Cohomological Weight Shiftings for Automorphic Forms on Definite Quaternion Algebras

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May 29, 2012

Abstract

Let \( F/\mathbb{Q} \) be a totally real field extension of degree \( g \) and let \( D \) be a definite quaternion algebra with center \( F \). Fix an odd prime \( p \) which is unramified in \( F \) and \( D \). We produce weight shiftings between \((\text{mod } p)\) automorphic forms on \( D^+ \) of a fixed level \( U \). When the starting weight does not contain any \((2, \ldots, 2)\)-block, we obtain these shiftings via maps induced in cohomology by intertwining operators acting on \( \overline{\mathbb{F}}_p\)-representations of \( GL_2(\mathcal{O}_F/p\mathcal{O}_F) \). We construct two families of such operators, each of cardinality \( g^2 \), and we produce between others weight shiftings by cyclic permutations of the blocks \((p^r, 0, \ldots, 0, \pm 1, 0, \ldots, 0)\) where the number of zeros between \( p^r \) and \( \pm 1 \) depends upon the value of the integer \( r \). In particular, we produce shiftings by \((p, \pm 1, 0, \ldots, 0)\).

Contents

1 Introduction 2

I Weight shiftings for \( GL_2(\mathbb{F}_q) \)-modules 8

2 Untwisted \( GL_2(\mathbb{F}_q) \)-modules 8

3 Twisted \( GL_2(\mathbb{F}_q) \)-modules and intertwining operators for \( g > 1 \) 16

3.1 Identities in \( K_0(G) \) (II) 16
3.2 Determination of Jordan-Hölder constituents: the case $g > 1$ ......................................... 18
3.3 Families of intertwining operators for $g > 1$ ............................................................... 21
  3.3.1 Generalized Dickson invariants ................................................................. 21
  3.3.2 Generalized $D$-operators ................................................................. 25

II  Weight shiftings for automorphic forms ..................................................... 29

4  Shiftings for weights not containing $(2, \ldots, 2)$-blocks ................................. 29
  4.1 Some motivations: geometric Hilbert modular forms .............................. 30
  4.2 Automorphic forms on definite quaternion algebras .............................. 32
  4.3 Behavior of Hecke eigensystems under reduction modulo $\mathfrak{m}_R$ ....... 33
  4.4 Holomorphic weights ............................................................................... 35
  4.4.1 Some results on holomorphic weight shiftings .................................... 36
  4.4.2 Link with classical automorphic forms on $D^\times$ ......................... 38
  4.5 Holomorphic weight shiftings via generalized Dickson invariants and $D$-operators ................................................................. 39
  4.5.1 Main theorem ...................................................................................... 39
  4.5.2 Analysis of the case $f_j < 3$ .............................................................. 43

5  Shiftings for weights containing $(2, \ldots, 2)$-blocks ................................. 46

1  Introduction

Let $F$ be a totally real number field and $p$ be an odd prime which is unramified in $F$. Let $D$ be a definite quaternion algebra with center $F$ and assume that $D$ is split at all primes of $F$ above $p$. In this paper we produce congruences modulo $p$ between automorphic forms on $D^\times$ having fixed level and varying weights. Our interest in this matter is partially motivated by the study of the weight part of Serre’s modularity conjecture over a totally real field, as formulated in [4] and proven in many cases in [8] and [9]. We hope that our constructions could be further generalizable to algebraic groups other than $D^\times$.

Weight shiftings for modulo $p$ elliptic modular forms can be obtained via the classical theory of Hasse invariants and theta operators ([18]), and have been studied via cohomological methods by Ash-Stevens ([3]) and Edixhoven-Khare ([7]). In [16], the author studied cohomological weight shiftings of Hasse-type, adopting the viewpoint conceived by C. Khare on this matter. In [7], Edixhoven and Khare produce parallel weight shiftings by $p-1$ for automorphic forms on $D^\times$ of parallel weight two, under the assumption that $p$ is inert in $F$. In [8], Gee uses a construction of Kisin ([14]) relying on the classification of the irreducible admissible $\mathbb{F}_p$-representations of $GL_2(\mathbb{Q}_p)$ to produce some non-parallel weight shiftings by $p-1$ on forms on $D^\times$, assuming that $p$ is totally split in $F$: this is a crucial step in his proof of the weight conjecture of [4] in the totally split case. Weight shiftings for geometric Hilbert modular forms over $F$ can be obtained via operators constructed by Andreatta and Goren in [1].
As these results show, there are two possible approaches to the study of congruences between automorphic forms of different weights: geometric and cohomological. Assume that \( p > 3 \) and that \( N \geq 5 \) is coprime to \( p \). The theta operator and the Hasse invariant are geometrically defined operators acting on spaces of \((\text{mod } p)\) elliptic modular forms of level \( N \). The first operator induces a Hecke equivariant injection increasing weights by \( p - 1 \); the latter is a Hecke twist-equivariant map that shifts weights by \( p + 1 \). The geometric approach to weight shiftings passes through the generalization of these operators to spaces of geometric Hilbert modular forms over the totally real field \( F \). This is carried over by Goren in \([10]\), where \([F : \mathbb{Q}]\) partial Hasse invariants are constructed assuming \( p \) is unramified in \( F \), and by Andreatta-Goren in \([1]\), where generalized theta operators are considered and the unramifiedness assumption of \( p \) in \( F \) is dropped.

The Eichler-Shimura isomorphism translates the study of Hecke eigensystems of \((\text{mod } p)\) elliptic modular forms of weight \( k \geq 2 \) and level \( N \) into the study of the Hecke action on the cohomology group \( H^1(\Gamma_1(N), \text{Sym}^{k-2} \mathbb{F}_p) \). In \([3]\), Ash and Stevens identify a cohomological analogue of the theta operator in the map induced in cohomology by the Dickson polynomial \( \Theta_p = XY^p - X^pY \in \mathbb{F}_p[X,Y] \). For the cohomological counterpart of the Hasse invariant, we must restrict ourselves to work with \( p \)-small weights. Additionally, the case of weight two must be treated \textit{per se} (this dichotomy between weight two and weight larger than two will appear, \textit{mutatis mutandis}, also in this paper). In \([7]\), a cohomological analogue of the Hasse invariant acting upon weight two forms is constructed by studying a degeneracy map:

\[
H^1(\Gamma_1(N), \mathbb{F}_p)^2 \longrightarrow H^1(\Gamma_1(N) \cap \Gamma_0(p), \text{Sym}^{p-1} \mathbb{F}_p^2).
\]

A \( GL_2(\mathbb{F}_p) \)-equivariant derivation \( D \) of \( \mathbb{F}_p[X,Y] \) defined by Serre by:

\[
Df = X^p \partial_X f + Y^p \partial_Y f
\]

is used in \([16]\) to produce weight shiftings by \( p - 1 \) starting from forms of weight \( 2 < k \leq p + 1 \). The cokernels of the operators \( \Theta_p \) and \( D \) are related to the characteristic zero theory of representations of \( GL_2(\mathbb{F}_p) \) (cf. \([3],[16]\)).

Fred Diamond suggested to look for a generalization of the results of \([16]\) to other contexts. Following his suggestion, in this paper we construct weight shiftings for Hilbert modular forms in cohomological settings. The geometric picture had a motivational role in our study.

Let us mention first a few advantages of working with cohomology groups rather than with geometrically defined modular forms. First, by the Jacquet-Langlands correspondence, we are led to the more general study of weight shiftings for adelic automorphic forms on definite quaternion \( F \)-algebras, where \( F \) is a field as above. These adelic spaces (cf. \([12]\)) seem to be well suited for computations. Furthermore, their formation is compatible with base change (Proposition \([12]\)), while the formation of spaces of geometric Hilbert modular forms is not, in general: none of the geometrically constructed partial Hasse invariants lift to characteristic zero if \( F \neq \mathbb{Q} \). Finally, our methods produce a
larger variety of weight shiftings than the ones arising from the geometric setup (cf. Remark 4.12).

The paper is divided into two parts: in the first part, consisting of sections 2 and 3, we study weight shiftings for Serre’s weights, i.e., for (irreducible) \( \mathbb{F}_p \)-linear representations of \( GL_2(\mathcal{O}_F/p\mathcal{O}_F) \); in the second part, consisting of sections 4 and 5, we address the problem of weight shiftings for automorphic forms associated to the definite quaternion algebra \( D \) with center \( F \).

Set \([F : \mathbb{Q}] = g\). The main novelties of the paper consist in: (1) the introduction of \( g^2 \) generalized Dickson operators and \( g^2 \) generalized \( D \)-operators acting on \( \mathbb{F}_p[GL_2(\mathcal{O}_F/p\mathcal{O}_F)] \)-modules: these maps will induce cohomological generalizations of the theta operators and of the partial Hasse invariants, respectively; (2) the determination of many non-parallel weight shiftings for automorphic forms on \( D \times \) of a fixed level. In particular, we will answer a question of Diamond, as for any prime \( \mathfrak{p} \) of \( F \) above \( p \) we will produce weight shiftings that increase those entries of the weight parameter \( \vec{k} \) associated to the embeddings \( F \mathfrak{p} \oplus \mathfrak{p} \rightarrow \bar{\mathbb{Q}}_p \) - or by any cyclic permutation of this tuple, cf. 3.3.2.

Along the way, we will also obtain new identities between virtual modular representations of \( GL_2(\mathcal{O}_F/p\mathcal{O}_F) \) that will allow us to give an algorithm to compute the Jordan-Hölder constituents of any product of symmetric power representations of this group.

Let us now summarize the content of each section of the paper. Fix a positive integer \( g \) and set \( q = p^g \) and \( G = GL_2(\mathbb{F}_q) \). For any non-negative integer \( k \), define the \( \mathbb{F}_q[G] \)-module \( M_k = \text{Sym}^k \mathbb{F}_2^q \). In [20], after extending the definition of the \( M_k \)’s for \( k < 0 \) in a suitable way, Serre proves the following identity, valid in the Grothendieck ring of finitely generated \( \mathbb{F}_q[G] \)-modules for any integer \( k \):

\[
M_k - \det \cdot M_{k-(q+1)} = M_{k-(q-1)} - \det \cdot M_{k-2q}.
\]

The weight shiftings by \( q - 1 \) and by \( q + 1 \) appearing in the above formula are induced by the Dickson invariant \( \Theta_q \) and the derivation map \( D \) mentioned above. In section 2 we recall some constructions associated to these operators and some weight shiftings results for elliptic modular forms (cf. [16]).

In section 3 we derive the new identity (Corollary 3.3):

\[
M_k^i \cdot M_h^{i+1} - \det^{i+1} \cdot M_{k-p} \cdot M_{h-1} = M_k^{i} \cdot M_h^{i+1} - \det^{i+1} \cdot M_{k-2p} \cdot M_h^{i+1} \quad (1)
\]

valid for any \( h, k, i \in \mathbb{Z} \). Here the superscript \([i]\) indicates that the \( G \)-action is twisted by the \( i \)th power of the absolute Frobenius morphism of \( \mathbb{F}_q \). For \( g > 1 \), this identity allows us to explicitly compute the Jordan-Hölder factors of any virtual representations of the form \( \prod_{i=0}^{g-1} M_k^{[i]} \) (Theorem 3.7; cf. also [17]).

As for the case \( g = 1 \), also for \( g > 1 \) the periods appearing in (1), i.e., the cyclic permutations of the \( g \)-tuples \( (p, 1, 0, \ldots, 0) \) and \( (p, -1, 0, \ldots, 0) \), correspond to weight shiftings arising from \( G \)-equivariant operators. In 3.3.1 and 3.3.2 we define two families of such operators, each containing \( g^2 \) maps. For any
integers $\alpha, \beta$ subject to the constraints $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$, we construct generalized Dickson operators $\Theta^{[\bar{a}]}_{\beta}$ and generalized $D$-operators $D^{[\bar{a}]}_{\beta}$ giving rise, for any set of non-negative integers $k_0, \ldots, k_{g-1}$, to the $G$-modules monomorphism:

$$
\Theta^{[\bar{a}]}_{\beta} : \det F^* \otimes \bigotimes_i M^{[i]}_{k_i} \rightarrow \left( \bigotimes_{i \neq \alpha, \alpha + \beta} M^{[i]}_{k_i} \right) \otimes M^{[\alpha + \beta]}_{k_{\alpha + \beta} + p^{g - \beta}}
$$

and to the $G$-morphism:

$$
D^{[\bar{a}]}_{\beta} : \bigotimes_i M^{[i]}_{k_i} \rightarrow \left( \bigotimes_{i \neq \alpha, \alpha + \beta} M^{[i]}_{k_i} \right) \otimes M^{[\alpha + \beta]}_{k_{\alpha + \beta} + p^{g - \beta}}.
$$

We study some properties of these and other $G$-operators at the end of section 3.

In section 4 we use the above results to obtain weight shiftings for modulo $p$ automorphic forms on $D^\times$ having fixed level. We start by treating the case in which the tensor factors - corresponding to the prime decomposition of $p$ in $F$ - of the weight that we want to shift are all of dimension greater than one: this is what we call a weight not containing a $(2, \ldots, 2)$-block.

More precisely, write $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ and denote by $f_j$ the residual degree of $\mathfrak{p}_j$ over $p$. Let $\mathcal{O}$ be the ring of integers of the smallest unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ containing the image of $F$ under all the embeddings $F \rightarrow \overline{\mathbb{Q}}_p$; let $\mathbb{F}$ be the residue field of $\mathcal{O}$. Let $A$ be a topological $\mathbb{Z}_p$-algebra and denote by $\text{S}_{\tau, \psi}(U, A)$ the space of $A$-valued adelic automorphic forms on $D$ having level $U \subset (D \otimes F \mathbb{A}_F^\infty)^\times$, weight $\tau : U \rightarrow \text{Aut}(W_\tau)$ and Hecke character $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \rightarrow A^\times$. For any set $S$ of primes of $F$ containing the ramification set of $D$, the primes above $p$ and the primes $v$ for which $U_v$ is not a maximal compact subgroup of $D_v^\times$, the universal Hecke algebra $\mathbb{T}_{S, A}^{\text{uni}} = A[T_v, S_v : v \notin S]$ acts upon this space. We assume that $U$ is small enough.

We use the generalized Dickson and $D$-operators from section 3, together with some classical results of Ash-Stevens and Deligne-Serre, to produce congruences modulo $p$ between Hecke eigenforms arising from the spaces $\text{S}_{\tau, \psi}(U, \mathbb{Z}_p)$ for fixed $U$ and varying $\tau$. Some technical difficulties arise, as we are mainly interested in forms having holomorphic weight in the sense of 4.4, but, in general, our intertwining operators do not preserve holomorphicity. Furthermore, we need to make sure that when we transfer forms from weight $\tau$ to weight $\tau'$, we can lift the reduction of $\psi$ modulo the maximal ideal of $\mathbb{Z}_p$ to a compatible $\mathbb{Z}_p^\times$-valued Hecke character for $\tau'$.

One of the weight shiftings result we can prove is the following (Theorem 4.14). Assume that $\tau$ is the $O$-linear weight with holomorphic parameters $(\bar{k}, w) \in \mathbb{Z}_{\geq 2}^2 \times (2\mathbb{Z} + 1)$ and that $\psi$ is a Hecke character compatible with $\tau$. Let $f$ be the minimum of the residual degrees of the primes $\mathfrak{p}_j$ and fix an integer $\beta$ such that $1 \leq \beta \leq f$. For any integers $i$ and $j$ with $1 \leq j \leq r$ and $0 \leq i \leq f_j - 1$ choose $a^{(i)}_{(j)} \in \{ p^\beta - 1, p^\beta + 1 \}$ and set $\bar{a} = (a^{(1)}_{(j)}, \ldots, a^{(r)}_{(j)})$ with $\bar{a}^{(j)} = (a^{(j)}_{0}, \ldots, a^{(j)}_{f_j - 1})$, and $w' = w + (p^\beta - 1)$. Then:
**Theorem** Suppose that the weight \((\vec{k}, w)\) is \(p\)-small and generic, i.e., \(2 < k_i^{(j)} \leq p+1\) for all \(i, j\). Then, if \(\Omega\) is a Hecke eigensystem occurring in the space \(S_{\tau, \psi}(U, O)\), there is a finite local extension of discrete valuation rings \(O'/O\) and an \(O'\)-valued Hecke eigensystem \(\Omega'\) occurring in holomorphic weight \((\vec{k} + \vec{a}, w')\) and with associated Hecke character \(\psi'\) such that \(\Omega'(\bmod M_{O'}) = \Omega(\bmod M_O)\). The character \(\psi'\) is compatible with the weight \((\vec{k} + \vec{a}, w')\) and it can be chosen so that \(\bar{\psi}' = \bar{\psi}\).

More weight shifting results are proved in 4.5.2 under the assumption \(f_j < 3\) for all \(j\). The combinatorics involved in describing all the holomorphic weight shifting arising from the generalized Dickson and \(D\) operators becomes more complicated as the \(\mathbb{Z}_p\)-rank of \(O\) grows.

The techniques of sections 3 and 4 cannot be successfully applied to obtain weight shifting by \(p-1\) when starting from weights that contain at least a \((2,...,2)\)-block (for example parallel weight two). In section 5 we therefore represent a result due to Edixhoven and Khare ([7]):

**Theorem** Assume that \(\tau\) is an irreducible (non necessarily holomorphic) \(\mathbb{F}\)-linear weight with parameters \((\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g\) such that \(\vec{k}^{(j)} = \vec{2}\) for some \(1 \leq j \leq r\). Let \(\tau'\) be the \(\mathbb{F}\)-linear weight associated to the parameters \(\vec{k}' = (k^{(1)},...,k^{(j)} + p-\vec{1},...,k^{(r)})\) and \(\vec{w}' = \vec{w}\). For any non-Eisenstein maximal ideal \(\mathfrak{M}\) of \(T_{\text{univ}}^U\), there is an injective Hecke-equivariant \(\mathbb{F}\)-morphism:

\[S_{\tau}(U, \mathbb{F})_{\mathfrak{M}} \hookrightarrow S_{\tau'}(U, \mathbb{F})_{\mathfrak{M}}.\]

The proof of this result of Edixhoven and Khare relies on the determination of the \(T_{\text{univ}}^U\)-support of the kernel of a degeneracy map \(S_{\tau, \psi}(U, \mathbb{F})^2 \rightarrow S_{\tau, \psi}(U_0, \mathbb{F})\). We remark that in the above theorem the weight \(\tau\) is not assumed parallel. The weight shifting produced by repeatedly applying this theorem are not parallel, but parallel in blocks. We do not know if, starting from weight two, weight shifting by \(p-1\) which are not of this type are possible or if they can be obtained via the above methods.

**Conventions** Unless otherwise stated, in this paper all rings are assumed to have an identity element and are commutative. All the group representations are assumed to be left representations on a module of finite length over a fixed coefficient ring. The letter \(p\) always denotes a positive rational prime.

**Acknowledgements**

The problem of constructing cohomological weight shifting in the context of the Hasse invariant was conceived and suggested to me by Chandrashekhar Khare, to whom I am deeply indebted. His viewpoint on the problem is the core of [15], in which the elliptic case is considered, and it is therefore also the base of our study in the present paper. The idea of the existence of a possible
connection between cohomological weight shifting operators and integral models of some irreducible characteristic zero representations of $GL_2(\mathbb{F}_q)$, is entirely his.

I would like to further thank C. Khare for his constant, patient and generous advices during the years of my Ph.D. in UCLA, when this paper was written.

I would like to thank Jean-Pierre Serre, for defining the differential operator $D$ acting on the polynomial algebra $\mathbb{F}_q[X,Y]$, which plays a crucial role in [16] and in the present paper.

I am very grateful to Fred Diamond, who suggested to look into generalizations of the results of [16], and in particular into the existence of cohomological weight shiftings "by $(p, -1, 0, ..., 0)$". This induced me to construct and study the operators $D_{\beta\alpha}^{[\nu]}$ appearing in this paper.

I would like to express my gratitude to Claus Sorensen, for sharing with me his unpublished results on cohomological weight shiftings for modular forms having trivial weight. I mostly do not address this case in the present paper, and the results of Edixhoven-Khare ([17]) and Sorensen complete in this sense the picture of weight shiftings considered here.

I would like to thank Don Blasius, Haruzo Hida, Gordan Savin, and Jacques Tilouine for their precious comments and questions on the topics studied in this paper.
Part I

Weight shiftings for
$GL_2(\mathbb{F}_q)$-modules

2 Untwisted $GL_2(\mathbb{F}_q)$-modules

Fix a rational prime $p$, a positive integer $g$, and set $q = p^g$. Denote by $\mathbb{F}_q$ a finite field with $q$ elements and fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$; denote by $\sigma \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_p)$ the arithmetic Frobenius element. Let $G = GL_2(\mathbb{F}_q)$ and let $M$ be a representation of $G$ over $\mathbb{F}_q$; for any $n \in \mathbb{Z}$, the Frobenius element $\sigma^n$ induces a map $G \to G$ obtained by applying $\sigma^n$ to each entry of the matrices in $G$: composing this map with the action of $G$ on $M$, we give to the latter a new structure of $G$-module, that is denoted $M^{[n]}$ and called the $n$th Frobenius twist of $M$. If $f : M \to N$ is a $G$-homomorphism and $n \in \mathbb{Z}$, denote by $f^{[n]} : M^{[n]} \to N^{[n]}$ the map defined by $f^{[n]}(x) = f(x)$ for all $x \in M^{[n]}$: $f^{[n]}$ is a $G$-homomorphism.

Let $M_1$ denote the standard representation of $G$ on $\mathbb{F}_q^2$ and, for any positive integer $k$, define $M_k = \text{Sym}^k M_1$ to be the $k$th symmetric power of $M_1$. We identify $M_k$ with the $\mathbb{F}_q$-vector space of homogeneous polynomials over $\mathbb{F}_q$ in two variables and of degree $k$, endowed with the action of $G$ induced by:

$$
\begin{pmatrix}
  a & b \\
  c & d 
\end{pmatrix} \cdot X = aX + cY, 
\begin{pmatrix}
  a & b \\
  c & d 
\end{pmatrix} \cdot Y = bX + dY.
$$

We set $M_0$ to be the trivial representation of $G$. Denote by $\det : G \to \mathbb{F}_q^\times$ the determinant character of $G$, so that $\det^{[n]} = \det \sigma^n$.

Recall (cf. [21], [22] §13) that the irreducible representations of $G$ over $\mathbb{F}_q$ are all and only of the form:

$$
\det^m \otimes_{\mathbb{F}_q} \bigotimes_{i=0}^{g-1} M_{k_i}^{[i]},
$$

where $k_0, ..., k_{g-1}$ and $m$ are integers such that $0 \leq k_i \leq p-1$ for $i = 0, ..., g-1$, $0 \leq m < q-1$, and all the tensor products are over $\mathbb{F}_q$. The above representations are pairwise non-isomorphic.

We denote by $K_0(G)$ the Grothendieck group of finitely generated $\mathbb{F}_q[G]$-modules: it can be identified with the free abelian group generated by the isomorphism classes of irreducible representations of $G$ over $\mathbb{F}_q$ ([19]). If $M$ is an $\mathbb{F}_q[G]$-module, we denote by $[M]$ its class in $K_0(G)$ and set $e = [\det]$; if no confusion arises we also write $M$ to denote $[M]$. Tensor product over $\mathbb{F}_q$ induces on $K_0(G)$ a structure of commutative ring with identity; we denote the product in $K_0(G)$ by $\cdot$ or by juxtaposition.
2.1 Identities in $K_0(G)$ (I)

We present some identities between virtual representations in $K_0(G)$ that we will need later.

