Unified Overview of Matrix-Monotonic Optimization for MIMO Transceivers

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Abstract—Matrix-monotonic optimization exploits the monotonic nature of positive semi-definite matrices to derive optimal diagonalizable structures for the matrix variables of matrix-variate optimization problems. Based on the optimal structures derived, the associated optimization problems can be substantially simplified and underlying physical insights can also be revealed. In this paper, a comprehensive overview of the applications of matrix-monotonic optimization to multiple-input multiple-output (MIMO) transceiver design is provided under various power constraints, and matrix-monotonic optimization is investigated for various types of channel state information (CSI) condition. Specifically, three cases are investigated: 1) both the transmitter and receiver have imperfect CSI; 2) perfect CSI is available at the receiver but the transmitter has no CSI; 3) perfect CSI is available at the receiver but the channel estimation error at the transmitter is norm-bounded. In all three cases, the matrix-monotonic optimization framework can be used for deriving the optimal structures of the optimal matrix variables. Furthermore, based on the proposed framework, three specific applications are given under three types of power constraints. The first is transceiver optimization for the multi-user MIMO uplink, the second is signal compression in distributed sensor networks, and the third is robust transceiver optimization of multi-hop amplify-and-forward cooperative networks.

I. MOTIVATIONS

Antenna arrays constitute a promising technique of realizing both a high bandwidth efficiency and high power efficiency in multiple-input multiple-output (MIMO) communication systems [1]–[13]. Transceiver optimization is of critical importance for fulfilling the potential of MIMO communication systems [11], [14]. MIMO transceiver optimization hinges on numerous factors, including their implementation issues, the availability of channel state information (CSI) and their system architectures. More specifically, MIMO transceivers can be classified into linear transceivers [11] and nonlinear transceivers [15], [16]. According to the different levels of CSI knowledge, MIMO transceiver designs can be classified into designs relying on perfect CSI [8]–[11] and designs having partial CSI [13], [17]–[21]. Finally, according to the system architecture, transceiver optimization can be used for point-to-point systems [8], [22], for multi-user (MU) MIMO systems [23]–[25], for distributed MIMO systems [26], [27], and for cooperative MIMO systems [28], [29].

In all the above-mentioned multiple antenna aided systems, the corresponding optimization variables become matrix variables [30]. As a result, optimization relying on matrix variables plays an important role in MIMO systems [31]. Optimization relying on matrix variables is generally very challenging and such problems are much more difficult to solve than their counterparts with vector variables or scalar variables, because matrix variable based optimization usually involves complex matrix operations, such as the calculation of the determinants, inverses, matrix decompositions and so on. Furthermore, because of spatial multiplexing gains, MIMO systems are capable of supporting multiple data streams. This fact makes transceiver optimization problems inherently multi-objective optimization problems. For example, given a limited transmit power, any specific transceiver optimization is a tradeoff between the performance of different data streams. This is the reason why there exists a rich body of work addressing various different MIMO transceiver designs [11], [14].

Any transceiver optimization problem hinges on the fundamental elements of the objective function and the specific optimization tools used for finding the extremities of the objective function. The more components the objective function has, the larger the research space becomes, which often makes a full hard utilization. A third related component is constituted by the constraints. The most widely used objective functions or performance metrics of MIMO transceiver optimization include the classic mean square error (MSE) minimization, signal to interference plus noise ratio (SINR) maximization or capacity maximization, bit error rate (BER) minimization, etc. [11]. Different performance metrics reflect different design preferences and different tradeoffs among the transmitted data streams. Transceiver optimization problems using different performance metrics imposes different degrees of difficulty to solve. Furthermore, different objective functions also correspond to different implementation strategies resulting in, for example, linear transceivers, nonlinear transceivers using Tomlinson-Harashima precoding (THP) or decision feedback equalizer (DFE) etc. [30]. Suffice to say that the specific choice of the objective function has a more substantial impact on the overall MIMO design than that of the tools used for optimizing it.

On the other hand, there are many different types of power constraints, such as the sum power constraint [30], per-antenna power constraint [23], [32], [33], shaping constraint [34],...
cognitive constraint, energy harvesting constraint, etc. [33], [35]. The most widely used power constraint is the sum power constraint requiring the sum of the powers at all the transmit antennas to be lower than a threshold. In communication systems, usually each antenna has its own amplifier [23]. Therefore, the per-antenna power constraint is more practical than the sum power constraint. However, the per-antenna power constraint is more challenging to consider than the sum power constraint [33]. The existing literature has revealed that if different transmit antennas have the same statistics, the performance gain of considering the more challenging per-antenna power constraint based design over using the simpler sum power constraint design is negligible. Thus, under the scenario of similar statistics for different transmit antennas, the sum power constraint is an effective modeling technique. It is worth noting however that in some cases, as in distributed antenna systems or heterogeneous networks, different antennas have significantly different statistics, and thus the per-antenna power constraint cannot be replaced by the sum power constraint without a significant performance loss. Moreover, considering other practical constraints, such as signal variances or the peak-to-average-ratio, joint power constraints or other types of constraints have to be taken into account [35].

It can be readily seen from the existing literature [14], [30], [33] that the underlying design principles for various transceiver optimization problems are almost the same. Generally, the main idea is taking advantage of the specific structure of the underlying optimization problem to simplify the transceiver optimization. Optimization theory plays an important role in MIMO transceiver optimization, and in the past decade many elegant results have been derived based on convex optimization theory [26], [31], [36]. Deriving optimal structures is critical in transceiver optimization [5], [10], [11]. Clearly, a general-purpose optimal structure that can cover every MIMO transceiver optimization problems does not exist, and most the research has been focused on finding an optimal diagonalizable structure for MIMO transceiver optimization. This is because based on the optimal diagonalizable structures of the MIMO transceivers, the corresponding optimization problems can be substantially simplified and deeply underlying physical insights can also be revealed [5], [10], [11]. Again, optimization variables of MIMO transceiver designs are generally matrix variables. Matrix-monotonic optimization exploits the monotonic nature of positive semi-definite matrices to derive optimal structures of the matrix variables in the underlying optimization problems [30], [33], [34]. Based on matrix-monotonic optimization, the matrix variables can be substantially simplified into vector variables. The optimal structures delivered by matrix-monotonic optimization, therefore, greatly simplify complicated MIMO transceiver designs and make the underlying physical interpretation more transparent. From a matrix-monotonic optimization perspective, MIMO transceiver optimization problems relying on different objective functions and power constraints can be unified and, therefore, their associated optimal structures can be derived using the same matrix-monotonic optimization tool. Exploiting matrix-monotonic optimization is a powerful mathematical tool conceived for solving challenging matrix-variate transceiver optimization problems. To the best of our knowledge, a comprehensive review of matrix-monotonic optimization is still missing in the open literature and this fact motivates us to write this tutorial.

This paper offers a comprehensive overview for matrix-monotonic optimization under various power constraints. Matrix monotonic optimization problem with various levels of CSI is investigated in depth. The whole big picture of this tutorial is given in Fig. 1. Our main contributions are listed as follows.

- We present the framework of matrix-monotonic optimization that unifies MIMO transceiver optimization problems relying on diverse objective functions and power constraints. In contrast to most of the existing works, various power constraints are considered and investigated in our framework, which include the sum power constraint, multiple weighted power constraints, joint power constraints and shaping constraints. In other words, the matrix-monotonic optimization framework proposed in this paper unifies both the families of linear and nonlinear MIMO transceiver optimization under the sum power constraint, shaping constraint, joint power constraints and multiple weighted power constraints.

- Moreover, robust MIMO transceiver optimization relying on partial CSI under various power constraints is investigated based on the matrix-monotonic optimization framework. Specifically, the following three cases are investigated:
  1) Both the transmitter and receiver have only imperfect CSI,
  2) The receiver has perfect CSI but the transmitter has no CSI,
  3) The receiver has perfect CSI but the channel estimate available at the transmitter is subject to a certain uncertainty norm-bounded error.

Although having imperfect CSI makes the MIMO transceiver optimization more complex and challenging, the proposed matrix-monotonic optimization framework is still capable of deriving the underlying optimal structures.

- Three important applications are provided to show how to extend the family of single matrix-variate matrix-monotonic optimization to multiple matrix-variate matrix-monotonic optimization. Specifically, the first application of multiple matrix-variate matrix-monotonic optimization involves a multi-user MIMO uplink under three different power constraints, i.e., the shaping constraint, joint power constraint and multiple weighted power constraints. The second is signal compression under three different power constraints in the context of distributed sensor networks. The third is robust transceiver optimization under three different power constraints for multi-hop amplify-and-forward (AF) MIMO cooperative networks. Even for these complex optimization problems associated with a high number of matrix variables, the matrix-monotonic optimization framework works well.

The remainder of this paper is organized as follows. In Sec-
In Section II, we present the fundamentals of the matrix-monotonic optimization framework. Then Section III investigates classic Bayesian robust matrix-monotonic optimization for robust transceiver design when the channel estimation errors are Gaussian distributed. In Section IV, stochastic robust matrix-monotonic optimization is investigated for MIMO transceiver optimization where the receiver has perfect CSI but the transmitter knows only the channel statistics. Section V is devoted to worst case matrix-monotonic optimization, which focuses on transceiver optimization in the face of norm-bounded channel estimation errors. Moreover, we extend our discussions from point-to-point MIMO systems to the MU-MIMO uplink in Section VI. Compression matrix optimization for distributed sensor networks is discussed based on matrix-monotonic optimization in Section VII. In Section VIII, robust transceiver optimization is proposed for multi-hop AF MIMO relaying networks separately under shaping constraints, joint power constraints and multiple weighted power constraints. Several numerical results are given in Section IX, and our conclusions are offered in Section X.

The following notational conventions are adopted throughout our discussions. The normal-faced letters denote scalars, while bold-faced lower-case and upper-case letters denote vectors and matrices, respectively. $Z^H$, $\text{Tr}(Z)$ and $|Z|$ denote the Hermitian transpose, trace and determinant of complex matrix $Z$, respectively. Statistical expectation is denoted by $\mathbb{E}\{\cdot\}$, and $a^+ = \max\{0, a\}$, while $(\cdot)^T$ denotes the vector/matrix transpose operator. $Z^+$ is the Hermitian square root of $Z$ which is positive semi-definite. The $i$th largest eigenvalue of $Z$ is denoted by $\lambda_i(Z)$, and the $i$th-row and $j$th-column element of $Z$ is denoted by $[Z]_{i,j}$, while $d[Z]$ denotes the vector consisting of the diagonal elements of $Z$ and $\text{diag}\{\{A_k\}_{k=1}^K\}$ denotes the block diagonal matrix whose diagonal sub-matrices are $A_1, \ldots, A_K$. Additionally, the $i$th element of a vector $z$ is denoted by $[z]_i$. The identity matrix of appropriate dimension is denoted by $I$, and $\otimes$ is the Kronecker product. In this paper, $\Lambda$ always denotes a diagonal matrix, and the expressions $\Lambda \diagdown$ and $\Lambda \diagup$ represent a rectangular or square diagonal matrix with the diagonal elements in descending order and ascending order, respectively.

II. FUNDAMENTALS OF MATRIX-MONOTONIC OPTIMIZATION

An optimization problem with a real-valued objective function $f_0(\cdot)$ that depends on a complex matrix variable $X$ is generally formulated as

$$\min_X f_0(X),$$

subject to $\psi_i(X) \leq 0, 1 \leq i \leq I,$

(1)

where $\psi_i(\cdot), 1 \leq i \leq I,$ are the constraint functions. A wide range of optimization problems can be cast in this optimization framework, including the classic MIMO transceiver optimization [14], training designs [30], MIMO radar waveform...
 Objective function \( - \text{Tr} \left( \left( Q_X^H F^H \Pi F Q_X + \Phi \right)^{-1} \left( A^H (Q_X^H F^H \Pi F Q_X + \alpha I)^{-1} A \right) \right) \)

Optimum \( U_{F\Pi F} U^H_\Phi \)

Obj. 3

Objective function \( \log \left| \left( Q_X^H F^H \Pi F Q_X + \alpha I \right)^{-1} A + \Phi \right| \)

Optimum \( U_{F\Pi F} U^H_A + \Phi_A \)

Obj. 4

Optimization [30], etc. In order to analyze the properties of this generic optimization problem, we first discuss two of its basic components, namely, the objective function and the constraints, separately.

### A. Objective Functions

The objective function reflects the cost or utility of the optimization problem. In this paper, all the optimization problems discussed are formulated with the objective of minimizing a cost function. Let us now discuss the commonly used objective functions, as listed in the left-column of Table I.

For transceiver optimization, the capacity is one of the most important performance metrics. For training optimization, the mutual information is also an important performance metric as it reflects the correlation between the estimated parameters and the true parameters. In these cases, the objective function is given by **Obj. 1**, where the complex matrix variable \( X \) is expressed by \( X = F Q_X \) with \( F \) defining the auxiliary matrix variable and \( Q_X \) a unitary matrix, while \( \Pi \) and \( \Phi \) are constant positive semi-definite matrices which have different physical meanings for different systems. The MSE is another important performance metric for transceiver or training optimization, which reflects how accurately a signal can be recovered rather than how much information can be transmitted. For the optimization problem of sum MSE minimization, the objective function is given in the form of **Obj. 2**.

Generally, the MSE formulation for linear transceiver optimization is determined by the specific signal model considered. For example, in a dual-hop AF MIMO relaying network, the MSE minimization has **Obj. 3**, where \( \alpha \) is a positive scalar and \( A \) is a constant complex matrix. Similarly, the capacity maximization for a dual-hop AF MIMO relaying network aims at minimizing the objective function **Obj. 4**. For linear transceiver optimization, to realize different levels of fairness between different transmitted data streams, a general objective function can be formulated as an additive Schur-convex function [14] or additive Schur-concave function [14] of the diagonal elements of the MSE matrix, which are given by **Obj. 5.1** and **Obj. 5.2**, respectively. The additive Schur-convex function \( f_{\text{concave}}(\cdot) \) and the additive Schur-concave function \( f_{\text{concave}}^+(\cdot) \) represent different levels of fairness among the diagonal elements of the data MSE matrix.

