Asynchronous Stochastic Gradient Descent with Variance Reduction for Non-Convex Optimization

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Abstract

We provide the first theoretical analysis on the convergence rate of the asynchronous stochastic variance reduced gradient (SVRG) descent algorithm on non-convex optimization. Recent studies have shown that the asynchronous stochastic gradient descent (SGD) based algorithms with variance reduction converge with a linear convergent rate on convex problems. However, there is no work to analyze asynchronous SGD with variance reduction technique on non-convex problem. In this paper, we study two asynchronous parallel implementations of SVRG: one is on a distributed memory system and the other is on a shared memory system. We provide the theoretical analysis that both algorithms can obtain a convergence rate of $O(1/T)$, and linear speed up is achievable if the number of workers is upper bounded.

1 Introduction

We consider the following non-convex finite-sum problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where $f(x)$ and $f_i(x)$ are Lipschitz smooth. In this paper, we use $F_n$ to denote all functions of the form above. Due to its efficiency and effectiveness, stochastic gradient descent method has been widely used to solve this kind of problem. However, because we use $\nabla f_i(x)$ to simulate full gradient, variance of stochastic gradient and decreasing learning rate lead to a slow convergence rate $O(1/\sqrt{T})$ for convex problem. Recently, variance reduced SGD algorithms [8, 18, 5] have gained much attention to solve machine learning problems like (1). These methods can achieve linear convergence rate on convex problems. In [3, 16], they analyze variance reduced stochastic gradient methods on non-convex problem, and prove that a sublinear convergence rate $O(1/T)$ can be obtained.

Although a faster convergence rate can be achieved using variance reduction technique, sequential method on one single machine may be still not enough to solve large-scale problem. There is growing interest in asynchronous distributed machine learning and optimization [15, 11, 12, 19, 20, 2, 7, 13, 14, 4]. The key idea of asynchronous
parallelism is to allow workers to work independently and have no need to synchronization. In general, there are mainly two distributed architecture categories, one is shared memory architecture [15, 20] and the other is distributed memory architecture [12, 19]. [12] considered the asynchronous stochastic gradient descent on non-convex problem, however they did not use any variance reduction technique, which can gain much acceleration. [19, 20, 17] proposed distributed variance reduced stochastic gradient method, and prove a linear convergence rate can be obtained on convex problem. However, there is no theoretical analysis on the corresponding non-convex situation.

To fill these gaps, in this paper, we focus on asynchronous stochastic gradient descent with variance reduction on non-convex optimization. Two different algorithms and analysis are proposed in this paper as per two different distributed categories, one for shared memory system (multicore, multiGPU) and the other one for distributed memory system. The key difference between these two categories lies on that distributed memory system can ensure the atomicity of reading and writing the whole vector of $x$, while the shared memory system can usually just ensure atomic reading and writing on a single coordinate of $x$. We apply asynchronous SVRG on two different systems and analyze that both of them can get an ergodic convergence rate $O(1/T)$. Besides, we also prove that the linear speedup is achievable if the number of workers is upper bounded.

We list our main contributions as follows:

- Our asynchronous SVRG on distributed memory system improve the earlier convergence rate analysis of ASYSG-CON for non-convex optimization in [12] and extend the asynchronous distributed semi-stochastic gradient optimization [19] to non-convex case. We obtain a non-linear convergence rate $O(1/T)$ on non-convex problem.

- Our asynchronous SVRG on shared memory system improve the earlier convergence rate analysis of ASYSG-INCON for non-convex optimization in [12] and extend the AysSVRG[20] to non-convex case. We obtain a non-linear convergence rate $O(1/T)$ on non-convex problem.

2 Background

In convex case, $f(x) - f(x^*)$ or $\|x - x^*\|$ are used as convergence criterion. Unfortunately, due to the fact that we just focus on non-convex problem, such criterion can not be used in this case. Following [12, 16], we use the weighted average of the $\ell_2$ norm of all gradients $\|\nabla f(x)\|^2$ as metric. Although $f(x) - f(x^*)$, $\|x - x^*\|$ and $\|\nabla f(x)\|^2$ are not comparable, they can be assumed to be in the same order [6].

For further analysis, throughout this paper, we make the following assumptions for problem [1]. All of them are very common assumptions in the theoretical analysis of stochastic gradient algorithms.

Assumption 1 We assume the following conditions holds,

- **Independence:** All random samples $i_t$ are selected independent to each other.

- **Unbiased Gradient:** The stochastic gradient $\nabla f_{i_t}(x)$ is unbiased,

$$E[\nabla f_{i_t}(x)] = \nabla f(x)$$ (2)
• **Lipschitz Gradient:** We say $f(x)$ is $L$-smooth if there is a constant $L$ such that

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||$$

(3)

Throughout, we assume that the function $f_i(x)$ are $L$-smooth, so that $||\nabla f_i(x) - \nabla f_i(y)|| \leq L||x - y||$

• **Bounded Delay:** Delay variable $\tau$ are bounded: $\max \tau \leq \Delta$.

## 3 Asynchronous SVRG for Distributed Memory System

In this section, we propose asynchronous SVRG algorithm for distributed memory system, and analyze its convergence rate.

### 3.1 Algorithm Description

In each iteration, the parameter $x$ is updated through the following update rule,

$$x_{t+1}^{s+1} = x_t^{s+1} - \eta v_t^{s+1}$$

(4)

where learning rate $\eta$ is constant, $v_t^{s+1}$ represents the variance reduced gradient

$$v_t^{s+1} = \nabla f_i_t (x_t^{s+1} - \tau) - \nabla f_i_t (\tilde{x}_s) + \nabla f (\tilde{x}_s)$$

(5)

where $i_t$ denotes index of sample, $\tau$ denotes time delay, $\tilde{x}_s$ denotes snapshot of $x$ after $m$ iterations.

We summarize the asynchronous stochastic gradient method with variance reduction on distributed memory system in the following algorithm.

**Algorithm 1 AsySVRG 1**

Initialize $x^0 \in \mathbb{R}^d$.

for $s = 0, 1, 2, ..., S - 1$ do

$\tilde{x}_s \leftarrow x^s$;

Compute full gradient $\nabla f (\tilde{x}_s) \leftarrow \frac{1}{n} \sum_{i=1}^{n} \nabla f_i (\tilde{x}_s)$;

for $t = 0, 1, 2, ..., m - 1$ do

Randomly select $i_t$ from $\{1, ..., n\}$;

Compute the update vector: $v_t^{s+1} \leftarrow \nabla f_i_t (x_t^{s+1} - \tau) - \nabla f_i_t (\tilde{x}_s) + \nabla f (\tilde{x}_s)$

Update $x_{t+1}^{s+1} \leftarrow x_t^{s+1} - \eta v_t^{s+1}$

end for

$x_{t+1}^{s+1} \leftarrow x_m$

end for

### 3.2 Convergence Analysis

**Assumption 2** For distributed memory architecture specifically,

- $x_{t-\tau}$ denotes old parameter, where $\tau \leq \Delta$. 
The intuition of variance reduced SGD methods is to reduce the variance of stochastic gradients. To analyze its convergence, it is nontrivial to obtain an upper bound of $\ell_2$ norm of $||v_i^{t+1}||^2$.

**Lemma 3.1** For the definition of the variance reduced gradient $v_i^{t+1}$ in Equation (5), and we define:

$$u_i^{t+1} = \left( \nabla f_{i_t}(x_i^{t+1}) - \nabla f_{i_t}(\bar{x}^*) + \nabla f(\bar{x}^*) \right)$$

We have the following inequality:

$$\sum_{t=0}^{m-1} \mathbb{E} [||u_t^{t+1}||^2] \leq \frac{2}{1 - 2L^2\Delta^2\eta^2} \sum_{t=0}^{m-1} \mathbb{E} [||u_t^{t+1}||^2]$$

Furthermore, we are able to show the convergence rate of Algorithm 1 based on Lemma 3.1 as follows.

