Collective excitations of a BCS superfluid in the presence of two sublattices

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We consider a generic Hamiltonian that is suitable for describing a uniform BCS superfluid on a lattice with a two-point basis, and study its collective excitations at zero temperature. For this purpose, we first derive an effective-Gaussian action for the pairing fluctuations, and then extract the low-energy dispersion relations for the in-phase Goldstone and out-of-phase Legget modes along with the corresponding amplitude (i.e., the so-called Higgs) ones. We find that while the Goldstone mode is gapless at zero momentum and propagating in general, the Leggett mode becomes undamped only with sufficiently strong interactions. Furthermore, we show that, in addition to the conventional contribution that is controlled by the energy of the Bloch bands, the velocity of the Goldstone mode has a geometric contribution that is governed by the quantum metric tensor of the Bloch states. Our results suggest that the latter contribution dominates the velocity when the former one becomes negligible for a narrow- or a flat-band.

I. INTRODUCTION

The deeper connection between the quantum geometry of the underlying Bloch states and the superfluid (SF) phase stiffness tensor or the often-called SF weight of some multi-band Fermi SFs has recently been revealed in the literature [1–3]. It turns out that the SF stiffness tensor of a multi-band SF has two physically distinct mechanisms: while the conventional contribution is due to the intraband processes and has a direct counterpart in the one-band models, the geometric contribution is due to the interband processes, and therefore, is exclusive to the multi-band models. In the particular case of a uniform BCS superfluid with two underlying sublattices [2, 4, 5], this is such that the geometric contribution is controlled by the so-called quantum metric tensor of the underlying Bloch states [3].

Furthermore, in the context of spin-orbit coupled Fermi SFs in continuum, we recently showed that the quantum metric tensor of the underlying helicity states has also a partial control over the low-energy collective excitations of the system at zero temperature [6]. Motivated by this result and earlier works [2, 4, 5], here we perform a similar collective-mode analysis to the case of a generic Hamiltonian that is suitable for describing a uniform BCS SF on a lattice with a two-point basis. Allowing that the SF order parameter may fluctuate (around its uniform value) independently on the two sublattices, there are two phase and two amplitude modes which are associated with the total and relative fluctuations of the phase and amplitude degrees of freedom. For instance, the in-phase fluctuations are phonon-like and correspond to the Goldstone mode, and the out-of-phase fluctuations are exciton-like and correspond to the Leggett mode. Thus, in comparison to the Goldstone mode that is considered in Ref. [3], the presence of a Leggett mode makes the current analysis somewhat more cumbersome.

We find that while the Goldstone mode is gapless at zero momentum and propagating in general, the low-energy Leggett mode becomes undamped only with sufficiently strong interactions. More importantly, by identifying the quantum-metric contribution to the Goldstone mode, we show that this geometric effect is complementary to the recent works on the geometric contribution to the SF stiffness tensor [2, 4, 5], i.e., they are both controlled by the effective-mass tensor of the SF carriers. This suggests that an analogous contribution to the collective excitations must be present in many other multi-band systems including the twisted bilayer graphene [10–12].

The rest of the paper is organized as follows. In Sec. II we first derive an effective-Gaussian action for the pairing fluctuations, then extract the low-energy dispersion relations for the collective modes, and then benchmark our generic results with those of the honeycomb literature [13, 14]. In Sec. III we show that the velocity of the Goldstone mode has a geometric contribution that can be traced back to the same origin as the recent works on the SF stiffness tensor [2, 3]. The paper ends with a summary of our conclusions in Sec. IV.

II. EFFECTIVE-ACTION APPROACH

In this section, we first introduce a generic lattice Hamiltonian that is suitable for describing a uniform BCS SF with two underlying sublattices, and then extract its collective excitations from an effective action that is derived up to the Gaussian order in the fluctuations of the SF order parameter.

A. Hamiltonian

Having a general single-particle Hamiltonian on a lattice with a two-point basis in mind, we consider

$$H = \sum_{\sigma k} \langle c_{\sigma A k} c_{\sigma B k}^\dagger \rangle \left( c_{\sigma A k} \tau_0 + d_k \cdot \tau \right) \left( c_{\sigma B k} \right) - U \sum_{Skk'q} c_{S, k^q}^\dagger c_{S, -k^q} \tau_0 c_{S, -k^q + q}^\dagger c_{S, k^q},$$

