We introduce a spectral transform for the finite relativistic Toda lattice (RTL) in generalized form. In the nonrelativistic case, Moser constructed a spectral transform from the spectral theory of symmetric Jacobi matrices. Here we use a non-symmetric generalized eigenvalue problem for a pair of bidiagonal matrices \((L, M)\) to define the spectral transform for the RTL. The inverse spectral transform is described in terms of a terminating T-fraction. The generalized eigenvalues are constants of motion and the auxiliary spectral data have explicit time evolution. Using the connection with the theory of Laurent orthogonal polynomials, we study the long-time behaviour of the RTL. As in the case of the Toda lattice the matrix entries have asymptotic limits. We show that \(L\) tends to an upper Hessenberg matrix with the generalized eigenvalues sorted on the diagonal, while \(M\) tends to the identity matrix.

1. Introduction

The relativistic Toda lattice (RTL) was introduced by Ruijsenaars \([17]\) and studied in \([1, 2, 4, 14, 18]\), for a review see \([12]\). The finite RTL is defined by the system of equations

\[
\ddot{q}_n = \epsilon^2 \dot{q}_n \left( \frac{\exp(q_{n-1} - q_n)}{1 + \epsilon^2 \exp(q_{n-1} - q_n)} - \frac{\exp(q_n - q_{n+1})}{1 + \epsilon^2 \exp(q_n - q_{n+1})} \right),
\]

with \(N \in \mathbb{N}, 1 \leq n \leq N\) and the convention \(q_0 \equiv -\infty\) and \(q_{N+1} \equiv +\infty\). Here the \(q_n\) are functions in the time parameter \(t\). The system (1.1) is the Newtonian form of a Hamiltonian system with Hamiltonian

\[
H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{n=1}^{N} e^{\eta_n} h(q_{n-1} - q_n) h(q_n - q_{n+1}),
\]

where \(h(x) = \sqrt{1 + \epsilon^2 e^x}\). Setting \(\dot{q}_n = \dot{Q}_n + 1/\epsilon\) and letting \(\epsilon \to 0\), one can easily check that (1.1) reduces to the equations of motion of the finite nonrelativistic Toda lattice,

\[
\ddot{Q}_n = \exp(Q_{n-1} - Q_n) - \exp(Q_n - Q_{n+1}), \quad 1 \leq n \leq N,
\]

so that (1.1) is a one-parameter perturbation.
In analogy with the Flaschka variables for the nonrelativistic Toda lattice one can use a (non-invertible) change of variables to prove integrability. Bruschi and Ragnisco [1, 2], see also [12, 18], obtain the two forms

\[
\begin{aligned}
\dot{a}_n &= \frac{b_n}{a_{n+1}} - \frac{b_{n-1}}{a_{n-1}}, \\
\dot{b}_n &= b_n \left( \frac{1}{a_n} - \frac{1}{a_{n+1}} \right),
\end{aligned}
\tag{1.2}
\]

and

\[
\begin{aligned}
\dot{a}_n &= a_n (b_{n-1} - b_n), \\
\dot{b}_n &= b_n (a_n - a_{n+1} + a_{n-1} - a_{n+1}),
\end{aligned}
\tag{1.3}
\]

both with \(a_n > 0\) for \(1 \leq n \leq N\), \(b_n > 0\) for \(1 \leq n \leq N - 1\), and \(b_0 \equiv 0\), \(b_N \equiv 0\). The systems (1.2) and (1.3) can also be written in matrix form. Define two bidiagonal matrices \(L\) and \(M\) by

\[
L = \begin{pmatrix}
a_1 & 1 & 0 & \cdots & \cdots & 0 \\
0 & a_2 & 1 & \cdots & \vdots \\
0 & 0 & a_3 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & a_{N-1} & 1 \\
0 & \cdots & \cdots & \cdots & 0 & a_N
\end{pmatrix}, \\
M = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 \\
-b_1 & 1 & 0 & \cdots & \vdots \\
0 & -b_2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -b_{N-1} & 1
\end{pmatrix}, \tag{1.4}
\]

Then Suris [13] noted that (1.2) can be written in the Lax form

\[
\begin{aligned}
\dot{L} &= LA - BL \\
\dot{M} &= MA - BM,
\end{aligned}
\tag{1.5}
\]

where

\[
A = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\frac{b_1}{a_2} & 0 & \cdots & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{b_{N-1}}{a_N} & 0
\end{pmatrix}, \\
B = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\frac{b_2}{a_3} & 0 & \cdots & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{b_{N-1}}{a_N} & 0
\end{pmatrix}. \tag{1.6}
\]

Then \(A = -(L^{-1}M)_-\) and \(B = -(ML^{-1})_-\), where we use \(X_-\) to denote the strictly lower triangular part of \(X\). The other system (1.3) can also be written in the form (1.3)
but now $A = -(M^{-1}L)_-$ and $B = -(LM^{-1})_-$, see Remark 3.4 below. In Section 3.1 we show that (1.2) and (1.3) are special cases of a generalized form of the finite RTL, namely

$$\begin{cases}
\dot{L} = \left(F(LM^{-1})\right)_- L - L \left(F(M^{-1}L)\right)_-
\dot{M} = \left(F(LM^{-1})\right)_- M - M \left(F(M^{-1}L)\right)_-
\end{cases} \quad (1.7)$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is an arbitrary function on $\mathbb{R}^+$. For $F(x) = 1/x$, we obtain (1.2) and for $F(x) = x$, we obtain (1.3).

The main goal of this paper is to solve the finite RTL in its general form (1.7) with the aid of a direct and inverse spectral transform, in a similar way as was done by Moser [13] for the nonrelativistic case. While the spectral transform for the nonrelativistic case is based on the spectral theory of tridiagonal (Jacobi) matrices and is connected with orthogonal polynomials, we will show that in the relativistic case, the spectral transform is based on the spectral theory of pairs of bidiagonal matrices (1.4) and Laurent orthogonal polynomials [10, 16, 20, 6]. See Figure 1 for the general scheme of the direct and inverse spectral problem.

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![Figure 1. The scheme of the direct and inverse spectral problem.](image)

To the best of our knowledge, the only transform methods that have been explicitly introduced in the study of the RTL are the ones by Ragnisco and Bruschi [14] for the periodic RTL, by Ruijsenaars [17], and the scattering transform of Cosentino [4] for the infinite RTL.

The Laurent orthogonal polynomials $P_n$ are defined by

$$P_0 \equiv 1, \quad P_n(z) = \det(zM_n - L_n), \quad 1 \leq n \leq N, \quad (1.8)$$

where $L_n$ and $M_n$ are the $n \times n$ upper left corner blocks of $L$ and $M$, respectively. They satisfy the recurrence relation

$$P_n(z) = (z - a_n)P_{n-1}(z) - b_{n-1}zP_{n-2}(z), \quad 1 \leq n \leq N,$$
with \( P_0 \equiv 1, \ P_{-1} \equiv 0 \). Under the assumption that \( a_n > 0 \) and \( b_n > 0 \), it is known that the zeros of these polynomials are simple, real and positive, see [3, 10, 11, 16] and Lemma 2.1 below. The first set of spectral data we choose are the ordered zeros of the polynomial \( P_N(z) = \det(zM - L) \) which we denote by \( 0 < \lambda_1 < \cdots < \lambda_N \). Then the \( \lambda_j \) are also the eigenvalues for the generalized eigenvalue problem of the pair \( (L, M) \)
\[
L \vec{e} = \lambda M \vec{e}.
\] (1.9)
To obtain our second set of spectral data we associate with each \( \lambda_j \) the value
\[
w_j = (\vec{u}_j^T M \vec{v}_j)^{-1},
\]
where \( \vec{u}_j \) and \( \vec{v}_j \) are the left and right eigenvectors of the generalized eigenvalue problem (1.9), respectively, normalized to have their first component equal to 1. We prove in Section 2.1 that \( w_j > 0 \) and \( \sum_{j=1}^{N} w_j = 1 \). The direct spectral transform
\[
\left\{ a_1, \ldots, a_N > 0 \atop b_1, \ldots, b_{N-1} > 0 \right\} \rightarrow \left\{ 0 < \lambda_1 < \cdots < \lambda_N \atop w_1, \ldots, w_N > 0, \ \sum_{j=1}^{N} w_j = 1 \right\}
\]
is invertible. Using T-fractions [6, 11], we explain in Section 2.2 how to obtain the \( a_n \) and \( b_n \) when the spectral data \( \lambda_j \) and \( w_j \) are given. We also note that the polynomials \( P_n \) satisfy
\[
\sum_{j=1}^{N} P_n(\lambda_j) P_m(\lambda_j) \frac{w_j}{\lambda_j} = 0, \quad \text{if} \ n > m,
\]
which is known as Laurent orthogonality [3, 9, 10, 16].

