Simple quantum model for light depolarization

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Depolarization of quantum fields is handled through a master equation of the Lindblad type. The specific feature of the model is that it couples dispersive to a randomly distributed atomic bath. The depolarizing dynamics emerging from this approach is analyzed for relevant states.

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I. INTRODUCTION

Polarization of light is a fascinating concept that has deserved a lot of attention over the years. Today, the field is witnessing a renewed interest because of the fast developments in optics, both on the applications and on the more fundamental aspects. Moreover, the polarization of a field is a robust characteristic, which is relatively simple to manipulate without inducing more than marginal losses. Therefore, it is not surprising that many experiments at the forefront of quantum optics use polarization states [1, 2, 3, 4].

The Stokes parameters provide, perhaps, the most convenient description of polarization in classical optics and can also be extended in a natural way to the quantum domain [5, 6]. They have the drawback in that they measure only second-order correlations of the field amplitudes: while this may suffice for classical problems, higher-order correlations are important for the realm of quantum optics. This is the basic reason why the Stokes parameters do not distinguish between quantum states having quite different polarization properties [7, 8]. Despite these subtleties, the Stokes parameters still provide the most commonly used definition of a degree of polarization [9, 10, 11, 12], as we discuss in Sec. II.

The degree of polarization is often desired to reach its maximum value, as well for classical as for quantum communication. The term depolarization has come to mean the decrease of the degree of polarization of a light beam when traversing an optical system. Our intuition strongly suggests that depolarization is due to decoherence. Roughly speaking, two basic mechanisms have been proposed to explain classically this phenomenon [13]: decorrelation of the phases of the electric field vector and selective absorption of polarization states. The first cause is far more interesting to examine when the microscopic mechanisms responsible of depolarization are analyzed.

To deal with this decoherence, some ingenious and practical tricks have been proposed, such as coupling each mode to a beam splitter where it is mixed with a vacuum [14]. We wish to look at this problem from a more fundamental point of view. Of course, one can find in the literature a number of pure dephasing models in which the environment is represented as a scattering process, which does not lead to a change of populations [15, 16, 17, 18, 19]. For many applications, a good knowledge of dephasing is of utmost importance. This holds most prominently for quantum information processing, where the operation completely relies on the presence of coherence. It is hardly surprising that a lot of attention has been paid to dephasing in systems such as quantum dots [20, 21, 22, 23], Josephson junctions [24, 25, 26], or general quantum registers [27], to cite only a few relevant examples.

However, the study of depolarization is not only a question of pure dephasing dynamics. Polarization has an additional su(2) invariance that leads to a natural structure of invariant subspaces. On physical grounds, we argue that this structure must be preserved in the evolution, which makes previous models fail in this case. The main goal of this paper is precisely to provide a simple approach that goes around this drawback and provides a picture of the mechanisms involved in the depolarization processes. Since in classical optics the randomization is produced as light propagates in a material medium, our main idea is to couple the field modes dispersively to a randomly distributed atomic bath: the resulting master equation has a quite appealing structure that is examined in some detail in Sec. III for the case of single-mode fields. Finally, in Sec. IV we extend the formalism to multimode fields, while our conclusions are summarized in Sec. V.

II. POLARIZATION AND SU(2) INvariance

In the interest of retaining as much clarity as possible, we first recall some well-known facts about the polarization structure of quantum fields. Since the formalism applies to fields of arbitrary wavefronts and frequencies, we consider a free transverse electromagnetic field with $m$ spatiotemporal modes, whose positive-frequency part
is

$$E^+(r, t) = i \sum_{j=1}^{m} \sum_{s=\pm} \sqrt{\frac{\hbar \omega_j}{2\epsilon_0}} a_{js} \mathbf{u}_{js}(r) \exp(-i\omega_j t).$$

(2.1)

In the Dirac quantization scheme the modes are plane waves

$$\mathbf{u}_{js}(r) = \frac{1}{\sqrt{V}} \mathbf{e}_{js} \exp(i\mathbf{k}_j \cdot \mathbf{r})$$

