Comments on two papers by Galliano Valent, concerning integrable Hamiltonian systems admitting quartic and cubic integrals.

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July 2, 2018

Abstract

In this note we comment on two recently published papers by G. Valent:
The first is the preprint "On a Class of Integrable Systems with a quartic First Integral, Arxiv: 1304-5859. April 22, (2013)". We show that the two integrable Hamiltonian systems introduced in this reprint as original results are not new. They are special cases of two systems introduced by the present author in 2006 in two papers [6] and [5].
The second paper is "On a Class of Integrable Systems with a Cubic First Integral, Commun. Math. Phys. 299, 631–649 (2010),
In that paper two integrable hamiltonian systems admitting a cubic integral were introduced. Those systems were referred to as original results by Tsiganov in [12], Vershikov and Tsiganov in [13], Bialy and Mironov in [15] and by Gibbons et al in [14]. We show that those systems are not new. Both can be obtained as special cases of one system introduced by us in [4] (2002) and one of them is a special case of a much earlier version [1] published 24 years earlier.

1 Introduction

In the last few decades, a great interest arose in integrable systems on various types of two-dimensional manifolds. Only few works were devoted to devise new methods for constructing such systems. The most successful method seems to be the one introduced by the present author in 1986 [1]. Until now, this method has led to the construction of a large collection of integrable 2D systems including the richest and biggest ever known ones: 41 time-irreversible systems with a complementary quadratic integral [2], [3]; two systems with a cubic integral [1].
and three systems with an integral of the fourth degree \[5\], \[6\] and \[7\]. One of those, called by us "master", is the biggest ever-known 2D integrable system, in the sense that it involves 21 free parameters.

One of the strong points in our method is the use of Lagrangian formulation and some point transformations as well as a change of the time parametrization preserving this formalism. This led to the formation of a small number of differential equations for the system determination and in some cases facilitated the process of solution.

Integrable systems constitute a rare exception in hamiltonian dynamics in general. One must pay due attention to make sure that newly found systems do not repeat already known ones. In the last few years, several trials were made to construct new integrable 2D systems using Hamiltonian formalism and (ansatz)es suited for search for certain special forms of quadratic and cubic integrals. Unfortunately, some authors have rediscovered certain integrable systems in Hamiltonian form and made no effort to find out that those systems were obtained significantly earlier, but in some other forms. Some authors give either incomplete or erroneous references; and other authors do not provide due references, leaving the reader with the impression that provided results are new. Some of the most notable examples are in the works \[8\]-\[9\].

2 Two systems with a quartic integral in Valent’s paper \[9\]

In a recent work \[9\] Valent introduced two integrable Hamiltonian systems which, in his words, ”generalize, to some extent, the results on integrable geodesic flows on two dimensional manifolds with a quartic first integral in the framework laid down by Selivanova and Hadeler”.

As a motivation of the method used by him in \[8\] and \[9\] Valent says "a more direct analysis of the differential system leading to integrability revealed to be also successful". In fact, this is the same method used in our earlier works including \[1\] and, in particular, in \[6\], where it has led to more general systems than those given in \[9\]. Apart from this, Valent’s conclusions about restoring integrable cases of rigid body motions are not only repetitions of earlier results, but also very special cases of them.

2.1 The first system of \[9\]

That is given as a Hamiltonian system in theorem 2 of \[9\]. Its Lagrangian equivalent is characterized by the Lagrangian
\[ L = \frac{1}{2} \beta^2 \left( \frac{\dot{x}^2}{F} + \frac{\varphi^2}{\sqrt{F}} \right) - \frac{1}{2\beta^2} \left( k \sqrt{FG'}(x) \cos \varphi + lg(x) + mx + n \right) \] (1)

where
\[
\beta^2 = b_0 + \alpha x, \\
F = x^4 + c_2 x^2 + c_1 x + c_0, \\
G(x) = \sqrt{F} - x^2 - \frac{c_2}{2}
\]

