Optimal radio labelings of graphs

Devsi Bantva*
Lukhdhirji Engineering College, Morvi - 363 642
Gujarat (India)

Abstract

Let \( N \) be the set of positive integers. A radio labeling of a graph \( G \) is a mapping \( \varphi : V(G) \to N \cup \{0\} \) such that the inequality \( |\varphi(u) - \varphi(v)| \geq \text{diam}(G) + 1 - d(u, v) \) holds for every pair of distinct vertices \( u, v \) of \( G \), where \( \text{diam}(G) \) and \( d(u, v) \) are the diameter of \( G \) and distance between \( u \) and \( v \) in \( G \), respectively. The radio number \( \text{rn}(G) \) of \( G \) is the smallest number \( k \) such that \( G \) has radio labeling \( \varphi \) with \( \max\{\varphi(v) : v \in V(G)\} = k \). Das et al. [Discrete Math. 340(2017) 855–861] gave a technique to find a lower bound for the radio number of graphs. In [Algorithms and Discrete Applied Mathematics: CALDAM 2019, Lecture Notes in Computer Science 11394, springer, Cham, 2019, 161–173], Bantva modified this technique for finding an improved lower bound on the radio number of graphs and gave a necessary and sufficient condition to achieve the improved lower bound. In this paper, one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graphs is given. Using this result, the radio number of the Cartesian product of a path and a wheel graphs is determined.

Keywords: Radio labeling, radio number, wheel graph, Cartesian product of graphs.

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1 Introduction

The channel assignment problem is the problem to assign channels to each TV or radio transmitters such that the interference constraints are satisfied and the use of spectrum is minimized. The problem was first introduced by Hale [11] in 1980. The interference between transmitters is closely related to geographic location of the transmitters. The closer the transmitters are, the higher the interference is and vice-versa. Hence, the frequency difference between two radio channels assigned to radio transmitters is in the inverse proportion to the distance between two transmitters. Initially only two level interference, namely high and low, was considered and accordingly, two transmitters are called very close and close, respectively. In a private communication with Griggs during 1988, Robert proposed a variation of the channel assignment problem in which close transmitters must receive different channels and very close transmitters must receive channels that are at least two apart. The problem is studied by mathematicians using graphs labeling approach.

In a graph, the transmitters are represented by vertices and two vertices are adjacent if two transmitters are very close and distance two apart if they are close. The problem of assignment of channels to transmitters is associated with graph labeling problem. Motivated through this problem, Griggs and Yeh introduced \( L(2,1) \)-labeling (or distance two labeling) in [9] as follows:

*E-mail : devsi.bantva@gmail.com (Devsi Bantva)
An \(L(2,1)\)-labeling of a graph \(G = (V(G), E(G))\) is a function \(\varphi\) from the vertex set \(V(G)\) to the set of non-negative integers such that \(|\varphi(u) - \varphi(v)| \geq 2\) if \(d(u, v) = 1\) and \(|\varphi(u) - \varphi(v)| \geq 1\) if \(d(u, v) = 2\), where \(d(u, v)\) is the distance between \(u\) and \(v\) in \(G\). The span of \(\varphi\) is defined as \(\text{span}(\varphi) = \max\{|\varphi(u) - \varphi(v)| : u, v \in V(G)\}\). The \(\lambda\)-number, denoted by \(\lambda(G)\), is defined as the minimum span over all \(L(2,1)\)-labelings of \(G\). The \(L(2,1)\)-labeling and other distance two labeling problems have been studied by many researchers in the past two and half decades; for example, see the survey articles \([4, 20]\).

In \([5, 6]\), Chartrand et al. extended the constraint on distance from two to the largest possible distance; the diameter of graph \(G\) and introduced the concept of radio labeling as follows:

**Definition 1.1.** A radio labeling of a graph \(G\) is a mapping \(\varphi : V(G) \to \mathbb{N} \cup \{0\}\) (\(\mathbb{N}\) is the set of natural numbers) such that the following is satisfied for every pair of distinct vertices \(u, v \in V(G)\),

\[
|\varphi(u) - \varphi(v)| \geq \text{diam}(G) + 1 - d(u, v). \tag{1}
\]

The assigned integer \(\varphi(u)\) is called the label of \(u\) under \(\varphi\) and, the span of \(\varphi\) is defined as

\[
\text{span}(\varphi) = \max\{|\varphi(u) - \varphi(v)| : u, v \in V(G)\}.
\]

The radio number of \(G\), denoted by \(\text{rn}(G)\), is defined as

\[
\text{rn}(G) := \min_{\varphi} \{\text{span}(\varphi)\}
\]

with minimum taken over all radio labelings \(\varphi\) of \(G\). A radio labeling \(\varphi\) is optimal if \(\text{span}(\varphi) = \text{rn}(G)\).