(1)
where $c_{\mathbf{k} \ell \mathbf{S} \ell}^\dagger (c_{\mathbf{k} \ell \mathbf{S} \ell})$ creates (annihilates) a spin-$\sigma$ fermion on sublattice $S \in \{A, B\}$ with quasi-momentum $\mathbf{k}$, i.e., in units of $\hbar$→ 1 the Planck constant. In the first line where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and $\tau_0$ is a $2 \times 2$ identity matrix, $\epsilon_{\mathbf{k}}$ is due to the intra-sublattice hoppings, and $\mu$ is the chemical potential. The inter-sublattice hoppings are taken into account by the second term where $\tau = \sum \tau_{\mathbf{i}}$ is a vector of Pauli matrices for the sublattice sector, and the sublattice-coupling field $\mathbf{d}_k = \sum_i \mathbf{d}_i$ is a generic one with $\mathbf{i}$ denoting a unit vector along the $i = (x, y, z)$ direction. Thus, the single-particle problem is described by the Hamiltonian density $h_{\mathbf{k}} = \epsilon_{\mathbf{k}} \tau_0 + \mathbf{d}_k \cdot \tau$, leading to a two-band energy spectrum $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} + \pm \mathbf{d}_k$, where $\mathbf{d}_k = |\mathbf{d}_k|$, where $s = \pm$ labels the upper/lower bands. For instance, in the case of a honeycomb lattice [4], one finds $\epsilon_{\mathbf{k}} = -2t' \cos(\sqrt{3}k_x a) - 4t' \cos(\sqrt{3}k_y a/2) \cos(3k_y a/2)$, $d_{\mathbf{k}} = t \cos(k_x a) - 2t \cos(t_x a/2) \cos(\sqrt{3}k_y a/2)$, $d_{\mathbf{k}}^y = t \sin(k_x a) - 2t \sin(k_x a/2) \cos(\sqrt{3}k_y a/2)$, and $d_{\mathbf{k}}^x = 0$. Similarly, in the case of a Mielke lattice [3], one finds $\epsilon_{\mathbf{k}} = -2(t + t') \cos(k_x a) \cos(k_y a)$, $d_{\mathbf{k}}^x = -2t \cos(k_x a) - 2t \cos(k_y a)$, $d_{\mathbf{k}}^y = 2(t' - t') \sin(k_x a) \sin(k_y a)$, and $d_{\mathbf{k}}^x = 0$. Note that while $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k} \ell \mathbf{S} \ell} = \epsilon_{\mathbf{k} \ell \mathbf{S} \ell}$ and the $\tau_z$ field $d_{\mathbf{k}}^z = d_{-\mathbf{k} \ell \mathbf{S} \ell}^z = d_{\mathbf{k} \ell \mathbf{S} \ell}^z$ are parity even functions of both $k_x$ and $k_y$, and the $\tau_\ell$ field $d_{\mathbf{k}}^x = d_{-\mathbf{k} \ell \mathbf{S} \ell}^x = -d_{\mathbf{k} \ell \mathbf{S} \ell}^x$ is an even (odd) function of both $k_x$ and $k_y$, the $\tau_y$ field $d_{\mathbf{k}}^y = -d_{-\mathbf{k} \ell \mathbf{S} \ell}^y = d_{\mathbf{k} \ell \mathbf{S} \ell}^y$ is an even (odd) function of either $k_x$ or $k_y$. In the second line of Eq. (1), $U \geq 0$ contributes to the strength of the on-site attraction between $\uparrow$ and $\downarrow$ particles, and we decouple this quartic term (in the fermionic degrees of freedom) using the Grassmann functional-integral formalism [13, 16]. For this purpose, we first express the partition function $Z = \int [\mathcal{D}c, c] e^{-\mathcal{S}}$ with the associated action $\mathcal{S} = \int_0^T d\tau (\sum_{\mathbf{k} \ell \mathbf{S} \ell} c_{\mathbf{k} \ell \mathbf{S} \ell}^\dagger(\tau) \partial_\tau c_{\mathbf{k} \ell \mathbf{S} \ell}(\tau) + H(\tau))$, where $T$ is the temperature in units of $k_B$→ 1 the Boltzmann constant. Then, we introduce a Hubbard-Stratranovich transformation at the expense of introducing a complex bosonic field $\Delta_{\mathbf{q}}$, and integrate out the remaining terms that are quadratic in the fermionic degrees of freedom. This leads to $Z = \int [\mathcal{D}[\Delta^+, \Delta] e^{-S_{\text{eff}}}]$, where $S_{\text{eff}}$ plays the role of a fluctuating order parameter for pairing, and $S_{\text{eff}}$ is the effective bosonic action for the resultant pairs of fermions. Here, the collective index $\mathbf{q} = (\mathbf{q}, \nu)$ denotes both the pair momentum $\mathbf{q}$ and the bosonic Matsubara frequency $\nu = 2\pi n T$. Finally, by decomposing $\Delta_{\mathbf{q}} = \Delta_0 + \Lambda_{\mathbf{q}}$ in terms of a $g$-independent stationary field $\Delta_0$ and $g$-dependent fluctuations around it, one may in principle obtain $S_{\text{eff}}$ at the desired order in $\Lambda_{\mathbf{q}}$. Note that $\Delta_0$ is uniform for the entire lattice.

In this paper we include only the first nontrivial term and obtain the effective-Gaussian action $S_{\text{Gauss}} = S_0 + S_2$, as the first-order term $S_1$ trivially vanishes due to the saddle-point condition discussed next.

B. Saddle-point approximation

The effective-action approach is a standard tool in many-body physics, and it leads to $S_0 = \Delta_0^2/(2U) + [1/(N_i T)] \sum_{\mathbf{k} \ell \mathbf{S} \ell} \xi_{\mathbf{k} \ell \mathbf{S} \ell} - (1/N_i) \sum_k \ln[\det(G_k^{-1}/T)]$, where the collective index $k = (\mathbf{k}, \omega_n)$ denotes both the particle momentum $\mathbf{k}$ and the fermionic Matsubara frequency $\omega_n = (2\nu + 1)\pi T$. Here, $N_i$ is the number of lattice sites, and $G_k^{-1} = \omega_n 1 - H_k^0$ is the inverse Green’s function for the mean-field Hamiltonian density $H_k^0$, i.e.,

$$G_k^{-1} = \begin{bmatrix} (i\omega_n - \xi_{\mathbf{k}})\tau_0 - \mathbf{d}_k \cdot \tau & -\Delta_0 \tau_0 \\ -\Delta_0 \tau_0 & (i\omega_n + \xi_{\mathbf{k}})\tau_0 + \mathbf{d}_k \cdot \tau \end{bmatrix}. $$

Here, we use $\epsilon_{-\mathbf{k}} = \epsilon_{\mathbf{k}}$ and $\mathbf{d}_{-\mathbf{k}} \cdot \tau^* = \mathbf{d}_k \cdot \tau$, and the Hamiltonian is given by $H_0 = \sum_k \Psi_k^\dagger H_0^0 \Psi_k$ where $\Psi_k = (c_{\mathbf{k} \ell \mathbf{A} \ell}^\dagger, c_{\mathbf{k} \ell \mathbf{B} \ell}^\dagger)$. After the summation over $\omega_n$, we obtain

$$S_0 = \frac{\Delta_0^2}{2U} + \frac{1}{N_i} \sum_{\mathbf{k} \ell \mathbf{S} \ell} \left\{ \frac{\xi_{\mathbf{k} \ell \mathbf{S} \ell} - E_{\mathbf{k} \ell \mathbf{S} \ell}}{T} + 2\ln[f(-E_{\mathbf{k} \ell \mathbf{S} \ell})] \right\}, \quad (2)$$

where $\xi_{\mathbf{k} \ell \mathbf{S} \ell} = \epsilon_{\mathbf{k} \ell \mathbf{S} \ell} - \mu$, $E_{\mathbf{k} \ell \mathbf{S} \ell} = \sqrt{\xi_{\mathbf{k} \ell \mathbf{S} \ell}^2 + \Delta_0^2}$ is the quasiparticle energy spectrum, and $f(x) = 1/(e^{x/T} + 1)$ is the Fermi-Dirac distribution.

The saddle-point order parameter $\Delta_0$ can also be expressed as $\Delta_0 = U(c_{\mathbf{k} \ell \mathbf{A} \ell} c_{-\mathbf{k} \ell \mathbf{A} \ell})$ with $\langle \ldots \rangle$ denoting a thermal average, and we take it to be a real parameter throughout the paper without the loss of generality. Using the saddle-point condition $\partial S_0/\partial \Delta_0 = 0$ for the action, and the thermodynamic relation $N_0 = -T \partial S_0/\partial \mu$ for the number of particles, we find [4, 5, 13]

$$\frac{1}{U} = \frac{1}{N_i} \sum_{\mathbf{k} \ell \mathbf{S} \ell} \left\{ 1 - \frac{2f(E_{\mathbf{k} \ell \mathbf{S} \ell})}{E_{\mathbf{k} \ell \mathbf{S} \ell}} \right\}, \quad (3)$$

$$N_0 = \sum_{\mathbf{k} \ell \mathbf{S} \ell} \left\{ \frac{1}{2} - \frac{\xi_{\mathbf{k} \ell \mathbf{S} \ell}}{2E_{\mathbf{k} \ell \mathbf{S} \ell}} \left[ 1 - 2f(E_{\mathbf{k} \ell \mathbf{S} \ell}) \right] \right\}. \quad (4)$$

In order to evaluate the collective excitations, we need self-consistent solutions for $\Delta_0$ and $\mu$ as a function of $U$ and hopping parameters. In addition, for the $T = 0$ of interest in this paper, these mean-field solutions turns out to be sufficient for a qualitative description of the many-body problem.

