TWISTED DEFORMATION QUANTIZATION OF ALGEBRAIC VARIETIES

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Abstract. Let $X$ be a smooth algebraic variety over a field of characteristic $0$. We introduce the notion of twisted associative (resp. Poisson) deformation of the structure sheaf $\mathcal{O}_X$. These are stack-like versions of usual deformations. We prove that there is a twisted quantization map from twisted Poisson deformations to twisted associative deformations, which is canonical and bijective on equivalence classes.

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0. Introduction

Let $K$ be a field of characteristic $0$, and let $X$ be a smooth algebraic variety over $K$, with structure sheaf $\mathcal{O}_X$. Suppose $R$ is a parameter algebra over $K$; namely $R$ is a complete local noetherian commutative $K$-algebra, with maximal ideal $m$ and residue field $R/m = K$. The main example is $R = K[[\hbar]]$, the formal power series ring in the variable $\hbar$. An associative $R$-deformation of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat $m$-adically complete associative $R$-algebras on $X$, with an isomorphism $K \otimes_R \mathcal{A} \cong \mathcal{O}_X$, called an augmentation. Similarly, a Poisson $R$-deformation of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat $m$-adically complete commutative Poisson $R$-algebras on $X$, with an augmentation to $\mathcal{O}_X$.

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Such deformations could be sheaf-theoretically trivial, meaning that $A \cong R \hat{\otimes}_K \mathcal{O}_X$, endowed with either an associative multiplication (called a star product), or a Poisson bracket. This is what happens in the differentiable setup (i.e. when $X$ is a $C^\infty$ manifold and $K = \mathbb{R}$). But in the algebro-geometric setup the sheaf $A$ could be very complicated – indeed, all classical commutative deformations of $\mathcal{O}_X$ are special cases of both associative and Poisson deformations.

In this paper we introduce the notion of twisted associative (resp. Poisson) $R$-deformation of $\mathcal{O}_X$. A twisted deformation (or either kind) is a stack-like version of an ordinary deformation. The precise definition is given in Section 5, where we discuss twisted objects in stacks (Definition 5.10). But to give an idea, let us say that a twisted deformation $A$ can be described as a collection of locally defined deformations $A_i$, each living on an open set $U_i$ of $X$, that are glued together in a loose way. We should also say that an associative $R$-deformation $A$ is an $R$-linear stack of algebroids, in the sense of [Ko2]. Indeed, the reason for introducing twisted deformations is to have a Poisson analogue of a stack of algebroids.

There is a notion of twisted gauge equivalence between twisted associative (resp. Poisson) $R$-deformation of $\mathcal{O}_X$. A twisted deformation $A$ induces a first order bracket $\{-,\}_A$ on $\mathcal{O}_X$ (see Definition 5.13).

Here is the main result of our paper (repeated in greater detail as Theorem 12.7):

**Theorem 0.1.** Let $\mathbb{K}$ be a field containing the real numbers, let $R$ be a parameter $\mathbb{K}$-algebra, and let $X$ be a smooth algebraic variety over $\mathbb{K}$. Then there is a canonical bijection of sets

\[
tw.\text{quant} : \{\text{twisted Poisson } R\text{-deformations of } \mathcal{O}_X\} \xrightarrow{\cong} \{\text{twisted associative } R\text{-deformations of } \mathcal{O}_X\}
\]

This bijection is called the twisted quantization map. It preserves first order brackets, and commutes with homomorphisms $R \rightarrow R'$ and étale morphisms $X' \rightarrow X$.

Here is a corollary (repeated as Corollary 12.8):

**Corollary 0.2.** Assume $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then there is a canonical bijection

\[
\text{quant} : \{\text{Poisson } R\text{-deformations of } \mathcal{O}_X\} \xrightarrow{\cong} \{\text{associative } R\text{-deformations of } \mathcal{O}_X\}
\]

This bijection preserves first order brackets, and commutes with homomorphisms $R \rightarrow R'$ and étale morphisms $X' \rightarrow X$.

Theorem 0.1 and Corollary 0.2 have their roots in Kontsevich’s paper [Ko2], where it was first suggested that a Poisson bracket on $\mathcal{O}_X$ can be quantized to a stack of algebroids.

Here is an outline of the paper. Throughout $\mathbb{K}$ is a field of characteristic 0.

In Sections 1-2 we study $R$-deformations of $\mathcal{O}_X$ in a rather wide context: $X$ is a topological space, and $\mathcal{O}_X$ is a sheaf of commutative $\mathbb{K}$-algebras on it. We give basic definitions and a few results.

Twisted deformations are introduced in Sections 3-5. Actually we work in greater generality (which hopefully helps to simplify the discussion): we define the notion of category with inner gauge groups. In such a category one can talk about twisted objects. This can be geometrized to a stack $\mathcal{P}$ of categories with inner gauge groups on a topological space $X$. A twisted object $\mathcal{A}$ in $\mathcal{P}$ is, roughly speaking, a collection of local objects $\mathcal{A}_i$ in $\mathcal{P}$, that are glued together by a gerbe $\mathcal{G}$, called the gauge gerbe
of $\mathcal{A}$. The way all this is related to twisted deformations is that we can take $P$ to be the stack on $X$ such that, for an open set $U \subset X$, the category $P(U)$ is the category of associative (resp. Poisson) $R$-deformations of $O_U$.

In Section 6 we discuss decomposition of twisted objects on an open covering. This is similar to the way gerbes decompose. The important result here is Theorem 6.12 which says that twisted associative (resp. Poisson) $R$-deformations of $O_X$ decompose on $O_X$-acyclic open coverings. It relies on our work on pronilpotent gerbes in [Ye4]. The obstruction theory developed in [Ye4] allows to determine if a twisted deformation $\mathcal{A}$ is really twisted, i.e. if it is not twisted equivalent to an ordinary deformation.

Sections 7-8 are about the role of the DG Lie algebras $T_{\text{poly}}(C)$ and $D_{\text{nor}}\text{poly}(C)$ in deformations of $C$. Here $C$ is a smooth $\mathbb{K}$-algebra (in the sense of algebraic geometry, namely $X := \text{Spec} \ C$ is a smooth affine algebraic variety over $\mathbb{K}$). We review some older results. Among the new results is Theorem 8.5 regarding differential gauge transformations.

In Sections 9-11 we show how twisted associative (resp. Poisson) $R$-deformations of $O_X$ can be encoded in terms of additive descent data in cosimplicial DG Lie algebras. Here $X$ is a smooth algebraic variety over $\mathbb{K}$. These cosimplicial DG Lie algebras are obtained from the sheaves $D_{\text{nor}}\text{poly},X$ (resp. $T_{\text{poly},X}$) by an affine open covering and a Čech construction. One consequence (Theorem 10.1) is that given an étale morphism of varieties $g : X' \to X$ and a twisted $R$-deformation $\mathcal{A}$ of $O_X$, there is an induced twisted $R$-deformation $\mathcal{A}'$ of $O_{X'}$.

A crucial result is Theorem 11.2. It says, roughly, that an additive descent datum in a cosimplicial DG Lie algebra $g$ is the same as a solution of the Maurer-Cartan equation in the Thom-Sullivan normalization of $g$. Theorem 11.2 is proved in our paper [Ye7], which is still in preparation (but an outline of the proof can be found in Remark 11.3).

In Section 12 we state and prove the main result, namely Theorem 12.7. The proof is an assembly of many other results from this paper, together with an important result from [Ye1] concerning deformation quantization on the level of sheaves (recalled here as Theorem 12.3). We also list several questions regarding the structure of twisted associative deformations and the behavior of the twisted quantization map. Perhaps the most intriguing one is Question 12.10 about the quantization of symplectic Poisson brackets on Calabi-Yau surfaces.

One consequence of Theorem 6.12 is that in the differentiable setup (and $\mathbb{K} = \mathbb{R}$) there are no really twisted $R$-deformations. However, when $X$ is a complex analytic manifold (and $\mathbb{K} = \mathbb{C}$), there do exist really twisted $R$-deformations. Presumably our methods (with minor adjustments) should work also for complex analytic manifolds.

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1. DEFORMATIONS OF ALGEBRAS

In this section we give the basic definitions and a few initial results. By default, associative algebras are assumed to be unital, and commutative algebras are assumed to be associative (and unital).
**Definition 1.1.** Let $\mathbb{K}$ be a field. A parameter $\mathbb{K}$-algebra is a complete local noetherian commutative $\mathbb{K}$-algebra $R$, with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = \mathbb{K}$. We sometimes say that $(R, \mathfrak{m})$ is a parameter $\mathbb{K}$-algebra. For $i \geq 0$ we let $R_i := R/\mathfrak{m}^{i+1}$. The ideal $\mathfrak{m}$ is called a parameter ideal over $\mathbb{K}$. The $\mathbb{K}$-algebra homomorphism $R \to \mathbb{K}$ is called the augmentation of $R$.

Note that $R$ can be recovered from $\mathfrak{m}$, since $R = \mathbb{K} \oplus \mathfrak{m}$ as $\mathbb{K}$-modules, with the obvious multiplication.

**Example 1.2.** The most important parameter algebra in deformation theory is $\mathbb{K}[[h]]$, the ring of formal power series in the variable $h$. A $\mathbb{K}[[h]]$-deformation (see below) is sometimes called a “1-parameter formal deformation”.

Let $M$ be an $R$-module. For any $i \geq 0$ there is a canonical bijection $R_i \otimes_R M \cong M/\mathfrak{m}^{i+1}M$. The $\mathfrak{m}$-adic completion of $M$ is the $R$-module $\hat{M} := \varprojlim_i (R_i \otimes_R M)$. The module $M$ is called m-adically complete if the canonical homomorphism $M \to \hat{M}$ is bijective. (Some texts, including [CA], use the expression “separated and complete”.) Given a $\mathbb{K}$-module $V$ we let $R \hat{\otimes}_\mathbb{K} V := R \otimes_{\mathbb{K}} \hat{V}$.

**Definition 1.3.** Let $(R, \mathfrak{m})$ be a parameter $\mathbb{K}$-algebra. An m-adic system of $R$-modules is a collection $\{M_i\}_{i \in \mathbb{N}}$ of $R$-modules, together with a collection $\{\psi_i\}_{i \in \mathbb{N}}$ of homomorphisms $\psi_i : M_{i+1} \to M_i$. The conditions are:

(i) For every $i$ one has $\mathfrak{m}^{i+1}M_i = 0$. Thus $M_i$ is an $R_i$-module.

(ii) For every $i$ the $R_i$-linear homomorphism $R_i \otimes_{R_{i+1}} M_{i+1} \to M_i$ induced by $\psi_i$ is an isomorphism.

Often the collection of homomorphisms $\{\psi_i\}_{i \in \mathbb{N}}$ remains implicit. The following (not so well known) facts will be important for us.

**Proposition 1.4.** Let $(R, \mathfrak{m})$ be a parameter $\mathbb{K}$-algebra, and let $M$ be an $R$-module. Define $M_i := R_i \otimes_R M$. The following conditions are equivalent:

(i) There is an isomorphism of $R$-modules $M \cong R \hat{\otimes}_\mathbb{K} V$ for some $\mathbb{K}$-module $V$.

(ii) The $R$-module $M$ is flat and m-adically complete.

(iii) The $R$-module $M$ is m-adically complete, and for every $\mathbb{K}$-linear homomorphism $M_0 \to M$ splitting the canonical surjection $M \to M_0$, the induced $R$-linear homomorphism $R \hat{\otimes}_\mathbb{K} M_0 \to M$ is bijective.

(iv) There is an m-adic system of $R$-modules $\{N_i\}_{i \in \mathbb{N}}$, such that each $N_i$ is flat over $R_i$, and an isomorphism of $R$-modules $M \cong \varprojlim_i N_i$.

Moreover, when these conditions hold, the induced homomorphisms

$$R_i \hat{\otimes}_\mathbb{K} V \to M_i \to N_i$$

are bijective for every $i$.

**Proof.** See [Ye5 Corollary 2.12, Theorem 1.12 and Theorem 2.10].

**Remark 1.5.** If $R$ is not artinian and the $\mathbb{K}$-module $V$ is not finitely generated, then the $R$-module $M \cong R \hat{\otimes}_\mathbb{K} V$ is not free. In [Ye5] we called $M$ an “m-adically free $R$-module”. It is a projective object in the additive category of complete $R$-modules; and it is topologically projective in the sense of [EGA I].

For the parameter algebra $R = \mathbb{K}[[h]]$ the proposition was proved in [CFT] Lemma A.1. A thorough discussion of completions of infinitely generated modules can be found in [St Chapter 2].

Consider the following setup:

**Setup 1.6.** $\mathbb{K}$ is a field; $(R, \mathfrak{m})$ is a parameter $\mathbb{K}$-algebra; and $C$ is a commutative $\mathbb{K}$-algebra.
Suppose $A$ is an $R$-algebra. We say $A$ is $m$-adically complete, or flat, if it is so as an $R$-module. We view $C$ as an $R$-algebra via the augmentation homomorphism $R \to \mathbb{K}$. Hence giving an $R$-algebra homomorphism $A \to C$ is the same as giving a $\mathbb{K}$-algebra homomorphism $\mathbb{K} \otimes_R A \to C$.

**Definition 1.7.** Assume setup [1.6]. An associative $R$-deformation of $C$ is a flat $m$-adically complete associative $R$-algebra $A$, together with a $\mathbb{K}$-algebra isomorphism $\psi : \mathbb{K} \otimes_R A \to C$, called an augmentation.

Given another such deformation $A'$, a gauge transformation $g : A \to A'$ is a unital $R$-algebra isomorphism that commutes with the augmentations to $C$.

We denote by $\text{AssDef}(R, C)$ the category of associative $R$-deformations of $C$, where the morphisms are gauge transformations.

Suppose $A$ is an associative $R$-deformation of $C$, with unit element $1_A$. Due to Proposition [1.4] there exists an isomorphism of $R$-modules $R \otimes \hat{C} \to A$, sending $1_R \otimes 1_C \mapsto 1_A$.

Let $A$ be a commutative $R$-algebra. An $R$-bilinear Poisson bracket on $A$ is an $R$-bilinear function

$$\{-, -\} : A \times A \to A$$

which is a Lie bracket (i.e. it is antisymmetric and satisfies the Jacobi identity), and also is a derivation in each of its arguments. The pair $(A, \{-, -\})$ is called a Poisson $R$-algebra. A homomorphism of Poisson $R$-algebras $f : A \to A'$ is an algebra homomorphism that respects the Poisson brackets.

**Definition 1.8.** Assume Setup [1.6] We consider $C$ as a Poisson $\mathbb{K}$-algebra with the zero bracket. A Poisson $R$-deformation of $C$ is a flat $m$-adically complete Poisson $R$-algebra $A$, together with an isomorphism of Poisson $\mathbb{K}$-algebras $\psi : \mathbb{K} \otimes_R A \to C$, called an augmentation.

Given another such deformation $A'$, a gauge transformation $g : A \to A'$ is an $R$-algebra isomorphism that respects the Poisson brackets and commutes with the augmentations to $C$.

We denote by $\text{PoisDef}(R, C)$ the category of Poisson $R$-deformations of $C$, where the morphisms are gauge transformations.

The categories $\text{AssDef}(R, C)$ and $\text{PoisDef}(R, C)$ are of course groupoids (namely all morphisms are invertible).

**Remark 1.9.** If the ring $C$ is noetherian, then any Poisson or associative $R$-deformation of $C$ is also a (left and right) noetherian ring. See [KSS] or [CA]. We are not going to need this fact.

Suppose $(R', m')$ is another parameter $\mathbb{K}$-algebra, and let $R'_i := R'/m'^{i+1}$. Let $\sigma : R \to R'$ be a $\mathbb{K}$-algebra homomorphism. Then $\sigma(m) \subset m'$, and so for every $i$ there is an induced homomorphism $R_i \to R'_i$. Given an $R$-module $M$ we let

$$R' \hat{\otimes}_R M := \lim_{\substack{i \to -1}} (R'_i \otimes_R M).$$

This is the $m'$-adic completion of the $R'$-module $R' \hat{\otimes}_R M$.

**Proposition 1.10.** Let $A$ be an associative (resp. Poisson) $R$-deformation of $C$, let $R'$ be another parameter $\mathbb{K}$-algebra, let $\sigma : R \to R'$ be a $\mathbb{K}$-algebra homomorphism, and let $A := R' \hat{\otimes}_R A$. Then $A'$ has a unique structure of associative (resp. Poisson) $R'$-deformation of $C$, such that the canonical homomorphism $A \to A'$ is a homomorphism of $R$-algebras (resp. Poisson $R$-algebras).

**Proof.** Let $A'_i := R'_i \otimes_R A$. This is a flat $R'_i$-module, and it has an induced $R'_i$-bilinear multiplication (resp. Poisson bracket). Thus $A'_i$ is an $R'_i$-deformation of $C$. 
In the limit, the $R'$-module $A' = \lim_{\leftarrow i} A'_i$ has an induced $R'$-bilinear multiplication (resp. Poisson bracket). And by Proposition [4] it is an $R'$-deformation of $C$. □

Let $C'$ be another commutative $\mathbb{K}$-algebra, and let $\tau : C \to C'$ be a homomorphism. We say that $C'$ is a principal localization of $C$ if there is a $C$-algebra isomorphism $C' \cong C_s = C[s^{-1}]$ for some element $s \in C$.

**Theorem 1.11.** Let $R$ be a parameter $\mathbb{K}$-algebra, let $C$ be a commutative $\mathbb{K}$-algebra, and let $A$ be a Poisson (resp. associative) $R$-deformation of $C$. Suppose $\tau : C \to C'$ is a principal localization. Then:

1. There exists a Poisson (resp. associative) $R$-deformation $A'$ of $C'$, together with a homomorphism $g : A \to A'$ of Poisson (resp. associative) $R$-algebras which lifts $\tau : C \to C'$.

2. Suppose $\tau' : C' \to C''$ is a homomorphism of commutative $\mathbb{K}$-algebras, $A''$ is a Poisson (resp. associative) $R$-deformation of $C''$, and $h : A \to A''$ is a homomorphism of Poisson (resp. associative) $R$-algebras which lifts $\tau' \circ \tau : C \to C''$. Then there is a unique homomorphism of Poisson (resp. associative) $R$-algebras $g' : A' \to A''$ such that $h = g' \circ g$.

When we say that $g : A \to A'$ lifts $\tau : C \to C'$, we mean that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & A' \\
\downarrow & & \downarrow \\
C & \xrightarrow{\tau} & C'
\end{array}
$$

in which the vertical arrows are the augmentations, is commutative. Observe that by part (2), the pair $(A', g)$ in part (1) is unique up to a unique gauge transformation.

For the proof we need the next lemma on Ore localization of noncommutative rings [MR]. Recall that a subset $S$ of a ring $A$ is called a denominator set if it is multiplicatively closed, and satisfies the left and right torsion and Ore conditions. If $S$ is a denominator set, then $A$ can be localized with respect to $S$. Namely there is a ring $A_S$, called the ring of fractions, with a ring homomorphism $A \to A_S$. The elements of $S$ become invertible in $A_S$, and $A_S$ is universal for this property: every element $b \in A_S$ can be written as $b = a_1s_1^{-1}s_2^{-1}a_2$, with $a_1, a_2 \in A$ and $s_1, s_2 \in S$; and $A_S$ is flat over $A$ (on both sides).

**Lemma 1.13.** Let $A$ be a ring, with nilpotent two-sided ideal $a$. Assume the ring $\text{gr}_a A = \bigoplus_{i \geq 0} a^i/a^{i+1}$ is commutative. Let $s$ be some element of $A$.

1. The set $\{s^j\}_{j \geq 0}$ is a denominator set in $A$. We denote by $A_s$ the resulting ring of fractions.

2. Let $A := A/a = \text{gr}_a^0 A$, let $\bar{s}$ be the image of $s$ in $\bar{A}$, and let $\bar{a}_s$ be the kernel of the canonical ring surjection $A_s \to \bar{A}_s$. Then $\bar{a}_s = aA_s = A_s \bar{a}$, and this is a nilpotent ideal.

3. Let $a$ be any element of $A$, with image $\bar{a} \in \bar{A}$. Then $a$ is invertible in $A_s$ if and only if $\bar{a}$ is invertible in $\bar{A}_s$.

**Proof.** (1) This is a variant of [YZ Corollary 5.18]. We view $A$ as a bimodule over the ring $\mathbb{Z}[s]$. Since the $a$-adic filtration is finite, and $\text{gr}_a A$ is commutative, it follows from [YZ Lemma 5.9] that $A$ is evenly localizable to $\mathbb{Z}[s, s^{-1}]$. According to [YZ Theorem 5.11] the set $\{s^j\}_{j \geq 0}$ is a denominator set in $A$. Moreover, $A_s \cong A \otimes_{\mathbb{Z}[s]} \mathbb{Z}[s, s^{-1}]$ as left $A$-modules.

(2) Since $A \to A_s$ is flat it follows that $a_s = aA_s = A_s a$. By induction on $i$ one then shows that $(a_s)^i = a^iA_s$; and hence $a_s$ is nilpotent.
(3) We prove only the nontrivial part. Suppose \( \tilde{a} \) is invertible in \( \tilde{A}_s \). So \( \tilde{a} \tilde{b} = 1 \) for some \( \tilde{b} \in \tilde{A}_s \). Thus \( \tilde{a} \tilde{b} = 1 - \epsilon \in \tilde{A}_s \), where \( \epsilon \in \tilde{a}_s \). Since the ideal \( \tilde{a}_s \) is nilpotent, the element \( 1 - \epsilon \) is invertible in \( \tilde{A}_s \). This proves that \( \tilde{a} \) has a right inverse. Similarly for a left inverse. \( \square \)

Proof of Theorem 1.11

The proof is in several steps.

Step 1. Consider the associative case, and assume \( R \) is artinian (i.e. \( \mathfrak{m} \) is nilpotent).

Take an element \( s \in C \) such that \( C' \cong C_s \). Choose some lifting \( \tilde{s} \in A \) of \( s \).

According to Lemma 1.13 there is a ring of fractions \( A_\tilde{s} \) of \( A \), gotten by inverting \( \tilde{s} \) on one side, and \( \mathbb{K} \otimes_R A_\tilde{s} \cong C' \). Since \( R \) is central in \( A \), it is also central in \( A_\tilde{s} \).

And since \( A_\tilde{s} \) is flat over \( A \), it is also flat over \( R \). We see that \( A_\tilde{s} \) is an associative \( R \)-deformation of \( C' \), and the homomorphism \( g : A \to A_\tilde{s} \) lifts \( C \to C_s \).

Now suppose we are in the situation of part (2). Since \( (\tau' \circ \tau)(s) \) is invertible in \( C'' \), Lemma 1.13 (3) says that the element \( h(s) \) is invertible in \( A'' \). Therefore there is a unique \( A \)-ring homomorphism \( g' : A_\tilde{s} \to A'' \) such that \( h = g' \circ g \).

Step 2. \( R \) is still artinian, but now we are in the Poisson case. So \( A \) is a Poisson \( R \)-deformation of \( C \).

From the previous step we obtain a flat commutative \( R \)-algebra \( A' \), such that \( \mathbb{K} \otimes_R A' \cong C' \), together with a homomorphism \( g : A \to A' \). The pair \((A', g)\) is unique for this property. We have to address the Poisson bracket.

Take an element \( \tilde{s} \in A \) like in Step 1; so \( A' \cong A_\tilde{s} \). There is a unique biderivation on the commutative ring \( A' \) that extends the given Poisson bracket \( \{-, -\} \) on \( A \); it has the usual explicit formula for the derivative of a fraction. And it is straightforward to check that this biderivation is anti-symmetric and satisfies the Jacobi identity. Hence \( A' \) becomes a Poisson \( R \)-deformation of \( C' \), uniquely.

In the situation of part (2), we know (from step 1) that there is a unique \( A \)-algebra homomorphism \( g' : A' \to A'' \) such that \( h = g' \circ g \). The formula for the Poisson bracket on \( A' \) shows that \( g' \) is a homomorphism of Poisson algebras.

Step 3. Finally we allow \( R \) to be noetherian, and look at both cases together. Then \( R \cong \lim_{\to i} R_i \), and, letting \( A_i := R_i \otimes_R A \), we have \( A \cong \lim_{\to i} A_i \). By the previous steps for every \( i \) there is an \( R_i \)-deformation \( A'_i \) of \( C' \).

Due to uniqueness these form an inverse system, and we take \( A' := \lim_{\to i} A'_i \). By Proposition 2.5 this is an \( R \)-deformation of \( C' \).

Part (2) is proved similarly by nilpotent approximations. \( \square \)

Corollary 1.14. Let \( \tau : C \to C' \) be a principal localization of commutative \( \mathbb{K} \)-algebras, and let \( \sigma : R \to R' \) be a homomorphism of parameter \( \mathbb{K} \)-algebras. Then there are functors

\[
\text{ind}_{\sigma, \tau} : \text{PoisDef}(R, C) \to \text{PoisDef}(R', C')
\]

and

\[
\text{ind}_{\sigma, \tau} : \text{AssDef}(R, C) \to \text{AssDef}(R', C').
\]

Proof. This is a combination of Proposition 1.10 and Theorem 1.11. \( \square \)

2. Deformations of Sheaves of Algebras

Let \( X \) be a topological space, and let \( \mathcal{O}_X \) be a sheaf of commutative \( \mathbb{K} \)-algebras on \( X \). In this section we define the notions of associative and Poisson \( R \)-deformations of the sheaf \( \mathcal{O}_X \).

First we need some properties of sheaves on \( X \). Suppose \( \mathcal{U} = \{U_k\}_{k \in K} \) is a collection of open sets in \( X \). For \( k_0, \ldots, k_m \in K \) we write

\[
U_{k_0, \ldots, k_m} := U_{k_0} \cap \ldots \cap U_{k_m}.
\]

Definition 2.1. Let \( \mathcal{N} \) be a sheaf of abelian groups on the topological space \( X \).
(1) An open set $U \subset X$ will be called $\mathcal{N}$-acyclic if the derived functor sheaf cohomology satisfies $H^i(U,\mathcal{N}) = 0$ for all $i > 0$.

(2) Now suppose $U = \{U_k\}_{k \in K}$ is a collection of open sets in $X$. We say that the collection $U$ is $\mathcal{N}$-acyclic if all the finite intersections $U_{k_0,\ldots,k_m}$ are $\mathcal{N}$-acyclic.

(3) We say that there are enough $\mathcal{N}$-acyclic open coverings of $X$ if for any open set $U \subset X$, and any open covering $U$ of $U$, there exists an $\mathcal{N}$-acyclic open covering $U'$ of $U$ which refines $U$.

**Example 2.2.** Here are a few typical examples of a topological space $X$, and a sheaf $\mathcal{N}$, such that there are enough $\mathcal{N}$-acyclic open coverings of $X$.

1. $X$ is an algebraic variety over a field $\mathbb{k}$ (i.e. an integral finite type separated $\mathbb{k}$-scheme), with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module. Then any affine open set $U$ is $\mathcal{N}$-acyclic, and any affine open covering of $X$ is $\mathcal{N}$-acyclic.

2. $X$ is a complex analytic manifold, with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is an $\mathcal{O}_X$-module. Then any Stein open set $U$ is $\mathcal{N}$-acyclic, and any Stein open covering of $X$ is $\mathcal{N}$-acyclic.

3. $X$ is a differentiable manifold, with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is any $\mathcal{O}_X$-module. Then any open set $U$ is $\mathcal{N}$-acyclic, and any open covering of $X$ is $\mathcal{N}$-acyclic.

4. $X$ is a differentiable manifold, and $\mathcal{N}$ is a locally constant sheaf of abelian groups. Then any sufficiently small simply connected open set $U$ is $\mathcal{N}$-acyclic. There are enough $\mathcal{N}$-acyclic open coverings.

**Remark 2.3.** For the purposes of this section it suffices to require only the vanishing of $H^1(U,\mathcal{N})$. But considering the examples above, we see that the stronger requirement of acyclicity is not too restrictive. Cf. also [KS3].

Let $R$ be a commutative ring. Recall that a sheaf $\mathcal{M}$ of $R$-modules on $X$ is called flat if for every point $x \in X$ the stalk $\mathcal{M}_x$ is a flat $R$-module.

Given a ring homomorphism $R \to R'$, the sheaf $R' \otimes_R \mathcal{M}$ is the sheaf associated to the presheaf $U \mapsto R' \otimes_R \Gamma(U,\mathcal{M})$, for open sets $U \subset X$. If $\{\mathcal{M}_i\}_{i \in I}$ is an inverse system of sheaves on $X$, then $\lim_{\to i} \mathcal{M}_i$ is the sheaf $U \mapsto \lim_{\to i} \Gamma(U,\mathcal{M}_i)$.

Now suppose $m$ is an ideal of $R$. For $i \geq 0$ we let $R_i := R/m^{i+1}$. By combining the operations above one defines the $m$-adic completion of a sheaf of $R$-modules $\mathcal{M}$ to be

$$\hat{\mathcal{M}} := \lim_{\to i} (R_i \otimes_R \mathcal{M}).$$

The sheaf $\mathcal{M}$ is called $m$-adically complete if the canonical sheaf homomorphism $\mathcal{M} \to \hat{\mathcal{M}}$ is an isomorphism.

We define $m^i \mathcal{M}$ to be the sheaf associated to the presheaf $U \mapsto m^i \Gamma(U,\mathcal{M})$ for open sets $U \subset X$; it is a subsheaf of $\mathcal{M}$. Next we define

$$\text{gr}_m^i \mathcal{M} := m^i \mathcal{M}/m^{i+1} \mathcal{M}$$

and $\text{gr}_m \mathcal{M} := \bigoplus_{i \geq 0} \text{gr}_m^i \mathcal{M}$. The latter is a sheaf of $\text{gr}_m R$-modules.

**Proposition 2.4.** Let $\mathbb{k}$ be a field, let $(R, m)$ be a parameter $\mathbb{k}$-algebra, let $X$ be a topological space, and let $\mathcal{M}$ be a sheaf of $R$-modules on $X$. Assume that $\mathcal{M}$ is flat and $m$-adically complete over $R$. Let $\mathcal{M}_i := R_i \otimes_R \mathcal{M}$.

1. The canonical sheaf homomorphism

$$\left(\text{gr}_m R\right) \otimes_\mathbb{k} \mathcal{M}_0 \to \text{gr}_m \mathcal{M}$$

is an isomorphism.
(2) Let $U$ be an $\mathcal{M}_0$-acyclic open set of $X$. Then the $R$-module $\Gamma(U, \mathcal{M})$ is flat and $m$-adically complete, and for every $i$ the canonical homomorphism

$$R_i \otimes_R \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{M}_i)$$

is bijective.

Proof. For part (1), we first note that the stalks at any point $x \in X$ satisfy $(\text{gr}_m \mathcal{M})_x \cong \text{gr}_m (\mathcal{M}_x)$. Now we can use flatness and [CA, Theorem III.5.1]. Part (2) is [Ye5, Theorem 3.6]. □

An $m$-adic system of sheaves $R$-modules on $X$ is the sheaf version of what is defined in Definition 1.3.

**Proposition 2.5.** Let $\mathbb{K}$ be a field, $(R, m)$ a parameter $\mathbb{K}$-algebra, $X$ a topological space, and $\{\mathcal{M}_i\}_{i \in \mathbb{N}}$ an $m$-adic system of sheaves of $R$-modules on $X$. Assume that $X$ has enough $\mathcal{M}_0$-acyclic open coverings, and that each $\mathcal{M}_i$ is flat over $R_i$. Then $\mathcal{M} := \lim_{\leftarrow i} \mathcal{M}_i$ is a flat and $m$-adically complete sheaf of $R$-modules, and the canonical homomorphisms $R_i \otimes_R \mathcal{M} \to \mathcal{M}_i$ are isomorphisms.

