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Thermal noise of beam splitters in laser gravitational wave detectors

Johannes Dickmann,1, *, Stefanie Kroker,1,2 Yuri Levin,3,4,5 Ronny Nawrodt,6 and Sergey Vyatchanin7,8

1Physikalisch-Technische Bundesanstalt, Bundesallee 100, 38116 Braunschweig, Germany
2Technische Universität Braunschweig, LENA Laboratory for Emerging Nanometrology, Pockelsstraße 14, 38106 Braunschweig, Germany
3Department of Physics, Columbia University, 704 Papin Hall, 538 West 120th st, New York, NY 10027, USA
4Center for Computational Astrophysics, Flatiron Institute, 162 5th Ave, New York, NY 10010, USA
5School of Physics and Astronomy, Monash University, Clayton, VIC 3800, Australia
6Physikalisches Institut, Universität Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany
7Faculty of Physics, M.V. Lomonosov Moscow State University, Moscow 119991, Russia
8Quantum Technology Centre, M.V. Lomonosov Moscow State University, Moscow, Russia

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We present the calculation of thermal noise in interferometric gravitational-wave detectors due to the thermal fluctuations of the beam splitter (BS). This work makes use of a recently developed method of the analysis of thermal noise in mirrors from first principles, based on the fluctuation dissipation theorem. The evaluation of BS thermal noise is carried out for the two different gravitational wave observatories, GEO600 and the Advanced Laser Interferometer Gravitational Wave Observatory (aLIGO). The analysis evaluates thermal noise from both the substrate and the optical reflective and antireflective stacks located on the BS surface. We demonstrate that the fluctuations of both reflecting and anti-reflecting surfaces significantly contribute to the total thermal noise of the BS. The oscillating intensity pattern couples small-scale distortions of the surface to the overall phase readout, and therefore increases the overall thermal noise. In the case of aLIGO, the BS contribution is with 0.3% negligibly small. At a frequency of 500 Hz, the BS causes about 10% of GEO600’s sensitivity limit. BS noise impairs the feasible sensitivity of the GEO-HF design proposal by about 50%.

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I. INTRODUCTION

The first direct detections of gravitational waves by the advanced LIGO detectors [1–4] have opened the era of gravitational wave astronomy. These fascinating detection results are a consequence of technology breakthroughs allowing us to measure tiny displacements of $\sim$10$^{-18}$ m of macroscopic test masses [5–10]. It is the thermal noise of the test masses that sets a severe limitation for the detector sensitivity in their most sensitive frequency band from 50 Hz to 2000 Hz [11–14]. In current detectors, a major source for thermal noise is Brownian noise in substrate and coating of the BS [15] in mirror coatings. The thermally induced random stresses in the coatings and the substrates of the test masses produce random deformations of their surfaces, which are detected as thermal noise at the interferometer output.

A gravitational wave induces a variation of the frequency of the main mode in the arm cavity of the interferometer, which is registered as a phase shift of a monochromatic optical wave reflected from the cavity. Concurrently, the thermal noise of the mirror’s surface also randomly changes the eigenfrequency of the mode masking the gravitational wave signal. The same considerations hold true for the thermal noise of the beam splitter (BS) in the interferometer. Usually, the beam splitter is a cylindrical plate made of an optically transparent material. One surface is covered by a reflecting (R) coating and the other by an anti-reflecting (AR) one. The thermal fluctuations of both surfaces also change the interferometer’s eigenfrequency. In previous works, BS noise was estimated by the approximation of BS as infinitely thin plates [16] and in the simplifications of small light beam radii, compared to the BS size and light at normal incidence [17].

In this paper, we present the accurate computation of BS thermal noise in gravitational wave detectors from first principles, following the approach formulated in [18]. In combination with the fluctuation-dissipation theorem (FDT) [19–21] this approach allows the accurate computation of the thermal noise resulting from thermal fluctuations of both R and AR surfaces of BS with light beams of finite size and oblique incidence. A similar approach has been used in [22, 23] for computing thermal noise in reflective gratings and in [24] for evaluating the influence of an absorbing layer on the resonant frequencies and Q-factors of spherical microresonators.

Here we apply our approach for the calculation of Brownian noise in substrate and coating of the BS for arbitrary polarizations of the light. Our estimates show that the contributions of thermoelastic [12] and thermorefractive [13, 25] noise are subdominant to the Brownian noise computed here for frequencies between 100 and 4000 Hz.

*Electronic address: johannes.dickmann@ptb.de
II. MODEL AND STATEMENT OF THE PROBLEM

We consider the simplified model of the GEO600 interferometer as shown in Fig. 1. We assume that the length of north (east) arm differs from a multiple of the laser wavelength $\lambda$ by small displacements $x_n$ ($x_e$):

$$L_n = n\lambda + x_n, \quad L_e = \ell\lambda + x_e, \quad L_n \simeq L_e \simeq L,$$  \hspace{1cm} (2.1)

where $n$ and $\ell$ are integer numbers. We also assume that the length $L$ of north and east arm is much larger than the length $\ell_w$ of west arm and length $\ell_s$ of the south arm (see notations in Fig. 1):

$$L \gg \ell_s, \ell_w.$$  \hspace{1cm} (2.2)

To begin with, we assume that the power recycling (PR) and signal recycling (SR) mirrors are perfectly reflecting. In the case of perfectly tuned arms, we have two optical modes: the “west” mode (the standing wave is in the west arm and is absent in the south arm) and the “south” mode (the standing wave is in the south arm and absent in the west arm). The displacements $x_e$, $x_n$ of the end mirrors produce a coupling between the two modes. Thus, the Hamiltonian $\mathcal{H}$ for the two modes is written in the following form (for details see \[26\]):

$$\mathcal{H} = \hbar \omega_w a_w^* a_w \left(1 - \frac{x_e}{L}\right) + \hbar \omega_s a_s^* a_s \left(1 - \frac{x_n}{L}\right)$$

$$+ \hbar \sqrt{\omega_s \omega_w} (a_s^* a_w + a_w^* a_s) \frac{x_e - x_n}{L},$$  \hspace{1cm} (2.3)

Instead of the partial field coordinates $a_e$, $a_w$ for two coupled modes (oscillators), we introduce eigen (normal) coordinates $b_{\pm}$ and rewrite the Hamiltonian as follows:

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-, \quad \mathcal{H}_{\pm} = \hbar \omega_{\pm} b_{\pm}^* b_{\pm},$$  \hspace{1cm} (2.6a)

$$b_{\pm} = a_w \pm a_s \sqrt{\frac{\omega_s}{\omega_w}}, \quad \omega_{\pm} = \omega_0 \left(1 \pm \frac{x_{\pm}}{L}\right).$$  \hspace{1cm} (2.6b)

The $b_+$ mode in the east arm is called "east" mode (it is absent in the north arm) and the $b_-$ mode is called "north" mode. In the relations (2.6), there are two independent oscillators and for each of them we calculate the adiabatic invariants $I_{\pm}$ and the resulting frequency variations and variations of the energies $\mathcal{E}_{\pm}$:

$$I_+ = \frac{\mathcal{E}_+}{\omega_+}, \quad I_- = \frac{\mathcal{E}_-}{\omega_-},$$  \hspace{1cm} (2.7a)

$$\frac{\Delta \omega_+}{\omega_+} = \frac{\Delta \mathcal{E}_+}{\mathcal{E}_+}, \quad \frac{\Delta \omega_-}{\omega_-} = \frac{\Delta \mathcal{E}_-}{\mathcal{E}_-}.\hspace{1cm} (2.7b)$$

The adiabatic invariant of an oscillator is conserved, if its frequency changes very slowly compared to its oscillation period (here: the period of the optical oscillations). We are now interested in the effective small changes of $x_-$ and $x_+$, created by small perturbations of the BS surface. In our case, variations of the eigenfrequencies above can be written using (2.5, 2.6b):

$$\frac{\Delta \omega_+}{\omega_+} = -\frac{x_+}{L} + \frac{x_-}{L}, \quad \frac{\Delta \omega_-}{\omega_-} = -\frac{x_+}{L} - \frac{x_-}{L},$$  \hspace{1cm} (2.7c)

By using the Eqs. (2.7c) and (2.7b), we find the effective relative displacements:

$$\frac{x_+}{L} = \frac{\Delta \mathcal{E}_+}{\mathcal{E}_+} + \frac{\Delta \mathcal{E}_-}{\mathcal{E}_-},$$  \hspace{1cm} (2.7d)

$$\frac{x_-}{L} = \frac{\Delta \mathcal{E}_+}{\mathcal{E}_+} - \frac{\Delta \mathcal{E}_-}{\mathcal{E}_-}.\hspace{1cm} (2.7e)$$

Let us assume that a small perturbation of the BS surface appears slowly on the initially flat surface. This will change $\Delta \mathcal{E}_+$ and $\Delta \mathcal{E}_-$, which can be computed from the
elastic energy stored in the BS due to the applied ponderomotive light pressures. As the surface perturbations are small, we can apply the presented approximations and calculate the pressures with Maxwell stress tensor of the unperturbed field in the region around the surface deformation.

