ERGODIC BEHAVIORS OF COMPOSITION OPERATORS ACTING ON SPACE OF BOUNDED HOLOMORPHIC FUNCTIONS

HAMZEH KESHAVARZI, KARIM HEDAYATIAN

Abstract. We completely characterize the mean ergodic composition operators on $H^\infty(\mathbb{B}_n)$. In particular, we show that a composition operator acting on this space is mean ergodic if and only if it is uniformly mean ergodic.

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1. Introduction and main results

The purpose of this paper is to prove the following theorem:

Theorem 1.1. Let $\varphi$ be a holomorphic self-map of $\mathbb{B}_n$. Then, the following statements are equivalent.

(i) $C_\varphi$ is mean ergodic on $H^\infty(\mathbb{B}_n)$.
(ii) $C_\varphi$ is uniformly mean ergodic on $H^\infty(\mathbb{B}_n)$.
(iii) $\varphi$ has a fixed point in $\mathbb{B}_n$ and there is a $k \in \mathbb{N}$ such that $\|\varphi_k - \rho_\varphi\|_\infty \to 0$ as $j \to \infty$.

Where $\rho_\varphi$ is the holomorphic retraction associated with $\varphi$ and is defined below. We prove this theorem in two parts: Theorems 1.3 and 1.4. Moreover, Theorem 1.2 plays a key role in our method. However, we believe that Theorem 1.2 has an independent interest.

Throughout the paper, $n$ is a fixed positive integer. Here is some notations:

- $\mathbb{C}$: the complex plane.
- $\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}$: the unit ball of $\mathbb{C}^n$.
- $D = \mathbb{B}_1$: the unit disk in $\mathbb{C}$.
- $H(\mathbb{B}_n)$: the space of all holomorphic functions from $\mathbb{B}_n$ into $\mathbb{C}$
- $H^\infty(\mathbb{B}_n)$: the subspace of all bounded functions in $H(\mathbb{B}_n)$.
- $\text{Hol}(\mathbb{B}_n, \mathbb{B}_n)$: the set of all holomorphic self-maps of $\mathbb{B}_n$

Consider $\varphi \in \text{Hol}(\mathbb{B}_n, \mathbb{B}_n)$. The iterates of $\varphi$ are the functions $\varphi_k := \varphi \circ (k) \circ \varphi$. We denote by $\varphi^i$, $1 \leq i \leq n$ the components of $\varphi$, that is, $\varphi = (\varphi^1, ..., \varphi^n)$ where $\varphi^i : \mathbb{B}_n \to \mathbb{C}$ are holomorphic functions. Moreover, the composition operator $C_\varphi$ on $H(\mathbb{B}_n)$ is defined as $C_\varphi f = f \circ \varphi$.

When we say that $\rho \in \text{Hol}(\mathbb{B}_n, \mathbb{B}_n)$ is holomorphic retraction, it means that it is an idempotent, that is, $\rho_2 = \rho$. Clearly, if $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ be holomorphic such that the sequence of its iterates converges to a holomorphic function $h : \mathbb{B}_n \to \mathbb{B}_n$. Then, $h_2 = h$, that is, $h$ is a holomorphic retraction of $\mathbb{B}_n$. For more details about the holomorphic self-maps of the unit ball and their iterates see [1, Chapter 2].
Let \( \varphi : \mathbb{B}_n \to \mathbb{B}_n \) be holomorphic and have an interior fixed point. Then, from [1, Theorem 2.1.29 and Proposition 2.2.30], there exist a unique submanifold \( M_\varphi \) of \( \mathbb{B}_n \) and a unique holomorphic retraction \( \rho_\varphi : \mathbb{B}_n \to M_\varphi \) such that every limit point \( h \in Hol(\mathbb{B}_n, \mathbb{B}_n) \) of \( \{ \varphi_j \} \) is of the form \( h = \gamma \circ \rho_\varphi \), where \( \gamma \) is an automorphism of \( M_\varphi \). Moreover, even \( \rho_\varphi \) is a limit point of the sequence \( \{ \varphi_j \} \). This implies that \( \rho_\varphi \circ \varphi = \varphi \circ \rho_\varphi \).

Let \( \{ e_1, \ldots, e_n \} \) be the standard basis of \( \mathbb{C}^n \).

**Theorem 1.2.** Let \( \varphi \) be a holomorphic self-map of the unit ball with converging iterates and \( \varphi(0) = 0 \). Then, there is an invertible matrix \( V \) so that:

\[
V^{-1}\varphi_j V = \left( (V^{-1}\varphi_j V)^1, \ldots, (V^{-1}\varphi_j V)^s \right) \oplus P_{n-s},
\]

where \( \dim M_\varphi = n - s \), the functions \( (V^{-1}\varphi_j V)^1, \ldots, (V^{-1}\varphi_j V)^s \) are the components of \( V^{-1}\varphi_j V \), and \( P_{n-s} \) is the orthogonal projection from \( \mathbb{C}^n \) onto \( U = e_{s+1} \oplus \ldots \oplus e_n \). Moreover, \( V^{-1}\varphi_j V \) converges to \( P_{n-s} \) uniformly on the compact subsets of \( V^{-1}\mathbb{B}_n \).

