A ZERO DENSITY RESULT FOR THE RIEMANN ZETA FUNCTION

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Abstract. In this article, we prove an explicit bound for $N(\sigma, T)$, the number of zeros of the Riemann zeta function satisfying $\Re s \geq \sigma$ and $0 \leq \Im s \leq T$. This result provides a significant improvement to Rosser’s bound for $N(T)$ when used for estimating prime counting functions.

1. Introduction

In recent years, it has become apparent that explicit results concerning prime numbers are required to solve important problems in number theory. In particular, the impressive works of Ramaré [18], Tao [30], and Helfgott [13] related to Goldbach’s conjecture highlight the need of better explicit bounds for finite sums over primes. For instance, they make use of [4], [21], [22], [24], [25], [26], [28]. Moreover articles of Rosser and Schoenfeld ([24], [25], [26], [27], [28]), Dusart ([5], [6], [7], [8]), and Ramaré and Rumely [23] are extensively used in a wide range of fields including Diophantine approximation, cryptography, and computer science. These results on primes rely heavily on explicit estimates of sums over the non-trivial zeros of the Riemann zeta function. More precisely, they rely on three key ingredients: a numerical verification of the Riemann Hypothesis (RH), an explicit zero-free region, and explicit bounds for the number of zeros in the critical strip up to a fixed height $T$.

In 1986, van de Lune et al. [34] established that RH had been verified for all zeros $\zeta$ verifying $|\Im \zeta| \leq H_0$ with $H_0 = 545 439 823$. In 2011, Platt [15] [16] proved that $H_0 = 30 610 046 000$ is admissible. Previously, Wedeniwski [34] in 2001 and Gourdon [11] in 2004 had announced higher values for $H_0$. As Platt’s computations are more rigorous (he employs interval arithmetic), we decide to use his value throughout this article:

$$H_0 = 3.061 \cdot 10^{10}.$$ 

For the latest explicit results about zero-free regions for the Riemann zeta function, we refer the reader to [14] and [10].

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Let $\sigma \geq 0.55$. We consider $N(\sigma, T)$, the number of zeros of the Riemann zeta function in the region $\sigma \leq \Re s \leq 1$ and $0 \leq \Im s \leq T$. Trivially we have that $N(\sigma, T) = 0$ for all $T \leq H_0$. We prove here an explicit bound for $N(\sigma, T)$ valid in the range $T \geq H_0$.

**Theorem 1.1.** Let $\sigma \geq 0.55$ and $T \geq H_0$. Let $\sigma_0$ and $H$ such that $0.5208 < \sigma_0 < 0.9723, \sigma_0 < \sigma,$ and $10^3 \leq H \leq H_0$. Then there exist $b_1, b_2, b_3$, positive constants depending on $\sigma, \sigma_0, H$, such that:

$$N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3.$$  

The $b_i$'s are defined in (6.3).

We rewrite this as $N(\sigma, T) \leq c_1T + c_2 \log T + c_3$, for $T \geq H_0$. Numerical values of the $b_i$'s and $c_i$'s are recorded at the end of this article in Table 1. For example, for $\sigma \geq 17/20$ and $T = H_0 + 1$, we have

$$N(\sigma, T) \leq 0.5561T + 0.7586 \log T - 268.658.$$  

Let $N(T)$ be the number of non-trivial zeros $\zeta$ with imaginary part $0 \leq \Im \zeta \leq T$. We recall that Rosser [24] proved

$$|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8}| \leq a \log T + b \log \log T + c,$$

with $a = 0.137$, $b = 0.443$, $c = 1.588$. Note that Rosser’s result got recently improved by Trudgian [33, Corollary 1] with $a = 0.111$, $b = 0.275$, $c = 2.450$. A trivial bound for $N(\sigma, T)$ follows from the inequalities $N(\sigma, T) \leq \frac{1}{2}N(T)$ and (1.1):

$$N(\sigma, T) \leq \frac{T}{4\pi} \log \left(\frac{T}{2\pi e}\right)(1 + o(1)).$$

Note that when $T$ is asymptotically large, then a factor of $\log T$ is saved. Moreover, we have $c_1 \sim \frac{\log(\log(2\sigma_0))}{4\pi(\sigma - \sigma_0)}$ where $\sigma_0$ is a parameter which value can be chosen to make $c_1$ as small as possible. Another feature of Theorem 1.1 is the factor $T - H$: when $T$ is near $H_0$, we choose $H$ to be close to $H_0$ so as to make $N(\sigma, T)$ of size $\log H_0$. This saves a factor of size $H_0$. As an example, for $\sigma \geq 17/20$ and $T = H_0 + 1$, we choose $H = H_0 - 1$ and $\sigma_0$ as in Table 1 and obtain $N(\sigma, H_0 + 1) \leq 156$ while (1.1) gives $5.2 \cdot 10^{10}$ (with either Rosser’s or Trudgian’s values).

The key motivation for establishing Theorem 1.1 is to use it in place of (1.1) and thus to provide improved explicit bounds for Chebyshev’s prime counting functions. We prove in [9] that, for all $x \geq e^b$,

$$|\psi(x) - x| \leq \epsilon_b x,$$  

(1.2)
where $b$ is a fixed positive constant, and $\epsilon_b$ is an effective positive constant. For example, for $x \geq e^{50}$ we obtain $\epsilon_{50} = 9.461 \cdot 10^{-10}$ while Dusart [7, Theorem 2] obtained $0.905 \cdot 10^{-7}$.

