Interference effects in isolated Josephson junction arrays with geometric symmetries

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As the size of a Josephson junction is reduced, charging effects become important and the superconducting phase across the link turns into a periodic quantum variable. Isolated Josephson junction arrays are described in terms of such periodic quantum variables and thus exhibit pronounced quantum interference effects arising from paths with different winding numbers (Aharonov–Casher effects). These interference effects have strong implications for the excitation spectrum of the array which are relevant in applications of superconducting junction arrays for quantum computing. The interference effects are most pronounced in arrays composed of identical junctions and possessing geometric symmetries; they may be controlled by either external gate potentials or by adding/removing charge to/from the array. Here we consider a loop of $N$ identical junctions encircling one half superconducting quantum of magnetic flux. In this system, the ground state is found to be non-degenerate if the total number of Cooper pairs on the array is divisible by $N$, and doubly degenerate otherwise (after the stray charges are compensated by the gate voltages).

\section{1. Introduction}

Josephson junction arrays are excellent tools for exploring quantum mechanical behavior in a wide range of parameter space. The charge and phase on each island provide a set of conjugate quantum variables, allowing for dual descriptions of the array either in terms of charges (Cooper pairs) hopping between the islands or in terms of vortices hopping between the plaquettes of the array. This opens the door for the setup and manipulation of interesting quantum interference effects in Josephson junction arrays: in a magnetic field, charges pick up additional Aharonov–Bohm phases and hence the properties of the array depend on the field strength. In the dual language, vortices moving around islands gain phases proportional to the average charges on the islands (Aharonov–Casher phases). These features have recently been used in various proposals for solid state implementations of qubits for quantum computing, based on either the charged or phase degree of freedom in Josephson junction arrays.

In this paper we study an interference effect in electrically isolated Josephson junction arrays which renders the ground state and excitation spectrum sensitive to the total charge on the array. This effect combines the dual descriptions in terms of charge or phase: \textit{i)} in the limit where the charging energy $E_C$ is much larger than the Josephson energy $E_J$, the fluctuations of charge on the islands are small and the system is equivalent to strongly repulsive bosons (Cooper pairs) hopping between islands; the total charge then determines the number of such bosons and hence the structure of the spectrum. \textit{ii)} In the opposite limit of large Josephson energy $E_J$, the phase fluctuations are small and the spectrum is determined by phase tunneling between classical minima of the Josephson energy; the total charge on the array then determines the interference between different tunneling trajectories via the Aharonov–Casher effect. These interference effects, where the ground state and the excitations may change degeneracies depending on the total charge on the array, are most pronounced in small arrays with geometric symmetries; for the symmetric loops considered in this paper they coincide in the charge- and phase-dominated limits and persist at arbitrary ratio of the Josephson- to charging energy. In mathematical terms, the total charge on the array enters its symmetry group producing a central extension. The irreducible representations are then classified by the total charge, and their dimensions (and associated level degeneracies) depend on the charge on the array.

In experimental realizations of Josephson junction arrays, the charges and potentials on the islands are affected by a number of external factors such as differences

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{A superconducting phase qubit consists of a small inductance loop made from $N$ superconducting islands connected by Josephson junctions and capacitively coupled to each other and to the ground (the case $N=3$ is shown here). The loop is placed in an external magnetic field producing one half superconducting flux quantum through the loop.}
\end{figure}
in island size, background charges, charges localized at
impurities, etc. All these factors produce random charge
offsets on the islands which distort the quantum inter-
ference pattern (see e.g. the discussion of offset charges
in Ref. [1]). In order to observe the quantum inter-
ference effects predicted in this paper the potentials of the
islands should be adjustable via external capacitive gates.
The offset charges should then be compensated by ap-
propriate tuning of the gate voltages; such a procedure
was successfully carried out in a transport measurement
involving two islands [1] and should also be possible for
the system considered in this paper. We also assume the
absence of quasiparticles on the islands. Firstly, this re-
quires the temperature to be sufficiently low to eliminate
thermal quasiparticles. Secondly, the total number of
electrons on the array must be even in order to exclude
unpaired charge due to the parity effect.

