ON THE AFFINE ANALOGUE OF JACK’S
AND MACDONALD’S POLYNOMIALS

Pavel I. Etingof, Alexander A. Kirillov, Jr.

Department of Mathematics
Yale University
New Haven, CT 06520, USA

e-mail: etingof@math.harvard.edu, kirillov@math.yale.edu

Introduction.

Jack’s and Macdonald’s polynomials are an important class of symmetric functions associated to root systems. In this paper we define and study an analogue of Jack’s and Macdonald’s polynomials for affine root systems. Our approach is based on representation theory of affine Lie algebras and quantum affine algebras, and follows the ideas of our recent papers [EK1,EK2,EK3].

We start with a review of the theory of Jack (Jacobi) polynomials associated with the root system of a simple Lie algebra $\mathfrak{g}$. This theory was described in the papers of Heckman and Opdam [HO,H1,O1,O2]. In these papers, Jack’s polynomials are defined as a basis in the space of Weyl group invariant trigonometric polynomials which 1) differs from the basis of orbitsums by a triangular matrix (with respect to the standard partial ordering on dominant integral weights) with ones on the diagonal, and 2) is an eigenbasis for a certain second order differential operator (the Sutherland-Olshanetsky-Perelomov operator, [Su,OP]). It turns out that these conditions determine Jack’s polynomials uniquely. Orbitsums and characters for $\mathfrak{g}$ turn out to be special cases of Jack’s polynomials. These polynomials have a $q$-deformation, which is called Macdonald’s polynomials; they have been introduced by I. Macdonald in his papers [M1, M2] and have been intensively studied since that time.

We generalize the definition of Jack’s polynomials to the case of affine root systems. We assign such a polynomial to every dominant integral weight of the affine root system. It is done in the same way as for the usual root systems: the only thing one has to do is replace the Sutherland operator by its affine analogue. This analogue is constructed in the same way as for usual root systems, and it turns out to be (after specialization of level) a parabolic differential operator whose coefficients are elliptic functions. This operator was introduced in [EK3] (for the root system $A_{n-1}$) and is closely related to the Sutherland operator with elliptic coefficients considered in [OP], but is more general. Analogously to the finite-dimensional case, orbitsums and characters (of integrable modules) for the affine Lie algebra $\hat{\mathfrak{g}}$ are special cases of affine Jack’s polynomials.
For orbit sums and characters of affine Lie algebras, there is a beautiful theory of modular invariance described in [K]. We generalize this theory to general affine Jack’s polynomials. It turns out that the finite-dimensional space spanned by the Jack’s polynomials of a given level is modular invariant with a certain weight. Moreover, as in the character case, the representation of the modular group in this space is (conjecturedly) unitary, with respect to a quite nontrivial inner product which generalizes the Macdonald inner product. This inner product coincides with the inner product on conformal blocks of the Wess-Zumino-Witten conformal field theory, and its existence still remains a conjecture.

However, we show that unlike the character case, the image of the corresponding projective representation of the modular group may be infinite.

For the root system $A_{n-1}$, it is possible to give an interpretation of Jack’s and Macdonald’s polynomials in terms of representation theory of the Lie algebra $\mathfrak{sl}_n$ and quantum group $U_q(\mathfrak{sl}_n)$, respectively [EK1,EK2]. More specifically, Macdonald’s polynomials are interpreted as certain (renormalized) vector-valued characters (traces of intertwiners) for quantum groups – a notion generalizing the usual characters. Analogously, in this paper we show that for the root system $\hat{A}_{n-1}$ the affine Jack’s polynomials defined as eigenfunctions of a certain second order differential operator can be represented as renormalized traces of intertwiners between certain representations of the affine Lie algebra $\hat{\mathfrak{g}}$. This proof is analogous to the one given in [EK2] for the finite-dimensional case. Finally, we define the affine Macdonald’s polynomials (i.e. $q$-deformed Jack’s polynomials) for the root system $\hat{A}_{n-1}$ to be renormalized traces of intertwiners for quantum affine algebras, and formulate (as a conjecture) the affine analogue of the Macdonald special value identities from [M2].

The paper is organized as follows. In Section 1, we give basic definitions concerning root systems. In Section 2, we define the Sutherland operator and its eigenfunctions (Jack’s polynomials) and quote some known results about them. In Section 3, we construct Jack’s polynomials for the root system $A_{n-1}$ via representation theory of $\mathfrak{sl}_n$. In Section 4, we make basic definitions concerning affine root systems. In Section 5, we define the affine analogue of the group algebra of the weight lattice. In Section 6, we define and study the affine Calogero-Sutherland operator and introduce the affine Jack’s polynomials. In Section 7, we construct the affine Jack’s polynomials via traces of intertwiners for $\hat{\mathfrak{sl}}_n$. In Section 8, we give a complex-analytic description of the affine Jack’s polynomials. In Section 9, we study modular properties of the affine Jack’s polynomials. In Section 10, we give a brief introduction to the Wess-Zumino-Witten model and formulate a conjecture on the unitarity of the action of the modular group on affine Jack’s polynomials. In Section 11, we define the affine Macdonald’s polynomials, and conjecture that an affine analogue of the Macdonald special value formula is true. Also, in this section we discuss the extension of the results of the previous sections to non-integer values of the central charge of the (quantum) affine algebra. Finally, Section 12 is devoted to the discussion of some interesting problems which still remain open.

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1. Basic definitions: finite-dimensional case.

We let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \), \( \mathfrak{h} \) be the Cartan subalgebra, \( \dim \mathfrak{h} = r \) be the rank of \( \mathfrak{g} \), \( R \subset \mathfrak{h}^* \) be the root system, \( Q \) be the lattice in \( \mathfrak{h}^* \) spanned by the roots, and \( W \) be the Weyl group. Let us fix a decomposition of \( R \) into positive and negative roots: \( R = R^+ \sqcup R^- \). Let \( \alpha_1, \ldots, \alpha_r \in R^+ \) be the basis of simple roots. Then we have the positive cone \( Q^+ = \bigoplus \mathbb{Z}_+ \alpha_i \subset Q \), and the highest root \( \theta \in R^+ \) such that \( \theta - \alpha \in Q^+ \) for any root \( \alpha \).

Let \( \langle \cdot, \cdot \rangle \) be a \( W \)-invariant bilinear symmetric form in \( \mathfrak{h}^* \) normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for long roots. This form gives an identification \( \nu: \mathfrak{h} \simeq \mathfrak{h}^* \) and thus can also be considered as a bilinear form on \( \mathfrak{h} \).

Let \( \alpha \in R \). Define the dual root \( \alpha^\vee \in \mathfrak{h} \) by \( \alpha^\vee = \frac{2\nu(\alpha)}{\langle \alpha, \alpha \rangle} \), in other words, \( \langle \alpha^\vee, \beta \rangle = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \) for any \( \beta \in \mathfrak{h}^* \), where \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( \mathfrak{h} \) and \( \mathfrak{h}^* \). Then \( R^\vee = \{ \alpha^\vee \in R \} \) is also a root system with the basis of simple roots given by \( \alpha_i^\vee \). Let us define the dual root lattice \( Q^\vee = \bigoplus \mathbb{Z} \alpha_i^\vee \).

**Lemma 1.1.** For any \( \lambda, \mu \in Q^\vee \), \( (\lambda, \mu) \in \mathbb{Z} \) and \( (\lambda, \lambda) \in 2\mathbb{Z} \).

Let us define the weight lattice \( P = \{ \lambda \in \mathfrak{h}^* | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in R \} \) and the cone of dominant weights \( P^+ = \{ \lambda \in \mathfrak{h}^* | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+ \} \). Obviously, \( Q \subset P \); it also follows from Lemma 1.1 that \( \nu(Q^\vee) \subset P \). Note that we have a partial ordering on \( P \): \( \lambda \leq \mu \) if \( \mu - \lambda \in Q^+ \). Also, it will be often used in the future that the action of \( W \) preserves \( P \) and that \( P^+ \) is the fundamental domain for this action: each \( W \)-orbit in \( P \) contains one and only one point from \( P^+ \).

Finally, let \( \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \), then \( \langle \rho, \alpha^\vee \rangle = 1 \) and thus \( \rho \in P^+ \). We define dual Coxeter number for \( \mathfrak{g} \) by

\[
(1.1) \quad h^\vee = (\rho, \theta) + 1 = \langle \rho, \theta^\vee \rangle + 1
\]

2. Sutherland operator. In this section we briefly summarize the known results about the diagonalization of Sutherland operator; most results in this section are due to Heckman and Opdam ([HO; H1; O1; O2]).

Denote by \( \mathbb{C}[P] \) the group algebra of the weight lattice; its basis is formed by formal exponentials \( e^\lambda, \lambda \in P \). We say that \( f \in \mathbb{C}[P] \) has the highest term \( e^\lambda \) if \( f = e^\lambda + \sum_{\mu < \lambda} c_\mu e^\mu \). We will also use the notation \( f = e^\lambda + \) lower order terms or just \( f = e^\lambda + \ldots \) in this case. Let \( \mathcal{R} \) be the ring obtained by adjoining to \( \mathbb{C}[P] \) the expressions of the form \( (e^\alpha - 1)^{-1}, \alpha \in R \). Note that the elements of \( \mathcal{R} \) may be considered as functions: if one replaces formal exponential \( e^\lambda \) by the function on \( \mathfrak{h} \) given by \( e^\lambda(h) = e^{2\pi i (\alpha, h)} \), the elements of \( \mathcal{R} \) become functions on the real torus \( T = \mathfrak{h}_R/Q^\vee, \mathfrak{h}_R = \bigoplus \mathbb{R} \alpha_i \) with singularities on the hypersurfaces \( e^\alpha(h) = 1, \alpha \in R \). However, we will use the formal language as far as possible.

Let us fix some non-negative integer \( k \) and consider the following differential operator in \( \mathcal{R} \):

\[
(2.1) \quad L = L_k = \Delta - k(k - 1) \sum_{\alpha \in R^+} \frac{\langle \alpha, \alpha \rangle}{(e^{\alpha/2} - e^{-\alpha/2})^2},
\]

where \( \Delta \) is Laplace’s operator: \( \Delta = \sum \partial^2_{x_i} \), \( x_i \) being an orthonormal basis in \( \mathfrak{h} \), and \( \partial_x e^\lambda = \langle x, \lambda \rangle e^\lambda \) with an obvious extension to \( \mathcal{R} \). This operator for the root
system $A_n$ was introduced by Sutherland ([Su]) and for an arbitrary root system by Olshanetsky and Perelomov ([OP]) as a Hamiltonian of an integrable quantum system. We will call $L$ the Sutherland operator.

