Rates of convergence for Smoluchowski’s coagulation equations

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Abstract

We establish nearly optimal rates of convergence to self-similar solutions of Smoluchowski’s coagulation equation with kernels \( K = 2, x + y, \) and \( xy. \) The method is a simple analogue of the Berry-Esséen theorem in classical probability and requires minimal assumptions on the initial data, namely that of an extra finite moment condition. For each kernel it is shown that the convergence rate is achieved in the case of monodisperse initial data.

Keywords: coagulation, coarsening, dynamic scaling, self-similarity, Berry-Esséen theorem

1 Introduction

Smoluchowski’s coagulation equation

\[ \frac{\partial n(t,x)}{\partial t} = \frac{1}{2} \int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy \]

is a fundamental mean-field model for cluster growth that arises in a wide range of fields, including physical chemistry, astrophysics, and the dynamics of biological systems (see [2] for a review). Here, \( n(t,x) \) is the density of the number distribution of clusters of size \( x \in (0, \infty) \) at time \( t \geq 0 \) and \( K(x,y) \) is a symmetric rate kernel. In the case of discrete sizes \( l \in \{1, 2, \cdots \} \) an analogous equation for the coefficients \( n_l(t) \) of the number distribution is

\[ \frac{\partial n_l(t)}{\partial t} = \frac{1}{2} \sum_{j=1}^{l-1} \kappa_{l-j,j} n_{l-j}(t)n_j(t) - \sum_{j=1}^\infty \kappa_{l,j} n_l(t)n_j(t), \]

with \( \kappa_{l,j} = K(l,j). \) The continuous and discrete cases can be considered together via the weak formulation of Smoluchowski’s coagulation equation, given in terms of a moment identity for the number distribution \( n(t, dx) \):

\[ \frac{\partial}{\partial t} \int_{(0, \infty)} \phi(x)n(t, dx) = \frac{1}{2} \int_{(0, \infty)} \int_{(0, \infty)} (\phi(x+y) - \phi(y) - \phi(x))K(x,y)n(t, dy)n(t, dx), \]

where \( \phi \) is a suitable test function. We direct the interested reader to [7, 8] for further discussion on the well-posedness and asymptotic behavior of measure-valued solutions \( n(t, dx) \)

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to (1.3). A general survey of existing literature in stochastic coalescence is given in [2], with more recent work reviewed in [9]. Note that throughout this paper we try to keep the same notation as in [8].

The present work is restricted to the homogeneous, ‘solvable’ kernels $K = 2, x + y, xy$, for which $K(\alpha x, \alpha y) = \alpha^\gamma K(x, y)$ with $\gamma = 0, 1, 2$. It has been shown in [7] that (1.3) admits a one-parameter family of self-similar solutions whose domains of attraction under dynamic scaling are characterized by the tails of the initial data. In particular, there is a unique self-similar solution with finite $(\gamma + 1)$th moment, which has exponentially decaying tails and attracts all initial data that satisfy this finite moment condition. In [8], it was shown that with an additional integrability hypothesis on the Fourier transform of the initial data one has uniform convergence of densities (in the continuous case) or coefficients (in the discrete case) to a self-similar solution as $t$ approaches the time horizon $T_\gamma$, where $T_\gamma = \infty$ for $\gamma = 0, 1$ and $T_\gamma = 1$ for $\gamma = 2$. The proof of this result is similar to that of uniform convergence of densities in the central limit theorem (see Feller [6], Section XV.5).

Here we establish near-optimal $L^\infty$-rates of convergence to the exponentially decaying self-similar solutions of Smoluchowski’s coagulation equation with $K = 2, x + y, xy$. Continuing the analogy with the classical central limit theorem (CLT), this result corresponds to the Berry-Esseen theorem for rates of convergence to the normal law ([6], XVI.5). The method is simple and robust: it holds for all initial distributions, requiring only an additional finite moment condition beyond those needed for well-posedness and convergence to the self-similar profile. In particular, it is true when the initial distribution has a density or is a lattice measure. Our work improves upon recent results of Cañizo, Mischler, and Mouhot [4] as it holds for a large class of initial data and for all of the solvable kernels. More broadly, it once again demonstrates the utility of applying methods from classical probability to study asymptotic behavior under rescaling for integral equations of convolution type. Additional estimates that make use of the present work, such as large deviations theorems, will be developed elsewhere.

We present the results for the solvable kernels in a unified framework. With $t_0 = 1$ for $\gamma = 0$, $t_0 = 0$ for $\gamma = 1, 2$, and $n_0(dx) = n(t_0, dx)$, define the moments $\mu_j = \int_{(0, \infty)} x^j n_0(dx)$ of the initial data. Assume that the $\gamma$th and $(\gamma + 1)$th moments are finite, for which we scale $x$ and $n_0$ so that $\mu_\gamma = \mu_{\gamma+1} = 1$. As shown in [8], finiteness of the $\gamma$th moment ensures well-posedness of (1.3), while finiteness of the $(\gamma + 1)$th moment guarantees that the initial data are in the domain of attraction of the self-similar solution with exponential decaying tails. These self-similar solutions are absolutely continuous and are explicitly given in terms of their densities (see [2, 7, 8]) by

\begin{equation}
(1.4)
n(t, x) = \frac{m_\gamma(t)}{\lambda_\gamma(t)\gamma+1} \hat{n}_{*,\gamma} \left( \frac{x}{\lambda_\gamma(t)} \right).
\end{equation}

Here, the profiles $\hat{n}_{*,\gamma}$ for $\hat{x} \geq 0$ take the form

\begin{equation}
(1.5)
\hat{n}_{*,0}(\hat{x}) = e^{-\hat{x}}, \quad \hat{x}\hat{n}_{*,1}(\hat{x}) = \hat{x}^2 \hat{n}_{*,2}(\hat{x}) = \frac{1}{\sqrt{2\pi}} \hat{x}^{-1/2} e^{-\hat{x}/2}
\end{equation}

and the time-dependent moments $m_\gamma$ and scalings $\lambda_\gamma$ are

\begin{equation}
(1.6)
m_0(t) = t^{-1}, \quad m_1(t) = 1, \quad m_2(t) = (1 - t)^{-1}
\end{equation}

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where 

\[ C(n) \]  

Then for data we demonstrate that exponential decay rates are achieved in the case of monodisperse initial we make no effort to optimize the constant in front of the exponential term. We also define the corresponding distribution functions

\[ (1.7) \]

\[ \lambda_0(t) = t, \quad \lambda_1(t) = e^{2t}, \quad \lambda_2(t) = (1 - t)^{-2}. \]

The main result of our work is as follows. Define the time parameter \( \tau_\gamma(t) = \int_0^t m_\gamma(s)ds \), explicitly given by

\[ (1.8) \]

\[ \tau_0(t) = \log t, \quad \tau_1(t) = t, \quad \tau_2(t) = \log(1 - t)^{-1}. \]

If we assume finiteness of the \((\gamma + 2)\)th moment of the initial data, we have exponentially fast convergence to self-similar form:

**Theorem 1.1.** Let \( n_0 \) be a positive measure with \( \int (0, \infty) x^\gamma n_0(dx) = \int (0, \infty) x^{\gamma + 1}n_0(dx) = 1 \). Suppose that the additional finite moment assumption \( \int (0, \infty) x^{\gamma + 2}n_0(dx) < \infty \) holds and let \( n(t, dx) \) be the measure-valued solution to Smoluchowski’s equation with \( K = 2, x + y, \) or \( xy \) and initial data \( n_0(dx) \). With rescaled solution

\[ (1.9) \]

\[ \hat{n}(\tau_\gamma, d\hat{x}) := \frac{\lambda_\gamma(t)^{\gamma}}{m_\gamma(t)} n(t, \lambda_\gamma(t)d\hat{x}) \]

define the corresponding distribution functions

\[ (1.10) \]

\[ F_\gamma(\tau_\gamma, \hat{x}) = \int_{(0, \hat{x})} \hat{y}^\gamma \hat{n}(\tau_\gamma, d\hat{y}), \quad F_\gamma^{\ast}(\hat{x}) = \int_{(0, \hat{x})} \hat{y}^\gamma \hat{n}_\gamma^{\ast}(\hat{y})d\hat{y}. \]

Then for \( \tau_\gamma \in [0, \infty), \)

\[ \sup_{\hat{x} > 0} |F_\gamma(\tau_\gamma, \hat{x}) - F_\gamma^{\ast}(\hat{x})| \leq C(\mu_{\gamma + 2})(1 + \tau_\gamma)e^{-\tau_\gamma} \]

where \( C(\mu_{\gamma + 2}) \) is a constant that depends only on \( \mu_{\gamma + 2} \).

