Quantum Knots and Riemann Hypothesis

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Abstract

In this paper we propose a quantum gauge system from which we construct generalized Wilson loops which will be as quantum knots. From quantum knots we give a classification table of knots where knots are one-to-one assigned with an integer such that prime knots are bijectively assigned with prime numbers and the prime number 2 corresponds to the trefoil knot.

Then by considering the quantum knots as periodic orbits of the quantum system and by the identity of knots with integers and an approach which is similar to the quantum chaos approach of Berry and Keating we derive a trace formula which may be called the von Mangoldt-Selberg-Gutzwiller trace formula. From this trace formula we then give a proof of the Riemann Hypothesis. For our proof of the Riemann Hypothesis we show that the Hilbert-Polya conjecture holds that there is a self-adjoint operator for the nontrivial zeros of the Riemann zeta function and this operator is the Virasoro energy operator with central charge $c = \frac{1}{2}$. Our approach for proving the Riemann Hypothesis can also be extended to prove the Extended Riemann Hypothesis. We also investigate the relation of our approach for proving the Riemann Hypothesis with the Random Matrix Theory for $L$-functions.

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1 Introduction

It is well known that the Jones polynomial as a knot invariant can be derived from a quantum Chern-Simon gauge field theory [1][2]. Inspired by this work in this paper we shall also propose a quantum gauge model. In this quantum model we generalize the way of defining Wilson loops to construct generalized Wilson loops which will be as quantum knots. From quantum knots we give a classification table of knots where knots are one-to-one assigned with an integer such that prime knots are bijectively assigned with prime numbers and the prime number 2 corresponds to the trefoil knot.

Then by considering the quantum knots as periodic orbits of the quantum system and by the identity of knots with integers and an approach which is similar to the quantum chaos approach of Berry and Keating we derive a trace formula which may be called the von Mangoldt-Selberg-Gutzwiller trace formula. From this trace formula we then give a proof of the Riemann Hypothesis [3]-[17].

From the quantum gauge model we first define the classical Wilson loop and Wilson line. Then from the quantum gauge model we derive a definition for the generator of the Wilson line.

Then we derive two quantum Knizhnik-Zamolodchikov (KZ) equations which are dual to each other for the product of quantum Wilson lines. This quantum KZ equation in dual form may be regarded as a quantum Yang-Mill equation as analogous to the classical Yang-Mill equation derived from the classical Yang-Mill theory since this quantum KZ equation is as the basic quantum equation derived from the quantum gauge model. Solutions of this quantum Yang-Mill equation are then used to construct generalized Wilson loops which are as quantum knots (These quantum knots may be regarded as solitons as similar to the instantons of the classical Yang-Mill equation). In deriving this quantum KZ equation we first derive a conformal field theory consisting of the Kac-Moody algebra and the Virasoro energy operator and Virasoro algebra.

Then from the quantum knots we derive a knot invariant. From this knot invariant we give the classification table of knots.
Then the quantum knots as the periodic orbits of the quantum gauge system and the identity of prime knots with prime numbers are as the two basic ingredients for proving the Riemann Hypothesis.

For our proof of the Riemann Hypothesis we show that the Hilbert-Polya conjecture holds that there is a self-adjoint operator for the nontrivial zeros of the Riemann zeta function and this operator is the Virasoro energy operator with central charge \( c = \frac{1}{2} \) \([18]-[19]\). Our approach for proving the Riemann Hypothesis can also be extended to prove the Extended Riemann Hypothesis. We also investigate the relation of our approach for proving the Riemann Hypothesis with the Random Matrix Theory for \( L \)-functions \([20]-[30]\).

This paper is organized as follows. In section 2 we give a brief description of a quantum gauge model of electrodynamics and its nonabelian generalization. In this paper we shall consider a nonabelian generalization with a \( SU(2) \) gauge symmetry. With this quantum model in section 3 we introduce the definition of classical Wilson loop and Wilson line.

In section 4 we derive the definition of the generator of the Wilson line. From this definition in section 4 and 5 we derive a conformal field theory which includes the Virasoro energy operator and Virasoro algebra, the affine Kac-Moody algebra and the quantum KZ equation in dual form. In section 6 we compute the solutions of the quantum KZ equation in dual form. In section 7 we compute the quantum Wilson lines. In section 8 we represent the braiding of two pieces of curves by defining the braiding of two quantum Wilson lines. By this representation in section 10 we define the generalized Wilson loop which will be as a quantum knot. In section 9 we compute the quantum Wilson loop. In section 10 we define generalized Wilson loops which will be shown to have properties of the corresponding knot diagram and will be regarded as quantum knots. In section 11 we give some examples of generalized Wilson loops and show that they have the properties of the corresponding knot diagram and thus may be regarded as quantum knots. In section 12 we show that this generalized Wilson loop is a complete copy of the corresponding knot diagram and thus we may call a generalized Wilson loop as a quantum knot. From quantum knots we have a knot invariant of the form \( Tr R^{-m} W(z, z) \) where \( W(z, z) \) denotes a quantum Wilson loop and \( R \) is the braiding invariant and is the monodromy of the quantum KZ equation and \( m \) is an integer. We show that this knot invariant classifies knots and that knots can be one-to-one assigned with the integer \( m \). In section 13 we give more computations of quantum knots and their knot invariant. Then in section 14 and 15 with the integer \( m \) we give a classification table of knots where we show that prime knots (and only prime knots) are assigned with prime integer \( m \).

Then in section 16 by using the classification table of knots and by considering the quantum knots as the periodic orbits of the quantum gauge model we then, by using the quantum chaos approach of Gutzwiller, Berry and Keating and the von Mangoldt-Selberg approach of proving the Riemann Hypothesis, derive the von Mangoldt-Selberg-Gutzwiller trace formula. From this trace formula we then prove the Riemann Hypothesis. We also generalize this approach to prove the Extended Riemann Hypothesis. In section 17 we determine that the central charge \( c \) for the Riemann zeta function and the Dirichlet \( L \)-functions is equal to \( \frac{1}{2} \). Then in section 18 by our approach for proving Riemann Hypothesis we give a commutative diagram relating Virasoro energy operator and Virasoro algebra, automorphic (or modular) forms and \( L \)-functions. In section 19 we show a connection of our approach with the Random Matrix Theory approach for \( L \)-functions by showing that this connection is from the conformal field theory.

2 A Quantum Gauge Model

We shall first establish a quantum gauge model. This quantum gauge model will be as a physical motivation for introducing operators which will be called Wilson loop and Wilson line as analogous to the Wilson loops in the existing quantum field theories. Then the definition of classical Wilson loop and Wilson line and the definition of a generator \( J \) of the Wilson line will be as the basis of the mathematical foundation of this paper (In order to simplify the mathematics of this paper we treat this quantum gauge model as a physical motivation instead of as the mathematical foundation of this paper).

We shall show that the generator \( J \) gives an affine Kac-Moody algebra and a Virasoro energy operator \( T \) with central charge \( c \). From \( J \) and \( T \) we shall derive the quantum KZ equation in dual form which will be regarded as the quantum Yang-Mills equation. From this quantum KZ equation we then construct
generalized Wilson loops which will be as quantum knots.

Let us construct a quantum gauge model, as follows. In probability theory we have the Wiener measure \( \nu \) which is a measure on the space \( C[t_0, t_1] \) of continuous functions \([41]\). This measure is a well defined mathematical theory for the Brownian motion and it may be symbolically written in the following form:

\[
d\nu = e^{-L_0} dx
\]  

(1)

where \( L_0 := \frac{1}{2} \int_{t_0}^{t_1} \left( \frac{d\lambda^2}{dx^2} \right)^2 dt \) is the energy integral of the Brownian particle and \( dx = \frac{1}{N} \prod_i dx(t) \) is symbolically a product of Lebesgue measures \( dx(t) \) and \( N \) is a normalized constant.

Once the Wiener measure is defined we may then define other measures on \( C[t_0, t_1] \) as follows\([41]\).

\[
d\nu_1 = e^{-\frac{1}{2} \int_{t_0}^{t_1} V dt} dr
\]  

(2)

Under some condition on \( V \) we have that \( \nu_1 \) is well defined on \( C[t_0, t_1] \). Let us call \([2]\) as the Feynmann-Kac formula \([41]\).

Let us then follow this formula to construct a quantum model electrodynamics, as follows. Then similar to the formula \([2]\) we construct a quantum model of electrodynamics from the following energy integral:

\[
- \int_{s_0}^{s_1} D s := \int_{s_0}^{s_1} \left( \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} \right)^* \left( \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} \right) + \left( \frac{dZ^2}{dx} + ie_0 A_z Z^* \right) \left( \frac{dZ}{dx} - ie_0 A_z Z \right) ds
\]  

(3)

where the complex variable \( Z = Z(z(s)) \) and the real variables \( A_1 = A_1(z(s)) \) and \( A_2 = A_2(z(s)) \) are continuous functions in a form that they are in terms of a (continuously differentiable) curve \( z(s) = (x^1(s), x^2(s)) \), \( s_0 \leq s \leq s_1 \), \( z(s_0) = z(s_1) \) in the complex plane where \( s \) is a parameter representing the proper time in relativity. The complex variable \( Z = Z(z(s)) \) represents a field of matter (such as the electron) \((Z^* \) denotes its complex conjugate) and the real variables \( A_1 = A_1(z(s)) \) and \( A_2 = A_2(z(s)) \) represent a connection (or the gauge field of the photon), and \( A_z = \sum_{j=1}^{2} A_j \frac{d^2 z}{dx^j} \), and \( e_0 \) denotes the bare electric charge.

The integral \([3]\) has the following gauge symmetry:

\[
Z'(z(s)) := Z(z(s)) e^{ie_0 a(z(s))}, \quad A_j'(z(s)) := A_j(z(s)) + \frac{\partial a}{\partial x_j}, \quad j = 1, 2
\]  

(4)

where \( a = a(z) \) is a continuously differentiable real-valued function of \( z \).

As the Wiener measure is based on the Banach space \( C[s_0, s_1] \), the above gauge model is based on the Banach space \( C^3[s_0, s_1] \) of continuous functions \( Z(z(s)), A_j(z(s)), j = 1, 2, s_0 \leq s \leq s_1 \) on the one dimensional interval \([s_0, s_1]\).

Similar to the differential \( dx \) of the Wiener measure, the differential of this gauge model is of the form \( dA_z dZ^* dZ \).

Since \( D \) is positive and the model is one dimensional we have that this gauge model is similar to the Wiener measure except that this gauge model has a gauge symmetry.

An advantage of this gauge model is that it is free of the difficulty of the degenerate degree of freedom of the gauge symmetry where the two degrees of \( A_1, A_2 \) is combined into one degree of \( A_z \). In the physics literature the usual way to treat the degenerate degree of freedom of gauge symmetry is to introduce a gauge fixing condition to eliminate the degenerate degree of freedom \([42]\). In this gauge model because it is free of the difficulty of the degenerate degree of freedom of the gauge symmetry that the mathematical difficulty of introducing gauge fixing condition can be avoid.

We have the following theorem:

**Theorem 1** The above gauge model is a measure defined on the Banach space \( C^3[s_0, s_1] \).

**Proof.** For this gauge model let us choose gauges such that the above gauge model is a measure defined on the Banach space \( C^3[s_0, s_1] \). As an example let us choose a gauge \((A_1, A_2)\) such that the following form holds:

\[
\frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} = \rho \frac{dA_z(z(s))}{ds}
\]  

(5)
for some constant $\rho > 0$. Then the energy integral is of the following form:

$$-\int_{s_0}^{s_1} Dds = -\int_{s_0}^{s_1} \left( \frac{dA_k}{ds} \right)^2 + \left( \frac{dA_0}{ds} + i\epsilon_0 A_z Z^* \right) \left( \frac{dA_0}{ds} - i\epsilon_0 A_z Z \right) ds$$  \hspace{1cm} (6)

Then the singular pure gauge $(A_1, A_2)$ part of the gauge model is of the form of the Wiener measure and thus is a well defined measure.

On the other hand the $Z$ part does not have the singularity of the gauge model and it gives a measure of the form of the above Feynman-Kac type. Thus from the energy integral (6) we have a well defined measure on the Banach space $C^3[0, s_1]$. This proves the theorem.

We remark that this model is not formulated with the four-dimensional space-time but is formulated with the one dimensional proper time in relativity theory. This gets rid of the difficulty of ultraviolet divergences of the four-dimensional space-time quantum field theory.

Similar to the usual Yang-Mills gauge theory we can generalize this gauge model with $U(1)$ gauge symmetry to nonabelian gauge models. As an illustration let us consider $SU(2)$ gauge symmetry. Similar to (3) we can develop a nonabelian gauge model as similar to that for the above abelian gauge model. We have that (7) is invariant under the following gauge transformation:

$$L := \int_{s_0}^{s_1} \frac{1}{2} Tr(D_1 A_2 - D_2 A_1)^*(D_1 A_2 - D_2 A_1) + (D_0 Z^*)(D_0 Z) ds$$ \hspace{1cm} (7)

where $Z = (z_1, z_2)^T$ is a two dimensional complex vector; $A_j = \sum_{k=0}^{3} A_k^j (k = 1, 2)$ where $A_k^j$ denotes a component of a gauge field $A^k$; $t^k = iT^k$ denotes a generator of $SU(2) \otimes U(1)$ where $T^k$ denotes a self-adjoint generator of $SU(2) \otimes U(1)$ (here for simplicity we choose a convention that the complex $i$ is absorbed by $t^k$ and $T^k$ is absorbed by $A_j$; and the notation $A_j$ is with a little confusion with the notation $A_j$ in the above formulation of (3) where $A_j$, $j = 1, 2$ are real valued); and $D_j = \frac{\partial}{\partial x_j} - A_j$ for $j = 1, 2$; and $D_0 = \frac{d}{dt} - A_z$ where $A_z = \sum_{j=1}^{2} A_j \frac{dx^j}{ds}$.

From (7) we can develop a nonabelian gauge model as similar to that for the above abelian gauge model. We have that (7) is invariant under the following gauge transformation:

$$Z^i(z(s)) := U(a(z(s)))Z(z(s))$$

$$A_j^i(z(s)) := U(a(z(s)))A_j(z(s))U^{-1}(a(z(s))) + U(a(z(s)))\frac{dU^{-1}}{dz^j}(a(z(s))), \quad j = 1, 2$$ \hspace{1cm} (8)

where $U(a(z(s))) = e^{a(z(s))}$; $a(z(s)) = \sum_k a^k(z(s))e_0 t_k$ for some functions $a^k$. We shall mainly consider the case that $a$ is a function of the form $a(z(s)) = \sum_k \text{Re} \omega^k(z(s))e_0 t_k$ where $\omega^k$ are analytic functions of $z$ (We let $\omega(z(s)) := \sum_k \omega^k(z(s))e_0 t_k$ and we write $a(z) = \text{Re} \omega(z)$).

Then as the above abelian case, this gauge model is a well defined measure on the Banach space of continuous functions $Z(z(s))$, $A_j(z(s))$, $j = 1, 2$, $s_0 \leq s \leq s_1$ on the one dimensional interval $[s_0, s_1]$.

From this gauge model we shall derive the quantum KZ equation in dual form which will be regarded as a quantum Yang-Mills equation. From this quantum KZ equation in dual form we then construct quantum knots. From quantum knots we then prove the Riemann Hypothesis.

3 Dirac-Wilson loop

Similar to the Wilson loop in quantum field theory [2] from our quantum theory we introduce an analogue of Wilson loop, as follows (We shall also call a Wilson loop as a Dirac-Wilson loop).

**Definition.** A classical Wilson loop $W_{DC}(\gamma)$ is defined by:

$$W_{DC}(\gamma) := W_C(z_0, z_1) := Pe^\int \gamma A_j dx^j$$ \hspace{1cm} (9)
where $D$ denotes a representation of $SU(2)$ and $C$ means classical (For simplicity we shall omit the notation $D$: $W_C(\gamma) := W_D(\gamma)$; $\gamma = z(\cdot)$ is a closed curve; $z_0 = z_1$ is a starting and end point of $\gamma = z(\cdot)$; and the quantum gauge model is based on $z(\cdot)$ as specific in the above section (We keep in mind that the definition of the classical Wilson loop $W_C(z_0, z_1)$ is defined and depended on the whole curve $z(\cdot)$). As usual the notation $P$ in the definition of $W_C(\gamma)$ denotes a path-ordered product:

$$Pe^{\int_c A_i dx^i} := \lim_{\max_k (r_{k+1} - r_k) \to 0} \prod_{k=0}^n e^{\sum_{j=1}^n A_j(z(r_k))\Delta x^j(r_k))} \quad (10)$$

where $s_0 = r_1 < r_2 < \cdots < r_n = s_1$ is a partition of $[s_0, s_1]$.

**Remarks.**

1. We extend the definition of $W_C(z(\cdot))$ to the case that $z(\cdot)$ is not a closed curve with $z_0 \neq z_1$. When $z(\cdot)$ is not a closed curve we shall call $W_C(z_0, z_1)$ as a Wilson line.

2. We shall use the notation $W(z_0, z_1)$ to denote the quantum version of the classical Wilson line $W_C(z_0, z_1)$ and we call $W(z_0, z_1)$ as the quantum Wilson line.

We first have the following theorem on $W_C(z_0, z_1)$:

**Theorem 2** For a given continuous path $A_i$, $i = 1, 2$ on $[s_0, s_1]$ the Wilson line $W_C(z_0, z_1)$ exists on this path and has the following transition property:

$$W_C(z_0, z_1) = W_C(z_0, z) W_C(z, z_1) \quad (11)$$

where $W_C(z_0, z_1)$ denotes the Wilson line of a curve $z(\cdot)$ which is with $z_0$ as the starting point and $z_1$ as the ending point and $z$ is a point on $z(\cdot)$ between $z_0$ and $z_1$.

**Proof.** We have that $W_C(z_0, z_1)$ is a limit (whenever exists) of ordered product of $e^{A_i \Delta x^i}$ and thus can be written in the following form:

$$W_C(z_0, z_1) = I + \int_{s''}^{s'} A_i(z(s)) \frac{dx^i(z)}{ds} ds + \int_{s'}^{s''} \int_{s''}^{s'} A_i(z(s_1)) \frac{dx^i(z_1)}{ds_1} ds_1 A_i(z(s_2)) \frac{dx^i(z_2)}{ds_2} ds_2 + \cdots \quad (12)$$

where $z(s') = z_0$ and $z(s'') = z_1$. Then since $A_i$ are continuous on $[s', s'']$ and $x^i(z(\cdot))$ are continuously differentiable on $[s', s'']$ we have that the series in (12) is absolutely convergent. Thus the Wilson line $W_C(z_0, z_1)$ exists. Then since $W_C(z_0, z_1)$ is the limit of ordered product we can write $W_C(z_0, z_1)$ in the form $W_C(z_0, z) W_C(z, z_1)$ by dividing $z(\cdot)$ into two parts at $z$. This proves the theorem. \textcircled{\small o}

By following the usual approach of deriving a chiral symmetry from a gauge transformation of a gauge field we have the following chiral symmetry which is derived by applying an analytic gauge transformation with an analytic function $\omega$ for the transformation:

**Theorem 3** Under an analytic gauge transformation with an analytic function $\omega$ we have the following symmetry:

$$W_C(z_0, z_1) \mapsto W_C^\omega(z_0, z_1) = U(\omega(z_1)) W_C(z_0, z_1) U^{-1}(\omega(z_0)) \quad (13)$$

where $W_C^\omega(z_0, z_1)$ is a Wilson line with gauge field:

$$A'_\mu = U(z) \frac{\partial U^{-1}(z)}{\partial x^\mu} + U(z) A_\mu U^{-1}(z) \quad (14)$$

**Proof.** Let a gauge transformation $U(\omega(z))$ be with an analytic variation $\omega$ of the form $\omega = \sum_k \omega^k e_0 t^k$ with the generators $e_0 t^k$.

Then let us prove the symmetry (13) as follows. Let $U(z) := U(\omega(z(s)))$ and $U^{-1}(z + dz) \approx U^{-1}(z) + \frac{\partial U^{-1}(z)}{\partial x^\mu} dz^\mu$ where $dz = (dx^1, dx^2)$. Following (10) we have

$$U(z)(1 + dx^\mu A_\mu) U^{-1}(z + dz) = U(z) U^{-1}(z + dz) + dx^\mu U(z) A_\mu U^{-1}(z + dz) \approx 1 + U(z) \frac{\partial U^{-1}(z)}{\partial x^\mu} dx^\mu + dx^\mu U(z) A_\mu U^{-1}(z + dz) \quad (15)$$

$$= 1 + dx^\mu A'_\mu$$
From [15] we have that [18] holds since [18] is the limit of ordered product in which the right-side factor \(U^{-1}(z_i + dz_i)\) in [15] is canceled by the left-side factor \(U(z_{i+1})\) of [15] where \(z_{i+1} = z_i + dz_i\). This proves the theorem. □

As analogous to the WZW model in conformal field theory [18][19], from the above symmetry we have the following formulas for the variations \(\delta_\omega W_C\) (where the notation \(\delta_\omega\) means the variation with respect to the variations \(\omega^k\)), and \(\delta_\omega W_C\) with respect to this symmetry ([18] p.621):

\[
\delta_\omega W_C(z, z') = W_C(z, z')\omega(z), \quad \delta_\omega W_C(z, z') = -\omega'(z')W_C(z, z') \tag{16}
\]

where \(z\) and \(z'\) are independent variables and \(\omega'(z') = \omega(z)\) when \(z' = z\). In [16] the variation \(\delta_\omega\) is with respect to the \(z\) variable while the variation \(\delta_\omega\) is with respect to the \(z'\) variable. This two-side-variations when \(z \neq z'\) can be derived as follows. For the left variation we may let \(\omega\) be analytic in a neighborhood of \(z\) and extended as a continuously differentiable function to a neighborhood of \(z'\) such that \(\omega(z') = 0\) in this neighborhood of \(z'\). Then from [13] we have that the first formula of (16) holds. Similarly we may let \(\omega'\) be analytic in a neighborhood of \(z'\) and extended as a continuously differentiable function to a neighborhood of \(z\) such that \(\omega'(z) = 0\) in this neighborhood of \(z\). Then we have that the second formula of (16) holds.

In terms of an analytic variation \(\omega\) of the form \(\omega = \sum_k \omega^k t^k\) where the generators \(t^k\) are without the charge \(e_0\), from [17] we have the following formulas for the variations \(\delta_\omega W_C\) and \(\delta_\omega W_C\):

\[
\delta_\omega W_C(z, z') = W_C(z, z')\omega(z)e_0, \quad \delta_\omega W_C(z, z') = -e_0\omega'(z')W_C(z, z') \tag{17}
\]

### 4 Affine Kac-Moody algebra

We shall derive a quantum loop algebra (or the affine Kac-Moody algebra) structure from the Wilson line \(W_C(z, z')\) for the generator \(J\) of \(W(z, z')\). To this end let us first consider the classical case. Since \(W_C(z, z')\) is constructed from \(SU(2)\) we have that the mapping \(z \rightarrow W_C(z, z')\) (We consider \(W_C(z, z')\) as a function of \(z\) with \(z'\) being fixed) has a loop group structure [14][45]. For a loop group we have the following generators:

\[
J_n^a = t^a z^n \quad n = 0, \pm 1, \pm 2, \ldots \tag{18}
\]

These generators satisfy the following algebra:

\[
[J_m^a, J_n^b] = i f_{abc} J_m^c + n \tag{19}
\]

This is the so called loop algebra [14][45]. Let us then introduce the following generating function \(J\):

\[
J(w) = \sum_a J^a(w) = \sum_a j^a(w)t^a \tag{20}
\]

where we define

\[
j^a(w) = j^a(w) := \sum_{n=-\infty}^{\infty} J_n^a(z)(w-z)^{-n-1} \tag{21}
\]

From \(J\) we have

\[
J_n^a = \frac{1}{2\pi i} \oint z \, dw (w-z)^n j^a(w) \tag{22}
\]

where \(\oint_z\) denotes a closed contour integral with center \(z\). This formula can be interpreted as that \(J\) is the generator of the loop group and that \(J_n^a\) is the directional generator in the direction \(\omega^a(w) = (w-z)^n\). We may generalize [22] to the following directional generator:

\[
\frac{1}{2\pi i} \oint z \, dw \omega(w) J(w) \tag{23}
\]

where the analytic function

\[
\omega(w) = \sum_a \omega^a(w)t^a \tag{24}
\]
is regarded as a variation direction and we define

$$\omega(w)J(w) := \sum_a \omega^a(w)J^a$$  \hspace{1cm} (25)$$

Then since $W_C(z,z') \in SU(2)$, from the variational formula (26) for the loop algebra of the loop group of $SU(2)$ we have that the variation of $W_C(z,z')$ in the direction $\omega(w)$ is given by

$$W_C(z,z') \frac{1}{2\pi i} \oint_z dw \omega(w)J(w)$$  \hspace{1cm} (26)$$

Now let us consider the quantum case which is based on the quantum gauge theory in section 2. For this quantum case we shall define a quantum generator $J$ which is analogous to the $J$ in (20). We shall choose the equations (37) and (38) as the equations for defining the quantum generator $J$. Let us first give a formal derivation of the equation (37), as follows. Let us consider the following functional

$$\langle W(z,z')B(z) \rangle := \int dA_z dZ^* dZe^{-L}W_C(z,z')B_C(z)$$  \hspace{1cm} (27)$$

where $A_z(z,s) = \sum_{j=1}^2 A_j \frac{dz^j}{dz^i}$ and $B_C(z)$ denotes a field from the quantum gauge theory (We first let $z'$ be fixed as a parameter) and $B(z)$ is the corresponding quantum operator.

Let us do a calculus of variation on this integral to derive a variational equation by applying a gauge transformation on (27) as follows (We remark that such variational equations are usually called the Ward identity in the physics literature).

