On basis images for the digital image representation

Gorbachev V.N., Denisov L.A., Kaynarova E.M., Metelev I.K. 1, Yakovleva E.S. 2

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1High School of Printing and Mediatechnology of St. Petersburg State University of Industrial Technology and Design, St. Petersburg, Russia
2St. Petersburg State University, St. Petersburg, Russia
Abstract

Digital array orthogonal transformations that can be presented as a decomposition over basis items or basis images are considered. The orthogonal transform provides digital data scattering, a process of pixel energy redistributing, that is illustrated with the help of basis images. Data scattering plays important role for applications as image coding and watermarking. We established a simple quantum analogues of basis images. They are representations of quantum operators that describe transition of single particle between its states.

Considering basis images as items of a matrix, we introduced a block matrix that is suitable for orthogonal transforms of multi-dimensional arrays such as block vector, components of which are matrices. We present an orthogonal transform that produces correlation between arrays. Due to correlation new feature of data scattering was found. A presented detection algorithm is an example of how it can be used in frequency domain watermarking.
0.1 Introduction

A digital image has various representations and some of them are required by applications. Many useful representations are produced by orthogonal transforms that are powerful tools of image processing. Well known examples are JPEG and JPEG2000 lossy compression formats based on DCT (Discrete Cosine Transform) and DWT (Discrete Wavelet Transform). For the image compression problem block based DCT and DWT techniques are developed [1] and generalized to non-separable transforms [2].

Orthogonal transform produces scattering of digital data, a process that redistributes pixel energy of transformed image. It is useful for protection the hiding data in steganography, when a message is embedded into image. The hidden data is scattered among all digital cover image and becomes more robust to lossy data compression and some statistical attacks [3].

The orthogonal transform of images may be considered as a decomposition over matrices known as basis matrices [4]. Being some kind of grayscale images, the basis matrices look attractive and they are often reproduced by textbooks [5]. We will also call these matrices basis images.

In this paper we study basis images. We focus on the following questions: color and wavelet basis images, orthogonal transform by matrix of basis images and their quantum analogues. For color images the solution is directly achieved by considering three-dimensional orthogonal transform but for wavelets the solution is not so simple. The reason is that in practice, DWT is calculated by algorithms using signal processing techniques instead of orthogonal transforms. Nevertheless these algorithms can be used to calculate wavelet basis images. So it was found for various wavelets that the basis has a block structure similar to DWT coefficients [6].

Basis images may be considered as items of a matrix. We introduced such a matrix, it is orthogonal and suitable for transforms of multi-dimensional arrays such as a block vectors consisting of matrices. In this case there is a large number of degrees of freedom that may be correlated by the transformation. The correlation results in new features of the orthogonal transform for data scattering.
The orthogonal transform

Indeed, in the standard image transform a given image pixel maps into all pixels of the transformed image. To retrieve it back all the transformed image pixels are required rather than one of them. A new feature is that retrieval can be made from a single pixel only due to correlation between arrays. A detection algorithm illustrates how this feature may be used for the frequency domain watermarking.

The paper is organized as follows. First, the orthogonal transform and the data scattering and basis images are considered. Then a matrix consisting of basis images and the orthogonal transform of multi-dimensional arrays are introduced. Next, an example of the scheme for frequency domain watermarking is presented.

0.2 The orthogonal transform

The orthogonal transform can scatter digital data.

**The orthogonal matrix** A square matrix $U$ of real items is orthogonal if

$$UU^T = 1 \text{ or } U^TU = 1. \quad (1)$$

Columns of this matrix $u_m$ and rows $u^T_n$ are orthonormal vectors

$$\langle u_m u_n \rangle = \delta_{mn},$$
$$\langle u^T_m u^T_n \rangle = \delta_{mn}, \quad (2)$$

where $\langle xy \rangle$ denotes scalar product of two vectors.

**Scattering.** We will study data scattering that can be illustrated by orthogonal transform of vectors.

Let us assume that $f = \{f_k\}, k = 1, \ldots N$ is a vector and $U$ is an $N \times N$ orthogonal matrix. Taking into account that $f = UU^TF$, we find orthogonal transform of vector $f$

$$f = Ug,$$
$$g = U^Tf, \quad (3)$$

where vector $g = \{g_p\}, p = 1, \ldots N$ is often called a representation of $f$. In matrix form these equations look as follows

$$f_k = \sum_p U_{kp}g_p,$$
$$g_p = \sum_k U_{kp}f_k. \quad (4)$$

As a result two points concerning data scattering can be made.

