Smoothed Noise and Mexican Hat Coupling Produce Pattern in a Stochastic Neural Field

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(Dated: January 16, 2018)

The formation of pattern in biological systems may be modeled by a set of reaction-diffusion equations. A diffusion-type coupling operator biologically significant in neuroscience is a difference of Gaussian functions (Mexican Hat operator) used as a spatial-convolution kernel. We are interested in the difference among behaviors of stochastic neural field equations, namely space-time stochastic differential-integral equations, and similar deterministic ones. We explore, quantitatively, how the parameters of our model that measure the shape of the coupling kernel, coupling strength, and aspects of the spatially-smoothed space-time noise, control the pattern in the resulting evolving random field. We find that a spatial pattern that is damped in time in a deterministic system may be sustained and amplified by stochasticity, most strikingly at an optimal spatio-temporal noise level. In addition, we find that spatially-smoothed noise alone causes pattern formation even without spatial coupling.

Keywords: neural field equation, difference of Gaussian coupling, stochastic process, spatial pattern, spatially-smoothed noise.

PACS: 89.75.-k, 89.75.Da, 05.45.Xt, 87.18.-h

MSC: 34C15, 35Q70

I. INTRODUCTION

In this paper we explore the formation of pattern by a stochastic neural field equation with simple damping as its reaction term. We have been shown by Hutt and colleagues [1] that spatial pattern, embedded in deterministic space-time dynamics but immediately damped, may be excited by noise. Butler and Goldenfeld [2, 3], and McKane, Biancalani and Rogers [4] also showed the existence of excitable spatial modes that, when noise was added, were revealed in power spectral densities. The knowledge of this possible noise-facilitation source of pattern, observed in biological systems, motivated us to begin, with a basic example, to explore how certain sample path properties of evolving stochastic neural fields depend on the parameters that control strength of coupling or local interaction in the field, noise intensity, and the extent of local sharing of noise.

The sample path properties we look at are: how pattern begins to appear when coupling is small and how it grows with coupling strength, and how pattern grows with noise level when coupling is fixed, becoming distinct and then less so with "too much noise." We explore how parameters controlling spatial smoothing of noise and field interaction combine to produce pattern. A striking result is that spatial smoothing of noise, alone, without direct field interaction, produces pattern.

We find that visual inspection of the time evolution of one-dimensional fields yields insight about typical sample path behavior of the stochastic process and inspires the formulation of the quantitative questions. We use two measures of spatial pattern. One is the power spectral density as a stochastic process in time. The second is a function of space, which we call $F$, that allows direct observation of dominant frequency, again a stochastic process in time. We believe that display of sample paths of both of these processes is innovative here, and that they will prove to be valuable in future studies of our model and of other stochastic neural fields.

What has come to be called a neural field equation is an integro-differential equation of the form

$$dY(t,x) = \left[-Y(t,x) + \int_{\mathbb{R}} w(x,y)S(Y(t,y))dy\right]dt, \tag{1}$$

where $Y(t,x)$ is an $\mathbb{R}^{1}$-valued state variable for a neural system, $w(x,y)$ is a coupling operator, for example the Mexican Hat convolution kernel, and $S$ is a scaling functional, typically a sigmoid, which keeps $Y(t,x)$ bounded in the diffusion term. For reasons to be explained shortly, we take $S$ to be the identity within a certain bounding range. We study a stochastic version of (1),

$$dY(t,x) = \left[-Y(t,x) + \int_{\mathbb{R}} cw(x,y)S(Y(t,y))dy\right]dt + \sigma dG(t,x), \tag{2}$$

where $\sigma$ is a constant diffusion coefficient, $c \geq 1$ is a constant, which we will call the coupling strength, and

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$G(t, x)$ is a Gaussian process that may depend on both time, $t$, and space, $x$. In our study of \([2]\), space is discretized, so that each value of $x$ represents the location of a component of the neural field equation \([2]\). We will use the word ‘component’ in this sense in the remainder of the paper.

Recently, the study of such equations has been put on a rigorous footing by Faugeras and Inglis \([5]\). Conveniently for the study of neural field equations that generate spatial patterns, they singled out the difference-of-Gaussians coupling operator (often called the Mexican Hat operator) as one that satisfies established conditions for the existence and uniqueness of a solution. As will be seen, this coupling operator generates pattern when used in equations like \([2]\). Sometimes the Laplacian has been used as a coupling operator, but used alone it does not generate spatial patterns \([6]\).

