Protecting the Protected Group: Circumventing Harmful Fairness

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Abstract

Machine Learning (ML) algorithms shape our lives. Banks use them to determine if we are good borrowers; IT companies delegate them recruitment decisions; police apply ML for crime-prediction, and judges base their verdicts on ML. However, real-world examples show that such automated decisions tend to discriminate against protected groups. This potential discrimination generated a huge hype both in media and in the research community. Quite a few formal notions of fairness were proposed, which take a form of constraints a “fair” algorithm must satisfy. We focus on scenarios where fairness is imposed on a self-interested party (e.g., a bank that maximizes its revenue). We find that the disadvantaged protected group can be worse off after imposing a fairness constraint. We introduce a family of Welfare-Equalizing fairness constraints that equalize per-capita welfare of protected groups, and include Demographic Parity and Equal Opportunity as particular cases. In this family, we characterize conditions under which the fairness constraint helps the disadvantaged group. We also characterize the structure of the optimal Welfare-Equalizing classifier for the self-interested party, and provide an algorithm to compute it. Overall, our Welfare-Equalizing fairness approach provides a unified framework for discussing fairness in classification in the presence of a self-interested party.

1 Introduction

At first glance, algorithms may seem obviously free of human biases as sexism or racism. However, in many situations they are not: the automated recruiting tool used by Amazon was favoring men [9]; judges in the US use the COMPAS algorithm to estimate the probability that the defendant will re-offend while this algorithm is biased against black people [21]. See O’Neill [27] for many more examples. These challenges call for imposing fairness constraints on algorithms design and, in particular, on ML classifiers. Naturally, fairness is costly as confirmed by the literature on the price of fairness (see, e.g., [2] and [4]); hence, it is crucial to understand who would incur these costs before imposing any fairness constraints. It is not surprising that imposing constraints on the decision-maker (the bank giving loans in our canonical example) may harm the well-being of the decision-maker. For instance, Corbett-Davies et al. [7] study this effect in the context of fair classification.

The possibility that fairness harms the well-being of those whom it is designed to protect, i.e., that it can harm the well-being of the disadvantaged group, seems counter-intuitive. However, keep

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in mind that the objective of the decision-maker is different from helping the disadvantaged group. Specifically, if we impose a fairness constraint on a revenue-maximizing bank, it will adjust its policy to accommodate the constraint but will still maximize its revenue. As a result, it will try to reallocate the “price of fairness” to the borrowers, and, possibly, to the disadvantaged subgroup. Indeed, real-life examples show that fairness can be harmful to welfare, e.g., Doleac and Hansen [9] show that the “ban the box” policy, adopted by the United States and preventing employers from seeing criminal backgrounds of applicants, decreased the chances of discriminated minorities (blacks and poor) to get a job.

In this paper, we take a utilitarian approach and model utility explicitly, by quantifying the well-being of agents for either label they obtain (i.e., whether they get the loan or not). Further, we model a self-interested decision-maker, who aims to maximize a quite general revenue-like objective function subject to the imposed fairness constraint. We construct fairness criteria that are driven by welfare, through the notion of agents’ utility function. We introduce a broad family of Welfare-Equalizing fairness concepts, equalizing the welfare among protected groups. As we show, special cases from this family include popular fairness concepts (Demographic Parity and Equal Opportunity) that are obtained by aligning their statistical requirements with a particular selection of an agent utility function.

Welfare-Equalizing fairness gives a general recipe to define a fairness concept specially tailored for a particular problem instance if there is some understanding of how different outcomes affect the well-being of agents. For example, it suggests how to naturally extend the existing fairness notions if we have some additional information about borrowers (e.g., loans may differ in the amount of money, duration, payment schemes, and so forth). In contrast to Heidari, Gummadi, and Krause [16], who unify existing fairness notions based on theories of justice from political philosophy, our unified framework of Welfare-Equalizing fairness allows one to effectively analyze the properties of optimal fair rules and their implications for the well-being of protected groups.

1.1 Our contribution

• Why is welfare important for fairness? Most of the previous literature study fairness constraints that address welfare only implicitly, and are given in the language of classification – using a statistical metric [19, 35]. For instance, Equal Opportunity equalizes the true positive rates of the groups. In contrast, we draw on the approach of recent work [7, 15, 16], and focus on welfare as the key ingredient of fairness. We argue that utilities and welfare are intrinsic to fair and just treatment, and hence cannot be neglected. Our paper is the first one to use the language of utilities in full force, allowing us to analyze the structural properties of a broad family of rules. Focusing on welfare allows us to investigate the downstream effects of imposing fairness constraints.

• Can fairness constraints harm the disadvantaged group? We study the implications of the class of Welfare-Equalizing fairness concepts and other fairness concepts on the disadvantaged group. We call the protected group having the lowest welfare before imposing a fairness constraint disadvantaged and say that the fairness constraint harms the protected group if imposing the constraint decreases its welfare. Our main result is that Welfare-Equalizing fairness helps the disadvantaged group. In comparison, the fairness constraint of Unawareness may harm both

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1The idea of maximizing the minimal welfare of the protected group was suggested in [15, 16] as a possible approach to fairness. We strengthen this desideratum by asking for equalizing the welfare of subgroups defined by a protected attribute. This new normative condition allows one to separate the constraint of fairness from the selfish objective of the decision-maker and thus allows one to analyze how the decisions change after imposing this constraint.
protected groups and the bank; Demographic Parity and Equal Opportunity could be harmful if their statistical constraint and the agent utility function are misaligned. This result turns out to be robust—If we use one welfare to define the fairness constraint and another one to measure its effect on the well-being of agents, fairness helps the disadvantaged group provided that the two welfare functions agree on which group is disadvantaged.

**Is there a way to compute the optimal Welfare-Equalizing classifier?** In order to have an impact on real-world applications, a fairness criterion must be efficiently computable. In Section 5, we provide efficient algorithms for two complementary cases, complete and incomplete information about agent utilities. Citizen utility is an important factor in any legislation process and is typically determined by domain experts to some granularity. In that case, the utility function is dictated to the bank by the legislator. For the case of incomplete information, we assume that agent utility is unknown, but can be learned from data. The utility is a real-valued function, and an extensive body of literature deals with estimating real-valued function (e.g., linear regression). We show how such estimators can be incorporated to learn an approximately optimal Welfare-Equalizing classifier, in the absence of full knowledge about the utilities. The approximation factor of the algorithm is given in terms of the quality of the estimators, a novel approach recently introduced for other problems in the ML space [13, 24, 30].

### 1.2 Paper structure

In Section 2 we present our formal model. We introduce Welfare-Equalizing fairness in Section 3, and Subsection 3.1 is devoted to characterizing the optimal policy of the bank (a self-interested decision-maker) when a Welfare-Equalizing fairness constraint is imposed on its behavior. Section 4 deals with the implication of fairness constraints. We first show that existing notions can harm the disadvantaged group. Then, in Subsection 4.1 we develop the general theory for Welfare-Equalizing classifiers, and prove that they always help the disadvantaged group. Finally, Subsection 5 describes how to compute an optimal Welfare-Equalizing classifier in a wide range of settings.

### 2 Model

We consider a general classification problem, where agents have ex-ante non-observable “quality” correlated with observable attributes. A classifier is predicting “quality” based on statistical data; misclassification is costly while a good guess is profitable. We will keep using a metaphor of a bank that predicts the reliability of a borrower and makes a lending decision; however, the same setting captures student admissions, recruiting, assessing the recidivism risk for a criminal, and so on.

We assume that each potential borrower (henceforth borrower) is associated with a pair of observable attributes \((X, A) \in X \times \{0, 1\}\). Here \(A\) is the binary \(^2\) protected attribute (e.g., gender) and \(X \in \mathcal{X}\) encodes all other characteristics of a borrower, e.g., employment history, salary, education, assets and so on. We do not make any assumptions on \(\mathcal{X}\). We also assume that each borrower has a utility for receiving the loan or not receiving it. Since utility is part of our approach to fairness, we defer its formal definition to Section 3.

\(^2\)The assumption of dichotomy of \(A\) is made for simplicity. Extension of all results to a non-binary case (ethnicity) is straightforward. A more general setting where protected classes are defined by combinations of attributes (“a single black women with 3 children”) can also be captured by introducing an auxiliary non-binary protected attribute that represents membership to each of the protected groups.
The statistical characteristics of the population are described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; so $X = X(\omega)$ and $A = A(\omega)$ are random variables on $\Omega$. Another random variable $Y = Y(\omega) \in \{0,1\}$ describes whether a given borrower will pay back if he or she is given the money. By its nature, $Y$ is unobservable. We call borrowers with $Y = 1$ and $Y = 0$, good and bad, respectively.

Further, we assume that the bank knows the joint distribution of $(X, A, Y)$ from historical data. In particular, it knows the exact conditional probability of being a good borrower given the attributes; we will denote it by $p(x, a) = \mathbb{P}(Y = 1 \mid X = x, A = a)$.

