Quasi-Spherical Gravitational Collapse in higher dimension and the effect of equation of state

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Gravitational collapse in (n+2) dimensional quasi-spherical space-time is studied for a fluid with non-vanishing radial pressure. An exact analytic solution is obtained (ignoring the arbitrary integration function) for the equation of state \( p_r = (\gamma - 1)\rho \). The singularity is studied locally by comparing the time of formation of apparent horizon and the central shell focusing singularity while the global nature of the final fate of collapse is characterized by the existence of radial null geodesic. It is revealed that the end state of collapse for D dimension with equation of state \( p = -\rho \), for \( (D-1) \) dimensional dust and \( (D-2) \) dimension with equation of state \( p = \rho \) are identical.

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I. INTRODUCTION

Gravitational collapse is an important and challenging issue in Einstein gravity, particularly after the formation of famous singularity theorems [1] and cosmic censorship conjecture (CCC)[2]. Also from the perspective of black hole physics and its astrophysical implications, it is interesting to know the final outcome of gravitational collapse [3] in the background of general relativity. The singularity theorems only tells us about the generic property of space-times in classical general relativity, but it cannot provide us about the detailed features of the singularities whether an external observer can visualize the singularity or not. Moreover, the CCC is incomplete [4,5] in the sense that there is no formal proof of it in one hand and on the other hand there are counter examples of it. It should be noted that choice of initial data [6] has a role in characterizing the final state of collapse.

Though a lot of works have been done for dust collapse [7-12] both for spherical and quasi-spherical models [13-18] in four and higher dimensional space-times but there is not much progress in studying gravitational collapse in quasi-spherical space for perfect fluid or matter with anisotropic pressure [19]. In recent past Dadhich etal [20] examined the role of the equation of state in characterizing the final state of spherical collapse and showed a similarity among collapsing processes in different dimensions with different eq. of state \( p_r = (\gamma - 1)\rho \) in higher dimensional space-time.

II. HIGHER DIMENSIONAL SZEKERES’ MODEL

The quasi-spherical Szekeres’ space-time in (n+2)dimension has the metric ansatz (in the comoving coordinates)

\[
 ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^{n} dx_i^2
\]

(1)

where \( \alpha \) and \( \beta \) being functions of all space-time co-ordinates are of the form

\[
 e^\alpha = R' + R \nu', \quad e^\beta = R e^\nu \quad \text{with} \quad R = R(r, t) \quad \text{and} \quad \nu = \nu(r, x_1, ..., x_n)
\]

(2)

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For the matterfield with anisotropic pressure i.e., $T_{\mu}^\nu = \text{diag}(\rho, -p_r, -p_T, \ldots, -p_T)$, we have from the Einstein equations (after some manipulation)

\[
\begin{align*}
\rho &= \frac{\rho'}{\mu'\mu} \\
p_r &= -\frac{\rho'}{\mu'} \\
\mu &= e^\beta
\end{align*}
\]

(3)

\[F(r,t) = \frac{n}{2} R^{n-1} e^{(n+1)\nu} (\dot{R}^2 - f(r))\]

(4)

while from conservation equation $T_{\mu}^\nu_{\nu} = 0$ we get

\[
\begin{align*}
\dot{\rho} + \dot{\alpha} (\rho + p_r) + n \dot{\beta} (p_T + \rho) &= 0 \\
p'_r + n \beta' (p_r - p_T) &= 0
\end{align*}
\]

(5)

and $\alpha_{x_i} (p_r - p_T) = \frac{\partial}{\partial x^i} p_r$, $i = 1, 2, \ldots, n$

Now for simplicity, we assume $p_T = 0$ and $p_r = (\gamma - 1) \rho, 0 < \gamma < 2$. As a result, conservation equations simplify to

\[
\begin{align*}
(i) \; \rho &= \rho_0 (r, x_1, x_2, \ldots, x_n) e^{-(\gamma \alpha + n \beta)} \\
(ii) \; \alpha_{x_i} &= 0 \text{ i.e. } \alpha \text{ is independent of } x_i \text{'s} \\
(iii) \; p_r &= p_0 (t, x_1, \ldots, x_n) e^{-n \beta}
\end{align*}
\]

(6)

Assuming for simplicity,

\[
p_r = 0 \text{ and } p_r = (\gamma - 1) \rho, \; 0 < \gamma < 2,
\]

(7)

and integrating the conservation equations we get (choosing the arbitrary integration functions appropriately)

\[
\rho(r,t) = \rho_0 \frac{e^{-\gamma \alpha}}{n \rho_0}
\]

(8)

and $\alpha = \alpha(r,t)$ (i.e. $\alpha$ is independent of $x_i$'s)

From equations (3), (4) and (7) we have the evolution equation of $R$ as

\[
\dot{R}^2 = \frac{2}{n} R^{1-n} [\rho_0 R^{1-\gamma} C(r)^{-\gamma} + g(r)]
\]

(9)

where $g(r)$ is an arbitrary integration function and assuming $C(r) = \frac{\dot{R}'}{R'} + \nu'$ to be independent of ‘t’.

