Neutrino mass matrix in triplet Higgs models with $A_4$ symmetry

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Abstract

We consider triplet Higgs model with $A_4$ symmetry to generate the neutrino mass matrix. The tribimaximal form of the neutrino mixing matrix can be naturally obtained. Imposing the neutrino oscillation data, we show that 1) both normal and inverted mass hierarchy are allowed, 2) there is a lower bound on the lightest neutrino mass and the effective mass for neutrinoless double beta decay, 3) the non-vanishing $\theta_{13}$ can be accommodated by considering small perturbation, 4) $\theta_{\text{atm}}$ should be very close to $\pi/4$ even after perturbation.

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1 Introduction

During last decade we have seen the firm evidences that the neutrinos have masses and mixings which may indicate the new physics (NP) beyond the Standard Model (SM). The solar, atmospheric and reactor neutrino experiments have measured the mass squared differences and mixing angles \([1]\) as shown in Table 1. From the data in Table 1 only, we do not know the mass hierarchy, the absolute neutrino masses, or whether neutrinos are Majorana or Dirac particles. The future experiments like the neutrinoless double beta decay or Tritium decay may give further information on the nature of neutrinos. Therefore, it is important to get a model which predicts the mass matrix of neutrinos.

The current experimental data for the neutrino oscillations suggest that the mixing matrix, the Pontecorvo-Maki-Nakagawa-Sakata (MNS) matrix, is tribimaximal at the zeroth order \([2]\):

\[
|U_{\nu}|^2 = \begin{pmatrix}
\frac{2}{3} & \frac{1}{3} & 0 \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2}
\end{pmatrix}.
\]

One of natural flavor symmetries which give the tribimaximal MNS matrix is the \(A_4\) symmetry \([3, 4]\). In addition, it was shown that the effective mass parameter relevant to the neutrinoless double beta decay can be predicted in a model with \(A_4\) flavor symmetry \([4]\).

The tribimaximal mixing can also be naturally realized in the triplet Higgs model.

| Parameter | Best fit | 3-\(\sigma\) c.l. range |
|-----------|----------|------------------------|
| \(\Delta m^2_{\odot}(10^{-5}\text{eV}^2)\) | \(7.65^{+0.23}_{-0.20}\) | 7.05 – 8.34 |
| \(\Delta m^2_{\text{atm}}(10^{-3}\text{eV}^2)\) | \(2.40^{+0.12}_{-0.11}\) | 2.07 – 2.75 |
| \(\sin^2(\theta_{\odot})\) | 0.304^{+0.022}_{-0.016} | 0.25 – 0.37 |
| \(\sin^2(\theta_{\text{atm}})\) | 0.50^{+0.07}_{-0.06} | 0.36 – 0.67 |
| \(\sin^2(\theta_{13})\) | 0.01^{+0.016}_{-0.011} | \(\leq 0.056\) |

Table 1: The neutrino oscillation data.
The triplet Higgs model, a TeV scale NP model, can generate Majorana neutrino masses. With $A_4$ symmetry the Higgs triplet model naturally gives tribimaximal mixings to neutrinos. The predicted doubly charged Higgs ($H^{±±}$), if light, can be found in the CERN Large Hadron Collider (LHC) experiment [5].

In this Letter we study a triplet Higgs model with $A_4$ flavor symmetry which predicts not only the neutrinoless double beta decay but also all the neutrino masses. The Letter is organized as follows: In Section 2 we introduce our model. The numerical analysis for the neutrino masses and mixings and the predictions for the neutrinoless double beta decay are done in Section 3. We conclude in Section 4.