Proof. This is [Ye5, Corollary 3.10]. □

**Corollary 2.6.** Let $R$, $X$ and $\mathcal{M}$ be as in Proposition 2.4. Assume that $X$ has enough $\mathcal{M}_0$-acyclic open coverings. Let $(R', m')$ be another parameter $\mathbb{K}$-algebra and $\sigma : R \to R'$ a $\mathbb{K}$-algebra homomorphism. Define $\mathcal{M}' := R' \otimes_R \mathcal{M}$ and

$$\mathcal{M}' := R' \otimes_R \mathcal{M} = \lim_{\leftarrow i} \mathcal{M}'_i.$$

Then $\mathcal{M}'$ is a flat and $m'$-adically complete sheaf of $R'$-modules, the canonical homomorphisms $R'_i \otimes_{R'} \mathcal{M}' \to \mathcal{M}'_i$ are isomorphisms, and $X$ has enough $\mathcal{M}_0'$-acyclic open coverings.

Proof. This follows from Proposition 2.5. Cf. also [Ye5, Corollary 3.11]. □

Because of these results, for deformations we work in the following setup:

**Setup 2.7.** $\mathbb{K}$ is a field; $(R, m)$ is a parameter $\mathbb{K}$-algebra (Definition 1.1); $X$ is a topological space; and $\mathcal{O}_X$ is a sheaf of commutative $\mathbb{K}$-algebras on $X$. The assumption is that $X$ has enough $\mathcal{O}_X$-acyclic open coverings.

By Example 2.2 this is a reasonable assumption.

**Definition 2.8.** Assume Setup 2.7. An associative $R$-deformation of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat $m$-adically complete associative $R$-algebras on $X$, together with an isomorphism of sheaves of $\mathbb{K}$-algebras $\psi : \mathbb{K} \otimes_R \mathcal{A} \to \mathcal{O}_X$, called an augmentation.

Suppose $\mathcal{A}'$ is another associative $R$-deformation of $\mathcal{O}_X$. A gauge transformation $g : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of sheaves of unital $R$-algebras that commutes with the augmentations to $\mathcal{O}_X$.

We denote by $\text{AssDef}(R, \mathcal{O}_X)$ the category of all associative $R$-deformations of $\mathcal{O}_X$, where the morphisms are gauge transformations.

**Remark 2.9.** Suppose $\text{char} \mathbb{K} = 0$, $(X, \mathcal{O}_X)$ is a smooth algebraic variety over $\mathbb{K}$, and $R = \mathbb{K}[[h]]$. In our earlier paper [Ye1] we referred to an associative $R$-deformation of $\mathcal{O}_X$ as a “deformation quantization of $\mathcal{O}_X$”. In retrospect this name seems inappropriate, and hence the new name used here.

Another, more substantial, change is that in [Ye1, Definition 1.6] we required that the associative deformation $\mathcal{A}$ shall be endowed with a differential structure. This turns out to be redundant – see Corollary 8.7.
Definition 2.10. Assume Setup 2.3. We view \( \mathcal{O}_X \) as a sheaf of Poisson \( \mathbb{K} \)-algebras with the zero bracket. A Poisson \( R \)-deformation of \( \mathcal{O}_X \) is a sheaf \( \mathcal{A} \) of flat \( m \)-adically complete commutative Poisson \( R \)-algebras on \( X \), together with an isomorphism of Poisson \( \mathbb{K} \)-algebras \( \psi : \mathbb{K} \otimes_R \mathcal{A} \rightarrow \mathcal{O}_X \), called an augmentation.

Suppose \( \mathcal{A}' \) is another Poisson \( R \)-deformation of \( \mathcal{O}_X \). A gauge transformation \( g : \mathcal{A} \rightarrow \mathcal{A}' \) is an isomorphism of sheaves of Poisson \( R \)-algebras that commutes with the augmentations to \( \mathcal{O}_X \).

We denote by \( \text{PoisDef}(R, \mathcal{O}_X) \) the category of all Poisson \( R \)-deformations of \( \mathcal{O}_X \), where the morphisms are gauge transformations.

Proposition 2.11. Let \( \mathcal{A} \) be a Poisson (resp. associative) \( R \)-deformation of \( \mathcal{O}_X \), and let \( U \) be an \( \mathcal{O}_X \)-acyclic open set of \( X \). Then \( A := \Gamma(X, \mathcal{A}) \) is a Poisson (resp. associative) \( R \)-deformation of \( C := \Gamma(X, \mathcal{O}_X) \).

Proof. This is immediate from Proposition 2.3. \( \square \)

Here is a converse to Proposition 2.11 in the affine algebro-geometric setting.

Theorem 2.12. Let \( R \) be a parameter \( \mathbb{K} \)-algebra, let \( X \) be an affine algebraic variety over \( \mathbb{K} \), with structure sheaf \( \mathcal{O}_X \), and let \( C := \Gamma(X, \mathcal{O}_X) \).

1. Let \( \mathcal{A} \) be a Poisson (resp. associative) \( R \)-deformation of \( C \). Then there exists a Poisson (resp. associative) \( R \)-deformation \( \mathcal{A} \) of \( \mathcal{O}_X \), together with a gauge transformation of deformations

\[
g : \mathcal{A} \rightarrow \Gamma(X, \mathcal{A}).
\]

2. Let \( \mathcal{A} \) and \( \mathcal{A}' \) be Poisson (resp. associative) \( R \)-deformations of \( \mathcal{O}_X \), and let

\[
h : \Gamma(X, \mathcal{A}) \rightarrow \Gamma(X, \mathcal{A}')
\]

gauge transformation of deformations. Then there is a unique gauge transformation of deformations \( h : \mathcal{A} \rightarrow \mathcal{A}' \) such that \( \Gamma(X, \tilde{h}) = h \).

Note that part (2) implies that the pair \( (\mathcal{A}, g) \) of part (1) is unique up to a unique gauge transformation.

Proof. The proof is in several steps.

Step 1. Assume \( R \) is artinian. For an element \( s \in C \) we denote by \( X_s \) the affine open set \( \{ x \in X \mid s(x) \neq 0 \} \); and we call it a principal open set. Note that \( \Gamma(X_s, \mathcal{O}_X) \cong C_s \). By Theorem 1.11 there is a deformation \( A_s \) of \( C_s \), unique up to a unique gauge transformation.

Now suppose \( t \) is another element of \( C \), and \( X_t \subset X_s \). Then we have \( \mathbb{K} \)-algebra homomorphisms \( C \rightarrow C_s \rightarrow C_t \). Again by Theorem 1.11 there is a unique homomorphism of deformations \( A_s \rightarrow A_t \) that’s compatible with the homomorphisms from \( A \).

By this process we obtain a presheaf of \( R \)-algebras on the principal affine open sets of \( X \). Since these open sets are a basis of the topology of \( X \), this gives rise to a sheaf of Poisson (resp. associative) \( R \)-algebras on \( X \), which we denote by \( \mathcal{A} \).

Step 2. \( R \) is still artinian. Let \( \mathcal{A} \) be the sheaf of algebras from the first step. Take a point \( x \in X \). Then the stalk \( \mathcal{A}_x \cong \lim_{\rightarrow} A_s \), the limit taken over the elements \( s \in C \) such that \( x \in X_s \). This shows that \( \mathcal{A}_x \) is a flat \( R \)-module; and hence the sheaf \( \mathcal{A} \) is flat. We conclude that \( \mathcal{A} \) is an \( R \)-deformation of \( \mathcal{O}_X \).

Now look at the \( R \)-algebra homomorphism \( g : \mathcal{A} \rightarrow \Gamma(X, \mathcal{A}) \). Since both are flat \( R \)-algebras augmented to \( C \), it follows that \( g \) is an isomorphism.

Step 3. Here we handle part (2), still with \( R \) artinian. Suppose \( \mathcal{A} \) and \( \mathcal{A}' \) are two \( R \)-deformations of \( \mathcal{O}_X \). Write \( A := \Gamma(X, \mathcal{A}) \) and \( A' := \Gamma(X, \mathcal{A}') \). We are given a gauge transformation \( h : \mathcal{A} \rightarrow \mathcal{A}' \). Take \( s \in C \). Since \( C \rightarrow C_s \) is a principal
localization, and both $\Gamma(X_s, \mathcal{A})$ and $\Gamma(X_s, \mathcal{A}')$ are $R$-deformations of $C_s$. Theorem 1.11 says that there is a unique gauge transformation $\Gamma(X_s, \mathcal{A}) \xrightarrow{\sim} \Gamma(X_s, \mathcal{A}')$ that’s compatible with the homomorphisms from $A$. In this way we obtain an isomorphism of sheaves $\hat{h} : \mathcal{A} \to \mathcal{A}'$ extending $h$; and it is unique.

Step 4. Finally we allow $R$ to be noetherian. Then $R \cong \lim_{\leftarrow} R_i$, and, letting $A_i := R_i \otimes_R A$, we have $A \cong \lim_{\leftarrow} A_i$. By the previous steps for every $i$ there is an $R_i$-deformation $A_i$. Due to uniqueness these form an inverse system, and we take $A := \lim_{\leftarrow} A_i$. By Proposition 2.5 this is an $R$-deformation of $O_X$.

Part (2) is also proved by nilpotent approximation. □

Corollary 2.13. In the situation of Theorem 2.12, there are equivalences of categories

$$\Gamma(X, -) : \text{AssDef}(R, O_X) \to \text{AssDef}(R, C)$$


and

$$\Gamma(X, -) : \text{PoisDef}(R, O_X) \to \text{PoisDef}(R, C).$$

Proof. Combine Proposition 2.11 and Theorem 2.12. □

We now revert to the more general Setup 2.7.

Proposition 2.14. Let $\mathcal{A}$ be a Poisson (resp. associative) $R$-deformation of $O_X$, let $R'$ be another parameter $K$-algebra, and let $\sigma : R \to R'$ a $K$-algebra homomorphism. Define $\mathcal{A}' := R' \otimes_R \mathcal{A}$. Then $\mathcal{A}'$ is a Poisson (resp. associative) $R'$-deformation of $O_X$.

Proof. The sheaf $\mathcal{A}'$ has an induced $R'$-bilinear Poisson bracket (resp. multiplication). By Corollary 2.6 the various conditions for a deformation are satisfied. □

Hence in the situation of this proposition, we get functors

$$\text{ind}_\sigma : \text{AssDef}(R, O_X) \to \text{AssDef}(R', O_X)$$


and

$$\text{ind}_\sigma : \text{PoisDef}(R, O_X) \to \text{PoisDef}(R', O_X).$$

We end this section with a discussion of first order brackets. Suppose $\mathcal{A}$ is an $R$-deformation of $O_X$. Since $\mathcal{A}$ is flat over $R$, and we have the augmentation $\psi : K \otimes_R \mathcal{A} \xrightarrow{\sim} O_X$, there is an induced $K$-linear isomorphism

$$m\mathcal{A}/m^2\mathcal{A} \cong (m/m^2) \otimes_K O_X.$$

This gives rise to a homomorphism

$$\psi^1 : m\mathcal{A} \to (m/m^2) \otimes_K O_X.$$

Suppose $\mathcal{A}$ is an associative deformation, with multiplication $\star$. Given local sections $a_1, a_2 \in \mathcal{A}$, the commutator satisfies

$$a_1 \star a_2 - a_2 \star a_1 \in m\mathcal{A}.$$ 

Hence we get

$$\psi^1(a_1 \star a_2 - a_2 \star a_1) \in (m/m^2) \otimes_K O_X.$$ 

Likewise if $\mathcal{A}$ is a Poisson deformation, with Poisson bracket $\{-,-\}$, then we have

$$\psi^1(\{a_1, a_2\}) \in (m/m^2) \otimes_K O_X.$$
**Lemma 2.15.** Let \( \mathcal{A} \) be associative (resp. Poisson) \( R \)-deformation of \( \mathcal{O}_X \). Assume \( \text{char } \mathbb{K} = 0 \). There is a unique \( \mathbb{K} \)-bilinear sheaf morphism
\[
\{ -, - \}_\mathcal{A} : \mathcal{O}_X \times \mathcal{O}_X \to (\mathfrak{m}/\mathfrak{m}^2) \otimes_\mathbb{K} \mathcal{O}_X,
\]
having the following property. Given local sections \( c_1, c_2 \in \mathcal{O}_X \), choose local liftings \( a_1, a_2 \in \mathcal{A} \) relative to the augmentation \( \psi : \mathcal{A} \to \mathcal{O}_X \). Then
\[
\{ c_1, c_2 \}_\mathcal{A} = \psi^1 \left( \frac{1}{\hbar} (a_1 \star a_2 - a_2 \star a_1) \right)
\]
or
\[
\{ c_1, c_2 \}_\mathcal{A} = \psi^1 (\{ a_1, a_2 \}),
\]
as the case may be.

**Proof.** This is a variant of the usual calculation in deformation theory. The only thing to notice is that it makes sense for sheaves. \( \square \)

**Definition 2.16.** Assume \( \text{char } \mathbb{K} = 0 \). The *first order bracket* of \( \mathcal{A} \) is the \( \mathbb{K} \)-bilinear sheaf morphism \( \{ -, - \}_\mathcal{A} \) in the lemma above.

**Proposition 2.17.**

(1) The first order bracket is gauge invariant. Namely if \( \mathcal{A} \) and \( \mathcal{B} \) are gauge equivalent \( R \)-deformations of \( \mathcal{O}_X \), then \( \{ -, - \}_\mathcal{A} = \{ -, - \}_\mathcal{B} \).

(2) The bracket \( \{ -, - \}_\mathcal{A} \) is a biderivation of \( \mathcal{O}_X \)-modules.

(3) Suppose \( R = \mathbb{K}[[\hbar]] \). Using the isomorphism \( \hbar^{-1} : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{K} \), we get a bilinear function
\[
\{ -, - \}_\mathcal{A} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X.
\]
Then this is a Poisson bracket on \( \mathcal{O}_X \).

**Proof.** All these statements are easy local calculations. \( \square \)

### 3. Twisted Objects in a Category

In this section we present some categorical constructions. These constructions will be made geometric in Section 5 where categories will be replaced by stacks of categories on a topological space.

First we must establish some set theoretical background, in order to avoid paradoxical phenomena. Recall that in set theory all mathematical objects and operations are interpreted as sets, with suitable additional properties. Following [SGA4-I], [ML], [DP] we fix a *Grothendieck universe* \( U \), which is a set closed under standard set theoretical operations, and is large enough such that the objects of interest for us (the field \( \mathbb{K} \), the space \( X \), the sheaf \( \mathcal{O}_X \) etc. from Sections 1-2) are elements of \( U \). We refer to elements of \( U \) as *small sets*. A category \( \mathcal{C} \) such that \( \text{Ob}(\mathcal{C}) \in U \), and \( \text{Hom}_\mathcal{C}(C_0, C_1) \in U \) for every pair \( C_0, C_1 \in \text{Ob}(\mathcal{C}) \), is called a *small category*.

By \( \text{Set} \) we refer the category of small sets; thus in effect \( \text{Ob}(\text{Set}) = U \). Likewise \( \text{Grp} \), \( \text{Mod} A \) etc. refer to the categories of small groups, small \( A \)-modules (over a small ring \( A \)) etc. A category \( \mathcal{C} \) such that \( \text{Ob}(\mathcal{C}) \subseteq U \), and \( \text{Hom}_\mathcal{C}(C_0, C_1) \in U \) for every pair \( C_0, C_1 \in \text{Ob}(\mathcal{C}) \), is called a \( U \)-category. Thus \( \text{Set} \) is a \( U \)-category, but it is not small.

Next we introduce a bigger universe \( V \), such that \( U \in V \). Then \( \text{Ob}(\text{Set}), \text{Ob}(\text{Grp}), \ldots \in V \). In order to distinguish between them, we call \( U \) the small universe, and \( V \) the large universe. The set of all \( U \)-categories is denoted by \( \text{Cat} \). Note that \( \text{Cat} \) is a \( V \)-category, but (this is the whole point!) it is not a \( U \)-category.

By default sets, groups etc. will be assumed to be small; and categories will be assumed to be \( U \)-categories.
Recall that a *groupoid* is a category $G$ in which all morphisms are invertible. We shall sometimes denote objects of a small groupoid $G$ by the letters $i, j, \ldots$; this is because we want to view the objects as indices, enumerating the collection of groups $\text{Hom}_G(i, i)$ and the collection of sets $\text{Hom}_G(i, j)$. We say that $G$ is *nonempty* if $\text{Ob}(G) \neq \emptyset$, and that $G$ is *connected* if $\text{Hom}_G(i, j) \neq \emptyset$ for all $i, j \in \text{Ob}(G)$.

Given a category $C$ and objects $C, D \in \text{Ob}(C)$, we sometimes write

$$C(C, D) := \text{Hom}_C(C, D),$$

the set of morphisms from $C$ to $D$. We also write

$$C^\times(C, D) := \text{Isom}_C(C, D),$$

the set of invertible morphisms from $C$ to $D$. Note that $C^\times$ is a groupoid (with set of objects $\text{Ob}(C^\times) = \text{Ob}(C)$). We say that $C^\times$ is *connected by isomorphisms* if and only if the set $\text{Ob}(C^\times)$ has one element.

**Definition 3.1.** Given a category $C$, we denote by $\overline{\text{Ob}}(C)$ the set of isomorphism classes of objects.

Note that $\overline{\text{Ob}}(C) = \overline{\text{Ob}}(C^\times)$; and $C$ is nonempty and connected by isomorphisms if and only if the set $\overline{\text{Ob}}(C)$ has one element.

**Definition 3.2.** Let $C$ be a category and $C, D \in \text{Ob}(C)$. Suppose $\phi \in \text{Isom}_C(C, D)$. We define a bijection

$$\text{Ad}_C(\phi) : \text{Hom}_C(C, C) \to \text{Hom}_C(D, D)$$

by

$$\text{Ad}_C(\phi)(\psi) := \phi \circ \psi \circ \phi^{-1}.$$  

**Example 3.3.** Suppose $G$ is a group, which we make into a one-object groupoid $G$, with $\text{Ob}(G) := \{0\}$, and $\text{Hom}_G(0, 0) := G$. Then for any $g \in G$ the bijection $\text{Ad}_G(g) : G \to G$ is conjugation in the group.

Recall that a monoid is a semigroup with unit. We denote the category of monoids, with unit preserving homomorphisms, by $\text{Monoid}$. The category $\text{Grp}$ of groups is viewed as a full subcategory of $\text{Monoid}$.

Let $C$ be a category. Then, in a tautological sort of way, there is a functor

$$\text{End}_C : C^\times \to \text{Monoid},$$

where

$$\text{End}_C(C) := \text{Hom}_C(C, C)$$

for an object $C \in \text{Ob}(C)$. Given another object $D \in \text{Ob}(C)$, and an isomorphism $\phi \in \text{Isom}_C(C, D)$, we let

$$\text{End}_C(\phi) := \text{Ad}_C(\phi) : \text{End}_C(C) \to \text{End}_C(D).$$

The functor $\text{End}_C$ has a subfunctor

$$\text{Aut}_C : C^\times \to \text{Grp},$$

which we might also denote by $\text{End}_C^\times$.

**Example 3.5.** Let $G$ be some groupoid. Then $G = G^\times$, and $\text{End}_G = \text{Aut}_G$ as functors $G^\times \to \text{Grp}$.

Suppose $F : C \to D$ is a functor between categories. Then there is a corresponding natural transformation

$$\text{End}_F : \text{End}_C \Rightarrow \text{End}_D \circ F$$
between functors $C^\times \to \text{Monoid}$. (We denote natural transformations by $\Rightarrow$; see Section 4 for an explanation.) The formula for $\text{End}_F$ is this: given an object $C \in \text{Ob}(C)$, the monoid homomorphism

$$\text{End}_F(C) : \text{End}_C(C) \to \text{End}_D(FC)$$

is $\text{End}_F(C)(\phi) = F(\phi)$. This natural transformation restricts to the group valued subfunctors

$$(3.6) \quad \text{Aut}_F : \text{Aut}_C \Rightarrow \text{Aut}_D \circ F.$$

Throughout this section $R$ is some commutative ring. Here is a definition due to Kontsevich [Ko2].

**Definition 3.7.** An $R$-linear algebroid is a small $R$-linear category $A$, which is nonempty and connected by isomorphisms.

**Example 3.8.** Take an associative $R$-algebra $A$. Then there is an $R$-linear algebroid $A$, with $\text{Ob}(A) := \{0\}$, and $A(0,0) := A$.

Here is a more interesting example, coming from Morita theory.

**Example 3.9.** Let $C$ be an $R$-linear abelian category with infinite direct sums. An object $P \in \text{Ob}(C)$ is called compact if for any collection $\{C_j\}_{j \in J} \subset \text{Ob}(C)$, indexed by some set $J$, the canonical homomorphism

$$\bigoplus_{j \in J} \text{Hom}_C(P, C_j) \to \text{Hom}_C(P, \bigoplus_{j \in J} C_j)$$

is bijective.

Suppose we are given a collection $\{P_i\}_{i \in I}$ of objects of $C$, indexed by a nonempty set $I$, such that these objects are all isomorphic. Let $A$ be the category with $\text{Ob}(A) := I$, $A(i,j) := \text{Hom}_C(P_i, P_j)$, and composition rule coming from $C$. Then $A$ is an $R$-linear algebroid.

Let us now assume that each $P_i$ is a compact projective generator of $C$. And let us denote by $\text{Mod}^\text{op} A$ the category of $R$-linear functors $M : A^{\text{op}} \to \text{Mod} R$, which we call right $A$-modules. For any $C \in \text{Ob}(C)$ there is a right $A$-module $M_C := \text{Hom}_C(-, C)$. Then the $R$-linear functor $C \mapsto M_C$ is an equivalence of categories $C \to \text{Mod}^\text{op} A$.

**Example 3.10.** Take an $R$-linear algebroid $A$, and define $G := A^\times$. Let $\text{Assoc}(R)$ denote the category of associative $R$-algebras. Then $\text{End}_A$ is a functor

$$\text{End}_A : G \to \text{Assoc}(R).$$

We are going to see what information, in addition to the groupoid $G$ and the functor $\text{End}_A$ in the example above, is needed to reconstruct the algebroid $A$. This will enable us to treat other kinds of mathematical structures that are similar to algebroids.

**Definition 3.11.** Let $P$ be a category. An inner gauge group structure on $P$ is a functor

$$\text{IG} : P \to \text{Grp},$$

together with a natural transformation

$$\text{ig} : \text{IG} \Rightarrow \text{Aut}_P$$

between functors $P^{\times} \to \text{Grp}$, such that the conditions below hold for any $A \in \text{Ob}(P)$:

1. The group homomorphism

$$\text{ig}(A) : \text{IG}(A) \to \text{Aut}_P(A)$$

is $\text{Aut}_P(A)$-equivariant. Here $\text{Aut}_P(A)$ acts on $\text{IG}(A)$ by functoriality, and on itself by conjugation.
(ii) The composed group homomorphism

$$\text{IG}(A) \xrightarrow{\text{ig}(A)} \text{Aut}_P(A) \xrightarrow{\text{Aut}_{\text{IG}}(A)} \text{Aut}_{\text{Grp}}(\text{IG}(A))$$

is the conjugation action of $\text{IG}(A)$ on itself.

Note that the group homomorphism $\text{Aut}_{\text{IG}}(A)$ is an instance of the natural transformation (3.6).

For an object $A$ of $P$, the group $\text{IG}(A)$ is called the group of inner gauge transformations of $A$. For an element $g \in \text{IG}(A)$, the automorphism $\text{ig}(A)(g)$ of $A$ is called an inner gauge transformation. The data $(P, \text{IG}, \text{ig})$ is called a category with inner gauge groups.

Remark 3.12. The conditions in Definition 3.11 say that the pair of groups $(\text{Aut}_P(A), \text{IG}(A))$ is a crossed module. This notion appears in several recent papers on gerbes (e.g. [BM]), and on higher gauge theory in mathematical physics (e.g. [BS]).

Here are some examples.

Example 3.13. Take the category $\text{Grp}$ of groups. For a group $G$ let $\text{IG}(G) := G$, and for an element $g \in G$ let $\text{ig}(G)(g) := \text{Ad}_G(g)$, i.e. conjugation.

Example 3.14. Take the category $P := \text{Assoc}(R)$. The group of inner gauge transformations of an associative $R$-algebra $A$ is the group $\text{IG}(A) := A^\times$ of invertible elements. It is functorial, since a ring homomorphism $f : A \to B$ sends invertible elements to invertible elements. The inner gauge transformation $\text{ig}(A)(g)$, for $g \in \text{IG}(A)$, is conjugation by this invertible element, namely

$$\text{ig}(A)(g)(a) = g \star a \star g^{-1},$$

where $\star$ is the multiplication in $A$. The conditions are very easy to verify.

Example 3.15. Take the category $P := \text{AssDef}(R,C)$ from Definition 1.7. The group of inner gauge transformations of an algebra $A \in P$ is the group

$$\text{IG}(A) := \{ g \in A \mid g \equiv 1 \mod m \} \subset A^\times.$$

The inner gauge transformation $\text{ig}(A)(g)$ is the same as in Example 3.14.

Before we go on, a reminder on nilpotent Lie theory in characteristic 0. Suppose $g$ is a finite dimensional nilpotent Lie algebra over a field $K$ of characteristic 0. Then there is an associated unipotent algebraic group $\exp(g)$, together with an abstract exponential map

$$\exp : g \to \exp(g).$$

The group $\exp(g)$ depends functorially on the Lie algebra $g$. See [Ho] for details. Now by passing to direct limits, the group $\exp(g)$ makes sense for any nilpotent Lie algebra $g$ over $K$. And by passing to inverse limits we can consider the group $\exp(g)$ for any pronilpotent Lie algebra $g$.

Example 3.16. Take the category $P := \text{PoisDef}(R,C)$ from Definition 1.8, and assume $\text{char} K = 0$. Consider an algebra $A \in \text{PoisDef}(R,C)$, with Poisson bracket $\{-,-\}$. The $R$-submodule $mA \subset A$ is then a pronilpotent Lie algebra over $K$ with respect to the Poisson bracket. We define the group of inner gauge transformations to be

$$\text{IG}(A) := \exp(mA).$$

An element $a$ of the Lie algebra $mA$ acts on the the Poisson algebra $A$ by the hamiltonian derivation

$$\text{ad}_A(a)(b) := \{a,b\}.$$


Hence for the element
\[ g := \exp(a) \in \exp(mA) \]
there is an induced automorphism
\[ \exp(\text{ad}_A)(g) := \exp(\text{ad}_A(a)) \]
of the Poisson algebra \( A \), called a *formal hamiltonian flow*. We define
\[ \text{ig}(A)(g) := \exp(\text{ad}_A(g)). \]

Here the details are a bit harder to verify (cf. [Hu, Section 2.3]), but indeed this is also an inner gauge group structure.

**Example 3.17.** Associative \( R \)-deformations in characteristic 0 can also be expressed in terms of nilpotent Lie theory. Indeed, suppose \( A \in \text{AssDef}(R,C) \). The \( R \)-submodule \( mA \subset A \) is a pronilpotent Lie algebra over \( K \) with respect to the Lie bracket
\[ [a,b] := a \star b - b \star a, \]
where \( \star \) denotes the multiplication in \( A \). Here the abstract group \( \exp(mA) \) is canonically identified with the multiplicative group \( \text{IG}(A) \subset A^\times \) from Example 3.15 and the abstract exponential map becomes
\[ \exp(a) = \sum_{i \geq 0} \frac{1}{i!} a \star \cdots \star a. \]

An element \( a \in mA \) acts on the ring \( A \) by the derivation
\[ \text{ad}_A(a) := [a, -]. \]
One can calculate that for \( g := \exp(a) \in \text{IG}(A) \) the induced automorphism \( \exp(\text{ad}_A(a)) \) of \( A \) is conjugation by the invertible element \( g \); cf. [Hu, Section 2.3].

Here is a result on the structure of the inner gauge groups of \( R \)-deformations.

**Proposition 3.18.** Let \( A \) be an \( R \)-deformation of \( C \) (associative or Poisson). Write \( G := \text{IG}(A) = \exp(mA) \), and \( N_p := \exp(mp+1A) \) for \( p \geq 0 \). Then for any \( p \) the subgroup \( N_p \subset G \) is normal, the extension of groups
\[ 1 \to N_p/N_{p+1} \to G/N_{p+1} \to G/N_p \to 1 \]
is central, and
\[ N_p/N_{p+1} \cong (mA/mp+1) \otimes_K C \]
as abelian groups. Moreover, the canonical homomorphism
\[ G \to \varinjlim_{p} G/N_p \]
is bijective.

**Proof.** For any \( p \) we have \( G/N_p \cong \exp(mA/mp+1A) \) and
\[ N_p/N_{p+1} \cong \exp(mp+1A/mp+2A). \]

We now return to the general theory.

**Definition 3.19.** Let \((P, \text{IG}, \text{ig})\) be a category with inner gauge groups. A *twisted object* of \((P, \text{IG}, \text{ig})\) is the data \((G, A, \text{cp})\), consisting of:

1. A small connected nonempty groupoid \( G \), called the *gauge groupoid*.
2. A functor \( A : G \to P \), called the *representation*.
3. A natural isomorphism
\[ \text{cp} : \text{Aut}_G \cong \text{IG} \circ A \]
of functors \( G \to \text{Grp} \), called the *coupling isomorphism*. 
The condition is:

\[ \text{(\ast)} \text{ The diagram} \]

\[
\begin{array}{ccc}
\text{Aut}_G & \xrightarrow{\text{cp}} & IG \circ A \\
\downarrow \text{\text{Aut}_A} & & \downarrow \text{\text{1}_A} \\
\text{Aut}_P & \xrightarrow{\text{\text{cp}}} & \text{A}
\end{array}
\]

\[ \text{of natural transformations between functors } G \to \text{Grp} \text{ is commutative.} \]

The set of twisted objects of \((P, IG, ig)\) is denoted by \(\text{TwOb}(P, IG, ig)\), or just by \(\text{TwOb}(P)\) if there is no danger of confusion. Similarly, we often refer to the twisted object \((G, A, \text{cp})\) just as \(A\). Since \(\text{Ob}(G) \subseteq U\) and \(\text{Ob}(P) \subseteq U\) (the small universe), it follows that the twisted object \((G, A, \text{cp})\) is an element of the small universe \(U\).

The definition above is terribly formal and almost impossible to understand. So here is what is really means. For any \(i \in \text{Ob}(G)\) there is an object \(A_i := A(i) \in \text{Ob}(P)\). Thus we are given a collection \(\{A_i\}_{i \in \text{Ob}(G)}\) of objects of \(P\). For any arrow \(g : i \to j\) in the groupoid \(G\) there is given an isomorphism \(A(g) : A_i \to A_j\) in \(P\). This tells us how we may try to identify the objects \(A_i\) and \(A_j\).

For any index \(i\) there is given a group isomorphism (the coupling)

\[ \text{cp} : G(i,i) = \text{Aut}_G(i) \xrightarrow{\cong} (IG \circ A)(i) = IG(A_i). \]

It forces the groupoid \(G\) to be comprised of inner gauge groups. But now an element \(g \in G(i,i)\) has two possible actions on the object \(A_i\): it can act as \(A(g)\), or it can act as \(ig(A_i)(\text{cp}(g))\). Condition \(\text{(\ast)}\) says that these two actions coincide.

A twisted object of \(\text{Assoc}(R)\) will be referred to as a \textit{twisted associative }\(R\)-\textit{algebra}. Likewise a twisted object of \(\text{AssDef}(R, C)\) will be called a \textit{twisted associative }\(R\)-\textit{deformation of }\(C\), and similarly for the Poisson case.

**Example 3.20.** Suppose \(A\) is an object of \(P\). We can turn it into a twisted object of \(P\) as follows. Let \(G\) be the one object groupoid, say \(\text{Ob}(G) := \{0\}\), with \(G(0,0) := IG(A)\). The functor \(A : G \to P\) is \(A(0) := A\) and \(A(g) := ig(g)\). The coupling isomorphism \(\text{cp}\) is the identity of \(IG(A)\). We refer to \(A\) as the \textit{twisted object generated by }\(A\).