It is clear that an optimal radio labeling \(\varphi\) always assign 0 to some vertex and hence the span of \(\varphi\) is the maximum integer assigned by \(\varphi\). A radio labeling is a one-to-one integral function from \(V(G)\) to the set of non-negative integers. Therefore, any radio labeling \(\varphi\) induces an ordering \(O_{\varphi}(V(G)) := (x_0, x_1, \ldots, x_{p-1})\) of \(V(G)\) such that \(0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \text{span}(\varphi)\), where \(p = |V(G)|\). It is clear that if \(\varphi\) is an optimal radio labeling then \(\text{span}(\varphi) \leq \text{span}(\psi)\) for any other radio labeling \(\psi\) of \(G\).

A radio labeling problem is recognized as one of the tough graph labeling problems. In \([5, 6]\), Chartrand et al. gave an upper bound for the radio number of paths and cycles. Liu and Zhu determined the exact radio number for paths and cycles in \([15]\). Even determining the radio number for basic graph families like paths and cycles was challenging. In \([16, 17, 18]\), Vaidya and Bantva determine the radio number for total graph of paths, strong product \(P_2\) with \(P_n\) and linear cacti. The radio number of trees remain the focus of many researchers in recent years. In \([10]\), Halász and Tuza determine the radio number of level-wise regular trees. In \([13]\), Li et al. determine the radio number of complete \(m\)-ary trees. In \([14]\), Liu gave a lower bound for the radio number of trees and, a necessary and sufficient condition to achieve the lower bound; the author presented a class of trees, namely spiders, achieving this lower bound. In \([3]\), Bantva et al. gave a different necessary and sufficient condition to achieve this lower bound and presented banana trees, firecrackers trees and a special class of caterpillars achieving this lower bound. Recently, Bantva and Liu gave a lower bound for the radio number of block graphs and three
necessary and sufficient condition to achieve the lower bound in [2]. They also discussed the radio number of line graph of trees and block graphs. Liu et al. also studied the radio \( k \)-labeling of trees in [7]. In [8], Das et al. gave a technique to find a lower bound for the radio number of graphs. In [1], Bantva improved this technique to find a lower bound for the radio number of graphs and gave a necessary and sufficient condition to achieve the improved lower bound. Using these results, the author determined the radio number of the Cartesian product of paths and Peterson graph.

In this paper, one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graphs given in [1] is established. Some subgraphs of a given graph \( G \) are characterized such that if the radio number of \( G \) achieves the lower bound given in [1] then these subgraphs also achieve the lower bound. Using these results, the radio number of the Cartesian product of the path graphs with wheel, star and the friendship graphs are determined.

2 Preliminaries

The book [19] is followed for standard graph theoretic terms and notation. Only simple finite connected graphs are considered throughout this paper. The distance \( d_G(u, v) \) between two vertices \( u \) and \( v \) is the least length of a path joining \( u \) and \( v \) in a graph \( G \). The suffix is dropped whenever the graph \( G \) is clear in the context. The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is \( \max\{d_G(u, v) : u, v \in V(G)\} \). The neighborhood of \( v \in G \), written as \( N(v) \), is the set of vertices adjacent to \( v \). Let \( S \subseteq V(G) \). Define \( N(S) = \{u \in V(G) : u \text{ is adjacent to } v \in S\} \) and \( \text{diam}(S) = \max\{d_G(u, v) : u, v \in S\} \). It is clear that if \( |S| = 1 \) then \( \text{diam}(S) = 0 \).

Let \( H \) be an induced connected subgraph of \( G \) with \( \text{diam}(H) = k \). Define layers \( L_i \) of graph \( G \) with respect to subgraph \( H \) as follows: Set \( L_0 = V(H) \) and \( L_1 = N(L_0) \). Recursively define \( L_{i+1} = N(L_i) \) for \( 1 \leq i \leq h - 1 \), where

\[
L(G) := \sum_{i=1}^{h} |L_i|i.
\]

For a graph \( G \), define

\[
\delta(G) = \begin{cases} 
1, & \text{if } |L_0| = 1; \\
0, & \text{if } |L_0| \geq 2.
\end{cases}
\]

Let \( G \) be any connected graph then for any \( u, v \in V(G) \), note that the distance between \( u \) and \( v \) in a graph \( G \) satisfies the following inequality:

\[
d(u, v) \leq d(u, L_0) + d(v, L_0) + \text{diam}(L_0).
\]
In [8], Das et al. gave a technique to find a lower bound for the radio number of graphs. In [1], Bantva improved this technique and gave a lower bound for the radio number of graphs which is given in the following theorem.

