SO and USp Kähler and Hyper-Kähler Quotients and Lumps

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Abstract

We study non-linear $\sigma$ models whose target spaces are the Higgs phases of supersymmetric SO and USp gauge theories by using the Kähler and hyper-Kähler quotient constructions. We obtain the explicit Kähler potentials and develop an expansion formula to make use of the obtained potentials from which we also calculate the curvatures of the manifolds. The 1/2 BPS lumps in the $U(1) \times SO$ and $U(1) \times USp$ Kähler quotients and their effective descriptions are also studied. In this connection, a general relation between moduli spaces of vortices and lumps is discussed. We find a new singular limit of the lumps with non-vanishing sizes in addition to the ordinary small lump singularity. The former is due to the existence of singular submanifolds in the target spaces.

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1 Introduction

The target space of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric non-linear sigma models (NL$\sigma$M), with four and eight supercharges, must be Kähler [1] and hyper-Kähler [2], respectively. By using this fact, the notion of the hyper-Kähler quotient was first found in physics [3, 4] and was later formulated mathematically [5]. (We recommend Ref. [6] as a review for physicists). A $U(1)$ hyper-Kähler quotient [3] recovers the Calabi metric [7] on the cotangent bundle over the projective space, $T^*\mathbb{C}P^{N-1}$, while its $U(N_C)$ generalization leads to the cotangent bundle over the complex Grassmann manifold, $T^*G_{N,N_C}$ [4]. The hyper-Kähler manifolds also appear in the moduli spaces of Bogomol’nyi-Prasad-Sommerfield (BPS) solitons such as Yang-Mills instantons [8, 9, 10] and BPS monopoles [11]. The hyper-Kähler quotient offers a powerful tool to construct these hyper-Kähler manifolds: instanton moduli spaces [8] and monopole moduli spaces [12]. Gravitational instantons [13, 14], Yang-Mills instantons on gravitational instantons [15] and toric hyper-Kähler manifolds [16] are all constructed using the hyper-Kähler quotient.

The Higgs branch of $\mathcal{N} = 2$ supersymmetric QCD is hyper-Kähler. The low energy effective theory on the Higgs branch is described by an $\mathcal{N} = 2$ NL$\sigma$M on the hyper-Kähler manifold [17, 18, 19]. In the cases of an $SU(N_C)$ or a $U(N_C)$ gauge theory with hypermultiplets charged commonly under $U(1)$, the explicit metrics on the Higgs branch and their Kähler potentials are known explicitly. The latter is nothing but the Lindström-Roček metric [4]. A $U(1) \times U(1)$ gauge theory with three hypermultiplets of certain charges for instance gives the space: $T^*F_n$ with $F_n$ being the Hirzebruch surface [20]. The Higgs branches of quiver gauge theories are gravitational instantons and Yang-Mills instantons on gravitational instantons [13, 15]. However, to our knowledge, the ones of an $SO$ or a $USp$ gauge theory has not been explicitly derived yet (except for $SO(2) \simeq U(1)$ and $USp(2) \simeq SU(2)$), which was an open question in [9].

The first purpose of this paper is to explicitly construct the metric and its Kähler potential on the Higgs branch of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theories with gauge groups $SO(N_C)$ and $USp(2M_C)$ or $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$. The vacua of $\mathcal{N} = 1$ supersymmetric gauge theories are determined by the $D$-term condition, $D = 0$, while those of $\mathcal{N} = 2$ theories are determined by both the $D$-term and the $F$-term conditions, $D = F = 0$. The moduli space of vacua is obtained by the space of solutions to these constraints modulo gauge groups, $\{D = 0\} / G$ and $\{D = F = 0\} / G$ for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ models, respectively. In the superfield formalism, solving the $D$-term condition and modding out the gauge group $G$, can be done simultaneously because the gauge symmetry is in fact complexified to $G^C$. As a bonus the Kähler potentials are directly obtained in the superfield formalism. Although the $D$-term conditions of $SU(N_C)$ and
$U(N_C)$ gauge groups can be solved in components easily, those of $SO(N_C)$ and $USp(2M_C)$ are difficult to solve. To our knowledge this has not been done yet. We use the superfield formalism to solve the $D$-term conditions for $SO(N_C)$ and $USp(2M_C)$ gauge groups by introducing a trick. Namely, we relax the algebra of the vector superfields $V$ from $\mathfrak{so}(N_C)$ and $\mathfrak{usp}(N_C = 2M_C)$ to $\mathfrak{u}(N_C)$ and then introduce a Lagrange multiplier to restrict the algebra of $V$ to $\mathfrak{so}(N_C)$ and $\mathfrak{usp}(2M_C)$. We then successfully solve the superfield equations to obtain the resultant Kähler potentials.

There exists another method to obtain the moduli space of vacua, which is more familiar in the literature; It is an algebro-geometrical method in the geometric invariant theory [21], in which one prepares holomorphic gauge invariants made of the original chiral superfields and looks for algebraic constraints among them. This method has been widely used in the studies of $\mathcal{N} = 1$ supersymmetric gauge theories [22, 23, 24]. See [25, 26] for recent developments. In particular in Ref. [26], the moduli spaces of vacua of $\mathcal{N} = 1$ supersymmetric $SO(N_C)$ and $USp(2M_C)$ gauge theories are found to be Calabi-Yau cones over certain weighted projective spaces. According to us, a weak point of the geometric invariant theory is that one has to solve algebraic constraints among invariants in order to calculate geometric quantities such as the metric and the curvature etc.

Compared with this situation our method provides the Kähler potentials directly. We rewrite them in terms of holomorphic gauge invariants. Furthermore, we calculate the metrics and the curvatures by expanding the Kähler potentials. We confirm that a singularity appears in the moduli space of vacua when the gauge symmetry is partly recovered, as expected. We then study the case of $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ gauge theories. Finally, we calculate $SO(N_C)$ and $USp(2M_C)$ hyper-Kähler quotients and obtain their Kähler potentials explicitly. Although only the lowest dimensional case $USp(2) \simeq SU(2)$ has been known so far [6], the higher dimensional cases are new.

The second purpose of this paper is concerned with sigma model lumps, or sigma model instantons. A lump solution was first found in the $O(3)$ sigma model, or the $\mathbb{C}P^1$ model [27]. It was then generalized to the $\mathbb{C}P^n$ model [28], the Grassmann model [29], and other Kähler coset spaces [30]. Lumps are topological solitons associated with $\pi_2(M)$ with $M$ being the target Kähler manifold. Their energy saturates the BPS bound of the topological charge written as the Kähler form of $M$ pulled-back to the two-dimensional space.\footnote{In the case of hyper-Kähler manifolds there exist triplets of complex structures and Kähler forms. Accordingly it has recently been found that there exists a BPS bound written by the sum of three different Kähler forms to three different planes in the three dimensional space [31].} The lump solutions preserve
half of supersymmetry, when embedded into supersymmetric theories. The dynamics of lumps was studied \cite{32} by the moduli space (geodesic) approximation. Lumps are related to vortices in gauge theories as follows. $U(1)$ gauge theories coupled to several Higgs fields often admit semi-local vortex-strings \cite{33}. In the strong gauge coupling limit, gauge theories reduce to NL\sigma Ms whose target space is the moduli space of vacua in the gauge theories, and in this limit, semi-local strings reduce to lump-strings. For instance, a $U(1)$ gauge theory coupled to two charged Higgs fields reduces to the $\mathbb{C}P^1$ model, while the semi-local vortex-strings in Ref. \cite{33} reduce to the $\mathbb{C}P^1$ lumps \cite{34}. In the gauge theories at finite coupling, the large distance behavior of semi-local strings is well approximated by lump solutions. The sizes or widths of semi-local strings are moduli of the solution in the BPS limit, and accordingly, the lumps also possess size moduli. When the size modulus of a semi-local string vanishes, the solution reduces to the Abrikosov-Nielsen-Olesen (ANO) vortex \cite{35} which is called a local vortex. This limit corresponds to a singular configuration in the NL\sigma M, which is called the small lump singularity. Lumps and semi-local strings are also candidates of cosmic strings, see e.g. Ref. \cite{36}, and appear also in recent studies of D-brane inflation etc. \cite{37}.

Recently, there has been much progress on non-Abelian vortices in $U(N_C)$ gauge theories \cite{38,39}. These vortices are naturally 1/2-BPS in $\mathcal{N} = 2$ supersymmetric theories. When the number of flavors $N_F$ is equal to the number of colors $N_C$, the theory admits local non-Abelian vortices. Each of them carries orientational moduli $\mathbb{C}P^{N_C-1}$ in the internal space. The determination of the full moduli space of multiple local vortices with arbitrary positions and arbitrary orientations were achieved in field theory \cite{40} by introducing the method of the “moduli matrix” \cite{41,42}. All the moduli parameters are contained in the moduli matrix, which is a holomorphic matrix of the same size as the Higgs fields, and the moduli space has been shown to coincide with the one \cite{38,43} conjectured in string theory. The dynamics of two non-Abelian vortices has been studied in the moduli space approximation \cite{44} by using the general formula for the effective action of BPS solitons \cite{45}. Many interesting aspects of non-Abelian vortices are reviewed in Refs. \cite{42,46,47,48}. For instance, monopoles (Yang-Mills instantons) become kinks \cite{49,50} ($\mathbb{C}P^{N_C-1}$-lumps \cite{50,51}) in the effective field theory of a vortex-string. Intriguing is also the flux matching between non-Abelian vortices and non-Abelian monopoles and the applications are very interesting in the connection with non-Abelian duality etc. \cite{52,47}. Furthermore, non-Abelian vortices in $\mathcal{N} = 1$ supersymmetric theories have been studied in Refs. \cite{53}. A dyonic extension of non-Abelian vortices has been studied recently in Ref. \cite{54}.

In the case of a $U(N_C)$ gauge theory, semi-local vortices exist when the number of flavors $N_F$ is larger than the number of colors $N_C$ \cite{55}. At strong gauge coupling, the $U(N_C)$ gauge theory...
reduces to the Grassmann sigma model on $Gr_{N_F,N_C} = SU(N_F)/[SU(N_F - N_C) \times SU(N_C) \times U(1)]$. It has been demonstrated in Ref. [56] that non-Abelian semi-local strings in a $U(N_C)$

gauge theory reduce to the Grassmann lumps at large distance. One interesting aspect of these

lumps (semi-local vortices) is the (non-)normalizability of zero modes. It has been shown in

Ref. [55] that all moduli parameters of a single lump are non-normalizable except for its position

moduli. Orientational moduli in the internal space for local vortices are in fact non-normalizable

in this case. However, in the limit of vanishing size modulus, normalizable orientational zero

modes appear [56]. More interestingly, for $k = 2$ lumps (semi-local vortices), their “relative”

orientational moduli are normalizable although their “overall” orientational moduli are non-

normalizable [44, 56].

After the discovery of the $U(N_C)$ non-Abelian vortices [38, 39], one remarkable new develop-

ment is an extension to vortices in $U(1) \times SO(N_C)$ gauge theories [57] and $U(1) \times G'$ gauge theories

with an arbitrary simple group $G'$ [58]. This was done by imposing $G'$ invariant constraints on

the moduli matrix, and the conditions for the local vortices in these theories have been found.

In this paper we focus on BPS lumps related to semi-local vortices in the $U(1) \times SO(N_C)$ and

$U(1) \times USp(2M_C)$ gauge theories, which is the second purpose of this paper. We make a connec-

tion between the lump moduli spaces and the vortex moduli spaces and on this course, introduce

the moduli matrix, in which we have the formalism to explicitly construct 1/2 BPS lumps in the

class of $U(1) \times G'$ gauge theories. The explicit examples we make are with $U(1) \times SO(2M_C)$ and

$U(1) \times USp(4)$. Interestingly, there is a crucial difference between the $U(N_C)$ and $U(1) \times SO(N_C)$
or $U(1) \times USp(2M_C)$ theories, which is that in the latter two, even for $N_F = N_C$, semi-local vortex

strings appear which is not the case for $U(N_C)$.

We examine the (non-)normalizability of the moduli parameters of lumps in the $U(1) \times

SO(N_C)$ and $U(1) \times USp(2M_C)$ Kähler quotients. In the case of a single lump solution, all

moduli parameters in both the models are non-normalizable except for the center of mass. This

is parallel to the case of the $U(N_C)$ Kähler quotient [55, 56].

This paper is organized as follows. In Sec. 2 we will make a short review on the $SU(N_C)$

and $U(N_C)$ Kähler quotients and also the $U(N_C)$ hyper-Kähler quotient while we will turn our

attention to the $SO(N_C)$ and $USp(2M_C)$ and also $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ Kähler

quotients in Sec. 3 furthermore construct the metrics, an expansion of the metric around their

vacuum expectation values and compute the corresponding curvatures. Then we make use of the

technology with some explicit examples. Finally, we lift the construction to the hyper-Kähler

quotient case of $SO(N_C)$ and $USp(2M_C)$ gauge theories. In Sec. 4 will consider the NLσM lumps,

first by general considerations of gauge theories with $U(1) \times G'$ gauge groups with $G'$ being an
arbitrary simple group. Then we make a connection between the moduli spaces of the lumps in these theories with the moduli spaces of the vortices. Finally, we construct the lumps with the target spaces which we constructed in Sec. 3, make effective descriptions of those, and identify the non-normalizable modes. In Sec. 5 we conclude and discuss further developments. Moreover, we have left various theorems and proofs used in the text for Appendix A, a uniqueness proof in Appendix B and a deformed Kähler potential for $USp(2M_C)$ in Appendix C.

2 The $SU(N_C)$ and $U(N_C)$ (Hyper-)Kähler Quotients: A Review

2.1 The $SU(N_C)$ and $U(N_C)$ Kähler Quotients

Let us first give a brief review on the $SU(N_C)$ Kähler quotient. We start with the $\mathcal{N} = 1$ $SU(N_C)$ supersymmetric Yang-Mills theory with $N_F$ chiral superfields $Q$ (i.e. an $N_C$-by-$N_F$ matrix) in the fundamental representation of $SU(N_C)$. Denote the $SU(N_C)$ vector multiplet by a superfield $V'$, then a Kähler potential for the system is

$$K_{SU(N_C)} = \text{Tr} \left[ QQ^\dagger e^{-V'} \right]. \quad (2.1)$$

We have used a matrix notation and the trace is taken over the color indices. The Lagrangian is invariant under the complexification of the gauge group, $SU(N_C)^C = SL(N_C, \mathbb{C})$, given by

$$Q \rightarrow e^{i\lambda'} Q, \quad e^{V'} \rightarrow e^{i\lambda'} e^{V'} e^{-i\lambda^\dagger}, \quad e^{i\lambda'} \in SU(N_C)^C. \quad (2.2)$$

We do not consider any superpotentials here.

As is well-known, the kinetic term of the vector supermultiplet $\int d^2 \theta \, W^\alpha W_\alpha/4g^2 + \text{c.c.}$ includes a so-called $D$-term potential in the Wess-Zumino gauge, in which $SU(N_C)^C$ is fixed to $SU(N_C)$

$$V_D = \frac{g^2}{2} \left( D^A \right)^2, \quad D^A = \text{Tr}_F \left( Q^w T^A Q^w \right), \quad (2.3)$$

where $T^A$ are $SU(N_C)$ generators and $Q^w$ is $Q$ in the Wess-Zumino gauge. The vacuum condition $D^A = 0$ ($D$-flatness) allows both for an unbroken phase and the Higgs phase. It implies that $Q^w Q^w \propto 1_{N_C}$ holds in the vacuum. On the Higgs branch (rank $Q^w = N_C$), the gauge fields acquire masses of the order $g \langle Q \rangle$ by the Higgs mechanism. If we restrict ourselves to energies much below the mass scale, we can omit the massive gauge fields. In order to get a low energy effective theory, it will prove useful to consider a limit where the gauge coupling is taken to
infinity: \( g \to \infty \). In this limit, the vector multiplet becomes infinitely massive and loses the kinetic term. Thus, it reduces to merely an auxiliary field. At the same time the \( D \)-term potential forces \( Q_{wz} \) to take a value in the vacuum \( D^A = 0 \). Thus, the low energy effective theory is a non-linear sigma model (NL\( \sigma \)M), whose target space is the vacuum of the gauge theory

\[
\mathcal{M}_{SU(N_C)} = \{ Q_{wz} \mid Q_{wz} Q_{wz}^\dagger \propto 1_{N_C} , \, \text{rank} \, Q_{wz} = N_C \} / SU(N_C).
\]  

The real dimension of the manifold is \( 2N_C N_F - (N_C - 1)^2 - (N_C - 1) = 2N_C (N_F - N_C) + 2 \).

Before fixing the complexified gauge symmetry \( SU(N_C)^C \), for example by the Wess-Zumino gauge as above, we can take the strong coupling limit. This gives another description of the non-linear sigma model. The Lagrangian consists of only one term i.e. Eq. (2.1). We do not have the \( D \)-term conditions anymore, however, instead we have the complex fields \( Q \) and the complexified gauge group \( SU(N_C)^C \). The target space is expressed by

\[
\mathcal{M}_{SU(N_C)} = \{ Q \mid \text{rank} \, Q = N_C \} / SU(N_C)^C.
\]  

In order for this quotient to be well-defined, the action of \( SU(N_C)^C \) must be free on \( Q \). Namely, the gauge symmetry should be completely broken, thus we are going to study the full Higgs phase. The complex dimension of the manifold is \( N_C N_F - (N_C^2 - 1) = N_C (N_F - N_C) + 1 \), which coincides with the dimension of (2.4). The two expressions (2.4) and (2.5) of the target space are identical. One can find a relation between them by solving the equations of motion for \( V' \). It determines the traceless part as \( QQ^\dagger e^{-V'} \propto 1_{N_C} \). Taking \( \text{Tr} \, V' = 0 \) into account, \( V' \) is uniquely determined as

\[
V' = \log QQ^\dagger - \frac{1}{N_C} 1_{N_C} \log \det(QQ^\dagger),
\]  

if and only if \( \text{rank} \, Q \) is the maximum, which means the full Higgs phase. Then we find an explicit map from the quotient (2.5) to the vacuum configuration (2.4):

\[
Q_{wz} = e^{-V'/2} Q = \left[ \det(QQ^\dagger) \right]^{\frac{1}{2N_C}} \frac{1}{\sqrt{QQ^\dagger}} Q.
\]  

There exists still another way to express the same NL\( \sigma \)M. As explained above, the target space is nothing but the classical moduli space of vacua of the original supersymmetric gauge theory. As discussed in Ref. [21] it can be described by holomorphic invariants of the complexified gauge group. Hence, the Kähler potential on the NL\( \sigma \)M should be expressed in terms of such holomorphic invariants. The holomorphic invariants of \( SU(N_C)^C \) are the baryon operators

\[
B^{(A_1 \cdots A_{N_C})} \equiv \det \, Q^{(A_1 \cdots A_{N_C})} = \epsilon^{i_1 \cdots i_{N_C}} Q_{i_1 A_1} \cdots Q_{i_{N_C} A_{N_C}},
\]  

(2.8)
where \( Q^{(A_1\ldots A_{N_C})} \) denotes an \( N_C \)-by-\( N_C \) minor matrix of \( N_C \)-by-\( N_F \) matrix \( Q \) as \( (Q^{(A)})^j_i = Q_{ij}^{(A)} \). We often abbreviate the label \( A_1 \cdots A_{N_C} \) as \( \langle A \rangle \). The important point is that all the \( B^{(A)} \)'s are not independent and they satisfy the so-called Plücker relations

\[
B^{(A_1\ldots A_{N_C-1}[B_1)} B^{(B_2\ldots B_{N_C+1})]} = 0. \tag{2.9}
\]

Furthermore, the condition for having the full Higgs phase requires that at least one of the \( B^{(A)} \)'s must take a non-zero value. Actually, we can reconstruct \( Q \) modulo \( SU(N_C) \) gauge symmetry by solving the Plücker relations with one non-zero \( B^{(A)} \) as the starting point. That is, the holomorphic invariants with the Plücker relations give us the same information as the two descriptions above. Hence, the target space is also expressed as

\[
\mathcal{M}_{SU(N_C)} = \{ B^{(A)} \mid \text{Plücker relations (2.9)} \} - \{ B^{(A)} = 0 , \forall \langle A \rangle \}. \tag{2.10}
\]

Let us show the metric on the target space. It can be derived from the Kähler potential (2.1) and is represented by

\[
K_{SU(N_C)} = N_C \left[ \det(QQ^\dagger) \right]^{\frac{1}{N_C}} = N_C \left( \sum_{\langle A \rangle} |B^{(A)}|^2 \right)^{\frac{1}{N_C}}. \tag{2.11}
\]

The appearance of the \( N_C \)th root reflects the fact that the \( U(1) \) charge of the invariants is \( N_C \), as we will see soon. Notice that the (partial) Coulomb phase \( \det(QQ^\dagger) = 0 \) shrinks to a point of the target manifold from the point of view of the NLσM and a trace of this fact is seen as the \( \mathbb{Z}_{N_C} \) conifold singularity at that point. In a simple example with \( N_F = N_C \), one can find the NLσM on an orbifold \( \mathbb{C}/\mathbb{Z}_{N_C} \). At the singularity, the vector multiplet becomes massless and the gauge symmetry is restored. We have to take all the massless fields into account there, namely we cannot restrict ourselves to the NLσM, but we have to return to the original gauge theory.

