STABILITY CONDITIONS ON THREEFOLDS WITH VANISHING
CHERN CLASSES

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ABSTRACT. We prove the Bogomolov-Gieseker type inequality conjectured by
Bayer, Macrì and Toda for threefolds with semistable tangent bundles and
vanishing Chern classes in any characteristic, which was originally proved by
Bayer, Macrì and Stellari in characteristic zero. This gives the existence of
Bridgeland stability conditions on such threefolds. As applications, we obtain
Reider type theorem and confirm Fujita’s conjecture for such threefolds in any
characteristic.

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1. INTRODUCTION

Since Bridgeland’s introduction in [7], stability conditions for triangulated cate-
gories have drawn a lot of attentions, and have been investigated intensively. The
existence of stability conditions on three-dimensional varieties is often considered
the biggest open problem in the theory of Bridgeland stability conditions.

In [4], Bayer, Macrì and Toda introduced a conjectural construction of Bridge-
land stability conditions for any projective threefold. Here the problem was reduced
to proving a Bogomolov-Gieseker type inequality for the third Chern character of
tilt-stable objects. It has been shown to hold for some Fano 3-folds [24, 27, 13, 6, 26],
abelian 3-folds [22, 23, 3], étale quotients of abelian 3-folds [3], toric threefolds
[6], product threefolds of projective spaces and abelian varieties [12] and quintic
threefolds [19]. However, counterexamples of the original Bogomolov-Gieseker type
inequality are found (see [28]). The modification of the original inequality for any
Fano threefolds is proved in [6, 26], and it still implies the existence of stability
conditions on such threefolds. Recently, Yuchen Liu [20] showed the existence
of stability conditions on product varieties. His method is different from that of
Bayer-Macrì-Toda.
In this paper, we prove the original Bogomolov-Gieseker type inequality for threefolds with semistable tangent bundles and vanishing Chern classes in any characteristic. This gives the existence of Bridgeland stability conditions on such threefolds.

**Theorem 1.1.** Let $X$ be a smooth projective threefold defined over an algebraically closed field $k$, and let $H$ be an ample divisor on $X$. Assume that $K_X \sim_{num} 0$, $Hc_2(X) = 0$ and $T_X$ is $\mu_H$-semistable. Then for any $\nu_{\alpha,\beta}$-stable object $E$ with $\nu_{\alpha,\beta}(E) = 0$, we have

$$\text{ch}_3^\beta(E) \leq \frac{\alpha^2}{6} H^2 \text{ch}_1^\beta(E).$$

By [29, Theorem 2] and [16, Theorem 4.1], one sees that all the Chern classes of $X$ are vanishing under the assumptions in Theorem 1.1. In characteristic zero, a well known consequence of Yau’s proof of Calabi’s conjecture shows that $X$ has a finite étale cover by an abelian variety if and only if $K_X \sim_{num} 0$ and $Hc_2(X) = 0$. And in this case, the semistability assumption of $T_X$ is automatically satisfied. Thus if $\text{char}(k) = 0$, Theorem 1.1 is a consequence of [3, Theorem 1.1] which showed the same inequality for abelian threefolds.

In positive characteristic not much is known about the characterizing projective varieties with vanishing Chern classes. And there are threefolds with vanishing Chern classes which do not have a finite étale cover by an abelian variety (see, e.g., [15, Section 7.3]). Hence in some sense, Theorem 1.1 is new in positive characteristic. The semistable assumption of $T_X$ in the theorem guarantees the classical Bogomolov-Gieseker inequality to be satisfied on $X$, so that the $\nu_{\alpha,\beta}$-stability is well defined.

The strategy of the proof is the following. In the case of $\text{char}(k) = p > 0$ we compute the Euler characteristic $\chi(O_X, (F^n)^* E)$ of the pullback of $E$ by the $n$-th iteration of the Frobenius morphism. By the Riemann-Roch theorem, one sees that $\chi(O_X, (F^n)^* E)$ is a polynomial of degree $3n$ with respect to $p$ and its leading coefficient is $\text{ch}_3(E)$. On the other hand, using the tilt-stability of the Frobenius pushforward of some locally free sheaves (see Proposition 3.3), we can show that $\text{ext}^i(O_X, (F^n)^* E) = O(p^{2n})$, for even $i$. Taking $n \to +\infty$, we obtain an inequality for the third Chern characters of $E$. The characteristic zero case follows from the standard spreading out technique.

**Applications.** Theorem 1.1 and [5, Theorem 4.1] give the following Reider type theorem:

**Corollary 1.2.** Under the situation of Theorem 1.1, fix a non-negative integer $\alpha$. Let $L$ be an ample divisor on $X$ satisfying

1. $L^3 > 49\alpha$;
2. $L^2D \geq 7\alpha$ for every integral divisor class $D$ with $L^2D > 0$ and $LD^2 < \alpha$;
3. $LC \geq 3\alpha$ for any curve $C \subset X$.

Then $H^1(X, I_Z(K_X + L)) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length $\alpha$. In particular, Kodaira’s vanishing theorem $H^1(X, O_X(K_X + M)) = 0$ holds for any ample divisor $M$ on $X$.

**Remark 1.3.** Theorem 4.1 in [5] was only showed when $\alpha > 0$, but the same proof works for $\alpha = 0$.

Setting $\alpha = 1$ or $\alpha = 2$, we confirm Fujita’s conjecture for such $X$ in any characteristic.
Corollary 1.4. Under the situation of Theorem 1.1, let $L$ be an ample divisor on $X$. Then

1. $\mathcal{O}_X(K_X + mL)$ is globally generated for $m \geq 4$.
2. $\mathcal{O}_X(K_X + mL)$ is very ample for $m \geq 5$.

Corollary 1.5. Under the situation of Theorem 1.1, let $c$ be the minimum positive value of $H^2D$ for integral divisor $D$. If $Q$ is a $\mu_H$-stable sheaf with $H^2c_1(Q) = c$, then

$$3c \text{ch}_3(Q) \leq 2(H \text{ch}_2(Q))^2.$$ 

We refer to [3, Example 4.4] for a proof and more discussion.

In [15], Langer proved that for a non-uniruled threefold $X$ with $K_X \sim_{\text{num}} 0$, the tangent bundle of $X$ is strongly $\mu_H$-semistable for every ample divisor $H$. Hence Theorem 1.1, Corollary 1.2, Corollary 1.4 and Corollary 1.5 hold for a non-uniruled threefold $X$ with $K_X \sim_{\text{num}} 0$ and $Hc_2(X) = 0$.

Organization of the paper. Our paper is organized as follows. In Section 2, we review basic notions and properties of some classical stabilities for coherent sheaves, tilt-stability, the conjectural inequality proposed in [4, 3]. Then in Section 3, we show the tilt-stability of the Frobenius pushforward of some locally free sheaves (see Proposition 3.3). Theorem 1.1 will be proved in Section 4.