In order to make our study more motivated and accessible for mathematical biologists who are familiar with the seminal work of J. D. Murray on biological pattern formation, we begin with an example from his classic 1989 book ‘Mathematical Biology’ \([7]\). There, in Chapter 16, Murray showed us that the formation of spatial patterns in biological systems may be modeled by a set of deterministic reaction-diffusion equations, such as a neural field equation \([1]\). In Murray’s example, the diffusion-type coupling term was a difference of Gaussian functions used as a convolution kernel. In the next section, following Murray \([4, 5]\), we compute conditions for the existence of excitable spatial frequencies of a spatial pattern in a deterministic model \([1]\) with $w(x)$ a Mexican Hat kernel. We will see that in this basic deterministic model the spatial pattern generated by the excitable modes is damped for a range of parameter values, whereas added noise reveals and sustains the damped spatial patterns generated by excitable modes.

What is new here is our explorations of properties of stochastic sample paths (with graphical depictions) from the evolving random field of the neural field equation as parameters are varied. Also new is the result that smoothed noise, by itself, can create spatial patterns similar to those created by excitable spatial modes created by the Mexican Hat coupling. We explore the relation between the coupling constant, $c$, and a parameter of noise smoothing, the standard deviation of the Gaussian smoothing kernel, $\eta$, upon which the noise correlation length depends. We also illustrate the relation between the noise strength and the noise correlation length in their effects on the spatial pattern created by the noise smoothing alone (i.e., when $c = 0$).

II. EXCITABLE MODES OF THE NEURAL FIELD EQUATION

As in Murray \([7]\) we chose the convolution coupling kernel $w(x)$ in \([1]\) to be a difference of Gaussian functions, written

$$w(x) = b_1 \exp \left[ - \left( \frac{x}{d_1} \right)^2 \right] - b_2 \exp \left[ - \left( \frac{x}{d_2} \right)^2 \right],$$

where $b_1, b_2$ are the heights of the Gaussian functions at $z = 0$, and $d_1, d_2$ are their dispersions. We let the reaction part of the reaction-diffusion-type model \([1]\) be the simple damping $-Y(t, x)$, the generic reaction term appearing in \([5]\). We include the stochastic term, $\sigma dG(t, x)$ in \([2]\) to obtain a generic stochastic neural field equation. Particular choices of the functional $S$ and of the space-time noise will complete the description of our model.

The biological role of $S$ has been related to the fact that neurons can fire only below their maximum rate, which is theoretically around 500 Hz but practically below about 200 Hz. Our choice of $S$, convenient for our computations while also maintaining a bound, is:

$$S(Y(t - \Delta t, x)) = \begin{cases} Y(t - \Delta t, x) & \text{if } |Y(t - \Delta t, x)| \leq Y_{\text{max}}, \\ 0.9Y(t - \Delta t, x) & \text{if } |Y(t - \Delta t, x)| > Y_{\text{max}}. \end{cases}$$

By choosing $Y_{\text{max}} = 5$ we have a wide parameter range within which to carry out our numerical study. In addition, when parameters are within this range we find (numerically) that $Y(t - \Delta t, x)$ never exceeds $Y_{\text{max}}$, so that we can carry out our stochastic analysis by approximating $S$ by the identity. Such a functional satisfies the Lipschitz and boundedness conditions under which Faugeras and Inglis \([5]\) assure us of the existence and uniqueness of a non-constant solution to the stochastic neural field equation \([2]\).

A different, and probably more frequent, choice of $S$ is the logistic function, $S(Y) = 1/(1 + \exp(-Y))$ (e.g., \([9]\)). Although the logistic is a mathematical model of a soft threshold for neural firing, and thus appropriate in this context, it complicates the stochastic analysis of the relation between the coupling constant and the noise smoothing parameter. It does give very similar numerical results (not shown) to the ones we present later, and so can be considered to be analogous for our purposes.

An alternative to inserting a functional, $S$, to keep the process defined by \([2]\) stochastically bounded, is to replace the simple damping function by a cubic reaction function as in \([6, 10, 12]\). For the sake of simplicity, here we use \([2]\) instead.

A. Wave Numbers and Fourier Amplitudes

In our illustrative computations we chose a form for $w(x)$, the difference of the two Gaussian functions, that expresses the common biological observation that there is excitation within a small neighborhood around each location, and inhibition in a somewhat larger neighborhood around the excitation. Another way to achieve this
effect is to multiply by a Gaussian function and then operate with a Laplacian [13], yet another alternative operator that involves a squared Laplacian is used in [3]. We chose the Mexican Hat because it combines the clear and insightful treatment of Murray [7] with the foundational justification, in terms of existence and uniqueness of solutions to general neural field equations that include this operator, of Faugeras and Inglis [3]. The existence and identity of excitable spatial modes that lead to spatial patterns in neural fields has been studied also using other approaches (see, e.g., [1] and for a review see [14]).

