GLOBAL DIMENSION FUNCTION ON STABILITY CONDITIONS
AND GEPNER EQUATIONS

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Abstract. We study the global dimension function \( \text{gldim} : \text{Aut} \ D / \mathbb{C} \rightarrow \mathbb{R}_{\geq 0} \)
on a quotient of the space of Bridgeland stability conditions on a triangulated category \( D \) as well as Toda’s Gepner equation \( \Phi(\sigma) = s \cdot \sigma \) for some \( \sigma \in \text{Stab} \ D \) and \( (\Phi, s) \in \text{Aut} \ D \times \mathbb{C} ).

For the bounded derived category \( \mathcal{D}^b(\mathbb{K} Q) \) of a Dynkin quiver \( Q \), we show that there is a unique minimal point \( \sigma_G \) of \( \text{gldim} \) (up to the \( \mathbb{C} \)-action), with value \( 1 - 2/h \), which is the solution of the Gepner equation \( \tau(\sigma) = (-2/h) \cdot \sigma \). Here \( \tau \) is the Auslander-Reiten functor and \( h \) is the Coxeter number. This solution \( \sigma_G \) was constructed by Kajiura-Saito-Takahashi. We also show that for an acyclic non-Dynkin quiver \( Q \), the minimal value of \( \text{gldim} \) is 1.

Our philosophy is that the infimum of \( \text{gldim} \) on \( \text{Stab} \ D \) is the global dimension for the triangulated category \( D \). We explain how this notion could shed light on the contractibility conjecture of the space of stability conditions.

Key words: global dimension, stability conditions, Gepner equations

1. Introduction

1.1. Global dimension for triangulated categories. A Bridgeland stability condition \( \sigma = (Z, P) \) on a triangulated category \( D \) consists of a central charge \( Z \in \text{Hom}(K(D), \mathbb{C}) \) and a slicing \( P = \{ P(\phi) \mid \phi \in \mathbb{R} \} \), which is a collection of additive/abelian subcategories. All stability conditions on \( D \) form a complex manifold \( \text{Stab} \ D \) ([B1]), which is a homological invariant of \( D \). The study of spaces of stability conditions already has many applications in several topics, such Donaldson-Thomas theory, cluster theory and (homological) mirror symmetry (cf. e.g. [BS, DHKK, KS]).

With the following two motivations:

1°. to understand the conjectural Frobenuis-Saito structure on the space of stability conditions, cf. [I, BQS],

2°. to understand the theory that realizes stability conditions on Fukaya type categories via quadratic differentials on Riemann surfaces, cf. [BS].

We have introduced \( q \)-deformations of stability conditions in [IQ1]. There, we have also introduced a function, the global dimension function,

\[
\text{gldim} \ P = \sup \{ \phi_2 - \phi_1 \mid \text{Hom}(P(\phi_1), P(\phi_2)) \neq 0 \}
\]
on the set of (slicings of) stability conditions \( \sigma = (Z, P) \). This function plays a key role in the existence of \( q \)-deformations of stability conditions.
In the case where $D = D^b(A)$ for some algebra $A$ and where its canonical heart $H = \text{mod } A$ is concentrated in one slice $P(\phi_0)$ (which implies that $P(\phi)$ vanishes if $\phi - \phi_0 \notin \mathbb{Z}$), we have

$$\text{gldim } P = \text{gldim } H = \text{gldim } A,$$

where gldim $A$ is the classical global dimension of the algebra $A$. Our philosophy is that the infimum of gldim on $\text{Stab } D$ is the categorical global dimension for a triangulated category $D$, i.e.

$$\text{gldim } D : = \inf \text{gldim}(\text{Stab } D).$$

For instance, when the category $D$ is the bounded derived category

$$D^\infty(Q) : = D^b(Q)$$

of a Dynkin quiver $Q$, we have

$$\text{gldim } D^\infty(Q) = 1 - 2/h,$$

where $h$ is the Coxeter number. This refines the usual global dimension of the path algebra of a Dynkin quiver.

1.2. Gepner equations. A Gepner equation on the space $\text{Stab } D$ is an equation of the form

$$\Phi(\sigma) = s \cdot \sigma,$$

where the parameters $(\Phi, s) \in \text{Aut } D \times \mathbb{C}$ consist of an autoequivalence and a complex number.

In the setting of [IQ1, IQ2], the category $D = D_X$ admits a distinguished autoequivalence $X$ which yields an isomorphism between the Grothendieck group $K(D_X)$ and $(\mathbb{Z}[q, q^{-1}])^{\oplus n}$. Then the Gepner equation $X(\sigma) = s \cdot \sigma$ defines what we call $q$-stability conditions on $D_X$.

In the usual setting, where one has $K(D) \cong \mathbb{Z}^n$, Gepner equations where studied by Toda [To1, To2, To3], who was interested in finding stability conditions with a symmetry property. In this case, a solution to such an equation is called a Gepner point. It is an orbifold point in $\text{Aut } \setminus \text{Stab } D / \mathbb{C}$. Toda’s motivation came from the study of Donaldson–Thomas invariants, in particular for B-branes on Landau-Ginzburg models associated to a superpotential. Toda constructed the central charge of a conjectural Gepner point $\sigma_G$ for categories $D = \text{HMF}(f)$ that arise as homotopy categories of graded matrix factorizations.

In the Dyknin case, i.e. when $\text{HMF}(f)$ is derived equivalent to $D^\infty(Q)$ for a Dynkin quiver $Q$, Kajiura-Saito-Takahashi [KST1] constructed such a Gepner point $\sigma_G$, a solution of equation (4.8), in the sense of Toda (actually before Toda’s work). The non trivial but very interesting fact is that $\sigma_G$ is actually the only solution of the Gepner equation $\tau(\sigma) = (-2/h) \cdot \sigma$, where $\tau$ is the Auslander-Reiten functor and $h$ is the Coxeter number.

This stability condition $\sigma_G$ is the most stable one, in the sense that
• $\sigma_G$ is totally stable, i.e. every indecomposable object is stable. Moreover, it is the only stability condition, up to the $\mathbb{C}$-action, where the function $\text{gldim}$ takes its minimal value $1 - 2/h$.

• The phases between stable objects are spread out evenly, cf. Figure 2, where the diagonals of a regular $(n+1)$-gon give the central charges of all the indecomposable objects in the $A_n$-case.

The notion of total (semi)stability is related to a conjecture of Reineke [R], which asserts the module category of any Dynkin quiver admits a totally (semi)stable stability condition (see [Q2, § 7.4] for a partial answer). We show that a stability condition $\sigma$ is totally semistable if and only if $\text{gldim} \sigma = 1$ (Proposition 3.5).

It would be interesting to calculate all possible Gepner points, say in the case where $\mathcal{D} = \mathcal{D}_\infty(Q)$. For instance, when the parameter $s$ in $(\Phi, s)$ is zero, then Gepner points correspond to stability conditions for certain species which are obtained from $Q$ by folding (cf. [CQ]).

1.3. Contents. The paper is organized as follows.

In Section 2, we collect basic facts about stability conditions. In Section 3, we introduce the global dimension function $\text{gldim}$ and study its general properties. In particular, we discuss its relation with total stability and Serre functors. For a type A quiver, we give a description of all total stability conditions in Proposition 3.6 and an explicit formula for calculating $\text{gldim}$ in such a situation (cf. Proposition 3.8).

In Section 4 and Section 5, we study the case of the bounded derived category $\mathcal{D}_\infty(Q)$ of an acyclic quiver $Q$. In Theorem 4.7, we show that when $Q$ is a Dynkin quiver, the minimal value $1 - 2/h$ of $\text{gldim}$ is attained by a unique solution of the Gepner equation (4.13). Moreover, in Theorem 4.8, we show that the image of $\text{Stab} \mathcal{D}_\infty(Q)/\mathbb{C}$ under $\text{gldim}$ is $[1 - \frac{2}{h}, +\infty)$ for a Dynkin quiver $Q$; and in Theorem 5.2, we show that the image of $\text{gldim}$ is $[1, +\infty)$ for a non-Dynkin acyclic quiver $Q$.

In Section 6, we discuss the potential application of $\text{gldim}$ to the contractibility conjecture for spaces of stability conditions.

In Appendix A, we calculate the Gepner points in the spaces of $q$-stability conditions for Dynkin quivers.

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2. Preliminaries

2.1. Bridgeland stability conditions. Following Bridgeland [B1], we recall the notion of stability conditions on a triangulated category. In this paper, $\mathcal{D}$ is a triangulated category with Grothendieck group $K(\mathcal{D})$ and we usually assume that $K(\mathcal{D}) \cong \mathbb{Z}^n$ for some $n$. We denote by $\text{Ind} \mathcal{D}$ the set of (isomorphism classes of) indecomposable objects in $\mathcal{D}$.

