A CURVATURE ESTIMATE FOR STABLE MARGINALY OUTER TRAPPED HYPERSURFACE WITH A FREE BOUNDARY

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ABSTRACT. A marginally outer trapped hypersurface is a generalization of minimal hypersurfaces originated from general relativity. We show a curvature estimate for stable marginally outer trapped hypersurfaces up to the free boundary satisfying a uniform area bound. Our proof is based on an iteration argument. The curvature estimate was previously known via a blowup argument for stable minimal hypersurfaces.

1. Introduction

Let $M^n$ be a spacelike submanifold in $S^{n+1,1}$ and $l^\pm$ be the two independent future directed null sections of the normal bundle of $M$ with corresponding null second fundamental form or shear tensor $\chi^\pm$. The traces of $\chi^\pm$ are called the null expansions which we denote by $\theta^\pm$.

**Definition 1.1.** The submanifold $M$ is called marginally outer (inner) trapped if
\begin{equation}
\theta^\pm = 0.
\end{equation}
We call $M$ a MOTS (MITS) in short.

We will only consider the case when $M$ sits in a spacelike hypersurface $N^{n+1} \subset S^{n+1,1}$. Let $\tau$ be the future timelike normal of $N$, $\hat{g}$ be the induced metric on $N$ and $p$ the second fundamental form of $N$ with respect to $\tau$ in $S^{n+1,1}$. The triple $(N^{n+1}, \hat{g}, p)$ is usually referred as an initial data set. A rather natural choice of $l^\pm$ in this situation is $\nu \pm \tau$ where $\nu$ is the outward pointing normal of $M$ in $N$. Let $e_i$ be an orthonormal frame of the tangent space of $M$, then
\begin{equation}
\chi^\pm = \sum_{i=1}^n \langle \nabla_{e_i}(\nu \pm \tau), e_i \rangle,
\end{equation}
an we have the null expansion is given by
\begin{equation}
\theta^\pm = H \pm \text{tr}_M p,
\end{equation}
where $H$ is the mean curvature of $M$ in $N$ and $\text{tr}_M p$ is the trace of the projection of $p$ to $M$.

The MOTS equation is then a prescribed mean curvature equation. When $N$ is a time-symmetric Cauchy hypersurface in $S^{n+1,1}$ i.e. $p \equiv 0$, then the MOTS is simply minimal in $N$. The concept of stability is central in the theory of MOTS which extends the stability of stable minimal hypersurfaces.

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Definition 1.2 ([AMS05]). We say that a MOTS $M$ is stable if there is an outward infinitesimal deformation which does not decrease $\theta^+$. To write the notion analytically, there exists a non-zero function $f \geq 0$ such that
\begin{equation}
\delta f \theta^+ \geq 0.
\end{equation}

For the construction of marginally outer trapped surface, see [Eic09 AM09 An20 AH21 RS22]. The notion of a free boundary (more generally, capillary) MOTS as well as the stability was introduced by Alaee-Lesourd-Yau [ALY20]. Recall the definition of a free boundary stable marginally outer trapped hypersurface and its stability.

Definition 1.3 ([ALY20]). A hypersurface $M$ with $\partial M \neq \emptyset$ is said to be a free boundary marginally outer trapped hypersurface if $\theta^+ = 0$ and $\partial M$ meets $\partial N$ orthogonally. It is said to be stable if there exists a non-zero $f \geq 0$ such that
\begin{equation}
\delta f \theta^+ \geq 0 \text{ in } M, \delta f \nu \langle \eta, \nu \rangle = 0 \text{ on } \partial M,
\end{equation}
where $\eta$ be the outward normal of $\partial N$ in $N$.

The equation (1.4) can be written down in terms of inequalities for $f$ as done in Lemma 3.1. The angle forming by the two vector fields $\eta$ and $\nu$ is call the contact angle of $\partial M$ and $\partial N$. Geometrically, the infinitesimal deformation in (1.4) fixes the contact angle of $\partial M$ and $\partial N$.

To simplify, we assume that $N$ lies in a larger manifold $\tilde{N}$ of the same dimension with boundary such that any point $x \in \partial N$ there exists a geodesic ball $B(x, r) \subset \tilde{N}$. To define the geodesic balls centered at the boundary point is easier than the balls defined via the Fermi coordinates (see [LZ21 Appendix A]).

Now we state our main result which is the following curvature estimate for stable marginally outer trapped hypersurface assuming a volume bound.

Theorem 1.4. Let $2 \leq n \leq 5$, if $M^n$ is a stable free boundary marginally outer trapped hypersurface in $(N, \tilde{g}, p)$ satisfying a uniform volume bound
\begin{equation}
\text{vol}(B(x, r) \cap M) \leq C_MR^n,
\end{equation}
for all $x \in \tilde{M}$ and all $0 < r \leq r_0$, then there exists a bound on the curvature
\begin{equation}
|A|(x) \leq \frac{C}{r}
\end{equation}
with $C$ depending only on $C_M$, $|p|_{C^1}$, $|Rm|_{C^0}$, $\text{inj}(\tilde{N}, g)^{-1}$ and $|d|_{C^3}$. Here $Rm$ is the curvature operator of $\tilde{N}$, $d$ is the distance function to $\partial N$ in $\tilde{N}$ and $\text{inj}(\tilde{N}, g)$ is the injective radius of $\tilde{N}$.

Remark 1.5. The curvature estimate holds as well if we replace the extrinsic balls with intrinsic balls in (1.5). The dependence on $|d|_{C^3}$ is actually dependence on $|b|_{C^1}$ where $b$ is the second fundamental form of the boundary $\partial N$ in $N$. We write in this way because $b$ is simply the Hessian of $d$.

Guang-Li-Zhou [GLZ20] showed a curvature estimate for stable minimal hypersurfaces using blow-up argument. Our approach involves deriving a Simons inequality for a perturbation of the second fundamental form (see Definition 2.2), combining with the stability (1.4) for the stable free boundary MOTS. It is in spirit closer to the works Schoen-Simon-Yau [SSY75] and Andersson-Metzger [AM10]. Compared to the work [GLZ20], the constant in our curvature estimate (1.6) explicitly depends on the geometric quantities of $N$, $\partial N$ and the volume bound. The
technical difference between our work and [AM10] is that we do not consider shear tensor $\chi^\pm$. The reason lies in the presence of the free boundary.

Schoen-Simon [SS81] generalized the curvature estimate [SSY75] for embedded stable minimal hypersurfaces to any dimension. However, Schoen-Simon theory for high dimensional stable free boundary minimal hypersurfaces is not seen in literature yet.

We would also like to mention another important problem: the curvature estimate for immersed stable minimal or marginally outer trapped surfaces with a free boundary (see [GLZ20, Conjecture 1.4]) without area bound. This is the free boundary analog of Schoen’s curvature estimate [Sch83, CM02]. It is also quite interesting to seek a curvature estimate for immersed stable capillary surfaces in an arbitrary manifold with boundary.

The paper is organized as follows:

In Section 2, we collect basics on Simons identity, the perturbation of the second fundamental form and its boundary derivatives. In Section 3 we derive some integral estimates only by the stability condition (1.4). In Section 4 we calculate a Simons inequality for the perturbation of the second fundamental form. In Section 5 via a de Giorgi iteration, we conclude the proof for the pointwise curvature estimate. In the Appendix A we record the Sobolev inequality on free boundary hypersurfaces.

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2. Preliminaries

First, we fix some notations used in this article. Let $K$ be the Riemann curvature operator of $N$, $R$ be that of $M$, $\hat{\nabla}$ be the Levi-Civita connection of $(N, \hat{g})$, $D$ be the induced connection on $\partial N$, and $\nabla$ the induced connection on $M$.