Since we are primarily concerned with an asymptotic rate of decay to self-similarity, we make no effort to optimize the constant in front of the exponential term. We also demonstrate that exponential decay rates are achieved in the case of monodisperse initial data \( n_0(dx) = \delta_1(dx) \). Specifically, we show that for these initial conditions

\[ \sup_{\hat{x} > 0} |F_\gamma(\tau_\gamma, \hat{x}) - F_\gamma^{\ast}(\hat{x})| = O(e^{-\tau_\gamma}). \]

The method of proof for these results requires us to work in the Fourier domain. To do so we make use of a smoothing argument given by Feller [6], XVI.3, which for completeness we restate here without proof. The distribution functions in (1.10) satisfy the assumptions of the lemma.

**Lemma 1.2.** Let \( F \) and \( F_\gamma \) be probability distribution functions and assume \( |F_\gamma'(x)| \leq 1 \). Consider the Fourier transforms of their corresponding measures,

\[ u(ik) = \int_{(-\infty, \infty)} e^{-ikx}F(dx), \quad u_\gamma(ik) = \int_{(-\infty, \infty)} e^{-ikx}F_\gamma(dx). \]

Assume that \( u'(0) = u_\gamma'(0) \) —that is, equality of the first moment of \( F(dx) \) and \( F_\gamma(dx) \). With the mollifier

\[ \psi_T(x) = \frac{1 - \cos(Tx)}{\pi Tx^2} \]

\[ 3 \]
define
\[ \Delta(x) = F(x) - F_s(x), \quad \Delta_T(x) = \psi_T(x) * \Delta. \]

Then
\[ (1.11) \quad \sup_x |\Delta(x)| \leq 2 \sup_x |\Delta_T(x)| + \frac{24}{\pi T}. \]

In terms of Fourier transforms, (1.11) is
\[ (1.12) \quad \sup_x |F(x) - F_s(x)| \leq \frac{1}{\pi} \sup_x \left| \int_{-T}^{T} e^{ikx} (u(ik) - u_*(ik)) \, dk \right| + \frac{24}{\pi T}. \]

In the following sections, we will need to approximate Fourier transforms near the origin by their Taylor expansions. To do so, we will use the following basic result also due to Feller (see [6], XV.4 and the subsequent appendix). Let \( C \) be the open left- and right-half of the complex plane, and denote by \( C_+ \) and \( C_- \) their respective closures. Let \( F(dx) \) be a probability measure supported on \([0, \infty)\). \( F(dx) \) is then characterized by its Fourier-Laplace transform
\[ (1.13) \quad u(s) = \int_{[0, \infty)} e^{-sx} F(dx), \quad s \in \mathbb{C}_+, \]
which is approximated as follows:

**Lemma 1.3.** Suppose \( F(dx) \) is a probability measure on \([0, \infty)\) with finite \( \gamma \)th moment
\[ (1.14) \quad M_\gamma := \int_{[0, \infty)} x^\gamma F(dx) < \infty, \]
where \( \gamma \geq 1 \) is an integer. Then for \( \xi, s \in \mathbb{C}_+ \), the Fourier-Laplace transform (1.13) satisfies
\[ (1.15) \quad \left| u(\xi + s) - \left( u(\xi) + u'(\xi) s + \cdots + u^{(\gamma-1)}(\xi) \frac{s^{\gamma-1}}{\gamma!} \right) \right| \leq \frac{M_\gamma}{\gamma!} |s|^\gamma. \]

**Proof.** We begin by showing that for \( s \in \mathbb{C}_- \),
\[ (1.16) \quad \left| e^s - \left( 1 + s \frac{1}{1!} + \cdots + s^{\gamma-1} \frac{1}{(\gamma-1)!} \right) \right| \leq \frac{|s|^\gamma}{\gamma!}. \]

Define \( g_1(s) = \int_0^s e^z \, dz \) and \( g_\gamma(s) = \int_0^s g_{\gamma-1}(z) \, dz \), where the integrals are taken over the straight line segment beginning at the origin and ending and \( s \). A simple calculation shows that \( g_\gamma(s) = e^s - \left( 1 + \frac{s}{1!} + \cdots + \frac{s^{\gamma-1}}{(\gamma-1)!} \right) \). Then (1.16) follows by induction. To see this, note that
\[ |g_1(s)| = \left| s \int_0^1 e^{\tau s} \, d\tau \right| \leq |s|, \quad s \in \mathbb{C}_- \]
since \( |e^{\tau s}| \leq 1 \). Assuming \( |g_{\gamma-1}(s)| \leq \frac{|s|^{\gamma-1}}{(\gamma-1)!} \), we obtain the desired estimate:
\[ |g_\gamma(s)| = \left| s \int_0^1 g_{\gamma-1}(\tau s) \, d\tau \right| \leq \frac{|s|^\gamma}{(\gamma-1)!} \int_0^1 \tau^{\gamma-1} \, d\tau = \frac{|s|^\gamma}{\gamma!}. \]
Equation (1.15) now follows directly. Using (1.16) and the definition (1.14) yields
\[
\left| u(\xi + s) - \left( u(\xi) + \frac{u'(\xi)}{1!} s + \cdots + \frac{u^{(\gamma-1)}(\xi)}{(\gamma-1)!} s^{\gamma-1} \right) \right| \\
= \left| \int_{[0,\infty)} e^{-\xi x} \left\{ e^{-sx} - \left( 1 - \frac{sx}{1!} + \cdots + (-1)^{\gamma-1} \frac{s^{\gamma-1}}{(\gamma-1)!} \right) \right\} F(dx) \right| \leq \frac{M_\gamma}{\gamma!} |s|^\gamma.
\]

Taking \( \xi = 0 \) in (1.15) and using that \( M_\gamma = (-1)^\gamma u^{(\gamma)}(0) \), gives the particularly useful expression
\[
(1.17) \quad \left| u(s) - \left( 1 - \frac{M_1}{1!} s + \cdots + (-1)^{\gamma-1} \frac{M_{\gamma-1}}{(\gamma-1)!} s^{\gamma-1} \right) \right| \leq \frac{M_\gamma}{\gamma!} |s|^\gamma, \quad s \in \mathbb{C}_+.
\]

Let us outline the remainder of the paper. We first prove Theorem 1.1 in the case of constant coagulation kernel in Section 2, where the method of proof is simpler and more easily applied. The same method is then used in Section 3 to establish the result for the additive kernel. Finally, the result for the multiplicative kernel is given in Section 4 by transforming to the case of additive kernel through a well-known change of variables.

2 Rate of convergence for the constant kernel \( K = 2 \)

In this section we restate and prove Theorem 1.1 for the constant kernel case. That is, we show:

**Theorem 2.1.** Let \( n_0 \) be a positive measure such that \( \int_{[0,\infty)} n_0(dx) = \int_{[0,\infty)} x n_0(dx) = 1 \), and \( \mu_2 = \int_{[0,\infty)} x^2 n_0(dx) < \infty \). With \( \tau = \log t \) and in terms of the rescaling (1.6-1.7), let \( \hat{n}(\tau,d\hat{x}) \) be the rescaled solution (1.9) to Smoluchowski’s equation with \( K = 2 \) and initial data \( n_0(dx) \). Then with
\[
F(\tau,\hat{x}) = \int_{[0,\hat{x}]} \hat{n}(\tau,d\hat{y}), \quad F_*(\hat{x}) = \int_{[0,\hat{x}]} \hat{n}_*(0,d\hat{y})
\]
and \( \tau \in [0,\infty) \),
\[
(2.1) \quad \sup_{\hat{x} > 0} |F(\tau,\hat{x}) - F_*(\hat{x})| \leq C(\mu_2)(1 + \tau)e^{-\tau}
\]
where \( C(\mu_2) \) is a constant that depends only on \( \mu_2 \).

2.1 Proof for the constant kernel

The proof is can be summarized as follows. For \( s \in \mathbb{C}_+ \) define the Fourier-Laplace transforms of the rescaled number measures as
\[
u_1(s) = \int_{[0,\infty)} e^{-s\hat{x}} \hat{n}(\tau,d\hat{x}), \quad \nu_*(s) = \int_{[0,\infty)} e^{-s\hat{x}} \hat{n}_*(0,d\hat{x})
\]
and let $u_0(s) = u(0, s)$. By (1.12), we have that

\begin{equation}
\sup_{\hat{x} > 0} |F(\tau, \hat{x}) - F_s(\hat{x})| \leq \frac{1}{\pi} \sup_{\hat{x} > 0} \left| \int_{-iT}^{iT} e^{\sigma \hat{x}} (u(\tau, \sigma) - u_*(\sigma)) d\sigma \right| + \frac{24}{\pi T}
\end{equation}

The uniform rate of convergence of the distribution function is thus given by the convergence rate of $u(\tau, s)$ to $u_*(s)$. As shown in [7, 8], $u$ satisfies a first-order PDE that can be solved in terms of $u_0$ by the method of characteristics in $\mathbb{C}_+$. The finiteness condition on $\mu_2$ is a statement about the decay of the tail of $\hat{n}_0$, and hence, about the regularity of $u_0$ near the origin. Using (1.17) to estimate the difference $u_0(s) - u_*(s)$ near $s = 0$, we can propagate this estimate outward at the growth rate of the characteristics. Combining this with Lemma 1.2 gives us the desired result.