Let $(A_1, A_2, Z)$ be regarded as a coordinate system of the integral (27). Under a gauge transformation (regarded as a change of coordinate) with gauge function $a(z(s))$ this coordinate is changed to another coordinate denoted by $(A'_1, A'_2, Z')$. As similar to the usual change of variable for integration we have that the integral (27) is unchanged under a change of variable and we have the following equality:

$$\int dA_z dZ^* dZe^{-L}W'_C(z,z')B'_C(z) = \int dA_z dZ^* dZe^{-L}W_C(z,z')B_C(z)$$  \hspace{1cm} (28)$$

where $W'_C(z,z')$ denotes the Wilson line based on $A'_1$ and $A'_2$ and similarly $B'_C(z)$ denotes the field obtained from $B_C(z)$ with $(A_1, A_2, Z)$ replaced by $(A'_1, A'_2, Z')$.

Then it can be shown that the differential is unchanged under a gauge transformation (42):

$$dA'_z dZ'^* dZ' = dA_z dZ^* dZ$$  \hspace{1cm} (29)$$

Also by the gauge invariance property the factor $e^{-L}$ is unchanged under a gauge transformation. Thus from (28) we have

$$0 = \langle W'(z,z')B'(z) \rangle - \langle W(z,z')B(z) \rangle$$  \hspace{1cm} (30)$$

where the correlation notation $\langle \cdot \rangle$ denotes the integral with respect to the differential (or the measure of the gauge model):

$$e^{-L}dA_z dZ^* dZ$$  \hspace{1cm} (31)$$

We can now carry out the calculus of variation. From the gauge transformation we have the formula:

$$W'_C(z,z') = U(a(z))W_C(z,z')U^{-1}(a(z'))$$  \hspace{1cm} (32)$$

where $a(z) = \Re \omega(z)$. This gauge transformation gives a variation of $W_C(z,z')$ with the gauge function $a(z)$ as the variational direction $a$ in the variational formulas (29) and (20). Thus analogous to the variational formula (20) we have that the variation of $W(z,z')$ under this gauge transformation is given by

$$W(z,z') \frac{1}{2\pi i} \oint_z dwa(w)J(w)$$  \hspace{1cm} (33)$$
where the generator $J$ for this variation is to be specific. This $J$ will be a quantum generator which generalizes the classical generator $J$ in (26). Thus under a gauge transformation with gauge function $a(z)$ from (30) we have the following variational equation:

$$0 = (W(z, z')[\delta_a B(z) + \frac{1}{2\pi i} \oint_{z} dw a(w) J(w) B(z)])$$

(34)

where $\delta_a B(z)$ denotes the variation of the field $B(z)$ in the direction $a(z)$. From this equation an ansatz of $J$ is that $J$ satisfies the following equation:

$$W(z, z')[\delta_a B(z) + \frac{1}{2\pi i} \oint_{z} dw a(w) J(w) B(z)] = 0$$

(35)

From this equation we have the following variational equation:

$$\delta_a B(z) = \frac{-1}{2\pi i} \oint_{z} dw a(w) J(w) B(z)$$

(36)

where $a(w) J(w)$ is defined by (25). This completes the calculus of variation.

From (36) we have the following equation for defining the generator $J$:

$$\delta_\omega B(z) = \frac{-1}{2\pi i} \oint_{z} dw \omega(w) J(w) B(z)$$

(37)

where we generalize the direction $a(z) = Re \omega(z)$ to the analytic variation direction $\omega(z)$ in (24).

Let us then add one more condition for determining the quantum generator $J$ in (37). As analogous to the WZW model in conformal field theory [18][19][46] let us consider a classical generator $J_C$ given by

$$J_C(z) := -k_0 W^{-1}_C(z, z') \partial_{z'} W_C(z, z')$$

(38)

where we define $\partial_z = \partial_{z^1} + i \partial_{z^2}$ and we set $z' = z$ after the differentiation with respect to $z$; $k_0 > 0$ is a constant which is fixed when the $J_C$ is determined to be of the form (38) and the minus sign is chosen by convention. In the WZW model [18][46] the $J_C$ of the form (38) is the generator of the chiral symmetry of the WZW model. We can write the $J_C$ in (38) in the following form:

$$J_C(w) = \sum_a J_C^a(w) = \sum_a j_C^a(w) e_0 t^a$$

(39)

Thus the quantum version $J$ of $J_C$ is of the following form:

$$J(w) = \sum_a J^a(w) = \sum_a j^a(w) e_0 t^a$$

(40)

where $j^a(w)$ are quantum generators.

We see that the generators $t^a$ of SU(2) appear in this form of $J_C$ and $J$ and this form is analogous to the classical generator $J$ in (26). This shows that this $J$ is a possible candidate for the quantum generator $J$ in (37).

Then from (13) and (38) we have that the variation $\delta_\omega J_C$ of the generator $J_C$ in (38) is given by [18] (p.622) [46]:

$$\delta_\omega J_C = [J_C, \omega] - k_0 \partial_{z'} \omega$$

(41)

Then the quantum generator $J$ is required also to have this property (41) of the classical generator $J_C$:

$$\delta_\omega J = [J, \omega] - k_0 \partial_{z'} \omega$$

(42)

Now we show that this quantum generator $J$ in (37) and (42) can be uniquely solved. From (37) and (42) we have that $J$ satisfies the following relation of current algebra [18][19][46]:

$$J^a(w) J^b(z) = \frac{k_0 \delta_{ab}}{(w - z)^2} + \sum_c i f_{abc} \frac{J^c(z)}{(w - z)}$$

(43)
where as a convention the regular term of the product $J^a(w)J^b(z)$ is omitted. Then by following \[13\] from \[13\] and \[16\] we can show that the $J^a$ in \[20\] for the corresponding Laurent series of the quantum generator $J$ satisfy the following Kac-Moody algebra:

$$[J^a_{m},J^b_{n}] = if_{abc}J^c_{m+n} + k_0m\delta_{ab}\delta_{m+n,0} \quad (44)$$

We remark that the constant $k_0$ is called the central extension or the level of the Kac-Moody algebra.

Remark. Let us also consider the other side of the above chiral symmetry. Similar to the $J$ in \[35\] we define a generator $J'_C$ by:

$$J'_C(z') = k_0\partial_{z'}W_C(z, z')W_C^{-1}(z, z') \quad (45)$$

where after differentiation with respect to $z'$ we set $z = z'$. Let us then consider the following formal correlation:

$$\langle B(z')W(z, z') \rangle := \int dAdZ^{\ast}dZB(z')W_C(z, z') \quad (46)$$

where $z$ is fixed. By an approach similar to the above derivation of \[37\] we have the following variational equation:

$$\delta_{\omega'}B(z') = -\frac{1}{2\pi i} \oint_{z'} dwB(z')J'(w)\omega'(w) \quad (47)$$

Then similar to \[12\] we also have

$$\delta_{\omega'}J' = [J', \omega'] - k_0\partial_{z'}\omega' \quad (48)$$

Then from \[12\] and \[18\] we can derive the current algebra and the Kac-Moody algebra for $J'$ which are of the same form of \[13\] and \[16\]. From this we have $J' = J$. \diamond

5 Quantum Knizhnik-Zamolodchikov Equation In Dual Form

With the above current algebra $J$ and the formula \[37\] we can now follow the usual approach in conformal field theory to derive a quantum Knizhnik-Zamolodchikov (KZ) equation for the product of primary fields in a conformal field theory \[13\] \[19\] \[16\]. We shall derive the KZ equation for the product of $n$ Wilson lines $W(z, z')$. Here an important point is that from the two sides of $W(z, z')$ we can derive two quantum KZ equations which are dual to each other. These two quantum KZ equations are different from the usual KZ equation in that they are equations for the quantum operators $W(z, z')$ while the usual KZ equation is for the correlations of quantum operators.

With this difference the following derivation of KZ equation for deriving these two quantum KZ equations is well known in conformal field theory \[13\] \[19\]. The reader may skip this derivation of KZ equation and just look at the form of the Virasoro energy operator $T(z)$ and the Virasoro algebra and the form of these two quantum KZ equations.

Let us first consider the following quantum version of \[17\] with $W_C(z, z')$ replaced by the quantum $W(z, z')$:

$$\delta_{\omega}W(z, z') = W(z, z')\omega(z)e_0, \quad \delta_{\omega'}W(z, z') = -e_0\omega'(z')W(z, z') \quad (49)$$

From \[37\] and \[49\] we have:

$$J^a(z)W(w, w') = \frac{-e_0f^aW(w, w')}{z - w} \quad (50)$$

where as a convention the regular term of the product $J^a(z)W(w, w')$ is omitted.

Following \[18\] and \[19\] let us define a Virasoro energy operator (also called an energy-momentum tensor) $T(z)$ by:

$$T(z) := \frac{1}{2(k_0 + g_0)} \sum_a :J^a(z)J^a(z): \quad (51)$$
where \( g_0 \) is the dual Coxeter number of the gauge group \([18]\). In \([34]\) the symbol : \( J^a(z)J^a(z) : \) denotes the normal ordering of the operator \( J^a(z)J^a(z) \) which can be defined as follows \([18][19]\). Let a product of operators \( A(z)B(w) \) be written in the following Laurent series form:

\[
A(z)B(w) = \sum_{n=-\infty}^{\infty} a_n(w)(z-w)^n
\]  

(52)

The singular part of (52) is called the contraction of \( A(z)B(w) \) and will be denoted by \( \langle \rangle \). Then the term \( a_0(w) \) is called the normal ordering of \( A(z)B(w) \) and we denote \( a_0(w) \) by : \( A(w)B(w) : \).

The above definition of the energy operator \( T(z) \) is called the Sugawara construction \([18]\). We first have the following well known theorem on \( T(z) \) in conformal field theory \([18]\):

**Theorem 4** The operator product \( T(z)T(w) \) is given by the following formula:

\[
T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\]  

(53)

for some constant \( c = \frac{4k_0}{k_0 + g_0} \) and as a convention we omit the regular term of this product.

**Proof.** In \([18]\) there is a detail proof of this theorem where the constant \( \frac{1}{2(k_0 + g_0)} \) is shown to be a normalization constant such that this theorem holds. Here we want to remark that the formula (43) for the product \( J^a(z)J^b(w) \) is used for the proof of this theorem.

From this theorem we then have the following Virasoro algebra of the mode expansion of \( T(z) \) \([18][19]\):

**Theorem 5** Let us write \( T(z) \) in the following Laurent series form:

\[
T(z) = \sum_{n=-\infty}^{\infty} (z-w)^{-n-2}L_n(w)
\]  

(54)

This means that the modes \( L_n(w) \) are defined by

\[
L_n(w) := \frac{1}{2\pi i} \oint_{w} dz (z-w)^{n+1}T(z)
\]  

(55)

Then we have that \( L_n \) form a Virasoro algebra:

\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}
\]  

(56)

From the formula (43) for the product \( J^a(z)J^b(w) \) we have the following operator product expansion \([18]\):

\[
\langle \rangle T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}
\]  

(57)

Then we have the following operator product of \( T(z) \) with an operator \( A(w) \):

\[
T(z)A(w) = \sum_{n=-\infty}^{\infty} (z-w)^{-n-2}L_nA(w)
\]  

(58)

From (57) and (58) we have that \( L_{-1}J^a(w) = \partial J^a(w) \) and \( L_{-1} = \frac{\partial}{\partial z} \). Thus we have

\[
L_{-1}W(w, w') = \frac{\partial W(w, w')}{\partial w}
\]  

(59)
On the other hand as shown in [18], by using the Laurent series expansion of $J^a(z)$ in the section on Kac-Moody algebra we can compute the normal ordering : $J^a(z)J^a(z)$ : from which we have the Laurent series expansion of $T(z)$ with $L_{-1}$ given by [18]:

$$L_{-1} = \frac{1}{2(k_0 + g_0)} \sum_a \left( \sum_{m \leq -1} J_m^a J_{-1-m}^a + \sum_{m \geq 0} J_{-1-m}^a J_m^a \right)$$

(60)

where since $J_m^a$ and $J_{-1-m}^a$ commute each other the ordering of them is irrelevant.

From (60) we then have

$$L_{-1} W(w, w') = \frac{1}{2(k_0 + g_0)} \sum_a \sum_{m \leq -1} J_m^a(w) J_{-1-m}^a(w) + \sum_{m \geq 0} J_{-1-m}^a(w) J_m^a(w) W(w, w')$$

(61)

since $J_m^a W(w, w') = 0$ for $m > 0$ by (60) and the quantum version of $J_m^a$ in [22] (where the $\sum_a$ is omitted).

It follows from (59) and (61) that we have: the following equality:

$$\partial_w W(w, w') = \frac{1}{(k_0 + g_0)} J_{-1}^a(w) J_0^a(w) W(w, w')$$

(62)

Then from (50) and the quantum version of $J_0^a$ in [22] we have:

$$J_0^a(w) W(w, w') = -e_0 t^a W(w, w')$$

(63)

From (62) and (63) we then have

$$\partial_z W(z, z') = \frac{-e_0}{k_0 + g_0} J_{-1}^a(z) t^a W(z, z')$$

(64)

Now let us consider a product of $n$ Wilson lines: $W(z_1, z'_1) \cdots W(z_n, z'_n)$. Let this product be represented as a tensor product when $z_i$ and $z'_j$, $i, j = 1, \ldots, n$ are all independent variables. Then from (64) we have

$$\partial_z W(z_1, z'_1) \cdots W(z_n, z'_n) = \frac{-e_0}{k_0 + g_0} J_{-1}(z) t^a W(z_1, z'_1) \cdots W(z_n, z'_n)$$

(65)

where the second equality is from the definition of tensor product for which we define

$$t^a W(z_1, z'_1) \cdots W(z_n, z'_n) := W(z_1, z'_1) \cdots [t^a W(z_1, z'_1)] \cdots W(z_n, z'_n)$$

(66)

With this formula (65) we can now follow [18] and [19] to derive the KZ equation. For a easy reference let us present this derivation in [18] and [19] as follows. From the Laurent series of $J^a$ we have

$$J_{-1}(z) = \frac{1}{2\pi i} \oint_{z_i} \frac{dz}{z - z_i} J^a(z)$$

(67)

where the line integral is on a contour encircling $z_i$. We let this contour be a circle with center $z_i$ and with small radius such that all other $z_j$ of $W(z_j, z'_j)$ for $j = 1, \ldots, n, j \neq i$, are excluded. Then we have:

$$J_{-1}(z_i) W(z_1, z'_1) \cdots W(z_n, z'_n)$$

$$= \frac{1}{2\pi i} \oint_{z_i} \frac{dz}{z - z_i} J^a(z) W(z_1, z'_1) \cdots W(z_n, z'_n)$$

(68)

$$= \frac{1}{2\pi i} \oint_{z_i} \frac{dz}{z - z_i} \sum_{j=1}^n W(z_1, z'_1) \cdots \left[ \frac{-e_0 t^a}{z - z_j} W(z_j, z'_j) \right] \cdots W(z_n, z'_n)$$

(69)
where the second equality is from the JW product formula (50). Then we have that (65) is equal to:

\[ \sum_{j=1, j \neq i}^{n} \frac{-e_0 t^a}{z_i - z_j} W(z_1, z'_1) \cdots W(z_n, z'_n) \]

(69)

where we have used the definition of tensor product.

From (69) and by applying (65) to \( z_i \) for \( i = 1, \ldots, n \) we have the following Knizhnik-Zamolodchikov equation [18] [19] [46]:

\[ \partial_{z_i} W(z_1, z'_1) \cdots W(z_n, z'_n) = \frac{e_0^2}{k_0 + g_0} \sum_{j \neq i}^{n} W(z_1, z'_1) \cdots \frac{t^a \otimes t^a}{z_i - z_j} W(z_1, z'_1) \cdots W(z_n, z'_n) \]

(70)

for \( i = 1, \ldots, n \). We remark that in (70) we have defined \( t^a := t^a \) and

\[ t^a \otimes t^a W(z_1, z'_1) \cdots W(z_n, z'_n) := W(z_1, z'_1) \cdots [t^a W(z_i, z'_i)] \cdots [t^a W(z_j, z'_j)] \cdots W(z_n, z'_n) \]

(71)

It is interesting and important that we also have another KZ equation with respect to the \( z'_i \) variables. The derivation of this KZ equation is dual to the above derivation in that the operator products and their corresponding variables are with reverse order to that in the above derivation.

From (47) and (49) we have a \( W, J' \) operator product given by

\[ W(w, w') J^a(z') = \frac{-W(w, w') e_0 t^a}{w' - z'} \]

(72)

where we have omitted the regular term of the product. Then similar to the above derivation of the KZ equation from (72) we can then derive the following Knizhnik-Zamolodchikov equation which is dual to (70):

\[ \partial_{z'_i} W(z_1, z'_1) \cdots W(z_n, z'_n) = \frac{e_0^2}{k_0 + g_0} \sum_{j \neq i}^{n} W(z_1, z'_1) \cdots \frac{t^a \otimes t^a}{z'_j - z'_i} W(z_1, z'_1) \cdots W(z_n, z'_n) \]

(73)

for \( i = 1, \ldots, n \) where we have defined:

\[ W(z_1, z'_1) \cdots W(z_n, z'_n) t^a \otimes t^a := W(z_1, z'_1) \cdots [W(z_i, z'_i) t^a] \cdots [W(z_j, z'_j) t^a] \cdots W(z_n, z'_n) \]

(74)

**Remark.** From the generator \( J \) and the Kac-Moody algebra we have derived a quantum KZ equation in dual form. This quantum KZ equation in dual form may be considered as a quantum Yang-Mills equation since it is analogous to the classical Yang-Mills equation which is derived from the classical Yang-Mills gauge model. This quantum KZ equation in dual form will be as the starting point for the construction of quantum knots.

6 Solving Quantum KZ Equation In Dual Form

Let us consider the following product of two quantum Wilson lines:

\[ G(z_1, z_2, z_3, z_4) := W(z_1, z_2) W(z_3, z_4) \]

(75)

where the two quantum Wilson lines \( W(z_1, z_2) \) and \( W(z_3, z_4) \) represent two pieces of curves starting at \( z_1 \) and \( z_3 \) and ending at \( z_2 \) and \( z_4 \) respectively.

We have that this product \( G \) satisfies the KZ equation for the variables \( z_1, z_3 \) and satisfies the dual KZ equation for the variables \( z_2 \) and \( z_4 \). Then by solving the two-variables-KZ equation in (70) we have that a form of \( G \) is given by [32] [33] [31]:

\[ e^{-i \log |z(z_1 - z_3)|} C_1 \]

(76)
where $t := \frac{e_{1}^{2}}{g_{1}+g_{2}} \sum_{a} t^{a} \otimes t^{a}$ and $C_{1}$ denotes a constant matrix which is independent of the variable $z_{1} - z_{3}$.

We see that $G$ is a multivalued analytic function where the determination of the $\pm$ sign depended on the choice of the branch.

Similarly by solving the dual two-variable-KZ equation in (73) we have that $G$ is of the form

$$C_{2} e^{i \log[\pm(z_{1} - z_{3})]}$$

(77)

where $C_{2}$ denotes a constant matrix which is independent of the variable $z_{4} - z_{2}$.

From (76), (77) and we let $C_{1} = A e^{i \log[\pm(z_{1} - z_{3})]}$, $C_{2} = e^{-i \log[\pm(z_{1} - z_{3})]} A$ where $A$ is a constant matrix we have that $G$ is given by

$$G(z_{1}, z_{2}, z_{3}, z_{4}) = e^{-i \log[\pm(z_{1} - z_{3})]} A e^{i \log[\pm(z_{4} - z_{2})]}$$

(78)

where at the singular case that $z_{1} = z_{3}$ we simply define $\log[\pm(z_{1} - z_{3})] = 0$. Similarly for $z_{2} = z_{4}$.

Let us find a form of the initial operator $A$. We notice that there are two operators $\Phi_{\pm}(z_{1} - z_{3}) := e^{-i \log[\pm(z_{1} - z_{3})]}$ and $\Psi_{\pm}(z_{4} - z_{2}) := e^{-i \log[\pm(z_{4} - z_{2})]}$ acting on the two sides of $A$ respectively where the two independent variables $z_{1}, z_{3}$ of $\Phi_{\pm}$ are mixedly from the two quantum Wilson lines $W(z_{1}, z_{2})$ and $W(z_{2}, z_{4})$ respectively and the two independent variables $z_{2}, z_{4}$ of $\Psi_{\pm}$ are mixedly from the two quantum Wilson lines $W(z_{1}, z_{2})$ and $W(z_{3}, z_{4})$ respectively. From this we determine the form of $A$ as follows.

Let $D$ denote a representation of $SU(2)$. Let $D(g)$ represent an element $g$ of $SU(2)$ and let $D(g) \otimes D(g)$ denote the tensor product representation of $SU(2)$. Then in the KZ equation we define

$$[t^{a} \otimes t^{a}][D(g_{1}) \otimes D(g_{1})] \otimes [D(g_{2}) \otimes D(g_{2})] := [t^{a} D(g_{1}) \otimes D(g_{1})] \otimes [t^{a} D(g_{2}) \otimes D(g_{2})]$$

(79)

and

$$[D(g_{1}) \otimes D(g_{1})] \otimes [D(g_{2}) \otimes D(g_{2})][t^{a} \otimes t^{a}] := [D(g_{1}) \otimes D(g_{1})]t^{a} \otimes [D(g_{2}) \otimes D(g_{2})]t^{a}$$

(80)

Then we let $U(a)$ denote the universal enveloping algebra where $a$ denotes an algebra which is formed by the Lie algebra $su(2)$ and the identity matrix.

Now let the initial operator $A$ be of the form $A_{1} \otimes A_{2} \otimes A_{3} \otimes A_{4}$ where $A_{i}, i = 1, ..., 4$ denote operators taking values in $U(a)$. In this case we have that in (78) the operator $\Phi_{\pm}(z_{1} - z_{3}) := e^{-i \log[\pm(z_{1} - z_{3})]}$ acts on $A$ from the left via the following formula:

$$t^{a} \otimes t^{a} A = [t^{a} A_{1}] \otimes A_{2} \otimes [t^{a} A_{3}] \otimes A_{4}$$

(81)

Similarly the operator $\Psi_{\pm}(z_{4} - z_{2}) := e^{i \log[\pm(z_{4} - z_{2})]}$ in (78) acts on $A$ from the right via the following formula:

$$A t^{a} \otimes t^{a} = A_{1} \otimes [A_{2} t^{a}] \otimes A_{3} \otimes [A_{4} t^{a}]$$

(82)

We may generalize the above tensor product of two quantum Wilson lines as follows. Let us consider a tensor product of $n$ quantum Wilson lines: $W(z_{1}, z'_{1}) \cdots W(z_{n}, z'_{n})$ where the variables $z_{i}, z'_{i}$ are all independent. By solving the two KZ equations we have that this tensor product is given by:

$$W(z_{1}, z'_{1}) \cdots W(z_{n}, z'_{n}) = \prod_{i} \Phi_{\pm}(z_{i} - z_{j}) A \prod_{i} \Psi_{\pm}(z'_{i} - z'_{j})$$

(83)

where $\prod_{i} \Phi_{\pm}(z_{i} - z_{j})$ or $\Psi_{\pm}(z'_{i} - z'_{j})$ for $i, j = 1, ..., n$ where $i \neq j$. In (83) the initial operator $A$ is represented as a tensor product of operators $A_{ij}, i, j = 1, ..., n$ where each $A_{ij}$ is of the form of the initial operator $A$ in the above tensor product of two-Wilson-lines case and is acted by $\Phi_{\pm}(z_{i} - z_{j})$ or $\Psi_{\pm}(z'_{i} - z'_{j})$ on its two sides respectively.

### 7 Computation of Quantum Wilson Lines

Let us consider the following product of two quantum Wilson lines:

$$G(z_{1}, z_{2}, z_{3}, z_{4}) := W(z_{1}, z_{2}) W(z_{3}, z_{4})$$

(84)
where the two quantum Wilson lines $W(z_1, z_2)$ and $W(z_2, z_4)$ represent two pieces of curves starting at $z_1$ and $z_3$ and ending at $z_2$ and $z_4$ respectively. As shown in the above section we have that $G$ is given by the following formula:

$$G(z_1, z_2, z_3, z_4) = e^{-i \log[\pm(z_1-z_3)]} A e^{i \log[\pm(z_4-z_2)]}$$

where the product is a 4-tensor.

Let us set $z_2 = z_3$. Then the 4-tensor $W(z_1, z_2)W(z_3, z_4)$ is reduced to the 2-tensor $W(z_1, z_2)W(z_2, z_4)$. By using (56) the 2-tensor $W(z_1, z_2)W(z_2, z_4)$ is given by:

$$W(z_1, z_2)W(z_2, z_4) = e^{-i \log[\pm(z_1-z_2)]} A_{144} e^{i \log[\pm(z_4-z_2)]}$$

where $A_{14} = A_1 \otimes A_4$ is a 2-tensor reduced from the 4-tensor $A = A_1 \otimes A_2 \otimes A_3 \otimes A_4$ in (55). In this reduction the $i$ operator of $\Phi = e^{-i \log[\pm(z_1-z_2)]}$ acting on the left side of $A_1$ and $A_3$ in $A$ is reduced to acting on the left side of $A_1$ and $A_4$ in $A_{14}$. Similarly the $i$ operator of $\Psi = e^{-i \log[\pm(z_4-z_2)]}$ acting on the right side of $A_2$ and $A_4$ in $A$ is reduced to acting on the right side of $A_1$ and $A_4$ in $A_{14}$.

Then since $i$ is a 2-tensor operator we have that $i$ is a matrix acting on the two sides of the 2-tensor $A_{14}$ which is also as a matrix with the same dimension as $i$. Thus $\Phi$ and $\Psi$ are as matrices of the same dimension as the matrix $A_{14}$ acting on $A_{14}$ by the usual matrix operation. Then since $i$ is a Casimir operator for the 2-tensor group representation of $SU(2)$ we have that $\Phi$ and $\Psi$ commute with $A_{14}$ since $\Phi$ and $\Psi$ are exponentials of $i$ (We remark that $\Phi$ and $\Psi$ are in general not commute with the 4-tensor initial operator $A$). Thus we have

$$e^{-i \log[\pm(z_1-z_2)]} A_{144} e^{i \log[\pm(z_4-z_2)]} = e^{-i \log[\pm(z_1-z_2)]} e^{i \log[\pm(z_4-z_2)]} A_{14}$$

(87)

We let $W(z_1, z_2)W(z_2, z_4)$ be as a representation of the quantum Wilson line $W(z_1, z_4)$ and we write $W(z_1, z_4) = W(z_1, z_2)W(z_2, z_4)$. Then we have the following representation of $W(z_1, z_4)$:

$$W(z_1, z_4) = W(z_1, w_1)W(w_1, z_4) = e^{-i \log[\pm(z_1-w_1)]} e^{i \log[\pm(z_4-w_1)]} A_{14}$$

(88)

This representation of the quantum Wilson line $W(z_1, z_4)$ means that the line (or path) with end points $z_1$ and $z_4$ is specified that it passes the intermediate point $w_1 = z_2$. This representation shows the quantum nature that the path is not specified at other intermediate points except the intermediate point $w_1 = z_2$. This unspecification of the path is of the same quantum nature of the Feynmann path description of quantum mechanics.