Negative weights  We extend the definition of $M_k \in K_0(G)$ for $k < 0$ in a way that is coherent with Brauer character computations, as suggested by Serre in [20]. We briefly explain this: a more detailed account of what follows is contained in [10] 2.1.

Let $G = GL_2$ as an algebraic group over $\mathbb{F}_q$, and let $T \subset G$ be the maximal split torus of diagonal matrices. Identify the character group $X(T)$ of $T$ with $\mathbb{Z}^2$ in the usual way, so that the roots associated to $(G, T)$ are $(1, -1)$ and $(-1, 1)$; fix a choice of positive root $\alpha = (1, -1)$. The corresponding Borel subgroup $B$ is the group of upper triangular matrices in $G$; we denote by $B^-$ the opposite Borel subgroup. For a fixed $\lambda \in X(T)$, let $M_\lambda$ be the one dimensional left $B^-$-module on which $B^-$ acts (through $T$) via the character $\lambda$. Denote by $\operatorname{ind}_{H^-}^G M_\lambda$ the left $G$-module given by algebraic induction from $B^-$ to $G$ of $M_\lambda$. Define the following generalization of the dual Weyl module for $\lambda$ (cf. [12], II.4.2):

$$W(\lambda) = \sum_{i \geq 0} (-1)^i \cdot R^i \operatorname{ind}_{B^-}^G (M_\lambda),$$

where $R^i \operatorname{ind}_{B^-}^G (\cdot)$ denote the $i$th right derived functor of $\operatorname{ind}_{B^-}^G (\cdot)$. $W(\lambda)$ is an element of the Grothendieck group $K_0(G)$ of $G$, because each $R^i \operatorname{ind}_{B^-}^G (M_\lambda)$ is a finite dimensional $G$-module, and $R^i \operatorname{ind}_{B^-}^G (M_\lambda)$ is zero for $i > 1$ ([12], II.4.2). For $\lambda_k = (k, 0) \in X(T)$ with $k$ any integer we have:

$$R^i \operatorname{ind}_{B^-}^G (M_{\lambda_k}) \simeq H^i(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k)).$$

If $k \geq 0$, $H^1(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k)) = 0$ so that $W(\lambda_k) = H^0(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k)) = \operatorname{Sym}^k \mathbb{F}_q^{\times}$; if $k < 0$ we have $H^0(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k)) = 0$ and $W(\lambda_k) = -H^1(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k))$; the canonical perfect pairing of $G$-modules:

$$H^0(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(-k - 2)) \times H^1(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(k)) \to H^1(\mathbb{P}_{\mathbb{F}_q}^1, \mathcal{O}(-2)) \simeq \det^{-1} \otimes \mathbb{G}_m,$$

brings naturally to the following:

**Definition 2.1** Let $k < 0$ be an integer. Define the element $M_k$ of the Grothendieck group $K_0(G)$ of $G$ over $\mathbb{F}_q$ by:

$$M_k = \begin{cases} 0 & \text{if } k = -1 \\ -e^{1+k} \cdot M_{-k-2} & \text{if } k \leq -2 \end{cases}.$$

**Lemma 2.2** For any $k \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k + e^{1+k} \cdot M_{-k-2} = 0. \quad (\Delta_{g,k})$$

9
Weight shifting by \( q \pm 1 \) Let us fix an embedding \( \iota : \mathbb{F}_q^2 \to M_2(\mathbb{F}_q) \) corresponding to a choice of \( \mathbb{F}_q \)-basis for the degree 2 extension of \( \mathbb{F}_q \) inside \( \mathbb{F}_q \). Let \( \mathbb{Q}_p \) be a fixed algebraic closure of the \( p \)-adic field \( \mathbb{Q}_p \) and let us fix an isomorphism between \( \mathbb{F}_q \) and the residue field of the ring of integers \( \mathbb{Z}_p \) of \( \mathbb{Q}_p \): denoting by \( \chi : \mathbb{F}_q^\times \to \mathbb{Z}_p^\times \) the corresponding Teichmüller character, the Brauer character \( G_{reg} \to \mathbb{Q}_p \) of the representations \( M_k \) \( (k \geq 1) \) is given as follows:

\[
\begin{pmatrix}
a \\
b
\end{pmatrix} \mapsto (k + 1)\chi(a)^k, \quad a \in \mathbb{F}_q^\times \\
\begin{pmatrix}
a \\
b
\end{pmatrix} \mapsto \frac{\chi(a)^{k+1} - \chi(b)^{k+1}}{\chi(a) - \chi(b)}, \quad a, b \in \mathbb{F}_q^\times, a \neq b \\
\iota(c) \mapsto \frac{\chi(c)^{q(k+1)} - \chi(c)^{k+1}}{\chi(c)^q - \chi(c)}, \quad c \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^\times.
\]

Using the above formulae, the following is proved in [20]:

**Lemma 2.3** For any \( k \in \mathbb{Z} \) we have in \( K_0(G) \) the identity:

\[ M_k - e \cdot M_{k-(q+1)} = M_{k-(q-1)} - e \cdot M_{k-2q}. \]  

\( \text{(Σ}_{q,k} \) 

Product formula It is a result of Glover that for any positive integers \( n, m \) there exists a short exact sequence of \( \mathbb{F}_q[SL_2(\mathbb{F}_q)] \)-modules of the form:

\[ 0 \to M_{n-1} \otimes_{\mathbb{F}_q} M_{m-1} \xrightarrow{j} M_n \otimes_{\mathbb{F}_q} M_m \xrightarrow{\pi} M_{n+m} \to 0, \]

where \( j \) is induced by the assignment \( u \otimes v \to uX \otimes vX - vX \otimes vX \) and \( \pi \) is induced by multiplication inside the algebra \( \mathbb{F}_q[X,Y] \). The following is an easy extension to \( GL_2 \) of Glover’s result:

**Lemma 2.4** For any \( n, m \in \mathbb{Z} \) we have in \( K_0(G) \) the identity:

\[ M_n M_m = M_{n+m} + e M_{n-1} M_{m-1}. \]  

\( \text{(Π}_{g,n,m} \) 

**Proof** Let \( \tau \) be the Brauer character of the virtual representation \( M_n M_m - M_{n+m} - e M_{n-1} M_{m-1} \). Let \( a, b \in \mathbb{F}_q^\times \) such that \( a \neq b \); denote by \( \hat{x} \) the Teichmüller lift of \( x \in \mathbb{F}_q^\times \) taken via \( \chi \). We have:

\[
\tau \left( \begin{pmatrix} a \\ a \end{pmatrix} \right) = (n + 1)(m + 1)a^{n+m} - (n + m + 1)a^{n+m} +
\]

\[
-\hat{a}^2 \cdot nma^{(n-1)+(m-1)};
\]

\[
\tau \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) = \frac{\hat{a}^{n+1} - \hat{b}^{n+1}}{\hat{a} - \hat{b}} \frac{a^{m+1} - b^{m+1}}{a - b} - \frac{\hat{a}^{n+m+1} - \hat{b}^{n+m+1}}{\hat{a} - \hat{b}} +
\]

\[
-\hat{a}\hat{b}(\hat{a}^{n} - \hat{b}^{n})(\hat{a}^{m} - \hat{b}^{m})/(\hat{a} - \hat{b})^2.
\]
Both these expressions are trivially zero. If $c \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^\times$ and $\iota : \mathbb{F}_q^2 \to M_2(\mathbb{F}_q)$ is as above, then $\det \iota(c) = c^{1+q}$, so that:

$$\tau(\iota(c)) = \frac{\tilde{c}^{q(n+1)} - \tilde{c}^{q(n+1)}}{\tilde{c}^q - \tilde{c}} \frac{\tilde{c}^{m+1} - \tilde{c}^{m+1}}{\tilde{c}^q - \tilde{c}} + \frac{\tilde{c}^{q(n+m+1)} - \tilde{c}^{q(n+m+1)}}{\tilde{c}^q - \tilde{c}} \frac{\tilde{c}^{1+q}(\tilde{c}^{m+1} - \tilde{c}^{m+1})}{(\tilde{c}^q - \tilde{c})^2},$$

and this is also zero. As $\tau$ is identically zero on $G^{reg}$, $M_nM_m - M_{n+m} - eM_{n-1}M_{m-1}$ is the zero element of $K_0(G)$. $\blacksquare$

We summarize the three identities obtained so far:

**Proposition 2.5** Let $q = p^g$ ($g \geq 1$) and let $k, n, m \in \mathbb{Z}$. The following identities hold in $K_0(G)$:

$$M_k = -e^{1+k} \cdot M_{-k-2} \quad \quad (\Delta_{g,k})$$

$$M_k - e \cdot M_{k-(q+1)} = M_{k-(q-1)} - e \cdot M_{k-2q} \quad \quad (\Sigma_{g,k})$$

$$M_nM_m = M_{n+m} + eM_{n-1}M_{m-1} \quad \quad (\Pi_{g,k})$$

### 2.2 Intertwining operators for the periods $q - 1$ and $q + 1$

Recall that the irreducible complex representations of $G$ (of dimension larger than one) that are not twists of the Steinberg representation are of two types: the principal series representations, having dimension $q+1$ and obtained by inducing to $G$ characters of the Borel subgroup of $G$, and the cuspidal representations, having dimension $q-1$ and characterized by the property that they do not occur as a factor of a principal series.

The two periods $q + 1$ and $q - 1$ appear in the identity $(\Sigma_{g,k})$ and suggest the existence of intertwining operators that shift weights by $q + 1$ and $q - 1$ respectively; furthermore one expects these operators to give a bridge between the modular representations of $G$ and the above mentioned characteristic zero representations of $G$. We recall below the known results on this matter, as the operators we introduce here will be the starting point of the generalizations considered in the following sections (cf. [33]).

**The period** $q + 1$ Let $k > q$ be an integer and let $\Theta_q = XY^q - X^qY \in \mathbb{F}_q[X,Y]$. (Dickson proved that this polynomial is one of the two generators of the ring of $SL_2(\mathbb{F}_q)$-invariants in the symmetric algebra $\text{Sym}^* \mathbb{F}_q^2$, so we will call it the Dickson invariant). Let us denote by $\Theta_q$ also the $G$-equivariant map $\det \otimes M_{k-(q+1)} \to M_k$ given by multiplication by $\Theta_q$. 

11
Proposition 2.6 For \( k > q \), there is an exact sequence of \( G \)-modules:

\[
0 \to \det \otimes M_{k-(q+1)} \xrightarrow{\Theta_q} M_k \to \text{Ind}_B^G (\eta^k) \to 0,
\]

where \( B \) is the subgroup of \( G \) consisting of upper triangular matrices, and \( \eta \) is the character of \( B \) defined extending the character \( \text{diag}(a,b) \mapsto a \) of the standard maximal torus of \( G \). Furthermore, for any integer \( \lambda \geq 0 \) there are isomorphisms of \( G \)-modules:

\[
\frac{M_k}{\det \otimes M_{k-(q+1)}} \cong \frac{M_{k+\lambda(q-1)}}{\det \otimes M_{k+\lambda(q-1)-(q+1)}},
\]

where the inclusion \( \det \otimes M_{k+\lambda(q-1)-(q+1)} \hookrightarrow M_{k+\lambda(q-1)} \) is induced by the multiplication by \( \Theta_q \).

**Proof** The above result is standard; cf. [16], Proposition 2.7. ■

The period \( q - 1 \) The period \( q - 1 \) is studied in [16]: here we just recall the main result proved there. The starting point is the \( G \)-equivariant derivation \( D : \mathbb{F}_q[X,Y] \to \mathbb{F}_q[X,Y] \) defined by Serre as:

\[
D : \ f(X,Y) \mapsto X^q \frac{\partial f}{\partial X}(X,Y) + Y^q \frac{\partial f}{\partial Y}(X,Y).
\]

This map defines by restriction an intertwining operator \( M_k \to M_{k+(q-1)} \) for any \( k \geq 0 \), giving rise to a weight shifting by \( q - 1 \). The kernel of \( D \) is often non trivial ([16], Proposition 3.3), and \( D \) captures essential properties related to the existence or non-existence of embeddings of \( G \)-modules of the form \( M_k \to M_{k+(q-1)} \) ([16], Proposition 3.5 and Proposition 3.6).

We now assume, for the rest of this paragraph, that \( p \) is an odd prime. If we restricted ourselves to weights \( 2 \leq k \leq p - 1 \) we have the following exact sequence:

\[
0 \to \det \otimes M_{k-2} \xrightarrow{\overline{\Theta}_q} \frac{M_{k+(q-1)}}{D(M_k)} \to \ker \overline{\Theta}_q \to 0,
\]

where \( \overline{\Theta}_q = \Theta_q (\text{mod} D(M_k)) \) is induced by the Dickson invariant.

**Theorem 2.7** Let \( q \neq 2, 2 \leq k \leq p - 1 \) with \( k \neq \frac{q+1}{2} \) and let us denote by \( \Xi(\chi^k) \) the cuspidal \( \mathbb{F}_q \)-representation of \( G \) associated to the \( k \)th-power of the Teichmüller character \( \chi \). Let \( C \) be the Deligne-Lusztig variety of \( \text{SL}_2/\mathbb{F}_q \). There exists a canonical \( W(\mathbb{F}_q) \)-integral model

\[
\hat{\Xi}(\chi^k) := H^1_{\text{cris}}(C/\mathbb{F}_q)_{-k}
\]

of \( \Xi(\chi^k) \), arising from the \((-k)\)-eigenspace of the first crystalline cohomology group of \( C/\mathbb{F}_q \), such that there is an isomorphism of \( \mathbb{F}_q[G] \)-modules:

\[
\frac{M_{k+(q-1)}}{D(M_k)} \cong \hat{\Xi}(\chi^k) \otimes_{W(\mathbb{F}_q)} \mathbb{F}_q.
\]

(Here the \((-k)\)-eigenspace of \( H^1_{\text{cris}}(C/\mathbb{F}_q) \) is computed with respect to the natural action of \( \ker(Nm_{\mathbb{F}_q^{1/q}}/\mathbb{F}_q) \) on \( H^1_{\text{cris}}(C/\mathbb{F}_q) \)).

**Proof** [16], Theorem 4.2. ■
2.3 Determination of Jordan-Hölder constituents: the case $g = 1$

Assume $g$ is any positive integer. For convenience, we give the following non standard definition:

**Definition 2.8** Let $M \in K_0(G)$ be of the form $M = e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$ where $m,k_0,\ldots,k_{g-1} \in \mathbb{Z}$. We say that the Jordan-Hölder factors of $M$ can be computed in the standard form (using $(\Delta), (\Sigma)$ and $(\Pi)$) if, by applying finitely many times the identities of Proposition 2.5 together with the identities $\varepsilon^q-1 = 1$ and $\sigma^q = 1$, we can write $M$ as:

$$M = \sum_{j \in J} n_j \left( e^{m_j} \prod_{i=0}^{g-1} M_{k_i}^{[j]} \right),$$

where $J$ is a finite set and for any $j \in J$ we have $n_j, m_j, k_0^{[j]}, \ldots, k_{g-1}^{[j]} \in \mathbb{Z}$ such that $n_j \neq 0$, $0 \leq m_j < q-1$, $0 \leq k_0^{[j]}, \ldots, k_{g-1}^{[j]} \leq p-1$ and, if $j \neq j'$ then $(m_j, k_0^{[j]}, \ldots, k_{g-1}^{[j]}) \neq (m_{j'}, k_0^{[j']}, \ldots, k_{g-1}^{[j']})$. (Notice that the integers $n_j, m_j, k_0^{[j]}, \ldots, k_{g-1}^{[j]}$ are uniquely determined by $M$).

Similarly one defines the notion of computability in standard form for an element of $K_0(G)$ that is given as an algebraic sum of products of elements of the form $M = e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$. Also, in an obvious way, one defines computability in standard form using any subset or superset of the identities $(\Delta), (\Sigma)$ and $(\Pi)$ (together with the identities $\varepsilon^q-1 = 1$ and $\sigma^q = 1$).

**Lemma 2.9** Let $g$ be any positive integer and let $n, m \in \mathbb{Z}$ such that $n, m \geq 0$. By applying $(\Pi_g)$ we obtain the following identity in $K_0(G)$:

$$M_n M_m = \sum_{i=0}^{\min\{n,m\}} e^i M_{n+m-2i}.$$

**Proof** We induct on $n$. For $n = 0$ the statement is true; for $n \geq 0$ we have, assuming $m > 0$:

$$M_{n+1} M_m = M_{n+m+1} + e M_n M_{m-1} = M_{n+m+1} + \sum_{i=0}^{\min\{n,m-1\}} e^{i+1} M_{n+m-2i} = M_{n+m+1} + \sum_{i=1}^{\min\{n+1,m\}} e^i M_{n+1+m-2i} = \sum_{i=0}^{\min\{n+1,m\}} e^i M_{n+1+m-2i}. \blacksquare$$

**Corollary 2.10** For any positive integer $t$ and any integers $n_1, \ldots, n_t \geq 0$ we have:

$$\prod_{i=1}^{t} M_{n_i} = \sum_{\alpha \in A} e^{s_{\alpha}} M_{r_{\alpha}},$$

where $A$ is a finite set and $s_{\alpha}, r_{\alpha} \geq 0$ for any $\alpha \in A$. 

13
Proof It follows from applying (Π₀) and inducting on t. ■

The following proposition guarantees that, if $g = 1$, $(Δ₁)$ and $(Σ₁)$ are enough to compute explicitly the Jordan-Hölder factors of any of the modules $M_k$ for $k \in \mathbb{Z}$.

**Proposition 2.11** Let $g = 1$. For any $m, k \in \mathbb{Z}$, we can compute the Jordan-Hölder factors of $e^mM_k$ in the standard form, using $(Δ₁)$ and $(Σ₁)$. Furthermore, by using also $(Π₁)$, we can compute the Jordan-Hölder factors in the standard form for any algebraic sum of products of $e^mM_k$’s.

**Proof** The second assertion in the statement of the proposition follows from the first one, together with Lemma 2.20. To prove the first assertion, we can assume $m = 0$ and, using $(Δ₁)$, we also suppose $k \geq 0$. Write $k = np + r$ where $n$ is a non-negative integer and $r$ is an integer such that $0 \leq r \leq p - 1$. We induct on $n$.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ is fixed and that we can compute the Jordan-Hölder factors of $M_k$ in the standard form, using $(Δ₁), (Σ₁)$ and $(Π₁)$, for any $k$ of the form $k = np + r'$ where $0 \leq r' \leq n - 1$ and $0 \leq r' \leq p - 1$. If $0 \leq r \leq p - 1$ we have, applying $(Σ₁)$, that $M_{np+r} = M_{(n-1)p+r+1} + e(M_{(n-1)p+r-1} - M_{(n-2)p+r})$. If $r \neq 0, p - 1$ we are done by induction assumption. If $r = 0$, then $M_{np} = M_{(n-1)p+1} + e(M_{(n-2)p+p+1} - M_{(n-2)p})$ and we are done. If $r = p - 1$, just notice that $M_{(n-1)p+p+1} = M_{np} = M_{(n-1)p+1} + e(M_{(n-2)p+p+1} - M_{(n-2)p+1})$. (When $n = 1$ one sometimes needs to apply $(Δ₁)$ to canonically compute the constituents of the virtual representations appearing in these identities). ■

### 2.4 Application to elliptic modular forms

In this section we summarize some weight shifting results for elliptic modular forms modulo $p$ in terms of cohomology of groups; the main references are [3], [7], [16]. We assume $p > 3$; by a modular form mod $p$ we mean the reduction modulo $p$ of a form in characteristic zero - as defined by Serre and Swinnerton-Dyer, unless otherwise specified.

Let $N \geq 5$ be a positive integer not divisible by $p$ and denote by $M_k(N, \overline{F}_p)$ the $\overline{F}_p$-vector space of mod $p$ modular forms for the group $Γ_1(N)$ having weight $k \geq 2$ and with coefficients in $\overline{F}_p$; the Hecke algebra $H_N$, generated over $\overline{F}_p$ by the operators $T_l$ for $p \nmid l$, acts on this space. The $q$-expansion homomorphism is an injective map $M_k(N, \overline{F}_p) \hookrightarrow \overline{F}_p[[q]]$.

The theta operator $Θ : M_k(N, \overline{F}_p) \to M_{k+(p+1)}(N, \overline{F}_p)$ is defined on $q$-expansion by the formula $Θ(\sum_n a_n q^n) = \sum_n n a_n q^n$; it satisfies $ΘT_l = (T_l) Θ$ for any prime $l \neq p$ ($T_l \in H_N$). Denote by $E_{p-1}$ the normalized form of the classical characteristic zero Eisenstein series whose $q$-expansion is given by:

$$E_{p-1}(q) = 1 - 2(p-1)/B_{p-1} \sum_n \sigma_{p-2}(n)q^n;$$

then $E_{p-1} \in M_{p-1}(1, \mathbb{Z}(p))$ and $E_{p-1} \equiv 1 \pmod{p\mathbb{Z}(p)[[q]]}$, as $2ζ(2-p)^{-1} \equiv 0 \pmod{p}$ by the Clausen-von Staudt theorem. Multiplication by the reduction
mod $p$ of $E_{p-1}$ gives rise to a Hecke-equivariant map $M_k(N, \mathbb{F}_p) \to M_{k+(p-1)}(N, \mathbb{F}_p)$, that we refer to as the Hasse invariant.

In view of the Eichler-Shimura isomorphism, the study of Hecke eigensystems of mod $p$ modular forms of weight $k \geq 2$ and level $N$ leads to the study of the eigenvalues of the Hecke algebra $\mathcal{H}_N$ acting on the cohomology group $H^1(\Gamma_1(N), M_{k-2})$, where $\Gamma_1(N)$ acts on $M_{k-2}$ via its reduction mod $p$, and the action of $\mathcal{H}_N$ comes from the $G$-action on $M_{k-2}$ and it is defined as in [3]. The weight shiftings realized on the spaces of modular forms by the theta operator and the Hasse invariant have cohomological counterparts. In [3], Ash and Stevens identifies a group-theoretical analogue of the $\Theta$-operator in the Hecke-equivariant map induced in cohomology by the Dickson invariant (cf. [22]):

$$\Theta_{p,*} : H^1(\Gamma_1(N), \det \otimes M_{k-2}) \to H^1(\Gamma_1(N), M_{k+p-1}).$$

Here the twisting by det on the left hand side is a manifestation of the fact that the $\Theta$ operator on spaces of modular forms is twist-Hecke-equivariant.

Edixhoven and Khare identifies in [7] a cohomological analogue of the Hasse invariant in the case $k = 2$ by studying the degeneracy map $H^1(\Gamma_1(N), M_0)^{\otimes 2} \to H^1(\Gamma_1(N) \cap \Gamma_0(p), M_{p-1})$. In [10], the $D$-derivation defined in [22] is used to allow weight shifting by $p - 1$ for $3 \leq k \leq p + 1$:

**Theorem 2.12** Let $\mathfrak{M}$ be a non-Eisenstein maximal ideal of the Hecke algebra $\mathcal{H}_N$. If $k \geq 0$ and $H^1(\Gamma_1(N), M_k)_{\mathfrak{M}} \neq 0$, then also $H^1(\Gamma_1(N), M_{k+(p-1)})_{\mathfrak{M}} \neq 0$. If $0 \leq k \leq p - 1$, there is a Hecke-equivariant embedding $H^1(\Gamma_1(N), M_k)_{\mathfrak{M}} \to H^1(\Gamma_1(N), M_{k+p-1})_{\mathfrak{M}}$ that is induced by the derivation $D$ if $0 < k \leq p - 1$, and is the map defined in [7] if $k = 0$.

