On $QCD_2$ from supergravity and mass gaps in $QCD$

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Abstract

As a test of the conjectured $QCD$/supergravity duality, we consider mass gaps in the supergravity construction of $QCD_2$. We find a mass gap in the dual field theory both when using non-rotating and rotating black D2-branes as backgrounds in the supergravity construction of $QCD_2$. So, since pure $QCD_2$ does not have a mass gap, the dual field theory of the supergravity construction of $QCD_2$ cannot be pure $QCD_2$. Considering the mass scales in the dual field theory of the supergravity construction of $QCD_2$, we find that this is explainable both in the case of the non-rotating background and of the rotating background. In particular, the mass gap in the case of the rotating background can be explained using results of the large angular momentum limit of euclidean rotating branes, obtained recently by Cvetic and Gubser. We furthermore remark on the possible implications for the mass gaps in the supergravity constructions of $QCD_3$ and $QCD_4$.

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1 Introduction and summary

In [1] Maldacena conjectured a duality between large $N$ superconformal field theories with maximal supersymmetry and superstring theory or M-theory on certain anti-de Sitter backgrounds. As an extension of this, Witten conjectured in [2] an approach to study large $N$ non-conformal and non-supersymmetric field theories such as pure $QCD$ (in this paper pure $QCD$ means pure Yang-Mills theory) at strong coupling using string theory on certain background geometries. According to this conjectured duality, the large $N$ expansion of the field theory should be identified to the perturbative expansion of the string theory, and the strong coupling expansion of the field theory should be identified to the $\mathcal{g}^2 \alpha'$ expansion of the string theory. In particular, one could hope to study $QCD$ for large $N$ and for large 't Hooft coupling $\lambda = g^2 N$ via a dual supergravity background, being the semiclassical approximation to weakly coupled string theory.

This idea has received much attention recently and it is found that the supergravity description of $QCD$ has many qualitative similarities with the expectations one has of pure $QCD$ at strong coupling. However, already from the beginning it has been clear that to really get to $QCD$, one must take a limit where the supergravity approximation breaks down, and one must instead consider at least tree-level string theory in a rather singular background [2, 4]. But, until a better understanding of string theory in these backgrounds is reached, one can try to probe how far the supergravity can be used in understanding field theories at large $N$.

One of the aspects of $QCD$ that has been discussed is the existence of a mass gap for glueball states. Witten initiated the study of this question in [2], and it has been studied further in several recent papers [8, 6, 5, 9]. The upshot of these papers is that $QCD_3$ ($QCD_4$) is constructed from a background of $N$ black D3-branes (D4-branes) in the field theory limit and that the existence of the mass gap can be shown by considering the dilaton fluctuation equation in the particular background, since the dilaton couples to the $\text{tr}(F^2)$ term in the expansion of the Born-Infeld action.

As pointed out in [5, 10], there are problems in this procedure for finding a mass gap in pure $QCD$ from the supergravity construction of $QCD$. The main problem is that the mass gap is of the same order as the cutoff in the theory. An attempt to solve this was first proposed in [10] by introducing rotating black branes, and was later extended in [11]. The claim of these papers is that for certain limits of the angular momentum, the scale of the cutoff should decouple from the scale of the mass gap. This would mean that the fermion mass scale $M_f$ and the mass gap scale $M_{gap}$ obey $M_{gap} < \ll M_f$. However, as argued in [12], this does not seem to hold since some of the fermions do not decouple from the mass gap scale in the considered limit. So, for both the non-rotating and rotating backgrounds it is not certain whether a non-zero eigenvalue of the dilaton fluctuation equation corresponds to a mass gap in pure $QCD$.

\footnote{For a recent review of this, see [3].}

\footnote{For further discussion see [3] and references therein.}
In this paper, we study the supergravity construction of $QCD_2$ for two main purposes:

1) Until now, the proposed $QCD/supergravity$ duality has almost exclusively been used on $QCD_3$ and $QCD_4$, so one of the purposes of this paper is to test the proposed duality in the case of pure $QCD_2$ (In $\text{QCD}_2$ from supergravity has been studied, but not the particular aspects that we consider in this paper).

2) The pure $QCD_2$ theory on a plane has no glueball mass gap, since the gluons have no degrees of freedom. Thus, contrary to the $QCD_3$ and $QCD_4$ cases, the supergravity description of $QCD_2$ is not supposed to predict the existence of a mass gap. So, in this paper, we will try to test whether a mass gap is predicted by the supergravity description.

In this paper, we consider both the non-rotating (see section 2) and rotating (see section 3) supergravity backgrounds for studying $QCD_2$. The non-rotation background consists of $N$ black D2-branes in the field theory limit and the rotating background consists of $N$ rotating black D2-branes in the field theory limit.

