BOSONIZATION OF FRIENDLY LIE BIALGEBRAS

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Abstract. We introduce friendly preadditive symmetric monoidal categories and study Lie bialgebras in this context. We prove that bosonization can be done consistently in this framework. In the last part of the paper we present explicit examples and indicate a deep relationship between certain friendly Lie bialgebras and Nichols algebras over abelian groups.

1. Introduction

Since their first appearance, Lie bialgebras and their generalizations fascinate a large community of mathematicians and physicists. Motivated by Sklyanin’s classical r-matrix, Drinfeld [4] and Semenov-Tian-Shansky [24] used Lie bialgebras to study Poisson Lie groups. Publications, problems and questions of Drinfeld had a great impact on further development of the field. Between 1996 and 2008, Etingof and Kazhdan proved in a series of papers that Lie bialgebras can be quantized, with particular emphasis on Kac-Moody and vertex algebras [11, 12, 13, 14, 15].

One of the main ideas of their approach was to quantize the whole category of modules rather than the universal enveloping algebra of the Lie algebra only, and then to conclude the existence of a quantized (quasi-)Hopf algebra using Tannakian reconstruction.

The deformation theory of Lie bialgebras remained until today a vital area of research. The class of examples has been extended successively among others by Andruskiewitsch, Enriquez, Geer, Halbout, Hurle, Majid, and Makhlouf to include quasi-Lie bialgebras, Γ-Lie bialgebras, Lie superbialgebras, color Lie bialgebras, and braided Lie bialgebras. For more details we refer to the papers [16, 17, 18, 19, 20]. A classification of Lie bialgebras over current algebras was achieved in [22].

Parallelly, several attempts have been made to study (among others) Lie algebras in very general categorical contexts, e.g. [10, 17, 23]. A good source towards this is the work [17] of Buchberger and Fuchs, where basic concepts on Lie algebras in (pre)additive symmetric monoidal categories have been worked out pleasingly in detail. Although there seems to be no clear agreement about the setting in the largest possible generality, there is a clear desire to search for this. E.g. Goyvaerts and Vercruysse write in [17]: Motivated by the way that the field of Hopf algebras benefited from the interaction with the field of monoidal categories (see e.g. [25]) on one hand, and the strong relationship between Hopf algebras and Lie algebras on the other hand, the natural question arose whether it is possible to study Lie algebras within the framework of monoidal categories, and whether Lie theory could also benefit from this viewpoint. In the same vein, Buchberger and Fuchs write:
... we insist on imposing relevant conditions directly on the underlying category. A benefit of the abstraction inherent in the categorical point of view is that it allows one to neatly separate features that apply only to a subclass of examples from those which are essential for the concepts and results in question and are thereby generic.

Our work is built partially on the point of view illustrated in the previous paragraph. As already observed by Majid in [20, Def. 2.2, Th. 3.7], the concept of a semidirect sum (also called bisum by Majid) of Lie bialgebras requires the introduction of a significantly more general framework and the appending of an additional term in the Lie bialgebra axiom. Motivated by the bosonization theory of braided Hopf algebras in the largest currently known categorical setting in [18, Ch. 3], we introduce the notion of a friendly category, which is nothing but a preadditive symmetric monoidal category together with a handshake. This gives us a very general and novel categorical framework and, quite surprisingly to the authors, allows a consistent discussion of bosonization. In the category of vector spaces only the trivial handshake exists, which explains partially why our structure was not discovered earlier. It should also be mentioned that most ideas regarding bosonization, except the use of the handshake, are already available in [20]. Majid himself writes that due to lack of examples (this was around the year 2000) it is not clear to him which framework is most suitable for his presentation. Also, Majid is not explaining how bosonization works for his braided Lie bialgebras.

Our second main motivation for the present work was an observation on some examples of Nichols algebras over some abelian groups. In [2] new examples of pointed Hopf algebras of finite Gelfand-Kirillov dimension appeared and were described very explicitly. During a visit of Hector Peña Pollastri in February 2022 in Marburg, he and the first named author observed in some of these examples an appearance of Lie bialgebras in that context. We do not aim in this paper a detailed analysis of the precise connection. Nevertheless, we present some examples of Lie bialgebras in some friendly categories with non-trivial handshake — more precisely, in the category of crossed modules over some abelian coabelian Lie bialgebras, with possibly non-trivial braiding — and point out the Nichols algebras they are related to.

The paper is organized as follows. Section 2 introduces friendly preadditive symmetric monoidal categories. In Sections 3 and 4 we recall basic definitions of Lie algebras and Lie coalgebras in symmetric monoidal categories. Lie bialgebras in friendly preadditive symmetric monoidal categories appear in Section 5. In Section 6 (see Theorems 6.1 and 6.2) we discuss bosonization of Lie bialgebras in friendly categories. Concrete examples (the Jordan plane, the super Jordan plane, and the Laistrygonians) are discussed in Section 7.

2. Symmetric monoidal categories with additional structure

In this paper, Lie bialgebras and variations of them will be objects in symmetric monoidal categories with additional structure. For convenience we will assume that the category is strict. In the literature such an assumption is not unusual, see e.g. [3]. In this section the most important properties of such categories are collected, which will be used freely in the sequel. Typically we write $\tau$ for the
symmetry of such a category and use occasionally the leg notation:
\[ \tau_{i(i+1)} = \text{id}^{\otimes i-1} \otimes \tau \otimes \text{id}^{\otimes j} \in \text{End}(V^{\otimes i+j+1}) \]
for all objects \( V \) and all \( i \geq 1, j \geq 0 \).

Recall that a preadditive category is a category where the morphisms between any two objects form an abelian group, and composition of morphisms satisfies the distributive law. A preadditive symmetric monoidal category is a symmetric monoidal category which is preadditive and the tensor functor is additive, that is, tensor product and addition of morphisms satisfy the distributive laws.

For our purpose we will need preadditive symmetric monoidal categories with an additional ingredient.

**Definition 2.1.** Let \( C \) be a preadditive symmetric monoidal category with identity object \( I \) and symmetry \( \tau \), together with a natural transformation \( \eta \)

\[ \eta = (\eta_{X,Y})_{X,Y \in C} \]

from the monoidal functor \( \otimes : C \times C \to C \) to itself. Assume that \( \eta \) satisfies the following conditions:

\[
\begin{align*}
\eta_{X,Y} \otimes Z &= \eta_{X,Y} \otimes \text{id}_Z + (\tau_{Y,X} \otimes \text{id}_Z)(\text{id}_Y \otimes \eta_{X,Z})(\tau_{X,Y} \otimes \text{id}_Z), \\
\eta_{Y,X} \tau_{X,Y} &= \tau_{X,Y} \eta_{X,Y}
\end{align*}
\]

for all \( X, Y, Z \in C \).

Then we say that \((C, \eta)\) is a friendly preadditive symmetric monoidal category, and call \( \eta \) its **handshake**.

Note that each preadditive symmetric monoidal category is friendly with \( \eta_{X,Y} = 0 \) for all \( X, Y \in C \). We are going to show how Lie bialgebras and bosonization give rise to friendly preadditive symmetric monoidal categories with non-zero handshake.

**Remark 2.2.**

1. The axioms of a friendly preadditive symmetric monoidal category imply in particular that

\[
\begin{align*}
\eta_{I,X} &= 0 = \eta_{X,I}, \\
\eta_{X,Y} \otimes Z &= \eta_{X,Y} \otimes \text{id}_Z + (\tau_{Y,X} \otimes \text{id}_Z)(\text{id}_Y \otimes \eta_{X,Z})(\tau_{X,Y} \otimes \text{id}_Z), \\
\eta_{Y,X} \tau_{X,Y} &= \tau_{X,Y} \eta_{X,Y}
\end{align*}
\]

for all \( X, Y, Z \in C \). Indeed, Equation (2.1) with \( Y = Z = I \) says that

\[ \eta_{X,I} = \eta_{X,I} + \eta_{X,I} \]

and hence \( \eta_{X,I} = 0 \). The rest follows by using also Equation (2.2).
2. Let \( C \) be the (preadditive symmetric monoidal) category Vec with the flip as symmetry. Let \( \eta \) be a natural transformation from the monoidal functor of \( C \) to itself. By using rank one linear maps between (non-zero) vector spaces, it follows quickly that there exists a scalar \( \lambda \) such that \( \eta_{X,Y} = \lambda \text{id}_{X \otimes Y} \) for all vector spaces \( X, Y \). Thus (2.3) implies that the only handshake of Vec is the zero natural transformation.
3. Let \( k \) be a field, let \( G \) be an abelian monoid, and let \( C = {}^kG \mathcal{M} \) be the (preadditive symmetric monoidal) category of \( kG \)-comodules (or, equivalently, \( G \)-graded vector spaces over \( k \)). Again, the symmetry of the category is the flip. By generalizing the arguments for Vec, one concludes that the
natural transformations from the monoidal functor of $C$ to itself correspond to maps $\chi : G \times G \to \mathbb{k}$ such that
$$\eta_{X,Y} = \chi(g,h) \text{id}_X \otimes \text{id}_Y$$
for all $g, h \in G$ and homogeneous objects $X$ of $G$-degree $g$ and $Y$ of degree $h$.

In this setting, $\eta$ is a handshake, that is, a natural transformation satisfying Equations (2.1) and (2.2), if and only if $\chi$ is a symmetric additive bicharacter of $G$ with values in $\mathbb{k}$, that is,
$$\chi(h_1, h_2) = \chi(h_2, h_1), \quad \chi(g, h_1 + h_2) = \chi(g, h_1) + \chi(g, h_2)$$
for all $g, h_1, h_2 \in G$.

3. Lie algebras in symmetric categories

Let $C$ be a preadditive symmetric monoidal category. A Lie algebra in $C$ is a pair $(g, \beta)$, where $g$ is an object in $C$ and $\beta : g \otimes g \to g$ is a morphism in $C$ such that

(1) $\beta(id + \tau) = 0$ ($\beta$ is antisymmetric) and
(2) $\beta(\beta \otimes id - (id \otimes \beta)(id - \tau_1)) = 0$ (Jacobi identity).

