Center-of-mass motion in many-body theory of BEC

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The method of generating a family of new solutions starting from any wave function satisfying the nonlinear Schrödinger equation in a harmonic potential proposed recently [J. J. García-Ripoll, V. M. Pérez-García, and V. Vekslerchik, Phys. Rev. E 64, 056602 (2001)] is extended to many-body theory of mutually interacting particles. Our method is based on a generalization of the displacement operator known in quantum optics and results in the separation of the center of mass motion from the internal dynamics of many-body systems. The center of mass motion is analyzed for an anisotropic rotating trap and a region of instability for intermediate rotational velocities is predicted.

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I. INTRODUCTION

In a recent paper García-Ripoll, Pérez-García, and Vekslerchik [1] have studied the properties of the solution of the nonlinear Schrödinger equation in a time-dependent, anisotropic harmonic potential

\[ i\hbar \partial_t \psi (\mathbf{r}, t) = \left( -\frac{1}{2} \Delta + \frac{1}{2} \mathbf{r} \cdot \dot{A}(t) \cdot \mathbf{r} + G(|\psi(\mathbf{r}, t)|) \right) \psi (\mathbf{r}, t), \tag{1} \]

where \( \dot{A}(t) \) is an arbitrary symmetric positive \( 3 \times 3 \) matrix and \( G \) is an arbitrary real function. They have shown that from every solution of this equation one may obtain the whole family of new solutions by a translation accompanied by a change of the phase

\[ \psi(\mathbf{r}, t) \rightarrow \psi'(\mathbf{r}, t) = \psi(\mathbf{r} - \mathbf{R}(t), t)e^{i\theta(\mathbf{r}, t)}, \tag{2} \]

provided \( \mathbf{R}(t) \) is a solution of the classical equation of motion in the harmonic potential

\[ \frac{d^2 \mathbf{R}(t)}{dt^2} = -\dot{A}(t) \cdot \mathbf{R}(t) \tag{3} \]

and the phase \( \theta(\mathbf{r}, t) \) is given by the formula

\[ \theta(\mathbf{r}, t) = \mathbf{r} \cdot \left( \frac{d\mathbf{R}(t)}{dt} - f(t) \right), \tag{4} \]

where \( f(t) \) is the classical action calculated along the trajectory \( \mathbf{R}(t) \) (a factor of \(-1/2\) is missing in Eq. (12) of Ref. [1])

\[ f(t) = \frac{1}{2} \int_0^t dt \left( \frac{d\mathbf{R}(t)}{dt} - \mathbf{R}(t) \cdot \dot{A}(t) \cdot \mathbf{R}(t) \right). \tag{5} \]

This result implies that the motion of the center of the wave packet is governed by the classical equation (3) and it decouples from the internal motion that determines the shape of the wave packet. It has been argued in [1] that such a decoupling may be of significance for the dynamics of Bose-Einstein condensates, since they are often described in the mean-field approximation by the nonlinear Schrödinger equation.

In this paper we extend the conclusions of Ref. [1] to the general case of a mutually interacting many-body quantum system. Our generalization is inspired by the observation that the new solutions (2) may be generated from the initial solution by the displacement operator well known in quantum optics [2, 3]

\[ e^{i(\hat{\mathbf{P}}(t) - \hat{\mathbf{R}}(t))}, \tag{6} \]

where \( \hat{\mathbf{r}} \) and \( \hat{\mathbf{p}} = -i\nabla \) are the quantum mechanical position and momentum operators whereas \( \mathbf{R}(t) \) and \( \mathbf{P}(t) \) represent the phase-space trajectory of a classical particle in the harmonic potential. In order to show that the transformation

\[ \psi(\mathbf{r}, t) \rightarrow \psi'(\mathbf{r}, t) = e^{i(\hat{\mathbf{r}} \cdot \mathbf{P}(t) - \hat{\mathbf{r}} \cdot \mathbf{P}(t))} \psi(\mathbf{r}, t) \tag{7} \]

