Domination and Closure

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January 14, 2015

Abstract

An expansive, monotone operator is dominating; if it is also idempotent it is a closure operator. Although they have distinct properties, these two kinds of discrete operators are also intertwined. Every closure is dominating; every dominating operator embodies a closure. Both can be the basis of continuous set transformations. Dominating operators that exhibit categorical pull-back constitute a Galois connection and must be antimatroid closure operators. Applications involving social networks and learning spaces are suggested.

keywords: antimatroid, operator, pull-back, category, closure, domination.

1 Introduction

The concept of “domination” is an important one in graph theory, where a set of nodes “dominates” its neighbors. An extensive treatment can be found in [7, 8, 28], and an interesting historical application in [26]. Concepts of “closure” arise in many contexts, including topology, algebra, and its closely related concept of “convexity” [2].

This paper develops both concepts in terms of general, discrete set systems, where Δ and φ are dominating and closure operators respectively. In Section 3, we develop the connection between domination and closure that seems to be largely unexplored. Every domination operator gives rise to a closure operator, and every closure operator is dominating.

Section 4 explores the properties of domination and closure under functional transformation, especially continuous transformation, and briefly reviews the definition of closure in terms of Galois transformations, or connections.

It is natural to regard collections of functions, whether operators or transformations, as a category. In Section 5 we develop this theme and introduce the category Dom to denote all domination morphisms over discrete sets of S. We show that if a subcategory C ⊂ Dom exhibits the pull-back property, then it consists of only antimatroid closure operators.

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We believe this approach to the study of set systems, including directed and undirected networks, solely in terms of set-valued operators and the representation of set system dynamics by set-valued transformations from one discrete set system, $S$, to another, $S'$, may be original.

## 2 Set Systems

Let $S$ be any finite set. By a set system, $S$, we mean a collection of subsets of $2^S$, the power set of $S$, together with various operators, $\alpha, \beta, \ldots$ defined on this collection. Elements of the ground set, $S$, we denote by lower case letters $x, y, z$. In the general theory, the nature of these members is unimportant; although in some applications they can be significant.

The sets of $S$ are denoted by upper case letters, $X, Y, Z$. We assume that $S = \bigcup_{X \in S} X$. Sets can also be denoted by their constituent members, such as $\{x, y, z\}$, or more simply by $xyz$. One can regard $xyz$ to be the label of a set. Whenever we reference an element, such as $x \in Y$, it can be interpreted as $\{x\} \cap Y \neq \emptyset$. This is a paper about sets, their properties and their transformations. The cardinality of a set $Y$ is denoted $|Y|$.

### 2.1 Operators on $S$

An unary operator $\alpha$ on $S$ is a function defined on the sets of $S$, that is, for all $Y \in S$, $Y.\alpha \in S$. Operators are expressed in suffix notation because they are "set-valued"\footnote{Here we follow a convention that is more often used by algebraists \cite{33}. Set valued functions/transformations, $f$, are presented in suffix notation, e.g. $S' = S.f$; single valued functions, $f$, on set elements are denoted by prefix notation, e.g. $e' = f(e)$.} An operator $\alpha$ is said to be:

- **contractive**, if $Y.\alpha \subseteq Y$;
- **expansive**, if $Y \subseteq Y.\alpha$;
- **monotone**, if $X \subseteq Y$ implies $X.\alpha \subseteq Y.\alpha$;
- **idempotent**, if $Y.\alpha.\alpha = Y.\alpha$;
- **path independent** if $(X.\alpha \cup Y.\alpha).\alpha = (X \cup Y).\alpha$.

Contractive operators are often called choice operators \cite{14 27}. Path independent choice operators are important in economic theory \cite{12 16}. Operators, $\Delta$, that are expansive and monotone we call domination (or dominating) operators. They are central to this paper.

If $X.\alpha = Y.\alpha = Z$ then $X$ and $Y$ are said to be **generators** of $Z$ (with respect to $\alpha$). A set $Y$ is said to be a **minimal** generator of $Z$ if for all $X \subset Y$, $X.\alpha \neq Z$. The operator $\alpha$ is said to be **uniquely generated** if for all $Z$, $Y.\alpha = Z$ implies there exists a unique minimal generator $X \subseteq Y$, such that $X.\alpha = Z$. When $X.\alpha = Y.\alpha$, we say $X$ and $Y$ are $\alpha$-equivalent, denoted $X =_\alpha Y$. 

2.2 Extended Operators

An operator $\alpha$ is said to be extended if for all $Y \in S$, $Y.\alpha = \bigcup_{y \in Y} \{y\}.\alpha$. That is, $\alpha$ has been extended from its definition on singleton subsets. This is, perhaps, the most common way of defining set-valued operators. It is not difficult to show that:

**Proposition 2.1** If $\alpha$ is an extended operator, then for all $z \in Y.\alpha$, there exists $y \in Y$ such that $z \in \{y\}.\alpha$.

**Corollary 2.2** If $\alpha$ is an extended operator, then $\alpha$ is a monotone operator.

**Corollary 2.3** If $\alpha$ is an extended operator, then $\emptyset.\alpha = \emptyset$.

If an operator $\alpha$ is extended, then it can be visualized as a simple graph with $(x, y) \in E$ if and only if $y \in \{x\}.\alpha$. We call this a graphic representation.