**Theorem 2.1.** [1] Let \( G \) be a simple connected graph of order \( p \), diameter \( d \) and \( L_0 \subseteq V(G) \). Denote \( k = \text{diam}(L_0) \) and \( \delta = \delta(G) \). Then

\[
\text{rn}(G) \geq (p - 1)(d - k + 1) + \delta - 2L(G).
\]

(3)

Although, both the lower bounds given in [8] and [1] seems to be identical in notation but the difference lies in fixing the \( L_0 \). In [8], Das et al. set a vertex or a clique of graph \( G \) as \( L_0 \) while Bantva set all vertices of an induced subgraph \( H \) of \( G \) as \( L_0 \) with the property that two non-adjacent vertices of \( V(H) \) have distance equal to \( \text{diam}(L_0) \). The readers may notice that this improved technique gives a better lower bound for the radio number of graphs, which is sharp for some classes of graph. The author of [1] presented one such class of graphs, which consists of the Cartesian product of the path graph and Peterson graph in [1]. In this paper, the condition to fix \( L_0 \) is further relaxed as follows. Set \( L_0 = V(H) \), where \( H \) is a connected induced subgraph of \( G \) with the property that the vertices of \( G \) can be ordered as \( x_0, x_1, \ldots, x_{p-1} \) such that \( d(x_i, x_{i+1}) = d(x_i, L_0) + d(x_{i+1}, L_0) + \text{diam}(L_0) \) for \( 0 \leq i \leq p - 2 \).

In [1], Bantva also gave a necessary and sufficient condition (given in the next theorem) to achieve the lower bound (3) for the radio number of graphs.

**Theorem 2.2.** [1] Let \( G \) be a simple connected graph of order \( p \), diameter \( d \) and \( L_0 \) is as described earlier. Denote \( k = \text{diam}(L_0) \) and \( \delta = \delta(G) \). Then

\[
\text{rn}(G) = (p - 1)(d - k + 1) + \delta - 2L(G)
\]

(4)

holds if and only if there exists a radio labeling \( \varphi \) with \( 0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \text{span}(\varphi) = \text{rn}(G) \) such that all the following hold for \( 0 \leq i \leq p - 1 \):

(a) \( d(x_i, x_{i+1}) = d(x_i, L_0) + d(x_{i+1}, L_0) + k \),

(b) \( x_0, x_{p-1} \in L_0 \) if \( |L_0| \geq 2 \) and \( x_0 \in L_0 \), \( x_{p-1} \in L_1 \) if \( |L_0| = 1 \),

(c) \( \varphi(x_0) = 0 \) and \( \varphi(x_{i+1}) = \varphi(x_i) + d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k \).

3 Main Result

In this Section, we give one more useful necessary and sufficient condition to achieve the improved lower bound for the radio number of graph given in [1], which rely only on the ordering of vertices of a graph.

**Theorem 3.1.** Let \( G \) be a simple connected graph of order \( p \), diameter \( d \geq 2 \) and \( L_0 \) is fixed in \( G \) as described earlier. Denote \( k = \text{diam}(L_0) \) and \( \delta = \delta(G) \). Then

\[
\text{rn}(G) = (p - 1)(d - k + 1) + \delta - 2L(G)
\]

(5)

holds if and only if there exists an ordering \( O(V(G)) := (x_0, x_1, \ldots, x_{p-1}) \) of \( V(G) \) such that the following conditions are satisfy.
(a) \( d(x_0, L_0) + d(x_{p-1}, L_0) = 1 \) if \( |L_0| = 1 \) and \( d(x_0, L_0) + d(x_{p-1}, L_0) = 0 \) if \( |L_0| \geq 2 \); 

(b) the distance between two vertices \( x_i \) and \( x_j \) (\( 0 \leq i < j \leq p - 1 \)) satisfy

\[
d(x_i, x_j) \geq \sum_{t=i}^{j-1} (d(x_t, L_0) + d(x_{t+1}, L_0) + k - d - 1) + d + 1. \tag{6}
\]

Moreover, under the conditions (a) and (b), the mapping \( \varphi \) defined by

\[
\varphi(x_0) = 0, \tag{7}
\]

\[
\varphi(x_{i+1}) = \varphi(x_i) + d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k, \; 0 \leq i \leq p - 2 \tag{8}
\]

is an optimal radio labeling of \( G \).

**Proof.** Necessity: Suppose that (5) holds then there exists an optimal radio labeling \( \varphi \) of \( G \) which induces an ordering \( O_\varphi(V(G)) := (x_0, x_1, \ldots, x_{p-1}) \) of \( V(G) \) with \( 0 = \varphi(x_0) < \varphi(x_1) < \ldots < \varphi(x_{p-1}) = \text{span}(\varphi) = \text{rn}(G) \) such that the conditions (a)-(c) of Theorem 2.2 hold. By Theorem 2.2 (b), it is clear that \( d(x_0, L_0) + d(x_{p-1}, L_0) = 1 \) when \( |L_0| = 1 \) and \( d(x_0, L_0) + d(x_{p-1}, L_0) = 0 \) when \( |L_0| \geq 2 \). By Theorem 2.2 (c), for any two vertices \( x_i \) and \( x_j \) (\( j > i \)) in ordering \( O_\varphi(V(G)) := (x_0, x_1, \ldots, x_{p-1}) \) of \( V(G) \), we obtain

\[
\varphi(x_j) - \varphi(x_i) = \sum_{t=i}^{j-1} (d + 1 - d(x_t, L_0) - d(x_{t+1}, L_0) - k).
\]