When nonlinear transceivers are chosen for improving the BER performance at the cost of increased complexity, e.g., THP or DFE, the objective functions of the transceiver optimization can be formulated as a multiplicative Schur-convex function or a multiplicative Schur-concave function of the vector consisting of the squared diagonal elements of the Cholesky-decomposition triangular matrix of the MSE matrix, that is, **Obj. 6.1** and **Obj. 6.2**, respectively, where \( L \) is a lower triangular matrix. The multiplicative Schur-convex function \( f_{\text{convex}}(\cdot) \) and the multiplicative Schur-concave function \( f_{\text{concave}}(\cdot) \) reflect different levels of fairness among the different data streams, i.e., different tradeoffs among the performance of different data steams.

In wireless communication designs, even for the same system or the same optimization problem, the mathematical formulae are not unique. More specifically, for the relative

| Index | Objective function | Optimum \( Q_X \) |
|-------|-------------------|--------------------|
| Obj. 1 | \( - \log \left| Q_X^H F^H \Pi F Q_X + \Phi \right| \) | \( U_{F\Pi F} U^H_\Phi \) |
| Obj. 2 | \( \text{Tr} \left( \left( Q_X^H F^H \Pi F Q_X + \Phi \right)^{-1} \right) \) | \( U_{F\Pi F} U^H_\Phi \) |
| Obj. 3 | \( \log \left| A^H \left( Q_X^H F^H \Pi F Q_X + \alpha I \right)^{-1} A + \Phi \right| \) | \( U_{F\Pi F} U^H_A + \Phi_A \) |
| Obj. 4 | \( f_{\text{convex}} \left( d^2(L) \right) \) | \( U_{F\Pi F} U^H_GMD \) |
| Obj. 5.1 | \( f_{\text{concave}} \left( d^2(L) \right) \) | \( U_{F\Pi F} U^H_A \) (High SNR) |
| Obj. 5.2 | \( \left( Q_X^H F^H \Pi F Q_X + \alpha I \right)^{-1} \) | \( U_{F\Pi F} U^H_A \) (High SNR) |
| Obj. 6.1 | \( \left( Q_X^H F^H \Pi F Q_X A + \Phi \right)^{-1} \) | \( U_{F\Pi F} U^H_A + \Phi_A \) |
| Obj. 6.2 | \( \left( Q_X^H F^H \Pi F Q_X A + \alpha I \right)^{-1} \) | \( U_{F\Pi F} U^H_A \) (High SNR) |
information maximization, we have the alternative objective function \textbf{Obj. 7}. Similarly, the sum MSE minimization has the alternative objective function \textbf{Obj. 8}. Moreover, the weighted MSE minimization can be considered as a general extension of the sum MSE minimization by introducing a weighting matrix, which has the objective function \textbf{Obj. 9}.

As discussed in the existing literature, some MIMO system optimization problems may involve Kronecker products [30]. The optimization problems relying on Kronecker product usually look very complicated. In this paper, the pair of optimization problems relying on either the matrix determinant or on the the matrix trace are discussed that involve Kronecker products. Based on \textbf{Obj. 1}, we have the extended Kronecker structured objective function \textbf{Obj. 10}, which is equivalent to \textbf{Obj. 11}. It can readily be seen that with the choice of \( \Sigma_1 = \Sigma_2 \), \textbf{Obj. 10} and \textbf{Obj. 11} are equivalent to \textbf{Obj. 1}. In this paper, we also consider a more general case in which \( \Sigma_1 \) and \( \Sigma_2 \) have the same eigenvalue decomposition (EVD) unitary matrix. Under this assumption and based on \textbf{Obj. 2}, we have the extended Kronecker structured objective function \textbf{Obj. 12}, which is equivalent to \textbf{Obj. 13}. Similarly, based on \textbf{Obj. 3}, we have the objective function \textbf{Obj. 14}, which is also equivalent to \textbf{Obj. 15}. In our following discussions involving \textbf{Obj. 10} to \textbf{Obj. 15}, it is always assumed that \( \Sigma_1 \) and \( \Sigma_2 \) have the same EVD unitary matrix.

\section*{B. Constraint Functions}

In practical communication system designs, typically the associated optimization problems have constraints, and these constraints have different physical meanings for different communication systems.

The most natural constraints are the power constraints, since practical amplifiers have certain maximum transmit power thresholds. The simplest power constraint, is the sum power constraint which can be expressed as

\[ \textbf{Constraint 1:} \quad \text{Tr}(XX^H) \leq P. \quad (2) \]

With the sum power constraint, the optimization problems associated with training sequence designs or transceiver designs are subjected to the constraint of the power sum of all the transmit antennas. In practical systems, each antenna has its own power amplifier and, therefore, the per-antenna power constraints or individual power constraints provide a more reasonable power constraint model, which is expressed as

\[ \textbf{Constraint 2:} \quad [XX^H]_{n,n} \leq P_n, \quad n = 1, \cdots, N, \quad (3) \]

where we have assumed that the number of transmit antennas is \( N \) and the matrix variable \( X \) has \( N \) rows. The per-antenna power constraint (3) may be more practical but it does not include the sum power constraint (2) as its special case.

In sophisticated communication networks, the constraints are not limited to reflect the maximum power constraints at the transmit antennas for the desired signal but they also reflect many other constraints such as the interference constraints between adjacent links. A more general power constraint is the following one having multiple weighted components

\[ \textbf{Constraint 3:} \quad \text{Tr}(\Omega_i XX^H) \leq P_i, \quad i = 1, \cdots, I, \quad (4) \]

where \( I \) is the number of weighted power constraints. \textbf{Constraint 3} is more general than \textbf{Constraint 1} and \textbf{Constraint 2}. The constraint model (4) includes the sum power constraint (2) and per-antenna power constraint (3) as its special cases. Specifically, by choosing \( I = 1 \) and \( \Omega_1 = I \), this power constraint model becomes the sum power constraint (2). Furthermore, when \( I = N \) and \( \Omega_i \) is the matrix whose \( i \)th diagonal element is one and all the other elements are zeros, this model is exactly the per-antenna power constraint (3).

From the subspace theory viewpoint, the interference constraint can be modeled as a constraint in a positive semi-definite matrix space. A classic example is the shaping constraint [34], which is formulated as the following matrix inequality.

\[ \textbf{Constraint 4:} \quad XX^H \preceq R. \quad (5) \]

From matrix inequality theory, this constraint is equivalent to

\[ \text{Tr}(\Omega_i XX^H) \leq \text{Tr}(\Omega_i R_i), \quad (6) \]

for any positive semi-definite matrix \( \Omega_i \). Based on this fact, we can argue that the shaping constraint represents a special case of the multiple weighted power constraint. A simplified version of \textbf{Constraint 4} is the constraints imposed on the eigenvalues of the covariance matrix \( XX^H \) formulated as

\[ \textbf{Constraint 5:} \quad \lambda_i(XX^H) \leq \tau_i. \quad (7) \]

A widely used eigenvalue constraint is the constraint on the maximum eigenvalue, \( \lambda_1(XX^H) \leq \tau_1 \), which is equivalent to

\[ XX^H \preceq \tau_1 I. \quad (8) \]

This constraint can be used together with the sum power constraint to limit the transmitter’s peak power. This is because most of the existing power constraints are based on statistical averages, while from a practical implementation perspective, the power constraint is an instantaneous constraint instead of being an average one. This kind of combined power constraint is termed as the joint power constraint, which is expressed as

\[ \textbf{Constraint 6:} \quad \text{Tr}(XX^H) \leq P, \]

\[ XX^H \preceq \tau_1 I. \quad (9) \]

In cognitive radio communications, the interference imposed by the secondary user on the primary user must be smaller than a threshold and this constraint can be written in the following form

\[ \textbf{Constraint 7:} \quad \text{Tr}(H_c XX^H H_c^H) \preceq \tau_C, \quad (10) \]

where \( H_c \) is the channel matrix between the secondary user and primary user, while \( \tau_C \) is the interference threshold. This kind of constraint is also a special case of \textbf{Constraint 3}.

Before turning attention to discuss the optimization problem (1), two fundamental definitions are first introduced.

\textbf{Definition 1} A constraint \( \psi(X) \leq 0 \) is a left unitary invariant constraint if we have

\[ \psi(QLX) = \psi(X), \quad (11) \]
where $Q_L$ is an arbitrary unitary matrix.

**Definition 2** A constraint $\psi(X) \leq 0$ is a right unitary invariant constraint if we have

$$\psi(XQ_R) = \psi(X),$$

where $Q_R$ is an arbitrary unitary matrix.

It is worth noting that all the constraints discussed above are right unitary invariant. Therefore, we can focus our attention on the family of right unitary invariant constraints only. In particular, we will focus our attention on the shaping constraint, joint power constraints and multiple weighted power constraints.

**C. Matrix-Monotonic Optimization**

Based on the above discussions, the generic optimization problem of MIMO systems can be formulated as

$$\text{Opt. 1.1: } \min_X f(X^H \Pi X),$$

s.t. $\psi_j(X) \leq 0, 1 \leq j \leq I.$

Since the constraints are right unitary invariant, we introduce the auxiliary matrix variable $F$ and express the original matrix variable $X$ as

$$X = FQ_X,$$

where $Q_X$ is an arbitrary unitary matrix. Based on (14), the optimization problem (13) can be reformulated as

$$\min_{F,Q_X} f(Q_X^H F^H \Pi F Q_X),$$

s.t. $\psi_j(FQ_X) = \psi_j(F) \leq 0, 1 \leq j \leq I.$

Note that the constraints do not depend on $Q_X$. Therefore, the optimal $Q_X$ is independent of the constraints.

1) Optimization of $Q_X$: Generally, there are two basic approaches to optimize $Q_X$. The first one is based on the basic matrix inequality and the other is based on majorization theory.

**Basic Matrix Inequalities** Typically, the extreme values of basic matrix operations e.g., trace, determinant, etc., are functions of the eigenvalues of the matrices involved. Given the positive semi-definite matrices $C \in \mathbb{C}^{N \times N}$ and $D \in \mathbb{C}^{N \times N}$, we consider the following EVDs

$$C = UC\Lambda_C U_C^H \text{ with } \Lambda_C \prec \mathbf{I},$$

$$D = UD\Lambda_D U_D^H \text{ with } \Lambda_D \succ \mathbf{I},$$

where $\Lambda_D$ and $\Lambda_D$ consist of the eigenvalues of $D$ arranged in descending order and ascending order, while $U_D$ and $\bar{U}_D$ contain the corresponding eigenvectors of $D$, respectively. Then we have the four basic matrix inequalities, ranging from (19) to (22), shown at the bottom of this page. Furthermore, in both Matrix Inequality 1 and Matrix Inequality 2, the left equality holds when $UC = UD$, and the right equality holds when $UC = U_D$; while in both Matrix Inequality 3 and Matrix Inequality 4, the left equality holds when $UC = U_D$, and the right equality holds when $UC = U_D$.

**Majorization Theory** Majorization theory constitutes an important branch of matrix equality theory [31], [37]. We have the following two important definitions.

**Definition 3 ([37])** For two vectors $x, y \in \mathbb{R}^N$, $x$ is said to be majorized by $y$, denoted as $x \prec y$, when the following inequalities are satisfied: $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for $1 \leq k \leq N-1$, and $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$, where $\sum$ denotes a mathematical operator.

In the following, we only consider the addition and product operators of $\sum = \sum$ and $\sum = \prod$.

**Definition 4 ([37])** A real-valued function $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ is additively or multiplicatively Schur-convex for any $x, y$ in the feasible set, $x \prec y \Rightarrow \phi(x) \leq \phi(y)$. On the other hand, $\phi$ is additively or multiplicatively Schur-concave when $x \prec y \Rightarrow \phi(x) \geq \phi(y)$.

**Optimal $Q_X$** Based on the basic matrix inequalities and majorization theory together with the following EVDs (23) to (26) and the singular value decomposition (SVD) (27)

$$F^H \Pi F = U_{\Pi} F^H U_{\Pi}^H \text{ with } \Lambda_{\Pi} \prec \mathbf{I},$$

$$\Phi = U_{\Phi} \Lambda_{\Phi} U_{\Phi}^H \text{ with } \Lambda_{\Phi} \prec \mathbf{I},$$

$$A \Phi^{-1} A^H = U_{A \Phi} \Lambda_{A \Phi} U_{A \Phi}^H \text{ with } \Lambda_{A \Phi} \prec \mathbf{I},$$

$$A = U_A V_A^H \text{ with } \Lambda_A \succ \mathbf{I},$$

the optimal unitary matrices $Q_X$ corresponding to the various objective functions can be derived and they are listed in the right column of Table I. In the SVD (27), $A_A$ contains the singular values of $A$, while $U_A$ and $V_A$ are the corresponding left and right unitary matrices, respectively.

In Table I, the unitary $U_{\text{DFT}}$ for Obj. 5.1 is a discrete Fourier transform (DFT) matrix, and $U_{\text{GMD}}$ for Obj. 6.1 is the unitary matrix that makes the diagonal elements of $L$ identical, that is, $U_{\text{GMD}}$ is the right unitary matrix of the geometric mean decomposition (GMD) of $(Q_X^H F^H \Pi F Q_X + \alpha I)^{-0.5}$. It is
also worth highlighting that for Obj. 8 and Obj. 9, in general, the closed-form optimal $Q_X$ cannot be derived, and only the approximated optimal solutions can be obtained at high signal-to-noise ratio (SNR) conditions.