Despite a very rich history of asymptotic results, there were almost no explicit bounds for $N(\sigma, T)$. Ramaré proved in an unpublished manuscript [19] that, for $T \geq 2000$, $Q \geq 10$, and $T \geq Q$,

$$\sum_{q \leq Q} \sum_{\chi \mod q} N(\sigma, T, \chi) \leq 157 (Q^5 T^3)^{1-\sigma} \log^{4-\sigma}(Q^2 T) + 6Q^2 \log^2(Q^2 T),$$

where $\sum_{\chi \mod q}$ denotes the sum over primitive Dirichlet characters $\chi$ to the modulus $q$, and $N(\sigma, T, \chi)$ counts the number of zeros $\rho$ of the Dirichlet $L$-function $L(s, \chi)$ satisfying $\sigma < \Re \rho < 1$ and $0 < \Im \rho < T$. Taking $Q = 10$ and restricting the left sum to $q = 1$, it follows that

$$N(\sigma, T) \leq 157(100000 T^3)^{1-\sigma} \log^{4-\sigma}(100 T) + 600 \log^2(100 T).$$

Our main theorem improves Ramaré’s result for certain values of $\sigma$ and $T$: he obtains $N(17/20, 10 \cdot H_0) \leq 2.675 \cdot 10^{12}$ while we have $N(17/20, 10 \cdot H_0) \leq 3.404 \cdot 10^{10}$. In 2010, Cheng [3] obtained the weaker result:

$$N(\sigma, T) \leq 453\,472.54 T^{8/3(1-\sigma)(\log T)},$$

for all $\sigma \geq 5/8$ and $T \geq \exp(\exp(18)) \simeq 10^{28.515.762}$. His method is based on Ford’s [10] effective version of Korobov-Vinogradov’s bound for the Riemann zeta function. He applied (1.4) to deduce explicit results on primes between consecutive cubes. Note that Cheng’s result is not valid in the region $T \leq \exp(\exp(18))$ while most applications require bounds for $T$ as small as $H_0$.

In order to prove Theorem 1.1 we establish two intermediate theorems about $\zeta(s)$ in the critical strip: an effective version of a Dirichlet polynomial approximation, and an explicit estimate for the second moment.

**Theorem 1.2.** Let $t_0 > 0$, $s = \sigma + it$ with $\sigma \geq 1/2$, $t \geq t_0$ and $c > \frac{1}{2\pi}$. Then

$$\zeta(s) = \sum_{1 \leq n < ct} \frac{1}{n^s} + R(s)$$

with $|R(s)| \leq C(\sigma, c)t^{-\sigma}$, and

$$C(\sigma, c) = \left( c + \frac{1}{2} + \frac{3\sqrt{1+1/t_0^2}}{2\pi} \left( \frac{\zeta(2)}{2\pi c} + 1 + \frac{1}{2\pi c - 1} \right) \right)c^{-\sigma}.$$

We apply the theorem for $c = 1$ and for $t_0$ the height of the first zero of zeta.
Corollary 1.3. Let $\sigma \geq 1/2$ and $t \geq 14.1347$. Then

\begin{equation}
\zeta(s) - \sum_{1 \leq n < t} \frac{1}{n^s} \leq c_0 t^{-\sigma}, \quad \text{where } c_0 = 2.1946.
\end{equation}

This is to compare to Proposition 1 of Cheng [2] who obtained $5.505$ instead of $2.1946 t^{-\sigma}$. When $\sigma \geq 1/2$ and $0 \leq t \leq 15$, a Mathematica computation gives us that $|\zeta(s) - \sum_{1 \leq n < t} n^{-s}| \leq 43 t^{-\sigma}$.

Theorem 1.4. Let $0.5208 < \sigma_0 < 0.9723$ and $10^3 \leq H \leq H_0$. We define

\begin{align}
\epsilon_1(\sigma_0, H) &= \frac{4H_0}{H_0 - H} \left( \frac{\log H_0 H_0^{1-2\sigma_0}}{2(1 - \sigma_0)} - \frac{(2\sigma_0 - 1) \log H_0}{2(1 - \sigma_0)} \right. \\
&\quad + \max \left( \frac{1}{2(1 - \sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2}, H_0^{-2\sigma_0} \right) + \frac{2 - \sigma_0}{2(1 - \sigma_0)^2} H_0^{1 - 2\sigma_0} \\
&\quad - \frac{\sigma_0 H_0^{-\sigma_0}}{2(1 - \sigma_0)^2} + \frac{H_0^{-2\sigma_0}}{2(2\sigma_0 - 1)} + \frac{H_0^{-2\sigma_0 - 1}}{2},
\end{align}

\begin{align}
\epsilon_2(\sigma_0, H) &= \frac{c_0^2}{2\sigma_0 - 1} \frac{H^{-(2\sigma_0 - 1)} - H_0^{-(2\sigma_0 - 1)}}{H_0 - H},
\end{align}

\begin{align}
\epsilon_3(\sigma_0, H) &= 2\sqrt{\epsilon_2(\sigma_0, H)(\zeta(2\sigma_0) + \epsilon_1(\sigma_0, H))},
\end{align}

\begin{align}
\mathcal{E}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3.
\end{align}

Then, for all $T \geq H_0$, we have

\begin{align}
\frac{1}{T - H} \int_T^H |\zeta(\sigma_0 + it)|^2 \, dt &\leq \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H), \\
\text{and } \int_T^H \log |\zeta(\sigma_0 + it)| \, dt &\leq \frac{T - H}{2} \log \left( \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right).
\end{align}