It can be easily verified that this system is a special case of the system introduced in 2006 in our work [6] with the Lagrangian

\[
L = \frac{1}{2} \left[ \mu \alpha_2 p^2 + \alpha_1 p + \alpha \right] - \alpha_2 \xi^2 \\
+ \left( \mu \alpha_2 p^2 + \alpha_1 p + \alpha - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right)^2 \\
- \mu h_3 p^2 - h_1 p - h_2 \\
\frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{16 \mu (\mu \alpha_2 p^2 + \alpha_1 p + \alpha - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0})} A \cos(\sqrt{\mu} \xi) \right] (2)
\]

From this Lagrangian we can get (1) simply by enforcing the conditions
\[
\mu = 1, A = -16k, \\
\alpha_0 = \alpha, \alpha_1 = b_0, \alpha_2 = 0, \\
h_1 = 2m, h_2 = 2n - lc_2, h_3 = 2l
\] (3)

and renaming the variables \( p \rightarrow x, \xi \rightarrow \varphi \).

Valent's Lagrangian (1) contains only 6 free parameters, i.e. two parameters less than our Lagrangian (2) involving 8 parameters.

Our paper [6] containing (2) and published 7 years earlier than [9] is not mentioned at all in [9].

2.2 The second system of [9]
That is given as a Hamiltonian system in theorem 3 of [9]. Its Lagrangian equivalent is characterized by the Lagrangian

\[ L = \frac{1}{2}r\left(-1 + \frac{x^2 + d}{\sqrt{R}}(\frac{\dot{x}^2}{\sqrt{R}} + \dot{\varphi}^2)\right) - \frac{1}{2\beta^2}(k\sqrt{F}\alpha\cos\varphi + lG(x) + mx + n) \]  

(4)

where

\[ F = x^4 - 4rx^3 \alpha + 2(d + 2r^2 \alpha^2 - 2rc_2)x^2 - 4c_1 rx + d^2 - 4c_0 r, \]
\[ G = \alpha x + c_2 - \beta^2, \]
\[ \beta^2 = \frac{(x^2 + d) - \sqrt{F}}{2r}. \]

This system is obtained from our "master" system of [5], whose 21-parameter Lagrangian is

\[ L = \frac{1}{2}\Lambda\left[\frac{\dot{p}^2}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\right] \]
\[-\frac{1}{\Lambda}\left[\frac{h_0 b_3 p^3 + h_3 p^2 + b_2 p + h_1}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{h_0 a_3 q^3 + h_3 q^2 + b_5 q + h_4}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\right] \]
\[+h_0\left[\frac{q(4a_4 p^3 + 3a_3 p^2 + 2a_2 p + a_1)}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{p(4a_4 q^3 + 3b_3 q^2 + 2b_2 q + b_1)}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\right] \]

(5)

where

\[ \Lambda = \left[\frac{\alpha_0 b_3 p^3 + a_3 p^2 + a_2 p + a_1}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{\alpha_0 a_3 q^3 + a_3 q^2 + a_5 q + a_4}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\right] \]
\[+\alpha_0\left[\frac{q(4a_4 p^3 + 3a_3 p^2 + 2a_2 p + a_1)}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{p(4a_4 q^3 + 3b_3 q^2 + 2b_2 q + b_1)}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}}\right] \]

by imposing on the last system the conditions

\[ a_4 = 1, b_2 = -2, b_0 = 1, b_3 = b_1 = 0, \]
\[ a_0 = -4rc_0 + d^2, a_1 = -4c_1 r, a_3 = -4\alpha r, \]
\[ a_2 = -4c_2 r + 2d + 4\alpha^2 r^2, \]
\[ \alpha_0 = \alpha_2 = \alpha_5 = 0, \alpha_1 = \frac{d}{r}, \alpha_3 = \frac{1}{r} = -\alpha_4, \]
\[ h_0 = \frac{k}{2r}, h_1 = 2n + 2lc_2 - \frac{ld}{r}, \]
\[ h_2 = 2m + 2\alpha r, -h_3 = h_4 = \frac{1}{r}, h_5 = 2\alpha k \]  

(6)
and changing the variables $p \rightarrow x, q \rightarrow \cos \varphi$. That is a restriction of the 21-parameter system to a 10-parameters one. As this case is the most involved, we provide in the appendix a MAPLE code which can be used to verify the conclusion of the present subsection concerning the Lagrangians $\text{(11)}$ and $\text{(10)}$.