Then let us consider another representation of the quantum Wilson line $W(z_1, z_4)$. We consider $W(z_1, w_1)W(w_1, w_2)W(w_2, z_4)$ which is obtained from the tensor $W(z_1, w_1)W(w_1, w_2)W(w_2, z_4)$ by two reductions where $z_j$, $w_j$, $w_j$, $j = 1, 2$ are independent variables. For this representation we have:

$$W(z_1, w_1)W(w_1, w_2)W(w_2, z_4) = e^{-i \log[\pm(z_1-w_1)]} e^{-i \log[\pm(z_1-w_2)]} e^{i \log[\pm(z_4-w_1)]} e^{i \log[\pm(z_4-w_2)]} A_{14}$$

(89)

This representation of the quantum Wilson line $W(z_1, z_4)$ means that the line (or path) with end points $z_1$ and $z_4$ is specified that it passes the intermediate points $w_1$ and $w_2$. This representation shows the quantum nature that the path is not specified at other intermediate points except the intermediate points $w_1$ and $w_2$. This unspecification of the path is of the same quantum nature of the Feynmann path description of quantum mechanics.

Similarly we may represent the quantum Wilson line $W(z_1, z_4)$ by path with end points $z_1$ and $z_4$ and is specified only to pass at finitely many intermediate points. Then we let the quantum Wilson line $W(z_1, z_4)$ as an equivalent class of all these representations. Thus we may write $W(z_1, z_4) = W(z_1, w_1)W(w_1, z_4) = W(z_1, w_1)W(w_1, w_2)W(w_2, z_4) = \cdots$.

Remark. Since $A_{14}$ is a 2-tensor we have that a natural group representation for the Wilson line $W(z_1, z_4)$ is the 2-tensor group representation of the group $SU(2)$. 

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8 Representing Braiding of Curves by Quantum Wilson Lines

Consider again the product \( G(z_1, z_2, z_3, z_4) = W(z_1, z_2)W(z_3, z_4) \). We have that \( G \) is a multivalued analytic function where the determination of the \( \pm \) sign depended on the choice of the branch.

Let the two pieces of curves be crossing at \( w \). Then we have \( W(z_1, z_2) = W(z_1, w)W(w, z_2) \) and \( W(z_3, z_4) = W(z_3, w)W(w, z_4) \). Thus we have

\[
W(z_1, z_2)W(z_3, z_4) = W(z_1, w)W(w, z_2)W(z_3, w)W(w, z_4)
\]  

(90)

If we interchange \( z_1 \) and \( z_3 \), then from (90) we have the following ordering:

\[
W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4)
\]  

(91)

Now let us choose a branch. Suppose that these two curves are cut from a knot and that following the orientation of a knot the curve represented by \( W(z_1, z_2) \) is before the curve represented by \( W(z_3, z_4) \). Then we fix a branch such that the angle of \( z_3 - z_1 \) minus the angle of \( z_1 - z_3 \) is equal to \( \pi \).

**Remark.** We may use other representations of \( W(z_1, z_2)W(z_3, z_4) \). For example we may use the following representation:

\[
W(z_1, w)W(w, z_2)W(z_3, w)W(w, z_4) = e^{-i\log(z_1 - z_3)}Ae^{i\log(z_4 - z_2)}
\]  

(92)

Then if we interchange \( z_1 \) and \( z_3 \) we have

\[
W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) = e^{-i\log(-z_1 - z_3)}Ae^{i\log(z_4 - z_2)}
\]  

(93)

From (92) and (93) as a choice of branch we have

\[
W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) = RW(z_1, w)W(z_3, w)W(w, z_4)
\]  

(94)

where \( R = e^{-i\pi} \) is the monodromy of the KZ equation. In (94) \( z_1 \) and \( z_3 \) denote two points on a closed curve such that along the direction of the curve the point \( z_1 \) is before the point \( z_3 \) and in this case we choose a branch such that the angle of \( z_3 - z_1 \) minus the angle of \( z_1 - z_3 \) is equal to \( \pi \).

Now from (94) we can take a convention that the ordering (91) represents that the curve represented by \( W(z_1, z_2) \) is undercrossing the curve represented by \( W(z_3, z_4) \) while (90) represents zero crossing of these two curves.

Similarly from the dual KZ equation as a choice of branch which is consistent with the above formula we have

\[
W(z_1, w)W(w, z_4)W(z_3, w)W(w, z_2) = W(z_1, w)W(w, z_2)W(z_3, w)W(w, z_4)R^{-1}
\]  

(96)

where \( z_2 \) is before \( z_4 \). We take a convention that the ordering in (96) represents that the curve represented by \( W(z_1, z_2) \) is undercrossing the curve represented by \( W(z_3, z_4) \). Here along the orientation of a closed curve the piece of curve represented by \( W(z_1, z_2) \) is before the piece of curve represented by \( W(z_3, z_4) \). In this case since the angle of \( z_3 - z_1 \) minus the angle of \( z_1 - z_3 \) is equal to \( \pi \) we have that the angle of \( z_4 - z_2 \) minus the angle of \( z_2 - z_4 \) is also equal to \( \pi \) and this gives the \( R^{-1} \) in this formula (96).

From (94) and (96) we have

\[
W(z_3, z_4)W(z_1, z_2) = RW(z_1, z_2)W(z_3, z_4)R^{-1}
\]  

(97)

where \( z_1 \) and \( z_2 \) denote the end points of a curve which is before a curve with end points \( z_3 \) and \( z_4 \). From (97) we see that the algebraic structure of these quantum Wilson lines \( W(z', z') \) is analogous to the quasi-triangular quantum group [15, 31].
9 Computation of Quantum Wilson Loop

Let us consider again the quantum Wilson line \( W(z_1, z_4) = W(z_1, z_2)W(z_2, z_4) \). Let us set \( z_1 = z_4 \). In this case the quantum Wilson line forms a closed loop. Now in \( [87] \) with \( z_1 = z_4 \) we have that \( e^{-i \log \pm (z_1 - z_4)} \) and \( e^{i \log \pm (z_1 - z_2)} \) which come from the two-side KZ equations cancel each other and from the multivalued property of the log function we have

\[
W(z_1, z_1) = R^n A_{14} \quad n = 0, \pm 1, \pm 2, ... \quad (98)
\]

where \( R = e^{-in\hat{t}} \) is the monodromy of the KZ equation \([31] \).

**Remark.** It is clear that if we use other representation of the quantum Wilson loop \( W(z_1, z_1) \) (such as the representation \( W(z_1, z_1) = W(z_1, w_1)W(w_1, w_2)W(w_2, z_1) \)) then we will get the same result as \( [98] \).

**Remark.** For simplicity we shall drop the subscript of \( A_{14} \) in \( [98] \) and simply write \( A_{14} = A \).

10 Defining Quantum Knots and Knot Invariant

Now we have that the quantum Wilson loop \( W(z_1, z_1) \) corresponds to a closed curve in the complex plane with starting and ending point \( z_1 \). Let this quantum Wilson loop \( W(z_1, z_1) \) represents the unknot. We shall call \( W(z_1, z_1) \) as the quantum unknot. Then from \( [98] \) we have the following invariant for the unknot:

\[
\text{Tr}W(z_1, z_1) = \text{Tr}R^n A \quad n = 0, \pm 1, \pm 2, ... \quad (99)
\]

where \( A = A_{14} \) is a 2-tensor constant matrix operator.

In the following let us extend the definition \([99]\) to a knot invariant for nontrivial knots. Let \( W(z_1, z_j) \) represent a piece of curve with starting point \( z_1 \) and ending point \( z_j \). Then we let

\[
W(z_1, z_2)W(z_3, z_4) \quad (100)
\]

represent two pieces of uncrossing curve. Then by interchanging \( z_1 \) and \( z_3 \) we have

\[
W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) \quad (101)
\]

represent the curve specified by \( W(z_1, z_2) \) upcrossing the curve specified by \( W(z_3, z_4) \).

Now for a given knot diagram we may cut it into a sum of parts which are formed by two pieces of curves crossing each other. Each of these parts is represented by \( [101] \) (For a knot diagram of the unknot with zero crossings we simply do not need to cut the knot diagram). Then we define the trace of a knot with a given knot diagram by the following form:

\[
\text{Tr} \cdots W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) \cdots \quad (102)
\]

where we use \( [101] \) to represent the state of the two pieces of curves specified by \( W(z_1, z_2) \) and \( W(z_3, z_4) \). The \( \cdots \) means the product of a sequence of parts represented by \( [101] \) according to the state of each part. The ordering of the sequence in \( [102] \) follows the ordering of the parts given by the orientation of the knot diagram. We shall call the sequence of crossings in the trace \( [102] \) as the generalized Wilson loop of the knot diagram. For the knot diagram of the unknot with zero crossings we simply let it be \( W(z, z) \) and call it the quantum Wilson loop.

We shall show that the generalized Wilson loop of a knot diagram has all the properties of the knot diagram in the following let us first consider some examples to illustrate the way to define \( [102] \) and the

11 Examples of Quantum Knots

Before the proof that a generalized Wilson loop of a knot diagram has all the properties of the knot diagram in the following let us first consider some examples to illustrate the way to define \( [102] \) and the
way of applying the braiding formulas (94), (96) and (97) to equivalently transform (102) to a simple expression of the form $Tr R^{-m} W(z, z)$ where $m$ is an integer.

Let us first consider the knot in Fig.1. For this knot we have that (102) is given by

$$Tr W(z_2, w) W(w, z_2) W(z_1, w) W(w, z_1)$$

where the product of quantum Wilson lines is from the definition (101) represented a crossing at $w$. In applying (101) we let $z_1$ be the starting and the ending point.

Then we have that (103) is equal to

$$Tr W(z_2, w) W(z_1, w) W(w, z_1) W(z_2, w)$$

$$= Tr RW(z_1, w) W(w, z_2) R^{-1} RW(z_2, w) W(w, z_1) R^{-1}$$

$$= Tr W(z_1, z_2) W(z_2, z_1)$$

where we have used (97). We see that (104) is just the knot invariant (99) of the unknot. Thus the knot in Fig.1 is with the same knot invariant of the unknot and this agrees with the fact that this knot is topologically equivalent to the unknot.

Let us then consider a trefoil knot in Fig.2a. By (101) and similar to the above examples we have that the definition (102) for this knot is given by:

$$Tr W(z_4, w_1) W(w_1, z_2) W(z_1, w_1) W(w_1, z_5) \cdot W(z_2, w_2) W(w_2, z_6)$$

$$W(z_5, w_2) W(w_2, z_3) \cdot W(z_6, w_3) W(w_3, z_4) W(z_3, w_3) W(w_3, z_1)$$

$$= Tr W(z_4, w_1) RW(z_1, w_1) W(w_1, z_2) R^{-1} W(w_1, z_5) \cdot W(z_2, w_2) RW(z_5, w_2)$$

$$W(w_2, z_6) R^{-1} W(w_2, z_3) \cdot W(z_6, w_3) RW(z_3, w_3) W(w_3, z_4) W(z_3, w_3) R^{-1} W(w_3, z_1)$$

$$= Tr W(z_4, w_1) RW(z_1, z_2) R^{-1} W(w_1, z_5) \cdot W(z_2, w_2) RW(z_5, z_6) R^{-1} W(w_2, z_3)$$

$$W(z_6, w_3) RW(z_3, z_4) R^{-1} W(w_3, z_1)$$

$$= Tr W(z_4, w_1) RW(z_1, z_2) W(z_2, w_2) W(w_1, z_5) W(z_5, z_6) R^{-1} W(w_2, z_3)$$

$$W(z_6, w_3) RW(z_3, z_4) R^{-1} W(w_3, z_1)$$

$$= Tr W(z_4, w_2) RW(z_1, z_2) W(w_1, z_6) R^{-1} W(w_2, z_3)$$

$$W(z_6, w_3) RW(z_3, z_4) R^{-1} W(w_3, z_1)$$

where we have repeatedly used (97). Then we have that (105) is equal to:

$$Tr W(z_6, w_3) W(w_3, z_1) W(z_3, w_3) W(w_3, z_6) W(z_1, z_3)$$

$$= Tr RW(z_3, w_3) W(w_3, z_1) W(z_6, w_3) W(w_3, z_6) W(z_1, z_3)$$

$$= Tr RW(z_3, w_3) RW(z_6, w_3) W(w_3, z_1) R^{-1} W(w_3, z_6) W(z_1, z_3)$$

$$= Tr W(z_3, w_4) RW(z_6, z_1) R^{-1} W(w_3, z_6) W(z_1, z_3)$$

$$= Tr RW(z_3, w_4) RW(z_6, z_3)$$

$$= Tr RW(z_3, w_4) W(w_3, z_6) W(z_6, z_3)$$

$$= Tr RW(z_3, w_4)$$

$$= 4.$$
where we have used (92) and (97). This is as a knot invariant for the trefoil knot in Fig. 2a.

Then let us consider the trefoil knot in Fig. 2b which is the mirror image of the trefoil knot in Fig. 2a. The definition (102) for this knot is given by:

\[
TrW(z_5, z_1)W(z_2, w_1)W(w_1, z_5)W(z_1, w_1)W(w_1, z_2) = TrW(z_5, z_1)W(z_2, w_1)W(w_1, z_2)R^{-1}W(w_1, z_5)R^{-1}
\]

where similar to (105) we have repeatedly used (97). Then we have that (107) is equal to:

\[
TrW(z_5, z_1)W(z_2, w_1)W(w_1, z_5)W(z_1, w_1)W(w_1, z_2) = TrW(z_5, z_1)W(z_2, w_1)R^{-1}W(w_1, z_5)R^{-1}W(w_1, z_2)
\]

where we have used (106) and (97). This is as a knot invariant for the trefoil knot in Fig. 2b. We notice that the knot invariants for the two trefoil knots are different. This shows that these two trefoil knots are not topologically equivalent.

More examples of the above quantum knots and knot invariants will be given in a following section.

12 Generalized Wilson Loops as Quantum Knots

Let us now show that the generalized Wilson loop of a knot diagram has all the properties of the knot diagram and that (102) is a knot invariant. To this end let us first consider the structure of a knot. Let \( K \) be a knot. Then a knot diagram of \( K \) consists of a sequence of crossings of two pieces of curves cut from the knot \( K \) where the ordering of the crossings can be determined by the orientation of the knot \( K \). As an example we may consider the two trefoil knots in the above section. Each trefoil knot is represented by three crossings of two pieces of curves. These three crossings are ordered by the orientation of the trefoil knot starting at \( z_1 \). Let us denote these three crossings by 1, 2 and 3. Then the sequence of these three crossings is given by 123. On the other hand if the ordering of the three crossings starts from other \( z_i \) on the knot diagram then we have sequences 231 and 312. All these sequences give the same knot diagram and they can be transformed to each other by circling as follows:

\[
123 \rightarrow 123(1) = 231 \rightarrow 231(2) = 312 \rightarrow 312(3) = 123 \rightarrow \cdots
\]

where \((x)\) means that the number \( x \) is to be moved to the \((x)\) position as indicated. Let us call (109) as the circling property of the trefoil knot.
As one more example let us consider the figure-eight knot in Fig.3. The simplest knot diagram of this knot has four crossings.

Starting at $z_1$ let us denote these crossings by 1, 2, 3 and 4. Then we have the following circling property of the figure-eight knot:

\[
1234 \rightarrow 1234(2) = 1342 \rightarrow 1342(1) = 3421 \rightarrow 3421(4) = 3214 \rightarrow 3214(3) = 2143 \rightarrow 2143(1) = 2431 \rightarrow 2431(2) = 4312 \rightarrow 4312(3) = 4123 \rightarrow 4123(4) = 1234 \rightarrow \cdots
\]

(110)

We notice that in this circling of the figure-eight knot there are subcirclings.

In summary we have that a knot diagram of a knot $K$ can be characterized as a finite sequence of crossings of curves which are cut from the knot diagram where the ordering of the crossings is derived from the orientation of the knot diagram and has a circling property for which (109) and (110) are examples.

Now let us represent a knot diagram of a knot $K$ by a sequence of products of Wilson lines representing crossings as in the above section. Let us call these products of Wilson lines by the term $W$-product. Then we call this sequence of $W$-products as the generalized Wilson loop of the knot diagram of a knot $K$.

Let us consider the following two $W$-products:

\[
W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) \quad \text{and} \quad W(z_1, z_2)W(z_3, z_4)
\]

(111)

In the above section we have shown that these two $W$-products faithfully represent two oriented pieces of curves crossing or not crossing each other where $W(z_1, z_2)$ and $W(z_3, z_4)$ represent these two pieces of curves.

Now there is a natural ordering of the $W$-products of crossings derived from the orientation of a knot as follows. Let $W(z_1, z_2)$ and $W(z_3, z_4)$ represent two pieces of curves where the piece of curve represented by $W(z_1, z_2)$ is before the piece of curve represented by $W(z_3, z_4)$ according to the orientation of a knot.

Then the ordering of these two pieces of curves can be represented by the product $W(z_1, z_2)W(z_3, z_4)$. Now let 1 and 2 denote two $W$-products of crossings where we let 1 before 2 according to the orientation of a knot. Then from the ordering of pieces of curves we have that the product 12 represents the ordering of the two crossings 1 and 2.

Now let a knot diagram of a knot $K$ be given. Let the crossings of this knot diagram be denoted by $1, 2, \cdots, n$ and let this knot diagram be characterized by the sequence of crossings $123 \cdots n$ which is formed according to the orientation of this knot diagram. On the other hand let us for simplicity also denote the corresponding $W$-products of crossings by $1, 2, \cdots, n$. Then the whole product of $W$-products of crossings $123 \cdots n$ represents the sequence $123 \cdots n$ of crossings which is identified with the knot diagram. This whole product $123 \cdots n$ of $W$-products of crossings is the generalized Wilson loop of the knot diagram and we denote it by $W(K)$. In the following let us show that this generalized Wilson loop $W(K)$ has the circling property of the sequences of crossings of the knot diagram. It then follows that this generalized Wilson loop represents all the properties of the sequence $123 \cdots n$ of crossings of the knot diagram. Then since this sequence $123 \cdots n$ of crossings of the knot diagram is identified with the knot diagram we have that this generalized Wilson loop $W(K)$ can be identified with the knot diagram. We have the following theorem.
Theorem 6 Each knot $K$ can be faithfully represented by its generalized Wilson loop $W(K)$ in the sense that if two knot diagrams have the same generalized Wilson loop then these two knot diagrams must be topologically equivalent.

Proof. Let us show that the generalized Wilson loop $W(K)$ of a knot diagram of $K$ has the circling property. Let us consider a product $W(z_1, z_2)W(z_3, z_4)$ where we first let $z_1, z_2, z_3$ and $z_4$ be all independent. By solving the two KZ equations as shown in the above sections we have

$$W(z_1, z_2)W(z_3, z_4) = e^{-i\log[z(z_1-z_2)]}Ae^{i\log[z(z_2-z_4)]} \quad (112)$$

where the initial operator $A$ is a 4-tensor as shown in the above sections. The sign $\pm$ in (112) reflects that solutions of the KZ equations are complex multi-valued functions. (We remark that the 4-tensor initial operator $A$ in general may not commute with $\Phi_{\pm}(z_1 - z_2) = e^{-i\log[\pm(z_1-z_2)]}$ and $\Psi_{\pm}(z_1 - z_2) = e^{i\log[\pm(z_1-z_2)]}$).

Then the interchange of $W(z_1, z_2)$ and $W(z_3, z_4)$ corresponds to that $z_1$ and $z_3$ interchange their positions and $z_2$ and $z_4$ interchange their positions respectively. This interchange gives a pair of sign changes:

$$(z_3 - z_1) \rightarrow (z_1 - z_3) \quad \text{and} \quad (z_2 - z_4) \rightarrow (z_4 - z_2) \quad (113)$$

From this we have that $W(z_3, z_4)W(z_1, z_2)$ is given by

$$W(z_3, z_4)W(z_1, z_2) = e^{-i\log[z(z_1-z_3)]}Ae^{i\log[z(z_4-z_2)]} \quad (114)$$

Now let us set $z_2 = z_3$ and $z_1 = z_4$ such that the two products in (112) and (114) form a closed loop. In this case we have that $\Phi_{\pm}(z_3 - z_1) = R\Phi_{\pm}(z_1 - z_3)$. Then since $W(z_1, z_2)$ and $W(z_3, z_4)$ represent two lines with $z_1, z_2$ and $z_3, z_4$ as starting and ending points respectively we have that the sign change $z_2 - z_4 \rightarrow z_4 - z_2$ also gives the same $i\pi$ difference from the multivalued function log. Thus we have that $\Psi_{\pm}(z_4 - z_2) = R^{-1}\Psi_{\pm}(z_2 - z_4)$. It follows from this pair of sign changes and that $A$ commutes with $\Phi_{\pm}$ and $\Psi_{\pm}$, we have that $W(z_1, z_2)W(z_3, z_4) = W(z_3, z_4)W(z_1, z_2)$ when $z_2 = z_3$ and $z_1 = z_4$. This proves the simplest circling property of generalized Wilson loops.

We remark that in the above proof the pair of sign changes gives two factors $R$ and $R^{-1}$ which cancel each other and gives the circling property. We shall later apply the same reason of pair sign changes to get the general circling property. We also remark that the proof of this circling property is based on the same reason as the derivation of the braiding formulas (94), (96) and (97) as shown in the above sections.

Let us consider a product of $n$ quantum Wilson lines $W(z_i, z'_i), i = 1, \ldots, n$, with the property that the end points $z_i, z'_i$ of these quantum Wilson lines are connected to form a closed loop. From the analysis in the above sections we have that this product is reduced from a tensor product to a 2-tensor. It then follows from (83) that this product is of the following form:

$$\prod_{ij} \Phi_{\pm}(z_i - z_j)A \prod_{ij} \Psi_{\pm}(z'_i - z'_j) \quad (115)$$

where the initial operator $A$ is reduced to a 2-tensor and that the $\pm$ signs of $\Phi_{\pm}(z_i - z_j)$ and $\Psi_{\pm}(z_i - z_j)$ are to be determined. Then since $\Phi_{\pm}(z_i - z_j)$ and $\Psi_{\pm}(z_i - z_j)$ commute with $A$ we can write (115) in the form

$$\prod_{ij} \Phi_{\pm}(z_i - z_j) \prod_{ij} \Psi_{\pm}(z'_i - z'_j)A \quad (116)$$

where $i \neq j$. From this formula let us derive the general circling property as follows.

Let us consider two generalized Wilson lines denoted by 1 and 2 respectively. Here by the term generalized Wilson line we mean a product of quantum Wilson lines with two open ends. As a simple
example let us consider the product \( W(z, z_1)W(z_2, z) \). By definition this is a generalized Wilson line with two open ends \( z_1 \) and \( z_2 \) (\( z \) is not an open end). Suppose that the two open ends of 1 and 2 are connected. Then we want to show that \( 12 = 21 \). This identity is a generalization of the above interchange of \( W(z_1, z_2) \) and \( W(z_3, z_4) \) with \( z_2 = z_3 \) and \( z_1 = z_4 \).

Because 12 and 21 form closed loops we have that 12 and 21 are products of quantum Wilson lines \( W(u_i, u_k) \) (where \( u_i \) and \( u_k \) denote some \( z_p \) or \( w_q \) where we use \( w_q \) to denote crossing points) such that for each pair of variables \( u_i \) and \( u_j \) appearing at the left side of \( W(u_i, u_k) \) and \( W(u_j, u_i) \) there is exactly one pair of variables \( u_i \) and \( u_j \) appearing at the right side of \( W(u_j, u_i) \) and \( W(u_i, u_j) \). Thus in the formula (116) (with the variables \( z, z' \) in (116) denoted by variables \( u \)) we have that the factors \( \Phi_{\pm}(u_i - u_j) \) and \( \Psi_{\pm}(u_i - u_j) \) appear in pairs.

As in the above case we have that the interchange of the open ends of 12 and 21 interchanges 12 to 21. This interchange gives changes of the factors \( \Phi_{\pm}(u_i - u_j) \) and \( \Psi_{\pm}(u_i - u_j) \) as follows.

Let \( z_1 \) and \( z_2 \) be the open ends of 1 and \( z_3 \) and \( z_4 \) be the open ends of 2 such that \( z_1 = z_4 \) and \( z_2 = z_3 \). Consider a factor \( \Phi_{\pm}(z_1 - z_3) \). The interchange of \( z_1 \) and \( z_3 \) interchanges this factor to \( \Phi_{\pm}(z_3 - z_1) \). Then there is another factor \( \Psi_{\pm}(z_2 - z_4) \). The interchange of \( z_2 \) and \( z_4 \) interchanges this factor to \( \Psi_{\pm}(z_4 - z_2) \). Thus this is a pair of sign changes. By the same reason and the consistent choice of branch as in the above case we have that the formula (116) is unchanged. This shows that \( 12 = 21 \).

Then let us consider a factor \( \Phi_{\pm}(u_i - u_j) \) of the form \( \Phi_{\pm}(z_1 - u_j) \) where \( u_i \) and \( u_j \) is not an open end. Corresponding to this factor we have the factor \( \Phi_{\pm}(z_1 - u_i) \) of \( \Phi_{\pm}(z_1 - u_j) \). The interchange of \( z_1 \) and \( z_3 \) we have that \( \Phi_{\pm}(z_1 - u_j) \) and \( \Phi_{\pm}(z_3 - u_j) \) change to \( \Phi_{\pm}(z_3 - u_j) \) and \( \Phi_{\pm}(z_1 - u_j) \) respectively which gives no change to the formula (116). A similar result holds for the interchange of \( z_2 \) and \( z_4 \) for factors \( \Psi_{\pm}(z_2 - u_j) \) and \( \Psi_{\pm}(z_4 - u_j) \).

