Upper bounds for Fourier decay rates of fractal measures

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ABSTRACT
For spherical and parabolic averages of the Fourier transform of fractal measures, we obtain new upper bounds on rates of decay by an ‘intermediate dimension’ trick.

1. Introduction
This paper is concerned with average decay rates of the Fourier transform of fractal measures. First recall the notation of ‘α-dimensional’ [11].

\[ c_\alpha(\mu) := \sup_{x \in \mathbb{R}^d, r > 0} \frac{\mu(B(x, r))}{r^\alpha} < \infty. \]

Let \( S \) be a bounded hypersurface in \( \mathbb{R}^d \) with everywhere non-vanishing Gaussian curvature and let \( d\sigma \) be the induced Lebesgue measure on \( S \). We use \( \beta_d(\alpha, S) \) to denote the average Fourier decay rate of fractal measures, which is defined as the supremum of the numbers \( \beta \) for which

\[ \| \hat{\mu}(R \cdot) \|_{L^2(S, d\sigma)}^2 \lesssim c_\alpha(\mu) \| R^{-\beta}, \]

whenever \( R > 1 \) and \( \mu \) is \( \alpha \)-dimensional. In this paper, we will focus on the case \( S \) is the unit sphere \( S^{d-1} \) or the truncated paraboloid \( \mathbb{P}^{d-1} \).

The problem of identifying the value of \( \beta_d(\alpha, S^{d-1}) \) was proposed by Mattila [13], and it relates to the classical distance set conjecture of Falconer [7].

In dimension two, the exact decay rates are known:

\[ \beta_2(\alpha, S) = \begin{cases} 
\alpha, & \alpha \in (0, 1/2], 
1/2, & \alpha \in [1/2, 1], 
\alpha/2, & \alpha \in [1, 2], 
\end{cases} \text{ (Mattila [12])} \]

In higher dimensions, it is known that \( \beta_d(\alpha, S) = \alpha \) in the range \( \alpha \in (0, d-1/2) \), but \( \beta_d(\alpha, S) \) is still a mystery for \( d-1/2 < \alpha < d \). The current best lower bounds are

\[ \beta_d(\alpha, S) \geq \begin{cases} 
\alpha, & \alpha \in (0, d-1/2], 
(d-1)/d, & \alpha \in [d-1/2, d/2], 
(d-1)\alpha/d, & \alpha \in [d/2, d]. 
\end{cases} \text{ (Mattila [12])} \]

\( \beta_d(\alpha, S) \) is still unknown for \( \alpha \in (d-1/2, d) \), but it is known that

\[ \beta_d(\alpha, S) \geq \begin{cases} 
\alpha, & \alpha \in (0, d-1/2], 
(d-1)/d, & \alpha \in [d-1/2, d/2], 
(d-1)\alpha/d, & \alpha \in [d/2, d]. 
\end{cases} \text{ (Mattila [12])} \]

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We remark that the above results were originally computed for either $S^{d-1}$ or $P^{d-1}$. It is, however, implicit in the arguments given in [4, 6, 12, 15] that the same estimates hold for any bounded hypersurface $S$ with everywhere non-vanishing Gaussian curvature (see, for example, [3] for a generalization of [6] to a class of hypersurfaces).

Unlike the results for lower bounds, the upper bounds for decay rates are usually obtained by constructing explicit examples and thus the results depend on the hypersurface $S$. The previous best results before this paper are summarized as follows: for the unit sphere, when $d = 3$,

$$\beta_3(\alpha, S^2) \leq \begin{cases} \alpha, & \alpha \in (0, 1], \\ \frac{\alpha + 1}{2}, & \alpha \in [1, 3], \end{cases} \quad \text{([14, Chapter 15.2])}$$

and when $d \geq 4$,

$$\beta_d(\alpha, S^{d-1}) \leq \begin{cases} \alpha, & \alpha \in \left(0, \frac{d}{2}\right], \\ \alpha - 1 + \frac{2(d - \alpha)}{d}, & \alpha \in \left[\frac{d}{2}, d\right], \end{cases} \quad \text{([14])}$$

for the truncated paraboloid and $d \geq 3$,

$$\beta_d(\alpha, P^{d-1}) \leq \begin{cases} \alpha, & \alpha \in \left(0, \frac{d - 1}{2}\right], \\ \frac{(d - 1)(\alpha + 1)}{d + 1}, & \alpha \in \left[\frac{d - 1}{2}, d\right], \end{cases} \quad \text{([14])}$$

for the truncated paraboloid and $d \geq 3$.

It is worth mentioning that when $\alpha = d - 1$, one can find a better upper bound of $\frac{(d - 1)^2}{d}$ by examining an example of Bourgain [2] carefully. As this upper bound coincides with the lower bound established in [4, 6], the exact decay rate can be determined in this case:

$$\beta_d(d - 1, P^{d-1}) = \frac{(d - 1)^2}{d}.$$

Bourgain’s example is a Schrödinger solution essentially supported in a small neighborhood of a hyperplane. Recently, the authors of [5] extended Bourgain’s idea to intermediate dimensions and disproved Schrödinger maximal estimates in certain range. In this paper, we further explore this ‘intermediate dimension’ trick to adapt the examples from [1, 11] and obtain improved upper bounds of Fourier decay rates.

We first state the results for spheres. For convenience of notation, we introduce the following functions $\kappa_1$ and $\kappa_2$:

$$\kappa_1(m; \alpha, d) := \frac{d - m}{d - m/2 - \alpha}, \quad \kappa_2(m; \alpha, d) := \frac{d - \alpha}{2(d - m)}.$$

(1.2)

For $\kappa_1$ and $\kappa_2$, we are only interested in the cases that $\alpha \in (d/2, d)$ and $m$ is an integer with $0 < m < d/2$. In this range, for fixed $\alpha$ and $d$, as $m$ increases, $\kappa_1(m; \alpha, d)$ decreases and $\kappa_2(m; d, \alpha)$ increases.

**Theorem 1.1.** Let $d \geq 4$ and $\alpha \in (d/2, d)$. Then

$$\beta_d(\alpha, S^{d-1}) \leq \alpha - 1 + 2\kappa(\alpha, d),$$

where $\kappa(\alpha, d)$ is defined as

$$\kappa(\alpha, d) := \max\left\{\kappa_1(m; \alpha, d), \kappa_2(m; \alpha, d)\right\}.$$
where $\kappa(\alpha, d)$ is given as follows:

(a) for $\alpha \in [d - 1, d)$,

$$\kappa(\alpha, d) = \kappa_2(1; \alpha, d) = \frac{d - \alpha}{2(d - 1)};$$

(b) for $\alpha \in [d - j, d - j + 1]$ with $j = 2, 3, \ldots, \lfloor \frac{d}{2} \rfloor$,

$$\kappa(\alpha, d) = \min \{\kappa_1(j - 1; \alpha, d), \kappa_2(j; \alpha, d)\} = \begin{cases} \kappa_2(j; \alpha, d), & d - j \leq \alpha \leq d - j + \frac{d - 2j}{d - j - 1}, \\ \kappa_1(j - 1; \alpha, d), & d - j + \frac{d - 2j}{d - j - 1} \leq \alpha \leq d - j + 1; \end{cases}$$

(c) for $d$ even and $\alpha \in (\frac{d}{2}, \frac{d}{2} + 1]$, 

$$\kappa(\alpha, d) = \kappa_1\left(\frac{d}{2} - 1; \alpha, d\right) = \frac{3d + 2 - 4\alpha}{2(d + 2)};$$