is the same as that given by [3], we apply the Baker-Hausdorff formula (using the canonical commutation re-
lations between \( \hat{r} \) and \( \hat{p} \) to write the displacement operator in the form

\[
e^{i(\hat{r} \cdot \mathbf{P}(t) - \hat{p} \cdot \mathbf{R}(t))} = e^{-i\mathbf{R}(t) \cdot \mathbf{P}(t)/2} e^{i\hat{r} \cdot \mathbf{P}(t)} e^{-i\hat{p} \cdot \mathbf{R}(t)}.
\]  \( \text{(8)} \)

The last term on the r.h.s shifts the argument of the wave function by \(-\mathbf{R}(t)\) and the middle term produces the first term of the phase \(\hat{r} \cdot \mathbf{P}(t)\). Finally, we note that the action \(f(t)\) after integration by parts reduces to the boundary terms

\[
f(t) = \left. \frac{1}{2} \frac{d\mathbf{R}(t)}{dt} \cdot \mathbf{R}(t) \right|_0^t.
\]  \( \text{(9)} \)

Therefore, the first term on the r.h.s. of \(\text{(8)}\) reproduces the second term of the phase \(\hat{r} \cdot \mathbf{P}(t)\) (up to an irrelevant constant phase given by the lower boundary term).

The action of the displacement operator on a solution of a nonlinear Schrödinger equation leads to a shift of the center of the wave packet. The vector \(\mathbf{R}(t)\) that determines this shift is a solution of the classical equations of motion in the harmonic potential. Therefore, the center of the wave packet always follows a classical trajectory. In the present paper, we carry out a similar analysis for an interacting many-body system. The role of the center of the wave packet is played by the center of mass operator of the whole system.

In Section II we construct the transformation analogous to \(\text{(7)}\) in the many-body theory of a system of mutually interacting bosons (or fermions) and we show that it also transforms solutions of the Schrödinger equation into new solutions of this equation. In Section III we show that all these properties are a result of a complete decoupling of the center of mass dynamics from the internal dynamics of the system confined in a harmonic potential. This decoupling has been noticed before (cf., for example, Ref. \(\text{(3)}\)). In Section IV we analyze the spectrum of stationary states for a rotating trap and we point out that the decoupling causes the splitting of the spectrum. We also show that for an anisotropic trap there always exists a region of instability at intermediate rotational velocities.

II. DISPLACEMENT OPERATOR IN MANY-BODY THEORY

The many-body theory of interacting atoms will be described within the formalism of second quantization. We shall use systematically the Schrödinger picture — all operators will be time independent. The Hamiltonian of the system of atoms contained in a harmonic trap (in natural units, \(m = 1, \ h = 1\)) has the form

\[
\hat{H}(t) = \frac{1}{2} \int d^3r \hat{\psi}^\dagger(r) \left( -\nabla \cdot \nabla + \mathbf{r} \cdot \hat{A}(t) \cdot \mathbf{r} \right) \hat{\psi}(r) + \frac{1}{2} \int d^3r \int d^3r' \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r') V(r - r') \hat{\psi}(r') \hat{\psi}(r),
\]  \( \text{(10)} \)

where \(\hat{A}(t)\) is the matrix of the harmonic potential of the previous Section and \(V(r - r')\) is an arbitrary two-particle interaction potential. The field operators \(\hat{\psi}(r)\) and \(\hat{\psi}^\dagger(r)\) that annihilate and create particles at the point \(r\) obey the standard Bose-Einstein or Fermi-Dirac commutation relations (we disregard the spin degrees of freedom)

\[
\left[ \hat{\psi}^\dagger(r), \hat{\psi}(r') \right] = \delta^{(3)}(r - r').
\]  \( \text{(11)} \)