It follows that under the interchange of the open ends of 1 and 2 we have the pairs of sign changes from which the formula (116) is unchanged. This shows that \( 12 = 21 \).

Then we consider two generalized Wilson products of crossings which are products of crossings with four open ends respectively. Let us again denote them by 1 and 2. Each such generalized Wilson crossing can be regarded as the crossing of two generalized Wilson lines. Then the interchange of two open ends of the two generalized lines of 1 with the two open ends of the two generalized lines of 2 respectively interchanges 12 to 21. Then let us suppose that the open ends of these two Wilson products are connected in such a way that the products 12 and 21 form closed loops. In this case we want to show that \( 12 = 21 \) which is a circling property of a knot diagram. The proof of this equality is again similar to the above cases. In this case we also have that the interchange of the open ends of the two generalized Wilson crossings gives pairs of sign changes of the factors \( \Phi_{\pm}(u_i - u_j) \) and \( \Psi_{\pm}(u_i - u_j) \) in 12 and 21. Then by using (116) we have 12 = 21.

Let us then consider two generalized Wilson products of crossings denoted by 1 and 2 with open ends connected in such a way that two open ends of 1 (of the four open ends of 1) are connected to two open ends of 2 to form a closed loop. We want to prove that \( 12 = 21 \). This will give the subcircling property.

Since a closed loop is formed we have that each open end of 1 or of 2 is connected to a closed loop. In this case as the above cases we have that the products 12 is with the initial operator \( \mathbf{A} \) being a 2-tensor since the open ends of 1 or 2 do not cause \( \mathbf{A} \) to be a tensor with tensor degree more than 2 by their connection to the closed loop. Indeed, let \( z \) be an open end of 1 or 2. Then it is an end point of a quantum Wilson line \( W(z, z') \) which is a part of 1 and 2 such that \( z' \) is on the closed loop formed by 1 and 2. Then we have that this quantum Wilson line \( W(z, z') \) is connected with the closed loop at \( z' \). Since the loop is closed from the open end \( z \) we can go continuously along the closed loop to the open end of other quantum Wilson lines connected to the closed loop. It follows that the open end \( z \) gives no additional tensor degree to the initial operator \( \mathbf{A} \) for the product 12 or 21 and that the initial operator \( \mathbf{A} \) is still as the initial operator for the closed loop that it is a 2-tensor. (This is the same reason that in the above section on the computation of quantum Wilson loop we have that two quantum Wilson lines \( W(z_1, z_2) \) and \( W(z_2, z_4) \) connected at \( z_2 \) with two open ends \( z_1 \) and \( z_4 \) is with the same 2-tensor intial operator \( \mathbf{A} \) as the case that the two quantum Wilson lines \( W(z_1, z_2) \) and \( W(z_2, z_4) \) form a closed loop with \( z_1 = z_4 \).

Now since \( \mathbf{A} \) is a 2-tensor we have that \( \mathbf{A}, \Phi_{\pm} \) and \( \Psi_{\pm} \) are as matrices of the same dimension. In this case we have that \( \mathbf{A} \) commutes with \( \Phi_{\pm} \) and \( \Psi_{\pm} \). Then by interchange the open ends of 1 with open ends
of 2 we interchange 12 to 21. This interchange again gives pairs of sign changes. Then since the initial
operator \(A\) commutes with \(\Phi_\pm\) and \(\Psi_\pm\) we have that \(12 = 21\), as was to be proved. Then we let 12 and
21 be connected to another generalized Wilson product of crossing denoted by 3 to form a closed loop.
Then from 12 = 21 we have 312 = 321 and 123 = 213. This gives the subcircling property of generalized
Wilson loops. This subcircling property has been illustrated in the knot diagram of the eight-eight knot.
Then from a case in the above we also have the circling property 321 = 213 between 3 and 21.

Continuing in this way we have the circling or subcircling properties for generalized Wilson loops
whenever the open ends of a product of generalized Wilson lines or crossings are connected in such a way
that among the open ends a closed loop is formed. This shows that the generalized Wilson loop of a knot
diagram has the circling property of the knot diagram. With this circling property it then follows that
the generalized Wilson loop of a knot diagram completely describes the structure of the knot diagram.

Now since the generalized Wilson loop of a knot diagram is a complete copy of this knot diagram
we have that two knot diagrams which can be equivalently moved to each other if and only if the
corresponding generalized Wilson loops can be equivalently moved to each other. Thus we have that
if two knot diagrams have the same generalized Wilson loop then these two knot diagrams must be
equivalent. This proves the theorem.

**Examples of generalized Wilson loops.** As an example of generalized Wilson loops let us consider
the trefoil knots. Starting at \(z_1\) let the W-product of crossings be denoted by 1, 2 and 3. Then we have the
following circling property of the generalized Wilson loops of the trefoil knots:

\[
123 = 123(1) = 231 = 231(2) = 312 = 312(3) = 123 = \cdots
\]

(117)

As one more example let us consider the figure-eight knot. Starting at \(z_1\) let the W-product of crossings
be denoted by 1, 2, 3 and 4. Then we have the following circling property of the generalized Wilson loop
of the figure-eight knot:

\[
1234 = 1234(2) = 1342 = 3421 = 3421(4) = 3214 = 3214(3) = 2143 = 2143(1) = 2431 = 2431(2) = 4312 =
4312(3) = 4123 = 4123(4) = 1234 = \cdots
\]

(118)

\(\diamondsuit\)

**Definition.** We may call a generalized Wilson loop of a knot diagram as a quantum knot since by
the above theorem this generalized Wilson loop is a complete copy of the knot diagram. \(\diamondsuit\)

From the above theorem we have the following theorem.

**Theorem 7** Let \(W(K)\) denote the generalized Wilson loop of a knot \(K\). Then we can write \(W(K)\) in
the form \(R^{-m}W(C) = R^{-m}W(z_1, z_1)\) for some integer \(m\) where \(C\) denotes a trivial knot and \(W(C) =
W(z_1, z_1)\) denotes a Wilson loop on \(C\) with starting point \(z_1\) and ending point \(z_1\). From this form we
have that the trace \(Tr R^{-m}\) is a knot invariant which classifies knots. Thus knots can be classified by
the integer \(m\).

**Proof.** Since a generalized Wilson loop \(W(K)\) is in a closed and connected form we have that a
generalized Wilson loop \(W(K)\) can be of the form \([116]\). Thus from the multivalued property of the log
function and the two-side cancelation in \([116]\) we have that \(W(K)\) can be of the following (multivalued)
form

\[
W(K) = R^{-k}A
\]

(119)

for some integer \(k, k = 0, \pm 1, \pm 2, \pm 3, \ldots\). Furthermore for nontrivial knot \(K\) there are some factors \(R^{-k}\) of
\(R^{-k}\) coming from the braidings of Wilson lines ( for which the generalized Wilson loop \(W(K)\) is formed)
by braiding operations such as \([114]\) and \([115]\). Thus we can write the integer \(k\) in the form \(k = m + n\) for
some integer \(m\) and for some integer \(n, n = 0, \pm 1, \pm 2, \ldots\) where \(n\) is obtained by the two-side cancelations
in such a way that the cancelations are obtained when the Wilson lines of the knot diagram for \(K\) are
connected together to form a Wilson loop \(W(C)\) where \(C\) is a closed curve which is as an unknot and is
of the same form as the knot diagram for \(K\) when this knot diagram of \(K\) is considered only as a closed
curve in the plane (such that the upcrossings and undercrossings are changed to let \(K\) be the unknot \(C\)).
From this we have \( W(C) = R^{-n}A \) for \( n = 0, \pm 1, \pm 2, \ldots \). Thus \( W(K) \) can be written in the following form for some \( m \):

\[
W(K) = R^{-m}W(C)
\]  

(120)

This number \( m \) is unique since if there is another number \( m_1 \) such that \( W(K) = R^{-m_1}W(C) \) then we have the equality:

\[
R^{-m}W(C) = W(K) = R^{-m_1}W(C)
\]

(121)

This shows that \( R^{-m} = R^{-m_1} \) and thus \( m_1 = m \).

From (120) we also have

\[
TrW(K) = TrR^{-m}W(C)
\]

(122)

for some integer \( m \) and that \( TrR^{-m}W(C) \) is a knot invariant.

Then let us show that the invariant \( TrR^{-m}W(C) \) classifies knots. Let \( K_1 \) and \( K_2 \) be two knots with the same invariant \( TrR^{-m}W(C) \). Then \( K_1 \) and \( K_2 \) are both with the same invariant \( R^{-m}W(C) \) where the trace is omitted. Then by the above formula (120) we have

\[
W(K_1) = R^{-m}W(C) = W(K_2)
\]

(123)

Thus \( W(K_1) \) and \( W(K_2) \) can be transformed to each other. Thus \( K_1 \) and \( K_2 \) are equivalent. Thus the invariant \( TrR^{-m}W(C) \) classifies knots. It follows that the invariant \( TrR^{-m} \) classifies knots and thus knots can be classified by the integer \( m \), as was to be proved. •

13 More Computations of Knot Invariant

In this section let us give more computations of the knot invariant \( TrR^{-m} \). We shall show by computation (with the chosen braiding formulae) that the figure-eight knot \( 4_1 \) is assigned with the number \( m = 3 \) and two composite knots composed by two trefoil knots (with the names reef knot and granny knot and denoted by \( 3_1 \times 3_1 \) and \( 3_1 \times 3_1 \) respectively) are assigned with the numbers \( -m = 4 \) and \( -m = 9 \) respectively. The computation is quite tedious. In the next section we shall have a more efficient way to determine the integer \( m \). Readers may skip this section for the first reading.

Let us first consider the figure-eight knot. From the figure of this knot in a above section we have that the knot invariant of this knot is given by:

\[
TrW(z_6, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_7) W(z_2, w_2)W(w_2, z_6)W(z_5, w_2)W(w_2, z_3) W(z_8, w_3)W(w_3, z_4)W(z_3, w_3)W(w_3, z_1) W(z_4, w_4)W(w_4, z_8)W(z_7, w_4)W(w_4, z_5)
\]

(124)

In the above computation we have chosen \( z_1 \) as the staring point (By the circling property we may choose any point as the starting point). By repeatedly applying the braiding formulas (94), (96) and (97) we have that this invariant is equal to:

\[
TrR^{-3}W(w_2, z_3)W(z_8, w_2)W(z_3, z_8)
\]

(125)

Then we have that (125) is equal to

\[
TrW(z_3, z_8)R^{-3}W(w_2, z)W(z, z_3)W(z_8, z_1)W(z_1, w_2)
\]

(126)

where \( W(w_2, z_3) = W(w_2, z)W(z, z_3) \) with \( z \) being a point on the line represented by \( W(w_2, z_3) \) and that \( W(z_8, w_2) = W(z_8, z_1)W(z_1, w_2) \). Since \( z_1 \) is as the starting and ending point and is an intermediate point we have the following braiding formula:

\[
R^{-1}W(z_8, z_1)W(z_1, w_2)W(w_2, z)W(z, z_3)R^{-1}
\]

(127)
Thus we have that (125) is equal to
\[
\begin{align*}
TrW(z_3, z_8)R^{-3}R^{-1}W(z_8, w_2)W(w_2, z_3)R^{-1} \\
= TrW(z_3, z_8)R^{-4}W(z_8, z_3)R^{-1} \\
=: TrW(z_3, z_8)R^{-4}\bar{W}(z_8, z_3)
\end{align*}
\]  
(128)

Then in (128) we have that

\[
\begin{align*}
\bar{W}(z_8, z_3) \\
= W(z_8, z_3)R^{-1} \\
= W(z_8, z_1)W(z_1, w_1)W(w_1, z_2)W(z_2, z_3)R^{-1} \\
= W(z_8, z_1)W(z_2, z_3)W(w_1, z_2)W(z_1, w_1)RR^{-1} \\
= W(z_8, z_1)W(z_2, z_3)W(w_1, z_2)W(z_1, w_1)
\end{align*}
\]  
(129)

This shows that \(\bar{W}(z_8, z_3)\) is a generalized Wilson line. Then since generalized Wilson lines are with the same braiding formulas as Wilson lines we have that by a braiding formula similar to (127) (for \(z_1\) as the starting and ending point and as an intermediate point) the formula (128) is equal to:

\[
\begin{align*}
TrR^{-4}\bar{W}(z_8, z_3)W(z_3, z_8) \\
= TrR^{-4}RW(z_3, z_8)\bar{W}(z_8, z_3)R \\
= TrR^{-3}W(z_3, z_8)W(z_3, z_3) \\
= TrR^{-3}W(z_3, z_3)
\end{align*}
\]  
(130)

where the first equality is by a braiding formula which is similar to the braiding formula (127). This is the knot invariant for the figure-eight knot and we have that \(m = 3\) for this knot.

Let us then consider the composite knot \(3_1 \ast 3_1\) in Fig.4. The trace of the generalized loop of this knot is given by (In Fig.4 one of the two \(w_3\) should be \(w_1)\):

\[
\begin{align*}
TrW(z_4, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_3); \\
W(z_2, w_2)W(w_2, z_6)W(z_5, z_2)W(w_2, z_3); \\
W(z_3, w_1)W(w_1, z_3)W(z_4, z_1)W(w_1, z_2); \\
W(z_5, w_2)W(w_2, z_1)W(z_2, z_2)W(w_2, z_6); \\
W(z_3, w_3)W(z_3, z_1)W(z_6, z_3)W(z_3, z_4); \\
W(z_6, w_3)W(w_3, z_4)W(z_1, z_3)W(w_3, z_1)
\end{align*}
\]  
(131)

By repeatedly applying braiding formula (127) we have that this invariant is equal to

\[
\begin{align*}
TrW(z_1, w_1)W(z_5, z_1)W(w_2, z_5) \\
= TrW(z_1, w_2)W(w_2, w_2)W(z_5, w_2)W(w_2, z_1)W(w_1, z_5) \\
= TrW(z_1, w_2)W(w_2, z_1)W(z_5, w_2)W(w_2, w_2)R^4W(w_2, z_5)
\end{align*}
\]

where the braiding of \(W(w_2, z_1)\) and \(W(w_2, w_2)\) gives \(R^4\). This braiding formula comes from the fact that the Wilson line \(W(w_2, w_2)\) represents a curve with end points \(w_2\) and \(w_2\) such that one and a half
loop is formed which cannot be removed because the end point \( w'_2 \) is attached to this curve itself to form the closed loop. This closed loop gives a \( 3\pi \) phase angle which is a topological effect. Thus while the usual braiding of two pieces of curves gives \( R \) which is of a \( \pi \) phase angle we have that the braiding of \( W(w_2, z_1) \) and \( W(w_2, w'_2) \) gives \( R \) and an additional \( 3\pi \) phase angle and thus gives \( R^4 \).

Then we have that \( 131 \) is equal to

\[
\text{Tr}W(w'_2, z_5)W(z_1, w_2)W(w_2, z_1)W(z_5, w_2)W(w_2, w'_2)R^4
\]

\[
= \text{Tr}W(w'_2, z_5)W(z_1, w_2)RW(z_5, w_2)W(w_2, z_1)R^{-1}W(w_2, w'_2)R^4
\]

\[
= \text{Tr}RW(z_1, w_2)W(w_2, z_5)W(z_5, z_1)R^{-1}W(w_2, w'_2)R^4
\]

\[
= \text{Tr}RW(z_1, w_2)W(w_2, z_1)R^{-1}W(w_2, w'_2)R^4
\]

\[
= \text{Tr}W(w'_2, z_1)W(z_1, w_2)W(w_2, w'_2)R^4
\]

\[
= \text{Tr}W(w'_2, w_2)R^4
\]

This is the invariant of \( 3_1 \times 3_1 \). Thus we have that \(-m = 4\) for \( 3_1 \times 3_1 \).

Let us then consider the composite knot \( 3_1 \times 3_1 \) in Fig.5. We have that the trace of the generalized Wilson loop of \( 3_1 \times 3_1 \) is given by (In Fig.5 one of the two \( w_3 \) should be \( w'_1 \))

\[
\text{Tr}W(z_4, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_5)\cdot W(z_2, w_2)W(w_2, z_6)W(z_5, w_2)W(w_2, z_1)\cdot W(z_4, w_1)W(w_1, z_2)W(z_3, w_1)W(w_1, z_5)\cdot W(z_2, w_2)W(w_2, z_6)W(z_5, w_2)W(w_2, z_3)\cdot W(z_6, w_3)W(w_3, z_4)W(z_3, w_3)W(w_3, z_1)\cdot W(z_6, w_3)W(w_3, z_4)W(z_1, w_3)W(w_3, z_1)
\]

Fig.5

By repeatedly applying braiding formulas \( 94, 96 \), and \( 97 \) we have that this invariant is equal to

\[
\text{Tr}RW(z'_1, w_1)R^2W(z'_4, z'_6)W(w_1, z'_4)W(z'_6, z'_1)
\]

(135)

where the quantum Wilson line \( W(w_1, z'_4) \) represents the piece of curve which starts at \( w_1 \) and goes through \( z_5, z_6, z_1 \) and ends at \( z'_4 \). This curve includes a one and a half loop which cannot be removed since \( w_1 \) is attached to this curve to form the loop. This is of the same case as that in the knot \( 3_1 \times 3_1 \). This is a topological property which gives a \( 3\pi \) phase angle.

We have that \( 135 \) is equal to

\[
\text{Tr}RW(z_1, w_1)R^2W(z'_4, w'_1)W(w'_1, z'_6)W(w_1, w'_1)W(w'_1, z'_4)W(z'_6, z'_1)
\]

(136)

where the piece of curve represented by quantum Wilson line \( W(w_1, w'_1) \) also contains the closed loop. Now let this knot \( 3_1 \times 3_1 \) be starting and ending at \( z'_6 \). Then by the braiding formula on \( W(w_1, w'_1) \) and
\[ W(z'_4, w'_1) \] as in the case of the knot \( 3_1 \times 3_1 \) we have that (138) is equal to

\[
TrRW(z'_1, w_1)R^2
\]

\[
R^4W(w_1, w'_1)W(w'_1, z'_4)W(z'_4, w'_1)W(w'_1, z'_1)W(z'_1, z'_4)
\]

\[
= TrRW(z'_1, w_1)R^6
\]

\[ W(w_1, w'_1)RW(z'_4, w'_1)W(w'_1, z'_6)W^{-1}(w'_1, z'_4)W(z'_6, z'_1) \]

\[
= TrRW(z'_1, w_1)R^6
\]

\[ W(w_1, w'_1)RW(z'_4, z'_6)W(z'_6, z'_1)W(w'_1, z'_4)R^{-1} \]

\[
= TrRW(z'_1, w_1)R^6
\]

\[ W(w_1, w'_1)RW(z'_4, z'_1)W(w'_1, z'_4)R^{-1} \]

\[
= TrW(z'_1, w_1)R^6W(w_1, w'_1)RW(z'_4, z'_1)W(w'_1, z'_4)
\]

where we have repeatedly applied the braiding formula (97).

Now let \( z'_4 \) be the starting and ending point. Then we have that (137) is equal to

\[
TrW(z'_1, w_1)R^6W(w_1, w'_1)W(w'_1, z'_4)W(z'_4, z'_1)R
\]

\[
= TrW(z'_1, w_1)R^6W(w_1, z'_1)R \]

\[
= TrW(z'_1, w_1)R^6W(w_1, w'_1)W(w'_1, z'_4)W(z'_4, w'_1)W(z'_4, z'_1)R \]

\[
= TrW(z'_1, w_1)R^6W(w_1, w'_1)W(w'_1, z'_1)W(z'_4, w'_1)W(w'_1, z'_4) \]

\[
= TrRW(z'_1, w_1)RW(z'_1, w_1)
\]

where \( W(w_1, z'_1) \) denotes the following generalized Wilson line:

\[
W(w_1, w'_1)W(w'_1, z'_1)W(z'_4, w'_1)W(w'_1, z'_4)
\]

(139)

Then by the same braiding formula for generalized Wilson lines as that for Wilson lines (with \( z'_4 \) as the starting and ending point and as an intermediate point) we have that (138) is equal to:

\[
TrRW(z'_1, w_1)RW(z'_1, w_1)
\]

\[
R = TrRW(z'_1, w_1)RW(z'_1, w_1)
\]

\[
R = TrRW(z'_1, w_1)RW(z'_1, w_1)
\]

This is the knot invariant for the knot \( 3_1 \times 3_1 \). Thus we have that \(-m = 9\) for the knot \( 3_1 \times 3_1 \). Then we have that the image of \( 3_1 \times 3_1 \) is with the knot invariant \( TrRW^{-9}W(z'_1, z'_1) \).

### 14 A Classification Table of Knots I

In the above sections the computations of the knot invariant \( TrRW^{-m} \) is tedious. In this section let us use another method to determine the integer \( m \) without carrying out the tedious computations. We shall use only the connected sum operation on knots to find out the integer \( m \). For simplicity we use the positive integer \( |m| \) to form a classification table of knots where \( m \) is assigned to a knot while \(-m\) is assigned to its mirror image if the knot is not equivalent to its mirror image. Our main references on the connected sum operation on knots are 34–39.

Let \( \ast \) denote the connected sum of two knots such that the resulting total number of alternating crossings is equal to the sum of alternating crossings of each of the two knots minus 2. As an example we have the reef knot (or the square knot) \( 3_1 \ast 3_1 \) which is a composite knot composed with the knot \( 3_1 \) and its mirror image as in Fig.4. This square knot has 6 crossings and 4 alternating crossings. Then let \( \times \) denote the connected sum for two knots such that the resulting total number of alternating crossings
is equal to the sum of alternating crossings of each of the two knots. As an example we have the granny knot $3_1 \times 3_1$ which is a composite knot composed with two identical knots $3_1$ as in Fig.5 (For simplicity we use one notation $3_1$ to denote both the trefoil knot and its mirror image though these two knots are nonequivalent). This knot has 6 alternating crossings which is equal to the total number of crossings. We have that the two operations $\times$ and $\times$ satisfy the commutative law and the associative law $[34]-[39]$. Further for each knot there is a unique factorization of this knot into a $\times$ and $\times$ operation of prime knots which is similar to the unique factorization of a number into a product of prime numbers $[34]-[39]$. We shall show that there is a deeper connection between these two factorizations.

We shall show that we can establish a classification table of knots where each knot is assigned with a number such that prime knots are bijectively assigned with prime numbers such that the prime number 2 corresponds to the trefoil knot (The trefoil knot will be assigned with the number 1 and is related to the prime number 2). We have shown by computation that the knot $3_1$ is with $m = 1$, the knot $4_1$ is with $m = 3$. Thus there are no knots assigned with the number 2 since other knots are with crossings more than these two knots. We have shown by computation that the knot $3_1 \times 3_1$ is assigned with the number 4. Thus we have $1 \times 1 = 4$ (Since knots are assigned with integers we may regard the $\times$ and $\times$ as operations on the set of numbers). This shows that the number 1 plays the role of the number 2. Thus while the knot $3_1$ is with $m = 1$ we may regard this $m = 1$ as the even prime number 2. We shall have more to say about this phenomenon of 1 and 2. This phenomenon reflects that the operation $\times$ has partial properties of addition and multiplication where $m = 1$ is assigned to $3_1$ for addition while $3_1$ plays the role of 2 is for multiplication. The aim of this section is to find out a table of the relation between knots and numbers by using only the operations $\times$ and $\times$ on knots and by using the following data as the initial step for induction:

**Initial data for induction:** The prime knot $3_1$ is assigned with the number 1 and it also plays the role of 2. This means that the number 2 is not assigned to other knots and is left for the prime knot $3_1$. 

**Remark.** We shall say that the prime knot $3_1$ is assigned with the number 1 and is related to the prime number 2.

We shall give an induction on the number $n$ of $2^n$ for establishing the table. For each induction step on $n$ because of the special role of the trefoil knot $3_1$ we let the composite knot $3_1^n$ obtained by repeatedly taking $\times$ operation $n - 1$ times on the trefoil knot $3_1$ be assigned with the number $2^n$ in this induction.

Let us first give the following table relating knots and numbers up to $2^5$ as a guide for the induction for establishing the whole classification table of knots:

| Type of Knot | Assigned number $|m|$ | Type of Knot | Assigned number $|m|$ |
|--------------|------------------|--------------|------------------|
| $3_1$        | 1                | $6_3$        | 17               |
| $4_1$        | 2                | $3_1 \times 4_1$ | 18               |
| $3_1 \times 3_1$ | 3            | $7_1$        | 19               |
| $5_1$        | 4                | $4_1 \times 5_1$ | 20               |
| $3_1 \times 4_1$ | 5            | $4_1 \times (3_1 \times 4_1)$ | 21               |
| $5_2$        | 6                | $4_1 \times 5_2$ | 22               |
| $3_1 \times 3_1 \times 3_1$ | 7         | $7_2$        | 23               |
| $3_1 \times 3_1$ | 8            | $3_1 \times (3_1 \times 3_1)$ | 24               |
| $3_1 \times 5_1$ | 9            | $3_1 \times (3_1 \times 5_1)$ | 25               |
| $6_1$        | 10               | $3_1 \times 6_1$ | 26               |
| $3_1 \times 5_2$ | 11           | $3_1 \times (3_1 \times 5_2)$ | 27               |
| $3_1 \times 6_2$ | 12           | $3_1 \times 6_2$ | 28               |
| $6_2$        | 13               | $7_3$        | 29               |
| $4_1 \times 4_1$ | 14           | $(3_1 \times 3_1) \times (3_1 \times 4_1)$ | 30               |
| $4_1 \times (3_1 \times 3_1)$ | 15          | $7_4$        | 31               |
| $(3_1 \times 3_1) \times (3_1 \times 3_1)$ | 16     | $(3_1 \times 3_1) \times (3_1 \times 3_1 \times 3_1)$ | 32               |

From this table we see that the $\times$ operation is similar to the usual multiplication $\times$ on numbers. Without the $\times$ operation this $\times$ operation would be exactly the usual multipilcation on numbers if this $\times$ operation
Suppose that $K$ is determined by other conditions. Thus this factorization of noncomparable in the sense that this factorization gives no preordering property and that the ordering other forms of these two knots are not for determining their ordering. Suppose that loss of generality let us suppose that $K$. This factorization is the only factorization that might change the preordering that $K$. We shall see that this preordering gives the comparable property in the above table.