(2.2)

defined in some large volume V (which may be taken to be infinity later). For definiteness, we shall work in the base vectors $\mathbf{e}_{js}$ corresponding to circular polarizations (and $s$ takes the values $\pm$). The annihilation and creation operators $a_{js}$ and $a_{js}^\dagger$ obey the familiar commutation relations

$$[a_{js}, a_{j's'}^\dagger] = \delta_{jj'} \delta_{ss'}. \quad (2.3)$$

As pointed out by Karassiov [28], there are specific observables that characterize proper polarization properties of fields of the form (2.1). They correspond to the generators of the group SU(2) of polarization gauge invariance. In the circular polarization basis they can be expressed as

$$J_+ = \sum_{j=1}^{m} a_{j+}^\dagger a_{j-}, \quad J_- = \sum_{j=1}^{m} a_{j-}^\dagger a_{j+},$$

$$J_z = \frac{1}{2} \sum_{j=1}^{m} (a_{j+}^\dagger a_{j+} - a_{j-}^\dagger a_{j-}).$$

(2.4)

They indeed furnish a Schwinger representation of the su(2) algebra:

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \quad (2.5)$$

The Casimir operator is

$$J^2 = J_+^2 + \frac{1}{2} (J_+ J_- + J_- J_+) = \frac{N}{2} \left( \frac{N}{2} + \mathbb{1} \right). \quad (2.6)$$

The operator

$$N = \sum_{j=1}^{m} (a_{j+}^\dagger a_{j+} + a_{j-}^\dagger a_{j-})$$

(2.7)

represents the total number of photons and satisfies

$$[N, J] = 0, \quad (2.8)$$

where $J = (J_x, J_y, J_z)$, with $J_\pm = J_x \pm iJ_y$. The total Hilbert space splits in this way in invariant subspaces of dimension $N + 1$. Since there is no risk of confusion, we denote by the same letter the operator [24] and its eigenvalue, which is the total number of photons.

The quantities $J$ are then measurable in photon-counting experiments [24]. In fact, for a single-mode field they coincide up to a factor $1/2$ with the Stokes operators [30]. It is thus natural to extend this identification and define

$$J = \frac{1}{2} \mathbf{S}$$

(2.9)

for $m$-mode fields. This allows one to parallel the classical idea and define the degree of polarization as

$$P = \frac{\sqrt{\langle \mathbf{S}^2 \rangle}}{\langle N \rangle}. \quad (2.10)$$

Note that $s = \langle \mathbf{S} \rangle / \langle N \rangle$ is the polarization vector in classical optics, which defines the Poincaré sphere. Equation (2.10) can then be interpreted as the distance from the point represented by $s$ to the origin, which is associated with the unpolarized light.

To highlight the SU(2) invariance of polarization, it has been proposed by several authors [31, 32, 33, 34, 35] to define unpolarized light as the field states that remain invariant under any linear polarization transformation, which in experimental terms can be accomplished with a combination of phase plates and rotators (that produce rotations of the electric field components around the propagation axis). Any state satisfying the invariance condition will also fulfill the classical definition of an unpolarized state, but the converse is not true.

The density operator of such quantum unpolarized states can be written as

$$\rho_{unpol} = \bigoplus_N r_N \mathbb{1}_N,$$

(2.11)

where $\mathbb{1}_N$ is the identity operator in the invariant subspace with $N$ photons and all $r_N$ are real and positive constants. The condition of unit trace imposes

$$\sum_N (N + 1) r_N = 1. \quad (2.12)$$

The vacuum state is the only pure quantum state that is unpolarized, and the unpolarized mixed states are totally mixed in each subspace. Note that in each invariant subspace, $\rho_{unpol}$ is fully random: we claim that in any physical depolarization process this fundamental structure of invariant subspaces must be preserved.

III. LIGHT DEPOLARIZATION: SINGLE-MODE FIELDS

A. Master equation for pure dephasing processes

We begin by focusing our attention on a single-mode field coupled to a bath system. The total Hamiltonian is

$$H = H_{\text{field}} + H_{\text{bath}} + H_{\text{int}}.$$  

(3.1)

Here $H_{\text{field}}$ is the Hamiltonian of the mode under consideration

$$H_{\text{field}} = \hbar \omega \sum_{s=\pm} a_s^\dagger a_s,$$  

(3.2)
while $H_{\text{bath}}$ describes the free evolution of the environment. We do not make any hypothesis on the precise kind of bath, and only assume that it is so large that its statistical properties are unaffected by the coupling with the system.

