Constraining $f(T)$ Theories with the Varying Gravitational Constant

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ABSTRACT

As is well known, a varying effective gravitational “constant” is one of the common features of most modified gravity theories. Of course, as a modified gravity theory, $f(T)$ theory is not an exception. Noting that the observational constraint on the varying gravitational “constant” is very tight, in the present work we try to constrain $f(T)$ theories with the varying gravitational “constant”. We find that the allowed model parameter $n$ or $\beta$ has been significantly shrunk to a very narrow range around zero. In fact, the results improve the previous constraints by an order of magnitude.

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I. INTRODUCTION

Motivated by the well-known large number hypothesis proposed in 1937 [1], the varying fundamental “constants” have remained as one of the unfading subjects for decades. In fact, the gravitational constant $G$ was the first one whose constancy was questioned by Dirac [1].

As a pure gravitational phenomenon, the variation of the gravitational “constant” does not affect the local physics (e.g. the atomic transitions or the nuclear physics), and hence most constraints on it are obtained from systems in which gravity is non-negligible, such as the motion of the bodies of solar system and astrophysical systems [2]. Following [2], here we briefly review the current observational constraints on the varying gravitational “constant”. The first type of constraints arises from solar system. The latest analysis of the lunar laser ranging experiment [3] gives the constraint

$$|\dot{G}/G| \leq 1.3 \times 10^{-12} \text{ yr}^{-1},$$

(1)

where a dot denotes the derivative with respect to the time $t$. In [4], the combination of Mariner 10, Mercury and Venus ranging data gives

$$|\dot{G}/G| \leq 2 \times 10^{-12} \text{ yr}^{-1}.$$

(2)

In [5], the ranging data from Viking landers on Mars lead to the constraint

$$|\dot{G}/G| \leq 6 \times 10^{-12} \text{ yr}^{-1}.$$

(3)

The second type of constraints arises from pulsar timing. In [6], the constraints from PSR B1913+16 and PSR B1855+09 are given by

$$|\dot{G}/G| \leq 9 \times 10^{-12} \text{ yr}^{-1},$$

(4)

and

$$|\dot{G}/G| \leq 2.7 \times 10^{-11} \text{ yr}^{-1},$$

(5)

respectively. On the other hand, the constraint from PSR J0437-4715 reads [7]

$$|\dot{G}/G| < 2.3 \times 10^{-11} \text{ yr}^{-1}.$$

(6)

The third type is the stellar constraints. In [8], the ages of globular clusters give the constraint

$$|\dot{G}/G| \leq 3.5 \times 10^{-11} \text{ yr}^{-1}.$$

(7)

The constraint from helioseismology is given by [9]

$$|\dot{G}/G| < 1.6 \times 10^{-12} \text{ yr}^{-1}.$$

(8)

The seismology of white dwarf G117-B15A [10] gives the constraint

$$|\dot{G}/G| < 4.1 \times 10^{-11} \text{ yr}^{-1}.$$

(9)

In [11], the constraint from the cooling of white dwarfs reads

$$|\dot{G}/G| < 2 \times 10^{-11} \text{ yr}^{-1}.$$

(10)

In [12], the light curves of supernovae give the constraint

$$|\dot{G}/G| \leq 4.8 \times 10^{-12} \text{ yr}^{-1}.$$

(11)

Finally, one can also obtain the cosmological constraints on the varying $G$ from cosmic microwave background (CMB) and big bang nucleosynthesis (BBN). However, this type of cosmological constraints on...
the varying \( G \) heavily relies on not only the assumption of the whole history of \( G(t) \) but also many other complicated factors. Therefore, it is very difficult to state a definitive constraint \([2]\). So, here we do not consider this type of the cosmological constraints on the varying gravitational “constant”.

Obviously, the currently tightest observational constraint on the varying gravitational “constant” is the one given in Eq. (1), namely, \( |\dot{G}/G| \leq 1.3 \times 10^{-12} \text{ yr}^{-1} \). So, in the present work we only use this tightest observational constraint given in Eq. (1) to constrain \( f(T) \) theories.