In the following we study the simplest setup exhibiting
these interference effects, a closed low-inductance loop
with \( N \geq 3 \) identical Josephson junctions and pierced by
one-half superconducting flux quantum (Fig. 1). Such
loops have been proposed as possible realizations for
superconducting phase quantum bits [8] and a successful
quantum superposition of the two qubit states has been
reported recently [9]. While in the original qubit design
the loop has been made asymmetric in order to suppress
the interference of different tunneling paths, it is this
interference that we study in the present paper where
we assume the islands and contacts to be nearly iden-
tical (the required precision will be specified at the end
of the paper). The ground state degeneracy of such a
symmetric Josephson-junction loop is found to depend
periodically on the total charge \( Q \) with period \( 2eN \), \( N \)
is the number of islands in the loop. In particular, in
the phase-dominated regime \( E_J \gg E_C \) the Josephson
energy of the fully frustrated loop exhibits two equiva-
lent minima describing states with currents circulating
in opposite directions. Tunneling between these minima
produces a splitting \( \Delta \) between these levels which is
determined by the charges \( q_i \) (in units of \( 2e \)) induced on
the islands,

\[
\Delta = \Delta_0 \left| 1 + e^{2\pi i q_1} + e^{2\pi i (q_1+q_2)} + \ldots + e^{2\pi i (q_1+\ldots+q_{N-1})} \right|, \tag{1}
\]

and which vanishes for \( (Q/2e) \neq 0 \pmod{N} \) (the sum of
all induced charges \( q_i \) equals \( Q \), and the expression
is symmetric under circular permutation of the islands).
This result is the consequence of the interference of dif-
f erent tunneling paths connecting the two minima where
the relative phases of the tunneling amplitudes depend
on the charges \( q_i \) (a similar effect was predicted for the
S-S-S double junction in Ref. [10]). The induced charges
\( q_i \) may be tuned by either gate voltages (redistributing
the existing charge) or by adding/removing extra charge
to/from the array. E.g. suppose that all charges \( q_i \) have
been set to zero by fine tuning the gate voltages, thus
maximizing the splitting \( \Delta \). Then adding (or remov-
ing) a charge \( Q \) at fixed relative potentials adds a value
\( Q/2eN \) to each induced charge \( q_i \) and closes the gap \( \Delta \):
the splitting is present when the charge \( Q/2e \) is divis-
ible by \( N \) and absent otherwise. A similar result applies
for the opposite limit \( E_C \gg E_J \) where the ground state
is non-degenerate with the next excited state an energy
\( E_C \) away if \( Q/2e \) is divisible by \( N \) (‘insulating’ state),
while it is degenerate with an excitation gap of order \( E_J \)
if \( Q/2e \neq 0 \pmod{N} \) (‘metallic’ state). Indeed, this
\( Q \)-dependence of the ground state degeneracy will be ex-
plained using symmetry considerations valid in the entire
range of couplings \( E_J/E_C \).

The paper is organized as follows. In Section II, we
define the model and prove the periodic dependence of
the excitation spectrum on the total charge \( Q \), for a sym-
metric loop. In Sections III and IV, we treat the limits
\( E_J \gg E_C \) and \( E_J \ll E_C \), respectively. Section V con-
tains the analysis of the symmetries of the loop. Finally,
in Section VI we discuss the physical requirements for
observing the charge-dependent interference effects.

II. MODEL AND PERIODIC Q-DEPENDENCE
OF THE SPECTRUM

The Hamiltonian describing the qubit in Fig. 1 takes
the form

\[
H = \frac{1}{2} \sum_{ij} Q_i (C^{-1})_{ij} Q_j + \sum_i V_i Q_i + \sum_i U_i (\varphi_{i+1} - \varphi_i - a_{i,i+1}). \tag{2}
\]

