Ground state hyperfine structure in muonic lithium ions

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Abstract

On the basis of perturbation theory in the fine structure constant \(\alpha\) and the mass ratio of the electron and muon, we calculate the one-loop vacuum polarization, electron vertex corrections, nuclear structure and recoil corrections of the hyperfine splitting of the ground state in muonic lithium ions \((\mu e \ 7\text{Li})^+\) and \((\mu e \ 3\text{Li})^+\). We obtain complete results for small hyperfine splittings of the ground state in \((\mu e \ 7\text{Li})^+\) of \(\Delta \nu_1 = 21572.16\) MHz and \(\Delta \nu_2 = 14152.56\) MHz and in \((\mu e \ 3\text{Li})^+\) \(\Delta \nu_1 = 21733.06\) MHz and \(\Delta \nu_2 = 13994.35\) MHz, which can be regarded as a reliable estimates for comparison with future experimental data.

Keywords: hyperfine structure, light muonic atoms, quantum electrodynamics

1. Introduction

Muonic lithium ions \((\mu e \ 7\text{Li})^+\) and \((\mu e \ 3\text{Li})^+\) are the simplest three-body atoms, consisting of one electron, a negatively charged muon and a positively charged nucleus, \(7\text{Li}\) or \(3\text{Li}\). The lifetime of muonic atoms is determined by the muon lifetime \(\tau_\mu = 2.19703(4) \cdot 10^{-6}\) s. It is longer than the time for atomic processes, so the muon has the time to make a number of transitions between energy levels which are accompanied by \(\gamma\)-radiation. These three-particle systems have a complicated ground state hyperfine structure which arises due to the interaction of the magnetic moments of the electron, muon and nucleus. Light muonic atoms are unique laboratories for the precise determination of nuclear properties such as the nuclear charge radius [1, 2]. In the last few years we have observed the essential progress achieved by the Charge Radius Experiment with Muonic Atoms (CREMA) collaboration in the study of the energy structure in muonic hydrogen. The measurement of the Lamb shift (2P–2S) and hyperfine splitting (HFS) of the 2S-state allows the more precise determination of the value of the proton charge radius \(r_p = 0.84087(39)\) fm, the Zemach radius \(r_\gamma = 1.082(37)\) fm and the magnetic radius \(r_M = 0.87(6)\) fm. The obtained value for the proton charge radius \(r_p\) is an order of magnitude more precise than the 2010 CODATA value which was derived using different methods including hydrogen spectroscopy. It differs from the CODATA value by \(7\sigma\). Note that the Zemach radius of the proton \(r_\gamma = 1.045(16)\) fm and magnetic radius \(r_M = 0.778(29)\) fm were previously obtained more accurately by comparing experimental data with the predictions for hydrogen hyperfine splitting [3]. Similar measurements are also being performed in the case of muonic deuterium and ions of muonic helium, the results of which are intended for publication. Light muonic atoms are important for checking the standard model and bound state theory in quantum electrodynamics, and in the search for exotic interactions of elementary particles. Thus, for example, muonic systems can be used in the search for Lorentz and CPT symmetry violations [4].

HFS of the ground state of the muonic helium atom \((\mu e \ 3\text{He})\) was measured many years ago with sufficiently high accuracy by [5]. This is the only experiment to date on muonic three-particle systems. In turn, the theoretical investigation of the energy spectrum of the muonic helium atom and other three-particle systems has achieved much success under two approaches [6–13]. The first approach [6, 7, 11] is based on the perturbation theory (PT) for the Schrödinger equation. In this case there is an analytical
solution for the three-particle bound state wave function in the initial approximation. Using this, a calculation of different corrections to the HFS can be performed. Another approach [9, 10, 12, 14, 15] is built on the variational method which allows the numerical calculation of the energy levels in three-particle systems with high accuracy. In order to find an arrangement of the low lying energy levels with high precision we should take into account different corrections to the particle interaction operator. First of all these corrections are related to recoil effects, vacuum polarization (VP) and nuclear structure effects. The aim of determining an analytical calculation of HFS in a muonic helium atom including an excited state was realized in [6, 7, 10–12, 16]. It allowed one to present HFS in an analytical form as a series into small parameters existing in the task. In this work we aim to extend this approach to muonic lithium ions, which are of potential interest for experimental study. So, the purpose of this paper is to provide a detailed calculation of the HFSs for the systems ($\mu e^6\text{Li}^+$, ($\mu e^3\text{Li}^+$).

The bound particles in muonic lithium ions have different masses $m_0 \ll m_j \ll m_L$. As a result the muon and Li nucleus comprise the pseudonucleus ($\mu e^6\text{Li}^+$) and the muonic lithium ion looks like a two-particle system in the first approximation. The three-particle bound system ($\mu e^6\text{Li}^+$) is described by the Hamiltonian:

$$H = H_0 + \Delta H + \Delta H_{\text{rec}} + \Delta H_{\text{VP}} + \Delta H_{\text{int}} + \Delta H_{\text{vert}},$$

$$H_0 = -\frac{1}{2M_\mu} \nabla_\mu^2 - \frac{1}{2M_e} \nabla_e^2 - \frac{3\alpha}{x_\mu} - \frac{2\alpha}{x_e},$$

$$\Delta H = \frac{\alpha}{x_{\mu e}} - \frac{\alpha}{x_e}, \quad \Delta H_{\text{rec}} = \frac{1}{M_{\mu i}} \nabla_\mu \cdot \nabla_e,$$  

(1)

where $x_\mu$ and $x_e$ are the muon and electron coordinates relative to the lithium nucleus, and $M_e = m_e M_{Li}/(m_e + M_{Li})$ and $M_\mu = m_\mu M_{Li}/(m_\mu + M_{Li})$ are the reduced masses of the subsystems ($e e^6\text{Li}^+$) and ($\mu e^6\text{Li}^+$). The Hamiltonian terms $\Delta H_{\text{VP}}, \Delta H_{\text{int}}$ and $\Delta H_{\text{vert}}$ which describe the VP, structure and vertex corrections, are constructed below. In the initial approximation the wave function of the ground state has the form:

$$\Psi_0(x_\mu, x_e) = \psi_0(x_\mu) \psi_0(x_e) = \frac{1}{\pi} \left( 6 \alpha^2 M_\mu M_e \right)^{1/2} e^{-3\alpha M_\mu x_\mu} e^{-2\alpha M_e x_e},$$

(3)

As follows from the structure of the Hamiltonian presented in (1)–(2), we include in the basic Hamiltonian $H_0$ only part of the Coulomb electron–nucleus interaction. The remainder is considered as a perturbation as is the Coulomb muon–electron interaction. In this way we can explore an analytical method for the calculation of hyperfine structure based on PT. An analytical solution for the wave function (3) allows us to obtain the perturbation contributions in two small parameters $\alpha$ and $M_\mu/M_e$ as demonstrated below. The corrections due to electron–muon interaction and mass polarization term (2) are considered in the second order of PT in subsequent sections.

They are first presented in an analytical integral form and then calculated analytically or numerically.

The basic contribution to the hyperfine interaction in the ground state of ($\mu e^6\text{Li}^+$) is determined by the following Hamiltonian:

$$\Delta H_{\text{HFS}} = \frac{2\pi\alpha}{3} \frac{g_e g_N}{m_\mu m_\rho} \left( S_\mu \cdot \mathbf{I} \right) \delta(x_\mu)$$

$$- \frac{2\pi\alpha}{3} \frac{g_e g_N}{m_\mu m_\rho} \left( S_\mu \cdot S_e \right) \delta(x_\mu - x_e)$$

$$+ \frac{2\pi\alpha}{3} \frac{g_e g_N}{m_\mu m_\rho} \left( S_e \cdot \mathbf{I} \right) \delta(x_e),$$

(4)

where $g_e$, $g_\mu$, and $g_N$ are the gyromagnetic factors of the electron, muon and nucleus. The total spin of the three spin particles can be either 2, 1 or 0 for ($\mu e^6\text{Li}^+$) and 5/2, 3/2 or 1/2 for ($\mu e^3\text{Li}^+$).

HFS of the energy levels in muonic lithium ions is determined by the following matrix elements:

$$\nu = \left\{ \Delta H_{\text{HFS}} \right\} = a \left\{ \mathbf{I} \cdot S_\mu \right\} - b \left\{ S_\mu \cdot S_e \right\} + c \left\{ S_e \cdot \mathbf{I} \right\},$$

(5)

where the spin-space expectation values can be calculated using the following basic transformation [17]:

$$\Psi_{S_\mu, S_N} = \sum_{S_{N\mu}} (-1)^{S_{N\mu} + S_e} \sqrt{(2S_{N\mu} + 1)(2S_e + 1)}$$

$$\times \left\{ S_e \quad S_N \quad S_{N\mu} \right\} \Psi_{S_{N\mu}, S_N},$$

(6)

$S_{N\mu}$ is the spin in the muon–nucleus subsystem, $S_N$ is the spin in the electron–nucleus subsystem and $S$ is total angular momentum. The properties of the 6j-symbols can also be found in [17]. As follows from (4) and (5), the basic contributions to coefficients $a$, $b$ and $c$ are the following:

$$a_0 = \frac{2\pi\alpha}{3} \frac{8 S_\mu S_N}{m_\rho m_\mu} \left\{ \delta(x_\mu) \right\},$$

$$b_0 = \frac{2\pi\alpha}{3} \frac{g_e g_\mu}{m_\mu m_e} \left\{ \delta(x_\mu - x_e) \right\},$$

$$c_0 = \frac{2\pi\alpha}{3} \frac{g_e g_N}{m_\mu m_\rho} \left\{ \delta(x_e) \right\},$$

(7)

where $\langle \ldots \rangle$ denotes the expectation value in coordinate space over wave functions (3). We have to take into account the numerical values of the gyromagnetic factors $g_e = 2$ for the coefficient $b$, $g_e = 2(1 + \kappa_e) = 2(1 + 1.1596521811(74) \cdot 10^{-3})$, and for the coefficient $c$, $g_e = 2(1 + \kappa_e) = 2 \cdot (1 + 1.16592069(60) \cdot 10^{-3})$, $g_N(\text{Li}) = 0.822047$, $g_N(\text{Li}) = 2.170951$.