We collect a few facts which would be used frequently later in the work.

2.1. Simons identity. We recall the Simons’ identity.

**Theorem 2.1.** For any hypersurface $M$ in $N$ the second fundamental form $h_{ij}$ satisfies the identity

$$
\Delta h_{ij} = \nabla_i \nabla_j H - \nabla_j K_{n+1,ki} - \nabla_k K_{n+1,ij}^k - K_{kij}^l h_{il} - K_{kijl}^k h_{il} - |A|^2 h_{ij} + h_{il} h_{jl} H.
$$

**(2.1)**

**Proof.** The identity is due to [Sim68], see also [SSY75, (1.19)-(1.20)]. In these papers the identity is not in the form that we will need, so we give a quick derivation for the convenience of the reader. We pick an orthonormal frame $\{e_i\}_{1 \leq i \leq n}$ on $TM$. We use the Einstein summation convention where the summation is done on repeated indices.

First, by Codazzi equation,

$$
\Delta h_{ij} = \nabla_k \nabla_i h_{ij} = \nabla_k \nabla_j h_{ik} = \nabla_k K_{jki,n+1}.
$$


By commuting covariant derivatives and the Gauss equation,
\[
\nabla_k \nabla_j h_{ik} = \nabla_j \nabla_k h_{ik} - R_{kjil} h_{kl} - R_{kjkl} h_{il} = \nabla_j \nabla_k h_{ik} - (K_{kjil} + h_{ji} h_{kl} - h_{jl} h_{ik}) h_{kl} - (K_{kjkl} + h_{jk} h_{kl} - h_{jl} h_{kk}) h_{il}
\]
\[
= \nabla_j \nabla_k h_{ik} - K_{kjil} h_{kl} - K_{kjkl} h_{il} - |A|^2 h_{ij} + h_{il} h_{jl} H.
\]

Applying the Codazzi equation on \(\nabla_j \nabla_k h_{ik}\) again,
\[
\nabla_j \nabla_k h_{ik} = \nabla_j \nabla_i h_{kk} - \nabla_j K_{ikk,n} + 1 = \nabla_j \nabla_i H + \nabla_j K_{ikk,n} + 1.
\]

Collecting all the above we obtained the desired identity for \(\Delta h_{ij}\).

2.2. Perturbed second fundamental form. Next, we collect some facts about the perturbation we are going to use throughout the paper.

**Definition 2.2** (\[Ede16\]). Extend and fix \(k\) and \(\eta\) to be defined on all of \(M\). Define the perturbed second fundamental form \(\bar{h}_{ij}\) of \(M\) to be
\[
\bar{h}_{ij} = h_{ij} + \bar{T}_{ij} := h_{ij} + T_{ij\nu} + \Lambda_0 g_{ij}
\]
where \(T\) is a 3-tensor define on \(N\) by
\[
T(X, Y, Z) = b(X, Z) g(Y, \eta) + b(Y, Z) g(X, \eta),
\]
and \(D_0\) is a constant depending only on \(|b|_{C^0}\) such that
\[
T(X, X, \nu) + \Lambda_0 \geq 1 + |p|_{C^0}
\]
for any unit vector \(X\). The perturbed null second fundamental form or shear tensor is given by
\[
\bar{\chi}_{ij} = \bar{h}_{ij} + p_{ij}.
\]

The most significant feature of the perturbed second fundamental form is that \(\bar{h}(\eta, \cdot)\) vanishes when restricted to \(\partial M\) and that the perturbed second fundamental form \(\bar{h}\) is comparable with the original second fundamental form and with good control of the boundary derivatives. These are presented in the following two lemmas and Proposition 2.6.

**Lemma 2.3.** Let \(M\) be a hypersurface in \(N\) not necessarily a MOTS, \(T\) be a 3-tensor on \(N\), and \(T_{ij\nu}\) be the 2-tensor \(T(\cdot, \cdot, \nu)\) restricted to \(TM\), then
\[
|\nabla T_{ij\nu}| \leq c_1 (1 + |A|).
\]
The constant \(c_1 > 0\) depends only on \(|T|_{C^1}\). And for any 2-tensor \(p\) on \(N\),
\[
|\nabla p| \leq c_2 (1 + |A|),
\]
The constant \(c_2 > 0\) depends on \(|p|_{C^1}\).

**Proof.** The computations are tensorial, we may assume the simplifications that there is an orthonormal frame such that \(\nabla_i e_j = 0\) at some point \(x \in M\), and hence
Proposition 2.6. Let $\nabla_i e_j = -h_{ij}\nu$ and $\nabla_i \nu = h_{ij}e_j$ at $x$. With these simplifications, we have
\[
(\nabla_k T)(e_i, e_j, \nu) = \nabla_k (T(e_i, e_j, \nu)) = (\nabla_k T)(e_i, e_j, \nu) - \nabla_k (T(e_i, e_j, \nu) + T(e_i, \nabla_k e_j, \nu) + T(e_i, e_j, \nabla_k \nu)
\]
\[
= (\nabla_k T)(e_i, e_j, \nu) - h_{ik} T_{\nu j\nu} - h_{jk} T_{i\nu\nu} + h_{ki} T_{ij\nu}.
\]
So (2.3) follows with $c_1$ depending on $|T|_{C^1}$. Similar calculation gives
\[
(\nabla_k p)(e_i, e_j) = (\nabla_k p)(e_i, e_j) - h_{ik} p_{j\nu} - h_{jk} p_{i\nu}.
\]
The bound on $|\nabla p|$ then easily follows. 

Remark 2.4. See also [Ede16, Proposition 5.1] for the computation of $\nabla^2 T_{ij\nu}$ in $\mathbb{R}^n$.

Lemma 2.5. For a marginally outer trapped hypersurface $M$,
\[
|\bar{A}| \geq 1, |A| \leq c_1 + |\bar{A}|, |\nabla A| \leq c_2(|\nabla \bar{A}| + |\bar{A}|).
\]
Here the constant $c_1 > 0$ depends only on $|b|_{C^0}$ and $|\eta|_{C^0}$, and $c_2$ depends on $|b|_{C^3}$ and $|\eta|_{C^3}$.

Proof. First by (2.2),
\[
\bar{H} \geq H + n + n |p|_{C^0} \geq n.
\]
So $|\bar{A}| \geq \frac{|B|}{n} \geq 1$,
\[
|A| \leq |\bar{A}| + |\bar{A}| \leq |\bar{A}| + c_1
\]
and
\[
|\nabla A| \leq |\nabla \bar{A}| + |\nabla \bar{A}| \leq c(1 + |A|) \leq c_2(|\nabla \bar{A}| + |\bar{A}|).
\]
The dependence of constants $c_1$ and $c_2$ are easy to track. 

2.3. Boundary derivatives. Let $\partial_i$ and $\partial_j$ are coordinate vector fields on $M$ such that $\eta = \partial_1$ along $\partial M$. We calculate a few important boundary derivatives of the second fundamental form namely $\nabla_i h_{11}$ and $\nabla_i h_{ij}$ where $i, j > 2$.

