Restricting Specht modules of cyclotomic Hecke algebras

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Abstract. This paper proves that the restriction of a Specht module for a (degenerate or non-degenerate) cyclotomic Hecke algebra, or KLR algebra, of type $A$ has a Specht filtration.

1. Introduction

It is a well-known and classical result in the semisimple representation theory of the symmetric group $S_n$, for $n \geq 1$, that the restriction of a Specht module is the direct sum of Specht modules indexed by certain partitions of $n - 1$. In general, only the weaker result holds that the restriction of a Specht module has a filtration by smaller Specht modules. That is, a filtration such that each subquotient is isomorphic to a Specht module.

All of these results generalise to the cyclotomic Hecke algebras of type $A$, which are certain deformations of the group algebra of the complex reflection group $S_n \wr (\mathbb{Z}/\ell \mathbb{Z})$, for $n, \ell \geq 1$. In the semisimple case, the restriction of a Specht module for a cyclotomic Hecke algebra was described by Ariki and Koike [3] using seminormal forms. In the non-semisimple case, the corresponding restriction theorem exists in the literature [4, Proposition 1.9], however, the proof given in [4], and repeated in [23, Proposition 6.1], is incomplete. Ariki [2, Lemma 13.2] has proved the weaker statement in the Grothendieck group.

Using the Murphy basis of the Specht module $S^\lambda$ for $H_n$, it is easy to construct an explicit $H_{n-1}$-module filtration of $S^\lambda$ that looks like a Specht filtration of $S^\lambda$. Unfortunately, as Goodman points out, it is not obvious that the subquotients in this filtration are isomorphic to Specht modules. This note gives a self-contained proof that the restriction of a Specht module has a Specht filtration (see Theorem 4.1 for a more precise statement).

Brundan, Kleshchev and Wang [10, Theorem 4.11] extended the Specht module restriction theorem to the graded setting, where this result is an important ingredient in the proof of the Ariki-Brundan-Kleshchev categorification theorem [9, Theorem 5.14]. The proof of the graded restriction theorem given in [10] uses, in an essential way, the construction in [4]. Moreover, the arguments of [10] only apply in the non-degenerate case and they require the ground-ring to be a field. We prove the graded restriction theorem for graded Specht modules of the cyclotomic KLR algebras of type $A$ defined over an integral domain, thus recovering and extending the results of [10].

The layout of this paper is as follows. Section 2 defines the cyclotomic Hecke algebras of type $A$ and sets up the combinatorial framework that is used throughout the paper. Section 3 gives a self-contained proof of the known branching rule for the Specht modules in the semisimple case using the theory of seminormal forms. Section 4 using the result from the semisimple case to show that the restriction of a Specht module for a cyclotomic
Hecke algebra of type $A$ has a Specht filtration, which we describe explicitly. Finally, Section 5 introduces the cyclotomic KLR algebras of type $A$, and their graded Specht modules. We show that the restriction of a graded Specht module has a filtration by graded Specht modules, up to shift, and proves analogous results for the dual graded Specht modules.

The corresponding results for inducing Specht modules can be found in [16, 24, 27].

2. ARIKI-KOIKE ALGEBRAS

This section introduces the cyclotomic Hecke algebras of type $A$. Throughout this note we fix positive integers $\ell$ and $n$ and let $\mathcal{S}_n$ be the symmetric group of degree $n$. For $1 \leq r < n$ let $s_r = (r, r + 1) \in \mathcal{S}_n$. Then $s_1, \ldots, s_{n-1}$ are the standard Coxeter generators of $\mathcal{S}_n$.

We use a slight variation on Ariki and Koike’s original definition of the cyclotomic Hecke algebras of type $A$; compare with [3, Definition 3.1] and [6, Definition 2.1]. The advantage of the definition below is that it allows us to treat the so-called degenerate and non-degenerate cases simultaneously, which correspond to $\xi^2 = 1$ and $\xi^2 \neq 1$, respectively. Previously these cases were treated separately.

Fix a commutative integral domain $\mathcal{A}$, with 1.

2.1. Definition (Hu-Mathas [17, Definition 2.2]). The cyclotomic Hecke algebra of type $A$, with Hecke parameter $\xi \in \mathcal{A}^\times$ and cyclotomic parameters $Q_1, \ldots, Q_\ell \in \mathcal{A}$, is the initial associative $\mathcal{A}$-algebra $\mathcal{H}_n = \mathcal{H}_n^A = \mathcal{H}_n(A, \xi, Q_1, \ldots, Q_\ell)$ with generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ and relations

\[ \prod_{r=1}^n (L_1 - Q_r) = 0, \quad (T_r + \xi^{-1})(T_r - \xi) = 0, \quad L_{r+1} = T_rL_rT_r + T_r, \]

\[ L_rL_t = L_tL_r, \quad T_rT_s = T_sT_r \text{ if } |r-s| > 1, \]

\[ T_rT_{s+1}T_r = T_{s+1}T_rT_{s+1}, \quad T_rL_t = L_rT_r \text{ if } t \neq r, r+1, \]

where $1 \leq r < n$, $1 \leq s < n-1$ and $1 \leq t \leq n$.

The arguments of Ariki and Koike [3, Theorem 3.10] show that $\mathcal{H}_n$ is free as an $\mathcal{A}$-module with basis $\{L_1^{a_1} \cdots L_n^{a_n}T_w | 0 \leq a_1, \ldots, a_n < \ell \text{ and } w \in \mathcal{S}_n\}$, where $T_w = T_{w_1} \cdots T_{w_\ell}$ if $w = s_{r_1} \cdots s_{r_\ell} \in \mathcal{S}_n$ is a reduced expression (that is, $k$ is minimal).

The Ariki-Koike basis theorem implies that there is a natural embedding of $\mathcal{H}_{n-1}$ in $\mathcal{H}_n$ and that $\mathcal{H}_n$ is free as an $\mathcal{H}_{n-1}$-module of rank $\ell n!$. If $M$ is an $\mathcal{H}_n$-module let

\[ \text{Res} M : \mathcal{H}_n-\text{Mod} \rightarrow \mathcal{H}_{n-1}-\text{Mod} \]

be the restriction functor from category of $\mathcal{H}_n$-modules to $\mathcal{H}_{n-1}$-modules. The functor $\text{Res}$ is exact.

The algebra $\mathcal{H}_n$ has another basis that is better adapted to the representation theory of $\mathcal{H}_n$. Recall that a partition of $n$ is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of non-negative integers such that $|\lambda| = \sum \lambda_i = n$. A multipartition, or $\ell$-partition, of $n$ is an ordered $\ell$-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of partitions such that $|\lambda^{(i)}| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$.

Let $P_n$ be the set of multipartitions of $n$. If $\lambda, \mu \in P_n$ then $\lambda$ dominates $\mu$, written $\lambda \geq \mu$, if

\[ \sum_{c=1}^{k-1} |\lambda^{(c)}| + \sum_{r=1}^{s} \lambda^{(k)}_r \geq \sum_{c=1}^{k-1} |\mu^{(c)}| + \sum_{r=1}^{s} \mu^{(k)}_r, \]

for $1 \leq k \leq \ell$ and for all $s \geq 1$. Dominance is a partial order on $P_n$.