1. Evolution of characteristics: As shown in [7], the Fourier-Laplace transform $u(\tau, s)$ solves the PDE

\begin{equation}
\partial_\tau u + s \partial_s u = -u(1 - u).
\end{equation}

This equation can be derived directly from (1.3), or from (1.1) if the initial number measure has a density as in [8]. Fix an initial point $s_0 \in \mathbb{C}_+$. The solution to (2.3) is given in terms of characteristics $s(\tau; 0, s_0)$ starting from $s_0$ at $\tau = 0$. The characteristics satisfy

\begin{equation}
\frac{ds}{d\tau} = s, \quad s(0; 0, s_0) = s_0 \in \mathbb{C}_+,
\end{equation}

whose solution is

\begin{equation}
s(\tau; 0, s_0) = e^\tau s_0 \in \mathbb{C}_+.
\end{equation}

Note that the characteristics are independent of the initial data $u_0$. Geometrically, they are rays emanating from the origin at speed $e^\tau$. In particular, the map $s_0 \mapsto s$ leaves the imaginary axis invariant. Along characteristics we have

\begin{equation}
\frac{du}{d\tau} = -u(1 - u),
\end{equation}

which can be integrated to yield the solution in terms of $s_0$:

\begin{equation}
u(\tau, s) = e^{-\tau} \frac{u_0(s_0)}{1 - u_0(s_0)(1 - e^{-\tau})}.
\end{equation}

The Fourier-Laplace transform of the self-similar solution,

\begin{equation}
u_*(s) = \frac{1}{1 + s},
\end{equation}

can also be expressed in terms of characteristics. Fixing $s_0^* \in \mathbb{C}_+$, the characteristics $s^*(\tau; 0, s_0^*)$ corresponding to the solution of (2.3) with initial data $u_*$ are

\begin{equation}s^*(\tau; 0, s_0^*) = e^\tau s_0^* \in \mathbb{C}_+.
\end{equation}
These characteristics are identical to (2.5), as expected. Since \( u_\ast(s) \) is a time-independent solution to (2.3),

\[
(2.9) \quad u_\ast(s) = e^{-\tau} \frac{u_\ast(s_\ast_0)}{1 - u_\ast(s_\ast_0)(1 - e^{-\tau})}.
\]

2. Estimates near the origin: Now we utilize the additional finite moment assumption

\[
\mu_2 := \int_{(0,\infty)} \hat{x}^2 \hat{n}_0(dx) = \int_{(0,\infty)} x^2 n_0(dx) < \infty
\]

and that

\[
2 = \int_{(0,\infty)} x^2 \hat{n}_\ast(x)dx = \int_{(0,\infty)} x^2 \hat{n}_\ast(x)dx
\]

to obtain approximations for \( u_0 \) and \( u_\ast \), respectively, near the origin. By (1.17),

\[
(2.10) \quad |u_0(s) - (1 - s)| \leq \frac{\mu_2}{2} |s|^2, \quad |u_\ast(s) - (1 - s)| \leq |s|^2, \quad s \in \mathbb{C}_+.
\]

We can also estimate \( u(\tau, s) \) near the origin using the time-dependent moment

\[
\hat{m}_2(\tau) = \int_{(0,\infty)} \hat{x}^2 \hat{n}(\tau, dx) = \int_{(0,\infty)} x^2 n(e^\tau, dx).
\]

From (1.3) (or (1.1) if \( n(t, dx) \) has a density) it follows that \( \partial_t \hat{m}_2(\tau) = 2 - \hat{m}_2(\tau) \), so

\[
\hat{m}_2(\tau) = 2 + (\mu_2 - 2)e^{-\tau}.
\]

Therefore, (1.17) implies that for all \( \tau \geq 0 \),

\[
(2.11) \quad |u(\tau, s) - (1 - s)| \leq \frac{\hat{m}_2(\tau)}{2} |s|^2 \leq \frac{\mu_2}{2} |s|^2, \quad s \in \mathbb{C}_+.
\]

3. Propagation of estimates: Consider the r.h.s. of (2.2) with \( T = \delta e^\tau \), where \( \delta > 0 \) is a length scale characterizing the region about the origin where the difference between \( u_0 \) and \( u_\ast \) is small by (2.10). Explicitly, define

\[
(2.12) \quad 2\delta = \sqrt{1 + 2 \left(1 + \frac{\mu_2}{2}\right)^{-1} - 1}
\]

so that \( (1 + \mu_2/2) \delta = (1 + \delta)^{-1}/2 \). As expected, if \( \mu_2 \to \infty \) then \( \delta \to 0 \).

4. Backward characteristics: Denote the common set of characteristics for \( u \) and \( u_\ast \) by \( \sigma(\tau; 0, \sigma_0) \) with \( \sigma_0 \in \mathbb{C}_+ \). Now define the backward characteristic \( \sigma_0(\tau, \sigma_0) \) as the inverse of the forward mapping \( \sigma_0 \mapsto \sigma(\tau; 0, \sigma_0) \). That is, for \( \omega \in \mathbb{C}_+ \),

\[
(2.13) \quad \sigma_0(\tau, \omega) = \sigma(0; \tau, \omega)
\]

Also note that by (2.5) or (2.8), the Jacobian of the transformation \( \sigma_0 \mapsto \sigma \) is

\[
(2.14) \quad \frac{d\sigma}{d\sigma_0} = e^{\tau}.
\]
We now estimate (2.2) by considering separately the integral near and away from the origin.

5. Bounds for $0 \leq |\sigma| \leq \delta e^{-\tau}$: Applying (2.10) and (2.11), it is simple to see that

\[
\left| \int_{0 \leq |\sigma| \leq \delta} e^{\sigma \hat{x}} \frac{e^{\sigma \hat{x}}}{|\sigma|} (u(\tau, \sigma) - u_*(\sigma)) d\sigma \right| \leq 2 \int_{0}^{\delta} \frac{1}{|\sigma|} \left| u(\tau, \sigma) - u_*(\sigma) \right| d|\sigma| \\
\leq \left( 1 + \frac{\mu_2}{2} \right) \delta^2 e^{-2\tau}.
\]

(2.15)

6. Bounds for $\delta e^{-\tau} \leq |\sigma| \leq \delta e^\tau$: Substituting (2.6), (2.9) into the r.h.s. of (2.2) and changing variables to $\sigma_0$ with (2.5) and (2.14) we have that

\[
\int_{\delta e^{-\tau} \leq |\sigma| \leq \delta e^\tau} e^{\sigma \hat{x}} \frac{e^{\sigma \hat{x}}}{\sigma} (u_0(\sigma_0) - u_*(\sigma_0)) d\sigma_0 \\
= \int_{\delta e^{-2\tau} \leq |\sigma_0| \leq \delta} \frac{e^{\sigma \hat{x};0,\sigma_0} \hat{x}}{\sigma(\tau;0,\sigma_0)} (1 - u_*(\sigma_0)(1 - e^{-\tau}))(1 - u_0(\sigma_0)(1 - e^{-\tau})) d\sigma_0
\]

Since $\sigma_0$ is pure imaginary in the region of integration, (2.7) implies

\[
|1 - u_*(\sigma_0)(1 - e^{-\tau})| = \left| \frac{e^{-\tau} + \sigma_0}{1 + \sigma_0} \right| \geq \frac{|\sigma_0|}{1 + \delta}
\]

and by (2.10),

\[
|1 - u_0(\sigma_0)(1 - e^{-\tau})| \geq |1 - u_*(\sigma_0)(1 - e^{-\tau})| - \left( 1 + \frac{\mu_2}{2} \right) |\sigma_0|^2 \geq \frac{1}{2} \cdot \frac{|\sigma_0|}{1 + \delta}.
\]

Therefore,

\[
\left| \int_{\delta e^{-2\tau} \leq |\sigma_0| \leq \delta} \frac{e^{\sigma \hat{x};0,\sigma_0}}{\sigma(\tau;0,\sigma_0)} (1 - u_*(\sigma_0)(1 - e^{-\tau}))(1 - u_0(\sigma_0)(1 - e^{-\tau})) d\sigma_0 \right| \\
\leq 4 \left( 1 + \frac{\mu_2}{2} \right) (1 + \delta)^2 e^{-\tau} \int_{\delta e^{-2\tau}}^{\delta} \frac{1}{|\sigma_0|} d|\sigma_0| \\
= 8 \left( 1 + \frac{\mu_2}{2} \right) (1 + \delta)^2 \tau e^{-\tau}.
\]