C. Gaussian fluctuations

Going beyond the saddle-point action $S_0$, we calculate the first nontrivial term in the expansion, and find $S_2 = \sum_{\mathbf{q} \ell \mathbf{S} \ell} |\Delta_{\mathbf{q}}|^2/(2TU) + [1/(2N_i T)] \text{Tr} \sum_{\mathbf{k} \ell \mathbf{S} \ell} \sum_{\mathbf{k} \ell' \mathbf{S} \ell'} G_k \Sigma_{\mathbf{q}} G_{k+\mathbf{q}} \Sigma_{\mathbf{q}}^{-}$, where $\text{Tr}$ denotes a trace over the sublattice and spin
sectors. The matrix elements of $G_k$ can be written as

$$G_{11}^k = \frac{1}{2} \sum_s \frac{i\omega + \xi_{sk}}{(i\omega)^2 - E_{sk}^2} \left( \tau_0 + s\hat{d}_k \cdot \tau \right),$$

$$G_{22}^k = \frac{1}{2} \sum_s \frac{i\omega - \xi_{sk}}{(i\omega)^2 - E_{sk}^2} \left( \tau_0 + s\hat{d}_k \cdot \tau \right),$$

$$G_{12}^k = \frac{1}{2} \sum_s \frac{\Delta_{0s}}{(i\omega)^2 - E_{sk}^2} \left( \tau_0 + s\hat{d}_k \cdot \tau \right),$$

where $\hat{d}_k = d_k/dk$, and $G_{21}^k = G_{12}^k$. In addition, the matrix elements of the fluctuation field $\Sigma_q$ are $\Sigma^q_{11} = \Sigma^q_{22} = 0$, $\Sigma^q_{12} = -\Lambda T \tau_0 - \Lambda R \tau_z$, and $\Sigma^{*q}_{21} = -\Lambda^* T - q \tau_0 - \Lambda^* R - q \tau_z$. Motivated by the earlier works on two-band SFs, we define $\Lambda T_q = (\Lambda A_q + \Lambda B_q)/2$ for the total and $\Lambda R_q = (\Lambda A_q - \Lambda B_q)/2$ for the relative fluctuations.

After the summation over $\omega$, we obtain $S_2 = \lceil 1/(2N(T)) \rceil \sum_q \Lambda^* q M_q \Lambda_q$, where $\Lambda^* q = (\Lambda T_q \Lambda T_q - q \Lambda R_q \Lambda R_q - q)$ is a vector of fluctuation fields and $M_q = \left( \frac{T_{q,E}}{C_q} R_q \right)$ stands for the inverse fluctuation propagator. Here, while the submatrices $T_q$ and $R_q$ describe the purely total and purely relative fluctuations, respectively, the submatrix $C_q$ is responsible for their coupling. The submatrix $C_q^*$ is related to $C_q$ via a complex conjugate acting only on the multiplying factors, i.e., its matrix elements are determined by Eq. (12) but with $[d_i + d_i' + i(d_i d_i' - d_i d_i')]$.

In order to simplify their expressions, we denote $\xi_{sk}$ by $\xi, \xi_{sk} + q$ by $\xi', E_{sk}$ by $E, E_{sk} + q$ by $E'$, $s\hat{d}_k/dk$ by $d_i, s\hat{d}_k/dk + q$ by $d_i'$, and define the functions $u^2 = (1 + \xi/E)/2, u^2 = (1 + \xi'/E')/2, \nu = (1 - \xi/E)/2, \nu' = (1 - \xi'/E')/2, f = 1/(e^{E/T} + 1)$, and $f' = 1/(e^{E'/T} + 1)$. In addition, we also define

$$r_1 = (1 - f - f') \left( \frac{u^2\nu^2}{\nu + E - E'} - \frac{u^2\nu^2}{\nu + E + E'} \right),$$

$$+ (f - f') \left( \frac{\nu u v}{\nu + E - E'} - \frac{\nu u v}{\nu + E + E'} \right),$$

$$r_2 = (1 - f - f') \left( \frac{u^2\nu^2}{\nu + E - E'} - \frac{u^2\nu^2}{\nu + E - E'} \right),$$

$$+ (f - f') \left( \frac{\nu u v}{\nu + E - E'} - \frac{\nu u v}{\nu + E - E'} \right),$$

for a compact presentation of the matrix elements of $M_q$ as well. Using these simpler notations and definitions, we find

$$T_{q_j} = \frac{1}{U} \left[ \frac{1}{2N} \sum_{ss'} \delta_{jj} + \frac{1}{2N} \sum_{ss'} r_j(1 + d_x d'_x + d_y d'_y + d_z d'_z), \right],$$

$$R_{q_j} = \frac{1}{U} \left[ \frac{1}{2N} \sum_{ss'} r_j(1 - d_x d'_x - d_y d'_y + d_z d'_z), \right],$$

$$C_{q_j} = \frac{1}{2N} \sum_{ss'} r_j[d_x + d'_x - i(d_x d'_y - d_y d'_x)], \right],$$

where $\delta_{ij}$ is the Kronecker-delta [17]. The remaining elements of $T_q$, $R_q$, and $C_q$ are all related to the given ones as follows: $T_{q} = T_{q_j}^1, R_{q} = R_{q_j}^1$, $T_{q_j}^2 = R_{q_j}^2, C_{q_j}^{1*} = C_{q_j}^{1*}$, $C_{q_j}^{1*} = C_{q_j}^{1*}$, and $C_{q_j}^{1*} = C_{q_j}^{1*}$. Here, the complex conjugate again acts only on the multiplying factor of Eq. (12). We note that while $T_{q_j}^1$ and $R_{q_j}^1$ are even both under $q \rightarrow -q$ and $iv_n \rightarrow -iv_n$, $C_{q_j}^{1*}$ is even only under $iv_n \rightarrow -iv_n$, and $T_{q_j}^1$ and $R_{q_j}^1$ are even only under $q \rightarrow -q$. In addition, we also note that a familiar factor $d_x d'_x + d_y d'_y + d_z d'_z = s \hat{d}_k \cdot \hat{d}_k + q$ is appearing in the elements of $T_q$ [9].

