Energy Distribution in $f(R)$ Gravity

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Abstract

The well-known energy problem is discussed in $f(R)$ theory of gravity. We use the generalized Landau-Lifshitz energy-momentum complex in the framework of metric $f(R)$ gravity to evaluate the energy density of plane symmetric solutions for some general $f(R)$ models. In particular, this quantity is found for some popular choices of $f(R)$ models. The constant scalar curvature condition and the stability condition for these models are also discussed. Further, we investigate the energy distribution of cosmic string spacetime.

Keywords: $f(R)$ gravity, Generalized Landau-Lifshitz EMC, Energy Density.

1 Introduction

The energy localization has been a thorny problem since the Einstein era. Several attempts have been made to find a general and unique tensor representation for the energy-momentum. Einstein was the first who tried to solve this problem by introducing energy-momentum pseudo tensors. He established the energy-momentum conservation laws given by [1]

$$
\frac{\partial}{\partial x^\nu}\left\{\sqrt{-g}(T^\nu_\mu + t^\nu_\mu)\right\} = 0, \quad (\mu, \nu = 0, 1, 2, 3),
$$

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where \( T^\nu_{\mu} \) is the energy-momentum density of matter and \( t^\nu_{\mu} \) represents the energy-momentum density of gravitation. It is mentioned here that \( t^\nu_{\mu} \) is not a tensor quantity rather it is the gravitational field pseudo-tensor. Komar [2] gave a set of energy-momentum covariant conservation laws and developed their relationship to the generators of infinitesimal coordinate transformations. Bergmann [3]–[5] contributed greatly to the fundamental nature of conservation laws.

Landau-Lifshitz introduced [6] the energy-momentum complex by using the geodesic coordinate system at some particular point of space. Many other people like Tolman [7], Papapetrou [8], Bergmann [9], Goldberg [10], Møller [11] and Weinberg [12] developed their own energy-momentum complexes. All these prescriptions, except Møller, are restricted to perform calculations in Cartesian coordinates only. Also, we cannot define angular momentum with the help of these prescriptions. This idea of energy-momentum pseudo-tensors was severely criticized by some people. Even it was quoted in a famous book [13] that: Anyone who seeks for a general formula for "local gravitational energy-momentum" is asking for the right answer to the wrong question. Misner et al. [13] showed that energy can only be localized in spherical systems. But later on, Cooperstock and Sarracino [14] proved that if energy is localizable for spherical systems, then it can be localized in any system. Bondi [15] argued that a non-localizable form of energy is not allowed in General Relativity.

The idea of quasi-local energy was proposed by some authors [16]–[19]. In this method, we can use any coordinate system while finding the quasi-local masses to obtain the energy-momentum of a curved spacetime. Chang et al. [20] proved that every energy-momentum complex can be associated with a particular Hamiltonian boundary term. Thus the energy-momentum complexes may also be considered as quasi-local. Virbhadra and his collaborators [21] verified for asymptotically flat spacetimes that different energy-momentum complexes could give the same result for a given spacetime. They also found encouraging results for the case of asymptotically non-flat spacetimes by using different energy-momentum complexes. Senovilla [22] constructed super-energy tensors for arbitrary fields in any dimension. These tensors had good mathematical and physical properties, and in general, the completely timelike component of these super-energy tensors had the mathematical features of an energy density.

It might be interesting if this problem could be explored in the alternative theories of gravity. Recently, some work about energy-momentum has been
investigated in teleparallel theory of gravity \cite{23-25} with the hope that this problem may be settled down in this theory. For this purpose, the teleparallel versions of Møller, Bergmann, Einstein and Landau-Lifshitz prescriptions are derived. Sharif and Jamil \cite{26} used these prescriptions to explore the energy-momentum distribution for particular spacetimes. It is concluded that results are consistent in some cases but no general conclusion can be deduced.