(d) for $d$ odd and $\alpha \in (\frac{d}{2}, \frac{d + 1}{2}]$,

$$\kappa(\alpha, d) = \kappa_1\left(\frac{d}{2} - 1; \alpha, d\right) = \frac{3d + 1 - 4\alpha}{2(d + 1)}.$$
(b) for \( \alpha \in [d - j, d - j + 1] \) with \( 2 \leq j \leq \lfloor \frac{d+1}{3} \rfloor \),
\[
\tilde{\kappa}(\alpha, d) = \min \{ \kappa_3(j - 1; \alpha, d), \kappa_4(j; \alpha, d) \}
\]
\[
= \begin{cases} 
\kappa_4(j; \alpha, d), & d - j \leq \alpha \leq d - j + \frac{d - 2j + 1}{d - j}, \\
\kappa_3(j - 1; \alpha, d), & d - j + \frac{d - 2j + 1}{d - j} \leq \alpha \leq d - j + 1;
\end{cases}
\]

(c) for \( \alpha \in [d - j, d - j + 1] \) with \( j = \lfloor \frac{d+1}{3} \rfloor + 1 \),
\[
\tilde{\kappa}(\alpha, d) = \kappa_3 \left( \frac{d + 1}{3} \right) ; \alpha, d \right);
\]

(d) for \( \alpha \in [d - j, d - j + 1] \) with \( \lfloor \frac{d+1}{3} \rfloor + 2 \leq j \leq \lfloor \frac{d}{2} \rfloor \),
\[
\tilde{\kappa}(\alpha, d)
\]
\[
= \min \{ \kappa_3(j - 2; \alpha, d), \max \{ \kappa_3(j - 1; \alpha, d), \kappa_5(j - 1; \alpha, d) \}, \kappa_5(j; \alpha, d) \}
\]
\[
= \begin{cases} 
\min \{ \kappa_3(j - 1; \alpha, d), \kappa_5(j; \alpha, d) \}, & d - j \leq \alpha \leq d - j + \frac{2(d - 2j + 1)}{d - j - 2}, \\
\min \{ \kappa_3(j - 2; \alpha, d), \kappa_5(j - 1; \alpha, d) \}, & d - j + \frac{2(d - 2j + 1)}{d - j - 2} \leq \alpha \leq d - j + 1;
\end{cases}
\]

(e) for \( d \) odd, \( d \geq 7 \) and \( \alpha \in (\frac{d-1}{2}, \frac{d+1}{2}] \),
\[
\tilde{\kappa}(\alpha, d) = \kappa_3 \left( \frac{d - 3}{2} \right) = \frac{3d + 3 - 4\alpha}{2(d + 5)} ;
\]

(f) for \( d \) even and \( \alpha \in (\frac{d-1}{2}, \frac{d}{2}] \),
\[
\tilde{\kappa}(\alpha, d) = \kappa_3 \left( \frac{d}{2} - 1; \alpha, d \right) = \frac{3d + 2 - 4\alpha}{2(d + 4)} .
\]

Note that the previous best upper bound from [1] is equivalent to saying that for \( d \geq 3 \) and \( \alpha \in (\frac{d-1}{2}, d) \),
\[
\beta_d(\alpha, \mathbb{R}^d) \leq \frac{(d - 1)(\alpha + 1)}{d + 1} = \alpha - 1 + 2\kappa_3(0; \alpha, d).
\]
Since \( \kappa_3(m; \alpha, d) \) is a decreasing function of \( m \) and
\[
\kappa_4(m; \alpha, d) < \kappa_3(0; \alpha, d) \quad \text{for} \quad m < \frac{d + 1}{2},
\]
we see that Theorem 1.2 is an improvement in the whole range stated in the theorem.

**Remark 1.3.** It is straightforward to check \( \tilde{\kappa}(\alpha, d) < \kappa(\alpha, d) \). In other words, the examples for parabolic decay rates are better than those for spherical decay rates.

By combining part (a) of Theorem 1.2 and the lower bounds from [4, 6], we can now determine the exact value of the parabolic Fourier decay rates for \( \alpha \in [d - 1, d) \). We record this result in the following corollary.
Corollary 1.4. Let \( d - 1 \leq \alpha < d \) and \( d \geq 3 \). Then
\[
\beta_d(\alpha, \mathbb{P}^{d-1}) = \alpha - 1 + \frac{d - \alpha}{d} = \frac{(d - 1)\alpha}{d}.
\]

Remark 1.5. To get a feeling about the numerology in Theorem 1.2, let us explicitly write out \( \tilde{\kappa}(\alpha, d) \) with \( \alpha \in \left( \frac{d-1}{2}, d - 1 \right] \) for some small values of \( d \). This will also be useful in the next remark.

- For \( d = 3, 4 \),
  \[
  \tilde{\kappa}(\alpha, d) = \kappa_3(1; \alpha, d) = \frac{2d - 1 - 2\alpha}{2d}, \quad \frac{d - 1}{2} < \alpha \leq d - 1.
  \]
- For \( d = 5, 6, 7 \),
  \[
  \tilde{\kappa}(\alpha, d) = \begin{cases} 
  \kappa_3(2; \alpha, d) = \frac{d - 1 - \alpha}{d - 1}, & \frac{d - 1}{2} < \alpha \leq d - 2, \\
  \kappa_4(2; \alpha, d) = \frac{d - \alpha}{2(d - 1)}, & d - 2 \leq \alpha \leq d - 2 + \frac{d - 3}{d - 2}, \\
  \kappa_3(1; \alpha, d) = \frac{2d - 1 - 2\alpha}{2d}, & d - 2 + \frac{d - 3}{d - 2} \leq \alpha \leq d - 1.
  \end{cases}
  \]

The situation becomes more complicated for larger \( d \), and \( \kappa_5(m; \alpha, d) \) will also come into play when \( d \) is large enough.

Remark 1.6. Let us see what we can tell about Falconer’s distance set conjecture from our new theorems.

(a) For \( \alpha \) close to and greater than \( d/2 \), Theorem 1.2 tells us that \( \beta_d(\alpha, \mathbb{P}^{d-1}) \leq \alpha - 1 + 2\tilde{\kappa}(\alpha, d) \), where
\[
\tilde{\kappa}(\alpha, 3) = \kappa_3(1; \alpha, 3) = \frac{5 - 2\alpha}{6}, \quad \tilde{\kappa}(\alpha, 5) = \kappa_3(2; \alpha, 5) = \frac{4 - \alpha}{4}, \\
\tilde{\kappa}(\alpha, d) = \kappa_3\left(\frac{d - 3}{2}; \alpha, d\right) = \frac{3d + 3 - 4\alpha}{2(d + 5)} \quad \text{for } d \text{ odd and } d \geq 7,
\]
and
\[
\tilde{\kappa}(\alpha, d) = \kappa_3\left(\frac{d - 1}{2}; \alpha, d\right) = \frac{3d + 2 - 4\alpha}{2(d + 4)} \quad \text{for } d \text{ even and } d \geq 4.
\]

(b) According to a famous scheme developed by Mattila, the Fourier decay rates of fractal measures and Falconer’s conjecture are related as follows (see, for example, [4]). Suppose that (1.1) holds for \( S = \mathbb{S}^{d-1} \) with some \( \beta \geq d - \alpha \). Then Falconer’s distance set conjecture holds for \( \alpha \), that is, for any compact subset \( E \) of \( \mathbb{R}^d \),
\[
\dim(E) > \alpha \Rightarrow |\Delta(E)| > 0,
\]
where \(| \cdot |\) denotes the Lebesgue measure, \( \dim(\cdot) \) is the Hausdorff dimension and \( \Delta(E) \) is the distance set given by \( \Delta(E) = \{|x - y| : x, y \in E\} \). The threshold for \( \alpha \) in Falconer’s conjecture is \( d/2 \).