**Proof** The first statement and the second statement for $k \neq 0$ are proved in [16], Proposition 5.1; the second statement in the case $k = 0$ is treated in [7], 2. ■

Notice that the above theorem cannot be deduced only by the existence of the map $D$, as for $k = 0$ the virtual representation $M_{p-1} - M_0$ is not positive in $K_0(G)$. A similar situation will occur later on when we will consider the more general case of Hilbert modular forms (cf. section 5).
3 Twisted $GL_2(\mathbb{F}_q)$-modules and intertwining operators for $g > 1$

We keep the notation of the previous section, so that $p$ is a prime number, $g$ a positive integer, and $q = p^g$; we denote by $\mathbb{F}_q$ a finite field with $q$ elements and we fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$; we let $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ be the arithmetic Frobenius element and $G = GL_2(\mathbb{F}_q)$. If $k \in \mathbb{Z}$, $M_k^{[1]}$ is its $i$th Frobenius twist of the virtual representation $M_k$, for any integer $i$.

3.1 Identities in $K_0(G)$ (II)

None of the identities in $K_0(G)$ appearing in Proposition 2.5 contains a Frobenius twist; this implies that, while $(\Delta_g), (\Sigma_g), (\Pi_g)$ are all we need to compute the Jordan-Hölder factors of products of virtual representations of the form $M_k (k \in \mathbb{Z})$ when $g = 1$ (Proposition 2.11), these same three families of identities are not enough to work out such a computation when $g > 1$. For example, when $g > 1$, the Jordan-Hölder factors of $M_p$ are $\{M_1^{[1]}, eM_{p-2}\}$ and they cannot be found using $(\Sigma_g)$.

**Proposition 3.1** Let $g \geq 1$. For any $k \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k = M_{k-p}M_1^{[1]} - e^pM_{k-2p}. \quad (\Phi_{g,k})$$

**Proof** Fix an embedding $\iota : \mathbb{F}_q^x \rightarrow M_2(\mathbb{F}_q)$ and denote by $\tilde{x} \in \mathbb{Z}_p$ the Teichmüller lift of $x \in \overline{\mathbb{F}}_q^x$, taken via the Teichmüller character we previously fixed. Let $\tau$ be the Brauer character of the virtual representation $M_k - M_{k-p}M_1^{[1]} + e^pM_{k-2p}$. Let $a, b \in \mathbb{F}_q^x$ such that $a \neq b$. We have:

$$\tau\left(\begin{array}{c} a \\ a \end{array}\right) = (k+1)\tilde{a}^k - (k-p+1)\tilde{a}^{k-p} \cdot 2\tilde{a}^p + \tilde{a}^{2p} \cdot (k-2p+1)\tilde{a}^{k-2p};$$

$$\tau\left(\begin{array}{c} a \\ b \end{array}\right) = \frac{\tilde{a}^{k+1} - \tilde{b}^{k+1}}{\tilde{a} - \tilde{b}} - \frac{\tilde{a}^{k-p+1} - \tilde{b}^{k-p+1}}{\tilde{a} - \tilde{b}} \frac{\tilde{a}^{2p} - \tilde{b}^{2p}}{\tilde{a} - \tilde{b}} + \frac{\tilde{a}^{2p}\tilde{a}^{k-2p+1} - \tilde{b}^{k-2p+1}}{\tilde{a} - \tilde{b}}.$$

Both these expressions are zero. If $c \in \mathbb{F}_q^x \setminus \mathbb{F}_q^x$ then $\det \iota (c) = c^{1+q}$; also notice that $\text{tr}(\iota (c) ; M_1^{[1]}) = \text{tr}(\iota (c)^q ; M_1) = \text{tr} \iota (c)^q = e^q + e^{pq}$, so that:

$$\tau(\iota (c)) = \frac{\tilde{c}^{(k+1)} - \tilde{c}^{k+1}}{\tilde{c} - \tilde{c}} - \frac{\tilde{c}^{(k-p+1)} - \tilde{c}^{k-p+1}}{\tilde{c} - \tilde{c}} (e^q + e^{pq}) + \frac{\tilde{c}^{(1+q)p} \tilde{c}^{(k-2p+1)} - \tilde{c}^{k-2p+1}}{\tilde{c} - \tilde{c}}.$$
and this is also zero. As $\tau$ is identically zero on $G^{reg}$, $M_k - M_{k-p}M_{k+1}^1 + e^p M_{k-2p}$ is the zero element of $K_0(G)$.

Corollary 3.2 Let $g \geq 1$; for any $k, h \in \mathbb{Z}$ the following identity holds in $K_0(G)$:

$$M_k M_h^1 - e^p M_{k-p} M_{h+1}^1 = M_{k-p} M_{h+1}^1 - e^p M_{k-2p} M_{h}^1.$$  \tag{$\Phi'_{g,k,h}$}

**Proof** Multiplying $(\Phi_{g,k})$ by $M_h^1$ we obtain the identity

$$M_k M_h^1 = M_{k-p} (M_1 M_h)^1 - e^p M_{k-2p} M_h^1.$$  

Applying $(\Pi_{g,1,h})$ and distributing the Frobenius action, we deduce that the left hand side of this equation equals $M_{k-p} \left( M_{h+1}^1 + e^p M_{h-1}^1 \right) - e^p M_{k-2p} M_h^1$. ■

Corollary 3.3 Let $g \geq 1$. For any $k, h, i \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k^{[i]} M_h^{[i+1]} - e^{p^{i+1}} M_{k-p}^{[i]} M_{h-1}^{[i+1]} = M_{k-p}^{[i]} M_{h+1}^{[i+1]} - e^{p^{i+1}} M_{k-2p}^{[i]} M_h^{[i+1]}.$$  

**Proof** Just apply $i$th Frobenius twist to $(\Phi'_{g,k,h})$. ■

Remark 3.4 1. By applying the product formula, one sees that $(\Phi_1)$ and $(\Sigma_1)$ are equivalent.

2. Equation $(\Phi'_g)$ ($g > 1$) has a structure similar to equation $(\Sigma_1)$: the weight shiftings appearing in the latter are by $p + 1$ and $p - 1$ (corresponding respectively to the degree of the Dickson invariant and of Serre’s derivation map); in equation $(\Phi'_g)$, the weight shiftings occurring are by $(p, 1, 0, \ldots, 0)$ and $(p, -1, 0, \ldots, 0)$ - the commas separate the shifting constants for tensor factors corresponding to Frobenius twistings by $\sigma^0, \sigma^1, \ldots, \sigma^{g-1}$. In this sense we can think of $(\Phi'_g)$ as a generalization of $(\Sigma_1)$ for $g > 1$.

3. The reason for which only three (possibly) non-zero terms appear in $(\Phi_g)$ instead of four - as one could have expected by looking at $(\Sigma_1)$, is that by applying weight-shifting of $(p, 1, 0, \ldots, 0)$ to $M_k$ we obtain $e^p M_k - M_{k-1}$ that is the zero module: this phenomenon cannot happen when $g = 1$.

4. The reason for which, when $g > 1$, we were expecting an identity in $K_0(G)$ involving weight shiftings by $(p, \pm 1, 0, \ldots, 0)$ (and cyclic permutations of this) resides in the existence of the partial Hasse invariants and theta operators acting on spaces of mod $p$ Hilbert modular forms of genus $g$. Also, for good reasons we do not have weight shiftings by $(\pm 1, p, 0, \ldots, 0)$, as long as $g > 2$: cf. 4.7.
3.2 Determination of Jordan-Hölder constituents: the case $g > 1$

We know show that equations $(\Delta_g), (\Phi_g), (\Pi_g)$ are enough to compute the Jordan-Hölder constituents of products of virtual representations of the form $e^n \prod_{i=0}^{g-1} M_{k_i}^{[i]}(m, k_0, \ldots, k_g \in \mathbb{Z})$.

**Lemma 3.5** Let $g \geq 1$; for any $k \in \mathbb{Z}$, we can compute the Jordan-Hölder factors of $M_k$ in the standard form, using $(\Delta_g), (\Phi_g), (\Pi_g)$.

**Proof** By applying $(\Delta_g)$ if necessary we can assume $k \geq 0$. If $g = 1$, the lemma follows from the last remark and Proposition 2.5. For $g \geq 2$, write $k = np + r$ where $n, r \in \mathbb{Z}$ are such that $n \geq 0$ and $0 \leq r \leq p − 1$. We induct on $n$.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ is fixed and that we can compute the Jordan-Hölder factors of $M_k$ in the standard form, using $(\Delta_g), (\Phi_g)$ and $(\Pi_g)$, for any $k$ of the form $k = n'p + r'$ where $0 \leq n' \leq n − 1$ and $0 \leq r' \leq p − 1$. We have $M_{n'p + r'} = M_{(n-1)p+r}M^{[1]}_1 - e^pM_{(n-2)p+r}$ by $(\Phi_g)$; the Jordan-Hölder factors of $e^pM_{(n-2)p+r}$ can be computed in the standard form by induction (if $n = 1$ then $M_{(n-2)p+r} = -e^{1-p+r}M_{p-r-2}$ by $(\Delta_g)$).

Also, by induction we have an algorithm that allows us to write $M_{(n-1)p+r} = \sum_{i \in I} J_i$, where $I$ is a finite set and each $J_i$ is of the form $e^n \prod_{i=1}^{g-1} M_{k_i}^{[i]}$ for some integers $m, k_0, \ldots, k_{g-1}$ such that $0 \leq m < q − 1$, $0 \leq k_0, \ldots, k_{g-1} \leq p − 1$. It is therefore enough to show that we can compute the factors of $\prod_{i=0}^{g-1} M_{k_i}^{[i]} M_1^{[1]}$ in standard form, where $0 \leq m < q − 1$, $0 \leq k_0, \ldots, k_{g-1} \leq p − 1$. The product formula gives:

$$\left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) M_1^{[1]} = \left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) M_{k_1+1}^{[1]} + e^p \left(\prod_{i=1}^{g-1} M_{k_i}^{[i]}\right) M_{k_1}^{[1]}.$$

If $k_1 \neq p - 1$, each of the two summands is either a Jordan-Hölder factor in standard form, or it is zero. Otherwise we are left with the determination of the constituents of the first summand. If $g = 2$ the latter equals $M_{k_0}M_{p}^{[1]} = M_{k_0}M_{p}^{[1]} + e^pM_{k_0}M_{p-2}^{[1]} = M_{k_0+1} + eM_{k_0-1} + e^pM_{k_0}M_{p-2}$ and this is not in standard form if and only if $k_0 = p - 1$, in which case we can compute the constituents of $M_{k_0+1} = M_{p}$ in standard form by using $(\Phi_g)$: $M_{p} = M_{1}^{[1]} + eM_{p-2}$.

Assume now $g > 2$ and $k_1 = p - 1$. We have, applying $(\Phi_g)$:

$$\left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) M_{p}^{[1]} = \left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) (M_{k_2}M_1)^{[2]} + e^p \left(\prod_{i=1}^{g-1} M_{k_i}^{[i]}\right) M_{p-2}^{[1]}.$$  

The second summand is already in standard form; for the first summand we have:

$$\left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) (M_{k_2}M_1)^{[2]} = \left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) M_{k_2+1}^{[2]} + e^{p^2} \left(\prod_{i=0}^{g-1} M_{k_i}^{[i]}\right) M_{k_2}^{[2]}.$$
If \( k_2 \neq p - 1 \), each of the two summands is either a Jordan-Hölder factor in standard form, or it is zero. Otherwise we are left with the determination of the constituents of the first summand. We proceed as before, distinguishing the cases \( g = 3 \) and \( g > 3 \). It is easily seen by induction that the algorithm produces the Jordan-Hölder factors of the virtual representations appearing in each step as long as \( k_i \neq p - 1 \) for some \( 1 \leq i \leq g - 1 \). If \( k_1 = \ldots = k_{g-1} = p - 1 \), we are left with the determination the Jordan-Hölder factors of \( M_{k_0}M_p^{[g-1]} = M_{k_0}(M_1 + e^{p-1}M_{p-2}) \). By the product formula, we just need to find the constituents of \( M_{k_0+1} \): if \( k_0 \neq p - 1 \) this is an irreducible representation; otherwise \( M_p = M_1^{[1]} + eM_{p-2} \) and we are done.

**Corollary 3.6** Let \( g \geq 1 \). Then \((\Delta_g), (\Phi_g), (\Pi_g)\) imply \((\Sigma_g)\).

**Proof** By the previous lemma, we can compute the Jordan-Hölder factors of each summand appearing in \((\Sigma_g)\) (in standard form). Since we know a priori that the Jordan-Hölder factors appearing in the right and left hand sides of \((\Sigma_g)\) have to appear with the same multiplicities, \((\Sigma_g)\) is a consequence of \((\Delta_g), (\Phi_g), (\Pi_g)\).

Notice that we were able to show that \((\Delta_g), (\Phi_g), (\Pi_g)\) imply \((\Sigma_g)\) because we knew already that \((\Sigma_g)\) was true. It does not seem to be an easy task to directly deduce Serre’s relation from the set \((\Delta_g), (\Phi_g), (\Pi_g)\). Serre’s relation will allow sometimes to bypass long computations involving Frobenius twists - this will turn out to be useful in \([17]\) for \( g > 1 \), where we will give an explicit presentation of the ring \( K_0(G) \).

We can finally prove:

**Theorem 3.7** Let \( g \geq 1 \). Using \((\Delta_g), (\Phi_g), (\Pi_g)\) we can compute the Jordan-Hölder factors in the standard form for any algebraic sum of products of virtual representations of the form \( e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]} \) \((m, k_0, \ldots, k_{g-1} \in \mathbb{Z})\).

**Proof** If \( g = 1 \), this is just Proposition 2.3. Assume \( g \geq 2 \); by applications of \((\Delta_g)\) and of Lemma 2.10 it is enough to prove that we can compute the Jordan-Hölder factors in the standard form for the representation \( M = \bigotimes_{i=0}^{g-1} M_{k_i}^{[i]} \) \((k_0, \ldots, k_{g-1} \geq 0)\). We induct on \( \dim K_qM \). If \( \dim K_qM = 1 \), we are done, otherwise we distinguish two cases.

**Case 1:** There is some \( i, 0 \leq i \leq g - 1 \), such that \( M_{k_i} \) is reducible.

By applying an appropriate Frobenius twist, we can assume without loss of generality that \( M_{k_0} \) is reducible. By the previous lemma, we can compute the Jordan-Hölder factors of \( M_{k_0} \) in the standard form, say \( M_{k_0} = \sum_{h \in I} J_h \) in \( K_0(G) \), where \( I \) is a finite set with at least two elements and each \( J_h \) is a non-zero composition factor of \( M_{k_0} \), written in standard form. It is then enough to compute in standard form the constituents of \( J_h \prod_{i=1}^{g-1} M_{k_i}^{[i]} \) for each \( h \in I \). Fix
an element $h \in I$; up to twisting by a power of $e$ we can assume $J_h = \prod_{i=0}^{g-1} M[i]_i$, where $0 \leq r_0, ..., r_{g-1} \leq p - 1$, so that an application of Lemma 2.9 gives:

\[
J_h \prod_{i=1}^{g-1} M[i]_i = M_0 \prod_{i=1}^{g-1} (M_r M[k_i])^{[i]} = \sum_{j=0}^{\min(r, k_i)} e^{j p} M[r_i + k_i - 2j] = \sum_{j_i=0}^{\min(r, k_i)} e^{s(j_1, ..., j_{g-1})} M_0 M[r_i + k_i - 2j_1, ..., M[r_{g-1} + k_{g-1} - 2j_{g-1}],
\]

where $s(j_1, ..., j_{g-1}) \in \mathbb{Z}$ and the last summation is over the $g - 1$ indices $j_1, ..., j_{g-1}$. Since

\[
\dim_q \left( M_0 \otimes M[1]_{r_1 + k_1 - 2j_1} \otimes ... \otimes M^{[g-1]}_{r_{g-1} + k_{g-1} - 2j_{g-1}} \right) < \dim_q M
\]

for any value of $j_1, ..., j_{g-1}$, by induction assumption we can compute the Jordan-Hölder constituents of $J_h \prod_{i=1}^{g-1} M[i]_i$ in the standard form.

**Case 2:** For any $i$, $0 \leq i \leq g - 1$, the representation $M[k_i]$ is irreducible.

By the previous lemma, we can assume - up to twistings by powers of $e$ - that we have written $M[k_i] = \prod_{j=0}^{g-1} M[j]_{r_j}$ for any $0 \leq i \leq g - 1$, where $0 \leq r_0(i), ..., r_{g-1}(i) \leq p - 1$. Then:

\[
M = \prod_{i=0}^{g-1} M[i]_i = \prod_{j=0}^{g-1} \left( \prod_{i=0}^{g-1} M[j]_{r_i} \right)^{[j]}.
\]

Applying only the product formula (cf. Corollary 2.10), we can write:

\[
\left( \prod_{i=0}^{g-1} M[i]_{r_i} \right)^{[j]} = \sum_{\alpha_j \in A_j} e^{s_{\alpha_j}} M[r_\alpha],
\]

where, for any $0 \leq j \leq g - 1$, $A_j$ is a non-empty finite set and $s_{\alpha_j}, r_\alpha \geq 0$ for $\alpha_j \in A_j$. Combining (1) and (2) we obtain:

\[
M = \sum_{\alpha_j \in A_j} e^{s(\alpha_0, ..., \alpha_{g-1})} M[r_\alpha_0] M[1]_{r_\alpha_1} ... M^{[g-1]}_{r_\alpha_{g-1}},
\]

where $s(\alpha_0, ..., \alpha_{g-1}) \in \mathbb{Z}$ and the summation is over the $g$-tuples $(\alpha_0, ..., \alpha_{g-1}) \in A_0 \times ... \times A_{g-1}$. If each of the sets $A_0, ..., A_{g-1}$ contains exactly one element, then for any $0 \leq j \leq g - 1$, at most one element in $(r^{(0)}_j, ..., r^{(g-1)}_j)$ is positive.

Indeed, if this were not the case, there would be some $j$ such that $r^{(a)}_j, r^{(b)}_j > 0$ for some $a, b$ with $0 \leq a < b \leq g - 1$; then by only applying the product formula we would obtain:

\[
\left( \prod_{i=0}^{g-1} M[i]_{r_i} \right)^{[j]} = \left( \prod_{i=0}^{g-1} M[i]_{r_i} \right) \left( M[r^{(a)}_j + r^{(b)}_j] + e M[r^{(a)}_j - 1] M[r^{(b)}_j - 1] \right)^{[j]}.
\]

Since $r^{(a)}_j - 1, r^{(b)}_j - 1 \geq 0$, the left hand side above contains at least two non-zero summand, contradicting the fact that by only applying the product
We conclude that if each of the sets $A_0, \ldots, A_{g-1}$ contains exactly one element, then $M$ is irreducible and (1) is the standard Jordan-H"{o}lder form of $M$. If there is $0 \leq j \leq g-1$ such that $A_j$ has at least two elements, then in (3) at least two non-zero terms appear, so that each of the summand of (3) has dimension strictly less than $\dim \mathbb{F}_q M$, and by induction we are done. ■

3.3 Families of intertwining operators for $g > 1$

For $g = 1$, one has available two intertwining operators acting on $\mathbb{F}_p[G]$-modules and shifting weights by $p \pm 1$, namely the Dickson invariant $\Theta_q$ and the derivation map $D$ (cf. 2.2). For $g > 1$, equation (Φ$_g$) and the existence of partial Hasse invariants and theta operators acting on spaces of mod $p$ Hilbert modular forms (cf. [1]) suggest that there should be other intertwining operators between modular representations of $G$, generalizing $\Theta_q$ and $D$. In this section we will construct such operators.

Unless otherwise specified, we will always assume $g > 1$, and we will consider all the tensor product over $\mathbb{F}_q$ ($q = p^g$).

3.3.1 Generalized Dickson invariants

**Definition 3.8** For any integer $\beta$ such that $1 \leq \beta \leq g - 1$, the (non-twisted) **generalized $\beta$th Dickson operator** is the element

$$\Theta_\beta = X \otimes Y^{p^\beta} - Y \otimes X^{p^\beta}$$

of the $G$-module $M_1 \otimes M^{[\beta]}$.

For integers $\alpha, \beta$ such that $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$, the **$\alpha$-twisted generalized $\beta$th Dickson operator** is the element

$$\Theta^\alpha_\beta = X \otimes Y^{p^\beta} - Y \otimes X^{p^\beta}$$

of the $G$-module $M_1^{[\alpha]} \otimes M^{[\alpha + \beta]}_{p^\beta}$.

**Lemma 3.9** Let $k, h$ be two non-negative integers and let $\alpha, \beta$ be two integers such that $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$. Multiplication by $\Theta^\alpha_\beta$ in the $\mathbb{F}_q[G]$-algebra $\mathbb{F}_q[X,Y][\alpha] \otimes \mathbb{F}_q[X,Y][\alpha + \beta]$ induces an injective $G$-homomorphism:

$$\Theta^\alpha_\beta : \det^{[\alpha]} \otimes M^{[\alpha]}_k \otimes M^{[\alpha + \beta]}_{h+1} \to M^{[\alpha]}_{k+1} \otimes M^{[\alpha + \beta]}_{h+p^\beta}.$$

**Proof** We can assume $\alpha = 0$. To prove $G$-equivariance of the map $\Theta_\beta$, it is enough to show that $\gamma \Theta_\beta = \det \gamma \cdot \Theta_\beta$ for all $\gamma \in G$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$; then
\( \gamma \Theta_\beta \) equals:
\[
(aX + cY) \otimes (b^p X + d^p Y)^{p^{\gamma - \beta}} - (bX + dY) \otimes (a^p X + c^p Y)^{p^{\gamma - \beta}}
\]
\[
= (aX + cY) \otimes (b^q X^{p^{\gamma - \beta}} + d^q Y^{p^{\gamma - \beta}}) - (bX + dY) \otimes (a^q X^{p^{\gamma - \beta}} + c^q Y^{p^{\gamma - \beta}})
\]
\[
= (aX + cY) \otimes (b X^{p^{\gamma - \beta}} + d Y^{p^{\gamma - \beta}}) - (bX + dY) \otimes (a X^{p^{\gamma - \beta}} + c Y^{p^{\gamma - \beta}})
\]
\[
= adX \otimes Y^{p^{\gamma - \beta}} + bcX \otimes X^{p^{\gamma - \beta}} - bcX \otimes Y^{p^{\gamma - \beta}} - adY \otimes X^{p^{\gamma - \beta}}
\]
\[
= \det \gamma \cdot \left( X \otimes Y^{p^{\gamma - \beta}} - Y \otimes X^{p^{\gamma - \beta}} \right)
\]
\[
= \det \gamma \cdot \Theta_\beta.
\]

To show injectivity of \( \Theta_\beta \), notice that there is an isomorphism of \( \mathbb{F}_q[G] \)-algebras \( \mathbb{F}_q[X, Y] \otimes \mathbb{F}_q[X, Y][\beta] \cong \mathbb{F}_q[Z, W, T^p, U^{p^\beta}] \) obtained by sending the ordered tuple \((X \otimes 1, Y \otimes 1, 1 \otimes X, 1 \otimes Y)\) into the ordered tuple \((Z, W, T^p, U^{p^\beta})\), were we are letting \( G \) acts on \( Z, W, T, U \) as follows: for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \)
\[
\gamma Z = aZ + cW,
\]
\[
\gamma W = bZ + dW,
\]
\[
\gamma T = aT + cU,
\]
\[
\gamma U = bT + dU.
\]
Under the above identification, the map \( \Theta_\beta \) corresponds to multiplication by \( ZU^q - WT^q \) on \( \mathbb{F}_q[Z, W, T^p, U^{p^\beta}] \), and it is therefore injective. ■

In addition to the above operators, the classical Dickson invariant also gives rise to an intertwining map:

**Proposition 3.10** Let \( k \) be a non-negative integer and let \( \alpha \) be an integer such that \( 0 \leq \alpha \leq g - 1 \). Let \( \Theta^{[\alpha]} = XY^q - YX^q \) be the classical Dickson invariant, viewed as an element of \( M_{q+1}^{[\alpha]} \). Multiplication by \( \Theta^{[\alpha]} \) in the \( \mathbb{F}_q[G] \)-algebra \( \mathbb{F}_q[X, Y]^{[\alpha]} \) induces an injective \( G \)-homomorphism:
\[
\Theta^{[\alpha]} : \det^{p^\alpha} \otimes M_k^{[\alpha]} \rightarrow M_k^{[\alpha + (q+1)\beta]}.
\]

**Proof** This follows from section 2.2. ■

Notice that the operators \( \Theta_\beta^{[\alpha]} \) and \( \Theta^{[\alpha]} \) \((0 \leq \alpha \leq g - 1, 1 \leq \beta \leq g - 1)\) pairwise commute, as it follows by seeing them as multiplication by polynomials in some polynomial algebra over \( \mathbb{F}_q \) (cf. the end of the proof of Lemma 3.5).