A more interesting example is:

**Example 3.21.** Suppose \(A\) is an \(R\)-linear algebroid. We are going to turn it into a twisted object of \(\text{Assoc}(R)\), where the inner gauge group structure is from Example 3.14. Consider the groupoid \(G := A^\times\), and the functor

\[ A := \text{End}_A : G \to \text{Assoc}(R) \]

from Example 3.10. So \(A(i) = A(i,i)\) for \(i \in \text{Ob}(G) = \text{Ob}(A)\). The coupling isomorphism is the identity

\[ \text{cp} : G(i,i) \xrightarrow{\cong} A(i,i)^\times = IG(A(i)). \]

**Definition 3.22.** Let \((G, A, \text{cp})\) and \((G', A', \text{cp}')\) be twisted objects in a category with inner gauge groups \((P, IG, ig)\). A \textit{twisted gauge transformation}

\[ (F_{\text{gau}}, F_{\text{rep}}) : (G, A, \text{cp}) \to (G', A', \text{cp}') \]

consists of an equivalence (of groupoids) \(F_{\text{gau}} : G \to G'\), and an natural isomorphism

\[ F_{\text{rep}} : A \xrightarrow{\cong} A' \circ F_{\text{gau}} \]
of functors $G \to \text{Grp}$. The condition is that the diagram

$$
\begin{array}{ccc}
\text{Aut}_{G} & \overset{\text{cp}}{\longrightarrow} & \text{IG} \circ A \\
\downarrow \text{Aut}_{F_{\text{gaus}}} & & \downarrow \text{IG} \circ F_{\text{rep}} \\
\text{Aut}_{G'} \circ F_{\text{gaus}} & \overset{\text{cp}'}{\longrightarrow} & \text{IG} \circ A' \circ F_{\text{gaus}}
\end{array}
$$

is commutative.

Let us spell out what this definition means for an object $i \in \text{Ob}(G)$. Let $i' := F_{\text{gaus}}(i) \in \text{Ob}(G)'$. Then there is an isomorphism $F_{\text{rep}} : A(i) \sim A'(i')$ in $P$, and the diagram

$$
\begin{array}{ccc}
G(i, i) & \overset{\text{cp}}{\longrightarrow} & \text{IG}(A(i)) \\
\downarrow F_{\text{gaus}} & & \downarrow \text{IG}(F_{\text{rep}}) \\
G'(i', i') & \overset{\text{cp}'}{\longrightarrow} & \text{IG}(A'(i'))
\end{array}
$$

in $\text{Grp}$ is commutative.

**Proposition 3.23.** Twisted gauge transformations form an equivalence relation on the set $\text{TwOb}(P)$.

**Proof.** This is an exercise in functors. \qed

We refer to this equivalence relation a *twisted gauge equivalence*, and we write

$$
\overline{\text{TwOb}(P)} := \frac{\text{TwOb}(P)}{\text{twisted gauge equivalence}}.
$$

**Remark 3.24.** One can introduce composition between twisted gauge transformations. (Indeed, that is needed to prove transitivity in the proposition above.) With this composition $\overline{\text{TwOb}(P)}$ becomes a $V$-category ($V$ is the large universe).

Furthermore one can introduce the notion of 2-isomorphism between twisted gauge transformations. In this way the $\overline{\text{TwOb}(P)}$ becomes a 2-groupoid (in a weak sense). However we shall not need this refined structure.

**Remark 3.25.** If one examines things a little, it becomes evident that a twisted object $(G, A, \text{cp})$ in $P$ is twisted gauge equivalent to the twisted object generated by $A(i) \in P$, for any $i \in \text{Ob}(G)$ (as in Example 3.20). Thus the whole concept is quite uninteresting.

However, in the geometric context (see Section 5), where the category $P$ is replaced by a stack of categories $P$ on a topological space $X$, the concept becomes interesting: really twisted objects (Definition 5.17) appear.

**Remark 3.26.** Concerning Example 3.21, we will see in Section 5 that stacks of $R$-linear algebroids on a topological space $X$ are the same as twisted sheaves of associative $R$-algebras on $X$.

**Remark 3.27.** In case $R = \mathbb{K}[[h]]$, the ring of formal power series in a variable $h$, and $P$ is either $\text{AssDef}(R, C)$ or $\text{PoisDef}(R, C)$, then condition $(\ast)$ in Definition 3.19 is redundant. This is because given a group homomorphism $\text{IG}(A_1) \to \text{IG}(A_2)$ for $A_1, A_2 \in \text{Ob}(P)$, there is at most one gauge transformation $A_1 \to A_2$ extending it. However if $R$ is not an integral domain (e.g. $R = \mathbb{K}[h]/(h^5)$), then things might be more complicated.
4. Stacks on a Topological Space

In this section we give a reminder on stacks on a topological space, that will serve to establish notation. A lengthier discussion of stacks and 2-categories (with the same conventions) can be found in [Ye4 Sections 1-2]. See also [ML Section XII.3], [DP] and [KS2, Chapter 19].

The most helpful (but imprecise) description of a stack on \( X \) in that this is the geometrization of the notion of a category, in the same way that a sheaf is the geometrization of the notion of a set.

In order to be precise we must first talk about 2-categories. Recall that a 2-category \( \mathcal{C} \) is a category enriched in categories. This means that there is a set \( \text{Ob}(\mathcal{C}) \), whose elements are called objects of \( \mathcal{C} \). For every pair of elements \( C_0, C_1 \in \text{Ob}(\mathcal{C}) \) there is a category \( \mathcal{C}(C_0, C_1) \). The objects of the category \( \mathcal{C}(C_0, C_1) \) are called 1-morphisms of \( \mathcal{C} \), and the morphisms of \( \mathcal{C}(C_0, C_1) \) are called 2-morphisms of \( \mathcal{C} \).

For every triple of elements \( C_0, C_1, C_2 \in \text{Ob}(\mathcal{C}) \) there is a bifunctor

\[
\mathcal{C}(C_0, C_1) \times \mathcal{C}(C_1, C_2) \rightarrow \mathcal{C}(C_0, C_2),
\]

called horizontal composition, and denoted by \( \circ \). Horizontal composition has to be associative (as a bifunctor, namely in its action on 1-morphisms and 2-morphisms).

For any element \( C \in \text{Ob}(\mathcal{C}) \) there is a distinguished 1-morphism \( 1_C \in \text{Ob}(\mathcal{C}(C, C)) \).

Horizontal composition with \( 1_C \), on either side, is required to be the identity functor.

If \( F \in \text{Ob}(\mathcal{C}(C_0, C_1)) \), then we say that \( F \) is a 1-morphism from \( C_0 \) to \( C_1 \), and we denote this by \( F : C_0 \rightarrow C_1 \). Next suppose \( F, G : C_0 \rightarrow C_1 \), and \( \eta \in \text{Hom}_{\mathcal{C}(C_0, C_1)}(F, G) \). Then we say that \( \eta \) is a 2-morphism from \( F \) to \( G \), and we denote this by \( \eta : F \Rightarrow G \). The composition in \( \mathcal{C}(C_0, C_1) \) is called vertical composition, and it is denoted by \( \ast \). The identity 2-morphism of a 1-morphism \( F \) is denoted by \( 1_F \).

Regarding set theoretical issues for 2-categories, recall that \( U \) is the small universe, and \( V \) the large universe. We assume that the following hold for a 2-category \( \mathcal{C} : \text{Ob}(\mathcal{C}) \subset V ; \text{Hom}_\mathcal{C}(C_0, C_1) \in V \) for any pair of objects \( C_0, C_1 \); and

\[
\text{Hom}_\mathcal{C}(C_0, C_1)(F, G) \in U
\]

for any pair of 1-morphisms \( F, G : C_0 \rightarrow C_1 \).

Note that if we forget the 2-morphisms (and the vertical composition) in \( \mathcal{C} \), then \( \mathcal{C} \) becomes a \( V \)-category.

The most important 2-category is \( \textbf{Cat} \). Recall that \( \textbf{Cat} \) was defined to be the set of all \( V \)-categories. We turn \( \textbf{Cat} \) into a 2-category by taking \( \textbf{Cat}(C_0, C_1) \) to be the category of all functors \( F : C_0 \rightarrow C_1 \). The morphisms in \( \textbf{Cat}(C_0, C_1) \), i.e. the 2-morphisms, are the natural transformations \( \zeta : F \Rightarrow G \). Horizontal composition is defined to be composition of functors.

Let \( X \) be a topological space. We denote by \( \textbf{Open} X \) the category of open sets of \( X \), where a morphism \( V \rightarrow U \) is an inclusion \( V \subset U \). A prestack on \( X \) is a (strict) pseudofunctor

\[
\mathcal{G} : (\textbf{Open} X)^{\text{op}} \rightarrow \textbf{Cat}.
\]

This means that for any open set \( U \subset X \) there is a category \( \mathcal{G}(U) \). There is a restriction functor

\[
\text{rest}_{U_1/U_0}^\mathcal{G} : \mathcal{G}(U_0) \rightarrow \mathcal{G}(U_1)
\]

for any inclusion \( U_1 \subset U_0 \) of open sets. And there are composition isomorphisms

\[
\gamma_{U_2/U_1/U_0}^\mathcal{G} : \text{rest}_{U_2/U_1}^\mathcal{G} \circ \text{rest}_{U_1/U_0}^\mathcal{G} \cong \text{rest}_{U_2/U_0}^\mathcal{G}
\]

for a double inclusion \( U_2 \subset U_1 \subset U_0 \). The conditions are: \( \text{rest}_{U/U}^\mathcal{G} = 1_{\mathcal{G}(U)} \), the identity functor of the category \( \mathcal{G}(U) \); \( \gamma_{U/U/U}^\mathcal{G} = 1_{1_{\mathcal{G}(U)}} \), the identity automorphism.
of the functor $1_{\mathcal{G}(U)}$; and

$$
\gamma_{U_0/U_2/U_0} \circ \gamma_{U_2/U_1/U_0} = \gamma_{U_0/U_1/U_0} \circ \gamma_{U_0/U_2/U_1}
$$

for a triple inclusion $U_3 \subset U_2 \subset U_1 \subset U_0$.

We denote by $\text{PreStack} X$ the set of prestacks on $X$. Since the category $\text{Open} X$ is small, it follows that $\text{PreStack} X \subset \text{V}$. The set $\text{PreStack} X$ has a structure of 2-category, which we now describe. Suppose $\mathcal{G}$ and $\mathcal{H}$ are prestacks on $X$. A 1-morphism of prestacks $F : \mathcal{G} \to \mathcal{H}$ is a 1-morphism between these pseudofunctors. Thus there is a functor

$$F(U) : \mathcal{G}(U) \to \mathcal{H}(U)$$

for any open set $U$, together with an isomorphism of functors

$$\psi^F_{U_1/U_0} : F(U_1) \circ \text{rest}^\mathcal{G}_{U_1/U_0} \cong \text{rest}^\mathcal{H}_{U_1/U_0} \circ F(U_0)$$

for any inclusion $U_1 \subset U_0$ of open sets. Note that this isomorphism is between objects in the category $\text{Cat}(\mathcal{G}(U_0), \mathcal{H}(U_1))$. The isomorphisms $\psi^F_{U_1/U_0}$ are required to satisfy the condition

$$\psi^F_{U_2/U_0} \circ \gamma_{U_2/U_1/U_0} = \gamma_{U_2/U_1/U_0} \circ \psi^F_{U_2/U_1} \circ \psi^F_{U_1/U_0}$$

for a double inclusion $U_2 \subset U_1 \subset U_0$. This equality is as isomorphisms

$$F(U_2) \circ \text{rest}^\mathcal{G}_{U_2/U_1} \circ \text{rest}^\mathcal{G}_{U_1/U_0} \cong \text{rest}^\mathcal{G}_{U_2/U_0} \circ F(U_0)$$

in the category $\text{Cat}(\mathcal{G}(U_0), \mathcal{H}(U_2))$.

The composition of 1-morphisms of prestacks $\mathcal{G} \xrightarrow{E} \mathcal{H} \xrightarrow{F} \mathcal{K}$ is denoted by $E \circ F$. The formula for this composition is obvious.

When no confusion can arise we sometimes say “functor of prestacks” instead of “1-morphism of prestacks”. The reason is that we want to think of a prestack as a generalization of a category. (Indeed when the space $X$ has only one point, then there is no distinction between these notions.)

Suppose $E, F : \mathcal{G} \to \mathcal{H}$ are 1-morphisms between prestacks. A 2-morphism $\eta : E \Rightarrow F$ consists of a morphism $\eta_U : E(U) \to F(U)$ of functors $\mathcal{G}(U) \to \mathcal{H}(U)$ for every open set $U$. The condition is

$$\eta_{U_1} \circ \psi^E_{U_1/U_0} = \psi^F_{U_1/U_0} \circ \eta_{U_2}$$

for an inclusion $U_1 \subset U_0$. This is equality as morphisms of functors

$$\text{rest}^\mathcal{H}_{U_1/U_0} \circ E(U_0) \Rightarrow F(U_1) \circ \text{rest}^\mathcal{H}_{U_1/U_0},$$

and these live in $\text{Cat}(\mathcal{G}(U_0), \mathcal{H}(U_1))$.

Given another 1-morphism $D : \mathcal{G} \to \mathcal{H}$, and a 2-morphism $\zeta : D \Rightarrow E$, the composition with $a$ is denoted by $\eta \ast \zeta : D \Rightarrow F$. Again the formula for this composition is obvious. We sometimes say “natural transformation” instead of “2-morphism” in this context.

As in any 2-category, we can say when a functor of prestacks $F : \mathcal{G} \to \mathcal{H}$ (i.e. a 1-morphism in $\text{PreStack} X$) is an equivalence. This just means that there is a functor of prestacks $E : \mathcal{H} \to \mathcal{G}$, and natural isomorphisms (i.e. 2-isomorphisms) $E \circ F \cong 1_{\mathcal{G}}$ and $F \circ E \cong 1_{\mathcal{H}}$. But here there is also a geometric characterization: $F$ is an equivalence if and only if for any open set $U \subset X$ the functor $F(U) : \mathcal{G}(U) \to \mathcal{H}(U)$ is an equivalence.

Suppose $\mathcal{G}$ is a prestack on $X$. Take an open set $U \subset X$ and two objects $i, j \in \text{Ob} \mathcal{G}(U)$. There is a presheaf of sets $\mathcal{G}(i, j)$ on $U$, defined as follows. For an open set $V \subset U$ we define the set

$$\mathcal{G}(i, j)(V) := \text{Hom}_{\mathcal{G}(U)}(\text{rest}^\mathcal{G}_{V/U}(i), \text{rest}^\mathcal{G}_{V/U}(j)).$$
For an inclusion $V_1 \subset V_0 \subset U$ of open sets, the restriction function

$$\text{rest}^G(i,j)_{V_1/V_0} : G(i,j)(V_0) \to G(i,j)(V_1)$$

is the composed function

$$\text{Hom}_{G(V_0)}(\text{rest}^G_{V_0/U}(i), \text{rest}^G_{V_0/U}(j))$$

$$\xrightarrow{\text{rest}^G_{V_1/V_0}} \text{Hom}_{G(V_1)}((\text{rest}^G_{V_1/V_0} \circ \text{rest}^G_{V_0/U})(i), (\text{rest}^G_{V_1/V_0} \circ \text{rest}^G_{V_0/U})(j))$$

$$\xrightarrow{\gamma^{V_1/V_0/U}} \text{Hom}_{G(V_1)}((\text{rest}^G_{V_1/U}(i), \text{rest}^G_{V_1/U}(j)).$$

Condition (4.1) ensures that

$$\text{rest}^G(i,j)_{V_1/V_0} \circ \text{rest}^G(i,j)_{V_1/V_0} = \text{rest}^G(i,j)_{V_1/V_0}$$

for an inclusion $V_2 \subset V_1 \subset V_0 \subset U$. Note that the set of sections of this presheaf is

$$\Gamma(V, G(i,j)) = G(V)(i,j).$$

From now on we shall usually write $i|_V$ instead of $\text{rest}^G_{V_0/U}(i)$, for a local object $i \in \text{Ob}(G(U))$: and $g|_{V_1}$ instead $\text{rest}^G(i,j)_{V_1/V_0}(g)$, for a local morphism $g \in G(i,j)(V_0)$.

We usually omit reference to the restriction functors $\text{rest}^G_{V_0/U}$ altogether. Another convention is that from now on we denote morphisms in the local categories $G(U)$ by “$\circ$”, and not by “$\ast$”, as might be suggested by the discussion of 2-categories above.

A prestack $G$ is called a stack if it satisfies descent for morphisms and descent for objects. The first condition says that the presheaves $G(i,j)$ are all sheaves. The second condition says that given an open set $U$, an open covering $U = \bigcup_{k \in K} U_k$, objects $i_k \in \text{Ob}(G(U_k))$, and isomorphisms

$$g_{i_0,k_1} \in G(U_{i_0,k_1})(i_{i_0,i_1} \restriction U_{i_0,k_1})$$

that satisfy

$$\gamma_{i_0,k_1,k_2} \circ g_{i_0,k_1} \restriction U_{i_0,k_1} \cdot k_2 = g_{i_0,k_2} \restriction U_{i_0,k_1} \cdot k_2,$$

there exists an object $i \in G(U)$, and isomorphisms $g_k \in G(U_k)(i|_{U_k}, i_k)$, such that

$$g_{i_0,k_1} \circ g_{i_0,k_1} \restriction U_{i_0,k_1} = g_{i_1,k_1} \restriction U_{i_0,k_1}.$$

(By the first condition this object $i$ is unique up to a unique isomorphism.)

Here are several examples, that will reappear later in the paper.

**Example 4.4.** On any open set $U \subset X$ we have the category $\text{Grp}U$ of sheaves of groups on $U$. For an inclusion $U_1 \subset U_0$ we have a functor $\text{Grp}U_1 \to \text{Grp}U_0$, namely the usual restriction of sheaves $G \mapsto G|_{U_1}$. And for $U_2 \subset U_1$ we have $G|_{U_2} = (G|_{U_1})|_{U_2}$. Thus we get a prestack $\text{Grp}X$ on $X$ with $(\text{Grp}X)(U) = \text{Grp}U$. It is easy to check that this is actually a stack, which we call the stack of sheaves of groups on $X$.

**Example 4.5.** Take a commutative ring $R$. For an open set $U \subset X$ denote by $\text{Assoc}(R,U)$ the category of sheaves of associative $R$-algebras on $U$. Let $\text{Assoc}(R,X)$ be the stack $U \mapsto \text{Assoc}(R,U)$. We call it the stack of sheaves of associative $R$-algebras on $X$.

**Example 4.6.** Suppose $A$ is a sheaf of rings on $X$. On any open set $U \subset X$ there is the category $\text{Mod}A|_U$ of sheaves of left $A$-modules on $U$. Like in the previous examples we get a stack $\text{Mod}A$ on $X$, with $(\text{Mod}A)(U) = \text{Mod}A|_U$.
Suppose $F : \mathcal{G} \to \mathcal{H}$ is a morphism of stacks. We call $F$ a weak epimorphism if it is locally essentially surjective on objects, and surjective on isomorphism sheaves. The first condition says that for any open set $U \subset X$, object $j \in \text{Ob} \mathcal{H}(U)$ and point $x \in U$, there is an open set $V$ with $x \in V \subset U$, an object $i \in \text{Ob} \mathcal{G}(V)$, and an isomorphism $h : F(i) \cong j$ in $\mathcal{H}(V)$. The second condition says that for any $i,j \in \text{Ob} \mathcal{G}(U)$ the map of sheaves of sets

\[(4.7) \quad F : \mathcal{G}(i,j) \to \mathcal{H}(F(i),F(j))\]

is surjective.

A weak equivalence of stacks is a weak epimorphism $F : \mathcal{G} \to \mathcal{H}$, such that the maps in $[4.7]$ are all isomorphisms of sheaves.

There is a stackification operation, which is analogous to sheafification: to any prestack $\mathcal{G}$ one assigns a stack $\tilde{\mathcal{G}}$, with a morphism of prestacks $F : \mathcal{G} \to \tilde{\mathcal{G}}$. These have the following universal property: given any stack $\mathcal{H}$ and morphism $E : \mathcal{G} \to \mathcal{H}$, there is a morphism $\tilde{E} : \tilde{\mathcal{G}} \to \mathcal{H}$, unique up to 2-isomorphism, such that $E \cong \tilde{E} \circ F$.

We denote by $\text{Stack} X$ the full sub 2-category of $\text{PreStack} X$ gotten by taking all stacks, all 1-morphisms between stacks, and all 2-morphisms between these 1-morphisms.

By a prestack of groupoids on $X$ we mean a prestack $\mathcal{G}$ such that each of the categories $\mathcal{G}(U)$ is a groupoid. If $\mathcal{G}$ is a prestack of groupoids, then the associated stack $\tilde{\mathcal{G}}$ is a stack of groupoids. We say that $\mathcal{G}$ is small if each of the groupoids $\mathcal{G}(U)$ is small. In this case $\tilde{\mathcal{G}}$ is also small.

We shall be interested in gerbes, which are stacks of groupoids that are locally nonempty and locally connected. The first condition says that any point $x \in X$ has an open neighborhood $U$ such that $\text{Ob} \mathcal{G}(U) \neq \emptyset$. The second condition says that for any $i,j \in \text{Ob} \mathcal{G}(U)$ and any $x \in X$, there is an open set $V$ such that $x \in V \subset U$, and $\mathcal{G}(V)(i,j) \neq \emptyset$.

Let $\mathcal{G}$ be a sheaf of groups on $X$. By a left $\mathcal{G}$-torsor on $X$ we mean a sheaf of sets $S$, with a left $\mathcal{G}$-action, such that $S$ is locally nonempty (i.e. each point $x \in X$ has an open neighborhood $U$ such that $S(U) \neq \emptyset$), and for any $s \in S(U)$ the morphism of sheaves of sets $\mathcal{G}|_U \to S|_U$, $g \mapsto g \cdot s$, is an isomorphism. The torsor $S$ is trivial if $S(X) \neq \emptyset$.

Suppose $\mathcal{G}$ is a gerbe on $X$. Given an open set $U \subset X$ and $i \in \text{Ob} \mathcal{G}(U)$, there is a sheaf of groups $\mathcal{G}(i,i)$ on $U$. If $j \in \text{Ob} \mathcal{G}(U)$ is some other object, then the sheaf of sets $\mathcal{G}(i,j)$ is a $\mathcal{G}(j,j)-\mathcal{G}(i,i)$-bitorsor. Namely, forgetting the left action by $\mathcal{G}(j,j)$, the sheaf $\mathcal{G}(i,j)$ is a right $\mathcal{G}(i,i)$-torsor; and vice versa.

It is not hard to see that a morphism of gerbes $F : \mathcal{G} \to \mathcal{H}$ is an equivalence iff it is a weak equivalence.

We denote by $\text{Gerbe} X$ the full sub 2-category of $\text{Stack} X$ gotten by taking all gerbes, all 1-morphisms between gerbes, and all 2-morphisms between these 1-morphisms.

**Proposition 4.8.** Let $\mathcal{G}$ be a stack on $X$. Then the prestack of groupoids $\mathcal{G}^\times$, defined by $U \mapsto \mathcal{G}(U)^\times$, is a stack.

**Proof.** Given two local objects $i,j \in \text{Ob} \mathcal{G}(U)$, the sub-presheaf $\mathcal{G}(i,j)^\times \subset \mathcal{G}(i,j)$ of invertible arrows is a sheaf. Hence $\mathcal{G}^\times$ has descent for morphisms.

Since descent for objects is determined in terms of isomorphisms, it follows that $\mathcal{G}^\times$ has descent for objects.

Let $U$ be an open set of $X$. Given a stack $\mathcal{G}$ on $X$, its restriction to $U$ is the stack $\mathcal{G}|_U$ on $U$ such that $(\mathcal{G}|_U)(V) = \mathcal{G}(V)$ for any open set $V \subset U$. In this way we get a 2-functor $\text{Stack} X \to \text{Stack} U$. If $\mathcal{G}$ is a gerbe then so is $\mathcal{G}|_U$. 

\[\square\]
5. Twisted Sheaves

Let $X$ be a topological space. We want to have a geometric analog of the notion of category $P$ with inner gauge group structure. So we imitate the definitions from Section 3.

Let $P$ be a stack on $X$. For an open set $U \subset X$ and an object $A \in \text{Ob} \, P(U)$, we refer to $A$ as a local object of $P$, or a sheaf in $P$. The example to keep in mind in Example 4.5.

Consider an open set $U \subset X$ and a local object $A \in \text{Ob} \, P(U)$. We denote by $	ext{Aut}_{P}(A)$ the sheaf of groups on $U$ such that

\[ \Gamma(V, \text{Aut}_{P}(A)) = \text{Aut}_{P}(V)(A|_{V}) \]

for an open set $V \subset U$. An isomorphism $\phi : A \to B$ in $P(U)$ induces an isomorphism of sheaves of groups

\[ \text{Aut}_{P}(\phi) := \text{Ad}_{P}(\phi) : \text{Aut}_{P}(A) \to \text{Aut}_{P}(B). \]

In this way we get a functor

\[ \text{Aut}_{P} : P(U)^{\times} \to \text{Grp} \, U. \]

As we let $U$ vary, we obtain a functor of stacks

\[ \text{Aut}_{P} : P^{\times} \to \text{Grp} \, X \]

(cf. Example 4.3).

**Definition 5.1.** Let $P$ be a stack on $X$. An inner gauge group structure on $P$ is a functor of stacks $\text{IG} : P \to \text{Grp} \, X$, together with a natural transformation $\text{ig} : \text{IG} \Rightarrow \text{Aut}_{P}$ between functors of stacks $P^{\times} \to \text{Grp} \, X$. The condition is that for every open set $U \subset X$, the corresponding data $(P(U), \text{IG}, \text{ig})$ is an inner gauge group structure on the category $P(U)$, as in Definition 3.11.

We say that $(P, \text{IG}, \text{ig})$ is a stack with inner gauge groups on $X$.

Here are several examples.

**Example 5.2.** Take the stack $P := \text{Grp} \, X$ from Example 4.3 and $\text{IG}$ and $\text{ig}$ as in Example 3.13.

**Example 5.3.** Take the stack $P := \text{Assoc}(R, X)$ from Example 4.5 and $\text{IG}$ and $\text{ig}$ as in Example 3.14.

We now work in this setup:

**Setup 5.4.** $K$ is a field of characteristic 0; $(R, \mathfrak{m})$ is a parameter $K$-algebra (see Definition 1.1); $X$ is a topological space; and $\mathcal{O}_{X}$ is a sheaf of commutative $K$-algebras on $X$. We assume that $X$ has enough $\mathcal{O}_{X}$-acyclic open coverings (see Definition 2.1).

This is Setup 2.7 plus the condition that $\text{char} \, K = 0$.

For any open set $U \subset X$ there is the category $\text{AssDef}(R, \mathcal{O}_{U})$ of associative $R$-deformations of $\mathcal{O}_{U}$; see Definition 2.8. Note that if $U_{1} \subset U_{0}$ is an inclusion of open sets, then by restriction of sheaves we have a functor

\[ \text{rest}_{U_{1}/U_{0}} : \text{AssDef}(R, \mathcal{O}_{U_{0}}) \to \text{AssDef}(R, \mathcal{O}_{U_{1}}). \]

Thus we get a prestack of groupoids $\text{AssDef}(R, \mathcal{O}_{X})$, where the composition isomorphisms $\gamma_{U_{2}/U_{1}/U_{0}}$ are the identities. Since the nature of these deformations is local, it follows that the prestack $\text{AssDef}(R, \mathcal{O}_{X})$ is a stack.
Here is the third example of a stack with inner gauge groups.

**Example 5.5.** Assume Setup 5.4 and take the stack $\mathcal{P} := \text{AssDef}(R, \mathcal{O}_X)$. Given a local deformation

$$\mathcal{A} \in \text{AssDef}(R, \mathcal{O}_U) = \mathcal{P}(U)$$

on some open set $U \subset X$, consider the sheaf of groups $\text{IG}(\mathcal{A})$ on $U$ defined by

$$\Gamma(V, \text{IG}(\mathcal{A})) := \{ a \in \Gamma(V, \mathcal{A}) | a \equiv 1 \text{ mod } m \}$$

for $V \subset U$. This is a functor of stacks $\text{IG} : \mathcal{P} \to \text{Grp}_X$. Note that there is an isomorphism of sheaves of groups

$$\text{IG}(\mathcal{A}) \cong \exp(m \mathcal{A})$$

see Example 3.17.

Next let $g \in \Gamma(V, \text{IG}(\mathcal{A}))$. Then we have an automorphism $\text{ig}(g)$ of the sheaf $\mathcal{A}|_V$, with formula

$$\text{ig}(g)(a) := \text{Ad}_\mathcal{A}(g)(a) = g \cdot a \cdot g^{-1}$$

for a local section $a \in \mathcal{A}|_V$. In this way we get a natural transformation of functors $\text{ig} : \text{IG} \to \text{Aut}_\mathcal{P}$.

Like in the associative case, we get a stack $\text{PoisDef}(R, \mathcal{O}_X)$ on $X$, with

$$\text{PoisDef}(R, \mathcal{O}_X)(U) = \text{PoisDef}(R, \mathcal{O}_U)$$

for an open set $U$; see Definition 2.10. Here is the fourth example of a stack with inner gauge groups.

**Example 5.6.** Assume Setup 5.4 and take the stack $\mathcal{P} := \text{PoisDef}(R, \mathcal{O}_X)$. Given a local deformation

$$\mathcal{A} \in \text{PoisDef}(R, \mathcal{O}_U) = \mathcal{P}(U)$$

on some open set $U \subset X$, consider the sheaf of pronilpotent Lie algebras $m \mathcal{A}$ on $U$, as in Example 3.16. The abstract exponential operation gives rise to a sheaf of groups $\text{IG}(\mathcal{A})$ on $U$, namely

$$\Gamma(V, \text{IG}(\mathcal{A})) := \exp(\Gamma(V, m \mathcal{A}))$$

for $V \subset U$. This is a functor of stacks $\text{IG} : \mathcal{P} \to \text{Grp}_X$.

Next let $g = \exp(a) \in \Gamma(V, \text{IG}(\mathcal{A}))$. Then we have an automorphism $\text{ig}(g)$ of the sheaf $\mathcal{A}|_V$ with formula

$$\text{ig}(g)(a) := \exp(\text{ad}_\mathcal{A}(a)).$$

In this way we get a natural transformation of functors $\text{ig} : \text{IG} \to \text{Aut}_\mathcal{P}$. This is a natural transformation of functors $\text{ig} : \text{IG} \to \text{Aut}_\mathcal{P}$.

The next structural result will be used later.

**Proposition 5.7.** Assume Setup 5.4 Let $U \subset X$ be an open set, and let $\mathcal{A}$ be an (associative or Poisson) $R$-deformation of $\mathcal{O}_U$. Write $\mathcal{G} := \text{IG}(\mathcal{A}) = \exp(m \mathcal{A})$, and $\mathcal{N}_p := \exp(m^{p+1} \mathcal{A})$ for $p \geq 0$. So $\mathcal{N}_p$ is a sheaf of normal subgroups of $\mathcal{G}$, and $\mathcal{N}_p/\mathcal{N}_{p+1}$ is abelian.