1. Every item of $f$ transforms into all items of $g$ with the weight $U_{kp}f_k$, where $p = 1, 2, \ldots$.

2. To get $f_k$, we need to know all items of $g$.

Let us assume that the data are hidden in $f_k$, for example, by steganography and is distributed among all the digital space of $g$ by an orthogonal transform. The data can be extracted, we need all space of $g$ in spite of every point the data have. Formally the problem is to find $f_k$ for given $g_p$ and the orthogonal
Here and later we will consider data scattering as mapping
\[ f_k \mapsto \{ g_1, g_2, \ldots \}. \]  

It is clear that due to symmetry, the vector \( f \) may be replaced with \( g \). The considered features are true for the orthogonal transforms of matrices and other multi-dimensional arrays.

Data scattering can be directly demonstrated by the orthogonal transform of a set of basic vectors. Let us consider a set of vectors and each of them has a nonzero component \( e_k = \{ \delta_{kn} \}, k, n = 1, \ldots, N \). The vectors are known to be unit vectors and form a standard basis:

\[
\begin{align*}
e_1 &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, & e_2 &= \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, & \ldots e_N &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.
\end{align*}
\]

An orthogonal matrix \( U \) transforms the standard basis into another basis consisting of the columns of \( U \):

\[ u_k = U e_k. \]

This equation shows that a single nonzero item of \( e_k \) distributes among a column \( u_k \)

\[
e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} u_{1k} \\ \vdots \\ u_{kk} \\ \vdots \\ u_{Nk} \end{bmatrix} = u_k.
\]

Since the column has at least two non zero items this transformation can be considered as scattering.

Scattering may result in energy concentration, a process that is important for applications. The array energy, defined as the sum of all components squared, is preserved under orthogonal transforms. Due to scattering, the energy can be distributed into a small amount of components, that is a base of coding in the image compression field. It depends on the orthogonal matrix, regardless of whether the energy would be concentrated or not. It is known that DCT, WHT (Walsh Hadamard Transform) and KLT (Karhunen Loeve Transform) can concentrate the image energy, if image is not random, but DST (Discrete Sine Transform) can’t do it [5].

### 0.3 Basis images

The orthogonal transform of a matrix and three-dimensional array provides decomposition over the grayscale and color basis images.

**Representation of matrix.** Let \( F = \{ F_{mn} \} \) be a real rectangular \( M \times N \) matrix, that corresponds to a grayscale image. We introduce two orthogonal matrices \( U = \{ U_{mn} \} \) and \( V = \{ V_{pk} \} \) of \( M \times M \) and \( N \times N \). Then taking into account, that \( F = U U^T F V V^T \), we find

\[
\begin{align*}
F &= U G V^T, \\
G &= U^T F V.
\end{align*}
\]
Basis images

where $G$ is a $M \times N$ matrix. Let us assume that $F$ is an image in a spatial domain (that is the image as we see it). Matrix $G$ is usually called a frequency representation of $F$ or an image in frequency domain. The frequency domain image may look senseless, however the orthogonal transform is reversible and the original image can always be retrieved.

Using the matrix form of (7), for example,

$$F_{xy} = \sum_{kp} U_{xk} G_{kp} (V^T)_{py} = \sum_{kp} (u_k \otimes v_p)_{xy} G_{kp},$$

we get a decomposition over tensor products of rows and columns of the matrices $U$ and $V$. Here and later we assume $U = V$ and $M = N$ that is a more interesting case. Then the decomposition produced by the orthogonal transformation takes the form

$$F = \sum_{k,p} (u_k \otimes u_p) G_{kp},$$

$$G = \sum_{x,y} (u_x^T \otimes u_y^T) F_{xy}.$$  

(8)