A. Calculations of the effective relative displacement ($x_/L$)

For the calculation of the terms in (2.7c) it is convenient to use the eigenamplitudes $b_\pm$ (2.6b). The work performed against the pressure is equal to the change of energy in the cavity taken with opposite sign. The light pressure $p_i$ acting on dielectric media’s surface may be calculated by means of Maxwell stress tensor $\sigma_{ij}$ inside and outside the material:

$$\sigma^i_{ij} = \frac{1}{4\pi} \left( \epsilon E_i E_j + H_i H_j - \frac{\epsilon E^2 + H^2}{2} \delta_{ij} \right).$$

The pressure in (2.8) is directed along the outer normal of the surface. For example, for a reflecting surface of the BS, the stress tensor is equal to the pressure acting along (η) axis (see Fig. 1). In contrast, the stress tensor calculated for fields directly beneath the reflecting surface (inside the BS) is equal to the pressure acting along the (−η) axis (compare Fig. 2).

The total normal pressure is equal to the difference of the pressures inside and outside the BS at the surface:

$$p_i^\perp = p_i^\text{outer} - p_i^\text{inside}. \quad \text{(2.9)}$$

The relative energy variations $\Delta E_\pm$ can be calculated as the work of ponderomotive light forces resulting from the total normal pressure:

$$\Delta E_\pm = -\int F_\pm(\vec{r}_\perp) u(\vec{r}_\perp) d\vec{r}_\perp, \quad F_\pm = \frac{p_\pm(\vec{r}_\perp)}{E_\pm}, \quad \text{(2.10)}$$

where $u(\vec{r}_\perp)$ is a small perturbation of the BS surface in normal direction, depending on the coordinate $\vec{r}_\perp$ on the BS surfaces. The averaging functions $F_\pm$ are defined by the pressures $p_\pm$ calculated for the corresponding modes and have to be applied to both sides of the BS, as will be shown later. For small perturbations $\Delta E \ll E$ the functions $F_\pm$ depend only on geometric factors, i.e. surface perturbations, radius of light beam, refractive index and thickness of the BS. They do not depend on the mode amplitudes $b_\pm$, because the energies $E_\pm$ and the pressures $p_\pm$ are both proportional to $b_\pm^2$.

B. Brownian Noise Power Spectral density

Following the FDT [19–21], we have to apply virtual pressures $p(\vec{r}) = F_0 p_\pm(\vec{r}) \sin \omega t$, oscillating with an angular frequency $\omega$, to calculate the mean dissipated power $W(\omega)$ in the BS and to finally calculate the Brownian noise power spectral density $S_{BS}(\omega)$:

$$S_{BS}(\omega) = \frac{8k_BT \omega_{\phi}}{\omega^2 F_0^2}, \quad \text{(2.11)}$$

where $k_B$ is the Boltzmann constant and $T$ is the absolute temperature. In this paper, we are interested in structural Brownian noise [27]. In the model of structural loss the dissipated power $W$ is defined by the phenomenological loss angle $\phi$:

$$W = E_\omega \phi, \quad \text{(2.12)}$$

where $E$ is the mean elastic energy stored in the BS. The pressures on the beam splitter $p_\pm$ have striped patterns [18, 28] i.e. which are proportional to $\cos^2 k\xi/\sqrt{2}$ ($k \equiv \omega_0/c$, $c$ is the speed of light). Hence, the applied pressure can be divided into two parts: 1) a smooth (non-striped) pressure $P_{sm}$ and 2) a striped pressure $P_{str} \sim \cos \sqrt{2}k\xi$. The elastic energies for the problems 1 and 2 can be calculated separately. Indeed, the total energy can be calculated as integral over both sides of the BS:

$$E = \frac{1}{2} \int \left[ P_{sm}(\vec{r}_\perp) + P_{str}(\vec{r}_\perp) \right] d\vec{r}_\perp \times \left[ u_{sm}(\vec{r}_\perp) + u_{str}(\vec{r}_\perp) \right] d\vec{r}_\perp$$

$$E_{sm} + E_{str} + E_\infty, \quad \text{(2.13b)}$$

$$E_{str} = \frac{1}{2} \int P_{str}(\vec{r}_\perp) u_{str}(\vec{r}_\perp) d\vec{r}_\perp \quad \text{(2.13c)}$$

$$E_{sm} = \frac{1}{2} \int P_{str}(\vec{r}_\perp) u_{str}(\vec{r}_\perp) d\vec{r}_\perp \quad \text{(2.13d)}$$

$$E_\infty = \frac{1}{2} \int \left[ P_{sm}(\vec{r}_\perp) u_{str}(\vec{r}_\perp) + P_{str}(\vec{r}_\perp) u_{sm}(\vec{r}_\perp) \right] d\vec{r}_\perp \quad \text{(2.13e)}$$

where $u_{sm}(\vec{r}_\perp)$ and $u_{str}(\vec{r}_\perp)$ are surface displacements caused by smooth and striped pressures, respectively. Obviously, the cross term of energy $E_\infty$ will be negligibly small after integration over the surface, due to the fast oscillating multiplier $\cos \sqrt{2}k\xi$. To our best knowledge, there is no approach to solve problem 1 analytically.

**FIG. 2:** Maxwell stress tensor $\sigma_{nm}$ leads to pressure $p_n$ along outer normal $\vec{n}$ to the boundary.
Hence, we solve it numerically using the finite element tool COMSOL Multiphysics [29]. In contrast, problem 2 can be solved analytically using the well-known solution for a half infinite elastic media [28]. This method can be applied, because the pressure contribution with fast spatial oscillation, characterized by the wave vector \( k_0 \), leads only to deformations located close to the surface. In particular, the elasticity divergence \( \Theta = u_{xx} + u_{yy} + u_{zz} \) decreases along the \( z \)-axis normal to the surface as \( \sim e^{-kz} \).