The bank makes lending decisions based on $A$ and $X$ but without observing $Y$. It uses a classifier $c: \mathcal{X} \times \{0, 1\} \to [0, 1]$ where $c(x, a)$ is the probability of giving a loan to a population of borrowers with $X = x$ and $A = a$. Each loan given to a good borrower brings a revenue of $\alpha_+(X) > 0$ to the bank while each bad borrower leads to a loss of $\alpha_-(X) > 0$; we assume that $\alpha_\pm$ are bounded functions of $X$. The bank aims to maximize its revenue, which depends on the choice of a classifier $c$, and is defined by

$$R(c) = \mathbb{E}[c(X, A)(\alpha_+(X)Y - \alpha_-(X)(1 - Y))].$$

To ease notation, we define $t(x)$ and $r(x, a)$ such that for every $a \in \{0, 1\}$, $x \in \mathcal{X}$

$$t(x) := \frac{\alpha_-(x)}{\alpha_+(x) + \alpha_-(x)},$$
$$r(x, a) := (\alpha_+(x) + \alpha_-(x))(p(x, a) - t(x));$$

hence, we can rewrite Equation (1) by taking conditional expectation with respect to $A, X$ as

$$R(c) = \mathbb{E}[c(X, A)(\alpha_+(X)p(X, A) - \alpha_-(X)(1 - p(X, A))]]$$
$$= \mathbb{E}[r(X, A)c(X, A)].$$

### 2.1 Optimal unconstrained classifier

We now exemplify our setting and notation, by considering the optimal classifier. If the bank is free to choose any classifier, then the optimal $c$ maximizing $R(c)$ given in Equation (2) has a simple form [7]. Only borrowers with $r(x, a) > 0$ are profitable for the bank, which is equivalent to the probability of paying back $p(x, a)$ being greater than $t(x)$. Consequently, the optimal lending policy is given by the threshold classifier $c^\ast_{\text{unc}}$; all borrowers with $p(x, a) > t(x)$ get loans ($c^\ast_{\text{unc}}(x, a) = 1$) and all borrowers with $p(x, a) \leq t(x)$ are rejected ($c^\ast_{\text{unc}}(x, a) = 0$). 4

This threshold behavior resembles the widespread usage of the credit score: a loan is given if the score is above some threshold. The probability $p(x, a)$ can thus be interpreted as an ideal credit score for unconstrained banks.

The following example illustrates that the optimal unconstrained behavior of the bank can discriminate one of the groups even if groups are of equal size and contain the same fraction of good borrowers.

3In contrast to the rest of the literature, we allow dependence of bank’s revenue on non-protected attribute $X$. This becomes important if $X$ also encodes the type of loan a client is applying for, e.g., different borrowers may need a different amount of money and thus bring a different revenue/loss.

4For definiteness, we assume that if the bank is indifferent between the two decisions (the knife-edge case $p(x, a) = t(x)$), it chooses the one with less loans given (e.g., this policy minimizes paperwork).
Example 1. Let $A$ and $X$ be binary, and let all four combinations be equally likely. The probability $p(x,a)$ of being a good borrower is given by the matrix

\[
\begin{array}{cc}
X = 0 & X = 1 \\
A = 0 & 0.4 & 0.6 \\
A = 1 & 0 & 1
\end{array}
\]

In the group \( \{A = 0\} \), the attribute $X$ poorly separates good and bad borrowers, while it is a perfect predictor of creditworthiness for \( \{A = 1\} \). If losses from a defaulting client are equal to the revenue from two borrowers paying back, e.g. $\alpha_- = 2 \cdot \alpha_+$, then we get the threshold $t = \frac{2}{3} \approx 0.66$; thus, the optimal policy of the bank is giving loans to applicants with $(X,A) = (1,1)$ only. As a result, no loans are given in the group \( \{A = 0\} \), although the prior distribution shows that in total $1/2$ of borrowers from each of the groups \( \{A = 0\} \) and \( \{A = 1\} \) are good. This contradicts intuitive understanding of fairness.

3 Welfare-Equalizing Fairness

The idea of a welfare-equalizing approach is to find the outcome that aligns the well-being of all the parties. The well-being of individuals is measured using a utility function (we assume that each agent can assign a numerical value to measure how valuable each outcome is for her) and well-being of groups using social welfare which aggregates utilities of the members. Welfare-equalizing, our approach to fairness, has a long history in normative economics [29, 33] (where it is known under the name of egalitarianism) and political philosophy [31]; it was used for fair resource allocation without money transfers [22], in the field of cooperative games [10] and bargaining problems [18].

In order to apply a welfare-equalizing approach to our problem, we shall first agree on how to measure the welfare of individuals. We assume that each individual obtains a utility of $u_+$ for receiving the loan and $u_-$ if the loan application is rejected. These utilities $u_+$ and $u_-$, on which we elaborate shortly, can depend on $X$ and $A$ (see example below, where $X$ contains the amount of money the individual wishes to borrow), but what is more critical — they must depend on the non-observable quality of borrower, namely $u_{\pm} = u_{\pm}(x,a,y)$.

Indeed, while good borrowers benefit from receiving loans, bad borrowers do not. As argued in Liu et al. [23], when bad borrowers are given a loan, they lose the borrowed money at the end (the bank will foreclose the property bought) and also harm their credit history.

For a given utility-function $u = (u_+, u_-)$ and a classifier $c$, we define the utilitarian welfare of the population as

\[
W_{u,c} = \mathbb{E}[u_+(X,A,Y)c(X,A) + u_-(X,A,Y)(1-c(X,A))].
\]  

(3)

Further, we denote by $W_{u,c}(a)$ the welfare of individuals in the protected group $a$, for $a \in \{0,1\}$, which is simply the conditional expectation of the same expression given $A = a$. We are now ready to define the Welfare-Equalizing fairness condition.

Definition 2. Given a utility-function $u = (u_+, u_-)$, a classifier $c$ is $u$-Welfare-Equalizing if

\[
W_{u,c}(0) = W_{u,c}(1),
\]

(4)

i.e., if $c$ equalizes the welfare among the two protected groups. The set of all such classifiers is denoted by $WE(u)$. 

5
Welfare-Equalizing fairness (hereinafter denoted by WE) allows one to analyze (existing) fairness concepts in a unified way and also opens some new possibilities. For instance, 5

- The fairness concept of Demographic Parity (e.g., [1, 11]), hereinafter denoted DP, imposes a constraint on the outcomes of decisions within the two groups: it requires that the fraction of those who receive loans in the two groups must be the same. Formally, a classifier $c$ satisfies DP if
  \[ \mathbb{E}[c(X, A) \mid A = 0] = \mathbb{E}[c(X, A) \mid A = 1]. \]
  It is a special case of WE fairness with $u_+ ≡ 1$ and $u_- ≡ 0$.

- Motivated by drawbacks of DP, Hardt et al. [14] concluded that good and bad borrowers within protected groups must be treated separately and introduced the concept of Equal Opportunity (hereinafter EO). Under this fairness concept, the fraction of good borrowers who get loans must be the same in the two subgroups. Formally, a classifier $c$ satisfies EO if
  \[ \mathbb{E}[c(X, A) \mid Y = 1, A = 0] = \mathbb{E}[c(X, A) \mid Y = 1, A = 1]. \]
  We recover EO by setting $u_+(y, a) = y \cdot \beta_a$ and $u_- ≡ 0$. The coefficients $\beta_a = 1/\mathbb{E}[Y \mid A = a]$ normalize the maximal possible welfare in each group to 1. Such a rescaling is known under the name of “relative welfare” and is commonly used in economics to make welfare or utilities among groups comparable [18].

- Borrowers can differ in the amount of money $m$ they need. We can assume that information about $m$ is encoded in $X$, so $m = m(X)$. Then, a straightforward generalization of EO is the following concept of Heterogeneous-EO given by $u_+(x, a, y) = y \cdot m(x) \cdot \beta_a$ with $\beta_a = 1/\mathbb{E}[m(x) \mid A = a]$. We can capture any other heterogeneity in a similar way (e.g., different interest rates, time-period, and payment schedules).

Notice that in all the above examples $u_-$ is identically zero. This is not a coincidence: without loss of generality, $u_- ≡ 0$ for any “reasonable” WE fairness concept. Typically, and under all the fairness criteria above, the zero classifier $c ≡ 0$ (giving no loans at all) is considered fair. The following lemma, which is formally proved in the appendix, shows that if the zero classifier is $u$-WE, then we can shift the utilities to have $u_- = 0$.

**Lemma 3.** If the zero classifier belongs to $\text{WE}(u)$ with $u = (u_+, u_-)$, then $\text{WE}(u) = \text{WE}(u')$, where $u' = (u'_+, 0)$ with $u'_+ = u_+ - u_-$. 

Hence, $u_- ≡ 0$ is a normalization-condition: prior to borrowing money everybody is at zero utility level, and then the WE approach equalizes utilitarian gains from borrowing. Motivated by Lemma 3, we henceforth assume that

\[ u = (u_+, 0) \quad \text{and} \quad u_+ ≥ 0. \]

Non-negativity of $u_+$ can be regarded as a rationality assumption on borrowers: no rational agent applies for a loan if she/he expects that getting the loan brings negative utility while not getting gives 0. In what follows, we drop “+” in $u_+$ and identify $u_+$ and $u = (u_+, 0)$ for brevity; so $W_{u,c} = \mathbb{E}[u(X, A, Y)c(X, A)] = \mathbb{E}[u_+(X, A, Y)c(X, A)]$.

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5We provide a brief introduction to some popular fairness concept in Section 10.
### 3.1 Structural properties of the optimal welfare-equalizing classifier

In this subsection, we explore the structure of optimal WE classifiers. The computational problem is deferred to Section 5.

For a fixed $u$, we denote by $c^*_{WE(u)}$ the classifier that maximizes the bank’s revenue $R(c)$ (see Equation (1)) among all $u$-WE classifiers $c \in WE(u)$. The set of $u$-WE classifiers is non-empty since $0 \in WE(u)$ and therefore the bank’s optimization problem is well-defined. The following proposition shows that any such classifier is a generalized threshold classifier.

