THE INTERSECTION GRAPH OF AN ORIENTABLE GENERIC SURFACE

DORON BEN HADAR

Abstract. I answer an open question left by Gui-Song Li in [1]. The intersection graph $M(i)$ of a generic surface $i : F \to S^3$ is the set of values which are either singularities or intersections. It is a multigraph whose edges are transverse intersections of two surfaces and whose vertices are triple intersections and cross-caps. $M(i)$ has an additional structure which Li called "a daisy graph." If $F$ is oriented then the orientation further refines $M(i)$’s structure into what Li called an "arrowed daisy graph."

Li left the open question "which arrowed daisy graphs can be realized as the intersection graph of an oriented generic surface?" The main theorem of this article will answer this. I will also provide some generalizations and extensions to this theorem in sections 4 and 5.

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1. The structure of the intersection graph

Definition 1.1. 1) A proper generic surface in a 3-manifold $M$ is a smooth mapping $i : F \to M$, where $F$ is a compact surface (called the underlying surface) and each value of $i$ has a neighbourhood $U$ in $M$ such that $U \cap i(F)$ looks like one of the pictures in figure 1 (The purple part is $\partial M$.)
Figure 1

[1A], [1B] and [1C] are (respectively) a regular value, a double value and a triple value. Locally they look like the intersection of 1, 2 or 3 of the coordinate planes in a neighbourhood of the origin in \( \mathbb{R}^3 \). [1D] is a cross-cap, locally, it looks like the renowned "Whitney’s umbrella". [1E] is a regular boundary (RB) value and [1F] is a double boundary (DB) value, they look like the intersection of 1 or 2 of the \([xy]\) and \([yz]\) planes with the boundary (the \([xz]\) plane) in a neighbourhood of the origin inside the upper half space.

Proper generic surfaces are “proper functions” in the sense that \( i(\partial F) = i(F) \cap \partial M \). They are “generic” in the sense that every smooth and proper function from a compact surface to \( M \) can be turned into a proper generic surface via an arbitrarily small perturbation. They are also “stable” in the sense that a small enough perturbation can only change \( i \) up to an isotopy of \( F \) and \( M \). If \( F \) is closed we can call \( i \) a ”closed generic surface”.

2) A co-orientation on \( i \) is a continuous choice, for every non cross-cap \( x \), of a normal vector in \( T_{i(x)}M \) that is orthogonal to \( D_i(T_x F) \). If \( M \) is oriented then there is a 1-1 correspondence between orientations on \( F \) and co-orientation on \( i \). It matches each orientation on \( F \) with the normal \( \overrightarrow{n} \) for which \((D_i(\overrightarrow{v}_1), D_i(\overrightarrow{v}_2), \overrightarrow{n})\) upholds the orientation of \( M \) whenever \((\overrightarrow{v}_1, \overrightarrow{v}_2)\) upholds that of \( F \).

The set \( M(i) = \{ p \in i(F) | 1 < |i^{-1}(p)| \} \) is called the intersection set of \( i \). It is the set of all values which are not regular nor RB. As we can see in figure 1B (in orange), the neighbourhood in \( M(i) \) of a double value is a "line segment" made from double values, which I call "a double line". These segments form long lines called "double arcs", and \( M(i) \) is the union of all these arcs. As we can see in figures 1D and 1F, such an arc may end in a cross-cap or a DB value on each side of it, but it can also close up into a circle. We call arcs of the latter kind "closed" and of the former type "open". In figure 1C we see that 3 "segments" of double arc (marked in orange, red and green) intersect in each triple value. It’s possible that these segments come from the same double arc which intersects itself. Double arcs are immersed but not necessarily embedded 1-manifolds.

From another point of view, \( M(i) \) can be regarded as a graph whose vertices are the triple values (degree 6), cross-caps and DB values (degree 1) of \( i \) and whose edges are the segments of double line between two vertices. In addition to this graph, \( M(i) \) may contain several "double circles" - closed double arcs that do not pass through triple values. They are not made of the vertices and edges of the
aforementioned "graph," and so they need to be accounted for separately. In graph diagrams of $M(i)$ they appear as circles, disjointed from the graph.

**Definition 1.2.**

1) Each edge of $M(i)$ has two ends. I call them "half edges." When an arc passes through a triple value, one half-edge enters the triple value and another half-edge (from either the same edge or a different one) exists it. We say that these two half-edges are "consecutive."

2) Notice that each of the three arcs that pass through a triple value is: a) made from the intersection of two of the three planes that intersect in the value, and b) intersect the remaining plane transversally. If $i$ is co-oriented then, as in figure 2, the normal arrows on this last plane point towards one direction on this arc or, equivalently, towards one of the two consecutive half-edges. We say that the half-edge the arrow points toward is the "preferred" one of the two.

![Figure 2](image)

All this information gives $M(i)$ the structures of what Li calls a daisy graph and an arrowed daisy graph:

**Definition 1.3.**

1) A daisy graph (DG) $(V, E, n, B, C)$ is a 5-tuple where $(V, E)$ is multigraph whose vertices are all of degree 1 or 6. $n$ is a non-negative integer. For the DG of a generic surface, $n$ will indicate the number of double circles the surface has. In diagrams, we draw $n$ circles next to the graph. $B$ is a subset of the set of degree 1 vertexes that, for the DG of a generic surface, will be the set of DB values (and the other degree 1 vertices are the cross caps.) In diagrams, we indicated the vertices of this set in purple.