We introduce the matrices

$$a_{kp} = u_k \otimes u_p,$$

$$d_{xy} = u_x^T \otimes u_y^T,$$

that we call basis images. There are $N^2$ basis images of size $N \times N$, every image pixel is a product of two items of the orthogonal matrix $U$

$$a_{kp}(x,y) = U_{xk} U_{yp}.$$  

Color basis images. A color RGB image is a three-dimensional array and similar to matrices it provides a decomposition over basis images. Let $T = \{T_{mnz}\}$ be a three-dimensional array of $M \times N \times Z$. The orthogonal transform of $T$ can be achieved by three orthogonal matrices $U$, $V$ and $W$. The matrices have size $M \times M$, $N \times N$ and $Z \times Z$ respectively. Similarly to (8) the array $T$ can be presented as follows

$$T = \sum_{kps} (u_k \otimes v_p \otimes w_s) t_{kps},$$

(10)

where $w_s$, $s = 1, \ldots, Z$ is a column of the matrix $W$. Tensor products

$$t_{kps} = u_k \otimes v_p \otimes w_s = a_{kp} \otimes w_s$$

produce a basis, the basis items are

$$t_{kps}(m,n,q) = a_{kp}(m,n)w_q.$$  

(11)

In general a three-dimensional array can not be a color image. The color RGB image is described by three matrices $R$, $G$ and $B$ of equal dimensions, say $M \times N$. Matrices are concatenated in a $N \times N \times 3$ array

$$C = cat(3, R, G, B),$$
where \textit{cat} is concatenation. Here we use notation of MATLAB, it means that \(C_{mn1} = R_{mn}\), \(C_{mn2} = G_{mn}\) and \(C_{mn3} = B_{mn}\).

Let us assume that in (10) \(W\) is a matrix of size \(3 \times 3\) and introduce color basis images
\[
\psi_{kps} = \text{cat}(3, r_{kps}, g_{kps}, b_{kps}).
\]
Using (11) we find the color channels \(r_{kps} = a_{kp}w_1s\), \(g_{kps} = a_{kp}w_2s\) and \(b_{kps} = a_{kp}w_3s\). Full basis has \(M \cdot N\) color items. As a result we get the decomposition of RGB images over basis color images
\[
C = \sum_{kps} \text{cat}(3, r_{kps}, g_{kps}, b_{kps})\psi_{kps}.
\]

0.4 \textbf{Properties of basis images}

Being tensor products of orthogonal matrix columns and rows the basis images have properties that follow from orthogonality, and they have a simple analogue came from quantum mechanics.

\textbf{Properties.} Now let us consider the basis images \(a_{kp}\), if \(U = V\), properties of \(d_{xy}\) are the same.

1. The matrix product of two basis images is another basis image
\[
a_{kp} \cdot a_{mn} = a_{kn}\delta_{pm}.
\]

2. The scalar product
\[
\langle a_{kp}, a_{mn} \rangle = \delta_{km}\delta_{pn},
\]
where the scalar product of matrices is \(\langle A, B \rangle = \sum_{mn} A_{mn}B_{mn}\).

3. The sum of diagonal elements, trace
\[
\sum_k a_{kk} = 1,
\]
\[
\sum_x a_{kp}(x, x) = \delta_{kp}.
\]

It follows that \(\sum_k a_{kk}(xy) = \delta_{xy}\).

Analysing these properties we came to the conclusion that basis images are orthonormal. This observation allows us to consider the orthogonal transform (8) as a standard decomposition over the orthonormal basis. Is is obvious that the first equation takes the form
\[
F = \sum_{k,p} a_{kp}G_{kp},
\]
where \(G_{kp} = \langle F, a_{kp} \rangle\).

\textbf{Generation of basis images.} There are at least two ways to get basis images. The first is to use its definitions. In this case the orthogonal matrix has to be given. The second way follows from orthogonal transform of the basis images.
Let us focus on the second approach. Let $F = a_{kp}$ be in equation (14). Then we find the basis image representation of the form $G_{k,p} = \delta_{ka}\delta_{pb}$. It means that the matrix $G$ has one nonzero pixel, it is equal to 1 and its position is $(a,b)$. So, the orthogonal transform of a basis image is a binary matrix of unit brightness. We denote such unit matrix as

$$e_{ab} = \{\delta_{ka}\delta_{pb}\},$$

where $k, p = 1, \ldots, N$. Then the next relations are true

$$a_{ab} = U e_{ab} U^T,$$
$$d_{ab} = U^T e_{ab} U.$$  

These equations are two-dimensional analogue of (6) and they have a simple meaning. So together with the unit vectors $e_k$ the unit matrices $e_{ab}$ form a standard basis and the orthogonal transform of the basis is a set of basis images $a_{ab}$.

Indeed, with the help of the standard basis any matrix can be presented in the following form

$$G = \sum_{kp} G_{kp} e_{kp}.$$ 

Then we get the decomposition given by (14), using the orthogonal transform and taking into account (16).