2.3 Dominating Operators

Throughout this paper we concentrate on expansive, monotone operators which we called domination operators, $\Delta$, in Section 2.1. We call $Y.\Delta$ the region dominated by $Y$. If $S$ denotes the nodes of a network $\mathcal{N}$, one can define $X.\Delta = X \cup \{y | \exists x \in X, (x, y) \in E\}$. $X$ is said to “dominate” $X.\Delta$ and there is a large literature, called “domination theory”, devoted to the combinatorial properties of the minimal generators, $X$, when $X.\Delta = S$ [8]. Whence the term “domination” operator. However, domination operators need not be graphically representable. Consider the following $\Delta$ defined on $S = \{a, b, c, d\}$. For the singleton sets, let $\{a\}.\Delta = \{ac\}$, $\{b\}.\Delta = \{bc\}$, $\{c\}.\Delta = \{cd\}$, and $\{d\}.\Delta = \{d\}$. Except for $\{ab\}$, let $Y.\Delta = \bigcup_{y \in Y} \{y\}.\Delta$, but let $\{ab\}.\Delta = \{abcd\}$. This is not a simple extension of $\{a\}.\Delta$ and $\{b\}.\Delta$; yet $\Delta$ is well defined. It should not be surprising that a larger set might have a larger radius of domination.

If the expansive, monotone operator, $\Delta$, is also idempotent, it is called a closure operator, $\varphi$.

It is sometimes convenient to distinguish that part of a dominated region $Y.\Delta$ from its generator $Y$. We call $Y.\eta = Y.\Delta - Y$ the dominated neighborhood of $Y$. Observe that, as an operator, $\eta$ is not expansive and generally is not monotone.

3 Closure Operators

The concept of closure appears to be an important theme in many discrete systems [22, 24]. A closure system can be defined by simply enumerating a collection, $\mathcal{C}$, of sets which are said to be closed. The union of all the subsets of $S$ is assumed to be in $\mathcal{C}$. The only other
constraint is that if the sets $X$ and $Y$ are in $C$, i.e. are closed, then $X \cap Y \in C$, must be closed.

Given such a collection, $C$, of closed sets we can then define a closure operator, $\varphi$, on $S$ by letting $Y.\varphi$ denote the smallest set $C_i \in C$ such that $Y \subseteq C_i$. Since $C$ is closed under intersection, $\varphi$ is single valued and well defined. It is well known \cite{18,21} that this definition of closure is equivalent to the one given in section 2.1, that is “an operator $\varphi$ is a closure operator if and only if

- $Y \subseteq Y.\varphi$, expansive;
- $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$, monotone; and
- $Y.\varphi.\varphi = Y.\varphi$, idempotent”.

Consequently, every closure operator, $\varphi$, is a dominating operator because it’s monotone and expansive. A dominating operator, $\Delta$, is a closure operator, $\varphi$ only if it is idempotent.

For any monotone operator, $\alpha, \Delta$ or $\varphi$, we have

$$(X \cap Y).\alpha \subseteq X.\alpha \cap Y.\alpha,$$

$$(X.\alpha \cup Y.\alpha \subseteq (X \cup Y).\alpha.$$  

**Proposition 3.1** Let $\varphi$ be an idempotent dominating operator $\Delta$. If $y \in X.\varphi$ then $\{y\}.\eta \subseteq X.\Delta$.

**Proof:** Suppose $\exists X$ and $y, z$, with $y \in X.\varphi, z \in \{y\}.\eta$, but $z \notin X.\Delta$. Then $z \in X.\Delta.\Delta$ contradicting the idempotency of $\Delta$.

Proposition 3.1 effectively asserts that for a dominating operator, $\Delta$, to be a closure operator, $\varphi$ must be “transitively closed”. Still a third characterization of closure systems can be found in \cite{16};

**Proposition 3.2** An expansive operator $\varphi$ is a closure operator if and only if $\varphi$ is path independent.

Let $Y$ be closed, a closure operator, $\varphi$, is said to be:

- **matroid** if $x, z \notin Y$ then $z \in (Y \cup \{x\}).\varphi$ implies $x \in (Y \cup \{z\}).\varphi$;
- **antimatroid** if $x, z \notin Y$ then $z \in (Y \cup \{x\}).\varphi$ implies $x \notin (Y \cup \{z\}).\varphi$;
- **topological** if $\emptyset.\varphi = \emptyset$ and $(X \cup Y).\varphi = X.\varphi \cup Y.\varphi$.

The first two expressions on the right are also known as the “exchange” and “anti-exchange” axioms.

Matroids are generalizations of linear independent structures \cite{13}. Matroid closure is usually denoted by the “spanning operator”, $\sigma$, \cite{29,32}. Antimatroids are typically viewed as convex geometries, \cite{3,11}, where closure is the convex hull operator, sometimes denoted by $h$, \cite{3,11}.

A closure operator, $\varphi$, is said to be **finitely generated** if every closed set has finite generators. Since we assume $S$ is finite, all operators will be finitely generated. In \cite{19}, it is shown that:
Proposition 3.3 Let \( S \) be finitely generated and let \( \varphi \) be antimatroid. If \( X \) and \( Y \) are generators of a closed set \( Z \), then \( X \cap Y \) is a generator of \( Z \).