Identify a multipartition $\lambda$ with its diagram, which is the set of nodes

\[ \{(l, r, c) | 1 \leq \lambda^{(l)}_r \leq c \text{ and } 1 \leq l \leq \ell\}. \]

More generally a node is any element of $\{1, \ldots, \ell\} \times \mathbb{N}^2$, which we order lexicographically. We picture the diagram of a multipartition as an array of boxes in the plane, allowing us to
talk of the components, rows and columns of $\lambda$. For example, if $\lambda = (3, 2^3 | 2^2)$ then

$$\lambda = \begin{pmatrix}
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}
\end{pmatrix}$$

has three components and $\lambda(2) = (1^3)$ has three rows and one column.

A **removable** node of $\lambda$ is any node $\rho \in \lambda$ such that $\lambda - \rho := \lambda \setminus \{\rho\}$ is (the diagram of) a multipartition. Write $\mu \rightarrow \lambda$ if $\mu$ is obtained from $\lambda$ by removing a removable node. Similarly, a node $\alpha \notin \lambda$ is an **addable** node if $\lambda \cup \{\alpha\}$ is a multipartition.

The removable nodes of $\lambda$ are totally ordered by the lexicographic order. As a consequence, the set of multipartitions $\{\mu \in \mathcal{P}_{n-1} \mid \mu \rightarrow \lambda\}$ is totally ordered by dominance.

If $\lambda \in \mathcal{P}_n$ then a $\lambda$-tableau is a function $t: \lambda \rightarrow \{1, 2, \ldots, n\}$. Write $\text{Shape}(t) = \lambda$ if $t$ is a $\lambda$-tableau. For convenience, identify $t = (t^{(1)}, \ldots, t^{(\ell)})$ with a labeling of the diagram $\lambda$ by elements of $\{1, \ldots, n\}$ in the obvious way. In this way we talk of the rows, columns and components of $t$.

A **standard $\lambda$-tableau** is a $\lambda$-tableau $t$ such that the entries in each row of $t^{(k)}$ increase from left to right and the entries in each column of $t^{(k)}$ increase from top to bottom, for $1 \leq k \leq \ell$. Let $\text{Std}(\lambda)$ be the set of standard $\lambda$-tableaux and set $\text{Std}(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda)$ and $\text{Std}^2(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda) \times \text{Std}(\lambda)$.

If $t$ is a standard tableau and $1 \leq m \leq n$ let $t_{ij}$ be the subtableau of $t$ that contains the numbers $1, 2, \ldots, m$. It is easy to see that $t_{ij}$ is standard because $t$ is standard. Extend the dominance ordering to $\text{Std}(\mathcal{P}_n)$ by declaring that $s \trianglerighteq t$ if $\text{Shape}(s_{ij}) \trianglerighteq \text{Shape}(t_{ij})$, for $1 \leq m \leq n$. As we will need it often, set $t_{ij} = t_{ij(n-1)}$.

Using the observations in the last paragraph, it is easy to verify the following well-known combinatorial fact, which underpins all of the results in this note.

2.2. **Lemma.** Suppose that $\lambda \in \mathcal{P}_n$. Then there is a bijection

$$\text{Std}(\lambda) \xrightarrow{\sim} \bigcup_{\mu \rightarrow \lambda, \mu \in \mathcal{P}_n} \text{Std}(\mu) \quad \text{(disjoint union)},$$

given by $t \mapsto t_{\lambda}$.

Let $t^\lambda$ be the unique standard $\lambda$-tableau such that $t^\lambda \trianglerighteq t$, for all $t \in \text{Std}(\lambda)$. More explicitly, the numbers $1, 2, \ldots, n$ increase from left to right, and then top to bottom, in each component of $t^\lambda$ with the numbers in $t^{(r)}$ being less than the numbers in $t^{(s)}$ whenever $1 \leq r < s \leq \ell$. The symmetric group $\mathfrak{S}_n$ acts on the right on the tableaux with entries in $\{1, 2, \ldots, n\}$ by permuting their entries. If $t$ is a standard $\lambda$-tableau let $d(t) \in \mathfrak{S}_n$ be the unique permutation such that $t = t^\lambda d(t)$.

For each multipartition $\lambda \in \mathcal{P}_n$, define $m_\lambda = u_\lambda x_\lambda$, where

$$u_\lambda = \prod_{s=2}^{\ell} \prod_{m=1}^{\lceil \lambda^{(s)} \rceil - 1} \prod_{s=2}^{\ell} \prod_{m=1}^{\lceil \lambda^{(s)} \rceil - 1} (L_m - Q_s) \quad \text{and} \quad x_\lambda = \sum_{w \in \mathfrak{S}_n} \xi^d(t) T_w.$$

(This corrects the definition of $u_\lambda$ given in [25, §1.5].)

Let $\ast$ be the unique anti-isomorphism of $\mathcal{H}_n$ that fixes all of the generators in Definition 2.1. In particular, $T_{w^\ast} = T_{w^{-1}}$. Define the **Murphy basis** element $m_{st}^\lambda = T_{d(t)}^* m_\lambda T_{d(t)}$, for $s, t \in \text{Std}(\lambda)$. By [11, Theorem 3.26] and [5, Theorem 6.3],

$$(2.3) \quad \{m_{st} \mid s, t \in \text{Std}(\lambda) \text{ and } \lambda \in \mathcal{P}_n\}$$

is a cellular basis of $\mathcal{H}_n$, in the sense of Graham and Lehrer [13]. Consequently, if $\mathcal{H}_n^{\lambda \lambda}$ is the $A$-module spanned by $\{m_{st} \mid s, t \in \text{Std}(\mu) \text{ for some } \mu \in \mathcal{P}_n \text{ with } \mu \trianglerighteq \lambda\}$, then $\mathcal{H}_n^{\lambda \lambda}$ is a two-sided ideal of $\mathcal{H}_n$. 


The Specht module $S^\lambda$ is the right $H_n$-submodule of $H_n/\mathcal{O}_n^\lambda$ generated by $m_\lambda + \mathcal{O}_n^\lambda$. It follows from the general theory of cellular algebras that $S^\lambda$ is free as an $A$-module with basis $\{m_t \mid t \in \text{Std}(\lambda)\}$, where $m_t = m_t + \mathcal{O}_n^\lambda$ for $t \in \text{Std}(\lambda)$. We write $S^\lambda = S^\lambda_A$ when we want to emphasize that $S^\lambda$ is an $A$-module.

Let $M$ be an $H_n$-module. Then $M$ has a Specht filtration if there exists a filtration
\[ 0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M \]
and multipartitions $\lambda_1, \ldots, \lambda_\ell$ such that $M_r/M_{r-1} \cong S^{\lambda_r}$, for $r = 1, \ldots, \ell$. The main results of this paper say that if $\lambda \in \mathcal{P}_n$ then restriction of $S^\lambda$ to $H_n^{-1}$ has a Specht filtration.