Theorem 10 Let two knots be of the form $K_1 \ast K_2$ and $K_1 \ast K_3$ where $K_2$ and $K_3$ are prime knots. Then we have the factorization $K_1 \ast K_2 = K_4 \ast K_5$ where $K_4 < K_5$ and $K_4$ and $K_5$ are prime knots. Then we have the factorization $K_1 \ast K_3 = (K_5 \ast K_2)$ and $K_1 \ast K_3 = K_5 \ast (K_4 \ast K_3)$. This factorization is the only factorization that might change the preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$. Then if $K_2 \neq K_4$ or $K_3 \neq K_3$ with this factorization the two knots $K_1 \ast K_2$ and $K_1 \ast K_3$ are noncomparable in the sense that this factorization gives no preordering property and that the ordering of these two knots is determined by other conditions. Thus this factorization of $K_1 \ast K_2$ and $K_1 \ast K_3$ is

is regarded as an operation on numbers. From this table we see that comparable composite knots (in a sense from the table and we shall discuss this point later) are grouped in each of the intervals between two prime numbers. It is interesting that in each interval composite numbers are one-to-one assigned to the comparable composite knots while prime numbers are one-to-one assigned to prime knots. Here a main point is to introduce the $\times$ operation while keeping composite knots correspond to composite numbers and prime knots correspond to prime numbers. To this end we need to have rooms at the positions of composite numbers for the introduction of composite knots obtained by the $\times$ operation. We shall show that these rooms can be obtained by using the special property of the trefoil knot which is assigned with the number 1 (for the addition property of the $\ast$ and $\times$ operations) while this trefoil knot is similar to the number 2 for the multiplication property of the $\ast$ operation.

Let us then carry out the induction steps for obtaining the whole table. To this end let us investigate in more detail the above comparable properties of knots. We have the following definitions and theorems.

**Definition.** We write $K_1 < K_2$ if $K_1$ is before $K_2$ in the ordering of knots; i.e. the number assigned to $K_1$ is less than the number assigned to $K_2$.

**Definition (Preordering).** Let two knots be written in the form $K_1 \ast K_2$ and $K_1 \ast K_3$ where we have determined the ordering of $K_2$ and $K_3$. Then we say that $K_1 \ast K_2$ and $K_1 \ast K_3$ are in a preordering in the sense that we put the ordering of these two knots to follow the ordering of $K_2$ and $K_3$. If this preordering is not changed by conditions from other preorderings on these two knots (which are from other factorization forms of these two knots) then this preordering becomes the ordering of these two knots. We shall see that this preordering gives the comparable property in the above table. 

**Remark.** a) If $K_1$ is the unknot then we have $K_1 \ast K_2 = K_2$ and $K_1 \ast K_3 = K_3$ and thus the ordering of $K_1 \ast K_2$ and $K_1 \ast K_3$ follows the ordering of $K_2$ and $K_3$.

b) We can also similarly define the preordering of two knots $K_1 \times K_2$ and $K_1 \times K_3$ with the $\times$ operation.

We have the following theorem.

**Theorem 8** Consider two knots of the form $K_1 \ast K_2$ and $K_1 \ast K_3$ where $K_1$, $K_2$ and $K_3$ are prime knots such that $K_2 < K_3$. Then we have $K_1 \ast K_2 < K_1 \ast K_3$.

**Proof.** Since $K_1$, $K_2$ and $K_3$ are prime knots there are no other factorization forms of the two knots $K_1 \ast K_2$ and $K_1 \ast K_3$. Thus these two forms of the two knots are the only way to give preordering to the two knots and thus there are no other conditions to change the preordering given by this factorization form of the two knots. Thus we have that $K_2 < K_3$ implies $K_1 \ast K_2 < K_1 \ast K_3$.

**Theorem 9** Suppose two knots are written in the form $K_1 \ast K_2$ and $K_1 \ast K_3$ for determining their ordering and that the other forms of these two knots are not for determining their ordering. Suppose that $K_2 < K_3$. Then we have $K_1 \ast K_2 < K_1 \ast K_3$.

**Proof.** The proof of this theorem is similar to the proof of the above theorem. Since the other factorization forms are not for the determination of the ordering of the two knots in the factorization form $K_1 \ast K_2$ and $K_1 \ast K_3$ we have that the preordering of these two knots in this factorization form becomes the ordering of these two knots. Thus we have $K_1 \ast K_2 < K_1 \ast K_3$.

As a generalization of theorem 8 we have the following theorem.

**Theorem 10** Let two knots be of the form $K_1 \ast K_2$ and $K_1 \ast K_3$ where $K_2$ and $K_3$ are prime knots. Suppose that $K_2 < K_3$. Then we have $K_1 \ast K_2 < K_1 \ast K_3$.

**Proof.** We have the preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$. Then since $K_2$ and $K_3$ are prime knots we have that the other preordering of $K_1 \ast K_2$ and $K_1 \ast K_3$ can only from the factorization of $K_1$. Without loss of generality let us suppose that $K_1$ is of the form $K_1 = K_4 \ast K_5$ where $K_4 < K_5$ and $K_4$ and $K_5$ are prime knots. Then we have the factorization $K_1 \ast K_2 = K_4 \ast (K_5 \ast K_2)$ and $K_1 \ast K_3 = K_5 \ast (K_4 \ast K_3)$. This factorization is the only factorization that might change the preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$. Then if $K_2 \neq K_4$ or $K_3 \neq K_3$ with this factorization the two knots $K_1 \ast K_2$ and $K_1 \ast K_3$ are noncomparable in the sense that this factorization gives no preordering property and that the ordering of these two knots is determined by other conditions. Thus this factorization of $K_1 \ast K_2$ and $K_1 \ast K_3$ is
not for the determination of the ordering of $K_1 \ast K_2$ and $K_1 \ast K_3$. Thus the preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$ is the ordering of $K_1 \times K_2$ and $K_1 \times K_3$. On the other hand if $K_2 = K_4$ and $K_3 = K_5$ then this factorization gives the same preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$. Thus for this case the preordering that $K_1 \ast K_2$ is before $K_1 \ast K_3$ is also the ordering of $K_1 \ast K_2$ and $K_1 \ast K_3$. Thus we have $K_1 \ast K_2 < K_1 \ast K_3$. ◦

In addition to the above theorems we have the following theorems.

**Theorem 11** Consider two knots of the form $K_1 \times K_2$ and $K_1 \times K_3$ where $K_1$, $K_2$ and $K_3$ are prime knots such that $K_2 < K_3$. Then we have $K_1 \times K_2 < K_1 \times K_3$.

**Proof.** By using a preordering property for knots with $\times$ operation as similar to that for knots with $\ast$ operation we have that the proof of this theorem is similar to the proof of the above theorems. ◦

**Theorem 12** Let two knots be of the form $K_1 \times K_2$ and $K_1 \times K_3$ where $K_2$ and $K_3$ are prime knots. Suppose that $K_2 < K_3$. Then we have $K_1 \times K_2 < K_1 \times K_3$.

**Proof.** The proof of this theorem is also similar to the proof of the theorem [10] ◦

**Remark.** These two theorems will be used for introducing and ordering knots involved with a $\times$ operation which will have the effect of pushing out composite knots with the property of jumping over (to be defined) such that knots are assigned with a prime number if and only if the knot is a prime knot. ◦

**Remark.** The above theorems on preordering are used to find out the ordering of knots.

Let us investigate more on the property of preordering. We consider the following

**Definition (Preordering sequences).** At the $n$th induction step let the prime knot $3_1$ take a $\ast$ operation with the previous $(n-1)$th step. We call this obtained sequence of composite knots as a preordering sequence. Thus from the ordering of the $(n-1)$th step we have a sequence of composite knots which will be for the construction of the $n$th step.

Then we let the prime knot $4_1$ (or the knot assigned with a prime number which is 3 in the 2nd step as can be seen from the above table) take a $\ast$ operation with the previous $(n-2)$th step. From this we get a sequence of composite knots for constructing the $n$th step. Then we let the prime knots $5_1$ and $5_2$ (which are prime knots in the same step assigned with a prime number which is 5 or 7 in the 3rd step as can be seen from the above table) take a $\ast$ operation with the previous $(n-3)$th step respectively. From this we get two sequences for constructing the $n$th step.

Continuing in this way until the sequences are obtained by a prime knot in the $(n-1)$th step taking a $\ast$ operation with the step $n = 1$ where the prime knot is assigned with a prime number in the $(n-1)$th step by induction (By induction each prime number greater than 2 will be assigned to a prime knot).

We call these obtained sequences of composite knots as the preordering sequences of composite knots for constructing the $n$th step. Also we call the sequences truncated from these preordering sequences as preordering subsequences of composite knots for constructing the $n$th step. ◦

We first have the following lemma on preordering sequence.

**Lemma 1** Let $K$ be a knot in a preordering sequence of the $n$th step. Then there exists a room for this $K$ in the $n$th step in the sense that this $K$ corresponds to a number in the $n$th step or in the $(n-1)$th step.

**Proof.** Let $K$ be of the form $K = 3_1 \ast K_1$ where $K_1$ is a knot in the previous $(n-1)$th step. By induction we have that $K_1$ is assigned with a number $a$ which is the position of $K_1$ in the previous $(n-1)$th step. Then since $3_1$ corresponds to the number 2 we have that $K$ corresponds to the number $2 \cdot a$ in the $n$th step (We remark that $K$ may not be assigned with the number $2 \cdot a$). Thus there exists a room for this $K$ in the $n$th step.

Then let $K$ be of the form $K = 4_1 \ast K_2$ where $K_2$ is a knot in the previous $(n-2)$th step. By induction we have that $K_2$ is assigned with a number $b$ which is the position of $K_2$ in the previous $(n-2)$th step. Since $4_1$ is by induction assigned with the prime number 3 we have $3 \cdot b > 3 \cdot 2^{n-3} > 2 \cdot 2^{n-3} = 2^{n-2}$. Also we have $3 \cdot b < 3 \cdot 2^{n-2} < 2^2 \cdot 2^{n-2} = 2^n$. Thus there exists a room for this $K$ in the $(n-1)$th step or the $n$th step.
Continuing in this way we have that this lemma holds. ⊤

**Remark.** By using this lemma we shall construct each nth step of the classification table by first filling the nth step with the preordering subsequences of the nth step. ⊤

**Remark.** When the number corresponding to the knot \( K \) in the above proof is not in the nth step we have that the knot \( K \) in the preordering sequences of the nth step has the function of pushing a knot \( K' \) out of the nth step where this knot \( K' \) is related to a number in the nth step in order for the knot \( K \) to be filled into the nth step.

As an example in the above table the knot \( K = 4_1 \times 5_1 \) (related to the number \( 3 \cdot 5 \)) in a preordering sequence of the 5th step pushes the knot \( K' = 5_1 \times 5_1 \) related to the number \( 5 \cdot 5 \) in the 5th step out of the 5th step. This relation of pushing out is by the chain \( 3 \cdot 5 \rightarrow 2 \cdot 5 \rightarrow 5 \).

As another example in the above table the knot \( K = 3_1 \times (3_1 \times 3_1) \) (corresponded to the number \( 2 \cdot 9 \)) in a preordering sequence of the 5th step pushes the knot \( K' = 3_1 \times (4_1 \times 5_1) \) related to the number \( 2 \cdot 3 \cdot 5 \) in the 5th step out of the 5th step. This relation of pushing out is by the chain \( 2 \cdot 9 \rightarrow 2 \cdot 3 \cdot 5 \).

**Lemma 2** For \( n \geq 2 \) the preordering subsequences for the nth step can cover the whole nth step.

**Proof.** For \( n = 2 \) we have one preordering sequence with number of knots = \( 2^0 \) which is obtained by the prime knot \( 3_1 \) taking \( * \) operation with the step \( n = 2 - 1 = 1 \). In addition we have the knot \( 3_1 \times 3_1 \) which is assigned at the position of \( 2^n, n = 2 \) by the induction procedure. Then since the total rooms of this step \( n = 2 \) is \( 2^1 \) we have that these two knots cover this step \( n = 2 \).

For \( n = 3 \) we have one preordering sequence with number of knots = \( 2^1 \) which is obtained by the prime knot \( 3_1 \) taking \( * \) operation with the step \( 3 - 1 = 2 \). This sequence cover half of this step \( n = 3 \) which is with \( 2^{3-1} = 2^2 \) rooms. Then we have one more preordering sequence which is obtained by the knot \( 4_1 \) taking \( * \) operation with step \( n = 1 \) giving the number \( 2^0 = 1 \) of knots. This covers half of the remaining rooms of the step \( n = 3 \) which is with \( 2^{2-1} = 2^1 \) rooms. Then in addition we have the knot \( 3_1 \times 3_1 \) which is assigned at the position of \( 2^n, n = 2 \) by the induction procedure. The total of these four knots thus cover the step \( n = 3 \).

For the nth step we have one preordering sequence with the number of knots = \( 2^{n-2} \) which is obtained by the prime knot \( 3_1 \) taking \( * \) operation with the \( n - 1 \)th step. This sequence cover half of this nth step which is with \( 2^{n-1} \) rooms. Then we have a preordering sequence which is obtained by the knot \( 4_1 \) taking \( * \) operation with the \( (n - 2) \)th step giving the number \( 2^{n-3} \) of knots. This covers half of the remaining rooms of the nth step which is with the remaining \( 2^{n-2} \) rooms. Then we have one preordering sequence obtained by picking a prime knot (e.g. \( 5_1 \)) which by induction is assigned with a prime number (e.g. the number 5) taking \( * \) operation with the \( (n - 3) \)th step. Continue in this way until the knot \( 3_1 \) is by induction assigned at the position of \( 2^n \). The total number of these knots is \( 2^{n-1} \) and thus cover this nth step. This proves the lemma. ⊤

**Remark.** Since there will have more than one prime number in the kth steps \( (k > 2) \) in the covering of the nth step there will have knots from the preordering sequences in repeat and in overlapping. These knots in repeat and in overlapping may be deleted when the ordering of the subsequences of the preordering sequences has been determined for the covering of the nth step.

Also in the preordering sequences some knots which are in repeat and are not used for the covering of the nth step will be omitted when the ordering of the subsequences of the preordering sequences has been determined for the covering of the nth step. ⊤

Let us then introduce another definition for constructing the classification table of knots.

**Definition (Jumping over of the first kind).** At an induction nth step consider a knot \( K' \) and the knot \( K = 3_1 \) which is a \( * \) product of \( n \) knots \( 3_1 \). \( K' \) is said to jump over \( K \), denoted by \( K \prec K' \), if exist \( K_2 \) and \( K_3 \) such that \( K' = K_2 \times K_3 \) and for any \( K_0, K_1 \) such that \( K = K_0 \times K_1 \) where \( K_0, K_1, K_2 \) and \( K_3 \) are not equal to \( 3_1 \) we have

\[
2^{n_0} < p_1 \cdots p_{n_2}, \quad 2^{n_1} > q_1 \cdots q_{n_3}
\]

or vice versa

\[
2^{n_0} > p_1 \cdots p_{n_2}, \quad 2^{n_1} < q_1 \cdots q_{n_3}
\]

(141) (142)
where $2^{n_0}, 2^{n_1}$ are the numbers assigned to $K_0$ and $K_1$ respectively ($n_0 + n_1 = n$) and

$$K_2 = K_{p_1} \cdots K_{p_{n_2}} \quad K_3 = K_{q_1} \cdots K_{q_{n_3}} \quad (143)$$

where $K_{p_1}, K_{q_j}$ are prime knots which have been assigned with prime integers $p_i, q_j$ respectively; and the following inequality holds:

$$2^n = 2^{n_0 + n_1} > p_1 \cdots p_{n_2} \cdot q_1 \cdots q_{n_3} \quad (144)$$

Let us call this definition as the property of jumping over of the first kind.

We remark that the definition of jumping over of the first kind is a generalization of the above ordering of $4_1 \times 5_1$ and $3_1 \times 3_1 \times 3_1 \times 3_1$ in the above table in the step $n = 4$ of $2^4$. Let us consider some examples of this definition. Consider the knots $K' = K_2 \times K_3 = 4_1 \times 5_1$ and $K = 3_1 \times 3_1 \times 3_1 \times 3_1$. For any $K_0, K_1$ which are not equal to $3_1$ such that $K = K_0 \times K_1$ we have $2^{n_0} < 5$ and $2^{n_1} > 3$ (or vice versa) where $3, 5$ are the numbers of $4_1$ and $5_1$ respectively. Thus we have that $(3_1 \times 3_1) \times (3_1 \times 3_1) < 4_1 \times 5_1$.

As another example we have that $3_1 \times (3_1 \times 3_1) \times (3_1 \times 3_1) < 5_1 \times 5_1, 4_1 \times 4_1 \times 4_1, \text{and } 3_1 \times (4_1 \times 5_1)$.

**A Remark on Notation.** At the $n$th step let a composite knot of the form $K_1 \times K_2 \times \cdots \times K_q$ where each $K_i$ is a prime knot such that $K_1$ is assigned with a prime number $p_i$ in the previous $n - 1$ steps. Then in general $K_1 \times K_2 \times \cdots \times K_q$ is not assigned with the number $p_1 \cdots p_q$. However with a little confusion and for notation convenience we shall sometimes use the notation $p_1 \cdots p_n$ to denote the knot $K_1 \times K_2 \times \cdots \times K_q$ and we say that this knot is related to the number $p_1 \cdots p_n$ (as similar to the knot $3_1$ which is related to the number 2 but is assigned with the number 1) and we keep in mind that the knot $K_1 \times K_2 \times \cdots \times K_q$ may not be assigned with the number $p_1 \cdots p_n$. With this notation then we may say that the composite number $3 \cdot 5$ jumps over the number $2^4$ which means that the composite knot $4_1 \times 5_1$ jumps over the knot $3_1 \times 3_1 \times 3_1 \times 3_1$.

**Definition (Jumping over of the general kind).** At the $n$th step let a composite knot $K'$ be related with a number $p_1 \cdot p_2 \cdots p_m$ where the number $p_1 \cdot p_2 \cdots p_m$ is in the $n$th step. Then we say that the knot $K'$ (or the number $p_1 \cdot p_2 \cdots p_m$) is of jumping over of the general kind (with respect to the knot $K$) in the definition of the jumping over of the first kind and we also write $K \prec K'$ if $K$ satisfies one of the following conditions:

1) $K'$ (or the number related to $K'$) is of jumping over of the first kind; or

2) There exists a $p_i$ (for simplicity let it be $p_1$) and a prime number $q$ such that $p_1$ and $q$ are in the same step $k$ for some $k$ and $q$ is the largest prime number in this step such that the numbers $p_1 \cdot p_2 \cdots p_m$ and $q \cdot p_2 \cdots p_m$ are also in the same step and that the knot $K'_q$ related with $q \cdot p_2 \cdots p_m$ is of jumping over of the first kind.

**Remark.** The condition 2) is a natural generalization of 1) that if $K'$ and the knot $K'_q$ are as in 2) then they are both in the preordering sequences of an induction $n$th step or both not. Then since $K'_q$ is of jumping over into an $(n + 1)$th induction step and thus is not in the preordering sequences of the induction $n$th step we have that $K'$ is also of jumping over into this $(n + 1)$th induction step (even if $K'$ is not of jumping over of the first kind). This means that $K'$ is of jumping over of the general kind.

**Example of jumping over of the general kind.** At an induction step let $K'$ be represented by $11 \cdot 5 \cdot 5$ (where we let $p_1 = 11$) and let $K'_q$ be represented by $13 \cdot 5 \cdot 5$ (where we let $q = 13$). Then $K'_q$ is of jumping over of the first kind. Thus we have that $K'$ is of jumping over (of the general kind).

We shall show that if $K = 3_1^n \prec K'$ then we can set $K = 3_1^n < K'$. Thus we have, in the above first example, $(3_1 \times 3_1) \times (3_1 \times 3_1) < 4_1 \times 5_1$ while $2^4 > 3 \cdot 5$. From this property we shall have rooms for the introduction of the $\times$ operation such that composite numbers are assigned to composite knots and prime numbers are assigned to prime knots. We have the following theorem.

**Theorem 13** If $K = 3_1^n \prec K'$ then it is consistent with the preordering property that $K = 3_1^n < K'$ for setting up the table.

For proving this theorem let us first prove the following lemma.

**Lemma 3** The preordering sequences for the construction of the $n$th step do not have knots of jumping over of the general kind.
Proof of the lemma. It is clear that the preordering sequence obtained by the $3_1$ taking a $\ast$ operation with the previous $(n - 1)$th step has no knots with the jumping over of the first kind property since $3_1$ is corresponded with the number 2 and the previous $(n - 1)$th step has no knots with the jumping over of the first kind property for this $(n - 1)$th step. Then preordering sequence obtained by the $4_1$ taking a $\ast$ operation with the previous $(n - 2)$th step has no knots with the jump over of the first kind property since $4_1$ is assigned with the number 3 and $3 < 2^2$ and the previous $(n - 2)$th step has no knots with the jumping over of the first kind property for this $(n - 2)$th step. Continuing in this way we have that all the knots in these preordering sequences do not satisfy the property of jumping over of the first kind. Then let us show that these preordering sequences have no knots with the property of jumping over of the general kind. Suppose this is not true. Then there exists a knot with the property of jumping over of the general kind and let this knot be represented by a number of the form $p_1 \cdot p_2 \cdots p_m$ as in the definition of jumping over of the general kind such that there exists a prime number $q$ and that $p_1$ and $q$ are in the same step $k$ for some $k$ and $q$ is the largest prime number in this step such that the numbers $p_1 \cdot p_2 \cdots p_m$ and $q \cdot p_2 \cdots p_m$ are also in the same step and the knot $K_q$ represented by $q \cdot p_2 \cdots p_m$ is of jumping over of the first kind. Then since $p_1$ and $q$ are in the same step $k$ we have that the two knots represented by $p_1 \cdot p_2 \cdots p_m$ and $q \cdot p_2 \cdots p_m$ are elements of two preordering sequences for the construction of the same $n$th step. Now since we have shown that the preordering sequences for the construction of the $n$th step do not have knots of jumping over of the first kind we have that this is a contradiction. This proves the lemma.

Proof of the theorem. By the above lemma if $K = 3_1^n \prec K'$ then $K'$ is not in the preordering sequences for the $n$th step and thus is pushed out from the $n$th step by the preordering sequences for the $n$th step and thus we have $K = 3_1^n \prec K'$, as was to be proved.

Remark. We remark that there may exist knots (or numbers related to the knots) which are not in the preordering sequences and are not of jumping over. An example of such special knot is the knot $4_1 * 5_1 * 5_1$ related with $3 \cdot 5 \cdot 5$ (but is not assigned with this number).

Definition. When there exists a knot which is not in the preordering sequences of the $n$th step and is not of jumping over we put this knot back into the $n$th step to join the preordering sequences for the filling and covering of the $n$th step if there are rooms to be filled in the $n$th step after the filling with the preordering sequences. Let us call the preordering sequences together with the knots which are not in the preordering sequences of the $n$th step and are not of jumping over as the generalized preordering sequences (for the filling and covering of the $n$th step).

Remark. By using the generalized preordering sequences for the covering of the $n$th step we have that the knots (or the number related to the knots) in the $n$th step pushed out of the $n$th step by the generalized preordering sequences are just the knots of jumping over (of the general kind).

Then we also have the following theorem.

Theorem 14 At each $n$th step ($n > 3$) in the covering of the $n$th step ($n > 3$) with the generalized preordering sequences there are rooms for introducing new knots with the $\times$ operations.

Proof. We want to show that at each $n$th step ($n > 3$) there are rooms for introducing new knots with the $\times$ operations. At $n = 4$ we have shown that there is the room at the position 9 for introducing the knot $3_1 \times 3_1$ with the $\times$ operation. Let us suppose that this property holds at an induction step $n - 1$. Let us then consider the induction step $n$. For each $n$ because of the relation between 1 and 2 for $3_1$ as a part of the induction step $n$ the number $2^n$ is assigned to the knot $3_1^n$ which is a $\ast$ product of $n$ $3_1$. Then we want to show that for this induction step $n$ by using the $\prec$ property we have rooms for introducing the $\times$ operation. Let $K'$ be a knot such that $3_1^n \prec K'$ and $K' = K_2 \ast K_3$ is as in the definition of $\prec$ of jumping over of the first kind such that $p_1 \cdots p_{n_2} \cdot q_1 \cdots q_{n_3} < 2^{n-1}$ (e.g. for $n - 1 = 4$ we have $K^4 = 3_1 \ast 3_1 \ast 3_1 \ast 3_1$ and $K' = K_2 \ast K_3 = 4_1 \ast 5_1$). Then let us consider $K'' = (3_1 \ast K_2) \ast K_3$. Clearly we have $3_1^n \prec K''$. Thus for each $K'$ we have a $K''$ such that $3_1^n \prec K''$. Clearly all these $K''$ are different.

Then from $K'$ let us construct more $K''$, as follows. Let $K'$ be a knot of jumping over of the first kind. Let $p_1 \cdots p_{n_2}$ and $q_1 \cdots q_{n_3}$ be as in the definition of jumping over of the first kind. Then as in the definition of jumping over of the first kind (w.l.o.g) we let

$$2^{n_2} < p_1 \cdots p_{n_2} \quad \text{and} \quad 2^{n_3} > q_1 \cdots q_{n_3} \quad (145)$$

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Then we have

\[2^{n_0+1} < (2 \cdot p_1 \cdots p_{n_2}) - 1 \quad \text{and} \quad 2^{n_1} > q_1 \cdots q_{n_3}\] (146)

Also it is trivial that we have \(2^{n_0} < (2 \cdot p_1 \cdots p_{n_2}) - 1\) and \(2^{n_1+1} > q_1 \cdots q_{n_3}\). This shows that \(3_1^n \prec K'' := K_{2a} \ast K_3\) where \(K_{2a}\) denotes the knot with the number \((2 \cdot p_1 \cdots p_{n_2}) - 1\) as in the definition of jumping over of the first kind (We remark that this \(K''\) corresponds to the knot \(4_1 \ast (4_1 \ast 4_1)\) in the above induction step where \(K_{2a} = 4_1 \ast 4_1\) is with the number \(2 \cdot 5 - 1 = 3 \cdot 3\)).