The \( f(R) \) theory of gravity is another alternative theory of gravity which has received much attention in recent years due to its cosmologically important \( f(R) \) models. These models include higher order curvature invariants as function of Ricci scalar. It has been shown \cite{27-29} that some \( f(R) \) models pass solar system test. In particular, Nojiri and Odintsov \cite{27} proposed \( f(R) \) models with negative and positive powers of the curvature. It is shown that the terms with positive powers of the curvature provide the inflationary epoch while the terms with negative powers of the curvature serve as effective dark energy, supporting current cosmic acceleration. They also discussed the consistency of some \( f(R) \) models which include the terms involving logarithm of scalar curvature. Cognola et al. \cite{30} introduced a class of exponential \( f(R) \) models. They proved that these models passed all local tests, including stability of spherical body solution, non-violation of Newton’s law, and generation of a very heavy positive mass for the additional scalar degree of freedom. Amendola et al. \cite{31} derived the conditions under which dark energy \( f(R) \) models are cosmologically viable. Thus \( f(R) \) theory of gravity seems attractive due to cosmologically important \( f(R) \) models. It is hoped that the issue of energy-momentum localization can be addressed in this theory.

In a recent paper, Multamäki et al. \cite{32} studied energy-momentum complexes in this theory. They generalized the Landau-Lifshitz prescription to calculate energy-momentum in the framework of metric \( f(R) \) gravity. As an important special case, they evaluated the energy density for the Schwarzschild de Sitter spacetime. Bertolami and Sequeira \cite{33} discussed some \( f(R) \) models and studied them from the point of view of the energy conditions and of their stability under the Dolgov-Kawasaki criterion.

In this paper, we investigate energy distribution of some static plane symmetric solutions \cite{34} using the generalized Landau-Lifshitz energy momentum complex. We also explore energy density of cosmic string spacetime. These results are also found for some important \( f(R) \) models. The stability and constant scalar curvature conditions of these models are also discussed. The pa-
per is organized as follows: In section 2, we give a brief introduction about the field equations and the generalized Landau-Lifshitz energy-momentum complex in the context of metric $f(R)$ gravity. In sections 3 and 4, the energy distribution of plane symmetric solutions and cosmic string spacetime are found respectively using the generalized Landau-Lifshitz energy-momentum complex. In the last section, we summarize and conclude the results.

2 Generalized Landau-Lifshitz Energy-Momentum Complex

The $f(R)$ theory of gravity modifies or generalizes the general theory of relativity. The action for $f(R)$ gravity is

$$S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right).$$

(1)

Here $f(R)$ is a general function of the Ricci scalar. We note that this action is obtained by replacing $R$ with $f(R)$ in the standard Einstein-Hilbert action. The corresponding field equations are found by varying this action with respect to the metric tensor $g_{\mu\nu}$

$$F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu},$$

(2)

where

$$F(R) \equiv \frac{df(R)}{dR}, \quad \Box \equiv \nabla^\mu \nabla_\mu$$

(3)

and $\nabla_\mu$ represent the covariant derivative. Contracting the field equations, we get

$$F(R) R - 2f(R) + 3 \Box F(R) = \kappa T$$

(4)

and in vacuum, this reduces to

$$F(R) R - 2f(R) + 3 \Box F(R) = 0.$$  

(5)

Equation (5) gives a relationship between $f(R)$ and $F(R)$. This equation shows that any metric with constant scalar curvature, say $R = R_0$, is a solution of the contracted equation (5) as long as the following equation holds

$$F(R_0) R_0 - 2f(R_0) = 0.$$  

(6)
This gives the condition of constant scalar curvature. For non-vacuum case, this is given by
\[ F(R_0)R_0 - 2f(R_0) = \kappa T. \] (7)
These conditions are very important because these are used to check the acceptability of \( (R) \) models. This assumption of constant scalar curvature was firstly used by Cognola et al. [35] to investigate the solutions in \( f(R) \) gravity.