Thus choosing $g(r) = 0$, $R$ has the explicit solution

\[
t - t_i = \left( \frac{\sqrt{2 n}}{n + \gamma} \right) \frac{C(r)^{\frac{n}{n-\gamma}}}{\sqrt{\rho_0 (r)}} \left[ R^{\frac{n+\gamma}{n}} - R^{\frac{n-\gamma}{n}} \right]
\]

(10)

The hypersurface $t = t_s(r)$ describing shell focusing singularity is characterized by $R(t_s(r), r) = 0$ i.e.

\[
t_s(r) - t_i = \left( \frac{\sqrt{2 n}}{n + \gamma} \right) \frac{C(r)^{\frac{n}{n-\gamma}}}{\sqrt{\rho_0 (r)}} r^{\frac{n+\gamma}{n}}
\]

(11)
Now for regularity near the central singularity (i.e., \( r = 0 \)) we assume

\[
C(r) = \sum_{j=0}^{\infty} C_j r^j \\
\rho_0(r) = \sum_{j=0}^{\infty} \rho_j r^{n+\gamma+j}
\]

(12)

Then from (11) the time for the central shell focusing singularity is given by

\[
t_0 = \lim_{r \to 0} t_s(r) = t_i + \frac{\sqrt{2n}}{(n+\gamma) \sqrt{\rho_0}} C_{n/2}^{\gamma/2}
\]

(13)

The apparent horizon is characterized by \( \dot{R}^2 = 1 \) and if \( t_{ah} \) is the time of formation of apparent horizon then

\[
 t_{ah}(r) - t_0 = \frac{\sqrt{2n}}{(n+\gamma) \sqrt{\rho_0}} \left[ \frac{C_{n/2}^{\gamma/2}}{\sqrt{\rho_0}} (B r + O(r^2)) - A r^q (1 + O(r)) \right]
\]

(14)

where, \( B = \frac{1}{2} \left( \frac{\gamma C_t}{C_0} - \frac{2}{\rho_0} \right), \quad A = \left( \frac{2}{\rho_0} \right)^{q/2} \), \( n = \frac{n+\gamma}{n+\gamma-2} (> 1) \).

We note that if \( B < 0 \) then \( t_{ah}(r) < t_0 \) i.e., trapped surface forms earlier than the formation of singularity so we can not visualize it (not naked) but if \( B > 0 \) then singularity appears before the formation of apparent horizon so it is possible to visualize the singularity (naked). It should be mentioned that \( t_{ah} > t_0 \) is the necessary condition for the visibility of the singularity (locally) while \( t_{ah} \leq t_0 \) is the sufficient condition for the formation of a black hole (globally).

Further, if \( C_1 = 0 = \rho_1 \) (i.e., \( a = 0 \)) and \( q < 2 \) \( (q > 1) \) then from (14) the dominant term is \( A r^q \) which appears in negative sign in the expression for \( (t_{ah}(r) - t_0) \). Hence the collapse always leads to the formation of a black hole. Now the restriction \( q < 2 \) leads to \( n + \gamma > 4 \) which implies \( D = n + 2 > 6 \) for \( \gamma = 0, D > 5 \) for \( \gamma = 1 \) and \( D > 4 \) for \( \gamma = 2 \). Therefore, the validity of cosmic censorship conjecture in ‘D’ or higher dimension depends on the equation of state.

