Invariant scalar product on extended Poincaré algebra

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Abstract

Two methods can be used to calculate explicitly the Killing form on a Lie algebra. The first one is a direct calculation of the traces of the generators in a matrix representation of the algebra, and the second one is the usage of the group invariance of the scalar product. We use both methods in our calculation of the scalar product on the extended Poincaré algebra \( LG(P) \) in order to have a cross check of our results. The algebra is infinite-dimensional and requires careful treatment of the infinities. The scalar product on the extended algebra \( LG(P) \) found by both methods coincides and the important conclusion which follows is that Poincaré generators are orthogonal to the gauge generators.

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1. Introduction

The algebra is defined as follows [1–3]:

\[
[P^\mu, P^\nu] = 0,
\]

\[
[M^{\mu\nu}, P^\rho] = i(\eta^{\lambda\nu} P^\mu - \eta^{\lambda\mu} P^\nu),
\]

\[
[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\lambda\rho} M^{\nu\lambda} - \eta^{\lambda\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\rho\lambda} - \eta^{\nu\rho} M^{\lambda\lambda}).
\] (1.1)

\[
[P^\mu, L'^{\lambda_{1} \cdots \lambda_s}_{a}] = 0,
\]

\[
[M^{\mu\nu}, L'^{\lambda_{1} \cdots \lambda_s}_{a}] = i(\eta^{\lambda\nu} L'^{\mu\lambda_{2} \cdots \lambda_s}_{a} - \eta^{\lambda\mu} L'^{\nu\lambda_{2} \cdots \lambda_s}_{a} + \cdots + \eta^{\nu\lambda} L'^{\lambda_{2} \cdots \lambda_{s-1} \mu}_{a} - \eta^{\mu\lambda} L'^{\lambda_{2} \cdots \lambda_{s-1} \nu}_{a})
\] (1.2)

\[
[L'^{\lambda_{1} \cdots \lambda_s}_{a}, L'^{\lambda_{1} \cdots \lambda_s}_{b}] = i f^{\nu}_{ab} L'^{\lambda_{1} \cdots \lambda_s}_{c} \quad (\mu, \nu, \lambda, = 0, 1, 2, 3; s = 0, 1, 2, \ldots),
\] (1.3)

where the flat space-time metric is \( \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1) \) and \( f^{\nu}_{ab} \) are the structure constant of a compact Lie algebra \( L_G \). One can check that all Jacoby identities are satisfied and we have an example of a fully consistent algebra. The algebra \( L_G(P) \) incorporates the Poincaré
algebra $L_P$ and an internal algebra $L_G$ in a nontrivial way, which is different from the direct product. The generators $L_a^{\lambda_1\ldots\lambda_s}$ have a nonzero commutation relation with $M^{\mu\nu}$, which means that the generators of this new symmetry have nontrivial Lorentz transformation and carry a spin different from zero. The generators $L_a^{\lambda_1\ldots\lambda_s}$ in (1.3) commute to themselves forming an infinite series of commutators of current subalgebra which cannot be truncated, so that the index $s$ runs from zero to infinity. We have here an example of an infinitely-dimensional current subalgebra [4, 5].

The algebra is invariant with respect to the following gauge transformations:

$$L_a^{\lambda_1\ldots\lambda_s} \rightarrow L_a^{\lambda_1\ldots\lambda_s} + \sum_1^2 P^{\mu_1} L_a^{\lambda_1\ldots\lambda_{s-1}\lambda_{s+1}...\lambda_2} + \sum_{2}^5 P^{\mu_2} L_a^{\lambda_1\ldots\lambda_s} + \ldots + P^{\mu_5} L_a^{\lambda_1\ldots\lambda_s},$$

$$M^{\mu\nu} \rightarrow M^{\mu\nu}, \quad P^\lambda \rightarrow P^\lambda,$$

where the sums $\sum_1, \sum_2, \ldots$ are over all inequivalent index permutations. The algebra $L_G(P)$ has representation in terms of differential operators of the form:

$$P^\mu = k^\mu,$$

$$M^{\mu\nu} = i\left(k^\mu \frac{\partial}{\partial k^\nu} - k^\nu \frac{\partial}{\partial k^\mu}\right) + i\left(e^\mu \frac{\partial}{\partial e^\nu} - e^\nu \frac{\partial}{\partial e^\mu}\right),$$

$$L_a^{\lambda_1\ldots\lambda_s} = e^{\lambda_1} \ldots e^{\lambda_s} \otimes L_a,$$

where $e^\lambda \in M^4$ is a translationally invariant space-time vector.

It is worthwhile to compare the above extension of the Poincaré algebra with the one which has been considered long ago by Ogievetski, Ivanov and others in a series of articles [6–9]. These authors were developing the idea that gauge bosons such as the photon, the Yang–Mills quanta and the graviton are Goldstone particles and therefore the spontaneously broken symmetry is more profound and general concept than the gauge symmetry [7]. In their approach the gauge transformations were considered as constant parameter transformations of a group which has an infinite number of generators $Q_a^{\mu_1\ldots\mu_s} = x^{\mu_1} \ldots x^{\mu_s} L_a$ together with the Poincaré generators

$$P^\mu = i \frac{\partial}{\partial x^\mu}, \quad M^{\mu\nu} = i \left(x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}\right),$$

$$Q_a^{\mu_1\ldots\mu_s} = x^{\mu_1} \ldots x^{\mu_s} L_a.$$

The generators $Q_a^{\mu_1\ldots\mu_s}$ are not translationally invariant because $[P^\mu, Q_a^{\mu_1\ldots\mu_s}] \neq 0$, therefore they are essentially different from the generators $L_a^{\lambda_1\ldots\lambda_s}$ with their vanishing commutators $[P^\mu, L_a^{\lambda_1\ldots\lambda_s}] = 0$ in (1.2). Still there is a similarity between the generators $L_a^{\lambda_1\ldots\lambda_s}$ and $Q_a^{\mu_1\ldots\mu_s}$.