**Example.** WHT basis images. The $2 \times 2$ orthogonal WHT matrix known also as Hadamard matrix consists of plus 1 and minus 1

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$  

In optics this matrix describes so called 50% beam splitter, a linear optical element often used in experiments to split the beam into two parts. Four basis images $a_{kp}$, denoted as tensor product of columns, have the following form

$$a_{11} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad a_{12} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$
$$a_{21} = \frac{1}{2} \begin{bmatrix} 1 \ -1 \\ -1 & -1 \end{bmatrix}, \quad a_{22} = \frac{1}{2} \begin{bmatrix} 1 \ -1 \\ -1 & 1 \end{bmatrix}.$$ 

The determinant of every matrix equals to 0 and the matrices are non invertable. The matrices can be generated from a unit matrix by WHT:

$$H : \quad e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \stackrel{WHT}{\Rightarrow} \quad \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = a_{11}.$$  

This equation illustrates relations between the basis images and the standard two-dimensional basis. But what is more interesting, the equation demonstrates scattering of digital data (5). So, a nonzero pixel of the unit matrix transforms into a basis images of a matrix with only nonzero pixels.

As a result, basis images can be produced by transformation of unit matrices.

**The quantum analogue.** The presented features allow us to consider basis images as a representation of quantum operators. These operators describe transitions of a physical system between its states.
or levels.

Let us assume that \{\langle k \rangle\} and \{\langle q \rangle\} are two basis of a single particle Hilbert space
\[
\sum_k |k\rangle\langle k| = 1, \\
\sum_q |q\rangle\langle q| = 1,
\]
where \(k \in Z = \{1, 2, \ldots\}, q \in Q = \{x, y, \ldots\}\). Let the overlapping integrals be real
\[
\langle k|q\rangle^* = \langle q|k\rangle.
\] (19)

Then we find a real matrix \(\hat{U}_{qk} = \langle q|k\rangle\) that is orthogonal because \(Z\) and \(Q\) are complete basis. The following operator
\[
|k\rangle\langle p| = \hat{a}_{kp},
\] (20)
where \(k, p \in Z\), describes transition from the state or level \(|p\rangle\) into level \(|k\rangle\). If \(k = p\), this operator is known as projection operator.

Using \(Q\), the introduced operator (20) can be presents as a real matrix
\[
\langle x|\hat{a}_{kp}|y\rangle = a_{kp}(xy),
\]
where \(x, y \in Q\). It is not difficult to understand, that these matrices are basis images, considered above. Using \(Z\) we can present any single particle operator \(\hat{F}\) as follows
\[
\hat{F} = \sum_{kp} |k\rangle\langle p|\langle k|\hat{F}|p\rangle.
\]

Operator \(\hat{F}\) can be written as a matrix using \(Q\) and (19), then the right part of this equation takes the form (14). As result we find that some of representations of single particle operators can be considered as basis grayscale images.

0.5 Basis wavelet images

Basis images can be generated by DWT. In calculation the DWT techniques do not use matrix methods and the basis wavelet images can be achieved by transform of standard basis.

Wavelet coefficients. The DWT coefficients have a block structure due to orthogonal matrix \(U\). In case of single level transform this matrix consists of two parts \(L\) and \(H\) known as low and high frequency blocks. Let \(G\) be a frequency representation of a \(N \times N\) grayscale image \(F = UGU^T\). Applying the MATLAB notation, we write DWT as follows
\[
G = \text{dwt}(F) = \begin{bmatrix} cA & cH \\ cV & cD \end{bmatrix}, \quad (21)
\]
\[
F = \text{idwt}(cA, cH, cV, cD).
\]

Here the introduced blocks \(cA, cH, cV\) and \(cD\) — are approximation coefficients, horizontal, vertical and diagonal details or \(LL, LH, HL\) and \(HH\) frequency bands.
The DWT coefficient matrix $G$ can be considered as a three-dimensional array $G = \{G_{kpz}\}$ of size $N/2 \times N/2 \times 4$. Index $z = 1, 2, 3, 4$ labels the $cA$, $cH$, $cV$ and $cD$ blocks, for example, $G_{kbp_1} = cA_{kp}$.