The generalized Landau-Lifshitz energy-momentum complex (EMC) is given by [32]
\[ \tau^{\mu\nu} = f'(R_0)\tau_{LL}^{\mu\nu} + \frac{1}{6\kappa}(f'(R_0)R_0 - f(R_0))\frac{\partial}{\partial x^\lambda}(g^{\mu\nu}x^\lambda - g^{\mu\lambda}x^\nu), \] (8)
where \( \tau_{LL}^{\mu\nu} \) is the Landau-Lifshitz EMC evaluated in the framework of General Relativity and \( \kappa = 8\pi G \). Its 00-component turns out to be
\[ \tau_{00}^{\mu\nu} = f'(R_0)\tau_{00}^{LL} + \frac{1}{6\kappa}(f'(R_0)R_0 - f(R_0))(\frac{\partial}{\partial x^i}g^{00}x^i + 3g^{00}), \] (9)
where \( \tau_{00}^{LL} \) is\[ \tau_{00}^{00} = (-g)(T_{00} + t_{00}^{00}) \] (10)
and \( t_{00}^{00} \) can be obtained from the following expression
\[ t_{00}^{00} = \frac{1}{2\kappa}[2(\Gamma_\alpha^\gamma\Gamma_\beta^\delta - \Gamma_\alpha^\gamma\Gamma_\beta^\gamma - \Gamma_\alpha^\gamma\Gamma_\beta^\delta)(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}) + \Gamma_\alpha^\gamma\Gamma_\beta^\delta(G_\alpha^\nu\Gamma_\beta^\gamma + \Gamma_\beta^\nu\Gamma_\alpha^\gamma - \Gamma_\beta^\nu\Gamma_\alpha^\delta - \Gamma_\gamma^\nu\Gamma_\alpha^\beta + \Gamma_\gamma^\nu\Gamma_\beta^\alpha - \Gamma_\beta^\nu\Gamma_\gamma^\delta) + \Gamma_\alpha^\gamma\Gamma_\beta^\delta(G_\alpha^\nu\Gamma_\beta^\gamma + \Gamma_\beta^\nu\Gamma_\alpha^\gamma - \Gamma_\beta^\nu\Gamma_\alpha^\delta - \Gamma_\gamma^\nu\Gamma_\alpha^\beta + \Gamma_\gamma^\nu\Gamma_\beta^\alpha - \Gamma_\beta^\nu\Gamma_\gamma^\delta) + g^{\alpha\beta}\Gamma_\gamma^\delta(G_\alpha^\nu\Gamma_\beta^\gamma + \Gamma_\beta^\nu\Gamma_\alpha^\gamma - \Gamma_\beta^\nu\Gamma_\gamma^\delta - \Gamma_\gamma^\nu\Gamma_\alpha^\beta + \Gamma_\gamma^\nu\Gamma_\beta^\alpha - \Gamma_\beta^\nu\Gamma_\gamma^\delta)]. \] (11)
It is mentioned here that Eq. (8) is the generalized formula of Landau-Lifshitz energy-momentum complex valid for constant scalar curvature. It would be worthwhile to mention here that we need cartesian coordinates to use this formula as some energy momentum pseudo-tensors are calculated in cartesian coordinates only.

## 3 Energy Distribution of Plane Symmetric Solutions

This section is used to evaluate energy density of some plane symmetric solutions found in \( f(R) \) gravity [34]. For this purpose, we use the generalized
Landau-Lifshitz energy-momentum complex valid for spacetimes that have constant scalar curvature.

3.1 Energy Density of the 1st Solution

The first vacuum solution (Taub’s metric) is given by

\[ ds^2 = k_1 x^{\frac{x}{2}} dt^2 - dx^2 - k_2 x^{\frac{x}{2}} (dy^2 + dz^2), \]  

(12)

where \( k_1 \) and \( k_2 \) are arbitrary constants. The corresponding 00-component takes the form

\[ \tau^{00} = f'(R_0) \tau_{LL}^{00} + \frac{11}{18 \kappa k_1} (f'(R_0) R_0 - f(R_0)) (x)^{\frac{x}{2}} \]  

(13)

whereas \( \tau_{LL}^{00} \) becomes

\[ \tau_{LL}^{00} = -\frac{1}{\kappa} \left( \frac{5k_2 x^{\frac{x}{2}}}{3} \right). \]  

(14)

Thus the 00-component of the generalized Landau-Lifshitz EMC becomes

\[ \tau^{00} = \frac{-5k_2 x^{\frac{x}{2}}}{3 \kappa} f'(R_0) + \frac{11}{18 \kappa k_1} (f'(R_0) R_0 - f(R_0)) (x)^{\frac{x}{2}}. \]  

(15)

Now we use \( f(R) \) model to evaluate this component. It is mentioned here that we have some restrictions for the choice of \( f(R) \) model when \( R = 0 \). For example, we cannot use a model including a logarithmic function of the Ricci scalar and also a model which is a linear superposition of \( R^m \), where \( m \) is any positive integer. Thus we take the \( f(R) \) model \[36\] as follows

\[ f(R) = R + \epsilon R^2, \]  

(16)

where \( \epsilon \) is any positive real number. Consequently, the 00-component of the generalized Landau-Lifshitz EMC reduces to

\[ \tau^{00} = \frac{-5k_2 x^{\frac{x}{2}}}{3 \kappa}. \]  

(17)