It was not mentioned at all in $\text{(9)}$ that the system in theorem 3 of that work is a special case of $\text{(5)}$ published 7 years earlier in $\text{[5]}$.

### 3 Two systems with a cubic integral in Valent’s paper $\text{[8]}$

The content of this section was submitted to "Communications in Mathematical Physics" as a comment. After reviewing the comment, the Editor decided that it is sufficient to publish the Erratum $\text{[19]}$, acknowledging the situation without referring to our comment. As the same situation is now repeated, we provide the full comment below.

In this section we consider in detail the two systems with a cubic integral pointed out by Valent in $\text{[8]}$ as new findings. Those systems were further referred to as original results in $\text{[13], [15]}$ and $\text{[14]}$. We show that both systems are special cases of one of the systems announced much earlier in our work $\text{[4]}$ and one of them is equivalent to a special case constructed 24 years earlier in $\text{[1]}$.

Works like $\text{[8]}$ (as well as $\text{[16]}$, which was briefly commented in $\text{[3]}$ and $\text{[10]}$ commented in detail in $\text{[11]}$), have brought some confusion concerning literature in the field of integrable systems, and we feel some light must be shed on the present, seemingly chaotic situation created by the commented paper and other works citing it.

Valent’s systems and their Lagrangian analogs

The Hamiltonian characterizing the first system of $\text{[8]}$, which we denote by $H_{v1}$, can be explicitly expressed using relations (14) and(15) of that work as

$$H_{v1} = \frac{1}{2}\left[(c_3^2 + 3c_0c_3 - 2\rho_0)p_\zeta^2 + \frac{-3\zeta^4 - 18c_0\zeta^2 + 24\rho_0\zeta + 9c_0^2}{4(\zeta^3 + 3c_0\zeta - 2\rho_0)}p_\varphi^2\right] + \lambda\sqrt{\zeta^3 + 3c_0\zeta - 2\rho_0}\cos\varphi + m\zeta$$

(7)

The corresponding Lagrangian, which we denote by $L_{v1}$, in the same generalized coordinates $\zeta, \varphi$ is

$$L_{v1} = \frac{1}{2}\left[\frac{\dot{\zeta}^2}{\zeta^3 + 3c_0\zeta - 2\rho_0} + \frac{4(\zeta^3 + 3c_0\zeta - 2\rho_0)}{3(-3\zeta^4 - 18c_0\zeta^2 + 24\rho_0\zeta + 9c_0^2)}\varphi^2\right] - \lambda\sqrt{\zeta^3 + 3c_0\zeta - 2\rho_0}\cos\varphi - m\zeta$$

(8)

The second system in Valent’s work $\text{[8]}$, as follows from formulas (39)-(40) of that work, has the Hamiltonian

$$H_{v2} = \frac{1}{2}\left[(c_3\zeta^3 + c_2\zeta^2 + c_1\zeta + c_0)p_\zeta^2\right]$$
The corresponding Lagrangian, which we denote by $L_{v2}$, in the same generalized coordinates $\zeta, \phi$ is

$$
L_{v2} = \frac{1}{2} \left( \frac{\dot{\zeta}^2}{c_3 \zeta^3 + c_2 \zeta^2 + c_1 \zeta + c_0} + \frac{4 \zeta (c_3 \zeta^3 + c_2 \zeta^2 + c_1 \zeta + c_0) \dot{\phi}^2}{4c_3^2 \zeta^4 - 4c_2 c_3 \zeta^3 - 6c_1 c_3 \zeta^2 - 12c_0 c_3 \zeta + c_1^2 - 4c_0 c_2} \right)
$$