Let \( X \) be a Banach space and \( T : X \to X \) be an operator. Then, We say that \( T \) is mean ergodic if

\[
M_j(T) = \frac{1}{j} \sum_{i=1}^{j} T^i.
\]

converges to a bounded operator defined on \( X \) for the strong operator topology. Uniformly mean ergodicity will define in a same way with convergence in the operator norm.

Lotz [13] proved that: If \( X \) is a Grothendieck Banach space with Dunford-Pettis property (GDP space), and \( T \in L(X) \) satisfies \( \|T^n/n\| \to 0 \), then \( T \) is mean ergodic if and only if it is uniformly mean ergodic. For the definition of GDP spaces see [13, Pages 208-209].

For some work on the mean ergodicity of composition operators see [2, 3, 4, 6, 10, 11, 12]. The (uniformly) mean ergodicity of composition operators on \( H^\infty(\mathbb{D}) \) have been characterized in [4]. It is well-known that \( H^\infty(\mathbb{D}) \) is a GDP space. Thus, a composition operator, acting on \( H^\infty(\mathbb{D}) \), is mean ergodic if and only if it is uniformly mean ergodic. However, we do not know whether \( H^\infty(\mathbb{B}_n) \) is a GDP space or not. In [12], the first author has proved that if \( \varphi \) is a holomorphic self-map of the unit ball with converging iterates and an interior fixed point, then the mean ergodicity and the uniformly mean ergodicity of \( C_\varphi \) are equivalent. In the following theorem, we give this equivalence for all \( \varphi \in Hol(\mathbb{B}_n, \mathbb{B}_n) \) with an interior fixed point.

**Theorem 1.3.** Let \( \varphi \) be a holomorphic self-map of the unit ball with a fixed point in \( \mathbb{B}_n \). Then, the following statements are equivalent.

(i) \( C_\varphi \) is mean ergodic on \( H^\infty(\mathbb{B}_n) \).
(ii) \( C_\varphi \) is uniformly mean ergodic on \( H^\infty(\mathbb{B}_n) \).
(iii) There is a \( k \in \mathbb{N} \) such that \( \|\varphi_{kj} - \rho_\varphi\|_\infty \to 0 \), as \( j \to \infty \).

As the final result, we prove that every holomorphic self-map of \( \mathbb{B}_n \) which has no interior fixed point induces a composition that is not mean ergodic on \( H^\infty(\mathbb{B}_n) \). This theorem gives the answer to [12, Question 3.16].

**Theorem 1.4.** Let the holomorphic function \( \varphi : \mathbb{B}_n \to \mathbb{B}_n \) has no interior fixed point. Then, \( C_\varphi \) is not mean ergodic on \( H^\infty(\mathbb{B}_n) \).
2. Basic results

Every automorphism \( \varphi \) of \( \mathbb{B}_n \) is of the form \( \varphi = U \varphi_a = \varphi_b V \), where \( U \) and \( V \) are unitary matrices of \( \mathbb{C}^n \) and

\[
\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B}_n,
\]

where \( a \neq 0 \), \( s_a = \sqrt{1 - |a|^2} \), \( P_a \) is the projection from \( \mathbb{C}^n \) onto the subspace \( \langle a \rangle \) spanned by \( a \), and \( Q_a \) is the projection from \( \mathbb{C}^n \) onto \( \mathbb{C}^n \ominus \langle a \rangle \). Clearly, \( \varphi_a(0) = a \), \( \varphi_a(a) = 0 \), and \( \varphi_a \circ \varphi_a(z) = z \). It is well-known that an automorphism \( \varphi \) of \( \mathbb{B}_n \) is a unitary matrix of \( \mathbb{C}^n \) if and only if \( \varphi(0) = 0 \).

Let \( \Omega \) be a strongly pseudoconvex bounded domain. The infinitesimal Kobayashi metric \( F_K : \Omega \times \mathbb{C}^n \to [0, \infty) \) is defined as:

\[
F_K(z, w) = \inf \left\{ C > 0 : \exists f \in H(\mathbb{D}, \Omega) \text{ with } f(0) = z, \ f'(0) = \frac{w}{C} \right\},
\]

where \( H(\mathbb{D}, \Omega) \) is the space of analytic functions from \( \mathbb{D} \) to \( \Omega \). Let \( \gamma : [0, 1] \to \Omega \) be a \( C^1 \)-curve. The Kobayashi length of \( \gamma \) is defined as:

\[
L_K(\gamma) = \int_0^1 F_K(\gamma(t), \gamma'(t)) dt.
\]

For \( z, w \in \Omega \), the Kobayashi metric function is defined as:

\[
k_\Omega(z, w) = \inf \left\{ L_K(\gamma) : \gamma \text{ is } C^1 \text{- curve with } \gamma(0) = z \text{ and } \gamma(1) = w \right\}.
\]

If \( \Omega \) and \( \Lambda \) are two strongly pseudoconvex bounded domains and \( \varphi : \Omega \to \Lambda \) is a holomorphic function, then from [1, Proposition 2.3.1], we have:

\[
k_\Lambda(\varphi(z), \varphi(w)) \leq k_\Omega(z, w), \quad \forall z, w \in \Omega.
\]

Thus, \( k_\Omega \) is invariant under automorphisms, that is,

\[
k_\Omega(\varphi(z), \varphi(w)) = k_\Omega(z, w),
\]

for all \( z, w \in \mathbb{B}_n \) and \( \varphi : \Omega \to \Omega \) is an automorphism.