This singularity (that is, the Coulomb phase) is removed once the overall \( U(1) \) phase is gauged and the so-called Fayet-Iliopoulos (FI) parameter \( \xi (> 0) \) is introduced for that \( U(1) \). Let us consider a \( U(1) \times SU(N_C) \) gauge theory. Still we neglect the kinetic terms associated with the vector multiplet, such that the vector multiplet is an auxiliary superfield. The Kähler potential is given by

\[
K_{U(1) \times SU(N_C)} = \text{Tr} \left[ QQ^\dagger e^{-V_e} e^{-V} \right] + \xi V_e = e^{-V_e} K_{SU(N_C)} + \xi V_e, \tag{2.12}
\]

where \( V_e \) is a \( U(1) \) vector supermultiplet and the chiral fields \( Q \) have \( U(1) \) charge +1. The \( D \)-flatness condition for the overall \( U(1) \) implies that \( Q_{wz} Q_{zw}^{\dagger} = \frac{\xi}{N_C} 1_{N_C} \). The target space of the NLσM becomes a compact space; the complex Grassmannian manifold \( \mathcal{M}_{U(1) \times SU(N_C)} = \)
$Gr_{N_F,N_C} \simeq SU(N_F)/[SU(N_F - N_C) \times SU(N_C) \times U(1)]$\footnote{The $U(N_C)$ Kähler quotient construction of the Grassmann manifold was first found in Ref. \cite{ref60} in the superfield formalism.} As in the case above, we have three different representations

$$\mathcal{M}_{U(1) \times SU(N_C)} = \left\{ Q_{wz} \mid Q_{wz} Q_{wz}^\dagger = \frac{\xi}{N_C} \mathbf{1}_{N_C} \right\}/(U(1) \times SU(N_C))$$

$$= \{ Q \mid \text{rank } Q = N_C \}/(U(1) \times SU(N_C))^C$$

$$= \left\{ \{ B^{(A)} \mid \text{Plücker relations} \right\} - \{ B^{(A)} = 0 , \forall (A) \} \right\}/U(1)^C. \quad (2.13)$$

A relation between $Q_{wz}$ and $Q$ is also found here by solving the equations of motion with respect to $V'$ and $V_e$. The solution for $V'$ is the same as Eq. (2.6) and the $U(1)$ part is then written as

$$V_e = \log (\xi^{-1} K_{SU(N_C)}). \quad (2.14)$$

Then the map from the quotient space to the vacuum configuration is given by

$$Q_{wz} = e^{-V'/2-V_e/2}Q = \sqrt{\frac{\xi}{N_C}} \sqrt{\frac{1}{QQ^\dagger}}Q. \quad (2.15)$$

The third expression in Eq. (2.13) shows the Plücker embedding of the Grassmannian space into a bigger space, the complex projective space $\mathbb{C}P^n$ with $n = \frac{N_F!}{N_C!(N_F - N_C)!} - 1$. The Kähler potential can now be expressed by

$$K_{U(1) \times SU(N_C)} = \frac{\xi}{N_C} \log \det (QQ^\dagger) = \frac{\xi}{N_C} \log \left( \sum_{(A)} |B^{(A)}|^2 \right). \quad (2.16)$$

The $1/N_C$ factor in front is the (inverse) $U(1)$ charge of the invariant $B^{(A)}$. The FI parameter plays an important role: it forces the gauge symmetry $U(1) \times SU(N_C)$ to be fully broken, namely it hides the singularity at the origin, where the gauge symmetry is recovered.

The Grassmannian manifold is one of the Hermitian symmetric spaces. NLσMs on all Hermitian symmetric spaces can be obtained by imposing proper holomorphic constraints from $F$-terms, by which Hermitian symmetric spaces are embedded into $\mathbb{C}P^{N_F-1}$ or the Grassmannian manifold $[61]$.

### 2.2 The $U(N_C)$ Hyper-Kähler Quotient

One can easily extend the above Kähler quotient to the hyper-Kähler quotient by considering a natural $\mathcal{N} = 2$ supersymmetric extension. Here we study the $U(1) \times SU(N_C)$ case.
potential and the superpotential are given by

\[ \tilde{K}_{U(1) \times SU(N_C)} = \text{Tr} \left[ QQ^\dagger e^{-V} e^{-V'} + \tilde{Q}^\dagger \tilde{Q} e^{V} e^{V'} \right] + \xi V , \tag{2.17} \]

\[ W = \text{Tr} \left[ Q \tilde{Q} \Sigma \right] , \tag{2.18} \]

respectively, where we have introduced \( N_F \) hypermultiplets \((Q, \tilde{Q}^\dagger)\) in the fundamental representation of \( U(N_C) \simeq U(1) \times SU(N_C) \) and \( U(N_C) \) vector superfields \((V, \Sigma) = (V' + V e_1, \Sigma)\). The complexified gauge transformation is given by

\[ Q \rightarrow e^{i \Lambda} Q , \quad \tilde{Q} \rightarrow \tilde{Q} e^{-i \Lambda} , \quad e^V \rightarrow e^{i \Lambda} e^V e^{-i \Lambda} , \quad \Sigma \rightarrow e^{i \Lambda} \Sigma e^{-i \Lambda} , \quad \Lambda \in GL(N_C, \mathbb{C}) . \tag{2.19} \]

The target space of the corresponding NL\( \sigma \)M is a hyper-Kähler manifold, namely the cotangent bundle \( T^* Gr_{N_F,N_C} \) over the complex Grassmannian manifold \( Gr_{N_F,N_C} \), endowed with the Lindström-Roček metric [4]. Let us obtain the Kähler potential with respect to \( Q, \tilde{Q} \) without choosing the Wess-Zumino gauge. The equations of motion for \( \Sigma \) and \( V \) are

\[ Q \tilde{Q} = 0 , \tag{2.20} \]

\[ -QQ^\dagger e^{-V} + e^V \tilde{Q}^\dagger \tilde{Q} + \frac{\xi}{N_C} 1_{N_C} = 0 . \tag{2.21} \]

The first equation implies that \( \tilde{Q} \) is orthogonal to \( Q \). The rank of \( Q \) must be \( N_C \) due to the positive FI parameter \( \xi \), while \( \tilde{Q} \) can be zero. Therefore \( Q \) (\( \tilde{Q} = 0 \)) parametrizes the base space \( Gr_{N_F,N_C} \) with the total space being the cotangent bundle over it. Let us count the complex dimensions of the target space: \( N_C N_F + N_F N_C - N_C^2 - N_C^2 = 2 N_C (N_F - N_C) \) where the first subtraction is the \( U(N_C)^C \) quotient and the second is the number of conditions given in Eq. (2.20).

In order to solve the second matrix equation, we first multiply by \( \sqrt{QQ^\dagger} e^{-V} \) from the left and by \( \sqrt{QQ^\dagger} \) from the right, such that the matrix equation becomes Hermitian

\[ X^2 - \frac{\xi}{N_C} X - \sqrt{QQ^\dagger} \tilde{Q} \sqrt{QQ^\dagger} = 0 , \quad X \equiv \sqrt{QQ^\dagger} e^{-V} \sqrt{QQ^\dagger} . \tag{2.22} \]

Therefore, using \( \text{det} QQ^\dagger \neq 0 \), we find the solution

\[ V = -\log \left[ \frac{1}{\sqrt{QQ^\dagger}} X \frac{1}{\sqrt{QQ^\dagger}} \right] , \]

with

\[ X = \frac{\xi}{2 N_C} 1_{N_C} + \sqrt{\sqrt{QQ^\dagger} \tilde{Q} \sqrt{QQ^\dagger}} + \frac{\xi^2}{4 N_C^2} 1_{N_C} . \tag{2.23} \]

\[^3\text{Note that the square root and the logarithm is uniquely defined for positive (semi-)definite Hermitian matrices. This point might be missed (at least in this context) in the physics literature so far.}\]
We will now switch to another description i.e. using holomorphic invariants. We have the following invariants of the $SU(N_C)^C$ gauge group

$$B^{(A)} = \det Q^{(A)} , \quad M = \tilde{Q}Q , \quad \left( \tilde{B}^{(A)} = \det \tilde{Q}^{(A)} \right). \quad (2.24)$$

In addition to the Plücker relations for the $B^{(A)}$'s, there are constraints on the mesonic invariant $M$

$$M_B^{[A_1 B^{(A_2 \cdots A_{N_C+1})}]} = 0 , \quad B^{(A_1 \cdots A_{N_C-1} A')} M_A^{B} = 0 . \quad (2.25)$$

Furthermore, $B^{(A)}$ (and $\tilde{B}^{(A)}$) are only defined up to $U(1)^C$ equivalence transformations. After reconstructing $Q$ from (some) non-vanishing $B^{(A)}$, we can reconstruct $\tilde{Q}$ from the first condition and find the constraint $Q \tilde{Q} = 0$ from the second. Therefore, these invariants and their constraints describe the same target space, $T^*Gr_{N_F, N_C}$. Plugging back the solution (2.23) into the Kähler potential (2.17), we obtain the Kähler potential in terms of these invariants [4, 6]

$$\tilde{K}_{U(1) \times SU(N_C)} = K_{U(1) \times SU(N_C)} + \frac{\xi}{N_C} \text{Tr}_F \left[ \sqrt{1 + \frac{4N_C^2}{\xi^2} MM^\dagger} \log \left( \sqrt{1 + \frac{4N_C^2}{\xi^2} MM^\dagger} \right) \right]. \quad (2.26)$$

We have used $A^\dagger A = MM^\dagger$ and the cyclic property of a trace, i.e. for $A = \sqrt{Q \tilde{Q}^\dagger}$

$$\text{Tr} \left[ f(AA^\dagger) - f(0_{N_C}) 1_{N_C} \right] = \text{Tr} \left[ f(A^\dagger A) - f(0_{N_F}) 1_{N_F} \right] . \quad (2.27)$$

This relation can be easily proved by expanding the function $f$ around $AA^\dagger = 0_{N_C}$. Recall that the logarithm and the square root of a positive (semi-)definite Hermitian matrix can be calculated by diagonalization and therefore the cyclic property works not only for polynomial functions but for any function $f(x)$.

The hyper-Kähler quotient construction of the cotangent bundle over the Grassmann manifold has been reviewed here. For $N_C = 1$, the $U(1)$ hyper-Kähler quotient reduces to the cotangent bundle over the complex projective space, $T^*CP^{N_F-1}$ [3], endowed with the Calabi metric [7]. The explicit Kähler potentials of the cotangent bundles over the other Hermitian symmetric spaces have recently been obtained by a rather different method [62]. It is an open question if these manifolds can be obtained as a certain hyper-Kähler quotient or not.

We will not repeat the derivation of the $SU(N_C)$ hyper-Kähler quotient here. Explicit expressions can be found in the literature, see for instance [6, 63]. It gives the cotangent bundle over the $SU(N_C)$ Kähler quotient derived in the last subsection.
3 The $SO(N_C)$ and $USp(2M_C)$ (Hyper-)Kähler Quotients

3.1 The $SO(N_C)$ and $USp(2M_C)$ Kähler Quotients

The Kähler potential for an $SO(N_C)$ or a $USp(2M_C)$ gauge theory is given by

$$K_{SO,USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} \right],$$

(3.1)

where $V'$ takes a value in the $so(N_C)$ or $usp(2M_C)$ algebra. The $D$-flatness conditions in the Wess-Zumino gauge are

$$D^A = \text{Tr}_F \left( Q_w^T T^A Q_w \right) = 0,$$

(3.2)

with $T_A$ being the generators in the Lie algebra of $SO$ or $USp$.

Instead of solving these equations explicitly, we will here discuss the breaking pattern of the gauge symmetry and the flat directions. For this we will use both the gauge and the global symmetry as is usually done. The vacuum expectation value of $Q_w$ in the case of $SO(N_C)$ can be put on the diagonal form after fixing both the local and the global symmetries as

$$Q_w^{SO(N_C)} = \begin{pmatrix} A_{N_C \times N_C} & 0_{N_C \times (N_F - N_C)} \end{pmatrix}, \quad \text{with} \quad A_{N_C \times N_C} = \text{diag}(a_1, a_2, \ldots, a_{N_C}),$$

(3.3)

where we have taken a normal basis for the $SO(N_C)$ group, namely $g^T g = 1_{N_C}$. Here all the parameters $a_i$ are taken to be real and positive, which indeed parametrize the flat directions of the Higgs branch. In generic points of the moduli space of vacua with non-degenerate $a_i$, the gauge symmetry is completely broken and the flavor symmetry $U(N_F)$ is broken to $U(N_F - N_C)$.

The moduli space of vacua can be locally written in generic points as

$$\mathcal{M}_{SO(N_C)} \simeq \mathbb{R}^{N_C}_{>0} \times \frac{U(N_F)}{U(N_F - N_C) \times (\mathbb{Z}_2)^{N_C - 1}}.$$  

(3.4)

Here the discrete unbroken group $(\mathbb{Z}_2)^{N_C - 1}$ has elements of $N_C$-by-$N_C$ diagonal matrices in the $SO(N_C)$ group elements acting from the left, which have an even number of $-1$ elements with the rest $1$, in addition to the same matrices embedded into the $U(N_F)$ group acting from the right. We see that the space is of cohomogeneity $N_C$, of which the isometry is $U(N_F)$ and the isotropy at generic points is $U(N_F - N_C)$. The coordinates of the coset space $U(N_F)/U(N_F - N_C)$ correspond to Nambu-Goldstone (NG) modes of the broken flavor symmetry, whereas the coordinates $\{a_i\}$ of the flat directions $\mathbb{R}^{N_C}_{>0}$ correspond to the so-called “quasi-Nambu-Goldstone” modes [6]. The quasi-NG modes do not correspond to a symmetry breaking but are ensured by supersymmetry. In general, the unbroken flavor symmetry, namely the isotropy of the space, changes from point
to point depending on the values of the parameters (the quasi-NG modes) \( a_i \)'s. When two parameters coincide, \( a_i = a_j, (i \neq j) \), a color-flavor locking \( SO(2) \) symmetry emerges. In such degenerate subspace on the manifold, the above coset space attached to \( \mathbb{R}^{N_C} \) shrinks to one with less dimension; 
\[
\mathcal{M}_{SO(N_C)} \sim \frac{\mathbb{R}^{N_C+1}}{U(N_F)} \times \frac{U(N_F)}{U(N_F - N_C)} \times \prod_i SO(n_i) \times (\mathbb{Z}_2)^{N_C - 1 - \sum(n_i - 1)} .
\]
In general, when \( n_i (i = 1, 2, \ldots, \text{and} \sum_i n_i \leq N_C) \) parameters among \( a_i \) coincide, the symmetry structure of the moduli space of vacua becomes
\[
\mathcal{M}_{SO(N_C)} \sim \frac{\mathbb{R}^{N_C + \frac{1}{2} \sum_i n_i(n_i - 1)}}{U(N_F)} \times \frac{U(N_F)}{U(N_F - N_C)} \times \prod_i SO(n_i) \times (\mathbb{Z}_2)^{N_C - 1 - \sum(n_i - 1)} . 
\] (3.5)

The most symmetric vacuum, when all parameters coincide, is realized as
\[
\mathcal{M}_{SO(N_C)} \sim \frac{\mathbb{R}^{N_C(N_C + 1)}}{U(N_F)} \times \frac{U(N_F)}{U(N_F - N_C)} \times SO(N_C) . 
\] (3.6)

This breaking pattern of the flavor symmetry is the one of non-supersymmetric \( SO(N_C) \) QCD \[68\]. The unbroken flavor symmetry in non-supersymmetric QCD is in general further broken down as in Eq. (3.4) or (3.5) in supersymmetric QCD.

No singularities appear in the moduli space even when the parameters coincide unless they vanish. The existence of the quasi-NG modes is strongly related to the emergence of the Coulomb phase. When one \( a_i \) vanishes, the NG part becomes \( U(N_F)/U(N_F - N_C + 1) \) but the gauge symmetry is still completely broken. Accordingly, no singularities appear. However, when any two of the \( a_i \)'s vanish, an \( SO(2) \) subgroup of the gauge symmetry is recovered and the NG part becomes \( U(N_F)/U(N_F - N_C + 2) \). (One expects a singularity on the manifold in the limit of two vanishing \( a_i \)'s). Thus, in the Higgs phase with completely broken gauge symmetry, the rank of \( Q_{wz} \) has to be greater than \( N_C - 2 \). In this paper we consider this latter case, the models with \( N_F \geq N_C - 1 \).

For the \( USp(2M_C) \) case it is known that the flat directions are parametrized by \[24, 19\]
\[
Q_{wz}^{USp(2M_C)} = 1_2 \otimes (A_{M_C} = M_C, 0_{M_C} = (M_F - M_C)) ,
\] (3.7)
where the number of flavors is even \( N_F = 2M_F \). Even in generic points with non-degenerate \( \{a_i\} \), color-flavor symmetries \( USp(2)^{M_C} \simeq SU(2)^{M_C} \) exist in the vacuum. Therefore, the moduli space of vacua can be locally written in generic points as
\[
\mathcal{M}_{USp(2M_C)} \simeq \frac{\mathbb{R}^{M_C}}{U(N_F - 2M_C) \times USp(2)^{M_C}} ,
\] (3.8)

\(^4\) Some quasi-NG modes change to NG modes reflecting further symmetry breaking. This change of quasi-NG and NG modes was pointed out in Ref. \[65\]. It was also observed in the moduli space of domain walls \[66\] and of non-Abelian vortices \[67\], where quasi-NG modes correspond to the positions of solitons. Here the notation “\( \# \)” is used for a local structure of the bundle \( F \times B \) with a fiber \( F \) and a base space \( B \). This is not globally true; once some values of \( \mathbb{R}^\#_0 \) change, the coset space changes in general.
except for submanifolds where the coset space shrinks. The resulting space is of cohomogeneity $M_C$. Again, when $n_i$ ($i = 1, 2, \cdots$, and $\sum_i n_i \leq M_C$) parameters among $a_i$ coincide, the symmetry structure becomes

$$
\mathcal{M}_{USp(2M_C)} \sim \frac{U(N_F)}{U(N_F - 2M_C) \times USp(2)^{M_C - \sum n_i} \times \prod_i USp(2n_i)}. \quad (3.9)
$$

The most symmetric vacuum, when all parameters coincide, is realized as

$$
\mathcal{M}_{USp(2M_C)} \sim \frac{U(N_F)}{U(N_F - 2M_C) \times USp(2M_C)}, \quad (3.10)
$$

whose breaking pattern is the one of non-supersymmetric $USp(2M_C)$ QCD. There are no singularities unless one of the parameters $a_i$ vanishes. In the case of $USp(2M_C)$ the complete broken gauge symmetry needs $M_F \geq M_C$.

Next we explicitly construct the Kähler potentials from the moduli space of vacua. The $D$-flatness conditions (3.2), however, are rather difficult to solve. Without taking the Wess-Zumino gauge, we can eliminate the superfield $V'$ directly within the superfield formalism by using a trick. To this end we note that $V'$ satisfies

$$
V'^\mathsf{T} J + J V' = 0 \leftrightarrow e^{-V'^\mathsf{T}} J e^{-V'} = J. \quad (3.11)
$$

Here the matrix $J$ is the invariant tensor of the $SO$ or $USp$ group, $g^T J g = J$ with $g \in SO(N_C), USp(2M_C)$, satisfying

$$
J^\mathsf{T} = \epsilon J, \quad J^\dagger J = 1_{N_C}, \quad \epsilon = \begin{cases} +1 & \text{for } SO(N_C), \\ -1 & \text{for } USp(N_C = 2M_C). \end{cases} \quad (3.12)
$$

We can choose the form of the invariant tensor $J$ as

$$
J^\pm_{M_C} \equiv \begin{pmatrix} 0_{M_C} & 1_{M_C} \\ \pm 1_{M_C} & 0_{M_C} \end{pmatrix}, \quad J_{M_C,\text{odd}} \equiv \begin{pmatrix} J^+_M & 0^\mathsf{T} \\ 0 & 1 \end{pmatrix}, \quad (3.13)
$$

where the last tensor is for the $SO(N_C = 2M_C+1)$ case. We will use these conventions throughout the paper unless otherwise stated.

We are now ready to eliminate $V'$ using the following trick. Let us first consider $V'$ taking a value in a larger algebra, namely $u(N_C)$ and then introduce an $N_C$-$N_C$ matrix of Lagrange

---

5 To our knowledge the $D$-flatness conditions are not solved in the case of an $SO$ or a $USp, \mathcal{N} = 1$ supersymmetric gauge theory.

