ON DEFINING CHARACTERISTIC REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

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ABSTRACT. We give parameterizations of the irreducible representations of finite groups of Lie type in their defining characteristic.

1. INTRODUCTION

We consider series of finite groups of Lie type which are specified by a root datum and a finite order automorphism of that root datum. For each power $q$ of a prime $p$ this determines a connected reductive algebraic group $G$ over $\overline{\mathbb{F}}_p$ (an algebraic closure of the field with $p$ elements) and a group of fixed points $G^F$ of a Frobenius morphism $F : G \to G$, up to isomorphism.

We are interested in a parameterization of the irreducible modules of $G^F$ over $\overline{\mathbb{F}}_p$.

A known solution to this task is to use that the groups $G^F$ are groups with a split $(B,N)$-pair of characteristic $p$. There exists a parameterization of the irreducible representations over an algebraically closed field of characteristic $p$ of such groups. For details we refer to the description by Curtis in [2, B, Thm.5.7]. In our setup it would be very technical to construct the data for this parameterization from the given root datum with Frobenius action. That parameterization looks very different for different Frobenius actions on the same algebraic group.

In the literature on representations of connected reductive algebraic groups and finite groups of Lie type in their defining characteristic most authors restrict their descriptions to the case of simply connected algebraic groups and the finite groups of Lie type arising from these.

In this case there is a nice combinatorial parameterization of the absolutely irreducible modules of the algebraic group by the set of dominant weights. The irreducible modules of the finite groups are restrictions of those of the algebraic group and Steinberg [19] described a nice subset of dominant weights which yields representatives of the isomorphism classes of these modules. A generalization to connected reductive groups with simply connected derived group can be found in [11, App.1.3].

Jantzen considers in [14] general connected reductive algebraic groups but does not consider the finite groups of Lie type. In the general case it is no longer true that all irreducible representations of the finite groups are restrictions from the algebraic group.

In this paper we give a parameterization of the irreducible representations in defining characteristic for arbitrary finite groups of Lie type. It is very concrete and computable starting from the given root datum for the algebraic group and Frobenius action on the root datum. The description will not become more complicated for twisted Frobenius actions.

Here is an overview of the content of the other sections of this paper. Section 2 contains a description of our setup. We describe how root data and Frobenius actions on root data...
can be represented and how to compute certain related data. Some of the results in this section may be of independent interest. For example, we describe a construction of a certain covering group of an arbitrary connected reductive group, which generalizes the well known simply connected coverings of semisimple groups (see 2.5).

In section 3 we first recall the results about defining characteristic representations of the algebraic groups and the finite groups of Lie type arising from simply connected semisimple groups which we have mentioned above. Then we state our main result in theorem 3.5 where we consider arbitrary finite groups of Lie type. In the end of that section we work out an example in some detail (certain centralizers of semisimple elements in exceptional groups of type $E_8$).

In section 4 we give a more detailed description of the parameter sets in our main theorem for finite groups of Lie type arising from any simple connected reductive group. As an application of these results we work out the number of semisimple conjugacy classes for all of these finite groups. The results of this application were obtained before by the first named author with a completely different proof. The new proof given here is more elementary.

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2. Root data for finite groups of Lie type

2.1. Connected reductive algebraic groups. Let $G$ be a connected reductive group over an algebraically closed field $k$. We recall how $G$ is determined by a root datum, for more details we refer to [18, 7.4, 9.6].

For each maximal torus $T$ of $G$ there is an associated root datum $\Psi = (X, R, Y, R^\vee)$ which together with $k$ determines $G$ up to isomorphism. Here, $X$ and $Y$ are the character and cocharacter groups of $T$, respectively, both isomorphic to $\mathbb{Z}^r$ for some $r$ called the rank of $G$ (or of $T$ or of $\Psi$). These are in duality via a natural pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$. Here $R$ is a finite subset of $X$, called the roots. There is a bijection $\vee : R \to R^\vee \subset Y$, $\alpha \mapsto \alpha^\vee$, to the set $R^\vee$ of coroots, such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$.

Each $\alpha \in R$ defines reflections $s_\alpha : X \to X$, $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$, and $s_\alpha^\vee : Y \to Y$, $y \mapsto y - \langle \alpha, y \rangle \alpha^\vee$. The group $W$ generated by all $s_\alpha$ is called the Weyl group of $G$ or $\Psi$, it is isomorphic to the group $W^\vee$ generated by the $s_\alpha^\vee$. We have $RW = R$ and $R^\vee W^\vee = R^\vee$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset R$ be a set of simple roots, that is each root is a linear combination of simple ones with either non-negative or non-positive coefficients. The integer $l$ is called the semisimple rank of $G$ and $\Psi$. The set $\{s_\alpha \mid \alpha \in \Delta\}$ is a set of Coxeter generators of $W$ and $\Delta$ is linearly independent as subset of $X \otimes \mathbb{Q}$. The matrix $C \in \mathbb{Z}^{l \times l}$, $C_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ for $1 \leq i, j \leq l$ is called the Cartan matrix of $\Psi$. We have $\Delta W = R$. The matrix $C$ is the Cartan matrix of a crystallographic root system, the set $\Delta$ can be reordered such that $C$ has a block diagonal form whose diagonal blocks are in the list given in [5, 3.6]. Cartan matrices can be encoded in a compact way by Dynkin diagrams, this is also explained in [5, 3.6].
We now introduce a compact description of a root datum which is useful to specify a root datum and for computations. This is for example used in the GAP [17] programs of the CHEVIE [10] project.

Given \( \Psi = (X, R, Y, R^\vee) \) we can choose \( \mathbb{Z} \)-bases of \( X \) and \( Y \) which are dual to each other and represent elements of \( x \in X \) and \( y \in Y \) by their coordinate row vectors with respect to these bases (so, we have \( \langle x, y \rangle = y^\top x \), where \( ^\top \) means the transpose).

For \( \Delta \) as above we define matrices \( A, A^\vee \in \mathbb{Z}^{l \times r} \) where the \( i \)-th row of \( A \) contains the coordinates of \( \alpha_i \) and the \( i \)-th row of \( A^\vee \) those of \( \alpha_i^\vee \).

From \( A \) and \( A^\vee \) we can compute the whole root datum: the \( i \)-th rows of the two matrices determine the generators \( s_{\alpha_i} \) and \( s_{\alpha_i}^\vee \) of \( W \) and \( W^\vee \), and the orbits of the rows of \( A \) under \( W \) yield \( R \) (and similarly for \( R^\vee \)). The product \( C = A^\vee A^\vee \in \mathbb{Z}^{l \times l} \) is the Cartan matrix of \( \Psi \).

Vice versa, let \( A, A^\vee \in \mathbb{Z}^{l \times r} \) be two matrices such that \( C = A^\vee A^\vee \in \mathbb{Z}^{l \times l} \) is the Cartan matrix of a crystallographic root system, and let \( \bar{k} \) be an algebraically closed field. Then there exists a connected reductive algebraic group over \( \bar{k} \) which yields \( (A, A^\vee) \) as described above (use [18, 7.4.1, 9.5.1, 10.1]).

**Definition 2.1.** We call a pair of matrices \((A, A^\vee) \in (\mathbb{Z}^{l \times r})^2 \) root datum matrices if \( C = A^\vee A^\vee \in \mathbb{Z}^{l \times l} \) is the Cartan matrix of a crystallographic root system.

**Remark 2.2.** (a) Fixing the type of a root datum via a Cartan matrix \( C \in \mathbb{Z}^{l \times l} \) (or, equivalently, a Dynkin diagram), the corresponding connected reductive groups of adjoint type are described by the root datum matrices \((\text{Id}_l, C)\) (the simple roots are a basis of \( X \)), and the corresponding groups of simply-connected type are described by \((C^\vee, \text{Id}_l)\) (the simple coroots are a basis of \( Y \)).

(b) For \( i = 1, 2 \) let \( G_i \) be a connected reductive group over \( \bar{k} \) with a maximal torus \( T_i \). Let \((A_i, A_i^\vee)\) be corresponding root datum matrices. Then the direct product \( G_1 \times G_2 \) can be described with respect to the maximal torus \( T_1 \times T_2 \) by root datum matrices \((A, A^\vee)\) where \( A \) and \( A^\vee \) are block diagonal with diagonal blocks \( A_1, A_2 \) and \( A_1^\vee, A_2^\vee \), respectively.

The following observation will be useful later. We formulate it with roots, there is a similar statement for the coroots.

**Lemma 2.3.** Let \( \Psi = (X, R, Y, R^\vee) \) be a root datum and \( \Delta \subset R \) a set of simple roots.

(a) The Cartan matrix \( C = ((\alpha_j, \alpha_i^\vee))_{i,j} \) or, equivalently, the Dynkin diagram of \( \Psi \) labeled by \( \Delta \) determines the set \( R \) of roots as linear combinations of those in \( \Delta \).

(b) The set of roots \( R \) as linear combinations of \( \Delta \) determines the Cartan matrix \( C \) or, equivalently, the Dynkin diagram of \( \Psi \).

**Proof.** (a) The set \( R \) is the union of orbits of \( \Delta \) under the Weyl group \( W \) which is generated by the \( s_{\alpha} \) with \( \alpha \in \Delta \). For \( \beta \in R \) (which is a \( \mathbb{Z} \)-linear combination of \( \Delta \)) we have \( s_{\alpha}(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \), so the action of \( s_{\alpha} \) on the \( \mathbb{Z} \)-lattice spanned by \( \Delta \) is completely determined by the Cartan matrix.

(b) For any two simple roots \( \alpha_i, \alpha_j \in \Delta \) the subset of \( R \) consisting of linear combinations of \( \alpha_i \) and \( \alpha_j \) is the same as the sub-root system spanned by these two roots (see [12, Prop. in 1.10]).

So, to find the bond between \( \alpha_i \) and \( \alpha_j \) in the Dynkin diagram labeled by \( \Delta \) we look at the subset of positive roots in \( R \) which are linear combinations of \( \alpha_i \) and \( \alpha_j \). There are 2, 3, 4 or 6 such roots, corresponding to no, a single, a double or a triple bond (types \( 2A_1, A_2, B_2, G_2 \)), respectively. In the last two cases an arrow must be added pointing to the shorter root, this is the one occurring with the largest coefficient in the linear combinations. \( \square \)
2.2. Homomorphisms of root data. We recall some information from [14, II 1.13–1.15].
For $i = 1, 2$ let $G_i$ be connected reductive groups over the same algebraically closed field $k$, with maximal tori $T_i$ and corresponding root data $\Psi_i = (X_i, R_i, Y_i, R_i^\vee)$.