Here, \( \varphi_i \) are the phases on the islands, \( Q_i \equiv -i\partial/\partial \varphi_i \)
s are the charge operators conjugate to \( \varphi_i \) (we measure \( Q \)
in units of \( 2e \) from now on), \( (C^{-1})_{ij} \) is the inverse
 capacitance matrix, \( V_i \) are the gate voltages applied to the
islands, \( U_i \) are the Josephson energies of the junctions,
and \( a_{i,i+1} \) is the electromagnetic vector potential induced
by the external magnetic field \( i+1 \) in the indices should
be understood modulo \( N \). This Hamiltonian acts on the
wave functions \( \Psi(\varphi_1, \ldots, \varphi_N) \) which are periodic in
all their variables,

\[
\Psi(\varphi_1, \ldots, \varphi_i + 2\pi, \ldots, \varphi_N) = \Psi(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_N). \tag{3}
\]

In general, the boundary conditions (3) may contain ar-
bbitary phase shifts \( e^{im} \) incorporating the effect of back-
ground charges. They can be manipulated by the gate
voltages \( V_i \) and we assume them to vanish through ap-
propriate fine tuning \( E_C \) (this is equivalent to adjusting
the zero positions of the gate voltages). We assume that
this tuning is performed once in the beginning of the ex-
periment and later measure the gate voltages relative to
these reference values.
Since the potential term in the Hamiltonian \( H \) contains only phase differences \( \phi_i - \phi_j \), it has the symmetry of rotating all the phases by the same angle: 
\[ \Psi(\phi_1, \ldots, \phi_N) \mapsto \Psi(\phi_1 + \delta \phi, \ldots, \phi_N + \delta \phi). \]
Equivalently, the total charge \( Q = \sum_i Q_i \) is conserved (i.e., the charge \( Q \) commutes with the Hamiltonian \( H \)) and we may project the Hilbert space onto the subspace with a given total charge \( Q \) before diagonalization, implying the transformation rule
\[
\Psi(\phi_1 + \delta \phi, \ldots, \phi_N + \delta \phi) = e^{i \delta \phi Q} \Psi(\phi_1, \ldots, \phi_N) \quad (4)
\]
for the simultaneous rotation of all the phases by \( \delta \phi \).

From the periodicity of \( \Psi \), it immediately follows that \( Q \) is integer, i.e., the total charge must be a multiple of \( 2e \) (this is in fact an implicit assumption when writing the Hamiltonian \( H \) in terms of phases only). In the following we shall discuss the symmetric loop, postponing the effects of asymmetry till the end of the paper. Here, by symmetry we mean \( (C^{-1})_{ij} = (C^{-1})_{i+k,j+k} \) for the Coulomb term and equality of all Josephson terms \( U_i(\phi) \).

This implies that the loop is invariant under circular permutation of the islands. Also, if the flux through the loop is exactly one half flux quantum, the loop is invariant under ‘flips’ changing the direction of the current.

The excitation spectrum of the symmetric loop periodically depends on the total charge \( Q \) with the period \( N \) (up to overall shifts): the unitary operator
\[
U : \Psi \mapsto \Psi e^{iQ(\phi_1 + \ldots + \phi_N)} \quad (5)
\]
increases the charge on all the islands by one, \( U^{-1}Q_i U = Q_i + 1 \), and therefore the sector with total charge \( Q \) maps onto the one with total charge \( Q + N \). On the other hand,
\[
U^{-1}HU = H + \frac{1}{2} \sum_{ij} (C^{-1})_{ij} Q_j + \sum_i V_i.
\]

If all the islands are equivalent, then the sum \( \sum_i (C^{-1})_{ij} \) is independent of \( j \) and
\[
U^{-1}HU = H + \frac{N}{C} \left( Q + \frac{N}{2} \right) + \sum_i V_i, \quad (6)
\]
where \( C = N \left[ \sum_i (C^{-1})_{ij} \right]^{-1} = \sum_{ij} C_{ij} \) is the total capacitance of the loop. Thus the operator \( U \) maps the eigenfunctions of the Hamiltonian in the sector with charge \( Q \) onto eigenfunctions in the sector with charge \( Q + N \) shifting them in energy by a constant as given by (5), thus proving our statement about the periodic \( Q \)-dependence of the excitation spectrum.