An example of the infinite-dimensional algebra which contains Lorentz subalgebra and the high-rank multispinor generators $Q^{\alpha_1\ldots\alpha_s\beta_1\ldots\beta_s}$ were considered in the paper of Vasiliev [10]. It is a generalization of the $L_{\text{osp}(4)}$ algebra to the infinite-dimensional associative algebra $L_{\text{hs}(4)}$ spanned by high order polynomials constructed from mutually conjugated spinor oscillators. In this algebra there are no internal charges $L_a$ associated with the generators $Q_a^{\mu_1\ldots\alpha_1\beta_1\ldots\beta_s}$.

The details of the calculations concerning the traces of the generators on $L_{\text{hs}(4)}$ can be found in [10, 11].

For the purposes of constructing field-theoretical models based on a larger group of symmetry one should define the notion of a trace for the fields taking values on the extended algebra and in particular on $L_G(P)$. In the next section we shall recollect the useful formulae defining the structure of the invariant scalar product both for the compact and noncompact finite-dimensional Lie algebras and in particular the invariant scalar product of the Poincaré generators. Two methods can be used to calculate explicitly the Killing form. The first one is a direct calculation of the traces of the generators in a matrix representation of the algebra,
and the second one is the usage of the group invariance of the scalar product. We shall use both methods in our calculation of the scalar product on the extended Poincaré algebra $L_G(P)$ in order to have a cross check of our final result. The reason is that the algebra is infinite-dimensional and requires careful treatment of the infinities. In section 3 we shall use the explicit matrix representation of the $L_G(P)$ algebra generators to calculate the traces and in section 4 we shall use group invariance of the scalar product. The results for the scalar product on the extended algebra $L_G(P)$ found by both methods coincide and are presented in the following table:

\[
L_G: \quad \langle L_a; L_b \rangle = \delta_{ab}, \quad (1.6)
\]

\[
L_P: \quad \langle P^\mu; P^\nu \rangle = 0,
\]

\[
\langle M_{\mu\nu}; P_\lambda \rangle = 0,
\]

\[
\langle M^{\mu\nu}; M^{\rho\sigma} \rangle = \eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}, \quad (1.7)
\]

\[
L_G(P): \quad \langle P^\mu; L^i_a \rangle = 0,
\]

\[
\langle M^{\mu\nu}; L^i_a \rangle = 0,
\]

\[
\langle L_a; L^i_b \rangle = 0,
\]

\[
\langle L^i_a; L^j_b \rangle = \delta_{ij} \eta^{k_1k_2},
\]

\[
\langle L_a; L^j_b \rangle = \delta_{ab} \eta^{k_1k_2},
\]

\[
\langle L^i_a; L^j_b \rangle = 0,
\]

\[
\langle L^i_a; L^{j+1}_b \rangle = 0, \quad s = 0, 1, 2, 3, \ldots
\]

\[
\langle L^{j-1}_a; L^{j+1}_b \rangle = \delta_{ab} \eta^{k_1k_2} \ldots \eta^{k_{2s-1}k_{2s}}, \quad (1.9)
\]

The Killing forms on the internal $L_G$ and on the Poincaré $L_P$ subalgebras are well known (1.6), (1.7). The important conclusion which follows from the above result is that Poincaré generators $P^\mu, M^{\mu\nu}$ are orthogonal to the gauge generators $L^i_a, (1.8)$. The last formulas (1.9) represent the Killing form on the current algebra (1.3).

2. Invariant Killing forms

Our intention is to define the invariant scalar product on the extended algebra $L_G(P)$. Let us recollect the structure of the invariant scalar product in the cases of finite-dimensional Lie algebras $L_G$ [12]. The scalar product of arbitrary two elements $X = X^a L_a$ and $Y = Y^a L_a$ of the algebra is defined as a trace of the generators in the adjoint representation ($L_a)_a^d = i f_{ad}^c$:

\[
\langle X; Y \rangle = \text{tr}(ad X ad Y) = X^a (L_a)_a^d Y^b (L_b)_c^d = X^a i f_{ad}^c Y^b i f_{bc}^d = g_{ab} X^a Y^b, \quad (2.1)
\]

where the associated Cartan metric $g_{ab}$ is

\[
g_{ab} = i f_{ad}^c i f_{bc}^d. \quad (2.2)
\]

If $X$ and $Y$ are the single generators we shall have

\[
\langle L_a; L_b \rangle = g_{ab}. \quad (2.3)
\]

The scalar product, if defined as a trace of the generators, depends only on the scales set by choice of a given representation and not on a particular representation used, therefore it is convenient to take the generators in the adjoint representation and express the scalar product in terms of the structure constants, as it is in (2.1) and (2.2).
The last two expressions can be used to find the explicit form of the scalar product for a specific algebra. So the defined scalar product is symmetric and bilinear:
\[
\langle X; Y \rangle = \langle Y; X \rangle, \quad \langle aX + \beta Y; Z \rangle = a \langle X; Z \rangle + \beta \langle Y; Z \rangle,
\]
where \( X = X^a L_a, Y = Y^a L_a, Z = Z^a L_a \) and is invariant under the action of the group \( \langle gX g^{-1}; gY g^{-1} \rangle = \langle X; Y \rangle \quad g \in G \).

For infinitesimal group elements it is equivalent to the expression [5, 12]:
\[
\langle [X, Y]; Z \rangle + \langle Y; [X, Z] \rangle = 0 \quad X, Y, Z \in L_G, \tag{2.4}
\]
which can also be used to define the invariant scalar product on the algebra. In the subsequent sections we shall use both methods to find scalar product on the extended algebra \( L_G(P) \). Because the representations of this algebra are infinite-dimensional, it is important to have an independent check by both methods, that is, by direct computation of traces (2.1) and by using the group invariance of the scalar product (2.4).