**Definition 2.1.** Let $\mathcal{D}$ be a triangulated category. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ called the central charge and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ called the slicing satisfying the following conditions:

(a) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi \phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ ($i = 1, 2$), then $\text{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,
(d) for each object $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_l$ (2.1) and a collection of exact triangles (known as the HN-filtration)

\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{l-1} \rightarrow E_l = E
\]

with $A_i \in \mathcal{P}(\phi_i)$ for all $i$.

The categories $\mathcal{P}(\phi)$ are then abelian. Their non zero objects are called semistable of phase $\phi$ and their simple objects stable of phase $\phi$. For a semistable object $E \in \mathcal{P}(\phi)$, denote by $\phi(E) := \phi$ its phase. Finally, for any interval $J \subset \mathbb{R}$, denote by $\mathcal{P}(J)$ the subcategories consisting of objects whose HN-filtrations have factors with phases in $J$.

For a stability condition $\sigma = (Z, \mathcal{P})$, we introduce the set of semistable classes $C^{ss}(\sigma) \subset K(\mathcal{D})$ by

\[
C^{ss}(\sigma) := \{ \alpha \in K(\mathcal{D}) \mid \text{there is a semistable object } E \in \mathcal{D} \text{ such that } [E] = \alpha \}.
\]

We always assume that our stability conditions satisfy the following condition, called the support property [KS]. Let $\| \cdot \|$ be some norm on $K(\mathcal{D}) \otimes \mathbb{R}$. A stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if there is some constant $C > 0$ such that

\[
C \cdot |Z(\alpha)| > \|\alpha\|
\]

for all $\alpha \in C^{ss}(\sigma)$.

There is a natural $\mathbb{C}$-action on the set $\text{Stab}(\mathcal{D})$ of all stability conditions on $\mathcal{D}$, namely:

\[
s \cdot (Z, \mathcal{P}) = (Z \cdot e^{-i\pi s}, \mathcal{P}_{\Re(s)}),
\]

where $\mathcal{P}_x(\phi) = \mathcal{P}(\phi + x)$. There is also a natural action on $\text{Stab}(\mathcal{D})$ by the group of autoequivalences $\text{Aut}(\mathcal{D})$, namely:

\[
\Phi(Z, \mathcal{P}) = (Z \circ \Phi^{-1}, \Phi(\mathcal{P})).
\]
2.2. **Gepner equation.** Following Toda [To1], we introduce Gepner equations on the space of stability conditions.

**Definition 2.2** (Toda). A Gepner equation on \( \text{Stab} \mathcal{D} \) is

\[
\Phi(\sigma) = s \cdot \sigma,
\]

where \((\Phi, s) \in (\text{Aut} \mathcal{D}) \times \mathbb{C}\).

A Gepner point is a solution of some Gepner equation.

Note that a Gepner point is an orbifold point of \( \text{Aut}(\mathcal{D}) \setminus \text{Stab} \mathcal{D} / \mathbb{C} \).

### 3. Global dimension functions

3.1. **Global dimension functions.** Motivated by the study of Calabi-Yau-X categories and \( q \)-stability conditions in [IQ1], we introduce a function on (certain quotient spaces of) spaces of stability conditions. It is to be viewed as a real-valued generalization of the usual notion of global dimension for algebras or abelian categories.

**Definition 3.1.** Let \( \mathcal{P} \) be a slicing on a triangulated category \( \mathcal{D} \). Define the global dimension of \( \mathcal{P} \) by

\[
gldim \mathcal{P} = \sup \{ \phi_2 - \phi_1 \mid \text{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0 \} \in \mathbb{R}_{\geq 0} \cup \{ +\infty \}.
\]

(3.1)

The global dimension of a stability condition \( \sigma = (Z, \mathcal{P}) \) is defined to be \( gldim \mathcal{P} \) for its slicing. We say that \( \mathcal{P} \) (or \( \sigma \)) is \( gldim \)-reachable if there are phases \( \phi_1 \) and \( \phi_2 \) such that

\[
\text{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0 \quad \text{and} \quad gldim \mathcal{P} = \phi_2 - \phi_1.
\]

For an algebra \( A \), let \( \mathcal{P}_A \) be the canonical slicing on \( D^b(A) \), i.e. we have \( \mathcal{P}_A(0) = \text{mod} A \) and \( \mathcal{P}_A(0, 1) = \emptyset \). Then we have \( gldim A = gldim \mathcal{P}_A \). We have the following facts (cf. [IQ1, § 2]). For any triangulated category \( \mathcal{D} \),

- \( gldim \) is a continuous function on \( \text{Stab} \mathcal{D} \) ([IQ1, Lem. 2.23]).
- \( gldim \) is an invariant under the \( \mathbb{C} \)-action and the action of \( \text{Aut} \mathcal{D} \).

Thus, we have a function

\[
gldim : \text{Aut} \setminus \text{Stab} \mathcal{D} / \mathbb{C} \to \mathbb{R}_{\geq 0}.
\]

(3.2)

Here \( \mathbb{R}_{\geq 0} \) includes \( +\infty \). We will sometimes also view \( gldim \) as a function on \( \text{Stab} \mathcal{D} \) or on \( \text{Stab} \mathcal{D} / \mathbb{C} \).

3.2. **Total (semi)stability.** We recall the notion of total (semi)stability, which is motivated by [Q2, Conjecture 7.13] due to Reineke [R].

**Definition 3.2.** [Q2] A stability condition \( \sigma \) on \( \mathcal{D} \) is totally (semi)stable, if every indecomposable object is (semi)stable with respect to \( \sigma \). We will call such \( \sigma \) a total stability condition.

**Lemma 3.3.** If \( gldim \sigma \leq 1 \), then \( \sigma \) is totally semistable. Moreover, if the inequality is strict, then \( \sigma \) is totally stable.
Proof. Suppose that \( E \in \text{Ind} \mathcal{D} \) is not totally semistable. Then it admits an HN-filtration (2.1) for some \( l > 1 \). So there is a triangle \( A_1 \to E \to E' \xrightarrow{\alpha} A_1[1] \) such that \( \alpha \in \text{Hom}(E', A_1[1]) \) does not vanish and \( E' \) admits a filtration with factors \((A_2, \ldots, A_l)\). Therefore, we have \( \text{Hom}(A_k, A_1[1]) \neq 0 \) for some \( 2 \leq k \leq l \), which implies
\[
\text{gldim} \sigma \geq \phi_{\sigma}(A_1[1]) - \phi_{\sigma}(A_k) = \phi_1 + 1 - \phi_k > 1. \tag{3.3}
\]
Thus the first assertion holds.

For the second assertion, take a semistable but not stable object \( E \in \mathcal{P}(\phi) \). Then it admits a filtration with factors in \( \mathcal{P}(\phi) \) (i.e. all stable subquotients have phase \( \phi \)). Applying the same argument as above we get the same calculation as in (3.3) with \( \geq 1 \) at the end, which implies the second assertion. \( \square \)

Lemma 3.4. If \( \sigma \) is totally semistable, then \( \text{gldim} \sigma \leq 1 \). Moreover, if \( \sigma \) is totally stable and \( \text{gldim} \)-reachable, then \( \text{gldim} \sigma < 1 \).

Proof. Suppose that \( \text{gldim} \sigma > 1 \), then there are semistable objects \( E_1 \) and \( E_2[1] \) with phase \( \phi_i = \phi_{\sigma}(E_i) \) such that \( \text{Hom}(E_1, E_2[1]) \neq 0, \phi_2 - \phi_1 > 0 \).

Hence there is an object \( X \) that sits in the triangle
\[
E_2 \to X \to E_1 \xrightarrow{\alpha} E_2[1] \tag{3.4}
\]
with \( \alpha \neq 0 \).

By the uniqueness of the HN-filtration, (3.4) is the HN-filtration of \( X \). As the filtration is of length greater than one the object \( X \) is non-semistable, hence decomposable (by total semistability). By assumption, the indecomposable factors of \( X \) are all semistable. Hence they form a splitting HN-filtration of \( X \) different from that in (3.4). This contradicts the uniqueness of the HN-filtration.

Similarly for the second statement. The only difference is that when \( \sigma \) is totally stable, one can only deduce \( \phi_2 - \phi_1 < 1 \) for any \( \text{Hom}(M_1, M_2) \neq 0, M_i \in \mathcal{P}(\phi_i) \). So we need the \( \text{gldim} \)-reachable assumption to get \( \text{gldim} \sigma < 1 \). \( \square \)

Combining the lemmas above, we have the following.

Proposition 3.5. \( \sigma \) is totally semistable if and only if \( \text{gldim} \sigma \leq 1 \). Moreover, when \( \sigma \) is \( \text{gldim} \)-reachable, then \( \sigma \) is totally stable if and only if \( \text{gldim} \sigma < 1 \).