**Remark 3.11** Let us fix a convention that will make the notation easier in the sequel. For non-negative integers \( k_0, \ldots, k_{g-1} \), the \( G \)-module \( M_{k_0} \otimes \cdots \otimes M_{k_{g-1}} \) will be identified with the \( G \)-module obtained by permuting in any possible way the tensor factors. Also, for integers \( \alpha, \beta \) and any \( G \)-module \( M \), the notation \( M^{[\alpha + \beta]} \) will denote the \( \gamma \)th Frobenius twist of \( M \), where \( \gamma \) is the smallest non-negative integer such that \( \gamma \equiv \alpha + \beta \mod g \).
We can summarize the above results as follows:

**Theorem 3.12** Let us fix non-negative integers $k_0, ..., k_{g-1}$. For any integers $\alpha, \beta$ subject to the constraints $0 \leq \alpha \leq g-1$ and $1 \leq \beta \leq g-1$, there are pairwise commuting injective $G$-intertwining operators as follows:

$\Theta^{[\alpha]} : \det^g \otimes \bigotimes_i M_{k_i}^{[i]} \rightarrow \left( \bigotimes_{i \neq \alpha, \alpha + \beta} M_{k_i}^{[i]} \right) \otimes M_{k_{\alpha+1}}^{[\alpha]} \otimes M_{k_{\alpha+\beta}+p^{g-\beta}}^{[\alpha+\beta]}$;

$\Theta^{[\alpha]} : \det^g \otimes \bigotimes_i M_{k_i}^{[i]} \rightarrow \left( \bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes M_{k_{\alpha}+(q+1)}^{[\alpha]}$;

where the tensor products indices run over the integers $i$ such that $0 \leq i \leq g-1$, unless otherwise specified.

**Remark 3.13** The operators $\Theta_{g-1}^{[\alpha]}$ for $0 \leq \alpha \leq g-1$ give, under suitable assumptions, cohomological analogues of the theta operators defined in [1] in the context of Hilbert modular forms. We do not know of any geometric interpretation of the other generalized Dickson operators.

We can picture the weight shiftings allowed by the $g(g-1) + g = g^2$ generalized Dickson operators with the following self-explanatory tables:

| $\Theta_{g-1}$ | $1, p^{g-1}, 0, 0, ..., 0, 0$ |
|----------------|----------------------------|
| $\Theta_{g-1}^{[1]}$ | $0, 1, p^{g-1}, 0, ..., 0, 0$ |
| $\Theta_{g-1}^{[2]}$ | $0, 0, 0, p^{g-1}, ..., 0, 0$ |
| $\Theta_{g-1}^{[g-2]}$ | $0, 0, 0, 0, ..., 1, p^{g-1}$ |
| $\Theta_{g-1}^{[g-1]}$ | $0, p^{g-1}, 0, 0, ..., 0, 1$ |
| $\Theta_2$ | $1, 0, p^{g-2}, 0, ..., 0, 0$ |
| $\Theta_2^{[1]}$ | $0, 1, 0, p^{g-2}, ..., 0, 0$ |
| $\Theta_2^{[2]}$ | $0, 0, 1, 0, ..., 0, 0$ |
| $\Theta_2^{[g-2]}$ | $0, 0, 0, 0, ..., 1, 0$ |
| $\Theta_2^{[g-1]}$ | $0, p^{g-2}, 0, 0, ..., 0, 1$ |

| $\Theta_{g-1}$ | $1, 0, 0, ..., 0, p$ |
|----------------|-------------------|
| $\Theta_{g-1}^{[1]}$ | $p, 1, 0, ..., 0, 0$ |
| $\Theta_{g-1}^{[2]}$ | $0, p, 1, 0, ..., 0, 0$ |
| $\Theta_{g-1}^{[g-2]}$ | $0, 0, 0, 0, ..., 1, 0$ |
| $\Theta_{g-1}^{[g-1]}$ | $0, 0, 0, 0, ..., p, 1$ |
| $\Theta$ | $g+1, 0, 0, ..., 0, 0$ |
| $\Theta^{[1]}$ | $0, q+1, 0, ..., 0, 0$ |
| $\Theta^{[2]}$ | $0, 0, q+1, ..., 0, 0$ |
| $\Theta^{[g-2]}$ | $0, 0, 0, ..., 1, q+1$ |
| $\Theta^{[g-1]}$ | $0, 0, 0, ..., q+1$ |

For example, if $g = 2$ the generalized Dickson operators give all and only the weight shiftings of the form:

$$a_1(1, p) + a_2(p, 1) + a_3(0, p^2 + 1) + a_4(p^2 + 1, 0),$$
for any non-negative integers \(a_1, a_2, a_3, a_4\). For \(g > 2\) a new phenomenon occurs, as the operators do not allow weight shiftings of the form:

\[
(1, p, 0, ..., 0, 0), (0, 1, p, ..., 0, 0), ..., (0, 0, 0, ..., 1, p), (p, 0, 0, ..., 0, 1).
\]

This happens not because of limitations intrinsic to our intertwining maps, but because of the structure of \(G\)-modules:

**Proposition 3.14** Assume \(g > 2\) and let \(k, h\) be integers such that \(0 \leq k, h \leq p - 1\). For any integer \(\alpha\) such that \(0 \leq \alpha \leq g - 1\) and any integer \(m\), there are no \(G\)-module embeddings \(\det^m \otimes M_k^{[\alpha]} \otimes M_h^{[\alpha+1]} \to M_k^{[\alpha]} \otimes M_h^{(\alpha+1)}\).

**Proof** It is enough to prove the non existence of embeddings for \(\alpha = 0\). Using \((\Phi_g)\) and \((\Delta_g)\) we have, in \(K_0(G)\):

\[
M_{h+p}^{[1]} = M_h^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-2}^{[1]}.
\]

If \(k \neq p-1\), as \(g > 2\), we deduce that the Jordan-Hölder factors of \(M_{k+1} \otimes M_{h+p}^{[1]}\) are \(M_{k+1} \otimes M_h^{[1]} \otimes M_1^{[2]}\) and \(\det^{p(h+1)} \otimes M_{k+1} \otimes M_{p-h-2}^{[1]}\), unless \(h = p - 1\), in which case only the first factor occurs. None of these factors coincides with \(\det^m \otimes M_k \otimes M_h^{[1]}\).

If \(k = p-1\), write \(M_p = M_1^{[1]} + e M_{p-2}\) in \(K_0(G)\). Applying \((\Pi_g)\) we obtain:

\[
M_p M_{h+p}^{[1]} = \left( M_1^{[1]} + e M_{p-2} \right) \left( M_h^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-2}^{[1]} \right)
\]

\[
= M_h^{[1]} M_1^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-1}^{[1]} + e^{p(h+2)} M_{p-h-3}^{[1]}
\]

\[
= e M_{p-2} M_1^{[1]} M_1^{[2]} + e^{p(h+1)+1} M_{p-2} M_{p-h-2}^{[1]}.
\]

If \(h \neq p-1\), the above formula shows that none of the Jordan-Hölder factors of \(M_p \otimes M_{h+p}^{[1]}\) equals \(\det^m \otimes M_{p-2} \otimes M_h^{[1]}\). If \(h = p-1\), we have:

\[
M_p M_{2p-1}^{[1]} = M_1^{[2]} M_1^{[1]} M_1^{[2]} + e^{p} M_{p-2} M_1^{[2]} + e^{p} M_{p-2} M_1^{[2]} + e M_{p-2} M_{p-1} M_1^{[2]}
\]

\[
= M_1^{[2]} + e^2 + 2 e M_{p-2} M_1^{[2]} + e M_{p-2} M_{p-1} M_1^{[2]},
\]

and \(\det^m \otimes M_{p-2} \otimes M_h^{[1]}\) is not a constituent of \(M_p \otimes M_{2p-1}^{[1]}\) if \(p \neq 2\). If \(p = 2\), decomposing \(M_2^{[2]}\) we get to the same conclusion. ■

We conclude this section by noticing the following consequence of Proposition 2.6

**Proposition 3.15** Let us fix non-negative integers \(k_0, \ldots, k_{g-1}\). For any integer \(\alpha\) such that \(0 \leq \alpha \leq g-1\) consider the \(G\)-map \(\Theta^{[\alpha]} : \det^\alpha \otimes \bigotimes_i M_{k_i}^{[\alpha]} \to \left( \bigotimes_{i \neq \alpha} M_{k_i}^{[\alpha]} \right) \otimes M_{k_\alpha+(q+1)}^{[\alpha]}\). We have:

\[
\coker \Theta^{[\alpha]} \simeq \left( \bigotimes_{i \neq \alpha} M_{k_i}^{[\alpha]} \right) \otimes \left[ \text{Ind}_G^H \left( \eta^{k_\alpha+2} \right) \right]^{[\alpha]},
\]

24
where $B$ is the subgroup of $G$ consisting of upper triangular matrices, and $\eta$ is the character of $B$ defined extending the character $\text{diag}(a, b) \mapsto a$ of the standard maximal torus of $G$.

**Remark 3.16** Even though the Jordan-Hölder constituents of $\ker \Theta^{[\alpha]}_3$ can be explicitly computed using the results of this paper, we do not know of any interesting description of the cokernel of the operators $\Theta^{[\alpha]}_3$.

### 3.3.2 Generalized $D$-operators

Let us denote by $\partial_X$ (resp. $\partial_Y$) the operator of partial derivation with respect to $X$ (resp. $Y$) acting on the polynomial algebra $\mathbb{F}_q[X, Y]$; if $f \in \mathbb{F}_q[X, Y]$, denote by the same symbol the $\mathbb{F}_q$-vector space endomorphism of $\mathbb{F}_q[X, Y]$ induced by multiplication by $f$. The operators $\partial_X \otimes f, \partial_Y \otimes f, f \otimes \partial_X$ and $f \otimes \partial_Y$ are therefore derivation of the $\mathbb{F}_q$-algebra $\mathbb{F}_q[X, Y] \otimes \mathbb{F}_q[X, Y]$.

**Definition 3.17** Let $k, h$ be two non-negative integers. For any integer $\beta$ such that $1 \leq \beta \leq g - 1$, the (non-twisted) **generalized $\beta$th $D$-operator** is the $\mathbb{F}_q$-vector space homomorphism:

$$D_\beta = \partial_X \otimes X^{p^\beta - \beta} + \partial_Y \otimes Y^{p^\beta - \beta} : M_k \otimes M_h^{[\beta]} \longrightarrow M_{k-1} \otimes M_h^{[\beta+\beta]}.$$

For any integers $\alpha, \beta$ such that $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$, the **generalized $\alpha$th $D$-operator** is the $\mathbb{F}_q$-vector space homomorphism:

$$D^{[\alpha]}_\beta = \partial_X \otimes X^{p^\beta - \beta} + \partial_Y \otimes Y^{p^\beta - \beta} : M_k^{[\alpha]} \otimes M_h^{[\beta+\beta]} \longrightarrow M_{k-1}^{[\alpha]} \otimes M_h^{[\beta+\beta]}.$$

**Lemma 3.18** Let $k, h$ be two non-negative integers and let $\alpha, \beta$ be integers such that $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$. The operator $D^{[\alpha]}_\beta : M_k^{[\alpha]} \otimes M_h^{[\beta+\beta]} \longrightarrow M_{k-1}^{[\alpha]} \otimes M_h^{[\beta+\beta]}$ is a $G$-homomorphism; it is injective if $0 < k \leq p - 1$ and $0 \leq h \leq p - 1$.

**Proof** By twisting, we can assume that $\alpha = 0$. Fix $f_1 \in M_k, f_2 \in M_h^{[\beta]}$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$; denote by $\gamma^{\sigma^\beta}$ the matrix $\begin{pmatrix} a^{\sigma^\beta} & b^{\sigma^\beta} \\ c^{\sigma^\beta} & d^{\sigma^\beta} \end{pmatrix}$, where $\sigma$ denotes the arithmetic Frobenius element of $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. $D_\beta (\gamma(f_1 \otimes f_2))$ equals:

$$[a \cdot (\partial_X f_1) (\gamma X, \gamma Y) + b \cdot (\partial_Y f_1) (\gamma X, \gamma Y)] \otimes X^{p^\beta - \beta} f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) + [c \cdot (\partial_X f_1) (\gamma X, \gamma Y) + d \cdot (\partial_Y f_1) (\gamma X, \gamma Y)] \otimes Y^{p^\beta - \beta} f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) = (\partial_X f_1) (\gamma X, \gamma Y) \otimes (aX^{p^\beta - \beta} + cY^{p^\beta - \beta}) f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) + (\partial_Y f_1) (\gamma X, \gamma Y) \otimes (bX^{p^\beta - \beta} + dY^{p^\beta - \beta}) f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) = \gamma D_\beta (f_1 \otimes f_2).$$

For the injectivity statement, notice that if $0 < k \leq p - 1$ and $0 \leq h \leq p - 1$, then $M_k \otimes M_h^{[\beta]}$ is an irreducible $G$-module, so it is enough to show that $D_\beta$ is
non-zero on $M_k \otimes M_k^{[\beta]}$. We have $D_\beta(X^k \otimes X^h) = kX^{k-1} \otimes X^{h+p^{\alpha}-\beta}$, and this is non-zero as $k$ is prime with $p$. 

In addition to the above operators, the $D$-map defined by Serre also gives an intertwining map:

**Proposition 3.19** Let $k$ be a non-negative integer and let $\alpha$ be an integer such that $0 \leq \alpha \leq g-1$. Then the Frobenius twists of Serre’s operator $D^{[\alpha]} = X^q\partial_X + Y^q\partial_Y$ define $G$-homomorphisms:

$$D^{[\alpha]} : M_k^{[\alpha]} \rightarrow M_{k+(q-1)}^{[\alpha]}$$

which are injective if $1 \leq k \leq p-1$.

**Proof** After twisting, we can assume $\alpha = 0$. The result then follows from section 2.2 and the irreducibility of $M_k^{[\alpha]}$ in the range $1 \leq k \leq p-1$. 

We can summarize the above results as follows:

**Theorem 3.20** Let us fix non-negative integers $k_0, \ldots, k_{g-1}$. For any integers $\alpha, \beta$ subject to the constraints $0 \leq \alpha \leq g-1$ and $1 \leq \beta \leq g-1$, there are $G$-intertwining operators as follows:

$$D^{[\alpha]} : \bigotimes_i M_{k_i}^{[\alpha]} \rightarrow \left( \bigotimes_{i \neq \alpha, \alpha+\beta} M_{k_i}^{[\alpha]} \right) \otimes M_{k_{\alpha+\beta}}^{[\alpha]} \otimes M_{k_{\alpha+\beta}+p^{\alpha}-\beta}^{[\alpha+\beta]};$$

$$D^{[\alpha]} : \bigotimes_i M_{k_i}^{[\alpha]} \rightarrow \left( \bigotimes_{i \neq \alpha} M_{k_i}^{[\alpha]} \right) \otimes M_{k_{\alpha+\beta}+(q-1)}^{[\alpha]};$$

where the tensor product indices run over the integers $i$ such that $0 \leq i \leq g-1$, unless otherwise specified. If $0 < k_{\alpha+\beta} \leq p-1$, then $D^{[\alpha]}$ is injective; if in addition $0 \leq k_{\alpha+\beta} \leq p-1$, then $D^{[\alpha]}$ is injective.

**Remark 3.21** The operators $D^{[\alpha]}_g$ for $0 \leq \alpha \leq g-1$ give, under suitable assumptions, cohomological analogues of the partial Hasse invariants defined in \cite{[1]} in the context of mod $p$ Hilbert modular forms. We do not know of any geometric interpretation of the other $D$-maps introduced above.

We can picture the weight shiftings allowed by the $g(g-1) + g = g^2$ generalized $D$-maps as follows:

| $D_1$   | $(−1, p^{g−1}, 0, 0, ..., 0, 0)$ |
|---------|----------------------------------|
| $D_1^{[1]}$ | $(0, −1, p^{g−1}, 0, ..., 0, 0)$ |
| $D_2^{[2]}$ | $(0, 0, −1, p^{g−1}, ..., 0, 0)$ |
| ... | ... |
| $D_1^{[g−2]}$ | $(0, 0, 0, ..., −1, p^{g−1})$ |
| $D_1^{[g−1]}$ | $(p^{g−1}, 0, 0, 0, ..., 0, −1)$ |
| $D_2$   | $(−1, 0, p^{g−2}, 0, ..., 0, 0)$ |
| $D_2^{[1]}$ | $(0, −1, 0, p^{g−2}, ..., 0, 0)$ |
| $D_2^{[2]}$ | $(0, 0, −1, 0, ..., 0, 0)$ |
| ... | ... |
| $D_2^{[g−2]}$ | $(p^{g−2}, 0, 0, 0, ..., −1, 0)$ |
| $D_2^{[g−1]}$ | $(0, p^{g−2}, 0, 0, 0, ..., −1)$ |
Similarly to what happened for the generalized Dickson operators, the non existence of shiftings of the form

\((-1, p, 0, ..., 0, 0), (0, -1, p, ..., 0, 0), ..., (0, 0, 0, ..., -1, p), (p, 0, 0, ..., 0, -1)\) when \(g > 2\) is a consequence of the structure of the irreducible \(G\)-modules:

**Proposition 3.22** Assume \(g > 2\) and let \(k, h\) be integers such that \(0 \leq k, h \leq p - 1\). For any integer \(\alpha\) such that \(0 \leq \alpha \leq g - 1\) and any integer \(m\), there are no \(G\)-module embeddings \(\det^m \otimes M_1^{[\alpha]} \otimes M_h^{[\alpha + 1]} \rightarrow M_{k - 1}^{[\alpha]} \otimes M_{h + p}^{[\alpha + 1]}\).

**Proof** It is enough to consider the case \(\alpha = 0\); we can also assume that \(k \neq 0\). Using formulae \((\Phi_g)\) and \((\Delta_g)\) we have, in \(K_0(G)\):

\[M_{k + p}^{[1]} = M_h^{[1]} M_1^{[2]} + e^{p(h + 1)} M_{p - h - 2}^{[1]}\]

As \(g > 2\), the Jordan-Hölder factors of \(M_{k - 1} \otimes M_{h + p}^{[1]}\) are \(M_{k - 1} \otimes M_h^{[1]} \otimes M_1^{[2]}\) and \(\det^{p(h + 1)} \otimes M_{k - 1} \otimes M_p^{[1]}\) unless \(h = p - 1\), in which case only the first factor occurs: none of these factors coincides with \(\det^m \otimes M_k \otimes M_h^{[1]}\).

We conclude by noticing the following consequence of Theorem 2.4

**Proposition 3.23** Let us fix non-negative integers \(k_0, ..., k_{g - 1}\); let \(\alpha\) be an integer such that \(0 \leq \alpha \leq g - 1\) and assume \(2 \leq k_\alpha \leq p - 1\), \(k_\alpha \neq \frac{p + 1}{2}\). Consider the injective \(G\)-map \(D^{[\alpha]} : D^{[\alpha]} : \bigotimes_{i \neq 0} M_{k_i}^{[\alpha]} \rightarrow \left(\bigotimes_{i \neq 0} M_{k_i}^{[\alpha]}\right) \otimes M_\alpha^{[\alpha + (q - 1)]}\). We have:

\[\text{coker } D^{[\alpha]} \simeq \left(\bigotimes_{i \neq 0} M_{k_i}^{[\alpha]}\right) \otimes \left[\Xi \left(\chi^{k_\alpha}\right)\right]^{[\alpha]}\]

where: \(\Xi \left(\chi^{k_\alpha}\right) = H^1_{\text{cris}}(\mathcal{C}/\mathbb{F}_q - \text{kappa} \otimes W(\mathbb{F}_q) \mathbb{F}_q, \mathcal{C} \text{ is the Deligne-Lusztig variety of SL}_2/\mathbb{F}_q\) and the \((-k_\alpha)\)-eigenspace of \(H^1_{\text{cris}}(\mathcal{C}/\mathbb{F}_q)\) is computed with respect to the natural action of \(\ker(\text{Nm}_{\mathbb{F}_q^*/\mathbb{F}_q^*})\) on \(H^1_{\text{cris}}(\mathcal{C}/\mathbb{F}_q)\). (Here \(W(\mathbb{F}_q)\) denotes the ring of Witt vectors of \(\mathbb{F}_q\)).
Remark 3.24 We do not know of any interesting description of the cokernel of the operators $D_{\alpha}^{[\beta]}$. The Jordan-Hölder constituents of $\text{coker} D_{\beta}^{[\alpha]}$ can be explicitly computed using the results of this paper.
Part II
Weight shiftings for automorphic forms

We apply the results of the previous sections to obtain weight shiftings for automorphic forms on definite quaternion algebras whose center is a totally real field $F$ unramified at the prime $p > 2$. In section 4 we mostly treat the case in which the tensor factors - corresponding to the prime decomposition of $p$ in $F$ - of the weight that we want to shift are all of dimension greater than one: this is what we call a weight not containing a $(2, \ldots, 2)$-block. In section 5 we consider shiftings for irreducible weights that contain a $(2, \ldots, 2)$-block.

4 Shiftings for weights not containing $(2, \ldots, 2)$-blocks

Let us fix some notation that will be used throughout this section and the next one. Let $F$ be a totally real number field of degree $g$ over $\mathbb{Q}$, and let $p > 2$ be a rational prime which is unramified in $F/\mathbb{Q}$. Denote by $\mathcal{O}_F$ the ring of integers of $F$ and write:

$$p\mathcal{O}_F = \prod_{j=1}^{r} \mathfrak{P}_j,$$

where the $\mathfrak{P}_j$'s are distinct maximal ideals of $\mathcal{O}_F$.

