On Automata with Boundary

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February 1, 2008

Abstract

We present a theory of automata with boundary for designing, modelling and analysing distributed systems. Notions of behaviour, design and simulation appropriate to the theory are defined. The problem of model checking for deadlock detection is discussed, and an algorithm for state space reduction in exhaustive search, based on the theory presented here, is described. Three examples of the application of the theory are given, one in the course of the development of the ideas and two as illustrative examples of the use of the theory.

1 Introduction

In this paper, we shall present an introduction to and overview of the theory of automata with boundary – a transition system based approach to the problem of designing, modelling and analyzing distributed systems. The treatment explicitly models the boundaries between subsystems, across which they communicate when part of a larger system. We describe notions of comparison and simulation
of automata that allow abstractions to be performed in a compositional manner. We give a notion of behaviour for these automata which works fluidly with the operations used to construct systems and the simulations used to abstract systems. Finally, we observe that this approach explicitly captures the design of a system as an element of the theory, and we describe some of the advantages of this.

In section 2, we present a general introduction to automata with boundary, behaviour, the operations used to construct systems from subsystems, and the notion of a design. This section develops the example of the dining philosophers in conjunction with the theory to show the reader how this well known example is treated using the ideas of this paper.

In section 3, we address the problem of model checking, specifically focusing on deadlock detection. We describe an algorithm (the minimal introspective sub-system algorithm) which utilises the design information for the system to assist in reducing the search space required for exhaustive model checking techniques. The section concludes with a description of the behaviour of the algorithm when applied to the dining philosophers example.

In section 4, we define comparison and simulation of automata, and describe the sense in which they are compositional, and the connection with behaviour and deadlock detection. We also give examples of simulations related to the dining philosophers, and indicate how simulations can be used to assist model checking, or to theoretically analyze systems of interest.

In section 5, we give two further examples of the theory – the design of a simple scheduler, and a message acknowledgement protocol. This section further illustrates the use of the theory and indicates the range of applicability.

The underlying mathematical formalism of the approach is that of category theory (see [15] or [22]) and more specifically Cartesian bicategories (see [3]). We stress, however, that no background in category theory is required in this paper – all the required definitions and results are stated in terms of transition systems, although we occasionally indicate in passing some key connections.

This paper forms part of an ongoing research project to develop a compositional theory of distributed systems using categorical structures. For an overview of the work of the project and the breadth of interpretation of ‘distributed system’, see [13]. The material in the current paper stems from work on the bicategory Span(Graph) : in [12], this bicategory was shown to have precise connections with the algebra of transition systems of Arnold and Nivat ([4]), and with the process algebras of Hoare ([7]); and, in [11], this bicategory was shown to be expressive enough to model place/transition nets ([18]).

2 Automata with Boundary

The goal of this section is to present automata with boundary as a model for systems composed from a number of communicating parts. The boundaries form an integral part of our theory – all interactions of a system with its environment
occur across its boundaries. We discuss *behaviours* of an automaton, and how these behaviours appear on boundaries of the automaton. Requirements on automata can be expressed by restricting the behaviours of an automaton as they appear on given boundaries. We describe the operations *bind*, *feedback*, and *product* of automata with boundary, enabling larger systems to be built by combining smaller systems in a way that interacts well with behaviours. Importantly, these operations can be represented pictorially as *designs*, allowing us to give high level views of systems.

As we introduce the basic definitions, we shall present an example illustrating the concepts being defined. We shall use the example of the dining philosophers, much favoured amongst works on distributed systems.

### 2.1 Automata with Boundary

The use of finite state automata to model transition systems has a long history (see [2]). In this paper, an automaton will consist of a reflexive graph plus other data. A *reflexive graph* consists of a directed graph (with parallel edges and loops allowed), together with a specified *reflexive edge* $v \rightarrow v$ for each vertex $v$ of the graph. We do not insist that our automata be finite, but all the examples we present are finite. We do restrict our attention to finite automata when considering model checking.

A crucial element of our theory is that of *boundary*. All boundaries are typed by the kind of synchronization actions which can occur across the boundary. By an *action set* we mean a finite set $X$ with a distinguished element, denoted $\cdot$. We refer to elements of $X$ as *actions*, and $\cdot$ as the trivial, or reflexive action.

An *automaton with boundary* $(S; (X_i, \mu_i)_{i \in I})$ consists of the following data:

1. A reflexive graph $S$, called the *state space* of the automaton, whose vertices are termed *states* and whose edges are termed *motions*.

2. A finite set $I$ indexing the boundaries of the automaton.

3. For each $i \in I$, a *boundary* $(X_i, \mu_i)$ consisting of an action set $X_i$ and a labelling $\mu_i(e)$ of each motion $e$ by an element of $X_i$, such that $\mu_i(e)$ is trivial if $e$ is reflexive.

Note that the state space of the automaton is the reflexive graph $S$ – it includes not only the states but the motions of the automaton, which provide the cohesion of the states to justify the terminology of a space. We have in mind that the reflexive edges of the state space $S$ are *idling* motions - see the comments after the definition of simulation (section 4.2) for a further exploration of this view.

The labelling of the motions indicates the actions on the boundaries which accompany given motions, and we require that if the automaton is idling, then it is idling on each boundary. Of course, nontrivial motions (i.e., motions which are not reflexive), may still idle on some or all boundaries. Those motions idling on all boundaries are called *internal* motions. They reflect the ability of an automaton to change state without this being reflected in its interaction with the environment.
We shall occasionally say a motion of $S$ performs an action on a boundary to mean that it is labelled by the specified action on the specified boundary. Note also that while a boundary consists both of the action set $X$ and the labelling of motions $\mu$, we shall often speak of the boundary $X$ when no confusion arises.

We shall say the type of a boundary $(X, \mu)$ to mean the action set $X$.

It is also worth noting that to give an action set is precisely to give a reflexive graph with one vertex. That is, the action set may be considered to be an automaton with trivial state. While not explored in this paper, a more general theory of this kind can relax this restriction, and allow one to calculate with boundaries which possess internal state.

2.1.1 Two boundary Automata

When dealing with an automaton, we typically focus temporarily on a subfamily of the boundaries over which some operation is being performed – for example the gluing of boundaries (i.e., binding – see section 2.3.1 below). Given a subset $J$ of the set $I$ indexing the boundaries of $S$, we may write $S$ as an arrow

$$\prod_{j \in J} X_j \xrightarrow{S} \prod_{k \in I \setminus J} X_k$$

and picture the motions of $S$ as being labelled in the two products via tupling of the labelling on individual boundaries. We shall typically abbreviate to just $S: X \to Y$ when we wish to emphasize the division, rather than the particular boundaries. In this case, we shall term $X$ the left boundary, and $Y$ the right boundary, of $S$. The passage from the product of $X_i$’s to the single object $X$ may be seen as the collection of a bundle of wires into a single cable for purposes of hierarchical design. This view gives the connection between automata with boundary and the theory of bicategories (see [10]).

This passage between multiple boundary and two boundary automata allows for a more natural and workable definition of the operations on automata, without sacrificing either expressive power or precision.

2.1.2 Pictorial representation of Automata

When describing automata, we typically draw pictures by drawing the state space of the automaton, with edges labelled to indicate the actions performed by the motions. For an automaton $S: X \to Y$, we write the label $(x|y)$ to indicate the motion performs the action $x$ on the left boundary and the action $y$ on the right boundary. We shall omit drawing reflexive edges, as they add no information to the picture – however the existence of these edges is crucial, as they allow subautomata in bound systems to act independently (see section 2.3.1 below).

We can depict automata with other than two boundaries in a similar manner. For an automaton with boundaries indexed by $I$, the labels on motions are $I$-tuples with entries drawn from the types of the corresponding boundary.
In this case, we make explicit the correspondence between tuple entries and boundaries.

2.1.3 Automata for the Dining Philosophers

For the example of the dining philosophers, we shall present two automata with boundary – a philosopher and a fork. Each philosopher will have two boundaries (the left and the right fork from her perspective), and each fork will likewise have two boundaries (the philosophers who can manipulate the fork). We shall thus have an action set $L$, and and automata with boundary $P: L \to L$ (a philosopher) and $Q: L \to L$ (a fork).

The action set $L$ consists of the actions which a philosopher and a fork jointly perform. A given nontrivial action on which a philosopher and a fork synchronize consists of the fork being either picked up or put down. We model this by taking the action set $L = \{-, lock, unlock\}$.

The philosopher $P$ is shown in figure 1. The philosopher has four states, corresponding to whether she is attempting to acquire her left fork, acquire her right fork, relinquish her left fork, or relinquish her right fork. The motions between these states are labelled by the boundary actions performed by the motion.

The fork $Q$ is shown in figure 2. The fork has three states, corresponding to whether it is unacquired (state $u$), acquired by its left boundary (state $l$), or acquired by its right boundary (state $r$). Once again, the motions are labelled by the boundary actions they perform.
2.2 Behaviour of Automata

For an automaton $S: X \rightarrow Y$ and a state $v_0$ of $S$, a behaviour $\beta$ of $S$ with initial state $v_0$ is a sequence of motions $(e_0, e_1, \ldots)$ (finite or infinite) of $S$ such that $s(e_0) = v_0$ and $s(e_{k+1}) = t(e_k)$ (for all appropriate $k$). That is, a behaviour of $S$ is nothing more than a path in its state space. We write such behaviours

$$\beta = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \ldots$$

Given a boundary $(X, \mu)$ of $S$, any behaviour of $S$ is reflected on the boundary via $\mu$. Precisely, by the appearance of a behaviour $\beta = (e_0, e_1, \ldots)$ on $X$ we mean the sequence of actions $(\mu(e_0), \mu(e_1), \ldots)$. We say a behaviour of $S$ on a boundary $X$ to mean a sequence which is the appearance of some behaviour of $S$ on $X$. Finally, we shall refer to reduced appearances and behaviours on boundaries to mean sequences obtained by eliding all trivial actions from an appearance or behaviour. It is crucial to note that a reduced appearance or behaviour need not be an actual appearance or behaviour, as nontrivial motions of $S$ may appear to be trivial actions on a given boundary.