In the construction of the displacement operator in many-body theory (following our earlier works \(\text{(1)}\)) we employ the operators of the total number of particles \(\hat{N}\), of the total position \(\mathbf{R}\), and of the total momentum \(\mathbf{P}\):

\[
\hat{N} = \int d^3r \hat{\psi}^\dagger(r) \hat{\psi}(r), \quad \mathbf{R} = \int d^3r \hat{\psi}^\dagger(r) \mathbf{r} \hat{\psi}(r), \quad \mathbf{P} = -i \int d^3r \hat{\psi}^\dagger(r) \nabla \hat{\psi}(r).
\]  \( \text{(12a)} \), \( \text{(12b)} \), \( \text{(12c)} \)

The number of particles operator \(\hat{N}\) is a constant of motion and it commutes with \(\mathbf{R}\) and \(\mathbf{P}\). The operators \(\mathbf{R}\) and \(\mathbf{P}\) satisfy the commutation relations

\[
\left[ \hat{R}_k, \hat{P}_l \right] = i\hat{N} \delta_{kl}.
\]  \( \text{(13)} \)

We define the many-body unitary displacement operator \(\hat{D}(t)\) by the same general formula \(\text{(5)}\) but with second-quantized operators \(\hat{\mathbf{R}}\) and \(\hat{\mathbf{P}}\) playing now the role of position and momentum,

\[
\hat{D}(t) = e^{i(\hat{\mathbf{R}} \cdot \mathbf{P}(t) - \hat{\mathbf{P}} \cdot \mathbf{R}(t))}.
\]  \( \text{(14)} \)

We would like to note that the new displacement operator \(\hat{D}(t)\) does not share all the properties of the Glauber displacement operator in quantum optics. Namely, it does not generate a coherent state when acting on the vacuum state but leaves the vacuum state unchanged \(\hat{D}(t)|0\rangle = |0\rangle\). This is due to the fact that the many-body vacuum state is not an exact counterpart of the QED vacuum.

In the proof of the invariance of the Schrödinger equation

\[
i\partial_t |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle
\]  \( \text{(15)} \)

under the action of \(\hat{D}(t)\) we shall again employ the Baker-Hausdorff decomposition of the displacement operator:

\[
e^{i(\hat{\mathbf{R}} \cdot \mathbf{P}(t) - \hat{\mathbf{P}} \cdot \mathbf{R}(t))} = e^{-i\hat{\mathbf{N}} C(t)} e^{i\hat{\mathbf{R}} \cdot \mathbf{P}(t)} e^{-i\hat{\mathbf{P}} \cdot \mathbf{R}(t)},
\]  \( \text{(16)} \)

where \(C(t) = \mathbf{R}(t) \cdot \mathbf{P}(t)/2\). Next we observe that the three factors on the r.h.s. of this formula act in the following way on the annihilation operators

\[
e^{i\hat{\mathbf{N}} C(t)} \hat{\psi}(r) e^{-i\hat{\mathbf{N}} C(t)} = e^{-i\hat{\mathbf{N}} C(t)} \hat{\psi}(r),
\]  \( \text{(17a)} \)

\[
e^{-i\hat{\mathbf{R}} \cdot \mathbf{P}(t)} \hat{\psi}(r) e^{i\hat{\mathbf{R}} \cdot \mathbf{P}(t)} = e^{i\hat{\mathbf{R}} \cdot \mathbf{P}(t)} \hat{\psi}(r),
\]  \( \text{(17b)} \)

\[
e^{i\hat{\mathbf{P}} \cdot \mathbf{R}(t)} \hat{\psi}(r) e^{-i\hat{\mathbf{P}} \cdot \mathbf{R}(t)} = \hat{\psi}(r - \mathbf{R}(t)),
\]  \( \text{(17c)} \)
where we have used the following commutation relations between the operators $\hat{N}$, $\hat{R}$, and $\hat{P}$ and the annihilation operators,

\[
\begin{align*}
\left[ \hat{N}, \hat{\psi}(\mathbf{r}) \right] &= -\hat{\psi}(\mathbf{r}), \quad (18a) \\
\left[ \hat{R}, \hat{\psi}(\mathbf{r}) \right] &= -\mathbf{r} \hat{\psi}(\mathbf{r}), \quad (18b) \\
\left[ \hat{P}, \hat{\psi}(\mathbf{r}) \right] &= i\nabla \hat{\psi}(\mathbf{r}). \quad (18c)
\end{align*}
\]