It is clear that all these more \(K''\) are different from the above \(K''\) constructed by the above method of taking a \(\ast\) operation with \(3_1\). Thus there are more \(K''\) than \(K'\). Thus at this \(n\)th step there are rooms for introducing new knots with the \(\times\) operations. This proves the theorem. \(\Diamond\)

**Remark.** In the proof of the above theorem we have a way to construct the knots \(K''\) by replacing a number \(a\) with the number \(2a - 1\). There is another way of constructing the knots \(K''\) by replacing a number \(b\) with the number \(2b + 1\). For this way we need to check that the number related to \(K''\) is in the \((n - 1)\)th step for \(K''\) of jumping over into the \(n\)th step.

As an example let us consider the knot \(K' = 4_1 \ast 4_1 \ast 4_1\) of jumping over into the 6th step with the following data:

\[2^3 < 3 \cdot 3 \quad \text{and} \quad 2^2 > 3\] (147)

From this data we have:

\[2^{3+1} < 2 \cdot 3 \cdot 3 - 1 = 17 \quad \text{and} \quad 2^2 > 3\] (148)

This data gives a knot \(K''\) with the related number 3 \cdot 17.

On the other hand from the data (147) we have:

\[2^2 < 3 \cdot 3 \quad \text{and} \quad 2^{2+1} > 2 \cdot 3 + 1\] (149)

Since \((3 \cdot 3)(2 \cdot 3 + 1) = (2 \cdot 5 - 1)(2 \cdot 3 + 1) = 2 \cdot 5 \cdot 2 \cdot 3 + 2 \cdot 2 - 1 < 2 \cdot 2 \cdot 2^4 - 1 < 2^6\) we have that the knot \(K'' = 4_1 \ast 4_1 \ast 5_2\) related with the number 3 \cdot 3 \cdot 7 is of jumping over into the 7th step (We shall show that \(5_2\) is assigned with the number 7). \(\Diamond\)

**Remark.** The above theorem shows that at each \(n\)th step there are rooms for introducing new knots with the \(\times\) operations and thus we may establish a one-to-one correspondence of knots and numbers such that prime knots are bijectively assigned with prime numbers. Further to this theorem we have the following main theorem:

**Theorem 15** A classification table of knots can be formed (as partly described by the above table up to \(2^n\) with \(n = 5\)) by induction on the number \(2^n\) such that knots are one-to-one assigned with an integer and prime knots are bijectively assigned with prime numbers such that the prime number 2 corresponds to the trefoil knot. This assignment is onto the set of positive integers except 2 where the trefoil knot is assigned with 1 and is related to 2 and at each \(n\)th induction step of the number \(2^n\) there are rooms for introducing new knots with the \(\times\) operations only.

Further this assignment of knots to numbers for the \(n\)th induction step effectively includes the determination of the distribution of prime numbers in the \(n\)th induction step and is by induction determined by this assignment for the previous \(n - 1\) induction steps.

Furthermore this assignment of knots to numbers for the \(n\)th induction step is by induction formed by the assignments (or structures) of the previous steps which are as the preordering sequences in the forming of the assignment (or structure) of the \(n\)th induction step such that the preordering sequences are arranged consistently with the ordering in the forming of the assignment (or structure) of the \(n\)th induction step. Thus all the properties of the structures of the previous steps are extended as properties of the structure of the \(n\)th induction step.

**Remark.** Let us also call this assignment of knots to numbers as the structure of numbers obtained by assigning numbers to knots. This structure of numbers is the original number system together with the one-to-one assignment of numbers to knots.

**Proof.** By the above lemmas and theorems we have that the generalized preordering sequences have the function of pushing out those composite knots of jumping over from the \(n\)th step. It follows that
for step $n > 3$ there must exist chains of transitions whose initial states are composite knots in repeat (to be replaced by the new composite knots with the $\times$ operations only); or the knots jumping over into this $n$th step from the previous $(n - 1)$th step or the knots in the preordering sequences with the $\times$ operations; such that the composite knots of jumping over are pushed out from the $n$th step by these chains. These chains are obtained by ordering the subsequences of the preordering sequences such that the preordering property holds in the $n$th step. Further the intermediate states of the chains must be positions of composite numbers. This is because that if a chain is transited to an intermediate state which is a position of prime number then there are no composite knots related with this prime number and thus this chain can not be transited to the next state and is stayed at the intermediate state forever and thus the chain can not push out the composite knot of jumping over. Then when a composite knot is transited to the position of an intermediate state (which is a position of composite number as has just been proved) this knot is definitely assigned with this composite number. Then when a composite knot which is in repeat is transited to the position of an intermediate state this knot is also definitely assigned with this composite number. It follows that when the chains are completed we have that the ordering of the subsequences of preordering sequences is determined.

Then the remaining knots (which are not at the transition states of the chains) which are not in repeat are definitely assigned with the number of the position of these knots in the $n$th step. For these knots the numbers of positions assigned to them are just the number related to them respectively.

Then the remaining knots (which are not at the transition states of the chains) which are in repeat must be replaced by new prime knots because of the repeat and that no other composite knots related with numbers in this $n$th step in the generalized preordering sequences can be used to replace the remaining knots. This means that the numbers of the positions of these remaining knots in repeat are prime numbers in this $n$th step. This is because that if the number of the position assigned to the new prime knot is a composite number then the composite knot related with this composite number is either in a transition state or is not in transition. If the composite knot is not in transition then the composite number related to this composite knot is just the number assigning to this composite knot and since this number is also assigned to the new prime knot that this is a contradiction. Then if this composite knot is in transition state then this means that the remaining knot is also in transition state and this is a contradiction since by definition the remaining knot is not at the transition states of the chains.

Thus prime numbers in the $n$th step are assigned and are only assigned to prime knots which replace the remaining knots in repeat in the $n$th step. Thus from the preordering sequences we have determined the positions (i.e. the distribution) of prime numbers in the $n$th step. Now since the preordering sequences are constructed by the previous steps we have shown that the basic structure (in the sense of above proof) of this assignment of knots with numbers for the $n$th step (including the determination of the distribution of prime numbers in the $n$th step) is determined by this assignment of knots with numbers for the previous $n - 1$ steps. In other words we have that the basic structure of the $n$th induction step is determined by the structure of the previous $n - 1$ steps.

In summary, we have shown that the basic structure of the $n$th induction step is formed by the structures of the previous steps which are as the preordering sequences in the forming of the structure of the $n$th induction step such that the preordering sequences are arranged consistently with the ordering in the forming of the structure of the $n$th induction step.

To complete the proof of this theorem let us show that at each $n$th induction step ($n > 3$) there are rooms for introducing new composite knots with the $\times$ operations only and we can determine the ordering of these composite knots with the $\times$ operations only in each $n$th induction step.

In the above proof we have shown that the basic structure of the $n$th induction step is determined by the structure of the previous steps such that the positions of the composite knots with the $\times$ operations only in the $n$th induction step are determined by the structures of the previous steps. Then by the preordering of these composite knots with the $\times$ operations only (with the $\ast$ operation replaced by the $\times$ operation), these positions can be fitted for the correct composite knots with the $\times$ operations only constructed (by the $\times$ operations) by knots in the previous steps, just as the preordering of those composite knots with the $\ast$ operations in the previous steps. Thus for this $n$th induction step the introducing and ordering of composite knots with the $\times$ operations only is also determined by the structures of the previous $n - 1$ steps.
Since the basic structure of the \( n \)th induction step is formed by the structures of the previous steps which are as the preordering sequences in the forming of the structure of the \( n \)th induction step such that the preordering sequences are arranged consistently with the ordering in the forming of the structure of the \( n \)th induction step, and that the structure of an induction step consists of the properties of the structure of this induction step, we have that all the properties of the structures of the previous steps are extended as properties of the structure of the \( n \)th induction step.

Thus the ordering properties of composite knots with the \( \times \) operations only in the previous steps are extended as ordering properties of the composite knots with the \( \times \) operations only in the \( n \)th induction step. (These ordering properties of the composite knots with the \( \times \) operations only can be used to find out the correct composite knots with the \( \times \) operations only to be assigned at the correct positions in the \( n \)th step).

With this fact let us then show that at each \( n \)th induction step \( (n \geq 3) \) there are rooms for introducing new composite knots with the \( \times \) operations only. As in the proof of the theorem \([14]\) we first construct more \( K'' \) by the method following \([145]\). Let us start at the step \( n = 4 \). For this step we have the knot \( K' = 4_1 \ast 5_1 \) jumps over into the step \( n = 5 \). For this \( K' \) we have the following data as in \([145]\):

\[
2^2 < 5 \quad \text{and} \quad 2^2 > 3
\]

From \([150]\) we construct a \( K'' \) for the step \( n = 5 \) by the following data:

\[
2^{2+1} < 2 \cdot 5 - 1 = 3 \cdot 3 \quad \text{and} \quad 2^2 > 3 \]

This data gives one more \( K'' = 4_1 \ast 4_1 \ast 4_1 \). Then from \([150]\) we construct one more \( K'' \) for the step \( n = 5 \) by the following data:

\[
2^3 > 5 \quad \text{and} \quad 2^{1+1} < 2 \cdot 3 - 1 = 5
\]

This data gives one more \( K'' = 5_1 \ast 5_1 \). Thus in this step \( n = 5 \) there are two rooms for the two knots \( K' = 4_1 \ast 5_1 \) and \( 3_1 \ast (3_1 \ast 3_1) \) coming from the preordering sequences and there exists exactly one room for introducing a new composite knot with the \( \times \) operations only (Recall that we also have a \( K'' = 3_1 \ast 4_1 \ast 5_1 \)). From the ordering of knots in the previous steps we determine that \( 3_1 \ast 4_1 \) is the composite knot with the \( \times \) operations only for this step. Thus at the 4th and 5th steps we can and only can introduce exactly one composite knot with the \( \times \) operations only and they are the knots \( 3_1 \ast 3_1 \) and \( 3_1 \ast 4_1 \) respectively. This shows that at the 4th and the 5th steps we can determine the number of prime knots with the minimal number of crossings = 3 and = 4 respectively (These two prime knots are denoted by \( 3_1 \ast 3_1 \) and \( 3_1 \ast 4_1 \) respectively and we do not distinguish knots with their mirror images for this determination of the ordering of knots with the \( \times \) operations only. This also shows that there are rooms for introducing new composite knots with the \( \times \) operations only in the 4th and 5th steps).

Then since this property is extended as a property in the 6th step we can thus determine that the 6th step is a step for introducing new composite knots with the \( \times \) operations only of the form \( 3_1 \ast 5_1 \) where \( 5_1 \) denotes a prime knot with the minimal number of crossings = 5 (and thus there are rooms for introducing new composite knots with the \( \times \) operations only in this 6th step). Also since the properties in the 4th and 5th steps are extended as properties in the 6th step we can determine the number of prime knots with the minimal number of crossings = 5 by the knots of the form \( 3_1 \ast 5_1 \) as this is a property of knots with the \( \times \) operations only in the 4th and 5th steps (In the classification table in the next section we show that there are exactly two prime knots of the form \( 3_1 \ast 5_1 \) and \( 3_1 \ast 5_2 \) in the 6th step whose ordering are determined by the preordering property of knots and the structure of the 6th step. This thus shows that there are exactly two prime knots with the minimal number of crossings = 5 and they are denoted by \( 3_1 \ast 5_1 \) and \( 3_1 \ast 5_2 \) respectively).

Then since the properties of the 4th, 5th and 6th steps are extended as properties in the 7th step we can determine that the 7th step is a step for introducing new composite knots with the \( \times \) operations only of the form \( 3_1 \ast 6_1 \) where \( 6_1 \) denotes a prime knot with the minimal number of crossings = 6 (and thus there are rooms for introducing new composite knots with the \( \times \) operations only in this 7th step). Also since the properties in the 4th, 5th and 6th steps are extended as properties in the 7th step we can determine the number of prime knots with the minimal number of crossings = 6 by the knots of the form

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$3_1 \times 6_1$ as this is a property of knots with the $\times$ operations only in the 4th, 5th and 6th steps (In the classification table in the next section we show that there are exactly three composite knots of the form $3_1 \times 6_1$, $3_1 \times 6_2$ and $3_1 \times 6_3$ in the 7th step whose ordering are determined by the preordering property of knots and the structure of the 7th step. This thus shows that there are exactly three prime knots with the minimal number of crossings = 6 and they are denoted by $6_1$, $6_2$ and $6_3$ respectively).

Continuing in this way we thus have that at each $n$th induction step ($n > 3$) we can determine the number of prime knots with the minimal number of crossings = $n - 1$ and there are rooms for introducing new composite knots with the $\times$ operations only. This proves the theorem. $\diamond$

**Example.** Let us consider the above classification table up to $2^5$ (with $n$ up to 5) as an example. For the induction step at $n = 2$ (or at $2^2$) we have one preordering sequence obtained by letting $3_1$ to take a $\star$ operation with the step $n = 1$ (For the step $n = 1$ the number $2^1$ is related to the trefoil knot $3_1$): $3_1 \times 3_1$. Then we fill the step $n = 2$ with this preordering sequence and we have the following ordering of knots for this step $n = 2$:

$$3_1 \times 3_1, 3_1 \times 3_1$$  \hspace{1cm} (153)

where the first $3_1 \times 3_1$ placed at the position 3 is the preordering sequence while the second $3_1 \times 3_1$ placed at the position $2^2$ is required by the induction procedure. For this step there is no numbers of jumping over. Then we have that the first $3_1 \times 3_1$ is a repeat of the second $3_1 \times 3_1$. Thus this repeat one must be replaced by a new prime knot. Let us choose the prime knot $4_1$ to be this new prime knot since $4_1$ is the smallest of prime knots other than the trefoil knot. Then this new prime knot must be at the position of a prime number, as we have proved in the above theorem. Thus we have determined that 3 is a prime number in this step $n = 2$ by using the structure of numbers of step $n = 1$ which is only with the prime number 2.

Then for the induction step at $n = 3$ (or at $2^3$) we have two preordering sequence obtained by letting $4_1$ to take a $\star$ operation with the step $n = 1$ and by letting $3_1$ to take a $\star$ operation with the step $n = 2$:

$$4_1 \times 3_1, 3_1 \times 3_1 \times 3_1 \times (3_1 \times 3_1)$$  \hspace{1cm} (154)

where the first knot is the preordering sequence obtained by letting $4_1$ to take a $\star$ operation with the step $n = 1$ and the second and third knots is the preordering sequence obtained by letting $3_1$ to take a $\star$ operation with the step $n = 2$.

For this step there is no numbers of jumping over and thus there are no chains of transition. Thus the ordering of the above three knots in this step follow the usual ordering of numbers. Thus the number assigned to the knot $4_1 \times 3_1 = 3_1 \times 4_1$ must be assigned with a number less than that of $3_1 \times 3_1 \times 3_1$ by the ordering of $3_1 \times 4_1$ and $3_1 \times 3_1 \times 3_1$ in the second preordering sequence. By this ordering of the two preordering sequences we have that the step $n = 3$ is of the following form:

$$4_1 \times 3_1, 3_1 \times 4_1, 3_1 \times (3_1 \times 3_1), 3_1 \times 3_1 \times 3_1$$  \hspace{1cm} (155)

where the fourth knot $3_1 \times 3_1 \times 3_1$ is put at the position of $2^3$ and is assigned with the number $2^3$ as required by the induction procedure. Thus the third knot $3_1 \times (3_1 \times 3_1)$ is a repeated one and thus must be replaced by a prime knot and the position of this prime knot is determined to be a prime number. Thus we have determined that the number 7 is a prime number. Then since there are no chains of transition we have that the composite knot $3_1 \times 4_1$ must be assigned with the number related to this knot and this number is $2 \cdot 3 = 6$. Thus the composite knot $3_1 \times 4_1$ is at the position of 6 and that the first knot $4_1 \times 3_1$ is a repeat of the second knot and thus must be replaced by a prime knot. Then since this prime knot is at the position of 5 we have that 5 is determined to be a prime number. Now the two prime knots at 5 and 7 must be the prime knots $5_1$ and $5_2$ respectively since these two knots are the smallest prime knots other than $3_1$ and $4_1$ (We may just put in two prime knots and then later determine what these two knots will be. If we put in other prime knots then this will not change the distribution of prime numbers determined by the structure of numbers of the previous steps and it is only that the prime knots are assigned with incorrect prime numbers. Further as shown in the above proof by using knots of the form $3_1 \times 5_1$ we can determine that there are exactly two prime knots with minimal number of crossings = 5 and they are denoted by $5_1$ and $5_2$ respectively. From this we can then determine that these two prime
knots are $5_1$ and $5_2$). Thus we have the following ordering for $n = 3$:

$$5_1 < 3_1 \times 4_1 < 5_2 < 3_1 \times 3_1 \times 3_1$$  \hspace{1cm} (156)$$

where $5_1$ is assigned with the prime number 5 and $5_2$ is assigned with the prime number 7. This gives the induction step $n = 3$. For this step there is no knot with $\times$ operation since there is no knots of jumping over.

Let us then consider the step $n = 4$ (or $2^4$). For this step we have the following three preordering sequences obtained from the steps $n = 1, 2, 3$:

$$5_1 \times 3_1;$$
$$4_1 \times 4_1, 4_1 \times 3_1 \times 3_1;$$
$$3_1 \times 5_1, 3_1 \times 3_1 \times 4_1, 3_1 \times 5_2, 3_1 \times 3_1 \times 3_1 \times 3_1;$$  \hspace{1cm} (157)$$

where the third sequence is obtained by taking $\times$ operation of the knot $3_1$ with step $n = 3$ while the third sequence is obtained by taking $\times$ operation of the knot $4_1$ with the step $n = 2$ and the first sequence is obtained by taking $\times$ operation of the knot $5_1$ with step $n = 1$. Then as required by the induction procedure the knot $3_1 \times 3_1 \times 3_1 \times 3_1$ is assigned at the position of $2^4$. The total number of knots in (157) plus this knot is exactly $2^3$ which is the total number of this step $n = 4$.

**Remark.** We have one more preordering sequence obtained by taking $\times$ operation of the knot $5_2$ with step $n = 1$. This preordering sequence gives the knot $5_1 \times 3_1$. However since the knots in (157) and the knot $3_1 \times 3_1 \times 3_1 \times 3_1$ assigned at the position of $2^4$ are enough for covering this step $n = 4$ and that the knot $5_1 \times 3_1$ of this preordering sequence is a repeat of the knot $5_1 \times 3_1$ in (157) that this preordering sequence obtained by taking $\times$ operation of the knot $5_2$ with step $n = 1$ can be omitted.

Then to find the chains of transition for this step let us order the three preordering sequences with the following ordering where we rewrite the preordering sequences in column form and the knot $3_1 \times 3_1 \times 3_1 \times 3_1$ assigned at the position of $2^4$ is put to follow the three sequences:

$$5_1 \times 3_1;$$
$$3_1 \times 5_1;$$
$$3_1 \times 3_1 \times 4_1;$$
$$3_1 \times 5_2;$$
$$3_1 \times 3_1 \times 3_1 \times 3_1;$$
$$4_1 \times 4_1;$$
$$4_1 \times 3_1 \times 3_1;$$
$$3_1 \times 3_1 \times 3_1 \times 3_1$$  \hspace{1cm} (158)$$

We notice that this column exactly fills the step $n = 4$.

For this step we have that the number $3 \cdot 5$ (or the knot $4_1 \times 5_1$ related with $3 \cdot 5$ ) is of jumping over. From (158) we have the following chain of transition for pushing out $4_1 \times 5_1$ at $3 \cdot 5$ by a knot with the $\times$ operation replacing the repeated knot $5_1 \times 3_1$ at the position of $9 = 3 \cdot 3$:

$$3_1 \times 3_1(\text{at} 3 \cdot 3) \rightarrow 4_1 \times 4_1(\text{at} 2 \cdot 7) \rightarrow 3_1 \times 5_2(\text{at} 2 \cdot 2 \cdot 3) \rightarrow 3_1 \times 3_1 \times 4_1(\text{at} 3 \cdot 5) \rightarrow 4_1 \times 5_1(\text{pushed out})$$  \hspace{1cm} (159)$$

where we choose the knot $3_1 \times 3_1$ as the knot with the $\times$ operation since $3_1 \times 3_1$ is the smallest one of such knots. For this chain the intermediate states are at positions of composite numbers $2 \cdot 7$, $2 \cdot 2 \cdot 3$ and $3 \cdot 5$. Thus the knots in this chain at the positions of these composite numbers are assigned with these composite numbers respectively.

Then once this chain of pushing out $4_1 \times 5_1$ at $3 \cdot 5$ is set up we have that the other knots in repeat must by replaced by prime knots and that their positions must be prime numbers. These positions are at 11 and 13 and thus 11 and 13 are determined to be prime numbers (The knot $3_1 \times 3_1 \times 3_1 \times 3_1$ at the end of this step must be assigned with $2^4 = 16$ by the induction procedure and thus the knot at 13 is a repeat). Then the new prime knots $6_1$ and $6_2$ are suitable knots corresponding to the prime numbers 11 and 13 respectively since they are the smallest prime knots other than $3_1$, $4_1$, $5_1$ and $5_2$ (As the above induction step we may just put in two prime knots and then later determine what these two prime knots
will be. As shown in the above proof by using knots of the form $3_1 \times 6_i$, we can determine that there are exactly three prime knots with minimal number of crossings $= 6$ and they are denoted by $6_1$, $6_2$ and $6_3$ respectively. From this we can then determine that these two prime knots are $6_1$ and $6_2$.

This completes the step $n = 4$. Thus the structure of numbers of this step (including distribution of prime numbers in this step) is determined by the structure of numbers of the previous induction steps.

Let us then consider the step $n = 5$. For this step we have the following four preordering sequences from the previous steps $n = 1, 2, 3, 4$:

$$6_1 \times 3_1$$

and

$$5_2 \times 4_1,$$  
$$5_2 \times (3_1 \times 3_1)$$

and

$$4_1 \times 5_1,$$  
$$4_1 \times (3_1 \times 4_1),$$  
$$4_1 \times 5_2,$$  
$$4_1 \times (3_1 \times 3_1 \times 3_1)$$

and

$$3_1 \times (3_1 \times 3_1),$$  
$$3_1 \times (3_1 \times 5_1),$$  
$$3_1 \times 6_1,$$  
$$3_1 \times (3_1 \times 5_2),$$  
$$3_1 \times 6_2,$$  
$$3_1 \times (4_1 \times 4_1),$$  
$$3_1 \times (3_1 \times 3_1 \times 4_1),$$  
$$3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1)$$

The total number of knots (including repeat) in the above sequences plus the knot $3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1$ to be assigned at the position of $2^5$ exactly cover this $n = 5$ step.

**Remark.** As similar to the step $n = 4$ two preordering sequences $5_1 \times 4_1, 5_1 \times 3_1 \times 3_1$ and $6_2 \times 3_1$ are omitted since these sequences are with knots which are repeats of the knots in the above preordering sequences.

Then to find the chains of transition for this step let us order these four preordering sequences with the following ordering where the knot $3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1$ assigned at the position of $2^5$ is put to follow the four sequences:

$$6_1 \times 3_1;$$  
$$5_2 \times 4_1;$$  
$$5_2 \times 3_1 \times 3_1;$$  
$$4_1 \times 5_1;$$  
$$4_1 \times (3_1 \times 4_1);$$  
$$4_1 \times 5_2;$$  
$$4_1 \times (3_1 \times 3_1 \times 3_1);$$  
$$3_1 \times (3_1 \times 3_1);$$  
$$3_1 \times (3_1 \times 5_1);$$  
$$3_1 \times 6_1;$$  
$$3_1 \times (3_1 \times 5_2);$$  
$$3_1 \times 6_2;$$  
$$3_1 \times (4_1 \times 4_1);$$  
$$3_1 \times (3_1 \times 3_1 \times 4_1);$$  
$$3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1);$$  
$$(3_1 \times 3_1) \times 3_1 \times 3_1 \times 3_1$$

For this step we have three composite knots $3_1 \times (4_1 \times 5_1), 5_1 \times 5_1$ and $4_1 \times (4_1 \times 4_1)$ (related with $2 \cdot 3 \cdot 5, 5 \cdot 5$ and $3 \cdot 3 \cdot 3$ respectively) of jumping over and there are two new knots $4_1 \times 5_1$ and $3_1 \times (3_1 \times 3_1)$
coming from the previous step. Thus there is a room for the introduction of new knot obtained only by the \( \times \) operation. Then this new knot must be the composite knot \( 3_1 \times 4_1 \) since besides the composite knot \( 3_1 \times 3_1 \) it is the smallest of composite knots of this kind.

From (164) there is a chain of transition given by \( 18 \rightarrow 21 \rightarrow 22 \rightarrow 26 \rightarrow 28 \rightarrow 27 \) and the composite knot \( 4_1 \times (4_1 \times 4_1) \) related with \( 27 = 3 \cdot 3 \cdot 3 \) is pushed out into the next step by the composite knot \( 5_2 \times 4_1 \) at the starting position 18. Then this repeated knot must be replaced by a new composite knot obtained by the \( \times \) operation only and this new composite knot must be the knot \( 3_1 \times 4_1 \).

Then the composite knots at the intermediate states are assigned with the numbers of these states respectively.

In addition to the above chain there are two more chains: \( 24 \rightarrow 30 \) and \( 20 \rightarrow 25 \). The chain \( 24 \rightarrow 30 \) starts from \( 3_1 \times (3_1 \times 3_1) \) at 24 and the composite knot \( 3_1 \times (4_1 \times 5_1) \) at 30 is pushed out by the composite knot \( 3_1 \times (3_1 \times 4_1) \). Then the chain \( 20 \rightarrow 25 \) starts from \( 4_1 \times 5_1 \) at 20 and the composite knot \( 5_1 \times 5_1 \) at 25 is pushed out by the composite knot \( 3_1 \times (3_1 \times 5_1) \).

Then the knots \( 3_1 \times (3_1 \times 4_1) \) and \( 3_1 \times (3_1 \times 5_1) \) at the intermediate states of these two chains are assigned with the numbers \( 30 = 2 \cdot 3 \cdot 5 \) and \( 25 = 5 \cdot 5 \) respectively.