In this paper, we find that a mass gap is predicted for pure $QCD_2$ for both non-rotating and rotating backgrounds. So, for pure $QCD_2$ the supergravity construction of $QCD_2$ has totally different properties than the field theory it was supposed to be dual to. We argue in section 2 and 3 that this is explainable when considering the mass scales in the field theories dual to the supergravity backgrounds. In section 4 we present the conclusions.

### 2 Mass gap from non-rotating black D2-branes

To study $QCD_2$ from supergravity we must construct an appropriate supergravity background which is dual to a two dimensional non-supersymmetric field theory. To do this, we start by considering $N$ D2-branes in the field theory limit[13]

\[
U \equiv \frac{r}{l_s^2}, \quad l_s \to 0
\]

The dual field theory on the D2-brane world-volume has Yang Mills coupling constant

\[
g_{YM_3}^2 = g_s l_s^{-1}
\]

In order to trust the supergravity description, we must have[13]

\[
1 \ll g_{YM_3}^2NU^{-1} \ll N^{4/5}
\]

So, we have that $N \gg 1$ and $g_{YM_3}^2N >> U$ which means that we consider the large $N$ limit of the $SU(N)$ gauge theory on the world-volume of the D2-branes, at strong coupling. Since it is the large $N$ limit, we introduce the ’t Hooft coupling

\[
\lambda_3 \equiv g_{YM_3}^2N
\]
The field theory limit of \( N \) black D2-branes gives the following solution

\[
\frac{ds^2}{L_s^2} = \frac{U^{5/2}}{\sqrt{6\pi^2\lambda_3}} \left[ (1 - \frac{U_0^5}{U^5}) dt^2 + (dy^1)^2 + (dy^2)^2 \right] \\
+ \frac{\sqrt{6\pi^2\lambda_3}}{U^{5/2}} \left( 1 - \frac{U_0^5}{U^5} \right)^{-1} dU^2 + \frac{\sqrt{6\pi^2\lambda_3}}{U^3} d\Omega_6^2
\]

with the dilaton field given by

\[
e^\phi = g_{YM}^2 (6\pi^2\lambda_3)^{1/4} U^{-5/4}
\]

This solution has temperature

\[
T = \frac{5}{4\pi} \frac{U_0^{3/2}}{\sqrt{6\pi^2\lambda_3}}
\]

One can also start from the 11 dimensional supergravity background \( AdS_4 \times S^7 \), with \( AdS_4 \) being an Anti-de Sitter black hole in 4 dimensions (this approach to \( QCD_2 \) was suggested in [7]). However, compactifying the \( S^7 \) on a circle in the usual M-theory/Type IIA duality manner, one gets the above solution.

The world-volume theory of black D2-branes is a \( 2 + 1 \) dimensional theory described at low energies by the Born-Infeld action. It is a non-supersymmetric theory, since the D2-branes are non-extremal and the fermions acquire masses of order the temperature \( T \), even before quantum corrections[2]. So, we have a 2 dimensional non-supersymmetric field theory if we consider energies much smaller than the temperature. We introduce the two-dimensional Yang-Mills coupling constant

\[
g_{YM_2}^2 = T g_{YM_3}^2
\]

and the 't Hooft coupling

\[
\lambda_2 = T \lambda_3
\]

This is the appropriate coupling constants for the dual 2 dimensional field theory.

The coupling of the world-volume fields with the dilaton can be found by expanding the Born-Infeld action, as done in [4]. From this, one finds that the dilaton couples to \( \text{tr}(F^2) \). If we consider the dilaton fluctuation \( h \) on the background field \( \phi \) stated above, we get the dilaton fluctuation equation

\[
\partial_\mu \left( e^{-2\phi} \sqrt{g} g^{\mu\nu} \partial_\nu h \right) = 0
\]

Introducing the Ansatz

\[
h = f(z)e^{ik_1 y^1 + ik_2 y^2}
\]

where \( z \equiv U/U_0 \), equation (5) yields

\[
\frac{e^{2\phi}}{\sqrt{g}} g y^1 y^1 U_0^{-1} \left( e^{-2\phi} \sqrt{g} g U U_0^{-1} f \right)' - k^2 f = 0
\]
We identify \( k^2 \equiv k_1^2 + k_2^2 \) with minus the two-dimensional mass: \( m^2 = -k^2 \). Reading the metric components from (2) we get

\[
z^{-1} (z^0 - z) f' + \frac{25}{16\pi^2 T^2} m^2 f = 0 \tag{8}
\]

The dilaton fluctuations are imposed to be regular at \( z = 1 \), and also to be square integrable with respect to the metric (3). Together, these boundary conditions are sufficient to make a well-posed Sturm-Liouville problem which can be solved by standard methods, such as a WKB approximation or a series solution. The result is that \( m \) has a discrete spectrum, with a first eigenvalue of \( m \approx \pi T \) (this fact can also be deduced from (3)).