The expectation value (5) is the $4 \times 4$ matrix corresponding to different values of total spin and muon–nucleus spin: ($S = 0$, $S_{N\mu} = \frac{1}{2}$), ($S = 1$, $S_{N\mu} = \frac{1}{2}$), ($S = 1$, $S_{N\mu} = \frac{3}{2}$), ($S = 2$, $S_{N\mu} = \frac{1}{2}$) for the ion ($\mu e^3\text{Li}^+$); and ($S = \frac{1}{2}$, $S_{N\mu} = \frac{1}{2}$), ($S = \frac{3}{2}$, $S_{N\mu} = 1$), ($S = \frac{1}{2}$, $S_{N\mu} = 1$), ($S = \frac{3}{2}$, $S_{N\mu} = 2$), ($S = \frac{5}{2}$, $S_{N\mu} = 2$) for...
the ion \((\mu e ^3\text{Li})^+\). After its diagonalization we obtain four energy eigenvalues \(\nu_i\). In the case of muonic lithium ions we have relations \(a \gg b\) and \(a \gg c\). So, small HFS intervals \(\Delta \nu_i\) related to the experiment can be written with good accuracy in the simple form:

\[
\Delta \nu_1^{HFS} (\mu e ^3\text{Li}) = \frac{2(b - 2c)}{3} + O \left( \frac{b}{a}, \frac{c}{a} \right),
\]

\[
\Delta \nu_2^{HFS} (\mu e ^3\text{Li}) = \frac{b + 4c}{3} + O \left( \frac{b}{a}, \frac{c}{a} \right),
\]

\[
\Delta \nu_1^{HFS} (\mu e ^3\text{Li}) = \frac{5(b - 3c)}{8} + O \left( \frac{b}{a}, \frac{c}{a} \right),
\]

\[
\Delta \nu_2^{HFS} (\mu e ^3\text{Li}) = \frac{3(b + 5c)}{8} + O \left( \frac{b}{a}, \frac{c}{a} \right).
\]

For the angular momentum of the muon–nucleus subsystem \(S_{JN} = 3/2\) and \(S_{JN} = 1/2\) \((\mu e ^3\text{Li})^+\) the HFS intervals (8) between states with total angular momentum \(S = 2, 1\) and \(S = 1, 0\) arise from magnetic interaction between the electron and pseudonucleus \((\mu e ^3\text{Li})^+\). The same situation is valid for HFS intervals (9) for \((\mu e ^3\text{Li})^+\). A schematic diagram of the HFS in muonic lithium ions is presented in figure 1.

In first order PT the basic contributions to the coefficients \(b\) and \(c\) (7) can be calculated analytically using (3) (hereafter the upper and lower values correspond to \((\mu e ^3\text{Li})^+\) and \((\mu e ^3\text{Li})^+\):)

\[
b_0 = \frac{2\pi \alpha}{3} \frac{g_e g_\mu}{m_e m_\mu} \int \psi^*(x_e, x_\mu) \delta(x_e - x_\mu) \psi(x_e, x_\mu) \times dx_e dx_\mu = \nu_F \left( \frac{1}{1 + 2M_e 3M_\mu} \right).
\]

\[
= \nu_F \left[ 1 + \kappa_\mu + \left( 1 + \kappa_\mu \right) \left( -\frac{2M_e}{M_\mu} + \frac{8 M_e^2}{3 M_\mu^2} \right) \right].
\]

\[
\nu_F = \frac{64 M_e^2 \alpha^4}{3 m_e m_\mu} = \left\{ \begin{array}{ll}
36140.290 \text{ MHz} & \text{for } \nu_F \\
36141.701 \text{ MHz} & \text{for } \nu_F
\end{array} \right.
\]

\[
c_0 = \frac{2\pi \alpha \frac{g_e g_N}{3 m_e m_\mu}} \int \psi^*(x_e, x_\mu) \delta(x_e - x_\mu) \psi(x_e, x_\mu) dx_e dx_\mu
\]

\[
= \nu_F \left( \frac{m_\mu}{m_e} \frac{g_e g_N}{m_\mu} \right) = \left\{ \begin{array}{ll}
1674.700 \text{ MHz} & \text{for } c_0 \\
4422.900 \text{ MHz} & \text{for } c_0
\end{array} \right.
\]

where we have extracted in square brackets the Fermi energy \(\nu_F\), muon anomalous magnetic moment correction \(\kappa_\mu\), and recoil terms. Their corresponding numerical values for two lithium ions are presented in table 1.

Note that, as we determine contributions to the energy spectrum numerically, corresponding results are presented with an accuracy of 0.001 MHz. We express further the HFS contributions in the frequency unit using the relation \(\Delta E^{HFS} = 2 \pi \hbar \Delta \nu^{HFS}\). Modern numerical values of fundamental physical constants are taken from [18–20]: the electron mass \(m_e = 0.510998928(11) \times 10^{-3}\) GeV, the muon mass \(m_\mu = 0.1056583715(35)\) GeV, the fine structure constant \(\alpha^{-1} = 137.035999074(44)\), the proton mass \(m_p = 0.938272046(21)\) GeV, the magnetic moments of the Li nucleus in nuclear magnetons \(\mu ^1\text{Li} = 0.8224073(6)\) and \(\mu ^3\text{Li} = 3.2564272\) for the masses of the Li nucleus \(M ^1\text{Li} = 5.60152\) GeV and \(M ^3\text{Li} = 6.53383\) GeV, the muon anomalous magnetic moment \(\kappa_\mu = 1.659209163(1) \times 10^{-3}\), and the electron anomalous magnetic moment \(\kappa_e = 1.15965218076\) for the muonic helium atom we present here corresponding results for muonic lithium ions. Some of these corrections have already appeared in the equation (10). In second order PT (SOPT) we also have the contribution to HFS which contains the necessary order corrections.

2. Recoil corrections

Let us consider a calculation of important recoil corrections of orders \(\alpha^4 M_e^4/3 m_e m_\mu\) and \(\alpha^4 M_e^2/3 m_e m_\mu\). Using the basic relations obtained in [6] for the muonic helium atom we have here corresponding results for muonic lithium ions.
Dividing the sum over muon states into two parts with $n = 0$ and $n \neq 0$ we obtain for the first part:

$$b_1(n = 0) = \frac{4\pi\alpha}{3} \frac{g_e g_i}{m_e m_p} \int \left[ \psi_{\mu0}(\mathbf{x}_e) \right]^2 \psi_{\alpha0}(\mathbf{x}_e) \times \Delta H\left(\mathbf{x}_e, \mathbf{x}_p, \mathbf{x}_\mu, \mathbf{x}_\mu'\right),$$

(12)

where the reduced Coulomb Green’s function has the form:

$$\tilde{G}\left(\mathbf{x}_e, \mathbf{x}_p; \mathbf{x}_\mu, \mathbf{x}_\mu'\right) = \sum_{n, n' \neq 0} \frac{\psi_{\mu0}(\mathbf{x}_e) \psi_{\alpha0}(\mathbf{x}_p) \psi_{\mu0}^*(\mathbf{x}_\mu') \psi_{\alpha0}^*(\mathbf{x}_\mu)}{E_{\alpha0} - E_{\mu0} - E_{\alpha0}' - E_{\mu0}'},$$

(13)

Dividing the sum over muon states into two parts with $n = 0$ and $n \neq 0$ we obtain for the first part:

$$b_1(n = 0) = \frac{4\pi\alpha}{3} \frac{g_e g_i}{m_e m_p} \int \left[ \psi_{\mu0}(\mathbf{x}_e) \right]^2 \psi_{\alpha0}(\mathbf{x}_e) \times \Delta H\left(\mathbf{x}_e, \mathbf{x}_p, \mathbf{x}_\mu, \mathbf{x}_\mu'\right),$$

(12)

where the reduced Coulomb Green’s function has the form:

$$\tilde{G}\left(\mathbf{x}_e, \mathbf{x}_p; \mathbf{x}_\mu, \mathbf{x}_\mu'\right) = \sum_{n, n' \neq 0} \frac{\psi_{\mu0}(\mathbf{x}_e) \psi_{\alpha0}(\mathbf{x}_p) \psi_{\mu0}^*(\mathbf{x}_\mu') \psi_{\alpha0}^*(\mathbf{x}_\mu)}{E_{\alpha0} - E_{\mu0} - E_{\alpha0}' - E_{\mu0}'},$$

(13)

For the subsequent integration over the coordinates in (14) we use the compact expression of the electron reduced Coulomb Green’s function obtained in [21]:

$$G_e(\mathbf{x}_e, \mathbf{x}_p) = \sum_{n, n' \neq 0} \frac{\psi_{\mu0}(\mathbf{x}_e) \psi_{\alpha0}(\mathbf{x}_p) \psi_{\mu0}^*(\mathbf{x}_\mu') \psi_{\alpha0}^*(\mathbf{x}_\mu)}{E_{\alpha0} - E_{\mu0} - E_{\alpha0}' - E_{\mu0}'}.$$
The second contribution to $b$ corresponding to muon excited states is equal to

$$b_1(n \neq 0) = \frac{4\pi\alpha}{3} \frac{g_e g_\mu}{m_e m_\mu} \int \psi_{\mu0}^*(x_3) \psi_{\mu0}(x_3)$$

$$\times \sum_{n \neq 0} \psi_{\mu0}(x_3) \psi_{\mu0}(x_2) G_{e}(x_3, x_1, z)$$

$$\times \left[ \frac{\alpha}{|x_2 - x_1|} - \frac{\alpha}{x_0} \right] \psi_{\mu0}(x_2) \psi_{\mu0}(x_1) dx_1 dx_2 dx_3,$$  \hspace{1cm} (18)

where the electron Coulomb Green’s function

$$G_e(x_3, x_1, z) = \sum_{n=0}^\infty \psi_{\mu0}^*(x_3) \psi_{\mu0}(x_1)$$

$$= \sum_{n=0}^\infty \psi_{\mu0}(x_3) \psi_{\mu0}(x_1)$$

$$= \frac{Me^2}{2\pi|x_3 - x_1|},$$  \hspace{1cm} (19)