Proposition 3.4 If \( S \) is finitely generated, then \( \varphi \) is antimatroid if and only if \( \varphi \) is uniquely generated.

A counter example is presented in [19] to show that the condition of finite generation is necessary.

3.1 Dominated Closure

It is evident that the domination operator \( \Delta \) and closure operator \( \varphi \) are closely related. For example, if \( Y.\Delta = Y.\Delta \) for all \( Y \), then \( \Delta \) is a closure operator. But, in general, \( Y.\Delta \subset Y.\Delta \). In this section, we explore the close relationship between these two operators even further.

We define dominated closure (sometimes denoted by \( \varphi_{\Delta} \)) to be:

\[
Y.\varphi_{\Delta} = \bigcup_{Z \subseteq Y.\Delta} \{ Z | Z.\Delta \subseteq Y.\Delta \}
\]

(1)

Here it is apparent that, \( Y \subseteq Y.\varphi \subseteq Y.\Delta \), as was shown in Proposition 3.1.

Although in general \( Y.\Delta \neq Y.\Delta \), we have

Proposition 3.5 For all \( Y \), \( Y.\varphi.\Delta = Y.\Delta \).

Proof: Let \( z \in Y.\varphi.\Delta \). If \( z \in Y.\varphi \), then since \( Y.\varphi \subseteq Y.\Delta \), we are done. So assume \( \exists y \in Y.\varphi, z \in \{ y \}.\Delta \). But, \( y \in Y.\varphi \) implies \( \{ y \}.\Delta \subseteq Y.\Delta \) so \( z \in Y.\Delta \). By monotonicity, \( Y.\Delta \subseteq Y.\varphi.\Delta \), so equality follows.

The operators, \( \Delta \) and \( \varphi \) are not in general commutative, since \( Y.\varphi.\Delta = Y.\Delta \subset Y.\Delta.\varphi \) as shown by the following example. Let \( S = \{ abc \} \) and let \( Y = \{ a \} \), where \( \{ a \}.\Delta = \{ ab \} \), \( \{ b \}.\Delta = \{ bc \} \), so \( \{ a \}.\varphi.\Delta = \{ a \}.\Delta = \{ ab \} \) as postulated by Proposition 3.5, but if \( \{ ab \}.\varphi = \{ ab \} \), \( \{ abc \} \) as postulated by Proposition 3.5.

Proposition 3.6 An operator \( \varphi \) is a closure operator if and only if there exists a dominating operator, \( \Delta \), related to \( \varphi \) by (1).

Proof: If \( \varphi \) is a closure operator, then let \( \Delta = \varphi \). Readily, \( \Delta \) is monotone, expansive because \( \varphi \) is. Let \( Z \subseteq Y.\Delta \). Since \( Y.\varphi.\varphi = Y.\varphi \), \( Z.\Delta = Z.\varphi \subseteq Y.\varphi = Y.\Delta \) satisfying equation (1).

Conversely, let \( \Delta \) be any monotone, expansive operator, and let \( \varphi \) be defined by (1). Monotonicity and expansivity follow from \( Y \subseteq Y.\varphi \subseteq Y.\Delta \). We need only show idempotency. Readily, \( Y.\varphi \subseteq Y.\varphi.\varphi \). By Prop. 3.5, \( Y.\varphi.\Delta \subseteq Y.\Delta \). So, \( Y.\varphi.\varphi = \bigcup_{Z \subseteq Y.\varphi.\Delta} \{ Z | Z.\Delta \subseteq Y.\varphi.\Delta \} \subseteq \bigcup_{Z \subseteq Y.\Delta} \{ Z | Z.\Delta \subseteq Y.\Delta \} = Y.\varphi \).
Proposition 3.7 X is a $\Delta$-generator of $Y.\Delta$ if and only if $X$ is a $\varphi_\Delta$-generator of $Y.\varphi_\Delta$.

Proof: Suppose $X$ is a $\Delta$-generator of $Y.\Delta$, so $X.\Delta = Y.\Delta$. By (1), $X.\varphi_\Delta = \bigcup_{Z \subseteq X.\Delta} \{Z.\Delta \subseteq X.\Delta\} = \bigcup_{Z \subseteq Y.\Delta} \{Z.\Delta \subseteq Y.\Delta\} = Y.\varphi_\Delta$.

Conversely, let $X$ be a $\varphi_\Delta$-generator of $Y.\varphi_\Delta$ and assume $X$ is not a $\Delta$-generator of $Y.\Delta$, so $X.\Delta \neq Y.\Delta$. Let $Z_0 = X.\Delta - Y.\Delta$ (or else $Y.\Delta - X.\Delta$). $Z_0 \not\subseteq Y.\Delta$ implies $Z_0 \not\subseteq \bigcup_{Z \subseteq Y.\Delta} \{Z.\Delta \subseteq Y.\Delta\} = Y.\varphi_\Delta$ contradicting assumption that $X$ is a $\varphi_\Delta$-generator of $Y.\varphi_\Delta$.

Proposition 3.8 A dominating operator, $\Delta$, is itself a closure operator, $\varphi$, if and only if $X \subseteq Y.\Delta$ implies $X.\Delta \subseteq Y.\Delta$.