## 2 The Higgs triplet model with $A_4$ symmetry

The conventional method to generate the neutrino masses is to introduce the heavy Majorana right-handed neutrinos (type-I seesaw). However, it is difficult to test this case experimentally. An alternative way to give Majorana masses to neutrinos is to introduce $SU(2)_L$ Higgs triplet $\Delta$ with $U(1)_Y$ charge 1 in the SM (type-II seesaw) [8]. Via Yukawa interaction the Higgs triplet model can provide Majorana masses to neutrinos if the neutral component $\Delta^0$ gets very small vev [7]. The Yukawa interaction is given by

$$\mathcal{L}_Y = \frac{1}{2} h_{ij} L^T_i C \tau_2 \Delta L_j + H.c,$$

(2)

where $h_{ij}$ is a complex symmetric coupling matrix, $L_i = (\nu_i, l_i)_L$ is a left-handed lepton doublet, $C$ is the Dirac charge conjugation operator, and $\tau_2$ is a Pauli matrix. The Higgs triplet can be decomposed as follows:

$$\Delta = \begin{pmatrix} \Delta^+ / \sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+ / \sqrt{2} \end{pmatrix}. \quad (3)$$

Then the neutrino mass matrix is written in terms of the vev as

$$(M_\nu)_{ij} = h_{ij} \langle \Delta^0 \rangle.$$  \quad (4)$$

There are many attempts to obtain the form of neutrino mass matrix suggested by Table 1 in the framework of $A_4$ symmetry in the literature [3, 4]. The group $A_4$ has three
| Fields | $e^c$ | $L$ | $\phi$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\Delta$ |
|--------|------|-----|-------|-------|-------|------|------|
| $A_4$  | 1, 1', 1'' | 3   | 3     | 1     | 1'    | 1''  | 3     |
| $SU(2)_L$ | 1    | 2   | 2     | 3     | 3     | 3    | 3     |
| $U(1)_Y$ | 1    | −1/2 | 1/2   | 1     | 1     | 1    | 1     |

Table 2: The assignments of $A_4$ and the $SU(2)_L \times U(1)_Y$ representations in our model.

Singlet representations, $1, 1', 1''$, and a triplet representation, $3$. Their tensor products are decomposed as

$$3 \otimes 3 = 3_s \oplus 3_a \oplus 1 \oplus 1' \oplus 1'', \quad 1' \otimes 1'' = 1.$$  \hspace{1cm} \text{(5)}

For two vectors $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ transforming as $3$, the first rule in the above equation states that

$$\begin{align*}
(3 \otimes 3)_3 &= (x_2y_3 + x_3y_2, x_3y_1 + x_1y_3, x_1y_2 + x_2y_1), \\
(3 \otimes 3)_3' &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1), \\
(3 \otimes 3)_1 &= x_1y_1 + x_2y_2 + x_3y_3, \\
(3 \otimes 3)_1' &= x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3, \\
(3 \otimes 3)_1'' &= x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3. 
\end{align*}$$ \hspace{1cm} \text{(6)}

We assign the three-dimensional representation $3$ of $A_4$ to the doublet Higgs $\phi$ and the triplet Higgs $\Delta$ as in Table 2. To implement $A_4$ flavor symmetric Lagrangian and get a neutrino mass matrix consistent with the experiments, we need to introduce additional Higgs multiplets. Here we introduce three more triplet Higgs $\chi_i$ ($i = 1, 2, 3$) like [4], which are assigned $1, 1', 1''$ of $A_4$. We assign $1, 1', 1''$ to the right-handed leptons and $3$ to the left-handed leptons.

Then not all the couplings in (2) but only the following $A_4$-symmetric Yukawa interactions are allowed:

$$\mathcal{L} = -\lambda e\overline{e}R(\phi^\dagger L)_1 - \lambda' e\overline{e}R(\phi^\dagger L)_1' - \lambda'' e\overline{e}R(\phi^\dagger L)_1'' + \frac{1}{2} \lambda_1 L^T C i \tau_2 \Delta L + \frac{1}{2} \lambda_2 (L^T C i \tau_2 \chi_1 L)_1 + \frac{1}{2} \lambda_3 (L^T C i \tau_2 \chi_2 L)_1' + \frac{1}{2} \lambda_3 (L^T C i \tau_2 \chi_3 L)_1''$$
where the subscripts $1, 1', 1''$ represent the transformation rules of the $(\phi^\dagger L)$ pair in the first line and of the $(L^TL)$ pair in the second line. The resulting lepton and neutrino mass matrices have the following form

