On homeomorphisms of Cantor space that induce only the trivial Turing automorphism

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Abstract

To determine whether there is a nontrivial automorphism of the Turing degrees remains a major open problem of computability theory. Past results have limited how nontrivial automorphisms could possibly be, and ruled out that an automorphism might be induced by a function on integers.

A homeomorphism $F$ of the Cantor space is said to induce a uniform map mod finite if for each $a \in \omega$ there is a $b \in \omega$ such that for all $X$ and $Y$, if $X(n) = Y(n)$ for all $n \geq b$ then $F(X)(m) = F(Y)(m)$ for all $m \geq a$.

We show that if such an $F$ induces an automorphism $\pi$ of the Turing degrees, then $\pi$ is trivial. This generalizes, and provides an easier proof of, the past result where $F$ was assumed to be induced by a permutation of $\omega$.

Dedicated to the celebration of the work of Theodore A. Slaman and W. Hugh Woodin

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1 Introduction

Let $\mathcal{D}_T$ denote the set of Turing degrees and let $\leq$ denote its ordering. This article gives a partial answer to the following famous question.

Question 1. Does there exist a nontrivial automorphism of $\mathcal{D}_T$?

Definition 2. A bijection $\pi : \mathcal{D}_T \to \mathcal{D}_T$ is an automorphism of $\mathcal{D}_T$ if for all $x, y \in \mathcal{D}_T$, $x \leq y$ iff $\pi(x) \leq \pi(y)$. If moreover there exists an $x$ with $\pi(x) \neq x$ then $\pi$ is nontrivial.

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Question 1 has a long history. Already in 1977, Jockusch and Solovay [2] showed that each jump-preserving automorphism of the Turing degrees is the identity above $\mathbf{0}^{(4)}$. Nerode and Shore 1980 [4] showed that each automorphism (not necessarily jump-preserving) is equal to the identity on some cone $\{ a : a \geq b \}$. Slaman and Woodin [5] showed that each automorphism is equal to the identity on the cone above $\mathbf{0}''$. and [5, 6] that $\text{Aut}(\mathcal{D})$ is countable.

An obstacle to reducing the base of the cone to $\mathbf{0}'$ and ultimately $\mathbf{0}$ is that Turing reducibility is $\Sigma_3^0$, but not $\Pi_2^0$ or $\Sigma_2^0$ in the sense of descriptive set theory.

In the other direction, S. Barry Cooper [1] claimed to construct a nontrivial automorphism, induced by a discontinuous function on $\omega^\omega$, itself induced by a function on $\omega^{<\omega}$. That claim was not independently verified. In [3] we attacked the problem by ruling out a certain simple but natural possibility: automorphisms induced by permutations of finite objects. We showed that no permutation of $\omega$ represents a nontrivial automorphism of the Turing degrees. That proof was too complicated, in a way, and did not extend from $D_T$ down to $D_m$. Here we give a more direct proof using the shift map $n \mapsto n + 1$. Our proof here will generalize to a certain class of homeomorphisms, distinct from the class of such homeomorphisms that the result in [3] generalizes to.

## 2 Excluding permutations by recursion

**Lemma 3.** Suppose $\theta : \omega \to \omega$ is a bijection such that for all computable $f : \omega \to \omega$, $\theta^{-1} \circ f \circ \theta$ is computable. Then $\theta$ is computable.

*Proof.* Let $f(n) = n + 1$ and $k(n) = (\theta^{-1} \circ f \circ \theta)(n)$. Then for any $m$, $k(\theta^{-1}(m)) = \theta^{-1}(f(m))$ and so we compute $\theta^{-1}$ by recursion:

$$\theta^{-1}(m + 1) = k(\theta^{-1}(m)) \quad \square$$

**Lemma 4.** Suppose $\sigma \in \mathcal{2}^{<\omega}$, $g : \omega \to \omega$, and $\Phi$ is a Turing functional, satisfying

$$(\forall n)(\forall \tau \geq \sigma)(\exists \rho \geq \tau)(\Phi^\rho(n) \downarrow \text{ and } \rho \circ g(n) = \Phi^\rho(n)). \quad (1)$$

For any $\rho \geq \sigma$ and $n$, if $\Phi^\rho(n) \downarrow$, then $g(n) < |\rho|$.

*Proof.* If instead $g(n) \geq |\rho|$ then $\rho(g(n))$ is undefined. So let $\tau \geq \rho$, $\tau(g(n)) = 1 - \Phi^\rho(n)$. This $\tau$ violates (1). \quad \square

**Lemma 5.** If $g : \omega \to \omega$ is injective and $\Phi$ is a Turing functional such that

$$\{ B : B \circ g = \Phi^B \}$$

is nonmeager, then $g$ is computable.