### III. CALCULATIONS OF PRESSURES

We assume all light beams in the interferometer to have an amplitude Gaussian distribution over the cross section:

\[
f_0(r_\perp) = \frac{e^{-r_\perp^2/2r_0^2}}{\sqrt{\pi r_0^2}} , \quad \int |f_0|^2 \, d\vec{r}_\perp = 1 , \quad (3.1)
\]

where \( d\vec{r}_\perp \equiv r_\perp \, dr_\perp \, d\phi \). For gravitational wave detectors we can assume that the beam radius \( r_0 \) is large:

\[
r_0 \gg \lambda = \frac{2\pi}{k} , \quad (3.2)
\]

where \( \lambda \) is the wavelength. Hence, we can consider the wave in the cavity as a plane wave with an amplitude multiplied by the Gaussian factor \( f_0 \) omitting terms of higher orders \( \sim 1/kr_0 \) (see for example [30–32]). For the calculations, we introduce the following relations between the coordinates \((x_e, x_n)\) and \((\xi, \eta)\) (see Fig. 1):

\[
x_e = \frac{\xi - \eta}{\sqrt{2}} , \quad x_n = \frac{\xi + \eta}{\sqrt{2}} , \quad (3.3)
\]

\[
\xi = x_e + x_n , \quad \eta = -x_e + x_n . \quad (3.4)
\]

Inside the BS the light propagates with an angle \( \alpha \) with respect to the BS axis (see Fig. 4):

\[
\sin \alpha = \frac{1}{n\sqrt{2}} , \quad 2a = h \tan \alpha . \quad (3.5)
\]

On the AR surface there are two centers of Gaussian distributions separated by distance \( 2a \) (see Fig. 4). Inside the BS, the beams propagate along the axes \( y_e, y_n \). These coordinates may be recalculated from the coordinates \((\xi, \eta)\):

\[
y_e = \xi \sin \alpha + \eta \cos \alpha , \quad (3.6)
\]

\[
y_n = \xi \sin \alpha - \eta \cos \alpha . \quad (3.7)
\]

We calculate the fields and pressures for different polarizations of traveling waves: S-polarization (vector of electric field is normal to the plane of figure) and p-polarization (vector of magnetic field is normal to the plane of figure). As a result, after simple but cumbersome calculations presented in Appendix A, we obtain the averaging functions on both surfaces of the BS and for each polarization orientation. We present these functions in the following 2 subsections.

#### A. S-polarization summary

For the GEO600 interferometer and s-polarization, we obtain the following equation for the eigenfrequency shift, i.e. effective relative displacement \( x_\perp / L \). For the reflecting surface R of the BS we retrieve using (2.7e) and (A6f, A21)

\[
\frac{x_\perp}{L} \bigg|_{rS} = \int \frac{F^s_1(\vec{r}_\perp)}{L} u_\perp(\vec{r}_\perp) \, d\vec{r}_\perp , \quad (3.8)
\]

\[
F^s_1(\vec{r}_\perp) \equiv \frac{L}{\varepsilon_+} \left( \frac{p^{s+}}{\varepsilon_+} - \frac{p^{s-}}{\varepsilon_-} \right) \quad (3.9)
\]

\[
= \frac{f_1^2}{2L} \left\{ \left[ 2n - 1 + \frac{1}{n} \right] \left[ 1 + 2 \sqrt{2(n - 1)} \right] \cos 2k_0 \xi \right\} , \quad (3.10)
\]

where we use the notations of (3.5) and

\[
k_0 = \frac{k}{\sqrt{2}} , \quad (3.11)
\]

\[
f_1 = \frac{1}{\sqrt{\pi r_0^2}} \exp \left( -\frac{z^2 + 0.5\xi^2}{2r_0^2} \right) . \quad (3.12)
\]

For the anti-reflecting surfaces AR of the BS we get with (2.7e) and (A13, A27):

\[
\frac{x_\perp}{L} \bigg|_{arS} = \int \frac{F^s_8(\vec{r}_\perp)}{L} u_\perp(\vec{r}_\perp) \, d\vec{r}_\perp , \quad (3.13)
\]

\[
F^s_8(\vec{r}_\perp) \equiv \left( \frac{p^{s+}}{\varepsilon_+} - \frac{p^{s-}}{\varepsilon_-} \right) \quad (3.14)
\]

\[
= \frac{1}{2L} \left\{ \left( f_2^2 \right) \left[ 1 - 2n + \frac{1}{n} \right] \right. \quad (3.15)
\]

\[
+ 2\sqrt{2(n - 1)} f_2 f_1 \cos k_0 (\xi_+ + \xi_-) \quad (3.15)
\]

\[
- f_2^2 \left[ 1 - \frac{1}{n} \right] \cos 2k_0 \xi . \quad (3.15)
\]

Here, we use the following notations (3.5, 3.11, 3.12) and definitions:

\[
f_\pm = \sqrt{\frac{1}{\pi r_0^2}} \exp \left( -\frac{z^2 + 0.5(\xi \mp a)^2}{2r_0^2} \right) , \quad (3.16)
\]

\[
\xi_\pm \equiv \xi \mp a . \quad (3.17)
\]

#### B. P-polarization summary

For the GEO600 interferometer and p-polarization, we obtain the following equations for the eigenfrequency shift, i.e. effective \( x_\perp / L \). For the surfaces R we get using (2.7e) and (A35, A51)

\[
\frac{x_\perp}{L} \bigg|_{rP} = \int \frac{F^p_1(\vec{r}_\perp)}{L} u_\perp(\vec{r}_\perp) \, d\vec{r}_\perp , \quad (3.18)
\]
\[ F^p_r(r_\perp) = \left( \frac{p^p_+}{\mathcal{E}_+} - \frac{p^p_-}{\mathcal{E}_-} \right) \]  
(3.19)

\[ = \frac{f^2_p}{2L} \left\{ \left[ 2n - \frac{1}{n} + 1 \right] \cos \sqrt{2}(n - 1) \right\}. \]
(3.20)

For the surface AR we get using (2.7e) and (A43, A57)

\[ \frac{x_-}{L} = \int F^a_r(r_\perp) u_\perp(r_\perp) \, dr_\perp, \]
(3.21)

\[ \frac{F^a_r(r_\perp)}{L} = \left( \frac{p^{ap}_+}{\mathcal{E}_+} - \frac{p^{ap}_-}{\mathcal{E}_-} \right) \]  
(3.22)

\[ = \frac{1}{2L} \left\{ \left( f^2_+ \right) \left[ 1 - 2n + \frac{1}{n} \right] \right\} \]  
(3.23)

\[ - 2\sqrt{2}(n - 1)f_+ f_- \cos \kappa(\xi_+ + \xi_-) \]

\[ + \left( f^2_+ \right) \left[ 1 - \frac{1}{n} \right] \cos 2k_0 \xi_+ \right\}. \]

We see that the "smooth" parts of the pressure acting on R and AR surfaces are the same both for p- and s-polarizations (compare (3.10) with (3.20) and (3.15) with (3.23)). However, the "striped" contributions have opposite signs for s- and p-polarization while being equal in their absolute value. Hence, for non-polarized light and for the case of equal s- and p-polarized field amplitudes, the "striped" term vanishes.

C. Generalization for the aLIGO interferometers

The results in subsections IIIA and IIIB are obtained for the GEO600 configuration without input mirrors (IM), shown by dashed rectangles in Fig. 1. For the calculation of the mode energies \( \mathcal{E}_\pm \), we use the fact, that the amplitudes of the fields on BS are nearly the same as in the arms, and in addition we utilize the approximation (2.2). For advanced LIGO, we have to recalculate the energies \( \mathcal{E}_\pm \) by taking into account, that the mean amplitudes in the arms are by a factor of approximately \( 2/\sqrt{T_{IM}} \) larger than on the BS. Here \( T_{IM} \) represents the power transmittance of the IM. Hence, we can generalize, for example, formula (3.10) for s-polarization on the R surface by the transformation:

\[ F^a_r|_{\text{GEO}} = F^a_r|_{\text{aLIGO}} \times \frac{T_{IM}}{4}. \]
(3.24)

Obviously, the other formulas (3.15, 3.20, 3.23) can be generalized for advanced LIGO using the same transformation (3.24).

