Structural Routability of $n$-Pairs Information Networks

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Abstract Information does not generally behave like a flow in communication networks with multiple sources and sinks. However, it is often conceptually and practically useful to be able to associate separate data streams with each source-sink pair, with only routing and no coding performed at the network nodes. This raises the question of whether there is a nontrivial class of network topologies for which achievability is always equivalent to ‘routability’, for any combination of source signals and positive channel capacities. This chapter considers possibly cyclic, directed, errorless networks with $n$ source-sink pairs and mutually independent source signals. The concept of downward matching is introduced and it is shown that, if the network topology is downward matched, then a given combination of source signals, demand rates and channel capacities is achievable if and only if there is a feasible multicommodity flow.

1 Introduction

In an $n$-pairs (or multiple unicast) communication network, $n$ source signals must be conveyed to their corresponding sinks without exceeding any channel capacities. Until quite recently, the belief was that this was possible iff there existed a routing solution, i.e. if every bit generated by a source could be carried without modification, over channels and through network nodes, until it reached the sink. At a macroscopic level, this is equivalent to presuming the existence of a feasible multicommodity flow [8].

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However, in [11, 2], an example was constructed of a multi-source, multi-sink communication network that did not admit a routing solution, but became admissible if nodes could perform modulo-2 arithmetic on incoming bits. This counter-intuitive result started the field of network coding, in which nodes are permitted to not just route incoming bits, but also to perform functions on them so as to better exploit the network structure and the available channel capacities.

It is now known that the achievable capacity regions of networks with multiple sources and sinks are not generally given by feasible multicommodity flows. In [1], n-pairs networks were constructed with coding capacity much larger than the routing capacity. Other related work includes [5], in which a necessary and sufficient condition for broadcasting correlated sources over erroneous channels was found, and [9], in which linear network coding was shown to achieve capacity for a multicast network.

Notwithstanding the power of network codes, routing/multicommodity flow solutions are appealing in several respects. Most obviously they are simpler, because network nodes are not required to perform extra mathematical operations on arriving bits. In addition, because different data streams are not ‘hashed’ together by means of some function, there is arguably less potential for cross-talk between different source-sink pairs, arising for instance from nonidealities during implementation in the physical layer. Furthermore, being able to treat information as a conservative fluid-flow could potentially provide a simple basis to analyse communication requirements in areas outside traditional multiterminal information theory, e.g. networked feedback control and multi-agent coordination/consensus problems - see, e.g. [3].

These considerations raise the natural questions of whether there is a general class of network topologies on which achievability is always equivalent to the existence of a feasible multicommodity flow, and of how to characterise this class. This chapter derives a partial answer to this question for possibly cyclic, directed, errorless networks with n source-sink pairs and mutually independent source signals. The communication objective is relaxed so that, instead of reconstructing the source signals with negligible probability of error at the sinks, the aim is for the marginal information rate supplied to each sink to exceed a specified positive demand.

The structural concept of downward matching (Def. 3.6) is introduced, and the main result (Thm. 3.1) is that if the network topology is downward matched then the network is structurally routable, i.e. any given combination of source signals, demand rates and channel capacities is achievable iff the network supports a feasible multicommodity flow.

This property inheres solely in the topology of the network and suits situations where channels, switches, transceivers and interfaces are expensive to set up and difficult to move, or where channel capacities, output demands and source-signal statistics are unknown. The proof technique for the necessity argument relies on the iterative generation of an informationally feasible multicommodity flow (Def. 4.2). It is emphasised that downward matching is a more general condition than the notion of ‘triangularisability’ that was introduced in the brief, conference version [10] of this chapter.
The downward matching condition is not generally easy to check in arbitrary n-pairs networks. Nonetheless, Theorem 3.1 is potentially useful because it defines a non-trivial class of directed network topologies for which achievability is always equivalent to the existence of a feasible multicommodity flow. On these structures, information can indeed be treated as if it was indeed an incompressible fluid flow. Several examples are then provided in Sect. 5 to illustrate the applicability of Theorem 3.1.

Note that although downward matching is sufficient to guarantee that routing can always achieve the full capacity of a network, it is not necessary, and the important question of finding a more general - or even tight - structural condition remains open.

1.1 Notation and Basic Terminology

For convenience, the basic notation and terminology used in this chapter are described below.

- The set of nonnegative integers (i.e. whole numbers) is denoted $\mathbb{W}$ and the set of positive integers (i.e. natural numbers) by $\mathbb{N}$.
- A contiguous set $\{i,i+1,\ldots,j\}$ of integers is denoted $[i:j]$.
- Random variables (rv’s) are written in upper case and their realizations are indicated in corresponding lower case.
- Sets are written in boldface, apart from common ones such as $\mathbb{R}$ and $\mathbb{W}$ that are conventionally written in blackboard boldface.
- The set operation $A \setminus B$ denotes $A \cap B^c$.
- A random signal or process $(F(k))_{k=0}^\infty$ is denoted $F$. With a mild abuse of notation, the finite sequence $(F(k))_{k=0}^t$ is denoted $F(0:t)$.
- Given a subscripted rv or signal $F_j$, with $j$ belonging to a countable set $J$, $F_J$ denotes an ordered tuple $(F_j)_{j \in J}$, arranged according to the order on $J$.
- The entropy of a discrete-valued rv $E$ is denoted $H[E] \geq 0$ and the differential entropy of a continuous-valued rv $F$ (with absolutely continuous distribution) is written $h[F] \in \mathbb{R}$. The conditional versions thereof are denoted $H[E|\cdot]$ and $h[F|\cdot]$ respectively.
- The mutual information in rv’s $E$ and $F$ is denoted $I[E;F] \geq 0$, which is defined as $H[E] - H[E|F]$ if $E$ is discrete-valued. If $E$ is continuous-valued, $H$ is replaced with the differential entropy $h$.
- The conditional mutual information in rv’s $E,F$ given $G$ is denoted $I[E;F|G]$, which is defined as $H[E|G] - H[E|F,G]$ if $E$ is discrete-valued. If $E$ is continuous-valued, $H$ is replaced with $h$.
- If $E,F$ are random processes and $E$ is discrete-valued, then the entropy rates of $E$, and the conditional entropy rate of $E$ given (past and present) $F$ are respectively defined as
The initial vertex of an arc is called its tail and the terminal vertex, its head.

A walk in a digraph is an alternating sequence \( \alpha = (\nu_1, \alpha_1, \nu_2, \alpha_2, \ldots, \alpha_k, \nu_{k+1}) \), \( k \geq 0 \), of vertices and arcs, beginning and ending in vertices, s.t. each arc \( \alpha_j \) connects the vertex \( \nu_j \) to \( \nu_{j+1} \). Each vertex \( \nu_j \) and arc \( \alpha_j \) in the sequence is said to be in the walk; with a minor abuse of notation, this is denoted \( \nu_j \in \alpha \).

A path is a walk that passes through no vertex more than once, including the initial one.

An undirected path is an alternating sequence \( \omega = (\nu_1, \alpha_1, \nu_2, \alpha_2, \ldots, \alpha_k, \nu_{k+1}) \), \( k \geq 0 \), of vertices and arcs, beginning and ending in vertices, s.t. no vertex is repeated and where each arc \( \alpha_j \) connects the vertex \( \nu_j \) to \( \nu_{j+1} \), or \( \nu_{j+1} \) to \( \nu_j \).

A cycle is a walk in which the initial and final vertices are identical, but every other vertex occurs once.

A subpath of a path \( (\nu_1, \alpha_1, \nu_2, \alpha_2, \ldots, \alpha_k, \nu_{k+1}) \) is a segment \( (\nu_l, \alpha_l, \nu_{l+1}, \ldots, \nu_j) \) of it, where \( 1 \leq l \leq j \leq k + 1 \).

A vertex \( \nu \) is said to be reachable from another vertex \( \mu \), denoted \( \mu \rightarrow \nu \), if \( \exists \) a path leading from \( \mu \) to \( \nu \). Equivalently, it is said that \( \mu \) can reach \( \nu \). The same terminology and notation apply, with analogous meaning, for pairs of arcs as well as mixed pairs of arcs and vertices. E.g. given an arc \( \beta \), \( \mu \rightarrow \beta \) means that there is a path from the vertex \( \mu \) to the tail of \( \beta \).

Similarly, a (vertex- or arc-)set \( W \) is said to be reachable from another set \( U \), denoted \( U \rightarrow W \), if there is an element of \( W \) that is reachable from an element of \( U \); equivalently, it is said that \( U \) can reach \( W \).

\[
\begin{align*}
H_\omega[E] & := \lim_{t \to \infty} \frac{H[E(0:t)]}{t + 1}, \\
H_\omega^\infty[E] & := \lim_{t \to \infty} \frac{H[E(0:t)]}{t + 1}, \\
H_\omega[E|F] & := \lim_{t \to \infty} \frac{H[E(0:t)|F(0:t)]}{t + 1},
\end{align*}
\]

If \( E \) is continuous-valued, \( H \) is simply replaced with \( h \).

