Fractal Interpolation and Integration over Two-Dimensional Triangular Meshes

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Abstract. The fractal interpolation techniques are powerful alternatives to classical interpolation methods in case of complex irregular data such as financial time series, seismic data and biological signals. The main goal of this study is to apply fractal interpolation techniques over two-dimensional irregular meshes. We partition the polygonal interpolation domain into triangular regions and consider affine contraction mappings from triangular regions to triangular patches of the mesh. Calculation of the fractal interpolation coefficients is reduced to solving a linear system of equations. Freely chosen scaling variables of the iterated function system transformations provide flexibility. Additionally, the constructed iterated function system coefficients for the interpolation can be used directly to evaluate two-dimensional numerical integrals over the domain. We provide basic error results via numerical simulations.

1. Introduction

Different kinds of interpolation techniques such as trigonometric, spline and rational have been proposed to model discrete data by means of a continuous function in the literature of numerical analysis and approximation theory [1, 2]. Barnsley introduced the fractal interpolation function (FIF) based on the theory of iterated function system (IFS) in 1986 [3]. The FIFs have been generalized to multivariate FIFs generated by using higher dimensional or recurrent IFSs, to the hidden variable IFSs produced by projecting a vector valued IFS to lower dimensional space, and to the Hermite or spline FIFs constructed by using Hermite or spline functions [4, 5, 6, 7].

The main goal of this study is to apply fractal interpolation techniques over two-dimensional irregular meshes. We partition the polygonal interpolation domain into triangular regions and consider affine contraction mappings from triangular regions to triangular patches of the mesh. The main formalism will be presented in Section 2. Section 3 presents numerical integration of the interpolant over the domain. Section 4 is devoted to the numerical simulations of the proposed method. The paper will be concluded in Section 5 with some remarks.

2. Fractal Interpolation over Two-Dimensional Triangular Meshes

Let \( \mathcal{P} = \{ P_i = (x_i, y_i), \ i = 1, \ldots, N \} \) be a given set of points including the vertices \( \tilde{P}_j = (\tilde{x}_j, \tilde{y}_j), \ j = 1, 2, 3 \) of a triangular domain \( \Omega \subset \mathbb{R}^2 \) as shown in Figure 1a. A triangulation \( T(\Omega) \) of \( \Omega \) over \( \mathcal{P} \) is a partition of \( \Omega \) such that \( \Omega = \bigcup_{i=0}^{K} \Omega^i \) where \( \Omega^i \) are non-overlapping triangular regions whose vertices are elements of \( \mathcal{P} \). Let \( f: \Omega \to \mathbb{R} \) be an interpolation function.
such that \( f(x_i, y_i) = z_i \in \mathbb{R} \). \( f \) can be constructed in two steps: (i) Construction of an IFS \( \mathcal{I} = \{ w_i : \Omega \times \mathbb{R} \mapsto \Omega \times \mathbb{R}, i = 1, \ldots, K \} \) where \( w_i \) is contraction mapping with contractivity factor \( \sigma_i \) and the attractor of \( \mathcal{I} \) is the graph \( G \) of the function \( f \). (ii) Construction of a contraction mapping \( M : C(\Omega) \mapsto C(\Omega) \) where \( C(\Omega) \) is the set of continuous functions defined over \( \Omega \) such that the fixed point of \( M \) is \( f \).

\[
\begin{align*}
\Omega & = \{ (x, y, z) \mid x, y, z \in \mathbb{R}, 0 < \alpha_i < 1, i = 1, \ldots, K \} \\
\Omega^i & = \{ (x, y, z) \mid x, y, z \in \mathbb{R}, 0 < \alpha_i, i = 1, \ldots, K \} \\
& \text{Let} \\
w_i(x, y, z) &= (L_i(x, y), F_i(x, y, z)) = \begin{bmatrix}
\alpha^1_i \\ \alpha^2_i \\ 0 \\
\alpha^3_i \\ \alpha^4_i \\ 0 \\
\alpha^5_i \\ \alpha^6_i \\ \alpha^7_i
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \begin{bmatrix}
\beta^1_i \\
\beta^2_i
\end{bmatrix}, i = 1, \ldots, K. \quad (1)
\end{align*}
\]

The contraction mappings \( L_i : \Omega \mapsto \Omega^i \) transform larger triangle \( \Omega \) to smaller triangles \( \Omega^i \) by

\[
L_i(x, y) = \begin{bmatrix}
\alpha^1_i \\ \alpha^2_i \\ \alpha^3_i \\ \alpha^4_i \\ \alpha^5_i \\ \alpha^6_i \\
0 \\ 0 \\ \alpha^7_i
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \begin{bmatrix}
\beta^1_i \\
\beta^2_i
\end{bmatrix}. \quad (2)
\]

\( L_i \) satisfies the conditions \( L_i(\tilde{x}_j, \tilde{y}_j) = (x^j_i, y^j_i) \) where \( (x^j_i, y^j_i) \) are the vertices of \( \Omega^i \). The contraction mapping \( F_i : \Omega \times \mathbb{R} \mapsto \mathbb{R} \) is defined as

\[
F_i(x, y, z) = \alpha^1_i x + \alpha^2_i y + \alpha^3_i z + \beta^4_i z \quad \text{where} \quad F_i(\tilde{x}_j, \tilde{y}_j, \tilde{z}_j) = z^j_i, \quad j = 1, 2, 3. \quad (3)
\]

