THE HODGE-INDEX THEOREM FOR ARITHMETIC INTERSECTIONS OVER FUNCTION FIELDS

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Abstract. In one of the fundamental results of Arakelov theory, Faltings and Hriljac (independently) proved the Hodge-index theorem for arithmetic surfaces by relating the intersection pairing to the negative of the Neron-Tate height pairing. More recently, Moriwaki and Yuan–Zhang generalized this to higher dimension. In this paper, we extend these results to projective varieties over transcendence degree one function fields. The new challenge is dealing with non-constant but numerically trivial line bundles coming from the constant field. As an application, we also prove a rigidity theorem for preperiodic points of polarized algebraic dynamical systems over global function fields.

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1. Introduction

The Hodge Index Theorem states classically that the divisor intersection pairing on an algebraic surface has signature $+1, -1, \ldots, -1$. The corresponding result for line bundles on arithmetic surfaces, i.e. relative curves over the ring of integers of a number field, was proven independently by Faltings [Fal84] and Hriljac [Hri85], and is a fundamental result in Arakelov Theory. More recently, Moriwaki [Mor96] extended this to higher dimensional arithmetic varieties, and Yuan and Zhang [YZ17] proved a hodge index theorem for adelic metrized line bundles over $\mathbb{Q}$.

In their work, Yuan and Zhang also conjectured that a similar result should hold over function fields. Here we prove their conjecture. Our theorem is actually stated slightly differently, using the $K/k$-image instead of the $K/k$-trace. This is more canonical, as our map $j$ doesn’t rely on a non-canonical isogeny between the trace and image to define the numerically trivial part of the intersection pairing. Properties of the trace and image are discussed further in Section 2.4.

Let $k$ be any field of arbitrary characteristic, and let $K = K(B)$ be the function field of $B$, a smooth, projective curve over $k$. Let $\pi : X \to \text{Spec}(K)$ be a normal, integral, projective variety of dimension $n \geq 1$. We will consider the group $\hat{\text{Pic}}(X)$ of adelic metrized line bundles on $X$ in the sense of [Zha95]; definitions will be recalled in Section 2.1. An important aspect of this theory is that an adelic metrized line bundle $L$ defines a height function $h_L$ of both points and closed subvarieties of $X$ via intersections. Since an adelic metric can be given, for example, by a line bundle on a model $X \to B$ of $X$, this setting also covers arithmetic varieties over $B$, in the same way that Yuan and Zhang’s work over number fields encompasses Arakelov’s setting of arithmetic varieties over the spectrum of the ring of integers of a number field.

The main new challenge working over function fields instead of number fields comes from the need to keep track of the $K/k$-trace and image. If the Albanese variety of $X$ has a non-trivial quotient which is the base change of an abelian variety defined over $k$, called the $K/k$-image, then pulling back the Albanese and quotient maps give a subgroup of $\text{Pic}(X)$ defined over $k$. The map $j$ defined below extends this construction to $\hat{\text{Pic}}(X)$.
This subgroup consists of non-trivial metrized line bundles which are all numerically trivial.
Thus in the proof of the results below, which require knowing when the intersection product
is degenerate, one must make sure that the arguments are functorial with the \( K/k \)-image.

1.1. **Statement of Results.** Assume \( K \) is large enough so that \( X(K) \) is non-empty, and
then we may fix an Albanese variety and morphism \( i : X \to \text{Alb}(X) \). We will need to
differentiate line bundles which come from the constant field \( k \), and so we define a map
\[
j : \text{Pic}^0 \left( \text{Im}_{K/k}(\text{Alb}(X)) \right)_Q \to \hat{\text{Pic}}(X)_Q
\]
given as the composition
\[
\text{Pic}^0 \left( \text{Im}_{K/k}(\text{Alb}(X)) \right)_Q \to \text{Pic} \left( \text{Im}_{K/k}(\text{Alb}(X))_K \right)_Q \to \hat{\text{Pic}}(X)_Q,
\]
where \( \text{Im}_{K/k} \) is Chow’s \( K/k \)-image functor, the second map is the pullback of projection
onto the first factor, and the last map is the pullback of the composition of the \( K/k \)-image
and Albanese maps, \( \lambda_{K/k} \circ i \). To shorten notation, define
\[
\hat{\text{Pic}}_{ck}^\text{im} (X)_Q := j \left( \text{Pic}^0 \left( \text{Im}_{K/k}(\text{Alb}(X)) \right)_Q \right)
\]
to be the image of the map in \( \hat{\text{Pic}}(X)_Q \). We can now state our main theorem:

**Theorem 1.** (Arithmetic Hodge-Index Theorem for function fields) Let \( \overline{M} \) be an integrable
adelic \( \mathbb{Q} \)-line bundle on \( X \) and \( \overline{T}_1, \ldots, \overline{T}_{n-1} \) nef adelic \( \mathbb{Q} \)-line bundles on \( X \). Suppose that
\( M \cdot L_1 \ldots L_{n-1} = 0 \) and that each \( L_i \) is big. Then
\[
\overline{M}^2 \cdot \overline{T}_1 \ldots \overline{T}_{n-1} \leq 0.
\]
If every \( \overline{T}_i \) is arithmetically positive, and \( \overline{M} \) is \( \overline{T}_i \)-bounded for every \( i \), then
\[
\overline{M}^2 \cdot \overline{T}_1 \ldots \overline{T}_{n-1} = 0
\]
if and only if
\[
\overline{M} \in \pi^* \hat{\text{Pic}}(K)_Q + \hat{\text{Pic}}_{ck}^\text{im} (X)_Q.
\]
Note the important case that when \( k \) is finite, \( \hat{\text{Pic}}_{ck}^\text{im} (X)_Q \) is zero.
Call a metrized line bundle $\overline{M}$ on $X$ numerically trivial if

$$\overline{M} \cdot \overline{M}_1 \cdots \overline{M}_n = 0$$

for every choice of metrized line bundles $\overline{M}_1, \ldots, \overline{M}_n$. The classical Hodge-index theorem says that the only divisors on a surface with zero self intersection are the numerically trivial divisors. We show that that is nearly, but not quite the case here:

**Theorem 2.** The following three subgroups of $\widehat{\text{Pic}}(X)$ are equal:

1. The numerically trivial elements of $\widehat{\text{Pic}}(X)_{\mathbb{Q}}$.
2. The set of $M \in \widehat{\text{Pic}}(X)_{\mathbb{Q}}$ such that the height function $h_M$ is zero on $X(\mathbb{K})$.
3. $\pi^*\text{Pic}^0(K)_{\mathbb{Q}} + \overline{\text{Pic}}_{\mathbb{Q}}^0(k)(X)$, where $\text{Pic}^0(K)_{\mathbb{Q}}$ is defined to be the elements of $\widehat{\text{Pic}}(K)_{\mathbb{Q}}$ with arithmetic degree zero.

Define $\text{Pic}^\tau(X)$ to be the group of isomorphism classes of numerically trivial line bundles on $X$. We also state an $\mathbb{R}$-linear version of this theorem, which more closely resembles the classical Hodge-Index Theorem on the signature of the intersection pairing:

**Theorem 3.** Let $M \in \text{Pic}^\tau(X)_{\mathbb{R}}$, and let $L_1, \ldots, L_{n-1} \in \text{Pic}(X)_{\mathbb{Q}}$ be nef. Pick any adelic metrics on $L_1, \ldots, L_{n-1}$ and any flat adelic metric on $M$, and then

$$\langle M, M \rangle_{L_1, \ldots, L_{n-1}} := \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}$$

is a well-defined quadratic form, independent of the choice of metrics, and

$$\langle M, M \rangle_{L_1, \ldots, L_{n-1}} \leq 0.$$

Further, if every $L_i$ is ample, then equality holds if and only if $M \in \text{Pic}^\text{im}_k(X)_{\mathbb{R}}$.

Flat metrics are a particular type of adelic metric on numerically trivial line bundles, and their definition, existence, and construction will be discussed further in Section 2.3.

1.2. **Dynamics Application.** A polarized algebraic dynamical system on $X$ consists of a morphism $f : X \to X$ and an ample line bundle $L$ on $X$ with a fixed isomorphism $f^* L \sim q L$
for some rational $q > 1$. We denote by $f^n$ the $n$-th iterate of $f$ for any non-negative integer $n$, where $f^0$ denotes the identity map. An important invariant studied for such systems is the set of preperiodic points,

$$\text{Prep}(f) := \{ x \in X(K) | f^m(x) = f^n(x) \text{ for some integers } n > m \geq 0 \}.$$

Fakhruddin [Fak03, Theorem 5.1], shows that Prep($f$) is always Zariski dense in $X$ (over any field, not just the fields considered here). We compare two different polarized algebraic dynamical systems on $X$.

Here we state our theorem for global function fields only, but we in fact prove a more general theorem on the rigidity of height zero points over any function field; see Theorem 25. When $K$ is the function field of a curve over a finite field, points of height zero are all preperiodic points, but this can be false over fields where Northcott finiteness does not hold.

**Theorem 4.** Let $X$ be a projective variety over the function field $K$ of a curve over a finite field, and let $f$ and $g$ be two polarized algebraic dynamical systems on $X$. Let $Z$ be the Zariski closure of Prep($f$) $\cap$ Prep($g$) in $X$. Then

$$\text{Prep}(f) \cap Z(K) = \text{Prep}(g) \cap Z(K).$$

Such a theorem has been proven previously for rational functions on $\mathbb{P}^1(\mathbb{C})$ by Baker and DeMarco [BD11] and for polarized algebraic dynamical systems over number fields by Yuan and Zhang [YZ17]. A particular case of interest for our theorem is when Prep($f$) $\cap$ Prep($g$) is dense, in which case the theorem says that $f$ and $g$ have the same preperiodic points.