*Proof.* By assumption, it is not the case that

$$(\forall \sigma)(\exists n)(\exists \tau \geq \sigma)(\forall \rho \geq \tau)(\Phi^\rho(n) \downarrow \to \rho \circ g(n) \neq \Phi^\rho(n)).$$

So we have

$$(\exists \sigma)(\forall n)(\forall \tau \geq \sigma)(\exists \rho \geq \tau)(\Phi^\rho(n) \downarrow \text{ and } \rho \circ g(n) = \Phi^\rho(n)).$$
Pick such a $\sigma$: then $\Theta$ cannot make a mistake above $\sigma$, and we can always extend to get the right answer.

As finite data we assume we know the values of $n$ and $g(n)$ for which $g(n) < |\sigma|$. We compute the value $g(n)$ as follows.

Check the finite database of $(k, g(k)) : g(k) < |\sigma|$, and output $g(n)$ is found. Otherwise we know $g(n) \geq |\sigma|$.

By dovetailing computations, find a $\rho_0 \geq \sigma$ such that $\Theta^{\rho_0}(n)$. By Lemma 3 we have that $g(n) < |\rho_0|$. Thus, $g(n) \in I$ where $I$ is the closed interval $[|\sigma|, |\rho_0| - 1]$. Let $a \in I, b \in I, a < b$. It suffices to show how to eliminate either $a$ or $b$ as a candidate for being equal to $g(n)$.

Let $\tau > \sigma$ be such that $\tau(a) \neq \tau(b)$ and let $\rho \geq \tau$ be such that $\Theta^{\rho}(n) \downarrow$. Then mark as eliminated whichever $c \in [a, b]$ makes $\rho(c) \neq \Theta^{\rho}(n)$. Thus we one-by-one eliminate all $a \in I$ until only one candidate remains.

**Theorem 6.** No permutation of the integers can induce a nontrivial automorphism of $D_r$, for any reducibility $\leq_r$ between $\leq_1$ and $\leq_T$.

**Proof.** Suppose $\Theta : \omega \to \omega$ is a permutation (bijection) and consider any injective recursive $f$. For any $B$, let $A = B \circ \theta^{-1} \circ f$ (so $\pi \in A$ iff $f(x) \in B \circ \theta^{-1}$). We have $A \leq_1 B \circ \theta^{-1}$ and so by assumption $A \circ \theta \leq_T B \circ \theta^{-1} \circ \theta = B$. This gives $(B \circ \theta^{-1} \circ f) \circ \theta = \Theta^B$ for some Turing functional $\Theta$. Let $g = \theta^{-1} \circ f \circ \theta$. Since for each $B$ and $f$ there exists such a $\Theta$, for each $f$ there must be some $\Theta$ such that the $G_\Theta$ set

$$\{B : B \circ g = \Theta^B\}$$

is nonmeager. By Lemma 5 $g$ is computable. By Lemma 3 $\Theta$ is computable.

But this means that for any $A, \Theta \leq_1 A$, so that the represented automorphism $\pi$ is everywhere-decreasing: $\pi(x) \leq_1 x$. Applying this to $\pi^{-1}$ we get $\pi(x) \equiv_1 x$.  

**3 Excluding maps on Cantor space mod finite**

Making computability-theoretic uniformity assumptions is an easy way to rule out certain possible Turing automorphisms, but we will not discuss that further as we are more interested in uniformity of a simpler, or purely combinatorial, kind.

**Definition 7.** Let $[a, \infty) = \{n \in \omega : a \leq n\}$. A map $\Theta$ uniformly induces a map on $2^\omega$ mod finite if for each $a$ there is a $b$ such that for all $A, B$, if $A \uparrow [b, \infty) = B \uparrow [b, \infty)$ then $F(A) \uparrow [a, \infty) = F(B) \uparrow [a, \infty]$.

Some continuous maps $F : 2^\omega \to 2^\omega$ induce maps $\tilde{F} : 2^\omega / =^* \to 2^\omega / =^*$ but do not have the uniform property:

**Example 8.** Let us code an alphabet of size 4 into $2^\omega$. Then $\Phi^X$ will look for the first 2 in $X$. When it appears, if $X(0) = 0$, then output 2; if $X(0) = 1$, then output 3. In other words,

$$\Phi^X(n) = \begin{cases} X(n) + X(0) & \text{if } n \text{ is the minimal } m \text{ such that } X(m) = 2, \\ X(n) & \text{otherwise} \end{cases}$$

Then $\Phi^X =^* X$ for all $X$, so $A =^* B$ implies $\Phi^A =^* \Phi^B$, but not uniformly.
Since homeomorphisms have finite use, $\pi$ is computable. Then we can recursively compute $\Phi$. Let $n$ tables that simply return the answer to "$\pi$?"$

\Sigma$ is the Turing functional for a continuous map $\Psi : 2^\omega \to 2^\omega$ associated with truth tables. That is,$$f(x) = a \implies \Theta^{-1} f \circ \Theta.$$Lemma 9. If $F(A) = A \circ f$ for a permutation $f : \omega \to \omega$ then $F$ induces a uniform map mod finite.