As mentioned earlier, our (1) is equivalent to Murray’s (16.2) because the threshold in (4) is not exceeded, and so S in (1) can be taken as the identity function. Murray [7] showed us how to find conditions on parameters $b_1, b_2, d_1, d_2$ in (3) under which solutions to his (16.2), and thus our (1) under these conditions, have excitable modes. For convenience, we choose $b_2 = d_1 = 1$. So that $w(k)$ is indeed the Mexican Hat operator we need $b_1/b_2 > 1$ and $d_2/d_1 > 1$. The equations Murray [7] derived for $W(k)$, the Fourier transform of $w(x)$, and $k_{\text{max}}$, the wave number for which $W(k)$ is largest, under these conditions are:

$$W(k) = \sqrt{\pi} \left[ b_1 d_1 \exp \left( -\frac{(d_1k)^2}{4} \right) - b_2 d_2 \exp \left( -\frac{(d_2k)^2}{4} \right) \right],$$  

$$k_{\text{max}} = \left( 4(d_2^2 - d_1^2)^{-1} \ln \left[ \frac{b_2}{b_1} \left( \frac{d_2}{d_1} \right)^3 \right] \right)^{0.5}.$$  

We also choose $d_2 > b_1$ for $c = 1$. With this constraint, $W(k) < 0$ implies $b_1 d_1 - b_2 d_2 < 0$, $W(k)$ has the form [5], and $k_{\text{max}}$ exists [7].

The ansatz $Y(t, x) = \exp(\lambda t + jkx)$, where $j = \sqrt{-1}$, for a solution of (1) leads, for each $k$, to the eigenvalue (growth rate in time of wave number $k$)

$$\lambda(k) = -1 + \int w(x) e^{j k x} dx = -1 + W(k).$$  

There exist excitable modes, or wave numbers, $k$, if and only if $\lambda(k_{\text{max}}) = -1 + W(k_{\text{max}}) > 0$ or, equivalently, $W(k_{\text{max}}) > 1$.

In Section I we introduced the parameter $c$, the strength of the coupling accomplished by $w(x)$. Changing the coupling strength, $c$, changes the heights of the two Gaussians composing $w(x)$, [3] without changing their dispersions: $cw(x) = cb_1 \exp(-(x/d_1)^2) - cb_2 \exp(-(x/d_2)^2)$.

Multiplying the Mexican Hat operator, and hence also its Fourier transform, by $c \geq 1$, we have excitable wave numbers for (2) if $cW(k_{\text{max}}) > 1$. Figure 1 shows the range of values of $b_1$ and $d_2$ for which $W(k_{\text{max}}) > 1$. For parameters $d_2 > b_1$ where $W(k_{\text{max}}) > 1$ (above both blue curve and red line) there are excitable wave numbers, whereas when $W(k_{\text{max}}) < 1$ there are none. In Figure 2 we see the effect of coupling strength, $c$, on the presence of excitable wave numbers for (2). As $c$ increases, the range of values of $d_2$ at which these wave numbers appear increases. It can happen that $0 < W(k_{\text{max}}) < 1$, in which case there would be no excitable wave numbers with $c = 1$. But, if $c > 1$ then $cW(k_{\text{max}})$ can be greater than 1, in which case there will be excitable wave numbers. Figure 3 shows how this happens for a particular set of values $b_1 = 1.5, d_2 = 1.6$ and $c = 1$ or $c = 2$. It is clear that as $c$ increases, the range of excitable $k$ numbers also increases, limited by the interval of $k$ where $W(k) > 0$.

![FIG. 1. Region of $b_1, d_2$ parameter space for which $W(k_{\text{max}}) > 1$. Because of restrictions on the parameter values (see Section II A) only the hatched area above both the red line ($d_2 = b_1$) and the blue line ($W(k_{\text{max}}) > 1$) represent excitable wave numbers. Blue curve is interpolated by Matlab routine from discrete computations.](image-url)
there are excitable wave numbers (of k) no excitable wave numbers for these values of the parameters k for various values of other in the brain the more similar is the set of synaptic connections. This is because the nearer neurons are to each input noise may be shared in a neighborhood of locations. This turns out to have unexpected consequences for spatial pattern formation. Because the Fourier transform of a Gaussian is again a Gaussian but with role of the variance inverted, if $\eta$ is larger, and thus the smoothing kernel wider, the spatial spectrum of $Y(t, x)$ is narrower and concentrated closer to 0.

In Section V we explore by simulation the effect on the emerging spatial pattern of changing the standard deviation, $\eta$, of the Gaussian noise smoother for $c = 0, 5, 9$.

