Reconstructing AdS/CFT

Laurent Freidel

Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo, N2L 2Y5, Canada.
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In this note we clarify the dictionary between pure quantum gravity on the bulk in the presence of a cosmological constant and a CFT on the boundary. We show for instance that there is a general correspondence between quantum gravity “radial states” and a pair of CFT’s. Restricting to one CFT is argued to correspond to states possessing an asymptotic infinity. This point of view allows us to address the problem of reconstructing the bulk from the boundary. And in the second part of this paper we present an explicit formula which gives, from the partition function of any 2 dimensional conformal field theory, a wave functional solution to the 3-dimensional Wheeler-DeWitt equation. This establishes at the quantum level a precise dictionary between 2d CFT and pure gravity.

I. INTRODUCTION

AdS-CFT is a deep and fascinating correspondence between a bulk theory of gravity and a boundary Conformal Field Theory which was first conjectured in the context of string theory [1]. The purpose of this note is to study from the point of view of quantum gravity the relationship between a theory of pure quantum gravity in the bulk and a boundary CFT. The main question we want to address here is the question : What is AdS/CFT from the point of view of background independent quantum gravity?

Trying to answer this leads to many related questionings. An obvious one is: Assuming that we have a theory of quantum gravity in the bulk, how can we prove the AdS/CFT correspondence? For instance, in 3 dimension a large body of work has led us to the conviction that 3d pure quantum gravity is a theory which can be consistently defined\(^1\). Can we then prove or disprove the recent proposal made by Witten for a CFT dual to pure 3d gravity [4]? Clearly, answering this question amounts to ask what is the precise dictionary between AdS Quantum gravity and boundary CFT?

It is sometimes claimed that the CFT provides a definition of what a background independent quantum gravity theory in the bulk should be or that solving quantum gravity amounts to find the CFT it is dual to [4]. Is it true? If so is there a way to reconstruct the bulk from the boundary? How? What should be the equation driving the reconstruction?

Concerning the correspondence one may ask: Is the correspondence one to one between one theory of QG to one CFT? or is it one to many? If so, how can we talk about the CFT

\(^1\) By this we mean that we can give a definition of the path integral, define certain transition amplitude, some set of observables and solve the Wheeler-DeWitt equation, at least in the context of compact manifolds and also in the presence of particles [2]. There have been a wealth of techniques developed from Chern-Simons quantization, spin foam approach, t’Hooft approach etc... which have somehow different range of applicability but agree when they can be applied to the same problem (see [3] and ref therein).
associated to quantum gravity? And how can we think of proving the correspondence?

Another concern sometimes raised is that gravity is background independent, is it in conflict or not with restricting to asymptotically AdS space? Can we formulate such a restriction starting from a background independent approach of quantum gravity?

Finally, one may wonder: Is AdS/CFT a property of string theory in certain backgrounds only or is it a genuine and intrinsic property of any theory of quantum gravity whatever is its realization?

There is a vast literature on the subject of AdS/CFT, however we feel that only a relative small portion of it focuses on the questions asked here: First there is the key original reference of Witten [5] where the meaning of the correspondence is proposed; the body of work by Skenderis et al. [6, 7, 8] developing the semi-classical understanding of the correspondence and the holographic renormalisation group; the central claim of Verlinde’s and deBoer [9] that there is a relationship between the Hamilton-Jacobi equation and the holographic renormalisation group, and a Schrödinger picture sketched by Maldacena [10] in the de Sitter context.

Despite these key references, it is hard to find one reference addressing all the questions asked here in a systematic fashion. One of the purpose of this note is to try to fill this gap and present a unified interpretation of the AdS/CFT correspondence from the Quantum gravity point of view. We hope that even specialists will learn something new about AdS/CFT from our presentation.

In the second part of this paper and using the understanding of AdS/CFT developed in the first part we give an explicit reconstruction formula of the bulk amplitude given a CFT in the 2+1 dimensional case.

In section II of this note we present some well known fact about AdS/CFT and the original formulation of the AdS/CFT correspondence. We then identify three problems with this original formulation and study independently the equations satisfied by the bulk gravity and the boundary CFT. In section III we show how these puzzles can be resolved by looking at the asymptotics of the gravity amplitude. And we reach several key and sometimes unusual conclusions about the AdS/CFT correspondence in section III C. In section IV we present the general features, in any dimension, of the reconstruction of the bulk from the boundary CFT. Moreover we give, in dimension 2+1, an explicit reconstruction formula (69) for the quantum gravity amplitude given any boundary CFT. In the last sections V, VI we give the proof of this reconstruction formula.

II. ON THE ADS-CFT CORRESPONDENCE

We want in this section to review some known fact about this correspondence and clarify some puzzles concerning the precise dictionary expected between AdS and CFT.

At the semi-classical level this correspondence can be stated quite precisely [5, 11] by looking at space-times which are asymptotically AdS and by relating the value of the classical on-shell action of bulk fields with the expectation value of corresponding operators in the CFT. The main focus of this paper is the case of Lorentzian AdS manifolds, but some of the results in the following section can be also stated for Lorentzian dS manifolds, so we will consider both cases simultaneously.

Suppose that we have a background Lorentzian spacetime $M, g$ of dimension $d+1$ solution
of Einstein equations with a cosmological constant $\Lambda$

$$R_{\mu\nu}(g) = -\epsilon \frac{d}{\ell^2} g_{\mu\nu}, \quad \Lambda = -\epsilon \frac{d(d-1)}{2\ell^2}$$

with $\epsilon = -1$ for dS and $\epsilon = +1$ for AdS. This manifold is conformally compact if there exists a defining function $\rho$ such that $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on $\partial M$ and such that the conformally equivalent metric

$$\ell^2 \bar{g} = \rho^2 g$$

extends smoothly to the boundary of $M$. The Einstein equations imply that

$$\bar{g}^\mu\nu \partial_\mu \rho \partial_\nu \rho = \epsilon$$

on $\partial M$. The conformal infinity is therefore Lorentzian in AdS and Riemannian in dS. It is always possible to chose a defining function $\rho$ such that the relation (2) is valid not only on $\partial M$ but also in a neighborhood of $\partial M$. For instance, when $M$ is conformally compact we can take as a defining function $\ell \rho(x) = d^R_M(x, x)$, the distance of $x$ from $\partial M$ with respect to $\bar{g}$. In this coordinates the metric can be written

$$ds^2 = \frac{\ell^2}{\rho^2} (\epsilon d\rho^2 + \gamma_\rho), \quad ds^2 = (\epsilon dr^2 + \ell^2 e^{2r} \gamma_r), \quad \rho = \exp \left(-\frac{r}{\ell} \right)$$

where $\rho^{-2} \gamma_\rho$ is the metric induced on the surfaces $\Sigma_\rho$ with $\rho = cste$ and $\rho = 0$ defines the conformal infinity of $M$. The surfaces $\Sigma_\rho$ are spacelike in the dS case and timelike in AdS case. In both cases $\epsilon = n^2$ is the square norm of the normal vector to $\Sigma_\rho$. In the AdS case $r$ is the radial geodesic distance for the physical metric $g$.

The normal vector and extrinsic curvature of the surfaces $\Sigma_\rho$ are given by

$$n = \partial_r = -\frac{\rho}{\ell} \partial_\rho, \quad K^\alpha_\mu = h^{\alpha\alpha} \nabla_\alpha n_\mu = \frac{1}{2} h^{\alpha\alpha} \mathcal{L} n g_{\alpha\mu}$$

where $h_{\mu\nu} = g_{\mu\nu} - \epsilon \eta_{\mu\nu}$. This extrinsic curvature can be easily computed in our coordinates and this gives

$$\ell K^j_i = (\delta^j_i - \rho (\gamma_\rho^{-1} \partial_\rho \gamma_\rho)_j^i) = \delta^j_i + O(\rho^2).$$

One can easily see that $\partial_\rho \gamma_\rho |_{\rho=0} = 0$ thus the extrinsic curvature tensor is proportional to the identity operator up to order $\rho^2$ at infinity, this fact plays a key role in the AdS-CFT correspondence even at the quantum level. In order to see why lets consider wave function depending on the metric $\gamma_{ij}$ on $\Sigma_\rho$ and lets consider a radial evolution\footnote{Of course this evolution is timelike in the dS case} of this wave function towards the conformal infinity

$$\partial_r \Psi(\gamma_\rho) = \int_{\Sigma_\rho} (\partial_r \gamma_{ij} \delta \Psi) \sim_{\rho=0} \frac{2}{\ell} \int_{\Sigma_0} \gamma_{ij} \delta \Psi.$$

The last operation becomes just the operation of conformal rescaling. Thus thanks to the presence of the cosmological constant a radial evolution in the bulk is equal near infinity to a conformal transformation at the boundary. Clearly, infinity is left invariant by the radial evolution and therefore one expect conformal invariance of the physics described by $\Psi$ at
asymptotic infinity. As we will see more precisely, this is the essence of the correspondence between bulk gravity and boundary CFT.

The AdS/CFT correspondence as originally stated by Witten [5, 11] is a relation between the partition function of quantum gravity with fixed boundary data in the bulk and the generating functional of connected correlation functions of the CFT on the boundary. Namely, let $\Phi_i = \Phi_i, g_{\mu\nu}, ...$ denote fields that propagate in the bulk and let $\phi_i$ denote the asymptotic boundary value of this field on $\Sigma$ by $\phi_i$. On the gravity side, one can define the amplitude

$$\Psi_{\Sigma_\rho}(\phi_i) = \int_{\Phi_i|\Sigma_\rho = \phi_i} D\Phi_i e^{iS_{B,M}(\Phi_i)} \tag{7}$$

Where $S_{B,M}$ is the bulk action and the bulk partition function is evaluated with Dirichlet boundary condition on the fields. It is important to note that if we were in asymptotically flat space, asymptotic infinity would be null ($\partial M = \mathcal{I}^+ \cup \mathcal{I}^-$ in flat space) but we can still define the same type of Dirichlet amplitudes if we go a bit away from infinity and take a spacelike slice. What has been established a long time ago [12] but which is not often stated explicitly is that this Dirichlet amplitude is, in this case, nothing but the S-matrix functional. The usual S-matrix elements can be obtained from this functional by taking derivative of the S-matrix. It is then tantalizing to call the amplitude $\Psi$ the “(Ad)S-matrix functional”. This object is naturally related to the quantum effective action $\Gamma(\Phi_i)$ which depends on the bulk fields and is obtained by performing the path integral in the presence of background bulk fields $\Phi_i$. Given the quantum effective action $\Gamma(\Phi_i)$ we evaluate it on-shell and compute its Hamilton-Jacobi functional

$$S(\phi_i) \equiv \Gamma(\Phi_i)|_{\delta \Phi_i = 0, \Phi_i|_{\partial M} = \phi_i}. \tag{8}$$

This quantum Hamilton-Jacobi functional is then (in flat space) the generating functional for connected S-matrix elements: $\Psi(\phi_i) = e^{iS(\phi_i)}$, the bulk amplitude is thus an on-shell amplitude.

On the other hand, on the CFT side one associates to each boundary field $\phi_i$ a primary operator $\hat{O}_i$ of the CFT. Different fields $\phi_i$ are characterized not only by their tensorial structure but also by their properties under conformal transformation of the boundary metric. Namely when $\gamma \rightarrow \rho^{-2\gamma}$ the fields transform as $\phi_i \rightarrow \rho^{\Delta_i} \phi_i$, $\Delta_i$ is then the conformal dimension of the operator $O_i$. Given the CFT one can define the generating functional of connected correlation functions

$$Z_{CFT}(\phi_i) = \langle \hat{O}_i \rangle \tag{9}$$

Note that this is an off-shell amplitude. The AdS/CFT correspondence is the statement that there is an equality

$$\Psi_{\Sigma_0}(\phi_i) = Z_{CFT}(\phi_i). \tag{10}$$

A. Three puzzles

This original formulation is however not precise enough and leads to several puzzles. Formulating and resolving these puzzles is a key part of establishing a proper understanding of the deep nature of AdS/CFT correspondence and will allow us to propose a bulk reconstruction formula. We can clearly identify three main problems with the original formulation:
1-The first puzzle is purely technical in nature and has been identified since the beginning. It comes from the fact that the evaluation of the (Ad)S-matrix is at conformal infinity (it is computed by taking a limit $\rho \to 0$ towards asymptotic infinity) there are, even at the classical level, infinities arising in its evaluation that need to be taken care of and substracted.

2-The second puzzle is more conceptual but as important as the previous one. It comes from the fact that the formulation given here is in term of a background metric since it usually explicitly refers to the spacetime slicing given by $\rho$. We will be interested in quantum gravity, and in quantum gravity the metric is a dynamical object and cannot be fixed beforehand. Even more, in quantum gravity the quantum spacetime is represented by the knowledge of the wave function $\Psi$. The formulation of the correspondence should therefore be independent of a choice of a background metric. But then where is the asymptotic boundary?

3-Finally, and this is at first sight the most serious puzzle, the two objects in (10) do not satisfy the same equations! One is a solution of Wheeler-deWitt equation which is a second order differential equation and the other a solution of a conformal Ward identity which is a first order equation. How could one have equivalence between a second and first order differential system?

Hopefully this three puzzles are related and can be resolved all together using the idea of the so called “holographic renormalisation group" which is thus a key and central ingredient for the precise formulation of the AdS/CFT correspondence. We review here some of the results of the holographic renormalisation group [6, 7] but we hope to give a new and fresh perspective on some of the results and on the resolutions of these puzzles that will allows us to go further.

Let us start to analyse first what type of equations $\Psi$ and $Z_{CFT}$ are supposed to satisfy. We will look from now on to the case of pure gravity.