A homomorphism from $\Psi_1$ to $\Psi_2$ is given by a $\mathbb{Z}$-linear map $f : X_2 \to X_1$ such that $f$ induces a bijection $R_2 \to R_1$ and its dual map $f^\vee : Y_1 \to Y_2$ induces a bijection $R_2^\vee \to R_1^\vee$.

For each such homomorphism of root data there is a homomorphism $\phi : G_1 \to G_2$ that maps $T_1 \to T_2$ such that $\phi |_{T_1}$ induces $f^\vee : Y_1 \to Y_2$ and $\ker \phi \leq Z(G_1) \leq T_1$, where $Z(G_1)$ denotes the center of $G_1$. More precisely, $\ker \phi = \{ t \in T_1 \mid f(x)(t) = 1 \text{ for all } x \in X_2 \}$.

The map $\phi$ is surjective if and only if $Y_2/f^\vee(Y_1)$ is finite, and $\phi$ is an isomorphism if and only if $f^\vee$ (or $f$) is invertible.

Moreover, $\phi$ is called an isogeny if it is surjective and has a finite kernel, that is $f^\vee$ maps $Y_1$ injectively onto a finite index subgroup of $Y_2$.

The root datum associated to a connected reductive group is unique up to isomorphism.

To construct homomorphisms of root data we will use the following lemma.

Lemma 2.4. A homomorphism of root data is determined by a $\mathbb{Z}$-linear map $f^\vee : Y_1 \to Y_2$ which induces a bijection $R_1^\vee \to R_2^\vee$ and which maps $R_1^\perp$ to $R_2^\perp$, where $R_1^\perp = \{ y \in Y_1 \mid \langle \alpha, y \rangle = 0 \text{ for } \alpha \in R_1 \}$.

Proof. The given map $f^\vee : Y_1 \to Y_2$ induces its dual map $f : X_2 \to X_1$ as follows. For $x_2 \in X_2$ the image $f(x_2) \in X_1$ is the unique element such that $\langle f(x_2), y_1 \rangle = \langle f^\vee(y_1), x_2 \rangle$ for all $y_1 \in Y_1$.

We need to show that the map $f$ induces a bijection from $R_2 \to R_1$.

Let $\Delta_1 = \{ \alpha_1, \ldots, \alpha_l \}$ be a set of simple roots in $R_1$ and $\Delta_1^\vee = \{ \alpha_1^\vee, \ldots, \alpha_l^\vee \}$ be the corresponding coroots. Since $f^\vee$ is $\mathbb{Z}$-linear and induces a bijection $R_1^\vee \to R_2^\vee$, it must map $\Delta_1^\vee$ to a set of simple coroots of $R_2^\vee$. So, there is a set of simple roots $\Delta_2 = \{ \beta_1, \ldots, \beta_l \}$ of $R_2$ such that $f^\vee(\alpha_i) = \beta_j^\vee$ for $1 \leq j \leq l$.

Now we use lemma 2.3(b) to conclude that the Cartan matrices of $\Psi_1$ and $\Psi_2$ are the same, more precisely $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \beta_i, \beta_j^\vee \rangle$ for all $1 \leq i, j \leq l$.

We show the lemma by checking that $f(\beta_i) = \alpha_i$ for $1 \leq i \leq l$.

Note that $QY_1 = Q\Delta_1^\vee \oplus QR_1^\perp$ because the Cartan matrix of $\Psi_1$ has full rank $l$. So, we can show that $f(\beta_i) = \alpha_i$ by showing that $\langle f(\beta_i), \alpha_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle$ for $1 \leq j \leq l$ and that $\langle f(\beta_i), y \rangle = \langle \alpha_i, y \rangle$ for all $y \in R_1^\perp$.

The first follows because the Cartan matrices of $\Psi_1$ and $\Psi_2$ are the same: $\langle f(\beta_i), \alpha_j^\vee \rangle = \langle \beta_i, f^\vee(\alpha_j^\vee) \rangle = \langle \beta_i, \beta_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle$.

The second follows because $f^\vee(y) \in R_2^\perp$ for $y \in R_1^\perp$: $\langle f(\beta_i), y \rangle = \langle \beta_i, f^\vee(y) \rangle = 0 = \langle \alpha_i, y \rangle$.

Of course, there is also a similar version of the lemma where the roles of $X_i$ and $Y_i$ are interchanged.

2.3. Frobenius morphisms. From now we assume that our field $k = \overline{\mathbb{F}}_p$ is an algebraic closure of the finite prime field with $p$ elements, and that $G$ is defined over the finite subfield $\mathbb{F}_q \leq k$ with $q$ elements. We refer to [8, Chapter 3] for an explanation of this notion. There is a corresponding Frobenius morphism $F : G \to G$. We consider the root datum of $G$ with respect to a maximal torus $T$ that is contained in a Borel subgroup $B$ with $F(B) = B$ and $F(T) = T$. Then $F$ induces a map on $X$ which is of the form $qF_0$ where $F_0$ defines an automorphism of root data of finite order which permutes the set of
ons of defining characteristic representations of finite reductive groups

2.4. A covering group. A semisimple group $G$ has a covering by a simply connected group. In this subsection we explicitly construct such a covering $\tilde{G}$ for general connected reductive $G$. If $F$ is a Frobenius morphism on $G$ we also construct a Frobenius morphism $\tilde{F}$ on $G$ which induces $F$ on $G$.

Proposition 2.5. Let $G$ be a connected reductive group, defined over $\mathbb{F}_q$ with Frobenius morphism $F$. Let the root datum $\Psi = (X, R, Y, R^\vee)$ of $G$ and $F$ be described by root datum matrices $(A, A^\vee)$ and $F_0$ as above.

There are root datum matrices $(\tilde{A}, \tilde{A}^\vee)$ and an automorphism $\pi : \tilde{F}_0$ of finite order of the corresponding root datum $\tilde{\Psi} = (\tilde{X}, \tilde{R}, \tilde{Y}, \tilde{R}^\vee)$, and a homomorphism $\tilde{\Psi} \to \Psi$ with the following properties.

(a) The connected reductive group $\tilde{G}$ over $\mathbb{F}_p$ determined by $\tilde{\Psi}$ is a direct product of simple simply connected groups and a central torus $\mathbb{Z}^0$.

(b) $\pi$ induces an isogeny $\pi : \tilde{G} \to G$.

(c) $\pi$ induces an isomorphism from $\mathbb{Z}^0$ to the connected center $\mathbb{Z}^0$ of $G$.

(d) There is a Frobenius morphism $\tilde{F} : \tilde{G} \to G$ corresponding to $F_0$ and $q$ which induces $F$ on $G$.

Proof. We first construct $\tilde{A}$ and $\tilde{A}^\vee$. Let $C = A^\vee A^\tau \in \mathbb{Z}^{l \times l}$ be the Cartan matrix of $\Psi$ and $r$ be the rank of $\Psi$. Let $\tilde{A} = (C^u | 0) \in \mathbb{Z}^{l \times r}$ be the matrix with $C$ as the first $l$ columns and $r - l$ zero columns, and similarly let $\tilde{A}^\vee = (1d_l | 0) \in \mathbb{Z}^{l \times r}$. Then $(\tilde{A}, \tilde{A}^\vee)$ are root datum matrices because $\tilde{A}^\vee \tilde{A}^\tau = C$, so determine a root datum $\tilde{\Psi} = (\tilde{X}, \tilde{R}, \tilde{Y}, \tilde{R}^\vee)$ and a connected reductive group $\tilde{G}$ over $\mathbb{F}_p$. After reordering of the simple roots $\tilde{A}$ and $\tilde{A}^\vee$ have block diagonal form, the blocks corresponding to the simple components of $\tilde{G}$. So, it is the root datum of a direct product of the simple components and a torus. The simple components are simply-connected, see remark 2.2.

Let $B \in \mathbb{Z}^{(r - l) \times r}$ be a matrix whose rows describe a $\mathbb{Z}$-basis of $R^\perp \subseteq Y$. Then the matrix $M^u = \begin{pmatrix} A^\vee & B \end{pmatrix} \in \mathbb{Z}^{r \times r}$ describes a $\mathbb{Z}$-linear map $f^\vee : \tilde{Y} \to Y$ which defines a homomorphism of root data: It maps the simple coroots to simple coroots and so by lemma 2.3(a) the coroots $\tilde{R}^\vee$ to $R^\vee$ (the root data have the same Cartan matrix). And it induces an isomorphism $\tilde{R}^\perp \to R^\perp$. Hence we can use Lemma 2.4.
We can compute $B$ as follows: Its rows are a $\mathbb{Z}$-basis of the set of solutions $y \in \mathbb{Z}^r$ of $yA^\nu = 0$. With the Smith normal form algorithm we can compute invertible integer matrices $P$ and $Q$ such that $PAQ$ has diagonal form, so the last $r - l$ columns of $AQ$ are zero (and the first $l$ columns are $\mathbb{Q}$-linearly independent). We can take the last $r - l$ rows of $Q\alpha$ as matrix $B$.

The map $f^\gamma : \tilde{Y} \to Y$ is injective, its image is generated by $R^\gamma$ and $R^\perp$. So, the image is invariant under $F_0^\mu$ and we can define $\tilde{F}_0^\mu : \tilde{Y} \to \tilde{Y}$ by $\tilde{y}\tilde{F}_0^\mu = f^{\gamma^{-1}}(f^\gamma(\bar{y}))F_0^\mu = \bar{y}M^\mu F_0^\mu M^{-\mu}$. This defines an automorphism of finite order of $\Psi$. Now, $\Psi, F_0^\mu$ and a prime power $q$ determine a reductive $G$, defined over $\mathbb{F}_q$ with Frobenius morphism $\tilde{F}$. We have a surjective homomorphism $\pi : G \to G$, and $\tilde{F}$ induces a Frobenius morphism $F'$ on $G$, which induces $F_0^\mu$ on $Y$. So, modifying $\tilde{F}$ by a conjugation with an appropriate torus element we can assume that $F$ induces $F$. on $G$.

The kernel $K := \ker(\pi)$ of the covering $\pi : \tilde{G} \to G$ is finite because $M$ (and $M^\mu$) have full $\mathbb{Q}$-rank $r$, so $XM \leq \tilde{X}$ is of finite index. We have $R^\perp = Y(\mathbb{Z}^0)$ and $R^\perp = Y(\mathbb{Z}^0)$ and since $f^\gamma$ induces an isomorphism between these two lattices, the homomorphism $\pi$ induces an isomorphism $\tilde{Z}^0 \to Z^0$. In subsection 2.5 we show how to compute the kernel of $\pi$ explicitly.