**III. LEVEL SPLITTING IN THE \( E_J \gg E_C \) LIMIT**

In the phase-dominated regime with \( E_J \gg E_C \) the low-energy states of the Josephson junction loop are determined by the classical minima of the Josephson energy as corrected by weak quantum tunneling (due to the finite charging energy). Here, we consider a symmetric loop pierced by half a quantum of magnetic flux, \( \sum_i a_{i,i+1} = \pi \), and we choose to work in a gauge with \( a_{i,i+1} = \pi/N \) for all \( i \). In the limit \( E_J \gg E_C \) the only constraint on the potential \( U_i(\phi) \) is the double degeneracy of the total Josephson energy as a function of the phases \( \phi_i \) (e.g., this requirement is satisfied for tunneling junctions with \( U_i(\phi) = -E_J \cos \phi \) and \( N \geq 3 \)). The two potential minima are determined by the phase configurations \( \phi_i = 0 \) and \( \phi_i = (2\pi/N)i \) (and all configurations obtained from these two by the continuous symmetry \( \phi_i \mapsto \phi_i + \delta \phi \)). These two minima involve different directions of the Josephson current, circulating the loop clockwise or counter-clockwise. The continuous symmetry \( \phi_i \mapsto \phi_i + \delta \phi, \quad i = 1, \ldots, N \) implies the quantization of the total charge \( Q \) as discussed above.

In the following it will be convenient to incorporate the voltage terms \( V_i Q_i \) into the quadratic kinetic term via a shift \( Q_i \mapsto Q_i - q_i \), where
\[
q_i = \frac{C_{ij}}{N} \left( V_i - \frac{\sum_k V_k}{N} \right) + \frac{Q}{N} \quad (7)
\]
are the mean charges induced on the islands (we also include the \( i \)-independent contribution from the total charge \( Q \) for further convenience). This gauge transformation eliminates the \( V_i Q_i \) terms in the Hamiltonian.
ables are corresponding to the other minimum. Accordingly, there along each of these trajectories shows that they may additionally follow the result (1) for the level splitting (the level splitting in the double-well problem is proportional to the absolute value of the tunneling amplitude).

IV. LEVEL SPLITTING IN THE $E_J \ll E_C$ LIMIT

In the charge-dominated limit $E_J \ll E_C$, it is convenient to work in the charge representation where the operators $Q_i$ are diagonal and the Josephson term in the Hamiltonian (4) takes the form (we restrict the discussion to tunneling junctions with $U_i(\varphi) = -E_J \cos(\varphi)$)

$$H_J = -\frac{E_J}{2} \sum_i \left( L_i^+ L_{i+1} e^{i\pi/n} + h.c. \right),$$

with $L_i^+$ ($L_i^-$) the charge raising (lowering) operators on the $i$-th island, $L_i^\dagger Q_j |\{Q_i = \pm 1\}$. The periodicity in $Q$ allows us to restrict our analysis to the $N$ charge sectors $Q = 0, \ldots, N-1$. Below we first neglect the coupling $E_J$ and find the ground states of the Coulomb part of the Hamiltonian; hopping between the islands is then perturbatively included in a second step.

We start with a diagonal matrix $(C^{-1})_{ij}$ and ignore the capacitive coupling of the junctions, i.e., the islands are coupled only to the ground but not to each other. In that case, the ground state of the Coulomb part of the Hamiltonian is $C^2_N$-fold degenerate ($C^2_N = N!/Q!(N-Q)!$ enumerates the number of ways to distribute $Q$ particles among $N$ sites without double occupancy). Second, we solve the Hamiltonian (11) projected onto this $C^2_N$-dimensional subspace in order to find the level splitting at finite $E_J$. This is easily done by observing the equivalence of this problem to the tight-binding model for hardcore bosons on the circle with $N$ sites. Mapping to free