(2.17)

7. Rate of convergence: Substituting (2.15) and (2.17) into (2.2), we obtain the desired result (2.1).

2.2 Near-optimality of the rate for $K = 2$

We demonstrate that the exponential rate is achieved for monodisperse initial data. Using the explicit solution given in [2] the solution with $K = 2$ and initial data $n_0(dx)$ is

\[
n(t, dx) = \sum_{j=1}^{\infty} t^{-2} \left( 1 - \frac{1}{t} \right)^{j-1} \delta_j(dx), \quad t \in [1, \infty).
\]

(2.18)
Here, $\delta_j(dx)$ is the Dirac measure centered at $j \in \mathbb{N}$. In similarity variables (1.9) the distribution function for $\hat{n}(\tau, \hat{x})$ corresponding to (2.18) is

$$F(\tau, \hat{x}) = e^{-\tau} \sum_{1 \leq j < e^\tau} (1 - e^{-\tau})^{j-1} = 1 - (1 - e^{-\tau})^{[e^\tau] - 1}$$

where $[\hat{x}] = \min\{j \in \mathbb{N} : j \geq \hat{x}\}$. The limiting distribution is

$$F_*(\hat{x}) = 1 - e^{-\hat{x}}.$$

We have that

$$F(\tau, \hat{x}) - F_*(\hat{x}) = e^{-\hat{x}} \left(1 - e^{\alpha(\tau, \hat{x})}\right) = e^{-\hat{x}} \left(\alpha(\tau, \hat{x}) + O(\alpha^2(\tau, \hat{x}))\right)$$

where

$$\alpha(\tau, \hat{x}) = \hat{x} + ([e^\tau \hat{x}] - 1) \log(1 - e^{-\tau}) = O\left(\left(1 + \frac{\hat{x}}{2}\right) e^{-\tau}\right)$$

uniformly in $\hat{x}$ as $\tau \to \infty$. Since $0 \leq (1 + \hat{x}/2)e^{-\hat{x}} \leq 1$, substituting (2.20) in (2.19) gives the desired result:

$$\sup_{\hat{x} > 0} |F(\tau, \hat{x}) - F_*(\hat{x})| = O(e^{-\tau}).$$

3 Rate of convergence for the additive kernel $K = x + y$

We now consider the case of the additive kernel:

**Theorem 3.1.** Let $n_0$ be a positive measure, $\int_{(0,\infty)} x n_0(dx) = \int_{(0,\infty)} x^2 n_0(dx) = 1$, and $\mu_3 = \int_{(0,\infty)} x^3 n_0(dx) < \infty$. With $\hat{n}(t,d\hat{x})$ the rescaled solution to Smoluchowski’s equation with $K = x + y$ and initial data $n_0(dx)$ let

$$F(t, \hat{x}) = \int_{(0,\hat{x}]} \hat{y} \hat{n}(t, d\hat{y}), \quad F_*(\hat{x}) = \int_{(0,\hat{x}]} \hat{y} n_*(\hat{y}) d\hat{y}.$$

Then for $t \in [0, \infty)$,

$$\sup_{\hat{x} > 0} |F(t, \hat{x}) - F_*(\hat{x})| \leq C(\mu_3)(1 + t)e^{-t}$$

where $C(\mu_3)$ is a constant that depends only on $\mu_3$.

3.1 Proof for the additive kernel

The analysis for the additive kernel is identical in spirit to that of the constant kernel but is complicated by the fact that the characteristics of the Fourier-Laplace transform are no longer rays. We use a contour deformation argument to overcome this difficulty.

To start, for $s \in \mathbb{C}_+$ define

$$\varphi(t, s) = \int_{(0,\infty)} (1 - e^{-s\hat{x}}) \hat{n}(t, d\hat{x}), \quad \varphi_s(s) = \int_{(0,\infty)} (1 - e^{-s\hat{x}}) \hat{n}_*(\hat{x}) d\hat{x}$$
and let $\varphi_0(s) = \varphi(0, s)$. We consider these quantities instead of standard Laplace transforms since the number measure need not be integrable near the origin. Another motivation is probabilistic: $\varphi$ and $\varphi_*$ are characteristic exponents of a subordinator with no drift, as given by the Lévy-Khintchine formula [3]. We have that the Fourier-Laplace transforms of the rescaled mass measures satisfy

$$u(t, s) = \partial_s \varphi(t, s) = \int_{(0, \infty)} e^{-s \hat{x}} \hat{n}(t, d\hat{x}), \quad u_*(s) = \int_{(0, \infty)} e^{-s \hat{x}} \hat{n}_*(d\hat{x})$$

with $u_0(s) = u(0, s)$. By (1.12),

$$\sup_{\hat{x} > 0} |F(t, \hat{x}) - F_*(\hat{x})| \leq \frac{1}{\pi} \sup_{\hat{x} > 0} \left| \int_{-iT}^{iT} \frac{e^{\sigma \hat{x}}}{\sigma} (u(t, \sigma) - u_*(\sigma)) d\sigma \right| + \frac{24}{\pi T}.$$  

1. **Evolution of characteristics:** The evolution of $\varphi$ and $u$ is given by

$$\partial_t \varphi + (2s - \varphi) \partial_s \varphi = \varphi$$

$$\partial_t u + (2s - \varphi) \partial_s u = -u(1-u).$$

As with the constant kernel, these equations can be derived from (1.3), or from (1.1) if the initial number measure has a density [8]. Fixing an initial point $s_0 \in \mathbb{C}_+$, the unique solution to (3.3-3.4) is given in terms of the characteristics $s(t; 0, s_0)$ which satisfy

$$\frac{ds}{dt} = 2s - \varphi, \quad s(0; 0, s_0) = s_0 \in \mathbb{C}_+.$$  

Along these characteristic curves we have that

$$\frac{d\varphi}{dt} = \varphi, \quad \frac{du}{dt} = -u(1-u)$$

so that by integrating:

$$\varphi(t, s) = e^t \varphi_0(s_0), \quad u(t, s) = e^{-t} \frac{u_0(s_0)}{1 - u_0(s_0)(1 - e^{-t})}.$$  

These solutions are analytic, as shown in [7]. The explicit form for the characteristics $s_0 \mapsto s$, which depend on the initial data through $\varphi_0$, is then

$$s(t; 0, s_0) = e^{2t} (s_0 - \varphi_0(s_0)(1 - e^{-t})).$$

This can be combined with the expression (3.6) for $\varphi$ to give

$$\frac{\varphi(t, s)}{s} = e^{-t} \frac{(\varphi_0(s_0)/s_0)}{1 - (\varphi_0(s_0)/s_0)(1 - e^{-t})}.$$  

Note that since $|u_0(s_0)| \leq 1$ and $|\varphi_0(s_0)| \leq |s_0|$ from (3.6) and (3.8) we have

$$|\varphi(t, s)| \leq |s|, \quad |u(t, s)| \leq 1.$$
for \( s(t; 0, s_0) \) such that \( s_0 \in \bar{C}_+ \) and \( t \geq 0 \).

The Fourier-Laplace transform

\[
(3.10) \quad u_*(s) = \frac{1}{\sqrt{1 + 2s}}
\]

of the self-similar solution can also be stated in terms of characteristics. In contrast to the constant kernel case, these characteristics differ from those for the solution \( u(t, s) \) with initial data \( u_0 \). Fixing \( s_0^* \in \bar{C} \), this distinct set of characteristics \( s_0^* \mapsto s^* \) satisfies

\[
(3.11) \quad s^*(t; 0, s_0^*) = e^{2t}(s_0^* - \varphi_*(s_0^*)(1 - e^{-t}))
\]

with \( \varphi_*(s) = \sqrt{1 + 2s} - 1 \). Since \( u_*(s) \) is a stationary solution along these characteristics, we necessarily have that

\[
(3.12) \quad u_*(s) = e^{-t} \frac{u_*(s_0^*)}{1 - u_*(s_0^*)(1 - e^{-t})}.
\]

2. Estimates near the origin: We use the additional finite moment assumption

\[\mu_3 = \int_{(0, \infty)} \hat{x}^3 n_0(dx) = \int_{(0, \infty)} x^3 n_0(dx) < \infty\]

along with

\[3 = \int_{(0, \infty)} \hat{x}^3 n_*(x)dx = \int_{(0, \infty)} x^3 n_*(x)dx\]

to approximate \( u_0 \) and \( u_* \) near the origin. By (1.17),

\[
(3.13) \quad |u_0(s) - (1 - s)| \leq \frac{\mu_3}{2}|s|^2, \quad |u_*(s) - (1 - s)| \leq \frac{3}{2}|s|^2, \quad s \in \bar{C}_+.
\]