As an example let us consider the general linear algebra \( L_{gl(n)} \) which is defined by the following commutation relation [12]:
\[
[L_{ij}, L_{kl}] = i(\delta_{jk} L_{il} - \delta_{il} L_{jk}), \tag{2.5}
\]
i, j, k, ..., = 1, 2, ..., n, with its structure constants
\[
f^{km}_{ij} = \delta^i_j \delta^k_m - \delta^k_i \delta^m_j,
\]
so that for the Cartan metric (2.2) one can get
\[
\langle L_{ij}; L_{pr} \rangle = g_{(ij)}p_{r} = if^{km}_{ij} f^{kl}_{pr} = -2n \delta_{ij} \delta_{rp} + 2 \delta_{ij} \delta_{pr}, \tag{2.7}
\]
or, equivalently,
\[
\langle X; Y \rangle = \langle X^{ij} L_{ij}; Y^{op} L_{op} \rangle = -2n \text{tr}(XY) + 2 \text{tr}(X) \text{tr}(Y). \tag{2.8}
\]

For the \( L_{so(p, q)} \) algebra with its commutation relation
\[
[M_{AB}, M_{CD}] = i(\eta_{AD} M_{BC} - \eta_{AC} M_{BD} + \eta_{BC} M_{AD} - \eta_{BD} M_{AC}), \tag{2.9}
\]
where \( \eta_{AB} = \text{diag}(+1, \ldots, +1, -1, \ldots, -1) \) is a diagonal matrix with \( p \) minuses and \( q \) pluses, we have
\[
f_{AB}^{KL} C = \delta^K_B \eta_{AD} \delta^L_C - \delta^K_B \eta_{AC} \delta^L_D + \delta^K_A \eta_{BC} \delta^L_D - \delta^K_A \eta_{BD} \delta^L_C,
\]
and using (2.2) one can get the following expression for the Cartan metric [12]:
\[
\langle M_{AB}; M_{CD} \rangle = g_{AB} g_{CD} = (-2p - 2q + 4)(\eta_{AD} \eta_{BC} - \eta_{AC} \eta_{BD}), \tag{2.10}
\]
or, equivalently,
\[
\langle X; Y \rangle = \langle X^{AB} M_{AB}; Y^{CD} M_{CD} \rangle = (-p - q + 2) \text{tr}(X - \tilde{X}) \cdot (Y - \tilde{Y}). \tag{2.11}
\]

Because the Poincaré algebra \( L_P \) can be considered as a contraction of the de Sitter algebra \( L_{so(3, 2)} \) with \( \eta_{AB} = \text{diag}(+ - - -) \), \( A, B = (0, 1, 2, 3, 4) \) by taking \( M_{\mu} = R P_{\mu} \) and \( R \to \infty \), where \( \mu, \nu, \ldots = (0, 1, 2, 3) \), we can calculate the scalar product on \( L_P \) using invariant scalar product (2.10):
\[
\langle P^\mu; P^\nu \rangle = \frac{1}{R^2} (M_{\mu}; M_{\nu}) = 6 \frac{R^2}{R^2} \eta_{\mu \nu} \to 0
\]
\[
\langle M_{\mu} P^\nu \rangle = \frac{1}{R} (M_{\mu} M_{\nu}) = -6 \frac{R}{R} (\eta_{\mu \lambda} \eta_{\nu \delta} - \eta_{\nu \lambda} \eta_{\mu \delta}) = 0
\]
\[
\langle M_{\mu} P^\nu M_{\nu} \rangle = 6 (\eta_{\mu \lambda} \eta_{\nu \rho} - \eta_{\nu \lambda} \eta_{\mu \rho}) = 0.
\]
Thus the scalar product of the Poincaré generators is defined as follows:
\[ \langle P_\mu ; P_\nu \rangle = 0 \]
\[ L_P: \langle M_{\mu \nu} ; P_\lambda \rangle = 0 \]
\[ \langle M_{\mu \nu} ; M_{\kappa \rho} \rangle = 6 (\eta_{\mu \kappa} \eta_{\nu \rho} - \eta_{\mu \rho} \eta_{\nu \kappa}) \].
(2.12)

If the Cartan metric \( g_{ab} \) of the semi-simple algebra \( L_G \) is positive definite, then the corresponding algebra is compact and one can choose the basis of the generators \( \{ L_a \} \) of \( L_G \) so that \( g_{ab} = \delta_{ab} \) and define the covariant rank-3 tensor as
\[ f_{abc} = f_{dab} g^{dc} = f_{ac}^{\ d} \).

It is antisymmetric over \( (a, b) \) and, as one can see from the relation
\[ f_{abc} = f_{dab} g^{dc} = f_{dab} f_{dm} f_{cn} + f_{dab} f_{ma} f_{cn}, \]
it is symmetric over cyclic permutations of \( (a, b, c) \), and thus it is totally antisymmetric. In summary, for a compact semi-simple Lie algebra \( L_G \) one can choose a basis of the generators \( \{ L_a \} \) such that the generators are orthogonal \([12]\]
\[ \langle L_a ; L_b \rangle = \delta_{ab} \]
and the structure constants \( f_{abc} \) are totally antisymmetric. Below we shall consider the case of the \( SU(N) \) algebra with its traceless Hermitian generators \( \text{tr}(L_a) = 0 \).