Secondly, if the arbitrary function \( g(r) \) is non-zero then the evolution equation (9) have complicated solution of the form

\[
t - t_i = \sqrt{\frac{2n}{g(r)}} \frac{1}{n+1} \left\{ \frac{n+1}{r^{n+\gamma}} \frac{2F_1[a, \frac{1}{2}, a+1, -\frac{C(r)\gamma r^{-\gamma} \rho_0(r)}{g(r)}] - R^{n+\gamma} \frac{2F_1[a, \frac{1}{2}, a+1, -\frac{C(r)\gamma r^{-\gamma} \rho_0(r)}{g(r)}]}{R^{n+\gamma} \frac{2F_1[a, \frac{1}{2}, b+1, -\frac{C(r)\gamma r^{-\gamma} g(r)}{\rho_0(r)}]}}{r^{n+\gamma}} \right\}
\]

\[
0 < \gamma < 1 \quad \text{with} \quad a = \frac{n+1}{2(1-\gamma)}
\]

and for \( 1 < \gamma < 2 \)

\[
t - t_i = \sqrt{\frac{2n}{\rho_0(r)}} \frac{C(r)^{\gamma/2}}{n+\gamma} \left\{ \frac{r^{n+\gamma}}{r^{n+\gamma}} \frac{2F_1[b, \frac{1}{2}, b+1, -\frac{C(r)\gamma r^{-\gamma} g(r)}{\rho_0(r)}] - R^{n+\gamma} \frac{2F_1[b, \frac{1}{2}, b+1, -\frac{C(r)\gamma r^{-\gamma} g(r)}{\rho_0(r)}]}{R^{n+\gamma} \frac{2F_1[b, \frac{1}{2}, b+1, -\frac{C(r)\gamma r^{-\gamma} g(r)}{\rho_0(r)}]}}}{r^{n+\gamma}} \right\}
\]

with \( b = \frac{n+\gamma}{2(\gamma-1)} \).

Thus proceeding as above the time difference between the form of trapped surface and the central singularity is given by

\[
t_{ah} - t_0 = \frac{r}{(n+1)^{\gamma/2} g_0} \left[ 2F_1[a, \frac{1}{2}, a+1, -\frac{C_0^{-\gamma} \rho_0}{g_0}] g_1 + \frac{(n+1) C_0^{-1-\gamma} (c_1 g_0 \rho_0 + C_0 g_0 \rho_0 - g_0 C_0 \rho_1)}{(3 + n - 2\gamma) g_0} \right] \times
\]

\[
2F_1[a + 1, \frac{3}{2}, a + 2, -\frac{C_0^{-\gamma} \rho_0}{g_0}] + 0(r^2) - 2 \left( \frac{2g_0/(n+1)^{\gamma/2}}{g_0} \right) 2F_1[a, \frac{1}{2}, a + 1, -\frac{2g_0^{1-\gamma} \rho_0 C_0^{-\gamma}}{g_0}] r^{2(1-\gamma) + \frac{n+1}{2}}
\]
for $0 < \gamma < 1$

and

$$t_{ah} - t_0 = \frac{rC_0^{\gamma/2}}{(n + \gamma)\rho_0^{3/2}} \left[ -2F_1\left[b, \frac{1}{2}, b + 1, -\frac{C_0^2g_0}{\rho_0}\right] \rho_1 + \frac{\rho_0\gamma}{C_0} 2F_1\left[b, \frac{1}{2}, b + 1, -\frac{C_0^2g_0}{\rho_0}\right] \gamma C_0 - \frac{(n + \gamma)C_0^{\gamma - 1}}{(n + 3\gamma - 2)\rho_0} \times 

\right]$$

$$2F_1\left[b + 1, \frac{3}{2}, b + 2, -\frac{C_0^2g_0}{\rho_0}\right] \left(\gamma C_1g_0\rho_0 + C_0g_1\rho_0 - g_0C_0\rho_1\right) + 0(r^2) - \frac{2}{n + \gamma} \left(\frac{2g_0}{n}\right) \frac{\rho_0^{\frac{\gamma - 1}{n}}} {C_0^{\gamma/2} \rho_0} \times 

\right]$$

$$2F_1\left[b, \frac{1}{2}, b + 1, \frac{C_0^2g_0}{\rho_0}\right] \left(\frac{2g_0}{n}\right)^{\frac{\gamma - 1}{n}} r^{\frac{n + \gamma}{n - 1} + \frac{2(n - 1)}{n - 1}}$$

for $1 < \gamma < 2$.

From the above time difference, it is not easy to speculate the end state of collapse (naked singularity or black hole) as it involves so many arbitrary (or initial) constants. However, if it so happens that the coefficient of $r$ vanishes then the leading order term in $r$ may depend on $\gamma$ if the following inequations are satisfied

$$1 < \frac{2(1 - \gamma)}{n - 1} + \frac{n + 1}{n - 1} < 2, \quad \text{for} \quad 0 \leq \gamma \leq 1$$

$$1 < \frac{n + \gamma}{n - 1} + \frac{2(\gamma - 1)}{n - 1} < 2, \quad \text{for} \quad 1 \leq \gamma \leq 2$$

In that case as before we always have black hole as the final state of collapse. The above inequalities show that the dimension parameter $n$ depends strongly on $\gamma$. In particular, for $\gamma = 1$ we have $n > 3$ i.e., we always have black hole solutions for six and higher dimension which is in agreement with result in dust collapse [11]. For different equation of state the restriction on $n$ is as follows:

| TABLE-I |
|---------|
| $\gamma$ | 0 | 1 | 4/3 | 2 |
| $n$     | 6 | 4 | 5 | 7 |

The table shows the least value of $n$ for different value of the equation of state parameter $\gamma$ in formation of black hole.