If we proceeded in the same way as in [2, 3, 4, 5] for the \( QCD_3 \) and \( QCD_4 \) cases, then we would predict a mass gap in pure \( QCD_2 \). But, as stated in the introduction, pure \( QCD_2 \) on a plane has no mass gap so this procedure cannot be correct. The reason for this comes from the fact that both the mass gap and the supersymmetry restoration energy are of order the temperature \( T \). This means that we cannot expect the fermions to decouple in the field theory for which we have computed the mass gap. So, it seems that we are not predicting a mass gap in pure \( QCD_2 \) but rather in a theory containing fermions. This also seems to suggest that one cannot conclude anything about the existence of a mass gap in \( QCD_3 \) and \( QCD_4 \) from supergravity using non-rotating black D-branes, since these theories also predict a mass gap of order the temperature (this was first noted in [3, 4]).

3 Mass gap from rotating black D2-branes

An attempt to solve the problem that the mass gap scale and the fermion mass scale are of the same order was pointed out by Russo in [10], and developed further in [11]. By considering rotating D-branes it was found that one could decouple the mass gap scale in \( QCD_3 \) and \( QCD_4 \) from the scale of the temperature in the limit of very high angular momenta.

A non-rotating black D2-brane has an \( SO(6) \) rotational symmetry in the transverse space, so since we can break \( SO(6) \) into \( SO(2) \times SO(2) \times SO(2) \), we can have 3 angular momentum parameters, \( l_1, l_2 \) and \( l_3 \). However, as we shall see, we only need one angular momentum \( l = l_1 \) to decouple the scale of the mass gap from the scale of the temperature. The rotating black D2-brane solution can be obtained from the rotating black hole solutions in an 8 dimensional space-time as done in [14]. To transform this solution with Minkowski signature to a solution with euclidean signature, one uses the transformation \( t \rightarrow -it \) and \( l \rightarrow il \). The field theory limit for \( N \) rotating D2-branes is [10, 11]

\[
U \equiv \frac{r}{l_s}, \quad a \equiv \frac{l}{l_s}, \quad l_s \rightarrow 0
\]
This gives the metric

\[
\frac{ds^2}{l_s^2} = \frac{\sqrt{\Delta} U^{5/2}}{\sqrt{6\pi^2\lambda_3}} \left[ (1 - \frac{U_0^5}{\Delta U^5}) dt^2 + (dy^1)^2 + (dy^2)^2 \right] 
+ \frac{\sqrt{6\pi^2\lambda_3\sqrt{\Delta}}}{U^{5/2}} \left( 1 - \frac{a^2}{U^2} - \frac{U_0^5}{U^5} \right)^{-1} dU^2 
+ \frac{\sqrt{6\pi^2\lambda_3}}{\sqrt{\Delta} U} \left[ \Delta d\theta^2 + \cos^2 \theta d\psi_1^2 \right] 
+ \cos^2 \theta \cos^2 \psi_1 d\psi_2^2 + \left( 1 - \frac{a^2}{U^2} \right) \sin^2 \theta d\phi_1^2 
+ \cos^2 \theta \cos^2 \psi_1 \sin^2 \psi_1 d\phi_2^2 \right] - 2\frac{U_0^{5/2}}{\sqrt{\Delta} U^{5/2}} a \sin^2 \theta dt d\phi_1 \tag{9}
\]

where

\[
\Delta = 1 - \frac{a^2 \cos^2 \theta}{U^2} \tag{10}
\]

The dilaton field is given by

\[
e^\phi = 9^{-2} \lambda_3 (6\pi^2) \frac{1/4}{1/4} \Delta^{-1/4} U^{-5/4} \tag{11}
\]

The temperature can be computed from (9) to be

\[
T = \frac{1}{4\pi} \frac{1}{\sqrt{6\pi^2\lambda_3}} \frac{U_+^4}{U_0^4} \left( 3 + 2 \frac{U_0^5}{U_+^5} \right) \tag{12}
\]

where \( U_+ \equiv r_+/l_s^2 \) and \( r_+ \) is the radius of the outer horizon. The square root of the determinant of the metric (9) is found to be

\[
\sqrt{g} = l_s^{10} 6\pi^2\lambda_3 U \cos^4 \theta \cos^2 \psi_1 \sin^2 \psi_1 \sin \theta \tag{13}
\]

It is remarkable that the dilaton fluctuation equation (5) admits an angle-independent Ansatz of the same form as (6), for which it yields

\[
z^{-1} \left( z^6 (1 - a^2 U_0^{-2} z^{-2} - z^{-5}) f' \right)' + \frac{6\pi^2 \lambda_3}{U_0^3} m^2 f = 0 \tag{14}
\]

which reduces to (8) when \( a = 0 \) as it should. The solutions of this equation are again constrained to be regular at the horizon and to be square-integrable with respect to the metric (9), and they can be found by means of a WKB method \[11\].

