Twisted Kähler–Einstein metrics in big classes

Tamás Darvas (University of Maryland)
Kewei Zhang (Beijing Normal University)

Abstract

We prove existence of twisted Kähler–Einstein metrics in big cohomology classes, using a divisorial stability condition. In particular, when $-K_X$ is big, we obtain a uniform Yau–Tian–Donaldson existence theorem for Kähler–Einstein metrics. To achieve this, we build up from scratch the theory of Fujita–Odaka type delta invariants in the transcendental big setting, using pluripotential theory. We do not use the K-energy in our arguments, and our techniques provide a simple roadmap to prove Yau–Tian–Donaldson existence theorems for Kähler–Einstein type metrics, that only needs convexity of the appropriate Ding energy. As an application, we give a simplified proof of Li–Tian–Wang’s existence theorem in the log Fano setting.

Keywords: Kähler–Einstein metric, geodesic ray, Yau–Tian–Donaldson theorem.

1 Introduction

The study of canonical metrics in Kähler geometry goes back to early work of Calabi [24] and Yau’s solution of the Calabi conjecture [81]. The different versions of the Yau–Tian–Donaldson (YTD) conjecture predict that existence of canonical metrics is equivalent to various algebraic stability conditions of the underlying manifold. We refer to the textbooks [71, 74] for an introduction to this exciting topic.

Among canonical metrics, Kähler–Einstein (KE) type metrics are of particular interest, whose study revealed numerous deep connections between differential geometry and algebraic geometry. Here we develop a transcendental pluripotential theoretic approach to KE metrics, which not only gives simplified proofs of some existing YTD type theorems in the literature but also allows one to treat KE metrics even in transcendental big cohomology classes. More specifically, we will study divisorial stability and geodesic stability of big classes, and obtain a uniform Yau–Tian–Donaldson type existence theorem for twisted Kähler–Einstein metrics in this general setting.

We recall the definition of the Ricci curvature and KE type metrics in the big context. Let $(X, \omega)$ be a compact Kähler manifold, and $\theta$ a smooth closed $(1, 1)$-form representing a big cohomology class $\{\theta\}$ in $H^{1,1}(X, \mathbb{R})$. All volumes and measures in this work are interpreted in the non-pluripolar context of [19] (see Section 2.1).

Given $u \in \mathcal{E}(X, \theta)$ (see Definition 2.1) suppose that the non-pluripolar measure $\theta_u^n := (\theta + dd^c u)^n$ is absolutely continuous with respect to $\omega^n$. If $\log (\theta_u^n/\omega^n)$ is integrable, then we introduce $\text{Ric} \theta_u$ as the following current, with the convention $dd^c := i\partial \bar{\partial}/2\pi$:

$$\text{Ric} \theta_u = \text{Ric} \omega - dd^c \log \frac{\theta_u^n}{\omega^n}.$$  \hspace{1cm} (1)
Now we define twisted Kähler–Einstein currents/metrics. Let $\eta$ be a smooth closed $(1,1)$-form representing the cohomology class for which the following decomposition holds:
\[ c_1(-K_X) = \{\theta\} + \{\eta\}. \]
Hodge theory provides $f \in C^\infty(X)$ such that $\text{Ric} \omega = \theta + \eta + dd^c f$. Let $\psi$ be a quasiplorisubharmonic function on $X$ and $\eta_\psi := \eta + dd^c \psi$ will be our twisting. We want to find $u \in \text{PSH}(X, \theta)$, with minimal singularity type, satisfying the $\eta_\psi$-twisted KE equation:
\[ \text{Ric} \theta_u = \theta u + \eta_\psi. \]

Using the identity $\text{Ric} \omega = \theta + \eta + dd^c f$ and (1) we reduce the twisted KE equation to the following highly degenerate complex Monge–Ampère equation:
\[ (\theta + dd^c u)^n = e^{-u + f - \psi} \omega^n. \] (2)

Already in the Kähler case it is well known that the above equation is not always solvable, with a whole host of algebraic criteria available, under the umbrella of K-stability.

In the Fano case, when $\{\theta\} = c_1(-K_X)$, YTD type existence theorems were proved using the continuity method/Cheeger–Colding–Tian theory [26, 39, 75, 76], Kähler–Ricci flow [27] and the variational/non-Archimedean method [9]. Recently the second author gave a proof using Kähler quantization [82].

In the singular setting of log-Fano pairs, YTD type existence theorems were proved in [64, 66, 67]. After a resolution of singularities, this case is roughly equivalent with looking for twisted KE metrics in big and semi-positive classes.

Regarding greater generality, we circumvent the difficulties that typically arise in the big setting by developing an analytic approach to divisorial stability via pluripotential theory. Of course, the original definition of K-stability going back to Tian [72] and extended by Donaldson [50] is not applicable, as our big class $\{\theta\}$ is not necessarily induced by a line bundle. However the more recent interpretation of K-stability in terms of divisorial data, going back to Fujita [53] and Li [63], adapts naturally to our context. Indeed, based on the Blum–Jonsson interpretation of the Fujita–Odaka delta invariant [15, 22, 54], rooted in the non-Archimedean approach to K-stability (see [2, Definition 7.2]), we define the twisted delta invariant of our data:
\[ \delta_\psi(\{\theta\}) := \inf_E \frac{A_\psi(E)}{S_\theta(E)}. \] (3)

Here $A_\psi(E) := A_X(E) - \nu(\psi, E)$ and the infimum is taken over all prime divisors $E$ over $X$, i.e., $E$ is a prime divisor inside a Kähler manifold $Y$ with $\pi : Y \to X$ being a proper bimeromorphic map. Recall that a prime divisor inside $Y$ is an irreducible analytic set of codimension 1.

When $X$ is projective, one can equivalently consider projective birational morphisms $\pi$, as explained in [33]. Here the log discrepancy $A_X(E)$ of $E$ is $A_X(E) := 1 + \text{coeff}_E(K_Y - \pi^*K_X)$, and $\nu(\psi, E)$ denotes the Lelong number of $\pi^*\psi$ at a very generic point of $E$ (see [13]). The expected Lelong number $S_\theta(E)$ of $\{\theta\}$ along $E$ is defined by
\[ S_\theta(E) := \frac{1}{\text{vol}(\{\theta\})} \int_0^{\tau_\theta(E)} \text{vol}(\pi^*\theta - x(E)) dx, \]
where $\tau_\theta(E) := \sup\{\tau \in \mathbb{R} : \{\pi^*\theta\} - \tau \{E\} \text{ is big}\}$ is the pseudoeffective threshold. We emphasize that the volume function $\text{vol}(\cdot)$ is understood in the sense of [10], but
it coincides with the algebraic volume in case \( \{\theta\} \) is in the Néron–Severi space \([19, Proposition \, 1.18]\). Hence our delta invariant extends the one by Fujita–Odaka to the transcendental case. When \( \eta = 0 \) and \( \psi = 0 \), we use the much simpler notation \( \delta := \delta_0 \).

**Definition 1.1.** We say \((X, \{\theta\}, \eta_\psi)\) is \(\eta_\psi\)-twisted uniformly K-stable if \(\delta_\psi(\{\theta\}) > 1\).

Some remarks are in order, to motivate this definition. In the Fano case, it is known by \([15, Theorem \, B]\) that \(\delta(-K_X) > 1\) if and only if \((X, -K_X)\) is uniformly K-stable/Ding stable (as introduced in \([21, 44]/[5, 9]\)). Recent work of Liu–Xu–Zhuang \([67]\) shows that \(\delta(-K_X) > 1\) if and only if \((X, -K_X)\) is K-stable. This condition is further equivalent with \([2]\) having a unique solution \([9, Theorem \, A]\). The main result of this paper is establishing the historically “harder” direction of this equivalence in the big context, in the presence of positive twisting current:

**Theorem 1.2.** Suppose that \(\eta_\psi \geq 0\). If \(\delta_\psi(\{\theta\}) > 1\) then \([2]\) has a solution \(u \in \mathrm{PSH}(X, \theta)\) with minimal singularity, i.e., \(\text{Ric} \, \theta_u = \theta_u + \eta_\psi\).

We note the following corollary, in the particular case when \(c_1(-K_X) = \{\theta\}\) and \(\psi = 0:\)

**Corollary 1.3.** Suppose that \(-K_X\) is big. If \(\delta(-K_X) > 1\) then \([2]\) has a solution \(u \in \mathrm{PSH}(X, \theta)\) with minimal singularity, i.e., \(\theta_u\) is a KE metric satisfying \(\text{Ric} \, \theta_u = \theta_u\).

When \(X\) is Fano, the argument of Corollary 1.3 gives a novel pluripotential theoretic proof of the classical (uniform) YTD conjecture, in addition to the ones mentioned above. In fact, as we shall see in \(\S 6.2\) the KE metrics found in Corollary 1.3 are actually in one-to-one correspondence to the KE metrics on the ‘ample model’ of \((X, -K_X)\).

One can produce examples with \(-K_X\) big and \(\delta(-K_X) > 1\). Indeed, let \(V\) be a del Pezzo surface with klt singularities and let \(X \xrightarrow{\pi} V\) be the minimal resolution of \(V\). Then \(-K_X = -\pi^*K_V + \sum a_i E_i\), where \(a_i \geq 0\), \(E_i\)'s are \(\pi\)-exceptional curves, and \((E_i \cdot E_j)\) is negative definite. Note that \(-K_X\) is big and \(-K_X = -\pi^*K_V + \sum a_i E_i\) is the Zariski decomposition of \(-K_X\). Also, \(-K_X\) is not nef when the singularities of \(V\) are worse than Du Val. Now if \(\delta(-K_V) > 1\), then one can show that \(\delta(-K_X) > 1\) as well. Examples of such \(V\) are constructed in \([25]\). Hence Corollary 1.3 is applicable in this case.

We expect that the twisted KE metric found in Theorem 1.2 is unique, an open question going back to Berndtsson \([11, page \, 4]\). Conversely, if a unique twisted KE metric exists, one can show that \(\delta_\psi(-K_X) > 1\) (see Proposition 5.9). A similar result has been independently obtained by \([43]\), who show that if a unique twisted KE metric exists then \((X, -K_X)\) is uniformly Ding stable in terms of test configurations.

We believe the novelty of our work partly lies in the adaptability and robust nature of our transcendental techniques. Whenever one can prove the convexity of Ding functionals of certain canonical KE metrics, our methods yield a YTD type existence theorem in terms of the appropriate delta invariant. To provide evidence, we give a much simplified proof of the main result of Li–Tian–Wang \([66]\) using our techniques. Remarkably, after the first version of our work appeared on the arXiv, in \([79]\) C. Xu noticed that by using results of Birkar–Cascini–Hacon–McKernan \([14]\) one can reduce uniform K-stability of \((X, -K_X)\) to uniform K-stability of a \(\mathbb{Q}\)-Fano ample model, which gives an algebraic proof of our Corollary 1.3, in addition to the transcendental one given in this work. In Section 6 we discuss the above connections in detail.

Compared to the seminal work of Berman–Boucksom–Jonsson \([3]\), we do not require the convexity of K-energy functionals, nor do we need to approximate geodesic rays via
test configurations. This is essential to our approach, as the K-energy functional is known to be convex only in the big and nef case [48]. In the general big case, even a suitable definition of K-energy is still missing to the authors’ knowledge.

Our main ingredients are the valuative criteria for integrability [3,20], (see [18] for an excellent learning source), the Guan–Zhou openness theorem [56], the relative pluri-potential theory developed in [31,32,34], and the Ross–Witt Nyström correspondence between (sub)geodesic rays and test curves introduced in [69], and studied further in [30,38].

That \( \eta \psi \geq 0 \) guarantees that the twisted Ding functional is convex, as follows from the main result of [13]. This is used in our proof of Theorem 1.2 in one specific point, but the rest of the argument works in much more general context, allowing to treat many different problems in the literature at the same time, as we now point out.

To start, we consider a general quasi-plurisubharmonic (qpsh) function \( \psi \) on \( X \) that does not necessarily satisfy \( \eta \psi \geq 0 \). In addition, let \( \chi \) be another qpsh function on \( X \) with analytic singularity type, and consider the following more general equation of twisted KE type, for \( u \in \text{PSH}(X,\theta) \) with minimal singularity type:

\[
(\theta + dd^c u)^n = e^{-u+\chi-\psi} \omega^n.
\]

(4)

To make sense of this equation, we must assume that \( \int_X e^{\chi-\psi} \omega^n < \infty \), i.e., \( \chi - \psi \) is klt. Clearly this is more general than (2), as any \( f \in C^\infty(X) \) is a qpsh function. When treating canonical metrics, it is natural to consider some type of continuity method. This is the case here as well, as we consider the following family of equations, for \( \lambda > 0 \):

\[
(\theta + dd^c u)^n = e^{-\lambda u+\chi-\psi} \omega^n.
\]

(5)

At least for small \( \lambda > 0 \) the Guan–Zhou openness theorem [56] gives \( \int_X e^{-\lambda v+\chi-\psi} \omega^n < \infty \) for all \( v \in E^1(X,\theta) \) (See Theorem 2.2 and Proposition 2.5 below).

The expression on the right hand side of (5) often appears in the literature and, after adding a constant, it is convenient to introduce the following Radon probability measure:

\[
\mu := e^{\chi-\psi} \omega^n.
\]

(6)

As in [8], such measures \( \mu \) are called tame. Following [49], one defines a functional, whose Euler-Lagrange equation is exactly (5). This is \( D^\lambda_\mu : E^1(X,\theta) \to \mathbb{R} \), the \( \lambda \)-Ding functional:

\[
D^\lambda_\mu(\varphi) = -\frac{1}{\lambda} \log \int_X e^{-\lambda \varphi} d\mu - I_\theta(\varphi) \text{ for } \varphi \in E^1(X,\theta),
\]

(7)

where \( I_\theta(\cdot) \) is the Monge-Ampère energy (see (12)). Our starting point is the following result that gives a formula for the slope of the \( \lambda \)-Ding functional along subgeodesic rays:

**Theorem 1.4.** Let \( (0,\infty) \ni t \mapsto u_t \in E^1(X,\theta) \) be a sublinear subgeodesic ray. Then

\[
\liminf_{t \to \infty} \frac{D^\lambda_\mu(u_t)}{t} = -\lim_{t \to \infty} \frac{I_\theta(u_t)}{t} + \sup \{ \tau \in \mathbb{R} : \int_X e^{-\lambda \hat{\varphi}} d\mu < \infty \}.
\]

(8)

For the definition of subgeodesic rays \( \{u_t\} \), and their Legendre transforms \( \{\hat{u}_\tau\} \), see Definition 3.1 and (18). Our \( \lambda \)-Ding functional is typically not convex along subgeodesics and we overcome this difficulty by building on ideas from [35, Section 4] to prove (5). We refer to [3, Theorem 1.3] and [4, Theorem 5.4] for similar flavour results.
On the heels of the above theorem it is convenient to introduce
\[ D_\mu^\lambda \{ u_t \} := \liminf_{t \to \infty} \frac{D_\mu^\lambda(u_t)}{t}, \]
and we will call this expression the radial \( \lambda \)-Ding functional of the subgeodesic ray \( \{ u_t \} \).

Next we introduce a more general delta invariant associated to our data:
\[ \delta_\mu = \delta_\mu(\{ \theta \}) := \inf_E A_{X, \psi}(E) / S_\theta(E), \tag{9} \]
where \( A_{X, \psi}(E) := A_X(E) + \nu(\chi, E) - \nu(\psi, E) \) and the inf is taken over prime divisors \( E \) in smooth bimeromorphic models over \( X \). Note that \( A_{X, \psi}(E) > 0 \) by [9, Theorem B.5].

The main technical ingredient in the proof of Theorem 1.2 is the following result, relating \( \delta_\mu \) to geodesic semistability of the \( \lambda \)-Ding functionals:

**Theorem 1.5.** With the notation from above we have
\[ \delta_\mu = \sup \{ \lambda > 0 \mid D_\mu^\lambda \{ u_t \} \geq 0 \text{ for all sublinear subgeodesic ray } u_t \in \mathcal{E}^1(X, \theta) \}. \]

When \( \{ \theta \} \) is ample and \( \chi = 0 \), this result is proved in [82] using quantization. See also [80, Proposition 4.5] for a similar statement in the log Fano setting, and [23, Theorem 3.16] for a related result about ample classes, formulated using non-Archimedean language. Theorem 1.5 generalizes these results to big cohomology classes in \( H^{1,1}(X, \mathbb{R}) \), yielding a universal analytic interpretation of the delta invariant.

Regarding the much more developed theory of KE metrics on manifolds of general type we note the important work [19], that studies KE equations when \( K_X \) is big, in which case a solution always exists. See [10, 52] for synergies between singular KE metrics and the minimal model program on varieties of general type. Regarding exciting developments on csck metrics we refer to [23, 46] for connections with divisorial notions of K-stability.

Regarding transcendental K-stability in the Kähler setting, we mention the works [47, 51] exploring a cohomological notion of stability.

**Future directions.** To stay brief, we cannot treat several interesting related questions.

Though the condition \( \eta \geq 0 \) allows one to conjecture a Bando–Mabuchi uniqueness theorem [11] for twisted KE metrics [11, page 4], it does come with a slight limitation. Indeed, when (2) is solvable, a positive klt current will exist in \( c_1(-K_X) \). By Nadel vanishing, this implies that \( H^2(X, O_X) = 0 \), hence \( \mathbb{R} \)-divisors exhaust the whole space \( H^{1,1}(X, \mathbb{R}) \). Since neither Theorem 1.4 nor 1.5 uses this numerical condition, we wonder if \( \eta_0 \geq 0 \) can be omitted from Theorem 1.2 as well. In this direction, [82, Theorem 2.3] treats the ample case, suggesting that this is possible.

As pointed out earlier, our definition of the delta invariant is rooted in the non-Archimedean approach to K–stability. As a result, our treatment has an algebraic/non-Archimedean interpretation that can be expanded when \( \{ \theta \} \) is the Chern class of a big line bundle. We will explore this in the companion paper [37].

As is well known, in the (log) Fano case \( \delta > 1 \) implies that \( X \) can not contain any nontrivial vector fields. See [3] for a related work using Gibbs stability in the big case. It is desirable to extend our treatment to allow for a non-trivial automorphism group as well. This should be possible after incorporating ideas from the works [59, 64].

Lastly, our approach seems adaptable to the case of canonical KE metrics with prescribed singularity type, as recently studied by Trusiani [77]. If the Ross–Witt Nyström correspondence could be extended to this context, we suspect that the analogue of Theorem 1.2 could be established, as the convexity of the corresponding Ding functional is known in that setting.
that locally can be written as $u$ is smooth around $u$ and the Bedford-Taylor in the local setting [2, 3], it has been pointed out in [19] that the sequence theorem of Demailly there are plenty of $\theta$ type as $V_u$. A function and we distinguish the potential with minimal singularity: $u \equiv \in \mathbb{C}$. We assume that $\rho = x \theta$.

In this short subsection we recall the basics of finite energy pluripotential theory in big cohomology classes. For a more thorough treatment we refer to the articles [19], [30], or the recent textbook [57].