3. Killing form on \( L_G(\mathcal{P}) \) algebra

Now let us define the invariant scalar product on the extended algebra \( L_G(\mathcal{P}) \). As we mentioned above, there are two ways by which we can define a scalar product. The first one is a direct calculation of the traces of the generators in the matrix representations of the algebra as in (2.1), and the second one is the usage of the group invariance of the scalar product (2.4). We shall proceed with the explicit matrix representation of the \( L_G(\mathcal{P}) \) generators. These representations have been constructed in \([2, 3]\). It has the representation of the following form:
\[ P_\mu = k_\mu, \]
\[ M_{\mu \nu} = i \left( k_\mu \frac{\partial}{\partial k_\nu} - k_\nu \frac{\partial}{\partial k_\mu} \right) + i \left( \xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu} \right) \]
\[ L_a^{\lambda_1...\lambda_s} = \xi^{\lambda_1} \ldots \xi^{\lambda_s} \otimes L_a, \]
(3.1)
where the vector space is parameterized by momentum coordinates \( k^\mu \) and translationally invariant vector variables \( \xi^{\mu} \):
\[ \Psi(k^\mu, \xi^\nu). \]
(3.2)
The irreducible representations can be obtained from (3.1) by imposing invariant constraints on the vector space of functions (3.2) of the following form \([13–16]\):
\[ k^2 = 0, \quad k^\mu \xi_\mu = 0, \quad \xi^2 = -1. \]
(3.3)
These equations have a unique solution
\[ \xi^\mu = \xi k^\mu + e^\mu_1 \cos \varphi + e^\mu_2 \sin \varphi, \]
(3.4)
where \( e^\mu_1 = (0, 1, 0, 0), e^\mu_2 = (0, 0, 1, 0) \) when \( k^\mu = \omega(1, 0, 0, 1) \). The invariant subspace of functions (3.1) now reduces to the form
\[ \Psi(k^\mu, \xi^\nu) \delta(k^2) \delta(k \cdot \xi) \delta(\xi^2 + 1) = \Phi(k^\mu, \varphi, \xi, \cdot), \]
(3.5)
1 One should use the Jacobi identity \( f_{ab} f_{cd} f_{de} = -f_{ab} f_{cd} f_{de} = f_{ad} f_{bm} + f_{bd} f_{ma} f_{de} \).
where $\xi$ and $\varphi$ remain as independent variables on the cylinder $\varphi \in S^1$, $\xi \in R^1$. The generators of the little group $L [13, 16, 17]$, which leave the fixed momentum $k^0 = \omega(1, 0, 0, 1)$ invariant, form the $E(2)$ algebra:

$$[h, \pi^\prime] = i\pi^\prime$, $[h, \pi^\prime] = -i\pi^\prime$, $[\pi^\prime, \pi^\prime] = 0,$

where $h = M_{12}, \pi^\prime = M_{10} + M_{13}, \pi^\prime = M_{20} + M_{23}$. Notice that the transformations which are generated by $L^\pm_{\lambda - \lambda s}$ also leave the manifold of states with fixed momentum invariant, so that we have to add them to the little algebra $L [2, 3]$:

$$h, \pi^\prime, \pi^\prime, L^\pm_{\lambda - \lambda s}.$$

The representation of the little algebra $L$ in terms of differential operators is of the form

$$L^\pm_{\lambda - \lambda s} = \sum_{i=1}^f i k^\mu_{\lambda i} + e^\mu_{\lambda i} \cos \varphi + e^\mu_{\lambda i} \sin \varphi \right) \otimes L^a.$$

This is a purely transversal representation in the sense that

$$k_a L^a_{\lambda - \lambda s} = 0, s = 1, 2, \ldots$$

Below we shall essentially use the operator representation (3.6) and (3.7) to calculate matrix elements and traces of the operator products. It is also important to know the helicity content of the gauge operators $L^\pm_{\lambda - \lambda s}$. The Poincaré generators $\pi^\lambda = \pi^\prime \pm \pi^\prime$ carry helicities $h = (1, -1)$.

The $L^\pm_{\lambda s} = L^s_{\lambda s} \pm i L^s_{\lambda s}$ carry helicities $h = (1, -1)$, as seen from $[h, L^\pm_{\lambda s}] = \pm L^\pm_{\lambda s}$. The rank-2 generators $L^\pm_{\lambda s}, L^\pm_{\lambda s}, L^\pm_{\lambda s}$

$$L^s_{\lambda s} = L^s_{11} + 2iL^s_{12} = L^s_{22}, \quad L^\pm_{\lambda s} = L^\pm_{11} + L^\pm_{22}, \quad L^\pm_{\lambda s} = L^\pm_{11} - 2iL^\pm_{12} - L^\pm_{22},$$

carry helicities $h = (2, 0, -2)$ because $[h, L^\pm_{\lambda s}] = \pm 2L^\pm_{\lambda s}, [h, L^\pm_{\lambda s}] = 0$. In general the rank-$s$ ($L^{\pm \mp}_{\lambda s}, \ldots, L^{-\pm}_{\lambda s}$) generators carry helicities in the following range:

$$h = (s, s - 2, \ldots, -s + 2, -s),$$

in total $s + 1$ states. This result proves that the algebra $L_G(P)$ has representations which describe propagation of the high helicity charged states.

Having in hand the explicit representation of the $L_G(P)$ generators we can find their matrix elements and calculate the corresponding traces of the generators. As a basis of functions on a cylinder $\varphi \in S^1$, $\xi \in R^1$ we shall take (see the appendix for details)

$$|n, m\rangle = \frac{1}{\sqrt{2\pi}} \exp i\frac{\xi^2}{2} \sqrt{\frac{m!}{\pi}} H_m(\xi) = \frac{1}{\sqrt{2\pi}} \exp i\xi \varphi_m(\xi),$$

where $H_m(\xi)$ are Hermite polynomials and the trace should be defined as

$$\langle A; B \rangle = \sum_{n,m} \int_0^\pi \int_{-\infty}^\infty d\varphi \, d\xi \, \text{tr}(\langle n, m\rangle A|B|n, m)$$

$$= \sum_{n,m} \left( \sum_{n,m} \int_0^\pi \int_{-\infty}^\infty d\varphi \, d\xi \, \text{tr}(\langle n, m\rangle A|B|n, m) \right) \langle r, l | B | n, m \rangle.$$
The traces of the Poincaré generators (3.6) can be easily computed:\footnote{In the subsequent calculations of the traces we shall not write explicitly summation and integration symbols over all variables involved in the trace, but the summations and integrations will be implied.}

\[
\left< h; \pi' \right> = \langle n, m | h | r, l \rangle \langle r, l | \pi' | n, m \rangle = r \delta_n, r \delta_m, l \frac{1}{2\omega} (\delta_{r,n+1} + \delta_{r,n-1})(\sqrt{2m} \delta_{l,m-1} - \sqrt{2(m+1)} \delta_{l,m+1}) = 0,
\]

and in a similar way \( \left< h; \pi'' \right> = \langle \pi'; \pi'' \rangle = 0 \). This explicit calculation confirms the previous result (2.12), which we had for the Poincaré generators. Indeed, the Little algebra generators are defined as \( h = M_{12}, \pi' = M_{10} + M_{13}, \pi'' = M_{20} + M_{23} \) and if one takes into account that the scalar product between gauge generators is given by (2.12) we shall see that \( h, \pi' \) and \( \pi'' \) are indeed orthogonal.