![Fig. 3. The tight-binding spectrum on the circle with periodic (a) and anti-periodic (b) boundary conditions applying for even and odd $Q$, respectively. The solid circles denote filled states at the bottom of the band. For $Q$ not divisible by $N$, the ground state is doubly degenerate. $Q$ divisible by $N$ corresponds to an empty (or filled) band.](image-url)
fermions and taking into account the flux through the
loop and the boundary conditions on the circle, one finds
that the projected $H_J$ describes a tight binding model
for $Q$ free fermions on a circle with $N$ sites and peri-
dodic boundary conditions for even $Q$, while anti-periodic
boundary conditions apply for odd $Q$ (Fig. 3). This
implies that the ground state is non-degenerate with a gap
of order $E_C$ if $Q$ is divisible by $N$ (the ‘insulating state’
with an empty band) and doubly degenerate (with the
lowest excitation at an energy of the order of the intra-
band level spacing $E_J/N$ above the degenerate ground
state) otherwise.

Turning on the junction capacitance makes the parti-
cles (Cooper pairs) repel each other and they tend to ar-
range in configurations with maximal separation between
them. The number of such configurations is generally less
(or equal) than $C_N^Q$. We conjecture that even in this case
the ground state level is degenerate unless $Q$ is divisible
by $N$. We do not have a rigorous proof of this statement
but have verified it for $N = 3,4,5$ (in fact, for $N = 3$
the off-diagonal elements of the matrix $(C^{-1})_{ij}$ may be
incorporated in a constant term $\propto Q^2$ and do not change
the properties of the system). The degeneracy may, of
course, be explicitly verified for any given $N$ and $Q$.

V. SYMMETRY ANALYSIS

In the previous sections, we have seen that the ground-
state degeneracy depends on the divisibility of the total
charge $Q$ by the number of islands $N$ and coincides in
the two limits $E_J \ll E_C$ and $E_J \gg E_C$. In this section
we show that this degeneracy is a consequence of the
geometric symmetries of the loop and remains exact be-
yond the perturbation theory around these limiting cases.
We shall classify the irreducible representations of the
symmetry group of the loop, and identify the representa-
tions corresponding to the ground state. In both limits,
the ground states correspond to the same representation,
which implies that the degeneracy is also preserved in
the intermediate parameter range, unless level-crossing
occurs. We verify the absence of level-crossing in the
several simplest cases numerically by exact diagonaliza-
tion.

The level degeneracies for general values of $E_J/E_C$ are
determined by the symmetry group of the Josephson-
junction loop. The geometric symmetries are described
by the dihedral group $D_N$ consisting of cyclic permuta-
tions $T$ of the islands (preserving the loop orientation;
these are equivalent to a $N$-fold rotation axis) and of re-
fections $R$ about diameters of the loop drawn through
islands or through junctions, see Fig. 4. The reflections $R$
reverse the flux through the loop and belong to its sym-
metry group for flux zero or half-quantum. The dihedral
group $D_N$ involves $2N$ symmetry operations; it may be
characterized with the defining relations

\begin{align}
T^N &= 1, \\
R^2 &= 1, \\
(TR)^2 &= 1.
\end{align}

These relations are obeyed by the operators represent-
ing $R$ and $T$ at zero magnetic flux through the loop.
On the other hand, for the case of half a flux quan-
tum piercing the loop the symmetry operators preserving
the Hamiltonian need to be supplemented by additional
gauge transformations. As a consequence, the above re-
lations are modified, with the appearance of additional
phase shifts. Explicitly, the operators corresponding to
rotations $T$ and reflections $R$ have the form

\begin{align}
T\psi(\varphi_1, \ldots, \varphi_N) &= \psi(\varphi_2, \ldots, \varphi_N, \varphi_1), \\
R\psi(\varphi_1, \ldots, \varphi_N) &= \psi(\varphi_N, \varphi_{N-1} + 2\pi/N, \ldots, \\
& \quad \varphi_1 + 2\pi(N-1)/N),
\end{align}

where the shifts $(2\pi/N)k$ of the phase $\varphi_{N-k}$ compensate
for the external vector potential in the Hamiltonian (12)
and guarantee its invariance. This modifies the relation
(13), producing an additional phase shift:

\begin{equation}
R^2 = \exp(-2\pi i Q/N).
\end{equation}

The simplest way to make use of symmetry arguments is
for the case $N$ even and $Q$ odd: we show that all states
are degenerate in this case. Let us assume the opposite,
i.e., that the eigenstate $|\psi\rangle$ is non-degenerate. Acting
with $T$ and $R$ on $|\psi\rangle$ we reproduce the state up to phases $t$
and $r$ which satisfy the relations $t^N = (tr)^2 = 1$ and $r^2 = 
\exp(-2\pi i Q/N)$; this set of equations is inconsistent for
$N$ even and $Q$ odd, hence all levels indeed are degenerate.