Fix an integer $j$ with $1 \leq j \leq r$. Let $f_j$ be the residual degree of $\mathfrak{P}_j$ over $p\mathbb{Z}$, so that $\mathbb{F}_{\mathfrak{P}_j} := \mathcal{O}_F/\mathfrak{P}_j$ is an extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of degree $f_j$. Let $\mathcal{F}_{\mathfrak{P}_j}$ be the completion of $F$ at $\mathfrak{P}_j$, and denote by $\mathcal{O}_{\mathcal{F}_{\mathfrak{P}_j}}$ its ring of integers. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$; let $n$ be the positive least common multiple of the integers $f_1, \ldots, f_r$ and let $E$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ having degree $n$ over $\mathbb{Q}_p$, so that $\text{Hom}(F, \overline{\mathbb{Q}}_p) = \text{Hom}(F, E)$. Denote by $\mathcal{O}$ the ring of integers of $E$ and let $\mathbb{F}$ be its residue field. Let $\sigma$ be the arithmetic Frobenius of the extension $E/\mathbb{Q}_p$. Set:

$$\text{Hom}(\mathbb{F}_{\mathfrak{P}_j}, E) = \{ \sigma_i^{(j)} : 0 \leq i \leq f_j - 1 \},$$

where the labeling is chosen so that, for any $i$, we have:

$$\sigma \circ \sigma_i^{(j)} = \sigma_{i+1}^{(j)}.$$

Here the subscripts read modulo $f_j$ and in the range $0 \leq i \leq f_j - 1$.

Denote by a bar the analogous morphisms for the residue fields, so that $\overline{\sigma}$ is the arithmetic Frobenius of the extension $\mathbb{F}/\mathbb{F}_p$, and:

$$\text{Hom}(\mathbb{F}_{\mathfrak{P}_j}, \mathbb{F}) = \{ \overline{\sigma_i^{(j)}} : 0 \leq i \leq f_j - 1 \}$$
are labeled so that:

$$\sigma \circ \sigma_{f_j}^{(i)} = \sigma_{f_{j+1}}^{(i)}$$

where the subscripts read modulo $f_j$ and in the range $0 \leq i \leq f_j - 1.$

We let $\mathcal{A}_F$ be the topological ring of adèles of $F,$ and we denote by $\mathcal{A}_F^\infty$ the subring of finite adèles. We let $\mathcal{M}_{F,f}$ (resp. $\mathcal{M}_{F,\infty}$) be the set of finite (resp. infinite) places of $F$ and we identify $\mathcal{M}_{F,f}$ with the set of maximal ideals of $\mathcal{O}_F.$

### 4.1 Some motivations: geometric Hilbert modular forms

Denote by $d_F$ the discriminant of $F/\mathbb{Q}$ and fix a fractional ideal $\mathfrak{a}$ of $F$ with its natural positive cone $\mathfrak{a}^+,$ so that $(\mathfrak{a}, \mathfrak{a}^+)$ represents an element in the strict class group of $F.$ Let $N \geq 4$ be an integer and recall that, by previous assumptions, $p$ does not divide $d_F.$ Let $S$ be a scheme over $\text{Spec}(\mathbb{Z} \left[ \frac{1}{d_F} \right]).$

There is an $S$-scheme $\mathcal{M}$ parametrizing isomorphism classes $[(A, \lambda, \iota, \varepsilon)/T/S]$ of $(\mathfrak{a}, \mathfrak{a}^+)$-polarized Hilbert-Blumenthal abelian $T$-schemes $(A, \lambda)$ of relative dimension $g$ ($T$ is an $S$-scheme), endowed with real multiplication $\iota$ by $\mathcal{O}_F,$ $\mu_N$-level structure $\varepsilon,$ and satisfying the Deligne-Pappas condition (or, equivalently since $d_F$ is invertible in $S,$ satisfying the Rapoport condition). $\mathcal{M}$ has relative dimension $g$ over $S$ and is geometrically irreducible; see [5] and [1] for more details.

Let $\mathbb{G} = \text{Res}_{\mathbb{O}_F/\mathbb{Z}}(\mathbb{G}_{m, \mathbb{O}_F})$ be the Weil restriction to $\mathbb{Z}$ of the algebraic $\mathcal{O}_F$-group $\mathbb{G}_{m, \mathbb{O}_F}.$ For any scheme $T,$ denote by $\mathbb{X}_T = \text{Hom}(\mathbb{G}_T, \mathbb{G}_{m, T})$ the group of characters of the base change $\mathbb{G}_T$ of $\mathbb{G}$ to $T.$ If $S$ is the scheme over $\text{Spec}(\mathbb{Z} \left[ \frac{1}{d_F} \right])$ fixed above, a geometric $(\mathfrak{a}, \mathfrak{a}^+)$-polarized Hilbert modular form $f$ over $S$ having weight $\chi \in \mathbb{X}_S$ and level $\mu_N$ is a rule that assigns to any affine scheme $\text{Spec}(R) \to S,$ any $R$-point $[(A, \lambda, \iota, \varepsilon)/R/S]$ of $\mathcal{M},$ and any generator $\omega$ of the $R \otimes_{\mathbb{Z}} \mathcal{O}_F$-module $\mathcal{O}_A^{1}/R,$ an element $f(A, \lambda, \iota, \varepsilon, \omega) \in R$ such that:

$$f(A, \lambda, \iota, \varepsilon, \omega^{-1} \varepsilon) = \chi(\alpha) \cdot f(A, \lambda, \iota, \varepsilon, \omega)$$

for $\alpha \in \mathbb{G}(R),$ and such that some compatibility conditions are satisfied (cf. [1], 5). We denote by $M_\chi(\mu_N, S)$ the $\Gamma(S, \mathcal{O}_S)$-module of such functions.

We remark that the formation of spaces of geometric Hilbert modular forms does not commute with base change: for example, if $g > 1$ and $1 \leq j \leq r,$ $0 \leq i \leq f_j - 1,$ the $(j, i)$th partial Hasse invariant that we will consider below is a non-zero, non-cuspidal modular forms over $\text{Spec} \mathbb{F}_{p_j}$ that cannot be lifted to a modular forms over $\text{Spec} \mathcal{O}_F$: the natural reduction morphism $M_\chi(\mu_N, \mathcal{O}_F) \to M_\chi(\mu_N, \mathbb{F}_{p_j})$ is in general not surjective.

Assume $g > 1$ for the rest of this paragraph. We consider modular forms over $S = \text{Spec}(\mathbb{F}).$ The labeling of the embeddings $\sigma_f^{(j)}$ for $1 \leq j \leq r$ and $0 \leq i \leq f_j - 1$ induces a canonical splitting:

$$\mathbb{G}_F = \bigoplus_{j=1}^r \left( \text{Res}_{\mathbb{F}_{p_j}/\mathbb{F}}(\mathbb{G}_{m, \mathbb{F}_{p_j}}) \times \text{Spec} \mathbb{F} \to \text{Spec} \mathbb{F} \right)$$

$$= \bigoplus_{j=1}^r \mathbb{G}_f^{(j)} : \mathbb{F}_{p_j} \to \mathbb{G}_{m, \mathbb{F}},$$

30
such that the projection $\chi_{(j,i)}$ of $G_{\mathbb{F}}$ onto the $(j,i)$th factor is induced by $\sigma^{(j)}_i$. The character group $X_{\mathbb{F}}$ of $G_{\mathbb{F}}$ is the free $\mathbb{Z}$-module or rank $g$ generated by these projections. A geometric Hilbert modular form over Spec($\mathbb{F}$) whose weight is $\prod_{j=1}^r \prod_{i=0}^{f_j-1} \chi_{(j,i)}$ for some $a_{(j)}^i \in \mathbb{Z}$ is also said to have weight vector $\vec{a} = (\vec{a}^{(1)}, \ldots, \vec{a}^{(r)})$ where $\vec{a}^{(j)} = (a_0^{(j)}, \ldots, a_{f_j-1}^{(j)})$ for $1 \leq j \leq r$.

Theorem 2.1 of [10] shows that, for any $1 \leq j \leq r$ and $0 \leq i \leq f_j - 1$, there is an $(a, a^+)$-polarized Hilbert modular form $h_{(j,i)}$ over Spec($\mathbb{F}$) having weight $\chi_{(j,i)}^p$ and level $1$, whose $q$-expansion at every $(a, a^+)$-polarized unramified $\mathbb{F}_p$-rational cusp is one. $h_{(j,i)}$ is called the $(j,i)$th partial Hasse invariant. As mentioned earlier, the forms $h_{(j,i)}$ are not liftable to characteristic zero; even the total Hasse invariant, i.e., the form $h = \prod_{(j,i)} h_{(j,i)}$, having parallel weight $(p-1, p-1, \ldots, p-1)$, is not always liftable to characteristic zero (cf. Proposition 3.1 in [10]).

As a consequence of the existence of the partial Hasse invariants, one can produce (geometric) weight shifting. More precisely, fix an integer $j$ such that $1 \leq j \leq r$ and assume $\chi \in X_{\mathbb{F}}$ is such that $M_{\chi} (\mu_N, \mathbb{F}) \neq 0$; denote the weight vector associated to $\chi$ by $\vec{a} = (\vec{a}^{(1)}, \ldots, \vec{a}^{(r)})$. Multiplication by $h_{(j,i)}$ for an integer $i$ such that $0 \leq i \leq f_j - 1$ induces a Hecke injection (weight shifting) of $M_{\chi} (\mu_N, \mathbb{F})$ into $M_{\chi'} (\mu_N, \mathbb{F})$, where the weight vector associated to $\chi'$ is $\vec{a} + \vec{t}$ and $\vec{t} = (\vec{t}^{(1)}, \ldots, \vec{t}^{(r)})$ is such that $\vec{t}^{(r)} = \vec{0}$ if $r \neq j$, while $\vec{t}^{(j)}$ is one of the following $f_j$-tuples:

$$
\begin{align*}
(-1, 0, 0, \ldots, 0, p) & \text{ if } i = 0, \\
(p, -1, 0, \ldots, 0, 0) & \text{ if } i = 1, \\
(0, p, -1, \ldots, 0, 0) & \text{ if } i = 2, \\
& \ldots \\
(0, 0, 0, \ldots, p, -1) & \text{ if } i = f_j - 1.
\end{align*}
$$

In this case, we will say that $h_{(j,i)}$ induces a weight shifting by $\vec{t}$.

In [13] 2.5. and [1] 12, generalized theta operators acting on spaces of geometric Hilbert modular forms over Spec($\mathbb{F}$) are defined, allowing additional weight shifting. For example, if $p$ is inert in $F/\mathbb{Q}$, these operators induce shifting by the vectors:

$$
\begin{align*}
(1, 0, 0, \ldots, 0, p), \\
(p, 1, 0, \ldots, 0, 0), \\
(0, p, 1, \ldots, 0, 0), \\
& \ldots \\
(0, 0, 0, \ldots, p, 1).
\end{align*}
$$

The reader will notice that the two sets of weight shifting vectors described above are contained in the sets of weight shifting vectors produced in 3.3.1 and 3.3.2 for $\mathbb{F}_p$-representation of $GL_2(\mathbb{F})$. Exploiting the adelic definition of
Hilbert modular forms, we will see that all the geometric weight shiftings can be obtained as cohomological weight shiftings via the operators considered in Section 3. The purely cohomological picture will be richer, as more shiftings will be allowed. The formation of spaces of adelic automorphic forms on definite quaternion algebra will have the big advantage of being compatible with base changes, under suitable assumptions (Proposition 4.2). Finally, our cohomological weight shiftings translate into weight shiftings for \((\bmod p)\) Galois representations arising from automorphic forms on \(GL_2(A_F)\).

4.2 Automorphic forms on definite quaternion algebras

We recall the definition and some properties of automorphic forms on definite quaternion algebras over totally real number fields. The exposition follows [24] and [15]; cf. also [23].

Fix a finite set \(\Sigma \subseteq \mathfrak{M}_{F,f}\) that is disjoint from the set of places of \(F\) lying above \(p\) and such that \(\#\Sigma + [F : \mathbb{Q}] \equiv 0 \pmod{2}\). Let \(D\) be a quaternion algebra over \(F\) whose ramification set is \(\mathfrak{M}_F, \infty \cup \Sigma\). Let \(\mathcal{O}_D\) be a fixed maximal order of \(D\) and for any \(v \in \mathfrak{M}_{F,f} - \Sigma\) fix ring isomorphisms \((\mathcal{O}_D)^\times_v \cong M_2(\mathcal{O}_{F_v})\).

Let \(U\) be a compact open subgroup of \((D \otimes_F \mathbb{A}_F)^\times\) such that:

1. \(U = \prod_{v \in \mathfrak{M}_{F,f}} U_v\), where \(U_v\) is a subgroup of \((\mathcal{O}_D)^\times_v\);
2. \(U_v = (\mathcal{O}_D)^\times_v\) if \(v \in \Sigma\);
3. if \(v|p\), then \(U_v = GL_2(\mathcal{O}_{F_v})\).

Let \(A\) be a topological \(\mathbb{Z}_p\)-algebra. Let \(v\) be a place of \(F\) above \(p\), say \(v = v_j := \mathfrak{P}_j\) for some integer \(j\) such that \(1 \leq j \leq r\); let \(W_{\tau_j}\) be a free \(A\)-module of finite rank and fix a continuous homomorphism \(\tau_j : U_{v_j} = GL_2(\mathcal{O}_{F_{v_j}}) \rightarrow \text{Aut}(W_{\tau_j})\), where \(\text{Aut}(W_{\tau_j})\) is the group of continuous \(A\)-linear automorphisms of \(W_{\tau_j}\). Let \(W_\tau = \bigotimes_{j=1}^{r} W_{\tau_j}\), where the tensor products are over \(A\), and denote by \(\tau\) the corresponding group homomorphism \(\tau : \prod_{j=1}^{r} U_{v_j} \rightarrow \text{Aut}(W_\tau)\). If no confusion arises, we also denote by \(\tau\) the action of \(U\) on \(W_\tau\) induced by precomposing the latter morphism with the natural projection \(U \rightarrow \prod_{j=1}^{r} U_{v_j}\).

For \(A\) as above, let \(\psi : (\mathbb{A}_F^\infty \times F^\times) / A^\times \rightarrow A^\times\) be a continuous character such that, for any \(v \in \mathfrak{M}_{F,f}\):

\[
\tau_{|U_v \cap \mathcal{O}_F^\times_v} (u) = \psi^{-1}(u) \cdot \text{Id}_{W_\tau}, \quad \text{for all } u \in U_v \cap \mathcal{O}_F^\times_v.
\]

We say that such a Hecke character \(\psi\) is compatible with \(\tau\).

**Definition 4.1** For \(D, U, A, \tau, W_\tau\) and \(\psi\) as above, the space \(S_{\tau,\psi}(U, A)\) of automorphic forms on \(D\) having level \(U\), weight \(\tau\), character \(\psi\) and coefficients in \(A\) is the \(A\)-module consisting of all the functions:

\[
f : D^\times \setminus (D \otimes_F \mathbb{A}_F^\infty)^\times \rightarrow W_\tau
\]
satisfying:

(a) \( f(gu) = \tau(u)^{-1}f(g) \) for all \( g \in (D \otimes_F \mathbb{A}_F^\infty)^{\times} \) and all \( u \in U \);

(b) \( f(gz) = \psi(z)f(g) \) for all \( g \in (D \otimes_F \mathbb{A}_F^\infty)^{\times} \) and all \( z \in (\mathbb{A}_F^\infty)^{\times} \).

As in [12], we will always assume, unless otherwise stated, that for all \( t \in (D \otimes_F \mathbb{A}_F^\infty)^{\times} \), the finite group \((U \cdot (\mathbb{A}_F^\infty)^{\times} \cap t^{-1}D^\times)/F^\times \) has order prime to \( p \). This assumption is automatically satisfied if \( U \) is sufficiently small, as Lemma 1.1. of [24] implies that in this case \((U \cdot (\mathbb{A}_F^\infty)^{\times} \cap t^{-1}D^\times)/F^\times \) is a 2-group. We obtain as a consequence (cf. [24], Corollary 1.2):

**Proposition 4.2** Let \( B \) a topological \( A \)-algebra. Then the natural morphism \( S_{\tau,\psi}(U,A) \otimes_A B \to S_{\tau,\psi}(U,B) \) is an isomorphism of \( B \)-modules.

Define a left action of \((D \otimes_F \mathbb{A}_F^\infty)^{\times} \) on the set of functions \( D^\times \setminus (D \otimes_F \mathbb{A}_F^\infty)^{\times} \to W_\tau \) by setting \( (gf)(x) := f(xg) \) for all \( g,x \in (D \otimes_F \mathbb{A}_F^\infty)^{\times} \). Let \( S \) be a set of primes of \( F \) containing the ramification set of \( D \), the primes above \( p \) and the primes \( v \) for which \( U_v \) is not a maximal compact subgroup of \( D_v^\times \). Let \( T_{S,A}^{univ} = A[T_v,S_v : v \notin S] \) be the commutative polynomial \( A \)-algebra in the indicated indeterminates. For each finite place \( v \notin S \), let \( \varpi_v \) a fixed uniformizer for \( F_v \). \( S_{\tau,\psi}(U,A) \) has a natural action of \( T_{S,A}^{univ} \), with \( S_v \) acting via the double coset \( U \left( \frac{\varpi_v}{\varpi_v} \right) U \) and \( T_v \) via \( U \left( \frac{\varpi_v}{1} \right) U \) (cf. [24], 1); this action does not depend upon the choices of uniformizers that we made. The image of \( T_{S,A}^{univ} \) in the ring of \( A \)-module endomorphisms of \( S_{\tau,\psi}(U,A) \) is the Hecke algebra \( T_{S,A} \) acting on \( S_{\tau,\psi}(U,A) \). The isomorphism of Proposition 4.2 is Hecke equivariant.

### 4.3 Behavior of Hecke eigensystems under reduction modulo \( \mathfrak{M}_R \)

For a discrete valuation ring \( R \), we will denote by \( \mathfrak{M}_R \) its maximal ideal. If the residual characteristic of \( R \) is \( p > 0 \) and no confusion arises, we will also improperly refer to reduction modulo \( \mathfrak{M}_R \) as reduction modulo \( p \). If \( T \) is a commutative algebra, a system of eigenvalues of \( T \) with values in \( \overline{R} \) is a set theoretic map \( \Omega : T \to \overline{R} \); the reduction of \( \Omega \) modulo \( p \), denoted \( \overline{\Omega} \), is the function obtained by composing \( \Omega \) with the reduction morphism \( R \to \overline{R} \). Let \( RT = R \otimes_T T \); if \( M \) is an \( RT \)-module, we say that a system of eigenvalues \( \overline{\Omega} : T \to \overline{\overline{R}} \) occurring in \( M \) if there is a non-zero element \( m \in M \) such that \( Tm = \overline{\Omega(T)}m \) for all \( T \in T \). Such a non-zero \( m \) is called an \( \overline{\Omega} \)-eigenvector.

Fixing \( R \) and \( T \) as above. We have:

**Lemma 4.3** Let \( M \) be an \( RT \)-module which is finitely generated over \( R \). If \( \overline{\Omega} : T \to \overline{R} \) is a system of eigenvalues of \( T \) occurring in \( M \), then \( \overline{\Omega} : T \to \overline{R} \) is a system of eigenvalues of \( T \) occurring in \( \overline{M} := M \otimes_R \overline{\overline{R}} \).

**Proof** Cf. [2], Proposition 1.2.3. ■
Lemma 4.4 Let $M$ be an $\mathcal{RT}$-module which is finite and free over $R$. Let $\Omega : \mathcal{T} \to \frac{R}{\mathfrak{m}_R}$ be a system of eigenvalues of $\mathcal{T}$ occurring in $M = M \otimes_R \frac{R}{\mathfrak{m}_R}$. There exists a finite extension of discrete valuation rings $R'/R$ such that $\mathfrak{m}_R \cap R' = \mathfrak{m}_{R'}$ and a system of eigenvalues $\Omega' : \mathcal{T} \to R'$ of $\mathcal{T}$ occurring in $M \otimes_R R'$ such that, for all $T \in \mathcal{T}$, $\Omega'(T)(\text{mod } \mathfrak{m}_{R'}) = \Omega(T)$ in $\frac{R'}{\mathfrak{m}_{R'}}$. (Here we view $\frac{R}{\mathfrak{m}_R} \subseteq \frac{R'}{\mathfrak{m}_{R'}}$ by the given embedding $R \subseteq R'$.)

Proof Cf. [4], Lemme 6.11. A generalization of the result is given in [2], Proposition 1.2.2. ■

Let $D$, $u$, $\tau$, $W_\tau$ and $\psi$ be as in [3] and set $A = \mathcal{O}$. In particular, we assume that $\psi$ is compatible with $(\tau, W_{\tau})$, $U$ is small enough and $p$ is odd. Denote by a bar the operation of tensoring over $\mathcal{O}$ with $\mathcal{F}$. From now on, unless otherwise stated, we assume fixed a set $S$ of primes of $F$ containing the ramification set of $D$, the primes above $p$ and the primes $v$ for which $U_v$ is not a maximal compact subgroup of $D_v^\times$. The Hecke eigensystems considered below will always be with respect to the Hecke algebra $\mathfrak{Z}_{\mathcal{S}, A'}$ for some topological $\mathbb{Z}_p$-algebra $A'$.

Proposition 4.5 Fix an $\mathcal{O}$-valued weight $(\tau', W_{\tau'})$ together with a compatible Hecke character $\psi' : (\mathbb{A}_F^\infty/\mathcal{O}^\times) \to \mathcal{O}^\times$ such that $\bar{\psi}' = \bar{\psi}$. Let $\varphi : (\bar{\tau}, W_{\tau}) \to (\bar{\tau}', W_{\tau'})$ be a non-zero intertwining operator for $\mathcal{F}$-representations of $\mathcal{U}$. $\varphi$ induces a Hecke equivariant map $\varphi_* : S_{\bar{\tau}, \bar{\psi}}(U, \mathcal{F}) \to S_{\bar{\tau}', \bar{\psi}}(U, \mathcal{F})$.

