \cong K(\hat{X}) \to K(Y). \]
Its composition with both of the arrows \( j^*_i \) in (4.2.23) are equal. Thus by the definition of the homotopy coequalizers, we deduce the commutative diagram of spaces in the homotopy category
\[
\begin{array}{ccc}
G^q(\hat{X}, Y, n) & \xrightarrow{G^q(\hat{X}, Y)} & K(Y).
\end{array}
\]

These \( G^q(\hat{X}, Y, n) \) and \( G(\hat{X}, Y, n) \) over \( n \geq 0 \) form cubical spaces.

**Definition 4.2.8.** We define
\[
G^q(\hat{X}, Y) := [n \mapsto G^q(\hat{X}, Y, n)],
\]
the geometric realizations.

By construction, we have the induced morphisms
\[
G^{q+1}(\hat{X}, Y) \to G^q(\hat{X}, Y) \to G(\hat{X}, Y) \to K(Y)
\]
in the homotopy category.

### 4.3. Moving lemma and general pull-backs

Combining the above construction with the discussions in §3, we would like to have the “mod \( Y \)-equivalence” version of Theorem 3.5.7. This is eventually proven in Theorem 4.2.9 below. Along the way, we go over part of the constructions in §3 to check that they respect the mod \( Y \)-equivalences. We discuss these first.

The following is a minor variant of Definition 4.2.6 for the triangulated subcategory \( D^{\text{coh}}_{\text{coh}, \{\hat{X}_1\}}(\hat{X}_2, n) \subset D^{\text{coh}}_{\text{coh}}(\hat{X}_2, n) \) (see Definition 3.2.1):

**Definition 4.3.1.** Take the homotopy coequalizer to define
\[
G^q_{\{\hat{X}_1\}}(\hat{X}_2, Y, n) := \text{hocoeq} \left( S^q(D_{\hat{X}_2}, Y, n) \Rightarrow G^q(\hat{X}_2, n) \xleftarrow{w.e.} G^q_{\{\hat{X}_1\}}(\hat{X}_2, n) \right),
\]
where \( w.e. \) is a weak-equivalence given by Corollary 3.4.3. Similarly we also define \( G_{\{\hat{X}_1\}}(\hat{X}_2, Y, n) \).

As before, we take the geometric realizations to define
\[
G^q_{\{\hat{X}_1\}}(\hat{X}_2, Y) := [n \mapsto G^q_{\{\hat{X}_1\}}(\hat{X}_2, Y, n)],
\]
and similarly we define \( G_{\{\hat{X}_1\}}(\hat{X}_2, Y) \).

The following is the mod \( Y \)-version of Corollary 3.4.3 for the \( K \)-spaces mod \( Y \):

**Lemma 4.3.2.** Let \( Y \) be an affine \( k \)-scheme of finite type. Suppose we have a sequence of closed immersions \( Y \hookrightarrow X_1 \hookrightarrow X_2 \), where \( X_1, X_2 \) are equidimensional smooth \( k \)-schemes. They induce closed immersions \( Y \hookrightarrow \hat{X}_1 \hookrightarrow \hat{X}_2 \), where \( \hat{X}_i \) is the completion of \( X_i \) along \( Y \).

Then for \( n \geq 0 \), the zigzag of functors
\[
D^{\text{coh}}_{\text{coh}, \{\hat{X}_1\}}(\hat{X}_2, n) \xleftarrow{\sim} D^{\text{coh}}_{\text{coh}}(\hat{X}_2, n) \xrightarrow{\sim} D^{\text{coh}}_{\text{coh}}(\hat{X}_1, n),
\]
induces morphisms of the spaces
\[
G^q(\hat{X}_2, Y, n) \xleftarrow{\sim} G^q_{\{\hat{X}_1\}}(\hat{X}_2, Y, n) \xrightarrow{\sim} G^q(\hat{X}_1, Y, n).
\]
where the first arrow $i$ is a weak-equivalence. Consequently, we have the Gysin morphisms of spaces in the homotopy category

\[(4.3.5)\]
\[
\begin{align*}
\tilde{\rho} &: G^q_!(\hat{X}_2, Y, n) \to G^q_!(\hat{X}_1, Y, n), \quad n \geq 0, \\
\tilde{\rho} &: G^q_!(\hat{X}_2, Y) \to G^q_!(\hat{X}_1, Y).
\end{align*}
\]

Proof. By Proposition 3.2.5, the first arrow $\tilde{\rho}$ is essentially surjective so that the morphism $G^q_!(\hat{X}_2, n) \to G^q_!(\hat{X}_1, n)$ is a weak-equivalence by Theorems 1.9.1, 1.9.8, p.263, p.271. Thus the morphism $G^q_!(\hat{X}_2, Y) \to G^q_!(\hat{X}_1, Y)$ (see Definition 4.3.1) of the homotopy coequalizers must also be a weak-equivalence. This gives the first morphism in (4.3.5).

Taking the geometric realizations of the cubical spaces over $n \geq 0$, we deduce the second morphism in (4.3.5), and the lemma follows. \[\square\]

We are now ready to prove the following:

**Theorem 4.3.6.** Let $g : Y_1 \to Y_2$ be a morphism of affine $k$-schemes of finite type. Let $Y_i \to X_i$ be a closed immersion into an equidimensional smooth $k$-scheme for $i = 1, 2$. Let $\hat{X}_i$ be the completion of $X_i$ along $Y_i$. Suppose there is a morphism $f : X_1 \to X_2$ such that the diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{g} & Y_2
\end{array}
\]

commutes. Then for $q \geq 0$, $n \geq 0$, there exist the induced morphisms in the homotopy category

\[(4.3.7)\]
\[
\tilde{f}^* : G^q(\hat{X}_2, Y_2, n) \to G^q(\hat{X}_1, Y_1, n), \quad \tilde{f}^* : G^q(\hat{X}_2, Y_2) \to G^q(\hat{X}_1, Y_1),
\]

where $G^q(\hat{X}, Y, n)$ and $G^q(\hat{X}, Y)$ are as in Definitions 4.2.6 and 4.2.8.

Proof. We are going to repeat part of the argument of Theorem 3.5.7. We continue to use the diagrams and the notations there.

From the commutative diagram of formal schemes (3.5.10), we deduced the pull-back functors in (3.5.11)

\[(4.3.8)\]
\[
\begin{align*}
\tilde{\mu}^*_2 &: D_\text{coh}^q(\hat{X}_2, n) \to D_\text{coh}^q(\hat{X}_{12}, n), \\
\alpha^* &: D_\text{coh}^q(\hat{X}_{12}, n) \to D_\text{coh}^q(\hat{X}_{12}, n),
\end{align*}
\]

and the morphisms of spaces in (3.5.12)

\[(4.3.9)\]
\[
\begin{align*}
\tilde{\mu}^*_2 &: G^q(\hat{X}_2, n) \to G^q(\hat{X}_{12}, n), \\
\alpha^* &: G^q(\hat{X}_{12}, n) \to G^q(\hat{X}_{12}, n).
\end{align*}
\]

We need to check that the functors in (4.3.8) respect the mod equivalences.

Suppose we are given a pair $(\mathcal{M}_1, \mathcal{M}_2)$ of coherent $\mathcal{O}_{\hat{X}_2 \times \square^n}$-modules such that we have an isomorphism in the derived category of $\mathcal{O}_{\hat{X}_2 \times \square^n}$-modules

\[(4.3.10)\]
\[
\mathcal{M}_1 \otimes_{\mathcal{O}_{\hat{X}_2 \times \square^n}} L \mathcal{O}_{\hat{X}_2 \times \square^n} \simeq \mathcal{M}_2 \otimes_{\mathcal{O}_{\hat{X}_2 \times \square^n}} L \mathcal{O}_{\hat{X}_2 \times \square^n}.
\]

Since $\tilde{\mu}^*_2$ is flat, applying the flat pull-back $\tilde{\mu}^*_2$ to (4.3.10), we deduce an isomorphism