To ease notation, we denote by $\overline{u}(x, a)$ the average utility of a borrower associated with $(x, a)$,

$$\overline{u}(x, a) = \mathbb{E}[u \mid X = x, A = a] = u(x, a, 1)p(x, a) + u(x, a, 0)(1 - p(x, a)).$$

Further, we denote by $R^*_a(w)$ the maximal revenue that the bank could extract from the group $\{A = a\}$ at the welfare level $W_{u,c}(a) = w$. Formally,

$$R^*_a(w) = \max_{c: \mathcal{X} \to [0,1], \mathbb{E}[r(X, A) \cdot c(X) \mid A = a]} \mathbb{E}[r(X, A) \cdot c(X) \mid A = a].$$

**Proposition 4.** The optimal $u$-WE classifier $c^*_{WE(u)}$ exists and has the following form:

$$c^*_{WE(u)}(x, a) = \begin{cases} 
1 & r(x, a) > \lambda_a \overline{u}(x, a) \\
0 & r(x, a) < \lambda_a \overline{u}(x, a) \\
\tau_a(x) & r(x, a) = \lambda_a \overline{u}(x, a) 
\end{cases} \quad (5)$$

Group-dependent thresholds $\lambda_a$, $a \in \{0, 1\}$ belong to the super-gradient\(^6\) of the subgroup-optimal revenue $R^*_a(w)$ (a concave function of $w$) computed at the welfare level $w^*$ maximizing the total bank’s revenue $\mathbb{P}(A = 0)R_0^*(w) + \mathbb{P}(A = 1)R_1^*(w)$.

Functions $\tau : \mathcal{X} \to [0,1]$ are arbitrary\(^7\) up to the constraint that $c^*_{WE(u)}$ provides the desired welfare level $w^*$ for both groups: $w^* = W_{u,c^*_{WE(u)}}(0) = W_{u,c^*_{WE(u)}}(1)$.

**Proof sketch of Proposition 4.** The revenue maximization over $c \in WE(u)$ can be represented as a two-stage procedure: 1) find the revenue-maximizing classifier in each of the subgroups $A = a$ given the welfare level $w$; 2) optimize over $w$. The welfare constraint in (1) can be internalized using the Lagrangian approach; the corresponding Lagrange multipliers $\lambda_a$ are equal to the “shadow prices”, i.e., the derivatives of the value functions $R^*_a(w)$ with respect to $w$. This provides the threshold structure for the optimal classifier from 1) inherited by $c^*_{WE(u)}$. Since the resulting linear program is infinite-dimensional and $R^*_a(w)$ may be non-differentiable, the formal proof requires some functional-analytic arguments presented in the appendix. □

**Examples** Proposition 4 immediately provides the optimal classifiers under DP and EO: for the former, it is given by the “additive perturbation” of the optimal unconstrained classifier, while the latter is obtained by “multiplicative perturbation.”

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\(^6\)For a concave function $f = f(t)$, $t \in [t_0, t_1]$, the super-gradient $\partial f$ is the set of all $q \in \mathbb{R}$ such that $f(t') \leq f(t) + q(t' - t)$ for all $t'$. If $f$ is continuous, then for any $t$ the super-gradient is non-empty, see Rockafellar [32].

\(^7\)In particular, there always exists $c^*_{WE(u)}$ with constant $\tau_a$, i.e., independent of $x$. 

7
More precisely, for the optimal classifier $c_{\text{parity}}^*$ satisfying DP, there exist group-dependent constants $\lambda_a \in \mathbb{R}, a \in \{0, 1\}$ such that
\[
c_{\text{parity}}(x, a) = \begin{cases} 
1 & p(x, a) > t(x, a) + \frac{\lambda_a}{\alpha_+(x) + \alpha_-(x)} \alpha_+(x) + \alpha_-(x) \\
0 & p(x, a) < t(x, a) + \frac{\lambda_a}{\alpha_+(x) + \alpha_-(x)} \alpha_+(x) + \alpha_-(x) 
\end{cases}.
\]
Indeed, this recovers the result of Corbett-Davies et al. [7]. For the optimal EO classifier $c_{\text{opportunity}}^*$, there exist $\lambda_a \in \mathbb{R}, a \in \{0, 1\}$ such that
\[
c_{\text{opportunity}}^*(x, a) = \begin{cases} 
1 & p(x, a) \left(1 - \frac{\lambda_a}{\alpha_+(x) + \alpha_-(x)}\right) > t(x, a) \\
0 & p(x, a) \left(1 - \frac{\lambda_a}{\alpha_+(x) + \alpha_-(x)}\right) < t(x, a)
\end{cases}.
\]

4 Implications of Fairness Constraints

In this section we discuss the implications of applying WE fairness. We call a group $\{A = a\}$ the disadvantaged protected group (or $u$-disadvantaged to emphasize the dependence on $u$) if under the optimal unconstrained classifier, the welfare of $\{A = a\}$ is lower than welfare of the other group $\{A = 1 - a\}$. Formally,
\[
W_{u,c_{\text{inc}}^*}(a) < W_{u,c_{\text{inc}}^*}(1 - a),
\]
where $W$ is defined in Equation (3). We say that a fairness constraint harms the group $\{A = a\}$ if $W_{u,c^*}(a) < W_{u,c_{\text{inc}}^*}(a)$, where $c^*$ is the optimal classifier after imposing that fairness constraint. In order to put the analysis of WE fairness in context, we first demonstrate that popular fairness constraints can be harmful.

Unawareness is perhaps the most intuitive fairness criterion. A classifier $c$ satisfies unawareness if $c(x, 0) = c(x, 1)$ for all $x \in \mathcal{X}$. Informally, to deliver a “fair” outcome to both groups $A = 0$ and
A = 1, the classifier c must ignore the protected attribute A. The shortcomings of Unawareness are well-known (see, e.g., [6, 12, 34]). As we formally show in the appendix, Unawareness can make all three parties strictly worse off: both groups and the bank, regardless of the utility function u.

The DP improves upon Unawareness. It never harms both groups, as it shifts welfare from one group to another (see Theorem 6 below). However, it can harm the disadvantaged one. Due to lack of space, we defer the worst-case examples for DP and Unawareness to the appendix, and focus on EO.

Example 5. Consider the utility-function \( u(x, a, y) \equiv 1 \), i.e., all borrowers equally benefit from receiving loans. Assume that \( \mathcal{X} = \{0, 1, 2\} \), and all combinations of \((x, a)\) have the same probability of \( \frac{1}{6} \). Furthermore, assume that the fraction \( p(x, a) \) of good borrowers is given by the matrix

\[
\begin{array}{cccc}
    x = 0 & x = 1 & x = 2 \\
    a = 0 & 3/4 & 3/4 & 1/4 \\
    a = 1 & 1 & 0 & 0
\end{array}
\]

In addition, let \( \alpha_+(x) = 2 \) and \( \alpha_-(x) = 3 \) for every \( x \in \mathcal{X} \). The unconstrained threshold thus equals \( t(x) = \frac{3}{2} \).

Under the optimal unconstrained classifier, \( c^*_{\text{inc}} \), all the good borrowers from \( \{A = 1\} \) receive loans. In \( \{A = 0\} \), since \( t(x) = \frac{3}{2} > \frac{1}{4} \), loans are given to borrowers with \( x = 0 \) and \( x = 1 \) only. As a result, the group \( \{A = 1\} \) is disadvantaged: \( W_{u, c^*_{\text{inc}}}(0) = \frac{3}{2} \) compared to \( W_{u, c^*_{\text{inc}}}(1) = \frac{1}{2} \).

On the other hand, the proportion of loans given to good borrowers in \( \{A = 0\} \) equals \( \mathbb{E}[c^*_{\text{inc}}(X, A) \mid Y = 1, A = 0] = \frac{2}{3} \), but \( \mathbb{E}[c^*_{\text{inc}}(X, A) \mid Y = 1, A = 1] = 1 \). To equalize the proportion of loans given to good borrowers between the two groups, the EO classifier must either increase the amount of loans given to \( \{A = 0\} \) by approving some applications of \( x = 2 \) or decrease the number of loans given to \( \{A = 1\} \). But giving loans to \((X, A) = (2, 0)\) is too costly: the cost \( \frac{3}{4} \alpha_- - \frac{1}{4} \alpha_+ \) is not compensated by the benefit \( 1 \cdot \alpha_+ \) from giving the same amount of loans to good borrowers \((X, A) = (0, 1)\). Therefore, the optimal EO classifier coincides with \( c^*_{\text{inc}} \) in \( \{A = 0\} \) and gives less loans to \((X, A) = (0, 1): c^*_{\text{opportunity}}(1, 0) = \frac{6}{7} \), equalizing the proportion of loans to good borrowers under both groups. Hence, \( W_{u, c^*_{\text{opportunity}}}(1) = \frac{2}{7} < W_{u, c^*_{\text{inc}}}(1) = \frac{1}{3} \) and the disadvantaged group is harmed by EO.

### 4.1 Welfare-Equalizing fairness protects the disadvantaged group

Having demonstrated that fairness constraints can be harmful for those they try to protect, we now develop the general theory for WE fairness concepts. We show that if the concept of fairness matches the way we measure the welfare of borrowers, then WE fairness always helps the disadvantaged group, on the expense of the advantaged group.

Theorem 6. The optimal \( u\)-WE classifier makes the \( u\)-disadvantaged protected group \( A = a \) weakly better off at the expense of the advantaged group. Formally,

\[
W_{u, c^*_{\text{inc}}}(a) < W_{u, c^*_{\text{inc}}}(\overline{a}) \implies \begin{cases} 
W_{u, c^*_{\text{inc}}}(a) \leq W_{u, c^*_{\text{WE}(u)}}(a) \\
W_{u, c^*_{\text{inc}}}(\overline{a}) \geq W_{u, c^*_{\text{WE}(u)}}(\overline{a})
\end{cases}.
\]

Moreover, any borrower from the \( u\)-disadvantaged group who receives a loan under the unconstrained
classifier, receives it under optimal u-WE. Formally, for all \( x \in X \) it holds that
\[
W_{u,c_{unc}^*}(a) < W_{u,c_{unc}^*}(1 - a) \implies c_{unc}^*(x, a) \leq c_{WE(u)}^*(x, a).
\]

**Proof of Theorem 6.** By Proposition 4, the welfare \( w^* = W_{u,c_{WE(u)}^*}(0) = W_{u,c_{WE(u)}^*}(1) \) achieved by u-WE classifier maximizes the total revenue \( \mathbb{P}(A = 0)R_0^*(w) + \mathbb{P}(A = 1)R_1^*(w) \) as a function of welfare level \( w \). The sub-group revenues \( R_a^*(w) \), \( a \in \{0, 1\} \) are concave functions and thus \( w^* \) lies between their maxima. These maxima are attained at welfare of the optimal unconstrained classifier and thus \( w^* \) is between \( W_{u,c_{unc}^*}(a) \), \( a \in \{0, 1\} \). See Figure 1 for illustration.