Lastly, for each triple value $v$ we divide its 6 half-edges into 3 pairs. $C(v)$ is the set of these pairs and, formally, $C$ itself is a function from the set of degree 6 vertices to the power set of the power set of the set of half-edges. In the DG of a generic surface the degree 6 vertices will be the triple values and these pairs will be the pairs of consecutive half-edges. In diagrams, we draw degree 6-vertices as intersections of 3 line segments such that each segment is cut into a pair of consecutive half-edges. (See Figures 3A and 3B for some examples of DG’s.)

2) An arrowed daisy graph (ADG) $(V, E, n, B, C, A)$ is a 6-tuple where $(V, E, n, B, C)$ is a DG and $A$ is a function from $\bigcup_v C(v)$ to the set of half-edges that assigns each pair of consecutive half-edges a chosen half-edge $A(p)$ from the pair $p$. In the ADG of a co-oriented generic surface this will be the preferred half-edge. In diagrams, we mark this half-edge with an arrow. (See Figure 3C, 3D and 3E for some examples of ADG’s.)
Remark 1.4. 1) Li defines DG’s and ADG’s in the same way in [1], but his don’t have cross-caps or DB values. As a result, all the double arcs of Li’s graphs are closed.

2) Despite the similarity to graphs on surfaces, daisy graphs are not planar. One arc can go “above” another. I mark it like a crossing in a knot (See figure 3 b,d,e) to avoid confusion, but it is not a real crossing - it does not matter which arc is above and which is below.

In [1] Li used ADG’s to prove that ”a DG (with no DB values or cross-caps) can be realized as the intersection graph of some proper generic immersion $i : F \rightarrow S^3$ where $F$ is a closed orientable surface” iff ”every (closed) arc is made of an even number of edges.” However, he hadn’t find which ADG’s are realizable (an ADG can be realized via a generic surface with a co-orientation) and left it as an open question. The main purpose of this article is to answer this in a generalized capacity. I begin with a definition:

Definition 1.5. A grading of an ADG is a choice of a number $g(e) \in \mathbb{Z}$ (called the grade) for every edge $e$ such that, at each triple value $v$, all the edges that have ”unpreferred ends” around $v$ have the same grade $a(v)$ and all edges that have ”preferred ends” around $v$ have the grade $a(v) + 1$.

And now I can formulate the main theorem:

Theorem 1.6. An ADG is realizable as the intersection graph of a proper generic surface $i : F \rightarrow D^3$, with an orientation on $F$ that induces the arrows iff it is gradable. Furthermore, we can make sure $F$ is connected and we can replace $D^3$ with any orientable 3-manifold $M$ with a boundary for which $H_1(M; \mathbb{Z})$ is periodic (all its elements are torsion). Lastly, if the ADG has no DB values, we may also replace $D^3$ with $S^3$ or any boundaryless $M$ with a periodic $H_1$.

Remark 1.7. 1) From this theorem we can deduce an extension of Li’s theorem which applies to DG’s with cross-caps and DB values. If a DG is realizable via an orientable generic surface then the orientation gives the DG an ADG structure (arrows) and this ADG is realizable and therefore gradable. The grading of each following edge on an arc will have a different parity than the grading of the previous edge and, in particular, closed arcs must have an even number of edges on them.

On the other hand, given a DG that upholds this condition (every closed arc must have an even number of edges) we can give the DG a ”short grading” - numbering the edges with only 0 and 1 so that consecutive edge have different numbers (clearly the only obstruction to this are closed arcs with an odd number of edges.) To match this, we give every triple value arrows that point toward the edges with the number...
1. This will make the DG a graded ADG that can be realized as the intersection graph of some closed generic surface, and in particular, will mean that the DG is realizable.

2) Given any grading of an ADG, we can get another grading by adding a constant to the grade of every edge in a chosen connected component of the graph.

3) To see if an ADG is gradable, we try to grade it then see if we encounter obstructions. Specifically, we take a spanning forest of the graph and choose one edge in each tree. Using (2) we can assume WLOG that the grade of these chosen edges should be 0. If any chosen edge \( e \) ends in a triple value, then the grade of any edge \( f \) in the spanning forest, that ends in this same triple value, must have the following values: 
   \[ g(f) = g(e) + 1 \] if \( f \) is preferred at the triple value and \( e \) is not, 
   \[ g(f) = g(e) - 1 \] if it is the other way around, and 
   \[ g(f) = g(e) \] if, either both of them are preferred, or both of them are not.

We go over the whole spanning forest and every time we reach a new triple value we use this law to find the grade of all the edges that end in it. Next, we try to do this for the edges outside the spanning forest, but since each of these edges closes a cycle in the graph, both ends of the edge must lie on triple values we already visited. Each of these triple values will impose a grade number on the edge. If they both give it the same number then this should indeed be the grade of the edge, otherwise we found an obstruction and the ADG is not gradable.