6 Two arbitrary choices of the invariant tensor are related by an appropriate unitary transformation $u : J' = u^T J u$. Correspondingly, the elements of the gauge group for different choices of the invariant tensor are related by $g' = u^T g u$. See Appendix A.1.
multipliers\textsuperscript{7} \lambda to restrict $V'$ to take a value in the $\mathfrak{so}(N_C)$ or the $\mathfrak{usp}(N_C = 2M_C)$ subalgebra:

$$K_{SO,USp} = \text{Tr} \left[ Q Q^\dagger e^{-V'} + \lambda \left( e^{-V'^T} J e^{-V'} - J \right) \right],$$

(3.14)

where $Q$ are $N_F$ chiral superfields as earlier and $V'$ is a vector superfield of $U(N_C)$. The added term breaks the complexified gauge transformation to $SO(N_C), USp(2M_C)$ and the equation of motion for $\lambda$ gives the constraint (3.11) which reduces the Kähler potential (3.14) back to (3.1).

Instead, we will take another path and eliminate $V'$. The equation of motion for $V'$ takes the form

$$QQ^\dagger e^{-V'} + (\lambda + \epsilon \lambda^T) J = 0,$$

(3.15)

where we have used (3.11). Combining (3.15) with its transpose: $e^{-V'^T}Q^*Q^T + J(\lambda + \epsilon \lambda^T) = 0$, then $\lambda$ can be eliminated:

$$QQ^\dagger e^{-V'} = e^{V'} J^T Q^* Q^T J.$$

(3.16)

Furthermore, in order to make the equation Hermitian, we multiply by $\sqrt{Q Q^\dagger e^{-V'}}$ from the left and by $\sqrt{Q Q^\dagger}$ from the right as in the previous case

$$X^2 = \left( Q^T J \sqrt{Q Q^\dagger} \right)^\dagger \left( Q^T J \sqrt{Q Q^\dagger} \right), \quad X \equiv \sqrt{Q Q^\dagger e^{-V'}} \sqrt{Q Q^\dagger}.$$

(3.17)

This equation uniquely gives a positive definite matrix $X$, by means of its square root. We can thus uniquely obtain $V'$ from this $X$, if and only if the holomorphic invariants $M \equiv Q^T J Q$ satisfy rank $M > N_C - 2$, that is, if and only if the vacuum is in the full Higgs phase. See Appendix B for a uniqueness proof, in the case of rank $M = N_C - 1$. It is possible to switch to $Q_{\text{wz}}$ from $Q$ by the complexified gauge transformation $Q_{\text{wz}} = u'^{-1}Q$ with $u'u'^\dagger = e^{V'}$. Without using an explicit solution for $V'$, we obtain the Kähler potential of the NL$\sigma$M

$$K_{SO,USp} = \text{Tr} X = \text{Tr} \sqrt{\left( Q^T J \sqrt{Q Q^\dagger} \right)^\dagger \left( Q^T J \sqrt{Q Q^\dagger} \right)}.$$

(3.18)

Thus we have obtained the explicit Kähler potentials.

Now we can naturally switch to another expression for this NL$\sigma$M in terms of the holomorphic gauge invariants. With the help of $\text{Tr} \sqrt{AA^\dagger} = \text{Tr}_F \sqrt{A^\dagger A}$, one can rewrite the Kähler potential (3.18) as

$$K_{SO,USp} = \text{Tr}_F \sqrt{MM^\dagger}, \quad M^\dagger = \epsilon M, \quad \epsilon = e^{V'^T} J \lambda e^{-V'^T} J.$$

(3.19)

\textsuperscript{7} Hermiticity of $\lambda$ is defined so that $\lambda e^{-V'^T} J$ is a vector superfield, that is, $\lambda^\dagger = e^{V'^T} J \lambda e^{-V'^T} J$. 

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where $M$ is nothing but the holomorphic invariants of the gauge symmetry
\[ M \equiv Q^T J Q , \quad B^{(A)} \equiv \det Q^{(A)} . \tag{3.20} \]

The first one is the “mesonic” invariant while the second is the “baryonic” one which appears for $N_F \geq N_C$. The two kinds of invariants should be subject to constraints in order to correctly describe the $\text{NL}_\sigma M$. There are relations between the mesons and the baryons:
\[ \text{SO}(N_C) : \det(J) \ B^{(A)} B^{(B)} = \det M^{(A)(B)} , \tag{3.21} \]
\[ \text{USp}(2M_C) : \text{Pf}(J) \ B^{(A)} = \text{Pf} M^{(A)(A)} . \tag{3.22} \]

where the $N_C$-by-$N_C$ matrix $M^{(A)(B)}$ is a minor matrix defined by $(M^{(A)(B)})^{ij} = M^{A_iB_j}$. The Plücker relations among the baryonic invariants $B^{(A)}$ are derived from the above relation. Actually, from the invariants $M$ and $B^{(A)}$ with the constraints we can reconstruct $Q$ modulo the complexified gauge symmetry as follows. By using an algorithm similar to the Cholesky decomposition of an Hermitian matrix, we can show that

An arbitrary $n$-by-$n$ (anti-)symmetric complex matrix $X$ can always be decomposed as $X = p^T J p$ with a rank($X$)-by-$n$ matrix $p$. \( \text{(3.23)} \)

See Appendix A.3 for a proof of this statement. In the USp case, with a decomposition of the meson $M$, we can completely reconstruct $Q$ modulo $\text{USp}(2M_C)$ transformations. This fact corresponds to the fact that there are no independent baryons $B^{(A)}$ in this $\text{USp}(2M_C)$ theory and only the meson fields describe the full Higgs phase
\[ \mathcal{M}_{\text{USp}} = \{ M \mid M \in N_F\text{-by-}N_F \text{ matrix, } M^T = -M, \quad \text{rank } M = 2M_C \} . \tag{3.24} \]

On the contrary, in the $\text{SO}(N_C)$ case, a decomposition of $M$ gives $Q$ modulo $O(N_C)^C$ and one finds two candidates for $Q$ since $\mathbb{Z}_2 \simeq O^C/\text{SO}^C$ which is fixed by the sign of the baryons. \[ ^8 \]
Therefore we have to take the degrees of freedom of the baryons into account to consider the full Higgs phase
\[ \mathcal{M}_{\text{SO}} = \{ M, B^{(A)} \mid M : \text{symmetric } N_F\text{-by-}N_F, \quad \text{Eq. } (3.21), \quad N_C - 1 \leq \text{rank } M \leq N_C \} . \tag{3.25} \]

For large $N_C$, it is a hard task to obtain an explicit metric from the formula \( (3.19) \), since we need to calculate the eigenvalues of $MM^\dagger$. Let us, therefore, consider expanding the Kähler potential \( (3.19) \) in terms of infinitesimal coordinates around a point. Note that the meson field

\[ ^8 \text{In the case of rank } M = N_C - 1, \ g \in \mathbb{Z}_2 \text{ acts trivially on } Q \text{ as } g \ Q = Q, \text{ although all the baryons vanish.} \]
\( M \) for \( SO(N_C) \), which is a symmetric matrix, can be always diagonalized by using the flavor symmetry \( U(N_F) \) as

\[
M_{\text{SO}}^{\text{vev}} \equiv u M u^T = \text{diag}(\mu_1, \mu_2, \cdots, \mu_{N_C}, 0, \cdots),
\]

with \( u \in U(N_F) \) and parameters \( \mu_i \in \mathbb{R}_{\geq 0} \) are square roots of the eigenvalues of \( M M^\dagger \). The meson field \( M \) in the \( USp(2M_C) \) case, which is an anti-symmetric matrix, can be also diagonalized as

\[
M_{\text{USp}}^{\text{vev}} \equiv u M u^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \text{diag}(\mu_1, \mu_2, \cdots, \mu_{M_C}, 0, \cdots).
\]

See Appendix A.3 for the proof. These vacuum configurations in both the cases, \( M_{\text{vev}} = M_{\text{SO}}^{\text{vev}}, M_{\text{vev}}^{\text{USp}} \), are summarized as

\[
(M_{\text{vev}})_{ij} = \mu_i (J)_{ij} = (J)_{ij} \mu_j,
\]

where we take the invariant tensors as \( (J)_{ij} = \delta_{ij} \) for the \( SO(N_C) \) case, and \( (J)_{ij} = \delta_{i,M_F,j} - \delta_{i,j,M_F} \) and \( \mu_{i,M_F} \equiv \mu_i, (1 \leq i \leq M_F) \) in the case of \( USp(N_C = 2M_C) \).

For simplicity, let us concentrate on the \( SO(N_C) \) case with \( N_C = N_F \), and consider generic points of the manifold with rank\( (M_{\text{vev}}) = N_C \), that is, \( \mu_i > 0 \) for all \( i \). In this case, there are no constraints for the meson field locally, and thus, the meson field \( M \) can be treated as coordinates parametrizing the manifold locally. It is convenient to consider a small fluctuation \( \phi = M - M_{\text{vev}} \) around the vacua \( M_{\text{vev}} \) and expand the formula (3.19) with respect to \( \phi \). The following formula is useful to expand a function \( f(X) \) of a matrix \( X \) in a trace around \( X = X_0 \),

\[
\text{Tr}[f(X_0 + \delta X)] = \frac{1}{2\pi i} \oint_{\mathcal{C}} d\lambda \ f(\lambda) \text{Tr} \left[ \frac{1}{\lambda 1 - X_0 - \delta X} \right]
= \text{Tr}[f(X_0)] + \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_{\mathcal{C}} d\lambda \ f'(\lambda) \text{Tr} \left[ \left( \frac{1}{\lambda 1 - X_0} \delta X \right)^n \right],
\]

where the closed path \( \mathcal{C} \) surrounds all eigenvalues of \( f(X) \) on the real positive axis but no singularities of \( f(\lambda) \). We set \( f(\lambda) = \sqrt{\lambda} \) and

\[
X = MM^\dagger, \quad X_0 = \text{diag}(\mu_1^2, \cdots, \mu_{N_C}^2), \quad \delta X = M_{\text{vev}}^{\dagger} + \phi M_{\text{vev}}^{\dagger} + \phi^{\dagger} \phi.
\]

Since \( f(\lambda) = \sqrt{\lambda} \) has a branch point at the origin, the eigenvalues \( \mu_i \) cannot be zero in this formula. To proceed the calculation, we need to perform the integrations

\[
A_n(\mu_1, \cdots, \mu_n) \equiv \frac{1}{2\pi i} \oint d\lambda \prod_{i=1}^{n} \frac{1}{\lambda - \mu_i^2}.
\]

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The results of the integrations can be expressed in terms of the elementary symmetric polynomials, $C_{k_1 k_2 \ldots k_n}^{(m)}$, $(m \leq n)$ defined by

$$\prod_{i=1}^{n}(t + \mu_{k_i}) = \sum_{m=0}^{n} C_{k_1 \ldots k_n}^{(m)} t^{n-m}, \quad P_{k_1 \ldots k_n} \equiv \prod_{m>n} (\mu_{k_m} + \mu_{k_n}),$$

where we also use a symmetric polynomial $P_{k_1 \ldots k_n}$. The first few integrations give

$$A_1(\mu_1) = \frac{1}{\mu_1}, \quad A_2(\mu_1, \mu_2) = -\frac{1}{\mu_1 \mu_2 (\mu_1 + \mu_2)},$$

$$A_3(\mu_1, \mu_2, \mu_3) = \frac{C_{123}^{(1)}}{C_{123}^{(3)} P_{123}} = \frac{\mu_1 + \mu_2 + \mu_3}{\mu_1 \mu_2 \mu_3 (\mu_1 + \mu_2) (\mu_2 + \mu_3) (\mu_3 + \mu_1)},$$

$$A_4(\mu_1, \mu_2, \mu_3, \mu_4) = -\frac{C_{1234}^{(1)} C_{1234}^{(2)}}{C_{1234}^{(4)} P_{1234}}.$$

After this preparation, we obtain the first few terms of the expansion of the Kähler potential as

$$K_{SO} = \frac{1}{2} \sum_{i,j} \frac{\phi_{ij} \phi_{ji}}{\mu_i + \mu_j} - \frac{1}{2} \sum_{i,j,k} \frac{\mu_i \phi_{ij} \phi_{jk} \phi_{ki}}{(\mu_i + \mu_j) (\mu_j + \mu_k) (\mu_k + \mu_i)} + \text{c.c.}$$

$$+ \frac{1}{2} \sum_{i,j,k,l} \frac{\mu_j \mu_k C_{ijkl}^{(1)}}{P_{ijkl}} \phi_{ij} \phi_{jk} \phi_{kl} \phi_{li} + \text{c.c.}$$

$$+ \frac{1}{2} \sum_{i,j,k,l} \frac{\mu_j \mu_k C_{ijkl}^{(1)}}{P_{ijkl}} \phi_{ij} \phi_{jk} \phi_{kl} \phi_{li} - \frac{1}{4} \sum_{i,j,k,l} \frac{C_{ijkl}^{(3)}}{P_{ijkl}} \phi_{ij} \phi_{jk} \phi_{kl} \phi_{li}$$

$$+ \text{Kähler trf.} + \mathcal{O}(\phi^5).$$

A coordinate singularity emerges in the limit $\mu_i \to 0$ since the expansion formula (3.29) is not applicable for $\mu_i = 0$. The above result gives enough information to calculate the scalar curvature $R$ of the manifold at $M = M_{\text{vev}}$ in the $SO(N_C)$ case, with a Kähler metric $g_{IJ}$

$$R|_{\phi=0} = -2 g^{IJ} \partial_I \partial_J \log g \bigg|_{\phi=0}$$

$$= 2 \sum_{i>j} \left( \frac{1}{\mu_i + \mu_j} + \sum_{k} \frac{\mu_k}{(\mu_k + \mu_i)(\mu_k + \mu_j)} \right) > 0,$$

where the indices $I, \bar{J}$ label the components as $\phi^I = \phi_{ij}$, $(i \geq j)$. This result shows that the coordinate singularity with rank($M_{\text{ev}}$) = $N_C - 1$ can be removed by taking appropriate coordinates and, on the other hand, the submanifold with rank($M_{\text{ev}}$) < $N_C - 1$ is a curvature singularity of the manifold. That is, the curvature singularity lies in the region corresponding to the Coulomb phase of the original gauge theory, as we expected.
The expansion of the Kähler potential in the $USp(2M_C)$ case, we obtain the result (3.34) with the substitution $\phi \to \phi J^\dagger$, $\phi^\dagger \to J\phi^\dagger$ and the curvature obtained using this expanded potential reads

$$R|_{\phi=0} = 4 \sum_{i>j}^{M_C} \left( \frac{1}{\mu_i + \mu_j} + \sum_k^{M_C} \frac{4\mu_k}{(\mu_k + \mu_i)(\mu_k + \mu_j)} \right) > 0 . \quad (3.36)$$

This result shows that the submanifold with $\text{rank}(M_{\text{vev}}) < 2(M_C - 1)$ is a curvature singularity of the manifold. This expansion, however, does not reveal the singularity appearing at $\text{rank}(M_{\text{vev}}) = 2(M_C - 1)$. To detect this singularity, we consider a deformation of the Kähler potential

$$K_{\text{USp,deformed}} = \text{Tr} \sqrt{MM^\dagger} + \varepsilon^2 , \quad (3.37)$$

and make a similar expansion (see Appendix C). Taking now only one eigenvalue, say $\mu_1 \to 0$ we find a term in the scalar curvature

$$\lim_{\mu_1 \to 0} R|_{\phi=0} \supset \frac{2}{\varepsilon} , \quad (3.38)$$

which shows the presence of a singularity for one vanishing eigenvalue, that is corresponding to an unbroken $USp(2) \simeq SU(2)$ symmetry.

### 3.2 The $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ Kähler Quotients

Next, we would like to consider a Kähler quotient with gauging an overall $U(1)$ phase in addition to the $SO(N_C)$ or $USp(2M_C)$ gauge symmetry. We turn on the FI $D$-term associated with the additional $U(1)$ gauge group. The Kähler potential can be written as

$$K_{U(1)\times(SO,USp)} = \text{Tr} \left[ QQ^T e^{-V^e} e^{-V_e} + \lambda \left( e^{-V^T} e^{-V^e} - J \right) \right] + \xi V_e , \quad (3.39)$$

where $V_e$ is the vector multiplet of the additional $U(1)$ gauge field. We have already solved the $SO(N_C)$ and $USp(2M_C)$ part in the previous section, so the Kähler potential can be rewritten as

$$K_{U(1)\times(SO,USp)} = \text{Tr} \left[ \sqrt{MM^\dagger} \right] e^{-V_e} + \xi V_e . \quad (3.40)$$

The equation of motion for $V_e$ can be solved by $V_e = \log \left[ \text{Tr} \left( \sqrt{MM^\dagger} \right) / \xi \right]$. Plugging this into the Kähler potential, we obtain

$$K_{U(1)\times(SO,USp)} = \xi \log \left[ \text{Tr} \left( \sqrt{MM^\dagger} \right) \right] , \quad M \equiv Q^T JQ . \quad (3.41)$$
In the case of \( N_C = N_F \), we can expand the Kähler potential around a point \( M = M_{vev} \) by using the same method as in Sec 3.1:

\[
K_{U(1) \times (SO,USp)} = \frac{\xi}{2 \sum_{k=1}^{N_C} \mu_k} \left( \sum_{i,j} \frac{\phi_{ij}(\phi_{ij})^\dagger}{\mu_i + \mu_j} - \frac{1}{2} \sum_{k=1}^{N_C} \left| \sum_{i=1}^{N_C} (J_i^\dagger \phi)_{ii} \right|^2 \right) + \text{Kähler trf.} + \mathcal{O}(\phi^3) \, .
\]

(3.42)

Here we can confirm that the mode \( \phi \propto M_{vev} \) corresponding to \( U(1)^C \) is not effective in this Kähler potential. Therefore, with the constraint \( \text{Tr} [\phi J^\dagger] = 0 \), we can write the Kähler potential to fourth order as

\[
K_{U(1) \times (SO,USp)} = \sum_{k=1}^{N_C} \frac{\xi}{\mu_k} \left( K_{SO,USp} - \frac{1}{8} \sum_{i,j} \phi_{ij} \phi_{ji}^\dagger \sum_{i,j} \phi_{ij} \phi_{ji}^\dagger \mu_i + \mu_j \right) ^2 + \text{Kähler trf.} + \mathcal{O}(\phi^5) \, .
\]

(3.43)

from which we obtain the curvatures as

\[
\xi R_{U(1) \times (SO,USp)} = R_{(SO,USp)} \sum_{i=1}^{N_C} \mu_i + 2 \hat{N}_e (\hat{N}_e + 1) \, ,
\]

(3.44)

where \( \hat{N}_e \) is the complex dimension of the manifold

\[
\hat{N}_e \equiv \dim_{\mathbb{C}} \mathcal{M}^{\text{vacuum}}_{U(1) \times (SO,USp)} = \frac{N_C(N_C + \epsilon)}{2} - 1 \, , \quad \epsilon = \begin{cases} +1 & \text{for } SO \\ -1 & \text{for } USp \end{cases} .
\]

(3.45)

A typical property of these theories is the existence of curvature singularities of the Kähler manifold. Since the Coulomb phase attached to the Higgs phase in the original gauge theory is strongly related to a singularity, the curvature singularity with \( 0 < \text{rank}(M) < N_C - 1 \) still survives after the \( U(1) \) gauging for the case of \( N_C \geq 3 \), while gauging \( U(1) \) in the \( SU(N_C) \) case removes the singularity.

### 3.3 Examples

#### 3.3.1 The \( SO(2) \) Quotient (SQED) and the \( U(1) \times SO(2) \) Quotient

The first example is \( SO(2) \) with \( N_F = 1 \). We have a complexified gauge symmetry \( SO(2)^C \), so the corresponding target space is

\[
\mathcal{M}^{SO(2)}_{N_F=1} = Q/\sim \, , \quad Q \sim g'Q \, , \quad g' \in SO(2)^C \, ,
\]

(3.46)
where \( Q = (Q_+, Q_-)^T \). In general, matrices in \( SO(2)^C \) can be expressed as
\[
g' = \begin{pmatrix} v' & 0 \\ 0 & 1/v' \end{pmatrix}, \quad v' \in \mathbb{C}^*.
\]
This simply shows the fact that \( SO(2) \simeq U(1) \) under which \( Q_+ \) has charge +1 while \( Q_- \) has charge \(-1\). This is nothing else than supersymmetric QED. The target space apparently seems to be a weighted complex projective space which is not a Hausdorff space
\[
\mathcal{M}_{N_F=1}^{SO(2)} = W\mathbb{C}P^1_{(1,-1)}. \tag{3.48}
\]
However, we have to be careful. Sick points \((Q_+, Q_-) = (Q_+, 0), (0, Q_-)\) for \( Q_+ \neq 0 \) and \( Q_- \neq 0 \) are forbidden by the \( D \)-term condition \(|Q_+|^2 - |Q_-|^2 = 0\) in the Wess-Zumino gauge. To understand the true well-defined target space, we take the holomorphic invariant of this model to be
\[
M = 2Q_+Q_- \quad \text{.} \tag{3.49}
\]
This is a good coordinate on the target space and the Kähler potential is given by
\[
K_{N_F=1}^{SO(2)} = |M| \quad \text{.} \tag{3.50}
\]
There is a conical singularity at the origin and the true target space is
\[
\mathcal{M}_{N_F=1}^{SO(2)} = \mathbb{C}/\mathbb{Z}_2 \quad \text{.} \tag{3.51}
\]
At the singularity, the gauge symmetry is restored and the vector multiplet obtains a massless field. In general, singularities in a classical moduli space leads to the appearance of some massless fields. Kähler potentials usually acquire quantum corrections and they may make such classical singular manifolds regular.