3. THE SEMISIMPLE CASE

The aim of this note is to understand how the Specht modules of the cyclotomic Hecke algebras of type $A$ behave under restriction. We start by considering the easiest case when $H_n$ is semisimple. In this case, the Specht module restriction theorem is classical, building from Young’s foundational work on the symmetric group from 1901 [28] to the results of Ariki and Koike [3]. We give the proof here both because it is not very difficult and because everything that follows builds upon this result.

First, we need to define content functions for tableaux. If $k \in \mathbb{Z}$ define the $\zeta$-quantum integer to be the scalar
\[
[k]_\zeta = \begin{cases} 
\zeta + \zeta^3 + \cdots + \zeta^{2k-1}, & \text{if } k \geq 0, \\
-\zeta^{2k-1}(-1)_\zeta, & \text{if } k < 0.
\end{cases}
\]
If $t \in \text{Std}(\mathcal{P}_n)$ is a standard tableau, and $1 \leq k \leq n$, then the content of $k$ in $t$ is
\[ c_k(t) = \zeta^{2(c-r)}Q_{l} + [c-r]_\zeta \quad \text{where} \quad (l, r, c) = t^{-1}(k). \]
The condition $(l, r, c) = t^{-1}(k)$ means that $k$ is in row $r$ and column $c$ of $t^{(i)}$.

The main property of the Murphy basis that we need is that the elements $L_1, \ldots, L_n$ act upon this basis triangularly. More precisely, we have the following:

3.1. Proposition ([18, Proposition 3.7] and [5, Lemma 6.6]). Suppose that $\lambda \in \mathcal{P}_n$, $t \in \text{Std}(\lambda)$ and $1 \leq k \leq n$. Then
\[ m_tL_k = c_k(t)m_t + \sum_{m \vdash t} r_s m_s, \]
for some $r_s \in A$.

The semisimplicity of $H_n$ is determined by the Poincaré polynomial of $H_n$:
\[ P_{H_n} = [1]_\zeta[2]_\zeta \cdots [n]_\zeta \prod_{1 \leq k < l \leq t} \prod_{-n < m < n} (\zeta^{2m}Q_k + [m]_\zeta - Q_l) \in A. \]
For example, if $\zeta^2 = 1$ then $H_n \cong A\mathfrak{S}_n$ and $P_{H_n} = n!$.

For symmetric groups, the next result is well-known. The extension of this result to $H_n$ when $\zeta^2 \neq 1$ is due to Ariki [1]. (The order of our exposition is misleading, however, because the most natural way to prove the equivalence of parts (a) and (b) in Proposition 3.2 is to use Theorem 3.3 below.)

3.2. Proposition (Ariki [1], Ariki, Mathas, Rui [5, Theorem 6.11]). Suppose that $A$ is a field. Then the following are equivalent:

a) The algebra $H_n$ is semisimple.

b) If $s, t \in \text{Std}(\mathcal{P}_n)$ then $s = t$ if and only if $c_m(s) = c_m(t)$ for $1 \leq m \leq n$.

c) The Poincaré polynomial $P_{H_n}$ is non-zero.
For the remainder of this section we assume that \( P_{\mathcal{H}_n} \) is invertible in \( \mathcal{A} \). The assumption that \( P_{\mathcal{H}_n} \) is a unit is useful precisely because it implies that \( c_m(t) - c_m(s) \) is invertible in \( \mathcal{A} \) whenever \( s, t \in \text{Std}(\mathcal{P}_n) \) and \( c_m(s) \neq c_m(t) \), for some \( m \). This follows because \( c_m(t) - c_m(s) \) divides \( P_{\mathcal{H}_n} \), up to a power of \( \xi \). Consequently, if \( t \in \text{Std}(\mathcal{P}_n) \) then

\[
F_t = \prod_{m=1}^{n} \prod_{\substack{s \in \text{Std}(\mathcal{P}_n) \\ c_m(s) \neq c_m(t)}} \frac{L_m - c_m(t)}{c_m(t) - c_m(s)} \in \mathcal{H}_n.
\]

If \( s, t \in \text{Std}(\lambda) \) define \( f_{st} = F_s^m a_s F_t \). Then \( \{ f_{st} \} \) is a \textit{seminormal basis} of \( \mathcal{H}_n \), in the sense of \cite[Definition 3.7]{17}. The next result summaries that main proper ties of such bases.

For the symmetric groups the following result goes back to Young \cite{28}. For the Hecke algebras of types \( A \) and \( B \) this was essentially proved by Hoefsmit \cite{14}. The general cyclotomic case was established by Ariki and Koike \cite{3}. Here we follow the exposition of \cite{17,25}.

### 3.3. Theorem

Suppose that \( P_{\mathcal{H}_n} \in \mathcal{A}^\times \). Then \( \{ f_{st} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n \} \) is a basis of \( \mathcal{H}_n \) and the following hold:

a) If \( (s, t) \in \text{Std}^2(\mathcal{P}_n) \) and \( 1 \leq m \leq n \) then

\[ f_{st} L_m = c_m(t) f_{st} \quad \text{and} \quad L_m f_{st} = c_m(s) f_{st}. \]

b) If \( (s, t) \in \text{Std}^2(\mathcal{P}_n) \) and \( 1 \leq r < n \) then

\[ f_{st} T_r = \frac{1 + (\xi - 1)}{c_r(t)} f_{st} + \alpha_r(t) f_{st}, \]

where \( v = (r, r + 1) \) and

\[
\alpha_r(t) = \begin{cases} 
1, & \text{if } t \triangleright v, \\
\left(1 - \frac{c_r(t) + \xi c_r(v)}{c_r(t) - c_r(v)} \right) - \frac{1 - \xi}{1 + \xi}, & \text{otherwise}.
\end{cases}
\]

c) If \( \lambda \in \mathcal{P}_n \) then \( S^\lambda \cong f_{st} \mathcal{H}_n \), for any \( s, t \in \text{Std}(\lambda) \).

d) If \( \mathcal{A} \) is a field then \( \mathcal{H}_n \) is a split semisimple algebra and \( \{ S^\lambda \mid \lambda \in \mathcal{P}_n \} \) is a complete set of pairwise non-isomorphic irreducible \( \mathcal{H}_n \)-modules.

Seminormal bases are most commonly given for Specht modules rather than for the regular representation. Parts (a), (c) and (d) of Theorem 3.3 are contained in \cite[Theorem 3.9]{17}, although most of Theorem 3.3 can be deduced from results in \cite{3,5}. As shown in \cite[Proposition 3.11]{17}, part (b) follows because, by (2.3) and Proposition 3.1, if \( (s, t) \in \text{Std}^2(\mathcal{P}_n) \) then

\[
f_{st} = F_s^m a_s F_t = m_{st} + \sum_{(u, v) \in \text{Std}^2(\mathcal{P}_n)} r_{uv} m_{uv}, \quad \text{for some } r_{uv} \in \mathcal{A},
\]

where \( r_{uv} \neq 0 \) only if either \( \text{Shape}(u) \triangleright \text{Shape}(v) \) or \( (u, v) \neq (s, t) \), \( \text{Shape}(u) = \text{Shape}(v) \) and \( u \triangleright s \) and \( v \triangleright t \).