In the interaction picture and under the usual weak-coupling and Markov approximations, the master equation takes the general form [17]

$$\dot{\varrho}(t) = -\frac{i}{\hbar} H_{\text{int}}(t) \varrho(t) + \frac{1}{\hbar^2} \int_0^\infty dt \left[ [H_{\text{int}}(t), [H_{\text{int}}(t-\tau), \varrho(t) \otimes \varrho_{\text{B}}]] \right],$$

where $\varrho_{\text{B}}$ is the partial trace over the bath variables and $\varrho_{\text{int}}(t) = \varrho(t) \otimes \varrho_{\text{B}}$ is the density operator for the total system. In addition, we take the mode and the bath initially independent. Henceforth, $\varrho(t)$ will denote the reduced density operator for the field.

A common way to couple the system to the bath is through the basic interaction Hamiltonian

$$H_{\text{int}} = \hbar \sum_{\lambda} \sum_{s=\pm} (\kappa_\lambda \Gamma_\lambda a_s^\dagger a_s + \kappa_\lambda^* \Gamma_\lambda^\dagger a_s^\dagger a_s),$$

where $\kappa_\lambda$ are coupling constants, $\Gamma_\lambda$ and $\Gamma_\lambda^\dagger$ are annihilation and creation operators for bath quanta and the sum over $\lambda$ runs over all the accessible bath modes. Note that a photon can lose energy by creating a bath quantum, and conversely.

We assume for simplicity that the bath is at zero temperature, which is tantamount to neglecting stimulated processes. The master equation (3.3) is then

$$\dot{\varrho} = \sum_s \frac{\gamma_s}{2} \mathcal{L}[a_s^\dagger a_s] \varrho,$$

where $\mathcal{L}[C_s]$ are the Lindblad superoperators [32]

$$\mathcal{L}[C_s] \varrho = 2 C_s \varrho C_s^\dagger - \{C_s^\dagger C_s, \varrho\},$$

and $\gamma_s$ are the decoherence rates associated with each polarization. As was recognized earlier [37, 38], this form of the master equation is completely positive, which is the only dynamical evolution of an open system ensuring that the state of the system remains physically valid at all times.

Equation (3.3) corresponds to two standard independent Lindblad decaying processes, one for each basic polarization. Since the coupling (3.4) transfers energy between the system and the bath, the net result is a damping of the beam intensity. In fact, one can check that, irrespective of the initial conditions, the stationary state is always $\varrho(t \to \infty) = |0, 0\rangle \langle 0, 0|$, where $|0, 0\rangle = |0\rangle_+ |0\rangle_- \varrho$ is the vacuum state for both circular polarizations. Obviously, this cannot describe depolarizing processes.

As we have anticipated in the Introduction, depolarization implies a pure dephasing process. In mathematical terms, this means that the interaction must commute with the system Hamiltonian, so no energy is transferred and only the phase changes. Various theoretical scenarios have been proposed to that end. Apart from minor details, all of them can be modeled by an interaction Hamiltonian of the type [17]

$$H_{\text{int}} = \sum_\lambda \sum_{s=\pm} (\kappa_\lambda \Gamma_\lambda a_s^\dagger a_s + \kappa_\lambda^* \Gamma_\lambda^\dagger a_s^\dagger a_s),$$

which can be viewed as a scattering process in which a bath quantum can be absorbed or emitted, but the number of photons in each polarization is preserved.

Assuming again a zero-temperature bath, the master equation for (3.7) can be written as

$$\dot{\varrho} = \sum_s \frac{\gamma_s}{2} \mathcal{L}[a_s^\dagger a_s] \varrho.$$

To illustrate the physics embodied in this equation, let us focus on the fully quantum regime and consider a one-photon state. We are then within a two-dimensional invariant subspace, whose basis will be labeled

$$|+\rangle = |1\rangle_+ |0\rangle_-, \quad |-\rangle = |0\rangle_+ |1\rangle_- .$$