**Theorem 3.2** Suppose $f \in \mathcal{F}_n$, $x \in \mathbb{R}^d$. We define:

$$c_t = c_{t+1}(1 + \eta \beta_t + \frac{4L^2\eta^2}{1 - 2L^2\Delta^2\eta^2}) + \frac{4L^2}{1 - 2L^2\Delta^2\eta^2}(\frac{L^2\Delta^2\eta^3}{2} + \frac{\eta^2 L}{2})$$

$$\Gamma_t = \frac{\eta}{2} - \frac{4}{1 - 2L^2\Delta^2\eta^2}(\frac{L^2\Delta^2\eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1}\eta^2),$$

where $c_m = 0$, learning rate $\eta_t = \eta > 0$ is constant, $\beta_t = \beta > 0$, such that $\Gamma_t > 0, \forall t \in [0, m - 1]$, $T$ denotes total iteration. Define $\gamma = \min_t \Gamma_t$, $x^*$ is the optimal solution. Then, we have the following ergodic convergence rate for iteration $T$:

$$\frac{1}{T} \sum_{s=0}^{T-1} \sum_{t=0}^{m-1} \mathbb{E} [||\nabla f(x_t^{s+1})||^2] \leq \frac{\mathbb{E} [f(x^0) - f(x^*)]}{T\gamma}$$

We note that $\gamma$ depends on $n, L, \Delta$. To clarify its dependence, we simply set $\eta$ and $\beta$ and achieve the following theorem.

**Theorem 3.3** Suppose $f \in \mathcal{F}_n$. Let $\eta = \frac{\rho_{\min}}{L\Delta^2}$, where $0 < u_0 < 1$ and $0 < \alpha \leq 1$, $\beta = 2L$, $m = \lfloor \frac{\alpha}{2u_0} \rfloor$, $T$ denotes total iteration. Then there exists universal constant $u_0, \sigma$, such that it holds that $\gamma \geq \frac{\sigma}{L\Delta^2}$ in Theorem 3.2 and if time delay $\Delta$ is upper bounded,

$$\Delta^2 \leq \min\left\{ \frac{1}{2u_0}, \frac{3 - 28u_0}{28u_0^2} \right\}$$

then we have:

$$\frac{1}{T} \sum_{s=0}^{T-1} \sum_{t=0}^{m-1} \mathbb{E} [||\nabla f(x_t^{s+1})||^2] \leq \frac{L\eta^3 \mathbb{E} [f(\bar{x}^0) - f(\bar{x}^*)]}{T\sigma}$$

Since this rate does not depend on the delay parameter $\Delta$, the negative effect of using old values of $x$ for stochastic gradient evaluation vanishes asymptotically, namely, we can achieve linear speedup when we increase the number of workers.
3.3 Mini-Batch Extension

In this section, we extend Algorithm 1 to mini-batch version. Mini-batch strategy is widely used in distributed computing, and it not only greatly reduces the communication costs and can also reduce the variance of stochastic gradient. We use a mini-batch $I_t$ of size $b$, and gradient $v_t^{s+1}$ in Algorithm 1 can be replaced with the following function,

$$v_t^{s+1} = \frac{1}{|I_t|} \sum_{i_t \in I_t} (\nabla f_{i_t}(x_t^{s+1}) - \nabla f_{i_t}(\bar{x}^s) + \nabla f(\bar{x}))$$

$i_t$ denotes index of sample, $\tau_i$ denotes time delay for each sample $i$ and mini-batch size $|I_t| = b$.

**Lemma 3.4** For definition of the variance reduced gradient $v_t^{s+1}$ in Equation (14), and we define:

$$u_t^{s+1} = \frac{1}{|I_t|} \sum_{i_t \in I_t} (\nabla f_{i_t}(x_t^{s+1}) - \nabla f_{i_t}(\bar{x}^s) + \nabla f(\bar{x}))$$

where $|I_t| = b$, we have the following inequality:

$$\sum_{t=0}^{m-1} \mathbb{E} \left[ ||v_t^{s+1}||^2 \right] \leq \frac{2}{1 - 2L^2 \Delta^2 \eta^2} \sum_{t=0}^{m-1} \mathbb{E} \left[ ||u_t^{s+1}||^2 \right]$$

$$\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \leq 2\mathbb{E} \left[ ||\nabla f(x_t^{s+1})||^2 \right] + \frac{2L^2}{b} \mathbb{E} \left[ ||x_t^{s+1} - \bar{x}^s||^2 \right]$$

**Theorem 3.5** Suppose $f \in F_n$. Let $c_m = 0$, learning rate $\eta_t = \eta > 0$ is constant, $\beta_t = \beta > 0$, $b$ denotes the size of mini-batch. We define

$$c_t = c_{t+1} + \eta \beta_t + \frac{4L^2 \eta^2}{(1 - 2L^2 \Delta^2 \eta^2)b} + \frac{4L^2}{(1 - 2L^2 \Delta^2 \eta^2)b} \left( \frac{L^2 \Delta^2 \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right)$$

$$\Gamma_t = \frac{\eta}{2} - \frac{4}{(1 - 2L^2 \Delta^2 \eta^2)} \left( \frac{L^2 \Delta^2 \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right)$$

such that $\Gamma_t > 0$ for $0 \leq t \leq m - 1$. Define $\gamma = \min_t \Gamma_t$, $x^*$ is the optimal solution. Then, we have the following ergodic convergence rate for iteration $T$:

$$\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \left[ ||\nabla f(x_t^{s+1})||^2 \right] \leq \frac{\mathbb{E} \left[ f(x^0) - f(x^*) \right]}{T \gamma}$$

**Theorem 3.6** Suppose $f \in F$. Let $\eta_t = \eta = \frac{\alpha b}{\ln \alpha}$, where $0 < \alpha_0 < 1$ and $0 < \alpha \leq 1$, $\beta = 2L$, $m = \lceil \frac{T}{6 \ln \eta} \rceil$ and $T$ is total iteration. If the time delay $\Delta$ is upper bounded by

$$\Delta^2 < \min \left\{ \frac{1}{2u_0 b^4}, \frac{3 - 28u_0 b}{28u_0^2 b^2} \right\}$$

then there exists universal constant $u_0, \sigma$, such that it holds that: $\gamma \geq \frac{\alpha b}{\ln \eta}$ in Theorem 3.3 and

$$\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \left[ ||\nabla f(x_t^{s+1})||^2 \right] \leq \frac{L \ln \eta \mathbb{E} \left[ f(x^0) - f(\bar{x}) \right]}{b T \sigma}$$

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4 Asynchronous SVRG for Shared Memory System

In this section, we propose asynchronous SVRG algorithm for shared memory system, and analyze its convergence rate.

4.1 Algorithm Description

Following the setting in [12], we define one iteration as a modification on any single component of \(x\) in the shared memory. We use \(x_{t+1}^s\) to denote the value of parameter \(x\) in the shared memory after \((mt + t)\) iterations, and \(\hat{x}_{t+1}^s\) to denote the value of parameter used to compute current gradient.

\[
(x_{t+1}^s)_k = (x_{t+1})_k - \eta(v_{t+1})_{k_t}, \quad k_t \in \{1, ..., d\}, \quad \eta \text{ is constant.}
\]

where \(k_t\) is the index of parameter in \(x\), \(k_t \in \{1, ..., d\}\), and learning rate \(\eta\) is constant.

\[
\hat{x}_{t+1}^s = x_{t+1}^s - \sum_{j \in J(t)} (x_{j+1} - x_j)
\]

\[
v_{t+1}^s = \nabla f_{i_t}(\hat{x}_{t+1}^s) - \nabla f_{i_t}(\hat{x}^s) + \nabla f(\hat{x}^s)
\]

\(i_t\) denotes index of sample, \(J(t) \in \{t - 1, ..., t - \Delta\}\) is a subset of index numbers of previous iterations, \(\Delta\) is the upper bound of time delay. The definition of \(\hat{x}_{t+1}^s\) is different from the analysis in [15], where \(\hat{x}_{t+1}^s\) is assumed to be some earlier state of \(x\) in the shared memory like in distributed memory system. However, it is not true in practice.