In fact, it is well known that a varying effective gravitational “constant” is one of the common features of many modified gravity theories \([2]\), for example, \( f(R) \) theory, scalar-tensor theory (including Brans-Dicke theory and Galileon theory), braneworld scenarios (such as DGP, RSI and RSII), \( f(G) \) theory (\( G \) is the Gauss-Bonnet term), Horava-Lifshitz theory, MOND and TeVeS theories (see e.g. \([13,14]\) for reviews). Recently, a new modified gravity theory, namely the so-called \( f(T) \) theory, attracted much attention in the community, where \( T \) is the torsion scalar. Similar to many modified gravity theories, the effective gravitational “constant” is also varying in \( f(T) \) theory. Therefore, it is of interest to constrain \( f(T) \) theory with the varying gravitational “constant”. This is what we need to do in the present work.

In Sec. II we briefly review the key points of \( f(T) \) theory. In Sec. III A in order to compare with the observational constraint on the varying gravitational “constant”, we give the corresponding formula of \( |G_{\text{eff}}/G_{\text{eff}}| \) for a general \( f(T) \) theory. In Secs. III B and III C we constrain two concrete \( f(T) \) theories with the varying gravitational “constant”, respectively. Finally, a brief conclusion is given in Sec. IV.

II. A BRIEF REVIEW OF \( f(T) \) THEORY

\( f(T) \) theory is a generalization of the teleparallel gravity originally proposed by Einstein \([18,19]\). In teleparallel gravity, the Weitzenb"{o}ck connection is used, rather than the Levi-Civita connection which is used in general relativity. Following \([20,21]\), here we briefly review the key points of \( f(T) \) theory. The orthonormal tetrad components \( e_i(x^\mu) \) relate to the metric through

\[
g_{\mu \nu} = \eta_{ij} e^i_\mu e^j_\nu, \quad (12)
\]

where Latin \( i, j \) are indices running over 0, 1, 2, 3 for the tangent space of the manifold, and Greek \( \mu, \nu \) are the coordinate indices on the manifold, also running over 0, 1, 2, 3. In \( f(T) \) theory, the gravitational action is given by

\[
S_T = \frac{1}{2\kappa^2} \int d^4x \sqrt{|e|} \left[ T + f(T) \right], \quad (13)
\]

where \( \kappa^2 \equiv 8\pi G_N \) (\( G_N \) is the Newton constant), and \( \sqrt{|e|} = \text{det} (e^i_\mu) = \sqrt{-g} \). It is worth noting that in the literature, \( T + f(T) \) in Eq. (13) could be instead replaced by \( f(T) \), and hence one should be aware of the correspondence between these two formalisms. The torsion scalar \( T \) is defined by

\[
T = S^\rho_{\mu \nu} T^\rho_{\mu \nu}, \quad (14)
\]

where

\[
T^\rho_{\mu \nu} \equiv -e^i_\mu (\partial_\rho e^i_\nu - \partial_\nu e^i_\rho), \quad (15)
\]

\[
K^\rho_{\mu \nu} \equiv -\frac{1}{2} (T^\rho_{\mu \nu} - T^\mu_{\rho \nu} - T^\nu_{\rho \mu}), \quad (16)
\]

\[
S^\rho_{\mu \nu} \equiv \frac{1}{2} (K^\rho_{\mu \nu} + \delta^\rho_{\mu} T^\theta_{\nu \theta} - \delta^\rho_{\nu} T^\theta_{\theta \mu}), \quad (17)
\]

In this work, we consider a spatially flat Friedmann-Robertson-Walker (FRW) universe. In this case, one easily finds that \([20,21]\)

\[
T = -6H^2, \quad (18)
\]

where \( H \equiv \dot{a}/a \) is the Hubble parameter; \( a = (1 + z)^{-1} \) is the scale factor (we have set \( a_0 = 1 \), whereas the subscript “0” indicates the present value of the corresponding quantity); \( z \) is the redshift. Therefore,
one can use \( T \) and \( H \) interchangeably. The modified Friedmann equation and Raychaudhuri equation are given by \[21–23\]

\[
H^2 = \frac{\kappa^2}{3} \rho - \frac{f}{6} - 2H^2 f_T ,
\]

(19)

\[
\left( H^2 \right)' = \frac{2\kappa^2 p + 6H^2 + f + 12H^2 f_T}{24H^2 f_T T - 2 - 2f_T} ,
\]

(20)

where \( f_T = \partial f/\partial T \), a prime denotes the derivative with respect to \( \ln a \), and \( \rho, p \) are the total energy density and pressure, respectively. In an universe with only dust matter, \( p = p_m = 0 \) and \( \rho = \rho_m \).