In Section 3.2 we describe the time evolution of the spectral data for the finite RTL in the generalized form (1.7) with positive initial data. We show that the \( \lambda_j \) are time independent and that
\[
w_j(t) = \frac{w_j(0)e^{-F(\lambda_j)t}}{\sum_{k=1}^{N} w_k(0)e^{-F(\lambda_k)t}}, \quad 1 \leq j \leq N.
\]
Finally, in Section 4 we investigate the behaviour of the RTL for \( t \rightarrow \pm \infty \). We assume that \( F \) is strictly increasing. Then
\[
\lim_{t \rightarrow \pm \infty} b_n(t) = 0 \quad \text{(1.10)}
\]
and
\[
\lim_{t \rightarrow -\infty} a_n(t) = \lambda_{N-n+1}, \quad \lim_{t \rightarrow +\infty} a_n(t) = \lambda_n, \quad 1 \leq n \leq N \quad \text{(1.11)}
\]
Similar limits hold for the case that \( F \) is strictly decreasing. This sorting property of the RTL indicates that the choice of the spectral data \( \lambda_1 < \cdots < \lambda_N \) is very natural. In Theorems 4.3 and 4.4 we establish the precise rates of convergence of (1.10) and (1.11). This is similar to Deift et al. [3] who obtained precise rates of convergence for the SVD flow (Singular Value Decomposition).
2. The direct and inverse spectral problem

In this section we introduce a transformation from matrix data $a_n > 0$, $b_n > 0$ to spectral data $\lambda_j$, $w_j$. We prove that this spectral transform is a bijection between $(\mathbb{R}^+)^{2N-1}$ and its image \( \{ 0 < \lambda_1 < \cdots < \lambda_N, \ w_j > 0, \ \sum_{j=1}^N w_j = 1 \} \).

2.1. The direct spectral problem

We start from positive matrix data $a_n, 1 \leq n \leq N$, and $b_n, 1 \leq n \leq N - 1$, and construct the bidiagonal matrices $L$ and $M$ as in (1.4).

Define the finite set of monic polynomials $P_n$ as

\[ P_0 \equiv 1, \quad P_n(z) = \det(zM_n - L_n), \quad 1 \leq n \leq N \]

where $L_n$ and $M_n$ are obtained from the matrices $L$ and $M$, respectively, by deleting the last $N - n$ rows and columns. These polynomials satisfy the recurrence relation

\[ P_n(z) = (z - a_n)P_{n-1}(z) - b_{n-1}z P_{n-2}(z), \quad 1 \leq n \leq N, \]

(2.1) as can be easily seen by expanding the determinant $\det(zM_n - L_n)$ first by the last column and then by the last row. The initial conditions for (2.1) are $P_0 \equiv 1$ and $P_{-1} \equiv 0$. The zeros of these polynomials behave in a way which is similar to the behaviour of zeros of polynomials which are orthogonal on the positive real line. The following lemma can be found in [9, 10, 11, 16], but we include a proof for completeness.

Lemma 2.1 The zeros of the polynomials $P_n(z) = \det(zM_n - L_n)$ are real, simple, positive (0 < $z_{1,n} < \cdots < z_{n,n}$) and have the interlacing property, which means that

\[ 0 < z_{1,n} < z_{1,n-1} < z_{2,n} < z_{2,n-1} < \cdots < z_{n-1,n-1} < z_{n,n}. \]

Proof. The lemma is true for the case $n = 1$ because $P_1(z) = z - a_1$ and $a_1 > 0$. Suppose that the lemma is true for some $n < N$. By the interlacing property we then know that $P_{n-1}$ changes sign between any two consecutive zeros of $P_n$. Evaluating the recurrence

\[ P_{n+1}(z) = (z - a_{n+1})P_n(z) - b_nz P_{n-1}(z) \]

at a zero $z_{j,n}$ of $P_n$, we get $P_{n+1}(z_{j,n}) = -b_n z_{j,n} P_{n-1}(z_{j,n})$. Since $b_n > 0$ and $z_{j,n} > 0$, we see that $P_{n+1}$ and $P_{n-1}$ have opposite signs at the zeros of $P_n$. It follows that also $P_{n+1}$ changes sign between any two consecutive zeros of $P_n$, and therefore it has at least one zero in each of the intervals $(z_{j,n}, z_{j+1,n})$, $1 \leq j \leq n - 1$. Since $P_{n-1}(z_{n,n})$ is positive, it also follows that $P_{n+1}(z_{n,n})$ is negative. Since $P_{n+1}$ is a monic polynomial, it must have a zero in $(z_{n,n}, \infty)$.

To complete the proof we will show that there is also a zero in $(0, z_{1,n})$. Evaluating the recurrence at 0, we get $P_{n+1}(0) = -a_{n+1}P_n(0)$. Since $P_n$ is a monic polynomial of degree $n$ having all its zeros on the positive real line, we find that $(-1)^n P_n(0) > 0$. Thus $(-1)^n P_{n+1}(0) < 0$. Since $P_{n-1}$ is a monic polynomial of degree $n - 1$ with all its zeros
to the right of $z_{1,n}$, we obtain $(-1)^nP_{n-1}(z_{1,n}) < 0$. Since $P_{n+1}$ and $P_{n-1}$ differ in sign at $z_{1,n}$, we get $(-1)^nP_{n+1}(z_{1,n}) > 0$. So $P_{n+1}$ has opposite signs in 0 and $z_{1,n}$, which shows that there is indeed a zero in $(0, z_{1,n})$. The lemma now follows by mathematical induction. \hfill \Box

Remark 2.2 An analogous reasoning as in Lemma 2.1 can be applied to the polynomials $\Delta_n(z) = \det(z\tilde{M}_n - \tilde{L}_n)$, $\Delta_0 \equiv 1$, where $\tilde{L}_n$ and $\tilde{M}_n$ are obtained from the matrices $L$ and $M$ by deleting the first $N-n$ rows and columns, respectively. These polynomials satisfy the recurrence
\begin{equation}
\Delta_n(z) = (z - a_{N-n+1})\Delta_{n-1}(z) - b_{N-n+1}z\Delta_{n-2}(z),
\end{equation}
for $1 \leq n \leq N$, with initial conditions $\Delta_0 \equiv 1$ and $\Delta_{-1} \equiv 0$. So also the zeros of these polynomials are real, simple and positive and have the interlacing property.

Consider now the generalized eigenvalue problem of the pair $(L, M)$
\begin{equation}
L\vec{x} = \lambda M\vec{x}.
\end{equation}
The eigenvalues are the zeros of the polynomial $P_N(\lambda) = \det(\lambda M - L)$. From Lemma 2.1 we know that these eigenvalues are real, simple and positive. We denote them by
\[0 < \lambda_1 < \cdots < \lambda_N < \infty.\]
We use $\vec{v}_j$ to denote the right eigenvector corresponding to the eigenvalue $\lambda_j$, normalized so that the first component is equal to 1. From the bidiagonal structure of the matrices $M$ and $L$ it easily follows that the first component of any eigenvector of (2.3) is non-zero, so that this normalization is always possible. We use $\vec{u}_j$ to denote the corresponding left eigenvector, also normalized to have the first component equal to 1. We view $\vec{u}_j$ as a column vector so that $\vec{u}_j^T L = \lambda_j \vec{u}_j^T M$. We collect the eigenvectors in the matrices
\[V = (\vec{v}_1 \cdots \vec{v}_N) \quad \text{and} \quad U = (\vec{u}_1 \cdots \vec{u}_N).\]
A left and right eigenvector of this generalized eigenvalue problem, corresponding to different eigenvalues, are $M$-orthogonal, that is, $\vec{u}_j^T M \vec{v}_k = 0$ whenever $j \neq k$. This implies that $U^T M V$ is a diagonal matrix. Since the three matrices $U$, $M$, and $V$ are invertible, the diagonal elements of $U^T M V$ are non-zero, and we can make the following definition.

Definition 2.3 Let $U$ and $V$ be as above. Then we define
\begin{equation}
w_j = \frac{1}{(U^T M V)_{jj}} = \left(\vec{u}_j^T M \vec{v}_j\right)^{-1}, \quad 1 \leq j \leq N.
\end{equation}
Now we have defined the spectral data $\lambda_j$ and $w_j$ for $1 \leq j \leq N$. To complete the description of the direct spectral transform we show that $w_j > 0$ and $\sum_{j=1}^N w_j = 1$. To prove this we introduce the Weyl (or resolvent) function
\[f(z) = \vec{e}_1^T (z M - L)^{-1} \vec{e}_1 = \left((z M - L)^{-1}\right)_{1,1}, \quad z \in \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N\},\]
with $\vec{e}_1$ the first unit vector in $\mathbb{R}^N$. 