IV. CALCULATION OF THE ELASTIC ENERGY

To calculate the spectral density of the Brownian BS noise in terms of relative displacements \( x_-/L \) for s-polarization, we have to apply virtual pressures \( p_{rs} \) to the R surface of the BS using (3.10) and \( p_{ars} \) to the AR surface using (3.15):

\[ p_{rs} = F_0 F^a_r(r) \sin \omega t, \quad p_{ars} = F_0 F^a_{ars}(r) \sin \omega t, \]
(4.1)

where \( F_0 \) is a constant, see (2.11). For the p-polarization we should use (3.20, 3.23). Then we calculate the mean elastic energy \( E \) stored in the BS and substitute this result into (2.11) by taking (2.12) into account. Since the elastic problem cannot be solved analytically, we perform the computations numerically using COMSOL [29]. We use the parameters for the BS of GEO600 and advanced LIGO listed in Table I. For the following considerations, we account only for the smooth contributions of the pressures (4.1), which are equal for both polarizations (see (3.10, 3.15, 3.20, 3.23). For the solution of the elastic problem, we have to fulfill two conditions: a) the sum of all external forces and b) the total torque of all external forces should be equal to zero. However, the pressures integrated over the R and the AR surfaces are not equal to zero, i.e. the total force acting on the BS is not equal to zero. Hence, in analogy to inertial forces we have to apply an additional volume force \( f \), which is [33]:

\[ f = -\frac{1}{\pi R^2 h} \int \left[ p_r(r_\perp) + p_{ars}(r_\perp) \right] \, dS = \]
\[ = -\frac{\sqrt{2}F_0}{\pi R^2 h} \times (1 - \epsilon_p), \]
(4.2)

where the integration is performed over the area of the BS, having radius \( R \) and height \( h \). In approximation \( R \to \infty \), the coefficient \( \epsilon_p \) is zero. For the parameters of

| Parameters | aLIGO | GEO600 |
|-----------|-------|--------|
| Radius \( R \) of BS, m | 0.1875 | 0.13 |
| Height \( h \) of BS, m | 0.064 | 0.08 |
| Refractive index \( n_{SiO_2} \) | 1.45 | 1.45 |
| \( a = \frac{h}{\sqrt{2}r_0} \) | 0.036 | 0.045 |
| Radius \( w_0 = \sqrt{2}r_0 \) of light beam on BS, m \(^a\) | 0.06 | 0.0088 |
| Young’s modulus \( E_{SiO_2} \), GPa | 73.1 | 73.1 |
| Poisson’s ratio \( \nu_{SiO_2} \) | 0.17 | 0.17 |
| Density \( \rho_{SiO_2} \), kg/m\(^3\) | 2203 | 2203 |
| Loss angle \( \phi_{SiO_2} \) | 10\(^{-8}\) | 10\(^{-8}\) |
| Power transmittance of input mirrors in arms of aLIGO | \( 5 \times 10^{-3} \) | — |

\(^a\)\( r_0 \) is the beam radius for the intensity distribution, which is proportional to \( \sim \exp[-r^2/r_0^2] \), whereas \( w_0 \) is the radius for the amplitude distribution, which is proportional to \( \sim \exp[-r^2/w_0^2] \). So \( w_0 = \sqrt{2}r_0 \).
advanced LIGO and GEO600 in Table I, the numerical computations yield:

\[ \epsilon_f^{\text{LIGO}} \simeq -0.000142, \quad (4.4) \]
\[ \epsilon_f^{\text{GEO}} \simeq 3.3 \times 10^{-16}. \quad (4.5) \]

The total torque of the external forces is not zero, because the centre of \( p_{\text{ar}} \) is shifted from the symmetrical cylinder axis by distance \( a \). Hence, we have to introduce an additional volume force \( f_{\text{add}} \sim y / R \) in order to eliminate effective torques:

\[ f_{\text{add}} = (1 - \epsilon_T) \frac{\sqrt{2} F_0}{\pi R^2 h} \left( 1 - 2n + \frac{1}{n} \right) \times \frac{2ay}{R^2}, \quad (4.6) \]

where the small coefficient \( \epsilon_T \) results from the finite dimension of the BS. In the approximation \( R \to \infty \), we retrieve \( \epsilon_T \to 0 \). For the parameters in Table I, numerical computations lead to:

\[ \epsilon_T^{\text{LIGO}} \simeq 0.00576962, \quad (4.7) \]
\[ \epsilon_T^{\text{GEO}} \simeq 1.19 \times 10^{-13}. \quad (4.8) \]

1. Substrate thermal Brownian noise of the BS

In order to calculate Brownian noise from thermal fluctuation in the BS substrate, we calculate the mean elastic energy stored in the BS. Using COMSOL, we performed numerical calculations of the mean elastic energy for parameters in Table I:

\[ E_{\text{aLIGO}} = 3.98 \times 10^{-10} J \times \frac{F_0^2}{N^2}, \quad (4.9) \]
\[ E_{\text{GEO}} = 1.97 \times 10^{-9} J \times \frac{F_0^2}{N^2}. \quad (4.10) \]

Using (2.11), we compute the power spectral density \( S_{\text{BS}} \) of the BS Brownian noise, recalculated to the differential coordinate \( x_- \):

\[ \sqrt{S_{\text{BS},(\omega)} / L}^{\text{LIGO}} \simeq 4.5 \times 10^{-27} \frac{1}{\sqrt{\text{Hz}}}, \quad (4.11) \]
\[ \sqrt{S_{\text{BS},(\omega)} / L}^{\text{GEO}} \simeq 2.7 \times 10^{-23} \frac{1}{\sqrt{\text{Hz}}}, \quad (4.12) \]

at the angular frequency \( \omega = 2\pi \times 100 \text{s}^{-1} \) for advanced LIGO and GEO600, respectively. For advanced LIGO we have taken the transformation (3.24) into account. The current sensitivity of advanced LIGO [8] is about \( \sqrt{S_{\omega}} / L \simeq 5 \times 10^{-23} 1/\sqrt{\text{Hz}} \), for the future cryogenic LIGO Voyager [34] the planned sensitivity is about \( \sqrt{S_{\omega}} / L \simeq 8 \times 10^{-25} 1/\sqrt{\text{Hz}} \). Since the substrate BS noise is substantially smaller than this sensitivity, LIGO Voyager will not be limited by it. The current sensitivity of GEO600 [10] is about \( \sqrt{S_{\omega}} / L \simeq 3 \times 10^{-22} 1/\sqrt{\text{Hz}} \). It is about 10 times larger than BS noise evaluated here.

2. Coating thermal Brownian noise of the BS

The Brownian noise of the R and AR coatings has to be calculated separately, because the loss angle of the alternating layers of \( Ta_2O_5 \) and \( SiO_2 \) are much larger than the loss of the substrate. These coatings are required to provide the optical function of the reflective and anti-reflective BS surfaces. For advanced LIGO, the parameters of the BS coatings are listed in Table II. The parameters of the GEO600 coatings are not published. Therefore, we assume the optical coating design to be the same as for LIGO (see Table II). We calculate the energy stored in the coatings with the assumption that the coatings are thin compared to the BS thickness. Thus, the strains in the layers are approximately the same as on the upper side of the substrate. We use the following expression for the volume density \( w \) of elastic energy in the layer [27, 28]:

\[ w_n = \frac{1}{2} \frac{(1 + \nu)(1 - 2\nu) \sigma_{zz}}{Y(1 - \nu)} + \frac{Y}{4(1 - \nu)} (u_{xx} + u_{yy})^2 + \frac{Y}{4(1 + \nu)} (u_{xx} - u_{yy})^2 \quad (4.13) \]

Here \( \sigma_{zz} \) is the normal component of the stress tensor, \( Y \) and \( \nu \) are Young’s modulus and Poisson’s ratio of the layers (made of \( Ta_2O_5 \) or \( SiO_2 \)), whereas \( u_{ij} \) are the tangent components of the strain tensors, which are the same as for the substrate’s surface. The total energy stored in the \( SiO_2 \) and \( Ta_2O_5 \) layers of the R and the AR coatings can be calculated as:

\[ E_{\text{coat}} = h_r \int w_n |_{SiO_2} dS_r \quad (4.14) \]
\[ + h_r \int w_n |_{Ta_2O_5} dS_r \quad (4.15) \]
\[ + h_{ar} \int w_n |_{ar \, SiO_2} dS_r \quad (4.16) \]
\[ + h_{ar} \int w_n |_{ar \, Ta_2O_5} dS_r \quad (4.17) \]

where the integration is carried out over the R and AR surfaces. Performing a numeric integration using \( u_{ij} \), ob-

| Parameters          | SiO₂   | Ta₂O₅ |
|---------------------|--------|-------|
| Reflective index    | 1.45   | 2.1   |
| Young modulus, GPa  | 73.1   | 140   |
| Poisson ratio       | 0.17   | 0.23  |
| Loss angle          | \( 1 \times 10^{-4} \) | \( 4 \times 10^{-4} \) |
| R coating total thickness \( h_r \), nm | 523    | 320   |
| AR coating total thickness \( h_{ar} \), nm | 517    | 359   |
V. CONCLUSION

In this contribution we applied the direct method of thermal noise calculations from first principles formulated in [18], to the computation of Brownian thermal noise of beam splitters in gravitational wave interferometers. We have demonstrated how the light pressure on both reflective and anti-reflective surfaces contributes to the total thermal noise of the BS. To this end, we took finite sized BS substrates and the coating contributions into account. An important new ingredient on our calculations is taking into account the stripped pattern of the form-factor that represents the sensitivity of the interferometers readout to the BS surface displacement. The pattern is due to the standing waves inside each of the arms. The striped contribution of the pressure turns out to be negligibly small for the substrate Brownian noise. But it provides an increase of the total spectral density of coating Brownian noise by about 50% (see Sec. IVB of [28]).

