Shaping topological properties of the band structures in a shaken optical lattice

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To realize band structures with non-trivial topological properties in an optical lattice is an exciting topic in current studies on ultra cold atoms. Here we point out that this lofty goal can be achieved by using a simple scheme of shaking an optical lattice, which is directly applicable in current experiments. The photon-assistant band hybridization leads to the production of an effective spin-orbit coupling, in which the band index represents the pseudospin. When this spin-orbit coupling has finite strengths along multiple directions, non-trivial topological structures emerge in the Brillouin zone, such as topological defects with a winding number 1 or 2 in a shaken square lattice. The shaken lattice also allows one to study the transition between two band structures with distinct topological properties.

The study on topological matters is one of the most important themes in condensed matter physics in the past a few years[1,2]. When non-trivial topology exists in the band structures of certain solid materials, a wide range of novel topological matters arise. Whereas the effort of searching for such materials in solids has been continuously growing, there have been great interests in realizing topological matters using the highly controllable ultra cold atoms[3–7]. It is hoped that the atomic counterpart will provide physicists not only a perfect simulator of electronic systems, but also opportunities to create new types of topological matters.

From the experimental side, the realization of synthetic spin-orbit coupling(SOC) using the Raman scheme is an exciting development[8–13]. As SOC is a key ingredient in many topological matters, synthetic SOC opens the door for accessing topological matters in ultra cold atoms. However, a shortcoming of the current scheme is that SOC exists along only one spatial direction. As it is in general requires a spin-orbit coupling with finite strengths along multiple directions for creating a high-dimensional topological matter, an experimental realization of such novel matters has not been achieved yet in ultra cold atoms. To implement theoretical proposals in the literature, further technique advances are required.

Both theoretical and experimental interests on shaken optical lattices have been arising recently[14–20]. It has been shown that such a scheme allows one to manipulate both the magnitude and the sign of tunneling constants so that a gauge field or interesting single-particle dispersions may emerge in a lattice. In this Letter, we point out that shaken lattices provide physicists an unprecedented opportunity to create a fully controllable “SOC” with finite strengths along multiple directions. Here, band indices play the role of the “spin” degree of freedom. By shaking a simple square lattice, which does not exhibit interesting topological properties at the stationary state, photon-assistant band hybridization creates an effective SOC Hamiltonian at the Γ and M point in the Brillouin zone(BZ),

\[ H = A(k_x^2 - k_y^2)\sigma_z + (Bk_xk_y + C)\sigma_x + D\sigma_y, \] (1)

where \( A, B, C, D \) are momentum-independent constants. Whereas \( A \) is mainly controlled by the static lattice, \( B, C, D \) are well tunable in the shaken lattice. The Hamiltonian in Eq.(1) can be classified to two categories. Case 1 \( B = 0 \). Eq.(1) then has a similar formalism with that realized by the Raman scheme in continuum[8–13]. As spin-momentum locking exists along only one direction, this type of SOC does not give rise to interesting topological properties of the band structure.

Case 2 \( B \neq 0 \). When \( C = D = 0 \), it corresponds to a SOC of \( d \)-wave nature, as \( H = Ak^2 \cos(2\theta_k)\sigma_z + Bk^2 \sin(2\theta_k)\sigma_x \), where \( \theta_k = \arg\{k_x + ik_y\} \). It has been shown that this model could be used to produce a topological semimetal[21]. This model is also relevant in the studies of crystalline topological insulators[22]. As we will show, a shaken square lattice not only allows one to create such a novel SOC, but also offers a unique opportunity for studying the transition between two band structures with distinct topological properties when the microscopic parameters in our system, including the shaking frequency, amplitude and phase shift, continuously change.

The potential of a shaken square lattice with a periodicity \( T \) is written as

\[ V(r,t) = V \sum_{i=x,y} \cos^2(k_0r_i + \frac{\phi_i(t)}{2}) \] (2)

where \( r_x = x, r_y = y, k_0 = \pi/d, d \) is the lattice spacing, \( \phi_i(t) = f \cos(\omega t + \varphi_i) \). \( f \) is the shaking amplitude, \( \omega = 2\pi/T \) is the frequency, and \( \varphi_i \) is the phase of the shaking along the \( i = x, y \) direction, as shown in Fig.(1A). For convenience, we set \( \varphi_x = 0 \) and \( \varphi_y = \varphi \). Different phase shift \( \varphi \) leads to different shaking modes. For instance, \( \varphi = 0 \) and \( \varphi = \pi/2 \) correspond to a linear shaking mode along the diagonal direction and a cyclic mode respectively. A one-dimensional version of Eq.(2) has already been used in C. Chin’s group[16]. Our proposal in two dimensions is directly applicable in current experiments.