III. SPACE-TIME INPUT NOISE

In modeling a spatially discrete stochastic neural field, a default choice (e.g., [17]) has been to introduce an independent Brownian component for each $x$. If the locations are tightly packed, however, as in neural tissue, the same input noise may be shared in a neighborhood of locations. This is because the nearer neurons are to each other in the brain the more similar is the set of synaptic inputs they receive, and also because neurons communicate more extensively with each other the closer they are to each other [18, 19]. We are interested in the effect on spatial patterns of the size of the neighborhood in which some of the same input noise is felt. In our exploration we have included cases where noise is independent at each location, $x$, and cases where noise is averaged over neighborhoods of various sizes. We refer to this as spatial smoothing of noise. It has been shown that such noise allows for (H"older) continuous solutions to a broad class of stochastic integro-differential equations, including equations such as (2) [20, 22].

In our case, for spatial smoothing we used a Gaussian kernel. The Gaussian smoothing kernel was convolved with i.i.d. Gaussian noises at each iteration of the evolving spatial field. This choice of noise smoothing kernel is well-motivated biologically, as the number and strength of synaptic connections between neighboring neurons tends to decline with the distance between them [18]. Given that shared noise would be mostly synaptic noise [19, 22], the amount of shared noise between two neurons should decline with the amount of effective connectivity between them.

Spatial smoothing of Gaussian noise by a Gaussian kernel as just described creates a Gaussian field. Its spatial correlation has the form for each $t$ (see [24])

$$E[Y(t, x)Y(t, x')] = \frac{1}{\eta} \exp \left( -\frac{(x_i - x_j)^2}{\eta^2} \right).$$

(8)

This means that as we increase $\eta$ we also increase the correlation length, i.e. the distance in our neural field over which substantial sharing of noise occurs. This turns out to have unexpected consequences for spatial pattern formation. Because the Fourier transform of a Gaussian is again a Gaussian but with role of the variance inverted, if $\eta$ is larger, and thus the smoothing kernel wider, the spatial spectrum of $Y(t, x)$ is narrower and concentrated closer to 0.

In Section IV we explore by simulation the effect on the emerging spatial pattern of changing the standard deviation, $\eta$, of the Gaussian noise smoother for $c = 0, 5, 9$.

IV. SIMULATION IMPLEMENTATION

In order to study the space-time neural field equation (2), we implemented a discrete (both space and time) approximation using the Euler-Maruyama procedure for obtaining a numerical solution of stochastic difference equations. This approach has been shown to converge rapidly to a close approximation of the continuous solution [23]. We solved the stochastic difference equation corresponding to (2) using 128 values of $x$ in a 1D spatial array, where each point in the spatial array corresponds to one component of (2). We implemented a periodic boundary so that the values of $x$ and the corresponding component of (2), can be thought of as forming a discrete ring of length $L = 128$, iteratively for 10,000 time iterations.
steps ($\Delta t$) of 0.00005. The initial value for each component was $0.5 + 0.001(y \in U(0,1))$. The initial perturbation from the constant 0.5 is necessary to observe patterns in the evolving coupled field in the absence of noise [12]. Without the initial perturbation from 0.5 the coupled deterministic field damps uniformly to zero because each identical component receives identical input from its neighbors. We implemented the function $S(Y)$ as discussed in Section III with threshold $Y_{\text{max}} = 5.0$. The 128 components were coupled by the Mexican Hat integral operator (3) with various parameter values $b_1, b_2, d_1, d_2$, noise strength $\sigma$ times independent samples from a standard Gaussian distribution, and coupling strength, $c$, as discussed in Section II A. Our Mexican Hat operator was 31 spatial locations wide; that is, it was truncated so that it operated only on 31 of the 128 components around each position in the wrapped 1D spatial array. We chose this width because (a) each single Mexican Hat-type neural coupling in a neural system should span only a fraction of the size of the system, and (b) 31 spatial locations, while being only a fraction of the system size of 128, is large enough to model a Mexican Hat operator that couples several excitatory and inhibitory components. A "biologically" based choice would depend on the relative size of an observed neural field to the span of a neighborhood of a neural location that includes both excitatory and inhibitory neighbors.

We implemented space-time noise as described in Section III. That is, on each iteration independent samples of Gaussian noise for each component were combined for each component by a normalized Gaussian kernel, with standard deviation $\eta$, centered at that component and defined over the entire ring. In our simulations, presented in Section IV we used a fairly small range of values for $\eta$, 0.001, 0.1, 0.2, 0.4 corresponding roughly to kernels effectively about 1 (i.e., no smoothing), 13, 25, and 47 spatial locations wide.