**Block structure of basis and basis images.** To calculate basis images we use equation (16)

$$a_{kp} = U e_{kp} U^T.$$  

According to (21) indexes $(k, p)$ belong to one of the blocks $cA$, $cH$, $cV$ or $cD$. Let $(k, p) \in cD$, so there is a set of basis items

$$E_{(kpD)} = \text{idwt}(O, O, O, e_{kp}),$$

where $O$ — is a $N/2 \times N/2$ matrix of zeros. Here the upper indexes are in brackets to label number of the matrices instead of indicating the pixel position. In other words, we perform an orthogonal transformation of the unit block matrix

$$E_{(kpD)} \mapsto \begin{bmatrix} O & O \\ O & e_{kp} \end{bmatrix}.$$  

The total number of basis images of $E_D = \{E_{(kpD)}\}$ is $N^2/4$, every image is a $N \times N$ matrix. It is important to note that the equation (16) gives solution by Matlab functions `dwt` and `idwt`. The reason is that in practice the DWT calculations are often based on the filter function techniques [9]. These techniques were developed for signal processing without referring to the orthogonal matrix $U$. Usually wavelets are introduced numerically or by recurrent equations so the calculation of $U$ is a problem (except, for example, the Haar wavelet).

Using the block coefficients $cD$ and $E_{kpD}$ we can achieve an approximation of original image

$$D = \sum_{kp} cD_{kp} E_{(kpD)},$$

This image has diagonal details only.

The wavelet coefficient structure results in basis of four blocks. The blocks refer to $cA$, $cH$, $cV$ and $cD$ similarly to (22)

$$\{\{E_A\}, \{E_H\}, \{E_V\}, \{E_D\}\}.$$

Every block has $N^2/4$ basis $N \times N$ images. As a result the representation over the wavelet basis images looks as follows

$$F = \sum_{kp} \left(cA_{kp} E_{(kpA)} + cH_{kp} E_{(kpH)} + cV_{kp} E_{(kpV)} + cD_{kp} E_{(kpD)}\right).$$

Indeed, the considered above function `dwt` can produce another basis. For this case in accordance with (21) every basis images has a block structure

$$J_{(xy)} = dwt(e_{xy}) = \begin{bmatrix} J_{(xyA)} \\ J_{(xyV)} \end{bmatrix}.$$

$$J_{(xy)} = \begin{bmatrix} J_{(xyA)} & J_{(xyH)} \\ J_{(xyV)} & J_{(xyD)} \end{bmatrix},$$
0.6 A block matrix

Basis images may be items of a matrix that can be orthogonal.

**A matrix of basis images.** Consider a square $N \times N$ matrix, which elements are basis images

$$b = \{b_{mn}\},$$

$$b_{mn} = a_{i_{mn}}.$$  \hspace{1cm} (23)

Elements of $b$ do not commute. The introduced matrix is a four-dimensional array, consisting of $(N \times N) \times (N \times N)$ elements

$$K_{kpxy} = a_{kp}(x,y) = U_{xk}U_{yp}.$$  \hspace{1cm}

The matrix $b$ has the following important feature:

$$bb = 1.$$  \hspace{1cm} (24)

So considering the matrix elements we find

$$(bb)_{mn} = \sum_k b_{mk}b_{kn} = \sum_k a_{km}a_{nk} = \delta_{mn} \sum_k a_{kk} = \delta_{mn}.$$  \hspace{1cm}

Indeed, matrix $\beta$, which elements are basis images, $\beta_{mn} = a_{mn}$, doesn’t have the property given by (24). In this case $\beta \beta = N \beta$.

The **biorthogonal decomposition.** The equation (24) tells that the matrix $b$ has rows orthogonal to columns

$$\langle b^T_{mn}b_{nk}\rangle = \delta_{mn}.$$  \hspace{1cm} (25)

However, the rows are not orthogonal vectors themselves and similarly to columns. An orthonormal basis is obtained from rows and columns. The basis is known to be biorthogonal or biorthonormal [8] and it can be used to represent digital arrays.

Let us consider a vector $f = \{f_k\}, k = 1, \ldots, N$. Using (24), we find

$$f = bb f = bg,$$

$$g = bf,$$  \hspace{1cm} (26)

where the introduced vector $g = \{g_p\}, p = 1, \ldots, N$ is a representation of $f$. To focus on the particular feature of transform (26), we introduce decomposition of vectors $f$ and $g$ over columns of matrix $b$

$$f = \sum_k b_k g_k,$$

In contrast to orthogonal transform, the coefficients $g_k$ are denoted by rows but not by columns

$$g_k = \langle b_k^T, f\rangle.$$  \hspace{1cm}

That is a biorthogonal decomposition.