Let \( \beta \) from \( \mathbb{B}_n \times \mathbb{B}_n \) to \([0, \infty)\) be the Bergman metric. From [1, Corollary 2.3.6], the Kobayashi metric and the Bergman metric coincide on \( \mathbb{B}_n \). We have:

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.
\]

We shall denote by \( B(a, r) \) the Bergman ball centered at \( a \in \mathbb{B}_n \) with radius \( r > 0 \), that is,

\[
B(a, r) = \{ z \in \mathbb{B}_n : \beta(a, z) < r \}.
\]

It is well-known (see [1, page 134]) that \( B(a, r) \) is the ellipsoid

\[
\frac{|P_\alpha(\zeta) - a_r|^2}{R^2 s^2} + \frac{|Q_\alpha(\zeta)|^2}{R^2 s} < 1,
\]

where \( R = \tanh r, a_r = \frac{1 - R^2}{1 - |a|^2} a \) and \( s = \frac{1}{1 - |a|^2} \).

Let \( P_k \) be the space homogeneous polynomial \( P : \mathbb{B}_n \to \mathbb{C} \) of degree \( k \). The Taylor series expansions of functions in \( H^\infty(\mathbb{B}_n) \) yield a direct sum decomposition of

\[
H^\infty(\mathbb{B}_n) = P_0 \oplus P_1 \oplus \ldots \oplus P_m \oplus R_m;
\]
where the remaining space $R_m$ consists of the functions $h \in H^\infty(\mathbb{B}_n)$ such that $|h(z)|/\|z\|^m$ is bounded for $z$ near 0. Similarly, $f : \mathbb{B}_n \to \mathbb{C}^n$ admits a homogeneous expansion:

$$f(z) = \sum_{k=0}^{\infty} F_k(z) = f(0) + f'(0)z + \ldots,$$

where all $n$ component functions of each $F_k$ are homogeneous polynomial of degree $k$.

It should be noted that $d_z \varphi = \varphi'(z)$. Note that $d_z \varphi$ is a matrix:

$$d_z \varphi := \begin{bmatrix} \frac{\partial \varphi^1}{\partial z_1} & \ldots & \frac{\partial \varphi^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi^n}{\partial z_1} & \ldots & \frac{\partial \varphi^n}{\partial z_n} \end{bmatrix} (z).$$

3. Proof of Theorem 1.2

Let $n - s$ be the dimension of $M_\varphi$.

If $s = 0$, then from [1, Proposition 2.2.14] and [12, Proposition 3.8], $\varphi$ is a unitary matrix. Since the iterates of $\varphi$ are convergent, $\varphi$ is the identity matrix. If $s = n$, then from [1, Theorem 2.2.32], $M_\varphi = \{0\}$ and $\rho_\varphi \equiv 0$. Therefore, for $s = 0$ or $n$, the result is obtained by considering $V$ as the identity matrix.

Thus, let $1 \leq s \leq n - 1$. We give the proof in three steps:

**Step 1.** There is an invertible matrix $V$ so that $V^{-1}d_0 \rho V = P_{n-s}$.

Proof. Recall that $P_{n-s}$ is the orthogonal projection from $\mathbb{C}^n$ onto $e_{s+1} \oplus \ldots \oplus e_n$.

Let $V$ be an invertible matrix so that $V^{-1}d_0 \rho V$ be the Jordan canonical form of $d_0 \rho$. Since, $\rho^2 = \rho$ and $\rho(0) = 0$, the matrix $d_0 \rho$ is also an idempotent. Thus, the eigenvalues of $V^{-1}d_0 \rho V$ are in $\{0, 1\}$. Note that since $\rho(\mathbb{B}_n) = M$ and $\rho$ is identity on $M$, it is easy to show that 0 and 1 will be repeated $s$ and $n - s$ times as the eigenvalues of $d_0 \rho$, respectively.

We have

$$V^{-1}d_0 \rho V = J_1(0) \oplus \ldots \oplus J_k(0) \oplus I_1(1) \oplus \ldots \oplus I_1(1),$$

where $J_1(0)$ and $I_1(1)$ are the blocks associated with the eigenvalues 0 and 1, respectively. Now since $d_0 \rho$ is an idempotent, the blocks $J_1(0)$ and $I_1(1)$ must be $1 \times 1$. That is,

$$V^{-1}d_0 \rho V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix},$$

where $I_{n-s}$ is the $(n - s) \times (n - s)$ identity matrix. Hence, $V^{-1}d_0 \rho V = P_{n-s}$. □

From [1, Theorem 2.1.21], we know that if $f : \mathbb{B}_n \to \mathbb{B}_n$ is holomorphic, $f(0) = 0$ and $d_0 f$ is identity, then so is $f$. In the next step, we want to show that if $d_0 f = 0_s \oplus I_{n-s}$, then $f = 0_s \oplus I_{n-s}$.

