Spectrum generating algebra and coherent states of the $C_\lambda$-extended oscillator

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Abstract

$C_\lambda$-extended oscillator algebras, generalizing the Calogero-Vasiliev algebra, where $C_\lambda$ is the cyclic group of order $\lambda$, have recently proved very useful in the context of supersymmetric quantum mechanics and some of its variants. Here we determine the spectrum generating algebra of the $C_\lambda$-extended oscillator. We then construct its coherent states, study their nonclassical properties, and compare the latter with those of standard $\lambda$-photon coherent states, which are obtained as a special case. Finally, we briefly review some other types of coherent states associated with the $C_\lambda$-extended oscillator.

1 Introduction

Coherent states (CS) of the harmonic oscillator [1] are known to have properties similar to those of the classical radiation field. They may be defined in various ways, for instance as eigenstates of the oscillator annihilation operator $b$. With the corresponding creation operator $b^\dagger$ and the number operator $N_b \equiv b^\dagger b$, the latter satisfies the commutation relations

$$[N_b, b^\dagger] = b^\dagger, \quad [N_b, b] = -b, \quad [b, b^\dagger] = I. \quad (1)$$

In contrast, generalized CS associated with various algebras [4] may have some nonclassical properties, such as photon antibunching or sub-Poissonian photon statistics, and squeezing. As examples of such CS, we may quote the eigenstates of $b^2$, which were introduced as even and odd CS or cat states [3], and are a special case of generalized CS associated with the Lie algebra $\text{su}(1,1)$ [4]. We may also mention the eigenstates of $b^\lambda (\lambda > 2)$ or kitten states [5], which may be generated in $\lambda$-photon processes.

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Other examples are provided by nonlinear CS associated with a deformed oscillator (or \( f \)-oscillator). The latter is defined in terms of creation, annihilation, and number operators, \( a^{\dagger} = f(N) b^{\dagger}, a = b f(N), N = N b, \) satisfying the commutation relations \([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\)

\[
[N, a^{\dagger}] = a^{\dagger}, \quad [N, a] = -a, \quad [a, a^{\dagger}] = G(N),
\]

(2)

where \( f \) is some Hermitian operator-valued function of the number operator and \( G(N) = (N + 1)f^2(N + 1) - N f^2(N) \). Nonlinear CS, defined as eigenstates of \( a \) \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\), of \( a^2 \) \([10, 11]\), or of an arbitrary power \( a^\lambda (\lambda > 2) \) \([11]\), have been considered in connection with nonclassical properties. It has been shown that for a particular class of nonlinearities the first ones are useful in the description of a trapped ion \([8]\).

In the present communication, we shall consider some multiphoton CS, which may be associated with the recently introduced \( C_\lambda \)-extended oscillator \([12]\). The latter may be considered as a deformed oscillator with a \( \mathbb{Z}_\lambda \)-graded Fock space and has proved very useful in the context of supersymmetric quantum mechanics and some of its variants \([12, 13]\). In particular, we shall deal here in detail with CS of the \( C_\lambda \)-extended oscillator spectrum generating algebra \([14]\), which are a special case of the CS of Ref. \([11]\) and exhibit some nonclassical properties.

## 2 The \( C_\lambda \)-extended oscillator algebra

The \( C_\lambda \)-extended oscillator algebra (where \( C_\lambda = \mathbb{Z}_\lambda \) is the cyclic group of order \( \lambda \)) was introduced as a generalization of the Calogero-Vasiliev algebra, defined by \([12]\)

\[
[N, a^{\dagger}] = a^{\dagger}, \quad [a, a^{\dagger}] = I + \alpha_0 K, \quad \{K, a^{\dagger}\} = 0,
\]

(3)

and their Hermitian conjugates, where \( \alpha_0 \) is some real parameter subject to the condition \( \alpha_0 > -1 \), and \( K \) is some Hermitian operator. The latter may be realized as \( K = (-1)^N \), so that the second equation in \((3)\) becomes equivalent to \([a, a^{\dagger}] = I + \alpha_0 P_0 + \alpha_1 P_1 \), where \( \alpha_0 + \alpha_1 = 0 \) and \( P_0 = \frac{1}{\lambda} [I + (-1)^N], P_1 = \frac{1}{\lambda} [I - (-1)^N] \) project on the even and odd subspaces of the Fock space \( \mathcal{F} \), respectively.