The second example is \( U(1) \times SO(2) \) with \( N_F = 1 \). We turn on the FI parameters \( \xi \) and we have
\[
\mathcal{M}_{N_F=1}^{U(1) \times SO(2)} = Q/\sim, \quad Q \sim V_eV'Q, \quad V_e \in U(1)^C, \quad V' \in SO(2)^C. \tag{3.52}
\]
We can explicitly show that
\[
g_e^{-1}g' = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}^* \quad . \tag{3.53}
\]
Here we impose that the gauge symmetry \( U(1) \times SO(2) \) is free, such that \(|Q| \neq 0\). Hence, the target space is just one point.
Next, let us consider $N_F = 2$ with the $SO(2)$ and the $U(1) \times SO(2)$ gauge groups. The scalar field is a 2 by 2 complex matrix

$$Q = \begin{pmatrix} Q_{+1} & Q_{+2} \\ Q_{-1} & Q_{-2} \end{pmatrix} \equiv \begin{pmatrix} \vec{Q}_+ \\ \vec{Q}_- \end{pmatrix}.$$  \hfill (3.54)

The holomorphic invariants of the $SO(2)$ part are on the form

$$M_{SO(2)} = \{ Q^T J Q, \det Q \} = \left\{ \begin{array}{c} 2Q_{-1}Q_{+1} \\ Q_{+1}Q_{-2} + Q_{+2}Q_{-1} \end{array} \right\}, \quad Q_{+1}Q_{-2} - Q_{+2}Q_{-1}). \hfill (3.55)$$

We have to remove the points $\vec{Q}_+ = 0$ and $\vec{Q}_- = 0$, where all the holomorphic invariants vanish $M = 0$. The moduli spaces of vacua turn out to be

$$M_{SO(2)}^{N_F=2} = \mathbb{C}P^2_{(1,-1,-1)} - \{ M_{SO(2)} = 0 \} = ((\mathbb{C}^2)^* \times (\mathbb{C}^2)^*)/\mathbb{C}^* , \hfill (3.56)$$

$$M_{U(1) \times SO(2)}^{N_F=2} = ((\mathbb{C}^2)^*/\mathbb{C}^*) \times ((\mathbb{C}^2)^*/\mathbb{C}^*) = \mathbb{C}P^1 \times \mathbb{C}P^1. \hfill (3.57)$$

Since positive real eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \sqrt{\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\lambda_2}}$, the Kähler potential can be easily shown to be

$$K_{SO(2)}^{N_F=2} = \sqrt{\text{Tr} M M^\dagger} + 2\sqrt{\det M M^\dagger} = 2\sqrt{\lvert \vec{Q}_+ \rvert^2 \lvert \vec{Q}_- \rvert^2} , \hfill (3.58)$$

$$K_{U(1) \times SO(2)}^{N_F=2} = \frac{\xi}{2} \log \lvert \vec{Q}_+ \rvert^2 + \frac{\xi}{2} \log \lvert \vec{Q}_- \rvert^2 . \hfill (3.59)$$

The prefactor $\xi/2$ in Eq. (3.59) will turn out to have a significant difference from the usual prefactor $\xi$ of the Kähler potential for usual $\mathbb{C}P^1$, see Eq. (2.16), when we will consider 1/2 BPS solitons.

It is straightforward to extend this to the case with generic $N_F$. The manifolds are on the form

$$M_{SO(2)}^{N_F} = \mathbb{C}P^{2N_F-2N_F-1}_{(1,-1,-1)} - \{ M_{SO(2)} = 0 \} = ((\mathbb{C}^{N_F})^* \times (\mathbb{C}^{N_F})^*)/\mathbb{C}^* , \hfill (3.60)$$

$$M_{U(1) \times SO(2)}^{N_F} = ((\mathbb{C}^{N_F})^*/\mathbb{C}^*) \times ((\mathbb{C}^{N_F})^*/\mathbb{C}^*) = \mathbb{C}P^{N_F-1} \times \mathbb{C}P^{N_F-1}. \hfill (3.61)$$

The Kähler potential for the latter manifold can be obtained by merely replacing the two vectors $Q_{1,2}$ by $N_F$ vectors in Eq. (3.59). Then the meson field becomes an $N_F$-by-$N_F$ matrix, however, only two eigenvalues $\lambda_1, \lambda_2$ of $M M^\dagger$ take non-zero values and in this case we have the following identity

$$\det(\lambda_1 I_{N_F} - MM^\dagger) = \lambda^{N_F-2} \det(\lambda_2 - (QQ^\dagger) J^\dagger (QQ^\dagger)^T J) . \hfill (3.62)$$
From this characteristic polynomial, we can read off
\[ \lambda_1 + \lambda_2 = 2|\vec{Q}_+|^2|\vec{Q}_-|^2 + 2|\vec{Q}_+\vec{Q}_-^\dagger|^2, \quad \lambda_1\lambda_2 = \left(|\vec{Q}_+|^2|\vec{Q}_-|^2 - |\vec{Q}_+\vec{Q}_-^\dagger|^2\right)^2. \] (3.63)

Therefore, we find also in the case of \( N_F \) flavors
\[ K_{N_F}^{SO(2)} = \sqrt{\lambda_1} + \sqrt{\lambda_2} = 2\sqrt{|\vec{Q}_+|^2|\vec{Q}_-|^2}. \] (3.64)

### 3.3.2 The \( U Sp(2) \) Quotient

This case completely reduces to the \( SU(2) \) case with \( N_F \) flavors. It is not difficult to show that only two eigenvalues \( \lambda_1, \lambda_2 \) of \( MM^\dagger \) take non-zero values and they coincide
\[ \lambda_1 = \lambda_2 = \frac{1}{2} \text{Tr}[MM^\dagger] = \text{det}(QQ^\dagger), \] (3.65)
and this indeed yields the Kähler potential for the \( SU(2) \) case
\[ K_{N_F}^{USp(2)\approx SU(2)} = \text{Tr}[\sqrt{MM^\dagger}] = 2\sqrt{\text{det}(QQ^\dagger)}. \] (3.66)

We find explicitly the \( \mathbb{Z}_2 \)-conifold singularity at the origin in this model.

### 3.3.3 The \( U Sp(4) \) Quotient

By “diagonalizing” \( M \) by \( M_{ij} = \mu_iJ_{ij} \), we find two non-vanishing eigenvalues both with multiplicity two, that is \( \lambda_1 = \lambda_3 = \mu_1^2 \) and \( \lambda_2 = \lambda_4 = \mu_2^2 \) and they can be written as
\[ \lambda_1 + \lambda_2 = \frac{1}{2} \text{Tr}[MM^\dagger], \quad \lambda_1\lambda_2 = \sum_{(A)} |P_{(A)}|^2, \] (3.67)
where \( P_{(A)} \) is the Pfaffian of a minor matrix
\[ P_{(A_1A_2A_3A_4)} \equiv 3M_{A_1[A_2}M_{A_3A_4]} \cdot \] (3.68)

In this case where we have \( U Sp(4) \) i.e. \( M_C = 2 \), thus it can be written as
\[ \sum_{(A)} |P_{(A)}|^2 = \frac{1}{8} \left(\text{Tr}[MM^\dagger]\right)^2 - \frac{1}{4} \text{Tr}[(MM^\dagger)^2]. \] (3.69)

Since the right hand sides of both the equations in Eq. (3.67) are invariant under the flavor transformation performing the diagonalization, we find for generic number of flavors \( N_F \)
\[ K_{N_F}^{USp(4)} = 2 \left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right) = 2\sqrt{\frac{1}{2} \text{Tr}[MM^\dagger] + 2\sum_{(A)} |P_{(A)}|^2}. \] (3.70)
Considering a minimal case with $M_F = M_C = 2$, with the following parametrization

$$M = \begin{pmatrix} 
0 & \phi_1 & \phi_2 & \phi_3 \\
-\phi_1 & 0 & \chi_3 & -\chi_2 \\
-\phi_2 & -\chi_3 & 0 & \chi_1 \\
-\phi_3 & \chi_2 & -\chi_1 & 0 
\end{pmatrix}, \quad (3.71)$$

we find $\text{Pf}(M) = \vec{\phi} \cdot \vec{\chi}$ and the simple form of the Kähler potential

$$K_{U\text{Sp}(4)}^{N_F=4} = 2 \sqrt{\frac{1}{2} \text{Tr}[MM^\dagger]} + 2|\text{Pf}(M)| = 2 \sqrt{|\vec{\phi}|^2 + |\vec{\chi}|^2 + 2|\vec{\phi} \cdot \vec{\chi}|}. \quad (3.72)$$

Manifestly, we can observe an orbifold singularity on the submanifold

$$|\vec{\phi}|^2 + |\vec{\chi}|^2 \neq 0, \quad \text{Pf}(M) = \vec{\phi} \cdot \vec{\chi} = 0, \quad (3.73)$$

of which the rank is $2M_C - 2 = 2$, since the $\text{Pf}(M) \in \mathbb{C}$ is an appropriate coordinate describing the orthogonal direction to the submanifold and the term $\sqrt{|\text{Pf}(M)|^2}$ emerges in the potential. In a generic region away from this singular submanifold, the scalar curvature is given by

$$R = \frac{20}{\sqrt{|\vec{\phi}|^2 + |\vec{\chi}|^2 + 2|\vec{\phi} \cdot \vec{\chi}|}}, \quad (3.74)$$

and is finite even in the vicinity of the submanifold.

### 3.3.4 The $SO(3)$ Quotient

The Kähler quotient for $SO(3)$ with $N_F$ flavors reads

$$K_{N_F}^{SO(3)} = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}, \quad (3.75)$$

and it is obtained by solving the following algebraic equations

$$(K^2 - A_1)^2 = 4A_2 + 8\sqrt{A_3}K, \quad (3.76)$$

where the definitions are

$$A_1 \equiv \lambda_1 + \lambda_2 + \lambda_3 = \text{Tr}[MM^\dagger],$$
$$A_2 \equiv \lambda_1\lambda_2 + \lambda_3\lambda_2 + \lambda_3\lambda_1 = \frac{1}{2}(\text{Tr}[MM^\dagger])^2 - \frac{1}{2}\text{Tr}[(MM^\dagger)^2],$$
$$A_3 \equiv \lambda_1\lambda_2\lambda_3. \quad (3.77)$$
A solution with a real number satisfying $K^2 \geq A_1 > 0$ should be unique. Here $\sqrt{A_3}$ does not imply a singularity immediately. In the case of $N_F = N_C = 3$, we can rewrite it in terms of the baryon field $B$ as

$$\sqrt{A_3} = \sqrt{\det(MM^\dagger)} = \sqrt{\det M^2} = |B|^2,$$

(3.78)

and around the submanifold with $B = 0$, $B$ is an appropriate coordinate around the submanifold. With $K_0 = |B|^2 = 0$, we find

$$K_{N_F=3}^{SO} = K_0 + \frac{2|B|^2}{K_0^2 - A_1} + O(|B|^4).$$

(3.79)

Since $K_0^2 - A_1 = 0$ implies that $A_2 = |B|^2 = 0$, which in turn implies that rank $M \geq N_C - 2 = 1$, this expansion tells us that the submanifold with rank $M = N_C - 1 = 2$ is not singular.

Let us now consider this simple example of $SO(3)$ with $N_F = 2$. The result of the Kähler potential is the same as in the $SO(2)$ case with $N_F = 2$

$$K_{N_F=2}^{SO} = \sqrt{\text{Tr} MM^\dagger + 2|\det M|}.$$

(3.80)

### 3.4 The $SO(N_C)$ and $USp(2M_C)$ Hyper-Kähler Quotients

Our next task is lifting up the $SO(N_C)$ and $USp(2M_C)$ Kähler quotients of the previous subsection to the hyper-Kähler quotients as we did for the $U(N_C)$ (hyper-)Kähler quotient in Sec. 2. We leave the issues of the hyper-Kähler quotients of $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ for the end of this section. In order to construct the $SO(N_C)$, $USp(2M_C)$ hyper-Kähler quotient we need to consider $\mathcal{N} = 2$ hypermultiplets. Hence, we consider an $\mathcal{N} = 2$ extension of the $\mathcal{N} = 1$ Kähler potential (3.14), together with the superpotential

$$\tilde{K}_{SO,USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} + Q^\dagger \tilde{Q} e^{V'} + \lambda \left( e^{-V'T} J e^{-V'} - J \right) \right],$$

(3.81)

$$W = \text{Tr} \left[ Q \tilde{Q} \Sigma' + \chi \left( \Sigma'^T J + J \Sigma' \right) \right],$$

(3.82)

where $(V', \Sigma')$ denote the $SO(N_C)$ or $USp(2M_C)$ vector multiplets, $(Q, \tilde{Q}^\dagger)$ are $N_F$ hypermultiplets in the fundamental representation of $SO(N_C)$ or $USp(2M_C)$, and $(\lambda, \chi)$ are the Lagrange multipliers which are $N_C$-by-$N_C$ matrix valued superfields.

We can rewrite the Kähler potential (3.81) as follows

$$\tilde{K}_{SO,USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} + J^T e^{-V'} J \tilde{Q}^T \tilde{Q} \right] = \text{Tr} \left[ QQ^\dagger e^{-V'} \right], \quad Q \equiv \left( Q, J \tilde{Q}^T \right),$$

(3.83)
where we have used $e^{V_T} = J^T e^{-V'} J$. This Kähler potential is nothing but the $\mathcal{N} = 1$ Kähler potential of $SO(N_C)$ and $USp(2M_C)$ with $Q$, a set of $2N_F$ chiral superfields. We can straightforwardly borrow the result of Sec. 3.1 and hence the Kähler potential reads

$$K_{SO,USp} = \text{Tr} \left[ \sqrt{\mathcal{M} \mathcal{M}^\dagger} \right], \quad \mathcal{M} = Q^T J Q. \tag{3.84}$$

The constraint coming from the superpotential (3.82) is

$$QQ' = JQ'Q^T = 0, \quad \text{with} \quad J \equiv \begin{pmatrix} 0 & 1_{N_F} \\ -\epsilon & 0 \end{pmatrix}. \tag{3.85}$$

Therefore, we again find the constraints for the meson field $\mathcal{M}$

$$\mathcal{M}^T = \epsilon \mathcal{M}, \quad \mathcal{M} J \mathcal{M} = 0, \quad N_C - 2 \leq \text{rank} \mathcal{M} \leq N_C. \tag{3.86}$$

As is well-known, the $SO(N_C)$ case has a $USp(2N_F)$ flavor symmetry while the $USp(2M_C)$ case has an $SO(2N_F)$ flavor symmetry. Therefore the $USp(2N_F)$ and $SO(2N_F)$ isometries act on the $SO(N_C)$ and $USp(2M_C)$ hyper-Kähler quotients, respectively. The resultant spaces can be written locally in generic points as

$$\mathcal{M}_{SO(N_C)}^{HK} \simeq \mathbb{R}_{>0}^{N_C} \times \frac{USp(2N_F)}{USp(2N_F - 2N_C) \times (\mathbb{Z}_2)^{N_C-1}} \supset \mathbb{R}_{>0}^{N_C} \times \frac{U(N_F)}{U(N_F - N_C) \times (\mathbb{Z}_2)^{N_C-1}}, \tag{3.87}$$

$$\mathcal{M}_{USp(2M_C)}^{HK} \simeq \mathbb{R}_{>0}^{M_C} \times \frac{SO(2N_F)}{SO(2N_F - 4M_C) \times Usp(2)^{M_C}} \supset \mathbb{R}_{>0}^{M_C} \times \frac{U(N_F)}{U(N_F - 2M_C) \times Usp(2)^{M_C}}, \tag{3.88}$$

for the $SO(N_C)$ and $USp(2M_C)$ hyper-Kähler quotients, respectively. These are hyper-Kähler spaces of cohomogeneity $N_C$ and $M_C$, respectively. The rightest ones denote the corresponding $SO(N_C)$ and $USp(2M_C)$ Kähler quotients given in Eqs. (3.4) and (3.8), respectively. These Kähler spaces are special Lagrangian subspaces of the hyper-Kähler spaces. As in the Kähler cases (3.4) and (3.8), the isotropy (unbroken flavor symmetry) changes from point to point. It is enhanced when some eigenvalues coincide.

Let us make a comment on the relation to the instanton moduli space. In Eq. (3.88) the simplest case of the $USp(2) \simeq SU(2)$ hyper-Kähler quotient was previously found in [6] to be

$$\mathcal{M}_{USp(2)\simeq SU(2)}^{HK} \simeq \mathbb{R}_{>0} \times \frac{SO(2N_F)}{SO(2N_F - 4) \times USp(2)}. \tag{3.89}$$

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9 Any smooth hyper-Kähler manifold of cohomogeneity one, must be the cotangent bundle over the projective space, $T^* \mathbb{C}P^{N_F-1}$ or flat space [69]. For the $U(1)$ hyper-Kähler quotient with $N_F$ flavors, the space is of cohomogeneity one: $\mathbb{R}_{>0} \times SU(N_F)/SU(N_F - 2)$. This space is blown up to a smooth manifold $T^* \mathbb{C}P^{N_F-1}$ once the FI parameters are introduced for the $U(1)$ gauge group. The result of Ref. [69] implies that hyper-Kähler spaces of cohomogeneity one in Eqs. (3.87) and (3.88) must have a singularity.
This is a hyper-Kähler cone and is particularly important because the single instanton moduli space of an $SO(2N_F)$ gauge theory is the direct product of this space and $\mathbb{C}^2$ i.e. the position. Here $\mathbb{R}_{>0}$ parametrizes the size while the coset part parametrizes the orientation of a single BPST instanton embedded into the $SO(2N_F)$ gauge group. The moduli space of $k$ instantons in $SO(N_C)$ and $USp(2M_C)$ gauge theories are known to be given by $USp(2k)$ and $O(k)$ hyper-Kähler quotients, respectively \[9, 10\]. Compared with our spaces in Eqs. \[3.87\] and \[3.88\], the instanton moduli spaces contain adjoint fields of $USp(2k)$ and $O(k)$ too and thus are larger. Inclusion of adjoint fields remains as a difficult but important problem.

Before closing this section we make a comment on the hyper-Kähler quotient of $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$. We succeeded in constructing the hyper-Kähler quotient of $SO(N_C)$ and $USp(2M_C)$ thanks to the fact that $J\tilde{Q}^T$ is in the fundamental representation, which is the same representation as $Q$. Although, we want to make use of the same strategy for $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ as before, $J\tilde{Q}^T$ still has charge $-1$ with respect to the $U(1)$ gauge symmetry while $Q$ has $U(1)$ charge $+1$. Therefore, it is not easy to construct the $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ and we will not solve this problem in this article.

## 4 1/2 BPS Configurations: NL$\sigma$M Lumps

In this section we will study NL$\sigma$M lumps which are 1/2 BPS configurations. Lumps are stringy topological textures extending for instance in the $x^3$ direction in $d = 1 + 3$ dimensional spacetime and are supported by the non-trivial second homotopy group $\pi_2(M)$ associated with a holomorphic map from the 2 dimensional spatial plane $z = x_1 + ix_2$ to a 2-cycle of the target space of the NL$\sigma$M. We will consider the $\mathbb{C}$-plane together with the point at infinity, that is $z \in \mathbb{C} \cup \{\infty\} \simeq S^2$, which is mapped into the target space. Lumps in non-supersymmetric $SO(N_C)$ theories were studied in Ref. \[68\] where the second homotopy group is $\pi_2[SU(N_C)/SO(N_C)] \simeq \mathbb{Z}_2$ and therefore those lumps are non-BPS. Here we do not consider this type of lumps. We will first study BPS lumps in the NL$\sigma$M of $U(1) \times G'$ Kähler quotients in general, then we investigate lumps in the case of $G' = SO, USp$ which have been constructed in previous sections.

### 4.1 Lumps in $U(1) \times G'$ Kähler Quotients

In the NL$\sigma$M of $U(1) \times G'$ Kähler quotients, (inhomogeneous) complex coordinates $\{\phi^a\}$ of the Kähler manifold, which are the lowest scalar components of the chiral superfields, are given by
some set of holomorphic $G'$ invariants $I^i$ modulo $U(1)^C$, namely $\phi^\alpha \in \{I^i\}/U(1)^C$. Static lump solutions can be obtained by just imposing $\phi^\alpha$ to be a holomorphic function with respect to $z$

$$\phi^\alpha(t, z, \bar{z}, x^3) \rightarrow \phi^\alpha(z; \varphi^i) ,$$

where $\varphi^i$ denote complex constants. The tension of the lumps can be obtained by plugging the solution back into the Lagrangian

$$T = 2 \int d^2 x \, K_{\alpha\bar{\beta}}(\phi, \bar{\phi}) \frac{\partial \phi^\alpha}{\partial \phi} \frac{\partial \bar{\phi}^{\bar{\beta}}}{\partial \bar{\phi}} \bigg|_{\phi \rightarrow \phi(z)} = 2 \int d^2 x \, \bar{\partial} \partial K(\phi, \bar{\phi}) \bigg|_{\phi \rightarrow \phi(z)} ,$$

where $K$ is the Kähler potential and $K_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} K$ is the Kähler metric. We would like to stress that all the parameters $\varphi^i$ are nothing but the moduli parameters of the 1/2 BPS lumps.