Notation. Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of arbitrary characteristic. We denote by $T_X$ and $\Omega^1_X$ the tangent bundle and cotangent bundle of $X$, respectively. $K_X$ and $\omega_X$ denote the canonical divisor and canonical sheaf of $X$, respectively. We write $c_i(X) := c_i(T_X)$ for the $i$-th Chern class of $X$, and say $X$ has vanishing Chern classes if all the $c_i(X)$’s are numerically equivalent to zero. Numerical equivalence of two divisors $D_1, D_2$ on $X$ is denoted by $D_1 \sim_{\text{num}} D_2$. For a triangulated category $\mathcal{D}$, we write $K(\mathcal{D})$ for the Grothendieck group of $\mathcal{D}$.

Let $\pi : X \to S$ be a flat morphism of Noetherian schemes and $s \in S$ be a point. We denote by $X_s = X \times_S \text{Spec } k(s)$ the fibre of $\pi$ over $s$, where $k(s)$ is residue field of $s$. We write $X_{\bar{k}} = X \times_S \text{Spec } \bar{k}(s)$ for the geometric fibre of $\pi$ over $s$, here $\bar{k}(s)$ is the algebraic closure of $k(s)$. We denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on $X$. Given $E \in \mathcal{D}^b(X)$, we write $E_s$ (resp., $E_{\bar{k}}$) for the pullback to the field $k(s)$ (resp., $\bar{k}(s)$).

We write $H^j(E)$ ($j \in \mathbb{Z}$) for the cohomology sheaves of a complex $E \in \mathcal{D}^b(X)$. We also write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the cohomology groups of a sheaf $F \in \text{Coh}(X)$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively. For a real number $d$, we denote by $[d]$ the small least integer $\geq d$.

Convention. Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p > 0$. Let $X^{(1)} = X \times_{\text{Spec } k} \text{Spec } k$, where the product is taken over the absolute Frobenius morphism on $\text{Spec } k$. Then the factorization of the absolute Frobenius morphism $F : X \to X$ gives the geometric Frobenius morphism $F_g : X \to X^{(1)}$.

The variety $X^{(1)}$ is not isomorphic to $X$ as a $k$-variety, but $X^{(1)}$ is isomorphic to $X$ as a scheme since $F : \text{Spec } k \to \text{Spec } k$ is an isomorphism. Hence any geometric statement on the objects in $\mathcal{D}^b(X)$ is equivalent to the corresponding statement on the objects in $\mathcal{D}^b(X^{(1)})$. For this reason, we shall abuse notation and not distinguish between $X$ and $X^{(1)}$. 
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2. Preliminaries

Throughout this section, we let $X$ be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field $k$ of arbitrary characteristic and $H$ be a fixed ample divisor on $X$. We will review some basic notions of stability for coherent sheaves, the weak Bridgeland stability conditions and Bogomolov-Gieseker type inequalities.

2.1. Stability for sheaves. For any $\mathbb{R}$-divisor $D$ on $X$, we define the twisted Chern character $\text{ch}^D = e^{-D} \text{ch}$. More explicitly, we have

$$
\begin{align*}
\text{ch}^0 &= \text{rk}, & \text{ch}^2 &= \text{ch}_2 - D \text{ch}_1 + \frac{D^2}{2} \text{ch}_0, \\
\text{ch}^1 &= -D \text{ch}_0, & \text{ch}^3 &= -D \text{ch}_2 + \frac{D^2}{2} \text{ch}_1 - \frac{D^3}{6} \text{ch}_0.
\end{align*}
$$

The first important notion of stability for a sheaf is slope stability, also known as Mumford stability. We define the slope $\mu_{H,D}$ of a coherent sheaf $E \in \text{Coh}(X)$ by

$$
\mu_{H,D}(E) = \begin{cases} 
+\infty, & \text{if } \text{ch}^0(E) = 0, \\
\frac{\mu^{n-1}_E\text{ch}^D(E)}{\mu^E}, & \text{otherwise}.
\end{cases}
$$

Definition 2.1. A coherent sheaf $E$ on $X$ is $\mu_{H,D}$-(semi)stable (or slope-(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have $\mu_{H,D}(F) < (\leq) \mu_{H,D}(E/F)$.

We say a $\mu_{H,D}$-semistable sheaf $E$ is strongly $\mu_{H,D}$-semistable if either $\text{char } k = 0$ or $\text{char } k > 0$ and all the Frobenius pull backs of $E$ are $\mu_{H,D}$-semistable.

Note that $\mu_{H,D}$ only differs from $\mu_H := \mu_{H,0}$ by a constant, thus $\mu_{H,D}$-stability and $\mu_H$-stability coincide. Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to $\mu_{H,D}$-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

$$
0 = E_0 \subset E_1 \subset \cdots \subset E_m = E
$$

such that: $G_i := E_i/E_{i-1}$ is $\mu_{H,D}$-semistable, and $\mu_{H,D}(G_1) > \cdots > \mu_{H,D}(G_m)$. We set $\mu^+_E(D) := \mu_{H,D}(G_1)$ and $\mu^-_E(D) := \mu_{H,D}(G_m)$.

2.2. Weak Bridgeland stability conditions. The notion of “weak Bridgeland stability condition” and its variant “very weak Bridgeland stability condition” have been introduced in [33, Section 2] and [3, Definition 12.1], respectively. We will use a slightly different notion in order to adapt our situation. The main difference is the rotation of the half-plane in $\mathbb{C}$.

Definition 2.2. A weak Bridgeland stability condition on $X$ is a pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}^b(X)$, and $Z : K(\mathcal{D}^b(X)) \to \mathbb{C}$ is a group homomorphism (called central charge) such that
\( Z \) satisfies the following positivity property for any \( E \in \mathcal{A} \):

\[
Z(E) \in \{ re^{i\phi} : r \geq 0, 0 < \phi \leq 1 \}.
\]

Every non-zero object in \( \mathcal{A} \) has a Harder-Narasimhan filtration in \( \mathcal{A} \) with respect to \( \nu_Z \)-stability, here the slope \( \nu_Z \) of an object \( E \in \mathcal{A} \) is defined by

\[
\nu_Z(E) = \begin{cases} 
+\infty, & \text{if } \Im Z(E) = 0, \\
\frac{\Re Z(E)}{\Im Z(E)}, & \text{otherwise}.
\end{cases}
\]

Let \( \alpha > 0 \) and \( \beta \) be two real numbers. We will construct a family of weak Bridgeland stability conditions on \( X \) that depends on these two parameters. For brevity, we write \( \text{ch}^\beta \) for the twisted Chern character \( \text{ch}^{\beta H} \).