In Algorithm 2 we summarize the asynchronous SVRG on shared memory system.

**Algorithm 2 AsySVRG 2**

Initialize \(x^0 \in \mathbb{R}^d\).

for \(s = 0, 1, 2, ..., S - 1\) do

\(\bar{x}^s \leftarrow x^s\);

Compute full gradient \(\nabla f(\bar{x}^s) \leftarrow \frac{1}{n} \sum_{i=1}^{n} \nabla f_{i_t}(\bar{x}^s)\);

for \(t = 0, 1, 2, ..., m - 1\) do

Randomly select \(i_t\) from \(\{1, ..., n\}\);

Compute the update vector; \(v_{t+1}^s \leftarrow \nabla f_{i_t}(x_{t+1}^s) - \nabla f_{i_t}(\bar{x}^s) + \nabla f(x^s)\)

Randomly select \(k_t\) from \(\{1, ..., d\}\)

Update \((x_{t+1}^s)_{k_t} \leftarrow (x_{t+1}^s)_{k_t} - \eta(v_{t+1}^s)_{k_t}\)

end for

\(x_{t+1}^s \leftarrow x_m^s\)

end for
4.2 Convergence Analysis

As per the definition of $\hat{x}_{t+1}$ above, the time delay assumption can be represented as follows:

Assumption 3 For shared memory architecture specifically,

- $\hat{x}_{t+1} = x_{t+1} - \sum_{j \in J(t)} (x_{t+1} - x_j)$ denotes old parameter, where $J(t) \subset \{t-1, \ldots, t-\Delta\}$.

In this case, we can also get a upper bound of $||x_{t+1}||^2$.

Lemma 4.1 For the definition of the variance reduced gradient $v_t$ in Equation (25), and we define:

$$v_t = \nabla f_t(x_t) - \nabla f_t(\hat{x}) + \nabla f(\hat{x})$$

We have the following inequality,

$$\sum_{t=0}^{m-1} \mathbb{E} [||v_t||^2] \leq \frac{2d}{d - 2L^2\Delta^2\eta^2} \sum_{t=0}^{m-1} \mathbb{E} [||u_t||^2]$$

Furthermore, the convergence rate of Algorithm 2 is as follows:

Theorem 4.2 Suppose $f \in F_n$, $x \in \mathbb{R}^d$. We define,

$$c_t = c_{t+1} \left(1 + \frac{\eta \beta_t}{d} + \frac{4L^2\eta^2}{d - 2L^2\Delta^2\eta^2} + \frac{4L^2}{d - 2L^2\Delta^2\eta^2} \left(\frac{L^2\Delta^2\eta^3}{2d} + \frac{\eta^2L}{2} + c_{t+1}\eta^2\right)\right)$$

$$\Gamma_t = \frac{\eta}{2} - \frac{4}{d - 2L^2\Delta^2\eta^2} \left(\frac{L^2\Delta^2\eta^3}{2d} + \frac{\eta^2L}{2} + c_{t+1}\eta^2\right),$$

where $c_{m} = 0$, learning rate $\eta_t = \eta > 0$ is constant, $\beta_t = \beta > 0$, such that $\Gamma_t > 0, \forall t \in [0, m-1]$, $T$ denotes total iteration. Defining $\gamma = \min_t \Gamma_t$, $x^*$ as the optimal solution, then we have the following ergodic convergence rate for iteration $T$:

$$\frac{1}{T} \sum_{s=0}^{T-1} \sum_{t=0}^{m-1} \mathbb{E} [||\nabla f(x_{t+1})||^2] \leq \frac{\mathbb{E} [f(x^0) - f(x^*)]}{T\gamma}$$

So far, we can conclude that this algorithm follows a sublinear convergence rate $O(1/T)$. To further illustrate the dependence of $\gamma$, we have the following theorem.

Theorem 4.3 Suppose $f \in F_n$. Let $\eta = \frac{\mu}{L\eta^2}$, where $0 < u_0 < 1$ and $0 < \alpha \leq 1$, $\beta = 2L$, $m = \lceil n^\alpha u_0 \rceil$, $T$ denotes total iteration. Then there exists universal constant $u_0, \sigma$, such that it holds that $\gamma \geq \frac{\sigma}{L\eta^2}$ in Theorem 4.2 and if time delay has an upper bound

$$\Delta^2 < \min\left\{\frac{d}{2u_0}, \frac{3d - 28u_0d}{28u_0^2}\right\}$$
Then we have
\[
\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E \left[ ||\nabla f(x_t^{s+1})||^2 \right] \leq \frac{dLm\alpha E \left[ f(\tilde{x}) - f(x^*) \right]}{T\sigma} \tag{33}
\]

Since this rate does not depend on the delay parameter \(\Delta\), the negative effect of using old values of \(x\) for stochastic gradient evaluation vanishes asymptotically, namely, we can achieve linear speedup when we increase the number of workers.

### 4.3 Mini-Batch Extension

In this section, we extend Algorithm 2 to mini-batch version. We use a mini-batch \(I_t\) of size \(b\), and gradient \(v_t^{s+1}\) in Algorithm 2 can be replaced with the following function,

\[
v_t^{s+1} = \frac{1}{|I_t|} \sum_{i_t \in I_t} \left( \nabla f_i(x_t^{s+1}) - \nabla f_i(x^*) + \nabla f(\tilde{x}) \right) \tag{34}
\]

\(x_t^{s+1}\) means the parameter used to compute gradient with sample \(i_t\), \(i_t\) denotes index of sample, \(J(t) \in \{t-1, \ldots, t-\Delta\}\) is a subset of index numbers of previous iterations, \(\Delta\) is the upper bound of time delay.

**Lemma 4.4** For the definition of the variance reduced gradient \(v_t^{s+1}\) in Equation (34), and we define:

\[
u_t^{s+1} = \left( \nabla f_i(x_t^{s+1}) - \nabla f_i(x^*) + \nabla f(\tilde{x}) \right) \tag{35}
\]

We have the following inequality:

\[
\sum_{t=0}^{m-1} E \left[ ||u_t^{s+1}||^2 \right] \leq \frac{2d}{d - 2L^2 \Delta^2 \eta^2} \sum_{t=0}^{m-1} E \left[ ||u_t^{s+1}||^2 \right] \tag{36}
\]

\[
E \left[ ||u_t^{s+1}||^2 \right] \leq 2E \left[ ||\nabla f(x_t^{s+1})||^2 \right] + \frac{2L^2}{b} E \left[ ||x_t^{s+1} - \tilde{x}^*||^2 \right] \tag{37}
\]

**Theorem 4.5** Suppose \(f \in \mathcal{F}\). Let \(c_m = 0\), learning rate \(\eta_t = \eta > 0\) is constant, \(\beta_t = \beta > 0\), \(b\) denotes the size of mini-batch. We define:

\[
c_t = c_t+1 \left( 1 + \frac{\eta \beta_t}{d} + \frac{4L^2 \eta^2}{(d - 2L^2 \Delta^2 \eta^2)b} \right) + \frac{4L^2}{(d - 2L^2 \Delta^2 \eta^2)b} \left( \frac{L^2 \Delta^2 \eta^3}{2d} + \frac{\eta^2 L}{2} \right) \tag{38}
\]

\[
\Gamma_t = \frac{2L^2}{d - 2L^2 \Delta^2 \eta^2} \left( \frac{L^2 \Delta^2 \eta^3}{2d} + \frac{\eta^2 L}{2} + c_t+1 \eta^2 \right) \tag{39}
\]

such that \(\Gamma_t > 0\) for \(0 \leq t \leq m-1\). Define \(\gamma = \min_t \Gamma_t\), \(x^*\) is the optimal solution. Then, we have the following ergodic convergence rate for iteration \(T\),

\[
\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E \left[ ||\nabla f(x_t^{s+1})||^2 \right] \leq \frac{E \left[ f(x^0) - f(x^*) \right]}{T\gamma} \tag{40}
\]
Theorem 4.6 Suppose $f \in F$. Let $\eta_t = \eta = \frac{u_0 b}{L \alpha}$, where $0 < u_0 < 1$ and $0 < \alpha \leq 1$, $\beta = 2L$, $m = \lfloor \frac{dn^6}{b u_0 b} \rfloor$ and $T$ is total iteration. If time delay $\Delta$ is upper bounded by

$$\Delta^2 < \min \{ \frac{d}{2u_0 b}, \frac{3d - 28u_0 b d}{28u_0 b^2} \}$$

(41)