### B. Gravity equations

Let us first analyse the gravity sector. In this case we are interested in the following functional of a metric $\gamma$ on a d dimensional space $\Sigma$. In the case this manifold is the topological boundary of a d+1 dimensional manifold $M$, $\partial M = \Sigma$, we consider

$$\Psi_\Sigma(\gamma) = \int_{g|_{\partial M} = \gamma} Dg \ e^{i S_M(g)}$$  \hspace{1cm} (11)

This is indeed a formal expression and the problem of quantum gravity is to make sense of it. One can hope that there is a precise definition of this object in string theory or non perturbative gravity which is consistent. In our case, we will later work in 2 + 1 gravity where a non perturbative definition of Lorentzian quantum gravity and proposal for this partition function exists. In higher dimension we can also think about this integral to be defined at one loop in which case it is perfectly meaningful, keeping in mind that we expect any theory of quantum gravity to be in agreement with the one loop results.

In order to write down the Einstein action associated with a $d+1$ dimensional manifold $M$ having a boundary $\Sigma$, lets introduce some notations. We denoted by $g_{\mu\nu}$ the metric on $M$, $n^\mu$ the unit vector normal to the boundary $\Sigma$. We have that $n^2 = -1$ if $\Sigma$ is spacelike (dS case) and $n^2 = +1$, if $\Sigma$ is timelike (AdS case). In both case we denote $\epsilon = n^2$. We also
denote $\gamma_{\mu \nu} = g_{\mu \nu} - \epsilon n_{\mu} n_{\nu}$ the boundary metric on $\Sigma$ and
\[ K_{\mu \nu} = \gamma_{\mu}^{\rho} \nabla_{\rho} n_{\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu \nu} \] (12) is the extrinsic curvature tensor. The action is
\[ S_M(g) = - \left( \frac{1}{2\kappa} \int_M \, d^{d+1}x \sqrt{g} \left( R(g) - 2\Lambda \right) + \frac{\epsilon}{\kappa} \int_{\Sigma} \, d^d x \sqrt{\gamma} K \right) \] (13) where\(^3\)
\[ \Lambda = -\epsilon \frac{d(d-1)}{2l^2}, \quad \kappa \equiv 8\pi G, \quad K = K_{\mu \nu} \gamma^{\mu \nu} \] and $l$ is the cosmological scale. The boundary term\(^4\) is necessary in order to have a well defined variational principle\[^{[13]}\]. Indeed for an on-shell\(^5\) variation we have
\[ \delta S = \frac{1}{2\kappa} \int_{\Sigma} \sqrt{\gamma} \Pi^{ab} \delta \gamma_{ab}, \quad \Pi^{ab} = \epsilon (K^{ab} - \gamma^{ab} K) \] (14)
One can also easily show that under a bulk diffeomorphism $\delta \xi g_{\mu \nu} = L_\xi g_{\mu \nu}$ the action transforms as
\[ \delta_\xi S_M(g) = - \frac{1}{2\kappa} \int_{\partial M} \, d^dx \xi_n \left( \sqrt{\gamma} \left( R(g) - 2\Lambda \right) + 2\mathcal{L}_n (\sqrt{\gamma} K) \right) \] (15)
\[ = - \frac{1}{2\kappa} \int_{\partial M} \, d^dx \sqrt{\gamma} \xi_n \left( R(\gamma) - 2\Lambda + \epsilon (K^2 - K^{ab} K_{ab}) \right) \] (16)
where $\xi_n = \xi^\mu n_\mu$, and in the last equality we have used the Gauss-Codazzi equations\(^6\) to write the bulk Lagrangian in terms of the boundary metric and extrinsic curvature tensor. Using
\[ R(\gamma) = R^a_\mu_{\nu a} - 2\partial_\mu \Gamma_{\nu a}^a - \Gamma_{\nu a}^b \Gamma_{a\mu}^b + (\mu \leftrightarrow \nu). \] This convention ensure that the Euclidean sphere as a positive scalar curvature, and that AdS has a negative scalar curvature.
\[ \text{If the boundary conditions is such that the boundary metric vary discontinuously or change signature we need to include additional boundary terms}^{[13, 14]} \text{to the action of the form} \]
\[ + \frac{1}{2\kappa} \int d^{d-1} \sqrt{\gamma} \Theta \]
where $\Theta$ is the angle or boost parameter along the $d - 1$ joints at which the normal change signature or vary discontinuously.
\[ \text{We have chosen the overall sign of} \ S_M \ \text{so that when we couple to matter field the total action reads} \ S_M + S_m \ \text{for instance for scalar field we take} \]
\[ S_m(\phi) = \frac{1}{2} \int \sqrt{g} \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2. \]
The energy momentum tensor being such that $\delta S_m = \int_M \sqrt{\gamma} T_{\mu \nu} \delta g^{\mu \nu}$.
\[ \text{We mean the identity} \]
\[ R(g) + 2\epsilon \nabla_\mu (n^{\mu} K) = R(\gamma) + \epsilon (K^2 - K^{ab} K_{ab}) + 2\epsilon \nabla_\alpha a^\alpha. \] (17)
where $a^\alpha = \nabla_\alpha n_a$ is the acceleration and $g_{\mu \nu} = \gamma_{\mu \nu} - \epsilon n_{\mu} n_{\nu}$.\(^6\)
the invariance under diffeomorphism of the measure of integration we can express the change of the action under diffeomorphism by a boundary transformation $\delta \gamma_{ab} = 2\nabla_a \delta b + \xi_a 2 K_{ab}$. Using this and (14,16) one gets the Ward identities expressing the invariance of $\Psi(\gamma)$ under bulk diffeomorphisms. Let denote

$$\hat{\Pi}^a_b \equiv \frac{\delta}{\delta \gamma_{ab}(x)}, \quad \hat{\Pi}^a \equiv \gamma^{ac} \hat{\Pi}_{cb}, \quad \hat{\Pi} \equiv \hat{\Pi}_c$$

Note that thanks to (14) the action of the functional derivative on $\Psi$ amounts to compute expectation value of $L_n \gamma$ that is

$$\hat{\Pi}^a_b \Psi_{\Sigma}(\gamma) = \frac{i\epsilon}{\kappa} \langle K^{ab}(x) - \gamma^{ab} K(x) \rangle_{\gamma}$$

where the expectation value is taken with respect to the gravity measure (11). The Ward identity expressing the invariance of $\Psi(\gamma)$ under bulk diffeomorphism can be written $H \Psi_{\Sigma} = H_a \Psi_{\Sigma} = 0$ where

$$H_b = \nabla_a \hat{\Pi}^a_b$$

$$H = -\epsilon \kappa^2 : \left( \hat{\Pi}^a \hat{\Pi}_b - \frac{\hat{\Pi}^2}{d-1} \right) : + R(\gamma) + \epsilon \frac{d(d-1)}{l^2}$$

This are similar to the usual Hamiltonian constraint equations where $H_b$ is the generator of infinitesimal diffeomorphism. Usually, that is in the hamiltonian picture, these equation are constraint equations which are written only for the case when $\gamma$ is a spacelike boundary metric in a Lorentzian manifold. But as we have just seen these equations are Ward identities expressing the invariance of the gravity partition function under bulk diffeomorphisms, they can be derived even in a generalized context where the boundary is Lorentzian. In the deSitter case since the boundary surfaces are taken to be spacelike these are exactly the usual Hamiltonian equations. However in the AdS case (20) is no longer constraint equations but an evolution equation, we will call it (20) the “radial” Wheeler-deWitt equation. Note also that often, what is consider in the literature is the case of Euclidean hyperbolic gravity where $i S_M$ is replaced by $-S_M$ and $\gamma$ is Euclidian. We get in this case a similar equation which can be obtained from the AdS radial equation by the “Wick rotation” $\hat{\Pi}^{ab} \rightarrow i \hat{\Pi}^{ab}$, this amounts to change the sign of the kinetic term in (20).

Note however that since in the AdS case we are dealing with the radial Wheeler-deWitt equation, this means we do not expect an a priori relation between Euclidian and Lorentzian gravity in the AdS case. This is in stark contrast with the usual case where we can expect some correspondence between Euclidian and Lorentzian solutions of Wheeler-deWitt equation. Let us expand a bit on this point; and lets denote the kinetic and potential term by $T \equiv \hat{\Pi}^a \hat{\Pi}_b - \frac{\hat{\Pi}^2}{d-1}$ and $V = R(\gamma) - 2 \Lambda$. We also denote by $H_L$ the usual hamiltonian constraint operator associated with a slicing by constant time surface in a Lorentzian manifold and by $H_E$ the constraint associated to an arbitrary slicing in an Euclidean manifold, these two operators are related since

$$H_L = \kappa^2 T + V, \quad H_E = -\kappa^2 T + V$$

In both cases the state $\Psi_{L,E}(\gamma)$ annihilated by $H_{L,E}$ is a functional of a $d$ dimensional Riemannian metric. Now suppose that $\psi_E(\gamma) = \exp\left(-\frac{1}{\kappa} \Gamma(\gamma, \kappa)\right)$, $\Gamma(\gamma, \kappa) \equiv \sum_{n=0}^{\infty} \kappa^n \Gamma_n(\gamma)$, with
$\Gamma_n$ real, is a solution of the Euclidean hamiltonian constraints then $\psi_L(\gamma) = \exp(\frac{i}{\kappa} \Gamma(\gamma, i\kappa))$ is a solution of the Lorentzian constraints. So there is a clear mapping between the lorentzian and euclidean sector in this case\(^7\) This is the reason behind Hartle-Hawking proposal \(^{[16]}\).

Now in the case of AdS the constraint satisfied by the AdS-matrix functional is of the form

$$H^{(radial)}_L = -\kappa^2 T + V,$$

moreover the boundary metric is lorentzian instead of Riemannian, since it correspond to a radial slicing. There is no longer, in this case, any simple correspondence between the Lorentzian and Euclidean solution. The only correspondence one can naively think of is to analytically continue $\Psi_L(\gamma)$ into a functional of non degenerate complex metrics and evaluate it on a Riemannian section. Since $H^{(radial)}_L$ and $H_E$ are the same this will give a solution of $H_E$. However this solution will be of the form $\exp(\frac{i}{\kappa} \Gamma(\gamma_E))$ and not of the form $\exp(-\frac{1}{\kappa} \Gamma(\gamma_E))$. So this will give a solution of Riemannian quantum gravity but not of Euclidian gravity. One can otherwise try to perform a Wick rotation $t \rightarrow it$ of the action $\Gamma(\gamma)$. Such a rotation can be performed meaningfully if $\gamma$ possesses at least one timelike killing vector field. This might give a prescription for $\Psi_E(\gamma)$ in terms of $\Psi_L(\gamma)$ for stationary $\gamma$. Since there exists no preferred timelike Killing vector field for a general $\gamma$ it is not really clear how to extend this prescription meaningfully to general metrics. Any such extension would amount to pick a particular gauge and an associated preferred time coordinate. Even if we do so it is even less clear weather such an extension, if it exists, maps solutions of $H_L$ to solutions of $H_E$ (We know that such extension do not generally maps classical euclidean solutions to Lorentzian classical solutions \(^8\)). This lead us to the conclusion that one cannot expect beforehand a deep relationship between Euclidean and Lorentzian quantum gravity in the AdS/CFT correspondence as it is often (if not always) assumed. One may be able to establish such Euclidian/Lorentzian correspondence in some limited regime (perturbation around static boundary space, for instance) but so far this is still an interesting open question.

The dots $\ldots$ in equation (20) denotes normal ordering terms necessary to define the kinetic term of the hamiltonian. From the derivation of this equation, one sees that the action of the kinetic term on $\Psi(g)$ leads to evaluation of the two point function at coincident point and the associate divergence needs to be subtracted following a prescription initially designed by Symanzik \(^{[18]}\). Namely if we denote by $G_D^{\alpha\beta\mu\nu}(x, y)$ the two point function $\langle g_{\alpha\beta}(x)g_{\mu\nu}(y) \rangle$ calculated with Dirichlet boundary\(^9\) condition and $G^{abcd} = 1/2(\gamma^{ac}\gamma^{bd} + \gamma^{bc}\gamma^{ad}) - \gamma^{ab}\gamma^{cd}$ the

\[\text{References:}\]
\(^7\) In other words $\Psi_L(\gamma) \equiv \int_{g|_{\Sigma}=\gamma} Dg e^{iS(L)(g)}$ and $\Psi_E(\gamma) \equiv \int_{g|_{\Sigma}=\gamma} Dg e^{-S(E)(g)}$ formally satisfy the same WdW equation. Here $S(L), (resp. S(E))$ denotes the Lorentzian (resp. Euclidian) action and $\gamma$ is a Riemannian metric.
\(^8\) Except in three dimensions where it is possible to find such extension since Lorentzian and Euclidian classical spacetime are respectively quotient of $AdS_3$ (resp. $H_3$) by discrete subgroup $G$ of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ (resp. $SL(2, \mathbb{C})$). Fixing a set of free generators of $G \subset SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ we can analytically continue these to a set of generator of $SL(2, \mathbb{C})$ \(^{[17]}\). Such an extension is uniquely defined only for static spacetime but in general there is an infinite number of inequivalent extensions. Moreover any such extension generally maps many Lorentzian solutions to the same Euclidean solution.
\(^9\) More precisely we should fix Dirichlet conditions $g_{ab} = \gamma_{ab}$ on the metric components tangential to the boundary. And we should use Neuman conditions induced by the gauge fixing on the other components $g_{0a}$ which depend on the normal direction to the boundary \(^{[19]}\), (see appendix).
boundary supermetric. One consider the boundary to boundary propagator

\[ K^{ab}_{cd}(x, y) = G^{mn} \left( \mathcal{L}_n G_{mned}(x, y) \mathcal{L}_n \right)_{x, y \in \partial M} \]

where \( \mathcal{L}_n \) denotes the Lie derivative with respect to the normal to the boundary. Now the boundary to boundary propagator can be split into a part which is singular in the coincident limit and a regular part. The singular part \( K_S \) of this propagator needs to be subtracted from the kinetic term of the Hamiltonian in order to get a well defined Hamiltonian this is what the normal order stands for.