\[
M_l = \begin{pmatrix}
\lambda_e v_1 & \lambda_e v_2 & \lambda_e v_3 \\
\lambda'_e v_1 & \omega \lambda'_e v_2 & \omega^2 \lambda'_e v_3 \\
\lambda''_e v_1 & \omega^2 \lambda''_e v_2 & \omega \lambda''_e v_3
\end{pmatrix},
\]

\[
M_\nu = \begin{pmatrix}
a + b + c & f & e \\
f & a + \omega^2 b + \omega c & d \\
e & d & a + \omega b + \omega^2 c
\end{pmatrix},
\]

where $v_i = \langle \phi_i \rangle$ ($i = 1, 2, 3$) and

\[
a = \lambda_1 \langle \chi_1 \rangle, \quad b = \lambda_2 \langle \chi_2 \rangle, \quad c = \lambda_3 \langle \chi_3 \rangle,
\]

\[
d = \lambda_\Delta \langle \Delta_1 \rangle, \quad e = \lambda_\Delta \langle \Delta_2 \rangle, \quad f = \lambda_\Delta \langle \Delta_3 \rangle.
\]

We assume $v_1 = v_2 = v_3$, and $d = e = f$, which can be guaranteed by a residual symmetry like $Z_3$. For simplicity we impose additional assumption: $b = c$ which naturally gives the maximal atmospheric neutrino mixing, although it is not required by any symmetry. It is straightforward to extend our analysis to the case $b \neq c$. We will consider the effect of small perturbation of $c - b$ in Section 3.2.

Then the lepton mass matrix, $M_l$, can be diagonalized by rotating the left-handed lepton by the unitary matrix

\[
U(\omega) = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}.
\]

The neutrino mass matrix is diagonalized by the transformation

\[
U_\nu^T M_\nu U_\nu = M_\nu^{\text{diag}},
\]

where $U_\nu$ is a unitary matrix which can be decomposed into three successive rotation
matrices: $U_\nu = V_{23}V_{13}V_{12}$ \cite{1}. The form of the neutrino mass matrix when $b = c$ is given by

$$M_\nu = \begin{pmatrix} a + 2b & d & d \\ d & a - b & d \\ d & d & a - b \end{pmatrix}. \quad (13)$$

The resulting MNS matrix is the product of $U(\omega)$ and $U_\nu$:

$$U^{\text{MNS}} = U(\omega)U_\nu. \quad (14)$$

Alternatively we can work in the basis where the mass matrix of the charged lepton is diagonal. In this basis the neutrino mass matrix is in the form:

$$M^\text{eff}_\nu = U(\omega)^\dagger M_\nu U(\omega) = \begin{pmatrix} a + 2d & b & b \\ b & b & a - d \\ b & a - d & b \end{pmatrix}, \quad (15)$$

such that

$$U^{\text{MNS}}_\nu M^\text{eff}_\nu U^{\text{MNS}}_\nu = M^\text{diag}_\nu. \quad (16)$$

Since the neutrino mass matrix is symmetric under the exchange of $\nu_\mu \leftrightarrow \nu_\tau$, we get a bimaximal $2 - 3$ mixing and a vanishing $1 - 3$ mixing

$$\theta_{23} = \pi/4, \quad \theta_{13} = 0. \quad (17)$$

The (15) does not automatically generate the desired mixing angle $\theta_{12}$ for the tribimaximal mixing matrix. The form of (1), however, can be achieved in a wide region of parameter space. This can be seen from the fact that the $U^{\text{MNS}}_\nu$ obtained in (16) can be decomposed in general as

$$U^{\text{MNS}}_\nu = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}) U^{\text{st}}_{\text{MNS}} = P(\beta) U^{\text{st}}_{\text{MNS}}, \quad (18)$$

where $U^{\text{st}}_{\text{MNS}}$ is the physically measurable matrix which includes the Majorana phases in the standard form \cite{1},

$$U^{\text{st}}_{\text{MNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1). \quad (19)$$
The phase matrix, $P(\beta)$, in (18) can be absorbed into the right-handed charged lepton sector, and therefore, is unphysical. However, since the $M_{\nu_{\text{eff}}}$ is complex matrix in general, the phase matrix $P(\beta)$ is generally allowed, and it makes the allowed parameter space much more wider than the real matrix case. The condition for the tribimaximal mixing including the unphysical $\beta$’s can be written as

$$d = -\frac{(e^{i\beta_1} - e^{i\beta_2})[a(e^{i\beta_1} + e^{i\beta_2}) + be^{i\beta_2}]}{2e^{2i\beta_1} + e^{2i\beta_2}},$$

(20)

where the phases $\beta_1, \beta_2$ can take arbitrary values. The $d = 0$ is a special case to give the tribimaximal mixing matrix.