Next we reexpress the fluctuation fields $\Lambda T(R) = \alpha_{T(R)} q \cos(\gamma T(R))$ in terms of real functions $\alpha_{T(R)}$ and $\gamma T(R)$; and associate $\lambda_{T(R)}(q) = \sqrt{2} \alpha_{T(R)} q \cos(\gamma T(R))$ with the amplitude degrees of freedom and $\theta_{T(R)}(q) = \sqrt{2} \Lambda T(R) q \sin(\gamma T(R))$ with the phase ones in the small $\gamma T(R) q$ limit. Such a unitary transformation can be achieved by [13] [16]

$$\tilde{\Lambda}_q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 & 1 & -i & 0 & 0 \ 1 & -i & 0 & 0 & 1 & i & 0 & 0 \ 0 & 0 & 1 & i & 0 & 0 & 1 & -i \ 0 & 0 & -i & 1 & 0 & 0 & -i & 1 \ \end{pmatrix} \begin{pmatrix} \lambda_{T_q} \\
\theta_{T_q} \\
\lambda_{R_q} \\
\theta_{R_q} \end{pmatrix},$$

where $\lambda_{T(R)}(q)$ and $\theta_{T(R)}(q)$ are real functions. Furthermore, assuming $\lambda_{T(R)} = \lambda_{T(R)}(q)$ and $\alpha_{T(R)} = \theta_{T(R)}(q)$, we finally obtain the desired action

$$S_2 = \frac{1}{2T} \sum_q \left( \lambda_{T_q} \theta_{T_q} \lambda_{R_q} \theta_{R_q} \right) \begin{pmatrix} T_{q,1} + T_{q,2} & T_{q,1}^* & T_{q,2} & T_{q,2}^* \ T_{q,1} & T_{q,1}^* & T_{q,2} & T_{q,2}^* \ T_{q,1}^* & T_{q,2}^* & T_{q,1} & T_{q,2} \ T_{q,2}^* & T_{q,2}^* & T_{q,2} & T_{q,2}^* \ \end{pmatrix} \begin{pmatrix} C_{q,E} + C_{q,o} & C_{q,O} & i C_{q,E} & i C_{q,O} \ C_{q,E} + C_{q,o} & C_{q,O} & i C_{q,E} & i C_{q,O} \ i C_{q,E} & i C_{q,O} & C_{q,E} - C_{q,o} & C_{q,O} - C_{q,1} \ i C_{q,E} & i C_{q,O} & C_{q,E} - C_{q,o} & C_{q,O} - C_{q,1} \ \end{pmatrix} \begin{pmatrix} \lambda_{T_q} \\
\theta_{T_q} \\
\lambda_{R_q} \\
\theta_{R_q} \end{pmatrix}.$$
as the so-called Higgs mode. Therefore, we conclude that

\[ T_{q,E}^{11} = T_{q,k}^{11} + T_{q,O}^{11}, \]

where \( T_{q,k}^{11} = (T_{q}^{11} + T_{q}^{22})/2 \) is the even and \( T_{q,O}^{11} = (T_{q}^{11} - T_{q}^{22})/2 \) is the odd part.

Having derived the effective-Gaussian action, next we are ready to analyze it in detail, and extract the collective modes of the system.

D. Collective excitations at \( T = 0 \)

The dispersions \( \omega_q \) for the collective modes are determined by the poles of the propagator matrix \( M_q^{-1} \) for the pair fluctuation fields, by setting \( \det M_q = 0 \) after an analytic continuation \( \nu_n \to \omega + i0^+ \) to the real axis.

Since the quasiparticle-quasihole terms with the prefactor \((f - f')\) have the usual Landau singularity for \( q \to 0 \), causing the collective modes to decay into the two-quasiparticle continuum, a small \( q \) expansion is possible only in two cases: (i) just below the critical SF transition temperature provided that \( \Delta_0 \to 0 \ll |\omega| \), and (ii) at \( T = 0 \) provided that \( |\omega| \ll \min(E + E') \). In this work, we are interested in the latter case, and by setting \( T = 0 \) in Eqs. (8) and (9), we find

\[
T_{q,E}^{11} = \frac{1}{U} + \frac{1}{2N} \sum_{ss'k} \frac{\langle \xi_s' + \xi_E' \rangle (E + E')}{\langle [\nu_n]^2 - (E + E')^2 \rangle} \times (1 + d_x d'_x + d_y d'_y + d_z d'_z),
\]

\[
T_{q,O}^{11} = \frac{1}{2N} \sum_{ss'k} \frac{\langle \xi_s' + \xi_E' \rangle \nu_n}{\langle [\nu_n]^2 - (E + E')^2 \rangle} \times (1 + d_x d'_x + d_y d'_y + d_z d'_z),
\]

\[
T_{q}^{12} = -\frac{1}{2N} \sum_{ss'k} \frac{\Delta_0^2 (E + E')}{\langle [\nu_n]^2 - (E + E')^2 \rangle} \times (1 + d_x d'_x + d_y d'_y + d_z d'_z).
\]

The matrix elements of \( \mathbf{R}_q \) and \( \mathbf{C}_q \) sectors have similar forms except for the multiplying factors in the second lines.

We note that, while in the limit when \( q \to 0 \), the multiplying factor for \( \mathbf{C}_q \) directly vanishes, i.e., \( d_x d'_x - i(d_x d'_y - d_y d'_x) \to 0 \), for the honeycomb lattice, these terms sum to 0 for the Mielke lattice as \( d_{ik}^2 \) is an odd function of both \( k_x \) and \( k_y \), suggesting that the total and relative fluctuations are always uncoupled. In addition, in the limit when \( q \to 0 \) and \( \nu_n \to 0 \), we find \( T_{q,E}^{11} - T_{q}^{12} \to 1/U - \sum_k 1/(2N \epsilon_{sk}) = 0 \) due to the saddle-point condition, suggesting that the total phase mode is gapless at \( q = 0 \) and we identify it as a Goldstone mode. Similarly, in the limit when \( q \to 0 \) and \( |\nu_n| \to 2\Delta_0 \), we find \( T_{q,E}^{11} + T_{q}^{12} \to 0 \) due again to the saddle-point condition, suggesting that the total amplitude mode is gapped with \( 2\Delta_0 \) (i.e., this holds only in the weakly-interacting BCS limit for which the amplitude and phase fields are weakly coupled thanks to the negligible contribution from \( T_{q,O}^{11} \)) at \( q = 0 \) and we identify it as the so-called Higgs mode. Therefore, we conclude that the \( q \to 0 \) limit is consistent with our physical intuition.