To determine the level degeneracy and to treat the case
of arbitrary $N$ and $Q$, we classify the representations of
the symmetry group. To take into account the phase
shift in (17), we include such phase shifts as new central
elements $\{Z^N|_{n=1}\}$ in the group. The resulting set of
defining relations is

\begin{align}
Z^N &= T^N = (TR)^2 = 1, \\
R^2 &= Z^{-1}, \\
ZR &= RZ, \\
ZT &= TZ
\end{align}
Indeed, the overall phase of the wave function has no representation of the symmetry group, but a more general

$D^{(\omega)}$ involve the roots $t_\mu = \exp(2\pi i\mu/N)$, with $\mu = 1, \ldots, (N-1)/2$.

$$Z = \exp(2\pi iQ/N)$$

This formula establishes the relation between the representations of $ZD_N$ and the total charge $Q$ on the array.

Here we should mention that central extensions of symmetry groups are common in quantum mechanics. Indeed, the overall phase of the wave function has no physical meaning. Therefore the operator representing the product of two symmetry operations must equal the product of the two operators representing each of these operations only up to an overall phase factor. In other words, quantum mechanics admits not only linear representations of the symmetry group, but a more general class of projective representations. At the same time any projective representation of a group corresponds to a linear representation of a central extension of this group. Physical examples are numerous, including half-integer spin (projectively representing the rotation group), magnetic translations (projectively representing geometric translations), and anyons (projectively representing the braid group). An example resembling the analysis in the present paper is given by Kalatsky and Pokrovsky, in their discussion of the spectrum of large spins in external crystal electric fields; the degeneracies then are described in terms of projective representations of a finite symmetry group which depend on the spin assuming integer or half-integer values.

In the following we first review the irreducible representations of $D_N$ and then discuss how it is modified when $D_N$ is extended to $ZD_N$.

Consider first the case with an odd number of islands $N$. Using the defining relations (12)–(14) we arrange the $2N$ elements of the group $D_N$ into the $(N + 3)/2$ conjugacy classes: $\{E\}, \{T^n, T^{-n}\}_{n=1}^{(N-1)/2}$, and $\{T^n R_{1m=0}^{N-1}\}$. Firstly, we can construct two one-dimensional representations $D^{(\omega)}$ where all operators are represented by numbers. In these representations, the value of $T$ is determined uniquely: $D_T^{(1)} = 1$, while $R$ may take two values: $D_R^{(1)} = \pm 1$. In addition, we can find $(N-1)/2$ representations $D^{(\omega)}$ with dimensionality two. The explicit form of $T$ and $R$ in these representations is:

$$D_T^{(\omega)} = \begin{pmatrix} t_\mu & 0 \\ 0 & t_\mu^{-1} \end{pmatrix}, \quad D_R^{(\omega)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (20)$$
where the parameter $\mu$ labeling representations takes integer values $1, \ldots, (N-1)/2$, and $t_\mu = \exp(2\pi i \mu/N)$. We thus have found all the $(N+3)/2$ irreducible representations; they are listed in the character TABLE I.