Assume $\varphi$ is injective: then if $\Omega$ is a Hecke eigensystem occurring in $S_{\bar{\tau}, \bar{\psi}}(U, \mathcal{O})$, there is a finite extension of $E$, with ring of integer $\mathcal{O}'$ such that $\mathfrak{m}_{\mathcal{O}' \cap \mathcal{O}} = \mathfrak{m}_{\mathcal{O}}$, and there is a Hecke eigensystem $\Omega'$ occurring in $S_{\bar{\tau}', \bar{\psi}'}(U, \mathcal{O}')$ such that:

$$\Omega'(\text{mod } \mathfrak{m}_{\mathcal{O}'}) = \Omega(\text{mod } \mathfrak{m}_{\mathcal{O}}) \quad \text{in } \frac{\mathcal{O}'}{\mathfrak{m}_{\mathcal{O}'}}.$$

Proof For $f \in S_{\bar{\tau}, \bar{\psi}}(U, \mathcal{F})$ set $\varphi_*(f) := \varphi \circ f$. If $g \in (D \otimes_F \mathbb{A}_F^\infty/\mathcal{O})^\times$, $u \in U$ and $z \in (\mathbb{A}_F^\infty)^\times$ we have:

$$\varphi_*(f)(gu) = \varphi(f(gu)) = \varphi(\bar{\tau}(u^{-1})f(g)) = \bar{\tau}'(u)^{-1}\varphi(f(g)),$$

$$\varphi_*(f)(gz) = \varphi(f(gz)) = \bar{\psi}(z)f(g) = \bar{\psi}(z)\varphi(f(g)).$$

Since $\bar{\psi}' = \bar{\psi}$, we have that $\bar{\tau}'$ and $\bar{\psi}$ are compatible and we conclude that $\varphi_*(f) \in S_{\bar{\tau}', \bar{\psi}'}(U, \mathcal{F})$. If $g, x \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, we have:

$$(g \cdot \varphi_*(f))(x) = (\varphi \circ f)(gx)$$

$$= (\varphi \circ (g \cdot f))(x)$$

$$= (\varphi_*(g \cdot f))(x),$$

so that $\varphi$ is Hecke-equivariant. Assume now that $\varphi$ is injective and notice that this implies the injectivity of $\varphi_*$. Let $\Omega$ be a Hecke eigensystem occurring in
the finite $\mathcal{O}$-module with Hecke action $S_{\tau,\psi}(U, \mathcal{O})$; by Proposition 4.2, reduction modulo $p$ induces a Hecke equivariant surjection:

$$
\pi: S_{\tau,\psi}(U, \mathcal{O}) \longrightarrow S_{\tau,\psi}(U, \mathbb{F}).
$$

By Lemma 4.3 the Hecke eigensystem $\bar{\Omega} := \Omega(\text{mod } \mathcal{O})$ occurs in $S_{\tau,\psi}(U, \mathbb{F})$, and hence in $S_{\tau,\psi}(U, \mathcal{O})$ as $\varphi_\tau$ is Hecke equivariant and injective. Now, applying Lemma 4.4 to the Hecke equivariant surjection $S_{\tau,\psi}(U, \mathcal{O}) \rightarrow S_{\tau,\psi}(U, \mathbb{F})$, we deduce the existence of a finite extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ such that $\mathcal{M}_{\mathcal{O}} \cap \mathcal{O} = \mathcal{M}_{\mathcal{O}}$, and of a Hecke eigensystem $\Omega' : T_{\mathcal{O}} \rightarrow \mathcal{O}'$ occurring in $S_{\tau,\psi'}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'$ whose reduction modulo $\mathcal{M}_{\mathcal{O}}$ has value in $\mathbb{F} \subset \mathcal{O}'_{\mathcal{O}_{\mathcal{M}}}$ and coincide with $\bar{\Omega}$. By Proposition 4.2, $S_{\tau,\psi'}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}' \simeq S_{\tau,\psi}(U, \mathcal{O}')$ as Hecke modules, and we are done. 

\section{4.4 Holomorphic weights}

For any integer $j$ such that $1 \leq j \leq r$ let us fix two tuples $\bar{k}(j) = (k_0^j, \ldots, k_{f_j - 1}^j) \in \mathbb{Z}_{\geq 2}^{f_j}$ and $\bar{w}(j) = (w_0^j, \ldots, w_{f_j - 1}^j) \in \mathbb{Z}^{f_j}$. Define the finite free $\mathcal{O}$-module with $GL_2(\mathcal{O})$-action:

$$
W_{(\bar{k}(j), \bar{w}(j))} := \bigotimes_{i=0}^{f_j - 1} \text{Sym}^{k_{i}^j - 2} \mathcal{O}^2 \otimes \det^{w_{i}^j}
$$

where the tensor products are over $\mathcal{O}$.

If we let the group $GL_2(\mathcal{O}_{F_{\varphi_j}})$ act on the tensor factor $\text{Sym}^{k_{i}^j - 2} \mathcal{O}^2 \otimes \det^{w_{i}^j}$ ($0 \leq i \leq f_j - 1$) via the embedding $GL_2(\mathcal{O}_{F_{\varphi_j}}) \rightarrow GL_2(\mathcal{O})$ induced by $\sigma_i = \sigma^0 \circ \sigma_i^j$, $W_{(\bar{k}(j), \bar{w}(j))}$ can be seen as a representation of $GL_2(\mathcal{O}_{F_{\varphi_j}})$. We convene of viewing $GL_2(\mathcal{O}_{F_{\varphi_j}})$ as a subgroup of $GL_2(\mathcal{O})$ via the embedding $\sigma_0^j$, and we write the $GL_2(\mathcal{O}_{F_{\varphi_j}})$-representation $W_{(\bar{k}(j), \bar{w}(j))}$ as:

$$
W_{(\bar{k}(j), \bar{w}(j))} = \bigotimes_{i=0}^{f_j - 1} \left(\text{Sym}^{k_{i}^j - 2} \mathcal{O}^2 \otimes \det^{w_{i}^j}\right)^{[i]},
$$

where the superscript $[i]$ indicates twisting by the $i$th power of the Frobenius element $\sigma$. In the sequel, unless otherwise stated, we always view $GL_2(\mathcal{O}_{F_{\varphi_j}}) \subseteq GL_2(\mathcal{O})$ via $\sigma_0^j$.

Denote by $\tau_{(\bar{k}, \bar{w})}$ the continuous action of $GL_2(\mathcal{O}_{F_{\varphi_j}})$ on $W_{(\bar{k}(j), \bar{w}(j))}$ and let $\tau_{(\bar{k}, \bar{w})} = \bigotimes_{j=1}^{r} \tau_{(\bar{k}(j), \bar{w}(j))}$, where the tensor products are over $\mathcal{O}$ and $\bar{k} = (\bar{k}(1), \ldots, \bar{k}(r))$. We have:

$$
\tau_{(\bar{k}, \bar{w})} : \prod_{j=1}^{r} GL_2(\mathcal{O}_{F_{\varphi_j}}) \rightarrow \text{Aut} W_{(\bar{k}, \bar{w})},
$$

with $W_{(\bar{k}, \bar{w})} = \bigotimes_{j=1}^{r} W_{(\bar{k}(j), \bar{w}(j))}$ (tensor product over $\mathcal{O}$).
If there is some integer \( j \) such that \( \tilde{k}^{(j)} = (2, \ldots, 2) \), we say that the weight \( \tau_{(\tilde{k},\tilde{w})} \) contains a \((2,\ldots,2)\)-block relative to the prime \( \mathfrak{q}_j \). This terminology is not standard but it is used throughout the paper.

We say that \( \tau_{(\tilde{k},\tilde{w})} \) is a holomorphic weight if there exists an integer \( w \) such that:

\[
k_i^{(j)} + 2w_i^{(j)} - 1 = w
\]

for all \( 1 \leq j \leq r \) and all \( 0 \leq i \leq f_j - 1 \) (cf. [11]).

The pair \((\tilde{k},\tilde{w})\) \( \in \mathbb{Z}_\geq 2 \times \mathbb{Z}^g \) is called the parameter pair for \( \tau_{(\tilde{k},\tilde{w})} \). If \( \tau_{(\tilde{k},\tilde{w})} \) is a holomorphic weight, it is also determined by the parameter pair \((k,w)\) \( \in \mathbb{Z}_\geq 2 \times \mathbb{Z} \), with \( w \) as in [11].

### 4.4.1 Some results on holomorphic weight shiftings

**Lemma 4.6** Let us view the holomorphic weight \( \tau_{(\tilde{k},\tilde{w})} \) as an \( \mathcal{O} \)-representation of the fixed level \( U \subset (D \otimes \mathbb{A}_F^\times)^\times \). A Hecke character \( \psi: (\mathbb{A}_F^\times) / F^\times \rightarrow \mathcal{O}^\times \) is compatible with \( \tau_{(\tilde{k},\tilde{w})} \), if and only if the following two conditions are satisfied:

(a) \( \psi(u) = 1 \) for all \( u \in U_v \cap \mathcal{O}_F^\times \), where \( v \in \mathcal{M}_{F,j} \) and \( v \nmid \mathfrak{p} \);

(b) \( \psi(u) = (\text{Nm}_{F_{\mathfrak{q}_j}/\mathbb{Q}_p}(u))^{1-w} \) for all \( u \in \mathcal{O}_F^\times \), where \( 1 \leq j \leq r \).

**Proof** The reason for condition (a) is clear, as the representation \( \tau_{(\tilde{k},\tilde{w})} \) factors through \( \prod_{j=1}^r \text{GL}_2(\mathcal{O}_{F_{\mathfrak{q}_j}}) \). Let \( j \) be such that \( 1 \leq j \leq r \) and fix \( u \in \mathcal{O}_{F_{\mathfrak{q}_j}} \); recall that we embed \( \mathcal{O}_{F_{\mathfrak{q}_j}} \) in \( \mathcal{O} \) via \( \sigma^{(j)} \). The matrix \( (u\ u) \in \text{GL}_2(\mathcal{O}_{F_{\mathfrak{q}_j}}) \) acts on \( W_{(\tilde{k},\tilde{w}),\omega} \) as the automorphism:

\[
\bigotimes_{i=0}^{f_j-1} \left( \sigma^i(u)^{k_i^{(j)}-2+2w_i^{(j)}} \cdot \text{Id}_i \right) = \bigotimes_{i=0}^{f_j-1} \sigma^i(u)^{w-1} \cdot \text{Id}_i
\]

\[
= \left( \text{Nm}_{F_{\mathfrak{q}_j}/\mathbb{Q}_p}(u) \right)^{w-1} \cdot \bigotimes_{i=0}^{f_j-1} \text{Id}_i
\]

\[
= \left( \text{Nm}_{F_{\mathfrak{q}_j}/\mathbb{Q}_p}(u) \right)^{w-1} \cdot \text{Id}_{W_{(\tilde{k},\tilde{w}),\omega}},
\]

where \( \text{Id}_i \) denotes the identity map of the \( \mathcal{O} \)-vector space:

\[
\left( \text{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}} \right)[l],
\]

and we used the assumption that the local extension \( F_{\mathfrak{q}_j}/\mathbb{Q}_p \) is unramified with Galois group generated by the restriction of \( \sigma \) to \( F_{\mathfrak{q}_j} \). The result now follows, as we need to have \( \tau_{\text{Nm}_{F_{\mathfrak{q}_j}/\mathbb{Q}_p}(u)} = \psi^{-1}(u) \cdot \text{Id}_{W_{(k,w)}} \). \( \blacksquare \)

**Lemma 4.7** Let \( w \) be an even integer. Then there exists a continuous character \( \psi: (\mathbb{A}_F^\times) / F^\times \rightarrow \mathbb{Z}_p^\times \) such that:
(a) $\psi(u) = 1$ for all $u \in \mathcal{O}_F^\times$, where $v \in \mathfrak{M}_{F,f}$ and $v|\beta$;
(b) $\psi(u) = \left(\text{Nm}_{F_p/v_p}(u)\right)^w$ for all $u \in \mathcal{O}_{F_p}^\times$, where $1 \leq j \leq r$.

**Proof** The adèles norm map $(\mathbb{A}_F^\times) \to (\mathbb{A}_Q^\times)^\times$ induces a continuous homomorphism $\text{Nm} : (\mathbb{A}_F^\times) / F^\times \to (\mathbb{A}_Q^\times)^\times / Q^\times$. The group-theoretic decomposition $(\mathbb{A}_Q^\times)^\times = Q^\times \tilde{\mathbb{Z}}^\times$ induces a continuous isomorphism $\beta : (\mathbb{A}_Q^\times)^\times / Q^\times \to \tilde{\mathbb{Z}}^\times / \langle -1 \rangle$. Finally, the map $\prod_{l} \mathbb{Z}_l^\times \to \mathbb{Z}_p^\times$ defined by sending the tuple $(a_l) \in \prod_l \mathbb{Z}_l^\times$ into $a_p^w \in \mathbb{Z}_p^\times$ defines a continuous homomorphism $\alpha : \tilde{\mathbb{Z}}^\times / \langle -1 \rangle \to \mathbb{Z}_p^\times$ since $w$ is even. We check that the composition $\psi := \alpha \circ \beta \circ \text{Nm}$ is a Hecke character with the desired properties.

Assume $v = \mathfrak{p}_j \mathfrak{p}$ and view a fixed $u \in \mathcal{O}_{F_{\mathfrak{p}_j}}^\times$ as an element of $(\mathbb{A}_F^\times)^\times$ whose $v$-component is $u$ and whose $v'$-component is $1$ for all finite places $v' \neq v$ of $F$. Then $\text{Nm}(u \cdot F^\times) = \text{Nm}_{F_p/v_p}(u) \cdot Q^\times$, where we identify $\text{Nm}_{F_p/v_p}(u)$ with the adèle of $Q$ whose $p$-component is the $p$-adic unit $\text{Nm}_{F_p/v_p}(u) \in \mathbb{Z}_p^\times$ and whose other components are equal to $1$. Then $(\alpha \circ \beta) \left(\text{Nm}_{F_p/v_p}(u) \cdot Q^\times\right) = \left(\text{Nm}_{F_p/v_p}(u)\right)^w \in \mathbb{Z}_p^\times$.

Assume $v$ is a finite place of $F$ lying above some rational prime $l \neq p$ and let $u \in \mathcal{O}_F^\times$ viewed as an element of $(\mathbb{A}_F^\times)^\times$ in the usual way. Write $\text{Nm}(u \cdot F^\times) = \text{Nm}_{F_p/v_p}(u) \cdot Q^\times$; since the $p$-component of $\text{Nm}_{F_p/v_p}(u) \in \tilde{\mathbb{Z}}^\times$ is trivial, $\psi(u) = 1$. ■

Set $A = \mathcal{O}$ and let $D, U, (\tau, W_\tau)$ and $\psi$ be as in 4.2.

**Proposition 4.8** Assume $\tau = \tau_{(k,w)}$ and $\tau' = \tau_{(k',w')}$ are holomorphic $A$-linear weights for automorphic forms on $D$, with $w \equiv w' \dbmod{p-1}$ and $w$ odd. Assume that $\tau_{(k,w)}$ and $\psi$ are compatible and that $\tau'_{(k,w)}$ is isomorphic to an $F$-linear $U$-subrepresentation of $\tau'_{(k',w')}$. Then:

(a) There is a Hecke character $\psi' : (\mathbb{A}_F^\times)^\times / F^\times \to \mathcal{O}^\times$ which is compatible with $\tau_{(k',w')}$ and such that $\psi' = \psi$;
(b) For any Hecke eigensystem $\Omega$ occurring in $S_{\tau,\psi}(U, \mathcal{O})$ there is a finite extension of discrete valuation rings $\mathcal{O}' / \mathcal{O}$ with $\mathfrak{M}_{O'} \cap \mathcal{O} = \mathfrak{M}_\mathcal{O}$ and a Hecke eigensystem $\Omega'$ occurring in $S_{\tau,\psi}(U, \mathcal{O}')$ such that $\Omega' \dbmod{\mathfrak{M}_{O'}} = \Omega \dbmod{\mathfrak{M}_\mathcal{O}}$.

**Proof** Since $p > 2$, the integer $1 - w'$ is even. By Lemma 4.7 there exists a Hecke character $\psi'' : (\mathbb{A}_F^\times)^\times / F^\times \to \mathbb{Z}_p^\times \subset \mathcal{O}^\times$ such that $\psi''(u) = 1$ for all $v \in \mathfrak{M}_{F,f}$ not lying above $p$ and all $u \in \mathcal{O}_F^\times$, and $\psi''(u) = \left(\text{Nm}_{F_p/v_p}(u)\right)^{1 - w'}$ for $u \in \mathcal{O}_{F_p}^\times$, $1 \leq j \leq r).$ By Lemma 4.6 $\psi''$ is compatible with $\tau_{(k',w')}$. 37
Let \( \alpha \) denote the reduction modulo \( \mathfrak{M}_\mathcal{O} \) of the Hecke character \( \psi^{-1}\psi'' \). Since \( w \equiv w' \pmod{p - 1} \), by the compatibility of \( \psi \) with \( \tau_{(k,w)} \) and by the construction of \( \psi'' \), the continuous character \( \alpha \) is trivial on the open subgroup

\[
\prod_{v \mid p} \left( U_v \cap \mathcal{O}_{F_v}^\times \right) \times \prod_{j=1}^r \mathcal{O}_{F_{p^j}}^\times
\]

of \( (\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times \). Therefore \( \alpha \) factors through a finite discrete quotient of \( (\mathbb{A}_F^\times)^\times \).

In particular, the Teichmüller lift \( \tilde{\alpha} \) of \( \alpha \) is a continuous character \( (\mathbb{A}_F^\times)^\times / F^\times \to \mathcal{O}^\times \).

The \( \mathcal{O}^\times \)-valued Hecke character \( \psi' := \psi''\tilde{\alpha}^{-1} \) is compatible with \( \tau_{(\vec{k},w')} \) and satisfies \( \tilde{\psi}' = \tilde{\psi} \), so that (a) is proved.

Part (b) follows by applying Proposition 4.5 with \( \psi' \) chosen as in (a).

### 4.4.2 Link with classical automorphic forms on \( D^\times \)

To conclude this paragraph, we make explicit the link between adelic automorphic forms for the algebraic \( \mathbb{Q} \)-group \( \mathbb{D} \) associated to \( D^\times \).

Set \( A = E \) and let \( \tau : \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p^j}}) \to \text{Aut}(\mathcal{W}_\tau) \) be a weight for adelic automorphic forms on \( D \) as considered in 4.2. Suppose \( \mathcal{W}_\tau = \mathcal{W}_{\tau^\text{alg}} \otimes_E \mathcal{W}_{\tau^\text{sm}} \), where \( \mathcal{W}_{\tau^\text{alg}} \) is a smooth irreducible \( E \)-representation of \( \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p^j}}) \), and

\[
\mathcal{W}_{\tau^\text{alg}} = \bigotimes_{j=1}^r \bigotimes_{i=0}^{f_j-1} \left( \text{Sym}^k \mathcal{E} \otimes \det \mathcal{E} \right)_{[i]}^{[i]}
\]

is an irreducible algebraic representation of \( \mathcal{D}(\mathbb{Q}_p) = (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times = \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p^j}}) \). We assume that \( k_i^{(j)} + 2w_i^{(j)} \) equals some fixed integer \( w \) for all \( 1 \leq j \leq r \) and all \( 0 \leq i \leq f_j - 1 \).

Recall that, as usual, we see \( \mathcal{F}_{p^j} \) embedded in \( E \) via \( \sigma_0^{(j)} \) for \( 1 \leq j \leq r \); we can also write \( \mathcal{W}_{\tau^\text{alg}} = \bigotimes_{\sigma_0^{(j)}} \mathcal{W}_{\tau^\text{alg}} \) be a Hecke character compatible with \( \tau \).

Fix an isomorphism \( \mathcal{Q}_p \cong \mathbb{C} \), inducing an embedding \( E \hookrightarrow \mathbb{C} \). View \( W_{\tau^\text{alg}} \subseteq \mathcal{W}_{\tau^\text{alg}} \otimes_{\mathcal{E}} \mathbb{C} \) as a complex representation of \( \mathbb{D}(\mathbb{R}) := (D \otimes_{\mathbb{R}} \mathbb{R})^\times \subset \mathcal{D}(\mathbb{C}) \cong \mathcal{D}(\mathbb{Q}_p) \) (resp. of \( \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p^j}}) \)). Let \( \mathcal{W}_c := \mathcal{W}_{\tau^\text{alg}} \otimes_{\mathcal{E}} \mathbb{C} \) be the corresponding complex representation of \( \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p^j}}) \times \prod_{v \mid \infty} (\mathcal{O}_{D_v})^\times \).

Let \( U' \) be a compact open subgroup of \( (D \otimes_F \mathbb{A}_F^\times)^\times \) such that \( U' = \prod_{v \not\mid p} U'_v \), where \( U'_v = U_v \) if \( v \not\mid p \) and, for \( v \mid p \), \( U'_v \subseteq GL_2(\mathcal{O}_{F_{p^j}}) \) acts trivially on \( \mathcal{W}_{\tau^\text{alg}} \).

Denote by \( \mathcal{C}^\infty(D^\times \backslash (D \otimes_F \mathbb{A}_F^\times)^\times / U') \) the complex vector space of smooth functions \( f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\times)^\times \to \mathbb{C} \) which are invariant by the action of \( U' \). Let \( W_{\tau^c}^* \) be the \( \mathbb{C} \)-linear dual of \( W_{\tau^c} \).

Define a map:

\[
\alpha : S_{\tau,\psi}(U,E) \to \text{Hom}_{(D \otimes_{\mathbb{Q}} \mathbb{R})^\times} \left( W_{\tau^c}, \mathcal{C}^\infty(D^\times \backslash (D \otimes_F \mathbb{A}_F^\times)^\times / U') \right)
\]

by sending \( f \in S_{\tau,\psi}(U,E) \) to the assignment:

\[
w^* \mapsto (g \mapsto w^*(\tau^\text{alg}\tau^{-1}_c \tau^\text{alg}(g_p) f(g^\infty))
\]

38
where \( w^* \in W^*_c \) and \( g \in (D \otimes \mathbb{A}_F)^\times \). We have the following (cf. [15], 3.1.14):

**Proposition 4.9** The map \( \alpha \) identifies \( S_{\tau, \psi}(U, E) \otimes E \mathbb{C} \) with a space of automorphic forms for the group \( D^\times \) having central character \( \psi_C \) given by \( \psi_C(g) = \text{Nm}_{F/Q}(g_{\infty})^{1-w} \text{Nm}_{F/Q}(g_p)^{w-1} \psi(g^\infty) \) for \( g \in (D \otimes \mathbb{A}_F)^\times \).

If \( \pi = \bigotimes_v \pi_v \) is an irreducible automorphic representation for the group \( D^\times \), then \( \pi \) is generated by an element in \( \alpha(f)(W^*_C) \) for some \( f \in S_{\tau, \psi}(U, E') \), some \( U \) small enough and some \( E' \supseteq E \) big enough, if and only if \( \pi \approx W^*_\tau \) and \( \bigotimes_v \pi_v \) contains \( W^*_C \) as a representation of \( \prod_{j=1}^r GL_2(\mathcal{O}_{F_{\tau_j}}) \).

Assume furthermore that \( F/Q \) has even degree and that we chose \( \Sigma \) to be the empty set. Let \( \tau \) be a holomorphic weight with parameters \( (\vec{k}, w) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) and let \( \psi : \mathbb{A}_F^\times / F^\times \to \overline{\mathbb{Q}}_p^\times \) be a continuous character such that \( \psi(a) = (\text{Nm} a)^{1-w} \) for all \( a \) contained inside an open subgroup of \( (F \otimes \mathbb{Q}_p)^\times \). Fix an isomorphism \( \overline{\mathbb{Q}}_p \approx \mathbb{C} \) as before.

As a consequence of the classical Jacquet-Langlands theorem, we can identify the complexification of the space \( S_{\tau, \psi}(U, \overline{\mathbb{Q}}_p) \) \( (\vec{k} \neq \vec{2}) \) with a space of regular algebraic cuspidal automorphic representations \( \pi \) of \( GL_2(\mathbb{A}_F) \) such that \( \pi_{\infty} \) has weight \( (\vec{k}, w) \) and \( \pi \) has central character \( \psi_{\infty} \). If \( \vec{k} = \vec{2} \) the identification works if we consider, instead of \( S_{\tau, \psi}(U, \overline{\mathbb{Q}}_p) \), the quotient of \( S_{\tau, \psi}(U, \overline{\mathbb{Q}}_p) \) by the subspace of functions factoring through the reduced norm. For a detailed formulation of these last facts, cf. Theorem 2.1 of [11] and Lemma 1.3 of [24].

### 4.5 Holomorphic weight shiftings via generalized Dickson invariants and \( D \)-operators

Let \( q \) be a power of \( p \). The intertwining operators between \( F_q \)-representations of \( GL_2(F_q) \) studied in Section 3 allow us to produce weight shiftings between spaces of automorphic forms having holomorphic weights.