We are also aware of several numerical works where the collective excitations of a BCS SF are analyzed on the two-dimensional honeycomb lattice \(^{12}^{12}\), and next we check the consistency of our generic results with those of the honeycomb literature.

1. Zhao and Paramekanti’s work

As a first benchmark, we consider the static limit when \( \nu_n \to 0 \), for which all of the matrix elements that couple amplitude and phase fields go to zero, i.e., \( \{T_{q,E}^{11}, T_{q,O}^{11}, C_{q,E}^{11}, C_{q,O}^{11}\} \to 0 \), making the analysis of \( S_2 \) a much simpler task. For instance, the phase fluctuations are described purely by the following action

\[
\left( \theta_{Tq}^{*} \theta_{Rq} \right) \left( \begin{array}{ccc} T_{q,E}^{11} - T_{q}^{12} & C_{q,E}^{11} - C_{q}^{12} \\ C_{q,E}^{11*} - C_{q}^{12*} & R_{q,E}^{11} - R_{q}^{12} \end{array} \right) \left( \theta_{Tq}^{*} \theta_{Rq} \right). \tag{18}
\]

In order to confirm that Eq. (18) reproduces the results of Ref. \(^{12}\), one first needs to reexpress their Eq. (4) in terms of the total and relative phases, and then match their matrix elements \( u_q \) and \( v_q \) in such a way that \( u_q - Re[v_q] = \Delta_0^2 (T_{11}^{11} - T_{12}^{12}) \), \( u_q - Re[v_q] = \Delta_0^2 (R_{11}^{11} - R_{12}^{12}) \), and \( Im[v_q] = -\Delta_0^2 (C_{11}^{11} - C_{12}^{12}) \). The origin of the prefactor \( \Delta_0^2 \) is due to the difference in the definitions of the fluctuations fields, i.e., they substitute \( \Delta_{2q} = \Delta_0 (\lambda_{qR} + i\theta_{qR}) \). We note that since \( d_{ik}^2 = 0 \) for the honeycomb model, their \( \gamma_{xk} = d_{ik} \) leads to \( \gamma_{xk}^* \gamma_{xk} = d_{ik}^2 d_{ik}^* + d_{ik}^* d_{ik} = 1/2 \). This expression corresponds to \( \{d_x d'_x + d_y d'_y + i(d_x d'_y - d_y d'_x)\}/(ss') \) in our notation. However, we note that there must be a typo in their Eq. (5), and the first term must read as \( \Delta_0^2 /U \) instead of \( 2\Delta_0^2 /U \). This is also evident from the discussion given below Eq. (17), i.e., noting that \( Im[v_q] \to 0 \) in the \( q \to 0 \) limit, \( u_q - Re[v_q] \) must also vanish in order to recover the total phase mode as a gapless Goldstone one. For completeness, the amplitude fluctuations are described purely by the following action

\[
\left( \lambda_{Tq} \lambda_{Rq}^* \right) \left( \begin{array}{ccc} T_{q,E}^{11} + T_{q}^{12} & C_{q,E}^{11} + C_{q}^{12} \\ C_{q,E}^{11*} + C_{q}^{12*} & R_{q,E}^{11} + R_{q}^{12} \end{array} \right) \left( \lambda_{Tq} \lambda_{Rq} \right). \tag{19}
\]

in the static limit.

2. Tsuchiya, Ganesh, and Nikuni’s work

As a second benchmark, we consider a two-band lattice whose energy bands are completely symmetric around the zero energy, i.e., \( \xi_{sk} = -\xi_{-s,\bar{k}} \), which requires that \( \mu = 0 \) and \( \epsilon_k = 0 \). For instance, this particular discussion is relevant in the context of a pair of Dirac cones at half filling. When this is the case, by setting \( \xi_{sk} = sd_{sk} \) and
\( E_{k} = \sqrt{d_{k}^2 + \Delta_{0}^2} = E_{k} \) in Eqs. (15)-(17), we find

\[
T_{q,E}^{11} = \frac{1}{U} + \frac{1}{N_{1}} \sum_{k} \left\{ \frac{E_{k} + E_{k+q}}{E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)} \times (E_{k} E_{k+q} + d_{k}^{\ast} d_{k+q}^{\ast} + d_{k+q}^{\ast} d_{k}^{\ast} + d_{k}^{\ast} d_{k+q}^{\ast}), \right\}
\]

(20)

\[
R_{q,E}^{11} = \frac{1}{U} + \frac{1}{N_{1}} \sum_{k} \left\{ \frac{E_{k} + E_{k+q}}{E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)} \times (E_{k} E_{k+q} + d_{k}^{\ast} d_{k+q}^{\ast} - d_{k+q}^{\ast} d_{k}^{\ast} + d_{k}^{\ast} d_{k+q}^{\ast}), \right\}
\]

(21)

\[
T_{q}^{12} = -\frac{1}{N_{1}} \sum_{k} \left\{ \frac{\Delta_{2}^{\ast} (E_{k} + E_{k+q})}{E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)} \right\}
\]

(22)

\[
C_{q,E}^{11} = -\frac{1}{N_{1}} \sum_{k} \left\{ \frac{(E_{k} + E_{k+q}) (d_{k}^{\ast} d_{k+q}^{\ast} - d_{k+q}^{\ast} d_{k}^{\ast})}{E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)} \right\}
\]

(23)

\[
C_{q,O}^{11} = \frac{1}{N_{1}} \sum_{k} \left\{ \frac{(d_{k}^{\ast} E_{k+q} + E_{k} d_{k+q}^{\ast}) \omega_{n}}{E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)} \right\}
\]

(24)

The remaining terms are such that \( R_{q}^{12} = T_{q}^{12} \), and \( T_{q,O}^{11} = R_{q,O}^{11} = C_{q,O}^{11} = 0 \). We also note that the saddle-point condition Eq. (3) becomes \( 1/U = \sum_{k} 1/(N_{1} E_{k}) \). Since \( C_{q,O}^{11} \) vanishes for the honeycomb lattice, and it sums to 0 for \( d_{k}^{\ast} \) that is odd in \( k_{x} \) or \( k_{y} \), we find for these cases that the amplitude and phase fields are completely decoupled, i.e., they are purely described by Eqs. (18) and (19). Setting the corresponding determinants to 0, we find

\[
\left\{ \frac{1}{U} + \sum_{k} \left\{ \frac{(E_{k} + E_{k+q}) (E_{k} E_{k+q} + \Delta_{0}^2 + d_{k}^{\ast} d_{k+q}^{\ast})^2}{N_{1} E_{k} E_{k+q} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)^2} \right\} \right\}
\]

(25)

for the poles of the propagator matrices given in Eqs. (18) and (19), where \( \pm \) is for the phase/amplitude modes. Note that since \( C_{q,E}^{11} \to 0 \) in the limit when \( q \to 0 \), the total and relative fields are not coupled, leading to a gapless Goldstone mode and a gapped Leggett mode as discussed below and in Sec. 11.