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ and $\theta$ a smooth closed $(1,1)$-form. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic (qpsh) if locally $u = \rho + \varphi$, where $\rho$ is smooth and $\varphi$ is a plurisubharmonic (psh) function. We say that $u$ is $\theta$-plurisubharmonic ($\theta$-psh) if it is qpsh and $\theta_u := \theta + dd^c u \geq 0$ as currents. We let $\text{PSH}(X, \theta)$ denote the space of $\theta$-psh functions on $X$.

The class $\{\theta\}$ is big if there exists $\psi \in \text{PSH}(X, \theta)$ satisfying $\theta \geq \varepsilon \omega$ for some $\varepsilon > 0$. We assume that $\{\theta\}$ is big throughout this paper, unless specified otherwise.

Given $u, v \in \text{PSH}(X, \theta)$, $u$ is more singular than $v$, (notation: $u \lessdot v$) if there exists $C \in \mathbb{R}$ such that $u \leq v + C$. The potential $u$ has the same singularity as $v$ (notation: $u \asymp v$) if $u \leq v$ and $v \leq u$. The classes $[u]$ of this latter equivalence relation are called singularity types. When $\{\theta\}$ is merely big, all elements of $\text{PSH}(X, \theta)$ are very singular, and we distinguish the potential with minimal singularity:

\[ V_\theta := \sup\{u \in \text{PSH}(X, \theta) \text{ such that } u \leq 0\}. \tag{10} \]

A function $u \in \text{PSH}(X, \theta)$ is said to have minimal singularity if it has the same singularity type as $V_\theta$, i.e., $[u] = [V_\theta]$.

We say that $[u]$ is an analytic singularity type if it has a representative $u \in \text{PSH}(X, \theta)$ that locally can be written as $u = c \log(\sum_j |f_j|^2) + g$, where $c > 0$, $g$ is a bounded function and the $f_j$ are a finite set of holomorphic functions. By a fundamental approximation theorem of Demailly there are plenty of $\theta$-psh functions with analytic singularity type.

The ample locus $\text{Amp}(\theta)$ of $\{\theta\}$ is the set of points $x \in X$ such that there exists $u \in \text{PSH}(X, \theta)$ with analytic singularity type, satisfying $\theta_u \geq \varepsilon \omega$ for some $\varepsilon > 0$ and that $u$ is smooth around $x$. In particular, $V_\theta$ is locally bounded on $\text{Amp}(\theta)$.

Let $\theta^1, ..., \theta^n$ be smooth closed $(1,1)$-forms and $\varphi_j \in \text{PSH}(X, \theta^j)$, $i = 1, ..., n$. Following Bedford-Taylor in the local setting [9, 10], it has been pointed out in [19] that the sequence
of positive measures

\[ \mathbb{1}_{\bigcap_j (\varphi_j > V_{\varphi_j} - k)} \theta_1^{\varphi_1} \land \ldots \land \theta^n_{\max(\varphi_n, V_{\varphi_n} - k)} \]

converges weakly to the so called non-pluripolar product \( \theta_1^{\varphi_1} \land \ldots \land \theta^n_{\varphi^n} \). The resulting positive Borel measure does not charge pluripolar sets. In the particular case when \( \varphi_1 = \varphi_2 = \ldots = \varphi_n = \varphi \) and \( \theta^1 = \ldots = \theta^n = \theta \) we will call \( \theta^\varphi \) the non-pluripolar Monge-Ampère measure of \( \varphi \), generalizing the usual notion of volume form, when \( \theta^\varphi \) is a smooth Kähler form.

As a consequence of Bedford-Taylor theory, the measures in (11) all have total mass less than \( \int_X \theta^n_u \), in particular, after letting \( k \to \infty \) we notice that \( \int_X \theta^n_u \leq \int_X \theta^n_{V_{\varphi}} \). What is more, it was proved in [78, Theorem 1.2] that for any \( u, v \in \text{PSH}(X, \theta) \) we have the following monotonicity result for the masses: if \( v \leq u \) then \( \int_X |u - V_{\varphi}| \theta^n_u < \infty \).

**Definition 2.1.** We say that \( u \in \text{PSH}(X, \theta) \) is a full mass potential (notation: \( u \in \mathcal{E}(X, \theta) \)) if \( \int_X \theta^n_u = \int_X \theta^n_{V_{\varphi}} =: \text{vol}(\{\theta\}) \). Moreover, we say that \( u \in \mathcal{E}(X, \theta) \) has finite energy (notation: \( u \in \mathcal{E}^1(X, \theta) \)) if \( \int_X |u - V_{\varphi}| \theta^n_u < \infty \).

The class \( \mathcal{E}^1(X, \theta) \) plays a central role in the variational theory of complex Monge-Ampère equations, as detailed in [8, 19] and later works. Here we only mention that the Monge-Ampère energy \( I_\theta \) naturally extends to this space with the usual formula:

\[
I_\theta(u) = \frac{1}{\text{vol}(\{\theta\})(n + 1)} \sum_{j=0}^n \int_X (u - V_{\varphi}) \theta^j_u \land \theta^{n-j}_{V_{\varphi}}, \quad u \in \mathcal{E}^1(X, \theta).
\]

It is upper semi-continuous (usc) with respect to the \( L^1 \) topology on \( \text{PSH}(X, \theta) \).

We recall some envelope notions that will be useful in this work. Given any \( f : X \to [-\infty, +\infty] \) the starting point is the envelope \( P_\theta(f) := \text{usc}(\text{sup}\{v \in \text{PSH}(X, \theta), v \leq f\}) \), where usc denotes the upper-semicontinuous regularization. Then, for \( u, v \in \text{PSH}(X, \theta) \) we can introduce the “rooftop envelope” \( P_\theta(u, v) := P_\theta(\text{min}(u, v)) \). This allows us to further introduce envelopes with respect to singularity type [69]:

\[
P_\theta[u](v) := \text{usc}\left( \lim_{C \to +\infty} P_\theta(u + C, v) \right).
\]

It is easy to see that \( P_\theta[u](v) \) depends on the singularity type \([u]\). When \( v = V_{\varphi} \), we simply write \( P[u] := P_\theta[u] := P_\theta[u](V_{\varphi}) \) and call this potential the envelope of the singularity type \([u]\). It follows from [32, Theorem 3.8] that \( \theta^n_{P[u]} \leq \mathbb{1}_{\{P[u] = 0\}} \theta^n \). Also, by [32, Proposition 2.3 and Remark 2.5] we have that \( \int_X \theta^n_{P[u]} = \int_X \theta^n_u \).

With the help of these envelopes one can define a complete metric on \( \mathcal{E}^1 \). Indeed, as pointed out in [33, Theorem 2.10], for \( u, v \in \mathcal{E}^1(X, \theta) \) we have that \( P(u, v) \in \mathcal{E}^1(X, \theta) \) and the following expression defines a complete metric on \( \mathcal{E}^1(X, \theta) \) [30, Theorem 1.1]:

\[
d_1(u, v) = I_\theta(u) + I_\theta(v) - I_\theta(P(u, v)).
\]

In addition, \( d_1 \)-convergence implies \( L^1 \)-convergence of qps potentials [30, Theorem 3.11].
2.2 Complex singularity exponents and openness

To start, we point out several notational differences compared to the closely related work of Berman–Boucksom–Jonsson [9]. Stemming from our choice $dd^c := i\partial\bar{\partial}/2\pi$, our Ding functionals (7) are free of factors of two compared to [9, page 4]. Moreover, for any qpsh function $\varphi$ on $X$ the complex singularity exponent attached to our tame measure $\mu := e^{\chi-\psi}\omega^n$ (recall (6)) is defined to be

$$c_\mu[\varphi] := \sup \left\{ \lambda > 0 : \int_X e^{-\lambda \varphi} d\mu < \infty \right\}.$$

And our definition of the Lelong number of a qpsh function $v$ at $x \in X$ is

$$\nu(v, x) := \sup \{ c > 0 \text{ s.t. } v(z) - c \log |z-x|^2 \text{ is bounded above near } x \}.$$ (13)

Given an irreducible analytic subset $E \subset X$, $\nu(v, E)$ is equal to the Lelong number of $v$ at a very general point of $E$ (by [70]).

On the level of potentials, our definition of singularity exponent and Lelong number differs from the ones used in [9] by a factor of two. However on the level of $(1,1)$-currents, all definitions agree due to our choice of $dd^c$. As an upside, using our terminology, all the formulas/inequalities we derive in this work contain no extra factors of two.

Openness theorems go back to the work of Berndtsson [12, Theorem 4.4] on Demailly–Kollár’s openness conjecture [43]. Not long after, Guan–Zhou [56] solved the strong openness conjecture of Demailly [42], which will be used multiple times in this work. The version below follows from (the proof of) [9, Corollary B.2] and the effective version of the strong openness theorem [56, §3.3] (see also [58, Main Theorem (2)] and [55, Corollary 1.2]):

**Theorem 2.2.** Suppose that $\lambda > 0$ and $u, u_j$ are qpsh functions on $X$ such that $u_j \nearrow u$ a.e., and $\int_X e^{-\lambda u} d\mu < \infty$. Then there exists $j_0$ such that $\int_X e^{-\lambda u_j} d\mu < \infty$ for $j \geq j_0$.

The above theorem implies that $c_\mu[\varphi]$ is always positive. The following result (that follows from [9, 20] and Guan–Zhou openness [56]) gives a more precise valuative characterization of $c_\mu[\cdot]$.

**Theorem 2.3.** For any qpsh function $\varphi$ on $X$ one has

$$c_\mu[\varphi] = \inf_E \frac{A_{\chi,\psi}(E)}{\nu(\varphi, E)},$$ (14)

where $A_{\chi,\psi}(E) := A_X(E) + \nu(\chi, E) - \nu(\psi, E)$ and the infimum is taken over all prime divisors $E$ in smooth bimeromorphic models over $X$.

If on the right hand side of (14) we have $\nu(\varphi, E) = 0$, then we use the convention $1/0 = \infty$. No ambiguity will arise from this, as follows from the proof below.

**Proof.** Let $\lambda \in (0, c_\mu[\varphi])$. By [9, Theorem B.5], for small enough $\varepsilon > 0$ we have

$$\nu(\chi, E) + A_X(E) \geq (1 + \varepsilon) \nu(\psi, E) + \lambda \nu(\varphi, E) \geq \nu(\psi, E) + (1 + \varepsilon) \lambda \nu(\varphi, E)$$

for any prime divisor $E$ over $X$. Letting $\varepsilon \searrow 0$ and then $\lambda \nearrow c_\mu[\varphi]$, we arrive at

$$c_\mu[\varphi] \leq \inf_E \frac{A_{\chi,\psi}(E)}{\nu(\varphi, E)}.$$
Recall that

\[ \nu(\chi, E) + A_X(E) \geq \nu(\psi, E) + (1 + \varepsilon)\lambda \nu(\varphi, E) \]

for all \( E \). On the other hand, we have \( \int_X e^{\chi - \psi}\omega^n = \int_X d\mu < \infty \). Using [3, Theorem B.5] again, for any small \( \tau > 0 \) we have that

\[ \nu(\chi, E) + A_X(E) \geq (1 + \tau)\nu(\psi, E) \]

holds for all \( E \). Summarizing, we find that

\[ (1 + \tau)(\nu(\chi, E) + A_X(E)) \geq (1 + \tau + \tau^2)\nu(\psi, E) + (1 + \varepsilon)\lambda \nu(\varphi, E). \]

Or equivalently,

\[ \nu(\chi, E) + A_X(E) \geq \frac{1 + \tau + \tau^2}{1 + \tau} \nu(\psi, E) + \frac{1 + \varepsilon}{1 + \tau} \lambda \nu(\varphi, E). \]

Finally, choosing \( \tau \) such that \( \tau < \varepsilon \), we get \( \nu(\chi, E) + A_X(E) \geq (1 + \varepsilon')(\nu(\psi, E) + \lambda \nu(\varphi, E)) \) for some \( \varepsilon' > 0 \), which holds for all \( E \). Using [3, Theorem B.5] one more time we obtain \( c_\mu[\varphi] \geq \lambda \), implying the other direction.

Theorem 2.3 has the following consequence.

**Lemma 2.4.** Let \( \{\psi_t\}_{t \in I} \) be a family of qph functions on \( X \), where \( I \subset \mathbb{R} \) is some connected open interval. Assume that \( I \ni t \mapsto \psi_t(x) \) is concave for any \( x \in X \). Then the map \( I \ni t \mapsto 1/c_\mu[\psi_t] \) is convex, so in particular it is also continuous.

**Proof.** For \( \tau_1, \tau_2 \in I \) and \( \lambda \in [0, 1] \), concavity reads \( \psi_{\lambda\tau_1 + (1-\lambda)\tau_2} \geq \lambda \psi_{\tau_1} + (1 - \lambda)\psi_{\tau_2} \). Let \( E \) be any prime divisor over \( X \). Then \( \nu(\psi_{\lambda\tau_1 + (1-\lambda)\tau_2}, E) \leq \lambda \nu(\psi_{\tau_1}, E) + (1 - \lambda)\nu(\psi_{\tau_2}, E) \), so that

\[ \frac{\nu(\psi_{\lambda\tau_1 + (1-\lambda)\tau_2}, E)}{A_X,\psi(E)} \leq \frac{\lambda \nu(\psi_{\tau_1}, E)}{A_X,\psi(E)} + \frac{(1 - \lambda)\nu(\psi_{\tau_2}, E)}{A_X,\psi(E)}. \]

Taking sup over all \( E \), by Theorem 2.3 we conclude the result.

Lastly we recall a consequence of Theorem 2.2 and [32, Theorem 1.3]:

**Proposition 2.5.** \( c_\mu[u] = c_\mu[P[u]] \) for any \( u \in \text{PSH}(X, \theta) \). In particular \( c_\mu[v] = c_\mu[V_\theta] \) for any \( v \in \mathcal{E}(X, \theta) \).

**Proof.** Recall that \( P(V_\theta, u + C) \succ P[u] \) a.e. As \( [P(V_\theta, u + C)] = [u] \) for any \( C \in \mathbb{R} \), Theorem 2.2 immediately gives that \( c_\mu[u] = c_\mu[P[u]] \). By [32, Theorem 1.3] we have that \( P[v] = V_\theta \) for any \( v \in \mathcal{E}(X, \theta) \). The last statement immediately follows.

### 3 The extended Ross–Witt Nyström correspondence

We give a precise correspondence between finite energy geodesic rays and certain maximal test curves in the big case. This theory was initiated by Ross and Witt Nyström in [60] in the Kähler case, and developed further in [30] and [38].
Definition 3.1. A sublinear subgeodesic ray is a subgeodesic $(0, \infty) \ni t \mapsto u_t \in \text{PSH}(X, \theta)$ (notation: \{u_t\}_t) such that $u_t \to_{L^1} u_0 := V_\theta$ as $t \to 0$, and there exists $C \in \mathbb{R}$ such that $u_t(x) \leq Ct$, $t \geq 0$, $x \in X$. In addition, \{u_t\}_t is of finite energy if $u_t \in \mathcal{E}^1(X, \theta)$, $t \geq 0$.

Using $t$-convexity, we obtain some immediate properties of sublinear subgeodesic rays:

**Lemma 3.2.** Suppose that \{u_t\}_t is a sublinear subgeodesic ray. Then the set \{u_t > -\infty\} is the same for any $t > 0$. In particular, for any $x \in X$ the curve $t \mapsto u_t(x)$ is either finite and convex on $(0, \infty)$, or equal to $-\infty$ on this interval.

A *psh geodesic ray* is a sublinear subgeodesic ray that additionally satisfies the following maximality property: for any $0 < a < b$, the subgeodesic $(0, 1) \ni t \mapsto v_{t}^{a,b} := u_{a(1-e)+bt} \in \text{PSH}(X, \theta)$ can be recovered in the following manner:

$$v_{t}^{a,b} := \sup_{h \in S} h_t, \quad t \in [0, 1],$$

where $S$ is the set of subgeodesics $(0, 1) \ni t \mapsto h_t \in \text{PSH}(X, \theta)$ with $\lim_{t \downarrow 0} h_t \leq u_a$ and $\lim_{t \uparrow 1} h_t \leq u_b$. The space of *finite energy (psh) geodesic rays* will be denoted by

$$\mathcal{R}^1(X, \theta).$$

**Remark 3.3.** Let \{u_t\}_t be a psh geodesic ray. Due to [30, (13)] the map $t \mapsto \sup_X u_t = \sup_X (u_t - V_\theta) = \sup_{\text{Amp}(\theta)} (u_t - V_\theta)$, $t > 0$ is linear, using an approximation argument via decreasing geodesic segments with minimal singularity type.

Making small tweaks to [69, Definition 5.1], we now give the definition of test curves:

**Definition 3.4.** A map $\mathbb{R} \ni \tau \mapsto \psi_\tau \in \text{PSH}(X, \theta)$ is a psh test curve, denoted \{\psi_\tau\}_\tau, if

(i) $\tau \mapsto \psi_\tau(x)$ is concave, decreasing and usc for any $x \in X$.

(ii) $\psi_\tau \equiv -\infty$ for all $\tau$ big enough, and $\psi_\tau$ increases a.e. to $V_\theta$ as $\tau \to -\infty$.

Note that this definition is more general than the one in [69] (where the authors only considered potentials with small unbounded locus), more general than the one in [30] (where the authors considered only bounded test curves, see below), and more general than the one in [38, Section 3.1] (where the authors only consider Kähler classes). Moreover, condition (ii) allows for the introduction of the following constant:

$$\tau^+_{\psi} := \inf\{\tau \in \mathbb{R} : \psi_\tau \equiv -\infty\}.$$  

We adopt the following convention: psh test curves will always be parametrized by $\tau$, whereas rays will be parametrized by $t$. Hence \{\psi_t\}_t will always refer to some kind of ray, whereas \{\phi_\tau\}_\tau will refer to some type of test curve. As pointed out below, rays and test curves are dual to each other, so one should think of the parameters $t$ and $\tau$ to be dual to each other as well.

**Definition 3.5.** A *psh test curve* \{\psi_\tau\}_\tau can have the following properties:

(i) \{\psi_\tau\}_\tau is maximal if $P[\psi_\tau] = \psi_\tau$ for any $\tau \in \mathbb{R}$.

(ii) \{\psi_\tau\}_\tau is a finite energy test curve if

$$\int_{-\infty}^{\tau^+_\psi} \left( \int_X \theta^n_{\psi_\tau} - \int_X \theta^n_{V_\theta} \right) \mathrm{d}\tau > -\infty.$$

(iii) We say \{\psi_\tau\}_\tau is bounded if $\psi_\tau = V_\theta$ for all $\tau$ negative enough. In this case, one can introduce the following constant, complementing [16]:

$$\tau^-_{\psi} := \sup\{\tau \in \mathbb{R} : \psi_\tau \equiv V_\theta\}.$$
In the above definition, we followed the convention $P[-\infty] = -\infty$. Note that bounded test curves are clearly of finite energy.