2) Optimization of $F$: For Opt. 1.1, given the optimal $Q_X$, the optimal solutions of $F$ fall in the Pareto optimal solution set of the following multi-objective optimization problem [30]

$$\text{Opt. 1.2: } \max_F \lambda(F^H F),$$

s.t. $\psi_j(F) \leq 0$, $1 \leq j \leq I$, (28)

where $\lambda(F^H F) = [\lambda_1(F^H F) \ldots \lambda_N(F^H F)]^T$. Clearly, the optimal structure of $F$ depends on both the objective function and on the constraints. As discussed in [30], deriving the optimal structure of $F$ for Opt. 1.2 corresponds to deriving the optimal structures of $F$ for Opt. 1.1 for various objectives functions, including Obj. 1 to Obj. 15.

Since $\psi_j(F)$ is right unitary invariant, Opt. 1.2 is equivalent to the following matrix-monotonic optimization problem

$$\text{Opt. 1.3: } \max_F F^H F,$$

s.t. $\psi_j(F) \leq 0$, $1 \leq j \leq I$. (29)

Generally, matrix-monotonic optimization maximizes a positive semi-definite matrix under certain power constraints. The optimal solutions of Opt. 1.1 for the objective functions Obj. 1 to Obj. 15 are all in the Pareto optimal solution set of Opt. 1.3. Since matrix-monotonic optimization derives the common structure of the Pareto optimal solution set of Opt. 1.3, the common optimal structures derived are exactly the structures of the optimal solutions of Opt. 1.1. By taking advantage of these optimal structures, Opt. 1.1 can be substantially simplified.

Interestingly, $F^H F$ can be interpreted as a matrix version SNR. Thus, based on Opt. 1.3 it can be concluded that various MIMO transceiver optimization problems maximize this matrix version SNR. When there are multiple data streams, maximizing the matrix version SNR inherently constitute a multi-objective optimization problem. In addition, each unitary matrix $Q_X$ corresponds to a specific implementation scheme. The focus of matrix-monotonic optimization is how to maximize the positive semi-definite matrix $F^H F$ under certain constraints. Different objective functions realize different tradeoffs among the multiple data streams, and matrix-monotonic optimization is a powerful tool that unifies the different constrained optimization problems with various objective functions, as illustrated in Fig. 2. Specifically, based on matrix-monotonic optimization, the common properties of these objective functions are revealed, which are reflected on the optimal diagonalizable structures.

These structures can transform complex optimization problems relying on matrix variables into much simpler ones with only vector variables. Thus case-by-case investigations for different objective functions are avoided. Since the optimal structure of $F$ also depends on the specific form of the constraints, in the following, three right unitary invariant constraints are investigated, namely, shaping constraint [34], joint power constraint [34] and multiple weighted power constraints [33].

Shaping Constraint For the shaping constraint, i.e., Constraint 4, Opt. 1.3 becomes the following optimization problem [34]

$$\text{Opt. 1.4: } \max_F F^H \Pi F,$$

s.t. $F F^H \leq R_s$. (30)

The following lemma reveals the optimal structure of $F$ for Opt. 1.4 with the shaping constraint.

**Lemma 1** When the rank of $R_s$ is not higher than the number of columns and the number of rows in $F$, the optimal solution $F_{\text{opt}}$ of Opt. 1.4 is a square root of $R_s$, i.e., $F_{\text{opt}} F_{\text{opt}}^H = R_s$.

**Proof 1** Since the constraint is right unitary invariant for $F$, the objective is equivalent to maximizing $\lambda(F^H F)$, which in turn is equivalent to maximizing $\lambda(\Pi^{1/2} F F^H \Pi^{1/2})$. It is plausible that when the rank of $R_s$ is not higher than the number of columns and the number of rows in $F$, the optimal solution $F_{\text{opt}}$ is a square root of $R_s$.

Joint Power Constraint Under the joint power constraint, Constraint 6, Opt. 1.3 can be rewritten as

$$\text{Opt. 1.5: } \max_F F^H F,$$

s.t. $\text{Tr}(F F^H) \leq P$, $F F^H \leq \tau I$. (31)

The optimal solution $F_{\text{opt}}$ for Opt. 1.5 is given in Lemma 2, and the proof can be found in [34].

**Lemma 2** For Opt. 1.5 with the joint power constraint, the Pareto optimal solutions satisfy the following structure

$$F_{\text{opt}} = U_{\Pi} \Lambda_F F^H_{\text{Arb}},$$

where the unitary matrix $U_{\Pi}$ is specified by the EVD

$$\Pi = U_{\Pi} \Lambda_{\Pi} U^H_{\Pi} \text{ with } \Lambda_{\Pi} \preceq \chi,$$

every diagonal element of the rectangular diagonal matrix $\Lambda_F$ is smaller than $\sqrt{\tau}$, and $U_{\text{Arb}}$ is an arbitrary unitary matrix having the appropriate dimension.

**Remark 1** For the optimization problem only under the sum power constraint, the optimal structure for $F_{\text{opt}}$ is also specified by (32), where the sum of the diagonal elements of $\Lambda_F$ is no larger than $P$. 

---

![Fig. 2. Illustration of matrix version SNR maximization in the cone of positive semi-definite matrices.](image-url)
Multiple Weighted Power Constraints Under the multiple weighted power constraints, Opt. 1.3 becomes

\[ \text{Opt. 1.6: } \max_F P \mathbf{H}^\text{H} \Pi F, \]
\[ \text{s.t. } \text{Tr}(\Omega_i F \mathbf{H}^\text{H}) \leq P_i, \quad 1 \leq i \leq I. \]

Note that the weighted power constraints include both the sum power constraint and per-antenna power constraints as its special cases. First define the auxiliary variable

\[ \tilde{F} = \left( \sum_{i=1}^{I} \alpha_i \Omega_i \right)^{\frac{1}{2}} F. \]

Lemma 3 The Pareto optimal solutions of Opt. 1.6 satisfy the following structure

\[ F_{\text{opt}} = \Omega^\frac{1}{2} \mathbf{U}_{\text{Arb}} \mathbf{F}_{\text{opt}} \mathbf{U}_{\text{Arb}}^\text{H}, \]

where \( \mathbf{U}_{\text{Arb}} \) is an arbitrary unitary matrix of appropriate dimension, \( \Omega = \sum_{i=1}^{I} \alpha_i \Omega_i \), the nonnegative scalars \( \alpha_i \) are the weighting factors that ensure that the constraints \( \text{Tr}(\Omega_i F \mathbf{H}^\text{H}) \leq P_i \) hold and they can be computed using classic subgradient methods, while the unitary matrix \( \mathbf{U}_{\text{opt}} \) is specified by the EVD

\[ \Omega^{-\frac{1}{2}} \Pi \Omega^{-\frac{1}{2}} = \mathbf{U}_{\text{opt}} \Lambda_{\text{opt}} \mathbf{U}_{\text{opt}}^\text{H} \text{ with } \Lambda_{\text{opt}} \succ 0. \]

Proof 2 See Appendix A

\[ \Omega^{-\frac{1}{2}} \Phi \Omega^{-\frac{1}{2}} = \mathbf{U}_{\text{opt}} \Lambda_{\text{opt}} \mathbf{U}_{\text{opt}}^\text{H} \text{ with } \Lambda_{\text{opt}} \succ 0. \]

D. Training and Transceiver Optimization

There exists a close relationship between training and transceiver optimization. In transceiver optimization, the channel matrix is usually assumed to be known and the signal vector is to be recovered. By contrast, in training optimization, the training signals (sequences) are assumed to be known, while the channel matrix has to be estimated. This ‘duality’ between transceiver optimization and training optimization means that most of the techniques developed for transceiver optimization can be used for training optimization. For example, under the sum power constraint, the training optimization and transceiver optimization can be unified into the same category of matrix-monotonic optimization [30]. Incidentally, as discussed in [30], training sequence optimization and waveform optimization for MIMO radar are very similar from the mathematical viewpoint. Thus the following discussions are also applicable to waveform optimization for MIMO radar.

The generic training optimization problem takes the following form:

\[ \text{Opt. 1.7: } \min_X f(\tilde{X} \Pi \tilde{X}^\text{H}), \]
\[ \text{s.t. } \psi_i(\tilde{X}) \leq 0, \quad 1 \leq i \leq I, \]

where \( \tilde{X} \) is the training sequence. Compared to the generic transceiver optimization problem Opt. 1.1, observe the different position of the Hermitian transpose operation in Opt. 1.7.

Under the sum power constraint, namely, for \( I = 1 \) and \( \psi_1(\tilde{X}) = \text{Tr}(\tilde{X} \tilde{X}^\text{H}) - P \), Opt. 1.7 becomes

\[ \min_X f(\tilde{X} \Pi \tilde{X}^\text{H}), \]
\[ \text{s.t. } \text{Tr}(\tilde{X} \tilde{X}^\text{H}) \leq P. \]

Observe that under the sum power constraint, Opt. 1.7 and Opt. 1.1 are exactly the same, when replacing \( \tilde{X}^\text{H} \) with \( \tilde{X} \) and noting that \( \text{Tr}(\tilde{X} \tilde{X}^\text{H}) = \text{Tr}(\tilde{X} \tilde{X}) \). This is why under the sum power constraint, training optimization is naturally very similar to transceiver optimization [38]-[42].

Under the multiple weighted power constraints, Opt. 1.7 becomes

\[ \min_X f(\tilde{X} \Pi \tilde{X}^\text{H}), \]
\[ \text{s.t. } \text{Tr}(\tilde{X} \tilde{X}^\text{H}) \leq P, \quad 1 \leq i \leq I. \]

However, under the multiple weighted power constraints, Opt. 1.7 and Opt. 1.1 are not the same upon replacing \( \tilde{X}^\text{H} \) with \( \tilde{X} \). This is because \( \text{Tr}(\tilde{X} \tilde{X}^\text{H} X) \neq \text{Tr}(\tilde{X} \tilde{X}^\text{H} \tilde{X}) \).

Similar to the transceiver optimization under multiple weighted power constraints shown in Appendix A, the training optimization (40) is also equivalent to

\[ \min_X f(\tilde{X} \Pi \tilde{X}^\text{H}), \]
\[ \text{s.t. } \text{Tr}(\tilde{X} \tilde{X}^\text{H}) \leq P, \quad 1 \leq i \leq I, \]

where the weights \( \alpha_i \), \( 1 \leq i \leq I \), ensure that the constraints \( \text{Tr}(\tilde{X} \tilde{X}^\text{H}) \leq P_i \) hold, and they can be computed using classic subgradient methods.

In this paper, our focus is on providing a comprehensive overview of transceiver optimization, and the detailed discussions on a training optimization is beyond the scope of this paper. The interested readers are referred to our work [43] for training optimization.

E. Advantages of Matrix-Monotonic Optimization

Again as summarized in Fig. 3, matrix-monotonic optimization theory can simplify the optimization problem relying on matrix variables into a much simpler one manipulating only vector variables. Using matrix-monotonic optimization, for example, the optimal structure of the matrix variable \( F \) can be derived and the remaining optimization problem becomes a much simpler one that optimizes the diagonal matrix \( \mathbf{A}_F \).

For the various objective functions and constraints discussed previously, the optimal solutions of the diagonal elements of the diagonal matrix \( \mathbf{A}_F \) are in fact diverse variants of classic water-filling solutions, which can be readily obtained straightforwardly based on the corresponding Karush-Kuhn-Tucker (KKT) conditions [44], [45].

In the existing literature, MIMO transceiver optimization problems are unified in the framework based on majorization theory [14]. Our work is different from this existing framework in two perspectives. Firstly, in [14], linear and nonlinear transceiver optimization is considered separately. In our work, they are considered in the same framework. Additionally, in our work, more objective functions are considered. More importantly, the shaping constraint, joint power constraint and multiple weighted power constraints are considered in our work instead of merely the sum power constraint.

For the multiple weighted power constraints, to the best of our knowledge, all the existing works are based on the KKT conditions. There are several limitations for these existing
Optimization relying on matrix variables under right unitary invariant constraints

Unitary Matrix Optimization

Matrix-Monotonic Optimization

Vector Variable Optimization

III. BAYESIAN ROBUST MATRIX-MONOTONIC OPTIMIZATION

In wireless communication systems, the channel parameters have to be estimated. However, due to the uncertainty introduced by both noise and the time-varying nature of wireless channels, channel estimation errors inevitably exist [19], and the true channel matrix \( \mathbf{H} \) can be expressed by the following Kronecker formula [20], [22]

\[
\mathbf{H} = \mathbf{\hat{H}} + \mathbf{H}_W \mathbf{\Psi}_W, \tag{42}
\]

where \( \mathbf{\hat{H}} \) is the estimated channel matrix and \( \mathbf{H}_W \mathbf{\Psi}_W \) is the channel estimation error, in which the elements of \( \mathbf{H}_W \) obey the independent and identical complex Gaussian distribution \( \mathcal{CN}(0, 1) \) and the covariance matrix \( \mathbf{\Psi} \) of the channel estimate is a function of both the training sequence and of the channel estimator [20], [22]. Based on (42), for Bayesian robust transceiver optimization, the matrix \( \Pi \) in the matrix-monotonic optimization can be expressed as [30]

\[
\Pi = \mathbf{\hat{H}}^H (\sigma_n^2 I + \text{Tr}(\mathbf{XX}^H \mathbf{\Psi}) I)^{-1} \mathbf{\hat{H}}, \tag{43}
\]

where \( \sigma_n^2 \) is the noise power in the data transmission.