(9)

The corresponding Lagrangian, which we denote by $L_y$, in the same generalized coordinates $\zeta, \phi$ is

$$
L_y = \frac{1}{2} \left[ 3(\gamma \nu + \delta) \frac{d\nu}{dt}^2 + \frac{(\gamma \nu + \delta)(4\nu^3 + 6\alpha \nu - \beta)}{(3\alpha^2 + 4\beta \nu - 12\alpha \nu^2 - 4\nu^4)} \frac{d\phi}{dt}^2 \right]
$$

$$+ C(2\nu^2 + \alpha) + D(4\nu^3 - 6\alpha \nu + 2\beta) \frac{d\phi}{dt}
$$

$$+ \frac{1}{3(\gamma \nu + \delta)} [3(a \nu + b) + \frac{2C^2 \nu + 6CD(2\nu^2 - \alpha) + 6D^2(4\nu^3 - \beta)}{3\alpha^2 + 4\beta \nu - 12\alpha \nu^2 - 4\nu^4}]
$$

$$+ 4\sqrt{4\nu^3 + 6\alpha \nu - \beta} \Phi(\phi) \right]
$$

(10)

where

$$
\Phi(\phi) = \begin{cases} 
A \cos \sqrt{\mu}(\phi - \phi_0) & \mu > 0 \\
A e^{\sqrt{-\mu} \phi} + B e^{-\sqrt{-\mu} \phi} & \mu < 0 
\end{cases}
$$

(12)

Specially notable is the presence in $L_y$ of the extra-parameters $C, D$ and $\mu$. $C, D$ invoke terms linear in the velocities and make our system time-irreversible. The parameter $\mu$ widens the range of applications of the system to special cases, for example, in different integrable dynamics of the rigid body. Negative values of $\mu$ characterize certain integrable particle dynamics of the Toda type [4].

We state the following notices on Valent’s work:

1. Both systems of Valent characterized by (7, 8) and (9, 10) are special cases of our system characterized by the 11-parameter Lagrangian (11).

In fact, the 4-parameter Lagrangian (8) can be obtained from (11) by enforcing the following restrictions on 7 parameters

$$C = D = \gamma = b = \phi_0 = 0, \delta = 4/3, \mu = 1
$$

(13)
and renaming the remaining 4 parameters

\[ \alpha = 2c_0, \beta = 8\rho_0, A = -\lambda/16, a = -4m/3 \] (14)

Similarly, (10) can be readily obtained from (11) by imposing the conditions

\[ C = D = a = \varphi_0 = 0, \mu = \gamma = 1, \int \text{roducing a change of the coordinate} \]

\[ v = \zeta - \delta, \text{ and renaming the parameters} \]

\[ \alpha = \frac{2(3c_1c_3 - c_2^2)}{9c_3^2}, \beta = \frac{-4(2c_2^3 + 27c_0c_2^2 - 9c_1c_2c_3)}{27c_3^3}, \delta = -\frac{c_2}{c_3}, A = -\frac{9c_3}{64q}, b = -\frac{2\beta_0}{2q} \] (15)

It is worth mentioning that the two systems (7, 9) were treated in [8] separately. In fact, Valent states at the end of §2 of [8] that "The special case \( q = 0 \) is rather difficult to obtain as the limit of the general case \( q \neq 0 \), so we will first work it out completely".