We recall the Legendre transform, that will establish the duality between various types of maximal test curves and geodesic rays. Given a convex function $f : [0, +\infty) \to \mathbb{R}$, its Legendre transform is defined as

$$\hat{f}(\tau) := \inf_{t \geq 0} (f(t) - t\tau) = \inf_{t > 0} (f(t) - t\tau), \quad \tau \in \mathbb{R}.$$ 

The (inverse) Legendre transform of a decreasing concave function $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is

$$\hat{g}(t) := \sup_{\tau \in \mathbb{R}} (g(\tau) + t\tau), \quad t \geq 0.$$ 

There is a sign difference in our choice of Legendre transform compared to the convex analysis literature, however this choice will be more suitable for our discussion.

We recall that, for every $\tau \in \mathbb{R}$ we have $\hat{g}(\tau) \geq g(\tau)$ with equality if and only if $g$ is additionally $\tau$-usc. Similarly, $\hat{f}(t) \leq f(t)$ for all $t \geq 0$ with equality if and only if $f$ is $t$-lsc. In general, $\hat{g}$ is the $\tau$-usc envelope of $g$, and $\hat{f}$ is the $t$-lsc envelope of $f$. These properties are called the involution property of the Legendre transform.

Starting with a psh test curve $\{\psi_\tau\}_\tau$, our goal is to construct a geodesic/subgeodesic ray by taking the $\tau$-inverse Legendre transform. The first step is the next proposition, essentially proved in [29]:

**Proposition 3.6.** Suppose $\{\psi_\tau\}_\tau$ is a psh test curve. Then $\sup_{\tau}(\psi_\tau(x) + t\tau)$ is usc with respect to $(t, x) \in (0, \infty) \times X$.

Since $\tau_+^+ < \infty$ and $\psi_\tau \leq V_\theta, \tau \in \mathbb{R}$, we note that $\sup_{\tau}(\psi_\tau + t\tau) \leq V_\theta + t\tau_+^+$ for $t \geq 0$. Even if true, it is not clear how to interpret upper semi-continuity at $(t, x) \in \{0\} \times X$. This is due to the fact that $V_\theta(x) = \sup_{\tau} \psi_\tau(x)$ a.e. $x \in X$, but not everywhere (!) in our definition of psh test curve.

**Proof.** Let $S = \{\text{Re } s > 0\}$ be the right open half plane. We consider the usual complexification of the inverse Legendre transform:

$$u(s, z) := \sup_{\tau} (\psi_\tau(z) + \tau \text{Re } s), \quad (s, z) \in S \times X.$$ 

Also, $u_t(x) := u(t, x) \leq V_\theta + t\tau_+^+ \leq tr_+^+, t > 0$. Clearly, usc $u \in \text{PSH}(S \times X, \pi^*\omega)$, where usc $u$ is the usc regularization of $u$ on $S \times X$, where $\pi : S \times X \to X$ is the natural projection. It will be enough to prove that usc $u = u$.

We introduce the set $E = \{u < \text{usc } u\} \subseteq S \times X$. As both $u$ and usc $u$ are $\mathbb{R}$-invariant in the imaginary direction of $S$, it follows that $E$ is also $\mathbb{R}$-invariant, i.e., there exists $B \subseteq (0, \infty) \times X$ such that $E = B \times \mathbb{R}$.

As $E$ has Monge–Ampère capacity zero, it follows that $E$ has Lebesgue measure zero. By Fubini’s theorem $B \subseteq (0, \infty) \times X$ has Lebesgue measure zero as well. For $z \in X$, we introduce the $z$-slices of $B$:

$$B_z = B \cap ((0, \infty) \times \{z\}).$$

By Fubini’s theorem again, we have that $B_z$ has Lebesgue measure 0 for all $z \in X \setminus F$, where $F \subseteq X$ is some set of Lebesgue measure 0.

11
By slightly increasing $F$, but not its zero Lebesgue measure, we can additionally assume that $u_0(z) > -\infty$ for all $t > 0$ and $z \in X \setminus F$ (indeed, at least one potential $\psi_t$ is not identically equal to $-\infty$).

Let $z \in X \setminus F$. We argue that $B_z$ is in fact empty. By our assumptions on $F$, both maps $t \mapsto u_t(z)$ and $t \mapsto (\text{usc } u)(t, z)$ are locally bounded and convex (hence continuous) on $(0, \infty)$. As they agree on the dense set $(0, \infty) \setminus B_z$, it follows that they have to be the same, hence $B_z = \emptyset$. This allows to conclude that

$$\inf_{t > 0} [u_t(x) - \tau t] = \chi_\tau := \inf_{t > 0} [(\text{usc } u)(t, x) - \tau t], \quad \tau \in \mathbb{R} \text{ and } z \in X \setminus F. \quad (17)$$

By duality of the Legendre transform $\psi_\tau(x) = \inf_{t > 0} [u_t(x) - t \tau]$ for all $x \in X$ and $\tau \in \mathbb{R}$ (using the $\tau$-usc property of $\tau \mapsto \psi_\tau$). From this and (17) it follows that $\psi_\tau = \chi_\tau$ a.e. on $X$, for all $\tau \in \mathbb{R}$. Since both $\psi_\tau$ and $\chi_\tau$ are $\theta$-psh (the former by definition, the latter by Kiselman’s minimum principle [41, Theorem I.7.5]), it follows that $\psi_\tau \equiv \chi_\tau$ for all $\tau \in \mathbb{R}$.

Consequently, applying the $\tau$-Legendre transform to the $\tau$-usc and $\tau$-concave curves $\tau \mapsto \psi_\tau$ and $\tau \mapsto \chi_\tau$, we obtain that $t \mapsto u_t(x)$ is the $t$-lsc envelope of $t \mapsto \text{usc } u(t, x)$ on $(0, \infty)$ for all $x \in X$. Hence, Lemma [30, 69] gives that $u_t(x) = \text{usc } u(t, x)$ on $(0, \infty) \times X$. \hfill $\Box$

Given a sublinear subgeodesic ray $\{\phi_t\}_t$ (psh test curve $\{\psi_t\}_t$), we can associate its (inverse) Legendre transform at $x \in X$ as

$$\hat{\phi}_\tau(x) := \inf_{t > 0} (\phi_t(x) - t \tau), \quad \tau \in \mathbb{R},$$
$$\tilde{\psi}_t(x) := \sup_{\tau \in \mathbb{R}} (\psi_\tau(x) + t \tau), \quad t > 0. \quad (18)$$

Our next theorem describes a duality between various types of rays and maximal test curves, extending various particular cases from [30, 69].

**Theorem 3.7.** The Legendre transform $\{\psi_t\}_\tau \mapsto \{\tilde{\psi}_t\}_t$ gives a bijective map with inverse $\{\phi_t\}_t \mapsto \{\hat{\phi}_\tau\}_\tau$ between:

(i) psh test curves and sublinear subgeodesic rays,
(ii) maximal psh test curves and psh geodesic rays,
(iii) [30, 69] maximal bounded test curves and geodesic rays with minimal singularity type.

In this case, we additionally have that $V_\emptyset + \tau^+ t \leq \tilde{\psi}_t \leq V_\emptyset + \tau^- t$, $t \geq 0$.

**Proof.** We prove (i). This is essentially [30, Proposition 4.4], where an important particular case was addressed. Let $\{\psi_t\}_t$ be a psh test curve. Then $\tilde{\psi}_t \in \text{PSH}(X, \theta)$ for all $t > 0$ due to Proposition 3.6. We also see that $\tilde{\psi}_t \leq V_\emptyset + t \tau^+_\psi$, and $\tilde{\psi}_t \to L_1 V_\emptyset$ as $t \to 0$, proving that $\{\tilde{\psi}_t\}_t$ is a sublinear subgeodesic.

For the reverse direction, let $\{\phi_t\}_t$ be a sublinear subgeodesic ray. Then $\hat{\phi}_\tau \in \text{PSH}(X, \theta)$ or $\hat{\phi}_\tau \equiv -\infty$ for any $\tau \in \mathbb{R}$ due to Kiselman’s minimum principle. By properties of Legendre transforms and Lemma 3.2, we get that $\tau \mapsto \hat{\phi}_\tau(x)$ is $\tau$-usc, $\tau$-concave and decreasing. Due to sublinearity of $\{\phi_t\}_t$, we get that $\hat{\phi}_\tau \equiv -\infty$ for $\tau$ big enough. Lastly $\psi_\tau \not\to V_\emptyset$ a.e. as $\tau \to -\infty$, since $\phi_t \to L_1 V_\emptyset$ as $t \to 0$.

We prove (ii). Let $\tau \in \mathbb{R}$ and $\{u_t\}_t$ a psh geodesic ray. From [29, Proposition 5.1] (that only uses the maximum principle [13]) we obtain that $\hat{u}_\tau = P[\hat{u}_{\tau}](V_\emptyset) = P[\hat{u}_{\tau}](0) = P[\hat{u}_{\tau}]$. Since $\{u_t\}_t$ is sublinear, the curve $\{\hat{u}_t\}_t$ is a maximal psh test curve.

Conversely, let $\{\psi_t\}_\tau$ be a maximal psh test curve. We will show that the sublinear subgeodesic $\{\tilde{\psi}_t\}_t$ is a psh geodesic ray. By elementary translation properties of the Legendre transform we can assume that $\tau^+ = 0$, in particular $\{\tilde{\psi}_t\}_t$ is $t$-decreasing.

12
Now assume by contradiction that \{\tilde{\psi}_t\}_t \text{ is not a psh geodesic ray. Comparing with (15), there exists } 0 < a < b \text{ such that }

\[ \tilde{\psi}_{(1-t)a+tb} \leq \chi_t := \sup_{h \in \mathcal{S}} h_t, \quad t \in [0, 1], \]

where \( \mathcal{S} \) is the set of subgeodesics \((a, b) \ni t \mapsto h_t \in \text{PSH}(X, \theta) \) satisfying \( \lim_{t \to a^+} h_t \leq \tilde{\psi}_a \) and \( \lim_{t \to b^-} h_t \leq \tilde{\psi}_b \). Now let \( \{\phi_t\}_t \) be the sublinear subgeodesic such that \( \phi_t := \tilde{\psi}_t \) for \( t \not\in (a, b) \) and \( \phi_{a-(1-t)+bt} := \chi_t \) otherwise.

Trivially, \( \tilde{\psi}_t \leq \phi_t \leq 0 \), hence by duality, \( \psi_\tau \leq \hat{\phi}_\tau \leq 0 \) and \( \tau^+ = \tau^+_\phi = 0 \). However, comparing with (15), we claim that \( \hat{\phi}_\tau \leq \psi_\tau + (a - b) \) for any \( \tau \in \mathbb{R} \). Since \( \tau^+ = \tau^+_\phi = 0 \), we only need to show this for \( \tau \leq 0 \). For such \( \tau \) we indeed have

\[ \inf_{t \in [a, b]} (\phi_t - t\tau) \leq \psi_b - b\tau = \tilde{\psi}_b - b\tau \leq \inf_{t \in [a, b]} (\psi_t - t\tau) + (a - b)\tau, \]

where in the last inequality we used that \( t \to \tilde{\psi}_t \) is decreasing.

By the maximality of \( \{\psi_\tau\}_\tau \), we obtain that \( \hat{\phi}_\tau \leq \hat{P}[\psi_\tau] = \psi_\tau \). As we already pointed out the reverse inequality above, we conclude that \( \hat{\phi}_\tau = \psi_\tau \). The (inverse) Legendre transform now gives that \( \psi_t = \phi_{\hat{\psi}_t} \) a contradiction. Hence \( \{\tilde{\psi}_t\}_t \) is a psh geodesic ray.

The duality of (iii) is simply [30, Theorem 1.3].

Given a finite energy (sub)geodesic ray \( \{u_t\}_t \), we know that \( t \to I_\theta(u_t) \) is (convex) linear [33, Theorem 3.12], allowing to introduce the following radial Monge–Ampère energy:

\[ I_\theta\{u_t\} := \lim_{t \to \infty} \frac{I_\theta(u_t)}{t}. \]  

(19)

Before proving the duality between maximal finite energy test curves and rays, we point out the following approximation result:

**Proposition 3.8.** Let \( \{u_t\}_t \) be a psh geodesic ray. Then there exists a sequence of geodesic rays \( \{u^j_t\}_t \) with minimal singularity type such that \( u^j_t \searrow u_t \) and \( \theta^n_{\hat{u}^j_t} \to \theta^n_{\hat{u}_t} \), weakly for \( j \to \infty \) for all \( t \geq 0 \) and \( \tau \in \mathbb{R} \).

**Proof.** In the Kähler case, this result is a particular case of [35, Theorem 4.5]. As the argument in the big case is virtually the same, we will be brief, and only point out how the sequence of approximating rays \( \{u^j_t\}_t \) is constructed.

We can assume without loss of generality that \( \tau^+_a = 0 \). For \( j \in \mathbb{N}, \tau < 0 \), set

\[ \psi^j_\tau(x) := (1 - \max \left( 0, 1 + \frac{\tau}{j} \right)) V_\theta + \max \left( 0, 1 + \frac{\tau}{j} \right) \hat{u}_\tau, \quad \text{and} \quad \hat{u}^j_\tau := P[\psi^j_\tau]. \]

We define \( \hat{u}^j_0 := \lim_{\tau \to 0^-} \hat{u}^j_\tau \).

Since \( \tau \to \hat{u}_\tau \) is \( \tau \)-concave, \( \tau \)-decreasing, and \( \hat{u}_\tau \leq V_\theta \), it is elementary to see that \( \tau \to \psi^j_\tau \) is also \( \tau \)-concave and \( \tau \)-decreasing. By elementary properties of \( P[\cdot] \) we get that \( \tau \to \hat{u}^j_\tau \) is also \( \tau \)-concave and \( \tau \)-decreasing (see the proof of [31, Proposition 4.6]).

Arguing the same way as in the proof of [33, Theorem 4.5] we conclude that \( u^j_t \searrow u_t \), \( \hat{u}_\tau \searrow \hat{u}_\tau \), and \( \theta^n_{\hat{u}^j_t} \to \theta^n_{\hat{u}_t} \), weakly for \( j \to \infty \) for all \( t \geq 0, \tau \in \mathbb{R} \).

Finally we prove the Ross–Witt Nyström correspondence between maximal finite energy test curves and the finite energy rays of \( \mathcal{R}^1(X, \theta) \).
Theorem 3.9. The Legendre transform \( \{ \psi_\tau \}_\tau \mapsto \{ \tilde{\psi}_t \}_t \) gives a bijective map with inverse \( \{ \phi_t \}_t \mapsto \{ \tilde{\phi}_\tau \}_\tau \) between maximal finite energy test curves and finite energy geodesic rays. In this case, we additionally have that

\[
I_\theta \{ \tilde{\psi}_t \} = \frac{1}{\text{vol}(\{ \theta \})} \int_{-\infty}^{\tau^+} \left( \int_X \theta^n_{\psi_{\tau}} - \int_X \theta^n_{V_\psi} \right) \, d\tau + \tau^+_\psi. \tag{20}
\]

Proof. As previously, we may assume that \( \tau^+_\psi = 0 \). As a preliminary result, in Proposition 3.10 below we prove (20) for bounded maximal test curves.

Given a finite energy maximal test curve \( \{ \psi_\tau \}_\tau \), we know that \( \{ \tilde{\psi}_t \}_t \) is a psh geodesic ray. By Proposition 3.8 above there exists geodesic rays \( \{ \psi^k \}_k \) with minimal singularity type such that \( \tilde{\psi}^k \searrow \tilde{\psi}_t \) for any \( t \geq 0 \), and \( \int_X \theta^n_{\psi^k} \searrow \int_X \theta^n_{\psi_{\tau}} \), for any \( \tau < \tau^+_\psi = \tau^+_\psi = 0 \).

By basic properties of the Monge–Ampère energy we have that \( I_\theta \{ \psi^k \} = I_\theta \{ \tilde{\psi}^k \} \rightarrow I_\theta \{ \tilde{\psi}_t \} \). By Proposition 3.10 below

\[
I_\theta \{ \tilde{\psi}^k \} = \frac{1}{\text{vol}(\{ \theta \})} \int_{-\infty}^{0} \left( \int_X \theta^n_{\psi^k} - \int_X \theta^n_{V_\psi} \right) \, d\tau.
\]

The right hand side is bounded from below, since \( \{ \psi_\tau \}_\tau \) is a finite energy test curve. Since \( \int_X \theta^n_{\psi^k} \searrow \int_X \theta^n_{\psi_{\tau}} \), we can take the limit on both the left and right hand side, to arrive at (20), also implying that \( \{ \tilde{\psi}_t \}_t \) is a finite energy geodesic ray.

Conversely, assume that \( \{ \tilde{\phi}_t \}_t \) is a finite energy geodesic ray, with decreasing approximating sequence of rays \( \{ \phi^k \}_k \), as detailed above. For similar reasons we have

\[
I_\theta \{ \tilde{\phi}^k \} = \frac{1}{\text{vol}(\{ \theta \})} \int_{-\infty}^{0} \left( \int_X \theta^n_{\phi^k} - \int_X \theta^n_{V_\phi} \right) \, d\tau.
\]

Since \( I_\theta \{ \phi^k \} \searrow I_\theta \{ \phi_t \} \), the monotone convergence theorem gives that (20) holds for \( \{ \tilde{\phi}_t \}_t \), finishing the proof. \( \Box \)

As promised, to complete the argument of Theorem 3.7 we prove the next proposition, whose argument can be extracted from [69, Section 6] with additional references to [32]. We recall the precise details here as the results of [69] were proved in the context of potentials with small unbounded locus.