For the rest of this article $H$, $T$, $\sigma_0$, and $\sigma$ satisfy

\begin{align}
H_0 &= 3.061 \cdot 10^{10}, 10^3 \leq H \leq H_0 \leq T, 0.5208 < \sigma_0 < 0.9723, \sigma_1 = 1.5002, \sigma_0 < \sigma < \sigma_1.
\end{align}

2. Approximate formula for $\zeta(\sigma + it)$ - Proof of Theorem 1.2

Let $s = \sigma + it$ with $1/2 < \sigma < 1$ and $t \geq 2$. Let $x = ct$ with $c > \frac{1}{2\pi}$, and let $N$ be a positive integer. Theorem 1.2 gives an explicit version of an approximation formula for zeta, as proven by Hardy and Littlewood in [12].

Proof. We start with the classical identity [31, equation 3.5.3]

\begin{align}
\zeta(s) - \sum_{1 \leq n < x} \frac{1}{n^s} = \sum_{x \leq n \leq N} \frac{1}{n^s} + s \int_N^\infty (u) \frac{du}{u^{s+1}} - \frac{N^{1-s}}{1-s} - \frac{1}{2} N^{-s},
\end{align}
Thus where \( ((u)) = [u] - u + 1/2 \). The summation formula \([31\text{, equation 2.1.2}]
\]
gives
\[
\sum_{x \leq n < N} \frac{1}{n^s} = \int_x^N \frac{du}{u^s} - \frac{((x))}{x^s} + \int_x^N \frac{((u))}{u^{1+s}} du = \frac{N^{1-s} - x^{1-s}}{1-s} - \frac{((x))}{x^s} + s \int_x^N \frac{((u))}{u^{1+s}} du.
\]
We have the bounds
\[
\left| \frac{x^{1-s}}{1-s} \right| \leq \frac{x^{1-\sigma}}{t}, \quad \left| \frac{((x))}{x^s} \right| \leq \frac{x^{-\sigma}}{2}, \quad \left| s \int_N^\infty \frac{((u))}{u^{s+1}} du \right| \leq \left| s \right| \frac{1}{2} \int_N^\infty \frac{1}{u^{\sigma+1}} du = \frac{|s|}{2\sigma N^\sigma}.
\]

Thus
\[
(2.2) \quad \left| \zeta(s) - \sum_{n \leq N < x} \frac{1}{n^s} \right| \leq x^{1-\sigma}t^{-1} + \frac{x^{-\sigma}}{2} + \left| s \int_x^N \frac{((u))}{u^{1+s}} du \right| + \frac{|s|}{2\sigma} N^{-\sigma} + \frac{1}{2} N^{-\sigma}.
\]

The choice \( x = ct \) is made to balance the error term \( x^{1-\sigma}t^{-1} + \frac{x^{-\sigma}}{2} \). We appeal to the Fourier series of \( ((x)) \) to obtain a smaller bound for the integral expression. For \( u \notin \mathbb{N} \), we have \([31\text{, p. 74}]
\]
\[
((u)) = [u] - u + 1/2 = \sum_{\nu=1}^{\infty} \frac{\sin(2\pi \nu u)}{\nu}.
\]

Lebesgue’s bounded convergence theorem applies, and we can exchange the order of the integral and the summation. We obtain
\[
(2.3) \quad \int_x^N \frac{((u))}{u^{1+s}} du = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_x^N \frac{\sin(2\pi \nu u)}{u^{1+s}} du = \sum_{\nu=1}^{\infty} \frac{I(\nu) - I(-\nu)}{\nu},
\]

where the integral \( I \) is given by
\[
(2.4) \quad I(h) = \frac{1}{2\pi i} \int_x^N e^{2\pi i (hu - \frac{t \log u}{2\pi})} du = \frac{1}{2\pi} \int_x^N F(h, u) d(e^{2\pi i (f(u) + hu)})
\]

with \( F(h, u) = \frac{u^{-\sigma}}{t - 2\pi uh} \) and \( f(u) = -\frac{t \log u}{2\pi} \). Since \( \frac{\partial}{\partial u} F(h, u) = u^{-\sigma} - \sigma tu^{-1} + 2\pi h (\sigma + 1) \),

it is easy to check that \( F(-\nu, u) \) is positive and decreases with \( u \), and that \( F(\nu, u) \) is negative and increases with \( u \).

We now apply the second mean value theorem from \([32\text{, section 12.3}]
\]

**Lemma 2.1.** If \( j(x) \) is integrable over \((a, b)\), and \( \phi(x) \) is positive, bounded, and non-increasing, then there exists \( \xi \in (a, b) \) such that
\[
\int_a^b \phi(x) j(x) dx = \phi(a + 0) \int_a^\xi j(x) dx.
\]