1.3. **Outline of Paper and sketch of methods.** Definitions and basic properties of adelic metrized line bundles and Chow’s $K/k$-image and trace are recalled in Section 2. Additionally, this section includes technical lemmas, such as the existence of flat metrics, which will be needed throughout the paper.

Our main Hodge-Index theorem and its $\mathbb{R}$-linear variant are proven in Section 3. We begin with the case of $X$ being a curve. Decomposing adelic metrized line bundles into flat and vertical pieces, and addressing intersections of the vertical parts using the local Hodge-Index
Theorem of [YZ17], Theorem 2.1, we reduce to the case of flat metrics. Then, following the methods of Faltings [Fal84] and Hriljac [Hri85], we relate the intersection pairing to the Néron-Tate height pairing on the Jacobian variety of $X$, and complete the result for curves using properties of heights on the Jacobian.

Next we prove the inequality part of Theorem 1 by induction on the dimension of $X$, using a Bertini-type theorem of Seidenberg [Sei50] to find sections which cut out nice subvarieties of $X$. Along the way we prove a Cauchy-Schwarz inequality for this intersection pairing. Theorem 2 and the equality part of Theorem 1 are proved similarly by induction, where we again decompose into flat and vertical metrics and must show that the $K/k$-trace and image functors behave nicely when restricted to a subvariety. Finally, Theorem 3 is easily deduced from Theorem 1 and its proof.

Section 4 proves the application of our result to polarized algebraic dynamical systems. We first describe and prove the existence of admissible metrics for a given polarized algebraic dynamical system, which generalize flat metrics. Next we show that intersecting with an admissible metrized line bundle can be used to define a height function on $X$ which is zero exactly at the preperiodic points of the system. This transforms the rigidity statement on preperiodic points into a statement of the equality of two different height functions defined by intersections, which is proved using the Hodge-index Theorem.

2. Preliminaries

Here we introduce the definitions, basic properties, and lemmas which will be needed throughout the paper.

2.1. Adelic metrized line bundles. The core theory used in this paper is built on local intersection theory as developed by Gubler [Gub98, Gub07b], Chambert-Loir [CL06], Chambert-Loir—Thuillier [CLT09], and Zhang [Zha95]. We briefly summarize some of the key points.

Let $K$ be a complete non-Archimedean field with non-trivial absolute value $|·|$. Denote the valuation ring of $K$ by

$$K^\circ := \{a \in K : |a| \leq 1\},$$
and its maximal ideal
\[ K^{\circ\circ} := \{ a \in K : |a| < 1 \}, \]
so that \( \overline{K} := K^\circ / K^{\circ\circ} \) is the residue field.

Let \( X \) be a variety over \( K \) and denote by \( X^{an} \) its Berkovich analytification as in [Ber90]. For \( x \in X^{an} \), write \( K(x) \) for the residue field of \( x \). Given a line bundle \( L \) on \( X \), it also has an analytification \( L^{an} \) as a line bundle on \( X^{an} \).

**Definition 5.** (Metrized line bundle) A continuous metric \( \| \cdot \| \) on \( L \) consists of a \( K(x) \)-metric \( \| \cdot \|_x \) on \( L^{an}(x) \) for every \( x \in X^{an} \), where this collection of metrics is continuous in the sense that for every rational section \( s \) of \( L \), the map \( X^{an} \to \mathbb{R} \) defined by \( x \mapsto \| s(x) \|_x \) is continuous away from the poles of \( s \). We call \( L \) with a continuous metric a metrized line bundle and denote this by \( \overline{L} = (L, \| \cdot \|) \).

An important example of a continuous metric is a model metric: Let \( X \) be a model of \( X \) over \( K^\circ \), i.e. a projective, flat, finitely presented, integral scheme over \( \text{Spec} \, K^\circ \) whose generic fiber \( X_K \) is isomorphic to \( X \), and let \( L \) be a line bundle on \( X \) whose generic fiber \( L_K \) is isomorphic to \( L \). Then we can define a continuous metric on \( L \) by specifying that for any trivialization \( L_{\mathcal{U}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{U}} \) on an open set \( \mathcal{U} \subset X \) given by a rational section \( \ell \), we have \( \| \ell(x) \|_x = 1 \) for any \( x \) reducing to \( \mathcal{U}_K \) in the reduction \( \overline{X} \) over \( \overline{K} \). We now define several important properties and notations.

**Definition 6.** Let \( \overline{L} = (L, \| \cdot \|) \) and \( \overline{M} \) be metrized line bundles on \( X \).

1. A model metric is nef if it is given by a relatively nef line bundle on \( X \).
2. Call both \( \overline{L} \) and \( \| \cdot \| \) nef if \( \| \cdot \| \) is equal to a uniform limit of nef model metrics.
3. \( \overline{L} \) is arithmetically positive if it is nef and \( L \) is ample.
4. \( \overline{L} \) is integrable if it can be written as \( \overline{L} = \overline{L}_1 - \overline{L}_2 \) with \( \overline{L}_1 \) and \( \overline{L}_2 \) nef.
5. \( \overline{M} \) is \( \overline{L} \)-bounded if there exists a positive integer \( m \) such that \( mL + \overline{M} \) and \( mL - \overline{M} \) are both nef.
6. \( \overline{L} \) is vertical if it is integrable and \( L \cong \mathcal{O}_X \).
7. \( \overline{L} \) is constant if it is isometric to the pull-back of a metrized line bundle on \( \text{Spec} \, K \).
(8) $\widehat{\text{Pic}}(X)$ is defined to be the category of integrable metrized line bundles, with morphisms given by isometries.

(9) $\widehat{\text{Pic}}(X)$ is defined to be the group of isometry classes of integrable metrized line bundles.

Remark. When we say a line bundle is relatively ample or nef, we always mean with respect to the structure morphism, here $X \to \text{Spec } K^0$. A concise discussion of the important aspects of relative ampleness and nefness can be found in [Laz04, Chapter 1.7].

We also have a local intersection theory for metrized line bundles on $X$. Let $Z$ be a $d$-dimensional cycle on $X$, let $L_0, \ldots, L_d$ be integral metrized line bundles on $X$, and $\ell_0, \ldots, \ell_d$ sections of each respectively such that

$$\left( \bigcap_i \text{div}(\ell_i) \right) \cap |Z| = \emptyset,$$

where $|Z|$ means the underlying topological space of the cycle $Z$. Then $Z$ has a local height $\widehat{\text{div}}(\ell_0) \cdots \widehat{\text{div}}(\ell_d) \cdot [Z]$ with the following properties:

1. The local height is linear in $\widehat{\text{div}}(\ell_i)$ and $Z$.
2. For fixed sections, it is continuous with respect to the metrics.
3. When $L_i$ has a model metric given by $L_i$, the height is given by classical intersections:

$$\widehat{\text{div}}(\ell_0) \cdots \widehat{\text{div}}(\ell_d) \cdot [Z] = \text{div}_X(\ell_0) \cdots \text{div}_X(\ell_d) \cdot [Z],$$

where $Z$ is the Zariski closure of $Z$ in $X$.

4. If the support of $\text{div}(\ell_0)$ contains no component of $Z$, there is a measure $c_1(T_1) \cdots c_1(T_d) \delta_Z$ on $X^{an}$ which allows the local height to be computed inductively as

$$\widehat{\text{div}}(\ell_0) \cdots \widehat{\text{div}}(\ell_d) \cdot [Z] = \widehat{\text{div}}(\ell_1) \cdots \widehat{\text{div}}(\ell_d) \cdot [\text{div}(\ell_0) \cdot Z]$$

$$- \int_{X^{an}} \log ||\ell_0(x)||_x c_1(T_1) \cdots c_1(T_d) \delta_Z.$$ 

This notation is meant to suggest that $c_1(T_i)$ should be thought of as the arithmetic version of the classical Chern form $c_1(L_i)$. 
(5) If \( L_0|_{Z_j} \cong \mathcal{O}_{Z_j} \) and \( c_1(L_1) \cdots c_1(L_d) \cdot [Z_j] = 0 \) for every irreducible component \( Z_j \) of \( Z \), then this pairing does not depend on the choice of sections, so we may simply write

\[
\mathcal{L}_0 \cdots \mathcal{L}_d \cdot Z = \hat{\text{div}}(\ell_0) \cdots \hat{\text{div}}(\ell_d) \cdot \left[ Z \right].
\]

When \( Z = X \), we typically omit \( Z \) in all of the above notation.

We now move to the global theory, which is built from the local theory at each place, discussing first models and then adelic metrized line bundles, which will be related similarly to the above. Return to the original setting, where \( k \) is any field, \( B \) is a smooth projective curve over \( k \), \( K = K(B) \) is its function field, and \( \pi : X \to \text{Spec}(K) \) is a normal, integral, projective variety.