Proof: Assume the condition holds. Since $\Delta$ is monotone, expansive we need only show $\Delta$ is idempotent. But readily, $Y.\Delta.\Delta = \bigcup_{X \subseteq Y.\Delta} \{X.\Delta \subseteq Y.\Delta\} = \bigcup_{X \subseteq Y.\Delta} \{X \subseteq Y.\Delta\} = Y.\Delta$.

Conversely, if $\Delta$ is idempotent, it is a closure operator by definition.

Several set systems have the property that $X \subseteq Y.\Delta$ implies $X.\Delta \subseteq Y.\Delta$. Let $P$ be a partially ordered set and let $Y.\Delta = \{x \mid x \leq y \in Y\}$. Readily $X \subseteq Y.\Delta$ implies $X.\Delta \subseteq Y.\Delta$, so $\Delta$ is a closure operator, $\varphi$. It has been called “downset” closure [19] [22]. The maximal elements in $Y$ constitute a unique generator; it is antimatroid.

One can use $\Delta$ to construct a large variety of closure systems $S$. As an example, let $P$ be any set, which we will augment with a special element, *. Let $Y \subseteq P$, and define $Y.\Delta = Y \cup \{\ast\}$ and let $\{\ast\}.\Delta = \{\ast\}$. Then $Y.\varphi = Y \cup \{\ast\}$ and $\{\ast\}.\varphi = \{\ast\}$. One can optionally let $\emptyset.\varphi = \{\ast\}$ or $\emptyset.\varphi = \emptyset$. It is apparent that $X \subseteq Y.\Delta$ implies $X.\Delta \subseteq Y.\Delta$, so $\Delta$ is a closure operator. No subset of $P$ is closed, and $Y \subseteq P$ is the unique minimal generator of the closed set of the form $Y \cup \{\ast\}$. Either $\emptyset$ or $\{\ast\}$ could be the minimal generator of $\{\ast\}$. In either case, $S$ is an antimatroid closure space. We call $S$, so defined, a “star space”.

For a simple example where $X \subseteq Y.\Delta$ need not imply $X.\Delta \subseteq Y.\Delta$, consider $S = \mathbb{Z}$, the integers. Define $\{y\}.\Delta = \{x \leq y \mid y \text{ is even} \}$ and $\{z \geq y \mid y \text{ is odd} \}$. Let $Y = \{1, 2\}$. Readily, $2 \in \{1\}.\Delta = \{i \mid i \geq 1\}$, but $\{2\}.\Delta = \{j \mid j \leq 2\} \not\subseteq \{1\}.\Delta$.

4 Transformations

A transformation $f$ is a function that maps the sets of one set system $S$ into another set system $S'$. If $S' = S$, then a transformation is just an operator on $S$. More often, however, $S'$ has a different internal structure than $S$. Frequently, $f$ represents “change” in a dynamic set system. Because the domain, and codomain, of a transformation $f$ is a collection of sets, including the empty set $\emptyset$, an expression such as $Y.\Delta = \emptyset'$ is well defined. Similarly one can have $\emptyset.\Delta = Y.\emptyset'$ if we have a functional notation for sets entering, or leaving, a set.

\footnote{\textsuperscript{2} Normally, we do not distinguish between $\emptyset$ and $\emptyset'$. The empty set is the empty set. We do so here only for emphasis.}
system altogether.

4.1 Monotone Transformations

A transformation \( S \xrightarrow{f} S' \) is said to be **monotone** if \( X \subseteq Y \) in \( S \) implies \( X.f \subseteq Y.f \) in \( S' \). Monotonicity seems to be absolutely basic to transformations and is assumed throughout this paper. No other property is. Monotonicity ensures that if \( Y.f = \emptyset \) then for all \( X \subseteq Y \), \( X.f = \emptyset' \), and if \( \emptyset.f = Y' \) then for all \( Z' \subseteq Y' \), \( \emptyset'.f = Z' \), so \( Z'.f^{-1} = \emptyset \). Readily,

**Proposition 4.1** The composition \( f \cdot g \) of monotone transformations is monotone.

4.2 Continuous Transformations

A transformation \( S \xrightarrow{f} S' \) is said to be **continuous** with respect to an operator, \( \alpha \), or more simply \( \alpha \)-continuous, if for all sets \( Y \in S \), \( Y.\alpha.f \subseteq Y.f.\alpha' \). In the referenced literature, continuity is only considered with respect to a closure operator, \( \varphi \). This is reasonable; but as the following propositions show, it can be generalized.

**Proposition 4.2** Let \( \alpha \) be any monotone operator. If \( S \xrightarrow{f} S' \) and \( S' \xrightarrow{g} S'' \) are monotone \( \alpha \)-continuous transformations then \( S \xrightarrow{f \cdot g} S'' \) is \( \alpha \)-continuous

**Proof:** Since \( f \) is continuous w.r.t. \( \alpha \), \( Y.\alpha.f \subseteq Y.f.\alpha' \). Since \( g \) is monotone, \( Y.\alpha.f.g \subseteq Y.f.\alpha'.g \). And finally \( g \) \( \alpha \)-continuous yields \( Y.\alpha.f.g \subseteq Y.f.\alpha'.g \subseteq Y.f.g.\alpha'' \).