\[
\tilde{\mu}^*_2(\mathcal{M}_1) \otimes_{\tilde{\mu}^*_2 \mathcal{O}_{\hat{X}_2 \times \square^n}} \tilde{\mu}^*_2 \mathcal{O}_{\hat{X}_2 \times \square^n} \simeq \tilde{\mu}^*_2(\mathcal{M}_2) \otimes_{\tilde{\mu}^*_2 \mathcal{O}_{\hat{X}_2 \times \square^n}} \tilde{\mu}^*_2 \mathcal{O}_{\hat{X}_2 \times \square^n},
\]
which implies
\[(4.3.11) \quad \widetilde{p}_2^\ast (\mathcal{M}_1) \otimes_{\mathcal{O}_{X_{12} \times \Box^n}} L\mathcal{O}_Y \times \Box^n \simeq \widetilde{p}_2^\ast (\mathcal{M}_2) \otimes_{\mathcal{O}_{X_{12} \times \Box^n}} L\mathcal{O}_Y \times \Box^n \]
via the natural morphism $\widetilde{p}_2^\ast \mathcal{O}_X \times \Box^n \to \mathcal{O}_{X_{12} \times \Box^n}$. We also have the natural morphism $\widetilde{p}_2^\ast \mathcal{O}_X \times \Box^n \to \mathcal{O}_{Y_{12} \times \Box^n}$, where $Y_{12} = Y_1 \times Y_2$. Applying the derived functor $(-) \otimes_{\mathcal{O}_{X_{12} \times \Box^n}} \mathcal{O}_{Y_{12} \times \Box^n}$ to \[(4.3.11),\] we deduce an isomorphism in the derived category of $\mathcal{O}_{Y_{12} \times \Box^n}$-modules
\[
\widetilde{p}_2^\ast (\mathcal{M}_1) \otimes_{\mathcal{O}_{X_{12} \times \Box^n}} \mathcal{O}_{Y_{12} \times \Box^n} \simeq \widetilde{p}_2^\ast (\mathcal{M}_2) \otimes_{\mathcal{O}_{X_{12} \times \Box^n}} \mathcal{O}_{Y_{12} \times \Box^n}.
\]
Thus, $\widetilde{p}_2^\ast$ of \[(4.3.8)\] induces the set map $L^\ast (\tilde{X}_2, Y_2, n) \to L^\ast (\tilde{X}_{12}, Y_{12}, n)$, which in turn induces
\[
\widetilde{p}_2^\ast : V^q(D_{\tilde{X}_2}, Y_2, n) \to V^q(D_{\tilde{X}_{12}}, Y_{12}, n).
\]
Applying the Waldhausen $K$-spaces, together with \[(4.3.9),\] we deduce
\[
S^q(D_{\tilde{X}_2}, Y_2, n) \xrightarrow{\tilde{j}_1^\ast} G^q(\tilde{X}_2, n) \xleftarrow{\widetilde{p}_2^\ast} S^q(D_{\tilde{X}_{12}}, Y_{12}, n) \xrightarrow{\tilde{j}_2^\ast} G^q(\tilde{X}_{12}, n),
\]
and taking the homotopy coequalizers of the horizontal arrows, for each $n \geq 0$, we deduce the morphism in the homotopy category
\[(4.3.12) \quad \widetilde{p}_2^\ast : G^q(\tilde{X}_2, Y_2, n) \to G^q(\tilde{X}_{12}, Y_{12}, n).\]
The above \[(4.3.12)\] over all $n \geq 0$ forms a morphism of cubical objects in the homotopy category. Taking the geometric realizations, we deduce the morphism in the homotopy category
\[(4.3.13) \quad \tilde{p}_2^\ast : G^q(\tilde{X}_2, Y_2) \to G^q(\tilde{X}_{12}, Y_{12}).\]
Since the completion morphism $\alpha$ in \[(3.5.10)\] is flat, we similarly deduce
\[(4.3.14) \quad \begin{cases} 
\alpha^\ast : G^q(\tilde{X}_{12}, Y_{12}, n) \to G^q(\tilde{X}_{12}', Y_{12}, n), \\
\alpha^\ast : G^q(\tilde{X}_{12}, Y_{12}) \to G^q(\tilde{X}_{12}', Y_{12}).
\end{cases}\]
On the other hand, by Lemma \[(4.3.2)\] from $\tilde{\iota}$ in \[(3.5.10),\] we deduce the following morphisms in the homotopy category
\[(4.3.15) \quad \begin{cases} 
\tilde{\iota} : G^q(\tilde{X}_{12}, Y_{12}, n) \to G^q(\tilde{X}_{1}, Y_{1}, n), \\
\tilde{\iota} : G^q(\tilde{X}_{12}, Y_{12}) \to G^q(\tilde{X}_{1}, Y_{1}).
\end{cases}\]
Composing \[(4.3.13), (4.3.14)\] and \[(4.3.15),\] we thus deduce the desired morphisms \[(4.3.7)\] as
\[
\begin{cases} 
\tilde{f}^\ast = \tilde{p}_2^\ast \circ \alpha^\ast \circ \tilde{\iota} : G^q(\tilde{X}_2, Y_2, n) \to G^q(\tilde{X}_1, Y_1, n), \\
\tilde{f}^\ast = \tilde{p}_2^\ast \circ \alpha^\ast \circ \tilde{\iota} : G^q(\tilde{X}_2, Y_2) \to G^q(\tilde{X}_1, Y_1).
\end{cases}\]
This proves the theorem. \qed

We can compose the above morphisms, too:

**Lemma 4.3.16.** Let $g_1 : Y_1 \to Y_2$ and $g_2 : Y_2 \to Y_3$ be morphisms of affine $k$-schemes of finite type. Suppose we have closed immersions $Y_i \hookrightarrow X_i$ into equidi- mensional smooth affine $k$-schemes for $i = 1, 2, 3$, and there are morphisms $f_1$ :
called systems of local embeddings

Thomason \[36,\] system of local embeddings is called a

The extended Čech construction.

5.1. Definition 5.1.1. Let \( Y \) be a \( k \)-scheme of finite type. A finite collection \( \mathcal{U} := \{(U_i, X_i)\}_{i \in A} \) consisting of

1. the underlying quasi-affine open cover \( \mathcal{U} := \{U_i\}_{i \in A} \) of \( Y \), and
2. a closed immersion \( U_i \hookrightarrow X_i \) into an equidimensional smooth \( k \)-scheme for each \( i \in A \),

is called a system of local embeddings for \( Y \).

N.B. For the rest of the paper, we will use the systems \( \mathcal{U} \) such that each \( U_i \) is affine only, though often we don’t say so explicitly.

For \( p \geq 0 \), let \( I = (i_0, \ldots, i_p) \in \Lambda^{p+1} \). We define \( U_I := U_{i_0} \cap \cdots \cap U_{i_p} \) and \( X_I := X_{i_0} \times \cdots \times X_{i_p} \). Consider the diagonal closed immersion \( U_I \hookrightarrow X_I \). Let \( \hat{X}_I \) be the completion of \( X_I \) along \( U_I \). Since each \( U_{i_0} \) is affine, so is \( U_I \), because \( Y \) is assumed to be separated over \( k \).

For the (usual) open cover \( \mathcal{U} \) of \( Y \), the usual Čech cosimplicial space in the sense of R. Thomason \[36, \S 1\] is defined to be:

\[
\mathcal{K}^*_{\mathcal{U}} := \left\{ \prod_{i_0 \in A} K(U_{i_0}) \overset{\subset}{\to} \prod_{i_0 \in A} K(U_I) \overset{\subset}{\to} \prod_{i_0 \in A^2} K(U_I) \cdots \right\}.
\]
Here, when \( U \subset Y \) is open, \( \mathcal{K}(U) \) is the usual Waldhausen algebraic \( K \)-theory space of the scheme \( U \), i.e., it is the Waldhausen \( K \)-space of the triangulated category of perfect complexes on \( U \).

We want to have its parallel construction for \( U \), involving the completions \( \hat{X}_I \) over the multi-indices \( I \in \Lambda^{p+1} \) and \( p \geq 0 \):

For each \( 0 \leq j \leq p \), let \( I'_j := (i_0, \ldots, i_j, \ldots, i_p) \in \Lambda^p \), i.e., we omit \( i_j \). The open inclusion \( U_I \hookrightarrow U_{I'_j} \) and the projection \( X_I \to X_{I'_j} \) form the commutative diagram

\[
\begin{array}{ccc}
X_I & \longrightarrow & X_{I'_j} \\
\uparrow & & \uparrow \\
U_I & \longrightarrow & U_{I'_j}.
\end{array}
\]

By Theorem 4.3.6, this diagram induces the morphism

\[
(5.1.3) \quad \mathcal{G}^q(\hat{X}_{I'_j}, U_{I'_j}) \to \mathcal{G}^q(\hat{X}_I, U_I)
\]

in the homotopy category, where \( \mathcal{G}^q(-,-) \) are as in Definition 4.3.7.

Thus for \( 0 \leq j \leq p \), we have the coface maps in the homotopy category

\[
(5.1.4) \quad \delta_j : \prod_{I \in \Lambda^p} \mathcal{G}^q(\hat{X}_I, U_I) \to \prod_{I \in \Lambda^{p+1}} \mathcal{G}^q(\hat{X}_I, U_I),
\]

defined as follows: for each component at \( I \in \Lambda^{p+1} \) of the right hand side, \( \delta_j \) is given first by taking the projection to the factor \( \mathcal{G}^q(\hat{X}_{I'_j}, U_{I'_j}) \) for \( I'_j \in \Lambda^p \) of the left hand side, and then applying (5.1.3).

We have the codegeneracies as well: for a given \( I = (i_0, \ldots, i_{p-1}) \in \Lambda^p \) and for each \( 0 \leq j \leq p \), consider \( I''_j := (i_0, \ldots, i_j, i_j, \ldots, i_{p-1}) \in \Lambda^{p+1} \), i.e., we insert a copy of \( i_j \) at the \((j+1)\)-th place, and shift the remaining ones to the next position. In this case \( U_I = U_{I''_j} \), while we have the diagonal closed immersion \( X_I \hookrightarrow X_{I''_j} \) induced from \( X_{i_j} \hookrightarrow X_{i_j} \times X_{i_j} \). They form the commutative diagram

\[
\begin{array}{ccc}
X_I & \longrightarrow & X_{I''_j} \\
\uparrow & & \uparrow \\
U_I & \longrightarrow & U_{I''_j}.
\end{array}
\]

Thus by the Theorem 4.3.6, this diagram induces the morphism

\[
(5.1.5) \quad \mathcal{G}^q(\hat{X}_{I''_j}, U_{I''_j}) \to \mathcal{G}^q(\hat{X}_I, U_I)
\]

in the homotopy category. Thus for \( 0 \leq j \leq p-1 \), we have the codegeneracy maps

\[
(5.1.6) \quad s_j : \prod_{I \in \Lambda^{p+1}} \mathcal{G}^q(\hat{X}_I, U_I) \to \prod_{I \in \Lambda^p} \mathcal{G}^q(\hat{X}_I, U_I)
\]

in the homotopy category, which is defined as follows: for the component at \( I \in \Lambda^p \) of the right hand side, \( s_j \) is given first by taking the projection to the factor \( \mathcal{G}^q(\hat{X}_{I''_j}, U_{I''_j}) \) for \( I''_j \in \Lambda^{p+1} \) of the left hand side, and then applying (5.1.5).