The biorthogonal decompositions are applied in the wavelet field. So, to perform **dwt** (21) and inverse transform **idwt**, we need two different wavelets. An example is the Cohen-Daubechies-Feauveau wavelet.
or biorthogonal 9/7 wavelet that is used in JPEG 2000.

**Orthogonality.** Is the matrix $b$ orthogonal? The answer is not clear because $b$ is a four-dimensional array. However, we can refer to the array primitives and consider rows and columns consisting of the rows and columns of the basis images. Let $r_k$ be a block row. It has items $b_{k1}, b_{k2}, \ldots, b_{kN}$ of basis images $a_{1k}, a_{2k}, \ldots, a_{Nk}$. Selecting a row $x$ of every basis image, we get a row $r_{kx}$. This may be done for a column as well. Introduced rows and columns will be orthonormal vectors. This is a reason to consider the block matrix $b$ as an orthogonal matrix.

Indeed, this result follows from the definition of the transposing operation. In case of block matrix $Z$ it can be presented as follows

$$Z^T = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z^T_{11} & Z^T_{21} \\ Z^T_{12} & Z^T_{22} \end{bmatrix}.$$

### 0.7 The block based representation

The orthogonal matrix of basis images provides a block based representation of multi-dimensional arrays.

**Representation.** The block based representation follows from the equations (26), if they are written in the matrix form

$$f_k = \sum_k a_{pk}g_p,$$
$$g_p = \sum_k a_{kp}f_k.$$  \hspace{1cm} (27)

Here $f$ and $g$ are two block vectors

$$f = (f_1, f_2, \ldots, f_N),$$
$$g = (g_1, g_2, \ldots, g_N).$$

items of which may be chosen as vectors, matrices etc.

Let us assume that $f_k = \{f_k(x, y)\}$ and $g_p = \{g_p(x, y)\}$ are $N \times Q$ matrices. Restrictions on $Q$ will be established later. For this case the equations (27) take the following form

$$f_k(x, y) = \sum_{p, z} a_{pk}(x, z)g_p(z, y),$$
$$g_p(x, y) = \sum_{k, z} a_{kp}(x, z)f_k(z, y).$$  \hspace{1cm} (28)

It is important to notice that index $y$ plays minor role in these equations and from the formal point of view it is unnecessary. It means that $f_k$ and $g_p$ have to be not less than one-dimensional arrays. Then for considered matrices we find the following condition $Q \geq 1$.

The unnecessary index indicates that there is more space to which matrices $a_{kp}$ do not belong. From the physical point of view we have two systems, for example, atoms and light. Both systems are described by its observations that can be represented by matrices that, however, affect its Hilbert spaces. To describe elements of different spaces, e.g. two matrices $A$ and $B$ a tensor product is introduced $A \otimes B$. 
0.8 Non separability and scattering

The block based representation leads to new features of data scattering and has a quantum analogue.

**Correlation.** Formally, the block based representation looks as one-dimensional transform and we find properties given by for data scattering. However, due to large number of degrees of freedom, scattering obtains new features.

Let us assume that both arrays $f_k$ and $g_p$ are block matrices consisting of other matrices. They are four-dimensional arrays that we specify by four indexes $(x, y, \alpha, \beta)$. Let, in contrast to $g_p$, the array $f_k$ be dependant on the last pair of indexes only

$$f_k = \delta_{xy} \psi_k(\alpha, \beta),$$
$$g_p = g_p(x, y, \alpha, \beta).$$

Under these conditions the equations take the following form

$$1 \otimes \psi_k = \sum_p (a_{pk} \otimes 1) g_p,$$
$$g_p = \sum_k a_{kp} \otimes \psi_k,$$

(29)

where $1 \otimes \psi_k = f_k$.

An important fact follows that the array $g_p$ is non-separable. We will use the term separable as divisibility, when the variables are factorized. For example, the function $F(x, y) = \cos(x) \cos(y)$ is separable over $x$ and $y$ and the function $\Phi(x, y) = \cos(x + y)$ is not. In our case we focus on two pairs of variables, a pair $x, y$, that describe basis images $a_{pk}(x, y)$, and pair $\alpha, \beta$. From this point of view, the array $f_k$ is separable in contrast to $g_p$, that is a non-separable array, because it is a sum of products

$$g_p(x, y, \alpha, \beta) = a_{1p}(x, y)\psi_1(\alpha, \beta) + a_{2p}(x, y)\psi_2(\alpha, \beta) + \ldots$$

(30)

Non separability is a kind of correlation. Now this is a correlation between the matrices from different spaces, the basis images and the matrices $\psi_k$.