Making use of the well known identity for Bessel function \( \exp[x(\zeta - \zeta^{-1})/2] = \sum_{n=-\infty}^{\infty} J_n(x)\zeta^n \), the lattice
potential can be rewritten as $V(r,t) = \sum_n \tilde{V}_n(r)e^{in\omega t}$,

$$\tilde{V}_n(r) = \begin{cases} \frac{n}{2} V J_n(f) \left( \cos(2k_0x) + e^{in\varphi} \cos(2k_0y) \right) & n \in \text{even} \\ \frac{n+1}{2} V J_n(f) \left( \sin(2k_0x) + e^{in\varphi} \sin(2k_0y) \right) & n \in \text{odd} \end{cases}$$

where $V_0$ represents the static lattice, and $\tilde{V}_n(r,t)$ is a dynamically induced lattice potential that excites the system by a multiple-photon energy $n\hbar\omega$, as shown in Fig. (1B). Eq. (3) contains a number of important microscopic parameters for manipulating the topology of the band structure. First, $\omega$ controls which bands shall be hybridized at resonance. Second, the parity of $\tilde{V}_n(r)$ is $(-1)^n$, which gives rise to distinct properties of the band hybridization for even and odd values of $n$. Third, the relative strengths of $\tilde{V}_n(r,t)$ are determined by the shaking amplitude $f$, as $\tilde{V}_n(r) \sim J_n(f)$. Though to choose a small $f$ shall minimize heating in a shaken lattice, this fact does provide experimentalists a further degree of freedom for manipulating the resultant SOC. Finally, as we will show, $\varphi$ controls the relative strength of the SOC along the $x$ and $y$ directions.

As $V(r,t) = V(r,t + T)$ is a time-periodic potential, we apply the Floquet theorem, which tells us that the solution of the Schrodinger equation could be written as $\Psi(r,t) = e^{-i\epsilon t}\Phi(r,t)$, where $\epsilon$ is the quasienergy and the Floquet mode $\Phi(r,t)$ satisfies $\Phi(r,t+T) = \Phi(r,t)$. We expand the Floquet mode as $\Phi(r,t) = \sum_{m,n} \phi_{mn}(r)e^{i\omega t}$, where $\phi_{mn}$ are time-independent constants, and $\phi_{mn}(r)$ is the Bloch wave function of the static lattice $V_0(r)$ with band index $m$ and crystal momentum $k$. The standard Floquet-matrix representation may be expressed as

$$\sum_{m',n'} \left( \Omega_{m,n',m'} + (\epsilon_{m'n'} + n'\hbar\omega)\delta_{m,n'}\delta_{m,m'} \right) e_{m'n'} = \epsilon_{mn},$$

where $\Omega_{m,n',m'} = \int \phi_{mn}(r)v_{mn}(r)\phi_{m'n'}(r)$. Since $V_n(r)$ has the same lattice spacing with the static one, matrix elements between Bloch wave functions with different $k$ vanish. The physical meaning of Eq. (4) is apparent. A band of the static lattice could absorb or emit $n$ photons and form a sequence of side bands, due to the driving potential $V_{n\neq0}(r,t)$. This photon-assistant process may make a resonance between certain side bands possible. For convenience, we use the notation $(m,n)$ to represent the dynamically generated $n$th side band of the band $m$ of the static lattice. The coupling between two side bands through $\Omega_{mn}$ will be referred to as a $n$-photon process.