Let \( \mathcal{X} \) be a model for \( X \), meaning that \( \mathcal{X} \to B \) is integral, projective, and flat, and the generic fiber \( \mathcal{X}_K \) is isomorphic to \( X \). Given an integral subvariety \( \mathcal{Y} \) of dimension \( d + 1 \) in \( \mathcal{X} \) and line bundles \( \mathcal{L}_0, \ldots, \mathcal{L}_d \) on \( \mathcal{X} \) each with a respective section \( \ell_0, \ldots, \ell_d \) such that their common support has empty intersection with \( \mathcal{Y}_K \), the arithmetic intersection pairing on \( \text{Pic}(\mathcal{X}) \) is defined locally as

\[
\mathcal{L}_0 \cdots \mathcal{L}_d \cdot \mathcal{Y} := \hat{\text{div}}(\ell_0) \cdots \hat{\text{div}}(\ell_d) \cdot \left[ \mathcal{Y} \right] := \sum_{\nu} \left( \hat{\text{div}}(\ell_0) \cdots \hat{\text{div}}(\ell_d) \cdot \left[ \mathcal{Y} \right] \right)_\nu,
\]

where \( \nu \) ranges over the closed points (places) of \( B \), and

\[
\left( \hat{\text{div}}(\ell_0) \cdots \hat{\text{div}}(\ell_d) \cdot \left[ \mathcal{Y} \right] \right)_\nu
\]

means the local intersection number after base-change to the complete field \( K_\nu \). As the notation suggests, this doesn’t depend on the choice of sections. Again we typically drop \( \mathcal{Y} \) in the notation if \( \mathcal{Y} = X \), and when \( \mathcal{X} \) is one dimensional, we call \( \hat{\text{deg}}(\mathcal{L}) := \mathcal{L}_0 \cdot \mathcal{X} \) the arithmetic degree of \( \mathcal{L}_0 \).

Note importantly that this arithmetic intersection theory for \( \mathcal{X} \to B \) is not the same as the classical intersection theory given by viewing \( \mathcal{X} \) as a variety over the field \( k \).
Given a line bundle $L$ on $X$ we call a line bundle $\mathcal{L}$ on $X$ a model for $L$ provided that $\mathcal{L}_K \cong L$. For each place $\nu$ of $B$, completing with respect to $\nu$ induces a model over $K_{\nu}$ and a model metric $\| \cdot \|_{\mathcal{L},\nu}$ of $L_{\nu}^{an}$ on $X_{\nu}^{an} := X_{K_{\nu}}^{an}$. We can then define

**Definition 7.** The collection $\| \cdot \|_{\mathcal{L},A} = \{\| \cdot \|_{\mathcal{L},\nu}\}$ of continuous metrics for every place $\nu$ of $B$ given by $(X, \mathcal{L})$ is called a model adelic metric on $L$. More generally, an adelic metric $\| \cdot \|_A$ on $L$ is a collection of continuous metrics $\| \cdot \|_{\nu}$ of $L_{\nu}^{an}$ on $X_{\nu}^{an}$ for every place $\nu$, which agrees with some model adelic metric at all but finitely many places. A line bundle on $X$ with an adelic metric is called an adelic metrized line bundle, and is denoted $\mathcal{T} = (L, || \cdot ||_A)$.

We extend our local definitions of properties of metrized line bundles to the global case.

**Definition 8.** Let $\mathcal{T}$ be an adelic metrized line bundle.

1. $\mathcal{T}$ is nef if it is equal to a uniform limit of model metrics induced by nef line bundles on models of $X$.
2. $\mathcal{T}$ is integrable if it can be written as $\mathcal{T} = \mathcal{T}_1 - \mathcal{T}_2$, where each $\mathcal{T}_i$ is nef.
3. $\mathcal{T}$ is arithmetically positive if $L$ is ample and $\mathcal{T} - \pi^*N$ is nef for some adelic metrized line bundle $\tilde{N}$ on $\text{Spec} K$ with $\widehat{\deg}(\tilde{N}) > 0$.
4. $\mathcal{T}$-bounded, vertical, constant, $\widehat{\text{Pic}}(X)$, and $\widehat{\text{Pic}}(X)$ are all defined word-for-word as in Definition 6.

**Remark.** We need not fix a model for $X$ when making these definitions; it is not a problem to take a uniform limit of model metrics coming from different models for both $X$ and $L$.

**Remark.** In the definition of arithmetically positive, we’ve thus far only defined the arithmetic degree in the model case, but every adelic metrized line bundle on $\text{Spec} K$ has a model metric, so we may use that definition. The definition is also resolved in the following material.

**Remark.** To avoid confusion, note that the preceding definitions are specified globally, and are not equivalent to requiring that the local property of the same name holds at every fiber. In fact, since relative amplitude (resp. nefness) holds if and only if the restriction to every
Each property in the adelic setting implies that the corresponding property holds locally at every place, but the converse is false, as the differences and uniform limits at each place may not come from a global difference or uniform limit.

Global intersections are defined similarly to the model case, except with the local metrics given explicitly by the adelic metric instead of induced by a model. Given a $d$-dimensional integral subvariety $Z$ of $X$, integrable adelic metrized line bundles $\mathcal{T}_0, \ldots, \mathcal{T}_d$ with respective sections $\ell_0, \ldots, \ell_d$ with empty common intersection with $Z$, their intersection is

$$\mathcal{T}_0 \cdots \mathcal{T}_d \cdot Z := \hat{\text{div}}(\ell_0) \cdots \hat{\text{div}}(\ell_d) \cdot [Z] = \sum_{\nu} \hat{\text{div}}(\ell_0|_{X_\nu}) \cdots \hat{\text{div}}(\ell_d|_{X_\nu}) \cdot [Z|_{X_\nu}],$$

where again this is independent of the choice of sections. Summing the local induction formula at each place produces a global induction formula: letting $\ell_0$ be a rational section of $\mathcal{T}_0$ whose support does not contain $Z$,

$$\mathcal{T}_0 \cdots \mathcal{T}_d \cdot Z = \mathcal{T}_1 \cdots \mathcal{T}_d \cdot (Z \cdot \hat{\text{div}}(\ell_0)) - \sum_{\nu} \int_{X_\nu^n} \log ||\ell_0(x)||_\nu c_1(\mathcal{T}_1, \nu) \cdots c_1(\mathcal{T}_d, \nu) \delta Z|_{X_\nu}.$$

As before, we drop $Z$ when $Z = X$, and when $X$ is zero-dimensional, we call $\hat{\text{deg}}(\mathcal{T}_0) := \mathcal{T}_0 \cdot X$ the arithmetic degree of $\mathcal{T}_0$.

**Definition 9.** An adelic metrized line bundle $\mathcal{M}$ on $X$ of dimension $n$ is called numerically trivial if for any $\mathcal{T}_1, \ldots, \mathcal{T}_n \in \widehat{\text{Pic}}(X)$,

$$\mathcal{M} \cdot \mathcal{T}_1 \cdots \mathcal{T}_n = 0.$$

Call two adelic metrized line bundles numerically equivalent if their difference is numerically trivial.

As an important example, observe that $\pi^*\widehat{\text{Pic}}^{0}(K) + \widehat{\text{Pic}}^{\text{im}}_k (X)$ is numerically trivial; Theorem 1 says that this is the entire numerically trivial subgroup of $\widehat{\text{Pic}}(X)$.

**2.2. Heights of points and subvarieties.** An important application of the intersection theory of adelic metrized line bundles is to define height functions.
Definition 10. Let $\overline{N} \in \hat{\text{Pic}}(X)$. We define the height of a point $x \in X(K)$ by

$$h_{\overline{N}}(x) := \frac{1}{[K(x) : K]} \overline{N} \cdot \hat{x},$$

where $\hat{x}$ is the image of $x$ in $X$ via $X_{K(x)} \to X_K = X$.

In addition to the height of a point, we can use $\overline{N}$ to define the height and the essential minimum of a subvariety:

Definition 11. Let $d = \dim Y$. The height of $Y$ with respect to $\overline{N}$ is defined to be

$$h_{\overline{N}}(Y) := \frac{(\overline{N}|_Y)^{d+1}}{(d+1)(\overline{N}|_Y)^d}$$

and the essential minimum of $Y$ with respect to $\overline{N}$ is

$$\lambda_1(Y, \overline{N}) := \sup_{U \subset Y \text{ open}} \left( \inf_{x \in U(K)} h_{\overline{N}|_Y}(x) \right).$$

When $\overline{N}$ is nef, by the successive minima of Zhang [Zha95], Theorem 1.1, and proven in the function field setting by Gubler [Gub07a], Theorem 4.1, we have

$$\lambda_1(Y, \overline{N}) \geq h_{\overline{N}}(Y) \geq 0.$$  

2.3. Flat metrics. Adelic metrized line bundles with flat metrics form an especially nice class of adelic metrized line bundles. We will often be able to split a metrized line bundle into a bundle with a flat metric plus a vertical bundle, and then work with each of these separately, as flatness will tell us that these have trivial intersection.

Definition 12. Let $X$ be a projective variety over a complete field $K$, and let $\overline{L}$ be a metrized line bundle on $X$. Then $\overline{L}$ is flat if for any morphism $f : C \to X$ of a projective curve over $K$ into $X$, we have $c_1(f^*\overline{L}) = 0$ on the Berkovich analytification $C^an$. If now $X$ is a projective variety over a global field and $\overline{T}$ an adelic metrized line bundle on $X$, call $\overline{T}$ flat provided it is flat at every place.
Note that if \( \mathcal{T} \) is flat, \( L \) must be numerically trivial, as
\[
\deg(L|_C) = \int_{C_{\an}} c_1(\mathcal{T}|_C) = 0.
\]

Additionally, we define admissible metrics, a notion which we will generalize in Section 4.

**Definition 13.** Given an abelian variety \( A \) over \( K \) and a metrized line bundle \( \mathcal{L} \) on \( A \), call \( \mathcal{L} \) admissible if \( [2]^*\mathcal{L} \cong 2\mathcal{L} \).

These two definitions will be related in the proof of the following lemma

**Lemma 14.** Let \( L \) be a numerically trivial line bundle on a projective, normal variety \( X \) over a global function field \( K \). Then \( L \) has a flat metric, which is unique up to constant multiple.

When \( X \) is a curve this lemma has a much simpler proof using linear algebra; see for example [Hri85, Theorem 1.3].