The formulas for the creation operators are obtained by hermitian conjugation. With the help of Eqs. (17) we may calculate the action of the displacement operator on the Hamiltonian

\[
\hat{D}(t) \Psi(\mathbf{r}, t) = \left( e^{i\theta(\mathbf{r}, t)} \right) \Psi(\mathbf{r}, t),
\]

where the phase $\theta(\mathbf{r}, t)$ is given by the same formula [1] as in the one-particle theory. Thus, to each solution of the many-body Schrödinger equation there corresponds the whole family of solutions labeled by the classical trajectories of the center of mass.

The interaction term in the Hamiltonian does not change because it is invariant under translation and under the change of the phase of the field operators. The last ingredient needed to prove the invariance of the Schrödinger equation is the following transformation formula for the time derivative

\[
\hat{D}^\dagger(t) i\partial_t \hat{D}(t) = i\partial_t + \frac{\hat{N}}{2} \left\{ \frac{d}{dt} (\hat{\mathbf{R}}(t) \cdot \hat{\mathbf{P}}(t)) \right\},
\]

where \( \hat{D}(t) \) is the many-body Schrödinger equation there corresponds the whole family of solutions labeled by the classical trajectories of the center of mass.

The new many-body wave functions are obtained from the original wave function by exactly the same transformation as in the one-particle case: a time-dependent shift of its arguments and a multiplication by the same phase factors. The transformation formula [23] acquires a particularly simple form in the special case, when the initial wave function describes a state in which the center of mass is not entangled with the internal degrees of freedom,

\[
\Psi(\mathbf{r}, t) = \Phi(\mathbf{p}, t) \Phi(\xi_1, \xi_2, \ldots, \xi_{n-1}, t),
\]
where $\rho = (r_1 + r_2 + \cdots + r_n)/n$ and $\xi_1, \xi_2, \ldots, \xi_{n-1}$ are some relative coordinates, invariant under the translations $r_i \rightarrow r_i + a$. Then, the transformation (23) affects only the center of mass wave function

$$\Phi(\rho, t) \rightarrow e^{i\theta(\rho, t)} \Phi(\rho - \mathbf{R}(t), t),$$

leaving the wave function of the internal motion intact. A general many-body wave function can always be written as a sum of product wave functions (24). The action of the displacement operator on a general wave function describing a state in which the center of mass motion is entangled with the internal motion gives

$$\hat{D}(t)\Psi(r_1, r_2, \ldots, r_n, t) = e^{i\theta(\rho)}
\times \sum_k \Phi_k(\rho - \mathbf{R}(t), t) \Phi_k(\xi_1, \xi_2, \ldots, \xi_{n-1}, t). \quad (26)$$

In this case the integration over the center of mass position (either $\rho$ or $\mathbf{R}(t)$) introduced in Ref. 3 leads to a mixed internal state of the condensate. The results presented in this Section are closely related to the fact that, as is shown in the next Section, the total Hamiltonian may be split into the Hamiltonian of the center of mass and the Hamiltonian of the internal motion.

III. DECOUPLING OF THE CENTER OF MASS MOTION

The separability of the center of mass dynamics from the internal dynamics in a harmonic trap follows directly from the form (10) of the Hamiltonian. Indeed, this Hamiltonian may be written as a sum of two commuting parts: the Hamiltonian of the center of mass motion $\hat{H}_{CM}(t)$ and the Hamiltonian $\hat{H}_I(t)$ describing the internal structure of the condensate,