Then there exists universal constant $u_0$, $\sigma$, such that it holds that $\gamma \geq \sigma b \ln \alpha$ in Theorem 3.5 and

$$\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} [||\nabla f(x^s_{t+1})||^2] \leq \frac{dL_n^a \mathbb{E} [f(\tilde{x}^0) - f(\tilde{x}^*)]}{T \sigma b}$$

(42)

5 Experiments

In this section, we perform experiments on distributed-memory architecture and shared-memory architecture respectively. One of the main purpose of our experiments is to validate the faster convergence rate of asynchronous SVRG method, and the other purpose is to demonstrate its linear speedup property. The speedup we consider in this paper is running time speedup when they reach similar performance, e.g. training loss function value. Given $T$ workers, running time speedup is defined as,

$$\text{Running time speedup of } T \text{ workers} = \frac{\text{Running time for the serial computation}}{\text{Running time of using } T \text{ workers}}$$

(43)

5.1 Shared-memory Architecture

We conduct experiment on a machine which has 2 sockets, and each socket has 18 cores. OpenMP library\footnote{https://openmp.org} is used to handle shared-memory parallelism. We consider the multi-class problem on MNIST dataset\footnote{10}, and use 10,000 training samples and 2,000 testing samples in the experiment. Each image sample is a vector of 784 pixels. We construct a toy three-layer neural network ($784 \times 100 \times 10$), where ReLU activation function is used in the hidden layer and there are 10 classes in MNIST dataset. We train this neural network with softmax loss function, and $\ell_2$ regularization with weight $C = 10^{-3}$. We set mini-batch size $|I_t| = 10$, and inner iteration length $m = 1,000$. Updating only one component in $x$ in each iteration is too time consuming, therefore we randomly select and update 1,000 components in each iteration.

We compare following three methods in the experiment:

- SGD: We implement stochastic gradient descent (SGD) algorithm and train with the best tuned learning rate. In our experiment, we use polynomial learning rate $\eta = \alpha \frac{1}{(1+s)^\beta}$, where $\alpha$ denotes initial learning rate and we tune it from $\{1e^{-2}, 5e^{-2}, 1e^{-3}, 5e^{-3}, 1e^{-4}, 5e^{-4}, 1e^{-5}, 5e^{-5}\}$, $\beta$ is a variable in $\{0, 0.1, ..., 1\}$ and $s$ denotes the epoch number.
- SVRG: We also implement stochastic gradient descent with variance reduction (SVRG) method and train with the best tuned constant learning rate $\alpha$.
- SGD-SVRG: SVRG method is sensitive to initial point, and we apply SVRG on a pre-trained model using SGD. In the experiment, we use the pre-trained model after 10 iterations of SGD method.
We test three compared methods on MNIST dataset, and each method trained with best tuned learning rate. Figure 1 shows the convergence rate of each method. We compare three criterion in this experiment, loss function value on training dataset, training error rate, and testing error rate. Figure 1a shows the curves of loss function on training dataset, it is clear that SGD method converges faster than SVRG method in the first 20 iterations, and after that, SVRG method outperforms SGD. SGD_SVRG method initializes with a pre-trained model, it has the best convergence rate. We are able to draw the same conclusion from Figure 1b and Figure 1c.

We also evaluate SVRG method with different number of threads, and Figure 2 presents the result of our experiment. In Figure 2a, we plot curves for each method when they get similar training loss value. As we can see, the more threads we use in the computation, the less time we need to achieve a similar performance. This phenomenon is reasonable, because iterations in a loop can be divided into multiple parts, and each thread handles one subset independently. The ideal result of parallel computation is linear speedup, namely if we use $K$ threads, its working time should be $\frac{1}{K}$ of the time when we just use a single thread. Figure 2c shows the ideal speedup and actual speedup in our experiment. We can find out that a almost linear speedup is achievable when we increase thread numbers. When the number of threads exceeds a threshold, performance tends to degrade.

![Figure 1](image1.png)

**Figure 1:** Comparison of three methods (SGD, SVRG, SGD_SVRG) on MNIST dataset. Figure 1a shows the convergence of loss function value on training dataset. Figure 1b shows the convergence of training error rate and Figure 1c shows the convergence of test error rate.

### 5.2 Distributed-memory Architecture

We conduct distributed-memory architecture experiment on Amazon AWS platform\(^2\), and each node is a t2.micro instance with one CPU. Each server and worker takes a single node. The point to point communication between server and workers are handled by MPICH library\(^3\). CIFAR-10 dataset\(^4\) has 10 classes of color image $32 \times 32 \times 3$. We use 20000 samples as training data and 4000 samples as testing data. We use a pre-trained CNN model in TensorFlow tutorial\(^1\), and extract features from second fully connected layer. Thus, each sample is a vector of size 384. We construct a three-layer fully connected neural network $(384 \times 50 \times 10)$. We train this model with softmax loss function, and $\ell_2$ regularization with weight $C = 1e^{-4}$. In this experiment, mini-batch size $|I_t| = 10$, and the inner loop length $m = 2,000$. Similar to the compared

\(^2\)https://aws.amazon.com/
\(^3\)http://www.mpich.org/
methods in shared-memory architecture, we implement SGD method with polynomial learning rate, SVRG with constant learning rate. SGD_SVRG method is initialized with parameters learned after 1 epoch of SGD method.

At first, we train our model on CIFAR-10 dataset with three compared methods, and each method is with a best tuned learning rate. Performances of all three methods are presented in Figure 3. In Figure 3a, the curves show that SGD is fast in the first few iterations, and then, SVRG-based method will outperform it due to learning rate issue. As mentioned in [16], SVRG is more sensitive than SGD to the initial point, so using a pre-trained model is really helpful. It is obvious that SGD_SVRG has better convergence rate than SVRG method. We can also draw the same conclusion from training error curves with respect to data passes in Figure 3b. Figure 3c represents that the test error performances of three compared methods are comparable.

We also test SVRG method with different number of workers, and Figure 4 illustrates the results of our experiment. It is easy to draw a conclusion that when the number of workers increases, we can get a near linear speedup, and when the number gets larger, the speedup tends to be worse.
Figure 4: Asynchronous stochastic gradient descent method with variance reduction runs on multiple machines from 1 to 10. The curves in Figure 4a shows the convergence of training loss value with respect to time. The curves in Figure 4b shows the convergence of error rate on testing data. Figure 4c represents the running time speedup when using different workers, where the dashed line denotes ideal linear speedup.

6 Conclusion

In this paper, we propose and analyze two different asynchronous stochastic gradient descent with variance reduction on non-convex optimization as per two different distributed categories, one for shared memory system and the other one for distributed memory system. We also extend these two methods to mini-batch version. We analyze and prove that both of them can get an ergodic convergence rate $O(1/T)$ and a linear speedup is achievable if the number of workers is upper bounded.