Suppose $V \subset U$ is an $\mathcal{O}_X$-acyclic open set. Let $\mathcal{A} := \Gamma(V, \mathcal{A})$, which by Proposition 2.11 is an $R$-deformation of $C := \Gamma(V, \mathcal{O}_X)$, and let $\mathcal{N}_p := m^{p+1} \mathcal{A}$. Then:

1. The cohomology groups $H^i(V, \mathcal{N}_p/\mathcal{N}_{p+1})$ are trivial for all $p \geq 0$ and $i \geq 0$.
2. The canonical homomorphisms $\mathcal{N}_p \to \Gamma(V, \mathcal{N}_p)$ and $\mathcal{N}_p/\mathcal{N}_q \to \Gamma(V, \mathcal{N}_p/\mathcal{N}_q)$ are bijective for all $q \geq p \geq 0$. 

Proof. Since \( \exp : m^{p+1}A \to N_p \) and
\[
\exp : m^{p+1}A/m^{q+1}A \to N_p/N_q
\]
are isomorphisms of sheaves of sets, and
\[
\exp : m^{p+1}A/m^{p+2}A \to N_p/N_{p+1}
\]
is an isomorphism of sheaves of abelian groups, we are allowed to switch from groups to Lie algebras. Namely, it suffices to prove assertions (1) and (2) for \( N_p := m^{p+1}A \) and \( N_p := m^{p+1}A \).

For assertion (1) we use the isomorphism
\[
N_p/N_{p+2} \cong (m^{p+1}/m^{p+1}) \otimes_{\mathbb{K}} \mathcal{O}_U
\]
of Proposition 2.4(1) to deduce
\[
H^i(V, N_p/N_{p+2}) \cong (m^{p+1}/m^{p+1}) \otimes_{\mathbb{K}} H^i(V, \mathcal{O}_U) = 0
\]
for \( i > 0 \).

For assertion (2) let \( R_p := R/m^{p+1} \), \( A_p := R_p \otimes R A \) and \( A_p := \Gamma(V, A_p) \).

Consider the commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & m^{p+1}A \\
\downarrow{\alpha} & & \downarrow{=} \\
\Gamma(V, m^{p+1}A) & \longrightarrow & \Gamma(V, A)
\end{array}
\]

The bottom row is exact since \( \Gamma(V, -) \) is left exact; and the top row is exact by Proposition 2.4(2), which says that \( A_p \cong R_p \otimes \mathbb{K} A \). We conclude that \( \alpha \) is bijective, and thus \( N_p = \Gamma(V, N_p) \).

For the same reasons as above we have a commutative diagram with exact rows like (5.8), but with \( A_q \) instead of \( A \), and \( A_q \) instead of \( A \). Since \( N_p/N_q \cong m^{p+1}A_q \), it follows that \( N_p/N_q = \Gamma(V, N_p/N_q) \).

\[\square\]

Definition 5.9. Let \((P, IG, ig)\) be a stack with inner gauge groups on \( X \). A twisted object of \((P, IG, ig)\), or a twisted sheaf in \((P, IG, ig)\), is the following data:

1. A small gerbe \( \mathcal{G} \) on \( X \), called the gauge gerbe.
2. A functor of stacks \( \mathcal{A} : \mathcal{G} \to P \), called the representation.
3. A natural isomorphism
\[
\text{cp} : \text{Aut}_\mathcal{G} \cong \text{IG} \circ \mathcal{A}
\]

between functors of stacks \( \mathcal{G} \to \text{Grp} \mathcal{X} \), called the coupling isomorphism.

The condition is:

\((*)\) The diagram
\[
\begin{array}{ccc}
\text{Aut}_\mathcal{G} & \longrightarrow & \text{IG} \circ \mathcal{A} \\
\downarrow{\text{cp}} & & \downarrow{=} \\
\text{Aut}_\mathcal{A} & \longrightarrow & \text{IG} \circ 1_\mathcal{A}
\end{array}
\]

of natural transformations between functors of stacks \( \mathcal{G} \to \text{Grp} \mathcal{X} \), is commutative.

We refer to this twisted object by \((\mathcal{G}, \mathcal{A}, \text{cp})\). The set of twisted objects in \((P, IG, ig)\) is denoted by \(\text{TwOb}(P, IG, ig)\).
What this definition amounts to is that on every open set $U$ the triple $(\mathcal{G}(U), \mathcal{A}, \text{cp})$ is almost a twisted object in the category with inner gauge groups $(\mathcal{P}(U), \text{IG}, \text{ig})$; the exception is that the groupoid $\mathcal{G}(U)$ might be empty or disconnected. These triples restrict correctly to smaller open sets.

In other words, to any local object $i \in \text{Ob} \mathcal{G}(U)$ on an open set $U \subset X$ we attach an object $\mathcal{A}(i) \in \text{Ob} \mathcal{P}(U)$, which we can also view as a a sheaf on $U$ (since $\mathcal{P}$ is a stack). To any other object $j \in \text{Ob} \mathcal{G}(U)$ and any arrow $g \in \mathcal{G}(U)(i, j)$ we attach an isomorphism

$$\mathcal{A}(g) = \text{ig}(\text{cp}(g)) : \mathcal{A}(i) \overset{\sim}{\rightarrow} \mathcal{A}(j)$$

in $\mathcal{P}(U)$. The various locally defined isomorphisms $\mathcal{A}(g)$ are related by the composition rule in the gerbe $\mathcal{G}$.

When there is no danger of confusion we write $\mathcal{A}$ instead of $(\mathcal{G}, \mathcal{A}, \text{cp})$, and $\text{TwOb}(\mathcal{P})$ instead of $\text{TwOb}(\mathcal{P}, \text{IG}, \text{ig})$. An object $\mathcal{A}(i)$, for some open set $U \subset X$ and $i \in \text{Ob} \mathcal{G}(U)$, is called a local object belonging to $\mathcal{A}$, or a sheaf belonging to $\mathcal{A}$.

We can finally define twisted deformations.

**Definition 5.10.** Assume Setup 5.4.

1. A twisted object of the stack with inner gauge groups $\text{AssDef}(\mathcal{R}, \mathcal{O}_X)$ is called a twisted associative $R$-deformation of $\mathcal{O}_X$.
2. A twisted object of the stack with inner gauge groups $\text{PoisDef}(\mathcal{R}, \mathcal{O}_X)$ is called a twisted Poisson $R$-deformation of $\mathcal{O}_X$.

**Definition 5.11.** Let $(\mathcal{P}, \text{IG}, \text{ig})$ be a stack with inner gauge groups on $X$, and let $(\mathcal{G}, \mathcal{A}, \text{cp})$ and $(\mathcal{G}', \mathcal{A}', \text{cp}')$ be twisted objects in $\mathcal{P}$. A twisted gauge transformation

$$(F_{\text{gau}}, F_{\text{rep}}) : (\mathcal{G}, \mathcal{A}, \text{cp}) \rightarrow (\mathcal{G}', \mathcal{A}', \text{cp}')$$

consists of an equivalence of stacks $F_{\text{ger}} : \mathcal{G} \rightarrow \mathcal{G}'$, and an isomorphism $F_{\text{rep}} : \mathcal{A} \overset{\sim}{\rightarrow} \mathcal{A}' \circ F_{\text{gau}}$ of functors of stacks from $\mathcal{G}$ to $\text{Grp} X$. The condition is that the diagram

$$\begin{array}{ccc}
\text{Aut}_{\mathcal{G}} & \overset{\text{cp}}{\longrightarrow} & \text{IG} \circ \mathcal{A} \\
\text{Aut}_{F_{\text{gau}}} \downarrow & & \downarrow \text{1}_{\text{IG}} \circ F_{\text{rep}} \\
\text{Aut}_{\mathcal{G}'} \circ F_{\text{ger}} & \overset{\text{cp}'}{\longrightarrow} & \text{IG} \circ \mathcal{A}' \circ F_{\text{gau}}
\end{array}$$

of natural transformations of functors of stacks $\mathcal{G} \rightarrow \text{Grp} X$ is commutative.

Thus for every open set $U \subset X$ there is a twisted gauge transformation

$$(F_{\text{gau}}, F_{\text{rep}}) : (\mathcal{G}(U), \mathcal{A}, \text{cp}) \rightarrow (\mathcal{G}'(U), \mathcal{A}', \text{cp}')$$

as in Definition 3.22 and these are compatible with restriction to smaller open sets.

As in Proposition 3.23 we have:

**Proposition 5.12.** Twisted gauge transformations form an equivalence relation on the set $\text{TwOb}(\mathcal{P}, \text{IG}, \text{ig})$.

**Remark 5.24.** applies here too.

**Definition 5.13.** The equivalence relation given by twisted gauge transformations is called twisted gauge equivalence, and we write

$$\text{TwOb}(\mathcal{P}, \text{IG}, \text{ig}) :\overset{\text{twisted gauge equivalence}}{\longrightarrow} \text{TwOb}(\mathcal{P}, \text{IG}, \text{ig}).$$

**Example 5.14.** Take the stack with inner gauge groups $\mathcal{P} := \text{Grp} X$ from Definition 5.2. A twisted sheaf in $\mathcal{P}$ is just a gerbe; and hence $\text{TwOb}(\mathcal{P}) = \text{Gerbe} X$.

Let $U \subset X$ be an open set. By restriction to $U$ we have a stack with inner gauge groups $(\mathcal{P}|_U, \text{IG}, \text{ig})$ on $U$. 
Definition 5.15. Let \((\mathcal{G}, \mathcal{A}, \text{cp})\) be a twisted object of \((\mathcal{P}, \mathcal{I}G, \text{ig})\). Its restriction to \(U\) is the twisted object \((\mathcal{G}|_U, \mathcal{A}, \text{cp})\) of \((\mathcal{P}|_U, \mathcal{I}G, \text{ig})\).

We sometimes write \(\mathcal{A}|_U\) instead of \((\mathcal{G}|_U, \mathcal{A}, \text{cp})\). The operation \(\mathcal{A} \mapsto \mathcal{A}|_U\) is a function
\[
\text{TwOb}(\mathcal{P}, \mathcal{I}G, \text{ig}) \to \text{TwOb}(\mathcal{P}|_U, \mathcal{I}G, \text{ig})
\]
that respects twisted gauge equivalence.

Suppose \(U \subset X\) is an open set, and \(\mathcal{A}\) is an associative or Poisson \(R\)-deformation of \(\mathcal{O}_U\). In Definition 5.16 we defined the first order bracket
\[
\{-,-\}_\mathcal{A} : \mathcal{O}_U \times \mathcal{O}_U \to (m/m^2) \otimes_k \mathcal{O}_U.
\]
By Proposition 2.17 this is gauge invariant. Therefore the next definition makes sense:

Definition 5.16. Let \((\mathcal{G}, \mathcal{A}, \text{cp})\) be a twisted associative (resp. Poisson) \(R\)-deformation of \(\mathcal{O}_X\). We define the first order bracket of \(\mathcal{A}\) to be the unique \(K\)-bilinear sheaf morphism
\[
\{-,-\}_\mathcal{A} : \mathcal{O}_X \times \mathcal{O}_X \to (m/m^2) \otimes_k \mathcal{O}_X
\]
having this property:

\(\ast\) Let \(i \in \text{Ob} \mathcal{G}(U)\), for some open set \(U \subset X\), and let \(\mathcal{A} := \mathcal{A}(i)\) be the corresponding \(R\)-deformation of \(\mathcal{O}_U\). Then the restriction of \(\{-,-\}_\mathcal{A}\) to \(U\) equals \(\{-,-\}_\mathcal{A}|_U\).

Again, Proposition 2.17 implies that if \(\mathcal{A}\) and \(\mathcal{A}'\) are twisted associative (resp. Poisson) \(R\)-deformations of \(\mathcal{O}_X\) which are twisted gauge equivalent, then
\[
\{-,-\}_\mathcal{A} = \{-,-\}_\mathcal{A}'.
\]
This means that we can talk about the first order bracket of an element of \(\text{TwOb}(\text{AssDef}(R, \mathcal{O}_X))\) or \(\text{TwOb}(\text{PoisDef}(R, \mathcal{O}_X))\).

Definition 5.17. Let \((\mathcal{G}, \mathcal{A}, \text{cp})\) be a twisted sheaf in some stack \(\mathcal{P}\). We say that \((\mathcal{G}, \mathcal{A}, \text{cp})\) is really twisted if there are no global sheaves belonging to it; namely if \(\text{Ob} \mathcal{G}(X) = \emptyset\).

Sometimes there are obstruction classes that determine whether a twisted sheaf is really twisted (see [Ye4], and the proof of Theorem 6.12 below).

Proposition 5.18. In the situation of Setup 5.1, let \(\sigma : R \to R'\) be a homomorphism of parameter algebras. We consider the following stacks with inner gauge groups on \(X\) in the two cases:

\(\ast\) The associative case, in which \(\mathcal{P}(R, X) := \text{AssDef}(R, \mathcal{O}_X)\) and \(\mathcal{P}(R', X) := \text{AssDef}(R', \mathcal{O}_X)\).

\(\ast\) The Poisson case, in which \(\mathcal{P}(R, X) := \text{PoisDef}(R, \mathcal{O}_X)\) and \(\mathcal{P}(R', X) := \text{PoisDef}(R', \mathcal{O}_X)\).

In both cases there is a function
\[
\text{ind}_\sigma : \text{TwOb}(\mathcal{P}(R, X)) \to \text{TwOb}(\mathcal{P}(R', X))
\]
with these properties:

\(\ast\) Functoriality: given another homomorphism of parameter algebras \(\sigma' : R' \to R''\), one has
\[
\text{ind}_{\sigma'} \circ \text{ind}_\sigma = \text{ind}_{\sigma' \circ \sigma}.
\]
If \(R' = R\) and \(\sigma = 1_R\), then \(\text{ind}_\sigma\) is the identity of \(\text{TwOb}(\mathcal{P}(R, X))\).
(ii) Let \((\mathcal{G}, \mathcal{A}, \text{cp}) \in \text{TwOb}(\mathcal{P}(R, X))\), and suppose \(i \in \text{Ob} \mathcal{G}(U)\) for some open set \(U\). Consider \[
\left(\mathcal{G}', \mathcal{A}', \text{cp}'\right) := \text{ind}_r(\mathcal{G}, \mathcal{A}, \text{cp}) \in \text{TwOb}(\mathcal{P}(R', X)).
\]
Then there is an object \(i' \in \text{Ob} \mathcal{G}'(U)\), such that \[
\mathcal{A}'(i') \cong R' \otimes_R \mathcal{A}(i)
\]
in \(\mathcal{P}(R', X)(U)\).

**Proof.** Let \((\mathcal{G}, \mathcal{A}, \text{cp})\) be a twisted \(R\)-deformation. Take an open set \(U\) and an object \(i \in \text{Ob} \mathcal{G}(U)\). Let \(\mathcal{A}_i := \mathcal{A}(i)\) be the corresponding \(R\)-deformation of \(\mathcal{O}_U\).

By Proposition 2.14 there is an induced \(R'\)-deformation \(\mathcal{A}_i' := R' \otimes_R \mathcal{A}_i\) of \(\mathcal{O}_U\). Let \(\mathcal{G}_i'(i, i) := IG(\mathcal{A}_i)\), which is a sheaf of groups on \(U\).

Now take another open set \(V\), and an object \(j \in \text{Ob} \mathcal{G}(V)\). For any point \(x \in U \cap V\) there is an open set \(W\) such that \(x \in W \subset U \cap V\), with a morphism \(g \in \mathcal{G}(W)(i, j)\). The \(R\)-linear gauge transformation \(\mathcal{A}(g) : \mathcal{A}_i|_W \xrightarrow{\sim} \mathcal{A}_j|_W\) induces, by base change to \(R'\), an \(R'\)-linear gauge transformation \(\mathcal{A}'(g) : \mathcal{A}_i'|_W \xrightarrow{\sim} \mathcal{A}_j'|_W\). The gauge transformation \(\mathcal{A}'(g)\) generates a \(\mathcal{G}'(j, j)\)-\(\mathcal{G}'(i, i)\)-bitorsor on \(W\). As \(W\) and \(g\) vary, these local patches agree, and thus we obtain a \(\mathcal{G}'(j, j)\)-\(\mathcal{G}'(i, i)\)-bitorsor \(\mathcal{G}_i'(i, j)\) on \(U \cap V\).

Next consider three open sets \(U_0, U_1, U_2\), and objects \(i_k \in \text{Ob} \mathcal{G}(U_k)\). The composition in the gerbe \(\mathcal{G}\) induces a map of sheaves of sets

\[
\mathcal{G}_i'(i_0, i_1)|_{U_0,1,2} \times \mathcal{G}_i'(i_1, i_2)|_{U_0,1,2} \rightarrow \mathcal{G}_i'(i_0, i_2)|_{U_0,1,2}.
\]

This composition rule is associative, and thus we obtain a prestack of groupoids \(\mathcal{G}'\) on \(X\) (with the same open sets as the gerbe \(\mathcal{G}\)). The assignments \(\mathcal{A}' : i \mapsto \mathcal{A}_i'\) and \(\mathcal{A}' : g \mapsto \mathcal{A}'(g)\) above form a functor of prestacks \(\mathcal{A}' : \mathcal{G}' \rightarrow \mathcal{P}(R', X)\), with tautological coupling isomorphism \(\text{cp}'\).

Let \(\tilde{\mathcal{G}}'\) be the stackification of \(\mathcal{G}'\). This is a gerbe on \(X\), and there is an induced functor of stacks \(\tilde{\mathcal{A}}' : \tilde{\mathcal{G}}' \rightarrow \mathcal{P}(R', X)\), and an induced coupling isomorphism \(\text{cp}'\).

We now define

\[
\text{ind}_r(\mathcal{A}) := (\tilde{\mathcal{G}}', \tilde{\mathcal{A}}', \text{cp}') \in \text{TwOb}(\mathcal{P}(R', X)).
\]

The fact that this construction respects twisted gauge equivalence is clear. Properties (i)-(ii) are clear too.

Let \(R\) be a commutative ring, and let \(X\) be a topological space. Recall that a *stack of \(R\)-linear algebroids* on \(X\) is a stack \(\mathcal{B}\) of \(R\)-linear categories that is locally nonempty and locally connected by isomorphisms (see [Ko2]). The set of all \(R\)-linear stacks of algebroids on \(X\) is denoted by \(\text{Algebroid}(R, X)\). Given \(\mathcal{B}, \mathcal{B}' \in \text{Algebroid}(R, X)\), we consider \(R\)-linear weak equivalences of stacks \(\mathcal{F} : \mathcal{B} \rightarrow \mathcal{B}'\); see Section 2.

Here is a result of some interest – it says that \(R\)-linear algebroids are the same as twisted sheaves of associative \(R\)-algebras.

**Proposition 5.19.** Let \(R\) be a commutative ring, and let \(X\) be a topological space. Then there is a bijection of sets

\[
\frac{\text{Algebroid}(R, X)}{\text{weak equivalence}} \cong \frac{\text{TwOb}(\text{Assoc}(R, X))}{\text{twisted gauge equivalence}}
\]

functorial in \(R\).

**Proof.** Take an \(R\)-linear stack of algebroids \(\mathcal{B}\). Then the stack of groupoids \(\mathcal{G} := \mathcal{B}^\circ\) (see Proposition 1.8) is a gerbe. We define the functor of stacks \(\mathcal{A} : \mathcal{G} \rightarrow \text{Assoc}(R, X)\) to be \(\mathcal{A} := \text{End}_\mathcal{B}\). Tautologically we get \(\text{cp} : \text{Aut}_\mathcal{G} \cong IG \circ \mathcal{A}\). In
this way we get a twisted associative algebra \((\mathcal{G}, \mathcal{A}, \text{cp})\). It is not hard to see that this construction respects the equivalence relations. The reverse direction is done similarly. \(\square\)

**Remark 5.20.** In the paper [KS3] the authors use the term *DQ algebroid* to denote a \(K[[\hbar]]\)-linear algebroid \(\mathcal{A}\) that locally looks like an associative \(K[[\hbar]]\)-deformation of \(\mathcal{O}_X\). This is very close to being a twisted associative \(K[[\hbar]]\)-deformation of \(\mathcal{O}_X\) in our sense. Indeed, \(\mathcal{A}\) is a twisted object of the stack \(\text{AssDef}(R, \mathcal{O}_X)\), but for a slightly different inner gauge group structure: for an associative deformation \(A\) in our sense. Indeed, \(\mathcal{A}\) is a twisted object of the stack \(\text{AssDef}(R, \mathcal{O}_X)\), but for an associative deformation \(\mathcal{A}\), the inner gauge group \(\text{IG}(\mathcal{A})\) is defined to be the whole group of invertible elements \(\mathcal{A}^\times\), and not just those elements congruent to 1 modulo \(\hbar\). As a consequence one gets 0-th order obstruction classes in \(H^1(X, \mathcal{O}_X^\times)\) (cf. Theorem 0.12 and Ye4 Theorems 4.7 and 4.16). We thank P. Polesello for explaining this subtlety to us.

**Remark 5.21.** Suppose \(\mathcal{A}\) is a twisted associative \(R\)-deformation of \(\mathcal{O}_X\). One can consider the stack of \(R\)-linear abelian categories \(\text{Coh}\mathcal{A}\) of coherent left \(\mathcal{A}\)-modules. It is a deformation of the stack \(\text{Coh}\mathcal{O}_X\). The twisted associative deformation \(\mathcal{A}\) can be recovered from the stack \(\text{Coh}\mathcal{A}\); and in fact these two notions of deformation are equivalent (it is a kind of geometric Morita theory; cf. Example 5.9). See the papers [Ko2, LV, Lo, KS3] and the last chapter of the book [KS2]. We do not know a similar interpretation of twisted Poisson deformations.

### 6. Multiplicative Descent Data

In this section we study the decomposition of twisted objects on open coverings.

**Definition 6.1.** Let \(\mathcal{G}\) be a gerbe on a topological space \(X\), and let \(U = \{U_k\}_{k \in K}\) be an open covering of \(X\). We say that \(U\) trivializes \(\mathcal{G}\) if it is possible to find an object \(i_k \in \text{Ob} \mathcal{G}(U_k)\), for every \(k \in K\), such that for every \(k_0, k_1 \in K\) the set of isomorphisms \(\mathcal{G}(U_{k_0, k_1})(i_{k_0, k_1})\) is nonempty.

It is well known that trivialized gerbes have a description in terms of descent data (certain nonabelian 2-cocycles). See [GI] or [Br2]. We shall see that the same is true for twisted sheaves.

**Definition 6.2.** Let \((\mathcal{P}, \text{ig}, \text{ig})\) be a stack with inner gauge groups on a topological space \(X\), and let \(U\) be an open covering of \(X\).

1. Let \((\mathcal{G}, \mathcal{A}, \text{cp})\) be a twisted object of \(\mathcal{P}\). We say that \(U\) trivializes \((\mathcal{G}, \mathcal{A}, \text{cp})\) if it trivializes the gauge gerbe \(\mathcal{G}\).

2. We say that \(U\) trivializes the stack \(\mathcal{P}\) if it trivializes all twisted objects of \(\mathcal{P}\).

**Remark 6.3.** In general there is no reason to expect that such trivializing open coverings should exist. On the other hand, if we were to consider hypercoverings, then there are always trivializations. Cf. [Br2] Section 5.

We shall see in Corollary 6.16 that for the stacks with inner gauge groups \(\text{AssDef}(R, \mathcal{O}_X)\) and \(\text{PoisDef}(R, \mathcal{O}_X)\) there do exist trivializing open coverings.

We denote by \(\text{TwOb}(\mathcal{P})^U\) the set of twisted objects of \(\mathcal{P}\) that are trivialized by the open covering \(U\).

Let \(U = \{U_k\}_{k \in K}\) be an open covering of \(X\). Recall that a refinement of \(U\) is an open covering \(U' = \{U'_k\}_{k \in K'}\) of \(X\), together with a function \(\rho : K' \to K\), such that \(U'_k \subset U_{\rho(k)}\) for any \(k \in K'\). Sometimes we say that \(\rho : U' \to U\) is a refinement.

If a gerbe \(\mathcal{G}\) trivializes on an open covering \(U\), and \(\rho : U' \to U\) is a refinement, then \(\mathcal{G}\) also trivializes on \(U'\). Hence there is an inclusion \(\text{TwOb}(\mathcal{P})^U \subset \text{TwOb}(\mathcal{P})^{U'}\).
Figure 1. Multiplicative descent datum on an open covering \( \{U_0, U_1, U_2\} \) of a topological space.

Let \( \mathcal{A}, \mathcal{A}' \in \text{TwOb}(\mathcal{P}) \). Assume that \( \mathcal{A} \) is trivialized by some open covering \( \mathcal{U} \), and that \( \mathcal{A}' \) is twisted gauge equivalent to \( \mathcal{A} \). This means that the corresponding gauge gerbes \( \mathcal{G} \) and \( \mathcal{G}' \) are equivalent. It follows that \( \mathcal{A}' \) is also trivialized by \( \mathcal{U} \).

Let us write \( \text{TwOb}(\mathcal{P})^\mathcal{U} := \text{TwOb}(\mathcal{P})^\mathcal{U}_{\text{twisted gauge equivalence}} \).

**Definition 6.4.** Let \( (\mathcal{G}, \mathcal{A}, \text{cp}) \) be a stack with inner gauge groups on a topological space \( X \), and let \( \mathcal{U} = \{U_k\}_{k \in K} \) be an open covering of \( X \). A multiplicative descent datum is a collection

\[
d = (\{\mathcal{A}_k\}_{k \in K}, \{g_{k_0,k_1}\}_{k_0,k_1 \in K}, \{a_{k_0,k_1,k_2}\}_{k_0,k_1,k_2 \in K})
\]

where

\[
\mathcal{A}_k \in \text{Ob} \mathcal{P}(U_k),
\]

\[
g_{k_0,k_1} \in \mathcal{P}(U_{k_0,k_1})^\times (\mathcal{A}_{k_0}, \mathcal{A}_{k_1})
\]

and

\[
a_{k_0,k_1,k_2} \in \Gamma(U_{k_0,k_1,k_2}, \text{IG} (\mathcal{A}_{k_0})).
\]

The conditions are as follows:

(i) (Normalization) \( g_{k,k} = 1 \), \( g_{k_1,k_0} \circ g_{k_0,k_1} = 1 \), \( a_{k_0,k_1,k_2} = a^{-1}_{k_0,k_2,k_1} \) and
\( \text{IG}(g_{k_0,k_1})(a_{k_0,k_1,k_2}) = a_{k_1,k_2,k_0} \).

(ii) (Failure of 1-cocycle)

\[
g_{k_2,k_0} \circ g_{k_1,k_2} \circ g_{k_0,k_1} = \text{ig}(a_{k_0,k_1,k_2})
\]

in \( \mathcal{P}(U_{k_0,k_1,k_2})^\times (\mathcal{A}_{k_0}, \mathcal{A}_{k_1}) \).

(iii) (Twisted 2-cocycle)

\[
a^{-1}_{k_0,k_1,k_3} \cdot a_{k_0,k_2,k_3} \cdot a_{k_0,k_1,k_2} = \text{IG}(g_{k_0,k_1})(a_{k_1,k_2,k_3})
\]

in \( \Gamma(U_{k_0,k_1,k_2,k_3}, \text{IG} (\mathcal{A}_{k_0})). \)

We denote by \( \text{MDD}(\mathcal{P}, \mathcal{U}) \) the set of all multiplicative descent data.

See Figure 1 for an illustration.
Definition 6.5. Let
\[ d = \left( \{ A_k \}_{k \in K}, \{ (g_{k_0,k_1})_{k_0,k_1 \in K}, \{ a_{k_0,k_1,k_2} \}_{k_0,k_1,k_2 \in K} \right) \]
and
\[ d' = \left( \{ A'_k \}_{k \in K}, \{ (g'_{k_0,k_1})_{k_0,k_1 \in K}, \{ a'_{k_0,k_1,k_2} \}_{k_0,k_1,k_2 \in K} \right) \]
be multiplicative descent data for the stack with inner gauge groups \( P \) and the open covering \( U = \{ U_k \}_{k \in K} \). A twisted gauge transformation \( d \to d' \) is a collection
\[ \{ (h_k)_{k \in K}, \{ (b_{k_0,k_1})_{k_0,k_1 \in K} \}, \]
where \( h_k \in P(U_k) \times (A_k, A'_k) \) and \( b_{k_0,k_1} \in \Gamma(U_{k_0,k_1}, IG(A_{k_0})) \). The conditions are
\[ g'_{k_0,k_1} \circ h_{k_0} \circ g_{k_0,k_1} \circ \lg(b_{k_0,k_1}) \]
and
\[ IG(h_{k_0}^{-1})(a'_{k_0,k_1,k_2}) = b_{k_0,k_1} \cdot IG(g_{k_0,k_1}^{-1})(b_{k_1,k_2} \cdot a_{k_0,k_1,k_2} \cdot b_{k_0,k_2}^{-1}) \]

Remark 6.6. The similarity between our notion of multiplicative descent data and the usual notion of descent data for gerbes (cf. [Br2]) is no coincidence. Indeed, as shown in Example 5.14, gerbes are an instance of twisted sheaves. Likewise for gauge transformations between descent data.

In [Ko2] this kind of data, for \( P = \text{Assoc}(R, X) \), is called a combinatorial description of algebroids (cf. Proposition 5.19).

Note that in the paper [BGNT] the authors refer to a multiplicative descent datum as a “stack”. This is not too much of an abuse, in view of Proposition 6.9 below.

Proposition 6.7. Twisted gauge transformations form an equivalence relation on the set \( \text{MDD}(P, U) \).

Proof. This is rather easy, yet tedious, calculation, almost identical to the case of gerbes; see [Br2] Section 5].

We write
\[ \text{MDD}(P, U) := \text{MDD}(P, U) \text{ twisted gauge equivalence} \]

Remark 6.8. The set \( \text{MDD}(P, U) \) has a structure of 2-groupoid, in which the 1-morphisms are the twisted gauge transformations. Cf. Remark 3.23.

The following proposition is basically the same as the well-known result for gerbes; cf. [Br2] Section 5].

Proposition 6.9. Let \( P \) be a stack with inner gauge groups on \( X \), and let \( U \) be an open covering of \( X \). Then there is a bijection of sets
\[ \text{dec} : \text{TwOb}(P)^U \to \text{MDD}(P, U) \]
called decomposition, with an explicit formula.

Proof. Write \( U = \{ U_k \}_{k \in K} \), and choose an ordering on the set \( K \).

Let \( (\mathcal{G}, \mathcal{A}, \text{cp}) \) be a twisted sheaf in \( P \) that trivializes on \( U \). Choose objects \( i_k \in \text{Ob} \mathcal{G}(U_k) \) and isomorphisms \( g_{k_0,k_1} \in \mathcal{G}(U_{k_0,k_1})(i_{k_0}, i_{k_1}) \) as in Definition 6.1. For \( k_0 < k_1 \) let \( g_{k_0,k_1}^n := g_{k_0,k_1} \) and \( g_{k_1,k_0}^n := g_{k_1,k_0}^{-1} \). Also let \( g^0_{k,k} := 1 \in \mathcal{G}(U_k)(i_k, i_k) \).

(The letter “n” stands for “normalized”.) Next let \( A_{k_0} := \mathcal{A}(i_{k_0}) \in \text{Ob} \mathcal{P}(U_{k_0}), \)
\[ g_{k_0,k_1} := \mathcal{A}(g_{k_0,k_1}^n) \in \mathcal{P}(U_{k_0}) \times (A_{k_0}, A_{k_1}) \]
and
\[ a_{k_0,k_1,k_2} := \text{cp}(g^0_{k_2,k_0} \circ g^0_{k_2,k_1} \circ g^0_{k_0,k_1}) \in \Gamma(U_{k_0,k_1,k_2}, IG(A_{k_0})) \]
for every \( k_0, k_1, k_2 \in K \). It is straightforward to check that
\[ d := \left( \{ A_k \}_{k \in K}, \{ g_{k_0,k_1} \}_{k_0,k_1 \in K}, \{ a_{k_0,k_1,k_2} \}_{k_0,k_1,k_2 \in K} \right) \]
is a multiplicative descent datum.