- If \( E, F \) and \( G \) are random processes, then the mutual information rates of \( E \) and \( F \), and the conditional mutual information rate of \( E \) and \( F \) given (past and present) \( G \) are respectively defined as

\[
\begin{align*}
I_\omega[E;F] & := \lim_{t \to \infty} \frac{I[E(0:t);F(0:t)]}{t + 1}, \\
I_\omega^\infty[E;F] & := \lim_{t \to \infty} \frac{I[E(0:t);F(0:t)]}{t + 1}, \\
I_\omega[E;F|G] & := \lim_{t \to \infty} \frac{I[E(0:t);F(0:t)|G(0:t)]}{t + 1}.
\end{align*}
\]
• The notation \( \text{OUT}(U) \) \((\text{IN}(U))\) denotes the set of arcs that have tails (resp. heads) in a vertex set \( U \subseteq V \) but heads (tails) \( \in V \setminus U \). If \( U \) is a singleton \( \{ \mu \} \), the braces are removed for notational compactness.

2 Problem Formulation

A multiterminal network of unidirectional, point-to-point channels may be modelled using a digraph \((V,A)\), where the vertex set \( V \) represents information sources, sinks, repeaters, routers etc., and the arc set \( A \) indicates the directions of any channels between network nodes. As usual with digraphs, it is assumed that no arc leaves and enters the same vertex, and that at most one arc leads from the first to the second element of any given ordered pair of vertices. In other words, every arc in \( A \) may be uniquely identified with a tuple \((\mu, \nu) \in V^2\), with \( \mu \neq \nu \). It is also assumed that the digraph is connected, i.e. there is an undirected path between any distinct pair of vertices.

In an \( n \)-pairs information network, the locations of sources and sinks are respectively represented by disjoint sets \( S = \{ \sigma_1, \ldots, \sigma_n \} \) and \( T = \{ \tau_1, \ldots, \tau_n \} \) of distinct vertices in \( V \), with each source \( \sigma_i \) aiming to communicate to exactly one sink \( \tau_i \). It is assumed that \( \sigma_i \rightarrow \tau_i \). The set of source-sink pairs \((\sigma_i, \tau_i)\), \( i \in [1:n] \), is denoted \( P \subset S \times T \). Without loss of generality, it is assumed that every source (sink) has no in-coming (resp. out-going) arcs and exactly one out-going (in-coming) arc.

The boundary \( \partial V \) of the network is the set \( S \cup T \) of source and sink vertices, and its interior is \( \text{int} V := V \setminus \partial V \).

Each channel in the network can transfer information errorlessly up to a maximum rate, as specified by a positive arc-capacity \( c_\alpha \in \mathbb{R}_{>0} \). In some situations, it may be natural to assign infinite capacity to certain arcs and the set of all such arcs is denoted \( A_{\infty} \subset A \). In particular, the arcs leaving sources are by convention assigned infinite capacity. The set of finite-capacity arcs is written \( A_f = A \setminus A_{\infty} \), with associated arc-capacity vector \( c := (c_\alpha)_{\alpha \in A_f} \in \mathbb{R}_{>0}^{|A_f|} \). The structure of the \( n \)-pairs information network is defined as the tuple \( \Sigma = (V, A_f, A_{\infty}, P) \).

The communication signals in the channels are represented by a vector \( S = (S_\alpha)_{\alpha \in A} \) of random processes called arc signals, with \( S_\alpha : \mathbb{W} \rightarrow \mathbb{R}^{m_\alpha} \) taken to be discrete-valued \( \forall \alpha \in A_f \). In particular, the arc signals leaving sources and entering sinks respectively represent the exogeneous inputs and outputs of the information network. For convenience, the input signal \( S_{\text{OUT}(\sigma_i)} \) generated by the \( i \)-th source \( \sigma_i \in S \) is called \( X_i \), and the output signal \( S_{\text{IN}(\tau_i)} \) entering the \( i \)-th sink \( \tau_i \in T \) is called \( Y_i \). It is assumed throughout this chapter that the signals \( X_1, X_2, \ldots, X_n \) are mutually

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2 Such digraphs are sometimes called simple.
3 If a source or sink were actually connected to multiple nodes in the network, it would be represented in the digraph by an auxiliary vertex connected by an arc (of infinite capacity) with a multiply-connected vertex.
4 For instance, when a single network node is represented as two “virtual” vertices connected by an arc of unbounded capacity.
independent, though each may itself be a correlated process, and that the signals on finite-capacity arcs are discrete-valued.

The arc-signal vector $S$ is assumed to have the following property:

**Definition 2.1 (Setwise Causality and Signal Graphs).** An arc-signal vector $S$ is called **setwise causal** on a structure $\Sigma = (V, A_f, A_\infty, P)$ if all arc signals leaving any internal vertex set $U \subseteq \text{int} V$ are causally determined by those entering it. That is,

$$\forall U \subseteq \text{int} V, \exists \text{ an operator } g_U \text{ s.t. } S_{\text{OUT}(U)}(t) = g_U(t, S_{\text{IN}(U)}(0:t)), \forall t \in \mathbb{W}. \quad (1)$$

The tuple $\langle \Sigma, S \rangle$ is then called a **signal graph**.

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**Remarks:** Setwise causality is a generalisation of the fundamental concept of well-posedness in feedback control theory. In acyclic digraphs (i.e. in which every walk is a path), it is equivalent to causality at every internal vertex. However, feedback signals may be present in cyclic digraphs, in which case vertex-wise causality would not suffice to guarantee that outgoing signals from an arbitrary vertex set are uniquely and causally determined by incoming ones as required in (1). Stronger assumptions would be needed, e.g. a positive time-delay at every vertex.

In an $n$-pairs network, information must be conveyed from each source $\sigma_i \in S$ to its corresponding sink $\tau_i$ so as to achieve some goal, without exceeding any channel capacities. In this chapter, the goal is to guarantee that the marginal information rates supplied to the sinks exceed specified **demands**. This leads to the following definition:

**Definition 2.2 (Achievability).** Consider an $n$-pairs information network with structure $\Sigma$, source-signal vector $X$, arc-capacity vector $c \in \mathbb{R}_{>0}^{|A_f|}$ and demand vector $d := (d_i)_{i=1}^n \in \mathbb{R}_{>0}^n$. The tuple $\langle \Sigma, X, c, d \rangle$ is called achievable if $\exists$ a setwise-causal (Def. 2.1) arc-signal vector $S$ s.t.

$$S_{\text{OUT}(\sigma_i)} = X_i, \forall i \in [1:n], \quad (2)$$

$$I_\infty[Y_i;X_i] \geq d_i, \forall i \in [1:n], \quad (3)$$

$$I^\alpha[S_\alpha,X] \leq c_\alpha, \forall \alpha \in A_\infty. \quad (4)$$

Such an $S$ is called a **solution** to the $n$-pairs information network problem $\langle \Sigma, X, c, d \rangle$. The demand vector $d$ is then called achievable on $\langle \Sigma, X, c \rangle$; $c$ is called achievable on $\langle \Sigma, X, d \rangle$; and $\langle X, c, d \rangle$ is called achievable on $\Sigma$.

The closure of the set of achievable demand vectors $d$ is called the **demand region** $D \subseteq \mathbb{R}_{\geq0}^n$ of $\langle \Sigma, X, c \rangle$.

◊

**Remarks:** The objective is a relaxation of the usual aim of reconstructing the source signals reliably or within some specified distortion level. This allows the standard assumption that each source signal is stationary or i.i.d. to be dropped. Note
that when $X_i$ is discrete-valued, $d_i = H_{\infty}[X_i]$ is necessary for reliable reconstruction at $\tau_i$.

Also note, it is implicit in Def. 2.2 that the input-output operators $g_\nu$ at every internal vertex $\nu$ may be freely designed to yield a solution $S$, as long as setwise causality (Def. 2.1) is respected.