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A Proof of Lemma 4.4

Proof 1 (Proof of Lemma 4.4) As per the definition of \(v_t^{s+1}\) and \(u_t^{s+1}\),

\[
\mathbb{E} \left[ ||v_t^{s+1}||^2 \right] = \mathbb{E} \left[ ||v_t^{s+1} - u_t^{s+1} + u_t^{s+1}||^2 \right] \\
\leq 2\mathbb{E} \left[ ||v_t^{s+1} - u_t^{s+1}||^2 \right] + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \\
= 2\mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_t} \nabla f_i(x_t^{s+1}) - \nabla f_i(x_t^{s+1}) ||^2 \right] + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \\
\leq \frac{2L^2}{b} \sum_{i \in I_t} \mathbb{E} \left[ ||x_t^{s+1} - x_t^{s+1}||^2 \right] + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \\
\leq \frac{2L^2}{b} \sum_{i \in I_t} \left( \left\| \sum_{j \in J(t, i)} (x_j^{s+1} - x_j^{s+1}) k_j \right\|^2 \right) + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \\
\leq \frac{2L^2 \Delta^2 \eta^2}{bd} \sum_{i \in I_t} \sum_{j \in J(t, i)} \mathbb{E} \left[ ||v_j^{s+1}||^2 \right] + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right]
\tag{44}
\]

where the first, third and last inequality follows from \(||a_1 + \ldots + a_n||^2 \leq n \sum ||a_i||^2\).

Second inequality follows from Lipschitz smoothness of \(f(x)\). Then sum over \(\mathbb{E} \left[ ||v_t^{s+1}||^2 \right]\) in one epoch, we get the following inequality,

\[
\sum_{t=0}^{m-1} \mathbb{E} \left[ ||v_t^{s+1}||^2 \right] \leq \sum_{t=0}^{m-1} \left[ \frac{2L^2 \Delta^2 \eta^2}{bd} \sum_{i \in I_t} \sum_{j \in J(t, i)} \mathbb{E} \left[ ||v_j^{s+1}||^2 \right] + 2\mathbb{E} \left[ ||u_t^{s+1}||^2 \right] \right]
\leq \frac{2L^2 \Delta^2 \eta^2}{d} \sum_{t=0}^{m-1} \mathbb{E} \left[ ||v_t^{s+1}||^2 \right] + 2 \sum_{t=0}^{m-1} \mathbb{E} \left[ ||u_t^{s+1}||^2 \right]
\tag{45}
\]

Thus, if \(d - 2L^2 \Delta^2 \eta^2 > 0\), then \(||v_t^{s+1}||^2\) is upper bounded by \(||u_t^{s+1}||^2\),

\[
\sum_{t=0}^{m-1} \mathbb{E} \left[ ||v_t^{s+1}||^2 \right] \leq \frac{2d}{d - 2L^2 \Delta^2 \eta^2} \sum_{t=0}^{m-1} \mathbb{E} \left[ ||u_t^{s+1}||^2 \right]
\tag{46}
\]

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Besides, we can have an upper bound of $\mathbb{E}[||u_t^{s+1}||^2]$.

$$
\mathbb{E}[||u_t^{s+1}||^2] = \mathbb{E}\left[\frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(x_s^t - \nabla f(x_s^t)) \right]^2
$$

$$
\leq 2\mathbb{E}\left[\left(\frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(x_s^t - \nabla f(x_s^t))) \right)^2 \right] + 2\mathbb{E}\left[||\nabla f(x_t^{s+1})||^2\right]
$$

$$
\leq 2\mathbb{E}[||\nabla f(x_t^{s+1})||^2] + 2\mathbb{E}\left[\left(\frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(x_s^t - \nabla f(x_s^t))) \right)^2 \right] + \frac{2L^2}{b} \mathbb{E}[||x_t^{s+1} - x_s^t||^2]
$$

(47)

where third inequality follows from $\mathbb{E}[||\xi - \mathbb{E}[\xi]||^2] \leq \mathbb{E}[||\xi||^2]$.

**B Proof of Theorem 4.5**

**Proof 2 (Proof of Theorem 4.5)** At first, we derive the upper bound of $\mathbb{E}[||x_t^{s+1} - x_s^t||^2]$.

$$
\mathbb{E}[||x_t^{s+1} - x_s^t||^2] = \mathbb{E}[||x_t^{s+1} - x_t^{s+1} + x_t^{s+1} - x_s^t||^2]
$$

$$
= \mathbb{E}[||x_t^{s+1} - x_t^{s+1}||^2 + ||x_t^{s+1} - x_s^t||^2 + 2(x_t^{s+1} - x_t^{s+1}, x_t^{s+1} - x_s^t)]
$$

$$
= \mathbb{E}\left[\frac{\eta^2}{d} ||v_t^{s+1}||^2 + ||x_t^{s+1} - x_s^t||^2 - \frac{2\eta}{d} \left(\frac{1}{b} \sum_{i \in I_t} \nabla f_i(x_t^{s+1}), x_t^{s+1} - x_s^t\right)\right]
$$

$$
\leq \frac{\eta^2}{d} \mathbb{E}[||v_t^{s+1}||^2] + \frac{2\eta}{d} \mathbb{E}\left[\left(\frac{1}{b} \sum_{i \in I_t} \nabla f_i(x_t^{s+1})\right)^2 + \frac{\beta_t}{2} ||x_t^{s+1} - x_s^t||^2\right]
$$

$$
+ \mathbb{E}[||x_t^{s+1} - x_s^t||^2]
$$

$$
= \frac{\eta^2}{d} \mathbb{E}[||v_t^{s+1}||^2] + \frac{\eta}{d\beta_t} \mathbb{E}\left[||b \sum_{i \in I_t} \nabla f_i(x_t^{s+1})||^2\right] + (1 + \frac{\eta\beta_t}{d}) \mathbb{E}[||x_t^{s+1} - x_s^t||^2]
$$

(48)

where the inequality follows from $(a, b) \leq \frac{1}{2}(a^2 + b^2)$. Then we know that $\mathbb{E}[f(x_t^{s+1} + 1)]$ is also upper bounded,

$$
\mathbb{E}[f(x_t^{s+1} + 1)] \leq \mathbb{E}\left[f(x_t^{s+1}) + \langle \nabla f(x_t^{s+1}), x_t^{s+1} - x_t^{s+1} + L ||x_t^{s+1} - x_t^{s+1}||^2\right]
$$

$$
= \mathbb{E}[f(x_t^{s+1})] - \frac{\eta L}{d} \mathbb{E}\left[\left(\nabla f(x_t^{s+1})\right)^2 \frac{1}{b} \sum_{i \in I_t} \nabla f_i(x_t^{s+1})\right] + \frac{\eta^2 L}{2d} \mathbb{E}[||v_t^{s+1}||^2]
$$

$$
= \mathbb{E}[f(x_t^{s+1})] - \frac{\eta}{2d} \mathbb{E}[||\nabla f(x_t^{s+1})||^2 + \frac{1}{b} \sum_{i \in I_t} ||\nabla f_i(x_t^{s+1})||^2]
$$

$$
= \mathbb{E}\left[f(x_t^{s+1}) - \frac{1}{b} \sum_{i \in I_t} \nabla f_i(x_t^{s+1})\right] + \frac{\eta^2 L}{2d} \mathbb{E}[||v_t^{s+1}||^2]
$$

(49)
where the first inequality follows from Lipschitz continuity of \( f(x) \).