Let us introduce the Weyl denominator

\begin{equation}
\delta = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})
\end{equation}

and define the following version of the Sutherland operator:

\begin{equation}
M_k = \delta^{-k} (L_k - k^2(\rho, \rho)) \delta^k.
\end{equation}

**Lemma 2.1.** ([HO])

1. \begin{equation}
M_k = \Delta - k \sum_{\alpha \in R^+} \frac{1 + e^\alpha}{1 - e^\alpha} \partial_\alpha
\end{equation}

(for brevity, we write $\partial_\alpha$ instead of $\partial_{\nu(\alpha)}$)

2. Both $L_k, M_k$ commute with the action of the Weyl group.

3. $M_k$ preserves the algebra $A = \mathbb{C}[P]^W \subset R$.

Let us introduce the basis of orbitsums in $A$:

\begin{equation}
m_\lambda = \sum_{\mu \in W\lambda} e^{\mu}, \quad \lambda \in P^+.
\end{equation}

**Lemma 2.2.**

\begin{equation}
M_k m_\lambda = (\lambda, \lambda + 2k\rho)m_\lambda + \sum_{\mu < \lambda, \mu \in P^+} c_{\lambda\mu} m_\mu
\end{equation}

Now we can consider the eigenfunction problem for $M_k$. Let us consider the action of $M_k$ in the finite-dimensional space spanned by $m_\mu$ with $\mu \leq \lambda$. Then the eigenvalue $(\lambda, \lambda + 2k\rho)$ has multiplicity one in this space due to the following trivial but very useful fact:

**Lemma 2.3.** Let $\lambda, \mu \in P^+, \mu < \lambda$. Then $(\mu + \rho, \mu + \rho) < (\lambda + \rho, \lambda + \rho)$.

Thus, we can give the following definition:

**Definition.** Jack’s polynomials $J_\lambda, \lambda \in P^+$ are the elements of $\mathbb{C}[P]^W$ defined by the following conditions:

1. $J_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu$

2. $M_k J_\lambda = (\lambda, \lambda + 2k\rho)J_\lambda$
Due to Lemma 2.3, these properties determine \( J_\lambda \) uniquely. Note that this definition is valid for any complex \( k \), and \( J_\lambda \) are rational in \( k \).

**Remark.** In [H1], these polynomials are called Jacobi polynomials associated with the root system \( R \); however, we prefer to call them Jack’s polynomials, since for the root system \( A_n \) they are known under this name.

Let us introduce inner product in \( A \). Let

\[
\langle f, g \rangle_0 = \frac{1}{|W|} [f \bar{g}]_0,
\]

where \([ \ ]_0\) is the constant term of a polynomial, and the bar involution is defined by \( e^{\bar{\lambda}} = e^{-\lambda} \). More generally, let

\[
\langle f, g \rangle_k = \langle f \delta^k, g \delta^k \rangle_0.
\]

**Lemma 2.4.** \( M_k \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_k \)

*Proof.* This is equivalent to saying that \( L_k \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_0 \), which is obvious.

**Corollary.** \( \langle J_\lambda, J_\mu \rangle_k = 0 \) if \( \lambda < \mu \).

In fact, one has a much stronger result:

**Theorem 2.5.** [H1] \( \langle J_\lambda, J_\mu \rangle = 0 \) if \( \lambda \neq \mu \).

We will prove this theorem for the root system \( A_n \) later by a different method.

Finally, let us consider the algebra \( \mathcal{L} \) of all \( W \)-invariant differential operators in \( \mathfrak{h} \) with coefficients in the ring \( \mathcal{R} \) which commute with \( M_k \). It is proved in [HO; O1; O2] that this algebra is in fact isomorphic to the algebra of \( W \)-invariant polynomials in \( \mathfrak{h} \). Since it is known that the latter one is a free polynomial algebra with generators \( p_1, \ldots, p_r \), \( \deg p_i = d_i \), we see that \( \mathcal{L} \) is also a free polynomial algebra generated by some differential operators \( D_i, D_1 = M_k \). Thus, we can formulate the eigenvalue problem: fix a sequence \( \Lambda_1, \ldots, \Lambda_r \) and find the common eigenfunction of \( D_i \) with eigenvalues \( \lambda_i \):

\[
D_i \psi = \Lambda_i \psi, i = 1 \ldots r.
\]

In an appropriate class of functions, this system always has a solution, and the number of solutions is equal to the order of the Weyl group. However, if we are looking for a non-zero polynomial \( W \)-invariant solution, i.e., \( \psi \in \mathbb{C}[P]^W \), then there is at most one solution for every \( \Lambda = (\Lambda_1, \ldots, \Lambda_r) \) and if it exists, it is precisely the Jack’s polynomial \( J_\lambda \) defined above; the corresponding eigenvalues are \( \Lambda_i = p_i(\lambda + k\rho), \lambda \in P^+ \).

**3. Construction of Jack’s polynomials via representation theory.** In this section we show how one can construct the Jack’s polynomials using the representation theory of \( \mathfrak{g} \) for \( \mathfrak{g} = gl_n \). All the results in this section are proved in the papers ([E; EK2]), so we give them here without proofs.
For \( \lambda \in P^+ \), let us denote by \( L_\lambda \) the irreducible finite-dimensional module over \( g \) with the highest weight \( \lambda \); also, let us denote by \( V[\mu] \) the subspace of weight \( \mu \) in \( V \), and let us fix a highest-weight vector \( v_\lambda \in V[\lambda] \). Let us consider the \( g \)-intertwining operators of the form

\[
\Phi: L_\lambda \rightarrow L_\lambda \otimes U,
\]

where \( U \) is an arbitrary module over \( g \).

Let us define the generalized character \( \chi \) for such an intertwiner by

\[
(3.2) \quad \chi_\Phi = \sum_{\mu \in P} e^{\mu} \text{Tr}|V[\mu]|\Phi
\]

This is an element of \( \mathbb{C}[P] \otimes U \); in fact, it is easy to see that it takes values only in the zero-weight subspace \( U[0] \). As in the previous section, we can consider it as a function on \( h \), which is equivalent to writing

\[
(3.3) \quad \chi_\Phi(h) = \text{Tr}_V(\Phi e^{2\pi i h}).
\]

One of the main results of the paper [E] is the following theorem:

**Theorem 3.1.** If \( \Phi \) is an intertwiner of the form (3.1) then the generalized character \( \chi_\Phi \) satisfies the following equation:

\[
(3.4) \quad \left( \Delta - 2 \sum_{\alpha \in R^+} \frac{1}{(e^{\alpha/2} - e^{-\alpha/2})^2} e^{\alpha} f_\alpha \right) (\chi_\delta) = (\lambda + \rho, \lambda + \rho) \chi_\delta
\]

Another important result is the orthogonality of the generalized characters:

**Theorem 3.2.** [EK1; EK2] If \( \lambda, \mu \in P^+ \), \( \lambda \neq \mu \), \( \Phi_1: L_\lambda \rightarrow L_\lambda \otimes U \), \( \Phi_2: L_\mu \rightarrow L_\mu \otimes U \) are nonzero intertwiners, then the characters \( \chi_{\Phi_1} \) and \( \chi_{\Phi_2} \) are orthogonal:

\[
\sigma(\langle \chi_{\Phi_1}, \chi_{\Phi_2} \rangle_1) = 0,
\]

where \( \sigma : U \leq U \rightarrow \mathbb{C} \) is the Shapovalov form.

Now, let us be more specific. From now till the end of this section, we only consider the Lie algebra \( \mathfrak{sl}_n \), i.e., the root system \( A_{n-1} \). Let us fix a positive integer \( k \) (later it will be the same \( k \) we considered in Section 2) and take \( U = S^{(k-1)n} \mathbb{C}^n \). Then \( U[0] \) is one-dimensional, and for every \( \alpha \in R^+ \), \( e_\alpha f_\alpha |_{U[0]} = k(k-1) \).

**Lemma 3.3.** Let \( \mu \in P^+ \). A non-zero intertwiner

\[
\Phi: L_\mu \rightarrow L_\mu \otimes U
\]

exists iff \( \mu = (k-1)\rho + \lambda, \lambda \in P^+ \); if it exists, it is unique up to a scalar. We will denote such an intertwiner by \( \Phi_\lambda \).

Proof of this lemma is a standard exercise, which we leave to the reader.

Since \( U[0] \cong \mathbb{C} \), we can consider \( \chi_\Phi \) as a scalar-valued function; we choose this identification in such a way that

\[
\chi_\lambda = \chi_{\Phi_\lambda} = e^{\lambda+(k-1)\rho} + \text{lower order terms}.
\]

Now we quote two results from [EK2]:
Theorem 3.4.  \[ (3.5) \]
\[ \chi_0 = \delta^{k-1}. \]

Theorem 3.5. \( \chi_\lambda \) is divisible by \( \chi_0 \), and the ratio is the Jack’s polynomial:
\[ \frac{\chi_\lambda}{\chi_0} = J_\lambda. \]

Proof. We briefly outline the proof, since it is very instructive. First, we prove by induction in \( k \) that \( \chi_\lambda \) is divisible by \( \chi_0 \) and the ratio is a symmetric polynomial with the highest term \( e^\lambda \). Next, it follows from Theorems 3.1 and 3.4 that
\[ (\Delta - 2k(k - 1) \sum_{\alpha \in \hat{R}^+} \frac{1}{(e^{\alpha/2} - e^{-\alpha/2})}) \left( \frac{\chi_\lambda}{\chi_0} \delta \right) = (\lambda + k\rho, \lambda + k\rho) \frac{\chi_\lambda}{\chi_0} \delta^k. \]

Comparing this with the formulas for operators \( L_k \) and \( M_k \) in Section 1, we see that \( \chi_\lambda/\chi_0 \) is the Jack’s polynomial \( J_\lambda \). \( \square \)

Corollary. \( \langle J_\lambda, J_\mu \rangle_k = 0 \) if \( \lambda \neq \mu \).

Proof. This follows immediately from Theorems 3.2, 3.4 and 3.5.

4. Basic definitions: affine case. Here we review the notations and facts about affine Lie algebras and root systems. All of them can be found in [K]. As a rule, we will use hat (\( \hat{\cdot} \)) in the notations of affine analogues of finite-dimensional objects.

Let \( \hat{g} \) be the affine Lie algebra corresponding to \( g \):
\[ \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus Cc \oplus Cd, \]
with the commutation rule given by
\[ [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n\delta_{m,-n}(x, y)c \]
\[ c \text{ is central} \]
\[ [d, x \otimes t^n] = nx \otimes t^n \]

(4.1)

Similarly to the finite-dimensional case, we define Cartan subalgebra \( \hat{h} = h \oplus Cc \oplus Cd \), \( \hat{h}^* = \hat{h}^* \oplus C\delta \oplus C\varepsilon \), where \( \langle \varepsilon, h \oplus Cd \rangle = \langle \delta, h \oplus Cc \rangle = 0, \langle \delta, d \rangle = 1, \langle \varepsilon, c \rangle = 1 \).

It will be convenient to consider affine hyperplanes \( \hat{h}^*_K = h^* \oplus C\delta + K\varepsilon, K \in \mathbb{C} \); we will refer to the elements of \( \hat{h}^*_K \) as having level \( K \).