One can check straightforwardly (but with tedious calculations) that the above cofaces in (5.1.4) and codegeneracies (5.1.6) indeed satisfy the axioms of a cosimplicial object in the homotopy category of spaces. So, we define:
Definition 5.1.7. Let \( Y \in \text{Sch}_k \). We have the following cosimplicial space in the homotopy category
\[
(5.1.8) \quad \tilde{G}^n_{\mathcal{U}} := \left\{ \prod_{i_0 \in \Lambda} G^q(\tilde{X}_{i_0}, U_{i_0}) \supseteq \prod_{i \in \Lambda^2} G^q(\tilde{X}_i, U_i) \supseteq \prod_{i \in \Lambda^3} G^q(\tilde{X}_i, U_i) \cdots \right\}.
\]

Let \( p \geq 0 \). Note that for each \( i \in \Lambda^{p+1} \), we have the closed immersion \( U_i \hookrightarrow \tilde{X}_i \), and it induces \( G^q(\tilde{X}_i, U_i) \to K(U_i) \) in the homotopy category. Comparing (5.1.2) and (5.1.8) over various \( q \geq 0 \), we obtain the induced morphisms of cosimplicial objects in the homotopy category
\[
(5.1.9) \quad \cdots \to \tilde{G}^n_{\mathcal{U}} \to \tilde{G}^{n-1}_{\mathcal{U}} \to \cdots \to \tilde{G}^0_{\mathcal{U}} \to K_{\mathcal{U}},
\]
where the last \( K_{\mathcal{U}} \) is as in (5.1.2). \( \square \)

Definition 5.1.10. Let \( Y \in \text{Sch}_k \) and let \( \mathcal{U} \) be a system of local embeddings for \( Y \). For the above cosimplicial objects \( \tilde{G}^n_{\mathcal{U}} \) in Definition 5.1.7 define the \( \check{C}ech \) hypercohomology space with respect to the system \( \mathcal{U} \) to be
\[
\check{H}(\mathcal{U}, \tilde{G}^n) := \text{holim}_\Delta \tilde{G}^n_{\mathcal{U}}.
\]
Recall that for (5.1.2), we similarly had (see R. Thomason \[36\], (1.5), p.443)
\[
\check{H}(\mathcal{|U|}, K) := \text{holim}_{\Delta} K_{|\mathcal{U}|}.
\]
The morphisms (5.1.9) induce the tower
\[
(5.1.11) \quad \cdots \to \check{H}(\mathcal{U}, \tilde{G}^n) \to \check{H}(\mathcal{U}, \tilde{G}^{n-1}) \to \cdots \to \check{H}(\mathcal{U}, \tilde{G}^0) \to \check{H}(\mathcal{|U|}, K)
\]
in the homotopy category. \( \square \)

Remark 5.1.12. Note that we have the natural morphism \( K(Y) \to \check{H}(\mathcal{|U|}, K) \), and it is a weak-equivalence. See Thomason–Trobaugh \[37\] Theorem 8.4, Exercise 8.5-(a), pp.373-374. \( \square \)

5.2. Refinements. For the systems of local embeddings in Definition 5.1.1 we have the following notion of refinements. This is more flexible than a similar notion considered in \[22\] Remark, p.28, and identical to the corresponding notion considered in \[32\] §6:

Definition 5.2.1. Let \( Y \in \text{Sch}_k \). Let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) and \( \mathcal{V} = \{(V_j, X_j)\}_{j \in \Lambda'} \) be systems of local embeddings for \( Y \). We say that \( \mathcal{V} \) is a refinement system of \( \mathcal{U} \), if there is a set map \( \lambda : \Lambda' \to \Lambda \) such that
\begin{itemize}
  \item[(1)] for each \( j \in \Lambda' \), we have \( V_j \subset U_{\lambda(j)} \), and
  \item[(2)] there is a morphism \( f_j : X'_j \to X_{\lambda(j)} \) that restricts to \( V_j \hookrightarrow U_{\lambda(j)} \).
\end{itemize}

Some of the basic lemmas in the subsection are also in \[32\] §6. We included their arguments here for self-containedness of this article.

Lemma 5.2.2. Let \( Y \in \text{Sch}_k \). Let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) and \( \mathcal{V} = \{(V_j, X_j)\}_{j \in \Lambda'} \) be systems of local embeddings for \( Y \). Suppose all \( U_i \) and \( V_j \) are affine open in \( Y \).

Then there exists a common refinement system \( \mathcal{W} \) of both \( \mathcal{U} \) and \( \mathcal{V} \).

Proof. Let \( \Lambda'' := \Lambda \times \Lambda' \). For \( (i, j) \in \Lambda'' \), let \( W_{ij} := U_i \cap V_j \). Since \( U_i, V_j \) are affine, so is \( W_{ij} \).

Note that we have a closed immersion \( U_i \hookrightarrow X_i \), and \( W_{ij} \) is open in \( U_i \). Since the subspace topology on \( U_i \) from \( X_i \) agrees with the topology of \( U_i \), there is an open subscheme \( X'_{ij} \subset X_i \) such that we have the induced closed immersion \( W_{ij} \to X'_{ij} \). Similarly, \( W_{ij} \) is open in \( V_j \) as well, so that there is an open subscheme \( X'_{ij} \subset X_j' \) such that we have the induced closed immersion \( W_{ij} \hookrightarrow X'_{ij} \).
The two closed immersions \( W_{ij} \hookrightarrow X_{ij}, X'_{ij} \) induce the diagonal closed immersion \( W_{ij} \hookrightarrow X''_{ij} := X_{ij} \times_{k} X'_{ij} \). Let \( \mathcal{W} := \{ (W_{ij}, X''_{ij})_{(i,j) \in \Lambda''} \} \). This is a system of local embeddings for \( Y \).

Consider the projection maps \( \lambda, \lambda' : \Lambda'' = \Lambda \times \Lambda' \to \Lambda, \Lambda' \) of the index sets, respectively. The corresponding morphisms

\[
\begin{align*}
X''_{ij} &= X_{ij} \times X'_{ij} \to X_{ij} \overset{op}{\to} X_i, \\
X''_{ij} &= X_{ij} \times X'_{ij} \to X'_{ij} \overset{op}{\to} X'_i
\end{align*}
\]

are given by projections followed by open immersions. Hence \( \mathcal{W} \) is a refinement of both \( \mathcal{U} \) and \( \mathcal{V} \).

\[\Box\]

\textbf{Lemma 5.2.3.} Let \( Y \in \text{Sch}_k \). Let \( \mathcal{U} = \{ (U_i, X_i) \}_{i \in \Lambda} \) and \( \mathcal{V} = \{ (V_j, X'_j) \}_{j \in \Lambda'} \) be systems of local embeddings for \( Y \) such that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \) for a set map \( \lambda : \Lambda' \to \Lambda \).

Then there exists a morphism of cosimplicial objects

\[
(5.2.4) \quad \lambda^* : \widehat{\mathcal{G}}^\bullet_\mathcal{U} \to \widehat{\mathcal{G}}^\bullet_\mathcal{V}
\]

in the homotopy category. In particular, we have the induced morphism of Čech hypercohomology spaces in the homotopy category

\[
\lambda^* : \widehat{\mathcal{H}}(\mathcal{U}, \widehat{\mathcal{G}}^\bullet) \to \widehat{\mathcal{H}}(\mathcal{V}, \widehat{\mathcal{G}}^\bullet).
\]

Furthermore, this is compatible with the morphisms \( \widehat{\mathcal{G}}^p_\mathcal{U} \to \widehat{\mathcal{G}}^{p-1}_\mathcal{U} \), where \( (-) = \mathcal{U}, \mathcal{V} \).

\textbf{Proof.} For \( p \geq 0 \), let \( J \in (\Lambda')^{p+1} \) so that \( \lambda(J) \in \Lambda^{p+1} \). Then we have the associated commutative diagram

\[
\begin{array}{ccc}
X'_J & \longrightarrow & X_{\lambda(J)} \\
\downarrow & & \downarrow \\
V_J & \longrightarrow & U_{\lambda(J)},
\end{array}
\]

which induces (by Theorem 4.3.6) the morphism of spaces

\[
(5.2.5) \quad \mathcal{G}^q(\hat{X}_{\lambda(J)}, U_{\lambda(J)}) \to \mathcal{G}^q(\hat{X}'_J, V_J)
\]

in the homotopy category. They induce the morphism in the homotopy category

\[
\lambda_p^* : \prod_{I \in \Lambda^{p+1}} \mathcal{G}^q(\hat{X}_I, U_I) \to \prod_{J \in (\Lambda')^{p+1}} \mathcal{G}^q(\hat{X}'_J, V_J),
\]

which is defined as follows: for the \( J \)-th factor on the right hand side, first taking the projection to the factor at \( \lambda(J) \in \Lambda^{p+1} \) on the left hand side, and then applying \( (5.2.5) \). One checks that these morphisms over \( p \geq 0 \) are compatible with the cofaces and the codegeneracies. Thus we have the morphism \( (5.2.4) \) of cosimplicial objects.

The second part follows by applying the homotopy limits of \( (5.2.4) \) over \( \Delta \) as in Definition 5.1.10. The last part of the lemma is immediate and we omit details. \( \Box \)

\textbf{Lemma 5.2.6.} Let \( Y \in \text{Sch}_k \). Let \( \mathcal{U} = \{ (U_i, X_i) \}_{i \in \Lambda} \) and \( \mathcal{V} = \{ (V_j, X'_j) \}_{j \in \Lambda'} \) be systems of local embeddings for \( Y \) such that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \) for two different set maps \( \lambda, \lambda' : \Lambda' \to \Lambda \).

Then the maps \( \lambda^* \) and \( (\lambda')^* \) are homotopic to each other.

\textbf{Proof.} This is standard. cf. [36, Lemma 1.20, p.446]. \( \Box \)
5.3. The filtrations. Finally, in §5.3 we define two filtrations on the algebraic $K$-theory of $Y \in \text{Sch}_k$.

Definition 5.3.1. Let $Y \in \text{Sch}_k$. Define in the homotopy category

$$\hat{G}_Y^q := \text{hocolim}_U \hat{H}(U, \hat{G}^q)$$

(see Definition 5.1.10 for $\hat{H}(U, \hat{G}^q)$), where the homotopy colimit is taken over all systems $U$ of local embeddings for $Y$. By Lemmas 5.2.2, 5.2.3 and 5.2.6, the homotopy colimit over $U$ is well-defined up to weak-equivalence.