When partitioning \( \mathcal{F} \) into \( \lambda \) subspaces \( \mathcal{F}_\mu \equiv \{ |k \lambda +\mu \rangle \mid k = 0, 1, \ldots \}, \mu = 0, 1, \ldots, \lambda - 1 \), instead of two, the Calogero-Vasiliev algebra is replaced by the \( C_\lambda \)-extended oscillator algebra, defined by \([12]\)

\[
[N, a^{\dagger}] = a^{\dagger}, \quad [a, a^{\dagger}] = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, \quad a^{\dagger} P_\mu = P_{\mu+1} a^{\dagger},
\]

(4)

and their Hermitian conjugates, where \( P_\mu = \lambda^{-1} \sum_{\nu=0}^{\lambda-1} \exp[2\pi i \nu(N-\mu)/\lambda] \) projects on \( \mathcal{F}_\mu \), \( \sum_{\mu=0}^{\lambda-1} P_\mu = I \), and \( \alpha_\mu \) are some real parameters subject to the conditions \( \sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0 \) and \( \sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \mu = 1, 2, \ldots, \lambda - 1 \). Taking this form of \( P_\mu \) into account, it is clear that the \( C_\lambda \)-extended oscillator algebra \([14]\) is a special case of deformed oscillator algebra, as defined in \([2]\).
The operators $N, a^\dagger, a$ are related to each other through the structure function $F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu$, $\beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu$, which is a fundamental concept of deformed oscillators. $a^\dagger a = F(N)$, $aa^\dagger = F(N + 1)$ [8] [4]. Comparing with Eq. (2), we get $f(N) = (F(N)/N)^{1/2}$.

The Fock space basis states $|n\rangle = |k\lambda + \mu\rangle = N_{n-1/2}^{-} (a^\dagger)^n |0\rangle$, where $a|0\rangle = 0$, $k = 0, 1, \ldots$, and $\mu = 0, 1, \ldots, \lambda - 1$, satisfy the relations

$$N|n\rangle = n|n\rangle, \quad a^\dagger|n\rangle = \sqrt{F(n + 1)} |n + 1\rangle, \quad a|n\rangle = \sqrt{F(n)} |n - 1\rangle.$$  (5)

Due to the restrictions on the range of the parameters $\alpha_\mu$ given below Eq. (4), $F(\mu) = \beta_\mu + \mu > 0$ so that all the states $|n\rangle$ are well defined.

The $C_\lambda$-extended oscillator Hamiltonian is defined by [12]

$$H_0 = \frac{1}{\lambda} \{ a, a^\dagger \}.$$  (6)

Its eigenstates are the states $|n\rangle = |k\lambda + \mu\rangle$ and their eigenvalues are given by $E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}$, where $\gamma_\mu \equiv \frac{1}{2} (\beta_\mu + \beta_{\mu+1})$. In each $\mathcal{F}_\mu$ subspace of $\mathcal{F}$, the spectrum of $H_0$ is harmonic, but the $\lambda$ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \ldots, \lambda - 1$, are shifted with respect to each other by some amounts depending upon the parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-1}$.