Proposition 3.10. Suppose that \( \{ \psi_\tau \}_\tau \) is a bounded maximal test curve with \( \tau^+_\psi = 0 \). Then

\[
\frac{I_\theta \{ \tilde{\psi}_t \}}{t} = \frac{1}{\text{vol}(\{ \theta \})} \int_{-\infty}^{0} \left( \int_X \theta^n_{\psi_{\tau}} - \int_X \theta^n_{V_\psi} \right) \, d\tau, \quad t > 0. \tag{21}
\]

Proof. Without loss of generality we assume that \( \text{vol}(\{ \theta \}) = 1 \). For \( N \in \mathbb{Z}_+, M \in \mathbb{Z} \) and \( t > 0 \), we introduce the following:

\[
\tilde{\psi}^{N,M}_t := \max_{k \leq M} \left( \psi^k_{t/2^N} + tk/2^N \right). \tag{22}
\]

It is clear that \( \tilde{\psi}^{N,M}_t \in \text{PSH}(X, \theta) \) has minimal singularity type, since it is a maximum of a finite collection of \( \theta \)-psh potentials (indeed, \( \{ \psi_\tau \}_\tau \) is a bounded test curve). Following [69, Lemma 6.6], we now argue that

\[
\frac{t}{2^N} \int_X \theta^n_{\psi_{(M+1)/2^N}} \leq I_\theta \{ \tilde{\psi}^{N,M+1}_t \} - I_\theta \{ \tilde{\psi}^{N,M}_t \} \leq \frac{t}{2^N} \int_X \theta^n_{\psi_{M/2^N}}. \tag{23}
\]

14
Indeed, by [30, Theorem 2.4(iii)],

$$
\int_X (\tilde{\psi}_{t \psi t}^{N, M+1} - \tilde{\psi}_{t \psi t}^{N, M}) \theta^n_{t \psi t, \psi t, M+1} \leq I_{\theta}(\tilde{\psi}_{t \psi t}^{N, M+1}) - I_{\theta}(\tilde{\psi}_{t \psi t}^{N, M}) \leq \int_X (\tilde{\psi}_{t \psi t}^{N, M+1} - \tilde{\psi}_{t \psi t}^{N, M}) \theta^n_{t \psi t, \psi t, M}.
$$

(24)

Clearly $\tilde{\psi}_{t \psi t}^{N, M+1} \geq \tilde{\psi}_{t \psi t}^{N, M}$. Using the $\tau$-concavity of $\tau \mapsto \psi_{t \psi t}$ we claim that

$$
U_t := \{ \tilde{\psi}_{t \psi t}^{N, M+1} - \tilde{\psi}_{t \psi t}^{N, M} > 0 \} \cap \text{Amp}(\theta) = \{ (\psi_{t(M+1)/2} + t \frac{M + 1}{2N} > \psi_{t/2}^t + t \frac{k}{2N}, \text{ for all } k \leq M \}
$$

(25)

Indeed, if $\psi_{t(M+1)/2}^t > \tilde{\psi}_{t \psi t}^{N, M}(x)$ for some $x \in \text{Amp}(\theta)$, then the maximum defining $\tilde{\psi}_{t \psi t}^{N, M+1}$ (see (22)) is 'uniquely' realized for $k = M + 1$, or equivalently,

$$
\psi_{t(M+1)/2}^t(x) + t \frac{M + 1}{2N} > \psi_{t/2}^t(x) + t \frac{k}{2N}, \text{ for all } k \leq M.
$$

(26)

The inequality for $k = M$ is equivalent to

$$
\psi_{t(M+1)/2}^t(x) + t \frac{M}{2N} > \psi_{t/2}^t(x) + t \frac{k}{2N}, \text{ for all } k \leq M - 1.
$$

(27)

hence the set on the left hand side of (25) is contained on the right hand side. To argue conversely, by the above, we only have to prove that (27) implies (26). This is where concavity of $\tau \mapsto u_{t}(x) + t\tau$ is used for some fixed $x \in \text{Amp}(\theta)$ (c.f. [69, Lemma 6.5]). More precisely, (27) implies that

$$
\psi_{t(M+1)/2}^t(x) + t \frac{M}{2N} > \psi_{t/2}^t(x) + t \frac{k}{2N}, \text{ for all } k \leq M - 1.
$$

(28)

Indeed, if for some $k \leq M - 1$ the above inequality fails, then so does the $\tau$-concavity of $\tau \mapsto \psi_{t}(x) + t\tau$, evaluated at the interior point $M/2^N$ of the interval $[k/2^N, (M+1)/2^N]$. This proves the claimed identity (25). Moreover, (26) and (28) together imply that

$$
\tilde{\psi}_{t \psi t}^{N, M+1} = \psi_{t(M+1)/2}^t + t(M + 1)/2^N, \quad \tilde{\psi}_{t \psi t}^{N, M} = \psi_{M/2}^t + tM/2^N \text{ on } U_t.
$$

(29)

Since $U_t$ is an open set in the plurifine topology, we have $\theta^n_{\psi_{t(M+1)/2}^t} |_{U_t} = \theta^n_{\tilde{\psi}_{t \psi t}^{N, M+1}} |_{U_t}$ and $\theta^n_{\psi_{M/2}^t} |_{U_t} = \theta^n_{\tilde{\psi}_{t \psi t}^{N, M}} |_{U_t}$ (see [34, Lemma 2.1]).

Recall that the measure $\theta^n_{\psi_{t(M+1)/2}^t}$ is supported on the set $\{ \psi_{t(M+1)/2}^t = 0 \}$, by [32, Theorem 3.8]. Since $\{ \psi_{t(M+1)/2}^t = 0 \} \subseteq U_t$ and $\{ \psi_{t(M+1)/2}^t = 0 \} \subset \{ \psi_{M/2}^t = 0 \}$, the first inequality of (24) implies the first inequality of (23):

$$
\frac{t}{2N} \int_X \theta^n_{\psi_{t(M+1)/2}^t} - \frac{t}{2N} \int_{\{ \psi_{t(M+1)/2}^t = 0 \}} \theta^n_{\psi_{t(M+1)/2}^t} \leq \int_{U_t} (\tilde{\psi}_{t \psi t}^{N, M+1} - \tilde{\psi}_{t \psi t}^{N, M}) \theta^n_{\tilde{\psi}_{t \psi t}^{N, M+1}} \leq I_{\theta}(\tilde{\psi}_{t \psi t}^{N, M+1}) - I_{\theta}(\tilde{\psi}_{t \psi t}^{N, M}).
$$

We also have that

$$
\int_X (\tilde{\psi}_{t \psi t}^{N, M+1} - \tilde{\psi}_{t \psi t}^{N, M}) \theta^n_{\tilde{\psi}_{t \psi t}^{N, M+1}} = \int_{U_t} (\psi_{t(M+1)/2}^t - \psi_{M/2}^t) \theta^n_{\psi_{M/2}^t} \leq \frac{t}{2N} \int_X \theta^n_{\psi_{M/2}^t}
$$

since $\psi_{t(M+1)/2}^t \leq \psi_{M/2}^t$, which gives the second inequality of (23).
Fixing $N$, let $M = [2^N \tau^-_\psi] \in \mathbb{Z}$. Repeated application of (25) gives us

$$
\sum_{M+1 \leq j \leq 0} \frac{t}{2^N} \int_X \theta^n_{\tau_j/2^N} \leq I_0(\bar{\psi}^{N,0}_t) - I_0(\bar{\psi}^{N,M}_t) \leq \sum_{M \leq j \leq -1} \frac{t}{2^N} \int_X \theta^n_{\tau_j/2^N}.
$$

Since $M \leq 2^N \tau^-_\psi$ we have that $\bar{\psi}^{N,M}_t = \psi_{M/2^N} + tM/2^N = V_0 + tM/2^N$. As a result $I_0(\bar{\psi}^{N,M}_t) = tM/2^N$, and we obtain

$$
\sum_{j=M+1}^{0} \frac{t}{2^N} \left( \int_X \theta^n_{\tau_j/2^N} - \int_X \theta^n_{V_0} \right) \leq I_0(\bar{\psi}^{N,0}_t) \leq -\sum_{j=M}^{-1} \frac{t}{2^N} \left( \int_X \theta^n_{\tau_j/2^N} - \int_X \theta^n_{V_0} \right).
$$

We have Riemann sums on both the left and right of the above inequality. Using Lemma 3.11 below, and the fact that $\bar{\psi}^{N,0} \nearrow \bar{\psi}_t$ a.e., it is possible to let $N \to \infty$ and obtain (21), as desired. \hfill \Box

**Lemma 3.11.** Suppose that $\{\psi_t\}_\tau$ is a psh test curve. Then $\tau \mapsto \int_X \theta^n_{\psi_t} > 0$ is a continuous function for $\tau \in (-\infty, \tau^+_\psi)$.\hfill \Box

The proof of this result will be omitted, as it is exactly the same as [38, Lemma 3.9], that deals with the Kähler case.

Lastly, we recall the maximization $\{v_t\}_t$ of a finite energy sublinear subgeodesic ray $\{u_t\}_t$, which goes back to [30, Proposition 4.6] and is determined by the formula:

$$\hat{v}_\tau := P[\hat{u}_\tau], \quad \tau < \tau^+_{\hat{u}},
$$

(30)\hfill \Box

$\hat{v}_{\tau^-} = \lim_{\tau \nearrow \tau^-} \hat{v}_\tau$ and $\hat{v}_\tau = -\infty$ for $\tau > \tau^+_{\hat{u}}$. By the lemma below $\{\hat{v}_\tau\}_\tau$ is a maximal finite energy test curve and by Theorem 3.7 we immediately have that $\{v_t\}_t$ is the smallest geodesic ray, satisfying $v_t \geq u_t, \ t > 0$.

**Lemma 3.12.** $\{\hat{v}_\tau\}_\tau$ constructed above is a maximal psh test curve.

**Proof.** The argument is very similar to [30, Proposition 4.6], so we will be again brief. By the definition of $P[\cdot]$ we see that $\tau \mapsto \hat{v}_\tau$ is $\tau$-concave. By the proof of [30, Lemma 4.3] we see that $\tau \mapsto v_t(x)$ is $\tau$-usc for any $x \in X$.

It remains to argue the maximality of $\{\hat{v}_\tau\}_\tau$. To start, we fix $\tau_1 \in (-\infty, \tau^+_{\hat{u}})$. Since $v_t \not\nearrow V_0$ as $\tau \to -\infty$, by [32, Theorem 2.1] we have that $\int_X \theta^n_{v_{\tau_0}} > 0$ for some $\tau_0 < \tau_1$. There exists $\alpha \in (0, 1)$ such that $\tau_1 = \alpha \tau_0 + (1 - \alpha) \tau^+_{\hat{u}}$. By $\tau$-concavity, we have that

$$
\hat{v}_{\tau_1} \geq \alpha \hat{v}_{\tau_0} + (1 - \alpha) \hat{v}_{\tau^+_{\hat{u}}}.
$$

By [78, Theorem 1.2] and multilinearity of non-pluripolar products we obtain $\int_X \theta^n_{\hat{u}_{\tau_1}} \geq \alpha^n \int_X \theta^n_{\hat{u}_{\tau_0}} > 0$. As a result, we can apply [30, Lemma 4.7] to conclude that $\hat{v}_{\tau_1} = P[\hat{v}_{\tau_1}]$. Lastly, we address maximality in the case $\tau := \tau^+_{\hat{u}}$. If $s < \tau = \tau^+_{\hat{u}}$, then by the above we can write $P[\hat{v}_s] \leq P[\hat{v}_{\tau}] = \hat{v}_s$. Letting $s \nearrow \tau = \tau^+_{\hat{u}}$, we obtain that $P[\hat{v}_\tau] \leq \hat{v}_\tau$. Since the reverse inequality is trivial, we get $P[v_\tau] = v_\tau$, finishing the proof. \hfill \Box

Finally, we note the following identity for the radial Monge–Ampère energy (recall [19]) of a finite energy subgeodesic and its maximization. It implies that the Legendre transform $\{\hat{u}_\tau\}_\tau$ of a finite energy sublinear subgeodesic ray $\{u_t\}_t$ is a finite energy test curve (recall Definition 3.5 ii)). The reverse correspondence however might not be true.
Proposition 3.13. Let \( \{u_t\}_t \) be a finite energy sublinear subgeodesic ray and \( \{v_t\}_t \) be its maximization constructed in (30). Then we have

\[
I_\theta \{v_t\} = I_\theta \{u_t\} = \frac{1}{\text{vol}(\{\theta\})} \int_{-\infty}^{t^+_\theta} \left( \int_X \theta_{u_t}^{\theta} - \int_X \theta_{v_t}^{\theta} \right) d\tau + \tau_{u_t}^+.
\]

Proof. That \( I_\theta \{v_t\} = \frac{1}{\text{vol}(\{\theta\})} \int_{-\infty}^{t^+_\theta} \left( \int_X \theta_{u_t}^{\theta} - \int_X \theta_{v_t}^{\theta} \right) d\tau + \tau_{u_t}^+ \) follows from (20) and the fact that \( \int_X \theta_{v_t}^{\theta} = \int_X \theta_{p(t)}^{\theta} = \int_X \theta_{u_t}^{\theta} \) (Proposition 2.3 and Remark 2.5).

To finish the proof, we utilize an alternative construction for the maximization \( \{v_t\}_t \).

Namely, for \( k \in \mathbb{N} \), let \( \{v_t^k\}_t \) be the sublinear subgeodesic ray such that \( v_t^k = u_t \) for \( t \geq k \) and \([0,k]\) \( \ni t \mapsto v_t^k \in \mathcal{E}_1(X,\theta) \) is the finite energy geodesic segment joining \( V_\theta \) and \( u_k \).

By the comparison principle we obtain that \( \{v_t^k\}_t \) is indeed a sublinear subgeodesic ray, moreover \( v_t \geq v_t^k \geq u_t \). Since the \( k \)-limit of \( \{v_t^k\}_t \) is the smallest finite energy geodesic ray dominating \( \{u_t\}_t \), we obtain that \( v_t^k \nearrow v_t \). Hence \( I_\theta \{v_t\} \geq I_\theta \{u_t\} = \lim_k I_\theta \{v_t^k\} \geq \lim_k I_\theta \{v_t^k\} = I_\theta \{v_t\} = I_\theta \{v_t\} \), where we used \( I_\theta \{u_t\} = I_\theta \{v_t^k\} \) for all \( k \), \( t \mapsto I_\theta \{v_t^k\} \) is convex and \( t \mapsto I_\theta \{v_t\} \) is linear. So \( I_\theta \{v_t\} = I_\theta \{u_t\} \), finishing the proof.

\[ \square \]

4 Delta invariant and geodesic semistability

In this section we prove Theorem 1.4 and Theorem 1.5. Let \( \mu := e^{\chi-\psi} \) be the tame measure defined in (30) and fix \( \lambda \in (0,c_*[V_\theta]) \). Here we recall that \( \chi,\psi \) are qpsf functions on \( X \) with \( \chi \) assumed to have analytic singularity type.

Our first result gives a precise formula for the slope of the \( \lambda \)-Ding functional (see (17)) along subgeodesic rays, rooted in the ideas of [38, Section 4]:

Theorem 4.1 (=Theorem 1.4). Let \( \{u_t\}_t \subset \mathcal{E}_1(X,\theta) \) be a sublinear subgeodesic ray. Then

\[
\liminf_{t \to \infty} \frac{D_\mu^\lambda(u_t)}{t} = -I_\theta \{u_t\} + \sup \{ \tau : \int_X e^{-\lambda\hat{u}_\tau} d\mu < \infty \} = -I_\theta \{u_t\} + \sup \{ \tau : c_\mu[\hat{u}_\tau] \geq \lambda \}.
\]

Proof. By basic properties of the Legendre transform, \( \sup_X (u_t - V_\theta) \nearrow \tau_{u_t}^+ \), as \( t \to \infty \). Hence, after adding a \( t \)-linear term to \( u_t \) we can assume that \( \sup_X (u_t - V_\theta) / t \nearrow \tau_{u_t}^+ = 0 \). In particular, \( t \mapsto u_t \) is \( t \)-decreasing.

Comparing (19) and the definition \( D_\mu^\lambda \), it is enough to investigate the following radial functional:

\[
\mathcal{L}_\mu^\lambda \{u_t\} := \liminf_{t \to \infty} \frac{-1}{\lambda t} \log \int_X e^{-\lambda u_t} d\mu.
\]

It amounts to showing that

\[
\mathcal{L}_\mu^\lambda \{u_t\} = \tau_D := \sup \{ \tau : \int_X e^{-\lambda\hat{u}_\tau} d\mu < \infty \}.
\]

We note that \( \tau_D > -\infty \) by Theorem 2.22 since \( \hat{u}_\tau \nearrow V_\theta \) in \( L^1 \) as \( \tau \to -\infty \) and \( \int_X e^{-\lambda V_\theta} d\mu < \infty \) by our assumption on \( \lambda \).

Let \( \tau < \tau_D \) and \( C := \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty \). For any \( t \geq 0 \) we have \( \hat{u}_\tau \leq u_t - \tau \), so \( C \geq \int_X e^{-\lambda(u_t - \tau)} d\mu \). As a result, \( \mathcal{L}_\mu^\lambda \{u_t\} \geq \tau \), hence

\[
\mathcal{L}_\mu^\lambda \{u_t\} \geq \tau_D.
\]

17
Now we prove the reverse inequality. We fix \( p < \mathcal{L}_\mu^\lambda\{u_t\} \) and \( \epsilon > 0 \) satisfying \( p + \epsilon < \mathcal{L}_\mu^\lambda\{u_t\} \). We can find \( t_0 > 0 \) such that

\[
\int_X e^{-\lambda u_t} d\mu < e^{-(p+\epsilon)\lambda t}, \quad t \geq t_0.
\]

Hence \( \int_0^\infty e^{p\lambda t} \int_X e^{-\lambda u_t} d\mu \ dt < \infty \). By Fubini–Tonelli this is equivalent to

\[
\int_X \int_0^\infty e^{\lambda(p-t)u_t} d\mu \ dt < \infty. \quad (33)
\]

Before proceeding further, we notice that the following estimate holds, using the fact that \( u_t \leq V_\theta + t \sup_X (u_t - V_\theta) \leq V_\theta \leq 0 \):

\[
p < \mathcal{L}_\mu^\lambda\{u_t\} \leq 0.
\]

As a result, \( \hat{u}_p = \inf(u_t - pt) \) is not identically equal to \( -\infty \), since \( \tau_\lambda^+ = 0 \). Let \( x \in X \) such that \( \hat{u}_\mu(x), \chi(x) \) and \( \psi(x) \) are finite. By definition of \( \hat{u}_p \), we can find \( t_0 > 0 \) so that \( \hat{u}_p(x) + 1 \geq u_{t_0}(x) - pt_0 \). Since \( t \mapsto u_t(x) \) is decreasing, we have \( \hat{u}_p(x) - p + 1 \geq u_t(x) - pt \) for \( t \in [t_0, t_0 + 1] \). Hence

\[
\int_0^\infty e^{\lambda(p-t)(u_t(x) + \chi(x) - \psi(x))} dt \geq \int_{t_0}^{t_0+1} e^{\lambda(p-t)(u_t(x) + \chi(x) - \psi(x))} dt
\]

\[
\geq \int_{t_0}^{t_0+1} e^{-\lambda \hat{u}_p(x)} e^{\lambda(p-1)+\chi(x)-\psi(x)} dt \geq e^{\lambda(p-1)} e^{-\lambda \hat{u}_p(x)} e^{\chi(x)-\psi(x)}.
\]

By this and (33) we obtain that \( \int_X e^{-\lambda \hat{u}_p} d\mu < \infty \), hence \( p \leq \tau_D \), concluding the proof. \( \square \)

Motivated by the above result, we introduce

\[
\mathcal{D}_\mu^\lambda\{u_t\} := \liminf_{t \to \infty} \frac{\mathcal{D}_\mu^\lambda\{u_t\}}{t},
\]

and we will call this expression the radial \( \lambda \)-Ding functional of the finite energy sublinear subgeodesic ray \( \{u_t\}_t \).

**Remark 4.2.** In the Fano case, when the geodesic ray \( \{u_t\}_t \) is induced by a filtration of the section ring, \( \mathcal{D}_\mu^\lambda\{u_t\} \) actually coincides with the \( \beta_\lambda \)-invariant introduced in [80, §4].