As mentioned in the introduction, it was once thought that a network was achievable iff it admitted a routing solution. In the present context, this is equivalent to presuming the existence of an $(X, c, d)$-feasible multicommodity flow, i.e. of a non-negative tuple $f = (f_\alpha, j)_{\alpha \in A, j \in [1:n]} \in \mathbb{R}^{|A|n}_{\geq 0}$, of bit-rates on each arc associated with every source-sink pair, s.t.

\[
\sum_{j=1}^{n} f_\alpha, j \leq c_\alpha, \quad \forall \alpha \in A_f \quad \text{(capacity bound),} \\
\quad \text{for any } j \in [1 : n] \text{ and } \nu \in V \setminus (\{\sigma_j\} \cup \{\tau_j\}).
\]

Via an explicit counter-example, the article [2] showed that this intuitive notion was incorrect, i.e. that although the existence of a feasible multicommodity flow is sufficient for achievability, it is not generally necessary. This laid the foundations for network coding, in which nodes are permitted to not just route incoming bits, but also to perform functions on them. Nonetheless, routing/multicommodity-flow solutions have certain virtues, as discussed in Sect. 1. This chapter poses the question: is there a general class of $n$-pairs information network structures $\Sigma$ for which the achievability of $(X, c, d)$ is equivalent to the existence of an $(X, c, d)$-feasible multicommodity flow $f$ (5–8)?

Any $n$-pairs information network structure $\Sigma$ can support $(X, c, d)$-feasible multicommodity flows if the arc-capacities and source entropy rates are sufficiently larger than the demands (and provided that each sink is reachable from its source). However, there are examples of structures on which an $(X, c, d)$-feasible multicommodity flow does not exist if arc-capacities are reduced or demands increased, even though $(X, c, d)$ is still achievable (see Sect. 5).

The aim of this chapter is to isolate certain structural properties that ensure routability over all achievable combinations of $(X, c, d)$. Such properties would inhere solely in $\Sigma$, situating situations in which channels, switches, transceivers and interfaces are expensive to set up and difficult to move, and/or where channel capacities, output demands and source-signal statistics are variable or unknown.

\footnote{5 ignoring differences in the definition of achievability}
3 Preliminary Notions and Main Result

Before proceeding, several new graph-theoretic notions and related results are needed. Throughout this section, \( \Sigma = (\mathcal{V}, \mathcal{A}, \mathcal{P}) \equiv (\mathcal{V}, \mathcal{A}_f, \mathcal{A}_w, \mathcal{P}) \) is the structure of an \( n \)-pairs information network as described in Sect. 2, and \( \Gamma = (\Sigma, \mathcal{S}) \) is its setwise-causal signal graph (Def. 2.1), with source- and sink-signal vectors \( X \) and \( Y \).

The result below is intuitive:

**Lemma 3.1.** The arc signal \( S_\alpha \) on any arc \( \alpha \in \mathcal{A} \) is causally determined by the source-signal vector \( X_K \), where \( K \subseteq [1:n] \) is the index set of all sources that can reach \( \alpha \).

**Proof.** Let \( U \subseteq \text{int} \mathcal{V} \) be the set of all internal vertices that can reach \( \alpha \). Observe that the unique tail \( \mu \in \mathcal{V} \) of any \( \beta \in \text{IN}(U) \) can automatically reach \( \alpha \) and thus would be \( \in U \) if it were internal. However, as \( \mu \) is outside \( U \), by definition of \( \text{IN}(U) \), it must therefore be a boundary vertex and specifically a source \( \sigma_i \) with \( i \in K \). As \( \alpha \in \text{OUT}(U) \), the proof then follows from setwise causality (Defn. 2.1). \( \square \)

Next, some largely familiar concepts are revisited. A path in an \( n \)-pairs information network that goes from a source \( \sigma_i \) to its sink \( \tau_i \) is called an \( i \)-path and the set of all \( i \)-paths is called an \( i \)-bundle. In other words, the \( i \)-bundle is the set of all acyclic paths via which information can be routed from \( \sigma_i \) to \( \tau_i \). Given a set \( J \subseteq [1:n] \), the set of all \( i \)-paths with \( i \in J \) is called a \( J \)-bundle (not the same as the set of all \( \{ \sigma_i : i \in J \} \leadsto \{ \tau_i : i \in J \} \)-paths, which contains it). Let \((\mathcal{V}^J, \mathcal{A}^J)\) denote the subgraph formed by all the vertices and arcs in the \( J \)-bundle. In particular, \((\mathcal{V}^i, \mathcal{A}^i)\) is the subgraph formed by the \( i \)-bundle. A vertex set \( U \subseteq \mathcal{V}^i \) such that \( \sigma_i \in U \) and \( \tau_i \notin U \) is called an \( i \)-cut.

The following notions are nonstandard.

**Definition 3.1 (J-Disjointness).** Given an index set \( J \subseteq [1:n] \), an arc set \( B \subseteq \mathcal{A} \) is \( J \)-disjoint if each path in the \( J \)-bundle passes through at most one arc in \( B \). If \( J = \{ i \} \) for some \( i \in [1:n] \), then \( B \) is called \( i \)-disjoint.

\( \diamond \)

**Remarks:** Empty and singleton arc-sets are automatically \( J \)-disjoint, and every \( B \subseteq \mathcal{A} \) is \( \emptyset \)-disjoint. As an aside, the family of \( J \)-disjoint sets forms a so-called independence system on \( \mathcal{A} \).

**Definition 3.2 (Chokes and Downward Wholeness).**

1. Given vertex sets \( W, Z \subseteq \mathcal{V} \), an arc set \( B \subseteq \mathcal{A} \) is a \( W \leadsto Z \)-choke if all paths from the vertex set \( W \) to \( Z \) pass through \( B \).
2. An \( i \)-choke \( B' \) is described as downward whole if all \( \sigma_j \leadsto \tau_i \)-paths pass through it, \( \forall j \in [1 : i - 1] \) s.t. \( \sigma_j \leadsto B' \).

\( \diamond \)
These notions are required in the following two lemmas. The first is intuitive, though some care is required in the proof when applying setwise causality (Def. 2.1).

**Lemma 3.2.** Let $B \subseteq A$ be a $\{\sigma_j : j \in J\}$-choke (Def. 3.2), where $J \subseteq [1 : n]$. Then the arc signal $Y_i$ entering the sink $\tau_i$ is causally determined by the arc-signal vector $S_B$ on the choke and the source-signal vector $X_J$, where $J = [1 : n] \setminus J$. \(\square\)

**Proof.** See Appendix 1.

Downward whole $i$-chokes are useful because for each $k \in [1 : i - 1]$, either no $\sigma_k \leadsto \tau_i$-walk passes through them, or all of them do. This renders them ‘immune’ to any information about $X_1, \ldots, X_{i-1}$ that is transmitted to the sink $\tau_i$ without passing through them, leading to a type of data-processing inequality that will be extremely useful later:

**Lemma 3.3 (Data Processing on Downward Whole $i$-Chokes).** Let $S$ be a setwise causal arc-signal vector (Def. 2.1) on $\Sigma$, with source- and sink-signal vectors $X$ and $Y$. If $B \subseteq A$ is a downward whole $i$-choke (Def. 3.2), then

$$I_{\infty} [X_i; Y_i | X_{[i+1:n]}] \leq I_{\infty} [X_i; S_B | X_{[i+1:n]}].$$

**Proof.** Let $K$ be the set of all source indices $j \in [1 : n]$ such that $\sigma_j \leadsto B$, and define $J := K \cap [1 : i]$ and $J' := [1 : i] \setminus J$. From Def. 3.2, $J \ni i$ and all $\{\sigma_j : j \in J\}$-walks pass through $B$. By Lemma 3.2, $Y_i$ is thus causally determined by $S_B$ and $X_{J'} \equiv (X_{J'}, X_{[i+1:n]})$, i.e., \(\forall t \in \mathbb{W},\)

$$Y_i(0 : t) = g(t, S_B(0 : t), X_{J'}(0 : t), X_{[i+1:n]}(0 : t)), \quad g \text{ is deterministic, time-varying mapping.}$$

Furthermore, by Lemma 3.1, $S_B$ is causally determined by $X_K$, and hence by $X_J$ and $X_{[i+1:n]}$. So the argument $X_{J'}(0 : t)$ in the RHS of the equation above must be independent of $(S_B(0 : t), X_{[i+1:n]}(0 : t), X_i(0 : t))$. From this, it follows that that

$$X_i(0 : t) \leftrightarrow (S_B(0 : t), X_{[i+1:n]}(0 : t)) \leftrightarrow (Y_i(0 : t), X_{[i+1:n]}(0 : t))$$

forms a Markov chain. By the data processing inequality,

$$I [X_i(0 : t); S_B(0 : t) | X_{[i+1:n]}(0 : t)] \leq I [X_i(0 : t); S_B(0 : t), X_{[i+1:n]}(0 : t)]$$

$$\geq I [X_i(0 : t); Y_i(0 : t), X_{[i+1:n]}(0 : t)]$$

$$= I [X_i(0 : t); Y_i(0 : t) | X_{[i+1:n]}(0 : t)],$$

where the equalities are due to the mutual independence of the source signals. Dividing by $t + 1$ and taking inferior limits completes the proof. \(\square\)

The next definition describes families of arc-sets that satisfy certain recursive structural properties. These properties are needed later to inductively extract informationally feasible multicommodity flows (Def. 4.2).
Definition 3.3 ($i$-Downward Matched Sets). For each $i \in [1 : n]$, the family $\mathcal{D}_i$ consists of all arc sets $E \subseteq A$ such that either $E \cap A^{[1:i]} = \emptyset$, or else $E$ contains

1. a (possibly empty) $i$-disjoint set (Def. 3.1) $E_1 \supseteq A^i \cap E$,
2. a (possibly empty) $\mathcal{D}_{i-1}$-set $E_2 \supseteq A^{[1:i-1]} \cap E$,
3. and a downward whole $i$-choke (Def. 3.2) $E_3$,

with $\mathcal{D}_0 := \emptyset$.