An automaton gives rise to a relation between behaviours on its boundaries. Given a behaviour of $S$ on the boundary $X$ and a behaviour of $S$ on the boundary $Y$, we may say these are related if they are the appearances of the same behaviour of $S$ on the boundaries in question. It is typical to specify a system by requesting this be a specific relation, or by requesting properties of this relation. For example, performing a given set of actions on the keypad of an automatic teller machine (one of its boundaries) is required to result in cash being dispensed (the action performed by the machine on the boundary with the cash dispenser) and in an amount being deducted from the user’s account (the action performed by the machine on its boundary with the bank’s account record).

For an automaton $S: X \rightarrow Y$ and a given state $v$ of $S$, by the subautomaton of $S$ reachable from $v$ we mean the automaton $S': X \rightarrow Y$ with states those states $v'$ of $S$ such that exists a behaviour of $S$ of the form

$$v \rightarrow \ldots \rightarrow v'$$

That is to say, there is a path of motions of $S$ from $v$ to $v'$. The motions of $S'$ are all motions of $S$ between states of $S'$, and the boundaries and labelling of $S'$ are inherited directly from $S$.

We shall often assign to an automaton $S$ of interest an initial state – that is, a specified state $v_0$ of $S$. In this case we speak merely of behaviours of $S$ to mean behaviours of $S$ with initial state $v_0$, and the reachable subautomaton to mean the subautomaton reachable from $v_0$.

The interpretation of time implied by this treatment of behaviour is that of ‘discretized continuous time’ – there is an underlying continuous time which is being discretely approximated by some fixed time interval. An aspect of this continuity is contained in the use of the motions - each motion represents an atomic transition with the same duration. We distinguish this from a purely
‘discrete time’, in which the motions are atomic processes which may have different durations.

Thus while synchronization may be viewed as a real world process which takes variable time - we model an instance of such a synchronization by a behaviour consisting of internal motions bracketed by atomic synchronizing transitions of the same duration.

2.2.1 Behaviour for the Dining Philosophers

Returning to the dining philosopher example of section 2.1.3, we choose the state 0 of the philosopher (figure 9) to be the initial state. A behaviour of the philosopher then consists of a repeating sequence of the cycle “lock left boundary”, “lock right boundary”, “unlock left boundary”, “unlock right boundary”, possibly interspersed with reflexive edges. The reduced appearance on a given boundary is simply an alternating sequence of “lock”, “unlock” actions.

With the state $u$ as initial, a behaviour of the fork (figure 3) consists of a sequence of “lock boundary”, “unlock boundary” pairs, with the boundary possibly differing from pair to pair, and again possibly interspersed with reflexive edges. Again, the reduced appearance on a given boundary is an alternating sequence of “lock”, “unlock” actions.

2.3 Operations: Binding, Feedback and Product

We describe three operations which may be used to construct new automata with boundary from old. In each case, we have a diagrammatic view of the operation, which should be considered to be a design – an expression in variable, or unimplemented, automata. It is an important feature of the methodology presented here that we can depict operations on systems without depicting the internals of the systems, thus allowing hierarchical design. The connection between the operations discussed here and Hoare’s parallel operation is discussed in [12], section 4.

For each operation, we describe the effect of the operation on behaviours, in the sense that we describe the behaviours of the new system in terms of the behaviours of the given automata. It is an important feature of our theory that the operations on automata work fluidly with the notion of behaviour described in section 2.2.

Each operation is described here for two boundary automata – as noted in section 2.1.1 this is sufficient to describe it for all automata.

2.3.1 Binding

The first operation we consider is binding. Given two automata with a common boundary, say $S: X \rightarrow Y$ and $T: Y \rightarrow Z$, we can produce a new automaton, their binding, denoted $S \cdot T$. A state of $S \cdot T$ is a pair $(v, w)$, where $v$ is a state of $S$ and $w$ is a state of $T$. A motion $(v, w) \rightarrow (v', w')$ of $S \cdot T$ consists of a pair $(e, f)$, where $e: v \rightarrow v'$ is a motion of $S$ and $f: w \rightarrow w'$ is a motion
of $T$, and such that $e$ and $f$ perform the same action on the boundary $Y$. A given motion $(e, f)$ of $S \cdot T : X \rightarrow Z$ is labelled on the boundary $X$ by the action $e$ performs on $X$ (in $S$), and is labelled on the boundary $Y$ by the action $f$ performs on $Z$ (in $T$). If each of $S$ and $T$ have initial states $v_0$ and $w_0$ respectively, we take $(v_0, w_0)$ as the initial state of the binding.

The binding $S \cdot T$ thus has states the Cartesian product of the states of $S$ and $T$, but motions the subset of the Cartesian product of motions consisting of those on which the automata $S$ and $T$ synchronize on the common boundary $Y$. The reflexive motions of $T$ allow $S$ to move independently of $T$, provided $S$ is performing trivial actions on the common boundary.

Binding models two automata communicating by synchronizing on a common boundary. We draw diagrams of bound systems by connecting the boundaries of the automata being bound.

We draw the binding of two automata $S : X \rightarrow Y$ and $T : Y \rightarrow Z$ as follows:

\[
\begin{array}{c}
X \\
S \\
Y \\
T \\
Z \\
\end{array}
\]

Importantly, binding interacts well with the behaviours of automata:

**Proposition 1** Let $S : X \rightarrow Y$ and $T : Y \rightarrow Z$ be automata with boundary. To give a behaviour $\beta$ of $S \cdot T$ is precisely to give a behaviour $\gamma$ of $S$ and a behaviour $\delta$ of $T$ such that $\gamma$ and $\delta$ have the same appearance on $Y$.

### 2.3.2 Feedback

Given an automaton $S : X \times Y \rightarrow Y \times Z$, we can “bind $S$ with itself”. This operation, called feedback, and denoted $\text{fb}_Y(S)$, is used to form closed systems by connecting boundaries. We define $\text{fb}_Y(S) : X \rightarrow Z$ to be the automaton with states precisely those of $S$, and motions those motions $e$ of $S$ such that $e$ performs the same action on the factor $Y$ of its left boundary and the factor $Y$ of its right boundary. This yields an automaton with left boundary $X$ and right boundary $Z$, where the labelling is inherited from $S$ in the obvious manner. If $S$ has an initial state $v_0$, we take $v_0$ as the initial state of the fed back automaton.

Just as with binding, feedback may be presented diagrammatically by connecting the fed back boundaries:

\[
\begin{array}{c}
X \\
S \\
Y \\
Z \\
\end{array}
\]

Again, behaviours of the fed back automaton are easy to calculate:

**Proposition 2** Let $S : X \times Y \rightarrow Y \times Z$ be an automaton with boundary. To give a behaviour $\beta$ of $\text{fb}_Y(S)$ is precisely to give a behaviour $\gamma$ of $S$ such that $\gamma$
has the same appearance on the factor Y of the left boundary as on the factor Y of the right boundary.

2.3.3 Product

Given two automata $S: X \rightarrow Y$ and $T: Z \rightarrow W$, we define the product of $S$ and $T$, an automaton $S \times T: X \times Z \rightarrow Y \times W$, in the obvious way – form the Cartesian product of the states (resp. motions) of $S$ and $T$ to obtain the states (resp. motions) of $S \times T$. Note that the boundaries are likewise formed by Cartesian product – the product automaton has boundaries those of $S$ and those of $T$. The labelling of motion $(e, f)$ is obtained from the labellings of $e$ and $f$. If each of $S$ and $T$ have initial states $v_0$ and $w_0$ respectively, we take $(v_0, w_0)$ as the initial state of the product.

The product models combining two automata in parallel with no communication between them. As with binding, we note that the reflexive actions in the automata allow the automata to act independently.

Diagrammatically, products are shown as follows:

\[
\begin{array}{c}
\text{X} \\
\text{S} \\
\text{Y} \\
\text{Z} \\
\text{T} \\
\text{W}
\end{array}
\]

Behaviours of the product are easily characterized:

**Proposition 3** Let $S: X \rightarrow Y$ and $T: Z \rightarrow W$ be automata with boundary. To give a behaviour $\beta$ of $S \times T$ is precisely to give a behaviour $\gamma$ of $S$ and a behaviour $\delta$ of $T$.

2.3.4 Structural Automata

In addition to the operations on automata, there are a number of constant operations, or “structural automata” which are useful for constructing systems. Two examples of note are the identity on a given action set $X$ (figure 3) and the diagonal on a given action set $X$ (figure 4).

\[
\begin{array}{c}
\text{X} \\
\text{(x|x)} \\
\text{C} \\
\text{X}
\end{array}
\]

Figure 3: The identity automaton on $X$

The identity automaton on $X$ has two boundaries of type $X$. It has a single state, and one motion for each action $x$ of $X$, which is labelled by $x$ on each boundary. The reflexive motion is the motion corresponding to the reflexive
action $- \in X$. As its name suggests, the identity automaton is the identity for binding on $X$. One particular use of identities is to connect similar boundaries by a single wire in a composed system.

![Diagram of the diagonal automaton on $X$](attachment:diagram.png)

Figure 4: The diagonal automaton on $X$

The diagonal automaton on $X$ has three boundaries of type $X$. It has a single state, and one motion for each action $x$ of $X$, which is labelled by $x$ on each boundary. The reflexive motion is the motion corresponding to the reflexive action $- \in X$. The diagonal automaton is useful for splitting a wire synchronously.