Further, the stability condition \[37\], \( \frac{1}{\epsilon (1 + 2 \epsilon R_0)} = \frac{1}{\epsilon} > 0 \), for the solution is satisfied.
3.2 Energy Density of the 2nd Solution

The second vacuum solution is
\[ ds^2 = (bx + bc)^2 dt^2 - dx^2 - e^{2a}(dy^2 + dz^2). \]

The corresponding 00-component is
\[ \tau^{00} = f'(R_0)\tau_{LL}^{00} + \frac{1}{6\kappa} (f'(R_0) R_0 - f(R_0)) \left( \frac{2x}{(x+c)^3} + \frac{3}{b^2(x+c)^2} \right) \] (18)

while \( \tau_{LL}^{00} \) becomes
\[ \tau_{LL}^{00} = \frac{1}{\kappa} \left( \frac{-e^{4a}}{(x+c)^2} \right). \] (19)

Thus the 00-component of the generalized Landau-Lifshitz EMC turns out to be
\[ \tau^{00} = \frac{1}{\kappa} \left( \frac{-e^{4a}}{(x+c)^2} \right) f'(R_0) + \frac{1}{6\kappa} (f'(R_0) R_0 - f(R_0)) \left( \frac{2x}{(x+c)^3} + \frac{3}{b^2(x+c)^2} \right). \] (20)

For a particular \( f(R) \) model,
\[ f(R) = R + \epsilon R^2, \] (21)

this reduces to
\[ \tau^{00} = \frac{1}{\kappa} \left( \frac{-e^{4a}}{(x+c)^2} \right). \] (22)

3.3 Energy Density of the 3rd Solution

The third solution (\( R \neq 0 \)) corresponds to anti deSitter metric and is given by
\[ ds^2 = e^{2(c_1 x + c_2)} (dt^2 - dy^2 - dz^2) - dx^2. \]

Here the 00-component becomes
\[ \tau^{00} = f'(R_0)\tau_{LL}^{00} + \frac{1}{6\kappa} (f'(R_0) R_0 - f(R_0)) \left( \frac{3 - 2c_1 x}{e^{2(c_1 x + c_2)}} \right) \] (23)

and also it follows that
\[ \tau_{LL}^{00} = \frac{1}{\kappa} \left( -5c_1^2 e^{4(c_1 x + c_2)} \right). \] (24)
Thus we obtain
\[
\tau^{00} = \frac{1}{\kappa}(-5c_1^2e^{4(c_1x+c_2)})f'(R_0) + \frac{1}{6\kappa}(f'(R_0)R_0 - f(R_0))(\frac{3-2c_1x}{e^{2(c_1x+c_2)}}).
\] (25)

Now we discuss an important \( f(R) \) model given by [27] \[
f(R) = R - \frac{a}{R} - bR^2. \] (26)

For \( R \equiv R_0 = 12c_1^2 \), we have
\[
f(R_0) = 12c_1^2 - \frac{a}{12c_1^2} - 144bc_1^4. \] (27)

Inserting this value and its derivative in Eq.(25), we get
\[
\tau^{00} = \frac{1}{\kappa}(-5c_1^2e^{4(c_1x+c_2)})(1+\frac{a}{144c_1^2} - 24bc_1^2) + \frac{1}{6\kappa}(\frac{a}{6c_1^2} - 144bc_1^4)(\frac{3-2c_1x}{e^{2(c_1x+c_2)}}). \] (28)

It is mentioned here that this \( f(R) \) model satisfy the constant scalar condition, i.e. \( F(R_0)R_0 - 2f(R_0) = 0 \) which implies that \( a = 48c_1^4 \). The stability condition [32], i.e. \( f''(R_0) \leq 0 \) yields \[
a + b(R_0)^3 \geq 0. \]

Since \( R_0 = 12c_1^2 \) and \( a = 48c_1^4 \), it follows that \[
1 + 36bc_1^2 \geq 0. \] (29)

Thus the model is acceptable.