**Step 2.** For the matrix $V$, obtained in step 1, we have $V^{-1}\rho V = P_{n-s}$.

Proof. Let $V^{-1}\rho V \neq P_{n-s}$. Consider the function $\psi = V^{-1}\rho V - P_{n-s} : \mathbb{B}_n \to \mathbb{C}^n$. Since $d_0 \psi = V^{-1}d_0 \rho V - d_0 P_{n-s} = 0$, $\psi(0) = 0$, but $\psi \neq 0$, we can write:

$$V^{-1}\rho V(z) = P_{n-s}(z) + F_k(z) + \sum_{j=k+1}^{\infty} F_j(z),$$
where \( F_k \) is a homogeneous polynomial of degree \( k \geq 2 \). In summation, \( F_j \) is zero or a homogeneous polynomial of degree \( j \).

Note that every component of a homogeneous polynomial of degree \( j \) is a summation of polynomials

\[
z^m = z_1^{m_1} \cdots z_n^{m_n},
\]

where \( z = (z_1, \ldots, z_n) \), \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \), and \( m_1 + \ldots + m_n = j \). Thus, for \( j \geq k \) if \( F_j = (F_j^1, \ldots, F_j^n) \) is non-zero, then each component of \( F_j(V^{-1} \rho V(z)) \) is a summation of polynomials

\[
\left(P_{n-s}(z) + F_k(z) + \sum_{j=k+1}^{\infty} F_j(z)\right)^m
= \left(F_k^1(z) + \sum_{j=k+1}^{\infty} F_j^1(z)\right)^{m_1} \cdots \left(F_k^n(z) + \sum_{j=k+1}^{\infty} F_j^n(z)\right)^{m_n}
\times \left(z_{s+1} + F_k^{s+1}(z) + \sum_{j=k+1}^{\infty} F_j^{s+1}(z)\right)^{m_{s+1}} \cdots \left(z_n + F_k^n(z) + \sum_{j=k+1}^{\infty} F_j^n(z)\right)^{m_n}
\]

Thus, from the above statement and the assumption \( 1 \leq s \leq n - 1 \), if \( F_j \) is non-zero for \( j \geq k \), then each component of \( F_j(V^{-1} \rho V(z)) \) is a polynomial with a degree greater than or equal to:

\[ km_1 + \ldots + km_s + m_{s+1} + \ldots + m_n. \]

On the other hand, since \( k \geq 2 \), we have

\[ km_1 + \ldots + km_s + m_{s+1} + \ldots + m_n > \sum_{i=1}^{n} m_i = j. \]

Thus,

\[ V^{-1} \rho^2 V(z) = P_{n-s}(z) + P_{n-s}F_k(z) + \sum_{j=k+1}^{\infty} G_j(z), \]

where each \( G_j \) is zero or a homogeneous polynomial of degree \( j \). Since \( \rho^2 = \rho \), we must have \( F_k = P_{n-s}F_k \) which contradicts the assumption that \( s \neq 0, n \).

Indeed, we proved the following result in steps 1 and 2 as well as the paragraph before them:

**Corollary 3.1.** Every holomorphic retraction \( \rho \) on \( \mathbb{B}_n \) which fixes the origin is a matrix.

**Step 3.** \( (V^{-1} \varphi_j V)^i(z^1, \ldots, z^n) = z^i, \) for \( i = s + 1, \ldots, n \) and \( j \in \mathbb{N} \).

**Proof.** From step 2,

\[ (3.1) \quad V^{-1}(\rho \circ \varphi)V = V^{-1} \circ \rho \circ V(V^{-1} \circ \varphi \circ V) = 0_s \oplus \begin{bmatrix} (V^{-1} \varphi V)^{s+1}(z) \\ \vdots \\ (V^{-1} \varphi V)^n(z) \end{bmatrix}. \]

Moreover, since \( \rho \) and \( \varphi \circ \rho \) are the limit points of the convergent sequence \( \{\varphi_j\} \), we have:

\[ (3.2) \quad V^{-1} \rho \circ \varphi V = V^{-1} \varphi \circ \rho V = V^{-1} \rho V. \]
Thus, 3.1, 3.2, and step 2 imply that
\[
\begin{bmatrix}
(V^{-1}\varphi V)^{s+1}(z) \\
\vdots \\
(V^{-1}\varphi V)^n(z)
\end{bmatrix} = \begin{bmatrix}
z^{s+1} \\
\vdots \\
z^n
\end{bmatrix}.
\]
Again, by a similar argument, we can see that \(\rho \circ \varphi_j = \varphi_j \circ \rho = \rho\). Thus,
\[
\begin{bmatrix}
(V^{-1}\varphi_j V)^{s+1}(z) \\
\vdots \\
(V^{-1}\varphi_j V)^n(z)
\end{bmatrix} = \begin{bmatrix}
z^{s+1} \\
\vdots \\
z^n
\end{bmatrix}.
\]
The proof is complete. \(\square\)

4. PROOF OF THEOREM 1.3

If \(\varphi\) has an interior fixed point \(a \in \mathbb{B}\), then \(\psi := \varphi_a \circ \varphi \circ \varphi_a\) is a holomorphic self-map of \(\mathbb{B}_n\) that \(\psi(0) = 0\). Hence, without loss of generality, we assume that \(\varphi(0) = 0\). (ii) \(\Rightarrow\) (i) is obvious.