As a result, the generic Bayesian robust matrix-variate optimization can be formulated as [30]

**Opt. 2.1:** \[
\min_{\mathbf{X}} f(\mathbf{X}^H \mathbf{\hat{H}}^H \mathbf{K}^{-1} \mathbf{\hat{H}} \mathbf{X}),
\]

\[
\text{s.t. } \mathbf{K} = \sigma_n^2 I + \text{Tr}(\mathbf{XX}^H \mathbf{\Psi}) I, \tag{44}
\]

\[
\psi_i(\mathbf{X}) \leq 0, \ 1 \leq i \leq I.
\]

As discussed in [30], after introducing the transformation \( \mathbf{X} = F Q \mathbf{X} \) and recalling that the constraints \( \psi_i(\cdot) \) are right unitary invariant, **Opt. 2.1** is transferred equivalently to the following matrix-monotonic optimization problem:

**Opt. 2.2:** \[
\max_{F} F^H \mathbf{\hat{H}}^H \mathbf{K}^{-1} \mathbf{\hat{H}} F,
\]

\[
\text{s.t. } \mathbf{K} = \sigma_n^2 I + \text{Tr}(\mathbf{FF}^H \mathbf{\Psi}) I, \tag{45}
\]

\[
\psi_i(F) \leq 0, \ 1 \leq i \leq I.
\]

Here the matrix \( F^H \mathbf{\hat{H}}^H \mathbf{K}^{-1} \mathbf{\hat{H}} F \) can be regarded as an extended SNR matrix in the presence of channel estimation errors, and this kind of matrix-monotonic optimization is named as robust matrix-monotonic optimization in [30]. In the following, we discuss the optimal solutions of this robust matrix-monotonic optimization problem under specific power constraints.

1) **Shaping Constraint:** Consider the shaping constraint of

\[
\psi(F) = \mathbf{FF}^H - \mathbf{R}_a, \tag{46}
\]

and assuming \( \mathbf{\Psi} = 0 \), the Pareto optimal solution \( F_{\text{opt}} \) of **Opt. 2.2** is given in Lemma 1. For the case of \( \mathbf{\Psi} \propto I \), it can be shown that for the optimal solution \( F_{\text{opt}} \), we have \( \text{Tr}(F_{\text{opt}} F_{\text{opt}}^H) = \text{Tr}(\mathbf{R}_a) \), which means that \( \mathbf{K} \) is constant. Therefore, the Pareto optimal solution of **Opt. 2.2** is also given by Lemma 1.

2) **Joint Power Constraint:** Next consider the joint power constraint specified by

\[
\psi_1(F) = \text{Tr}(\mathbf{FF}^H) - P,
\]

\[
\psi_2(F) = F^H F - \tau I. \tag{47}
\]

For the perfect CSI case associated with \( \mathbf{\Psi} = 0 \), the Pareto optimal solutions of **Opt. 2.2** are specified by Lemma 2. When \( \mathbf{\Psi} \propto I \) and \( \psi_1(F) \leq 0 \) is active at the optimal solutions \( F_{\text{opt}} \), the Pareto optimal solutions of **Opt. 2.2** also satisfy the structure given in Lemma 2, since in this case \( \mathbf{K} \) is constant.

3) **Multiple Weighted Power Constraints:** When the multiple weighted power constraints are used, we have

\[
\text{Tr}(\Omega_{\text{opt}} F F^H) \leq P_i, \ 1 \leq i \leq I. \tag{48}
\]
From $\text{Tr}(\Omega FF^H) \leq P_i$, it is readily seen that the following inequality holds
\[
\text{Tr}((\sigma_i^2\Omega_i + P_i\Psi) FF^H) = \text{Tr}(\sigma_i^2\Omega_i FF^H) + P_i\text{Tr}(\Psi FF^H) 
\leq \sigma_i^2P_i + P_i\text{Tr}(\Psi FF^H). \tag{49}
\]
Hence $\text{Tr}(\Omega FF^H) \leq P_i$ is equivalent to
\[
\frac{\text{Tr}((\sigma_i^2\Omega_i + P_i\Psi) FF^H)}{\sigma_i^2 + \text{Tr}(\Psi FF^H)} \leq P_i. \tag{50}
\]
As a result, the Bayesian robust matrix-monotonic optimization problem (45) is equivalent to the following problem

**Opt. 2.3:** \[ \max_F \quad F^H \hat{H}^H K_n^{-1} \hat{H} F, \]
\[ \text{s.t.} \quad K_n = \sigma_i^2I + \text{Tr}(\Psi FF^H)I, \quad \text{tr}(\sigma_i^2\Omega_i + P_i\Psi FF^H) \leq P_i, \quad 1 \leq i \leq I. \tag{51} \]

By defining the auxiliary matrix variable
\[
\hat{F} = \frac{1}{[\sigma_i^2 + \text{Tr}(\Psi FF^H)]^{1/2}}, \tag{52}
\]
the optimization problem (51) can be simplified to:

**Opt. 2.4:** \[ \max_F \quad \hat{F}^H \hat{H}^H \hat{H} \hat{F}, \]
\[ \text{s.t.} \quad \text{tr}(\sigma_i^2\Omega_i + P_i\Psi \hat{F} \hat{F}^H) \leq P_i, \quad 1 \leq i \leq I. \tag{53} \]

Similar to the proof of Lemma 3, specifically to (150) in Appendix A, the above optimization problem is equivalent to

**Opt. 2.5:** \[ \max_F \quad \hat{F}^H \hat{H} \hat{F}, \]
\[ \text{s.t.} \quad \text{tr}(\Omega \hat{F} \hat{F}^H) \leq \sum_{i=1}^I P_i, \tag{54} \]

where
\[
\Omega = \sum_{i=1}^I \alpha_i(\sigma_i^2\Omega_i + P_i\Psi). \tag{55}
\]

According to Lemma 3, the Pareto optimal solutions $\hat{F}_{\text{opt}}$ of Opt. 2.5 satisfy the following structure
\[
\hat{F}_{\text{opt}} = \hat{H} \hat{F}_{\text{opt}} = \hat{H} \hat{F}, \tag{56}
\]
where the unitary matrix $\hat{F}$ is specified by the SVD of:
\[
\hat{H} = \frac{1}{\sqrt{\text{Tr}(\hat{F} \hat{F}^H)}} \hat{H}_{\text{opt}}, \tag{57}
\]
and we have $\hat{F} = \hat{H} \hat{F}_{\text{opt}}$. From (52), we have $[\sigma_i^2 + \text{Tr}(\sigma_i^2\Omega_i + P_i\Psi) \hat{F} \hat{F}^H]^{1/2} = \hat{F}$ and based on this we have the following equation
\[
[\sigma_i^2 + \text{Tr}(\Psi FF^H)]^2 \text{Tr}(\Psi FF^H) + \sigma_i^2 = \text{Tr}(\Psi FF^H) + \sigma_i^2. \tag{58}
\]
This yields
\[
\sigma_i^2 + \text{Tr}(\Psi FF^H) = \frac{\sigma_i^2}{1 - \text{Tr}(\Psi FF^H)} \tag{59}
\]

Thus, given the Pareto optimal $\hat{F}_{\text{opt}}$, the Pareto optimal $F_{\text{opt}}$ is expressed as
\[
F_{\text{opt}} = \sqrt{\frac{\sigma_i^2}{1 - \text{Tr}(\Psi F_{\text{opt}} F_{\text{opt}}^H)}} \hat{F}_{\text{opt}}. \tag{60}
\]

Given (60) and (56), we arrive at the following lemma.

**Lemma 4** The Pareto optimal solutions $F_{\text{opt}}$ of Opt. 2.2 under the multiple weighted power constraints satisfy the following structure
\[
F_{\text{opt}} = \frac{\sigma_n \hat{V}_H \hat{A}_F \hat{U}_H^H}{[1 - \text{Tr}(\hat{F} \hat{F}^H)]^{1/2}}. \tag{61}
\]

The robust optimal structure under the multiple weighted power constraints given in Lemma 4 is significantly different from the existing conclusions previously designed for the robust solutions under the sum power constraints [30] and for the transceiver designs relying on perfect CSI under the per-antenna power constraints [33].

**IV. STOCHASTIC ROBUST MATRIX-MONOTONIC OPTIMIZATION**

When the receiver has perfect CSI, but the transmitter only has the statistics of the CSI, the corresponding stochastic robust matrix-monotonic optimization can be formulated as [17]

**Opt. 3.1:** \[ \min_X \quad \mathbb{E}_H \{f(X^H H^H R_n^{-1} H X)\}, \]
\[ \text{s.t.} \quad \psi_i(X) \leq 0, \quad 1 \leq i \leq I, \tag{62} \]

where $R_n$ is the noise covariance matrix. For this kind of optimization problems, the objective function is an average value over the distribution of the channel matrix $H$. Generally, the channel matrix can be decomposed as [17], [18]
\[
H = \Sigma^{1/2} H_{\text{W}}, \Psi^{1/2}, \tag{63}
\]
where the elements of $H_{\text{W}}$ follows the independent and identical complex Gaussian distribution $CN(0,1)$, while $\Sigma$ and $\Psi$ are the row and column correlation matrices, respectively. For MIMO systems, $\Sigma$ is the spatial correlation matrix of the receiver antenna array, while $\Psi$ is the spatial correlation matrix of the transmitter antenna array. Since the constraints are right unitary invariant, **Opt. 3.1** can be expressed as

**Opt. 3.2:** \[ \min_F \quad \mathbb{E}_H \{f(Q_X^H F^H H^H R_n^{-1} H F Q_X)\}, \]
\[ \text{s.t.} \quad \psi_i(F) \leq 0, \quad 1 \leq i \leq I. \tag{64} \]

In this tutorial, we concentrate our attention on the unitary invariant objective functions obeying
\[
f(Q_X^H F^H H^H R_n^{-1} H F Q_X) = f(F^H H^H R_n^{-1} H F). \tag{65}
\]

Obviously, the objective functions of both the capacity maximization and sum MSE minimization both satisfy this property.

The stochastic matrix-monotonic optimization naturally aims at optimizing the distribution of the random matrix $\Sigma^{1/2} H_{\text{W}} \Psi^{1/2} F$, based on the channel model (63). Therefore, **Opt. 3.2** can be rewritten as **Opt. 3.3** of (66) at the top of the next page, where $p(H_{\text{W}})$ is the probability density function (PDF) of $H_{\text{W}}$. Based on the fact that the objective function is unitary invariant, the objective function of **Opt. 3.3** can be
**Opt. 3.3:** \[ \min_F \int f \left( (\Sigma H W \Psi F) H R_n^{-1} \left( \Sigma H W \Psi F \right) \right) p(H_W) dH_W, \]
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I, \]  
\[ (66) \]

**Opt. 3.4:** \[ \min_F \int f \left( \lambda \left( \left( \Psi F H \Sigma H W \Psi F \right) H R_n^{-1} \left( \Psi F H \Sigma H W \Psi F \right) \right) \right) p(H_W) dH_W, \]  
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I, \]  
\[ (67) \]

**Opt. 3.5:** \[ \min_F \int f \left( \lambda \left( R_n^{-1} \Sigma H W \Psi F H \Sigma H W \Psi F \right) \right) p(H_W) dH_W, \]  
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I. \]  
\[ (68) \]

considered as a function of the eigenvalues and, therefore, **Opt. 3.3** is equivalent to **Opt. 3.4** of (67) given at the top of the next page. Clearly, **Opt. 3.4** can be rewritten as **Opt. 3.5** of (68) given at the top of the next page.

In the stochastic matrix-monotonic optimization problem (68), only the inner term \( \Psi F H \Sigma H W \Psi F \) contains the matrix variable \( F \) to be optimized. Note that multiplying a unitary matrix on the right side of \( H_W \) does not change its distribution. This means that key to the optimization of **Opt. 3.5** is to maximize the eigenvalues of \( \Psi F H \Sigma H W \Psi F \). Hence, this stochastic matrix monotonic optimization problem is equivalent to

**Opt. 3.6:** \[ \max_F \lambda \left( \Psi F H \Psi \right), \]
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I. \]  
\[ (69) \]

As discussed previously in Section II-C, the above multi-objective optimization problem is equivalent to the following matrix-monotonic optimization problem

**Opt. 3.7:** \[ \max_F F H \Psi, \]
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I. \]  
\[ (70) \]

Again, we discuss the Pareto optimal solutions of this stochastic robust matrix-monotonic optimization problem under three specific power constraints, respectively.

1) **Shaping Constraint:** Under the shaping constraint of \( \psi(F) = FF^H - R_n \), the optimal solution \( F_{\text{opt}} \) to **Opt. 3.7** is specified by Lemma 1. Specifically, when the rank of \( R_n \) is not higher than the number of columns and the number of rows in \( F \), \( F_{\text{opt}} \) is a square root of \( R_n \).

2) **Joint Power Constraint:** Clearly, under the joint power constraint (47), **Opt. 3.7** is identical to **Opt. 1.5** with \( \Pi = \Psi \). Therefore, the Pareto optimal solutions \( F_{\text{opt}} \) of **Opt. 3.7** under the joint power constraint are defined exactly in Lemma 2 by simply replacing \( \Pi \) in (32) and (33) with \( \Psi \).

3) **Multiple Weighted Power Constraints:** Obviously, under the multiple weighted power constraints (48), the Pareto optimal solutions \( F_{\text{opt}} \) of **Opt. 3.7** are specified by Lemma 3, where \( \Pi \) should be replaced by \( \Psi \).