Individual guarantees follow from the threshold structure in Equation (5) of \( c_{WE(u)}^* \); since \( w^* \) is above the maximum of \( R_a^*(w) \), the subgradient contains \( \lambda_a \leq 0 \) and thus \( c_{unc}^*(x, a) \) (which corresponds to zero \( \lambda_a \)) is below \( c_{WE(u)}^*(x, a) \).

Theorem 6 emphasizes an important aspect of WE fairness: the well-being of the disadvantaged protected group will always (weakly) increase under the optimal classifier after imposing WE fairness, while the well-being of the advantageous protected group will always (weakly) decrease; hence, the optimal u-WE classifier balances the well-being of the three parties: the two groups and the bank.

Ultimately, we note that in some cases the exact borrowers’ utility function \( u \) is unknown, and a theoretical approximation \( \tilde{u} \) is used instead. One criticism of our model could be that, under the unobservable utility scenario, the WE fairness is not meaningful. Theorem 6 ensures that such a fairness concept weakly increases \( u \)-welfare of the \( \tilde{u} \)-disadvantaged group, but we would like to ensure that the same holds with respect to “real” but unobserved utilities \( u \).

**Corollary 7.** If \( \tilde{u} \) and \( u \) agree on which group is disadvantaged, then the \( \tilde{u} \)-WE classifier weakly increases \( u \)-welfare of the disadvantaged group, i.e.,
\[
\begin{align*}
W_{\tilde{u},c_{unc}^*}(a) &< W_{\tilde{u},c_{unc}^*}(1 - a) \\
W_{u,c_{unc}^*}(a) &< W_{u,c_{unc}^*}(1 - a)
\end{align*}
\implies W_{u,c_{unc}^*}(a) \leq W_{u,c_{WE(\tilde{u})}^*}(a).
\]

In fact, beyond the robustness guarantee for WE fairness for a particular instance, Corollary 7 also elucidates the benefits of other fairness criteria and characterizes when they could benefit the disadvantaged group. For instance, let \( u \) be an arbitrary utility function such that under the unconstrained classifier the group \( \{A = 0\} \) is disadvantaged, i.e., \( W_{u,c_{unc}^*}(0) = W_{u,c_{unc}^*}(1) \) and focus on DP. Recall that DP coincides with a WE fairness classifier for \( \tilde{u} \equiv 1 \). Consequently, Corollary 7 guarantees that DP will improve the well-being (w.r.t. the arbitrary utility \( u \)) of the disadvantaged group once applied, provided that it perceives \( \{A = 0\} \) as the disadvantaged group, namely if \( \mathbb{E}[c_{unc}^*(X, A) | A = 0] < \mathbb{E}[c_{unc}^*(X, A) | A = 1] \). Since the identity of the disadvantaged group is typically known, this simple characterization provides a sufficient condition for circumventing harmful fairness.

### 5 Computing Optimal Welfare-Equalizing Classifiers

In this subsection, we provide evidence for the applicability of our approach by developing tools for computing optimal WE classifiers. We first assume complete information about the problem (e.g.,

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8While our paper is focused on a group-notions of fairness, we stress that this result provides stronger “individual” guarantees that no particular agent from the disadvantaged group will be harmed by fairness.
maximize \[ \sum_{x,a} P(X = x, A = a) \cdot r(x, a) \cdot c(x, a) \]

subject to

\[ \sum_x P(X = x \mid A = 0) \cdot \overline{u}(x, 0) \cdot c(x, 0) = \sum_x P(X = x \mid A = 1) \cdot \overline{u}(x, 1) \cdot c(x, 1), \]

\[ 0 \leq c(x, a) \leq 1, x \in X, a \in A. \]

Figure 2: Linear programming formulation. The objective function of the LP is \( R(c) \), the bank’s revenue, which is maximized over the class of functions \( \{ X \times A \rightarrow [0, 1] \} \). The first constraint is welfare-equalizing (see Equation (4)): the welfare of the group \( \{ A = 0 \} \) should be equal to the welfare of \( \{ A = 1 \} \). The remaining constraints are range constraints, assuring that the classifier is in segment \([0, 1]\).

the utility \( u \), revenues/losses \( \alpha \pm \), and the probability distribution \( P \) are known), and then relax this assumption. Due to space considerations and our desire to focus on the conceptual assets of the paper, we defer most of the analysis to the appendix, as well as an elaborated version of formal statements.

We assume throughout this subsection that \( 0 < W_{u,c*_{\text{unc}}}(0) < W_{u,c*_{\text{unc}}}(1) \), i.e., the optimal unconstrained classifier provides non-zero utility to both groups and the group \( \{ A = 0 \} \) is disadvantaged. The positivity condition allows to avoid degenerate cases. To ease readability, we also assume that \( \max_{x,a} |r(x, a)| = 1 \) and \( \max_{x,a} \overline{u}(x, a) = 1 \).

For a fixed utility \( u \), we denote by \( c^*_{\text{WE}(u)} \) the classifier that maximizes the bank’s revenue \( R(c) \) (see Equation (1)) among all \( u\)-WE classifiers \( c \in \text{WE}(u) \). The set of \( u\)-WE classifiers is non-empty since \( 0 \in \text{WE}(u) \) and therefore the bank’s optimization problem is well-defined. The optimal WE-classifier is the solution to the linear program (LP) given in Figure 2. It maximizes the revenue under the constraint of equal welfare, and hence for a medium size set of attributes \( X \) (say several thousands) we can compute \( c^*_{\text{WE}(u)} \) explicitly by standard LP-methods.

For large sets of attributes, e.g., multidimensional or continuous, the size of the LP “explodes”, and a different approach should be taken. In this case, we use the structural insights from the previous subsection and in particular Proposition 4: the optimal WE-classifier \( c^*_{\text{WE}(u)} \) is parameterized by the two thresholds \( \lambda_a \) for \( a \in \{0, 1\} \); therefore, to compute it we can restrict our attention to a finite-dimensional parametric family of classifiers. Due to the large-scale nature of the problem, we shall seek for efficient algorithms for computing approximately optimal WE classifiers, where these approximated solutions are defined as follows.

**Definition 8.** A classifier \( c \) is \((\varepsilon, \varepsilon')\) optimal WE if \( R(c) \geq R(c^*_{\text{WE}}) - \varepsilon \) and \(|W_{u,c}(0) - W_{u,c}(1)| \leq \varepsilon'\).

Notice that such classifiers are doubly approximate: they approximate the revenue of the (exact) optimal WE classifier, and also approximately equalize the welfare of the two classes. The next proposition shows that we can efficiently find a classifier that approximates the revenue and (exactly) equalizes the welfare.

**Proposition 9.** Assume that \( \alpha_+, \alpha_- \), and \( u \) are known. Further, assume that the bank has access to an oracle that computes expectations w.r.t. \( P \) in constant time. Then, for any small enough positive \( \varepsilon \), an \((\varepsilon, 0)\) optimal WE-classifier can be computed in \( O \left( \log^2 \left( \frac{1}{\varepsilon} \right) \right) \) time.
Proof sketch of Proposition 9. Similarly to the proof of Proposition 4, we represent the optimization problem as finding an approximately-optimal marginal classifier for a given welfare level \( w \) in each subgroup and then determining revenue-maximizing \( w \).

For a given welfare level \( w \), the revenue-maximizing classifiers and the revenue \( R^*(w) \) itself can be approximated by the binary search over thresholds \( \lambda_a \). Since \( R^* \) is a concave function, the revenue-maximizing \( w \) can be determined by the Golden ratio or the ternary search as in Hardt et al. [14]. The technical difficulty is to bound the number of steps needed for binary search to converge even though a small change in \( \lambda_a \) can result in massive change both in welfare and revenue.

As we show in the appendix, the assumption on the expectation oracle can be relaxed by sampling to obtain an approximately optimal WE classifier.

Next, we relax the full information assumption. An extensive body of literature deals with estimating real-valued functions, be it linear regression or more involved techniques, for instance, deep learning. Recall that \( \bar{u} \) and \( r \) are real-valued functions; hence, we rely on existing methods to estimate \( \bar{u} \) and \( r \). We can thus provide performance guarantees, which depend on the quality of those estimators, a concept adopted recently for several other ML problems [25, 24, 30].

Assume that the bank has estimators \( \hat{u} \) and \( \hat{r} \) for \( \bar{u} \) and \( r \), respectively. The following result quantifies the approximation factors that can be achieved in this case in terms of the estimation errors of \( \hat{u} \) and \( \hat{r} \).

**Proposition 10.** Fix a small \( \delta > 0 \) and assume that the bank has access to a sample of \((X, A, Y, \alpha_\pm, u)\) and to estimators \( \hat{u} \) and \( \hat{r} \) such that \( \mathbb{E}[|\hat{u} - \bar{u}|] \leq \eta_u \) and \( \mathbb{E}[|\hat{r} - r|] \leq \eta_r \) with small enough \( \eta_u \) and \( \eta_r \). Then, an \((\varepsilon, \varepsilon)\) optimal WE classifier with

\[
\varepsilon = 2 \sqrt{6 \left( \frac{1}{\mathbb{P}(A = 0)} + \frac{1}{\mathbb{P}(A = 1)} \right) \max\{\eta_u, \eta_r\}}
\]

can be computed with probability \( 1 - \delta \) on a sample of size \( O \left( \frac{1}{\max\{\eta_u, \eta_r\}} \left( \log \frac{1}{\delta} + \log \log \frac{1}{\max\{\eta_u, \eta_r\}} \right) \right) \).