Similarly, for a fixed $n \geq 0$, if we use $\hat{G}^q(\hat{x}_I, U_I, n)$ of Definition 4.2.6 in (5.1.3) instead of the geometric realization $\hat{G}^q(\hat{x}_I, U_I)$, and modify accordingly the construction in Definitions 5.1.7 and 5.1.10 for $U$, $q \geq 0$, $n \geq 0$ we obtain $\hat{H}(U, \hat{G}^q, n)$ as well as

$$\hat{G}_{Y,n}^q := \text{hocolim}_U \hat{H}(U, \hat{G}^q, n)$$

in the homotopy category.

Note that for each system $U$ and a refinement $V$ of $U$, we have the following commutative diagram in the homotopy category

$$\cdots \hat{H}(U, \hat{G}^q) \longrightarrow \cdots \hat{H}(U, \hat{G}^0) \longrightarrow \hat{H}(\{U\}, K) \xrightarrow{w.e.} K(Y)$$

$$\cdots \hat{H}(V, \hat{G}^q) \longrightarrow \cdots \hat{H}(V, \hat{G}^0) \longrightarrow \hat{H}(\{V\}, K),$$

where $w.e.$ stands for weak-equivalences (Remark 5.1.12). So, after taking the homotopy colimits over $U$ and inverting weak-equivalences, we deduce morphisms in the homotopy category

(5.3.2) \[ \cdots \to \hat{G}_{Y}^q \to \hat{G}_{Y}^{q-1} \to \cdots \to \hat{G}_{Y}^{0} \to K(Y). \]

Similarly, if we repeat the above arguments with $\hat{H}(U, \hat{G}^q, n)$ instead of $\hat{H}(U, \hat{G}^q)$ for $n \geq 0$, we have

(5.3.3) \[ \cdots \to \hat{G}_{Y,n}^q \to \hat{G}_{Y,n}^{q-1} \to \cdots \to \hat{G}_{Y,n}^{0} \to K(Y). \]

For (5.3.3), we will be primarily interested in the case $n = 0$.

Definition 5.3.4. Let $Y \in \text{Sch}_k$ and let $q \geq 1$ be an integer. Define

$$F_{\text{cnv}}^q K_n(Y) := \text{Im} \left( \pi_n(\hat{G}^q_{Y,0}) \to \pi_n(K(Y)) \right) = \lim_{U} \text{Im} \left( \pi_n(\hat{H}(U, \hat{G}^q, 0)) \to \pi_n(K(Y)) \right),$$

and

$$F_{\text{m,cnv}}^q K_n(Y) := \text{Im} \left( \pi_n(\hat{G}^q_{Y}) \to \pi_n(K(Y)) \right) = \lim_{U} \text{Im} \left( \pi_n(\hat{H}(U, \hat{G}^0)) \to \pi_n(K(Y)) \right).$$

For $q \leq 0$, we define

$$F_{\text{cnv}}^q K_n(Y) := K_n(Y), \quad \text{and} \quad F_{\text{m,cnv}}^q K_n(Y) := K_n(Y).$$

By construction, they form decreasing filtrations

$$K_n(Y) = F_{\text{cnv}}^0 K_n(Y) \supset F_{\text{cnv}}^1 K_n(Y) \supset F_{\text{cnv}}^2 K_n(Y) \supset \cdots,$$

and

$$K_n(Y) = F_{\text{m,cnv}}^0 K_n(Y) \supset F_{\text{m,cnv}}^1 K_n(Y) \supset F_{\text{m,cnv}}^2 K_n(Y) \supset \cdots,$$

which we call the coniveau filtration and the motivic coniveau filtration on $K_n(Y)$, respectively.

By construction, these filtrations are independent of the choice of a particular system $U$ of local embeddings for $Y$. \( \square \)
5.4. Consistency for the coniveau filtration. In [5.4] we check the consistency of the coniveau filtration $F^*_\text{cnv}$ on $K_n(Y)$ in Definition 5.3.4 when $Y$ is smooth.

Consider the tower of spaces constructed by D. Quillen [33 §7.5 Theorem 5.4]:
\[
\cdots \to K(M_q(Y)) \to K(M_{q-1}(Y)) \to \cdots \to K(M_1(Y)) \to K(M_0(Y)) = K(Y),
\]
where $M_q(Y)$ is the exact category consisting of coherent $\mathcal{O}_Y$-modules with the supports of codimension $\geq q$.

**Proposition 5.4.2.** Let $Y$ be an equidimensional smooth $k$-scheme, and let $q \geq 1$.

Then there exists a commutative diagram in the homotopy category
\[
\begin{array}{cccc}
\cdots & K(M_q(Y)) & \to & K(M_{q-1}(Y)) & \to \cdots & K(M_1(Y)) & \to & K(Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & w.e. \\
\cdots & \hat{G}^q_{Y,0} & \to & \hat{G}^{q-1}_{Y,0} & \to \cdots & \hat{G}^1_{Y,0} & \to & K(Y),
\end{array}
\]
where the vertical maps except the last one are weak-equivalences induced by the inclusions $M_q(Y) \subseteq D^q_{\text{coh}}(Y)$.

In particular, the coniveau filtration in Definition 5.3.4 coincides with the classical coniveau filtration on $K_n(Y)$ of Quillen.

**Proof.** Since the classical coniveau filtration $F^q_{\text{top}}K_n(Y)$ on $K_n(Y)$ is defined [33 §7.5 Theorem 5.4] as the image of the map $K_n(M_q(Y)) \to K_n(Y)$ induced by applying $\pi_n$ to (5.4.1), it suffices to show the existence of the diagram (5.4.3), where the vertical maps are weak-equivalences.

Let $\mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda}$ be a system of local embeddings for $Y$, with each $U_i \subseteq Y$ is affine open. Since $Y$ is equidimensional and smooth over $k$, each open set $U_i \subseteq Y$ of the underlying open cover $|\mathcal{U}| = \{U_i\}_{i \in \Lambda}$, is also equidimensional and smooth, so that the identity map $U_i \to U_i$ gives a closed immersion into an equidimensional smooth $k$-scheme.

Hence the collection $\mathcal{U}_0 := \{(U_i, U_i)\}_{i \in \Lambda}$ is a system of local embeddings for $Y$, which is also a refinement of $\mathcal{U}$. Furthermore, $\hat{\mathcal{U}}_I = \hat{\mathcal{U}}$ for each $I \in \Lambda^{p+1}$, so by Remark 5.2.7 we obtain $\hat{\mathcal{G}}^q(U_I, U_I, 0) = \hat{\mathcal{G}}^q(U_I, 0)$ in Definition 5.4.6. Thus, $\hat{\mathcal{H}}(\mathcal{U}_0, \hat{\mathcal{G}}^q) = \hat{\mathcal{H}}(\overline{\mathcal{U}}, \hat{\mathcal{G}}^q(-, 0))$ for the presheaf on $\mathcal{Y}_{\text{zar}}$ that assigns $U \mapsto \hat{\mathcal{G}}^q(U, 0)$, where $\hat{\mathcal{G}}^q(U, 0)$ is as in 5.1.5 applied to the scheme $U$ seen as a formal scheme.

Now, we consider the presheaf $U \mapsto K(M_q(U))$ on $\mathcal{Y}_{\text{zar}}$. Observe that the natural map $K(M_q(Y)) \to \hat{\mathcal{H}}(\mathcal{U}, K(M_q(-)))$ is a weak-equivalence; this is standard, but one can argue also by combining the computation of Quillen [33 (5.5) in Theorem 5.4 §7.5] and the descending induction argument of [37 (10.3.13), (10.3.14), p.386] applied to $\hat{\mathcal{H}}$ instead of $\mathcal{H}_{\text{zar}}$. We shrink details.

Therefore, it suffices to see that for each equidimensional smooth $k$-scheme $U$ the inclusions $M_q(U) \subseteq D^q_{\text{coh}}(U)$ induce weak-equivalences $K(M_q(U)) \to \hat{\mathcal{G}}^q(U, 0)$, which fit into a commutative diagram in the homotopy category:
\[
\begin{array}{cccc}
\cdots & K(M_q(U)) & \to & K(M_{q-1}(U)) & \to \cdots & K(M_1(U)) & \to & K(U) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & w.e. \\
\cdots & \hat{\mathcal{G}}^q(U, 0) & \to & \hat{\mathcal{G}}^{q-1}(U, 0) & \to \cdots & \hat{\mathcal{G}}^1(U, 0) & \to & K(U).
\end{array}
\]

To check this, we observe that $D^q_{\text{coh}}(U, 0) = D^0_{\text{coh}}(U)$ (see Remark 5.1.4), and therefore the inclusion $M_q(U) \subseteq D^q_{\text{coh}}(U)$ induces the desired weak-equivalence by
Invariance of $K$ in the homotopy category, where the last weak-equivalence follows from the Definition 5.5.7.

The commutativity of the diagram is clear since the horizontal maps are induced by the inclusion of the corresponding supports. □

5.5. **Consistency for the motivic coniveau filtration.** Now in §5.5, we proceed to check the consistency when $Y$ is smooth for the motivic coniveau filtration $F^\bullet_{\text{m.cv}}$ on $K_n(Y)$ of Definition 5.5.1.

First let’s consider the cubical version of the homotopy coniveau tower of M. Levine [26 §2.1] for Quillen’s $K$-theory in what follows.

**Definition 5.5.1.** Let $n \geq 0$. Let $S_Y^{[q]}(n)$ be the set of closed subsets $Z$ of $Y \times \Box^n$ of codimension $\geq q$, such that

\[(5.5.2)\] \[\text{codim}_{Y \times F} Z \cap (Y \times F) \geq q\]

for each face $F \subset \Box^n$ (cf. Definition 2.4.3-\((\text{GP})\)). □

**Remark 5.5.3.** We note that there is a subtle difference between Definition 5.5.1 and Definition 2.4.3-\((\text{GP})\). Recall that in the latter, when $Z$ in $Y \times \Box^n$ has the codimension $c$, then we want

\[(5.5.4)\] \[\text{codim}_{Y \times F}(Z \cap (Y \times F)) \geq c,\]

for each face $F \subset \Box^n$. However, when an irreducible set $Z$ is in $S_Y^{[q]}(n)$, and its codimension $c$ in $Y \times \Box^n$ is $> q$, then we have (5.5.2), but it does not imply (5.5.4) in general.