Lemma 2.4 The Weyl function is given by

\[ f(z) = \sum_{j=1}^{N} \frac{w_j}{z - \lambda_j}. \]

Proof. Note that by definition of \( w_j \), we have \( \vec{u}_j^T M \vec{v}_k = \frac{1}{w_j} \delta_{j,k} \), and so

\[ \tilde{u}_j^T L \vec{v}_k = \lambda_j \tilde{u}_j^T M \vec{v}_k = \frac{\lambda_j}{w_j} \delta_{j,k}, \quad 1 \leq j, k \leq N. \]

In matrix form this is \( U^T M V = \text{diag} \left( \frac{1}{w_1}, \ldots, \frac{1}{w_N} \right) \) and \( U^T LV = \text{diag} \left( \frac{\lambda_1}{w_1}, \ldots, \frac{\lambda_N}{w_N} \right) \).

Combining these two equations, we find that

\[ U^T (z M - L) V = \text{diag} \left( \frac{z - \lambda_1}{w_1}, \ldots, \frac{z - \lambda_N}{w_N} \right). \]

From this we readily obtain that

\[ (zM - L)^{-1} = V \text{diag} \left( \frac{w_1}{z - \lambda_1}, \ldots, \frac{w_N}{z - \lambda_N} \right) U^T. \] (2.5)

We now recall that the eigenvectors \( \vec{u}_j \) and \( \vec{v}_j \) are normalized with their first component equal to 1. Thus the first rows of the matrices \( U \) and \( V \) contain only ones. Then (2.5) gives that

\[ f(z) = \left( (zM - L)^{-1} \right)_{1,1} = \sum_{j=1}^{N} \frac{w_j}{z - \lambda_j}. \]

The lemma shows that the \( \lambda_j \) are simple poles of the Weyl function and that the \( w_j \) are the corresponding residues. A second representation of \( f \) is in terms of the polynomials \( \Delta_n \) introduced in Remark 2.2.

Lemma 2.5 The Weyl function \( f \) can be written as \( f(z) = \frac{\Delta_{N-1}(z)}{\Delta_N(z)}. \)

Proof. This follows immediately from Cramer’s rule. \( \square \)

As a corollary we get

Corollary 2.6 We have \( w_j > 0 \) for every \( j \) and \( \sum_{j=1}^{N} w_j = 1. \)

Proof. If we combine Lemma 2.4 and Lemma 2.5, we get

\[ \frac{\Delta_{N-1}(z)}{\Delta_N(z)} = \sum_{j=1}^{N} \frac{w_j}{z - \lambda_j}. \] (2.6)

From Remark 2.2 we know that the zeros of \( \Delta_{N-1} \) and \( \Delta_N \) are interlacing. Since \( \Delta_{N-1} \) and \( \Delta_N \) are monic polynomials, it then follows that \( w_j > 0 \) for every \( j \). The fact that \( \sum_{j=1}^{N} w_j = 1 \) follows by multiplying (2.6) by \( z \) and then letting \( z \to \infty. \) \( \square \)
We now have the direct spectral transform
\[(\mathbb{R}^+)^{2N-1} \rightarrow \Lambda : (a_1, \ldots, a_N, b_1, \ldots, b_{N-1}) \mapsto (\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N)\]
where
\[
\Lambda = \left\{ (\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N) \mid \begin{array}{l}
0 < \lambda_1 < \cdots < \lambda_N < \infty, \\
w_j > 0, \sum_{j=1}^N w_j = 1
\end{array} \right\}. \tag{2.7}
\]

We conclude this paragraph by establishing an explicit representation of the Weyl function in terms of the given matrix data. This representation is also the key to the inverse spectral transform. It turns out that the function \(zf(z)\) can be written as a special continued fraction, known as a T-fraction, see [11, 6].

**Lemma 2.7** The Weyl function \(f\) can be written as
\[
f(z) = \frac{\Delta_{N-1}(z)}{\Delta_N(z)} = \frac{1}{z - a_1 - \frac{b_2 z}{z - a_2 - \frac{b_3 z}{z - a_3 - \cdots}}}. \tag{2.8}
\]

**Proof.** Let \(r_k(z) = \frac{\Delta_{k-1}(z)}{\Delta_k(z)}\) for \(1 \leq k \leq N\). Since \(\Delta_0(z) = 1\) and \(\Delta_1(z) = z - a_N\) we have \(r_1(z) = \frac{1}{z - a_N}\). From the recurrence (2.2) satisfied by the polynomials \(\Delta_k\), we get that
\[
r_k(z) = \frac{1}{z - a_{N-k+1} - z b_{N-k+2} r_{k-1}(z)}. 
\]
Then it is easy to show by induction that
\[
r_k(z) = \frac{1}{z - a_{N-k+1} - \frac{b_{N-k+2} z}{z - a_{N-k+2} - \frac{b_{N-k+3} z}{z - a_{N-k+3} - \cdots}}}. \tag{2.9}
\]
Taking \(k = N\) in (2.9) we obtain the lemma. \(\square\)

From Lemma 2.4 and Lemma 2.7 it is easy to see that the spectral transform is injective. In the next section we show that it is also surjective.

### 2.2. The inverse spectral problem

We now start from spectral data \((\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N)\) in \(\Lambda\), see (2.7), and show that there exist positive matrix data \((a_1, \ldots, a_N, b_1, \ldots, b_{N-1})\) which correspond to the given spectral data. We also give a method to construct these matrix data. In view of Lemma
it is our task to develop $\sum_{j=1}^{N} \frac{w_j}{z-\lambda_j}$ into a continued fraction of the form (2.8) with positive $a_n$ and $b_n$, that is, we want

$$\sum_{j=1}^{N} \frac{w_j}{z-\lambda_j} = \frac{1}{z - a_1 - \frac{b_1z}{z - a_2 - \frac{b_2z}{z - a_3 - \ldots - \frac{b_{N-1}z}{z - a_N}}}.$$  \hspace{1cm} (2.10)

**Theorem 2.8** For given spectral data $(\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N)$ in $\Lambda$ there exist positive $a_n$ and $b_n$ such that (2.10) holds.

**Proof.** Let us say that $f \in W_N$ if $f$ is of the form

$$f(z) = \sum_{j=1}^{N} \frac{w_j}{z-\lambda_j},$$  \hspace{1cm} (2.11)

with $0 < \lambda_1 < \cdots < \lambda_N$, $w_j > 0$, and $\sum_{j=1}^{N} w_j = 1$. We will prove that for $f \in W_N$, there exist $a, b > 0$ and $g \in W_{N-1}$ so that

$$f(z) = \frac{1}{z - a - bzg(z)}.$$  

The theorem then follows by repeated application of this fact.

To prove the claim, we observe that $f$ is a rational function which we can write as

$$f(z) = \frac{p_{N-1}(z)}{q_N(z)},$$

with $p_{N-1}$ and $q_N$ monic polynomials of degrees $N-1$ and $N$, respectively (here we use the fact that $\sum_{j=1}^{N} w_j = 1$). Then there exist $a, b \in \mathbb{R}$ and a monic polynomial $p_{N-2}$ of degree at most $N-2$ such that

$$q_N(z) = (z - a)p_{N-1}(z) - bzp_{N-2}(z).$$  \hspace{1cm} (2.12)

Taking $z = 0$ in (2.12) we find

$$a = -\frac{q_N(0)}{p_{N-1}(0)} = \left(\sum_{j=1}^{N} \frac{w_j}{\lambda_j}\right)^{-1},$$  \hspace{1cm} (2.13)

which is positive, since all $\lambda_j$ and $w_j$ are positive. To find $b$, we write $-q_N(z) + (z - a)p_{N-1}(z) = bzp_{N-2}(z)$ and compute the coefficient of $z^{N-1}$ on the left-hand side. Since $q_N(z) = \prod_{j=1}^{N} (z - \lambda_j)$ and $p_{N-1}(z) = \sum_{k=1}^{N} w_k \prod_{j \neq k} (z - \lambda_j)$, this coefficient is

$$\sum_{j=1}^{N} \lambda_j - \sum_{k=1}^{N} w_k \sum_{j=1, j \neq k}^{N} \lambda_j - a = \sum_{j=1}^{N} \lambda_j \left(1 - \sum_{k=1, k \neq j}^{N} w_k\right) - a$$

$$= \sum_{j=1}^{N} w_j \lambda_j - \left(\sum_{j=1}^{N} \frac{w_j}{\lambda_j}\right)^{-1}.$$
In the last step we used (2.13) and \( \sum_{j=1}^{N} w_j = 1 \). Next, from Jensen’s inequality
\[
\phi \left( \int z d\mu(z) \right) < \int \phi(z) d\mu(z)
\]
applied to the strictly convex function \( \phi(z) = 1/z \) and the probability measure \( \mu = \sum_{j=1}^{N} w_j \delta_{\lambda_j} \), we get that
\[
\sum_{j=1}^{N} w_j \lambda_j - \left( \sum_{j=1}^{N} \frac{w_j}{\lambda_j} \right)^{-1} > 0.
\]
(2.14)
So the coefficient of \( z^{N-1} \) in \( b z p_{N-2}(z) \) is positive. Since \( p_{N-2} \) is a monic polynomial of degree \( \leq N - 2 \), it then follows that \( p_{N-2} \) has exact degree \( N - 2 \) and that \( b \) is equal to (2.14). Hence \( b > 0 \). If we now define \( g = p_{N-2}/p_{N-1} \) then we have that
\[
f(z) = \frac{1}{z - a - bzg(z)}, \quad \text{with } a, b > 0.
\]