Lemma 2.6. Let $G, G$ be connected reductive groups with a surjective homomorphism $\pi : \tilde{G} \to G$ and central kernel $K$. Let $\tilde{F}$ be a Frobenius morphism of $G$ with $\tilde{F}(K) = K$ and $F$ be the induced Frobenius morphism on $G$. The induced map $\pi : \tilde{G}/G \to G/F$ is in general not surjective. We define $\mathcal{L}(K) = \{z^{-1}\tilde{F}(z) \mid z \in K\}$. Then $\pi(\tilde{G}/G)$ is a normal subgroup of $G/F$ and there is a natural isomorphism

$\tilde{G}/G \cong \mathcal{L}(K)$.

Proof. We first show that $\pi(\tilde{G}/G)$ is normal. Let $h \in \tilde{G}/G$, $g \in G/F$ and $\bar{g} \in \tilde{G}$ with $\pi(\bar{g}) = g$. Then $\tilde{F}(\bar{g}) = \bar{g}z$ for some $z \in K$. It follows $\tilde{F}(\bar{g}^{-1}) = z^{-1}\bar{g}^{-1}$. Hence $\tilde{F}(\bar{g}^{-1}h\bar{g}) = z^{-1}\bar{g}^{-1}h\bar{g}z = \bar{g}^{-1}h\bar{g}$ because $K$ and so $z$ is central. This shows that $g^{-1}\pi(h)g = \pi(\bar{g}^{-1}h\bar{g}) \in \pi(\tilde{G}/G)$.

Since the group $K$ as subgroup of the center of $G$ is abelian if follows that $\mathcal{L}(K)$ is a subgroup of $G$.

We have $G \cong \tilde{G}/G$ and for $g \in G$ we have $gK \in (\tilde{G}/G)/F \cong G/F$ if and only if $g^{-1}\tilde{F}(g) \in K$. We consider the map

$\tilde{G}/G \cong (\tilde{G}/G)/F \to K/\mathcal{L}(K), \quad gK \mapsto g^{-1}\tilde{F}(g)\mathcal{L}(K)$.

Since $K$ is central, it is easy to check that this is a well-defined isomorphism. The Lang-Steinberg theorem (for $G$) shows that this map is surjective. An element $gK$ is in the kernel of this map if and only if $gK$ contains an element of $\tilde{G}/G$.

2.5. Torus elements. Given a root datum $\Psi = (X, R, Y, R^\vee)$ for $G$ and $T$, we can recover $T$ by the isomorphism $T \cong Y \otimes \mathbb{Z} \mathbb{F}_p^\times$. Via some fixed isomorphism we identify the multiplicative group $\mathbb{F}_p^\times$ with the additive group $\mathbb{Q}_{p'}/\mathbb{Z}$ of elements of $p'$-order in $\mathbb{Q}/\mathbb{Z}$. See [6, 3.1] for more details.

Choosing dual bases of $X$ and $Y$, we can describe $\Psi$ by root datum matrices $(A, A^\vee)$ and identify $T \cong Y \otimes \mathbb{Z} (\mathbb{Q}_{p'}/\mathbb{Z})$ with $t$-tuples of elements in $\mathbb{Q}_{p'}/\mathbb{Z}$. In this setup we can compute $y(c)$ for $y \in Y$ and $c \in \mathbb{Q}_{p'}/\mathbb{Z}$, and apply $x \in X$ and $F$ to $t \in T = (\mathbb{Q}_{p'}/\mathbb{Z})^r$ as follows:

$y(c) = c \cdot y, \quad x(t) = tx^r \in \mathbb{Q}_{p'}/\mathbb{Z}, \quad F(t) = qtF_0^{\alpha} \in T$. 
The center of $G$ is the intersection of the kernels of all (simple) roots in $T$. We can compute it as the solutions $t \in T$ of the system of equations $tA^u = 0 \in (\mathbb{Q}_p'/\mathbb{Z})^l$. The $F$-fixed points $T^F$ of $T$ are the solutions $t \in T$ of the system of equations $t(qF_0^u - \text{Id}_r) = 0 \in (\mathbb{Q}_p'/\mathbb{Z})^r$.

We consider the isogeny from proposition 2.5, $\pi : \tilde{G} \to G$. In the proof of the proposition we have computed a matrix $M$ describing the map $f : X \to \tilde{X}$ for the corresponding homomorphism of root data.

We can compute the kernel $K$ of $\pi$ as set of solutions $t \in T$ of the system of equations

$$tM^u = 0 \in (\mathbb{Q}_p'/\mathbb{Z})^r.$$ 

(The $\mathbb{Z}$-span of the rows of $M$ is the image $f(X) \leq \tilde{X}$.) And we can compute the $\tilde{F}$-action on the elements $t \in K$ by

$$\tilde{F}(t) = qt\tilde{F}_0^u.$$ 

This yields an explicit description of the elements in $K$, $K^{\tilde{F}}$ and $\mathcal{L}(K)$.

2.6. The derived subgroup. We will also need to consider the derived group $G'$ of $G$ and the quotient torus $G/G'$. We use the description in [18, 8.1.9] or [14, 1.18].

The images of all coroots generate $T \cap G'$ and a character $x \in X$ has $T \cap G'$ in its kernel if and only if $x \in (R^\vee)^\perp$.

We compute a matrix $D \in \text{GL}_r(\mathbb{Z})$ such that the last $l - r$ columns of $A^\vee D$ are zero (for example by the Smith normal form algorithm). Instead of $(A, A^\vee)$ and $F_0$ we then consider the isomorphic data $(AD^{-u}, A^\vee D)$ and $D^u F_0 D^{-u}$.

Now we get root datum matrices for $G'$ by taking the first $l$ columns of $AD^{-u}$ and $A^\vee D$, and the restriction of $F$ to $G'$ is described by the upper left $l \times l$ corner of $D^u F_0 D^{-u}$.

Furthermore, the lower right $(r-l) \times (r-l)$ corner of $D^u F_0 D^{-u}$ describes the Frobenius action induced on the torus $G/G'$.

Lemma 2.7. Let $\pi : \tilde{G} = \tilde{G}' \times Z^0 \to G$ be the covering and $\tilde{F}$ be the Frobenius morphism of $\tilde{G}$ as constructed in proposition 2.5.

Then $\pi(\tilde{G}^{\tilde{F}}) \leq G^F$ is a normal subgroup, and the quotient $G^F / \pi(\tilde{G}^{\tilde{F}})$ is an abelian group of order prime to $p$.

Proof. That $\pi(\tilde{G}^{\tilde{F}}) \leq G^F$ is normal can be shown as in the proof of lemma 2.6, using that $G'$ is normal in $G$.

In [8, proof of 13.20] it is shown that $G^F = T^F \cdot \pi(\tilde{G}^{\tilde{F}})$. So, the quotient $G^F / \pi(\tilde{G}^{\tilde{F}})$ is isomorphic to $T^F / (T^F \cap \pi(\tilde{G}^{\tilde{F}}))$, hence it is abelian and of order prime to $p$. 

3. Irreducible representations in defining characteristic

In this section we consider (finite dimensional rational) irreducible representations of our connected reductive algebraic groups $G$ and the finite groups of Lie type $G^F$ over the defining field $\bar{k} = \mathbb{F}_p$ of $G$. As before, let $\Psi = (X, R, Y, R^\vee)$ be the root datum of $G$ and $F_0 : X \to X$ and $q$ be the finite order automorphism and the prime power determined by $F$.

3.1. Representations of connected reductive groups. In this subsection $\bar{k}$ can be any algebraically closed field. We fix a set $\Delta \subset R$ of simple roots. The set

$$X_+ = \{ x \in X \mid \langle x, \alpha^\vee \rangle \geq 0 \text{ for } \alpha \in \Delta \}$$ 

is called the set of dominant weights of $G$ (or $\Psi$).
One can associate to each irreducible representation of $G$ over $\bar{k}$ a highest weight $\lambda \in X_+$. Chevalley proved the following basic theorem, see [14, 2.7]:

**Theorem 3.1.** Associating the highest weight induces a bijection from the isomorphism classes of irreducible representations of $G$ over $\bar{k}$ to the set $X_+$ of dominant weights.

For $\lambda \in X_+$ we denote by $L(\lambda)$ the corresponding irreducible module, and by $\rho_\lambda$ the corresponding representation.

### 3.2. Finite groups of Lie type, simply connected case

In this subsection we assume that $G$ is semisimple and of simply-connected type. In this case the simple coroots are a $\mathbb{Z}$-basis of $Y$. The elements of the dual basis $\{\omega_1, \ldots, \omega_l\} \subset X$ are called the fundamental weights. For a positive integer $b$ we call the subset

$$ X_b = \{x \in X_+ \mid \langle x, \alpha^\vee \rangle < b \text{ for all } \alpha \in \Delta \} = \{a_1\omega_1 + \ldots + a_l\omega_l \mid 0 \leq a_i < b \text{ for } 1 \leq i \leq l \} $$

of dominant weights the set of $b$-restricted weights.

Steinberg proved the following theorem, see [19, 13.3]. Although we have excluded the cases of Suzuki and Ree groups from our general setup we include them in this theorem.

**Theorem 3.2.** (a) The restrictions of $\rho_\lambda$ with $\lambda \in X_q$ to $G^F$ remain irreducible. This induces a bijection from the isomorphism classes of irreducible representations of $G^F$ over $\bar{F}_q$.

(b) Let $G$ be of type $B_2$, $F_4$ or $G_2$, $q^2$ be an odd power of $p = 2$, 2 or 3, respectively, and $F_0$ be of order 2. We consider the set $X'_q$ of dominant weights $\sum_{i=1}^l a_i\omega_i$ with $0 \leq a_i < q\sqrt{p}$ if $\alpha_i$ is a short simple root and $0 \leq a_i < q/\sqrt{p}$ otherwise. Then the restrictions of $\rho_\lambda$ with $\lambda \in X'_q$ induce a bijection from $X'_q$ to the isomorphism classes of irreducible representations of $G^F$ over $\bar{F}_q$.

### 3.3. Finite tori

Let $G = T$ be a torus. The irreducible representations of $T$ are the characters $X(T)$. Let $F_0 : X \to X$ be the finite order automorphism induced by $F$, and let $m \in \mathbb{N}$ such that $F_0^m = \text{Id}$. We have $T^F \leq T^{F^m}$. Since these groups are abelian, each irreducible representation of $T^F$ can be extended to one of $T^{F^m}$, see [13, 5.5]. Using subsection 2.5 we see that the group $T^{F^m}$ is isomorphic to a direct product of $r$ cyclic groups of order $p^m - 1$. And restriction yields a bijection from the set of characters $\{\rho_\lambda \mid \lambda \in X_q^{m-1}\}$ of $T$ to the set of irreducible characters over $\bar{F}_q$ of the finite group $T^{F^m}$.

