On Integrable Structure behind the Generalized WDVV Equations

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ABSTRACT

In the theory of quantum cohomologies the WDVV equations imply integrability of the system \((I∂_µ - zC_µ)ψ = 0\). However, in generic situation – of which an example is provided by the Seiberg-Witten theory – there is no distinguished direction (like \(t^0\)) in the moduli space, and such equations for \(ψ\) appear inconsistent. Instead they are substituted by \((C_µ∂_ν - C_ν∂_µ)ψ \sim (F_µ∂_ν - F_ν∂_µ)ψ = 0\), where matrices \((F_µ)_{αβ} = \partial_α∂_β∂_µF\).

1 Quantum Cohomologies (a brief summary)

The WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations \([1]\) are an important ingredient of the theory of quantum cohomologies (2d topological \(σ\)-models) and play a role in the formulation of mirror transform. The central object in these studies is the prepotential: a function of “time”-variables \(F(t^α)\), which satisfies the WDVV equations:

\[ C_µC_ν = C_νC_µ, \quad \forall µ, ν. \tag{1} \]

Here \(C_µ\) are matrices,

\[ (C_µ)_{β}^γ = η^{αγ}(F_µ)_{γβ}, \quad (F_µ)_{αβ} = \frac{∂^3F}{∂τ^α∂τ^β∂τ^µ} \tag{2} \]

1 If the prepotential is interpreted as the “quantum” deformation of the generating function of intersection numbers on some manifold \(M\) (a Gromov-Witten functional for \(M\)), the variables \(t^α\) are associated with “observables” \(φ_α\) – the elements of the cohomology ring \(H^*(M)\). Basically,

\[ F(t) = \langle \exp \left( \sum_{α=1}^{\dim H^*(M)} t^αφ_α \right) \rangle_0 \]
and the “metric"

\[ \eta_{\alpha\beta}^{(0)} = (F_0)_{\alpha\beta} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta}, \quad \eta^{\alpha\beta} \eta_{\beta\gamma} = \delta_{\alpha}^{\gamma}. \tag{3} \]

In conventional theory of quantum cohomologies there is a distinguished variable \( t^0 \) (associated with the unity \( \phi_0 = I \) in the ring \( H^* (M) \)), such that the metric \( \eta = \eta^{(0)} = F_0 \) in (3) is constant:

\[ \frac{\partial \eta}{\partial t^\alpha} = 0 \quad \tag{4} \]

As a corollary, the matrices \( F_\mu \) and \( C_\mu \) are independent of \( t^0 \). In these circumstances the set of WDVV equations (1) together with the relations (2) – saying that the “structure constants” \( C_\mu \) are essentially the third derivatives of a single function \( F(t) \) – implies the consistency condition

\[ [D_\mu (z), D_\nu (z)] = 0 \quad \forall \mu, \nu \tag{5} \]

for the set of differential equations (2)

\[ D_\mu (z) \psi_z = \left( I \frac{\partial}{\partial t^\mu} - z C_\mu (t) \right) \psi_z (t) = 0 \quad \left( \partial_\mu \psi_z^\alpha = z C_{\mu\beta} \psi_z^\beta \right) \tag{6} \]

with arbitrary “spectral parameter” \( z \).

This reveals an integrable (Whitham-like) structure behind the conventional WDVV equations. (Direct interpretation of (6 is in terms of deformations of the Hodge structures on Kahler manifolds.)

The “Baker-Ahiezer vector-function” \( \psi_z (t) \) has various interpretations.