V. WORST CASE ROBUST MATRIX-MONOTONIC OPTIMIZATION

When the channel estimation error is norm bounded, i.e., \( \| \Delta H \|_F \leq \gamma \) with \( \cdot \|_F \) denoting the matrix Frobenius norm, the worst case (min-max) criterion is a widely used performance metric for robust designs [21]. Under norm bounded channel errors, the robust matrix-monotonic optimization problem can be formulated as

**Opt. 4.1:** \[ \max_{\Delta H} \min_F F H \left( \hat{H} - \Delta H \right) H R_n^{-1} \left( \hat{H} - \Delta H \right) F, \]
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I, \]  
\[ \| \Delta H \|_F \leq \gamma. \]  
\[ (71) \]

Generally speaking, the worst case or min-max robust matrix monotonic optimization is quite challenging to solve [21]. Similar to [21], the min-max matrix-monotonic optimization **Opt. 4.1** is first transferred to the following problem that minimizes the trace of the weighted objective function:

**Opt. 4.2:** \[ \min_{\Delta H} \max_F \text{Tr} \left( WF H \left( \hat{H} - \Delta H \right) H R_n^{-1} \left( \hat{H} - \Delta H \right) F \right), \]
\[ \text{s.t. } \psi_i(F) \leq 0, \quad 1 \leq i \leq I, \]  
\[ \| \Delta H \|_F \leq \gamma, \]  
\[ (72) \]

where \( W \) is a weighting matrix. It is worth highlighting that **Opt. 4.2** is more general than the min-max robust matrix monotonic optimization studied in [21], since here diverse constraints are considered.

As there are two matrix variables \( F \) and \( \Delta H \), natural dictates to derive \( \Delta H \) as a function of \( F \) first. The optimization with respect to \( \Delta H \) is defined by

**Opt. 4.3:** \[ \min_{\Delta H} \text{Tr} \left( WF H \left( \hat{H} - \Delta H \right) H R_n^{-1} \left( \hat{H} - \Delta H \right) F \right), \]
\[ \text{s.t. } \| \Delta H \|_F^2 \leq \gamma^2, \]  
\[ (73) \]

whose Lagrangian is

\[ L(\Delta H, t) = \text{Tr} \left( WF H \left( \hat{H} - \Delta H \right) H R_n^{-1} \left( \hat{H} - \Delta H \right) F \right) + t \left( \text{Tr} \left( \Delta H \Delta H^H \right) - \gamma^2 \right), \]  
\[ (74) \]

where \( t \) is the Lagrange multiplier corresponding to the norm bounded constraint in **Opt. 4.3**. We will assume that the noise is uncorrelated and, therefore, we have:

\[ R_n = \sigma_n^2 I. \]  
\[ (75) \]

Based on (74) and (75), the corresponding KKT conditions can be derived. Using the first KKT condition, we obtain

\[ (\hat{H} - \Delta H) FW F H = \sigma_n^2 t \Delta H. \]  
\[ (76) \]
Opt. 4.4: \[
\max_F \text{Tr} \left( \frac{1}{\sigma_n} F W F^H \left( I - F W F^H (F W F^H + \sigma_n^2 t I)^{-1} \right)^{H} \right) \right),
\]
\[
\text{s.t. } P_i(F) \leq 0, \ 1 \leq i \leq I.
\]

From (76), \( \Delta H \) can be solved:

\[
\Delta H = \hat{H} F W F^H (F W F^H + \sigma_n^2 t I)^{-1},
\]

where \( t \) can be computed using the classic bisection search.

By substituting (77) into the objective function of Opt. 4.1 and noting (75), the worst case robust matrix monotonic optimization problem Opt. 4.1 can be rewritten as Opt. 4.4 of (78) given at the top of the next page. Note that the objective function of Opt. 4.4 is a monotonically increasing function with respect to the matrix \( F W F^H \).

1) **Shaping Constraint:** With the shaping constraint (46) in behind and upon choosing \( W = I \), the optimal solution \( F_{\text{opt}} \) of Opt. 4.4 is also specified by Lemma 1. That is, when the rank of \( R_e \) is not larger than the number of columns and the number of rows in \( F \), \( F_{\text{opt}} \) of Opt. 4.4 is a square root of \( R_e \).

2) **Joint Power Constraint:** Based on **Matrix Inequality 1** and Lemma 2, the optimal solution \( F_{\text{opt}} \) is given in Lemma 5.

**Lemma 5** For Opt. 4.4 under the joint power constraint of (47), the optimal solution satisfies the following structure

\[
F_{\text{opt}} = U_{\hat{H}} \Lambda_F U_{\hat{W}}^H,
\]

where the unitary matrices \( U_{\hat{H}} \) and \( U_{\hat{W}} \) are specified by the following EVDs

\[
\hat{H}^H \hat{H} = U_{\hat{H}} \Lambda_{\hat{H}} U_{\hat{H}}^H \text{ with } \Lambda_{\hat{H}} \downarrow \Lambda,
\]

\[
W = U_{\hat{W}} \Lambda_W U_{\hat{W}}^H \text{ with } \Lambda_W \downarrow \Lambda,
\]

while every diagonal element of the rectangular diagonal matrix \( \Lambda_F \) is smaller than \( \sqrt{\tau} \).

3) **Multiple Weighted Power Constraints:** With the multiple weighted power constraints of (48), the optimal solution of Opt. 4.4 satisfies the following structure

\[
F_{\text{opt}} = U_F \Lambda_F U_{\hat{W}}^H.
\]

Unfortunately, in this case, the optimal unitary matrix \( U_F \) cannot be derived in closed-form.

We consider to approximate the multiple weighted power constraints under the following constraints

\[
\psi_i(F) = \text{Tr} \left( F \sum_i F^H \right) - P_i, \ 1 \leq i \leq I,
\]

where the matrices \( \sum_i, \ 1 \leq i \leq I \), are predefined appropriately. We admit that this approximation is a somewhat ad-hoc approximation but based on this approximation the optimal structure can be derived in closed-form.

**Lemma 6** For Opt. 4.4 associated with the approximate multiple weighted power constraints (83), the optimal solution satisfies the following structure

\[
F = U_{\hat{H}} \Lambda_F U_{\hat{W}}^H \sum_i^{-\frac{1}{2}},
\]

where

\[
\sum = \sum_{i=1}^I \alpha_i \sum_i,
\]

and the unitary matrix \( U_{\hat{W}} \) is specified by the following EVD

\[
\sum^{-\frac{1}{2}} W \sum^{-\frac{1}{2}} = U_{\hat{W}} \Lambda_{\hat{W}} U_{\hat{W}}^H \text{ with } \Lambda_{\hat{W}} \downarrow \Lambda.
\]

**Remark 2** Our solution for min-max optimization is more general than the existing solution given in [21], while our optimization logic goes beyond that of [21].

The matrix-monotonic optimization discussed above involves only a single matrix variable. A natural question is how to extend the single-matrix-variate matrix-monotonic optimization to multiple-matrix-variate matrix-monotonic optimization. Generally, this kind of extension is quite challenging and should be investigated case by case. In the following, three specific applications are considered to show how to exploit specific structures to convert multiple-matrix-variate optimization problems into single-matrix-variate ones. The structures of the three applications are most representative of

1) Projected structure: After appropriate transformations, for a given matrix variate, the other matrix variates are contained in the equivalent noise covariance matrix.

2) Block diagonal structure: All matrix variables are contained in a single matrix which is of block diagonal structure.

3) Cascade structure: The terms containing individual matrix variates are multiplied with each other. These terms are connected by unitary matrices.

In the following, we use three applications to discuss these three specific multiple-matrix-variate optimization problems.

VI. MU-MIMO UPLINK COMMUNICATIONS

A. Capacity Maximization

The first application scenario for the matrix monotonic optimization theory is found in MU MIMO uplink commu-
Opt. 5.1: \[
\min_{\{F_k\}} - \log \left| R_n + \sum_{k=1}^{K} H_k F_k H_k^H \right|
\]
\[
s.t. \quad \psi_{k,i}(F_k) \leq 0, 1 \leq i \leq I_k, 1 \leq k \leq K,
\]
where \( H_k \) is the MIMO channel matrix between the \( k \)th user and the BS, \( F_k \) is the precoding matrix at the \( k \)th user, and \( R_n \) is the additive noise's covariance matrix at the BS. Different from the work in [23], the power constraints considered in our work are more general than the per-antenna power constraints in [23]. The objective function of Opt. 5.1 satisfies the following property, which can be exploited to optimize the multiple matrix variables
\[
\log \left| R_n + \sum_{k=1}^{K} H_k F_k H_k^H \right|
= \log \left| I + H_k F_k H_k^H \right|^{-1} \left( R_n + \sum_{j \neq k} H_j F_j H_j^H \right) + \log \left| K_{nk} \right|
\]
where we have
\[
K_{nk} = R_n + \sum_{j \neq k} H_j F_j H_j^H.
\]
The matrix \( F_k H_k^H K_{nk}^{-1} H_k F_k \) can be interpreted as the matrix version SNR for the \( k \)th user [33]. From the multi-objective optimization viewpoint, the optimal solutions of Opt. 5.1 belong to the Pareto optimal solution sets of the following optimization problems for \( 1 \leq k \leq K \)
Opt. 5.2: \[
\max_{F_k} \quad F_k H_k^H K_{nk}^{-1} H_k F_k
\]
\[
s.t. \quad K_{nk} = R_n + \sum_{j \neq k} H_j F_j H_j^H,
\]
\[
\psi_{k,i}(F_k) \leq 0, i = 1, \ldots, I_k.
\]

1) Shaping Constraint: We have \( I_k = 1 \) and
\[
\psi_{k,1}(F_k) = F_k H_k^H - R_{nk}.
\]

2) Joint Power Constraint: We have \( I_k = 2 \) and
\[
\psi_{k,2}(F_k) = \text{Tr}(F_k F_k^H) - P_k,
\]
\[
\psi_{k,2}(F_k) = F_k F_k^H - \tau_k I.
\]

Based on Lemma 2, we readily conclude that for \( 1 \leq k \leq K \), the optimal solution \( F_{opt,k} \) of Opt. 5.2 satisfies the following structure
\[
F_{opt,k} = V_{H_k} \Lambda_{F_k} U_{\Lambda H_k}^H,
\]
where the unitary matrix \( V_{H_k} \) is defined based on the SVD
\[
K_{nk}^{-1} H_k F_k = U_{H_k} \Lambda_{H_k} V_{H_k}^H, \quad \Lambda_{H_k} \geq \chi,
\]
and every diagonal element of the rectangular diagonal matrix \( \Lambda_{F_k} \) is smaller than \( \sqrt{\tau_k} \).

3) Multiple Weighted Power Constraints: In this case, we have
\[
\psi_{k,i}(F_k) = \text{Tr}(\Omega_{k,i} F_k F_k^H) - P_{k,i},
\]

Then based on Lemma 3, we conclude that for \( 1 \leq k \leq K \), the optimal solution \( F_{opt,k} \) satisfies the following structure
\[
F_{opt,k} = \Omega_{k}^{1/2} V_{H_k} \Lambda_{F_k} U_{\Lambda H_k}^H,
\]
where the unitary matrix \( V_{H_k} \) is defined by the following SVD
\[
K_{nk}^{-1/2} H_k F_k = U_{H_k} \Lambda_{H_k} V_{H_k}^H, \quad \Lambda_{H_k} \geq \chi.
\]
and the matrix \( \Omega_k \) is defined as
\[
\Omega_k = \sum_{i=1}^{I_k} \alpha_{k,i} \Omega_{k,i}.
\]

Note that similar to (35), the auxiliary variable \( \tilde{F}_k \) is given by
\[
\tilde{F}_k = \left( \sum_{i=1}^{I_k} \alpha_{k,i} \Omega_{k,i} \right)^{1/2} F_k.
\]

B. MSE Minimization

For an MU-MIMO uplink, the signal received at the BS can be expressed as
\[
y = \text{diag}\{\{H_k\}_K\} \text{diag}\{\{F_k\}_K\} \left[ s_1^T \cdots s_K^T \right]^T + n.
\]
Based on a joint linear equalizer \( G \) to recover \( \left[ s_1^T \cdots s_K^T \right]^T \), i.e.,
\[
G y = \left[ s_1^T \cdots s_K^T \right]^T,
\]
the linear transceiver optimization based on the sum MSE minimization can be formulated as:
Opt. 5.3: \[
\min_{\{F_k\}} \text{Tr} \left( \left( I + R_n^{-1} \sum_{k=1}^{K} H_k F_k H_k^H \right)^{-1} \right),
\]
s.t. \( \psi_{k,i}(F_k) \leq 0, 1 \leq i \leq I_k, 1 \leq k \leq K \).

It is worth noting that taking \( F_k F_k^H \), \( 1 \leq k \leq K \), as the new matrix variables and relaxing the rank constraint, the optimization problem Opt. 5.3 becomes a convex optimization problem under shaping constraint, joint power constraint or multiple weighted power constraints. Based on the Schur
complement [47], this optimization problem can be transferred into a standard semidefinite programming (SDP) problem [47]. Then Opt. 5.3 can be solved by classic interior point algorithms. However, the dimension of the multiple matrix variables \( \{ F_k F_k^H \}_{k=1}^K \) is huge, and the computational complexity of solving Opt. 5.3 in this way may be excessively high.