By part (c) of the theorem, \( S^\lambda \cong f_{st} \mathcal{H}_n \) for any \( s, t \in \text{Std}(\lambda) \). The module \( f_{st} \mathcal{H}_n \) has basis \( \{ f_{uv} \mid u \in \text{Std}(\lambda) \} \) and if \( u \in \text{Std}(\lambda) \) then \( f_{st} \mathcal{H}_n \cong f_{ut} \mathcal{H}_n \) with an isomorphism being given by \( f_{st} \mapsto f_{ut} \) by Theorem 3.3. In view of (3.4), we can identify \( S^\lambda \) with the right ideal \( f_{st} \mathcal{H}_n \) of \( \mathcal{H}_n \), which has basis \( \{ f_{uv} \mid u \in \text{Std}(\lambda) \} \).

An almost immediate consequence of Theorem 3.3 is the branching rule for the Specht modules in the semisimple case. Recall that \( t_i = t_{i(n-1)} \), for \( t \in \text{Std}(\mathcal{P}_n) \).

### 3.5. Corollary

(Ariki-Koike \cite[Corollary 3.12]{3}). Suppose that \( P_{\mathcal{H}_n} \in \mathcal{A}^\times \) and fix a multipartition \( \lambda \in \mathcal{P}_n \). Then

\[
\text{Res } S^\lambda \cong \bigoplus_{\mu \in \mathcal{P}_{\lambda^{-1}}} S^\mu,
\]
as $\mathcal{H}_{n-1}$-modules. An explicit isomorphism is given by $f_{t^1} \mapsto f_{t^1_1}$, where $t \in \text{Std}(\lambda)$ and $\mu = \text{Shape}(t_1)$.

**Proof.** As above, we may assume that $S^\lambda$ has basis $\{f_{t^1} \mid t \in \text{Std}(\lambda)\}$. If $\mu \in \mathcal{P}_{n-1}$ and $\mu \rightarrow \lambda$ let $S^\mu \rightarrow \lambda$ be the vector subspace of $S^\lambda$ with basis $\{f_{t^1_1} \mid t_1 \in \text{Std}(\mu)\}$. By Lemma 2.2, there are vector space isomorphisms

$$S^\lambda \cong 
\bigoplus_{\mu \in \mathcal{P}_{n-1}} S^{\mu \rightarrow \lambda} \cong 
\bigoplus_{\mu \in \mathcal{P}_{n-1}} S^\mu,$$

where the second isomorphism is determined by $f_{t^1} \mapsto f_{t^1_1}$, for $t \in \text{Std}(\lambda)$ and where $\mu = \text{Shape}(t_1)$. By parts (a)–(c) of Theorem 3.3, this map restricts to an isomorphism $S^{\mu \rightarrow \lambda} \cong S^\mu$ of $\mathcal{H}_{n-1}$-modules whenever $\mu \in \mathcal{P}_{n-1}$ and $\mu \rightarrow \lambda$. We are done. □

We have shown that Corollary 3.5 holds whenever $P_{\mathcal{H}_n}$ is invertible in $A$. In particular, this result does not require the coefficient ring to be a field.

4. **The non-semisimple case**

This section proves the first main theorem of this note, which is the restriction theorem for the Specht modules over an integral domain. This generalises Corollary 3.5, the branching rule for the Specht modules in the semisimple case. The main difference is that when $P_{\mathcal{H}_n}$ is invertible then $\text{Res} S^\lambda$ is isomorphic to a direct sum of “smaller” Specht modules whereas, in general, the restriction of a Specht module only has a Specht filtration.

The next result appears in the literature as [4, Proposition 1.9], however, the proof given there is incomplete. The first paragraph of the proof of Theorem 4.1, which is given below, is essentially the same as the proof of this result given in [4]. The remaining paragraphs complete the argument from [4] by showing that the subquotients in the filtration we construct are isomorphic to appropriate Specht modules.

Recall from Section 2 that $>$ is the lexicographic order on the set $\{1, \ldots, \ell\} \times \mathbb{N}^2$ of nodes. Consequently, if $A$ and $B$ are removable nodes of $\lambda \in \mathcal{P}_n$ then $A > B$ only if $A$ is in a latter component of $\lambda$ than $B$, or $A$ and $B$ are in the same component and $A$ is in a latter row.

4.1. **Theorem.** Suppose that $\lambda \in \mathcal{P}_n$ and let $A_1 > \cdots > A_z$ be the removable nodes of $\lambda$, ordered lexicographically. Then $\text{Res} S^\lambda$ has an $\mathcal{H}_{n-1}$-module filtration

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{z-1,\lambda} \subset S^{z,\lambda} = \text{Res} S^\lambda$$

such that $S^{r,\lambda}/S^{r-1,\lambda} \cong S^{\lambda-A_r}$, for $1 \leq r \leq z$.

**Proof.** Let $\mu_r = \lambda - A_r$, for $1 \leq r \leq z$. Then $\{\mu_1, \ldots, \mu_z\} = \{\mu \in \mathcal{P}_{n-1} \mid \mu \rightarrow \lambda\}$ and $\mu_1 \supset \cdots \supset \mu_z$. Recall from Section 2 that $\{m_t \mid t \in \text{Std}(\lambda)\}$ is a basis of $S^\lambda$. Motivated by the proof of Corollary 3.5, for $1 \leq r \leq z$ define $S^{r,\lambda}$ to be the $A$-submodule of $S^\lambda$ with basis $\{m_t \mid \text{Shape}(t_1) \supseteq \mu_r\}$ and set $S^{0,\lambda} = 0$. Then $S^\lambda$ has an $A$-module filtration

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{z-1,\lambda} \subset S^{z,\lambda} = S^\lambda.$$

We claim that this is the required $\mathcal{H}_{n-1}$-module filtration of $S^\lambda$. By Proposition 3.1, if $1 \leq r \leq z$ then the $A$-module $S^{r,\lambda}$ is stable under the action of $L_1, \ldots, L_n$. As remarked in [4], it is possible to show that $S^{r,\lambda}$ is an $\mathcal{H}_{n-1}$-module using [11, Proposition 3.18] (in the cases when $\xi^2 \neq 1$), however, we will prove this directly below.

Fix $r$ with $1 \leq r \leq z$. To complete the proof of the theorem it is enough to show that $S^{r,\lambda} \cong S^{r,\lambda}/S^{r-1,\lambda}$ as $\mathcal{H}_{n-1}$-modules. Let $\theta_r : S^{r,\lambda}/S^{r-1,\lambda} \rightarrow S^{r,\lambda}$ be the unique $A$-linear map such that

$$\theta_r(m_t + S^{r-1,\lambda}) = m_t,$$

for $t \in \text{Std}(\lambda)$ with $t_1 \in \text{Std}(\mu_r)$.