Obviously, if \( f(T) = \text{const.} \), \( f(T) \) theory reduces to the well-known \( \Lambda \)CDM model.

In fact, \( f(T) \) theory was firstly used to drive inflation by Ferraro and Fiorini \[24, 25\]. Later, Bengochea and Ferraro \[20\], as well as Linder \[21\], proposed to use \( f(T) \) theory to drive the current accelerated expansion of our universe without invoking dark energy. Very soon, \( f(T) \) theory attracted much attention in the community. We refer to e.g. \[26–31, 36\] for relevant work.

III. CONSTRAINING \( f(T) \) THEORIES WITH THE VARYING GRAVITATIONAL "CONSTANT"

A. \(|\dot{G}_{\text{eff}}/G_{\text{eff}}| \) for a general \( f(T) \) theory

As is well known, a varying effective gravitational “constant” is one of the common features of most modified gravity theories \[2\]. Of course, as a modified gravity theory, \( f(T) \) theory is not an exception. In fact, the effective gravitational “constant” of \( f(T) \) theory has been derived in \[30\], namely

\[
G_{\text{eff}} = \frac{G_N}{1 + f_T} .
\]

(21)

Obviously, if \( f(T) \) is a linear function of \( T \), namely \( f(T) = \alpha T \), the effective gravitational constant is just rescaled to be \( G_{\text{eff}} = G_N/(1 + \alpha) \), which is still constant in time. However, in general \( G_{\text{eff}} \) is varying for any non-linear \( f(T) \). From Eq. (21), it is easy to get

\[
\left| \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \right| = \left| \frac{f_{TT} \dot{T}}{1 + f_T} \right| .
\]

(22)

Using Eq. (18) and the universal relation \( \dot{x} = -(1 + z)H(dx/dz) \) for any function \( x \), we have

\[
\dot{T} = -(1 + z) T_0 H \frac{dE^2}{dz} ,
\]

(23)

where \( E^2 \equiv T/T_0 = H^2/H_0^2 \). Substituting \( f_T, f_{TT} \) and Eq. (23) into Eq. (22) and setting \( z = 0 \), the present value of \(|\dot{G}_{\text{eff}}/G_{\text{eff}}| \) is on hand (note that the observational constrain given in Eq. (1) is obtained from the lunar laser ranging experiment, at redshift \( z = 0 \)). It is easy to see that the Hubble constant \( H_0 \) will appear in the final result of \(|\dot{G}_{\text{eff}}/G_{\text{eff}}| \) through \( \dot{T} \) at redshift \( z = 0 \). According to \[32\],

\[
H_0 = 100 \ h \ \text{km/s/Mpc} = 1.02275 \times 10^{-10} \ h \ \text{yr}^{-1} .
\]

(24)

Note that we use the units \( h = c = 1 \) throughout this work. It is easy to see that the Hubble constant \( H_0 \) is a suitable measurement of the present value of \(|\dot{G}_{\text{eff}}/G_{\text{eff}}| \). Very recently, the SHOES (Supernovae and H0 for the Equation of State) Hubble Space Telescope program \[33, 34\] released its latest model-independent measurement of Hubble constant, namely \( h = 0.738 \pm 0.024 \) (with only 3.3% uncertainty) \[34\]. Therefore, we adopt \( h = 0.738 \) in the following.