(c) Suppose we plan to approach Falconer’s conjecture using the above relation. Assume (1.1) also holds for \( S = \mathbb{P}^{d-1} \) with the same \( \beta \geq d - \alpha \). (This is the case in all previous works [4, 6, 12, 15].) Then Theorem 1.2 tells us that the best possible threshold for \( \alpha \) one could get using Mattila’s scheme is
\[
\frac{7}{4} = \frac{3}{2} + \frac{1}{4} \quad \text{when } d = 3, \quad \frac{8}{3} = \frac{5}{2} + \frac{1}{6} \quad \text{when } d = 5,
\]
\[ \frac{d}{2} + \frac{1}{d+3} \text{ when } d \text{ odd and } d \geq 7, \quad \frac{d}{2} + \frac{1}{d+2} \text{ when } d \text{ even and } d \geq 4. \]

This suggests that new approach (for example, [9, 10]) may be needed to fully resolve Falconer’s conjecture.

**NOTATION.** We write \( A \lesssim B \) if \( A \leq CB \) for some absolute constant \( C \), \( A \sim B \) if \( A \lesssim B \) and \( B \lesssim A \), and \( A \lessapprox B \) if \( A \leq C R^s B \) for any \( c > 0, R > 1 \). Let \( c = 1/1000 \) be fixed. By \( \rho \)-lattice points in \( \mathbb{R}^d \) we mean the points in \( \rho \mathbb{Z}^d \). Let \( \mathcal{B}^d(x,r) \) denote the ball centered at \( x \), of radius \( r \), in \( \mathbb{R}^d \).

### 2. Proof of Theorem 1.1: spherical decay rates

Let \( \mu \) be \( \alpha \)-dimensional. Given a function \( g \) on the unit ball \( \mathcal{B}^d(0,1) \), we can write \( g = g_1 - g_2 + i(g_3 - g_4) \), where each component \( g_j \) is positive. Then by considering the positive measures \( g_1 \mu \), the estimate (1.1) tells us that

\[
\|\tilde{g}_\mu (R \cdot)\|_{L^2(S)}^2 \lesssim c_\alpha \mu \|R^{-\beta}g\|_{L^\infty}^2.
\]

Thus, by duality, we are looking for an upper bound for the \( \beta \) such that

\[
\|E_S f(R \cdot)\|_{L^1(\mu)} \lesssim R^{-\beta/2} \sqrt{c_\alpha \mu} \|f\|_{L^2(S)},
\]

where

\[
E_S f(x) = (f d\sigma)^\vee(x) = \frac{1}{(2\pi)^{d/2}} \int_S e^{i\omega \cdot x} f(\omega) d\sigma(\omega).
\]

This example is adapted from that of [11]. Let \( c = 1/1000 \) be a fixed small constant and \( 0 < \kappa < 1/2 \). The exact value of \( \kappa \) will be chosen later. Let \( 1 \leq m < d/2 \) and \( d \geq 4 \). Denote

\[
x = (x_1, \ldots, x_d) = (x', x'') \in \mathcal{B}^d(0,1),
\]

\[
\xi = (\xi_1, \ldots, \xi_d) = (\xi', \xi'') \in S^{d-1},
\]

where

\[
x' = (x_1, \ldots, x_m), \quad x'' = (x_{m+1}, \ldots, x_d),
\]

\[
\xi' = (\xi_1, \ldots, \xi_m), \quad \xi'' = (\xi_{m+1}, \ldots, \xi_d).
\]

For \( S = S^{d-1} \), the unit sphere in \( \mathbb{R}^d \), we write \( E_S f(Rx) \) as

\[
E f(Rx) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} e^{iRx' \cdot \xi' + Rx'' \cdot \xi''} f(\xi) d\sigma(\xi).
\]

To prove Theorem 1.1, we will test the estimate (2.5) on the characteristic function \( f(\xi) = \chi_{\Omega}(\xi) \), where the set \( \Omega \) is defined by

\[
\Omega := \left[ B^m(0,cR^{-1/2}) \times (\Gamma + B^{d-m}(0,cR^{-1})) \right] \cap S^{d-1},
\]

and

\[
\Gamma := \{ \omega \in S^{d-m-1} : R^\omega \omega \in 2\pi \mathbb{Z}^{d-m} \}.
\]

So, we have that \( \|f\|_2 = \sigma(\Omega)^{1/2} \).

It is well known (see, for example, a survey about lattice points on spheres [8]) that for \( d - m \geq 2 \), there holds

\[
\# \Gamma \gtrsim R^{\kappa(d-m-2)},
\]

for a sequence of \( R \) tending to \( \infty \). We will focus on such values of \( R \). Note that, in the definition of \( \Omega \), each point in \( \Gamma \) gives us a small patch on \( S^{d-1} \), which has size \( \sim R^{-1/2} \) in \( m \) dimension.
and $\sim R^{-1}$ in each of the other $(d - m - 1)$ dimensions. Therefore,

$$\sigma(\Omega) \gtrsim R^{\kappa(d-m-2)-\frac{m}{2}-(d-m-1)} = R^{\kappa(d-m-2)-d+\frac{m}{2}+1}.\quad (2.9)$$

Next, we define a set $\Lambda$ in $B^d(0, 1)$ by

$$\Lambda := \left[B^m(0, cR^{-1/2}) \times (R^{-1}2^{d-m} + B^{d-m}(0, cR^{-1}))\right] \cap B^d(0, 1).\quad (2.10)$$

The idea is that for $x \in \Lambda$, the phase of the integrand in (2.6) is sufficiently close to $2\pi i \mathbb{Z}$, and so there is little cancelation (see Lemma 2.1). Now define $\mu$ by

$$d\mu = \chi_{\Lambda} dx,$$

where $dx$ is the Lebesgue measure in $\mathbb{R}^d$. From the definition it follows that

$$\|\mu\| = |\Lambda| \quad \text{and} \quad |\Lambda| \sim R^{-m/2}(R^{1-\kappa}R^{-1})^{d-m} = R^{-\kappa(d-m)-m/2}.\quad (2.12)$$

We need the following two lemmas, whose proofs are postponed.

**Lemma 2.1.** For $f$ given above,

$$|Ef(Rx)| \sim \sigma(\Omega), \quad \forall x \in \Lambda.\quad (2.13)$$

**Lemma 2.2.** By taking

$$\kappa = \begin{cases} \kappa_1(m; \alpha, d) = \frac{d-m/2-\alpha}{d-m}, & \alpha \in \left(\frac{d}{2}, d-m\right) \\ \kappa_2(m; \alpha, d) = \frac{d-\alpha}{2(d-m)}, & \alpha \in [d-m, d), \end{cases}\quad (2.14)$$

we have

$$c_\alpha(\mu) \sim R^{\alpha-d}.\quad (2.15)$$

By plugging in (2.9), (2.12), (2.13) and (2.15), we obtain

$$\frac{\|Ef(Rx)\|_{L^1(\mu)}}{\sqrt{c_\alpha(\mu)\|\mu\|\|f\|_2}} \sim \frac{\sigma(\Omega)|\Lambda|}{R^{(\alpha-d)/2}|\Lambda|^{1/2}\sigma(\Omega)^{1/2}} \gtrsim R^{\kappa \frac{1+\alpha}{2}}.$$ 

Comparing the above with (2.5), letting $R$ tend to infinity and taking $\beta$ sufficiently close to $\beta_d(\alpha, S^{d-1})$, we see that

$$\beta_d(\alpha, S^{d-1}) \leq \alpha - 1 + 2\kappa,$$

where $\kappa$ is given as in (2.14). To prove Theorem 1.1, we just take suitable $m$ for different values of $\alpha$. It follows directly from (2.14) that we can choose $\kappa$ as follows.