Unlike the multiple-matrix-variate matrix-monotonic optimization Opt. 5.1, which can be readily transferred into the multiple single-matrix-variate matrix-monotonic optimization Opt. 5.2, it is not clear how to transfer Opt. 5.3 into several single-matrix-variate matrix-monotonic optimization similar to Opt. 5.2. Therefore, we change the joint signal recovering strategy (101) at the BS. Rather, we adopt the individual linear equalizers \( G_k \) to recover the individual signals \( s_k \), i.e.,

\[
G_k y = \hat{s}_k, \quad 1 \leq k \leq K.
\]

The corresponding MSE matrix derived for the \( k \)th user becomes

\[
(I + F_k H_k^H K_{nk}^{-1} H_k F_k)^{-1},
\]

with \( K_{nk}^{-1} \) defined in (89). In other words, for each user, the optimization is selfishly maximizing the equivalent matrix SNR \( F_k H_k^H K_{nk}^{-1} H_k F_k \). Thus the corresponding linear transceiver optimization problem can be formulated as follows:

\[
\text{Opt. 5.4: } \max_{F_k} f_k \left( d \left( I + F_k H_k^H K_{nk}^{-1} H_k F_k \right)^{-1} \right),
\]

s.t. \( K_{nk} = R_n + \sum_{j \neq k} H_j F_j H_j^H \),

(105)

where \( f_k(\cdot) \) is an additively Schur-convex function [14]. Thus, based on ‘approximating’ the joint equalizer \( G \) by the individual equalizers \( G_k \) for \( 1 \leq k \leq K \), the multiple-matrix-variate matrix-monotonic optimization problem Opt. 5.3 can be ‘approximately’ cast into several single-matrix-variate matrix-monotonic optimization of the form Opt. 5.4.

Remark 3 It can be seen that for the MU-MIMO uplink application, the MSE minimization and the capacity maximization can be unified into the same matrix-monotonic optimization framework.

VII. SIGNAL COMPRESSION FOR DISTRIBUTED SENSOR NETWORKS

In the distributed sensor network illustrated in Fig. 5, the \( K \) sensors transmit their individual signals to the fusion center. Specifically, the \( k \)th sensor transmits its signal \( x_k \) to the fusion center, when the channel between the \( k \)th sensor and the fusion center is \( H_k \). The fusion center recovers the transmitted signals \( x_k \) for \( 1 \leq k \leq K \). In contrast to the scenario of MU-MIMO communications, there exist correlations among \( x_k \) [27], and the correlation matrix is denoted by

\[
C_x = \mathbb{E} \left\{ [x_1^T \cdots x_K^T]^T [x_1^T \cdots x_K^T]^* \right\}.
\]

Note that the correlations among the signals makes the optimization approach of this application totally different from that of the MU-MIMO application.

Based on the performance metric of mutual information maximization, the signal compression can be formulated as Opt. 6.1 [27], given at the bottom of this page, where \( F_k \) is the signal compression matrix at the \( k \)th sensor, \( R_{x_k} \) is the covariance matrix of the signal \( x_k \) transmitted from the \( k \)th sensor, and \( R_{n_k} \) is the covariance matrix of the additive noise \( n_k \) with the \( k \)th sensor signal received at the fusion center. Note that if all the sensors send signals at the same frequency, all the \( R_{n_k} \) are identical. If the sensors use different frequency bands, the noise covariance matrices \( R_{n_k} \) are different.

Note that in [27], only the simple sum power constraint is considered, while in our work the more general multiple weighted linear power constraints are taken into account. In other words, the result derived in this section for signal compression in distributed sensor networks is novel.

For the general correlation matrix \( C_x \), it is difficult to directly decouple the optimization problem. A natural choice is to take advantage of alternating optimization algorithms among \( F_k \) for \( 1 \leq k \leq K \). To simplify the derivation, a permutation matrix \( P \) is first introduced, which reorders the block diagonal matrix \( \{ F_k^H H_k^H R_{n_k}^{-1} H_k F_k \}_{k=1}^K \) so that the following equality holds

\[
P \text{diag} \left\{ \{ F_k^H H_k^H R_{n_k}^{-1} H_k F_k \}_{k=1}^K \right\} P^H
= \begin{bmatrix}
F_k^H H_k^H R_{n_k}^{-1} H_k F_k & 0 \\
0 & \Xi
\end{bmatrix}.
\]

Note that a permutation matrix is also a unitary matrix. By further exploiting the properties of matrix determinants, Opt. 6.1 becomes equivalent to Opt. 6.2 of (109).

\[
\text{Opt. 6.1: } \min_{\{ F_k \}_{k=1}^K} - \log \left| C_x^{-1} + \text{diag} \left\{ \{ F_k^H H_k^H R_{n_k}^{-1} H_k F_k \}_{k=1}^K \right\} \right|,
\]

s.t. \( \psi_{k,i}(F_k R_{x_k}^H) \leq 0, \quad 1 \leq i \leq I_k, \quad 1 \leq k \leq K \),

(107)

\[
\text{Opt. 6.2: } \min_{\{ F_k \}_{k=1}^K} - \log \left| PC_x^{-1} P^H + P \text{diag} \left\{ \{ F_k^H H_k^H R_{n_k}^{-1} H_k F_k \}_{k=1}^K \right\} P^H \right|,
\]

s.t. \( \psi_{k,i}(F_k R_{x_k}^H) \leq 0, \quad 1 \leq i \leq I_k, \quad 1 \leq k \leq K \).
In order to simplify Opt. 6.2, we divide \( \mathbf{PC}_a^{-1}\mathbf{PH} \) into
\[
\mathbf{PC}_a^{-1}\mathbf{PH} = \begin{bmatrix}
P_{1,1} & P_{1,2} \\
P_{2,1} & P_{2,2}
\end{bmatrix} \tag{110}
\]
Combining (108) and (110) leads to
\[
\mathbf{PC}_a^{-1}\mathbf{PH} + \mathbf{P}_{\text{diag}} \left\{ \left\{ \mathbf{F}_{k}^H \mathbf{H}_{k}^H \mathbf{R}_{m_k}^{-1} \mathbf{H}_k \mathbf{F}_k \right\}_{k=1}^{K} \right\} \mathbf{PH} =
\begin{bmatrix}
P_{1,1} + F_{k}^H H_{k}^H R_{m_k}^{-1} H_k F_k & P_{1,2} \\
P_{2,1} & P_{2,2} + \Xi
\end{bmatrix} \tag{111}
\]
Further exploiting the fundamental properties of matrix determinants [27], [48], we have the following equality
\[
\begin{bmatrix}
P_{1,1} + F_{k}^H H_{k}^H R_{m_k}^{-1} H_k F_k & P_{1,2} \\
P_{2,1} & P_{2,2} + \Xi
\end{bmatrix} =
\begin{bmatrix}
P_{2,2} + \Xi & F_{k}^H H_{k}^H R_{m_k}^{-1} H_k F_k + P_k
\end{bmatrix}, \tag{112}
\]
where
\[
P_k = P_{1,1} - P_{1,2} (P_{2,2} + \Xi)^{-1} P_{2,1}.
\]
Based on (112), the alternating optimization of \( F_k \) for \( 1 \leq k \leq K \) can be performed. Specifically, the optimization problem Opt. 6.1 is transferred into: for \( 1 \leq k \leq K \),
\[
\text{Opt. 6.3:} \quad \min_{F_k} \quad - \log \left| P_k + F_{k}^H H_{k}^H R_{m_k}^{-1} H_k F_k \right|,
\]
s.t. \( \psi_{k,i}(F_k) \leq 0, 1 \leq i \leq I_k \). \tag{144}

It can be seen that by exploiting its block diagonal structure, the multiple-matrix-variate matrix-monotonic optimization of Opt. 6.1 is transferred into several single-matrix-variate matrix-monotonic optimization problems of the form of Opt. 6.3.

For \( 1 \leq k \leq K \), by introducing the auxiliary variable
\[
\bar{F}_k = F_k \bar{R}_{\alpha_k}^{-\frac{1}{2}},
\]
the optimization problem Opt. 6.3 is transferred into:
\[
\text{Opt. 6.4:} \quad \min_{F_k} \quad - \log \left| \bar{R}_{\alpha_k}^{-\frac{1}{2}} P_k \bar{R}_{\alpha_k}^{-\frac{1}{2}} + \bar{F}_{k}^H H_{k}^H R_{m_k}^{-1} H_k \bar{F}_k \right|,
\]
s.t. \( \psi_{k,i}(\bar{F}_k) \leq 0, 1 \leq i \leq I_k \). \tag{116}

It is worth noting that Opt. 6.4 is Opt. 1.1 with Obj. 1. Specifically, the optimal solutions of Opt. 6.4 are the Pareto optimal solutions of the following matrix-monotonic optimization problem:
\[
\text{Opt. 6.5:} \quad \max_{F_k} \quad \tilde{F}_{k}^H H_{k}^H R_{m_k}^{-1} H_k \tilde{F}_k,
\]
s.t. \( \psi_{k,i}(\tilde{F}_k) \leq 0, 1 \leq i \leq I_k \). \tag{117}

Based on the fundamental results of the previous sections derived for matrix-monotonic optimization, we have the following results.

1) Shaping Constraint: \( I_k = 1 \) and
\[
\psi_{k,1}(\tilde{F}_k) = \tilde{F}_k \tilde{F}_k^H - \bar{R}_{\alpha_k},
\]
When the rank of \( \bar{R}_{\alpha_k} \) is not higher than the number of columns and the number of rows in \( \tilde{F}_k \), the optimal solution \( \tilde{F}_{\text{opt}, k} \) is a square root of \( \bar{R}_{\alpha_k} \). Clearly, \( F_{\text{opt}, k} = \tilde{F}_{\text{opt}, k} \bar{R}_{\alpha_k}^{-\frac{1}{2}} \).

2) Joint Power Constraints: We have
\[
\psi_{k,1}(\tilde{F}_k) = \text{Tr}(\tilde{F}_k \tilde{F}_k^H) - P_k,
\]
where every diagonal element of the rectangular diagonal matrix \( \Lambda_{\tilde{F}_k} \) is smaller than \( \tau_k \).

3) Multiple Weighted Power Constraints: We have
\[
\psi_{k,i}(\tilde{F}_k) = \text{Tr}(\Omega_k \tilde{F}_k \tilde{F}_k^H) - P_{k,i},
\]
where \( \Omega_k \) is given by (98), while \( \bar{V}_{\alpha_k} \) and \( \bar{U}_{\alpha_k}^H \) are defined by the following SVD and EVD, respectively,
\[
\bar{R}_{\alpha_k}^{-\frac{1}{2}} H_{k} \bar{H}_{k}^{-\frac{1}{2}} \Omega_k^{-\frac{1}{2}} = \bar{U}_{\alpha_k} \Lambda_{\bar{H}_{k}} \bar{V}_{\alpha_k}^H \quad \text{with} \quad \Lambda_{\bar{H}_{k}} \prec \alpha_k,
\]
\[
\bar{R}_{\alpha_k}^{-\frac{1}{2}} P_k \bar{R}_{\alpha_k}^{-\frac{1}{2}} = \bar{P}_k \bar{R}_{\alpha_k} \Lambda_{\bar{P}_k} \bar{R}_{\alpha_k} \bar{P}_k^H.
\]

VIII. MULTI-HOP AF MIMO RELAYING NETWORKS

Multi-hop relaying communication [34] is one of the most important enabling technique for future flexible and high-throughput communications, such as machine-to-machine, device-to-device, vehicle-to-vehicle or satellite communications. The key idea behind multi-hop communications is to deploy multiple relays to realize the communications between the source node and destination node. Before presenting our third application of transceiver optimization for multi-hop communications, we first highlight the difference between our work presented in this section and the previous conclusions in [33], [34].

- We consider a more general power constraint which includes both the per-antenna power constraint in [33] and the shaping constraints in [34] as its special cases.
- Critically, the channel estimation errors are realistically taken into account in our work. By contrast, in [33] the CSI is assumed to be perfectly known.
TABLE II

| Index | Objective function | Optimal $Q_1$ |
|-------|--------------------|---------------|
| Obj. 1 | $\log \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C = I \right)$ | $Q_{\text{opt.}1} = V_A U_H^{\text{arb}}$ |
| Obj. 2 | $\text{Tr} \left( \Phi^H \Phi \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C = I \right) \right)$ | $Q_{\text{opt.}1} = V_A U_H^{U}$ |
| Obj. 3 | $f^\text{Convex}_{\text{A-Schur}} \left( d \left[ \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C = I \right) \right] \right)$ | $Q_{\text{opt.}1} = V_A U_H^{\text{DFT}}$ |
| Obj. 4 | $f^\text{Convex}_{\text{A-Schur}} \left( d \left[ \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C = I \right) \right] \right)$ | $Q_{\text{opt.}1} = V_A U_H^{\text{GMD}}$ |
| Obj. 5 | $f^\text{Convex}_{\text{A-Schur}} \left( d \left[ \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C \right) \right] \right)$ | $Q_{\text{opt.}1} = V_A U_H^{\text{GMD}}$ |
| Obj. 6 | $f^\text{Convex}_{\text{A-Schur}} \left( d \left[ \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C \right) \right] \right)$ | $Q_{\text{opt.}1} = V_A U_H^{\text{GMD}}$ |

To the best of our knowledge, the robust transceiver optimization for multi-hop communications even under the per-antenna power constraint is still a largely open problem in the existing literature. Therefore, the results presented in this section is novel and significant.