### 3 Spectrum generating algebra of the $C_\lambda$-extended oscillator

One can generate the whole spectrum of the $C_\lambda$-extended oscillator Hamiltonian (5) from the eigenstates $|\mu\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, by using the operators [14]

$$J_+ = \frac{1}{\lambda} (a^\dagger)^\lambda, \quad J_- = \frac{1}{\lambda} a^\lambda, \quad J_0 = \frac{1}{\lambda} H_0 = \frac{1}{2\lambda} \{ a, a^\dagger \}.$$  (7)

They satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = f(J_0, P_\mu), \quad [J_0, P_\mu] = [J_\pm, P_\mu] = 0,$$  (8)

where $f(J_0, P_\mu)$ (which has nothing to do with the function $f(N)$ of Eq. (2)) is a $(\lambda - 1)$th-degree polynomial in $J_0$ with $P_\mu$-dependent coefficients, $f(J_0, P_\mu) = \sum_{i=0}^{\lambda-1} s_i(P_\mu) J_0^i$. The spectrum generating algebra (SGA) of the $C_\lambda$-extended oscillator is therefore a $C_\lambda$-extended polynomial deformation of su(1,1); in each $\mathcal{F}_\mu$ subspace, it reduces to a standard polynomial deformation of su(1,1) [5].

Its Casimir operator can be written as

$$C = J_- J_+ + h(J_0, P_\mu) = J_+ J_- + h(J_0, P_\mu) - f(J_0, P_\mu),$$  (9)

where $h(J_0, P_\mu)$ is a $\lambda$th-degree polynomial in $J_0$ with $P_\mu$-dependent coefficients, $h(J_0, P_\mu) = \sum_{i=0}^{\lambda} t_i(P_\mu) J_0^i$. Each $\mathcal{F}_\mu$ subspace is the carrier space of a unitary irreducible representation (unirrep) of the SGA, characterized by an eigenvalue $c_\mu$ of
\[ C, \text{ and by the lowest eigenvalue } (\mu + \gamma + \frac{1}{2})/\lambda \text{ of } J_0. \] The explicit expressions of \( f(J_0, P_\mu), h(J_0, P_\mu), \) and \( c_\mu \) are given in Ref. [14].

For \( \lambda = 2 \), for which the \( C_\lambda \)-extended oscillator algebra reduces to the Calogero-Vasiliev algebra, the SGA (7), (8) reduces to the Lie algebra su(1,1), for which \( f(J_0) = -2J_0, h(J_0) = -J_0(J_0 + 1), \) and \( c = (1 + \alpha_\mu)(3 - \alpha_\mu)/16 \) [16].

Nonlinearities make their appearance for \( \lambda = 3 \), for which
\[
\begin{align*}
 f(J_0, P_\mu) &= -9J_0^2 - J_0 \sum_\mu (\alpha_\mu + 2\alpha_{\mu+1})P_\mu - \frac{1}{12} \sum_\mu (1 + \alpha_\mu)(5 - \alpha_\mu)P_\mu, \\
 h(J_0, P_\mu) &= -J_0 \left[ 3J_0^2 + \frac{1}{2} J_0 \sum_\mu (9 + \alpha_\mu + 2\alpha_{\mu+1})P_\mu + \frac{1}{12} \sum_\mu (23 + 10\alpha_\mu \right. \\
 & \left. + 12\alpha_{\mu+1} - \alpha_\mu^2)P_\mu \right], \\
 c_\mu &= \frac{1}{12}(1 + \alpha_\mu)(5 - \alpha_\mu)(3 + \alpha_\mu + 2\alpha_{\mu+1}).
\end{align*}
\] (10)

For \( \alpha_\mu = 0 \) corresponding to \( a^\dagger = b^\dagger, a = b \), the operators (9) close a polynomial deformation of su(1,1), with \( f(J_0) \) and \( h(J_0) \) expressed in terms of some binomial coefficients and Stirling numbers [14].