2 A limited version of the Lagrangian (11) obtained 24 years earlier than the publication of [8] may be expressed from formulas (33) of [1], after the usual time change, as

\[ L_{\eta_0} = \frac{1}{2}\left[ \frac{\dot{\eta}^2}{\mu(\eta^3 + 12\alpha\eta - 12)} + \frac{4(\eta^3 + 12\alpha\eta - 12)\dot{\varphi}^2}{3(48\alpha^2 + 48\eta - 24\alpha \eta^2 - \eta^4)} \right] + \frac{\beta}{3}\sqrt{\eta^3 + 12\alpha\eta - 12}\cos(\sqrt{\mu}\varphi) + \frac{p_0}{2\eta} \] (16)

It is not hard to verify that the Lagrangian \( L_{\eta_1} \) of Valent’s first system can be obtained from (10) by setting \( \mu = 1 \), rescaling the generalized coordinate \( \eta \) in (16) by the relation

\[ \eta = \sqrt[3]{\frac{6}{\rho_0}} \] (17)

and renaming the parameters according to the formulas

\[ \alpha = \sqrt[3]{\frac{9}{16\rho_0}}c_0, \beta = -\frac{3\lambda}{8}\sqrt{6*\rho_0^5}, p_0 = -\frac{m}{4}\sqrt{6*\rho_0^2} \] (18)

4 The equation (26) of Valent’s commented paper attributed to Selivanova [17] is the special case (\( \mu = 1 \)) of equation (28) of our 1986 paper [1]. The solution of this equation in the form (27) of [5], claimed by Valent to be new, is exactly the same as the solution given, 24 years earlier, by the first equations of (33) and (34) of [1] for \( \mu = 1 \). The same equation with more details on the process of solution is given in [4], where also Selivanova’s work was commented in §3.
5 In the last page of the commented paper, while he counts known related integrable systems, the author of [8] mentions Goriachev-Chaplygin’s, Dullin-Matveev’s and Goriachev’s cases, but without reference what sover for the first and third cases. In fact, both those cases were obtained for the first time as special cases of a more general integrable system admitting a cubic integral in our 1986 work [1]. In our later paper [4] both systems were significantly generalized by adding several parameters invoking additional potential and gyroscopic forces.

6 In spite of the facts that:

(a) our works [4] and [1] were published 8 and 24 years earlier than the commented article [8],

(b) our works [4] and [1] contain essential results of [8] as special cases,

(c) the authors of the work [18], which is cited in [8] as ref. 4, devote a special section (§4. Remark) at the end of their article to acknowledge that the integrable system studied by them is contained in our results in [4],

neither of our works is mentioned in [8].

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4 Appendix

The MAPLE code below is used to directly and explicitly show how to get the second case of Valent's work [9] (see section 2.2 above) as special case of our MASTER system:

\[ \begin{align*}
Lv := & -1/2*ft^2*(-x^2-2d)*(-x^2-2d)*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2
-4*r*c1*x+d^2-4*r*c0)/(x^4-4*r*alpha*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2-4*r*c1*x+d^2-4*r*c0)/r
+1/2*xt^2*(-x^4-4*r*alpha*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2-4*r*c1*x+d^2-4*r*c0)/(1/2)+x^2+d+(-4*r*alpha*x^3-4*x^2-4*r*c1*x+d^2-4*r*c0)/r
+1/2*xt^2*(-x^4-4*r*alpha*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2-4*r*c1*x+d^2-4*r*c0)/(1/2)+x^2+d+(-4*r*alpha*x^3-4*x^2-4*r*c1*x+d^2-4*r*c0)/r
+1/2*xt^2*(-x^4-4*r*alpha*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2-4*r*c1*x+d^2-4*r*c0)/(1/2)+x^2+d+(-4*r*alpha*x^3-4*x^2-4*r*c1*x+d^2-4*r*c0)/r
+1/2*xt^2*(-x^4-4*r*alpha*x^3+(-4*c2*r+2*d+4*alpha^2*r^2)*x^2-4*r*c1*x+d^2-4*r*c0)/(1/2)+x^2+d+(-4*r*alpha*x^3-4*x^2-4*r*c1*x+d^2-4*r*c0)/r
\end{align*} \]