Given \( 0 < \alpha^k_i < 1 \) for all \( i, \alpha^k_i, k = 1, \ldots, 6, i = 1, \ldots, K \) can be solved using

\[
\begin{align*}
\alpha^1_i \tilde{x}_j + \alpha^2_i \tilde{y}_j + \beta^4_i &= x^j_i \\
\alpha^3_i \tilde{x}_j + \alpha^4_i \tilde{y}_j + \beta^5_i &= y^j_i \\
\alpha^5_i \tilde{x}_j + \alpha^6_i \tilde{y}_j + \beta^6_i &= z^j_i - \alpha^7_i \tilde{z}_j
\end{align*} \quad (4)
\]

It can be shown that the functional transformation \( M \) such that \( (Mg)(x, y) = F_i(L_i^{-1}(x, y), g(L_i^{-1}(x, y)), x, y \in \Omega^i, i = 1, \ldots, K \) is a contraction mapping and has a unique fixed point \( f \) which interpolates the given data set [3]. Clearly, \( f = \lim_{n \to \infty} M^{\infty n}(g) \) for all \( g \in C(\Omega). \)

The method is not restricted to triangular regions. Given a polygonal region \( \Gamma \), it can be partitioned into triangular regions \( \Omega_l, l = 1, \ldots, L \) as shown in Figure 1b. Then, the same procedure can be used to construct a piecewise defined interpolant \( f : \Gamma \mapsto \mathbb{R} \) such that \( f(x, y) = f_l(x, y) \) for \( (x, y) \in \Omega_l \) where \( f_l : \Omega_l \mapsto \mathbb{R} \) is the interpolant constructed for \( \Omega_l \) as presented.
3. Integration
The coefficients of the constructed IFS for interpolation can be used to evaluate two-dimensional numerical integrals of the form \( \int_{\Omega} f(x,y) \, dxdy \) directly [3]. Since \( f \) is the unique fixed point of \( M \),

\[
I = \int_{\Omega} f(x,y) \, dxdy = \sum_{n=1}^{L} \int_{\Omega_n} f(x,y) \, dxdy = \sum_{n=1}^{L} \int_{\Omega_n} Mf(x,y) \, dxdy \\
= \sum_{n=1}^{L} \int_{\Omega} F_n(L_n^{-1}(x,y), f(L_n^{-1}(x,y))) \, dxdy
\]

Using the coordinate transformation \((u,v) = L_n^{-1}(x,y)\), (5) reduces to

\[
I = \sum_{n=1}^{L} \int_{\Omega} F_n(u,v,f(u,v)) \Delta_n dudv = \sum_{n=1}^{L} \int_{\Omega} (\alpha_n^u u + \alpha_n^v v + \alpha_n^f f(u,v) + \beta_n^3) \Delta_n dudv = \frac{A}{1-B}
\]

\[
\Delta_n = det(L_n) = \alpha_1^\alpha_2^\alpha_3^, \quad A = \sum_{n=1}^{L} \Delta_n \alpha_1^n, \quad B = \sum_{n=1}^{L} \int_{\Omega} (\alpha_n^u u + \alpha_n^v v + \beta_n^3) dudv
\]

4. Illustrative Example
Consider the Ackley function [8]

\[
h(x,y) = -20 \exp \left[ -0.2 \sqrt{0.5(x^2 + y^2)} \right] - \exp \left[ 0.5(\cos(2\pi x_i) + \cos(2\pi y_i)) \right] + \exp(1) + 20
\]

over a hexagonal domain \( \Gamma \) in the plane as shown in Figure 2a. The piecewise defined interpolant \( f : \Gamma \rightarrow \mathbb{R} \) is constructed according to aforementioned method by using a mesh having 2933 points generated by Gmsh software [9]. Choosing the vertical scaling parameters \( \alpha_{t,l} = \alpha = 0.001, t = 1, \ldots, L, i = 1, \ldots, K_l \) where \( K_l \) is the number of triangles in \( \Omega_l \), fractal interpolation function is approximated using \( g^{n+1} = M(g^n) \) where \( g^0(x,y) = 0 \) for \( n = 10 \) iterations. We calculated relative interpolation errors \( e_r(x_i,y_i) = |f(x_i,y_i) - h(x_i,y_i)|/|h(x_i,y_i)| \times 100 \) over refined mesh dividing each edge of the mesh by 2 containing \( M = 60006 \) points. Figure 2b depicts the interpolation function \( f \) with triangularized \( \Gamma \). The relative errors are shown in Figure 2c. As a simple figure of merit, we calculated the mean square error (MSE)

\[
e_m = \sum_{i=1}^{M} \frac{1}{M} (f(x_i,y_i) - h(x_i,y_i))^2
\]

in terms of vertical scaling parameter \( \alpha \in (0.001, 0.5) \).

It can be seen that interpolation is quite robust with respect to the choice of \( \alpha \) as seen in Figure 2d. On the other hand, when \( \alpha \) is close to 0, the fractal interpolation approaches to crude linear interpolation losing fine features of the data.

5. Conclusion
In this study, we applied the fractal interpolation techniques over two-dimensional irregular triangular meshes. It is straightforward to extend the proposed approach to polygonal domains. The quality of the approximation can be adjusted via free vertical scaling parameters of underlying affine transformations. The fractal interpolation coefficients of the iterated function systems can be used to evaluate numerical integrals over the interpolation domain directly. The fractal approximation over triangular meshes can be useful for especially representing and processing the data in geographical information systems, biomedical signal processing, bivariate financial market and as such.
Figure 2: Fractal interpolation of the data set \( \{(x_i, y_i, z_i) : z_i = h(x_i, y_i), \ i = 1, \ldots, N\} \), for 10 iterations and \( \alpha_{i,l}^7 = 0.001, \ l = 1, \ldots, L, \ i = 1, \ldots, N \). (a) The graph of the Ackley function \( h \). (b) Interpolated values. (c) Relative interpolation error (%) (d) Mean square error (MSE) versus \( \alpha \).

6. References

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