**Proof.** First suppose \( A \) is an abelian variety. Replacing \( L \) by a power if necessary, we may assume \( L \) is algebraically trivial, in which case we have an isomorphism \( \phi : [2]^*L \cong 2L \).

Take any metric \( || \cdot ||_1 \) on \( L \). Then Tate’s limiting argument defines an admissible metric on \( L \) as the limit of
\[
|| \cdot ||_n := \phi^*[2]^*|| \cdot ||_1 \cdot ||_n^{-1}
\]
as \( n \to \infty \). [Zha95, Theorem 2.2, shows that this limit converges uniformly to an admissible adelic metric \( || \cdot ||_0 \) on \( L \), and that this is the unique admissible metric on \( L \) up to constant multiples.

Now let \( C \to A \) be a smooth projective curve in \( A \). After a translation, we can fix a point \( x_0 \in C(K) \) which maps to \( 0 \in A \). By the universal property of the Jacobian, \( C \to A \) factors through the Jacobian map \( C \to \text{Jac}(C) \) taking \( x_0 \to 0 \), and the pullback of \( (L, || \cdot ||_0) \) to \( \text{Jac}(C) \) is also admissible. Then by Remark 3.14 of [Gub07b], \( c_1(L, || \cdot ||_0) = 0 \), and hence \( A \) has a flat metric. By taking the tensor product of this metric with the inverse of any other flat metric on \( L \), uniqueness up to constant multiple is reduced to showing that \( ||1|| \)
is constant for any flat metric on $\mathcal{O}_X$. Any two points on $X$ are connected by a curve; let $D$ be its normalization. Then $||1||$ is constant by local Hodge-Index Theorem in dimension one at each place.

Now choose a point $x_0 \in X(K)$ and recall the Albanese map $i : X \to \text{Alb}(X)$ taking $x$ to $0$. $L$ corresponds to a point $\xi \in \text{Pic}_{X/K}(K) = \text{Alb}(X)^\vee$. By definition, $L$ is (isomorphic to) the Poincare bundle $P$ on $\text{Alb}(X) \times \text{Alb}(X)^\vee$ restricted to $\text{Alb}(X) \times \{\xi\}$, then pulled back through

$$i \times \text{id} : X \times \text{Alb}(X)^\vee \to \text{Alb}(X) \times \text{Alb}(X)^\vee.$$ 

$P|_{\text{Alb}(X) \times \{\xi\}}$ is algebraically trivial, and hence has a flat metric. But the pullback of a flat metric is also flat, so this defines a flat metric for $L$.

□

The main use of flat metrics is the following lemma:

**Lemma 15.** Let $K$ be a complete non-archimedean field, and $X \to \text{Spec} K$ a geometrically connected, normal, projective variety of dimension $n$, with a flat metrized line bundle $M$. Then given any integrable metrized line bundles $L_1, \ldots, L_{n-1}$ on $X$,

$$c_1(M)c_1(L_1) \cdots c_1(L_{n-1}) = 0.$$ 

This is proved in [Gub07b], and also follows directly from the definition of a flat metric and the induction formula for local intersections. In particular, it implies that the intersection of a flat metrized line bundle with a vertical metrized line bundle is zero.

2.4. **Chow’s $K/k$-trace and image.** Proofs of the following can be found in [Lan83] and [Con06]. Let $A$ be an abelian variety defined over $K$. The $K/k$-image $(\text{Im}_{K/k}(A), \lambda)$ consists of an abelian variety $\text{Im}(A)$ over $k$ and a surjective morphism

$$\lambda : A \to \text{Im}_{K/k}(A)_K$$
with the following universal property: If \( V \) is an abelian variety defined over \( k \), and \( \phi : A \to V_K \) a morphism, then \( \phi \) factors through \( \lambda \). Provided the fields \( K \) and \( k \) are clear, we will usually drop the \( K/k \) subscript and just write \( \operatorname{Im}(A) \).

The \( K/k \)-trace is \( (\operatorname{Tr}_{K/k}(A), \tau) \) where \( \operatorname{Tr}_{K/k}(A) \) is an abelian variety over \( k \), and

\[
\tau : \operatorname{Tr}_{K/k}(A)_K \to A
\]

is universal among all morphisms from \( k \)-abelian varieties to \( A \). Again we will often drop the \( K/k \) when the fields are unambiguous. The image can be thought of as the largest quotient of \( A \) that can be defined over \( k \) and the trace the largest abelian subvariety that can be defined over \( k \). This heuristic is literally true in characteristic zero, but in positive characteristic additional care is required for the trace; see [Con06], Section 6 for details.

These constructions are dual to each other in the sense that

\[
\operatorname{Tr}(A^\vee) = \operatorname{Im}(A)^\vee,
\]

and the image and trace are isogenous via the composition \( \lambda \circ \tau \) (descended to the \( k \)-varieties).

Given a morphism of abelian varieties \( f : A \to B \), we get morphisms \( f_{\operatorname{Tr}} : \operatorname{Tr}(A) \to \operatorname{Tr}(B) \) and \( f_{\operatorname{Im}} : \operatorname{Im}(A) \to \operatorname{Im}(B) \) commuting with \( \tau \) and \( \lambda \). By slight abuse of notation, given a morphism \( f : X \to Y \) of (not necessarily abelian) varieties, the Albanese functor gives us a morphism of abelian varieties \( \operatorname{Alb}(X) \to \operatorname{Alb}(Y) \), and we also denote by \( f_{\operatorname{Tr}} \) and \( f_{\operatorname{Im}} \) the descent of this morphism to the trace and image.

3. Proof of Hodge-Index Theorem

3.1. Curves. Assume \( \dim(X) = 1 \). Let \( M \in \hat{\operatorname{Pic}}(X) \) (as opposed to \( \hat{\operatorname{Pic}}(X)_{\mathbb{Q}} \)). Then the theorem discusses the self-intersection \( M^2 \) when \( \deg M = 0 \). By Lemma [13] \( M \) has a flat metric \( M_0 = (M, \| \cdot \|_0) \).

Let \( \mathcal{N} \) be the vertical line bundle defined by

\[
\overline{M} = M_0 + \mathcal{N}.
\]
Since $\mathcal{M}_0$ is flat, $\mathcal{M}_0 \cdot \mathcal{N} = 0$ so that

$$\mathcal{M}^2 = \mathcal{M}_0^2 + \mathcal{N}^2 = \mathcal{M}_0^2 + \sum \mathcal{N}_{\nu}^2,$$

where $\mathcal{N}_{\nu}$ is the restriction of $\mathcal{N}$ to $X_{\nu} := X \otimes_K K_{\nu}$ for each place $\nu$ of $K$ (i.e closed point of $B$). Now $\mathcal{N}_{\nu}^2 \leq 0$ with equality if and only if $\mathcal{N}_{\nu}$ is constant by the local hodge index theorem, [YZ17] Theorem 2.1 Hence

$$\sum \mathcal{N}_{\nu}^2 \leq 0$$

with equality if and only if $\mathcal{N} \in \pi^* \widehat{\text{Pic}}(K)$.

Next, we consider $\mathcal{M}_0$. Let $\mathcal{X} \to B$ be a model for $X \to \text{Spec}(K)$. Since $\mathcal{X}$ is a surface over $k$ we may assume $\mathcal{X}$ is nonsingular. In fact $\| \cdot \|_0$ is a model metric induced by a model $\mathcal{M}$ of $M$ on $\mathcal{X}$ perpendicular to $\widehat{\text{Pic}}(\mathcal{X})_{\text{vert}}$. Such $\mathcal{M}$ can be constructed by starting with any model then adjusting only finitely many vertical fibers, and a straightforward linear algebra argument shows that this can always be achieved; see [Hri85], for example.

We have a natural correspondence taking $M$ to an element of $\text{Pic}^0(X) = \text{Jac}(X)(K) = \text{Alb}(X)(K)$. Then Faltings [Fal84] and Hriljac [Hri85] (in the number field case) and Gubler [Gub03] (verifying the work of the previous in the setting of M-fields, which includes the function fields considered here) show that

$$\mathcal{M}^2 = -2h_{\text{NT}}(M),$$

where $h_{\text{NT}}(M)$ is the Néron-Tate height of the corresponding point on the Jacobian. In fact, since the intersection pairing and the Néron-Tate pairing are both bilinear, we have more generally for flat metrized $\mathcal{D}, \mathcal{E}$ that

$$\mathcal{D} \cdot \mathcal{E} = -2(D, E)_{\text{NT}},$$

and choosing an embedding of $X$ into its Jacobian,

$$h_{\mathcal{D}}(x) = \mathcal{D} \cdot x = -2(D, x)_{\text{NT}}.$$
Since the Néron-Tate height is non-negative, this proves the inequality part Theorem 1 and Theorem 2 and the equality part of Theorem 1 are obtained by describing when the Néron-Tate pairing is degenerate.