That the composition of \( \alpha \)-continuous transformations is continuous when \( \alpha \) is a closure operator has already been shown in [19], where a counter example is provided to demonstrate the necessity of having \( g \) be monotone. They also show that the collection of all monotone, \( \varphi \)-continuous transformations forms a concrete category, \( \text{MCont} \), [19]. By Proposition 3.5 \( Y.\varphi.\Delta = Y.\Delta \subseteq Y.\Delta.\varphi \), so \( \Delta \) is monotone, \( \varphi \)-continuous, and thus a member of \( \text{MCont} \). And since \( \varphi \) is idempotent, \( \varphi \) is trivially \( \varphi \)-continuous, as well.

**Proposition 4.3** Let \( \Delta \) be a dominating operator and let \( S \xrightarrow{f} S' \) be monotone. Then \( f \) is \( \Delta \)-continuous if and only if \( X.\Delta = Y.\Delta \) implies \( X.f.\Delta' = Y.f.\Delta' \).

**Proof:** Let \( f \) be \( \Delta \)-continuous, and let \( X.\Delta = Y.\Delta \), so \( X = Y.\Delta \). By monotonicity and continuity, \( X.f \subseteq X.\Delta.f = Y.\Delta.f \subseteq Y.f.\Delta' \). Similarly, \( Y.f \subseteq X.f.\Delta' \). Since \( Y.f.\Delta' \) is the smallest \( \Delta \)-set containing \( X.f \) and \( X.f.\Delta' \) is the smallest \( \Delta \)-set containing \( Y.f \), \( X.f.\Delta' = Y.f.\Delta' \).

Conversely, assume \( f \) is not \( \Delta \)-continuous. So there exists \( Y \) with \( Y.\Delta.f \nsubseteq Y.f.\Delta' \). Let \( X \in Y.\Delta^{-1} \). \( X.f \subseteq X.\Delta.f = Y.\Delta.f \nsubseteq Y.f.\Delta' \), so \( X.f.\Delta' \neq Y.f.\Delta' \), contradicting the condition.

Thus, the image of a generator under a continuous transformation is again a generator. However, if \( X \in Y.\Delta^{-1} \) is a minimal generator, Proposition 4.3 only shows that \( X.f \) is still a generator of \( Y.f.\Delta' \); it need not be minimal.
A transformation $S \xrightarrow{f} S'$ is \textbf{$\Delta$-surjective} if for all $\Delta$-sets $Y'$, there exists a set $Y \subseteq S$ such that $Y.f = Y'$.

**Proposition 4.4** Let $f$ be monotone, $\Delta$-continuous and $\Delta$-surjective, then for all $\Delta$-sets $Y'$ in $S'$, there exists a set $Y \subseteq S$ such that $Y.f = Y'$.

**Proof:** Since $f$ is $\Delta$-surjective, $\exists Y, Y.f = Y'$. But, by monotonicity and $\Delta$-continuity, $Y.f \subseteq Y.\Delta.f \subseteq Y.f.\Delta' = Y'$. So, $Y.\Delta.f = Y'$.

**Proposition 4.5** Let $S \xrightarrow{f} S', S' \xrightarrow{g} S''$ be monotone, $\Delta$-continuous transformations. If both $f$ and $g$ are $\Delta$-surjective, then so is $S \xrightarrow{f \cdot g} S''$.

**Proof:** Because the composition of $\Delta$-continuous transformations is $\Delta$-continuous, we need only consider surjectivity. Let $Y''$ be a $\Delta$-set in $S''$. Since $g$ is surjective, $\exists Y' \subseteq S', Y'.g = Y''$. Because, $g$ is continuous we may assume, by Prop. 4.4 that $Y'$ is an $\Delta$-set. Thus, by surjectivity of $f$, $\exists Y \subseteq S, Y.f = Y'$. Consequently, $f \cdot g$ is $\Delta$-surjective.

A transformation $S \xrightarrow{f} S'$ is said to be \textbf{$\alpha$-preserving} if $Y.\alpha.f = Y.f.\alpha'$. An $\alpha$-preserving map takes $\alpha$-sets onto $\alpha$-sets.

**Proposition 4.6** Let $\Delta$ be a dominating operator and let $f$ be monotone. $f$ is $\Delta$-preserving if and only if for all $Y, Y.f.\Delta' \subseteq Y.\Delta.f$.

**Proof:** Assume, $Y.f.\Delta' \subseteq Y.\Delta.f$. Let $Y = Y.\Delta$, so $Y.f.\Delta' \subseteq Y.\Delta.f = Y.f$. Readily $Y.f \subseteq Y.\Delta.f'$ so $Y.f = Y.f.\Delta'$ and $f$ is $\Delta$-preserving.

Now assume $\Delta$ is idempotent and that $f$ is $\Delta$-preserving. By monotonicity of $f$, $Y \subseteq Y.\Delta$ implies $Y.f \subseteq Y.\Delta.f$ and and since $\Delta$ is monotone, $Y.f.\Delta' \subseteq Y.\Delta.f.\Delta'$. Idempotency implies $Y.\Delta.\Delta = Y.\Delta$. Since $f$ is $\Delta$-preserving, $Y.\Delta.f.\Delta' = Y.\Delta.f$ so $Y.f.\Delta' \subseteq Y.\Delta.f$.