**Scattering.** Due to the property of the scalar product of basis matrices, we find that

$$\sum_{xy} a_{kp}(x, y) g_p(x, y, \alpha, \beta) = \psi_k(\alpha, \beta)$$

or

$$\langle a_{kp}, g_p \rangle = \psi_k.$$

(31)

For data scattering this result tells us the following. The component $\psi_k$ scatters into every $g_p$ with its weight $a_{kp}$ and it may be established from every item $g_p$

$$\psi_k \rightarrow \{g_1, g_2, \ldots, \},$$
$$\psi_k \leftarrow g_p.$$  

This is a new property and it is usually impossible. The property arises from non separability produced by orthogonal transform of the block matrix $b$. The transform results in correlation between the set of
Non separability and scattering

basis images and the input matrices.

**A quantum analogue.** The block-based representation can be introduced for a three particle quantum system.

Let us consider a three particle operator given by

\[ \hat{c} = \sum_{k,p} |k\rangle\langle p| \otimes |p\rangle\langle k| \otimes 1, \]

where \( k, p \in \mathbb{Z} = \{1, 2, \ldots \} \) and \( \{|k\}\) is a single particle basis. The operator \( \hat{c} \) is Hermitian and unitary

\[ \hat{c} = \hat{c}^\dagger, \]
\[ \hat{c}\hat{c} = 1. \]

Let us note that two particle operator

\[ \hat{b} = \sum_{k,p} |k\rangle\langle p| \otimes |k\rangle\langle p| = \hat{b}^\dagger \]

is a quantum analogue of the matrix \( b \), given by (23).

Let us introduce three particle operators \( \hat{f} \) and \( \hat{g} \) that are equal up to orthogonal transform given by \( \hat{c} \)

\[ \hat{f} = \hat{c}\hat{g}, \]
\[ \hat{g} = \hat{c}\hat{f}. \]

These equations can be written in a block form. Introducing the matrix elements over particle 1 for operators \( \hat{f} \) and \( \hat{g} \) way we get two operators of particle 2 and 3, which we denote as

\[ 1\langle k|\hat{f}|m\rangle_1 = \hat{f}_k, \]
\[ 1\langle p|\hat{g}|m\rangle_1 = \hat{g}_p, \]

where \( k, p, m \in \mathbb{Z} \). Then we have the block representation

\[ \hat{f}_k = \sum_p (\hat{a}_{pk} \otimes 1)\hat{g}_p, \]
\[ \hat{g}_p = \sum_k (\hat{a}_{kp} \otimes 1)\hat{f}_k. \]

Let \( \hat{f}_k \) be the operator of particle 3 only, \( \hat{f}_k = 1 \otimes \hat{\psi}_k \), then we find \( \hat{f}_k \) being a two particle non-separable operator

\[ \hat{g}_p = a_{1p} \otimes \hat{\psi}_1 + a_{2p} \otimes \hat{\psi}_2 \ldots \]

This equation is a quantum analogue of (31) found for digital data scattering. It is obvious, that

\[ Sp_2 (a_{kp}\hat{g}_p) = \hat{\psi}_k, \]

where the average refers to particle 2.
### 0.9 A steganographic scheme

The block-based representation may be useful for frequency domain steganographic technique.

**Scheme.** Let the digital data $f$ be images in a spatial domain and $g$ be its representation in a frequency domain. Any standard frequency embedding scheme has the following steps.

- Transform data into the frequency domain $f \rightarrow g$ and embed a message $M$ using an algorithm $g \rightarrow g_M = emb(g, M, K)$, where $K$ is a set of parameters with a possible secret key.
- Transform data into the spatial domain $g_M \rightarrow f_M$ and send it to a receiver via the communication channel.
- Extract the embedded message using detection algorithm $f_M \rightarrow M = det(f_M, K)$.

The scheme includes transformations

$$ f \rightarrow g \rightarrow g_M \rightarrow f_M \rightarrow g_M \rightarrow M. $$

Indeed, the transform $g_M \rightarrow f_M$ can scatter the embedded data among the spatial domain. Scattering may result in more robust of hidden data to degradation due to various transformations. An example is a JPEG lossy compression, that stores image in a graphical format. By decreasing the image redundancy, the lossy compression introduces changes into embedded data that exploits the redundancy. So, there is a trade between the compression and the quality of the extracted information. The higher the compression level is, the worse the quality is.