Now the remaining repeated composite knots at the positions 17, 19, 23, 29, 31 must be replaced by new prime knots and thus 17, 19, 23, 29, 31 are determined to be prime numbers and they are determined by the prime numbers in the previous induction steps. Then we may follow the usual table of knots to determine that the new prime knots for the prime numbers 17, 19, 23, 29, 31 are \( 6_3, 7_1, 7_2, 7_3 \) and \( 7_4 \) respectively (As the above induction steps we may just put in five prime knots and then later determine what these five prime knots will be. As shown in the above proof by using knots of the form \( 3_1 \times 7 \) we can determine the number of prime knots with minimal number of crossings = 7. From this we can then determine these five prime knots).

In summary we have the following form of the step \( n = 5 \):

\[
\begin{align*}
6_3 \\
3_1 \times 4_1 \\
7_1 \\
4_1 \times 5_1 \\
4_1 \times (3_1 \times 4_1) \\
4_1 \times 5_2 \\
7_2 \\
3_1 \times (3_1 \times 3_1) \\
3_1 \times (3_1 \times 5_1) \\
3_1 \times 6_1 \\
3_1 \times (3_1 \times 5_2) \\
3_1 \times 6_2 \\
7_3 \\
3_1 \times (3_1 \times 3_1 \times 4_1) \\
7_4 \\
(3_1 \times 3_1) \times 3_1 \times 3_1 \times 3_1 \\
\end{align*}
\]

This completes the induction step at \( n = 5 \). We have that the structure of numbers of this step (including distribution of prime numbers in this step) is determined by the structure of numbers of the previous induction steps. 

15 A Classification Table of Knots II

Following the above classification table up to \( 2^5 \) let us in this section give the table up to \( 2^7 \). Again we shall see from the table that the preordering property is clear. At the 7th step there is a special composite knot \( 4_1 \times 5_1 \times 5_1 \) which is not of jumping over and is not in the preordering sequences (On the other hand the knot \( 5_1 \times 5_1 \times 5_1 \) is of jumping over).
We remark again that it is interesting that (by the ordering of composite knots with the $\times$ operation only) at the 6th step we require exactly two prime knots with minimal number of crossings = 5 to form the two composite knots obtained by the $\times$ operation only. From this we can determine the number of prime knots with minimal number of crossings = 5 without using the actual construction of these prime knots. We then denote these two prime knots by $5_1$ and $5_2$ respectively and the two composite knots obtained by the $\times$ operation only by $3_1 \times 5_1$ and $3_1 \times 5_2$ respectively. Similarly at the 7th step we can determine that there are exactly three prime knots with minimal number of crossings = 6 and we denote these three prime knots by $6_1$ and $6_2$ and $6_3$ respectively. These three prime knots give the composite knots $3_1 \times 6_1$, $3_1 \times 6_2$ and $3_1 \times 6_3$ respectively. We can then expect that at the next 8th step we may determine that the number of prime knots with minimal number of crossings = 7 is 7 and then at the next 9th step the number of prime knots with minimal number of crossings = 8 is 21, and so on; as we know from the well known table of prime knots [39]. Here the point is that we can determine the number of prime knots with the same minimal number of crossings without using the actual construction of these prime knots (and by using only the classification table of knots).

| Type of Knot | Assigned number $|m|$ | Repeated Knots being replaced |
|--------------|---------------------|-----------------------------|
| $3_1 \times 6_3$ | 33 |  |
| $3_1 \times (3_1 \times 4_1)$ | 34 |  |
| $3_1 \times 7_1$ | 35 |  |
| $3_1 \times 5_1$ | 36 | $3_1 \times (4_1 \times 5_1)$ |
| $7_5$ | 37 | $3_1 \times (4_1 \times 3_1 \times 4_1)$ |
| $3_1 \times 5_2$ | 38 | $3_1 \times (4_1 \times 5_2)$ |
| $3_1 \times 7_2$ | 39 |  |
| $3_1 \times (3_1 \times 3_1 \times 3_1)$ | 40 |  |
| $7_6$ | 41 | $3_1 \times (3_1 \times 3_1 \times 3_1 \times 5_1)$ |
| $5_1 \times 5_1$ | 42 |  |
| $7_7$ | 43 | $5_1 \times (3_1 \times 4_1)$ |
| $5_1 \times 5_2$ | 44 |  |
| $4_1 \times 4_1$ | 45 | $5_1 \times (3_1 \times 3_1 \times 3_1), 5_2 \times (3_1 \times 4_1)$ |
| $5_2 \times 5_2$ | 46 |  |
| $8_1$ | 47 | $5_2 \times (3_1 \times 3_1 \times 3_1), 4_1 \times (3_1 \times 3_1)$ |
| $4_1 \times (3_1 \times 5_1)$ | 48 |  |
| $4_1 \times 6_1$ | 49 |  |
| $4_1 \times (3_1 \times 5_2)$ | 50 |  |
| $4_1 \times 6_2$ | 51 |  |
| $4_1 \times (4_1 \times 4_1)$ | 52 |  |
| $8_2$ | 53 | $4_1 \times (4_1 \times 3_1 \times 3_1)$ |
| $3_1 \times (3_1 \times 3_1 \times 5_1)$ | 54 |  |
| $3_1 \times (3_1 \times 6_1)$ | 55 |  |
| $3_1 \times (3_1 \times 3_1 \times 5_2)$ | 56 |  |
| $3_1 \times (3_1 \times 6_2)$ | 57 |  |
| $3_1 \times 7_3$ | 58 |  |
| $8_3$ | 59 | $3_1 \times (3_1 \times 3_1 \times 3_1 \times 4_1)$ |
| $3_1 \times 7_4$ | 60 |  |
| $8_4$ | 61 | $3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1)$ |
| $4_1 \times (4_1 \times 3_1 \times 3_1)$ | 62 |  |
| $4_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1)$ | 63 |  |
| $3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1)$ | 64 |  |
| Type of Knot | Assigned number | Repeated Knots being replaced |
|--------------|-----------------|-----------------------------|
| $3_1 \times (3_1 \times 6_3)$ | 61 | $3_1 \times (3_1 \times 3_1 \times 4_1)$ |
| $3_1 \times (3_1 \times 3_1)$ | 65 | $3_1 \times (3_1 \times 7_1)$ |
| $4_1 \times 5_1$ | 66 | $3_1 \times (3_1 \times 4_1 \times 5_1)$ |
| $4_1 \times (3_1 \times 4_1)$ | 66 | $3_1 \times (3_1 \times 4_1 \times 3_1 \times 4_1)$ |
| $4_1 \times 5_2$ | 70 | $3_1 \times (3_1 \times 4_1 \times 5_2)$ |
| $5_1 \times 6_1$ | 70 | $3_1 \times (3_1 \times 7_1)$ |
| $5_1 \times (3_1 \times 5_2)$ | 72 | $3_1 \times (3_1 \times 4_1 \times 5_1)$ |
| $5_2 \times 5_1$ | 72 | $4_1 \times (3_1 \times 4_1)$ |
| $5_2 \times (3_1 \times 5_2)$ | 73 | $5_1 \times (3_1 \times 4_1)$ |
| $5_2 \times 6_2$ | 73 | $5_1 \times (3_1 \times 4_1)$ |
| $8_1 \times 7_1$ | 79 | $5_1 \times (4_1 \times 4_1)$ |
| $4_1 \times 7_1$ | 80 | $5_1 \times (3_1 \times 5_1)$ |
| $4_1 \times 7_2$ | 83 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 5_1)$ | 84 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 5_2)$ | 85 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 6_1)$ | 86 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 6_2)$ | 87 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 7_1)$ | 88 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 7_2)$ | 89 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 3_1 \times 3_1)$ | 90 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 3_1 \times 5_1)$ | 91 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 3_1 \times 5_2)$ | 92 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 6_1)$ | 93 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 6_2)$ | 94 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 7_3)$ | 95 | $5_2 \times (4_1 \times 4_1)$ |
| $4_1 \times (3_1 \times 3_1 \times 3_1)$ | 96 | $5_2 \times (4_1 \times 4_1)$ |
| Type of Knot                                             | Assigned number $|n|$ | Repeated Knots being replaced                                      |
|---------------------------------------------------------|------------------|---------------------------------------------------------------|
| $8_{11}$                                                | 97               | $4_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1)$ |
| $4_1 \times (5_1 \times 5_1)$                          | 98               | $3_1 \times (3_1 \times 6_3)$                                 |
| $3_1 \times (3_1 \times 3_1 \times 4_1)$               | 99               |                                                               |
| $3_1 \times (3_1 \times 7_1)$                          | 100              |                                                               |
| $8_{12}$                                                | 101              | $3_1 \times (3_1 \times 5_1)$                                 |
| $3_1 \times 7_5$                                       | 102              |                                                               |
| $8_{13}$                                                | 103              | $3_1 \times (3_1 \times 5_2)$                                 |
| $3_1 \times (3_1 \times 7_2)$                          | 104              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1)$   | 105              |                                                               |
| $3_1 \times 7_6$                                       | 106              |                                                               |
| $8_{14}$                                                | 107              | $3_1 \times (5_1 \times 5_1)$                                 |
| $3_1 \times 7_7$                                       | 108              |                                                               |
| $8_{15}$                                                | 109              | $3_1 \times (5_1 \times 5_2)$                                 |
| $3_1 \times 6_1$                                       | 110              | $3_1 \times (5_1 \times 5_2)$                                 |
| $3_1 \times (3_1 \times 5_2)$                          | 111              | $3_1 \times (4_1 \times 4_1)$                                 |
| $3_1 \times 6_2$                                       | 112              | $3_1 \times (5_2 \times 5_2)$                                 |
| $8_{16}$                                                | 113              | $3_1 \times (5_2 \times 5_2)$                                 |
| $3_1 \times 8_1$                                       | 114              |                                                               |
| $3_1 \times 6_3$                                       | 115              | $3_1 \times (4_1 \times 3_1 \times 5_1), 3_1 \times (4_1 \times 4_1 \times 4_1)$ |
| $3_1 \times 8_2$                                       | 116              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 3_1 \times 5_1)$   | 117              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 6_1)$              | 118              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 3_1 \times 5_1)$   | 119              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 6_2)$              | 120              |                                                               |
| $3_1 \times (3_1 \times 3_1 \times 7_3)$              | 121              |                                                               |
| $3_1 \times 8_3$                                       | 122              |                                                               |
| $3_1 \times (3_1 \times 7_4)$                          | 123              |                                                               |
| $3_1 \times 8_4$                                       | 124              |                                                               |
| $3_1 \times (3_1 \times 4_1)$                          | 125              | $3_1 \times (4_1 \times 4_1 \times 3_1 \times 3_1)$          |
| $3_1 \times (4_1 \times 3_1 \times 3_1 \times 3_1)$   | 126              |                                                               |
| $8_{17}$                                                | 127              | $3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1)$ |
| $3_1 \times (3_1 \times 3_1 \times 3_1 \times 3_1 \times 3_1)$ | 128              |                                                               |

16 Proof of the Riemann Hypothesis

From the above table we have that quantum prime knots can be identified with prime numbers such that they have the same distribution in the classification table of knots by integers. On the other hand we have that quantum knots are periodic orbits of the quantum gauge dynamical system in the beginning sections. From these two facts we can now follow the approach of quantum chaos to derive a trace formula to prove the Riemann Hypothesis\cite{12}\cite{13}. The reason is that, as pointing out by Berry and Keating in the quantum chaos approach of proving the Riemann Hypothesis\cite{12}\cite{13}, if one can find an energy operator of a quantum dynamical system and the periodic orbits of the dynamical system (classical or quantum) relating to the energy operator for giving a trace formula then one may use the Hilbert-Polya method to prove the Riemann Hypothesis. Here with the quantum knots as periodic orbits of the quantum gauge dynamical system and with the quantum prime knots identified with the prime numbers we can now follow the approach of quantum chaos to derive a trace formula to prove the Riemann Hypothesis, as follows.

Let us first derive a general expression of the Green’s function of the quantum gauge dynamical system, by the usual methods for Green’s function, as follows. Let $T(z)$ be the Virasoro operator in the
Sugawara construction form of the quantum system as described in the beginning sections ($T(z)$ depends on a central charge $c > 0$). This Virasoro operator is for the construction of the quantum knots and thus is as the energy operator of these quantum knots.

Remark. In quantum field theory $T(z)$ is usually called the energy-momentum tensor as it is usually derived such that its components are as the densities of energy and momentums operators. However in this conformal field theory $T(z)$ is as a one component operator and that there is no distinction of energy and momentum as $z$ is for the complex Euclidean plane that there is no distinction as that of time and space. Thus we have that in this conformal field theory $T(z)$ is as the density of energy operator as there is no distinction of energy and momentums.

Further in the usual conformal field theory which is based on the two dimensional integral $\int \int$ the Virasoro algebra $L_0 = \frac{1}{2\pi i} \oint z T(z) dz$ is regarded as the energy operator and $T(z)$ is as the density of energy operator which gives $L_0$. Here our quantum gauge model is based on the one dimensional integral $\int_{s_0}^{s_1}$ and thus as the usual quantum mechanics (which is also based on one dimensional time integral $\int_{t_0}^{t_1}$) the energy operator is just the density of energy operator which is the operator $T(z)$. Thus the Virasoro operator $T(z)$ is the energy operator of the quantum gauge system.

Let $E_j, j = 1, 2, 3, \ldots$ be a sequence of positive eigenvalues of the Virasoro operator $T(z)$:

$$H(z) \phi_j(z) = E_j \phi_j(z), \quad j = 1, 2, 3, \ldots \quad (166)$$

Then following the usual approach for constructing Green’s functions we can write the Green’s function $G$ for this sequence of eigenvalues in the following form:

$$G(z, z', E) = \sum_j \frac{\phi_j(z) \phi_j(z')}{E - E_j} \quad (167)$$

where $E$ is the energy variable.

We remark that for each sequence of eigenvalues of a quantum system there is a corresponding Green’s function of the form (167). The most complete Green’s function of the quantum system is the usual Green’s function which includes all the eigenvalues of the energy operator of the quantum system.

We can further generalize the Green’s function to the form that $E$ is a negative variable with $E_j$ replaced by $-E_j$ and then $E$ is extended as a complex variable with nonzero real part.

In quantum physics, the green’s function $G(z, z', E)$ of a quantum system is as the amplitude for the transition from the state $z$ to the state $z'$. Then by letting $z = z'$ (and $E$ is extended as a complex variable with nonzero real part) and taking integration on $z$ we have the following trace formula:

$$\int dz G(z, z, E) = \sum_j \frac{1}{E - E_j} \quad (168)$$

whichever the sum of this trace formula exists, where we set a normalized condition on the eigenfunctions $\phi_j$ such that $\int d\bar{z} \phi_j(\bar{z}) \phi_j(\bar{z}) = 1$ (For a notation simplicity we omit a trace operation notation $Tr$ on $\phi_j(z) \phi_j(\bar{z})$).

Then we have the following trace function:

$$\sum_j \frac{1}{E - E_j - i\epsilon} \quad (169)$$

where $\epsilon > 0$ is a small nonzero parameter and $E$ is a real energy variable. In (169) the function:

$$\frac{1}{E - E_j - i\epsilon} \quad (170)$$

is called as the propagator (or the Green’s function in the momentum space).
Then, for \( z = z' \), the transition orbit from \( z \) to \( z' \) is a closed orbit. Thus, as the trace function in quantum chaos \([11][12][13]\), this trace function is as a sum of amplitudes of periodic (i.e. closed) orbits of the quantum gauge system with the energy operator \( T(z) \).

Then, since the Green’s function is the basic function of the Virasoro operator \( T(z) \) giving the amplitude of the transition orbits from \( z \) to \( z' \), we have that the trace function \([109]\) is the basic function of giving sums of amplitudes of periodic (i.e. closed) orbits of the quantum gauge system. Thus we have the following theorem:

**Theorem 16** Let \( E_j, j = 1, 2, 3, \ldots \) be a sequence of positive eigenvalues of the Virasoro operator \( T(z) \). Then the trace function \([109]\) is a sum of amplitudes of periodic (i.e. closed) orbits of the quantum gauge system with the energy operator \( T(z) \). This trace of amplitudes corresponds to the sequence of eigenvalues \( E_j, j = 1, 2, 3, \ldots \) such that each eigenvalue \( E_j \) appearing with the same weight.

Further the trace function \([109]\) is the basic function giving sums of amplitudes of periodic (i.e. closed) orbits of the quantum gauge system.

Now the quantum knots are as periodic orbits of the quantum gauge system with the Virasoro operator \( T(z) \) as the energy operator. Thus we have that the trace function \([109]\) is a sum of the amplitudes of quantum knots.

Then since quantum knots (which are not the quantum unknot) can be factorized as a connected sum of quantum prime knots, as the probability amplitudes in quantum theory, the amplitudes of quantum knots (or periodic orbits) are as a product of the corresponding amplitudes of quantum prime knots of the connected sum of the quantum knots. Thus the amplitude of a quantum knot (which is not the quantum unknot) is determined by the amplitudes of quantum prime knots. Thus the trace function \([109]\) is only as a sum of the amplitudes of quantum prime knots (and the quantum unknot). Thus we have the following theorem:

**Theorem 17** Let \( E_j, j = 1, 2, 3, \ldots \) be a sequence of positive eigenvalues of the Virasoro operator \( T(z) \). Then, corresponding to this sequence of eigenvalues, the trace function \([109]\) is a sum of amplitudes of quantum prime knots and the quantum unknot.

Further the trace function \([109]\) is the basic function giving sums of amplitudes of quantum prime knots and the quantum unknot of the quantum gauge system.

Then, following the treatment in quantum chaos \([10]-[17]\), when \( E \) is the real energy variable let us write the propagator in the following form\([10]\):

\[
\frac{1}{E - E_j - i\epsilon} = \frac{E - E_j}{(E - E_j)^2 + \epsilon^2} + i\frac{\epsilon}{(E - E_j)^2 + \epsilon^2}
\]

Then we have \([10]\):

\[
\lim_{\epsilon \to 0} \frac{1}{E - E_j - i\epsilon} = P\left(\frac{1}{E - E_j}\right) - i\pi\delta(E - E_j)
\]

where the real part is a principal-part integral \( P \) and the imaginary part is the delta function \( \pi\delta(E - E_j) \). Then we can write the trace formula \([108]\) in the following form:

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \text{Im} \int dz G(z, z, E - i\epsilon) = \sum_j \delta(E - E_j)
\]

As the theory of quantum chaos \([10]-[17]\), let us call the trace function:

\[
d(E) := \sum_j \delta(E - E_j)
\]

as the density of states with real variable \( E \).

From the trace formula \([173]\) we shall get a more explicit form of the trace formula expressed as a sum of the amplitudes of quantum knots (This is analogous to the trace formula of quantum chaos where the
periodic orbits are classical while here the periodic orbits are the quantum prime knots of the quantum nature \([10]-[17]\).

We have the following theorems:

**Theorem 18** Let \(E_j, j = 1, 2, 3, \ldots\) be a sequence of positive eigenvalues of the Virasoro operator \(T(z)\). Then \(d(E)\) is a sum of amplitudes of quantum prime knots and the quantum unknot.

**Proof.** From the sequence \(E_j, j = 1, 2, 3, \ldots\) of eigenvalues of \(T(z)\) we have the trace function \((169)\) with real variable \(E\). This function is a sum of the wave amplitudes of quantum prime knots (and the quantum unknot). Then from \((172)\) the imaginary part of this function gives the density of states \(d(E)\). Thus the density of states \(d(E)\) is also a sum of the wave amplitudes of quantum prime knots (and the quantum unknot).

**Definition.** Let \(f\) be a function of complex variables with nonzero imaginary part. \(f\) is said to be a weak analytic continuation of \(d(E)\) if \(\lim_{\epsilon \to 0} \text{Im}(E - i\epsilon) = d(E)\) for real variables \(E\).

**Theorem 19** Let \(E_j, j = 1, 2, 3, \ldots\) be a sequence of positive eigenvalues of the Virasoro operator \(T(z)\). Let \(f\) be a weak analytic continuation of the density of states \(d(E)\) in \((174)\) and \(f\) gives a sum of (complex) amplitudes of quantum prime knots and the quantum unknot for complex variable \(E\) with nonzero imaginary part. Then \(f\) is the following trace function:

\[
\sum_j \frac{1}{E - E_j}
\]

(175)

where \(E\) is a complex variable with nonzero imaginary part.

**Proof.** The trace function \((175)\) is the extension of the trace function \((169)\) with the variable \(E - i\epsilon\) (where \(E\) is a real variable) extended to a complex variable with nonzero imaginary part. Thus this function \((175)\) gives a sum of amplitudes of quantum prime knots and the quantum unknot since the trace function \((169)\) gives a sum of amplitudes of quantum prime knots and the quantum unknot.

Then from \((172)\) the imaginary part of this trace function \((175)\) gives the density of states \(d(E)\) since the variable \(E - i\epsilon\) (where \(E\) is a real variable) is a complex variable with nonzero imaginary part. Thus the trace function \((175)\) is a weak analytic continuation of the density of states \(d(E)\), whenever this trace function \((175)\) exists.

Then, for the given sequence of eigenvalues \(E_j, j = 1, 2, 3, \ldots\), the trace function \((169)\) is the basic function of the Virasoro operator \(T(z)\) giving the sums of amplitudes of quantum prime knots and the quantum unknot and giving the density of states \(d(E)\). Thus, for a given weak analytic continuation function \(f\) of the density of states \(d(E)\) such that \(f\) gives a sum of (complex) amplitudes of quantum prime knots and the quantum unknot for complex variable \(E\) with nonzero imaginary part, \(f(E - i\epsilon)\) (where \(E\) is a real variable) must be the trace function \((169)\). It follows that, as a function of a complex variable with nonzero imaginary part extended from the variable \(E - i\epsilon\) (where \(E\) is a real variable), this function \(f\) must be the trace function \((175)\). This proves the theorem. \(\Box\)

Let us then use another approach to find out the amplitudes of quantum prime knots (and the quantum unknot).

From the above sections we have the classification table of knots which gives a correspondence of quantum knots and numbers such that the distribution of the quantum prime knots in this correspondence is the same distribution as that of the prime numbers and prime quantum knots are one-to-one corresponded to prime numbers (We may let the trefoil knot be related to the prime number 2). Then we may use this correspondence of quantum knots and numbers (which are topological invariant of knots) to find out the amplitude of quantum prime knots, as follows.

Let us first consider the case (which will correspond to the Riemann zeta function) that each quantum knot is counted with the same weight (This corresponds to that each number is counted with the same weight and thus prime numbers are with the usual counting function of prime numbers \(\pi(x)\) corresponding to the Riemann zeta function). For this case let us derive the amplitude of quantum prime knots (and the quantum unknot).
Starting from the counting function of prime numbers \( \pi(x) \) it is well known that we can derive its relation with the Riemann zeta function \( \zeta \) and we have the following well known von Mangoldt-Selberg formula \([3] - [9]\):

\[
N(T) = \sum_{p < T} \log p = \left( \frac{1}{2} \arg(iT - \frac{1}{2}) + \frac{1}{2} \arg \Gamma(\frac{3}{4} + \frac{iT}{2}) - \frac{1}{12} T \log \pi \right) + \lim_{\varepsilon \to 0} \text{Im} \log \zeta(\frac{1}{2} + iT + \varepsilon)
\]

\[
= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(\frac{1}{T}) + \lim_{\varepsilon \to 0} \text{Im} \log \zeta(\frac{1}{2} + iT + \varepsilon)
\]

\[
= \left( \frac{1}{2} \text{Im} \log(iT - \frac{1}{2}) + \frac{1}{2} \text{Im} \log \Gamma(\frac{3}{4} + \frac{iT}{2}) - \frac{1}{2\pi} T \log \pi \right) + \lim_{\varepsilon \to 0} \text{Im} \log \zeta(\frac{1}{2} + iT + \varepsilon)
\]

(176)

where \( N(T) \) denotes the number of zeros in the critical strip with height \( T \) of the Riemann’s zeta function \( \zeta \) and a branch of \( \log \) is chosen to be continuous such that \( N_{fl}(0) = 0 \) \([3] - [9]\). Then by using the Euler product form of the zeta function the fluctuation term \( N_{fl} \) can be written in the following form \([12]\):

\[
N_{fl}(T) := \lim_{\varepsilon \to 0} \text{Im} \log \zeta(\frac{1}{2} + iT + \varepsilon) = -\frac{1}{\pi} \text{Im} \sum_p \log(1 - p^{-\frac{1}{2} + iT})
\]

(177)

where the \( \sum_p \) is a sum over all prime numbers \( p \). Differentiating with respect to \( T \) we have

\[
N'(T) = -\frac{1}{\pi} \text{Im} \frac{i}{iT - \frac{1}{2}} + \frac{1}{\pi} \text{Im} \frac{i\Gamma'(\frac{3}{4} + \frac{iT}{2})}{2\Gamma(\frac{3}{4} + \frac{iT}{2})} - \frac{1}{2\pi} \log \pi - \frac{1}{\pi} \text{Im} \sum_p \frac{i \log p}{p^{\frac{1}{2} + iT} - 1}
\]

where the \( \sum_p \) is a sum over all prime numbers \( p \) which is as the fluctuation part of \( N'(T) \).

Since \( N(T) \) is a counting function we have that \( N'(T) \) is the sum of a sequence of delta functions concentrating at a sequence of nonnegative numbers. Thus \( N'(T) \) is of the form of the right hand side of the trace formula \([173]\).

On the other hand we have that in \([178]\) each term in the sum \( \sum_p \) represents the amplitude of the corresponding prime number \( p \). The other part of the right hand side of \([178]\) is as the average of \( N'(T) \) \([12]\). Then by the identity of prime knots with prime numbers that prime knots and prime numbers have the same distribution in the classification table of knots which gives a one-to-one correspondence of prime knots and prime numbers and that the prime numbers are topological invariant of the corresponding prime knots (and thus the prime numbers are the topological properties of the corresponding prime knots) we have that there is a distribution of amplitudes to prime numbers then the corresponding prime knots also have this distribution of amplitudes (and vise versa). Thus if there is a distribution of amplitudes to prime numbers then there must exist a Virasoro energy operator with a central charge \( c > 0 \) such that the corresponding quantum prime knots also have this distribution of amplitudes.