Note that \( \varphi \) is a radio labeling of \( G \) and so \( \varphi(x_j) - \varphi(x_i) \geq d + 1 - d(x_i, x_j) \). Substituting this in above equation, we obtain

\[
d(x_i, x_j) \geq \sum_{t=i}^{j-1} (d(x_t, L_0) + d(x_{t+1}, L_0) + k - d - 1) + d + 1. \tag{9}
\]

Sufficiency: Suppose that an ordering \( O(V(G)) := (x_0, x_1, \ldots, x_{p-1}) \) of \( V(G) \) satisfies conditions (a)-(b) of hypothesis and \( \varphi \) is defined by (7) and (8). Note that it is enough to prove that \( \varphi \) is a radio labeling with span equal to the right-hand side of (5). Let \( x_i \) and \( x_j \) (\( 0 \leq i < j \leq p - 1 \)) be two arbitrary vertices then by (8) and using (9), we have

\[
\varphi(x_j) - \varphi(x_i) = (j - i)(d + 1) - \sum_{t=i}^{j-1} (d(x_t, L_0) + d(x_{t+1}, L_0) + k)
\]

\[
\geq d + 1 - d(x_i, x_j)
\]

and hence \( \varphi \) is a radio labeling. The span of \( \varphi \) is given by

\[
\text{span}(\varphi) = \varphi(x_{p-1}) - \varphi(x_0)
\]

\[
= \sum_{i=0}^{p-2} (\varphi(x_{i+1}) - \varphi(x_i))
\]
\[
\sum_{i=0}^{p-2} (d + 1 - d(x_i, L_0) - d(x_{i+1}, L_0) - k)
\]
\[
= (p - 1)(d + 1) - \sum_{i=0}^{p-2} (d(x_i, L_0) + d(x_{i+1}, L_0) + k)
\]
\[
= (p - 1)(d - k + 1) - 2L(G) + d(x_0, L_0) + d(x_{p-1}, L_0)
\]
\[
= (p - 1)(d - k + 1) + \delta - 2L(G).
\]

Therefore, \(\text{rn}(G) \leq (p - 1)(d - k + 1) + \delta - 2L(G)\). This together with (3) implies (5).

A graph with no cycle is called an acyclic graph. A forest is an acyclic graph. A tree is a connected acyclic graph. A spanning subgraph of a graph \(G\) is a subgraph with vertex set \(V(G)\). Let \(H\) be a connected proper induced subgraph of a graph \(G\). A spanning subgraph rooted at \(H\) of a graph \(G\) is a subgraph \(G_H\) of \(G\) with vertex set \(V(G_H) = V(G)\) and \(G_H(V(H)) = H\). A spanning tree rooted at \(H\) of a graph \(G\), denoted by \(T_H\), is a spanning subgraph rooted at \(H\) of \(G\) such that \(T_H \setminus H\) is a forest. A spanning tree \(T_H\) rooted at \(H\) is called minimum distance spanning tree rooted at \(H\) if \(L(T_H) = L(G)\), denoted by \(T_H^m\).

**Observation 3.1.** Let \(G\) be a simple connected graph of order \(p\), diameter \(d \geq 2\) and \(L_0\) is fixed in \(G\) as described earlier. Let \(T_{L_0}^m\) be a minimum distance spanning tree rooted at \(L_0\) of \(G\). Then

(a) \(\text{diam}(T_{L_0}^m) = \text{diam}(G)\),

(b) \(d_{T_{L_0}^m}(u, L_0) = d_G(u, L_0)\) for all \(u \in V(T_{L_0}^m)\),

(c) \(L(T_{L_0}^m) = L(G)\),

(d) \(d_{T_{L_0}^m}(u, v) \geq d_G(u, v)\) for all \(u, v \in V(G)\).

**Theorem 3.2.** Let \(G\) be a simple connected graph of order \(p\), diameter \(d \geq 2\) and \(L_0\) is fixed in \(G\) as described earlier. Denote \(\text{diam}(L_0) = k\). If \(\text{rn}(G)\) attains a lower bound given in (3) then \(\text{rn}(T_{L_0}^m)\) attains a lower bound given in (3) and \(\text{rn}(T_{L_0}^m) = \text{rn}(G)\).