In order to illustrate spatial patterns, for each set of parameter values we display a representative realization of the paths of all 128 components of (2), distributed evenly over the ring, with the ring flattened out. We also display for those realizations the spatial power spectral density (PSD) as a stochastic process in $t$, and a second measure we term $F$. $F$ is a function of $t$ and $x$, written as

$$F(t, x) = \frac{1}{Tm} \sum_{s \in \text{timeblock}} \sum_{t=1}^{T} \sum_{y=1}^{m} |Y(s, y + x) - Y(s, y)|$$

(9)

where $x$ is a spatial offset and $m$ is the distance across the array for which we are computing $F$. In our computations $m$ was fixed at $m = 64$ because the period of the spatial pattern never exceeded this value. In the computation $x$ is increased progressively across the spatial array. Thus, whenever the difference $|Y(t, y + x) - Y(t, y)|$ is large, the value of $F$ is correspondingly increased. Local maxima in the plot of $F(t, x)$ occur wherever the spatial offset $x$ matches half the period of a spatial pattern. The presence of clear maxima in $F$ indicates the presence of a periodic spatial pattern, and the form of the pattern in $F$ displays the pattern in the $Y(t, x)$ but sometimes more clearly because of the averaging implicit in the computation of $F$. For both the spatial PSD and the computation of $F$ we coarse-grained in time, considering 500-iteration time blocks: 1-500, 750-1250, 1750-2250, ..., 9501-10000. For the spatial PSD we averaged the amplitude of each component over the 500-iteration block and then computed the PSD on the resulting spatial array. In computing $F(t, x)$ we averaged over $T = 500$ iterations as in (9).

V. RESULTS

A. Excitable wave numbers in the simulation with $c = 1$

The solutions to the array of 128 coupled components form cyclical spatial patterns, which will be displayed in the next subsections (the reader might wish to look ahead at some of these figures to get an impression of what a spatial cycle looks like). Figure 4 shows the number of spatial cycles across the ring of 128 spatial locations. A spatial cycle in $Y(t, x)$ is defined here as a periodic change in $Y(t, x)$ over $x$ for a single period, i.e. the value of $Y(t, x)$ goes through a maximum and a minimum across space, $x$, at a specific time or for an average over a time interval. The number of spatial cycles is the number of distinct maximum-minimum pairs occurring over the entire ring. This number is also measured by the maximum in the PSD when it is computed using the length of the signal in space as $L = 128$, as it is here, and by the 1/2 the distance between successive peaks in the value of $F$.

Here we took $c = 1, \sigma = 0$, i.e., no noise was added. Clearly, the number of spatial cycles in the pattern is related to the value of $k_{\text{max}}$ in a stepwise fashion, and the value of $k_{\text{max}}$ in turn results from particular combinations of the parameters $b_1, b_2, d_1, d_2$ according to (6). For values of $c > 1$ the increase in the range of excitable $k$ numbers can result in an increase in the number of cycles for any given value of parameters $b_1, b_2, d_1, d_2$ (see Figure 5 and cf. [7]). Note that values of $k_{\text{max}}$ outside those shown in Figure 4 do not have excitable modes with $c = 1$. When excitable modes are created by making $c > 1$ for $k_{\text{max}} > 1.6$ the number of cycles should be equal to or greater than 7. We will find that for $b_1 = 1.1, d_2 = 1.2$, and thus $k_{\text{max}} = 2.063$, which we have chosen to explore in what follows, 7 or more cycles are observed.

To understand why the number of spatial cycles appears to jump in discrete steps as $k_{\text{max}}$ increases we need to consider the entire set of excitatory modes created by a particular Mexican Hat operator. In Figure 4 we have varied the parameters in (6) so that the dominant mode, $k_{\text{max}}$, increases. The wrapping of the spatial array constrains the number of cycles to be an integer value. This
constraint creates a range of values of $k_{\text{max}}$ that produces the same number of spatial cycles, yielding steps in the graph of Figure 4. This wrapping constraint and our rather coarse sampling of space also limit our ability to completely characterize the spatial power spectrum in these simulations.