3.3. Totally stable stability conditions for type A quiver. Let us describe all total stability conditions for type A quiver and give explicit formula of global dimension function in such a case. Note that the \( \mathbb{C} \)-actions preserve total stability. Consider the \( A_n \) quiver with straight orientation
\[
Q = \overrightarrow{A_n} : 1 \leftarrow 2 \leftarrow \cdots \leftarrow n. \tag{3.5}
\]

For \( 1 \leq i \leq j \leq n \), denote by \( M_{ij} \) the indecomposable object in \( \text{mod} kQ \) that corresponds to the representation \( \{V_k, f_a \mid k \in Q_0, a \in Q_1 \} \) of \( Q \) with \( V_k = k \) if \( i \leq k \leq j \) and
\[ V_k = 0 \text{ otherwise. So } S_i = M_{ii} \text{ are the simple objects in mod } kQ. \] The Auslander-Reiten quiver of mod \( kQ \) is illustrated as follows.

Denote by \( \text{ToSt}(A_n) \) the subspace in \( \text{Stab} D_\infty(A_n) \) consisting of total stability conditions. Denote by \( \text{Poly}(m) \) be the set of convex \( m \)-gon on \( \mathbb{R}^2 = \mathbb{C} \) with vertices \( P_0, P_1, \ldots, P_m = P_0 \) in anticlockwise order satisfying \( P_0 = 0 \) and \( P_1 = 1 \).

**Proposition 3.6.** There is a natural bijection
\[ \mathfrak{Z} : \text{ToSt}(A_n)/\mathbb{C} \to \text{Poly}(n + 1), \]
sending a stability condition \( \sigma \) to an \( (n + 1) \)-gon \( P \). such that the oriented diagonals of \( P \) gives the central charges, with respect to \( \sigma \), of indecomposable objects in \( D_\infty(A_n) \).

**Figure 1.** Convex hexagon for a total stability condition on \( D_\infty(A_5) \)

**Proof.** Given a total stability condition \( \sigma \) on \( D_\infty(A_n) \), we can assume \( Z(S_1) = 1 \) with phase \( \phi_\sigma(S_1) = 0 \) up to the \( \mathbb{C} \)-action. Then \( \sigma \) is uniquely determined by \( Z_i = Z(S_i) \) as we have
\[ Z(M_{ij}) = Z_i + \cdots + Z_j. \]
Let $P_i$ be the point in $\mathbb{R}^2$ that corresponds to $Z(M_{1i})$, $1 \leq i \leq n$. We claim that \{${P_i \mid 0 \leq i \leq n}$\} forms a convex polygon $\mathbf{P}$. To see this, we only need to show that the angle $P_{i-1}P_iP_{i+1}$ is in $(0, \pi)$. This is equivalent to

$$
\begin{cases}
\phi_\sigma(M_{1n}[−1]) < \phi_\sigma(S_1) < \phi_\sigma(M_{1n}[−1]) + 1, \\
\phi_\sigma(S_i) < \phi_\sigma(S_{i+1}) < \phi_\sigma(S_i) + 1, & 1 \leq i \leq n, \\
\phi_\sigma(S_n) < \phi_\sigma(M_{1n}[1]) < \phi_\sigma(S_n) + 1,
\end{cases}
$$

(3.7)

since $Z_i = \overrightarrow{P_{i-1}P_i}$. Now, as there are non-zero morphisms in

$$
\begin{gathered}
\text{Hom}(M_{1n}[−1], M_{2n}[−1]), \quad \text{Hom}(M_{2n}[−1], S_1) \quad \text{and} \quad \text{Hom}(S_1, M_{1n}), \\
\text{Hom}(S_i, M_{i,i+1}), \quad \text{Hom}(M_{i,i+1}, S_{i+1}) \quad \text{and} \quad \text{Hom}(S_{i+1}, S_i[1]), \\
\text{Hom}(S_n, M_{1,n−1}[1]), \quad \text{Hom}(M_{1,n−1}, M_{1n}) \quad \text{and} \quad \text{Hom}(M_{1n}, S_n),
\end{gathered}
$$

where all the objects are stable, we have (3.7). Hence we obtain an injective map $\mathfrak{g}$: $\text{ToSt}(A_n)/\mathcal{C} \to \text{Poly}(n+1)$.

What is left to show, is that $\mathfrak{g}$ is surjective. Given a convex $(n+1)$-gon $\mathbf{P}$ with vertices $P_i$, we claim that there is a total stability condition $\sigma = (Z, \mathcal{P})$ satisfying

$$
\begin{cases}
Z(M_{ij}) = \overrightarrow{P_{i−1}P_j}, \\
\phi_\sigma(M_{ij}) = (\text{arg } \overrightarrow{P_{i−1}P_j})/\pi,
\end{cases}
$$

(3.8)

where $\text{arg}$ takes values in $[0, 2\pi)$. In fact, we only need to check that for any arrow $E \to F$ in the AR-quiver of $\mathcal{D}_\infty(A_n)$, one has $\phi_\sigma(E) < \phi_\sigma(F)$. There are three types of such arrows (up to shift):

$$
M_{ij} \to M_{i+1,j}, \quad M_{ij} \to M_{i,j+1} \quad \text{and} \quad M_{kn} \to M_{1,k−1}[1],
$$

for $1 \leq i \leq j \leq n$ and $2 \leq k \leq n$. But

$$
\begin{gathered}
(\text{arg } \overrightarrow{P_{i−1}P_j})/\pi < (\text{arg } \overrightarrow{P_{i−1}P_j})/\pi, \\
(\text{arg } \overrightarrow{P_{i−1}P_j})/\pi < (\text{arg } \overrightarrow{P_{i−1}P_{j+1}})/\pi \quad \text{and} \\
(\text{arg } \overrightarrow{P_{k−1}P_n})/\pi < (\text{arg } \overrightarrow{P_0P_{k−1}})/\pi + 1,
\end{gathered}
$$

indeed follow from the convexity of $\mathbf{P}$. \hfill \Box

**Remark 3.7.** Note that in the proof of the proposition above, the heart of the total stability condition $\sigma$ is usually NOT mod $kQ$ for $Q = \overrightarrow{A_n}$ in (3.5). Moreover, given any orientation of a quiver $Q$ of type $A_n$, there always exits a total stability condition $\sigma$ with heart mod $kQ$ [BGMS]. But this is not our focus.

**Proposition 3.8.** Let $\sigma \in \text{ToSt}(A_n)$ and $\mathbf{P} = \mathfrak{g}(\sigma)$ be the corresponding $(n+1)$-gon. Then

$$
gldim \sigma = \frac{1}{\pi} \max \{\text{arg } \overrightarrow{P_jP_i} − \text{arg } \overrightarrow{P_{i+1}P_{j+1}} \mid 0 \leq i < j \leq n\}
$$

(3.9)

where $P_{n+1} = P_0$ (cf. angle $\alpha$ in Figure 1 and angles in Figure 2).
Proof. By Auslander-Reiten duality, we have \( \text{Hom}(E, F) = D \text{Hom}(F, \tau(E[1])) \), where \( \tau \) is the Auslander-Reiten functor. As any indecomposable is stable, we have
\[
gldim \sigma = \max \{ \phi(\tau(E[1])) - \phi(E) \mid \text{indecomposable } E \}.
\]
On the other hand, any indecomposable \( E \) in \( D_\infty(A_n) \) is of the form \( M_{ij}[m] \) for \( 1 \leq i \leq j \leq n \) and \( m \in \mathbb{Z} \). Then formula above becomes (3.9) using (3.8).

3.4. Calabi-Yau case.

Definition 3.9. For a stability condition \( \sigma \), we say that an auto-equivalence \( \Phi \) is \( \sigma \)-semistable if \( \Phi(E) \) is \( \sigma \)-semistable for every \( \sigma \)-semistable object \( E \). If in addition \( \phi_\sigma(\Phi(E)) - \phi_\sigma(E) = s \) for every \( \sigma \)-semistable object \( E \), then we say that \( \phi_\sigma(\Phi) \) exists and equals \( s \).

Clearly, the shift functor \([1]\) is semistable with respect to any stability condition. Denote by \( S \) the Serre functor of \( D \) (provided it exists), that is the auto-equivalence in \( \text{Aut } D \) satisfying
\[
S: \text{Hom}(L, M) \xrightarrow{\sim} D \text{Hom}(M, S(L)).
\]
(3.10)

Proposition 3.10. Let \( \sigma \in \text{Stab } D \). If \( \phi_\sigma(S) \) exists, then \( gldim \sigma = \phi_\sigma(S) \).

