Prefix-Free Coding for LQG Control

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Abstract

In this work, we develop quantization and variable-length source codecs for the feedback links in linear–quadratic–Gaussian (LQG) control systems. We prove that for any fixed control performance, the approaches we propose nearly achieve lower bounds on communication cost that have been established in prior work. In particular, we refine the analysis of a classical achievability approach with an eye towards more practical details. Notably, in the prior literature the source codecs used to demonstrate the (near) achievability of these lower bounds are often implicitly assumed to be time-varying. For single-input single-output (SISO) plants, we prove that it suffices to consider time-invariant quantization and source coding. This result follows from analyzing the long-term stochastic behavior of the system’s quantized measurements and reconstruction errors. To our knowledge, this time-invariant achievability result is the first in the literature.

Index Terms

Control systems, control with communication constraints, LQG, source coding..

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I. INTRODUCTION

In this work we consider LQG control over communication networks. Our motivation is a scenario where measurements from a remote sensor platform are conveyed wirelessly to a controller. In such a system, the bitrate of the feedback channel can be tied directly to the amount of physical layer resources (e.g., time, bandwidth, and power) that must be allocated to attain satisfactory control performance. Such resources are inherently scarce. This motivates approaches to control that minimize communication overhead; potentially enabling, for example, automated factories where many agents share the communication medium [1].

We attack this problem via data compression; we develop quantizers and variable-length codecs for the LQG feedback link. We consider a setup where at each discrete timestep an encoder, co-located with a sensor that can fully observe the plant, conveys a variable-length packet of bits to a decoder co-located with the controller. We introduce various prefix constraints that can be imposed on the packets. Such constraints allow the decoder, and possibly other users sharing the same communication medium, to uniquely identify the end of the encoder’s transmission and can enable efficient resource sharing. The packet bitrate provides a notion of communication cost. We prove that for a fixed control performance, the approaches we propose nearly achieve known lower bounds on the minimum achievable bitrate. This bitrate is interpreted as the minimum rate of reliable feedback communication necessary to achieve the control objective.

We first overview the contributions of this manuscript which contextualizes the subsequent literature review.

A. Our Contributions

Our main contributions are summarized as follows:

1) For multiple-input multiple output (MIMO) plants, we demonstrate that a system consisting of a dithered element-wise uniform quantizer followed by a time-varying Shannon–Fano–Elias (SFE) lossless source codec achieves a time-average expected bitrate within a few
bits of previously established directed information lower bounds. We generalize the upper bounds on expected codeword length from [2] to include the practically relevant case when the SFE codec ignores the dither signal. We also demonstrate that the space-filling loss in [2] can be reduced by $1/2$ of a bit per plant dimension.

2) For single-input SISO plants, we use tools from ergodic theory to prove that a bitrate equivalent to the approach in [2] is achievable with a time-invariant quantization and prefix-free codec. Nominally, neither the quantizer, nor the mapping from quantized measurements to codewords is time varying. To our knowledge, this is the first such result in the literature.

B. Literature Review

This work considers minimum bitrate LQG control via dithered uniform quantization and variable length coding. An early paper to consider stabilizing a linear system with uniformly quantized feedback measurements was [3]. Considering a deterministic system, [3] analyzed the long-term behavior of the chaotic dynamics of the state vector using results from ergodic theory. The problem of stabilizing a Gauss–Markov plant over a feedback channel with a random, time-varying rate was considered in [4]. In the scalar case, a necessary and sufficient condition (e.g., a quantizer and codec design) for stabilization was derived. In contrast, we consider the problem of attaining a fixed control cost with variable-length coding (the number of bits to be transmitted at each time is chosen by the encoder, not by nature) uniform random variables shared between the encoder and decoder to effectively whiten the reconstruction error [7]. At every timestep, [5]. This work follows from a model for LQG control with minimum rate variable-length feedback from [5]. Considering scalar plants, [5] lower bounded the time-average expected bitrate of a prefix-free source codec inserted into an LQG feedback channel in terms of Massey’s directed information (DI) [6]. This motivated a rate-distortion problem for the tradeoff between (a lower bound on) communication cost, quantified by the DI, and LQG control performance. Using entropy-coded dithered quantization (ECDQ), [5] showed that the DI lower bound was nearly
achievable. ECDQ uses uniform quantizers and a sequence of independent, identically distributed (IID) assumed that the quantized measurements were encoded into an SFE style prefix-free code; in particular, the codeword length analysis assumed that the SFE codec was designed optimally at each timestep and used the side information (SI) provided by the shared dither sequence. This followed from analysis in [7]. While using the SI allows for a reduction in codeword length (replacing an unconditional entropy with a conditioned one), it generally introduces practical difficulties; requiring that the codebook be adapted contingent on the realization of the dither.

The quantizer and source codec designs we propose follow from analyzing a DI/LQG cost rate-distortion function. While the proof of the DI-based bitrate lower bound in [5] purported to apply to systems using dithering, an error was discovered in [8]. For slightly different problem formulations, revised proofs in [8] (see also [9]) and [10] established that the DI lower bound on time average bitrate holds even when the encoder and decoder share randomness. The rate-distortion formulation in [5] was extended to MIMO plants in [11]. In particular, [11] analyzed the optimization over a randomized encoder and decoder policy space. This lead to a formulation of an optimal test channel consisting of an “encoder” that conveys a linear/Gaussian plant measurement to a “decoder/controller” consisting of a Kalman filter and certainty equivalent controller. The minimal DI attainable for any limit on LQG control performance was shown to be a convex log-determinant optimization. In [12], via [13], the DI lower bound for prefix-free codes was extended to the more general class of uniquely decodable codes. Analytical lower bounds on the DI cost as a function of control performance were also derived. The lower bounds in [12] are applicable to plants with non-Gaussian process noise. A rate-distortion lower bound on the bitrate required to asymptotically estimate the state of an uncontrolled system is considered in [14] and [15]. The rate-distortion is solved via dynamic programming and reverse waterfilling.

While quantization/coding approaches in [2], [12], and [15] can be shown to nearly achieve respective lower bounds for MIMO systems, they generally rely on time-varying lossless source codecs. In [2], the near minimum bitrate approach from [5] was extended to MIMO plants. In [2],
linear measurements, dithered element-wise uniform quantization, Kalman filters, and certainty equivalent control are used to develop a system with discrete feedback from plant to controller but where all other system variables have identical means and covariances to those in [11]'s optimal test channel. This ensures that the LQG performance is equivalent to that in the test channel, and leads to an asymptotic bound on the conditional entropy of the quantizer output (given the dither) within a few bits of the DI lower bound. This result proved that conveying the quantized measurements from the encoder to the decoder via a time-varying SFE codec that accounts for the dither asymptotically achieves a time-average bitrate near the lower bound. An achievability approach not relying on dithered quantization was provided in [12]. The approach in [12] uses lattice quantization and entropy coding. In particular, using a bound on the output entropy of a lattice quantizer from [16], [12] demonstrates that the entropy of quantized innovations is close to a corresponding lower bound in the high rate(strict control cost) regime.

In the achievability approaches in [5], [2] and [12], the upper bounds on rate are developed in terms of the output entropy of a quantizer. While a lossless codec can be used to encode the quantizations into a variable-length binary string without delay and with an expected length close to this entropy, this generally requires the codec to be adapted, at every timestep, to the probability distribution of the quantizer output. This complication is compounded in [5] and [2], as the source codec must be adapted to the conditional probability distribution of the quantizer output given the dither. Even if the random variables describing the quantizer output converge in some sense, a codec adapted to the limiting distribution will not necessarily yield a time-average bitrate close to known lower bounds. In sources with countably infinite alphabets (like the quantizers employed in [2] and [12]) weak convergence does not imply convergence in the sense of Kullback-Leibler (KL) divergence. Convergence in the KL sense is required to ensure no asymptotic redundancy (see e.g., [17]).

Some work on fixed-rate feedback control is also relevant [18] [19]. In [18], fixed-rate quantizers were designed to minimize control cost. Using a Lloyd-Max style quantizer design,
an optimal greedy control policy was developed that demonstrated competitive performance \cite{18}. Stabilization via an adaptive fixed-rate quantizer was considered in \cite{19}. Using tools from ergodic theory, \cite{19} analyzed the long-term behavior of the state and quantizer parameters and proved the existence of limiting distributions. It was shown that for a particular quantizer, finite control cost was achieved. In the work presented here, we will use similar ergodicity results to prove time-invariant achievability results for variable-length coding under a control cost constraint.

In this work, we refine the analysis on the dithered quantizer output entropy from \cite{2}. While we still employ dithered quantization, we develop a bound on the unconditioned output entropy of the quantizer. This provides an analysis of a far simpler approach to lossless entropy coding. As in \cite{3}, we use ergodic theory to analyze the long-term behavior of the system, albeit with a stochastic plant and an eye to optimize control and communication costs. We demonstrate that, for SISO systems, it is sufficient to consider time-invariant entropy codecs to encode the quantizer output. We use results from \cite{20} to prove the existence of an invariant measure for the Markov chain describing the quantizer input, and then use theorems from \cite{21} to verify that the chain converges to the invariant measure and has an ergodic property. This gives a guarantee on the time-average codeword length of a zero-delay SFE–style source codec designed assuming the quantizer output is induced by the invariant measure. Finally, we prove that KL divergence between the true quantizer output and the output induced by the invariant measure tends to zero. This proves a guarantee on the time average expected codeword length. We show that the time-invariant approach satisfies a stronger prefix constraint with respect to prior approaches.

**Notation and Organization:** Constant scalars and vectors are denoted by lower-case letters $x$. Matrices are denoted by capital letters $X$ and the $m \times m$ identity matrix by $I_{m \times m}$. Random scalars or vectors are written in boldface $\mathbf{x}$. If $\mathbf{x}$ is a vector, $[\mathbf{x}]_i$ denotes its $i^{th}$ element. We write $\mathbf{a} \perp \mathbf{b}$ to denote that $\mathbf{a}$ and $\mathbf{b}$ are independent. We write $\mathbf{a} \overset{a.s.}{=} \mathbf{b}$ if $P_{\mathbf{a},\mathbf{b}}[\mathbf{a} = \mathbf{b}] = 1$, and define $\overset{a.s.}{\geq} \mathbf{b}$, $\overset{a.s.}{<} \mathbf{b}$, etc. analogously. Denote the set of finite-length binary strings $\{0, 1\}^*$. For time domain sequences, let $\{x_t\}$ denote $(x_0, x_1, \ldots)$, $x^b_a = (x_a, \ldots, x_b)$ if $b \geq a$, and $x^b_a = \emptyset$
otherwise. We let \( x^b = x^b_0 \). For a topological space \( S \), let \( \mathcal{B}(S) \) denote the standard Borel \( \sigma \)-algebra of \( S \). For Euclidean spaces, let \( \lambda \) denote the Lebesgue measure (e.g., if \( S = \mathbb{R}^n \), then for \( B \in \mathcal{B}(S) \), \( \lambda(B) \) is the volume of \( B \) in \( \mathbb{R}^n \)).

In Section II we formulate the problem of LQG control with minimum rate prefix-free coding in the feedback link. Section III restates the rate-distortion formulation and overviews the optimal test channel from [11]. Our main results are in Section IV. We begin by overviewing the achievability approach and its key ingredients in Section IV-A. Our time-varying availability approach for MIMO plants is discussed in Section IV-B. Our time-invariant approach for scalar plants is given in Section IV-C. Some proofs from Section IV-C are relegated to Appendix A in the online supplementary material. We conclude in Section V.

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider the system model depicted in Fig. 1. We consider a time-invariant MIMO plant controlled via a feedback model where communication occurs over an ideal (delay and error free) binary channel. The plant is fully observable to an encoder/sensor block, which conveys a variable-length binary codeword \( a_t \in \{0, 1\}^* \) over the channel to a combined decoder/controller. Upon receipt of the codeword, the decoder/controller designs the control input. Denote the state vector \( x_t \in \mathbb{R}^m \), the control input \( u_t \in \mathbb{R}^u \), and let \( w_t \sim \mathcal{N}(0, W) \) denote processes noise assumed to be IID over time. We assume \( W \succ 0_{m \times m} \), i.e., the process noise covariance is full rank. We assume assume that \( x_0 \sim \mathcal{N}(0, X_0) \) for some \( X_0 \succeq 0 \). For \( A \in \mathbb{R}^{m \times m} \) the system matrix and \( B \in \mathbb{R}^{m \times u} \) the feedback gain matrix, for \( t \geq 0 \) the plant dynamics are given by

\[
x_{t+1} = Ax_t + Bu_t + w_t.
\]

To ensure finite control cost is attainable, we assume \((A, B)\) are stabilizable.