First, we consider \( I(-\nu) \). We separate the real and imaginary part in \( d(e^{2\pi i (f(u) + hu)}) \) in \((2.4)\) and we apply the Lemma for \( \phi(u) = F(-\nu, u) \). We
consider $j(u)du = d(\cos(2\pi(f(u) - \nu u)))$, and $j(u)du = d(\sin(2\pi(f(u) - \nu u)))$ respectively. We obtain that there exist $\xi_1, \xi_2 \in (x, N)$ such that

$$2\pi I(-\nu) = F(-\nu, x) \cos(2\pi(f(\xi_1) - \nu \xi_1)) - F(-\nu, x) e^{2\pi i (f(x) - \nu x)}$$

$$+ i F(-\nu, x) \sin(2\pi(f(\xi_2) - \nu \xi_2)).$$

It follows that

$$|I(-\nu)| \leq \frac{3}{2\pi} F(-\nu, x) = \frac{3}{2\pi} \frac{(ct)^{-\sigma}}{t + 2\pi \nu} \leq \frac{3}{(2\pi)^2} \frac{c^{-\sigma} t^{-\sigma - 1}}{\nu}.$$  \hspace{1cm} (2.5)

A similar argument applies to $I(\nu)$. We obtain

$$|I(\nu)| \leq -\frac{3}{2\pi} F(\nu, ct) = \frac{3}{2\pi} \frac{(ct)^{-\sigma}}{2\pi \nu t - t} \leq \begin{cases} \frac{3}{2\pi} \frac{c^{-\sigma} t^{-\sigma - 1}}{\nu} & \text{if } \nu \geq 2, \\ \frac{3}{2\pi} \frac{c^{-\sigma} t^{-\sigma - 1}}{2\pi c - 1} & \text{if } \nu = 1. \end{cases}$$  \hspace{1cm} (2.6)

Using the simplification $\sum_{\nu=2}^\infty \frac{1}{\nu(\nu-1)} = 1$, $\sum_{\nu=1}^\infty \frac{1}{\nu^2} = \zeta(2)$, and $|s| \leq \sqrt{1 + 1/t^2}$, we put together (2.3), (2.5), and (2.6), and obtain the bound

$$\left| s \int_x^N \frac{(u)}{u^{1+s}} du \right| \leq |s| \sum_{\nu=1}^\infty \frac{|I(\nu)| + |I(-\nu)|}{\nu} \leq 3 \sqrt{1 + 1/t^2} \frac{1}{2\pi} \left( 1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c} \right) c^{-\sigma} t^{-\sigma}.$$  \hspace{1cm} (2.7)

Letting $N \to \infty$, inequality (2.2) becomes

$$\left| \zeta(s) - \sum_{1 \leq n < ct} \frac{1}{n^s} \right| \leq \left( c + \frac{1}{2} + \frac{3 \sqrt{1 + 1/t^2}}{2\pi} \left( 1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c} \right) \right) (ct)^{-\sigma}. \hspace{1cm} \Box$$

**Remark 2.2.** A careful reading of Cheng’s proof shows that his error term has size $O(t^{1-2\sigma})$, instead of our $O(t^{-\sigma})$. This comes from the fact that he bounds directly the terms $N^{1-s}_{n < ct}$, instead of eliminating them as we did.

3. **Explicit upper bound for the second moment of zeta - Proof of Theorem 1.4**

We recall that $\sigma_0, T, H$ are as in (1.11). By Theorem 1.2 we have the identity

$$\frac{1}{T-H} \int_H^T |\zeta(\sigma_0 + it)|^2 \, dt = D(\sigma_0, T, H) + E_1(\sigma_0, T, H) + E_2(\sigma_0, T, H) + E_3(\sigma_0, T, H),$$  \hspace{1cm} (3.1)
where

\[ D(\sigma_0, T, H) = \frac{1}{T-H} \int_H^T \sum_{1 \leq n < t} \frac{1}{n^{2\sigma_0}} dt, \]

\[ E_1(\sigma_0, T, H) = \frac{2}{T-H} \int_H^T \sum_{1 \leq n < m < t} \frac{\cos(t \log(m/n))}{(nm)^{\sigma_0}} dt, \]

\[ E_2(\sigma_0, T, H) = \frac{1}{T-H} \int_H^T |R(\sigma_0 + it)|^2 dt, \]

\[ E_3(\sigma_0, T, H) = 2 \frac{T-H}{T-H} \Re \int_H^T \sum_{1 \leq n < t} R(\sigma_0 + it) \frac{R}{n^{\sigma_0+it}} dt. \]

We recall here some basic inequalities that we use throughout the following argument. Let \( A, B \in \mathbb{N}. \) If \( f \) is decreasing and positive, then

\[ \sum_{A \leq j \leq B} f(j) \leq f(A) + \int_A^B f(u) du. \]

For \( \sigma_0 > 1/2, \) we bound trivially the diagonal term:

\[ D(\sigma_0, T, H) \leq \zeta(2\sigma_0). \]

We interchange summation order in the off-diagonal terms \( E_1(\sigma_0, T, H) \) and use the fact that \( \int_u^v \cos(at) dt \leq \frac{2}{a} \) when \( a \neq 0: \)

\[ E_1(\sigma_0, T, H) \leq \frac{4}{T-H} \sum_{1 \leq n < m < T} \frac{(nm)^{-\sigma_0}}{\log(m/n)}. \]

We use the fact that, for \( \lambda > 1 \) and \( \sigma < 1, \) \( \frac{1}{\log \lambda} \leq 1 + \frac{\lambda^{1-\sigma}}{\lambda-1}. \) Taking \( \lambda = \frac{m}{n}, \)

we obtain

\[ E_1(\sigma_0, T, H) \leq \frac{4}{T-H} \sum_{1 \leq n < m < T} (nm)^{-\sigma_0} + \frac{4}{T-H} \sum_{1 \leq n < m < T} \frac{m^{1-2\sigma_0}}{m-n}. \]