### 2.3.5 Binding Philosophers and Forks

The binding $P \cdot Q$ of a single philosopher and a single fork is shown in figure 5 – we have conserved space a little by abbreviating lock and unlock to l and u respectively. The initial state of the bound system is the state $(0, u)$. There are several points to note about the bound system:

- Motions where the automata have synchronized have become internal motions (i.e., motions that perform trivial actions on both boundaries). In general, synchronizing two motions which perform trivial actions on all boundaries that are not being synchronized will produce an internal motion.
- Not all states are reachable from the initial state. Thus one often considers the reachable subautomaton of a bound system.
- The model allows for true concurrency, not just interleaving semantics. A motion of the bound system such as that labelled $(1,1):(0,u) \rightarrow (1,r)$ is truly concurrent, in that the philosopher and the fork change state simultaneously.

The binding $P \cdot Q$ allows a philosopher and a fork to synchronize on their common boundary by locking and unlocking.
Figure 5: The binding of a philosopher and a fork
2.4 Designs and Systems

The design diagrams we have given for the operations are more than a guide to the intuition behind the operations. These diagrams form a precise algebra for constructing designs. Given a stock of variables for automata with boundaries of given type, we can draw a diagram by juxtaposing automata and connecting boundaries with wires for the operations of binding, feedback and product – such a diagram is an expression for an automata, which can be evaluated given automata values for the variables. Such an expression is called a design.

2.4.1 The Geometry of Designs

Considering designs as expressions in a precise algebra with the operations of section 2.3, one should “parenthesize” such expressions to indicate the desired order of evaluation.

Given two automata \( S: X \to Y \) and \( T: X \to Y \), we say they are isomorphic if there is a bijection between states of \( S \) and states of \( T \) and a bijection between motions of \( S \) and motions of \( T \) which respect the source and target of motions and the labelling of motions on the boundaries. One can then prove propositions justifying the diagrammatic manipulations one would like to carry out, and alleviate the need to parenthesize diagrams in most situations.

For example, one can easily prove that binding is associative (up to isomorphism of automata). Given automata \( S: X \to Y \), \( T: Y \to Z \) and \( U: Z \to W \), we have that

\[
\begin{align*}
X & \xrightarrow{S} Y \xrightarrow{T} Z \xrightarrow{U} W \\
\cong & \ \ \\
X & \xrightarrow{S} Y \xrightarrow{T} Z \xrightarrow{U} W
\end{align*}
\]

where the dotted boxes indicate the order of binding. Symbolically, \((S \cdot T) \cdot U\) and \(S \cdot (T \cdot U)\) are isomorphic.

Thus we can draw diagrams when binding many systems with no risk of confusion. Of course, binding is not the only operation we consider, and one can prove propositions relating the different operations:

**Proposition 4** Given automata with boundary \( S: X \to Y \) and \( T: Y \to Z \), we
have that

In symbols, the automata \( S \times T \) and \( S \cdot T \) are isomorphic.

The following result is termed the middle four interchange law:

**Proposition 5** Given automata with boundary \( S: X \to Y, T: Y \to Z, Q: U \to V, \) and \( R: V \to W, \) we have that

\[
\begin{array}{c}
X \\
\hdashline \\
S \\
\hdashline \\
T \\
\hdashline \\
Y \\
\hdashline \\
\text{Z} \\
\hline \\
\text{U} \\
\hdashline \\
Q \\
\hdashline \\
V \\
\hdashline \\
\text{R} \\
\hline \\
\text{W} \\
\end{array}
\]

\[
\cong
\]

\[
\begin{array}{c}
X \\
\hdashline \\
S \\
\hdashline \\
T \\
\hdashline \\
Z \\
\hline \\
\text{U} \\
\hdashline \\
Q \\
\hdashline \\
V \\
\hdashline \\
\text{R} \\
\hline \\
\text{W} \\
\end{array}
\]

In symbols, the automata \((S \cdot T) \times (Q \cdot R)\) and \((S \times Q) \cdot (T \times R)\) are isomorphic.

It is worth noting that the geometry of designs is a purely combinatorial geometry – the wires are a mechanism for denoting the connection between boundaries, and the curvature and crossing of wires has no effect on the geometry of the design. A precise combinatorial model of designs in an appropriate mathematical context shall be described in a forthcoming paper [6].
2.4.2 Systems are designs with implementation

By a *system* we mean a design together with an assignment of an automaton with boundary to each variable in the design, these assignments being compatible with the operations used to evaluate designs. An instance of a variable automaton occurring in a given design is termed a *component* of the system. Each system has an associated automaton, called the *composite automaton*, or the *evaluation* of the system, obtained by realising the operations of the design on the assigned automata in accordance with the definition of the operations in section 2.3.

While the evaluation of a system may in some sense be seen as the goal of design, inasmuch as the problem of design is to produce an automaton with specific properties, the system itself is far more important from the point of view of analysis. Retaining the design of the final automaton in the system allows us to utilise facts about the construction of the system in order to analyse it – in section 3.3 we shall give an algorithm which uses design information to assist model checking. Given the effort typically devoted to design of a system in real world terms, it seems only rational that a theory of distributed systems retain designs as an element which is both precisely represented and capable of being computed with. For even though the ultimate (external customer) deliverable of a development effort is the compiled code (= evaluated automaton), a development effort is also expected to deliver a design to maintenance engineers in a usable (= analyzable) form.

2.4.3 Subsystems

Given a system, by a *subsystem* we mean some subset of the components of the system. Such a subsystem gives rise to an automaton via evaluation, we shall usually abuse terminology and use the term subsystem for this evaluation also.

For both binding and product, the states of the constructed automaton are pairs of states of the automata being operated upon. In the case of feedback, the states are states of the feedback automaton. Hence each state of the evaluation of a system gives rise to a state of the automaton associated with each component. We shall refer to states of the automata associated with components as *local states*, and by contrast refer to a state $v$ of the evaluation of the system as a *global state* of the system.

Further, given a subsystem of some larger system, each state of the evaluation of the system gives rise to a state of the subsystem in the obvious way (subsystem states are tuples of the local states of the components which form the subsystem). We refer to a state of a subsystem arising in this manner as a *local state* of the subsystem.

Similar remarks hold for motions and behaviours, and we shall thus use the terms *local motion*, *global motion*, *local behaviour*, and *global behaviour* for the corresponding concepts.
2.4.4 Dining Philosopher Systems

For any \( n \in \mathbb{N} \), we can form the composite automaton \( \text{fb}_L((P \cdot Q)^n) \) - this automaton has no boundaries, and models a ring of \( n \) philosophers with their \( n \) intervening forks. For example, figure 6 shows a design for a ring of three philosophers with their forks. This design, together with with assignment of \( P \) and \( Q \) to the automata of figures \( \text{[1]} \) and \( \text{[2]} \) respectively, comprise a system of dining philosophers.

\[
\begin{array}{cccc}
P & Q & P & Q \\
\end{array}
\]

Figure 6: A ring of 3 philosophers and their forks

2.5 Linear Automata

A motion \( e \) of an automaton with boundary \( (S, (X_i, \mu_i)_{i \in I}) \) is said to be **linear** if the action \( \mu_i(e) \) performed by \( e \) on the \( i \)'th boundary is nontrivial for at most one \( i \in I \). The automaton \( S \) itself is said to be **linear** if every motion of \( S \) is linear.

That is, a linear automaton is an automaton that interacts with at most one boundary in a given state. Linear automata have their boundaries decoupled, in the sense that they never require simultaneity on distinct boundaries. Another point of view is that linear automata are those modelling systems for which *interleaving* semantics are sufficient.

We note that even if the automata \( S \) and \( T \) are linear, the binding \( S \cdot T \) and the product \( S \times T \) may be nonlinear. For example, the binding of a philosopher and a fork (each a linear automaton) produces the nonlinear automaton of figure 6.

2.5.1 Linearizable Automata

An automaton with boundary \( (S, (X_i, \mu_i)_{i \in I}) \) is **linearizable** if, for each motion \( e : v \to w \) of \( S \) and given total order on \( I \), we can find a behaviour

\[
v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \ldots \xrightarrow{e_n} v_n = w
\]

of \( S \) such that

(i) each \( e_k \) is linear

(ii) if \( k, l \in [n] \) are such that \( \mu_i(e_k) \) and \( \mu_j(e_l) \) are nontrivial and \( i \leq j \) in the total order on \( I \), then we have \( k \leq l \)
(iii) if \( i \in I \) is such that \( \mu_i(e) \) is nontrivial, then there exists a \( k \in \{1,\ldots,n\} \) such that \( \mu_i(e) = \mu_i(e_k) \).

where \( \{1,\ldots,n\} \). Note that condition (ii) implies that distinct \( e_k \)'s cannot both perform nontrivial actions on the same boundary, i.e. that the existence in \( (iii) \) is unique.

Less symbolically, linearizable automata are those for which any nonlinear motion \( e \) can be refined into a series of linear motions with any desired ordering on the actions carried out simultaneously by \( e \).

Given linear automata \( S \) and \( T \), we observed above their binding and product need not be linear. It is however the case that they will be linearizable. A linearizable automaton can be linearized by considering only the linear motions. For example, linearizing the subautomaton of figure 5 reachable from the initial state produces the automaton shown in figure 7.

![Figure 7: The linearized reachable binding of a philosopher and a fork](image)

It should be noted that one cannot restrict attention solely to linear automata, as the operation of feedback presented in section 2.3.2 is not well suited to linear automata – the only motions in a feedback linear automaton are those which are trivial on the feedback boundaries. Further, the structural components described in section 2.3.4 are not linear automata.