#### 4.5.1 Main theorem

Let us set \( A = \mathcal{O} \) and let \( D, U, (\tau, W_\tau) \) and \( \psi \) be as in 3.2. Recall in particular that \( U \) is small enough, and that \( \psi \) is compatible with \( \tau \). For simplicity, if \( \tau \) is a holomorphic weight with parameters \( (\vec{k}, w) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) and \( f \in S_{\tau, \psi}(U, \mathcal{O}) \), we also say that \( f \) has weight \( (\vec{k}, w) \) or, sometimes, that \( f \) has weight \( \vec{k} \). Recall that we write \( \vec{k} = (\vec{k}^{(1)}, ..., \vec{k}^{(r)}) \) with \( \vec{k}^{(j)} = (k^{(j)}_0, ..., k^{(j)}_{f_j}) \in \mathbb{Z}_{\geq 2}^{f_j} \) for \( 1 \leq j \leq r \), and that we define the vector \( \vec{w}^{(j)} = (w^{(j)}_0, ..., w^{(j)}_{f_j - 1}) \in \mathbb{Z}^{f_j} \) by the relations \( k^{(j)}_i + 2w^{(j)}_i - 1 = w \), for all \( 0 \leq i \leq f_j - 1 \).

**Theorem 4.10** Assume \( \tau \) is a holomorphic \( \mathcal{O} \)-linear weight with parameters \( (\vec{k}, w) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z} \) with \( w \) odd. Let \( f = \min\{f_1, ..., f_r\} \) and fix an integer \( \beta \) such that \( 1 \leq \beta \leq f \). For any integers \( i, j \) with \( 1 \leq j \leq r \) and \( 0 \leq i \leq f_j - 1 \) choose:

\[
a_i^{(j)} \in \{p^\beta - 1, p^\beta + 1\}.
\]
Set $\vec{a} = (\vec{a}^{(1)}, \ldots, \vec{a}^{(r)})$ with $\vec{a}^{(j)} = (a_0^{(j)}, \ldots, a_{f_j-1}^{(j)})$, and let $w' = w + (p^\beta - 1)$. Assume at least one of the following conditions is satisfied:

\((*)\) Let $j$ be any integer such that $1 \leq j \leq r$ and $\beta < f_j$. Then for any $i$ with $0 \leq i \leq f_j - 1$ and $a_i^{(j)} = p^\beta - 1$, we have that $2 < k_i^{(j)} \leq p + 1$, $2 \leq k_i^{(j)} + f_j - \beta \leq p + 1$ and if $i' \neq i$ is another integer such that $0 \leq i' \leq f_j - 1$ and $a_i^{(j)} = p^\beta - 1$, we also have $i \neq i' - \beta (\text{mod } f_j)$.

Let $j$ be any integer such that $1 \leq j \leq r$ and $\beta = f_j$. Then for any $i$ with $0 \leq i \leq f_j - 1$ and $a_i^{(j)} = p^\beta - 1$, we have that $2 < k_i^{(j)} \leq p + 1$.

\((**)\) The weight $(\vec{k}, w)$ is $p$-small and generic, i.e., $2 < k_i^{(j)} \leq p + 1$ for all $i, j$.

Let $\psi : (\mathbb{A}_F^\infty)^* / F^\times \to \mathcal{O}^\times$ be a Hecke character compatible with $\tau$. Then, if $\Omega$ is a Hecke eigensystem occurring in the space $S_{\tau, \psi}(U, \mathcal{O})$, there is a finite local extension of discrete valuation rings $\mathcal{O}' / \mathcal{O}$ and an $\mathcal{O}'$-valued Hecke eigensystem $\Omega'$ occurring in holomorphic weight $(\vec{k} + \vec{a}, w')$ and with associated Hecke character $\psi'$ such that:

$$\Omega' \equiv 1 \mod \mathcal{M}_{\mathcal{O}'}, \quad \psi' \equiv \tilde{\psi}.$$ 

The character $\psi'$ is compatible with the weight $(\vec{k} + \vec{a}, w')$ and it can be chosen so that $\tilde{\psi} = \bar{\tau}$. 

**Proof** Recall that $\tau$ is the $\mathcal{O}$-linear representation $\tau : \prod_{j=1}^r GL_2(\mathcal{O}_{F_{p_j}}) \to \text{Aut } W$, where $W = \bigotimes_{j=1}^r W_j$, $W_j = \bigotimes_{i=0}^{f_j - 1} (\text{Sym}^{k_i^{(j)} - 2} \mathcal{O}^2 \otimes \det w_i^{(j)})^{[i]}$, $k_i^{(j)} + 2w_i^{(j)} - 1 = w$. The group $GL_2(\mathcal{O}_{F_{p_j}})$ acts on $W_j$ via the action on $W_j$ induced by the embedding $\sigma_0^{(j)} : GL_2(\mathcal{O}_{F_{p_j}}) \to GL_2(\mathcal{O})$. The superscript $[i]$ indicates twisting by the $i$th power of the arithmetic Frobenius element of $\text{Gal}(E / \mathbb{Q}_p)$.

The $\mathbb{F}$-linear representation $W_j := W_j \otimes \mathbb{F}$ of $GL_2(\mathcal{O}_{F_{p_j}})$ factors through the reduction map $GL_2(\mathcal{O}_{F_{p_j}}) \to GL_2(\mathbb{F}_{p_j})$; using the notation introduced in Section 3.3.1 we can identify $W_j$ with the $\mathbb{F}[GL_2(\mathbb{F}_{p_j})]$-module

$$W_j = \bigotimes_{i=0}^{f_j - 1} \left( M_{k_i^{(j)} - 2} \otimes \det w_i^{(j)} \right)^{[i]},$$

where we see $GL_2(\mathbb{F}_{p_j}) \to GL_2(\mathbb{F})$ via $\sigma_0^{(j)}$, and the superscript $[i]$ indicates twisting by the $i$th power of the arithmetic Frobenius element of $\text{Gal}(\mathbb{F} / \mathbb{F}_p)$.

For any fixed integer $j$, $1 \leq j \leq r$, let $T_{f_j} = \{ i : a_i^{(j)} = p^\beta + 1 \}$ and $D_{f_j} = \{ i : a_i^{(j)} = p^\beta - 1 \}$. For $i \in T_{f_j}$ set $\vartheta_i^{(j)} := \Theta_{f_j - \beta}^{[i]}$ if $\beta < f_j$ and $\vartheta_i^{(j)} := \Theta^{[i]}$ if $\beta = f_j$, where $\Theta_{f_j - \beta}^{[i]}$ and $\Theta^{[i]}$ are the generalized Dickson invariants for the group $GL_2(\mathbb{F}_{p_j}) \simeq GL_2(\mathbb{F}_{p_{f_j}})$ as defined in 3.3.1 For $i \in D_{f_j}$ set $\delta_i^{(j)} := D_{f_j - \beta}^{[i]}$. 

40
if \( \beta < f_j \) and \( \sigma_j := D[\beta] \) if \( \beta = f_j \), where \( D_{f_j-\beta} \) and \( D[\beta] \) are the generalized \( D \)-operators for \( GL_2(\mathbb{F}_q) \) defined in \( \S 3.2 \). Set:

\[
\Lambda_j = \left( \bigotimes_{i \in T_j} \vartheta_i^{(j)} \right) \circ \left( \bigotimes_{i \in D_j} \delta_i^{(j)} \right),
\]

where the symbol \( \bigotimes \) denotes composition of functions, and each of the two composition factors above is computed by ordering \( T_j \) and \( D_j \) in the natural way. As seen in section \( \S 3 \), the operators \( \vartheta_i^{(j)} \) and \( \delta_i^{(j)} \) give rise to morphisms of \( \mathbb{F}_q[GL_2(\mathbb{F}_q)] \)-modules, and hence to morphisms of \( \mathbb{F}[GL_2(\mathbb{F}_q)] \)-modules via the scalar extension \( \bar{\sigma}_0^{(j)} : \mathbb{F}_q \rightarrow \mathbb{F} \). We deduce that \( \Lambda_j \) induces a \( GL_2(\mathbb{F}_q) \)-equivariant and \( \mathbb{F} \)-linear morphism:

\[
\Lambda_j : \bar{W}_j \rightarrow \bar{W}'_j,
\]

where \( \bar{W}'_j \) is the \( \mathbb{F}[GL_2(\mathbb{F}_q)] \)-module:

\[
\bar{W}'_j : = \bigotimes_{i \in T_j} \left( M_{k_i^{(j)}+(p^3+1)-2} \otimes \det w_i^{(j)} \right)^{[i]} \otimes \bigotimes_{i \in D_j} \left( M_{k_i^{(j)}+(p^3-1)-2} \otimes \det w_i^{(j)} \right)^{[i]}.
\]

Indeed, by Theorem \( \S 3.12 \), \( \Theta_{f_j-\beta}^{(i)} \) increases \( k_i^{(j)} \) by \( 1 \), \( k_i^{(j)} \) by \( p^3 \), \( w_i^{(j)} \) by \( -1 \), and does not change \( k_s^{(j)} \) for \( s \neq i, i + f_j - \beta \) or \( w_s^{(j)} \) for \( s \neq i \); \( \Theta[i] \) increases \( k_i^{(j)} \) by \( p^3 + 1 \), \( w_i^{(j)} \) by \( -1 \), and does not change \( k_s^{(j)} \) or \( w_s^{(j)} \) for \( s \neq i \). On the other hand, by Theorem \( \S 3.20 \), the operator \( D_{f_j-\beta}^{[i]} \) increases \( k_i^{(j)} \) by \( -1 \), \( k_i^{(j)} \) by \( p^3 \), and does not change \( k_s^{(j)} \) for \( s \neq i, i + f_j - \beta \) or \( w_s^{(j)} \) for \( s \neq i \); \( D[i] \) increases \( k_i^{(j)} \) by \( p^3 - 1 \), and does not change \( k_s^{(j)} \) for \( s \neq i \) or \( w_s^{(j)} \) for any \( s \).

By Theorem \( \S 3.12 \), \( \bigotimes_{i \in T_j} \vartheta_i^{(j)} \) is injective. If \( (*) \) is satisfied, the injectivity statement of Theorem \( \S 3.20 \) implies that \( \bigotimes_{i \in D_j} \delta_i^{(j)} \) is injective on \( \bar{W}_j \). The image of \( \bigotimes_{i = 0}^{f_j-1} \left( X^{k_i^{(j)}-2} \otimes 1 \right)^{[i]} \) in \( \bar{W}_j \) under \( \bigotimes_{i \in D_j} \delta_i^{(j)} \) is easily seen to be of the form \( \prod_{i \in D_j} (k_i^{(j)} - 2) \cdot u \) for some non-zero \( u \in \bar{W}_j \). If \( (**) \), holds, \( \prod_{i \in D_j} (k_i^{(j)} - 2) \) is non-zero in \( \mathbb{F} \) and, being \( \bar{W}_j \) an irreducible representation of \( GL_2(\mathbb{F}_q) \), we deduce that \( \bigotimes_{i \in D_j} \delta_i^{(j)} \) is injective on \( \bar{W}_j \). We conclude that under assumptions \( (*) \) or \( (**) \), all the maps \( \Lambda_j \) for \( 1 \leq j \leq r \) are injective.

Let \( b_i^{(j)} = -1 \) if \( i \in T_j \) and \( b_i^{(j)} = 0 \) if \( i \in D_j \). Define the \( \mathcal{O}[GL_2(\mathcal{O}_{\mathbb{F}_q})] \)-module:

\[
W_j' = \bigotimes_{i = 0}^{f_j-1} \left( \text{Sym}^{k_i^{(j)}+a_i^{(j)}-2} \mathcal{O}^2 \otimes \det w_i^{(j)} \right)^{[i]},
\]

so that \( W_j' \otimes_{\mathcal{O}} \mathbb{F} = \bar{W}_j' \) as \( \mathbb{F} \)-representations of \( GL_2(\mathcal{O}_{\mathbb{F}_q}) \) or, equivalently, of \( GL_2(\mathbb{F}_q) \). Set \( W' = \bigotimes_{j=1}^{r} W_j' \) and denote by \( \tau' \) the action of \( U \) on \( W' \) induced by the projection \( U \rightarrow \prod_{j=1}^{r} GL_2(\mathbb{F}_q) \). Let \( w' = w + (p^3 - 1) \); for
all the values of $i$ and $j$ for which the following integers are defined, we have $k_i^{(j)} + a_i^{(j)} \geq k_i^{(j)} \geq 2$ and:

$$
\begin{align*}
(k_i^{(j)} + a_i^{(j)}) + 2(w_i^{(j)} + b_i^{(j)}) - 1 \\
= (k_i^{(j)} + 2w_i^{(j)} - 1) + p^\beta - 1 \\
= w + (p^\beta - 1).
\end{align*}
$$

Therefore $\tau'$ is a holomorphic weight for automorphic forms on $D$ with parameters $(\vec{k} + \vec{a}, w') \in \mathbb{Z}^{g} \times \mathbb{Z}$.

The injections $\Lambda_j (1 \leq j \leq r)$ constructed above allow us to see $\tilde{W} = \bigotimes_{j=1}^{r} W_j$ as an $\mathbb{F}$-linear $U$-subrepresentation of $W' = \bigotimes_{j=1}^{r} W'_j$. Since $w$ is odd and $w \equiv w' \pmod{p-1}$, we can apply Proposition 4.8. We conclude that there exists a Hecke character $\psi' : (\mathbb{A}_F^\times / F^\times) \to \mathcal{O}_\mathcal{O}^\times$ compatible with $\tau'$ and such that $\tilde{\psi}' = \tilde{\psi}$; furthermore, for any Hecke eigensystem $\Omega$ occurring in $S_{\tau, \psi'}(U, \mathcal{O})$ there is a finite extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ with $\mathcal{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathcal{M}_{\mathcal{O}}$ and a Hecke eigensystem $\Omega'$ occurring in $S_{\tau', \psi'}(U, \mathcal{O}')$ such that $\Omega' \pmod{\mathcal{M}_{\mathcal{O}'}} = \Omega \pmod{\mathcal{M}_{\mathcal{O}}}$.

**Corollary 4.11** Under the same notation and assumptions of Theorem 4.10, any $\mathbb{F}_p$-linear continuous Galois representation arising from a Hecke eigenform in $S_{\tau, \psi'}(U, \mathcal{O})$, where $\tau$ is a holomorphic weight of parameter $\vec{k}$, also arises from an eigenform in $S_{\tau', \psi'}(U, \mathbb{Z}_p)$, where $\tau'$ is a holomorphic weight of parameters $\vec{k} + \vec{a}$ and $\psi'$ is some $\mathcal{O}_\mathcal{O}^\times$-valued Hecke character compatible with $\tau'$ and such that $\tilde{\psi}' = \tilde{\psi}$.

**Remark 4.12** We remark what follows:

1. Condition $(\ast)$ of Theorem 4.10 is true if, for example, for any $j$ with $1 \leq j \leq r$, there is at most one $i$, $0 \leq i \leq f_j - 1$, such that $a_i^{(j)} = p^\beta - 1$, and for these values of $i$ and $j$ we have $2 < k_i^{(j)} \leq p+1$ and $2 \leq k_{i+f_j-\beta}^{(j)} \leq p+1$.

2. The reason for which in the above result we limit $a_i^{(j)}$ to be in the set \{\$p^\beta - 1, p^\beta + 1\} for all $i, j$ is that we want to preserve the holomorphicity of the weights of the automorphic forms involved. More weight shiftings are possible using the generalized Dickson and $D$-operators if we do not impose the holomorphicity condition. On the other side, we will see in Section 4.5.2 that when $g > 1$ our operators allow more holomorphic weight shiftings than the ones described in the Theorem 4.10.

3. As a consequence of Remark 3.13 and Remark 3.21, the above result gives rise in general to more holomorphic weight shiftings than the ones obtained by the theory of generalized theta operators and Hasse invariants for geometric (mod $p$) Hilbert modular forms (cf. 4.7).
4.5.2 Analysis of the case \( f_j < 3 \)

We determine additional holomorphic weight shiftings using the generalized Dickson and \( D \)-operators. Since the combinatorics involved in the computations becomes very complicated as \( \max\{ f_1, ..., f_r \} \) grows, we assume that \( f_j < 3 \) for all \( j \). A procedure similar to the one described below could be applied in greater generality.

The most interesting cases for us arise when some of the residue degrees \( f_j \) equal two, so that we assume without loss of generality \( g = 2 \), \( p\mathcal{O}_F = \mathfrak{P} \) and \( [\mathbb{F}_p : \mathbb{F}_p] = 2 \). We maintain the notation introduced at the beginning of the section, but since \( r = 1 \) we drop the index \( j \) wherever it appeared before. Notice that we have \( E = \mathbb{F}_p \), \( \mathfrak{F} = \mathbb{F}_p \) and we can assume that \( \sigma_0 \) (resp. \( \overline{\sigma}_0 \)) is the identity automorphism of \( \mathbb{F}_p \) (resp. \( \mathbb{F}_p \)).

Fix \( \vec{k} = (k_0, k_1) \in \mathbb{Z}^2 \) and \( \vec{w} = (w_0, w_1) \in \mathbb{Z}^2 \) such that \( k_i + 2w_i - 1 = w \) (\( i = 0, 1 \)) with \( w \) odd. Let \((\tau, W)\) be the holomorphic \( \mathcal{O} \)-linear weight with parameters \((\vec{k}, \vec{w})\), so that the reduction modulo \( \mathfrak{P} \) of \( W \) is the \( \mathbb{F}_p[GL_2(\mathbb{F}_p)] \)-module:

\[
\overline{W} = (M_{k_0-2} \otimes \det^{w_0}) \otimes (M_{k_1-2} \otimes \det^{w_1})^{[1]}.
\]

Fix non-negative integers \( n, m, r, s, t, u, v, z \) and let:

\[
\Lambda = \Theta^{[1, u]} \circ \Theta^{[0, t]} \circ \Theta^{[1, m]} \circ \Theta^{[0, n]} \circ D^{[1, z]} \circ D^{[0, r]} \circ D^{[1, s]} \circ D^{[0, t]},
\]

where the above operators are defined as in [3.3.1] and [3.3.2]. \( \Lambda \) defines a \( \mathbb{F}_p[GL_2(\mathbb{F}_p)] \)-homomorphism having source \( \overline{W} \) as long as \( r + 2 \leq k_0 \) and \( s + 2 \leq k_1 + pr \); we assume therefore:

\[
\left\{ \begin{array}{l}
r + 2 \leq k_0 \\
s + 2 \leq k_1.
\end{array} \right.
\]

\( (\ddagger') \)

If \( (\ddagger') \) holds, we have \( \Lambda : \overline{W} \rightarrow W' \), where:

\[
W' = (M_{k'_0-2} \otimes \det^{w'_0}) \otimes (M_{k'_1-2} \otimes \det^{w'_1})^{[1]},
\]

with:

\[
\begin{align*}
k'_0 &= k_0 + n + pm - r + ps + (p^2 + 1)t + (p^2 - 1)v \\
k'_1 &= k_1 + pm + m + pr - s + (p^2 + 1)u + (p^2 - 1)z \\
w'_0 &= w_0 - n - t + \alpha(p^2 - 1) \\
w'_1 &= w_1 - m - u + \beta(p^2 - 1).
\end{align*}
\]

Here \( \alpha, \beta \) can be chosen to be any integers, as \( \det^{p^2-1} = 1 \) on \( GL_2(\mathbb{F}_p) \).

Assume that the following are satisfied:

\( (A) \) relations \( (\ddagger') \) hold and \( \Lambda \) is injective;

\( (B) \) \( k'_0, k'_1 \geq 2 \);
(C) \( k'_0 + 2w'_0 - 1 = k'_1 + 2w'_1 - 1 =: w' \); 

(D) \( w \equiv w'(\text{mod } p - 1) \).

Then we can apply Proposition [44] to obtain holomorphic weight shiftings for Hecke eigensystems associated to automorphic forms on \( D \). We therefore want to translate the above four conditions into relations between the integral parameters \( k, w, \alpha, \beta, u, m, r, s, t, u, v, z \).

We easily see that:

\[
\begin{align*}
\bar{k}'_0 + 2w'_0 - 1 &= w - n + pm - r + ps + (p^2 - 1)(t + v + 2\alpha) \\
\bar{k}'_1 + 2w'_1 - 1 &= w + pm - m + pr - s + (p^2 - 1)(u + z + 2\beta),
\end{align*}
\]

so that condition (C) is equivalent to:

\[
(m - n) + (s - r) + (p - 1)((t - u) + (v - z) + 2(\alpha - \beta)) = 0. \tag{♠}
\]

Computing \( m \) and \( r \) from (♣) we obtain:

\[
\begin{align*}
k'_0 &= k_0 + (p + 1)(n + t) + (p - 1)(r + v) + p(p - 1)(u + z + 2(\beta - \alpha)) \\
k'_1 &= k_1 + (p + 1)(m + u) + (p - 1)(s + z) + p(p - 1)(t + v + 2(\alpha - \beta)) \\
w' &= w + (p - 1)(n + t + r + v + 2\alpha + p(u + z + 2\beta)).
\end{align*}
\]

Condition (D) is then automatically satisfied, and (B) holds if \( \alpha = \beta \). If \( r = s = v = z = 0 \), condition (A) is satisfied, as the generalized Dickson invariants induce injective morphisms of \( \mathbb{F}_q[GL_2(\mathbb{F}_q)] \)-modules, and (♠') is then a consequence of \( k_0, k_1 \geq 2 \).

We claim that if \( 2 < k_0, k_1 \leq p + 1 \) and the non-negative integers \( r, s, v, z \) satisfy:

\[
\begin{align*}
\begin{cases}
  r + v + 2 &\leq k_0 \\
s + z + 2 &\leq k_1,
\end{cases}
\end{align*}
\]

then (A) holds. (Notice that (♠) implies (♠')). To prove this, first observe that if \( 2 < k_0, k_1 \leq p + 1 \), then \( \tilde{W} \) is irreducible for the action of \( GL_2(\mathbb{F}_q) \), so we only need to show that under the above assumptions \( \Lambda \neq 0 \). Write \( a = k_0 - 2 \) and \( b = k_1 - 2 \) and denote for simplicity the element \( (X^a \otimes 1) \otimes (X^b \otimes 1)^{[1]} \) of \( \tilde{W} \) by \( X^a \otimes X^b \). We have:

\[
\left( D^{[1], z} \circ D^{[0], v} \circ D_1^{[1], s} \circ D_1^{[0], r} \right) (X^a \otimes X^b) = c \cdot X^{a-r+ps+(p^2-1)v} \otimes X^{b+pr-s+(p^2-1)z}, \tag{♣}
\]

where:

\[
c = \frac{a!}{(a-r)!} \cdot \frac{(b+pr)!}{(b+pr-s)!} \cdot \frac{(a-r+ps)!}{(a-r+ps-v)!} \cdot \frac{(b+pr-s)!}{(b+pr-s-z)!} \pmod{p}.
\]

The exponents in the right hand side of (♣) and the integers in the above formula for \( c \) are non-negative under the assumption (♠). Since \( 0 < a, b \leq p - 1 \), (♠) also
implies that $p$ does not divide the integer $\frac{a!}{(a-r)^!} \cdot \frac{(b+r)^!}{(b-s+r)^!}$. Assume $r + v \leq a$ and $v > 0$; if $p$ divided the integer:

$$\frac{(a - r + ps)!}{(a - r + ps - v)!} = \prod_{j=0}^{v-1} (a - r + ps - j),$$

then $p$ would divide $a - r - j$ for some $0 \leq j \leq v - 1$, which is impossible as $1 \leq a - r - j \leq p - 1$. Similarly, if $s + z \leq b$ we see that $p$ does not divide $\frac{(b-s+pr-z)^!}{(b-s+pr-z)^!}$.