The term $(-\alpha/x_0)$ does not contribute due to the orthogonality of muon wave functions. In order to perform an analytical integration in (18) we use a replacement of $G_e$ by the free electron Green’s function [6]:

$$G_e(x_3, x_1, E_{\mu0} + E_{e0} - E_{\mu0})$$

$$\rightarrow G_{e0}(x_3, x_1, E_{\mu0} + E_{e0} - E_{\mu0})$$

$$= \frac{Me^2 e^{-|x_3 - x_1|}}{2\pi|x_3 - x_1|},$$  \hspace{1cm} (20)

where $\beta = \sqrt{2Me^2 (E_{\mu0} - E_{e0} - E_{\mu0})}$. Moreover, we replace the electron wave functions in (18) with their values at the origin $\psi_{\mu0}(0)$. The omitted terms in this approximation can give contributions of second order in $\frac{M_\mu}{M_e}$. The results of numerical integration presented in [6] for muonic helium show that these corrections are numerically small. After used approximations an analytical integration over coordinate $x_i$ gives the result:

$$\int \frac{e^{-\beta|x_3 - x_1|}}{|x_3 - x_1|} dx_1$$

$$= \frac{\pi}{\beta^2} \left[ 1 + 2 \left| 1 - \frac{2}{\beta} |x_3 - x_1| \right| + \cdots \right],$$  \hspace{1cm} (21)

where an expansion of the exponent $e^{-\beta|x_3 - x_1|}$ over $\beta$ is used. It is equivalent to an expansion in powers of $\sqrt{M_e/M_\mu}$. Whereas the first term $\beta^{-2}$ does not contribute, the second term in (19) yields $-\nu_F \frac{3M_M^2}{2\alpha M_e}$. In addition, the third term in (21) the leads to the following integral:

$$\int \psi_{\mu0}^*(x_3) \sum_n \sqrt{2M_e} (E_{\mu0} - E_{\mu0})$$

$$\times \psi_{\mu0}(x_3) \psi_{\mu0}(x_2) (x_2 \cdot x_1) \psi_{\mu0}(x_2) dx_1 dx_2 dx_3$$

$$= \frac{1}{3\alpha M_e} \left( \frac{M_e}{M_\mu} \right)^{1/2} S_{1/2},$$  \hspace{1cm} (22)

where we define

$$S_{1/2} = \sum_n \left( \frac{E_{\mu0} - E_{\mu0}}{R_\mu} \right)^{1/2} H_{\mu0}(\frac{X}{a_\mu} |\mu\mu|) \cdot$$  \hspace{1cm} (23)

Discrete and continuum state contributions to (23) are correspondingly equal [22, 23]:

$$S_{1/2}^{dis} = \sum_n 2^n \left( n + \frac{2}{3} \right)^{-1/2} = 1.90695 ..., \hspace{1cm} (24)$$

$$S_{1/2}^{cont} = \int_0^\infty 2^k dk \left( \delta_n + 1 \right)^{1/2} \left( 1 + ik \right)^{1/2}$$

$$\times 1.03111 ..., \hspace{1cm} (25)$$

where $R_\mu = \frac{3M_M^2}{2\alpha M_e}$. Summing the corrections in the first and SOPT, we obtain the total recoil correction to the coefficient $b$ in order $\alpha^2$:

$$b_{rec} = \nu_F \left( 1 + \nu_F \right) \left[ -3M_e M_\mu + 8 \frac{M_e^2}{9 M_\mu} \ln \frac{M_\mu}{2 M_e} + \frac{4}{9} \left( \frac{M_e}{M_\mu} \right)^{1/2} \right]$$

$$\times S_{1/2}$$

There are similar contributions to the coefficient $c$ in SOPT. In order to obtain these we have to use $\Delta H_{SPT}(x_e) = \frac{2\alpha M_e}{3 M_\mu} \delta(x_e)$ in the general expression (12). After evident simplifications the recoil correction to $c$ can be written as:

$$c_1 = \frac{4\pi\alpha}{3} \frac{g_e g_\mu}{m_e m_\mu} \sum_n \psi_{\mu0}(0) \psi_{\mu0}(x_3)$$

$$\times \left[ \frac{1}{4\alpha M_e x} - \ln \frac{4\alpha M_e x}{2} + C - 2\alpha M_e x \right].$$  \hspace{1cm} (27)

The result of an analytical integration is presented as an expansion in $M_e/M_\mu$:

$$c_1 = \left[ \frac{M_e}{M_\mu} + 8 \frac{M_e^2}{9 M_\mu} \left( \frac{1}{4} + \frac{3}{2} - \ln \frac{M_\mu}{M_e} \right) \right]$$

$$= 8.467 \text{ MHz}.$$

$$22.302 \text{ MHz}.$$  \hspace{1cm} (28)
3. The effects of VP

The VP effects lead to the appearance of new terms in the Hamiltonian which we denote by $\Delta H_{\text{VP}}$ in (1). The ratio of the electron Compton wave length to the Bohr radius in the subsystem ($\mu^{2}_{Li}$) : $Zm_{e}/m_e = 2.96185...$ is not a small value. So, we cannot use for the calculation of VP effects an expansion over $\alpha$. In this section we present a calculation of VP corrections to hyperfine structure in the first and second orders of PT. A modification of the Coulomb potentials due to VP effects is described by the following relations [24, 25]:

$$
\Delta V_{\text{VP}}^\alpha(x_e) = \frac{\alpha \pi}{3} \int x_e^{\infty} \rho(\xi) \left( -\frac{3\alpha}{x_e} \right) e^{-2\alpha \xi \xi_e} d\xi,
$$

$$
\rho(\xi) = \frac{\gamma E - 1}{\xi^4},
$$

$$
\Delta V_{\text{VP}}^\alpha(x_{\mu}) = \frac{\alpha \pi}{3} \int x_{\mu}^{\infty} \rho(\xi) \left( \frac{3\alpha}{x_{\mu}} \right) e^{-2\alpha \xi \xi_{\mu}} d\xi,
$$

$$
\Delta V_{\text{VP}}^\alpha(x_e - x_{\mu}) = \frac{\alpha \pi}{3} \int x_{\mu}^{\infty} \rho(\xi) \left( \frac{3\alpha}{x_{\mu}} \right) e^{-2\alpha \xi \xi_{\mu}} d\xi,
$$

where $x_{\mu} = |x_e - x_{\mu}|$. These terms give contributions to the hyperfine structure in the SOPT and are discussed below. The VP correction in the first order PT is connected with the modification of HFS part of the Hamiltonian (4) (the amplitude in figure 2(a)). It can be written in integral form in the coordinate representation [26]:

$$
\Delta V_{\text{VP}}^{\text{HFS,eq}}(x_{\mu}) = -\frac{8\alpha^2}{9m_e m_{\mu}} \left( S_e \cdot S_{\mu} \right) \frac{\alpha}{3\pi} \int x_{\mu}^{\infty} \rho(\xi) d\xi \left[ \pi \delta(x_e - x_{\mu}) - \frac{m_e^2 \xi^2}{x_{\mu}^4} e^{-2m_e \xi \xi_{\mu}} \right],
$$

$$
\Delta V_{\text{VP}}^{\text{HFS,eq}}(x_e) = \frac{8\alpha^2}{9m_e m_p} \left( S_e \cdot I \right) \frac{\alpha}{3\pi} \int x_e^{\infty} \rho(\xi) d\xi \left[ \pi \delta(x_e - x_{\mu}) - \frac{m_e^2 \xi^2}{x_e^4} e^{-2m_e \xi \xi_{\mu}} \right].
$$

The matrix element of the potential (32) over the wave function (3) gives the necessary contribution to the coefficient

$$
b_{\text{VP}} = \frac{\pi \rho(\xi)}{9m_e m_{\mu}} \int x_{\mu}^{\infty} \rho(\xi) d\xi \left[ \pi \delta(x_e - x_{\mu}) - \frac{m_e^2 \xi^2}{x_{\mu}^4} e^{-2m_e \xi \xi_{\mu}} \right].
$$

$$
I_1 = \int dx_e \int dx_{\mu} e^{-6m_e \xi \xi_e} e^{-4m_{\mu} \xi \xi_{\mu}} \pi \delta(x_e - x_{\mu})
$$

$$
= \frac{\pi^2}{(3\alpha M_p)^4} \frac{1}{1 + \frac{2M_e}{3M_{\mu}}},
$$

$$
I_2 = \int dx_e \int dx_{\mu} e^{-6m_e \xi \xi_e} e^{-4m_{\mu} \xi \xi_{\mu}} \pi \delta(x_e - x_{\mu})
$$

$$
= \frac{\pi^2 m_e^2 \xi^2}{(3\alpha M_p)^4} \left[ \frac{4M_e^2}{9M_p^2} \left( 1 + \frac{m_e \xi}{3M_{\mu} \alpha} \right)^2 + \frac{M_e}{M_p} \left( 6 + \frac{4m_e \xi}{3M_{\mu} \alpha} \right) \left( 1 + \frac{m_e \xi}{3M_{\mu} \alpha} \right) \right].
$$

They are separately divergent in the subsequent integration over the spectral parameter $\xi$. But their sum is finite and can be written as follows:

$$
b_{\text{VP}} = \nu_F \frac{2\alpha M_e}{9\pi M_p} \left[ \frac{2M_e}{3M_{\mu} \alpha} + \frac{2m_e \xi}{3M_{\mu} \alpha} + \frac{m_e \xi}{3M_{\mu} \alpha} \right] \left( 1 + \frac{m_e \xi}{3M_{\mu} \alpha} \right) \left( \frac{2M_e}{3M_{\mu} \alpha} + \frac{m_e \xi}{3M_{\mu} \alpha} \right) \left( 1 + \frac{m_e \xi}{3M_{\mu} \alpha} \right)
$$

$$
= \left\{ \begin{array}{ll}
0.701 \text{ MHz} & \\
0.706 \text{ MHz}, & \\
\end{array} \right.
$$

Two small parameters $\alpha$ and $M_{\mu}/M_{\mu}$ determine the order of this contribution and are written explicitly in (37). The correction $b_{\text{VP}}$ has the fifth order in $\alpha$ and the first order in $M_{\mu}/M_{\mu}$. The muon VP contribution to HFS is negligibly small. The two-loop VP correction to the hyperfine structure is suppressed relative to the one-loop VP contribution by the factor $\alpha^3/\pi$. Thus at the present level of accuracy we can neglect this correction because its numerical value is small. Higher orders of PT which contain one-loop VP and the Coulomb interaction (2) give recoil corrections of order...
\[ \nu_{\text{VP}} \frac{\alpha c_{\text{VP}}}{M_0} \alpha \frac{m_e}{M} \]  

Such contributions are included in the theoretical error.