To understand how the parameters of the Mexican Hat operator determine the number of cycles in a spatial pattern we need to consider how the operator is implemented. The parameters $d_1$, $d_2$ determine the relative widths of the two Gaussians in $w(x)$, but the resulting Mexican Hat operator can be spread over different numbers of spatial locations (components) in the ring array. Wider or narrower implementation of the operator in the simulation, for example spreading an operator with the same values of $d_1$, $d_2$ over 31 as opposed to 15 spatial locations, results in fewer or more cycles, respectively, for a given value of $k_{\text{max}}$. This would model, for example, different extents of the operator (different extents of effective connectivity) in a neural field. In order to implement a simulation relating the parameters of the Mexican Hat to a specific neural model, it is necessary to specify the relation between the spatial scale of the neural field and the spatial scale of the operator. For example, in the case of stripe formation in the visual cortex discussed by Murray [7], one would need to specify the relationship between the spatial scales of the visual cortex and those of the Mexican Hat operator. Because we modeled a generic spatial array with no particular spatial scale, and wrapped boundaries, we cannot provide a meaningful explicit expression relating the parameters of the Mexican Hat operator to the number of cycles observed in our simulations. We discuss this further in Section VI.

![FIG. 4. Number of spatial cycles in 128 spatial locations as a function of $k_{\text{max}}$ from [8] for $c = 1$. Here we chose $b_1$, $d_2$, computed $k_{\text{max}}$, and observed the number of spatial cycles after 10,000 iterations in the simulation.](image1.png)

**B. Damping in deterministic solutions**

Figure 5 displays solutions to (2) with $\sigma = 0$, i.e., solutions to the deterministic version of (2) for the 128 spatial replicants, $b_1 = 1.1$, $d_2 = 1.2$. Although all 128 equations began with a small, random, perturbation from 0.5, they all damp very quickly to very small amplitudes for $c = 1$. There is no indication in either the spatial PSD or $F$ plots of any spatial pattern. This is to be expected because $0 < W(k_{\text{max}}) < 1$ (see Figure 3). Even for $c = 5$, however, $Y(t, x)$ damps very quickly even though $W(k_{\text{max}}) \gg 1$, and there is very little indication of a spatial pattern in the PSD or $F$ plots. Finally, for $c = 15$ there is an emerging spatial pattern of 8 spatial cycles for the $Y(t, x)$ amplitude, PSD, and $F$. Even larger values of $c$ show this pattern more clearly.

It appears that, between $c = 5$ and $c = 15$, there is a minimum value of $c$ such that the pattern arises out of the damping and persists. For different pairs $b_1$, $d_2$ in our Mexican Hat function, this value of $c$ would be different, and the range of values of $c$ for which the pattern would be dominated by damping would also be different. We speculate that as the pair $b_1$, $d_2$ gives rise to values of $W(k_{\text{max}})$ closer to 1, this range would be narrower. It is possible that this result could be computed using the methods of [10, 11] (see Section VI C).

![FIG. 5. Damping of spatial patterns for which $W(k_{\text{max}}) > 1$. Top row: amplitude $Y(t, x)$; middle row: $F$; bottom row: spatial PSD. As $c$ (indicated at top of figure) increases the spatial pattern becomes more apparent. Here $b_1 = 1.1$, $d_2 = 1.2$, $\sigma = 0$ and thus noise smoothing is irrelevant. Here $k_{\text{max}} = 2.063$ and the expected number of cycles in $2\pi$ is 7 or more (does not appear in Figure 4 because for $c = 1$ there are no excitable modes for $b_1 = 1.1$, $d_2 = 1.2$).](image2.png)

**C. Effect of added noise**

Figure 6 displays the effects of adding i.i.d. Gaussian noise to the neural field equation with the same parameters for $w(x)$ as in Figure 5 except that in Figure 6...
c = 5. We would expect, then, that little or no indication of spatial pattern would be apparent when \( \sigma = 0 \), and that is indeed the case (left column). When a small amount of Gaussian noise \( \sigma = 0.5, \eta = 0.001 \) (i.e., no noise smoothing) was added on each iteration, a spatial pattern is evident. More noise, \( \sigma = 1.0 \), makes the pattern more apparent. Thus, noise can both reveal weak, unobservable, spatial patterns, and sustain them at observable levels across time. The dominant wave number of the spatial pattern does not depend on the standard deviation of the noise, \( \sigma \). It does depend, however, on the standard deviation of the smoothing kernel as will be seen.

Spatially smoothing the noise makes the spatial pattern induced by the Mexican Hat coupling more apparent (Figure 7). It appears that there might be an interaction between smoothing width and coupling strength, as the case with \( \eta = 0.1 \) appears to give the cleanest spatial pattern. Larger \( \eta \) seems to impose a lower spatial frequency upon the one created by the Mexican Hat operator. This can be seen in both PSD and F values.

D. Interaction between coupling strength, \( c \), and noise correlation length, \( \eta \)

Figure 8 displays some aspects of the interaction between the smoothed noise and the coupling strength, \( c \), for the same values of \( b_1, d_2, \sigma = 0.5 \) and the standard deviation of Gaussian smoothing of noise, \( \eta \). It is clear that, among those shown, the combination \( c = 9, \eta = 0.2 \) reveals the cleanest evolving pattern. This is confirmed by the spatial PSD and F measures (not shown). More Gaussian smoothing seems to produce pattern more reliably for larger values of \( c \).