If we were to make another choice of objects \( i'_k \in \text{Ob}\mathcal{G}(U_k) \) and isomorphisms 
\( g'_{k_0,k_1} \in \mathcal{G}(U_{k_0},k_1)(i'_{k_0},i'_{k_1}) \) above, then the resulting multiplicative descent datum 
\( d' \) would be gauge equivalent to \( d \), again by a twisted gauge transformation that can be written down. Thus we get a well-defined function 
\[
\text{dec} : \text{TwOb}(\mathcal{P})^U \to \text{MDD}(\mathcal{P}, U).
\]

In order to show that this is a bijection, we construct an inverse. Given a multiplicative descent datum \( d \) as in (6.10), we first construct a “pre twisted sheaf”, namely the following creature. Consider the prestack of groupoids \( \mathcal{G} \), with 
\[
\text{Ob} \mathcal{G}(V) := \{ k \in K \mid V \subset U_k \}
\]
for an open set \( V \subset X \). For two objects \( k_0, k_1 \in \text{Ob} \mathcal{G}(V) \), the sheaf of morphisms 
\( \mathcal{G}(k_0,k_1) \) on \( V \) is the IG(\( A_{k_0} \)) -IG(\( A_{k_1} \)) -bitorsor on \( V \) generated by IG(\( \tilde{g}_{k_0,k_1} \)). The composition rule in the prestack of groupoids is given by the \( a_{k_0,k_1,k_2} \). Note that there is a tautological coupling isomorphism 
\( \psi : \mathcal{G}(k,k) \cong \text{IG}(A_k) \).

Now we take the gerbe \( \tilde{\mathcal{G}} \) associated to the prestack \( \mathcal{G} \). The only difference is that the gerbe \( \tilde{\mathcal{G}} \) has new local objects, gotten by gluing together compatible pieces of local objects of the prestack \( \mathcal{G} \). For such a new local object, say \( i \), we can attach a sheaf \( A_i \) in \( \mathcal{P} \), by using the same gluing information that defined \( i \). At the same time we construct the coupling isomorphism 
\( \psi : \tilde{\mathcal{G}}(i,i) \cong \text{IG}(A_i) \). The resulting creature is now a twisted sheaf in \( \mathcal{P} \).

It remains to check that the operation above is inverse (up to twisted gauge equivalence) to \( \text{dec} \); but this is straightforward.

\[\text{Example 6.11.}\] Let \( A_0 \in \mathcal{P}(X) \). Consider the open covering \( U = \{ U_0 \} \), with 
\( U_0 := X \). Take \( g_{0,0} := 1 \) and \( a_{0,0,0} := 1 \). From the proposition we get a twisted sheaf \( \mathcal{A} \), which we refer to as the twisted sheaf generated by \( A_0 \). Conversely, any twisted sheaf \( \mathcal{A} \) which is not really twisted arises in this way (up to twisted gauge equivalence).

**Theorem 6.12.** Assume Setup [5.4] We consider the two cases:

* The associative case, in which \( \mathcal{P}(R,X) \) is the stack with inner gauge groups 
  \( \text{AssDef}(R,\mathcal{O}_X) \) on \( X \).

* The Poisson case, in which \( \mathcal{P}(R,X) \) is the stack with inner gauge groups 
  \( \text{PoisDef}(R,\mathcal{O}_X) \) on \( X \).

Let \( (\mathcal{G}, \mathcal{A}, \psi) \) be a twisted object in \( \mathcal{P}(R,X) \), and let \( U \) be an open set of \( X \).

1. If \( H^2(U,\mathcal{O}_X) = 0 \) then the groupoid \( \mathcal{G}(U) \) is nonempty.
2. If \( H^1(U,\mathcal{O}_X) = 0 \) then the groupoid \( \mathcal{G}(U) \) is connected.

**Proof.** Let \( i \) be a local object of the gauge gerbe \( \mathcal{G} \), defined on some open set 
\( U \subset X \). Let’s write \( \mathcal{A}_i := \mathcal{A}(i) \in \mathcal{P}(U) \), which is an \( R \)-deformation of \( \mathcal{O}_U \). There is an isomorphism of sheaves of groups 
\( \psi : \mathcal{G}(i,i) \cong \text{IG}(\mathcal{A}_i) \) on \( U \). By definition 
\( \text{IG}(\mathcal{A}_i) = \exp(m\mathcal{A}_i) \), and hence for any \( p \in \mathbb{N} \) we get a sheaf of normal subgroups 
\( \mathcal{N}_p(i) := \psi^{-1}(\exp(m^{p+1}\mathcal{A}_i)) \subset \mathcal{G}(i,i) \).

Since \( \exp(m\mathcal{A}_i) \) is pronilpotent, we see that \( \mathcal{G}(i,i) \) is complete with respect to the nilpotent filtration \( \{ \mathcal{N}_p(i) \}_p \).

Next suppose \( j \) is another object of \( \mathcal{G}(U) \), and \( g \in \mathcal{G}(U)(i,j) \). Since \( \mathcal{A}(g) : \mathcal{A}_i \to \mathcal{A}_j \) is an \( R \)-linear sheaf isomorphism, and since \( \psi : \text{Aut}\mathcal{G} \cong \text{IG} \circ \mathcal{A} \) is a natural isomorphism of functors, it follows that \( \text{Ad}(g)(\mathcal{N}_p(i)) = \mathcal{N}_p(j) \). This says that for fixed \( p \), the collection \( \{ \mathcal{N}_p(i) \} \) is a normal collection of subgroups of the
gerbe $\mathcal{G}$, in the sense of [Ye4, Definition 3.2]. Moreover, there is a central extension of gerbes

$$1 \to \mathcal{N}_p / \mathcal{N}_{p+1} \to \mathcal{G} / \mathcal{N}_{p+1} \xrightarrow{F} \mathcal{G} / \mathcal{N}_p \to 1,$$

and an isomorphism of sheaves of abelian groups on $X$

$$\mathcal{N}_p / \mathcal{N}_{p+1} \cong (m^{p+1} / m^{p+2}) \otimes_k \mathcal{O}_X.$$ 

See [Ye4, Definition 3.11]. As we let $p$ vary, we have a nilpotent filtration $\{\mathcal{N}_p\}_{p \in \mathbb{N}}$ of the gerbe $\mathcal{G}$, and it is complete with respect to this filtration. See [Ye4, Definition 6.5].

Let $U$ and $A_i$ be as above, and let $V \subset U$ be an $\mathcal{O}_X$-acyclic open set. According to Proposition 5.7, the set $V$ is acyclic with respect to the nilpotent filtration $\{\mathcal{N}_p(i)\}_{p \geq 0}$ of the sheaf of groups $G(i, i)$, in the sense of [Ye4, Definition 6.2].

Now take an open covering $U = \{U_k\}_{k \in K}$ of $X$ such that $\text{Ob} \mathcal{G}(U_k) \neq \emptyset$ for every $k \in K$. This is possible because $\mathcal{G}$ is locally nonempty. Since $X$ has enough $\mathcal{O}_X$-acyclic open coverings, we can find an open covering $U' = \{U'_k\}_{k \in K'}$, which refines $U$, and such that each finite intersection $U'_k \cap \cdots \cap U'_l$ is $\mathcal{O}_X$-acyclic. Note that $\text{Ob} \mathcal{G}(U'_k, \cdots, U'_l) \neq \emptyset$. The discussion in the previous paragraph (for $V := U'_k$) tells us that the covering $U'$ is acyclic with respect to the nilpotent filtration $\{\mathcal{N}_p\}_{p \in \mathbb{N}}$ of the gerbe $\mathcal{G}$. We conclude that there are enough acyclic coverings with respect to $\{\mathcal{N}_p\}_{p \in \mathbb{N}}$, in the sense of [Ye4, Definition 6.9].

Finally let $U$ be an open set of $X$. Then

$$H^q(U, \mathcal{N}_p / \mathcal{N}_{p+1}) \cong (m^{p+1} / m^{p+2}) \otimes_k H^q(U, \mathcal{O}_X).$$

According to [Ye4, Theorem 6.10], if $H^2(U, \mathcal{N}_p / \mathcal{N}_{p+1})$ is trivial for all $p \geq 0$, then then the groupoid $\mathcal{G}(U)$ is nonempty; and if $H^1(U, \mathcal{N}_p / \mathcal{N}_{p+1})$ is trivial for all $p \geq 0$, then then the groupoid $\mathcal{G}(U)$ is connected. $\square$

Suppose $U = \{U_k\}_{k \in K}$ and $U' = \{U'_k\}_{k \in K'}$ are open coverings of $X$, and $\rho : U' \to U$ is a refinement. Then there is a function

$$\rho^* : \text{MDD}(\mathcal{P}(R, X), U) \to \text{MDD}(\mathcal{P}(R, X), U').$$

The formula is obvious: say $d = (\{A_k\}_{k \in K}, \ldots)$; then $\rho^*(d) = (\{A'_k\}_{k \in K'}, \ldots)$, where $A'_k := A_{\rho(k)}(U'_k)$, etc. It is easy to see that this function preserves the equivalence relation.

Let $\sigma : R \to R'$ be a homomorphism of parameter algebras. Given a descent datum $d = (\{A_k\}, \ldots) \in \text{MDD}(\mathcal{P}(U), U)$, let $A'_k := R' \otimes_R A_k$, which is an $R'$-deformation of $A_k$. There are induced $R'$-linear gauge transformations $g'_{k_0, k_1}$ and induced gauge elements $a'_{k_0, k_1, k_2}$, and together these make up a descent datum $\sigma(d) = (\{A'_k\}, \ldots)$. This construction is a function

$$\sigma : \text{MDD}(\mathcal{P}(R, X), U) \to \text{MDD}(\mathcal{P}(R', X), U),$$

which respects twisted gauge equivalence.

**Corollary 6.16.** In the situation of Theorem 6.12 suppose that $U$ is an $\mathcal{O}_X$-acyclic open covering of $X$. Then there is a bijection

$$\text{dec} : \text{TwOb}(\mathcal{P}(R, X)) \cong \text{MDD}(\mathcal{P}(R, X), U).$$
If $\sigma : R \to R'$ is a homomorphism of parameter algebras, $U'$ is another $\mathcal{O}_X$-acyclic open covering of $X$, and $p : U' \to U$ is a refinement, then the diagram

$$\xymatrix{ \text{TwOb}(\mathcal{P}(R, X)) \ar[r]^{\text{dec}} \ar[d]_{\text{ind}_{\sigma}} & \text{MDD}(\mathcal{P}(R, X), U) \ar[d]^{\sigma \circ p^*} \\
\text{TwOb}(\mathcal{P}(R', X)) \ar[r]^{\text{dec}} & \text{MDD}(\mathcal{P}(R', X), U')}
$$

is commutative. Here $\text{ind}_{\sigma}$ is the function from Proposition 5.18.

**Proof.** According to Theorem 6.11 the open covering $U$ trivializes the stack with inner gauge groups $\mathcal{P}(R, X)$. Hence the decomposition of Proposition 6.9 applies to the whole set $\text{TwOb}(\mathcal{P}(R, X))$.

The second assertion is proved by comparing the explicit construction of the function $\text{dec}$ in the proof of Proposition 6.9 to the explicit construction of the function $\text{ind}_{\sigma}$ in the proof of Proposition 5.18. $\square$

**Example 6.17.** It is easy to construct an example of a commutative associative (or Poisson) $\mathbb{K}[\![h]\!]$-deformation of $\mathcal{O}_X$ that is really twisted. Take an algebraic variety $X$ with nonzero cohomology class $c \in H^2(X, \mathcal{O}_X)$. Let $U$ be an affine open covering of $X$, and let $\{e_{a_{k_0,k_1,k_2}}\}$ be a normalized Čech 2-cocycle representing $c$ on this covering. Now consider the multiplicative descent datum $\{\mathcal{A}_k; \{g_{k_0,k_1}\}, \{a_{k_0,k_1,k_2}\}\}$ with $\mathcal{A}_k := \mathcal{O}_X[\![h]\!]$, $g_{k_0,k_1} := 1$ and $a_{k_0,k_1,k_2} := \exp(hc_{k_0,k_1,k_2})$.

The resulting twisted deformation $\mathcal{A}$ will have obstruction class $c$ in the first order central extension. More precisely, in the central extension of gerbes $\mathcal{G}/\mathcal{N}_1(X)$, with $p = 0$, the obstruction class for the unique (up to isomorphism) object $j$ of $(\mathcal{G}/\mathcal{N}_1(X))$ is

$$c_1^2(j) = ch \in H^2(X, (m/m^2) \otimes_{\mathbb{K}} \mathcal{O}_X).$$

Hence $\text{Ob}((\mathcal{G}/\mathcal{N}_1(X))) = \emptyset$, implying that $\text{Ob}(\mathcal{G}(X)) = \emptyset$.

Recall that for a category $\mathcal{G}$ we denote by $\text{Ob}(\mathcal{G})$ the set of isomorphism classes of objects.

**Corollary 6.18.** In the situation of Theorem 6.12 suppose that

$$H^2(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = 0.$$

Let us denote by $\mathcal{P}(R, X)$ either of the categories $\text{AssDef}(R, \mathcal{O}_X)$ or $\text{PoisDef}(R, \mathcal{O}_X)$, as the case may be. Then the function

$$\text{Ob}(\mathcal{P}(R, X)) \to \text{TwOb}(\mathcal{P}(R, X))$$

constructed in Example 6.11 is a bijection.

**Proof.** Let $(\mathcal{G}, \mathcal{A}, \text{cp})$ be a twisted object of $\mathcal{P}(R, X)$. By Theorem 6.12(1) there exists an object $i \in \text{Ob}(\mathcal{G}(X))$. Let $\mathcal{A}_i := \mathcal{A}(i) \in \text{Ob}(\mathcal{P}(X))$ be the corresponding deformation. Then $\mathcal{A}$ is twisted gauge equivalent to the twisted object generated by $\mathcal{A}_i$.

Now suppose $\mathcal{A}_0, \mathcal{A}'_0 \in \text{Ob}(\mathcal{P}(X))$ are such that the corresponding twisted objects $\mathcal{A}, \mathcal{A}' \in \text{TwOb}(\mathcal{P})$ are twisted equivalent. Let

$$(F_{\text{gau}}, F_{\text{rep}}) : (\mathcal{G}, \mathcal{A}, \text{cp}) \to (\mathcal{G}', \mathcal{A}', \text{cp}')$$

be a twisted gauge equivalence. Now $0 \in \text{Ob}(\mathcal{G}(X))$, so there is an object $i := F_{\text{gau}}(0) \in \text{Ob}(\mathcal{G}'(X))$, and an isomorphism $F_{\text{rep}} : \mathcal{A}_0 \cong \mathcal{A}'(i)$ in $\mathcal{P}(X)$. 


On the other hand, since the groupoid $\mathcal{G}'(X)$ is connected, there is some isomorphism $g : i \cong 0$ in it. Therefore we get an isomorphism $\mathcal{A}'(g) : \mathcal{A}'(i) \cong \mathcal{A}'(0) = \mathcal{A}_0$ in $\mathcal{P}(X)$. We see that $\mathcal{A}_0 \cong \mathcal{A}_0'$ in $\mathcal{P}(X)$.

7. DG Lie Algebras and Deformations

In this section we study differential properties of deformations of commutative algebras.

Suppose $k$ is a field of characteristic 0, and $(R, m)$ is a parameter $k$-algebra. Let $g = \bigoplus_{p \in \mathbb{Z}} g^p$ be a DG (differential graded) Lie algebra over $k$, with differential $d$ and Lie bracket $[\cdot, \cdot]$. We define the extended DG Lie algebra $R \hat{\otimes}_K g$ as follows. For each $p$ we let

$$R \hat{\otimes}_K g^p := \lim_{\leftarrow i} (R/m^i) \otimes_k g^p,$$

and

$$R \hat{\otimes}_K g := \bigoplus_p R \hat{\otimes}_K g^p.$$

The differential and Lie bracket of $R \hat{\otimes}_K g$ are the $R$-linear extensions of those of $g$. Inside $R \hat{\otimes}_K g$ there is a closed sub DG Lie algebra $m \hat{\otimes}_K g$. See [Ye2] for a discussion of such completions (and the theory of dir-inv structures).

The Lie algebra $m \hat{\otimes}_K g^0$ is pronilpotent, and we denote by $\exp(m \hat{\otimes}_K g^0)$ the associated pronilpotent group. It is called the gauge group of $m \hat{\otimes}_K g$.

As usual, for any element $\gamma \in R \hat{\otimes}_K g$, we denote by $\text{ad}(\gamma)$ the $R$-linear operator on $R \hat{\otimes}_K g$ with formula $\text{ad}(\gamma)(\beta) := [\gamma, \beta]$. If $\gamma \in m \hat{\otimes}_K g^0$, and we write $g := \exp(\gamma) \in \exp(m \hat{\otimes}_K g^0)$, then we obtain an $R$-linear automorphism

$$\exp(\text{ad})(g) := \exp(\text{ad}(\gamma))$$

of $R \hat{\otimes}_K g$.

An $MC$ element in $m \hat{\otimes}_K g$ is an element $\beta \in m \hat{\otimes}_K g^1$ which satisfies the Maurer-Cartan equation

$$d(\beta) + \frac{1}{2}[\beta, \beta] = 0.$$

We denote by $\text{MC}(m \hat{\otimes}_K g)$ the set of $MC$ elements.

The Lie algebra $R \hat{\otimes}_K g^0$ acts on the $R$-module $R \hat{\otimes}_K g^1$ by the affine transformations

$$af(\gamma)(\beta) := d(\gamma) - \text{ad}(\gamma)(\beta) = d(\gamma) - [\gamma, \beta],$$

for $\gamma \in R \hat{\otimes}_K g^0$ and $\beta \in R \hat{\otimes}_K g^1$. This action integrates to an action $\exp(af)$ of the group $\exp(m \hat{\otimes}_K g^0)$. The group action $\exp(af)$ preserves the set $\text{MC}(m \hat{\otimes}_K g)$, and we write $\text{MC}(m \hat{\otimes}_K g)$ for the quotient set by this action.

Suppose $\mathfrak{h}$ is another DG Lie algebra, and $\phi : g \rightarrow \mathfrak{h}$ is a homomorphism of DG Lie algebras. There is an induced $R$-linear homomorphism $\phi_R : R \hat{\otimes}_K g \rightarrow R \hat{\otimes}_K \mathfrak{h}$ of DG Lie algebras, and an induced function

$$\text{MC}(\phi_R) : \text{MC}(m \hat{\otimes}_K g) \rightarrow \text{MC}(m \hat{\otimes}_K \mathfrak{h}).$$

If $\phi$ is a quasi-isomorphism then so is $\phi_R$, and on gauge equivalence classes of $MC$ elements we get a bijection

$$\text{MC}(\phi_R) : \text{MC}(m \hat{\otimes}_K g) \rightarrow \text{MC}(m \hat{\otimes}_K \mathfrak{h}).$$

See [Ye1, Ye3] and their references for details.

For an element $\beta \in R \hat{\otimes}_K g^1$ we let

$$d_\beta := d + \text{ad}(\beta),$$

which is an operator of degree 1 on $R \hat{\otimes}_K g$. Thus for $\alpha \in R \hat{\otimes}_K g$ one has $d_\beta(\alpha) = d(\alpha) + [\beta, \alpha]$. Moreover, for $\gamma \in R \hat{\otimes}_K g^0$ one has $d_\beta(\gamma) = af(\gamma)(\beta)$.

Definition 7.2. We say $g$ is a quantum type DG Lie algebra if $g^p = 0$ for $p < -1$. 


Proposition 7.3. Suppose $\mathfrak{g}$ is a quantum type DG Lie algebra. Let $\beta \in \text{MC}(\mathfrak{m} \hat{\otimes}_K \mathfrak{g})$.

1. The formula
   \[ [\alpha_1, \alpha_2] : = [d_\beta(\alpha_1), \alpha_2] \]
   defines an $R$-linear Lie bracket on $R \hat{\otimes}_K \mathfrak{g}^{-1}$. We denote the resulting Lie algebra by $(R \hat{\otimes}_K \mathfrak{g}^{-1})_\beta$. It has a pronilpotent Lie subalgebra $(\mathfrak{m} \hat{\otimes}_K \mathfrak{g}^{-1})_\beta$.

2. The function
   \[ d_\beta : (R \hat{\otimes}_K \mathfrak{g}^{-1})_\beta \to R \hat{\otimes}_K \mathfrak{g}^0 \]
   is an $R$-linear Lie algebra homomorphism.

3. Let $g \in \exp(\mathfrak{m} \hat{\otimes}_K \mathfrak{g}^0)$ and $\beta' := \exp(af)(\beta)$. Then
   \[ \exp(\text{ad}g) : (R \hat{\otimes}_K \mathfrak{g}^{-1})_\beta \to (R \hat{\otimes}_K \mathfrak{g}^{-1})_{\beta'} \]
   is an $R$-linear Lie algebra isomorphism.

Proof. See [Ge1, Section 2.3], where this structure is called the Deligne 2-groupoid of $\mathfrak{m} \hat{\otimes}_K \mathfrak{g}$.

Remark 7.4. If the differential $d$ vanishes on $\mathfrak{g}^{-1}$ (and this does happen in our work), then
   \[ [\alpha_1, \alpha_2] : = [[\beta, \alpha_1], \alpha_2] \]
   for $\beta \in \text{MC}(\mathfrak{m} \hat{\otimes}_K \mathfrak{g})$ and $\alpha_1, \alpha_2 \in R \hat{\otimes}_K \mathfrak{g}^{-1}$. Therefore the whole Lie algebra $(R \hat{\otimes}_K \mathfrak{g}^{-1})_\beta$ is pronilpotent.

Sometimes it is convenient to have a more explicit (but less canonical) way of describing the DG Lie algebra $\mathfrak{m} \hat{\otimes}_K \mathfrak{g}$. This is done via choice of filtered $K$-basis of $\mathfrak{m}$.

A filtered $K$-basis of a finitely generated $R$-module $M$ is a sequence $\{m_j\}_{j \geq 0}$ of elements of $M$ (finite if $M$ has finite length, and countable otherwise) whose symbols form a $K$-basis of the graded $K$-module
   \[ \text{gr}_m M = \bigoplus_{i \geq 0} m^i/M / m^{i+1} M. \]
It is easy to find such bases: simply choose a $K$-basis of $\text{gr}_m M$ consisting of homogeneous elements, and lift it to $M$. Once such a filtered basis is chosen, any element $m \in M$ has a unique convergent power series expansion $m = \sum_{j \geq 0} \lambda_j m_j$, with $\lambda_j \in K$.

Let us choose a filtered $K$-basis $\{r_j\}_{j \geq 0}$ of $R$, such that $r_0 = 1$. Then the sequence $\{r_j\}_{j \geq 1}$ is a filtered $K$-basis of $\mathfrak{m}$.

Example 7.5. For $R = K[[\hbar]]$ the obvious filtered basis is $r_j := \hbar^j$. In the paper [Ye1] we used the notation $g[[\hbar]]^+$ for the DG Lie algebra $\mathfrak{m} \hat{\otimes}_K \mathfrak{g}$ in this case.

Getting back to our DG Lie algebra $\mathfrak{g}$, any element $\gamma \in \mathfrak{m} \hat{\otimes}_K \mathfrak{g}^p$ can be uniquely expanded into a power series $\gamma = \sum_{j \geq 1} r_j \otimes \gamma_j$, with $\gamma_j \in \mathfrak{g}^p$. With this notation one has
   \[ d(\gamma) = \sum_{j \geq 1} r_j \otimes d(\gamma_j) \]
   and
   \[ [\gamma, \gamma'] = \sum_{j,k \geq 1} r_j r_k \otimes [\gamma_j, \gamma'_k]. \]
In the rest of this section we make the following assumption:
Setup 7.6. \( \mathbb{K} \) is a field of characteristic 0; \((R, m)\) is a parameter \(\mathbb{K}\)-algebra (see Definition 7.1); and \(C\) is a smooth integral commutative \(\mathbb{K}\)-algebra.

Note that \(\text{Spec } C\) is a smooth affine algebraic variety.

For Poisson deformations the relevant DG Lie algebra is the algebra of poly derivations

\[
T_{\text{poly}}(C) = \bigoplus_{p=0}^{n-1} T_{\text{poly}}^p(C)
\]

of \(C\) relative to \(\mathbb{K}\), where \(n := \dim C\). It is the exterior algebra over \(C\) of the module of derivations \(T(C)\), but with a shift in degrees:

\[
T_{\text{poly}}^p(C) = \wedge_{C}^{p+1} T(C).
\]

The differential is zero, and the Lie bracket is the Schouten-Nijenhuis bracket, that extends the usual Lie bracket on \(T(C) = T_{\text{poly}}^0(C)\), and its canonical action \(\text{ad}_C\) on \(C = T_{\text{poly}}^1(C)\) by derivations. The DG Lie algebra \(T_{\text{poly}}(C)\) is of course of quantum type.

Passing to the extended algebra \(R \widehat{\otimes}_\mathbb{K} T_{\text{poly}}(C)\), we have an action of the Lie algebra \(m \widehat{\otimes}_\mathbb{K} T_{\text{poly}}^0(C)\) on the commutative algebra \(A := R \widehat{\otimes}_\mathbb{K} C\) by \(R\)-linear derivations, which we denote by \(\text{ad}_A\). If we choose a filtered \(\mathbb{K}\)-basis \(\{r_j\}_{j \geq 1}\) of \(m\), then for \(\gamma = \sum_{j \geq 1} r_j \otimes \gamma_j\) and \(c \in C\) this action becomes

\[
\text{ad}_A(\gamma)(c) = \sum_{j \geq 1} r_j \otimes \text{ad}_C(\gamma_j)(c) \in m \widehat{\otimes}_\mathbb{K} C.
\]

Its exponential is an automorphism

\[
\exp(\text{ad}_A(\gamma)) = \sum_{i \geq 0} \frac{1}{i!} \underbrace{\text{ad}_A(\gamma) \circ \cdots \circ \text{ad}_A(\gamma)}_{i}
\]

of the \(R\)-module \(A = R \widehat{\otimes}_\mathbb{K} C\).

Likewise any element \(\beta \in m \widehat{\otimes}_\mathbb{K} T_{\text{poly}}^1(C)\) determines an antisymmetric \(R\)-bilinear function \(\{-,-\}_\beta\) on \(R \widehat{\otimes}_\mathbb{K} C\). If the expansion of \(\beta\) is \(\beta = \sum_{j \geq 1} r_j \otimes \beta_j\), then

\[
\{c_1, c_2\}_\beta := \sum_{j \geq 1} r_j \otimes \beta_j(c_1, c_2) \in m \widehat{\otimes}_\mathbb{K} C,
\]

where for \(\gamma_1, \gamma_2 \in T(C)\) and \(c_1, c_2 \in C\) we let

\[
(\gamma_1 \wedge \gamma_2)(c_1, c_2) := \frac{1}{2}(\text{ad}_C(\gamma_1)(c_2) \cdot \text{ad}_C(\gamma_2)(c_1) - \text{ad}_C(\gamma_1)(c_1) \cdot \text{ad}_C(\gamma_2)(c_2)).
\]

Definition 7.7. Consider the commutative \(R\)-algebra \(A := R \widehat{\otimes}_\mathbb{K} C\), with the obvious augmentation \(\psi : \mathbb{K} \otimes_R A \xrightarrow{\sim} C\).

(1) A formal Poisson bracket on \(A\) is an \(R\)-bilinear Poisson bracket that vanishes modulo \(m\).

(2) A gauge transformation of \(A\) (as \(R\)-algebra) is an \(R\)-algebra automorphism that commutes with the augmentation to \(C\).

According to Proposition 7.4, the commutative \(R\)-algebra \(A := R \widehat{\otimes}_\mathbb{K} C\) is flat and \(m\)-adically complete. Therefore, by endowing it with a formal Poisson bracket \(\beta\), we obtain a Poisson \(R\)-deformation of \(C\), and we denote this deformation by \(A_\beta\).

The next result, when combined with Proposition 7.9, summarizes the role of \(T_{\text{poly}}(C)\) in Poisson deformations.

Proposition 7.8. Consider the augmented commutative \(R\)-algebra \(A := R \widehat{\otimes}_\mathbb{K} C\).

(1) The formula

\[
\exp(\gamma) \mapsto \exp(\text{ad}_A(\gamma))
\]

determines a group isomorphism from \(\exp(m \widehat{\otimes}_\mathbb{K} T_{\text{poly}}^0(C))\) to the group of gauge transformations of \(A\) (as \(R\)-algebra).
(2) The formula

\[ \beta \mapsto \{-,-\}_\beta \]

determines a bijection from \( \text{MC}(\mathfrak{m} \otimes_K T_{\text{poly}}(C)) \) to the set of formal Poisson brackets on \( A \). For such \( \beta \) we denote by \( A_\beta \) the corresponding Poisson algebra.

(3) Let \( \beta, \beta' \in \text{MC}(\mathfrak{m} \otimes_K T_{\text{poly}}(C)) \), and let \( \gamma \in \mathfrak{m} \otimes_K T_{\text{poly}}^0(C) \). Then

\[ \beta' = \exp(\text{ad}(\gamma))(\beta) \]

if and only if

\[ \exp(\text{ad}(\gamma))^\circ : A_\beta \to A_{\beta'} \]

is a gauge transformation of Poisson deformations.

(4) For \( \beta \in \text{MC}(\mathfrak{m} \otimes_K T_{\text{poly}}(C)) \), one has

\[ \text{IG}(A_\beta) = \exp(\mathfrak{m} \otimes_K T_{\text{poly}}^{-1}(C))_{\beta'} \]

Proof. (1) By definition the operator \( \text{ad}(\gamma) \) is a pronilpotent derivation of the \( R \)-algebra \( A \). According to [Hu, Section 2.3] the operator \( \exp(\text{ad}(\gamma)) \) is an \( R \)-algebra automorphism of \( A \). Since

\[ \text{ad}_A : \mathfrak{m} \otimes_K T(C) \to \text{End}_R(A) \]

is an injective Lie algebra homomorphism, it follows that \( \exp(\text{ad}_A(-)) \) is an injective group homomorphism.

Now suppose \( g : A \to A \) is a gauge transformation. We will produce a sequence \( \gamma_i \in \mathfrak{m} \otimes_K T(C) \) such that \( g = \exp(\text{ad}(\gamma_i)) \) modulo \( m^{i+1} \). Then for \( \gamma := \lim_{i \to \infty} \gamma_i \) we have \( g = \exp(\text{ad}(\gamma)) \). Here is the construction. We start with \( \gamma_0 := 0 \) of course. Next assume then that we have \( \gamma_i \). There is a unique element

\[ \delta_{i+1} \in (m^{i+1}/m^{i+2}) \otimes_K \text{End}_K(C) \]

such that

\[ g \circ \exp(\text{ad}(\gamma_i))^{-1} = 1 + \delta_{i+1} \]

as automorphisms of the \( R_{i+1} \)-algebra \( A_{i+1} := R_{i+1} \otimes_K C \). The usual calculation shows that is a derivation, i.e.

\[ \delta_{i+1} \in (m^{i+1}/m^{i+2}) \otimes_K T(C). \]

Choose some lifting \( \hat{\delta}_{i+1} \in m^{i+1} \otimes_K T(C) \) of \( \delta_{i+1} \), and define \( \gamma_{i+1} := \gamma_i + \hat{\delta}_{i+1} \).

(2, 3) See [Ko1, paragraph 4.6.2] or [CKTB, paragraph 3.5.3].

(4) This is immediate from the definitions. \( \square \)

Proposition 7.9. Let \( A \) be a Poisson \( R \)-deformation of \( C \). Then there is an isomorphism of augmented commutative \( R \)-algebras \( R \otimes_K C \cong A \).