The density matrix in this two-dimensional subspace can be expressed as

$$\varrho = \frac{1}{2} \left( \begin{array}{cc} 1 + s_z & s_x - is_y \\ s_x + is_y & 1 - s_z \end{array} \right) = \frac{1}{2} (\mathbb{1} + s \cdot \sigma),$$

where $s$ is the polarization vector defined after Eq. (2.10) and can be written as $s = 2 \text{Tr}(\mathcal{O})$, $\sigma$ being the Pauli matrices. In this subspace the model (3.8) can be easily recast in terms of $s$; the final solution reads as

$$s_x(t) = s_x(0) e^{-i(\gamma_+ + \gamma_-) t/2},$$
$$s_y(t) = s_y(0) e^{-i(\gamma_+ + \gamma_-) t/2},$$
$$s_z(t) = s_z(0).$$

This interaction indeed preserves the invariant subspaces, but produces no thermalization: its effect is merely to maintain the occupation probabilities, while erasing all coherences. Although this can model appropriately many interesting phenomena, it fails to describe depolarizing effects, because depolarization leads to a uniform distribution of photons in each invariant subspace and not only to a pure dephasing. In other words, depolarization not only preserves the invariant subspaces, but the steady state in each one of them must be a diagonal state. We believe that these conditions are essential to ensure a correct description of depolarization.

B. Modeling depolarization with a nonresonant randomly-distributed atomic medium

Apart from the previous drawbacks, generic models of the type (3.7) suffer from the problem that one cannot provide a physically feasible mechanism leading to an interaction of that form. To solve these difficulties we shall take another route. We recall that in classical optics, the
electric field of the light experiences a decorrelation as it propagates through a material medium. If $\alpha_{\pm}$ denote the classical field-mode amplitudes, such a decorrelation can be taken into account by assuming that $\alpha_{\pm} \to \alpha_{\pm} e^{i \phi_{\pm}}$, where $\phi_{\pm}$ are random phases.

We wish to explore this picture in the quantum world. In consequence, we assume that the field propagates through a material medium represented by a collection of two-level atoms. Since there is no net energy transfer between the field and the atoms, their interaction must be necessarily dispersive. In short, our basic system is

$$H_{\text{sys}} = H_{\text{field}} + H_{\text{at}} + V, \quad (3.12)$$

where

$$H_{\text{at}} = \frac{1}{2} \sum_{\lambda} \hbar \omega_{\lambda} \sigma_{\lambda}^z,$$  \quad (3.13)

$$V = \hbar \sum_{\lambda} \sum_{s = \pm} (g_{\lambda s} \sigma_{\lambda}^- a_s^\dagger + g_{\lambda s}^* \sigma_{\lambda}^+ a_s).$$  \quad (3.14)

The form of the interaction $V$ assumes, as it is usually done, that the atoms interact with the field mode in the dipolar and rotating-wave approximations. In addition, the atoms are randomly distributed so the coupling constants $g_{\lambda s}$ have random phases.

If $\Delta_{\lambda} = \omega_{\lambda} - \omega$ is the detuning, we must consider the far-off-resonant regime $|g_{\lambda s}| \ll \Delta_{\lambda}$. In such a limit, we can adiabatically eliminate the nonresonant interactions in Eq. (3.13) and obtain the following effective Hamiltonian that describes our system:

$$H_{\text{sys}} \simeq \hbar \omega N + \sum_{\lambda} \frac{1}{2} \hbar \Omega_{\lambda} \sigma_{\lambda}^z$$

$$+ \sum_{\lambda, \lambda'} \sum_{s = \pm} \frac{1}{2} g_{\lambda s} g_{\lambda' s}^* \left( \frac{1}{\Delta_{\lambda}} + \frac{1}{\Delta_{\lambda'}} \right) \sigma_{\lambda}^z \sigma_{\lambda'}^z, \quad (3.15)$$

where

$$N = \sum_{s = \pm} a_s^\dagger a_s + \sum_{\lambda} \frac{1}{2} \sigma_{\lambda}^z.$$

is the conserved excitation number operator and

$$\Omega_{\lambda} = \Delta_{\lambda} + \delta_{\lambda} + \frac{\mathcal{J}_\lambda}{\Delta_{\lambda}},$$  \quad (3.16)

$$\mathcal{J}_\lambda = 2|g_{\lambda}|^2 (N + J_+ e^{i \varphi_{\lambda}} + J_- e^{-i \varphi_{\lambda}}).$$

Here we have written $g_{\lambda \pm} = |g_{\lambda}| e^{\pm i \varphi_{\lambda}}/2$, with $\varphi_{\lambda}$ being random phases. This reflects the fact that the relative phase between the atomic dipole and the field mode is random. The parameter

$$\delta_{\lambda} = \frac{|g_{\lambda}|^2}{\Delta_{\lambda}}$$  \quad (3.17)

represents just a small frequency shift.