Next, we move on to prove Theorem 1.5. We start with the following preliminary estimate regarding the \( \delta_\mu \) invariant defined in (9):

**Lemma 4.3.** We have \( c_\mu[V_\theta] \geq \delta_\mu \).

**Proof.** Let \( \pi : Y \to X \) be a smooth bimeromorphic model over \( X \) and \( E \) be a prime divisor in \( Y \). Recall that \( \tau_\theta(E) = \sup \{ \tau > 0 \text{ s.t. } \{\pi^*\theta\} - \tau E \text{ is a big class} \} \) is the pseudo-effective threshold of \( E \). Equivalently, one has

\[
\tau_\theta(E) = \sup_{u \in \operatorname{PSH}(X,\theta)} \nu(u, E).
\]

So in particular, \( \nu(V_\theta, E) \leq \tau_\theta(E) \). A well-known fact that we shall use is (see [16])

\[
\operatorname{vol}(\{\pi^*\theta\} - t\{E\}) = \operatorname{vol}(\{\theta\}), \quad t \in [0, \nu(V_\theta, E)].
\]

18
We reproduce the proof of this identity for the reader’s convenience. Let \( h_E \) be a smooth hermitian metric on \( \mathcal{O}_Y(E) \), with canonical section \( S_E \) and \( \Theta(h_E) := -dd^c \log h_E \). We notice that

\[
[V_{\pi^*\theta - \Theta(h_E)}] = [V_{\pi^*\theta} - t \log |S_E|_{h_E}^2] = [V_{\Theta} \circ \pi - t \log |S_E|_{h_E}^2], \quad t \in [0, \nu(V_{\Theta}, E)].
\]

This follows from the fact that the map \( U \mapsto U - t \log |S_E|_{h_E}^2 \) gives a monotone increasing bijection between \( \text{PSH}(Y, \pi^*\theta) \) and \( \text{PSH}(Y, \pi^*\theta - t\Theta(h_E)) \) for \( t \in [0, \nu(V_{\Theta}, E)] \), hence preserves potentials of minimal singularity type.

As a result, using \([78, \text{Theorem 1.2}]\), for \( t \in [0, \nu(V_{\Theta}, E)] \) we get that

\[
\text{vol}(\{\pi^*\theta\} - t\{E\}) = \int_Y (\pi^*\theta - t\Theta(h_E) + dd^c V_{\pi^*\theta - t\Theta(h_E)})^n
\]

\[
= \int_Y (\pi^*\theta - t\Theta(h_E) + dd^c (V_{\pi^*\theta} - t \log |S_E|_{h_E}^2))^n = \int_X \theta^n_v,
\]

where in the last line we used \( \Theta(h_E) + dd^c \log |S_E|_{h_E}^2 = 0 \) on \( Y \setminus E \). Then from \( \nu(V_{\Theta}, E) \leq \tau_0(E) \) and the above identity we obtain that

\[
S_{\theta}(E) := \frac{1}{\text{vol}(\{\theta\})} \int_0^{\tau_0(E)} \text{vol}(\{\pi^*\theta\} - x\{E\})dx \geq \nu(V_{\Theta}, E).
\]

(34)

Using Theorem\([223]\) this allows to conclude: \( c_\mu[V_{\Theta}] = \inf_E A_{\nu(V_{\Theta}, E)} \geq \inf_E A_{\delta_{\nu}(E)} = \delta_\mu \). \( \square \)

Let \( E \subset Y \xrightarrow{\tau} X \) be a prime divisor over \( X \). We shall make use of the following bounded test curve naturally associated to \( E \), going back to \([69]\):

\[
\hat{u}^E_\tau := \begin{cases} 
V_{\Theta}, & \tau \leq 0; \\
v^E_\tau, & 0 < \tau < \tau_0(E); \\
\lim_{\tau \to \tau_0(E)} v^E_\tau, & \tau = \tau_0(E); \\
-\infty, & \tau > \tau_0(E),
\end{cases}
\]

(35)

where \( v^E_\tau \) is given by \( v^E_\tau := \sup \{ \varphi \in \text{PSH}(X, \theta) | \nu(\varphi, E) \geq \tau, \varphi \leq 0 \} \). Using basic properties of Lelong numbers, one can show that \( v^E_\tau \in \text{PSH}(X, \theta) \) and that

\[
\nu(v^E_\tau, E) \geq \tau.
\]

(36)

The test curve \( \{\hat{u}^E_\tau\}_\tau \) is actually maximal, but we do not need this in what follows.

Let \( \{u^E_\tau\}_\tau \) be the finite energy (sub)geodesic ray associated to the test curve \( \{\hat{u}^E_\tau\}_\tau \). We then have the following observation.

**Lemma 4.4.** We have \( I_\theta\{u^E_\tau\} = S_{\theta}(E) \).

**Proof.** Let \( h_E \) be a smooth hermitian metric on \( \mathcal{O}(E) \), with canonical section \( S_E \) and \( \Theta(h_E) := -dd^c \log h_E \). Using \([36]\) we notice the following identity of singularity types:

\[
[p^*v^E_\tau - \tau \log |S_E|_{h_E}^2] = [V_{\pi^*\theta - \tau\Theta(h_E)}], \quad \tau \in [0, \tau_0(E)].
\]

This follows from the fact that the map \( U \mapsto U - \tau \log |S_E|_{h_E}^2 \) gives a monotone increasing bijection between \( \{w \in \text{PSH}(Y, \pi^*\theta), \nu(w, E) \geq \tau\} \) and \( \text{PSH}(Y, \pi^*\theta - \tau\Theta(h_E)) \), hence preserves qphs potentials of minimal singularity type in each set.
Since \( \Theta(h_E) + \dd_f \log |S_E|_{h,E}^2 = 0 \) on \( Y \setminus E \), we have that
\[
(\pi^*\theta - \tau \Theta(h_E) + \dd_f(\pi^*v^E_\tau - \tau \log |S_E|_{h,E}^2))^n = (\pi^*\theta + \dd_f(\pi^*v^E_\tau))^n.
\]

As a result, using [78, Theorem 1.2], we notice that for \( \tau < \tau_0(E) \) we have
\[
\int_X \theta^a_{\pi E} = \int_Y \pi^*\theta^*v^E_\tau = \text{vol}(\{\pi^*\theta\} - \tau \{E\}).
\]
That \( I_\theta(u^E_\tau) = S_\theta(E) \), follows after comparing (1) and (31).

Next we introduce a new valuative invariant attached to certain test curves. Given a sublinear subgeodesic ray \( \{u_\tau\}_\tau \subset \mathcal{E}^1(X, \theta) \), it follows from (31) that the Legendre transform \( \{\hat{u}_\tau\}_\tau \) of \( \{u_\tau\}_\tau \) is a finite energy test curve. For any prime divisor \( E \) over \( X \), we define the expected Lelong number \( \nu(\{\hat{u}_\tau\}_\tau, E) \) of the test curve \( \{\hat{u}_\tau\}_\tau \) along \( E \) to be
\[
\nu(\{\hat{u}_\tau\}_\tau, E) := \begin{cases} 
\nu(\{u_\tau\}_\tau, E), & \text{if } I_\theta(u_\tau) < \tau_\theta^+, \\
\nu(\{V_\theta\}_\theta, E), & \text{if } I_\theta(u_\tau) = \tau_\theta^+.
\end{cases}
\]

The following estimate for the expected Lelong number will play a key role, and should be compared with [54, Lemma 2.2].

**Proposition 4.5.** For any sublinear subgeodesic ray \( \{u_\tau\}_\tau \subset \mathcal{E}^1(X, \theta) \) and any prime divisor \( E \) over \( X \) we have
\[
\nu(\{\hat{u}_\tau\}_\tau, E) \leq S_\theta(E).
\]

**Proof.** Put \( f(\tau) := \nu(\{\hat{u}_\tau\}, E) \) for \( \tau \in (-\infty, \tau_\theta^+) \). Then it is clear that \( f(\tau) \) is a non-negative convex non-decreasing function defined on \(( -\infty, \tau_\theta^+) \). So in particular we may set \( f(-\infty) := \lim_{\tau \to -\infty} f(\tau) \). When \( I_\theta(u_\tau) = \tau_\theta^+ \), the desired inequality is \( \nu(\{\hat{u}_\tau\}, E) = \nu(V_\theta, E) \leq S_\theta(E) \), which is (34).

So in what follows we can assume that \( I_\theta(u_\tau) < \tau_\theta^+ \). We first consider the case when \( f(-\infty) = f(I_\theta(u_\tau)) \). This means that \( f(\tau) = a \in \mathbb{R} \) for \( \tau \in (-\infty, I_\theta(u_\tau)) \). Namely, \( \nu(\{\hat{u}_\tau\}, E) \equiv a \) for \( \tau \in (-\infty, I_\theta(u_\tau)) \). So by [78, Theorem 1.2] and (31) we have
\[
\int_X \theta^a_{\pi E} \leq \int_X \theta^a_{\pi E} = \text{vol}(\{\pi^*\theta\} - a \{E\}) \text{ for } \tau \in (-\infty, I_\theta(u_\tau)),
\]
where \( v^E_\tau \) is defined below (31). Thus, by (34)
\[
I_\theta(u_\tau) \leq \frac{1}{\text{vol}(\{\theta\})} \int_{-\infty}^{I_\theta(u_\tau)} \left( \text{vol}(\{\pi^*\theta\} - a \{E\}) - \text{vol}(\{\theta\}) \right) d\tau + \tau_\theta^+,
\]
which forces that \( \text{vol}(\{\pi^*\theta\} - a \{E\}) = \text{vol}(\{\theta\}) \) since \( \{\hat{u}_\tau\}_\tau \) has finite energy. So we have
\[
S_\theta(E) = \frac{1}{\text{vol}(\{\theta\})} \int_0^{I_\theta(E)} \text{vol}(\{\pi^*\theta\} - x \{E\}) dx \geq \frac{1}{\text{vol}(\{\theta\})} \int_0^a \text{vol}(\{\pi^*\theta\} - a \{E\}) dx = a,
\]
what we aimed to prove. Thus, we can assume that \( f(-\infty) < f(I_\theta(u_\tau)) \). Put
\[
b := f'(I_\theta(u_\tau)) := \lim_{h \to 0^+} \frac{f(I_\theta(u_\tau)) - f(I_\theta(u_\tau) - h)}{h},
\]
which has to be a finite positive number by the convexity of $f$. Now we put

$$g(\tau) := \begin{cases} 0, & \tau \in (-\infty, I_\theta \{u_t\} - b^{-1}f(I_\theta \{u_t\})) \\ b(\tau - I_\theta \{u_t\}) + f(I_\theta \{u_t\}), & \tau \in [I_\theta \{u_t\} - b^{-1}f(I_\theta \{u_t\}), \tau_+^u]. \end{cases}$$

Due to convexity, we have $f(\tau) \geq g(\tau), \tau \in (-\infty, \tau_+^u)$. By \[78, \text{Theorem 1.2}\] and \[37\],

$$\int_X \theta^u \leq \int_X \theta^u = \text{vol}(\{\pi^*\theta - g(\tau)\}E) \text{ for } \tau \in (-\infty, \tau_+^u).$$

Thus using \[31\] we have that

$$I_\theta \{u_t\} \leq \frac{1}{\text{vol}(\{\theta\})} \int_{-\infty}^{\tau_+^u} \left( \text{vol}(\{\pi^*\theta - g(\tau)\}E) - \text{vol}(\{\theta\}) \right) d\tau + \tau_+^u$$

$$\leq \frac{1}{\text{vol}(\{\theta\})} \int_{I_\theta \{u_t\} - b^{-1}f(I_\theta \{u_t\})}^{\tau_+^u} \left( \text{vol}(\{\pi^*\theta - g(\tau)\}E) - \text{vol}(\{\theta\}) \right) d\tau + \tau_+^u$$

$$\leq \frac{1}{\text{vol}(\{\theta\})} \int_0^{\tau_+^u} \text{vol}(\{\pi^*\theta - x\}E) dx + I_\theta \{u_t\} - b^{-1}f(I_\theta \{u_t\})$$

$$= b^{-1}(S_\theta(E) - f(I_\theta \{u_t\}) + I_\theta \{u_t\}).$$

Using the above inequality, since $b > 0$, we finally arrive at $\nu(\{\hat{u}_t\}_\tau, E) = \nu(\hat{u}_{I_\theta \{u_t\}}, E) = f(I_\theta \{u_t\}) \leq S_\theta(E)$, finishing the proof.

Inspired by the above, for any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^1(X, \theta)$ we put

$$c_\mu(\hat{u}_t) := \begin{cases} c_\mu[\hat{u}_{I_\theta \{u_t\}}], & \text{if } I_\theta \{u_t\} < \tau_+^u, \\ c_\mu[V_\theta], & \text{if } I_\theta \{u_t\} = \tau_+^u, \end{cases}$$

which is called the expected complex singularity exponent of $\{\hat{u}_t\}_\tau$.

We show the following analytic characterization of the $\delta$-invariant that should be compared with Fujita–Odaka’s basis-divisor formulation \[54\].

**Theorem 4.6.** We have

$$\delta_\mu = \inf_\{u_t\} c_\mu(\hat{u}_t),$$

where the inf is over all sublinear subgeodesic rays $\{u_t\}_t \subset \mathcal{E}^1(X, \theta)$.

This result implies that $\delta_\mu > 0$. Indeed, we have $\delta_\mu \geq \inf_{u \in \text{PSH}(X, \theta)} c_\mu[u]$. By strong openness \[56\] the same measure $\mu$ has $L^p$ density for some $p > 1$. So Hölder’s inequality and \[13, 73\] imply that $c_\mu[u] \geq \frac{p-1}{p} c_{\alpha^\mu}[u] > \alpha$ for some uniform $\alpha > 0$.

**Proof.** Given $\{u_t\}_t$, if $I_\theta \{u_t\} = \tau_+^u$, then by Lemma \[43\] $c_\mu(\hat{u}_t) = c_\mu[V_\theta] \geq \delta_\mu$. If $I_\theta \{u_t\} < \tau_+^u$, then Theorem \[2.5\] and Proposition \[4.3\] imply that

$$c_\mu(\hat{u}_t) = c_\mu[\hat{u}_{I_\theta \{u_t\}}] = \inf_E \frac{A_{X, \psi} \nu_{\{I_\theta \{u_t\}, E\}}}{\nu(\hat{u}_{I_\theta \{u_t\}}, E)} \geq \inf_E \frac{A_{X, \psi} \nu_{\{I_\theta \{u_t\}, E\}}}{S_\theta(E)} = \delta_\mu.$$ 

So we obtain that $\inf_\{u_t\} c_\mu(\hat{u}_t) \geq \delta_\mu$.

To see the reverse direction, let $E$ be any prime divisor over $X$ and let $\{\hat{u}^E_t\}_\tau$ be the bounded test curve constructed in \[35\]. Then by Lemma \[1.4\] we have $I_\theta \{u_t^E\} =$
$S_\theta(E)$, and $\tau^+_\rho = \tau_\theta(E)$. If $I_\theta\{u^E_t\} = \tau^+_\rho$, i.e., $S_\theta(E) = \tau_\theta(E)$, then $\hat{u}^E_\tau \in \mathcal{E}(X, \theta)$ for all $\tau \in (\infty, \tau_\theta(E))$ in light of (31). So [33, Theorem 1.1] and (36) imply that

$$\nu(V_\theta, E) = \nu(\hat{u}^E_\tau, E) \geq \tau \text{ for all } \tau \in (\infty, \tau_\theta(E)), \text{ and hence } \nu(V_\theta, E) = \tau_\theta(E) = S_\theta(E).$$

So by Theorem 2.8 again,

$$c_\mu \{\hat{u}^E_\tau\} = c_\mu [V_\theta] \leq \frac{A(X, \psi, E)}{\nu(V_\theta, E)} = \frac{A(X, \psi, E)}{S_\theta(E)},$$

If $I_\theta\{u^E_t\} < \tau^+_\rho$, i.e., $S_\theta(E) < \tau_\theta(E)$, then from (36) we also have $\nu(\hat{u}^E_\tau, E) \geq S_\theta(E)$ (this is actually an equality, in view of Proposition 4.5). The we derive that

$$c_\mu \{\hat{u}^E_\tau\} = c_\mu [\hat{u}^E_{I_\theta\{u^E_t\}}] = c_\mu [\hat{u}^E_{S_\theta(E)}] \leq \frac{A(X, \psi, E)}{\nu(\hat{u}^E_{S_\theta(E)}, E)} \leq \frac{A(X, \psi, E)}{S_\theta(E)},$$

by Lemma 4.4 and Theorem 2.8. This implies the reverse direction.

Now we are in the position to prove the following key result.

**Theorem 4.7** (=Theorem 1.3). One has

$$\delta_\mu = \sup \{\lambda > 0|\mathcal{D}_\mu^\lambda\{u_t\} \geq 0 \text{ for all sublinear subgeodesic ray } \{u_t\}_t \subset \mathcal{E}^1(X, \theta)\}. \quad (38)$$

**Proof.** Assume that $\mathcal{D}_\mu^\lambda\{u_t\} \geq 0$ holds for any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^1(X, \theta)$. Consider the associated test curve $\{\hat{u}_t\}_t$. Set for simplicity $\beta := \sup \{\tau : \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty\}. \text{ Then by Theorem 4.1 } \beta - I_\theta\{u_t\} = \mathcal{D}_\mu^\lambda\{u_t\} \geq 0, \text{ so that } \beta \geq I_\theta\{u_t\}. \text{ This means that for any } \tau < I_\theta\{u_t\}, \text{ } c_\mu [\hat{u}_\tau] \geq \lambda. \text{ If } I_\theta\{\hat{u}_\tau\} < \tau^+_\rho, \text{ then by Lemma 2.3 we also have } c_\mu [\{\hat{u}_\tau\}] = c_\mu [\{\hat{u}_{I_\theta\{u_t\}}\}] \geq \lambda. \text{ Taking inf over all such } \{u_t\}_t \text{ then yields } \delta_\mu \geq \lambda \text{ by the previous theorem. Hence, the right hand side of (38) is dominated by } \delta_\mu.$

It remains to argue that for any $\lambda \in (0, \delta_\mu), \mathcal{D}_\mu^\lambda\{u_t\} \geq 0$ holds for any sublinear subgeodesic ray $\{u_t\}_t \subset \mathcal{E}^1(X, \theta)$. To start we note by Proposition 2.5 and Lemma 1.3 that $\mathcal{D}_\mu^\lambda(v)$ is finite for any $v \in \mathcal{E}^1(X, \theta)$.

Consider the associated test curve $\{\hat{u}_t\}_t$ and put again $\beta := \sup \{\tau : \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty\}. \text{ First, if } I_\theta\{u_t\} = \tau^+_\rho, \text{ then in view of (31) } \hat{u}_\tau \text{ has full mass for any } \tau < \tau^+_\rho, \text{ so that } \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty \text{ for all } \tau \in (\infty, \tau^+_\rho) \text{ by Proposition 2.5 and Lemma 1.3. Thus by Theorem 4.1 } \mathcal{D}_\mu^\lambda\{u_t\} = \beta - I_\theta\{u_t\} = \tau^+_\rho - \tau^+_\rho = 0, \text{ as desired. Next, if } I_\theta\{u_t\} < \tau^+_\rho, \text{ then we can apply Theorem 4.6 to conclude that } c_\mu [\hat{u}_{I_\theta\{u_t\}}] \geq \delta_\mu > \lambda, \text{ so that } \beta \geq I_\theta\{u_t\}. \text{ Thus } \mathcal{D}_\mu^\lambda\{u_t\} = \beta - I_\theta\{u_t\} \geq I_\theta\{u_t\} - I_\theta\{u_t\} = 0, \text{ finishing the proof.}$

**Proposition 4.8.** For any $\lambda \in (0, \delta_\mu)$, then there exists $\varepsilon > 0$ such that

$$\mathcal{D}_\mu^\lambda\{w_t\} \geq -\varepsilon I_\theta\{w_t\}. \quad (39)$$

for all finite energy geodesic rays $\{w_t\}_t, t \in \mathcal{R}^1(X, \theta)$ with $\sup_X w_t = 0. \text{ In particular, if } \{w_t\}_t \text{ is also } d_1\text{-unit speed, then } \mathcal{D}_\mu^\lambda\{w_t\} \geq \varepsilon.$

**Proof.** Consider Tian’s $\alpha$-invariant [73] adapted to our setting:

$$\alpha_\mu := \sup \left\{ \alpha > 0 : \sup_{u \in \text{PSH}(X, \theta)} \int_X e^{-\alpha(u - \sup_X u)} d\mu < \infty \right\}, \quad (40)$$

which is positive. Indeed, pick $A > 0$ sufficiently large such that $\theta + A\omega$ is Kähler, so that $\text{PSH}(X, \theta) \subset \text{PSH}(X, \theta + A\omega)$. The positivity of $\alpha_\mu$ follows from [8, Proposition 1.4].