Again, we have a bilinear non-degenerate symmetric form \( \langle \cdot, \cdot \rangle \) on \( \hat{h}^* \) which coincides with previously defined on \( h^* \) and \( (\varepsilon, \delta) = 1, (\varepsilon, h^*) = (\delta, h^*) = (\varepsilon, \varepsilon) = (\delta, \delta) = 0 \). This gives an identification \( \nu: \hat{h} \simeq \hat{h}^* \), and a bilinear form on \( \hat{h} \) such that \( (c, d) = 1 \).

We define the root system \( \hat{R} = \{ \hat{\alpha} = \alpha + n\delta | \alpha \in R, n \in \mathbb{Z} \text{ or } \alpha = 0, n \in \mathbb{Z} \setminus \{0\} \} \).

Again, we have the notion of positive roots: \( \hat{R}^+ = \{ \hat{\alpha} = \alpha + n\delta \in \hat{R} | n > 0 \text{ or } n = 0, \alpha \in R^+ \} \) and the basis of simple roots \( \alpha_0 = -\theta + \delta, \alpha_1, \ldots, \alpha_r \).

Now, let us define the affine Weyl group \( \hat{W} \) as the group of transformations of \( \hat{h}^* \) generated by the reflections with respect to \( \alpha_i, i = 0 \ldots r \). This group preserves the bilinear form; also, it preserves each of the affine hyperplanes \( \hat{h}^*_K \). We have notion of sign of an element of \( \hat{W} \): \( \varepsilon(w) = (-1)^l \) if \( w \) is a product of \( l \) reflections.
Theorem 4.1. (see [K]) \( \hat{W} \cong W \rtimes Q^\vee \), where the action of \( W \) is the same as in the classical case, and the action of \( Q^\vee \) in \( \hat{h}_K^\ast \) is given by

\[
(4.2) \quad \alpha^\vee : \hat{\lambda} \mapsto \hat{\lambda} + K\nu(\alpha^\vee) - (\langle \hat{\lambda}, \alpha^\vee \rangle + \frac{1}{2} K(\alpha^\vee, \alpha^\vee)) \delta
\]

Now we can define the root lattice \( \hat{P} = P \oplus \mathbb{Z} \delta \oplus \mathbb{Z} \varepsilon \subset \hat{h}^\ast \) and the cone of dominant weights \( \hat{P}^+ = \{ \hat{\lambda} \in \hat{h}^\ast | \langle \hat{\lambda}, \alpha^\vee \rangle \in \mathbb{Z}_+, i = 0, \ldots, r \} \). We will also use the notation \( \hat{P}_K^+ = \hat{P}^+ \cap \hat{h}^+_K = \{ \lambda + n\delta + K\varepsilon | \lambda \in P^+, \langle \lambda, \theta^\vee \rangle \leq K \} \) for \( K \in \mathbb{Z}_+ \). Note the cone of dominant weights is invariant with respect to the translations along \( \delta \) direction, but if one factors this out then there is only a finite number of dominant weights for every level \( K \). Abusing the notations, we will write \( \hat{P}_K^+ = \{ \lambda \in P^+ | \langle \lambda, \theta \rangle \leq K \} \).

Also, we introduce the following affine analogue of \( \rho \):

\[
\hat{\rho} = \rho + h^\vee \varepsilon
\]

then \( \langle \hat{\rho}, \alpha_i^\vee \rangle = 1, i = 0, \ldots, r \) and thus \( \hat{\rho} \in \hat{P}^+ \).

Lemma 4.2. 1. \( \hat{W} \) preserves each \( \hat{P}_K = \hat{P} \cap \hat{h}_K^\ast \)

2. \( \hat{P}_K^+ \) is a fundamental domain for the action of \( \hat{W} \) in \( \hat{P}_K \) for \( K > 0 \).

5. The group algebra of the weight lattice. In this section we define our basic object of study – the algebra of \( \hat{W} \)-invariants of the (suitably completed) group algebra of \( \hat{P} \) and study its elementary properties, following the paper of Looijenga ([Lo]).

Let us consider the group algebra \( \mathbb{C}[\hat{P}] \), i.e. the algebra spanned by the formal exponentials \( e^{\hat{\lambda}}, \hat{\lambda} \in \hat{P} \). It is naturally \( \mathbb{Z} \)-graded: \( \mathbb{C}[\hat{P}] = \bigoplus_{K \in \mathbb{Z}} \mathbb{C}[\hat{P}_K] \). Consider the following completion:

\[
(5.1) \quad \overline{\mathbb{C}[\hat{P}_K]} = \{ \sum_{n=1}^{\infty} a_n e^{\hat{\lambda}_n} | \lim_{n \to \infty} (\hat{\lambda}_n, \hat{\rho}) = -\infty \}
\]

Then \( \overline{\mathbb{C}[\hat{P}]} = \bigoplus_K \overline{\mathbb{C}[\hat{P}_K]} \) is again a \( \mathbb{Z} \)-graded algebra. (This completion is chosen so to include the characters of Verma modules over \( \hat{g} \).)

However, we will use a smaller algebra \( A = \bigcap_{w \in \hat{W}} w \left( \overline{\mathbb{C}[\hat{P}]} \right) \), which is a natural analogue of the group algebra \( \mathbb{C}[P] \) for finite-dimensional case. In particular, we have a natural action of \( \hat{W} \) in \( A \). It is also \( \mathbb{Z} \)-graded: \( A_K = \bigcap_{w \in \hat{W}} w \left( \overline{\mathbb{C}[\hat{P}_K]} \right) \)

Now, let us consider the algebra of \( \hat{W} \)-invariants \( A_{\hat{W}} \subset A \). Abusing the language, we will call elements of \( A_{\hat{W}} \) invariant polynomials.

Example 1. For any \( \hat{\lambda} \in \hat{P}_K, K \geq 0 \) the orbit sum \( m_{\hat{\lambda}} = \sum_{\hat{\mu} \in \hat{W} \hat{\lambda}} e^{\hat{\mu}} \) belongs to \( A_{\hat{W}}^K \).

Example 2. If \( \hat{\lambda} \) is a dominant weight then the irreducible highest-weight module \( L_{\hat{\lambda}} \) with highest weight \( \hat{\lambda} \) is integrable, and its character belongs to \( A_{\hat{W}}^K \).

Obviously, \( A_{\hat{W}} \) is \( \mathbb{Z} \)-graded. Moreover, the following is well-known:
Lemma 5.1. \( A_0^W = 0 \) for \( K < 0 \), and \( A_0^W = \left\{ \sum_{n \leq n_0} a_n e^{n\delta}, a_n \in \mathbb{C} \right\} \)

Theorem 5.2. (cf. [Lo]) For every \( K \in \mathbb{Z}_+ \), the orbitsums \( m_{\lambda + K\varepsilon}, \lambda \in \mathcal{P}_K^+ \) form a basis of \( A_K^W \) over the field \( A_0^W \).

This theorem follows from the fact that \( \hat{P}_K^+ \) is a fundamental domain for the action of \( \hat{W} \) in \( \hat{P}_K \).

It will be convenient to introduce formal variable \( q = e^{-\delta} \); then every element of \( A_K^W \) can be written as a formal Laurent series in \( q \) with coefficients from \( \mathbb{C} \). In particular, in these notations \( A_0^W \simeq \mathbb{C}(q) \).

6. The affine Calogero-Sutherland operator. In this section we give the definition of the affine analogue of Sutherland operator, which we will call the affine Calogero-Sutherland operator. As before, we fix some positive integer \( k \).

First of all, let us define the analogue of the ring \( \mathcal{R} = \mathbb{C}[[\mathcal{P}]](1 - e^\alpha)^{-1} \), defined in Section 2. Consider the algebra \( \mathbb{C}[[\hat{P}]](1 - e^\alpha)^{-1} \), obtained by adjoining to \( \mathbb{C}[[\hat{P}]] \) the inverses of \( (1 - e^\alpha) \) for \( \alpha \in \hat{R} \) (no completion so far). Then we have a morphism

\[
\tau: \mathbb{C}[[\hat{P}]](1 - e^\alpha)^{-1} \to \mathbb{C}[[\hat{P}]],
\]

given by expanding \( (1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \ldots \) for \( \alpha \in \hat{R}^+ \). Note that the image is not in \( \mathcal{R} \).

Similarly, for every \( w \in \hat{W} \) we have

\[
\tau_w: \mathbb{C}[[\hat{P}]](1 - e^\alpha)^{-1} \to w\left( \mathbb{C}[[\hat{P}]] \right),
\]

given by expanding \( (1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + \ldots \) for \( \alpha \in w\hat{R}^+ \).

Define \( \hat{\mathcal{R}} = \left\{ \sum a_n | a_n \in \mathbb{C}[[\hat{P}]](1 - e^\alpha)^{-1}, \sum \tau_w(a_n) \text{ converges in } w\left( \mathbb{C}[[\hat{P}]] \right) \text{ for every } w \in \hat{W} \right\} \). This is the right analogue of the ring \( \mathcal{R} \) introduced in Section 2; for example, \( \sum_{\alpha \in \hat{R}^+} \frac{1}{1 - e^\alpha} \in \hat{\mathcal{R}} \). Note that there is a natural action of the Weyl group \( \hat{W} \) in \( \hat{\mathcal{R}} \); also note that this algebra has a natural \( \mathbb{Z} \)-grading given by level.

Definition. The Calogero-Sutherland operator for (affine) root system \( \hat{R} \) is the differential operator which acts in \( \hat{R}_K \) by the following formula (all the notations as before):

\[
(6.1) \quad \hat{L}_k = \Delta - 2Kq \frac{\partial}{\partial q} - k(k - 1) \sum_{\alpha \in \hat{R}_K^+ \atop n \in \mathbb{Z}} \frac{q^n e^{\alpha}}{(1 - q^n e^{\alpha})^2} (\alpha, \alpha).
\]

Remark. Note that for \( K = 0 \) this operator coincides (up to a constant) with the elliptic Calogero-Sutherland operator (see [OP]); compare with formula (8.5) below. In more general situation, operator (6.1) was introduced by Bernard ([B]).

Introducing the Laplace’s operator \( \hat{\Delta} \) on \( \mathfrak{h} \): \( \hat{\Delta} = \Delta + 2\partial_c \partial_d \), we can rewrite (6.1) as follows:
(6.2) \[ \hat{L} = \hat{\Delta} - k(k - 1) \sum_{\hat{\alpha} \in \hat{R}^+} \frac{1}{(e^{\hat{\alpha}/2} - e^{-\hat{\alpha}/2})^2} (\hat{\alpha}, \hat{\alpha}), \]

making \( \hat{L} \) an absolute analogue of (2.1). Note that \( \hat{L} \) obviously commutes with the action of \( \hat{W} \) in \( \hat{R} \).

Similarly to Section 2, define

(6.3) \[ \hat{\delta} = e^{\hat{\rho}} \prod_{\hat{\alpha} \in \hat{R}^+} (1 - e^{-\hat{\alpha}}). \]

This is an element of \( A \). It is known (see [Lo, K]) that \( \hat{\delta} \) is \( \hat{W} \)-antiinvariant; moreover, every \( \hat{W} \)-antiinvariant element of \( A \) has the form \( f\delta, f \in A^{\hat{W}} \).