The results show, that the BS noise is negligibly small for advanced LIGO. However, for GEO600, it accounts for about 10% of the current noise budget in the most sensitive frequency band. Furthermore, BS noise impairs the feasible sensitivity of the proposed GEO-HF design by about 50%. This is because the additional Fabry-Perot arm cavities in LIGO lead to smaller light powers at the BS, compared to the power circulating in the arms. Hence, the BS noise in comparison to the test mass noise is suppressed. Whereas in GEO600, the contribution of BS noise is much larger, because the light power at the BS is the same as at the test masses. Overall, the Brownian BS noise is not a critical issue for the current sensitivity of gravitational wave detectors. However, the presented approach is based on a first-principle method with the Hamiltonian as starting point. It is thus useful for noise computations in other interferometer topologies, for example Sagnac interferometers or other complex optical devices.

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1 We note that similar standing-wave patterns are important in computations of thermoelastic [12] and thermorefractive [13, 25] noise of the BS.
Appendix A: Calculation details for fields and pressures

In this appendix we present in detail the derivations of the functions \( f_{\pm} \), describing the pressure distributions on both BS sides for two ("plus" and "minus") modes defined by (2.6). We consider separately s- and p-polarization of the circulating light waves.

1. S-polarization, \( B_+ \) mode

For the pressure, we have to calculate the fields on the R and on AR surface inside and outside the BS. We begin with the \( B_+ \) mode and s-polarization, see Fig. 4, denoted by \( B_+ \) amplitude of the magnetic field inside the east arm, whereas the wave in north arm is absent. The \( z \)-axis is directed out of plane. The electric field components are denoted as \( E \), the magnetic ones by \( H \) and subscripts denote projections.

**Surface 1R.** For the complex amplitudes of electric and magnetic fields we obtain (see Fig. 4 and definition (3.11)):

\[
E_z = \frac{B_+}{\sqrt{2}} (e^{ikx} - e^{-ikx})_{\eta=0} = i\sqrt{2}B_+\sin(k_{bs}\xi),
\]

\[
H_\eta = -\frac{B_+}{\sqrt{2}} (e^{ikx} + e^{-ikx})_{\eta=0} = -\sqrt{2}B_+\cos(k_{bs}\xi),
\]

\[
H_\xi = H_\eta = \frac{H_\eta}{\sqrt{2}} = -B_+\cos(k_{bs}\xi).
\]

Using definition (2.8) we calculate the Maxwell stress tensor on surface 1R:

\[
\sigma_{\eta\eta} = -\frac{|B_+|^2}{2\pi}\sin^2(k_{bs}\xi). \tag{A2}
\]

Component \( \sigma_{\eta\eta} \) corresponds to the pressure acting along axis \( \eta \), hence \( p_\eta = \sigma_{\eta\eta} \). The spatial distribution of the pressure \( p_\eta \) should be restored in (A2), by projecting (3.1) on the reflecting surface R:

\[
p_\eta^{1R} = -\frac{|B_+|^2}{4\pi}(1 - \cos 2k_{bs}\xi) f_\perp^3, \tag{A3}
\]

see definition (3.12). The pressure consists of a smooth contribution and a fast oscillating one \( \sim \cos 2k_{bs}\xi \).

**Surface 2R (s-polarization, \( B_+ \) mode)**. The relationship between the electric and magnetic fields inside the material (marked by superscript \( c \)) and the light intensity is:

\[
|H'| = \sqrt{\varepsilon}|E'|, \quad I_\varepsilon = c \frac{|E'|^2}{4\pi}, \quad n = \sqrt{\varepsilon}. \tag{A4}
\]

The intensity inside and outside the BS of AR surface has to be conserved. Thus, for the amplitudes of the wave propagating along \( y_n^- \) and \( y_e^- \) axes and inside the BS:

\[
E_z = \frac{B_+}{\sqrt{n}\sqrt{2}} (e^{inky} - e^{-inky}) \tag{A5a}
\]

\[
H_\eta = \frac{B_+}{\sqrt{n}} (e^{inky} - e^{-inky}), \tag{A5b}
\]

\[
k_\alpha = nk\sin\alpha = k_{bs}, \tag{A5c}
\]

\[
\sin\alpha = \frac{1}{\sqrt{2}n}, \quad \cos\alpha = \sqrt{1 - \frac{1}{2n^2}}, \tag{A5d}
\]

\[
H_\eta = -\frac{B_+\sqrt{n}\sin\alpha}{\sqrt{2}} (e^{inky} + e^{-inky}) \tag{A5e}
\]

\[
- B_+\sqrt{n}\sin\alpha (e^{inky} + e^{-inky}), \tag{A5f}
\]

\[
H_\xi = B_+\sqrt{n}\cos\alpha (e^{inky} + e^{-inky}) \tag{A5g}
\]

\[
- B_+\sqrt{n}\cos\alpha (e^{inky} + e^{-inky}), \tag{A5h}
\]

We calculate the Maxwell stress tensor at the surface 2R, using definition (2.8) again:

\[
\sigma_{\eta\eta} = \frac{n|B_+|^2}{4\pi} \left\{ 3(1 + \cos 2\alpha) \right\} \tag{A6a}
\]

\[
- \left[ 3 - 3 \cos 2\alpha + 4\sqrt{2} \cos 2k_{bs}\xi \right].
\]

The spatial distribution of the pressure \( p_\eta \) should be restored in (A6a), by projecting (3.1) on the reflecting surface of the BS:

\[
p_\eta^{2R} = \frac{n|B_+|^2}{4\pi} \left\{ 3(1 + \cos 2\alpha) \right\} \tag{A6b}
\]
The total pressure applied to the surface \( R \) of BS (s-polarization, \( B_+ \) mode) is equal to the sum of (A3) and (A6b):

\[
\begin{align*}
p_{\eta}^{S+} &= p_{\eta}^{1rS+} + p_{\eta}^{2rS+} \\
&= \frac{|B_+|^2}{4\pi} \left\{ 3n(1 + \cos 2\alpha) - 1 \right\} \\
&\quad + \left( 1 - n \left[ 3 - 3\cos 2\alpha + 4\sqrt{2} \right] \right) \cos 2k_0b_\eta \xi, \\
\end{align*}
\]

We calculate \( \mathcal{E}_+ \) of this mode using the assumption (2.2) and the expression for the function \( f_0 \) (3.1):

\[
\mathcal{E}_+ = \frac{|B_+|^2}{\pi} \left( L + \frac{\ell_s + \ell_w}{2} \right) \approx \frac{|B_+|^2 L}{\pi}. \tag{A6e}
\]

Using definitions (2.10), we get:

\[
\begin{align*}
p_{\eta}^{1rS+} + p_{\eta}^{2rS+} &= \frac{f_2^2}{4\pi} \left\{ 3n(1 + \cos 2\alpha) - 1 \right\} \\
&\quad + \left( 1 - n \left[ 3 - 3\cos 2\alpha + 4\sqrt{2} \right] \right) \cos 2k_0b_\eta \xi. \\
\end{align*}
\]