4 Energy Distribution of Cosmic String Spacetime

The idea of big bang suggests that universe has expanded from a hot and dense initial condition at some finite time in the past. It is a general cosmological assumption that the universe has gone through a number of phase transitions at early stages of its evolution. The cosmic string spacetime has received serious attention in recent years due to their cosmological implications. Here we discuss energy distribution for this cosmological model [38] in \( f(R) \) gravity.
Consider the non-static cosmic string spacetime \[ds^2 = dt^2 - e^{2\sqrt{2}t}[d\rho^2 + (1 - 4GM)^2 d\phi^2 + dz^2]. \quad (30)\]

We write this metric in Cartesian coordinates as it is required for the generalized Landau-Lifshitz EMC. In Cartesian coordinates, this becomes

\[ds^2 = dt^2 - e^{\alpha t^2 x^2 + a^2 y^2} dx^2 - e^{\alpha t^2 y^2} dy^2 - e^{\alpha t^2 z^2 + 2e}\frac{y^2 - 1}{x^2 + y^2} x y dx dy, \quad (31)\]

where \(a = 1 - 4GM\) with \(G\) as the gravitational constant and \(M\) as mass per unit length of the string in the \(z\) direction and \(\alpha = 2\sqrt{\frac{\Lambda}{3}}\) with \(\Lambda\) as the cosmological constant. Also, the energy-momentum tensor is defined as

\[T^\nu_\mu = M \delta(x)\delta(y)\text{diag}(1, 0, 0, 1). \quad (32)\]

The Ricci scalar for this spacetime becomes

\[R = -3\alpha^2 = -4\Lambda. \quad (33)\]

Since the Ricci scalar is constant, we can find energy density of this model by using the generalized Landau-Lifshitz EMC. Its 00-component becomes

\[\tau^{00} = F(R_0)\tau_{LL}^{00} + \frac{1}{2\kappa}(F(R_0)R_0 - f(R_0))(g^{00}). \quad (34)\]

We can evaluate \(\tau_{LL}^{00}\) by using

\[\tau_{LL}^{00} = -g(\tau_{LL}^{00} + T^{00}). \quad (35)\]

After some manipulations, it follows that

\[\tau_{LL}^{00} = \frac{3}{4a} \left(\frac{\alpha^2}{a} \right)^2 \frac{(x^2 + a^2 y^2)(x^2 + a^2 x^2)}{(x^2 + y^2)^2} - \frac{1}{4a} \left(\frac{\alpha^2}{a} \right)^2 \frac{(a^2 - 1)(x^2 + a^2 y^2)}{x^2 + y^2} - \frac{x^3 y^5(a^2 - 1)^3}{a^2 e^{\alpha t(x^2 + y^2)^5}} + \frac{2x^4 y^4(a^2 - 1)^3}{a^2 e^{\alpha t(x^2 + y^2)^5}} + \frac{x^4 y^2(a^2 - 1)^2(x^2 + a^2 y^2)}{a^2 e^{\alpha t(x^2 + y^2)^5}} + \frac{1}{2\left(\frac{\alpha^2}{a} \right)^2} \frac{x^2 y^2(x^2 + a^2 y^2)(y^2 + a^2 x^2)(a^2 - 1)^2}{(x^2 + y^2)^4} \]

\[\tau_{LL}^{00} = -\frac{1}{2\left(\frac{\alpha^2}{a} \right)^2} \frac{(x^2 + a^2 y^2)^2(y^2 + a^2 x^2)^2}{(x^2 + y^2)^4}. \quad (36)\]
Also, $T^{00}$ is given by

$$T^{00} = M \delta(x) \delta(y). \quad (37)$$

Using these values in Eq. (35), it follows that

$$\tau^{00}_{LL} = a^2 e^{3\alpha t} \left[ - \frac{3}{4} \left( \frac{\alpha}{a} \right)^2 \frac{(x^2 + a^2 y^2)(y^2 + a^2 x^2)}{(x^2 + y^2)^2} - \frac{1}{4} \left( \frac{\alpha}{a} \right)^2 \frac{(a^2 - 1)(x^2 + a^2 y^2)}{x^2 + y^2} \right. \right.$$

$$- \frac{x^3 y^5 (a^2 - 1)^3}{a^2 e^{\alpha t}(x^2 + y^2)^5} - \frac{2 x^4 y^4 (a^2 - 1)^3}{a^2 e^{\alpha t}(x^2 + y^2)^5} + \frac{x^4 y^2 (a^2 - 1)^2 (x^2 + a^2 y^2)}{a^2 e^{\alpha t}(x^2 + y^2)^5}$$

$$+ \frac{1}{2} \left( \frac{\alpha}{a} \right)^2 \frac{x^2 y^2 (x^2 + a^2 y^2)(y^2 + a^2 x^2)(a^2 - 1)^2}{(x^2 + y^2)^4}$$

$$- \frac{1}{2} \left( \frac{\alpha}{a} \right)^2 \frac{(x^2 + a^2 y^2)^2 (y^2 + a^2 x^2)^2}{(x^2 + y^2)^4} + M \delta(x) \delta(y) \right] \quad (38)$$