### 4 Coherent states associated with the \( C_\lambda \)-extended oscillator spectrum generating algebra

As CS associated with the \( C_\lambda \)-extended oscillator SGA, let us consider generalizations of the Barut-Girardelle CS of su(1,1) [4], to which they will reduce in the case \( \lambda = 2 \). These are the eigenstates \(|z; \mu\rangle \) of the operator \( J_- \) defined in (7),
\[
J_-|z; \mu\rangle = z|z; \mu\rangle, \quad z \in \mathbb{C}, \quad \mu = 0, 1, \ldots, \lambda - 1. \] (11)

Here \( \mu \) distinguishes between the \( \lambda \) independent (and orthogonal) solutions of equation (11), belonging to the various subspaces \( \mathcal{F}_\mu \). The CS \(|z; \mu\rangle\) may be considered as special cases of the nonlinear CS of Ref. [11], since Eq. (11) is equivalent to \( a^\lambda|z; \mu\rangle = \lambda z|z; \mu\rangle \), for \( a = b f(N_b) \) and \( f(N_b) \) as given in Sec. 2.

It can be shown [14] that the states (11) satisfy Klauder’s minimal set of conditions for generalized CS [17]: they are normalizable, continuous in the label \( z \), and they allow a resolution of unity. The other discrete label \( \mu \) is analogous to the vector components of vector (or partially) CS [18].

The states \(|z; \mu\rangle\) can be written in either of the alternative forms
\[
|z; \mu\rangle = [N_\mu(|z|)]^{-1/2} \sum_{k=0}^\infty \left( \frac{z/\lambda^{(\lambda-2)/2}}{k!} \left( \Pi_{\nu=1}^\mu (\beta_\nu + 1)_k \right) \left( \prod_{\nu'=1+\mu}^{\lambda-1} (\beta_{\nu'})_k \right) \right)^{1/2} k^{\lambda + \mu}, \] (12)

\[
4
\]
\[ |z; \mu \rangle = [N_\mu(|z|)]^{-1/2} {}_0F_{\lambda-1} \left( \beta_1 + 1, \ldots, \beta_\mu + 1, \beta_{\mu+1}, \ldots, \beta_{\lambda-1}; z J_+ / \lambda^{\lambda-2} \right) |\mu\rangle, \]

where \( \beta_\mu \equiv (\beta_\mu + \mu) / \lambda \), \((a)_k \) denotes Pochhammer’s symbol, and the normalization factor \( N_\mu(|z|) \) can be expressed in terms of a generalized hypergeometric function,

\[ N_\mu(|z|) = {}_0F_{\lambda-1} \left( \beta_1 + 1, \ldots, \beta_\mu + 1, \beta_{\mu+1}, \ldots, \beta_{\lambda-1}; y \right), \quad y \equiv |z|^2 / \lambda^{\lambda-2}. \] (14)

Their unity resolution relation can be written as

\[ \sum_\mu \int d\rho_\mu (z; z^*) |z; \mu\rangle \langle z; \mu| = I, \] (15)

where \( d\rho_\mu (z, z^*) \) is a positive measure, given in terms of a generalized hypergeometric function and a Meijer \( G \) - function by

\[ d\rho_\mu (z, z^*) = {}_0F_{\lambda-1} \left( \beta_1 + 1, \ldots, \beta_\mu + 1, \beta_{\mu+1}, \ldots, \beta_{\lambda-1}; y \right) h_\mu(y) |z| d|z| d\phi, \]

\[ h_\mu(y) = \frac{G_{0\lambda}^0 \left( y | 0, \beta_1, \ldots, \beta_\mu, \beta_{\mu+1} - 1, \ldots, \beta_{\lambda-1} - 1 \right)}{\pi \lambda^{\lambda-2} \left( \prod_{\nu=\mu+1}^{\lambda-1} \Gamma(\beta_\nu + 1) \right) \left( \prod_{\nu'=\mu+1}^{\lambda-1} \Gamma(\beta_\nu') \right)}, \] (16)

with \( y \) defined in Eq. (14).

In the \( \lambda = 2 \) case, the functions \( {}_0F_1 \) and \( G_{02}^0 \) of Eqs. (13), (14), and (16) being proportional to modified Bessel functions \( I_{2\nu}(2|z|) \) and \( K_{2\nu}(2|z|) \), \( \nu = (\alpha_0 - 1 + 2\mu) / 2 \), respectively, the CS defined in (11) reduce to Barut-Girardello su(1,1) CS for the appropriate unirreps, as it should be.