We assume that the boundary of $z \rightarrow \infty$ is mapped to a single point $\phi^\alpha(z) \rightarrow \phi^\alpha_{\text{vev}}$ on the target space. Since the functions $\phi^\alpha(z)$ should be single valued, $\phi^\alpha(z)$ can be expressed with a finite number of poles as

$$\phi^\alpha(z) = \phi^\alpha_{\text{vev}} + \sum_{i=1}^k \frac{\phi_{\text{vev}}^\alpha}{z - z_i} + O(z^{-2}) .$$

Strictly speaking, we have to change patch of the target manifold at the poles to describe the solutions correctly. To describe the lump solutions, it is convenient to use the holomorphic $G'$ invariants $I^i$ satisfying the constraints as homogeneous coordinates. The holomorphic map is expressed by the homogeneous coordinates $I^i(z)$ which are holomorphic in $z$

$$I^i(z) = I^i_{\text{vev}} z^{n_i \nu} + O(z^{n_i \nu - 1}) ,$$

where $n_i$ is the $U(1)$ charge of the holomorphic $G'$ invariant $I^i$, and $\nu$ is some number. $I^i_{\text{vev}}$ denotes the vacuum expectation value of $I^i$ at spatial infinity. Since all $n_i \nu$ must take value in $\mathbb{Z}_{>0}$, we can express $\nu = k/n_0$ with the greatest common divisor (GCD) $n_0$ of $\{n_i\}$ and $k$ a non-negative integer. The integer $k$ will be found to be the topological winding number. These polynomials are basic tools to study lump solutions and their moduli, and $\phi^\alpha(z)$ can be written as ratios of these polynomials, namely $U(1)^C$ invariants, which are known as rational maps in the Abelian case.

There is a remark in store for constructing lump solutions. If a holomorphic map touches the unbroken phase of the original gauge theory at some point, the behavior of the lump is ill-defined there in terms of the NLÎσM. Generally speaking, as we will see in examples later, the lump configuration becomes singular at that point. Therefore, we have to exclude such singular configurations and all points in the base manifold $\mathbb{C}$ must be mapped to the full Higgs phase by the
holomorphic map \[(4.4)\]. We will denote this condition the **lump condition**. In other words, there exist limits where lump configurations become singular by varying the moduli parameters. For instance, the invariants $I_i(z)$ are prohibited from having common zeros by the lump condition. Since common zeros cannot be detected even in the vicinity of a corresponding point in the base space, an emergence of common zeros indicates a small lump singularity, which is well-known for lumps in the $\mathbb{C}P^n$ model. The lump condition requires non-vanishing size moduli there. As we will show in examples later, this situation implies the emergence of a local vortex. The lump condition is stronger than the condition of no common zeros in the invariants, except for the $U(N)$ case \[56\], where in fact both the conditions are equivalent. The difference between the two conditions above implies the existence of limits where a lump configuration becomes singular with a non-vanishing size. This is a typical property of lumps in a NL$\sigma$M with a singular submanifold. We will see explicit examples of this property later.

### 4.2 Lump Moduli Spaces vs. Vortex Moduli Spaces

As a NL$\sigma$M can be obtained in the strong gauge coupling limit of the gauge theory, lump solutions in such NL$\sigma$Ms can also be given as that limit of semi-local vortex solutions, whose configuration can smoothly be mapped to the Higgs phase. Therefore, lump solutions are closely related to semi-local vortices in the original gauge theory, even with a finite gauge coupling. Lumps in the $U(N_C)$ Kähler quotient, namely in the Grassmann sigma model, have been studied previously in Refs. \[29, 55, 56\]. In fact, the dimensions of both the moduli spaces coincide
\[
dim_{\mathbb{C}}\mathcal{M}_{U(N_C),N_F}^{k\text{-vortex}} = \dim_{\mathbb{C}}\mathcal{M}_{U(N_C),N_F}^{k\text{-lump}} = kN_F \quad \text{\[38, 42\].}
\]
It has been found that the moduli space of $k$ lumps in the Grassmann sigma model is identical to that of $k$ semi-local vortices with the lump condition in Ref. \[56\]. Hence, the inclusive relation is $\mathcal{M}_{U(N_C),N_F}^{k\text{-vortex}} \supset \mathcal{M}_{U(N_C),N_F}^{k\text{-lump}}$. The lump condition excludes subspaces of $\mathcal{M}_{U(N_C),N_F}^{k\text{-vortex}}$ corresponding to the minimal size vortices whose size is of order of the inverse gauge coupling.

In this section we will discuss the relation between moduli spaces for lump solutions and vortex solutions in the $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ cases. Here we take $N_C = N_F$ and $\det M_{\text{vev}} \neq 0$ for simplicity. The dimension of the moduli space of $k$ vortices in a $U(1) \times G'$ gauge theory ($N_F = N_C$) has been found to be \[58\]
\[
\dim_{\mathbb{C}}\mathcal{M}_{U(1) \times G'}^{k\text{-vortex}} = kN_C^2/n_0, \quad \text{\[4.5\]}
\]
with $N_C = 2M_C$ for $USp(2M_C)$. In the following, we will count the dimensions of the lump moduli spaces. (We will use the same characters for lowest scalar components of chiral superfields as for the superfields themselves).
In the $U(1) \times SO(2M_C)$ case ($N_C = 2M_C$), lump solutions with fixed boundary conditions are given by taking the following polynomials as the holomorphic invariants $I^i = \{M, B\}$ defined in (3.20). Their $U(1)$ charges are $\{2, 2M_C\}$, respectively. Thus, their GCD is $n_0 = 2$ and we find

$$M(z) = M_{\text{vev}} z^k + O(z^{k-1}) , \quad B(z) = B_{\text{vev}} z^{M_C} + O(z^{M_C-1}) ,$$

with $k \in \mathbb{Z}_{>0}$. Note that we should not neglect the baryon field $B$, although the baryon field $B$ is dependent on $M$. This is because the baryon field $B$ determined by $M(z)$ is not necessarily holomorphic everywhere in the complex plane $\mathbb{C}$:

$$\det(J)B(z)^2 = \det M(z) .$$

Generically, this gives $2kM_C$ constraints for moduli parameters. For instance, with a single lump solution in the $U(1) \times SO(2)$ case, a general form of $M(z)$ is given by setting $M_{\text{vev}} = \sigma_1$ and $k = 1$

$$M(z) = \begin{pmatrix} b & z - a \\ z - a & c \end{pmatrix} \rightarrow \det M(z) = bc - (z - a)^2 .$$

The constraint (4.7) requires $\det M(z)$ to be exactly a square of a polynomial and then we find the non-trivial conditions; $b = 0$ or $c = 0$ where the intersection point $b = c = 0$ is excluded by the lump condition. These two disconnected solutions correspond to two different types of lumps wrapping different $\mathbb{C}P^1$'s of $\mathcal{M}^{U(1) \times SO(2)} = \mathbb{C}P^1 \times \mathbb{C}P^1$ in Eq. (3.57). For generic $k$-lump configurations, we can count the degrees of freedom of the moduli parameters as

$$\dim_c \mathcal{M}^{k\text{-lump}}_{SO(2M_C)} = \#\text{moduli in } M(z) + \#\text{moduli in } B(z) - \#\text{constraints}$$

$$= k \frac{(2M_C)(2M_C + 1)}{2} + kM_C - 2kM_C = 2kM_C^2 .$$

In the $U(1) \times SO(2M_C + 1)$ case, the $U(1)$ charges of the invariants $\{M, B\}$ are $\{2, 2M_C + 1\}$. Hence their GCD is $n_0 = 1$ and lump solutions are given by the following polynomials

$$M(z) = M_{\text{vev}} z^{2k} + O(z^{2k-1}) , \quad B(z) = B_{\text{vev}} z^{(2M_C + 1)k} + O(z^{(2M_C + 1)k-1}) .$$

The dimension of the $k$-lump moduli space in this case is generically given by

$$\dim_c \mathcal{M}^{k\text{-lump}}_{SO(2M_C)} = 2k \frac{(2M_C + 1)(2M_C + 2)}{2} + k(2M_C + 1) - 2k(2M_C + 1) = k(2M_C + 1)^2$$

$$= k(2M_C + 1)^2 .$$

These two results are the same as those of the 1/2 BPS vortex moduli spaces derived from the index theorem [58], see Eq. (4.5). That is, at least for generic points of the lump moduli space, the moduli for the lump solutions are sufficient to describe the vortex moduli space in the original
gauge theory, and there are no internal moduli unlike the orientational moduli $\mathbb{C}P^{N_C-1}$ of the $U(N_C)$ case with $N_F = N_C$ flavors. This property is significantly different from the $U(N_C)$ case with the minimal number of flavors $N_F = N_C$, where only local vortices carrying the orientational moduli exist and the strong coupling limit of them are not lumps but singular objects of zero sizes.

In the $U(1) \times USp(2M_C)$ case, the baryon field is completely described by the meson fields and there are no constraints

$$M(z) = M_{\text{vev}} z^k + \mathcal{O}(z^{k-1}), \quad B(z) = (\text{Pf} J)^{-1} \text{Pf}(M(z)).$$

Therefore, the number of complex parameters in $M(z)$ is simply given by

$$\#\text{moduli in } M(z) = k \frac{2M_C(2M_C - 1)}{2} = \dim \mathcal{M}_{k-\text{vortex}}^{USp(2M_C)} - kM_C.$$  \hspace{1cm} (4.13)

Note that it is different from the dimensions of the vortex moduli space. This deficit number $M_C$ for each lump can be understood as follows. In this case, color-flavor symmetries $USp(2)^{M_C} \simeq SU(2)^{M_C}$ survive even at a generic point in the vacuum as we explained below Eq. (3.7). These surviving symmetries are broken in a vortex configuration and this means that the vortex configuration has orientational moduli $(\mathbb{C}P^1)^{M_C}$ as NG modes. These modes are expected to be localized in the Coulomb phase of the original gauge theory, which corresponds to the curvature singularity of the NL$\sigma$M, and therefore, cannot be detected as moduli of lump solutions in the NL$\sigma$M. Therefore, roughly speaking, we guess that

$$\mathcal{M}_{k-\text{vortex}}^{USp(2M_C)} \sim \mathcal{M}_{k-\text{singular lump}}^{USp(2M_C)} \times (\mathbb{C}P^1)^{kM_C},$$ \hspace{1cm} (4.14)

where $\mathcal{M}_{k-\text{singular lump}}^{USp(2M_C)}$ is the would-be lump moduli space which is parametrized by the complex parameters in the meson field $M(z)$. Emergence of these internal moduli is strongly related to singular configurations of lumps\footnote{This situation is similar to the case of a $U(N_C)$ gauge theory with $N_F = N_C$ flavors. The gauge theory has a non-Abelian vortex whose internal moduli space is $\mathbb{C}P^{N_C-1}$. But the strong gauge coupling limit yields a NL$\sigma$M of only a point and there are no lump solutions.}. Actually, to get regular solutions from lumps in any NL$\sigma$M, we have to require the lump condition, which means that the rank of the meson $M$ should be $2M_C$ everywhere in this $USp(2M_C)$ case. Therefore, no regular solutions exist in the case of $N_F = 2M_C$, because $\text{Pf} M$ are polynomials in $z$ with order $M_C k$ and thus has $kM_C$ zeros. We will show a concrete example in the next subsection. We expect that each of the orientational moduli $\mathbb{C}P^1$ are attached to such zeros and the deficit dimension of $\mathcal{M}_{k-\text{singular lump}}^{USp(2M_C)}$ should be strongly related to the non-existence of regular solutions. Regular lump solutions require the number of flavors to be greater than $2M_C$. 


In both cases of $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ gauge theories, additional NG zero modes can emerge as the moduli of vortex configurations if we choose special points as the vacuum, $M_{\text{vev}}(B_{\text{vev}})$. Especially, by choosing $M_{\text{vev}} = J (\mu_i = 1$ for all $i)$, the following moduli spaces for a single local vortex were found as \[58\]
\[ \mathcal{M}_{\text{vortex}}^{G',k=1} \supset \mathcal{M}_{\text{local vortex}}^{G',k=1} = \mathbb{C} \times \frac{G'}{U(M_C)} , \quad G' = SO(2M_C), USp(2M_C) , \] 
which cannot be moduli of single lump configurations.

To completely treat the vortex moduli, including internal moduli, we need to use the moduli matrix formalism \[42\]. This formalism is obtained by merely rewriting the holomorphic gauge invariants $M(z), B(z)$ in terms of the original chiral field $Q(z)$ whose components are also polynomials in the complex coordinate $z$\[11\]. The description of the lump solutions with respect to $Q(z)$ is redundant, since $Q(z)$ and $Q'(z)$ determine the same holomorphic map $M(z), B(z)$, if they are related by a complexified gauge transformation $Q'(z) = V(z)Q(z)$. Therefore we have the following equivalence relation, called the $V$-equivalence
\[ Q(z) \sim V(z)Q(z) , \quad V(z) \in U(1)^C \times SO(N_C)^C , \quad U(1)^C \times USp(2M_C)^C . \] 
The parameters contained in $Q(z)$ after gauge fixing, parametrize the moduli space of vortices. Conversely, all moduli of vortices including internal moduli are contained in $Q(z)$, and thus $Q(z)$ is denoted the moduli matrix. In this formalism the boundary conditions \[4.6\], \[4.10\] and \[4.12\] are interpreted as constraints for the moduli matrix $Q(z)$ \[58\]
\[ \begin{align*}
SO(2M_C), USp(2M_C) : & \quad Q^T(z)JQ(z) = M_{\text{vev}}z^k + \mathcal{O}(z^{k-1}) , \\
SO(2M_C + 1) : & \quad Q^T(z)JQ(z) = M_{\text{vev}}z^{2k} + \mathcal{O}(z^{2k-1}) .
\end{align*} \] 
The constraint \[4.7\] is of course automatically solved in this formalism. This formalism is apparently independent of the gauge coupling and it is well-defined to require the lump conditions to hold on the vortex moduli space. We expect that a submanifold of the $k$-vortex moduli space satisfying the lump condition is equivalent to the $k$-lump moduli space,
\[ \mathcal{M}^{k-\text{lump}} \simeq \{ a | a \in \mathcal{M}^{k-\text{vortex}}, \text{ the lump condition} \} . \] 

\[11\] The way to derive the moduli matrix here is slightly different from the way used in \[58\]. These two ways can be identified by considering BPS vortex solutions in the superfield formulation \[45\]. The key observation is that the gauge symmetry $G$ in the supersymmetric theory is complexified : $G^C$. Hence, the moduli matrix naturally appears in the superfield formulation, while if we fix $G^C$ in the Wess-Zumino gauge, the scalar field $Q_{\text{wz}}$ appears as the usual bosonic component in the Lagrangian. The moduli matrix is usually denoted by the symbol $H_0(z)$ in literature.
This expectation is quite natural and is enforced by the above observations by counting the dimensions. Because, if we can consider a NLσM as an approximation to the gauge theory with a strong but finite gauge coupling $g$, a lump solution should describe an approximate configuration of a vortex, whereas a steep configuration with a width of order $1/g\sqrt{\xi}$ is excluded by some UV cutoff $\Lambda < g\sqrt{\xi}$. Of course, to justify this expectation, we need to verify an equivalence between the two formalisms, the moduli matrix formalism and the holomorphic map (4.4) with the constraint on the invariants, under the lump condition. In examples of the next subsection, we just assume that this expectation is true. To construct lump solutions for large $N_F(N_C)$, the moduli matrix formalism is somewhat easier than treating $M(z)$, $B(z)$ as they are.

4.3 Lumps in $U(1) \times SO(2M_C)$ and $U(1) \times USp(2M_C)$ Kähler Quotients

4.3.1 BPS Lumps in the $U(1) \times SO(2M_C)$ Kähler Quotient

Let us start with the simplest example in which the gauge group is $U(1) \times SO(2)$ with two flavors $N_F = 2$. As we have studied in Sec. 3.3.1, the target space is $\mathbb{C}P^1 \times \mathbb{C}P^1$. Lump solutions are classified by a pair of integers $(k_+, k_-)$ given as

\[ \pi_2\left( \mathcal{M}_{U(1) \times SO(2)}^{N_F=2} \right) = \mathbb{Z} \times \mathbb{Z} \ni (k_+, k_-) . \tag{4.19} \]

A solution with $(k_+, k_-)$ lumps is given by

\[ Q(z) = \begin{pmatrix} Q_1^+(z) & Q_2^+(z) \\ Q_1^-(z) & Q_2^-(z) \end{pmatrix} , \tag{4.20} \]

where $Q_1^+(z), Q_1^-(z)$ are holomorphic functions of $z$ of degree $k_\pm$, respectively. One can verify that the tension is given by

\[ T = \int d^2x \ 2\partial\bar{\partial}K_{U(1) \times SO(2)} = \pi \xi (k_+ + k_-) \equiv \pi \xi k , \tag{4.21} \]

where $K_{U(1) \times SO(2)}$ is the Kähler potential given in Eq. (3.59). Interestingly, the tension of the minimal lump $(k_+, k_-) = (1, 0), (0, 1)$ is half of $2\pi \xi$ which is that of the minimal lump in the usual $\mathbb{C}P^1$ model. A similar observation has been obtained recently in Ref. [58].

Next, we would like to consider lump configurations in slightly more complicated models by considering general $U(1) \times SO(2M_C)$ Kähler quotients, where we set $M_C \geq 2$, $N_F = 2M_C$ and

\footnote{In the $U(1) \times USp$ and $U(1) \times SO$ cases, we have to verify that the meson field $M(z)$ whose elements are polynomials can be always decomposed in $Q(z)$ whose elements are also polynomials and furthermore that there is no degeneracy of moduli in the construction of $M(z)$ from $Q(z)$ under the lump condition. There is no known proof and it is expected to be technically complicated.}
As an example for $k = 1$, we take
\[
Q_{k=1} = \begin{pmatrix} z1_{MC} - A & C \\ 0 & 1_{MC} \end{pmatrix},
\]
where
\[
A = \text{diag}(z_1, z_2, \ldots, z_{MC}),
\]
\[
C = \text{diag}(c_1, c_2, \ldots, c_{MC}).
\]

These diagonal choices allow us to treat the invariants as if they were independent invariants of $M_{C}$ different $SO(2)$’s. Hence, one can easily find an $SO(2)$ part inside $M$ as
\[
\begin{pmatrix} (M)_{i,i} & (M)_{i,i+MC} \\ (M)_{i+MC,i} & (M)_{i+MC,i+MC} \end{pmatrix} = \begin{pmatrix} 0 & z - z_i \\ z - z_i & 2c_i \end{pmatrix}, \quad i = 1, 2, \ldots, M_{C},
\]
which satisfies the constraint (4.17). Note that non-zero parameters $c_i$ keep the rank $M \geq 2M_{C} - 1$, even at $z = z_i$. All their eigenvalues are also eigenvalues of $MM^\dag$
\[
\lambda_{i\pm} = |z - z_i|^2 + 2|c_i|^2 \pm 2|c_i|\sqrt{|z - z_i|^2 + |c_i|^2}.
\]

Thus, the Kähler potential in Eq. (3.41) becomes
\[
K = \xi \log \left[ \sum_{i=1}^{M} \left( \sqrt{\lambda_{i+}} + \sqrt{\lambda_{i-}} \right) \right] = \xi \log \left( 2 \sum_{i=1}^{M} \sqrt{|z - z_i|^2 + |c_i|^2} \right).
\]

The energy density is obtained by $E = 2\partial \bar{\partial}K$ with this Kähler potential and exhibits an interesting structure. It is proportional to the logarithm of the sum of the square root of $|P_i(z)|^2$, while the known Kähler potential of a $\mathbb{C}P^M$ lump is just the logarithm of the sum of $|P_i(z)|^2$. This difference gives us quite distinct configurations. If we take some $c_i$ to vanish, then we find that the energy density of the configuration becomes singular at $z = z_i$
\[
E = 2\xi \partial \bar{\partial} \log \left( \sqrt{|z - z_i|^2 + \cdots} \right) \sim \text{const.} \times \frac{1}{|z - z_i|} + \mathcal{O}(z^0).
\]

This is due to the curvature singularity which appears when the manifold becomes of rank $M = 2M_{C} - 2$, and in other words, violate the lump condition. Note that this singular configuration has a non-vanishing size, as we mentioned above. If we take all $z_i$’s and all $c_i$’s to be coincident, respectively, we find that the Kähler potential reduces to that of the minimal winding one in the $U(1) \times SO(2)$ model. This suggests that the trace part of $C$ determines the overall size of the configuration and the trace part of $A$ corresponds to the center of mass. As we will explain later, only this trace part of $A$ among the parameters is a normalizable mode in the effective action of the lump.