For the first sum, we complete the square

\[ \sum_{1 \leq n < m < T} (nm)^{-\sigma_0} = \frac{1}{2} \left( \sum_{k<T} k^{-\sigma_0} \right)^2 - \frac{1}{2} \sum_{k<T} k^{-2\sigma_0} = \frac{1}{2} \left( \sum_{k<T} k^{-\sigma_0} \right)^2 - \frac{1}{2} \left( \zeta(2\sigma_0) - \sum_{k \geq T} k^{-2\sigma_0} \right), \]

and use \((3.2)\) with \( f(t) = t^{-\sigma_0} \) and \( f(t) = t^{-2\sigma_0} \) to bound the resulting sums. We obtain

\[ \sum_{1 \leq n < m < T} (nm)^{-\sigma_0} \leq \frac{T^2(1-\sigma_0)}{2(1-\sigma_0)^2} \frac{\sigma_0 T^1-\sigma_0}{(1-\sigma_0)^2} + \frac{\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2} \zeta(2\sigma_0) - \frac{T^1-2\sigma_0}{2(1-2\sigma_0)} + \frac{T-2\sigma_0}{2}. \]

We consider \( k = m-n \) and separate variables in the second sum of \((3.4)\)
and use \((3.2)\), with \( f(t) = t^{1-2\sigma_0} \) and \( f(t) = t^{-1} \), to bound the resulting
Together with (3.4), (3.5) and (3.6), we obtain
\[ E_1(\sigma_0, T, H) \leq \frac{4T}{T-H} \left( \frac{\log T \cdot T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0 - 1}{2(1-\sigma_0)} \log T \right) + \frac{1-3\sigma_0 + 3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2} \zeta(2\sigma_0) \frac{1}{T} \]
\[ + \frac{2-\sigma_0}{2(1-\sigma_0)^2} T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{T^{-2\sigma_0}}{2(2\sigma_0 - 1)} + \frac{1}{2} T^{-2\sigma_0 - 1}. \]

We denote
\[ E_{11}(\sigma_0, T) = \frac{\log T \cdot T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0 - 1}{2(1-\sigma_0)} \log T, \]
\[ E_{12}(\sigma_0, T) = \frac{1-3\sigma_0 + 3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2} \zeta(2\sigma_0) \frac{1}{T}, \]
\[ E_{13}(\sigma_0, T) = \frac{2-\sigma_0}{2(1-\sigma_0)^2} T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2}, \]
\[ E_{14}(\sigma_0, T) = \frac{T^{-2\sigma_0}}{2(2\sigma_0 - 1)} + \frac{1}{2} T^{-2\sigma_0 - 1}, \]

and we now study their behavior with respect to \( T \geq H_0. \) It is immediate that \( E_{14} \) decreases with \( T. \) Considering the fact that \( \frac{1-3\sigma_0 + 3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2} \zeta(2\sigma_0) \) changes sign at \( \sigma_0 = 0.679785 \ldots, \) we obtain
\[ E_{12}(\sigma_0, T) \leq \max \left( 0, E_{12}(\sigma_0, H_0) \right). \]

For \( 0.5208 < \sigma_0 < 1, \) we find
\[ \frac{\partial E_{11}(\sigma_0, T)}{\partial T} = \frac{(1-2\sigma_0)(2\sigma_0 - 1)(\log T - 1) + 2(1-\sigma_0)}{2(1-\sigma_0)T^2} \leq 0, \]

and, when \( \sigma_0 \leq 0.9723, \) that
\[ \frac{\partial E_{13}(\sigma_0, T)}{\partial T} = \left( - \frac{(2-\sigma_0)(2\sigma_0 - 1)}{2} T^{1-\sigma_0} + \sigma_0^2 \right) \frac{T^{-1-\sigma_0}}{(1-\sigma_0)^2} \leq 0. \]

Thus \( E_{11}(\sigma_0, T) \) and \( E_{13}(\sigma_0, T) \) decrease with \( T \geq H_0. \) We conclude that, for \( T \geq H_0 \) and \( 0.5208 \leq \sigma_0 \leq 0.9723, \)
\[ (3.7) \ E_1(\sigma_0, T, H) \leq \frac{4H_0}{H_0 - H} \left( \frac{\log H_0 \cdot H_0^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0 - 1}{2(1-\sigma_0)} \log H_0 \right) \]
\[ + \max \left( 0, \frac{1-3\sigma_0 + 3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2} \right) \frac{\zeta(2\sigma_0)}{2} + \frac{2(2\sigma_0 - 1) + H_0^{-2\sigma_0 - 1}}{2}. \]
Theorem 1.2 gives
\[ E_2(\sigma_0, T, H) \leq \epsilon_0^2 \frac{1}{T - H} \int_H^T t^{-2\sigma_0} dt \leq \epsilon_0^2 \frac{H^{-(2\sigma_0-1)} - H_0^{-(2\sigma_0-1)}}{2\sigma_0 - 1}. \]

We use the Cauchy-Schwarz inequality to bound \( E_3 \):
\[ E_3(\sigma_0, T, H) \leq 2 \left( \frac{1}{T - H} \int_H^T |\Re R(s)|^2 dt \right)^{\frac{1}{2}} \left( \frac{1}{T - H} \int_H^T \left| \sum_{1 \leq n < t} \frac{1}{n^{\sigma_0+it}} \right|^2 dt \right)^{\frac{1}{2}} \]
\[ \leq 2 \sqrt{E_2(\sigma_0, T, H) (D(\sigma_0, T, H) + E_1(\sigma_0, T, H))} \]
\[ \leq 2 \sqrt{\epsilon_2(\sigma_0, H) (\zeta(2\sigma_0) + \epsilon_1(\sigma_0, H))}. \]