4.1. (iii) \(\Rightarrow\) (ii). Since \(\varphi_{kj} \to \rho\), from Theorem (1.2), there is an invertible matrix \(V\) so that

\[
V^{-1}\varphi_{kj} V = \left((V^{-1}\varphi_{kj} V)^1,\ldots,(V^{-1}\varphi_{kj} V)^s\right) \oplus P_{n-s},
\]
and

\[
V^{-1}\rho V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix} = P_{n-s}.
\]

From the continuity of \(V^{-1}\) and (iii), there is a \(C > 0\) so that

\[
\lim_{j \to \infty} \sup_{z \in V^{-1}\mathbb{B}_n} \left|\left((V^{-1}\varphi_{kj} V)^1,\ldots,(V^{-1}\varphi_{kj} V)^s\right)(z)\right| = \lim_{j \to \infty} \|V^{-1}(\varphi_{kj} - \rho)\|_{\infty}
\]

\[
\leq C \lim_{j \to \infty} \|\varphi_{kj} - \rho\|_{\infty} = 0.
\]

(4.1)

It is easy to see that \(V^{-1}\mathbb{B}_n\) is a taut manifold. Thus, from 2.2 we have:

\[
\sup_{z \in \mathbb{B}_n} \beta(\varphi_{kj}(z), \rho(z)) \leq \sup_{z \in \mathbb{B}_n} k_{V^{-1}\mathbb{B}_n}(V^{-1}\varphi_{kj}(z), V^{-1}\rho(z))
\]

\[
= \sup_{z \in \mathbb{B}_n} k_{V^{-1}\mathbb{B}_n}(V^{-1}\varphi_{kj} V(z), V^{-1}\rho V(z)).
\]

Hence, from [12, Lemma 4.1] and Equation 4.1, we obtain:

\[
\sup_{z \in \mathbb{B}_n} \beta(\varphi_{kj}(z), \rho(z)) \leq \sup_{z \in V^{-1}\mathbb{B}_n} \omega\left(\left|\left((V^{-1}\varphi_{kj} V)^1,\ldots,(V^{-1}\varphi_{kj} V)^s\right)\right|, 0\right)
\]

\[
= \frac{1}{2} \sup_{z \in V^{-1}\mathbb{B}_n} \tanh^{-1}\left(\left|\left((V^{-1}\varphi_{kj} V)^1,\ldots,(V^{-1}\varphi_{kj} V)^s\right)\right|\right) \to 0,
\]
as \(j \to \infty\). Therefore, (ii) follows from [12, Theorem 3.6].
4.2. (i)⇒ (iii). Before presenting the proof, we state some auxiliary results.

For $k > 0$ and $\zeta \in \partial \mathbb{B}_n$, we define the ellipsoid

$$E(k, \zeta) = \{ z \in \mathbb{B}_n : |1 - \langle z, \zeta \rangle|^2 \leq k(1 - |z|^2) \}.$$ 

Let $\rho$ be a holomorphic self-map of the unit ball and $\eta > 0$. Set

$$L(\rho, \eta) = \{ z \in \mathbb{B}_n, \beta(z, \rho(z)) \geq \eta \}.$$ 

The following lemma is an extension of [12, Lemma 3.9]. Since the proof is the same, we omit it.

**Lemma 4.1.** Let $\varphi$ be a holomorphic self-map of the unit ball, $\varphi(0) = 0$, and $\rho$ be the holomorphic retraction associated with $\varphi$. If $\eta > 0$ be such that $L(\rho, \eta) \neq \emptyset$, then there is some $A > 1$ such that

$$\frac{1 - |\varphi(z)|}{1 - |z|} > A, \quad \forall z \in L(\rho, \eta).$$

**Proposition 4.2.** $\beta(z, w) \geq \frac{1}{2}|z - w|$, for all $z, w \in \mathbb{B}_n$.

**Proof.** The case $z = w$ is clear. Let $z \neq w$. Then $\beta(z, w) = r > 0$. Note from (2.4) that $B(w, r)$ is the ellipsoid

$$\frac{|P_w(z) - w_R|^2}{R^2 s^2} + \frac{|Q_w(z)|^2}{R^2 s} < 1,$$

where

$$R = \tanh r = \frac{e^r - e^{-r}}{e^r + e^{-r}} < 1,$$

$$w_R = \frac{1 - R^2}{1 - R^2 |w|^2} R w$$

and $s = \frac{1 - |w|^2}{1 - R^2 |w|^2} < 1$. Thus,

$$\frac{|P_w(z) - w_R|^2}{R^2 s^2} + \frac{|Q_w(z)|^2}{R^2 s} = 1,$$

Since $s < 1$ and $Q_w(z)$ is orthogonal to $P_w(z)$ and $P_w(z) - w_R$, we obtain

$$|z - w_R|^2 = |P_w(z) - w_R|^2 + |Q_w(z)|^2$$

$$= R^2 s \left( \frac{|P_w(z) - w_R|^2}{R^2 s} + \frac{|Q_w(z)|^2}{R^2 s} \right)$$

$$< R^2 s \left( \frac{|P_w(z) - w_R|^2}{R^2 s^2} + \frac{|Q_w(z)|^2}{R^2 s} \right) = R^2 s.$$

From the mean value theorem, there is a $0 \leq t \leq r$ so that:

$$R = \tanh r = r \text{sech}^2 t \leq r.$$

Note that the last inequality comes from $\text{sech} t = \frac{2}{e^{2r} + e^{-2r}} \leq 1$.