By the Shioda-Tate Theorem [Shi99], explained explicitly in this context in [Ulm14], the zeros of the Néron-Tate height are exactly the $k$-points of the $K/k$-trace of $\Jac(X) = \Alb(X)$, embedded via

$$\Tr(\Alb(X))(k) = \Tr(\Alb(X))_K(K) \xrightarrow{\tau} \Jac(X)(K) \hookrightarrow \Pic(X) \to \widehat{\Pic}(X).$$

We verify that this is the same as the map $j$ in our theorem using the following diagram:

$$\begin{array}{ccccccc}
\Jac(X) & \xrightarrow{\lambda^*} & \Pic(X) & \xrightarrow{\lambda^*} & \cdots \\
\downarrow{\lambda^*} & & \downarrow{\lambda^*} & & \\
\Pic^0(\Im(Jac(X))) & \xrightarrow{\lambda^*} & \Pic(\Im(Jac(X))) & \xrightarrow{\lambda^*} & \cdots \\
\cdots & \xrightarrow{\lambda^*} & \Pic(X) & \xrightarrow{\lambda^*} & \widehat{\Pic}(X) \\
\cdots & \xrightarrow{\lambda^*} & \Pic(\Im(Jac(X)) \times_k B) & \xrightarrow{\lambda^*} & \widehat{\Pic}(\Im(Jac(X)))
\end{array}$$

Since the Jacobian is self-dual, $\Tr(\Jac(X)) \cong \Im(\Jac(X))^{\vee}$ and $\lambda^{\vee} = \tau$, so that

$$\begin{array}{ccc}
\Jac(X) & \xrightarrow{\tau} & \Im(\Jac(X))^{\vee} \\
\downarrow{\lambda^*} & & \\
\Tr(\Jac(X)) & \cong & \Pic^0(\Im(Jac(X)))
\end{array}$$

commutes. As this also extends $\mathbb{R}$-linearly, this completes the proof of Theorem 3 in dimension 1, and Theorems 1 and 2 when $\mathcal{M}_0$ is flat.

If $\mathcal{M}$ is vertical, then a section corresponds to a sum of components of fibers of $X \to B$, and $h_{\mathcal{M}}(x)$ is the intersection of this sum with the closure of $x$ in $X$. This is only constant for all $x \in X(K)$ if and only if the sum consists only of constant multiples of whole fibers $X_\nu$, in which case $\mathcal{M} = \pi^*N$ for some $N \in \widehat{\Pic}(K)$, and then the height is the arithmetic degree of $N$.

Finally, we consider heights $h_{\overline{\mathcal{M}}}$ where $\deg M \neq 0$. Suppose $\deg M > 0$. Then $M$ is ample, and by scaling the height we may assume $M$ is very ample so that it gives a $\deg M$
embedding into projective space. The height $h_{\overline{M}}$ differs only by a bounded function from the naïve height defined by this embedding. Repeatedly projecting down from a point, we get a degree $\deg M$ covering $X \to \mathbb{P}^1_K$. Thus since the height on $\mathbb{P}^1$ is unbounded, $h_{\overline{M}}$ is also unbounded. Finally, replacing $\overline{M}$ with $-\overline{M}$ covers the last remaining case where $\deg M < 0$.

3.2. Inequality. We now prove the inequality part of Theorem [1] by induction on $n = \dim X$, and get a version of the Cauchy-Schwarz inequality as a corollary. As in [YZ17], we may assume that each $\overline{L}_i$ is arithmetically positive (instead of just big) by a limiting argument. Additionally, by approximation, we may assume that $\overline{M}$ and each $\overline{L}_i$ are induced by models $\mathcal{X}, \mathcal{M}, \mathcal{L}_i$ of $X, M, L_i$ respectively. We must allow the possibility that $\mathcal{X}$ has isolated singularities, but we will assume $\mathcal{X}$ is normal, as we may simply replace it with its normalization if not.

First we prove the inequality

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \leq 0.$$ 

Replace $\overline{L}_{n-1}$ by a positive power if necessary, so that $\mathcal{L}_{n-1}$ may be assumed to be very ample over $k$. For the moment base change to $\overline{k}$ to guarantee that we are working over an infinite field, and then Seidenberg's Bertini-type theorem [Se50, Theorem 7], tells us that almost all sections of $\mathcal{L}_{n-1}$ cut out normal, integral subvarieties. We choose such a section $s$, cutting out a horizontal subvariety $\mathcal{Y}$. This section is defined over some finite extension of $k$; replace $k$ by that finite extension and now continue working over $k$. This finite base change doesn’t affect the intersection numbers nor the group $\widehat{\text{Pic}}^i_k(X) \otimes$, as the trace is compatible with base changes of the constant field, see [Con06, Theorem 6.8]. Then

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \mathcal{M}|_\mathcal{Y}^2 \cdot \mathcal{L}_1|_\mathcal{Y} \cdots \mathcal{L}_{n-2}|_\mathcal{Y} \leq 0,$$

where the inequality follows from the induction hypothesis. As a corollary, we have the following Cauchy-Schwarz inequality:
Corollary 16. Let $\mathcal{M}, \mathcal{M}'$ be two integral adelic line bundles on $X$, and let $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$ be nef adelic line bundles on $X$ such that

$$M \cdot L_1 \cdots L_{n-1} = M' \cdot L_1 \cdots L_{n-1} = 0.$$

Then

$$\left(\mathcal{M} \cdot \mathcal{M}' \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}\right)^2 \leq \left(\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}\right) \left(\mathcal{M}'^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}\right).$$

Proof. This follows from the inequality part of the Hodge-Index theorem proven above, and from the standard proof of the Cauchy-Schwarz inequality using the (negative semi-definite) inner product

$$\langle M, M' \rangle_{\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}} := \mathcal{M} \cdot \mathcal{M}' \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}.$$


3.3. Equality. Proceeding now to the equality part of Theorem 1, we add the assumptions that each $\mathcal{L}_i$ is arithmetically positive, that $\mathcal{M}$ is $\mathcal{L}_i$-bounded for all $i$, and that

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = 0.$$

Note that as a consequence of the Cauchy-Schwarz inequality above, the set of metrized line bundles $\mathcal{M}$ satisfying these properties forms a group.

By Lemma 3.7 of [YZ17] (this uses the fact that $\mathcal{L}_i$ is arithmetically positive), $M$ is numerically trivial on $X$. Thus it has a flat metric; let $\mathcal{M}_0 = (M, || \cdot ||)$ be flat. Then, similar to the curve case, $\mathcal{N} := \mathcal{M} - \mathcal{M}_0$ is vertical, and

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \mathcal{M}_0^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} + \mathcal{N}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}.$$

The inequality part of the hodge index theorem guarantees that both terms on the right are zero, and then by the local hodge index theorem at every place occurring in $\mathcal{N}$, we have $\mathcal{N} \in \widehat{\text{Pic}}(K)\mathbb{Q}$. Hence we are reduced to proving the statement in the flat metric case $\mathcal{M} = \mathcal{M}_0$. 
Lemma 17. Under the above conditions, and additionally assuming that $M$ is flat,

$$M^2 \cdot L_1 \cdots L_{n-2}|_Y = 0$$

for any closed integral subvariety $Y$ of codimension one in $X$.

Proof. Possibly replacing $L_{n-1}$ by a positive power, we can find a non-zero section $s$ of $L_{n-1}$ vanishing on $Y$. Write $\text{div}(s) = \sum_{i=1}^{t} a_i Y_i$, where the $a_i$s are positive integers, $Y_i$s are distinct integral subvarieties, and $Y_1 = Y$. Then by the induction formula of Chambert-Loir [CL06],

$$M^2 \cdot L_1 \cdots L_{n-1} = \sum_{i=1}^{t} a_i M|_{Y_i}^2 \cdot L_1|_{Y_i} \cdots L_{n-2}|_{Y_i} - \sum_v \int_{X^n} \log ||s||_v c_1(M)^2 c_1(L_1) \cdots c_1(L_{n-2}).$$

Since $M$ is flat, all the integrals are zero, and since by the Hodge-Index Theorem inequality each term of the first sum is non-positive, the claim follows. $\square$

Taking a general hyperplane section $Y$ of some very ample line bundle on $X$, Seidenberg’s Bertini theorem [Sei50] tells us $Y$ is normal, and then the above lemma tells us via the induction hypothesis that

$$M|_Y \in \pi^* \widehat{\text{Pic}}(K) \otimes \mathbb{Q} + \widehat{\text{Pic}}_k(Y) \otimes \mathbb{Q}$$

Write $M|_Y = M_1|_Y + M_2|_Y$, with $M_1|_Y \in \pi^* \widehat{\text{Pic}}(K) \otimes \mathbb{Q}$ and $M_2|_Y \in \widehat{\text{Pic}}_k(Y) \otimes \mathbb{Q}$, for some $M_1, M_2 \in \widehat{\text{Pic}}(X)$. To justify this notation, note that (replacing $M$ by a positive integer multiple if necessary), $M_1|_Y = \mathcal{O}_Y$, so that we may specify $M_1 = \mathcal{O}_X$ and give $M_1$ the same constant metric as $M_1|_Y$, and define $M_2 = M - M_1$. Since $M$ is numerically trivial, again replacing $M$ by a positive integer multiple if necessary, we may further assume $M$ is algebraically trivial, and then that $M_1, M_2 \in \text{Pic}^0(X)$.

Forgetting the metric structure, the map $j$ defined earlier gives us

$$\begin{align*}
\text{Pic}^0(\text{Im}(\text{Alb}(X)) \otimes \mathbb{Q}) &\longrightarrow \text{Pic}^0(\text{Im}(\text{Alb}(X)) \times_k K) \otimes \mathbb{Q} \\
&\longrightarrow \cdots \\
\text{Pic}^0(\text{Im}(\text{Alb}(Y)) \otimes \mathbb{Q}) &\longrightarrow \text{Pic}^0(\text{Im}(\text{Alb}(Y)) \times_k K) \otimes \mathbb{Q} \\
&\longrightarrow \cdots
\end{align*}$$
\[ \cdots \longrightarrow \Pic^0(\Alb(X))_\mathbb{Q} \xrightarrow{\sim} \Pic^0(X)_\mathbb{Q} \]
\[ \downarrow \quad \downarrow \]
\[ \cdots \longrightarrow \Pic^0(\Alb(Y))_\mathbb{Q} \xrightarrow{\sim} \Pic^0(Y)_\mathbb{Q} \]

where the vertical maps come from the pullback of \( Y \hookrightarrow X \) and its descent to the \( K/k \)-image. We show via the following lemma that \( M_2|_Y \) lifts to an element of \( \hat{\Pic}^\text{im}_k(X)_\mathbb{Q} \).