**Proof.** Since \(\text{rn}(G)\) attains a lower bound given in (3), there exists an ordering \(O(V(G)) = (x_0, x_1, \ldots, x_{p-1})\) of \(V(G)\) which satisfies conditions (a)-(b) of Theorem 3.1. Then by Observation 3.1 the same ordering of \(V(T_{L_0}^m) = V(G)\) satisfies conditions (a)-(b) of Theorem 3.1. Hence, \(\text{rn}(T_{L_0}^m)\) attains a lower bound given in (3). Since \(V(T_{L_0}^m) = V(G)\), \(\text{diam}(T_{L_0}^m) = \text{diam}(G)\) and \(L(T_{L_0}^m) = L(G)\), it is clear that \(\text{rn}(T_{L_0}^m) = \text{rn}(G)\).

**Theorem 3.3.** Let \(G\) be a simple connected graph of order \(p\), diameter \(d \geq 2\) and \(L_0\) is fixed in \(G\) as described earlier. Denote \(\text{diam}(L_0) = k\). Let \(G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_t = T_{L_0}^m\) be a sequence of subgraphs obtained by deleting edges in \(G\) to obtain \(T_{L_0}^m\). If \(\text{rn}(G)\) attains a lower bound given in (3) then for \(1 \leq i \leq t\), \(\text{rn}(G_i)\) attains a lower bound given in (3) and \(\text{rn}(G_i) = \text{rn}(G)\).

**Proof.** The proof is similar to the proof of Theorem 3.2.

Let \(G = (V(G), E(G))\) and \(H = (V(H), E(H))\) be two graphs. The Cartesian product of \(G\) and \(H\), denoted by \(G \square H\), is a graph with vertex set \(V(G \square H) = V(G) \times V(H)\), where two vertices...
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(g₁, h₁) and (g₂, h₂) are adjacent if g₁ = g₂ and h₁h₂ ∈ E(H), or h₁ = h₂ and g₁g₂ ∈ E(G). A path Pₘ on m vertices is a tree in which each vertex has degree at most 2. Denote the vertex set of Pₘ by V(Pₘ) = {u₁, u₂, . . . , uₘ} with E(Pₘ) = {uᵢuᵢ₊₁ : 1 ≤ i ≤ m − 1}. A wheel graph Wₙ is a graph obtained by joining each vertex of a cycle Cₙ to a new vertex v₀. Denote the vertex set of Wₙ by V(Wₙ) = {v₀, v₁, . . . , vₙ} with E(Wₙ) = {v₀v₁, v₀vₙ, vᵢvᵢ₊₁, v₁vₙ : 1 ≤ i ≤ n − 1}. Observe that the diameter of Pₘ□Wₙ is m + 1.

Theorem 3.4. Let m ≥ 3 and n ≥ 7 be any integers. Then

\[ \text{rn}(Pₘ□Wₙ) = \begin{cases} \frac{1}{2}(m²n + m² + 2m - 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m²n + m² + 2m + n - 1), & \text{if } m \text{ is odd.} \end{cases} \tag{10} \]

Proof. We consider the following two cases.

Case-1: m is even.

In this case, set \{(uₘ/2, v₀), (uₘ/2+1, v₀)\} of Pₘ□Wₙ as L₀ then diam(L₀) = k = 1 and the maximum level in Pₘ□Wₙ is h = m/2.

The order of Pₘ□Wₙ and L(Pₘ□Wₙ) are given by

\[ p := m(n + 1), \tag{11} \]

\[ L(Pₘ□Wₙ) := \frac{m}{4}mn + 2n + m - 2. \tag{12} \]

Substituting (11) and (12) into (3) we obtain the right-hand side of (10) which is a lower bound for the radio number of rn(Pₘ□Wₙ). We prove that this lower bound is tight. For this purpose, we give an ordering O(V(Pₘ□Wₙ)) := (x₀, x₁, . . . , xₚ₋₁) of V(Pₘ□Wₙ) which satisfies conditions (a) and (b) of Theorem 3.1.

Let τ and σ be two permutation defined on \{1, 2, ..., n\} as follows:

\[ \tau(j) = \begin{cases} n - 1, & \text{if } j = 1; \\ n, & \text{if } j = 2; \\ j - 2, & \text{if } 3 ≤ j ≤ n. \end{cases} \]

\[ \sigma(j) = \begin{cases} [j/4], & \text{if } j \equiv 1 \pmod{4}; \\ \sum_{t=0}^{k-2}[(n-t)/4] + [j/4], & \text{if } j \equiv k \pmod{4}, k = 2, 3, 4. \end{cases} \]

Using these two permutations we first rename \((uᵢ, vⱼ)(1 ≤ i ≤ m, 0 ≤ j ≤ n)\) as \((aᵣ, bₛ)\) as follows.