### 2.5.2 Atomic Motions

Consider a system in which each automaton assigned to variable of the design is linear. Given a global motion \( e \) of the system, we have a corresponding local motions \( e_c \) for each component \( c \) of the system. We say that the global motion \( e \) is an atomic motion if

(i) The local motions \( e_c \) are nontrivial for at most two components.

(ii) In the case that the local motions \( e_c \) and \( e_d \) are nontrivial for distinct components \( c \) and \( d \), then the components \( c \) and \( d \) have boundaries \((X, \mu_i)\) and \((X, \mu_j)\) (respectively) which are joined by a wire in the design, and for which \( \mu_i(e_c) = \mu_j(e_d) \) is nontrivial.

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That is to say, the atomic motions are those for which multiple components move nontrivially only in the event they are synchronizing on boundaries joined by the design.

Given a system comprised of linear automata, the subautomaton of the composite automaton with all states but only atomic motions is termed the atomic core of the system. The atomic core restricts attention to those motions which are not fortuitously simultaneous. We shall use this notion when discussing model checking for deadlock in section 3.3. We simply remark at this point that when considering a complete system, the atomic core allows sufficient motions to fully explore the system, in the sense that any state \( w \) of the system reachable from a state \( v \) is reachable via atomic motions. However, we cannot restrict attention to the atomic core prematurely, for feedback of systems relies on nonlocal simultaneity.

We shall not investigate linearity further at present, but merely note that one of the strengths of the theory presented here is that specific requirements for envisaged domains (e.g. interleaving semantics) can be carried as additional properties of, or structure on, the basic theory. Precisely what can be done to tailor the basic theory for application to a specific domain is an area for further interesting work. Further work on linear automata can be found in section 4.1 of [12], including a connection with the process algebras of Hoare.

3 Model Checking

In this section we turn our attention to the problem of model checking – verifying that a given system has certain properties. The property we shall examine in detail is that of deadlock. We give an algorithm for finding a subspace of the state space of a given automaton, such that if the automaton possesses a deadlock \( v \), then the subspace possesses \( v \). In the case of the dining philosophers, this subspace is only quadratically large in the number of philosophers. This result has also been achieved using stubborn sets (see [20]). We then indicate some examples where the algorithm does not give such a good result. In section 4.3, we shall show how to leverage the algorithm presented here to these cases using abstraction techniques.

For the remainder of this section, we restrict attention to finite automata.

3.1 Deadlock detection

A state \( v \) of a automaton \( S : X \rightarrow Y \) is said to be a deadlock state if the only motion with source \( v \) is the reflexive motion. For example, if the composite of three dining philosophers and their forks (figure 3) is evaluated, the state \( (1, r, 1, r, 1, r) \) is a reachable deadlock. Naively, to check for a reachable deadlock in an automaton, one must examine every reachable state and determine if it is a deadlock state. In the example of the dining philosophers, a ring of \( n \) philosophers has \( (4 \times 3)^n \) states, of which \( 3^n - 1 \) are reachable (for \( n \geq 2 \).
However, we can attempt to exploit the design of the system to simplify our search. The motivating case is the example of products of automata:

**Proposition 6** Let $S: X \to Y$ and $T: Z \to W$ be automata with given initial states $v_0$ and $w_0$. If the product $S \times T$ has a reachable deadlock, then it has a reachable deadlock of the form $(v^*, w^*)$ where $v^*$ is a deadlock of $S$ and $w^*$ is a deadlock of $T$.

Moreover, this deadlock is reachable by first considering those motions trivial in $T$, and then considering those motions trivial in $S$.

The content of the proposition is that it suffices to check the subautomaton of states of the form $(v, w_0)$ or $(v^*, w)$ when searching for a reachable deadlock. In general, this subautomaton has a number of states bounded by $\#S + \#T$, as opposed to the $\#S \times \#T$ states in the full product $S \times T$.

We define a **strong deadlock analysis** of an automaton $S: X \to Y$ with initial state $v$ to be a subautomaton $T$ of $S$ such that

(i) $T$ contains $v$

(ii) If $w$ is a deadlock state of $S$ reachable from $v$, then $w$ is in $T$.

One could also consider the notion of a **weak deadlock analysis**, where the second condition is replaced by the weaker condition

(ii’) If a deadlock state of $S$ is reachable from $v$, then $T$ contains a deadlock reachable from $v$.

Then the content of the proposition is that the subautomaton of $S \times T$ with states those states $(v, w)$ such that $v = v^*$ or $w = w_0$, and motions all motions between these states, comprises a weak deadlock analysis of the product. If we include all states $(v, w)$ such that either $v$ is a deadlock of $S$ or $w = w_0$ we obtain a strong deadlock analysis of the product.

### 3.2 Introspective Subsystems

Most systems of interest do not decompose as products as required for the deadlock analysis provided by proposition 6—some coupling is required in order for distributed parts of the system to communicate and achieve a common goal. However, many systems of interest do “locally decouple” in the sense that parts do not spend their entire time in communication with each other, and generally restrict their interaction to specific parts of the system at specific times.

Before presenting a notion of local decoupling appropriate to our theory, we mention again the example of the dining philosophers. Consider the reachable linearized subautomaton of the binding of a philosopher and a fork as shown in figure 7. From the state $(3, l)$ we have a motion to $(0, u)$ labelled $(-|-)$. Being an internal motion of the bound automaton and the only motion out of $(3, l)$, any behaviour from this state must use this motion, and the environment of the automaton cannot affect the viability of this motion. That is to say, the
philosopher and the fork have locally decoupled from the rest of any larger system they may be a part of. We would like to restrict our search for a deadlock by taking advantage of the fact this motion is independent of the action of the rest of the system.

Given an automaton $S$ with boundaries $(X_i, \mu_i)$, and a state $v$ of $S$, we say that $S$ is watching or looking at boundary $i$ in state $v$ if there is a motion $e: v \rightarrow w$ of $S$ such that $\mu_i(e)$ is nontrivial. Conversely, the automaton $S$ is ignoring boundary $i$ in state $v$ if every motion with source $v$ performs the trivial action on the boundary $X_i$.

For example, the philosopher of figure 1 is watching the left boundary in states 0 and 2, and watching the right boundary in states 1 and 3. The fork of figure 2 is watching the left boundary in states $l$ and $u$, and the right boundary in states $u$ and $r$.

**Proposition 7** Let $S: X \rightarrow Y$ and $T: Y \rightarrow Z$ be automata. Let $v$ be a nondeadlock state of $S$ such that $S$ is ignoring $Y$ in state $v$. For any state $w$ of $T$, if there is a behaviour of $S \cdot T$ with initial state $(v, w)$ which leads a deadlock $(v^*, w^*)$, then there is such a behaviour where the first nontrivial motion is trivial in $T$.

**Proof:** Suppose, by way of contradiction, this were not true. Let $(v, w)$ be a state of $S \cdot T$ such that a deadlock $(v^*, w^*)$ is reachable from $(v, w)$, but not by an initial nontrivial motion which is trivial in $T$. Let $\beta$ be a behaviour of $S \cdot T$ with initial state $(v, w)$ and reaching the deadlock. Write

$$\beta = (v_0, w_0) \xrightarrow{(f_1, g_1)} (v_1, w_1) \xrightarrow{(f_2, g_2)} (v_2, w_2) \ldots (v_n, w_n) = (v^*, w^*)$$

where $v_0 = v$ and $w_0 = w$.

We claim some $f_k$ is a nontrivial motion of $S$. If not, then $v_n = v$. Since $v$ is not a deadlock state of $S$, there is some motion with source $v$, say $e$ labelled $(x|y)$. Since $S$ is not looking at $Y$ in state $v$, it must be that $y = -$ is trivial. Hence we can extend $\beta$ with the motion $e$ in $S$ and the trivial motion in $T$, and thus $\beta$ did not reach a deadlock state, contrary to choice of $\beta$.

Let $k$ be minimal such that $f_k$ is a nontrivial motion in $S$. Note that $f_i$ is trivial for $i = 0, \ldots, k - 1$. Thus

$$v_0 = v_1 = \ldots = v_{k-1}$$

The triviality of $f_i$ for $i < k$ also implies that the action performed by $f_i$ on the boundary $Y$ is trivial. The action performed by $f_k$ on the boundary $Y$ is also trivial, since $S$ is not looking at $Y$ in state $v$. Thus, since $g_i$ synchronizes with $f_i$, we have that the action performed by $g_i$ on the boundary $Y$ is also trivial, for $i = 0, \ldots, k$. 


It is now evident that
\[
(v_0, w_0) = (v_{k-1}, w_0) \xrightarrow{(f_k, -)} (v_k, w_0) \xrightarrow{(-, g_1)} (v_k, w_1)
\]
\[
\xrightarrow{(-, g_2)} \ldots (v_k, w_{k-1}) \xrightarrow{(-, g_k)} (v_k, w_k)
\]
\[
\xrightarrow{(f_{k+1}, g_{k+1})} (v_{k+1}, w_{k+1}) \ldots (v_n, w_n) = (v^*, w^*)
\]
is a behaviour of \( S \cdot T \) with initial state \((v, w)\) that leads to the specified deadlock \((v^*, w^*)\), and that has first motion trivial in \( T \). Hence the desired contradiction.

For a system with composite automata \( S \) and a global state \( v \) of \( S \), a subsystem is said to be **introspective at** \( v \) if each component of the subsystem, when in the local state corresponding to \( v \), is ignoring every boundary on which it connects to components not in the subsystem.

**Proposition 8** Consider a global state \( v \) of a system and a subsystem that is introspective at \( v \) but not deadlocked when in the local state corresponding to \( v \). If a deadlock of the system is reachable from \( v \), it is reachable via a behaviour whose first nontrivial motion is trivial outside the given subsystem.