**Fig. 6.** Multi-hop cooperative AF MIMO relaying network.

The $K$-hop AF MIMO relaying network is illustrated in Fig. 6, where the source, denoted as node 0, communicates with the destination, represented by node $K$, with the help of the $(K - 1)$ relays, which are nodes 1 to $(K - 1)$. Let the signal to be sent by the source be denoted as $x_0$, which has the covariance matrix of $\sigma_{x_0}^2 I$. Then the signal model in the $k$th hop, for $1 \leq k \leq K$, can be expressed as

$$x_k = H_k S_k x_{k-1} + n_k,$$

where $x_k$ is the signal received by node $k$, $H_k$ is the channel matrix of the $k$th hop, and $n_k$ is the additive noise of the corresponding link with the covariance matrix $\sigma_{n_k}^2 I$, while $S_k$ is the forwarding matrix of node $(k - 1)$. Note that $S_1$ is the source’s transmit precoding matrix. When the channel estimation error is considered, the CSI of the $k$th hop is expressed as

$$H_k = \tilde{H}_k + H_{W,k} \Psi_k^{\frac{1}{2}},$$

where $\tilde{H}_k$ and $H_{W,k} \Psi_k^{\frac{1}{2}}$ are the estimated CSI and the channel estimation error of the $k$th hop, respectively. Furthermore, $\Psi_k$ is the covariance matrix of the channel estimate, and the elements of $H_{W,k}$ follow the independent and identical complex Gaussian distribution $CN(0, 1)$. For notational convenience, let us define the new variables $F_1 Q_1 = S_1$, with the associated unitary matrix $Q_1$, and $F_k Q_k$ for $2 \leq k \leq K$ as

$$F_k = S_k K_{n_k-1} M_{k-1} Q_k^H,$$

where $Q_k$ is the associated unitary matrix,

$$M_k = \left( K_{n_{k-1}} \tilde{H}_{k-1} F_{k-1} Q_{k-1}^H \tilde{H}_{k-1} K_{n_{k-1}} \right)^{\frac{1}{2}},$$

and clearly $M_1 = I$ and $K_{n_1} = I$. Based on these definitions, the MSE matrix of the data detection at the destination is expressed as [33], [34]

$$\Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K, C \right) = \sigma_{x_0}^2 C C^H - \sigma_{x_0}^2 C \left( \prod_{k=1}^K M_k^{\frac{1}{2}} K_{n_k} \tilde{H}_k F_k Q_k \right)^H \times \left( \prod_{k=1}^K M_k^{\frac{1}{2}} K_{n_k} \tilde{H}_k F_k Q_k \right) C^H.$$

For linear transceivers, $C = I$ is an identity matrix, while for nonlinear transceiver optimization, $C$ is a lower triangular matrix. Specifically, we assume that the size of $C$ is $N \times N$. Then, for nonlinear transceivers, the optimal $C$ satisfies [33]

$$C_{\text{opt.}} = \text{diag} \left( \{ |L|, i \} \right) L^{-1},$$

where $L$ is the triangular matrix of the Cholesky decomposition of the following matrix [33]

$$LL^H = \Phi_{\text{MSE}} \left( \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K \right)$$

$$= \sigma_{x_0}^2 I - \sigma_{x_0}^2 \left( \prod_{k=1}^K M_k^{\frac{1}{2}} K_{n_k} \tilde{H}_k F_k Q_k \right)^H \times \left( \prod_{k=1}^K M_k^{\frac{1}{2}} K_{n_k} \tilde{H}_k F_k Q_k \right),$$

which has the same diagonal elements. Based on the MSE matrix given in (130), both the linear and nonlinear transceiver
optimization problems [33], [34] can be unified into the general optimization problem (133), given at the bottom of this page. Various objective functions typically adopted for Opt. 7.1 are listed in Table II. The constraints $\psi_{k,i}(F_k) \leq 0$ are left unitary invariant, and the power constraint model of Opt. 7.1 is more general than the power constraint models considered in [30], [33], [34].

The optimal unitary matrices $Q_k$ can be derived based on majorization theory. Specifically, the optimal $Q_k$ for $k > 1$ are derived as [29], [33], [34]

$$Q_{opt,k} = V_{Ak} U_{Ak}^H,$$

where the unitary matrices $V_{Ak}$ and $U_{Ak}$ are defined by the following SVDs

$$M_k^{\frac{1}{2}} \tilde{H}_k Q_k = U_{Ak} \Lambda_k V_{Ak}^H$$

where $\Lambda_k$ denotes an arbitrary matrix having the appropriate dimension. The unitary matrix $U_W$ is the unitary matrix defined by the following EVD

$$W = U_W \Lambda_W U_W^H$$

The unitary matrix $U_{DFT}$ is a DFT matrix. Finally, the unitary matrix $\hat{U}_{GM}$ ensures that the triangular matrix of the Cholesky decomposition of $\hat{\Phi}_{MSE} = \{F_k\}_{k=1}^K, \{Q_k\}_{k=1}^K$ has the same diagonal elements [33].

Given the optimal $Q_{opt,k}$ and $C_{opt}$, the objective function of Opt. 7.1 can be rewritten as [34]

$$f \big( \Phi_{MSE} \big( \{F_k\}_{k=1}^K, \{Q_{opt,k}\}_{k=1}^K, C_{opt} \big) \big)$$

$$= \bigg( \sum_{i=1}^N \frac{\lambda_i(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k)}{1 + \lambda_i(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k)} \bigg)^{\frac{1}{2}}$$

where $f_{Eig}(\cdot)$ is a monotonically decreasing function with respect to the eigenvalue vector $\lambda(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k)$. Hence, given $Q_{opt,k}$ and $C_{opt}$, Opt. 7.1 is transformed into

$$\text{Opt. 7.2: } \min_{\{F_k\}_{k=1}^K} f_{Eig} \left( \left( \lambda(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k) \right)_{k=1}^K \right)$$

$$\text{subject to }$$

$$\begin{align*}
K_{nk} &= (\sigma_{nk}^2 + \text{Tr}(F_k^H F_k^H \Psi_k)) I, \\
\psi_{k,i}(F_k) &\leq 0, 1 \leq i \leq I_k, 1 \leq k \leq K.
\end{align*}$$

Since the objective function of Opt. 7.2 is a monotonically decreasing function of $\lambda(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k)$, it can be decoupled into the following sub-problems: for $1 \leq k \leq K$,

$$\text{Opt. 7.3: } \min_{F_k} \lambda(F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k),$$

$$\text{subject to }$$

$$\begin{align*}
K_{nk} &= (\sigma_{nk}^2 + \text{Tr}(F_k^H F_k^H \Psi_k)) I, \\
\psi_{k,i}(F_k) &\leq 0, 1 \leq i \leq I_k.
\end{align*}$$

Clearly, Opt. 7.3 is equivalent to the following matrix-monotonic optimization problem

$$\text{Opt. 7.4: } \min_{F_k} F_k^H \hat{H}_k^H K_{k-1}^{-1} \hat{H}_k F_k,$$

$$\text{subject to }$$

$$\begin{align*}
K_{nk} &= (\sigma_{nk}^2 + \text{Tr}(F_k^H F_k^H \Psi_k)) I, \\
\psi_{k,i}(F_k) &\leq 0, 1 \leq i \leq I_k.
\end{align*}$$

In this application, by exploiting its cascade structure, we are able to transfer the associated multiple-matrix-variate matrix-monotonic optimization problem into several single-matrix-variate matrix-monotonic optimization problems. Based on the fundamental results of the previous sections, we readily have the following results.

1) **Shaping Constraint:** The constraint is (91). Based on Lemma 1, we conclude that when the rank of $R_{nk}$ is not higher than the number of columns and the number of rows in $F_k$, the optimal solution $F_{opt,k}$ is a square root of $R_{nk}$.

2) **Joint Power Constraint:** The constraint is (92). Based on Lemma 2, we conclude that when $\Psi_k = 0$ or $\Psi_k \propto I$, the optimal solutions $F_{opt,k}$ satisfy the following structure

$$F_{opt,k} = V_{\tilde{H}_k} \Lambda_{\tilde{F}_k} U_{\Lambda_{nk,k}}^H,$$

where $V_{\tilde{H}_k}$ is the right unitary matrix of the SVD of $\tilde{H}_k$, i.e.,

$$\tilde{H}_k = U_{\tilde{H}_k} \Lambda_{\tilde{H}_k} V_{\tilde{H}_k}^H$$

and every diagonal element of the rectangular diagonal matrix $\Lambda_{\tilde{F}_k}$ is smaller than $\tau_k$.

3) **Multiple weighted power constraints:** With the constraints (95) and based on Lemma 3, we conclude that the optimal solutions $F_{opt,k}$ satisfy the following structure

$$F_{opt,k} = \sigma_{nk}^2 \tilde{H}_k \Lambda_{\tilde{F}_k} V_{\tilde{H}_k}^H$$

where the unitary matrix $V_{\tilde{H}_k}$ is defined by the SVD

$$\tilde{H}_k \Lambda_{\tilde{F}_k} = U_{\tilde{H}_k} \Lambda_{\tilde{H}_k} V_{\tilde{H}_k}^H$$

and the matrix $\tilde{F}_k$ is defined by

$$\tilde{F}_k = \sigma_{nk}^2 \sum_{i=1}^{I_k} \alpha_{k,i} (\Omega_{k,i} + P_{k,i} \Psi_k).$$

**IX. SIMULATION RESULTS AND DISCUSSIONS**

A. Two-user MIMO Uplink

We first consider the MU-MIMO uplink, where a pair of 4-antenna mobile users communicate with an 8-antenna BS. We define $P_k$ as the SNR for the $k$th user, where $P_k$ is the sum transmit power of the user $k$ and $\sigma_k^2$ is the noise power at each receive antenna of the BS. Without loss of generality, the same transmit power value is assumed for all the users, i.e., $P_1 = P_2$. Based on the Kronecker correlation model, the spatial correlation matrix $R_{tx}$ of the BS’s receive antennas and the spatial correlation matrix $R_{tx,k}$ of the $k$th user’s transmit antennas, where $k = 1, 2$, are specified respectively by $[R_{tx}]_{i,j} = r_i^{(i-j)}$ and $[R_{tx,k}]_{i,j} = r_i^{(i-j)}$. In the simulations, we further set $r_1 = r_2 = r_1$. Three power constraints,
namely, the shaping constraint, the joint power constraint and the per-antenna power constraints, are considered. For the shaping constraint, the widely used Kronecker correlation model of $[R_{nn}]_{i,j} = 0.6|^{i-j}|$ is employed. For the joint power constraint, the threshold is chosen as $\tau_k = 1.4$. For the per-antenna power constraints, the power limits for the four antennas of each user are set to 1.2, 1.2, 0.8 and 0.8, respectively.

It is worth highlighting that the transceiver optimization under these three power constraints can be transferred into convex optimization problems, which can be solved numerically using the CVX tool [49]. This approach however suffers from high computational complexity, especially for high dimensional antenna arrays. By contrast, our approach presented in Section VI provides the optimal closed-form solutions for the same transceiver optimization design problems. Fig. 7 compares the sum capacity performance as the function of the SNR for the proposed closed-form solutions and for the numerical optimization solutions computed by the CVX tool. It can be seen that our closed-form solutions have an identical performance to the solutions computed by the CVX tool.

### B. Dual-hop AF MIMO Relaying Network

A dual-hop AF MIMO relaying network is simulated, which consists of one source, one relay and one destination. All the nodes are equipped with 4 antennas. At the source and relay, per-antenna power constraints are imposed. Specifically, the power limits for the four antennas are set as 1, 1, 1 and 1, respectively. The SNR in each hop is defined as the ratio between the transmit power and the noise variance, i.e., $\text{SNR}_k = \frac{P_k}{N_k}$.

Without loss of generality, the SNRs in the both hops are assumed to be the same, namely, $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$.

In contrast to the existing works [33], [34], which consider the transceiver optimization unrealistically with the perfect CSI, in this paper, we focus on the robust transceiver optimization, which takes into account the channel estimation error. In the simulations, the estimated channel matrix is generated according to $\hat{H}_k = \tilde{H}_{W,k} W_k^\ast$ [30], where we have $[\Psi_k]_{i,j} = 0.6|^{i-j}|$. The elements of $\tilde{H}_{W,k}$ are independently identically distributed Gaussian random variables. In order to ensure that $\mathbb{E}\{[H_{i,j}]^2\} = 1$, $\forall i,j$, we set $\mathbb{E}\{[H_{W,k}]_{i,j}[H_{W,k}]_{i,j}^\ast\} = \sigma_{e_k}^2$ and $\mathbb{E}\{[\tilde{H}_{W,k}]_{i,j}[\tilde{H}_{W,k}]_{i,j}^\ast\} = 1 - \sigma_{e_k}^2$. Without loss of generality, we assume $\sigma_{e_1}^2 = \sigma_{e_2}^2 = \sigma_{e}^2$. It can be seen from Fig. 8 that our robust design achieves better capacity performance than the non-robust design of [33]. Furthermore, as expected, the performance gap between the robust and non-robust designs becomes larger as the channel estimation error increases.

### X. Conclusions

In this paper, a comprehensive overview for matrix-monotonic optimization has been given under various power constraints, including shaping constraint, joint power constraint and multiple weighted power constraints. Matrix-monotonic optimization of three different CSI scenarios have been investigated in depth, which are: 1) both transmitter and receiver have imperfect CSI; 2) perfect CSI is available at the receiver but the transmitter has no CSI; and 3) perfect CSI is available at the receiver, but the channel estimation error at the transmitter is norm-bounded. In all three cases, the matrix monotonic optimization framework has been used to derive closed-form optimal structures of the optimal matrix variables, which significantly simplifies the associated optimization problems and reveals a range of underlying physical insights. Furthermore, we have used the three applications, namely MU-MIMO uplink communications, signal compression for distributed sensor networks and multi-hop AF MIMO relaying networks to demonstrate how to transform the multiple-matrix-variate matrix-monotonic optimization problems into single-matrix-variate matrix-monotonic optimization problems by exploiting the underlying special physical structures.
APPENDIX

A. Proof of Lemma 3

Proof 3 Any Pareto optimal solution of Opt. 1.6, $F_{Pareto}$, is also a Pareto optimal solution of the following multi-objective optimization problem

$$
\begin{align*}
\min_{F} & \{ \text{Tr} \left( \Omega, F F^H \right) \}_{i=1}^I, \\
\text{s.t.} & \quad F^H \Pi F = F^H_{Pareto} \Pi F_{Pareto}.
\end{align*}
$$

(146)

Otherwise we can find a solution better than $F_{Pareto}$ and this contradicts the fact that $F_{Pareto}$ is Pareto optimal.