It remains to show that \( g \in \mathcal{W}_{N-1} \). Observe that the poles of \( g \) are the zeros of \( f \). From (2.11) with \( w_j > 0 \) it follows that the zeros of \( f \) interlace with its poles \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N \). So the poles of \( g \) are simple and positive. If we denote them by \( 0 < \lambda_1^* < \lambda_2^* < \cdots < \lambda_{N-1}^* \), then
\[
g(z) = \frac{p_{N-2}(z)}{p_{N-1}(z)} = \sum_{j=1}^{N-1} \frac{w_j^*}{z - \lambda_j^*},
\]
for certain residues \( w_j^* \) that satisfy \( \sum_{j=1}^{N-1} w_j^* = 1 \), because \( p_{N-2} \) and \( p_{N-1} \) are monic. We also have
\[
w_j^* = \lim_{z \to \lambda_j^*} (z - \lambda_j^*) g(z) = \lim_{z \to \lambda_j^*} \frac{z - \lambda_j^*}{b} \left( \frac{z - a - \frac{1}{zf(z)}}{z} \right)
\]
\[
= -\lim_{z \to \lambda_j^*} \frac{z - \lambda_j^*}{bzf(z)} = -\frac{1}{b\lambda_j^* f'(\lambda_j^*)}.
\]
Since \( b > 0, \lambda_j^* > 0, \) and \( f'(\lambda_j^*) = -\sum_{k=1}^{N} \frac{w_k}{(\lambda_j^* - \lambda_k)^2} < 0 \), we see that \( w_j^* > 0 \) for every \( j \).

This completes the proof of the theorem. \( \square \)

2.3. Laurent orthogonality

Suppose that \( \mu \) is a positive measure with support in \( \mathbb{R}^+ \), for which all the strong moments \( \int z^k d\mu(z), \ k \in \mathbb{Z}, \) exist. For \( n \) not exceeding the number of mass points in the support of \( \mu \), the monic Laurent orthogonal polynomial, \( p_n \), of degree \( n \) is defined as the monic orthogonal polynomial of degree \( n \) for the varying measure \( d\mu_n(z) = z^{-n} d\mu(z) \). So \( p_n \) satisfies the orthogonality relations
\[
\int p_n(z) z^k \frac{d\mu(z)}{z^n} = 0, \quad 0 \leq k \leq n - 1.
\]
Cochran and Cooper [3] and Jones et al. [8, 9, 10] showed that these polynomials satisfy a recurrence relation of the form
\[ p_n(z) = (z - \alpha_n)p_{n-1}(z) - \beta_{n-1}z p_{n-2}(z), \quad n \geq 1, \]
with initial conditions \( p_0 \equiv 1 \) and \( p_{-1} \equiv 0 \), and positive recurrence coefficients \( \alpha_n \) and \( \beta_{n-1} \). From the theory of orthogonal polynomials it follows that the zeros of \( p_n \) are simple, real and positive, since \( \mu \) is a measure on the positive real line. Some examples of (continuous) Laurent orthogonal polynomials are given in [15].

We now show that the polynomials \( P_n(z) = \det(zM_n - L_n) \) introduced in (1.8)
are the monic Laurent orthogonal polynomials with respect to the discrete probability measure
\[ \mu_N = \sum_{j=1}^{N} w_j \delta_{\lambda_j}, \]
see also [16] for a similar situation. These polynomials are also generated by the recurrence relation (2.1). This illustrates the connection between the spectral transform, proposed in Section 2.1, and the theory of Laurent orthogonal polynomials.

**Theorem 2.9** Let \( a_n > 0, 1 \leq n \leq N \), and \( b_n > 0, 1 \leq n \leq N - 1 \). Let the polynomials \( P_n \) be defined by the recurrence relation (2.1) with \( P_0 \equiv 1 \) and \( P_{-1} \equiv 0 \). Let \( \lambda_j \) and \( w_j \) be the spectral data associated with \( a_n, b_n \) as defined in Section 2.1. Then
\[ \sum_{j=1}^{N} P_m(\lambda_j) P_n(\lambda_j) \frac{w_j}{\lambda_j^n} \prod_{k=1}^{n} b_k = \delta_{m,n}, \quad 0 \leq m \leq n \leq N - 1. \]

**Proof.** Evaluating the recurrence relation (2.1) at \( \lambda_j \) (which is a zero of \( P_N \)) one can easily see that the right eigenvector \( \vec{v}_j \) for the generalized eigenvalue problem (2.3) is equal to
\[ \vec{v}_j = \left( 1, P_1(\lambda_j), \ldots, P_{N-1}(\lambda_j) \right)^T. \]

Also note that the left eigenvector \( \vec{u}_j \) can be written as
\[ \vec{u}_j = \left( 1, \frac{P_1(\lambda_j)}{\lambda_j b_1}, \frac{P_2(\lambda_j)}{\lambda_j^2 b_2 b_1}, \ldots, \frac{P_{N-1}(\lambda_j)}{\lambda_j^{N-1} b_{N-1} \ldots b_1} \right)^T. \]

Since \( U^T M V = \text{diag} \left( \frac{1}{w_1}, \ldots, \frac{1}{w_N} \right) \) by the definition of the \( w_j \), we have
\[ V \text{diag}(w_1, \ldots, w_N) U^T = M^{-1}. \]

Now note that \( M \) is lower triangular with ones on the diagonal, and therefore the same holds for \( M^{-1} \). So if \( 0 \leq m \leq n \leq N - 1 \) we have
\[ \left( V \text{diag}(w_1, \ldots, w_N) U^T \right)_{m+1,n+1} = \delta_{m,n}. \]

Using (2.15) and (2.16) we finally get
\[ \sum_{j=1}^{N} P_m(\lambda_j) P_n(\lambda_j) \frac{w_j}{\lambda_j^n} \prod_{k=1}^{n} b_k = \delta_{m,n}. \]
The relativistic Toda lattice and Laurent orthogonal polynomials

which proves the theorem.

The theorem suggests an alternative method for performing the inverse spectral transform. Knowing the spectral data, one can construct the monic Laurent orthogonal polynomials corresponding to the discrete measure \( \mu_N \). These polynomials satisfy a three term recurrence relation with coefficients equal to \( a_n \) and \( b_n \).

### 3. The generalized finite RTL and spectral evolution

In the introduction we have already mentioned that the finite RTL

\[
\begin{cases}
\dot{a}_n = \frac{b_n}{a_{n+1}} - \frac{b_{n-1}}{a_n}, & 1 \leq n \leq N, \\
\dot{b}_n = b_n \left( \frac{1}{a_n} - \frac{1}{a_{n+1}} \right), & 1 \leq n \leq N - 1,
\end{cases}
\]

with \( b_0 \equiv 0, b_N \equiv 0 \), can be written in Lax form representation \([18]\), namely

\[
\begin{cases}
\dot{L} = LA - BL, \\
\dot{M} = MA - BM,
\end{cases}
\]

(3.1)

with \( A = -(L^{-1} M)_- \) and \( B = -(M L^{-1})_- \). Here the matrices \( L \) and \( M \) contain the matrix data \( a_n \) and \( b_n \) and are defined in \([14]\). In this section we first show that this system can be generalized as in \([17]\). Then we will solve the finite RTL in its generalized form, using the spectral transform introduced in Section 2. We will find the explicit evolution of the spectral data.