Proof. Let $a \in \omega$. Let $b = \min\{f(n) : n \geq a\}$. Suppose $X(m) = Y(m)$ for all $m \geq b$. Then $X \circ f(n) = Y \circ f(n)$ for all $n \geq a$. □

Lemma 10. Suppose $\Theta : 2^\omega \to 2^\omega$ is a homeomorphism such that for all computable $f : \omega \to \omega$, the function $\Theta^{-1} f \circ \Theta$ is computable. Then $\Theta$ is computable.

Proof. Let $\pi_n : \omega \to \omega$ be the constant $n$ function. For $\pi^*_n : 2^\omega \to 2^\omega$, note that $\pi^*_n(A)(u) = (A \circ \pi_n)(u) = A(n)$, so $\pi^*_n(A) \in \{0^\omega, 1^\omega\}$. Let $f(n) = n + 1$. Note that

$$(\pi^*_n \circ f^*)(A)(u) = \pi^*_n(f^*(A))(u) = (f^*(A)) \circ \pi_n(u) = (f^*(A))(n)$$

so $\pi^*_n \circ f^* = \pi^*_{n+1}$. Also, $f \circ \pi_n = \pi_{n+1}$. Let

$$\Phi = \Theta^{-1} \circ f^* \circ \Theta.$$By assumption, $\Phi$ is computable. Then

$$\Theta \circ \Phi = f^* \circ \Theta$$

$$\pi^*_n \circ \Theta \circ \Phi = \pi^*_n \circ f^* \circ \Theta$$

$$\pi^*_n \circ \Theta \circ \Phi = \pi^*_{n+1} \circ \Theta$$

$$(\pi^*_n \circ \Theta) \circ \Phi = \pi^*_{n+1} \circ \Theta.$$Since homeomorphisms have finite use, $\pi^*_n \circ \Theta$ is just a finite amount of information, and so we can recursively compute $\pi^*_{n+1} \circ \Theta$ this way. □

Remark 11. In terms of Odifreddi’s notation where $\sigma_n$ is the $n$th truth table, $\Theta(A) = \{n : A \models \sigma_{\phi(n)}\}$ for some (not a priori computable) $t$, and $\pi^*_0 \circ \Theta(A) = \{n : A \models \sigma_{\phi(0)}\} \in \{\emptyset, \omega\}$. For a continuous map $\Psi : 2^\omega \to 2^\omega$, let the lower-case version $\psi : \omega \to \omega$ pick out the associated truth tables. That is,$$
\Psi(A) = \{n : A \models \sigma_{\phi(n)}\}.$$Let $\Phi(A) = \{n : A \models \sigma_{\phi(n)}\}$. Let $\Sigma(A) = \{n : A \models \sigma_n\}$. Then $\Phi(A) = \phi^*(\Sigma(A))$, since

$$\Phi(A)(n) = 1 \iff A \models \sigma_{\phi(n)},$$

and

$$\phi^*(\Sigma(A))(n) = 1 \iff \Sigma(A)(\phi(n)) = 1 \iff A \models \sigma_{\phi(n)}.$$The Turing functional $\Sigma$ is left-invertible as we can effectively pick out a list of truth tables that simply return the answer to “$n \in A$?” So we have $\Sigma^{-1}(A) = A$. However, $\Sigma$ is not onto (not every list of answers is a coherent list of answers to truth table
We may express this in oracle notation with $\Xi A_n = \pi^* n (A)$, $\Xi A_n (u) = A (n)$, as

$$\Xi^{\varnothing^1}_{n+1} = \Xi^{\varnothing^0_{n+1}}$$

$$\Xi^{\varnothing^1}_{n+1} (u) = \Xi^{\varnothing^0_{n+1}} (u)$$

$$\Theta^A (n + 1) = \Theta^A (n)$$

**Theorem 12.** Let $F$ be a homeomorphism of $2^n$ which is induced by a uniform map mod finite, $\Theta$. Then $F$ is computable.