\[ (aᵣ, bₛ) = \begin{cases} (uᵢ, vⱼ), & \text{if } 1 ≤ i ≤ m \text{ and } j = 0; \\ (uᵢ, vᵣτ(j)), & \text{if } 1 ≤ i ≤ m/2 \text{ and } 1 ≤ j ≤ n; \\ (uᵢ, vᵣσ(j)), & \text{if } m/2 < i ≤ m \text{ and } 1 ≤ j ≤ n. \end{cases} \]

We now define an ordering O(V(Pₘ□Wₙ)) := (x₀, x₁, . . . , xₚ₋₁) as follows: Let \(xᵣ := (aᵣ, bₛ)\), where

\[ t := \begin{cases} 2(m/2 - r)(n + 1) + 2s, & \text{if } 1 ≤ r ≤ m/2 \text{ and } 1 ≤ s ≤ n; \\ 2(m - r)(n + 1) + 2s - 1, & \text{if } m/2 < r ≤ m \text{ and } 1 ≤ s ≤ n; \\ 2(m/2 - r)(n + 1), & \text{if } 1 ≤ r ≤ m/2 \text{ and } s = 0; \\ 2(m - r + 1)(n + 1) - 1, & \text{if } m/2 < r ≤ m \text{ and } s = 0. \end{cases} \]

Then note that \(d(x₀, L₀) + d(x₋₁, L₀) = 0\). Hence, the condition (a) in Theorem 3.1 is satisfied.
We now define an ordering \( O(V(P_m \square W_n)) := (x_0, x_1, \ldots, x_{p-1}) \) satisfies (10).

Let \( x_i \) and \( x_j \) (\( 0 \leq i < j \leq p - 1 \)) be any two arbitrary vertices. Denote the right-hand side of (10) by \( E(i, j) \). Let \( O(V(P_m \square W_n)) := (x_0, x_1, \ldots, x_{p-1}) = U_0 \cup U_1 \cup \ldots \cup U_{m/2 - 1} \), where \( U_t := (x_{2t(n+1)}, x_{2t(n+1)+1}, \ldots, x_{2t(n+1)+2n+1}) \) for \( 0 \leq t \leq m/2 - 1 \). It is clear that

\[
d(x_i, L_0) + d(x_{i+1}, L_0) \leq (d+1)/2 \text{ for all } 0 \leq i \leq p - 2.
\]

Now if \( x_i \in U_a, x_j \in U_b \). If \( b > a + 1 \) then \( E(i, j) < 0 < d(x_i, x_j) \). If \( b = a + 1 \) then we consider the following two cases: (i) \( j = i + 2n \) and (ii) \( j \neq i + 2n \). If \( j = i + 2n \) then \( d(x_i, x_j) = 1 \) and in this case, \( E(i, j) < 0 < d(x_i, x_j) \) and if \( j \neq i + 2n \) then \( d(x_i, x_j) = 2 \) and in this case, \( E(i, j) \leq 2 \leq d(x_i, x_j) \). If \( x_i, x_j \in U_a \) then if \( x_i = x_{2t(n+1)} \) or \( x_j = x_{2t(n+1)+2n+1} \) \((0 \leq t \leq m/2 - 1) \) then \( E(i, j) \leq 1 \leq d(x_i, x_j) \). If \( x_{2t(n+1)} < x_i < x_j < x_{2t(n+1)+2n+1} \) then if \( d(x_i, x_j) = 1 \) then \( j - i \leq [n/2] + 1 \geq 3 \) and hence \( E(i, j) \leq 2 \leq d(x_i, x_j) \) which completes the proof of Claim-1.

Case-2: \( m \) is odd.

In this case, set \( \{(u_{(m+1)/2}, v_0)\} \) of \( P_m \square W_n \) as \( L_0 \) then \( \text{diam}(L_0) = k = 0 \) and the maximum level in \( P_m \square W_n \) is \( h = (m + 1)/2 \). The order of \( P_m \square W_n \) and \( L(P_m \square W_n) \) are given by

\[
p := m(n + 1)
\]

\[
L(P_m \square W_n) := \frac{1}{4}(m^2n + m^2 + 4mn - n - 1).
\]

Substituting (13) and (14) into (10) we obtain the right-hand side of (10) which is a lower bound for the radio number of \( \text{rn}(P_m \square W_n) \). We prove that this lower bound is tight. For this purpose, we give an ordering \( O(V(P_m \square W_n)) := (x_0, x_1, \ldots, x_{p-1}) \) of \( V(P_m \square W_n) \) which satisfies conditions (a) and (b) of Theorem 3.1

Let \( \tau \) and \( \sigma \) are as defined earlier in Case-1. Let \( \alpha \) be a permutation defined on \( \{1, 2, \ldots, n\} \) as follows:

\[
\alpha(j) = \begin{cases} 
  n - 3, & \text{if } j = 1; \\
  n - 2, & \text{if } j = 2; \\
  n - 1, & \text{if } j = 3; \\
  n, & \text{if } j = 4; \\
  j - 4, & \text{if } 5 \leq j \leq n.
\end{cases}
\]