Define

(6.4) \[ \hat{M}_k = \hat{\delta}^{-k} \circ (\hat{L}_k - k^2(\hat{\rho}, \hat{\rho})) \circ \hat{\delta}^k. \]

**Theorem 6.1.**

1. \( \hat{M}_k \) is a well defined operator in \( \hat{R} \).
2. \( \hat{M}_k = \hat{\Delta} - 2k \sum_{\hat{\alpha} \in \hat{R}^+} \frac{1}{1 - e^{\hat{\alpha}}} \partial_{\hat{\alpha}} + 2k \partial_{\hat{\rho}}. \)
3. \( \hat{M}_k \) commutes with the action of \( \hat{W} \).

**Proof.** It will be convenient to use vector fields in \( \hat{h} \), i.e. the elements of \( \hat{R} \otimes \hat{h} \). If \( f = \sum f_i \lambda_i, f_i \in \hat{R}, \lambda_i \in \hat{h} \) then we will denote by \( \partial_f = \sum f_i \partial_{\lambda_i} \) the corresponding differential operator. Then we have the following obvious formula:

(6.6) \[ \hat{\delta}^{-1} \circ \hat{\Delta} \circ \hat{\delta} = \hat{\Delta} + 2\partial_v + (\hat{\delta}^{-1} \hat{\Delta}(\hat{\delta})), \]

where

\[ v = \hat{\delta}^{-1} \text{grad } \hat{\delta} = \hat{\rho} + \sum_{\hat{\alpha} \in \hat{R}^+} \frac{e^{-\hat{\alpha}}}{1 - e^{-\hat{\alpha}}} \hat{\alpha}. \]

Note that \( \hat{W} \)-antiinvariance of \( \hat{\delta} \) implies \( \hat{W} \)-invariance of \( v \). Since \( \hat{\Delta}(\hat{\delta}) = (\hat{\rho}, \hat{\rho}) \hat{\delta} \), which follows from the denominator identity for affine root systems, this proves that \( \hat{\delta}^{-1} \hat{\Delta} \hat{\delta} \) is a well defined operator in \( \hat{R} \).

It is easy to prove by induction that

\[ \hat{\delta}^{-k} \circ \hat{\Delta} \circ \hat{\delta}^k = \hat{\Delta} + k(\hat{\rho}, \hat{\rho}) + 2k \partial_v + k(k - 1)\hat{\delta}^{-1}(\partial_v \hat{\delta}). \]

Obviously, \( \hat{\delta}^{-1}(\partial_v \hat{\delta}) = (v, v) \), where \((, )\) is the inner product of the vector fields, i.e., the inner product on \( \hat{h} \) extended by \( \hat{R} \)-linearity to \( \hat{R} \otimes \hat{h} \) (it has nothing to do with the inner product on polynomials!).
Lemma.

\[(v, v) = (\hat{\rho}, \hat{\rho}) + \sum_{\hat{\alpha} \in \hat{\mathcal{R}}^+} (\hat{\alpha}, \hat{\alpha}) \frac{1}{(e^{\hat{\alpha}/2} - e^{-\hat{\alpha}/2})^2}.\]

Proof. Let us consider \(X = (v, v) - (\hat{\rho}, \hat{\rho}) - \sum_{\hat{\alpha} \in \hat{\mathcal{R}}^+} (\hat{\alpha}, \hat{\alpha}) \frac{1}{(e^{\hat{\alpha}/2} - e^{-\hat{\alpha}/2})^2} \in \hat{\mathcal{R}}.\) Obviously, it is \(\hat{W}\)-invariant. Also, from the explicit expression for \(v\) it follows that \(X\) has only simple poles. Consider \(X\delta\). It is an element of \(\hat{\mathcal{R}}\) with no poles, thus it is an element of \(A\). Also, it is antiinvariant. Thus, \(X \in A\). Obviously, \(X \in A_0\); since \(X\) is \(\hat{W}\)-invariant, Lemma 5.1 implies that \(X \in C((q))\).

To complete the calculation, let us write the explicit expression for \(X\); then, let us expand it in a series in \(e^{-\hat{\alpha}}, \hat{\alpha} \in \hat{\mathcal{R}}^+\) (i.e., apply the map \(\tau\) defined above) and keep only the terms of the form \(e^{n\delta}\) in the expansion. This gives:

\[X = 2 \left( rh^\vee - \sum_{\alpha \in R^+} (\alpha, \alpha) \right) \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.\]

On the other hand, it is known that for any Lie algebra \(\mathfrak{g}\), \(\sum_{\alpha \in R^+} (\alpha, \alpha) = rh^\vee\). The simplest way to prove it is to consider the action of the Casimir element \(\Omega \in U\mathfrak{g}\) in the adjoint representation. On one hand, \(\Omega_{|\mathfrak{g}} = 2h^\vee \text{Id}_{\mathfrak{g}}\), and thus \(\text{Tr}_{|\mathfrak{g}} \Omega = 2rh^\vee\). On the other hand, it is easy to deduce from the formula \(\Omega = \sum_{\alpha \in R^+} e_\alpha f_\alpha + f_\alpha e_\alpha + \sum x_i^2\) that \(\text{Tr}_{|\mathfrak{g}} \Omega = 2 \sum_{\alpha \in R^+} (\alpha, \alpha)\). Thus, \(X = 0\).

This lemma together with previous results immediately implies statements 1 and 2 of the theorem. Statement 3 follows from \(\hat{W}\)-invariance of the operator \(L_k\). □

Note that Theorem 6.1 is a complete analogue of Lemma 2.1.

Remark 1. This technique is borrowed from [Ma].

Remark 2. In the simply-laced case the identity \(\sum (\alpha, \alpha) = rh^\vee\) becomes \(\dim \mathfrak{g} = r(h+1)\), where \(h\) is the Coxeter number for \(\mathfrak{g}\). This latter identity is well known and has a beautiful interpretation in terms of the Coxeter automorphism, due to Kostant.

Theorem 6.2. \(\hat{M}\) preserves the algebra of \(\hat{W}\)-invariant polynomials: \(\hat{M}A_{K}^{\hat{W}} \subset A_{K}^{\hat{W}}\). Moreover, its action is triangular:

\[\hat{M}m_\lambda = (\lambda, \lambda + 2k\hat{\rho})m_\lambda + \sum_{\mu < \lambda, \mu \in \hat{\mathcal{R}}_K} c_{\lambda \mu} m_\mu\]

Proof. Proof is based on the following lemma:

Lemma. Let \(f \in A^{\hat{W}}, f = e^{\hat{\lambda}} + \text{lower terms}, \hat{\alpha} \in \hat{\mathcal{R}}^+\). Then \(\frac{1}{1-e^{\hat{\alpha}}} \partial_{\hat{\alpha}} f\) is a well-defined element of \(A\) with highest term \(-(\hat{\lambda}, \hat{\alpha})e^{\hat{\lambda}-\hat{\alpha}}\).

Proof. The proof is based on the fact that due to \(\hat{W}\)-invariance, \(f\) contains terms \(e^{\hat{\mu}}\) and \(e^{\hat{\mu}-\mu, \hat{\alpha}}\) with equal coefficients, and on explicit calculation.

Now the statement of the theorem follows from Theorem 6.1. □
Our main objective will be the study of the eigenfunctions of action of this operator in $A^\hat{W}$, which we will call affine Jack’s polynomials. More precisely, let us consider the action of $\hat{M}$ in the linear space spanned by $m_{\hat{\mu}}$ with $\hat{\mu} \leq \hat{\lambda}$. This space is not finite-dimensional; however, one can still check that $\hat{M}$ has a unique eigenvector with eigenvalue $(\hat{\lambda}, \hat{\lambda} + 2k\hat{\rho})$ in this space (this is based on the affine analogue of Lemma 2.3). Thus, we adopt the following definition:

**Definition.** Affine Jack’s polynomials $\hat{J}_{\hat{\lambda}}, \hat{\lambda} \in \hat{P}^+$ are the elements of $A^\hat{W}$ defined by the following conditions:

1. $\hat{J}_{\hat{\lambda}} = m_{\hat{\lambda}} + \sum_{\hat{\mu} < \hat{\lambda}} c_{\hat{\lambda}, \hat{\mu}} m_{\hat{\mu}}$
2. $\hat{M}_k \hat{J}_{\hat{\lambda}} = (\hat{\lambda}, \hat{\lambda} + 2k\hat{\rho}) \hat{J}_{\hat{\lambda}}$

As was said above, these conditions determine $\hat{J}_{\hat{\lambda}}$ uniquely. Note that if $\hat{\mu} = \hat{\lambda} + n\delta$ then $J_{\hat{\mu}} = q^{-n} J_{\hat{\lambda}}$. Thus, it suffices to consider only the polynomials $J_{\hat{\lambda}}$ for $\hat{\lambda} = \lambda + K\varepsilon, \lambda \in P^+_K$.

7. Intertwiners and traces for affine Lie algebras.

In this section we introduce the tools we will use later to construct affine Jack’s polynomials.

First, we define the notion of evaluation representation. Let $V$ be a finite-dimensional module over $\mathfrak{g}$ and $z$ be a non-zero complex number. Then we can construct an evaluation representation of $\mathfrak{g}$ (not $\hat{\mathfrak{g}}$!) in $V$ by $\pi_V(z)(a \otimes t^n) = z^n \pi_V(a), \pi_V(z)(c) = 0$.

Note that $V(z)$ has no $\hat{P}$-gradation but has a natural $P$-gradation. We will be interested in intertwining operators

$\Phi: L_{\hat{\lambda}} \rightarrow \hat{L}_{\hat{\lambda}} \otimes V(z)$,

where $\hat{L}_{\hat{\lambda}}$ is the completion of the integrable highest-weight module with respect to the $d$-grading. To prove the existence of such intertwiners, we use the following well-known result (see, for example, the arguments in [TK], which work for general Lie algebra in the same way as for $\mathfrak{sl}_2$):

**Lemma 7.1.** The mapping $\Phi \mapsto \langle v_{\hat{\lambda}}, \Phi v_{\hat{\lambda}} \rangle$ establishes one-to-one correspondence between the space of all intertwiners of the form (7.1) and the subspace in $V[0]$ formed by the vectors $v$ such that $xv = 0$ for every $x \in U\mathfrak{g}^-$ such that $xv_{\hat{\lambda}} = 0$ in $L_{\hat{\lambda}}$.

Suppose that $\Phi$ is a non-zero intertwiner of the form (7.1). Then we define the “generalized character” $\chi_\Phi$ by

$\chi_\Phi = \sum_{\hat{\mu} \in \hat{P}} e^{\hat{\mu}} \text{Tr}|_{L_{\hat{\lambda}}[\hat{\mu}]} \Phi$.