**Proof.** The proof follows that of Theorem 6 with Lemma 10 playing the role of Lemma 3. Our new assumption of uniformity mod finite makes the proof of a lemma corresponding to Lemma 5 go through. □

**Example 13 (The inductive procedure in Lemma 10).** Suppose $\Phi$ is the truth table reduction given by $\Phi^A (n) = A (2n) \cdot A (2n + 1)$. Suppose $\Theta^A (0) = A (2) \rightarrow A (3)$. Then

$$\Theta^A (1) = \Theta^\Phi (0) = \Phi^A (2) \rightarrow \Phi^A (3)$$

$$= A (4) A (5) \rightarrow A (6) A (7).$$

Next,

$$\Theta^A (2) = \Theta^\Phi (1)$$

$$= \Phi^A (4) \Phi^A (5) \rightarrow \Phi^A (6) \Phi^A (7)$$

$$= A (8) A (9) A (10) A (11) \rightarrow A (12) A (13) A (14) A (15).$$

**Example 14 (Another example of Lemma 10).** Let $\Theta$ be given by

$$\langle \Theta^A (2n), \Theta^A (2n + 1) \rangle = \begin{cases} 
(1, 1) & \text{if } \langle A (2n), A (2n + 1) \rangle = \langle 0, 0 \rangle, \\
(0, 1) & \text{if } \langle A (2n), A (2n + 1) \rangle = \langle 0, 1 \rangle, \\
(1, 0) & \text{if } \langle A (2n), A (2n + 1) \rangle = \langle 1, 0 \rangle, \\
(0, 0) & \text{if } \langle A (2n), A (2n + 1) \rangle = \langle 1, 1 \rangle, \\
\langle \neg A (2n + 1), \neg A (2n) \rangle & \end{cases}$$
which satisfies $\Theta = \Theta^{-1}$. Then $\Phi = \Theta^{-1} \circ f^* \circ \Theta$ applied to the Thue-Morse sequence $t$ is given by:

$$t = 0110\ 1001\ 1001\ 0110\ldots$$

$$\rightarrow_{\Theta} = 0110\ 1001\ 1001\ 0110\ldots$$

$$\rightarrow_{f^*} = 1101\ 0011\ 0010\ 1101\ldots$$

$$\rightarrow_{\Theta^{-1}} = 0001\ 1100\ 1110\ 0001\ldots$$

Now $\Theta^A(0) = \neg A(1)$. And $\Theta^A(1) = \neg \Phi^A(1)$. In particular:

$$\Theta^A(0) = \neg t(1) = 0,$$

$$\Theta^A(1) = \Phi^A(0) = \neg \Phi^A(1) = 1,$$

$$\Theta^A(2) = \Phi^A(1) = \neg \Phi^A(0) = 1$$

We computed this using $\Phi$ above but we see that it is also correct for $\Theta$.

Actually $\Theta^A(n) = \neg A(n + (-1)^n \mod 2)$ so $f^*(\Theta^A(n)) = \neg A(n + 1 + (-1)^{n+1 \mod 2})$ and

$$\Theta^{-1}(f^*(\Theta^A))(n) = A\left([n + (-1)^n \mod 2] + 1 + (-1)^{[n + (-1)^n \mod 2] + 1 \mod 2}\right)$$

To simplify this, note that

$$[n + (-1)^n \mod 2] + 1 \mod 2 = n \mod 2$$

so

$$\Theta^{-1}(f^*(\Theta^A))(n) = A\left([n + (-1)^n \mod 2] + 1 + (-1)^n \mod 2\right)$$

$$= A(n + 3)[n \text{ even}], \quad A(n - 1)[n \text{ odd}]$$

and $\Theta^A(n)$ is the negation of $A(n + 1)[n \text{ even}], A(n - 1)[n \text{ odd}]$. And then the claim is that $\Theta^A(n + 1)$ is $\Theta^A(n)$.

**Example 15.** Let $\Theta(A) = A + K \mod 2$, where $K$ is your favorite non-computable set. Then $\Theta^{-1}(f^*(\Theta(A)))$ is the sequence $A(1) + K(1) + K(0), A(2) + K(2) + K(1),\ldots$ If this were a computable operator (sequence of truth tables) then so is the sequence $K(1) + K(0), K(2) + K(1),\ldots$ But then $K$ would be computable: start by knowing $K(0)$. Then $K(1) = (K(1) + K(0)) + K(0)$, and so on inductively.

**Remark 16.** Woodin mentioned on June 6, 2019 that he and Slaman may have shown, in unpublished work from the 1990s, that each automorphism of $\mathcal{D}_a$, the degrees of arithmetical reducibility, is represented by a continuous function outright. This gives some extra interest in a possible future $\mathcal{D}_a$ version of our results.


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