If $C$ is friendly, then a Lie algebra in $C$ is just a Lie algebra in the underlying preadditive symmetric monoidal category.

The categorical context implies directly some compatibility conditions between the braiding $\tau$ and the bracket $\beta$ of a Lie algebra $g$ in $C$. We will typically use freely these identities. Two of the most frequent identities are

$$\tau(\beta \otimes id)(id \otimes \tau) = (id \otimes \beta)(\tau \otimes id) : g \otimes V \otimes g \to V \otimes g$$

and

$$\tau(id \otimes \beta)(\tau \otimes id) = (\beta \otimes id)(id \otimes \tau) : g \otimes V \otimes g \to g \otimes V$$

for all objects $V \in C$.

Let $(f, \beta)$ be a Lie algebra in $C$. A left Lie module over $f$ in $C$ is a pair $(V, \alpha)$, where $V$ is an object in $C$ and $\alpha : f \otimes V \to V$ is a morphism in $C$ such that the diagram

\[
\begin{array}{ccc}
\text{f} \otimes \text{f} \otimes V & \xrightarrow{\text{id} - \tau_1} & \text{f} \otimes \text{f} \otimes V \\
\beta \otimes \text{id} & \downarrow & \text{id} \otimes \alpha \\
\text{f} \otimes V & \xrightarrow{\alpha} & V
\end{array}
\]

commutes. If we want to be more explicit, we use a notation for $\alpha$ indicating $V$, i.e. $\alpha = \alpha_V$. We write $\mathcal{C}$ for the category where the objects are left Lie modules over $f$ in $C$, and the morphisms between two objects $(V, \alpha_V)$ and $(W, \alpha_W)$ are the morphisms $f : V \to W$ in $C$ with

$$\alpha_W(id \otimes f) = f \alpha_V.$$  

The category $\mathcal{C}$ is preadditive, strict monoidal and symmetric, where the monoidal structure is given by the diagonal action

$$\alpha_V \otimes \alpha_W = \alpha_V \otimes \text{id}_W + (\text{id}_V \otimes \alpha_W)(\tau_{f,V} \otimes \text{id}_W)$$

for any $(V, \alpha_V), (W, \alpha_W)$ in $\mathcal{C}$, and the braiding of $\mathcal{C}$ is the braiding $\tau$ of $C$.

Note that each Lie algebra $(f, \beta)$ in $C$ is a left Lie module over $f$ in $\mathcal{C}$ with module structure $\alpha_f = \beta$. Therefore each tensor power of $f$ is a left Lie module over $f$ in $C$.  

Let \((f, \beta)\) be a Lie algebra in \(C\) and let \((g, \beta_g)\) be a Lie algebra in \(C\) with \(f\)-action \(\alpha_g\). Assume that the biproduct \(g \oplus f\) exists in \(C\). Then there is a unique morphism 
\[
\beta : (g \oplus f) \otimes (g \oplus f) \to g \oplus f
\]
in \(C\) such that 
\[
\beta(t_f \otimes t_f) = t_f \beta_f, \quad \beta(t_g \otimes t_g) = t_g \beta_g,
\]
and 
\[
\beta(t_f \otimes t_g) = t_g \alpha_g, \quad \beta(t_g \otimes t_f) = -t_g \alpha_g \tau_{g,f},
\]
where \(t_f : f \to g \oplus f\) and \(t_g : g \to g \oplus f\) are the canonical monomorphisms. Moreover, 
\((g \oplus f, \beta)\) is a Lie algebra in \(C\) and is called the semidirect sum of \(g\) and \(f\).

4. Lie coalgebras in symmetric categories

Let \(C\) be a preadditive symmetric monoidal category. A Lie coalgebra in \(C\) is a pair \((g, \delta)\), where \(g\) is an object in \(C\) and \(\delta : g \to g \otimes g\) is a morphism in \(C\) such that 
\[
(1) \quad (\id + \tau) \delta = 0 \quad (\delta\text{ is co-antisymmetric}) \quad \text{and}
\]
\[
(2) \quad (\delta \otimes \id - (\id - \tau_{12})(\id \otimes \delta)) \delta = 0 \quad (\text{co-Jacobi identity}).
\]
Similarly to Lie algebras, identities involving the braiding are often applied without explanation. Here are two frequently used identities for the cobracket \(\delta\) of a Lie coalgebra \(g\):
\[
(id \otimes \tau)(\delta \otimes \id) \tau = (\tau \otimes \id)(\id \otimes \delta) : V \otimes g \to g \otimes V \otimes g
\]
and 
\[
(\tau \otimes \id)(\id \otimes \delta) \tau = (id \otimes \tau)(\delta \otimes \id) : g \otimes V \to g \otimes V \otimes g
\]
for all objects \(V \in C\).

The study of Lie coalgebras has a long tradition, see e.g. [21]. Let \((f, \delta)\) be a Lie coalgebra in \(C\). A left Lie comodule over \(f\) in \(C\) is a pair \((V, \lambda)\), where \(V\) is an object in \(C\) and \(\lambda : V \to f \otimes V\) is a morphism in \(C\) such that the diagram 
\[
f \otimes f \otimes V \xrightarrow{id \otimes -\tau_{12}} f \otimes f \otimes V \xleftarrow{id \otimes \lambda} f \otimes V
\]
\[
\xleftarrow{\delta \otimes \id} f \otimes V \xrightarrow{\lambda} V
\]
commutes. We write \(\mathcal{C}\) for the category where the objects are left Lie comodules over \(f\) in \(C\), and the morphisms between two objects \((V, \lambda_V)\) and \((W, \lambda_W)\) are the morphisms \(g : V \to W\) in \(C\) such that 
\[
\lambda_W g = (id \otimes g)\lambda_V.
\]
The category \(\mathcal{C}\) is preabelian strict monoidal and symmetric, where the monoidal structure is given by the diagonal coaction on tensor products, 
\[
\lambda_V \otimes \lambda_W = \lambda_V \otimes id_W + (\tau_{V,f} \otimes id_W)(id_V \otimes \lambda_W)
\]
for any \((V, \lambda_V), (W, \lambda_W)\) in \(\mathcal{C}\), and the braiding of \(\mathcal{C}\) is the braiding of \(C\).

Note that each Lie coalgebra \((g, \delta)\) in \(C\) is a left Lie comodule over \(g\) in \(\mathcal{C}\) with comodule structure \(\lambda_g = \delta\). Therefore each tensor power of \(g\) is a left Lie comodule over \(g\) in \(C\).
Let \((f, \delta)\) be a Lie coalgebra in \(C\) and let \((g, \delta_g)\) be a Lie coalgebra in \(1C\) with \(f\)-coaction \(\lambda_f\). Assume that the biproduct \(g \oplus f\) of \(g\) and \(f\) exists in \(C\). Then there is a unique morphism
\[
\delta : g \oplus f \to (g \oplus f) \otimes (g \oplus f)
\]
in \(C\) such that
\[
(\pi_f \otimes \pi_f)\delta = \delta_f \pi_f, \quad (\pi_g \otimes \pi_g)\delta = \delta_g \pi_g,
\]
\[
(\pi_f \otimes \pi_g)\delta = \lambda_g \pi_g, \quad (\pi_g \otimes \pi_f)\delta = -\tau_{f,g} \lambda_g \pi_g,
\]
where \(\pi_g : g \oplus f \to g\) and \(\pi_f : g \oplus f \to f\) are the canonical epimorphisms. In particular,
\[
\delta_{\pi_f} = ((\iota_f \pi_f + \iota_g \pi_g) \otimes (\iota_f \pi_f + \iota_g \pi_g))\delta_{\pi_f} = (\iota_f \otimes \iota_f)\delta_{\pi_f},
\]
\[
\delta_{\pi_g} = (\iota_g \otimes \iota_g)\delta_{\pi_g} + (id - \tau)(\iota_f \otimes \iota_g)\delta_{\pi_g}.
\]
Moreover, \((g \oplus f, \delta)\) is a Lie coalgebra in \(C\) and is called the semidirect sum of \(g\) and \(f\).

5. Lie bialgebras in friendly symmetric categories

Let \(C = (C, \eta)\) be a friendly preadditive symmetric monoidal category. A Lie bialgebra in \(C\) is a triple \((g, \beta, \delta)\), where \((g, \beta)\) is a Lie algebra in \((\text{the underlying preadditive symmetric monoidal category}) C\), \((g, \delta)\) is a Lie coalgebra in \(C\), and
\[
\delta \beta = (id - \tau)(\beta \otimes id)(id \otimes \tau)(\delta \otimes id)(id - \tau) + (\tau - id) \eta
\]
as endomorphisms of \(g \otimes g\) in \(C\). The compatibility condition \((5.1)\) has many other equivalent formulations. One of them is the following:
\[
\delta \beta = (\beta \otimes id)(id \otimes \delta) + (id \otimes \beta)(\tau \otimes id)(id \otimes \delta)
\]
\[
+ (id \otimes \beta)(\delta \otimes id) + (\beta \otimes id)(id \otimes \tau)(\delta \otimes id) + (\tau - id) \eta.
\]

Remark 5.1. Our definition of a Lie bialgebra in a friendly preadditive symmetric monoidal category is a far reaching but very natural generalization of the notion of a Lie bialgebra. In this paper we even take the perspective that a categorical notion of a Lie bialgebra is only possible after fixing a handshake (which also may be zero) for the category.

Let \((f, \beta, \delta)\) be a Lie bialgebra in \(C\). A (left) crossed module over \(f\) in \(C\) is a triple \((V, \alpha, \lambda)\), where \((V, \alpha) \in \mathcal{C}\), \((V, \lambda) \in 1\mathcal{C}\), and
\[
\lambda : \beta \otimes (id \otimes \lambda) + (id \otimes \alpha)(\tau_f \otimes id)(id \otimes \lambda) + (id \otimes \alpha)(\delta \otimes id) - \eta
\]
as endomorphisms of \(f \otimes V\) in \(C\). Let \(\mathcal{C}\) denote the category of left crossed modules over \(f\) in \(C\), where morphisms are left Lie module and left Lie comodule morphisms in \(C\). Recall the diagonal action from Equation \((5.1)\) and the diagonal coaction from Equation \((4.1)\) of \(f\) on tensor products. The proof of the following lemma is straightforward and is left to the reader.