Now we are prepared to calculate the traces between Poincaré generators and the gauge generators \( L_{a}^{h_{1} \cdots h_{k}} \). We have

\[
\left< h; L_{a} \right> = \langle n, m | h | r, l \rangle \langle r, l | L_{a} | n, m \rangle = r \delta_n, r \delta_m, l \delta_{r,n} \delta_{r,m} \text{tr}(L_{a}) = 0,
\]

that is they are orthogonal. For the vector generator \( L_{a}^{\mu} \) we shall get

\[
\left< h; L_{a}^{\mu} \right> = \langle n, m | h | r, l \rangle \langle r, l | \xi k^\mu + e_1^\mu \cos \varphi + e_2^\mu \sin \varphi | n, m \rangle \text{tr}(L_{a}) = r \delta_n, r \delta_m, l \left( k^\mu \left( \frac{m}{2} \delta_{l,m-1} + \frac{(m+1)}{2} \delta_{l,m+1} \right) \delta_{r,n} + (e_1^\mu \delta_{r,n+1} + e_2^\mu \delta_{r,n-1}) \delta_{l,m} \right) \text{tr}(L_{a}) = 0,
\]

where \( 2e_1^\mu = e_1^\mu + ie_2^\mu \) and in general one can get convinced that generators \( h, \pi', \pi'' \) and gauge generators \( L_{a}^{h_{1} \cdots h_{k}} \) are orthogonal to each other:

\[
\left< h; L_{a}^{h_{1} \cdots h_{k}} \right> = 0,
\]

\[
L: \quad \langle \pi'; L_{a}^{h_{1} \cdots h_{k}} \rangle = 0,
\]

\[
\langle \pi''; L_{a}^{h_{1} \cdots h_{k}} \rangle = 0.
\]

This statement can be extended to all Lorentz generators since a state of the Hilbert space with fixed momentum \( k^\mu \) will transform to the state with momentum \( k' = \Lambda k \) if one applies the group operator \( U_{\Lambda} \) corresponding to the Lorentz transformation \( \Lambda_{\mu \nu} \). Therefore we have

\[
L_{G}(P): \quad \left[ M^{\mu \nu}; L_{a}^{h_{1} \cdots h_{k}} \right] = 0.
\]

Finally we have to calculate the scalar product between gauge generators \( L_{a}^{h_{1} \cdots h_{k}} \). For the compact Lie algebra \( L_{SU(N)} \) we have (2.13)

\[
L_{G}: \quad \langle L_{a}; L_{b} \rangle = \delta_{ab},
\]

and then with the first level generator \( L_{b}^{\mu} \)

\[
\langle L_{a}; L_{b}^{\mu} \rangle = \langle n, m | r, l | \xi k^\mu + e_1^\mu \cos \varphi + e_2^\mu \sin \varphi | n, m \rangle \langle L_{a}; L_{b} \rangle = r \delta_n, r \delta_m, l \left( k^\mu \left( \frac{m}{2} \delta_{l,m-1} + \frac{(m+1)}{2} \delta_{l,m+1} \right) \delta_{r,n} + (e_1^\mu \delta_{r,n+1} + e_2^\mu \delta_{r,n-1}) \delta_{l,m} \right) \delta_{ab} = 0.
\]

The scalar product between first and third level generators also nullifies, as one can see from the following calculation:

\[
\left< L_{a}; L_{b}^{h_{1} \cdots h_{k}} \right> = 0.
\]
\[ \langle L^\mu_a; L^\nu_b \rangle = \langle n, m | \xi k^\mu + e^\nu_1 \cos \varphi + e^\nu_2 \sin \varphi | r, l \rangle \\
\langle r, l | \xi k^\nu + e^\nu_1 \cos \varphi + e^\nu_2 \sin \varphi \rangle \langle \xi k^\mu + e^\nu_1 \cos \varphi + e^\nu_2 \sin \varphi | n, m \rangle \langle L^\nu_a; L^\mu_b \rangle \]
\[ = k^\mu \left( \frac{T}{2} \delta_{m,l-1} + \sqrt{\frac{(l+1)}{2}} \delta_{m,l+1} \right) \delta_{n,r} + (e^\nu_1 \delta_{n,r+1} + e^\nu_2 \delta_{n,r-1}) \delta_{m,l} \]
\[ \times k^\nu \left( \frac{m(m-1)}{4} \delta_{l,m-2} + \frac{2m+1}{2} \delta_{l,m} + \sqrt{\frac{(m+1)(m+2)}{4}} \delta_{l,m+2} \right) \delta_{r,n} \]
\[ + \frac{k^\nu}{\sqrt{2}} \delta_{l,m} \left( \frac{m}{2} \delta_{l,m-1} + \sqrt{\frac{(m+1)}{2}} \delta_{l,m+1} \right) \delta_{r,n} \]
\[ + (e^\nu_1 \delta_{n,r+1} + e^\nu_2 \delta_{r,n-1}) k^\nu \left( \frac{m}{2} \delta_{l,m-1} + \sqrt{\frac{(m+1)}{2}} \delta_{l,m+1} \right) \delta_{r,n} \]
\[ + \delta_{l,m} (e^\nu_1 e^\nu_1 \delta_{r,n+2} + (e^\nu_1 e^\nu_2 + e^\nu_2 e^\nu_1) \delta_{r,n} + (e^\nu_1 e^\nu_2 + e^\nu_2 e^\nu_1) \delta_{r,n-2}) \right) \delta_{ab} = 0. \tag{3.13} \]