In figure 4 we see a ADG with a grading (4A) and without one (4B). It has no grading since, according to the upper triple value, the red and green edges should have the same grading, and according to the lower, one they don’t.

4) Both ends of a loop (edge that starts and ends in the same vertex) in a gradable ADG must enter its vertex from the same direction, either with or against the arrow, as in figure 4D, and not like in figure 4C.

I will prove the ”only if” theorem in section 2 and the ”if” direction in section 3. In section 4 we will see which ADG’s are realizable in 3-manifolds whose first homology group is not periodic. In section 5 I will refine theorem by adding more information to the structure the intersection graph.

2. Winding numbers

**Definition 2.1.** Let \( i : F \to M \) be a proper generic surface in a 3-manifold \( M \).

1) A face (resp. body) of \( i \) is a connected component of \( i(F) \setminus M(i) \) (resp. \( M \setminus i(F) \)).

2) Each face \( V \) is an embedded surface in \( M \), and there is a body on each side of it. We say these two bodies are adjacent (via \( V \)). A priori, it is possible that these two bodies are in fact two parts of the same body and even that \( V \) is a one-sided surface. In these cases this body will be self adjacent, but this does not happen in any cases we are interested in.
3) If \( i \) has a co-orientation then each face \( V \) is two sided, and the arrows on the face point towards one of its two sides. We say that the body on the side the arrows points toward is "greater" (via \( V \)) than the body on the other side of \( V \).

4) A choice of "winding numbers" for \( i \) is a choice of an integer \( w(U) \in \mathbb{Z} \), for every body \( U \) of \( i \), such that if \( U_1 \) and \( U_2 \) are adjacent, and \( U_1 \) the greater of the two, then \( g(U_1) = g(U_2) + 1 \).

**Lemma 2.2.** If \( M \) is an orientable 3-manifold, \( H_1(M;\mathbb{Z}) \) is periodic, and \( i : F \rightarrow M \) is a co-oriented generic surface then \( i \) has a choice of winding numbers.

**Proof.** Pick one body \( U_0 \) to be "the exterior" of the surface and set \( w(U_0) = 0 \). Then define the winding numbers for every other body \( U \) like so:

Take a smooth path from \( U_0 \) to \( U \) that is in general position to \( i(F) \) only at faces, and does so transversally) and set \( w(U) \) be the signed number of times it crosses \( i(F) \), the number of times it intersects it in the direction of the co-orientation minus the times it crosses it against the co-orientation. This is well defined since any two such paths \( \alpha \) and \( \beta \) must give the same number. Otherwise, the composition \( \beta^{-1} * \alpha \) is a 1-cycle whose intersection number with the 2-cycle represented by \( i \) is non-zero. This implies that this 1-cycle is of infinite order in \( H_1(M;\mathbb{Z}) \) - contradicting the "periodicness" of this group.

It is also clear that if \( U_1 \) and \( U_2 \) are adjacent and \( U_1 \) is greater then \( g(U_1) = g(U_2) + 1 \). \( \square \)

**Remarks 2.3.**

1) It is clear that two different choices of "winding numbers" for \( i \) will differ by a constant, and that the one we created is unique in satisfying \( w(U_0) = 0 \).

2) We can do a similar process on a loop \( \gamma \) in \( \mathbb{R}^2 \) instead of a surface in a 3-manifold. If we choose the component \( U_0 \) of \( \mathbb{R}^2 \setminus \text{Im}(\gamma) \) to be the actual exterior then we will get usual winding numbers - \( w(U) \) will be a the number of times \( \gamma \) winds around a point in \( U \).

We can use the winding numbers to induce a grading as follows: The neighbourhood of a double value includes 4 bodies, with the possibility that some of them are in fact different parts of the same body. If the surface has a co-orientation and winding numbers then there is a number \( g \) such that two of these bodies have the WN \( g \), one has the WN \( g + 1 \) and one has the WN \( g - 1 \). Figure 5A depicts this:

Due to continuity, this will be the same value number \( g \) for all the double values on the same edge (or double circle). We call this number the grading of the edge, and name the grading of an edge \( e \) \( g(e) \). This is indeed a grading in the sense of definition 1.5. To prove this, we need to see that at every triple value of the surface all the preferred half-edges have the same grading, which is greater by 1 than the grading of all the unpreferred ones. We can see this in figure 5B, which depicts the winding numbers of the bodies around an arbitrary triple value. Indeed, we can
see that the preferred half-edges - the ones going up, left and outwards (toward the reader) have the grading \( g + 1 \), while the other edges have the grading \( g \). This proves the "only if" direction of theorem L.6.

3. Realising graded ADG's

To prove the "if" direction of theorem L.6 I will first prove a partial result. I will limit the discussion to connected ADG's with no DB values.

**Lemma 3.1.** Every connected, gradable ADG \( G \) without DB values has a closed generic surface \( i : F \to S^3 \) such that the intersection graph of \( i \) is equal, as an ADG, to \( G \).

**Remark 3.2.** We can also modify the generic surface to make \( F \) connected without affecting the intersection graph:

1) If \( i(F) \) is not connected then one of its components will contain the connected intersection graph and the rest will be (some) embedded connected surfaces in \( M \). We can delete the preimages of these components from \( F \), which will make \( i(F) \) connected.