We note that if we assume $\langle \Delta_{22}^0 \rangle = \langle \Delta_{33}^0 \rangle = 0$, the tribimaximal form can be obtained without further conditions among $a, b, d$. This case was considered in [6].

Our model is similar to the one considered in [4]. But their neutrino mass matrix is different from ours and can be obtained by exchanging $b \leftrightarrow d$. Phenomenologically the two are significantly different in that, for example, $d = 0$ gives the tribimaximal mixing matrix in our case, while the model in [4] cannot.

3 The numerical analysis and model predictions

3.1 The case $b = c$

As mentioned in the previous section, we get $\theta_{23} = \pi/4, \theta_{13} = 0$ in this case. More explicitly, with $U_{\text{MNS}} = U_{23}U_{13}U_{12}$, we get

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (21)

After rotating in the 23 and 13 plane, we can block-diagonalize the $M_{\nu_{\text{eff}}}$ as

$$U_{\text{MNS}}^T M_{\nu_{\text{eff}}} U_{\text{MNS}} = U_{12}^T \begin{pmatrix} a + 2d & \sqrt{2}b & 0 \\ \sqrt{2}b & a + b - d & 0 \\ 0 & 0 & -a + b + d \end{pmatrix} U_{12} = M_{\nu_{\text{diag}}}.$$  \hspace{1cm} (22)
We can read \( m_3^2 = |a^2 + b^2 + d^2| \) and obtain the remaining mass-squared eigenvalues, \( m_1^2, m_2^2 \), and the remaining mixing angle, \( \theta_{12} \), by squaring (22), i.e. \( U_{\text{MNS}}^\dagger M_\nu^\text{eff} M_\nu^\text{eff} U_{\text{MNS}} = (M_\nu^\text{diag})^2 \).

There can be spontaneous CP violation and the parameters \( a, b, d \) are in general complex numbers: 

\[
a = |a| e^{i\varphi_a}, \quad b = |b| e^{i\varphi_b}, \quad \text{and} \quad d = |d| e^{i\varphi_d}.
\]

However, one of the vev's can be made real by \( SU(2) \) rotation. So we can set \( \varphi_d \equiv 0 \) without loss of generality. Then it is straightforward to get the mixing angle \( \theta_{12} \) of \( U_{12} \) and the mass-squared eigenvalues:

\[
\begin{align*}
m_{12}^2 &= \frac{|d|^2}{2} |\Lambda \mp \Delta|, \\
m_3^2 &= |d|^2 \left( x^2 + y^2 + 1 - 2xy \cos \varphi_{ab} - 2x \cos \varphi_a + 2y \cos \varphi_b \right), \\
t_{12} &\equiv \tan \theta_{12} = \sqrt{\frac{\Delta + \Sigma}{\Delta - \Sigma}},
\end{align*}
\]

where \( x \equiv |a|/|d|, \ y \equiv |b|/|d|, \ \varphi_{ab} \equiv \varphi_a - \varphi_b \) and

\[
\begin{align*}
\Lambda &= 2x^2 + 5y^2 + 5 + 2xy \cos \varphi_{ab} + 2x \cos \varphi_a - 2y \cos \varphi_b, \\
\Sigma &= -y^2 + 3 - 2xy \cos \varphi_{ab} + 6x \cos \varphi_a + 2y \cos \varphi_b, \\
\Delta &= \sqrt{\Sigma^2 + 8(\epsilon_1^2 + \epsilon_2^2)},
\end{align*}
\]

where \( \epsilon_1 = y^2 + 2xy \cos \varphi_{ab} + y \cos \varphi_b \) and \( \epsilon_2 = 3y \sin \varphi_b \).