For generality, we assume that the encoder/sensor and the decoder/controller may be randomized. In Fig. 1 we assume that the encoder/sensor and decoder/controller share access to
a common random *dither signal*, \{d_t\}. The dither is assumed to be IID over time. In real-world systems, this *shared randomness* can be effectively accomplished using two synchronized pseudorandom number generators at the encoder and decoder. The encoder/sensor policy in Fig. 1 is a sequence of causally conditioned Borel measurable kernels denoted

\[ P_E[a_0^\infty|d_0^\infty, x_0^\infty] = \{ P_{E,t}[a_t|a_{t-1}, d_0^t, x_0^t] \}_t. \]  

(2)

The corresponding decoder/controller policy is given by

\[ P_C[u_0^\infty|a_0^\infty, d_0^\infty] = \{ P_{C,t}[u_t|a_0^t, d_0^t, u_0^{t-1}] \}_t. \]  

(3)

Note that under the dynamics (1), \( x_0^t \) is a deterministic function of \( x_0 \), \( a_{t-1}^t \), \( u_{t-1}^t \), and \( w_{t-1}^t \). We enforce conditional independence assumptions in the system model by a factorization of the one-step transition kernels for \( a_t, d_t, u_t, \) and \( w_t \). These are discussed in Fig. 1. For \( t \geq 0 \), we assume the kernels factorize via

\[
P[a_{t+1}, d_{t+1}, u_{t+1}, w_{t+1}|a_t^t, d_0^t, u_0^t, w_0^t, x_0] = P_{E,t+1}[a_{t+1}|a_0^t, d_0^{t+1}, x_0^t] P_{C,t+1}[u_{t+1}|a_{t+1}, d_0^{t+1}, u_0^t] P[d_{t+1}] P[w_{t+1}],
\]  

(4)

and that we have \( P[a_0, d_0, u_0, w_0|x_0] = P[d_0] P_{E,0}[a_0|x_0, d_0] P_{C,0}[u_0|a_0, d_0] P[w_0]. \)

![Diagram](image-url)

**Fig. 1.** The system model with dithering. The encoder policy allows the codeword \( a_t \) to be generated randomly given “all the information known to the encoder at time \( t \)”. When \( a_t \) arrives at the decoder, the decoder can randomly generate its control input given \( a_t \) as well as its previous knowledge. Notably, both the encoder and decoder share access to \( d_t \), an IID sequence generated “independently” of all past system variables.

The length of the codewords \( \{a_t\} \) quantifies the communication cost. For a codeword \( a_t \in \{0, 1\}^* \), denote its length in bits by \( \ell(a_t) \). The problem of interest is to minimize the time-average
expected bitrate subject to a constraint on control performance, quantified via the standard LQG cost. We will impose prefix constraints on the codewords \( \{a_t\} \). These constraints will allow the decoder (and possibly other agents sharing the same communication medium) to uniquely identify the end of the transmission from the encoder. Three possible prefix constraints that can be imposed on the system are the following:

**Prefix Constraint 1.** For any realizations \((a_{t-1}^0 = a_{t-2}^0, d_{t-1}^0, u_{t-1}^0 = u_{t-2}^0)\), for all distinct \(a_1, a_2 \in \{0, 1\}^* \) with \(P_{a_t | a_{t-1}^0, d_{t-1}^0, u_{t-1}^0}[a_t = a_1 | a_{t-1}^0 = a_0^{t-1}, d_0^t, u_0^{t-1} = u_0^{t-1}] > 0\) and \(P_{a_t | a_{t-1}^0, d_{t-1}^0, u_{t-1}^0}[a_t = a_2 | a_{t-1}^0 = a_0^{t-1}, d_0^t, u_0^{t-1} = u_0^{t-1}] > 0\), \(a_1\) is not a prefix of \(a_2\).

**Prefix Constraint 2.** For all distinct \(a_1, a_2 \in \{0, 1\}^* \) with \(P_{a_t | a_{t-1}^0}[a_t = a_1] > 0\) and \(P_{a_t | a_{t-1}^0}[a_t = a_2] > 0\), \(a_1\) is not a prefix of \(a_2\).

**Prefix Constraint 3.** For all \(i, j\) and distinct \(a_1, a_2 \in \{0, 1\}^* \) with \(P_{a_i | a_{i-1}^0}[a_i = a_1] > 0\) and \(P_{a_j | a_{j-1}^0}[a_j = a_2] > 0\), \(a_1\) is not a prefix of \(a_2\).

Prefix Constraints 1 and 2 were defined in [10]. In a sense Constraint 1 is the least strict. Constraint 1 allows any agent with knowledge of the information possessed by the decoder at time \(t\) to uniquely identify the end of the encoder’s transmission at time \(t\). A downside, however, is that this information may actually be necessary to determine the end of the codeword. This complicates the system architecture and may inhibit other agents from recognizing the end of the transmission. Constraint 2 is notionally stricter; nominally, it guarantees that any agent who knows the codebook used by the encoder at time \(t\) (precisely, the set \(\{b \in \{0, 1\}^* : \exists t \text{ s.t. } P_{a_t | a_{t-1}^0}[a_t = b] > 0\}\)) can uniquely identify the end of the transmission. Under Constraint 2 agents on the same network can identify the end of the transmission without knowing \((a_{t-1}^0, d_{t-1}^0, u_{t-1}^0)\). Constraint 3 is time-invariant. Any approach conforming to Constraint 3 certainly conforms to Constraint 2. Constraint 3 requires that the prefix condition holds across time, ensuring that any codeword at time \(t\) is not a prefix of any codeword used at time \(t + m\). Any user with knowledge of the set of binary strings \(\{b \in \{0, 1\}^* : \exists t \text{ s.t. } P_{a_t | a_{t-1}^0}[a_t = b] > 0\}\) can uniquely identify the
end of the transmission at any time $t$. Notably, to identify the end of the transmission, a user need not know the codebook used at time $t$, but only the strings lying in the union of codebooks across time. Note that Constraint 3 is satisfied if the same prefix-free code is used for all $t$.

We are interested in the optimization, for codewords conforming to Prefix Constraints 1–3:

$$\inf_{P,\Omega,\Pi} \frac{1}{T+1} \sum_{t=0}^{T} E[\ell(a_t)]$$

subject to

$$\limsup_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} E[\|x_{t+1}\|_Q^2 + \|u_t\|_\Phi^2] \leq \gamma,$$

where $Q \succeq 0$, $\Phi \succ 0$, and $\gamma$ is the maximum tolerable LQG cost. The minimization is over admissible sensor/encoder and decoder/controller policies described by (2) and (3). In Section III, we discuss a lower bound on (5) that applies to all encoder and decoder policies conforming to (4) and any of the Prefix Constraints 1–3. These bounds follow from [10] and [11].

III. RATE DISTORTION LOWER BOUND

We summarize the relevant results from [10] and [11] into the following theorem.

**Theorem III.1.** Let the minimum communication cost attained by the optimization in (5) for an LQG cost constraint $\gamma$ be denoted $\mathcal{L}(\gamma)$. Let $S$ be a stabilizing solution to the discrete algebraic Riccati equation (DARE) $A^TSA - S - A^TSB(B^TSA + \Phi)^{-1}B^TSA + Q = 0$, $K = -(B^TSA + \Phi)^{-1}B^TSA$, and $\Theta = K^T(B^TSA + \Phi)K$. Define the convex log-det optimization

$$\mathcal{R}(\gamma) = \left\{ \begin{array}{l}
\inf_{P,\Pi,\Omega \in \mathbb{R}^n \times \mathbb{R}^n} \frac{1}{2} \left( - \log_2 \det \Pi + \log_2 \det \Omega \right) \\
\text{s.t. } \text{Tr}(\Theta P) + \text{Tr}(WS) \leq \gamma, \ P \preceq APA^T + W. \end{array} \right. \ (6)$$

For a system model conforming to that of Fig. 1 and (4), if the codewords $\{a_t\}$ satisfy any of the Prefix Constraints 1–3 then we have $\mathcal{L}(\gamma) \geq \mathcal{R}(\gamma)$. 
The proof of Theorem III.1 follows immediately from [10] and [11] and recognizing that Constraint [3] is more stringent than Constraint [2]. The interpretation of the optimization in (6) is aided by the three-stage test channel illustrated in Fig. 2. The test channel consists of an “encoder” that conveys a linear/Gaussian plant measurement to a “decoder”/controller. The decoder consists of a Kalman filter to track the state, followed by a standard certainty equivalent linear feedback controller. Denote the minimizing $P$ from (6) by $\hat{P}$. Let $C \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{m \times m}$ $V \succ 0$ be any such matrices that satisfy

$$\hat{P}^{-1} = (A\hat{P}A^T + W)^{-1} - C^T V^{-1} C = 0.$$  

(7)

The decoder receives the measurement

$$y_t = Cx_t + v_t \text{ where } v_t \sim \mathcal{N}(0, V) \text{ IID}, v_t \perp x_0.$$  

(8)

Let $\hat{P}_+ = A\hat{P}A^T + W$ and let $J = \hat{P}_+ C^T (C\hat{P}_+ C^T + V)^{-1}$. Denote the filter’s sequence of prior and posterior state estimates as $\{\bar{x}_{t|t-1}\}$ and $\{\bar{x}_{t|t}\}$. Let $\bar{x}_{0|-1} = 0$. The filtering recursion is

$$\bar{x}_{t|t} = \bar{x}_{t|t-1} + J(y_t - C\bar{x}_{t|t-1}),$$  

(9a)

$$\bar{x}_{t|t-1} = A\bar{x}_{t-1|t-1} + Bu_{t-1}.$$  

(9b)

Define the prior and posterior error processes and their respective covariances via $\bar{e}_{t|t-1} = x_t - \bar{x}_{t|t-1}$, $\bar{P}_{t|t-1} = \mathbb{E}[\bar{e}_{t|t-1}\overline{\bar{e}_{t|t-1}^T}]$ and $\bar{e}_{t|t} = x_t - \bar{x}_{t|t}$, $\bar{P}_{t|t} = \mathbb{E}[\bar{e}_{t|t}\overline{\bar{e}_{t|t}^T}]$. Note that for all $t \geq 0$,
\[ E[\xi_{t|t-1}] = 0 \text{ and } E[\xi_{t|t}] = 0. \] A discrete Lyapunov equation can be used to establish that for any \( C \) satisfying (7), \( (A, C) \) is detectable. Since \( W > 0 \), \( (A, W^{1/2}) \) is stabilizable. This implies that

\[
\lim_{t \to \infty} \mathcal{P}_{t|t-1} = \hat{P}_+ \text{ and } \lim_{t \to \infty} \mathcal{P}_{t|t} = \hat{P}.
\] (10)

Recall \( K = -(B^T S B + R)^{-1} B^T S A \). The control input at time \( t \) given by \( u_t = K \hat{x}_{t|t} \). It can be shown (see [11]) that, in the architecture of Fig. 2 the control cost satisfies

\[
\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} E[\|x_{t+1}\|^2_Q + \|u_t\|^2_R] = \text{Tr}(SW) + \text{Tr}(\hat{P}\Theta) \leq \gamma,
\] (11)

where (12) follows as \( \hat{P} \) is a feasible solution of (6). The minimum of (6) is given by (see [11])

\[
R(\gamma) = \frac{1}{2} \log_2 \frac{\det \hat{P}_+}{\det \hat{P}}.
\] (13)

The minimum is unique for any minimizing \( C \) and \( V \). We reiterate that (13) lower bounds the communication cost achievable in the (original) architecture in Fig. 1. We will use (11), (13), and the test channel in that analysis of system we use to demonstrate achievability.

IV. UPPER BOUNDS (ACHIEVABILITY)

In this section, we present theoretical results demonstrating that, assuming access to a uniform dither signal in the architecture of Fig. 1 a uniform quantizer and time-varying source code that conforms to Constraint 1 achieves a bitrate that is nearly equal to the minimal DI lower bound obtained from the optimization (6). These results are a refinement of those from [2]. We derive the expression to upper bound the increase in communication cost incurred by enforcing Constraint 2 on the codebooks from [2]. For scalar systems, we present novel time-invariant achievability results, including an approach whose codebook conforms to Constraint 3. While these results require the use of a dither signal, the bounds on communication cost developed are shown to hold under this greatly simplified system architecture.
Fig. 3 illustrates an overview of the framework we will use to demonstrate achievability in this section. The approach conforms to the architecture in Fig. 1 with the dither signal chosen as an IID sequence of element-wise mutually independent uniform random vectors. Three key components of the framework in Fig. 3 include time-invariant Kalman filters, uniform scalar quantizers with subtractive dither, and SFE entropy source codecs. In the following subsection, we describe the latter two ingredients in the detail necessary to proceed with our analysis.

Fig. 3. The achievability architecture. The dither sequence \( \{d_t\} \) are shared random vectors that are IID uniform on \([-\Delta/2, \Delta/2]\). It may be used by the entropy codec, but need not be (we consider both cases).