6 Conclusions

In this paper, we draw on the economic approach to design a family of fairness constraints. In particular, we assume a selfish decision-maker and agents having a utility function, and design a fairness constraint under which the disadvantaged group is always better off. We sketched the structure of the optimal WE fairness for the bank, showing that our proposed family contains popular concepts as special cases. We also show that the WE fairness approached is robust to imperfect information about agent utility.

We see considerable scope for follow-up work. One prominent direction is to estimate the “price of fairness” and its distribution among the parties involved. How much of the bank’s revenue can drop due to imposing fairness constraints? How much can the welfare of an advantaged protected group decrease? Should the welfare of the advantaged group decrease uniformly over the group or some sub-groups tend to pay that price?
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7 Omitted Proofs

Proof of Lemma 3. For any classifier \( c \) we have \( W_{u,c}(a) = W_{u',c}(a) + \mathbb{E}[u_\cdot | A = a] \). Since \( 0 \in \text{WE}(u) \), the second summands coincide for \( a \in \{0, 1\} \): \( \mathbb{E}[u_\cdot | A = 0] = \mathbb{E}[u_\cdot | A = 1] \). Thus \( W_{u,c}(0) = W_{u,c}(1) \) if and only if \( W_{u',c}(0) = W_{u',c}(1) \), which is equivalent to \( \text{WE}(u) = \text{WE}(u') \). \( \square \)

Proof of Proposition 4. The revenue maximization under the WE constraint can be split into two subsequent maximization problems:

\[
\max_{c \in \text{WE}(u)} \max_{w} R(c) = \max_{c \in \mathcal{X} \times \{0, 1\}} \max_{w} \mathbb{P}(A = 0) R_0(c) + \mathbb{P}(A = 1) R_1(c). \]

First, the bank finds the revenue-maximizing classifier \( c_{a,w}^* : \mathcal{X} \rightarrow [0, 1] \) that maximizes the revenue \( R_a(c) = \mathbb{E}[r(X, A) \cdot c(X) | A = a] \) in the subgroup \( \{A = a\} \) given some welfare-level \( W_{u,c}(a) = w \). Then, the bank finds the optimal level \( w^* \) of \( w \) by maximizing the total revenue \( \mathbb{P}(A = 0) R_0^*(w) + \mathbb{P}(A = 1) R_1^*(w) \), whereas in the statement of the proposition, \( R_a^*(w) \) denotes \( R_a(c_{a,w}^*) \).

Thus, the optimal WE classifier \( c_{\text{WE}(a)}^*(x, a) \) equals \( c_{a,w^*}^*(x) \) provided that the optimization problems for \( c_{a,w}^* \) and \( w^* \) have a solution. The threshold representation in Equation (5) will follow from a similar representation for \( c_{a,w}^* \). Existence and the threshold representation of \( c_{a,w}^* \) is the subject of the next two subsections.

7.1 Existence of \( c_{a,w}^* \) and \( w^* \), concavity of \( R_a^*(w) \)

For a given welfare-level \( w \) of a group \( \{A = a\} \), the set of feasible classifiers \( F(w) = \{ c : \mathcal{X} \rightarrow [0, 1] | W_{u,c}(a) = w \} \) is non-empty if and only if \( w \in (0, \mathbb{E}[u \cdot | A = a]) \). Indeed, for any \( c \) we have \( W_{u,c}(a) \in (0, \mathbb{E}[u \cdot | A = a]) \), therefore, the set of feasible marginal classifiers \( F(w) \) is empty outside of this interval. Therefore, for any \( w \) inside, the constant classifier \( \frac{w}{\mathbb{E}[u \cdot | A = a]} \in F(w) \) and thus \( F(w) \) is non-empty.

Proof of Proposition 4. For any classifier \( c \) we have \( W_{u,c}(a) = W_{u',c}(a) + \mathbb{E}[u_\cdot | A = a] \). Since \( 0 \in \text{WE}(u) \), the second summands coincide for \( a \in \{0, 1\} : \mathbb{E}[u_\cdot | A = 0] = \mathbb{E}[u_\cdot | A = 1] \). Thus \( W_{u,c}(0) = W_{u,c}(1) \) if and only if \( W_{u',c}(0) = W_{u',c}(1) \), which is equivalent to \( \text{WE}(u) = \text{WE}(u') \). \( \square \)
The existence of the optimal \( w^* \), where
\[
w^* \in \arg \max_w \{ \mathbb{P}(A = 0) R_0^*(w) + \mathbb{P}(A = 0) R_1^*(w) \},
\]
follows from continuity of \( R_a^*(w) \) for \( w \in [0, \mathbb{E}[u \mid A = a]] \).

### 7.2 Threshold structure of the optimal marginal classifiers \( c_{a,w}^* \)

By concavity and continuity of \( R_a^*(w) \), the super-gradient at \( w^* \) is non-empty. Pick an element \( \lambda_a \) from it.

By the definition of super-gradient, for any \( w \in \mathbb{R} \) it holds that \( R_a^*(w) \leq R_a^*(w^*) + \lambda_a (w - w^*) \).

Equivalently, for the optimal classifier \( c_{a,w}^* \in F(w^*) \) and an arbitrary classifier \( c : \mathcal{X} \to [0,1] \) we have
\[
R_a(c) - \lambda_a W_{u,c}(a) \leq R_a(c_{a,w}^*) - \lambda_a W_{u,c_{a,w}^*}(a).
\]

In other words, the optimal classifier \( c_{a,w}^* \) maximizes \( R_a(c) - \lambda_a W_{u,c}(a) \) over all \( c : \mathcal{X} \to [0,1] \).

The converse is also true: any maximizer \( c \) gives an optimal classifier \( c_{a,w}^* \) provided that it belongs to \( F(w^*) \).

Since \( R_a(c) - \lambda_a W_{u,c}(a) = \mathbb{E} [(r(X,A) - \lambda_a \overline{u}(X,A)) c(X,A) \mid A = a] \),

the maximizer equals 1 when \( r(x,a) - \lambda_a \overline{u}(x,a) > 0 \) and equals 0 if the inequality has the strict opposite sign. The condition \( c \in F(w^*) \) imposes the constraint on otherwise arbitrary values of \( c(x,a) = \tau_a(x) \) for \( x \) with \( r(x,a) - \lambda_a \overline{u}(x,a) = 0 \).

\[ \square \]

### 8 Worst Cases for Popular Fairness Concepts from Section 4

In this section, we show that Unawareness, DP, and EQ can harm the disadvantaged groups. Each subsection is devoted to another fairness concept.

#### 8.1 Unawareness may harm both groups and the bank.

*Unawareness* is perhaps the most intuitive fairness criterion. A classifier \( c \) satisfies unawareness if \( c(x,0) = c(x,1) \) for all \( x \in \mathcal{X} \). Informally, to deliver a “fair” outcome to both groups \( A = 0 \) and \( A = 1 \), the classifier \( c \) must ignore the protected attribute \( A \). We now show that *Unawareness* can make all three parties strictly worse off: both groups and the bank, regardless of the utility function \( u \). The following example shows this phenomenon when information losses caused by Unawareness are significant: the interpretation of non-protected attributes \( X \) depends on the protected\(^9\) attribute \( A \) and without knowing \( A \) the classifier cannot achieve good separation of good and bad borrowers thus giving loans becomes too risky.

*Example* 11. Let \( u \geq 0 \) be any arbitrary utility function, and assume that \( \mathcal{X} = \{0,1\} \) and all the four combinations of attributes \((x,a)\) are equally likely in the population. The fraction \( p(x,a) \) of good borrowers is given by the following matrix:

\[
\begin{array}{cc}
x = 0 & x = 1 \\
a = 0 & 2/3 & 1/3 \\
a = 1 & 1/3 & 2/3 \\
\end{array}
\]

\(^9\)For example, having children correlates with spending more time at work for men and has negative correlation for women [28].
Notice that the fraction of good and bad borrowers is the same in both groups. Further, \( X = 1 \) is a positive signal about the quality of a borrower in the group \( \{ A = 0 \} \) and a negative one for \( \{ A = 1 \} \).

The optimal unconstrained classifier \( c^\ast_{\text{unc}} \) gives loans to agents with \( p(x,a) \geq t(x) \); thus, if the threshold \( t \) is between \( \frac{1}{3} \) and \( \frac{2}{3} \) (for example, \( t = \frac{2}{3} \) if the revenue \( \alpha_+ \) from giving money to a good borrower equals \( \frac{2}{3} \) of the losses \( \alpha_- \) from a bad one), then agents with \( (X,A) \) equal to \( (0,0) \) or \( (1,1) \) get loans under \( c^\ast_{\text{unc}} \). So one half of the members in each group receives loans and the bank gets a positive revenue.

After imposing Unawareness, the optimal classifier \( c^\ast_{\text{unaware}} \) compares the average fraction of good borrowers with a given \( x \), namely \( \overline{p}(x) = \mathbb{P}(A = 0)p(x,0) + \mathbb{P}(A = 1)p(x,1) \), with \( t(x) \). In our example \( \overline{p}(x) = \frac{1}{2} \) for every \( x \), and thus no loans are given for \( t > 1/2 \); thus, for \( t \in (1/2, 2/3) \) unawareness pushes the welfare of both groups to zero as well as the bank’s revenue.

### 8.2 DP may harm the disadvantaged protected group

The DP improves upon Unawareness. It never hurts both groups but can harm the disadvantaged one as illustrated by the following Example 12. We consider the benchmark case where the welfare is given by the fraction of good borrowers receiving loans, so \( u(x,a,y) = y \cdot \beta_a \) for \( \beta_a = 1/\mathbb{E}[Y|A=a] \).