- For $\alpha \in [d-1, d)$, $\kappa = \kappa_2(1; \alpha, d)$.
- For $d$ even and $\alpha \in (\frac{d}{2}, \frac{d}{2} + 1]$, $\kappa = \kappa_1(\frac{d}{2} - 1; \alpha, d)$.
- For $d$ odd and $\alpha \in (\frac{d}{2}, \frac{d+1}{2}]$, $\kappa = \kappa_1(\frac{d-1}{2}; \alpha, d)$.
- For $\alpha \in [d-j, d-j + 1]$ with $j = 2, 3, \ldots, \left[\frac{d-1}{2}\right]$, $\kappa = \min \{\kappa_1(j-1; \alpha, d), \kappa_2(j; \alpha, d)\}$.

It is straightforward to check that

$$\kappa_2(j; \alpha, d) \leq \kappa_1(j-1; \alpha, d) \quad \iff \quad \alpha \leq d - j + \frac{d - 2j}{d - j - 1}.$$ 

Also note that $0 < \frac{d-2j}{d-j-1} < 1$ in this case.
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This finishes the proof of Theorem 1.1 up to Lemmas 2.1 and 2.2.

2.1. Proof of Lemma 2.1

Since \( f = \chi_\Omega \), we have

\[
Ef(Rx) = \frac{1}{(2\pi)^{d/2}} \int_\Omega e^{i(Rx' \cdot \xi' + Rx'' \cdot \xi'')} \, d\sigma(\xi).
\]

So, it suffices to prove that

\[
Rx' \cdot \xi' + Rx'' \cdot \xi'' \in 2\pi\mathbb{Z} + \left( -\frac{1}{100}, \frac{1}{100} \right),
\]

provided that \( \xi \in \Omega \) and \( x \in \Lambda \). Indeed, by definitions of \( \Omega \) and \( \Lambda \), we write

\[
|\xi'| < cR^{-\frac{1}{2}}, \quad |x'| < cR^{-\frac{1}{2}}
\]

\[
\xi'' = 2\pi R^\kappa m + v, \quad \text{where} \quad m \in \mathbb{Z}^{d-m}, |m| < \frac{1}{2\pi} R^\kappa, |v| < cR^{-1},
\]

and

\[
x'' = R^\kappa \ell + u, \quad \text{where} \quad \ell \in \mathbb{Z}^{d-m}, |\ell| < R^{1-\kappa}, |u| < cR^{-1}.
\]

Then it is straightforward to verify that (2.16) holds.

- \(|Rx' \cdot \xi'| < RcR^{-1/2}cR^{-1/2} = c^2\).
- For \( Rx'' \cdot \xi'' \), we have

\[
Rx'' \cdot \xi'' = R(R^\kappa \ell + u) \cdot (2\pi R^\kappa m + v)
\]

\[
= 2\pi \ell \cdot m + R^\kappa \ell \cdot v + 2\pi R^{1-\kappa} u \cdot m + Ru \cdot v,
\]

where \( 2\pi \ell \cdot m \in 2\pi \mathbb{Z} \) and the other three terms are bounded by

\[
R^\kappa R^{1-\kappa} + R^{1-\kappa}cR^{-1}R^\kappa + RcR^{-1}cR^{-1} = c + c + c^2 R^{-1}.
\]

Therefore, (2.16) follows by taking \( c \) sufficiently small, say \( c = 1/1000 \).

2.2. Proof of Lemma 2.2

Recall that \( d\mu = \chi_\Lambda \, dx \) and \( \Lambda \) is defined by

\[
\Lambda := \left[ B^m(0, cR^{-1/2}) \times (R^\kappa \mathbb{Z}^{d-m} + B^{d-m}(0, cR^{-1})) \right] \cap B^d(0, 1).
\]

We aim to prove that

\[
c_\alpha(\mu) \sim R^{\alpha-d},
\]

by taking

\[
\kappa = \begin{cases} 
\kappa_1(m; \alpha,d) = \frac{d-m/2 - \alpha}{d-m}, & \alpha \in \left( \frac{d}{2}, d-m \right] \\
\kappa_2(m; \alpha,d) = \frac{d-\alpha}{2(d-m)}, & \alpha \in [d-m, d).
\end{cases}
\]

(2.18)

For convenience, we write

\[
c_\alpha(\mu) := \sup_{x \in \mathbb{R}^d, r > 0} \frac{\mu(B(x,r))}{r^\alpha} = \sup_{r > 0} c_\alpha(\mu, r),
\]

\[
\frac{1}{(2\pi)^{d/2}} \int_\Omega e^{i(Rx' \cdot \xi' + Rx'' \cdot \xi'')} \, d\sigma(\xi).
\]
where
\[ c_\alpha(\mu, r) := \sup_{x \in \mathbb{R}^d} \frac{\mu(B(x, r))}{r^\alpha}. \]

We will calculate \( c_\alpha(\mu, r) \) directly from (2.17). The important scales for \( r \) are ordered as follows:
\[ R^{-1} < R^{\kappa - 1} < R^{-1/2} < 1. \]

Now we calculate \( C_\alpha(\mu, r) \) for different values of \( r \).

- For \( 0 < r \leq R^{-1} \),
  \[ c_\alpha(\mu, r) \sim \frac{r^d}{r^\alpha} = r^{d-\alpha}. \]

  Since \( d-\alpha > 0 \), we have
  \[ \sup_{0 < r \leq R^{-1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1}) \sim R^{\alpha-d}. \] (2.19)

- For \( R^{-1} \leq r \leq R^{\kappa - 1} \),
  \[ c_\alpha(\mu, r) \sim \frac{r^m}{r^\alpha} \cdot \frac{R^{-(d-m)}}{r^\alpha} = r^{m-\alpha}R^{-(d-m)}. \]

  Since \( m < d/2 < \alpha \), we have
  \[ \sup_{R^{-1} \leq r \leq R^{\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1}). \] (2.20)

- For \( R^{\kappa - 1} \leq r \leq R^{-\frac{1}{2}} \),
  \[ c_\alpha(\mu, r) \sim \frac{r^m}{r^\alpha} \cdot \frac{(\frac{r}{R})^{d-m}}{r^\alpha} = r^{d-\alpha}R^{-\kappa(d-m)}. \]

  Since \( d-\alpha > 0 \), we have
  \[ \sup_{R^{\kappa - 1} \leq r \leq R^{-\frac{1}{2}}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-\frac{1}{2}}) \sim R^{-\frac{d-\alpha}{2}}R^{-\kappa(d-m)}. \] (2.21)

- For \( R^{-\frac{1}{2}} \leq r \leq 1 \),
  \[ c_\alpha(\mu, r) \sim \frac{r^{d-m/2}}{r^\alpha} \cdot \frac{(\frac{r}{R})^{d-m}}{r^\alpha} = r^{d-m-\alpha}R^{-\kappa(d-m)-m/2}. \]

  If \( \alpha \leq d-m \), we have
  \[ \sup_{R^{-1/2} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1) \sim R^{-\kappa(d-m)-\frac{m}{2}}, \] (2.22)

  and if \( \alpha \geq d-m \), we have
  \[ \sup_{R^{-1/2} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-\frac{1}{2}}). \] (2.23)

  It is also obvious that
  \[ \sup_{r \geq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1). \]

Therefore, for \( \alpha \leq d-m \), by combining (2.19), (2.20), (2.21) and (2.22), we can tell that
\[ c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, 1) \right\} \]
\[ \sim \max \left\{ R^{\alpha-d}, R^{-\kappa(d-m)-\frac{m}{2}} \right\} = R^{\alpha-d}, \]
provided that
\[ \kappa = \kappa_1(m; \alpha, d) = \frac{d - m/2 - \alpha}{d - m}. \]
And for \( \alpha \geq d - m \), by combining (2.19), (2.20), (2.21) and (2.23), we can tell that
\[ c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, R^{-\frac{d-\alpha}{d}}) \right\} \]
\[ \sim \max \left\{ R^\alpha - d, R^{-d - \alpha - \kappa(d-m)} \right\} = R^\alpha - d, \]
provided that
\[ \kappa = \kappa_2(m; \alpha, d) = \frac{d - \alpha}{2(d - m)}, \]
as desired. This completes the proof of Lemma 2.2.