#### 3.1. The generalized finite RTL

The generalized finite RTL is defined by the matrix differential equations

\[
\begin{cases}
\dot{L} = \left( F (LM^{-1}) \right)_- L - L \left( F (M^{-1} L) \right)_- \\
\dot{M} = \left( F (LM^{-1}) \right)_- M - M \left( F (M^{-1} L) \right)_-
\end{cases}
\]

(3.2)

where \( F : \mathbb{R}^+ \to \mathbb{R} \) is a real-valued function on \( \mathbb{R}^+ \). The matrices \( L \) and \( M \) contain the matrix data \( a_n \) and \( b_n \) as in \([14]\). We assume positive initial data \( a_n(0) > 0, b_n(0) > 0 \).

The system is of the form \([3.1]\) if we set

\[
A = -\left( F (M^{-1} L) \right)_- \quad \text{and} \quad B = -\left( F (LM^{-1}) \right)_-.
\]

(3.3)

The eigenvalues of \( LM^{-1} \) and \( M^{-1} L \) are equal to the generalized eigenvalues for the pair \( (L, M) \). We know that there are \( N \) distinct positive generalized eigenvalues. Hence the matrices \( LM^{-1} \) and \( M^{-1} L \) are diagonalizable, and in fact we have

\[
LM^{-1} = U^{-T} D U^T, \quad M^{-1} L = V^{-1} D V
\]

where \( D \) is a diagonal matrix containing the generalized eigenvalues \( \lambda_1 < \lambda_2 < \cdots < \lambda_N \). Thus

\[
F (LM^{-1}) = U^{-T} \text{diag}(F(\lambda_1), \ldots, F(\lambda_N)) U^T,
\]
and
\[ F \left( M^{-1}L \right) = V^{-1} \text{diag} \left( F(\lambda_1), \ldots, F(\lambda_N) \right) V, \]
see [7, Definition 6.2.4].

Of course one has to check that the structure of the matrices \( L \) and \( M \) is preserved by the differential equations (3.2). We will use the following lemma.

**Lemma 3.1** For every function \( F \) we have that
\[
LF \left( M^{-1}L \right) = F \left( LM^{-1} \right) L \quad \text{and} \quad MF \left( M^{-1}L \right) = F \left( LM^{-1} \right) M. \quad (3.4)
\]

**Proof.** For a monomial \( F(x) = x^n \), we have
\[
LF \left( M^{-1}L \right) = L \left( M^{-1}L \right)^n = \left( LM^{-1} \right)^n L = F \left( LM^{-1} \right) L
\]
and
\[
MF \left( M^{-1}L \right) = M \left( M^{-1}L \right)^n = \left( LM^{-1} \right)^n M = F \left( LM^{-1} \right) M.
\]

Then, by linearity, (3.4) holds for every polynomial \( F \). On the finite set of eigenvalues of \( LM^{-1} \) (which are the same as the eigenvalues of \( M^{-1}L \)) each function \( F \) is equal to a polynomial, so the equalities hold for every \( F \). \( \square \)

We now prove that the proposed generalisation of the finite RTL is justified.

**Theorem 3.2** The system of matrix differential equations (3.2) is well defined, which means that \( LA - BL \) is a diagonal matrix and \( MA - BM \) has only non-zero elements on the first subdiagonal, where
\[
A = - \left( F \left( M^{-1}L \right) \right)_- \quad \text{and} \quad B = - \left( F \left( LM^{-1} \right) \right)_-.
\]

**Proof.** Since \( L \) is upper Hessenberg and \( A \) and \( B \) are strictly lower triangular, one immediately sees that \( LA - BL \) is lower triangular. Now \( L \) is also upper triangular, so that
\[
\left( \left( F \left( LM^{-1} \right) \right)_- L \right)_- = \left( F \left( LM^{-1} \right) L \right)_-
\]
and
\[
\left( L \left( F \left( M^{-1}L \right) \right)_- \right)_- = \left( LF \left( M^{-1}L \right) \right)_-.
\]
This means that
\[
( LA - BL )_- = \left( F \left( LM^{-1} \right) L - LF \left( M^{-1}L \right) \right)_-.
\]
From Lemma 3.1, we then obtain that \( ( LA - BL )_- \) is the zero matrix. We have already mentioned that \( LA - BL \) is lower triangular, so we conclude that \( LA - BL \) is a diagonal matrix.

Since \( M \) is lower triangular and \( A \) and \( B \) are strictly lower triangular, \( MA - BM \) is strictly lower triangular. Then we note that only the first of the subdiagonals of \( M \) has non-zero elements. This implies that below the first subdiagonal,
\[
MA - BM = \left( F \left( LM^{-1} \right) \right)_- M - M \left( F \left( M^{-1}L \right) \right)_-
\]
agrees with
\[
F \left( LM^{-1} \right) M - MF \left( M^{-1}L \right) = O.
\]
The last equality follows from Lemma 3.1. Thus the only non-zero elements of \( MA - BM \) are on the first subdiagonal.

\[\square\]

**Remark 3.3** The generalized finite RLT (3.2) can also be written as
\[
\begin{align*}
\dot{a}_n &= \left( F \left( LM^{-1} \right) \right)_{n,n-1} - \left( F \left( M^{-1}L \right) \right)_{n+1,n}, \quad 1 \leq n \leq N, \\
\dot{b}_n &= \left( F \left( M^{-1}L \right) \right)_{n+1,n} - \left( F \left( LM^{-1} \right) \right)_{n,n}, \quad 1 \leq n \leq N - 1.
\end{align*}
\] (3.5)

**Remark 3.4** If we take \( F(x) = 1/x \) in (3.2) we obtain the first representation of the finite RTL (1.2). The other representation [1.3] found in the literature, is obtained by taking \( F(x) = x \). To see this, we note that from (1.4) we have
\[
M^{-1} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
\dot{b}_1 & 1 & 0 & & & \\
\ddot{b}_2 & \ddot{b}_1 & 1 & 0 & & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots & \\
& & & \ddots & \ddots & \ddots \\
\prod_{i=1}^{N-1} b_i & \cdots & \cdots & b_{N-1} b_{N-2} & b_{N-1} & 1
\end{pmatrix},
\]
so that \( (LM^{-1})_{i+1,i} = \dot{b}_i (b_{i+1} + a_i) \) and \( (M^{-1}L)_{i+1,i} = \ddot{b}_i (b_{i-1} + a_i) \) for \( i = 1, \ldots, N - 1 \).

So, if \( F(x) = x \), the equations (3.5) are
\[
\begin{align*}
\dot{a}_n &= b_n (n + a_n) - b_n (b_{n-1} + a_n) = a_n (b_{n-1} - b_n), \\
\hat{b}_n &= b_n (b_{n-1} + a_n) - b_n (b_{n+1} + a_{n+1}) = b_n (a_n - a_{n+1} + b_{n-1} - b_{n+1}),
\end{align*}
\]
with \( b_0 = b_N = 0 \). This corresponds to (1.3).

### 3.2. The evolution of the spectral data

We now describe the time evolution of the spectral data introduced in Section 2. The generalized RTL (3.2) can then be solved with the aid of this spectral transform which is a bijection between \( (\mathbb{R}^+)^{2N-1} \) and its image \( \{ 0 < \lambda_1 < \cdots < \lambda_N, w_j > 0, \sum_{j=1}^{N} w_j = 1 \} \). In particular we can start from positive initial matrix data \( a_n(0) \) and \( b_n(0) \) and follow the scheme in Figure II to find the matrix data at time \( t \in \mathbb{R} \).

Let \( L(t) \) and \( M(t) \) be the solutions of the generalized RTL (3.2) with
\[
A(t) = -\left( F \left( M^{-1}(t)L(t) \right) \right)_- \quad \text{and} \quad B(t) = -\left( F \left( L(t)M^{-1}(t) \right) \right)_-.
\] (3.6)

Let \( L_1 \) and \( L_2 \) be the unique solutions of the linear differential equations
\[
\dot{L}_1(t) = L_1(t)B(t), \quad L_1(0) = I,
\] (3.7)
and
\[ \dot{L}_2(t) = -A(t)L_2(t), \quad L_2(0) = I. \] (3.8)
Then \( L_1 \) and \( L_2 \) are lower triangular matrices with ones on the diagonal and therefore invertible. Furthermore
\[ \frac{d}{dt}(L_1L_2) = L_1BL_2 + L_1(LA - BL)L_2 - L_1LAL_2 = 0 \]
and similarly \( \frac{d}{dt}(L_1M_2) = 0. \) This means that
\[ L_1(t)L(t)L_2(t) = L(0) \quad \text{and} \quad L_1(t)M(t)L_2(t) = M(0). \] (3.9)

**Theorem 3.5** The eigenvalues \( \lambda_j \) of the generalized eigenvalue problem of the pair \((L, M)\) are constants of motion, which means
\[ \dot{\lambda}_j = 0, \quad 1 \leq j \leq N. \] (3.10)

**Proof.** From (3.9) it follows that for every \( z \),
\[ L_1(t)(zM(t) - L(t))L_2(t) = zM(0) - L(0). \]
Since \( \det(L_1(t)) = \det(L_2(t)) = 1 \) we see that
\[ \det(zM(t) - L(t)) = \det(zM(0) - L(0)). \]
This proves the theorem. \( \square \)

To find the evolution of the spectral data \( w_j \), we need the following lemma. For a matrix \( X \) we use \( X \geq \) to denote its upper triangular part. So \( X = X_\geq + X_\leq \).