A similar correction to the coefficient \( c \) of order \( \alpha^6 \) is calculated analytically using the potential (33) \((\alpha_1 = 2 \alpha M_e / m_e):

\[
c_{\text{VP}} = \nu_{\text{VP}} \frac{\alpha g_{\text{VP}} m_e}{6 \pi m_p} \left( 1 - \alpha_1^2 \left( \alpha_1 + \frac{3}{2} \right) + \frac{6 - 3 \alpha_1^2 + 6 \alpha_1^2 \arccos \alpha_1}{3 \alpha_1 \sqrt{1 - \alpha_1^2}} \right)
\]

\[
= \begin{cases} 
0.066 \text{ MHz} \\
0.175 \text{ MHz} 
\end{cases}
\]

(38)

The electron VP effect (the potentials (29)–(31)) gives the corrections in SOPT (the amplitude in figure 2(b)). The contribution of the electron–nucleus Coulomb interaction (29) to the HFS can be presented in the form:

\[
b_{\text{VP, SOPT}}^e = 4 \pi \alpha g_{\text{VP}} \frac{g}{3 m_e m_p} \int dx_1 \int dx_2 \frac{\alpha}{3 \pi} \int_0^\infty \rho(\xi) d\xi \int_0^\infty x_2^2 d\xi \\
\times \left( \psi_{\text{VP}}(x_1) \psi_{\text{VP}}(x_2) \psi_{\text{VP}}(x_3) \right) \\
\times \sum_{n \neq n'} \frac{\psi_{\text{VP}}(x_1) \psi_{\text{VP}}(x_2) \psi_{\text{VP}}(x_3)}{E_{n} - E_{n'} - E_{\text{VP}}}
\times \left( -3 \alpha \frac{\alpha}{x_1} \right) e^{-2\alpha x_1} \psi_{\text{VP}}(x_2) \psi_{\text{VP}}(x_1),
\]

(39)

where the indices on the coefficient \( b \) indicate the VP contribution in SOPT when the electron–nucleus Coulomb VP potential is considered. The summation in (39) is carried out over the complete system of the eigenstates of the electron and muon excluding the state with \( n, n' = 0 \). The evaluation of (39) can be carried out using the orthogonality condition for the muon wave functions:

\[
b_{\text{VP, SOPT}}^e = \nu_{\text{VP}} \frac{2 \alpha M_e^2}{9 \pi M_0} \int_0^\infty \rho(\xi) d\xi \int_0^\infty x_2^2 d\xi \\
\times \left[ \frac{3 M_0}{2 M_0 x_c} - \ln \left( \frac{2 M_0}{3 M_0 x_c} \right) \right] e^{-2\alpha x_1} \psi_{\text{VP}}(x_2) \psi_{\text{VP}}(x_1)
\]

\[
= \begin{cases} 
1.136 \text{ MHz} \\
1.137 \text{ MHz} 
\end{cases}
\]

It is necessary to emphasize that the transformation of the expression (39) into (40) is performed by means of (16).

The contribution (40) has the same order of magnitude \( O(\alpha^5 M_0 / M) \) as the previous correction (37) in the first order PT. The same evaluation can be performed in the case of the muon–nucleus Coulomb VP potential (30). The electron is in the intermediate 1S-state and the reduced Coulomb Green’s function of the system transforms to the Green’s function of the muon. For this case the correction of the operator (30) to the HFS (the coefficient \( b \)) is obtained in the form:

\[
b_{\text{VP, SOPT}}^\mu = \nu_{\text{VP}} \frac{\alpha}{3 \pi} \int_0^\infty \rho(\xi) d\xi \int_0^\infty x_2^2 d\xi \\
\times \left[ \frac{1}{x_2} - \ln x_2 - \ln x_3 + E(\xi, x_2) \\
+ \frac{7}{2} - 2 C - \frac{x_2 + x_3}{2} + \frac{1}{x_3} \right]
\]

\[
= \begin{cases} 
0.694 \text{ MHz} \\
0.693 \text{ MHz} 
\end{cases}
\]

(41)

The most difficult aspect for the computation is the VP correction to HFS which is determined by the operator (31) in SOPT. In this case we should consider the intermediate excited states both for the muon and the electron. This contribution is divided into two parts. The first part with the muon in the intermediate 1S-state has the form:

\[
b_{\text{VP, SOPT}}^\mu (n = 0) = 256 \alpha^2 (2 \alpha M_0)^3 (3 \alpha M_\mu)^3 \\
\times \int_0^\infty x_3^2 d\xi e^{-2\alpha (2 M_0 x_3)}
\times \int_0^\infty x_1^2 d\xi e^{-2\alpha M_\mu x_1} \int_0^\infty \rho(\xi) d\xi \Delta V_{\text{VP}, \mu}(x) G_E(x, x_1),
\]

(42)

where the auxiliary function \( V_{\text{VP}, \mu}(x) \) is equal

\[
\Delta V_{\text{VP}, \mu}(x_1) = \int dx_2 e^{-2\alpha x_2} \left( \frac{3 \alpha M_\mu}{\pi} \right)^3 \\
\times \frac{\alpha}{|x_1 - x_2|} e^{-2\alpha x_2} \xi^2 \\
= \frac{108 \alpha^3 M_\mu^2}{x_1 \left( 36 \alpha^2 M_\mu^2 - 4 M_0^2 \xi^2 \right)^2} \\
\times \left[ 12 \alpha M_\mu \left( e^{-2\alpha M_\mu x_1} - e^{-2\alpha M_\mu x_3} \right) \\
+ x_1 \left( 4 M_0^2 \xi^2 - 36 \alpha^2 M_\mu^2 \right) e^{-2\alpha M_\mu x_3} \right].
\]

(43)

Substituting (43) into (42) we obtain the result after numerical integration:

\[
b_{\text{VP, SOPT}}^\mu (n = 0) = \begin{cases} 
-0.310 \text{ MHz} \\
-0.311 \text{ MHz} 
\end{cases}
\]
The second part of the correction to HFS with VP coming from the electron–muon interaction can be written as follows:

\[ b_{\text{VP, SOPT}}(n \neq 0) = - \frac{4e^2}{9m_e m_p} \int dx_3 \]

\[ \times \int dx_2 \int_{-\infty}^{\infty} \rho(\xi) d\xi \psi_{\mu_0}^{\ast}(x_3) \psi_{\mu_0}(x_3) \]

\[ \times \sum_{n \neq 0} \psi_{\mu}^{\ast}(x_3) \psi_{\mu}(x_3) \]

\[ \times \frac{\alpha}{|x_3 - x_1|} e^{-2m_e |x_3 - x_1|} \psi_{\mu_0}(x_3) \psi_{\mu_0}(x_1), \quad (45) \]

where we have replaced the exact electron Coulomb Green’s function with the free electron Green’s function. Also neglecting the higher order recoil corrections we replace the electron wave functions by their values at the origin. Then the integration over \( x_3 \) can be performed analytically:

\[ J = \int dx_3 \frac{e^{-\beta |x_3 - x_1|} e^{-2m_e |x_3 - x_1|}}{x_3 - x_2} \]

\[ = \frac{4\pi}{|x_3 - x_2|} \beta^2 - 4m_e^2 \frac{\xi}{\xi^2} \]

\[ \times \left( \frac{1 - e^{-2m_e |x_3 - x_1|}}{2m_e^2 \xi^2 |x_1 - x_2|} - \frac{\beta}{2m_e^2 \xi^2} \right) \]

\[ \times \left( \frac{1 - e^{-2m_e |x_3 - x_1|}}{8m_e^2 \xi^4 |x_3 - x_2|} + \frac{\beta^2}{4m_e^2 \xi^2} + \cdots \right), \quad (46) \]

where an expansion of the first exponent in square brackets in powers of \( \beta |x_3 - x_1| \) is carried out. Further transformation is based on the completeness condition:

\[ \sum_{n \neq 0} \psi_{\mu}^{\ast}(x_3) \psi_{\mu}(x_3) = \delta(x_3 - x_2) - \psi_{\mu_0}(x_3) \psi_{\mu_0}^{\ast}(x_2). \quad (47) \]

The orthogonality of wave functions leads to the zero results for the second and fifth terms in the square brackets of (46). The first term in (46) gives the leading order contribution in two small parameters \( \alpha \) and \( M_e/M_p \) (\( \gamma = m_e \xi / 3\alpha M_p \)):

\[ b_{\text{VP, SOPT}}^{\mu\nu}(n \neq 0) = b_{11} + b_{12} \]

\[ = -0.432 \text{ MHz} \quad b_{11} = -3 \alpha^2 M_e \]

\[ -0.431 \text{ MHz} \quad b_{12} = \frac{2\alpha^2 M_e}{8m_e} \nu_F, \quad (48) \]

\[ b_{12} = \nu_F \frac{\alpha^2 M_e}{24\pi m_e} \int_{-1}^{\infty} \frac{\rho(\xi) d\xi}{(1 - \gamma^2)^2} \frac{[16 + 5(\gamma + 4) + 29]}{(1 + \gamma)^3}. \quad (49) \]