For \( \alpha_\mu = 0 \) corresponding to \( a^\dagger = b^\dagger \), \( a = b \), the CS defined in (11) reduce to the eigenstates of \( b^\lambda \) or standard \( \lambda \)-photon CS for the appropriate unirreps, as it should be.

\[ |z; \mu \rangle = [N_\mu(|z|)]^{-1/2} \sum_{k=0}^{\infty} \left( \frac{\mu!}{(k \lambda + \mu)!} \right)^{1/2} (\lambda z)^k |k \lambda + \mu\rangle, \] (17)

satisfying the resolution of unity (13) with \( h_\mu(y) \) given by

\[ h_\mu(y) = \lambda^{\mu-\lambda+2} (\pi \mu!)^{-1} y^{(\mu-\lambda+1) / \lambda} \exp \left( -\lambda y^{1/\lambda} \right). \] (18)

The states (17) can be rewritten in the alternative form

\[ |z; \mu \rangle = \left( \frac{\mu!}{E_{\lambda,\mu+1}(\lambda^2 |z|^2)} \right)^{1/2} E_{\lambda,\mu+1} \left( \lambda^2 z J_+ \right) |\mu\rangle, \] (19)

where \( E_{\alpha,\beta}(x) \equiv \sum_{k=0}^{\infty} x^k / \Gamma(\alpha k + \beta) \) is a generalized Mittag-Leffler function. Hence, they provide a simple example of the Mittag-Leffler CS considered in Ref. [1].
5 Nonclassical properties of coherent states

The CS $|z; \mu\rangle$ may be considered as exotic states in quantum optics. Their properties may be analyzed in two different ways, by considering either “real” photons, described by the operators $b^\dagger$, $b$ satisfying the canonical commutation relation, as given in Eq. (1), or “dressed” photons, described by the operators $a^\dagger$, $a$ of Eq. (2), which may appear in some phenomenological models explaining some non-intuitive observable phenomena.

5.1 Photon statistics

Since $N = N_b$, the photon number statistics is not affected by the choice made for the type of photons. A measure of its deviation from the Poisson distribution is the Mandel parameter

$$Q = \frac{\langle (\Delta N)^2 \rangle - \langle N \rangle}{\langle N \rangle}, \quad \Delta N \equiv N - \langle N \rangle,$$

which vanishes for the Poisson distribution, and is positive or negative according to whether the distribution is super-Poissonian (bunching effect) or sub-Poissonian (antibunching effect).

It is well known that for $\lambda = 2$, the standard even (resp. odd) CS, corresponding to $\alpha_0 = \alpha_1 = 0$ or $a^\dagger = b^\dagger$, $a = b$ and $\mu = 0$ (resp. $\mu = 1$), are characterized by a super-Poissonian (resp. sub-Poissonian) number distribution. It can be shown that for the even (resp. odd) CS associated with the Calogero-Vasiliev algebra, i.e., for $\lambda = 2$, $\alpha_0 = -\alpha_1 \neq 0$ and $\mu = 0$ (resp. $\mu = 1$), this trend is enhanced for positive (resp. negative) values of $\alpha_0$. However, as shown in Fig. 1, for negative (resp. positive) values of $\alpha_0$ and sufficiently high values of $|z|$, the opposite trend can be seen.

For higher values of $\lambda$, more or less similar results are obtained for $\mu = 0$, on one hand, and $\mu \neq 0$, on the other hand. However the behaviour of $Q$ becomes more complicated for intermediate values of $\mu$.