Proposition 2.6. Let $X$ and $Y$ be tangent vector fields to $\partial M$, then
\[
h(\eta, X) = -b(\nu, X).
\]
The normal derivatives of $h$ satisfies
\[
(\nabla_\eta h)(\eta, \eta) = \nabla_\eta H - \text{tr}_{\partial M}((\nabla_\eta h)(\cdot, \cdot)).
\]
Let $D$ be the induced connection on $\partial N$, then
\[
(\nabla_\eta h)(X, Y) = -K(Y, \eta, X, \nu) - K(Y, \nu, X, \eta)
\]
\[
- h(\nabla_Y \eta, X) - b(D_X \nu, Y) - (\nabla_\nu b)(X, Y)
\]
\[
+ b(X, Y)h_{\eta\nu} + A(X, Y)b_{\nu\nu}.
\]
Proof. By definitions of $h$, $k$ and the free boundary condition on $M$,

$$h(\eta, X) = \langle \bar{\nabla}_X \nu, \eta \rangle = -\langle \bar{\nabla}_X \eta, \nu \rangle = -B(\nu, X).$$

This proves (2.6). Note that $H$ vanishes identically, so

$$\nabla_\eta H = (g^{ij} - \eta^i \eta^j + \eta^i \eta^j) \nabla_\eta h_{ij},$$

and we get (2.7). We calculate now the two terms $\nabla_Y (h(\eta, X))$ and $\nabla_Y (k(\nu, X))$.

First,

$$\nabla_Y (h(\eta, X)) = (\nabla_Y h)(\eta, X) + h(\nabla_Y \eta, X) + h(\nabla_Y \eta, X) = (\nabla_Y h)(X, Y) + K(Y, \eta, X, \nu) + h(\nabla_Y \eta, X) + h(\eta, \nabla_Y X).$$

where in the last line we used the Codazzi equation. Similarly,

$$\nabla_Y (b(\nu, X)) = D_Y (b(\nu, X)) = (D_Y b)(\nu, X) + b(\nu, D_Y X) + b(D_Y \nu, X) = (D_Y b)(X, Y) + K(Y, \nu, X, \eta) + b(\nu, D_Y X) + b(D_Y \nu, X).$$

We see that

$$\nabla_Y X = (\bar{\nabla}_Y X)^{\partial M} + (\nabla_Y X, \eta) \eta$$

and similarly $D_Y X = (\bar{\nabla}_Y X)^{\partial M} - A(X, Y) \nu$. By the relation (2.6), we have

(2.9) \quad h(\eta, \nabla_Y X) + b(\nu, D_Y X) = -b(X, Y) h_{\eta\eta} - A(X, Y) b_{\nu\nu}.

We know from $\nabla_Y (h(\eta, X))$ and $\nabla_X (b(\nu, X))$ that

$$(\nabla_\eta h)(X, Y) = -K(Y, \eta, X, \nu) - h(\nabla_Y \eta, X) - h(\eta, \nabla_Y X) - (D_Y b)(X, Y) - K(Y, \nu, X, \eta) - b(\nu, \nabla_X Y) - b(D_X \nu, Y).$$

Using (2.9), we can drop the terms containing $\nabla_X Y$, and we obtain (2.8) for $$(\nabla_\eta h)(X, Y).$$

\[\square\]

Remark 2.7. See [Ede16, Lemma 6.1] for the case in $\mathbb{R}^n$, [LS16] for a special case where $\partial N$ is the sphere, and [HL20] for a 5-parameter perturbation.

Remark 2.8. Let $\{e_i\}$ (with $i$ in the appropriate range) be an orthonormal frame of $\partial M$, we have

$$h(\nabla_Y \eta, X) = \sum_i h(e_i, X) (\nabla_Y \eta, e_i) = \sum_i h(e_i, X) b(Y, e_i),$$

and similarly, $b(D_Y \nu, X) = \sum_i b(e_i, X) h(e_i, Y)$.

It is easy then to see the following.

Corollary 1. If $M$ is a free boundary MOTS, then

$$-c|\tilde{A}| \leq \partial_1 |\tilde{A}| \leq c|\tilde{A}|,$$

where $c > 0$ depends only on $|b|_{C^1}$.
Lemma 3.3. \(\phi \in M\) restricted to \(\partial M\).

Remark 3.2 \((3.1)\)

We know that \(|\bar{A}| > 0\). So

\[
\partial_1 |\bar{A}|^2 = \sum_{i,j \geq 1} 2\bar{h}_{ij} \nabla_1 \bar{h}_{ij}
\]

\[
= 2\bar{h}_{11} \nabla_1 \bar{h}_{11} + 2 \sum_{i,j \geq 2} \bar{h}_{ij} \nabla_1 \bar{h}_{ij} + 2 \sum_{i \geq 2} \bar{h}_{ij} \nabla_1 \bar{h}_{1i}
\]

where in the last line we have used \(\bar{h}_{11} \equiv 0\) along \(\partial M\). By the MOTS equation \((1.2)\) and \((2.4)\)

\[
|\nabla_n H| \leq |\nabla H| = |\nabla \text{tr}_M p| \leq c|A| \leq c|\bar{A}|.
\]

So from \((2.7)\) and \((2.8)\), we get

\[
|\nabla_1 \bar{h}_{11}| + |\nabla_1 \bar{h}_{ij}| \leq c(1 + |\bar{A}|).
\]

So due to \((2.3)\), we see \(|\partial_1 |A|| \leq c|\bar{A}|\). \(\square\)

The paper \([AM10]\) estimate the size of the shear \(|\chi|\), at a first glance, it is natural trying to estimate the perturbed shear tensor \(|\bar{\chi}|\) defined in \((2.2)\). However, when computing \(\partial_1 |\bar{\chi}|\), the term \(\sum_{i \geq 2} \bar{\chi}_{1i} \nabla_1 \bar{\chi}_{1i}\) is not favorable due to non-vanishing \(\bar{\chi}_{11}\). The boundary derivative \(\nabla_1 \bar{\chi}_{1i}\) then essentially requires an estimate on \(\nabla_1 \bar{h}_{1i}\) in terms of \(|A|\) and its estimate seems difficult to do. This is the reason that we consider directly the perturbed second fundamental form \(A\).

3. Stability inequality

The work of \([GS06]\) observed that the stability of a closed MOTS and the dominant energy condition implies some topological properties of the MOTS. The stability of \([GS06]\) used the Schoen-Yau’s rewrite of the stability and contains the scalar curvature of the MOTS. Hence, the stability \((1.3)\) is not well suited for curvature estimates. We will consider the less familiar form. To this end we recall the first variation of the null expansion \(\theta^+\) (see \([AEM10]\)) and the variation of the contact angle \([Sta96, ALY20]\).

Lemma 3.1. Let \(M\) be a free boundary MOTS, then the first variation of the null expansion \(\theta^+\) is given by

\[
\delta_{f^c} \theta^+ \big|_M = \mathcal{L} f := -\Delta f + 2S(\nabla f) - |A|^2 f + f \text{div} S - \langle h, p \rangle_M + f\mathcal{X}
\]

where \(\mathcal{X} = -\text{Ric}(\nu) + \nabla_\nu (\text{tr}_N p) - \text{div}_N p(\nu) + H p_{\nu\nu}\) and \(S\) is the 1-form \(k(\nu, \cdot)\) restricted to \(M\). Along the boundary \(\partial M\), the variation of \(\langle \eta, \nu \rangle\) is given by

\[
\delta_{f^c} \langle \eta, \nu \rangle = B f := -\nabla_\nu f + f b(\nu, \nu).
\]

Remark 3.2. Note that \(|\mathcal{X}| \leq C\) with \(C\) depending on \(|\text{Ric}|_{C^0}\) and \(|p|_{C^1}\).

We have an integral estimate for \(|\bar{A}|\) following from the stability.

Lemma 3.3. If \(M\) is a stable MOTS with a free boundary, the for all \(\varepsilon > 0\) and \(\phi \in C^0_{\infty}(M)\), the following inequality

\[
\int_M \phi^2 |\bar{A}|^2 \leq (1 + \varepsilon) \int_M |\nabla \phi|^2 + \left[ c_1 \int_M \phi^2 + c_2 \int_{\partial M} \phi^2 \right]
\]
holds where the constant \( c_1 \) depends on \( 1/\epsilon \), \(|\text{Ric}|_{C^0}, |p|_{C^1}\) and \( c_2 \) depends on \( 1/\epsilon \), \(|p|_{C^1}\) and \(|b|_{C^0}\).