Proof. Since \( 0 \neq D \text{Hom}(E, E) = \text{Hom}(E, S(E)) \) for any (semi)stable object \( E \), then
\[
\phi_\sigma(S(E)) - \phi_\sigma(E) = \phi_\sigma(S)
\]
which implies \( gldim \sigma \geq \phi_\sigma(S) \).

On the other hand, for any semistable object \( E' \) such that \( \text{Hom}(E, E') \neq 0 \), we have
\[
\text{Hom}(E', S(E)) = D \text{Hom}(E, E') \neq 0,
\]
which implies \( \phi_\sigma(E') \leq \phi_\sigma(S(E)) \). Thus,
\[
\phi_\sigma(E') - \phi_\sigma(E) \leq \phi_\sigma(S(E)) - \phi_\sigma(E) = \phi_\sigma(S)
\]
and \( gldim \sigma \leq \phi_\sigma(S) \). This completes the proof.

We have the following immediate consequence.

Corollary 3.11. If \( D \) is Calabi-Yau-\( N \) for some integer \( N \geq 1 \), i.e. \( S = [N] \), then \( \phi_\sigma(S) = N \) for any \( \sigma \). Thus \( gldim \equiv N \) on \( \text{Stab } D \).

Proof. Since \( \phi(E[1]) = \phi(E) + 1 \), \( \phi_\sigma(S) \) exists and equals \( N \). Then the assertion follows from the proposition above.

A remark is that Serre-invariant stability conditions are studied in [FP, PR], where the uniqueness of such stability conditions is proven in certain cases.
4. Global dimension for Dynkin quiver

4.1. Homotopy category of graded matrix factorizations. Consider a weighted graded polynomial ring

\[ A := \mathbb{C}[x_1, x_2, \ldots, x_m], \quad \deg x_i = a_i \]  

(4.1)

with \( a_1 \geq a_2 \geq \cdots \geq a_m \). Let \( f \in A \) be a homogeneous element of degree \( h \), known as superpotential such that \( (f = 0) \subset \mathbb{C}^n \) has an isolated singularity at the origin. Denote by \( P(k) \) the \( k \)-grading shifts of a graded \( A \)-module \( P = \bigoplus_{i \in \mathbb{Z}} P_i \), where \( P(k)_i = P_{i+k} \).

**Definition 4.1.** A graded matrix factorization of \( f \) is a \( \mathbb{Z}_2 \)-graded complex

\[ P^\bullet : P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(h) \]  

(4.2)

where \( P^i \) are graded free \( A \)-modules of finite rank and \( p^i \) are homomorphisms of graded \( A \)-modules, satisfying the following conditions:

\[ p^1 \circ p^0 = f \cdot \text{id}, \quad p^0(h) \circ p^1 = f \cdot \text{id}. \]

The category \( \text{HMF}(f) \) is defined to be the homotopy category of graded matrix factorizations of \( f \): whose objects consist of \( \mathbb{Z}_2 \)-graded complexes (4.2) and the set of morphisms are given by the commutative diagrams of graded \( A \)-modules modulo null-homotopic morphisms. The category \( \text{HMF}(f) \) is a triangulated category where the shift \([1]\) of (4.2) is

\[ P^\bullet[1] : P^1 \xrightarrow{-p^1} P^0(h) \xrightarrow{-p^0(h)} P^1(h) \]  

(4.3)

and the distinguished triangles are defined via the usual mapping cone constructions, cf. [KST1, Tol]. The grading shift functor \( P^\bullet \mapsto P^\bullet(1) \) induces the autoequivalence \( \tau \) of \( \text{HMF}(f) \) such that

\[ \tau^h = [2]. \]  

(4.4)

Furthermore, there exists a Serre functor \( S_f \) on \( \text{HMF}(f) \):

\[ S_f = \tau^{-\varepsilon}[m-2] \]  

(4.5)

where \( \varepsilon \) is the Gorenstein index defined by

\[ \varepsilon := \sum_{i=1}^m a_i - h \in \mathbb{Z}. \]  

(4.6)

**Remark 4.2.** We reserve \( \tau \) for Auslander-Reiten functor.

4.2. Toda’s conjecture. Consider a central charge \( Z_G \) on \( \text{HMF}(f) \), symbolically defined by

\[ Z_G(P^\bullet) := s\text{Tr}(e^{2\pi i/h} : P^\bullet \rightarrow P^\bullet) \]  

(4.7)

for a graded matrix factorization \( P^\bullet \) in (4.2). Here the \( e^{2\pi i/h} \)-action is induced by the \( \mathbb{Z} \)-grading on each \( P^i \) and \( s\text{Tr} \) means the super trace of the \( e^{2\pi i/h} \)-action with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( P^\bullet = P^0 \oplus P^1 \).
Conjecture 4.3. [To1] There is a stability condition \( \sigma_G = (Z_G, P_G) \) whose central charge is given by (4.7). Moreover, it satisfies the Gepner equation with respect to \( (\tau, 2/h) \) in \( \text{Stab HMF}(f) \). In other words, the equation

\[
\tau(\sigma) = 2h \cdot \sigma
\]  

admits a solution \( \mathbb{C} \cdot \sigma_G \).

4.3. Kajiura-Saito-Takahashi’s solution in the Dynkin case. In this section, we focus on \( m = 3, \varepsilon > 0 \) case where Kajiura-Saito-Takahashi [KST1] proved the triangle equivalence (4.10) and Conjecture 4.3.

Let \( Q \) be a Dynkin quiver, i.e. the underlying diagram of \( Q \) is one of the following graphs (with the labeling of the vertices)

\[
\begin{align*}
A_n & : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \\
D_n & : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n - 2 \longrightarrow n \\
E_{6,7,8} & : \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8
\end{align*}
\]

Let \( h = h_Q \) be the Coxeter number, which is

\[
h_Q = \begin{cases}
  n + 1, & \text{if } Q \text{ if of type } A_n; \\
  2(n - 1), & \text{if } Q \text{ if of type } D_n; \\
  12, 18, 30 & \text{if } Q \text{ if of type } E_6, E_7, E_8 \text{ respectively}. 
\end{cases}
\]

Denote by \( D_\infty(Q) = D^b(kQ) \) the bounded derived category of the path algebra \( kQ \) and by \( \tau \) its Auslander-Reiten functor. Let \( (x, y, z) = (x_1, x_2, x_3) \), \( (a, b, c) = (a_1, a_2, a_3) \).

Theorem 4.4. [KST1] There are triangle equivalences

\[
\Phi: \text{HMF}(f) \cong D_\infty(Q),
\]

for \( m = 3, \varepsilon > 0 \),

\[
(a, b, c; h) = \begin{cases}
  (1, b, n + 1 - b; n + 1), & Q = A_n(n \geq 1, 1 \leq b \leq n); \\
  (n - 2, 2, n - 1; 2(n + 1)), & Q = D_n, (n \geq 4); \\
  (4, 3, 6; 12), & Q = E_6; \\
  (6, 4, 9; 18), & Q = E_7; \\
  (10, 6, 15; 30), & Q = E_8; 
\end{cases}
\]

and

\[
f(x, y, z) = \begin{cases}
  x^{n+1} + yz, & Q = A_n(n \geq 1); \\
  x^2y + y^{n-1} + z^2, & Q = D_n, (n \geq 4); \\
  x^3 + y^4 + z^2, & Q = E_6; \\
  x^3 + xy^3 + z^2, & Q = E_7; \\
  x^3 + y^5 + z^2, & Q = E_8;
\end{cases}
\]
Moreover, $\Phi(\tau) = \tau^{-1}$, the Serre functor in $D_\infty(Q)$ is

$$S = [1] \circ \tau \quad \text{with} \quad \tau^h = [-2] \Leftrightarrow S^h = [h - 2].$$

(4.11)

and Conjecture 4.3 holds.

Remark 4.5. When $Q$ is of type $A_n$, the corresponding polygon is a regular $(n+1)$-gon (see Proposition 3.6) and any angle $\arg \overrightarrow{P_jP_i} - \arg \overrightarrow{P_{i+1}P_{j+1}}$ in (3.9) equals $\frac{n-1}{n+1}\pi$, cf. Figure 2 for $n = 6$ case.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Regular_heptagon.png}
\caption{Regular heptagon for $\sigma_G$ in $A_6$ case and angles equals $\pi \text{gldim} \sigma_G$}
\end{figure}

4.4. Uniqueness. Define

$$\text{gd}_Q = 1 - \frac{2}{h}. \quad (4.12)$$

We will show that $\text{gldim} D_\infty(Q) = \text{gd}_Q$. Moreover, we will prove the uniqueness of the solution of (4.8) (cf. [To1, § 2.9] for the uniqueness) and the uniqueness of the solution of equation $\text{gldim} \sigma = \text{gd}_Q$, up to the $C$-action.