3. Proof of Theorem 1.2: parabolic decay rates

This example is adapted from that of [1] in a similar way as in the previous section. Recall that \( c = 1/1000 \) is a fixed small constant. In this section, we will still use but redefine the notations \( f, \Omega \) and \( \Lambda \). Let \( 0 < \kappa < 1/2 \). Let \( d \geq 3 \) and \( 1 \leq m \leq d/2 \). In \( \kappa_3(m; \alpha, d) \) below, \( m < d/2 \), while in \( \kappa_4(m; \alpha, d) \) and \( \kappa_5(m; \alpha, d) \) below, \( m \) could be \( d/2 \). Denote
\[ x = (x_1, \ldots, x_d) = (x', x'', x_d) \in B^d(0,1), \]
\[ \xi = (\xi_1, \ldots, \xi_{d-1}) = (\xi', \xi'') \in B^{d-1}(0,1), \]
where
\[ x' = (x_1, \ldots, x_m), \quad x'' = (x_{m+1}, \ldots, x_{d-1}), \]
\[ \xi' = (\xi_1, \ldots, \xi_m), \quad \xi'' = (\xi_{m+1}, \ldots, \xi_{d-1}). \]

For \( S = \mathbb{R}^{d-1} \), the truncated paraboloid in \( \mathbb{R}^d \), we write \( E_S f(Rx) \) as
\[ E_S f(Rx) = \frac{1}{(2\pi)^{d/2}} \int_{B_{d-1}(0,1)} e^{iR(x'\cdot \xi' + x''\cdot \xi'') + x_d|\xi'| + x_d|\xi''|} f(\xi) \, d\xi. \] (3.24)

For simplicity, we denote \( B^d(0,r) \) by \( B^d_r \), and write the interval \((-r, r)\) as \( I_r \). To prove Theorem 1.2, we will test the estimate (2.5) on the characteristic function \( f(\xi) = \chi_{1}(\xi) \), where the set \( \Omega \) is defined by
\[ \Omega := [B^m_{\epsilon R^{-1/2}} \times (2\pi R^{-\kappa} Z^{d-m-1} + B^{d-m-1}_{\epsilon R^{-1}})] \cap B^{d-1}(0,1). \] (3.25)

By definition, we have
\[ \|f\|_2 = |\Omega|^{1/2} \quad \text{and} \quad |\Omega| \sim R^{(\kappa-1)(d-m-1)-m/2}. \] (3.26)

Next, we define a set \( \Lambda \) in \( B^d(0,1) \) by
\[ \Lambda := \left[ B^m_{\epsilon R^{-1/2}} \times (2\pi R^{-\kappa} Z^{d-m-1} + B^{d-m-1}_{\epsilon R^{-1}}) \times \left( \frac{1}{2\pi} R^{2\kappa-1} Z + I_{\epsilon R^{-1}} \right) \right] \cap B^d(0,1). \] (3.27)

Now, define \( \mu \) by
\[ d\mu = \chi_\Lambda dx, \] (3.28)
where \( dx \) is the Lebesgue measure in \( \mathbb{R}^d \). From the definition, it follows that
\[ \|\mu\| = |\Lambda| \quad \text{and} \quad |\Lambda| \sim R^{-m/2-\kappa(d-m-1)-2\kappa} = R^{-\kappa(d-m+1)-m/2}. \] (3.29)
Moreover, we have the following two lemmas, whose proofs are postponed.

**Lemma 3.1.** For $f$ given above,
\[ |Ef(Rx)| \sim |\Omega|, \quad \forall x \in \Lambda. \] (3.30)

**Lemma 3.2.** We have
\[ c_\alpha(\mu) \sim R^{\alpha-d}, \] (3.31)
by taking $\kappa$ as follows.

(a) If $1 \leq m \leq \frac{d+1}{3}$, then
\[
\kappa = \kappa_3(m; \alpha, d) \quad \text{for} \quad m \leq \alpha \leq d - m,
\] (3.32)
and
\[
\kappa = \kappa_4(m; \alpha, d) \quad \text{for} \quad d - m \leq \alpha < d.
\] (3.33)

(b) If $\frac{d+1}{3} < m < \frac{d}{2}$, then
\[
\kappa = \kappa_3(m; \alpha, d) \quad \text{for} \quad m \leq \alpha \leq d - m - 1,
\] (3.34)
and
\[
\kappa = \max \{ \kappa_3(m; \alpha, d), \kappa_5(m; \alpha, d) \} \quad \text{for} \quad d - m - 1 \leq \alpha \leq d - m - \frac{2(d - 2m - 1)}{d - m - 3},
\] (3.35)

\[
\kappa = \kappa_5(m; \alpha, d) \quad \text{for} \quad \alpha \leq d - m - \frac{2(d - 2m - 1)}{d - m - 3},
\] (3.36)
and
\[
\kappa = \kappa_4(m; \alpha, d) \quad \text{for} \quad \frac{d + m - 1}{2} \leq \alpha < d.
\] (3.37)

Moreover, (3.36) and (3.37) also hold when $m = \frac{d}{2}$.

By plugging in (3.26), (3.29), (3.30) and (3.31), we obtain
\[
\frac{\|Ef(R\cdot)\|_{L^1(d\mu)}}{\sqrt{c_\alpha(\mu)}} \|\mu\|_{\mathbb{P}_{d-1}} \sim \frac{|\Omega| |\Lambda|}{R^{(\alpha-d)/2} |\Lambda|^{1/2} |\Omega|^{1/2}} \sim R^{-\kappa + \frac{1}{2}}.
\]

Comparing the above with (2.5), letting $R$ tend to infinity and taking $\beta$ sufficiently close to $\beta_d(\alpha, \mathbb{P}^{d-1})$, we see that
\[
\beta_d(\alpha, \mathbb{P}^{d-1}) \leq \alpha - 1 + 2\kappa,
\] (3.38)
where $\kappa$ is given as in Lemma 3.2. To prove Theorem 1.2, we just take suitable $m$ for different values of $\alpha$.

- For $\alpha \in [d - 1, d)$, by (3.33) we can take
\[
\kappa = \kappa_4(1; \alpha, d).
\]
For $\alpha \in [d - j, d - j + 1]$ with $2 \leq j \leq \left\lfloor \frac{d + 1}{3} \right\rfloor$, by \eqref{eq:3.32} we can take $\kappa = \kappa_3(j - 1; \alpha, d)$, and by \eqref{eq:3.33} we can take $\kappa = \kappa_4(j; \alpha, d)$. Therefore, \eqref{eq:3.38} holds with

$$\kappa = \min \{ \kappa_3(j - 1; \alpha, d), \kappa_4(j; \alpha, d) \}.$$ 

It is straightforward to check that

$$\kappa_4(j; \alpha, d) \leq \kappa_3(j - 1; \alpha, d) \iff \alpha \leq d - j + \frac{d - 2j + 1}{d - j},$$