Since \( u_0 = \partial_s \varphi_0 \) implies

\[\varphi_0(s) = s \int_0^1 u_0(\beta s)d\beta,\]

together with an analogous expression for \( \varphi_* \) these estimates give that

\[
(3.14) \quad \left| \varphi_0(s) - \left( s - \frac{1}{2}s^2 \right) \right| \leq \frac{\mu_3}{6}|s|^3, \quad \left| \varphi_*(s) - \left( s - \frac{1}{2}s^2 \right) \right| \leq \frac{1}{2}|s|^3, \quad s \in \bar{C}_+.
\]

This yields a bound on the distance between the characteristics \( s, s^* \) starting at \( \sigma_0 \in \bar{C}_+ \):

\[
|s(t; 0, \sigma_0) - s^*(t; 0, \sigma_0)| = e^{2t}|\varphi_0(\sigma_0) - \varphi_*(\sigma_0)|(1 - e^{-t}) \leq \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (1 - e^{-t}) e^{2t}|\sigma_0|^3.
\]

With (3.9) we also have that

\[
(3.16) \quad |s^*(t; 0, \sigma_0)| = e^{2t}|\sigma_0 - \varphi_*(\sigma_0)(1 - e^{-t})| \leq 2e^{2t}|\sigma_0|.
\]
Finally, we estimate $u(t, s)$ itself near the origin using the time-dependent moment

$$\hat{m}_3(t) = \int_{(0, \infty)} x^3 \hat{n}(t, dx) = \int_{(0, \infty)} x^3 n(t, dx).$$

It is straightforward from (1.3) (or (1.1) if $n(t, dx)$ has a density) that $\partial_t \hat{m}_3(t) = 3 - \hat{m}_3(t)$, so

$$\hat{m}_3(t) = 3 + (\mu_3 - 3)e^{-t}.$$

Then (1.17) implies that for all $t \geq 0$,

$$|u(t, s) - (1 - s)| \leq \frac{\hat{m}_3(t)}{2}|s|^2 \leq \frac{\mu_3}{2}|s|^2.$$

3. Propagation of estimates: Upon inspection of (3.7) and (3.11) we see that the natural growth rate of both sets of characteristics is $e^t$ near the origin, and $e^{2t}$ at an $O(1)$ distance away from the origin. Explicitly, for small $\delta > 0$ note that

$$s(t; 0, i\delta) = e^{2t}(i\delta - \varphi_0(i\delta)(1 - e^{-t})) \approx e^{2t}\left(i\delta e^{-t} + \frac{1}{2}(i\delta)^2(1 - e^{-t})\right).$$

Bending this point onto the imaginary axis, define

$$T(t) = e^{2t}\left(\delta e^{-t} + \frac{1}{2}\delta^2(1 - e^{-t})\right).$$

We will bound the r.h.s. of (3.2) with this value for $T$, where $\delta > 0$ is a length scale characterizing the region about the origin where the difference between $u_0$ and $u_*$ is small by (3.13). To begin, let $\rho_3 = \max\{\mu_3, 3\}$ and define

$$\delta = \sqrt{1 + \frac{1}{2} \left(\frac{1}{4} + \frac{\rho_3}{6}\right)^{-1}} - 1$$

so that $(\frac{1}{4} + \frac{\rho_3}{6}) \delta = \frac{1}{2}(2 + \delta)^{-1}$. Note that if $\mu_3 \to \infty$ then $\delta \to 0$.

4. Backward characteristics: The main difficulty in the proof arises from the fact that characteristics are no longer rays as in the constant kernel case. We now discuss properties of the characteristic map which we will use later in the proof.

To begin, let $\Omega_t$ and $\Omega^*_t$ be the images of $\mathbb{C}_+$ under the maps $s_0 \mapsto s(t; 0, s_0)$ and $s^*_0 \mapsto s^*(t; 0, s^*_0)$ with $t \geq 0$. By Lemma 3.3 in [8], the image $\Gamma_t = \partial\Omega_t$ of the imaginary axis under $s_0 \mapsto s(t; 0, s_0)$ is a curve which passes through the origin and lies in the closed left half plane $\mathbb{C}^-$. This is easily seen by taking the real part of (3.7) with $s = ik$ and noting that

$$\text{Re } \varphi_0(ik) = \int_{(0, \infty)} (1 - \cos(kx))n_0(dx) \geq 0.$$ 

Similarly, the image $\Gamma^*_t = \partial\Omega^*_t$ of the imaginary axis under $s^*_0 \mapsto s^*(t; 0, s^*_0)$ passes through the origin and otherwise lies in the open left half plane $\mathbb{C}^-$ due to the continuity of $n_*(x)$. Furthermore, it has been shown that $\Gamma_t$ and $\Gamma^*_t$ are curves on which $\text{Re } s$ is a function of $\text{Im } s$. 

We also have by [8], Lemma 3.3(iii) that for \( t \geq 0, s_0 \mapsto s(t;0,s_0) \) is one-to-one from \( \mathbb{C}_+ \) onto \( \Omega_t \). It is analytic in \( \mathbb{C}_+ \), and its inverse map \( s \mapsto s_0 \) is analytic in \( \Omega_t \). In particular, this yields the analyticity of \( u(t,s) \) in \( \Omega_t \) (and analyticity of \( u_*(s) \) in \( \Omega_t^* \) by substituting \( s \) with \( s^* \)). As the positive real axis is invariant under \( s_0 \mapsto s(t;0,s_0) \), the continuity of the mapping implies that characteristics starting in the upper or lower half of \( \mathbb{C}_+ \) remain in the upper or lower half of the complex plane, respectively.

Next, define the backward characteristics \( s_0(t,s) \) and \( s_0^*(t,s) \) as inverses of the forward mappings \( s_0 \mapsto s(t;0,s_0) \) and \( s_0^* \mapsto s^*(t;0,s_0^*) \), respectively. Explicitly, for \( \omega \in \Omega_t \) and \( \omega^* \in \Omega_t^* \),

\[
(3.20) \quad s_0(t,\omega) = s(0; t, \omega), \quad s_0^*(t,\omega^*) = s^*(0; t, \omega^*).
\]

By (3.7) and (3.11), the Jacobians of the transformations \( s_0 \mapsto s \) and \( s \mapsto s^* \) are

\[
(3.21) \quad \frac{ds}{ds_0} = e^{2t} (1 - u_0(s_0))(1 - e^{-t}), \quad \frac{ds^*}{ds_0^*} = e^{2t} (1 - u_*(s_0^*))(1 - e^{-t}).
\]

We will now estimate (3.2) by considering separately the integral near and away from the origin. In doing so, we use the following convention for the remainder of the proof. The independent variable \( \sigma \) is used solely for points on the imaginary axis in the \( s, s^* \)-plane, while \( s_0 \) is used analogously in the \( s, s_0^* \)-plane.

5. **Bounds for \( 0 \leq |\sigma| \leq \delta e^{-t} \):** Applying (3.13) and (3.17), we get that

\[
\left| \int_{0 \leq |\sigma| \leq \delta e^{-t}} \frac{e^{\sigma \hat{x}}}{\sigma} (u(t,\sigma) - u_*(\sigma)) d\sigma \right| \leq 2 \int_0^{\delta e^{-t}} \frac{1}{|\sigma|} |u(t,\sigma) - u_*(\sigma)| d|\sigma| \leq \left( \frac{1}{2} + \frac{\mu_3}{6} \right) \delta^2 e^{-2t}.
\]

6. **Bounds for \( \delta e^{-t} \leq |\sigma| \leq T(t) \):** Since the flow of characteristics no longer leaves the imaginary axis invariant, we use a contour deformation argument (see [8] for a similar argument).

6a. **Contour deformation:** Without loss of generality, we work in the upper half of the complex plane. Bounds for quantities in the lower half plane are identical since \( u(t,s) = u(t,\bar{s}) \) and \( u_*(s) = u_*(\bar{s}) \). Take

\[
A' = i\delta e^{-2t}, \quad D' = i\delta
\]

to be the endpoints of a piece of the imaginary axis in the \( s_0, s_0^* \)-plane (see Figure 3.1). Let \( AD \) and \( A^*D^* \) be the images of \( A'D' \) under the mappings \( s_0 \mapsto s(t;0,s_0) \) and \( s_0^* \mapsto s^*(t;0,s_0^*) \). As discussed in step 4, \( AD \subset \Gamma_t \) and \( A^*D^* \subset \Gamma_t^* \) lie in the left half of the complex plane.