Lemma 5.2. The category \(\mathcal{C}\) is preadditive symmetric monoidal, where the identity is the identity of \(C\) with zero action and coaction, the action and the coaction of \(f\) on tensor products are diagonal, and the braiding is the braiding of \(C\).

We will improve Lemma \((5.2)\) in Proposition \((5.6)\) below.

Remark 5.3.
Lemma 5.4. Let $A$ be a Lie bialgebra in the friendly preadditive symmetric monoidal category $C$. For each pair $(V, \alpha_V, \lambda_V)$ and $(W, \alpha_W, \lambda_W)$ of objects in $C$ let

$$\zeta_{V,W} = (\alpha_W \otimes \text{id}_V)(\text{id}_f \otimes \tau_{V,W})(\lambda_V \otimes \text{id}_W) : V \otimes W \to W \otimes V.$$  

(1) For all morphisms $f : V_1 \to V_2$, $g : W_1 \to W_2$ in $C$, 

$$\zeta_{V_2,W_2}(f \otimes g) = (g \otimes f)\zeta_{V_1,W_1}.$$  

(2) For all $X, Y, Z \in C$,

$$\zeta_{X,Y,Z} = (\text{id}_f \otimes \tau_{X,Z})(\epsilon_{X,Y \otimes \text{id}Z}) + (\text{id}_V \otimes \epsilon_{X,Y \otimes \text{id}Z}),$$  

$$\zeta_{X,Y,Z} = (\epsilon_{X,Z \otimes \text{id}Y}) + (\tau_{X,Z \otimes \text{id}Y})(\text{id}_Y \otimes \epsilon_{Y,Z}).$$

Note that the morphisms $\zeta_{V,W}$ in the lemma are typically not morphisms in $C$, not even if $\eta = 0$. Nevertheless they are useful to discuss the morphisms in Lemma 5.5(2) below.

**Proof.** Both claims follow directly from the definitions (including the definitions of the $f$-action and of the $f$-coaction on a tensor product of two objects) and the naturality of the symmetry $\tau$. \hfill \square

Lemma 5.5. Let $A$ be a Lie bialgebra in the friendly preadditive symmetric monoidal category $C$. For each pair $(V, \alpha_V, \lambda_V)$ and $(W, \alpha_W, \lambda_W)$ in $C$ let

$$\zeta_{V,W} = (\alpha_V \otimes \text{id}_V)(\text{id}_f \otimes \tau_{V,W})(\lambda_V \otimes \text{id}_W) : V \otimes W \to W \otimes V$$ (as in Lemma 5.4),

$$\hat{\alpha}_{V,W} = (\alpha_V \otimes \alpha_W)(\text{id}_f \otimes \tau_{V,W})(\delta \otimes \text{id}_{V \otimes W}) : f \otimes V \otimes W \to V \otimes W,$$

$$\hat{\lambda}_{V,W} = (\beta \otimes \text{id}_{V \otimes W})(\text{id}_f \otimes \tau_{V,W})(\alpha \otimes \lambda_W) : V \otimes W \to f \otimes V \otimes W.$$

(1) For all $(V, \alpha_V, \lambda_V), (W, \alpha_W, \lambda_W) \in C$ the following equations hold.

$$\tau_{V,W} \hat{\alpha}_{V,W} = -\hat{\alpha}_{W,V}(\text{id}_f \otimes \tau_{W,V}),$$

$$\hat{\lambda}_{W,V} \tau_{V,W} = -(\text{id}_f \otimes \tau_{V,W})\lambda_{V,W},$$

$$\zeta_{V,W} \alpha_{V \otimes W} + \tau_{V,W} \lambda_{V,W} + (\alpha \otimes \text{id})(\eta \otimes \text{id})(\text{id}_X \otimes \text{id}_Y) = \alpha_{V \otimes W}(\text{id}_f \otimes \zeta_{V,W}),$$

$$\lambda_{W \otimes V} \hat{\zeta}_{V,W} + \hat{\lambda}_{W,V} \tau_{V,W} + (\eta \otimes \text{id})(\text{id}_X \otimes \text{id}_Y)(\text{id}_f \otimes \zeta_{V,W}) = (\text{id}_f \otimes \zeta_{V,W})\lambda_{V \otimes W}.$$

(2) The morphism

$$\hat{\eta}_{V,W} = \zeta_{V,W} \tau_{V,W} + \tau_{W,V} \hat{\zeta}_{V,W} + \eta_{V,W}$$

in $C$ is an endomorphism of $V \otimes W$ in $C$.  


Proof. (1) The equations

\[ \tau_{V,W} \hat{\alpha}_{V,W} = -\hat{\alpha}_{W,V} (id_f \otimes \tau_{V,W}), \quad \hat{\lambda}_{W,V} \tau_{V,W} = -(id_f \otimes \tau_{V,W}) \hat{\lambda}_{V,W} \]

follow from \( \tau \delta = -\delta \) and \( \beta \tau = -\beta \), respectively, and from the naturality of the symmetry \( \tau \).

Now we are going to prove the third equation. Recall that

\[ \alpha_{V \otimes W} = \alpha_V \otimes id_W + (id_V \otimes \alpha_W)(\tau_{W,V} \otimes id_W). \]

Therefore

\[
\begin{align*}
\zeta_{V,W} \alpha_{V \otimes W} &= (\alpha_W \otimes id_V)(id_f \otimes \tau_{V,W})(\lambda_V \otimes id_W) \\
&\quad \cdot (\alpha_V \otimes id_W + (id_V \otimes \alpha_W)(\tau_{V,W} \otimes id_W)) \\
&= (\alpha_W \otimes id_V)(id_f \otimes \tau_{V,W})(\lambda_V \otimes id_W) \\
&\quad - (\eta_{V,W} \otimes id_V) + (\beta \otimes id_V \otimes id_W)(\tau_{V,W} \otimes \lambda_V \otimes id_W) \\
&\quad + (id_V \otimes \alpha_W)(\tau_{W,V} \otimes id_W)(id_f \otimes \lambda_V \otimes id_W) \\
&\quad + (id_f \otimes \alpha_V \otimes id_W)(\delta \otimes id_V \otimes id_W) + (\lambda_V \otimes \alpha_W)(\tau_{V,W} \otimes id_W)
\end{align*}
\]

by the crossed module axiom (5.3). In the second term we rewrite \( \alpha_W(\beta \otimes id_W) \) using the Lie module axiom for \( (W, \alpha_W) \) and obtain that

\[
\begin{align*}
\zeta_{V,W} \alpha_{V \otimes W} &= (\alpha_W \otimes id_V)(id_f \otimes \tau_{V,W})(\lambda_V \otimes id_W) \\
&\quad - (\eta_{V,W} \otimes id_V) + (\beta \otimes id_V \otimes id_W)(\tau_{V,W} \otimes \lambda_V \otimes id_W) \\
&\quad + (id_V \otimes \alpha_W)(\tau_{W,V} \otimes id_W)(id_f \otimes \lambda_V \otimes id_W) \\
&\quad + (id_f \otimes \alpha_V \otimes id_W)(\delta \otimes id_V \otimes id_W) + (\lambda_V \otimes \alpha_W)(\tau_{V,W} \otimes id_W)
\end{align*}
\]

Now the last term cancels with part of the second term, and the fourth term is \( \hat{\alpha}_{W,V} (id_f \otimes \tau_{V,W}) = -\tau_{V,W} \hat{\alpha}_{V,W} \). It follows that

\[
\begin{align*}
\zeta_{V,W} \alpha_{V \otimes W} &= (\alpha_W \otimes id_V)(id_f \otimes \tau_{V,W})(\lambda_V \otimes id_W) \\
&\quad - (\eta_{V,W} \otimes id_V) + (\beta \otimes id_V \otimes id_W)(\tau_{V,W} \otimes \lambda_V \otimes id_W) \\
&\quad + (id_V \otimes \alpha_W)(\tau_{W,V} \otimes id_W)(id_f \otimes \lambda_V \otimes id_W) - \tau_{V,W} \hat{\alpha}_{V,W}
\end{align*}
\]

In the last expression, the sum of the second and the third term is \( \alpha_W \otimes V(id_f \otimes \zeta_{V,W}) \). This implies the third equation in part (1) of the lemma.

The fourth equation of part (1) of the lemma can be proven similarly.
(2) We prove that \( \tilde{\eta}_{V,W} \) is an endomorphism of the \( \mathfrak{f} \)-module \( V \otimes W \). By definition of \( \tilde{\eta}_{V,W} \) and by (1),

\[
\tilde{\eta}_{V,W} \alpha_{V \otimes W} = (\zeta_{V,V} \tau_{V,W} + \tau_{V,V} \zeta_{V,W} + \eta_{V,W}) \alpha_{V \otimes W}
\]

\[
= \zeta_{W,V} \alpha_{W \otimes V} (id_{W} \otimes \tau_{V,W}) + \tau_{W,V} (\alpha_{W \otimes V} (id_{W} \otimes \zeta_{V,W}) - \tau_{V,W} \alpha_{V \otimes W})
\]

\[
+ \tau (\alpha \otimes id)(id \otimes \tau)(-\eta_{V,W} \otimes \alpha_{W \otimes V}) + \eta_{V,W} \alpha_{V \otimes W}
\]

\[
= \alpha_{V \otimes W} (id_{W} \otimes \zeta_{W,V} \tau_{V,W}) + \alpha_{V \otimes W} (id_{W} \otimes \tau_{W,V} \zeta_{V,W}) - \alpha_{V \otimes W}
\]

\[
+ \tau (\alpha \otimes id)(id \otimes \tau)(-\eta_{V,W} \otimes \alpha_{W \otimes V}) + \eta_{V,W} \alpha_{V \otimes W}
\]

\[
= \alpha_{V \otimes W} (id_{W} \otimes \zeta_{W,V} \tau_{V,W}) - id_{W} \otimes \eta_{V,W}
\]

\[
- \tau (\alpha \otimes id)(id \otimes \tau)(\eta_{V,W} \otimes \alpha_{W \otimes V}) + \eta_{V,W} \alpha_{V \otimes W}.
\]