Before going further, we would like to say some words here. It is worth noting that the \( G_{\text{eff}} \) given in Eq. (21) was obtained by using linear theory \[30\]. However, the Earth-Moon system is far from linearity,
at least concerning the density contrast (we thank the anonymous referee for pointing out this issue). It is well known that via a conformal transformation, \( f(R) \) theory can be equivalent to General Relativity (GR) with a scalar field (which is coupled with matter) in Einstein frame \([13][17]\). In this case, one finds that \( f(R) \) theory can evade the local gravity tests through the so-called chameleon mechanism \([35]\), in which the gravity is effectively screened in the solar system. However, this way is not viable in \( f(T) \) theory unfortunately. In e.g. \([36]\), it is found that \( f(T) \) theory cannot be equivalent to teleparallel action plus a scalar field via a conformal transformation, because an additional scalar-torsion coupling term cannot be removed by the conformal transformation. This point makes \( f(T) \) and \( f(R) \) theories different. As a result, also in \([36]\), it is claimed that the chameleon mechanism might not work in \( f(T) \) theory, and then it might be hard to evade the solar system tests, unlike \( f(R) \) theory. In fact, we will find that \( f(T) \) theory is tightly constrained by the solar system tests in the following.

In the next subsections, we will consider two concrete \( f(T) \) theories, namely, \( f(T) = \mu f(-T)^n \) and \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \), which are the most popular \( f(T) \) theories discussed extensively in the literature (see e.g. \([20][22][25][31]\)).

### B. \( f(T) = \mu f(-T)^n \)

At first, we consider the case of \( f(T) = \mu f(-T)^n \), where \( \mu \) and \( n \) are both constants. This is the simplest model, and has been considered in most papers on \( f(T) \) theory. Obviously, if \( n = 0 \), it reduces to \( \Lambda \)CDM model. Substituting \( f(T) = \mu f(-T)^n \) into the modified Friedmann equation \([19]\), one easily finds that \( \mu \) is not an independent model parameter, namely \([21][22][31]\)

\[
\mu = \frac{1 - \Omega_{m0}}{2n - 1} \left( 6H_0^2 \right)^{1-n} = \frac{1 - \Omega_{m0}}{2n - 1} \left( -T_0 \right)^{1-n},
\]

where \( \Omega_{m0} \equiv \kappa^2 \rho_{m0}/(3H_0^2) \) is the present fractional energy density of dust matter. So, we have

\[
f(T) = \frac{1 - \Omega_{m0}}{2n - 1} \left( -T_0 \right) \left( \frac{T}{T_0} \right)^n,
\]

and then

\[
f_T = \frac{n(1 - \Omega_{m0})}{1 - 2n} E^{2(n-1)}, \quad f_{TT} = \frac{n(1 - \Omega_{m0})}{1 - 2n}, \quad f_{TTT} = \frac{n(n-1)(1 - \Omega_{m0}) E^{2(n-2)}}{1 - 2n}, \quad f_{TTT} = \frac{n(n-1)(1 - \Omega_{m0})}{(1 - 2n)T_0},
\]

where \( E^2 \equiv T/T_0 = H^2/T_0^2 \). Substituting \( f(T) = \mu f(-T)^n \) and Eq. (25) into the modified Friedmann equation \([19]\), we find that \([22][31]\)

\[
E^2 = \Omega_{m0} (1 + z)^3 + (1 - \Omega_{m0}) E^{2n}.
\]

Obviously, if \( n = 0 \), it reduces to the one of \( \Lambda \)CDM model. Differentiating Eq. (29) with respect to redshift \( z \), we obtain

\[
\frac{dE^2}{dz} = \frac{3\Omega_{m0}(1 + z)^2}{1 - n(1 - \Omega_{m0}) E^{2(n-1)}}.
\]
FIG. 1: The viable region in the \( \Omega_{m0} - n \) parameter space which can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), as well as the latest cosmological data (SNIa+BAO+CMB), for the case of \( f(T) = \mu(-T)^n \). See text for details.