By Lemma 2.2, $\theta_r$ is an $A$-module isomorphism. We claim that $\theta_r$ is an isomorphism of $\mathcal{H}_{n-1}$-modules.
Let \( Z = Z[v, v^{-1}, q_1, \ldots, q_\ell] \), where \( v, q_1, \ldots, q_\ell \) are indeterminates over \( \mathbb{Z} \), and consider the Hecke algebra \( H_n^Z = H_n(Z, v, q_1, \ldots, q_\ell) \). Then \( \mathcal{H}_n \cong H_n^Z \otimes_{\mathbb{Z}} A \) and \( S_A^\lambda \cong S_Z^\lambda \otimes_{\mathbb{Z}} A \), where we consider \( A \) as a \( Z \)-module by letting \( v \) act as multiplication by \( v \) and \( q_k \) as multiplication by \( Q_k \), for \( 1 \leq k \leq \ell \). Similarly, let \( H_n^{-1} \) be the obvious subalgebra of \( H_n^Z \) such that \( H_n^{-1} \cong H_n^{-1} \otimes_{\mathbb{Z}} A \). By definition, the map \( \theta \), commutes with base change. Therefore, in order to show that \( \theta_\tau \) is an \( \mathcal{H}_{n-1} \)-module homomorphism it is enough to consider the case when \( A = Z \). Let \( K = \mathbb{Q}(v, q_1, \ldots, q_\ell) \) be the field of fractions of \( Z \). Then \( S_K^\lambda \cong S_Z^\lambda \otimes Z \), so by considering \( S_Z^\lambda \) as a \( Z \)-submodule of \( S_K^\lambda \) it follows that \( \theta_\tau \) is a homomorphism of \( \mathcal{H}_n^{-1} \)-modules if and only if the induced map \( \theta_\tau \otimes 1_K \) over \( K \) is a homomorphism of \( \mathcal{H}_{n-1} \)-modules. Hence, we are reduced to the case when \( A = K \).

Assume now that \( A = K \) and \( \mathcal{H}_n = \mathcal{H}_n^K \). Then \( P_{\mathcal{H}_n^K} \neq 0 \), so \( \mathcal{H}_n^K \) has a seminormal basis \( \{f_t\} \) by Theorem 3.3. For \( t \in \text{Std}(\lambda) \) set \( f_t = f_{\lambda t} + \mathcal{H}_n^\lambda \). Then \( \{f_t \mid t \in \text{Std}(\lambda)\} \) is a basis of \( S_K^\lambda \). By (3.4), if \( t \in \text{Std}(\lambda) \) then

\[
f_t = m_t + \sum_{s \geq 1} a_s t_s \quad \text{for some } a_s \in K.
\]

Consequently, \( S_K^\lambda \) has basis \( \{t \mid t \in \text{Std}(\lambda) \text{ and Shape}(t_i) \geq \mu_i\} \). In particular, \( S_K^\lambda \) is an \( \mathcal{H}_{n-1} \)-submodule of \( S_K^\lambda \) by Theorem 3.3. Although we do not need this, this also implies that \( S_K^\nu \cong S_K^{\nu, -1} \oplus \cdots \oplus S_K^{\nu, -\lambda} \) as \( \mathcal{H}_{n-1} \)-modules, where we use the notation from the proof of Corollary 3.5.

By Corollary 3.5 the \( K \)-module isomorphism determined by

\[
\tilde{\theta} : S_K^\lambda \rightarrow \bigoplus_{\mu \in P_{a-1}} S_K^\mu : f_t \mapsto f_{\mu t}
\]

is an isomorphism of \( \mathcal{H}_{n-1} \)-modules. Moreover, since the transition matrix between the Murphy basis \( \{m_t\} \) and the seminormal basis \( \{f_t\} \) of \( S_K^\lambda \) is unitriangular, it follows by downwards induction on dominance that \( \tilde{\theta}(m_t) = m_{t_1} \), for all \( t \in \text{Std}(\lambda) \). By the last paragraph, \( \tilde{\theta} \) induces an \( \mathcal{H}_{n-1} \)-homomorphism \( \theta_\tau : S_K^\nu \otimes_{S_K^{\tau-1}} S_K^{\tau-1} \rightarrow S_K^\nu \) given by

\[
\tilde{\theta}(f_t + S_K^{\tau-1}) = \tilde{\theta}(f_t) = f_{t_1}, \quad \text{for } t \in \text{Std}(\lambda) \text{ with Shape}(t_i) = \mu_i.
\]

Therefore, \( \tilde{\theta}_\tau(m_t + S_K^{\tau-1}) = \tilde{\theta}(m_t) = \theta_\tau(m_t + S_K^{\tau-1}) \). Hence, \( \theta_\tau = \tilde{\theta}_\tau \) is an \( \mathcal{H}_{n-1} \)-module homomorphism as we wanted to show. Moreover, there is an isomorphism of \( \mathcal{H}_{n-1} \)-modules \( S_K^\nu \cong S_K^{\nu, -1} \otimes_{S_K^{\tau-1}} S_K^{\nu, -\lambda} \), for \( 1 \leq r \leq z \), so this completes the proof.

In level one (that is, when \( \ell = 1 \)), a more complicated proof of Theorem 4.1, which builds on a sketch by the author, can be found in [12].

5. Graded Specht modules

In this final section we consider the analogue of Theorem 4.1 for the graded Specht modules of the corresponding cyclotomic KLR (or quiver) Hecke algebra \( H_n^\lambda \). The cyclotomic KLR Hecke algebras [19, 26] are a family of \( \mathbb{Z} \)-graded algebras that arise in the categorification of quantum groups. For quivers of type \( A \), Brundan and Kleshchev [9] have shown that over a field the cyclotomic quiver Hecke algebras are isomorphic to the cyclotomic Hecke algebras.

Let us set the notation. Fix \( e \in \{2, 3, 4, \ldots\} \cup \{\infty\} \) and let \( \Gamma_e \) be the quiver with vertex set \( I = \mathbb{Z}/e\mathbb{Z} (\text{put } e\mathbb{Z} = \{0\} \text{ when } e = \infty) \), and with edges \( i \rightarrow i+1 \), for \( i \in I \). To the quiver \( \Gamma_e \), we attach the standard Lie theoretic data of a Cartan matrix \( (c_{ij})_{i,j \in I} \), fundamental weights \( \{\Lambda_i \mid i \in I\} \), positive weights \( P^+ = \sum_{i \in I} N_i \Lambda_i \) and positive roots \( Q^+ = \bigoplus_{i \in I} N_i \alpha_i \). Let \((\cdot, \cdot)\) be the bilinear form satisfying

\[
(\alpha_i, \alpha_j) = c_{ij} \quad \text{and} \quad (\Lambda_i, \alpha_j) = \delta_{ij}, \quad \text{for } i, j \in I.
\]
Let $Q^+_n = \{ \beta \in Q^+ | \sum_{i \in I} (\Lambda_i, \beta) = n \}$ and set $I^\beta = \{ i \in I^\beta | \beta = \alpha_{i_1} + \cdots + \alpha_{i_n} \}$ for $\beta \in Q^+_n$.