In the honeycomb case, Eq. (25) is in somewhat agreement with Ref. [14], i.e., our \( \pm \) results are similar to their expressions Eqs. (12) and (11), respectively, when their \( F = 0 \). This discrepancy is amusing given that the collective modes for the usual one-band models that are found from the Gaussian fluctuations and random-phase approximation are known to be consistent with each other. Furthermore, they conclude that the Goldstone and Leggett modes are both gapless and degenerate at \( q = 0 \).

When we set \( q = 0 \) in Eq. (25), we find two solutions for the phase modes and two solutions for the amplitude ones, which can be written, respectively, as

\[
0 = \frac{1}{N_{1}} \sum_{k} \frac{\omega_{n}^2}{E_{k} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)}
\]

(26)

\[
0 = \frac{1}{N_{1}} \sum_{k} \frac{\omega_{n}^2 - 4d_{k}^2}{E_{k} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)}
\]

(27)

\[
0 = \frac{1}{N_{1}} \sum_{k} \frac{\omega_{n}^2 - 4\Delta_{0}^2}{E_{k} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)}
\]

(28)

\[
0 = \frac{1}{N_{1}} \sum_{k} \frac{\omega_{n}^2 - 4d_{k}^2 + 4d_{k}^4 - 4\Delta_{0}^2}{E_{k} ([\omega_{n}]^2 - (E_{k} + E_{k+q})^2)}
\]

(29)

Here, Eq. (20) suggests that the total phase (Goldstone) mode is gapless when \( \omega_{n} \to 0 \), and Eq. (25) suggests that the total amplitude mode is gapless when \( |\omega_{n}| \to 2\Delta_{0} \). In addition, the relative phase (Leggett) mode is gapped as Eq. (27) is not satisfied for \( \omega_{n} \to 0 \), and assuming \( |\omega_{n}| \ll \min(2E_{k} = 2\Delta_{0} \), its finite frequency is determined by \( \omega_{n}^2 = \{ \sum_{k}[d_{k}^2 - (d_{k}^2)/E_{k}^2] \}/[\sum_{k}[\Delta_{0}^2 + (d_{k}^2)/4E_{k}^2] \}. This is in agreement with the low-frequency expansion that is presented in Sec. 1113 where \( \omega_{n}^2 \to \bar{P}/\bar{R} \) at \( q = 0 \). Applying a similar analysis to Eq. (29), we find that the finite frequency of the relative amplitude mode is determined by \( \omega_{n}^2 = \{ \sum_{k}[E_{k}^2 - (d_{k}^2)]/E_{k}^2 \}/[\sum_{k}[\Delta_{0}^2 + (d_{k}^2)/4E_{k}^2] \}, and it is much larger than \( 2\Delta_{0} \). This clearly suggests that this mode is always damped, and it decays into the two-quasiparticle continuum. For instance, in the strong-coupling BEC limit when \( \Delta_{0} \gg \max d_{k} \), these frequencies can be approximated by \( \omega_{n}^2 = (8/N_{1}) \sum_{k}[d_{k}^2 - (d_{k}^2)/E_{k}^2] \) for the undamped Leggett mode, and by \( \omega_{n}^2 = 2N_{1}\Delta_{0}^2/ \sum_{k}(d_{k}^2) \) for the damped relative amplitude one. The former result is consistent with the recent literature, where undamped Leggett modes are found for sufficiently strong interactions away from the weak-coupling BCS limit [13, 113]. As a final remark, setting \( d_{k}^2 = 0 \) in Eq. (29) for the honeycomb case, we simply find \( 0 = 1/U \), suggesting that the relative amplitude branch disappears at \( q = 0 \).

### III. GEOMETRIC INTERPRETATION

As discussed in Sec. 1111, the total and relative fluctuations turn out to be uncoupled from each other in the limit when \( q \to 0 \). Next we consider this limit, and discuss purely total and purely relative fluctuations in detail due to their analytical simplicity.

#### A. Purely total fluctuations

For this purpose, it is sufficient to take into account the following terms in the small \( q \) and \( \omega \) expansions: \( T_{q,E}^{11} + T_{q}^{12} = A + \sum_{ij} C_{ij} q_{i} q_{j} - D\omega^2 + \cdots; T_{q,E}^{11} - T_{q}^{12} = \sum_{ij} Q_{ij} q_{i} q_{j} - R\omega^2 + \cdots \); and \( T_{q,O}^{11} = -B\omega + \cdots \). Since \( B \neq 0 \) in general, it couples the total phase and total...
amplitude fields, and therefore, we derive a total phase
(amplitude)-only action by integrating out the total am-
plitude (phase) fields. This leads to a phonon-like gapless
in-phase (Goldstone) mode and an exciton-like gapped
amplitude (Higgs) mode [18]
\[ \omega_{Gq}^2 = \sum_{ij} \frac{Q_{ij}}{R + B^2/A} q_{ij}, \]  
\[ \omega_{Rq}^2 = \frac{A + B^2/R}{D} + \sum_{ij} \left( \frac{C_{ij}}{D} + \frac{B^2Q_{ij}/R}{B^2 + AR} \right) q_{ij}. \]  
Here, the nonkinetic coefficients are given by \( A = \sum_{sk} \Delta^2_d/(2N_iE^3_{sk}), \) \( B = \sum_{sk} \xi_{sk}/(4N_iE^3_{sk}), \) \( D = \sum_{sk} \xi_{sk}/(8N_iE^3_{sk}), \) and \( R = \sum_{sk} 1/(8N_iE^3_{sk}). \) We note that these expressions are simply summations over their
conventional counterparts for the usual one-band prob-
lem, i.e., they are due entirely to intraband mechanisms.
On the other hand, the kinetic coefficients have a
tensor structure, and they consist of both an intra-
band and an interband contribution in such a way that \( C_{ij} = C_{ij}^{\text{intra}} + C_{ij}^{\text{inter}} \) and \( Q_{ij} = Q_{ij}^{\text{intra}} + Q_{ij}^{\text{inter}}. \) A compact way to express these coefficients are
\[ C_{ij}^{\text{intra}} = \frac{1}{N_i} \sum_{sk} \frac{1}{8E^3_{sk}} \left( 1 - \frac{5\Delta^2_d E^2_{sk}}{E^3_{sk}} \right) \frac{\partial \xi_{sk}}{\partial k_i} \frac{\partial \xi_{sk}}{\partial k_j}, \]  
\[ Q_{ij}^{\text{intra}} = \frac{1}{N_i} \sum_{sk} \frac{1}{8E^3_{sk}} \left( \frac{\partial \xi_{sk}}{\partial k_i} \frac{\partial \xi_{sk}}{\partial k_j} \right), \]  
\[ C_{ij}^{\text{inter}} = -\frac{1}{N_i} \sum_{sk} \frac{d_k}{4s\xi_{sk}E_{sk}} \left( 1 + \frac{2\Delta^2_d}{d_k^2} \right) g^k_{ij}, \]  
\[ Q_{ij}^{\text{inter}} = -\frac{1}{N_i} \sum_{sk} \frac{d_k}{4s\xi_{sk}E_{sk}} g^k_{ij}. \]  
We again note that while Eqs. (32) and (33) can be expressed as a sum over their conventional counter-
parts, Eqs. (34) and (35) do not have counterparts in the usual one-band problem. It turns out that the inter-
band contributions are controlled by the quantum met-
ric tensor \( g^k_{ij} \) of the underlying quantum states in \( k \) space [35]. For our generic two-band lattice model,
the quantum metric tensor of the Bloch states can be writ-
ten as \( 2g^k_{ij} = -\partial d_k \cdot \partial d_k/(\partial k_i \partial k_j) \) or equivalently \( 2g^k_{ij} = (\partial d_k / \partial k_i) \cdot (\partial d_k / \partial k_j). \) Alternatively, it can be expressed as
\[ g^k_{ij} = \frac{1}{2d_k^2} \sum_{k,i,j} \frac{\partial d_k}{\partial k_i} \frac{\partial d_k}{\partial k_j} \cdot \frac{1}{2d_k^2} \frac{\partial d_k}{\partial k_i} \frac{\partial d_k}{\partial k_j}. \]  
without the loss of generality.