First, [3, 4], as a function of \( z \) it is a generating function of the correlators, linear in gravitational descendants (Morita-Mumford classes) \( c^n (\phi) \):

\[ \psi_{z,\rho}^\alpha (t) = \sum_{n=0}^{\infty} z^n \langle c^n (\phi_\rho) \phi^\alpha \phi^{\sum t^\beta} \rangle_0 \tag{7} \]

Thus \( \psi_z \) is an important part of the reconstruction of the full spherical prepotential \( F_0 (t^0_n) = \langle \exp \sum_{\alpha,n} t^0_n c^n (\phi_\alpha) \rangle_0, \#(\alpha) = \dim H^* (M), n = 0,1, \ldots \) – the generating function of correlators with arbitrary number of descendants. Original prepotential appears when all descendant time-variables vanish:

\[ F(t^0) = F_0 (t^0_n) |_{n z_j = 0} \]

When descendants are included, \( \psi_z \) satisfies a hierarchy of quadratic equations

\[ \frac{\partial}{\partial t^\mu} \psi_z^{\alpha,\rho} = \eta^{\alpha\gamma} \langle c^m (\phi_\mu) \phi_\gamma \phi_\beta \rangle \psi_z^{\beta,\rho} = \psi_z^{\beta,\rho} \int \frac{dy}{y^{m+1}} \frac{\partial}{\partial t^\alpha} \psi_y^{\alpha,\mu} \tag{8} \]
which is the “quasiclassical” limit of some full (i.e. possessing a group-theory interpretation in the spirit of [3]) integrable hierarchy – to which the full prepotential, the generating function of all correlators for all genera, is a solution.

Second, the function \( \psi_z \) usually possesses integral representations of the form

\[
\psi_z^\alpha(t) = \int_{\Gamma} \Omega_\alpha^z(t)
\]

along some cycles on some manifold \( \tilde{M} \) – which is interpreted as a mirror of \( M \).

In concrete examples (see, for example, [4]) this representation is implied by the hidden group-theory structure behind integrable system (6), which allows to interpret \( \psi_z \) as eigenfunctions of Casimir operators. Such eigenfunctions are well known to possess natural integrable representations, see [6] and references therein.

Clarification of these constructions, associating some (loop) algebra with a manifold, remains an interesting open question.

## 2 WDVV Equations in Seiberg-Witten Theory

The WDVV-like equations are now known to arise in a somewhat broader context than conventional quantum cohomologies. Namely, one can relax the condition (4) and study the WDVV equations in the situation when there is no distinguished modulus \( t^0 \) and no distinguished metric \( \eta^{(0)} \). Such situation arises, for example, in Seiberg-Witten theory [7] of low-energy effective actions for \( N = 2 \) SUSY Yang-Mills models in four and five dimensions. This theory is long known to involve integrable structures [8] and the prepotential (quasiclassical \( \tau \)-function) theory [9]. The WDVV-like equations arise in Seiberg-Witten theory in the form [10]:

\[
F_\mu F_\lambda^{-1} F_\nu = F_\nu F_\lambda^{-1} F_\mu, \quad \forall \lambda, \mu, \nu
\]  

\[
(F_\mu)_{\alpha \beta} = \partial_\alpha \partial_\beta \partial_\mu F
\]

i.e. the role of the metric \( \eta \) can be played by any matrix \( F_\lambda \) (actually, by any linear combination of such matrices). Accordingly, the mutually commuting matrices \( C_{\mu}^{(\lambda)} = F_\lambda^{-1} F_{\mu} \),

\[
\left[ C_{\mu}^{(\lambda)}, C_{\nu}^{(\lambda)} \right] = 0 \quad \forall \mu, \nu
\]

are now implicitly dependent on the choice of \( \lambda \).

However, since generically there is no constant (moduli-independent) matrix \( F_\lambda \), the generalized WDVV equations (10) no longer imply (6). This system of

\footnote{
To avoid confusion, the set (10) is not richer than (6), as it can seem: with any given \( \lambda \) immediately implies the equations for all other \( \lambda \).}
consistent equations is instead substituted by

\[ \left( \partial_\mu - C^{(\lambda)}_\mu \partial_\lambda \right) \psi = 0 \quad \forall \mu, \lambda \]  

(12)

or, in a more symmetric form,

\[ (F_\lambda \partial_\mu - F_\mu \partial_\lambda) \psi = 0 \quad \forall \mu, \lambda \]  

(13)