A. Key ingredients

1) Lossless entropy source codecs [22]: In this section we briefly outline some classical SFE approaches to prefix-free source coding. Let \( q \) denote a discrete random variable with (countable) range \( \mathbb{A} \). Assume, without loss of generality, that \( \mathbb{A} = \mathbb{N} \) (if the alphabet is countably infinite) or \( \mathbb{A} = \{0, 1, \ldots, r\} \), and that \( \mathbb{P}_q[q] > 0 \ \forall \ q \in \mathbb{A} \). Consider the problem of encoding \( q \) into a prefix-free codeword such that it can be recovered at a decoder. We consider first a notion of “prefix-free” that conforms with Constraint 2 from Section II

Define the function \( \overline{F}_q: \mathbb{A} \to (0, 1) \) based on the cumulative distribution function of \( q \)

\[
\overline{F}_q(q) = \mathbb{P}_q[q < q] + \frac{\mathbb{P}_q[q]}{2}. \tag{14}
\]
Define the “Fano–type” SFE code $C^F: \mathcal{A} \rightarrow \{0, 1\}^*$ as the function

$$C^F_q(q) = \text{the binary expansion of } F_q(q) \text{ truncated to } \lceil -\log_2(\mathbb{P}_q[q]) \rceil + 1 \text{ bits}. \quad (15)$$

It can be shown that for distinct $q_1, q_2 \in \mathcal{A}$, we have that $a_1 = C^F(q_1)$ is not a prefix of $a_2 = C^F(q_2)$ and vice versa [22, Chapter 5.9]. We have that

$$H(q) \leq \mathbb{E}_q[C^F_q(q)] \leq H(q) + 2. \quad (16)$$

It turns out that the overhead can be reduced by one bit if the probability mass function (PMF) of $q$ is “sorted” such that if $q_1 < q_2$ then $\mathbb{P}_q[q_1] \geq \mathbb{P}_q[q_2]$. Define the function $F_q: \mathcal{A} \rightarrow [0, 1)$ by $F_q(q) = \mathbb{P}_q[q < q]$, and define the “Shannon–type” code $C^S: \mathcal{A} \rightarrow \{0, 1\}^*$ by

$$C^S_q(q) = \text{the binary expansion of } F_q(q) \text{ truncated to } \lceil -\log_2(\mathbb{P}_q[q]) \rceil \text{ bits}. \quad (17)$$

It can be shown that for distinct $q_1, q_2 \in \mathcal{A}$, we have that $a_1 = C^S(q_1)$ is not a prefix $a_2 = C^S(q_2)$ and vice versa. The expected codeword length satisfies (see [22, Problem 5.28]):

$$H(q) \leq \mathbb{E}_q[C^S_q(q)] \leq H(q) + 1. \quad (18)$$

Even if $\mathcal{A} = \mathbb{N}$ (the PMF has countably infinite support) there always exists a bijection $s: \mathcal{A} \rightarrow \mathcal{A}$ that “re-indexes” the support of $q$ so that $\mathbb{P}_q[s(0)] \geq \mathbb{P}_q[s(1)] \geq \mathbb{P}_q[s(2)] \ldots$. Defining the random variable $\mathbf{q} = s(q)$, $\mathbf{q}$ can be encoded per the recipe (17) (see [13]). Upon receipt of the codeword, the decoder recovers $\mathbf{q}$ exactly and computes $s^{-1}(\mathbf{q})$ to recover $q$. Even though the bijection $s$ always exists, it can be extremely difficult (and computationally unreasonable) to find.

We now consider a codebook construction that more closely mirrors Constraint [1] from Section II. Let $s$ be a random variable on some general alphabet, say $\mathcal{X}$, representing SI. Assume that $s$ is shared by both the encoder and decoder. Given a realization $s = s$, both the encoder and decoder can construct the codebooks in [15] or (17) according to the conditional PMF of $\mathbb{P}_q|s[q|s = s]$. 
For example, replace (14) with \( F_{q|s}(q|s) = \mathbb{P}_q(q < q|s = s) + \frac{F_{q|s}[q|s]}{2} \), and replace the encoding function (15) with
\[
C_{q|s}^F(q|s) = \text{the binary expansion of } F_{q|s}(q|s) \text{ truncated to } \left\lceil -\log_2(\mathbb{P}_{q|s}[q|s]) \right\rceil + 1 \text{ bits.}
\] (19)

For any realization \( s \), the code (19) achieves a codeword length satisfying
\[
H(q|s = s) \leq \mathbb{E}_{q|s=s}[C_{q|s}^F(q|s)|s = s] \leq H(q|s = s) + 2.
\] Taking the expectation over realizations gives
\[
H(q|s) \leq \mathbb{E}_{q,s}[C_{q|s}^F(q|s)] \leq H(q|s) + 2.
\] (20)

The code \( C_{q|s}^F(q|s) \) is prefix-free for any realization of \( s \), i.e., if \( s = s \) for any distinct \( q_1 \) and \( q_2 \) in \( \mathcal{A} \), we have \( C_{q|s}^F(q_1|s) \) is not a prefix of \( C_{q|s}^F(q_2|s) \) and vice versa. There is an analogous generalization of the “sorted” version of the SFE code to the case with SI, although the sorting function must, in general, be computed for every realization of the SI. As before, the “sorted” version of the SFE code with SI reduces the upper-bound in (20) by one bit. We denote the “sorted”, or Shannon-type, SFE codec for \( q \) given the SI \( s \) by \( C_{q|s}^{SI} \). Even without sorting, the construction (19) is much more complex than the one in (15); essentially, a different codebook is used for every realization of the SI. Note that the difference between the upper-bounds applying with and without SI (e.g., compare (20) and (16)) is exactly \( H(q) - H(q|s) = I(q; s) \), i.e., the “benefit” of using SI is \( I(q; s) \) bits. If this quantity is small, it may be preferable to “ignore” the SI in exchange for reduced complexity. A salient feature of SFE coding, and, prefix-free entropy source coding in general, is that the mapping from source symbols (and SI) to codewords depends on the (conditional) PMF of the source symbols (given the SI). A mismatch between the PMF used to design an SFE codebook and the true underlying (conditional) PMF of the source symbols (given the SI) results in an increase in expected codeword length.

We conclude with a discussion of how these approaches can apply to the compression of stochastic processes. Define the discrete-valued, countably supported stochastic process \( \{q_t\} \) on the range \( \mathcal{A} \). Likewise, define the stochastic process \( \{s_t\} \) with a range on \( \mathcal{X} \). Assume that the processes are not identically distributed. We consider causal prefix-free encoding of \( \{q_t\} \) possibly
with the aid of the SI \(\{s_t\}\). We first consider *time-varying codebooks*. At every time \(t\), assume that after observing \(s_t\), the encoder encodes \(q_t\) via using as side-information SFE code like (19) (or its sorted counterpart) designed via the conditional PMF \(\mathbb{P}_{q_t|s_t}\). In other words, assume that either \(a_t = C_{q_t|s_t}^F(q_t|s_t)\) or \(a_t = C_{q_t|s_t}^S(q_t|s_t)\). At every time \(t\), the codeword length \(\ell_t = \ell(a_t)\) is a random variable. Given \(s_t\), the codewords are prefix-free and conform to Constraint 1. The time average expected codeword length satisfies

\[
\frac{1}{T+1} \sum_{i=0}^{T} H(q_t|s_t) \leq \frac{1}{T+1} \sum_{i=0}^{T} E[\ell_t] \leq \frac{1}{T+1} \sum_{i=0}^{T} H(q_t|s_t) + r,
\]

where \(r\) is either 1 or 2 bits depending on whether the “sorted” \((C_{q_t|s_t}^S)\) or “unsorted” \((C_{q_t|s_t}^F)\) version of SFE coding is used. Assume that at every time \(t\) the encoder ignores the SI, instead encoding \(q_t\) using a codebook like (15) or (17) designed via the marginal PMF \(\mathbb{P}_{q_t}\) (i.e., either \(a_t = C_{q_t}^F(q_t)\) or \(a_t = C_{q_t}^S(q_t)\)). The time-average communication cost satisfies

\[
\frac{1}{T+1} \sum_{i=0}^{T} H(q_t) \leq \frac{1}{T+1} \sum_{i=0}^{T} E[\ell_t] \leq \frac{1}{T+1} \sum_{i=0}^{T} H(q_t) + r,
\]

where \(r\) is either 1 or 2 bits. In this case Prefix Constraint 2 is satisfied. These approaches use time-varying codebooks; the mappings from source realizations to codewords (e.g., \(C_{q_t|s_t}^F, C_{q_t|s_t}^S\)) must be redesigned at every time \(t\) according to the (conditional) PMF of \(q_t\) (given \(s_t\)).

We now consider time-invariant lossless coding. Assume that an SFE codebook like (15) or (17) is designed for the source PMF \(\mathbb{P}_{q_t}\) (e.g., another PMF defined on \(\mathcal{A}\)) and that the same codebook is used at every time \(t\). In a precise sense, the codec is designed under the assumption that the \(\{q_t\}\) are independently identically distributed according to the PMF \(\mathbb{P}_{q_t}\). In this case either \(a_t = C_{q_t}^F(q_t)\) (without sorting), or \(a_t = C_{q_t}^S(q_t)\) (with sorting). Since the same codebook is used at every time \(t\), this approach conforms to the Prefix Constraint 3. By definition (see (15), (17)), we have

\[
-\log_2(\mathbb{P}_{q_t}[q_t]) \leq \ell_t \leq -\log_2(\mathbb{P}_{q_t}[q_t]) + r,
\]

where \(r\) is 1 or 2 depending on whether \(C_{q_t}^F\) or \(C_{q_t}^S\) is used. Thus, the time average expected codeword length satisfies

\[
\sum_{i=0}^{T} \frac{H(q_t) + D_{KL}(q_t||q)}{T+1} \leq \frac{\sum_{i=0}^{T} E[\ell_t]}{T+1} \leq \sum_{i=0}^{T} \frac{H(q_t) + D_{KL}(q_t||q)}{T+1} + r.
\]
Time-invariant prefix-free codebooks are attractive for many reasons, most of all because they need to be designed only once according to a single PMF. Furthermore, to use a time-varying “sorted” construction, in general each of the PMFs $P_{q_t}$ must be sorted to extract the 1 bit per timestep reduction in overhead. However, at every time $t$ a time-invariant codebook incurs an increase in codeword length given by $D_{KL}(q_t||q) \geq 0$. In some cases (and, indeed, in a case we will consider in the sequel) a PMF $P_q$ can be found such that $\limsup_{t \to \infty} D_{KL}(q_t||q) = 0$, ensuring that this redundancy is asymptotically negligible.

2) Uniform quantizers with subtractive dither: In this section, we introduce some key properties pertaining to element-wise uniform quantization with subtractive dither. For completeness, we will restate generalizations of key results from [7] that were derived in [2]. Let $\Delta Z^m$ denote the set of $m$-tuples of integer multiples of $\Delta$, e.g., $r \in \Delta Z^m$ if, for some $r_0, \ldots, r_{m-1} \in \mathbb{Z}$, $r = (r_0 \Delta, r_1 \Delta, \ldots, r_{m-1} \Delta)$. Define an element-wise uniform quantizer with stepsize $\Delta$ as $Q_\Delta: \mathbb{R}^m \to \Delta Z^m$ via

$$[Q_\Delta(x)]_i = k\Delta, \text{ if } [x]_i \in [k\Delta - \Delta/2, k\Delta + \Delta/2),$$

(24)

where $x \in \mathbb{R}^m$, $i \in \{0, \ldots, m - 1\}$. In other words, the uniform element-wise quantizer “independently” rounds each element of the vector $x$ to the nearest multiple of $\Delta$. Let $z$ be a random variable with range in $\mathbb{R}^m$. Let $\delta = [\delta_0, \ldots, \delta_{m-1}]^T$ be independent of $z$ and such that the $[\delta]_i$ are IID uniform on $[-\Delta/2, \Delta/2]$. When $z$ is quantized with an element-wise uniform quantizer with subtractive dither, the quantization is the random variable with range $\Delta Z^m$ defined by

$$q = Q_\Delta(z + \delta),$$

(25)

the reconstruction is defined as $\tilde{q} = q - \delta$, and the reconstruction error is $v = \tilde{q} - z$. The following lemma states useful properties exhibited by the reconstruction error in dithered element-wise uniform quantizers. The lemma originally appeared in [2, Lemma 1] and is proven in [23].
Lemma IV.1. Let \( z, \delta, q, \tilde{q}, \) and \( v \) be as defined above. Assume \( \mathbb{E}[z] \) and \( \mathbb{E}[zz^T] \) are finite (this implies \( h(z) < \infty \)). We have

i. The \( i^{th} \) element of the reconstruction error \([v]_i\) is uniformly distributed on the interval \([-\Delta/2, \Delta/2]\). The \( m \) elements of \( v \) are mutually independent, and \( v \) is independent of \( z \).

ii. Let \( n \) be a random vector with range \( \mathbb{R}^m \) such that the elements \([n]_i\) are IID uniformly distributed on \([-\Delta/2, \Delta/2]\). Let \( n \) be independent of \( z \) and let \( r = z + n \). We have in turn that \( H(q|\delta) = I(z; r) \) and \( I(z; r) = h(r) - h(n) \).

We next state a useful expression relating the differential entropy of an IID uniform random vector that of a Gaussian random vector with the same mean and covariance.

Lemma IV.2. Let \( n \) be as defined in Lemma IV.1. Let \( N \) be a diagonal matrix with \([N]_{i,i} = \Delta_i^2/12 \). Note by definition \( \mathbb{E}[n] = 0 \) and \( \mathbb{E}[nn^T] = N \). We have

\[
h(n) = h(n^G) - D_{\text{KL}}(n||n^G) = \frac{1}{2} \log_2 \left( \det(2\pi e N) \right) - \frac{m}{2} \log_2 \left( \frac{2\pi e}{12} \right). \tag{26}
\]

\[
h(n) = \frac{1}{2} \log_2 \left( \det(2\pi e N) \right) - \frac{m}{2} \log_2 \left( \frac{2\pi e}{12} \right). \tag{27}
\]

Proof: This result is a straightforward extension of [24, Lemma 2 (B9)] to the vector case. It can be shown directly by writing the standard expression for \( D_{\text{KL}}(n||n^G) \), then applying the “second order equivalence” between \( n \) and \( n^G \) and the definitions of \( h(n) \), \( h(n^G) \).