The naturality of \( \eta \) and the compatibility conditions of \( \eta \) and the monoidal structure imply that

\[
\eta_{V,W} \alpha_{V \otimes W} = \eta_{V,W} (\alpha_{V} \otimes id_{W} + id_{V} \otimes \alpha_{W})(\tau_{V} \otimes id_{W})
\]

\[
= (\alpha_{V} \otimes id_{W}) \eta_{V,W} \otimes id_{W} + (id_{V} \otimes \alpha_{W}) \eta_{V,W} \otimes id_{W}
\]

\[
= (\alpha_{V} \otimes id_{W})(id_{V} \otimes \eta_{V,W} + id_{V} \otimes \tau_{V,W})(\eta_{V,W} \otimes id_{W})(id_{V} \otimes \tau_{V,W})
\]

\[
+ (id_{V} \otimes \alpha_{W})(\eta_{V,W} \tau_{V,W} \otimes id_{W} + (\tau_{V,W} \otimes id_{W})(id_{V} \otimes \eta_{V,W})).
\]

After applying the distributive rule, we get four terms. The first and the fourth terms yield \( \alpha_{V \otimes W}(id_{W} \otimes \eta_{V,W}) \), and the third of the four terms is

\[
(id_{V} \otimes \alpha_{W})(\eta_{V,W} \tau_{V,W} \otimes id_{W}) = (id_{V} \otimes \alpha_{W})(\eta_{V,W} \tau_{V,W} \otimes id_{W})
\]

\[
= \tau_{V,W} (\alpha_{W} \otimes id_{V})(id_{V} \otimes \tau_{V,W})(\eta_{V,W} \otimes id_{W}).
\]

Now it is easy to confirm that \( \tilde{\eta}_{V,W} \alpha_{V \otimes W} = \alpha_{V \otimes W}(id_{W} \otimes \tilde{\eta}_{V,W}) \).

Similarly, \( \tilde{\eta}_{V,W} \) is an endomorphism of the \( \mathfrak{f} \)-comodule \( V \otimes W \), which proves the claim in (2). \qed

**Proposition 5.6.** Let \( (\beta, \delta) \) be a Lie bialgebra in the friendly preadditive symmetric monoidal category \( \mathcal{C} \), and for all \( V, W \in \mathcal{C} \) let \( \tilde{\eta}_{V,W} \) be the endomorphism of \( V \otimes W \in \mathcal{C} \) from Lemma 5.2(2). Then \( \mathcal{C} \) is a friendly preadditive symmetric monoidal category with handshake \( \tilde{\eta} \).

**Proof.** As noted in Lemma 5.2, \( \mathcal{C} \) is a preadditive symmetric monoidal category. The naturality of \( \tilde{\eta} \) follows directly from Lemma 5.1(1) and the naturality of \( \tau \) and \( \eta \). Thus it remains to verify Equations (2.1) and (2.2) for \( \tilde{\eta} \).

Equation \( \tau \tilde{\eta} = \tilde{\eta} \tau \) follows directly from the definition of \( \tilde{\eta} \), since \( \tau^{2} = id \) and \( \tau \eta = \eta \tau \).

We now prove Equation (2.1) for \( \tilde{\eta} \). Let \( X, Y, Z \in \mathcal{C} \). Then

\[
\tilde{\eta}_{X,Y \otimes Z} = \zeta_{Y \otimes Z,Y \otimes Z} \tau_{X,Y \otimes Z} + \tau_{Y \otimes Z,Y \otimes Z} \zeta_{X,Y \otimes Z} + \eta_{X,Y \otimes Z}.
\]
Then Lemma 5.4(2), the braiding axioms for $\tau$, and Equation (2.1) for $\eta$ imply that

$$\hat{\eta}_{X,Y} = ((\zeta \otimes \text{id})(\text{id} \otimes \tau) + (\tau \otimes \text{id})(\text{id} \otimes \zeta))((\tau \otimes \text{id})(\text{id} \otimes \tau) + (\text{id} \otimes \tau)(\text{id} \otimes \zeta))
+ \eta_{X,Y} \otimes \text{id}_Z + (\tau_{Y,X} \otimes \text{id}_Z)(\text{id}_Y \otimes \eta_{X,Z})(\tau_{X,Y} \otimes \text{id}_Z)
+ \zeta \tau \otimes \text{id} + (\tau \otimes \text{id})(\text{id} \otimes \zeta)(\tau \otimes \text{id})
+ \tau \zeta \otimes \text{id} + (\text{id} \otimes \tau)(\text{id} \otimes \tau)(\tau \otimes \text{id})
+ \eta_{X,Y} \otimes \text{id}_Z + (\tau_{Y,X} \otimes \text{id}_Z)(\text{id}_Y \otimes \eta_{X,Z})(\tau_{X,Y} \otimes \text{id}_Z)$$

and hence Equation (2.1) is fulfilled for $\hat{\eta}$. Thus the proof of the proposition is completed. \hfill \Box

**Remark 5.7.** Note that $\hat{\eta}$ in Proposition 5.6 is in general non-zero, even if $\eta = 0$.

6. **Bosonization of Lie bialgebras in friendly symmetric categories**

In this section let $\mathcal{C} = (\mathcal{C}, \eta)$ be a friendly preadditive symmetric monoidal category. We are going to use semidirect sum Lie algebras and semidirect sum Lie algebra modules from Proposition 5.6. Recall from Proposition 5.6, that for each Lie bialgebra $f$ in $\mathcal{C}$, $(f, \hat{\eta})$ is a friendly preadditive symmetric monoidal category.

**Theorem 6.1.** Let $\pi : g \to f$, $\gamma : f \to g$ be Lie bialgebra morphisms between two Lie bialgebras $(f, \beta_f, \delta_f)$ and $(g, \beta_g, \delta_g)$ in $\mathcal{C}$ such that $\pi \gamma = \text{id}_f$. Assume that $\pi$ has a kernel $\kappa : \mathfrak{k} \to g$ in $\mathcal{C}$. Let $\vartheta : g \to \mathfrak{k}$ be the morphism with $\kappa \vartheta = \text{id}_g - \gamma \pi$.

(a) The object $g$ in $\mathcal{C}$ together with the morphisms $\vartheta : g \to \mathfrak{k}$, $\pi : g \to f$, and $\kappa : \mathfrak{k} \to g$, $\gamma : f \to g$ is a biproduct of $\mathfrak{k}$ and $f$ in $\mathcal{C}$.  

(b) The object $\mathfrak{k} \in \mathcal{C}$ is a left crossed module over $f$ in $\mathcal{C}$, $\mathfrak{k} \in \mathfrak{f} \mathcal{C}$, via $f$-action $\alpha_\mathfrak{f}$ and $f$-coaction $\lambda_\mathfrak{f}$, where

$$\alpha_\mathfrak{f} = \vartheta \beta_g(\gamma \otimes \kappa), \quad \lambda_\mathfrak{f} = (\pi \otimes \vartheta) \delta_g \kappa.$$

Moreover $\mathfrak{k}$ is a Lie bialgebra in $(\mathfrak{f} \mathcal{C}, \hat{\eta})$ with bracket $\beta_\mathfrak{f}$ and cobracket $\delta_\mathfrak{f}$, where

$$\beta_\mathfrak{f} = \vartheta \beta_g(\kappa \otimes \kappa), \quad \delta_\mathfrak{f} = (\vartheta \otimes \vartheta) \delta_g \kappa.$$

**Proof.** (a) Existence and uniqueness of $\vartheta$ follow from the equation

$$\pi(\text{id}_g - \gamma \pi) = 0$$

and the choice of $\kappa$. Moreover, $\kappa \vartheta \kappa = \kappa$ since $\pi \gamma = 0$. Hence $\vartheta \kappa = \text{id}_\mathfrak{k}$ since $\kappa$ is a monomorphism. Similarly, $\vartheta \gamma = 0$ since $\kappa \vartheta \gamma = (\text{id}_g - \gamma \pi) \gamma = 0$. The equations $\pi \gamma = \text{id}_g$ and $\pi \kappa = 0$ are clear.
Similarly, \((\mathfrak{k}, \lambda_\tau)\) is a left Lie comodule over \(\mathfrak{k}\) in \(\mathcal{C}\). The compatibility condition \((5.3)\) between \(\alpha_k\) and \(\lambda_\tau\) follows from the Lie bialgebra axiom \((5.2)\) for \(g\) composed with \(\pi \otimes \vartheta\) from the left and with \(\gamma \otimes \kappa\) from the right. In this calculation the naturality of \(\eta\) has to be used. We conclude that \((\mathfrak{k}, \alpha_t, \lambda_\tau) \in \mathcal{C}\).

The Jacobi identity for \(\beta_g\) in the form
\[
\beta_g(id_g \otimes \beta_g) = \beta_g(\beta_g \otimes id_g)(id - \tau_{23})
\]
multiplied from the left with \(\vartheta\) and from the right with \(\gamma \otimes \kappa \otimes \kappa\), using the antisymmetry of \(\beta_g\) and equations between \(\gamma, \pi, \kappa,\) and \(\vartheta\) implies that \(\beta_g\) is a morphism in \(\mathcal{C}\). By multiplying the Lie bialgebra axiom \((5.2)\) for \(g\) from the left with \(\pi \otimes \vartheta\) and from the right with \(\kappa \otimes \kappa\) we conclude that \(\beta_t\) is a morphism in \(\mathcal{C}\). Similarly, \(\delta_t\) is a morphism in \(\mathcal{C}\).