**Remark 3.3.** All irreducible representations of a finite torus $T^F$ over $\bar{F}_q$ are restrictions of irreducible representations (characters) of the torus $T$.

In the general case there seems to be no nice description of a subset of $X$ which yields the pairwise different characters of $T^F$, for this one has to compute an explicit parameterization of $T^F$. This can be done by solving the system of equations $t(qF_0^u - \text{Id}_r) = 0 \in (\mathbb{Q}_p/\mathbb{Z})^r$, as explained in subsection 2.5 (we see that the order of $T^F$ is just the characteristic polynomial of $F_0$ evaluated at $q$).

### 3.4. Extending representations

**Proposition 3.4.** Let $G$ be a connected reductive group over $\bar{k}$ (= $\bar{F}_p$) with Frobenius morphism $F$. Let $H \leq G^F$ be a normal subgroup such that $G^F/H$ is an abelian group of order prime to $p$. Then each irreducible representation of $H$ over $\bar{k}$ can be extended to a representation of $G^F$. And each irreducible representation of $G^F$ over $\bar{k}$ restricts irreducibly to $H$. 

Proof. We use Clifford theory, see for example [1, 9.18]. It follows that the two state-
ments in the proposition are equivalent. We show the latter: the restriction of every irre-
ducible \( \bar{k}G^F \)-module \( V \) to \( H \) is irreducible.

The restriction is a direct sum of irreducible \( \bar{k}H \)-modules \( W_i \),
\[
V_H = \bigoplus_{i=1}^r W_i,
\]
we show \( r = 1 \). Let \( U \) be a Sylow-\( p \)-subgroup of \( G^F \). Then \( U \leq H \) because \( H \) has \( p' \)-index. The only simple \( \bar{k}U \)-module is the trivial module. Therefore, each \( W_i \) must have at least a one-dimensional subspace on which \( U \) acts trivially. So, \( V \) contains at least an \( r \)-dimensional subspace on which \( U \) acts trivially.

Now we use that the group \( G^F \) is a finite group with split \((B, N)\)-pair in characteristic \( p \), see [9, Cor. 4.2.5]. Thus we can apply a result by Richen and Curtis that says that the subspace of \( V \) fixed by \( U \) is one-dimensional, see [7, 4.3(c)]. Hence \( r = 1 \) and \( V_H \) is an irreducible \( \bar{k}H \)-module.

3.5. Parameterization of irreducible representations of finite groups of Lie type. We can now describe the main result of this paper.

As before, let \( G \) be a connected reductive group over \( \bar{k} \), defined over \( \mathbb{F}_q \) with corre-
sponding Frobenius morphism \( F \), given by root datum matrices \((A, A^\vee)\) and a finite order matrix \( F_0 \), as explained in section 2.

In proposition 2.5 we have constructed a covering \( \pi : \tilde{G} = \tilde{G}' \times Z^0 \to G \) and a Frobenius morphism \( \bar{F} \) of \( G \) inducing \( F \) on \( G \). We write \( K \) for the kernel of \( \pi \) and \( G' \) for the derived subgroup of \( G \).

Theorem 3.5. The irreducible representations of \( G^F \) over \( \bar{k} \) can be parameterized by the direct product of the following three sets

(A) the \( q \)-restricted weights of \( \tilde{G}' \) which have \( K^F \cap \tilde{G}' \) in their kernel,
(B) the group \( K^F \cap G' \),
(C) and the group \( (G/G')^F \).

Proof. This follows from Steinberg’s theorem 3.2 applied to \( G^tF \) and Clifford theory, see for example [1, 9.18]. We give more details.

We know from lemma 2.7 that \( \pi(\tilde{G}'^F) \) is a normal subgroup of \( G^F \) with abelian quo-
tient of order prime to \( p \). Thus we can apply proposition 3.4 to see that all irreducible \( \bar{k}G^F \)-modules are extensions of irreducible \( \bar{k}\pi(G^F) \)-modules. By Clifford theory the extensions of a fixed \( \bar{k}\pi(G^F) \)-module to \( G^F \) are parameterized by the group of linear characters of the quotient group \( G^F/\pi(G^F) \) which is isomorphic to the quotient group itself.

The irreducible representations of \( \pi(\tilde{G}'^F) \) can be interpreted as the irreducible repre-
sentations \( G^F \) which have \( K^F \cap \tilde{G}' \) in their kernel. And, since \( \tilde{G}' \) is simply connected, the irreducible representations of \( G^F \) are by theorem 3.2 parameterized by the \( q \)-restricted weights of \( G' \). An element \( x \in Z(\tilde{G}') \leq \tilde{T} \cap G' \) lies in the kernel of an irreducible representation with highest weight \( \lambda \in \tilde{X}(\tilde{T} \cap G') \) if its (only) eigenvalue is 1. This eigenvalue can be read off at the weight space of the highest weight by evaluating \( \lambda \) at \( z \). This shows that the irreducible representations of \( \pi(\tilde{G}'^F) \) can be parameterized by the set (A).

We can parameterize the characters \( \text{Hom}(G^F/\pi(\tilde{G}'^F), \bar{k}^\times) \) in two steps, first by the restriction to \( \text{Hom}(G'/\pi(\tilde{G}'^F), \bar{k}^\times) \) and then by the characters \( \text{Hom}(G^F/G^F, \bar{k}^\times) \).
(again by Clifford theory because all characters in \( \text{Hom}(G^F/\pi(\tilde{G}^F), \tilde{k}^\times) \) extend to \( G^F/\pi(\tilde{G}^F) \)). The latter yields our parameter set (C) using \( G^F/G^F_{1} \cong (G/G')^F \) which follows from the Lang-Steinberg theorem. The set (B) we get from the isomorphism \( G^F/\pi(\tilde{G}^F) \cong K^F/\pi(K') \). This follows from lemma 2.6 applied to the induced map \( \pi : \tilde{G} \to G \) which has the finite kernel \( K' = K \cap \tilde{G}' \). The lemma shows \( G^F/\pi(\tilde{G}^F) \cong K'/L(K') \). This last group is isomorphic to \( K^F = K^F \cap G' \) which follows from dualizing the exact sequence \( 1 \to K^F \to K' \to L(K') \to 1 \). 

We now indicate how to compute the parameter sets (A), (B) and (C). In proposition 2.5 we have constructed the root datum of \( \tilde{G} \) such that the first \( l \) coordinates and the last \( r-l \) coordinates of \( X \) and \( \tilde{Y} \) correspond to the factors of the direct product \( \tilde{T} = (T \cap \tilde{G}') \times \tilde{Z}^0 \). Thus it is easy to decide which elements of \( K^F \), computed as in subsection 2.5, are contained in \( \tilde{G}' \), this yields the set (B).

In 2.5 we have also shown how to evaluate a \( \lambda \in X(\tilde{T} \cap \tilde{G}') \) at a torus element. This way we can decide which \( \pi \)-restricted weights of \( \tilde{G}' \) have \( K^F \cap \tilde{G}' \) in their kernel. This determines the set (A).

For the set (C) we need to compute the structure of the abelian group \( G^F/G^F_{1} \cong (G/G')^F \). In subsection 2.6 we have described how to compute the \( F \)-action on the torus \( G/G' \). We can use 2.5 again to compute the \( F \)-fixed points of \( G/G' \).

**Remark 3.6.** (a) Assume that the derived group \( G' \) of \( G \) is simply connected. Then each irreducible \( \tilde{k}G^F \)-module is the restriction of an irreducible \( \tilde{k}G \)-module. In [11, App. 1.3] Herzog gives another parameterization in this case: Namely by all \( q \)-restricted weights of \( G \) (these are infinitely many if \( G \) is not semisimple) and showing that two \( q \)-restricted weights \( \lambda_1, \lambda_2 \) yield the same restriction to \( G^F \) if and only if \( \lambda_1 - \lambda_2 \in (q \cdot \text{id} - F_0)(\tilde{R}^\vee)^2 \).

(b) In general, not all irreducible \( \tilde{k}G^F \)-modules are restrictions of modules of the algebraic group \( G \). As an example consider \( G = PGL_{l+1}(\tilde{k}) \), the adjoint groups of type \( A_l \) with Frobenius map \( F \) such that \( G^F = PGL_{l+1}(q) \). For some prime powers \( q \) the finite group \( G^F \) has non-trivial \( \tilde{k} \)-representations of dimension 1. Such representations are not restrictions from \( G \) because \( G \) is perfect.

**Proof.** We show the first statement of (a) using our setup. If \( G' \) is simply connected our parameterization of irreducible \( \tilde{k}G^F \) modules in theorem 3.5 is particularly simple: the set (A) consists of all \( q \)-restricted weights of \( G' \), the group (B) is trivial and (C) is the finite torus \( (G/G')^F \).

Since \( G' \) is simply connected, \( X \) contains \( \omega_i \) with \( \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \) for \( 1 \leq i, j \leq l \). So, for each \( q \)-restricted weight \( \lambda' \) of \( G' \) there is a \( \lambda \in X \) such that the module \( L(\lambda) \) of \( G \) restricts to \( G' \) as \( L(\lambda') \). Together with Steinberg’s theorem 3.2 this shows that each irreducible representation \( \tilde{\rho} \) of \( \tilde{G}^F \) can be extended to a representation \( \rho \) of the algebraic group \( G \). All the other extensions of \( \tilde{\rho} \) to \( G^F \) are obtained by tensoring \( \tilde{\rho}|_{G^F} \) with the linear characters of \( G^F/G^F_{1} \). But these are also obtained as restrictions of linear characters of the algebraic group \( G/G' \) as we have seen in remark 3.3. □

**3.6. A variant.** As a variant of theorem 3.5 we could have first given a parameterization of the irreducible representations of \( \tilde{G}^F \) and use Clifford theory only for the quotient \( G^F/\pi(\tilde{G}^F) \). But our description in theorem 3.5 often leads to a more natural parameterization.