In Fig. 3, the entropy source codec encodes the output of a dithered uniform element-wise quantizer into a prefix-free codeword. Since the dither sequence is known to both the encoder and decoder, the encoder can select a codebook conditioned on the realization of \( \delta \). The Lemma IV.3 demonstrates that taking advantage of this SI reduces the codeword length by at most one bit per plant dimension.

Lemma IV.3. Let \( q, \delta \) and \( z \) be as defined in Lemma IV.1. We have \( H(q) - H(q|\delta) \leq m \).

Proof: Note that \( H(q) - H(q|\delta) = I(q; \delta) \). By the chain rule, we have that both
Let \( I((q,z); \delta) = I(q; \delta) + I(z; \delta|q) \) and also \( I((q,z); \delta) = I(z; \delta) + I(q; \delta|z) \). By the assumption of independence, \( I(z; \delta) = 0 \). Since \( I(z; \delta|q) \geq 0 \), we have \( I(q; \delta) \leq I(q; \delta|z) \). Since \( q \) is a discrete random variable that is a deterministic function of \( \delta \) and \( z \), we have \( I(q_t; \delta|z) = H(q|z) \).

Consider first the case \( m = 1 \). Say \( z = z \) for some arbitrary \( z \in \mathbb{R} \).

Let \( i \) be the unique \( i \in \mathbb{Z} \) such that \( z \in [i\Delta - \Delta/2, i\Delta + \Delta/2) \). Assume first that \( z \in [i\Delta, i\Delta + \Delta/2) \), i.e., \( z \) is in the “top half” of the quantization interval. Since \( \delta \in [-\Delta/2, \Delta/2] \), given \( z = z \) and (24), \( q \in \{i\Delta, (i + 1)\Delta\} \). In other words, if \( j \in \Delta \mathbb{Z} \) with \( j \neq \{i\Delta, (i + 1)\Delta\} \) then \( P_{q|z}[q = j|z = z] = 0 \). Thus, \( z \in [i\Delta, i\Delta + \Delta/2) \), then the “instantaneous” conditional entropy satisfies \( H(q|z = z) \leq 1 \) bit, since given \( z = z \), \( q \) has a support only over the 2-tuple \( \{i\Delta, (i+1)\Delta\} \). Likewise, if \( z \in [i\Delta - \Delta/2, i\Delta) \), we have, for all \( j \in \Delta \mathbb{Z} \) with \( j \neq \{(i-1)\Delta, i\Delta\} \) that \( P_{q|z}[q = j|z = z] = 0 \). Again, it follows that \( H(q|z = z) \leq 1 \). Since by definition \( E_{z \sim P(z)}[H(q|z = z)] = H(q|z) \), we have \( m = 1 \) that \( H(q_t|e_t) \leq 1 \). By the chain rule and the fact that conditioning reduces entropy, we have for arbitrary \( m \),

\[
H(q_t|e_t) \leq \sum_{k=0}^{m-1} H([q_t]_k|[e_t]_k) \leq m. \tag{28}
\]

To show (29), apply the result for \( m = 1 \) to each term of the form \( H([q_t]_k|[e_t]_k) \).

In the following proposition, we combine Lemmas IV.1 and IV.2 to bound the conditional entropy \( H(q|\delta) \) and, in turn, \( H(q) \) via Lemma IV.3

**Proposition IV.4.** Let \( z, \delta, q, \) and \( n \) be as defined in Lemma IV.1. Say \( E[z] = z_\mu \), and assume the covariance of \( z \) is \( E[(z - E[z])(z - E[z])^T] = Z \). Define a “Gaussian version” of \( z \) via \( z^G \sim \mathcal{N}(z_\mu, Z) \). By definition \( E[n] = 0 \) and \( E[nn^T] = N \), where \( N \) is a diagonal matrix with diagonal elements given by \( [N]_{i,i} = \Delta^2/12 \). Let \( n^G \sim \mathcal{N}(0, N) \) be independent of \( z^G \), and define \( r^G = z^G + n^G \). By definition, \( r^G \) is a jointly Gaussian random vector such that \( r^G \sim \mathcal{N}(z_\mu, Z+N) \)
and \( h(r^G) = \log_2(\det(2\pi e(N + Z))) \). We have

\[
H(q|\delta) = h(r) - h(n)
\]

\[
\leq h(r^G) - \frac{1}{2} \log_2(\det(2\pi eN)) + \frac{m}{2} \log_2 \left( \frac{2\pi e}{12} \right).
\]  

(30)

Furthermore,

\[
H(q) \leq h(r^G) - \frac{1}{2} \log_2(\det(2\pi eN)) + m \left( 1 + \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right) \right).
\]  

(32)

Proof: Equation (30) follows from Lemma [IV.1] and (31) follows from substituting (27) for \( h(n) \) and the fact that the Gaussian distribution maximizes differential entropy over all distributions with the same second moment matrix. Finally, (32) follows from Lemma [IV.3].

B. Time-variant achievability for MIMO plants

Having reviewed the key ingredients of the framework in Fig. 3, we now proceed to analyze the control and communication costs achieved by the system. This follows the analysis in [2], although with a refinement of the space-filling gap from \( \frac{1}{2} \log_2 \left( \frac{4\pi e}{12} \right) \) to \( \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right) \) bits per plant dimension (owing to a replacement of [2, Lemma 1 (c,d)] with Proposition [IV.4]).

Consider the system in Fig. 3. Define \( C \) and \( V \) to be chosen optimally via the rate-distortion formulation in (6). Since \( C \) and \( V \) are defined with respect to the minimizers of (6) via (7), we can assume, without loss of generality, that for some \( v > 0 \), \( V = vI_{m \times m} \) and that \( C \) satisfies \( \hat{P}^{-1} - (A\hat{P}A^T + W)^{-1} = C^TV^{-1}C \) (where \( \hat{P} \) minimizes (6)). With foresight, we choose the quantizer sensitivity and dither support to be \( \Delta = \sqrt{12v} \). Let \( \{x^G_t, x^G_t, u^G_t\} \) be a stochastic process that is identically distributed to the stochastic process \( \{x_t, u_t\} \) under the defined choice of \( C \) and \( V \) in the time-invariant three-stage separation architecture of Fig. 2. Let \( \{x^{NG}_t, u^{NG}_t\} \) denote the stochastic process \( \{x_t, u_t\} \) under the choice of encoder and decoder policy in Fig. 3. The superscripts are used to emphasize that in Fig. 2, all signals are jointly Gaussian, whereas in Fig. 3, they are not. Achievability is demonstrated by a proof that in Fig. 3 (an architecture
conforming to Fig. 1) the resulting \( \{x_t^{NG}, u_t^{NG}\} \) are equivalent to \( \{x_t^{G}, u_t^{G}\} \) to second order, followed by a bound on expected codeword length at every time \( t \). This will follow from our work in Section IV-A. In Fig. 3, both the encoder and the decoder operate identical time-invariant Kalman filters. We denote the a priori and a posteriori estimates computed by these filters as \( \bar{x}_t^{NG} \) and \( \bar{x}_t^{NG} \), and the corresponding estimator errors as \( \bar{e}_t^{NG} = x_t^{NG} - \bar{x}_t^{NG} \) and \( \bar{e}_t^{NG} = x_t^{NG} - \bar{x}_t^{NG} \). Denote the error covariance matrices \( P_{t, t-1}^{NG} = E[(\bar{e}_t^{NG})(\bar{e}_t^{NG})^T] - E[\bar{e}_t^{NG}](E[\bar{e}_t^{NG}])^T \) and \( P_t^{NG} = E[(\bar{e}_t^{NG})(\bar{e}_t^{NG})^T] - E[\bar{e}_t^{NG}](E[\bar{e}_t^{NG}])^T \). We denote the corresponding variables in Fig. 2 (see (9)) by \( \bar{x}_t^{G}, \bar{x}_t^{G}, \bar{e}_t^{G}, \bar{P}_t^{G}, \bar{e}_t^{G}, \bar{P}_t^{G} \). In both systems, the initial a priori estimates are defined to be \( \bar{x}_0^{NG} = \bar{x}_0^{G} = 0 \). As in the Gaussian architecture of Fig. 2, in Fig. 3 the control input is designed via certainty equivalence, i.e., the control inputs are given by

\[
\hat{u}_t^{NG} = K \bar{x}_t^{NG}. \tag{33}
\]

We denote the shared dither sequence in Fig. 3 by \( \{d_t\} \). The elements of the vector \( d_t \) are IID random variables uniformly distributed on \( [-\frac{\Delta}{2}, \frac{\Delta}{2}] \). We assume that the sequence \( \{d_t\} \) is IID over time and conforms to the conditional independence relationships implied by (4).

The full description of the closed loop system in Fig. 3 is necessarily recursive. Suppose that at time \( t \geq 1 \), both the encoder and decoder have access to \( \bar{x}_t^{NG} \). Since the encoder knows \( \bar{x}_t^{NG} \) and the control is certainty equivalent, the encoder can compute the control input \( \hat{u}_t^{NG} \). Both the encoder and decoder first compute the one-step prediction update via \( \bar{x}_t^{NG} = (A + BK) \bar{x}_{t-1}^{NG} \). The encoder first computes a quantized measurement innovation via

\[
q_t = Q\Delta(C\bar{e}_t^{NG} + d_t). \tag{34}
\]

The encoder then encodes \( q_t \) into the codeword \( a_t \) using an SFE code. For simplicity, in this subsection we assume that “sorted” or “Shannon–type” SFE codecs are used; if an unsorted SFE code is used, the upper bounds on expected codeword length will increase by one bit.

Recall that the \( d_t \) is known to both the encoder and decoder. Assume that the SFE codec used at time \( t \) uses this SI and is designed according to the conditional PMF \( P_{q_t|d_t} \) (i.e., \( a_t = \)}
$C_{q_t|d_t}^S(q_t|d_t)$ where $C_{q_t|d_t}^S$ is the “sorted version” of (19). At time $t$, the codeword length satisfies

$$\mathbb{E}[\ell(a_t)] \leq H(q_t|d_t) + 1. \quad (35)$$

This approach conforms to Prefix Constraint 1. If the SFE codec used at time $t$ ignores the SI provided by $d_t$ and is designed via the marginal PMF $P_{q_t}$ (e.g., $a_t = C_{q_t}^S(q_t)$), the approach conforms to Prefix Constraint 2 and the codeword length at time $t$ satisfies

$$\mathbb{E}[\ell(a_t)] \leq H(q_t) + 1. \quad (36)$$

Whether or not the encoder and decoder use the SI provided by $d_t$, the quantized value $q_t$ is recovered exactly at the decoder. The decoder then computes the reconstruction $\tilde{q}_t = q_t - d_t$.

Define the reconstruction error

$$v_{t}^{NG} = \tilde{q}_t - C\tilde{e}_{t|t-1}^{NG} \quad (37)$$

and note that by Lemma [IV.1] $v_{t}^{NG} \sim \text{Uniform}[-\Delta/2, \Delta/2]$ and $v_{t}^{NG}$ is independent of $C\tilde{e}_{t|t-1}^{NG}$.

The decoder then computes the centered measurement

$$y_{t}^{NG} = \tilde{q}_t + C\tilde{x}_{t|t-1}^{NG} \quad (38)$$

$$= C'(\tilde{e}_{t|t-1}^{NG} + \tilde{x}_{t|t-1}^{NG}) + v_{t}^{NG} \quad (39)$$

$$= C\tilde{x}_{t}^{NG} + v_{t}^{NG}. \quad (40)$$

By the definition of $y_{t}^{NG}$ and the fact that it is independent of $C\tilde{e}_{t|t-1}^{NG}$, $v_{t}^{NG}$ is independent of $\tilde{x}_{t}$. Furthermore, the elements of the vector $[v_{t}^{NG}]_i$ are IID uniform on $[-\Delta/2, \Delta/2]$. Thus, the decoder’s effective measurement model (38) is affine with measurement matrix $C$ and additive uniform noise with zero mean and covariance $V$. The encoder can also compute the centered measurement $y_{t}^{NG}$. Given this measurement, both the encoder and decoder compute the time-invariant Kalman filter update via (9a) obtaining

$$\tilde{x}_{t|t}^{NG} = \tilde{x}_{t|t-1}^{NG} + J(y_{t}^{NG} - C\tilde{x}_{t|t-1}^{NG}). \quad (41)$$
The control input is then computed via (33). Under this feedback arrangement, the sequence of reconstruction errors \( \{v_t^{NG}\} \) are independent and identically distributed over time, and thus satisfy 
\[
\mathbb{E}[v_{t+s}(v_t^{NG})^T] = V \mathbb{1}_{s=0}.
\]
Furthermore, \( v_t^{NG} \) is independent of \( (e_i^{NG})_0^t, (v_G)^{t-1}_0, w_t^{t-1} \). All of these follow from a straightforward corollary of the proof of [2, Lemma 1a]. In particular, given (4), this implies that for any \( t_1 \) and \( t_2 \), \( v_{t_1}^{NG} \) and \( w_{t_2} \) are pairwise independent. Thus, the measurement model (38) is identical to that in (8) to second order. Since 
\[
P_{0|0}^{NG} = P_{0|0}^G,
\]
\( \mathbb{E}[e_0^{NG}|_{0}^0] = 0 \), and \( \mathbb{E}[e_0^{NG}] = 0 \), it follows directly that for \( t \geq 0 \), \( \mathbb{E}[e_t^{NG}] = 0 \), \( \mathbb{E}[e_{t+1}^{NG}] = 0 \), 
\[
\mathcal{T}_{t|t}^{NG} = \mathcal{T}_{t|t}^G, \quad \mathcal{T}_{t+1|t}^{NG} = \mathcal{T}_{t+1|t}^G.
\]
Since they are identical, to clarify the notation we presently drop the superscripts for the estimator error covariances. Recall that since \((A,W^{1/2})\) is stabilizable and \((C,A)\) is detectable, we have that 
\[
\lim_{t \to \infty} P_{t|t} = \hat{P} \quad \text{and} \quad \lim_{t \to \infty} P_{t+1|t} = \hat{P}.
\]
In the following theorem, we use these results to quantify the control and communication costs incurred in the architecture in Fig. 3.