22
Fix \( \eta \in (\lambda, \delta_\mu) \) and \( \alpha \in (0, \min\{\lambda, \alpha_\mu\}) \). By Hölder’s inequality, for all \( w \in \mathcal{E}^1(X, \theta) \) we have

\[
\frac{1}{\lambda} \log \int_X e^{-\lambda w} \, d\mu \geq -\frac{\lambda - \alpha}{\lambda \eta - \alpha} \log \int_X e^{-\eta w} \, d\mu - \frac{\eta - \lambda}{\lambda \eta - \alpha} \log \int_X e^{-\alpha w} \, d\mu.
\]

Therefore, one can find \( C > 0 \) such that for all \( w \in \mathcal{E}^1(X, \theta) \) with \( \sup_X w = 0 \) we have

\[
\mathcal{D}_\mu^\lambda(w) \geq \eta \frac{(\lambda - \alpha)}{\lambda \eta - \alpha} \mathcal{D}_\mu^\eta(w) - \frac{\alpha(\eta - \lambda)}{\lambda \eta - \alpha} I_\theta(w) - C.
\]  

(41)

Putting \( w_t \) in the above inequality, dividing with \( t \) and letting \( t \to \infty \), we find that

\[
\mathcal{D}_\mu^\lambda\{w_t\} \geq \eta \frac{(\lambda - \alpha)}{\lambda \eta - \alpha} \mathcal{D}_\mu^\eta\{w_t\} - \frac{\alpha(\eta - \lambda)}{\lambda \eta - \alpha} I_\theta\{w_t\}.
\]

So we conclude (39) by Theorem 4.7. In case \( \{w_t\}_t \) is unit speed, we have that \( 1 = d_1(V_\theta, w_1) = -I_\theta(w_1) \). By linearity of \( t \mapsto I_\theta(w_t) \) the last statement also follows.

5 Twisted Kähler–Einstein metrics

5.1 Energy properness of the Ding functionals

As in the previous section, let \( \{\theta\} \) be a big class and \( \mu \) be a tame measure defined in (6). We will be interested in the following KE type complex Monge–Ampère equation:

\[
\theta^n_u = e^{-\lambda u} \, d\mu
\]  

(42)

for \( \lambda > 0 \). To make sense of the above equation, we need to further assume that

\[
\lambda \in (0, c_\mu[V_\theta])
\]

Due to Proposition 2.5 we have \( c_\mu[v] > \lambda \) for any \( v \in \mathcal{E}^1(X, \theta) \), hence the \( \lambda \)-twisted Ding energy \( \mathcal{D}_\mu^\lambda \) corresponding to (42) can be defined as follows, having an exponential part \( \mathcal{L}_\mu^\lambda \) and an energy part:

\[
\mathcal{D}_\mu^\lambda(u) := \mathcal{L}_\mu^\lambda(u) - I_\theta(u) = -\frac{1}{\lambda} \log \int_X e^{-\lambda u} \, d\mu - I_\theta(u) \text{ for } u \in \mathcal{E}^1(X, \theta).
\]  

(43)

Definition 5.1. We say that \( \mathcal{D}_\mu^\lambda \) is coercive/proper if there exists \( \varepsilon > 0 \) and \( C > 0 \),

\[
\mathcal{D}_\mu^\lambda(u) \geq \varepsilon (\sup_X u - I_\theta(u)) - C, \quad u \in \mathcal{E}^1(X, \theta).
\]

As is well known (cf. [1, 8]), properness of the twisted Ding energy \( \mathcal{D}_\mu^\lambda \) implies that (12) admits a finite energy solution:

Proposition 5.2. If \( \mathcal{D}_\mu^\lambda \) is proper, and \( \mathcal{L}_\mu^\lambda \) is \( L^1 \)-continuous, then \( \mathcal{D}_\mu^\lambda \) has a global minimizer in \( \mathcal{E}^1(X, \theta) \). Any such minimizer is a solution to (12), up to an additive constant.

We believe that the \( L^1 \)-continuity of \( \mathcal{L}_\mu^\lambda \) holds unconditionally. However to stay focused, we do not attempt to prove this here. We do point out later that this always holds in case of our applications, as an easy consequence of [13, Theorem 0.2(2)].
Proof. Let \( u_i \in \mathcal{E}(X, \theta) \) with \( \sup_X u_i = 0 \) be a minimizing sequence:
\[
\lim_{i \to \infty} \mathcal{D}_\mu(u_i) = \inf_{u \in \mathcal{E}(X, \theta)} \mathcal{D}_\mu(u).
\]
The properness condition implies that we can find \( C > 0 \) such that \( I_\theta(u_i) \geq -C \) for all \( i \). Then up to a subsequence one can find \( u_\infty \in \mathcal{E}(X, \theta) \) such that \( u_i \to u_\infty \) in \( L^1 \). Note that by our assumption \( \mathcal{D}_\mu \) is \( L^1 \)-lsc on \( \mathcal{E}(X, \theta) \). So \( u_\infty \) is a minimizer of \( \mathcal{D}_\mu \).

We now proceed to show that \( u_\infty \) solves (48). Indeed, for any \( f \in C^0(X, \mathbb{R}) \), we have
\[
\mathcal{D}_\mu(u_\infty) \leq \mathcal{D}_\mu(P_\theta(u_\infty + tf)) \leq -\frac{1}{\lambda} \log \int_X e^{-\lambda(u_\infty + tf)} \mu - I_\theta(P_\theta(u_\infty + tf))
\]
for all \( t \in \mathbb{R} \). So the expression \(-\frac{1}{\lambda} \log \int_X e^{-\lambda(u_\infty + tf)} \mu - I_\theta(P_\theta(u_\infty + tf))\) is minimized at \( t = 0 \). By [7, Lemma 4.2] we can take derivatives and conclude that
\[
\frac{\int_X f e^{-\lambda u_\infty} \mu}{\int_X e^{-\lambda u_\infty} \mu} = \frac{\int_X f \theta_n}{\text{vol}(\{\theta\})}
\]
for any \( f \in C^0(X, \mathbb{R}) \). This implies that \( \frac{e^{-\lambda u_\infty} \mu}{\int_X e^{-\lambda u_\infty} \mu} = \frac{\theta_n}{\text{vol}(\{\theta\})} \), finishing the argument.

Now we relate the properness of \( \lambda \)-twisted Ding functional to destabilizing geodesic rays. For \( \lambda = 1 \) a similar result appears in [6, Theorem 2.16] in the ample case. One of the novelties of our work is that the argument below does not use the (convexity of the) K-energy, allowing us to extend the scope of the result significantly.

Theorem 5.3. Let \( \lambda \in (0, c_\mu[V_\theta]) \). Assume that \( \mathcal{D}_\mu \) is not proper, \( \mathcal{L}_\mu \) is \( L^1 \)-continuous on \( \mathcal{E}(X, \theta) \), and \( \mathcal{D}_\mu \) is convex along the geodesics of \( \mathcal{E}(X, \theta) \). Then there exists a non-trivial finite energy geodesic ray \( \{v_t\}_t \in \mathcal{R}(X, \theta) \) such that \( \mathcal{D}_\mu \{v_t\} \leq 0 \).

Proof. Since \( \mathcal{L}_\mu(u) = \mathcal{D}_\mu(u) + I_\theta(u) \), we notice that \( \mathcal{L}_\mu \) is also convex along geodesics, as \( I_\theta \) is linear along them. Recall also that \( I_\theta \) is \( L^1 \)-usc [8], so \( \mathcal{D}_\mu \) is \( L^1 \)-lsc.

Since \( \mathcal{D}_\mu \) is not proper, there exists \( \varphi^j \in \mathcal{E}(X, \theta) \) with \( \sup_X \varphi^j = 0 \) satisfying
\[
\mathcal{D}_\mu(\varphi^j) \leq \frac{1}{j} d_1(V_\theta, \varphi^j) - j \text{ for } j \in \mathbb{N}_{>0},
\]
where \( d_1(V_\theta, \varphi^j) = I_\theta(V_\theta) - I_\theta(\varphi^j) = -I_\theta(\varphi^j) \). We claim that \( d_1(V_\theta, \varphi^j) \to \infty \). Indeed, suppose that \( I_\theta(\varphi^j) \geq -C \) for all \( j \). Since \( I_\theta \) is \( L^1 \)-usc, after possibly passing to a subsequence, one can find \( \varphi^\infty \in \mathcal{E}(X, \theta) \) with \( \varphi^j \to \varphi^\infty \) in the \( L^1 \) topology. Using that \( \mathcal{D}_\mu \) is \( L^1 \)-lsc, we deduce \( -\infty < \mathcal{D}_\mu(\varphi^\infty) \leq \liminf_j \mathcal{D}_\mu(\varphi^j) = -\infty \), a contradiction.

As a result, using \( d_1(V_\theta, \varphi^j) = -I_\theta(\varphi^j) \), we get that
\[
\limsup_j \frac{\mathcal{L}_\mu(\varphi^j)}{d_1(V_\theta, \varphi^j)} = \limsup_j \frac{\mathcal{D}_\mu(\varphi^j) - d_1(V_\theta, \varphi^j)}{d_1(V_\theta, \varphi^j)} \leq -1 \text{ and } d_1(V_\theta, \varphi^j) \to \infty.
\]

Let \( [0, d_1(V_\theta, \varphi^j)] \ni t \mapsto u^j_t \in \mathcal{E}(X, V_\theta) \) be the unit speed finite energy geodesic joining \( V_\theta, \varphi^j \). Due to the convexity of \( \mathcal{L}_\mu \) along such geodesic segments we have that
\[
\limsup_j \frac{\mathcal{L}_\mu(u^j_t)}{t} \leq -1, \quad d_1(V_\theta, u^j_t) = -I_\theta(u^j_t) = t \text{ for all } t \in [0, d_1(V_\theta, \varphi^j)].
\]
As the geodesic segments $t \to u_t$ emanate from $V_\theta$ we have $\sup_X (u_t - V_\theta) = \sup_X u_t = 0$, $t \in [0, d_1(V_\theta, \varphi^j)]$, as follows from Remark 3.3. It is well known that the condition $\sup_X u_t = 0$ implies that the $L^1$ norm on $X$ of $u_t$ is uniformly bounded with respect to $j, t$ (see for example [28, Lemma 3.45]). By Fubini’s theorem, the $L^1$ norm of $(t, x) \to u_t(x)$ is uniformly bounded on any compact subset $K \subset (0, \infty) \times X$ (for high enough $j$ we have $K \subset (0, d_1(V_\theta, \varphi^j)) \times X$).

Let $S_j := (0, d_1(V_\theta, \varphi^j)) + i\mathbb{R}$, $S := (0, \infty) + i\mathbb{R} \subset \mathbb{C}$. We get that
\[
U^j(s, x) := u_{Re s}(x) \in \text{PSH}(S_j \times X, \pi^*\theta).
\]

Due to $i\mathbb{R}$-invariance, on any compact subset of $S \times X$ the $L^1$-norms of $U^j$ are uniformly bounded. As a result, we can apply [40, Proposition I.5.9] to conclude existence of $U \in \text{PSH}(S \times X, \pi^*\theta)$ such that $U^j \to U$ with respect to $L^1_{loc}(S \times X)$. Since $U$ is $i\mathbb{R}$-invariant we obtain existence of a subgeodesic ray $\{u_t\}_t$ such that $U(s, x) = u_{Re s}(x)$.

Next we argue (among other things) that $\{u_t\}$ is actually a finite energy sublinear subgeodesic. By $i\mathbb{R}$-invariance we have that $u_t(x) \to u_t(x)$ with respect to $L^1_{loc}((0, \infty) \times X)$, hence by Fubini’s theorem we obtain that $u_t \to u_t$ with respect to $L^1(X)$, for all $t \in (0, \infty) \setminus E$, where $E$ is a set of Lebesgue measure zero. Thus
\[
\frac{\mathcal{L}^\lambda_{\mu}(u_t)}{t} \leq -1, \quad \sup_X u_t = 0, \quad 0 \geq I_\theta(u_t) \geq -t, \quad \text{for all } t \in (0, \infty) \setminus E, \tag{44}
\]
as $\mathcal{L}^\lambda_{\mu}$ is $L^1$-continuous, $\sup_X(\cdot)$ is $L^1$-continuous and $I_\theta$ is $L^1$-useful [8].

Let $l_j \in (0, \infty) \setminus E$ be a sequence converging to some $t \in (0, \infty)$. Due to $t$-convexity of $\{u_t\}_t$, and the fact that $\sup_X u_t = 0$, we must have that $u_t \to d_1(u_t, V_\theta) = -I_\theta(u_t) \to 0$ as $t \to 0$. As a result, we have that (44) must hold for all $t \in (0, \infty)$.

In addition, due to [30, Theorem 3.11], we see that $u_t \to V_\theta$ in $L^1$ as $t \to 0$, as we have $d_1(u_t, V_\theta) = -I_\theta(u_t) \to 0$ as $t \to 0$. By the definition of $\mathcal{L}^\lambda_{\mu}$ and (32) we get that
\[
\mathcal{L}^\lambda_{\mu}\{u_t\} = I_\theta\{u_t\} + D^\lambda_{\mu}\{u_t\} \leq -1.
\]

Let $\{v_t\}_t$ be the maximization of $\{u_t\}_t$ defined in (30). By (31) and (44), we have that $I_\theta\{v_t\} = I_\theta\{u_t\} \geq -1$. By Theorem 4.1 and Proposition 2.5 we then have that $D^\lambda_{\mu}\{v_t\} = D^\lambda_{\mu}\{u_t\} \leq 0$ and also $\mathcal{L}^\lambda_{\mu}\{v_t\} = \mathcal{L}^\lambda_{\mu}\{u_t\} \leq -1$.

To finish we argue that $\{v_t\}_t$ is non-trivial. Since $\sup_X v_t = \sup_X u_t = 0$, we only need to rule out that $v_t = V_\theta$, $t \geq 0$. If this were the case, then $\mathcal{L}^\lambda_{\mu}\{v_t\} = 0$ for all $t \geq 0$. This would imply that $0 = \mathcal{L}^\lambda_{\mu}\{v_t\} = \mathcal{L}^\lambda_{\mu}\{v_t\} \leq -1$, which is absurd. \[\square\]

Lastly, using a ‘thermodynamic argument’ (cf. [8, Proposition 4.11] and [83, Proposition 3.5]), one can show that properness of the $\lambda$–Ding functional is an open condition. This result will only be used to prove Proposition 5.9.

**Proposition 5.4.** If $D^\lambda_{\mu}$ is proper, then so is $D^{\lambda+\tau}$ for $\tau > 0$ small. In particular, $\delta_{\mu} > \lambda$.

**Proof.** Since $D^\lambda_{\mu}$ is coercive, there exist $\varepsilon > 0$ and $C_0 > 0$ such that
\[
-\frac{1}{\lambda} \log \int_X e^{-\lambda\nu}d\mu - I_\theta(v) \geq \varepsilon \left( \sup_X v - I_\theta(v) \right) - C_0 \quad \text{for all } v \in \mathcal{E}(X, \theta).
\]

Using that $\sup_X v = \sup_X (v - V_\theta)$, we derive
\[
\sup_X v - I_\theta(v) \geq \frac{1}{\text{vol}(\{\theta\})} \int_X (v - V_\theta)\theta^\nu - I_\theta(v) \geq \frac{1}{n} \left( I_\theta(v) - \frac{1}{\text{vol}(\{\theta\})} \int_X (v - V_\theta)\theta^\nu \right),
\]
where the last inequality follows from [13, (2.7)]. Thus we have, for any \( v \in \mathcal{E}^1(X, \theta) \),

\[
-\frac{1}{\lambda} \log \int_X e^{-\lambda u} d\mu - E_\theta(v) \geq \frac{\varepsilon}{n} \left( E_\theta(v) - \frac{1}{\text{vol}(\{\theta\})} \int_X (v - V_\theta) \theta^n \right) - C_0. \tag{45}
\]

Recall that \( \lambda < c_\mu[V_\theta] \), we may fix \( \varepsilon \) small enough so that \( \lambda(1 + 2\varepsilon/n) < c_\mu[V_\theta] \) and put

\[
C_1 := \int_X e^{-\lambda(1+\varepsilon/n)V_\theta} d\mu < \infty.
\]

Our goal is to find a constant \( \tau > 0 \) such that

\[
\sup_{\varphi \in \mathcal{E}^1(X, \theta)} \int_X e^{-\lambda(1+\tau)(\varphi - I_\theta(\varphi))} d\mu < \infty. \tag{46}
\]

Note that this is equivalent to \( \inf_{\varphi \in \mathcal{E}^1(X, \theta)} D^\lambda_X(\varphi) > -\infty \), which implies that \( \delta_\mu \geq \lambda(1+\tau) \) by Theorem [13]. Moreover, in view of (44), \( D^\lambda_X \) is proper for any \( \lambda' \in (0, \lambda(1+\tau)) \), hence finishing the proof.

So the rest of the argument is devoted to showing (46). Given any \( \varphi \in \mathcal{E}^1(X, \theta) \), Theorem B] ensures that one can find \( u \in \mathcal{E}^1(X, \theta) \) solving the equation

\[
\theta^n_u = e^{-\lambda(1+\varepsilon/n)\varphi + \frac{\lambda}{n}V_\theta} d\mu
\]

for some suitable normalization constant \( c \in \mathbb{R} \), since the above right hand side has \( L^p \) density for some \( p > 1 \) (recall \( \lambda(1+\varepsilon/n) < c_\mu[V_\theta] = c_\mu[\varphi] \) and openness [12, 56]).