In this Letter, we focus on the lowest three bands, $m = s, p_x, p_y$. Fig. (1C) provides two examples, in which the side band $(s,1)$ and $(s,2)$ become close with $p_x$ and $p_y$ band respectively. A straightforward calculation (see Supplementary Materials) shows that the matrix elements $\Omega_{n,k}$ is a constant if $n$ is odd, or $k$-dependent if $n$ is even, i.e.,

$$\begin{align*}
\Omega_{2l+1}^{s,p_x} &= i^{2l+1}\Omega_{2l}k_x, \\
\Omega_{2l+1}^{s,p_y} &= i^{2l+1}\Omega_{2l}k_y, \\
\Omega_{2l+1}^{s,p_z} &= i^{2l+2}\Omega_{2l+1}k, \\
\Omega_{2l+1}^{s,p_z} &= i^{2l+2}\Omega_{2l+1}k, \\
\end{align*}$$

where $l$ is an integer, and $d$ has been absorbed.
to \( k_{i=x,y} \), \( \Omega_{2f} = V J_2(f)(W_p R_i | \cos(2k_0 x)| W_{pR_i + d x} \), \( \Omega_{2f+1} = \frac{V}{2} J_2(f)(W_p R_i | \sin(2k_0 x)| W_{pR_i} \) are constants, \( W_p R_i \) is a Wannier function for the band \( m \) at the lattice site \( R_i \), and \( \hat{z} \) is the unit vector along the \( x \) direction. As we will show, such a \( k \)-dependance in the 2-photon process produces topological defects in the BZ, whereas the 1-photon resonance alone leads to a trivial band structure.

We perform a numerical calculation on the Floquet-matrix by including up to 9 side bands. Tight-binding model for the dispersions in the static lattice has been used, i.e., \( e^{0, s, k} = -t_s \cos k_x - t_s \cos k_y - \Delta, e^{0, p_x, k} = t_p \cos k_x - t_s \cos k_y, e^{0, p_y, k} = -t_s \cos k_x + t_p \cos k_y \), where \( t_s, t_p \) are the tunneling amplitudes. A typical band structure is shown in Fig (2 A). At \( \Gamma \) and \( M \) point, there are two nearly degenerate bands, the main contributions to which come from \((p_x, 0)\) and \((p_y, 0)\). Without hybridization with other bands in an ordinary static lattice, it is well known that the \( p_x \) and \( p_y \) bands are degenerate at \( \Gamma \) and \( M \) point. This degeneracy is lifted by turning on the hybridization with the side bands (\( s, n \)).

To understand the band structures, we apply the degenerate perturbation method around the \( \Gamma \) and \( M \) points. Treat a state in the nearly degenerate \( \Gamma \) point, there are \( \tilde{B}_0 \) and \( \tilde{B}_1 \) points respectively. The virtual hopping processes, as show in Fig (2 B), lead to a Hamiltonian which can be regarded as an effective SOC, \( H = B_k \cdot \hat{\sigma} \), where \( \sigma_{x,y,z} \) are the Pauli matrices. To be explicit, we obtain

\[
H = -(t_p + t_s)(\cos k_x - \cos k_y)\sigma_z/2 \\
+ (B_{x,e} \sin k_x \sin k_y + B_{x,o})\sigma_x \\
+ (B_{y,e} \sin k_x \sin k_y + B_{y,o})\sigma_y,
\]

where the subscript \( e \) and \( o \) represent the effective magnetic field induced by processes of even and odd number of photons. For the \( \sigma_z \) term, the main contribution comes from the energy difference between \((p_x, 0)\) and \((p_y, 0)\) bands, and a small correction \( B' \) from hybridization with other bands does not affect the results.(See Supplementary Materials). The transverse fields entirely come from the band hybridization,

\[
B_{x,e} = \sum_{n \in \text{even}} \frac{-\cos(n\phi)}{\cos^2(n\phi)} \text{O}_n^2 \\
B_{y,e} = \sum_{n \in \text{even}} \frac{\sin(n\phi)}{\cos^2(n\phi)} \text{O}_n^2.
\]

As for \( B_{x,o} \) and \( B_{y,o} \), the expressions are identical, with the summation over odd integers. In the leading order, \( B_{x,e} \) and \( B_{y,e} \) are momentum independent constants, as \( e^{0, s,k}, e^{0, p_x,k} \) and \( e^{0, p_y,k} \) may be replaced by their values at the \( \Gamma \) and \( M \) point in the numerator. These results explicitly tell one that the contribution of a side band to the \( B_k \) field depends on both \( \Omega_n \) and its energy separation to the \((p_x, 0)\) and \((p_y, 0)\) bands.