and

$$0 < \frac{d - 2j + 1}{d - j} < 1.$$

For $\alpha \in [d - j, d - j + 1]$ with $\left\lfloor \frac{d + 1}{3} \right\rfloor + 2 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor$, by applying \eqref{eq:3.32} when $j = \left\lfloor \frac{d + 1}{3} \right\rfloor + 2$ and applying \eqref{eq:3.34} otherwise we can take $\kappa = \kappa_3(j - 2; \alpha, d)$, by \eqref{eq:3.35} we can take $\kappa = \max\{\kappa_3(j - 1; \alpha, d), \kappa_5(j - 1; \alpha, d)\}$, and by \eqref{eq:3.36} we can take $\kappa = \kappa_5(j; \alpha, d)$. Therefore, \eqref{eq:3.38} holds if we choose $\kappa$ to be

$$\min \{ \kappa_3(j - 2; \alpha, d), \max \{ \kappa_3(j - 1; \alpha, d), \kappa_5(j - 1; \alpha, d)\}, \kappa_5(j; \alpha, d) \},$$

and \eqref{eq:3.35} tells us that this number is

$$\min \{ \kappa_3(j - 1; \alpha, d), \kappa_5(j; \alpha, d) \} \quad \text{for} \quad \alpha \leq d - j + \frac{2(d - 2j + 1)}{d - j - 2}$$

and

$$\min \{ \kappa_3(j - 2; \alpha, d), \kappa_5(j - 1; \alpha, d) \} \quad \text{for} \quad \alpha \geq d - j + \frac{2(d - 2j + 1)}{d - j - 2}.$$

For $d$ odd, $d \geq 7$ and $\alpha \in \left(\frac{d - 1}{2}, \frac{d + 1}{2}\right]$, by applying \eqref{eq:3.32} when $d = 7, 9, 11$ and applying \eqref{eq:3.34} when $d \geq 13$, we can take

$$\kappa = \kappa_3 \left( \frac{d - 3}{2}; \alpha, d \right).$$

Note that when $d = 3, 5$, the case $\alpha \in \left(\frac{d - 1}{2}, \frac{d + 1}{2}\right]$ is the same as the case $\alpha \in [d - j, d - j + 1]$ with $j = \left\lfloor \frac{d + 1}{3} \right\rfloor + 1$, and we have

$$\kappa = \kappa_3(1; \alpha, 3) \quad \text{for} \quad d = 3, \quad \text{and} \quad \kappa = \kappa_3(2; \alpha, 5) \quad \text{for} \quad d = 5.$$

For $d$ even and $\alpha \in \left(\frac{d - 1}{2}, \frac{d}{2}\right]$, by applying \eqref{eq:3.32} when $d = 4, 6, 8$ and applying \eqref{eq:3.34} when $d \geq 10$, we can take

$$\kappa = \kappa_3 \left( \frac{d - 1}{2}; \alpha, d \right).$$

Note that the above discussion covers all the cases $d \geq 3$ and $\alpha \in (\frac{d - 1}{2}, d)$ for Theorem 1.2. It remains to verify Lemmas 3.1 and 3.2, and we will do so in the following two subsections.

### 3.1. Proof of Lemma 3.1

Since $f = \chi_{\Omega}$, we have

$$Ef(Rx) = \frac{1}{(2\pi)^{d/2}} \int_{\Omega} e^{i R(x' \cdot \xi' + x'' \cdot \xi'' + x_d |\xi'|^2 + x_d|\xi''|^2)} \, d\xi.$$
So, it suffices to prove that
\[ R(x' \cdot \xi' + x'' \cdot \xi'' + x_d|\xi'|^2 + x_d|\xi''|^2) \in 2\pi \mathbb{Z} + \left(-\frac{1}{100}, \frac{1}{100}\right), \]  (3.39)
provided that \( \xi \in \Omega \) and \( x \in \Lambda \). Indeed, by definitions of \( \Omega \) and \( \Lambda \), we write
\[ |\xi'| < cR^{-\frac{1}{2}}, \quad |x'| < cR^{-\frac{1}{2}} \]
\[ \xi'' = 2\pi R^{-\kappa}m + v, \quad \text{where} \quad m \in \mathbb{Z}^{d-m-1}, |m| < \frac{1}{2\pi} R^{\kappa}, |v| < cR^{-1}, \]
\[ x'' = R^{\kappa-1}\ell + u, \quad \text{where} \quad \ell \in \mathbb{Z}^{d-m-1}, |\ell| < R^{1-\kappa}, |u| < cR^{-1}, \]
and
\[ x_d = \frac{1}{2\pi} R^{2\kappa-1}k + \varepsilon, \quad \text{where} \quad k \in \mathbb{Z}, |k| < 2\pi R^{1-2\kappa}, |\varepsilon| < cR^{-1}. \]

Let us look at the four components in (3.39) separately.

- For \( Rx' \cdot \xi' \), we have
  \[ |Rx'| < R^{cR^{-1/2}}cR^{-1/2} = c^2. \]
- Since \( |x_d| < 1 \),
  \[ |Rx_d| \xi'|^2 < R^{cR^{-1}} = c^2. \]
- For \( Rx'' \cdot \xi'' \), we have
  \[ Rx'' \cdot \xi'' = R(R^{\kappa-1}\ell + u) \cdot (2\pi R^{-\kappa}m + v) \]
  \[ = 2\pi \ell \cdot m + R^\kappa \ell \cdot v + 2\pi R^{1-\kappa}u \cdot m + Ru \cdot v, \]
  where \( 2\pi \ell \cdot m \in 2\pi \mathbb{Z} \) and the other three terms are bounded by
  \[ R^{\kappa} R^{1-\kappa} cR^{-1} + R^{1-\kappa} cR^{-1} R^{\kappa} + R^{cR^{-1}}cR^{-1} = c + c + c^2 R^{-1}. \]
- For \( Rx_d|\xi''|^2 \), we have
  \[ Rx_d|\xi''|^2 = R(\frac{1}{2\pi} R^{2\kappa-1}k + \varepsilon)(2\pi R^{\kappa}m + v) \cdot (2\pi R^{\kappa}m + v) \]
  \[ = 2\pi k|m|^2 + 2R^\kappa k(m \cdot v) + \frac{1}{2\pi} R^{2\kappa}k|v|^2 \]
  \[ + 4\pi^2 \varepsilon R^{1-2\kappa}|m|^2 + 4\pi \varepsilon R^{1-\kappa}(m \cdot v) + \varepsilon R|v|^2, \]
  where \( 2\pi k|m|^2 \in 2\pi \mathbb{Z} \) and the other five terms are bounded by
  \[ R^{\kappa} R^{1-2\kappa} cR^{-1} + R^{2\kappa} 2\pi R^{1-2\kappa}c^2 R^{-2} + cR^{-1} R^{1-2\kappa} R^{2\kappa} \]
  \[ + cR^{-1} R^{1-\kappa} cR^{-1} + cR^{-1} Rc^2 R^{-2} \]
  \[ = c + 2\pi c^2 R^{-1} + c + c^2 R^{-1} + c^3 R^{-2}. \]

Therefore, (3.39) follows by taking \( c \) sufficiently small, say \( c = 1/1000 \).

3.2. Proof of Lemma 3.2

Recall that \( d\mu = \chi_A \, dx \) and \( \Lambda \) is defined by
\[ \Lambda := \left[ B_{cR^{-1/2}}^m \times (R^{-1} \mathbb{Z}^{d-m-1} + B_{cR^{-1}}^{d-m-1}) \times \left( \frac{1}{2\pi} R^{2\kappa-1} \mathbb{Z} + I_{cR^{-1}} \right) \right] \cap B^d(0, 1). \]  (3.40)

We aim to prove that
\[ c_\alpha(\mu) \sim R^{\kappa-d}, \]
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by taking $\kappa$ as stated in Lemma 3.2.