Inserting all these values in Eq. (34), we have

$$\tau^{00} = \frac{1}{2\kappa} \left[ F(R_0) a^2 e^{3\alpha t} \left\{ - \frac{3}{4} \left( \frac{\alpha}{a} \right)^2 \frac{(x^2 + a^2 y^2)(y^2 + a^2 x^2)}{(x^2 + y^2)^2} \right. \right.$$

$$- \frac{1}{4} \left( \frac{\alpha}{a} \right)^2 \frac{(a^2 - 1)(x^2 + a^2 y^2)}{x^2 + y^2} \right. \right.$$

$$- \frac{x^3 y^5 (a^2 - 1)^3}{a^2 e^{\alpha t}(x^2 + y^2)^5} - \frac{2 x^4 y^4 (a^2 - 1)^3}{a^2 e^{\alpha t}(x^2 + y^2)^5} + \frac{x^4 y^2 (a^2 - 1)^2 (x^2 + a^2 y^2)}{a^2 e^{\alpha t}(x^2 + y^2)^5}$$

$$+ \frac{1}{2} \left( \frac{\alpha}{a} \right)^2 \frac{x^2 y^2 (x^2 + a^2 y^2)(y^2 + a^2 x^2)(a^2 - 1)^2}{(x^2 + y^2)^4}$$

$$- \frac{1}{2} \left( \frac{\alpha}{a} \right)^2 \frac{(x^2 + a^2 y^2)^2 (y^2 + a^2 x^2)^2}{(x^2 + y^2)^4} + M \delta(x) \delta(y) \right\} + \left( F(R_0) R_0 - f(R_0) \right) \right]. \quad (39)$$

Now we discuss a well-known special case for the choice of $f(R)$ model

$$f(R) = R - (-1)^{n-1} \frac{a}{R^n} + (-1)^{m-1} b R^m, \quad (40)$$

where $m$ and $n$ are positive integers. For $R \equiv R_0 = -4\Lambda$, we have

$$f(R_0) = -4\Lambda + \frac{a}{(4\Lambda)^n} - b(4\Lambda)^m \quad (41)$$

and

$$f'(R_0) = \frac{(4\Lambda)^{n+1} + an + bm(4\Lambda)^{m+n}}{(4\Lambda)^{n+1}}. \quad (42)$$
Inserting these values in Eq. (39), it follows that

\[
\tau_{00} = \frac{1}{2\kappa} \left[ \frac{(4\Lambda)^{n+1} + an + bm(4\Lambda)^{m+n}}{(4\Lambda)^{n+1}} \right] a^2 e^{3at} \left\{ -\frac{3}{4} \left( \frac{\alpha}{a} \right)^2 \right. \\
\times \frac{(x^2 + a^2 y^2)(y^2 + a^2 x^2)}{(x^2 + y^2)^2} - \frac{1}{4} \left( \frac{\alpha}{a} \right)^2 (a^2 - 1)(x^2 + a^2 y^2) \\
- \frac{x^3 y^5(a^2 - 1)^3}{a^2 e^{6at}(x^2 + y^2)^5} + \frac{\alpha^2 x^2 y^2(x^2 + a^2 y^2)(y^2 + a^2 x^2)(a^2 - 1)^2}{2a^2(x^2 + y^2)^4} \\
+ \frac{x^4 y^2(a^2 - 1)^2(x^2 + a^2 y^2)}{2a^2 e^{6at}(x^2 + y^2)^5} - \frac{\alpha^2 (x^2 + a^2 y^2)^3(y^2 + a^2 x^2)^2}{2a^2(x^2 + y^2)^4} \\
- \frac{2x^4 y^4(a^2 - 1)^3}{a^2 e^{6at}(x^2 + y^2)^5} + M\delta(x)\delta(y) \right\} + \frac{b(1 - m)(4\Lambda)^{m+n} - a(1 + n)}{(4\Lambda)^n}. \tag{43}
\]

This model must satisfy the constant curvature condition given by Eq. (7). Imposing this condition, we obtain

\[
a(n + 2) + b(m - 2)(4\Lambda)^{m+n} = (4\Lambda)^{n+1} - 2\kappa M\delta(x)\delta(y)(4\Lambda)^n. \tag{44}
\]

For a particular case, when \(m = 2\) or \(b = 0\), it reduces to

\[
a = \frac{(4\Lambda)^{n+1} - 2\kappa M\delta(x)\delta(y)(4\Lambda)^n}{n + 2}. \tag{45}
\]

It satisfies the constant scalar curvature condition which is necessary for the acceptability of the model.