\[ : \left( \hat{\Pi}^a_b \hat{\Pi}_b^a \right) \right|_{(x,x)} = G_{abcd} \hat{\Pi}^a_b \hat{\Pi}^c_d - K_S^{ab}(x,x) \]

This Schrödinger renormalisation procedure which was first proposed and analysed by Symanzik [18] in the context of scalar field theory, can be carried out loop order by loop order [20]. For instance at one loop, one can evaluate the coincident limit of \( K_S \) by heat kernel methods, This functional is a local functional of \( K^{ab} \) and \( g^{ab} \). The terms dependent on \( K^{ab} \) can be reabsorbed into a redefinition of the kinetic terms while the terms dependent on \( g^{ab} \) into a redefinition of the potential terms which cancels the divergence up to two loop.\(^{10} \)

The last point we want to stress concerning the Hamiltonian equation is the obvious fact that this is a second order differential equation. For comparison with the CFT equations, it is convenient to decompose the metric in terms of a Liouville field \( \phi \) and a determinant one \( \gamma = e^{2\phi} \), \( \det(\gamma) = 1 \). We define \( e^{-\delta \phi} \hat{P}_b^a \equiv \hat{P}_b^a - \frac{\delta}{d} \hat{\Pi} \) the traceless derivative operator that acts on \( \gamma \) only, preserves its unimodularity and commute with \( \delta/\delta \phi \). The Hamiltonian equation in this splitting can be written as

\[ \mathcal{H} = \frac{\kappa^2}{d(d-1)} \left( \frac{\delta}{\delta \phi} \right)^2 - \kappa^2 e^{-2d\phi} \hat{P}^2 + e^{-2\phi} R(\gamma) - 2(d-1) \left( \hat{\Box} \phi + (d-2)(\nabla^2 \phi) \right) + \frac{d(d-1)}{l^2} \]

where \( \hat{P}^2 = \hat{P}_b^a \hat{P}_a^b \). This equation is a relativistic equation ( if one think that \( \phi \) plays a role analogous to time) which controls the Liouville field dependence of \( \Psi(e^{2\phi} \gamma) \). Such dependence is in principle determined once both \( \Psi(\gamma) \) and \( \frac{\delta}{\delta \phi} \Psi(e^{2\phi} \gamma) \) are given. This means that one expect the existence of (highly non trivial) propagating kernels \( K_\phi, K_{\phi} \) such that

\[ \Psi(e^{2\phi} \gamma) = \int D\gamma' \left( K_\phi(\gamma, \gamma') \Psi(\gamma') + K_{\phi}(\gamma, \gamma') \frac{\delta \Psi}{\delta \phi}(\gamma') \right) \]

Taking \( e^{2\phi} = \rho^{-2} \) with \( \rho \) a spatial constant defines a one parameter Hamiltonian subgroup or a preferred radial evolution. In the case of deSitter this evolution is the usual cosmological evolution where time appears as the rescaling of the spatial slices.

\(^{10} \) In a renormalisable theory one can show that this procedure preserves the Schrödinger form of the Hamiltonian, that is the Hamiltonian operator is always at most quadratic in the functional derivatives \( \hat{\Pi}^{ab} \). This follows from the fact that the renormalisation factor which is constructed in terms of the boundary to boundary propagator contains at most two time derivative. In a theory of gravity this procedure make sense in dimension higher that 3 only if one fixes the number of loops and allows oneself to fix a increasing number of renormalisation constant.
C. CFT equations

The analysis of the CFT side is much simpler. Since we work with pure gravity we are only interested by the equation satisfied by the CFT partition function $Z_{CFT}(\gamma)$ which also depends like $\Psi$ on a $d$-dimensional metric $\gamma$. This CFT partition function is the generating functional of connected correlation function of the energy momentum tensor of the CFT since insertions of $T^{ab}$ can be obtained by derivative of $Z_{CFT}$ with respect to $\gamma_{ab}$. Such a CFT partition satisfies also two equations. The first one expresses the invariance under diffeomorphism

$$\nabla_a \hat{\Pi}_b Z_{CFT}(\gamma) = 0,$$

and the second is the conformal Ward identity

$$\hat{\Pi}_x Z_{CFT}(\gamma) = \frac{1}{\sqrt{\gamma}} \delta \frac{\delta}{\delta \phi(x)} Z_{CFT}(e^{2\phi} \gamma) \bigg|_{\phi=0} = i A_d(x) Z_{CFT}(\gamma)$$

where $A_d(x)$ is the anomaly. It is zero in odd dimensions, whereas in even dimension it can be expanded in terms of a basis of certain curvature invariants of dimension $2d$ which should satisfy the Wess-Zumino consistency condition \[21\]. In dimension 2 and 4 it is given by

$$A_2(x) = \frac{c}{24\pi} R(x)$$

$$A_4(x) = \frac{1}{16\pi} \left( aE(x) - cW^2(x) + \alpha \Box R(x) \right)$$

where $c$ and $a$ are the two central charges. $W^2$ is the square of the Weyl Tensor and $E$ is the Euler density. The term proportional to $\alpha$ is not an anomaly since it can be obtained from the variation of a local action $\int \sqrt{\gamma} R^2$.

$$W^2 = R^2_{abcd} - 2 R^2_{ab} + \frac{1}{3} R^2,$$

$$E = \left( \frac{1}{2} \epsilon^{ef}_{ab} R_{efcd} \right)^2 = R^2_{abcd} - 4 R^2_{ab} + R^2.$$ \(27\) \(28\)

One remark that when $a = c$ the 4 dimensional conformal anomaly simplifies and contains no square of the Riemann tensor. Interestingly this is exactly the anomaly that arise from the semi-classical AdS/CFT correspondence \[6, 22\].

The conformal Ward identity is only a first order equation which can be explicitly integrated out. In two dimensions, the integrated Ward identity is given by

$$Z_{CFT}(e^{2\phi} \gamma) = e^{\frac{\alpha}{2\pi}} S_L(\phi, \gamma) Z_{CFT}(\gamma)$$

where the Liouville action is

$$S_L(\phi, \gamma) = \int_{\Sigma} \sqrt{\gamma} (-\phi \Box \phi + \phi R(\gamma)).$$

In dimension 4, it is also possible to integrate out the anomaly. Quite remarkably, and this fact seems to have been unnoticed in the literature, in the gravitational case $a = c$ the
integrated anomaly is also of the Liouville form\[29\]

\[ S_L(\phi, \gamma) = \int_{\Sigma} \sqrt{\gamma} \left( \frac{1}{2} \phi \Box_4 \phi + \phi Q_4(\gamma) \right) \]  

(31)

where \( Q_4 \equiv \frac{1}{2} \Box P + P^2 - P_a P^a \) and \( \Box_4 \equiv \Box^2 + \nabla_a (2P g^{ab} - 4P^{ab}) \nabla_b \) with \( P_{ab} = \frac{1}{d-2} \left( R_{ab} - \frac{R}{2(d-1)} g_{ab} \right) \). These objects satisfy the key relation

\[ e^{4\phi} Q_4(e^{2\phi} \gamma) = \Box_4 \phi + Q_4(\gamma). \]  

(32)

The curvature \( Q_4 \) is part of a series of curvatures called Q-curvatures which are the object of intense study in the mathematical literature \[23\].

**D. 2d CFT**

It is important to remember that a general CFT of central charge \( c \) in dimension two is not only supposed to be invariant under infinitesimal diffeomorphisms as expressed by (23) but also under large diffeomorphism, in other words it should also be modular invariant.

In two dimensions, the conformal Ward identities are powerful enough to extract a wealth of information about energy-momentum tensor correlations. As we have seen the conformal anomaly can be integrated out in terms of the Liouville action (30). If we pick local coordinates \( u, \tilde{u} \) we can write the metric as

\[ ds^2 = e^{2\phi} (du + \mu d\tilde{u})(d\tilde{u} + \tilde{\mu} du) = e^{2\phi} \hat{g} \]  

(33)

where \( \mu, \tilde{\mu} \) are the so-called Beltrami differentials. And the Partition function decomposes as

\[ Z(e^{2\phi} \hat{g}) = e^{\frac{ic}{12}} S_L(\phi, \hat{g}) Z(\hat{g}). \]  

(34)

This decomposition is dependent on the choice of the local coordinate system and imply that the action of diffeomorphism on \( Z(\hat{g}) \) is anomalous.

What is quite remarkable and has been first shown by Verlinde \[24\], in a beautiful work, is that it is possible to chirally split the diffeomorphism anomaly if one add to the effective action the counterterm

\[ S_V(\mu, \tilde{\mu}) = \int \frac{1}{1 - \mu \tilde{\mu}} \left( \partial_u \mu \partial_u \tilde{\mu} - \frac{\tilde{\mu}}{2} (\partial_u \mu)^2 - \frac{\mu}{2} (\partial_u \tilde{\mu})^2 \right) \]  

(35)

Namely if one define \( S(\phi, \mu, \tilde{\mu}) = S_L(\phi, \hat{g}) + S_V(\mu, \tilde{\mu}) \) and define

\[ Z(e^{2\phi} \hat{g}) \equiv e^{\frac{ic}{12}} S(\phi, \mu, \tilde{\mu}) Z(\mu, \tilde{\mu}). \]  

(36)

Then \( Z(\mu, \tilde{\mu}) \) satisfies the Virasoro Ward identities

\[ (\partial_u - \mu \partial_{\tilde{u}} - 2\partial_u \mu) \frac{\delta Z}{\delta \mu(x)}(\mu, \tilde{\mu}) = \frac{ic}{12\pi} \partial_u^3 \mu(x) Z(\mu, \tilde{\mu}) \]  

(37)

\[ (\partial_{\tilde{u}} - \tilde{\mu} \partial_u - 2\partial_{\tilde{u}} \tilde{\mu}) \frac{\delta Z}{\delta \tilde{\mu}(x)}(\mu, \tilde{\mu}) = \frac{ic}{12\pi} \partial_{\tilde{u}}^3 \tilde{\mu}(x) Z(\mu, \tilde{\mu}). \]  

(38)

\[ 1 \] where the prefactor is now \( \frac{1}{24} \) and one need to chose \( \alpha = 4 \)
Since derivative with respect to $\mu$ inserts expectation value of the holomorphic component of the energy momentum tensor: $Z(\mu, \bar{\mu}) = \langle e^{i \pi R T_\mu} e^{i \pi R \bar{T}_{\bar{\mu}}} \rangle$, it is possible to show that these Virasoro ward identities are in fact equivalent to the usual Virasoro O.P.E.

$$T(z) T(w) \sim \frac{c/2}{(z - w)^4} + \left( \frac{2}{(z - w)^2} + \frac{\partial_w}{(z - w)} \right) T(w).$$

The space of holomorphic conformal block is defined to be the space of solutions of (37) and this implies that the partition function can be written as a sum of holomorphic times anti-holomorphic components

$$Z(g) = e^{\frac{c}{24} \pi S(\phi, \mu, \bar{\mu})} \sum_{I, \bar{I}} N^{I \bar{I}} \chi_I(\mu) \chi_{\bar{I}}(\bar{\mu}).$$

The only ingredient not fixed by the conformal symmetry is the value of $N^{I \bar{I}}$ which should be such that the CFT is modular invariant. If we assume that the spectra of the CFT is discrete and then $N^{I \bar{I}}$ should be positive integers. A general solution of the Ward identity can be obtained by linear combination of such irreducible CFTs.

III. PUZZLES RESOLUTION AND ADS/CFT DICTIONARY

Now that we understand the nature of both sides of the AdS/CFT correspondence in more details, we can present a resolution of the three puzzles presented earlier. The key puzzle to address first, and which naturally leads to the resolution of the other two, is the question about background independence. The gravity AdS-matrix is defined by

$$\Psi_\Sigma(\gamma) = \int_{g|\partial M = \gamma} Dg \ e^{i S_M(g)}$$

The key puzzling point is that such a partition function depends on a topological manifold $M$ with boundary and a choice of a boundary metric $\gamma$. However this amplitude does not depend on some auxiliary asymptotically AdS spacetime and corresponding bulk metric. We cannot tell for instance where is the boundary located in some asymptotic AdS space. But if so where is asymptotic infinity?