5.2 Squeezing effect

5.2.1 “Dressed” photons

Let us define the deformed quadratures $x$ and $p$ as

$$x = \frac{1}{\sqrt{2}} (a^\dagger + a), \quad p = \frac{i}{\sqrt{2}} (a^\dagger - a).$$

In any state belonging to $F_\mu$, their dispersions $\langle (\Delta x)^2 \rangle$ and $\langle (\Delta p)^2 \rangle$ satisfy the uncertainty relation

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{1}{4} |\langle [x, p] \rangle|^2 = \frac{\lambda^2}{4} (\bar{\beta}_{\mu+1} - \bar{\beta}_\mu)^2,$$

where $\bar{\beta}_\mu = \langle \beta_\mu \rangle$.
where the right-hand side becomes smaller than the conventional value 1/4 if $\alpha_0 < 0$
for $\mu = 0$ or $-2 < \alpha_\mu < 0$ for $\mu = 1, 2, \ldots$, or $\lambda - 1$.

In $\mathcal{F}_\mu$, the role of the vacuum state is played by the number state $|\mu\rangle = |0; \mu\rangle$,
which is annihilated by $J_-$. The corresponding dispersions are given by

$$
\langle (\Delta x)^2 \rangle_0 = \langle (\Delta p)^2 \rangle_0 = \frac{\lambda}{2} (\bar{\beta}_{\mu+1} + \bar{\beta}_\mu).
$$

Comparing with the uncertainty relation (22), we conclude that the state $|\mu\rangle$ satisfies
the minimum uncertainty property in $\mathcal{F}_\mu$, i.e., gives rise to the equality in (22), only
for $\mu = 0$ because $\bar{\beta}_0 = 0$ and $\bar{\beta}_\mu > 0$ for $\mu = 1, 2, \ldots$, $\lambda - 1$. On the other hand,
the dispersions in the vacuum may be smaller than the conventional value 1/2 for
$\mu = 0, 1, \ldots, \lambda - 2$.

Let us restrict ourselves to the CS $|z; 0\rangle$, which satisfies for $z = 0$ the minimum
uncertainty property. The quadrature $x$ (resp. $p$) is said to be squeezed to the second
order in $|z; 0\rangle$ if $X \equiv \langle (\Delta x)^2 \rangle / \langle (\Delta x)^2 \rangle_0$ (resp. $P \equiv \langle (\Delta p)^2 \rangle / \langle (\Delta p)^2 \rangle_0$) is less than one. Similarly, it is said to be squeezed to the fourth order if $Y \equiv \langle (\Delta x)^4 \rangle / \langle (\Delta x)^4 \rangle_0$
(resp. $Q \equiv \langle (\Delta p)^4 \rangle / \langle (\Delta p)^4 \rangle_0$) is less than one.

For $\lambda = 2$, $X$ and $P$, or $Y$ and $Q$, are related with each other by the transformation
$\Re z \rightarrow -\Re z$. Moreover $X$ and $Y$ are minimum for real, negative values of $z$.
On Fig. 2, they are displayed for such values. We note a large squeezing effect over
the whole range of real, negative values of $z$ for positive values of $\alpha_0$ (for which the
conventional uncertainty relation is respected).

For $\lambda > 2$, there is no second-order squeezing, but for $\lambda = 4$, a small fourth-order
squeezing is obtained in accordance with the results for standard $\lambda$-photon CS [5].

5.2.2 “Real” photons

Let us now define the quadratures $x$ and $p$ as

$$
x = \frac{1}{\sqrt{2}} \left( b^\dagger + b \right), \quad p = \frac{i}{\sqrt{2}} \left( b^\dagger - b \right).
$$

Their dispersions $\langle (\Delta x)^2 \rangle$ and $\langle (\Delta p)^2 \rangle$ satisfy the usual uncertainty relation. Considering again the CS $|z; 0\rangle$, on Fig. 3 we observe for the ratios $X$ and $P$ more or
less similar trends as noted in the case of “dressed” photons.