This is an element of $\mathbb{C}[\hat{P}] \otimes V[0]$ with the highest term $\langle v_{\hat{\lambda}}, \Phi v_{\hat{\lambda}} \rangle e^{\hat{\lambda}}$, and it is independent of $z$. Moreover, let us assume that $\hat{\lambda}$ is a dominant weight. Then $\chi_\Phi \in A$. Unless stated otherwise, in this chapter we always assume that $\hat{\lambda}$ is dominant.
Theorem 7.2. (see [EK3]) If \( \Phi \) is an intertwiner of the form (7.1), \( \hat{\lambda} = \lambda + a\delta + K\varepsilon \) then the generalized character \( \chi_{\Phi} \) satisfies the following equation:

\[
(7.3) \left( \Delta - 2(K + h^\vee)q \frac{\partial}{\partial q} - 2 \sum_{\alpha \in R^+, n \in \mathbb{Z}} \frac{q^n e^\alpha}{(1 - q^n e^\alpha)^2} e_\alpha f_\alpha \right) (\chi_{\Phi}\hat{\delta}) = (\hat{\lambda} + \hat{\rho}, \hat{\lambda} + \hat{\rho}) \chi_{\Phi}\hat{\delta}.
\]

As before, we restrict ourselves to the case \( g = sl_n \), and take \( V = S^{(k-1)n} \mathbb{C}^n \). Then \( V[0] \simeq \mathbb{C} \), and \( e_\alpha f_\alpha|_{V[0]} = k(k-1) \).

Proposition 7.3. A non-zero intertwiner

\[
\Phi: L_{\mu} \to \widehat{L}_{\mu} \otimes V(z)
\]

exists iff \( \hat{\mu} = (k-1)\hat{\rho} + \hat{\lambda}, \hat{\lambda} \in \hat{R}^+ \); if it exists, it is unique up to a scalar. We will denote such an intertwiner by \( \Phi_{\lambda} \).

Proof. The proof is based on Lemma 7.1.

Let us consider the traces \( \chi_{\hat{\lambda}} = \chi_{\Phi_{\lambda}} \). They take values in \( V[0] \), which is one-dimensional, and thus can be considered as scalar-valued; we choose this identification so that

\[
\chi_{\hat{\lambda}} = e^{\hat{\lambda} + (k-1)\hat{\rho}} + \text{lower terms}
\]

Proposition 7.4. For every real \( \hat{\alpha} \in \hat{R}^+ \), \( \chi_{\hat{\lambda}} \) is divisible by \( (1 - e^{-\hat{\alpha}})^{k-1} \) (divisibility is to be understood in the algebra \( A \)).

Proof. The proof is absolutely similar to the finite-dimensional case (cf. [EK2]), and is based on consideration of the traces of the form

\[
(7.4) \chi_{\hat{\lambda}}^F = \sum_{\hat{\mu} \in \hat{P}} e^{\hat{\mu}} \text{Tr}|_{L_{\hat{\lambda}}|\hat{\mu}} (\Phi_{\lambda} F),
\]

where \( F \) is an arbitrary element of \( U\hat{g} \). Let us take \( F = f_{\hat{\alpha}}^{k-1} \). Then, using the intertwining property of \( \Phi_{\hat{\lambda}} \) and the identity \( \Delta(f_{\hat{\alpha}}) = f_{\hat{\alpha}} \otimes 1 + 1 \otimes f_{\hat{\alpha}} \), we can prove by induction that

\[
\chi_{\hat{\lambda}}^F = \frac{f_{\hat{\alpha}}^{k-1} \chi_{\hat{\lambda}}}{(1 - e^{-\hat{\alpha}})^{k-1}}.
\]

Since \( V[0], V[-(k-1)\alpha] \) are one-dimensional, we can identify both of them with \( \mathbb{C} \); then \( f_{\hat{\alpha}}^{k-1}: V[0] \to V[-(k-1)\alpha] \) becomes a non-zero constant. On the other hand, it is easy to see that \( \chi_{\hat{\lambda}}^F \in A \), which proves the proposition. \( \square \)

Theorem 7.5.

\[
\chi_{0} = \hat{\delta}^{k-1}.
\]

Proof. Let us consider the ratio \( f = \chi_{0}/\hat{\delta}^{k-1} \). It follows from Proposition 7.4 that \( f \in A \). It has level zero and highest term 1. Moreover, similar arguments show that if we twist the order on \( \hat{P} \) by the action of the Weyl group: \( \hat{\lambda} \geq_w \hat{\mu} \) if \( \lambda - \mu \in w(Q^+) \), \( w \in \hat{W} \) then highest term of \( f \) with respect to any such twisted
ordering is still 1. This is only possible if $f \in \mathbb{C}((q))$. To complete the proof, we have to use the differential equation for the characters. Indeed, Theorem 7.2 implies that $\chi_\hat{\lambda}$ satisfies the following equation:

$$\hat{L}(\chi_\hat{\lambda}) = (\hat{\lambda} + k\hat{\rho}, \hat{\lambda} + k\hat{\rho})(\chi_\hat{\lambda}).$$

Substituting in this equation $\chi_0 = f(q)^{\hat{k}-1}$, we see that $f$ satisfies $\hat{M} f = 0$. Using formula (6.5) for $\hat{M}$ we get $2k\hbar^\gamma q^{\frac{\partial}{\partial q}} f = 0$, which is possible only if $f$ is a constant. Comparing highest terms of $\chi_0$ and $\hat{\delta}^{\hat{k}-1}$, we get the statement of the theorem. □

**Theorem 7.6.**

$$\frac{\chi_\hat{\lambda}}{\chi_0} = \hat{J}_\hat{\lambda}.$$  

**Proof.** Let us first prove that $\chi_\hat{\lambda}/\chi_0 \in \hat{A}_\hat{W}$. Consider the module $L = L_\hat{\lambda} \otimes L_{(k-1)\hat{\rho}}$. This module unitary (since both factors are unitary); thus, it is completely reducible and can be decomposed in a direct sum of the modules $L_{\hat{\mu}}$:

$$L = L_{(k-1)\hat{\rho} + \hat{\lambda}} + \sum_{\hat{\mu} \in \hat{P}^+ \atop \hat{\mu} < \hat{\lambda} + (k-1)\hat{\rho}} N_{\hat{\mu}} L_{\hat{\mu}}.$$

This sum is, of course, infinite; however, all the multiplicities are finite. In particular, this implies that the character of this module belongs to the algebra $\hat{A}_\hat{W}$ (that is, $\hat{W}$-invariants of completed group algebra of $\hat{P}$), so in a certain sense this sum converges.

Let us construct an intertwiner $\Psi: L \to L \otimes V(z)$ as $\Psi = \text{Id}_{L_\hat{\lambda}} \otimes \Phi_0$. Consider the corresponding trace $\chi_\Psi$. Then it follows from the decomposition of $L$ that

$$\chi_\Psi = \chi_\hat{\lambda} + \sum_{\hat{\mu} \in \hat{P}^+ \atop \hat{\mu} < \hat{\lambda}} a_{\hat{\lambda}\hat{\mu}} \chi_\hat{\mu}.$$  

On the other hand, $\chi_\Psi = \chi_0 \text{Ch} L_\hat{\lambda}$, where $\text{Ch} L_\hat{\lambda}$ is the ordinary character of the module $L_\hat{\lambda}$. Thus, dividing both sides by $\chi_0$ we get

$$\text{Ch} L_\hat{\lambda} = \sum_{\hat{\mu} \in \hat{P}^+} a_{\hat{\lambda}\hat{\mu}} \frac{\chi_\hat{\mu}}{\chi_0},$$

where $a_{\hat{\lambda}\hat{\mu}} \neq 0$ only if $\hat{\mu} \leq \hat{\lambda}$, and $a_{\hat{\lambda}\hat{\lambda}} = 1$. It is a trivial exercise in linear algebra to check that in this case the matrix $(a_{\hat{\lambda}\hat{\mu}})$ has an inverse: one can write

$$\frac{\chi_\hat{\lambda}}{\chi_0} = \sum_{\hat{\mu} \in \hat{P}^+} b_{\hat{\lambda}\hat{\mu}} \text{Ch} L_{\hat{\mu}},$$

and the coefficients $b_{\hat{\lambda}\hat{\mu}}$ satisfy the same conditions as $a_{\hat{\lambda}\hat{\mu}}$. Thus, $\chi_\hat{\lambda}/\chi_0 \in \hat{A}_\hat{W}$ and has highest term $e^{\hat{\lambda}}$.

We have proved that $\chi_\hat{\lambda}/\chi_0$ satisfies the first condition in the definition of the Jack’s polynomials. Now, Theorem 7.2 implies that $\hat{L}(\chi_\hat{\lambda}) = (\hat{\lambda} + k\hat{\rho}, \hat{\lambda} + k\hat{\rho})(\chi_\hat{\lambda})$.

Due to Theorem 7.5, this means that
\[
\hat{L}\left(\frac{\lambda_0 \delta^k}{\lambda_0}\right) = (\hat{\lambda} + k\hat{\rho}, \hat{\lambda} + k\hat{\rho})\left(\frac{\lambda_0 \delta^k}{\lambda_0}\right),
\]
which is precisely the definition of Jack’s polynomials. □

8. Normalized characters and their functional interpretation.

This section is of preparatory nature; its results will be used in the next section where we study the modular properties of the Jack’s polynomials.

First of all, to make our functions modular invariant we need to introduce some factors of the form \(q^t, t \in \mathbb{Q}\). Thus, we need to consider slightly more general setting than in the previous section. Namely, instead of the weight lattice \(\hat{P}\) we consider a bigger abelian group \(\hat{P}' = P + \mathbb{C}\delta + \mathbb{Z}\varepsilon\). Also, we can consider the algebra \(A' = \{\sum_{i=1}^{N} q^{a_i} f_i, a_i \in \mathbb{C}, f_i \in A\}\), and the subalgebra of \(\hat{W}\)-invariants in \(A'\) in a manner quite similar to the one of the previous section. All the results of Section 5 hold with obvious changes.

Define the normalized analogues of \(\hat{\rho}\) and \(\hat{\delta}\) as follows:

\[
\hat{\rho}' = \hat{\rho} - \frac{(\rho, \rho)}{2h^\vee}\delta = \rho - \frac{(\rho, \rho)}{2h^\vee}\delta + h^\vee\varepsilon,
\]
\[
\hat{\delta}' = e^{\hat{\rho}'} \prod_{\alpha \in \hat{R}^+} (1 - e^{-\hat{\alpha}}).
\]

This renormalization is chosen so that \(\hat{\Delta} \hat{\delta}' = 0\); another reason for this renormalization is that so defined \(\hat{\delta}'\) possesses nice modular properties (see below).