**Proof.** We reorder the terms in Lemma 3.1 as in [GS06], we see

\[
0 \leq f^{-1}Lf = \text{div} \left( S - \frac{S}{f} \right) - \left| S - \frac{S}{f} \right|^2 + |S|^2 - |A|^2 - \langle h, p \rangle_M + f \mathcal{X}.
\]

Multiplying the above with \( \phi^2 \) and integration by parts,

\[
\int_M \phi^2 |S - f^{-1}f|^2 + |A|^2)
\leq \int_{\partial M} \phi^2 \left( S - \frac{S}{f}, \eta \right) + \int_M |S|^2 - \langle h, p \rangle_M + \Lambda |\phi|^2 - 2 \left( S - \frac{S}{f}, \nabla \phi \right).
\]

Using Cauchy-Schwarz inequality,

\[
2 \left| \left( S - \frac{S}{f}, \nabla \phi \right) \right| \leq |S - \frac{S}{f}|^2 \phi^2 + |\nabla \phi|^2,
\]

and the boundary stability condition in (1.4),

\[
\partial M f = b(\nu, \nu) f,
\]

we obtain

\[
\int_M \phi^2 |A|^2 \leq \int_M |\nabla \phi|^2 + [c_1 - \langle h, p \rangle] \phi^2 + c_2 \int_{\partial M} \phi^2,
\]

where \( c_1 \) depends on \(|\text{Ric}|_{C^0}, |p|_{C^1}\) and \( c_2 \) depends on \(|p|_{C^1}\) and \(|b|_{C^0}\). Using (3.2)

\[
|\langle h, p \rangle_M| \leq \epsilon |A|^2 + \frac{1}{\epsilon} |p|^2,
\]

we get

\[
\int_M \phi^2 |A|^2 \leq (1 + \epsilon) \int_M |\nabla \phi|^2 + \left[ c_1 \int_M \phi^2 + c_2 \int_{\partial M} \phi^2 \right].
\]

Since \( h_{ij} = h_{ij} + T_{ij} \), we do the same as in [3,2] and we obtain the desired inequality for \( \int \phi^2 |A|^2 \) (with renaming of \( \epsilon \)).

Now we use a common trick to get rid of the boundary terms in (3.1).

**Corollary 2.** If \( M \) is a stable MOTS with a free boundary, the for all \( \epsilon > 0 \) and \( \phi \in C^\infty_0(M) \), the following inequality

\[
\int_M \phi^2 |\bar{A}|^2 \leq (1 + \epsilon) \int_M |\nabla \phi|^2 + c_1 \int_M \phi^2
\]

holds where the constant \( c_1 \) depends on \( 1/\epsilon \), \(|\text{Ric}|_{C^0}, |p|_{C^1}\) and \( c_2 \) depends on \( 1/\epsilon \), \(|p|_{C^1}\) and \(|b|_{C^0}\).

**Proof.** Since \( \phi \) is compactly supported, \( \langle Dd, \eta \rangle = 1 \) by the free boundary condition of \( M \), so by divergence theorem,

\[
\int_{\partial M} \phi^2 = \int_{\partial M} \phi^2 \langle Dd, \eta \rangle = \int_{\partial M} \text{div}_M (\phi^2 \nabla d) = \int_M 2\phi \langle \nabla \phi, \nabla d \rangle + \phi^2 \langle \nabla e_i, e_i \rangle.
\]
Note that $|H| \leq c$ from (1.2), so by decomposition of Hessian of $M$, 
\[
\sum_{i=1}^{n} \langle \nabla e_i, \nabla d, e_i \rangle = \sum_{i=1}^{n} (D^2 d)(e_i, e_i) - H \langle Dd, \nu \rangle
\]
is bounded by a constant depending on $|d|_{C^2}$ and $|p|_{C^0}$. So
\[
\int_{\partial M} \phi^2 \leq c \int_{M} \phi^2 + c \int_{M} \phi |\nabla \phi|.
\]
Using Cauchy-Schwarz inequality on the second term on the right and combining with (3.1), we have obtained (3.3). □

Lemma 3.4. Let $M$ be a stable free boundary MOTS, we have for any $\varepsilon > 0$ and $q \geq 2$, we have
\[
\int_{M} \phi^2 |\tilde{A}|^{q+2} \leq (1 + \varepsilon) \int_{M} \phi^2 |\tilde{A}|^{q+2} |\nabla |\tilde{A}||^2 + c_1 \int_{M} [\phi^2 + |\nabla \phi|^2] |\tilde{A}|^{q+2} + c_2 \int_{\partial M} \phi^2 |\tilde{A}|^{q+2}.
\]
The dependence of the constants $c_i$ are the same with Lemma 3.3.

Proof. Letting $\phi$ to be $\phi |\tilde{A}|^{q/2}$ in (3.1), we have
\[
\int_{M} \phi^2 |\tilde{A}|^{q+2} \leq (1 + \varepsilon) \int_{M} [\nabla (\phi |\tilde{A}|^{q/2})]^2 + c_1 \int_{M} \phi^2 |\tilde{A}|^{q+2} + c_2 \int_{\partial M} \phi^2 |\tilde{A}|^{q+2}.
\]
We estimate $|\nabla (\phi |\tilde{A}|^{q/2})|^2$ as follows:
\[
|\nabla (\phi |\tilde{A}|^{q/2})|^2 = |\tilde{A}|^{q/2} \nabla \phi + \phi \frac{q}{2} |\tilde{A}|^{q/2-1} \nabla |\tilde{A}||^2 \leq \frac{q^2}{4} (1 + \varepsilon) \phi^2 |\tilde{A}|^{q/2-2} |\nabla |\tilde{A}||^2 + c (\varepsilon^{-1}) |\nabla \phi|^2 |\tilde{A}|^q.
\]
Combining the above two inequalities and adjusting the value $\varepsilon$, we obtained the desired inequality. □

We show that integrals $\int_{M} \phi^2 |\tilde{A}|^q$ on the boundary can be transfered to the an integral in the interior via an application of the divergence theorem. It works for any hypersurface $M$.

Lemma 3.5. Let $M$ be any hypersurface, for any $q > 0$,
\[
\int_{\partial M} |\tilde{A}|^p \phi^2 \leq c \int_{M} \phi |\tilde{A}|^q (|\phi| |\tilde{A}| + |\phi| |\nabla |\tilde{A}|| + |\nabla \phi|),
\]
where the constant $c$ only depends on $p$ and $|d|_{C^2}$. 
Proof. Suppose now that $\phi$ is a function compactly supported, since $\langle Dd, \eta \rangle \equiv 1$ along $\partial M$. By the divergence theorem,

$$\int_{\partial M} |\bar{A}|^q \phi^2 = \int_{\partial M} |\bar{A}|^q \phi^2 \langle Dd, \eta \rangle = \int_M \text{div}_M (|\bar{A}|^q \phi^2 Dd) = \int_M |\bar{A}|^q \phi^2 \text{div}_M Dd + \int_M \phi^2 Dd \cdot \nabla |\bar{A}|^q + 2 \int_M |\bar{A}|^q \phi \nabla \phi \cdot Dd,$$

and then the lemma follows similarly as Corollary 2. □

4. Simons Inequality

4.1. Simons inequality of $\bar{A}$. First, we have a Simons type inequality for $\bar{A}$ of a marginally outer trapped hypersurface $M$. The Simons identity (Theorem 2.1) would be our starting point. We combine it with estimates of the perturbation. We use $A \ast B$ to denote linear combinations of contractions of $A \otimes B$ for convenience.