Generally when the total number of Lorentz indices in the scalar product is odd one finds that the scalar product vanishes: \( \langle L^\mu_a; L^\nu_b \rangle = 0 \). The scalar product is nonzero when the number of indices is even:
\[ \langle L^\mu_a; L^\nu_b \rangle = \langle n, m | \xi k^\mu + e^\nu_1 \cos \varphi + e^\nu_2 \sin \varphi | r, l \rangle \\
\langle r, l | \xi k^\nu + e^\nu_1 \cos \varphi + e^\nu_2 \sin \varphi | n, m \rangle \langle L^\nu_a; L^\mu_b \rangle \]
\[ = k^\mu \left( \frac{T}{2} \delta_{m,l-1} + \sqrt{\frac{(l+1)}{2}} \delta_{m,l+1} \right) \delta_{n,r} + (e^\nu_1 \delta_{n,r+1} + e^\nu_2 \delta_{n,r-1}) \delta_{m,l} \]
\[ \times k^\nu \left( \frac{m(m-1)}{4} \delta_{l,m-2} + \frac{2m+1}{2} \delta_{l,m} + \sqrt{\frac{(m+1)(m+2)}{4}} \delta_{l,m+2} \right) \delta_{r,n} \]
\[ = \delta_{ab} \sum_{n,m} \left[ k^\mu k^\nu (m+1/2) + e^\nu_1 e^\nu_1 + e^\nu_2 e^\nu_2 \right], \tag{3.14} \]

where the summation over \( n, m \) have been explicitly reintroduced in the last line. This sum can be regularized by defining the traces of the infinite-dimensional matrices in (3.10) by the exponential weight factor \(-12 \exp (-\epsilon |n| - \epsilon m)\), subtruction of the singular terms and taking the limit \( \epsilon \to 0 \):
\[ -12 \sum_{n=-\infty}^{\infty} e^{-\epsilon |n|} \sum_{m=0}^{\infty} e^{-\epsilon m} k^\mu k^\nu (m+1/2) + e^\nu_1 e^\nu_1 + e^\nu_2 e^\nu_2 = -2(k^\mu k^\nu + e^\nu_1 e^\nu_1 + e^\nu_2 e^\nu_2). \]

One should also average over all orientations of the momentum \( k^\mu \). Its rotation is generated by the application of the group operator \( U_\lambda \) corresponding to the Lorentz rotation with the group parameters \( \Lambda_{\mu \nu} \). This average is equal to \(-2(k^\mu k^\nu + e^\nu_1 e^\nu_1 + e^\nu_2 e^\nu_2) = \eta^{\mu \nu} \), thus we find that
\[ \langle L^\mu_a; L^\nu_b \rangle = \delta_{ab} \eta^{\mu \nu}. \tag{3.15} \]

In a similar way
\[ \langle L^\mu_a; L^\nu_b \rangle = \delta_{ab} \eta^{\mu \nu}. \tag{3.16} \]

We can summarize now the structure of the scalar product of the extended algebra \( L_G(P) \) in the following table:
\[ L_G: \quad \langle L_a^i; L_b^j \rangle = \delta_{ab}, \]
\[ L_P: \quad \langle P^\mu; P^\nu \rangle = 0 \]
\[ \langle M^{\mu\nu}; P_\lambda \rangle = 0 \]
\[ \langle M^{\mu\nu}; M^{\rho\sigma} \rangle = \eta^{\mu\lambda} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\lambda} \tag{3.17} \]
\[ L_G(P): \quad \langle P^\alpha; L_a^{\lambda_1\ldots\lambda_n} \rangle = 0, \]
\[ \langle M^{\mu\nu}; L_a^{\lambda_1\ldots\lambda_n} \rangle = 0, \tag{3.18} \]
\[ \langle L_a^i; L_b^j \rangle = 0, \]
\[ \langle L_a^i; L_b^{\lambda_1} \rangle = \delta_{ab} \eta^{\lambda_1\lambda_2}, \]
\[ \langle L_a^i; L_b^{\lambda_2} \rangle = \delta_{ab} \eta^{\lambda_1\lambda_2}, \]
\[ \langle L_a^i; L_b^{\lambda_1\lambda_2} \rangle = 0, \]
\[ \langle L_a^{\lambda_1\lambda_2}; L_b^{\lambda_1\lambda_3} \rangle = \delta_{ab} 2! (\eta^{\lambda_1\lambda_2} \eta^{\lambda_3\lambda_4} + \eta^{\lambda_1\lambda_3} \eta^{\lambda_2\lambda_4} + \eta^{\lambda_1\lambda_4} \eta^{\lambda_2\lambda_3}) \tag{3.19} \]

The most important conclusion which can be drawn upon above computation is that Poincaré generators \( P^\mu \), \( M^{\mu\nu} \) are orthogonal to the gauge generators \( L_a^{\lambda_1\ldots\lambda_n} \).

Our intention is to derive the expression for the scalar products, this time using the invariance of the scalar product under the action of group transformations which is expressed by the equation \( (2.4) \). New derivation will provide us with an independent cross check of the direct calculation of the traces of infinite-dimensional matrices which we performed in this section.