This is one reason why the proof of Lemma 5.5.19 later, is more complicated than one might hope at first sight. □

**Definition 5.5.5.** Given $Z \in S_Y^{[q]}(n)$, let $K^Z(Y \times \Box^n)$ denote the homotopy fiber of the map $j^* : K(Y \times \Box^n) \to K((Y \times \Box^n)\setminus Z)$ induced by the open immersion $j : (Y \times \Box^n)\setminus Z \to Y \times \Box^n$, where $K(-)$ is Quillen’s $K$-space functor.

Let $K^{[q]}(Y, n)$ be the filtered homotopy colimit

\[
K^{[q]}(Y, n) = \text{hocolim}_{Z \in S_Y^{[q]}(n)} K^Z(Y \times \Box^n).
\]

By pulling back along the cofaces and codegeneracies of the co-cubical scheme $(n \mapsto Y \times \Box^n)$, we obtain a cubical space $(n \mapsto K^{[q]}(Y, n))$, and we will write

\[
K^{[q]}(Y) := \{n \mapsto K^{[q]}(Y, n)\}
\]

for its (cubical) geometric realization, as in Remark 3.1.12. □

Via the inclusions of supports, we deduce the following tower of spaces

\[(5.5.6)\] \[\cdots \to K^{[q]}(Y) \to K^{[q-1]}(Y) \to \cdots \to K^{[1]}(Y) \to K^0(Y) \cong K(Y)\]

in the homotopy category, where the last weak-equivalence follows from the $A^1$-invariance of $K$-theory for smooth $k$-schemes.

In order to compare the cubical homotopy coniveau tower (5.5.6) with the simplicial one in M. Levine [26 §2.1], we consider the following intermediate “simplicial-cubical” construction.

**Definition 5.5.7.** Let $q, m, n \geq 0$ Let $S_Y^{[q]}(m, n)$ be the set of closed subsets $Z$ of $Y \times \Delta^m \times \Box^n$ of codimension $\geq q$ such that

\[
\text{codim}_{Y \times F_x \times F_c} Z \cap (Y \times F_x \times F_c) \geq q
\]

for each pair consists of a face $F_x \subset \Delta^m$ and a face $F_c \subset \Box^n$. 

[37] Theorem 1.9.8, p.271], [37] Theorem 1.11.7, p.279], and an argument parallel to [37] Lemmas 3.11, 3.12, Corollary 3.13, pp.316-317] that compares the Quillen $K$-theory with the Waldhausen $K$-theory. The commutativity of the diagram is clear since the horizontal maps are induced by the inclusion of the corresponding supports. □
Let $K^q(Y, m, n)$ be the filtered homotopy colimit
$$K^q(Y, m, n) = \operatorname{hocolim}_{\Delta^m \times \Box^n} K^q(Y \times \Delta^m \times \Box^n),$$
which gives a simplicial-cubical space. We will write
$$\begin{align*}
K^{q, (\infty)}(Y, n) &= |\mathbb{H} Y| \to K^q(Y, m, n)|, \\
K^{q, (\infty)}(Y, m) &= |\mathbb{H} Y| \to K^q(Y, m, n)|,
\end{align*}$$
for the geometric realizations of the simplicial and the cubical spaces, respectively.

Thus, we obtain a cubical (resp. simplicial) space $(n \mapsto K^{q, (\infty)}(Y, n))$ (resp. $(m \mapsto K^{q, (\infty)}(Y, m))$), and since colimits commute among themselves we conclude that the corresponding geometric realizations are isomorphic to each other in the homotopy category:
$$|\mathbb{H} Y| \mapsto K^{q, (\infty)}(Y, n)| \cong |\mathbb{H} Y| \mapsto K^{q, (\infty)}(Y, m)| := K^{q, (\infty)}(Y).$$

**Lemma 5.5.8.** Let $Y$ be an equidimensional smooth $k$-scheme, and let $q, m, n \geq 0$ be integers.

Then there are weak-equivalences

$$K^{q, ( \infty )}(Y) \to K^{q, (\infty)}(Y) \leftarrow K^{q}(Y),$$
that are natural in $Y$, where $K^{q}(Y)$ is the $q$-th term in the simplicial homotopy coniveau tower of M. Levine [26, §2.1].

**Proof.** We apply M. Levine [25, Proposition 3.3.4] for $A = k$ to conclude that there is a weak-equivalence $K^{q, (\infty)}(Y, n) \to K^{q}(Y \times \Box^n)$, which is natural in $Y$. Then [25, Theorem 3.3.5] implies that the degeneracies $K^{q, (\infty)}(Y, n) \to K^{q, (\infty)}(Y, n + 1)$ are weak-equivalences for $n \geq 1$, so all the $n$-cubes in the cubical space $(n \mapsto K^{q, (\infty)}(Y, n))$ are degenerate for $n \geq 1$. Hence the map from the 0-cubes into the geometric realization gives the first desired weak-equivalence of (5.5.9)

$$K^{q}(Y) = K^{q, (\infty)}(Y, 0) \to |\mathbb{H} Y| \mapsto K^{q, (\infty)}(Y, n)| = K^{q, (\infty)}(Y).$$

To construct the remaining weak-equivalence of (5.5.9), we use the cubical analogue of [25, Proposition 3.3.4, Theorem 3.3.5], *mutatis mutandis* and argue similarly. We shrink details.

As a result, we deduce the map from the 0-simplices into the geometric realization is the second desired weak-equivalence of (5.5.9):

$$K^{q, ( \infty )}(Y) \to K^{q, (\infty)}(Y, 0) \to |\mathbb{H} Y| \mapsto K^{q, (\infty)}(Y, m)| = K^{q, (\infty)}(Y).$$

Combining the weak-equivalences in (5.5.10) and (5.5.11), we have the lemma. \hfill \Box

We can now state the comparison result:

**Proposition 5.5.12.** Let $Y$ be an equidimensional smooth $k$-scheme, $q \geq 1$. Then there exists a commutative diagram in the homotopy category

$$\begin{align*}
\cdots &\to K^{q}(Y) \to K^{q-1}(Y) \to \cdots \to K^{1}(Y) \to K(Y) \\
\cdots &\to \mathcal{G}^{q}_{Y} \to \mathcal{G}^{q-1}_{Y} \to \cdots \to \mathcal{G}^{1}_{Y} \to \mathcal{K}(Y),
\end{align*}$$
where the vertical maps are weak-equivalences.

In particular, the motivic coniveau filtration in Definition [5.3.4] coincides with the homotopy coniveau filtration on $K_{*}(Y)$ of M. Levine [26, §2.1].
Proof. By Lemma 5.5.8 it suffices to show the existence of the commutative diagram (5.5.13), where the vertical maps are weak-equivalences.

Let $U = \{(U_i, X_i)\}_{i \in \Lambda}$ be a system of local embeddings for $Y$, where each $U_i$ is affine for $i \in \Lambda$. Since $Y$ is equidimensional and smooth over $k$, each open set $U_i \subset Y$ of the underlying open cover $|U| = \{(U_i)_{i \in \Lambda}$, is also equidimensional and smooth, so that the identity map $U_i \to U_i$ gives a closed immersion into an equidimensional smooth $k$-scheme.

Hence the collection $U_0 := \{(U_i, X_i)\}_{i \in \Lambda}$ is a system of local embeddings for $Y$, which is also a refinement of $U$. Furthermore, $U_I = U_I$ for each $I \in \Lambda^{n+1}$, so by Remark 4.2.7 we obtain $\hat{G}^q(U_I, U_I, n) = \hat{G}^q(U_I, n)$ for $n \geq 0$ and thus $\hat{G}^q(U_I, U_I) = \hat{G}^q(U_I)$.

Thus $\hat{h}(U_0, \hat{G}^q) = \hat{h}(U_0, \hat{G}^q)$, for the presheaf $U \mapsto \hat{G}^q(U)$ on $Z_{zar}$.

Now, we consider the presheaf $U \mapsto K^{[q]}(U)$ on $Z_{zar}$. Observe that the natural map $K^{[q]}(Y) \to \hat{h}(U_0, K^{[q]}(\_))$ is a weak-equivalence; this follows from Lemma 5.5.8 and M. Levine [26, Theorem 7.1.1].

Therefore, it suffices to see that for every smooth affine equidimensional $k$-scheme $U$ there exist weak-equivalences $\hat{G}^q(U) \to K^{[q]}(U)$ for $q \geq 0$, which fit into a commutative diagram in the homotopy category:

\[
\begin{array}{ccccccccc}
\cdots & \to & K^{[q]}(U) & \to & K^{[q-1]}(U) & \to & \cdots & \to & K^{[1]}(U) & \to & K(U) \\
| & & | & & | & & | & w.e. & | & | \\
\cdots & \to & \hat{G}^q(U) & \to & \hat{G}^{q-1}(U) & \to & \cdots & \to & \hat{G}^1(U) & \to & K(U).
\end{array}
\]

The existence of the vertical arrows follows by taking the geometric realizations for the weak-equivalences between the cubical spaces in Lemmas 5.5.15 and 5.5.19 proven separately below. The commutativity of the diagram is clear since the horizontal maps are induced by inclusion of supports which are compatible with the weak-equivalences in Lemmas 5.5.15 and 5.5.19.

The proof of the above Proposition 5.5.12 used the following Lemmas 5.5.15 and 5.5.19. We establish them in what follows.

**Definition 5.5.14.** Let $M_q(Y, n)$ be the exact category of coherent $O_Y \times \square^n$-modules with the supports in $S_q^{[n]}(n)$ defined in Definition 5.5.7, and let $D^{[q]}_{coh}(Y, n) \subseteq D_{coh}(Y \times \square^n)$ be the full triangulated subcategory generated by $M_q(Y, n)$.