Moreover, it is well known that the atoms decay irresversibly. This usually assigned to their interaction with the continuum of modes of an additional thermal electromagnetic environment. In such a case, the density matrix for the system [31,32] evolves according to [17]

$$\dot{\rho}_{\text{sys}}(t) = -i \frac{\hbar}{\gamma} [H_{\text{sys}}, \rho_{\text{sys}}]$$

$$+ \sum_{\lambda} \frac{\gamma_{\lambda}}{2} \left( (\bar{n}_{\lambda} + 1) \mathcal{L}[\sigma_{\lambda}^-] \rho_{\text{sys}} + \bar{n}_{\lambda} \mathcal{L}[\sigma_{\lambda}^+] \rho_{\text{sys}} \right), \quad (3.18)$$

where $\gamma_{\lambda}$ is the decay constant of the $\lambda$th atom due to its coupling to the thermal environment with $\bar{n}_{\lambda}$ excitations. As is well established [31,32,33], the properties of a random medium are well reproduced in the high-temperature limit: $\bar{n}_{\lambda} \gg 1$. In this limit, the effect of spontaneous emission can be disregarded in comparison with the stimulated emission processes. Emission into the reservoir and absorption from the reservoir therefore become identical; i.e., they balance each other in the stationary state: the emission and absorption processes depend solely on the initial population of the atomic state. Consequently, the steady-state reduced density operator is approximately given by a mixture of equally populated atomic states. The density matrix of the $\lambda$th atom then reads as

$$\rho_{\text{sys}} = \frac{1}{N + 1} \mathds{1},$$

where $\mathds{1}$ is the identity matrix.

As indicated in the Appendix, in this limit we can also adiabatically eliminate the atomic variables and obtain a master equation that, after averaging over the random phases, reads as

$$\dot{\rho} = -i [\omega N, \rho] + \frac{\gamma}{2} \mathcal{L}[N] \rho + \frac{\gamma}{2} \mathcal{L}[J_+] \rho + \frac{\gamma}{2} \mathcal{L}[J_-] \rho, \quad (3.20)$$

where $\rho(t) = \text{Tr}_{\text{at}}[\rho_{\text{sys}}(t)]$ is again the reduced density operator for the field mode and the decoherence rate $\gamma$ is

$$\gamma = 4 \sum_{\lambda} \frac{|g_{\lambda}|^4}{\gamma_{\lambda} \Delta_{\lambda}^3 \bar{n}_{\lambda}}. \quad (3.21)$$

Equation (3.20) is our central result. We observe that it preserves the $\text{SU}(2)$ invariant subspaces and the steady state in each $N$-photon subspaces is a completely random state

$$\rho(t \to \infty) = \frac{1}{N + 1} \mathds{1}. \quad (3.22)$$

This can be considered as the major advantage of our approach.

A rough estimate immediately shows that the depolarization rates (3.21) appearing in this model are very
small when compared with other typical system parameters, which is in agreement with the experimental observations.

The terms $\mathcal{L}[J_z] \rho$ describe depolarization in each invariant subspace, meanwhile the action of $\mathcal{L}[N] \rho$ therein is trivial. Nevertheless, this $\mathcal{L}[N] \rho$ is responsible for the relative phase decay between blocks of the density matrix corresponding to different excitation numbers.

To compare with the previous discussion in Eq. (3.11), we consider again the one-photon case. By recasting the depolarization master equation (3.20) in terms of $s$, one obtains the solution as

$$s_x(t) = s_x(0)e^{-\gamma t},$$

$$s_y(t) = s_y(0)e^{-\gamma t},$$

$$s_z(t) = s_z(0)e^{-2\gamma t},$$

which displays the aforementioned desirable properties.