The definitions of \( \epsilon_1, \epsilon_2, \epsilon_3 \) follow from (3.7), (3.8), and (3.9). The proof is achieved by putting together (3.1), (3.3), (3.7), (3.8), (3.9), and by applying the following bound for concave functions
\[ \int_H^T \log |\zeta(\sigma_0 + it)| dt \leq \frac{T - H}{2} \log \left( \frac{1}{T - H} \int_H^T |\zeta(\sigma_0 + it)|^2 dt \right). \]

4. A lower bound for \( \log |\zeta(s)| \) when \( \sigma > 1 \).

**Lemma 4.1.** Let \( 2 \leq H \leq T \) and \( \sigma_1 = 1.5002 \). Then
\[ \int_H^T \log |\zeta(\sigma_1 + it)| dt \geq -E_2, \text{ with } E_2 = 1.7655. \]

**Proof.** Let \( s = \sigma_1 + it \). It follows from the Euler product that
\[ \log |\zeta(s)| = \Re \sum_{n \geq 2} \frac{\Lambda(n)}{(\log n)^s}. \]

Thus
\[ \int_H^T \log |\zeta(\sigma_1 + it)| dt = \sum_{n \geq 2} \frac{\Lambda(n) \left( \sin(T \log n) - \sin(H \log n) \right)}{(\log n)^2 n^{\sigma_1}} \geq -2 \sum_{n \geq 2} \frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}}. \]

We truncate the sum at \( N_0 = 10^3 \) and bound the tail
\[ \sum_{n > N_0} \frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}} \leq \frac{1}{(\log N_0)^2} \left( -\frac{\zeta'(\sigma_1)}{\zeta(\sigma_1)} - \sum_{n \leq N_0} \frac{\Lambda(n)}{n^{\sigma_1}} \right). \]

We obtain
\[ \int_H^T \log |\zeta(\sigma_1 + it)| dt \geq -2 \left( \frac{-\zeta'(\sigma_1)}{(\log N_0)^2} + \sum_{n \leq N_0} \frac{\Lambda(n)}{n^{\sigma_1}} \left( \frac{1}{(\log n)^2} - \frac{1}{(\log N_0)^2} \right) \right), \]
and a numerical calculation with Maple gives the value for the above left term. \( \square \)
5. Explicit bounds for $\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau$.

Lemma 5.1. Let $\eta = 0.0001, \sigma_1 = 3/2 + 2\eta = 1.5002$. Let $\sigma_0, T, H$ satisfy $\sigma_0 < \sigma_1, 2 \leq H \leq T$. Then

$$\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau \leq E_3(\sigma_0) \log(HT) + E_4(\sigma_0, H)$$

with

$$E_3(\sigma_0) = \frac{\pi(1 + 2\eta)(\sigma_1 - \sigma_0)}{4 \log 2},$$

$$E_4(\sigma_0, H) = \frac{\pi(\sigma_1 - \sigma_0)}{\log 2} \log \left( \frac{3H + 3(1 + \eta)}{H - (1 + 2\eta)} \right) \frac{3(1 + \eta)/H + 1}{2\pi} \zeta(1 + \eta)^4 \zeta(2(1 + \eta))^2.$$

It suffices to bound an integral of the form

$$\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it) d\tau,$$

with $t \geq H$. We only make use of the convexity bound for $\zeta(s)$.

Proof. Let $\omega \in \mathbb{C}$ and $N \in \mathbb{N}$. Following Rosser’s modification of Backlund’s trick ([1] equation (32) and [24] page 223), we introduce $f_i(\omega) = \frac{1}{2} \left( \zeta(\omega + it)^N + \zeta(\omega - it)^N \right)$. We denote $n$ to be the number of real zeros of $f_i(\tau) = \Re \zeta(\tau + it)^N$ in the interval $\sigma_0 < \tau < \sigma_1$. The interval is split into $n + 1$ subintervals and on each of them arg $\zeta(\tau + it)^N$ changes by at most $\pi$. Thus

$$\left| \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it) d\tau \right| = \frac{1}{N} \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it)^N d\tau \leq \frac{(\sigma_1 - \sigma_0)(n + 1)\pi}{N}.$$

We denote $n(r)$ the number of zeros of $f_i$ in the circle centered at $1 + \eta + it$, and with radius $r$. For $r \geq 1/2 + \eta$, the segment $[\sigma_0, \sigma_1]$ is contained in $[1 + \eta - r, 1 + \eta + r]$, thus $n \leq n(r)$. The following version of Jensen’s formula [29] p. 137, equation (2)],

$$\log |f_i(1 + \eta)| + \int_0^{1 + 2\eta} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \log |f_i(1 + \eta + (1 + 2\eta)e^{i\theta})| d\theta,$$

allows us to deduce an upper bound for $n$:

$$n \leq \frac{1}{\log 2} \int_0^{1 + 2\eta} \frac{n(r)}{r} dr \leq \frac{1}{2\pi \log 2} \int_{-\pi/2}^{3\pi/2} \log |f_i(1 + \eta + (1 + 2\eta)e^{i\theta})| d\theta - \log |f_i(1 + \eta)| \frac{\log 2}{\log 2}.$$