**Lemma 5.5.15.** Let $Y$ be an equidimensional smooth $k$-scheme, and let $q, n \geq 0$. Then there are weak-equivalences:

\[ (5.5.16) \quad K(D^{[q]}_{coh}(Y, n)) \leftarrow K(M_q(Y, n)) \to K^{[q]}(Y, n), \]

that are natural in $Y$ and they form maps of cubical spaces as $n \geq 0$ varies, where the space $K^{[0]}(Y, n)$ is defined in Definition 5.5.3.

**Proof.** By Thomason-Trobaugh [37, Theorem 1.9.8, p.271], [37, Theorem 11.17, p.279], and an argument parallel to [37, Lemmas 3.11, 3.12, Corollary 3.13, pp.316-317], the inclusion $M_q(Y, n) \subseteq D^{[q]}_{coh}(Y, n)$ induces the left weak-equivalence of (5.5.16)

\[ K(M_q(Y, n)) \leftarrow K(D^{[q]}_{coh}(Y, n)). \]

It is natural in $Y$ because so is the inclusion $M_q(Y, n) \subseteq D^{[q]}_{coh}(Y, n)$. The compatibility with the faces and the degeneracies is immediate.
For the remaining map of (5.5.16), we apply D. Quillen \[33\] §2 (9), p.104 and [33] §7 Proposition 3.1, p.127 to obtain a weak-equivalence
\[
K(M_q(Y, n)) \xrightarrow{w,e} \lim_{Z \in \mathcal{S}^\triangleright(n)} G(Z).
\]
However, \(G(Z) = K^Z(Y \times \square^n)\) by the localization theorem for G-theory in D. Quillen \[33\] §5 Theorem 5, p.113. Thus, there is a weak-equivalence
\[
(5.5.17) \quad K(M_q(Y, n)) \xrightarrow{w,e} \lim_{Z \in \mathcal{S}^\triangleright(n)} K^Z(Y \times \square^n).
\]
Since this is a filtered colimit, it follows from D. Dugger \[11\] Proposition 7.3 that the right hand side of (5.5.17) computes the filtered homotopy colimit
\[
\hocolim_{Z \in \mathcal{S}^\triangleright(n)} K^Z(Y \times \square^n) = K^{[q]}(Y, n).
\]
This construction is also natural in \(Y\) and it is compatible with the faces and the degeneracies. \(\square\)

**Definition 5.5.18.** Let \(M_q^i(Y, n)\) be the exact category of coherent \(\mathcal{O}_{Y \times \square^n}\)-modules that are admissible of codimension \(\geq q\) (see Definition 5.1.13). Since \(Y\) is smooth, there is an inclusion \(M_q^i(Y, n) \subseteq M_q(Y, n)\) into the category of Definition 5.5.14. This induces a fully faithful triangulated functor \(i_n : \mathcal{D}^q_{\text{coh}}(Y, n) \rightarrow \mathcal{D}^q_{\text{coh}}(Y, n)\).

**Lemma 5.5.19.** Let \(U\) be an equidimensional smooth affine \(k\)-scheme, and let \(q\), \(n \geq 0\). Let \(\text{Spec}(R) = U \times \square^n\).

Then the triangulated functor \(i_n : \mathcal{D}^q_{\text{coh}}(U, n) \rightarrow \mathcal{D}^q_{\text{coh}}(U, n)\) is an equivalence of categories, so it induces a weak-equivalence
\[
i_n : \mathcal{G}^q(U, n) \rightarrow \mathcal{K}(\mathcal{D}^q_{\text{coh}}(U, n)).
\]
Furthermore, it is natural in \(U\) and forms a map of cubical spaces as \(n \geq 0\) varies.

**Proof.** The morphism \(i_n\) is natural in \(U\) because the triangulated functor \(i_n\) is induced by the inclusion \(M_q^i(U, n) \subseteq M_q(U, n)\), which is clearly natural. The compatibility with the faces and the degeneracies for \(i_n\) is immediate.

It follows from Thomason-Trobaugh \[34\] Theorems 1.9.1, 1.9.8, p.263, p.271 that \(i_n\) is a weak-equivalence provided that \(i_n\) is an equivalence of categories. Since \(i_n\) is fully faithful, it only remains to check that \(i_n\) is essentially surjective on objects.

Recall (Definition 5.5.14) that \(\mathcal{D}^q_{\text{coh}}(U, n)\) is the full triangulated subcategory of \(\mathcal{D}^q_{\text{coh}}(U \times \square^n)\) generated by coherent \(\mathcal{O}_{U \times \square^n}\)-modules with the supports in \(\mathcal{S}^\triangleright(n)\). Therefore if \(q = 0\), we indeed have the equality \(\mathcal{D}^q_{\text{coh}}(U, n) = \mathcal{D}^q_{\text{coh}}(U, n)\), thus \(\mathcal{G}^0(U, n) = \mathcal{K}(\mathcal{D}^0_{\text{coh}}(U, n))\).

We now assume \(q \geq 1\).

Since every coherent \(\mathcal{O}_{U \times \square^n}\)-module \(\mathcal{M}\) admits a finite decreasing filtration \(\mathcal{M}_i \subseteq \mathcal{M}\) such that each successive quotient \(\mathcal{M}_i/\mathcal{M}_{i+1} \cong \mathcal{O}_{Z_i}\) for an irreducible reduced closed subscheme \(Z_i\) of \(U \times \square^n\), where the set of such \(Z_i\) is uniquely determined counting multiplicities. Thus in order to show the required essential surjectivity, it is enough to verify the following two claims:

1. Suppose \(Z \in \mathcal{S}^q(U, n)\) is irreducible. Then there exists a closed subscheme \(Z' \subseteq Y \times \square^n\) such that \(Z \subseteq Z'\) and \([Z'] \in z^{=q}(U, n)\).
2. Let \(Z \subseteq Z'\) be irreducible reduced closed subschemes of \(U \times \square^n\) such that \([Z'] \in z^{=q}(U, n)\). Then \(\mathcal{O}_Z \in \mathcal{D}^q_{\text{coh}}(U, n)\). (N.B. We don’t claim \([Z] \in z^{=q}(U, n)\). See Remark 5.5.3)
We show that the two claims hold.

1. Since \( Z \in \mathcal{O}_U(n) \), \( Z \) is a closed subset of \( U \times \square^n \) of codimension \( c \geq q \), such that

\[
\text{(IF(q))} \quad \text{codim}_{U \times F} Z \cap (U \times F) \geq q \geq 1
\]

for each face \( F \subset \square^n \). If \( c = q \), by definition \( Z \) intersects \( U \times F \) properly for each face \( F \) of \( \square^n \), so \( [Z] \in z^{\geq q}(U, n) \).

In case \( c \geq q + 1 \), by induction we will construct \( f_1, \ldots, f_q \in R \) such that \( Z \subseteq V_i = V(f_1, \ldots, f_i) \) and \( [V_i] \in z^{\geq q}(U, n) \) for \( 1 \leq i \leq q \). (N.B. Here, the codimension of \( V_i \) should be necessarily \( i \).

Let \( J \subset R \) be an ideal such that \( V(J) = Z \) and let \( \{ \mathcal{P}_{0, r} \in \text{Spec} (R) \}_{r \in C_0} \) be the finite set of prime ideals such that \( \{ \mathcal{V}(\mathcal{P}_{0, r}) \}_{r \in C_0} \) is the set of irreducible components in the collection of closed subsets \( \{ U \times F \mid F \) is a face of \( \square^n \) \). By \textbf{[IF(q)]}, we deduce that \( J \not\subset \mathcal{P}_{0, r} \) for all \( r \in C_0 \), thus the prime avoidance (Lemma \textbf{3.3.4}) implies that there is some \( f_1 \in J \) such that \( f_1 \not\subset \mathcal{P}_{0, r} \) for all \( r \in C_0 \). Hence \( V_1 := V(f_1) \) satisfies the required conditions.

We proceed the induction steps. Let \( 1 \leq i < q \). Assume that there are \( f_1, \ldots, f_i \in R \), with the required properties. Let \( \{ \mathcal{P}_{i, r} \in \text{Spec} (R) \}_{r \in C_i} \) be the finite set of prime ideals such that \( \{ \mathcal{V}(\mathcal{P}_{i, r}) \}_{r \in C_i} \) is the set of irreducible components in the collection of closed subsets \( \{ V_i \cap (U \times F) \mid F \) is a face of \( \square^n \} \). Since \( \text{codim}_{U \times F} V_i \cap (U \times F) = i \leq q \), we deduce by \textbf{[IF(q)]} that \( J \not\subset \mathcal{P}_{i, r} \) for all \( r \in C_i \), thus the prime avoidance implies that there is some \( f_{i+1} \in J \) such that \( f_{i+1} \not\subset \mathcal{P}_{i, r} \) for all \( r \in C_i \). Hence \( V_{i+1} := V(f_1, \ldots, f_{i+1}) \) satisfies the required conditions. This proves (1).

2. If \( Z = Z' \), then the claim holds by definition, so we may assume \( Z \subset Z' \).

Let \( 0 \neq \mathcal{I} \subset \mathcal{O}_Z \) be the ideal sheaf such that \( \mathcal{O}_{Z'} / \mathcal{I} \cong \mathcal{O}_Z \). Consider the short exact sequence

\[
(5.5.20) \quad 0 \to \mathcal{I} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0,
\]

and observe that \( \mathcal{O}_{Z'} \in \mathcal{D}^q_{\text{coh}}(U, n) \) since we are given that \( [Z'] \in z^{\geq q}(U, n) \).

We first claim that \( [\mathcal{I}] \in z^{\geq q}(U, n) \).

To prove it, we switch to affine notations: let \( \mathcal{P} \in \text{Spec} (R) \) be such that \( \text{Spec} (R/\mathcal{P}) = Z' \), and let \( \mathcal{Q} \in \text{Spec} (R/\mathcal{P}) \) be such that \( \text{Spec} (R/\mathcal{Q}) = Z \). In particular, \( \mathcal{Q} = \mathcal{I} \). Then \( R/\mathcal{P} \) is an integral domain since \( Z' \) is reduced and irreducible.