**Proof:** Using the algebra of designs, organize the system as a composite of the given subsystem and its complement:

\[
\begin{array}{c}
\text{Introspective Subsystem} \\
\text{Rest of System}
\end{array}
\]

Now apply proposition 7 to the composite of the evaluation of the two subsystems.

Given an introspective but not deadlocked subsystem at each global state \( v \) of the composite automata \( S \) of a system, we can apply proposition 8 repeatedly to produce a strong deadlock analysis by including only those motions of \( S \) which are trivial outside the introspective subsystem associated with their source, and including only those states which are reachable from the initial state of \( S \) via the included motions.

### 3.3 Minimal Introspective Subsystem Analysis

It may be that, for a given design, the introspective subsystems are obvious, or designed in to the system so as to provide for more efficient checking. However, it is also desirable to automatically check a given system for absence of deadlock, exploiting the known design of the system to reduce the state space explosion associated with exhaustive model checking.

The idea of minimal introspective subsystem analysis is to guide the exploration of the state space via proposition 8. More precisely, we construct the
deadlock analysis of a given system suggested at the end of the previous section as we explore the state space, by choosing a minimal introspective subsystem at each state.

Let us fix for discussion a system with composite automaton $S$. Given a global state $v$ of $S$ and a component of the system, we can examine the automaton assigned to the component to determine which boundaries the automaton is looking at in the local state corresponding to $v$.

Given this information for each component and the design of the system, it is a simple matter to construct a non-deadlocked minimal introspective subsystem at $v$ – consider the directed graph with vertices the components and edges indicating that the component represented by the source is looking at the component represented by the target, and flood fill along edges from each vertex to find introspective subsystems.

This process determines, for each component, the minimal introspective subsystem containing that component. If the global state $v$ is not a (global) deadlock, then some nontrivial motion is possible. Hence at least one component, and thus the introspective subsystem containing it, is not deadlocked. We select the smallest of these subsystems which is not deadlocked in the local state corresponding to $v$.

We then explore the states of $S$ only along motions which are trivial outside the selected minimal introspective subsystem, looking for deadlock – proposition \ref{prop:subset} guarantees that a reachable deadlock is reachable via a motion trivial outside the introspective subsystem. By maintaining a list of visited states, we can ensure the algorithm terminates.

In the event the automata assigned to variables of the design are linear, it suffices to explore the states of $S$ only along atomic motions (see \ref{sec:atoms}) – in this case restricting to the subautomaton including only atomic motions does not alter the reachability of states.

### 3.3.1 Minimal Introspective Subsystem Analysis of Dining Philosophers

Let us consider the system consisting of a ring of $n$ philosophers and their $n$ intervening forks. Since each automata used in the system is linear, we can restrict our attention to atomic motions. We shall now walk through the application of the minimal introspective subsystem analysis algorithm proposed in section \ref{sec:analysis} for this system.

Initially, each philosopher is looking at the left boundary, and each fork is looking at both boundaries. So the only introspective subsystem is the entire system, giving $n$ atomic motions to be explored (each motion being one for which a given philosopher acquires their left fork).

Each state reached next has precisely one philosopher having obtained their left fork. Let us consider the state in which the philosopher $P_1$ has acquired his left fork. At this point, we see $P_1$ is now looking at his right boundary, and $P_2$ is looking at her left boundary, and the fork $F_1$ is looking at both boundaries. These three components thus form an introspective subsystem as required. A
moments thought shows that it is minimal, and a moments more that it is the only minimal introspective subsystem. There are only two nontrivial atomic motions in this subsystem – either \( P_1 \) acquires the fork or \( P_2 \) acquires the fork.

In the former case, the system comprised of \( P_1 \) and his left fork now constitute a minimal introspective subsystem – the philosopher is in state 2 attempting to relinquish the left fork, and the left fork is in state \( r \) having being acquired by the philosopher on its right. The single nontrivial atomic motion of the subsystem is to relinquish the fork. Following this, and by a similar analysis, the philosopher relinquishes his right fork. The system has now returned to the initial state, which is marked as checked.

In the latter case of \( P_2 \) acquiring the fork \( F_1 \), we apply similar reasoning to the competition between \( P_2 \) and \( P_3 \) to acquire \( F_2 \), as these three systems once again comprise the unique minimal introspective subsystem at the global state under consideration. In exploring the case that \( P_2 \) is successful, she will proceed to relinquish her left and right forks, and we return to the state where she is competing with \( P_1 \). In exploring the case \( P_3 \) is successful we examine the subsystem comprised of \( P_3 \), \( F_3 \) and \( P_4 \).

The algorithm continues in this manner, obtaining a minimal introspective subsystem comprising two philosophers and their intervening fork at each stage, and progressing in two ways – allowing one philosopher to run to completion, or moving to the competition for the next fork around the table. Eventually the deadlock in which each philosopher has acquired their left fork is found.

It is evident that after the initial state, we explore 3 states for each philosopher (as it moves through states 1, 2 and 3) bar the last. The exploration stops when the last philosopher acquires his left fork, and hence each philosopher has acquired their left fork and the system is deadlocked. Potentially then, we are required to explore the initial state, the final deadlock state, and \( 3(n - 1) \) for each choice of the initial \( n \) motions. Thus \( 3n^2 - 3n + 2 \) states are explored, a significant reduction on the \( 3^n - 1 \) reachable states in the system.

This result has been described using stubborn sets in \[19\], where the same polynomial for the number of states checked is computed.

It should be noted that there are systems very similar to the dining philosophers in which the above algorithm does not reduce the checking to a polynomial number of states. Replacing the philosophers by either the system shown in figure 8 or figure 9 results in a system for which the above algorithm searches an exponential number of states.

The alternative philosopher I, shown in figure 8, may be termed the non-deterministic philosopher. In this case, minimal introspective subsystem analysis of the composite system must check two branches as the minimal introspective subsystem under consideration moves around the table. Informally then, we see that an exponential number of states will be checked (although still significantly less than the total number of states of the system).

The alternative philosopher II, shown in figure 9, may be termed the double cover philosopher. In this case, when the philosopher returns to the state of wishing to acquire his left fork in the first instance, he is in the local state 4,
and not a searched state as in the basic example. Thus the algorithm arrives at many distinct states in which the minimal introspective subsystem is the entire system. There are $2^n$ reachable global states of this form, and it can be argued that the algorithm proposed above will visit all of them. Informally then we again have a situation where an exponential number of states are checked.

It is worth noting that both the nondeterministic and double cover philosophers can be abstracted to the philosopher of figure 1 in a sense which is made precise in section 4.2. This abstraction allows us to use the strong deadlock analysis with only polynomially many states to check the more complex systems (see section 4.3).

In this section we have outlined the principles of model checking for deadlock as manifested in our theory, and presented a very simple algorithm for reducing state space explosion in model checking. It should be emphasized that although simplistic, the algorithm does have demonstrably good behaviour on a particular system, and importantly exhibits the principle of using design information..
4 Simulation

In this section we introduce a compositional notion of simulation of automata which has a close relation to work on simulations and bisimulations (see [1], [4], and [8]). We will also indicate how simulations can be used to facilitate model checking.

4.1 Comparison of Automata

We begin by defining reflexive graph morphisms. Denote a reflexive graph $G$ by the pair $(V, E)$ comprising its set of vertices $V$ and its set of edges $E$. A reflexive graph morphism $f$ from $G = (V, E)$ to $G' = (V', E')$ consists of functions $f_V: V \to V'$ and $f_E: E \to E'$ such that sources and targets of edges are preserved, as are reflexive edges.

Suppose $(S, (X_i, \mu_i)_{i \in I})$ and $(T, (X_i, \nu_i)_{i \in I})$ are two automata with the same boundary action sets $(X_i)_{i \in I}$. A comparison $f$ from $S$ to $T$ is a reflexive graph morphism $f$ from $S$ to $T$ which preserves the actions on the boundaries—that is, for each $i \in I$, we have $\nu_i : f = \mu_i$. We shall term the function comprising the action of $f$ on states the state map, and the function comprising the action of $f$ on motions the motion map.

When we are writing automata in the form $X \to Y$ (as two boundary automata), such a comparison is denoted $f: S \Rightarrow T: X \to Y$, although typically we shall just write $f: S \Rightarrow T$ as the boundaries will be understood. In this section we will consider automata equipped with an initial state, and comparisons are asked to preserve initial states.

For example, let $P: L \to L$ be the philosopher described in section 2.1.3 (figure 1), let $P': L \to L$ be the alternative philosopher I depicted in figure 8, and let $P'': L \to L$ be the alternative philosopher II depicted in figure 9. There are unique comparisons $p: P' \Rightarrow P$, $q: P'' \Rightarrow P$ and $r: P'' \Rightarrow P''$, and these comparisons preserve the initial vertex 0.

Action sets, automata and comparisons form what is known as a discrete Cartesian bicategory (see [3]). Rather than recalling the definition of this complicated algebraic structure, we will only consider the operations relevant to this paper; namely, composition, binding, feedback and product of comparisons.

4.1.1 Composition

Given automata $R$, $S$, and $T: X \to Y$, and comparisons $f: R \Rightarrow S$ and $g: S \Rightarrow T$, we define the composite comparison $g \cdot f: R \Rightarrow T$. The composite has state map the composite of the state maps of $f$ and $g$ and motion map the composite of the motion maps of $f$ and $g$—both these latter composites being the usual composite of functions. It is routine to check that the composite, so defined, is a comparison $R \Rightarrow T$. 
In categorical terms, this composite is simply the composite of $f$ and $g$ in the category of reflexive graphs. In fact, for any two action sets $X$ and $Y$ we can form a category $\text{Aut}(X, Y)$. Its objects are automata of the form $S: X \to Y$ and its arrows are comparisons between these automata.