Since the constraint of (146) is equivalent to $U \Pi^\perp F = \Pi^\perp F_{Pareto}$, where $U$ is a suitable unitary matrix, the optimization problem (146) is equivalent to

$$
\begin{align*}
\min_{F} & \{ \text{Tr} \left( \Omega, F F^H \right) \}_{i=1}^I, \\
\text{s.t.} & \quad U \Pi^\perp F = \Pi^\perp F_{Pareto}.
\end{align*}
$$

(147)

In (147), the objective functions are quadratic functions and the constraint is a linear function with respect to $F$, which means that the multi-objective optimization problem (147) is convex [47]. Therefore, for any Pareto optimal solution of (147), there exist the weights $\alpha_i$, $1 \leq i \leq I$, for ensuring that the Pareto optimal solution can be computed via solving the following weighted sum optimization problem [47]

$$
\begin{align*}
\min_{F} & \sum_{i=1}^I \alpha_i \text{Tr} \left( \Omega, F F^H \right), \\
\text{s.t.} & \quad U \Pi^\perp F = \Pi^\perp F_{Pareto}.
\end{align*}
$$

(148)

Thus the whole Pareto optimal solution set of (146) can be solved via achieving the following optimization problem by changing the weights $\alpha_i$, $1 \leq i \leq I$.

$$
\begin{align*}
\min_{F} & \sum_{i=1}^I \alpha_i \text{Tr} \left( \Omega, F F^H \right), \\
\text{s.t.} & \quad F^H \Pi F = F^H_{Pareto} \Pi F_{Pareto}.
\end{align*}
$$

(149)

Then $F_{Pareto}$ is a Pareto optimal solution of the following optimization problem

$$
\begin{align*}
\max_{F} & \quad F^H \Pi F, \\
\text{s.t.} & \quad \text{Tr} \left( \sum_{i=1}^I \alpha_i \Omega, F F^H \right) \leq \sum_{i=1}^I P_i.
\end{align*}
$$

(150)

In a nutshell, for any Pareto optimal solution of Opt. 1.6, there exist the weights $\alpha_i$, $1 \leq i \leq I$, for ensuring that this Pareto optimal solution of Opt.1.6 is also the Pareto optimal solution of (150). Therefore, it can be concluded that any Pareto optimal solution of Opt. 1.6 satisfies the common structures of the Pareto optimal solutions of (150).

Next, we show that the Pareto optimal solutions of (150) own the same diagonalizable structure and thus this structure is also the optimal structure of the Pareto optimal solutions of Opt. 1.6. Noting the auxiliary variable $\tilde{F}$ of (35), the optimization (150) is transferred into:

$$
\begin{align*}
\max_{F} & \quad \tilde{F}^H (\Omega^{-\frac{1}{2}})^H \Pi (\Omega^{-\frac{1}{2}}) \tilde{F}, \\
\text{s.t.} & \quad \text{Tr} \left( \tilde{F} \tilde{F}^H \right) \leq P.
\end{align*}
$$

(151)

The Pareto optimal solution set of (151) consist of the optimal solutions of the following optimization problem for all the possible $\tilde{F}$ in that are in the sphere region of $\text{Tr} \left( F \tilde{F}^H \right) \leq P$:

$$
\begin{align*}
\max_{\tilde{F}} & \quad \alpha, \\
\text{s.t.} & \quad \tilde{F}^H (\Omega^{-\frac{1}{2}})^H \Pi (\Omega^{-\frac{1}{2}}) \tilde{F} = \alpha \tilde{F}^H_{in} (\Omega^{-\frac{1}{2}})^H \Pi (\Omega^{-\frac{1}{2}}) \tilde{F}_{in}, \\
\text{Tr} \left( \tilde{F} \tilde{F}^H \right) & \leq P.
\end{align*}
$$

(152)

The first constraint in (152) is equivalent to

$$
\Pi^\perp \Omega^{-\frac{1}{2}} \tilde{F} = \sqrt{\alpha} U \Pi \Omega^{-\frac{1}{2}} \tilde{F}_{in}.
$$

(153)

Using pseudo inverse, we have

$$
\left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)^\dagger \Pi^\perp \Omega^{-\frac{1}{2}} \tilde{F} = \sqrt{\alpha} \left( U \Pi \Omega^{-\frac{1}{2}} \right)^\dagger \tilde{F}_{in},
$$

(154)

based on which $\alpha$ is solved as

$$
\alpha = \frac{\| \left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)^\dagger \Pi^\perp \Omega^{-\frac{1}{2}} \tilde{F} \|^2_F}{\| \left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)^\dagger U \Pi \Omega^{-\frac{1}{2}} \tilde{F}_{in} \|^2_F}.
$$

(155)

Based on Matrix Inequality 1, the numerator of (155) satisfies

$$
\| \left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)^\dagger \Pi^\perp \Omega^{-\frac{1}{2}} \tilde{F} \|^2_F \geq \sum_j \lambda_j \left( \tilde{F} \tilde{F}^H \right),
$$

(156)

while its denominator satisfies

$$
\| \left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)^\dagger U \Pi \Omega^{-\frac{1}{2}} \tilde{F}_{in} \|^2_F \geq \sum_j \frac{\sigma_j^2 \left( \Pi \Omega^{-\frac{1}{2}} \tilde{F}_{in} \right)}{\sigma_j^2 \left( \Pi^\perp \Omega^{-\frac{1}{2}} \right)},
$$

(157)

where $\sigma_j^2 (A)$ denotes the jth singular value of $A$. Clearly, $\alpha$ attains the maximum value when the both equalities in (156) and (157) hold. For the optimal $\tilde{F}$ and $U$ together with the fact that for Opt. 1.6, the optimal $\tilde{F}$ is unitary invariant, we complete the proof.

REFERENCES

[1] S. Sugiuira, S. Chen, and L. Hanzo, “A universal space-time architecture for multiple-antenna aided systems,” IEEE Commun. Survey & Tutorials, vol. 14, no. 2, pp. 401–420, 2012.

[2] M. I. Kadir, S. Sugiuira, S. Chen, and L. Hanzo, “Unified MIMO-Multicarrier designs: A space-time keying approach,” IEEE Commun. Survey & Tutorials, vol. 17, no. 2, pp. 550–579, 2015.

[3] S. Sugiuira, S. Chen, and L. Hanzo, “MIMO-Aided near-capacity turbo transceiver: Taxonomy and performance versus complexity,” IEEE Commun. Survey & Tutorials, vol. 14, no. 2, pp. 421–422, 2012.

[4] S. Yang and L. Hanzo, “Fifty years of MIMO detection: The road to large-scale MIMO,” IEEE Commun. Survey & Tutorials, vol. 17, no. 4, pp. 1941–1988, 2015.

[5] J. Yang and S. Roy, “On joint transmitter and receiver optimization for multiple-input-multiple-output (MIMO) transmission systems,” IEEE Trans. Commun., vol. 42, no. 12, pp. 3221–3231, Dec. 1994.

[6] S. M. Alamouti, “A simple transmit diversity technique for wireless communications,” IEEE J. Sel. Areas Commun., vol. 16, no. 8, pp. 1451–1458, Oct. 1998.

[7] I. E. Telatar, “Capacity of multi-antenna Gaussian channels,” European Trans. Commun., vol. 10, no. 2, pp. 585–595, Nov./Dec. 1999.

[8] H. Sampath, P. Stoica, and A. Paulraj, “Generalized linear precoder and decoder design for MIMO channels using the weighted MMSE criterion,” IEEE Trans. Commun., vol. 49, no. 12, pp. 2198–2206, Dec. 2001.

[9] H. Sampath and A. Paulraj, “Linear precoding for space-time coded systems with known fading correlations,” IEEE Commun. Lett., vol. 6, no. 6, pp. 239–241, Jun. 2002.

[10] A. Scaglione, et al., “Optimal designs for space-time linear precoders and decoders,” IEEE Trans. Signal Process., vol. 50, no. 5, pp. 1051–1064, May 2002.
[11] D. P. Palomar, J. M. Cioffi, and M. A. Lagunas, “Joint Tx-Rx beamforming design for multicarrier MIMO channels: A unified framework for convex optimization,” IEEE Trans. Signal Process., vol. 51, no. 9, pp. 2381–2401, Sep. 2003.

[12] A. Feiten, R. Mathar, and S. Hanly, “Eigenvalue-based optimum-power allocation for Gaussian vector channels,” IEEE Trans. Infom. Theory, vol. 53, no. 6, pp. 2304–2309, Jun. 2007.

[13] A. Yadav, M. Junjti, and J. Lilleberg, “Linear precoder design for doubly correlated partially coherent fading MIMO channels,” IEEE Trans. Wirel. Commun., vol. 13, no. 7, pp. 3621–3635, Jul. 2014.

[14] D. P. Palomar and Y. Jiang, “MIMO transceiver designs via majorization theory,” Foundations and Trends in Commun. and Infor. Theory, vol. 3, no. 4–5, pp. 331–551, Jun. 2007.

[15] Y. Jiang, J. Li, and W. W. Hager, “Joint transceiver design for MIMO communications using geometric mean decomposition,” IEEE Trans. Signal Process., vol. 53, no. 10, pp. 3791–3803, Oct. 2005.

[16] W. Yao, S. Chen, and L. Hanzo, “A transceiver design based on uniform channel decomposition and MBER vector perturbation,” IEEE Trans. Veh. Technology, vol. 59, no. 6, pp. 3153–3159, Jul. 2010.

[17] S. A. Jafar and A. Goldsmith, “Multiple-antenna capacity in correlated Rayleigh fading with channel covariance information,” IEEE Trans. Wirel. Commun., vol. 4, no. 3, pp. 990–997, May 2005.

[18] S. A. Jafar and A. Goldsmith, “Transmitter optimization and optimality of beamforming for multiple antenna systems,” IEEE Trans. Wirel. Commun., vol. 3, no. 9, pp. 3025–3037, Sep. 2004.

[19] X. Zhang, D. P. Palomar, and B. Ottersten, “Statistically robust design of linear MIMO transceivers,” IEEE Trans. Signal Process., vol. 56, no. 8, pp. 3678–3689, Aug. 2008.

[20] M. Ding and S. D. Blostein, “MIMO minimum total MSE transceiver design with imperfect CSI at both ends,” IEEE Trans. Signal Process., vol. 57, no. 3, pp. 1141–1150, Mar. 2009.

[21] J. Wang, M. Bengtsson, B. Ottersten, and D. P. Palomar, “Robust MIMO precoding for several classes of channel uncertainty,” IEEE Trans. Signal Process., vol. 61, no. 12, pp. 3056–3070, Jun. 2013.

[22] A. Pastore, M. Joham, and J. R. Fonollosa, “A framework for joint design of pilot sequence and linear precoder,” IEEE Trans. Infor. Theory, vol. 62, no. 9, pp. 5059–5079, Sep. 2016.

[23] W. Yu and T. Lan, “Transceiver optimization for the multi-antenna downlink with per-antenna power constraints,” IEEE Trans. Signal Process., vol. 55, no. 6, pp. 2646–2660, Jun. 2007.

[24] S. Vishwanath, N. Jindal, and A. Goldsmith, “Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels,” IEEE Trans. Infom. Theory, vol. 49, no. 10, pp. 2658–2668, Oct. 2003.

[25] S. Serbetli and A. Yener, “Transceiver optimization for multiuser MIMO systems,” IEEE Trans. Signal Process., vol. 52, no. 1, pp. 214–226, Jan. 2004.

[26] Q. Shi, M. Razaviyayn, Z. Q. Luo, and C. He, “An iteratively weighted MMSE approach to distributed sum-utility maximization for a MIMO interfering broadcast channel,” IEEE Trans. Signal Process., vol. 59, no. 9, pp. 4331–4340, Sep. 2011.

[27] J. Fang, H. Li, Z. Chen, and Y. Gong, “Joint precoder design for distributed transmission of correlated sources in sensor networks,” IEEE Trans. Wirel. Commun., vol. 12, no. 6, pp. 2918–2929, Jun. 2013.

[28] C. Xing, et al., “A general robust linear transceiver design for amplify-and-forward multi-hop MIMO relaying systems,” IEEE Trans. Signal Process., vol. 61, no. 5, pp. 196–2129, Mar. 2013.

[29] C. Xing, M. Xia, F. Gao and Y.-C. Wu, “Robust transceiver with Tomlinson-Harashima precoding for amplify-and-forward MIMO relaying systems,” IEEE J. Sel. Areas Commun., vol. 30, no. 8, pp. 1370–1382, Sep. 2012.

[30] C. Xing, S. Ma, and Y. Zhou, “Matrix-monotonic optimization for MIMO systems,” IEEE Trans. Signal Process., vol. 63, no. 2, pp. 334–348, Jan. 2015.

[31] E. Jorswieck and H. Boche, “Majorization and matrix-monotone functions in wireless communications,” Foundations and Trends in Communication and Information Theory, vol. 3, no. 6, pp. 553–701, Jul. 2007.

[32] M. Vu, “MIMO capacity with per-antenna power constraint,” in Proc. ISIT’2011 (Houston, USA), Dec. 5–9, 2011, pp. 1–5.

[33] C. Xing, Y. Ma, Y. Zhou, and F. Gao, “Transceiver optimization for multi-hop communications with per-antenna power constraints,” IEEE Trans. Signal Process., vol. 64, no. 6, pp. 1519–1534, Mar. 2016.

[34] C. Xing, F. Gao, and Y. Zhou, “A framework for transceiver designs for multi-hop communications with covariance shaping constraints,” IEEE Trans. Signal Process., vol. 65, no. 15, pp. 3930–3945, Aug. 2017.

[35] J. Dai, C. Chang, W. Xu, and Z. Ye, “Linear precoder optimization for MIMO systems with joint power constraints,” IEEE Trans. Commun., vol. 60, no. 8, pp. 2240–2254, Aug. 2012.