Using permutations \( \alpha, \tau \) and \( \sigma \), we first rename \((u_i, v_j)(1 \leq i \leq m, 0 \leq j \leq n)\) as \((a_r, b_s)\) as follows:

\[
(a_r, b_s) = \begin{cases} 
  (u_i, v_j), & \text{if } 1 \leq i \leq m \text{ and } j = 0; \text{ or } i = m \text{ and } 1 \leq j \leq n; \\
  (u_i, v_{r(j)}), & \text{if } i = 1 \text{ and } 1 \leq j \leq n; \\
  (u_i, v_{\alpha(j)}), & \text{if } i = (m + 1)/2 \text{ and } 1 \leq j \leq n; \\
  (u_i, v_{\tau(j)}), & \text{if } 2 \leq i \leq (m - 1)/2 \text{ and } 1 \leq j \leq n; \\
  (u_i, v_{\sigma(j)}), & \text{if } (m + 3)/2 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n.
\end{cases}
\]

We now define an ordering \( O(V(P_m \square W_n)) := (x_0, x_1, \ldots, x_{p-1}) \) as follows: Let \( x_t := (a_r, b_s) \),
Let $x_i$ and $x_j$ (0 \leq i < j \leq p - 1) be any two arbitrary vertices. Denote the right-hand side of (5) by $E(i,j)$. Let $O(P_m \Box W_n) := U_1 \cup \ldots \cup U_{(m-1)/2}$, where $U_1 := (x_0,x_1,\ldots,x_{3(n+1)-1})$ and $U_{t+2} := (x_{2t(n+1)+3(n+1)},\ldots,x_{2t(n+1)+5(n+1)-1})$ for 0 \leq t \leq (m-5)/2. Let $x_i \in U_a$ and $x_j \in U_b$. Assume $a = b = 1$. In this case, if $j \geq i + 3$ then $E(i,j) \leq 0 < d(x_i,x_j)$. If $j = i + 2$ then note that $d(x_i,x_j) \geq (d+2)/2$ and hence $E(i,j) \leq 2 \leq d(x_i,x_j)$. Let $a = 1$ and $b > 1$. If $j \geq i + 3$ then $E(i,j) \leq 0 < d(x_i,x_j)$. If $j = i + 2$ then if $x_i = x_{3(n+1)-2}$ then note that $d(x_i,x_j) = (d-1)/2$ and $E(i,j) = (d-4)/2 < d(x_i,x_j)$. If $x_i = x_{3(n+1)-1}$ then note that $d(x_i,x_j) = 2$ and $E(i,j) = 1 < d(x_i,x_j)$. Let $a,b \geq 2$. If $b > a + 1$ then $E(i,j) < 0 < d(x_i,x_j)$. If $b = a + 1$ then we consider the following two cases: (i) $j \neq i + 2n$ and (ii) $j = i + 2n$. If $j = i + 2n$ then $d(x_i,x_j) = 1$ and in this case, $E(i,j) < 0 < d(x_i,x_j)$ and if $j \neq i + 2n$ then $d(x_i,x_j) = 2$ and in this case, $E(i,j) \leq 2 \leq d(x_i,x_j)$. If $x_i,x_j \in U_a$. If $x_j = x_{2t(n+1)+5(n+1)-2}$ or $x_j = x_{2t(n+1)+5(n+1)-1}$ (0 \leq t \leq (m-5)/2) then $E(i,j) \leq 1 \leq d(x_i,x_j)$; otherwise $E(i,j) \leq 2 \leq d(x_i,x_j)$ which completes the proof of Claim 2.

An n-star, denoted by $K_{1,n}$, is a tree consisting of $n$ leaves and another vertex joined to all leaves by edges. Denote the vertex set of $K_{1,n}$ by $V(K_{1,n}) = \{v_0,v_1,\ldots,v_n\}$ with $E(K_{1,n}) = \{v_0v_i : 1 \leq i \leq n\}$. A friendship graph $F_n$ is a graph obtained by identifying one vertex of $n$ copies of cycle $C_3$ with a common vertex. Denote the vertex set of $F_n$ by $V(F_n) = \{v_0,v_1,\ldots,v_{2n}\}$ with $E(F_n) = \{v_0v_i,v_0v_{n+i},v_{2i-1}v_{2i} : 1 \leq i \leq n\}$.