Substituting Eqs. (27), (28) and (30) into Eq. (23) and then Eq. (22), it is easy to see that at redshift \( z = 0 \), we have

\[
\left| \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \right| = \left| \frac{3n(n-1)(1 - \Omega_{m0})\Omega_{m0}H_0}{(1-n)^2 - n^2\Omega_{m0}} \right|^2.
\]

Therefore, if the model parameters \( \Omega_{m0} \) and \( n \) are given, we can correspondingly get the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) from Eq. (31). Note that in [22], this \( f(T) = \mu(-T)^n \) theory has been constrained by using the latest cosmological data, i.e., 557 Union2 type Ia supernovae (SNIa) dataset, baryon acoustic oscillation (BAO), and cosmic microwave background (CMB) data from WMAP7. The corresponding 2σ results are given by [22]

\[
\Omega_{m0} = 0.272^{+0.036}_{-0.032}, \quad n = 0.04^{+0.22}_{-0.33}.
\]

At first, we try to see whether the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) with the best-fit parameters of [22] and the corresponding 2σ edge can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), namely, \( |\dot{G}/G| \leq 1.3 \times 10^{-12} \text{ yr}^{-1} \). From Table II it is easy to see that none of them can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1). Also from Table II, we find that for a fixed \( n \), the smaller \( \Omega_{m0} \), the smaller \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) is, whereas \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) is somewhat correlated with the extent of \( n \) deviating from zero. Next, we try to find the viable region in the \( \Omega_{m0} - n \) parameter space which can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), as well as the latest cosmological data (SNIa+BAO+CMB). To this end, we scan the \( \Omega_{m0} - n \) parameter space within the 2σ region of [22], namely \( 0.240 \leq \Omega_{m0} \leq 0.308 \) and \( -0.29 \leq n \leq 0.26 \) (see Eq. [32]), and correspondingly calculate the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) at every scanned \( (\Omega_{m0}, n) \) point. The viable parameter region is determined by \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \leq 1.3 \times 10^{-12} \text{ yr}^{-1} \), and we present it in Fig. 1 which is approximately a trapezoid region. It is easy to see that the allowed \( \Omega_{m0} \) is still unchanged, i.e., \( 0.240 \leq \Omega_{m0} \leq 0.308 \). However, the allowed \( n \) is significantly shrunk from the original \( -0.29 \leq n \leq 0.26 \) of [22] to a very narrow range around zero, namely

\[
-0.032 \lesssim n \lesssim 0.030.
\]
This is the additional constraint on \( f(T) = \mu(-T)^n \) theory from the varying gravitational “constant”. Clearly, this result improves the previous constraint by an order of magnitude.

\[
C. \quad f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right)
\]

In this subsection, we consider the case of \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \), where \( \mu \) and \( \beta \) are both constants. Obviously, when \( \beta \to 0 \) we have \( f(T) \to \mu \beta T_0 = \text{const.} \), so it reduces to \( \Lambda \text{CDM} \) model. Substituting \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \) into the modified Friedmann equation (19), one easily finds that \( \mu \) is not an independent model parameter [21, 22, 31], i.e.,

\[
\mu = \frac{1 - \Omega_{m0}}{1 - (1 - 2\beta) e^\beta}.
\]

It is easy to obtain

\[
f_T = -\mu + \mu \left( 1 - \beta/E^2 \right) e^{\beta/E^2}, \quad f_{T0} = -\mu + \mu (1 - \beta) e^\beta,
\]

\[
f_{TT} = \frac{\mu \beta^2}{T_0 E_0} e^{\beta/E^2}, \quad f_{TT0} = \frac{\mu \beta^2}{T_0} e^\beta,
\]

where \( E^2 \equiv T/T_0 = H^2/H_0^2 \). On the other hand, substituting \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \) into the modified Friedmann equation (19), we find that [22, 31]

\[
E^2 = \Omega_{m0}(1 + z)^3 + \mu E^2 \left[ 1 - e^{\beta/E^2} + 2 \left( \frac{\beta}{E^2} \right) e^{\beta/E^2} \right].
\]

If \( \beta \to 0 \), we have \( \mu \beta \to 1 - \Omega_{m0} \) from Eq. (34), and hence Eq. (37) reduces to the one of \( \Lambda \text{CDM} \) model. Differentiating Eq. (37) with respect to redshift \( z \), we obtain

\[
\frac{dE^2}{dz} = \frac{3(1 + z)^2}{1 + \mu \left( 1 - \beta/E^2 + 2\beta^2/E^4 \right)e^{\beta/E^2} - 1}.
\]