In particular for the distribution of amplitudes of prime numbers in \([178]\) there must exist a Virasoro energy operator with a central charge \( c > 0 \) such that each term in the sum \( \sum_p \) is as the amplitude for the quantum prime knot corresponding to the prime number \( p \) and the other part is the amplitude for the quantum unknot.

Now since \( N'(T) \) is of the form of density of energy states at the right hand side of the trace formula \([173]\) we have that the distribution of amplitudes of the prime numbers \( p \) in \([178]\) is identified as the distribution of amplitudes of the corresponding quantum prime knots in the trace formula \([173]\) for a sequence of eigenvalues \( E_j, j = 1, 2, 3, \ldots \) of \( T(z) \).

It follows that there exists a Virasoro energy operator \( T(z) \) with a central charge \( c > 0 \) and a sequence of eigenvalues \( E_j, j = 1, 2, 3, \ldots \) of \( T(z) \) such that the right hand side of \([178]\) is given by the left hand side of the trace formula \([173]\) (with \( E = T \)) and we have the following more explicit form of the trace formula \([173]\):

\[
N'(E) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \text{Im} \int dz G(z, z, E - ie)
\]

\[
= \left[ \frac{1}{\pi} \text{Im} \frac{i}{E - \frac{1}{2}} + \frac{1}{\pi} \text{Im} \frac{i\Gamma'(\frac{3}{4} + \frac{iE}{2})}{2\Gamma(\frac{3}{4} + \frac{iE}{2})} - \frac{1}{2\pi} \log \pi \right] - \frac{1}{\pi} \text{Im} \sum_p \frac{i \log p}{p^{\frac{1}{2} + iE} - 1} = \sum_j \delta(E - E_j)
\]

(179)
where \( \int dz G(z, z, E) \) is given by the trace formula (168) with eigenvalues \( E_j, j = 1, 2, 3, \ldots \). We may call this formula (179) as the von Mangoldt-Selberg-Gutzwiller trace formula.

Then when \( E(= T) \) is a negative variable by symmetry we get the same formula as (179) with \( E_j, j = 1, 2, 3, \ldots \) replaced by \( E_j, j = -1, -2, -3, \ldots \) where we define \( E_{-j} := -E_j \) for \( j = 1, 2, 3, \ldots \). Then let \( E(= T) \) be extended as a complex variable such that \( \frac{1}{2} + iE(= \frac{1}{2} + iT) \) is a complex variable in the critical strip of the Riemann zeta function. Then this complex extension of the von Mangoldt-Selberg trace function (178) is of the same form as (178) (without the operation \( \text{Im} \)) with the real \( E \) replaced by the complex \( E \) with nonzero imaginary part. Thus, by the von Mangoldt-Selberg trace formula (179), this complex extension of the von Mangoldt-Selberg trace function (178) is a weak analytic continuation of the density of state \( d(E) \) of the Virasoro energy operator \( T(z) \).

Since the von Mangoldt-Selberg trace function (178) is a distribution of amplitudes of the prime numbers (and the number 1), this complex extension of the von Mangoldt-Selberg trace function (178) is a distribution of (complex) amplitudes of the quantum prime knots (and the quantum unknot), this complex extension of the von Mangoldt-Selberg trace function (178) is a weak analytic continuation of the density of states \( d(E) \) of the Virasoro operator \( T(z) \) and is a sum of (complex) amplitudes of the quantum prime knots (and the quantum unknot).

Thus by Theorem 20 this weak analytic continuation of the von Mangoldt-Selberg trace function (178) is the von Mangoldt-Selberg-Gutzwiller trace formula.

Then by Theorem 19 this weak analytic continuation of the von Mangoldt-Selberg trace function (178) is the trace function (175). Thus we have the following formula:

\[
\frac{i}{iE - \frac{1}{2}} + \frac{\Gamma'(\frac{s}{2} + \frac{1}{2}iE)}{2\Gamma(\frac{s}{2} + \frac{1}{2}iE)} - \frac{i}{2} \log \pi - \sum_{p} \frac{i \log p}{p^{rac{s}{2} + iE} - 1} = \sum_{j} \frac{1}{E - E_j} \tag{180}
\]

when \( E(= T) \) is extended as a complex variable (with nonzero imaginary part) such that \( \frac{1}{2} + iE(= \frac{1}{2} + iT) \) is a complex variable in the critical strip and the sum \( \sum_{j} \) is on \( j = \pm 1, \pm 2, \pm 3, \ldots \) and \( E_{-j} = -E_j \) and \( E_j, j = 1, 2, 3, \ldots \) are eigenvalues as in the von Mangoldt-Selberg-Gutzwiller trace formula (179).

Now since in (180) the only singularities of \( \frac{1}{s - \rho_j} \) are the eigenvalues \( E_j, j = \pm 1, \pm 2, \pm 3, \ldots \) we have that the only singularities of \( N'(T)(= N'(E)) \) in (180) (or (178)) are the eigenvalues \( E_j, j = \pm 1, \pm 2, \pm 3, \ldots \) when \( T(= E) \) is extended as a complex variable such that \( \frac{1}{2} + iT(= \frac{1}{2} + iE) \) is a complex variable in the critical strip.

On the other hand since \( N(E)(= N(T)) \) is in terms of \( \text{Im} \log \zeta(\frac{1}{2} + iT) \) from this result on the form of \( N'(E)(= N'(T)) \) we have that the nontrivial zeros \( \rho_j \) of the Riemann’s zeta function in the critical strip are of the form \( \rho_j = \frac{1}{2} + iE_j, j = \pm 1, \pm 2, \pm 3, \ldots \) where \( E_j, j = 1, 2, 3, \ldots \) are eigenvalues of the Virasoro energy operator \( H(z) \) and \( E_{-j} = -E_j, j = 1, 2, 3, \ldots \). This means that the Riemann Hypothesis holds. Thus we have the following theorem:

**Theorem 20 (Riemann Hypothesis)**

The nontrivial zeros of the Riemann’s zeta function all lie in the critical line \( Re \, z = \frac{1}{2} \).

**Remark.** We notice that the equation (180) can be written in the following form:

\[
\frac{1}{s - 1} + \frac{\Gamma'(\frac{s}{2} + \frac{1}{2})}{2\Gamma(\frac{s}{2} + \frac{1}{2})} - \frac{1}{2} \log \pi + \frac{\zeta'(s)}{\zeta(s)} = \sum_{j} \frac{1}{s - \rho_j} + \frac{1}{\rho_j} + B \tag{181}
\]

where \( s := z = \frac{1}{2} + iE \), \( \zeta'(s) = -\sum_{p} \frac{\log p}{p^s - 1} \), \( B = -\frac{1}{2} \gamma - 1 + \frac{1}{2} \log 4\pi \) with \( \gamma \) as the Euler constant, and \( \sum_{j} \frac{1}{\rho_j} + B = 0 \). This is just a well known formula derived from the Hadamard product formula for the zeta function \( \zeta \). Here is the point that we have shown that the nontrivial zeros \( \rho_j \) of the Riemann’s zeta function in the critical strip in this equation are of the form \( \rho_j = \frac{1}{2} + iE_j, j = \pm 1, \pm 2, \pm 3, \ldots \) where \( E_j, j = 1, 2, 3, \ldots \) are eigenvalues of the Virasoro energy operator \( T(z) \) and \( E_{-j} = -E_j, j = 1, 2, 3, \ldots \).
Let us then consider the generalization of the Riemann Hypothesis. For this generalization we suppose that there is a \( L \)-function which gives formula similar to the \( \zeta \) which gives amplitudes to prime numbers. Then by the same reason as above we have that the corresponding quantum prime knots must also have the same amplitudes. Thus there must exist a Virasoro energy operator \( T(z) \) with a central charge \( c > 0 \) and a sequence of eigenvalues \( E_j, j = 1, 2, 3, \ldots \) of \( T(z) \) such that the quantum prime knots have the same amplitudes as the prime numbers. Then as similar to the proof of the Riemann Hypothesis we have that the corresponding Riemann Hypothesis for this \( L \)-function holds that the nontrivial zeros of this \( L \)-function all lie in the critical line \( \Re z = \frac{1}{2} \).

As examples let us consider the Dirichlet \( L \)-functions. Let \( L(\chi, s) \) be a Dirichlet \( L \)-function where \( \chi \) denotes a Dirichlet character. Then \( L(\chi, s) \) can be written in the following Euler product form \[4\] [5]:

\[
L(\chi, s) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}
\]

(182)

where \( p \) denotes a prime number. By this Euler product form as similar to the derivation of \[4\] [5] we have that the corresponding formula of \( N'(T) \) for \( L(\chi, s) \) can be written as a sum where each term of the fluctuation part \( N'_f(T) \) corresponds to a prime number. Thus this formula of \( N'(T) \) for \( L(\chi, s) \) gives amplitudes to prime numbers. Thus by the above proof of Riemann Hypothesis for the Riemann zeta function we have that the corresponding Riemann Hypothesis for this \( L \)-function \( L(\chi, s) \) holds. This proves the following Generalized Riemann Hypothesis:

**Theorem 21 (Generalized Riemann Hypothesis)**

The nontrivial zeros of a Dirichlet \( L \)-function all lie in the critical line \( \Re z = \frac{1}{2} \).

As further examples let us consider Dedekind zeta functions \( \zeta_K(s) \) where \( K \) denotes a number field. We have that a Dedekind zeta function \( \zeta_K(s) \) can be written in the following Euler product form \[4\] [5]:

\[
\zeta_K(s) = \prod_{p \text{ split}} (1 - p^{-s})^{-2} \prod_{p \text{ inert}} (1 - p^{-2s})^{-1} \prod_{p \text{ ramified}} (1 - p^{-s})^{-1}
\]

(183)

where \( p \) denotes a prime number. By this Euler product form as similar to the derivation of \[4\] [5] we have that the corresponding formula of \( N'(T) \) for \( \zeta_K(s) \) can be written as a sum where each term of the fluctuation part \( N'_f(T) \) corresponds to a prime number. Thus this formula of \( N'(T) \) for \( \zeta_K(s) \) gives amplitudes to prime numbers. Thus by the above proof of Riemann Hypothesis for the Riemann zeta function we have that the corresponding Riemann Hypothesis for this \( L \)-function \( \zeta_K(s) \) holds. This proves the following Extended Riemann Hypothesis:

**Theorem 22 (Extended Riemann Hypothesis)**

The nontrivial zeros of a Dedekind zeta function all lie in the critical line \( \Re z = \frac{1}{2} \).

In the following section we show that for the Riemann zeta function and the Dirichlet \( L \)-functions we have that the central charge \( c = \frac{1}{2} \).

### 17 Determination of the Central Charge \( c \) for a \( L \)-function

Let us first show that for the Riemann zeta function and the Dirichlet \( L \)-functions we have that \( c = \frac{1}{2} \). By using the Poisson summation formula Riemann showed that the Riemann zeta function \( \zeta \) is related to the Jacobi \( \theta \) function by the following Mellin transform formula \[50\] [61]:

\[
\pi^{-s} \Gamma(s) \zeta(2s) = \int_0^\infty \left[ \frac{\theta(it) - 1}{2} \right]^s t dt
\]

(184)

We have that the Jacobi \( \theta \)-function is an automorphic (or modular) form with weight \( c = \frac{1}{2} \). We shall show that this weight is the same \( c \) for the central charge of the Virasoro algebra.

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For the Virasoro algebra \((L_j)_j=-\infty\) with central charge \(c = \frac{1}{2}\) we have the following formula for \(L_0\) \cite{18}:

\[
L_0 = \sum_{k>0} kb_{-k}b_k + \frac{1}{16}
\]  

(185)

where \(b_k, k > 0\) denote the creation operators of the Fermi field (We remark that this \(L_0\) is a sum with \(k > 0\)). From this formula by following the derivation of a modular invariant partition function of Fermi field in conformal field theory we have \cite{18} p.347:

\[
Tr(-1)^F q^{L_0 - \frac{c}{2}} = q^{\frac{c}{2} \sum_{n=1}^\infty (1 - q^n)} = \eta
\]  

(186)

where \(c = \frac{1}{2}\) is the central charge; \(F\) is the Fermion number: \(F = \sum_{k>0} F_k, F_k = b_{-k}b_k\) and \(\eta\) is the Dedekind eta function which is an automorphic form with weight \(\frac{1}{2}\) (We remark that this \(F\) is a sum with \(k > 0\) and is not with \(k \geq 0\) as the Fermion number in the derivation of the modular invariant partition function \cite{18} p.347). Thus we have that the \(\eta\)-function with weight \(\frac{1}{2}\) is derived from the Virasoro algebra with central charge \(c = \frac{1}{2}\).

Then we have the following formula relating the Dedekind \(\eta\)-function and the Jacobi \(\theta\)-function:

\[
\theta(\tau) = \frac{\eta^{2}(\tau+1)}{\eta(\tau+1)}
\]  

(187)

This shows that the weight \(c = \frac{1}{2}\) for the Jacobi \(\theta\)-function is the same \(c = \frac{1}{2}\) for the Dedekind \(\eta\) function. Now we have that the weight \(c = \frac{1}{2}\) for the Dedekind \(\eta\) function is the same \(c\) for the central charge of the corresponding Virasoro algebra and that the Dedekind \(\eta\) function is derived from the Virasoro algebra with the central charge \(c = \frac{1}{2}\). Thus we have that the weight \(c = \frac{1}{2}\) for the Jacobi \(\theta\) function is the same \(c\) for the central charge of the Virasoro algebra. Thus from \cite{18} we determine that the central charge \(c\) of the Virasoro algebra for the Riemann zeta function is equal to \(\frac{1}{2}\).

Similarly since the Dirichlet \(L\)-functions are related to the twisted Jacobi \(\theta\) functions twisted by the Dirichlet characters by using a Mellin transform as similar to \cite{18} we can show that the central charge \(c\) of the Virasoro algebra for the Dirichlet \(L\)-functions is equal to \(\frac{1}{2}\).

Then for a conformal field consists of \(n\) independent Fermi fields we have that the corresponding \(n\) product of the \(\eta\)-functions with weight \(\frac{1}{2}\) of each independent Fermi field is an automorphic (or modular) form which is with a positive integer (or half integer) \(c = \frac{n}{2}\) as the weight corresponding to the Virasoro algebra of the conformal field consists of \(n\) independent Fermi fields with the same \(c = \frac{n}{2}\) as the central charge. This shows that the positive weight \(c = \frac{n}{2}\) of the \(n\) product of the \(\eta\)-functions which is an automorphic (or modular) form is the central charge of the corresponding Virasoro operator.

Thus for a general \(L\)-function related to an automorphic form with positive integer (or half integer) \(c\) as the weight by a formula similar to \cite{18} we can determine that the central charge of the Virasoro algebra for this \(L\)-function is equal to the weight \(c\) of the automorphic form related to this \(L\)-function.

On the other hand by using Boson field from conformal field theory we have that the Virasoro algebra with central \(c = 1\) also corresponds to the automorphic form \(\frac{1}{2}\) with weight \(-1\) \cite{18}. Then for a conformal field consists of \(n\) independent Boson fields we have that the corresponding automorphic form with weight \(-c = -n\) corresponds to the Virasoro algebra with the positive central charge \(c = n\). Then from the relation between automorphic forms and \(L\)-functions as similar to \cite{18} we can determine that the central charge of the Virasoro operator for the \(L\)-functions corresponding to automorphic forms with negative weight \(-c = -n\) is equal to \(c = n\).

18 Commutative Diagram of Virasoro Operators and Algebras, Automorphic Forms and \(L\)-functions

By the proof of the Extended Riemann Hypothesis and the determination of central charge of Virasoro algebra for \(L\)-functions we have the following commutative diagram of Virasoro operators and algebras,
automorphic (or modular) forms and $L$-functions:

\[
\begin{array}{c}
\text{Virasoro energy operator} \\
\uparrow
\end{array} \quad \begin{array}{c}
\text{complex transform} \\
\leftarrow \rightarrow
\end{array} \quad \begin{array}{c}
\text{Virasoro algebra} \\
\uparrow
\end{array}
\]

\[
\begin{array}{c}
\text{eigenvalues of Virasoro energy operator} \\
\text{(or nontrivial zeros of$ L$-functions)}
\end{array} \quad \begin{array}{c}
\leftarrow \rightarrow
\end{array} \quad \begin{array}{c}
\text{eigenvalues of Virasoro algebra} \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
L\text{-functions} \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Mellin transform} \\
\leftarrow \rightarrow
\end{array} \quad \begin{array}{c}
\text{automorphic (or modular) forms} \\
\text{(or Fourier transform)}
\end{array}
\]

In this commutative diagram we use the term complex transform for the Virasoro operator $T(z)$ forming from the Virasoro algebra. This complex transform is analogous to the Fourier transform which gives the Poisson summation formula and Mellin transform.

In this commutative diagram we have that the links between Virasoro operator $T(z)$ and Virasoro algebra; between Virasoro algebra and automorphic (or modular) forms; and between automorphic forms and $L$-functions are well known. Here we complete this commutative diagram by establishing the link between the Virasoro operator and the $L$-functions which is established by the proof of the Extended Riemann Hypothesis.

19 Connection with Random Matrix Theory

It is well known that the Random Matrix Theory is related to the study of $L$-functions \textsuperscript{[20]-[30]}. By using our approach for proving the Riemann Hypothesis we may find a connection with the Random Matrix Theory. This connection gives an explanation for the relation of the Random Matrix Theory with the $L$-functions. Indeed we have proved that the Virasoro energy operators are the operators for the non-trivial zeros of the Riemann zeta function (and the related $L$-functions). Then we know that the Virasoro energy operators and the corresponding Virasoro algebras form the basis of conformal field theory. On the other hand a Random Matrix Theory can be formulated as a conformal field theory with a corresponding Virasoro algebra \textsuperscript{[48],[49]}. Thus the Virasoro energy operators and the corresponding Virasoro algebra can be as a basis of a Random Matrix Theory. Now since the Virasoro energy operators are the operators for the non-trivial zeros of the Riemann zeta function (and the related $L$-functions) we have that the Random Matrix Theory can be connected to the $L$-functions via the Virasoro energy operators and the corresponding Virasoro algebras. This connection thus gives an explanation for the mystery of success of the Random Matrix Theory for describing the $L$-functions.

20 Conclusion

In this paper we establish a quantum gauge model of knots. In this quantum model we generalize the way of defining Wilson loops to construct generalized Wilson loops which will be as quantum knots. From quantum knots we give a classification table of knots where knots are one-to-one assigned with an integer such that prime knots are bijectively assigned with prime numbers and the prime number 2 corresponds to the trefoil knot.

Then by considering the quantum knots as periodic orbits of the quantum model and by the identity of knots with integers (which is from the classification table of knots) we then derive a trace formula which may be called as the von Mangoldt-Selberg-Gutzwiller trace formula (which is different from the Selberg trace formula) for the Riemann zeta function. By using this von Mangoldt-Selberg-Gutzwiller trace formula and an approach which is similar to the quantum chaos approach of Berry and Keating we give a proof of the Riemann hypothesis. In this approach for proving the Riemann Hypothesis we show that the Hilber-Polya Conjecture holds that there is a self-adjoint operator which is the Virasoro energy operator with central charge $c = \frac{1}{2}$ such that the nontrivial zeros of the Riemann zeta function are from
the energy eigenvalues of this Virasoro energy operator. This proof of the Riemann Hypothesis can also be extended to prove the Extended Riemann Hypothesis.

In this proof of the Riemann Hypothesis the basic mathematical tool is the conformal field theory which consists of the Virasoro energy operator with central charge $c$ and the Virasoro algebra, the affine Kac-Moody algebra, the quantum Knizhnik-Zamolodchikov (KZ) equation in dual form and the generalized Wilson loops which are as quantum knots and are as solitons derived from the quantum KZ equation. This conformal field structure is related to the Random Matrix Theory for $L$-functions since the Random Matrix Theory can also be formulated as a conformal field theory.

References

[1] V.F.R. Jones, A polynomial invariant for knots via von Neuman algebras, Bull. Amer. Math. Soc. 12 103-111 (1985).
[2] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 351, 1989.
[3] A. Selberg, On the zeros of the zeta-function of Riemann, Der. Kong. Norske. Vidensk. Selsk. Forhand. 15, 59-62, 1945.
[4] E.C.Titchmarsh, The Theory of the Riemann Zeta Function, (Clarendon Press 1986).
[5] H.M.Edwards, Riemann's Zeta Function, (Academic Press 1974).
[6] N.Levinson, More than one third of the zeros of Riemann's zeta function are on $\sigma = \frac{1}{2}$, Ad. Math.13 pp.383-436, 1974.
[7] J.B. Conrey, At Least Two Fifths of the Zeros of the Riemann Zeta Function Are on the Critical Line, Bull. Amer. Math. Soc. 20, 79-81, 1989.
[8] J.B. Conrey, More than Two Fifths of the Zeros of the Riemann Zeta Function Are on the Critical Line, J. reine angew. Math. 399, 1-26, 1989.
[9] J.B. Conrey, The Riemann Hypothesis, Not. Amer. Math. Soc. 50, 341-353, 2003.
[10] M.C.Gutzwiller, Periodic orbits and classical quantization conditions, J. Math. Phys., 12, 343-358 (1971).
[11] M.C.Gutzwiller, Chaos in Classical and Quantum Mechanics, (Springer-Verlag 1990).
[12] M.V.Berry and J.P.Keating, The Riemann zeros and eigenvalue asymptotics, SIAM Review vol44 No. 2, 236-266 (1999).
[13] M.V.Berry and J.P.Keating, $H=xp$ and the Riemann zeros in Supersymmetry and Trace Formula: Chaos and Disorder, J.P.Keating, D.E.Khmelnitskii and I.V.Lerner,eds. Plenum, pp.355-367 (1988).
[14] M.V.Berry, Riemann’s zeta function: A model for quantum chaos? in Quantum Chaos and Statistical Nuclear Physics, T.H.Seligman and H.Nishioka, eds. Lecture Note in Physics 263, Springer Verlag, pp.1-17 (1986).
[15] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, Selecta.Math. (NS) 5, pp.29-106 (1999).
[16] C. Denninger, Some analogies between number theory and dynamical systems on foliated spaces, Proc. Int. Congress Math. Berlin Vol. I, pp.163-186 (1998).
[17] J.P.Keating, The Riemann zeta function and quantum chaology in Quantum Chaos, G.Casti, I.Guarneri and V.Smilansky, eds. North-Holland, pp.145-185 (1993).
[18] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, (Springer-Verlag 1997).

[19] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, (Cambridge University Press 1992).

[20] F.J. Dyson and M.L. Mehta, *Statistical theory of the energy levels of complex systems IV*, Journal of Mathematical Physics, 4 pp.701-712 (1963).

[21] H.L. Montgomery, *The pair correlation of zeros of the zeta function*, in Proc. Sympos. Pure Math. 24, pp.181-193 (1973).

[22] A.M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. 48 pp.273-308 (1987).

[23] E. Bogomolny, J.P. Keating, *Random matrix theory and the Riemann zeros I: three- and four-point correlations*, Nonlinearity, 8, 1115-1131, 1995.

[24] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Mathematics Journal 81, 269-322, 1996.

[25] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, *Integral moments of L-functions*, To appear in PLMS.

[26] N.M. Katz and P. Sarnak, *Zeros of Zeta Functions and Symmetry* Bulletin (New series) of American Mathematical Society 36 1, 1-26, 1999.

[27] N.M. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues, and Monodromy*, (Amer. Math. Soc., 1999).

[28] N. Snaith, *Random matrix theory and zeta functions*, Ph.D. thesis, University of Bristol, 2000.

[29] J.P. Keating and N.C. Snaith, *Random matrix theory and zeta(1/2 + it)*, Communications in Mathematical Physics 214, 57-89, 2000.

[30] J.P. Keating and N.C. Snaith, *Random Matrices and L-functions* J. of Phys A: Math. Gen. 36 2859-2881, 2003.

[31] V. Chari and A. Pressley, *A Guide to Quantum Groups*, (Cambridge University Press 1994).

[32] T. Kohno, *Monodromy representations of braid groups and Yang-Baxter equations*, Ann. Inst. Fourier (Grenoble) 37 (1987) 139-160.

[33] V. G. Drinfel’d, *Quasi-Hopf algebras*, Leningrad Math. J. 1 1419-57 (1990).

[34] C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, (W.H. Freeman, 1994).

[35] A. Kawauchi, *A Survey of Knot Theory*, (Birkhauser Verlag, 1996).

[36] W.B.R. Lickorish, *An Introduction to Knot Theory*, (Springer, 1997).

[37] C. Livingston, *Knot Theory*, (Mathematical Association of American, 1993).

[38] K. Murasugi, *Knot Theory and Its Applications*, (Birkhauser Verlag, 1997).

[39] D. Rolfsen, *Knots and Links*. (Publish or Perish, Inc. 1976).

[40] L. Kauffman, *Knots and Physics*, (World Scientific, 1993).

[41] J. Glimm and A. Jaffe, *Quantum Physics*, (Springer-Verlag, 1987).

[42] L.D. Faddev and V.N. Popov, Phys. Lett. 25B 29 (1967).
[43] J. Baez and J. Muniain, *Gauge Fields, Knots and Gravity*, (World Scientific 1994).

[44] D. Lust and S. Theisen, *Lectures on String Theory*, (Springer-Verlag 1989).

[45] A. Pressley and G. Segal, *Loop Groups*, (Clarendon Press 1986).

[46] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. B 247 (1984)83.

[47] J. F. Cornwell, *Group Theory in Physics*, (Academic Press 1984).

[48] I.K. Kostov, *Conformal Field Theory Techniques in Random Matrix Models* [hep-th/9907060](http://arxiv.org/abs/hep-th/9907060).

[49] A. Morozov *Matrix Models as Integrable Systems*. [hep-th/9502091](http://arxiv.org/abs/hep-th/9502091).

[50] B. Riemann, *Uber die Anzahl der Primzahlen unter einer gegebenen Grosse*, Mon. Not. Berlin Akad pp.671-680 (1859).

[51] S.S. Gelbert and S.D. Miller, *Riemann's Zeta Function and Beyond*, Bulletin( New series) of American Mathematical Society 4 no.1 pp.59-112 (2003).