**Lemma 3.6** Let \( R \) be the unique solution of the linear differential equations
\[ \dot{R}(t) = -\left(F(L(t)M^{-1}(t))\right) \geq R(t), \quad R(0) = I. \] (3.11)
Then \( R \) is upper triangular and we have
\[ L_1(t)R(t) = e^{-tF(L(0)M^{-1}(0))}, \] (3.12)
which gives an LR-factorization of \( e^{-tF(L(0)M^{-1}(0))} \).

**Proof.** Combining (3.7), (3.11), and (3.8), we obtain
\[ \frac{d}{dt}(L_1(t)R(t)) = L_1(t)B(t)R(t) - L_1(t)\left(F(L(t)M^{-1}(t))\right) \geq R(t) \]
\[ = -L_1(t)F(L(t)M^{-1}(t))R(t). \]
From (3.9) we get that
\[ F(L(t)M^{-1}(t)) = F(L_1^{-1}(t)L(0)M^{-1}(0)L_1(t)) \]
\[ = L_1^{-1}(t)F(L(0)M^{-1}(0))L_1(t) \]
(see [3, Definition 6.2.4]). Hence
\[ \frac{d}{dt}(L_1(t)R(t)) = -F(L(0)M^{-1}(0))L_1(t)R(t), \]
which proves (3.12) since \( L_1(0)R(0) = I. \) \( \square \)
The theorem below gives the evolution of the spectral data \( w_j \). Moser obtained a similar expression for his spectral data in the case of the nonrelativistic Toda lattice [13].

**Theorem 3.7** For the generalized RTL (3.2) with positive initial data \( a_n(0) > 0, b_n(0) > 0 \), the spectral data \( w_j \) have the time evolution

\[
w_j(t) = \frac{w_j(0)e^{-tF(\lambda_j)}}{\sum_{k=1}^{N} w_k(0)e^{-tF(\lambda_k)}}, \quad 1 \leq j \leq N. \tag{3.13}
\]

**Proof.** Set \( D = \text{diag}(\lambda_1, \ldots, \lambda_N) \), which is time independent. The matrix \( V(t) \) contains the right eigenvectors (with the first components equal to 1) of the generalized eigenvalue problem of the pair \((L(t), M(t))\). For \( t = 0 \) this means that \( L(0)V(0) = M(0)V(0)D \). Using (3.13) we then obtain

\[
L(t)L_2(t)V(0) = M(t)L_2(t)V(0)D.
\]

So the columns of the matrix \( L_2(t)V(0) \), which are \( L_2(t)v_j(0), 1 \leq j \leq N \), are right eigenvectors of the generalized eigenvalue problem of the pair \((L(t), M(t))\). Because \( L_2(t) \) is lower triangular with ones on the diagonal, and \( v_j(0) \) has first component equal to 1, we find that \( L_2v_j(0) \) has first component equal to 1. So we have

\[
v_j(t) = L_2(t)v_j(0), \quad 1 \leq j \leq N. \tag{3.14}
\]

Similarly we get from \( U(0)^TL(0) = DU(0)^TM(0) \) and (3.3)

\[
U(0)^TL_1(t)L(t) = DU(0)^TL_1(t)M(t).
\]

The \( j \)th column of the matrix \( L_1(t)^TU(0) \), which is \( L_1(t)^Tu_j(0) \), is then a left eigenvector of the generalized eigenvalue problem of the pair \((L(t), M(t))\) with eigenvalue \( \lambda_j \). So it is equal to \( u_j(t) \) up to a constant factor. We now look at the first component of \( L_1(t)^Tu_j(0) \). From Lemma 3.6 we get

\[
L_1(t) = e^{-tF(L(0)M^{-1}(0))}R^{-1}(t).
\]

Since \( R(t) \) is upper triangular, we obtain

\[
u_j(0)^TL_1(t)e_1 = u_j(0)^Te^{-tF(L(0)M^{-1}(0))}(R^{-1}(t))_{1,1}e_1. \tag{3.15}
\]

From the fact that \( u_j(0) \) is a left eigenvector of \( L(0)M^{-1}(0) \) with eigenvalue \( \lambda_j \), it follows that it is also a left eigenvector of \( e^{-tF(L(0)M^{-1}(0))} \) with eigenvalue \( e^{-tF(\lambda_j)} \). So

\[
u_j(0)^Te^{-tF(L(0)M^{-1}(0))} = e^{-tF(\lambda_j)}u_j(0)^T.
\]

Using this in (3.15) and noting that the first component of \( u_j(0) \) is 1, we see that

\[
u_j(0)^TL_1(t)e_1 = e^{-tF(\lambda_j)}(R^{-1}(t))_{1,1}u_j(t).
\]

From this we conclude

\[
L_1(t)^Tu_j(0) = e^{-tF(\lambda_j)}(R^{-1}(t))_{1,1}u_j(t), \quad 1 \leq j \leq N. \tag{3.16}
\]
Combining Definition 2.3, (3.14), (3.16), and (3.9) we now obtain

\[ w_j(t) = \left( \bar{u}_j(t)^T M(t) \bar{v}_j(t) \right)^{-1} = \left( \frac{e^{tF(\lambda_j)}}{(R^{-1}(t))_{1,1}} \bar{u}_j(0)^T L_1(t) M(t) L_2(t) \bar{v}_j(0) \right)^{-1} = (R^{-1}(t))_{1,1} e^{-tF(\lambda_j)} \left( \bar{u}_j(0)^T M(0) \bar{v}_j(0) \right)^{-1} = (R^{-1}(t))_{1,1} w_j(0) e^{-tF(\lambda_j)}. \]

Since the \( w_j(t) \) sum to 1 we have

\[ (R^{-1}(t))_{1,1} = \frac{1}{\sum_{k=1}^{N} w_k(0)e^{-tF(\lambda_k)}}, \]

which completes the proof of the theorem. \( \Box \)

**Remark 3.8** Standard existence and uniqueness results from the theory of ordinary differential equations, give that the generalized RTL (3.2) with positive initial data \( a_n(0) \) and \( b_n(0) \) has a unique solution in a time interval containing \( t = 0 \), so that \( a_n(t) > 0, b_n(t) > 0 \). The expressions (3.10) and (3.13) allow us to define spectral data for all \( t \in \mathbb{R} \) that remain in \( \{ 0 < \lambda_1 < \cdots < \lambda_N, w_j > 0, \sum_{j=1}^{N} w_j = 1 \} \). It then follows that the system (3.2) has a unique solution valid for all time, in which the matrix data are positive.

**3.3. Example**

As an example we take \( N = 5 \) and positive initial matrix data

\[
\begin{align*}
    a_1(0) &= 3, & b_1(0) &= 1, \\
    a_2(0) &= 12, & b_2(0) &= 6, \\
    a_3(0) &= 16, & b_3(0) &= 11, \\
    a_4(0) &= 7, & b_4(0) &= 5, \\
    a_5(0) &= 5.
\end{align*}
\]

We obtain the corresponding spectral data, defined in Section 2.1, by solving the generalized eigenvalue problem of the pair \((L(0), M(0))\). The results are

\[
\begin{align*}
    \lambda_1 &= 1.9812757881, & w_1(0) &= 0.0097186754 \\
    \lambda_2 &= 2.6941860907, & w_2(0) &= 0.8409233539 \\
    \lambda_3 &= 6.6927423653, & w_3(0) &= 0.0757415291 \\
    \lambda_4 &= 13.8305993379, & w_4(0) &= 0.0665694128 \\
    \lambda_5 &= 40.8011964181, & w_5(0) &= 0.0070470286.
\end{align*}
\]

We consider the case that \( F(x) = 1/x \). Knowing (3.10) and (3.13) we then compute the spectral data at several time values. For each of these values we construct the T-fraction (2.10) as in the proof of Theorem 2.8. This then gives the values of the matrix data. The results are shown in Figures 2 and 3. We observe that the \( b_n \) tend to 0 as \( t \to \pm \infty \).
It is also clear from Figure 2 that the $a_n$ have limits for $t \to \pm \infty$. In the next section we will show that these limits are in fact $\lambda_{6-n}$ and $\lambda_n$ as $t \to +\infty$, or $t \to -\infty$, respectively.