Lemma 4.6. $\text{gldim} \sigma_G = \text{gd}_Q$ for a Dynkin quiver $Q$.

Proof. Since $\tau(\sigma_G) = \varphi^{-1}(\sigma_G) = (-2/h) \cdot \sigma_G$, we have $S(\sigma_G) = \text{gd}_Q \cdot \sigma_G$, which implies that $\phi_{\sigma_G} (S) = \text{gd}_Q$. Then the lemma follows from Proposition 3.10. \qed

Theorem 4.7. Up to the $C$-action, $\sigma_G$ is the unique solution of (4.8), or equivalently,

$$\tau(\sigma) = \left(-\frac{2}{h}\right) \cdot \sigma. \quad (4.13)$$

Moreover, the minimal value of $\text{gldim}$ on $\text{Stab} D_\infty(Q)$ is $\text{gd}_Q$ and $\sigma_G$ is also the unique solution of $\text{gldim} \sigma = \text{gd}_Q$ up to the $C$-action.
Proof. First, we show that any solution $\sigma$ of (4.13) satisfies $\text{gldim} \sigma \leq \text{gd}_Q$. In fact, for any two semistable objects $E, F$ with $\text{Hom}(E, F) \neq 0$, we have $\text{Hom}(F, S(E)) \neq 0$. On the other hand, (4.13) implies that $S(E) = \tau(E[1])$ is also semistable. Thus we have

$$\phi_\sigma(E) \leq \phi_\sigma(F) \leq \phi_\sigma(S(E)) = \phi_\sigma(E) + 1 - \frac{2}{h},$$

which implies $\phi_\sigma(F) - \phi_\sigma(E) \leq 1 - 2/h$. Thus $\text{gldim} \sigma \leq \text{gd}_Q$.

Now consider any $\sigma = (Z, P)$ with $\text{gldim} \sigma \leq \text{gd}_Q$. We only need to show that it must be $\sigma_G$, up to the $C$-action, to complete the proof.

Since $\text{gldim} \sigma \leq \text{gd}_Q < 1$, Proposition 3.5 says that any indecomposable object is stable with respect to $\sigma$. For any $E \in \text{Ind} \mathcal{D}_\infty(Q)$, we have

$$\text{Hom}(S^{i-1}(E), S^i(E)) \neq 0 \quad \text{for } i = 1, \ldots, h.$$ 

Thus we have

$$h \cdot \text{gldim} \sigma \geq \sum_{i=1}^{h} (\phi_\sigma(S^i(E)) - \phi_\sigma(S^{i-1}(E))) = \phi_\sigma(S^h(E)) - \phi_\sigma(E) = \phi_\sigma(E[h-2]) - \phi_\sigma(E) = h - 2,$$

where we use (4.11). Hence for any $E$ we have $\phi_\sigma(S(E)) - \phi_\sigma(E) = 1 - 2/h$, i.e.

$$S(P) = P_{-2/h} \quad \text{or} \quad \tau(P) = P_{-2/h}, \quad (4.14)$$

where $P_\phi = P(\phi + x)$. Thus $\text{gldim} \sigma = \text{gd}_Q$. We process to show that $\sigma$ is unique up to the $C$-action and satisfies (4.13).

By Proposition 3.5, all indecomposables are stable with respect to $\sigma$. Consider a leaf $l$ (vertex with only one arrow in or out) of $Q$, which corresponds to a boundary $\tau$-orbit in the Auslander-Reiten quiver of $\mathcal{D}_\infty(Q)$. For instance, in Section 3.3, the leaf 1 in (3.5) corresponds to the last (red) dashed line in (3.6). Denote the objects in this $\tau$-orbit by $L_i$, $i \in \mathbb{Z}$ such that $\tau(L_{i+1}) = L_i$ (and hence $L_{i+h} = L_i[2]$). Denote the objects in its neighbour $\tau$-orbit by $K_i$, $i \in \mathbb{Z}$ such that $\tau(K_{i+1}) = K_i$ so that there are Auslander-Reiten triangles

$$L_i \rightarrow K_i \rightarrow L_{i+1} \rightarrow L_i[1].$$

By (4.14), we have

$$\phi_\sigma(L_{i+1}) - \phi_\sigma(L_i) = 2/h = \phi_\sigma(K_{i+1}) - \phi_\sigma(K_i) \quad (4.15)$$

for any $i \in \mathbb{Z}$. The triangles above imply that $Z(K_i) = Z(L_i) + Z(L_{i+1})$ and they are all stable, which means that $\phi_\sigma(K_i)$ is in the interval

$$(\phi_\sigma(L_i), \phi_\sigma(L_{i+1})) = (\phi_\sigma(L_i), \phi_\sigma(L_{i+1}) + 2/h).$$

Therefore

$$\begin{align*}
\begin{cases}
|Z(L_i)| > |Z(L_{i+1})| &\iff \phi_\sigma(K_i) < \phi_\sigma(L_{i+1}) + 1/h, \\
|Z(L_i)| = |Z(L_{i+1})| &\iff \phi_\sigma(K_i) = \phi_\sigma(L_{i+1}) + 1/h, \\
|Z(L_i)| < |Z(L_{i+1})| &\iff \phi_\sigma(K_i) > \phi_\sigma(L_{i+1}) + 1/h.
\end{cases}
\end{align*}$$

(4.16)
and one of these cases holds for all $i$ noticing the periodicity (4.15). However, $Z(L_{i+k}) = Z(L_i[2]) = Z(L_i)$. This forces that the second case in (4.16) occurs. Thus $\{Z(L_i), \phi_{\sigma}(L_i)\}$ satisfies (4.13), which also determines $\{Z(K_i), \phi_{\sigma}(K_i)\}$. Note that $\{Z(K_i), \phi_{\sigma}(K_i)\}$ also satisfies (4.13), and determines $\{Z(L_i), \phi_{\sigma}(L_i)\}$.

Recall that $L_i$ corresponds to a leaf $l$ in $Q$ and then $K_i$ corresponds to an adjacent vertex $k$ in $Q$. If $k$ is not a trivalent vertex, let $j \neq l$ be the other adjacent vertex of $k$.

It is straightforward to see that the central charges and phases of objects in the $\tau$-orbit corresponding to $j$ are determined by $\{Z(L_i), \phi_{\sigma}(L_i)\}$ and satisfy (4.13). We can keep this process until we reach a trivalent vertex of $Q$.

As we can apply this analysis to any leaves in $Q$, we deduce that the central charges and phases of objects in any given $\tau$-orbit of the Auslander-Reiten quiver of $\mathcal{D}_\infty(Q)$ satisfy (4.13) and they determine $\sigma = (Z, P)$ uniquely. Therefore, up to $\mathbb{C}$-action, $\sigma$ is unique. By Lemma 4.6, we conclude that $\sigma = s \cdot \sigma_G$ for some $s \in \mathbb{C}$. □

4.5. Range of gldim in the Dynkin case. We proceed to calculate the range of the global dimension function (3.2) on $\text{Stab} \mathcal{D}_\infty(Q)$. Note that as $\text{Ind} \mathcal{D}_\infty(Q)/[1]$ is finite, $\text{gldim} \sigma < +\infty$ for any $\sigma$.

**Theorem 4.8.** The range of gldim on $\text{Stab} \mathcal{D}_\infty(Q)/\mathbb{C}$ is $[\text{gd}_Q, +\infty)$. Moreover, (the orbit of) $\sigma_G$ is the unique point that achieves the minimal value $\text{gd}_Q$.

**Proof.** By Theorem 4.7, gldim has minimal value $\text{gd}_Q$, which is achieved by a unique stability condition in $\text{Stab} \mathcal{D}_\infty(Q)/\mathbb{C}$.

Note that gldim is continuous (cf. Section 3.1) and $\text{Stab} \mathcal{D}_\infty(Q)$ is connected (cf. e.g. appendix of [Q1]), we only need to show that there exists $\sigma$ with gldim $\sigma$ arbitrary big. This follows from the following facts:

- A heart $\mathcal{H}$ in $\mathcal{D}_\infty(Q)$ and a central charge $Z$ on its simples with values in the upper half plane $H = \{r \exp(i\pi\theta) \mid r \in \mathbb{R}_{>0}, 0 \leq \theta < 1\} \subset \mathbb{C}$ determine a stability condition $\sigma$.

- By definition, an arrow $S \xrightarrow{d} S'$ with degree $d$ in the Ext-quiver of a heart $\mathcal{H}$ corresponds to an element of a basis of $\text{Hom}(S, S'[k])$. Thus,

$$\text{gldim} \sigma \geq \phi_{\sigma}(S'[k]) - \phi_{\sigma}(S) > k - 1$$

for any $\sigma$ with heart $\mathcal{H}$.