### 4.1.2 Binding, feedback and product

We now describe how the operations on automata described in section 2.3 also apply to comparisons. In each case, the data for the operations consists of comparisons between data suitable for the corresponding automata operation.

Given automata $S: X \to Y$ and automata $U: Y \to Z$, together with comparisons $f: S \Rightarrow T$ and $g: U \Rightarrow V$, we define the binding of $f$ and $g$—a comparison $f \cdot g: S \cdot U \Rightarrow T \cdot V$.

As described in section 2.3.1, the state space of $S \cdot U$ has states consisting of pairs $(v, w)$ with $v$ a state of $S$ and $w$ a state of $U$, and motions consisting of pairs of motions performing the same action on the common boundary. The comparison $f \cdot g$ maps the state $(v, w)$ by mapping $v$ as $f$ does, and $w$ as $g$ does, with the obvious extension to motions.

The fact that $f$ and $g$ respect the actions of motions on boundaries implies that $f \cdot g$ maps motions of $S \cdot U$ to motions of $T \cdot V$, and it is routine to check we have defined a comparison.

Given automata $S$ and $T: X \times Y \to Y \times Z$ and a comparison $f: S \Rightarrow T$, we define the feedback of $f$—a comparison $\text{fb}_Y(f): \text{fb}_Y(S) \Rightarrow \text{fb}_Y(T)$.

The state space of $\text{fb}_Y(S)$ is that of $S$, but with motions only those that perform the same action on the two boundaries of $S$ of type $Y$. Since $f$ preserves the actions on boundaries, the image of such a motion under $f$ is a motion of $T$ performing the same action on the two boundaries of $T$ of type $Y$. Thus the comparison $f$ restricts to a comparison between the fed back automata, and this latter comparison is $\text{fb}_Y(f)$.

Given automata $S$ and $T: X \to Y$ and automata $U$ and $V$, together with comparisons $f: S \Rightarrow T$ and $g: U \Rightarrow V$, we define the product of $f$ and $g$—a comparison $f \times g: S \times U \Rightarrow T \times V$.

The product $f \times g$ has state (resp. motion) map the product of the state (resp. motion) maps of $f$ and $g$. That is, it operates on the state space of $S \times U$ componentwise.

### 4.2 Simulations

If $S: X \to Y$ is an automaton, let $\overline{S}: X \to Y$ denote the reachable subautomaton of $S$. Note that to give a comparison $f: S \Rightarrow T$ is just to give a comparison $f: \overline{S} \Rightarrow \overline{T}$.

By a simulation $f$ from $S: X \to Y$ to $T: X \to Y$ we mean a comparison $f: \overline{S} \Rightarrow \overline{T}$ such that $f$ satisfies the following ‘lifting property’: for all states $v$...
of $\overline{S}$ and all motions $e: f(v) \to w$ in $\overline{T}$, there exists a (finite) behaviour of $\overline{S}$

$$v = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \ldots \xrightarrow{e_n} v_{n+1}$$

such that

(i) for $0 \leq i \leq n - 1$, the motion $f(e_i)$ is the reflexive motion at $f(v)$

(ii) $f(e_n) = e$

We say that the automaton $T$ simulates $S$ via $f$, and write $f: S \leadsto T: X \to Y$, or merely $f: S \leadsto T$ if the boundaries are understood.

In light of this definition, we should revisit our view of reflexive motions as idling motions. More precisely, we emphasize that reflexive motions are idling at the level of abstraction of the automaton. When abstracting automata – that is, constructing comparisons and simulations – we may have cause to abstract away internal motions which are not germane to the analysis task at hand. Thus reflexive motions may be thought of as representing motions unimportant at the level of abstraction of the automaton, and not necessarily a strictly idle state of the process being modelled.

The proof of the following proposition is straightforward.

**Proposition 9** If there is a simulation $f: S \leadsto T$ then $S$ and $T$ have the same set of reduced appearances.

Of course, in the above proposition, we are only considering behaviours beginning at initial states. For the connection between this notion of simulation and notions of observational equivalences such as $\equiv_1$, the reader is referred to [8] and [9].

The comparisons of philosophers $p: P' \Rightarrow P$, $q: P'' \Rightarrow P$ and $r: P'' \Rightarrow P'$ mentioned above are examples of simulations. One class of trivial (but, nevertheless important) simulations is provided by the subautomata reachable from the initial states. That is, for every $S: X \to Y$, the identity graph morphism of $\overline{S}$ provides a simulation $1_S : S \leadsto S$.

**Proposition 10** Suppose $f: S \leadsto T$ is a simulation. If $v$ is a reachable deadlock of $S$ then $f(v)$ is a reachable deadlock of $T$.

The above proposition indicates that simulations may be used for detecting deadlocks: given an automaton $S$ we try to find a simulation $f: S \leadsto T$ where $\overline{T}$ has significantly less states than $\overline{S}$ (that is, $\overline{T}$ is a quotient of $\overline{S}$); we then look for deadlocks $v$ in $\overline{T}$; if $\overline{T}$ has no deadlocks, we conclude that neither does $S$; and if there are deadlocks $v$ in $\overline{T}$, we check to see if there are any among the states $w \in f^{-1}(v)$ of $\overline{S}$. We give an example of this process at the end of this section.

Action sets, automata and simulations also form a discrete Cartesian bicategory. The operations composition, binding, product and feedback of comparisons induce the same operations on simulations.
4.2.1 Composition

Given simulations \( f: R \sim S \) and \( g: S \sim T \) where \( R, S, \) and \( T: X \rightarrow Y \) are automata, there exists a simulation \( g \circ f: R \sim T \) called the composite of \( f \) and \( g \).

The composite is formed by composing the comparisons \( R \Rightarrow S \) and \( S \Rightarrow T \) being the data provided for \( f \) and \( g \). The requisite lifting property is easily established by first lifting via \( g \), and then lifting each component of this lifting via \( f \).

As was the case with comparisons, for any two action sets \( X \) and \( Y \) we can form a category; namely the category \( \text{Sim}(X, Y) \) whose objects are automata with left boundary \( X \) and right boundary \( Y \) and whose arrows are simulations between these automata.

4.2.2 Binding, feedback and product

Given automata \( S \) and \( T: X \rightarrow Y \) and automata \( U \) and \( V: Y \rightarrow Z \), together with simulations \( f: S \sim T \) and \( g: U \sim V \), we define a simulation \( f \times g: S \times U \sim T \times V \) called the binding of \( f \) and \( g \).

Given a state \((v, w)\) of \( S \times U \), it is clear that \( v \) is reachable in \( S \) and \( w \) is reachable in \( U \). Applying \( f \) to \( v \) and \( g \) to \( w \) thus yields a pair of states, and the reachability of \((v, w)\) implies this image pair is reachable in \( T \times V \). This defines the state map of \( f \times g \). The motion map is similarly obtained from \( f \) and \( g \).

The lifting property is obtained by lifting componentwise. Without loss of generality, the lifted paths have the same length (if not, extend the shorter path by prepending reflexive motions). All but the last motion in each lifting has a reflexive image, and thus will be a motion of \( S \times U \). The images of the final motion in each lifting perform a common action on the boundary \( Y \), since we are lifting a motion of \( T \times V \). Thus, since \( f \) and \( g \) are comparisons, the lifted motions also agree on their actions on the common boundary.

Given automata \( S \) and \( T: X \rightarrow Y \times Z \) and a simulation \( f: S \sim T \), we shall construct a simulation \( \text{fb}_Y(f): \text{fb}_Y(S) \sim \text{fb}_Y(T) \) called the feedback of \( f \).

Given a state \( v \) of \( \text{fb}_Y(S) \), we have a path from the initial state of \( S \) to \( v \) consisting only of motions performing the same action on the two boundaries of type \( Y \). Such \( v \) is clearly a state of \( \overline{S} \), and applying \( f \) then gives us a similar path in \( T \), and we see that \( f \) induces a comparison as required.

As in the case for binding, the lifting property follows from the property for \( f \) together with the fact that motions with reflexive image clearly perform the same action on the two boundaries of type \( Y \), and the final lifted motion agrees after application of \( f \) and hence before it, since \( f \) respects the actions on boundaries.

Given automata \( S \) and \( T: X \times W \rightarrow Y \times Z \), together with simulations \( f: S \sim T \) and \( g: U \sim V \), we define the product of \( f \) and \( g \), a simulation \( f \times g: S \times U \sim T \times V \).

Observe that \( \overline{S \times U} = \overline{S} \times \overline{U} \), and thus the product of the comparisons underlying the simulations \( f \) and \( g \) yields a comparison to underly \( f \times g \). The
lifting is performed componentwise, extending the shorter path by prepending reflexive motions if required.

It is a crucial aspect of the theory that the operations on designs lift to operations on simulations. Thus, given a design, we can abstract parts of the design (i.e. simulate them with simpler systems) and produce abstractions of the whole system. In the next section we shall indicate how this can be used to support model checking in the concrete example of the dining philosophers.

4.3 Simulations and Dining Philosophers

We now indicate how simulations may facilitate the task of model checking.

Consider the two alternative dining philosophers presented at the end of section 3.3.1 (figures 8 and 9). As noted there, the sizes of the state spaces of these systems which are explored by minimal introspective subsystem analysis grow exponentially with the number of philosophers \( n \).