To simplify the expressions, we apply a spin rotation about the \( z \) axis, \( e^{i\theta_{x,y}/2}\sigma_x e^{-i\theta_{x,y}/2} = \cos \theta_{x,y} + \sin \theta_{x,y}, e^{i\theta_{x,y}/2}\sigma_y e^{-i\theta_{x,y}/2} = -\sin \theta_{x,y} + \cos \theta_{x,y} \), where \( \tan(\theta) = -B_{x,e}/B_{x,o} \), so that \( B_{x,e} \sigma_x + B_{y,e} \sigma_y \rightarrow B_{x}\), where \( B_x = \sqrt{B_{x,e}^2 + B_{y,e}^2} \). Near the \( \Gamma \) point, \( \cos k_x - \cos k_y \sim (k_x^2 - k_y^2) \), the effective magnetic field \( B_k \) in Eq.(8) becomes

\[
B_k = (B_{k} k_x k_y + B_{x,o}, B_{y,o}, (t_p + t_s)(k_x^2 - k_y^2)/4)
\]

where \( B_{x,o} = B_{x,0} \cos \theta - B_{y,0} \sin \theta \) and \( B_{y,o} = B_{x,0} \sin \theta + B_{y,0} \cos \theta \). This leads to the expression for the Hamiltonian in Eq.(1). The formalism of the Hamiltonian near the \( M \) point is the same, with quantitatively different values of the three components of \( B_k \). Depending on the choice of \( \omega \) and \( f \), both Case 1 and Case 2 of Eq.(1) can be realized. To demonstrate the underlying physics, we focus on the 1- and 2-photon processes, as the hybridization through a high order photon process is in general much weaker, i.e., \( \Omega_n > 1 \sim J_n>(f) \ll \Omega_{1,2} \) for a small shaking amplitude \( f \).

**1-photon process** If one chooses a small shaking amplitude \( f \) so that \( \Omega_1 \gg \Omega_2 \), and tune the \((s, 1)\) band to be closest to the \((p_x, 0)\), \((p_y, 0)\) bands, the 1-photon process is dominant, which leads to \( B_{x,o} \gg B_{x} \). Eq.(6) becomes

\[
H = \pm \frac{t_s + t_p}{4}(k_x^2 - k_y^2)\sigma_z + \tilde{B}_{x,e} \sigma_x + \tilde{B}_{y,e} \sigma_y,
\]

where \( \pm \) corresponds to the \( \Gamma \) and \( M \) points respectively. Case 1 of Eq. (1) that is topologically trivial is then achieved.

**2-photon process** This could certainly be realized by choosing \( f \) to satisfy \( J_1(f) = 0 \) so that the 1-photon process is completely suppressed. Note the first zero of \( J_1(x) \) is already around 3.8. Such a large shaking amplitude may produce considerable heating in the system. One could alternatively choose a proper frequency so that \((s, 2)\) is the closest one to \((p_x, 0)\) and \((p_y, 0)\) bands so that its contribution to \( B_k \) is dominant, as shown in Fig (1C). In this case, Eq.(6) becomes

\[
H = \pm \frac{t_s + t_p}{4}(k_x^2 - k_y^2)\sigma_z + \tilde{B}_{x,e} k_x k_y \sigma_x
\]

Topological defects then emerge at the \( \Gamma \) and \( M \) points where the effective magnetic field \( B_k \) vanishes. If a closed loop in the momentum space around one of these two points , a winding number of \( \pm 2 \) of \( B_k \) is evident, as \( B_k \sim (\sin(2\theta_k), 0, \cos(2\theta_k)) \).

In general, both 1- and 2-photon processes contribute to the effective Hamiltonian. Eq. (5) allows one to investigate how the two band structures with distinct topological properties may evolve from one to the other when \( \omega \) continuously changes. To study the stability of the
topological defects against perturbation, we numerically solve the Floquet-matrix. We find that if the \((p_x, 0)\) and \((p_y, 0)\) bands are not degenerate with other side bands at the \(\Gamma\) and \(M\) point, a spin-1/2 description is sufficient for describing the eigenstates near these two points, as they are dominated by \((p_x, 0)\) and \((p_y, 0)\) bands. The spin eigen state is written as \((\cos(\alpha_k/2), e^{i\alpha_k}\sin(\alpha_k/2))\), from which an effective magnetic field \(B_k\) is constructed, as \(\alpha_k\) and \(\beta_k\) correspond to the direction of the unit vector \(\mathbf{B}_k/|\mathbf{B}_k|\) on the Bloch sphere and the energy splitting gives rise to the strength of \(B_k\).