We write $\zeta(1 + \eta + it) = Re^{i\phi}$. Thus $f_i(1 + \eta) = \Re \left( \zeta(1 + \eta + it)^N \right) = R^N \cos(N\phi)$. We choose a sequence of $N$’s such that $\lim_{N \to \infty} N\phi = 0$ (mod $2\pi$).
Thus
\[(5.5) \quad \log |f_t(1 + \eta)| = N \log((1 + o(1))R) = N \log((1 + o(1))|\zeta(1 + \eta + it)|) \]
\[\geq N \log \left(\frac{\zeta(2(1 + \eta))}{\zeta(1 + \eta)}\right) + o_N(1),\]
where \(o_N(1) \to 0\) when \(N \to \infty\). We now split the integral in the left term of inequality (5.4) depending on the sign of \(\cos \theta\). For \(\theta \in (-\pi/2, \pi/2)\), we use the trivial bound
\[|\zeta(1 + \eta + (1 + 2\eta)e^{i\theta} \pm it)| \leq \zeta(1 + \eta),\]
giving
\[(5.6) \quad \int_{-\pi/2}^{\pi/2} \log |f_t(1 + \eta + (1 + 2\eta)e^{i\theta})| \, d\theta \leq N \pi \log (\zeta(1 + \eta)).\]
For \(\theta \in (\pi/2, 3\pi/2)\), we use Rademacher’s bound \[17, equation (7.4)\]:
\[|\zeta(s)| \leq 3 \frac{|1 + s|}{|1 - s|} \left(\frac{|1 + s|}{2\pi}\right)^{1+\eta-2\Re s} \zeta(1 + \eta)\]
with \(s = 1 + \eta + (1 + 2\eta)e^{i\theta} \pm it\). Since
\[|1 + s| \leq t + 3(1 + \eta), \quad |1 - s| \geq |3ms| \geq t - (1 + 2\eta), \quad \text{and} \quad 0 \leq 1 + \eta - \Re s \leq 1 + 2\eta,\]
then
\[(5.7) \quad \int_{\pi/2}^{3\pi/2} \log |f_t(1 + \eta + (1 + 2\eta)e^{i\theta})| \, d\theta \]
\[\leq N \pi \log \left(3 \frac{t + 3(1 + \eta)}{t - (1 + 2\eta)} \left(\frac{t + 3(1 + \eta)}{2\pi}\right)^{1+2\eta} \zeta(1 + \eta)\right).\]
Together with (5.4), (5.5), (5.6), and (5.7), we deduce
\[(5.8) \quad n \leq \frac{N}{2 \log 2} \log \left(3 \frac{t + 3(1 + \eta)}{t - (1 + 2\eta)} \left(\frac{3(1 + \eta) + t}{2\pi}\right)^{1+2\eta} \zeta(1 + \eta)^4 \zeta(2(1 + \eta))^{-2}\right) + o_N(1)\]
\[\leq \frac{N(1 + 2\eta)}{4 \log 2} \log t + \frac{N}{2 \log 2} \log \left(3 \frac{t + 3(1 + \eta)}{t - (1 + 2\eta)} \left(\frac{3(1 + \eta)/t + 1}{2\pi}\right)^{1+2\eta} \zeta(1 + \eta)^4 \zeta(2(1 + \eta))^{-2}\right) + o_N(1).\]
Together with (5.3) and letting \(N \to \infty\), we obtain
\[\left|\int_{\sigma_1}^{\sigma_0} \arg \zeta(t + it) \, d\tau\right| \leq \frac{\pi(1 + 2\eta)(\sigma_1 - \sigma_0)}{4 \log 2} \log t \]
\[+ \frac{\pi(\sigma_1 - \sigma_0)}{2 \log 2} \log \left(3 \frac{t + 3(1 + \eta)}{t - (1 + 2\eta)} \left(\frac{3(1 + \eta)/t + 1}{2\pi}\right)^{1+2\eta} \zeta(1 + \eta)^4 \zeta(2(1 + \eta))^{-2}\right).\]
Observing that the second term decreases with \(t \geq H\) achieves the proof. \[\square\]
6. Explicit upper bounds for \(N(\sigma,T)\) - Proof of Theorem 1.1

Proof. We recall that \(\sigma, \sigma_0, \sigma_1, H\) and \(T\) satisfy (1.11). We consider the number \(N(\sigma,T)\) of zeros \(\rho = \beta + i\gamma\) of zeta in the rectangle \(\sigma < \beta < 1\) and \(H < \gamma < T\). Since \(N(\sigma,H) = 0\), we have

\[
N(\sigma,T) \leq \frac{1}{\sigma - \sigma_0} \int_{\sigma_0}^{\sigma_1} \left( N(\tau,T) - N(\tau,H) \right) d\tau.
\]

It follows from a lemma of Littlewood (see [31, (9.9.1)]) that

\[
\int_{\sigma_0}^{\sigma_1} \left( N(\tau,T) - N(\tau,H) \right) d\tau = -\frac{1}{2\pi i} \int_{\mathcal{R}} \log \zeta(s) ds,
\]

where \(\mathcal{R}\) is the rectangle with vertices \(\sigma_0 + iH, \sigma_1 + iH, \sigma_1 + iT,\) and \(\sigma_0 + iT\). Thus

\[
N(\sigma,T) \leq \frac{1}{2\pi(\sigma - \sigma_0)} \left( \int_H^T \log |\zeta(\sigma_0 + it)| dt + \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau \right. \]
\[
- \left. \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau - \int_H^T \log |\zeta(\sigma_1 + it)| dt \right).
\]