In fact, because we are dealing with quantum gravity the functional $\Psi$ represents the quantum spacetime. This quantum spacetime is however represented only through its dependence on the boundary metric $\gamma$. In a classical spacetime this boundary metric is the metric induced on a radial slice and changing the slice will change the induced boundary metric. In quantum gravity we just have to reverse this point of view: varying the metric $\gamma$ amounts to move the slice inside the spacetime represented by $\Psi$. The knowledge of $\Psi$ for all possible values of $\gamma$ allows us in principle to reconstruct the semi-classical spacetime it corresponds to. At the classical level we have seen that the metric induced on the radial slice $\Sigma_\rho$ is given near infinity by $\rho^{-2}\gamma$ where $\gamma$ is a representative of the conformal class of the metric induced on asymptotic infinity. The rescaling in $\rho$ amounts to move the slice towards infinity. The natural proposal is therefore that the asymptotic property of the spacetime encoded by $\Psi$ is given by the behavior of the wave functional $\Psi(\rho^{-2}\gamma)$ in the limit when $\rho \to 0$. So asymptotic infinity in this context is not a specific locus in a background metric but comes from the asymptotic behavior of the functional $\Psi(\gamma)$ under infinite rescaling. This is a simple but very important point that allows us to talk about asymptotically AdS spacetime in quantum gravity without having to introduce a background spacetime.
A. The holographic side of Wheeler-deWitt equation

At the classical level the properties of asymptotic infinity follow \textit{dynamically} from the Einstein equation once one assumes the existence of a conformal boundary. The idea is simply to follow the same reasoning at the quantum level by looking at the asymptotic behavior of a solution of the radial WdW equation. The main property we want to show in this section is the following:

Given $\Psi(\gamma)$ an arbitrary solution of the radial WDW equation (20) then its asymptotic behavior when $\rho \to 0$ is given by

$$
\Psi_\Sigma \left( \frac{\gamma}{\rho^2} \right) \sim e^{\pm \frac{1}{\kappa} S(d) \left( \frac{\gamma}{\rho^2} \right)} Z_+ \left( \frac{\gamma}{\rho^2} \right) + e^{-\frac{1}{\kappa} S(d) \left( \frac{\gamma}{\rho^2} \right)} Z_- \left( \frac{\gamma}{\rho^2} \right)
$$

where $S(d)(\gamma)$ is an explicit local action, containing terms of dimension at most $d$. This action is \textit{universal} in the sense that it is the same for all radial states $\Psi$. Moreover $Z_\pm(\gamma)$ are a pair of CFT’s solution of the conformal Ward identity (24).

$$
\frac{1}{\sqrt{\gamma}} \frac{\delta}{\delta \phi(x)} Z_\pm(e^{2\phi \gamma}) \bigg|_{\phi=0} = \pm i A_d(x) Z_\pm(\gamma)
$$

This means that we can easily extract the $\rho$ dependence of $Z_\pm$ and get

$$
\Psi_\Sigma \left( \frac{\gamma}{\rho^2} \right) \sim e^{\pm \frac{1}{\kappa} \tilde{S}(d)(\gamma)} Z_+(\gamma) + e^{-\frac{1}{\kappa} \tilde{S}(d)(\gamma)} Z_-(\gamma).
$$

where

$$
\tilde{S}(d)(\gamma) = S(d) \left( \frac{\gamma}{\rho^2} \right) + \kappa \ln \rho \int_{\Sigma} \sqrt{\gamma} A_d.
$$

The conformal Ward identity satisfied by $Z_\pm$ is a left over or more precisely an holographic in-print of the radial WdW equation onto asymptotic infinity. Moreover from (44) and the previous discussion it is clear that the CFT’s $Z_\pm$ can be thought simply as initial value data for the radial wave function $\Psi$. The only unusual feature being that this initial slice is at infinity instead of being in the bulk. The expansion (44) is also reminiscent of a Born-Oppenheimer expansion where infinity is treated as the “heavy” component and possesses a rapidly oscillating but localized phase factor, whereas the CFT is the leftover “light” component.

The local action can be computed in terms of a loop expansion, the order zero term is, as we will see, given

$$
S(d)(\gamma) = \frac{d-1}{\ell} \int_{\Sigma} \sqrt{\gamma} - \frac{\ell}{2(d-2)} \int_{\Sigma} \sqrt{\gamma} R(\gamma) + \frac{\ell^3}{2(d-4)} \int_{\Sigma} \sqrt{\gamma} \left( P_a^b P^a_b - PP \right) + \cdots
$$

This expression is exactly the same as the counterterm action appearing in the holographic renormalisation group analysis \cite{7, 25}. In dimension $d = 2$ only the first term of (45) is

\footnote{We have denoted $P_a^b \equiv \frac{1}{(d-2)} \left( R_a^b - \frac{R}{2(d-1)} \delta_a^b \right)$ so that the third lagrangian is $L_2 = \frac{1}{(d-2)^2} \left( R_a^b R^a_b - \frac{d}{2(d-1)} R^2 \right)$}
relevant in the limit \( \rho \to 0 \); in dimension \( d = 3, 4 \) the first two terms contribute; in dim \( 5, 6 \) the first three etc... One also notice that in even dimension, the integrated anomaly is exactly given by the residue of the pole that arises for \( d = 2n \) in the local expansion of \( S^{(d)} \).

The inclusion of loop corrections renormalises the coefficients appearing in this local expansion. This renormalisation can be explicitly computed at one loop and we will see later that in \( d = 2 \) it leads to a finite renormalisation of the central charge. Since only a finite number of terms are relevant in order to get the dominant asymptotic, it seems that the non renormalisability of gravity is not registered by the asymptotic expansion \([12]\), this is clearly something that deserves further study and a deeper understanding.

To see why a solution of the radial WdW equation has the asymptotic behavior \([12]\) one first perform the change of variable \( \gamma_{ab} \to \rho^{-2} \gamma_{ab}; \hat{\Pi}^a_b \to \rho^{-d} \frac{\delta}{\delta \gamma_{ab}} \) \( \equiv \rho^{-d} \hat{\Pi}^a_b \) in the hamiltonian constraints. Thus \( \Psi(\rho^{-2} \gamma) \equiv \Psi_\rho(\gamma) \) is a solution of \( \mathcal{H}_\rho \Psi_\rho(\gamma) = 0 \) with

\[
\mathcal{H}_\rho = -\epsilon \kappa^2 \rho^{2d} \hat{\Pi} \cdot \hat{\Pi} + \epsilon \frac{d(d-1)}{\ell^2} + \rho^2 R(\gamma)
\]  

where we have denoted \( \hat{\Pi} \cdot \hat{\Pi} \equiv \hat{\Pi}^a_a \hat{\Pi}^b_b - \frac{\rho^2}{d-1} \). We now look here only at the radial equation for AdS (\( \epsilon = +1 \)). The dS case is similar.

One sees that in the limit \( \rho \to 0 \) the curvature term becomes irrelevant. and the solution of the resulting equation, where it is neglected, is easily found. There are two such solutions given by

\[
\Psi^{(0)}_\rho(\gamma) = \exp \left( \pm \frac{i}{\kappa \rho^d} \frac{d-1}{\ell} \int_{\Sigma} \sqrt{\gamma} \right)
\]
a linear combination of which represents the dominant term in the asymptotic expansion of \( \Psi_\rho \). They satisfies

\[
\kappa^2 \rho^{2d} \hat{\Pi} \cdot \hat{\Pi} \Psi^{(0)}_\rho(\gamma) = \frac{d(d-1)}{\ell^2} \Psi^{(0)}_\rho(\gamma).
\]

We can now expand around the state associated with the sign + (the expansion around the other state is similar) : \( \Psi = \Psi^{(0)} \Psi^{(1)} \). This expansion is easily carried out if one use the fact that \( \rho^{d} \hat{\Pi}^a_b \Psi = \Psi^{(0)} \left( \frac{i}{\kappa \rho^d} \hat{\Sigma}_b^a + \rho^d \hat{\Pi}^a_b \right) \Psi^{(1)} \). By construction the argument of \( \Psi^{(1)} \) is subdominant \( (O(\rho^2)) \) compare to the one of \( \Psi^{(0)} \), thus \( (\Psi^{(1)})^{-1} \left( \rho^{d} \hat{\Pi}^a_b \Psi^{(1)} \right) \) is of order at least \( \rho^2 \). In this expansion the hamiltonian constraint becomes

\[
\frac{2\kappa}{\ell} i \rho^{d} \hat{\Pi} \Psi^{(1)} + \rho^2 R(\gamma) \Psi^{(1)} - \kappa^2 \rho^{2d} \hat{\Pi} \circ \hat{\Pi} \Psi^{(1)} = 0.
\]  

(47)

In the case \( d = 2 \) and in the limit \( \rho \to 0 \) the last term of the identity is negligible since \( \hat{\Pi}^a_b \Psi^{(1)} \) is \( O(1) \) and the equation reduces, as promised, to the Ward identity

\[
\hat{\Pi} \Psi^{(1)} = i \frac{\ell}{2\kappa} R(\gamma) \Psi^{(1)} = i \frac{c}{24\pi} R(\gamma) \Psi^{(1)}, \quad \text{with} \quad c = \frac{3\ell}{2G}.
\]  

(48)

To go to the next order, we continue the expansion \( \Psi = \Psi^{(0)} \Psi^{(1)} \Psi^{(2)} \) with

\[
\Psi^{(1)} = e^{-i \frac{\ell}{2\kappa(d-2)} \int_{\Sigma} \sqrt{\gamma} R(\gamma)}
\]
the equation for $\Psi^{(2)}$ is then\(^{13}\)

$$
\frac{i 2 \kappa}{\ell} \hat{\Pi} \Psi^{(2)} + \rho^{4-d} \left( R^a_b R^a_b - \frac{dR^2}{4(d-1)} \right) \Psi^{(2)} + \frac{i 2 \kappa \ell \rho^2}{d-2} \left( R^a_b \hat{\Pi}^a_b - \frac{2d-3}{2(d-1)} \hat{R} \hat{\Pi} \right) \Psi^{(2)} - \kappa^2 \rho^d \hat{\Pi} \circ \hat{\Pi} \Psi^{(2)} = 0
$$

In dimension 4 and in the limit $\rho \to 0$ it becomes again the Ward identity

$$
\hat{\Pi} \Psi^{(2)} = i \frac{\ell}{8 \kappa} \left( R^a_b R^a_b - \frac{R^2}{3} \right) \Psi^{(2)} \tag{49}
$$

This shows the main statement of this section.

**B. Solving the Hamilton-Jacobi equation**

The result we have just presented is very close in spirit to the seminal work of deBoer and Verlinde's \cite{9}. This suggests a much simpler and more systematic derivation of the local action that enters the asymptotic expansion at the semi-classical level. This derivation essentially amounts to solve the Hamilton-Jacobi equation in terms of a local expansion. Such an expansion have been studied already in \cite{26} and developed to a much deeper extent by Skenderis et Papadimitriou in \cite{8}. Here we propose an even simpler derivation of their result and gives a closed recursive equation which follows directly from the Hamilton-Jacobi equation.

We define functional derivative operators

$$
\hat{\Pi}^{ab} \equiv \frac{2}{\sqrt{|\gamma|}} \frac{\delta}{\delta g_{ab}(x)}, \quad \hat{\Pi}^a_b \equiv \gamma^{ac} \hat{\Pi}_{cb}, \quad \hat{\Pi}^c \equiv \hat{\Pi}^a_a, \quad \hat{K}^a_{b} \equiv \hat{\Pi}^a_b - \delta^a_b \frac{\hat{R}}{d-1}, \quad \Pi^a_b = \hat{K}^a_{b} - \delta^a_b K \tag{50}
$$

The Hamiltonian constraint (for the AdS case) is given by

$$
\mathcal{H} = -\kappa^2 \left( \hat{\Pi}^a_b \hat{\Pi}^b_a - \frac{\hat{R}^2}{d-1} \right) + \left( R(g) + \frac{d(d-1)}{\ell^2} \right) \tag{51}
$$

Looking for a state $\Psi = e^{i S(g)}$ solution of the Hamiltonian constraint and neglecting all quantum correction proportional to $\kappa$ we get the Hamilton-Jacobi equation. For future convenience we write down this equation in terms of the $\hat{K}$ operator instead of the canonical momenta operator. One of the reason is that when we act with this operator on $S$ we obtain the extrinsic curvature which has a clearer geometrical meaning. It is also convenient to introduce the short hand notation

$$
(K \circ K)_{\mu\nu} \equiv \gamma^{\alpha\beta}(K_{\mu\alpha}K_{\nu\beta} - K_{\mu\nu}K_{\alpha\beta}), \quad (K \circ K) \equiv \gamma^{\mu\nu}(K \circ K)_{\mu\nu} \tag{52}
$$

\(^{13}\) If one neglects some operator ordering ambiguity that should be taken into account as a one loop renormalisation effect
The corresponding Hamilton-Jacobi equation reads

\[
\left( \hat{K} (\tilde{S}) \circ \hat{K} (\tilde{S}) \right) (\gamma) + R(\gamma) + \frac{d(d-1)}{l^2} = 0. \tag{53}
\]

We look for an expansion of \( S(\gamma) \) in terms of functional over the metric having fixed conformal dimension, that is

\[
\tilde{S}(\gamma) = \sum_{n=0}^{\infty} S_n(\gamma), \quad \text{with} \quad S_n(\rho^{-2} \gamma) = \rho^{-d+2n} S_n(\gamma). \tag{54}
\]

This expansion can therefore be thought as a Taylor expansion in the parameter \( \rho \). Let's start to solve the equation at order 0. From

\[
\hat{K}^b_a \left( \int \sqrt{\gamma} \right) = \frac{1}{(1-d)} \delta^b_a, \quad (\delta \circ \delta) = d(1-d), \quad (\delta \circ K) = (1-d)K \tag{55}
\]

one easily sees that

\[
S_0(\gamma) = \pm \frac{(d-1)}{\ell} \int_\Sigma \sqrt{\gamma} \tag{56}
\]

is the solution at order 0. In order to match the classical expansion we choose the sign + that match the classical analysis that is \( \ell \hat{K}^b_a(\Psi) = \delta^b_a \Psi \). The other sector can be obtained by changing the sign of \( \ell \). We can now use this solution to start a local expansion of the Hamilton-Jacobi equation,

\[
\tilde{S}(\gamma) = S_0(\gamma) + S(\gamma)
\]

where \( \tilde{S} \) starts at order one. The equation now reads

\[
\frac{2}{\ell} \hat{\delta}^D_x S(\gamma) = \left( \hat{K}_x (S) \circ \hat{K}_x (S) \right) (\gamma) + R(\gamma(x)) \tag{57}
\]