Combining the above estimates, we deduce that:

$$|z - w| \leq |z - w_R| + |w_R - w|$$

$$< R \sqrt{s} + R^2 \left( \frac{1 - |w|^2}{1 - R^2 |w|^2} \right)$$

$$< 2R \leq 2r = 2\beta(z, w).$$

The proof is complete. \qed
Now we proceed to the proof of (i)⇒(iii). From [12, Lemma 3.3], there is a positive integer \( k \) so that \( \varphi_{kj} \rightarrow \rho \) uniformly on the compact subsets of \( \mathbb{B}_n \) and

\[
(4.2) \quad \lim_{j \to \infty} M_j(C_\varphi) = \frac{1}{k} \sum_{i=0}^{k-1} C_{\rho \circ \varphi_i},
\]

for the strong operator topology. Let (iii) not hold.

**Claim 4.3.** There is an \( \varepsilon > 0 \) so that \( \|\varphi_{kj} - \rho\|_\infty \geq \varepsilon \) for all \( j \).

**Proof.** Since (iii) does not hold, there is a sequence \( m_j \in \mathbb{N} \) such that \( \|\varphi_{km_j} - \rho\|_\infty \geq \varepsilon \) for all \( j \). Consider an arbitrary positive integer \( j \). Then, there is a \( j_0 \) so that \( m_{j_0} \geq j \). Thus, from the fact that \( \rho \circ \varphi_{kj} = \rho \) for all \( l \in \mathbb{N} \), we have:

\[
\varepsilon \leq \|\varphi_{km_{j_0}} - \rho\|_\infty = \|\varphi_{kj} \circ \varphi_{k(m_{j_0} - j)} - \rho \circ \varphi_{k(m_{j_0} - j)}\|_\infty = \sup_{z \in \mathbb{B}_n} |(\varphi_{kj} - \rho)(\varphi_{k(m_{j_0} - j)}(z))| \leq \|\varphi_{kj} - \rho\|_\infty.
\]

The proof is complete. \( \square \)

**Claim 4.4.** For every \( 0 < r < 1 \), we can find \( a \in \mathbb{B}_n \) and \( m \in \mathbb{N} \) such that:

\[
|\varphi_{2km}(a) - \rho(a)| \geq \varepsilon, \quad \text{and} \quad |\varphi_{km}(a)| > r.
\]

**Proof.** If the claim do not hold, then there is an \( 0 < r < 1 \) so that

\[
(4.3) \quad \sup\{|\varphi_{2kj}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |\varphi_{kj}(z)| > r\} \leq \varepsilon.
\]

for all \( j \in \mathbb{N} \). On the other hand, there is a \( j_0 \) so that

\[
(4.4) \quad \sup\{|\varphi_{kj_0}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |z| \leq r\} \leq \varepsilon.
\]

We have:

\[
\|\varphi_{2kj_0} - \rho\|_\infty = \max\left\{ \sup\{|\varphi_{2kj}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |\varphi_{kj}(z)| > r\}, \sup\{|\varphi_{2kj}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |\varphi_{kj}(z)| \leq r\} \right\}.
\]

From (4.3), the first supremum is less than or equal to \( \varepsilon \). For the second one, from the fact \( \rho = \rho \circ \varphi_{kj_0} \) and (4.4), we have

\[
\sup\{|\varphi_{2kj_0}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |\varphi_{kj_0}(z)| \leq r\}
\]

\[
= \sup\{|\varphi_{kj_0} \circ \varphi_{kj_0}(z) - \rho \circ \varphi_{kj_0}(z)|; \ z \in \mathbb{B}_n, \ |\varphi_{kj}(z)| \leq r\}
\]

\[
\leq \sup\{|\varphi_{kj_0}(z) - \rho(z)|; \ z \in \mathbb{B}_n, \ |z| \leq r\} \leq \varepsilon
\]

Therefore, \( \|\varphi_{2kj_0} - \rho\|_\infty \leq \varepsilon \), which contradicts Claim (4.3). \( \square \)

**Claim 4.5.** There are two sequences \( \{m_j\} \subseteq \mathbb{N} \) and \( \{a_j\} \subseteq \mathbb{B}_n \) and some \( f \) in \( H^\infty(\mathbb{B}_n) \) such that \( |\varphi_{2km_j}(a_j) - \rho(a_j)| \geq \varepsilon \) for all \( j \), and

\[
f \circ \rho \equiv 0, \quad f(\varphi_{l}(a_j)) = |\varphi_{2km_j}(a_j) - \rho(a_j)|^2, \quad 1 \leq l \leq km_j, \ \forall j \in \mathbb{N}.
\]
Proof. From Lemma 4.1, there is a constant $0 < a < 1$ such that if $\beta(z, \rho(z)) \geq \varepsilon/2$, then
\begin{equation}
\frac{1 - |z|}{1 - |\varphi(z)|} < a.
\end{equation}