The summary numerical value for \( b_{11} + b_{12} \) is included in Table 1. The calculation of other terms of the expression (46) in the HFS is also important. Taking the fourth term in (46), which is proportional to \( \beta^2 = 2M_e (E_{\mu 0} - E_{\mu}), \) we perform the sequence of transformations in the coordinate representation:

\[ \sum_{n=0}^{\infty} E_{\mu_0} \int dx_3 \int dx_2 \psi_{\mu_0}^{\ast}(x_3) \psi_{\mu_0}(x_3) \]

\[ \times \psi_{\mu_0}(x_2) \left| x_3 - x_2 \right| \psi_{\mu_0}(x_2) \]

\[ = \int dx_2 \int dx_3 \delta(x_1 - x_2) \]

\[ \left[ - \frac{\Sigma_3}{2M_p} \left| x_3 - x_2 \right| \psi_{\mu_0}(x_3) \right] \psi_{\mu_0}(x_2). \quad (50) \]

We have the divergent expression in (50) due to the presence of the \( \delta \)-function. The same divergence occurs in the other term with \( \beta^2 \) entering in the square brackets of (46). But their sum is finite and can be evaluated analytically with the following result:

\[ b_{\beta^2} = \nu_F \frac{3\alpha^2 M_e^2}{8m_e M_p} \left( 1 + \frac{5\alpha^2 M_e^2}{8m_e^2} \right) \quad (51) \]

Numerically this correction is essentially smaller than the leading order term. The other terms in (46) give negligibly small contributions.

The potential (29) with VP does not contain the muon coordinate. A corresponding contribution to the coefficient \( c \) in SOPT can be determined by setting \( n = 0 \) for the muon state in the Coulomb Green’s function. Moreover, the \( \delta(x_2) \) function in (4) leads to the appearance of the electron Green’s function with one zero argument. The value of HFS in this case is equal to

\[ c_{\text{VP, SOPT}} = \nu_F \frac{\alpha m_e g_N 8N}{4\pi m_p} \]

\[ \times \int_{-1}^{\infty} \frac{\rho(\xi) d\xi}{(1 - \gamma^2)^2} \frac{2d_1 + 2a_1 + 2\ln a_1 - 2}{2a_1^2} \]

\[ = \{ 0.104 \text{ MHz} \}

\[ 0.274 \text{ MHz} \quad (52) \]

The VP in the Coulomb muon–nucleus (\( \mu-N \)) interaction does not contribute to \( c \) in SOPT because of the muon wave function’s orthogonality. Let us calculate the correction to the coefficient \( c \) arising from (31) in SOPT. Only an intermediate muon state with \( n = 0 \) in the Green’s function gives the necessary contribution. By means of (27) we perform coordinate integration and express this correction in the form

\[ (\gamma = m_e \xi / 3\alpha M_p, \gamma_1 = 2M_e / 3M_p): \]

\[ c_{\text{VP, SOPT}}^{\mu\nu} = -\nu_F \frac{2\alpha m_e g_N M_e^2}{27\pi m_p M_p^2} \]

\[ \times \int_{-1}^{\infty} \frac{\rho(\xi) d\xi}{(1 - \gamma^2)^2} \int_{-1}^{\infty} x e^{-\gamma x} dx \]

\[ \times \left[ e^{-\gamma x} + e^{-x} + \frac{x}{2} e^{-x} \left( \gamma^2 - 1 \right) \right] \]

\[ \times \left[ \frac{1}{\gamma_1 x} - 2 \ln \gamma_1 x - C - \frac{\gamma_1 x}{2} \right] \]

\[ = \{ -0.018 \text{ MHz} \}

\[ -0.047 \text{ MHz} \}. \quad (53) \]
There is another contribution of SOPT in which the hyperfine VP potential enters as a perturbation (32)–(33) (see figure 3). The other perturbation potential in this case is determined by the first term of equation (2). Dividing the HFS correction of (33) into two parts we write the first part with \( n = 0 \) for the muon ground state. The second part with \( n \neq 0 \) contains excited muon states. The term with the \( \delta \)-function in (32) gives the following contribution to HFS at \( n = 0 \):

\[
b^{(1)}_{\text{VP, SOPT}}(n = 0) = \nu_{\text{F}} \frac{\alpha}{3\pi} \int_{-\infty}^{\infty} \rho(\xi) d\xi \frac{11M_e}{24M_\mu}. \tag{54}
\]

The integral in the spectral variable \( \xi \) is divergent. So, we should consider the contribution of the second term of the potential (32) to HFS which is determined by the following formula:

\[
b^{(2)}_{\text{VP, SOPT}}(n = 0) = \frac{16\alpha^2 m_e^2}{9\pi m_\mu m_\mu} \int_{1}^{\infty} \rho(\xi) \xi^2 d\xi \times \int \Delta V_1(x_3) \psi_{\text{d}}(x_3) \Delta V_1(x_3) \psi_{\text{d}}(x_3) dx_3,
\]

where \( \Delta V_1(x_3) \) is defined in (43) and \( \Delta V_2(x_3) \) in (15). Integrating in (55) over all coordinates we obtain the following result in the leading order in \( (M_\mu/M_\mu) \):

\[
b^{(2)}_{\text{VP, SOPT}}(n = 0) = -\nu_{\text{F}} \frac{m_e^2}{M_\mu} \frac{M_\mu^2}{216\pi M_\mu^2} \int_{1}^{\infty} \rho(\xi) \xi^2 \frac{32 + 63\gamma + 44\gamma^2 + 11\gamma^3}{(1 + \gamma)^4} d\xi. \tag{56}
\]

This integral also has the divergence at large values of the parameter \( \xi \). But the sum of integrals (54) and (56) is finite:

\[
b^{(1)}_{\text{VP, SOPT}}(n = 0) + b^{(2)}_{\text{VP, SOPT}}(n = 0) = \nu_{\text{F}} \frac{\alpha M_\mu}{72\pi m_\mu} \int_{1}^{\infty} \rho(\xi) d\xi \frac{11 + 12\gamma + 3\gamma^2}{(1 + \gamma)^4}
\]

\[
= \{ 0.067 \text{ MHz} \}
\]

Let us evaluate the terms in the coefficient \( b \) with \( n \neq 0 \). The \( \delta \)-like term of the potential (33) gives the following contribution to the HFS:

\[
b^{(2)}_{\text{VP, SOPT}}(n \neq 0) = \nu_{\text{F}} \frac{2\alpha M_\mu}{9\pi m_\mu} \int_{1}^{\infty} \rho(\xi) d\xi \frac{11 + 12\gamma + 3\gamma^2}{(1 + \gamma)^4}
\]

\[
\times \int \Delta V_1(x_3) \psi_{\text{d}}(x_3) \Delta V_1(x_3) \psi_{\text{d}}(x_3) dx_3,
\]

\[
\times \left( \frac{1}{|x_3 - x_4|} - \frac{1}{|x_4|} \right) \frac{G_e(x_4, x_4)}{x_4} \frac{e^{-2m_e|x_4|}}{x_4} \right). \tag{62}
\]

The absolute values of the calculated VP corrections (38), (42), (44), (45), (47), (57) and (61) are sufficiently large, but their summary contribution to the HFS (see table 1) is small because they have different signs.

The HFS interaction (33) gives the contributions to the coefficient \( c \) in SOPT. Since the muon coordinate does not enter into the potential (33), we set \( n = 0 \) for the muon intermediate states in the Green’s function. The basic formula for this correction is

\[
c_{\text{VP, SOPT}} = \frac{8\alpha^3 g_{\text{HFS}}}{9\pi m_\mu m_\mu} \int_{1}^{\infty} \rho(\xi) d\xi \int dx_3 \int dx_3
\]

\[
\times \left( \frac{1}{|x_3 - x_4|} - \frac{1}{|x_4|} \right) \frac{G_e(x_4, x_4)}{x_4} \frac{e^{-2m_e|x_4|}}{x_4} \right). \tag{62}
\]
Integrating over $\chi_3$ analytically as in (15) we divide (62) into two parts. The coordinate integration in the first term with the $\delta$-function is performed by means of (28). In the second term of (62) we use the electron Green’s function in the form (16). The summary result can be written in the integral form in the leading order in $M_e/M_p$:

$$c_{\text{VP, SOPT}} = \nu_F \frac{\alpha g_e m_e M_e}{18\pi m_p M_p} \times \int_{1}^{\infty} \rho(\xi) d\xi \left( \frac{3 + 2 m_e \xi}{3\alpha M_p} \right)^2 = \left\{ \begin{array}{l} 0.017 \text{ MHz} \smallskip \quad \text{0.044 MHz} \end{array} \right. \quad (63)$$

4. Nuclear structure and recoil effects

Another important type of correction to HFS of muonic lithium ions which we investigate in this work is determined by the nuclear structure and recoil [27–31]. We describe the charge and magnetic moment distributions of the Li nucleus by means of two form factors $G_E(k^2)$ and $G_M(k^2)$ for which we use the dipole parameterization:

$$G_E(k^2) = \frac{1}{\left(1 + \frac{k^2}{\Lambda^2}\right)^2}, \quad G_M(k^2) = \frac{G(0)}{\left(1 + \frac{k^2}{\Lambda^2}\right)^2}, \quad G(0) = \frac{g_N m}{Z m_p}, \quad (64)$$

where the parameter $\Lambda$ is related to the nuclear charge radius $r_N$: $\Lambda = \sqrt{12}/r_N$. In $1\gamma$-interaction the nuclear structure correction to the coefficient $c$ is determined by the amplitudes shown in figure 4. The purely point contribution in figure 4(b) leads to the HFS value (11). Then the nuclear structure correction is given by

$$c_{\text{str}}, 1\gamma = \nu_F \frac{g_N m}{4m_p} \left[ \int \frac{G_M(x)}{G_M(0)} e^{-\alpha M_e r} dx - 1 \right]$$

$$= \left\{ \begin{array}{l} -0.283 \text{ MHz} \smallskip \quad -0.707 \text{ MHz} \end{array} \right. \quad (65)$$