B. Purely relative fluctuations

Similar to Sec. [31] it may again be sufficient to take
into account the following terms in the small \( q \) and \( \omega \)
expansions: \( R_{q,E}^{11} + R_{q,E}^{12} = A + \sum_{ij} C_{ij} q_i q_j - D\omega^2 + \cdots; \) 
\( R_{q,O}^{11} = -\tilde{B} \omega + \cdots. \) None of these expansion coefficients have a
conventional counterpart in the usual one-band model.
For instance, the nonkinetic coefficients are given by
\[ \tilde{A}(\tilde{P}) = \frac{1}{U} - \frac{1}{N_i} \sum_{ss'k} \frac{\xi_{sk}\xi_{s'k}}{4E_{sk}E_{s'k}(E_{sk} + E_{s'k})} \Delta^2_{ss'}^2 x^k_{ss'}, \]  
\[ \tilde{D}(\tilde{R}) = \frac{1}{N_i} \sum_{ss'k} \frac{\xi_{sk}\xi_{s'k}}{4E_{sk}E_{s'k}(E_{sk} + E_{s'k})} \Delta^2_{ss'}^2 x^k_{ss'}, \]  
\[ \tilde{B} = \frac{1}{N_i} \sum_{ss'k} \frac{\xi_{sk}\xi_{s'k}}{4E_{sk}E_{s'k}(E_{sk} + E_{s'k})} \Delta^2_{ss'}^2 x^k_{ss'}, \]  
where we define \( x^k_{ss'} = 1 - s_{s'} (d_k^0)^2 + (d_k^0)^2 - (d_k^0)^2/d_k^2. \) The kinetic coefficients \( \tilde{C}_{ij} \) and \( \tilde{Q}_{ij} \) are more involved and not presented here.

This expansion suggests that the Leggett mode is
gapped as long as \( \tilde{P} \neq 0, \) and its finite frequency
is determined by \( \omega^2 = \tilde{P}^2/\tilde{R}, \) when the coupling between the relative phase and relative amplitude fields is negli-
gible. Here we note an intuitive result that \( \tilde{P} = 0 \) when the
two bands are identical, i.e., when the sublattice-coupling
field \( d_k = 0 \) vanishes and therefore \( \xi_{sk} = \xi_{s'k} = \xi = \mu. \) In addition, in the strong-coupling BEC limit when \( \Delta_0 \gg \max |\xi_{sk}|, \) we note that \( \tilde{P} \rightarrow 0 \) as well. This is because since \( \mu \ll 0 \) and \( |\mu| \gg \max |\xi_{sk}| \) in the dilute limit of
particles/holes when \( 0.5 - N_0/N_i \approx \pm 0.5, \) and \( |\mu| \approx 0 \) around half filling when \( N_0/N_i \approx 0.5, \) one can substitute
\( \xi_{sk} \rightarrow -\mu \) and \( E_{sk} \rightarrow \sqrt{\omega^2 + \Delta^2_0}. \) Thus, we conclude that the Leggett mode becomes undamped for sufficiently
strong interactions with a negligibly smaller gap in the
strong-coupling limit. This result is also intuitive given that
the sublattice structure of the non-interacting parti-
cles should not play a primary role in the regime of
tightly-bound molecules.

Since \( \tilde{B} \neq 0 \) in most cases, we derive a relative phase
(amplitude)-only action by integrating out the relative
amplitude (phase) fields. This leads to an exciton-like out-of-phase (Leggett) mode and an exciton-like higher-
energy amplitude (Higgs) mode [19]
\[ \omega_{L(H)}^2 = \frac{\tilde{B}^2 + \tilde{A} \tilde{R} + \tilde{P} \tilde{D}}{2 \tilde{D} \tilde{R}} = \tilde{W}, \]  
\[ + \sum_{ij} \frac{\tilde{C}_{ij} q_i q_j}{2 \tilde{D} \tilde{R}} \left( \frac{1}{\tilde{W}} \right) \]  
\[ + \frac{\tilde{Q}_{ij} q_i q_j}{2 \tilde{D} \tilde{R}} \left( \frac{1}{\tilde{W}} \right) \]  
where we define \( \tilde{W} = [(\tilde{B}^2 + \tilde{A} \tilde{R} + \tilde{P} \tilde{D})^2 - 4\tilde{A} \tilde{P} \tilde{D} \tilde{R}]^{1/2}. \) Here,
the leading nonzero contribution to \( \beta_\theta \) is approxi-
mately by \( \tilde{D} \tilde{R} \), and it must be replaced with the
proper factor coming from the higher-order expansion
coefficients in those exceptional cases when \( \tilde{D} = 0. \) One
such example is the honeycomb lattice that is consid-
ered in Sec. [31] for which case we find \( \tilde{B} = 0 \) and
set \( \tilde{W} = \tilde{A} \tilde{R} - \tilde{P} \tilde{D} \), where \( \tilde{A} = 1/U = \sum_k 1/(N_k E_k) \), 
\( \tilde{P} = \sum_k d_k^2/(N_k E_k^2) \), \( \tilde{R} = \sum_k \Delta_k^2/(4N_k E_k^2) \), and \( \tilde{D} = 0 \).