Proof. We write \( R_i := R/m^{i+1} \) for \( i \geq 0 \). Since \( C \) is formally smooth over \( K \), we can find a compatible family of \( K \)-algebra liftings \( C \to R_i \otimes_R A \) of the augmentation. Due to flatness the induced \( R_i \)-algebra homomorphisms \( R_i \otimes_K C \to R_i \otimes_R A \) are bijective. And because \( A \) is complete we get an isomorphism of augmented \( R \)-algebras \( R \otimes_K C \cong A \) in the limit. \( \square \)

The associative case is much more difficult. When dealing with associative deformations we view \( A := R \otimes_K C \) as an \( R \)-module. The augmentation \( A \to C \) is viewed as a homomorphism of \( R \)-modules, and there is a distinguished element \( 1_A := 1_R \otimes 1_C \in A \).

Definition 7.10. Consider the augmented \( R \)-module \( A := R \otimes_K C \), with distinguished element \( 1_A \).
(1) A star product on $A$ is an $R$-bilinear function
\[ \star : A \times A \to A \]
that makes $A$ into an associative $R$-algebra, with unit $1_A$, such that
\[ c_1 \star c_2 \equiv c_1 c_2 \mod m \]
for $c_1, c_2 \in C$.

(2) A gauge transformation of $A$ (as $R$-module) is an $R$-module automorphism
that commutes with the augmentation to $C$ and fixes the element $1_A$.

Given a star product $\star$ on $A$, we have an associative $R$-deformation of $C$.
If we choose a filtered $\mathbb{K}$-basis $\{r_j\}_{j \geq 1}$ of $m$, then we can express $\star$ as a power series
\[ c_1 \star c_2 = c_1 c_2 + \sum_{j \geq 1} r_j \beta_j(c_2, c_2) \in A, \]
where $\beta_j \in \text{Hom}_{\mathbb{K}}(C \otimes_{\mathbb{K}} C, C)$, and we identify an element $c \in C$ with the tensor $1_R \otimes c \in A = R \hat{\otimes}_{\mathbb{K}} C$. Likewise for a gauge transformation $g$ of the $R$-module $A$, we can expand $g$ into a power series
\[ g(c) = c + \sum_{j \geq 1} r_j \gamma_j(c) \in A, \]
where $\gamma_j \in \text{Hom}_{\mathbb{K}}(C, C)$.

Star products are controlled by a DG Lie algebra too. It is the shifted Hochschild cochain complex
\[ C_{\text{shc}}(C) = \bigoplus_{p \geq -1} C^p_{\text{shc}}(C), \]
where
\[ C^p_{\text{shc}}(C) := \text{Hom}_{\mathbb{K}}(C \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} C, C) \]
for $p \geq 0$, and $C^{-1}_{\text{shc}}(C) := C$. The differential is the shift of the Hochschild differential, and the Lie bracket is the Gerstenhaber bracket. (In our earlier paper [Ye2] we used the notation $C_{\text{dual}}(C)[1]$ for this DG Lie algebra.) Inside $C_{\text{shc}}(C)$ there is a sub DG Lie algebra $C^\text{nor}_{\text{shc}}(C)$, consisting of the normalized cochains. By definition a cochain $\phi \in C^p_{\text{shc}}(C)$ is normalized if either $p = -1$, or $p \geq 0$ and $\phi(c_1 \otimes \cdots \otimes c_{p+1}) = 0$ whenever $c_i = 1$ for some index $i$.

Given $\beta \in m \hat{\otimes}_{\mathbb{K}} C^\text{nor}_{\text{shc}}(C)$ we denote by $\star_\beta$ the $R$-bilinear function on the $R$-module $A := R \hat{\otimes}_{\mathbb{K}} C$ with formula
\[ c_1 \star_\beta c_2 := c_1 c_2 + \beta(c_1, c_2) \]
for $c_1, c_2 \in C$. And for $\gamma \in m \hat{\otimes}_{\mathbb{K}} C_{\text{shc}}^\text{nor,0}(C)$ we denote by $\text{ad}_A$ the $R$-linear function on $A$ such that
\[ \text{ad}_A(c) := [\gamma, c] = \gamma(c) \]
for $c \in C$.

We know that any associative $R$-deformation $A$ of $C$ is isomorphic, as augmented $R$-module, to $R \hat{\otimes}_{\mathbb{K}} C$. Like Proposition 7.8 we have:

**Proposition 7.11.** Consider the augmented $R$-module $A := R \hat{\otimes}_{\mathbb{K}} C$ with distinguished element $1_A$.

(1) The formula
\[ \exp(\gamma) \mapsto \exp(\text{ad}_A(\gamma)) \]

determines a group isomorphism from $\exp(m \hat{\otimes}_{\mathbb{K}} C_{\text{shc}}^\text{nor,0}(C))$ to the group of gauge transformations of the $R$-module $A$. 
(2) The formula
\[ \beta \mapsto \star \beta \]
determines a bijection from \( \text{MC}(\mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor}}(C)) \) to the set of star products on \( A \). For such \( \beta \) we denote by \( A_\beta \) the resulting associative \( R \)-algebra.

(3) Let \( \beta, \beta' \in \text{MC}(\mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor}}(C)) \), and let \( \gamma \in \mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor}}(C) \). Then
\[ \beta' = \exp(af(\gamma))(\beta) \]
if and only if
\[ \exp(\text{ad}_{A}(\gamma)) : A_\beta \rightarrow A_{\beta'} \]
is a gauge transformation of associative \( R \)-deformations of \( C \).

(4) For \( \beta \in \text{MC}(\mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor}}(C)) \), one has a canonical isomorphism of groups
\[ IG(A_\beta) \cong \exp(\mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor}}(C))_{\beta}. \]

Proof. (1) We have an injective Lie algebra homomorphism
\[ \text{ad}_A : \mathfrak{m} \hat{\otimes}_K C_{\text{shc}}^{\text{nor},0}(C) \rightarrow \text{End}_R(A) \]
whose image consists of pronilpotent endomorphisms. So the exponential is an injective group homomorphism. The proof of surjectivity here is similar to that of Proposition 7.8(1), so we won’t repeat it. The only point worth mentioning is that the automorphism \( \exp(\text{ad}_{A}(\gamma)) \) fixes \( 1_A \) if and only if \( \gamma \) is normalized.

(2, 3) See [Ko1, paragraphs 3.4.2 and 4.6.2] or [CKTB, Section 3.3]. Cf. also [Ye1, Propositions 3.20 and 3.21].

(4) See Example 3.17. \( \square \)

8. Differential Star Products

We continue with Setup 7.6. In this section we prove that associative deformations are actually controlled by a sub DG Lie algebra \( D_{\text{poly}}(C) \) of \( C_{\text{shc}}(C) \), which has better behavior.

Take a Hochschild cochain
\[ \phi : C^\otimes p = C \otimes_K \cdots \otimes_K C \rightarrow C \]
for some \( p \geq 1 \). The function \( \phi \) is called a poly differential operator if, when we view \( C^\otimes p \) as a \( K \)-algebra and \( C \) as a \( C^\otimes p \)-module, \( \phi \) is a differential operator. (In [Ye2] we used another, but equivalent, definition.) We denote by \( D_{\text{poly}}^{-1}(C) \) the set of such poly differential operators. And we let \( D_{\text{poly}}^{-1}(C) := C \). Then \( D_{\text{poly}}(C) \) is a sub DG Lie algebra of \( C_{\text{shc}}(C) \). We define a yet smaller DG Lie algebra
\[ D_{\text{poly}}^{\text{nor}}(C) := D_{\text{poly}}(C) \cap C_{\text{shc}}^{\text{nor}}(C), \]
whose elements are the normalized poly differential operators.

Definition 8.1. Consider the augmented \( R \)-module \( A := R \hat{\otimes}_K C \), with distinguished element \( 1_A \). Recall the bijections of Proposition 7.11(1-2).

(1) A gauge transformation \( g : A \rightarrow A \) is called a differential gauge transformation if \( \gamma := \log(g) \) belongs to \( \mathfrak{m} \hat{\otimes}_K D_{\text{poly}}^{\text{nor},0}(C) \).

(2) A star product \( \star \) on \( A \) is called a differential star product if the corresponding MC element \( \beta \) belongs to \( \mathfrak{m} \hat{\otimes}_K D_{\text{poly}}^{\text{nor},1}(C) \).
Theorem 8.2. Assume $R$ and $C$ are as in Setup \[\text{Setup 7.3}\]. Then any star product on the $R$-module $A := R \hat{\otimes}_K C$ is gauge equivalent to a differential star product. Namely, given a star product $\ast$ on $A$, there exists a gauge transformation $g : A \to A$, and a differential star product $\ast'$, such that

\[(8.3)\quad g(a_1 \ast a_2) = g(a_1) \ast' g(a_2)\]

for any $a_1, a_2 \in A$.

Proof. This is a mild generalization of \[\text{Ye2} \, \text{Proposition 8.1}\], which refers to $R = \mathbb{K}[[h]]$. According to \[\text{Ye2} \, \text{Corollary 4.12}\], the inclusion $\mathcal{D}_{\text{nor}}^\text{poly}(C) \to \mathcal{C}_{\text{shc}}^\text{nor}(C)$ is a quasi-isomorphism. Therefore we get a bijection

$$\text{MC}(m \hat{\otimes}_K \mathcal{D}_{\text{poly}}^\text{nor}(C)) \to \text{MC}(m \hat{\otimes}_K \mathcal{C}_{\text{shc}}^\text{nor}(C)).$$

Let $\beta \in \text{MC}(m \hat{\otimes}_K \mathcal{C}_{\text{shc}}^\text{nor}(C))$ be the element representing $\ast$; see Proposition \[\text{7.11\,2}\]. Next let $\beta' \in \text{MC}(m \hat{\otimes}_K \mathcal{D}_{\text{poly}}^\text{nor}(C))$ be an element that’s gauge equivalent to $\beta$. By Proposition \[\text{7.11\,3}\] we get a gauge transformation $g := \exp(\text{ad}_A(\gamma))$ which satisfies equation \[8.3\]. □

Remark 8.4. It should be noted that the proof of \[\text{Ye2} \, \text{Corollary 4.12}\] relies on the fact that $C$ is a smooth $\mathbb{K}$-algebra and $\text{char} \, \mathbb{K} = 0$. The result is most likely false otherwise.

We learned the next result from P. Etingof. It is very similar to \[\text{KS3} \, \text{Proposition 4.3}\].

Theorem 8.5. Assume $R$ and $C$ are as in Setup \[\text{Setup 7.3}\]. Suppose $\ast$ and $\ast'$ are two differential star products on the augmented $R$-module $A := C \hat{\otimes} R$, and $g$ is a gauge transformation of $A$ satisfying \[\text{8.3}\]. Then $g$ is a differential gauge transformation.

Proof. Let us choose a filtered $\mathbb{K}$-basis $\{r_i\}_{i \geq 0}$ of $R$, such that $r_0 = 1$, and $\text{ord}_m(r_i) \leq \text{ord}_m(r_{i+1})$. Denote by $\{\mu_{i,j,k}\}_{i,j,k \geq 0}$ the multiplication constants of the basis $\{r_i\}_{i \geq 0}$, i.e. the collection of elements of $\mathbb{K}$ such that

$$r_i \cdot r_j = \sum_k \mu_{i,j,k} r_k \in R.$$

Note that $\mu_{0,j,j} = \mu_{i,0;i} = 1$, and $\mu_{i,j,k} = 0$ if $i + j > k$.

The gauge transformation $g$ has an expansion

$$g = \sum_{i \geq 0} r_i \otimes \gamma_i,$$

with $\gamma_0 = 1_C$ and

$$\gamma_i \in \mathcal{C}_{\text{shc}}^\text{nor,0}(C) \subset \text{End}_K(C)$$

for $i \geq 1$. We will begin by showing that $\gamma_i$ are differential operators. This calculation is by induction on $i$, and it is almost identical to the proof of \[\text{KS3} \, \text{Proposition 4.3}\].

Let us denote by $\beta_i, \beta'_i \in \mathcal{D}_1^\text{poly}(C)$ the bidifferential operators such that

$$c \ast d = \sum_{i \geq 0} r_i \otimes \beta_i(c, d)$$

and

$$c \ast' d = \sum_{i \geq 0} r_i \otimes \beta'_i(c, d)$$

for all $c, d \in C$. Thus $\beta_0(c, d) = \beta_0'(c, d) = cd$, and $\beta_i, \beta'_i \in \mathcal{D}_i^\text{poly}(C)$ for $i \geq 1$. By expanding the two sides of \[8.3\] we get

$$g(c \ast d) = \sum_{i \geq 0} r_i \otimes \left( \sum_{j+k \leq i} \mu_{j,k;i} \gamma_k(\beta_j(c, d)) \right)$$
By plugging Corollary 8.7.
what we now know: on an associative variety over \(K\) we take the summand with \(k = i\) (and \(j = 0\)) in the left side of (8.6), and subtract from it the summand with \(j = m = i\) (and \(k = l = 0\)) in the right side of that equation. This yields

\[
\mu_{0,i; i} \gamma_i (\beta_0 (c, d)) - \mu_{i,0;i} \mu_{i,0;i} \gamma_i (\gamma_0 (c), \gamma_0 (d)) = \phi_i (c, d),
\]

where \(\phi_i (c, d)\) involves the bidifferential operators \(\beta_k, \beta'_k\), and the operators \(\gamma_j\) for \(j < i\), which are differential by the induction hypothesis. We see that \(\phi_i (c, d)\) is itself a bidifferential operator, say of order \(\leq m_i\) in each argument. And since \(\mu_{0,i;i} = 1\) etc., we have

\[
\gamma_i (cd) - \gamma_i (c)d = \phi_i (c, d).
\]

Now, letting \(c\) vary, the last equation reads

\[
[\gamma_i, d] = \phi_i (-, d) \in \text{End}_K (C).
\]

Hence \([\gamma_i, d]\) is a differential operator, also of order \(\leq m_i\). This is true for every \(d \in C\). By Grothendieck’s characterization of differential operators, it follows that \(\gamma_i\) is a differential operator (of order \(\leq m_i + 1\)).

Finally let us consider \(\log (g)\). We know that \(r_i \otimes \gamma_i \in \mathfrak{m} \otimes_K D^\text{nor,0} (C)\) for \(i \geq 1\). And \(\mathfrak{m} \otimes_K D^\text{nor,0} (C)\) is a closed (nonunital) subalgebra of the ring \(R \otimes_K \text{End}_K (C)\).

By plugging \(x := \sum r_i \otimes \gamma_i\) into the usual power series

\[
\log(1 + x) = x - \frac{1}{2} x^2 + \cdots
\]

we conclude that \(\log (g) \in \mathfrak{m} \otimes_K D^\text{nor,0} (C)\). \(\square\)

In [Ye1] Definitions 1.4, 1.8 we introduced the notion of differential structure on an associative-R-deformation \(A\) of \(\mathcal{O}_X\). We said there that one must stipulate the existence of such a differential structure, and uniqueness was not clear. Here is what we now know:

**Corollary 8.7.** Let \(K\) be a a field of characteristic 0, let \(X\) be a smooth algebraic variety over \(K\), and let \(A\) be an associative \(K[[\hbar]]\)-deformation of \(\mathcal{O}_X\). Then \(A\) admits a differential structure. Moreover, any two such differential structures are equivalent.

**Proof.** Choose any affine open covering \(U = \{U_0, \ldots, U_m\}\) of \(X\), and let \(C_i := \Gamma(U_i, \mathcal{O}_X)\). By Theorem 8.2 the deformation \(A_i := \Gamma(U_i, A)\) is isomorphic to \(C_i[[\hbar]]\), with some differential star product \(\ast_i\). According to Theorem 2.12(2) there is an isomorphism of sheaves of \(K\)-algebras \(\tau_i : O_{U_i}[[\hbar]] \cong A|_{U_i}\). In the terminology of [Ye1] Definition 1.2, this is a differential trivialization of \(A|_{U_i}\).

Consider a double intersection \(U_{i,j} = U_i \cap U_j\). By Theorem 8.3 the gauge transformation

\[
\tau_j^{-1} \circ \tau_i : \Gamma(U_{i,j}, \mathcal{O}_X)[[\hbar]] \cong \Gamma(U_{i,j}, \mathcal{O}_X)[[\hbar]]
\]

is differential. Hence the collection \(\{\tau_i\}\) is a differential structure on \(A\).

The uniqueness of this differential structure up to equivalence is also a consequence of Theorem 8.5. \(\square\)
9. Cosimplicial DG Lie Algebras and Descent Data

We begin with a quick review of cosimplicial theory. Let $\Delta$ denote the category of finite ordinals. The set of objects of $\Delta$ is the set $\mathbb{N}$ of natural numbers. Given $p,q \in \mathbb{N}$, the morphisms $\alpha : p \to q$ in $\Delta$ are order preserving functions

$$\alpha : \{0,\ldots,p\} \to \{0,\ldots,q\}.$$  

We denote this set of morphisms by $\Delta^q_p$. An element of $\Delta^q_p$ may be thought of as a sequence $i = (i_0,\ldots,i_p)$ of integers with $0 \leq i_0 \leq \cdots \leq i_p \leq q$. We call $\Delta^q := \{\Delta^q_p\}_{p\in \mathbb{N}}$ the $q$-dimensional combinatorial simplex, and an element $i \in \Delta^q$ is a $p$-dimensional face of $\Delta^q$. If $i_0 < \cdots < i_p$ then $i$ is said to be nondegenerate.

Let $C$ be some category. A cosimplicial object in $C$ is a functor $C : \Delta \to C$. We shall be interested in the category of DG Lie algebras over a field of characteristic $0$. The set of objects of $\Delta$ is a category of sets with structure (i.e. there is a faithful functor $\mathbb{N} \to \Delta$). For every $p \in \mathbb{N}$, the morphisms $\Delta^q_p \to \Delta^q_0$ are order preserving functions.

Let $C$ be some category. A cosimplicial object in $C$ is a functor $C : \Delta \to C$. We shall usually write $C^p := C(p) \in \text{Ob}(C)$, and leave the morphisms $C(\alpha) : C(p) \to C(q)$, for $\alpha \in \Delta^q_p$, implicit. Thus we shall refer to the cosimplicial object $C$ as $\{C^p\}_{p\in \mathbb{N}}$.

If $C$ is a category of sets with structure (i.e. there is a faithful functor $C \to \text{Set}$), then given a nondegenerate face $i = \alpha \in \Delta^q_p$ and an element $c \in C^p$, it will be convenient to write

$$c^i := C(\alpha)(c) \in C^q.$$  

The picture to keep in mind is of “the element $c$ pushed to the face $i$ of the simplex $\Delta^q$”. See Figure 2 for an illustration.

We shall be interested in the category $\text{DGLie}_K$ of differential graded Lie algebras over a field $K$ of characteristic $0$. A cosimplicial object $g$ of $\text{DGLie}_K$ will be called a cosimplicial DG Lie algebra. It consists of a collection $g = \{g^p\}_{p\in \mathbb{N}}$ of DG Lie algebras over $K$, and these homomorphisms satisfy the simplicial relations.

Given a parameter $K$-algebra $(R,m)$, there is an extended cosimplicial $R$-linear DG Lie algebra $m \hat{\otimes}_K g := \{m \hat{\otimes}_K g^p\}_{p\in \mathbb{N}}$, where

$$m \hat{\otimes}_K g^p := \bigoplus_{i \in \mathbb{Z}} m \hat{\otimes}_K g^{p,i}.$$  

**Definition 9.1.** A quantum type cosimplicial DG Lie algebra is a cosimplicial DG Lie algebra $g = \{g^p\}_{p\in \mathbb{N}}$ such that each $g^p$ is a quantum type DG Lie algebra.

In short, the condition is that $g^{p,i} = 0$ when $i < -1$.

Recall the notions of MC elements and gauge groups from Section 7.

**Definition 9.2.** Let $K$ be a field of characteristic $0$. Suppose $g$ is a quantum type cosimplicial DG Lie algebra, and $m$ is a parameter ideal, both over $K$. By an additive descent datum in $m \hat{\otimes}_K g$ we mean a triple of elements

$$\delta = (\delta^0,\delta^2,\delta^2),$$  

where

$$\delta^q \in m \hat{\otimes}_K g^{q,1-q},$$  

that satisfy the following conditions.

(i) The element $\beta := \delta^0 \in m \hat{\otimes}_K g^{0,1}$ is an MC element in the DG Lie algebra $m \hat{\otimes}_K g^0$.

(ii) Consider the vertices $(0),(1)$ in $\Delta^1$, and the elements

$$\beta^{(0)},\beta^{(1)} \in m \hat{\otimes}_K g^{1,1},$$  

which are MC elements in the DG Lie algebra $m \hat{\otimes}_K g^1$. Also consider the group element

$$g := \exp(\delta^1) \in \exp(m \hat{\otimes}_K g^{1,0}).$$
Figure 2. Illustration of conditions (ii) and (iii) in Definition 9.2.

The condition is that
\[
\exp(\alpha f)(\beta|^{(0)}) = \beta|^{(1)}
\]
in \( m \hat{\otimes}_K g^{1,1} \). See Figure 2.

(iii) Consider the vertex \( (0) \) in \( \Delta^2 \), and the corresponding MC element
\( \beta|^{(0)} \in m \hat{\otimes}_K g^{2,1} \).

There are 1-dimensional faces \( (0,1), (1,2), (0,2) \) in \( \Delta^2 \), and corresponding group elements
\( g|^{(0,1)}, g|^{(1,2)}, g|^{(0,2)} \in \exp(m \hat{\otimes}_K g^{2,0}) \).

The MC element \( \beta|^{(0)} \) determines a Lie algebra structure \( (m \hat{\otimes}_K g^{2,1})_{\beta|^{(0)}} \) on the \( K \)-module \( m \hat{\otimes}_K g^{2,1} \), and a group homomorphism
\( \exp(d_{\beta|^{(0)}}) : \exp(m \hat{\otimes}_K g^{2,1})_{\beta|^{(0)}} \to \exp(m \hat{\otimes}_K g^{2,0}) \).

Define the group element
\( a := \exp_{\beta|^{(0)}}(\delta^2) \in \exp(m \hat{\otimes}_K g^{2,1})_{\beta|^{(0)}} \).

The condition is that
\[
(g|^{(0,2)})^{-1} \cdot g|^{(1,2)} \cdot g|^{(0,1)} = \exp(d_{\beta|^{(0)}})(a)
\]
in the group \( \exp(m \hat{\otimes}_K g^{2,0}) \). See Figure 2.

(iv) (tetrahedron) Here we consider faces of \( \Delta^3 \) of dimensions 2, 1, 0. For any \( 0 \leq i < j < k \leq 3 \) there is a group element
\( a|^{(i,j,k)} \in \exp(m \hat{\otimes}_K g^{2,1})_{\beta|^{(0)}} \).

There is a group element
\( g|^{(0,1)} \in \exp(m \hat{\otimes}_K g^{3,0}) \),
and it induces a group isomorphism
\( \exp(\text{ad})(g|^{(0,1)}) : \exp(m \hat{\otimes}_K g^{3,1})_{\beta|^{(0)}} \cong \exp(m \hat{\otimes}_K g^{3,0})_{\beta|^{(1)}} \).

The condition is that
\[
(a|^{(0,1,3)})^{-1} \cdot a|^{(0,2,3)} \cdot a|^{(0,1,2)} = \exp(\text{ad})(g|^{(0,1)})^{-1}(a|^{(1,2,3)})
\]
in the group \( \exp(m \hat{\otimes}_K g^{3,1})_{\beta|^{(0)}} \). See Figure 3.

The set of additive descent data is denoted by ADD(\( m \hat{\otimes}_K g \)).
Figure 3. Illustration of condition (iv) in Definition 9.2, showing data on 3 faces of the tetrahedron. The remaining face (0, 1, 2) is shown in Figure 2.

Definition 9.3. Let $\delta = (\delta^0, \delta^1, \delta^2)$ and $\delta' = (\delta'^0, \delta'^1, \delta'^2)$ be additive descent data in $m \hat{\otimes} K g$. A twisted gauge transformation $\delta \rightarrow \delta'$ is a pair of elements $(\epsilon^0, \epsilon^1)$, with $\epsilon^q \in m \hat{\otimes} K g^{q-q}$, satisfying conditions (i)-(iii) below. We use the notation $\beta, g, a$ of definition 9.2, as well as $\beta', g', a'$, where $\beta' := \delta'^0$ etc. We also let

$$h := \exp(\epsilon^0) \in \exp(m \hat{\otimes} g^{0,0})$$

and

$$b := \exp_{\beta[0]}(\epsilon^1) \in \exp(m \hat{\otimes} g^{1,-1})_{\beta[0]}.$$

These are the conditions:

(i) There is equality $\exp(af)(h)(\beta) = \beta'$ in the set $m \hat{\otimes} g^{1,1}$.

(ii) There is equality $g' \cdot h|_{(0)} = h|^{(1)} \cdot g \cdot \exp(d_{\beta[0]})(b)$ in the group $\exp(m \hat{\otimes} g^{1,0})$.

(iii) There is equality $b|_{(0,1)} \cdot \exp(af)(g'|_{(0,1)})^{-1} b|^{(1,2)} \cdot a \cdot (b|_{(0,2)})^{-1} = \exp(af)(h|_{(0)})^{-1}(a')$ in the group $\exp(m \hat{\otimes} g^{2,-1})_{\beta[0]}$.

Proposition 9.4. Twisted gauge transformations form an equivalence relation on the set $\text{ADD}(m \hat{\otimes} g)$.

Proof. Same calculation as in the proof of Proposition 6.7. □

We denote by $\text{ADD}(m \hat{\otimes} g)$ the set of equivalence classes. A variant of Remark 6.8 applies to $\text{ADD}(m \hat{\otimes} g)$; cf. also Theorem 9.10 below.

Proposition 9.5. The sets $\text{ADD}(m \hat{\otimes} g)$ and $\text{ADD}(m \hat{\otimes} g)$ are functorial in $m$ and $g$.

Proof. Suppose $\sigma : m \rightarrow m'$ is a homomorphism of parameter ideals, and $\tau : g \rightarrow g'$ is a homomorphism of cosimplicial quantum type DG Lie algebras. There is an induced homomorphism of cosimplicial DG Lie algebras

$$\sigma \otimes \tau : m \hat{\otimes} g \rightarrow m' \hat{\otimes} g'.$$
Given $\delta = (\delta^0, \delta^2, \delta^2) \in \text{ADD}(m \otimes_k g)$, let
\[
\delta'^q := (\sigma \otimes \tau)(\delta^q) \in m' \otimes_k g'^{q, 1-q}
\]
and $\delta' := (\delta'^0, \delta'^1, \delta'^2)$. It is easy to see that $\delta' \in \text{ADD}(m' \otimes_k g')$.

Similarly for twisted gauge transformations.

\[\square\]

Suppose $X$ is a topological space. By an ordered open covering of $X$ we mean an open covering $U = \{U_k\}_{k \in K}$ in which the set $K$ is ordered. For a natural number $p$ we denote by $\Delta_p(K)$ the set of order preserving functions $k : \{0, \ldots, p\} \to K$. In other words, $k = (k_0, \ldots, k_p)$ with $k_0 \leq \cdots \leq k_p$. Given $\alpha \in \Delta_q^p$ we write $\alpha(k) := k \circ \alpha \in \Delta_q(K)$. Thus the collection $\{\Delta_p(K)\}_{p \in \mathbb{N}}$ is a simplicial set.

Suppose $\mathcal{G}$ is a sheaf of sets on $X$. Let us recall how to construct the associated Čech cosimplicial set $C(U, \mathcal{G})$. For $p \in \mathbb{N}$ we take the set
\[
C^p(U, \mathcal{G}) := \prod_{k \in \Delta_p(K)} \Gamma(U_k, \mathcal{G}).
\]
If $k \in \Delta_q^p(K)$ and $\alpha \in \Delta_q^p$, then there is an inclusion of open sets $U_k \subset U_{\alpha(k)}$, and by restriction there is an induced function $\alpha : C^p(U, \mathcal{G}) \to C^q(U, \mathcal{G})$.

Now suppose $U' = \{U_k\}_{k \in K'}$ is another ordered open covering of $X$, and $\rho : U' \to U$ is an ordered refinement, namely $\rho : K' \to K$ is an order preserving function such that $U_k \subset U_{\rho(k)}$ for all $k \in K'$. Then there is a map of cosimplicial sets
\[
\rho^* : C(U, \mathcal{G}) \to C(U', \mathcal{G}),
\]
with the obvious rule.

**Remark 9.7.** Ordered open coverings have the advantage that the associated Čech cosimplicial sets are smaller than the ones gotten from unordered coverings. This is the benefit of the “broken symmetry” imposed by the ordering of the index sets.

The disadvantage is that usually two ordered open coverings $U, U'$ of a space $X$ do not admit a common ordered refinement; namely there does not exist an ordered open covering $U''$, and ordered refinements $U' \to U$ and $U'' \to U'$.

However, any two ordered open coverings $U, U'$ can be effectively compared as follows. Say $U = \{U_k\}_{k \in K}$ and $U' = \{U'_k\}_{k \in K'}$. Define the ordered set $K'' := K \sqcup K'$, and the ordered open covering $U'' := \{U''_k\}_{k \in K''}$, where $U''_k := U_k$ if $k \in K$, and $U''_k := U'_k$ if $k \in K'$. We refer to $U''$ as the concatenation of $U$ and $U''$. There are obvious ordered refinements $U \to U''$ and $U' \to U''$.

Regarding Čech cohomology of a sheaf of abelian groups, the result is the same whether ordered or unordered coverings are used; cf. [Ha, Remark III.4.0.1].

If $\mathcal{G}$ is a sheaf of DG Lie algebras on $X$, then by letting
\[
g^{p,i} := C^p(U, \mathcal{G}^i)
\]
we obtain a cosimplicial DG Lie algebra $g$. An ordered refinement $\rho : U' \to U$ gives rise to a homomorphism of cosimplicial DG Lie algebras
\[
\rho^* : C(U, \mathcal{G}) \to C(U', \mathcal{G}).
\]

Consider the following setup:

**Setup 9.9.** $\mathbb{K}$ is a field of characteristic 0; $(R, m)$ is a parameter algebra over $\mathbb{K}$; and $X$ is a smooth algebraic variety over $\mathbb{K}$, with structure sheaf $\mathcal{O}_X$.

There are sheaves of DG Lie algebras $T_{\text{poly}, X}$, $\mathcal{D}_{\text{poly}, X}$ and $\mathcal{D}_{\text{poly}, X}^{\text{nor}}$ on $X$. The sheaves $T_{\text{poly}, X}^p$ are coherent $\mathcal{O}_X$-modules, and the sheaves $\mathcal{D}_{\text{poly}, X}^p$ and $\mathcal{D}_{\text{poly}, X}^{\text{nor}, p}$ are quasi-coherent $\mathcal{O}_X$-modules. The differentials of these DG Lie algebras are
For any affine open set $U = \text{Spec } C \subset X$ one has
\[
\Gamma(U, \mathcal{T}_{\text{poly}}) = \mathcal{T}_{\text{poly}}(C),
\]
and
\[
\Gamma(U, \mathcal{D}_{\text{poly}}) = \mathcal{D}_{\text{poly}}(C).
\]

See [Ye1, Proposition 3.18].