Two-photon amplitudes of the electron–nucleus (e–N) interaction (see figure 5) give the contribution to HFS of order $\alpha$. This can be presented in integral form in terms of the form factors $G_E$ and $G_M$ taking into account the subtraction term [30, 32]:

$$c_{\text{str}}, 2\gamma = \nu_F \frac{3\alpha M_e m \gamma_s g_N}{2\pi^2 m_p} \int \frac{d^3p}{p^3} \frac{G_M(p)}{G_M(0)} \left( G_E(p) - 1 \right), \quad (66)$$

where the subtraction term contains the magnetic form factor $G_M(p)$. Using the dipole parameterization (64) we can present the last integral in analytical form:

$$c_{\text{str}}, 2\gamma = \nu_F \frac{3\alpha M_e m \gamma_s g_N}{16m_p \lambda} \left\{ \begin{array}{l} -0.195 \text{ MHz} \smallskip \quad -0.486 \text{ MHz} \end{array} \right. \quad (67)$$

Other parts of the iteration contribution $\langle \gamma_1 \times G' \times V_{\text{HFS}} \rangle$ are used in SOPT (see figure 6).

Two different types of nuclear structure corrections to another coefficient $c$ in SOPT are presented in figure 6. The first contribution is determined by amplitudes in figure 6(a), (b) when the hyperfine part of the first perturbation is determined by magnetic form factor $G_M$ and the second perturbation is connected to the nucleus charge radius $r_N$:

$$\Delta V_{\text{str, e–N}}(r) = \frac{2}{3} \pi Z \delta^2(r). \quad (68)$$

This correction is described by the following general integral expression and has the numerical value ($a_2 = 4\alpha M_e/\lambda$):

$$c_{\text{str}}, c_{\text{SOPT}} = \nu_F \frac{\alpha^2 M_e^2 g_s g_N m}{m_p} \int_0^{\infty} x^2 dx e^{-x(1+a_2)} \left( -\ln a_2 x + \frac{5}{2} - C - \frac{1}{2} a_2 x \right)$$

$$= \left\{ \begin{array}{l} -0.0003 \text{ MHz} \smallskip \quad -0.0008 \text{ MHz} \end{array} \right. \quad (69)$$

Numerically, the contribution $c_{\text{str}}, \text{SOPT}$ is obtained by means of the charge radii of nucleus $\frac{1}{2}^3$Li $r_{\frac{1}{2}^3}$Li = 2.589(39) fm and $r_{\frac{1}{2}^3}$Li = 2.444(42) fm [33]. The second type of nuclear structure correction from amplitudes in figure 6(c), (d) is calculated by means of the potential $\Delta H$ (2) and the nucleus magnetic form factor. In the case of the amplitude in figure 6(c) we perform the integration over the muon coordinate in the muon state with $n = 0$ and present the correction to the coefficient $c$ as follows ($a_2 = 6\alpha M_e/\lambda$):

$$c_{\text{str}}, \text{SOPT} + c_1 = \nu_F \frac{2\alpha^2 M_e^2 m g_s g_N}{m_p a_2^2} \times \int_0^{\infty} x_1^2 dx_1 e^{-a_2 x_1} \int_0^{\infty} x_2 dx_2 \times \left( 1 + \frac{1}{2} a_2 x_2 \right) e^{-a_2 x_2}$$

$$\times \left[ \begin{array}{l} \frac{1}{a_2 x_2} - \ln(a_2 x_2) - \ln(a_2 x_2) + E(a_2 x) \smallskip + \frac{7}{2} - C - \frac{1}{2} a_2 (x_1 + x_2) + \frac{1 - e^{a_2 x_1}}{a_2 x_1} \end{array} \right]$$

$$= \left\{ \begin{array}{l} 8.314 \text{ MHz} \smallskip \quad 21.918 \text{ MHz} \end{array} \right. \quad (70)$$

After the subtraction of the point contribution $c_1$ (28) we obtain

$$c_{\text{str}}, \text{SOPT} = \left\{ \begin{array}{l} -0.153 \text{ MHz} \smallskip \quad -0.384 \text{ MHz} \end{array} \right. \quad (71)$$

There is a nuclear structure contribution to the coefficient $b$ in SOPT, which is presented in figure 7. For the Coulomb muon–nucleus interaction the structure correction takes the
An analytical integration over the coordinate $x_3$ in (72) can be performed using the expression for the muon Green’s function similar to (27). Expanding the result of the integration of order $O(\alpha^5)$ in the ratio $M_e/M_\mu$ we obtain:

$$b_{\mu-N}^{\mu-N} = \frac{32\pi^2\alpha^2}{3m_\mu m_\mu} \frac{1}{\sqrt{\pi}} \left(3\alpha M_\mu \right)^{3/2}$$

$$\times \int \psi^\dagger \psi(x_3) \left| \psi(x_0(x_3)) \right|^2 G_\mu(x_3, 0, E_{\mu0}).$$  \hspace{5cm} (72)

A similar approach can be used in the calculation of the structure correction to the electron–nucleus interaction. The electron also feels the distribution of the nucleus electric charge. The corresponding contribution of the nuclear structure effect on the hyperfine structure is determined by the formula:

$$b_{\mu-N}^{\mu-N} = \frac{32\pi^2\alpha^2}{3m_\mu m_\mu} \int \psi^\dagger \psi(x_3) \left| \psi^\dagger \psi(x_3) \right|^2$$

$$\times \psi(x_0(x_3)) G_e(x_3, \xi_0, E_{\mu0}) \psi(x_3) \delta(x_0).$$  \hspace{5cm} (74)
After an analytical integration in (74) we obtain the following expansion:

\[
b_{\text{rec}N} = -\nu_F 6\alpha^2 e^2 m_e r_0^2 \times \left[ 1 - \frac{4M_e}{3M_\mu} \ln \frac{2M_e}{3M_\mu} + \frac{4M_e^2}{9M_\mu^2} \left(6 \ln \frac{2M_e}{3M_\mu} - 4\right) + \ldots \right]
\]

\[
= \{-0.109 \text{ MHz}, -0.098 \text{ MHz}. \quad (75)
\]

The total nuclear structure contribution to the coefficient \(b\) which is equal to the sum of numerical values (73) and (75) is included in table 1.

The two-photon electron–muon interaction shown in figure 8 provides large recoil corrections. They were investigated in quantum electrodynamics in [10, 24, 34]. The leading order recoil contribution to the electron–muon interaction operator is determined by the following expression:

\[
\Delta \psi_{\text{HFS},-e}(x_i) = -\frac{8 \alpha^2}{m_\mu^2 - m_e^2} \ln \frac{m_\mu}{m_e} \delta(x_i).
\]

After averaging \(\Delta \psi_{\text{HFS},-e}\) over the wave functions (3) we obtain the recoil correction to the coefficient \(b\):

\[
b_{\text{rec},e} = \nu_F 3\alpha \frac{m_e m_\mu}{m_\mu - m_e} \ln \frac{m_\mu}{m_e} \left(1 + \frac{2M_e}{3M_\mu}\right)^3
\]

\[
= \{6.430 \text{ MHz}, 6.431 \text{ MHz}. \quad (77)
\]

There are also another two-photon interactions between the bound particles in muonic lithium ions. So, for example, one hyperfine photon transfers the interaction from the electron to the muon and another Coulomb photon from the electron to the nucleus (or from the muon to the nucleus). Assuming that these amplitudes give a smaller contribution to HFS we included them in the theoretical error.

The first-order recoil correction \(O(M_e/M_\mu)\) has a contribution from intermediate states in which the muon and electron are excited to P-states:

\[
\Delta b_{\text{rec,SOPT}} = \frac{32\alpha^3 M_e M_\mu}{m_e m_\mu M_\mu} \int dx_i
\]

\[
\times \int dx_2 \int dx_3 \psi^*_\mu(x_2) \psi^*_\mu(x_3) \psi_{\text{HFS}}(x_i) \psi_{\text{HFS}}(x_i)
\]

\[
\times \sum_{n,n',\alpha^0} \psi_{\mu}(x_2) \psi_{\mu}(x_3) \frac{e^{-b|\mathbf{n}|r_0}}{|x_i - x_i|} \left(\mathbf{n}_1 \cdot \mathbf{n}_2\right) \psi_{\text{HFS}}(x_2) \psi_{\text{HFS}}(x_3).
\]

(78)

In order to present an analytical estimate of this correction we transform (78) as in section 2, introducing the free electron Green’s function:

\[
\Delta b_{\text{rec,SOPT}} = \frac{16\alpha^3 M_e^2 M_\mu}{m_e m_\mu M_\mu} \int dx_i
\]

\[
\times \int dx_2 \int dx_3 \psi_{\mu}(x_2) \psi_{\mu}(x_3) \psi_{\text{HFS}}(x_i) \psi_{\text{HFS}}(x_i)
\]

\[
\times \sum_{n,n',\alpha^0} \psi_{\mu}(x_2) \psi_{\mu}(x_3) e^{-b|\mathbf{n}|r_0} \frac{e^{-b|\mathbf{n}|r_0}}{|x_i - x_i|} \left(\mathbf{n}_1 \cdot \mathbf{n}_2\right) \psi_{\text{HFS}}(x_2) \psi_{\text{HFS}}(x_3).
\]

(79)