This phenomenon of an optimal pair \((c, \eta)\) is reminiscent of stochastic resonance (or stochastic facilitation [29]), in which tuning of the noise strength and threshold yields optimum performance. Of course, the greater \( c \) is, the stronger the patterns, so this analogy only applies for situations where damping is sufficient that the pattern is only revealed and sustained by optimally smoothed noise. Smoothing of noise that is too broad imposes a lower frequency on the array, and interferes somewhat with the pattern created by the Mexican Hat operator, as seen in the case where \( c = 5, \eta = 0.2 \). We have not explored how the frequency of spatial cycles depends on the pair \((k_{max}, \eta)\) except that it increases with \( k_{max} \) when \( \eta = 0.001 \) (Figure 4), and decreases as \( \eta \) increases when \( c = 0 \) (Figure 9 and section V F).
E. Effect of spatial smoothing of noise alone

Figure 9 displays solutions to the stochastic neural field equation (2) with $c = 0$ and Gaussian-smoothed noise. That is, there is no coupling via the Mexican Hat operator. In these cases, however, $\eta$ has been varied, from 0.001 (i.i.d. noise to each component, no smoothing), to 0.1 and to 0.2. We observe that smoothing the noise itself creates a spatial pattern with frequency depending on $\eta$, effecting what we could term a noise-smoothing coupling. The spectrum decays exponentially toward the higher frequencies, as expected. With $\eta = 0.4$ the PSD is nearly all near a very low spatial frequency (not shown). Indeed, when the Gaussian smoother has significant weighting over the entire ring, $\eta \geq 1$, there is only one large bump in the path, and the PSD is concentrated at a spatial frequency of one cycle over $L$ (not shown). As with the other values of $\eta$, however, the bump moves around over time. Finally, the spatial pattern created by the smoothing kernel can be expected to interact with that created by the Mexican Hat operator to create a sustained and powerful standing wave when the noise smoothing is optimal, or to overwhelm the Mexican Hat pattern when noise smoothing is too great.

Pattern produced by Gaussian noise smoothing does have, however, one significant difference from the effect of Mexican Hat coupling. Gaussian noise smoothing with $c = 0$ tends to produce irregular bumps over space, and these bumps move around in the spatial array, making the evolving neural field resemble a chimera of emerging and fading pattern [27]. The Mexican Hat coupling with or without smoothing tends to produce stripes rather than bumps (Figures 6, 7), although the stripes do move around somewhat, especially when the noise is not optimally smoothed.

F. Interaction of noise strength and noise correlation length

Given that the smoothing of the noise by Gaussian kernel itself produces spatial pattern, it is relevant to investigate the dependence of this process on the overall amount of noise in the system, $\sigma$ together with $\eta$, in the absence of any other pattern-producing coupling. Figure 10 displays the interaction of these two parameters for a range of values of interest: $\sigma = 0.4, 0.8, 1.2, 1.6, 2.0$ and $\eta = 0.001, 0.01, 0.05, 0.1, 0.4$, corresponding to (approximate) correlation lengths of 1, 6, 13, 25, 47 components. Maximum PSD increases with noise strength, as expected, and decreases with correlation length. The noise correlation kernel was always normalized to have integral equal to one. It is clear that as correlation length increases the spatial frequency at which the maximum of the PSD occurs moves toward zero, as predicted in section III. At the extreme, only one large-amplitude "bump" (dominant spatial frequency of 1 cycle per $2\pi$) is seen in the ring. The effect of changes in noise strength seems to be limited to the low-correlation length region; for moderate to large correlation length (relative to the size of the ring) the pattern is driven toward low spatial frequencies regardless of noise strength.

VI. DISCUSSION

We have illustrated, in the context of a "standard" [5] stochastic neural field equation, (2), that a Mexican Hat convolution kernel produces spatial patterns that can be revealed and sustained by noise, whereas without noise the damping tends to conceal the pattern. Moreover, Gaussian-smoothed noise also produces a spatial pattern, and the two patterns interact. It has been known for some time that a Mexican Hat convolution kernel produces spatial patterns, similar to Turing patterns, in a variety of contexts (e.g., [7, 12]). It was known previously that noise can reveal and sustain such patterns when damping tends to make them vanish, much as quasicycles are revealed and sustained by noise in temporal stochastic processes. We have revealed some of the details of this process, showing sample paths of the transition between damping and emergence of spatial pattern, and the dependence of the exposed pattern on the strength of the Mexican Hat coupling and the spatial correlation length of the noise. A particular finding is that Gaussian noise smoothing can itself produce spatial patterns in the context of neural field equations, and likely in other contexts as well. In what follows we discuss some of the nuances of our findings and relate them to those of others working with similar models.
A. Mexican Hat parameters and spatial patterns