6 Concluding remarks

In the present contribution, we determined the SGA of the $C_\lambda$-extended oscillator
and studied some CS associated with it, namely the eigenstates of its lowering
generator $J_-$. Other types of CS may be considered and will be studied in a forthcoming
publication. Let us mention here two of them:
1. The eigenstates of the $C_\lambda$-extended oscillator annihilation operator $a$:

$$a |z; \mu\rangle = z |z; \mu\rangle.$$  \hfill (25)

These generalize the paraboson CS, which correspond to $\lambda = 2$. \hfill [20]

2. The solutions of the equation

$$\left[ a^{\lambda-\alpha} - z \left( a^\dagger \right)^\alpha \right] |z; \mu\rangle = 0, \quad \alpha = 0, 1, \ldots, \left[ \frac{\lambda}{2} \right], \quad \mu = 0, 1, \ldots, \lambda - \alpha - 1.$$  \hfill (26)

For $\alpha = 0$, these are the eigenstates of $a^\lambda$, which are directly related to those of $J_-$, considered here. Moreover, for $\lambda = 2$ and $\alpha = 1$, they reduce to the Perelomov $\text{su}(1,1)$ CS \hfill [2].

References

[1] R.J. Glauber, \textit{Phys. Rev.} \textbf{131} (1963) 2766.

[2] A.P. Perelomov, \textit{Generalized Coherent States and Their Applications} (Springer, Berlin, 1986).

[3] V.V. Dodonov, I.A. Malkin, V.I. Man’ko, \textit{Physica} \textbf{72} (1974) 597.

[4] A.O. Barut, L. Girardello, \textit{Commun. Math. Phys.} \textbf{21} (1971) 41.

[5] V. Bužek, I. Jex, Tran Quang, \textit{J. Mod. Opt.} \textbf{37} (1990) 159.

[6] C. Daskaloyannis, \textit{J. Phys. A} \textbf{24} (1991) L789.

[7] A.I. Solomon, \textit{Phys. Lett. A} \textbf{196} (1994) 29.

[8] R.L. de Matos Filho, W. Vogel, \textit{Phys. Rev. A} \textbf{54} (1996) 4560.

[9] V.I. Man’ko, G. Marmo, F. Zaccaria, E.C.G. Sudarshan, \textit{Phys. Scr.} \textbf{55} (1997) 528.

[10] S. Mancini, \textit{Phys. Lett. A} \textbf{233} (1997) 291; S. Sivakumar, \textit{Phys. Lett. A} \textbf{250} (1998) 257.

[11] X.-M. Liu, \textit{J. Phys. A} \textbf{32} (1999) 8685.

[12] C. Quesne, N. Vansteenkiste, \textit{Phys. Lett. A} \textbf{240} (1998) 21.

[13] C. Quesne, N. Vansteenkiste, \textit{Helv. Phys. Acta} \textbf{72} (1999) 71; $C_\lambda$-extended oscillator algebras and some of their deformations and applications to quantum mechanics, preprint \texttt{math-ph/0003029}, to be published in \textit{Int. J. Theor. Phys.}
[14] C. Quesne, *Phys. Lett. A* **272** (2000) 313.

[15] A.P. Polychronakos, *Mod. Phys. Lett. A* **5** (1990) 2325; M. Roček, *Phys. Lett. B* **255** (1991) 554.

[16] T. Brzeziński, I.L. Egusquiza, A.J. Macfarlane, *Phys. Lett. B* **311** (1993) 202.

[17] J. Klauder, *J. Math. Phys.* **4** (1963) 1058.

[18] J. Deenen, C. Quesne, *J. Math. Phys.* **25** (1984) 2354; D.J. Rowe, *J. Math. Phys.* **25** (1984) 2662.

[19] J.-M. Sixdeniers, K.A. Penson, A.I. Solomon, *J. Phys. A* **32** (1999) 7543.

[20] J.K. Sharma, C.L. Mehta, E.C.G. Sudarshan, *J. Math. Phys.* **19** (1978) 2089; J.K. Sharma, C.L. Mehta, N. Mukunda, E.C.G. Sudarshan, *J. Math. Phys.* **22** (1981) 78.