**Theorem IV.5.** Let \( C, V \in \mathbb{R}^{m \times m} \) be obtained from the minimizers of (6) via (7) with \( V = vI_{m \times m} \)

for some \( v > 0 \). If the quantizer sensitivity and dither signal are chosen so that 
\[
\Delta = \sqrt{12v},
\]
the architecture in Fig. 3 achieves the same control performance as the optimal system in Fig. 2. Assume that the system in Fig. 3 encodes \( q_t \) into the codeword \( \alpha_t \) given the SI \( d_t \) using a time-varying Shannon–type SFE codec designed per the conditional PMF \( P_{q_t|d_t} \) at all \( t \) (i.e., \( \alpha_t = C_{q_t|d_t}^{S}(q_t|d_t) \) for all \( t \)). The system will satisfy Prefix Constraint 1 and will attain a communication cost that satisfies

\[
\limsup_{t \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}[\ell(\alpha_t)] \leq \mathcal{R} + 1 + \frac{m}{2} \log_2 \left( \frac{2\pi e}{12} \right),
\]

where \( \mathcal{R} \) is the lower bound on minimum time-average coding rate obtained (6). Assume instead that the system encodes \( q_t \) into \( \alpha_t \) without using the SI via a time-varying Shannon–type SFE codec designed per the marginal PMF \( P_{q_t} \) at each \( t \) (i.e., \( \alpha_t = C_{q_t}^{S}(q_t) \) for all \( t \)). In this case,
the system will satisfy Prefix Constraint \(^2\) and will attain a communication cost that satisfies
\[
\limsup_{t \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} E[\ell(a_t)] \leq R + 1 + m \left(1 + \frac{1}{2} \log_2 \left(\frac{2\pi e}{12}\right)\right).
\] (43)

**Proof:** The certainty equivalent control and the convergence of the posterior estimator covariance matrix \(\text{cov}(e_\text{NG}_{t|t}) = \mathcal{P}_{t|t}\) to \(\hat{P}\) ensure that the control cost in Fig. 3 is identical to that in (11). At every time \(t\), by (35), the codeword length for the time-varying code that incorporates the SI satisfies
\[
E[\ell(a_t)] \leq H(q_t|d_t) + 1.
\] Note that the input to the dithered quantizer in (34) satisfies
\[
E[C_\text{e}_{\text{NG}_{t|t-1}}(e_{\text{NG}_{t|t-1}})^T C_t] = C_{\mathcal{P}_{t|t-1}} C_t T.
\] By Proposition IV.4, we have that
\[
H(q_t|d_t) \leq \frac{1}{2} \log_2 (\det(2\pi e (C_{\mathcal{P}_{t|t-1}} C_t + V))) - \frac{1}{2} \log_2 (\det(2\pi e V)) + \frac{m}{2} \log_2 \left(\frac{2\pi e}{12}\right).
\] (44)

Taking the limit supremum of both sides of (44) and applying (10) gives
\[
\limsup_{t \to \infty} H(q_t|d_t) \leq \frac{\log_2 (\det(2\pi e (C_{\hat{P}_{t|t-1}} C_t + V))) - \log_2 (\det(2\pi e V)) + \frac{m}{2} \log_2 \left(\frac{2\pi e}{12}\right)}{2}.
\] (45)

Recall that \(\hat{P}^{-1} = \hat{P}_+^{-1} + C^T V^{-1} C\) (see (7)). Using the matrix determinant lemma, then
\[
\det(C_{\hat{P}_+} C_t + V) = \det(\hat{P}_+) \det(V) \det(\hat{P}^{-1}).
\] (46)

Thus we have \(\limsup_{t \to \infty} H(q_t|d_t) \leq \frac{1}{2} \log_2 (\det(\hat{P}_+)) - \frac{1}{2} \log_2 (\det(\hat{P})) + \frac{m}{2} \log_2 \left(\frac{2\pi e}{12}\right)\). Recall that
\[
R = \frac{1}{2} \log_2 (\det(\hat{P}_+)) - \frac{1}{2} \log_2 (\det(\hat{P})).
\] Thus, by (35) and the Cesáro mean,
\[
\limsup_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} E[\ell_t] \leq \limsup_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} (H(q_t|d_t) + 1)
\] (47a)
\[
\leq \limsup_{T \to \infty} H(q_t|d_t) + 1
\] (47b)
\[
\leq \limsup_{T \to \infty} R + 1 + \frac{m}{2} \log_2 \left(\frac{2\pi e}{12}\right).
\] (47c)

The proof of (43) is analogous, given (36) and that \(H(q_t) \leq H(q_t|d_t) + m\) via Lemma IV.3. ■

Notably, both of the achievability results in Theorem IV.5 rely on dithering. The bounds on communication and control costs assume dithered quantization, allowing the decoder to whiten the reconstruction error. Furthermore, all of the achievability results of this section are time-varying in the sense that the lossless source codecs in Fig. 3 must (generally) change their
mappings from source symbols (and, for (42), SI sequences) to codewords at every \( t \). In the next section, we show that for scalar plants \((m = 1)\), the bound on communication cost (43) holds for a particular choice of time-invariant SFE codec.

C. Time-Invariant Achievability for SISO Plants

Here, we use tools from ergodic theory to demonstrate that, at least when \( m = 1 \), the results of Section IV-B can be achieved with time-invariant source codecs. This result follows from long-term analysis of the stochastic process \( \{ e_{NG}^t | t-1, d_t \} \). While the process of interest is \( \{ q_t, d_t \} \), it is clear from (34) that \( q_t \) is a measurable function of \( e_{NG}^t | t-1, d_t \). To simplify notation, in what follows let \( e_t = e_{NG}^t | t-1 \) and \( v_t = v_{NG}^t \). Some properties of \( \{ e_{NG}^t | t-1, d_t \} \) will be especially useful.

Let \( L = AJ \) and \( R = (A - LC) \). Recall the definition of the reconstruction error \( v_t \) from (37). By definition, \( \{ e_t \} \) obeys the recursion

\[
e_t = R e_{t-1} - L v_{t-1} + w_{t-1},
\]

and note that since \( x_0 = 0 \), \( e_0 \sim \mathcal{N}(0, X_0) \). Also recall that while \( v_t \) is a measurable function of \( (e_t, d_t) \), it is readily shown that \( v_t \perp (e_t^0, v_{t-1}^0, w_{t-1}^0) \). Since \((A, W^{1/2})\) is stabilizable and \((C, A)\) is detectable, \( R \) is stable, i.e., when \( m = 1 \), \(|R| < 1 \). With these in hand, we state the main result before deferring proofs of some key lemmas to Appendix A.

Theorem IV.6. There exists a conditional PMF \( \mathbb{P}_{q|d} : \mathbb{Z} \times [-\Delta/2, \Delta/2] \to [0, 1] \) such that:

1) Let \( C_{q|d}^F \) be as defined in (19) with respect to \( \mathbb{P}_{q|d} \). If the source codec in Fig. 3 encodes the quantization \( q_t \) with \( C_{q|d}^F \) given the dither \( d_t \) at every time \( t \) (i.e., \( a_t = C_{q|d}^F(q_t|d_t) \) for all \( t \)), the codewords will be prefix-free given \( d_t \) and their lengths will almost surely satisfy

\[
\lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} \ell(a_t) \leq \mathcal{R} + \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right) + 2.
\]

2) Denote the marginal PMF \( \mathbb{P}_q(q) = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \mathbb{P}_{q|d}(q|s)ds \). Let \( C_{q}^S \) be as defined in (17) with respect to \( \mathbb{P}_q \) (which we may assume is sorted without loss of generality). If the source
codec in Fig. 3 uses $C_S^q$ to encode the quantization $q_t$ at every time $t$ (i.e., $a_t = C_S^q(q_t)$ for all $t$), then the codewords will satisfy Prefix Constraint 3 and their lengths will almost surely satisfy

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{i=0}^{T} \ell(a_t) \leq R + \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right) + 2. \quad (50)$$

Furthermore, the time-average of expected codeword lengths satisfy

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{i=0}^{T} \mathbb{E}[\ell(a_t)] \leq R + \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right) + 2. \quad (51)$$

Regardless of what lossless source codecs are used in Fig. 3 we have almost surely that

$$\limsup_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \|x_{t+1}\|_Q^2 + \|u_t\|^2_R < \gamma. \quad (52)$$

Likewise, we have

$$\limsup_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}[\|x_{t+1}\|_Q^2] + \mathbb{E}[\|u_t\|^2_R] < \gamma. \quad (53)$$

Taking several technical lemmas for granted, we will prove Theorem IV.6 in the remainder of this section. The stochastic process $\{e_t, d_t\}$ is a time-homogeneous first order Markov chain on the state space $S = \mathbb{R} \otimes [-\Delta/2, \Delta/2]$ whose transitions can be defined via a well-defined conditional probability density function (PDF). To simplify the notation, we denote the PDF $f_{t+1|t} = f_{e_{t+1}, d_{t+1}|e_t, d_t}$. Define the function $M : (x, y) \in S \to \mathbb{R}$ via

$$M(x, y) = Rx - L(Q\Delta(Cx + y) - y - Cx). \quad (54)$$

The transition PDF $f_{t+1|t} : S \times S \to \mathbb{R}_+$ is

$$f_{t+1|t}(e_{t+1}, d_{t+1}|e_t, d_t) = \frac{e^{-(e_{t+1} - M(e_t, d_t))^2}}{\Delta} \frac{1}{\sqrt{2\pi W}}. \quad (55)$$

Analogously, the Markov chain can be defined by the transition kernel (or regular conditional probability) satisfying for $B \in \mathcal{B}(S)$ where $\mathcal{B}(S)$ is the Borel $\sigma-$algebra on $S$

$$\mathbb{P}_{e_{t+1}, d_{t+1}|e_t, d_t}[(e_{t+1}, d_{t+1}) \in B|e_t, d_t] \overset{a.s.}{=} \int \int \frac{1}{\Delta} \frac{e^{-(x-M(e_t, d_t))^2}}{\sqrt{2\pi W}} \, dx \, dy. \quad (56)$$
Both (55) and (56) follow from the state space model (1) and (48). This Markov chain has some useful properties. Namely, it admits an invariant measure and has an ergodic property. These results are summarized in the following technical lemmas, proven in Appendix A.

**Lemma IV.7.** The Markov chain on $S$ defined by (55) and (56) admits an invariant PDF; i.e., there exists a function $g_\infty: S \rightarrow \mathbb{R}^+$ such that

$$g_\infty(e_+, d_+) = \int \int f_{t+1}|(e_+, d_+|e, \delta)g_\infty(e, \delta)d\delta$$

and $g_\infty(e, d) > 0$ for all $(e, d) \in S$. In other words, the Markov chain admits an invariant probability measure $P_{e_\infty, d_\infty}: \mathcal{B}(S) \rightarrow [0, 1]$ defined by $P_{e_\infty, d_\infty}[B] = \int \int_{(e, \delta) \in B} g_\infty(e, \delta)d\delta$ that is equivalent to the Lebesgue measure on $S$ (i.e., $P_{e_\infty, d_\infty}$ is absolutely continuous with respect to $\lambda$ and vice versa).

For some intuition, note that if the initial conditions of a Markov chain are drawn from the invariant measure (e.g., $(e_0, d_0) \sim P_{e_\infty, d_\infty}$) then for $i \geq 1$ we will have $(e_i, d_i) \sim P_{e_\infty, d_\infty}$.