**4. Group invariance of the scalar product**

Let us first consider the Poincaré algebra \( L_P \). The invariant equation for the scalar product has the form \( (2.4) \)
\[ \langle X; [Y, Z] \rangle + \langle [X, Z] ; Y \rangle = 0 \quad X, Y, Z \in L_G, \]
and we shall take \( X = P^\mu \), \( Y = M^{\mu\nu} \) and \( Z = P^\lambda \). The above equation reduces to \( \langle P^\mu; [M^{\mu\nu}, P^\lambda] \rangle + \langle [P^\mu, P^\lambda]; M^{\mu\nu} \rangle = 0, \)
or using the definition of the commutation relations of the \( L_P \) algebra \( (1.1) \) we shall get
\[ \eta^{\nu\lambda} \langle P^\mu; P^\nu \rangle - \eta^{\mu\lambda} \langle P^\rho; P^\nu \rangle = 0. \tag{4.1} \]
At \( \nu = \lambda = 0, \mu = \rho = 1 \) we shall get \( \langle P^1; P^1 \rangle = 0 \), at \( \nu = \lambda = 1, \mu = \rho = 0 \) we shall get \( \langle P^0; P^0 \rangle = 0 \), at \( \nu = \lambda = 2, \mu = 1, \rho = 0 \) we shall get \( \langle P^2; P^1 \rangle = 0 \) and so on. This derivation confirms our previous calculation of the product of momentum generators \( (2.12) \).

Taking \( X = P^\mu \), \( Y = M^{\mu\nu} \) and \( Z = M^{\rho\sigma} \) the equation \( (2.4) \) takes the form
\[ \langle P^\mu; [M^{\mu\nu}, M^{\rho\sigma}] \rangle + \langle [P^\mu, M^{\rho\sigma}]; M^{\mu\nu} \rangle = 0, \]
or using the \( L_P \) algebra \( (1.1) \)
\[ \eta^{\mu\rho} \langle P^\nu; M^{\mu\nu} \rangle - \eta^{\nu\lambda} \langle P^\rho; M^{\mu\rho} \rangle + \eta^{\nu\lambda} \langle P^\sigma; M^{\mu\sigma} \rangle - \eta^{\mu\rho} \langle P^\nu; M^{\mu\nu} \rangle + \eta^{\nu\lambda} \langle P^\rho; M^{\mu\rho} \rangle - \eta^{\mu\rho} \langle P^\nu; M^{\mu\nu} \rangle = 0. \tag{4.2} \]
At \( \mu = \rho = 0, \sigma = 1, v = 2, \lambda = 2 \) we shall get \( \langle P^1; M^{23} \rangle = 0 \), at \( \mu = \rho = 1, \sigma = 0, v = 2, \lambda = 3 \) we shall get \( \langle P^3; M^{23} \rangle = 0 \) and so on. This confirms the second equation in (2.12).

Taking \( X = M^{\mu \delta}, Y = M^{\mu \nu} \) and \( Z = M^{\nu \rho} \), the equation (2.4) takes the form

\[
\langle M^{\mu \delta}; [M^{\mu \nu}, M^{\nu \rho}] \rangle + \langle [M^{\mu \delta}, M^{\nu \rho}]; M^{\mu \nu} \rangle = 0
\]

or using the commutators of the \( L_P \) algebra (1.1) we get

\[
\eta^{\mu \rho} \langle M^{\mu \delta}; M^{\nu \rho} \rangle - \eta^{\mu \lambda} \langle M^{\mu \delta}; M^{\nu \rho} \rangle + \eta^{\nu \lambda} \langle M^{\mu \delta}; M^{\nu \rho} \rangle - \eta^{\nu \rho} \langle M^{\mu \delta}; M^{\nu \rho} \rangle - \eta^{\mu \rho} \langle M^{\nu \delta}; M^{\nu \rho} \rangle + \eta^{\nu \delta} \langle M^{\mu \delta}; M^{\nu \rho} \rangle
\]

\[
- \eta^{\nu \lambda} \langle M^{\mu \delta}; M^{\nu \rho} \rangle - \eta^{\nu \rho} \langle M^{\mu \delta}; M^{\nu \rho} \rangle - \eta^{\mu \rho} \langle M^{\nu \delta}; M^{\nu \rho} \rangle + \eta^{\mu \delta} \langle M^{\nu \delta}; M^{\nu \rho} \rangle - \eta^{\nu \delta} \langle M^{\mu \delta}; M^{\nu \rho} \rangle + \eta^{\mu \delta} \langle M^{\nu \delta}; M^{\nu \rho} \rangle = 0. \tag{4.3}
\]

At \( \mu = \rho = 0, \sigma = 1, \delta = v = 2, \lambda = 3 \) we have \( \langle M^{12}; M^{23} \rangle = 0 \) and so on. This confirms the third equation in (2.12). Thus all relations for the scalar product in (3.17) are consistent with the requirement of group invariance.

Now let us consider the full algebra \( L_G(P) \) (1.1). In the case when \( X = M^{\mu \nu} \) and the other two operators are spacetime scalars \( Y = L_a \) and \( Z = L_b \), we have to consider the equation

\[
\langle M^{\mu \nu}; [L_a, L_b] \rangle + \langle [M^{\mu \nu}, L_b]; L_a \rangle = 0
\]

or using the commutations relations of the algebra (1.1) and in particular that \( [M^{\mu \nu}, L_a] = 0 \) the equation takes the form

\[
i f_{abc} \langle M^{\mu \nu}; L_c \rangle = 0,
\]

and we can conclude that the product of the operator \( M^{\mu \nu} \) with the spacetime scalar operators \( L_a \) is equal to zero

\[
\langle M^{\mu \nu}; L_a \rangle = 0. \tag{4.4}
\]

Next we shall consider the product of the \( M^{\mu \nu} \) with the spacetime vector operator \( Y = L_a \). For that let consider the case when \( X = M^{\mu \nu}, Y = L_a \) and \( Z = L_b \), so that we have to consider the equation

\[
\langle M^{\mu \nu}; [L_a, L_b] \rangle + \langle [M^{\mu \nu}, L_b]; L_a \rangle = 0,
\]

or using the definition of the commutators of the algebra (1.1) we get

\[
i f_{abc} \langle M^{\mu \nu}; L_c \rangle = 0,
\]

thus we conclude that the scalar product of the operator \( M^{\mu \nu} \) with the vector operator \( L_a \) is also equal to zero

\[
\langle M^{\mu \nu}; L_a \rangle = 0. \tag{4.5}
\]