The part of (3.2) that remains to be estimated is an integral over the imaginary axis in the \( s, s^* \)-plane, bounded away from the origin, with endpoints

\[
B = i\delta e^{-t}, \quad C = iT(t).
\]
To do so, split the integral into its individual components

\[ (3.23) \int_{BC} e^{\sigma \hat{x}} \sigma \frac{u(t, \sigma) - u_*(\sigma)}{\sigma} d\sigma = \int_{BC} e^{\sigma \hat{x}} \frac{u(t, \sigma)}{\sigma} d\sigma - \int_{BC} e^{\sigma \hat{x}} u_*(\sigma) d\sigma. \]

Since \( u \) and \( u_* \) are analytic in \( \Omega_t \) and \( \Omega'_t \), the integrands are analytic away from the origin and we can apply Cauchy’s theorem:

\[ (3.24) \int_{BC} e^{\sigma \hat{x}} \sigma u(t, \sigma) d\sigma = \int_{AD} e^{s \hat{x}} \frac{u(t, s)}{s} ds + \int_{BA} e^{s \hat{x}} \frac{u(t, s)}{s} ds + \int_{DC} e^{s \hat{x}} \frac{u(t, s)}{s} ds, \]

and

\[ (3.25) \int_{BC} e^{\sigma \hat{x}} u_*(\sigma) d\sigma = \int_{A^*D^*} e^{s^* \hat{x}} \frac{u_*(s^*)}{s^*} ds^* + \int_{BA^*} e^{s^* \hat{x}} \frac{u_*(s^*)}{s^*} ds^* + \int_{D^*C} e^{s^* \hat{x}} \frac{u_*(s^*)}{s^*} ds^*. \]

The error terms are integrals over contours that connect the endpoints of \( BC \) to those of \( AD \) and \( A^*D^* \), respectively. We take \( BA \) and \( BA^* \) to be straight line contours. Additionally, let \( DC \) and \( D^*C \) be such that their backward images

\[ (3.26) \quad D'C' = s_0(t, DC), \quad D'C^{*'} = s_0^*(t, D^*C) \]

are straight lines. We will estimate the integrals over these contours in (3.24-3.25) at the end of the section.

Figure 3.1: Endpoints of contours on imaginary axis are \( B = i\delta e^{-t}, C = iT(t) \) and \( A' = i\delta e^{-2t}, D' = i\delta \). On the left, \( A = s(t; 0, A'), A^* = s^*(t; 0, A'), D = s(t; 0, D'), D^* = s^*(t; 0, D') \). On the right, \( B' = s_0(t, B), B^{*'} = s_0^*(t, B), C' = s_0(t, C), C^{*'} = s_0^*(t, C) \).

Making the change of variables \( s \mapsto s_0 \) and \( s^* \mapsto s_0^* \) and using (3.6), (3.12), and (3.21), we have:

\[ (3.27) \int_{AD} e^{s \hat{x}} \frac{u(t, s)}{s} ds - \int_{A^*D^*} e^{s^* \hat{x}} \frac{u_*(s^*)}{s^*} ds^* \]

\[ = e^{-t} \int_{A'D'} \left( e^{2t} \frac{s(t; 0, \sigma_0) \sigma_0}{s(t; 0, \sigma_0)} u_0(\sigma_0) - e^{2t} \frac{s^*(t; 0, \sigma_0) \sigma_0}{s^*(t; 0, \sigma_0)} u_*(\sigma_0) \right) d\sigma_0. \]
Using the basic inequality $|a_1 b_1 - a_2 b_2| \leq |b_1 - b_2||a_1| + |a_1 - a_2||b_2|$, we can bound the magnitude of (3.27) by the sum $\Lambda_1 + \Lambda_2$, where

$$\Lambda_1 = e^{-t} \int_{\delta e^{-2t}}^{\delta} \left| \frac{e^{2t}}{s} u_0(\sigma_0) - \frac{e^{2t}}{s^*} u_*(\sigma_0) \right| |e^{s^2}| d|\sigma_0|$$

(3.28)

$$\Lambda_2 = e^{-t} \int_{\delta e^{-2t}}^{\delta} \left| e^{s^2} - e^{s^2^*} \right| \left| \frac{e^{2t}}{s^*} u_*(\sigma_0) \right| d|\sigma_0|$$

(3.29)

We now evaluate each of these terms. For $\Lambda_1$ the argument is nearly identical to that presented in Section 2.

6b. Bounds for $\Lambda_1$: Since $s \in \tilde{C}_+$, $|e^{s^2}| \leq 1$. Using the explicit form (3.7), (3.11) of the characteristics, we rearrange terms in the integrand of (3.28) to get

$$\left| \frac{e^{2t}}{s} u_0(\sigma_0) - \frac{e^{2t}}{s^*} u_*(\sigma_0) \right| = \left| \frac{1}{\sigma_0} \left( \frac{e^{-2t} s^* u_0(\sigma_0)}{\sigma_0} - \frac{e^{-2t} s u_*(\sigma_0)}{\sigma_0} \right) \right|$$

(3.30)

$$\leq \left| \frac{e^{-2t} s^*}{\sigma_0} - \frac{e^{-2t} s}{\sigma_0} \right| |u_*(\sigma_0)| + \left| u_0(\sigma_0) - u_*(\sigma_0) \right| \left| \frac{e^{-2t} s^*}{\sigma_0} \right|$$

Notice that here $\varphi(\sigma_0)/\sigma_0$ and $\varphi_*(\sigma_0)/\sigma_0$ play the same role as $u_0(\sigma_0)$ and $u_*(\sigma_0)$ do in (2.17). Using that $|u_*| \leq 1$ and the estimates (3.15), (3.16), the numerator in (3.30) satisfies

$$\leq 7 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) |\sigma_0|^2$$

(3.31)

To estimate the denominator of (3.30), we replace $\varphi(\sigma_0)$ and $\varphi_*(\sigma_0)$ by $2\sigma_0/(2 + \sigma_0)$. Since

$$\left| \frac{2s}{2 + s} - \left( s - \frac{1}{2} s^2 \right) \right| \leq \frac{1}{4} |s|^3, \quad s \in \tilde{C}_+,$$

with (3.9), (3.14), and (3.19) this implies that

$$\left| 1 - \frac{\varphi_0(\sigma_0)}{\sigma_0} (1 - e^{-t}) \right| \geq \left| \frac{2e^{-t} + \sigma_0}{2 + \sigma_0} \right| - \left( \frac{1}{4} + \frac{\mu_3}{6} \right) |\sigma_0|^2$$

(3.32)

$$\geq \frac{|\sigma_0|}{2 + \delta} - \left( \frac{1}{4} + \frac{\mu_3}{6} \right) |\sigma_0|^2 \geq \frac{1}{2} \cdot \frac{|\sigma_0|}{2 + \delta}.$$

The same argument holds with $\varphi_0$ replaced by $\varphi_*$ and $\mu_3$ replaced by 3, but $\delta$ unchanged:

$$\left| 1 - \frac{\varphi_*(\sigma_0)}{\sigma_0} (1 - e^{-t}) \right| \geq \frac{1}{2} \cdot \frac{|\sigma_0|}{2 + \delta}.$$
Therefore, using (3.33-3.32) and (3.31) in (3.30) yields

\[
e^{-t} \int_{\delta e^{-2t} \leq |\sigma_0| \leq \delta} \left| \frac{e^{2t}}{s} u_0(\sigma_0) - \frac{e^{2t}}{s^*} u_s(\sigma_0) \right| |e^{s^*\tilde{x}}| d|\sigma_0| \\
\leq 28 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (2 + \delta)^2 e^{-t} \int_{\delta e^{-2t}}^\delta |\sigma_0| d|\sigma_0| \\
= 56 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (2 + \delta)^2 t e^{-t}.
\]

(3.34)

6c. Bounds for \( \Lambda_2 \): Now we consider \( \Lambda_2 \). Using \( |u_s| \leq 1 \) and (3.11) we can rewrite this as

\[
\Lambda_2 \leq e^{-t} \int_{\delta e^{-2t}}^\delta \frac{|e^{s^*\tilde{x}} - e^{s^*\tilde{x}}|}{|\sigma_0|} \frac{1}{|1 - \varphi^*(\sigma_0)|} \frac{|\sigma_0|^3}{|1 - e^{-t}|} d\sigma_0.
\]

(3.35)

As noted in step 4, since \( \sigma_0 = ik \) with \( k \in \mathbb{R} \) nonzero, we have that \( \text{Re} \, s^* < 0 \). This implies that \( \sup_{\tilde{x} \geq 0} |\tilde{x} e^{s^*\tilde{x}}| \) is bounded above by \( 1/|\text{Re} \, s^*| \). Then

\[
|e^{s^*\tilde{x}} - e^{s^*\tilde{x}}| = |e^{s^*\tilde{x}}| |e^{(s-s^*)\tilde{x}} - 1| \leq |s - s^*| \sup_{\tilde{x} > 0} |\tilde{x} e^{s^*\tilde{x}}|
\]