A graded $A$-module or, more accurately a $\mathbb{Z}$-graded $A$-module, is an $A$-module $M$ with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that each summand $M_d$ is $A$-free and of finite rank. We always assume that only finitely many of the $M_d$ are non-zero. A graded $A$-algebra is an $A$-algebra $A = \bigoplus_{d \in \mathbb{Z}} A_d$ that is a graded module such that $A_c A_d \subseteq A_{c+d}$, for all $c, d \in \mathbb{Z}$. A graded $A$-module is an $A$-module that is graded and $M_c A_d \subseteq M_{c+d}$, for all $c, d \in \mathbb{Z}$. If $M$ is a graded $A$-module let $M$ be the module obtained by forgetting the grading. If $s \in \mathbb{Z}$ let $q^s M$, where $q$ is an indeterminate, be the graded $A$-module obtained by shifting the grading by $s$, so that $(q^s M)_d = M_{d-s}$. Two graded $A$-modules are isomorphic if there is a degree preserving isomorphism between them. For example, $M \cong q^s M$ if and only if $s = 0$.

5.1. Definition. Fix $e \in \{2, 3, 4, \ldots \} \cup \{ \infty \}$ and suppose that $\Lambda \in P^+$ and $\beta \in Q^+$. The cyclotomic quiver Hecke algebra $R^\Lambda_\beta$ is the unital associative $A$-algebra with generators

$$\{ \psi_1, \ldots, \psi_{n-1} \} \cup \{ y_1, \ldots, y_n \} \cup \{ e(i) | i \in I^\beta \}$$

and relations

$$y_1^{(\Lambda, \alpha_{i_1})} e(i) = 0, \quad e(i) e(j) = \delta_{ij} e(i), \quad \sum_{i \in I^\beta} e(i) = 1,$$

$$y_re(i) = e(i)y_r, \quad \psi_r e(i) = e(\text{s}.r \cdot i)\psi_r, \quad y_r y_s = y_s y_r,$$

$$\psi_r y_{r+1} e(i) = (\psi_r y_r + \delta_{i, r+1}) e(i), \quad y_{r+1} \psi_r e(i) = (\psi_r y_r + \delta_{i, r+1}) e(i),$$

$$\psi_r y_s = y_s \psi_r, \quad \text{if } s \neq r, r + 1,$$

$$\psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r - s| > 1,$$

$$\psi_r^n e(i) = Q_{i, i+1}(y_r, y_{r+1}) e(i),$$

$$(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_r) e(i) = \frac{Q_{i, i+2}(y_{r+2}, y_{r+1}) - Q_{i, i+1}(y_r, y_{r+1})}{y_{r+2} - y_r} e(i)$$

for $i, j \in I^\beta$ and all admissible $r$ and $s$ and where

$$Q_{ij}(u, v) = \begin{cases} (u - v)(v - u), & \text{if } i \equiv j, \\ (u - v), & \text{if } i \rightarrow j, \\ (v - u), & \text{if } i \leftarrow j, \\ 0, & \text{if } i = j, \\ 1, & \text{otherwise.} \end{cases}$$

For $n \geq 0$, define $\mathcal{R}^\Lambda_n = \bigoplus_{\beta \in Q^+_n} \mathcal{R}^\Lambda_\beta$.

The algebra $\mathcal{R}^\Lambda_n$ is a $\mathbb{Z}$-graded algebra with degree function determined by

$$\deg e(i) = 0, \quad \deg y_r = 2 \quad \text{and} \quad \deg \psi_s e(i) = (\alpha_{i_r}, \alpha_{i_{r+1}}),$$

for $i \in I^n$, $1 \leq r \leq n$ and $1 \leq r < n$.

Inspecting the relations in Definition 5.1, $\mathcal{R}^\Lambda_n$ has a unique anti-isomorphism $\gamma$ that fixes each of the generators.

To connect these algebras with the cyclotomic Hecke algebras, let $e \in \{2, 3, 4, \ldots \} \cup \{ \infty \}$ be minimal such that $[e]_k = 0$, or set $e = \infty$ if $[k]_e \neq 0$ for $k > 0$.

For $\Lambda \in P^+$ choose a multicharge $\kappa = (\kappa_1, \ldots, \kappa_\ell) \in \mathbb{Z}^\ell$ such that

$$(\Lambda, \alpha_i) = \# \{ 1 \leq k \leq \ell | \kappa_k \equiv i \pmod{e} \} \quad \text{for all } i \in I.$$ 

Let $\mathcal{H}^\Lambda_n$ be the cyclotomic Hecke algebra with parameters $Q_r = [k_i]_e$, for $1 \leq r \leq \ell$. Up to isomorphism, the algebra $\mathcal{H}^\Lambda_n$ depends only on $\Lambda$ and not on the choice of multicharge.

5.2. Theorem (Brundan and Kleshchev [8]). Suppose that $\mathcal{A}$ is a field. Then $\mathcal{H}^\Lambda_n \cong \mathcal{R}^\Lambda_n$. 
The isomorphism of Theorem 5.2 holds only over a field even though the cyclotomic KLR algebra $R_\lambda^\Lambda$ is defined over any ring. For each $\lambda \in \mathcal{P}_n$ there is a graded Specht module $S^\lambda$ such that $S^\lambda \cong S_k^\lambda$ as (ungraded) $R_\lambda^\Lambda$-modules. To define these, and to prove a graded version of Theorem 4.1, we need some notation.

Fix $i \in I$. A node $A = (l, r, c)$ is an $i$-node if $i = \kappa_l + r + c \in \mathbb{Z}$. Similarly, if $t \in \Std(P_n)$ and $1 \leq k \leq n$ then the $c$-residue of $k$ in $t$ is

$$\res_c(t) = \kappa_l + r + c \mathbb{Z} \quad \text{if} \quad t(l, r, c) = k.$$ 

The residue sequence of $t$ is $\res(t) = (\res_1(t), \ldots, \res_n(t)) \in I^n$. Set $I^\lambda = \res(t^\lambda)$.

For $t \in I$ and $\lambda \in \mathcal{P}_n$ let $\Add_t(\lambda)$ be the set of addable $i$-nodes for $\lambda$ and let $\Rem_t(\lambda)$ be the set of removable $i$-nodes. Order $\Add_t(\lambda)$ and $\Rem_t(\lambda)$ lexicographically by $<$. 

5.3. Definition (Brundan, Kleshchev and Wang [10, §3.5]). Suppose that $\lambda \in \mathcal{P}_n$ and let $A$ be an addable or removable $i$-node of $\lambda$. Define integers

$$d^\lambda_A = \# \{ B \in \Add_t(\lambda) \mid B > A \} - \# \{ B \in \Rem_t(\lambda) \mid B > A \},$$

$$d^\lambda_A = \# \{ B \in \Add_t(\lambda) \mid B < A \} - \# \{ B \in \Rem_t(\lambda) \mid B < A \}.$$

If $t \in \Std(\lambda)$ define the degree of $t$ inductively by setting $A = t^{-1}(n)$ and

$$\deg t = \begin{cases} \deg t_1 + d^\lambda_A, & \text{if } n > 0, \\ 0, & \text{if } n = 0. \end{cases}$$

Following [15, Definition 4.9], for $\lambda \in \mathcal{P}_n$ define $d^\lambda_k = \deg t^\lambda_{k+1} - \deg t^\lambda_{k-1}$, for $k = 1, 2, \ldots, n$. Define $e^\lambda = e(1^\lambda)$ and $y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_n^{\lambda_n}$. For $s, t \in \Std(\lambda)$ set

$$\psi_{st} = \psi^{\lambda}_{(s,t)} e^\lambda y^\lambda \psi_{d(t)},$$

where for each $w \in S_n$ we fix an arbitrary reduced expression $w = s_{r_1} \cdots s_{r_k}$ and define $\psi_w = \psi_{r_1} \cdots \psi_{r_k} \in S_n^\lambda$. In general, the elements $\{ \psi_{st} \}$ depend upon the choices of reduced expression for the elements of $S_n$. By definition, $(\psi_{st})^\pi = \psi_{ts}$.