There exists a torsion pair \( (T_{\beta H}, \mathcal{F}_{\beta H}) \) in \( \text{Coh}(X) \) defined as follows:

\[
T_{\beta H} = \{ E \in \text{Coh}(X) : \mu_{\beta H}^+(E) > \beta \}
\]

\[
\mathcal{F}_{\beta H} = \{ E \in \text{Coh}(X) : \mu_{\beta H}^-(E) \leq \beta \}
\]

Equivalently, \( T_{\beta H} \) and \( \mathcal{F}_{\beta H} \) are the extension-closed subcategories of \( \text{Coh}(X) \) generated by \( \mu_{\beta H}^H \)-stable sheaves of positive and non-positive slope, respectively.

**Definition 2.3.** We let \( \text{Coh}^{\beta H}(X) \subset \text{D}^b(X) \) be the extension-closure

\[
\text{Coh}^{\beta H}(X) = (T_{\beta H}, \mathcal{F}_{\beta H}[1]).
\]

By the general theory of torsion pairs and tilting \([9]\), \( \text{Coh}^{\beta H}(X) \) is the heart of a bounded t-structure on \( \text{D}^b(X) \); in particular, it is an abelian category. Consider the following central charge

\[
Z_{\alpha,\beta}(E) = H^{n-2} \left( \frac{\alpha^2 H^2}{2} \text{ch}^\beta_0(E) - \text{ch}^\beta_2(E) + iH \text{ch}^\beta_1(E) \right).
\]

We think of it as the composition

\[
Z_{\alpha,\beta} : \text{K}(\text{D}^b(X)) \xrightarrow{\text{ch}_H} \mathbb{Q}^3 \xrightarrow{z_{\alpha,\beta}} \mathbb{C},
\]

where the first map is given by

\[
\text{ch}_H(E) = (H^n \text{ch}_0(E), H^{n-1} \text{ch}_1(E), H^{n-2} \text{ch}_2(E)),
\]

and the second map is defined by

\[
z_{\alpha,\beta}(e_0, e_1, e_2) = \frac{1}{2} (\alpha^2 - \beta^2)e_0 + \beta e_1 - e_2 + i(e_1 - \beta e_0).
\]

**Definition 2.4.** We say \((X, H)\) satisfies Bogomolov's inequality, if

\[
H^{n-2} \Delta(E) := H^{n-2} \left( \text{ch}^2_1(E) - 2 \text{ch}_0(E) \text{ch}_2(E) \right) \geq 0
\]

for any \( \mu_H \)-semistable sheaf \( E \) on \( X \).

**Theorem 2.5.** If \((X, H)\) satisfies Bogomolov’s inequality, then for any \((\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}, \sigma_{\alpha,\beta} = (Z_{\alpha,\beta}, \text{Coh}^{\beta H}(X))\) is a weak Bridgeland stability condition.

**Proof.** The required assertion is proved in \([8, 1]\) for the surface case. For the threefold case, the conclusion is showed in \([4, 3]\). But the proof in \([3, \text{Appendix 2}]\) still works for the general case. \(\square\)

**Corollary 2.6.** Assume that either \( \text{char}(k) = 0 \) or \( T_X \) is \( \mu_H \)-semistable and \( K_X \sim_{\text{num}} 0 \). Then for any \((\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}, \sigma_{\alpha,\beta} = (Z_{\alpha,\beta}, \text{Coh}^{\beta H}(X))\) is a weak Bridgeland stability condition.
Proof. It is well known that Bogomolov’s inequality holds in characteristic zero (see [11, Theorem 3.4.1]). In positive characteristic Langer [13] proved that the same inequality holds for strongly $\mu_H$-semistable sheaves. Mehta and Ramanathan [25] showed that if $X$ satisfies $\mu_H^+(\Omega^1_X) \leq 0$, then all $\mu_H$-semistable sheaves on $X$ are strongly $\mu_H$-semistable. Thus Bogomolov’s inequality holds under our assumptions. □

We now suppose the assumption in the above Corollary holds. We write $\nu_{\alpha, \beta}$ for the slope function on $\Coh^{\beta H}(X)$ induced by $Z_{\alpha, \beta}$. Explicitly, for any $E \in \Coh^{\beta H}(X)$, one has

$$\nu_{\alpha, \beta}(E) = \begin{cases} +\infty, & \text{if } H^{n-1}c_1^\beta(E) = 0, \\ \frac{H^{n-2}c_2^\beta(E) - \frac{1}{2} \alpha^2 H^n c_0^\beta(E)}{H^{n-1}c_1^\beta(E)}, & \text{otherwise.} \end{cases}$$

Corollary 2.6 gives the notion of tilt-stability:

**Definition 2.7.** An object $E \in \Coh^{\beta H}(X)$ is tilt-(semi)stable (or $\nu_{\alpha, \beta}$-(semi)stable) if, for all non-trivial subobjects $F \hookrightarrow E$, we have $\nu_{\alpha, \beta}(F) < (\leq) \nu_{\alpha, \beta}(E/F)$.

For any $E \in \Coh^{\beta H}(X)$, the Harder-Narasimhan property gives a filtration in $\Coh^{\beta H}(X)$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that: $\mathcal{F}_k := E_k/E_{k-1}$ is $\nu_{\alpha, \beta}$-semistable with $\nu_{\alpha, \beta}(\mathcal{F}_1) > \cdots > \nu_{\alpha, \beta}(\mathcal{F}_m)$.

2.3. **Bogomolov-Gieseker type inequality.** We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [4, 3].

**Definition 2.8.** We define the generalized discriminant

$$\Delta^{\beta H}_H := (H^{n-1}c_1^\beta)^2 - 2H^n c_0^\beta (H^{n-2}c_2^\beta).$$

A short calculation shows

$$\Delta^{\beta H}_H = (H^{n-1}c_1^\beta)^2 - 2H^n c_0^\beta (H^{n-2}c_2^\beta) = \Delta_H.$$

Hence the generalized discriminant is independent of $\beta$.

**Theorem 2.9.** Under the assumption in Corollary 2.6, if $E \in \Coh^{\beta H}(X)$ is $\nu_{\alpha, \beta}$-semistable, then $\Delta_H(E) \geq 0$.

**Proof.** This inequality was proved in [4, Theorem 7.3.1] and [3, Theorem 3.5] on threefolds, but their proof works for the general case. □

**Conjecture 2.10 ([4, Conjecture 1.3.1]).** Assume that $n = 3$, char($k$) = 0 and $E \in \Coh^{\beta H}(X)$ is $\nu_{\alpha, \beta}$-semistable with $\nu_{\alpha, \beta}(E) = 0$. Then we have

$$\alpha^2 c_3^\beta(E) \leq \frac{\alpha^2}{6} H^2 c_1^\beta(E).$$

Such an inequality provides a way to construct Bridgeland stability conditions on threefolds. Recently, Schmidt [28] found a counterexample to Conjecture 2.10 when $X$ is the blowup at a point of $\mathbb{P}^3$. Therefore, the inequality (2.1) needs some modifications in general setting. See [26] and [6] for the recent progress.
Definition 2.11. Assume that $n = 3$ and $(X, H)$ satisfies the assumption in Corollary 2.6. For any object $E \in \text{Coh}^3H(X)$, we define

$$\overline{\beta}(E) = \begin{cases} \frac{H^2ch_1(E) - \sqrt{\Delta_H(E)}}{H^2ch_0(E)}, & \text{if } ch_0(E) \neq 0, \\ \frac{Hch_2(E)}{H^2ch_1(E)}, & \text{otherwise.} \end{cases}$$

Moreover, we say that $E$ is $\overline{\beta}$-(semi)stable, if it is $\nu_{\alpha, \beta}$-(semi)stable in an open neighborhood of $(0, \overline{\beta}(E))$ in $(\alpha, \beta)$-plane.