2) When \( i(F) \) is connected, if \( F \) has more than one connected component, then the images of some pair of connected components must intersect generically at a double line. The left part of figure 6 depicts this, where the vertical surface comes from one connected component of \( F \) and the horizontal comes from another. We may connect them via a handle in an orientation preserving way, as in the right part of figure 6, thereby decreasing the number of connected components of \( F \). We continue in this manner until \( F \) is connected.

![Figure 6](image)

I begin with the unique case where the ADG is a double circle. The generic surface from figure 7A has a single double circle as its intersection graph. It is the surface of revolution of the curve from figure 7B around the blue axis. Both figures have indication for the co-orientation. The intersection graph will be the revolution of the orange dot where the curve intersects itself, and will thus be a circle. The underlying surface is clearly a sphere.

![Figure 7](image)

Any other connected ADG is a "graph ADG" - it will have no double circles. In this case, we begin by constructing a part of the generic surface we want - the regular
neighbourhood of the intersection graph. We have an intuitive understanding of how such a neighbourhood should look like. Li defined something similar in [1] which he called a "cross-surface", and I will use the same notation.

**Definition 3.3.** Given a connected ADG \( G \) with no DB values, that is not a double circle, a "cross-surface" \( X_G \) of \( G \) is a shape in \( S^3 \) that is built via the following two steps:

1) For every triple value \( v \) of \( G \) we embed a copy of figure 8A in \( S^3 \). The shape is called called the "vertex neighbourhood" of \( v \). Similarly, for every cross cap \( v \) of \( G \), we embed a vertex neighbourhood that looks like figure 8B in \( S^3 \). We make sure the different vertex neighbourhood will be pairwise disjoint. These will be the neighbourhoods of the actual triple values and cross-caps of the surface we are constructing. It is important to remember which vertex neighbourhood correspond to which vertex of \( G \).

![Figure 8](image)

From the above we deduce that each such "orange zone" should correspond to a half-edge that ends in \( v \). Similarly to the way in which we identified each vertex neighbourhood with a vertex of \( G \), we identify each orange zone with a specific half-edge of the corresponding vertex. We want this association to reflect the ADG structure of \( G \) and so, in accordance with definitions 1.2 and 1.3, we make sure that for every triple value:

(a) Two orange zones on opposite sides of the values neighbourhood, like those marked red and green in figure 8C, will correspond to a pair of consecutive half-edges.

(b) In compliance with the little arrows on the vertex neighbourhoods, which represents the co-orientation the surface is aught to have, we need the zones that the arrows point toward - those marked with the number 1 in figure 8C, to correspond to the preferred half-edges. The other zones, marked with 0, will correspond to non-preferred half-edges.
2) In step 1 we realized the vertices of $G$. We want to move on to the edges. As mentioned above, the ends of each edge are inside vertex neighbourhoods and each such end is bounded by an orange zone. We want to add the "length of the edge" to our construction. This should be a double line, the intersection of two surfaces, as in figure 9A. So, for each edge $e$ of $G$ we take a copy of this shape and embed it in $S^3$. This embedding must follow the following rules:

(a) The shape in figure 9A is a bundle over a closed interval whose fibre looks like the "X" in figure 8B. We call this an "X-bundle." The boundary of this shape is composed of two parts: i) the fibres at the ends of the interval, coloured orange, and ii) the (union of the) ends of all the fibres, coloured blue. It is natural to identify the two orange fibres with the two ends (half-edges) of our edge $e$.

Up to now, we identified each half-edge of the ADG with both i) an "orange zone" on the boundary of the neighbourhood of some vertex and b) a fibre at the end of some X-bundle. We make sure to embed the X-bundles so that each end fibre coincides with the matching orange zone. Additionally, we make sure that the "length" of the X-bundle (the X-bundle minus the end fibres) is disjoint from the vertex neighbourhoods, and that X-bundles of different edges do not touch one another.

The resulting shape is the cross-surface. It is similar to a generic surface but it has a boundary - the union of all the "blue parts" of the boundaries of the vertex neighbourhoods and the X-bundles. The intersection graph of this "generic surface with a boundary" is clearly isomorphic (as a multigraph) to $G$ - we already identified each vertex / edge of it with a unique vertex / edge of $G$ and made sure that each edge ends in the vertices it should properly end in. Rule (a) from step (1) implies that this identification will preserve the consecutive pairs of half-edges. This means that the intersection graph is isomorphic to $G$ as a DG, not just as a multigraph.

(b) To have an ADG structure, we need the cross surface to have a co-orientation. Notice that both the vertex neighbourhoods and the X-bundles have arrows on them, representing co-orientations. When we embed the X-bundles, we want the co-orientations on them to match those on the vertex neighbourhoods, as in figure 10A, unlike figure 10B. This way they will merge into a continuous co-orientation on the entire cross surface.

Figure 10

Rule (b) from step (1) implies that the preferred half-edges of the intersection graph will correspond the preferred half-edges of $G$. This means that the intersection graph will be isomorphic to $G$ as an ADG as well.