We can show that \( \Lambda \geq \Delta \) (the equality sign holds for \( x = 27/32, y = (91 - \sqrt{4641})/128 \), and \( \varphi_a = \varphi_b = 0 \)). So the smallness of the solar mass difference, \( \Delta m_\odot^2 = \Delta m_{21}^2 = |d|^2 \Delta \), implies \( \Delta \approx 0 \). And this leads to \( \Sigma \approx \epsilon_1 \approx \epsilon_2 \approx 0 \). From this we get a “magic relation” for \( y = \mathcal{O}(1) \):

\[
\begin{align*}
\varphi_b &\approx 0, \quad y + 2x \cos \varphi_a + 1 \approx 0, \\
or \quad \varphi_b &\approx \pi, \quad y - 2x \cos \varphi_a - 1 \approx 0.
\end{align*}
\]

We note that the above conditions give \( \Delta \approx -3\Sigma \) or \( \tan \theta_{12} = 1/\sqrt{2} \). Then we obtain the tribimaximal mixing (1) without further conditions\(^3\). Although the magic relation (25) is not guaranteed by any symmetry, it is satisfied in a large parameter space, especially when the CP violating phase \( \varphi_a \) is allowed. In this approximation we get the following

\(^3\)If (25) holds exactly, \( t_{12} \) becomes undetermined.
mass-squared differences:
\[ \Delta m^2_{21} \equiv m_2^2 - m_1^2 \approx |d|^2 \Delta, \]
\[ \Delta m^2_{32} \equiv m_3^2 - m_2^2 \approx 6|d|^2 y \text{ (for } \varphi_b = 0), \text{ or } -6|d|^2 y \text{ (for } \varphi_b = \pi). \] (26)

Here \( \Delta m^2_{21} > 0 \) by definition and it can be identified with the \( \Delta m^2_{\text{sol}} \). However, the \( \Delta m^2_{32} > 0 \) can be either positive or negative: \( \Delta m^2_{32} = \Delta m^2_{\text{atm}} > 0 \) for \( \varphi_b \approx 0 \) (normal hierarchy) and \( \Delta m^2_{31} = -\Delta m^2_{\text{atm}} < 0 \) for \( \varphi_b \approx \pi \) (inverted hierarchy).

Now we get the numerical constraints on the five parameters, \(|d|, x, y, \varphi_a \) and \( \varphi_b \), from the experimental data in Table 1. In Fig. 1, we show a scattered plot in the \((\varphi_a, \varphi_b)\) plane. The blue (orange) color represents the case for the normal (inverted) hierarchy. We can see the allowed range of \( \varphi_b \) is quite restricted and looks almost like line (its thickness is about 0.02). This shows the magic relation (25) works quite well. For the normal hierarchy \( (\varphi_b \approx 0) \) \( \varphi_a \) is restricted in the region \( (\pi/2, 3\pi/2) \), since the \( \cos \varphi_a \approx -(y + 1)/2x < 0 \). by the magic relation. For the inverted hierarchy \( (\varphi_b \approx \pi) \), since \( \cos \varphi_a \approx (y - 1)/2x \), both signs are allowed in principle. However, the region in which both \( x \) and \( y \) are small is excluded and the minimum value of \( \cos \varphi_a \) allowed by the data is about \(-0.22\).

The very restricted range of \( \varphi_b \) signifies a fine-tuning. To avoid the fine-tuning problem, we set \( \varphi_b = 0, \pi \) identically for the normal and inverted hierarchy, respectively. To simplify expressions, we allow \( y \) to take negative values from now on, understanding that \( y > 0 \) \( (y < 0) \) implies normal (inverted) hierarchy. Then the expressions for the mixing angle \( \theta_{12} \) simplify greatly and give
\[ \tan 2\theta_{12} = \pm \frac{2\sqrt{2}|y|}{y - 3} \text{ for } \text{sign}(y + 2x \cos \varphi_a + 1) = \pm. \] (27)