The antisymmetry of \(\beta_t\) follows from the antisymmetry of \(\beta_g\) and the naturality of the symmetry \(\tau\):
\[
\beta_t \tau = \vartheta \beta_g(\gamma \otimes \kappa) = \vartheta \beta_g \tau(\kappa \otimes \kappa) = -\vartheta \beta_g(\kappa \otimes \kappa) = -\beta_t.
\]
The Jacobi identity for \(\beta_t\) can be concluded from the Jacobi identity of \(\beta_g\), the defining equation of \(\vartheta\), and from the equation \(\pi \beta_g(\kappa \otimes \kappa) = 0\). Similarly, \(\delta_t\) is co-antisymmetric and satisfies the co-Jacobi identity. Finally, the defining equation of \(\vartheta\) and the Lie bialgebra axiom \((5.1)\) for \(g\) imply that
\[
\delta_t \beta_t = (\vartheta \otimes \vartheta)(\delta_g \kappa \vartheta \beta_g)(\kappa \otimes \kappa)
\]
\[
= (\vartheta \otimes \vartheta)(\delta_g \beta_g)(\kappa \otimes \kappa)
\]
\[
= (id - \tau)(\vartheta \otimes \vartheta)(\beta_g \otimes id)(id \otimes \tau)(\delta_g \otimes id)(\kappa \otimes \kappa)(id - \tau)
\]
\[
+ (\tau - id)(\vartheta \otimes \vartheta)(\delta_g \otimes id)(\kappa \otimes \kappa).
\]
Now the defining equation of \(\vartheta\) is plugged in and the naturality of \(\eta\) is used to conclude that
\[
\delta_t \beta_t = (id - \tau)(\vartheta \otimes \vartheta)(\beta_g \otimes id)((k \vartheta + \gamma \pi) \otimes id_g \otimes \tau)(\delta_g \otimes id)(\kappa \otimes \kappa)(id - \tau)
\]
\[
+ (\tau - id)(\delta_g \otimes id)(\kappa \otimes \kappa)(id - \tau)
\]
\[
= (id - \tau)(\beta_t \otimes id)(id \otimes \tau)(\delta_t \otimes id) + (\alpha_k \otimes id)(id \otimes \tau)(\lambda_k \otimes id)(id - \tau)
\]
\[
+ (\tau - id)(\delta_g \otimes id)(\kappa \otimes \kappa).
\]
Now the definition of \(\delta_t\) implies that
\[
\delta_t \beta_t = (id - \tau)(\beta_t \otimes id)(id \otimes \tau)(\delta_t \otimes id)(id - \tau) + (\tau - id)(\delta_t \otimes id).
\]
Thus \(\mathfrak{t}\) satisfies Equation \((5.1)\), and the proof of the theorem is completed. \(\square\)

Now we prove a converse of Theorem \((6.1)\).
Theorem 6.2. Let \( \mathfrak{f} \) be a Lie bialgebra in \( \mathcal{C} \) and \( \mathfrak{k} \) be a Lie bialgebra in \( (\mathcal{C}, \tilde{\eta}) \). Assume that the biproduct \( \mathfrak{k} \oplus \mathfrak{f} \) exists in \( \mathcal{C} \). Then \( (\mathfrak{k} \oplus \mathfrak{f}, \beta, \delta) \) is a Lie bialgebra in \( \mathcal{C} \), where \( (\mathfrak{k} \oplus \mathfrak{f}, \beta) \) is the semidirect sum Lie algebra in \( \mathcal{C} \) and \( (\mathfrak{k} \oplus \mathfrak{f}, \delta) \) is the semidirect sum Lie coalgebra in \( \mathcal{C} \). This Lie bialgebra is called the bisum Lie bialgebra of \( \mathfrak{k} \) and \( \mathfrak{f} \) in \( \mathcal{C} \).

Note that the bisum Lie bialgebra in Theorem 6.2 satisfies the assumptions in Theorem 6.1 with \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{f} \) and with \( \pi : \mathfrak{g} \to \mathfrak{f} \), \( \gamma : \mathfrak{f} \to \mathfrak{g} \) the corresponding biproduct morphisms.

Proof. It only remains to prove that \( \beta \) and \( \delta \) satisfy the Lie bialgebra axiom (5.1). Since \( \beta \tau = -\beta \) and \( (\text{id} - \tau) \tau = -(\text{id} - \tau) \), it suffices to prove that (5.1) is satisfied on \( (\mathfrak{k} \otimes \mathfrak{f}) \oplus (\mathfrak{f} \otimes \mathfrak{f}) \).

Since \( \delta \iota_f = (\iota_f \otimes \iota_f)\delta_i \) by (4.3) and \( \beta(\iota_f \otimes \iota_f) = \iota_f \beta_f \) by (4.2), and since \( (\mathfrak{i}, \beta_i, \delta_i) \) is a Lie bialgebra in \( \mathcal{C} \), (5.1) is clearly satisfied on \( \mathfrak{i} \oplus \mathfrak{i} \). Moreover, on both sides of (5.1) the expression is invariant under composition with \( -\tau \) from the left or from the right. Therefore it remains to prove that (5.1) or (5.2) is satisfied after composing from the left with \( \pi \mathfrak{k} \otimes \pi \mathfrak{f} \) or with \( \pi \mathfrak{f} \otimes \pi \mathfrak{k} \), and from the right with \( \iota_f \otimes \iota_f \) or with \( \iota_f \otimes \iota_f \).

Let \( \alpha_f \) and \( \lambda_f \) denote the left Lie action on \( \mathfrak{k} \) by \( \mathfrak{f} \) and the left Lie action on \( \mathfrak{k} \) by \( \mathfrak{f} \), respectively. Then we obtain from (3.2) and (4.2) that

\[
(\pi_f \otimes \pi_f)(\delta \beta(\iota_f \otimes \iota_f)) = \delta \pi_f \iota_f \beta_f = \delta \beta_f.
\]

On the other hand, using further (3.2) and (4.2) and the naturality of \( \eta \), we obtain that

\[
(\pi_f \otimes \pi_f)(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\iota_f \otimes \iota_f) + (\pi_f \otimes \pi_f)(\tau - \text{id})\eta_{\mathfrak{k}}(\iota_f \otimes \iota_f)
\]

\[
= (\text{id} - \tau)(\pi_f \beta \otimes \pi_f)(\iota_f \pi_f \iota_f + \iota_f \pi_f \iota_f \otimes \tau)(\delta \iota_f \otimes \iota_f)(\iota_f - \tau) + (\tau - \text{id})\eta_{\mathfrak{k}}(\iota_f \otimes \iota_f)
\]

\[
= (\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\iota_f \otimes \iota_f)(\text{id} - \tau) + (\tau - \text{id})\eta_{\mathfrak{k}}(\iota_f \otimes \iota_f).
\]

Since (\alpha_f \otimes \text{id})(\text{id} \otimes \tau_{\mathfrak{k}, \mathfrak{f}})(\lambda_f \otimes \text{id})(\text{id} - \tau) = \zeta_{\mathfrak{k}, \mathfrak{f}} \eta_{\mathfrak{k}, \mathfrak{f}}), it follows from Lemma 5.5(2) that the last expression is equal to

\[
(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau) + (\tau \otimes \text{id} - \text{id} \otimes \text{id})\eta_{\mathfrak{k}, \mathfrak{f}}.
\]

Since \( \mathfrak{k} \) is a Lie bialgebra in \( (\mathcal{C}, \tilde{\eta}) \), (5.1) holds when composing from the left with \( \pi \mathfrak{i} \otimes \pi \mathfrak{k} \) and from the right with \( \iota_f \otimes \iota_f \).

Next we obtain from (3.2) and (4.2) that

\[
(\pi_f \otimes \pi_f)(\delta \beta(\iota_f \otimes \iota_f)) = \lambda_f \beta \iota_f \beta_f = \lambda_f \beta_f = (\text{id} \otimes \text{id})\eta_{\mathfrak{k}}(\iota_f \otimes \iota_f),
\]

since \( \beta_f : \mathfrak{k} \otimes \mathfrak{f} \to \mathfrak{k} \) is a Lie comodule morphism by assumption.

On the other hand,

\[
(\pi_f \otimes \pi_f)(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\iota_f \otimes \iota_f) + (\pi_f \otimes \pi_f)(\tau \otimes \text{id})\eta_{\mathfrak{k}}(\iota_f \otimes \iota_f)
\]

\[
= (\pi_f \otimes \pi_f)(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\iota_f \otimes \iota_f)(\text{id} - \tau)
\]
since $\eta$ is a natural transformation and $\pi t \xi = 0$. Since $\pi t \beta = \beta t (\pi f \otimes \pi f)$ and since $\tau(\beta \otimes \text{id})(\text{id} \otimes \tau) = (\text{id} \otimes \beta)(\tau \otimes \text{id})$, the latter expression can be rewritten as

$$
= (\beta t (\pi f \otimes \pi f) \otimes \pi f)(\text{id} \otimes \tau)(\delta t \otimes \xi t)(\text{id} - \tau)
- (\pi f \otimes \pi f)(\text{id} \otimes \beta)(\tau \delta t \otimes \xi t)(\text{id} - \tau)
= (\pi f \otimes \pi f)(\text{id} \otimes \beta)(\delta t \otimes \xi t)(\text{id} - \tau),
$$

where the last equation follows from $\pi f \xi t = 0$ and from $\tau \delta = -\delta$. For the next reformulation we conclude first from (4.2) that $(\pi f \otimes \pi f)\delta t = (\pi f \otimes \xi t \pi f)\delta t$. Hence the expression in (6.1) is equal to

$$(\text{id} \otimes \pi f \beta)(\text{id} \otimes \xi t \otimes \xi t)((\pi f \otimes \pi f)\delta t \otimes \text{id})(\text{id} - \tau) = (\text{id} \otimes \beta t)(\lambda t \otimes \text{id} \otimes \text{id})(\text{id} - \tau).$$

Since $\beta t \tau = -\beta t$, the latter expression is equal to $(\text{id} \otimes \beta t)(\lambda t \otimes \text{id} \otimes \text{id})$. Therefore (6.1) holds when composing from the left with $\pi f \otimes \pi f$ and from the right with $\xi t \otimes \xi t$.

Very similarly we conclude that (5.1) holds when composing from the left with $\pi t \otimes \pi t$ and from the right with $\xi t \otimes \xi t$.