The boundary of the cross-surface is the union of many embedded intervals in $S^3$ - the "blue parts" of the boundaries of the vertex neighbourhoods and the X-bundles. Since each end of every interval coincides with an end of one other interval, and the intervals do not otherwise intersect, their union is an embedded compact
1-manifold in $S^3$. The cross-surface induces an orientation on this 1-manifold, the usual orientation that an oriented manifold induces on its boundary. It is depicted in the left part of figure 11A.

I will show that the boundary of the cross surface is also the oriented boundary of an embedded surface, a "not necessarily connected" one, which is disjoint from the cross surface. It follows that the union of the cross surface and the embedded surface, with the orientation on the embedded surface reversed, will be a closed and oriented generic surface whose intersection graph will be isomorphic to $G$. This will prove lemma 3.1.

Figure 11

So, how do we prove that such an embedded surface exists? We begin by "thickening" the cross surface as in figure 11A. Figure 11A only shows how to do this to an X-bundle, but we can similarly do this to all the vertex neighbourhoods. This results in a handle body $H$ in $S^3$ and our 1-cycle is on its boundary. It will suffice to prove that the 1-cycle is the boundary of some embedded surface in the complement of $H$. All an embedded 1-manifold in the boundary of a 3-manifold needs in order to be the boundary of a properly embedded surface in the 3-manifold is to be a "boundary" in the homological sense. So all we need to prove is that the boundary of the cross surface is equal to 0 in $H_1(S^3 \setminus H; \mathbb{Z})$.

Given any loop $\gamma$ in the intersection graph, we can define a functional $f_\gamma : H_1(S^3 \setminus H; \mathbb{Z}) \to \mathbb{Z}$ such that $f(c)$ is the linking number of $\gamma$ and a representative of $c$. It is well defined since cycles in $S^3 \setminus H$ are disjoint from $\gamma$ and since the linking number of $\gamma$ with any boundary in $H_1(S^3 \setminus H)$ is 0, as the boundary bounds a surface in $S^3 \setminus H$ which is disjoint from $\gamma$.

If the genus of $G$, and therefore of the intersection graph and of $H$, is $n$, then the intersection graph has $n$ simple cycles $C_1, \ldots, C_n$ such that each cycle $C_i$ contains an edge $e_i$ that is not in any of the other cycles. For every cycle $C_i$ we can take a small meridian $m_i$ around the edge $C_i$ (as depicted in red in figure 11B.) It follows that $f_{C_i}(\{m_j\}) = \delta_{ij}$ where $\delta$ is the Kronecker delta function. Additionally, since $S^3 \setminus H$ is the compliment of an $n$-handle body, $H_1(S^3 \setminus H) \equiv \mathbb{Z}^n$. I will prove that:

**Lemma 3.4.** These meridian form a base of $H_1(S^3 \setminus H)$.

**Proof.** Firstly, we show that the meridians are independent. This is because a boundary $S^3 \setminus H$ in would have 0 as the linking number with every $e_i$, but the linking number of a non trivial combination $x = \sum a_i[m_i]$ with any $c_j$ will be $a_j$, and for some $j a_j \neq 0$. Secondly, notice that this imply that $N = \text{Span}_\mathbb{Z}\{[m_1], \ldots, [m_n]\}$ is a maximal lattice in $H_1(S^3 \setminus H) \equiv \mathbb{Z}^n$, and therefore has a finite index.

Thirdly, had $N$ been a strict subgroup of $H_1(S^3 \setminus H; \mathbb{Z})$, then there was an elements $y \in H_1(S^3 \setminus H; \mathbb{Z}) \setminus N$. Define $b_i = lk(y, c_i)$ and $y' = y - \sum_{i=1}^n b_i[m_i]$. $y'$ will have 0 as the linking number with every $e_i$, but it will not belong to $N$. The
finite index of $N$ implies that $ky' \in N$ for some $k$, but $lk(ky', c_i) = k0 = 0$ for all $i$, and thus $ky' = 0$. This means that $y'$ is a non-zero element of finite order in $H_1(S^3 \setminus H; \mathbb{Z}) \cong \mathbb{Z}^n$, but no such element exist. □

Lemma 3.5. Let $G$ be a connected ADG that has no DB values, is not a double circle, and is gradable. Then the linking number of the boundary of its cross surface with any simple cycle in the intersection graph of this cross surface is 0.

Proof. We can calculate the linking number of the boundary of the cross-surface with any loop that is disjoint from this boundary by doing the following: a) moving the loop homotopically until it is in general position competed to the cross surface, making sure that the loop does not intersect the 1-cycle during any part of the homotopy, and b) calculating the intersection number of the "moved" loop with the cross-surface.

Let $C$ be a simple cycle in the intersection graph that is composed of the (distinct) edges $e_1, ..., e_n = e_0$, and let $v_i$ be the vertex it passes from $e_{i-1}$ to $e_i$. Each $v_i$ is a triple value since it is not a degree 1 vertex. We want to move $C$ into general position compared to the cross-surface. We begin by pushing each edge $e_i$ away from its matching X-bundle in a direction that agrees with the co-orientation on both of the surfaces that intersect in this X-bundle, as in figure 12.