Proposition 4.6 is also proven in [19] where $\Delta$ is a closure operator $\varphi$.

**Corollary 4.7** A monotone transformation $f$ is both $\Delta$-continuous and $\Delta$-preserving, if and only if $Y.\Delta.f = Y.f.\Delta'$.

### 4.3 Galois Connections

Monotone transformations $S \xrightarrow{f} S'$ can provide another mechanism for defining closure operators on $S$.

Let $S \xrightarrow{f} S'$ and $S' \xrightarrow{g} S$ be monotone transformations. The composite $(f \cdot g)$ is called a \textbf{Galois connection} if for all $X \subseteq S, Y' \subseteq S'$.
Proposition 4.8 Let $S \xrightarrow{f} S' \xrightarrow{g} S$ be monotone. The following are equivalent statements.

(a) $(f \cdot g)$ is a Galois connection.
(b) For all $X \subseteq S$ and all $Y' \subseteq S'$, $X.f \subseteq Y' \subseteq Y'$.g.

Proof: (a) implies (b): Let $X.f \subseteq Y'$, so $g$ monotone implies $X.f.g \subseteq Y'.g$, thus $X \subseteq X.f.g \subseteq Y'.g$. Similarly, $X \subseteq Y'.g$ implies $X.f \subseteq Y'.g$.

(b) implies (a): Let $X.f = Y'$, so trivially $X.f \subseteq Y'$. By (b) $X \subseteq Y'.g$ implying $X \subseteq Y'.g \subseteq X.f.g$. $f \cdot g$ is expansive. Similarly, $X \subseteq Y'.g$ implies $g.f \subseteq Y'$.

Proposition 4.9 If $S \xrightarrow{f} S' \xrightarrow{g} S$ is a Galois connection, then $f$ and $g$ uniquely determine each other.

Proof: Let $f \cdot g$ be a Galois connection and suppose there exists $h$ such that $f \cdot h$ is also a Galois connection. We apply $h$ to $Y'.g.f \subseteq Y'$. $Y'.g \subseteq Y'.g.f.h$ because $f \cdot h$ is expansive, and $Y'.g.f.h \subseteq Y'.h$ since $h$ is monotone. So, $Y.g \subseteq Y'.h$. Applying $g$ to $Y'.h.f \subseteq Y'$ yields $Y'.h \subseteq Y'.g$, $\forall Y'$, so $h = g$.

A similar argument shows that $f$ must be unique given $g$.

Proposition 4.10 If $S \xrightarrow{f} S' \xrightarrow{g} S$ is a Galois connection, then $f \cdot g \cdot f = f$ and $g \cdot f \cdot g = g$.

Proof: Let $Y' = X.f$. Since $g \cdot f$ is contractive, $X.f = Y' \subseteq Y'.g. f = X.f.g.f$. However, $X \subseteq X.f.g$ implies $X.f \subseteq X.f.g.f$ by monotonicity, so $X.f = X.f.g.f, \forall X$.

Similarly, $Y'.g = Y.g.f.g, \forall Y' \subseteq S'$.

Corollary 4.11 If $f \cdot g$ is a Galois connection, then $S \xrightarrow{f,g} S$ is a closure operator on $S$.

Proof: Since $f \cdot g$ are Galois connected, $f \cdot g$ is expansive. $f$ and $g$ are monotone. Prop. 4.10 establishes that $f \cdot g$ is idempotent.

The preceding three propositions largely follow the development of Galois connections provided by Castellini in [1], except for changed notation.

A somewhat different development can be found in Ganter & Wille [6]. They choose to let $f$ and $g$ be “anti-monotone”, that is, $X_1 \subseteq X_2$ implies $X_1.f \supseteq X_2.f$ and $Y'_1 \subseteq Y'_2$ implies $Y'_1.g \supseteq Y'_2.g$ and $X \subseteq X.f.g, Y' \subseteq Y'.g.f$. With this definition of Galois connection, Proposition 4.8 must be rewritten as: “$f \cdot g$ is a Galois connection if and only if $Y' \subseteq X.f$ implies $X \subseteq Y'.g$ and conversely”. However, both approaches will yield Proposition 4.10.
and its corollary. With the latter definition, both \( f \cdot g \) and \( g \cdot f \) are expansive operators on \( S \), so both are closure operators. This is important for the subsequent development of “Formal Concept Analysis” in [6].

We prefer the development presented here, and in [1], because we can compose monotone transformations, so is easy to show that

**Proposition 4.12** Let \( S \xrightarrow{f} S' \xrightarrow{g} S \) and \( S' \xrightarrow{h} S'' \xrightarrow{k} S \) be Galois connections, then \( S \xrightarrow{f \cdot h} S'' \xrightarrow{k \cdot g} S \) is a Galois connection.

Let \( Y, \gamma \) denote the collection of minimal generators of \( Y, \Delta \). If \( \Delta \) is uniquely generated, then \( Y, \gamma \) is a well defined function.

**Proposition 4.13** If \( \Delta \) is uniquely generated, then \( \Delta \) and \( \gamma \) constitute a Galois connection on \( S \).