Finally, Equations (6.2) and (4.2) imply that

$$(\pi f \otimes \pi f)\delta t (\xi f \otimes \xi t) = (\pi f \otimes \pi f)\delta t t \pi f \beta (\xi f \otimes \xi t) = \lambda t \alpha t.$$ 

On the other hand, the expression

$$(\pi f \otimes \pi f)(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(\xi f \otimes \xi t)$$

$$+ (\pi f \otimes \pi f)(\tau - \text{id}) \eta f f f t (\xi f \otimes \xi t)$$

is the sum of the following five terms:

$$(\pi f \otimes \pi f)(\beta \otimes \text{id})(\delta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)$$

$$= (\pi f \beta \otimes \pi f)(\xi f \otimes \xi t)(\text{id} \otimes \tau)(\delta \otimes \text{id})(\xi f \otimes \xi t)$$

$$= (\pi f \otimes \pi f)(\beta \otimes \text{id})(\delta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)$$

$$= (\text{id} \otimes \alpha f)(\tau f f \otimes \text{id} \otimes \text{id})(\text{id} \otimes \lambda t),$$

$$= (\pi f \otimes \pi f)(\beta \otimes \text{id})(\delta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)$$

$$= (\pi f \otimes \pi f)(\text{id} \otimes \beta)(\delta \otimes \text{id})(\xi f \otimes \xi t)$$

$$= (\text{id} \otimes \alpha f)(\delta f \otimes \text{id} \otimes \text{id}),$$

$$= (\pi f \otimes \pi f)(\beta \otimes \text{id})(\delta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)$$

$$= (\beta f \otimes \text{id} \otimes \text{id})(\text{id} \otimes \lambda t),$$

$$= (\pi f \otimes \pi f)(\tau - \text{id}) \eta f f f t (\xi f \otimes \xi t) = -\eta f f t.$$ 

Thus, by the crossed module axiom for $f$, (5.1) holds when composing from the left with $\pi f \otimes \pi f$ and from the right with $\xi f \otimes \xi t$. This finishes the proof of the theorem. \hfill $\square$

7. Examples

In this section we present some non-trivial friendly Lie bialgebras. Each of these examples is related to certain color vector spaces.

Let $G$ be an abelian group. We omit the symbol for the group operation and write $gh$ for the product of two elements $g, h \in G$, $1$ for the neutral element of $G$, and $g^{-1}$ for the inverse of $g \in G$. Let $k$ be a field, $k^\times$ its subgroup of units, and let
\( \chi : G \times G \to k^\times \) be an antisymmetric bicharacter, also called composition factor. The latter means that

\[
\chi(g_1, g_2) = \chi(g_2, g_1) = 1, \quad \chi(g_1 g_2, h) = \chi(g_1, h) \chi(g_2, h).
\]

These equations imply in particular that \( \chi(g, g) = 1 \) for all \( g \in G \). Moreover, \( \chi \) defines a group homomorphism \( G \to k^\times \) via \( g \mapsto \chi(g, g) \).

A \( G \)-graded vector space is a vector space \( V \) with a direct sum decomposition \( V = \bigoplus_{g \in G} V_g \). A morphism between \( G \)-graded vector spaces \( V, W \) is a linear map \( f : V \to W \) with \( f(V_g) \subset W_g \) for all \( g \in G \). The category of \( G \)-graded vector spaces over \( k \) together with an antisymmetric bicharacter \( \chi : G \times G \to k^\times \) is commonly known as the category of \((G, \chi)\)-color vector spaces. It is a monoidal category, where

\[
(V \otimes W)_g = \bigoplus_{h \in G} V_h \otimes W_{h^{-1}g}
\]

for all \( G \)-graded vector spaces \( V, W \). Moreover, the presence of the antisymmetric bicharacter allows to attach a non-trivial symmetry to the category:

\[
\tau_{V,W} : V \otimes W \to W \otimes V, \quad \tau_{V,W}(v \otimes w) = \chi(g, h) w \otimes v
\]

for all \( v \in V_g, w \in W_h \). We view the category of \((G, \chi)\)-color vector spaces as a friendly preadditive symmetric monoidal category with symmetry \( \tau \) and with zero handshake, and abbreviate it by \( \mathcal{C} \).

Each object \( \mathfrak{f} \in \mathcal{C} \) is a Lie algebra with zero bracket, called abelian Lie algebra. Similarly, each object \( \mathfrak{f} \in \mathcal{C} \) is a Lie coalgebra with zero cobracket. We then say that \( \mathfrak{f} \) is coabelian. Our examples in this section will be friendly bialgebras in the category of crossed modules (see Proposition 5.6) over an abelian coabelian Lie bialgebra. For the discussion of the Laistrygonian examples the following lemma will be useful.

**Lemma 7.1.** Let \( \mathfrak{f} \) be a trivially \( G \)-graded coabelian Lie coalgebra in \( \mathcal{C} \). Let \( V \in \mathcal{C} \) and let \( (x_i)_{i \in I} \) be a vector space basis of \( V \) consisting of homogeneous elements with respect to the \( G \)-grading. Then for each family \( (f_i)_{i \in I} \) of elements in \( \mathfrak{f} \), the map

\[
\lambda : V \to \mathfrak{f} \otimes V, \quad \lambda(x_i) = f_i \otimes x_i
\]

defines a left Lie comodule structure over \( \mathfrak{f} \) on \( V \).

**Proof.** The assumptions imply that \( \lambda \) is a morphism in \( \mathcal{C} \), \( \delta \otimes \text{id} : \mathfrak{f} \otimes V \to \mathfrak{f} \otimes \mathfrak{f} \otimes V \) is zero, and

\[
((\text{id} - \tau) \otimes \text{id})(\text{id} \otimes \lambda) \lambda = 0.
\]

This implies the claim. \( \square \)

The Nichols algebras in [2] indicate the existence of several finite-dimensional friendly Lie bialgebras in the category of crossed modules over an abelian coabelian Lie bialgebra. We discuss here only two single examples (the Jordan and the super Jordan plane) and a series (the Laistrygonian examples) in detail. Other candidates are the super Laistrygonians as well as the (super) Endymion and Poseidon examples. Our presentation here is aimed to be explicit rather than being efficient. Nonetheless, we strongly believe that by adding more theory an appealing efficient presentation of large classes of examples is possible.
7.1. The Jordan plane. We start with a relatively simple example, where the benefit of the combination of so much fairly trivial structure may not be fully convincing at the first moment. Nevertheless, this example will appear later as a subobject in more complicated examples. Therefore it is worthwhile to study it separately.

Assume that \( G = \langle g \rangle \cong (\mathbb{Z},+) \) and \( \chi(g,g) = 1 \). Let \( \mathcal{C} \) be the category of \((G,\chi)\)-color vector spaces (with zero handshake). Let \( \mathfrak{k} = \mathbb{ks} \) be a one-dimensional \((G,\chi)\)-color vector space of \( G \)-degree 1, with the abelian and coabelian Lie bialgebra structure. Let \( J \in \mathcal{C} \) be a two-dimensional homogeneous \( G \)-graded vector space of \( G \)-degree \( g \) and let \( x_1, x_2 \) be a basis of \( J \).

Proposition 7.2. Let \( J \in \mathcal{C} \) with the following action and coaction of \( \mathfrak{k} \):

\[
\lambda(x) = s \otimes x \quad \text{for all } x \in J,
\]

and

\[
s \cdot x_1 = 0, \quad s \cdot x_2 = x_1,
\]

where \( s \cdot x = \alpha(s \otimes x) \) for all \( x \in J \). Then \( J \) is a friendly Lie bialgebra in \( \mathcal{C} \) with zero bracket, zero cobracket, and the natural handshake \( \hat{\eta} \) of \( \mathcal{C} \).

It is illuminating to put the definition of the friendly Lie bialgebra \( J \) next to the description of the Nichols algebra of the Jordan plane in \cite{Bourgain1998} Prop. 3.4. We call \( J \) the Lie bialgebra of the Jordan plane.

Proof. Since \( \tau(s \otimes s) = s \otimes s \), it is easy to see that \( (J,\alpha) \in \mathcal{C} \) and \( (J,\lambda) \in \mathcal{C} \). Since \( \mathfrak{k} \) is abelian and coabelian, the crossed module axiom \cite{Brown2006} is equivalent to

\[
\lambda(s \cdot x) = (\text{id} \otimes \alpha)(\tau_{\mathfrak{k},\mathfrak{k}} \otimes \text{id})(\text{id} \otimes \lambda)(s \otimes x) \quad \text{for all } x \in J,
\]

which is clearly satisfied. Thus \( J \in \mathcal{C} \).

The crossed module \( J \in \mathcal{C} \) is a Lie algebra with zero bracket and a Lie coalgebra with zero cobracket. The handshake of \( J \) in \( \mathcal{C} \) in the notation of Lemma \cite{Brown2006} is

\[
\hat{\eta}_{J,J}(x \otimes y) = (\zeta \tau + \tau \zeta)(x \otimes y) = s \cdot x \otimes y + x \otimes s \cdot y
\]

for all \( x, y \in J \). Since \( x_1 \otimes x_1, x_1 \otimes x_2 + x_2 \otimes x_1, x_2 \otimes x_2 \in \ker(\tau - \text{id}) \), it follows that

\[
(\tau - \text{id})(J \otimes J) = \mathbb{k}(x_1 \otimes x_2 - x_2 \otimes x_1), \quad (\tau - \text{id})\hat{\eta}_{J,J} = 0.
\]

Therefore the zero bracket and zero cobracket satisfy the Lie bialgebra axiom \cite{Brown2006}, and hence \( J \) is a friendly Lie bialgebra in \( \mathcal{C} \).

7.2. The super Jordan plane. This is our first example of a nonabelian non-
coabelian friendly Lie bialgebra structure. It is illuminating to put it next to the description of the Nichols algebra of the super Jordan plane in \cite{Bourgain1998} Prop. 3.5. We do not put any assumptions on the field \( \mathbb{k} \), not even on its characteristic, but make a comment on this after explaining the example in Proposition 7.3 below.