Conjecture 2.10 can be reduced as follows:

Theorem 2.12 ([3 Theorem 5.4]). Assume that $n = 3$, char$(k) = 0$ and for any $\overline{\beta}$-stable object $E \in \text{Coh}^3H(X)$ with $\overline{\beta}(E) \in [0, 1)$ and $ch_0(E) \geq 0$ the inequality

$$ch_3(E)(E) \leq 0$$

holds. Then Conjecture 2.10 holds.

3. Tilt-stability of Frobenius direct images

Throughout this section, we let $k$ be an algebraically closed field of characteristic $p > 0$ and $X$ be a smooth projective variety of dimension $n$ defined over $k$. We fix an ample divisor $H$ on $X$. Assume that $K_X \sim_{num} 0$, $H^{n-2}c_2(X) = 0$ and $T_X$ is $\mu_H$-semistable. Let $F : X \to X$ be the absolute Frobenius morphism. We will investigate the tilt-stability of $F_*E$ for a locally free sheaf $E$ on $X$.

Lemma 3.1. Let $E$ be a locally free sheaf on $X$. Then we have

$$H^{n-i}ch_i(F_*E) = p^{n-i}H^{n-i}ch_i(E)$$

for $i = 0, 1, 2$ and $\overline{\Delta}_H(F_*E) = p^{2n-2}\overline{\Delta}_H(E)$.

Proof. The similar computations have been done by the author in [31 Section 7]. We repeat them here for the reader’s convenience.

From the Grothendieck-Riemann-Roch theorem, it follows that

$$ch(F_*E) \text{td}(X) = F_*\left(ch(E) \text{td}(X)\right).$$

Since $\text{td}(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots$, the above equation implies

$$ch_0(F_*E) = p^nch_0(E),$$

$$\frac{1}{2}ch_0(F_*E)c_1 + ch_1(F_*E) = p^{n-1}\left(\frac{c_1}{2} + c_1(E)\right),$$

$$\frac{c_1^2 + c_2}{12}ch_0(F_*E) + \frac{c_1}{2}ch_1(F_*E) + ch_2(F_*E) = p^{n-2}\left(\frac{c_1^2 + c_2}{12} + \frac{c_1}{2}c_1(E) + ch_2(E)\right).$$

By our assumptions on $c_1$ and $c_2$, a simple computation shows $H^{n-i}ch_i(F_*E) = p^{n-i}H^{n-i}ch_i(E)$ for $i = 0, 1, 2$. Hence $\overline{\Delta}_H(F_*E) = p^{2n-2}\overline{\Delta}_H(E)$. □

Lemma 3.2. Let $E$ be $\mu_H$-semistable locally free sheaf on $X$. Then $F_*E$ is $\mu_H$-semistable.

Proof. Xiaotao Sun [32] proved that the stability of $F_*E$ depends on the stability of $T^i(\Omega^1_X)$, $0 \leq l \leq n(p - 1)$. On the other hand, by [24] Theorem 2.1 one sees that under our assumptions $\Omega^1_X$ and $E$ are strongly $\mu_H$-semistable. So is $E \otimes T^l(\Omega^1_X)$ for any $0 \leq l \leq n(p - 1)$. From [32] Theorem 4.8, it follows that $F_*E$ is $\mu_H$-semistable. □
Proposition 3.3. Let $m$ and $l$ be two integers. Let $L$ be a divisor on $X$ and $\mathcal{G}$ be a $\nu_{\alpha, \beta}$-semistable object in an open neighborhood of $(0, \beta_0)$ in $(\alpha, \beta)$-plane. Assume that $L \sim_{\text{num}} mH$, $l > 0$ and

$$\lim_{(\alpha, \beta) \to (0, \beta_0)} \nu_{\alpha, \beta}(\mathcal{G}) = 0.$$ 

Then

1. $\text{Hom}((\mathcal{F}_l)_{\ast} \mathcal{O}_X(L), \mathcal{G}) = 0$ if $\beta_0 < \frac{m}{p^l}$.

2. $\text{Hom}(\mathcal{G}, (\mathcal{F}_l)_{\ast} \mathcal{O}_X(L)[1]) = 0$ if $\beta_0 > \frac{m}{p^l}$.

Proof. By Lemma 3.1 and 3.2, one sees that $\mathcal{E} := (\mathcal{F}_l)_{\ast} \mathcal{O}_X(L)$ is $\mu_H$-semistable with

$$H^n \text{ch}_0(\mathcal{E}), H^{n-1} \text{ch}_1(\mathcal{E}), H^{n-2} \text{ch}_2(\mathcal{E})) = (p^l H^n, p^l(n-1) m H^n, \frac{1}{2} p^l(n-2) m^2 H^n).$$

This implies $\mu_H(\mathcal{E}) = \frac{p^l(n-1) m H^n}{p^l H^n} = \frac{m}{p^l}$ and $\overline{\Delta}_H(\mathcal{E}) = 0$. Consider its Jordan-Hölder filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{s-1} \subset \mathcal{E}_s = \mathcal{E},$$

and set $\mathcal{Q}_i$ be the $\mu_H$-stable sheaf $\mathcal{E}_i/\mathcal{E}_{i-1}$. It turns out that

$$\mu_H(\mathcal{Q}_i) = \mu_H(\mathcal{E}_s) = \frac{m}{p^l},$$

for any $i > 0$. By Bogomolov’s inequality for strongly semistable sheaves, we deduce that

$$0 = \frac{\overline{\Delta}_H(\mathcal{E}_s)}{H^n \text{rk } \mathcal{E}_s} = \frac{\mu_H(\mathcal{E}_s) H^{n-1} \text{ch}_1(\mathcal{E}_s) - 2 H^{n-2} \text{ch}_2(\mathcal{E}_s)}{H^n \text{rk } \mathcal{E}_s} = \mu_H(\mathcal{E}_s) \sum_{i=1}^{s} H^{n-1} \text{ch}_1(\mathcal{Q}_i) - 2 \sum_{i=1}^{s} H^{n-2} \text{ch}_2(\mathcal{Q}_i) = \sum_{i=1}^{s} \left( \mu_H(\mathcal{Q}_i) H^{n-1} \text{ch}_1(\mathcal{Q}_i) - 2 H^{n-2} \text{ch}_2(\mathcal{Q}_i) \right) = \sum_{i=1}^{s} \frac{\overline{\Delta}_H(\mathcal{Q}_i)}{H^n \text{rk } \mathcal{Q}_i} \geq 0,$$