**Lemma IV.8.** For $\lambda$ almost every initial condition, the $n$-step transition probabilities of the Markov chain defined by (55) and (56) converge in total variation to the invariant measure, i.e., for $\lambda$ almost every $(e_0, d_0)$

$$\lim_{t \to \infty} \sup_{B \in \mathcal{B}(S)} |P_{e_\infty, d_\infty}[B] - P_{e_t, d_t}[e_t, d_t] \in B| e_0 = e_0, d_0 = d_0| = 0.$$  

Furthermore, if $(e_0, d_0)$ are continuous random variables then for any function $\theta: S \rightarrow \mathbb{R}$ with

$$\int \int_{S} |\theta(e, \delta)|g_\infty(e, \delta)d\delta < \infty,$$

the “law of large numbers” applies to the process $\{e_i, d_i\}$ in the sense that

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{i=0}^{T} \theta(e_i, d_i) \overset{a.s.}{=} \mathbb{E}_{(e, d) \sim g_\infty}[\theta(e, d)].$$
Since \((e_0, d_0)\) are continuous random variables on \(\mathcal{S}\), the convergence in total variation implies that the sequence \(P_{e_t, d_t}\) converges weakly to \(P_{e_\infty, d_\infty}\). Lemmas IV.7 and IV.8 can be combined to prove the following useful facts relating to \(P_{e_\infty, d_\infty}\).

**Corollary IV.9.** Assume \((e, d) \sim P_{e_\infty, d_\infty}\). Denote the marginal PDF of \(e\) via

\[
g_{e_\infty}(e) = \int_{-\Delta/2}^{\Delta/2} g_{\infty}(e, \delta)d\delta. \tag{61}
\]

We have that \(e \perp d\) and \(d \sim \text{Uniform}[ -\Delta/2, \Delta/2]\). This implies that, \(g_{\infty}: \mathcal{S} \rightarrow \mathbb{R}\) factorizes via \(g_{\infty}(e, d) = \frac{g_{e_\infty}(e)}{\Delta}\) for \((e, d) \in \mathcal{S}\). Furthermore, we have \(\mathbb{E}[e] = 0\) and \(\mathbb{E}[e^2] = \hat{P}_+\).

**Proof:** If \(\mathcal{A}\) is an open interval in \(\mathbb{R}\) and \(\mathcal{D}\) an open interval in \([-\Delta/2, \Delta/2]\) then \(\mathcal{A} \times \mathcal{D} \in \mathcal{B}(\mathcal{S})\). Using the definition of the invariant PDF (57) and the formula for \(f_{t|1-t}\) from (55), it can be shown that if \(\mathcal{B} = \mathcal{A} \times \mathcal{D}\) then, \(P_{e, d}^{\mathcal{B}} = \int_{\mathcal{A}} g_{e_\infty}(e)de^{\lambda(D)}/\Delta\). By Dynkin’s \(\pi - \lambda\) theorem, this proves that \(e \perp d\) (see e.g., [27, Prop. 2.13]).

Define \(v = (Q_\Delta(Ce + d) - (Ce + d))\). By definition, \(M(e, d) = Re - Lv\). By the result just established, \(d\) is independent of \(e\) and uniform on \([-\Delta/2, \Delta/2]\), so the properties of dithered quantizers established in Section IV-A can be applied to \(s\). Namely, by Lemma IV.1 (i), \(v\) is uniform on \([-\Delta/2, \Delta/2]\) and \(v \perp e\). Applying the definition of the invariant PDF (57),

\[
\mathbb{E}[e^2] = \int_{(e,d)\in \mathcal{S}} e^2 g_{\infty}(e, \delta)de d\delta
\]

\[
= \int_{(e,d)\in \mathcal{S}} \int_{(s,t)\in \mathcal{S}} f_{t+1|t}(e, \delta|s, t)g_{\infty}(s, t)ds dt de d\delta \tag{63}
\]

\[
= \int_{(s,t)\in \mathcal{S}} \left( \int_{(e,d)\in \mathcal{S}} e^2 f_{t+1|t}(e, \delta|s, t)g_{\infty}(s, t)de d\delta \right) ds dt \tag{64}
\]

\[
= \int_{(s,t)\in \mathcal{S}} \left( W + (M(s, t))^2 \right) g_{\infty}(s, t)ds dt \tag{65}
\]

\[
= W + \mathbb{E}_{(e,d)\sim P_{e_\infty, d_\infty}} [ R^2 e^2 + L^2 v^2 - 2RL e v ] \tag{66}
\]

\[
= W + R^2 \mathbb{E}[e^2] + L^2 \frac{\Delta^2}{12} \tag{67}
\]

where (64) follows from the Fubini/Tonelli Theorem, (65) follows from (66) (i.e., since given
(e_{t-1}, d_{t-1}), e_t is normal with mean $M(e_{t-1}, d_{t-1})$ and variance $W$). (66) follows from (54) and the definition of $v$ above, and (67) follows from the properties of $v$ in Lemma IV.2. Recall that $V = \Delta^2/12$, and recognize that the identity (67) is a Lyaponov equation in $\mathbb{E}[e^2]$. This equation has a unique PSD solution [28, Prob. 4.9]. By definition, $\hat{P}_+$ satisfies the DARE

$$\hat{P}_+ = A \left( \hat{P}_+ - \hat{P}_+ C^T (C \hat{P}_+ C^T + V)^{-1} C \hat{P}_+ \right) A^T + W. \quad (68)$$

Substituting the explicit formula for $L$ and setting $\mathbb{E}[e^2] = \hat{P}_+$ in the right-hand side of (67) exactly recovers the right-hand side of (68). This proves the result. Since $e \in L^2$, we have $e \in L^1$. Given this, reductions analogous to (62) through (67) demonstrate that $\mathbb{E}[e] = R \mathbb{E}[e]$. Since $|R| < 1$, it must be that $\mathbb{E}[e] = 0$.

So long as the source codec used in Fig. 3 is lossless, the guarantee on control cost (52) follows. Lemma IV.8 additionally gives an almost sure guarantee on the control cost. We have

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^T \|x_{t+1}\|^2_Q + \|u_t\|^2_R \overset{\text{a.s.}}{=} \text{Tr}(\Theta \hat{P}) + \text{Tr}(WS).$$

Since $\text{Tr}(\Theta \hat{P}) + \text{Tr}(WS) < \gamma$, this proves (53). We now use Lemmas IV.7 and IV.8 to demonstrate that a time-invariant source codec constructed using the invariant measure $g_{\infty}$ achieves the same bound on codeword length that was derived in Section IV-B. Let $(e, d) \sim P_{e_{\infty}, d_{\infty}}$ and let $q = Q_{\Delta}(Ce + d)$. From Corollary IV.9 the conditional PMF of $P_{q|d}$: $(\Delta Z \times [-\Delta/2, \Delta/2]) \to [0, 1]$ is

$$P_{q|d}(q|d) = \int_{q+\frac{\Delta}{2} - d}^{q+\frac{\Delta}{2}} \frac{1}{|C|} g_{e_{\infty}} \left( \frac{x}{C} \right) dx, \text{ where } q = i\Delta \text{ for } i \in \mathbb{Z} \text{ and } d \in [-\Delta/2, \Delta/2]. \quad (69)$$

We show that if the lossless codec in Fig. 3 is designed assuming that the conditional PMF of $q_t$ given $d_t$ is (69) at all $t$, the system’s time-average bitrate almost surely attains the same bound as an unsorted version of (42). We also consider a codec that ignores the SI. Since by Corollary IV.9 $d$ is uniform on $[-\Delta/2, \Delta/2]$, the PMF of $q$ is given by

$$P_q(q) = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} P_{q|d}(q|d) d\delta. \quad (70)$$

If the source codec in Fig. 3 ignores the SI provided by the dither sequence $\{d_t\}$ and further assumes that the PMF of $q_t$ is given by (70) for at all $t$, the system’s time-average bitrate almost
surely satisfies the upper bound in (43). The system’s time average expected bitrate satisfies the same upper bound.

Consider first the case where the codec uses the SI provided by $d_t$, and that the Fano (unsorted) $C_{q|d}^F$ described in Theorem IV.6 is used. At every time $t$, the encoder transmits at most
\[
\ell_t = -\log_2(P_{q|d}(q_t|d_t)) + 2
\] bits (see Section IV-A1), and so $0 \leq \ell(C_{q|d}^F(q_t|d_t)) \leq \ell_t$ at all $t$. Thus, if $H(q|d) = \mathbb{E}[-\log_2(P_{q|d}(q|d))]$ is finite, Lemma IV.8 guarantees that
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{i=0}^{T} \ell_t \overset{a.s.}{\Rightarrow} \mathbb{E}_{e,d}[-\log_2(P_{q|d}(q|d))] + 2 \tag{71}
\]
\[
= H(q|d) + 2. \tag{72}
\]

We now derive an upper bound for $H(q|d)$ that mirrors (42). By Corollary IV.9, we have that $Ce + d$ has zero mean and variance $C^2\hat{P}_+ + V$. Using Proposition IV.4, we have
\[
H(q|d) \leq \frac{1}{2} \log_2\left(\frac{2\pi e (C^2\hat{P}_+ + V)}{2\pi e}\right) - \frac{1}{2} \log_2\left(\frac{2\pi e V}{2\pi e}\right) + \frac{1}{2} \log_2\left(\frac{2\pi e}{12}\right), \tag{73}
\]
Taking inspiration from (46), we recognize that $(C^2\hat{P}_+ + V) = \hat{P}_+ V \hat{P}^{-1}$. Thus
\[
H(q|d) \leq \frac{1}{2} \log_2(\hat{P}_+) - \frac{1}{2} \log_2(\hat{P}) + \frac{1}{2} \log_2\left(\frac{2\pi e}{12}\right) \tag{74}
\]
\[
\leq \mathcal{R} + \frac{1}{2} \log_2\left(\frac{2\pi e}{12}\right), \tag{75}
\]
where (75) follows from expressing $\mathcal{R}$ as in (13). Thus we have
\[
\limsup_{T \to \infty} \frac{1}{T} \sum_{i=0}^{T} \ell(C_{q|d}^F(q_i|d_i)) \leq \limsup_{T \to \infty} \frac{1}{T} \sum_{i=0}^{T} \ell_t \overset{a.s.}{\Rightarrow} \mathcal{R} + \frac{1}{2} \log_2\left(\frac{2\pi e}{12}\right) + 2. \tag{76}
\]
Note that the bound (76) is in terms of the instantaneous “sample average” of communication costs, rather than in terms of the time-average expected codeword length. This proves (49).

We can prove a result similar to (36) if the source codec is time-invariant and also ignores the SI provided by the dither. Assume the codec in Fig. 3 uses $C_q^S$ as described as in Theorem IV.6, i.e., assume a Shannon (sorted) codec designed with respect to (70). At every $t$, the encoder
transmits at most \( \ell_t = -\log_2(P_{q_t|q_t}) + 1 \) bits. Given Corollary IV.9 \( d \perp e \). Thus, given the definition of \( q \), Lemma IV.3 gives that \( H(q) \leq H(q|d) + 1 \). Then as before via Lemma IV.8

\[
\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ell_t \overset{a.s.}{=} H(q) + 1, \tag{78}
\]

and thus

\[
\limsup_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ell(C^S(q_t)) \overset{a.s.}{\leq} H(q) + 1 \tag{79}
\]

\[
\overset{a.s.}{\leq} 2 + \frac{1}{2} \log_2 \left( \frac{2\pi e}{12} \right), \tag{80}
\]

where (79) follows from the fact that \( \ell(C^S(q_t)) \leq \ell_t \) as well as (78). Finally, (80) follows from the bound (75) and the fact that \( H(q) - H(q|d) \leq 1 \). This proves (50).

Unfortunately, the guarantees (76) and (80) cannot immediately be generalized to bound the time-average of expected codeword lengths as in (51). We do have at every \( t \)

\[
\mathbb{E}[\ell(C^S(q_t))] \leq \mathbb{E}[\ell_t] \leq 1 + H(q_t) + D_{KL}(q_t||q). \tag{81}
\]

Via the properties of the invariant measure and a data processing inequality for f-divergences [29, Theorem 2.2 (6)] one can prove that the sequence of relative entropies is monotonically decreasing, i.e., that \( D_{KL}(q_t+1||q) \leq D_{KL}(q_t||q) \) for all \( t \). We now prove that \( \lim_{t \to \infty} D_{KL}(q_t||q) = 0 \).