Assume that \( G = \langle g \rangle \cong (\mathbb{Z},+) \) and \( \chi(g,g) = -1 \). Let \( \mathfrak{k} = \mathbb{ks} \) be a one-dimensional \((G,\chi)\)-color vector space of \( G \)-degree 1, with the abelian and coabelian Lie bialgebra structure. Let \( J^- \) be a four-dimensional homogeneous \( G \)-graded vector space with basis

\[
x_{11}, x_{12}, x_{21}, x_{22},
\]
where $x_{ij} \in J_{g_i}^-$ for all $i, j \in \{1, 2\}$. For convenience, we write $x_{i0} = 0$ for all $i \in \{1, 2\}$. When comparing $J^-$ with the super Jordan plane in \cite[Prop. 3.5]{[2]}, the generators $x_{i1}$, with $i \in \{1, 2\}$ here should be identified with the generators $x_i$ there.

The definition of $J^-$ implies that

$$\tau(x_{ij} \otimes x_{kl}) = \chi(g^i_*, g^k_*) x_{kl} \otimes x_{ij} = (-1)^{ik} x_{kl} \otimes x_{ij}$$

for all $i, j, k, l \in \{1, 2\}$.

**Proposition 7.3.** Let $J^- \in \hat{C}$ with the following action and coaction of $\hat{f}$:

\begin{align*}
(7.2) & \quad \lambda(x_{ij}) = is \otimes x_{ij} \text{ for all } i, j, \\
(7.3) & \quad s \cdot x_{ij} = x_{i, j-1} \text{ for all } i, j,
\end{align*}

where $s \cdot x = \alpha(s \otimes x)$ for all $x \in J^-$. Then $J^-$ is a friendly Lie bialgebra in $(\hat{C}, \tilde{\eta})$ with the natural (non-zero) handshake $\tilde{\eta}$ of $\hat{C}$ and the following bracket $\beta$ and cobracket $\delta$:

\begin{align*}
(7.4) & \quad \beta(x_{1i} \otimes x_{1j}) = (i + j - 2)x_{2, i+j-2}, \\
(7.5) & \quad \beta(x_{2i} \otimes x) = \beta(x \otimes x_{2i}) = 0, \\
(7.6) & \quad \delta(x_{1i}) = 0, \\
(7.7) & \quad \delta(x_{2i}) = (\tau - \text{id})(x_{11} \otimes x_{1i}) = -x_{11} \otimes x_{1i} - x_{11} \otimes x_{11}
\end{align*}

for all $i, j \in \{1, 2\}$, $x \in J^-_{g_i^*}$.

Using Kronecker’s delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

the definition of the bracket and cobracket can also be written as

\begin{align*}
(7.8) & \quad \beta(x_{ij} \otimes x_{kl}) = \delta_{ij} \delta_{kl} (j + l - 2)x_{2, j+l-2}, \\
(7.9) & \quad \delta(x_{ij}) = \delta_{ij} (\tau - \text{id})(x_{11} \otimes x_{1j}) = -\delta_{ij} x_{11} \otimes x_{1j} - \delta_{ij} x_{1j} \otimes x_{11}
\end{align*}

for all $i, j, k, l \in \{1, 2\}$.

We call $J^-$ the Lie bialgebra of the super Jordan plane.

Recall that we did not put any assumption on the field $k$. Assume now that the characteristic of $k$ is 2. Then $kx_{21}$ is a one-dimensional subobject of $J^-$ in $\hat{C}$ and a Lie ideal of $J^-$. Moreover, $\delta(x_{21}) = 0$, which allows to take the quotient $J^-/kx_{21}$. This does not work for other fields.

**Proof.** It is fairly clear that $(J^-, \alpha) \in \hat{g}C$ and $(J^-, \lambda) \in \hat{f}C$. Since $\hat{f}$ is abelian and coabelian, the crossed module axiom \cite[3]{[5]} is equivalent to

$$\lambda(s \cdot x) = (\text{id} \otimes \alpha)(\tau \otimes \text{id})(\text{id} \otimes \lambda)(s \otimes x) \quad \text{for all } x \in J^-,$$

which follows directly from $\alpha(s \otimes J_{g_i}^-) \subseteq J_{g_i^-}^-$ for all $i \in \{1, 2\}$. Thus $J^- \in \hat{f}C$. 


It is clear that $\beta$ and $\delta$ are morphisms in $\mathcal{C}$. Moreover, for all $i, j, k, l \in \{1, 2\}$ we obtain that

$$s \cdot \beta(x_{ij} \otimes x_{kl}) = s \cdot \delta^K_{i1} \delta^K_{k1} (j + l - 2)x_{j, l+1, t-2} = \delta^K_{i1} \delta^K_{j2} \delta^K_{k2} x_{21},$$

$$\beta(s \cdot (x_{ij} \otimes x_{kl})) = \beta(\delta^K_{j2} x_{ii} \otimes x_{kl} + \delta^K_{l2} x_{ij} \otimes x_{k1}) = \delta^K_{i1} \delta^K_{j1} \delta^K_{k2} x_{21},$$

$$s \cdot \delta(x_{ij}) = s \cdot (-\delta^K_{i1} x_{ii} \otimes x_{ij} - \delta^K_{l2} x_{ij} \otimes x_{i1}) = -2\delta^K_{j2} \delta^K_{l2} x_{11} \otimes x_{11},$$

$$\delta(s \cdot x_{ij}) = \delta(\delta^K_{j2} x_{ii}) = \delta^K_{j2} \delta^K_{l2} (-2)x_{11} \otimes x_{11}.$$ 

Hence $\beta$ and $\delta$ are morphisms in $\mathcal{C}$.

The antisymmetry of $\beta$ and the co-antisymmetry of $\delta$ are fairly obvious. Since the image of $\beta \otimes \text{id} - (\text{id} \otimes \beta)(\text{id} - \tau_{12})$ is contained in $J_g \otimes J' + J' \otimes J_g \subseteq \ker \beta$, $\beta$ satisfies the Jacobi identity. Similarly,

$$\delta(J^-) \subseteq J_g^- \otimes J_g^- \subseteq \ker (\delta \otimes \text{id} - (\text{id} - \tau_{12})(\text{id} \otimes \delta)).$$

Thus $(J^-, \beta)$ is a Lie algebra in $\mathcal{C}$ and $(J^-, \delta)$ is a Lie coalgebra in $\mathcal{C}$.

The handshake of $J^-$ in $\mathcal{C}$ in the notation of Lemma 5.5 is

$$\tilde{\eta}_{J^-,J^-}(x_{ij} \otimes x_{kl}) = (\eta + \tau \zeta)(x_{ij} \otimes x_{kl}) = k\delta \cdot x_{ij} \otimes x_{kl} + x_{ij} \otimes x_{kl}$$

$$= k\delta^K x_{i1} \otimes x_{kl} + 2\delta^K x_{ij} \otimes x_{kl}$$

for all $i, j, k, l \in \{1, 2\}$. In particular, $(\tau - \text{id})\tilde{\eta}_{J^-,J^-} \neq 0$.

We check the Lie bialgebra axiom (5.1) for $\beta$ and $\delta$.

For all $i, j, k, l \in \{1, 2\}$ we obtain directly from the definitions that

$$\delta^2(x_{ij} \otimes x_{kl}) = \delta^K_{i1} \delta^K_{k1} (j + l - 2)\delta(x_{j, l+1, t-2})$$

$$= \delta^K_{i1} \delta^K_{i1} (j + l - 2)(\tau - \text{id})(x_{11} \otimes x_{1, j+l, t-2}).$$

Moreover,

$$(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(x_{ij} \otimes x_{kl})$$

$$= (\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(x_{ij} \otimes x_{kl} - (-1)^{ik} x_{kl} \otimes x_{ij})$$

$$= (\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(x_{ij} \otimes x_{kl} - \delta^K x_{11} \otimes x_{11} \otimes x_{kl} \otimes x_{ij})$$

$$+ \delta^K x_{11} \otimes x_{11} \otimes x_{ij} \otimes x_{kl}$$

$$= (\text{id} - \tau)(\delta^K x_{11} \otimes x_{11} \otimes x_{ij} \otimes x_{kl} + \delta^K x_{12} x_{21} \otimes x_{ij} + \delta^K x_{12} x_{21} \otimes x_{ij}$$

$$- \delta^K x_{k2} x_{21} \otimes x_{ij} - \delta^K x_{k2} x_{21} \otimes x_{ij}).$$

Therefore the Lie bialgebra axiom for $\beta$ and $\delta$ is equivalent to

$$(j + l - 2)\delta^K x_{11} \otimes x_{1, j+l, t-2} + \delta^K x_{12} x_{21} \otimes x_{ij}$$

$$+ \delta^K x_{k2} x_{21} \otimes x_{ij} - \delta^K x_{k2} x_{21} \otimes x_{ij}$$

$$- \delta^K x_{k2} x_{21} \otimes x_{ij} - \delta^K x_{k2} x_{21} \otimes x_{ij} \in \ker (\tau - \text{id}).$$

For $i = k = 2$, Equation (7.10) is equivalent to

$$-2\delta^K x_{21} \otimes x_{2l} - 2\delta^K x_{2j} x_{21} \in \ker (\tau - \text{id}),$$

which is clear whenever $j = 1$ or $l = 1$ or $j = l$, and hence in all cases.
For \( i = 2, k = 1 \), Equation (7.10) is equivalent to
\[
\delta^K x_{21} \otimes x_{1j} + (j + l - 2) x_{2,j+l-2} \otimes x_{11} - \delta^K x_{21} \otimes x_{11} - 2\delta^K x_{2j} \otimes x_{11} \\
\in \ker(\tau - \text{id}).
\]

One checks for all possible pairs \((j, l)\) that the expression on the left hand side of the relation is in fact already 0.

For \( i = 1, k = 2 \), Equation (7.10) is equivalent to
\[
- \delta^K x_{21} \otimes x_{1l} - (j + l - 2) x_{2,j+l-2} \otimes x_{11} - 2\delta^K x_{11} \otimes x_{2l} - \delta^K x_{1j} \otimes x_{21} \\
\in \ker(\tau - \text{id}).
\]

By going through all four possibilities for the pair \((j, l)\) one checks that the relation holds.