It follows that

$$\frac{m}{p^l} \frac{H^{n-1} \text{ch}_1(\mathcal{Q}_i)}{H^n \text{ch}_0(\mathcal{Q}_i)} = \frac{2 H^{n-2} \text{ch}_2(\mathcal{Q}_i)}{H^{n-1} \text{ch}_1(\mathcal{Q}_i)} = \frac{m^2 - 2 \beta m p^l + (\beta^2 - \alpha^2) p^{2l}}{2 (p^l m - \beta p^{2l})} = \frac{(m - \beta p^l)^2 - \alpha^2 p^{2l}}{2p^l (m - \beta p^l)}.$$

and

$$\nu_{\alpha, \beta}(\mathcal{Q}_i) = \frac{m^2 - 2 \beta m p^l + (\beta^2 - \alpha^2) p^{2l}}{2(p^l m - \beta p^{2l})} = \frac{(m - \beta p^l)^2 - \alpha^2 p^{2l}}{2p^l (m - \beta p^l)}.$$ 

So

$$\lim_{(\alpha, \beta) \to (0, \beta_0)} \nu_{\alpha, \beta}(\mathcal{Q}_i) = \frac{1}{2} \left( \frac{m}{p^l} - \beta_0 \right).$$

On the other hand, by [33 Corollary 3.11] or [30 Theorem 1.3, 1.4] one sees that $\mathcal{Q}_i$ is $\nu_{\alpha, \beta}$-stable for any $\alpha > 0$, $\beta < \frac{m}{p^l}$ and $\mathcal{Q}_i[1]$ is $\nu_{\alpha, \beta}$-stable for any $\alpha > 0$, $\beta \geq \frac{m}{p^l}$. These imply that $\text{Hom}(\mathcal{Q}_i, \mathcal{G}) = 0$ if $\beta_0 < \frac{m}{p^l}$ and $\text{Hom}(\mathcal{G}, \mathcal{Q}_i[1]) = 0$ if $\beta_0 > \frac{m}{p^l}$. The conclusion of the proposition follows from

$$\text{Hom}((\mathcal{F}_l)_{\ast} \mathcal{O}_X(L), \mathcal{G}) \leq \sum_{i=1}^{s} \text{Hom}(\mathcal{Q}_i, \mathcal{G})$$
and
\[ \text{hom}(G, (F^i)_*O_X(L)[1]) \leq \sum_{i=1}^s \text{hom}(G, Q_i[1]). \]

4. The proof of the main theorem

In this section, we will proof Theorem 1.1. By Theorem 2.12, this will be done, if we can show the following:

**Theorem 4.1.** Under the situation of Theorem 1.1, let \( E \in \text{Coh}^{\beta H}(X) \) be a \( \beta \)-stable object with \( \beta(E) \in [0, 1) \) and \( \text{ch}_0(E) \geq 0 \). Then we have \( \text{ch}_3(E) \leq 0 \).

Since the statement of Theorem 4.1 is independent of scaling \( H \), we will assume throughout this section that \( H \) is very ample. In order to prove Theorem 4.1, we use the standard spreading out technique and Frobenius morphism.

In the case of \( \text{char}(k) = 0 \), there is a subring \( R \subset k \), finitely generated over \( \mathbb{Z} \), and a scheme \( \pi : X \to S = \text{Spec} R \) so that \( \pi \) is smooth, projective and \( X = X \times_R k \). We also have an object \( \mathcal{E} \in D^b(X) \) and a divisor \( \mathcal{H} \) on \( X \) such that \( E = \mathcal{E} \times_R k \) and \( \mathcal{H} = H \times_R k \). By the openness of semistability, one sees that \( X_s \) satisfies the assumptions in Theorem 1.1 for a general point \( s \in S \). Since the semistability of sheaves is preserved by field extensions, Bogomolov’s inequality holds for any \( \mu_{\mathcal{H}_s} \)-semistable sheaves on the fiber of \( \pi \) over a general point \( s \in S \). From [2, Proposition 25.3], it follows that for a general closed point \( s \in S \), \( E_s \in \text{Coh}^{\beta H_s}(X_s) \) is \( \beta \)-stable. By [2, Theorem 12.17], the same thing holds for the object \( E_s \in \text{Coh}^{\beta H_s}(X_s) \). Therefore we may further assume that \( \text{char}(k) = p > 0 \) and denote by \( F : X \to X \) the absolute Frobenius morphism.

4.1. **Proof of Theorem 4.1** integral case. Assume that \( \beta(E) = 0 \), i.e.,
\[ H \text{ch}_2(E) = 0 = K_X \text{ch}_2(E). \]

We want to show that \( \text{ch}_3(E) \leq 0 \).

We assume the contrary \( \text{ch}_3(E) > 0 \), and so \( \text{ch}_3(E) \geq 1 \). Since \( H^2 \text{ch}_1(\beta(E))(E) = H^2 \text{ch}_1(E) \geq 0 \) and \( \text{ch}_0(E) \geq 0 \), by using the Riemann-Roch theorem we can compute
\[ \chi(\mathcal{O}_X, (F^n)^*E) = p^{3n} \text{ch}_3(E) + O(p^{2n}) \geq p^{3n} + O(p^{2n}), \]
for any positive integer \( n \). On the other hand, since \( E \) is a two term complex concentrated in degree \(-1\) and \( 0 \), one sees
\[ \chi(\mathcal{O}_X, (F^n)^*E) \leq \text{hom}(\mathcal{O}_X, (F^n)^*E) + \text{ext}^2(\mathcal{O}_X, (F^n)^*E). \]

Our goal is to bound from above the right hand side of this inequality with a lower order in \( p^n \).