Now, let us define the renormalized operator

\[
\hat{M}' = \hat{\delta}'^{-k} \hat{L} \hat{\delta}'^k = \hat{M} - \frac{K k (\rho, \rho)}{h^\vee}.
\]

Finally, for \(K \in \mathbb{Z}_+, \lambda \in P^+_K\), (i.e., \((\lambda, \theta) \leq K\)) consider

\[
\hat{\lambda} = \lambda + K \varepsilon + \delta\left(\frac{(\rho, \rho)}{2h^\vee} - \frac{(\lambda + k \rho, \lambda + k \rho)}{2(K + k h^\vee)}\right).
\]

Now we can consider the generalized characters \(\chi_{\hat{\lambda}}\), defined in Section 7, and introduce the normalized Jack’s polynomials:

**Definition.** If \(K \in \mathbb{Z}_+, \lambda \in P^+_K\) then the Jack’s polynomial \(J_{\lambda,K}\) is given by

\[
J_{\lambda,K} = \frac{\chi_{\hat{\lambda}}}{\hat{\delta}'^{(k-1)}},
\]
where \(\hat{\lambda}\) is given by (8.2).

The results of the previous section imply that \(J_{\lambda,K}\) are invariant polynomials:

\[
J_{\lambda,K} \in A^{\hat{W}}, \text{ and that highest term of } J_{\lambda,K} \text{ is } e^{\lambda + K \varepsilon + \delta\left(\frac{(\rho, \rho)}{2h^\vee} - \frac{(\lambda + k \rho, \lambda + k \rho)}{2(K + k h^\vee)}\right)}.
\]

Note that for \(k = 1\) they are precisely the (usual) characters of integrable highest-weight modules, and the normalization coincides with that in [K, Chapter 13].

Moreover, it follows from Theorem 7.6 that the normalized Jack’s polynomials satisfy the following differential equation:

\[
\hat{M}' J_{\lambda,K} = 0.
\]
**Theorem 8.1.** The space of solutions of the equation $\hat{M}'f = 0$ in $A_{K}^{\hat{W}}$ is finite-dimensional, and the basis in the space of solutions is given by the normalized Jack's polynomials $J_{\lambda,K}$ (basis over $\mathbb{C}$, not over $A_{0}^{\hat{W}}$).

**Proof.** This theorem is quite standard and is based on the fact that each solution is uniquely determined by its highest term.

So far, all our constructions were purely algebraical; everything was considered as formal power series in $q$. However, in order to study modular properties we will need analytical approach. So, let us consider every $e^{\lambda} \in \mathbb{C}[\hat{P}']$ as a function on the domain $Y = \mathfrak{h} \times \mathbb{C} \times \mathcal{H}$, where $\mathcal{H}$ is the upper half-plane: $\mathcal{H} = \{ \tau \in \mathbb{C} \mid \text{Im } \tau > 0 \}$, by the following rule: if $\hat{\lambda} = \lambda + a\delta + K\varepsilon$ then put $e^{\hat{\lambda}}(h,u,\tau) = e^{2\pi i(\lambda,h)+K(u-a\tau)}$. Note that this agrees with our previous convention $e^{-\delta} = q$ if one lets $q = e^{2\pi i\tau}$.

Of course, we can't extend this rule to the completion $A$. However, it turns out that we can extend it to certain elements of $A'$, namely to the generalized characters:

**Theorem 8.2.** For every $\hat{\lambda} \in \hat{P}^+$, the Jack polynomial $J_{\hat{\lambda}}$ defined in Section 6, can be considered as an analytical function on $Y$ by the above rule (i.e., the corresponding series converges uniformly on compact sets in $Y$). The same is true for $\hat{\delta}'$.

**Proof.** The fastest way to prove this theorem is to use the defining differential equation for $J_{\hat{\lambda}}$. Indeed, due to Theorem 5.2 we can write $J_{\hat{\lambda}} = \sum_{\lambda \in \hat{P}^+} f_{\lambda}(q) m_{\lambda,K}$ for some $f_{\lambda} \in A'_{0}$. Substituting it in the defining differential equation for $J_{\hat{\lambda}}$ and using the fact that $m_{\lambda}$ are eigenfunctions of $\hat{\Delta}$, we get a system of ordinary differential equations for $f_{\lambda}$. It is easy to check that the coefficients of these equations will be analytical functions of $\tau$. Thus, we get that $f_{\lambda}$ will be analytical functions of $\tau$.

So, we can consider the generalized characters as functions on $Y$. Note that this is equivalent to writing:

$$
\chi_{\hat{\lambda}}(h,u,\tau) = e^{2\pi i K' u} \text{Tr}|L_{\mu,K'} (\Phi q L_{0} - c \pi e^{2\pi i h}),
$$

where $K' = K + (k - 1)\frac{\mu}{K'}$, $\mu = \lambda + (k - 1)\rho$, and $L_{0} = -d + \text{const}$ where the constant is chosen so that on the highest weight vector, $L_{0} v_{\mu,K'} = \frac{(\mu,\mu + 2\rho)}{2(K' + h'\rho)} v_{\mu,K'}$, and $c$ is the Virasoro central charge: $c = \frac{K' \dim \mathfrak{g}}{K' + h'\rho}$. Similar expressions appear in the Wess-Zumino-Witten model of conformal field theory.

We can define the action of the affine Weyl group $\hat{W}$ on $Y$ so that $e^{u \hat{\lambda}}(h,u,\tau) = e^{\hat{\lambda}}(w^{-1}(h,u,\tau))$. One easily checks that when restricted to the finite Weyl group $W$ this action coincides with the usual action of $W$ on $\mathfrak{h}$ (leaving $u,\tau$ invariant), and the action of $\alpha^{\vee} \in Q^{\vee}$ is given by

$$
\alpha^{\vee}(h,u,\tau) = (h - \alpha^{\vee} \tau, u + \frac{1}{2}(\alpha^{\vee},\alpha^{\vee}) \tau - (\alpha^{\vee},h),\tau).
$$

This implies

**Proposition 8.3.** Let $f \in A'_{K}$ be such that it gives an analytic function on $Y$. Then $f \in A'_{K}^{\hat{W}}$ iff the corresponding function on $Y$ satisfies the following conditions:

(a) $f(h + \alpha^{\vee},u,\tau) = f(h,u,\tau)$ for every $\alpha^{\vee} \in Q^{\vee}$.
(b) \( f(h + \tau \alpha^\vee, u + \frac{1}{2}(\alpha^\vee, \alpha^\vee) \tau + (\alpha^\vee, h), \tau) = f(h, u, \tau) \) for every \( \alpha^\vee \in Q^\vee \).

(c) \( f(h, u + a, \tau) = e^{2\pi i K a} f(h, u, \tau) \) for every \( a \in \mathbb{C} \).

(d) \( f(wh, u, \tau) = f(h, u, \tau) \) for every \( w \in W \).

Note that conditions (a)–(c) can be rewritten as follows: \( f(h, u, \tau) = e^{2\pi i K u} g(h, \tau) \), and \( g \) satisfies

\[
\begin{align*}
\varphi(x, \tau) &= \sum_{m \in \mathbb{Z}} \frac{q^m e^{2\pi i x}}{(1 - q^m e^{2\pi i x})^2} = \frac{1}{4\pi^2} \partial_x^2 \log \theta_1(x, \tau),
\theta_1(x, \tau) &= i q^{1/8} (e^{\pi i x} - e^{-\pi i x}) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i x})(1 - q^n e^{-2\pi i x})(1 - q^n) .
\end{align*}
\]

Note that the elliptic properties of \( \theta_1 \) imply that \( \varphi(x + 1, \tau) = \varphi(x + \tau, \tau) = \varphi(x, \tau) \); in fact, \( \varphi(x, \tau) = -\frac{1}{4\pi} \varphi(x, \tau) + c(\tau) \), where \( \varphi(x, \tau) \) is the Weierstrass function with periods 1, \( \tau \). Thus, the operator \( \hat{L} \) is well-defined on the torus \( \mathfrak{h}/(Q^\vee + \tau Q^\vee) \).

**Theorem 8.4.**

1. For any \( K \in \mathbb{Z}_+ \), the Jack’s polynomials \( \{ J_{\lambda + K \varepsilon} \}_{\lambda \in P_K^+} \) form a basis of the space of analytical functions on \( Y \) satisfying conditions (a)–(d) above over holomorphic functions of \( \tau \).
2. For any \( K \in \mathbb{Z}_+ \), the normalized Jack’s polynomials \( \{ \hat{J}_{\lambda, K} \}_{\lambda \in P_K^+} \) form a basis of the space of analytical functions on \( Y \) satisfying conditions (a)–(d) above and the condition \( \hat{M}' f = 0 \) over \( \mathbb{C} \).
9. Modular invariance.

Recall that the modular group $\Gamma = SL_2(\mathbb{Z})$ is generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfying the defining relations $(ST)^3 = S^2, S^2T = TS^2, S^4 = 1$. This group acts in a natural way on $Y$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (h, u, \tau) = \left( \frac{h}{c\tau + d} u - \frac{c(h, h)}{2(c\tau + d)}, \frac{a\tau + b}{c\tau + d} \right)$$

In particular,

$$T(h, u, \tau) = (h, u, \tau + 1)$$
$$S(h, u, \tau) = \left( \frac{h}{\tau}, u - \frac{(h, h)}{2\tau}, -\frac{1}{\tau} \right)$$

Also, for any $j \in \mathbb{C}$ we will define a right action of $\Gamma$ on functions on $Y$ as follows: if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then let

$$(f[\alpha])_j(h, u, \tau) = (c\tau + d)^{-j} f(\alpha(h, u, \tau)).$$

In fact, this is a projective action, which is related to the ambiguity in the choice of $(c\tau + d)^{-j}$ for non-integer $j$; to make it a true action one must consider a central extension of $SL_2(\mathbb{Z})$; we are not going into details here. We will call this action "an action of weight $j$".

Our main goal will be to find the behaviour of the (normalized) affine Jack’s polynomials under modular transformations. The first result in this direction is

**Theorem 9.1.** Fix $k, \kappa \in \mathbb{Z}_+$. Then the space of all solutions of the equation $\hat{L}f = 0$ in the space of theta-functions of level $\kappa$ on $Y$ (i.e., functions $f$ satisfying the conditions (a)–(c) of Proposition 8.3) is invariant under the action of $\Gamma$ of weight $j = \frac{r}{2} \left( 1 + \frac{k(k-1)h^\vee}{\kappa} \right)$.

**Proof.** It is easy to see that the operator $\hat{L}$ is invariant under the action of $T \in \Gamma$. Thus, to prove the theorem, it suffices to prove the following formula:

$$\hat{L}(f[S])_j = \tau^{-2}((\hat{L}g)[S])_j - \frac{1}{2\pi i \tau} (\kappa(r - 2j) + k(k - 1)rh^\vee) f[S]_j.$$ 

This is based on formula (8.5) for $\hat{L}$. Indeed, using modular properties of the theta-function (see, for example, [Mu]), we can show that $\varphi(z, \frac{1}{z}) = \tau^2 \varphi(x, \tau) + \frac{1}{2\pi i \tau}$.

Also, it is not too difficult to check that

$$\hat{\Delta}(f[S])_j = \tau^{-2}((\hat{\Delta}f)[S])_j - \frac{\kappa}{2\pi i \tau} (r - 2j)(f[S]_j),$$
where $\hat{\Delta} = 1/4\pi^2(-\sum \partial^2_{x_{\alpha}} + 2\partial_u \partial_r)$.