Finally, for \( i = 1, k = 1 \), Equation (7.10) is equivalent to
\[
(j + l - 2) x_{11} \otimes x_{1,j+l-2} - \delta^K x_{11} \otimes x_{1l} - \delta^K x_{1j} \otimes x_{11} \in \ker(\tau - \text{id}).
\]

Here, if \((j, l) \neq (2, 2)\), then the expression on the left hand side of the relation is already 0. If however \( j = l = 2 \), then the condition simplifies to
\[
x_{11} \otimes x_{12} - x_{12} \otimes x_{11} \in \ker(\tau - \text{id}),
\]
which is easily checked.

We conclude that \((J^-, \beta, \delta)\) satisfies the Lie bialgebra axiom and hence it is a friendly Lie bialgebra in \( \hat{(1, h)} \), and the proof is completed. \( \square \)

7.3. The Laistrygonian Lie bialgebras. In this section we use a larger abelian coabelian Lie bialgebra \( f \), but the general approach remains similar. We identify an infinite family of friendly Lie bialgebras, see Proposition 7.4 below, each of them containing the Jordan plane as a subobject.

Assume that \( G = \langle g, h \rangle \cong (\mathbb{Z}^2, +) \) and
\[
\chi(g, g) = \chi(h, h) = 1.
\]

Let \( C \) be the category of \((G, \chi)\)-color vector spaces (with zero handshake) over a field \( k \) of characteristic \( \neq 2 \). Let
\[
f = f_1 = ks + kt
\]
be a two-dimensional \((G, \chi)\)-color vector space of \( G \)-degree 1, with the abelian and coabelian Lie bialgebra structure:
\[
\beta_f = 0, \quad \delta_f = 0.
\]

Let \( G \in \mathbb{N}_0 \) and let
\[
L(1, G) = L(1, G)_g \oplus \bigoplus_{k=0}^G L(1, G)_{g^k} \in C,
\]
\[
L(1, G)_g = kx_1 + kx_2, \quad L(1, G)_{g^k} = kz_k \quad \text{for all } 0 \leq k \leq G,
\]
be a \( G + 3 \)-dimensional \( G \)-graded vector space. Let \( L_0(1, G) \) be the subspace of \( L(1, G) \) spanned by \( x_1 \) and \( z_k \), \( 0 \leq k \leq G \). Clearly, \( L_0(1, G) \in C \).

Proposition 7.4. Let \( G \in \mathbb{N}_0 \). Then \( L(1, G) \in \hat{1}C \) with the following action and coaction of \( f \):
\[
\lambda(x_1) = s \otimes x_1, \quad \lambda(x_2) = s \otimes x_2, \quad \lambda(z_k) = (ks + t) \otimes z_k
\]
for all $0 \leq k \leq \mathcal{G}$, and

$$s \cdot x_2 = x_1, \quad t \cdot x_2 = -\frac{G}{2} x_1, \quad s \cdot x = t \cdot x = 0$$

for all $x \in L_0(1, \mathcal{G})$, where $f \cdot x = \alpha(f \otimes x)$ for all $f \in \mathfrak{f}$ and $x \in L(1, \mathcal{G})$. Moreover, $L(1, \mathcal{G})$ is a friendly Lie bialgebra in the category $(\mathcal{C}, \mathfrak{n})$ with the natural (non-zero) handshake $\tilde{\eta}$ of $\mathcal{C}$, bracket $\beta$ and cobracket $\delta$, where

$$\beta(x \otimes y) = 0 \quad \text{for all } x, y \in L_0(1, \mathcal{G}),$$

$$\beta(x_2 \otimes x_1) = \beta(x_2 \otimes x_2) = \beta(x_1 \otimes x_2) = 0,$$

$$\beta(x_2 \otimes z_k) = -\beta(\tau(x_2 \otimes z_k)) = z_{k+1},$$

$$\delta(x_1) = \delta(x_2) = 0,$$

$$\delta(z_k) = \frac{k(k-1-G)}{2}(\tau - \text{id})(x_1 \otimes z_{k-1})$$

for all $0 \leq k \leq \mathcal{G}$ (with the convention $z_{\mathcal{G}+1} = 0$).

It is illuminating to put the Lie bialgebras $L(1, \mathcal{G})$ next to the description of the Laistrygonian Nichols algebras in [2, Prop. 4.17]. We call the friendly Lie bialgebras $L(1, \mathcal{G})$ Laistrygonian Lie bialgebras.

**Proof.** By Lemma 7.1, $\lambda$ defines a Lie coaction of $\mathfrak{f}$ on $L(1, \mathcal{G})$. It is easy to check that the action of $s$ and of $t$ on $L(1, \mathcal{G})$ commute, and hence $\alpha$ defines a Lie module structure of $\mathfrak{f}$ on $L(1, \mathcal{G})$. The crossed module axiom (5.3) applied to $f \otimes x$ with $f \in \mathfrak{f}$ and $x \in L_0(1, \mathcal{G})$ is satisfied trivially, since all terms are zero. The axiom for $f \otimes x_2$ with $f \in \{s, t\}$ is easily checked. Thus $L(1, \mathcal{G}) \in \mathcal{C}$ and $L_0(1, \mathcal{G}) \in \mathcal{C}$.

Clearly, $\beta$ is a morphism in $\mathcal{C}$ and $\beta \tau = -\beta$. One checks quickly that $\beta$ is a morphism in $\mathcal{C}$. A straightforward calculation shows that $\beta$ satisfies the Jacobi identity. For example, for all $x \in L_0(1, \mathcal{G})$ one obtains that

$$\beta((\text{id} \otimes \beta)(\text{id} + \tau)(x_2 \otimes x_2 \otimes x)) = \beta(x_2 \otimes \beta((\text{id} + \tau)(x_2 \otimes x))) + \beta((\text{id} \otimes \beta)(\tau)(x_2 \otimes x_2 \otimes x)) = 0$$

by the antisymmetry of $\beta$ and since $\beta(x_2 \otimes x_2) = 0$. Hence $(L(1, \mathcal{G}), \beta)$ is a Lie algebra in $\mathcal{C}$.

It is fairly obvious that $\delta$ is a morphism in $\mathcal{C}$. By definition, $\tau \delta = -\tau$. The co-Jacobi identity clearly holds when evaluated at $x_1$ or $x_2$. On $z_k$ with $0 \leq k \leq \mathcal{G}$ it follows by using the definition of $\delta$ and the equation $(\tau - \text{id})(x_1 \otimes x_1) = 0$. Hence $(L(1, \mathcal{G}), \delta)$ is a Lie coalgebra in $\mathcal{C}$.

For the handshake we obtain from Lemma 5.9 that

$$\tilde{\eta}(x \otimes y) = 0 \quad \text{for all } x, y \in L_0(1, \mathcal{G}),$$

$$\tilde{\eta}(x_2 \otimes x_2) = x_1 \otimes x_2 + x_2 \otimes x_1,$$

$$\tilde{\eta}(x_2 \otimes z_k) = (k - \frac{G}{2})x_1 \otimes z_k$$

for all $0 \leq k \leq \mathcal{G}$. In particular, $\tilde{\eta}$ is non-zero.

Finally, we verify the Lie bialgebra axiom (5.1) on the given basis vectors of $L(1, \mathcal{G})$. To do so, we introduce first the total order

$$x_1 < x_2 < z_0 < z_1 < \cdots < z_{\mathcal{G}}.$$
on our basis. Since both sides of (5.1) are invariant under multiplication with $-\tau$ from the right, it suffices to check the Lie bialgebra axiom for tensors $x \otimes y$ of basis vectors $x,y$ with $x \leq y$. Moreover, the Lie bialgebra axiom has been already verified on $kx_1 + kx_2$ when studying the Lie bialgebra of the Jordan plane. Now for all $0 \leq k \leq \mathcal{G} \cap \mathcal{L}$ we have that 

$$\delta \beta(x_1 \otimes z_k) = \delta(0) = 0, \quad (\tau - \text{id})\eta(x_1 \otimes z_k) = 0,$$

and

$$(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(x_1 \otimes z_k)$$

$$\in k(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(x_1 \otimes z_{k-1} \otimes x_1)$$

$$= k(\text{id} - \tau)(\beta \otimes \text{id})(x_1 \otimes x_1 \otimes z_{k-1}) = 0.$$ 

Similarly, for all $0 \leq k \leq l \leq \mathcal{G}$,

$$\delta \beta(z_k \otimes z_l) = \delta(0) = 0, \quad (\tau - \text{id})\eta(z_k \otimes z_l) = 0,$$

and

$$(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(z_k \otimes z_l)$$

$$\in (\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(x_1 \otimes L_0(1, \mathcal{G}) \otimes L_0(1, \mathcal{G})) = 0.$$ 

Lastly, for all $0 \leq k \leq \mathcal{G}$ (with $z_{-1} = z_{\mathcal{G}+1} = 0$) we obtain that 

$$\delta \beta(x_2 \otimes z_k) = \delta(z_{k+1}) = \frac{(k+1)(k - \mathcal{G})}{2}(\tau - \text{id})(x_1 \otimes z_k),$$

$$(\tau - \text{id})\eta(x_2 \otimes z_k) = (k - \frac{\mathcal{G}}{2})(\tau - \text{id})x_1 \otimes z_k,$$

and

$$(\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(\text{id} - \tau)(x_2 \otimes z_k)$$

$$= (\text{id} - \tau)(\beta \otimes \text{id})(\delta \otimes \text{id})(-x_2 \otimes z_k)$$

$$= \frac{k(k - 1 - \mathcal{G})}{2}(\text{id} - \tau)(\beta \otimes \text{id})(x_2 \otimes x_1 \otimes z_{k-1})$$

$$= \frac{k(k - 1 - \mathcal{G})}{2}(\tau - \text{id})(\tau \otimes \beta)(\tau \otimes \text{id})(x_2 \otimes x_1 \otimes z_{k-1})$$

$$= \frac{k(k - 1 - \mathcal{G})}{2}(\tau - \text{id})(x_1 \otimes z_k).$$

These equations imply directly that (5.1) is fulfilled, completing the proof of Proposition 7.4. \qed

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