**Data scattering in the spatial domain.** Let us consider data scattering in the block based representation (27), assuming $k, p = 1, 2$. Let a message be embedded into $g_2 \rightarrow g_{2M}$. Then two spatial items will be changed

$$ f_1 \rightarrow f_{1M} = a_{11}f_1 + a_{21}g_{2M}, $$
$$ f_2 \rightarrow f_{2M} = a_{12}f_2 + a_{22}g_{2M}. $$

To extract the message, we need $g_{2M}$ or two items $f_{1M}$ and $f_{2M}$

$$ g_{2M} = a_{12}f_{1M} + a_{22}f_{2M}. $$

The equation is a basis for the detection algorithm

$$ det(f_{1M}, f_{2M}, K) \rightarrow M. $$

Data scattering means that all spatial items were changed after embedding and all items are required for detection. Any frequency domain watermarking technique has these properties regardless of whether it use the block based representation or not. However, the representation leads to new features appearance.

**Embedding.** Let us assume that both vectors $f$ and $g$ have two components. Also $f_k = 1 \otimes \psi_k$, where $\psi_k$ is an image in the spatial domain, $k = 1, 2$. In accordance to (29), the frequency representation $g$ consists of pair of four-dimensional arrays

$$ g_1 = a_{11} \otimes \psi_1 + a_{21} \otimes \psi_2, $$
$$ g_2 = a_{12} \otimes \psi_1 + a_{22} \otimes \psi_2. $$
Let the embedding algorithm replace $\psi_k$ with messages
\begin{align*}
g_1 \to g_1M &= a_{11} \otimes M_1 + a_{21} \otimes M_2, \\
g_2 \to g_2M &= a_{12} \otimes M_3 + a_{22} \otimes M_4,
\end{align*}
where four matrices $M_1, \ldots, M_4$ are introduced messages. The main feature of this algorithm is store the ability to the structure of the array that holds a set of tensor products including basis images. In the spatial domain we have
\begin{align*}
f_1M &= a_{11} \otimes M_1 + a_{22} \otimes M_3, \\
f_2M &= a_{11} \otimes M_2 + a_{22} \otimes M_4.
\end{align*}
This allows us to exploit the equation (31) for detection. Then the embedded messages can be extracted, if the component $f_1M$ or the component $f_2M$ is given
\begin{equation}
f_1M \to M_1 = \langle a_{11}, f_1M \rangle.
\end{equation}
For this case the detection algorithm works as follows
\begin{align*}
det(f_1M, K) &\to M_1, M_3, \\
det(f_2M, K) &\to M_2, M_4.
\end{align*}
Let us note that it differs from the standard algorithm \[33\] that needs two spatial items instead of one. Moreover there is a difference between this and the spatial domain embedding. We assume that message is embedded into spatial components $\psi_k \to \psi_{kM}, k = 1, 2$. It is clear that two messages can be embedded only. In the frequency domain there are four messages that may be embedded. But what is more important, these four messages can be distinguished. This fact plays a key role in detection and arises from coupling the messages and basis images to be orthogonal and hence to be well distinguished.

Taking into account the considered quantum analogues, we admit that the presented scheme can be extended to quantum mechanic fields.

\section{Conclusions}

1. Orthogonal transform provides decomposition over basis items or basis images that have a simple quantum analogue. So, they are a representation of single particle operators that describe transitions of a particle between its states.

2. Grayscale, color and wavelet basis images can be introduced for decomposition of two- and three-dimensional arrays.

3. Basis images can be achieved by orthogonal transform of a standard basis that is a set of unit vectors, unit matrices and etc. This fact illustrates digital data scattering, a process of redistributing pixel energy.

4. Due to scattering, energy can be concentrated in small amount of items or, in contrast, be spread. Both cases are interesting for applications. For example, in lossy compression scattering allows to extract the image redundancy, in watermarking it can increase the robustness of a watermark.
5. A block matrix of basis images may be orthogonal and suitable for transformation of multi-
dimensional arrays. Different degrees of freedom can be correlated by this transform and non
separable arrays can be produced. As a result, in this way, new features of scattering appears.
These features may be used whole executing detection algorithms in frequency domain watermark-
ing.
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