**Example 12.** The non-protected attribute is ternary, \( X = \{0,1,2\} \) and all combinations of \( (x,a) \) have the same probability of \( \frac{1}{6} \). The fraction \( p(x,a) \) of good borrowers is given by the following matrix

\[
\begin{array}{ccc}
x = 0 & x = 1 & x = 2 \\
ad = 0 & 3/4 & 3/4 & 1/4 \\
ad = 1 & 1 & 0 & 0
\end{array}
\]

We see that the fraction of good borrowers is higher in the group \( \{A = 0\} \): \( \frac{7}{12} \) against \( \frac{1}{3} \). However, for \( \{A = 1\} \), good borrowers and bad borrowers are perfectly separated by \( X \).

We assume that \( \alpha_+ = 2 \) and \( \alpha_- = 3 \), i.e., costs of false-positives are \( \frac{3}{2} \) times higher than the benefit from true-positives. The threshold \( t = \frac{\alpha_-}{\alpha_+ + \alpha_-} \) thus equals \( \frac{3}{5} \).

Under optimal unconstrained classifier \( c^\ast_{\text{unc}} \), all the good borrowers from the second group receive loans (for any \( t \)). In \( \{A = 0\} \), for \( t \in \left( \frac{1}{2}, \frac{3}{5} \right) \), loans are given to agents with \( x = 0 \) and \( x = 1 \) only. As a result, for \( t = \frac{3}{5} \), the group \( \{A = 0\} \) is disadvantaged: only \( \frac{6}{7} = W_{x,c^\ast_{\text{unc}}} \) of good borrowers there get loans compared to all good borrowers in \( \{A = 1\} \).

Next, consider an optimal classifier \( c^\ast_{\text{parity}} \) satisfying DP. Since \( \alpha_- > \alpha_+ \), costs of giving loans to \( (x,a) \in \{(1,1),(2,1)\} \) cannot be compensated by the revenue derived from allocating the same amount of loans to good borrowers; hence, in the group \( \{A = 1\} \), \( c^\ast_{\text{parity}} \) gives money to all agents with \( x = 0 \) while no loans to \( \{(1,1),(2,1)\} \). In order to satisfy parity constraint in the group \( \{A = 0\} \) we have \( \sum_x c^\ast_{\text{parity}}(x,0) = 1 \). This amount of 1 can be arbitrarily divided among the two equally profitable subgroups \( x = 0 \) and \( x = 1 \); for example, \( c^\ast_{\text{parity}}(0,0) = 1 \) and \( c^\ast_{\text{parity}}(1,0) = 0 \) (thus no agent from \( (1,0) \) gets money) or \( c^\ast_{\text{parity}}(0,0) = 1/2 \) and \( c^\ast_{\text{parity}}(1,0) = 1/2 \) (agents from \( (0,0) \) and \( (1,0) \) receive loans with probability \( 1/2 \)).

We see that under the constraint of DP, loan applications of good borrowers from the disadvantaged group are approved less likely than before imposing the fairness constraint! Agents from the other group, \( \{A = 1\} \), feel no impact after imposing the fairness constraint.
8.3 EO helps the disadvantaged group

In a case the utility function is \( u(x, a, y) = y \cdot \beta_a \) with \( \beta_a = 1/\mathbb{E}[Y|A=a] \), EO always makes good borrowers from the disadvantaged group weakly better off. This improvement follows from Theorem 6 – because our choice of a utility function is aligned with the statistical constraint of the fairness concept (here EO). However, EO is not robust to other selections of a utility function.

Example 13. Assume that \( u(x, a, y) \equiv 1 \), \( X = \{0, 1, 2\} \) with the same distribution of good/bad borrowers as in Example 12, and \( \frac{5}{8} < t < \frac{3}{4} \). The group \( \{A = 1\} \) is disadvantaged: \( W_{u,c^{\text{unc}}}(0) = \frac{2}{3} \) and \( W_{u,c^{\text{unc}}}(1) = \frac{1}{3} \).

EO classifier, in order to equalize the number of loans given to good borrowers, must either increase the amount of loans given to \( \{A = 0\} \) by approving some applications from \( x = 2 \) or decrease the number of loans given to \( \{A = 1\} \). For \( t > \frac{5}{8} \) giving loans to \( (x, a) = (2, 0) \) is too costly: the cost \( \frac{3}{4} \alpha - \frac{1}{4} \alpha_x \) is not compensated by the benefit \( 1 \cdot \alpha_+ \) from giving the same amount of loans to good borrowers \( (0, 1) \). Therefore, the optimal EO classifier coincides with \( c^{\text{unc}} \) in \( \{A = 0\} \) and gives less loans to \( (0, 1) \): \( c^{\text{opportunity}}(1, 0) = \frac{6}{7} \). Hence \( W_{u,c^{\text{opportunity}}}(1) = \frac{2}{7} < W_{u,c^{\text{unc}}}(1) \) and the disadvantaged group is harmed by EO.

9 Algorithms for Approximately-Optimal WE-Classifiers

In this section we develop algorithms for finding approximately optimal WE-classifiers (Recall Definition 8), augmenting Section 5.

9.1 Complete information

We first discuss computing approximately-optimal classifier when the distribution \( \mathbb{P} \) and functions \( \alpha_+, \alpha_- \), and \( u \) are perfectly known and later relax it by assuming that the bank has access to a finite sample only. In particular, the bank can compute \( r(x, a) = \mathbb{E}[\alpha_+ Y - \alpha_- (1-Y) | X = x, A = a] \) and \( \overline{u}(x, a) = \mathbb{E}[u_+(X, A, Y) | X = x, A = a] \), namely, the average revenue and the average utility for a given \( (x, a) \) pair.

The proposition we prove next mirrors Proposition 9.

Proposition 14 (Full version of Proposition 9). Assume that \( \alpha_+, \alpha_- \), and \( u \) are known. Further, assume that the bank has access to an oracle that computes expectations w.r.t. \( \mathbb{P} \) in constant time. Then, for any positive \( \varepsilon \) such that

\[
\varepsilon < \frac{W_{u,c^{\text{unc}}}(0)}{6 \max_{x,a} \overline{u}(x, a)} \min \left\{ \frac{\mathbb{P}(A = 0)}{\mathbb{P}(A = 1)}, \frac{\mathbb{P}(A = 1)}{\mathbb{P}(A = 0)} \right\}
\]

an \((\varepsilon \cdot \max_{x,a} |r(x, a)|, 0)\) optimal WE-classifier can be computed in \( O(\log^2 (\frac{1}{\varepsilon})) \), uniformly over all other parameters.

Proof of Proposition 14. We can assume w.l.o.g. that \( \max_{x,a} \overline{u}(x, a) = \max_{x,a} |r(x, a)| = 1 \) by changing the scale (by doing that we exclude a trivial case of zero \( r \), where any classifier is revenue-maximizing). Similarly to the proof of Proposition 4, we represent the optimization problem as finding an approximately-optimal marginal classifier for a given welfare level \( w \) in each subgroup, and then determining the revenue-maximizing \( w \).
Lemma 15 below guarantees that for each subgroup \( \{ A = a \} \) and welfare level \( w \), a marginal classifier with welfare \( w \) and revenue at least \( R^*_a(w) - \varepsilon \) can be computed in \( O(\log(\frac{1}{\varepsilon})) \) time by binary search over thresholds \( \lambda_a \). We verify below that the upper bound on \( \varepsilon \) ensures that none of \( \{ \lambda_0, \lambda_1 \} \) exceeds \( \frac{\varepsilon}{2} \) (a technical condition of the lemma).

Therefore, for a given \( w \) we can also compute the optimal total revenue \( R^*(w) = \mathbb{P}(A = 0)R^*_0(w) + \mathbb{P}(A = 1)R^*_1(w) \) up to \( \frac{\varepsilon}{2} \) in \( O(\log(\frac{1}{\varepsilon})) \). The total revenue \( R^* \) is a continuous concave function of the welfare level \( w \) (see the proof of Proposition 4) and thus it can be approximately maximized by the Golden-ratio search or by the ternary search as in Hardt et al. [14].

For the moment, assume that we can compute \( R^*(w) \) exactly. Then \( O(\log \frac{1}{\varepsilon}) \) iterations of the search will find \( w \) such that \( R^*(w) \geq R^*(w^*) - \varepsilon \) and, therefore, provides an \( \varepsilon \)-optimal WE-classifier. Since all \( O(\log \frac{1}{\varepsilon}) \) iterations of the search algorithm involve computing \( R^* \) at a new point, the total run-time is \( O(\log \frac{1}{\varepsilon} \cdot \log \frac{1}{\varepsilon}) \).

In order to satisfy the technical condition of Lemma 15, we need one more twist: the search should start on the smaller interval

\[
I = \left[ \max_{a \in \{0,1\}} w_{\frac{\varepsilon}{2}}(a), \min_{a \in \{0,1\}} w_{\frac{\varepsilon}{2}}(a) \right] \cap [0,b],
\]

where \( w_{\lambda}(a) \) denotes a point \( w \) such that the super-gradient of \( R^*_a(w) \) contains \( \lambda \) (a point \( w_{\lambda}(a) \) can be easily computed as in the proof of Lemma 15). By the definition of \( I \), any \( w \in I \) satisfies the condition of Lemma 15 and it remains to check that the Golden-ratio search, restricted to \( I \), still approximates the optimum \( w^* \). Put differently, it is enough to show that \( w^* \in I \). At \( w^* \), First order conditions imply \( 0 \in \partial R^*(w^*) = \mathbb{P}(A = 0)\partial R^*_0(w^*) + \mathbb{P}(A = 1)\partial R^*_1(w^*) \). Therefore, there exist \( \lambda_a \in \partial R^*_a(w^*) \) such that \( \mathbb{P}(A = 0)|\lambda_0| = \mathbb{P}(A = 1)|\lambda_1| \). Pick \( \lambda_a \) that is non-negative. Then \( \lambda_a \cdot w^* \leq 2 \); otherwise, \( R^*_a(0) \leq R^*_a(w^*) + \lambda_a(0 - w^*) < -1 \) which contradicts the normalization of \( r \).