(3.36)

where the last inequality follows from (3.11) and (3.15). We now make use of the explicit form \( \varphi^*(\sigma_0) = \sqrt{1 + 2\sigma_0} - 1 \) obtained from (3.10). Since \( re^{i\theta} = 1 + 2ik \in \mathbb{C}_+ \) with \( r = \sqrt{1 + 4k^2} \) and \( |\theta| < \pi/2 \),

\[
\text{Re} \, \sqrt{1 + 2ik} = \sqrt{r} \cos \left( \frac{\theta}{2} \right) = \sqrt{\frac{1}{2} (r + 1)}
\]

by using a double-angle formula and \( \text{Re}(\cos(\theta)) = 1 \). To summarize,

\[
\text{Re} \, \varphi^*(\sigma_0) = \sqrt{\frac{1}{2} \left( 1 + \sqrt{1 + 4|\sigma_0|^2} \right)} - 1.
\]

Defining \( g(y) = \sqrt{\frac{1}{2} (1 + \sqrt{1 + y})} - 1 \), a routine calculation shows that \( g''(y) > 0 \) for all \( y > 0 \). It follows by Taylor’s theorem that \( g(y) \geq y/8 - 5y^2/128 \). Therefore, if \( |\sigma_0| \leq \delta \) then \( \text{Re} \, \varphi^*(\sigma_0) = g(4|\sigma_0|^2) \) satisfies

\[
|\text{Re} \, \varphi^*(\sigma_0)| \geq \left( \frac{1}{2} - \frac{5}{8} \delta^2 \right) |\sigma_0|^2 \geq \frac{1}{4} |\sigma_0|^2
\]

(3.37)

since \( \delta^2 < 2/5 \) by (3.19). Combining (3.36) and (3.37) yields

\[
\frac{|e^{s^*\tilde{x}} - e^{s^*\tilde{x}}|}{|\sigma_0|} \leq 4 \left( \frac{1}{2} + \frac{\mu_3}{6} \right).
\]

(3.38)
For the remaining term in the integrand of (3.35) we use the estimate (3.33). Putting this together with (3.38) we conclude that

\[ e^{-t} \int_{\delta e^{-2t}}^{\delta} \frac{|e^{s \hat{x}} - e^{s^* \hat{x}}|}{|s_0|} \frac{1}{1 - \frac{\varphi(s_0)}{\sigma_0}(1 - e^{-t})} d|\sigma_0| \]

\[ \leq 8 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (2 + \delta) e^{-t} \int_{\delta e^{-2t}}^{\delta} \frac{1}{|s_0|} d|\sigma_0| \]

(3.39)

\[ = 16 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (2 + \delta)te^{-t}. \]

6d. Bounds for \( \Lambda_1 + \Lambda_2 \): Combining (3.34) and (3.39), we obtain a bound for (3.28-3.29):

(3.40)

\[ \Lambda_1 + \Lambda_2 \leq 64 \left( \frac{1}{2} + \frac{\mu_3}{6} \right) (2 + \delta)^2 te^{-t}. \]

7. Error terms: Lastly, we bound the two error terms in (3.24) obtained from the contour deformation argument. The corresponding error terms in (3.25) are bounded by an argument identical to the one given below with only constants differing.

First we consider the term

(3.41)

\[ \left| \int_{BA} \frac{e^{s \hat{x}}}{s} u(t, s) ds \right| \leq |BA| \sup_{BA} \left( |s|^{-1} \right). \]

Here we have used that for \( s \in \bar{\Gamma}_t \), both \( |e^{s \hat{x}}| \) and \( |u(t, s)| \) are bounded by 1. From the explicit expression for the endpoints we have that

\[ |BA| = |e^{2t}(i\delta e^{-2t} - \varphi_0(i\delta e^{-2t})(1 - e^{-t})) - i\delta e^{-t}|. \]

Substituting \( \varphi_0(i\delta e^{-2t}) \) with \( i\delta e^{-2t} \) and using the bound

(3.42)

\[ |\varphi_0(s_0) - s_0| \leq \frac{1}{2} |s_0|^2, \quad s_0 \in \bar{C}_+ \]

derived from (1.17), we find that \( |BA| \leq \frac{1}{2} \delta^2 e^{-2t} \).

For the remaining term, write \( s \in BA \) as an interpolation of the endpoints:

\[ s = \beta(e^{2t}(i\delta e^{-2t} - \varphi_0(i\delta e^{-2t})(1 - e^{-t})) + (1 - \beta)i\delta e^{-t}), \quad \beta \in [0, 1]. \]

Again, substituting \( \varphi_0(i\delta e^{-2t}) \) by \( i\delta e^{-2t} \) and using (3.42) implies that for \( s \in BA \),

\[ |s| \geq \delta e^{-t} - \frac{\beta}{2} \delta^2 e^{-2t} \geq \frac{1}{2} \delta e^{-t}. \]

Putting this together with the previous bound, we conclude that

(3.43)

\[ \left| \int_{BA} \frac{e^{s \hat{x}}}{s} u(t, s) ds \right| \leq \delta e^{-t}. \]

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Now we estimate the second error term. Changing variables from \(s\) to \(s_0\),
\[
\left| \int_{DC} \frac{e^{s\hat{x}}}{s} u(t, s) ds \right| = e^{-t} \left| \int_{D'C'} \frac{e^{s(t;0,s_0)\hat{x}}}{s_0} \frac{u_0(s_0)}{1 - \frac{\varphi_0(s_0)}{s_0}(1 - e^{-t})} ds_0 \right|
\leq e^{-t}|D'C'| \sup_{D'C'} \left| \frac{|s_0|^{-1}}{} \right| \cdot \sup_{D'C'} \left( 1 - \frac{\varphi_0(s_0)}{s_0}(1 - e^{-t}) \right)^{-1}.
\]
(3.44)

The endpoints of the line segment \(D'C'\) are \(i\delta\) and \(\pi_0 := s_0(t, iT(t))\). By (3.7) and (3.18), the latter satisfies
\[
(3.45) \quad i \left( \delta e^{-t} + \frac{1}{2}\delta^2(1 - e^{-t}) \right) = \pi_0 - \varphi_0(\pi_0)(1 - e^{-t}).
\]

Properties of the characteristic map (see step 4) imply that this equation has a unique solution in the upper half of \(C_+\). Rewriting the r.h.s. as \(\pi_0 e^{-t} + (\pi_0 - \varphi_0(\pi_0))(1 - e^{-t})\), we take absolute values and use (3.42) to get that
\[
\delta e^{-t} + \frac{1}{2}\delta^2(1 - e^{-t}) \leq |\pi_0| e^{-t} + \frac{1}{2}|\pi_0|^2(1 - e^{-t}).
\]

Since \(h(y) = \beta y + (1 - \beta)\frac{1}{2}y^2\) is strictly increasing for positive \(y\) and any \(\beta \in [0, 1]\), this implies that \(|\pi_0| \geq \delta\). The endpoints of the line \(D'C'\) are therefore at least a distance \(\delta\) away from the origin, so \(\sup_{D'C'} |s_0|^{-1} \leq \sqrt{2\delta^{-1}} =: C_1(\delta)\).

We also have an upper bound for \(|\pi_0|\) as follows. Since the mass measure \(\hat{x}u_0(d\hat{x})\) is not concentrated at \(\hat{x} = 0\), \(|u_0|\) is bounded away from 1 in an annulus about the origin. Then there exists an \(\varepsilon = \varepsilon(\delta) > 0\) such that for all \(s_0 \in D'C'\),
\[
\left| \frac{\varphi_0(s_0)}{s_0} \right| \leq \int_0^1 |u_0(\beta s_0)| d\beta \leq 1 - \varepsilon.
\]

This shows that
\[
(3.46) \quad \sup_{D'C'} \left( 1 - \frac{\varphi_0(s_0)}{s_0}(1 - e^{-t}) \right)^{-1} \leq \varepsilon^{-1}.
\]

Rewriting the r.h.s. of (3.45) as \(\pi_0 \left( 1 - \frac{(\varphi_0(\pi_0)/\pi_0)(1 - e^{-t})}{\pi_0} \right)\), we find
\[
|\pi_0| = \left( \delta e^{-t} + \frac{1}{2}\delta^2(1 - e^{-t}) \right) \left| 1 - \frac{\varphi_0(\pi_0)}{\pi_0}(1 - e^{-t}) \right|^{-1} \leq \frac{\delta + \frac{1}{2}\delta^2}{\varepsilon}.
\]

Thus \(|D'C'|\) is bounded above by some constant \(C_2(\delta)\).