**Bound on** \( \text{hom}(\mathcal{O}_X, (F^n)^*E) \)

We want to show
\[ (4.1) \quad \text{hom}(\mathcal{O}_X, (F^n)^*E) = O(p^{2n}). \]
By \[3\] Lemma 7.1, we have the exact triangle in \(D^b(X)\)
\[(F^n)^* E \otimes \mathcal{O}_X (-H) \to (F^n)^* E \to ((F^n)^* E) \otimes \mathcal{O}_Y,
\]
where \(Y\) is a general smooth surface in \([H]\). It follows that
\[\text{hom}(\mathcal{O}_X, (F^n)^* E) \leq \text{hom}(\mathcal{O}_X, (F^n)^* E \otimes \mathcal{O}_X (-H)) + \text{hom}(\mathcal{O}_X, ((F^n)^* E) \otimes \mathcal{O}_Y).\]

By Serre duality and adjointness between \((F^n)^*\) and \((F^n)_*\), one obtains
\[\text{hom}(\mathcal{O}_X, (F^n)^* E \otimes \mathcal{O}_X (-H)) = \text{hom}((F^n)_* \mathcal{O}_X (H + K_X), E \otimes \omega_X).\]

Since \(K_X \sim_{\text{num}} 0\), Proposition \[3\] gives \(\text{hom}((F^n)_* \mathcal{O}_X (H + K_X), E \otimes \omega_X) = 0\). Thus we have
\[\text{hom}(\mathcal{O}_X, (F^n)^* E) \leq \text{hom}(\mathcal{O}_X, ((F^n)^* E) \otimes \mathcal{O}_Y).\]

We then consider the cohomology sheaves of \(E\) and the exact triangle in \(D^b(X)\)
\[\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E).\]

Since \(Y\) is general, \[3\] Lemma 7.1 gives
\[\text{hom}(\mathcal{O}_X, ((F^n)^* E) \otimes \mathcal{O}_Y) \leq h^0\left( ((F^n)^* \mathcal{H}^0(E) \right) |_Y) + h^1\left( ((F^n)^* \mathcal{H}^{-1}(E) \right) |_Y).\]

The bound \(4.1\) will then follow from the following lemma.

**Lemma 4.2.** Let \(Q\) be a sheaf and \(L\) be a line bundles on \(X\). Let \(Y\) be a general smooth surface in the very ample linear system \([bH]\), where \(b\) is a positive integer. Then for any \(0 \leq i \leq 2\), there are rational numbers \(a_i \) \((1 \leq i \leq 6)\) which are independent of \(n\) and \(L\) such that
\[h^i(Y, ((F^n)^* Q \otimes L)|_Y) \leq a_1 p^{2n} + a_2 \mu_H (L)p^n + a_3 p^n + a_4 \mu_H (L)^2 + a_5 \mu_H (L) + a_6.\]

**Proof.** We denote by \(F_Y\) the absolute Frobenius morphism of \(Y\) and assume first that \(Q\) is torsion free. Take a positive integer \(a\) such that \(T_Y(aH)|_Y\) is globally generated. Since \(((F^n)^* Q)|_Y = (F^n)_*(Q)|_Y\), by \[13\] Corollary 2.5, one obtains that
\[
\begin{align*}
\mu^+_{H|_Y} \left( ((F^n)^* Q) |_Y \right) &\leq p^n \mu^+_{H|_Y} (Q|_Y) + \frac{p^n (\text{rk } Q - 1)}{p - 1} abH^3 \\
\bar{\mu}^+_{H|_Y} \left( ((F^n)^* Q) |_Y \right) &\geq p^n \bar{\mu}^+_{H|_Y} (Q|_Y) - \frac{p^n (\text{rk } Q - 1)}{p - 1} abH^3.
\end{align*}
\]

Hence
\[
\mu^+ := \mu^+_{H|_Y} \left( ((F^n)^* Q \otimes L) |_Y \right) \leq p^n \mu^+_{H|_Y} (Q|_Y) + \frac{p^n (\text{rk } Q - 1)}{p - 1} abH^3 + \mu_{H|_Y} (L|_Y)
\]
\[= p^n \mu^+_{H|_Y} (Q|_Y) + \frac{p^n (\text{rk } Q - 1)}{p - 1} abH^3 + \mu_H (L)
\]

From Langer’s estimation \[14\] Theorem 3.3, it follows that
\[
h^0(Y, ((F^n)^* Q \otimes L)|_Y)
\]
\[
\leq \begin{cases} 
\frac{(\text{rk } Q)bH^3}{2} \left( \mu^+ + f(\text{rk } Q) + 2 \right) \left( \mu^+ + f(\text{rk } Q) + 1 \right), & \text{if } \mu^+ \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]
\[
\leq b_1 p^{2n} + b_2 \mu_H (L)p^n + b_3 p^n + b_4 \mu_H (L)^2 + b_5 \mu_H (L) + b_6
\]
where \(f(\text{rk } Q) = -1 + \sum_{i=1}^{\text{rk } Q} \frac{1}{i}\) and \(b_i\)’s are independent of \(n\) and \(L\).
The $h^2$-estimate follows similarly, by using Serre Duality. For $h^1$, the Riemann-Roch theorem gives
\[
h^1(Y, ((F^n)^* Q \otimes \mathcal{L})|_Y) = h^0(Y, ((F^n)^* Q \otimes \mathcal{L})|_Y) + h^2(Y, ((F^n)^* Q \otimes \mathcal{L})|_Y) - \chi(Y, ((F^n)^* Q \otimes \mathcal{L})|_Y).
\]
It follows that the upper bound of $h^1$ has the same form as that of $h^0$. This finishes the proof in the torsion-free case. The proof for a general sheaf $Q$ is the same as that of [3, Lemma 7.3].

**Bound on** $\text{ext}^2 (\mathcal{O}_X, (F^n)^* E)$

This is similar to the previous case. We consider the exact triangle
\[
(F^n)^* E \to ((F^n)^* E) \otimes \mathcal{O}_X(H) \to ((F^n)^* E) \otimes \mathcal{O}_Y(H).
\]
By Proposition 3.3, Serre duality and the adjointness, one obtains
\[
\text{ext}^2 (\mathcal{O}_X, ((F^n)^* E) \otimes \mathcal{O}_X(H)) = \text{ext}^1 ((F^n)^* E, \omega_X(\neg H))
= \text{ext}^1 (E, (F^n)_*(\omega_X(\neg H)))
= \hom (E, (F^n)_*(\omega_X(\neg H))[1])
= 0.
\]
Thus Lemma 4.2 gives
\[
\text{ext}^2 (\mathcal{O}_X, (F^n)^* E) \leq \text{ext}^1 (\mathcal{O}_X, ((F^n)^* E) \otimes \mathcal{O}_Y(H))
\leq h^1((F^n)^* H^0(E) \otimes \mathcal{O}_Y(H))
+ h^2((F^n)^* H^{-1}(E) \otimes \mathcal{O}_Y(H))
= O(p^{2n}).
\]

In conclusion, we have
\[
p^{3n} + O(p^{2n}) \leq \chi(\mathcal{O}_X, (F^n)^* E) \leq O(p^{2n}),
\]
which gives the required contradiction for $n$ sufficiently large.