**Corollary 3.1.** Let $m \geq 3$ and $n \geq 7$ be any integers. Then

$$\text{rn}(P_m \Box K_{1,n}) = \begin{cases} \frac{1}{2}(m^2n+m^2+2m-2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m^2n+m^2+2m+n-1), & \text{if } m \text{ is odd.} \end{cases}$$

**Proof.** Observe that $P_m \Box K_{1,n}$ can be regarded as a subgraph of $P_m \Box W_n$ with identical $L_0 = \{(u_{m/2},v_0),(u_{m+1/2},v_0)\}$ when $m$ is even and $L_0 = \{(u_{m+1/2},v_0)\}$ when $m$ is odd and hence by Theorem 5.3, the radio number of $P_m \Box W_n$ and $P_m \Box K_{1,n}$ are identical. \hfill \Box

**Corollary 3.2.** Let $m \geq 3$ and $n \geq 4$ be any integers. Then

$$\text{rn}(P_m \Box F_n) = \begin{cases} \frac{1}{2}(2m^2n+m^2+2m-2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(2m^2n+m^2+2m+2n-1), & \text{if } m \text{ is odd.} \end{cases}$$

**Proof.** Observe that $P_m \Box F_n$ can be regarded as a subgraph of $P_m \Box W_n$ with identical $L_0 = \{(u_{m/2},v_0)\}$ when $m$ is even and $L_0 = \{(u_{m+1/2},v_0)\}$ when $m$ is odd and hence by Theorem 5.3, the radio number of $P_m \Box W_n$ and $P_m \Box F_n$ are identical. \hfill \Box
Proof. Observe that \( P_m \Box F_n \) can be regarded as a subgraph of \( P_m \Box W_{2n} \) with identical \( L_0 = \{(u_m/2, v_0), (u_m/2+1, v_0)\} \) when \( m \) is even and \( L_0 = \{(u_{(m+1)/2}, v_0)\} \) when \( m \) is odd and hence by Theorem 3.3, the radio number of \( P_m \Box W_{2n} \) and \( P_m \Box F_n \) are identical. \qed

**Example 3.1.** In Table 1, an ordering of vertices and the corresponding optimal radio labeling of \( P_7 \Box W_7 \) is shown.

Table 1: An ordering and optimal radio labeling for vertices of \( P_7 \Box W_7 \).

| (\( u_i, v_j \)) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-----------------|----|----|----|----|----|----|----|
| 0               | 72 | 139| 206| 0  | 133| 200| 69 |
| 1               | 51 | 104| 171| 37 | 76 | 143| 5  |
| 2               | 60 | 120| 187| 46 | 92 | 159| 14 |
| 3               | 6  | 80 | 147| 55 | 108| 175| 23 |
| 4               | 15 | 96 | 163| 64 | 124| 191| 30 |
| 5               | 24 | 112| 179| 10 | 84 | 151| 41 |
| 6               | 33 | 128| 195| 19 | 100| 167| 50 |
| 7               | 42 | 88 | 155| 28 | 116| 183| 59 |

**Example 3.2.** In Table 2, an ordering of vertices and the corresponding optimal radio labeling of \( P_3 \Box W_7 \) is shown.

Table 2: An ordering and optimal radio labeling for vertices of \( P_3 \Box W_7 \).

| (\( u_i, v_j \)) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|
| 0               | 201| 134| 67 | 0  | 263| 196| 129| 62 |
| 1               | 234| 167| 100| 33 | 206| 139| 72 | 5  |
| 2               | 250| 183| 116| 49 | 222| 155| 88 | 21 |
| 3               | 210| 143| 76 | 9  | 238| 171| 104| 37 |
| 4               | 226| 159| 92 | 25 | 254| 187| 120| 53 |
| 5               | 242| 175| 108| 41 | 214| 147| 80 | 13 |
| 6               | 258| 191| 124| 57 | 230| 163| 96 | 29 |
| 7               | 218| 151| 84 | 17 | 246| 179| 112| 45 |

4 Concluding Remarks

In [12], Kim et al. determined the radio number of Cartesian product of paths and complete graph \( P_m \Box K_n \) as follows:

**Theorem 4.1.** [12] Let \( m \geq 4 \) and \( n \geq 3 \) be integers. Then

\[
\text{rn}(P_m \Box K_n) = \begin{cases} \frac{1}{2}(m^2n - 2m + 2), & \text{if } m \text{ is even;} \\ \frac{1}{2}(m^2n - 2m + n + 2), & \text{if } m \text{ is odd.} \end{cases}
\]  (15)

Theorem 4.1 can also be proved using Theorem 3.1. The order and total level of \( P_m \Box K_n \) are given by

\[
p := mn
\]  (16)

\[
L(P_m \Box K_n) := \begin{cases} \frac{1}{2}(mn(m-2)), & \text{if } m \text{ is even;} \\ \frac{1}{4}(m^2 - 1)n, & \text{if } m \text{ is odd.} \end{cases}
\]  (17)
Substituting (16) and (17) in (3) we obtained the right-hand side of (15) which is a lower bound for the radio number of $P_m \square K_n$. Now it is easy to prove that the radio labeling given in [12] satisfies conditions (a) and (b) of Theorem 3.1 and hence the right-hand side of (3) is exact radio number for $P_m \square K_n$ which is (15) in the present case.

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