Where we have introduced the operator \( \hat{\delta}^D_x \) that generates local conformal rescaling namely

\[
\frac{2}{\sqrt{\gamma}} g^{ab} \frac{\delta}{\delta g_{ab}(x)} \equiv \hat{\delta}^D_x F(\gamma) = \left. \frac{1}{\sqrt{\gamma}} \frac{\delta F(e^{2\phi})}{\delta \phi(x)} \right|_{\phi=0} \tag{58}
\]

If one integrate this operator over \( \Sigma \) one obtain the conformal dimension operator, which can in turn be written as a differential operator \(-\rho \partial_{\rho}\). Using this we can now get, quite remarkably, a closed equation for \( S \) which is the result we were looking for

\[
2 \partial_r S(e^{2\phi} \gamma) = \int_{\Sigma} \sqrt{\gamma(x)} \left( \hat{K}_x (S) \circ \hat{K}_x (S) \right) (e^{2\phi} \gamma) + e^{(d-2)\phi} \int_{\Sigma} \sqrt{\gamma} R(\gamma). \tag{59}
\]

We can write this equation in terms of the expansion coefficients

\[
S_1(\gamma) = \frac{\ell}{2(d-2)} \int_{\Sigma} \sqrt{\gamma} R(\gamma) \tag{60}
\]

\[
2(d-2n)S_n(\gamma) = \ell \sum_{m=1}^{n-1} \int_{\Sigma} \sqrt{\gamma} \left( \hat{K}_x (S_m) \circ \hat{K}_x (S_{n-m}) \right) (\gamma) \tag{61}
\]
We can recursively solve these equations, the procedure is straightforward but increasingly tedious. It is however quite simpler than the original way of computing the counterterm action \[6, 7\]. Using the results for \(S_1\) we can compute the extrinsic curvature at first order and putting it back in (61) compute \(S_2\), denoting \(K_{(n)}^b \equiv \hat{K}^b_n(S_n)\) one obtains

\[
K_{(1)}^b_a = -\frac{\ell}{(d-2)} \left( R^b_a - \frac{R}{2(d-1)} \delta^b_a \right) \equiv -\ell P^b_a \tag{62}
\]

\[
S_2 = \frac{\ell^3}{2(d-4)} \int_\Sigma \sqrt{\gamma} \left( P^b_a P^a_b - PP \right). \tag{63}
\]

C. AdS/CFT dictionary

The asymptotic expansion (42) is the key formula expressing a deep relationship between bulk quantum gravity and boundary CFT. It shows that a generic radial state solution of the radial Wheeler-deWitt equation generically corresponds to two different CFTs. And that these CFTs can be understood as initial value data, on a slice at infinity, for the wave function solution of radial WdW.

The fact that there are two CFTs is in sharp contrast with the usual interpretation of AdS/CFT where one usually assumes the existence of only one CFT associated with quantum gravity in the bulk. This surprising feature follows immediately from the fact that the WdW equation is second order whereas the Ward identity is first order so that we need two independent pieces of initial data to specify a solution of WdW equation. Note that we have emphasized here the AdS case, but as we have already said all the derivations and conclusions are true also in the deSitter case. In the deSitter case, since one can have two asymptotic region, it is customary to associate two CFTs to a gravity state \[27\], what we are emphasizing is that this conclusion should hold generically in AdS too.

Another surprising, and disturbing at first sight, feature of our derivation is the fact that contrarily to the classical case we haven’t assume at any moment that the state \(\Psi\) corresponds to a spacetime which admits an asymptotic infinity. In fact, the proof given in the previous section is valid for any solution of the quantum gravity equation of motion. Among these there should be ones which correspond semi-classically to AdS compact spacetime. This means that the AdS/CFT correspondence can be in principle extended even to spacetime having no classical asymptotic infinity as long as we accept having to deal with pairs of CFTs. From the quantum point of view there is nothing deeply surprising about it since we know that a wave function do carry some information about classically forbidden region\[14\].

\[14\] There is even in the mathematical literature an important result that points in the same direction: In the context of 3d classical gravity the problem of classification of all global hyperbolic Lorentzian AdS spacetimes with compact slice has been solved, single-handedly, in a remarkable work by G. Mess \[28\]. Remarkably, one of the key ingredient of the constructive proof given by Mess is to show that a 3d geometry can be mapped to a pair of points on Teichmüller space, that is to a pair of conformal 2d geometries. These conformal geometries lives in fact on the infinity of the universal cover of the compact spacetime under consideration. Since the spacetime is compact this infinity is fictitious from the point of view of the initial spacetime geometry even if it the most efficient way to describe the 3d geometrical information.
Now this lead immediately to a natural question: Which quantum states Ψ corresponds to spacetimes having an asymptotic infinity? If one look closer at the structure of the asymptotic one can see that the two terms that arises in the asymptotic expansion are distinguished by the value of the extrinsic curvature \( \hat{K}^b_a \Psi = \frac{\kappa}{i} \left( \hat{\Pi}^b_a - \delta^b_a \frac{\hat{\Pi}}{d-1} \right) \Psi \sim (\delta^b_a + \cdots) e^{iS^{(d)}(\gamma)} Z_+(\gamma) + (-\delta^b_a + \cdots) e^{-iS^{(d)}(\gamma)} Z_-(\gamma) \) (64)

where \( \cdots \) stands for terms of order \( O(\rho^2) \). The branch associated to \( Z_+ \) has a positive extrinsic curvature proportional to the identity when going to infinity (represented by the limit \( \gamma \sim \rho^{-2}, \rho \to 0 \)) which is what is expected in a classical spacetime with asymptotic infinity. The branch associated with \( Z_- \) has a negative extrinsic curvature when going to infinity. There is no classical interpretation of this branch. The only possible interpretation is to imagine that one has a slicing of spacetime where infinity is reached from the outside instead of the inside. In the Schrödinger picture, the branch \( Z_+ \) correspond to a wave outgoing at infinity and the branch \( Z_- \) to a wave in-going from infinity. Thus if \( Z_- \neq 0 \) the wave asymptotic is similar to a standing wave for a particle in a confining potential. This analogy and the extrinsic curvature computation tell us that a radial state can correspond to a semi-classical spacetime with asymptotic infinity if we have that \( Z_- = 0 \). From now on we will assume that an quantum AdS spacetime with asymptotic infinity is a solution of radial WdW equation where \( Z_- = 0 \). This is then consistent with the usual picture. However, at this stage this is only a plausible interpretation that \( Z_- = 0 \) amounts to having an asymptotic infinity. More work is clearly needed in order to show this in more detail and understand the deeper role of the pairs of CFT’s.

Another key point which follows from (42) is the fact that the correspondence between quantum gravity and Conformal Field theory is not one to one, that is there is not one CFT which is equivalent to the theory of quantum gravity, even if we restrict to \( Z_- = 0 \). The correspondence is indeed one to many. More precisely, since the CFT is understood as some initial value data for the quantum gravity radial states, it defines a radial state. In fact as we will see in the next section the correspondence should be such that for any boundary CFT there is a corresponding quantum gravity radial state solution of WdW equation as long as the central charge match. This will be argued for the case of general dimension and proven in the case \( d = 2 \). For instance, in the case \( d = 2 \) what this means is that the space of 2d CFTs with fixed central charge is isomorphic to the space of radial quantum gravity states.

This is not at odd with the usual interpretation in which a drastic change in the bulk (addition of branes or different internal boundary conditions) can be sometimes taken into

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15 Note that if we where working in the context of Euclidian AdS/CFT then the behavior of a radial WdW solution is

\[
\Psi_\Sigma \left( \frac{\gamma}{\rho^2} \right) \sim e^{-\frac{i}{2} \tilde{S}^{(d)}_\rho(\gamma)} Z_+(\gamma) + e^{\frac{i}{2} \tilde{S}^{(d)}_\rho(\gamma)} Z_-(\gamma).
\] (65)

where \( \tilde{S}^{(d)}_\rho(\gamma) \) is a positive functional near \( \rho = 0 \) and \( \frac{1}{\sqrt{\gamma}} \left. \frac{\delta}{\delta \phi(x)} Z_\pm(e^{2\rho \gamma}) \right|_{\phi=0} = \mp A_d(x) Z_\pm(\gamma) \). The universal prefactor associated with \( Z_+ \) is exponentially subdominant in the limit compare to the factor associated with \( Z_- \). The semi-classical evaluation of the Euclidean path integral corresponds to the sector \( Z_- = 0 \) for which \( \Psi_\Sigma \) is normalisable.
account in the CFT by adding a relevant or irrelevant operator which modify the hamiltonian of the original CFT. However, the picture here is even more dramatic than that (in the case of pure gravity and in the absence of SUSY\textsuperscript{16}), because scanning the space of CFT means changing the Hamiltonian with a given field content but it also means that we can change the field content dramatically as long as we preserve the central charge. It can also mean that we scan different CFT having quite different gauge symmetries.

To some extent AdS/CFT correspondence in the context of pure gravity is reminiscent of what happens in the quantum Hall effect. In this case the bulk system is made up of interacting electrons confined to a plane. Now, given a choice of filling fraction, magnetic field, etc... that fix a vacuum state in the bulk (e.g Laughlin state) there exists a boundary description of this system in terms of a chiral CFT. The chiral boundary theory depends heavily on the chosen bulk vacuum state. The difference here is that what is fixed in the bulk is a radial state and not an usual hamiltonian vacuum state. The physical meaning and interpretation of this radial state is not entirely clear, despite some attempts [30, 31]. Bringing light on this issue seems one of the key thing to develop in order to grasp the deeper meaning of AdS/CFT in the context of quantum gravity.

What we can say is that a radial state represents at the semi-classical level a spacetime or a superposition of spacetimes with fixed boundary topology. So it is somehow a definition of what we would call a quantum spacetime. At the quantum level it is not known precisely how this radial state is related to usual hamiltonian states of gravity, but what one can expect is that such a radial states are in one to one correspondence with density matrix, even the precise form of the isomorphism needs to be unraveled.

Since it is clear that there are many CFT’s given one quantum gravity theory, we are faced with the puzzle of understanding what does it means to find the CFT dual to a given theory of quantum gravity? For instance in [4] Witten made a proposal for the CFT dual to 3d gravity. How can we prove or disprove such a statement even in principle, if the correspondence is not one to one and assuming that we have a bulk definition of quantum gravity?

From the previous discussion we know that a CFT cannot define the full quantum gravity theory but at most a radial state of quantum gravity, thus what really make sense is to identify a particular radial state of quantum gravity and then construct the CFT associated to this preferred “vacuum” radial state. The question is then which radial state should one chose?

If we first think about this question in the Euclidean context, there is a preferred notion of such a radial state (see [32] for a 3d example). Let us recall that given a topological manifold $M$ with boundary $\Sigma$ and assuming that we have a definition of bulk quantum gravity (i.e that we can make sense of the path integral ) we can associate to such a topological manifold a unique radial state given by

$$\Psi_M(\gamma) = \int_{[\gamma]} Dg \ e^{iS_M(g)}$$

where $[\gamma]$ denotes a diffeomorphism equivalence class of boundary metric. More precisely Lets consider $D_\Sigma \equiv Diff_\Sigma/Diff_M(\Sigma)$ the space of diffeomorphisms of $\Sigma$ which do not

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\textsuperscript{16} For instance, the set all possible CFT in dimension 4, which have $N = 4$ SUSY and a fixed anomaly (which determine the gauge group) consist of only one candidate [29].
extend to diffeomorphism of \( M \) (\( D_\Sigma \) only contains diffeomorphism not connected to the identity). Then

\[
\Psi_M(\gamma) = \sum_{f \in D_\Sigma} \int_{g|\Sigma = f^*\gamma} \text{D}g \ e^{iS_M(g)}. \quad (67)
\]

The summation over non trivial boundary diffeomorphisms ensure the modular invariance of the boundary CFT.

This prescription assigns modular invariant states to spacetimes. It is then natural to propose that the preferred radial state one should study corresponds to the simplest manifold given a boundary topology. For instance, if \( \Sigma \) is a \( d \)-dimensional sphere one should take as \( M \) the \( d + 1 \) dimensional ball. If \( \Sigma \) is a \( d \)-dimensional torus, we should take as \( M \) the handlebody or plain torus which is the simplest topological manifold having \( \Sigma \) as a boundary. This defines a unique radial state of quantum gravity and a corresponding CFT that should be identified. If \( d = 2 \) and \( \Sigma \) is a genus \( g \) surface we can still define the notion of a handlebody \( H_\Sigma \) with boundary \( \Sigma \) and the corresponding state \( \Psi_{H_\Sigma} \) should allow us to identify the genus \( g \) CFT partition function.

The prescription for the Lorentzian case which is really the case of interest is a bit more difficult to identify precisely. The reason being that a radial state depends only on the timelike part of the boundary. However, in general one needs to specify what type of data one should use in the spacelike part of the boundary. For instance, if we have an AdS cylinder, it is natural to put as initial and final state the vacua state of the ADM Hamiltonian which will be identified as some time translation operator on the boundary. Thus the boundary amplitude, will not really be a partition function but a transition amplitude from initial to final CFT state.

In order to identify the CFT we would like to have access to a partition function instead. Moreover, the radial gravity state that one should use to identify the CFT should at least contain Black-Holes, so it is natural to consider \( M \) to be a manifold of topology the one of an eternal Black Hole solution in AdS. Such a space has trivial topology in the Bulk and possess two different asymptotic region which are both isometric to timelike cylinder on which one should fix the asymptotic conformal metric. There is an ambiguity to resolve onto how one should identify the two metrics associated with different timelike cylinders. The simplest and most natural prescription is to identify these two metrics at late and early time so that the asymptotic boundary is effectively a torus. This leads to a proposal (which should be tested) for a choice of a radial state of gravity which is needed if one want to be able to identify, even in principle, what is meant by the CFT associated with gravity.