Let $a_1$ in $\mathbb{B}_n$ be such that $|\varphi_{2k}(a_1) - \rho(a_1)| \geq \varepsilon$. Then, from Proposition (4.2), the fact that $\rho \circ \varphi_{kl} = \rho = \varphi_{l} \circ \rho$ for all $l \in \mathbb{N}$, and inequality (2.2), we obtain:
\[ \frac{\varepsilon}{2} \leq \beta(\varphi_{2k}(a_1), \rho(a_1)) = \beta(\varphi_{2k}(a_1), \rho \circ \varphi_{2k}(a_1)) \leq \beta(\varphi_{i}(a_1), \rho \circ \varphi_{i}(a_1)). \]
for all $1 \leq i \leq 2k$. Thus, from (4.5), we have
\[ \frac{1 - |\varphi_{i}(a_1)|}{1 - |\varphi_{i+1}(a_1)|} < a, \quad 1 \leq i \leq k - 1. \]

Put $m_1 = 1$. Using Claim 4.4, we can find $a_2 \in \mathbb{B}_n$ and $m_2 \in \mathbb{N}$ such that $|\varphi_{km_2}(a_2)|$ is large enough so that
\[ |\varphi_{2km_2}(a_2) - \rho(a_2)| \geq \varepsilon, \]
and
\[ \frac{1 - |\varphi_{km_2}(a_2)|}{1 - |\varphi_{i}(a_2)|} < a. \]

Again,
\[ \frac{\varepsilon}{2} \leq \beta(\varphi_{2km_2}(a_2), \rho(a_2)) \leq \beta(\varphi_{i}(a_2), \rho \circ \varphi_{i}(a_2)) \]
for all $0 \leq i \leq 2km_2$. Thus, from (4.5), we obtain:
\[ \frac{1 - |\varphi_{i}(a_2)|}{1 - |\varphi_{i+1}(a_2)|} < a, \quad 1 \leq i \leq km_2 - 1. \]

By repeating this process we will construct the sequence
\[
\begin{align*}
x_1 &= \varphi_k(a_1), & x_2 &= \varphi_{k-1}(a_1), & \ldots, & x_{km_1} &= \varphi(a_1) \\
x_{km_1+1} &= \varphi_{km_2}(a_2), & x_{km_1+2} &= \varphi_{km_2-1}(a_2), & \ldots, & x_{km_1+m_1} &= \varphi(a_2) \\
x_{km_1+m_1+1} &= \varphi_{km_3}(a_3), & x_{km_2+m_1+2} &= \varphi_{km_3-1}(a_3), & \ldots, & x_{km_3+m_2+m_1} &= \varphi(a_3) \\
\vdots & & \vdots & & \vdots & \ddots
\end{align*}
\]
which satisfies condition (i) of [12, Lemma 3.11]. Thus, there are some $M > 0$ and a sequence $\{f_{i,j}\}_{j,l=1}^{k,m} \subset H^\infty(\mathbb{B}_n)$ such that

(a) $f_{i,j}(\varphi_l(a_j)) = 1$, and $f_{i,j}(\varphi_r(a_s)) = 0$ whenever $l \neq r$ or $j \neq s$.

(b) $\sum_{j=1}^{\infty} \sum_{l=1}^{km_l} |f_{i,j}(z)| \leq M$, for all $z \in \mathbb{B}_n$.

Define
\[ f(z) = \sum_{j=1}^{\infty} \sum_{l=1}^{km_l} \langle \varphi_{2km_l-l}(z) - \rho \circ \varphi_{2km_l-l}(z), \varphi_{2km_l}(a_j) - \rho(a_j) \rangle f_{i,j}(z). \]

Hence, from the Lebesgue dominated convergence theorem, (a), (b), and the fact that $\rho \circ \varphi = \varphi \circ \rho$, we deduce that $f \in H^\infty(\mathbb{B}_n)$, $f(\rho) = 0$, and
\[ f(\varphi_l(a_j)) = |\varphi_{2km_l}(a_j) - \rho(a_j)|^2, \quad 1 \leq j < \infty, \quad 1 \leq l \leq km_j. \]

The proof is complete. \qed
Using Claim 4.5, we have:

\[
\left\| \frac{1}{m_j} \sum_{l=1}^{m_j} C_{\varphi_l} - \frac{1}{k} \sum_{i=0}^{k-1} C_{\rho \varphi_i} \right\| \geq \frac{1}{\|f\|_{\infty}} \left\| \frac{1}{m_j} \sum_{l=1}^{m_j} C_{\varphi_l} f - \frac{1}{k} \sum_{i=0}^{k-1} C_{\rho \varphi_i} f \right\|_{\infty} \geq \frac{1}{\|f\|_{\infty}} \frac{1}{m_j} \sum_{l=1}^{m_j} \left| f(\varphi_l(a_j)) - \frac{1}{k} \sum_{i=0}^{k-1} f(\rho \varphi_i(a_j)) \right| = \frac{1}{\|f\|_{\infty}} \frac{1}{m_j} \sum_{l=1}^{m_j} \left| \varphi_{2km_j}(a_j) - \rho(a_j) \right|^2 \geq \frac{\varepsilon^2}{\|f\|_{\infty}}.
\]

From the above estimate, we deduce that \( \{ M_j(C_\varphi) \}_{j=1}^\infty \) does not converge to \( \frac{1}{k} \sum_{i=0}^{k-1} C_{\rho \varphi_i} \) for the strong operator topology, which contradicts 4.2. Thus, (iii) holds.