We put

\[
H_\mu(u) := \frac{1}{\text{vol}(\{\theta\})} \int_X \log \left( \frac{\theta^n_u}{\text{vol}(\{\theta\})d\mu} \right) \theta^n_u.
\]

One can then easily check the following:

\[
H_\mu(u) \leq \frac{1}{\text{vol}(\{\theta\})} \int_X (-\lambda(1+\varepsilon/n)\varphi)e^{-\lambda(1+\varepsilon/n)\varphi} d\mu + A \leq A \int_X e^{-\lambda(1+\varepsilon/n)\varphi} \omega^n + A < \infty
\]

for some constant \( A > 0 \), where the first inequality is due to \( V_\theta \leq 0 \), in the penultimate estimate we have used that \( te^{-\frac{\lambda t^2}{2}} \) is always bounded for all \( t \geq 0 \), and in the last estimate we have used \( \lambda(1 + 2\varepsilon/n) < c_\mu[V_\theta] \) and Proposition 2.5. So \( H_\mu(u) \) does make sense. Moreover, Jensen’s inequality implies that, for any \( \alpha \in (0, \alpha_\mu) \) (recall (10)) and \( \phi \in \text{PSH}(X, \theta) \) with \( \sup_X \phi = 0 \),

\[
\int_X (-\alpha \phi) \theta^n_u \leq \log \int_X e^{-\alpha \phi} d\mu + H_\mu(u) < \infty.
\]

In other words, any \( \phi \in \text{PSH}(X, \theta) \) is \( L^1 \)-integrable against the measure \( \theta^n_u \).

Applying Jensen’s inequality once again, we deduce from (45) that

\[
\frac{1}{\text{vol}(\{\theta\})} \int_X u \theta^n_u + \frac{1}{\lambda} H_\mu(u) - I_\theta(u) \geq \frac{\varepsilon}{n} \left( I_\theta(u) - \frac{1}{\text{vol}(\{\theta\})} \int_X (u - V_\theta) \theta^n_u \right) - C_0.
\]

Namely,

\[
H_\mu(u) \geq \lambda(1 + \frac{\varepsilon}{n}) \left( I_\theta(u) - \frac{1}{\text{vol}(\{\theta\})} \int_X (u - V_\theta) \theta^n_u \right) - C_0 \lambda - \frac{\lambda}{\text{vol}(\{\theta\})} \int_X V_\theta \theta^n_u.
\]
Combining (47) with this estimate, we can write

$$\log \int_X e^{-\lambda(1+\frac{\epsilon}{n})(\varphi-I_\theta)(\varphi)+\frac{\epsilon}{n}V_\vartheta} d\mu = \log \int_X e^{-\lambda(1+\frac{\epsilon}{n})(\varphi)+\frac{\epsilon}{n}V_\vartheta} \frac{\theta^n}{\vol(\{\theta\})} + \lambda(1+\frac{\epsilon}{n})I_\theta(\varphi)$$

$$= \int_X \log \left( e^{-\lambda(1+\frac{\epsilon}{n})(\varphi)+\frac{\epsilon}{n}V_\vartheta} \frac{\theta^n}{\vol(\{\theta\})} \right) + \lambda(1+\frac{\epsilon}{n})I_\theta(\varphi)$$

$$= \int_X \left( -\lambda(1+\frac{\epsilon}{n})\varphi \frac{\theta^n}{\vol(\{\theta\})} - H_\mu(u) + \frac{\epsilon}{n} \int_X V_\vartheta \frac{\theta^n}{\vol(\{\theta\})} + \lambda(1+\frac{\epsilon}{n})I_\theta(\varphi) \right) \leq \lambda(1+\frac{\epsilon}{n}) \left( I_\theta(\varphi) - I_\theta(u) - \int_X (\varphi - u) \frac{\theta^n}{\vol(\{\theta\})} \right) + C_0 \lambda.$$

Now using that $I_\theta(\varphi) - I_\theta(u) \leq \frac{1}{\vol(\{\theta\})} \int_X (\varphi - u) \theta^n$ for $\varphi, u \in \mathcal{E}^1(X, \theta)$, (see e.g. Theorem 2.4(3))) we arrive at

$$\int_X e^{-\lambda(1+\frac{\epsilon}{n})(\varphi-I_\theta(\varphi)+\frac{\epsilon}{n}V_\vartheta} d\mu \leq e^{C_0 \lambda} \text{ for any } \varphi \in \mathcal{E}^1(X, \theta).$$

Now we are able to prove (46). Take $\tau := \frac{e^2}{1+2\epsilon/n}$, then $1+\tau = \frac{(1+\epsilon/n)^2}{1+2\epsilon/n} < 1+\frac{\epsilon}{n}$. So Hölder’s inequality with $p = \frac{1+2\epsilon/n}{1+\epsilon/n}$ and $q = \frac{1+2\epsilon/n}{\epsilon/n}$ implies that

$$\int_X e^{-\lambda(1+\tau)(\varphi-I_\theta(\varphi)+\frac{\epsilon}{n}V_\vartheta)} d\mu = \left( \int_X e^{-\lambda(1+\tau)(\varphi-I_\theta(\varphi)+\frac{\epsilon}{n}V_\vartheta)} \frac{\theta^n}{\vol(\{\theta\})} \right) \left( \int_X e^{-\lambda(1+\tau)\vartheta} \frac{\theta^n}{\vol(\{\theta\})} \right) \leq \left( \int_X e^{-\lambda(1+\tau)(\varphi-I_\theta(\varphi)+\frac{\epsilon}{n}V_\vartheta)} \frac{\theta^n}{\vol(\{\theta\})} \right)^\frac{1+\epsilon/n}{1+2\epsilon/n} \cdot \left( \int_X e^{-\lambda(1+\tau)\vartheta} \frac{\theta^n}{\vol(\{\theta\})} \right)^\frac{\epsilon/n}{1+2\epsilon/n}.$$

This completes the proof. \hfill \square

### 5.2 The proof of Theorem 1.2

In this part we return to the assumptions at the very beginning of this paper, that $c_1(-K_X)$ admits a decomposition $c_1(-K_X) = \{\theta\} + \{\eta\}$, where $\eta$ is a smooth representative of a pseudoeffective class and $\psi \in \text{PSH}(X, \eta)$. In particular, $-K_X$ is big (thus $X$ is projective). Further assume that $\eta + dd^c\psi \in \{\eta\}$ has klt singularities, i.e., $\int_X e^{-\psi}\omega^n < \infty$. Our goal is to search for a twisted KE potential $u \in \mathcal{E}^1(X, \theta)$ satisfying

$$\text{Ric} \theta_u = \theta_u + \eta_\psi.$$

This amounts to solving

$$\theta^n_u = e^{f-u-\psi}\omega^n,$$

with $f \in C^\infty(X)$ a Ricci potential satisfying $\text{Ric} \omega = \theta + \eta + dd^c f$. So in what follows we choose our tame measure $\mu$ to be

$$\mu := e^f-\psi\omega^n.$$

**Proposition 5.5.** Suppose that $c_\mu[V_\eta] > 1$ and that $\mathcal{D}_\mu^1$ is not proper. Then $\delta_\mu \leq 1$.  

27
Proof. Since $-K_X$ is big under our conditions, we have that $X$ is Moishezon [60], hence also projective [68]. Also, the condition $c_\mu[V_\theta] > 1$ implies that $c_\mu[v] > 1$ for any $v \in \mathcal{E}^1(X, \theta)$ (Proposition 2.5). As a result, given a finite energy geodesic $t \mapsto v_t$, both conditions of [13, Theorem 0.1] are satisfied, and we obtain that $t \mapsto \mathcal{D}_\mu^1(v_t)$ is convex.

By openness [12, 56] and [43, Theorem 0.2(2)] we get that $\mathcal{L}_\mu^1(u)$ (recall (43)) is $L^1$-continuous. Now, applying Theorem 5.3 we find a unit speed finite energy geodesic ray $\{w_t\}_t$ with sup$_X w_t = 0$ such that $\mathcal{D}_\mu^1\{w_t\} \leq 0$. Then by Proposition 4.8 one must have $\delta_\mu \leq 1$, as asserted.

A direct consequence is the following:

**Corollary 5.6** (=Theorem 1.2). If $\delta_\mu > 1$ then $\mathcal{D}_\mu^1$ is proper. In particular, (48) admits a solution $u \in \text{PSH}(X, \theta)$ with minimal singularity type.

Proof. Lemma 4.3 gives that $c_\mu[V_\theta] > 1$. By the previous result and its proof, $\mathcal{D}_\mu^1$ is proper and $\mathcal{L}_\mu^1$ is $L^1$-continuous. Thus, by Proposition 5.2 (48) admits a solution $u \in \mathcal{E}^1(X, \theta)$. That $u$ has minimal singularity type follows from Kolodziej’s estimate (see [19, Theorem B]), as the right hand side of (48) has $L^p$ density for some $p > 1$ by openness [12, 56].

**The (partial) converse.** In this paragraph we give a (partial) converse for Corollary 5.6 and show the role of uniqueness in a full converse. We begin with the next standard result, following [7, Theorem 6.6].

**Proposition 5.7.** If $u \in \mathcal{E}^1(X, \theta)$ is a solution to (48) then $u$ minimizes $\mathcal{D}_\mu^1$.

Proof. Given $u, v \in \mathcal{E}^1(X, \theta)$, let $[0, 1] \ni t \mapsto u_t \in \mathcal{E}^1(X, \theta)$ be the geodesic segment connecting $u_0 = u$ and $u_1 = v$, constructed in [30]. If $u$ satisfies $\theta_u^\mu = e^{-\mu}$ then $c_\mu[V_\theta] > 1$ by openness [12, 56], so $\mathcal{D}_\mu^1$ is well defined on $\mathcal{E}^1(X, \theta)$ by Proposition 2.5. We introduce

$$f(t) := \mathcal{D}_\mu^1(u_t), \ t \in [0, 1].$$

By $d_1$-continuity of the Ding energy and [13, Theorem 0.1], $f$ is continuous and convex.

We want to show that $\mathcal{D}_\mu^1(u) \leq \mathcal{D}_\mu^1(v)$. Note that $u$ has minimal singularity by [19]. We can assume that $v$ also has minimal singularity type, since such potentials are $d_1$-dense in $\mathcal{E}^1(X, \theta)$. By the above considerations, it is enough to argue that

$$\left. \frac{d}{dt} \right|_{t=0^+} f(t) \geq 0.$$

After adding a constant to $v$, we may further assume that $u > v$ on $X$. Consider $g(t) := \frac{u - v}{t}$ for $t \in (0, 1]$. Since $u, v$ have minimal singularity type, $t \mapsto u_t$ is uniformly $t$-Lipschitz. As a result, \( \lim_{t \to 0^+} g(t) =: \dot{u}_0^+ \) is uniformly bounded and well defined away from a pluripolar set. We notice that (by [30, Theorem 2.4(iii)])

$$\frac{I_\theta(u_t) - I_\theta(u)}{t} \leq \frac{1}{\text{vol}(\{\theta\})} \int_X g(t) \theta_u^\mu = \frac{1}{\text{vol}(\{\theta\})} \int_X g(t)e^{-\mu}d\mu.$$

Letting $t \to 0$ by monotone convergence we get

$$\left. \frac{d}{dt} \right|_{t=0^+} I_\theta(u_t) \leq \frac{1}{\text{vol}(\{\theta\})} \int_X \dot{u}_0^+ e^{-\mu}d\mu. \quad (49)$$
On the other hand, using dominated convergence we conclude that
\[
\lim_{t \to 0} \int_X e^{-u_t} d\mu - \int_X e^{-u} d\mu = \lim_{t \to 0} \int_X \frac{u - u_t}{t} e^{u - u} - \frac{1}{u - u_t} e^{-u} d\mu = \int_X (-\dot{u}_0) e^{-u} d\mu. \tag{50}
\]
Putting together (49) and (50), we conclude that
\[
\frac{d}{dt} \bigg|_{t=0} + D_1^\mu (u_t) \geq 0,
\]
finishing the proof.

We now record a partial converse to Corollary 5.6:

**Proposition 5.8.** Assume that (48) is solvable. Then \(\delta_\mu \geq 1\).

**Proof.** As follows from the previous result (and its argument), \(D_1^\mu\) is well defined and bounded from below on \(\mathcal{E}_1^1(X, \theta)\). This gives \(\delta_\mu \geq 1\) in light of Theorem 1.5.

We end this section with the following full converse to Corollary 5.6, provided the uniqueness of the solution.

**Proposition 5.9.** If the twisted KE metric found in Corollary 5.6 is unique, then \(\delta_\mu > 1\).

**Proof.** Indeed, if a unique twisted KE metric exists then the conditions of the properness/existence principle of the first named author and Y. Rubinstein \(\text{[36, Theorem 3.7]}\) (c.f. \(\text{[28, Theorem 4.7]}\)) are satisfied, implying that the Ding functional \(D_\mu^1\) is proper. Then Proposition 5.4 ensures that \(\delta_\mu > 1\), as asserted.

# 6 The case of Fano type varieties

In this section we show how our techniques yield a simplified proof of the YTD existence theorem of Li–Tian–Wang. Conversely, we also elaborate on the recent note of C. Xu, pointing out how the Li–Tian–Wang result together with deep results of the minimal model gives an alternative argument of our Corollary 1.3.

A log Fano pair \((Z, \Delta)\) consists of the following data: \(Z\) is a normal projective variety and \(\Delta\) an effective Weil divisor on \(Z\) such that \(-K_Z - \Delta\) is an ample \(\mathbb{Q}\)-Cartier divisor and that \((Z, \Delta)\) has klt singularities.

## 6.1 The YTD existence theorem of Li–Tian–Wang.

In what follows we give a simplified proof of the following result due to Li–Tian–Wang \(\text{[66]}\), using the techniques of our paper:

**Theorem 6.1.** The log Fano pair \((Z, \Delta)\) admits a KE metric if it is uniformly K-stable.

Put for simplicity \(L := -K_Z - \Delta\). Then the pair \((Z, \Delta)\) being uniformly K-stable means exactly that \(\delta_\Delta(L) > 1\) (cf. \(\text{[13]}\)), where

\[
\delta_\Delta(L) := \inf_E A_{Z, \Delta}(E) \overline{S}_L(E).
\]

Here the infimum is over all prime divisors \(E \subset Y \xrightarrow{\pi} Z\) over \(Z\), where \(Y\) is a smooth projective manifold and \(\pi\) is a projective birational morphism. The log discrepancy is \(A_{Z, \Delta}(E) := 1 + \operatorname{ord}_E(K_Y + \pi^* L)\) and the expected vanishing order \(S_L(E)\) is given by

\[
\overline{S}_L(E) := \frac{1}{\operatorname{vol}(L)} \int_0^{r_L(E)} \operatorname{vol}(\pi^* L - xE) dx,
\]
with \( \tau_L(E) := \sup\{x > 0 : \pi^* L - xE \text{ big}\} \) being the pseudoeffective threshold. The volume function \( \text{vol}(\cdot) \) here is understood in the sense of algebraic geometry (cf. \[62\]).

To prove that \((Z, \Delta)\) admits a KE metric, we use the variational framework in \[66\]. The goal is to argue that the corresponding Ding functional \( D \) defined in \[66, (40)\] for the pair \((Z, \Delta)\) is proper. Suppose on the contrary that \( D \) is not proper. To derive a contradiction out of this, the Li–Tian–Wang argument takes five steps; see \[66, \S 4.1-4.5\], where the first step showing the convexity of the Mabuchi functional is particularly difficult to establish.

However, relying on the results in this paper we can obtain a contradiction almost immediately. Indeed, take a log resolution \( X \xrightarrow{\pi} Z \) of the pair \((Z, \Delta)\), so in particular \( X \) is a smooth birational model over \( Z \). We take our big class \( \{\theta\} \) to be \( c_1(\sigma^* L) \) and let \( \mu = e^{\chi-\psi} \omega^n \) be the tame measure on \( X \) that is associated to the log Fano pair \((Z, \Delta)\) (see \[8, Lemma 3.2(i)\]), with \( \chi := \psi^+, \psi := \psi^- \) and \( \omega^n := d\psi \) for some Kähler form \( \omega \) on \( X \). Then we consider the following algebraic \( \delta \)-invariant:

\[
\tilde{\delta}_\mu(\sigma^* L) := \inf_E \frac{A_{\chi,\psi}(E)}{S_{\sigma^* L}(E)},
\]

where the inf is over all prime divisors in smooth birational models over \( X \). Note that this invariant is potentially different from the \( \delta_\mu \) invariant defined in \[59\], as the latter involves inf over all smooth bimeromorphic models over \( X \). However due to \[69, Theorem B.7\] and the non-pluripolar volume being the same as the algebraic volume in the Nerón–Severi space (see \[17\] and \[10, Proposition 1.18\]), we find that all the previous arguments in this paper hold without change when restricting to divisors in smooth birational models over \( X \). Hence, Theorem 4.3 (or simply Theorem 4.6) gives that

\[
\delta_\mu(\{\theta\}) = \tilde{\delta}_\mu(\sigma^* L).
\]

Moreover, as any prime divisor \( E \) over \( X \) is a prime divisor over \( Z \), one has \( A_{\chi,\psi}(E) = A_{Z,\Delta}(E) \) and \( S_{\sigma^* L}(E) = S_L(E) \). So we find that

\[
\delta_\mu(\{\theta\}) \geq \delta_\Delta(L).
\]

(They are actually equal, as by Hironaka’s theorem any prime divisor over \( Z \) can be viewed as a prime divisor over \( X \) as well.)

Next, consider the Ding functional \( D^1_\mu \) associated to the above chosen triple \((X, \{\theta\}, \mu)\).

It follows from Berndtsson’s convexity that \( D^1_\mu \) is convex along geodesics in \( \mathcal{E}^1(X, \theta) \) (see \[8, Theorem 11.1\] for the precise version that we need). Note also that \( V_\theta = 0 \) (as \( \{\theta\} \) is semipositive) and \( \int_X e^{-p\psi} \omega^n < \infty \) for some \( p > 1 \) (by \[8, Lemma 3.2(i)\]). So by Proposition 2.5 one has \( \int_X e^{-pu-\psi} \omega^n < \infty \) for any \( u \in \mathcal{E}^1(X, \theta) \). As a result, \[43, Theorem 0.2(2)\] implies that \( u \mapsto \int_X e^{-u-\psi} \omega^n \) is \( L^1 \)-continuous on \( \mathcal{E}^1(X, \theta) \). This argument also implies that \( L^1_\mu \) is \( L^1 \)-continuous (recall \[11\]), as \( \chi \) is bounded from above.

Moreover, \( D^1_\mu \) is not proper since we assumed the Ding functional \( D \) on \( Z \) is not proper. Then by Theorem 5.3 we obtain a unit speed finite energy geodesic ray \( \{w_t\}_t \) with \( \sup_X w_t = 0 \) such that \( D^1_\mu \{w_t\} \leq 0 \).

On the other hand, \((Z, \Delta)\) being uniformly K-stable implies that \( \delta_\mu(\{\theta\}) \geq \delta_\Delta(L) > 1 \). So we derive from Proposition 4.8 that \( D^1_\mu \{w_t\} > 0 \). This immediately gives us a contradiction, hence proving Theorem 6.1.
6.2 Connections with birational geometry

In this part we restrict ourselves to the case when \( \{ \theta \} = c_1(-K_X) \). So \( X \) is projective and \(-K_X\) is big. In this particular setting, following \[79\] closely, we point out how our work connects with birational geometry to give an algebraic proof of Corollary 1.3.