Recall that

$$c_\alpha(\mu) := \sup_{x \in \mathbb{R}^d, r > 0} \frac{\mu(B(x, r))}{r^\alpha} = \sup_{r > 0} c_\alpha(\mu, r),$$

where

$$c_\alpha(\mu, r) := \sup_{x \in \mathbb{R}^d} \frac{\mu(B(x, r))}{r^\alpha}.$$

We will calculate $c_\alpha(\mu, r)$ directly from (3.40). The important scales for $r$ are $R^{-1}, R^{\kappa - 1}, R^{2\kappa - 1}$ and $R^{-1/2}$. To compare the scales $R^{2\kappa - 1}$ and $R^{-1/2}$, we consider the two cases $\kappa \leq 1/4$ and $\kappa > 1/4$ separately.

Case I: $\kappa \leq 1/4$. In this case, the important scales for $r$ are ordered as follows:

$$R^{-1} < R^{\kappa - 1} < R^{2\kappa - 1} \leq R^{-1/2} < 1.$$

Now we calculate $C_\alpha(\mu, r)$ for different values of $r$.

- For $0 < r \leq R^{-1}$,

  $$c_\alpha(\mu, r) \sim \frac{r^d}{r^\alpha} = r^{d-\alpha}.\quad (3.41)$$

  Since $d - \alpha > 0$, we have

$$\sup_{0 < r \leq R^{-1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1}) \sim R^{\alpha - d}.\quad (3.41)$$

- For $R^{-1} \leq r \leq R^{\kappa - 1}$,

  $$c_\alpha(\mu, r) \sim \frac{r^m \cdot R^{-(d-m)} \cdot R^{-1}}{r^\alpha} = r^{m-\alpha} R^{-(d-m)}.\quad (3.42)$$

  If $\alpha \leq m$, we have

$$\sup_{R^{-1} \leq r \leq R^{\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{\kappa - 1}),\quad (3.42)$$

  and if $\alpha \geq m$, we have

$$\sup_{R^{-1} \leq r \leq R^{\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1}).\quad (3.43)$$

- For $R^{\kappa - 1} \leq r \leq R^{2\kappa - 1}$,

  $$c_\alpha(\mu, r) \sim \frac{r^m \cdot \left( \frac{r}{R^{\kappa-1}} \right)^{d-m-1} \cdot R^{-1}}{r^\alpha} = r^{d-1-\alpha} R^{-\kappa(d-m-1)-1}.\quad (3.44)$$

  If $\alpha \leq d - 1$, we have

$$\sup_{R^{\kappa - 1} \leq r \leq R^{2\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{2\kappa - 1}),\quad (3.44)$$

  and if $\alpha \geq d - 1$, we have

$$\sup_{R^{\kappa - 1} \leq r \leq R^{2\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{\kappa - 1}).\quad (3.45)$$

- For $R^{2\kappa - 1} \leq r \leq R^{-\frac{1}{2}}$,

  $$c_\alpha(\mu, r) \sim \frac{r^m \cdot \left( \frac{r}{R^{\kappa-1}} \right)^{d-m-1} \cdot \left( \frac{r}{R^{\kappa-1}} \right)^{R^{-1}}}{r^\alpha} = r^{d-\alpha} R^{-\kappa(d-m+1)}.\quad (3.45)$$
Since $d - \alpha > 0$, we have
\[
\sup_{R^{2-\alpha} \leq r \leq R^{-1/2}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1/2}) \sim R^{-\frac{d-\alpha}{2}} \kappa(d-m+1)
\] (3.46)

- For $R^{-\frac{1}{2}} \leq r \leq 1$,
\[
c_\alpha(\mu, r) \sim \frac{R^{-m/2} \cdot (\frac{r}{R^{2-\alpha}} R^{-1})^{d-m-1} (\frac{r}{R^{2-\alpha}} R^{-1})}{r^\alpha} = r^{d-m-\alpha} R^{-\kappa(d-m+1) - \frac{m}{2}}.
\]

If $\alpha \leq d - m$, we have
\[
\sup_{R^{-1/2} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1) \sim R^{-\kappa(d-m+1) - \frac{m}{2}},
\] (3.47)
and if $\alpha \geq d - m$, we have
\[
\sup_{R^{-1/2} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-\frac{1}{2}}).
\] (3.48)

It is also obvious that
\[
\sup_{r \geq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1).
\]

Therefore, for $m \leq \alpha \leq d - m$, by combining (3.41), (3.43), (3.44), (3.46) and (3.47), we can tell that
\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, 1) \right\}
\sim \max \left\{ R^{\alpha-d}, R^{-\kappa(d-m+1) - \frac{m}{2}} \right\} = R^{\alpha-d},
\]
provided that
\[
\kappa = \kappa_3(m; \alpha, d) = \frac{d - m/2 - \alpha}{d - m + 1}.
\]

For $d - m \leq \alpha \leq d - 1$, by combining (3.41), (3.43), (3.44), (3.46) and (3.48), we can tell that
\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, R^{-\frac{1}{2}}) \right\}
\sim \max \left\{ R^{\alpha-d}, R^{-\frac{d-\alpha}{2} \kappa(d-m+1)} \right\} = R^{\alpha-d},
\]
provided that
\[
\kappa = \kappa_4(m; \alpha, d) = \frac{d - \alpha}{2(d - m + 1)}.
\]

For $d - 1 \leq \alpha < d$, by combining (3.41), (3.43), (3.45), (3.46) and (3.48), we can tell that
\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, R^{-\frac{1}{2}}) \right\} \sim R^{\alpha-d},
\]
provided that
\[
\kappa = \kappa_4(m; \alpha, d).
\]

Note that the calculation of $c_\alpha(\mu)$ above is in the case $\kappa \leq 1/4$. While
\[
\kappa_3(m; \alpha, d) \leq \frac{1}{4} \iff \alpha \geq \frac{3d - m - 1}{4},
\]
and
\[
\kappa_4(m; \alpha, d) \leq \frac{1}{4} \iff \alpha \geq \frac{d + m - 1}{2}.
\]
Also note that
\[ m < \frac{3d - m - 1}{4} \quad \text{for} \quad d \geq 3, \]
\[ \frac{3d - m - 1}{4} \leq d - m \iff m \leq \frac{d + 1}{3}, \]
and
\[ d - m \geq \frac{d + m - 1}{2} \iff m \leq \frac{d + 1}{3}. \]
Therefore, in Case I we obtain \( c_\alpha(\mu) \sim R^{\alpha - d} \) by taking \( \kappa \) as follows.

- If \( 1 \leq m \leq \frac{d + 1}{3} \), then
  \[ \kappa = \kappa_3(m; \alpha, d) \quad \text{for} \quad \frac{3d - m - 1}{4} \leq \alpha \leq d - m, \]  
  and
  \[ \kappa = \kappa_4(m; \alpha, d) \quad \text{for} \quad d - m \leq \alpha < d. \]

- If \( \frac{d + 1}{3} < m \leq \frac{d}{2} \), then
  \[ \kappa = \kappa_4(m; \alpha, d) \quad \text{for} \quad \frac{d + m - 1}{2} \leq \alpha < d. \]

Case II: \( \kappa > \frac{1}{4} \). Note that, we have proved Lemma 3.2 for \( \alpha \geq d - 1 \) in Case I. Therefore, here we can assume that \( \alpha < d - 1 \). In this case, the important scales for \( r \) are ordered as follows:
\[ R^{-1} < R^{\kappa - 1} < R^{-1/2} < R^{2\kappa - 1} < 1. \]
Now we calculate \( C_\alpha(\mu, r) \) for different values of \( r \).