After that the integration over \(x_i\) and expansion in \(b\) (or in \(\sqrt{M_e/M_\mu}\)) give the result:

\[
\int dx_i (\mathbf{n}_1 \cdot \mathbf{n}_2) \frac{e^{-b|\mathbf{n}|r_0}}{|x_i - x_i|}
\]

\[
= 2\pi (\mathbf{n}_2 \cdot \mathbf{n}_1) \left[\frac{4x_i}{3b} - \frac{x_i^2}{2} + \frac{2bx_i^3}{15} + \ldots \right].
\]

(80)

Taking the first term in square brackets in (80) we perform an angular integration and introduce the dimensionless variables in integrals with radial wave functions:

\[
\delta b_{\text{rec,SOPT}} = \nu_F \frac{64M_e}{9M_\mu} \frac{M_\mu}{M_\mu} \sum_{n, \alpha^0} \frac{n}{\sqrt{n^2 - 1}} \int dx_2 \int dx_3 \psi_{\mu}(x_2) \psi_{\mu}(x_3) \psi_{\text{HFS}}(x_i) \psi_{\text{HFS}}(x_i)
\]

\[
\times \int dx_2 \int dx_3 \psi_{\mu}(x_2) \psi_{\mu}(x_3) \frac{e^{-b|\mathbf{n}|r_0}}{|x_i - x_i|} \left(\mathbf{n}_1 \cdot \mathbf{n}_2\right) \psi_{\text{HFS}}(x_2) \psi_{\text{HFS}}(x_3).
\]

(81)

The two contributions of the discrete and continuous spectra are the following:

\[
\delta b_{\text{rec,SOPT}}^{\text{disc}} = \nu_F \frac{64M_e}{9M_\mu} \frac{M_\mu}{M_\mu} \sum_{n, \alpha^0} \frac{n}{n^2 - 1} \int dx_2 \int dx_3 \psi_{\mu}(x_2) \psi_{\mu}(x_3) \psi_{\text{HFS}}(x_i) \psi_{\text{HFS}}(x_i)
\]

\[
\times \int dx_2 \int dx_3 \psi_{\mu}(x_2) \psi_{\mu}(x_3) \frac{e^{-b|\mathbf{n}|r_0}}{|x_i - x_i|} \left(\mathbf{n}_1 \cdot \mathbf{n}_2\right) \psi_{\text{HFS}}(x_2) \psi_{\text{HFS}}(x_3).
\]

\[
\delta b_{\text{rec,SOPT}}^{\text{cont}} = \nu_F \frac{64M_e}{9M_\mu} \frac{M_\mu}{M_\mu} \int_0^\infty \frac{ke^{-\frac{k}{b} \arctan(k)}}{(1 - e^{-2\pi/k})(k^2 + 1)^{3/2}}
\]

\[
= \{0.392 \text{ MHz}, 0.336 \text{ MHz}. \quad (82)
\]

(83)
5. Correction of the electron vertex function

The leading order $\alpha^4$ contribution to the hyperfine structure is related to the interaction operator (4) as discussed above. Among the many corrections to (4) there is a contribution of the electron vertex function, which is presented in figure 9(a).

First it is convenient to write this correction in the momentum representation:

$$\Delta_{\text{vert}}^{\text{HFS}}(k^2) = -\frac{8\alpha^2}{3m_e m_p} \left( \frac{\sigma_e \sigma_p}{4} \right) \left[ G_M^{(e)}(k^2) - 1 \right].$$  

(85)

where we take the factor $\alpha/\pi$ of the bracket in the expression $[G_M^{(e)}(k^2) - 1]$ containing the magnetic form factor $G_M^{(e)}(k^2)$ of the electron. A commonly used approximation for the magnetic form factor of the electron $G_M^{(e)}(k^2) \approx G_M^{(e)}(0) = 1 + \kappa_e$ is not applicable in this task. Since the typical photon momentum exchange is $k \sim \alpha M_p$, we cannot neglect it in $G_M^{(e)}(k^2)$ compared to the mass of the electron $m_e$. Therefore, we must use the exact expression for the Pauli form factor $g(k^2)$ ($G_M^{(e)}(k^2) - 1 \approx g(k^2)$) [25] trying to improve the estimate of the correction due to the electron anomalous magnetic moment.

Using the Fourier transform of the potential (85) we average the obtained expression over wave functions (4) and present the electron vertex correction to HFS in the form:

$$b_{\text{vert. vert}}(n = 0) = \nu_F \frac{2M_e^2}{3M_p M_{\text{Li}}} - \int_0^\infty k^2 dk \left[ G_M^{(e)}(k^2) - 1 \right] \times \left[ 1 + \left( \frac{m_e}{6\alpha M_p} \right)^2 k^2 \right]^{-1}$$

$$= \left\{ \begin{array}{l}
40.956 \text{ MHz} \\
40.958 \text{ MHz} 
\end{array} \right. \quad \text{(86)}$$

The contribution (86) has the order $\alpha^5$. The numerical value (86) is obtained after numerical integration with the one-loop expression of the electron magnetic form factor $G_M^{(e)}(k^2)$. When using the value $G_M^{(e)}(k^2 = 0)$ we obtain the electron vertex correction 41.959 MHz. Thus, the electron form factor in the one-loop approximation leads to the 1 MHz decrease of the vertex correction to the HFS in the $1\gamma$-interaction. Taking (85) as an additional perturbation potential we have to evaluate its contribution to HFS in SOPT (the diagram in fig 9(b)). The dashed line represents the Coulomb Hamiltonian $\Delta H$ (2). As previously, we can divide the total contribution of the amplitude in figure 9(b) into two parts which correspond to the muon ground state ($n = 0$) and muon excited intermediate states ($n \neq 0$). The first contribution with $n = 0$ takes the form:

$$b_{\text{vert. vert}}(n = 0) = \nu_F \frac{2M_e^2}{3\pi^2 m_e m_p} \times \int_0^\infty k^2 dk \left[ G_M^{(e)}(k^2) - 1 \right] \times \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{z}_0 \psi_{\phi_0}(\mathbf{x}_1) \psi_{\phi_0}(\mathbf{x}_2)$$

$$\Delta \tilde{V}(\mathbf{k}, \mathbf{x}_3) = \int d\mathbf{x}_1 d\mathbf{x}_2 \psi_{\phi_0}^*(\mathbf{x}_1) \sin(k [\mathbf{x}_3 - \mathbf{x}_4]) \psi_{\phi_0}(\mathbf{x}_4)$$

$$= \sin( k x_3 ) \frac{1}{1 + \frac{k^2}{(6\alpha M_p)^2}}.$$

(87)

(88)

After the substitution of the electron Green function (28) to (88) we transform this expression to integral form:

$$b_{\text{vert. vert}}(n = 0) = \nu_F \frac{2M_e^2}{3\pi^2 m_e m_p} \times \int_0^\infty k^2 dk \left[ G_M^{(e)}(k^2) - 1 \right] \times \left[ 1 + \left( \frac{m_e}{6\alpha M_p} \right)^2 k^2 \right]^2$$

$$= \left\{ \begin{array}{l}
40.956 \text{ MHz} \\
40.958 \text{ MHz} 
\end{array} \right. \quad \text{(86)}$$
All integrations over the coordinate \( x_1, x_3 \) are carried out analytically and final integration in \( k \) is performed numerically. Here we omit the intermediate expression before the integration in \( k \) because of its bulky form. The second part of the vertex contribution (figure 9(b)) with \( n \neq 0 \) can be converted to the following form after several simplifications which are discussed in section 2:

\[
\begin{align*}
\times & \int_0^\infty x_3 e^{2M_{e^{+}}x_3} \sin \left( \frac{m_e k}{6\alpha M_{\mu}} x_3 \right) dx_3 \\
\times & \int_0^\infty x_1 \left( 1 + \frac{x_1}{2} \right) e^{-x_1} \left( 1 + \frac{2M_{e^{+}}}{3M_{\mu}} \right) dx_1 \\
\times & \left[ \frac{3M_{\mu}}{2M_\mu x_\ge} - \ln \left( \frac{2M_{e^{+}}}{3M_{\mu}} x_\ge \right) \right] \\
= & \left\{ \begin{array}{l}
0.054 \text{ MHz} \\
0.054 \text{ MHz}
\end{array} \right. \quad (89)
\end{align*}
\]

It is useful to emphasize that the theoretical error in the summary contribution \( b_{1, \text{SOPT}}(n \neq 0) \) is determined by the factor \( \sqrt{M_{e^{+}}/M_{e}} \) connected with the omitted terms of the used expansion. It can amount to 10% of the total results of (91)–(92) which is a value near 0.010 MHz.

The electron vertex corrections investigated in this section have the order \( \alpha^5 \) in the HFS interval. The summary value of all obtained contributions in SOPT is equal to –0.056 MHz (\( \gamma \text{Li} \)) and (\( \gamma \text{Li} \)). Summing this number with the correction (86) we obtain the value 40.900 MHz. It differs significantly by 1.059 MHz from the result 41.959 MHz which was obtained in the approximation of vertex correction by the electron anomalous magnetic moment.

6. Summary and conclusion

In this work we have carried out analytical and numerical computation of HFS intervals in muonic lithium ions (\( \mu e^{-\gamma \text{Li}} \)) on the basis of the PT method suggested previously in the case of muonic helium in [6]. To increase the accuracy of the calculations we take into account several important corrections to HFS of the ground state of orders \( \alpha^5 \) and \( \alpha^6 \) connected with the VP, nuclear structure, recoil effects and electron vertex corrections. The numerical values of different contributions to hyperfine structure are presented in table 1.

Let us list a number of basic features of the calculations.

1. Muonic lithium atoms have a complicated hyperfine structure which appears due to the interaction of the magnetic moments of the three particles. We investigate small HFSs which can be important in experimental studies.

2. In this problem there are two small parameters, the fine structure constant and the ratio of particle masses, which can be used for the construction of the perturbation interactions. The basic contributions appear in orders \( \alpha^5 \) and \( \alpha^6 \) taking into account of first and second order recoil effects.