When \(-K_X\) is big, the anti-canonical ring \( R(X, -K_X) := \bigoplus_{m \geq 0} H^0(X, -mK_X) \) might not be finitely generated. Surprisingly, C. Xu recently observed that when \( \delta(-K_X) > 1 \), the ring \( R(X, -K_X) \) is indeed finitely generated:

**Lemma 6.2.** \[79, Lemma 3.1\] Assume that \(-K_X\) is big and \( \delta(X, -K_X) > 1 \). Then there exists an effective \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} -K_X \) such that the pair \((X, D)\) is klt. In particular \( X \) is a Mori dream space, hence \( R(X, -K_X) \) is finitely generated.

The last statement follows from \[14, Corollary 1.3.1\]. In what follows we point out how this result gives an alternative proof for Corollary 1.3. What is more, using Proposition 6.4, in this case we get a full characterization:

**Theorem 6.3.** Suppose that \( \{ \theta \} = c_1(-K_X) \) is big. Then \( \text{Ric} \theta_u = \theta_u \) has a unique solution \( u \in \text{PSH}(X, \theta) \) with minimal singularity if and only if \( \delta(-K_X) > 1 \).

By Proposition 6.9 we only need to argue existence and uniqueness. So we assume that \( \delta(-K_X) > 1 \) and let \( r > 0 \) be sufficiently divisible such that \( \bigoplus_{m \geq 1} H^0(X, -mrK_X) \) is generated by \( H^0(X, -rK_X) \). Standard results in birational geometry (see e.g., \[61\] Proposition 1.16\]) imply a number of facts that we now recall. First, \( \lvert -rK_X \rvert \) induces a birational map, say \( \phi : X \to Z \), from \( X \) to a normal projective variety \( Z \). Next, letting \( H \) denote the hyperplane class on \( Z \), then \(-rK_Z \sim H\), so in particular \(-K_Z\) is an ample \( \mathbb{Q} \)-line bundle. Moreover, letting \( X \xleftarrow{\pi} W \xrightarrow{\tau} Z \) be a resolution of the base locus of \( \lvert -rK_X \rvert \), one has a following Zariski decomposition \( \pi^* \lvert -mrK_X \rvert = \tau^* \lvert mH \rvert + mF \) for all \( m \geq 0 \), where \( F \geq 0 \) is a \( \tau \)-exceptional divisor on \( W \). So in particular, one has a Zariski decomposition for the big line bundle \(-K_X\):

\[-\pi^* K_X = -\tau^* K_Z + \frac{1}{r} F.\]  

(51)

Finally, one can show that \( Z \) has klt singularities. Indeed, it is enough to argue that \( A_Z(E) > 0 \) for any prime divisor \( E \) over \( Z \). For this, pick \( D \in \lvert -K_X \rvert \) such that \((X, D)\) is a klt pair, whose existence is guaranteed by Lemma 6.2. So one has \( A_X(E) - \text{ord}_E(D) > 0 \) for all \( E \). Using the Zariski decomposition (51) this implies that

\[A_Z(E) = A_W(E) + \text{ord}_E(K_W - \pi^* K_X) - \text{ord}_E(r^{-1} F) \geq A_W(E) + \text{ord}_E(K_W - \pi^* K_X) - \text{ord}_E(\pi^* D) = A_X(E) - \text{ord}_E(D) > 0.\]

So \( Z \) has klt singularities, as asserted. Therefore, \( Z \) is a \( \mathbb{Q} \)-Fano variety.

Using the Zariski decomposition (51) one more time, and ideas from \[8, 19\], we deduce the following result:

**Proposition 6.4.** The KE metrics in \( c_1(-K_X) \) are in one-to-one correspondence with the singular KE metrics in \( c_1(-K_Z) \).

**Proof.** We use the terminology underlying formula (51). Let \( \omega \in c_1(-K_Z) \) be a smooth background metric on \( Z \). Put \( \beta := \tau^* \omega \), which is a smooth semi-positive form in \( c_1(-\tau^* K_Z) \). We choose tame measures \( \mu_1 \) and \( \mu_2 \) on \( X \) and \( Z \) respectively in the sense
of \cite[Definition 3.1]{8}. More specifically, \( \mu_1 \) is determined by \( \theta \in c_1(-K_X) \) and \( \mu_2 \) by \( \omega \in c_1(-K_Z) \). One can naturally lift two measures to \( W \), which will be denoted by \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) respectively. We further assume that \( h \) is a smooth Hermitian metric on \( \mathcal{O}_W(F) \) so that \( \pi^*\theta = \beta + r^{-1}dd^c \log h \). Let \( S_F \) be the canonical section of \( \mathcal{O}_W(F) \).

There exists divisors \( E \) and \( E' \) on \( W \) respectively so that \( K_W = \pi^*K_Z + E \) and \( K_W = \pi^*K_X + E' \). Consequently, using \eqref{51} we obtain that \( E' = E + \frac{1}{r}F \). Using (the proof of) \cite[Lemma 3.2(1)]{8} we obtain that

\[
\tilde{\mu}_1 = e^C(|S_F|^2_h)^{1/r}\tilde{\mu}_2,
\]

where \( C \in \mathbb{R} \) is some suitable normalization constant.

Now, a KE metric in \( c_1(-K_Z) \) corresponds to a bounded potential \( u \in PSH(W, \beta) \) such that \((\beta + dd^c u)^n = e^{C_1-u}\tilde{\mu}_2\) for some constant \( C_1 \in \mathbb{R} \). Let \( v := u + r^{-1} \log |S_F|^2_h \), then \( v \in PSH(W, \pi^*\theta) \) has minimal singularity type and solves

\[
(\pi^*\theta + dd^c v)^n = e^{C_1-C-\tilde{u}}1, \]

which gives a KE metric in \( c_1(-K_X) \). Conversely, by the same argument, a KE metric in \( c_1(-K_X) \) also gives rise to a KE metric in \( c_1(-K_Z) \).

Thanks to \eqref{51}, it is easy to see that the inequality \( \delta_{mr}(-K_Z) \geq \delta_{mr}(-K_X) \) holds as long as one has \( \delta_{mr}(-K_X) \geq 1 \) (here we are using Fujita–Odaka’s quantized delta invariant \[54\]). So by \cite[Theorem 4.4]{15} we can let \( m \to \infty \) to conclude the following result, that is part of \cite[Theorem 1.2]{7}:

\[
\delta(Z, -K_Z) \geq \delta(X, -K_X) > 1. \tag{52}
\]

Putting Proposition \ref{6.1} and \ref{52} together it is enough to show that \( Z \) has a unique KE metric when \( \delta(-K_Z) > 1 \). This latter condition implies that \( Z \) is uniformly K-stable, so \( Z \) does not support any non-trivial holomorphic vector field. Theorem \ref{6.1} and \cite[Theorem 11.1]{8} now yield existence of a unique KE metric on \( Z \), as desired.

\section*{References}

[1] S. Bando and T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions, Algebraic geometry, Sendai, 1985. Volume 10, Adv. Stud. Pure Math. Pages 11–40. North-Holland, Amsterdam, 1987.

[2] E. Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. Acta Mathematica, 149:1–40, 1982.

[3] E. Bedford and B. A. Taylor. The dirichlet problem for a complex Monge–Ampère equation. Inventiones mathematicae, 37(1):1–44, 1976.

[4] E. Bedford and B. A. Taylor. Fine topology, σild boundary, and \((dd^c)^n\). J. Funct. Anal., 72(2):225–251, 1987.

[5] R. J. Berman. K-polystability of Q-Fano varieties admitting Kähler–Einstein metrics. Inventiones mathematicae, 203(3):973–1025, 2016.

[6] R. J. Berman. Measure preserving holomorphic vector fields, invariant anti-canonical divisors and Gibbs stability. Anal. Math., 48(2):347–375, 2022.

[7] R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi. A variational approach to complex monge-ampère equations. Publications mathématiques de l’IHÉS:1–67, 2013.

[8] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. Journal für die reine und angewandte Mathematik (Crelles Journal), 751:27–89, 2019.
[9] R. J. Berman, S. Boucksom, and M. Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. *Journal of the American Mathematical Society*, 34(3):605–652, 2021.

[10] R. J. Berman and H. Guenancia. Kähler-Einstein metrics on stable varieties and log canonical pairs. *Geometric and Functional Analysis*, 24(6):1683–1730, 2014.

[11] B. Berndtsson. A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. *Inventiones mathematicae*, 200(1):149–200, 2015.

[12] B. Berndtsson. The openness conjecture and complex Brunn-Minkowski inequalities, *Complex geometry and dynamics*. Volume 10, Abel Symp. Pages 29–44. Springer, Cham, 2015.

[13] B. Berndtsson and M. Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. *Duke Mathematical Journal*, 145(2):341–378, 2008.

[14] C. Birken, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *Journal of the American Mathematical Society*, 23(2):405–468, 2010.

[15] H. Blum and M. Jonsson. Thresholds, valuations, and K-stability. *Adv. in Math.*, 365:107062, 57, 2020.

[16] S. Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Annales scientifiques de l’École normale supérieure*, 37(1):45–76, 2004.

[17] S. Boucksom. On the volume of a line bundle. *International Journal of Mathematics*, 13(10):1043–1063, 2002.

[18] S. Boucksom. Singularities of plurisubharmonic functions and multiplier ideals. [http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf](http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf), 2017.

[19] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge–Ampère equations in big cohomology classes. *Acta mathematica*, 205(2):199–262, 2010.

[20] S. Boucksom, C. Favre, and M. Jonsson. Valuations and plurisubharmonic singularities. *Publications of the Research Institute for Mathematical Sciences*, 44(2):449–494, 2008.

[21] S. Boucksom, T. Hisamoto, and M. Jonsson. Uniform K-stability, Duistermaat–Heckman measures and singularities of pairs. *Annales de l’Institut Fourier*, 67(2):743–841, 2017.

[22] S. Boucksom and M. Jonsson. A non-Archimedean approach to K-stability, II: Divisorial stability and openness. *J. Reine Angew. Math.*, 805:1–53, 2023.

[23] E. Calabi. On Kähler manifolds with vanishing canonical class. *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*. Volume 12. 1957, pages 78–89.

[24] I. Cheltsov, J. Park, and C. Shramov. Delta invariants of singular del Pezzo surfaces. *Journal of Geometric Analysis*, 31(3):2354–2382, 2021.

[25] X. Chen, S. Donaldson, and S. Sun. Kähler–Einstein metrics on Fano manifolds. I–III. *Journal of the American Mathematical Society*, 28(1):183–234, 235–278, 239–278, 2015.

[26] X. Chen, S. Sun, and B. Wang. Kähler-Ricci flow, Kähler-Einstein metric, and K-stability. *Geometry & Topology*, 22(6):3145–3173, 2018.

[27] T. Darvas. Geometric pluripotential theory on Kähler manifolds. *Advances in complex geometry. A symposium in honor of S. Lefschetz*. Volume 12. 1957, pages 78–89.

[28] T. Darvas. Weak geodesic rays in the space of Kähler potentials and the class $\mathcal{E}(X,\omega_0)$. *Journal of the Institute of Mathematics of Jussieu*, 16(4):837–858, 2017.

[29] T. Darvas, E. Di Nezza, and C. H. Lu. $L^1$ metric geometry of big cohomology classes. *Annales de l’Institut Fourier*, 68(7):3053–3086, 2018.

[30] T. Darvas, E. Di Nezza, and C. H. Lu. Log-concavity of volume and complex Monge–Ampère equations with prescribed singularity. *Mathematische Annalen*, 379(1-2):95–132, 2021.

[31] T. Darvas, E. Di Nezza, and C. H. Lu. Monotonicity of non-pluripolar products and complex Monge–Ampère equations with prescribed singularity. *Analysis in PDE*, 11(8):2049–2087, 2018.

[32] T. Darvas, E. Di Nezza, and C. H. Lu. On the singularity type of full mass currents in big cohomology classes. *Compositio Mathematica*, 154(2):380–409, 2018.

[33] T. Darvas, E. Di Nezza, and C. H. Lu. The metric geometry of singularity types. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2021(771):137–170, 2021.

[34] T. Darvas and C. H. Lu. Geodesic stability, the space of rays and uniform convexity in Mabuchi geometry. *Geometry & Topology*, 24(4):1907–1967, 2020.
[36] T. Darvas and Y. Rubinstein. Tian’s properness conjectures and Finsler geometry of the space of Kähler metrics. *Journal of the American Mathematical Society*, 30(2):347–387, 2017.

[37] T. Darvas, M. Xia, and K. Zhang. A transcendental approach to non-Archimedean metrics of pseudoeffective classes, 2023. To appear.

[38] T. Darvas and M. Xia. The closures of test configurations and algebraic singularity types. *Advances in Mathematics*, 397:Paper No. 108198, 56, 2022.

[39] V. Datar and G. Székelyhidi. Kähler–Einstein metrics along the smooth continuity method. *Geometric and Functional Analysis*, 26(4):975–1010, 2016.

[40] J.-P. Demailly. Analytic methods in algebraic geometry. International Press Somerville, MA, 2012.

[41] J.-P. Demailly. Complex analytic and differential geometry. https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, 2012.

[42] J.-P. Demailly. Multiplier ideal sheaves and analytic methods in algebraic geometry, *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*. Volume 6, ICTP Lect. Notes, pages 1–148. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[43] J.-P. Demailly and J. Kollár. Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Annales Scientifiques de l’École Normale Supérieure (4)*, 34(4):525–556, 2001.

[44] R. Dervan. Uniform stability of twisted constant scalar curvature Kähler metrics. *International Mathematics Research Notices*, 2016(15):4728–4783, 2016.

[45] R. Dervan and R. Reboulet. Ding stability and Kähler-Einstein metrics on manifolds with big anticanonical class, 2022. arXiv:2209.08952.

[46] R. Dervan and E. Legendre. Valuative stability of polarised varieties. *Math. Ann.*, 385(1-2):357–391, 2023.

[47] R. Dervan and J. Ross. K-stability for Kähler manifolds. *Math. Res. Lett.*, 24(3):689–739, 2017. issn: 1073-2780.

[48] E. Di Nezza and C. H. Lu. Geodesic distance and Monge-Ampère measures on contact sets. *Anal. Math.*, 48(2):451–488, 2022.

[49] W. Y. Ding. Remarks on the existence problem of positive Kähler-Einstein metrics. *Mathematische Annalen*, 282(3):463–471, 1988.

[50] S. Donaldson. Scalar curvature and stability of toric varieties. *Journal of Differential Geometry*, 62(2):289–349, 2002.

[51] Z. S. Dyrefelt. K-semistability of cscK manifolds with transcendental cohomology class. *The Journal of Geometric Analysis*:1–34, 2016.

[52] P. Eyssidieux, V. Guedj, and A. Zeriahi. Singular Kähler–Einstein metrics. *Journal of the American Mathematical Society*, 22(3):607–639, 2009.

[53] K. Fujita. A valuative criterion for uniform K-stability of Q-Fano varieties. *Journal für die reine und angewandte Mathematik*, 2019(751):309–338, 2019.

[54] K. Fujita and Y. Odaka. On the K-stability of Fano varieties and anticanonical divisors. *Tohoku Mathematical Journal*, 70(4):511–521, 2018.

[55] Q. Guan, Z. Li, and X. Zhou. Estimation of weighted $L^2$ norm related to Demailly’s strong openness conjecture, 2016. arXiv:1603.05733 [math.CV].

[56] Q. Guan and X. Zhou. A proof of Demailly’s strong openness conjecture. *Annals of Mathematics (2)*, 182(2):605–616, 2015.

[57] V. Guedj and A. Zeriahi. Degenerate complex Monge-Ampère equations, volume 26 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2017.

[58] P. H. Hiep. The weighted log canonical threshold. *C. R. Math. Acad. Sci. Paris*, 352(4), 2014.

[59] T. Hisamoto. Stability and coercivity for toric polarization. 2016. arXiv:1610.07998.

[60] S. Ji and B. Shiffman. Properties of compact complex manifolds carrying closed positive currents. *J. Geom. Anal.*, 3(1):37–61, 1993.

[61] J. Kollár. Singularities of the minimal model program, volume 200. Cambridge University Press, 2013.
[62] R. Lazarsfeld. Positivity in algebraic geometry. I, volume 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[63] C. Li. K-semistability is equivariant volume minimization. Duke Mathematical Journal, 166(16):3147–3218, 2017.

[64] C. Li. G-uniform stability and Kähler-Einstein metrics on Fano varieties. Inventiones Mathematicae, 227(2):661–744, 2022.

[65] C. Li, G. Tian, and F. Wang. On the Yau-Tian-Donaldson conjecture for singular Fano varieties. Communications on Pure and Applied Mathematics, 74(8):1748–1800, 2021.

[66] C. Li, G. Tian, and F. Wang. The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties. Peking Math. J., 5(2):383–426, 2022.

[67] Y. Liu, C. Xu, and Z. Zhuang. Finite generation for valuations computing stability thresholds and applications to K-stability. Annals of Mathematics (2), 196(2):507–566, 2022.

[68] B. G. Moishezon. On $n$-dimensional compact complex manifolds having $n$ algebraically independent meromorphic functions. I. Izv. Akad. Nauk SSSR Ser. Mat., 30:133–174, 1966.

[69] J. Ross and D. Witt Nyström. Analytic test configurations and geodesic rays. Journal of Symplectic Geometry, 12(1):125–169, 2014.

[70] Y. T. Siu. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. Inventiones Mathematicae, 27:53–156, 1974. issn: 0020-9910.

[71] G. Székelyhidi. An introduction to extremal Kähler metrics, volume 152 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2014.

[72] G. Tian. Kähler–Einstein metrics with positive scalar curvature. Inventiones Mathematicae, 130(1):1–37, 1997.

[73] G. Tian. On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. Inventiones mathematicae, 89(2):225–246, 1987.

[74] G. Tian. Canonical metrics in Kähler geometry. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.

[75] G. Tian. K-stability and Kähler-Einstein metrics. Communications on Pure and Applied Mathematics, 68(7):1085–1156, 2015.

[76] G. Tian and F. Wang. On the existence of conic Kähler-Einstein metrics. Advances in Mathematics, 375, 2020.

[77] A. Trusiani. Kähler-Einstein metrics with prescribed singularities on Fano manifolds. J. Reine Angew. Math., 793:1–57, 2022.

[78] D. Witt Nyström. Monotonicity of non-pluripolar Monge–Ampère masses. Indiana University Mathematics Journal, 68(2):579–591, 2019.

[79] C. Xu. K-stability for varieties with a big anticanonical class. Épijournal Géom. Algébrique:Art. 7, 9, 2023.

[80] C. Xu and Z. Zhuang. On positivity of the CM line bundle on K-moduli spaces. Annals of Mathematics (2), 192(3):1005–1068, 2020.

[81] S.-T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I. Communications on pure and applied mathematics, 31(3):339–411, 1978.

[82] K. Zhang. A quantization proof of the uniform Yau-Tian-Donaldson conjecture. J. Eur. Math. Soc. (2023), DOI 10.4171/JEMS/1373.

[83] K. Zhang. Continuity of delta invariants and twisted Kähler–Einstein metrics. Advances in Mathematics, 388:107888, 2021.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND
tdarvas@umd.edu

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY
kwzhang@bnu.edu.cn