A single lump in $U(1) \times SO(2M_{C} + 1)$ might be almost the same as the coincident $k = 2$ lumps in $SO(2M_{C})$. However we will not discuss this case in detail.
4.3.2 BPS Lumps in the $U(1) \times USp(2M_C)$ Kähler Quotient

Let us first examine a lump solution in the $U(1) \times USp(2)$ theory with $N_F = 2$. In this case, however, we obtain only local vortices and cannot observe regular lumps in the NL$\sigma$M since the vacuum is just a point. After fixing the gauge, the chiral field can be expressed as

$$Q(z) = \begin{pmatrix} z - a & 0 \\ b & 1 \end{pmatrix}. \quad (4.27)$$

This matrix yields

$$M = (z - a) J, \quad K = \frac{\xi}{2} \log |z - a|^2. \quad (4.28)$$

At the center of the vortex, the rank of $M$ always reduces to zero, where the $U(1)$ gauge symmetry is restored. Therefore, solutions are always singular at that point, because we know that $USp(2) \simeq SU(2)$ and the $U(2)$ model with 2 flavors admits only local vortices rather than semi-local vortices which reduce to lumps in the NL$\sigma$M limit. Indeed, the parameter $b$ which does not appear in $M$ is the orientational modulus of local vortex in the original $U(1) \times USp(2)$ gauge theory and describes $\mathbb{C}P^1$.

As we have mentioned, lump solutions in the case of $M_C = M_F$ always have singular points in the configurations. The simplest non-trivial example for a regular lump is obtained in the case of $U(1) \times USp(4)$ with 6 flavors. A lump (vortex) solution in this case, with the minimal winding ($k = 1$) has $M_CN_F = 12$ complex parameters. Let us consider the following field configuration as a typical minimal example of $k = 1$;

$$Q(z) = \begin{pmatrix} z - z_+ & 0 & 0 & c & a_+ & 0 \\ 0 & z - z_- & -c & 0 & 0 & a_- \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad (4.29)$$

which gives the following characteristic polynomial

$$\det(\lambda - MM^\dagger) = \lambda^2 \left( \lambda^2 - (R_+^2 + R_-^2 + 4|c|^2)\lambda + R_+^2R_-^2 \right)^2, \quad (4.30)$$

with $R_{\pm} = \sqrt{|z - z_{\pm}|^2 + |a_{\pm}|^2}$. Then the energy density of the configuration $\mathcal{E}$ is given by

$$\mathcal{E} = 2\partial\bar{\partial}K_{U(1) \times USp(4)}|_{sol} = \xi \partial\bar{\partial} \log \left((R_+ + R_-)^2 + 4|c|^2\right). \quad (4.31)$$

This configuration is regular everywhere as long as $a_{\pm} \neq 0$, that is, it satisfies the lump condition. If we choose $a_+ = a_-$ and $z_+ = z_-$, it corresponds to a $\mathbb{C}P^2$ single lump solution.

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4.4 Effective Action of Lumps

Now we have a great advantage thanks to the above superfield formulation of the NLσM. A supersymmetric low energy effective theory on the 1/2 BPS lumps is immediately obtained merely by plugging the 1/2 BPS solution \( (4.1) \) into the Kähler potential which we have obtained in the previous section after promoting the moduli parameters \( \phi \) to fields on the lump world-volume

\[
\phi^\alpha(t, z, \bar{z}, x^3) \rightarrow \phi^\alpha(z; \varphi^i(t, x^3)).
\] (4.32)

The resulting (effective) expression for the Kähler potential is

\[
K_{\text{lump}} = \int dzd\bar{z} \ K\left(\phi(z, \varphi^i(t, x^3)), \ \phi^\dagger(\bar{z}, \bar{\varphi}^i(t, x^3))\right).
\] (4.33)

Let us make a simple example of the \( \mathbb{C}P^1 \) sigma model which is the strong coupling limit of a \( U(1) \) gauge theory with \( N_F = 2 \) flavors \( Q = (Q_1, Q_2) \). In this case, \( Q_1 \) and \( Q_2 \) themselves play the role of the holomorphic invariants \( I^i \) and the inhomogeneous coordinate is given by \( \phi = Q_2/Q_1 \). We fix the \( U(1)^C \) symmetry in such a way that \( Q \) is expressed by

\[
Q = (1, b).
\] (4.34)

From Eq. (2.16), the Kähler potential and the corresponding Lagrangian are of the form

\[
K = \xi \log(1 + |b|^2), \quad \mathcal{L} = \xi \frac{|\partial_\mu b|^2}{(1 + |b|^2)^2}.
\] (4.35)

A single 1/2 BPS lump solution in this model is given by

\[
Q(z) = (z - z_0, a) \quad \leftrightarrow \quad \phi = \frac{a}{z - z_0},
\] (4.36)

where \( z_0 \) corresponds to the position of the lump and \( a \) is its transverse size and phase moduli.

To obtain the effective theory of the lump, one needs to promote the moduli matrix as follows

\[
Q(z) = (z - z_0, a) \quad \Rightarrow \quad Q(t, z) = (z - z_0(t), a(t)).
\] (4.37)

Plugging this into the formal expression (4.33), we get the effective theory

\[
\mathcal{L}^{\text{eff}}_{\infty} = \pi \xi |\dot{z}_0(t)|^2,
\] (4.39)

where \( 2\pi \xi \) is the tension of the minimal winding solution.
4.5 Identifying Non-normalizable Modes

We can determine which parameters in $Q(z)$ are localized on lumps and normalizable, and which parameters are non-normalizable. If there exists a divergence in the Kähler potential which cannot be removed by the Kähler transformations, it indicates that the moduli parameters included in the divergent terms are non-normalizable. Let us substitute an expansion of the lump solution with respect to $z^{-1}$

$$\phi^\alpha(z) = \phi^\alpha_{\text{vev}} + \frac{\chi^\alpha}{z} + O(z^{-2}) , \quad \chi^\alpha = \sum_{i=1}^{k} \phi^\alpha_i ,$$

into the Kähler potential (4.33) and expand it as well

$$K_{\text{lump}} = \lim_{L \to \infty} \int_{|z| \leq L} d^2x \left[ K(\phi^\alpha_{\text{vev}}, \bar{\phi}^\bar{\beta}_{\text{vev}}) + \frac{1}{z} \partial_{\bar{\alpha}} K \chi^{\bar{\alpha}} + \frac{1}{\bar{z}} \bar{\partial}_{\alpha} \bar{K} \bar{\chi}^{\alpha} + \frac{1}{|z|^2} \partial_{\alpha} \bar{\partial}_{\bar{\beta}} K \chi^{\alpha} \bar{\chi}^{\bar{\beta}} + O(|z|^{-3}) \right]$$

$$= \lim_{L \to \infty} \left[ 2\pi L^2 K(\phi^\alpha_{\text{vev}}, \bar{\phi}^\bar{\beta}_{\text{vev}}) + 2\pi \log L \partial_{\alpha} \bar{\partial}_{\bar{\beta}} K(\phi^\alpha_{\text{vev}}, \bar{\phi}^\bar{\beta}_{\text{vev}}) \chi^{\alpha} \bar{\chi}^{\bar{\beta}} + O(1) \right] , \tag{4.41}$$

where $L$ is an infrared cutoff. Thus we can conclude that the moduli parameters included in $\{\phi^\alpha_{\text{vev}}, \chi^\alpha\}$ are all non-normalizable and the others are normalizable. The modulus $a$ in the last subsection is a typical example of $\chi^\alpha$.

For instance, let us take a look at the example (4.23) of the solution for single lumps in the $U(1) \times SO(2M_C)$ case. The meson field $M(z)$ has the following elements : $(z - z_i)$ and $2c_i$. One can partly construct inhomogeneous coordinates of the manifold in this case by taking ratios from pairs of the elements,

$$\phi^i = \frac{2c_i}{z - z_{M_C}} = \frac{2c_i}{z} + O(z^{-2}) , \quad \text{for } 1 \leq i \leq M_C ,$$

$$\phi^{i+M_C} = \frac{z - z_i}{z - z_{M_C}} = 1 - \frac{z_i - z_{M_C}}{z} + O(z^{-2}) , \quad \text{for } 1 \leq i \leq M_C - 1 . \tag{4.42}$$

Thus the moduli $c_i$ and $z_i - z_{M_C}$ are non-normalizable. The only normalizable modulus is $\sum_{i=1}^{M_C} z_i/M_C$ which is the center of mass. This fact is a result of the Kähler metric (3.42) where the trace part of the meson field $M$ does not contribute to the metric. Generally speaking, all moduli of a single lump in the $U(1) \times SO(2M_C)$ and $U(1) \times USp(2M_C)$ theories are non-normalizable except for the center of mass and the orientational moduli of local vortex.

5 Conclusion and Discussion

We have explicitly constructed the Kähler potentials for NLσMs describing the Higgs phase of $\mathcal{N} = 1$ supersymmetric $SO(N_C)$ and $USp(2M_C)$ gauge theories. The key point in the construction
lies in the use of taking the gauge symmetry to be $U(N_C)$ and restricting the algebra down to either $\mathfrak{so}(N_C)$ or $\mathfrak{usp}(2M_C)$ with Lagrange multipliers. The result is written both in terms of the component fields and the holomorphic invariants, i.e. the mesons and the baryons of the theories. Because the obtained result is difficult to manage in practice in the large $N_C \ (N_F)$ limit, we have developed an expansion around the vacuum expectation values of the meson field, and obtained the scalar curvature of both theories, i.e. $SO(N_C)$ and $USp(2M_C)$. Furthermore, have made the same considerations for the case of $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$, and obtained the Kähler potential, metric, expansion and curvature also in these cases.

Following the same strategy as in the Kähler quotient case, we have been able to obtain the hyper-Kähler quotient in the case of $SO(N_C)$ and $USp(2M_C)$ gauge theories, simply by rewriting the fields by means of the algebra to fields with $2N_F$ flavors all in the fundamental representation and we confirm the flavor symmetry of the $SO(N_C)$ hyper-Kähler quotient to be $USp(2N_F)$ and for $USp(2M_C)$ it is $SO(2N_F)$.

A significant feature of those NL$\sigma$Ms, is that a point in the target space can reach within a finite distance submanifolds corresponding to unbroken phases of the gauge theories. We have observed that a curvature singularity emerges there. If we consider a generic gauge group with a generic representation as the original gauge theory, we can observe such singularities in many NL$\sigma$Ms unlike the well-known $U(N)$ (Grassmannian) case. The NL$\sigma$Ms we have considered here can be regarded as test cases for those theories.

In the second part of the paper we have studied the $1/2$ BPS, NL$\sigma$M lumps in $U(1) \times G'$ gauge theories and observed that we can construct lump solutions straightforwardly if the Kähler potential for the NL$\sigma$M is given in terms of holomorphic invariants of $G'$. We found that counting the dimension of these (regular) lump moduli spaces gives the same result for the semi-local vortex moduli space in the case of $SO(N_C)$ and $USp(2M_C)$ theories. This fact enforces our natural expectation that those moduli spaces are homeomorphic to each other except in the subspaces where the lump condition is violated. Furthermore, by considering effective actions within our formalism for the NL$\sigma$M lumps, we have obtained a conventional method to clarify the non-normalizability of the moduli parameters in general cases. By using this, we can conclude that in both the cases of $U(1) \times SO(2M_C)$ and $U(1) \times USp(2M_C)$ Kähler quotients, all moduli parameters of a single regular lump are non-normalizable except for the center of mass.

An important observation of lump configurations in $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ theories is the existence of a singularity in the target manifold. In those theories, a lump configuration becomes singular without taking the zero size limit, simply if the configuration touches the singularity of the manifold, whereas a lump in the $U(N)$ case is always regular with a finite
size and becomes singular only in the zero size limit. Especially, in the case of \(U(1) \times USp(2M_C)\) with \(N_F = 2M_C\), only singular solutions (with a finite or zero size) exist.

It is an important problem to determine the second homotopy group \(\pi_2(M_{U(1) \times (SO,U Sp)})\) in the case of \(U(1) \times SO(N_C)\) and \(U(1) \times USp(2M_C)\) theories. To support stability of lumps in those models, we expect that

\[
\pi_2(M_{U(1) \times SO(N_C)}) \simeq \mathbb{Z} \times \mathbb{Z}_2, \quad \pi_2(M_{U(1) \times USp(2M_C)}) \simeq \mathbb{Z},
\]

(5.1)

where the \(\mathbb{Z}_2\) charge for the \(U(1) \times SO(N_C)\) case is naturally expected, since the corresponding local vortices have their charges due to \(\pi_1(U(1) \times SO(N_C)/\mathbb{Z}_2) = \mathbb{Z} \times \mathbb{Z}_2\). To determine the homotopy group in these cases is a complicated task since we have to take non-trivial directions of cohomogeneity into account, and a further study of the moduli space of lumps beyond counting dimensions also is needed. This problem still remains as a future problem. The relation between our solutions and the lumps in non-supersymmetric \(SO(N)\) QCD is, therefore, unclear so far. In their case, the lumps are supported by the homotopy group \(\pi_2(SU(N_F)/SO(N_F)) \simeq \mathbb{Z}_2\). Therefore, these lumps are non-BPS. In our case, the gauge coupling constants for \(SO(N_C)\) and \(U(1)\) could be different although we did not consider it. Let \(g\) and \(e\) be the gauge couplings of the \(SO(N_C)\) and \(U(1)\) gauge groups, respectively. We have taken the strong gauge coupling limit for both the couplings, \(g, e \to \infty\), in which case the gauge theory reduces to the NL\(\sigma\)M of the \(U(1) \times SO(N_C)\) Kähler quotient. Without taking the strong coupling limit for \(e\), the size (width) \(1/e \sqrt{\xi}\) for the “Abelian” vortices becomes larger as the \(U(1)\) gauge coupling \(e\) becomes smaller. In the limit of vanishing \(e\), we expect that they disappear and only non-BPS \(\mathbb{Z}_2\) lumps remain. It is important to clarify this point which also remains as a future problem.

Besides these problems, there are many interesting future problems in the following.

In certain models it has been proposed that the moduli space of vacua admits a Ricci-flat (non-compact Calabi-Yau) metric \([26]\). In the case of the \(SU(N_C)\) Kähler quotient, a Ricci-flat metric was obtained by deforming the Kähler potential \([2.1]\) of the original \(SU(N_C)\) gauge theory to \(K = f(\text{Tr}[QQ^\dagger e^{-V}])\) with an unknown function \(f\), and solving the Ricci-flat condition (the Monge-Ampere equation) for \(f\). The metric turns out to be the canonical line bundle over the Grassmann manifold \(Gr_{N_F,N_C}\). It is certainly worthwhile to construct a Ricci-flat metric also on the \(SO\) and \(USp\) Kähler quotients. The expansion \([3.34]\) should be enough to determine the unknown function \(f\) with a Kähler potential \(K = f(\text{Tr}[\sqrt{MM^i}])\).

An extension to hyper-Kähler quotients with other gauge groups, namely exceptional groups is also an interesting future problem. As in Eq. \([3.14]\) for \(SO(N_C)\) and \(USp(2M_C)\) Kähler quotients, Kähler quotients may be achieved by introducing a proper constraint. For instance for
a $E_6$ quotient, $\Gamma_{ijk} (e^{V'})^i_l (e^{V'})^j_m (e^{V'})^k_n - \Gamma_{lmn} = 0$ is a candidate constraint to embed $E_6$ into $U(27)$, where $\Gamma_{ijk}$ is the third-rank invariant symmetric tensor of $E_6$. This will be achieved by introducing a Lagrange multiplier $\lambda_{lmn}$ belonging to the rank-3 anti-symmetric representation. Since the study of vortices in $U(1) \times G'$ with $G'$ being exceptional groups has been raised in [58], lumps in these Kähler quotients are also interesting subjects to be studied.

We should also consider hyper-Kähler quotients for other representations. In particular, including adjoint fields into our work is important because the resultant spaces appear as multi-instanton moduli spaces of $SO(N_C)$ and $USp(2M_C)$ gauge theories.

In the case of the $\mathcal{N} = 2$ hyper-Kähler NL$\sigma$M, the only possible potential consistent with eight supercharges is written as the square of a tri-holomorphic Killing vector [72]. The explicit potentials can be found for instance for $T^*\mathbb{C}P^{N-1}$ [73, 74], toric hyper-Kähler manifolds [75], $T^*Gr_{N,M}$ [63] and $T^*F_n$ [20]. In terms of the hyper-Kähler quotients these potentials are obtained as usual masses of hypermultiplets in the corresponding $\mathcal{N} = 2$ supersymmetric gauge theories [63]. For this massive deformed hyper-Kähler NL$\sigma$M one can construct domain walls which are the other fundamental 1/2 BPS objects; 1/2 BPS domain wall solutions in the $U(N_C)$ hyper-Kähler quotient, namely $T^*Gr_{N,N_C}$, see [11]. Constructing a massive deformation and domain wall solutions in $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ hyper-Kähler quotients remains as future problems.

Time-dependent stationary solutions, called Q-lumps [76], are also BPS states in a NL$\sigma$M with a potential. Q-lumps were constructed in the $\mathbb{C}P^1$ model [76], the Grassmann sigma model ($U(N_C)$ Kähler quotient) [77], and the asymptotically Euclidean spaces [31]. It is one of the possible extensions to construct Q-lumps in $U(1) \times SO(N_C)$ and $U(1) \times USp(2M_C)$ Kähler quotients.

Finally, many extensions and applications of the present works include: dynamics of lumps [32], cosmic lump strings [36] [37] [33] and especially their reconnection [44], composite states like triple lump-string intersections [31] and lump-strings stretched between domain walls [41], and the Seiberg-like duality [56].

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A Various Theorems and Their Proofs

A.1 $SO(2M_C), USp(2M_C)$ Groups and Their Invariant Tensors

Let us define the following sets of $n$-by-$n$ matrices for $\epsilon = \pm 1$

$$\text{Inv}_\epsilon(n) \equiv \{ J \mid J^T = \epsilon J, \ J^\dagger J = 1_n \} . \quad \text{(A.1)}$$

That is, elements of $\text{Inv}_\epsilon(n)$ are (anti)symmetric and unitary.

**Proposition:** For arbitrary $A \in \text{Inv}_+(2)$, there exists a 2-by-2 unitary matrix $u$ such that

$$A = u^T u . \quad \text{(A.2)}$$

**Proof:** A general solution of $A$ is given by

$$A = e^{i\lambda} \begin{pmatrix} e^{i\rho} \cos \theta & i \sin \theta \\ i \sin \theta & e^{-i\rho} \cos \theta \end{pmatrix} = e^{i(\lambda_1 + \rho \sigma_3)} (\cos \theta 1_2 + i \sigma_1 \sin \theta) e^{i(\lambda_2 + \rho \sigma_3)} = u^T u , \quad \text{(A.3)}$$

with $u = e^{i\theta_1} e^{i(\lambda_1 + \rho \sigma_3)} \in U(2)$. ■

**Theorem 1-s:** An arbitrary $A \in \text{Inv}_+(n)$ can be written as

$$A = u^T u , \quad \text{(A.4)}$$

with an $n$-by-$n$ unitary matrix $u$. ■

Therefore we find,

$$\text{Inv}_+(n) \simeq U(n)/O(n) . \quad \text{(A.5)}$$
Proof 1-s: It is easy to show that an arbitrary symmetric matrix can be rewritten as

\[
A \rightarrow A' = u' A u'^T = \begin{pmatrix}
|a_1| & b_1 & 0 & 0 & \cdots \\
b_1 & |a_2| & b_2 & 0 & \cdots \\
0 & b_2 & \ddots & \ddots & \\
0 & 0 & \ddots & \ddots & \\
\vdots & \vdots & & & \\
\end{pmatrix} \in \text{Inv}_+(n),
\]

(A.6)

with an unitary matrix \(u'\). The matrix \(A'\) is also a unitary matrix and this fact leads to \(b_1 = 0\) or \(b_2 = 0\). Therefore

\[
A' = \begin{pmatrix}
1 & 0 \\
0 & A_{(n-1)}
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
A_{(2)} & 0 \\
0 & A_{(n-2)}
\end{pmatrix},
\]

(A.7)

where \(A_{(m)} \in \text{Inv}_+(m)\). Recursively, we find \(A'\) takes a block-diagonal form which diagonal elements are 1 or 2-by-2 symmetric unitary matrices. By using Proposition (A.2), we can show that there exists a unitary matrix \(\tilde{u}\) such that \(\tilde{u} A \tilde{u}^T = \mathbf{1}_n\), that is, there exists a unitary matrix \(u\) such that \(A = u^T u\).

By using a similar algorithm, we can show that

Theorem 1-a: An arbitrary \(A \in \text{Inv}_{-}(2m)\) can be rewritten as

\[
A = u^T J_m^- u, \quad J_m^- = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \otimes \mathbf{1}_m,
\]

(A.8)

with an appropriate unitary matrix, \(u\), \((uu^\dagger = \mathbf{1}_{2m})\).