Using these alternative philosophers with the design of the usual dining philosopher systems, we may construct corresponding systems \( \text{fb}_L((P' \cdot Q)^n) \) and \( \text{fb}_L((P'' \cdot Q)^n) \). We have already noted, however, that there are two simulations \( p: P' \sim P \) and \( q: P'' \sim P \). Thus using the operations of section 4.2.2 we can construct simulations

\[
\hat{p} = \text{fb}_L((p \cdot 1_Q)^n) \cdot \text{fb}_L((P' \cdot Q)^n) \sim \text{fb}_L((P \cdot Q)^n)
\]

and

\[
\hat{q} = \text{fb}_L((q \cdot 1_Q)^n) \cdot \text{fb}_L((P'' \cdot Q)^n) \sim \text{fb}_L((P \cdot Q)^n).
\]

Now apply the minimal introspective subsystem analysis to the standard philosopher system (recall the explored state space grows only quadratically with the number \( n \)). This analysis will find the unique deadlock \( d \) of \( \text{fb}_L((P \cdot Q)^n) \). Recall from section 3.3.1 that this state \( d \) corresponds to each fork being in state \( r \) and each philosopher being in state 1.

We now know that the only deadlocks of the alternative philosopher systems are contained in \( \hat{p}^{-1}(d) \) and \( \hat{q}^{-1}(d) \). It is easy to calculate these sets of states – for example, a state in \( \hat{p}^{-1}(d) \) corresponds to each fork being in state \( r \) and each philosopher being in state 1 or 1'. In fact, each \( v \in \hat{p}^{-1}(d) \) and each \( w \in \hat{q}^{-1}(d) \) is a deadlock of \( \text{fb}_L((P' \cdot Q)^n) \) and \( \text{fb}_L((P'' \cdot Q)^n) \) respectively, and these are the only deadlocks of these systems.

What if we want to analyse the dining philosopher system for arbitrary \( n \)? With the use of software tools (such a tool is currently being specified and prototyped by the authors), it is reasonably straightforward to construct an automaton \( R : L \rightarrow L \), together with a pair of simulations \( f_2: (P \cdot Q)^2 \sim R \) and \( f': P \cdot Q \cdot R \sim R \).
The compositionality of simulations allows us to deduce that for any $n \geq 2$, there is a simulation $f_n': (P \cdot Q)^n \leadsto R$. We define, inductively, for $n \geq 2$

$$f_{n+1}': (P \cdot Q)^{n+1} = P \cdot Q \cdot (P \cdot Q)^n \leadsto P \cdot Q \cdot R \leadsto R$$

where the first simulation is $\frac{1}{n+1}f_n$ and the second simulation is $f'$.

In other words (from the the point of view of observational equivalence and checking for deadlocks) we can replace a composed sequence of philosophers and forks of any length by the simple system $R$.

In the case of checking the dining philosopher system for deadlocks, we first form the simulation $g = \text{fb}_L(f_n) : \text{fb}_L((P \cdot Q)^n) \leadsto \text{fb}_L(R)$ and then note that the automaton $\text{fb}_L(R)$ has a unique deadlock $c$. It is easy to check that the only vertex $v$ of $\text{fb}_L((P \cdot Q)^n)$ with the property that $g(v) = c$ is that corresponding to each philosopher being in state 1 and each fork in state $r$, allowing us to conclude that this is the only deadlock of $\text{fb}_L((P \cdot Q)^n)$.

## 5 Further Examples

In this section we shall present two further examples of systems composed from automata with boundary. We model a scheduler, which is responsible for ensuring certain execution order properties in a collection of concurrent systems.

We also present a model of processes communicating via a channel, and indicate how communication protocols may be modelled as systems of automata with boundary. The goal of this section is not to present any deep insights into the systems we model, but to demonstrate the expressive power of the methodology, and the process of design within the methodology.

### 5.1 Scheduling

We have in mind a system which controls the execution of a number of processes in order to meet certain specifications. Each process has a certain part of its execution, called the \textit{controlled} section, which is of interest to the scheduler. This system is also used as an example of the calculus described in [16].

Our processes $P_1, \ldots, P_n$ each have one boundary, over which they will communicate with the scheduler. We shall write $C = \{-, \text{begin}, \text{end}\}$ for the action set of these boundaries – the process synchronizes with a \text{begin} to indicate it is entering its controlled section, and synchronizes with a \text{end} to indicate it is leaving its controlled section. Every behaviour of each process must alternate \text{begin} and \text{end} actions on its boundary. We shall model the processes then with an automaton as shown in figure 11.

In light of proposition 10 and the constructions of section 4.2.2, any analysis we perform on systems involving $P$ for deadlock will lift to the corresponding systems using more complex processes, provided only that these processes are simulated by $P$.

In fact, one could argue that the property of being simulated by $P$ can be taken as a definition of the kind of process we are interested in – for the existence
of such a simulation says precisely that the process has states of two kinds (those mapping to 0 and those mapping to 1), and that it transits between these states by actions observable on the boundary as \texttt{begin} and \texttt{end}.

### 5.1.1 Design of the Scheduler

A scheduler $S$ for $n$ processes of this kind has $n + 1$ boundaries, each of type $C$. The scheduler will be connected to the processes $P_i$ on $n$ of its boundaries, with the last boundary being reserved for control of the scheduler – allowing an external agent to start and stop the scheduler.

We shall construct a scheduler which forces the processes to \texttt{begin} their controlled sections in a fixed, cyclic, order. That is to say, we wish to control the processes such that in a global behaviour of the system, the first nontrivial local motion of one of the process components $P_i$ is \texttt{begin} by $P_1$. The next nontrivial local motion by a process component is \texttt{begin} by $P_2$, and so on.

Following the design of [16], we shall construct the scheduler from a number of smaller automata. We shall use $n$ copies of an automaton $N$, called a \texttt{notifier}. Each notifier starts a single process and records the completion of its controlled section. We also have a single master automaton $M$, which responds to outside control. The notifiers will pass a token around a circle. When receiving the token, a notifier ensures its process \texttt{begin}s its controlled section, and then passes the token on. The master automaton hands the token out when it \texttt{begin}s, and will only \texttt{end} when it holds the token, at which point it will not pass it on until another \texttt{begin} action occurs.

Each notifier is a copy of an automaton $N$, which has a boundary of type $C$, and two boundaries of type $G = \{\text{go, go}\}$ over which they will synchronize with each other. The notifier is shown in figure 11 – and edge labelled $(a|b|c)$ indicates the action $a$ on the left boundary $G$, the action $b$ on the lower boundary $C$, and the action $c$ on the right boundary $G$.

Beginning in state 0, a notifier waits for a \texttt{go} from its left hand side. It \texttt{begin}s its process, and then passes the \texttt{go} to its right side. At this point, it waits to synchronize with a \texttt{go} from its left and a \texttt{end} from its process, before allowing the process to start again.

The master $M$ likewise has a boundary of type $C$, and two boundaries of type $G$. It is shown in figure 12 with the same labelling convention as the notifier.

When the master receives a \texttt{begin} on its control boundary $C$, it sends a \texttt{go} to the right, and then waits for a \texttt{go} on its left. After receiving the \texttt{go}, it either
passes it on or ends, at which point it must begin again before passing go to its right.

The scheduler proper is then constructed by composing \( n \) notifiers and 1 master in a cycle. The resultant automaton has \( n + 1 \) boundaries of type \( C \). The \( n \) boundaries arising from the notifiers are connected to the processes we wish to control. The remaining boundary, arising from the master, is connected to the external control mechanism. Figure 13 shows the design of the final composite in the case \( n = 3 \).

In order to analyse this system below, we shall close the final boundary by binding a controlling process (an automaton of the form of \( P \)) to the remaining boundary of this design.

5.1.2 Analysis of the Scheduler

We shall now briefly indicate how one could analyze the scheduler system using the methodology described in this paper.

Consider the design of the system as shown in figure 13, and with an additional controlling process \( P \) bound to the remaining boundary.

We begin by evaluating the binding \( N \cdot P \). We shall not draw this binding in full, but we note that it is simulated by the automaton \( Q \) shown in figure 14.

Thus the binding \( N \cdot P \) is simulated by a system which performs the go action on its boundaries alternately. The image on the initial state under the simulation is the state 0 in the automaton \( Q \).

Now consider the master system. When bound with a controlling process of the form of \( P \), the resultant system is also simulated by \( Q \), although this time with initial state having image 1 in \( Q \).

Thus the evaluation of the system of figure 13 is simulated by a ring of \( n + 1 \) copies of \( Q \), with initial state having precisely one copy of \( Q \) in the local state 1.
Figure 12: The master $M$

It is easy to see the evaluation of such a system is isomorphic to the automaton with states length $n + 1$ cyclic strings of 0s and 1s, and motions being those which replace a 10 substring with a 01 substring. Consider the subautomaton reachable from an initial state being a cyclic string containing precisely 1 local state of 1: This is clearly a (graph theoretic) cycle of $n + 1$ states. Hence the system is deadlock free.

By keeping track of the simulations indicated in the preceding discussion, one can deduce more from the constructed simulation, including the desired behaviour in terms of the order of entering the controlled section for each controlled process.

5.2 Communication Protocols

5.2.1 Notation

For this section, we shall introduce an abbreviated notation for drawing automata which allows for easier depiction of automata where many states are similar.

As described here, the notation merely gives a compact representation of certain automata of interest. However, the authors intend to more fully explore this notation, with a view to allowing specification of automata using abstract data types via the interpretations of [21] and [5].

Let us consider an automaton $X \rightarrow Y$. Our pictures will be graphs, with the following additional data:

1. associated to each vertex is a given a set $V$. We typically abuse notation by denoting and referring to the vertex as $V$, provided no confusion arises.