There is an important caveat to keep in mind in this derivation: as we have seen, from a radial state of the type \( \Psi_{H_\Sigma} \) we can extract the CFT partition function \( Z_+(\gamma) \) which satisfies all the axioms that a CFT partition function should satisfy. However, it is not clear a priori that such a CFT partition function is irreducible. That is we haven’t shown that such a CFT partition function cannot be written a linear sum of different CFTs: \( Z_+(\gamma) = \sum_i a_i Z_i(\gamma) \) where \( Z_i \) are individual CFTs associated with a conformally invariant Hamiltonian \( H_i \).

IV. BULK RECONSTRUCTING KERNEL

In the previous section we have seen that given a radial quantum gravity state \( \Psi \) we can extract from it a CFT \( Z_+ \) (we restrict to gravity states determined by one CFT) by looking
at its asymptotic behavior. The key question now is can we reconstruct the bulk geometry, hence $\Psi$, from the knowledge of the Boundary CFT?

The answer to this question is clearly positive: First, as shown in section II B since $\Phi$ is solution of an Hamiltonian equation we can reconstruct it from its knowledge on a given initial slice (see (22)); moreover we can interpret $Z_+$ as the specification of $\Psi$ on a slice situated at infinity. One expect, therefore, the existence of a reconstructing kernel which allows us to “evolve” $\Psi$ from infinity to the the interior. This means that there exist a Kernel $K(\gamma, \gamma')$ which is state independent and should intertwine the action of the radial hamiltonian constraint with the action of the Ward identity operator. That is $K$ should be such that

$$
\Psi_Z(\gamma) = \int D\gamma' K(\gamma, \gamma') Z(\gamma') , \quad \mathcal{H} \Psi_Z = 0 \quad \text{and} \quad \Psi_Z(\rho^{-2}\gamma) \sim e^{iS(\rho^{-2}\gamma)} Z(\gamma), \quad (68)
$$
given any solution $Z$ of the conformal ward identity.

It is often said or assumed [9] that this bulk reconstruction is equivalent to a renormalisation group analysis hence the name “holographic renormalisation”. However in the light of our analysis we don’t feel that this is a very accurate description: It is true that a change of scale in the boundary (going from small to big scale) amounts to probing more and more the interior of the bulk [33]. In fact this comes from the original picture that a rescaling of $\gamma$ as an argument of $\Psi$ can be interpreted as moving the slicing which defines $\gamma$ radially. Moreover, (this follows from (68) and we will this more explicitly soon) the kernel $K$ is picked around $\gamma = \gamma'$ so a a bulk metric rescaling amounts to a boundary metric rescaling. However, it is incorrect to think that the equation governing the radial evolution is just a renormalisation group equation (which is first order). The renormalisation group equation (or conformal Ward identity) governs finite rescaling of the boundary CFT. In order to probe the bulk we need to achieve infinite rescaling and the corresponding equation is the full WdW equation which is second order. In this sense the radial Wheeler de Witt equation can be thought as an extension or a completion of the renormalisation group equation beyond its usual range of validity. And the Kernel $K$ is needed in order to convert boundary rescaling into bulk radial motion.

### A. Reconstructing Kernel in 2+1 dimensions

We now present an explicit reconstruction formula for the wave functional in terms of the boundary CFT in the case of three dimensional gravity. In order to do so it is convenient to label the 2-dimensional geometries by frame fields $e^\pm = e^\pm_\mu dx^\mu$ and $E^\pm$ such that

$$
d s^2 = e^+ - e^- , \quad d s'^2 = E^+ E^- .
$$

We also denote

$$
e \equiv e^+ \wedge e^-
$$

the 2d volume two form\textsuperscript{17}.

\textsuperscript{17} The normalisation is such that $e = 2\sqrt{\gamma}$. 
The main claim of this paper is that if $Z_c(E)$ is a solution of the 2d Ward identity then the wave functional $\Psi(e^\pm)$:

$$\Psi(e^\pm) = \exp\left(\frac{ik}{4\pi} \int e\right) \int DE \exp\left(-\frac{ik}{2\pi} \int (E^+ - e^+) \wedge (E^- - e^-)\right) Z_c(E)$$

(69)

is a solution of the three dimensional Wheeler-deWitt equation. Here $k \equiv \frac{\ell}{\pi \rho}$ and the relationship between $k$ and $c$ is

$$c = 1 + 6k.$$  

(70)

The functional measure is given in term of the reparametrisation and Lorentz gauge invariant distance on the space of frame

$$(\delta E, \delta E) = \int \delta E^+ \wedge \delta E^- \int D(\delta E) e^{\frac{i}{2\pi} (\delta E, \delta E)} \equiv 1.$$  

(71)

Before giving a proof of this statement lets us consider the behavior of such a state $\Psi$ under a rescaling of the metric:

$$\Psi\left(\frac{e^\pm}{\rho}\right) = \exp\left(\frac{ik}{4\pi \rho^2} \int e\right) \int DE \left(\frac{E}{\rho}\right) \exp\left(-\frac{ik}{2\pi} \int (E^+ - e^+) \wedge (E^- - e^-)\right) Z_c(\rho^{-1} E).$$

We can use the the conformal anomaly equation to extract the $\rho$ dependence in $Z_c(E)$ hence

$$\Psi\left(\frac{e^\pm}{\rho}\right) = \exp\left(\frac{ik}{4\pi \rho^2} \int e\right) \rho^{\frac{ic}{28\pi}} \int DE \exp\left(-\frac{ik}{2\pi} \int (E^+ - e^+) \wedge (E^- - e^-)\right) Z_c(E)$$

In the limit $\rho \rightarrow 0$ the term in the integrand become a delta functional imposing $E = e$. It is convenient to make the change of variable $E \rightarrow e + \rho E$ hence we obtain

$$\Psi\left(\frac{e^\pm}{\rho}\right) = \exp\left(\frac{ik}{4\pi \rho^2} \int e\right) \rho^{\frac{ic}{28\pi}} \int DE \exp\left(-\frac{ik}{2\pi} \int E^+ \wedge E^-\right) Z_c(e + \rho E)$$

thus in the limit we get

$$\Psi\left(\frac{e^\pm}{\rho}\right) \sim \mathcal{N} \exp\left(\frac{ik}{4\pi \rho^2} \int e + \ln \rho \frac{ic}{24\pi} \int e R\right) Z_c(e)$$

(72)

which agree with the holographic renormalisation in the semi-classical limit $c \sim 6k$. Here $\mathcal{N} = \int DE \exp\left(-\frac{ik}{2\pi} \int E^+ \wedge E^-\right) = 1.$

Another intriguing limit to consider, is the opposite limit $\lim_{\rho \rightarrow 0} \psi(\rho e)$ which amounts to look at the value of $\psi$ for the singular metric $e = 0$. In this case it is convenient to introduce the parameterization

$$e^+ = e^{\varphi + \alpha}(du + \mu d\tilde{u}), \quad e^- = e^{\varphi - \alpha}(du + \tilde{\mu} d\tilde{u})$$

In this limit, the dependence on $e$ formally drops out of the integral and the resulting integral is invariant under 2d diffeomorphism and lorentz gauge transformation. This means that this limit is singular and we can factor out an infinite gauge volume factor $V = \text{Vol}(\text{Diff}_\Sigma)\text{Vol}(\text{Lorentz})$. The resulting integral contains an integral over the Liouville field and the moduli $m$. It is given by

$$\Psi(\rho e^\pm) \sim_{\rho = 0} V \int D\phi Dm D\tilde{m} e^{\frac{i}{2\pi} (\phi (S_L(\phi, g_m) + \mu) \sqrt{g_m})} Z_c(g_m)$$

(73)
with \((c - 26)\mu \equiv 24k\). The integral is exactly the definition of non-critical string theory associated with the CFT \(Z_\beta\). Of course, since the prefactor is infinite this is a extremely singular limit but it is still interesting to see non critical string theory arising from an attempt to look at the wave function deeply in the bulk. It would be interesting to understand what is the meaning of this property.

V. GRAVITY IN THE FIRST ORDER FORMALISM

Since in the proof of (69) we use the first order formulation of gravity, we review in this section the formulation of gravity in the first order formalism and show its equivalence at the hamiltonian level with the WdW equation discussed in the bulk of the paper. The action of gravity in the metric formalism is given by

\[
S = \frac{1}{16\pi G} \int_M \sqrt{g} \left( R(g) + \frac{2}{l^2} \right) + \frac{1}{8\pi G} \int_{\partial M} \sqrt{g} K
\]  

(74)

In the first order formalism (Cartan-Weyl formulation of gravity) the dynamical variables are a frame field \(e^i\) and an \(SL(2,\mathbb{R})\) connection \(\omega^i\) and the bulk action is given by

\[
S_G = \frac{1}{8\pi G} \int_M dt \int_{\Sigma} \text{tr} \left( e_u \partial_t \omega^u - e_\bar{u} \partial_t \omega_\bar{u} + 2e_t H(e, \omega) + \omega_t G(e, \omega) \right) d\bar{u} u
\]  

(75)

where \(R^i(\omega) = d\omega^a + \epsilon^{abc} \omega_b \wedge \omega_c\), \(\Lambda = -1/l^2\), and the metric is given by \(ds^2 = e^i \eta_{ij} e^j\). From this Cartan-Weyl formulation, one recover the second order formulation if one solves for the connection. In order to do the hamiltonian analysis, we introduce a ‘time’ slice and coordinates \(t, u = x + y, \bar{u} = x - y\).

\[
S_G = \frac{1}{8\pi G} \int dt \int_{\Sigma} \text{tr} \left( e_u \partial_t \omega^u - e_\bar{u} \partial_t \omega_\bar{u} + 2e_t H(e, \omega) + \omega_t G(e, \omega) \right) d\bar{u} u
\]  

(76)

where the constraints are

\[
H(e, \omega) \equiv \partial_u \omega^u - \partial_\bar{u} \omega_\bar{u} - [\omega_u, \omega_\bar{u}] - \Lambda [e_u, e_\bar{u}]  
\]  

(77)

\[
G(e, \omega) \equiv \partial_u e_\bar{u} - \partial_\bar{u} e_u + [\omega_u, e_\bar{u}] - [\omega_\bar{u}, e_u] 
\]  

(78)

where \(e = e^iT_i\), with \([T_i, T_j] = \epsilon_{ijk}T^k\), \(\text{tr}(T_i T_j) = \frac{1}{2} \eta_{ij}\) \((+++\) signature).

What we remark first is the fact that there are three lie algebra elements \(e^i\) parameterizing the 2d metric \(g_{ab} = e^i_a e^j_b \eta_{ij}\) on \(\Sigma\). In order to make contact with the hamiltonian formulation in terms of variable we introduce a Cartan decomposition of the \(SL(2,\mathbb{R})\) algebra, where the generators are \(T_3 = \frac{1}{2}\sigma_3\), \(T_\pm = \frac{1}{2\sqrt{2}} \sigma_\pm\) which satisfies \([T_3, T_\pm] = \pm T_\pm\), \([T_+, T_-] = T_3\) and \(\text{tr}(T_3 T_3) = \frac{1}{4} = \text{tr}(T_+ T_-)\). We decompose the metric in terms of a vector \(n \equiv e^3\) and a 2d frame field \(e^\pm\)

\[
e^\mu_\ell \equiv n_\mu T_3 + e^\mu_+ T_+ + e^\mu_- T_- 
\]

Similarly we decompose the connection in terms of a \(U(1)\) connection \(\omega\) and the variables conjugated to \(e^\pm\). More precisely we denote

\[
\omega_\mu \equiv \omega_\mu^3, \quad \pi^u_\pm \equiv \omega_\mu^\pm, \quad \pi^u_\pm \equiv -\omega_\mu^\mp. 
\]
The commutators are given by

\[
[\pi^\mu(x), e^b_\nu(y)] = \delta_\nu^b \frac{4\pi}{ik} \delta^{(2)}(x-y),
\]

\[
[\omega_\mu(x), n_\nu(y)] = \frac{4\pi}{ik} \delta^{(2)}(x-y), \quad [\omega_\mu(x), n_\nu(y)] = -\frac{4\pi}{ik} \delta^{(2)}(x-y)
\]

where \(a, b = \pm\) and \(\mu = u, v\).

We can write the constraints in terms of these variables, one obtains

\[
G^3 = \pi^\mu e_\mu^+ - \pi^\mu e_\mu^- + \partial_u n_\bar{u} - \partial_\bar{u} n_u
\]

\[
G^+ = \nabla_\mu e_\mu^+ - \nabla_\bar{u} e_\bar{u}^+ + \pi^\mu n_\mu
\]

\[
G^- = \nabla_\mu e_\mu^- - \nabla_\bar{u} e_\bar{u}^- - \pi^\mu n_\mu
\]

These represent as we will see generators of \(SL(2, \mathbb{R})\) gauge transformations. We also have

\[
-H^\pm = \nabla_\mu \pi^\mu_\pm - n_\mu e^\pm_\mu + n_\bar{u} e^\pm_\bar{u}
\]

These generate two dimensional diffeomorphism and finally the hamiltonian constraint

\[
H^3 = -(\pi^u_+ \pi^u_- - \pi^\bar{u}_+ \pi^\bar{u}_-) + (e^+_{\bar{u}} e^-_{\bar{u}} - e^+_u e^-_u) + R(\omega)
\]

where \(R(\omega) = \partial_{\bar{u}} \omega_u - \partial_u \omega_{\bar{u}}\) is the curvature of the \((U(1))\) connection.