- Recall from [KQ, Def. 3.7 and Def. 5.11] there is the notion of iterated simple (HRS-)tilting. By [KQ, Thm. 5.9] and [Q2, Thm A.6] one can iterated forward tilt a simple of any heart in $\mathcal{D}_\infty(Q)$. Hence the corresponding Ext quiver of the heart $\mathcal{H}$ can have arbitrary large degrees. □

The fact that the gldim can be arbitrary large can be also deduced from [M, Lem. 3.19] easily.
Example 4.9. In the case when $Q$ is $A_2$ quiver, the description of $\text{Stab} \mathcal{D}_\infty(A_2)$ is explicit in [BQS, Q1]. The shadow area in Figure 3 is the fundamental domain for $\text{Aut} \backslash \text{Stab} \mathcal{D}_\infty(A_2)/\mathbb{C}$. Note that the ratio of the central charges of two of the semistable objects equals $e^{\pi iz}$ for $z = x + yi$ and

$$l_{\pm} = \{z \mid x \in (\frac{1}{2}, \frac{2}{3}], y\pi = \mp \ln(-2 \cos x\pi)\}$$

are the boundaries. And we have the formula to calculate $\text{gldim} \sigma$ via $x$-axis in the fundamental domain:

$$x = 1 - \text{gldim} \sigma.$$ 

5. Global dimension for hereditary case

A triangulated category $\mathcal{D}$ is hereditary if it admits a heart $\mathcal{H}$ such that

$$\text{Hom}^{\geq 2}(\mathcal{H}, \mathcal{H}) = 0. \tag{5.1}$$

We call a heart $\mathcal{H}$ of $\mathcal{D}$ hereditary if (5.1) holds. The following is straightforward.

Lemma 5.1. Let $\mathcal{H}$ be a hereditary heart of $\mathcal{D}$. If $\mathcal{H}$ is finite (i.e. is a length category with finitely many simple objects), then $\text{gldim} \mathcal{D} \leq 1$.

Proof. Take a stability condition $\sigma = (Z, \mathcal{P})$ with heart $\mathcal{H} = \mathcal{P}(0, 1]$, such that the central charge $Z(S)$ is a negative real number for any simple $S \in \text{Sim} \mathcal{H}$. In other words, we have $\mathcal{H} = \mathcal{P}(1)$. Thus

$$\mathcal{P}(\phi) \neq \emptyset \iff \phi \in \mathbb{Z}.$$
Noticing (5.1), we have \( \text{Hom}(\mathcal{P}(i), \mathcal{P}(j)) \neq 0 \) if and only if \( i, j \in \mathbb{Z} \) and \( i \leq j \leq 1 \). Thus \( \text{gldim} \sigma \leq 1 \) and the lemma follows. \( \square \)

5.1. Acyclic quiver case.

**Theorem 5.2.** Suppose that \( Q \) is an acyclic quiver, which is not of Dynkin type. Then the range of \( \text{gldim} \) on \( \text{Aut} \setminus \text{Stab} \mathcal{D}_\infty(Q)/\mathbb{C} \) is \([1, +\infty)\).

**Proof.** \( \mathcal{D}_\infty(Q) \) admits a hereditary heart \( \mathcal{H}_Q := \text{mod} kQ \), which is finite/algebraic. By Lemma 5.1, we see that \( \min \text{gldim} \leq 1 \). Now suppose that \( \text{gldim} \sigma \leq 1 \) for some \( \sigma \in \text{Stab} \mathcal{D}_\infty(Q) \). By Proposition 3.5, every indecomposable is semistable. However, when \( Q \) is not of Dynkin type, there are indecomposable objects with non-trivial self-extension, which implies \( \text{gldim} \geq 1 \). Thus, \( \text{gldim} \sigma = 1 \). Any value in \((1, +\infty)\) is reachable is similar to the proof in Theorem 4.8. What is left to show is that \( \text{gldim} \sigma < +\infty \) for any \( \sigma \).

Let \( \sigma = (Z, \mathcal{P}) \in \text{Stab} \mathcal{D}_\infty(Q) \) with heart \( \mathcal{H} = \mathcal{P}(0, 1] \). There exists an integer \( L >> 1 \) such that any simple of \( \mathcal{H}_Q \) and hence \( \mathcal{H}_Q \) is in \( \mathcal{P}(-L, L) \). Then we deduce that \( \mathcal{H} \cap \mathcal{H}_Q[L'] = \emptyset \) for integer \( |L'| \geq L \). As \( \text{Ind} \mathcal{D}_\infty(Q) = \bigcup_{m \in \mathbb{Z}} \text{Ind} \mathcal{H}_Q[m] \), we deduce that \( \mathcal{H} \subset \langle \mathcal{H}_Q[m] \mid |m| < L \rangle \).

Thus (5.1) implies \( \text{Hom}^{>2L}(\mathcal{H}, \mathcal{H}) = 0 \) and hence \( \text{gldim} \sigma \leq 2L \). \( \square \)

5.2. Example: Kronecker quiver. Notice that we have a derived equivalence

\[ \mathcal{D}^b(\text{coh} \mathbb{P}^1) \cong \mathcal{D}_\infty(K_2), \]

where \( K_2 \) is the Kronecker quiver (two vertices with double arrows). Similar to \( A_2 \) case (cf. Figure 3), we have the following fundamental domain \( K_0 \) in Figure 4 for \( \text{Aut} \setminus \text{Stab} \mathcal{D}^b(\text{coh} \mathbb{P}^1)/\mathbb{C} \) (see [Ok] and [Q1, §7.5.2]). Note that the ratio of the central charges of two of the semistable objects equals \( e^{\pi i} \) for \( z = x + yi \) and

\[
\begin{align*}
    k_1 &= \{ z = x + yi \mid x \in \left(\frac{1}{4}, 1\right), y \pi = \ln(-\cos x \pi) \}, \\
    k_0 &= \{ z = x + yi \mid x \in \left(\frac{1}{2}, 1\right), y \pi = -\ln(-\cos x \pi) \},
\end{align*}
\]

are the boundaries lines of \( K_0 \) in the figure. One can calculate the following.

**Proposition 5.3.** \( \text{gldim}^{-1}(1) \subset K_0 \) is exactly the area in Figure 4 bounded by \( k_0, k_1 \) and \( y \)-axis. Outside this area, \( x \)-axis gives \( 1 - \text{gldim} \). Thus we have

\[ \text{gldim} = \max\{1 - x, 1\}. \]

**Proof.** By the description of \( \text{Aut} \setminus \text{Stab} \mathcal{D}^b(\text{coh} \mathbb{P}^1)/\mathbb{C} \) (see [Ok, Q1] for details), this is a direct calculation. \( \square \)
In this section, we explain the potential application of global dimension function to the contractibility conjecture of spaces of stability conditions.

6.1. Dynkin case. Let $Q$ be a Dynkin quiver.

**Conjecture 6.1.** Regard $\text{gldim}$ as a function on $\text{Stab} \mathcal{D}_\infty(Q) / \mathbb{C}$. Then the preimage $\text{gldim}^{-1}(\text{gd}_Q, s)$ contracts to the Gepner point $\sigma_G$ for any real number $s > \text{gd}_Q$.

A much wild guess is the following.

**Question 6.1.** The function $\text{gldim}$ on $\text{Stab} \mathcal{D}_\infty(Q) / \mathbb{C}$ is piecewise differentiable and behaves like a Morse function with a unique critical point $\sigma_G$.

It is also interesting to consider the stability conditions such that every indecomposable is stable. Such a subspace of $\text{Stab} \mathcal{D}_\infty(Q)$ is closely related to the fundamental domain of $\text{Stab} \mathcal{D}_{\text{fd}}(\Gamma_2 Q)$ for the Calabi-Yau-2 case (or Kleinian singularities), cf. [B2].

**Question 6.2.** It is also interesting to study the space of total stability conditions, or $\text{ToSt}(Q)$ for type $D$ and $E$. 

---

**Figure 4.** $\text{Aut} \ \text{Stab} \mathcal{D}_\infty(K_2) / \mathbb{C} = \left( \text{Stab} \mathcal{D}_\infty(K_2) / \mathbb{C} \right) / \mathbb{Z}$
In fact, this question has been just solved in a follow-up work [QZ].

6.2. Coherent sheaves on $\mathbb{P}^n$. Let $\mathcal{D}^b(\text{coh}\mathbb{P}^n)$ be the derived category of coherent sheaves on $\mathbb{P}^n$ for some $n \geq 1$. Recall that for $n = 1$, we have (5.2).

We have the following conjecture.