By Theorem 6, \( w^* \geq W_{u,c}_{\text{inc}}(0); \) hence, \( \lambda_a \leq \frac{2}{w_{u,c}_{\text{inc}}(0)} \). The upper bound on \( \varepsilon \) implies that both \( |\lambda_0| \) and \( |\lambda_1| \) are bounded by \( \frac{1}{\varepsilon} \) and thus \( w^* \) belongs to \( I \).

The next lemma is an auxiliary technical result used in the proof of Proposition 14. It shows that approximately optimal marginal classifier for a group \( \{ A = a \} \) with a given welfare level \( w \) can be found by binary search over thresholds. The threshold \( \lambda_a \) can also be thought as a super-gradient of the marginal revenue \( R^*_a \). The super-gradient of a concave function may “explode” if the point is close to the boundary. This is why we need an additional technical assumption that the super-gradient of \( R^*_a \) at \( w \) is not too big.

**Lemma 15.** Under the assumptions of Proposition 14, for a given \( a \in \{0,1\} \), \( \varepsilon > 0 \), and \( w \in [0, \mathbb{E}[\sum_{a \in \{0,1\}} |r(x,a)|] \) such that the super-gradient \( \partial R^*_a(w) \) contains \( \lambda \) with \( |\lambda| < \frac{\varepsilon}{2} \), a marginal classifier for a subgroup \( \{ A = a \} \) with welfare level \( w \) and revenue of at least \( R^*_a(w) - \varepsilon \cdot \max_{x,a} |r(x,a)| \) can be computed in \( O(\log(\frac{1}{\varepsilon})) \).
Proof of Lemma 15. We can assume w.l.o.g. that \( \max_{x,a} |r(x,a)| = 1 \) (similar arguments otherwise). Consider the threshold classifier \( c_\lambda \) such that \( c_\lambda(x) = 1 \) if \( r(x,a) \geq \lambda \pi(x,a) \) and equals zero otherwise. For any \( \lambda \in \mathbb{R} \), this classifier generates the optimal subgroup revenue among all the classifiers having the same welfare level \( w_\lambda(a) = \mathbb{E}[\pi(X,A) c_\lambda(X) | A = a] \), i.e., \( R_a(c_\lambda) \) equals \( R^*_a(w_\lambda) \); moreover, \( \lambda \) belongs to the super-gradient of the concave function \( R^*_a(w) \) at \( w_\lambda \) (see the proof of Proposition 4).

Note that \( w_\lambda \) is a decreasing function of \( \lambda \) and thus we can use binary search in order to find a \( \lambda \) such that \( w_\lambda \) is close to \( w \). By the assumption on \( w \) we can restrict the search to \( \lambda \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}] \) and thus after \( O \left( \log \frac{1}{\varepsilon} \right) \) steps find \( \lambda \) and \( \lambda' \) such that \( w_\lambda \leq w \leq w_{\lambda'} \) and \( |\lambda - \lambda'| \leq \varepsilon \). We can treat \( w \) as the convex combination \( \theta w_\lambda + (1-\theta)w_{\lambda'} \), \( \theta \in [0,1] \), and consider a classifier \( c = \theta c_\lambda + (1-\theta)c_{\lambda'} \).

By the construction, \( c \) has the right welfare level of \( w \).

Let us check that \( R_a(c) \geq R^*_a(w) - \varepsilon \). For any concave function \( h = h(z) \), \( z_1, z_2 \in \mathbb{R} \), and \( \theta \in [0,1] \), it holds that

\[
\theta h(z_1) + (1 - \theta)h(z_2) \geq h(\theta z_1 + (1 - \theta)z_2) - |z_1 - z_2| \cdot |g_1 - g_2|,
\]

where \( g_k \) is any super-gradient of \( h \) at \( z_k \). The revenue of \( c \) is given by the convex combination of values \( \theta R^*_a(w_\lambda) + (1 - \theta)R^*_a(w_{\lambda'}) \), while \( R^*_a(w) \) is the value at the convex combination \( w = \theta w_\lambda + (1 - \theta)w_{\lambda'} \). Applying the above inequality, we get \( R_a(c) \geq R^*_a(w) - |w_\lambda - w_{\lambda'}| \cdot |\lambda - \lambda'| \); hence, \( R_a(c) \geq R^*_a(w) - \varepsilon \).

\[\square\]

9.2 Incomplete information

The assumption of expectation oracle from Proposition 14, as well as the exact knowledge of \( \pi \) and \( r \), can be relaxed. This is the subject of the following Proposition 16, which mirrors Proposition 10.

Proposition 16 (Full version of Proposition 10). Fix \( \delta > 0 \) and assume that the bank has access to a sample of \((X,A,Y,\alpha_{\pm},u)\) and to estimators \( \hat{\pi}(x,a) \) and \( \hat{r}(x,a) \) such that \( \mathbb{E}[|\hat{\pi} - \pi|] \leq \eta_\pi \cdot \max_{x,a} \pi(x,a) \) and \( \mathbb{E}[|\hat{r} - r|] \leq \eta_r \cdot \max_{x,a} |r(x,a)| \). Let \( \varepsilon \) such that

\[
\varepsilon > 2 \sqrt{6 \left( \frac{1}{\mathbb{P}(A = 0)} + \frac{1}{\mathbb{P}(A = 1)} \max\{\eta_\pi, \eta_r\} \right)},
\]

which also satisfies the upper bound in Inequality (6). Then, an \( \left( \varepsilon \cdot \max_{x,a} r(x,a), \varepsilon \cdot \max_{x,a} \pi(x,a) \right) \) optimal WE classifier can be computed with probability at least \( 1 - \delta \) on a sample of size

\[
O \left( \frac{1}{\varepsilon^2} \left( \log \frac{1}{\delta} + \log \log \frac{1}{\varepsilon} \right) \right).
\]

The proof of the proposition is deferred to the end of the subsection, and it relies on three lemmas. First, in Lemma 19 we prove that having a sample access to the distribution is enough for constructing \( (\varepsilon, \varepsilon) \) optimal WE classifier of a particular functional form.

Definition 17. A classifier \( c \) is a classifier in the reduced form\(^{10}\) if \( c = c(r(x,a), \pi(x,a)) \) (i.e., \( c \) depends on \( x \) and \( a \) only through \( r(x,a) \) and \( \pi(x,a) \)).

\(^{10}\)Note that both optimal and \( \varepsilon \)-optimal classifier from Propositions 4 and 9 have reduced form.
However, the constructed classifier turns out to be sensitive to small errors in $r$ and $\overline{\pi}$. In Lemma 20, we describe the smoothing technique that allows to ensure that the outcome of classification is robust to small perturbations of $r$ and $\overline{\pi}$ in the following sense.

**Definition 18.** A reduced-form classifier $c$ is $\epsilon$-robust if

$$ |c(r, \overline{\pi}) - c(r', \overline{\pi}')} | \leq \frac{1}{\epsilon} \left( \frac{|r - r'|}{\max_{x,a} |r(x,a)|} + \frac{|\overline{\pi} - \overline{\pi}'|}{\max_{x,a} \overline{\pi}(x,a)} \right). $$

Finally, Lemma 21 bounds the losses in revenue and welfare of an $\epsilon$-robust classifier that relies on estimators of $\overline{\pi}$ and $r$ instead of the exact values.

**Lemma 19.** For $\delta > 0$ and $\epsilon > 0$ satisfying Inequality (6), a reduced form of an optimal WE classifier can be computed with probability at least $1 - \delta$ on a sample of size

$$ O\left( \frac{1}{\epsilon^2} \left( \log \frac{1}{\delta} + \log \log \frac{1}{\epsilon} \right) \right). $$

**Proof of Lemma 19.** We mimic the algorithm from Proposition 14 but change all the exact expectations to empirical averages.

By re-scaling, we assume $\max_{x,a} |r(x,a)| = \max_{x,a} u(x,a) = 1$. The algorithm from Proposition 14 computes the following conditional expectations $O\left( \log \frac{1}{\epsilon} \right)$ times: welfares $\mathbb{E}[u(X, A, Y)c_{\lambda}(X) | A = a]$ and revenues $\mathbb{E}[r(X, A)c_{\lambda}(X) | A = a] = \mathbb{E}[(\alpha_+(X)Y - \alpha_-(X)(1 - Y))c_{\lambda}(X) | A = a]$. By the Chernoff bound, for i.i.d. random variables $\xi_1, ..., \xi_K$ in $[-1, 1]$ we have $\mathbb{P}\left( \left| \frac{1}{K} \sum_{k=1}^{K} \xi_k \right| > \epsilon \right) \leq 2 \exp\left( -\frac{K\epsilon^2}{2} \right)$. Hence, we ensure that each particular expectation is computed with accuracy $\epsilon$ with probability $1 - \delta'$ using a sample of size $O\left( \frac{1}{\epsilon^2} \log \frac{1}{\delta'} \right)$ conditional on $\{A = a\}$. In order to have such a conditional sample, the unconditional sample must be $\frac{1}{\mathbb{P}(A = a)}$-times bigger. By the union bound, selecting $\delta'$ such that $\delta' \cdot O\left( \log \frac{1}{\delta} \right) \leq \delta$ we guarantee that with probability at least $1 - \delta$ all $O\left( \log \frac{1}{\epsilon} \right)$ expectations are computed with precision $\epsilon$ throughout the run of the algorithm.

**Lemma 20.** The result of Lemma 19 remain true if we additionally ask the computed classifier to be $\frac{\epsilon}{3}$-robust.