Lemma 4.1. Let $M$ be a MOTS, then

$$h_{ij} \Delta h_{ij} \geq \bar{h} \ast (\nabla K + \nabla^2 H + \Delta \bar{T}) - |\bar{A}|^4 - c(|\bar{A}|^3 + |\nabla \bar{A}| |\bar{A}|),$$

where the constant $c$ depends only on $|p|_{C^1}, |b|_{C^1}$ and $|K|_{C^0}$.

Proof. First, we estimate $H$ and $\nabla H$. The mean curvature $H$ itself is easy, and an upper bound follows from the MOTS equation (1.1),

$$|H| = | - \text{tr}_M p| \leq n |p|_{C^0}.$$

By (2.1), we have

$$|\nabla_i H| = | - \nabla_i \text{tr}_M p| = |g^{j} \nabla_i g_{jk} p_{kj}| \leq c(1 + |A|).$$

Then using these estimates in the Simons identity (2.1), we have that

$$h_{ij} \Delta h_{ij} \geq - h_{ij} (\nabla_j K_{n+1,kik} + \nabla_k K_{n+1,ijk}) + h_{ij} \nabla_i \nabla_j H - |A|^4 - c(1 + |A| + |A|^2 + |A|^3 + |\nabla A| |A|).$$

Now we consider the perturbation,

$$\bar{h}_{ij} \Delta \bar{h}_{ij} = (h_{ij} + \bar{T}_{ij})(\Delta h_{ij} + \Delta \bar{T}_{ij}) = h_{ij} \Delta h_{ij} + \bar{T}_{ij} \Delta h_{ij} + (h_{ij} + \bar{T}_{ij}) \Delta \bar{T}_{ij} = h_{ij} \Delta h_{ij} + \bar{T}_{ij} \Delta h_{ij} + \bar{h} \ast \Delta \bar{T}.$$
By combining the two inequalities in the above, we see that

\[
\bar{h}_{ij} \Delta \bar{h}_{ij} \\
\geq h_{ij}(-\nabla_j K_{n+1,kik} - \nabla_k K_{n+1,ijk} + \nabla_i \nabla_j H) - |A|^4 \\
- c(1 + |A|^3 + |\nabla A||A|)
\]

\[
+ \bar{T}_{ij}(-\nabla_j K_{n+1,kik} - \nabla_k K_{n+1,ijk} + h_{ij} \nabla_i \nabla_j H) \\
- c(|A|^3 + |A|^3) + \bar{h} \Delta \bar{T} \\
\geq \bar{h} \ast (\nabla K + \nabla^2 H + \Delta \bar{T}) - |A|^4 - c(1 + |A|^3 + |\nabla A||A|).
\]

We use Cauchy-Schwarz inequality to absorb $|A|$ and $|A|^2$ into $|A|^3$, and note that $\bar{h} = h + \bar{T}$, so

\[
\bar{h}_{ij} \Delta \bar{h}_{ij} \geq \bar{h} \ast (\nabla K + \nabla^2 H + \Delta \bar{T}) - |A|^4 - c(1 + |A|^3 + |\nabla A||A|).
\]

After applying (2.5), we obtain our desired inequality. 

The following Kato type inequality for $\bar{A}$ is standard when applying Schoen-Simon-Yau’s iterative arguments for curvature estimates. For completeness, we include the proof. The Kato type inequality asserts a lower bound of $|\nabla_k \bar{h}_{ij}|^2$ in terms of $|\nabla |\bar{A}||^2$. Our proof is taken from [SSY75].

**Lemma 4.2.** Let $M$ be a MOTS, then we have

\[
\sum_{ij} |\nabla_k \bar{h}_{ij}|^2 - |\nabla |\bar{A}||^2 \geq \frac{1}{(1+2m)} |\nabla |\bar{A}||^2 - c(1 + |\bar{A}|^2),
\]

where $c > 0$ depends on $\frac{1}{2}, |K|_{C^0}$, $|p|_{C^1}$ and $|b|_{C^1}$.

**Proof.** Let $\mathcal{T} = |\nabla \bar{A}|^2 - |\nabla |\bar{A}||^2$. We compute

\[
|\bar{A}|^2 \mathcal{T} \\
= |\bar{A}|^2 |\nabla \bar{A}|^2 - \frac{1}{4} |\nabla |\bar{A}||^2 \\
= \sum_{i,j,k,l,m} (\bar{h}_{ij} \nabla_k \bar{h}_{ml}) - \sum_k (\sum_{ij} \bar{h}_{ij} \nabla_k \bar{h}_{ij})^2 \\
= \frac{1}{2} \sum_{i,j,k,l,m} (\bar{h}_{ij} \nabla_k \bar{h}_{ml} - \bar{h}_{ml} \nabla_k \bar{h}_{ij})^2.
\]

By choosing basis of $TM$ we can assume that $\bar{h}_{ij}$ is diagonal and

\[
\bar{h}_{ij} = \lambda_i \delta_{ij}.
\]
Using (4.4), we have
\[
\sum_{i,j,k,l,m} (\bar{h}_{ij} \nabla_k \bar{h}_{ml} - \bar{h}_{ml} \nabla_k \bar{h}_{ij})^2
= \sum_{i,m,l,k} (\bar{h}_{ii} \nabla_k \bar{h}_{ml} - \bar{h}_{ml} \nabla_k \bar{h}_{ii})^2
\geq \left( \sum_{i \not= j,k} \bar{h}_{ii}^2 \right) \sum_{m,l} |\nabla_k \bar{h}_{ij}|^2
\geq 2 \left( \sum_{m,l} \bar{h}_{ml}^2 \right) \sum_{i \not= j,k} |\nabla_k \bar{h}_{ij}|^2.
\]

Hence,
\[
T = |\nabla \bar{A}|^2 - |\nabla|\bar{A}|^2 \geq \sum_{i \not= j,k} |\nabla_k \bar{h}_{ij}|^2.
\]

Observe that the proof of the above inequality only uses symmetry properties of \( \bar{h}_{ij} \). We estimate \( \sum_{i \not= j,k} |\nabla_k \bar{h}_{ij}|^2 \) as follows:
\[
\sum_{i \not= j,k} |\nabla_k \bar{h}_{ij}|^2 \geq \sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2 + |\nabla_j \bar{h}_{ij}|^2 = 2 \sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2.
\]

Since
\[
\sqrt{\sum_{i \not= j} |\nabla_j \bar{h}_{ii}|^2} \leq \sqrt{\sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2} + \sqrt{\sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2},
\]
and by Codazzi equation and the definition of \( \bar{h} \) on the first term on the right hand side, we have
\[
\sqrt{\sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2} \geq \sqrt{\sum_{i \not= j} |\nabla_j \bar{h}_{ij}|^2} - \sqrt{\sum_{i \not= j} (K_{n+1,iji} + \nabla_i \bar{T}_{ij} - \nabla_j \bar{T}_{ij})^2}.
\]

With the bound on \( K \) by \( |K|_{C^0} \) and (2.3) in the above, we obtain
\[
\sqrt{\sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2} \geq \sqrt{\sum_{i \not= j} |\nabla_j \bar{h}_{ij}|^2 - c(1 + |A|)}.
\]