\[
\mathbb{E} \left[ \left\| \nabla f(x_{s+1}^t) - \frac{1}{b} \sum_{i \in I_t} \nabla f(\hat{x}_{s+1}^t) \right\|^2 \right] \leq \frac{L^2}{b} \sum_{i \in I_t} \mathbb{E} \left[ \left\| x_{s+1}^t - \hat{x}_{s+1}^t \right\|^2 \right]
\]

\[
= \frac{L^2}{b} \sum_{i \in I_t} \mathbb{E} \left[ \left\| \sum_{j \in J(t,i)} (x_{s+1}^t - x_{j+1}^t) \right\|^2 \right]
\]

\[
\leq \frac{L^2 \Delta}{b} \sum_{i \in I_t} \sum_{j \in J(t,i)} \mathbb{E} \left[ \left\| x_{s+1}^t - x_{j+1}^t \right\|^2 \right]
\]

\[
\leq \frac{L^2 \Delta \eta^2}{bd} \sum_{i \in I_t} \sum_{j \in J(t,i)} \mathbb{E} \left[ \left\| v_{s+1}^t \right\|^2 \right]
\]  (50)

where the first inequality follows from Lipschitz continuity of \( f(x) \). \( \Delta \) denotes the upper bound of time delay. From (49) and (50), it is to derive the following inequality,

\[
\mathbb{E} \left[ f(x_{s+1}^{t+1}) \right] \leq \mathbb{E} \left[ f(x_{s+1}^t) \right] - \frac{\eta}{2d} \mathbb{E} \left[ \left\| \nabla f(x_{s+1}^t) \right\|^2 \right] - \frac{\eta}{2d} \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i \in I_t} \nabla f(\hat{x}_{s+1}^t) \right\|^2 \right]
\]

\[
+ \frac{\eta^2 L}{2d} \mathbb{E} \left[ \left\| \xi_{s+1}^t \right\|^2 \right] + \frac{L^2 \Delta \eta^3}{2bd^2} \sum_{i \in I_t} \sum_{j \in J(t,i)} \mathbb{E} \left[ \left\| v_{j+1}^t \right\|^2 \right]
\]  (51)

Define Lyapunov function,

\[
R_{s+1}^{t+1} = \mathbb{E} \left[ f(x_{s+1}^{t+1}) + c_t \left\| x_{s+1}^t - \tilde{x}^s \right\|^2 \right]
\]  (52)
As per the definition of Lyapunov function, and inequalities in (48) and (51),

\[
R_{t+1}^+ = \mathbb{E} [f(x_{t+1}^+) + c_{t+1} \|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
\leq \mathbb{E} [f(x_{t+1}^+)] - \frac{\eta}{2d} \mathbb{E} [\|\nabla f(x_{t+1}^+)^2\|] + \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
= \mathbb{E} [f(x_{t+1}^+)] - \frac{\eta}{2d} \mathbb{E} [\|\nabla f(x_{t+1}^+)^2\|] + \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
+ \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + \frac{\eta \beta_i}{d} \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
\leq \sum_{t=0}^{m-1} R_{t+1}^+ \leq \sum_{t=0}^{m-1} \left[ \mathbb{E} [f(x_{t+1}^+)] - \frac{\eta}{2d} \mathbb{E} [\|\nabla f(x_{t+1}^+)^2\|] + \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
+ \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + \frac{\eta \beta_i}{d} \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
\leq \sum_{t=0}^{m-1} \left[ \mathbb{E} [f(x_{t+1}^+)] - \frac{\eta}{2d} \mathbb{E} [\|\nabla f(x_{t+1}^+)^2\|] + \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
+ \frac{L^2 \Delta \eta^3}{2bd^2} \sum \sum \mathbb{E} [\|v_{t+1}^+\|^2] + \frac{\eta \beta_i}{d} \mathbb{E} [\|v_{t+1}^+\|^2] + c_{t+1} \left(1 + \frac{\eta \beta_i}{d}\right) \mathbb{E} [\|x_{t+1}^+ - \bar{x}\|^2]
\]

\[
\leq \sum_{t=0}^{m-1} R_{t+1}^+ - \sum_{t=0}^{m-1} \left[ \mathbb{E} [\nabla f(x_{t+1}^+)^2] \right]
\]

(53)
have, satisfy when $n$

In the final inequality, we constrain that $d n < 0$

$\Gamma_t = \frac{\eta}{2d} - \frac{4}{d - 2L^2 \Delta^2 \eta^2} \left( \frac{L^2 \Delta^2 \eta^3}{2d} + \frac{\eta^2 L}{2} + c_t + 1 \right) \eta^2$ (56)

Setting $c_m = 0$, $x^{s+1} = x^{s+1}$, and $\gamma = \min \Gamma_t$, then $R_m^{*+1} = \mathbb{E}[f(x_m^{s+1})] = \mathbb{E}[f(\tilde{x}^{s+1})]$ and $R_0^{*+1} = \mathbb{E}[f(x_0^{s+1})] = \mathbb{E}[f(\tilde{x}^s)]$. Thus we can get,

$$\sum_{t=0}^{m-1} \mathbb{E}[||\nabla f(x_t^{s+1})||^2] \leq \frac{\mathbb{E}[f(\tilde{x}^s) - f(\tilde{x}^{s+1})]}{\gamma}$$

Summing up all epochs, and define $x^0$ as initial point and $x^*$ as optimal solution, we have the final inequality,

$$\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E}[||\nabla f(x_t^{s+1})||^2] \leq \frac{\mathbb{E}[f(x^0) - f(x^*)]}{T \gamma}$$

**C Proof of Theorem 4.6**

**Proof 3 (Proof of Theorem 4.6)** Setting $c_m = 0$, $\eta_t = \eta = \frac{u_0 b}{L \nu \omega}$, $\beta_t = \beta = 2L$, $0 < u_0 < 1$, and $0 < \alpha < 1$.

$$\theta = \frac{\eta \beta}{d} + \frac{4L^2 \eta^2}{(d - 2L^2 \Delta^2 \eta^2) b} = \frac{2u_0 b}{d \nu \omega} + \frac{4u_0^2 b}{d \nu \omega - 2 \Delta^2 u_0^2 b^2} \leq \frac{6u_0 b}{d \nu \omega}$$

In the final inequality, we constrain that $d n^\alpha \leq d n^2 \Delta^2 u_0^2 b^2$, and it is easy to satisfy when $n$ is large. We set $m = \frac{d n^\alpha}{6u_0 b}$, and from the recurrence formula of $c_t$, we have,

$$c_0 = \frac{2L^2 b}{(d - 2L^2 \Delta^2 \eta^2) b} \left( \frac{L^2 \Delta^2 \eta^3}{d} + \frac{\eta^2 L}{2} \right) \frac{(1 + \theta)^m - 1}{\theta}$$

$$= \frac{2L b \Delta^2}{(d - 2L^2 \Delta^2 \eta^2) \left( \frac{2u_0 b}{d \nu \omega} + \frac{4u_0^2 \Delta^2}{d \nu \omega - 2 \Delta^2 u_0^2 b^2} \right) d} \frac{(1 + \theta)^m - 1}{\theta}$$

$$\leq \frac{L(u_0 b \Delta^2 b)}{3d} \frac{(1 + \theta)^m - 1}{\theta}$$

$$\leq \frac{L(u_0 b \Delta^2 b + d)}{3d} (e - 1)$$

(60)
Proof 4 (Proof of Lemma 3.4) As per the definition of \( v_{t+1}^* \) and \( u_{t+1}^* \),
\[
\begin{align*}
\mathbb{E} \left[ \|v_{t+1}^*\|^2 \right] &= \mathbb{E} \left[ \|v_{t+1}^* - u_{t+1}^* + u_{t+1}^*\|^2 \right] \\
&\leq 2 \mathbb{E} \left[ \|v_{t+1}^* - u_{t+1}^*\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \\
&= 2 \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_{t+1}^*) - \nabla f_i(x_{t+1}^*)) \right\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \\
&\leq \frac{2L^2}{b} \sum_{i \in I_t} \mathbb{E} \left[ \|x_{t+1}^* - x_{t+1}^*\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \\
&\leq \frac{2L^2}{b} \sum_{i \in I_t} \mathbb{E} \left[ \left\| \sum_{j=t}^{t-1} (x_{j+1}^* - x_{j+1}^*) \right\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \\
&\leq \frac{2L^2 \Delta^2 \eta^2}{b} \sum_{i \in I_t} \sum_{j=t}^{t-1} \mathbb{E} \left[ \|v_{j+1}^*\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right]
\end{align*}
\]
where the first inequality follows from \( \|a + b\| \leq 2\|a\|^2 + 2\|b\|^2 \), second inequality follows from Lipschitz continuity of \( f(x) \). Then, we sum up \( u_{t+1}^* \) from 0 to \( m - 1 \),
\[
\sum_{t=0}^{m-1} \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \leq \sum_{t=0}^{m-1} \left[ \frac{2L^2 \Delta^2 \eta^2}{b} \sum_{i \in I_t} \sum_{j=t}^{t-1} \mathbb{E} \left[ \|v_{j+1}^*\|^2 \right] + 2 \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right] \right] \\
\leq 2L^2 \Delta^2 \eta^2 \sum_{t=0}^{m-1} \mathbb{E} \left[ \|v_{t+1}^*\|^2 \right] + 2 \sum_{t=0}^{m-1} \mathbb{E} \left[ \|u_{t+1}^*\|^2 \right]
\]
In the following proof, we constrain \( 1 - 2L^2\Delta^2\eta^2 > 0 \), so we can get the inequality,