Therefore we find

\[
\text{Inv}_{-}(2m) \simeq U(2m)/USp(2m).
\]

(A.9)

A choice of \(J_e \in \text{Inv}_{e}(n)\) defines a subgroup \(G_e(J_e)\) of \(U(n)\) as

\[
G_e(J_e) = \{ g \in U(n) \mid g^T J g = J \}.
\]

(A.10)

Conversely, we can say that \(J_e\) is an invariant tensor of \(G_e(J_e)\).

Corollary 1: Arbitrary two elements \(J, J' \in \text{Inv}_{e}(n)\) are related to each other with appropriate unitary matrix \(u\) as, \(J' = u J u^T\) and corresponding group \(G_e(J)\) and \(G_e(J')\) are isomorphic to each other.

Therefore, from (A.4) and (A.8) we find that \(G_{+}(J_+)\) is isomorphic to \(O(n)\) and \(G_{-}(J_-)\) is isomorphic to \(USp(n = 2m)\).
A.2 Diagonalization of the Vacuum Configuration

**Theorem 2-s:** Let us consider an arbitrary \( n \)-by-\( m \) \((n \leq m)\) matrix \( Q \) satisfying
\[
QQ^\dagger = (QQ^\dagger)^T.
\] (A.11)
Then \( Q \) is always decomposed as
\[
Q = O \begin{pmatrix}
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{pmatrix} U ,
\] (A.12)
where \( O \in SO(n) \) with \( J = 1_n \) and \( U \in U(m) \).

**Proof 2-s:** Since \( QQ^\dagger \) is symmetric and Hermitian, \( QQ^\dagger \) is a real symmetric matrix. Therefore it can be diagonalized as \( QQ^\dagger = O\Lambda^2O^T \) with \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \) with \( \lambda_i \in \mathbb{R}_{\geq 0} \).

**Theorem 2-a:** Let us consider an arbitrary \( 2n \)-by-\( m \) \((2n \leq m)\) matrix \( Q \) satisfying
\[
JQQ^\dagger = (QQ^\dagger)^T J ,
\] (A.13)
with \( J = i\sigma_2 \otimes 1_n \). Then \( Q \) can always be decomposed as
\[
Q = O \begin{pmatrix}
\Lambda & 0 \\
\vdots & \ddots \\
0 & \cdots & 0
\end{pmatrix} U ,
\] (A.14)
where \( O \in USp(2n) \) and \( U \in U(m) \) and \( \Lambda = 1_2 \otimes \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \) with \( \lambda_i \in \mathbb{R}_{\geq 0} \).

**Proof 2-a:** The Hermitian positive semi-definite matrix \( QQ^\dagger \) is always diagonalized as \( QQ^\dagger = u\Lambda^2u^\dagger \) with an appropriate unitary matrix \( u \in U(2n) \). Then the condition tells us that \( X = u^TJu \) commutes with \( \Lambda^2 \), \([X, \Lambda^2] = 0 \). We can set \( \Lambda \) to be positive semi-definite, then \([X, \Lambda] = 0 \). Furthermore, we find \( XX^\dagger = 1_{2n} \) and \( X^T = -X \). According to Theorem 1-a, thus, \( X \) turns out to be \( X = u^TJu = J \) by taking an appropriate \( u \). This means \( u \) is an element of \( USp(2n) \). Here \( \Lambda \) takes a form \( 1_2 \otimes \Lambda' \), since \([\Lambda, J] = 0 \).

A.3 Diagonalization of a Non-Hermitian (Anti)symmetric Matrix

**Theorem 3:** An arbitrary \( n \)-by-\( n \) (anti)symmetric matrix \( M \), (that is, \( M^T = \epsilon M \)) can be written in a block-diagonal form as
\[
M = u \begin{pmatrix}
|\mu(1)||J(1) \\
|\mu(2)||J(2) \\
& \ddots
\end{pmatrix} u^T ,
\] (A.15)
where \( J_k \in \text{Inv}_e(n_k) \) and \( n = \sum_k n_k \).

**Proof 3:** \( M M^\dagger \) is an Hermitian matrix and thus, can always be diagonalized as
\[
MM^\dagger = u \text{ diag } (|\mu(1)|^2 1_{n_1}, |\mu(2)|^2 1_{n_2}, \cdots) u^\dagger,
\]
with a unitary matrix \( u \) and \( |\mu(i)| < |\mu(i+1)| \). Therefore, \( \tilde{M} \equiv u^\dagger M u^* \) satisfies
\[
\tilde{M} \tilde{M}^\dagger = \text{ diag } (|\mu(1)|^2 1_{n_1}, |\mu(2)|^2 1_{n_2}, \cdots) = (\tilde{M} \tilde{M}^\dagger)^T = \tilde{M}^\dagger \tilde{M}.
\]
Note that \( \tilde{M}^T = \epsilon \tilde{M} \). This equation means that \( \tilde{M} \) is a normal matrix \( [\tilde{M}, \tilde{M}^\dagger] = 0 \) and can be diagonalized as
\[
\tilde{M} = \tilde{u} \text{ diag } (\mu_1, \mu_2, \cdots) \tilde{u}^\dagger,
\]
with a unitary matrix \( \tilde{u} \). By substituting this form to Eq. (A.17), we find that
\[
|\mu(1)|^2 = |\mu_1|^2 = |\mu_2|^2 = \cdots, |\mu(2)|^2 = |\mu_{n_1+1}|^2 = \cdots, |\mu(3)|^2 = \cdots.
\]
and \( \tilde{u} \) should take a block-diagonal form as
\[
\tilde{u} = \text{ diag}(u(1), u(2), \cdots),
\]
where \( u(k) \) is an \( n_k \)-by-\( n_k \) unitary matrix. Therefore, \( \tilde{M} \) also takes block-diagonal form as
\[
\tilde{M} = \text{ diag } (|\mu(1)| J(1), |\mu(2)| J(2), \cdots).
\]

The meson field is always 'diagonalized' by fixing the flavor symmetry. Combining Theorem 1-s(1-a) with Theorem 3, we find the following corollaries.

**Corollary 3-s:** An arbitrary symmetric matrix \( M \) can be diagonalized
\[
M = u m u^T, \quad m = \text{ diag } (|\mu_1|, |\mu_2|, \cdots),
\]
with a unitary matrix \( u \).

**Corollary 3-a:** An arbitrary anti-symmetric matrix \( M \) can be diagonalized
\[
M = u m u^T, \quad m = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \otimes \text{ diag } (|\mu_1|, |\mu_2|, \cdots),
\]
with a unitary matrix \( u \).

**Corollary 3':** An arbitrary \( n \)-by-\( n \) (anti-)symmetric matrix \( M \) can be decomposed as
\[
M = Q^T J Q.
\]
where $Q$ is an $n$-by-$m$ matrix and $J \in \text{Inv}_e(m)$ with $m = \text{rank}(M)$. ■

The (anti)symmetric matrix $M$ breaks the $U(n)$ symmetry $M \to uMu^T$ as

\[
U(n) \to \left\{ \begin{array}{l}
U(n_0) \times O(n_1) \times O(n_2) \times \cdots \\
U(n_0) \times USp(2m_1) \times USp(2m_2) \times \cdots
\end{array} \right.,
\]

where $n_0$ is a number of zero-eigenvalues of $M$.

### B Non-trivial Uniqueness Proof

In this section, we prove the uniqueness of the solution to Eq. (3.17). Here we consider the $SO(N_C)$ case. We can always write the $N_C$-by-$N_F$ matrix $Q$ as $Q = (\hat{Q}, 0)U$, $U \in U(N_F)$, (B.1) up to $U(N_C)$ transformation which rotates the columns of the $N_C$-by-$N_C$ matrix of $\hat{Q}$. We can show that for $\hat{M} \equiv \hat{Q}^T J \hat{Q}$

\[
\text{rank } \hat{M} = N_C \iff \text{rank } \hat{Q} = N_C,
\]

\[
\text{rank } \hat{M} = N_C - 1 \Rightarrow \text{rank } \hat{Q} = N_C - 1,
\]

(B.2)
since $\text{det } \hat{M} = \text{det } J(\text{det } \hat{Q})^2$ and $N_C \geq \text{rank } \hat{Q} \geq \text{rank } \hat{M}$ is always satisfied.

#### B.1 Solution with rank $M = N_C$

If the rank of $M \equiv Q^T J Q$ is $N_C$, then $\hat{M}$ also has rank $N_C$. Therefore rank $\hat{Q} = N_C$, namely $\hat{Q}$ is invertible and

\[
U_Q \equiv \hat{Q}^{-1} \sqrt{\hat{Q} \hat{Q}^\dagger},
\]

(B.3)
is a unitary matrix, $U_Q \in U(N_C)$. In terms of this unitary matrix, we rewrite Eq. (3.17) as

\[
X = \sqrt{QQ^\dagger e^{-V'}} \sqrt{QQ^\dagger} = U_Q^\dagger \hat{Q}^\dagger e^{-V'} \hat{Q} U_Q,
\]

\[
X^2 = \left( Q^T J \sqrt{QQ^\dagger} \right)^\dagger Q^T J \sqrt{QQ^\dagger} = U_Q^\dagger \hat{M}^\dagger \hat{Q} U_Q = U_Q^\dagger \hat{M}^\dagger \hat{M} U_Q.
\]

(B.4)

Since $\hat{Q}$ and $\hat{M}$ are invertible, we find a unique solution of $V'$

\[
V' = \log \left( \frac{\hat{Q}}{\sqrt{\hat{M}^\dagger \hat{M}}} \hat{Q}^\dagger \right).
\]

(B.5)
B.2 Solution with rank $M = N_C - 1$

In this case rank $\hat{Q} = N_C - 1$, we can use the $U(N_C)$ rotation so that the $N_C$-by-$N_C$ matrix $\hat{Q}$ takes the form

$$\hat{Q} = \begin{pmatrix} 0 & \cdots & 0 \\ \hat{Q} & \vdots & \hat{Q} \\ 0 & \cdots & 0 \end{pmatrix}, \quad (B.6)$$

where $\hat{Q}$ is an $N_C$-by-$(N_C - 1)$ matrix. We can introduce an $N_C$-component vector $p$ such that

$$p^T J \hat{Q} = p^T J Q = 0, \quad p^T J p = 1, \quad (B.7)$$

and the following $N_C$-by-$N_C$ matrix has the maximal rank

$$R \equiv (\hat{Q}, p) \in GL(N_C, \mathbb{C}). \quad (B.8)$$

Note that with a given $\hat{Q}$, the column vector $p$ is uniquely determined up to sign. Since $R$ is invertible, $e^{V'}$ can be decomposed as

$$e^{V'} = R \begin{pmatrix} B & c \\ c^T & a \end{pmatrix} R^\dagger. \quad (B.9)$$

Here, $B$ is an $(N_C - 1)$-by-$(N_C - 1)$ Hermitian matrix and $a$ is a real parameter. Eq.(3.16) can be rewritten as

$$e^{V'^T} J Q Q^\dagger = Q^* Q^T J e^{V'}. \quad (B.10)$$

Substituting the above decomposition and multiplying $R^T J^*$ from the left and $J^T R^*$ from the right, we find that

$$B^T \tilde{M} = \tilde{M} B, \quad c = 0. \quad (B.11)$$

From the condition for $e^{V'} \in SO(N_C)$, we find the following equations

$$a^2 = 1, \quad \tilde{M}^T B^T \tilde{M} B = 1_{N_C - 1}. \quad (B.12)$$

Note that we can say that $B$ and $a$ are positive definite since $c = 0$. Combining the above two equations, we obtain

$$B = \frac{1}{\sqrt{\tilde{M}^T \tilde{M}}}, \quad a = 1. \quad (B.13)$$

45
Therefore we finally find a unique solution
\[
e^V' = \hat{Q} \frac{1}{\sqrt{M^\dagger M}} \hat{Q}^\dagger + pp^\dagger.
\] (B.14)

Note that \(pp^\dagger\) is uniquely determined for a given \(\hat{Q}\), namely for a given \(Q\). Even if we could construct a similar solution for \(V'\) in the case of rank \(M < N_C - 1\), it is obviously expected that a matrix corresponding to \(pp^\dagger\) would not be unique. These results exactly reflect the appearance of a partial Coulomb phase in the case of rank \(M < N_C - 1\).

## C Deformed Kähler Potential for USp(2MC)

The expansion of the deformed Kähler potential of Eq. (3.37) reads
\[
K_{USp,\text{deformed}} = \frac{1}{2} \sum_{i,j} \frac{1}{\mu_i' + \mu_j'} \left[ 1 + \frac{\varepsilon^2}{\mu_i' \mu_j'} \right] \phi_{ij} \phi_{ji}^\dagger \\
- \frac{1}{2} \sum_{i,j,k} \frac{\mu_i}{\mu_i' + \mu_j'} (\mu_i' + \mu_k')(\mu_j' + \mu_k) \left[ 1 + \frac{\varepsilon^2 \mu_i' + \mu_j' + \mu_k'}{\mu_i' \mu_j' \mu_k'} \right] \phi_{ij} \phi_{jk}^\dagger (\phi J)^\dagger_{ki} + \text{c.c.} \\
+ \sum_{i,j,k,l} X(\varepsilon)_{ijkl}(\phi J)^\dagger_{ij}(\phi J)^\dagger_{jk} \phi_{kl}^\dagger \phi_{li}^\dagger + \text{c.c.} \\
+ \frac{1}{2} \sum_{i,j,k,l} \mu_{ij} \mu_{kl} \frac{C_{ijkl}^{(1)'}}{P_{ijkl}'} \left[ C_{ijkl}^{(1)'2} C_{ijkl}^{(2)'2} - C_{ijkl}^{(3)'2} \right] (\phi J)^\dagger_{ij} \phi_{jk}^\dagger \phi_{kl}^\dagger (\phi J^\dagger)^{\dagger}_{li} \\
- \frac{1}{4} \sum_{i,j,k,l} \frac{C_{ijkl}^{(3)'2}}{P_{ijkl}'} + 2\varepsilon^2 \sum_{i,j,k,l} \frac{C_{ijkl}^{(1)'2}}{P_{ijkl}'} + 4\varepsilon^4 \sum_{i,j,k,l} \frac{C_{ijkl}^{(3)'2}}{P_{ijkl}'} \phi_{ij} \phi_{jk} (\phi J^\dagger)^{\dagger}_{li} + \text{Kähler trfs.} + \mathcal{O}(\phi^5),
\]
(C.1)

where \(\mu_i'^2 \equiv \mu_i^2 + \varepsilon^2\). The resulting curvature is
\[
R|_{\phi=0} = -2 \sum_{i}^{MC} \frac{\mu_i^6 + \varepsilon^2 \mu_i'^4 - \varepsilon^4 17 \mu_i'^2 - \varepsilon^6 7}{2\mu_i'(\varepsilon^2 + \mu_i'^2)^3} \\
- 2 \sum_{i,j}^{MC} \frac{\mu_i'^4 \mu_j'^4}{(\mu_i' + \mu_j')(\varepsilon^2 + \mu_i'^2)(\varepsilon^2 + \mu_j'^2)(\varepsilon^2 + \mu_i' \mu_j')^2} \\
+ 2\varepsilon^2 \sum_{i,j}^{MC} \frac{\mu_i'^2 \mu_j'^2 (\mu_i'^2 + \mu_j'^2)}{(\mu_i' + \mu_j')(\varepsilon^2 + \mu_i'^2)(\varepsilon^2 + \mu_j'^2)(\varepsilon^2 + \mu_i' \mu_j')^2} \\
- 4\varepsilon^4 \sum_{i,j}^{MC} \frac{\mu_i'^2 (6\mu_i'^2 + 9\mu_i' \mu_j' + 5\mu_j'^2)}{(\mu_i' + \mu_j')(\varepsilon^2 + \mu_i'^2)(\varepsilon^2 + \mu_j'^2)(\varepsilon^2 + \mu_i' \mu_j')^2}.
\]
\[ -4\varepsilon^6 \sum_{i,j}^{MC} \frac{\mu_i' \left(5\mu_i'^2 + 15\mu_i'\mu_j' + 13\mu_j'^2\right)}{\mu_j'\left(\mu_i' + \mu_j'\right)\left(\varepsilon^2 + \mu_i'^2\right)\left(\varepsilon^2 + \mu_j'^2\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)^2} \]

\[ -2\varepsilon^8 \sum_{i,j}^{MC} \frac{10\mu_i' + 13\mu_j'}{\mu_j'\left(\mu_i' + \mu_j'\right)\left(\varepsilon^2 + \mu_i'^2\right)\left(\varepsilon^2 + \mu_j'^2\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)^2} \]

\[ + 16\varepsilon^6 \sum_{i,j,k}^{MC} \frac{\mu_i^3\mu_j^3\mu_k^3}{\left(\mu_i' + \mu_j'\right)^2\left(\mu_j' + \mu_i'\right)\left(\mu_i' + \mu_k'\right)\left(\mu_j' + \mu_k'\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)\left(\varepsilon^2 + \mu_k'\mu_j'\right)\left(\varepsilon^2 + \mu_i'\mu_k'\right)^2} \]

\[ + 16\varepsilon^2 \sum_{i,j,k}^{MC} \frac{\mu_i'\mu_j'\mu_k'}{\left(\mu_i' + \mu_j'\right)^2\left(\mu_j' + \mu_i'\right)\left(\mu_i' + \mu_k'\right)\left(\mu_j' + \mu_k'\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)\left(\varepsilon^2 + \mu_k'\mu_j'\right)\left(\varepsilon^2 + \mu_i'\mu_k'\right)^2} \]

\[ + 16\varepsilon^4 \sum_{i,j,k}^{MC} \frac{\mu_i'\mu_j'\mu_k'}{\left(\mu_i' + \mu_j'\right)^2\left(\mu_j' + \mu_i'\right)\left(\mu_i' + \mu_k'\right)\left(\mu_j' + \mu_k'\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)\left(\varepsilon^2 + \mu_k'\mu_j'\right)\left(\varepsilon^2 + \mu_i'\mu_k'\right)^2} \]

\[ + 16\varepsilon^6 \sum_{i,j,k}^{MC} \frac{\mu_i^2\mu_j^2\mu_k^2}{\left(\mu_i' + \mu_j'\right)^2\left(\mu_j' + \mu_i'\right)\left(\mu_i' + \mu_k'\right)\left(\mu_j' + \mu_k'\right)\left(\varepsilon^2 + \mu_i'\mu_j'\right)\left(\varepsilon^2 + \mu_k'\mu_j'\right)\left(\varepsilon^2 + \mu_i'\mu_k'\right)^2} \]

\[ \text{References} \]

[1] B. Zumino, “Supersymmetry And Kahler Manifolds,” Phys. Lett. B 87, 203 (1979).

[2] L. Alvarez-Gaume and D. Z. Freedman, “Geometrical Structure And Ultraviolet Finiteness In The Supersymmetric Sigma Model,” Commun. Math. Phys. 80, 443 (1981).

[3] T. L. Curtright and D. Z. Freedman, “Nonlinear Sigma Models With Extended Supersymmetry In Four-Dimensions,” Phys. Lett. B 90, 71 (1980) [Erratum-ibid. B 91, 487 (1980)]; L. Alvarez-Gaume and D. Z. Freedman, “Ricci Flat Kahler Manifolds And Supersymmetry,” Phys. Lett. B 94, 171 (1980); M. Roček and P. K. Townsend, “Three Loop Finiteness Of The N=4 Supersymmetric Nonlinear Sigma Model,” Phys. Lett. B 96, 72 (1980).

[4] U. Lindström and M. Roček, “Scalar Tensor Duality And N=1, N=2 Nonlinear Sigma Models,” Nucl. Phys. B 222, 285 (1983).

[5] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkahler Metrics and Supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[6] I. Antoniadis and B. Pioline, “Higgs branch, hyperKaehler quotient and duality in SUSY N = 2 Yang-Mills theories,” Int. J. Mod. Phys. A 12, 4907 (1997) [arXiv:hep-th/9607058].

[7] E. Calabi, “Métriques kähleriennes et fibrés holomorphes,” Ann. Scient. Ec. Norm. Sup. 12, 269 (1979).

[8] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, “Construction of instantons,” Phys. Lett. A 65, 185 (1978).
[9] N. H. Christ, E. J. Weinberg and N. K. Stanton, “General self-dual Yang-Mills solutions,” Phys. Rev. D 18, 2013 (1978); E. Corrigan, D. B. Fairlie, S. Templeton and P. Goddard, “A Green’s Function For The General Selfdual Gauge Field,” Nucl. Phys. B 140, 31 (1978).

[10] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, “Multi-instanton calculus and the AdS/CFT correspondence in N = 4 Nucl. Phys. B 552, 88 (1999) [arXiv:hep-th/9901128].

[11] M. F. Atiyah and N. J. Hitchin, “Low-Energy Scattering Of Nonabelian Monopoles,” Phys. Lett. A 107, 21 (1985); “Low-energy scattering of nonAbelian magnetic monopoles,” Phil. Trans. Roy. Soc. Lond. A 315, 459 (1985).