To proceed we have to consider the product of \( M^{\mu \nu} \) with the higher rank gauge operator \( L_a^{\lambda_1 \lambda_2 \lambda_3} \). Thus let us consider the case when \( X = M^{\mu \nu}, Y = L_a^{\lambda_1 \lambda_2 \lambda_3} \) and \( Z = L_b \) the invariance equation is

\[
\langle M^{\mu \nu}; [L_a^{\lambda_1 \lambda_2 \lambda_3}, L_b] \rangle + \langle [M^{\mu \nu}, L_b]; L_a^{\lambda_1 \lambda_2 \lambda_3} \rangle = 0
\]

and using the definition of the commutations relations of the algebra (1.1) we get

\[
\langle M^{\mu \nu}; L_a^{\lambda_1 \lambda_2 \lambda_3} \rangle = 0, \tag{4.6}
\]

which is consistent with our previous direct calculation (3.12).

We are interested now to find the scalar products between gauge generators. In the case when \( X = M^{\mu \nu}, Y = L_a \) and \( Z = L_b \) we have the equation

\[
\langle M^{\mu \nu}; [L_a, L_b] \rangle + \langle [M^{\mu \nu}, L_b]; L_a \rangle = 0
\]

or using the definition of the commutators we get

\[
i f_{abc} \langle M^{\mu \nu}; L_c \rangle + \eta^{\rho \lambda} \langle L_a^{\rho \lambda}; L_b \rangle - \eta^{\sigma \rho} \langle L_a^{\sigma \rho}; L_b \rangle = 0
\]
and because \( \langle M^{\mu \nu}; L_0^2 \rangle = 0 \) (4.6) we conclude that the scalar product of the operator \( L_0 \) with the vector operator \( L_0^2 \) is equal to zero

\[
\langle L_0^2; L_0^2 \rangle = 0.
\]  

(4.7)

Considering the case \( X = M^{\mu \nu}, Y = L_0 \) and \( Z = L_0^{h_{1,2}} \) we have the equation

\[
\langle M^{\mu \nu}; [L_0, L_0^{h_{1,2}}] \rangle + \langle [M^{\mu \nu}, L_0^{h_{1,2}}]; L_0 \rangle = 0,
\]

or

\[
i f_{abc} \langle M^{\mu \nu}; L_0^{h_{1,2}} \rangle + \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle - \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle + \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle - \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle = 0
\]

and because \( \langle M^{\mu \nu}; L_0^{h_{1,2}} \rangle \) we shall get

\[
\eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle = \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle + \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle - \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0 \rangle = 0.
\]

It follows then that

\[
\langle L_0^{h_{1,2}}; L_0 \rangle \sim \eta^{h_{1,2}} c_{ab} \]  

(4.8)

where \( c_{ab} \) is an arbitrary rank-2 tensor. Considering the case \( X = L_0^{h_{1,2}}, Y = L_0^{h_{1,2}} \) and \( Z = M^{\mu \nu} \) we have the equation

\[
\langle L_0^{h_{1,2}}; [L_0^{h_{1,2}}, M^{\mu \nu}] \rangle + \langle [L_0^{h_{1,2}}, M^{\mu \nu}]; L_0^{h_{1,2}} \rangle = 0,
\]

or

\[
\eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle - \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle + \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle - \eta^{h_{1,2}} \langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle = 0
\]

and it follows that

\[
\langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle \sim \eta^{h_{1,2}} c_{ab}'.
\]  

(4.9)

Taking in the last two cases instead of \( M^{\mu \nu} \) the operator \( L_0 \) one can see that \( \langle L_0^{h_{1,2}}; L_0 \rangle \) and \( \langle L_0^{h_{1,2}}; L_0^{h_{1,2}} \rangle \) are the isotropic tensors proportional to \( c_{ab} \), where \( c \) is arbitrary. This consideration proves that the invariance equation allows nonzero solutions for these scalar products, but it cannot fix the value of the constant \( c \), which is in general representation-dependent constant. Thus reconfirming the results (3.15) and (3.16). In a similar way one can calculate the structure of the scalar products between the higher-rank gauge generators and confirm the result (3.19).

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**Appendix**

The useful properties of the normalized Hermite polynomials

\[
\psi_m(\xi) = \frac{\exp \left(-\xi^2/2\right)}{\sqrt{2^m m! \sqrt{\pi}}} H_m(\xi), \quad \psi_m' = \sqrt{2m} \psi_{m-1} - \sqrt{2(m+1)} \psi_{m+1}
\]  

(A.1)
are
\[ \langle l | m \rangle = \int_{-\infty}^{\infty} \psi_l(\xi) \psi_m(\xi) d\xi = \delta_{l,m}, \]
\[ \langle l | d \frac{d}{d\xi} | m \rangle = \int_{-\infty}^{\infty} \psi_l(\xi) \psi_m'(\xi) d\xi = \sqrt{2m} \delta_{l,m-1} - \sqrt{2(m+1)} \delta_{l,m+1}, \]
\[ \langle l | \xi | m \rangle = \int_{-\infty}^{\infty} \psi_l(\xi) \xi \psi_m(\xi) d\xi = \sqrt{m} \delta_{l,m-1} + \frac{(m+1)}{2} \delta_{l,m+1}, \]
\[ \langle l | \xi^2 | m \rangle = \int_{-\infty}^{\infty} \psi_l(\xi) \xi^2 \psi_m(\xi) d\xi = \sqrt{m} \frac{m-1}{2} \delta_{l,m-2} + \frac{(2m+1)}{2} \delta_{l,m} + \frac{(m+1)}{2} \frac{(m+2)}{2} \delta_{l,m+2}. \]

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