**Theorem 9.10.** Let $(R, m)$ and $X$ be as in Setup [9.9] and let $U$ be a finite affine ordered open covering of $X$. We consider two cases:

(*) The associative case, in which
\[
\mathcal{P}(R, X) := \text{AssDef}(R, \mathcal{O}_X) \quad \text{and} \quad \mathfrak{g}(U) := \mathcal{C}(U, \mathcal{D}_{\text{poly}, X}).
\]

(*) The Poisson case, in which
\[
\mathcal{P}(R, X) := \text{PoisDef}(R, \mathcal{O}_X) \quad \text{and} \quad \mathfrak{g}(U) := \mathcal{C}(U, \mathcal{T}_{\text{poly}, X}).
\]

In either case there is a function
\[
\exp : \text{ADD}(m \otimes_K \mathfrak{g}(U)) \to \text{MDD}(\mathcal{P}(R, X), U),
\]
which respects twisted gauge equivalences, and induces a bijection of sets
\[
\exp : \text{ADD}(m \otimes_K \mathfrak{g}(U)) \to \text{MDD}(\mathcal{P}(R, X), U).
\]

Moreover, if $U'$ is another finite affine ordered open covering, $\rho : U' \to U$ is an ordered refinement, and $\sigma : (R, m) \to (R', m')$ is a homomorphism of parameter algebras, then the diagram
\[
\begin{array}{ccc}
\text{ADD}(m \otimes_K \mathfrak{g}(U)) & \xrightarrow{\exp} & \text{MDD}(\mathcal{P}(R, X), U) \\
\downarrow{\sigma \otimes \rho^*} & & \downarrow{\sigma \circ \rho^*} \\
\text{ADD}(m' \otimes_K \mathfrak{g}(U')) & \xrightarrow{\exp} & \text{MDD}(\mathcal{P}(R', X), U')
\end{array}
\]
is commutative. Here the left vertical arrow is a combination of [9.8] and Proposition [9.5]. The right vertical arrow is a combination of [6.15] and [6.14].

**Proof.** Let us denote by $\mathcal{G}$ the sheaf of DG Lie algebras in either case, and let $U = \{U_k\}_{k \in K}$. Suppose $\delta = (\delta^0, \delta^1, \delta^2)$ is an additive descent datum. Consider $\delta^0$. We know that
\[
\delta^0 = \{\beta_k\}_{k \in K} \in m \otimes_K \prod_{k \in K} \Gamma(U_k, \mathcal{G}^1).
\]

For any $k$ the element $\beta_k$ is an MC element in $m \otimes_K \Gamma(U_k, \mathcal{G})$; so it determines a differential star product (or formal Poisson bracket, as the case may be) on the sheaf $\mathcal{A}_k := R \otimes_K \mathcal{O}_{U_k}$. This is a deformation of $\mathcal{O}_{U_k}$.

Next consider $\delta^1$. We have
\[
\delta^1 = \{\gamma_{k_0, k_1}\} \in \Gamma(U_{k_0, k_1}, \mathcal{G}^0).
\]

For $k_0 < k_1$ let
\[
g_{k_0, k_1} := \exp(\text{ad}(\gamma_{k_0, k_1})),
\]
which is a gauge transformation $\mathcal{A}_{k_0} |_{U_{k_0, k_1}} \to \mathcal{A}_{k_1} |_{U_{k_0, k_1}}$. For $k_0 > k_1$ we take $g_{k_0, k_1} := g^{-1}_{k_1, k_0}$. And we take $g_{k, k}$ to be the identity automorphism of $\mathcal{A}_k$.

Lastly consider $\delta^2$. We have
\[
\delta^2 = \{\alpha_{k_0, k_1, k_2}\} \in \prod_{(k_0, k_1, k_2) \in \Delta_2(K)} \Gamma(U_{k_0, k_1, k_2}, \mathcal{G}^{-1}).
\]
For $k_0 < k_1 < k_2$ let

$$a_{k_0,k_1,k_2} := \exp_{\beta_{k_0}}(a_{k_0,k_1,k_2});$$

this is an element of $\Gamma(U_{k_0,k_1,k_2}, \mathcal{I}_G(\mathcal{A}_{k_0}))$. For other triples $(k_0, k_1, k_2)$ we define $a_{k_0,k_1,k_2}$ so as to preserve normalization. This is possible because of conditions (iii), (iv) of Definition 9.2.

We have thus constructed a datum

$$(9.11) \quad d := \{(\mathcal{A}_k)_{k \in K}, \{g_{k_0,k_1}\}_{k_0,k_1 \in K}, \{a_{k_0,k_1,k_2}\}_{k_0,k_1,k_2 \in K}\}.$$

It is straightforward to verify that $d$ is indeed a multiplicative decent datum. It is also clear that the function $\exp : \delta \mapsto d$ depends functorially on $R$ and $U$.

A twisted gauge transformation $(\epsilon^0, \epsilon^1) : \delta \rightarrow \delta'$ determines a twisted gauge transformation

$$\{(h_k), (b_{k_0,k_1})\} : \exp(\delta) \rightarrow \exp(\delta'),$$

where $h_k$ is defined like $g_{k_0,k_1}$ above, and $b_{k_0,k_1}$ is defined like $a_{k_0,k_1,k_2}$. All twisted gauge transformations $\exp(\delta) \rightarrow \exp(\delta')$ arise this way. Therefore we get an injection

$$(9.12) \quad \exp : \text{ADD}(m \otimes_k g(U)) \rightarrow \text{MDD}(P(R, X), U).$$

It remains to prove that (9.12) is surjective. So let $d$ be a multiplicative descent datum as in (9.11), and assume we are in the associative case. According to Corollary 2.13 and Theorem 8.2, there is a gauge transformation $\mathcal{A}_k \cong (R \otimes_k \mathcal{O}_{U_k})_{\beta_k}$, where $\beta_k$ is some differential star product. By Theorem 8.5 the gauge transformation

$$g_{k_0,k_1} : \mathcal{A}_{k_0} |_{U_{k_0,k_1}} \rightarrow \mathcal{A}_{k_1} |_{U_{k_0,k_1}}$$

becomes a differential gauge transformation

$$\exp(\text{ad}(\gamma_{k_0,k_1})) : (R \otimes_k \mathcal{O}_{U_{k_0,k_1}})_{\beta_{k_0}} \rightarrow (R \otimes_k \mathcal{O}_{U_{k_0,k_1}})_{\beta_{k_1}}.$$

The elements $\alpha_{k_0,k_1,k_2}$ are obtained similarly. The result is an additive descent datum $\delta$ satisfying $\exp(\delta) = d$.

In the Poisson case the proof is similar, using Corollary 2.13 and Propositions 7.9 and 7.8. \qed

**Remark 9.13.** More generally, if $\mathcal{G}$ is any sheaf of quantum type DG Lie algebras on a topological space $X$, $U$ is an ordered open covering of $X$, $g(U) := C(U, \mathcal{G})$ and $\delta \in \text{ADD}(m \otimes_k g(U))$, then $\exp(\delta)$ is a multiplicative descent datum for a gerbe $\mathcal{H}$ on $X$. For an index $k \in K$ there is an object $k \in \text{Ob} \mathcal{H}(U_k)$, and its sheaf of automorphisms is $\mathcal{H}(k,k) = \exp(m \otimes_k \mathcal{G}^{-1})_{\beta_k}$.

It might be interesting to study the kind of gerbes that arise in this way.

10. Étale Morphisms

Suppose $g : X' \rightarrow X$ is a map of topological spaces, $U = \{U_k\}_{k \in K}$ is an ordered open covering of the space $X$, and $U' = \{U'_k\}_{k \in K'}$ is an ordered open covering of $X'$. A *morphism of ordered coverings extending $g$* is an order preserving function $\rho : K' \rightarrow K$, such that $g(U'_k) \subset U_{\rho(k)}$ for every $k \in K'$. We also say that $\rho : U' \rightarrow U$ is a morphism of ordered coverings. If $\mathcal{M}$ (resp. $\mathcal{M}'$) is a sheaf of abelian groups on $X$ (resp. $X'$), and $\phi : \mathcal{M} \rightarrow g_* \mathcal{M}'$ is a homomorphism of sheaves of groups on $X$, then there is an induced homomorphism of cosimplicial abelian groups

$$C(\rho, \phi) : C(U, \mathcal{M}) \rightarrow C(U', \mathcal{M}').$$

See [9.6] for our conventions regarding cosimplicial sets.

Let us return to the algebro-geometric setup, namely to Setup 9.11. Suppose $g : X' \rightarrow X$ is an étale morphism between smooth varieties over $\mathbb{K}$. It follows from
[Ye2] Proposition 4.6] that there are induced homomorphisms of sheaves of DG Lie algebras
\[ g^* : \mathcal{T}_{\text{poly},X} \to g_* \mathcal{T}_{\text{poly},X'} \]
and
\[ g^* : \mathcal{D}_{\text{poly},X}^{\text{nor}} \to g_* \mathcal{D}_{\text{poly},X'}^{\text{nor}} \]
on \( X \), extending the ring homomorphism \( g^* : \mathcal{O}_X \to g_* \mathcal{O}_{X'} \). Given an ordered finite affine open covering \( U \) (resp. \( U' \)) of \( X \) (resp. \( X' \)), and a morphism of ordered coverings \( \rho : U' \to U \) extending \( g \), there is an induced homomorphism of cosimplicial DG Lie algebras \( \rho^* : \mathcal{G}(U) \to \mathcal{G}(U') \). Here we use the notation of Theorem 9.10 with obvious modifications; e.g. \( g(U') := C(U', \mathcal{T}_{\text{poly},X'}) \) in the Poisson case.

The first order bracket of a twisted deformation was defined in Definition 5.16.

**Theorem 10.1.** Let \( \mathbb{K} \) be a field of characteristic 0, let \( g : X' \to X \) be an étale morphism between smooth algebraic varieties over \( \mathbb{K} \), and let \( \sigma : (R, \mathfrak{m}) \to (R', \mathfrak{m}') \) be a homomorphism between parameter \( \mathbb{K} \)-algebras. We use the notation of Theorem 9.10 with obvious modifications pertaining to the variety \( X' \) and the algebra \( R' \).

Then, both in the Poisson case and in the associative case, there is a function
\[ \text{ind}_{\sigma,g} : \text{TwOb}(\mathcal{P}(R, X)) \to \text{TwOb}(\mathcal{P}(R', X')) \]
with these properties:

(i) Transitivity: if \( g' : X'' \to X' \) and \( \sigma' : R' \to R'' \) are other morphisms of the same kinds, then
\[ \text{ind}_{\sigma',\sigma \circ g g'} = \text{ind}_{\sigma', \sigma'} \circ \text{ind}_{\sigma, g} \]  

(ii) Suppose \( U \) (resp. \( U' \)) is an ordered finite affine open covering of \( X \) (resp. \( X' \)), and \( \rho : U' \to U \) is a morphism of ordered coverings extending \( g \). Then the diagram of sets
\[ \begin{array}{c}
\text{ADD}(m \otimes_{\mathbb{K}} g(U)) \xrightarrow{\text{exp}} \text{MDD}(\mathcal{P}(R, X), U) \xleftarrow{\text{dec}} \text{TwOb}(\mathcal{P}(R, X)) \\
\sigma \otimes \rho^* \downarrow \quad \quad \quad \downarrow \text{ind}_{\sigma,g}
\end{array} \]
\[ \begin{array}{c}
\text{ADD}(m' \otimes_{\mathbb{K}} g(U')) \xrightarrow{\text{exp}} \text{MDD}(\mathcal{P}(R', X'), U') \xleftarrow{\text{dec}} \text{TwOb}(\mathcal{P}(R', X'))
\end{array} \]
is commutative. Here \( \text{dec} \) is the decomposition of Corollary 6.16.

(iii) If \( X' = U \) is an open set of \( X \), and \( g : U \to X \) is the inclusion, then for any \( \mathcal{A} \in \text{TwOb}(\mathcal{P}(R, X)) \) there is an isomorphism
\[ \text{ind}_{\sigma,g}(\mathcal{A}) \cong \text{ind}_{\sigma}(\mathcal{A}|_{U'}) \]
in \( \text{TwOb}(\mathcal{P}(R', U)) \). Here \( \mathcal{A}|_{U} \) is from Definition 5.15 and \( \text{ind}_{\sigma} \) is from Proposition 5.18.

(iv) The function \( \text{ind}_{\sigma,g} \) respects first order brackets.

What condition (iv) says is as follows. Say \( \mathcal{A} \in \text{TwOb}(\mathcal{P}(R, X)) \) and
\[ \mathcal{A}' := \text{ind}_{\sigma,g}(\mathcal{A}) \in \text{TwOb}(\mathcal{P}(R', X')). \]
Then for local sections \( c_1, c_2 \in \mathcal{O}_X \) one has
\[ (\sigma \otimes g^*)(\{c_1, c_2\} \mathcal{A}) = \{g^*(c_1), g^*(c_2)\} \mathcal{A}' \]
as local sections of \( (m'/m'^2) \otimes_{\mathbb{K}} \mathcal{O}_{X'} \).

Note that since the horizontal arrows in property (ii) of the theorem are bijections, this property completely determines the function \( \text{ind}_{\sigma,g} \).
Proof. Choose coverings $U$ and $U'$, and a morphism of coverings $\rho : U' \to U$, as in property (ii). This is possible of course. Define

$$\text{ind}_{\sigma,g} : \text{TwOb}(\mathcal{P}(R,X)) \to \text{TwOb}(\mathcal{P}(R',X'))$$

to be the unique function that makes the diagram commute.

To prove properties (i)-(ii) it suffices to show that the function $\text{ind}_{\sigma,g}$ is independent of the choices made above. So let $V$ be another such covering of $X$, let $V'$ be another such covering of $X'$, and let $\theta : V' \to V$ be a morphism of ordered coverings extending $g$. As explained in Remark 9.7, we may assume that there is a commutative diagram of morphisms of ordered coverings

$$
\begin{array}{ccc}
U' & \xrightarrow{\rho} & U \\
\downarrow{\tau'} & & \downarrow{\tau} \\
V' & \xrightarrow{\theta} & V
\end{array}
$$

Let us consider a rectangular three dimensional diagram. Its rear face is this:

$$
\begin{array}{ccc}
\text{ADD}(m \otimes_{K} g(V)) & \xrightarrow{\exp} & \text{MDD}(\mathcal{P}(R,X), V) \\
\downarrow{1_{m} \otimes \tau'} & & \downarrow{1_{m} \otimes \tau} \\
\text{ADD}(m \otimes_{K} g(U)) & \xrightarrow{\exp} & \text{MDD}(\mathcal{P}(R,X), U)
\end{array}
\longleftrightarrow
\begin{array}{c}
\text{TwOb}(\mathcal{P}(R,X))
\end{array}
$$

The front face is the same as the front, but with the replacements $X \rightsquigarrow X'$, $R \rightsquigarrow R'$ etc. The left face is

$$
\begin{array}{ccc}
\text{ADD}(m \otimes_{K} g(V)) & \xrightarrow{\sigma \otimes \theta^*} & \text{ADD}(m' \otimes_{K} g(V')) \\
\downarrow{1_{m} \otimes \tau'} & & \downarrow{1_{m'} \otimes \tau'^*} \\
\text{ADD}(m \otimes_{K} g(U)) & \xrightarrow{\sigma \otimes \rho^*} & \text{ADD}(m' \otimes_{K} g(U'))
\end{array}
$$

The bottom face is the diagram in property (ii), and the top is the same, but with the replacement $U \rightsquigarrow V$. By Corollary 9.11 and Theorem 9.11, the front and rear diagrams are commutative. The functoriality of ADD (Proposition 9.5) implies that the left face is commutative. By definition the top and bottom diagrams are commutative. It follows that the right face is commutative; and hence the functions “$\text{ind}_{\sigma,g}$” determined by $\rho$ and $\theta$ are the same.

Property (iii) is obvious from the construction of $\text{ind}_{\sigma,g}$.

Finally let’s prove property (iv). Take a twisted deformation $\mathcal{A}$ on $X$, and let $\mathcal{A}' := \text{ind}_{\sigma,g}(\mathcal{A})$. Say $U = \{U_k\}_{k \in K}$ and $U' = \{U'_k\}_{k' \in K'}$. Choose an index $k' \in K'$, and let $k := \rho(k') \in K$. Write $C := \Gamma(U_k, \mathcal{O}_X)$ and $C' := \Gamma(U'_k, \mathcal{O}_{X'})$; so we have an étale ring homomorphism $g^* : C \to C'$. The first order bracket $\{-,-\}_{\mathcal{A}}$ is encoded locally (on the open set $U_k$) by an MC element $\beta \in m \otimes_{K} \mathcal{T}_{poly}(C)$; and its action on $R \otimes_{K} C = R \otimes_{K} \mathcal{T}_{poly}^{-1}(C)$ is via the Lie bracket. Now we have a homomorphism of DG Lie algebras

$$
g^* : m \otimes_{K} \mathcal{T}_{poly}(C) \to m' \otimes_{K} \mathcal{T}_{poly}(C').
$$

Let $\beta' := g^*(\beta) \in m' \otimes_{K} \mathcal{T}_{poly}^{-1}(C')$. Then the first order bracket of $\mathcal{A}'$ is encoded by $\beta'$, and we see that equation (12.2) holds.

The same line of reasoning gives a similar result for usual deformations, which we state a bit loosely:
Theorem 10.3. In the situation of Theorem 10.1, let us write $P(R, X)$ for either $\text{AssDef}(R, \mathcal{O}_X)$ or $\text{PoisDef}(R, \mathcal{O}_X)$, as the case may be. Also let us write $P(R', X')$ for the corresponding set of $R'$-deformations of $\mathcal{O}_{X'}$. Then there is a function

$$\text{ind}_{\sigma, g} : \text{Ob}(P(R, X)) \to \text{Ob}(P(R', X')),$$

enjoying the obvious analogues of properties (i)-(iv) in Theorem 10.1.

Proof. Let $U$ be some finite ordered affine open covering of $X$. Define $\text{MDD}_1(P(R, X), U)$ to be the subset of $\text{MDD}(P(R, X), U)$ consisting of multiplicative descent data

$$d = (\{A_k\}, \{g_{k_0, k_1}\}, \{a_{k_0, k_1, k_2}\})$$

(cf. Definition 6.4) such that $a_{k_0, k_1, k_2} = 1$. We consider gauge transformations between elements of $\text{MDD}_1(P(R, X), U)$; these are the twisted gauge transformations $\left(\{b_k\}, \{b_{k_0, k_1}\}\right)$ of Definition 6.5 such that $b_{k_0, k_1} = 1$. We get an equivalence relation on $\text{MDD}_1(P(R, X), U)$, and the quotient is denoted by $\text{MDD}_1(P(R, X), U)$.

Because of descent for sheaves (gluing a sheaf $\mathcal{A} \in \text{Ob}(P(R, X))$ from data on the open covering) we get a canonical bijection

$$\text{MDD}_1(P(R, X), U) \cong \text{Ob}(P(R, X)).$$

Next let $\text{ADD}_1(m \hat{\otimes}_K g(U))$ to be the subset of $\text{ADD}(m \hat{\otimes}_K g(U))$ consisting of additive descent data $(\delta^0, \delta^2)$ (cf. Definition 9.2) such that $\delta_2 = 0$. We consider gauge transformations between elements of $\text{ADD}_1(m \hat{\otimes}_K g(U))$: these are the twisted gauge transformations $(\epsilon^0, \epsilon^1)$ of Definition 9.2 such that $\epsilon^1 = 0$. We get an equivalence relation on $\text{ADD}_1(m \hat{\otimes}_K g(U))$, and the quotient is denoted by $\text{ADD}_1(m \hat{\otimes}_K g(U))$.

Like in Theorem 9.10 there is a canonical bijection

$$\exp : \text{ADD}_1(m \hat{\otimes}_K g(U)) \cong \text{MDD}_1(P(R, X), U).$$

From here we can proceed like in the proof of Theorem 10.1. □

11. Commutative Cochains

Let $K$ be a field of characteristic 0. For $q \in \mathbb{N}$ we denote by $\Delta^q_K$ the $q$-dimensional geometric simplex over $K$. This is the affine scheme

$$\Delta^q_K := \text{Spec} \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1),$$

where $t_0, \ldots, t_q$ are variables. The collection $\{\Delta^q_K\}_{q \in \mathbb{N}}$ is a cosimplicial scheme. If $K = \mathbb{R}$, then the set $\Delta^q_K(\mathbb{R}_{\geq 0})$ of $\mathbb{R}$-points with nonnegative coordinates is the usual realization of the combinatorial simplex $\Delta^q$.

Let $X = \text{Spec} \mathcal{C}$ be an affine $\mathbb{K}$-scheme. We denote by $\Omega_X^p$ the sheaf of differential $p$-forms on $X$ (relative to $\mathbb{K}$), and we write

$$\Omega^p(X) := \Gamma(X, \Omega_X^p) = \Omega^p_X.$$ 

The direct sum

$$\Omega(X) := \bigoplus_{p \geq 0} \Omega^p(X)$$

is a super-commutative DG algebra (the de Rham complex). A morphism $f : X \to Y$ of affine schemes gives rise to a homomorphism of DG algebras $f^* : \Omega(X) \to \Omega(Y)$. In this way we obtain the simplicial DG algebra $\{\Omega(\Delta^q_K)\}_{q \in \mathbb{N}}$.

We denote by

$$\int_{\Delta^q_K} : \Omega^q(\Delta^q_K) \to \mathbb{K}$$

for
the $\mathbb{K}$-linear homomorphism that for forms defined over $\mathbb{Q}$ coincides with integration on the compact manifold $\Delta^q_{\mathbb{R}_{>0}}$. E.g.

$$\int_{\Delta^q} dt_1 \wedge \cdots \wedge dt_q = \frac{1}{q!}.$$

Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial $\mathbb{K}$-module. For $p \in \mathbb{N}$ we define $\hat{N}^p M$ to be the subset of

$$\prod_{i \in \mathbb{N}} (\Omega^p(\Delta^k_{\mathbb{K}}) \otimes_{\mathbb{K}} M^i)$$

consisting of the sequences $\{u_i\}_{i \in \mathbb{N}}$, $u_i \in \Omega^p(\Delta^k_{\mathbb{K}}) \otimes_{\mathbb{K}} M^i$, such that

$$(1 \otimes \alpha)(u_k) = (\alpha \otimes 1)(u_i) \in \Omega^p(\Delta^k_{\mathbb{K}}) \otimes_{\mathbb{K}} M^i$$

for every $\alpha \in \Delta^k_{\mathbb{K}}$.

The de Rham differential induces a differential $\hat{N}^p M \to \hat{N}^{p+1} M$. The resulting DG $\mathbb{K}$-module $\hat{N}M := \bigoplus_p \hat{N}^p M$ is called the Thom-Sullivan normalization of $M$, or the complex of commutative cochains of $M$. In this way we get a functor

$$\hat{N} : \text{Cosimp Mod } \mathbb{K} \to \text{DGMod } \mathbb{K}.$$

Now let $\mathfrak{g} = \{\mathfrak{g}^p\}_{p \in \mathbb{N}}$ be a cosimplicial DG Lie algebra. Thus for every $p$ there is a DG Lie algebra $\mathfrak{g}^p = \bigoplus_{q \in \mathbb{Z}} \mathfrak{g}^{p,q}$. And for every $q$ there is a cosimplicial $\mathbb{K}$-module $\mathfrak{g}^{p,q} := \{\mathfrak{g}^{p,q}\}_{p \in \mathbb{N}}$. Let $\hat{N}^p \mathfrak{g} := \hat{N}^p \mathfrak{g}^{p,\cdot}$, which is a $\mathbb{K}$-module. Next let $\hat{N}^1 \mathfrak{g} := \bigoplus_{p+q = 1} \hat{N}^p \mathfrak{g}$ and $\hat{\mathfrak{g}} := \bigoplus_i \hat{N}^i \mathfrak{g}$. The latter is a DG Lie algebra. If $\mathfrak{g}$ is a quantum type cosimplicial DG Lie algebra, then $\hat{\mathfrak{g}}$ is a quantum type DG Lie algebra. See [Ye7] Section 4.1 for details.

Suppose $\mathfrak{g}$ is a quantum type cosimplicial DG Lie algebra, and $\mathfrak{m}$ is a parameter ideal (Definition 11.1). Let $\beta \in \mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{N}^1 \mathfrak{g}$. Then

$$\beta = \beta^0 + \beta^1 + \beta^2,$$

with $\beta^p \in \mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{N}^{1-p} \mathfrak{g}$. Using the inclusion

$$\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{N}^{1-p} \mathfrak{g} \subset \prod_{i \in \mathbb{N}} (\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \Omega^p(\Delta^1_{\mathbb{K}}) \hat{\otimes}_{\mathbb{K}} \mathfrak{g}^{1-1-p})$$

we can express $\beta^p$ as a sequence $\beta^p = \{\beta^p_i\}_{i \in \mathbb{N}}$, with

$$(11.1) \quad \beta^p_i \in \mathfrak{m} \hat{\otimes}_{\mathbb{K}} \Omega^p(\Delta^1_{\mathbb{K}}) \hat{\otimes}_{\mathbb{K}} \mathfrak{g}^{1-1-p}.$$

The next result, which is crucial for this paper, is proved in [Ye7]:

**Theorem 11.2 (Ye7).** Let $\mathbb{K}$ be a field of characteristic 0, let $\mathfrak{g}$ be a quantum type cosimplicial DG Lie algebra over $\mathbb{K}$, and let $\mathfrak{m}$ be a parameter ideal over $\mathbb{K}$. Then there is a function

$$\int : \text{MC}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{\mathfrak{g}}) \to \text{ADD}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \mathfrak{g})$$

with these properties:

(i) The function $\int$ is functorial in $\mathfrak{g}$ and $\mathfrak{m}$. Namely if $\sigma : \mathfrak{m} \to \mathfrak{m}'$ is a homomorphism of parameter ideals, and $\tau : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of cosimplicial DG Lie algebras, then the diagram

$$\begin{array}{ccc}
\text{MC}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{\mathfrak{g}}) & \xrightarrow{\int} & \text{ADD}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \mathfrak{g}) \\
\sigma \circ \tau & \downarrow & \sigma \circ \tau \\
\text{MC}(\mathfrak{m}' \hat{\otimes}_{\mathbb{K}} \hat{\mathfrak{g}}') & \xrightarrow{\int} & \text{ADD}(\mathfrak{m}' \hat{\otimes}_{\mathbb{K}} \mathfrak{g}')
\end{array}$$

is commutative.

(ii) The function $\int$ respects twisted gauge equivalences, and induces a bijection

$$\int : \text{MC}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \hat{\mathfrak{g}}) \to \text{ADD}(\mathfrak{m} \hat{\otimes}_{\mathbb{K}} \mathfrak{g}).$$
(iii) Assume $m^2 = 0$. Let $\beta \in MC(m \hat{\otimes}_K N\mathfrak{g})$, and let $\beta^p$ be its components as in formula (11.1) above. Then

$$\text{int}(\beta) = (\delta^0, \delta^1, \delta^2),$$

where

$$\delta^0 := \beta^0_0 \in m \hat{\otimes}_K g^{0,1},$$

$$\delta^1 := \int_{\Delta^1} \beta^1_1 \in m \hat{\otimes}_K g^{1,0},$$

and

$$\delta^2 := \int_{\Delta^2} \beta^2_2 \in m \hat{\otimes}_K g^{2,-1}.$$

**Remark 11.3.** Here is a brief outline of the proof of Theorem 11.2.

In [Ye6] we develop a theory of nonabelian multiplicative integration on surfaces. This is done in the context of Lie crossed modules over $\mathbb{R}$ (cf. Remark 3.12). We construct a well-defined multiplicative integral of a connection-curvature pair, and prove a 3-dimensional Stoke’s Theorem. These results appear to be of independent interest for differential geometry.

In the paper [Ye7] we prove that for nilpotent Lie groups, the multiplicative integration of [Ye6] is algebraic. This, together with functoriality, allows us to treat pronilpotent Lie algebras over any field $K$ of characteristic 0. The resulting formulas are then used to construct the function $\text{int}$ in Theorem 11.2 with its functoriality. The hardest thing to verify is condition (iv) of Definition 9.2 (the tetrahedron axiom), and this turns out to be a consequence of the nonabelian 3-dimensional Stoke’s Theorem. Once we have the function $\text{int}$, proving properties (ii-iii) is not too hard, by induction on the length of $m$.

**Remark 11.4.** It should be noted that if one only wants to obtain a bijection $\text{int}$ as in property (ii) of Theorem 11.2 satisfying property (iii), then an inductive argument (similar to the proof of [BGNT, Proposition 3.3.1]) is sufficient. But we could not use this method alone to prove functoriality of $\text{int}$. Perhaps this could be deduced from [BGNT, Proposition 3.4.1]; however we did not understand the proof given there. See also the related paper [Gr2].

Let $X$ be a smooth algebraic variety over $K$, with ordered finite affine open covering $U = \{U_0, \ldots, U_m\}$. Let $\mathcal{M}$ be a sheaf of $\mathbb{K}$-modules on $X$. The Čech cosimplicial construction $C(U, \mathcal{M})$ from (9.6) can be sheafified. For a sequence $i = (i_0, \ldots, i_p) \in \Delta^m_p$ we denote by

$$g_i : U_i = U_{i_0} \cap \cdots \cap U_{i_p} \to X$$

the inclusion of this (affine) open set. We then define the sheaf

$$C^p_U(M) := \prod_{i \in \Delta^m_i} g_{i*} g^{-1}_i \mathcal{M}.$$  

(In the paper [Ye3] this sheaf was denoted by $C^p(U, \mathcal{M})$.) The collection $C_U(\mathcal{M}) := \{C^p_U(\mathcal{M})\}_{p \in \mathbb{N}}$ is a cosimplicial sheaf on $X$. Note that for any open set $V \subset X$ we get a cosimplicial $\mathbb{K}$-module

$$\Gamma(V, C_U(\mathcal{M})) := \{\Gamma(V, C^p_U(\mathcal{M}))\}_{p \in \mathbb{N}}.$$  

In particular, for $V = X$ we get

$$\Gamma(X, C_U(\mathcal{M})) = C(U, \mathcal{M})$$

as in (9.10).

There is a sheaf of DG $\mathbb{K}$-modules $\tilde{NC}_U(\mathcal{M})$ on $X$, such that

$$\Gamma(V, \tilde{NC}_U(\mathcal{M})) = \tilde{N} \Gamma(V, C_U(\mathcal{M}))$$
for any open set $V \subset X$. Moreover, there is a quasi-isomorphism of sheaves $\mathcal{M} \to \check{\text{NC}}_U(\mathcal{M})$, which is functorial in $\mathcal{M}$. We call $\check{\text{NC}}_U(\mathcal{M})$ the \textit{commutative Čech resolution of} $\mathcal{M}$. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, then globally this induces an isomorphism
\[(11.5) \quad R\Gamma(X, \mathcal{M}) \cong \Gamma(X, \check{\text{NC}}_U(\mathcal{M})) = \check{\text{NC}}(U, \mathcal{M})\]
in the derived category $D(\text{Mod} \mathbb{K})$. See [Ye3, Section 3].

On the variety $X$ one has the sheaf $\mathcal{P}_X$ of principal parts (the formal completion of $\mathcal{O}_{X \times X}$ along the diagonal), which we consider as an $\mathcal{O}_X$-bimodule. The sheaf $\mathcal{P}_X$ is equipped with the Grothendieck connection
\[\nabla : \mathcal{P}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X.\]
This connection gives rise to a sheaf of right DG $\mathcal{O}_X$-modules $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, there is a DG $\mathbb{K}$-module
\[\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}.\]
The Čech construction gives a cosimplicial sheaf of DG $\mathbb{K}$-modules $\mathcal{C}U(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$.

The \textit{mixed resolution} of $\mathcal{M}$ is by definition the complex of sheaves
\[\text{Mix}_U(\mathcal{M}) := \hat{\check{\text{NC}}}_U(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}),\]
together with the quasi-isomorphism $\mathcal{M} \to \text{Mix}_U(\mathcal{M})$. Here we use the complete variant $\hat{\check{N}}$ of the commutative cochain functor $\check{N}$, because of the adic topology on the sheaf $\mathcal{P}_X$; see [Ye3, Section 3] for details.

The mixed resolution factors through the commutative Čech resolution; so we get functorial quasi-isomorphisms of complexes of sheaves
\[(11.6) \quad \mathcal{M} \to \check{\text{NC}}_U(\mathcal{M}) \to \text{Mix}_U(\mathcal{M})\]
on $X$. Globally we obtain a quasi-isomorphism of complexes of $\mathbb{K}$-modules
\[(11.7) \quad \check{\text{NC}}(U, \mathcal{M}) \to \Gamma(X, \text{Mix}_U(\mathcal{M})).\]

The constructions above can be easily extended to the case when $\mathcal{M}$ is a bounded below complex of quasi-coherent sheaves, instead of a single sheaf, by totalizing double complexes. We still have the quasi-isomorphisms $(11.6)$, and globally there is a quasi-isomorphism $(11.7)$, and a derived category isomorphism $(11.5)$.