**Conjecture 6.2.** The minimal value of gldim on $\text{Stab} \mathcal{D}^b(\text{coh}\mathbb{P}^n)$ is $n$.

This conjecture has been confirmed by Kikuta-Ougchi-Takahashi, see [KOT, Thm. 4.2 and Ex. 2.5], where they show that $\text{gldim} \mathcal{D}^b(\text{coh}\mathbb{P}^n) \geq n$. The equality follows immediately as there is a stability condition $\sigma$ with $\text{gldim} = n$, where the heart of $\sigma$ is the Beilinson heart $\mathcal{H}$ with simples

$$\text{Sim} \mathcal{H} = \mathcal{O}[n], \mathcal{O}(1)[n-1], \ldots, \mathcal{O}(n)$$

and central charges $Z(\mathcal{O}(i)[n-i]) = 1$.

Moreover, recall the following definition of geometric stability conditions.

**Definition 6.3.** A stability condition $\sigma$ on $\mathcal{D}^b(\text{coh}\mathbb{P}^n)$ is geometric if all skyscraper sheaves $\mathcal{O}_x$ are $\sigma$-stable with the same phase.

Note that the conjecture above holds for $n = 1$ by the calculation in Section 5.2. In the follow-up work [FLLQ], we prove the following conjecture. One expects similar result should also hold for $n > 2$.

**Conjecture 6.4.** Regard gldim as a function on $\text{Stab} \mathcal{D}^b(\text{coh}\mathbb{P}^2)$. Then $\text{gldim}^{-1}(2)$ consists of almost all geometric stability conditions and $\text{gldim}^{-1}[2, s)$ contracts $\text{gldim}^{-1}(2)$, for any real number $s > 2$.

**Appendix A.** $q$-stability conditions on Calabi-Yau-$\mathbb{X}$ categories

Let $\mathcal{D}_\mathbb{X}$ be a triangulated category with a distinguish auto-equivalence

$$\mathbb{X}: \mathcal{D}_\mathbb{X} \to \mathcal{D}_\mathbb{X}.$$  

We will write $E[l]|\mathbb{X}|$ instead of $\mathbb{X}^l(E)$ for $l \in \mathbb{Z}$ and $E \in \mathcal{D}_\mathbb{X}$. Set

$$R = \mathbb{Z}[q^{\pm 1}]$$

and define the $R$-action on $\mathcal{K}(\mathcal{D}_\mathbb{X})$ by

$$q^n \cdot [E] := [E[n]|\mathbb{X}|].$$

Then $\mathcal{K}(\mathcal{D}_\mathbb{X})$ has an $R$-module structure. Let $\text{Aut} \mathcal{D}_\mathbb{X}$ be the group of auto-equivalences of $\mathcal{D}_\mathbb{X}$ that commute with $\mathbb{X}$ and $\text{Hom}^\mathbb{Z}(M, N) := \bigoplus_{k,l \in \mathbb{Z}} \text{Hom}(M, N[k + l|\mathbb{Z}|]).$
A.1. Calabi-Yau-$X$ categories from quivers. For an acyclic quiver $Q$, denote by $\Gamma_X Q$ the Calabi-Yau-$X$ Ginzburg differential $\mathbb{Z} \oplus \mathbb{Z}[X]$ graded algebra of $Q$, that is constructed as follows (cf. [IQ1] and [K, Sec. 7.2]):

- Let $\overline{Q}$ be the graded quiver whose vertex set is $Q_0$ and whose arrows are: the arrows in $Q$ with degree 0; an arrow $a^* : j \rightarrow i$ with degree $2 - X$ for each arrow $a : i \rightarrow j$ in $Q$; a loop $e^* : i \rightarrow i$ with degree $1 - X$ for each vertex $e$ in $Q$.
- The underlying graded algebra of $\Gamma_X Q$ is the completion of the graded path algebra $kQ$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $Q$.
- The differential of $\Gamma_X Q$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following (non-zero) values on the arrows of $Q$:

$$d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*].$$

Write $\mathcal{D}_X(Q)$ for $\mathcal{D}_{fd}(\text{mod } \Gamma_X Q)$, the finite dimensional derived category of $\Gamma_X Q$, which admits the Serre functor $X$ that corresponds to grading shift of $(0,1)$.

Note that there is an embedding

$$\mathcal{L} : \mathcal{D}_\infty(Q) \rightarrow \mathcal{D}_X(Q),$$

(A.1)

induced from the projection $\Gamma_X Q \rightarrow kQ$, which is a Lagrangian immersion, in the sense that

$$\text{RHom}_{\mathcal{D}_X(Q)}(\mathcal{L}(L), \mathcal{L}(M)) = \text{RHom}_{\mathcal{D}_\infty(Q)}(L, M) \oplus D \text{RHom}_{\mathcal{D}_\infty(Q)}(M, L)[-X],$$

(A.2)

Moreover, $\mathcal{L}$ also induces an $R$-isomorphism

$$\mathcal{L}_* : K(\mathcal{D}_X(Q)) \cong_R K(\mathcal{D}_\infty(Q)) \otimes R = R^n,$$

(A.3)

where the simple $kQ$-modules provide a $\mathbb{Z}$-basis for $K(\mathcal{D}_\infty(Q)) \cong \mathbb{Z}^n$ and the simple $\Gamma_X Q$-modules provide an $R$-basis for $K(\mathcal{D}_X(Q))$.

By abuse of notation, we will not distinguish objects in $\mathcal{D}_\infty(Q)$ and their images in $\mathcal{D}_X(Q)$ (under the fixed canonical Lagrangian immersion $\mathcal{L}$).

A.2. $q$-Stability conditions. We recall $q$-stability conditions from [IQ1].

**Definition A.1.** [IQ1, Def. 3.4] Suppose that $\mathcal{D}$ is a triangulated category with Grothendieck group $K(\mathcal{D}_X) \cong_R R^n$. An $q$-stability condition consists of a (Bridgeland) stability condition $\sigma = (Z, P)$ on $\mathcal{D}_X$ and a complex number $s \in \mathbb{C}$, satisfying

$$X(\sigma) = s \cdot \sigma.$$  

(A.4)

We may write $\sigma[X]$ for $X(\sigma)$. Denote by $Q\text{Stab}_s \mathcal{D}_X$, the space of $q$-stability conditions consisting of $(\sigma, s)$ with $q$-support property.

We have the following results.
Proposition A.2. [IQ1, Thm. 3.10] The projection map defined by taking central charges

\[ \pi_Z: \text{QStab}_s \mathcal{D}_X \rightarrow \text{Hom}_R(K(\mathcal{D}_X), \mathbb{C}_s), \quad (Z, \mathcal{P}) \mapsto Z \]  

is a local isomorphism of topological spaces. In particular, \( \pi \) induces a complex structure on \( \text{QStab}_s \mathcal{D}_X \).

Theorem A.3. [IQ1, Thm. 5.9] Let \( \sigma = (Z, \mathcal{P}) \) be a stability condition on \( \text{Stab} \mathcal{D}_\infty(Q) \) and let \( s \in \mathbb{C} \) satisfies

\[ \text{Re}(s) \geq \text{gldim} \sigma + 1. \]

Then \( \mathcal{L} \) induces a \( q \)-stability condition \( (\sigma^\mathcal{L}_s, s) \) such that \( \sigma^\mathcal{L}_s = (Z^\mathcal{L}_s, \mathcal{P}^\mathcal{L}_s) \) is defined as

- \( Z^\mathcal{L}_s = q_s \circ (Z \otimes 1): K(\mathcal{D}_X(Q)) \rightarrow \mathbb{C} \) (recall that we have (A.3)), where \( q_s \) is the specialization
  \[ q_s: \mathbb{C}[q, q^{-1}] \rightarrow \mathbb{C}, \quad q \mapsto e^{i\pi s}. \]

- \( \mathcal{P}^\mathcal{L}_s(\phi) = \langle \mathcal{P}(\phi + k \text{Re}(s))[k \mathcal{X}] | k \in \mathbb{Z} \rangle. \)

As we have calculate the image of gldim on \( \text{Stab} \mathcal{D}_\infty(Q) \) in Theorem 4.8, a direct corollary is the following.

Corollary A.4. Let \( Q \) be a Dynkin quiver and \( \text{Re}(s) \geq \text{gd}_Q + 1. \) Then \( \text{QStab}_s \mathcal{D}_X(Q) \) is non-empty.

In particular, for any \( \text{Re}(s) \geq \text{gd}_Q + 1 \), denote by \( \sigma^\mathcal{L}_{G,s} \) the induced \( q \)-stability condition in \( \text{QStab}_s \mathcal{D}_X(Q) \) from the Gepner point \( \sigma_G \in \text{Stab} \mathcal{D}_\infty(Q) \) with \( \text{gldim} \sigma_G = \text{gd}_Q \). In the rest of this section, we shall prove that \( \sigma^\mathcal{L}_{G,s} \) is a Gepner point of \( \text{QStab}_s \mathcal{D}_X \), in the sense that it satisfies one more Gepner equation (2.2) other than (A.4).