This form of the constraints are not exactly the same as the one in the metric formalism. The reason being that there is an extra pair of canonical variables \(n_\mu, \omega_\mu\) compare to the metric formulation. When the covector \(n_\mu = e^3_\mu\) is not equal to zero the 2-dimensional metric induced on \(\Sigma\) is not \(ds^2 = e^+ e^-\) but it is given by

\[
\frac{1}{2}n_\mu n_\nu \right d x^\mu d x^\nu
\]

However there are also an additional sets of constraints \(G^\pm\) which generates \(SL(2, \mathbb{R})\) gauge transformations of the triplet \(e^+, e^-, n\): Let define \(G(X) \equiv \int (X_G^+ - X_G^-)\), we can compute its action on \(e, n\)

\[
[G(X), e^\pm_\mu] = -\frac{4\pi}{ik} n_\mu X_-, \quad [G(X), e^-_\mu] = \frac{4\pi}{ik} n_\mu X_+
\]

\[
[G(X), n_\mu] = \frac{4\pi}{ik} (e^-_\mu X_- - e^+_\mu X_+).
\]

Using these transformations and if we now choose the restriction that the metric is timelike we can always go to a Lorentz frame in which \(n_\mu = 0\), which we call the radial gauge.

One may be surprised here that we choose the 2d metric to be timelike since this is unusual but it is perfectly consistent, nothing in the formalism prevent us from choosing at will the signature of the induced metric on \(\Sigma\) since this is just a restriction on the field configurations which is consistent with the dynamics. In order to recover the usual hamiltonian case for which \(\Sigma\) is spacelike we just have to chose a basis of generators where \(T_3\) is timelike.
achieved by making the replacements $n_{\mu} \rightarrow in_{\mu}$, $\omega_{\mu} \rightarrow i\omega_{\mu}$ together with the replacement of the null coordinates $u, \bar{u}$ by complex coordinates $u = z, \bar{u} = \bar{z}$ and the change of canonical momenta $\pi^\pm_{\mu} \rightarrow i\pi^\pm_{\mu}$. In this case $e^+_z$ and $e^-_z$ are complex conjugate. Note that we could also consider the case of Riemannian gravity which just correspond to a Wick rotation of the coordinates $u, \bar{u} \rightarrow z, \bar{z}$. In the following we nevertheless continue to deal with the Lorentzian AdS case where $u, \bar{u}$ are real null coordinates and $e^\pm_u, e^\pm_{\bar{u}}$ are real fields.

In order to make the link with the metric formulation we therefore have to chose the time gauge $n_{\mu} = 0$. That is suppose that we have a wave functional $\Psi(e^+, e^-, n)$ which satisfies the constraints $G^+ \Psi = G^- \Psi = 0$ we can then defined a functional $\hat{\Psi}(e^+, e^-) \equiv \Psi(e^+, e^-, n)|_{n=0}$. In order to implement this gauge we have however to compute what is the residual action of $\omega$, which is the field conjugated to $n$, on $\hat{\Psi}$. In order to do so we define the spin connection $\omega_{\mu}(e)$ which satisfies $de^\pm + \omega(\pm e^\pm) = 0$ or explicitly

$$
\begin{align*}
\partial_u e^+_u - \partial_{\bar{u}} e^+_u &= \omega_u(e) e^+_u - \omega_u(e) \overline{e^+_u}, \\
\partial_u e^-_u - \partial_{\bar{u}} e^-_u &= \omega_u(e) e^-_u - \omega_u(e) \overline{e^-_u}.
\end{align*}
$$

We assume that the 2d induced metric is invertible and we denote $e^\mu_u$ the 2d inverse frame field $e^\mu_u e^\nu_b = \delta^\mu_b$. We can compute

$$
e^\mu_+ G^+ - e^\mu_- G^- = \epsilon^{\mu\nu} (\omega_\nu - \omega_\nu(e)) - n_\nu(e^\mu_+ \pi^\nu_- + e^\mu_- \pi^\nu_+).$$

Taking the action of this operator on a gauge invariant functional satisfying the constraints $G^+ \hat{\Psi} = G^- \hat{\Psi} = 0$ one conclude

$$
\omega_u \Psi(e^+, e^-, n)|_{n=0} = \frac{4\pi}{ik} \epsilon_{\mu\nu} \frac{\delta}{\delta n_\nu} \Psi(e^+, e^-, n)|_{n=0} = \omega_u(e) \Psi(e^+, e^-, 0)
$$

Not surprisingly one obtains that in the radial gauge the action of $\omega_{\mu}$ is therefore given by multiplication by the spin connection.

One has a residual gauge constraints $G^a$ which reads in the radial gauge

$$(\pi^+ - \pi^-) \hat{\Psi}(e^+, e^-) = 0$$

hence this imply that $\pi$ is a symmetric tensor. This equation is satisfied if $\Psi$ depends on $e^\pm$ only via the metric $\gamma_{\mu\nu} = e^\mu_+ e^-_\nu$.

Thus in the time gauge and once we solve all the gauge constraints $G^a = 0$ the wave equation is a functional of the metric solution of $H^a \Psi = 0$ which can easily be seen to be just the usual metric diffeomorphism and hamiltonian constraints.

VI. PROOF

We can now present the proof of the reconstruction formula (99). Let consider the hamiltonian constraint in the first order formalism

$$
\hat{H} = \hat{T} + R(e), \quad \hat{T} \equiv -(\hat{\pi}_+^u \hat{n}_-^u - \hat{\pi}_-^u \hat{n}_+^u) + (e^+_u e^-_u - e^+_u e^-_u)
$$

where

$$
\hat{\pi}_\pm^u = \frac{4\pi}{ik} \frac{\delta}{\delta e^\pm_u}, \quad \hat{n}_\pm^u = \frac{4\pi}{ik} \frac{\delta}{\delta e^\pm_u}
$$
and \( R(e) = R(\omega(e)) \). One note here that there is no operator ordering ambiguity in the definition of \( \hat{T} \). One introduce a notation for the kernel entering the integral formula (69)

\[
K(e, E) \equiv \exp \left( -\frac{ik}{2\pi} \int_{\Sigma} (E^+ - e^+) \wedge (E^- - e^-) \right)
\]  

(95)

On decomposes the proof if two parts: One looks separately at how the action of \( \hat{T} \) and the action of \( R(e) \) on \( \Psi(e) \) is reflected onto \( Z_c(E) \).

A. Kinetic term

One starts with the action

\[
\hat{\pi}_u^u \int \det(e) = \pm \frac{4\pi}{ik} e_\alpha^\pm, \quad \hat{\pi}_\pm \int \det(e) = \mp \frac{4\pi}{ik} e_\alpha^\mp,
\]

where \( \det(e) = e_\alpha^+ e_\alpha^- - e_\alpha^- e_\alpha^+ \).

One can commute the action of \( \hat{\pi}_u^u \hat{\pi}_\pm \) through the first term in (69)

\[
\left[ \hat{\pi}_u^u \hat{\pi}_\pm - e_\alpha^+ e_\alpha^- \right] \Psi(e) = \exp \left( \frac{ik}{4\pi} \int_{\Sigma} \det(e) \right) \int DE Z_c(E) \times \left[ \left( \hat{\pi}_u^u + e_\alpha^- \right) \left( \hat{\pi}_\pm + e_\alpha^+ \right) - e_\alpha^+ e_\alpha^- \right] K(e, E)
\]  

(96)

we expand the operator as follows

\[
\left[ \left( \hat{\pi}_u^u + e_\alpha^- \right) \left( \hat{\pi}_\pm + e_\alpha^+ \right) - e_\alpha^+ e_\alpha^- \right] = \frac{1}{2} \hat{\pi}_u^u \left( \hat{\pi}_u^u + 2e_\alpha^+ \right) + \frac{1}{2} \hat{\pi}_u^u \left( \hat{\pi}_u^u + 2e_\alpha^- \right) - [\hat{\pi}_u^u, e_\alpha^-]
\]  

(97)

We can now use the following identities to convert differential operator of \( e \) into differential operators acting on \( E \)

\[
\left( \hat{\pi}_\pm + 2e_\alpha^+ \right) K(e, E) = 2E_u^+ K(e, E)
\]

\[
\left( \hat{\pi}_u^u + \frac{4\pi}{ik} \frac{\delta}{\delta E_u^+} \right) K(e, E) = 0
\]

Doing the same manipulation for the second term in (97) one gets a contribution

\[
-\frac{4\pi}{ik} \left[ E_u^+ \frac{\delta}{\delta E_u^+} + E_a^- \frac{\delta}{\delta E_a^-} \right] K(e, E).
\]

If one symmetrize this expression one need to commute some terms \( \delta/\delta E_u^+ \) with \( E_u^+ \), this commutators cancels exactly the third term in (97). We are then left with

\[
\left[ \hat{\pi}_u^u \hat{\pi}_\pm - e_\alpha^+ e_\alpha^- \right] \Psi(e) = \exp \left( \frac{ik}{4\pi} \int_{\Sigma} \det(e) \right) \int DE Z_c(E) \times
\]

\[
-\frac{2\pi}{ik} \left[ E_u^+ \frac{\delta}{\delta E_u^+} + E_a^- \frac{\delta}{\delta E_a^-} + \frac{\delta}{\delta E_u^+} E_u^+ + \frac{\delta}{\delta E_a^-} E_a^- \right] K(e, E)
\]  

(98)

Doing similar manipulations for the operator \( \hat{\pi}_u^u \hat{\pi}_\pm - e_\alpha^+ e_\alpha^- \) leads to the conclusion

\[
\hat{T} \Psi(e) = \exp \left( \frac{ik}{4\pi} \int_{\Sigma} \det(e) \right) \int DE Z_c(E) \frac{2\pi}{ik} \left[ \frac{\delta}{\delta \phi} - \left( \frac{\delta}{\delta \phi} \right)^\dagger \right] K(e, E)
\]
where we denote the infinitesimal conformal rescaling
\[
\frac{\delta}{\delta \phi} \equiv \left[ \frac{E^a_{\mu}}{\delta E^a_{\mu}} \right], \quad -\left( \frac{\delta}{\delta \phi} \right)^\dagger = \left[ \frac{\delta}{\delta E^a_{\mu}} E^a_{\mu} \right]
\]

The measure of integration is not invariant under rescaling, following the argument of David-Distler-Kawai \[34\], and accordingly \(\frac{\delta}{\delta \phi}\) is not anti-hermitic. Let us introduce the parameterization
\[
e^+ = e^{\phi + \alpha} \hat{e}_\mu^+, \quad \hat{e}_\mu^+ \equiv (du + \mu d\bar{u}), \quad e^- = e^{\phi - \alpha} \hat{e}_\mu^-, \quad \hat{e}_\mu^- \equiv (du + \bar{\mu} d\bar{u})
\]
The measure can be split into \(DE = D\phi D\alpha D\hat{g}_{\mu,\bar{\mu}}\). The problem is that the measure for \(\phi\) and \(\alpha\) depends non-linearly on \(\delta e, \delta e\) = \(\int \hat{e} e^{2\phi}(\delta \phi^2 - \delta \alpha^2)\) (99)

if one focus on variation of \(\phi, \alpha\) only. The idea of David, Distler and Kawai \[34\] is to express this measure in terms of a translation invariant measure in \(\phi\) and \(\alpha\) that we denote \(D_0\phi\). What as been show by computing the jacobian of the transformation and invoking self consistent condition \[33\] is that we can express the original measure in terms of a translationally invariant measure provided we introduce a jacobian which is proportional to the exponential of the Liouville action. That is
\[
DE = D_0\phi D_0\alpha D\hat{g}_{\mu,\bar{\mu}} \exp \left(-\frac{2i}{24\pi} S_L(\phi, g_{\mu,\bar{\mu}})\right)
\]
The factor 2 comes from the fact that both \(\phi\) and \(\alpha\) contributes to the jacobian. This imply that \(\delta/\delta \phi\) is not an hermitian operator but such that
\[
\left( \frac{\delta}{\delta \phi} \right)^\dagger = -\frac{\delta}{\delta \phi} + \frac{i}{12\pi} R(E)
\]
We finally obtain
\[
\hat{T}\Psi (e) = \exp \left( \frac{ik}{4\pi} \int_{\Sigma} \det(e) \right) \int DE K(e, E) \frac{4\pi}{ik} \left( -\frac{\delta}{\delta \phi} + \frac{i}{24\pi} R(E) \right) Z_c(E)
\]