Surface 3AR (s-polarization, \( B_+ \) mode). The distributions of fields from south and east arm are shifted by \( \pm \alpha \) from the center as shown in Fig. 4, \( 2a = h \tan \alpha \). Using definitions (3.16, 3.17) we obtain:

\[
\begin{align*}
E_z &= \frac{B_+}{\sqrt{2}} \left( e^{ik_x a} - e^{-ik_x a} \right) f_- \\
&\quad + B_+ \left( e^{ik_x a} - e^{-ik_x a} \right) f_+, \tag{A7a} \\
E_z|_{\eta=-h} &= i\sqrt{2}B_+ \left( \sin \left( \frac{k_\xi}{\sqrt{2}} \right) f_- \\
&\quad + \sin \left( \frac{k_\xi}{\sqrt{2}} \right) \sqrt{2}f_+ \right), \tag{A7b} \\
H_\xi &= \frac{B_+}{\sqrt{2}} \left( e^{ik_x a} + e^{-ik_x a} \right) f_- \sqrt{2} \\
&\quad - B_+ \left( e^{ik_x a} + e^{-ik_x a} \right) f_+ \sqrt{2} \tag{A7c} \\
H_\xi|_{\eta=-h} &= B_+ \left( \cos \left( \frac{k_\xi}{\sqrt{2}} \right) f_- \\
&\quad - \cos \left( \frac{k_\xi}{\sqrt{2}} \right) \sqrt{2}f_+ \right), \tag{A7d} \\
H_\eta &= -\frac{B_+}{\sqrt{2}} \left( e^{ik_x a} + e^{-ik_x a} \right) f_- \sqrt{2} \\
&\quad - B_+ \left( e^{ik_x a} + e^{-ik_x a} \right) f_+ \sqrt{2} \tag{A7e} \\
H_\eta|_{\eta=-h} &= -B_+ \cos \left( \frac{k_\xi}{\sqrt{2}} \right) f_- \\
&\quad + B_+ \cos \left( \frac{k_\xi}{\sqrt{2}} \right) \sqrt{2}f_+. \tag{A7g}
\end{align*}
\]

Maxwell stress tensor at the surface 2AR of the BS:

\[
\begin{align*}
\sigma_{\eta \eta} &= \frac{n|B_+|^2}{2\pi} \left\{ -(f_2^2 + 2f_3^2) \cos^2 \alpha \right. \\
&\quad + \left( f_2^2 \cos 2\kappa_- + 2f_3^2 \cos 2k_0 \xi \right) \sin^2 \alpha \right. \\
&\quad + \left. 2\sqrt{2}f_- f_+ \cos k_0 b_\eta (\xi_+ + \xi_-) \right\}. \tag{A11}
\end{align*}
\]

Pressure along the \( \eta \)-axis: \( p_{\eta}^{2arS+} = \sigma_{\eta \eta} \). So, the total pressure along the \( \eta \)-axis on surface 2AR of the BS (s-polarization, \( B_+ \) mode) is equal to:

\[
\begin{align*}
p_{\eta}^{2arS+} + p_{\eta}^{3arS+} &= \frac{1}{4L} \left\{ f_2^2 + 2f_3^2 \right\} \left[ 1 + \frac{1}{n} - 2n \right] \\
&\quad + 4\sqrt{2}(n-1)f_- f_+ \cos k_0 (\xi_+ + \xi_-) \\
&\quad + \left( f_2^2 \cos 2k_0 \xi_- + 2f_3^2 \cos 2k_0 \xi_+ \right) \left( -1 + \frac{1}{n} \right) \right\}. \tag{A13}
\end{align*}
\]
2. S-polarization. $B_-$ mode

The geometry is shown in Fig. 5, the $z$–axis is directed out of the plane of figure. $B_-$ indicates the amplitude of the magnetic field in the north arm, the wave is absent in the east arm.

Surface 1R of BS (s-polarization, $B_-$ mode):

$$E_z = \frac{B_-}{\sqrt{2}} (e^{ikx_n} - e^{-ikx_n}) + B_- (e^{ikx_n} - e^{-ikx_n}),$$  \hspace{1cm} (A14a)

$$E_{z|\eta=0} = i\sqrt{2} B_- (\sqrt{2} - 1) \sin k_{bs},$$  \hspace{1cm} (A14b)

$$H_{\eta|\eta=0} = (1 - \sqrt{2}) B_- \cos k_{bs},$$  \hspace{1cm} (A14c)

$$H_{\xi|\eta=0} = (1 + \sqrt{2}) B_- \cos k_{bs},$$  \hspace{1cm} (A14d)

Maxwell stress tensor on surface 1R:

$$\sigma_{\eta\eta} = -\frac{|B_-|^2}{4\pi} \left(3 + 4\sqrt{2} - 3\right) \cos 2k_{bs}\xi.$$  \hspace{1cm} (A15)

The component $\sigma_{\eta\eta}$ corresponds to the pressure acting along the $\eta$–axis, hence $p_{\eta} = \sigma_{\eta\eta}$. A negative sign in (A15) means, that the field pushes the BS. The spatial distribution of the pressure $p_{\eta}$ has to be restored in (A15), by projecting (3.1) on the reflecting surface of the BS:

$$p_{\eta}^{1rS-} = -\frac{|B_-|^2}{4\pi} \left(3 + 4\sqrt{2} - 3\right) \cos 2k_{bs}\xi f^2_\perp,$$  \hspace{1cm} (A16)

$f_\perp$ is defined by (3.12).

Surface 2R of BS (s-polarization, $B_-$ mode):

$$E_z = \frac{B_-}{\sqrt{2}\sqrt{n}} (e^{iky_n} - e^{-iky_n}),$$  \hspace{1cm} (A17a)

$$H_{\eta|\eta=0} = -\sqrt{\frac{2}{\sqrt{n}}} B_- \sin k_{bs}\xi,$$  \hspace{1cm} (A17b)

$$H_{\xi|\eta=0} = \sqrt{\frac{2}{\sqrt{n}}} B_- \cos k_{bs}\xi,$$  \hspace{1cm} (A17c)

Maxwell stress tensor on surface 2R:

$$\sigma_{\eta\eta} = -\frac{2|B_-|^2}{4\pi} \left(-\cos^2 \alpha + \sin^2 \alpha \cos 2k_{bs}\xi\right).$$  \hspace{1cm} (A18)

Here, the component $\sigma_{\eta\eta}$ corresponds to the pressure acting along the $\eta$–axis, hence $p_{\eta} = -\sigma_{\eta\eta}$. The spatial distribution of the pressure $p_{\eta}$ has to be restored in (A15), by projecting (3.1) on the reflecting surface of the BS:

$$p_{\eta}^{2rS-} = \frac{2|B_-|^2}{4\pi} \left(\cos^2 \alpha - \sin^2 \alpha \cos 2k_{bs}\xi\right)f^2_\perp.$$  \hspace{1cm} (A19)

The total pressure acting on surface R (s-polarization, $B_-$ mode) reads:

$$p_{\eta}^{1s-} = p_{\eta}^{1rS-} + p_{\eta}^{2rS-},$$  \hspace{1cm} (A20)

$$p_{\eta}^{2s-} = \frac{f^2_\perp}{4L} (2\cos^2 \alpha - 3 - \left[4\sqrt{2} - 3 + 2n \sin^2 \alpha\right] \cos 2k_{bs}\xi).$$  \hspace{1cm} (A21)

On surface 2AR of the BS (s-polarization, $B_-$ mode):

Apparently, in this case the fields will be the same, i.e. equation (A17) can be applied. Hence, equation (A18) for the stress tensor may be applied, but in this case the component $\sigma_{\eta\eta}$ corresponds to pressure acting along the $\eta$–axis and $p_{\eta} = \sigma_{\eta\eta}$. The spatial distribution of the pressure $p_{\eta}$ on the anti-reflecting surface of the BS is equal to:

$$p_{\eta}^{2arS-} = \frac{2|B_-|^2}{4\pi} \left(-\cos^2 \alpha + \sin^2 \alpha \cos 2k_{bs}\xi\right)f^2.$$

Thus, $p_{\eta}^{2arS-} = -p_{\eta}^{2rS-}$ (with substitution $f_\perp$ instead of $f_\perp$).