**Lemma IV.10.** We have \( \lim_{t \to \infty} D_{KL}(q_t||q) = 0 \).

**Proof:** We first develop a bound on \( D_{KL}(q_t||q) \) that applies to any \( t \). It can be readily shown via Jensen’s inequality that if \( a, b \) are random variables that are absolutely continuous with respect to Lebesgue measure such that \( a \) is absolutely continuous with respect to \( b \), then \( D_{KL}(Q_{\Delta}(a)||Q_{\Delta}(b)) \leq D_{KL}(a||b) \). Letting \( (e, d) \sim P_{e \infty, d \infty} \), we then have \( D_{KL}(q_t||q) \leq D_{KL}(Ce_t + d_t||Ce + d) \). Recalling that \( d_t \) and \( d \) are identically distributed and that both \( e_t \perp d_t \) and \( e \perp d \), the data processing inequality for KL divergences (see [29, Theorem 2.2 (6)]) gives that

\[
D_{KL}(Ce_t + d_t||Ce + d) \leq D_{KL}(e_t||e). \tag{82}
\]
The proof follows from bounding $D_{KL}(e_t||e)$. Let $\{\nu_t\}$ denote an IID sequence of random variables uniformly distributed on $[-\Delta/2, \Delta/2]$. Likewise, let $\{\omega_t\}$ be IID with $\omega_t \sim \mathcal{N}(0, W)$, and let $\lambda \sim \mathcal{N}(0, X_0)$. Assume $\{\omega_t\}$, $\{\nu_t\}$, and $\lambda$ are mutually independent. Let “$D$” denote “equality in distribution”, e.g., we write $a \overset{D}{=} b$ to imply $a$ and $b$ are identically distributed. By the recursive definition of $\{e_t\}$ in (48), we have

$$e_t \overset{D}{=} R^t \lambda + \sum_{i=0}^{t-1} R^i (\omega_i - L \nu_i). \quad (83)$$

Likewise, by definition of $e$, we have

$$e \overset{D}{=} \lim_{t \to \infty} R^t \lambda + \sum_{i=0}^{t-1} R^i (\omega_i - L \nu_i) \quad (84)$$

$$\overset{D}{=} \lim_{t \to \infty} \sum_{i=0}^{t-1} R^i (\omega_i - L \nu_i), \quad (85)$$

which follows from the convergence in total variation (in turn implying weak convergence) given by Lemma IV.8. Define the random variables

$$g_{\leq t} = \sum_{i=0}^{t-1} R^i \omega_i, \quad u_{\leq t} = -\sum_{i=0}^{t-1} R^i L \nu_i, \quad \text{and} \quad s_{>t} = \lim_{r \to \infty} \sum_{i=t}^{r} R^i (\omega_i - L \nu_i),$$

where we note that the limit is well defined by Kolmogorov’s two-series theorem. By definition,

$$e_t \overset{D}{=} R^t \lambda + g_{\leq t} + u_{\leq t} \quad \text{and} \quad e \overset{D}{=} g_{\leq t} + u_{\leq t} + s_{>t}. \quad (86)$$

Note that $g_{\leq t} \sim \mathcal{N}(0, W \frac{1-R^t}{1-R^2})$. We have

$$D_{KL}(e_t||e) = D_{KL}(R^t \lambda + g_{\leq t} + u_{\leq t}||g_{\leq t} + u_{\leq t} + s_{>t}) \quad (87)$$

$$\leq D_{KL}(R^t \lambda + g_{\leq t}||g_{\leq t} + s_{>t}) \quad (88)$$

$$\leq D_{KL}(R^t \lambda + g_{\leq t}||g_{\leq t} + s_{>t}||s_{>t}), \quad (89)$$

where (88) follows from the data processing inequality for f-divergences and (89) follows since conditioning increases KL divergence (see [29] Theorem 2.2 (5)).
Given $s_{>t} = s$ (89) simplifies to a KL divergence between Gaussians. Let $\sigma^2_t = W \frac{1-R^2_t}{1-R^2} \text{ and } \sigma^2_t = \sigma^2_t + R^{2t}X_0$. Since $\lambda \perp g_{\leq t}$ by construction, $R^t\lambda + g_{\leq t} \sim \mathcal{N}(0, \sigma^2_t)$. Recall that $g_{\leq t} \perp s_{>t}$ by construction. Thus, we have

$$D_{KL}(R^t\lambda + g_{\leq t} || g_{\leq t} + s_{>t} | s_{>t} = s) = D_{KL}(\mathcal{N}(0, \sigma^2_t) || \mathcal{N}(s, \sigma^2_t)) = \frac{1}{2} \log_2 \left( \frac{\sigma^2_t}{\sigma^2_t} \right) + \frac{\sigma^2_t + (s)^2}{2\sigma^2_t} - \frac{1}{2}.$$ 

So we have that

$$D_{KL}(R^t\lambda + g_{\leq t} || g_{\leq t} + s_{>t} | s_{>t}) = \mathbb{E}_{s_{>t}} \left[ \frac{1}{2} \log_2 \left( \frac{\sigma^2_t}{\sigma^2_t} \right) + \frac{\sigma^2_t + (s_{>t})^2}{2\sigma^2_t} - \frac{1}{2} \right].$$ (90)

By Fatou’s lemma, then

$$\mathbb{E}[s^2_{>t}] \leq \liminf_{r \to \infty} \mathbb{E} \left[ \sum_{i=t}^r R^t(\omega_i - L\nu_i)^2 \right] \leq \frac{R^{2t}}{1-R^2} \left( W + L^2 \frac{\Delta^2}{12} \right),$$ (91)

where we used the fact that the $w_i - L\nu_i$ are IID. Thus, since $\lim_{t \to \infty} \sigma^2_t = \lim_{t \to \infty} \sigma^2_t$ and $\lim_{t \to \infty} \mathbb{E}[s^2_{>t}] = 0$, we have from (90) that $\lim_{t \to \infty} D_{KL}(R^t\lambda + g_{\leq t} || g_{\leq t} + s_{>t} | s_{>t}) = 0$. Since $0 \leq D_{KL}(q_t || q) \leq D_{KL}(R^t\lambda + g_{\leq t} || g_{\leq t} + u_{>t} | u_{>t})$, the result is proven.

Since $D_{KL}(q_t || q)$ tends to zero, using (81) and the Cesáro mean immediately proves (51).

V. CONCLUSION

In this work we demonstrated that dithered quantization enables near minimum bitrate prefix-free feedback in LQG control systems. We showed that, for scalar systems, the quantizer output can be encoded using a fixed, time-invariant source code without any asymptotic redundancy.

There are several interesting opportunities for future work. We conjecture that an extension of our time-invariant achievability argument to MIMO plants and to nonsingular codes can be accomplished without much additional proof technology albeit with some additional analytical complications. It would also be interesting to examine adaptive zero-delay source coding codecs.
in our present context; it seems likely that the properties of the invariant measure established in Section IV-C may be useful in analyzing the asymptotic redundancy of such approaches. One could also explore the ergodic properties of the quantizer output in the achievability approach proposed in [12]; this could lead to a dither-free time-invariant achievability result.

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A. Proof of Lemma IV.7

We prove the existence of the invariant PDF using results from [20]. Formally speaking, we use the results of [20] to verify that Markov chain described by (56) has an invariant measure that is equivalent to the Lebesgue measure (i.e., it has a PDF that is strictly positive). Generally speaking, when restating definitions and theorems from [20], we will not do so in full generality but rather adapt them to the present setting. We begin with a definition.

Definition A.1 ([20, Definition 5]). A set $F \in \mathcal{B}(S)$ is called weakly transient with respect to the Markov kernel (56) if there exists a sequence of positive integers $n_1 < n_2 < \ldots$ such that

$$
\sum_{i=1}^{\infty} P_{e_{n_i},d_{n_i}|e_0,d_0}[F|e_0 = e_0, d_0 = d_0] < \infty
$$

holds for $\lambda$ almost-every $(e_0, d_0)$.

A key result from [20] is the following.

Theorem A.2 ([20, Theorem 5]). There exists an invariant PDF $g_\infty$ satisfying (57) and $g_\infty(a, b) > 0$ for all $(a, b) \in S$ if and only if every weakly transient set $F$ has $\lambda(F) = 0$.

We prove that the invariant PDF exists by demonstrating that under the Markov model (56), any weakly transient set must have Lebesgue measure 0. Recall from the discussion in Section IV-A2 that the reconstruction at time $t$ is given by

$$
\tilde{q}_t = Q_\Delta(Ce_t + d_t) - d_t
$$

and the reconstruction error is then $v_t = \tilde{q}_t - Ce_t$. We have the following
Lemma A.3. For all \( n \geq 1 \) we have
\[ f_{e_n|e_0,d_0,v_1^{n-1}}(e_n|e_0,d_0,v_1^{n-1}) = \frac{e^{-(e_n-\mu_n)^2/(2\sigma_n^2)}}{\sqrt{2\pi\sigma_n^2}}, \]  
where
\[ \mu_n(e_0,d_0,v_1^{n-1}) = R^{n-1}M(e_0,d_0) - \sum_{i=0}^{n-2} R^iL v_{n-1-i}, \]  
and
\[ \sigma_n^2 = \sum_{i=0}^{n-1} (R^2)^i W, \]
where \( M \) was defined in (54), and by convention \( v_1^z = \emptyset \) if \( z \leq 0 \), \( \sigma_1^2 = W \), and \( \mu_1 = M(e_0,d_0) \).

Proof: The proof follows from induction on \( n \). The base case for \( n = 1 \) is readily established from (55) after marginalizing over \( d_1 \). Assume the formula (95) holds for \( n = k - 1 \). We demonstrate that it must hold for \( n = k \). We have
\[ f_{e_k|e_0,d_0,v_1^{k-1}}(e_k|e_0,d_0,v_1^{k-1}) = \frac{f_{e_k,v_{k-1}|e_0,d_0,v_1^{k-2}}(e_k,v_{k-1}|e_0,d_0,v_1^{k-2})}{f_{v_{k-1}|e_0,d_0,v_1^{k-2}}(v_{k-1}|e_0,d_0,v_1^{k-2})} \]  
It is immediate that \( v_{k-1} \) is conditionally independent of \( (v_1^{k-2},e_0,d_0) \) given \( e_{k-1} \). By Lemma [IV.1] \( v_{k-1} \) is (pairwise) independent of \( e_{k-1} \). Together, these imply
\[ f_{v_{k-1}|e_{k-1},e_0,d_0,v_1^{k-2}}(v_{k-1}|e_{k-1},e_0,d_0,v_1^{k-2}) = f_{v_{k-1}}(v_{k-1}). \]  
Thus we have
\[ f_{v_{k-1}|e_0,d_0,v_1^{k-2}}(v_{k-1}|e_0,d_0,v_1^{k-2}) = \int_R f_{v_{k-1}}(v_{k-1}) f_{e_{k-1}|e_0,d_0,v_1^{k-2}}(e_{k-1}|e_0,d_0,v_1^{k-2})de_{k-1} = f_{v_{k-1}}(v_{k-1}). \]
We also have (suppressing the implicit dependence on realizations)
\[ f_{e_k,v_{k-1}|e_0,d_0,v_1^{k-2}} = \int_R f_{e_k|e_{k-1},e_0,d_0,v_1^{k-1}} f_{v_{k-1}|e_{k-1},e_0,d_0,v_1^{k-2}} f_{e_{k-1}|e_0,d_0,v_1^{k-2}}de_{k-1} \]
\[ = f_{v_{k-1}} \int_R f_{e_k|e_{k-1},e_0,d_0,v_1^{k-1}} f_{e_{k-1}|e_0,d_0,v_1^{k-2}}de_{k-1} \]
Thus, substituting (100) and (102) into (98) we can write
\[ f_{e_k|e_0,d_0,v_1^{k-1}} = \int_R f_{e_k|e_{k-1},e_0,d_0,v_1^{k-1}} f_{e_{k-1}|e_0,d_0,v_1^{k-2}}de_{k-1}. \]
From the recursion relationship (48), we have that
\[
f_{e_k|e_{k-1}, e_0, d_0, v_1^{k-1}}(e_k|e_{k-1}, e_0, d_0, v_1^{k-1}) = f_{e_k|e_{k-1}, v_{k-1}}(e_k|e_{k-1}, v_{k-1})
\]
\[= e^{-\frac{(e_k-(Re_{k-1}-Lv_{k-1}))^2}{2W}} \sqrt{\frac{1}{2\pi W}}, \tag{104}
\]
and by the inductive assumption we have
\[
f_{e_{k-1}|e_0, d_0, v_1^{k-2}}(e_{k-1}|e_0, d_0, v_1^{k-2}) = e^{-\frac{(e_{k-1}-\mu_{k-1}(e_0, d_0, v_1^{k-2}))^2}{2\sigma_{k-1}^2}} \sqrt{\frac{1}{2\pi \sigma_{k-1}^2}}. \tag{105}
\]

Thus, the integration in (103) is essentially a convolution of two Gaussians. Carrying out the integration by the standard method of completing-the-square in the exponent gives
\[
f_{e_k|e_0, d_0, v_1^{k-1}}(e_k|e_0, d_0, v_1^{k-1}) = e^{-\frac{(e_k-(R\mu_{k-1}(e_0, d_0, v_1^{k-2})-Lv_{k-1}))^2}{2W+R^2\sigma_{k-1}^2}} \sqrt{\frac{1}{2\pi (W + R^2\sigma_{k-1}^2)}}. \tag{106}
\]

Substituting the assumed formulas (95) for \(\sigma_{k-1}^2\) and \(\mu_{k-1}(e_0, d_0, v_1^{k-2})\) into
\[
\sigma_k^2 = W + \sigma_{k-1}^2 R^2 \tag{107}
\]
and
\[
\mu_k(e_0, d_0, v_1^{k-1}) = R\mu_{k-1}(e_0, d_0, v_1^{k-2}) - Lv_{k-1} \tag{108a}
\]

exactly recovers the formula (95) predicts for \(n = k\).

With this in hand, we prove that the Markov process \(\{e_t, d_t\}\) satisfies the hypothesis of Theorem A.2.

**Lemma A.4.** All sets \(F \in \mathcal{B}(\Sigma)\) that are weakly transient with respect to the Markov kernel (56) have \(\lambda(F) = 0\).