![Figure 12](image)

We need to continue this "pushing" at the vertex neighbourhood of each $v_i$. In figures 13-15 we see how we push away the half-edges from their original position. The half-edges we push are coloured green, and the arrows on them indicate the direction of the cycle - the half-edge whose arrow points toward (resp. away from) the triple value is a part of $e_{i-1}$ (resp. $e_i$). Continuity dictates that we always push in the direction indicated by the orientations on the surface as we did in figure 12, and we can see that figures 13-15 comply with this.

Each of the three figures depict a different situation with regards to which of the two half-edges, if any, is preferred at $v_i$. Figure 13 depicts the case where both the half-edges are preferred, and we can see that when we push $C$ away from the cross-surface it will not intersect the cross surface at the neighbourhood of $v_i$.

![Figure 13](image)

In figure 14 we see that the half-edge that is a part of $e_{i-1}$, the one entering the triple value, is not preferred, and the half-edge that is a part of $e_i$, the one exiting
the triple value, is preferred. In this case we see that when we push away $C$ it will intersect the cross-surface once, and it will do so agreeing with the direction of the co-orientation on the surface (that is why there is a little $+1$ next to the intersection.)

Figure 14

Figure 14 depicts the case where the two half-edges are not consecutive, but even if they were, the same thing would happen - $C$ would intersect the cross-surface once, in agreement with the co-orientation. The only difference would be that the half-edge that was exiting $v_i$ would have continued leftwards instead of turning outwards towards the reader. Furthermore, had the half-edge coming from $e_{i-1}$ been preferred and the one coming from $e_i$ hadn’t then the pushing would still occur as in figure 14 except that the arrows on the green line will point the other way. In this case $C$ will still intersect the cross surface once after the pushing, but it will be against the direction on the co-orientation.

Lastly, figure 15 deals with the case in which both half-edges are not preferred. We see that, after the pushing, $C$ will intersect the cross-surface twice in the neighbourhood of $v_i$. One intersection, marked $+1$, is in the direction of the co-orientation, and the other intersection, marked $-1$, is against it.

Figure 15

Let $G$ be a grading of the intersection graph. Since $e_{i-1}$ and $e_i$ share a vertex the difference between their grading is at most 1. If $g(e_i) - g(e_{i-1}) = 1$ (resp. $-1$) then $e_i$ (resp $e_{i-1}$) is preferred and $e_{i-1}$ (resp. $e_i$) isn’t, and we have seen that in this case the signed number of intersections between the ”pushed away” $C$ and the cross surface is 1 (resp. $-1$). If $g(e_i) - g(e_{i-1}) = 0$ then either both $e_i$ and $e_{i-1}$ are preferred, in which cases $C$ does not intersect the cross surface around $v_i$, or they are both unpreferred, in which case they intersect once with and once against the co-orientation.

It follows that in all cases the signed number of intersections between the pushed $C$ and the cross surface around $v_i$ is equal to $g(e_i) - g(e_{i-1})$. Since this pushed $C$ is in general position to the cross surface, and they only intersect at the vertex neighbourhoods of the $v_i$’s, their intersection number is $\sum_{i=1}^{n} (g(e_i) - g(e_{i-1})) = g(e_n) - g(e_0) = 0$. Since $C$ did not cross the the boundary of the cross surface during the pushing, this (0) is equal to the linking number of $C$ and the boundary.
We have proved lemmas 3.5 and 3.1 can now prove the "if" direction of theorem 1.6:

Proof. a) We first prove that a gradable ADG $G$ with no DB values can be realized in any 3-manifold $M$, via a generic surface with a connected underlying surface: $G$ has a finite number of connected components $G_1, ..., G_r$, all of which are gradable ADG's with no DB values. By lemma 3.1 each $G_k$ can be realized via a closed generic surface $i_k : F_k \to S^3$, and by remark 3.2 we can assume the $F_k$'s are connected.

Pick a face $v_k$ of each $i_k$. The co-orientation on $v_k$ point towards a on body $U_k$. Pick a point in $U_k$ and remove it from $S^3$ in order to regard $i_k$ as a surface is $\mathbb{R}^3 \equiv S^3 \setminus \{p\}$, whose exterior body is $U_k$ (minus a point.) Embed these $r$ copies of $\mathbb{R}^3$ in $M$, next to each other, to get a generic surface that realizes $G$ in $M$.

The surface has one "exterior body" $A$ that contains all the $U_k$, and all the co-orientations on all the $V_k$'s will point towards this body. We can connect each $V_k$ to $V_{k+1}$ (for $k < r$) with a handle going through $A$ as in figure 18 (ignore the "A" and "B" at the bottom of the drawing.) This changes the underlying surface from a disjoint union of all the $F_k$'s to their connected sum.

b) We move on the ADG's with DB values, which we need to embed in any 3-manifold with a boundary. Given such an ADG $G$, define a similar ADG $G'$ where each DB value is replaced with a cross-cap. Realize $G'$, via (a), with a closed generic surface $i : F \to S^3$ for which $F$ is connected.