4. The sorting property of the generalized finite RTL

In this section we suppose that $F$ is strictly increasing and we investigate the behaviour of the matrix data of the generalized finite RTL when $t \to \pm \infty$. We will show that
\[
\lim_{t \to \pm \infty} b_n(t) = 0, \quad 1 \leq n \leq N - 1, \tag{4.1}
\]
and
\[
\lim_{t \to +\infty} a_n(t) = \lambda_n, \quad \lim_{t \to -\infty} a_n(t) = \lambda_{N-n+1}, \quad 1 \leq n \leq N. \tag{4.2}
\]
We also establish the precise rates of convergence of \((4.1)\) and \((4.2)\). For the SVD flow (and the related nonrelativistic Toda lattice) this was obtained in \cite{5}. Note that in view of the sorting property in \((4.2)\) the choice of spectral data $\lambda_1 < \cdots < \lambda_N$ appears to be very natural. In the limit $t \to \pm \infty$, the diagonal entries $a_n(t)$ of the matrix $L(t)$ tend to the eigenvalues and the eigenvalues are sorted in increasing order (as $t \to +\infty$) or in decreasing order (as $t \to -\infty$).

The case when $F$ is strictly decreasing can be studied in the same way as we will do for strictly increasing $F$. In this case, the $b_n(t)$ tend to 0, and the $a_n(t)$ tend to the eigenvalues in decreasing order as $t \to +\infty$, and in increasing order as $t \to -\infty$.

Our proof of \((4.1)\) and \((4.2)\) is based on properties of the Laurent orthogonal polynomials $P_n$. To indicate the time dependence we write $P_n(z, t)$. These polynomials are defined by the recurrence relation
\[
P_n(z, t) = (z - a_n(t))P_{n-1}(z, t) - b_{n-1}(t)zP_{n-2}(z, t), \tag{4.3}
\]
with \( P_0 \equiv 1 \) and \( P_{-1} \equiv 0 \). For the polynomials \( P_n(z, t) \) we find the following limits as \( t \to \pm \infty \).

**Lemma 4.1** Assume \( F \) is strictly increasing. Then we have for \( 1 \leq n \leq N \),

\[
\lim_{t \to +\infty} P_n(z, t) = \prod_{k=1}^{n} (z - \lambda_k) \quad \text{and} \quad \lim_{t \to -\infty} P_n(z, t) = \prod_{k=N-n+1}^{N} (z - \lambda_k).
\]

**Proof.** First we investigate what happens when \( t \to +\infty \). In Theorem 2.9 we showed that the polynomial \( P_n(z, t) \) is orthogonal with respect to the discrete measure \( \sum_{j=1}^{N} w_j(t) \delta_{\lambda_j} \). It is well-known that monic orthogonal polynomials minimize the weighted \( L^2 \) norm, see \([19, \text{Section 3.1}]\). In our case this means that the minimum of

\[
\sum_{j=1}^{N} \left( q_n(\lambda_j) \right)^2 \frac{w_j(t)}{\lambda_j^n}
\]

(4.5)

taken over all monic polynomials \( q_n \) of degree \( n \), is attained by \( P_n(z, t) \). Take \( q_n(z) = \prod_{k=1}^{n} (z - \lambda_k) \), then (since \( \lambda_j > 0 \) and \( w_j(t) > 0 \)) we obtain for \( 1 \leq i \leq N \),

\[
\left( P_n(\lambda_i, t) \right)^2 \frac{w_i(t)}{\lambda_i^n} \leq \sum_{j=1}^{N} \left( P_n(\lambda_j, t) \right)^2 \frac{w_j(t)}{\lambda_j^n}
\]

\[
\leq \sum_{j=1}^{N} \left( q_n(\lambda_j) \right)^2 \frac{w_j(t)}{\lambda_j^n}
\]

\[
= \sum_{j=n+1}^{N} \left( \prod_{k=1}^{n} (\lambda_j - \lambda_k) \right)^2 \frac{w_j(t)}{\lambda_j^n}.
\]

(4.6)
From Theorem 3.7 we get for the ratio of the spectral data $w_j(t)$ that
\[ \frac{w_j(t)}{w_k(t)} = \frac{w_j(0)}{w_k(0)} e^{-t\left(F(\lambda_j) - F(\lambda_k)\right)}, \quad 1 \leq j, k \leq N. \tag{4.7} \]

Since $F$ is strictly increasing, we obtain from (4.7),
\[ \lim_{t \to +\infty} \frac{w_j(t)}{w_k(t)} = 0, \quad 1 \leq k < j \leq N. \tag{4.8} \]

So (4.6) gives for every $1 \leq i \leq N$,
\[ \left(P_n(\lambda_i, t)\right)^2 w_i(t) \leq \left(\prod_{k=1}^{n} (\lambda_{n+1} - \lambda_k)\right)^2 \frac{w_{n+1}(t)}{\lambda_{n+1}^n} \left(1 + o(1)\right), \quad \text{as} \ t \to +\infty. \tag{4.9} \]

For $1 \leq i \leq n$, we get from (4.8) that $w_{n+1}(t)$ tends to zero as $t \to +\infty$. So from (4.9) we conclude that
\[ \lim_{t \to +\infty} P_n(\lambda_i, t) = 0, \quad 1 \leq i \leq n. \]

Then the first limit in (4.4) follows, since $P_n(z, t)$ is a monic polynomial of degree $n$ for every $t$. Similarly we can prove the second limit when $t \to -\infty$, by taking
\[ q_n(z) = \prod_{i=N-n+1}^{N} (z - \lambda_i) \]
in (4.3). \qed

Starting from the recurrence relation (4.3), the limits of the polynomials $P_n(z, t)$ as $t \to \pm\infty$ give rise to the limits of the matrix data (4.1) and (4.2). However, we also want to establish the precise rate of convergence. We then need the following lemma.

**Lemma 4.2** Let $0 \leq n \leq N - 1$. Then for $1 \leq l \leq n + 1$,
\[ \prod_{k=1}^{n} b_k(t) = P_n(\lambda_l, t) \left(\prod_{i=1, i \neq l}^{n+1} (\lambda_l - \lambda_i)\right) \frac{w_l(t)}{\lambda_l^n} \left(1 + o(1)\right), \quad \text{as} \ t \to +\infty, \tag{4.10} \]
and for $N - n \leq l \leq N$,
\[ \prod_{k=1}^{n} b_k(t) = P_n(\lambda_l, t) \left(\prod_{i=N-n, i \neq l}^{N} (\lambda_l - \lambda_i)\right) \frac{w_l(t)}{\lambda_l^n} \left(1 + o(1)\right), \quad \text{as} \ t \to -\infty. \tag{4.11} \]

**Proof.** From Theorem 2.3 it follows that for every monic polynomial $Q_n$ of degree $n$, we have
\[ \sum_{j=1}^{N} P_n(\lambda_j, t)Q_n(\lambda_j) \frac{w_j(t)}{\lambda_j^n} = \prod_{k=1}^{n} b_k(t), \]
where $0 \leq n \leq N - 1$. If we take $1 \leq l \leq n + 1$ and
\[ Q_n(z) = \prod_{i=1, i \neq l}^{n+1} (z - \lambda_i), \]
then we obtain
\[ \prod_{k=1}^{n} b_k(t) = P_{n}(\lambda_l, t) \left( \prod_{i=1, i \neq l}^{n+1} (\lambda_l - \lambda_i) \right) \frac{w_l(t)}{\lambda_l^n} \]
\[ + \sum_{j=n+2}^{N} P_{n}(\lambda_j, t) \left( \prod_{i=1, i \neq l}^{n+1} (\lambda_j - \lambda_i) \right) \frac{w_j(t)}{\lambda_j^n}. \tag{4.12} \]

From Lemma 4.1 we see that
\[ \lim_{t \to +\infty} P_{n}(\lambda_j, t) \neq 0, \quad n + 1 \leq j \leq N. \tag{4.13} \]

Taking \( l = n + 1 \) in (4.12) and letting \( t \to +\infty \), we get with (4.13) and (4.8) that
\[ \prod_{k=1}^{n} b_k(t) = P_{n}(\lambda_{n+1}, t) \left( \prod_{i=1}^{n} (\lambda_l - \lambda_i) \right) \frac{w_{n+1}(t)}{\lambda_l^n} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty, \tag{4.14} \]

which is (4.10) with \( l = n + 1 \). So, for large \( t \), \( \prod_{k=1}^{n} b_k(t) \) behaves like \( w_{n+1}(t) \) up to a constant. Knowing this, we get (4.10) for \( 1 \leq l \leq n \) from (4.12), (4.13) and (4.8). The proof of (4.11) is similar. \( \square \)

From Lemma 4.2 we can easily establish the rates of convergence of \( P_{n}(\lambda_l, t), 1 \leq l \leq n \), as \( t \to +\infty \), and of \( P_{n}(\lambda_l, t), N - n + 1 \leq l \leq N \), as \( t \to -\infty \).