Hu and the author [15] proved that if $A$ is a field then $\{ \psi_{st} \}$ is a graded cellular basis of $R_\lambda^\Lambda$, where graded cellularity is a natural extension of the definition of cellular algebras to the graded setting. In particular, the existence of a graded cellular basis automatically gives graded Specht modules. In fact, $R_\lambda^\Lambda$ is a graded cellular algebra over any integral domain.

5.4. Theorem (Li [21, Theorem 1.1]). Suppose that $\Lambda \in P^+$ and let $A$ be an integral domain. Then $\{ \psi_{st} \mid s, t \in \Std(\lambda) \}$ for some $\lambda \in \mathcal{P}_n$ is a graded cellular basis of $R_\lambda^\Lambda$. In particular, $\deg \psi_{st} = \deg s + \deg t$, for $s, t \in \Std(\lambda)$ and $A \in \mathcal{P}_n$.

Following [15], we now define the graded Specht modules although we need to be slightly careful to choose the “correct” grading shift. In more detail, if $\lambda \in \mathcal{P}_n$ let $S_\lambda^\Lambda$ be the submodule of $S_\lambda^\Lambda$ spanned by $\{ \psi_{at} \mid s, t \in \Std(\mu) \}$ where $\mu \triangleright \lambda$. Then $S^\Lambda_{\lambda}$ is a two-sided ideal of $S_{\lambda}^\Lambda$. The graded Specht module is the $S^\Lambda_{\lambda}$-module

$$S^\Lambda = q^{-\deg t^\lambda} (\psi_{t^\lambda} e^\Lambda + S^\Lambda_{\lambda}) S^\Lambda_{\lambda},$$

Set $\psi_t = \psi_{t^\lambda} + S^\Lambda_{\lambda} \in S^\Lambda$. Then $\{ \psi_t \mid t \in \Std(\lambda) \}$ is a homogeneous basis of $S^\Lambda$, where the grading is given by $\deg \psi_t = \deg t$, for $t \in \Std(\lambda)$. By Theorem 5.4, the Specht module $S^\Lambda$ is defined over an integral domain. The integral graded Specht module can also be defined using [20], which gives homogeneous relations for the graded Specht modules.

Importantly, if $A$ is a field and $\lambda \in \mathcal{P}_n$ then $S^\Lambda_\lambda \cong S_\lambda^\Lambda$ as (ungraded) $R^\Lambda_\lambda$-modules, so it is natural to ask for a graded analogue of Theorem 4.1. In the non-degenerate case ($\xi^2 \neq 1$), and when working over a field, this result appears as Brundan, Kleshchev and Wang [10, Theorem 4.11], however, their proof relies on the arguments of [4]. We give a self-contained proof of this result that applies over an integral domain.
5.5. Theorem. Suppose that $A$ is an integral domain, $\lambda \in \mathcal{P}_n$ and let $A_1 > \cdots > A_z$ be the removable nodes of $\lambda$, ordered lexicographically. Then $\text{Res} S^\lambda$ has an $R_{n-1}$-module filtration

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{z-1,\lambda} \subset S^z = \text{Res} S^\lambda$$

such that $S^{r,\lambda}/S^{r-1,\lambda} \cong q^{d_r}, \lambda-A_r$, for $1 \leq r \leq z$.

The special case of Theorem 5.5 when $A$ is a field, and $e$ is not equal to the characteristic, recovers [10, Theorem 4.11].

Proof. The proof follows the same strategy as the proof of Theorem 4.1. For $1 \leq r \leq n$ let $\mu_r = \lambda - A_r$ and define $S^{r,\lambda}$ to be the $A$-submodule of $S^\lambda$ with basis

$$\{\psi_1 | \text{Shape}(t_1) \cong \mu_r\}.$$ 

Then, as an $A$-module, $S^\lambda$ has a filtration $0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{z,\lambda} = S^\lambda$. By Lemma 2.2 and Definition 5.3, there are graded $A$-module isomorphisms

$$\theta_r : S^{r,\lambda}/S^{r-1,\lambda} \longrightarrow q^{d_r}, S^{\lambda-A_r}; \psi_1 + S^{r-1,\lambda} \rightarrow \psi_t$$

for $1 \leq r \leq z$. By definition, $\theta_r$ is homogeneous, however, it is not clear that it is a $R_{n-1}$-module homomorphism.

Fix $r$ with $1 \leq r \leq z$. We claim that $\theta_r$ is an isomorphism of graded $R_{n-1}$-modules. By Theorem 5.4, or [20, Corollary 6.24], $S^\lambda_A \cong S^\lambda_{Z} \otimes_A A$. By definition, $\theta_r$ commutes with base change so it is enough to prove the theorem in the special case when $A = Z$. Let $\xi$ be a primitive $e$th root of unity, or an indeterminate over $Q$ if $e = \infty$, and let $K = Q(\xi)$. Then $S^\lambda_{Z}$ embeds in $S^\lambda_K \cong S^\lambda_Z \otimes_Z K$, so it is enough to prove the theorem over $K$. Now, by Theorem 5.2, $R_{n-1} \otimes_Z K \cong H_n^K$, so we are reduced to showing that $\theta_r$ is an $H_n^K$-module homomorphism. To complete the proof we need one more observation: by [10, Proposition 4.5], or [15, Lemma 5.4], there exist non-zero scalars $c_t \in K$ such that

$$\psi_1 = c_t m_t + \sum_{v \in \text{Std}(\lambda)} r_v m_v, \quad \text{for } r_v \in K.$$ 

That is, the transition matrix between the $\psi$-basis of $S^\lambda_{K}$ and the Murphy basis is triangular.

It follows that the map $\theta_r$, defined above, coincides exactly with the map $\theta_r$ defined in the proof of Theorem 4.1. In particular, $\theta_r$ is a (homogeneous) $H_n^K$-module homomorphism, as we needed to show. \qed

5.6. Remarks.

a) The argument above proves Theorem 5.5 by reducing to the situation considered in Theorem 4.1 that, in turn, reduces to the semisimple case of Corollary 3.5. Using the framework developed in [17] it is possible to deduce Theorem 5.5 directly from Corollary 3.5.

b) In the special case when $A$ is a field, Theorem 5.5 can be deduced almost directly from Theorem 4.1 since the transition matrix between the $\psi$-basis and the Murphy basis of the Specht module is unitriangular. We use the integral form of $S^\lambda$ in the proof of Theorem 5.5 only to obtain the result over an integral domain.

c) Because of Definition 2.1, the proof of Theorem 5.5 simultaneously treats both the degenerate and non-degenerate cases. Only the non-degenerate case was considered in [10].