Take a small ball around each of the cross-caps that replaces a DB value of $G$, as in figure 16A. Figure 16B depicts the intersection of the surface with the boundary of the ball. It is an "8-figure" as in figure 16C, and the orange dot (the intersection in the 8-figure) is the intersection of the boundary with the intersection graph. If we remove this ball from $S^3$, then instead of ending in the cross cap, the edge will end in the orange dot in the 8-figure, which will become a DB value. It follows that after removing all these balls the intersection graph will be ADG isomorphic to $G$.

We now have a generic surface, in $S^3$ minus some number $n \geq 1$ of balls, whose underlying surface is connected and whose intersection graph is ADG isomorphic to $G$. We can think of our 3-manifold as $D^3$ minus $n - 1$ balls and we want to turn it into $D^3$.

For this, pick a path from the boundary of any of these $n - 1$ balls into the exterior boundary of our $D^3$ and make sure the path is in general position to the generic surface - it may intersect it only at faces and will do transversally. Thicken these paths into narrow 1-handles and remove them from our 3-manifold. The resulting 3-manifold will be diffeomorphic to $D^3$. This will cut from the surface its intersection with the 1-handles, but these will be small discs disjoint from the
intersection graph, so removing them will not change the intersection graph nor make the underlying surface not connected.

We now have a generic surface in $D^3$ whose underlying surface is connected and whose intersection graph is $G$. We can remove a point from the boundary of $D^3$, making it diffeomorphic to the closed half space $\{(x, y, z) \in \mathbb{R}^3 | z \geq 0\}$ which can be properly embedded in any 3-manifold with a boundary. This finishes the proof. □

We now know which ADG’s can be realized via a generic surface in an oriented 3-manifold with a periodic first homology group. But what about 3-manifolds with a non-periodic $H_1$? We answer this in the next section, although we have to limit ourselves to 3-manifolds that are also compact:

4. Infinite homology

**Theorem 4.1.** If $M$ is an oriented, compact and boundaryless 3-manifold with an infinite first homology group, then any ADG $G$ with no DB values can be realized as the intersection graph of an oriented generic surface in $M$. If $M$ has a boundary then any ADG $G$ can be realized in $M$. As in theorem 1.6 we can require the underlying surface of the generic surface to be connected.

We prove two lemmas first:

**Lemma 4.2.** $M$ has a connected, compact, oriented and properly embedded surface $S \subseteq M$ that is non-dividing ($M \setminus S$ is connected.)

**Proof.** $H_2(M; \mathbb{Z})$ is generated by 2-cycles of the form $[S]$ where $S \subseteq M$ is connected, compact, oriented and properly embedded surfaces. If the statement of the lemma is false then each such surface divides $M$ into two connected components and will therefore be a boundary in $H_2(M; \mathbb{Z})$. This implies that $H_2(M; \mathbb{Z}) \equiv \{0\}$. Via Poincaré’s duality $\{0\} \equiv H_2(M; \mathbb{Z})/\text{Tor}(H_2(M; \mathbb{Z})) \equiv H_1(M; \mathbb{Z})/\text{Tor}(H_1(M; \mathbb{Z})).$ That implies that every element of $H_1(M; \mathbb{Z})$ is of finite order, contradicting the assumption. □

**Lemma 4.3.** If $G$ is gradable, then there is a generic surface $i : F \rightarrow M$ which realizes $G$ and for which $M \setminus i(F)$ is connected (equivalently, $i$ has only one body.)

**Proof.** Take the generic surface $S \subseteq M$ from lemma 4.2 and a subset $M' \subseteq M$ that is disjoint from $S$ and is homomorphic to a half-space (if $M$ has a boundary) or to $\mathbb{R}^3$ (if it does not.) By theorem 1.6 there is a generic surface $i : F \rightarrow M'$ which realizes $G$ and for which $F$ is connected. Connect some face $V$ of the generic surface to $S$ with a handle as in figure 17 (the handle does not intersect $i(F)$ or $S$.) If needed, reverse the co-orientation of $S$ so that the resulting surface will be continuously co-oriented.

![Figure17](image-url)
We now have a new generic surface $i' : F\#S \to M$ whose intersection graph is still isomorphic to $G'$. Since $S$ was non-dividing, the connected sum of $V$ and $S$ is a face of this surface that has the same body $A$ on both sides (As indicated by the green path in figure 17, which does not intersect the surface). If this is the surface's only body we are done. If not, we will decrease the number of bodies.

Let $B$ be another body of the surface that is adjacent to $A$. Connect the face $W$ which separates $A$ and $B$ to the face $V\#S$ with a path that goes trough $A$ and does not intersect out generic surface except at the ends of the path. Since $V\#S$ has $A$ on both sides, we can approach it from either side we want. If the arrows on $W$ points toward $A$ (resp. $B$) make sure the path enters $V\#S$ from the direction the arrows point towards (resp. point away from). We then attach the faces $V$ and $W$ with a handle that runs along this path. Figure 18 depicts the case there the arrows on $W$ points towards $A$. Reverse the direction of all arrow to see the other case.