The irreducible representations of the extended dihedral group $ZD_N$ are found in an analogous way. Assume $N$ odd first. Then $ZD_N$ contains $N(N+3)/2$ conjugacy classes: $\{Z^{n_1}T^{m_1}, Z^{n_2}T^{m_2} \ldots \} n_1, m_1 = 1, \ldots, N/2$ and $\{Z^{n_1}T^{m_1}R^{n_1}R^{m_1} \ldots \} n_1, m_1 = 1, \ldots, N/2$. The construction of the irreducible representations again follows the scheme described above: for one-dimensional representations $D^{(\pm Q)}$ we find 2$N$ solutions: $D^T_{\pm Q} = \sqrt{Z}$ and $D^R_{\pm Q} = \pm 1/\sqrt{Z}$, where we choose that branch of the square root which puts $\sqrt{Z}$ onto one of the roots $\exp(2\pi i n/N)$ for odd $N$ either $\sqrt{Z}$ or $-\sqrt{Z}$ belongs to the set $\{\exp(2\pi i n/N)\}_{n=1}^{N-1}$, see Fig. 3. The remaining two-dimensional representations $D^{(\mu Q)}$ may again be constructed explicitly:

$$D_T^{(\mu Q)} = \sqrt{Z} \begin{pmatrix} t_\mu & 0 \\ 0 & t_\mu^{-1} \end{pmatrix}, \quad D_R^{(\mu Q)} = \begin{pmatrix} 0 & 1 \\ Z^{-1} & 0 \end{pmatrix},$$

where for $\sqrt{Z}$ we again take one of the roots $\exp(2\pi i n/N)$, the parameter $\mu$ takes integer values $1, \ldots, (N-1)/2$, and $t_\mu = \exp(2\pi i \mu/N)$. With $2N$ one-dimensional and $N(N-1)/2$ two-dimensional representations we then have constructed all irreducible representations of $ZD_N$. The result is summarized in TABLE II.

A similar analysis for the qubit loop with an even number $N$ of junctions produces the character TABLES III and IV for the groups $D_N$ and $ZD_N$ respectively.

In summary, the energy levels of the Josephson junction loop may be classified according to the irreducible representations of its symmetry group $ZD_N$ at a given $Q$. The representation tables (Tables II and IV) look slightly different for odd and even $N$. All representations are either one- or two-dimensional. Given an odd $N$, for any value of $Q$, there are two one-dimensional representations and $(N-1)/2$ two-dimensional representations. Given an even $N$, for even $Q$ we have four one-dimensional and $(N-2)/2$ two-dimensional representations, while for odd $Q$ there are no one-dimensional and $N/2$ two-dimensional representations. The representations may be distinguished by the eigenvalues of the operator $T$ involved, see Fig. 3.

Connecting back to our physical problem we can find the degeneracy of the states: for example, when $N$ is even and $Q$ is odd (see Fig. 3(c)) all states (including the ground state) are doubly degenerate. More work is needed to determine whether the ground state is degenerate in the other cases. Inspection of the ground state wave functions in the limits $E_J \gg E_C$ and $E_J \ll E_C$ shows that they transform with a representation involving the eigenvalues $T = 1$ and $T = \exp(2\pi i Q/N)$. Indeed, in the $E_J \gg E_C$ limit, the state $\{\phi_i = -\pi/N \}$ is a $T = 1$ eigenstate, while the state represented by the points $\{\phi_i = \pi/N, i \neq k; \phi_k = \pi/N - 2\pi \}$, $k = 1, \ldots, N$ by virtue of (20) belongs to the eigenvalue $T = \exp(2\pi i Q/N)$. In the limit $E_J \ll E_C$, $T$ measures the total momentum of the repulsive bosons which (for a diagonal capacitance matrix) equals the total momentum of the tight-binding fermions. The latter is easily read off Fig. 3 and accounting for the shift $\pi Q/N$ due to the boundary condition (originating from the half-quantum flux) we again obtain the two eigenvalues $T = 1$ and $T = \exp(2\pi i Q/N)$ for the two lowest states. Thus if $Q \neq 0 \pmod{N}$ the two lowest states combine into a
two-dimensional representation (and the same in both limits), resulting in a doubly degenerate ground state. For $Q$ divisible by $N$, the ground state corresponds to a one-dimensional representation of $ZD_N$, i.e., it is non-degenerate. As the ratio $E_J/E_C$ is swept from one limit to the other, the ground state continuously evolves preserving its degeneracy, unless a level-crossing occurs with a level of different symmetry. We have confirmed numerically that such a level crossing does not occur for the cases $N = 3$ and $N = 4$ using a diagonal capacitance matrix $(C^{-1})_{ij} = E_C\delta_{ij}$, and vanishing gate voltages $V_i = 0$. The ground-state energy at each value of $E_C/E_J$ is subtracted. (a) in the $Q = 0$ sector where the ground state is non-degenerate; (b) in the $Q = 1$ sector with the ground state doubly degenerate. Note the large (small) gap of order $E_C$ ($E_J/N$) for the insulating state with $Q = 0$ (the metallic state with $Q = 1$) in the limit $E_C/E_J \rightarrow \infty$.