**Lemma 18.** Let \( f : A \to B \) be a morphism of abelian varieties defined over \( K \). In the commutative diagram
\[
\begin{array}{ccc}
\text{Tr}(A)(k)_\mathbb{Q} & \xrightarrow{\tau_A} & A(K)_\mathbb{Q} \\
\downarrow{f_{\text{Tr}}} & & \downarrow{f} \\
\text{Tr}(B)(k)_\mathbb{Q} & \xrightarrow{\tau_B} & B(K)_\mathbb{Q}
\end{array}
\]

\((f \circ \tau_A)(\text{Tr}(A)(k)_\mathbb{Q})\) is equal to \( f(A(K)_\mathbb{Q}) \cap \tau_B(\text{Tr}(B)(k)_\mathbb{Q}) \).

By the duality of the \( K/k \)-image and trace, we may let \( A \) be the abelian variety \( \Pic^0(\Alb(X)) = \Alb(X)^\vee \) and \( B = \Alb(Y)^\vee \) so that this lemma proves the existence of a lift of \( M_2|_Y \). We now prove the lemma:

**Proof.** To shorten notation, we will drop writing the map \( \tau_A \) and consider \( \text{Tr}(A)(k) \) directly as a subgroup of \( A(K) \) (and similarly for \( B \)). First reduce to the case where \( f \) is surjective: let \( B' \) be the image of \( f \), an abelian subvariety of \( B \). By Poincaré reducibility, \( B \) is isogenous to \( B' \times B'' \), for some abelian variety \( B'' \). Then \( \text{Tr}(B) \) is isogenous to \( \text{Tr}(B') \times \text{Tr}(B'') \), and the intersection of \( \text{Tr}(B')(k) \times \text{Tr}(B'')(k) \) with \( B'(K) \) is just \( \text{Tr}(B')(k) \).

Now assume \( f \) is surjective. We can find abelian subvarieties \( A' \subset A \) and \( B' \subset B \), and abelian varieties \( A'', B'' \) such that \( A \) is isogenous to \( A' \times A'' \), \( \text{Tr}(A') = \text{Tr}(A) \), and \( \text{Tr}(A'') = 0 \), and similarly for \( B \). Then \( f \) induces a surjection \( A' \to B' \), but \( A' \) is isogenous to \( \text{Tr}(A')_K \), and \( B' \) is isogenous to \( \text{Tr}(B')_K \), so we get a surjection \( \text{Tr}(A)(k) \to \text{Tr}(B)(k) \), proving the lemma. \( \square \)

Hence we may lift \( M_2|_Y \) to an element \( \overline{M}_2 \in \hat{\Pic}^\text{im}_k(X)_\mathbb{Q} \), and we must have
\[
\overline{M}_2 - \overline{M}_2' \in \ker \left( \hat{\Pic}(X) \to \hat{\Pic}(Y) \right).
\]
Since $\text{Pic}^0(X) \to \text{Pic}^0(Y)$ has finite kernel, replacing $M$ with a positive integer multiple, we may assume $M_2 - M'_2 = \mathcal{O}_X$, since it must be zero in $\text{Pic}^0(X)_\mathbb{Q}$. Additionally, by the Cauchy-Schwarz inequality \[ (M_2 - M'_2)^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = (M - M_1 - M'_2)^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = 0, \] so that by the local Hodge-Index Theorem the metric must be constant at each place and $M_2 - M'_2 \in \pi^* \hat{\text{Pic}}(K)_\mathbb{Q}$. This means that $M = (M_1 + M_2 - M'_2) + M'_2 \in \pi^* \hat{\text{Pic}}(K)_\mathbb{Q} + \hat{\text{Pic}}^\text{im}_k (X)_\mathbb{Q}$.

This proves that when $M$ is $L_i$ bounded and $L_i$ is arithmetically positive for all $i$, then

$M^2 \cdot L_1 \cdots L_{n-1} = 0.$

if and only if $M \in \pi^* \hat{\text{Pic}}(K)_\mathbb{Q} + \hat{\text{Pic}}^\text{im}_k (X)_\mathbb{Q}$.

But in fact we’ve shown something more general, namely that if $Y$ is a general hyperplane section of $X$, the preimage under $\hat{\text{Pic}}(X) \to \hat{\text{Pic}}(Y)$ of $\pi^* \hat{\text{Pic}}^0 (K)_\mathbb{Q} + \hat{\text{Pic}}^\text{im}_k (Y)_\mathbb{Q}$ is exactly $\pi^* \hat{\text{Pic}}^0 (K)_\mathbb{Q} + \hat{\text{Pic}}^\text{im}_k (X)_\mathbb{Q}$. Thus, by cutting down $X$ by general hyperplane sections to a curve, we also prove Theorem 2.

Finally, we note that the above arguments also prove the equality part of Theorem 3. Given $M \in \text{Pic}^\tau (X)_\mathbb{R}$ and $L_i$ nef, $M$ has a ($\mathbb{R}$-linear sum of) flat metric $\overline{M}$ as proven above, and each $L_i$ can be extended to a nef adelic metrized bundle $\overline{L}_i$. Lemma 17 works just the same in this $\mathbb{R}$-linear setting, and then by induction, we can assume $\overline{M}|_Y \in \hat{\text{Pic}}^\text{im}_k (Y)_\mathbb{R}$. Lemma 18 is also the same in the $\mathbb{R}$-linear instead of $\mathbb{Q}$-linear setting, so that $\overline{M} \in \hat{\text{Pic}}^\text{im}_k (Y)_\mathbb{R}$ as desired.

### 4. Algebraic Dynamical Systems

We work in the same setting as before, where $K$ is the function field of a smooth projective curve $B$ over $k$, and $X$ is a projective variety over $K$. Suppose $(X, f, L)$ and $(X, g, M)$ are two polarized dynamical systems on $X$, so that $f$ and $g$ are endomorphisms of $X$, and $L$ and $M$ are ample line bundles such that $f^*L \cong qL$ and $g^*M \cong rM$ for some $q, r > 1$. 
Remark. If $X$ is not normal, we may replace $X$ by its normalization $\psi: X' \to X$, replace $f$ by the normalization $f': X' \to X'$ of $f \circ \psi$, and replace $L$ by $L' = \psi^*L$ to get a new polarized algebraic dynamical system $(X', f', L')$ with $\text{Prep}(f') = \psi^{-1}\text{Prep}(f)$, and similarly for $(X, g, M)$. Hence from here on out we assume without loss of generality that $X$ is normal.

Our main goal in this section is to prove a comparison theorem for the points with dynamical height 0 under $f$ and $g$, with an important corollary comparing the preperiodic points of $f$ and $g$ when $k$ is a finite field. We begin with general properties of polarized algebraic dynamical systems, then define the particular arithmetic dynamical heights involved before stating the theorem.

4.1. An $f^*$-splitting of the Néron-Severi sequence. $f^*$ preserves the exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \text{NS}(X) \to 0$$

defining the Néron-Severi group $\text{NS}(X)$, and the Néron-Severi Theorem tells us that $\text{NS}(X)$ is a finitely generated $\mathbb{Z}$-module. For arbitrary $k$, the $\mathbb{Z}$-module $\text{Pic}^0(X)$ need not be finitely generated, but by the Lang–Néron Theorem,

$$\text{Pic}^0(X)/\text{Tr}_{K/k}\text{Pic}^0(X) \cong \text{Pic}^0(X)/\text{Pic}^0(\text{Im}_{K/k}(\text{Alb}(X)))$$

is a finitely generated $\mathbb{Z}$-module. Note that the inclusion $\text{Pic}^0(\text{Im}(\text{Alb}(X))) \to \text{Pic}^0(X)$ is simply the map $j$ defined in the introduction, with the metric structure dropped. To shorten our notation, define

$$\text{Pic}_{tr}^0(X) := \text{Pic}^0(X)/\text{Tr}_{K/k}\text{Pic}^0(X),$$

$$\text{Pic}_{tr}(X) := \text{Pic}(X)/\text{Tr}_{K/k}\text{Pic}^0(X),$$

so that we have an exact sequence of finite-dimensional $\mathbb{C}$-vector spaces

$$0 \to \text{Pic}_{tr}^0(X) \to \text{Pic}_{tr}(X) \to \text{NS}(X) \to 0,$$

which is also an exact sequence of $f^*$-modules.
Lemma 19. The operator $f^*$ is semisimple on $\text{Pic}^0_{\text{tr}}(X)_{\mathbb{C}}$ with eigenvalues of absolute value $q^{1/2}$, and is semisimple on $\text{NS}(X)$ with eigenvalues of absolute value $q$.

Proof. As usual let $n = \dim X$. By the classical Hodge-index theorem [SGA71, Exposé XIII, Corollary 7.4], we can decompose $\text{NS}(X)_{\mathbb{R}}$ as

$$\text{NS}(X)_{\mathbb{R}} := \mathbb{R} L \oplus P(X), \quad P(X) := \{\xi \in \text{NS}(X)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\}$$

and define a negative definite pairing on $P(X)$ by

$$\langle \xi_1, \xi_2 \rangle := \xi_1 \cdot \xi_2 \cdot L^{n-2}.$$ 

The projection formula for intersection numbers applied to $L^n$ gives us $\deg f = q^n$, and then applied to this pairing, we have

$$\langle f^* \xi_1, f^* \xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle.$$ 

Hence $\frac{1}{q} f^*$ is orthogonal with respect to the negative of this pairing, and $\frac{1}{q} f^*$ is diagonalizable on $\text{NS}(X)_{\mathbb{C}}$ with eigenvalues all of absolute value 1.