Fix $i \in I$. If $j \in I_{n-1}$ let $j \vee i = (j_1, \ldots, j_{n-1}, i) \in I^n$ and define

$$c_{n-1,i} = \sum_{j \in I_{n-1}} e(j \vee i) \in R_{n-1}.$$
Using the relations, and Theorem 5.2, there is a non-unital homogeneous algebra embedding \( A_{n-1} \rightarrow A_n \) given by

\[
e(j) \mapsto e(j \lor i), \quad y_r \mapsto e_{n-1,i}y_r, \quad \text{and} \quad \psi_j \mapsto e_{n-1,i}\psi_j,
\]

for \( j \in I^{n-1}, 1 \leq r < n \) and \( 1 \leq s < n - 1 \). If \( i \in I \) and \( M \) is an \( A_{n-1} \)-module define \( i-\text{Res} M = Me_{n-1,i} \). Then there is an isomorphism of functors

\[
\text{Res} \cong \bigoplus_{i \in I} i-\text{Res}
\]
corresponding to the identity \( 1_{A_{n-1}} = \sum_{i \in I} e_{n-1,i} \). By [7, 8, 22], the blocks of \( A_{n-1} \) are labelled by \( P_n^+ \). For \( \beta \in P_n^+ \) let \( P_\beta = \{ \mu \in P_n \mid 1^\mu \in I^\beta \} \), for \( m \geq 0 \).

Applying the idempotent \( e_{n-1,i} \) to Theorem 5.5 gives the following:

**Corollary.** Let \( A \) be an integral domain. Suppose that \( i \in I \) and \( \lambda \in P_{\beta+\alpha} \), where \( \beta \in P_n^+ \). Let \( B_1 > \cdots > B_y \) be the removable \( i \)-nodes of \( \lambda \), ordered lexicographically. Then \( i-\text{Res} S^\lambda \) has an \( A_{n-1} \)-module filtration

\[
0 = S_0^\lambda \subset S_1^\lambda \subset \cdots \subset S_y^\lambda = \text{i-Res} S^\lambda
\]
such that \( S_{y,r}^\lambda / S_{y-1,r}^\lambda \cong q^{\lambda_i^y} S_{\beta}^{A_{n-1}} \) as \( A_{n-1} \)-modules for \( 1 \leq r \leq y \).

Let \( \text{Rep}(A_{n-1}) \) be the Grothendieck group of \( A_{n-1} \). If \( M \) is an \( A_{n-1} \)-module let \( [M] \) be its image in \( \text{Rep}(A_{n-1}) \). Then \( \text{Rep}(A_{n-1}) \) is free as a \( \mathbb{Z}[q, q^{-1}] \)-module, where \( [qM] = q[M] \).

By Theorem 5.5, if \( i \in I \) then

\[
[i-\text{Res} S^\lambda] = \sum_{r=1}^{\infty} q^{\lambda_i^y} [S_{\beta}^{A_{n-1}}].
\]

This result is used in [9, Proposition 5.5] to show that the cyclotomic Hecke algebras categorify the highest weight representations of the quantum affine special linear group.

Finally, the papers [16, 20] also define dual graded Specht modules \( S_\lambda \), for \( \lambda \in P_n \). These modules can be defined in exactly the same way as the graded Specht modules using a “dual” graded cellular basis \( \{ \psi'_k \} \) of \( A_{n-1} \). This allows all of the arguments in this section to be repeated, essentially word-for-word, to prove analogues of Theorem 5.5 and Corollary 5.7 for the dual graded Specht modules. We state only the dual Specht module version of Corollary 5.7.

**Corollary.** Let \( A \) be an integral domain. Suppose that \( i \in I \) and \( \lambda \in P_{\beta+\alpha} \), where \( \beta \in P_n^+ \). Let \( B_1 > \cdots > B_y \) be the removable \( i \)-nodes of \( \lambda \), ordered lexicographically. Then \( i-\text{Res} S_\lambda \) has an \( A_{n-1} \)-module filtration

\[
0 = S_0^\lambda \subset S_1^\lambda \subset \cdots \subset S_y^\lambda = i-\text{Res} S_\lambda
\]
such that \( S_{y,r}^\lambda / S_{y-1,r}^\lambda \cong q^{\lambda_i^y} S_{\beta}^{-A_{n-1}} \) as \( A_{n-1} \)-modules for \( 1 \leq r \leq y \).

**Proof.** Rather than repeating the arguments of this section using the \( \psi' \)-basis, which we have not defined, we deduce the result from Corollary 5.7. First, if \( M \) is an \( A_{n-1} \)-module let \( M^0 = \text{Hom}_A(M, A) \) be the graded dual of \( M \), where \( A \) is in degree 0. Define the defect of \( \lambda \) to be the non-negative integer \( d(\lambda) = (\lambda, \beta_\lambda) - \frac{1}{2}(\beta_\lambda, \beta_\lambda) \), where \( i\lambda = (i_1^\lambda, \ldots, i_n^\lambda) \) and \( \beta_\lambda = \sum_{\alpha=1}^{n} \alpha i_\alpha \in Q^+ \). We can now prove the corollary.

Let \( \{ S_{r}^\lambda \mid 0 \leq r \leq y \} \) be the filtration of \( S^\lambda \) given by Corollary 5.7 and set

\[
S_{y,r}^\lambda = q^{d(\lambda)/2} \sum_{\gamma} S_{\gamma}^\lambda / S_{\gamma-1,r}^\lambda
\]
for \( 0 \leq r \leq y \).

Then \( 0 = S_0^\lambda \subset S_1^\lambda \subset \cdots \subset S_y^\lambda \subset S_{y,r}^\lambda = i-\text{Res} S_\lambda \) is a filtration of \( S_\lambda \) as an \( A_{n-1} \)-module. Fix \( r \) with \( 1 \leq r \leq y \). By Corollary 5.7, \( S_{y-1,r}^\lambda / S_{y,r}^\lambda \cong q^{d(\lambda)} S_{\beta}^{-A_{n-1}} \), as \( A_{n-1} \)-modules, so there is a short exact sequence

\[
0 \rightarrow q^{d(\lambda)} S_{\beta}^{-A_{n-1}} \rightarrow S^\lambda / S_{y-1,r}^\lambda \rightarrow S^\lambda / S_{y,r}^\lambda \rightarrow 0.
\]
By [20, Theorem 7.25], if $\mu \in \mathcal{P}_m$, then $S_{\mu}^{e} \cong q^{\text{def } \mu} (S_{\mu})^\circ$ as $S_{m}^{\Lambda}$-modules. Therefore, taking duals and shifting the degree by $q^{\text{def } \Lambda}$, there is a short exact sequence

$$0 \rightarrow S_{r-1, \Lambda} \rightarrow S_{r, \Lambda} \rightarrow q^{\text{def } \Lambda - \text{def } (\lambda - B_r)} B_r C_{\lambda - B_r} \rightarrow 0.$$ 

This completes the proof because $d_{\lambda}^{\Lambda}$ is $\text{def } \lambda - \text{def } (\lambda - B_r) - d_{B_r}^{\Lambda}$ by [10, Lemma 3.11].

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