\[
\sum_{t=0}^{m-1} \mathbb{E} [||v_{t+1}^s||^2] \leq \frac{2}{1 - 2L^2\Delta^2\eta^2} \sum_{t=0}^{m-1} \mathbb{E} [||u_{t+1}^s||^2] \tag{65}
\]

The proof of upper bound of \( \mathbb{E} [||u_{t+1}^s||^2] \) is similar to the proof in Lemma 4.4.

## E Proof of Theorem 3.5

**Proof 5 (Proof of Theorem 3.5)**

\[
\mathbb{E} [||x_{t+1}^s - \bar{x}^s||^2] = \mathbb{E} [||x_{t+1}^s - x_t^s + x_t^s - \bar{x}^s||^2]
\]

\[
= \mathbb{E} [||x_{t+1}^s - x_t^s + x_t^s - \bar{x}^s||^2 + 2 \langle x_{t+1}^s - x_t^s, x_t^s - \bar{x}^s \rangle]
\]

\[
= \mathbb{E} [\eta^2||v_t^s||^2 + ||x_t^s - \bar{x}^s||^2 - 2\eta \left\{ \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-1}^i), x_t^s - \bar{x}^s \right\}]
\]

\[
\leq \eta^2\mathbb{E} [||v_t^s||^2] + 2\eta \mathbb{E} \left[ \frac{1}{2\beta_t} \sum_{i \in I_t} \nabla f(x_{t-1}^i) ||x_t^s - \bar{x}^s||^2 + \frac{\beta_t}{2} ||x_t^s - \bar{x}^s||^2 \right]
\]

\[
+ \mathbb{E} [||x_t^s - \bar{x}^s||^2]
\]

\[
= \eta^2\mathbb{E} [||v_t^s||^2] + (1 + \eta\beta_t) \mathbb{E} [||x_t^s - \bar{x}^s||^2] + \frac{\eta}{\beta_t} \mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-1}^i) ||x_t^s - \bar{x}^s||^2 \right]
\]

\[
\tag{66}
\]

where the first inequality follows \( 2 \langle a, b \rangle \leq ||a||^2 + ||b||^2 \)

\[
\mathbb{E} [f(x_{t+1}^s)] \leq \mathbb{E} \left[ f(x_t^s) + \langle \nabla f(x_t^s), x_{t+1}^s - x_t^s \rangle + \frac{L}{2} ||x_{t+1}^s - x_t^s||^2 \right]
\]

\[
= \mathbb{E} [f(x_t^s)] - \eta \mathbb{E} \left[ \langle \nabla f(x_t^s), \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-1}^i) \rangle \right] + \frac{\eta^2L}{2} \mathbb{E} [||v_t^s||^2]
\]

\[
= -\frac{\eta}{2} \mathbb{E} \left[ ||\nabla f(x_{t+1}^s)||^2 + ||\frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-1}^i)||^2 - ||\nabla f(x_{t+1}^s) - \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-1}^i)||^2 \right]
\]

\[
+ \mathbb{E} [f(x_t^s)] + \frac{\eta^2L}{2} \mathbb{E} [||v_t^s||^2]
\]

\[
\tag{67}
\]

where the first inequality follows from Lipschitz continuity of \( f(x) \).
\[ \| \nabla f(x_t^{s+1}) - \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-\tau_i}^{s+1}) \|^2 \leq \frac{1}{b} \sum_{i \in I_t} \| \nabla f(x_t^{s+1}) - \nabla f(x_{t-\tau_i}^{s+1}) \|^2 \]

\[ \leq \frac{L^2}{b} \sum_{i \in I_t} \| x_t^{s+1} - x_{t-\tau_i}^{s+1} \|^2 \]

\[ = \frac{L^2}{b} \sum_{i \in I_t} \| \sum_{j=1}^{t-1} (x_j^{s+1} - x_{j+1}^{s+1}) \|^2 \]

\[ \leq \frac{L^2 \Delta^2}{b} \sum_{i \in I_t} \sum_{j=t-\tau_i}^{t-1} \| x_j^{s+1} - x_{j+1}^{s+1} \|^2 \]

\[ = \frac{L^2 \Delta^2 \eta^2}{b} \sum_{i \in I_t} \sum_{j=t-\tau_i}^{t-1} \| v_j^{s+1} \|^2 \quad (68) \]

where the second inequality follows from Lipschitz continuity of \( f(x) \). \( \Delta \) denotes the upper bound of time delay, \( \tau \leq \Delta \). Above all, we have the following inequality,

\[ \mathbb{E} \left[ f(x_{t+1}^{s+1}) \right] \leq \mathbb{E} \left[ f(x_t^{s+1}) \right] - \frac{\eta}{2} \mathbb{E} \left[ \| \nabla f(x_t^{s+1}) \|^2 \right] - \frac{\eta}{2} \mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-\tau_i}^{s+1}) \|^2 \right] \]

\[ + \frac{\eta^2 L}{2} \mathbb{E} \left[ \| v_t^{s+1} \|^2 \right] + \frac{L^2 \Delta^2 \eta^3}{2b} \sum_{i \in I_t} \sum_{j=t-\tau_i}^{t-1} \mathbb{E} \left[ \| v_j^{s+1} \|^2 \right] \]

\[ + \theta_{t+1} \left[ \eta^2 \mathbb{E} \left[ \| v_t^{s+1} \|^2 \right] + (1 + \eta \beta_t) \mathbb{E} \left[ \| x_t^{s+1} - \tilde{x}_t^{s+1} \|^2 \right] + \frac{\eta}{\beta_t} \mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-\tau_i}^{s+1}) \|^2 \right] \right] \]

\[ = \mathbb{E} \left[ f(x_{t+1}^{s+1}) \right] - \frac{\eta}{2} \mathbb{E} \left[ \| \nabla f(x_t^{s+1}) \|^2 \right] - \frac{\eta}{2} - \frac{\eta c_{t+1} \eta}{\beta_t} \mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_t} \nabla f(x_{t-\tau_i}^{s+1}) \|^2 \right] \]

\[ + \frac{L^2 \Delta \eta^3}{2b} \sum_{i \in I_t} \sum_{j=t-\tau_i}^{t-1} \mathbb{E} \left[ \| v_j^{s+1} \|^2 \right] + \left( \frac{\eta^2 L}{2} + \theta_{t+1} \eta^2 \right) \mathbb{E} \left[ \| v_t^{s+1} \|^2 \right] \]

\[ + \theta_{t+1} (1 + \eta \beta_t) \mathbb{E} \left[ \| x_t^{s+1} - \tilde{x}_t^{s+1} \|^2 \right] \]

\[ \leq \mathbb{E} \left[ f(x_t^{s+1}) \right] + \frac{\eta}{2} \mathbb{E} \left[ \| \nabla f(x_t^{s+1}) \|^2 \right] + \frac{L^2 \Delta \eta^3}{2b} \sum_{i \in I_t} \sum_{j=t-\tau_i}^{t-1} \mathbb{E} \left[ \| v_j^{s+1} \|^2 \right] \]