For example, let \( G = GL_{l+1}(\tilde{k}) \) and \( g \equiv 1 \pmod{l+1} \). Then \( \tilde{G} = SL_{l+1}(\tilde{k}) \times Z^0 \) and the kernel of \( \pi, K = K^F, \) is cyclic of order \( l+1 \) and is isomorphic to \( G^F/\pi(G^F) \). The irreducible representations of \( \tilde{G}^F \cong SL_{l+1}(q) \times (Z^0)^\tilde{F} \) (the second factor is cyclic of
order $q - 1$) are easy to describe. But it is a bit complicated to describe the subset which has $K$ in its kernel. The quotient $G^F / \pi(G^F)$ is cyclic of order $l + 1$, so its irreducible representations are also easy to describe.

Our parameterization in theorem 3.5 is more natural in this example: The derived subgroup of $G$ and of $\tilde{G}$ is $\text{SL}_{l+1}(k)$ and so is simply connected. Hence we are in the situation of remark 3.6(a), our set (A) consists of all $q$-restricted dominant weights of $\text{SL}_{l+1}(k)$ and our set (B) is trivial. The set (C) corresponds to the $q - 1$ linear characters of $\text{GL}_{l+1}(q) / \text{SL}_{l+1}(q)$.

**Remark 3.7.** A variant of the main theorem 3.5 is also true if $G^F$ has Suzuki or Ree groups as components. Since the Suzuki and Ree groups have trivial center, we can assume that $G^F$ arises from an algebraic group such that the Suzuki and Ree components are coming from direct factors of $G$ of simply connected type. We can then deal with these components using theorem 3.2(b).

### 3.7. An example.

Let us consider as an example a reductive group $G$ which occurs as the centralizer of a semisimple element in the simple algebraic group of type $E_8$, equipped with a Frobenius morphism $F$. It is given by root datum matrices $A$, $A^\vee$, and a matrix $F_0$, as explained in sections 2.1 and 2.3:

$$A := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 1
\end{pmatrix}, \quad A^\vee := \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$F_0 := \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 1 \\
-2 & -3 & -4 & -6 & -5 & -3 & -2 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

We do not fix the $q$, but we want to investigate all finite groups of Lie type for any prime power $q$ which are determined by these data.

The group $G$ has rank 8 and semisimple rank 7. Looking at the Cartan matrix $A^\vee A^w$ we see that the pairs of simple roots number 1 and 3, number 4 and 5 and number 6 and 7 each span a sub-root system of type $A_2$, and root number 2 spans a subsystem of type $A_1$. The matrix $AF_0$ yields a permutation of the rows of $A$, the permutation is $(1, 3)(4, 6)(5, 7)$. Thus, the data describe groups $G^F$ which are central products of components of type $2A_2(q)$, $A_2(q^2)$, $A_1(q)$ and a finite torus of rank 1.

Now we look at the covering group $\tilde{G}$ of $G$ constructed in proposition 2.5. We do not need the matrices $\tilde{A}$ and $\tilde{A}^\vee$, but the matrix $M^w$ is essential which describes the homomorphism $Y \to Y$ that determines the covering $\tilde{\pi} : \tilde{G} \to G$. As described in the proof of proposition 2.5 we can compute a $\mathbb{Z}$-basis of $R^\perp$ by applying the Smith normal form algorithm to $A$. This yields invertible integer matrices $P$ and $Q$ such that $PAQ$ is of diagonal form (the diagonal entries are six times 1 and one 3). Then $M^w$ is given by the rows of $A^\vee$ and the last row of $Q^w$, the latter spans $R^\perp$. We furthermore need $\tilde{F}_0$ which defines the Frobenius morphism on the covering group $\tilde{G}$. We can compute it with $M^w$ as
\[ \tilde{F}_0^u = M^u F_0^u M^{-u}. \]

We get

\[
M^u = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}, \quad \tilde{F}_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Using these two matrices we can determine the finite kernel \( K \) of the covering \( \pi : \tilde{G} \to G \) and its \( \tilde{F} \)-fixed points \( K^F \), as explained in section 2.5. To find \( K \) we solve the system of equations

\[ t M^u = 0 \in (\mathbb{Q}_p')^\times. \]

To do so, we use again the Smith normal form algorithm to find matrices \( P, Q \in \text{GL}_r(\mathbb{Z}) \) such that \( P M^u Q \) is diagonal. The diagonal entries in our example are six times 1, 3 and 6. It is easy to write down the solutions of

\[ t_1(P M^u Q) = 0 \in (\mathbb{Q}_p')^\times. \]

If the \( i \)-th diagonal entry of the diagonal matrix is an integer \( n \) then the \( i \)-th entry of any solution \( t_1 \) has the form \( i/n_p' \) for one \( 0 \leq i < n_p' \) (where \( n_p' \) is the largest divisor of \( n \) prime to \( p \)). Having found all solutions \( t_1 \) of this last equation we get the solutions of the original equation as \( t = t_1 P \). In practice we first compute all solutions \( t \in (\mathbb{Q}/\mathbb{Z})^\times \), because we have not yet said anything about the \( q \) and so the \( p \). In our example we have 18 solutions for \( t_1 \) over \( \mathbb{Q}/\mathbb{Z} \), they have the form \( (0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{4}) \). And multiplying with \( P \) we get for \( t \) the 18 \( \mathbb{Z} \)-linear combinations of the two elements \( (\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0, \frac{1}{3}) \) and \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0, \frac{1}{3}) \).

We find \( K^F \) by applying the Frobenius \( F \) to the elements just found. This action is for any \( t \in T \) given by

\[ F(t) = t(q \tilde{F}_0^u). \]

To be able to evaluate this on \( t \in K \) we need to know the residue of \( q \) modulo all denominators of the coordinates of \( t \in K \). A common denominator of all these entries is

\[ m := \text{the largest elementary divisor of } M^u = \text{lcm(entries of Smith normal form of } M^u) = 6. \]

We still do not fix \( q \), but the remaining computations are done for any congruence class \( c \) of a prime power modulo \( m \) separately, assuming that \( q \equiv c \pmod{m} \). In our example we have to distinguish the cases of \( q \equiv 1, 2, 3, 4, 5 \mod{6} \).

In cases \( c = 2 \) or 4 we have \( p = 2 \) (the prime dividing \( c \) and \( m \)), and in this case the kernel \( K \) only contains the 9 elements given above which are of order 1 or 3. Similarly, in case \( c = 3 \) we have \( p = 3 \) and \( K \) only contains the two elements of order 1 and 2. In the other cases \( K \) contains all 18 elements given above.

For the computation of \( K^F \) we comment on the case \( c = 2 \). Multiplying the elements of \( K \) by \( q \tilde{F}_0^u \) and using that \( q \equiv 2 \mod{6} \) we find that only the three multiples of \( (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, 0, 0) \) are mapped to themselves.

We need to decide which of these \( \tilde{F} \)-fixed elements lie in \( \tilde{G}' \). This is easy to see, because the 1 basis elements of \( X \) and \( Y \) correspond to the maximal torus of the semisimple factor and derived subgroup \( G' \). So, here the \( F \)-fixed elements of \( K \) all lie in \( K^F \cap \tilde{G}' \) because their last coordinate is 0.
Considering also the other cases for \( c \) we find that \( \tilde{K}^F \cap \tilde{G}' \) is cyclic of order 3 if \( q \equiv 2 \) or 5 (mod 6) and it is trivial in the other cases. We have found the group (B) for our parameterization of the irreducible representations of \( G^F \).

The elements of \( \tilde{K}^F \cap \tilde{G}' \) are also needed to find our parameter set (A). This consists of all \( q \)-restricted weights \( \lambda \in \tilde{X}^+ \) of \( \tilde{G}' \) which are trivial on \( \tilde{K}^F \cap \tilde{G}' \). This means

\[
t\lambda^\mathbb{F} = 0 \in \mathbb{Q}_p^r/\mathbb{Z} \text{ for all } t \in \tilde{K}^F \cap \tilde{G}'.
\]

These equations can be reformulated in terms of integers by multiplying with a common multiple \( m' \) of all denominators in \( t \in \tilde{K}^F \cap \tilde{G}' \) (a divisor of \( m \)). In our example we can multiply with \( m' = 3 \) and then consider the equations modulo \( m' \):

\[
(m't)\lambda^\mathbb{F} \equiv 0 \pmod{m'} \text{ for all } t \in \tilde{K}^F \cap \tilde{G}'.
\]

Writing all \((m't)\) for a set of generators \( t \in \tilde{K}^F \cap \tilde{G}' \) in one matrix we can further simplify the system of equations by computing the Hermite normal form \((mod m')\) of this matrix. In our example we get no non-trivial equation if \( c \not\in \{2, 5\} \). So, in these cases all \( q \)-restricted weights \( \lambda \) lie in our parameter set (A). If \( c = 2 \) or 5, the set (A) contains only those \( q \)-restricted \( \lambda \) which fulfill the single equation

\[
(1, 0, 2, 1, 2, 2, 1)\lambda^\mathbb{F} = 0 \pmod{3}.
\]

Using this equation it is easy to check for a concrete \( q \) and \( q \)-restricted weight if it is in the parameter set (A).

For general \( q \) we can also count the number of parameters in the set (A). For this we use the following trivial lemma.

**Lemma 3.8.** Let \( q, c, i, m \in \mathbb{N} \) with \( 0 \leq i, c < m \) and \( q \equiv c \pmod{m} \). Then the number of integers \( j \) with \( 0 \leq j < q-1 \) and \( j \equiv i \pmod{m} \) is \((q-c)/m\) if \( i \geq c \) and \((q-c)/m+1\) for \( i < c \).

This lemma can be applied recursively to count the sets (A). For example in the case \( q \equiv 2 \pmod{3} \) above we need to count the \( \lambda = (\lambda_1, \ldots, \lambda_7) \in \mathbb{Z}^7 \) with \( 0 \leq \lambda_i < q \) for \( i = 1, \ldots, 7 \) and \( 1 \cdot \lambda_1 + 0 \cdot \lambda_2 + \cdots + 1 \cdot \lambda_7 \equiv 0 \pmod{3} \).

From the lemma we can easily deduce how often each congruence class \( \pmod{3} \) is hit by \( 1 \cdot \lambda_1, 0 \cdot \lambda_2, \) and so on. Combining this it is easy to count how often each congruence class \( \pmod{3} \) is hit by \( 1 \cdot \lambda_1 + 0 \cdot \lambda_2 \). In the next step we find the numbers for the expressions \( 1 \cdot \lambda_1 + 0 \cdot \lambda_2 + 2 \cdot \lambda_3 \). Going on recursively, we find for each congruence class \( \pmod{3} \) the number of \( q \)-restricted \( \lambda \) with \( (1, 0, 2, 1, 2, 2, 1)\lambda^\mathbb{F} \) in that class. In particular, we find for the 0-class the number of \( q \)-restricted weights in (A), it is \((q^7 + 2q)/3\) for \( q = 2, 5 \pmod{6} \).