[12] G. W. Gibbons, R. Goto and P. Rychenkova, “HyperKaehler quotient construction of BPS monopole moduli spaces,” Commun. Math. Phys. 186, 585 (1997) [arXiv:hep-th/9608085].

[13] P. B. Kronheimer, “The Construction of ALE spaces as hyperKahler quotients,” J. Diff. Geom. 29, 665 (1989).

[14] U. Lindström, M. Roček and R. von Unge, “Hyperkaehler quotients and algebraic curves,” JHEP 0001, 022 (2000) [arXiv:hep-th/9908082].

[15] P. B. Kronheimer and H. Nakajima, “Yang-Mills instantons on ALE gravitational instantons”, Math. Ann. 288, 263 (1990) ; H. Nakajima, “Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras”, Duke Math. J. 76, 365 (1994) .

[16] R. Bielawski, “Complete hyperKaehler 4n-manifolds with n commuting tri-Hamiltonian vector fields,” Math. Ann. 314, 505 (1999) [arXiv:math.dg/9808134].

[17] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,” Nucl. Phys. B 431, 484 (1994) [arXiv:hep-th/9408099].

[18] P. C. Argyres, M. R. Plesser and N. Seiberg, “The Moduli Space of N=2 SUSY QCD and Duality in N=1 SUSY QCD,” Nucl. Phys. B 471, 159 (1996) [arXiv:hep-th/9603042].

[19] P. C. Argyres, M. R. Plesser and A. D. Shapere, “N = 2 moduli spaces and N = 1 dualities for SO(n(c)) and USp(2n(c)) super-QCD,” Nucl. Phys. B 483, 172 (1997) [arXiv:hep-th/9608129].

[20] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, “Global structure of moduli space for BPS walls,” Phys. Rev. D 71, 105009 (2005) [arXiv:hep-th/0503033].

[21] M. A. Luty and W. Taylor, “Varieties of vacua in classical supersymmetric gauge theories,” Phys. Rev. D 53, 3399 (1996) [arXiv:hep-th/9506098].

[22] K. A. Intriligator and N. Seiberg, “Lectures on supersymmetric gauge theories and electromagnetic duality,” Nucl. Phys. Proc. Suppl. 45BC, 1 (1996) [arXiv:hep-th/9509066].
[23] K. A. Intriligator and N. Seiberg, “Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric SO(N(c)) gauge theories,” Nucl. Phys. B 444, 125 (1995) [arXiv:hep-th/9503179].

[24] K. A. Intriligator and P. Pouliot, “Exact superpotentials, quantum vacua and duality in supersymmetric SP(N(c)) gauge theories,” Phys. Lett. B 353, 471 (1995) [arXiv:hep-th/9505006].

[25] J. Gray, A. Hanany, Y. H. He, V. Jejjala and N. Mekareeya, “SQCD: A Geometric Apercu,” JHEP 0805, 099 (2008) [arXiv:0803.4257 [hep-th]].

[26] A. Hanany and N. Mekareeya, “Counting Gauge Invariant Operators in SQCD with Classical Gauge Groups,” [arXiv:0805.3728 [hep-th]].

[27] A. M. Polyakov and A. A. Belavin, “Metastable States of Two-Dimensional Isotropic Ferromagnets,” JETP Lett. 22, 245 (1975) [Pisma Zh. Eksp. Teor. Fiz. 22, 503 (1975)].

[28] V. L. Golo and A. M. Perelomov, “Solution Of The Duality Equations For The Two-Dimensional SU(N) Invariant Chiral Model,” Phys. Lett. B 79, 112 (1978); A. D’Adda, M. Luscher and P. Di Vecchia, “A 1/N Expandable Series Of Nonlinear Sigma Models With Instantons,” Nucl. Phys. B 146, 63 (1978); A. M. Din and W. J. Zakrzewski, “General Classical Solutions In The Cp**(N-1) Model,” Nucl. Phys. B 174, 397 (1980).

[29] A. J. Macfarlane, “Generalizations Of Sigma Models And Cp(N) Models And Instantons,” Phys. Lett. B 82, 239 (1979); A. M. Din and W. J. Zakrzewski, “Classical Solutions In Grassmannian Sigma Models,” Lett. Math. Phys. 5, 553 (1981); “Further Properties Of Classical Solutions In Grassmannian Sigma Models,” Nucl. Phys. B 237, 461 (1984).

[30] A. M. Perelomov, “Chiral models: geometrical aspects,” Phys. Rept. 146, 135 (1987); A. Ya. Morozov, A. M. Perelomov and M. A. Shifman, “Exact Gell-Mann-Low Function Of Supersymmetric Kahler Sigma Models,” Nucl. Phys. B 248, 279 (1984).

[31] M. Naganuma, M. Nitta and N. Sakai, “BPS lumps and their intersections in N = 2 SUSY nonlinear sigma models,” Grav. Cosmol. 8, 129 (2002) [arXiv:hep-th/0108133]; R. Portugues and P. K. Townsend, “Sigma-model soliton intersections from exceptional calibrations,” JHEP 0204, 039 (2002) [arXiv:hep-th/0203181]; M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, “1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations,” Nucl. Phys. B 752, 140 (2006) [arXiv:hep-th/0506257].

[32] R. S. Ward, “Slowly Moving Lumps In The Cp**1 Model In (2+1)-Dimensions,” Phys. Lett. B 158, 424 (1985); R. Leese, “Low-energy scattering of solitons in the CP**1 model,” Nucl. Phys. B 344, 33 (1990).
[33] T. Vachaspati and A. Achucarro, “Semilocal cosmic strings,” Phys. Rev. D 44, 3067 (1991); A. Achucarro and T. Vachaspati, “Semilocal and electroweak strings,” Phys. Rept. 327, 347 (2000) [arXiv:hep-ph/9904229].

[34] M. Hindmarsh, “Existence and stability of semilocal strings,” Phys. Rev. Lett. 68, 1263 (1992); M. Hindmarsh, “Semilocal topological defects,” Nucl. Phys. B 392, 461 (1993) [arXiv:hep-ph/9206229]; J. Preskill, “Semilocal defects,” Phys. Rev. D 46, 4218 (1992) [arXiv:hep-ph/9206216].

[35] A. A. Abrikosov, “On the Magnetic properties of superconductors of the second group,” Sov. Phys. JETP 5, 1174 (1957) [Zh. Eksp. Teor. Fiz. 32, 1442 (1957)]; H. B. Nielsen and P. Olesen, “Vortex-line models for dual strings,” Nucl. Phys. B 61 (1973) 45.

[36] K. Benson and M. Bucher, “Skyrmions and semilocal strings in cosmology,” Nucl. Phys. B 406, 355 (1993) [arXiv:hep-ph/9304214]; A. Achucarro, J. Borrill and A. R. Liddle, “The formation rate of semilocal strings,” Phys. Rev. Lett. 82, 3742 (1999) [arXiv:hep-ph/9802306]; A. Achucarro, P. Salmi and J. Urrestilla, “Semilocal Cosmic String Networks,” Phys. Rev. D 75, 121703 (2007) [arXiv:astro-ph/0512487]; P. Laguna, V. Natcu, R. A. Matzner and T. Vachaspati, “Intercommutation of semilocal strings and skyrmions,” Phys. Rev. Lett. 98, 041602 (2007) [arXiv:hep-th/0604177]; J. Urrestilla, N. Bevis, M. Hindmarsh, M. Kunz and A. R. Liddle, “Cosmic microwave anisotropies from BPS semilocal strings,” JCAP 0807, 010 (2008) [arXiv:0711.1842 [astro-ph]]; R. A. Battye, B. Garbrecht and A. Pilaftsis, “Textures and Semi-Local Strings in SUSY Hybrid Inflation,” [arXiv:0807.1729 [hep-ph]].

[37] K. Dasgupta, J. P. Hsu, R. Kallosh, A. Linde and M. Zagermann, “D3/D7 brane inflation and semilocal strings,” JHEP 0408, 030 (2004) [arXiv:hep-th/0405247]; K. Dasgupta, H. Firouzjahi and R. Gwyn, “Lumps in the throat,” JHEP 0704, 093 (2007) [arXiv:hep-th/0702193]; K. Dasgupta, V. Hussin and A. Wissanji, “Quaternionic Kahler Manifolds, Constrained Instantons and the Magic Square: I,” Nucl. Phys. B 793, 34 (2008) [arXiv:0708.1023 [hep-th]].

[38] A. Hanany and D. Tong, “Vortices, instantons and branes,” JHEP 0307, 037 (2003) [arXiv:hep-th/0306150].

[39] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, “Nonabelian superconductors: Vortices and confinement in N = 2 SQCD,” Nucl. Phys. B 673, 187 (2003) [arXiv:hep-th/0307287].

[40] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Moduli space of non-Abelian vortices,” Phys. Rev. Lett. 96, 161601 (2006) [arXiv:hep-th/0511088]; M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, “Non-Abelian vortices of higher winding numbers,” Phys. Rev. D 74, 065021 (2006) [arXiv:hep-th/0607070].
[41] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “All exact solutions of a 1/4 Bogomol’nyi-Prasad-Sommerfield equation,” Phys. Rev. D 71, 065018 (2005) [arXiv:hep-th/0405129]; “Construction of non-Abelian walls and their complete moduli space,” Phys. Rev. Lett. 93, 161601 (2004) [arXiv:hep-th/0404198]; “Non-Abelian walls in supersymmetric gauge theories,” Phys. Rev. D 70, 125014 (2004) [arXiv:hep-th/0405194].

[42] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Solitons in the Higgs phase: The moduli matrix approach,” J. Phys. A 39, R315 (2006) [arXiv:hep-th/0602170]; “Solitons in supersymmetric gauge theories: Moduli matrix approach,” in the proceedings of 7th Workshop on Continuous Advances in QCD, Minneapolis, Minnesota, 11-14 May 2006, Published in *Minneapolis 2006, Continuous advances in QCD* 58-71 [arXiv:hep-th/0607225].

[43] K. Hashimoto and D. Tong, “Reconnection of non-abelian cosmic strings,” JCAP 0509, 004 (2005) [arXiv:hep-th/0506022]; R. Auzzi, M. Shifman and A. Young, “Composite non-Abelian Flux Tubes in N=2 SQCD,” Phys. Rev. D 73, 105012 (2006) [Erratum-ibid. D 76, 109901 (2007)] [arXiv:hep-th/0511150].

[44] M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi and W. Vinci, “Universal reconnection of non-Abelian cosmic strings,” Phys. Rev. Lett. 98, 091602 (2007) [arXiv:hep-th/0609214].

[45] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Manifestly supersymmetric effective Lagrangians on BPS solitons,” Phys. Rev. D 73, 125008 (2006) [arXiv:hep-th/0602289].

[46] D. Tong, “TASI lectures on solitons,” [arXiv:hep-th/0509216].

[47] K. Konishi, “The magnetic monopoles seventy-five years later,” Lect. Notes Phys. 737, 471 (2008) [arXiv:hep-th/0702102].

[48] M. Shifman and A. Young, “Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories,” Rev. Mod. Phys. 79, 1139 (2007) [arXiv:hep-th/0703267].

[49] D. Tong, “Monopoles in the Higgs phase,” Phys. Rev. D 69, 065003 (2004) [arXiv:hep-th/0307302]; M. Shifman and A. Young, “Non-Abelian string junctions as confined monopoles,” Phys. Rev. D 70, 045004 (2004) [arXiv:hep-th/0403149].

[50] A. Hanany and D. Tong, “Vortex strings and four-dimensional gauge dynamics,” JHEP 0404, 066 (2004) [arXiv:hep-th/0403158].

[51] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Instantons in the Higgs phase,” Phys. Rev. D 72, 025011 (2005) [arXiv:hep-th/0412048]; T. Fujimori, M. Nitta, K. Ohta, N. Sakai and M. Yamazaki, “Intersecting Solitons, Amoeba and Tropical Geometry,” arXiv:0805.1194 [hep-th], to appear in Phys. Rev. D.
[52] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, “Nonabelian monopoles and the vortices that confine them,” Nucl. Phys. B 686, 119 (2004) [arXiv:hep-th/0312233]; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and H. Murayama, “NonAbelian monopoles,” Nucl. Phys. B 701, 207 (2004) [arXiv:hep-th/0405070]; M. Eto et al., “Non-Abelian duality from vortex moduli: a dual model of color-confinement,” Nucl. Phys. B 780, 161 (2007) [arXiv:hep-th/0611313]; D. Dorigoni, K. Konishi and K. Ohashi, “Non-Abelian Vortices without Dynamical Abelianization,” [arXiv:0801.3284 [hep-th]].

[53] M. Shifman and A. Yung, “Non-abelian flux tubes in SQCD: Supersizing world-sheet supersymmetry,” Phys. Rev. D 72, 085017 (2005) [arXiv:hep-th/0501211]; A. Gorsky, M. Shifman and A. Yung, “N = 1 supersymmetric quantum chromodynamics: How confined non-Abelian monopoles emerge from quark condensation,” Phys. Rev. D 75, 065032 (2007) [arXiv:hep-th/0701040]; M. Edalati and D. Tong, “Heterotic vortex strings,” JHEP 0705, 005 (2007) [arXiv:hep-th/0703045]; D. Tong, “The quantum dynamics of heterotic vortex strings,” JHEP 0709, 022 (2007) [arXiv:hep-th/0703235]; M. Shifman and A. Yung, “Heterotic Flux Tubes in N=2 SQCD with N=1 Preserving Deformations,” Phys. Rev. D 77, 125016 (2008) [arXiv:0803.0158 [hep-th]]; M. Shifman and A. Yung, “Large-N Solution of the Heterotic N=(0,2) Two-Dimensional CP(N-1) Model,” Phys. Rev. D 77, 125017 (2008) [arXiv:0803.0698 [hep-th]].

[54] B. Collie, “Dyonic Non-Abelian Vortices,” [arXiv:0809.0394 [hep-th]].

[55] M. Shifman and A. Yung, “Non-Abelian semilocal strings in N = 2 supersymmetric QCD,” Phys. Rev. D 73, 125012 (2006) [arXiv:hep-th/0603134].

[56] M. Eto et al., “On the moduli space of semilocal strings and lumps,” Phys. Rev. D 76, 105002 (2007) [arXiv:0704.2218 [hep-th]].

[57] L. Ferretti, S. B. Gudnason and K. Konishi, “Non-Abelian vortices and monopoles in SO(N) theories,” Nucl. Phys. B 789, 84 (2008) [arXiv:0706.3854 [hep-th]].

[58] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, “Constructing Non-Abelian Vortices with Arbitrary Gauge Groups,” [arXiv:0802.1020 [hep-th]], to appear in Phys. Lett. B.

[59] P. Fayet and J. Iliopoulos, Phys. Lett. B 51, 461 (1974).

[60] S. Aoyama, “The Supersymmetric U(N,R) Sigma Model And Its 0(2) Extended Supersymmetry,” Nuovo Cim. A 57, 176 (1980).

[61] K. Higashijima and M. Nitta, “Supersymmetric nonlinear sigma models as gauge theories,” Prog. Theor. Phys. 103, 635 (2000) [arXiv:hep-th/9911139].
[62] S. J. Gates and S. M. Kuzenko, “The CNM-hypermultiplet nexus,” Nucl. Phys. B 543, 122 (1999) [arXiv:hep-th/9810137]; “4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kaehler manifolds,” Fortsch. Phys. 48, 115 (2000) [arXiv:hep-th/9903013]; S. J. Gates, T. Hubsch and S. M. Kuzenko, “CNM models, holomorphic functions and projective superspace C-maps,” Nucl. Phys. B 557, 443 (1999) [arXiv:hep-th/9902211]; M. Arai and M. Nitta, “Hyper-Kaehler sigma models on (co)tangent bundles with SO(n) isometry,” Nucl. Phys. B 745, 208 (2006) [arXiv:hep-th/0602277]; M. Arai, S. M. Kuzenko and U. Lindstrom, “Hyperkaehler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace,” JHEP 0702, 100 (2007) [arXiv:hep-th/0612174]; “Polar supermultiplets, Hermitian symmetric spaces and hyperkahler metrics,” JHEP 0712, 008 (2007) [arXiv:0709.2633 [hep-th]].

[63] M. Arai, M. Nitta and N. Sakai, “Vacua of massive hyper-Kaehler sigma models of non-Abelian quotient,” Prog. Theor. Phys. 113, 657 (2005) [arXiv:hep-th/0307274]; “Massive Hyper-Kaehler sigma models and BPS domain walls,” Phys. Atom. Nucl. 68, 1634 (2005) [Yad. Fiz. 68, 1698 (2005)] [arXiv:hep-th/0401102].

[64] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, “Structure Of Nonlinear Realization In Supersymmetric Theories,” Phys. Lett. B 138, 94 (1984); “Nonlinear Realization In Supersymmetric Theories,” Prog. Theor. Phys. 72, 313 (1984); “Nonlinear Realization In Supersymmetric Theories. 2,” Prog. Theor. Phys. 72, 1207 (1984); K. Higashijima, M. Nitta, K. Ohta and N. Ohta, “Low energy theorems in N = 1 supersymmetric theory,” Prog. Theor. Phys. 98, 1165 (1997) [arXiv:hep-th/9706219].

[65] A. C. W. Kotcheff and G. M. Shore, “Kahler sigma models from supersymmetric gauge theories,” Int. J. Mod. Phys. A 4, 4391 (1989); M. Nitta, “Moduli space of global symmetry in N = 1 supersymmetric theories and the quasi-Nambu-Goldstone bosons,” Int. J. Mod. Phys. A 14, 2397 (1999) [arXiv:hep-th/9805038].

[66] M. Eto, T. Fujimori, M. Nitta, K. Ohashi and N. Sakai, “Domain Walls with Non-Abelian Clouds,” Phys. Rev. D 77, 125008 (2008) [arXiv:0802.3135 [hep-th]].

[67] M. Eto, M. Nitta and N. Sakai, “Effective theory on non-Abelian vortices in six dimensions,” Nucl. Phys. B 701, 247 (2004) [arXiv:hep-th/0405161].

[68] K. M. Benson, A. V. Manohar and M. Saadi, “QCD flux tubes as sigma model relics,” Phys. Rev. Lett. 74, 1932 (1995) [arXiv:hep-th/9409042]; K. M. Benson and M. Saadi, “QCD flux tubes in a current algebra approach,” Phys. Rev. D 51, 3096 (1995) [arXiv:hep-th/9409109].

[69] A. Dancer and A Swann, “Hyperkähler metrics of cohomogeneity one,” J. Geom. Phys. 21, 218 (1997)
[70] K. Higashijima, T. Kimura and M. Nitta, “Ricci-flat Kaehler manifolds from supersymmetric gauge theories,” Nucl. Phys. B 623, 133 (2002) [arXiv:hep-th/0108084]; “Gauge theoretical construction of non-compact Calabi-Yau manifolds,” Annals Phys. 296, 347 (2002) [arXiv:hep-th/0110216].

[71] K. Higashijima, T. Kimura and M. Nitta, “Calabi-Yau manifolds of cohomogeneity one as complex line bundles,” Nucl. Phys. B 645, 438 (2002) [arXiv:hep-th/0202064].

[72] L. Alvarez-Gaume and D. Z. Freedman, “Potentials For The Supersymmetric Nonlinear Sigma Model,” Commun. Math. Phys. 91, 87 (1983); S. J. Gates, “Superspace Formulation Of New Nonlinear Sigma Models,” Nucl. Phys. B 238, 349 (1984).

[73] J. P. Gauntlett, D. Tong and P. K. Townsend, “Multi-domain walls in massive supersymmetric sigma-models,” Phys. Rev. D 64, 025010 (2001) [arXiv:hep-th/0012178].

[74] M. Arai, M. Naganuma, M. Nitta and N. Sakai, “Manifest supersymmetry for BPS walls in N = 2 nonlinear sigma models,” Nucl. Phys. B 652, 35 (2003) [arXiv:hep-th/0211103]; “BPS wall in N = 2 SUSY nonlinear sigma model with Eguchi-Hanson manifold,” in Garden of Quanta - In honor of Hiroshi Ezawa, Eds. by J. Arafune et al. (World Scientific Publishing Co. Pte. Ltd. Singapore, 2003) pp 299-325, [arXiv:hep-th/0302028].

[75] J. P. Gauntlett, D. Tong and P. K. Townsend, “Supersymmetric intersecting domain walls in massive hyper-Kaehler sigma models,” Phys. Rev. D 63, 085001 (2001) [arXiv:hep-th/0007124].

[76] R. A. Leese, “Q lumps and their interactions,” Nucl. Phys. B 366, 283 (1991); E. Abraham, “Nonlinear Sigma Models And Their Q Lump Solutions,” Phys. Lett. B 278, 291 (1992).

[77] D. Bak, S. O. Hahn, J. Lee and P. Oh, “Supersymmetric Q-lumps in the Grassmannian nonlinear sigma models,” Phys. Rev. D 75, 025004 (2007) [arXiv:hep-th/0610067]; H. Nakajima, P. Oh and S. Shin, “Quantum SUSY Algebra of Q-lump in the Massive Grassmannian Sigma Model,” arXiv:0808.1019 [hep-th].