Although Murray [7] writes an equation for the PSD of the Mexican Hat kernel [5], that equation does not allow any direct computation of the PSD of the spatial pattern created by its application in a specific setting such as in our simulations. Choices have to be made not only of the values of parameters of the Mexican Hat operator itself, but also of how it is to be discretized and implemented in a simulation or in a specific model, for example in a model of the visual cortex [7]. In this sense, Murray’s development is a framework for understanding how and where spatial patterns develop. Further work is necessary to obtain quantitative results in a specific system, e.g., to predict the number of cycles, or spatial frequency, that appears.

B. Spatially correlated noise-induced patterns

The fact that spatially-smoothed noise, i.e. noise that has non-zero correlation length in space, can by itself produce spatial patterns has, we believe, been unappreciated until now. When a Gaussian kernel is used to accomplish the smoothing it is clear that the scenario is similar to that of a coupling kernel being applied across space. That is, the Fourier transform of the Gaussian smoothing kernel predicts the PSD of the spatial pattern in a way similar to the way the Fourier transform of the Mexican Hat kernel predicts the excitable spatial modes of the coupling. In the case of the smoothing kernel, a greater spread of smoothing leads to a lower value of the spatial frequency at which the maximum PSD of the resulting pattern occurs, with smoothing that is significant over the entire lattice leading to a very low value for that spatial frequency. Clearly, the pattern that results from the interaction of the Mexican Hat coupling and the coupling by smoothing of noise is a combination of the respective spatial frequencies they induce. Which frequency dominates depends on the weighting of the respective operators and their extents with respect to the size of the system. Further work might well lead to an expression for the spatial frequency of the pattern resultant from any combination of Mexican Hat coupling with spatially-smoothed noise.

C. Relation to Other Work

There is an extensive literature on stochastic neural field equations. We have chosen a representative, assorted, sample from this literature in order to point out similarities and differences to the present work, and to suggest directions for future studies.

In [10, 11] there is a cubic reaction term that succeeds in keeping the process stochastically bounded. This is different from the thresholded identity function used in the present work. In [10] the coupling operator is $(1 + (\partial^2/\partial x^2))^2$, which has an effect similar to a Mexican Hat, whereas in [11] an effectively Mexican Hat operator, written differently, is used. Both in [10] and in [11] the noise is either uncorrelated spatially or, the other extreme, ”global fluctuations,” in which the same noise is added to all components of the neural field at each time point. This is in contrast to locally spatially smoothed noise, which is what we studied here. In these papers the analytic method of center manifold theory together with adiabatic elimination is used to obtain solutions to the neural field equation. This method, beyond the scope of our paper, may well produce further analytic results about the model we studied here.

The paper [12] studied a model that creates a moving front between states 0 and 1 using a cubic reaction term as in [10]. At the same time a Mexican Hat kernel together with a diffusion term creates a Turing pattern. Homogeneous solutions coexist with spatially periodic
states. There is no stochastic term, however, so the effects of noise in this model are unknown.

In [6], Gaussian white noise, as in [28] together with spatial coupling of the form \((K_2^2 + \nabla^2)^2\) and, again, a cubic reaction term, in a Stratonovich SDE, create patterns in \(\mathbb{R}^2\). These patterns resemble various highly regular patterns of vegetation that occur on slopes in semi-deserts around the world.

In Touboul’s paper [17] space-dependent delays are introduced. Again the noise is not smoothed across space. A relevant result is convergence-in-law of network equations. These are in continuous time and discrete in numbers of neurons and of populations, both of which increase to infinity, the ”neural-field limit” hypothesis. The limit is a particular McKean-Vlasov equation, a stochastic neural mean field equation with delays.

These other works suggest various additional ways to pursue the questions studied here. For example, one could insert a cubic reaction term in (2) instead of using the function \(S\) to maintain stochastic boundedness of the process. Additional analytic results might well be obtained using the center manifold theory as in [10] [11].

COMPETING INTERESTS

The authors declare that they have no competing interests.

AUTHOR’S CONTRIBUTIONS

Both authors contributed to the conceptualization and writing of the paper. The numerical simulations were accomplished by LMW.

ACKNOWLEDGEMENTS

Lawrence M. Ward was supported by Discovery Grant A9958 from NSERC of Canada.

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