Substituting Eqs. (35), (36) and (38) into Eq. (23) and then Eq. (22), it is easy to see that at redshift \( z = 0 \), we have

\[
\left| \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \right| = \left| \frac{3\beta^2 \Omega_{m0} e^\beta H_0}{1 - \mu + \mu (1 - \beta) e^\beta \left[ 1 - \mu + \mu (1 - \beta + 2\beta^2 e^\beta) \right]} \right|,
\]

where \( \mu \) have been given in Eq. (34). Therefore, if the model parameters \( \Omega_{m0} \) and \( \beta \) are given, we can correspondingly get the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) from Eq. (39). Note that in [22], this \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \) theory has been constrained by using the latest cosmological data, i.e., 557 Union2

| Description | Best fit | 2σ right edge | 2σ left edge | 2σ top edge | 2σ bottom edge |
|-------------|----------|---------------|--------------|-------------|---------------|
| \( \Omega_{m0}, \beta \) | (0.272, -0.02) | (0.308, -0.02) | (0.238, -0.02) | (0.272, 0.29) | (0.272, -0.22) |
| \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) | 0.919465 | 0.988932 | 0.842703 | 9.74454 | 13.8775 |

TABLE II: The present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) (in units of \( 10^{-12} \text{yr}^{-1} \)) with the best-fit parameters of [22] and the corresponding 2σ edge for the case of \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \).
FIG. 2: The viable region in the \( \Omega_{m0} - \beta \) parameter space which can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), as well as the latest cosmological data (SNIa+BAO+CMB), for the case of \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \). See text for details.

Again, we firstly try to see whether the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) with the best-fit parameters of [22] and the corresponding 2\( \sigma \) edge can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), namely, \( |\dot{G}/G| \leq 1.3 \times 10^{-12} \text{yr}^{-1} \). From Table II it is easy to see that the first three points (whose \( \beta \) is close to zero) can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), whereas the last two points (whose \( \beta \) is far away from zero) cannot. Also from Table II we find that for a fixed \( \beta \), the smaller \( \Omega_{m0} \), the smaller \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) is, whereas \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) is also somewhat correlated with the extent of \( \beta \) deviating from zero. Next, we try to find the viable region in the \( \Omega_{m0} - \beta \) parameter space which can satisfy the observational constraint on the varying gravitational “constant” given in Eq. (1), as well as the latest cosmological data (SNIa+BAO+CMB). To this end, we scan the \( \Omega_{m0} - \beta \) parameter space within the 2\( \sigma \) region of [22], namely, \( 0.238 \leq \Omega_{m0} \leq 0.308 \) and \(-0.22 \leq \beta \leq 0.29 \) (see Eq. (41)), and correspondingly calculate the present value of \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \) at every scanned \( (\Omega_{m0}, \beta) \) point. The viable parameter region is determined by \( |\dot{G}_{\text{eff}}/G_{\text{eff}}| \leq 1.3 \times 10^{-12} \text{yr}^{-1} \), and we present it in Fig. 2 which is approximately a trapezoid region. It is easy to see that the allowed \( \Omega_{m0} \) is still unchanged, i.e., \( 0.238 \leq \Omega_{m0} \leq 0.308 \). However, the allowed \( \beta \) is significantly shrunk from the original \(-0.22 \leq \beta \leq 0.29 \) of [22] to a very narrow range around zero, namely

\[ -0.030 \leq \beta \leq 0.033 . \]  

This is the additional constraint on \( f(T) = -\mu T \left( 1 - e^{\beta T_0/T} \right) \) theory from the varying gravitational “constant”. Clearly, this result also improves the previous constraint by an order of magnitude.

IV. CONCLUSION

As is well known, a varying effective gravitational “constant” is one of the common features of most modified gravity theories [2]. Of course, as a modified gravity theory, \( f(T) \) theory is not an exception.
Noting that the observational constraint on the varying gravitational “constant” is very tight, in the present work we tried to constrain \( f(T) \) theories with the varying gravitational “constant”. We found that the allowed model parameter \( n \) or \( \beta \) has been significantly shrunk to a very narrow range around zero. In fact, the results improve the previous constraints by an order of magnitude.

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