Since $\partial_u f = 2\pi i x f$ and $\sum_{\alpha \in R^+}(\alpha, \alpha) = rh^\vee$, we get the desired formula.

Next, we will need the following well-known fact (see [K]):

$$\hat{\delta}'[\alpha]_{r/2} = l(\alpha)\hat{\delta},$$

where the function (not a character) $l : \Gamma \to \mathbb{C}^\times$ is such that $l^{24} = 1$.

This gives us the following theorem:

**Theorem 9.2.** For a fixed $k \in \mathbb{Z}_+$, the linear span of the normalized Jack’s polynomials $J_{\lambda,K}, \lambda \in P^+_K$ is invariant under the action of $\Gamma$ of weight $j = -\frac{K(k-1)r}{2(K+kh^\vee)}$.

**Proof.** This is a corollary of Theorem 8.4, definition of $\hat{M}'$ and the previous theorem.

Thus, for every $K \in \mathbb{Z}_+$ we have a projective representation of $\Gamma$ in the finite-dimensional linear space $V_K = \bigoplus_{\lambda \in P^+_K} \mathbb{C} J_{\lambda,K}$; in fact, we have a family of representations of $\Gamma$ in the same space $V_K$, which are obtained for different values of $k$.

In general, these representations seem to be very interesting. First of all, note that it follows from the formula for the highest term of $J_{\lambda,K}$ that the eigenvalues of $T$ in such a representation are roots of unity of degree $N = M(K+kh^\vee)$, where $M$ is the smallest positive integer such that $(\lambda, \mu) \in \frac{1}{M}\mathbb{Z}$, $(\lambda, \lambda) \in \frac{2}{M}\mathbb{Z}$ for any $\lambda, \mu \in P$.

Thus, for fixed $K$ and large enough $k$ the order of $T$ tends to $\infty$. It is known that for ordinary characters, the representation of the modular group (considered as mapping $\Gamma \to PGL(V_K)$) is trivial on the congruence subgroup $\Gamma(N)$ and thus is in fact a representation of the finite group $\Gamma/\Gamma(N) \simeq SL_2(\mathbb{Z}/N\mathbb{Z})$. For generalized characters it is not so. The best we can say is the following trivial proposition:

**Proposition.** The representation of $\Gamma$ in $V_K$ is trivial on the normal subgroup $T(N)$, which by definition is the smallest normal subgroup containing $T^N$, $N = M(K+kh^\vee)$.

**Example.** Let $g = E_8$. Then for any fixed $K > 1$ coprime with $h^\vee = 30$ there is an infinite number of values of $k$ for which the image of $\Gamma$ in $PGL(V_K)$ is infinite.

**Proof.** This is based on the following theorem, due to Jordan (cf. [CR, §36]): for every fixed $n$ there is a constant $C(n)$ such that any finite subgroup in $PGL(n, \mathbb{C})$ has a commutative normal subgroup of index not exceeding $C(n)$. Now, fix $K$; take the constant $C = C(|P^+_K|)$ and choose $k$ such that all the prime factors of $N = K+kh^\vee$ are larger than $C$ (recall that $M = 1$ for $E_8$). Assume that the image of $\Gamma$ in $PGL(V_K)$ is finite; then it has a commutative normal subgroup $A$ of index $\leq C$. Since $T^N = 1$, the order of $T$ is relatively prime with the index of $A$. Thus, image of $T$ is in $A$. But the same is true for $T' = S^{-1}TS = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, since it is conjugate with $T$. On the other hand, $T$ and $T'$ generate $\Gamma$, and thus the image of $\Gamma$ is contained in $A$ and thus is commutative. But it is known that $|\Gamma/\Gamma| = 12$, and thus $\Gamma$ cannot have a commutative quotient of order greater than $12$. □

**Remark.** It seems plausible that for any $g$ and fixed sufficiently large $K$ the image is in fact infinite for all sufficiently large $k$. 

It is known that the quotient $\Gamma(N)/T(N) \simeq \pi_1(\Sigma_N)$, where $\Sigma_N$ is the modular curve: $\Sigma_N = \mathcal{H}/\Gamma(N)$. Thus representations of $\Gamma(N)/T(N)$ classify the flat connections in vector bundles over $\Sigma_N$, and representations of $\Gamma/T(N)$ classify the flat connections that are invariant with respect to the natural action of $SL_2(\mathbb{Z}/N\mathbb{Z})$ on $\Sigma_N$. It seems interesting to interpret our representation from this point of view.

Note that the usual characters from this point of view are trivial, so this phenomenon is specific for $k > 1$.

10. Unitarity and relation with conformal field theory.

This section is devoted to the discussion of the following conjecture.

**Conjecture 10.1.** For every $K, k \in \mathbb{Z}_+$ there exist positive real numbers $d_\lambda, \lambda \in P^+_K$ such that the above defined projective action of $\Gamma$ in the space $V_K$ is unitary with respect to the hermitian form in $V_K$ defined by $(\hat{J}_{\lambda,K}, \hat{J}_{\mu,K}) = d_\lambda \delta_{\lambda,\mu}$.

**Example.** For $k = 1$ this conjecture holds with $d_\lambda = 1$; the proof is based on the Weyl-Kac formula for the characters (see [K]).

**Corollary.** Conjecture 10.1 implies that the eigenvalues of the action in $V_K$ of any $x \in \Gamma$ lie have unit norm.

Note that this makes sense: though it is a projective representation, the corresponding cocycle takes values in the unit circle, and thus does not change the notion of unitarity.

This conjecture is motivated by the modular invariance in conformal field theory (CFT). We briefly outline the relation here; for detailed exposition, see, for example, [MS].

The Wess-Zumino-Witten model of conformal field theory is based on the integrable highest-weight representations of $\hat{g}$ of level $K$. The space of physical states in this theory is the Hilbert space

$$H = \bigoplus_{\lambda \in P^+_K} L_{\lambda,K} \otimes L^*_{\lambda*K},$$

where the involution $^*: P^+ \rightarrow P^+$ is defined by the condition that the finite-dimensional modules over $\mathfrak{g}$ are related by $V_{\lambda^*} = (V_\lambda)^*$.

The essence of the CFT is construction of the so-called amplitudes. We consider Riemann surfaces $\Sigma$ together with a finite number of marked points and local parameters at these points, divided into two subsets of “incoming” and “outgoing” points. To each marked point we associate a copy of the space $H$ and define the spaces $H_{in}, H_{out}$ which are just the tensor products of the spaces $H$ over all the incoming (resp., outgoing) points. Then we must construct the amplitudes, i.e., the operators $A(\Sigma): H_{in} \rightarrow H_{out}$, satisfying a number of axioms; the most important of them is the sewing axiom.

In the WZW model these amplitudes are defined as follows: first, we consider a slightly more general setting and consider Riemann surfaces with marked points (divided into “incoming” and “outgoing”), local parameters at these points and a choice of $\lambda \in P^+$ for every marked point. Then we construct a map

$$(10.1) \quad A_{\lambda_1,...,\mu_1,...} : L_{\lambda_1,K} \otimes L_{\lambda_2,K} \otimes \cdots \rightarrow W_{\lambda_1,...,\mu_1,...} \otimes L_{\mu_1,K} \otimes \cdots.$$
where the tensor products are taken over all incoming (resp., outgoing) points and $W$ is some finite-dimensional space, depending on $\Sigma, \lambda_i, \mu_j$, called the space of conformal blocks. Now we can construct the global amplitude $A$ as follows:

$$A = \bigoplus_{\lambda, \mu} (A_{\lambda_i, \mu_j}, A_{\lambda_i^*, \mu_j^*}),$$

and $(\cdot, \cdot)$ stands for a Hermitian form

$$W_{\lambda_i, \mu_j} \otimes W_{\lambda_i^*, \mu_j^*} \to \mathbb{C}.$$

Thus, to define the amplitudes we must define the spaces of conformal blocks along with the mappings (10.1) and pairing (10.3). The sewing axiom along with some other axioms of CFT says that all of these is defined uniquely as soon as it is defined for the sphere with three punctures at $0, z_0, \infty$ and local parameters $z, z - z_0, 1/z$. In this case, it is possible to define the space of conformal blocks explicitly. Namely, if $0$ is the incoming point with the weight $\lambda$ assigned to it and $\infty, z_0$ are outgoing points with the assigned weights $\mu, \nu$ respectively then one can define the corresponding space of conformal blocks as the space of all vertex operators, i.e., operators

$$\Phi : L_{\lambda, K} \to L_{\mu, K} \otimes L_{\nu, K}$$

satisfying certain commutation relations (cf. [MS,TK]). It is known that such an operator is uniquely defined by its restriction to the highest level (with respect to the $d$-grading) of the module $L_{\nu, K}$, and this restriction must be the intertwiner for $\hat{g}$ if we consider the highest level of $L_{\nu, K}$ as the evaluation representation of $\hat{g}$; thus, it is the same intertwining operator which we considered in Section 7. However, the question of defining the scalar product on the space of conformal blocks is much more subtle (cf. [FGK]). Nonetheless, it is generally believed that the following is true.

**Conjecture: consistency of WZW model.** *For a suitably defined inner product on the space of conformal blocks, the resulting conformal field theory is well-defined, i.e., the amplitudes (10.2) are uniquely determined by the complex structure on $\Sigma$ and do not depend on the choice of obtaining $\Sigma$ by gluing from three-punctured spheres, caps and cylinders.*

This is a very strong condition. It was shown in [MS] that it suffices to check consistency for the sphere and the torus. Moreover, for the sphere it is proved in the framework of vertex operator algebras. However, to the best of our knowledge, no satisfactory proof is known for consistency of WZW model on the torus; yet, some physical arguments (such as path integration) suggest that this is indeed true.

Returning to affine Jack’s polynomials, we can say that the unitarity conjecture 10.1 can be deduced from the conjecture on the consistency of the WZW model formulated above. Indeed, it can be easily checked that the unitarity of the action of $\Gamma$ is equivalent to modular invariance of the function

$$F(\tau) = \sum_{\lambda \in P_K^+} d_{\lambda} J_{\lambda, K}(0, 0, \tau) \overline{J_{\lambda, K}(0, 0, \tau)}.$$
This function is nothing but a certain component of the one-point correlation function on a torus for the Wess-Zumino-Witten conformal field theory, based on integrable representations of $\hat{\mathfrak{g}}$ of level $K+(k-1)h^\vee$, and thus, its modular invariance follows from consistency of the Wess-Zumino-Witten model which in particular implies that the correlation function does not depend on the choice of representation of the torus in the form $T = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, i.e., is modular invariant.