The degree of polarization (2.10) of this mode evolves then as

$$P(t) = [s_x^2(0) + s_y^2(0) + s_z^2(0)e^{-2\gamma t}]^{1/2}e^{-\gamma t}.$$  (3.24)

IV. LIGHT DEPOLARIZATION: MULTIMODE FIELDS

For the multimode case, it seems reasonable to assume that the depolarization is independent in each mode. The corresponding master equation can be thus simply written as

$$\dot{\rho} = -i \sum_{j=1}^{m} \omega_j [N_j, \rho] + \sum_{j=1}^{m} \frac{\gamma_j}{2} \{ \mathcal{L}[N_j] \rho + \mathcal{L}[J_{j+}] \rho + \mathcal{L}[J_{j-}] \rho \}. \quad (4.1)$$

One can check that the properties discussed in the previous section also hold in this multimode case.

For concreteness, henceforth we restrict our attention to the case of a two-mode field. We have then a representation of SU(2) $\otimes$ SU(2), and the whole Hilbert space can be decomposed into irreducible subspaces. For the case where there is one photon in each mode, we have, using the standard terminology of SU(2) representations, that $D^{1/2} \otimes D^{1/2} = D^0 \oplus D^1$, where $D^1$ is the invariant subspace with eigenvalue $j$.

The one-dimensional subspace $D^0$ is spanned by the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |+\rangle_1 |+\rangle_2), \quad (4.2)$$

while $D^1$ is spanned by the triplet states

$$|\psi^\uparrow\rangle = \frac{1}{\sqrt{2}}(|+_\rangle_1 |-_\rangle_2 + |-_\rangle_1 |+_\rangle_2), \quad |+_\rangle_1 |+_\rangle_2, \quad |-_\rangle_1 |-_\rangle_2. \quad (4.3)$$

Here the states $|\pm\rangle$ for each mode are given as in (3.3) and the subscripts 1 and 2 label the corresponding modes.

We take our system to be initially in a generic density matrix such as

$$\rho(0) = \left( \begin{array}{cccc}
\rho_{11}(0) & \rho_{12}(0) & \rho_{13}(0) & \rho_{14}(0) \\
\rho_{21}(0) & \rho_{22}(0) & \rho_{23}(0) & \rho_{24}(0) \\
\rho_{31}(0) & \rho_{32}(0) & \rho_{33}(0) & \rho_{34}(0) \\
\rho_{41}(0) & \rho_{42}(0) & \rho_{43}(0) & \rho_{44}(0)
\end{array} \right), \quad (4.4)$$

where we have employed the standard eigenbasis

$$|1\rangle = |+_\rangle_1 |+_\rangle_2, \quad |2\rangle = |+_\rangle_1 |-_\rangle_2, \quad |3\rangle = |-_\rangle_1 |+_\rangle_2, \quad |4\rangle = |-_\rangle_1 |-_\rangle_2. \quad (4.5)$$

After some computation, the solution of the master equation (4.1) can be shown to be

$$\rho_{21}(t) = \frac{1}{2} \left[ \rho_{21}(0)(1 + e^{-2\gamma_1 t}) + \rho_{23}(0)(1 - e^{-2\gamma_1 t})e^{-\gamma_2 t} \right],$$

$$\rho_{31}(t) = \frac{1}{2} \left[ \rho_{31}(0)(1 + e^{-2\gamma_1 t}) + \rho_{32}(0)(1 - e^{-2\gamma_1 t})e^{-\gamma_2 t} \right],$$

$$\rho_{41}(t) = \rho_{41}(0)e^{-\gamma_1 t},$$

$$\rho_{32}(t) = \rho_{32}(0)e^{-\gamma_1 t},$$

$$\rho_{42}(t) = \frac{1}{2} \left[ \rho_{42}(0)(1 + e^{-2\gamma_1 t}) + \rho_{43}(0)(1 - e^{-2\gamma_1 t})e^{-\gamma_2 t} \right],$$

$$\rho_{43}(t) = \frac{1}{2} \left[ \rho_{43}(0)(1 + e^{-2\gamma_1 t}) + \rho_{41}(0)(1 - e^{-2\gamma_1 t})e^{-\gamma_2 t} \right]. \quad (4.6)$$