Surface 3AR of BS (s-polarization, $B_-$ mode):

$$E_z = \frac{B_-}{\sqrt{2}} (e^{ikx_n} - e^{-ikx_n}),$$  \hspace{1cm} (A23a)

$$E_{z|\eta=0} = i\sqrt{2} B_- \sin k_{bs}\xi_-, $$  \hspace{1cm} (A23b)

$$H_{\eta|\eta=0} = -B_- \cos k_{bs}\xi_-, $$  \hspace{1cm} (A23c)

$$H_{\xi|\eta=0} = B_- \cos k_{bs}\xi_-, $$  \hspace{1cm} (A23d)

Maxwell stress tensor on surface 3AR:

$$\sigma_{\eta\eta} = -\frac{|B_-|^2}{4\pi} \left(1 - \cos 2k_{bs}\xi_\perp\right).$$  \hspace{1cm} (A24)

$$p_{\eta}^{3arS-} = \frac{|B_-|^2}{4\pi} \left(1 - \cos 2k_{bs}\xi_\perp\right)f^2.$$  \hspace{1cm} (A25)

Total pressure acting on surface AR (s-polarization, $B_-$ mode):

$$p_{\eta}^{3s-} = p_{\eta}^{2arS-} + p_{\eta}^{3arS-},$$  \hspace{1cm} (A26)
\[ p^*_q \frac{\partial S^*}{\partial E} = \frac{f^2}{4L} \left( \frac{1}{n} - 2n \right) + \left[ \frac{1}{n} - 1 \right] \cos 2k_b \xi. \]  

(A27)

\[ H_z = \frac{B_+}{\sqrt{2}} (e^{ikx_\eta} + e^{-ikx_\eta}), \]  

(A28a)

\[ H_z|_{\eta=0} = \sqrt{2} B_+ \cos k_b \xi, \]  

(A28b)

\[ E_n = \frac{B_+}{\sqrt{2}} (e^{ikx_\eta} - e^{-ikx_\eta})_{\eta=0}, \]  

(A28c)

\[ E_\xi = E_\eta = \frac{E_n}{\sqrt{2}} = iB_+ \sin k_b \xi. \]  

(A28d)

On surface 1R of BS (P-polarization, B_+ mode):

\[ H_z = \frac{\sqrt{n}B_+}{\sqrt{2}} (e^{iky_n} + e^{-iky_n}) \]  

(A31a)

\[ p^{_{\text{trP}}}_q = \frac{n|B_+|^2}{4\pi} \left( 1 + \cos \sqrt{2}k \xi \right) f_\perp. \]  

(A30)

On surface 2R (p-polarization, B_+ mode):

\[ H_z = \frac{\sqrt{n}B_+}{\sqrt{2}} (e^{iky_n} + e^{-iky_n}) \]  

\[ E_\xi = -\frac{E_n}{\sqrt{2}} = -iB_+ \sin k_b \xi. \]  

(A31b)

\[ E_y = \frac{B_+ \sin \alpha}{\sqrt{n}\sqrt{2}} (e^{iky_n} - e^{-iky_n}), \]  

(A31c)

\[ E_q|_{\eta=0} = \frac{i\sqrt{2}B_+ \sin \alpha}{\sqrt{n}} \left( \sqrt{2} + 1 \right) \sin k_b \xi. \]  

(A31d)

\[ E_\xi|_{\eta=0} = \frac{i\sqrt{2}B_+ \cos \alpha}{\sqrt{n}} \left( \sqrt{2} - 1 \right) B_+ \sin k_b \xi. \]  

(A31e)

Maxwell stress tensor on reflecting surface 2R:

\[ \sigma_{nq} = -\frac{n|B_+|^2}{4\pi} \left\{ 6 \cos^2 \alpha \right. \]  

\[ + \left( 2 \left[ \sqrt{2} + 1 \right]^2 - 6 \cos^2 \alpha \right) \cos 2k_b \xi \left( A32 \right) \]

Component \( \sigma_{nq} \) corresponds to pressure acting along the \( \eta \)-axis, hence \( p_q = \sigma_{nq} \). The spatial distribution of the pressure \( p_q \) should be restored in (A29), by projecting (3.1) on the reflecting surface of the BS by (3.12):

\[ p^{_{\text{trP}}}_q = \frac{n|B_+|^2}{4\pi} \left\{ 6 \cos^2 \alpha \right. \]  

\[ + \left( 2 \left[ \sqrt{2} + 1 \right]^2 - 6 \cos^2 \alpha \right) \cos 2k_b \xi \left( A33 \right) \]

Total pressure applied to surface R (p-polarization, B_+ mode) is equal to sum (A30) and (A33):

\[ p^{_{\text{trP}}}_q = p^{_{\text{trP}}}_q + p^{_{\text{trP}}}_q, \]  

(A34)

\[ p^{_{\text{trP}}}_q = \frac{f^2}{4L} \left\{ 6n \cos^2 \alpha - 1 \right. \]  

\[ + \left( 2n \left[ \sqrt{2} + 1 \right]^2 - 6n \cos^2 \alpha - 1 \right) \cos 2k_b \xi \left( A35 \right) \]

On surface 3AR (p-polarization, B_+ mode). Again we have to account that distributions of fields from south arm and east arm are shifted by \( \pm a \) (3.16) as shown in Fig. 6:

\[ H_z = \frac{B_+}{\sqrt{2}} (e^{ikx_n} + e^{-ikx_n}) f_\perp \]  

(A36a)

\[ + \frac{B_+}{\sqrt{2}} (e^{ikx_\eta} + e^{-ikx_\eta}) f_\perp, \]  

(A36b)

\[ E_\xi = -\frac{B_+}{\sqrt{2}} (e^{ikx_n} - e^{-ikx_n}) \frac{f_\perp}{\sqrt{2}} \]  

(A36c)

\[ H_z|_{\eta=-h} = \sqrt{2}B_+ \left( f_- \cos k_b \xi_\eta \right) + \sqrt{2} f_+ \cos k_b \xi_\eta \right), \]  

(A36b)

\[ E_\xi = -\frac{B_+}{\sqrt{2}} (e^{ikx_n} - e^{-ikx_n}) \frac{f_-}{\sqrt{2}} \]  

(A36c)

FIG. 6: B_+ mode, p-polarization, i.e. the electric field (blue arrows) is in the plane of figure, the magnetic field is perpendicular to the plane; fat dot means that it is directed out of plane, cross means into the plane.
Maxwell stress tensor on anti-reflecting surface 3AR:

\[ E_{\xi\eta=-h} = iB_+ \left( \sqrt{2} f_+ \sin k_{bs} \xi_+ - f_- \sin k_{bs} \xi_- \right), \]  
(A36d)

\[ E_\eta = \frac{B_+}{\sqrt{2}} \left( e^{ikx_+} - e^{-ikx_+} \right) f_-, \]  
(A36e)

\[ + B_+ \left( e^{ikx_+} - e^{-ikx_+} \right) f_+, \]  
\[ \quad + \frac{\sqrt{2}}{2} \left( f_- \sin k_{bs} \xi_- + \sqrt{2} f_+ \sin k_{bs} \xi_+ \right). \]  
(A36f)

Maxwell stress tensor on surface 2AR:

\[ \sigma_{\eta\eta} = -\frac{|B_+|^2}{4\pi} \left\{ f_-^2 + 2f_+^2 + 2\sqrt{2} f_- f_+ \cos k_{bs} (\xi_+ + \xi_-) \right\}, \]  
(A37)

\[ \sigma_{\eta\eta} = -\frac{|B_+|^2}{4\pi} \left\{ 1 + \frac{1}{n} - 2n \right\}, \]  
(A38)