Every member set of $\mathcal{D}_i$ is called $i$-downward matched.

The next concept describes a class of $i$-cuts that satisfy specific structural properties. These properties are of interest because they are exhibited by all min-cuts in a residual capacitated digraph that is constructed to prove the main result here (Thm. 3.1).

Definition 3.4 (Viable $i$-Cuts). Given an index $i \in [1 : n]$, an $i$-cut $U \subset V^i$ is called viable under the following conditions:

1. Every arc in $\text{OUT}(U) \cap A^i$ is finite-capacity.
2. There is an $i$-path that leaves $U$ without re-entering.
3. Each arc in $\text{OUT}(U) \cap A^i$ lies in an $i$-path that exits $U$ without re-entering or that lies in the $[1 : i-1]$-bundle.
4. Every vertex $v \in U$ lies on an undirected path $\pi$ from $\sigma_i$ to $v$ such that
   a. all vertices before $v$ on $\pi$ are in $U$, and
   b. every reverse-oriented arc in $\pi$ (i.e. pointing from $v$ to $\sigma_i$) lies on an $i$-path that does not re-enter $U$.

Definition 3.5 (Reverse Structure). Given a structure $\Sigma$, the reverse structure $\Sigma' := (V, A'_f, A'_\infty, P')$ is given by

$$A'_f := \{ (\mu', \nu') : (\nu', \mu') \in A_f \}, \quad \quad (9)$$
$$A'_\infty := \{ (\mu', \nu') : (\nu', \mu') \in A_\infty \}, \quad \quad (10)$$
$$P' := \{ (\sigma'_i, \tau'_i) : (\tau'_i, \sigma'_i) \in P \}. \quad \quad (11)$$

Remarks: This describes the $n$-pairs information network obtained by reversing arcs and swapping the roles of sources and sinks. The reverse structure is useful because a multicommodity flow on $\Sigma$ can be reversed in direction to yield one on $\Sigma'$, and vice-versa.

Definition 3.6 (Downward Matching). A structure $\Sigma$ is called downward matched if there is an ordering $(\sigma_1, \delta_1), \ldots, (\sigma_n, \delta_n)$ of the source-sink pairs so that for each $i \in [2 : n]$ and viable $i$-cut $U$ (Def. 3.4), the set $O^i = \text{OUT}(U) \cap A^i$ of outgoing arcs in the $i$-bundle contains
1. a (possibly empty) \((i-1)\)-downward matched arc set (Def. 3.3) \(B \supseteq O^{i-1} \cap A^{[i]}\)
2. and a downward-whole \(i\)-choke \(B'\) (Def. 3.2).

\[B \supseteq O^{i-1} \cap A^{[i]}\]

\[\text{Remark: Note that 1-pair structures are automatically downward matched, since the conditions above become empty.}\]

The main result of this chapter can now be stated:

**Theorem 3.1 (Downward Matching Implies Structural Routability).** Suppose an \(n\)-pairs information network has a structure \(\Sigma\) or reverse structure \(\Sigma'\) (Def. 3.5) that is downward matched (Def. 3.6). Then \(\Sigma\) is structurally routable, i.e. for any \((X, c, d)\), \((\Sigma, X, c, d)\) is achievable (Def. 2.2) iff an \((X, c, d)\)-feasible multicommodity flow \(f\) (5)–(8) exists on \(\Sigma\).

\[\text{Remarks: The downward matching structural condition (Def. 3.6) is not generally easy to verify for arbitrary } n \text{-pairs networks, since every viable } i \text{-cut must be found and checked. In addition, the } i \text{-downward matching condition (Def. 3.3) is expressed recursively. Nonetheless, this result is potentially useful because it defines a non-trivial class of directed network structures for which achievability is always equivalent to the existence of a feasible multicommodity flow. On these structures, information can be treated as if it was indeed an incompressible fluid flow.}\]

The proof of Thm. 3.1 is given in the next section. In Sect. 5, several network examples are discussed to illustrate the applicability of Thm. 3.1.

### 4 Proof of Theorem 3.1

In both the proofs of necessity and sufficiency, use will be made of the fact that \(\forall i \in [1 : n]\), any single-commodity flow \(q\) from \(\sigma_i\) to \(\tau_i\) in the structure \(\Sigma\) can be decomposed into a superposition of \(i\)-path flows and cycle flows (see e.g. [4], Thm. 3.3.1). That is, if \(\pi_{1,i}, \ldots, \pi_{n,i}\) are the distinct \(i\)-paths and \(\gamma_1, \ldots, \gamma_g\), the distinct cycles, then \(\exists \) numbers \(u_{1,i}, \ldots, u_{p,i} \geq 0\) and \(w_{1,i}, \ldots, w_{g,i} \geq 0\) s.t.

\[q_\alpha = \sum_{1 \leq k \leq p, \pi_k \ni \alpha} u_{k,i} + \sum_{1 \leq l \leq g, \gamma_l \ni \alpha} w_{l,i} \quad (12)\]

If \(w_{l,i} = 0\) for all \(l \in [1 : g]\), then \(q\) is called acyclic.

The proof of sufficiency in Sect. 4.2 is relatively straightforward. Given an \((X, c, d)\)-feasible multicommodity flow \(f\) (5)–(8) on \(\Sigma\), the decomposition (12) is used directly to devise a routing solution \(S\).

The proof of necessity in Sect. 4.1 is more difficult and involves induction, using the following building blocks.

**Definition 4.1 (J-Flow).** Given an index set \(J \subseteq [1 : n]\), a nonnegative tuple \(f = (f_{\alpha,j})_{\alpha \in A, j \in J} \in \mathbb{R}^{\geq 0}^{|A|\cdot|J|}\) is called a \(J\)-flow on the structure \(\Sigma\) if \(\forall j \in J\) and \(v \in V \setminus \{\sigma_j\} \cup \{\tau_j\}\),
\[
\sum_{\alpha \in \text{IN}(v)} f_{\alpha,j} = \sum_{\alpha \in \text{OUT}(v)} f_{\alpha,j} \quad (j\text{-flow conservation}),
\]

As a convention, the \(0\)-flow is defined as the empty sequence (\(\varnothing\)).

\[\diamond\]

**Remark:** A \(J\)-flow is a (possibly infeasible) multicommodity flow with source-sink pairs \((\sigma_j, \tau_j), j \in J\). If each \(j\)-flow \(f_{\alpha,j}\) is acyclic, \(\forall j \in J\), then \(f\) is called an **acyclic** \(J\)-flow.

The next concept is central to the proof of necessity. It defines a class of feasible \([1 : i]\)-flows that obey certain information-theoretic bounds when only the signals \(X_j, j \in [1 : i]\), need to be communicated.

**Definition 4.2 (Informational Feasibility).** Given \(i \in [1 : n]\) and a setwise-causal arc-signal vector \(S\) (Def. 2.1), a \([1 : i]\)-flow \(f \in \mathbb{R}^{\mid A_i}_{\geq 0}\) (Def. 4.1) is called **informationally feasible** on the signal graph \((\Sigma, S)\) if it satisfies the following conditions:

i) On every arc \(\alpha \in A_i\),

\[
\sum_{j=1}^{i} f_{j,\alpha} \leq c_{\alpha}.
\]

ii) On any \(i\)-downward matched arc set \(B\) (Def. 3.3),

\[
\sum_{\alpha \in B, j \in [1 : i]} f_{\alpha,j} \leq I_\infty [X_{[1:i]} ; S_B | X_{[i+1:n]}].
\]

iii) On arcs entering sinks and leaving sources,

\[
f_{\text{IN}(\tau_j),j} = f_{\text{OUT}(\sigma_j),j} = I_\infty [X_j ; Y_j | X_{[j+1:n]}], \quad \forall j \in [1 : i].
\]

\[\diamond\]

**Remark:** Note as well that the \(0\)-flow is informationally feasible, since the condition (16) disappears and (15) is trivially satisfied due to a zero left-hand side (LHS).

The following result concerning informationally feasible \([1 : n]\)-flows also plays a key role.