for the nondiagonal elements, and

$$\rho_{11}(t) = \frac{1}{4} \left( 1 + \left[ \left| \rho_{11}(0) + \rho_{22}(0) \right| - 1 \right] e^{-2\gamma_1 t} \right. \nonumber$$

$$\left. + \left[ \left| \rho_{11}(0) + \rho_{33}(0) \right| - 1 \right] e^{-2\gamma_2 t} \right),$$

$$\rho_{22}(t) = \frac{1}{4} \left( 1 + \left[ \left| \rho_{22}(0) + \rho_{11}(0) \right| - 1 \right] e^{-2\gamma_1 t} \right. \nonumber$$

$$\left. + \left[ \left| \rho_{22}(0) + \rho_{44}(0) \right| - 1 \right] e^{-2\gamma_2 t} \right),$$

$$\rho_{33}(t) = \frac{1}{4} \left( 1 + \left[ \left| \rho_{33}(0) + \rho_{44}(0) \right| - 1 \right] e^{-2\gamma_1 t} \right. \nonumber$$

$$\left. + \left[ \left| \rho_{33}(0) + \rho_{11}(0) \right| - 1 \right] e^{-2\gamma_2 t} \right), \quad (4.7)$$

for the diagonal ones. Here $\gamma_1$ and $\gamma_2$ are the decoherence rates for modes 1 and 2, respectively. Note that the trace condition gives $\rho_{44} = 1 - \rho_{11} - \rho_{22} - \rho_{33}$.

In this basis, the operators $J$ defined in Eq. (2.14), take the explicit form

$$J_x = \frac{1}{2} \left( \begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array} \right), \quad J_y = \frac{1}{2} \left( \begin{array}{cccc}
0 & -i & -i & 0 \\
i & 0 & 0 & -i \\
i & 0 & 0 & -i \\
0 & i & i & 0
\end{array} \right),$$
pure disentangled state such as any element of the basis (2.10) is now

\[ P(t) = \frac{1}{2} (e^{-2\gamma_1 t} + e^{-2\gamma_2 t}). \]  

(4.9)

This is a quite compact result. Consider, for example, a pure disentangled state such as any element of the basis (2.10), say \( \phi(0) = |1\rangle|1\rangle \). Using (4.10) and (4.11) we immediately get

\[ P(t) = \frac{1}{2} (e^{-2\gamma_1 t} + e^{-2\gamma_2 t}). \]  

(4.10)

On the other hand, for the maximally entangled Bell-like states (4.11)

\[ |\psi^\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

we have

\[ P(t) = \frac{1}{2} e^{-(\gamma_1 + \gamma_2) t}. \]  

(4.12)

In consequence, the degree of polarization (8.10) is now given by

\[ J_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]  

V. CONCLUDING REMARKS

What we expect to have accomplished in this paper is to present a comprehensive theory of the depolarization of quantum fields. In our model the field modes couple dispersively to a randomly distributed atomic bath: the resulting master equation has unique properties that we have explored in detail.

We hope that the tools introduced here could be of interest in treating depolarization in the fully quantum regime, especially for biphotons, which have strong implications in areas of futuristic technologies such as quantum computing, quantum cryptography, and quantum communications.

APPENDIX A: DERIVATION OF THE MASTER EQUATION (3.20)

Let us define the following operators

\[ N_\lambda^+ \phi = \sigma_\lambda^- \phi, \quad N_\lambda^- \phi = \sigma_\lambda^+ \phi, \]  

(A1)

\[ N_\lambda^0 \phi = \frac{1}{2} \sigma_\lambda^z \phi, \quad N_\lambda^\pm \phi = \pm i \sigma_\lambda^z \phi, \]  

(A2)

which satisfy the following commutation relations

\[ [N_\lambda^\pm, N_{\lambda'}^\pm] = \mp N_\lambda^0 \delta_{\lambda\lambda'}, \quad [N_\lambda^\pm, N_{\lambda'}^0] = \mp N_\lambda^\pm \delta_{\lambda\lambda'}, \quad [N_\lambda^0, N_{\lambda'}^0] = 0, \]  