Pressure along \( \eta \)-axis:

\[ p_{\eta\eta}^{2arP} = -\sigma_{\eta\eta}. \]  
(A39)

\[ H_z = \frac{\sqrt{n} B_+}{\sqrt{2}} \left( e^{iky_+} - e^{-iky_+} \right) f_- \]  
(A39a)

\[ + \sqrt{n} B_+ \left( e^{iky_+} - e^{-iky_+} \right) f_+, \]  
(A39b)

\[ E_\xi = -\frac{B_+}{\sqrt{2n}} \left( e^{iky_+} - e^{-iky_+} \right) f_- \cos \alpha \]  
(A39c)

\[ + B_+ \left( e^{iky_+} - e^{-iky_+} \right) f_+ \cos \alpha, \]  
(A39d)

\[ E_\xi|_{\eta=-h} = i\frac{\sqrt{2} B_+}{\sqrt{n}} \cos \alpha \]  
(A39e)

\[ \times \left( \sqrt{2} f_+ \sin k_{bs} \xi_+ - f_- \sin k_{bs} \xi_- \right), \]  
(A39f)

\[ E_\eta = \frac{B_+}{\sqrt{2n}} \left( e^{iky_+} - e^{-iky_+} \right) f_- \sin \alpha \]  
(A39g)

\[ + B_+ \left( e^{iky_+} - e^{-iky_+} \right) f_+ \sin \alpha, \]  
(A39h)

\[ E_\eta|_{\eta=-h} = \frac{\sqrt{2} B_+}{\sqrt{n}} \sin \alpha \left( f_- \sin k_{bs} \xi_- + \sqrt{2} f_+ \sin k_{bs} \xi_+ \right). \]  
(A39i)

Maxwell stress tensor on surface 1R:

\[ \sigma_{\eta\eta} = \frac{|B_-|^2}{2\pi} \left\{ 1 + 3 + 3\sqrt{2} \cos k_{bs} \xi_+ \right\}. \]  
(A40)

Pressure along \( \eta \)-axis:

\[ p_{\eta\eta}^{2arP} = \sigma_{\eta\eta}. \]  
(A41)

Total pressure along the \( \eta \)-axis acting on surface AR (p-polarization, \( B_+ \) mode) is equal to:

\[ p_{\eta\eta}^{2arP} = p_{\eta\eta}^{2arP} + p_{\eta\eta}^{3arP} \]  
(A42)

\[ p_{\eta\eta}^{2arP} = \frac{|B_+|^2}{4\pi} \left\{ \left( f_-^2 + 2f_+^2 \right) \left[ 1 + \frac{1}{n} - 2n \right] + (4(1-n)\sqrt{2} f_- f_+ \cos k_{bs} (\xi_- + \xi_+)) + (f_-^2 \cos 2k_{bs} \xi_- + 2f_+^2 \cos 2k_{bs} \xi_+) \left( 1 - \frac{1}{n} \right) \right\}. \]  
(A43)

4. **P-polarization. \( B_- \) mode**

**On surface 1R (p-polarization, \( B_- \) mode):**

\[ H_z = \frac{-B_-}{\sqrt{2}} \left( e^{ikx_+} + e^{-ikx_+} \right) \]  
(A44a)

\[ + B_- \left( e^{ikx_+} + e^{-ikx_+} \right), \]  
(A44b)

\[ E_\eta|_{\eta=0} = i \left( \sqrt{2} - 1 \right) B_- \sin k_{bs} \xi_-, \]  
(A44c)

\[ E_\xi|_{\eta=0} = -i \left( 1 + \sqrt{2} \right) B_- \sin k_{bs} \xi_-, \]  
(A44d)

Component \( \sigma_{\eta\eta} \) corresponds to the pressure acting along the \( \eta \)-axis, hence \( p_\eta = \sigma_{\eta\eta} \). The spatial distribution of the magnetic field is normal to the plane of figure, fat dot means that it is directed into the plane, cross means out of plane. \( B_- \) is the amplitude of the electric field in the north arm. Blue arrows indicate the direction of the electric field.
the pressure $p_\eta$ should be restored in (A45), by projecting (3.1) on surface 1R:

$$p_{\eta 1R}^- = -\frac{|B_-|^2}{4\pi} \left( 3 + 3 - 4\sqrt{2} \right) \cos 2k_\eta \xi \right) f_\eta^2.$$

On surface 2R ($s$-polarization, $B_-$ mode):

$$H_z = \frac{\sqrt{n} B_-}{\sqrt{2}} \left( e^{ik\eta} + e^{-ik\eta} \right),$$

$$H_z|_{\eta=0} = \sqrt{2n} B_- \cos k_\eta \xi,$$

$$E_{\eta}|_{\eta=0} = i \frac{\sqrt{2} B_-}{\sqrt{n}} \sin \alpha \sin k_\eta \xi,$$

$$E_{\xi}|_{\eta=0} = -i \frac{\sqrt{2} B_-}{\sqrt{n}} \cos \alpha \sin k_\eta \xi,$$

Maxwell stress tensor on surface 2R:

$$\sigma_{\eta\eta} = \frac{2n|B_-|^2}{4\pi} \left( -\cos^2 \alpha - \sin^2 \alpha \cos 2k_\eta \xi \right).$$

Here component $\sigma_{\eta\eta}$ corresponds to pressure acting along the $\eta$-axis, hence $p_\eta = -\sigma_{\eta\eta}$. The spatial distribution of the pressure $p_\eta$ should be restored in (A48), by projecting (3.1) on the reflecting surface of the BS:

$$p_{\eta 2R}^- = \frac{2n|B_-|^2}{4\pi} \left( \cos^2 \alpha + \sin^2 \alpha \cos 2k_\eta \xi \right) f_\eta^2.$$

Total pressure acting on surface R ($p$-polarization, $B_-$ mode):

$$p_{\eta R}^- = p_{\eta 1R}^- + p_{\eta 2R}^-,$$

$$\frac{p_{\eta R}^-}{E_-} = \frac{f_\eta^2}{4\eta} \left( 2n \cos^2 \alpha - 3 \right) \left[ 4\sqrt{2} - 3 + 2n \sin^2 \alpha \right] \cos 2k_\eta \xi \right) f_\eta^2.$$

On surface 2AR ($p$-polarization, $B_-$ mode):

$$p_{\eta 2AR}^- = \frac{2n|B_-|^2}{4\pi} \left( \cos^2 \alpha + \sin^2 \alpha \cos 2k_\eta \xi \right) f_\eta^2.$$

Obviously, $p_{\eta 2AR}^- = -p_{\eta 2R}^-$. 

On surface 3AR ($p$-polarization, $B_-$ mode):

$$H_z = \frac{B_-}{\sqrt{2}} \left( e^{ik\eta} + e^{-ik\eta} \right),$$

$$H_z|_{\eta=0} = \sqrt{2} B_- \cos k_\eta \xi,$$

$$E_{\eta}|_{\eta=0} = i B_- \sin k_\eta \xi,$$

$$E_{\xi}|_{\eta=0} = -i B_- \sin k_\eta \xi,$$

Maxwell stress tensor on surface 3AR:

$$\sigma_{\eta\eta} = \frac{|B_-|^2}{4\pi} \left( 1 + \cos 2k_\eta \xi \right)$$

$$p_{\eta 3AR}^- = \frac{|B_-|^2}{4\pi} \left( 1 + \cos 2k_\eta \xi \right) f_\eta^2.$$

Total pressure acting on surface AR ($p$-polarization, $B_-$ mode):

$$p_{\eta AR}^- = p_{\eta 2AR}^- + p_{\eta 3AR}^-,$$

$$\frac{p_{\eta AR}^-}{E_-} = \frac{f_\eta^2}{4\eta} \left( 1 - 2n \cos^2 \alpha \right) \left[ 1 - 2n \sin^2 \alpha \right] \cos 2k_\eta \xi \right) f_\eta^2.$$

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