2. associated to each edge $V \rightarrow W$ is a subset of $X \times V \times W \times Y$. We shall denote the subset by a label $(x(i)|v(i) \rightarrow w(i)|y(i))$ where $i$ ranges over some (typically implicit) indexing set.
Figure 13: The design of a three process scheduler

Figure 14: An automaton $Q$ simulating $N \cdot P$

An automaton is associated with such a graph as follows: A vertex denoted $V$ indicates a set of states indexed by $V$; an edge labelled $(x(i)v(i) \rightarrow w(i)y(i))$ indicates a family of motions $v(i) \rightarrow w(i)$ labelled $x(i)$ on the boundary $X$ and $y(i)$ on the boundary $Y$.

For example, given a set $M$ of messages, let us write $M^-$ for the boundary obtained by adjoining a trivial action to $M$. Figure 15 shows an automaton with boundaries $X = M^-$ and $Y = M^-$. The automaton has $M + 1$ states, and nontrivial motions of two kinds:

1. from the lone state of 1 to each state of $M$, this motion being labelled by the target state on the left boundary, and $-$ on the right boundary,

2. from each state of $M$ to the lone state of 1, this motion being labelled by $-$ on the left boundary and the source state on the right boundary.

Such an automaton is a simplistic delayed message passer – it synchronizes with its left boundary to obtain $m \in M$ (storing it internally by moving to an appropriate state), and then synchronizes with its right boundary to pass $m$ on (and forgetting the $m$ in the process). Considering the definition of binding of
automata, we note that this automaton is incapable of losing messages – bound systems synchronize on the boundary actions which represent passing/receiving a message to/from the automaton.

\[
\begin{array}{ccc}
M^- & 1 & M \\
(m \rightarrow m & & (m \rightarrow m \\
\rightarrow m & & \rightarrow m)
\end{array}
\]

Figure 15: A message passer automaton

5.2.2 Channels

By a channel \( C \) of type \( M \), for a given set of messages \( M \), we mean an automaton \( C: M^- \rightarrow M^- \). We have in mind that an action \( m \) being performed on the left boundary is sending the message \( m \) down the channel, and some time later the right boundary will perform the action \( m \) as the message emerges. However, we do not require these properties of a channel, as we wish to model channels which may lose, modify or reorder messages.

Given the above discussion of the message passer, we note that a synchronization with a channel is considered to be a tightly coupled interaction whereby the channel accepts a message from its boundary. Synchronization occurs when a message is transferred across the boundary. That is, our I/O is fundamentally blocking I/O.

Non-blocking I/O is modelled by having a automaton which can receive messages in any (or almost any) state. We note that this is an accurate model of non-blocking I/O. Such I/O is not distinguished in that it does not synchronize, but rather in that it synchronizes locally – that is, with lower layer processes in the local communication library rather than with a distant system.

To reconcile the tightly coupled nature of the synchronization in the binding operation with our desire to model channels which lose messages, we construct channels which literally lose messages – it is a property of the channel that a message which enters it may not emerge. By explicitly modelling that part of the system that loses messages, we can provide precise analyses of whether or not certain protocols lose messages.

Such a channel is shown in figure 16. This is a channel of type \( M \), which we shall refer to as a capacity 1 channel. If the channel is empty (in the state 1), an input transition of \( m \) results in the message \( m \) being stored (in one of the states \( M \)). This can later be read by an output transition, and the channel returns to the empty state. Any input messages supplied to the channel while it is full are simply lost.

Precisely speaking, we may consider any sequence \( \sigma \) of actions on the left boundary of the automaton. For any behaviour \( \beta \) of the automaton with reduced appearance on the left boundary being the given sequence \( \sigma \), the reduced
appearance on the right boundary is a subsequence of $\sigma$. Further, there exists a behaviour $\beta$ for which the reduced appearance on the right boundary is precisely the sequence $\sigma$.

5.2.3 Protocols

By a protocol $P$ of type $M$ implemented on a channel of type $N$, we mean a pair $(S, R)$ of automata, called the sender $S: M^- \rightarrow N^-$ and the receiver $R: N^- \rightarrow M^-$. Given a channel $C$ of type $N$, we can construct the channel $S \cdot C \cdot R$.

This latter channel is what is usually termed the virtual channel provided by the protocol. Given the definitions of this paper, it is in fact a channel, no less real for the fact it is built from simpler automata. The authors suggest the term “designed channel” to distinguish the latter channel from the former. A composite of this kind is shown in figure 17 – the term virtual channel arises by thinking of the dotted line in the upper diagram as a direct connection; the author’s point of view is that the dotted box in the lower diagram shows a designed channel constructed from the protocol and underlying channel.

One goal of protocol design is to construct the automata $S$ and $R$ in such a way that this virtual channel has better properties than the underlying channel $C$. Typically, we wish to show that given certain properties of the channel $C$, the virtual channel $S \cdot C \cdot R$ has certain other properties.

More generally, we may state the problem of protocols as follows: Given a family of channels $C_i$, a desired channel type $M$, and requirements on the behaviours of the desired virtual channel, we need to construct automata $S: M^- \rightarrow N^-$ and $R: N^- \rightarrow M^-$ such that the channel $S \cdot C \cdot R$ has the desired properties for some $C: N \rightarrow N$ selected from our family $C_i$. Note that the family $C_i$ models “the sorts of channels available to us at this level of abstraction”, and would typically be described as the closure under certain operations of certain basic channels – for example, any channel which is a product of capacity 1 channels of any type.

5.2.4 Message Acknowledgement

Given that the capacity 1 channel can lose messages, we might ask to establish a virtual channel solving this problem. One solution is to acknowledge sent messages. We shall use a channel from the receiver to the sender of type $A = \{\text{ack}\}$ to carry the acknowledgements. That is, we shall build a virtual channel of type $M$ from a channel of type $N = M \times A + M + A$. Note that the type $N$
of this channel should be thought of as the product of the types $M$ and $A$ in the sense that $N^{-} = M^{-} \times A^{-}$.

The sender and receiver automata $S$ and $R$ are shown in figures 18 and 19 respectively.

One can easily evaluate the binding $S \cdot R$ to determine the behaviour of the message acknowledgement protocol over a perfect channel.

However, we wish to analyze the protocol over a pair of capacity 1 channels running in opposite directions – a channel of type $M$ from $S$ to $R$ and a channel of type $A$ from $R$ to $S$. The design of the system we wish to analyze is shown in figure 20.

Given an automaton $S: X \rightarrow Y$, the automaton $S^{\text{op}}: Y \rightarrow X$ is constructed by interchanging the boundaries – we may call $S^{\text{op}}$ as the opposite of $S$. Thus the channel of interest in this context is the product $C_{M} \times C_{A}^{\text{op}}$ of a pair of capacity 1 channels of type $M$ and and $A$ respectively (running in opposite
Our goal now is to explain why the channel so constructed meets the design goals - that is, has only behaviours which have identical reduced appearances on each boundary. We do this by evaluating the design of figure 20 - the reachable part is shown in figure 21. It is clear that this channel is simulated by the message passers of figure 15 - map the states in the top row to the unique state of 1 in figure 15, and map the states in the bottom row to the corresponding states of $M$ in figure 13. Proposition 3 of section 4.2 now provides the desired result.

What happens if the channel of type $M$ being used is not capacity 1, but may in fact lose messages arbitrarily. Such a channel would be modelled by an automaton similar to that of figure 16 but with an additional transition labelled $(|\leq m|s \rightarrow |\leq m|)$ from the state 1 to itself. One can readily evaluate the design of figure 20 using this channel in place of $C_M$, and with a correspondingly modified channel in place of $C_A$. The result is the automaton of figure 22. In
this case the system deadlocks if a message or an acknowledgement is lost.

Figure 22: Message Acknowledgement Protocol over lossy channels

To repair this defect one typically uses timeouts and retransmission, but the analysis of such protocols, while of direct interest to the authors, is beyond the goals of the current paper. It is observed however, that timeouts could be modelled in the theory of automata with boundary by using timeout motions in the sender and receiver automata.

6 Conclusions and Future Directions

We have presented the basic theory of automata with boundary, together with examples designed to elucidate the presentation and show the scope of the theory described here. It is important to reiterate that the theory provides an algebra for constructing systems from primitive elements. One of the crucial aspects of this theory is the attempt to capture the design of a system as a precise
theoretical element, distinct from the system itself and its implementation.

The underlying mathematical formalism of the approach has been explored in [3], [9], [13], [10], and [12], and the interested reader is referred there. There is still work to be done in clarifying some details of the mathematics appropriate to the model, for example [6] and other papers in preparation by the authors. We note also a precise description of the application of the bicategory \text{Span(Graph)} to the domain of asynchronous circuit design is given in [14] and [23].

We have proposed an algorithm for model checking systems for deadlock which fits comfortably with the theory, and illustrates the principle that incorporation of designs as an element of the theory has benefits in other areas. As noted at the conclusion of section 3.3.1, this algorithm is simplistic and does not always perform well – more work in understanding the applications, limitations and possible evolution of algorithms based on these ideas is clearly warranted.

The theory supports abstraction of automata via the notions of comparison and simulation. The algebra used to construct systems extends to an algebra including the abstraction mechanisms, facilitating the construction of abstractions of larger systems. While the authors are still investigating the use of this technique, some indication of the benefits this approach yields are seen in section 4.3: abstractions may be used in conjunction with model checking to check larger systems; and the compositionality of the abstractions allows theoretical checking of families of systems.

In addition, the authors note that the combinatorial nature of the theory presented here makes it ideal for machine manipulation. As mentioned in section 4.3 the authors are presently prototyping tools designed to facilitate calculation in the algebra presented in this paper. It is hoped that such tools will allow calculation with larger models, such as several layers of a multilayer network protocol, both to demonstrate the applicability of the theory and to further refine the ideas presented in the current work.

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