(A2)

where \( \alpha, \beta = \pm, l, r \). Apart from their interesting algebraic properties, about which we are not concerned in this paper, the operators (A1) allow one to recast the master equation (3.18) in a more appropriate form. In fact in the high-temperature limit we can replace \( H_{sys} \) by the effective Hamiltonian (3.19) and we have

\[ \dot{\phi}_{sys} = - \sum_\lambda \gamma_\lambda \bar{n}_\lambda \phi_{sys} \]

\[ + \sum_\lambda \left[ \gamma_\lambda \bar{n}_\lambda (N_\lambda^+ + N_\lambda^-) - 2iA_\lambda^l N_\lambda^r + 2iA_\lambda^r N_\lambda^l \right] \phi_{sys}. \]  

(A3)

The dissipative term in this equation is much bigger than the Hamiltonian term. This allows us to adiabatically eliminate the atomic degrees of freedom. We can e.g. make use of the perturbative method proposed in Refs. (15) and (16) (see Ref. (17) for a complete account of the application of the method to effective master equations). This requires us to represent Eq. (A3) in terms of diagonal and raising-lowering operators. To this end, we first diagonalize the dissipative part of (A3) by applying the following \( \pi/2 \) rotation

\[ V = \exp \left[ \frac{\pi}{2} \sum_\lambda \frac{1}{2} (N_\lambda^+ - N_\lambda^-) \right], \]  

(A4)

which leads to

\[ \dot{\phi}_{sys} = - \sum_\lambda [\gamma_\lambda \bar{n}_\lambda + \gamma_\lambda \bar{n}_\lambda (N_\lambda^+ + N_\lambda^-) / 2 \]

\[ - i(A_\lambda^l - A_\lambda^r)(N_\lambda^+ + N_\lambda^-) \]

\[ + i(A_\lambda^r + A_\lambda^l)(N_\lambda^- - N_\lambda^+)] \phi_{sys}. \]  

(A5)

Next, we apply the following “small rotation”

\[ U = \exp \left[ -i \sum_\lambda \frac{A_\lambda^l - A_\lambda^r}{2\gamma_\lambda \bar{n}_\lambda} (N_\lambda^+ - N_\lambda^-) \right]. \]  

(A6)
After a lengthy but otherwise straightforward calculation and applying the inverse of the transformation \( A^\dagger \), one finally gets

\[
\dot{\hat{\rho}}_{\text{sys}} = -\sum_{\lambda} \{ \gamma_\lambda \hat{n}_\lambda [1 - N^+_\lambda - N^-_\lambda] \\
+ i(A^\dagger_\lambda + A^\dagger_\lambda)(N^+_\lambda - N^-_\lambda) \} \hat{\rho}_{\text{sys}} \\
- \sum_{\lambda} 2\gamma_\lambda \hat{n}_\lambda \left( \frac{A^\dagger_\lambda - A^\dagger_\lambda}{2\gamma_\lambda \hat{n}_\lambda} \right)^2 (N^+_\lambda + N^-_\lambda) \hat{\rho}_{\text{sys}} (A^\dagger)\]

where the field mode is expressed in a rotating frame. This transformed equation has the virtue of containing only diagonal terms so we can immediately trace over atomic variables to obtain the master equation for the density matrix of the mode \( \hat{\rho}(t) = \text{Tr}_{\text{atom}}[\hat{\rho}_{\text{sys}}(t)] \). Since each atom in the medium is in a statistical mixture, \( \hat{\rho}_\lambda = \frac{1}{2} \hat{1} \), the first term in \( A^\dagger \) vanishes and one obtains

\[
\dot{\hat{\rho}} = \sum_{\lambda} \frac{1}{2\gamma_{\lambda} \Delta_{\lambda}} \mathcal{L}[J_{\lambda}] \hat{\rho}, \quad (A8)
\]

where \( \mathcal{L}[J_{\lambda}] \) has been defined in Eq. (3.16). If we average over all the random phases \( \varphi_\lambda \) we get

\[
\dot{\hat{\rho}} = \frac{\gamma}{2} \mathcal{L}[n] \hat{\rho} + \frac{\gamma}{2} \mathcal{L}[J_+] \hat{\rho} + \frac{\gamma}{2} \mathcal{L}[J_-] \hat{\rho}, \quad (A9)
\]

with \( \gamma \) given in (3.21). This is precisely the master equation \( \mathcal{L}[\hat{J}] \), written in the interaction picture.

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