Thus \( \text{Ass}(R/\mathcal{P}) = \{ (0) \} \), the zero ideal of \( R/\mathcal{P} \). Since we have \( 0 \neq \mathcal{Q} \subset R/\mathcal{P} \), we have \( 0 \neq \text{Ass}(\mathcal{Q}) \subset \text{Ass}(R/\mathcal{P}) = \{ (0) \} \). In particular, \( \text{Ass}(\mathcal{Q}) = \{ (0) \} \).

Since \( 0 \subset R/\mathcal{Q} \) corresponds to \( Z' \), this means \( [\mathcal{I}] = [\mathcal{Q}] = [Z'] \), which is in \( z^{\geq q}(U, n) \), proving the claim.

This implies in particular that \( \mathcal{I} \in \mathcal{D}^q_{\text{coh}}(U, n) \). Since \( \mathcal{I}, \mathcal{O}_{Z'} \in \mathcal{D}^q_{\text{coh}}(U, n) \), from the exact sequence \( (5.5.20) \), we deduce that \( \mathcal{O}_Z \in \mathcal{D}^q_{\text{coh}}(U, n) \) (see e.g. A. Neeman [30], Remark 1.5.2]). This proves (2), and finishes the proof of the lemma.

5.6 Functoriality. We want to check that the two filtrations we defined in Definition \textbf{5.3.4} are functorial on \textbf{Sch}_k. We consider the following, which is also used in [22], §6. This also generalizes the notion of refinement of systems of local embeddings:

\textbf{Definition 5.6.1.} Let \( g : Y_1 \to Y_2 \) be a morphism in \textbf{Sch}_k. Let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) be a system of local embeddings for \( Y_2 \).

We say that a system \( \mathcal{V} = \{(V_j, X'_j)\}_{j \in \Lambda'} \) of local embeddings for \( Y_1 \) is associated to \( \mathcal{U} \) by \( g \), if there is a set map \( \lambda : \Lambda' \to \Lambda \) such that for each \( j \in \Lambda' \), we have \( g(V_j) \subset U_{\lambda(j)} \), and there is a morphism \( f_j : X'_j \to X_{\lambda(j)} \) such that \( f_j|_{V_j} = g|_{V_j} \).

Such systems do exist:
Lemma 5.6.2. Let \( g: Y_1 \to Y_2 \) be a morphism in \( \text{Sch}_k \). Let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) be an arbitrary system of local embeddings for \( Y_2 \).

Then there exists a system \( \mathcal{V} \) of local embeddings for \( Y_1 \) associated to \( \mathcal{U} \) by \( g \).

Proof. For each \( i \in \Lambda \), the open subset \( g^{-1}(U_i) \subset Y_1 \) may not be affine, but we can find a finite affine open cover \( \{V_{ij}\}_{j \in \Lambda'_i} \) of \( g^{-1}(U_i) \) for a finite set \( \Lambda'_i \). Let \( \Lambda' := \prod_{i \in \Lambda} \Lambda'_i \). Let \( \lambda: \Lambda' \to \Lambda \) be the projection set map given by sending \( j \in \Lambda'_i \) to \( i \).

For each \( V_{ij} \), choose a closed immersion \( i_{ij}: V_{ij} \hookrightarrow Z_{ij} \) into an equidimensional smooth \( k \)-scheme. Let \( X'_{ij} := Z_{ij} \times X_i \), and consider the closed immersion

\[
V_{ij} \xrightarrow{g_{ij}} V_{ij} \times U_i \hookrightarrow Z_{ij} \times X_i = X'_{ij},
\]

where we let \( g_{ij} := g|_{V_{ij}} \). Then \( \mathcal{V} := \{(V_{ij}, X'_{ij})\}_{i,j} \) is a system of local embeddings for \( Y_1 \). For the projection \( f_{ij}: X'_i = Z_{ij} \times X_i \to X_i \), we have \( f_{ij}|_{V_{ij}} = g_{ij} = g|_{V_{ij}} \). Thus \( \mathcal{V} \) is associated to \( \mathcal{U} \) by \( g \).

Lemma 5.6.3. Let \( g: Y_1 \to Y_2 \) be a morphism in \( \text{Sch}_k \) and let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) be a system of local embeddings for \( Y_2 \). Let \( \mathcal{V} = \{(V_{ij}, X'_{ij})\}_{j \in \Lambda'_{ij}} \) be a system of local embeddings for \( Y_1 \), that is associated to \( \mathcal{U} \) by \( g \) as in Lemma 5.6.2.

Then there exists a morphism of cosimplicial objects

\[
(5.6.4) \quad g_{\mathcal{U}, \mathcal{V}}: \tilde{G}_{\mathcal{U}} \to \tilde{G}_{\mathcal{V}}
\]
in the homotopy category. In particular, we have a morphism

\[
(5.6.5) \quad g_{\mathcal{U}, \mathcal{V}}: \tilde{H}(\mathcal{U}, \tilde{G}^q) \to \tilde{H}(\mathcal{V}, \tilde{G}^q)
\]
in the homotopy category.

Proof. Let \( p \geq 0 \) be an integer. Let \( J \in (\Lambda')^{p+1} \). Then we have the commutative diagram

\[
\begin{array}{ccc}
X'_i & \xrightarrow{f_{ij}} & X_{M(J)} \\
\downarrow & & \downarrow \\
V_{ij} & \xrightarrow{g_{ij}} & U_{M(J)}.
\end{array}
\]

By Theorem 4.3.6, this diagram induces a morphism

\[
(5.6.6) \quad \tilde{G}^q(\tilde{X}_{M(J)}, U_{M(J)}) \to \tilde{G}^q(\tilde{X}'_{ij}, V_{ij})
\]
in the homotopy category. These induce morphisms in the homotopy category for \( p \geq 0 \)

\[
(5.6.7) \quad \prod_{\ell \in \Lambda^{p+1}} \tilde{G}^q(\tilde{X}_{\ell}, U_{\ell}) \to \prod_{J \in (\Lambda')^{p+1}} \tilde{G}^q(\tilde{X}'_{ij}, V_{ij})
\]
defined as follows: for the \( J \)-th factor on the right hand side of (5.6.7), we first project the left hand side to the factor over \( \lambda(J) \in \Lambda^{p+1} \), and then apply (5.6.6). This gives (5.6.7). That they are compatible with the cofaces and codegeneracies are immediate, and we thus have a morphism of the cosimplicial objects (5.6.4). By taking the homotopy limits over \( \Delta \), we deduce (5.6.5).

We want to take homotopy colimits over \( \mathcal{V} \) and \( \mathcal{U} \) in (5.6.5). For this, we need:

Lemma 5.6.8. Let \( g: Y_1 \to Y_2 \) be a morphism in \( \text{Sch}_k \). Let \( \mathcal{U} = \{(U_i, X_i)\}_{i \in \Lambda} \) be a system of local embeddings for \( Y_2 \) and let \( \mathcal{V} = \{(V_{ij}, X'_{ij})\}_{j \in \Lambda'} \) be a system for \( Y_1 \).

Then there exists a system \( \mathcal{V}' \) of local embeddings for \( Y_1 \), which is

(1) a refinement of \( \mathcal{V} \) and

(2) associated to \( \mathcal{U} \) by \( g \).
Proof. First choose a system $W$ of local embeddings for $Y_1$, that is associated to $\mathcal{U}$ by $g$, by Lemma 5.6.2

If the system $W$ is already a refinement of $\mathcal{V}$, then take $\mathcal{V}' = W$.

If not, then by Lemma 5.2.2, find a common refinement $\mathcal{V}'$ of both systems $\mathcal{V}$ and $W$ for $Y_1$. Note that when $W$ is associated to $\mathcal{U}$ by $g$, then so is any refinement of $W$. Hence $\mathcal{V}'$ satisfies the above two conditions of the lemma. \hfill \Box

**Lemma 5.6.9.** Let $g : Y_1 \to Y_2$ be a morphism in $\text{Sch}_k$. Let $\mathcal{U}, \mathcal{U}'$ be systems of local embeddings for $Y_2$ and $\mathcal{V}, \mathcal{V}'$ be systems for $Y_1$, such that

1. $\mathcal{V}$ is associated to $\mathcal{U}$ by $g$,
2. $\mathcal{V}'$ is associated to $\mathcal{U}'$ by $g$,
3. $\mathcal{U}'$ is a refinement of $\mathcal{U}$,
4. $\mathcal{V}'$ is a refinement of $\mathcal{V}$.

Then the following diagram

\[
\begin{array}{ccc}
\check{H}(\mathcal{U}, \hat{G}^q) & \overset{g_{U,V}^*}{\longrightarrow} & \check{H}(\mathcal{V}, \hat{G}^q) \\
\downarrow & & \downarrow \\
\check{H}(\mathcal{U}', \hat{G}^q) & \overset{g_{U',V'}^*}{\longrightarrow} & \check{H}(\mathcal{V}', \hat{G}^q)
\end{array}
\]

in the homotopy category of spaces commutes, where the horizontal maps are supplied by Lemma 5.6.3 and the vertical maps are the refinement morphisms of Lemma 5.6.2.

**Proof.** It is elementary. We omit it. \hfill \Box

For a morphism $g : Y_1 \to Y_2$ in $\text{Sch}_k$, choose a system $\mathcal{U}$ of local embeddings for $Y_2$ and choose a system $\mathcal{V}$ for $Y_1$ associated to $\mathcal{U}$ by $g$ by Lemma 5.6.2 so that we have the map $g_{U,V}^* : \check{H}(\mathcal{U}, \hat{G}^q) \to \check{H}(\mathcal{V}, \hat{G}^q)$.

Note that a refinement $\mathcal{V}'$ of $\mathcal{V}$ again gives a system for $Y_1$ associated to $\mathcal{U}$ by $g$.