The bodies $A$ and $B$ have been connected into one body, and so we get a generic surface in $M$ that has one less body. This surface still realizes $G'$, have a connected underlying surface, and has a face with both bodies on the same side. Repeat this process until we get a surface with only one body. □

We can now prove theorem 4.1

**Proof.** Let $H$ be the graph part of $G - G$ without the double circles. We use induction on the genus of $H$. If the genus is 0 then $G$ is the union of a forest with some double circles, and remark [1.7.3] implies that it is gradable. By lemma 4.3 there is a generic surface in $M$ that realizes $G$, has a connected underlying surface, and has only one body.

If the genus of $H$ is positive, we pick an edge $e \in H$ such that $H \setminus \{e\}$ has a smaller genus, or equivalently such that the connected component of $H$ that contains $e$ is not divided when $e$ is removed. Note that both ends $e$ are on triple values, since cross-caps and DB values are of degree 1 and removing their single edge divides the graph.

Define a new ADG $G'$ like so: Start with a copy of $G$ and cut the edge $e$ in the middle. Instead of $e$ we get two “new edges” $e_1$ and $e_2$. Each $e_i$ has one end on a new cross-cap we added to the graph, while the other end “replaces” one of the ends of $e$ - it enters the triple value that the said end of $e$ was on, and it retains the ADG data - it is preferred if the iff the half-edge of $e$ was preferred, and it has the same consecutive half-edge. Figure 19 depicts the two possible ways to construct $G'$ from $G$. 
$H'$, the graph structure of $G'$, has a lower genus then $H$. We assume, via induction, that there is a generic surface in $M$ that: a) realizes $G'$, b) has a connected underlying surface, and c) has only one body. We will prove the same holds for $G$.

Look at the new cross-caps at the ends of $e_1$ and $e_2$. Change the surface in a small neighbourhood of each cross-cap as per figure 20A, deleting the cross-cap and leaving instead a "figure 8 boundary" of the surface.

This figure 8 boundary is depicted in figure 16C. Take a bundle over an interval whose fibres are "8-figures", as in figure 20B, and embed it in $M$ in such a way that its end-fibres coincide with the said "figure 8 boundaries" (in a way that preserves the arrows of the co-orientation.) Since the compliment of the surface (before we changed it) was connected, we can make sure that the bundle does not intersect the surface anywhere beside its ends.

This closes $e_1$ and $e_2$ into one edge, reversing the procedure that created $G'$ from $G$. It follows that the generic surface this created realizes $G$. It still has a connected underlying surface and only one body, and the proof follows via induction. □

5. Ordered daisy graphs

In this last section I will refine the graph structure of the intersection graph so that it better reflects its topology. Figure 21A depicts a neighbourhood of a triple value $v$ in a cross surface. We indexed the preferred half-edges there as $+1, +2, +3$ (and the corresponding unpreferred ones as $-1, -2, -3$.) To the end of each of the depicted half-edges is glued an end of some X-bundle. If for each $k = 1, 2, 3$ the "X-bundle ends" that are glued to the half-edges $\pm i$ were respectively glued instead to the half-edges $\pm \sigma(i)$ then we would still have gotten a "similar" cross-surface.
When I say that two cross surfaces are "similar" I mean that each has a neighbourhood in $M$ such that there is an orientation preserving homeomorphism between the neighbourhoods that sends one cross-surface to the other in a way that preserves the co-orientation on them. This homomorphism does not need to extend to all of $M$ since the cross surface retracts to the intersection graph and a graph with a positive genus can be knotted. Notice that when we defined cross surfaces in definition 3.3 we did not mind the exact way that the X-bundles where embedded in $S^3$, which clearly allows knotting.

The similarity that realizes $\sigma$ is a homeomorphism that acts as a rotation in the neighbourhood of the triple value (the rotation that sends figure 21A to figure 21B) and acts as the identity everywhere else. We cannot do the same with an odd permutation. We can define a similar homeomorphism via a reflection on the neighbourhood of the triple value (such as the reflection that sends figure 21A to figure 21C) but that will not preserve the orientation of $M$. The cross-surfaces may even not be homeomorphic as subsets of $M$. However, the intersection graphs of the two cross-surface will be ADG equivalent - they will have the same vertices and edges and the same preferred / consecutive half-edges. Figure 21C depicts this.

**Definition 5.1.** We can refine the graph structure of the intersection graph to reflect this information. We add, as a 7th entry to the 6-tuple of the ADG, a function that assigns each triple value an indexing of the preferred edges, which is unique up to an even permutation, such that using this indexing the triple value look like figure 21A (and not like figure 21C.) I call this the Ordered daisy graph (ODG) structure of the intersection graph.

Given an ODG, we can create a cross-surface that represents it like we did for ADG’s in definition 3.3. We just need to add the additional requirement that the X-bundles are attached to the triple values in the correct way. The benefit of using ODG’s is that two cross surfaces with the same ODG are similar. We can send each vertex neighbourhood / X-bundle in one cross surface homeomorphically to the corresponding vertex neighbourhood / X-bundle in the other cross surface in a way that preserves the gluings. Additionally, we can rephrase theorems 1.6 and 4.1 to consider ODG’s instead of ADG’s and they will still hold (the same proves will still work.)

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Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel 5290002

E-mail address: doron.ben@live.biu.ac.il

URL: math.biu.ac.il/node/641