**Lemma 4.1.** If the arc-signal vector \(S\) is a solution to the \(n\)-pairs information network problem \((\Sigma, X, c, d)\) (Def. 2.2), then any informationally feasible (Def. 4.2) \([1 : n]\)-flow \(f \in \mathbb{R}^{\mid A_i}_{\geq 0}\) on \((\Sigma, S)\) is an \((X, c, d)\)-feasible multicommodity flow on \(\Sigma\) (5)–(8).

**Proof.** See Appendix 2. \(\square\)

### 4.1 Necessity Proof for Theorem 3.1

Let the arc-signal vector \(S\) be a solution (Def. 2.2) to the \(n\)-pairs information network problem \((\Sigma, X, c, d)\). An informationally feasible \([1 : n]\)-flow (Def. 4.2) \(f^n\) will
be constructed, using upward induction. By Lemma 4.1, \( f^n \) will then be a feasible multicommodity flow, as desired.

Let \( \Sigma \) be downward matched (Def. 3.6) and suppose that \( f^{i-1} = (f_{\alpha,j})_{\alpha \in A, j \in [1:i-1]} \in \mathbb{R}_{\geq 0}^{A \times [1:i-1]} \) is an informationally feasible, acyclic \([1:i-1]\)-flow for some \( i \in [1:n] \), noting that the \( \emptyset \)-flow \( f^0 \) is informationally feasible. An \([1:i]\)-flow \((f_{\alpha,i})_{\alpha \in A} \in \mathbb{R}^A \) will be constructed in such a way that \( f^i \in \mathbb{R}^A \) will be an informationally feasible, acyclic \([1:i]\)-flow.

On any arc \( \alpha \in A \), let

\[
  r_\alpha := \begin{cases} 
  c_\alpha - \sum_{j=1}^{i-1} f_{\alpha,j} & \text{if } \alpha \in A_f \\
  \infty & \text{if } \alpha \in A_\infty \equiv A \setminus A_f 
  \end{cases}
\]  

be the residual capacity after subtracting the relevant components of \( f^{i-1} \). Note that \( r_\alpha \geq 0 \) since \( f^{i-1} \) is an informationally feasible \([1:i-1]\)-flow. The next step is to find an acyclic \([1:i]\)-flow (Def. 4.1) \( q \in \mathbb{R}^A \) from \( \sigma_i \mapsto \tau_i \) that is \( a) \leq \) the residual capacity on each arc, and \( b) \geq I_{\infty} [X_i;Y_i|X_{i+1:n}] \) on the arc entering \( \tau_i \). There are two mutually exclusive cases to consider.

### 4.1.1 1st Case: \( \exists \) an \( i \)-Path with No Finite-Capacity Arcs

Denote this \( i \)-path by \( \pi_e \), noting that \( r_\alpha = \infty \), \( \forall \alpha \in \pi_e \) by the 2nd line of (17). Set the \( i \)-path flows as

\[
u_k = \begin{cases} 
  I_\infty [X_i;Y_i|X_{i+1:n}] & \text{if } k = e \\
  0 & \text{otherwise}, \forall k \in [1:p], 
  \end{cases}
\]  

and the cycle flows equal to zero in the decomposition (12) (dropping the \( i \)-subscripts), so that

\[
  q_\alpha = \sum_{1 \leq k \leq p : \alpha \ni k} u_k, \forall \alpha \in A. 
\]  

Evidently \( q \) is acyclic and meets the residual capacity constraint on all arcs in \( A \). Furthermore, since every \( i \)-path passes through the single arc entering \( \tau_i \),

\[
  q_{IN(\tau_i)} = \sum_{1 \leq k \leq p} u_k \quad (19)
\]  

satisfying the conditional information constraint.

### 4.1.2 2nd Case: Every \( i \)-Path Has One or More Finite-Capacity Arcs

Observe first that for any arc set \( B \subseteq A_f \),
\sum_{\beta \in B} c_{\beta} \geq \sum_{\beta \in B} I^r[S_{\beta};X] \\
\equiv \sum_{\beta \in B} \lim_{t \to \infty} \frac{H[S_{\beta}(0:t)] - H[S_{\beta}(0:t)|X(0:t)]}{t+1} \\
= \sum_{\beta \in B} \lim_{t \to \infty} \frac{H[S_{\beta}(0:t)]}{t+1} \\
\geq \lim_{t \to \infty} \frac{1}{t+1} \sum_{\beta \in B} H[S_{\beta}(0:t)] \\
\geq \lim_{t \to \infty} \frac{H[S_B(0:t)]}{t+1} \\
\geq \lim_{t \to \infty} \frac{H[S_B(0:t)|X_{[i+1:n]}(0:t)]}{t+1} \\
= \lim_{t \to \infty} \frac{H[S_B(0:t)|X_{[i+1:n]}(0:t)] - H[S_B(0:t)|X(0:t)]}{t+1} \\
= \lim_{t \to \infty} \left( \frac{H[S_B(0:t)|X_{[i+1:n]}(0:t)] - H[S_B(0:t)|X(0:t)]}{t+1} + \frac{H[S_B(0:t)|X_{[i:n]}(0:t)] - H[S_B(0:t)|X(0:t)]}{t+1} \right) \\
= \lim_{t \to \infty} \left( \frac{I[S_B(0:t);X_i(0:t)|X_{[i+1:n]}(0:t)]}{t+1} + \frac{I[S_B(0:t);X_{[i:n]}(0:t)]}{t+1} \right) \\
\geq I^r[X_i;S_B|X_{[i+1:n]}] + I^r[X_{[i:n]};S_B|X_{[i:n]}]. 

In (21) and (24), the conditional discrete entropies are 0 since, by Lemma 3.1, $S_B(0:t)$ is a function of $X(0:t)$; the inequality (22) is due to the subadditivity of joint entropy; and (23) holds because conditioning cannot increase entropy.

Now, consider the residual capacitated digraph $(\mathcal{V}', \mathcal{A}', r_{\mathcal{A}'})$ formed by the $i$-bundle.\(^6\) Let $q$ be an acyclic maximal flow on it under the constraints

\[ 0 \leq q_\alpha \leq r_\alpha, \quad \forall \alpha \in \mathcal{A}'. \tag{26} \]

By the *Min-Cut Max-Flow Theorem* (see e.g. [4], Thm. 3.5.3.) \(\exists\) an $i$-cut $U \subset \mathcal{V}'$, consisting of every vertex $v \in \mathcal{V}'$ for which \(\exists\) an undirected path $\pi$ in $(\mathcal{V}', \mathcal{A}')$ from $\sigma_i$ to $v$ s.t.

\(^6\) Here, arcs are permitted to have $r_\alpha = 0$. 

[4] G.N. Nair. Structural Routability of $n$-Pairs Information Networks
• (Forward Slack) every forward-oriented arc $\alpha$ in $\pi$ (i.e. pointing from $\sigma_i$ to $\nu$) has $q_\alpha < r_\alpha$, and
• (Backward Flow) every backward-oriented arc $\alpha$ in $\pi$ (pointing from $\nu$ to $\sigma_i$) has $q_\alpha > 0$.

As a consequence of this,

$$q_\alpha = r_\alpha, \forall \alpha \in O^i := \text{OUT}(U) \cap A^i,$$

$$q_\alpha = 0, \forall \alpha \in I^i := \text{IN}(U) \cap A^i.$$  

Note also that since the cyclic flow components $w_1, \ldots, w_j$ in (12) are zero,

$$q_\alpha = \sum_{1 \leq k \leq p : \pi_k \ni \alpha} u_k, \forall \alpha \in A^i.$$  

The $i$-cut $U$ evidently depends on the residual capacity vector $r$. However, the following purely structural statements may be made about it:

1. Every arc in $O^i$ lies in $A^i$, i.e. is finite-capacity. Otherwise $q_\alpha \geq r_\alpha \geq \infty$, implying by (29) that $u_k = \infty$ on some $i$-path $\pi_k$, which is impossible since every $i$-path in this case travels over at least one finite-capacity arc.

2. Every arc $\alpha \in O^i$ is in an $i$-path that exits $U$ without re-entering, or else $\alpha$ is in the $[1 : i - 1]$-bundle. To see this, suppose that every $i$-path $\pi_k$ passing through $\alpha$ re-enters $U$. Evidently, it must then pass through some arc $\beta \in I^i$. By (28) $q_\beta = 0$, implying by virtue of (29) and nonnegativity that $u_k = 0$. From (27) and (28), this implies that $r_\alpha = 0$. As $c_\alpha > 0$, it must then hold that $f_{\alpha,j} > 0$ for some $j \in [1 : i - 1]$. As the $j$-flow $(f_{\alpha,j})_{\alpha \in A}$ is acyclic by construction, $\alpha$ must then lie on a $j$-path, by (12).