By invoking the elementary fact that the inequality \( \sqrt{a} \geq \sqrt{b} - \sqrt{c} \) implies \( a \geq \frac{b}{1 + \epsilon} - \frac{c}{\epsilon} \), we have
\[
\sum_{i \not= j,k} |\nabla_k \bar{h}_{ij}|^2
\geq 2 \sum_{i \not= j} |\nabla_i \bar{h}_{ij}|^2
\geq \frac{2}{1 + \epsilon} \sum_{i \not= j} |\nabla_j \bar{h}_{ij}|^2 - \frac{\epsilon}{\epsilon} (1 + |A|)^2.
\]
We then try to bound $\sum_{i \neq j} |\nabla_j \tilde{h}_{ii}|$ using $|\nabla |\tilde{A}| |$. By (4.14),

$$|\nabla |\tilde{A}| |^2 = |\tilde{A}|^{-2} \sum_k (\sum_{j,i} \tilde{h}_{ij} \nabla_k \tilde{h}_{ij})^2$$

$$= (\bar{\Omega}^2)^{-1} \sum_k (\sum_{i} \tilde{h}_{ii} \nabla_k \tilde{h}_{ii})^2$$

$$\leq \sum \left| \nabla_k \tilde{h}_{ii} \right|^2$$

$$= \sum_{i \neq k} \left| \nabla_k \tilde{h}_{ii} \right|^2 + \sum_i \left| \nabla_i \tilde{h}_{ii} \right|^2$$

$$= \sum_{i \neq k} \left| \nabla_k \tilde{h}_{ii} \right|^2 + \sum_i \left| \nabla_i \tilde{h}_{ii} + \nabla_i \bar{T}_{ii} \right|^2.$$ 

Because of $H = \sum_i h_{ii}$, the bounds (4.2) and (2.3),

$$|\nabla |\tilde{A}| |^2 = \sum \left| \nabla_k \tilde{h}_{ii} \right|^2 + \sum_i \left| \nabla_i \tilde{h}_{ii} + \nabla_i \bar{T}_{ii} \right|^2$$

$$= n \sum_{i \neq j} \left| \nabla_i \tilde{h}_{jj} \right|^2 + c|\nabla \tilde{A}| |A| + c|A|^2 + c|\nabla \tilde{A}| + c.$$ 

Therefore,

$$\sum_{i \neq j} \left| \nabla_k \tilde{h}_{ij} \right|^2 - |\nabla |\tilde{A}| |^2 \geq \sum \left| \nabla_k \tilde{h}_{ij} \right|^2$$

$$\geq \frac{1}{\bar{\xi}} \sum_{i \neq j} \left| \nabla_i \tilde{h}_{jj} \right|^2 - \xi (1 + |A|)^2$$

$$\geq \frac{2}{(1 + \varepsilon)n} |\nabla |\tilde{A}| |^2 - \frac{\xi}{\xi} (1 + |A|)^2 - (c|\nabla \tilde{A}| |A| + c|A|^2 + c|\nabla \tilde{A}| + c).$$ 

By absorbing $|A|$ into $|A|^2$, $|\nabla \tilde{A}|$ to the left, and (2.5), we obtained the desired inequality. 

**Corollary 3.** Let $M$ be a MOTS, then

$$|\nabla \tilde{A}|^2 - |\nabla |\tilde{A}| |^2 \geq \frac{1}{(1 + \varepsilon)n + 1} (|\nabla \tilde{A}|^2 + |\nabla |\tilde{A}| |^2) - c(1 + |\tilde{A}|^2),$$

with $c$ depending on the same constants as in Lemma 4.2.

5. Curvature estimates

5.1. $L^q$ curvature estimate.
Recall the Simons inequality (4.1), we have that
\[ \int_M \phi^2 |\overline{A}|^{q-2} |\nabla |\overline{A}||^2 \leq c \int_M (\phi^2 + |\nabla \phi|^2) |\overline{A}|^q, \]
and
\[ \int \phi^2 |\overline{A}|^{q+2} \leq c \int_M |\overline{A}|^{q} [|\nabla \phi|^2 + \phi^2], \]
where the constant \( c > 0 \) depends only on \( |K|_{C^0}, |p|_{C^1}, |b|_{C^1}, |\nabla|_{C^1} \) and \( |d|_{C^2} \).

Proof. We have respectively
\[ \Delta |\overline{A}|^2 = 2 |\overline{A}| \Delta |\overline{A}| + 2 |\nabla |\overline{A}||^2, \]
and
\[ \Delta |\overline{A}|^2 = 2 \overline{h}_{ij} \Delta \overline{h}_{ij} + 2 |\nabla \overline{A}|^2. \]

Subtracting the above two equations give
\[ - |\overline{A}| \Delta |\overline{A}| + |\nabla |\overline{A}||^2 - |\nabla |\overline{A}||^2 = - \overline{h}_{ij} \Delta \overline{h}_{ij}. \]

Recall the Simons inequality (4.1), we have that
\[ - |\overline{A}| \Delta |\overline{A}| + |\nabla |\overline{A}||^2 - |\nabla |\overline{A}||^2 \leq (\nabla K + \nabla^2 H + \Delta \overline{T}) \ast \overline{h} + c(1 + |\overline{A}|^3 + |\nabla |\overline{A}| |\overline{A}|) + |\overline{A}|^4. \]

Multiply this equation by \( \phi^2 |\overline{A}|^q \) and integrate. This yield
\[ \int_M -\phi^2 |\overline{A}|^{q-1} \Delta |\overline{A}| + \phi^2 |\overline{A}|^{q-2} (|\nabla |\overline{A}||^2 - |\nabla |\overline{A}||^2) \leq \int_M \phi^2 |\overline{A}|^{q+2} + c \phi^2 (1 + |\overline{A}|^3 + |\nabla |\overline{A}| |\overline{A}|) |\overline{A}|^{q-2} \]
\[ + \int_M \phi^2 |\overline{A}|^{q-2} (\nabla K + \nabla^2 H + \Delta \overline{T}) \ast \overline{h}. \]

Doing an integration by parts involving the Laplacian on the first line and \( \nabla K + \nabla^2 H + \Delta \overline{T} \) on the last line, performing some elementary estimates we find
\[ \int_M (q - 1) \phi^2 |\overline{A}|^{q-2} |\nabla |\overline{A}||^2 + \phi^2 |\overline{A}|^{q-2} (|\nabla |\overline{A}||^2 - |\nabla |\overline{A}||^2) \leq \int_M \phi^2 |\overline{A}|^{q+2} + c \phi^2 (1 + |\overline{A}|^3 + |\nabla |\overline{A}| |\overline{A}|) |\overline{A}|^{q-2} \]
\[ - \int_M 2 \phi |\overline{A}|^{q-2} (\nabla |\overline{A}|, \nabla \phi) + \int_{\partial M} \phi^2 |\overline{A}|^{q-1} \partial_n |\overline{A}| \]
\[ + c \int_M \phi |\nabla \phi| |\overline{A}|^{q-1} + \phi^2 U |\overline{A}|^{q-2} (|\nabla |\overline{A}| + |\nabla \overline{A}|) \]
\[ + \int_{\partial M} \phi^2 |\overline{A}|^{q-1} U. \]

Here
\[ U := |K| + |\nabla H| + |\nabla \overline{T}| \leq c |\overline{A}| \]

Proposition 5.1. Let \( M \) be a stable free boundary MOTS, we have that for \( q \in [2, 2 + \sqrt{\frac{2}{n}}], \)
\[ \int_M \phi^2 |\overline{A}|^{q-2} |\nabla |\overline{A}||^2 \leq c \int_M (\phi^2 + |\nabla \phi|^2) |\overline{A}|^q, \]
and
\[ (5.1) \int \phi^2 |\overline{A}|^{q+2} \leq c \int_M |\overline{A}|^{q} [|\nabla \phi|^2 + \phi^2], \]
where the constant \( c > 0 \) depends only on \( |K|_{C^0}, |p|_{C^1}, |b|_{C^1}, |\nabla|_{C^1} \) and \( |d|_{C^2} \).
due to (2.3) and (1.2). After applying (11) to the above, then (3.5), we obtain,
\[
\int_M (q - 1) \phi^2 |A|^{q - 2} \text{grad} |A|^2 + \phi^2 |A|^{q - 2} (|\text{grad} A|^2 - |\text{grad} |A||^2) \\
\leq \int_M \phi^2 |A|^{q + 2} + c \phi^2 (1 + |A|^3 + |\text{grad} |A||A||^{q - 2} \\
- \int_M 2 \phi |A|^{q - 2} (|\text{grad} |A||\text{grad} \phi) + c \int_M \phi^2 |A|^q \\
+ c \int_M \phi |\text{grad} \phi||A|^q + \phi^2 |A|^{q - 1} (|\text{grad} |A|| + |\text{grad} |A|)
\]