\textbf{Proposition 11.8.} Let $X$ be a smooth algebraic variety over $\mathbb{K}$, and let $U$ be a finite ordered affine open covering of $X$. Consider the commutative diagrams
\[T_{\text{poly}, X} \to \check{\text{NC}}_U(T_{\text{poly}, X}) \to \text{Mix}_U(T_{\text{poly}, X})\]
and
\[D^\text{nor}_{\text{poly}, X} \to \check{\text{NC}}_U(D^\text{nor}_{\text{poly}, X}) \to \text{Mix}_U(D^\text{nor}_{\text{poly}, X})\]
of quasi-isomorphisms of complexes of sheaves on $X$, gotten as instances of $(11.6)$. Then all the objects in these diagrams are sheaves of DG Lie algebras, and all the arrows are DG Lie algebra homomorphisms.

Note the similarity to [Ye1, Proposition 6.3].
Proof. Let $\mathcal{H}$ be some complete bounded below DG Lie algebra in $\text{Dir Inv Mod} \mathbb{K}_X$. This means that $\mathcal{H}$ is a sheaf of DG Lie algebras on $X$, with extra data consisting of filtrations, analogous to an adic topology; cf. [Ye3] Section 1. By [Ye3] Lemma 3.7 the complete commutative Čech resolution $\tilde{\mathcal{NC}_U(\mathcal{H})}$ has a structure of sheaf of DG Lie algebras, and the inclusion $\mathcal{H} \to \tilde{\mathcal{NC}_U(\mathcal{H})}$ is a DG Lie algebra quasi-isomorphism. This is functorial in $\mathcal{H}$.

Take $\mathcal{G}$ to be either $T_{\text{poly},X}$, $D^\text{mor}_{\text{poly},X}$ or $D_{\text{poly},X}$, which are all discrete as dir-inv modules. Then $\tilde{\mathcal{NC}_U(\mathcal{G})} = \tilde{\mathcal{NC}_U(\mathcal{G})}$, and we obtain the left portion of the diagrams in the proposition.

For the mixed resolutions things are more delicate. According to [Ye1] Proposition 5.4, the graded sheaf $\Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{G}$ is a complete DG Lie algebra in $\text{Dir Inv Mod} \mathbb{K}_X$ (no longer discrete – it has the adic topology of $\mathcal{P}_X$). And the canonical homomorphism

$$\mathcal{G} \to \Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{G}$$

is a DG Lie algebra homomorphism. Now we apply the functor $\tilde{\mathcal{NC}_U(-)}$ to obtain a DG Lie algebra quasi-isomorphism

$$\tilde{\mathcal{NC}_U(\mathcal{G})} = \tilde{\mathcal{NC}_U(\mathcal{G})} \to \tilde{\mathcal{NC}_U(\Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{G})} = \text{Mix}_U(\mathcal{G}).$$

\hfill \Box

Proposition 11.9. Suppose $g : X' \to X$ is an étale morphism of varieties, $U$ (resp. $U'$) is a finite ordered affine open covering of $X$ (resp. $X'$), and $\rho : U' \to U$ is a morphism of coverings extending $g$. Let us denote by $\mathcal{G}_X$ either of the sheaves $T_{\text{poly},X}$ or $D_{\text{poly},X}$ on $X$, and by $\mathcal{G}_X'$ the corresponding sheaf on $X'$. Then there is a homomorphism of sheaves of DG Lie algebras on $X$

$$\text{Mix}_\rho(g^*) : \text{Mix}_U(\mathcal{G}_X) \to g_* \text{Mix}_{U'}(\mathcal{G}_X').$$

This homomorphism extends the canonical homomorphism $g^* : \mathcal{G}_X \to g_* \mathcal{G}_X'$ from Section 10 and it is functorial in $(g, \rho)$.

Proof. There are canonical homomorphisms of sheaves $g^* : \mathcal{G}_X \to g_* \mathcal{G}_X'$, $g^* : \Omega_X \to g_* \Omega_X'$, and $g^* : P_X \to g_* P_X'$ on $X$. It remains to combine them and apply the cosimplicial operations. \hfill \Box

12. Twisted Deformation Quantization

In this section we state and prove the main result of the paper, namely Theorem 12.7. We work in the following setup:

Setup 12.1. $K$ is a field of characteristic $0$; $(R, m)$ is a parameter $K$-algebra (see Definition 12.1); and $X$ is a smooth algebraic variety over $K$, with structure sheaf $\mathcal{O}_X$.

Suppose $g$ and $h$ are DG Lie algebras. An $L_\infty$ morphism $\Psi : g \to h$ is a sequence $\Psi = \{\Psi_i\}_{i \geq 1}$ of $K$-multilinear functions $\Psi_i : \prod^i g \to h$, satisfying rather complicated equations (see [Ko1] or [Ye1] Definition 3.7). Here $\prod^i g$ denotes the $i$-th cartesian power. The homomorphism $\Psi_1 : g \to h$ is a homomorphism of DG $K$-modules, and it respects the Lie brackets up to the homotopy $\Psi_2$; and so on. Thus if $\Psi_i = 0$ for all $i \geq 2$, then $\Psi_1 : g \to h$ is a DG Lie algebra homomorphism.

If $\phi : g' \to g$ and $\theta : h \to h'$ are DG Lie algebra homomorphisms, then we get an $L_\infty$ morphism $\theta \circ \Psi \circ \phi : g' \to h'$, with components $\theta \circ \Psi_i \circ \prod^i (\phi)$.
Passing to extended DG Lie algebras, there is an induced $R$-multilinear $L_\infty$ morphism $\Psi_R = \{\Psi_{R,i}\}_{i \geq 1} : \mathfrak{m} \otimes K \mathfrak{g} \to \mathfrak{m} \otimes K \mathfrak{h}$, and a function $MC(\Psi_R) : MC(\mathfrak{m} \otimes K \mathfrak{g}) \to MC(\mathfrak{m} \otimes K \mathfrak{h})$ between MC sets, with explicit formula $MC(\Psi_R)(\beta) := \sum_{i \geq 1} \Psi_{R,i}(\beta, \ldots, \beta)$. If $\Psi$ is an $L_\infty$ quasi-isomorphism (namely if $\Psi_1$ is a quasi-isomorphism), then we get a bijection $MC(\Psi_R) : MC(\mathfrak{m} \otimes K \mathfrak{g}) \to MC(\mathfrak{m} \otimes K \mathfrak{h})$ on gauge equivalence classes. See [Kol] or [Ye1, Corollary 3.10].

Next let $G$ and $H$ be sheaves of DG Lie algebras on $X$. An $L_\infty$ morphism $\Psi : G \to H$ is a sequence $\Psi = \{\Psi_i\}_{i \geq 1}$ of $K$-multilinear sheaf morphisms $\Psi_i : \prod_i G \to H$, such that for any open set $U$ the sequence $\{\Gamma(U, \Psi_i)\}_{i \geq 1}$ is an $L_\infty$ morphism $\Gamma(U, G) \to \Gamma(U, H)$. We say that $\Psi$ is an $L_\infty$ quasi-isomorphism if $\Psi_1 : G \to H$ is a quasi-isomorphism of sheaves of $K$-modules.

Let $n$ be the dimension of $X$. Since $X$ is smooth, it is possible to find a finite ordered affine open covering $U_i = \{U_0, \ldots, U_m\}$, with an étale morphisms $s_i : U_i \to \mathbb{A}_K^n$. Let us write $s := \{s_0, \ldots, s_m\}$. We refer to $(U, s)$ succinctly as a covering with coordinates.

Suppose $g : X \to X'$ is an étale morphism of varieties, and we are given a covering with coordinates $(U', s')$ of $X'$. Say the covering $U'$ is indexed by $\{0, \ldots, m'\}$. A morphism of coverings with coordinates extending $g$ is a function $\rho : \{0, \ldots, m'\} \to \{0, \ldots, m\}$, such that $g(U'_i) \subset U_{\rho(i)}$ and $s_{\rho(i)} \circ g = s'_i$ for every $i \in \{0, \ldots, m'\}$. We indicate this morphism by $\rho : (U', s') \to (U, s)$.

The antisymmetrization homomorphism (also called the HKR map) $\nu : \mathcal{T}_{\text{poly}, X}^p \to \mathcal{D}_{\text{poly}, X}^p$ is defined as follows. For $p \geq 0$ we take $\nu(\xi_1 \wedge \cdots \wedge \xi_{p+1})(c_1, \ldots, c_{p+1}) := \frac{1}{(p+1)!} \sum_{\sigma} \text{sign}(\sigma) \xi_{\sigma(1)}(c_1) \cdots \xi_{\sigma(p+1)}(c_{p+1})$ for local sections $\xi_i \in \mathcal{T}_{\text{poly}, X}^0$ and $c_i \in \mathcal{O}_X$. The summation is over permutations $\sigma$ of the set $\{1, \ldots, p+1\}$. For $p = -1$ we let $\nu$ be the identity map of $\mathcal{O}_X$. According to [Ye2, Corollary 4.12] the homomorphism $\nu : \mathcal{T}_{\text{poly}, X} \to \mathcal{D}_{\text{poly}, X}$ is a quasi-isomorphism of sheaves of $\mathcal{O}_X$-modules. Note that $\nu$ does not respect the Lie brackets; but $H(\nu) : \mathcal{T}_{\text{poly}, X} \to H \mathcal{D}_{\text{poly}, X}$ is an isomorphism of sheaves of graded Lie algebras.

Recall the mixed resolutions $\mathcal{T}_{\text{poly}, X} \to \text{Mix}_U(\mathcal{T}_{\text{poly}, X})$ and $\mathcal{D}_{\text{poly}, X} \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X})$, which are quasi-isomorphisms of sheaves of DG Lie algebras (Proposition 11.8).

The next result is a slight improvement of [Ye1, Theorem 0.2]. A similar result is [VdB, Theorem 1.1].

**Theorem 12.3** ([Ye1, Theorem 0.2]). Let $X$ be a smooth algebraic variety over $K$, and assume $R \subset K$. Let $(U, s)$ be a covering with coordinates of $X$. Then:
(1) There is an $L_{\infty}$ quasi-isomorphism

$$\Psi_s = \{ \Psi_{s;i} \}_{i \geq 1} : \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X})$$

between sheaves of DG Lie algebras on $X$.

(2) The diagram of isomorphisms of sheaves of graded Lie algebras on $X$

$$\begin{array}{ccc}
\mathcal{T}_{\text{poly}, X} & \xrightarrow{H(\nu)} & \mathcal{D}_{\text{poly}, X} \\
\downarrow & & \downarrow \\
H \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) & \xrightarrow{H(\Psi_{s;1})} & H \text{Mix}_U(\mathcal{D}_{\text{poly}, X}),
\end{array}$$

in which the vertical arrows are the mixed resolutions, is commutative.

(3) Suppose $g : X \to X'$ is an étale morphism of varieties, $(U', s')$ is a covering with coordinates of $X'$, and $\rho : (U', s') \to (U, s)$ is a morphism of coverings with coordinates extending $g$. Then the diagram of $L_{\infty}$ morphisms on $X$

$$\begin{array}{ccc}
\text{Mix}_U(\mathcal{T}_{\text{poly}, X}) & \xrightarrow{\Psi_s} & \text{Mix}_U(\mathcal{D}_{\text{poly}, X}) \\
\text{Mix}_s(\rho_*) & \downarrow & \text{Mix}_s(\rho_*) \\
g_* \text{Mix}_{U'}(\mathcal{T}_{\text{poly}, X'}) & \xrightarrow{g_*(\Psi_{s'})} & g_* \text{Mix}_{U'}(\mathcal{D}_{\text{poly}, X'})
\end{array}$$

(cf. Proposition 11.9) is commutative.

Proof. Part (1) is the content of [Ye1] Theorem 0.2, which is repeated in greater detail as [Ye1] Erratum, Theorem 1.2. Part (3) is a direct consequence of the construction of the $L_{\infty}$ quasi-isomorphism $\Psi_s$ in the proof of [Ye1] Erratum, Theorem 1.2.

The idea for the proof of part (2) was communicated to us by M. Van den Bergh. Let $\mathcal{M}$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules. For $j \geq 0$ let

$$G^j \text{Mix}_U(\mathcal{M}) := \bigoplus_{i \geq j} \text{Mix}_U^i(\mathcal{M}).$$

This gives a decreasing filtration $G = \{ G^j \}_{j \geq 0}$ of $\text{Mix}_U(\mathcal{M})$ by subcomplexes. Note that $\text{gr}_G^p \text{Mix}_U(\mathcal{M}) = \text{Mix}_U^p(\mathcal{M})[-p]$ as complexes. The filtration $G$ gives rise to a convergent spectral sequence

$$E_1^{p,q}(\mathcal{M}) \Rightarrow H^{p+q} \text{Mix}_U(\mathcal{M}),$$

and its first page is

$$E_1^{p,q}(\mathcal{M}) = H^{p+q} \text{gr}_G^p \text{Mix}_U(\mathcal{M}) \cong \text{Mix}_U^p(H^q\mathcal{M}).$$

The differential $E_1^{p,q}(\mathcal{M}) \to E_1^{p+1,q}(\mathcal{M})$ is the differential of the mixed resolution. So from the quasi-isomorphism $\{ \text{Mix}_U^i(\mathcal{M}) \}_{i \geq 0}$, applied to the sheaf $H^q\mathcal{M}$, we get

$$E_2^{p,q}(\mathcal{M}) \cong \begin{cases} H^q\mathcal{M} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

We see that the spectral sequence collapses, and $E_\infty^{p,q}(\mathcal{M}) = E_2^{p,q}(\mathcal{M})$. In particular the induced filtration on the limit $H^{p+q} \text{Mix}_U(\mathcal{M})$ of the spectral sequence has only one nonzero jump (at level $G^0$).

According to [Ye1] Erratum, Theorem 1.2] the homomorphism of complexes

$$\Psi_{s;1} : \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X})$$

respects the filtrations $G$, and for every $p$ there is equality of homomorphisms of complexes

$$\text{gr}_G^p(\Psi_{s;1})[p] = \text{Mix}_U^p(\nu) : \text{Mix}_U^p(\mathcal{T}_{\text{poly}, X}) \to \text{Mix}_U^p(\mathcal{D}_{\text{poly}, X}).$$

Moreover, by [Ye3] Theorem 4.17, Mix$_U^p(\nu)$ is a quasi-isomorphism.
Since $\Psi_{*;1}$ respects the filtrations, there is an induced map of spectral sequences

$$E_{p,q}^r(\Psi) : E_{p,q}^r(T) \to E_{p,q}^r(D).$$

From (12.5) we see that there is an isomorphism in the first pages of the spectral sequences

$$E_{p,q}^1(\Psi) : \text{Mix}_p^p(T_{\text{poly}},X) \cong E_{p,q}^1(T) \cong \text{Mix}_p^p(H^q\mathcal{D}_{\text{poly}},X).$$

Since the differentials are the same, it follows that

$$E_{p,q}^2(\Psi) : E_{p,q}^2(T) \to E_{p,q}^2(D)$$

is an isomorphism.

Finally let’s examine the diagram of isomorphisms

$$\begin{array}{ccc}
T_{\text{poly},X}^q & \xrightarrow{H(\nu)} & H^q\mathcal{D}_{\text{poly},X} \\
\alpha \downarrow & & \alpha \downarrow \\
E_{2,q}^0(T) & \xrightarrow{E_{2,q}^0(\Psi)} & E_{2,q}^0(D) \\
\beta \downarrow & & \beta \downarrow \\
H^q\text{Mix}_U(T_{\text{poly},X}) & \xrightarrow{H^q(\Psi_{*;1})} & H^q\text{Mix}_U(D_{\text{poly},X}).
\end{array}$$

The arrows $\alpha$ come from (12.4); and the top square commutes because of (12.5). The arrows $\beta$ come from the collapse of the spectral sequence, and for this reason the bottom square is commutative.

We also need a slightly modified version of [Ye1, Theorem 0.1]. Observe that the variety $X$ is affine in this theorem. For the notation see Definitions 3.1, 2.8 and 2.10.

**Theorem 12.6.** Let $K$ be a field containing the real numbers, let $R$ be a parameter $K$-algebra, and let $X$ be an affine smooth algebraic variety over $K$. Then there is a bijection of sets

$$\text{quant} : \text{Ob}(\text{PoisDef}(R,\mathcal{O}_X)) \cong \text{Ob}(\text{AssDef}(R,\mathcal{O}_X))$$

called the quantization map. It preserves first order brackets, commutes with homomorphisms $R \to R'$ of parameter algebras, and commutes with étale morphisms $X' \to X$ of varieties.

**Proof.** The original result [Ye1, Theorem 0.1] was stated and proved for $R = K[[h]]$; but the modification to any parameter algebra $R$ is easy, using a filtered basis (cf. Example 7.5 and the proof of Theorem 8.5).

Here is the main result of our paper (the expanded form of Theorem 11.1):

**Theorem 12.7.** Let $K$ be a field containing the real numbers, let $R$ be a parameter $K$-algebra, and let $X$ be a smooth algebraic variety over $K$. Then there is a bijection of sets

$$\text{tw.quant} : \text{TwOb}(\text{PoisDef}(R,\mathcal{O}_X)) \cong \text{TwOb}(\text{AssDef}(R,\mathcal{O}_X))$$

(see Definitions 5.9 and 5.10) called the twisted quantization map, having these properties:
(i) If \( g : X' \to X \) is an étale morphism of varieties, and if \( \sigma : R \to R' \) is a homomorphism of parameter algebras, then the diagram

\[
\begin{array}{ccc}
\text{TwOb} \left( \text{PoisDef}(R, \mathcal{O}_X) \right) & \xrightarrow{\text{tw.quant}} & \text{TwOb} \left( \text{AssDef}(R, \mathcal{O}_X) \right) \\
\text{ind}_{s,g} & & \text{ind}_{s',g} \\
\text{TwOb} \left( \text{PoisDef}(R', \mathcal{O}_{X'}) \right) & \xrightarrow{\text{tw.quant}} & \text{TwOb} \left( \text{AssDef}(R', \mathcal{O}_{X'}) \right)
\end{array}
\]

(cf. Theorem 11.11) is commutative.

(ii) If \( X \) is affine, then under the bijections

\[
\text{Ob} \left( \mathcal{P}(X, R) \right) \cong \text{TwOb} \left( \mathcal{P}(X, R) \right)
\]

of Corollary 6.18 the twisted quantization map \( \text{tw.quant} \) coincides with the quantization map \( \text{quant} \) of Theorem 12.6.

(iii) The bijection \( \text{tw.quant} \) preserves first order brackets (see Definition 5.10). Namely if \( A \) is a twisted Poisson \( R \)-deformation of \( \mathcal{O}_X \), and \( B := \text{tw.quant}(A) \), then

\[
\left\{ -, - \right\}_A = \left\{ -, - \right\}_B.
\]

Proof. Let \((U, s)\) be a covering with coordinates of \( X \) (i.e. a finite ordered affine open covering with étale coordinate systems). According to Proposition 11.8 we get quasi-isomorphisms of sheaves of DG Lie algebras

\[
\tilde{NC}_U(T_{\text{poly},X}) \to \text{Mix}_U(T_{\text{poly},X})
\]

and

\[
\tilde{NC}_U(D_{\text{poly},X}^{\text{nor}}) \to \tilde{NC}_U(D_{\text{poly},X}) \to \text{Mix}_U(D_{\text{poly},X}).
\]

By taking global sections we obtain quasi-isomorphisms of DG Lie algebras

\[
\tilde{NC}(U, T_{\text{poly},X}) \to \Gamma(X, \text{Mix}_U(T_{\text{poly},X}))
\]

and

\[
\tilde{NC}(U, D_{\text{poly},X}^{\text{nor}}) \to \Gamma(X, \text{Mix}_U(D_{\text{poly},X})).
\]

According to Theorem 12.3 there is an \( L_\infty \) morphism

\[
\Psi_s : \Gamma(X, \text{Mix}_U(T_{\text{poly},X})) \to \Gamma(X, \text{Mix}_U(D_{\text{poly},X})).
\]

Because the mixed resolutions are acyclic for \( \Gamma(X, -) \), this is in fact a quasi-isomorphism. Let \( \Psi_{s,R} \) be its \( R \)-multilinear extension.

By Corollary 6.10 we get the bijections “dec” in Figure 4. Using Theorem 9.10 we get the bijections “exp”, where \( m \) is the maximal ideal of \( R \). Using Theorem 11.2 we get the bijections “int”. By combining formulas 11.17 and 7.1 we get the bijections “\( \nabla \)”. And by combining the \( L_\infty \) quasi-isomorphism \( \Psi_{s,R} \) with formula 12.2 we get the horizontal bijection in Figure 4. We define

\[
\text{tw.quant} : \text{TwOb} \left( \text{PoisDef}(R, \mathcal{O}_X) \right) \cong \text{TwOb} \left( \text{AssDef}(R, \mathcal{O}_X) \right)
\]

to be the unique bijection making this diagram commutative.

The whole diagram in Figure 4 is functorial w.r.t. \( R \). Suppose \( g : X' \to X \) is étale. Choose any affine open covering \( U' \) of \( X' \) refining \( U \); namely there is a morphism of coverings \( \rho : U' \to U \) extending \( g \) (this is easy). We get induced étale coordinate systems \( s' \) on \( X' \). Using Theorem 12.3(3) we see that the diagram in property (i) is commutative.

In order to show that the function \( \text{tw.quant} \) is independent of the of choice of \((U, s)\) we use the arguments in the proof of Theorem 10.1 together with Theorem 12.3(3).
When $X$ is affine the construction of the function $\text{tw.quant}$ agrees with the construction of the function quant in the proof of [Ye1, Erratum, Theorem 1.14]. This proves property (ii).

Regarding property (iii), it suffices to prove that the first order brackets $\{ -, - \}_A$ and $\{ -, - \}_g$ coincide on any affine open set $U \subset X$. Consider the étale morphism $g : U \to X$, and the twisted deformations $A|_U \cong \text{ind}_{\sigma,g}(A)$ and $B|_U \cong \text{ind}_{\sigma,g}(B)$, where $\sigma$ is the indentity automorphism of $R$. By property (i) we know that $\text{tw.quant}(A|_U) \cong B|_U$; By property (ii) we can replace the twisted deformations $A|_U$ and $B|_U$ with usual deformation $A$ and $B$ respectively, and those will satisfy $B \equiv \text{quant}(A)$. According to Theorem 10.1 the restriction of $\{ -, - \}_A$ to $U$ is $\{ -, - \}_A$, and likewise the restriction of $\{ -, - \}_g$ to $U$ is $\{ -, - \}_g$. But by Theorem 12.6 we have $\{ -, - \}_A = \{ -, - \}_g$.

$\square$

**Corollary 12.8.** If $H^1(X, O_X) = H^2(X, O_X) = 0$ then there is a bijection

$$\text{quant} : \bigcap \text{PoisDef}(R, O_X) \cong \bigcap \text{AssDef}(R, O_X),$$

which preserves first order brackets, commutes with homomorphisms $R \to R'$ of parameter algebras, and commutes with étale morphisms $X' \to X$ of varieties.

**Proof.** Combine Theorem 12.7 with Corollary 6.18 $\square$

We end the paper with a few questions. A twisted $\mathbb{K}[\hbar]$-deformation $A$ of $O_X$ is called *symplectic* if the first order bracket $\{ -, - \}_A$ is a symplectic Poisson bracket on $O_X$ (cf. Proposition 12.13).

**Question 12.9.** It is easy to construct an example of a commutative associative $\mathbb{K}[\hbar]$-deformation of $O_X$ that is really twisted – see Example 6.17. But does there exist a variety $X$, with a really twisted symplectic associative $\mathbb{K}[\hbar]$-deformation of $O_X$? Perhaps the results of [BK] can be useful here.

A more concrete (but perhaps much more challenging) question is:

**Question 12.10.** Let $X$ be a Calabi-Yau surface over $\mathbb{K}$ (e.g. an abelian surface or a K3 surface), and let $\alpha$ be a symplectic Poisson bracket on $O_X$ (namely any nonzero section of $\Gamma(X, \wedge^2 O_X T_X)$). Consider the Poisson $\mathbb{K}[\hbar]$-deformation $A := O_X[[\hbar]]$, with formal Poisson bracket $\hbar \alpha$, and let $\mathcal{A}$ be the corresponding twisted deformation
(see Example 6.11). Let $\mathcal{B} := \text{tw} \cdot \text{quant} (\mathcal{A})$. Is $\mathcal{B}$ really twisted? If so, what is the significance of this phenomenon? Note that the obstruction classes for $\mathcal{B}$ can be calculated explicitly; but these calculations look quite complicated. Kontsevich [private communication] appears to think that the twisted deformation $\mathcal{B}$ is really twisted, and he has an indirect argument for that.

**Question 12.11.** The construction of the $L_\infty$ quasi-isomorphism $\Psi_s$ in Theorem 12.3 relied on the explicit universal quantization formula of Kontsevich [Ko1]. This is the reason for the condition $\mathbb{R} \subset \mathbb{K}$. But suppose another quantization formula is used in the case of formal power series (e.g. a rational form, see [CV2]). Then the twisted quantization map $\text{tw} \cdot \text{quant}$ may change. Indeed, it is claimed by Kontsevich [Ko3] that the Grothendieck-Teichmüller group acts on the quantizations by changing the formality quasi-isomorphism (or in other words, the Drinfeld associator), and sometimes this action is nontrivial. The question is: does this action change the geometric nature of the resulting twisted associative deformation – namely can it change from being really twisted to being untwisted?

**Question 12.12.** The only “axioms” we have for the twisted quantization map $\text{tw} \cdot \text{quant}$ are invariance with respect to $R \to R'$, étale $X' \to X$, preservation of first order brackets, and behavior on affine open sets. Are there more such axioms, that will make the twisted quantization unique (given a choice of formality quasi-isomorphism)? A possible direction might be the work of Calaque and Van den Bergh on [CV2] on Hochschild cohomology and the Caldararu conjecture.

**References**

[BGNT] P. Bressler, A. Gorokhovsky, R. Nest and B. Tsygan, Deformation quantization of gerbes, Adv. Math. 214, Issue 1 (2007), 230-266.

[BK] R.V. Bezrukavnikov and D. Kaledin, Fedosov quantization in algebraic context, Mosc. Math. J. 4 (2004), Number 3, 559-592.

[BM] L. Breen and W. Messing, Differential geometry of gerbes, Adv. Math. 198 (2005), no. 2, 732-846.

[Br1] L. Breen, On the classification of 2-gerbes and 2-stacks, Astérisque 225 (1994).

[Br2] L. Breen, Notes on 1- and 2-gerbes, arXiv:math/0611317 at http://arxiv.org

[BS] J. Baez and U. Schreiber, Higher Gauge Theory, eprint arXiv:math/0511710 at http://arxiv.org

[CA] Bourbaki, “Commutative Algebra”, Chapters 1-7, Springer, 1989.

[Ca] D. Calaque, Ph.D. Thesis, available from author.

[CFT] A.S. Cattaneo, G. Felder and L. Tomassini, From local to global deformation quantization of Poisson manifolds, Duke Math. J. 115, Number 2 (2002), 329-352.

[CKTB] A. Cattaneo, B. Keller, C. Torossian and A. Bruguières, “Déformation, Quantification, Théorie de Lie”, Panoramas et Synthèses 20 (2005), Soc. Math. France.

[CV1] D. Calaque and M. Van den Bergh, Hochschild cohomology and Atiyah classes, eprint arXiv:0708.2225 at http://arxiv.org

[CV2] D. Calaque and M. Van den Bergh, Global formality at the $G_{\infty}$-level, eprint arXiv:0710.4510 at http://arxiv.org

[DP] A. D’Agnolo and P. Polesello, Stacks of twisted modules and integral transforms, eprint arXiv:math/0307387 at http://arxiv.org

[EGA I] A. Grothendieck and J. Dieudonné, “Éléments de Géometrie Algébrique I”, Springer, Berlin, 1971.

[Ge1] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, Duke Math. J. 111, Number 3 (2002), 535-560.

[Ge2] E. Getzler, Lie theory for nilpotent $L$-infinity algebras, eprint arXiv:math/0404003 at http://arxiv.org

[Gi] J. Giraud, “Cohomologie non abélienne”, Grundlehren der Math. Wiss. 179, Springer 1971.

[Ha] R. Hartshorne, ‘Algebraic Geometry”, Springer-Verlag, New-York, 1977.

[Ho] G. Hochschild, “Basic Theory of Algebraic Groups and Lie Algebras,” Springer, 1981.

[Hu] J.E. Humphreys, “Introduction to Lie Algebras and Representaion Theory”, GTM 9, Springer, 1972.
[Ko1] M. Kontsevich, Deformation Quantization of Poisson Manifolds, Lett. Math. Phys. 66 (2003), Number 3, 157-216.

[Ko2] M. Kontsevich, Deformation quantization of algebraic varieties, Lett. Math. Phys. 56 (2001), no. 3, 271-294.

[Ko3] M. Kontsevich, Operads and Motives in Deformation Quantization, Lett. Math. Phys. 48 (1999), Number 1, 35-72.

[KS1] M. Kashiwara and P. Schapira, “Sheaves on Manifolds”, Springer 1990.

[KS2] M. Kashiwara and P. Schapira, “Categories and Sheaves”, Springer 2006.

[KS3] M. Kashiwara and P. Schapira, Deformation quantization modules I: Finiteness and duality, eprint arXiv:0802.1245 at http://arxiv.org.

[Lo] W. Lowen, Algebroid prestacks and deformations of ringed spaces, eprint math.AG/0511197 at http://arxiv.org.

[LV] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories, Trans. AMS 358, Number 12 (2006), 5441-5483.

[ML] S. Mac Lane, “Categories for the Working Mathematician”, Springer 1978.

[MR] J.C. McConnell and J.C. Robson, “Noncommutative Noetherian Rings,” Wiley, Chichester, 1987.

[St] J.R. Strooker, “Homological Questions in Local Algebra”, Cambridge University Press, 1990.

[SGA4-1] M. Artin, A. Grothendieck and J.-L. Verdier, eds., “Séminaire de Géométrie Algébrique – Théorie des Topos et Cohomologie Étale des Schémas – Tome 1”, LNM 269, Springer.

[VdB] M. Van den Bergh, On global deformation quantization in the algebraic case, J. Algebra 315 (2006), 326-395.

[Ye1] A. Yekutieli, Deformation Quantization in Algebraic Geometry, Adv. Math. 198 (2005), 383-432. Erratum: Adv. Math. 217 (2008), 2897-2906.

[Ye2] A. Yekutieli, Continuous and Twisted L_infinity Morphisms, J. Pure Appl. Algebra 207 (2006), 575-606.

[Ye3] A. Yekutieli, Mixed Resolutions and Simplicial Sections, Israel J. Math. 162 (2007), 1-27.

[Ye4] A. Yekutieli, Central Extensions of Gerbes, eprint arXiv:0804.0083v3 at http://arxiv.org.

[Ye5] A. Yekutieli, On Flatness and Completion for Infinitely Generated Modules over Noetherian Rings, eprint arXiv:0902.4378 at http://arxiv.org.

[Ye6] A. Yekutieli, Nonabelian Multiplicative Integration on Surfaces, in preparation.

[Ye7] A. Yekutieli, Quantum Type DG Lie Algebras and Descent for Nonabelian Gerbes, in preparation.

[YZ] A. Yekutieli and J.J. Zhang, Dualizing Complexes and Perverse Modules over Differential Algebras, Compositio Math. 141 (2005), 620-654.

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