It is interesting to note that this expression is a function of \( y \) only and independent of \( x \) and \( \varphi_a \). The 3-\( \sigma \) allowed range of \( y \) can be read from Fig. 2 and is given by
\[ y > 12.7 \text{ for } (y + 2x \cos \varphi_a + 1) > 0 \]
\[ 1.14 < y < 1.70 \text{ for } (y + 2x \cos \varphi_a + 1) < 0 \] (28)
for normal hierarchy and
\[ y < -4.71 \text{ for } (y + 2x \cos \varphi_a + 1) < 0 \] (29)
Figure 1: Scatter plot in the $(\varphi_a, \varphi_b)$ plane. The blue (orange) color represents the case for the normal (inverted) hierarchy. The dashed black lines are $\varphi_a = \pi/2, 3\pi/2$.

for inverted hierarchy. It is impossible to satisfy the data for the solar mixing angle when $(y + 2x \cos \varphi_a + 1) > 0$.

Figure 2: Thick solid (dashed) curve: $\tan \theta_{12}$ as a function of $y$ for $y + 2x \cos \varphi_a + 1 > 0$ ($< 0$). Dashed horizontal lines: $3\sigma$ allowed range of $\tan \theta_{12}$, $0.577 < \tan \theta_{12} < 0.766$. The left (right) panel is for normal (inverted) hierarchy.

To consider the constraints from the mass-squared differences, we take the ratio
\[ |\Delta m_{32}^2|/\Delta m_{21}^2 \] because \(|d|^2\) is canceled in this case. Then Eqs. (23) and (24) give

\[ \rho \equiv \frac{\Delta m_{s0}^2}{\Delta m_{atm}^2} = \frac{\Delta m_{21}^2}{|\Delta m_{32}^2|} = \frac{|y + 2x \cos \varphi_a + 1|}{f(y)}, \] (30)

where

\[ f(y) = \frac{6|y|}{\sqrt{9 - 6y + 9y^2}}. \] (31)

Fig. 3 shows the plots of \( f(y) \).

![Figure 3: Solid (Dashed) curve: \( f(y) \) for normal (inverted) hierarchy.](image)

The experimental value of \( \rho = (3.19 \pm 0.19) \times 10^{-2} \) can be satisfied by tuning \( y + 2x \cos \varphi_a + 1 \approx 0 \).

Using the relation, \( 2x \cos \varphi_a + y + 1 \approx 0 \), we can get

\[ \frac{m_1^2}{\Delta m_{atm}^2} \simeq \frac{1}{6|y|} \left( \frac{y + 1}{2 \cos \varphi_a} \right)^2 + 2(y^2 - y + 1), \] (32)

in the leading order in \( \rho \). The minimum of \( m_1 \) is obtained when \( \cos^2 \varphi_a = 1 \) for both normal and inverted hierarchy. Fig. 4 shows \( (m_1)_{\text{min}}/\sqrt{\Delta m_{atm}^2} \), that is, the \( m_1/\sqrt{\Delta m_{atm}^2} \) as a function of \( y \) when \( \cos^2 \varphi_a = 1 \). It shows that there is a lower bound on the lightest neutrino mass, \( m_{1(3)} \gtrsim 0.03(0.05) \text{ eV} \) for the normal (inverted) hierarchy.
Figure 4: The mass ratio, \((m_1)_{\text{min}}/\sqrt{\Delta m_{\text{atm}}^2}\), as a function of \(y\) when \(\cos^2 \varphi_a = 1\). The thick solid curves are allowed by the current experimental data.

Now let us consider the bound on the effective Majorana mass for neutrinoless double-beta decay [4]. The amplitude of the neutrinoless double beta decay \(0\nu\beta\beta\) is proportional to the effective Majorana mass for the \(0\nu\beta\beta\) defined by

\[
|\langle m_{\beta\beta} \rangle| \equiv \left| \sum_i m_i (U_{\text{MNS}}^{*})_{ii}^2 \right| = |(M_{\nu})_{11}^\text{eff}| = |a + 2d| = |dx e^{i \varphi_a} + 2|.
\]  

(33)