Finally, putting these estimates together in (3.44) yields
\[
(3.47) \quad \left| \int_{DC} \frac{e^{s\hat{x}}}{s} u(t, s) ds \right| \leq C_3(\delta)e^{-t}.
\]
where \(C_3 := C_1 C_2 \varepsilon^{-1}\) depends on the initial data \(u_0\) only through \(\mu_3\).

8. **Rate of convergence:** Using (3.22) and (3.40) in (3.2) with the bounds (3.43) and 3.47 for the error terms, we obtain the desired result (3.1).
3.2 Monodisperse initial data with $K = x + y$

As shown in [2], the explicit solution with $K = x + y$ and initial data $n_0(dx) = \delta_1(dx)$ can be given in terms of the Borel-Tanner distribution

$$B(\lambda, j) = \lambda^{j-1} \frac{j!}{\lambda^j}, \quad 0 \leq \lambda \leq 1$$

by

$$(3.48) \quad n(t, dx) = \sum_{j=1}^{\infty} e^{-t} B(1 - e^{-t}, j) \delta_j(dx), \quad t \in [0, \infty).$$

In the probabilistic setting, $B(\lambda, j)$ is the distribution of the total population size $X_\lambda$ of a Galton-Watson branching process with one progenitor and Poisson($\lambda$) offspring distribution. Note that

$$\mathbb{E}[X_\lambda] = \sum_{j=1}^{\infty} j B(\lambda, j) = (1 - \lambda)^{-1}.$$ 

In similarity variables (1.9) the distribution function for $\hat{x} n(t, dx)$ corresponding to (3.48) is

$$(3.49) \quad F(t, \hat{x}) = e^{-2t} \sum_{1 \leq j < e^{2t} \hat{x}} e^j (1 - e^{-t})^{j-1} \frac{j! e^{-(1-e^{-t})j}}{j!}.$$

and the limiting distribution is

$$(3.50) \quad F_*(\hat{x}) = \int_{0}^{\hat{x}} \frac{1}{\sqrt{2\pi}} \hat{y}^{-1/2} e^{-\hat{y}^2/2} d\hat{y}.$$ 

We consider now $\sup_{\hat{x} > 0} |F(t, \hat{x}) - F_*(\hat{x})|$. Substituting Stirling’s formula for the factorial in (3.49) and multiplying by an extra factor of $1 - e^{-t}$, define

$$\Phi(t, \hat{x}) = e^{-2t} \sum_{1 \leq j < e^{2t} \hat{x}} (1 - e^{-t})^j \frac{j! e^{-(1-e^{-t})j}}{\sqrt{2\pi} j^{1+1/2} e^{-j}}.$$ 

In addition, define the Riemann sum corresponding to (3.50) by

$$\Phi_*(t, \hat{x}) = e^{-2t} \sum_{1 \leq j < e^{2t} \hat{x}} \frac{1}{\sqrt{2\pi}} (e^{-2t} j)^{-1/2} e^{-(e^{-2t}j)/2}.$$ 

Each of the differences

$$(3.51) \quad D_1 = F(t, \hat{x}) - \Phi(t, \hat{x})$$

$$D_2 = \Phi(t, x) - \Phi_*(t, \hat{x})$$

$$D_3 = \Phi_*(t, \hat{x}) - F_*(\hat{x})$$

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is estimated as follows. For $D_1$, we can replace $F(t, \hat{x})$ by $(1 - e^{-t})F(t, \hat{x})$ since the difference between these terms is $O(e^{-t})$. Utilizing the following error estimate for Stirling’s approximation [1]

$$ j! = \sqrt{2\pi} j^{j+\frac{1}{2}} e^{-j} e^{\varepsilon(j)}, \quad 0 < \varepsilon(j) < \frac{1}{12j}, $$

we find that

$$ D_1 = e^{-2t} \sum_{1 \leq j < e^{2t} \hat{x}} (1 - e^{-t}j^j e^{-(1-e^{-t})j}) \left(1 - e^{\varepsilon(j)}\right) + O(e^{-t}) $$

$$ = e^{-t}(1 - e^{-t})\mathbb{P}(X_1 - e^{-t} < e^{2t} \hat{x}) + O(e^{-t}). $$

To estimate $D_3$, we use that the Riemann sum corresponding to an integral of a differentiable function leads to an error proportional to the grid size. Since $F'_n(\hat{x}) \in C^1$ we have that $D_3 = O(e^{-2t})$. Finally, for the remaining term $D_2$ note that

$$ D_2 = e^{-2t} \sum_{1 \leq j < e^{2t} \hat{x}} \frac{1}{\sqrt{2\pi}} (e^{-2t}j)^{1/2} e^{-(e^{-2t}j)/2} \left(e^{\alpha(t,j)} - 1\right) $$

where

$$ \alpha(t, j) = -j(-e^{-t} - \log(1 - e^{-t})) + e^{-2t} j/2 = -\frac{1}{3} e^{-3t} j + O(e^{-4t}). $$

Therefore,

$$ D_2 = \frac{1}{3} e^{-t} \int_0^{\hat{x}} \frac{1}{\sqrt{2\pi}} \hat{y}^{1/2} e^{-\hat{y}/2} d\hat{y} + O(e^{-2t}). $$

Putting together these estimates for (3.51) we conclude that

$$ \sup_{\hat{x} > 0} |F(t, \hat{x}) - F_n(\hat{x})| = O(e^{-t}). $$

4 Rate of convergence for the multiplicative kernel $K = xy$

Lastly, we consider the case of the multiplicative kernel:

**Theorem 4.1.** Let $n_0$ be a positive measure such that $\int_{(0,\infty)} x^2 n_0(dx) = \int_{(0,\infty)} x^3 n_0(dx) = 1$, and $\mu_4 = \int_{(0,\infty)} x^4 n_0(dx) < \infty$. With $\tau = \log(1 - t)^{-1}$ and $\hat{n}(t, dx)$ the rescaled solution to Smoluchowski’s equation with $K = xy$ and initial data $n_0(dx)$, let

$$ F(\tau, \hat{x}) = \int_{(0,\hat{x}]} \hat{y}^2 \hat{n}(\tau, d\hat{y}), \quad F_n(\hat{x}) = \int_{(0,\hat{x}]} \hat{y}^2 \hat{n}_n(\hat{y}) d\hat{y}. $$

Then for $\tau \in [0, \infty)$,

$$ \sup_{\hat{x} > 0} |F(\tau, \hat{x}) - F_n(\hat{x})| \leq C(\mu_4)(1 + \tau) e^{-\tau} $$

where $C(\mu_4)$ is a constant that depends only on $\mu_4$. 

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4.1 Proof for the multiplicative kernel

The rate for the multiplicative kernel can be recovered from that of the additive kernel by a classical change of variable due to Drake [5] (and discussed in [8]). With initial data \( n_0(dx) \), the measure-valued solutions \( n_{\text{mul}}(t,dx) \) and \( n_{\text{add}}(t,dx) \) to (1.3) with \( K = xy \) and \( K = x + y \), respectively, are related by

\[
x n_{\text{mul}}(t,dx) = (1 - t)^{-1} n_{\text{add}}(\tau(t),dx), \quad t \in (0,1).
\]

Using the similarity variables

\[
\hat{x}_1 = \frac{x}{\lambda_1(t)}, \quad \hat{x}_2 = \frac{x}{\lambda_2(t)}
\]

given by (1.7) and the rescaled number distributions (1.9), we have for \( \tau \in [0,\infty) \) that

\[
x_2^2 n_{\text{mul}}(\tau,d\hat{x}_2) = \hat{x}_1 n_{\text{add}}(\tau,d\hat{x}_1).
\]

Therefore,

\[
F_2(\tau,\hat{x}) = \int_0^{\hat{x}} y_2^2 n_{\text{mul}}(\tau,d\hat{y}_2) = \int_0^{\hat{x}} \hat{y}_1 n_{\text{add}}(\tau,d\hat{y}_1) = F_1(\tau,\hat{x}).
\]

Since (1.5) implies that the limiting distributions satisfy \( F_{*,2}(\hat{x}) = F_{*,1}(\hat{x}) \), we combine this with the previous equation to get (4.1):

\[
\sup_{\hat{x} > 0} |F_2(\tau,\hat{x}) - F_{*,2}(\hat{x})| = \sup_{\hat{x} > 0} |F_1(\tau,\hat{x}) - F_{*,1}(\hat{x})| \leq C(\mu_4)(1 + \tau)e^{-\tau}.
\]

The constant \( C \) is the same as that obtained in the case \( K = x + y \), with \( \mu_3 \) replaced by \( \mu_4 \). Lastly, the convergence rate is nearly optimal for monodisperse initial data by making the change of variables to the additive kernel case.

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