4.2. **Proof of Theorem 4.1 rational case.** We assume that $\overline{\mathcal{B}(E)} \in \mathbb{Q} \setminus \mathbb{Z}$ and write $\overline{\mathcal{B}(E)} = \frac{1}{p^ru}$ with $p$ and $u$ coprime and $p^ru > v > 0$. By Euler’s theorem, we have
\[
p^{n\varphi(u)} \equiv 1 \mod u
\]
for any positive integer $n$, where $\varphi(u)$ is Euler’s totient function. This implies that $c_n := \frac{1}{u} n^{\varphi(u)}$ is an integer and
\[
\frac{c_n u}{p^n u^{\varphi(u)} + r} = (1 - \frac{1}{p^{n\varphi(u)}}) \overline{\mathcal{B}(E)}.
\]
Set $a_n = n\varphi(u) + r$. By using the Riemann-Roch theorem we can compute
\[
\chi(\mathcal{O}_X, (F^{a_n})^* E \otimes \mathcal{O}_X(-c_n vH)) = \chi_3 ((F^{a_n})^* E \otimes \mathcal{O}_X(-c_n vH)) + O(p^{2a_n})
= p^{3a_n} \left( \chi_3 c_n v/p^{2n} (E) \right) + O(p^{2a_n}).
\]
From (4.2), one obtains that
\[
\text{ch}^{c_n v/p^{an}}_3(E) = \text{ch}_3(1 - \frac{1}{p^{an}v}) \overline{\beta}(E)
\]
\[
= \text{ch}_3(\overline{\beta}(E)) + \left(\frac{\overline{\beta}(E)}{p^{an}v}\right)^2 H^2 \text{ch}_1(\overline{\beta}(E)) + \left(\frac{\overline{\beta}(E)}{p^{an}v}\right)^3 H^3 \text{ch}_0(\overline{\beta}(E)).
\]
Hence we deduce that
\[
\chi(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) = p^{3an} \left(\text{ch}^{c_n v/p^{an}}_3(E)\right) + O(p^{2an})
\]
\[
= p^{3an} \left(\text{ch}_3(\overline{\beta}(E))\right) + O(p^{2an}).
\]
and
\[
\chi(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) \leq \text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) + \text{ext}^2 (\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)).
\]
Since
\[
\frac{c_n v + p^r}{p^{an}} = \overline{\beta}(E) + (1 - \overline{\beta}(E)) \frac{1}{p^{an}v} > \overline{\beta}(E),
\]
from Proposition 3.3 it follows that
\[
\text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H - p^r H)) = \text{hom}((F^{an})^* \mathcal{O}_X(K_X + c_n v H + p^r H), E \otimes \omega_X)
\]
\[
= 0
\]
and
\[
\text{ext}^2 (\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) = \text{ext}^1 (E, (F^{an})^* \mathcal{O}_X(K_X + c_n v H))
\]
\[
= 0.
\]
Similar to the proof of (4.4), one obtains
\[
\text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) \leq \text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H - p^r H)) + \text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_Z(-c_n v H))
\]
\[
= \text{hom}(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_Z(-c_n v H))
\]
\[
\leq h^0((F^{an})^* \mathcal{H}^0(E) \otimes \mathcal{O}_Z(-c_n v H)) + h^1((F^{an})^* \mathcal{H}^{-1}(E) \otimes \mathcal{O}_Z(-c_n v H)) = O(p^{2an}),
\]
where Z is a general smooth surface in $|p^r H|$

In conclusion, we have
\[
p^{3an} \text{ch}_3(\overline{\beta}(E)) + O(p^{2an}) \leq \chi(\mathcal{O}_X, (F^{an})^* E \otimes \mathcal{O}_X(-c_n v H)) \leq O(p^{2an}).
\]
This gives $\text{ch}_3(\overline{\beta}(E)) \leq 0$ by taking $n \to +\infty$. 
4.3. Proof of Theorem 4.1, irrational case. We now assume that $\beta(E) \in \mathbb{R} \setminus \mathbb{Q}$.
By assumption, there exists $0 < \varepsilon < \beta(E)$ such that $E$ is $\nu_{\alpha, \beta}$-stable for all $(\alpha, \beta)$ in
$$V_\varepsilon := \{(\alpha, \beta) \in \mathbb{R}_0 \times \mathbb{R} : 0 < \alpha < \varepsilon, \beta(E) - \varepsilon < \beta < \beta(E) + \varepsilon\}.$$ 
By the Dirichlet approximation theorem, there exists a sequence $\beta_n = v_n p^r u_n \in \mathbb{Q}$ of rational numbers with $u_n > 0, v_n > 0, r_n \geq 0, u_n$ and $p$ coprime and $p^{r_n} u_n \to +\infty$ as $n \to +\infty$ such that
$$\left| \beta(E) - \frac{v_n}{p^{r_n} u_n} \right| < \frac{1}{p^{2r_n} u_n^2} < \varepsilon$$
for all $n$. As in the rational case, by Euler’s theorem, for any $m \geq 1$,
$$c_{mn} := \frac{p^{m\varphi(u_n)} - 1}{u_n}$$
is a positive integer. It turns out that
$$(1 - \frac{1}{p^{m\varphi(u_n)}})(\beta(E) - \frac{1}{p^{2r_n} u_n^2}) < \frac{c_{mn} v_n}{p^{m\varphi(u_n) + r_n}}$$
$$= (1 - \frac{1}{p^{m\varphi(u_n)}})\beta_n$$
$$< (1 - \frac{1}{p^{m\varphi(u_n)}})(\beta(E) + \frac{1}{p^{2r_n} u_n^2}).$$

Let $a_{mn} := m\varphi(u_n) + r_n$ and $Q_{mn} := (F^{a_{mn}})^* E \otimes O_X (-c_{mn} v_n H)$. We compute, for $m \gg 0$,
$$(4.4) \quad \chi(O_X, Q_{mn}) = \chi_3((F^{a_{mn}})^* E \otimes O_X (-c_{mn} v_n H)) + O(p^{2a_{mn}})$$
$$= p^{3a_{mn}} \chi_3^{c_{mn} v_n / p^{a_{mn}}} (E) + O(p^{2a_{mn}})$$
$$= p^{3a_{mn}} \left( \chi_3^\beta (E) + \frac{\beta_n}{p^{m\varphi(u_n)}} H \chi_2^\beta (E) \right.$$
$$+ \left. (\frac{\beta_n}{p^{m\varphi(u_n)}})^2 H^2 \chi_1^\beta (E) + \frac{\beta_n}{p^{m\varphi(u_n)}} H^3 \chi_0^\beta (E) \right) + O(p^{2a_{mn}})$$
$$\geq p^{3a_{mn}} \chi_3^\beta (E) + O(p^{2a_{mn}})$$
$$\geq p^{3a_{mn}} \chi_3^\beta (E) + O(p^{2a_{mn}}).$$
The last inequality follows since, by definition, $\chi_3^\beta (E)$ has a local minimum at $\beta = \beta(E)$. As in the previous case, we want to bound
$$(4.5) \quad \chi(O_X, Q_{mn}) \leq \text{hom} (O_X, Q_{mn}) + \text{ext}^2 (O_X, Q_{mn})$$
for $m \gg 0$ and $n \gg 0$. 