**Proof of Lemma 20.** We describe the smoothing technique as usual assuming $\max_{x,a} |r(x,a)| = \max_{x,a} \overline{\pi}(x,a) = 1$. To compute an $\frac{\epsilon}{3}$-robust classifier $c$, we consider an auxiliary “smoothed” classification problem with $x^0 = (x, \xi_u, \xi_r)$, where $\xi_u, \xi_r$ are uniformly distributed on $[0, \epsilon/3]$ and are independent from each other and from $(x, a)$, and define $\overline{\pi}^\circ(x^0, a) = \overline{\pi}(x, a) + \xi_u$ and $r^\circ(x^0, a) = r(x, a) + \xi_r$.

Next, we compute the reduced-form of an $(\epsilon/3, \epsilon/3)$ optimal WE classifier $c^\circ$ in the new problem (under $\overline{\pi}^\circ$) and define a reduced-form classifier in the original problem (under $\overline{\pi}$) by $c(r, \overline{\pi}) = \mathbb{E}_{(\xi_u, \xi_r)} c^\circ(r + \xi_r, \overline{\pi} + \xi_u)$. If we slightly perturb $r$ and $\overline{\pi}$, then $c^\circ$ is integrated over almost-coinciding squares in the $(r, \overline{\pi})$-plane and hence $|c(r, \overline{\pi}) - c(r', \overline{\pi}^')| \leq \frac{|r - r'| + |\overline{\pi} - \overline{\pi}'|}{\epsilon/3}$, i.e., we get robustness. The constructed classifier $c$ equalizes revenues up to $2\epsilon/3 \leq \epsilon$ since $u$ and $u^0$ differ by at most $\epsilon/3$. 

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It remains to check that $c$ generates $\varepsilon$-optimal revenue. The optimal classifier $c_{WE}^{*}$ in the original problem induces a classifier that equalizes welfare up to $\varepsilon/3$ and has revenue $R^\varepsilon(c_{WE}^{*}) \geq R(c_{WE}) - \varepsilon/3$ in the new problem. We get $R^\varepsilon(c_{opt}) + \varepsilon/3 \geq R^\varepsilon(c_{WE}^{*}) \geq R(c_{WE}) - \varepsilon/3$. Since $R(c) \geq R^\varepsilon(c_{opt}) - \varepsilon/3$, we obtain that $c$ generates $\varepsilon$-optimal revenue in the original problem.

\[\text{Lemma 21}. \text{ Suppose we have estimators } \hat{\pi} \text{ and } \hat{r} \text{ such that } \mathbb{E}[|\hat{\pi} - \pi|] \leq \eta_u \cdot \max_{x,a} u(x, a) \text{ and } \mathbb{E}[|\hat{r} - r|] \leq \eta_r \cdot \max_{x,a} |r(x, a)|. \text{ Further, let } c \text{ be a reduced form of a } \kappa\text{-robust } (\varepsilon, \varepsilon') \text{ optimal WE classifier and define } a \text{ the } c' \text{ as the version of } c \text{ that uses the estimators } \hat{\pi} \text{ and } \hat{r}, \text{ i.e., } c'(x, a) = c(\hat{\pi}, \hat{r}). \text{ Then, } c' \text{ is } (\hat{\varepsilon}, \hat{\varepsilon}') \text{ optimal with }
\]

$$\hat{\varepsilon} = \varepsilon + \frac{\max_{x,a} |r(x, a)|}{\kappa} (\eta_u + \eta_r)$$

and

$$\hat{\varepsilon}' = \varepsilon' + \frac{\max_{x,a} u(x, a)}{\kappa} (\eta_u + \eta_r) \left( \frac{1}{\mathbb{P}(A = 0)} + \frac{1}{\mathbb{P}(A = 1)} \right).$$

\[\text{Proof of Lemma 21}. \text{ By } \kappa\text{-robustness we get }
\]

$$\mathbb{E}[u(x, a)c(\hat{\pi}, \hat{r}) | A = a] - \mathbb{E}[u(x, a)c(\pi, r) | A = a] \leq \max_{x,a} u(x, a) \mathbb{E}[|c(\hat{\pi}, \hat{r}) - c(\pi, r)| | A = a]$$

$$\leq \max_{x,a} u(x, a) \mathbb{P}(A = a) \mathbb{E}[|\hat{r} - r|] \leq \frac{\max_{x,a} u(x, a)}{\kappa} \mathbb{P}(A = a) \mathbb{P}(A = a) (\eta_u + \eta_r);$$

thus, by applying the triangle inequality we get

$$|W_{u,c(\hat{\pi}, \hat{r})}(1) - W_{u,c(\pi, \pi)}(0)| \leq \varepsilon' + \frac{\max_{x,a} u(x, a)}{\kappa} (\eta_u + \eta_r) \left( \frac{1}{\mathbb{P}(A = 0)} + \frac{1}{\mathbb{P}(A = 1)} \right).$$

The argument for the revenue is similar but simpler since we need unconditional expectations only.

\[\text{Proof of Proposition 16}. \text{ As usual assume } \max_{x,a} \pi(x, a) = \max_{x,a} r(x, a) = 1. \text{ By Lemma 20, we can compute an } \frac{\varepsilon}{\kappa} \text{-robust } \left(\frac{\varepsilon}{\kappa}, \frac{\varepsilon}{\kappa}\right) \text{ optimal WE classifier } c \text{ in the reduced form on a sample of required size. The lower bound on } \varepsilon \text{ is chosen in such a way that } \frac{\varepsilon}{\kappa} (\eta_u + \eta_r) \left( \frac{1}{\mathbb{P}(A = 0)} + \frac{1}{\mathbb{P}(A = 1)} \right) \leq \frac{\varepsilon}{\kappa}\] and thus, by Lemma 21, the classifier $c(\hat{r}(x, a), \hat{\pi}(x, a))$ is an $(\varepsilon, \varepsilon)$ optimal WE classifier.

\[10 \text{ Popular Fairness Concepts} \]

In this subsection give an introductory to the most influential mathematical concepts of fairness from the literature, which aim to prevent discrimination. Importantly, observe that these fairness concepts impose constraints on the policy of the bank without explicitly considering the welfare of the borrowers. We shall revisit these concepts in Section 3 under our utilitarian approach.
**Unawareness**  One of the most intuitive concepts is *Unawareness*: to deliver a “fair” outcome to both groups $A = 0$ and $A = 1$, the classifier $c$ must ignore the protected attribute $A$. Formally, a classifier $c$ satisfies unawareness if $c(x, 0) = c(x, 1)$ for all $x \in X$.

At first glance, this natural idea is not innocuous: $X$ and $A$ can be dependent and thus information contained in $X$ can be used as a proxy for $A$, driving the disadvantaged group even worse off, see [9]. Despite its flaws, unawareness is promoted by layers in labor market, insurance market\textsuperscript{11}, and also by GDPR.\textsuperscript{12}

The optimal unaware classifier $c^*_{\text{unaware}}$ can be constructed by the same logic as the optimal unconstrained, [7]: an applicant receives money if the probability $P(Y = 1 | X)$ of being a good borrower given the observed attribute $X$ is above the threshold $t(X) = \frac{\alpha_-(X)}{\alpha_+(X) + \alpha_-(X)}$. Formally, $c^*_{\text{unaware}}(x, a) = c^*_{\text{unaware}}(x) = 1$ for borrowers $x$ such that $\bar{p}(x) = \mathbb{P}(A = 0)p(x, 0) + \mathbb{P}(A = 1)p(x, 1) > t(x)$ and $c^*_{\text{unaware}}(x) = 0$ if $\bar{p}(x) < t(x)$.

**Demographic Parity (DP)**  The fairness concept of DP, instead of reducing the information available to the decision-maker, imposes constraints on the outcomes of decisions within the two groups: it requires that the fraction of those who receive loans in the two groups must be the same. Formally, a classifier $c$ satisfies DP if

$$\mathbb{E}[c(X, A) \mid A = 0] = \mathbb{E}[c(X, A) \mid A = 1].$$

Although DP circumvents the “proxy” flaw of *Unawareness*, it is criticized [14]: when the two groups are different in their average creditworthiness, DP forbids the perfect classifier $C(X, A) = Y$ even if such classifier is feasible, i.e., quality $Y$ is a function of attributes $X$ and $A$.

**Equal Opportunity (EO)**  Motivated by drawbacks of DP, Hardt et al. [14] concluded that good and bad borrowers within protected groups must be treated separately and introduced the concept of EO. Under this fairness concept, the fraction of good borrowers who get loans must be the same in the two subgroups, i.e., DP is applied to smaller subgroups $\{Y = 1, A = 0\}$ and $\{Y = 1, A = 1\}$. Formally, a classifier $c$ satisfies EO if

$$\mathbb{E}[c(X, A) \mid Y = 1, A = 0] = \mathbb{E}[c(X, A) \mid Y = 1, A = 1].$$

In contrast to DP, it allows for perfect classifier $C(X, A) = Y$ if $Y$ is a function of observables, and does not artificially force a bank to give an equal amount of loans if the two groups significantly differ in the fraction of good borrowers\textsuperscript{13}; unlike *Unawareness*, it does not lead to information losses and has no “proxy” issues since it constrains the decision, not the data.

\textsuperscript{11}By the decision of the Court of Justice of the European Union, insurance premiums must be determined in a gender-blind way starting from December 2012. This initiative eventually increased the gap between premiums paid by females and males [5].

\textsuperscript{12}The General Data Protection Regulation is a law adopted by the European Union in 2016 that regulates data protection and privacy.

\textsuperscript{13}Think of rich and poor neighborhoods. We would like to be fair to their residents, but there is no reason why equalizing the number of loans given is “fair” if the performances of the two subgroups are different. Moreover, this might be considered discriminatory to the group with higher performance. However, the condition that for a good borrower, it must be equally easy to get a loan no matter what neighborhood she is from, is pretty natural.