C. SF phase stiffness tensor

At \( T = 0 \), we verify that the SF phase stiffness tensor \( D_{ij} \) is directly proportional to the kinetic coefficient \( Q_{ij} \) of the total phase fluctuations, i.e., \( D_{ij} = 8N/(\Delta_k^2/A)Q_{ij} \) with \( A \) the area of the lattice, in such a way that \( [2, 4, 5] \)

\[
D^\text{conv}_{ij} = \frac{\Delta_k^2}{A} \sum_{sk} \frac{1}{E_k^3} \frac{\partial \xi_{sk}}{\partial k_i} \frac{\partial \xi_{sk}}{\partial k_j}, \quad (41)
\]

\[
D^\text{geom}_{ij} = -\frac{2\Delta_k^2}{A} \sum_{sk} \frac{d_k}{\xi_{sk} E_{sk}} g_{ij}^k, \quad (42)
\]

In addition to those given in Sec. II D 1 and II D 2, this association may be considered as a third benchmark for the consistency of our results with the recent literature. The direct link between the quantum metric tensor and the SF stiffness tensor is relatively new in the literature \( [1, 2] \), revealing the geometric origin of superconductivity in the presence of other bands. This result is particularly illuminating for a narrow- or flat-band superconductivity for which the geometric contribution clearly dominates the SF stiffness tensor when the conventional one is negligible. Motivated by these works, there have been many studies on the subject exploring a variety of multi-band Hamiltonians, including most recently that of the twisted bilayer graphene \( [10 - 12] \).

Furthermore, it has been proposed that the quantum metric tensor has a partial control over all those SF properties that depend explicitly on the effective-mass tensor of the SF carriers, i.e., of the corresponding (two- or many-body) bound state \( [4, 5] \). In the context of two-band SFs, our finding Eq. (30) for the velocity of the Goldstone mode is in complete agreement with our earlier work \( [9] \), suggesting that an analogous contribution to the collective excitations must be present in many other multi-band systems as well.

IV. CONCLUSIONS

In summary, we considered a generic lattice Hamiltonian that is suitable for describing a uniform BCS SF with two underlying sublattices, and then extracted its collective excitations from an effective action that is derived up to the Gaussian order in the fluctuations of the SF order parameter. Allowing for independent fluctuations on the two sublattices, there are phonon-like in-phase (Goldstone) and exciton-like out-of-phase (Leggett) modes in this system. While the Goldstone mode is gapless at zero momentum and propagating in general, the Leggett mode becomes undamped only with sufficiently strong interactions. Furthermore, we showed that, in addition to the conventional contribution, the velocity of the Goldstone mode has a geometric contribution that is governed by the quantum metric tensor of the Bloch states. This suggests that the latter contribution dominates the velocity when the former one becomes negligible for a narrow- or a flat-band model. We traced the origin of the geometric contribution to the Goldstone mode back to the recent works on the geometric contribution to the SF stiffness tensor, and argued that these geometric effects are complementary to each other, i.e., they are both controlled by the effective-mass tensor of the SF carriers. This suggests that an analogous contribution to the collective excitations must be present in many other multi-band systems including the twisted bilayer graphene \( [10 - 12] \). As a further outlook, it is also worthwhile to study the damping of these collective excitations at finite temperatures \( [21] \).

Acknowledgments

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[17] Here the multiplying factors follow from the trace over the sublattice sector, where $\text{Tr}[\tau_0, s\sigma_i, \sigma_j] = 2(1 + d_i + d_j + d_i^2 + d_j^2)$ for the elements of the submatrix $T$, $\text{Tr}[\tau_0, s\sigma_i, \sigma_j, \sigma_k] = 2(1 - d_i + d_j^2 + d_i^2)$ for the elements of the submatrix $R$, and $\text{Tr}[\tau_0, s\sigma_i, \sigma_j, \sigma_k] = 2(d_i + d_j + i(d_i^2 - d_j^2))$ for the elements of the submatrix $C_i$.
[18] For the purely total fluctuations, one needs to derive the characteristic equation up to fourth order in the expansion, e.g., $\beta_2 \omega^2 + \sum_{ijkl} \beta_4^{ijkl} q_i q_j q_k q_l + \sum_{ij} \beta_5^{ij} q_i q_j = 0$, such that the Goldstone mode $\omega_{G}^{2} = \sum_{ij} \beta_4^{ij} q_i q_j$ is determined by $x_{G}^{ij} = -\beta_4^{ij} / \beta_2$, and the Higgs mode $\omega_{H}^{2} = \sum_{ij} \beta_5^{ij} q_i q_j$ is determined by $x_{H}^{ij} = -\beta_5^{ij} / \beta_2$. In the main text, while our quadratic expansion fully determines $x_{G}^{ij}$, i.e., our Eq. (30) is exact, we neglect the additional corrections to $\omega_{H}^{2}$ and $x_{H}^{ij}$ that are coming from the higher-order terms to $\beta_4^{ij}$ and $\beta_5^{ij}$. To be more precise, we substitute $\beta_4^{ij} = A Q_{ij}$, $\beta_5^{ij} = -A R - B^2$, $\beta_5^{ij} \approx -R C_{ij} - D Q_{ij}$, and $\beta_5 \approx D R$.
[19] For the purely relative fluctuations, one needs to derive the characteristic equation up to fourth order in the expansion, e.g., $\beta_2 \omega^4 + \sum_{ijkl} \beta_4^{ijkl} q_i q_j q_k q_l + \sum_{ij} \beta_5^{ij} \omega^2 q_i q_j + \beta_6 \omega^2 + \sum_{ijkl} \beta_7^{ijkl} q_i q_j q_k q_l = 0$, such that the Goldstone mode $\omega_{G}^{2} = \sum_{ij} \beta_4^{ij} q_i q_j$ is determined by $x_{G}^{ij} = -\beta_4^{ij} / \beta_2$, and the Higgs mode $\omega_{H}^{2} = \sum_{ij} \beta_5^{ij} q_i q_j$ is determined by $x_{H}^{ij} = -\beta_5^{ij} / \beta_2$. In the main text, while our quadratic expansion fully determines $x_{G}^{ij}$, i.e., our Eq. (30) is exact, we neglect the additional corrections to $\omega_{H}^{2}$ and $x_{H}^{ij}$ that are coming from the higher-order terms to $\beta_4^{ij}$ and $\beta_5^{ij}$. To be more precise, we substitute $\beta_4^{ij} = A Q_{ij}$, $\beta_5^{ij} = -A R - B^2$, $\beta_5^{ij} \approx -R C_{ij} - D Q_{ij}$, and $\beta_5 \approx D R$.
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