On $\text{Pic}^0(X)_{\mathbb{R}}$ we can define a pairing as follows: for $\xi_1, \xi_2 \in \text{Pic}^0(X)_{\mathbb{R}}$, let $\bar{\xi}_1, \bar{\xi}_2$ be flat metrized extensions, and let $\overline{L}$ be any integrable adelic line bundle extending $L$. Then define

$$\langle \xi_1, \xi_2 \rangle := \bar{\xi}_1 \cdot \bar{\xi}_2 \cdot \overline{L}^{n-1}.$$ 

By Theorem 5.19 of [YZ17], this is well defined regardless of the choices of metrics, and Theorem [H] establishes that this pairing is also negative definite. Since $\text{Tr}_{K/k} \text{Pic}^0(X)$ is numerically trivial, this pairing descends to $\text{Pic}^0_{\text{tr}}(X)_{\mathbb{R}}$.

Again applying the projection formula,

$$(f^* \bar{\xi}_1) \cdot (f^* \bar{\xi}_2) \cdot (f^* \overline{L})^{n-1} = q^n \langle \bar{\xi}_1, \bar{\xi}_2 \rangle \cdot \overline{L}^{n-1},$$
Since $f^*\bar{\xi}_i$ is still flat, and since we may replace $f^*\mathcal{L}$ by $q\mathcal{L}$ because the pairing is independent of the choice of metric on $L$, we have

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q\langle \xi_1, \xi_2 \rangle.$$ 

Hence, $q^{-\frac{1}{2}}f^*$ is orthogonal on $\text{Pic}^0_{tr}(X)_{\mathbb{R}}$ with respect to the negative of this pairing, making it diagonalizable with eigenvalues of absolute value 1 as a transformation on $\text{Pic}^0_{tr}(X)_{\mathbb{C}}$.

By the theorem,

$$0 \rightarrow \text{Pic}^0_{tr}(X)_{\mathbb{C}} \rightarrow \text{Pic}_{tr}(X)_{\mathbb{C}} \rightarrow \text{NS}(X)_{\mathbb{C}} \rightarrow 0$$

has a unique splitting as $f^*$-modules by a section

$$\ell_f : \text{NS}(X)_{\mathbb{C}} \rightarrow \text{Pic}_{tr}(X)_{\mathbb{C}}.$$

Let $P, Q \in \mathbb{Q}[T]$ be the minimal polynomials of $f^*$ on $\text{Pic}^0_{tr}(X)_{\mathbb{Q}}$ and $\text{NS}(X)_{\mathbb{Q}}$ respectively. Because the eigenvalues of $f^*$ are different on $\text{Pic}^0_{tr}(X)_{\mathbb{Q}}$ and $\text{NS}(X)_{\mathbb{Q}}$, the product $R = PQ$ must be irreducible, and therefore is the minimal polynomial of $f^*$ on $\text{Pic}_{tr}(X)_{\mathbb{Q}}$. Define

$$\text{Pic}_{tr,f}(X)_{\mathbb{Q}} := \ker R(f^*)|_{\text{Pic}_{tr}(X)_{\mathbb{Q}}}$$

and then this splitting can be given over $\mathbb{Q}$ as

$$\ell_f : \text{NS}(X)_{\mathbb{Q}} \rightarrow \text{Pic}_{tr,f}(X)_{\mathbb{Q}} \rightarrow \text{Pic}_{tr}(X)_{\mathbb{Q}}$$

4.2. Admissible metrics.

**Theorem 20.** The projection $\widehat{\text{Pic}}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(X)_{\mathbb{Q}}$ has a unique section $M \mapsto \overline{M}_f$ as $f^*$-modules, satisfying:

1. If $M \in \text{Pic}^0(X)_{\mathbb{Q}}$ then $\overline{M}_f$ is flat.
2. If $M \in \text{Pic}_f(X)_{\mathbb{Q}}$ is ample then $\overline{M}_f$ is nef.

Adelic metrized line bundles of the form $\overline{M}_f$ are called $f$-admissible.
Proof. Define $\widehat{\text{Pic}}(X)'$ to be the group of adelic line bundles on $X$ with continuous (but not necessarily integrable) metrics. This contains $\widehat{\text{Pic}}(X)$. We will show that the projection $\widehat{\text{Pic}}(X)'_Q \to \text{Pic}(X)_Q$ has a unique section, and then that properties 1 and 2 of the theorem hold for this section. Since $\text{Pic}^0(X)_Q$ and the ample elements of $\text{Pic}_f(X)_Q$ generate $\text{Pic}(X)_Q$, the section does in fact produce integrable metrics, proving the theorem.

The kernel of the projection $\widehat{\text{Pic}}(X)'_Q \to \text{Pic}(X)_Q$ is

$$D(X) = \widehat{\text{Pic}}(K)_Q \bigoplus_v C(X_v^{an}),$$

where $C(X_v^{an})$ is the ring of continuous $\mathbb{R}$-valued functions on $X_v^{an}$, via the association $\| \cdot \|_v \to -\log \|1\|_v$. Recall that $R = PQ$ was defined to be the minimal polynomial of $f^*$ on $\text{Pic}(X)_Q$ and now consider the action of $R(f^*)$ on $D(X)$.

**Lemma 21.** $R(f^*)$ is invertible on $D(X)$.

Proof. $f^*$ acts as the identity on $\widehat{\text{Pic}}(X)$, hence $R(f^*)$ acts as $R(1)$, and this is not zero because the roots of $R$ all have absolute value $q$ or $q^{\frac{1}{2}}$. So it suffices to show that $R(f^*)$ is invertible on $C(X)_C := (\bigoplus_v C(X_v^{an})) \otimes_\mathbb{R} \mathbb{C}$. Factor $R$ over $\mathbb{C}$ as

$$R(T) = a \prod_i \left(1 - \frac{T}{\lambda_i}\right),$$

where $a \neq 0$, and by lemma [19] $|\lambda_i|$ is either $q^{\frac{1}{2}}$ or $q$. $R(f^*)$ is invertible provided each term $1 - f^*/\lambda_i$ is, and each term has inverse

$$\left(1 - \frac{f^*}{\lambda_i}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{f^*}{\lambda_i}\right)^k,$$

provided this series converges absolutely with respect to the operator norm, which is defined with respect to the supremum norm $\| \cdot \|_{\text{sup}}$ on $C(X_v^{an})_\mathbb{C}$ for every place $v$. $f^*$ doesn’t change the supremum norm, so the operator norm of $f^*$ is 1, and

$$\left\| \left(\frac{f^*}{\lambda_i}\right)^k \right\| = \frac{1}{|\lambda_i|^k} \leq q^{-\frac{k}{2}},$$

so the series converges absolutely. \hfill \Box
Corollary 22. The exact sequence

$$0 \to D(X) \to \widehat{\text{Pic}}(X)_Q' \to \text{Pic}(X)_Q \to 0$$

has a unique $f^*$-equivariant splitting.

Proof. Define

$$E(X) := \ker \left( R(f^*) : \widehat{\text{Pic}}(X)_Q \to \widehat{\text{Pic}}(X)'_Q \right).$$

Since $R(f^*)$ kills all of $\text{Pic}(X)_Q$, this gives an $f^*$-invariant decomposition

$$\widehat{\text{Pic}}(X)'_Q = D(X) \bigoplus E(X)$$

such that the projection onto $\text{Pic}(X)$ gives an isomorphism $E(X) \xrightarrow{\sim} \text{Pic}(X)_Q$, whose inverse is the desired splitting.

We can write this down even more explicitly. For $M \in \text{Pic}(X)_Q$, let $\overline{M}$ be any choice of metric in $\widehat{\text{Pic}}(X)'_Q$. Then define

$$\overline{M}_f := \overline{M} - R(f^*)|_{D(X)}^{-1} R(f^*) \overline{M}.$$ 

\[\square\]

It now remains to show that this splitting satisfies (1) and (2). To start, suppose $M$ is in $\text{Pic}^0(X)_Q$. Let $x_0 \in \text{Prep}(f)$, then after replacing $f$ by an iterate and $K$ by a finite extension if necessary, we may assume that $x_0$ is a fixed point. Let $i : X \to \text{Alb}(X)$ be the Albanese map taking $x_0 \mapsto 0$, then $f^*$ and $i^*$ induce the following commutative diagram, where $f' := (f^*)^\lor$:

\[
\begin{array}{ccc}
\text{Pic}^0(\text{Alb}(X)) & \xrightarrow{i^*} & \text{Pic}^0(X) \\
\downarrow{(f')^*} & & \downarrow{f^*} \\
\text{Pic}^0(\text{Alb}(X)) & \xrightarrow{i^*} & \text{Pic}^0(X) \\
\downarrow{M \mapsto \overline{M}_f} & & \downarrow{M \mapsto \overline{M}_f} \\
\text{Pic}(\text{Alb}(X))' & \xrightarrow{i^*} & \widehat{\text{Pic}}(X)' \\
\end{array}
\]
Because this commutes, it suffices to show (1) for abelian varieties, as \( i^* \) takes \( M_{f'} \) to \( \overline{M}_f \), and the pullback of a flat metric is also flat. Now \([2]^* M = 2M\), and since \([2]\) commutes with \(f'\),

\[ [2]^* \overline{M}_{f'} = 2\overline{M}_{f'}, \]

so that as in the proof of Lemma 13, \( \overline{M}_{f'} \), and hence also \( \overline{M}_f \) is flat.