Finally, we need the set (C), the structure of \((G/G')^F\). Using subsection 2.6 we can find the matrix of \( F_0 \) acting on the characters of this torus via the transformation of the matrix \( A \) to Smith normal form. In our example we find the \( 1 \times 1 \) identity matrix. So the group of \( F \)-fixed points in this torus is cyclic of order \( q-1 \). In general the order of a finite torus is the characteristic polynomial of \( F_0 \) evaluated at \( q \). The precise structure of the finite abelian group for a specific \( q \) is found by the Smith normal form of the characteristic matrix at \( q \). See [6, Chapter 3] for more details.

To summarize: The parameter group (C) is for any \( q \) cyclic of order \( q-1 \). For \( q \equiv 2 \pmod{3} \) the parameter group (B) is of order 3 and the set (A) contains \((q^7 + 2q)/3\) weights. For \( q \equiv 0, 1 \pmod{3} \) the group (B) is trivial and the set (A) contains all \( q \)-restricted weights.
In this last section of the paper we want to apply our main theorem 3.5 to all finite groups of Lie type arising from simple algebraic groups $G$. As an application we determine the number of semisimple classes of these groups.

As before, we exclude here the Suzuki and Ree groups, in these cases the $q^2$, respectively $q^4$, irreducible representations were already described in theorem 3.2(b).

For each type of irreducible root system $R$, we choose a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subseteq R$. We fix a numbering of the simple roots via the Dynkin diagrams given in table 1. The node labelled by $i$ corresponds to the simple root $\alpha_i$ of $\Delta$. This is the labelling used in CHEVIE; see [10] (the often used Bourbaki labelling is different for types $B, C, D$, where it starts to count from the right side of the shown diagrams).

For a Frobenius morphism $F$ of $G$ we consider a root datum of $G$ with respect to a maximally split maximal torus. Then $F_0$ permutes the set of simple roots and induces a graph automorphism of the Dynkin diagram. This graph automorphism can be non-trivial in cases $A_l$ with $l \geq 2$, $D_l$ with $l \geq 4$ and $E_6$. We also write $F_\epsilon$ instead of $F$ in these cases with $\epsilon = 1$ in case of the trivial graph automorphism, $\epsilon = -1$ in case of the graph automorphism of order 2 (permuting nodes 1 and 2 in case $D_4l$) and $\epsilon = 3$ in case $D_4$ permuting the nodes with cycle $(1, 2, 4)$.

Let $G_{sc}$ be the simply-connected simple group of the same type as $G$. As explained in proposition 2.5 we have an isogeny $G_{sc} \to G$ with a central kernel $K$ and $G_{sc}$ has a Frobenius morphism that induces $F$ on $G$, we denote that also by $F$ or $F_\epsilon$.

As in the proof of proposition 2.5 we choose as root datum matrices for $G_{sc}$ the pair $(C^*, \text{Id})$, where $C$ is the Cartan matrix corresponding to the chosen numbering of $\Delta$. This means that in the root datum $(\hat{X}, \hat{R}, \hat{Y}, \hat{R}^*)$ of $G_{sc}$ we use the simple coroots as basis of $\hat{Y}$ and the fundamental weights as basis of $\hat{X}$. The matrix for $F_0$ is the permutation matrix for the graph automorphism induced by $F$. As before, we identify a maximal torus of $G_{sc}$ with $\hat{Y} \otimes \mathbb{Z} (\mathbb{Q}_p'/\mathbb{Z}) \cong (\mathbb{Q}_p'/\mathbb{Z})^l$.

4.1. A parameterization of the irreducible representations in defining characteristic.

Let $G$, $G_{sc}$ and $K$ be as above. We want to give a parameterization of the irreducible defining characteristic representations of $G^F$ by describing the parameter sets (A), (B) and (C) of theorem 3.5.
Since $G$ is semisimple we have $G = G'$ and so the group $(C)$ is trivial in all cases considered here.

The parameter sets $(A)$ and $(B)$ only depend on $K^F$, the $F$-fixed elements of the kernel of the isogeny $G_{sc} \to G$.

We now consider the possibilities for $G$, $K$ and $K^F$ for the various types of root systems separately.

We make use of the result in [16, §6.2] which describes explicitly the elements of the center $Z$ of $G_{sc}$ in all cases (as elements of $(\mathbb{Q}_{p'}/\mathbb{Z})^l$ as explained above).

For any positive integers $l$ and $q$, we will write $E_{l,q}$ for the set of tuples $(\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l$ such that $0 \leq \lambda_i < q$ for all $1 \leq i \leq l$.

4.1.1. Type $A_l$. The group $Z$ is cyclic of order $m = (l+1)p'$, generated by

$$z = \left( \frac{1}{m}, \frac{2}{m}, \ldots, \frac{l}{m} \right) \in (\mathbb{Q}_{p'}/\mathbb{Z})^l.$$  

For each divisor $e$ of $l+1$ there is an algebraic group $G$ such that the index of $ZR \leq X$ is $e$, we denote its type by $(A_l)_e$. So, $e = l+1$ yields the simply connected groups, isomorphic to $SL_{l+1}(\bar{k})$, and $e = 1$ yields the adjoint groups, isomorphic to $PGL_{l+1}(\bar{k})$.

Assume that $G$ is of type $(A_l)_e$.

Then $K$ is the subgroup of $Z$ of order $((l+1)/e)p' = m/e_{p'}$ (the group generated by $e_{p'}z$).

For the Frobenius morphism $F_i$ on $G_{sc}$ and $i \in \mathbb{Z}$ we have $F_i(iz) = iz$ if and only if $(q - \epsilon)i \in m\mathbb{Z}$ if and only if $(m/\gcd(m,q - \epsilon)) \mid i$.

Combining, we find that the group $K^{F_i}$ is the subgroup of $Z$ of order

$$d := \gcd(m/e_{p'}, \gcd(m,q - \epsilon)) = \gcd(m/e_{p'}, q - \epsilon) = \gcd((l+1)/e, q - \epsilon)$$

(the last equation because $q - \epsilon$ is prime to $p$).

We have found that the parameter group $(B)$ is cyclic of order $d$. The set $(A)$ consists of the $q$-restricted weights which are trivial on the generator $(m/d)z$ of $K^{F_i}$. These are the $(\lambda_1, \ldots, \lambda_l) \in E_{l,q}$ such that

$$\sum_{i=1}^l i\lambda_i \equiv 0 \mod d. \quad (1)$$

4.1.2. Types $B_l$ and $C_l$. In these two cases, the center $Z$ of $G_{sc}$ has order $m = \gcd(2, p+1)$ and has the following generators.

| Type  | Generator                  |
|-------|----------------------------|
| $B_l$ | $(\frac{1}{m}, 0, \ldots, 0)$ |
| $C_l$ | $(\frac{1}{m}, \frac{l-1}{m}, \ldots, \frac{2}{m}, 0, \frac{1}{m})$ |

There are two possibilities for $G$, the simply-connected type where $K$ and so $K^F$ are trivial, and the adjoint type where $K = Z$ and clearly $K^F = \bar{K}$ (since $K$ is of order 1 or 2).

So, when $p = 2$ or $G$ is simply-connected then the parameter group $(B)$ is trivial and the parameter set $(A)$ consists of all $q$-restricted weights. Otherwise, for odd $q$ and $G$ of adjoint type, the group $(B)$ is of order 2 and the parameter set $(A)$ consists of the weights
\((\lambda_1, \ldots, \lambda_l) \in \mathcal{E}_{l,q}\) satisfying the following equation. In the case of type \(B_l\), the equation is
\[
(2) \quad \lambda_1 \equiv 0 \mod 2,
\]
and in case of type \(C_l\) it depends on the parity of \(l\). This is
\[
(3) \quad \sum_{1 \leq i \leq l, \; i \text{ even}} \lambda_i \equiv 0 \mod 2 \quad \text{or} \quad \sum_{1 \leq i \leq l, \; i \text{ odd}} \lambda_i \equiv 0 \mod 2,
\]
according to \(l\) being odd or even.

4.1.3. Type \(D_l\), \(l \geq 4\). Assume that \(l = 2k + 1\) is odd. Then the center \(Z\) is cyclic of order \(m = 4\) for odd \(p\) and \(m = 1\) for \(p = 2\), and is generated by
\[
z = \left( \frac{1}{m}, \frac{3}{m}, \frac{2}{m}, 0, \frac{2}{m}, 0, \frac{2}{m}, \ldots, 0, \frac{2}{m} \right).
\]
There are three possibilities for \(G\), the simply-connected type for which \(K\) is trivial, or \(\mathbb{Z}R\) is of index 2 in \(X\), then \(G\) is isomorphic to \(SO_{2l}(k)\) and \(K\) is generated by \(2z\), or the group of adjoint type where \(K = Z\).

So, if \(p = 2\) or if \(G\) is simply-connected then the parameter group \((B)\) is trivial and \((A)\) consists of all \(q\)-restricted weights.

If \(p\) is odd and \(G\) of type \(SO\), then \(K^{F_q} = K\) (since \(2z\) is the only element of order 2 in \(Z\)), so the group \((B)\) is of order 2. In this case, \(G^F = SO_{2l}(q)\), the parameter set \((A)\) consists of the weights \((\lambda_1, \ldots, \lambda_l) \in \mathcal{E}_{l,q}\) such that
\[
(4) \quad \lambda_1 + \lambda_2 \equiv 0 \mod 2.
\]

Let \(p\) be odd and \(G\) be of adjoint type, then \(K = Z\). If \(q \equiv \epsilon \mod 4\) then \(K^{F_q} = K\), so the parameter group \((B)\) is cyclic of order 4 and the parameter set \((A)\) consists of the weights \((\lambda_1, \ldots, \lambda_l) \in \mathcal{E}_{l,q}\) such that
\[
(5) \quad \sum_{i=1}^{k} 2\lambda_{2i+1} \equiv \lambda_2 - \lambda_1 \mod 4.
\]
Otherwise, if \(q \equiv -\epsilon \mod 4\) then \(K^{F_q}\) is of order 2 and the parameter sets \((B)\) and \((A)\) are the same as in the SO-case.

4.1.4. Type \(D_l\), \(l \geq 4\). Assume now that \(l = 2k\) is even. Then \(Z\) is elementary abelian of order 4 if \(p\) is odd and trivial if \(p = 2\). If \(p\) is odd then \(Z\) is generated by
\[
z_1 = \left( \frac{1}{2}, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots, 0, \frac{1}{2} \right) \quad \text{and} \quad z_2 = \left( 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots, 0, \frac{1}{2} \right),
\]
if \(p = 2\) we set \(z_1 = z_2 = 1\).