**B. the potential**

We now turn to the action of the potential term on the wave function \(\Psi\). The curvature is
\[
R(e) = \partial_{\bar{\mu}} \omega_u(e) - \partial_u \omega_{\bar{\mu}}(e)
\]
where the spin connection is
\[
\det(e)^{-1} (e^-_u de^+ + e^+_u de^-) = \omega_{\bar{\mu}}(e) \quad (102)
\]
\[
\det(e)^{-1} (e^-_u de^+ + e^+_u de^-) = \omega_u(e) \quad (103)
\]
\[
\partial_u e^+_u - \partial_{\bar{\mu}} e^+_\mu \equiv de^\pm \quad (104)
\]
Let us define the operator
\[ \hat{\Pi}^a \equiv \frac{2\pi}{ik} \epsilon^{ab} \epsilon_{\mu\nu} \frac{\delta}{\delta E^b_{\nu}} \]
with \( \epsilon^{ab}, \epsilon_{\mu\nu} \) antisymmetric tensors normalized by \( \epsilon^{+-} = 1, \epsilon_{w\bar{w}} = 1 \). One can check that the operators \( E^a_{\mu} + \hat{\Pi}^a_{\mu} \) all commute with each others. Moreover, their action on the kernel gives
\[ \left( E^a_{\mu} + \hat{\Pi}^a_{\mu} \right) K(e, E) = e^a_{\mu} K(e, E) \]
therefore the action of \( R(e) \) on \( \Psi(e) \) is equivalent to the action of \( R(E + \hat{\Pi}) \) on the kernel. From the definition of the spin connection and being careful with the operator ordering one can express the difference
\[ \omega_{\mu}(E + \hat{\Pi}) - \omega_{\mu}(E) = \left( (\nabla\hat{\Pi}^+) E^+_{\mu} - (\nabla\hat{\Pi}^-) E^-_{\mu} \right) \text{det}(E + \hat{\Pi})^{-1}, \tag{105} \]
\[ \nabla\hat{\Pi}^\pm \equiv d\hat{\Pi} \pm \omega(E) \wedge \hat{\Pi}^\pm. \tag{106} \]
One might worried that the inverse of \( \text{det}(E + \hat{\Pi}) \) is a dangerous operator however one has to remember that it is acting on the exponential kernel and its action is just equal to the inverse of \( \text{det}(e) \) which is well defined. To go further we introduce \( \hat{D}^\pm_{\mu} \equiv E^\pm_{\mu} \nabla\Pi^\pm \) which are the generators of chiral diffeomorphisms:
\[ \int \xi^\mu \hat{D}^\pm_{\mu} E^a^\pm_{\alpha} = \mathcal{L}_\xi E^a^\pm_{\alpha} \pm \xi^\mu \omega_{\mu}(E) E^a^\pm_{\alpha} \tag{107} \]
where \( \mathcal{L}_\xi \) is the Lie derivative and the second term represent an infinitesimal Lorentz gauge transformation with parameter \( \xi^\mu \omega_{\mu} \). In the RHS of (105) we recognise the generators of diffeomorphism up to a commutator
\[ [\nabla\Pi^+(x), E^-_{\mu}(y)] = -\frac{2\pi}{ik} \nabla^+_{\mu} \delta^{(2)}(x, y), \quad [\nabla\Pi^-(x), E^+_{\mu}(y)] = \frac{2\pi}{ik} \nabla^-_{\mu} \delta^{(2)}(x, y) \]
with \( \nabla^\pm_{\mu} = \partial_{\mu} \pm \omega_{\mu} \).
Thus overall and after integration by part we have
\[ \omega_{\mu}(e)\Psi(e) = \exp \left( \frac{ik}{4\pi} \int_{\Sigma} \text{det}(e) \right) \left( \int DE K(e, E) \omega_{\mu}(E) Z_e(E) \right) + \int DE K(e, E) \text{det}(e)^{-1} \left( \hat{D}^+_{\mu} - \hat{D}^-_{\mu} + R_{\mu} \right) Z_e(E) \tag{108} \]
\[ + \int DE K(e, E) \text{det}(e)^{-1} \left( \hat{D}^+_{\mu} - \hat{D}^-_{\mu} + R_{\mu} \right) Z_e(E) \tag{109} \]
where \( \hat{D}^\pm_{\mu} \equiv E^\pm_{\mu} \nabla\Pi^\pm \) are the generators of chiral diffeomorphism Since \( Z_e(E) \) is invariant under Lorentz gauge transformation and diffeomorphisms it is annihilated by the action of \( \hat{D}^\pm_{\mu} \).
The last term is a commutator term
\[ R_{\mu} = \left( E^-_{\mu} \left( (\nabla\hat{\Pi}^-) + (\nabla\hat{\Pi}^+) \right) + E^+_{\mu} \left( (\nabla\hat{\Pi}^-) - (\nabla\hat{\Pi}^-) \right) \right) \tag{111} \]
The operator \( \int X_a \nabla\hat{\Pi}^a \) generates transformations
\[ \int X_a \nabla\hat{\Pi}^a E^\pm_{\mu} \equiv \delta_X E^\pm_{\mu} = \nabla_{\mu} X^a \tag{112} \]
the key point is that the distance \((71)\) we have chosen to define the integration measure is invariant under such transformations

\[
\delta X(e,e) = \int \delta X e^+ \wedge e^- + \int e^+ \wedge \delta X e^- = - \int X^+ \nabla \wedge e^- - \int X^- \nabla \wedge e^+ = 0 \tag{114}
\]

This means that the measure is invariant under such transformation hence \(\nabla \hat{\Pi}^a\) is an hermitian operator, thus \(R_\mu = 0\), which is the result we wanted to establish. This shows that

\[
\hat{H} \Psi(e) = \exp \left( \frac{ik}{4\pi} \int_\Sigma \text{det}(e) \right) \int DE K(e,E) \frac{4\pi}{ik} \left( - \frac{\delta}{\delta \phi} + \frac{i}{24\pi} (6k + 1) R(E) \right) Z_e(E)
\]

### VII. CONCLUSION

In this work we have given a new perspective on the relationship between quantum gravity in the bulk and CFT on the boundary in the context of a non-zero cosmological constant. As we have seen the AdS-matrix of gravity satisfies a radial Wheeler de Witt equation which is a second order equation. One has derived that the asymptotic of any solution of the radial WdW equation is given by a pair of CFTs. We have argued that restricting to one CFT amount to look at quantum spacetime which possess an asymptotic infinity. We have shown that the CFT can be interpreted as an initial value data for a radial quantum gravity state. Thus the AdS/CFT correspondence is one to many, that is it is a correspondence between radial states of quantum gravity and different CFTs. An explicit way to see this, is to give an explicit reconstruction formula of the radial quantum gravity state from the boundary CFT. This has been achieved in the context of 2+1 quantum gravity. From this work we can to some extent answer all the questions asked in the introduction.

Our work suggests several avenues which would need to be resolved in order to have a deeper understanding of the AdS/CFT correspondence. For instance, as we discussed in section II B it is far from obvious that there is a clear relationship between Euclidean and Lorentzian bulk gravity solutions; except when these solutions are evaluated at on a static boundary metric; since the usual correspondence between Euclidean and Lorentzian solution of Wheeler-deWitt equation do not hold anymore for the radial solution of WdW. However since one can always defined a Wick rotated boundary CFT, assuming it is unitary, one expect a correspondence between these solutions via the Boundary CFT. It would be extremely interesting to find out what is explicitly this mapping at the quantum gravity level.

Also in our derivation of the asymptotic form of the radial WdW equation we have argued that only a finite number of terms should be renormalised in the asymptotic expansion and we have shown this explicitly in the case of 2+1 dimensions. It would be however quite instructing to see how this work exactly by carrying out the Schrödinger renormalisation program of Symanzik in the context of gravity.

One open problem that arises here, concerns the meaning of the radial state of gravity. More precisely, it would be extremely useful to unravel the relationship between radial state and usual states of quantum gravity. One expect such a radial state to correspond to a gravitational density matrix, can we find the explicit form of this correspondence? Does a
radial state, hence the CFT, allows us to reconstruct the Hilbert space of quantum gravity or does it knows only about some specific subset of states?

We have given a prescription for the construction of the CFT associated with quantum gravity in the Lorentzian context. This needs to be further developed and one should do it explicitly in the context of 3d gravity. Also as we have seen, the asymptotic expansion shows that one expects to have a CFT corresponding to such a quantum gravity radial state. However we do not know whether this CFT is irreducible or given by a sum of irreducible CFT. This should be resolved.

Most of our analysis have been done in the context of AdS, but as we emphasized several time most of our results are also valid in the dS context. In that case we obtain a CFT partition function associated with an imaginary central charge. That is even if the boundary metric is Riemannian the CFT correspond to a quantum weight $e^{iS}$ and not a statistical weight $e^{-S}$. How can we interpret such a “imaginary” CFT, How is this related to the thermal properties of de Sitter space? These questions certainly needs to be developed further.

It would be interesting to understand the appearance of the non critical string amplitude in the singular limit of the reconstruction formula given in 2+1. Finally It would be extremely interesting to propose a bulk reconstruction formula analogous to (69) in the context of 4 dimensions. Of course a full quantum gravity one is hopeless at first sight but a semi-classical one might be within reach. We hope to come back to this issue in the near future.

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VIII. APPENDIX

A. One loop renormalisation

In order to perform the renormalisation of the hamiltonian at one loop we need to compute the divergent part of the boundary to boundary propagator at one loop. In order to do so lets start from the formal path integral.

$$\Psi_{\Sigma}(\gamma) = \int_{g|\partial M = \gamma} Dg \ e^{iS_M(g)} \quad (115)$$

In order to define this path integral we first split the bulk metric in terms of a background metric and a perturbation $g = \bar{g} + h$. $\bar{g}$ is a solution of the classical equations of motion

$$R_{\alpha\beta}(\bar{g}) = -\frac{d}{l^2} g_{\alpha\beta}, \quad \bar{g}_{ab}|_{\partial M} = \gamma_{ab}, \quad \bar{g}_{0b} = \delta_{0b}$$

where 0 refer to the coordinate transverse to the boundary. $h$ is the quantum field that we need to integrate over. It is such that its components parallel to the boundary satisfies Dirichlet boundary conditions $h_{ab}|_{\partial M} = 0$. The components transverse to the boundary are not restricted and should be integrated over. This transverse integration implement the
constraints. In order to make sense of the path integral we also need to perform a gauge fixing. The original gauge invariance can be divided between between the background and perturbation either as a background gauge invariance
\[ \delta B_\xi g_{\mu\nu} = 2 \nabla_\mu \xi_\nu, \quad \delta B_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} = \xi^\rho \nabla_\rho h_{\mu\nu} + 2 h_{\rho(\mu} \nabla_\nu) \xi^\rho \]
or as a gauge invariance
\[ \delta \xi \bar{g}_{\mu\nu} = 0, \quad \delta \xi h_{\mu\nu} = 2 \nabla_\mu \xi_\nu + \mathcal{L}_\xi h_{\mu\nu}, \quad \xi|_{\partial M} = 0 \]
where the covariant derivative is with respect to \( \bar{g} \) and all indices are raised with \( \bar{g} \). This gauge invariance of the action needs to be fixed. And we denote \( F^\mu_\nu(g) \) the corresponding gauge condition and \( J^\mu_\nu \) its variation with respect to diffeomorphisms
\[ \delta \xi F^\mu_\nu = J^\mu_\nu. \]
In order to respect the background diffeomorphism symmetry such a gauge should be invariant under the background gauge transformation \( \delta B_\xi F^\mu_\nu \sim F^\nu_\nu \). A convenient gauge is for instance the deDonder gauge
\[ F^\mu_\nu(h) = \nabla_\nu h_{\mu\nu} - \frac{1}{2} \nabla^\mu h, \quad J^\mu_\nu = \Box \delta^\rho_\nu + R^\mu_\nu + O(h) \]
The terms of order \( h \) will not contribute to one loop. Such a gauge fixing is taken into account by inserting the Faddev-Popov term
\[ \prod_{x,\mu} \delta(F^\mu(x)) \det(J^\mu_\nu(x)) \quad \text{or} \quad \frac{\det(J)}{(\det i\alpha \bar{g})^\frac{1}{2}} e^{i \int_\mathcal{M} \sqrt{\bar{g}} F^\mu_\nu \bar{g}_{\mu\nu} F^\nu_\nu} \]
or more conveniently by taking an average
\[ \int D\alpha e^{-\frac{1}{2} \int \sqrt{\alpha g_{\mu\nu} \alpha^\nu}} \prod_{x,\mu} \delta(F^\mu(x) - \alpha^\mu(x)) \det(J) \]
We can now, following Barvinsky [19], define the One loop corrected gravity wave function. We denote the bulk operator denoted \( \Box^{\alpha\beta,\mu\nu} = \delta^2 S/\delta g_{\alpha\beta} \delta g_{\mu\nu} \). The Green function \( G_{(ND)} \) entering the definition of the one loop partition function satisfies mixture of Dirichlet with Neuman Boundary conditions. The component tangential to the boundary satisfy Diriclet boundary conditions while the comonent transverse to the boundary satisfy Neumann conditions induced by the gauge fixing. Namely, denoting \( a, b \) indices tangential to the boundary we have
\[ \Box^{\alpha\beta,\mu\nu} G_{\mu\nu;\rho\sigma}(x, y) = \delta^{\alpha\beta}_{\rho\sigma} \delta(x, y) \]
\[ G_{ab,\mu\nu}(x, y)|_{x \in \partial M} = 0 \]
\[ \bar{F}^{\alpha\beta}_\mu G_{\alpha\beta,\mu\nu}(x, y)|_{x \in \partial M} = 0 \]
where the gauge condition is \( F^\mu(h) \equiv \bar{F}^{\alpha\beta}_\mu h_{\alpha\beta} \). The one-loop wave functional is given by
\[ \psi_{\Sigma}(\gamma) = \frac{\text{Det}\,D(J)}{(\text{Det}\,D_{\mathcal{M}}(\Box))^\frac{1}{2}} e^{i \int S^{(H, J)}(\gamma)} \]
where $S^{(HJ)}(\gamma)$ is the hamilton-jacobi functional and $\text{Det}_{DN}(\Box) \equiv \text{Det}^{-1}(G^{(DN)})$ while the ghost determinant is computed with Dirichlet conditions. These determinants should be regularised with the heat kernel methods and are such that the local singular terms are substracted. In the case of usual scalar field theory one can check that this renormalised wave function do satisfy the Schrödinger equation modified by the one loop substraction (21) of the Schrödinger operator (20). One expect the same result in the context of the WdW equation but the explicit check as not been performed yet.

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