Hence we have the induced morphism in the homotopy category

\[
g_{U,V}^* := \text{hocolim}_V g_{U,V}^* : \check{H}(\mathcal{U}, \hat{G}^q) \to \hat{G}^q_{Y_1}.
\]

On the other hand, by Lemmas 5.6.8 and 5.6.9, for a refinement $\mathcal{U}'$ of $\mathcal{U}$, the morphisms (5.6.10) for $\mathcal{U}$ and $\mathcal{U}'$ satisfy the commutative diagram

\[
\begin{array}{ccc}
\check{H}(\mathcal{U}, \hat{G}^q) & \overset{g_{U,V}^*}{\longrightarrow} & \hat{G}^q_{Y_1} \\
\downarrow & & \downarrow g_{U,V'}^* \\
\check{H}(\mathcal{U}', \hat{G}^q),
\end{array}
\]

where the vertical arrow is the morphism induced by the refinement. Hence this induces

\[
g^* := \text{hocolim}_{\mathcal{U}} g_{U,V}^* : \hat{G}^q_{Y_2} \to \hat{G}^q_{Y_1},
\]

which fits into the following commutative diagram in the homotopy category

\[
\begin{array}{cccccccc}
\cdots & \to & \hat{G}^q_{Y_2} & \to & \hat{G}^{q-1}_{Y_2} & \to & \cdots & \to & \hat{G}^1_{Y_2} & \to & \check{K}(Y_2) \\
& & \downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \downarrow g^* \\
\cdots & \to & \hat{G}^q_{Y_1} & \to & \hat{G}^{q-1}_{Y_1} & \to & \cdots & \to & \hat{G}^1_{Y_1} & \to & \check{K}(Y_1),
\end{array}
\]

(5.6.12)
because the maps in \((5.6.3)\) commute with the corresponding maps \(G^n(X, U) \to G^{n-1}(X, U)\) induced by the inclusion of supports \(D^\text{coh}(\mathcal{X}, n) \subseteq D^\text{coh}(\mathcal{X}, n)\).

Notice that the arguments in Lemmas \(5.6.3, 5.6.4\) apply \textit{mutatis mutandis} for the tower \((5.3.3)\), thus for each \(n \geq 0\) we obtain the following commutative diagram in the homotopy category:

\[
\begin{array}{cccccccc}
\cdots & \to & \widehat{\mathcal{C}}^n_{Y_2,n} & \to & \widehat{\mathcal{C}}^{n-1}_{Y_2,n} & \to & \cdots & \to & \widehat{\mathcal{C}}^0_{Y_2,n} & \to & K(Y_2) \\
\downarrow & & \downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \\
\cdots & \to & \widehat{\mathcal{C}}^n_{Y_1,n} & \to & \widehat{\mathcal{C}}^{n-1}_{Y_1,n} & \to & \cdots & \to & \widehat{\mathcal{C}}^0_{Y_1,n} & \to & K(Y_1) \\
\end{array}
\]

\((5.6.13)\)

**Definition 5.6.14.** The diagram \((5.6.13)\) with \(n = 0\) and the diagram \((5.6.12)\), respectively, induce the homomorphisms

\[
\begin{align*}
& g^* : \mathcal{C}^n_{\text{cnv}}(Y_2) \to \mathcal{C}^n_{\text{cnv}}(Y_1), \\
& g^* : \mathcal{C}^n_{\text{m.cnv}}(Y_2) \to \mathcal{C}^n_{\text{m.cnv}}(Y_1),
\end{align*}
\]

of abelian groups defined in Definition \((5.3.3)\). One notes that they give homomorphisms of filtered abelian groups.

To see the above \(g^*\) in Definition \((5.6.14)\) is functorial, let \(g_1 : Y_1 \to Y_2\) and \(g_2 : Y_2 \to Y_3\) be two morphisms. A question to check is whether \((g_2 \circ g_1)^* = g_1^* \circ g_2^*\). To see this, first choose an arbitrary system \(U_4\) of local embeddings for \(Y_3\). By Lemma \((5.6.4)\) there exists a system \(U_2\) for \(Y_2\) associated to \(U_4\) by \(g_2\), and applying Lemma \((5.6.2)\) again, there exists a system \(U_1\) for \(Y_1\) associated to \(U_2\) by \(g_1\). By definition, one sees that \(U_1\) is associated to \(U_4\) by \(g_2 \circ g_1\).

It follows from Lemma \((3.5.1)\) that \((g_2 \circ g_1)^* \mid_{U_1} = (g_2)^* \mid_{U_2} \circ (g_1)^* \mid_{U_1}\). Thus that \((g_2 \circ g_1)^* = g_1^* \circ g_2^*\) follows by taking homotopy colimits over \(U_1, U_2, U_3\), consecutively. We leave out details.

These discussions summarize into the following:

**Theorem 5.6.15.** The towers \((5.3.2), (5.3.3)\) in the homotopy category are functorial. More precisely, we have the following:

1. For each morphism \(g : Y_1 \to Y_2 \in \textbf{Sch}_k\), we have the commutative diagrams of towers in the homotopy category for \(n \geq 0\)

\[
\begin{array}{ccc}
\widehat{\mathcal{C}}^n_{Y_2,n} & \to & K(Y_2) \\
\downarrow g^* & & \downarrow g^* \\
\widehat{\mathcal{C}}^n_{Y_1,n} & \to & K(Y_1).
\end{array}
\]

2. For two morphisms \(g_1 : Y_1 \to Y_2\) and \(g_2 : Y_2 \to Y_3\) in \(\textbf{Sch}_k\), we have the equality in the homotopy category for \(n \geq 0\)

\[
\begin{align*}
& (g_2 \circ g_1)^* = g_1^* \circ g_2^* : \widehat{\mathcal{C}}^n_{Y_2,n} \to \widehat{\mathcal{C}}^n_{Y_1,n}, \\
& (g_2 \circ g_1)^* = g_1^* \circ g_2^* : \widehat{\mathcal{C}}^n_{Y_3,n} \to \widehat{\mathcal{C}}^n_{Y_1,n}.
\end{align*}
\]

Taking homotopy groups we conclude:

**Theorem 5.6.16.** The coniveau filtration (resp. motivic coniveau filtration) of Definition \((5.3.3)\) on the algebraic \(K\)-theory of schemes in \(\textbf{Sch}_k\) is functorial. More precisely, we have the following:
(1) For each morphism $g : Y_1 \to Y_2 \in \text{Sch}_k$, we have homomorphisms of filtered abelian groups

\[
\begin{align*}
  g^* : F_{\text{conv}}^n K_n(Y_2) &\to F_{\text{conv}}^n K_n(Y_1), \\
  g^* : F_{\text{m.conv}}^n K_n(Y_2) &\to F_{\text{m.conv}}^n K_n(Y_1).
\end{align*}
\]

(2) For two morphisms $g_1 : Y_1 \to Y_2$ and $g_2 : Y_2 \to Y_3$ in $\text{Sch}_k$, we have the equalities

\[
\begin{align*}
  (g_2 \circ g_1)^* &= g_2^* \circ g_1^* : F_{\text{conv}}^n K_n(Y_3) \to F_{\text{conv}}^n K_n(Y_1), \\
  (g_2 \circ g_1)^* &= g_2^* \circ g_1^* : F_{\text{m.conv}}^n K_n(Y_3) \to F_{\text{m.conv}}^n K_n(Y_1).
\end{align*}
\]

Recall that for a filtered abelian group $F^* A$, its $n$-th graded associated is the group $gr^n F^* A := F^n A/F^{n+1} A$. Its direct sum is written $gr^n F^* A := \bigoplus_n gr^n F^* A$. From Theorem 5.6.16 we deduce immediately:

**Theorem 5.6.17.** The graded associated of the coniveau filtration (resp. motivic coniveau filtration) on the algebraic $K$-theory of schemes in $\text{Sch}_k$ is functorial. More precisely, we have the following:

(1) For each morphism $g : Y_1 \to Y_2 \in \text{Sch}_k$, we have homomorphisms of filtered abelian groups

\[
\begin{align*}
  g^* : gr_{\text{conv}}^n K_n(Y_2) &\to gr_{\text{conv}}^n K_n(Y_1), \\
  g^* : gr_{\text{m.conv}}^n K_n(Y_2) &\to gr_{\text{m.conv}}^n K_n(Y_1).
\end{align*}
\]

(2) For two morphisms $g_1 : Y_1 \to Y_2$ and $g_2 : Y_2 \to Y_3$ in $\text{Sch}_k$, we have the equalities

\[
\begin{align*}
  (g_2 \circ g_1)^* &= g_2^* \circ g_1^* : gr_{\text{conv}}^n K_n(Y_3) \to gr_{\text{conv}}^n K_n(Y_1), \\
  (g_2 \circ g_1)^* &= g_2^* \circ g_1^* : gr_{\text{m.conv}}^n K_n(Y_3) \to gr_{\text{m.conv}}^n K_n(Y_1).
\end{align*}
\]

**Remark 5.6.18.** Note that we can apply Theorem 5.6.16 to morphisms $g : Y_1 \to Y_2$ between smooth irreducible quasi-projective $k$-schemes. We saw that the classical coniveau filtration and our new coniveau filtration coincide (Proposition 5.6.12) for each $Y_j$, $j = 1, 2$, so that the theorem shows that the pull-back $g^* : K_i(Y_2) \to K_i(Y_1)$ respects the classical coniveau filtration.

This is already known, e.g. see H. Gillet [15 Theorem 83, Lemma 84, p.283], which is based on the technique of deformation to the normal cone and $A^1$-invariance of the algebraic $K$-theory of regular schemes.

Our proof given in this article via formal schemes and a generalized Čech machine offers a new argument even for the classical smooth case, and it does not explicitly rely on the $A^1$-invariance.

More about the connections between cycles in terms of the new higher Chow groups of $\text{Sch}_k$ and the above motivic coniveau filtration on $K_i(Y)$ will be pursued in follow-up papers.

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