VI. REQUIREMENTS FOR INTERFERENCE OBSERVATION

The experimental observation of the quantum interference effect discussed in this paper relies on several requirements on the Josephson junction loop. First, we have to assume that all quasi-particles are frozen out, which defines an upper bound on the temperature roughly estimated as $T < T^* \sim \Delta_{sc} / \log(\nu_0 V \Delta_{sc})$, where $\nu_0$ is the density of states at the Fermi level, $\Delta_{sc}$ is the superconducting gap, and $V$ is the volume of the system. For a micrometer-size aluminium loop similar to that used in the experimental work of van der Wal et al. $T^* \sim \Delta_{sc}/15 \sim 100$ mK and can be easily achieved.

Second, we assume that no charge tunneling is possible onto the array. In practice, we may allow the array to be connected to an external reservoir via a large resistor (to be able to change the charge on the array by an overall shift of gate voltages). Its resistance then must be much larger than the characteristic resistance scale $R^* \sim (C\Delta)^{-1}$, where $C$ is the capacitance of the array, and $\Delta$ is the relevant energy scale (of the order of the splitting between the two lowest states). In the charge-dominated limit, $\Delta \sim E_C$ and $R^*$ is of the order of the resistance quantum $h/4e^2$, in agreement with the conventional condition for charge quantization in the Coulomb blockade setup. In the phase-dominated limit, $\Delta \ll E_C$ and $R^*$ is much larger than the resistance quantum. Also, under the condition $\Delta_{sc} > E_C^{\text{loop}}$ a ‘weak’ contact to the external world would allow an unpaired quasi-particle to escape from the loop, leaving only paired electrons in the system.

Third, the islands and the junctions are assumed to be identical. The precision to which this symmetry has to hold in the phase-dominated limit may be simply estimated from the condition that the tunneling actions agree up to small deviations of order one. The tunneling action scales as $S \sim (E_J/E_C)^{1/2}$ and therefore the required precision is $\delta(E_J/E_C), (\delta E_C/E_C) \ll S^{-1}$. For a detectable splitting $S$ must be not too large; for a qubit design, $S$ is typically taken to be of order 5−10, and these conditions may well be satisfied.

The required precision on the magnitude of the external magnetic field (or, equivalently, on the flux through the loop) may be estimated from the condition that the level splitting due to the deviation of the flux from $\Phi_0/2$ is much smaller than that from the tunneling between the qubit states. In the phase-dominated limit ($E_J \gg E_C$) this produces the condition $E_J\delta\Phi/\Phi_0 \ll \Delta$ and hence $\delta\Phi/\Phi_0 \ll S^{-1} e^{-S}$. In the opposite charge-dominated limit, it is sufficient to require that $\delta\Phi/\Phi_0 \ll 1$.

In conclusion we have discussed the spectral properties of symmetric Josephson junction loops, devices similar to those recently proposed as potential qubits for quantum computing. While in the charge-dominated regime with $E_C \gg E_J$ the dependence of the ground-state degeneracy on the total charge on the island is a
simple charging effect, this degeneracy derives from a subtle quantum interference in the phase-dominated limit $E_J \gg E_C$. Several proposals on solid state realizations of quantum bits belong to the latter limit. From our analysis it follows that the requirements for observing the charge dependence of the qubit-level splitting are similar to those for the qubit operation (plus symmetry requirements which are easy to satisfy). The interference effects studied in this paper then may serve as a good test of quantum coherence in such qubit designs.

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