5. Proof of Theorem 1.4

First, we define the sequence of operators \( T_j : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D}) \) as follows

\[
T_j f(z) := f \circ \varphi_j^1(z, 0, ..., 0),
\]

where \( \varphi_j^1 \) is the first component of \( \varphi_j \). Note that if we consider \( f \in H^\infty(\mathbb{D}) \) as a function in \( H^\infty(\mathbb{B}_n) \), then \( T_j f = C_{\varphi_j} f \). Thus, if \( C_{\varphi} \) is mean ergodic on \( H^\infty(\mathbb{B}_n) \), then

\[
N_j(\varphi) := \frac{1}{j} \sum_{i=1}^{j} T_i : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D}),
\]

converges for the strong operator topology.

We give the proof in two steps. In the first step, we show that if \( N_j(\varphi) \) is SOT-convergent, then it must converge in the norm operator. Then, in the second step, we prove that \( N_j(\varphi) \) does not converge in the norm operator. Therefore, the proof will be complete.

**Step 1.** From the ergodic theorem, \( M_j(\varphi) \) converges to a projection \( P \) so that \( PC_{\varphi} = C_{\varphi} P = P \). Since \( N_j(\varphi) \) converges for the strong operator topology to \( P \mid_{H^\infty(\mathbb{D})} \) and \( H^\infty(\mathbb{D}) \) is a GDP space, from [13, Theorem 2] the spectral radius of \( N_j(\varphi) - P \) converges to 0 as \( j \to \infty \). That is, \( I - N_j(\varphi) + P \) is invertible for a large enough \( j \).

Now, we show that \( I - T_1 + P \) is bounded below. If not, then there is a sequence of unit vectors \( \{ f_l \} \) in \( H^\infty(\mathbb{B}_n) \) so that:

\[
\|(I - T_1 + P) f_l\|_{\infty} \to 0 \quad \text{as} \quad j \to \infty.
\]

Since \( P = PT_1 = P^2 \), we obtain

\[
\|P f_l\|_{\infty} = \|P(I - T_1 + P) f_l\|_{\infty} \to 0 \quad \text{as} \quad j \to \infty.
\]

Thus,

\[
\|(I - T_1) f_l\|_{\infty} \to 0 \quad \text{as} \quad j \to \infty.
\]
Therefore,
\[
(I - N_j(\varphi) + P)f_l = (I - M_j(\varphi) + P)f_l
\]
\[
= \frac{1}{n} \sum_{i=1}^{j} (I - C\varphi_i)f_l + Pf_l
\]
\[
= \frac{1}{n} \sum_{i=1}^{j} (I + C\varphi + \ldots + C\varphi_{i-1})(I - C\varphi)f_l + Pf_l
\]
\[
= \frac{1}{n} \sum_{i=1}^{j} (I + T_1 + \ldots + T_{i-1})(I - T_1)f_l + Pf_l \to 0,
\]
as \(l \to \infty\). This contradicts the invertibility of \(I - N_j(\varphi) + P\).

Now, since \(I - T_1 + P\) is bounded below, there is a bounded operator \(S\) on \(H^\infty(\mathbb{D})\) so that \(S(I - T_1 + P) = I\). Therefore,
\[
(N_j(\varphi) - P) = S(I - T_1 + P)(N_j(\varphi) - P)
\]
\[
= S(I - C\varphi + P)(M_j(\varphi) - P)
\]
\[
= \frac{1}{j} S(C\varphi - C\varphi_{j+1}) \to 0,
\]
as \(j \to \infty\).

**Step 2.** The proof of this step is similar to that of [4, Theorem 3.6] and also [12, Theorem 3.14].

From [1, Theorem 2.2.31], there is a \(z_0 \in \partial \mathbb{B}_n\) such that \(\varphi_j \to z_0\) uniformly on the compact subsets of \(\mathbb{B}_n\). By a unitary equivalent, we can let \(z_0 = e_1\). Thus, if \(\varphi_j = (\varphi_{1j}, \ldots, \varphi_{nj})\), then \(\varphi_{1j} \to 1\) and \(\varphi_{ij} \to 0\) for \(2 \leq i \leq n\) uniformly on the compact subsets of \(\mathbb{B}_n\) as \(j \to \infty\).

Thus, if \(N_j(\varphi)\) converges in operator norm, then \(N_j(\varphi) \to K_1\) on \(A(\mathbb{D}) = H(\mathbb{D}) \cap \{f : \overline{\mathbb{D}} \to \mathbb{C}, \text{continuous}\}\), where \(K_1(f) = f(1)\) on \(A(\mathbb{D})\). The remaining of the proof is similar to that of [4, Theorem 3.6], by considering \(g(z) = \frac{1+\overline{z}}{2} \in A(\mathbb{B}_n)\).

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Hamzeh Keshavarzi
E-mail: Hamzehkeshavarzi67@gmail.com
Department of Mathematics, College of Sciences, Shiraz University, Shiraz, Iran.

Karim Hedayatian
E-mail: hedayati@shirazu.ac.ir
Department of Mathematics, College of Sciences, Shiraz University, Shiraz, Iran.