III. GEODESICS AND THE NATURE OF SINGULARITY

As before, using for simplicity $g(r) = 0 = f(r)$ the scale factor $R(t, r)$ has the simple explicit solution (choosing the initial time $t_i = 0$)

$$R = \left[r^{\frac{n + \gamma}{n}} - \frac{n + \gamma}{2n} \frac{\sqrt{\rho_0(r)}}{C(r)^{\gamma/2}} \right]^{\frac{1}{n + \gamma}}$$

Following the geodesic analysis of Joshi and Dwivedi [21] for TBL model, let us introduce the functions:
\[ X = \frac{R}{r^m} \]
\[ \xi = 2rB' \]
\[ \eta = \frac{rQ'}{Q} \]
\[ \zeta = \frac{2\mu^2}{m(n+\gamma-2)} \]
\[ Q = e^{-\nu} \]
\[ \Theta = \frac{1 - \xi}{r^{m(n+\gamma-2)/2}} \]

\[ (16) \]

with \( m \geq 1 \) and \( B(r) = \sqrt{\frac{\rho_{00}(r)}{C(r)^{\gamma/2}}} \).

In this section the visibility or non-visibility of the singularity is characterized by examining the possibility to have any outgoing null geodesics which are terminated in the past at the central singularity \( r = 0 \). Suppose this occurs at singularity \( t = t_0 \) at which \( R(t_0,0) = 0 \). For convenience, we consider the radial null geodesic, given by

\[ \frac{dt}{dr} = e^\alpha = R' + \gamma R' \]

Now choosing, \( U = r^m \), the above geodesic equation can be written as

\[ \frac{dR}{dU} = U(X, u) \]

where \( U(X, u) = (1 - \sqrt{\frac{\xi}{X^{n+\gamma-2}}})H + \frac{n}{m}\sqrt{\frac{\zeta}{X^{n+\gamma}}} \) with \( H = \frac{\xi X}{n+\gamma} + \frac{\Theta}{X^{n+\gamma-2}/2} \).

If we approach the singularity at \( R = 0, u = 0 \) along the radial null geodesic then the limiting behaviour of the function \( X \) is given by

\[ X_0 = \lim_{R \to 0, u \to 0} X = \lim_{R \to 0, u \to 0} \frac{R}{u} = \lim_{R \to 0, u \to 0} \frac{dR}{du} = \lim_{R \to 0, u \to 0} U(X, u) = U(X_0, 0) \]

This is a polynomial equation in \( X_0 \) with explicit form

\[ X_0^{n+\gamma-1}[m - \frac{\xi_0}{n+\gamma}] - \Theta_0 X_0^{(n+\gamma-2)/2} + \Theta_0 \zeta_0 + \sqrt{\Theta_0} X_0^{(n+\gamma)/2} \left[ \frac{\xi_0}{n+\gamma} - \eta_0 \right] = 0 \]

(20)

Here the suffix ‘0’ stands for the value of the variable at \( r = 0 \).

If it has at least one positive real root then it is possible to have a radial null geodesic outgoing from the central singularity and the singularity will be naked. The above polynomial equation shows that whenever ‘\( n \)’ occurs it is always with \( \gamma \) in the form \( n + \gamma \). So if both \( n \) and \( \gamma \) changes keeping \( n + \gamma \) to be invariant then the final fate of the singularity remains same i.e., if we have space-time dimension \( D = n + 2 \) and equation of state \( p = (\gamma - 1)\rho \) and we have another space-time dimension \( D = n + 1 \) and equation of state \( p = \gamma \rho \) then the end state of collapse will be same for both space time.
TABLE II