Fig. (3) shows a few typical topological structures of \(B_k\). When \(\varphi = 0\), \(B_k\) has only the \(x\) and \(z\) components. When \((s, 2)\) is the closest side band to \((p_x, 0)\) and \((p_y, 0)\) bands, topological defects are present. Interestingly, we find that, due to a finite \(\tilde{B}_{x,o}\) induced by the \((s, 1)\) band, the topological defects of winding number \(2\) at the \(\Gamma\) point splits to two ones with winding number \(1\) as shown in Fig (3 A). The same phenomenon occurs at the \(M\) point. This can be seen from the fact that \(B\) now vanishes at \((\pm k^*, \pm k^*)\), where \(k^* = \sqrt{|\tilde{B}_{x,o}|}\). Near these two points, \(B_k \sim (\tilde{k}_x + \tilde{k}_y, 0, \tilde{k}_z - k_y)\) that corresponds to a winding number \(1\), where \(k_{i=x,y} = k_i \pm k^*\). We have also verified that if one sets \(\Omega_{2t+1} = 0\), the splitting is absent and only defects of winding number \(2\) show up at the \(\Gamma\) and \(M\) points. In general cases with a finite \(\tilde{B}_{x,o}\), the winding number of the \(B\) field on a closed loop in the BZ depends on how many defects it encloses, as shown in Fig. (3 A). If one makes \((s, 1)\) to be more close to the \(p\) bands with changing \(\omega\), \(|\tilde{B}_{x,o}|\) and \(k^*\) increases, and the defect with winding number \(1\) splits from the \(\Gamma\) point gradually approaches the defect of winding number \(-1\) from the \(M\) point, as shown Fig. (3 B, C), and the topological structures eventually disappears. This establishes the evolution between two band structures with distinct topological properties. As the spin corresponds to the band index, the topological structure here and its evolution can be visualized in experiments using a variety of schemes[23–25].

It is worth pointing out that the value of \(\tilde{B}_{x,o}\) relies on the phase shift \(\varphi\). For the cyclic shaking, \(\varphi = \pi/2\), \(\tilde{B}_{x,o}\) is zero. This could be easily seen from Eq. (7) that \(B_{y,e} = B_{x,o} = 0\). Under this situation, changing the value of \(\omega\) only leads to a tilting of the spin along the \(y\) direction. The topological structure on the \(\sigma_x - \sigma_y\) plane is not affected, as shown in Fig. (3 D). For the cyclic shaking, the topological defects are therefore always stable.

All previous discussions can be directly applied to one dimension. For the 1-photon process that hybridizes \((s, 1)\) and \((p, 0)\), the Hamiltonian can be written as \(H = (t_x + t_y) \cos(2k_x)\sigma_z - \Omega_1\sigma_x\). Though this one-photon process could produce an interesting double-well structure in the momentum space[16], the topological properties is trivial. The Zak phase[26], which characterizes the winding number of the spin when \(k_x\) changes from \(-\pi\) to \(\pi\), is zero. In contrast, when the 2-photon process is dominant, the Hamiltonian can be written as \(H = (t_x + t_y) \cos(2k_x)\sigma_z - \Omega_2\sin(2k_x)\sigma_y\). It is interesting to note that this Hamiltonian is equivalent to that obtained in a tilted double-well lattice[7]. This Hamiltonian could produce two flat bands with a Zak phase \(\pm\pi\), which corresponds to a \(2\pi\) rotation of the spin on the \(x - z\) plane when \(k_x\) changes from \(-\pi\) to \(\pi\).

Whereas we have been focusing on a shaken square lattice in this Letter, the general principle of producing a multi-dimensional SOC using dynamically generated band hybridization could be straightforwardly generalized to other lattices. It is expected that the interplay between the tunable lattice geometry and the shaking will lead to fruitful results on shaping the topology of band structures in optical lattices in the near future.

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Note Near the completion of this manuscript, two preprints [arXiv:1402.3295 [arXiv:1402.4034] on topological band structures in shaken optical lattices have just appeared.

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010)
As the static state $V_0(r)$ is separable along the $x$ and $y$ directions, the two Wannier wave function can be written as $W_s(r) = w_0(x)w_0(y)$ and $W_{ps}(r) = w_1(x)w_0(y)$, where $w_0(x)$ and $w_1(x)$ are the lowest two Wannier functions for one dimensional lattice $V_0(x,0)$ or $V_0(0,x)$ respectively. Apparently, $V_s(x) = W_s(r)$ and $W_{ps}(x,y) = -W_{ps}(-x,y) = W_{ps}(x,-y)$.