We use Theorem 1.4, Lemma 4.1, and Lemma 5.1 respectively to bound these integrals:

\[
\int_H^T \log |\zeta(\sigma_0 + it)| dt \leq \frac{T - H}{2} \log \left( \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right),
\]

\[- \int_H^T \log |\zeta(\sigma_1 + it)| dt \leq \mathcal{E}_2,
\]

\[
\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iT) d\tau - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH) d\tau \leq \mathcal{E}_3(\sigma_0) \log(HT) + \mathcal{E}_4(\sigma_0, H),
\]

where the \(\mathcal{E}_i\)'s are defined respectively in (1.10), (4.1), (5.1), and (5.2). We obtain

\[
N(\sigma,T) \leq b_1(\sigma_0, H)(T - H) + b_2(\sigma_0, H) \log(TH) + b_3(\sigma_0, H),
\]

with

\[
b_1(\sigma_0, H) = \frac{\log \left( \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H) \right)}{4\pi(\sigma - \sigma_0)}, \quad b_2(\sigma_0, H) = \frac{\mathcal{E}_3(\sigma_0)}{2\pi(\sigma - \sigma_0)}, \quad b_3(\sigma_0, H) = \frac{\mathcal{E}_2 + \mathcal{E}_4(\sigma_0, H)}{2\pi(\sigma - \sigma_0)}.
\]

It follows

\[
N(\sigma,T) \leq c_1 T + c_2 \log T + c_3, \text{ with } c_1 = b_1, \quad c_2 = b_2, \quad c_3 = -b_1 H + b_2 \log H + b_3.
\]

\(\square\)
Table I records values of the $b_i$'s and $c_i$'s computed for $H_0 = 3.061 \cdot 10^{10}$. Specific choices of parameters $\sigma_0$ and $H$ are chosen in order to obtain good bounds for $N(\sigma, T)$ when $T$ is asymptotically large. The values of $\sigma_0, b_1, c_1, b_2,$ and $c_2$ displayed in the table are rounded up to 4 decimal places. We take the ceiling of the values of $H, b_3,$ and $c_3$.

Table 1. $N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3$ and $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$, for $T \geq H_0$.

| $\sigma$ | $\sigma_0$ | $H$       | $b_1 = c_1$ | $b_2 = c_2$ | $b_3$ | $c_3$   |
|---------|-----------|-----------|------------|------------|--------|--------|
| 0.60    | 0.5229    | 19399     | 4.2288     | 2.2841     | 333    | -81673 |
| 0.65    | 0.5552    | 40105     | 2.4361     | 1.7965     | 262    | -97414 |
| 0.70    | 0.5873    | 91470     | 1.4934     | 1.4609     | 213    | -136370|
| 0.75    | 0.6096    | 169119    | 1.0031     | 1.1442     | 167    | -169449|
| 0.76    | 0.6136    | 188973    | 0.9355     | 1.0921     | 160    | -176604|
| 0.77    | 0.6175    | 210645    | 0.8750     | 1.0437     | 153    | -184134|
| 0.78    | 0.6213    | 234346    | 0.8205     | 0.9986     | 146    | -192120|
| 0.79    | 0.6250    | 260321    | 0.7714     | 0.9566     | 140    | -200644|
| 0.80    | 0.6287    | 288853    | 0.7269     | 0.9176     | 134    | -209795|
| 0.81    | 0.6324    | 320270    | 0.6864     | 0.8812     | 129    | -219667|
| 0.82    | 0.6361    | 354951    | 0.6495     | 0.8473     | 124    | -230367|
| 0.83    | 0.6398    | 393341    | 0.6156     | 0.8157     | 119    | -242009|
| 0.84    | 0.6435    | 435955    | 0.5846     | 0.7862     | 115    | -254724|
| 0.85    | 0.6472    | 483393    | 0.5561     | 0.7586     | 111    | -268658|
| 0.86    | 0.6510    | 536357    | 0.5297     | 0.7327     | 107    | -283978|
| 0.87    | 0.6548    | 595670    | 0.5053     | 0.7085     | 104    | -300872|
| 0.88    | 0.6587    | 662291    | 0.4827     | 0.6857     | 101    | -319555|
| 0.89    | 0.6626    | 737343    | 0.4617     | 0.6644     | 97     | -340272|
| 0.90    | 0.6667    | 822142    | 0.4421     | 0.6443     | 95     | -363301|
| 0.91    | 0.6708    | 918225    | 0.4238     | 0.6253     | 92     | -388959|
| 0.92    | 0.6750    | 1027390   | 0.4066     | 0.6075     | 89     | -417606|
| 0.93    | 0.6793    | 1151729   | 0.3905     | 0.5906     | 87     | -449647|
| 0.94    | 0.6838    | 1293683   | 0.3754     | 0.5747     | 84     | -485543|
| 0.95    | 0.6883    | 1456079   | 0.3612     | 0.5596     | 82     | -525807|
| 0.96    | 0.6930    | 1642194   | 0.3478     | 0.5452     | 80     | -571018|
| 0.97    | 0.6977    | 1855803   | 0.3352     | 0.5316     | 78     | -621815|
| 0.98    | 0.7026    | 2101249   | 0.3232     | 0.5187     | 76     | -678911|
| 0.99    | 0.7077    | 2383498   | 0.3118     | 0.5063     | 74     | -743087|
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