3. There must be an $i$-path that leaves $U$ without re-entering. To see this, suppose in contradiction that every $i$-path re-enters $U$. By the preceding argument, all $i$-paths must then have associated acyclic flow components $u_k = 0$. Pick any $i$-path and let $\nu$ be the last vertex in $U$ that it traverses before leaving $U$ without further re-entry. Let $\omega$ denote its subpath from $\nu \sim \tau_i$. By the definition of $U$, there is an undirected path $\pi$ from $\sigma_i$ to $\nu$ such that all forward-oriented arcs in it are slack and all backward-oriented arcs carry strictly positive $q$-flow. Note also that all vertices before $\nu$ in $\pi$ must also lie in $U$, by construction. From (29), any backward arc in $\pi$ would have to carry an $i$-path flow component $u_k > 0$, which would be a contradiction. Consequently, all the arcs in $\pi$ must be forward-oriented, i.e. $\pi$ is a directed path in $U$ from $\sigma_i \sim \nu$. The concatenation of $\pi$ with $\omega$ then yields an $i$-path that leaves $U$ exactly once, a contradiction.

4. Finally, by construction of $U$, every vertex $\nu$ in it must lie on an undirected path $\pi$ from $\sigma_i$ to $\nu$ such that

a. every vertex before $\nu$ in $\pi$ is also in $U$ (since the subpath from $\sigma_i$ to $\nu$ automatically satisfies the defining forward-slack and backward-flow properties), and
b. every reverse-oriented arc in $\pi$ lies on an $i$-path that does not re-enter $U$ (since such arcs must by definition carry positive $q$-flow, and $i$-paths that re-enter $U$ carry zero $q$-flow).

In other words, $U$ is a viable $i$-cut (Def. 3.2). By downward matching (Def. 3.6), $O'$ contains an $(i-1)$-downward matched arc set (Def. 3.3) $B \supseteq O' \cap A_{i-1}$ and a downward whole $i$-choke (Def. 3.2) $B'$. Using $i$-flow conservation,

$$q_{\text{IN}(\tau_i)} = \sum_{\beta \in O'} q_\beta - \sum_{\alpha \in F} q_\alpha \sum_{\beta \in O'} r_\beta$$

$$= \sum_{\beta \in O'} c_\beta - \sum_{\beta \in O', j \in [1:i-1]} f_{\beta, j}$$

$$\geq L_\infty \left[ X_i; S_{O'} | X_{i+1:n} \right] + L_\infty \left[ X_{i+1:1}; S_{O'} | X_{i:n} \right] - \sum_{\beta \in B, j \in [1:i-1]} f_{j, \beta}$$

$$\geq L_\infty \left[ X_i; S_{B'} | X_{i+1:n} \right] + L_\infty \left[ X_{i+1:1}; S_B | X_{i:n} \right] - \sum_{\beta \in B, j \in [1:i-1]} f_{j, \beta}$$

$$\geq L_\infty \left[ X_i; S_B | X_{i+1:n} \right],$$

where the identity (30) holds because the acyclic $[1: i - 1]$ flow $f^{i-1}$ has zero-valued components on all arcs not in $A_{i-1}$, and (31) is due to the monotonicity of (conditional) information. As $B'$ is a downward whole $i$-choke, applying Lemma 3.3 to the RHS of (32) yields

$$q_{\text{IN}(\tau_i)} \geq L_\infty \left[ X_i; S_{B'} | X_{i+1:n} \right] \geq L_\infty \left[ X_i; Y_i | X_{i+1:n} \right],$$

as desired.

### 4.1.3 Construction of $f^t$ in Both Cases

For both cases above, let

$$f_{\alpha, i} := \frac{L_\infty \left[ X_i; Y_i | X_{i+1:n} \right]}{q_{\text{IN}(\tau_i)} \equiv v q_\alpha, \; \forall \alpha \in A},$$

where $v \in (0, 1]$. Clearly, $f_{\alpha, i}$ is still an acyclic $i$-flow since it just a scaled version of $q$. Furthermore,

$$\sum_{j=1}^{i} f_{\alpha, j} v q_\alpha + \sum_{j=1}^{i-1} f_{\alpha, j} \leq q_\alpha + \sum_{j=1}^{i-1} f_{\alpha, j} \leq c_\alpha.$$
It is next verified that $f^i = f_{A \times [1:j]}$ satisfies the remaining conditions \[15\]--\[16\] for an informationally feasible $[1 : i]$-flow. Let $E$ be any $i$-downward matched arc set if $E \cap A^{[1:j]} = \emptyset$, then \[15\] holds trivially with a zero LHS. Otherwise $E$ must contain sets $E_1, E_2$ and $E_3$ each satisfying the conditions listed in Def. \[3.3\]. Then

$$
\sum_{\eta \in E_1, j \in [1:i]} f_{\eta,j} = \sum_{\eta \in E} f_{\eta,j} + \sum_{\eta \in E, j \in [1:i-1]} f_{\eta,j} = \sum_{\eta \in E_1} f_{\eta,j} + \sum_{\eta \in E_2, j \in [1:i-1]} f_{\eta,j} \tag{35}
$$

By the $i$-disjointness of $E_1$, each $i$-path $\pi_p$ transits over at most one arc in it. The nonnegativity of the $i$-path flows $u_1, \ldots, u_p$ then implies that

$$
\sum_{\eta \in E_1} \left( \sum_{1 \leq k \leq p; \eta \ni \eta_k} u_k \right) = \sum_{\eta \in E_1} \left( \sum_{1 \leq k \leq p} u_k \right) \leq \sum_{1 \leq k \leq p} u_k = q_{IN}(\tau_i). \tag{37}
$$

Furthermore, since $E_2$ is $(i-1)$-downward matched,

$$
\sum_{\eta \in E_2, j \in [1:i-1]} f_{\eta,j} \leq L_\infty \left[ X_{[1:i-1]} ; S_{E_2} | X_{[i:n]} \right]. \tag{38}
$$

Substituting this and (37) into (36) then yields

$$
\sum_{\eta \in E_1, j \in [1:i]} f_{\eta,j} \leq v q_{IN}(\tau_i) + L_\infty \left[ X_{[1:i-1]} ; S_{E_2} | X_{[i:n]} \right]. \tag{39}
$$

where (39) is due the monotonicity of conditional information, and (38) follows from applying Lemma \[3.3\] on the downward whole $i$-choke $E_3$. This confirms that $f^i$ satisfies \[15\]. As $f^{-1}$ is an informationally feasible $[1 : i - 1]$ flow, \[16\] is satisfied $\forall j \in [1 : i - 1]$. By flow conservation,
\[ f_{\text{OUT}}(\sigma), i = f_{\text{IN}}(\tau), i \]

verifying (16) when \( j = i \). Thus \( f^i \) is an informationally feasible \([1 : i]\)-flow.

By upward induction, \( f_n \) is an informationally feasible \([1 : n]\)-flow. If the reverse structure \( \Sigma' \) is downward matched, the induction goes through identically, but with \((A'_f, A'_\infty, P')\) replacing \((A_f, A_\infty, P)\). The informationally feasible \([1 : n]\)-flow \( f''_n \) obtained on \( \Sigma' \) then yields an informationally feasible \([1 : n]\)-flow \( f''_n \) on \( \Sigma \), using \( f((\mu, \nu)), j = f''((\nu, \mu)), j \) \( \forall (\mu, \nu) \in A \) and \( j \in [1 : n] \).

By Lemma 4.1, \( f_n \) is automatically an \((X, c, d)\)-feasible multicommodity flow, as desired.

### 4.2 Sufficiency Proof for Theorem 3.1

The sufficiency part of Thm. 3.1 is easier to establish, since it is not difficult to see that the existence of a feasible multicommodity flow implies achievability. Thus only the key steps are provided below.

Suppose \( f \) is an \((X, c, d)\)-feasible multicommodity flow (5)–(8) on \( \Sigma \). In the decomposition (12) for each \( i \)-flow \( f_A, i \), no cycle flow can enter any sink, since it has no departing arcs. Consequently, the cycle flows may be taken to be zero in (12) without violating (5)–(8), yielding

\[ f_{\alpha, i} = \sum_{1 \leq k \leq p_i: \pi_k, i \ni \alpha} u_{k, i}, \quad (41) \]

where \( \pi_{1, i}, \ldots, \pi_{p_i, i} \) are the \( i \)-paths and \( u_{1, i}, \ldots, u_{p_i, i} \geq 0 \), the \( i \)-path flows.