If $n$ is odd, one sees that the integral in Eq. (11) is finite when taking $i = j$, due to the fact that $V_{2l+1}(r) = -V_{2l+1}(-r)$. This means that $V_{2l+1}(r)$ is able to couple the Wannier orbital $W_s(r - R_1)$ and $W_p(r - R_1)$ at the same lattice site $R_1$. Meanwhile, the integral in Eq. (11) is much smaller if $i \neq j$ because of the the small overlap of the Wannier wave functions at different lattice site. Therefore, $\mathcal{V}^{sp}_{2l+1,k}$ becomes a constant in the leading order,

$$\mathcal{V}^{sp}_{2l+1,k} = i^{2l+2}2\Omega_{2l+1}, \quad \mathcal{V}^{sp}_{2l,k} = i^{2l+1}2e^{i(2l+1)\phi}\Omega_{2l+1}$$

(12)

where $\Omega_{2l+1} = \frac{V}{2}J_{2l+1}(f)|W_{s,R_1}|\sin(2k_0x)|W_{p,R_1}|$.

If $n$ is even, the situation is very different. It is clear that $V_{2l}(r)$ is not able to couple the two Wannier orbital $W_s(r)$ and $W_p(r)$ at the same lattice site. The leading contribution to $\mathcal{V}^{sp}_{2l,k}$ therefore must come from the nearest neighbor ones. Through a simple calculation, one sees that

$$\mathcal{V}^{sp}_{2l,k} = i^{2l+1}\Omega_{2l} \sin(k_xd), \quad \mathcal{V}^{sp}_{2l,k} = i^{2l+1}e^{i2\phi}\Omega_{2l} \sin(k_yd),$$

(13)

where $\Omega_{2l} = VJ_{2l}(f)|W_{s,R_1}|\cos(2k_0x)|W_{p,R_1}| + dx|d\sin(k_xd)|d\sin(k_yd)|$ is the lattice spacing and $\hat{x}$ is the unit vector along the $x$ axis.

**Correction to $B_z$**

$$\Delta E' = -\sum_{n \in \text{even}} \left( \frac{\Omega_n^2 \sin^2(k_yd)}{\epsilon_{s,k}^0 + n\hbar\omega - \epsilon_{p,k}^0} - \frac{\Omega_n^2 \sin^2(k_xd)}{\epsilon_{s,k}^0 + n\hbar\omega - \epsilon_{p,k}^0} \right)$$

$$-\sum_{n \in \text{odd}} \left( \frac{\Omega_n^2 \sin^2(k_yd)}{\epsilon_{s,k}^0 + n\hbar\omega - \epsilon_{p,k}^0} = \frac{\Omega_n^2 \sin^2(k_xd)}{\epsilon_{s,k}^0 + n\hbar\omega - \epsilon_{p,k}^0} \right)$$

(14)

In the leading order, $\Delta E'$ may be written as

$$\Delta E' = -\sum_{n \in \text{even}} \frac{\Omega_n^2}{n\hbar\omega - \Delta} (\sin^2(k_yd) - \sin^2(k_xd))$$

$$-\sum_{n \in \text{odd}} \frac{\Omega_n^2}{(n\hbar\omega - \Delta)^2} (\cos(k_yd) - \cos(k_xd))$$

(15)

### Supplementary Materials

**Matrix elements** $\mathcal{V}^{sp}_{n,k}$, $\mathcal{V}^{sp}_{n,k}$

Take $s$ and $p_z$ band as an example, we expand the Bloch wave functions in the basis of Wannier functions $W_n(r)$, and rewrite $\mathcal{V}^{sp}_{n,k}$ as

$$\mathcal{V}^{sp}_{n,k} = \sum_{R,R_1} \int dW_s(r - R)W_{ps}(r - R_1)\mathcal{V}_n(r)e^{ik(R - R_1)}.$$
As both these two terms $\sim k_x^2 - k_y^2$ near the $\Gamma$ and $M$ points, they contribute a correction to the expression of $B_z$. Their coefficients in the parentheses are much smaller than $t_p + t_s$ in the small $\Omega_n$ limit and can be ignored.