Similar to (32), we get

\[
\frac{|\langle m_{\beta\beta} \rangle|^2}{\Delta m_{\text{atm}}^2} \simeq \frac{1}{6|y|} \left( \frac{y + 1}{2 \cos \varphi_a} \right)^2 - 2(y - 1),
\]

(34)

in the leading approximation in \(\rho\). The ratio \(|\langle m_{\beta\beta} \rangle|/\sqrt{\Delta m_{\text{atm}}^2}\) has lower bounds for the allowed range in \(y\) (thick parts in the solid blue curve in Fig. 5). Numerically we get

\[
|\langle m_{\beta\beta} \rangle|/\sqrt{\Delta m_{\text{atm}}^2} > 0.2 \quad \text{(for } 1.13 < y < 1.7), \quad > 0.57 \quad \text{(for } y > 13), \\
|\langle m_{\beta\beta} \rangle|/\sqrt{\Delta m_{\text{atm}}^2} > 0.72 \quad \text{(for } y < -4.5). \]

(35)

The model with \(A_4\) symmetry which has non-vanishing lower bound for \(|\langle m_{\beta\beta} \rangle|/\sqrt{\Delta m_{\text{atm}}^2}\) even in the case of normal hierarchy was considered in [4]. Numerically we get similar values to theirs.
Figure 5: The mass ratio, \((m_{\beta\beta})_{\text{min}}/\sqrt{\Delta m_{\text{atm}}^2}\), as a function of \(y\) when \(\cos^2 \varphi_a = 1\). The blue (red, orange) curve corresponds to the case \(b = c\) (\(\xi = 0.3, \xi = -0.3\)). The thick parts are allowed by the current experimental data.

### 3.2 The case \(b \neq c\)

As mentioned in Section 2, the case \(b = c\) is not guaranteed by any symmetry. In this section we extend to the case \(b \neq c\). The effective neutrino mass matrix given in (15) is now in the form:

\[
M_{\nu}^{\text{eff}} = U(\omega)^* M_\nu U(\omega)^\dagger = \begin{pmatrix}
    a + 2d & c & b \\
    c & b & a - d \\
    b & a - d & c
\end{pmatrix}.
\]  
(36)

Since \(b = c\) case can already give the tribimaximal mixing matrix, we can see \(b\) and \(c\) cannot be so different. Therefore, we can apply the time-independent perturbation theory to diagonalize the neutrino mass matrix and expand in powers of \(\xi \equiv (c - b)/2b\). Since \(b\) is assumed to be real, we also assume \(\xi\) to be real.

Since the 1st and 2nd eigenvalues giving the \(\Delta m_{\text{sol}}^2\) are quasi-degenerate, the blind application of the perturbation formula gives unreasonable results. To evade this problem
we diagonalized the 2 × 2 sub-matrix exactly. And then we applied the perturbation formula in the basis where the first two mass-squared eigenvalues are diagonal.

The solar mass-squared difference is obtained to be

\[ \Delta m_{\text{sol}}^2 \simeq d^2 |2x \cos \varphi_a + y + 1 + \xi y| \sqrt{9y^2 - 6y + 9}. \] (37)

This implies the “magic relation” corresponding to (25) is simply replaced by

\[ 2x \cos \varphi_a + y + 1 + \xi y \approx 0. \] (38)

The atmospheric mass-squared difference is

\[ \Delta m_{\text{atm}}^2 \simeq 6d^2 |y|(1 + \xi). \] (39)

Since \( \xi \ll 1, y > 0 \) \((y < 0)\) still gives normal (inverted) hierarchy.

The correction in \( \xi \) to the effective mass for the 0νββ, (34), is given by

\[ \frac{|\langle m_{\beta\beta} \rangle|^2}{\Delta m_{\text{atm}}^2} \simeq \frac{1}{6|y|(1 + \xi)} \left( \frac{y + 1}{2 \cos \varphi_a} \right)^2 - 2(y - 1) + 2y\xi \frac{y + 1}{(2 \cos \varphi_a)^2}. \] (40)