As \( \sum_{k=1}^{p_i} u_{k, i} \leq H_\infty[X_i] \), Slepian–Wolf-style coding ideas can be used, at the vertex succeeding each source \( \sigma_i \), to causally produce \( p_i \) mutually independent data-streams \( Z_{i, 1}, \ldots, Z_{i, p_i} \) from each source signal \( X_i \), with the \( k \)-th data-stream designed to have average bit-rate \( H^m[Z_{i, k}] = H_\infty[Z_{i, k}] = u_{k, i} \) and routed along the \( i \)-path \( \pi_{k, i} \). On every arc \( \alpha \in A \) apart from the ones leaving sources, the arc signal is then represented by a vector \( S_\alpha = (Z_{i, k})_{i \in [1, n], k \in [1, p_i]: \pi_{k, i} \ni \alpha} \) with mutually independent components.\(^7\) The arc signals leaving sources are set to the respective source signals to satisfy (2). It may be verified that \( S \) is setwise causal (Def. 2.1), since every signal \( Z_{i, k} \) is routed along an acyclic path. In addition, \( \forall \alpha \in A_f \),

\(^7\) If an arc is not on any \( i \)-path, then its arc signal may be taken to be a constant.
\[ I^{\infty}[S_{\alpha};X] = H^{\infty}[S_{\alpha}] = H^{\infty}\left[\left(Z_{i,k}\right)_{i\in[1:n],k\in[1:p_i]:\pi_{k,i} \ni \alpha}\right] \] (42)
\[ \leq \sum_{i\in[1:n],k\in[1:p_i]:\pi_{k,i} \ni \alpha} H^{\infty}[Z_{i,k}] \] (43)
\[ = \sum_{i\in[1:n],k\in[1:p_i]:\pi_{k,i} \ni \alpha} u_{k,i} \] (44)
\[ \leq c_{\alpha}, \]

where the 1st equality in (42) holds because \( S_{\alpha} \) is causally determined by \( X \) by Lemma 3.1 and (43) is due to the subadditivity of entropy. Furthermore,

\[ I^{\infty}[X_i;Y_i] \equiv I^{\infty}[X_i;S_{\text{IN}(\tau_i)}] = I^{\infty}[X_i;Z_{i,1},\ldots,Z_{i,p_i}] \]
\[ = H^{\infty}[Z_{i,1},\ldots,Z_{i,p_i}] = \sum_{k=1}^{p_i} H^{\infty}[Z_{i,k}] \]
\[ = \sum_{k=1}^{p_i} u_{k,i} \leq d_i, \]

where (44) is due to the fact that \( Z_{i,1},\ldots,Z_{i,p_i} \) are causally determined by \( X_i \), and are also mutually independent. Consequently, \( S \) is a solution to the \( n \)-pairs information network problem \((\Sigma, X, c, d)\), establishing achievability (Def. 2.2).

5 Examples

In this section, several examples are given to illustrate the applicability of Thm. 3.1. However, to begin with a well-known counterexample is discussed.

5.1 Butterfly Network

The first example, a 2-pairs butterfly network, is adapted from [11, 7] and depicted in Fig. 1. All its arcs have infinite capacities except on the arc from \( \omega_1 \) to \( \omega_2 \). The signals \( X_1 \) and \( X_2 \) are i.i.d. and binary-valued with \( H[X_1(t)] = H[X_2(t)] = 1 \), and the demands \( d_1 = d_2 = 1 \).

The vertex set \( U = \{\sigma_2, v_1, \omega_1\} \subset V_2 = \{\sigma_2, v_1, \omega_1, \omega_2, \mu_2, \tau_2\} \) is a 2-cut. Observe that

1. \( \text{OUT}(U) \cap A^2 = \{\omega_1, \omega_2\} \) is finite-capacity,
2. the 2-path \( (\sigma_2, v_1, \omega_1, \omega_2, \mu_2, \tau_2) \) leaves \( U \) without returning,
3. \( \text{OUT}(U) \cap A^2 = \{\omega_1, \omega_2\} \) lies in this non-returning 2-path, and
4. every vertex $\in U$ lies in an undirected path from $\sigma_2$ such that all its predecessors in the path are also in $U$, with no reverse-oriented arcs.

Consequently $U$ is a viable 2-cut (Def. 3.4). However, $\text{OUT}(U) \cap A^2 = \{(\omega_1, \omega_2)\}$ does not contain a downward whole $i$-choke (Def. 3.2), since one $\sigma_1 \rightarrow \tau_2$-path, $(\sigma_1, \mu_1, \omega_1, \omega_2, \mu_2, \tau_2)$, bypasses it but the other, $(\sigma_1, \mu_1, \omega_1, \omega_2, \mu_2, \tau_2)$, does not. Consequently, this network is not downward matched (Def. 3.6) and Thm. 3.1 does not apply.

This is in agreement with [7]. Observe that if $c = c_{\omega_1, \omega_2} = 2$, then $(X, c, d)$ is achievable by routing, since 1 unit of flow can be routed from $\sigma_1$ to $\tau_1$ and from $\sigma_2$ to $\tau_2$ via $(\omega_1, \omega_2)$, without exceeding its arc-capacity. However, if $c$ is reduced to 1, then the network can no longer support an $(X, c, d)$-feasible 2-commodity flow (5)–(8). Nonetheless, $(X, c, d)$ still remains achievable, by sending the signal $X_1$ on $(\mu_1, \mu_2)$, $X_2$ on $(\nu_1, \nu_2)$, and $Z = X_1 + X_2$ (mod 2) on $(\omega_1, \omega_2)$, $(\omega_2, \mu_2)$ and $(\omega_2, \nu_2)$, and setting $Y_2 = Z + X_1$ (mod 2) = $X_2$ and $Y_1 = Z + X_2$ (mod 2) = $X_1$.

5.2 Directed Lines

In this subsection, it is shown that downward matching (Def. 3.6) holds for directed lines, i.e. networks consisting of a cascaded sequence of arcs with $n$ source-sink pairs connected to the vertices. Theorem 3.1 then implies that $(X, c, d)$ is achievable on such a structure iff there exists an $(X, c, d)$-feasible multicommodity flow agreeing with these earlier works.

The argument proceeds by induction. For any $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the proposition that for all $n$-pairs directed lines,

i. downward matching holds (Def. 3.6), and

ii. for each $i \in [1:n]$, every arc in the $i$-bundle arc-set $A^i$ is $i$-downward matched (Def. 3.3).

Note that that all 1-pair directed lines trivially satisfy $\mathcal{P}_1$. Suppose that $\mathcal{P}_{n-1}$ holds for some $n \geq 2$. Without loss of generality assume that the line is directed to the
right and label the sources from left to right as $\sigma_1, \ldots, \sigma_{n-1}$. Observe that any $n$-pairs directed line can be constructed from an $(n-1)$-pairs one by attaching a source $\sigma_n$ somewhere after $\sigma_{n-1}$.

First, condition (i) above is checked. By hypothesis, conditions 1 and 2 in Def. 3.6 are satisfied $\forall i \in [2 : n-1]$, so it only remains to verify them for the case $i = n$. Notice that the outgoing arc-set of every viable $n$-cut consists of a single arc through which every $n$-path passes. Furthermore, every source $\sigma_i$, $i < n$ can reach $\tau_n$ only by passing through this arc. Thus every viable $n$-cut is a downward whole $n$-choke (Def. 3.2), satisfying condition 2. In addition, since each viable $n$-cut consists of a single arc, it is either completely in or out of the $[1 : n-1]$-bundle arc-set $A_{[1:n-1]}$. If it is out of $A_{[1:n-1]}$, then the set $B$ in condition 1 may be taken to be empty and is thus automatically $(n-1)$-downward matched. On the other hand, if it is in $A_{[1:n-1]}$, then by hypothesis it is $(n-1)$-downward matched, meeting condition 1.

Next, condition (ii) is established. By hypothesis, it holds for each $i \in [1 : n-1]$ so only needs to be verified for $i = n$. Consider any arc $\alpha \in A^n$ and observe that the singleton $E_1 = \{ \alpha \}$ is automatically $n$-disjoint (Def. 3.1). If $\alpha \in A_{[1:n-1]}$ then set $E_2 = \{ \alpha \}$, which is $(n-1)$-downward matched by hypothesis; otherwise set $E_2 = \emptyset$, which is $(n-1)$-downward matched by Def. 3.3. Finally, note that by the same reasoning as in the preceding paragraph, $E_3 = \{ \alpha \}$ is a downward whole $n$-choke. Thus all the three conditions of Def. 3.3 are satisfied when $i = n$, verifying (ii).

By induction, $\mathcal{P}_n$ holds for all $n \in \mathbb{N}$, and in particular, condition (ii) implies structural routability, by virtue of Thm. 3.1.

5.2.1 Directed Cycles

The argument above can be extended to show that downward matching is also satisfied by the class of directed $n$-pairs cycles that satisfy the property that for each index $i \in [2 : n]$ and every $j \in [1 : i-1]$, the source $\sigma_j$ occurs after $\tau_i$ and before $\sigma_i$, in the orientation of the cycle.