\[ + \left( \frac{\eta^2 L}{2} + \theta_{t+1} \eta^2 \right) \mathbb{E} \left[ \| v_t^{s+1} \|^2 \right] + \theta_{t+1} (1 + \eta \beta_t) \mathbb{E} \left[ \| x_t^{s+1} - \tilde{x}_t^{s+1} \|^2 \right] \quad (70) \]
In the final inequality, we make \( \frac{\eta}{2} - \frac{\Gamma_t}{\eta_t} \eta_t > 0 \). Then we sum over \( R^+_{t+1} \)

\[
\sum_{t=0}^{m-1} R^+_{t+1} \leq \sum_{t=0}^{m-1} \left[ E \left[ f(x^+_{t+1}) \right] - \frac{\eta}{2} E \left[ \| \nabla f(x^+_{t+1}) \|^2 \right] + \frac{L^2 \Delta \eta^3}{2b} \sum_{i \in I_t, j = -r_i}^{t-1} E \left[ ||v_j^{+1}||^2 \right] \\
+ \left( \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] + c_{t+1} \left( 1 + \eta \beta \right) E \left[ ||x^+_{t+1} - \tilde{x}^+||^2 \right] \\
\right] \\
\leq \sum_{t=0}^{m-1} \left[ E \left[ f(x^+_{t+1}) \right] - \frac{\eta}{2} E \left[ \| \nabla f(x^+_{t+1}) \|^2 \right] + c_{t+1} \left( 1 + \eta \beta \right) E \left[ ||x^+_{t+1} - \tilde{x}^+||^2 \right] \\
+ \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] \\
\right] \\
\leq \sum_{t=0}^{m-1} \left[ E \left[ f(x^+_{t+1}) \right] - \frac{\eta}{2} E \left[ \| \nabla f(x^+_{t+1}) \|^2 \right] + c_{t+1} \left( 1 + \eta \beta \right) E \left[ ||x^+_{t+1} - \tilde{x}^+||^2 \right] \\
+ \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] \\
\right] \\
= \sum_{t=0}^{m-1} R^+_{t+1} - \sum_{t=0}^{m-1} \left[ \Gamma_t E \left[ ||\nabla f(x^+_{t+1})||^2 \right] \right] \\
\leq \sum_{t=0}^{m-1} \left[ E \left[ f(x^+_{t+1}) \right] - \frac{\eta}{2} E \left[ \| \nabla f(x^+_{t+1}) \|^2 \right] + c_{t+1} \left( 1 + \eta \beta \right) E \left[ ||x^+_{t+1} - \tilde{x}^+||^2 \right] \\
+ \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] \\
\right] \\
\leq \frac{2}{1 - 2L^2 \Delta \eta^2} \left( \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] \\
\right] \\
\leq \frac{2}{1 - 2L^2 \Delta \eta^2} \left( \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) E \left[ ||v_j^{+1}||^2 \right] \\
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(71)

where the last inequality follows from Lemma 3.4, and we define

\[
c_t = c_{t+1} \left( 1 + \eta \beta + \frac{4L^2 \eta^2}{(1 - 2L^2 \Delta \eta^2) b} \right) + \frac{4L^2 \eta^2}{(1 - 2L^2 \Delta \eta^2) b} \left( \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) \\
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(72)

\[
\Gamma_t = \frac{\eta}{2} - \frac{4}{(1 - 2L^2 \Delta \eta^2) b} \left( \frac{L^2 \Delta \eta^3}{2} + \frac{\eta^2 L}{2} + c_{t+1} \eta^2 \right) \\
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(73)

We set \( c_m = 0 \), and \( \tilde{x}^+ = x^+_m \), and \( \gamma = \min \Gamma_t \), thus \( R^+_{m+1} = E \left[ f(x^+_{m+1}) \right] = E \left[ f(\tilde{x}^+) \right] \), and \( R^+_{0+1} = E \left[ f(x^+_{0+1}) \right] = E \left[ f(\tilde{x}^+) \right] \). Summing up all epochs, the following inequality holds,

\[
\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} E \left[ \| \nabla f(x^+_{t+1}) \|^2 \right] \leq \frac{E \left[ f(x^0) - f(\tilde{x}^+) \right]}{T \gamma} \\
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(74)

\section{F Proof of Theorem 3.6}

\textbf{Proof 6 (Proof of Theorem 3.6) Follows from the proof of Theorem 3.5} we let \( c_m = 0, \eta_t = \eta = \frac{u_0 b}{\lambda^2 m}, \beta_t = \beta = \frac{b}{2L}, 0 < u_0 < 1, \) and \( 0 < \alpha < 1 \). We define \( \theta \), and get its upper bound,

\[
\theta = \eta \beta + \frac{4L^2 \eta^2}{(1 - 2L^2 \Delta \eta^2) b} \\
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(75)
where we assume $n^{2\alpha} - 2\Delta^2 u_0^2 \geq n^\alpha$. We set $m = \lceil \frac{n^\alpha}{b u_0} \rceil$, from the recurrence formula between $c_t$ and $c_{t+1}$, $c_0$ is upper bounded,

$$
c_0 = \frac{2L^2}{(1 - 2L^2 \Delta^2 \eta^2)^b} \left( \frac{L^2 \Delta^2 \eta^3 + \eta^2 L}{2} \right) \frac{(1 + \theta)^m - 1}{\theta} $$

$$
\leq \frac{2L}{(1 - 2L^2 \Delta^2 \eta^2)^b} \left( \frac{u_0 \Delta^2 \eta^3}{n^\alpha} + \frac{u_0^2 \Delta^2 \eta^3}{n^\alpha} \right) \left( (1 + \theta)^m - 1 \right)
$$

$$
\leq \frac{L(u_0 b \Delta^2 + 1)}{3} \left( (1 + \theta)^m - 1 \right)
$$

$$
\leq \frac{L(u_0 b \Delta^2 + 1)}{3} \left( e - 1 \right)
$$

where the final inequality follows from that $(1 + \frac{1}{l})^l$ is increasing for $l > 0$, and

$$
\lim_{l \to \infty} (1 + \frac{1}{l})^l = e.
$$

From Theorem 3.5, we know that $c_0 < \frac{\alpha}{2} = L$, then $u_0 b \Delta^2 < \frac{\alpha}{2}$, $c_t$ is decreasing with respect to $t$, and $c_0$ is also upper bounded. Now, we can get a lower bound of $\gamma$.

$$
\gamma = \min_t \Gamma_t
$$

$$
\geq \frac{\eta}{2} - \frac{4}{(1 - 2L^2 \Delta^2 \eta^2)^b} \left( \frac{L^2 \Delta^2 \eta^3}{2} + \frac{\eta^2 L}{2} + c_0 \eta^2 \right)
$$

$$
\geq \frac{\eta}{2} - 4n^\alpha \left( \frac{L^2 \Delta^2 \eta^3}{2} + \frac{\eta^2 L}{2} + c_0 \eta^2 \right)
$$

$$
\geq \frac{1}{2} - \frac{14\Delta^2 u_0^2 b^2 + 14 u_0 b}{3} \eta
$$

$$
\geq \frac{\sigma b}{Ln^\alpha}
$$

(77)

There exists a small value $\sigma$ that the final inequality holds if $\frac{1}{2} > \frac{14\Delta^2 u_0^2 b^2 + 14 u_0 b}{3}$. So, if $\Delta^2$ has an upper bound $\Delta^2 < \min \left\{ \frac{1}{2n^\alpha} - \frac{3 - 28u_0 b^2}{28u_0 b^2} \right\}$, we can prove the final conclusion,

$$
\frac{1}{T} \sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \left[ \| \nabla f(x_{t+1}^s) \|^2 \right] \leq \frac{Ln^\alpha \mathbb{E} \left[ f(\tilde{x}^0) - f(\tilde{x}^*) \right]}{bT \sigma}
$$

(78)