It is easy to see that the operators with different \( \mu \) at the l.h.s. commute with each other:

\[
\left[ \left( \partial_\mu - C^{(\lambda)}_\mu \partial_\lambda \right), \left( \partial_\nu - C^{(\lambda)}_\nu \partial_\lambda \right) \right] = \left[ C^{(\lambda)}_\mu, C^{(\lambda)}_\nu \right] \partial^2_\lambda + \\
\left( (\partial_\nu C^{(\lambda)}_\mu) - (\partial_\mu C^{(\lambda)}_\nu) + C^{(\lambda)}_\mu (\partial_\lambda C^{(\lambda)}_\nu) - C^{(\lambda)}_\nu (\partial_\lambda C^{(\lambda)}_\mu) \right) \partial_\lambda
\]  

(14)

The first term at the r.h.s. vanishes due to the WDVV equations, and the second one can be seen to vanish if the definition of \( C^{(\lambda)}_\mu \) is used together with the fact that \( F_\mu \) are matrices, consisting of third derivatives. Eq. (13) is (12), multiplied by a matrix \( F_\lambda \) from the left.

It can still seem non-obvious that equations (13) are all consistent, i.e. that the vector \( \psi \) can be chosen in a \( \lambda \)-independent way. This follows from the relation:

\[
F_\mu \partial_\nu - F_\nu \partial_\mu = \\
F_\mu (\partial_\nu - C^{(\lambda)}_\nu \partial_\lambda) - F_\nu (\partial_\mu - C^{(\lambda)}_\mu \partial_\lambda)
\]  

(15)

In order to return back from the generic system (12) to (6), it is enough to choose \( \psi = e^{zt} \psi_z \), what is a self-consistent anzats when all the \( C^{(0)}_\mu \) are \( t^0 \)-independent.

There is no spectral parameter in the system (13), instead it is homogeneous (linear) in derivatives and possesses many solutions. They can be formally represented in the form:

\[
\hat{\psi}(t) = P \exp \int^t dt' C^{(\lambda)}_\mu (t') \partial_\lambda = \left\{ I + \left( \int^t dt_1 C^{(\lambda)}_\mu (t_1) \right) + \\
+ \int^t dt_1 C^{(\lambda)}_\mu (t_1) \partial_\lambda \left( \int^t dt_2 C^{(\lambda)}_\nu (t_2) \right) + \ldots \right\} \hat{\psi}(t)
\]  

(16)

and one can choose, for example, \( \hat{\psi}(t) = e^{zt} \). Then different terms of expansion of (13) in \( z \) are different solutions to (12).

3 As well as I understand such equations per se were studied as an alternative to (4) by B.Dubrovin (see ref. [1]) and other authors – but in the context of conventional quantum cohomology theory, with distinguished \( t^0 \)-direction.
As a simplest example, one can take

$$F = \frac{1}{2} \left( t_1^2 \log t_1 + t_2^2 \log t_2 + (t_1 - t_2)^2 \log(t_1 - t_2) \right)$$

(17)

which is the perturbative prepotential for $SU(3) N = 2$ SYM model in $4d$. Then the first few solution to (13) are:

$$
\psi = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} t_1^2 - 2t_1 t_2 \\ t_2 - 2t_1 t_2 \end{pmatrix}, \\
\psi = \begin{pmatrix} t_1^3 - 2t_1^2 t_2 \\ -t_1 t_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} t_1^3 - 2t_1^2 t_2 \\ -t_1^2 t_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} -t_1 t_2 \\ t_3 - 2t_1 t_2 \end{pmatrix},
$$

(18)

By the way, $\psi^\alpha = t^\alpha$ is always a solution to (13) – this follows immediately from the definition of $F_\mu$’s as the matrices of the 3-rd derivatives, which are symmetric under permutations of indices.

### 3 Conclusion and acknowledgements

The purpose of this letter is to explain that appropriate integrable structure on the moduli space exists behind the generalized WDVV equations, i.e. existence of a constant metric is not needed for such structure to emerge. I do not touch here neither interpretation, nor implications of this simple statement. They will be discussed elsewhere.

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