Here, for any \(q\), \(F_1\) is the identity on \(Z\), \(F_{-1}\) permutes \(z_1\) and \(z_2\), and in case \(l = 4\) the Frobenius \(F_3\) permutes \(z_1, z_2\) and \(z_1 + z_2\) cyclically.

If \(G\) is simply-connected or \(p = 2\), then \(K = K^F = 1\), the parameter group \((B)\) is trivial and the set \((A)\) consists of all \(q\)-restricted weights.

If the index of \(ZR\) in \(X\) is 2, there are two possibilities for \(G\). Either \(G\) has only Frobenius morphisms of type \(F_1\), then \(G\) is isomorphic to a half spin group \(HSpin_{2l}(k)\) and \(K = K^F\) is generated by \(z_1\) (or by \(z_2\)). In this case, for odd \(p\), the parameter group \((B)\) is of order 2 and the set \((A)\) consists of the weights \((\lambda_1, \ldots, \lambda_l) \in \mathcal{E}_{l,q}\) such that
\[
(6) \quad \lambda_2 + \lambda_4 + \cdots + \lambda_{2k} \equiv 0 \mod 2.
\]
Otherwise, $K$ is generated by $z_1 + z_2$ and $K = K^{F_1}$ for $\epsilon \in \{\pm 1\}$. Then $G$ is isomorphic to a special orthogonal group $SO_{2l}(k)$ and $G^{F_1}$ is isomorphic to $SO_{2l}(q)$. For odd $p$ the parameter group (B) is also of order 2 and the set (A) consists of the weights $(\lambda_1, \ldots, \lambda_l) \in E_{l,q}$ such that

$$\lambda_1 + \lambda_2 \equiv 0 \mod 2.$$  

The final possibility is that $G$ is of adjoint type and $K = Z$. Then $K^{F_2}$ is generated by $z_1 + z_2$ and for odd $p$ and $\epsilon = -1$ we get the same parameter sets (B) and (A) as in the SO-case. Furthermore, we have $K^{F_1} = K$, so for odd $p$ and $\epsilon = 1$ the parameter group (B) is elementary abelian of order 4 and the parameter set (A) consists of the weights $(\lambda_1, \ldots, \lambda_l) \in E_{l,q}$ such that

$$\begin{cases} 
\sum_{i=2}^{k} \lambda_{2i} \equiv \lambda_1 \mod 2, \\
\lambda_1 = \lambda_2 \mod 2.
\end{cases}$$

If $l = 4$ then $K^{F_3} = 1$ and we get the same parameterization as in the simply connected case for $F = F_3$.

4.1.5. Types $G_2$, $F_4$ and $E_6$. In these cases $Z$ and so $K = K^F$ and the parameter group (B) is trivial. The set (A) consists of all $q$-restricted weights.

4.1.6. Type $E_6$. The group $Z$ is cyclic of order $m = 3$ if $p \neq 3$ and $m = 1$ if $p = 3$, it is generated by $z = (\frac{1}{m}, 0, \frac{2}{m}, 0, \frac{1}{m}, \frac{2}{m})$. The group $G$ can either be simply-connected or of adjoint type.

If $G$ is simply-connected or $p = 3$ then $K = K^{F_1} = 1$, the parameter group (B) is trivial and (A) consists of all $q$-restricted weights.

If $G$ is of adjoint type then $K = Z$. We have $K^{F_1} = K$ if $q \equiv \epsilon \mod 3$. In that case the parameter group (B) is cyclic of order 3 and the parameter set (A) consists of the weights $(\lambda_1, \ldots, \lambda_6) \in E_{6,q}$ such that

$$\lambda_1 - \lambda_3 + \lambda_5 - \lambda_6 \equiv 0 \mod 3.$$  

For $q \equiv -\epsilon \mod 3$ we have $K^{F_2} = 1$ and the parameter sets are as in the simply-connected case.

4.1.7. Type $E_7$. The group $Z$ is cyclic of order $m = 2$ if $p \neq 2$ and $m = 1$ if $p = 2$, it is generated by $z = (0, \frac{1}{m}, 0, 0, \frac{1}{m}, 0, \frac{1}{m})$. The group $G$ is either simply-connected or of adjoint type.

If $G$ is simply-connected or if $p = 2$ then $K = K^F$ is trivial, the parameter group (B) is trivial and the set (A) consists of all $q$-restricted weights.

If $G$ is of adjoint type then $K = K^F = Z$. For odd $p$ the parameter group (B) is of order 2 and the set (A) consists of the weights $(\lambda_1, \ldots, \lambda_7) \in E_{7,q}$ such that

$$\lambda_2 + \lambda_5 + \lambda_7 \equiv 0 \mod 2.$$  

4.2. Application: number of semisimple classes. In this section, we will compute the number of isomorphism classes of irreducible $\overline{F}_q$-modules (or, equivalently, the number of semisimple classes) of the finite groups $G^F$ for all simple algebraic groups $G$ defined over $\overline{F}_q$. 


Theorem 4.1. (a) Let \( G \) be a connected reductive group of semisimple rank \( l \), such that its derived group \( G' \) is simply-connected. Let \( Z(G)^0 \) be the connected component of the center of \( G \). We assume that \( G \) is defined over \( \mathbb{F}_q \) and denote \( F : G \to G \) the corresponding Frobenius morphism. Then the number of semisimple conjugacy classes of \( G^F \) is \( q^{|(Z(G)^0)^F|} \). In particular, if \( G \) is simply-connected this number is \( q^l \).

(b) Now let \( G \) be a simple connected reductive group of rank \( l \), defined over \( \mathbb{F}_q \) with corresponding Frobenius morphism \( F \). Then the number of semisimple conjugacy classes of \( G^F \) is either \( q^l \), or it is given in Table 2.

Proof. (a) This follows from Theorem 3.5. Under the given assumptions the parameter set (B) is trivial, and the set (A) contains all \( q \)-restricted weights. The parameter set (C) contains \(|(G/G')^F| = |(Z(G)^0)^F| \) elements. See also [6, 3.7.6(ii)] for a completely different proof of this result.

(b) This will be shown in the rest of this section. Here the parameter group (C) is always trivial. We need to go through all the cases of subsection 4.1. Whenever the group (B) is non-trivial (B) and find the cardinalities of the sets (A) by counting the solutions of certain modular equations.

We denote by \( \Lambda \) the set of parameters (A) for the group \( G^F = (G_{sc}/K)^F \). By Theorem 3.5, the number of isomorphism classes of irreducible \( \mathbb{F}_q \)-modules of \( G^F \) is \( \left| K^F \right| \cdot |\Lambda| \).

The following lemma will be useful in several cases.

Lemma 4.2. Assume that \( q \) is odd, and for any positive integers \( n \) and \( \nu \in \{0, 1\} \), define

\[
E_{n, \nu} = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid 0 \leq \lambda_i \leq q - 1, \quad \sum_{i=1}^n \lambda_i \equiv \nu \mod 2 \right\}.
\]

Then, we have

\[
|E_{n, \nu}| = \frac{q^n + 1 - 2\nu}{2}
\]

Proof. This follows easily by induction on \( n \). \( \square \)

Types \( B_l \) and \( C_l \). We must consider the case that \( p \neq 2 \) and \( K = Z \). Then \( G \) is of adjoint type. If \( G \) is of type \( B_l \) we use equation (2) and obtain

\[
\Lambda = \{ (\lambda_1, \ldots, \lambda_l) \in \mathcal{E}_{l,q} \mid \lambda_i \in 2\mathbb{Z} \}.
\]

Thus, Lemma 4.2 gives \( |\Lambda| = q^{l-1} \cdot \frac{q^l + 1}{2} \). Now, since \( |K^F| = 2 \), the entry for case \( B_l \) in Table 2 follows.

If \( G \) is of type \( C_l \) with \( l = 2k \) (resp. \( l = 2k + 1 \)), then in equation (3) there are \( k \) summands (resp. \( k + 1 \) summands) in the sum. Hence, Lemma 4.2 gives

\[
|\Lambda| = q^k \cdot \frac{q^k + 1}{2} \quad (\text{resp. } q^k \cdot \frac{q^{k+1} + 1}{2}).
\]

Since \( k = \lfloor l/2 \rfloor \) and \( |K^F| = 2 \), the entry for type \( C_l \) in Table 2 follows.

Type \( D_l \). We only need to consider the case \( p \neq 2 \). First assume that \( l = 2k \). We compute the number of elements in the set (A) for \( G_{ad}^{F_{ad}} \) using Equation (8) as follows. If \( \lambda_1 \) is odd, then \( \lambda_2 \) is odd. This implies that \( \lambda_1 + \lambda_6 + \cdots + \lambda_{2k} \in 2\mathbb{Z} + 1 \). By Lemma 4.2, there are \( q^{k-1} \left( \frac{q+1}{2} \right)^2 \cdot \frac{q^{k-1} + 1}{2} \) such solutions. Similarly, there are \( q^{k-1} \left( \frac{q+1}{2} \right)^2 \cdot \frac{q^{k+1} + 1}{2} \)
solutions such that \( \lambda_1 \) is even. Therefore, we deduce that

\[
|\Lambda| = q^{k-1} \cdot \frac{q^{k+1} + 2q + q^{k-1}}{4}.
\]

In the same way, using lemma 4.2, we count the number of solutions of equations (7) and (6), giving \(|\Lambda|\) for \( \text{SO}_2(q) \) and \( \text{HSpin}_2(q) \), respectively.