**Proof:** \( Y, \gamma, \Delta = Y, \Delta \), so \( \gamma, \Delta \) is expansive. Because \( \Delta \) is uniquely generated, \( Y, \gamma \subseteq Y \), so \( Y, \Delta, \gamma \) is contractive.

Hence, by Corollary 4.11 any uniquely generated dominating operator \( \Delta \) is a closure operator, and by Proposition 3.4, it is antimatroid. If we suppose \( \Delta \) is not uniquely generated, say \( X \) and \( Z \) are minimal sets such that \( X, \Delta = Z, \Delta \), then \( X, \Delta, \gamma = X \cup Z \not\subseteq X \). \( \Delta, \gamma \) is not contractive.

## 5 Categorical Closure

Operators, such as \( \alpha, \Delta, \varphi \), can be naturally regarded as morphisms in a categorical sense. In this section we develop this way of approaching domination and closure. It is somewhat different from the approach found in [1], but is compatible with it. The following material has been largely derived from [9, 25]; the only substantive differences have been to change notation. All functions, or morphisms, will still be denoted using postfix notation. Proposition 5.1 is, we believe, original.

### 5.1 Review and Examples

Two categories which are usually presented in category text books are \( \mathbf{Set} \) and \( \mathbf{Porder} \). For us, the objects of \( \mathbf{Set} \) will be all possible finite sets.\(^3\) The morphisms of \( \mathbf{Set} \) consist of all functions \( f : X \rightarrow Y \) (or morphisms \( X \xrightarrow{f} Y \)), with the usual composition.

In the partial order category \( \mathbf{Porder} \), the objects consist of all finite sets, \( S \), with a partial order \( \leq_S \) on \( S \). Its morphisms are all order preserving (monotone) functions.

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\(^3\) To avoid logical paradoxes, MacLane, [13], lets the objects in \( \mathbf{Set} \) be all the sets in a fixed universe \( U \). These he calls *small* sets. Readily, our finite sets are small.
If $f : S \to T$, such that $x \leq_S y$ implies $x.f \leq_T y.f$. Checking that $f \cdot g$ is order preserving is usually left as an exercise.

These two well known categories are paradigms for the dominance and closure categories. Analogous to Set, is the category Pow whose objects consist of all finite power sets. Its morphisms are all total functions $2^S \xrightarrow{f} 2^T$ with the usual composition; $(\forall X \in 2^S)[X.(f \cdot g) = X.f.g]$ and the usual identity on $2^S$, $(\forall X \subseteq S)[X.id_S = X]$. It is not difficult to show this composition is well-defined and associative. The transformations of Section 4 are morphisms in Pow.

The category Pow would appear to be completely isomorphic to Set, with each $S \in$ Set replaced by $2^S \in$ Pow. But there are essential differences. An object $Y$ is said to be terminal in $\mathcal{C}$ if for every object $X$ in $\mathcal{C}$ there exists exactly one morphism $X \to Y$. As is well known, [15], every element $x$ is a terminal object of Set, since for all $S$, the function $f : S \to x$ must be unique. However, the singleton sets in Pow need not be terminal. To see this, let $Y = \{x, y\} \in 2^S$ and let $f, g$ be extended transformations, where $\{x\}f = \{x\}$, $\{y\}f = \emptyset$ and $\{x\}g = \emptyset$, $\{y\}g = \{x\}$. Then $S.f = S.g = \{x\}$, but $f \neq g$. So the singleton elements $\{x\}$ cannot be terminal in Pow. Only $\emptyset$ is terminal in Pow, and initial as well.

If we restrict the morphisms in Pow to be monotone, or order preserving, as in Section 4, that is, $X \subseteq Y$ implies $X.f \subseteq Y.f$ then we have an exact analogue to Porder, but over Pow, not Set. This category, which we call Trans, has the objects of Pow as its objects, and the collection of all monotone transformations $2^S \xrightarrow{f} 2^T$. It is simply a restriction on the morphisms of Pow. Proposition 4.1 asserts that composition in Trans is still well defined. Much of Section 4 is simply concerned with examining the properties of transformations $f$ in Trans. When the morphisms in Trans are of the form $2^S \xrightarrow{\alpha} 2^S$, which we have called “operators”, they constitute a category Opr $\subset$ Trans.

Our final example is that of Dom, the category of all expansive, monotone operators, $\Delta$ on power sets $2^S$; that is, the category of dominating operators. The objects of Dom are precisely those of Pow, that is $2^S$ partially ordered by $\subseteq$. Its morphisms are all expansive, monotone (order preserving) functions (operators) $\alpha : (S, \subseteq) \to (S, \subseteq)$, such that $X \subseteq Y$ implies $X.\alpha \subseteq Y.\alpha$ and $X \subseteq X.\alpha$. Readily, the identity operator $id_S$ preserves the order $\subseteq$. The usual composition, $\alpha \cdot \beta$ is order preserving. Readily, Dom $\subset$ Opr $\subset$ Trans $\subset$ Pow.