We have the following inequalities for any \(s < 2\) and any small \(\varepsilon > 0\) which follow from an application of the Young’s inequality, and we apply them to the above,
\[
|A|^{q + s} \leq \varepsilon |A|^{q + 2} + c(\varepsilon^{-1}, s), \\
|\text{grad} A| |A|^{q - 1} \leq \varepsilon |\text{grad} |A||^2 |A|^q + c(\varepsilon^{-1}) |A|^q, \\
\phi(|\text{grad} |A||\text{grad} \phi) \leq \varepsilon \phi^2 |\text{grad} |A||^2 + c(\varepsilon^{-1}) |\text{grad} \phi|^2, \\
\phi |\text{grad} \phi||A|^q \leq \varepsilon \phi^2 |A|^{q + 2} + c(\varepsilon^{-1}) |\text{grad} \phi|^2 |A|^q, \\
|\text{grad} |A|| + |\text{grad} |A| \leq \varepsilon (|\text{grad} |A||^2 + |\text{grad} |A|^2) + c(\varepsilon^{-1}) \\
\leq \varepsilon (|\text{grad} |A||^2 + |\text{grad} |A|^2) + c(\varepsilon^{-1}).
\]

Hence we obtain
\[
\int_M (q - 1) \phi^2 |A|^{q - 2} \text{grad} |A|^2 + \phi^2 |A|^{q - 2} (|\text{grad} A|^2 - |\text{grad} |A||^2) \\
\leq (1 + \varepsilon) \int_M \phi^2 |A|^{q + 2} + \varepsilon \int_M \phi^2 |A|^{q - 2} (|\text{grad} |A||^2 + |\text{grad} A|^2) \\
+ c \int_M (|\text{grad} \phi|^2 + \phi^2) |A|^q.
\]

We use the inequalities (3.4), (4.3) and (4.5), we obtain
\[
\int_M (q - 1 + \frac{2}{1 + \varepsilon} n - \varepsilon (1 + \varepsilon) n + 1) \phi^2 |A|^{q - 2} |\text{grad} |A||^2 \\
\leq (1 + \varepsilon)(1 + \varepsilon) \frac{2}{(1 + \varepsilon) n} \int_M \phi^2 |A|^{q - 2} |\text{grad} |A||^2 + c \int_M (|\text{grad} \phi|^2 + \phi^2) |A|^q.
\]

Since \(q \in [2, 2 + \sqrt{\frac{n}{2}})\) ensures that \(\frac{2}{n} < q - 1 + \frac{2}{n}\), by choosing \(\varepsilon\) sufficiently small, we can absorb the first term on the right to the left, and we obtain the desired inequality.

The second inequality which asserts a bound on \(\int \phi^2 |A|^{q + 2}\) follows by combining with Lemma 3.4. \(\square\)

**Theorem 5.2.** Let \(q \in [2, 2 + \sqrt{\frac{n}{2}})\), and \(M\) be a stable free boundary MOTS satisfying the volume bound (1.5), then
\[
(5.3) \quad \int_{B(x,r/4)} |A|^{q + 2} \leq cr^{n - 2 - q},
\]
where the constant \(c\) depends on \(\text{Ric} \, C^0, \, |p| C^1, \, |b| C^0, \, |d| C^2\) and the constant \(C_M\) in (1.5).
Proof. By letting $q = 2$ and $\phi$ be a standard cutoff, that is, $\phi = 0$ outside $B(x, r)$, $\phi = 1$ in $B(x, r/2)$, $|\nabla \phi| \leq \frac{c}{r}$. So using $\phi$ in \eqref{eq:1.2},

\begin{equation}
\int_{B(x, r/2)} |\bar{A}|^2 \leq c r^{-2} \int_{B(x, r)} |\bar{A}|^2 \leq c r^{-4+n}. \tag{5.4}
\end{equation}

From the $L^q$ estimate \eqref{eq:5.1}, we have

$$
\int_{B(x, r/4)} |\bar{A}|^{q+2} \leq c \int_{B(x, r/2)} |\bar{A}|^q.
$$

Note that $q < 2 + \sqrt{\frac{8}{n}} \leq 4$, so from Hölder inequality and the $L^4$ estimate \eqref{eq:5.4},

$$
\int_{B(x, r/4)} |\bar{A}|^{q+2} \leq c \left( \int_{B(x, r/2)} |\bar{A}|^4 \right)^{\frac{q}{4}} |B(x, r/2)|^{1-\frac{q}{4}} \leq c r^{-2-q+n}.
$$

This is our desired bound. \qed

Now we prove our main Theorem 1.4.

Proof of Theorem 1.4. We employ the iteration method of De Giorgi. Recall the Simons inequality \eqref{eq:4.1}, we have that

$$
-|\bar{A}|\Delta |\bar{A}| + |\nabla \bar{A}|^2 - |\nabla \bar{A}|^2 
\leq (\nabla K + \nabla^2 H + \nabla^2 \bar{T}) \ast \bar{h} + c(1 + |\bar{A}|^3 + |\nabla \bar{A}| |\bar{A}|) + |\bar{A}|^4.
$$

Applying \eqref{eq:4.2} and Cauchy-Schwarz inequality to absorb the term $|\nabla \bar{A}| |\bar{A}|$, and absorbing $|\bar{A}|^3$ into $|\bar{A}|^4$, we have that

$$
-\Delta |\bar{A}|^2 + c_1 |\nabla \bar{A}|^2 \leq (\nabla K + \nabla^2 H + \nabla^2 \bar{T}) \ast \bar{h} + c|\bar{A}|^4.
$$

Here $c_1$ is a positive constant. We multiply both sides by $\phi$, we have that

$$
- \int_M \phi \Delta |\bar{A}|^2 + c_1 \phi |\nabla \bar{A}|^2 \leq \int_M \phi (\nabla K + \nabla^2 H + \nabla^2 \bar{T}) \ast \bar{h} + c\phi |\bar{A}|^4.
$$

We use integration by parts on the first term on the right and the bound \eqref{eq:1.2} on $U$ to obtain a bound on the first term on the right:

$$
\int_M \phi (\nabla K + \nabla^2 H + \nabla^2 \bar{T}) \ast \bar{h}
= \int_{\partial M} \phi (K + \nabla H + \nabla \bar{T}) \ast \bar{h} \ast \eta - \int_{M} (K + \nabla H + \nabla \bar{T}) \ast (\bar{h} \nabla \phi + \phi \nabla \bar{h})
\leq c \int_{\partial M} \phi |\bar{A}|^2 + c \int_{M} |\nabla \phi| |\bar{A}|^2 + c \int_{M} \phi |\bar{A}| |\nabla \bar{A}|.
$$

Absorbing $|\nabla \bar{A}|$ using the Cauchy-Schwarz inequality

$$
|\bar{A}| |\nabla \bar{A}| \leq c |\nabla \bar{A}|^2 + \frac{1}{2c} |\bar{A}|^2,
$$

we have

$$
\int_M \phi |\bar{A}|^2 \leq c_2.
$$

This proves Theorem 1.4. \qed
absorbing $|\bar{A}|^2$ into $|\bar{A}|^4$, we have

$$- \int_M \phi \Delta |\bar{A}|^2 \leq c \int_{\partial M} \phi |\bar{A}|^2 + c \int_M |\nabla \phi||\bar{A}|^2 + \phi |\bar{A}|^4.$$  