The minimum values in \( \varphi_a \) are obtained for \( \varphi_a = \pi(0) \) for normal (inverted) hierarchy. These are plotted in Fig. 5 as a function of \( y \) for different values of \( \xi = 0, \pm 0.3 \). The allowed regions are drawn in thick lines. The value \(|\xi| \simeq 0.3\) is almost maximum allowed by the 3-σ range in \( s_{13} \) (see Figs. 6,7). Numerically the minimum values for the normal hierarchy are given by

\[ \left( \frac{|\langle m_{\beta\beta} \rangle|}{\sqrt{\Delta m_{\text{atm}}^2}} \right)_{\text{min}} = 0.1, 0.2, 0.34 \text{ (for } \xi = 0.3, 0, -0.3, \text{ resp.)}. \] (41)

Since \( \rho \ll 0.1 \), the correction to (40) in \( \rho \) does not change the results much.

For the mixing angle \( \theta_{12} \) we get a formula similar to (27),

\[ \tan 2\theta_{12} \simeq \pm \frac{2\sqrt{2}|y|}{y - 3} \text{ for } \text{sign}(y + 2x \cos \varphi_a + 1 + \xi y) = \pm. \] (42)

The most significant change from the case \( b = c \) is that the non-vanishing \( \theta_{13} \) and consequently \( \delta \) for non-trivial \( \varphi_a \) is allowed in \( b \neq c \) case. The expression for \( s_{13} \equiv \sin \theta_{13} \) is obtained by

\[ s_{13} \simeq \frac{|\xi|}{\sqrt{2}} \sqrt{1 + \left( \frac{(y + 1 + \xi y) \tan \varphi_a}{3} \right)^2}. \] (43)
The CP violating phase $\delta$ is given by

$$
\delta \simeq \tan^{-1} \left( \frac{(y + 1 + \xi y) \tan \varphi_a}{3} \right) \pmod{\pi}.
$$

(44)

We do not have a definite prediction for $s_{13}$. But the 3-$\sigma$ range in Table 1 can be accommodated. Fig. 6 (7) shows contours for the constant $s_{13}$ and $\delta$ for the normal (inverted) hierarchy case. We can see the 3-$\sigma$ range for the $s_{13}$ can be accommodated in the perturbative region for $\xi$. All the possible values of $\delta$ are allowed by the current experimental values, although relatively small values of $\delta$ are preferred.

A very interesting prediction is for $\theta_{23}$. Up to the first order in $\xi$, the prediction for $\theta_{23}$ is still $\pi/4$. The first correction appears in second order in $\xi$:

$$
\sin^2 \theta_{23} = 0.5 + O(\xi^2) \approx 0.5 \pm 0.01,
$$

(45)

for the 1-$\sigma$ allowed $\xi$ ($\xi \lesssim 0.1$) by $s_{13}$. Therefore if the experiments would confirm significant deviation from $\pi/4$ for $\theta_{23}$, our model would be ruled out.

Figure 6: Solid (dashed) curves: contours for the constant $s_{13}$ ($\delta$ (in degrees) ) for the normal hierarchy. The inner (outer) solid lines are contours for the 3-(1-)\(\sigma\) values of $s_{13}$. We take $y = 13$ ($y = 1.5$) for the left (right) panel.
Figure 7: The same with Fig. 6. But we take $y = -5$.

4 Conclusions

We studied a triplet Higgs model to generate Majorana neutrino masses and the mixing matrix in the framework of $A_4$ symmetry. With the assignments of $A_4$ representations given in Table 2, we see that

- The tribimaximal form of the neutrino mixing matrix can be naturally obtained for $b = c$.
- There is a lower bound on the lightest neutrino mass: $m_1 \gtrsim 0.03(0.05)$ eV for the normal (inverted) hierarchy.
- There is a lower bound on the effective mass for the neutrinoless double beta decay: $|\langle m_{\beta\beta} \rangle|/\sqrt{\Delta m^2_{\text{atm}}} > 0.2(0.57)$ for the normal (inverted) hierarchy.
- For $b \neq c$ case, we can accommodate the data for the $\theta_{13}$.
- Even for $b \neq c$ case, the prediction for the atmospheric mixing angle does not change much from $\theta_{23} = \pi/4$ and gives $\sin^2 \theta_{23} \approx 0.5 \pm 0.01$, which can be tested in near future.
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