Now we look at the matrix data \( a_n(t) \) and \( b_n(t) \). Lemmas 4.1 and 4.2 will help us to find their behaviour when \( t \to \pm \infty \).

**Theorem 4.3** For the generalized finite RTL (3.2) with \( F \) strictly increasing we have for \( 1 \leq n \leq N - 1 \),
\[ b_n(t) = \frac{w_{n+1}(0)}{w_n(0)^2} \left( \frac{\prod_{i=1}^{n} (\lambda_{n+1} - \lambda_i)}{\prod_{i=1}^{n-1} (\lambda_n - \lambda_i)} \right)^2 \exp \left( -t \left( F(\lambda_{n+1}) - F(\lambda_n) \right) \left( 1 + o(1) \right) \right), \tag{4.15} \]
as \( t \to +\infty \), and
\[ b_n(t) = \frac{w_{N-n}(0)}{w_{N-n+1}(0)^2} \left( \frac{\prod_{i=N-n+1}^{N} (\lambda_{N-n} - \lambda_i)}{\prod_{i=N-n+2}^{N} (\lambda_{N-n+1} - \lambda_i)} \right)^2 \exp \left( t \left( F(\lambda_{N-n+1}) - F(\lambda_{N-n}) \right) \left( 1 + o(1) \right) \right), \tag{4.16} \]
as \( t \to -\infty \).

**Proof.** We only give the proof for the case when \( t \to +\infty \). The proof of the case \( t \to -\infty \) is similar. Let \( 0 \leq n \leq N - 1 \). From Lemma 4.1 we have that
\[ \lim_{t \to +\infty} P_{n}(\lambda_{n+1}, t) = \prod_{i=1}^{n} (\lambda_{n+1} - \lambda_i). \]
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From (4.10) we have
\[ \prod_{k=1}^{n} b_k(t) = \left( \prod_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \right)^2 \frac{w_{n+1}(t)}{\lambda_{n+1}^n} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty. \quad (4.17) \]

Since
\[ \frac{w_{n+1}(t)}{w_n(t)} = \frac{w_{n+1}(0)}{w_n(0)} e^{-t \left( F(\lambda_{n+1}) - F(\lambda_n) \right)}, \]
(4.15) follows immediately from (4.17).

This theorem implies that the matrix data \( b_n(t) \) tend to 0 as \( t \to \pm \infty \) at an exponential rate. We now use the recurrence relation for the polynomials \( P_n(z, t) \) to compute the behaviour of \( a_n(t) \) as \( t \to \pm \infty \).

**Theorem 4.4** Let \( 1 \leq n \leq N \). For the generalized finite RTL \((3.2)\) with \( F \) strictly increasing we have
\[ \lambda_n - a_n(t) = \frac{\lambda_n}{\lambda_n - \lambda_{n+1}} b_n(t) \left( 1 + o(1) \right) - \frac{\lambda_n}{\lambda_{n-1} - \lambda_n} b_{n-1}(t) \left( 1 + o(1) \right), \quad (4.18) \]
as \( t \to +\infty \), and
\[ \lambda_{N-n+1} - a_n(t) = \frac{\lambda_{N-n+1}}{\lambda_{N-n+1} - \lambda_{N-n}} b_n(t) \left( 1 + o(1) \right) - \frac{\lambda_{N-n+1}}{\lambda_{N-n+2} - \lambda_{N-n+1}} b_{n-1}(t) \left( 1 + o(1) \right), \quad (4.19) \]
as \( t \to -\infty \). Recall that \( b_N \equiv 0, b_0 \equiv 0 \).

**Proof.** We will only prove (4.18) and in the proof we assume \( 2 \leq n \leq N - 1 \). The recurrence relation \((4.3)\) for the polynomials \( P_n(z, t) \) gives
\[ \lambda_n - a_n(t) = \lambda_n \frac{P_n(\lambda_n, t)}{\lambda_n b_n(t) P_{n-1}(\lambda_n, t)} b_n(t) + \lambda_n \frac{P_{n-2}(\lambda_n, t)}{P_{n-1}(\lambda_n, t)} b_{n-1}(t). \quad (4.20) \]

From Lemma 4.4 we obtain
\[ \frac{P_{n-2}(\lambda_n, t)}{P_{n-1}(\lambda_n, t)} = \frac{1}{\lambda_n - \lambda_{n-1}} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty. \quad (4.21) \]

From (4.10) we have
\[ \frac{P_n(\lambda_n, t)}{\lambda_n^n \prod_{k=1}^{n} b_k(t)} w_n(t) = \left( \prod_{i=1, i \neq n}^{n+1} (\lambda_n - \lambda_i) \right)^{-1} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty. \]

and
\[ \frac{P_{n-1}(\lambda_n, t)}{(\lambda_n)^{n-1} \prod_{k=1}^{n-1} b_k(t)} w_n(t) = \left( \prod_{i=1}^{n-1} (\lambda_n - \lambda_i) \right)^{-1} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty. \]

Taking the ratio of the last two expressions, we obtain
\[ \frac{P_n(\lambda_n, t)}{\lambda_n b_n(t) P_{n-1}(\lambda_n, t)} = \frac{1}{\lambda_n - \lambda_{n+1}} \left( 1 + o(1) \right), \quad \text{as } t \to +\infty. \quad (4.22) \]
Combining (4.20), (4.21) and (4.22) gives (4.18).

The cases $n = 1$ and $n = N$ are simpler, because then either the last or the first term on the right-hand side of (4.20) equals 0. The proof for the case $t \to -\infty$ is similar. 

Since we know from Theorem 4.3 that the $b_n(t)$ tend to 0 as $t \to \pm\infty$, Theorem 4.4 proves the limits (4.2). Because the $b_n(t)$ are exponentially small as $t \to \pm\infty$, we also see that $a_n(t)$ approaches its limit at an exponential rate.

**Remark 4.5** We return to the Newtonian form (1.1) of the finite RTL. In particular we look at the behaviour of the differences of the $q_n$ as $t \to \pm\infty$. In (1.2) it was shown that the change of variables

\[
\begin{align*}
  a_n &= \frac{h(q_{n-1} - q_n)e^{p_n}}{h(q_n - q_{n+1})}, & 1 \leq n \leq N, \\
  b_n &= \epsilon^2 \frac{\exp(q_n - q_{n+1} + p_n)h(q_{n-1} - q_n)}{h(q_n - q_{n+1})}, & 1 \leq n \leq N - 1,
\end{align*}
\]  

(4.23)

where $h(x) = \sqrt{1 + \epsilon^2 x^2}$, leads to (1.3). This system is of the form (3.2) with $F(x) = x$ (see Remark 3.4). Here the initial conditions $a_n(0)$ and $b_n(0)$ can be computed from the $q_n(0)$ and $p_n(0)$ (or $q_n(0)$ and $\dot{q}_n(0)$). From (1.23) we have $b_n = \epsilon^2 e^{-(q_{n+1} - q_n)}a_n$, so that

\[
q_{n+1}(t) - q_n(t) = \log\left(\frac{\epsilon^2 a_n(\pm\infty)}{b_n(t)}\right) + o(1), \quad \text{as} \quad t \to \pm\infty.
\]

The asymptotic results (4.1) and (4.2) for the matrix data then show that the differences between the $q_n$, namely $q_{n+1} - q_n$, tend to $+\infty$ for $t \to \pm\infty$. From Theorem 4.3 we also know that the $b_n$ decay at an exponential rate as $t \to \pm\infty$. So the $q_{n+1} - q_n$ have linear behaviour as $t \to \pm\infty$, where the intercept and the slope can be established explicitly. In particular we get

\[
q_{n+1}(t) - q_n(t) = t (\lambda_{n+1} - \lambda_n) + O(1), \quad \text{as} \quad t \to +\infty,
\]

and

\[
q_{n+1}(t) - q_n(t) = -t (\lambda_{N-n+1} - \lambda_{N-n}) + O(1), \quad \text{as} \quad t \to -\infty,
\]

where the values $\lambda_1 < \cdots < \lambda_N$ are obtained from the initial conditions $a_n(0)$ and $b_n(0)$.

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