We let \( l_0 = \left[ \frac{p^r \beta(E)}{p^{r+1} u_n} \right] + p^r \beta(E) \) and \( l_1 = \left[ \frac{p^r \beta(E)}{p^{r+1} u_n} \right] - p^r \beta(E) \). Then by (4.3) one has

\[
\frac{c_{mn} v_n + l_0}{p^{2r+1}} \geq (1 - \frac{1}{p^{m \varphi(\gamma)} v_n}) \beta(E) + \frac{l_0}{p^{2r+1}}
\]

and

\[
\frac{c_{mn} v_n - l_1}{p^{2r+1}} \geq (1 - \frac{1}{p^{m \varphi(\gamma)} v_n}) \beta(E) - \frac{l_1}{p^{2r+1}}
\]

Thus Proposition 3.3 gives

(4.6) \[ \text{hom}(O_X, Q_{mn}(-l_0 H)) \]

\[ = \text{hom} \left( O_X, (F^{mn})^* E \otimes O_X(-c_{mn} v_n H - l_0 H) \right) \]

\[ = \text{hom} \left( (F^{mn})^* O_X(K_X + c_{mn} v_n H + l_0 H), E \otimes \omega_X \right) \]

\[ = 0 \]

and

(4.7) \[ \text{ext}^2(O_X, Q_{mn}(l_1 H)) \]

\[ = \text{ext}^2 \left( O_X, (F^{mn})^* E \otimes O_X(-c_{mn} v_n H + l_1 H) \right) \]

\[ = \text{hom} \left( E, (F^{mn})^* O_X(K_X + c_{mn} v_n H - l_1 H)[1] \right) \]

\[ = 0 \]

Consider the exact triangle in \( D^b(\mathcal{X}_{nk}) \)

\[ Q_{mn}(-(j + 1)H) \rightarrow Q_{mn}(-jH) \rightarrow Q_{mn}(-jH) \otimes O_Y, \]

where \( 0 \leq j \leq l_0 - 1 \) and \( Y \) is a general smooth surface in \( |H| \). From 4.6, it follows that

\[ \text{hom}(O_X, Q_{mn}) \]

\[ \leq \text{hom}(O_X, Q_{mn}(-l_0 H)) + \sum_{j=0}^{l_0-1} \text{hom}(O_X, Q_{mn}(-jH) \otimes O_Y) \]

\[ = \sum_{j=0}^{l_0-1} \text{hom}(O_X, Q_{mn}(-jH) \otimes O_Y). \]
On the other hand, by Lemma 4.2 and the definition of $c_{mn}$, one sees for $m \gg 0$,

\[
\sum_{j=0}^{l_0-1} \text{hom} \left( \mathcal{O}_X, Q_{mn}(-jH) \otimes \mathcal{O}_Y \right) \\
\leq \sum_{j=0}^{l_0-1} \left( b_1 p^{2a_{mn}} + (b_2 p^{a_{mn}} + b_3) (c_{mn} v_n + j) + b_4 p^{a_{mn}} + b_5 (c_{mn} v_n + j)^2 + b_6 \right) \\
= \sum_{j=0}^{l_0-1} \left( b_1 p^{2a_{mn}} + b_2 (c_{mn} v_n + j) p^{a_{mn}} + b_5 (c_{mn} v_n + j)^2 \right) + O(p^{2a_{mn}}) \\
= l_0 (b_1 p^{2a_{mn}} + b_2 c_{mn} v_n p^{a_{mn}} + b_3 c_{mn}^2 v_n^2) + \frac{l_0 (l_0 - 1)}{2} (b_2 p^{a_{mn}} + 2 b_5 c_{mn} v_n) \\
+ \frac{b_5}{6} l_0 (l_0 - 1) (2l_0 - 1) + O(p^{2a_{mn}}) \\
= \frac{p^{a_{mn}}}{p^{2a_{mn}} u_n^2} (b_1 p^{2a_{mn}} + b_2 \beta_2 p^{2a_{mn}} + b_3 \beta_1^2 p^{2a_{mn}}) + \frac{p^{2a_{mn}}}{2 p^{4r_n u_n^4}} (b_2 p^{a_{mn}} + 2 b_5 \beta_2 p^{a_{mn}}) \\
+ \frac{b_5}{3} \frac{p^{3a_{mn}}}{p^{6r_n u_n^6}} + O(p^{2a_{mn}}),
\]

where $b_i$'s and $d_j$'s are independent of $m$ and $n$. Therefore for $m \gg 0$ we have

\[(4.8) \quad \text{hom} \left( \mathcal{O}_X, Q_{mn} \right) \leq \left( \frac{d_1}{p^{2r_n u_n^2}} + \frac{d_2}{p^{4r_n u_n^4}} + \frac{d_3}{p^{6r_n u_n^6}} \right) p^{3a_{mn}} + O(p^{2a_{mn}}).\]

To bound $\text{ext}^2 \left( \mathcal{O}_X, Q_{mn} \right)$, as before, we consider the exact triangle in $D^b(X)$

\[Q_{mn}((i-1)H) \to Q_{mn}(iH) \to Q_{mn}(iH) \otimes \mathcal{O}_Y,\]

where $1 \leq j \leq l_1$. From (4.7), it follows that

\[\text{ext}^2(\mathcal{O}_X, Q_{mn}) \leq \text{ext}^2(\mathcal{O}_X, Q_{mn}(l_1 H)) \sum_{j=1}^{l_1} \text{ext}^1(\mathcal{O}_X, Q_{mn}(jH) \otimes \mathcal{O}_Y) \]

\[= \sum_{j=1}^{l_1} \text{ext}^1(\mathcal{O}_X, Q_{mn}(jH) \otimes \mathcal{O}_Y).\]

As the same proof of (4.8), for $m \gg 0$ one obtains,

\[(4.9) \quad \text{ext}^2(\mathcal{O}_X, Q_{mn}) \leq \left( \frac{e_1}{p^{2r_n u_n^2}} + \frac{e_2}{p^{4r_n u_n^4}} + \frac{e_3}{p^{6r_n u_n^6}} \right) p^{3a_{mn}} + O(p^{2a_{mn}}),\]

where the constants $e_i$'s are independent of $m$ and $n$.

In conclusion, by (4.3), (4.5), (4.8) and (4.9), we obtain, for $m \gg 0$,

\[
\geq \chi(\mathcal{O}_X, Q_{mn}) \\
\geq p^{3a_{mn}} \chi(\mathcal{O}_X, E^3) + O(p^{2a_{mn}}).
\]
This implies
\[ \text{ch}_3^{(E)}(E) \leq \frac{d_1 + e_1}{p^{2\tau_n} u_1^2} + \frac{d_2 + e_2}{p^{4\tau_n} u_2^4} + \frac{d_3 + e_3}{p^{6\tau_n} u_3^6}. \]

Taking \( n \to +\infty \), we conclude that \( \text{ch}_3^{(E)}(E) \leq 0 \). This completes the proof of Theorem 4.1.

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