| γ | m | η₀ | ξ₀ | ζ₀ | Θ₀ | Positive roots (X₀) |
|---|---|---|---|---|---|---|
|   |   |   |   |   |   | 4D | 5D | 6D | 7D | 8D | 10D | 14D |
| 0 | - | .01 | .1 | .21 | .31 | .45 | .62 |
| 4/3 | 1 | -6 | .05 | .01 | -5 | .03 | .14 | .25 | .39 | .41 | .52 | .66 |
| 2 | .1 | .21 | .31 | .39 | .45 | .55 | .67 |
| 0 | - | - | - | - | - | - | - |
| 4/3 | 1 | 0.1 | 1 | 1 | 1 | 13.6 | - | - | - | - | - |
| 2 | - | - | - | - | - | - | - |
| 0 | 1.19 | .11 | .36 | .54 | .66 | .81 | - |
| 4/3 | 4 | 0 | 5 | 0.1 | 4 | .2 | .43 | .58 | .69 | .76 | - | - |
| 2 | .93 | .89 | .88 | .87 | .87 | - | - |
| 0 | 1.04 | .9 | .87 | .86 | .86 | - | - |
| 4/3 | 1 | 0 | 1 | 0.1 | 1 | .2 | .43 | .59 | .69 | .77 | - | - |
| 2 | .89 | .87 | .86 | .86 | .85 | - | - |
| 4/3 | 1 | 0 | 1 | 0.1 | 1 | .2 | .43 | .59 | .69 | .77 | - | - |
| 2 | .87 | .86 | .86 | .85 | - | - |

Due to the complicated nature of the above polynomial an exact analytic solution is not possible for X₀. So we study the roots by numerical methods. The above table (Table II) shows the dependence of the nature of the roots on the variation of the parameters involved. It is clear from the definition of the functions in eq. (16) that the parameter m, ξ₀, ζ₀ are always positive while η₀, Θ₀ may take both positive and negative values. However, it is not possible to find any definite conclusion due to the variation of the parameter Θ₀.

A. Strength of the naked singularity

The strength of a singularity (Tipler[22]) is characterized by examining the state of a body that falls in it. If all objects that fall into a singularity are destroyed by crushing or tidally stretching to zero volume then the singularity is called gravitationally strong otherwise it is called weak singularity. A precise mathematical definition was given by Clarke and Krolak [23]. According to them, a sufficient condition for a strong curvature singularity is that, for at least one non space-like geodesic with affine parameter \( \lambda \rightarrow 0 \) on approach to the singularity, we must have

\[
\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j > 0
\]

with \( K^i = \frac{dx^i}{d\lambda} \), the tangent vector to the radial null geodesic. For future-directed radial null geodesics that
originate from the naked singularity the above limit can be written as (using L'Hospital's rule)

$$\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j = \frac{n \zeta_0 (H_0 - \eta_0 X_0) \left( \xi_0 - \frac{1}{2} n(n+\gamma) \eta_0 \right)}{2 X_0^{n+\gamma-1} \left( N_0 + \eta_0 \sqrt{\zeta_0 X_0} \right)^2}$$

(22)

where $H_0 = H(X_0, 0), N_0 = N(X_0, 0)$.

The singularity is gravitationally strong in the sense of Tipler if

$$\xi_0 - \frac{1}{2} n(n+\gamma) \eta_0 > \max \left\{ 0, \frac{(n+\gamma) \Theta_0}{X_0} \right\}$$

or

$$\xi_0 - \frac{1}{2} n(n+\gamma) \eta_0 < \min \left\{ 0, \frac{(n+\gamma) \Theta_0}{X_0} \right\}$$

If the above condition is not satisfied for the values of the parameters then

$$\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j \leq 0$$

and the singularity may or may not be Tipler strong.

IV. DISCUSSIONS AND CONCLUDING REMARKS

The paper deals with gravitational collapse in $(n+2)$ dimensional quasi-spherical Szekeres' space-time for perfect fluid with barotropic equation of state $p_r = (\gamma - 1) \rho$. The role of the parameter ‘$\gamma$’ has been discussed in formation of naked singularity. It is observed that the end state of collapse (black hole or naked singularity) remains invariant as long as the sum $n + \gamma$ is fixed i.e., a perfect fluid with equation of state $p_r = \rho$ in ‘$n$’ dimension, dust in $(n+1)$ dimension and exotic matter with equation of state $p_r = -\rho$ in $(n+2)$ dimension are equivalent from the point of view of final state of collapse. This interesting feature has been shown both for local and global nature of the singularity. However, one may note that the above results are valid for the specific choice of $C(r)$ in the series for given in the eq.(12). Finally, the strength of the naked singularity has been examined using the criterion introduced by Tipler [22]. It is found that naked singularity may be a strong curvature singularity or not depending on the choice of the parameters at $r = 0$.

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