3. The VP effects are important in order to obtain theoretical splittings with high accuracy. They give rise
to the modification of the two-particle interaction potential which provides the $\alpha^5 M_e^5/M_p^5$-order corrections to the hyperfine structure. We take into account the VP corrections in the first and second orders of PT.

4. The electron vertex corrections to the coefficient $b$ should be considered with the exact account of the one-loop magnetic form factor of the electron because the characteristic momentum incoming into the electron vertex operator is of order of the electron mass.

5. Nuclear structure corrections to the ground state HFS are expressed in terms of electromagnetic form factors and the charge radius of two Li nuclei.

6. Relativistic correction is obtained by means of the expression from [7]:

$$
\Delta v_{\text{rel}} = \nu_{\text{F}} \left( 1 + \frac{3}{2} (Z_1 \alpha)^2 - \frac{1}{3} (Z_2 \alpha)^2 \right),
$$

which gives contributions to both coefficients $b$ and $c$ (see table 1).

Using the total numerical values of coefficients $b$ and $c$, presented in table 1, we find the following HFSs for muonic lithium ions: $\Delta v_1(\mu e^5\text{Li})^+ = 21572.160$ MHz and $\Delta v_2(\mu e^5\text{Li})^+ = 14152.560$ MHz; $\Delta v_1(\mu e^6\text{Li})^+ = 21733.056$ MHz and $\Delta v_2(\mu e^6\text{Li})^+ = 13994.345$ MHz. The calculation of hyperfine structure in three-particle muonic atoms (muonic helium, ions of muonic lithium) was performed in [12, 13] using a variational method. The second paper in [12] is devoted to muonic helium and the first paper in [12] contains an estimate of HFSs in muonic lithium ion. Later it this estimate was corrected in [13]. So, we make our comparison namely with the values obtained in [13]: $\Delta v_1(\mu e^5\text{Li})^+ = 21676.112$ MHz and $\Delta v_2(\mu e^5\text{Li})^+ = 14148.678$ MHz; and $\Delta v_1(\mu e^6\text{Li})^+ = 21729.22$ MHz $\Delta v_2(\mu e^6\text{Li})^+ = 13989.19$ MHz.

An analysis of the separate contributions to the hyperfine structure coefficients $b$ and $c$ in table 1 shows that relativistic and electron vertex corrections have large values. So, for example, the difference of our calculation in [13] for the electron vertex corrections is a value of order 1 MHz and the relativistic corrections amount to 6 MHz. The recoil contribution from $2\gamma$ exchange amplitudes has a similar value of $\sim 6$ MHz. As follows from the calculation in [13], only the expectation values of the $\delta$-functions are taken into account with very high accuracy, but different corrections to the leading order Hamiltonian are omitted. A nonrelativistic Hamiltonian is used in [13] and the electron vertex corrections are taken into account in terms of the anomalous magnetic moment. So, the difference in total results between our calculations and [13] arises first of all from these terms. The VP and nuclear structure corrections influence the total result to a smaller extent. The numerical values of the fundamental physical constants in our work and [13] coincide. It is also useful to compare our results with [13] in the order $\alpha^6$ with the account of electron vertex corrections in terms of electron anomalous magnetic moment. Such a comparison shows that the difference between our work and [13], which lies in the region 0.7 $\div$ 1.5 MHz for separate HFSs, remains. We consider that this is related to terms of order $\nu_{\text{F}} M_e^2/M_p^2$ which are not taken into account exactly in our work. We included this term in the total theoretical error. Further improvement of the results obtained in this work can be achieved in the calculation of second order corrections in two small parameters $\alpha$ and $M_e/M_p$.

The estimate of theoretical uncertainty can be performed in terms of the Fermi energy $\nu_{\text{F}}$ and small parameters $\alpha$ and the ratio of the particle masses. In our opinion, there exist several main sources for the theoretical errors. First of all, as we mentioned above in section 2, the recoil corrections of order $M_e^2/M_p^2$ are not taken into account exactly because of a replacement of the electron Green’s function by free one. Numerically this contribution can give 0.88 MHz. The second source of the error is related to contributions of order $\alpha^2 \nu_{\text{F}}$ which appear both from QED amplitudes and in higher orders of PT. In the case of two-particle bound states these corrections were calculated in [31, 35–37]. Considering that they should be studied more carefully for three-particle bound states we included a correction $\alpha^2 \nu_{\text{F}} \approx 1.92$ MHz in the theoretical error. Another part of the theoretical error is determined by the two-photon three-body exchange amplitudes mentioned above. They are of the fifth order over $\alpha$ and contain the recoil parameter $(m_e/m_p) \ln (m_e/m_p)$, so their possible numerical value can be equal to $\pm 0.22$ MHz. Thereby, the total theoretical uncertainty does not exceed $\pm 2.13$ MHz. To obtain this estimate we add the above mentioned uncertainties in quadrature.

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References

[1] Pohl R et al 2010 Nature 466 213
Antognini A et al 2013 Science 339 417
Pohl R, Gilman R, Miller G A and Pachucki K 2013 Annu. Rev. Nucl. Part. Sci. 63 175

[2] Nebel T et al 2012 Hyperfine Int. 212 195
Antognini A et al 2013 Ann. Phys. 331 127

[3] Volotka A V, Shabaev V M, Plunien G and Soff G 2005 Eur. Phys. J. D 33 23

[4] Drake G W F and Byer L L 1985 Phys. Rev. A 32 713
Gomes A H, Kostelecky V A and Vargas A J 2014 arXiv:1407:7748 [hep-ph]
Moumni M and Benslama A 2013 Int. J. Mod. Phys. A 28 1350139

[5] Gladish M et al 1983 Proc. 8th Int. Conf. on Atomic Physics ed I Lindgren, A Rosen and S Svanberg (New York: Plenum)
[6] Lakdawala S D and Mohr P 1981 Phys. Rev. A 24 2224
Lakdawala S D and Mohr P 1980 Phys. Rev. A 22 1572
Lakdawala S D and Mohr P 1984 Phys. Rev. A 29 1047
[7] Huang K-N and Hughes V W 1982 Phys. Rev. A 26 2330
Huang K-N and Hughes V W 1979 Phys. Rev. A 20 706
[8] Borie E 1979 Z. Phys. A 291 107
[9] Drachman R L 1983 J. Phys. B 16 L749
Drachman R L 1980 Phys. Rev. A 22 1755
[10] Chen M-K 1993 J. Phys. B 26 2263
Chen M-K 1990 J. Phys. B 23 4041
[11] Yakhotov V L and Amusia M Y 1994 J. Phys. B 27 3743
Yakhotov V L and Amusia M Y 1983 J. Phys. B 16 L71
[12] Frolov A M 2000 Phys. Rev. A 61 022509
Frolov A M 1998 Phys. Rev. A 57 2436
Frolov A M 2012 Phys. Lett. A 376 2548
[13] Frolov A M 2006 Phys. Lett. A 357 344
[14] Korobov V I 2000 Phys. Rev. A 61 064503
[15] Pachucki K 2001 Phys. Rev. A 63 032508
[16] Krutov A A and Martynenko A P 2008 Phys. Rev. A 78 032513
Krutov A A and Martynenko A P 2011 Eur. Phys. J. D 62 163
Krutov A A and Martynenko A P 2012 Phys. Rev. A 86 052501
[17] Sobelman I I 1972 Introduction to the Theory of Atomic Spectra (Oxford: Pergamon)
[18] Mohr P J, Taylor B N and Newell D B 2012 Rev. Mod. Phys. 84 1533
[19] Karshenboim S G 2013 Ann. Phys. (Berlin) 525 472
[20] Stone N J 2005 Atom. Data and Nucl. Data tables 90 75
[21] Hameka H F 1967 J. Chem. Phys. 47 2728
[22] Bethe H A and Salpeter E E 1957 Quantum Mechanics of One- and Two-Electron Atoms (Berlin: Springer)
Fock V A 1976 Principles of Quantum Mechanics (Moscow: Nauka)
[24] Eides M I, Grotch H and Shelyuto V A 2001 Phys. Rep. 342 62
Eides M I, Grotch H and Shelyuto V A 2007 Theory of light hydrogenic bound states Springer Tracts in Modern Physics 222 (Berlin: Springer)
[25] Berestetskii V B, Lifshits E M and Pitaevskii L P 1980 Quantum Electrodynamics (Moscow: Nauka)
[26] Martynenko A P 2005 Phys. Rev. A 71 022506
Martynenko A P 2005 JETP 101 1021
[27] Grotch H and Yennie D R 1969 Rev. Mod. Phys. 41 350
[28] Bodwin G T and Yennie D R 1988 Phys. Rev. D 37 498
[29] Faustov R N and Martynenko A P 2003 Phys. Atom. Nucl. 66 1719
Faustov R N and Martynenko A P 1999 JETP 88 672
Faustov R N and Martynenko A P 2004 JETP 98 39
[30] Martynenko A P 2007 Phys. Rev. A 76 012505
Martynenko A P 2008 JETP 106 691
[31] Faustov R N, Martynenko A P, Martynenko G A and Sorokin V V 2014 Phys. Lett. B 733 354
Faustov R N, Martynenko A P, Martynenko G A and Sorokin V V 2014 Phys. Rev. A 90 012520
[32] Faustov R N and Martynenko A P 2002 Phys. Atom. Nucl. 65 265
Faustov R N and Martynenko A P 2000 Phys. Atom. Nucl. 63 845
Faustov R N and Martynenko A P 2003 Phys. Rev. A 67 052506
[33] Nörtershäuser W, Neff T, Sanchez R and Sick I 2011 Phys. Rev. C 84 024307
[34] Arnowitt R 1953 Phys. Rev. 92 1002
[35] Brodsky S J and Erickson G W 1966 Phys. Rev. 148 26
[36] Kroll N and Pollack F 1951 Phys. Rev. 84 594
[37] Karplus R, Klein A and Schwinger J 1951 Phys. Rev. 84 597