However, in [6] it was shown that for any $n$-pairs directed cycles, network coding solutions are possible iff a feasible multicommodity flow exists. A similar result had also been proved in [7] for the case of a directed triangle with $n = 2$.

This indicates that downward matching is a conservative structural condition, basically due to the requirement that viable $i$-cuts contain downward whole $i$-chokes. It may be possible to loosen it further by exploiting notions of informational dominance [6] and d-separation [7].

5.3 A Cyclic Example

Next, consider the cyclic 2-pairs network of Fig. 2. The only 2-cuts $U^k$ with $\text{OUT}(U^k) \cap A^2 \subseteq A_i$ are $U^1 = \{ \sigma_2, \mu_1, \omega \}$, $U^2 = \{ \sigma_2, \mu_1, \omega, \nu \}$ and $U^3 = \{ \sigma_2, \mu_1, \omega, \nu_1, \nu_2 \}$.
For the first two 2-cuts,

$$\text{OUT}(U^k) \cap A^2 = \{ (\omega, \mu_2), (\mu_1, v_2) \}, \ k = 1, 2,$$

while \( \text{OUT}(U^3) \cap A^2 = \{ (\omega, \mu_2) \} \). Observe that

- There is a 2-path, \( \pi = (\sigma_2, \mu_1, \omega, \mu_2, \tau_2) \), that leaves each \( U^k \) without re-entering.
- The single arc \( (\omega, \mu_2) \) in \( \text{OUT}(U^3) \cap A^2 \) lies in the non-returning 2-path \( \pi \). The other arc \( (\mu_1, v_2) \) in \( \text{OUT}(U^k) \cap A^2, \ k = 1, 2 \), does not lie on a non-returning 2-path, but does lie on a 1-path.
- Every vertex in \( U^1 \) lies in an undirected path from \( \sigma_2 \) that does not leave \( U^1 \) and does not contain any backward-oriented arcs; the same for \( U^3 \).
- Every vertex \( v \) in \( U^2 \) lies in one undirected path from \( \sigma_2 \) that does not leave \( U^2 \). However, when \( v = v_1 \), this path contains a backward arc \( (v_1, \omega) \) that does not lie on a non-returning 2-path.

Consequently, the only viable 2-cuts (Def. 3.4) are \( U^1 \) and \( U^3 \).

Now, each arc set \( \text{OUT}(U^k) \cap A^2, \ k = 1, 3 \), contains an arc set \( B = \{ (\omega, \mu_2) \} \), which being a singleton is automatically 1-disjoint. Furthermore, both 2-paths and the only 1-path pass through it, as does the only \( \sigma_1 \leadsto \tau_2 \)-path. Thus \( B \) is a downward whole 1- and 2-choke, and so is downward matched (Def. 3.6). Consequently, by Thm. 3.1 any tuple \((X, c, d)\) is achievable iff there exists an \((X, c, d)\)-feasible multicommodity flow. After eliminating the flow variables, \((X, c, d)\) is thus achievable iff

\[
\begin{align*}
d_1 + d_2 & \leq c_{\omega, \mu_2}, \\
d_1 & \leq \min \left\{ H_\infty[X_1], c_{\mu_1, v_2} \right\}, \\
d_2 & \leq H_\infty[X_2].
\end{align*}
\]
6 Conclusion

This chapter examined the routability of possibly cyclic $n$-pairs information networks from a structural perspective. The concept of downward matching was introduced, and it was shown that for networks with downward matched structures, routability and achievability are equivalent, i.e. a given combination of source signals, demand rates and channel capacities is achievable iff the network supports a feasible multicommodity flow.

Downward matching is a conservative structural condition, and future work will focus on trying to relax it. The inductive nature of the proof of necessity here requires it directly, so any generalisation may need a very different analysis technique.

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Appendix 1- Proof of Lemma 3.2

Define $S_J := \{\sigma_j : j \in J\}$ and $S_{J^c} := \{\sigma_j : j \in J^c\}$, and let $\pi_1, \ldots, \pi_l$ be the different paths from $S_J$ to $\tau_i$, each of which must pass through $B$. Let $\gamma_k$ be the last arc $\in B$ on $\pi_k$; $\rho_k$, the subpath along $\pi_k$ from the head of $\gamma_k$ to $\tau_i$; and $V_k$, the set of all vertices $\neq \tau_i$ (therefore internal) on $\rho_k$. Define $U := \bigcup_{k=1}^l V_k$ and $C := \{\gamma_k : 1 \leq k \leq l\} \subseteq B$. Let $D := B \cap \text{IN}(U)$ and $E := B^c \cap \text{IN}(U)$.

It is next shown that no arc $\eta \in E$ is reachable from $S_J$. Let $\eta$ have tail $\nu$ and head $\mu$. Note that $\nu \notin U$ and $\mu \in U$, by definition of $\text{IN}(U)$. Plainly, $\mu$ must be the first vertex of a (possibly nonunique) path $\rho_k$ leading to $\tau_i$. In addition, $\mu$ has an incoming arc $\gamma_k \in D$. Suppose that there were to be a path $\pi$ from $S_J$ to $\nu$. There would then be a path $(\pi, \eta, \rho_k)$ from $S_J$ to $\tau_i$, via the arc $\eta \notin B$. By definition, this path would have to pass through $B$ at some stage, with the last arc of $B$ it passes through denoted $\gamma_m$. As $\eta \notin B$ and $\rho_k$ does not pass through $B$, it follows that $\gamma_m$ lies on $\pi$. Thus all non-$\tau_i$ vertices, including $\nu$ that are after the arc $\gamma_m$ on $\rho_m$, would have to be in $U$, contradicting the fact that $\nu \notin U$.

As the arc leading into $\tau_i$ is an element of $\text{OUT}(U)$, set-wise causality (1) implies that
\[ Y_i(t) \equiv \text{IN}(\tau_i)(t) \equiv g(U)(t, S_D(0:t), S_E(0:t)), \quad \forall t \in W. \quad (45) \]
As $E$ is unreachable from $S_J$, Lemma 3.1 implies that
\[ S_E(t) \equiv g(U)(t, X_J(0:t)), \quad \forall t \in W, \]
which substituted into (45) yields
\[ Y_i(t) = g'_U(t, S_D(0 : t), g_U(t, X_K(0 : t))) \]
\[ \equiv g'_U(t, S_D(0 : t), X_K(0 : t)), \forall t \in W. \]

Observing that \( D \subseteq B \) completes the proof.

Appendix 2 - Proof of Lemma 4.1

It will be shown that (13) with \( J = \{1 : n\} \), (14)–(16) with \( i = n \), and Def. 2.2 imply (5)–(8). First, observe that (13) with \( J = \{1 : n\} \) is identical to (8), and (14) with \( i = n \), to (5). Next, note that (16) with \( i = n \) yields

\[
 f_{IN(\tau_j), \varphi} \equiv \lim_{t \to \infty} \left\{ \frac{H[X_j(0 : t) | X_{j+1:n}(0 : t)]}{t+1} \right\} \]
\[
 = \lim_{t \to \infty} \left\{ \frac{H[X_j(0 : t)] - H[X_j(0 : t) | X_{j+1:n}(0 : t)]}{t+1} \right\} \]
\[
 = \lim_{t \to \infty} \left\{ \frac{I[X_j(0 : t); Y_j(0 : t), X_{j+1:n}(0 : t)]}{t+1} \right\} \]
\[
 \equiv \lim_{t \to \infty} I[X_j; Y_j, X_{j+1:n}] \geq d_j, \forall j \in [1 : n], \quad (46)
\]

yielding (6), where (46) follows from the mutual independence of \( X_1, \ldots, X_n \). (If \( X_j \) is continuous-valued then \( h \) replaces \( H \).)

Finally, if \( X_j \) is continuous-valued then \( H[X_j(0 : t)] = \infty, \forall t \in W \), so \( H_{\infty}[X_j] = \infty \) and (7) automatically holds. On the other hand, if \( X_j \) is discrete-valued then
\[ f_{\text{OUT}(\sigma_i),j}^{[16]} \geq \sum_{j=1}^{\infty} I_{\sigma_i}[X_j;Y_j|X_{j+1:n}] = \lim_{t \rightarrow \infty} \left\{ \frac{H[X_j(0:t)|X_{j+1:n}(0:t)]}{t+1} ight\} \]

\[ \leq \lim_{t \rightarrow \infty} \frac{H[X_j(0:t)] - H[X_j(0:t)|Y_j(0:t),X_{j+1:n}(0:t)]}{t+1} \]

\[ \leq \lim_{t \rightarrow \infty} \frac{H[X_j(0:t)]}{t+1} = H_{\sigma_i}[X_j], \quad (48) \]

where the bound in (48) is due to the nonnegativity of discrete entropy.

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