A.3. Center of the braid group.

Definition A.5. [BrSa] Denote by \( \text{Br}(Q) \) the braid group (a.k.a Artin group) associated to a Dynkin quiver \( Q \), which admits the following presentation

\[ \text{Br}(Q) = \langle b_i, i \in Q_0 | \text{Co}(b_i, b_j), \text{no arrows between } i \text{ and } j ; \text{Br}(b_j, b_k), \exists! \text{ arrow between } i \text{ and } j \rangle. \]

where \( \text{Co}(a, b) \) means the commutation relation \( ab = ba \) and \( \text{Br}(a, b) \) means the braid relation \( aba = bab \).

Provided the labeling of vertices as in (4.9), define

\[ \zeta_Q = b_n \circ ... \circ b_1, \quad (A.6) \]

\[ \delta_Q = \begin{cases} 1, & \text{if } Q \text{ is of type } A_n, D_{2l+1}, E_6; \\ 1/2, & \text{if } Q \text{ is of type } D_{2l}, E_7, E_8. \end{cases} \quad (A.7) \]

Then it is well-known that \( z_Q = \zeta_Q^{\delta_Q b_Q} \) generates of the center of the braid group \( \text{Br}(Q) \).
Definition A.6. An object $S$ in a CY-$X$ category $D$ is ($X$-)spherical if
\[
\text{Hom}(S, S[k + lX]) = \begin{cases} 
k, & \text{if } k = 0 \text{ and } l \in \{0, 1\}; 
0, & \text{otherwise,} 
\end{cases}
\]
and induces a twist functor $\Psi_S \in \text{Aut} D$, such that
\[
\Psi_S(X) = \text{Cone} \left( S \otimes \text{Hom}^2(S, X) \to X \right)
\]
with inverse
\[
\Psi_S^{-1}(X) = \text{Cone} \left( X \to S \otimes \text{Hom}^2(X, S)^\vee \right) [-1].
\]

In the case when $D_X(Q) = D_{fd}(\Gamma_X Q)$, any simple $S_i$ for $i \in Q_0$ in the canonical heart
\[
\mathcal{H}_Q^X = \text{mod } \Gamma_X Q
\]
is spherical and the spherical twist group is
\[
\text{ST}_X(Q) = \langle \Psi_{S_i} \mid i \in Q_0 \rangle \subset \text{Aut } D_X(Q).
\]

We have the following.

Theorem A.7. [IQ1, Thm. 6.6] There is a canonical isomorphism $\text{Br}(Q) \cong \text{ST}_X(Q)$, sending the standard generators to the standard ones.

Thus the generator $\zeta_Q$ of $Z(\text{Br}(Q))$ becomes
\[
\zeta_Q = \Psi_{S_n} \circ \cdots \circ \Psi_{S_1}.
\]

A.4. $X$-Auslander-Reiten functor. First, we prove the following lemma.

Lemma A.8. $L$ induces an injection $L_* : \text{Aut } D_{\infty}(Q) \to \text{Aut } D_X(Q)$ that fits into the commutative diagram
\[
\begin{array}{ccc}
D_{\infty}(Q) & \xrightarrow{\Phi} & D_{\infty}(Q) \\
\downarrow{\L} & & \downarrow{\L} \\
D_X(Q) & \xrightarrow{L_*(\Phi)} & D_X(Q),
\end{array}
\]
for any $\Phi \in \text{Aut } D_{\infty}(Q)$.

Proof. By Koszul duality,
\[
D_{\infty}(Q) = D_{fd}(kQ) \cong \text{per } \mathcal{E}_Q,
\]
where
\[
\mathcal{E}_Q = \text{RHom}_{D_{\infty}(Q)}(S_Q, S_Q)
\]
is the dg endomorphism algebra for $S_Q = \bigoplus S_i$. Similarly, we have
\[
D_X(Q) = D_{fd}(\Gamma_X Q) \cong \text{per } \mathcal{E}_Q^X,
\]
where
\[ E^X_Q = \text{RHom}^2_{D_X(Q)}(S_Q, S_Q) \]
is the differential \(\mathbb{Z}^2\)-graded endomorphism algebra. Then any auto-equivalence \(\Phi\) in \(\text{Aut} D_\infty(Q)\) maps \(E_Q\) to another dg endomorphism algebra
\[ \Phi(E_Q) = \text{RHom}^2_{D_X(Q)}(\Phi(S_Q), \Phi(S_Q)) \]
that \(\Phi\) can be realized as per \(E_Q \to \text{per} \Phi(E_Q)\). After applying \(L\) that passes to \(D_X(Q)\), we obtain an auto-equivalence
\[ L_* (\Phi) : \text{per} E^X_Q \to \text{per} L_* \left( \Phi(E^X_Q) \right) \].

Finally, if \(L_* (\Phi) = \text{id}\) preserves \(S_i\) in \(D_X(Q)\) (and the Homs between them), then \(\Phi\) preserves them in \(D_\infty(Q)\), which must be identity. \(\square\)

Now, let \(\tau^E_X = L_*(\tau) \in \text{Aut} D_X(Q)\). We have the following.

**Proposition A.9.** Let \(Q\) be a Dynkin quiver. Then
\[ \tau^E_X = [X - 2] \circ \zeta^E_Q \] (A.8)
satisfies \((\tau^E_X)^h = [-2]\).

**Proof.** The calculation is exactly the same as the Calabi-Yau-N case in the proof of [Q1, Prop. 6.4.1]. Note that the assumption there, i.e. the isomorphism \(\text{Br}(Q) \cong \text{ST}_N(Q)\), has been proved in Theorem A.7 (cf. [QW]). \(\square\)

Recall we have a Gepner point \(\sigma_G = (Z_G, P_G)\) on \(\text{Stab}_s D_\infty(Q)\) and it induces a \(q\)-stability condition \((\sigma^E_{G,s}, s)\) for \(\text{Re}(s) \geq gd_Q + 1\), where \(\sigma^E_{G,s} = (Z^E_s, P^E_s)\) is constructed in Theorem A.3.

**Theorem A.10.** \(\sigma^E_{G,s} \in \text{QStab}_s D_X(Q)\) satisfies the Gepner equation
\[ \tau^E_X(\sigma) = \left( -\frac{2}{h} \right) \cdot \sigma \] (A.9)
for \(\text{Re}(s) \geq gd_Q + 1\).

**Proof.** As \(\tau^E_X\) is induced from \(\tau\) via \(L\), we have \(\tau^E_X = \tau \otimes R\) on the Grotendieck groups. Thus, for the central charge we have
\[ Z^E_{G,s} \circ \tau^E_X = (q_s \circ (Z_G \otimes R)) \circ \tau^E_X = q_s \circ ((Z_G \otimes R) \circ \tau^E_X) = q_s \circ ((Z_G \circ \tau) \otimes R) = q_s \circ (e^{2\pi i/h} \cdot Z_G \otimes R) = e^{2\pi i/h} \cdot (q_s \circ (Z_G \otimes R)) = e^{2\pi i/h} \cdot Z^E_{G,s}, \]
where we use the Gepner property of $\sigma_G$ that $Z_G \circ \tau = e^{2\pi i/h} \cdot Z_G$. For the slicing, we have $\tau_X^G(P) = \tau(P)$ (recall that we identify $P(\phi) \subset D_\infty(Q)$ with its image in $D_X(Q)$ under $L$) and hence

$$
\tau_X^G(P_{G,s}^s(\phi)) = \langle \tau(P(\phi - k \text{Re}(s)))|kX| \ | k \in \mathbb{Z}\rangle
$$

where we use the Gepner property of $\sigma_G$ that $\tau(P) = P(-2/h)$.

Thus $\sigma_{G,s}^s$ satisfies (A.9).

□

Remark A.11. A reachable Lagrangian immersion $L'$, by definition, is of the form $L' = \Upsilon \circ L$ for $\Upsilon \in \text{Aut} D_X(Q)$, where $L$ is the fixed initial Lagrangian immersion in (A.1). By the construction in Theorem A.3, we have

$$
\sigma_{G,s}^s = \Upsilon \circ \sigma_{G,s}^s,
$$

which solves the equation

$$
\tau_X^G(\sigma) = (-\frac{2}{h}) \cdot \sigma,
$$

for

$$
\tau_X^G(L') = (L')_s(\tau) = \Upsilon \circ \tau_X^G \circ \Upsilon^{-1}.
$$

Therefore, all such Gepner points $\sigma_{G,s}^s$ correspond to the same point $\sigma_G$ in $\text{Stab} D_X(Q)/\mathbb{C}$.

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