Now we discuss another important \(f(R)\) model [39] given by

\[
f(R) = R - a \ln \left( \frac{|R|}{k} \right) + (-1)^{n-1}bR^n. \tag{46}
\]

In this case, the 00-component of the generalized Landau-Lifshitz EMC takes
the form

\[ \tau^{00} = \frac{1}{2\kappa} \left[ \frac{(4k\Lambda - a + bkn(4\Lambda)^n)}{4k\Lambda} \right] a^2 e^{3\alpha t} \left\{ -\frac{3}{4} \frac{\alpha}{a} \left( \frac{\Lambda}{a} \right)^2 - \frac{a^2}{(x^2 + y^2)^2} \right\} \]

\[ - \frac{1}{4} \frac{\alpha}{a} \frac{(a^2 - 1)(x^2 + a^2 y^2)}{x^2 + y^2} - \frac{x^3 y^5 (a^2 - 1)^3}{a^2 e^{\alpha t} (x^2 + y^2)^5} + \frac{x^4 y^2 (a^2 - 1)^2 (x^2 + a^2 y^2)}{a^2 e^{\alpha t} (x^2 + y^2)^5} \]

\[ + \frac{1}{2} \frac{\alpha}{a} \frac{x^2 y^2 (x^2 + a^2 y^2)(y^2 + a^2 x^2)(a^2 - 1)^2}{(x^2 + y^2)^4} + M \delta(x) \delta(y) \]

\[ + a \ln \left( \frac{4\Lambda}{k} \right) + \frac{a}{k} + b(1 - n)(4\Lambda)^n. \]  

(47)

Also, the constant scalar curvature condition gives

\[ 2a \ln \left( \frac{4\Lambda}{k} \right) - a + 4\Lambda = b(4\Lambda)^n (n - 2) + 2\kappa M \delta(x) \delta(y). \]  

(48)

For \( n = 2 \) or \( b = 0 \), this reduces to

\[ a = \frac{2\kappa M \delta(x) \delta(y) - 4\Lambda}{2 \ln \left( \frac{4\Lambda}{k} \right) - 1} \]  

(49)

which satisfies the constant curvature condition necessary for the acceptability of the model given by (46).

5 Summary and Conclusion

In this paper, the well-posed problem of energy-momentum localization has been discussed in the context of \( f(R) \) gravity. For this purpose, we use the generalized Landau-Lifshitz energy-momentum complex. We evaluate energy density of some static plane symmetric solutions by using this energy-momentum complex. The energy density of the cosmic string spacetime is also calculated. Further, this quantity is investigated for some important \( f(R) \) models. We have mainly considered two types of models, one with negative and positive powers of curvature and other including logarithmic term of curvature. These models have been found consistent with the solar system test and it has been shown that the model with negative and positive power of curvature unifies inflation and cosmic acceleration. The terms with
positive powers of curvature provide the inflationary stage while the terms with negative powers of curvature serves as an alternative for dark energy which is responsible for cosmic acceleration.

The model with logarithm term is also cosmologically important as it suggests due the logarithmic term which may be responsible for the current acceleration of the universe. It has been shown that the chosen $f(R)$ models satisfy the constant scalar curvature condition which is the necessary requirement for the validity of these models. We have also explored the stability condition for these models. The results (15), (20) and (25) show that the energy density expressions are well-defined in these cases. These results can reduce to GR by taking $f(R) = R$ in all the cases. We would like to mention here that we have calculated for the first time, to our knowledge, the energy density for a non-vacuum case (cosmic string spacetime) using the generalized Landau-Lifshitz energy-momentum complex.

This work adds some knowledge about the longstanding and crucial problem of the localization of energy. It gives the energy density expressions for different solutions with important $f(R)$ models which may help at some stage to overcome the theoretical difficulties in the cosmological and astrophysical context. It would be interesting to find the Landau-Lifshitz EMC for non-constant scalar curvature. The extension of other EMCs in the context of $f(R)$ gravity as well as in other versions of $f(R)$ gravity would also be worthwhile.

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