11. Quantum affine algebras and affine analogue of the Macdonald's polynomials. In this section we briefly outline how to define the generalized characters for quantum affine algebra. Let $U_p\hat{\mathfrak{sl}}_n$ be the quantum affine algebra, i.e., the quantization of the universal enveloping algebra of $\hat{\mathfrak{sl}}_n$ (see [Dr, J1, J2]). We use $p$ for the quantization parameter to avoid confusion with $q$ used in previous section for denoting the modular parameter of the torus. We can define for this algebra the notion of Verma module, irreducible highest-weight module etc. in the same manner as for usual affine algebra. As before, the modules $L_{\hat{\lambda}}$, $\hat{\lambda} \in \hat{P}^+$ are called integrable modules. Note that unlike the classical case, there is no natural action of (central extension of) $\hat{W}$ in integrable modules (though, of course, there is an action of $\hat{W}$ on the set of weights of $L_{\hat{\lambda}}$). Also, we can define the notion of evaluation representation $V(z)$, though it is much less obvious; it is based on the existence of the evaluation homomorphism $U_p\hat{\mathfrak{sl}}_n \to U_p\mathfrak{sl}_n$ (see [J2]).

Similar to Section 7, define intertwiners

$$\Phi^p_{\hat{\lambda}}: L_{\hat{\mu}} \to \hat{L}_{\hat{\mu}} \otimes V(z),$$

where $V$ is the deformation of representation of $\mathfrak{sl}_n$ in $S^{(k-1)n}\mathbb{C}^n$, and $\hat{\mu} = \hat{\lambda} + (k-1)\hat{\rho}, \hat{\lambda} \in \hat{P}^+$. Also, define the corresponding traces:

$$\chi_{\hat{\lambda}} = \sum_{\hat{\mu} \in \hat{P}} e^{\hat{\mu}} \text{Tr}_{L_{\hat{\mu}}[\hat{\nu}]} \Phi^p_{\hat{\lambda}}.$$

Most of the theory developed for these traces in Section 7 can be generalized to the quantum case with some changes. However, there are two major distinctions. First, these traces do not have any natural $\hat{W}$-symmetry; however, we never used the $\hat{W}$-symmetry of the traces in Section 7; to prove the symmetry of the ratio $\chi_{\hat{\lambda}}/\chi_0$ we only used $\hat{W}$-invariance of usual characters, which is still valid for quantum case. Another distinction is that we do not have any differential equation for these traces. In principle, one can say that the quantized traces satisfy some difference equation, which is an analogue of (7.3), but we do not know the explicit expression. However, we still have the following propositions.

**Theorem 11.1.**

$$\chi_0^p = f(p, q)e^{(k-1)\hat{\rho}} \prod_{j=1}^{k-1} \prod_{\hat{\alpha} \in \hat{R}^+} (1 - p^{2j} e^{-\hat{\alpha}}),$$

where $f$ is some formal power series in $q = e^{-\delta}$ whose coefficients are rational functions in $p$ and highest term is 1.

Proof of this theorem is quite similar to that of Theorem 7.2 and can be obtained by conjunction of arguments in the proofs of Theorem 7.2 and Proposition 4.1 in [EK2]; however, in the quantum case we do not have the differential equation for traces, and thus are unable to determine the factor $f(p, q)$. 
Theorem 11.2. The ratio $\hat{J}_\lambda^p = \chi_\lambda^p / \chi_0^p$ is a symmetric Laurent polynomial: $\hat{J}_\lambda^p \in A^\hat{W}$, and has highest term $e^\lambda$.

Again, the proof is a literal repetition of the proof of Theorem 7.3.

Definition. The polynomials $\hat{J}_\lambda^p = \chi_\lambda^p / \chi_0^p$ are called affine Macdonald’s polynomials.

This definition is motivated by the fact that the same construction for the finite-dimensional case (i.e., for representations of $U_p\mathfrak{g}$) gives the usual Macdonald’s polynomials. We believe that the above definition gives the right affine analogue of Macdonald’s polynomials. However, at this moment we are unable to define them as orthogonal with respect to a certain inner product or as eigenfunctions of an explicitly presented difference operator.

Note that among the identities satisfied by Macdonald’s polynomials (see [M2]) there is one that can be easily generalized to the affine case: this is the so-called special value identity, which for simply-laced case looks as follows: if one defines the evaluation map $\pi : e^\lambda \to p^{2(\lambda,k\rho)}$ then

$$\pi(P_\lambda) = p^{-2(\lambda,k\rho)} \prod_{\alpha \in R^+} \prod_{i=0}^{k-1} \frac{1 - p^{2(\alpha,\lambda+k\rho)+2i}}{1 - p^{2(\alpha,k\rho)+2i}}.$$

This leads us to the following conjecture:

Conjecture 11.3. The affine Macdonald’s polynomials $\hat{J}_\lambda^p$ satisfy the following identity:

$$\pi(\hat{J}_\lambda^p) = p^{-2(\hat{\lambda},k\hat{\rho})} \prod_{\hat{\alpha} \in R^+} \prod_{i=0}^{k-1} \frac{1 - p^{2(\hat{\alpha},\hat{\lambda}+k\hat{\rho})+2i}}{1 - p^{2(\hat{\alpha},k\hat{\rho})+2i}},$$

where, as before, the evaluation $\pi$ is defined by $\pi(e^{\hat{\lambda}}) = p^{2(\hat{\lambda},k\hat{\rho})}$.

Example. For $k = 1$ this formula can be easily proved using the Weyl-Kac character formula and the denominator identity for affine Lie algebras.

Finally, we briefly mention how to generalize the construction of the affine Macdonald’s polynomials to generic $k$ (not necessarily positive integer). This is done quite similar to the finite-dimensional case (see [EK2]). First of all, it is easy to show that the affine Macdonald’s polynomials have the form $P = \sum_\lambda f_\lambda(p,p^k)e^{\hat{\lambda}}$, where $f_\lambda(p,t)$ is a rational function of two variables; it does not depend on $k$. Thus, it is very easy to continue such a polynomial to arbitrary value of $k$ (provided that $f$ does not have a pole at this point). It is convenient to consider $k$ as a formal variable, i.e., consider $p$ and $p^k$ as algebraically independent; we will use this convention in the remaining part of the paper.

Consider the intertwining operator

$$\bar{\Phi}_\lambda : M_{\lambda+(k-1)\hat{\rho}} \to M_{\hat{\lambda}+(k-1)\hat{\rho}} \otimes U_k,$$

where $M_{\lambda+(k-1)\hat{\rho}}$ is Verma module and $U_k$ is the evaluation representation of $U_p\mathfrak{sl}_n$ obtained from the standard action of $U_p\mathfrak{sl}_n$ in the functions of the form
$(x_1 \ldots x_n)^{k-1} p(x)$, $p(x)$ being a Laurent polynomial in $x_i$ of total degree zero (see [EK2]). Since $k$ is a formal variable, $M_{\lambda+(k-1)\rho}$ is irreducible and thus such an intertwiner exists and is unique up to a constant. Similarly to Section 7, define the traces

\begin{equation}
\tilde{\chi}_\lambda = \sum_{\mu} e^{\mu} Tr M_{\lambda+(k-1)\rho}[\mu] (\tilde{\Phi}_\lambda)
\end{equation}

**Theorem 11.4.**

1. The ratio $\tilde{\chi}_\lambda / \tilde{\chi}_0$ is the affine Macdonald’s polynomial $\hat{J}_\lambda^p$.
2. \[\tilde{\chi}_0 = f(p,q) e^{(k-1)\hat{\rho}} \prod_{\alpha \in \hat{R}^+} \prod_{i=1}^{\infty} \frac{1 - p^{2i} e^{\alpha}}{1 - p^{2(k-1)+2i} e^{\alpha}}.\]

**Proof.** The proof is quite similar to the finite-dimensional case (see [EK2, Section 5]) and is based on the fact that for fixed $\lambda, \mu$ the weight subspace $L_{\lambda+(k-1)\rho}[\lambda+(k-1)\rho-\mu]$ (here $k$ is a positive integer) is for $k$ large enough isomorphic to the weight subspace $M_{\lambda+(k-1)\rho}[\lambda+(k-1)\rho-\mu]$, where $k$ is a formal variable, and the restriction to this subspace of the intertwiner $\Phi$, defined in Section 7 for $k \in \mathbb{Z}^+$, coincides for $k$ large enough with the specialization of the intertwining operator $\tilde{\Phi}$ defined above for the case when $k$ is a formal variable. On the other hand, if we have two rational functions of two variables $f(p,t), g(p,t)$ such that $f(p,p^k) = g(p,p^k)$ for all integer $k >> 0$ then $f = g$ as functions of two variables. We refer the reader to [EK2] for the details.

**12. Summary.**

Let us summarize what has been done so far. We defined the affine analogue of Jack’s and Macdonald’s polynomials (for the root system $\hat{A}_{n-1}$); we also checked that the affine Jack’s polynomials are eigenfunctions of the affine analogue of Sutherland operator. If we now turn back to the finite-dimensional case, we see that in the affine case there are still missing pieces of structure. First of all, we have not defined the inner product which would make these polynomials orthogonal. This is a deep problem; the trivial generalization of the finite-dimensional inner product is not convergent. However, there are good reasons to believe that one can regularize this inner product so to make sense of it. Indeed, if we write the affine analogue of Macdonald’s inner product identities, which give an explicit expression for $(P_\lambda, P_\lambda)_k$ in terms of the root system, then the product in the right-hand side has a pole as $p \to 1$ (recall that $p$ is the parameter of the quantization). However, for fixed $K, k$ the order of the pole is the same for all $\lambda$. This suggests that one can renormalize the trivial definition of the inner product (for a fixed level) and then prove the affine version of Macdonald’s inner product identities. In this case the right-hand side of these identities is expressed in terms of $\Gamma$-function at rational points. Moreover, there are certain arguments suggesting that this renormalized inner product should coincide with the inner product discussed in Section 10, i.e., the inner product known in CFT. Remarkably, in those few cases when the inner product on the space of conformal blocks of CFT is written down explicitly (see [MS, Appendix D]), the answer is also written as a ratio of gamma-functions of rational argument. This will be discussed in our future papers.
Finally, in the finite-dimensional case we had a large family of commuting differential (for \( q \neq 1 \), difference) operators. In the affine case similar operators can be defined (see \([E1]\)). However, these operators are not difference operators but rather infinite sums of difference operators.

If we pass to the critical level \((K + kh^{\vee} = 0)\) then the (local completion of) the universal enveloping algebra has a center. In this case the term containing \( q^{\partial \over \partial q} \) in the expression for the affine Sutherland operator drops, and we get the elliptic Calogero-Sutherland operator, which can be included in a family of commuting differential operators (see \([OP, Ch]\)), which can be obtained from the elements of the center (see \([E]\)). This suggests that the asymptotics of affine Jack’s polynomials in the limit \( K \) fixed, \( K + kh^{\vee} \rightarrow 0 \) should be very interesting. These asymptotics can be calculated by writing explicit integral formulas for the traces of intertwiners (similar to \([BF, EK3]\)) and finding the asymptotics by the saddle-point method. For the case \( g = sl_2 \) it was done in \([EK3]\); general case will be discussed in our future papers.

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