5.2 Pullbacks

Recall that the pullback of a pair of morphisms, $Y \xrightarrow{f} Z$ and $X \xrightarrow{g} Z$, is an object $W$ and two morphisms $W \xrightarrow{f'} X$, $W \xrightarrow{g'} Y$ such that $g' \cdot f = f' \cdot g$. Moreover, if there exist morphisms $V \xrightarrow{h_X} X$, $V \xrightarrow{h_Y} Y$, such that $h_Y \cdot f = h_X \cdot g$ then there exists a unique

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Note that “dom” is the standard categorical way of denoting the domain of a morphism, that is, if $\alpha : 2^S \to 2^T$ then $\text{dom} \alpha = 2^S$ and $\text{cod} \alpha = 2^T$, where $\text{cod}$ denotes codomain. We do not use the terminology dom or cod in this paper.
morphism $V \xrightarrow{u} W$ such that $h_Y = u \cdot g'$ and $h_X = u \cdot f'$.

**Proposition 5.1** Let $C \subseteq \text{Dom}$ be a subcategory. Its morphisms $\Delta$ are antimatroid closure operators if and only if $C$ exhibits the pullback property of Figure 1.

**Proof:** Suppose that the dominating morphisms $\Delta$ of $C \subseteq \text{Dom}$ are antimatroid closure operators. We must show that the pullback diagram of Figure 1 is satisfied. We assume that $\Delta_X : X \rightarrow Z$ and $\Delta_Y : Y \rightarrow Z$, so $X$ and $Y$ are generators of $Z$. Let $V$ be any set $V \subseteq X$, $V \subseteq Y$ such that $V.(\subseteq X \cdot \Delta_X) = Z$ and $V.(\subseteq Y \cdot \Delta_Y) = Z$, so $V$ is a generator of $Z$. By Prop. 3.3 $W = X \cap Y$ is the unique pullback of these two generators.

Conversely, let the morphisms $\Delta \in C$ exhibit the pullback property of Figure 1. $X$ and $Y$ are generators of $Z$. Since $V \xrightarrow{u} W$ is unique, by Prop. 3.4 $\Delta$ is uniquely generated. By Prop. 4.13 $\Delta$ must be a closure operator.

Figure 1: The pullback of antimatroid generators.

Observe that in Figure 1 the sets $V, W, X, Y$ are all generators of $Z$. It is known that pullback diagrams preserve monomorphisms and retractions [15].

It is conjectured that an analog of Proposition 5.1 is true as well, that is that a subcategory of $\text{Dom}$ satisfying the “push-out” property must consist of matroid closure operators.

6 **Summary**

There are many more dominating than closure operators\(^5\) in spite of Proposition 3.6 which established that for every dominating operator there exists a corresponding (not necessarily unique) closure operator. While dominating operators are more ubiquitous, for example most network operators associated with internet analysis are expansive; closure operators are more structured. By Proposition 4.13 only antimatroid closure operators can be uniquely generated. However, dominated closure has been used to reduce social networks in a way that preserves path connectivity [24].

\(^5\) If $|S| = n \geq 10$, there exist more than $n^n$ distinct antimatroid closure operators [20].
The dominated closure, \( \varphi_\Delta \) defined by (1) is often called a **neighborhood closure**, because the definition (1) can be rewritten as

\[
Y \varphi = Y \cup \bigcup_{\{z\} \subseteq Y \eta} \{\{z\}, \Delta \subseteq Y, \Delta\}
\]

which, in many cases, is computationally much more efficient.

Dominating operators can be easily computed. A common way, in practice, of defining a dominating operator and its dominated neighborhood is by an adjacency matrix, \( A \). For each row \( i \) of \( A \), if \( (i, k) \neq 0 \), then \( k \) is in the region dominated by \( i \), or \( \{k\} \subseteq \{i\}, \Delta \). Thus \( A \) can be the base of a graphic representation, \( G \), described in Section 2.2. If \( \Delta \), and \( A \), are symmetric, the graph \( G \) is undirected. In most applications, \( Y, \Delta \) is assumed to be \( \bigcup_{y \subseteq Y} \{\{y\}, \Delta\} \), or all elements directly connected to \( Y \) in \( G \), i.e. \( \Delta \) is an extended operator.

However, one might want much larger sets \( Z \) to have a much wider scope of dominance. Similarly, in \( A \times A \), \( (i, k) \neq 0 \), if \( k \) is two, or fewer, links from \( i \). It is still a dominance region. But, it is not graphically representable, nor is it extensible.

The term “domination” is well established in the graph theory literature, but we prefer to think of these expansive, outward accessing operators as “exploratory” operators, especially when viewing their role in social network analysis. Some educators have suggested that “knowledge spaces” might be modelled by closed sets. If so, “exploratory” closure, \( \varphi_\Delta \), becomes a plausible mechanism for “learning”. One can regard \( S \) as a universe of related “experiences” and “knowledge” to be a collection \( Y \) of such experiences, or skills. An experience \( f \) expands one’s knowledge if it is congruent with one’s existing knowledge set \( Y \); that is if \( \{f\}, \eta \subseteq Y, \eta \). That is, the connections of \( f \) are congruent with the connections of \( Y \); it makes sense.

Dominated closure may find other applications as well. In [23], it was shown that one can define “fuzzy” dominated closure, yet still retain many of the crisp properties of closure operators.

It is evident that many of the preceding results, which have been expressed in operator terminology, can be recast using graph terminology. In return, graph theory can provide a rich source of discrete set systems. In particular, domination [7, 8, 28] and closure [1, 5, 10] have been well studied and can provide many operator examples.

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