We conclude that $c \neq 0$ and hence $D^{[1],z} \circ D^{[0],u} \circ D^{[1],s} \circ D^{[0],r} \neq 0$. The injectivity of the generalized Dickson invariants implies then the claim.

Let us set $A = \mathcal{O} = \mathcal{O}_{F_p}$ and let $D, U$ be as in \[4.2\] The above considerations and Proposition \[4.8\] prove the following:

\textbf{Theorem 4.13} Assume $g = f = 2$ and let $\tau$ be an $\mathcal{O}$-linear holomorphic weight for automorphic forms on $D$ of parameters $(k_0, k_1; w) \in \mathbb{Z}_{\geq 2}^2 \times \mathbb{Z}$ with $w$ odd; let $\psi$ be a Hecke character compatible with $\tau$. Fix $\alpha \in \mathbb{Z}$ and non-negative integers $n, m, r, s, t, u, v$ and $z$. Assume at least one of the following two conditions is satisfied:

1. $r = s = v = z = 0$;
2. $2 < k_0, k_1 \leq p + 1$, and $r + v \leq k_0 - 2$, $s + z \leq k_1 - 2$.

Assume furthermore that the relation:

\[ (m-n) + (s-r) = (p-1) \cdot ((u-t) + (z-v)) \]  

(\textup{\large \(\spadesuit\)})

holds. Define:

\[
\begin{aligned}
k'_0 &= k_0 + (p+1)(n+t) + (p-1)(r+v) + p(p-1)(u+z) \\
k'_1 &= k_1 + (p+1)(m+u) + (p-1)(s+z) + p(p-1)(t+v) \\
w' &= w + (p-1)(n+t+r+v+2\alpha) + p(u+z+2\alpha).
\end{aligned}
\]

Then if $\Omega$ is a Hecke eigensystem occurring in $S_{r,v}(U, \mathcal{O})$, there is a finite local extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ and an $\mathcal{O}'$-valued Hecke eigensystem $\Omega'$ occurring in holomorphic weight $(k'_0, k'_1; w')$ and with associated Hecke character $\psi'$ such that:

$\Omega' \mod \mathfrak{m}_{\mathcal{O}'} = \Omega \mod \mathfrak{m}_{\mathcal{O}}$.

The character $\psi'$ is compatible with the weight $(k'_0, k'_1; w')$ and it can be chosen so that $\psi' = \psi$.

\textbf{Remark 4.14} Many of the weight shiftings produced by Theorem \[4.13\] do not arise from Theorem \[4.10\] or from the operators described in \[4.7\].
5 Shiftings for weights containing (2, ..., 2)-blocks

While the generalized Dickson invariants induce injective maps on the trivial \( F \)-representation of \( GL_2(F_{\mathfrak{q}_p}) \), the \( D \)-operators are identically zero on this module. Starting with automorphic forms whose weight contains a (2, ..., 2)-block (cf. definition in \[4\]), we can then produce weight shiftings through the operators \( \Theta_\alpha^{[\beta]} \) but we cannot always successfully use the operators \( D_\alpha^{[\beta]} \). On the other side, the study of weight shiftings "by \( p - 1 \)" for automorphic forms whose weight contains a (2, ..., 2)-block is motivated by the weight part of Serre’s modularity conjecture for totally real fields (cf. Remark \[5.4\] below).

In this section we present a result of Edixhoven and Khare (cf. \[7\]) to produce weight shiftings "by \( p - 1 \)" starting from forms whose weight is not necessarily parallel but contains (2, ..., 2)-blocks relative to some primes of \( F \) above \( p \). We always assume that \( p > 2 \) is unramified in the totally real number field \( F \).

We keep the notation introduced in \[4\] and we furthermore assume that \( F \) has even degree over \( \mathbb{Q} \) and that the quaternion \( F \)-algebra \( D \) is ramified at all and only the infinite places of \( F \), i.e., \( \Sigma = \emptyset \). We fix an isomorphism \( (D \otimes_F \mathbb{A}_F^\infty)^{\times} \cong GL_2(\mathbb{A}_F^\infty) \).

The symbols \( F \), \( U \), \( (\tau, W_\tau) \), \( \psi \), \( S \) and \( T_{S, \mathfrak{f}}^{\text{univ}} \) will have the same meaning as in \[4, 2\]. We assume that \( \tau \) is a (non necessarily holomorphic) \( F \)-linear weight with parameters \( (\vec{k}, \vec{\omega}) \in \mathbb{Z}_{\geq 2}^2 \times \mathbb{Z}_\ell \), where \( \vec{k} = (\vec{k}^{(1)}, ..., \vec{k}^{(r)}) \) and \( \vec{k}^{(j)} = (k_0^{(j)}, ..., k_j^{(j)}) \in \mathbb{Z}_{\geq 2}^j \); \( \vec{\omega} = (\omega^{(1)}, ..., \omega^{(r)}) \) and \( \vec{\omega}^{(j)} = (\omega_0^{(j)}, ..., \omega_j^{(j)}) \in \mathbb{Z}_\ell^j \), for \( 1 \leq j \leq r \).

We write \( W_\tau = \bigotimes_{j=1}^r W_{\tau_j} \), where \( W_{\tau_j} \) is the \( F \)-representation of \( GL_2(\mathcal{O}_{F_{\mathfrak{q}_j}}) \) defined by:

\[
W_{\tau_j} = \bigotimes_{i=0}^{j-1} \left( \text{Sym}^{k_i^{(j)}-2} \mathbb{F}^2 \otimes \det^{\omega_i^{(j)}} \right)^{[i]}.
\]

If the weight \( \tau \) is holomorphic, it is also determined by the pair \( (\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^2 \times \mathbb{Z}_\ell \), where \( k_i^{(j)} + 2\omega_i^{(j)} - 1 = \omega_i \) for all \( i \) and \( j \).

Choose a prime \( \mathfrak{q} \) of \( F \) above \( p \) and let \( \varpi \) be a fixed choice of uniformizer for the ring of integers of the completion of \( F \) at \( \mathfrak{q} \). We can assume, up to relabeling, that \( \mathfrak{q} = \mathfrak{q}_1 \). Define the matrix of \( GL_2(F_{\mathfrak{q}_1}) \):

\[
\Pi = \begin{pmatrix}
1 & 0 \\
0 & \varpi
\end{pmatrix},
\]

and view it as an element of \( GL_2(\mathbb{A}_F^\infty) \) whose components away from \( \mathfrak{q}_1 \) are trivial.

If \( g \) is an element of \( GL_2(\mathbb{A}_F^\infty) \) and \( Q \) is a finite set of finite places of \( F \), we denote by \( g^Q \) the element of \( GL_2(\mathbb{A}_F^\infty) \) whose components at each place of \( Q \) are trivial, and whose components away from \( Q \) coincide with those of \( g \). We let \( g_Q = g^Q \). A similar convention is used for subgroups of \( GL_2(\mathbb{A}_F^\infty) \) which are products of subgroups of \( GL_2(F_v) \) for \( v \) varying over the finite places of \( F \). In particular, by assumption we have \( U_p = GL_2(\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p) \).
We denote the action by right translation of $GL_2(\mathbb{A}_F)$ on $S_{\tau,\psi}(U, F)$ by a dot. Set:

$$U_0 = \left\{ u \in U : u \mathfrak{p}_1 \equiv \left( \begin{array}{cc} \ast & \ast \\ 0 & \ast \end{array} \right)(\mod \varpi) \right\}.$$ 

By restricting $\tau$ to $U_0$, we define $S_{\tau,\psi}(U_0, F)$ as in Definition [41], notice that the level of the automorphic forms belonging to this space is not prime-to-$p$.

We have the following result, which is a not-prime-to-$p$ version of Lemma 3.1 of [24]:

**Lemma 5.1** Assume that $\tau$ is an irreducible (non necessarily holomorphic) $F$-linear weight with parameters $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^2 \times \mathbb{Z}^9$ such that $\vec{k}(1) = \vec{2}$. Then the map:

$$\alpha : S_{\tau,\psi}(U, F) \oplus S_{\tau,\psi}(U, F) \longrightarrow S_{\tau,\psi}(U_0, F)$$

defined by:

$$(f_1, f_2) \mapsto f_1 + \Pi \cdot f_2$$

is a Hecke-equivariant $F$-morphism whose kernel is Eisenstein, i.e., the localization $(\ker \alpha)_{\mathfrak{m}}$ vanishes for all maximal ideals $\mathfrak{m}$ of $T_{S, \mathfrak{B}}^{\text{univ}}$ which are non-Eisenstein.

**Proof** It is straightforward to check that $\alpha$ is well defined, using the fact that $GL_2(\mathcal{O}_{F_{\mathfrak{p}_1}})$ acts on $W_{\tau,\psi}$ via an integral power of the determinant character raised to the power of $(\mod \varpi)$

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**Proof** It is straightforward to check that $\alpha$ is well defined, using the fact that $GL_2(\mathcal{O}_{F_{\mathfrak{p}_1}})$ acts on $W_{\tau,\psi}$ via an integral power of the (mod \varpi) determinant character. Also, $\alpha$ is equivariant for the action of the algebra $T_{S, \mathfrak{B}}^{\text{univ}}$.

Write $\Pi U \Pi^{-1} = U^{\mathfrak{p}_1} \times \Pi GL_2(\mathcal{O}_{F_{\mathfrak{p}_1}}) \Pi^{-1}$. Define an $F$-linear action of the subgroup $\Pi U \Pi^{-1}$ of $GL_2(\mathbb{A}_F)$ on $W_\tau$ by letting $U^{\mathfrak{p}_1}$ act on $\bigotimes_{j=0}^l \mathcal{W}_j$ via the restriction of $\tau$ to $U^{\mathfrak{p}_1}$, and by letting $\Pi GL_2(\mathcal{O}_{F_{\mathfrak{p}_1}}) \Pi^{-1}$ act on $W_{\tau,\psi}$ via the reduction modulo $\varpi$ of the determinant character raised to the power of $\sum_{j=0}^{l-1} w_j(1)$. Observe that this action is compatible with the given action $\tau$ of $U$ on $W_\tau$.

If $(f_1, f_2) \in \ker \alpha$, we see that $f_1(\mathfrak{g}u) = u^{-1} f_1(g)$ for all $u$ in $U$ and all $u$ in $\Pi U \Pi^{-1}$, so that $f_1(\mathfrak{g}u) = u^{-1} f_1(g)$ for every $u$ in $SL_2(\mathcal{O}_{\mathfrak{p}_1})U \subset GL_2(\mathbb{A}_F)$. Here $SL_2(\mathcal{O}_{\mathfrak{p}_1})$ acts on $W_{\tau,\psi}$ trivially.

Assume that $W_{\tau} \neq \{0\}$, i.e., that $W_{\tau} = F$ is the trivial representation of $U$. If $(f_1, f_2) \in \ker \alpha$, then $f_1$ is invariant under right translations by elements of $D^X U$; strong approximation for $SL_2$ then implies that $f_1$ is invariant under right translations by any element of $SL_2(\mathbb{A}_F)$, and hence it factors through the reduced norm map $D^X \backslash (D \otimes_F \mathbb{A}_F)^X \rightarrow F^X \backslash (\mathbb{A}_F)^X$. Since any maximal ideal of $T_{S, \mathfrak{B}}^{\text{univ}}$ in the support of the space of functions $D^X \backslash (D \otimes_F \mathbb{A}_F)^X \rightarrow W_{\tau}$ factoring through the reduced norm is Eisenstein, we obtain the desired result.

Assume now that $W_{\tau} = \{0\}$ and let $(f_1, f_2) \in \ker \alpha$. Using strong approximation, we see that for any $g \in GL_2(\mathbb{A}_F)$ and $u \in \Pi_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{p}_j}})$ we can find an element $\delta \in D^X \cap g SL_2(\mathcal{O}_{\mathfrak{p}_j})U g^{-1}$ such that for all $j = 1, \ldots, r$:

$$g_{\mathfrak{p}_j}^{-1} \delta g_{\mathfrak{p}_j} \in \mathfrak{w}_{\mathfrak{p}_j} + M_2(\mathfrak{P}_j).$$
In particular, we obtain:

\[ f_1(g) = f_1(\delta^{-1}g) = f_1(g^{-1}g) \]

and, since \( g^{-1}\delta^{-1}g \in SL_2(F_{\mathfrak p_1})U \):

\[ f_1(g) = (g^{-1}\delta g) f_1(g) = uf_1(g). \]

Since \( u \) is arbitrary, we conclude that \( f_1(g) \in W^U_\tau \) for any \( g \in GL_2(\mathbb A_F) \), so that \( f_1 = 0, f_2 = 0 \) and \( \alpha \) is injective. \( \blacksquare \)

Let \( \mathcal F_\tau \) denote the space consisting of all the functions \( f : D^\times \setminus (D \otimes_F \mathbb A_F)^\times \to W_\tau \), and define a left \( \mathbb F \)-linear action of \( U \) on \( \mathcal F_\tau \) by:

\[(uf)(g) = \tau(u)f(gu)\]

for all \( u \in U, \ g \in (D \otimes_F \mathbb A_F)^\times \) and \( f \in \mathcal F_\tau \). Set:

\[ S_\tau(U, \mathbb F) = H^0(U, \mathcal F_\tau). \]

In what follows, we work for simplicity with the spaces \( S_\tau(U, \mathbb F) \), forgetting about the action of the center of \( (D \otimes_F \mathbb A_F)^\times \) on \( \mathcal F_\tau \).

Following the proof of Proposition 1 at page 48 of [7], and using Lemma 5.1, we obtain the following result:

**Theorem 5.2** Assume that \( \tau \) is an irreducible (non necessarily holomorphic) \( \mathbb F \)-linear weight with parameters \( (\vec{k}, \vec{w}) \in \mathbb Z_{\geq 2}^r \times \mathbb Z^r \) such that \( \vec{k}(j) = \vec{2} \) for some \( 1 \leq j \leq r \). Let \( \tau' \) be the \( \mathbb F \)-linear weight associated to the parameters \( \vec{w}' = (\vec{k}(1), \ldots, \vec{k}(j) + p - 1, \ldots, \vec{k}(r)) \) and \( \vec{w}'' = \vec{w} \). For any non-Eisenstein maximal ideal \( \mathfrak M \) of \( T_{S_{\mathbb F}} \), there is an injective Hecke-equivariant \( \mathbb F \)-morphism:

\[ S_\tau(U, \mathbb F)[\mathfrak M] \hookrightarrow S_{\tau'}(U, \mathbb F)[\mathfrak M]. \]

**Proof** Assume without loss of generality that \( j = 1 \). Via the surjection \( U \to GL_2(F_{\mathfrak p_1}) \), the group \( U \) acts on the \( F_{\mathfrak p_1} \)-points \( \mathbb P^1(F_{\mathfrak p_1}) \) of the projective \( F_{\mathfrak p} \)-line, and we can identify the coset space \( U \setminus \mathbb P^1 \) with \( \mathbb P^1(F_{\mathfrak p_1}) \). Recall that we are viewing \( F_{\mathfrak p_1} \) as a subfield of \( \mathbb F \) via the fixed embedding \( \sigma_0^{(1)} \).

By Shapiro’s lemma applied to the pair \((U, \mathbb P^1)\) and the left \( \mathbb F[\mathbb P^1] \)-module \( \mathcal F_\tau \), we obtain an isomorphism:

\[ H^0(U, \mathcal F_\tau) \sim H^0(U, \mathcal F_\tau \otimes \mathbb F[\mathbb P^1(F_{\mathfrak p_1})]). \]  

Here \( U \) acts on \( \mathbb F[\mathbb P^1(F_{\mathfrak p_1})] = \{ \varphi : \mathbb P^1(F_{\mathfrak p_1}) \to \mathbb F \} \) via its quotient \( GL_2(F_{\mathfrak p_1}) \) and by the rule \( (u \varphi)(P) = \varphi(u^{-1}P) \) for \( u \in GL_2(F_{\mathfrak p_1}) \) and \( P \in \mathbb P^1(F_{\mathfrak p_1}) \). Furthermore \( U \) acts diagonally on \( \mathcal F_\tau \otimes \mathbb F[\mathbb P^1(F_{\mathfrak p_1})] \). By Lemma 1.1.4 of [2], the isomorphism \( (1) \) preserves the Hecke action on both sides.

By Lemma 2.6 of [16], there is an isomorphism of \( \mathbb F[GL_2(F_{\mathfrak p_1})] \)-modules:

\[ \mathbb F[\mathbb P^1(F_{\mathfrak p_1})] \simeq \mathbb F \oplus \text{Sym}^{6_{\mathfrak p_1}-1}(\mathbb F^2) = M_0 \oplus M_{6_{\mathfrak p_1}-1}. \]
inducing a surjection:

\[ H^0(U, F_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_p)]) \rightarrow H^0(U, F_\tau \otimes M_{p^{j-1}}). \]  

(2)

Observe that the composition of the restriction map \( H^0(U, F_\tau) \rightarrow H^0(U_0, F_\tau) \) with the surjection:

\[ H^0(U_0, F_\tau) \simeq H^0(U, F_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_p)]) \rightarrow H^0(U, F_\tau) \]

is given by \( f \mapsto \frac{1}{\prod_{i \in I_0} (1 \otimes u f)} \sum_{u \in U_0} 1 \otimes u f = 1 \otimes f \). This implies that the first summand of \( H^0(U, F_\tau) \otimes \mathbb{F} \) is identified via the map \( \alpha \) of Lemma [5.1] and the Shapiro isomorphism with the direct summand \( H^0(U, F_\tau) \) of \( H^0(U, F_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_p)]) \).

Using the map \( \alpha \), the Shapiro isomorphism, the projection [2], and the isomorphism of \( \mathbb{F}[GL_2(\mathbb{F}_p)] \)-modules \( M_{p^{j-1}} \simeq \bigotimes_{i=0}^{j-1} M_{p^{[i]}} \), we obtain a Hecke equivariant morphism:

\[ \beta : H^0(U, F_\tau) \otimes \mathbb{F} \rightarrow H^0(U, F_\tau \otimes \bigotimes_{i=0}^{j-1} M_{p^{[i]}}). \]

By Lemma [5.1] precomposing \( \beta \) with the injection \( H^0(U, F_\tau) \hookrightarrow H^0(U, F_\tau) \otimes \mathbb{F} \) given by \( f \mapsto (0, f) \) we obtain a Hecke equivariant injective morphism:

\[ H^0(U, F_\tau)_{\mathfrak{m}} \hookrightarrow H^0(U, F_\tau \otimes \bigotimes_{i=0}^{j-1} M_{p^{[i]}})_{\mathfrak{m}} \]  

(3)

for any non-Eisenstein maximal ideal \( \mathfrak{m} \) of \( T^{\text{univ}}_{S,F} \).

Let \( \tilde{\mathbf{k}} = (\tilde{p} + \tilde{1}, \tilde{p}^{(2)}, \ldots, \tilde{p}^{(r)}) \) and set \( \tilde{\omega} = \tilde{\omega}^\prime \). Observe that if \( \tau' \) is the representation of \( U \) associated to the parameters \( (\tilde{\mathbf{k}}, \tilde{\omega}) \) then \( W_{\tau'} \simeq W_{\tau} \otimes_{\mathbb{F}} \bigotimes_{i=0}^{j-1} M_{p^{[i]}} \).

The \( U \)-equivariant map \( F_\tau \otimes \bigotimes_{i=0}^{j-1} M_{p^{[i]}} \rightarrow F_{\tau'} \) induced by the assignment:

\[ f \otimes m \mapsto [g \mapsto f(g) \otimes m] \]

for \( g \in D^X \setminus (D \otimes_{\mathbb{F}} \mathbb{K}_{\mathfrak{F}}) \) is injective. We deduce that for any non-Eisenstein maximal ideal \( \mathfrak{m} \) of \( T^{\text{univ}}_{S,F} \), there is a Hecke equivariant monomorphism:

\[ H^0(U, F_\tau \otimes \bigotimes_{i=0}^{j-1} M_{p^{[i]}})_{\mathfrak{m}} \hookrightarrow H^0(U, F_{\tau'})_{\mathfrak{m}}. \]

Combining this with (3), we are done. \( \blacksquare \)

**Remark 5.3** Under the assumptions of the above theorem, \( \tau' \) is an irreducible representation of \( U \). This implies that, if the number of indices \( j \) such that \( \tilde{k}^{(j)} = 2 \) is larger than one, Theorem 5.2 can be further applied to obtain weight shiftings ”in blocks” by \( \frac{1}{p - 1} \).

**Remark 5.4** The content of Theorem 5.2 generalizes Lemma 4.6.8 of [3], which is proved in loc. cit. via Lemma 1.5.5 of [14].
The weight shifting produced by Theorem 5.2 is not in general of holomorphic type: for example, if \( r > 1 \) and \( \tau \) is holomorphic, then \( \tau' \) is never holomorphic. Nevertheless we have:

**Corollary 5.5** Assume that \( \tau \) is the irreducible holomorphic \( \mathbb{F} \)-linear weight with parameters \( (\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times (2\mathbb{Z} + 1) \). Let \( \tau' \) be the holomorphic weight associated to the parameters \( (p + \vec{1}, w + (p - 1)) \in \mathbb{Z}_{\geq 2}^g \times (2\mathbb{Z} + 1) \). For any non-Eisenstein maximal ideal \( \mathfrak{M} \) of \( T_{S, F}^{\text{univ}} \), there is an injective Hecke-equivariant \( \mathbb{F} \)-morphism:

\[
S_{\tau}(U, \mathbb{F})_{\mathfrak{M}} \hookrightarrow S_{\tau'}(U, \mathbb{F})_{\mathfrak{M}}.
\]

**Proof** Fix a non-Eisenstein maximal ideal \( \mathfrak{M} \) of \( T_{S, F}^{\text{univ}} \). Applying Theorem 5.2 \( r \) times we obtain a Hecke equivariant injection \( S_{\tau}(U, \mathbb{F})_{\mathfrak{M}} \hookrightarrow S_{\tau'}(U, \mathbb{F})_{\mathfrak{M}} \), where \( \tau' \) is the irreducible \( \mathbb{F} \)-linear weight with parameters \( (\vec{k}', w') \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g \) and each component of \( \vec{v}' \) equals the integer \( \frac{w - 1}{2} \). This weight is holomorphic with parameters \( (p + \vec{1}, w + (p - 1)) \).

In terms of Galois representations we obtain:

**Corollary 5.6** Assume that \( \tau \) is an irreducible (non necessarily holomorphic) \( \mathbb{F} \)-linear weight with parameters \( (\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g \) such that \( \vec{k}(j) = \vec{2} \) for some \( 1 \leq j \leq r \). Then an irreducible continuous representation \( \rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\bar{F}_p) \) arising from an automorphic eigenform on \( S_{\tau}(U, \mathbb{F}) \) also arises from an automorphic eigenform on \( S_{\tau'}(U, \mathbb{F}) \), where \( \tau' \) is the irreducible weight associated to the parameters \( \vec{k}' = (\vec{k}(1), ..., \vec{k}(j) + p - \vec{1}, ..., \vec{k}(r)) \) and \( w' = w \).

The Jacquet-Langlands correspondence and Corollary 5.5 imply the following (cf. [2]):

**Corollary 5.7** An irreducible continuous representation \( \rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\bar{F}_p) \) arising from a holomorphic Hilbert modular form of level \( U \subset GL_2(\mathbb{A}_F) \) and parallel weight \( \vec{2} \) also arises from a holomorphic Hilbert modular form of level \( U \) and parallel weight \( p + \vec{1} \).

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