- For \( 0 < r \leq R^{\kappa - 1} \), same as in Case I, if \( \alpha \leq m \) we have
  \[ \sup_{0 < r \leq R^{\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{\kappa - 1}), \]
  and if \( \alpha \geq m \) we have
  \[ \sup_{0 < r \leq R^{\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-1}) \sim R^{\alpha - d}. \]

- For \( R^{\kappa - 1} \leq r \leq R^{-\frac{1}{2}} \),
  \[ c_\alpha(\mu, r) \sim \frac{R^m \cdot \left( \frac{r}{R^{\kappa - 1}} R^{-1} \right)^{d-m-1} \cdot R^{-1}}{r^\alpha} = r^{d-1-\alpha} R^{-\kappa(d-m-1)-\frac{1}{2}}. \]
  Since \( \alpha < d - 1 \), we have
  \[ \sup_{R^{\kappa - 1} \leq r \leq R^{-1/2}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-\frac{1}{2}}) \sim R^{-\frac{d-\alpha}{2} - \kappa(d-m-1)-\frac{1}{2}}. \]

- For \( R^{-\frac{1}{2}} \leq r \leq R^{2\kappa - 1} \),
  \[ c_\alpha(\mu, r) \sim \frac{R^{-\frac{d}{2}} \cdot \left( \frac{r}{R^{2\kappa - 1}} R^{-1} \right)^{d-m-1} \cdot R^{-1}}{r^\alpha} = r^{d-m-1-\alpha} R^{-\kappa(d-m-1)-\frac{d}{2}-1}. \]
  If \( \alpha \leq d - m - 1 \), we have
  \[ \sup_{R^{-1/2} \leq r \leq R^{2\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{2\kappa - 1}), \]
and if $\alpha \geq d - m - 1$, we have
\[
\sup_{R^{-1/2} \leq r \leq R^{2\kappa - 1}} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{-\frac{1}{2}}). \tag{3.56}
\]

- For $R^{2\kappa - 1} \leq r \leq 1$,
\[
c_\alpha(\mu, r) \sim \frac{R^{-\frac{m}{2}} \cdot \left(\frac{r}{R}\right)^{d-m-1} \left(\frac{r}{R}\right)^{\kappa}}{r^{\alpha}} = r^{d-m-\alpha} R^{-\kappa(d-m+1)-\frac{m}{2}}.
\]

If $\alpha \leq d - m$, we have
\[
\sup_{R^{2\kappa - 1} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1) \sim R^{-\kappa(d-m+1)-\frac{m}{2}}. \tag{3.57}
\]

and if $\alpha \geq d - m$, we have
\[
\sup_{R^{2\kappa - 1} \leq r \leq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, R^{2\kappa - 1}). \tag{3.58}
\]

It is also obvious that
\[
\sup_{r \geq 1} c_\alpha(\mu, r) \sim c_\alpha(\mu, 1).
\]

Therefore, for $m \leq \alpha \leq d - m - 1$, by combining (3.53), (3.54), (3.55) and (3.57), we can tell that
\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, 1) \right\}
\]
\[
\sim \max \left\{ R^{\alpha-d}, R^{-\kappa(d-m+1)-\frac{m}{2}} \right\} = R^{\alpha-d},
\]
provided that
\[
\kappa = \kappa_3(m; \alpha, d) = \frac{d - m/2 - \alpha}{d - m + 1}.
\]

For $m \leq d - m - 1 \leq \alpha \leq d - m$ (and so $m < d/2$), by combining (3.53), (3.54), (3.56) and (3.57), we can tell that
\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, R^{-\frac{1}{2}}), c_\alpha(\mu, 1) \right\}
\]
\[
\sim \max \left\{ R^{\alpha-d}, R^{-\frac{m}{2}} - \kappa(d-m-1) - \frac{1}{2}, R^{-\kappa(d-m+1)-\frac{m}{2}} \right\} = R^{\alpha-d},
\]
provided that
\[
\kappa \geq \kappa_3(m; \alpha, d) \quad \text{and} \quad \kappa \geq \kappa_5(m; \alpha, d) = \frac{d - \alpha - 1}{2(d - m - 1)}.
\]

Therefore, we can take
\[
\kappa = \max \left\{ \kappa_3(m; \alpha, d), \kappa_5(m; \alpha, d) \right\}.
\]

While, by a direct calculation, if $m \leq \frac{d+1}{3}$, then
\[
\kappa = \kappa_3(m; \alpha, d), \quad \text{for} \quad d - m - 1 \leq \alpha \leq d - m;
\]

and if $m > \frac{d+1}{3}$, then
\[
\kappa = \begin{cases} 
\kappa_3(m; \alpha, d), & \text{for} \quad d - m - 1 \leq \alpha \leq d - m - 1 + \frac{2(d - 2m - 1)}{d - m - 3}, \\
\kappa_5(m; \alpha, d), & \text{for} \quad d - m - 1 + \frac{2(d - 2m - 1)}{d - m - 3} \leq \alpha \leq d - m. 
\end{cases}
\]
Next, for \( m \leq d - m \leq \alpha < d - 1 \) (and so \( m \leq d/2 \)), by combining (3.53), (3.54), (3.56) and (3.58), we can tell that

\[
c_\alpha(\mu) \sim \max \left\{ c_\alpha(\mu, R^{-1}), c_\alpha(\mu, R^{-\frac{1}{2}}) \right\} \sim R^{\alpha-d},
\]

provided that

\[
\kappa = \kappa_5(m; \alpha, d).
\]

Note that the calculation of \( c_\alpha(\mu) \) above is in the case \( \kappa \geq 1/4 \). While

\[
\kappa_3(m; \alpha, d) \geq \frac{1}{4} \iff \alpha \leq \frac{3d - m - 1}{4},
\]

and

\[
\kappa_5(m; \alpha, d) \geq \frac{1}{4} \iff \alpha \leq \frac{d + m - 1}{2}.
\]

Also, note that

\[
\frac{3d - m - 1}{4} \geq d - m - 1 \iff m \geq \frac{d - 3}{3},
\]

\[
\frac{3d - m - 1}{4} \geq d - m \iff m \geq \frac{d + 1}{3},
\]

and

\[
\frac{d + m - 1}{2} \geq d - m \iff m \geq \frac{d + 1}{3}.
\]

Therefore, in Case II we obtain \( c_\alpha(\mu) \sim R^{\alpha-d} \) by taking \( \kappa \) as follows.

- If \( 1 \leq m \leq \frac{d + 1}{3} \), then
  \[
  \kappa = \kappa_3(m; \alpha, d) \quad \text{for} \quad m \leq \alpha \leq \frac{3d - m - 1}{4}.
  \]
  (3.59)

- If \( \frac{d + 1}{3} < m < \frac{d}{2} \), then
  \[
  \kappa = \kappa_3(m; \alpha, d) \quad \text{for} \quad m \leq \alpha \leq d - m - 1,
  \]
  (3.60)

and

\[
\kappa = \max \{ \kappa_3(m; \alpha, d), \kappa_5(m; \alpha, d) \} \quad \text{for} \quad d - m - 1 \leq \alpha \leq d - m - 1
\]

(3.61)

\[
= \begin{cases} 
\kappa_3(m; \alpha, d), & d - m - 1 \leq \alpha \leq d - m - 1 + \frac{2(d - 2m - 1)}{d - m - 3}, \\
\kappa_5(m; \alpha, d), & d - m - 1 + \frac{2(d - 2m - 1)}{d - m - 3} \leq \alpha \leq d - m,
\end{cases}
\]

and

\[
\kappa = \kappa_5(m; \alpha, d) \quad \text{for} \quad d - m \leq \alpha \leq \frac{d + m - 1}{2}.
\]

(3.62)

And (3.62) also holds and is nontrivial when \( m = \frac{d}{2} \).

The proof of Lemma 3.2 is done by combining the conclusions from both Cases I and II.

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