**Proof:** We proceed via the contrapositive. Namely, we demonstrate that if \(F \in \mathcal{B}(\Sigma)\) has \(\lambda(F) > 0\), then \(F\) is not weakly transient with respect to the Markov kernel (56). Assume that
$F \in \mathcal{B}(\mathcal{S})$ has $\lambda(F) > 0$. We will prove that for every such $F$ and initial condition $e_0, d_0$ there exists $\epsilon > 0$ such that

$$\mathbb{P}_{e_n, d_n | e_0, d_0}[F|e_0 = e_0, d_0 = d_0] > \epsilon, \text{ for all } n \in \mathbb{N}_+.$$  \hfill (109)

This ensures that for every $e_0, d_0$ and subsequence $n_i \in \mathbb{N}$

$$\sum_{i=1}^{\infty} \mathbb{P}_{e_{n_i}, d_{n_i} | e_0, d_0}[F|e_0 = e_0, d_0 = d_0] = \infty.$$  \hfill (110)

Since $F \subset \mathcal{S}$ has positive Lebesgue measure, it follows that $F$ must contain an open ball properly contained in $\mathcal{S}$. In turn, this open ball must necessarily contain a closed rectangle, i.e., there exists a set $H \subset F$ such

$$H = \{(x, y) \in \mathbb{R}^2 : |x - x_c| \leq \frac{\delta}{2}, |y - y_c| \leq \frac{\delta}{2}\}$$  \hfill (111)

for some sufficiently small $\delta > 0$ and $(x_c, y_c) \in \mathcal{S}$. Note that $H$ has positive Lebesgue measure, and note that by countable additivity for all $n \in \mathbb{N}_+$

$$\mathbb{P}_{e_n, d_n | e_0, d_0}[F|e_0 = e_0, d_0 = d_0] \geq \mathbb{P}_{e_n, d_n | e_0, d_0}[H|e_0 = e_0, d_0 = d_0].$$  \hfill (112)

Consider a fixed $n \in \mathbb{N}_+$. It is obvious that

$$\mathbb{P}_{e_n, d_n | e_0, d_0}[H|e_0 = e_0, d_0 = d_0] \geq \delta^2 \inf_{(x, y) \in H} f_{n|0}(x, y|e_0 = e_0, d_0 = d_0).$$  \hfill (113)

We establish the result of the lemma by finding a lower bound for the infimum on the right-hand side of (113) that does not depend on $n$. Let $H_x = \{x \in \mathbb{R} : |x - x_c| \leq \frac{\delta}{2}\}$ denote the $x$ section of $H$. Denote the $n-$step transition PDF $f_{e_{t+n}, d_{t+n} | e_t, d_t} = f_{t+n|t}$. By the standard Chapman-Kolmogorov equations, we have

$$f_{t+n|t}(e_{t+n}, \delta_{t+n}|e_t, \delta_t) = \int_{\mathcal{S}^{n-1}} \prod_{i=1}^{n} f_{t+1|i}(e_{t+i}, \delta_{t+i}|e_{t+i-1}, \delta_{t+i-1})de_{t+1}d\delta_{t+1}\cdots de_{t+n-1}d\delta_{t+n-1}
= f_d(\delta_{t+n})f_{e_{t+n}|e_t, \delta_t}(e_{t+n}|e_t, \delta_t).$$  \hfill (114)
Recall that $f_{d_t}$ is a uniform density on $[-\Delta/2, \Delta/2]$. From the factorization of the n-step transition PDF (114) and the fact that $H$ is contained strictly inside $S$ we have

$$\inf_{(x,y) \in H} f_{n|0}(x,y|e_0, d_0) = \inf_{x \in H_x} \frac{f_{e_n|e_0,d_0}(x|e_0, d_0)}{\Delta}. \quad (115)$$

Note that by definition

$$f_{e_n|e_0,d_0}(x|e_0, d_0) = \mathbb{E}_{v_1^{n-1}|e_0=d_0} f_{e_n|v_1^{n-1},e_0,d_0}(x|v_1^{n-1}, e_0, d_0).[116]$$

Recall that by definition the reconstruction error satisfies $v_i \in [-\Delta/2, \Delta/2]$ for all $i$. Since “the minimum is less than or equal to the average”, we have

$$\inf_{x \in H_x} f_{e_n|e_0,d_0}(x|e_0, d_0) \geq \inf_{v_1^{n-1} \in [-\Delta/2, \Delta/2]^{(n-1)}} f_{e_n|v_1^{n-1},e_0,d_0}(x|v_1^{n-1}, e_0, d_0) \quad (117)$$

In Lemma A.3, we demonstrated that for any $n$, realizations of the reconstruction error $v_1^{n-1}$, and realizations of the initial conditions $e_0, d_0$ we have

$$f_{e_n|v_1^{n-1},e_0,d_0}(x|v_1^{n-1}, e_0, d_0) = \frac{e^{-(x-\mu_n(e_0,d_0,v_1^{n-1}))^2/(2\sigma_n^2)}}{\sqrt{2\pi\sigma_n^2}}. \quad (118)$$

From the expression for $\sigma_n^2$ in (95) and since $|R| < 1$ we have for all $n$

$$W \leq \sigma_n^2 \leq \frac{W}{1-|R|^2}. \quad (119)$$

From the triangle inequality and the fact that $|v_i| \leq \frac{\Delta}{2}$, we have

$$\mu_n(e_0, d_0, v_1^{n-1}) \leq |R|^{n-1} |M(e_0, d_0)| + \sum_{i=0}^{n-2} |R|^i L |v_{n-1-i}| \quad (120)$$

$$\leq |R|^{n-1} |M(e_0, d_0)| + \frac{1-|R|^{n-1}}{1-|R|} |L| \frac{\Delta}{2} \quad (121)$$

Since $|R| < 1$, for any $n$ we can loosen this bound to have

$$\mu_n(e_0, d_0, v_1^{n-1}) \leq |M(e_0, d_0)| + \frac{|L| \Delta}{2(1-|R|)}. \quad (122)$$

which does not depend on $n$. Likewise, we have the lower bound

$$-|M(e_0, d_0)| - \frac{|L| \Delta}{2(1-|R|)} \leq \mu_n(e_0, d_0, v_1^{n-1}). \quad (123)$$
Let
\[ \gamma(e_0, d_0) = |M(e_0, d_0)| + \frac{|L|\Delta}{2(1 - |R|)}. \] (124)

Thus, we have (see (117))
\[ \inf_{x \in H_x} e_n|v_1^{n-1}, e_0, d_0}(x|v_1^{n-1}, e_0, d_0) \geq \inf_{x \in H_x} \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma^2}. \] (125)

Recall \( H_x = \{ x \in R : |x - x_c| \leq \frac{\Delta}{2} \} \). So long as \( W > 0 \), clearly, for some \( \epsilon > 0 \) depending on \( e_0, d_0, \Delta, W, R, L, x_c, \) and \( \delta \) but explicitly not depending on \( n \) we have
\[ \inf_{x \in H_x} \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma^2} \geq \epsilon(e_0, d_0, \Delta, W, R, L, x_c, \delta). \] (126)

This essentially completes the proof. In summary, we have shown that for all \( n \in N \) for any set \( F \) with nonzero Lebesgue measure, and initial conditions \((e_0, d_0) \in S \) there exists \( x_c, \delta, \epsilon > 0 \) such that
\[ P_{e_n, d_n|e_0, d_0}[F|e_0 = e_0, d_0 = d_0] \geq P_{e_n, d_n|e_0, d_0}[H|e_0 = e_0, d_0 = d_0] \] (127)
\[ \geq \delta^2 \inf_{x \in H_x} \frac{f_{e_n|e_0, d_0}(x|e_0, d_0)}{\Delta} \] (128)
\[ \geq \inf_{x \in H_x} \frac{\delta^2}{\Delta} \frac{f_{e_n|v_1^{n-1}, e_0, d_0}(x|v_1^{n-1}, e_0, d_0)}{\Delta} \] (129)
\[ \geq \inf_{x \in H_x} \frac{\delta^2}{\Delta} \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma^2} \] (130)
\[ \geq \frac{\delta^2}{\Delta} \epsilon(e_0, d_0, \Delta, W, R, L, x_c, \delta) \] (131)
\[ > 0. \] (132)

Consider, for \( \{n_i\} \) a subsequence of \( N \), the infinite series
\[ \sum_{i=1}^{\infty} P_{e_{n_i}, d_{n_i}|e_0, d_0}[F|e_0 = e_0, d_0 = d_0]. \] (133)
Our result in (127) demonstrates that the terms in the sum in (133) are bounded from below by a positive number. Thus the series diverges. This implies that if \( F \) has nonzero Lebesgue measure, it cannot be weakly transient with respect to the Markov Kernel (56).

Combining Lemma A.4 with Theorem A.2 then proves Lemma IV.7.

B. Proof of Lemma IV.8

We now demonstrate that the Markov chain describing (jointly) the dither and innovation processes satisfies some ergodic properties. We begin again with some definitions and a key result from the survey [21].

**Definition A.5 ([21]).** A Markov chain \( \{z_i\} \) on some state space \( S \) is called \( \phi \)-irreducible if there exists a nonzero \( \sigma \)-finite measure \( \phi \) such that for all measurable \( A \subset S \) with \( \phi(A) > 0 \) and all initial conditions \( z_0 = z_0 \) with \( z_0 \in S \) we can find an integer \( n \) such that

\[
P_{z_n|z_0}(z_n \in A|z_0 = z_0) > 0. \tag{134}
\]

**Definition A.6 ([21]).** A Markov chain on \( S \) is called aperiodic if there does not exist \( d \geq 1 \) and disjoint nonempty measurable subsets \( Z_0, Z_1, \ldots Z_{d-1} \) such that when \( z_{n-1} \in Z_i \)

\[
P_{z_n|z_{n-1}}(z_n \in Z_{i+1 \text{ mod } d}|z_{n-1} = z_{n-1}) = 1. \tag{135}
\]

**Definition A.7 (Total Variation).** Define the total variation norm between two probability measures \( P_1 : \mathcal{B}(S) \to [0, 1] \) and \( P_2 : \mathcal{B}(S) \to [0, 1] \) defined on the same measure space via

\[
\|P_1 - P_2\|_{\text{T.V.}} \overset{\Delta}{=} \sup_{A \in \mathcal{S}} |P_1(A) - P_2(A)|. \tag{136}
\]

**Theorem A.8 ([21, Theorem 4]).** Consider a Markov chain \( \{r_i\} \) on a countably generated state-space that is aperiodic, \( \phi \)-irreducible, and admits an invariant measure \( P_\infty \) that is absolutely continuous with respect to Lebesgue measure. For \( \lambda \)-almost every initial condition \( r_0 \) we have

\[
\lim_{n \to \infty} \|P_{r_n|r_0}(r_n \in \circ|r_0 = r_0) - P_\infty(\circ)\|_{\text{T.V.}} = 0. \tag{137}
\]
Furthermore, the law of large numbers holds in the following sense. Assume the initial state of the chain $z_0$ is a random variable that is absolutely continuous with respect to $\lambda$. For all measurable functions $\eta$ such that $E_{r \sim P_\infty}[|\eta(r)|] < \infty$ and $E_{r_0}[|\eta(r_0)|] < \infty$ we have

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{i=0}^N \eta(r_i) = E_{r \sim P_\infty}[\eta(r)].$$

The Borel $\sigma$-algebra on $S$ is countably generated, and the Lebesgue measure on $S$ (denoted $\lambda$) is $\sigma$-finite. Thus, to guarantee that the n-step conditional probability measures for the Markov chain $\{e_t, d_t\}$ defined by (55) will converge to the stationary distribution in total variation, and to verify that the law of large numbers holds in the sense of (138), we can verify that the chain is $\lambda$-irreducible and aperiodic.

**Lemma A.9.** The Markov chain induced by (55) and (56) is $\lambda$-irreducible and aperiodic.

**Proof:** We first demonstrate $\lambda$-irreducibility. Let $A \subset S$ be any set of positive Lebesgue measure. Take $(e_0, \delta_0) \in S$. We have

$$P_{e_1,d_1|e_0,d_0}[(e_1, d_1) \in A|e_0 = e_0, d_0 = \delta_0] = \int_A f_{e_1+d_1|e_0,\delta_0}(e_1, \delta_1|e_0, \delta_0)de_1d\delta_1$$

$$= \int_A \frac{1}{\Delta} e^{-\frac{(e_1-M(e_0,d_0))^2}{2\pi W}}de_1d\delta_1$$

$$> 0$$

This established that taking $n = 1$ always allows us to satisfy the requirements for $\lambda$-irreducibility.

This allows the proof of aperiodicity to follow immediately. Assume that one has disjoint nonempty measurable subsets $S_1, S_2, \cdots \subset S$ satisfying the hypotheses of Definition [A.6]. Take $(e_t, d_t) \in S_1$. By assumption

$$P[(e_{t+1}, d_{t+1}) \in S_2|e_t = e_t, d_t = d_t] = 1.$$
contradiction, since the hypothesis of Definition A.6 is that \( S_1 \) is nonempty and \( S_1 \cap S_2 = \emptyset \).

Lemma A.9 verifies the hypothesis of Theorem A.8 and thus proves Lemma IV.8. Note that the convergence in total variation implies weak convergence (i.e., convergence in distribution).