It turns out that

\[
m^2 = \frac{\pi^2 k}{\xi^2} \left( k + \frac{2}{3} \right) \quad k = 1, 2, ... \tag{15}
\]

where the mass scale is set by

\[
\xi = \frac{2\pi \sqrt{6\pi^2 \lambda_3}}{5 U_0^{3/2}} \int_{z_+}^{\infty} \frac{dz}{\sqrt{z^5 - a^2 U_0^{-2} z^3 - 1}} \tag{16}
\]
where $z_+ = U_+/U_0$ is the angular momentum dependent horizon coordinate, given by the largest real solution of

$$z^5 - a^2U_0^{-2}z^3 - 1 = 0 \quad (17)$$

We shall now consider the limit $a >> U_0$ to show that the mass gap scale decouples from the scale of the temperature in this limit. For $a >> U_0$ we have $U_+ \sim a$ and from (16) we have $\xi \sim U_+^{-3/2}$ so we find that the $a >> U_0$ limits of (12) and (13) are

$$T \sim a^4, \quad m \sim a^{3/2} \quad (18)$$

respectively. This shows that the scale of the mass gap decouples from the scale of the temperature for $a >> U_0$.

Thus, if for a moment we suppose that the scalars and fermions in the D2-brane field theory get masses of order the temperature or higher, we are inevitably lead to the conclusion that pure $\text{QCD}_2$ has a mass gap, because of the decoupling shown above. Therefore, since pure $\text{QCD}_2$ does not have a mass gap, the conclusion must be that some of the fermions or scalars do not decouple from the mass gap scale.

This conclusion has also been reached in [12], where it was shown for rotating M2, M5 and D3-branes that some of the fermions get masses of order $\sqrt{1 - a/U_+ T}$. Since we consider M2-branes with the transverse space compactified on a circle, this also applies to our case. That some of the fermions are becoming lighter with higher angular momentum means that a fraction of the original supersymmetry in the non-rotating extremal case is restored for $a >> U_0$, as mentioned in [12]. In our case we get

$$\sqrt{1 - a/U_+ T} \sim a^{-5/2}T \sim a^{3/2} \quad (19)$$

So, some of the fermions have the same mass scale as the mass gap in this limit. This means that the mass gap that we have obtained for $a >> U_0$ is not a mass gap of pure $\text{QCD}_2$, but instead a mass gap in a theory containing fermions, just as in the non-rotating case. As stated in [12], this result seems to apply also to the pure $\text{QCD}_3$ and $\text{QCD}_4$ cases, so also in these cases it seems that one cannot establish the existence of a mass gap from rotating branes in supergravity.

4 Conclusions

In this paper we have considered the correspondence between $\text{QCD}_2$ and supergravity. To test this correspondence, we have tried to see if the supergravity construction of $\text{QCD}_2$ predicts a mass gap. We found that a mass gap was predicted using non-rotating and rotating black D2-branes as supergravity backgrounds. Since pure $\text{QCD}_2$ does not have a mass gap, this suggests that the supergravity construction of $\text{QCD}_2$ does not fully reproduce the dynamics of pure $\text{QCD}_2$. As argued in sections 2 and 3, the reason for this was that, for both non-rotating and rotating supergravity backgrounds, the dual field theory for which we found a mass gap contained fermions and was therefore not pure $\text{QCD}_2$. 
That we found a mass gap for rotating D2-branes can be seen as a confirmation of the results of [12], since their results precisely explain the discrepancy between the supergravity construction of \( QCD_2 \) and pure \( QCD_2 \).

As already stated, it seems that the problems for the mass gap calculation in the supergravity construction of \( QCD_2 \) also apply for the \( QCD_3 \) and \( QCD_4 \) cases, both for non-rotating and rotating D-branes. So, as concluded in [12], it seems that there is no a priori reason to believe that the supergravity constructions of \( QCD_3 \) and \( QCD_4 \) can say anything about the existence of mass gaps in pure \( QCD_3 \) and pure \( QCD_4 \).

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