Finally, we show that (2) also holds. This is proven in the arithmetic setting (i.e. when \( X \) is defined over a number field) in [YZ17, Theorem 4.9, Step 4], however the proof is a purely numerical argument on weighted sums of adelic line bundles, and applies identically in our geometric setting. Note that Lemma 5.7, on which the proof relies and which states that arithmetic ampleness is an open condition, is a more well-known result in the geometric setting, proven in [Laz04, Proposition 1.3.7, for example.

\[ \square \]

The above section also descends to a section \( \text{Pic}_r(X) \to \widehat{\text{Pic}}(X)/\widehat{\text{Pic}}_{im}(X) \), as by construction every element of \( \widehat{\text{Pic}}_{im}(X) \) has a flat metric. Thus, we have an \( f^* \)-equivariant linear map

\[ \widehat{\ell}_f : NS(X) \to (\widehat{\text{Pic}}(X)/\widehat{\text{Pic}}_{im}(X))_\mathbb{Q} \]

given by the composition of the section developed in Theorem 20 and the map just preceding it.

4.3. **Rigidity of height zero points and preperiodic points.** Heights given by \( f \)-admissible metrized line bundles have particularly nice properties and correspond to the dynamical canonical heights defined by Call-Silverman [CS93].

**Proposition 23.** Let \( M \in \text{Pic}(X) \). Then:

1. If \( f^* M = \lambda M \) for some \( \lambda \in \mathbb{Q} \), then \( f^* \overline{M}_f = \lambda M_f \) in \( \widehat{\text{Pic}}(X) \), and \( h_{\overline{M}_f}(f(\cdot)) = \lambda h_{M_f}(\cdot) \).

2. For \( x \in \text{Prep}(f) \), \( \overline{M}_f|_x \) is trivial in \( \widehat{\text{Pic}}(x) \), and in particular \( h_{\overline{M}_f} \) is zero on \( \text{Prep}(f) \).

Further, if \( M \) is ample and \( f^* M = \lambda M \) for some \( \lambda > 1 \) (in particular, if \( M = L \)), then:
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(3) \( h_{M_f}(x) \geq 0 \) for all \( x \in X(K) \).

Proof. (1) is clear from the statement of Theorem 20. For (2), let \( f^m(x) = f^n(x) \) for some \( m > n \geq 0 \). Consider the linear map of finite dimensional vector spaces

\[
(f^*)^m - (f^*)^n : \text{Pic}_{tr}(X)_\mathbb{Q} \to \text{Pic}_{tr}(X)_\mathbb{Q}
\]

Since \( f^* \) has eigenvalues with absolute value \( q, q^{1/2} > 1 \), this is an isomorphism, hence surjective. Then for some \( N \in \text{Pic}(X)_\mathbb{Q} \) we can write

\[
h_{M_f}(x) = h_{(f^*)^mN_f} - h_{(f^*)^nN_f}(x) = (f^*)^mN_f|_x - (f^*)^nN_f|_x = M_f|_{f^m(x)} - M_f|_{f^n(x)} = 0.
\]

When \( M \) is ample and \( f^*M = \lambda M \), let \( h_M \) be the Weil height (sometimes called naïve height) coming from \( M \), and define

\[
\hat{h}_{M_f} := \lim_{n \to \infty} \frac{h_M(f^n(x))}{n}.
\]

Call and Silverman \cite{CS93} show that this gives a well-defined canonical height function which agrees with our \( h_{M_f} \) defined via intersections. (3) is then clear as it holds for all canonical heights. \( \square \)

We can say more when \( K \) is a global function field, i.e. when \( k \) is finite.

**Proposition 24.** If \( M \in \text{Pic} X \) is ample and \( f^*M = \lambda M \) for some \( \lambda > 1 \), and \( k \) is a finite field, \( h_{M_f}(x) = 0 \) if and only if \( x \in \text{Prep}(f) \).

Proof. Suppose \( h_{M_f}(x) = \hat{h}_{M_f}(x) = 0 \) for some \( x \in X(K) \). Consider the set \( S = \{f^n(x)\}_{n \geq 0} \) of forward iterates of \( x \). Since \( f \) is defined over \( K \), we have

\[
[K(f^n(x)) : K] \leq [K(x) : K]
\]

and \( h_{M_f}(f^n(x)) = 0 \) for all \( n \geq 0 \). Hence \( S \) is a set of bounded height and bounded degree, and must be finite by the Northcott property for global function fields. This means the forward orbit of \( x \) is finite and \( x \) is preperiodic. \( \square \)

We can now state and prove our main theorem of this section.
Theorem 25. Let $Z_f := \{ x \in X(\overline{K}) | h_{\mathcal{L}_f}(x) = 0 \}$ be the set of height zero points with respect to $f$, $Z_g$ the same set for $g$, and let $Z$ be the Zariski closure of $Z_f \cap Z_g$ in $X$. Then

$$Z_f \cap Z(\overline{K}) = Z_g \cap Z(\overline{K}).$$

When $k$ is finite, $Z_f = \text{Prep}(f)$ and $Z_g = \text{Prep}(g)$, so Theorem 3 stated in the introduction follows as an immediate consequence. If $k$ is not finite, it is still true that $Z_f \cap \text{Prep}(f)$, but there may be height zero points with infinite forward orbit.

Proof. Let $Y$ be the normalization of an irreducible component of $Z$ and say $\dim Y = d$. Let $\xi$ be the image of $L$ in $\text{NS}(X)$. $\xi$ has two different lifts $\ell_f(\xi)$ and $\ell_g(\xi)$ to $\text{Pic}_{tr}(X)_Q$; let $L_f$ and $L_g$ be representatives in $\text{Pic}(X)_Q$ of these classes in $\text{Pic}_{tr}(X)_Q$. Since $L$ is one such choice of representative for $\ell_f(\xi)$ and ampleness is preserved by numerical equivalency, $L_f$ and $L_g$ must both be ample.

By Theorem 20 $L_f$ and $L_g$ have $f$- and $g$-admissible metrics, which we call $\mathcal{L}_f$ and $\mathcal{L}_g$ respectively. Both are nef. Their sum $\mathcal{N} := \mathcal{L}_f + \mathcal{L}_g$ is also nef, and defines a height function $h_{\mathcal{N}}$, which does not depend on the choice of representatives.

By construction, $Y$ has a dense set of points which have height zero under both $h_{\mathcal{L}_f}$ and $h_{\mathcal{L}_g}$, and hence a dense set of points for which $h_{\mathcal{N}}$ is zero. But then the successive minima means

$$\lambda_1(Y, \mathcal{N}) = h_{\mathcal{N}}(Y) = 0.$$

Rewriting the height of $Y$ in terms of intersections,

$$0 = (\mathcal{L}_f|_Y + \mathcal{L}_g|_Y)^{d+1} = \sum_{i=0}^{d+1} \binom{d+1}{i} \mathcal{L}_f|_Y^i \cdot \mathcal{L}_g|_Y^{d+1-i}. $$

Since both $\mathcal{L}_f|_Y$ and $\mathcal{L}_g|_Y$ are nef, every term in the sum on the right is non-negative, hence must be zero. Then

$$(\mathcal{L}_f|_Y - \mathcal{L}_g|_Y)^2 \cdot (\mathcal{L}_f|_Y + \mathcal{L}_g|_Y)^{d-1} = 0$$

as well, and since $L_f - L_g$ is numerically trivial,

$$(L_f|_Y - L_g|_Y) \cdot (L_f|_Y + L_g|_Y)^{d-1} = 0.$$
Since \((\mathcal{L}_f - \mathcal{L}_g)\) is clearly \((\mathcal{L}_f + \mathcal{L}_g)\)-bounded, we are nearly in the right setting to apply Theorem 11 except that \((\mathcal{L}_f|_Y + \mathcal{L}_g|_Y)\) is only nef, not necessarily arithmetically positive. To fix this, we simply adjust the metric by a small positive factor: let \(\mathcal{C} \in \hat{\text{Pic}}(K)\) with \(\hat{\deg}(\mathcal{C}) > 0\). Replacing \((\mathcal{L}_f|_Y + \mathcal{L}_g|_Y)\) by \((\mathcal{L}_f|_Y + \mathcal{L}_g|_Y + \pi^* \mathcal{C})\) all the conditions of the theorem are now satisfied, so that the theorem tells us

\[
(\mathcal{L}_f|_Y - \mathcal{L}_g|_Y) \in \hat{\text{Pic}}(K)_Q + \hat{\text{Pic}}^{*\text{im}}_k (Y)_Q.
\]

As described previously, \(\hat{\text{Pic}}^{*\text{im}}_k (Y)_Q\) is numerically trivial. Let \(\mathcal{P} \in \hat{\text{Pic}}(K)_Q\) such that

\[
(\mathcal{L}_f|_Y - \mathcal{L}_g|_Y) = \pi^* \mathcal{P} + \hat{\text{Pic}}^{*\text{im}}_k (Y)_Q.
\]

Then for any \(x \in \text{Prep}(f) \cap \text{Prep}(g)\), \(x^* \pi^* \mathcal{P} = 0 \in \hat{\text{Pic}}(x)_Q\), hence

\[
\mathcal{P} = 0 \in \hat{\text{Pic}}(K).
\]

We conclude from Theorem 2 that since \(\mathcal{L}_f|_Y\) and \(\mathcal{L}_g|_Y\) are numerically equivalent, they define the same height functions

\[
h_{\mathcal{L}_f} = h_{\mathcal{L}_f|_Y} = h_{\mathcal{L}_g|_Y} = h_{\mathcal{L}_g}
\]

on \(Y\). As both \(\text{Prep}(f)\) and \(\text{Prep}(g)\) are the points of \(Y\) with height zero under this height function, we are finished.

\[\square\]

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