Let $u = |\bar{A}|^2$, $v = \max\{u - k, 0\}$ and replacing $\phi$ by $\phi^2 v$ in the above,

$$- \int_M \phi^2 v \Delta u \leq c \int_{\partial M} \phi^2 v \partial_n v + c \int_M |\nabla (\phi^2 v)| u + \phi^2 v u |\bar{A}|^2.$$  

We can apply divergence theorem on the first term

$$- \int_M \phi^2 v \Delta u = - \int_{\partial M} \phi^2 v \partial_n v + \int_M \phi^2 |\nabla v|^2 + 2 \phi v (\nabla \phi, \nabla v).$$  

Note that $|\partial_n v| \leq cu$ due to (1), so

$$\int_M \phi^2 |\nabla v|^2 + 2 \phi v (\nabla \phi, \nabla v) + \int_M c_1 \phi^2 v |\nabla A|^2 \leq c_1 \int_{\partial M} \phi^2 v u + c_1 \int_M |\nabla (\phi^2 v)| u + \phi^2 v u |\bar{A}|^2.$$  

For the boundary term $\int_{\partial M} \phi^2 v u$, we use

$$\int_{\partial M} \phi^2 v u = \int_M \text{div}_M (\phi^2 v u \nabla d) = \int_M 2\phi (\nabla \phi, \nabla d) vu + \int_M \phi^2 u (\nabla v, \nabla d) + \int_M \phi^2 v (\nabla u, \nabla d) + \phi^2 uv \text{div}_M \nabla d$$

$$\leq c \int_M \phi |\nabla \phi| vu + \int_M \phi^2 u |\nabla v| + \int_M \phi^2 v |\nabla u| + \int_M \phi^2 vu$$

$$\leq c \int_M v^2 |\nabla \phi|^2 + c \int_M \phi^2 u^2 + \varepsilon \int_M \phi^2 |\nabla v|^2 + c(\varepsilon^{-1}) \int_M \phi^2 u^2$$

$$+ \varepsilon \int_M \phi^2 |\nabla v|^2 + c(\varepsilon^{-1}) \int_M \phi^2 v^2 + \int_M \phi^2 u^2$$

$$\leq \varepsilon \int_M \phi^2 |\nabla v|^2 + c \int_M v^2 |\nabla \phi|^2 + c \int_M k^2 \phi^2 + \int_M \phi^2 v^2.$$  

And the term $c \int_M |\nabla (\phi^2 v)| u$ is estimated as follows:

$$\int_M |\nabla (\phi^2 v)| u$$

$$\leq 2 \int_M \phi |\nabla \phi| vu + \int_M \phi^2 |\nabla v| u$$

$$\leq c \int_M |\nabla \phi|^2 u^2 + \int_M \phi^2 u^2 + \varepsilon \int_M \phi^2 |\nabla v|^2 + c(\varepsilon^{-1}) \int_M \phi^2 u^2$$

$$\leq \varepsilon \int_M \phi^2 |\nabla v|^2 + c \int_M v^2 |\nabla \phi|^2 + c \int_M k^2 \phi^2 + \int_M \phi^2 v^2.$$  

And

$$\int_M \phi^2 vu |\bar{A}|^2 \leq \int_M \phi^2 v^2 |\bar{A}|^2 + k \int_M \phi^2 v |\bar{A}|^2 \leq \frac{3}{2} \int_M \phi^2 v^2 |\bar{A}|^2 + \frac{k^2}{2} \int_M \phi^2 |\bar{A}|^2.$$
Using a bound on $|\bar{A}| \geq 1$,
\[
\int_M \phi^2 |\nabla v|^2 + 2\phi v \langle \nabla \phi, \nabla v \rangle \\
\leq 2\varepsilon \int_M \phi^2 |\nabla v|^2 + c \int_M v^2 |\nabla \phi|^2 + c \int_M k^2 |\bar{A}|^2 \phi^2 + \int_M \phi^2 v^2.
\]
Letting $\varepsilon = \frac{1}{4}$, then we have that
\[
\frac{1}{2} \int_M \phi^2 |\nabla v|^2 + 2\phi v \langle \nabla \phi, \nabla v \rangle \\
\leq c \int_M v^2 |\nabla \phi|^2 + c \int_M k^2 |\bar{A}|^2 \phi^2 + c \int_M \phi^2 v^2.
\]
Since
\[
|\nabla (\phi v)|^2 = \phi^2 |\nabla v|^2 + 2\phi v \langle \nabla \phi, \nabla v \rangle + v^2 |\nabla \phi|^2,
\]
and the Cauchy-Schwarz inequality $2\phi v \langle \nabla \phi, \nabla v \rangle \geq -\frac{1}{4} \phi^2 |\nabla v|^2 + 4v^2 |\nabla \phi|^2$, so
\[
\int_M |\nabla (\phi v)|^2 \leq c \int_M v^2 |\nabla \phi|^2 + c \int_M k^2 |\bar{A}|^2 \phi^2 + c \int_M \phi^2 v^2.
\]
Now the inequality is a good starting point for De Giorgi’s iteration scheme. One more ingredient we need is the Sobolev inequality (Theorem A.1). For the details, we refer to the [HL11, Chapter 4]. We obtain therefore an $L^2$ mean value inequality for $u$,
\[
\sup_{B(x,r/2)} u \leq cr^{-2} \int_{B(x,r)} u^2
\]
provided that $|\bar{A}| \in L^{q_1}(M)$ where $q_1 > n/2$. By the estimate (5.3), $|A'| \in L^{q_2}$ for any $4 \leq q_2 < 4 + \sqrt{\frac{8}{n}}$. Such $q_1$ exists only when $4 + \sqrt{\frac{8}{n}} > n$, that is, $n$ is in any dimension from 2 to 5. So when $2 \leq n \leq 5$,
\[
\sup_{B(x,r)} |\bar{A}| \leq \frac{\varepsilon}{7},
\]
finishing our proof the curvature estimate. Considering (2.5), we have the bound (1.6) for $|A|$.

**Appendix A. Sobolev inequalities**

Recall the following general Sobolev inequality for hypersurfaces.

**Theorem A.1** ([HS74]). Assume that $M$ is $n$-dimensional hypersurface in a manifold $\tilde{N}$, let $h$ be a nonnegative $C^1$ function on $M$ which vanishes on $\partial N$. Then
\[
\|h\|_{L_n^{q_1}(M)} \leq c \int_M |\nabla h| + h |H|,
\]
provided the measure of the support of $h$ is less than a constant $c_0 > 0$ which depends on $n$, the upper bound of the sectional curvature and the injective radius of $\tilde{N}$. Here $c$ depends only on $n$.

Going through the same reasoning as in [Ede16, Theorem 2.3], we obtain the following.
Theorem A.2. If $M$ meets $\partial N$ orthogonally and $v \in C^1(\bar{M})$ and the measure of the support of $v$ is less than $c_0$ (as in as Theorem A.1), then for any $1 \leq p < n$,
\[ \|v\|_{L^{p^*}(M)} \leq c(\|\nabla v\|_{L^p(M)} + \|Hv\|_{L^p(M)} + \|v\|_{L^p(M)}), \]
where $c$ is a constant depending only the dimension $n$, the exponent $p$, the distance function to $\partial N$, the upper bound of the sectional curvature and the injective radius of $\bar{N}$. The number $p^*$ is the critical Sobolev exponent, when $n > 2$, $p^* = \frac{np}{n-p}$ and when $n = 2$, $p^*$ could be any number greater than 2.

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