Suppose now that \( l = 2k + 1 \). Assume that \( K = Z \) and that \( q \equiv \epsilon \mod 4 \). By equation (5) we have to find the number of solutions \( (\lambda_1, \ldots, \lambda_{2k}) \in \mathcal{E}_{2k,q} \) of \( 2(x_3 + x_5 + \cdots + x_{2k+1}) + x_1 - x_2 \in 4\mathbb{Z} \). There are \( q^{k-1} \cdot \frac{q^{k+1}}{2} \) tuples \((\lambda_1, \ldots, \lambda_{2k-1})\) with \( 0 \leq \lambda_1 \leq q - 1 \), such that \((\lambda_3 + \lambda_5 + \cdots + \lambda_{2k+1})\) is even. For each tuple, we have to find the number of solutions of \( \lambda_1 - \lambda_2 \in 4\mathbb{Z} \). There are \( (\frac{q+3}{4})^2 + 3 \cdot (\frac{q-1}{4})^2 \) such solutions. Thus, there are

\[
n_0 = \frac{1}{2} \cdot q^{k-1} \cdot (q^{k+1} + 3q^k + q^2 + 3),
\]

solutions \((\lambda_1, \ldots, \lambda_{2k+1}) \in \mathcal{E}_{2k+1,q}\), such that \( \lambda_1 + \lambda_3 + \cdots + \lambda_{2k+1} \) is even. Similarly, there are \( n_1 = \frac{1}{8} \cdot q^{k-1} \cdot (q^{k+2} - q^k - q^2 + 1) \) solutions \((\lambda_1, \ldots, \lambda_{2k+1}) \in \mathcal{E}_{2k+1,q}\), such that \( \lambda_1 + \lambda_3 + \cdots + \lambda_{2k-1} \) is odd. Adding up we find for the case \( |K^F_\mathcal{A}| = 4 \) that

\[
|\Lambda| = n_0 + n_1 = q^{k-1} \cdot \frac{q^{k+2} + q^k + 2}{4}.
\]

Finally, again with lemma 4.2, we count the solutions of equation (4) and obtain \(|\Lambda|\) for the cases \( \text{SO}_2(q) \), and for the cases with \( G \) of adjoint type and \( q \equiv -\epsilon \mod 4 \).

**Types \( E_6 \) and \( E_7 \).** We only need to consider \( G \) of adjoint type. In type \( E_6 \) with \( q \equiv \epsilon \mod 3 \) we compute \(|\Lambda|\) using equation (9) and lemma 3.8. For type \( E_7 \) and odd \( p \) we conclude as above using equation (10) and lemma 4.2.

**Type \( A_1 \).** We have postponed this case because it is a bit trickier to derive a closed formula for the cardinality \(|\Lambda|\). We need to count the solutions of equation (1) to find the first line of table 2. Instead of counting the solutions of equation (1) we can introduce...
another coordinate \( \lambda_0 \), count the solutions of the equation

\[
\sum_{i=0}^{l} i \lambda_i \equiv 0 \mod d
\]

with \( 0 \leq \lambda_i < q \) for \( 0 \leq i \leq l \), and divide the result by \( q \).

The number of solutions of this modified equation follows from the following lemma applied with \( n = l + 1 \) and \( m = d \).

**Lemma 4.3.** Let \( n \geq 2 \) be an integer, and \( m \mid n \). Let \( \epsilon \in \{-1, 1\} \). For any positive integer \( q \) with \( m \mid (q - \epsilon) \) define \( E = \{0, \ldots, q - 1\} \) and write \( X = \{(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \in E^n \mid \sum_{i=0}^{n-1} i \lambda_i \equiv 0 \mod m\} \). Then

\[
|X| = \frac{1}{m} \sum_{d \mid m} \phi(d) q^{n/d}.
\]

**Proof.** The following proof was shown to us by Darij Grinberg, who generously allowed us to include it in this article.

For a non-negative integer \( t \) let

\[
X_t = \{(\lambda_0, \ldots, \lambda_{n-1}) \in E^n \mid \sum_{i=0}^{n-1} i \lambda_i = t\}.
\]

We want to investigate \( |X| = \sum_{t \in \mathbb{Z}/m} |X_t| \).

For integers \( j \) with \( 0 \leq j \leq n - 1 \) we consider the polynomials

\[
P_j(z) = 1 + z^j + z^{2j} + \ldots + z^{(q-1)j} \in \mathbb{C}[z].
\]

It is clear that \( |X_t| \) is the coefficient of \( z^t \) in the product of the \( P_j(z) \):

\[
P(z) = \prod_{j=0}^{n-1} P_j(z) = \sum_{t=0}^{\infty} |X_t| z^t.
\]

Now let \( \zeta \in \mathbb{C} \) be a primitive \( m \)-th root of unity. We will use repeatedly the fact that for \( t \in \mathbb{Z} \) the sum \( \sum_{k=0}^{m-1} \zeta^{tk} \) equals \( m \) if \( m \mid t \) and equals \( 0 \) otherwise (use the formula for geometric sums).

We evaluate

\[
\sum_{k=0}^{m-1} P(\zeta^k) = \sum_{k=0}^{m-1} \sum_{t=0}^{\infty} |X_t| \zeta^{kt} = \sum_{t=0}^{\infty} |X_t| \sum_{k=0}^{m-1} (\zeta^{t})^k = \sum_{t=0}^{\infty} |X_t| \cdot m = m|X|.
\]

From now we fix a \( k \in \mathbb{Z} \) with \( 0 \leq k \leq m - 1 \) and set \( d = \frac{m}{\gcd(m, k)} \).

We will show that

\[
P(\zeta^k) = q^{n/d}.
\]

This proves the lemma, because for \( d \mid m \) we have

\[
|\{0 \leq k < m \mid d = \gcd(m, k)\}| = \frac{m}{d} \cdot \{|0 \leq i < d \mid \gcd(i, d) = 1\}| = \phi(d).
\]

It is easy to evaluate each \( P_j(z) \) at \( \zeta^k \) for \( 0 \leq j \leq n - 1 \):

\[
P_j(\zeta^k) = 1 + \zeta^{jk} + (\zeta^{jk})^2 + \ldots + (\zeta^{jk})^{q-1} = \begin{cases} q, & \text{if } m \mid jk \\ 1, & \text{if } m \nmid jk \text{ and } \epsilon = 1 \\ -\zeta^{-kj}, & \text{if } m \nmid jk \text{ and } \epsilon = -1 \end{cases}
\]
because for $m \nmid j k$ every $m$ consecutive summands sum up to 0. In case $m \mid (q - 1)$ it remains the last summand which is 1. And in case $m \mid (q + 1)$ an additional summand $(\zeta^{j k})^q = \zeta^{-j k}$ cancels all the previous ones.

For an integer $k, d = m/\gcd(m, k)$ and $j \in \mathbb{Z}$ we have that $m \mid j k$ if and only if $d \mid j$.

Since $m \mid n$ we also have $d \mid n$ and so there are $n/d$ indices $j$ with $0 \leq j \leq n - 1$ and $P_j(\zeta^k) = q$.

In case $m \mid (q - 1)$ we have $P_j(\zeta^k) = 1$ for the remaining $j$ with $d \nmid j$. Taking the product we get

$$P(\zeta^k) = q^{n/d}.$$  

To see that the same is true in case $m \mid (q + 1)$ we must show that

$$\prod_{j=0}^{n-1} (-\zeta^{-j k}) = 1.$$  

The root of unity $\zeta^k$ and so also $\zeta^{-k}$ has order $d (= m/\gcd(m, k))$. So, if $d \mid j$ we have $(\zeta^{-k})^j = 1$ and we get

$$\prod_{j=0}^{n-1} (-\zeta^{-j k}) = (-1)^{n-n/d} \prod_{j=0}^{n-1} (\zeta^{-k})^j = (-1)^{n-n/d} \cdot (\zeta^{-k})^{n(n-1)/2}.$$  

Since $d \mid n$, we have: $d \nmid n(n-1)/2$ iff $(d \text{ is even and } n/d \text{ is odd})$ iff $(n/d) (d - 1) = n - n/d$ is odd, and in this case $(d/2) \mid n(n - 1)/2$ so that $(\zeta^{-k})^{n(n-1)/2} = -1$. This shows that the right hand side in the last displayed equation is always 1.

This proves the lemma.

\begin{remark}
The results given in table 2 are not new. They were worked out in [3] using sophisticated results from the ordinary representation theory of the groups $G^F$ in good characteristic. The completely different approach in this section is more elementary (and it works for arbitrary root data and characteristics).

For small rank groups, in particular the exceptional types, detailed parameterizations of all conjugacy classes were computed, this also yields the number of semisimple conjugacy classes, see [15].
\end{remark}

\section*{References}

[1] N. Blackburn and B. Huppert. \textit{Finite Groups II}. Springer Verlag, Berlin, 1981.

[2] A. Borel et al. \textit{Seminar on Algebraic Groups and Related Finite Groups}, volume 131 of \textit{Lecture Notes in Mathematics}. Springer Verlag, Berlin, 1970.

[3] O. Brunat. Counting $p'$-characters in finite reductive groups. \textit{J. Lond. Math. Soc.} (2), 81(3):544–562, 2010.

[4] M. Cabanes. Brauer morphism between modular Hecke algebras. \textit{J. Algebra}, 115(1):1–31, 1988.

[5] R. W. Carter. \textit{Simple Groups of Lie Type}. A Wiley-Interscience publication, London, 1972.

[6] R. W. Carter. \textit{Finite Groups of Lie Type - Conjugacy Classes and Complex Characters}. A Wiley-Interscience publication, Chichester, 1985.

[7] C. W. Curtis. Modular representations of finite groups with split $(B, N)$-pairs. In \textit{Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)}, pages 57–95. Springer, Berlin, 1970.

[8] F. Digne and J. Michel. \textit{Representations of Finite Groups of Lie Type}. Cambridge University Press, Cambridge, 1991.

[9] M. Geck. \textit{An introduction to algebraic geometry and algebraic groups}, volume 10 of \textit{Oxford Graduate Texts in Mathematics}. Oxford University Press, Oxford, 2003.
[10] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE—a system for computing and processing generic character tables. *Appl. Algebra Engrg. Comm. Comput.*, 7(3):175–210, 1996. Computational methods in Lie theory (Essen, 1994).

[11] F. Herzig. The weight in a Serre-type conjecture for tame $n$-dimensional Galois representations. *Duke Math. J.*, 149(1):37–116, 2009.

[12] J. E. Humphreys. *Reflections groups and Coxeter groups*, volume 29 of *Cambridge studies in advanced mathematics*. Cambridge Univ. Press, 1990.

[13] I. M. Isaacs. *Character theory of finite groups*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.

[14] J. C. Jantzen. *Representations of algebraic groups*. Mathematical surveys and monographs. American Mathematical Society, 2003.

[15] F. Lübeck. Numbers of conjugacy classes of finite groups of Lie type. http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/nrclasses/nrclasses.html.

[16] F. Lübeck. Small degree representations of finite Chevalley groups in defining characteristic. *LMS J. Comput. Math.*, 4:135–169 (electronic), 2001.

[17] M. Schönert et al. GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997.

[18] T. A. Springer. *Linear Algebraic Groups, Second Edition*, volume 9 of *Progress in Mathematics*. Birkhäuser, Boston, 1998.

[19] R. Steinberg. *Endomorphisms of linear algebraic groups*. Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968.

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