$$\hat{H}(t) = \hat{H}_{CM}(t) + \hat{H}_I(t), \quad (27)$$

$$\hat{H}_{CM}(t) = \frac{\hat{P} \cdot \hat{P}}{2N} + \frac{\hat{R} \cdot \hat{A}(t) \cdot \hat{R}}{2N}, \quad (28)$$

$$\hat{H}_I(t) = \frac{1}{2} \int d^3r \dot{\psi}^\dagger(r) \left(-\nabla_c \cdot \nabla_c + r_c \cdot C \cdot \hat{A}(t) \cdot r_c \right) \dot{\psi}(r)$$
$$+ \frac{1}{2} \int d^3r \int d^3r' \dot{\psi}^\dagger(r) \dot{\psi}^\dagger(r') V(r - r') \dot{\psi}(r') \dot{\psi}(r), \quad (29)$$

where $\nabla_c$ and $r_c$ are the one-particle operators shifted by the (normalized) center of mass operators

$$r_c = r - \hat{R}/\hat{N}, \quad -i\nabla_c = -i\nabla - \hat{P}/\hat{N}. \quad (30)$$

The sum (27) describes the dynamics of two independent systems since the center of mass operators $\hat{R}$ and $\hat{P}$ commute with the Hamiltonian $\hat{H}_I(t)$, despite the appearance of these operators in $\hat{H}_I(t)$. To prove this statement we observe that

$$\left[\hat{R}, \hat{H}_I(t)\right] = \left[\hat{R}, \hat{H}(t)\right] - \left[\hat{R}, \hat{H}_{CM}(t)\right] = \hat{P} - \hat{P} = 0. \quad (31)$$

Similarly one can show that $[\hat{P}, \hat{H}_I(t)] = 0$.

The many-body displacement operator constructed from the center of mass operators $\hat{R}$ and $\hat{P}$ commutes with $\hat{H}_I(t)$. Therefore, the transformation generated by the displacement operator does not change the internal state of the system; it only acts on the center of mass variables.

IV. SEPARABILITY OF THE SPECTRUM

The Hamiltonian (10) is time dependent because we have allowed for a time dependent trap potential. Such a Hamiltonian does not possess true stationary states. The most interesting case of a time-dependent harmonic potential occurs when the time dependence is caused by a rotation of the trap. Then, by going to the rotating frame we can eliminate the time dependence and study the spectral properties of the resulting Hamiltonian. The price to be paid for this simplification is the appearance of an extra term in the Hamiltonian — the scalar product $-\hat{\Omega} \cdot \hat{M}$ of the angular velocity $\hat{\Omega}$ and the total angular momentum of the system $\hat{M}$,

$$\hat{M} = -i \int d^3r \dot{\psi}^\dagger(r) (r \times \nabla) \dot{\psi}(r). \quad (32)$$

This term is responsible for the inertial forces: the Coriolis force and the centrifugal force. The total Hamiltonian in the rotating frame is

$$\hat{H}_R = \hat{H}_{CM} + \hat{H}_I, \quad (33)$$

$$\hat{H}_{CM} = \frac{\hat{P} \cdot \hat{P}}{2N} + \frac{\hat{R} \cdot \hat{A} \cdot \hat{R}}{2N} - \hat{\Omega} \cdot \hat{R}, \quad (34)$$

$$\hat{H}_I = \frac{1}{2} \int d^3r \dot{\psi}^\dagger(r) \left(-\nabla_c \cdot \nabla_c + r_c \cdot \hat{A} \cdot r_c \right) \dot{\psi}(r)$$
$$+ \frac{1}{2} \int d^3r \int d^3r' \dot{\psi}^\dagger(r) \dot{\psi}^\dagger(r') V(r - r') \dot{\psi}(r') \dot{\psi}(r)$$
$$+ i\hat{\Omega} \cdot \int d^3r \dot{\psi}^\dagger(r) (r_c \times \nabla_c) \dot{\psi}(r), \quad (35)$$

where $\hat{A}$ is the time independent matrix whose eigenvalues $a_x$, $a_y$, and $a_z$ are the squared frequencies of the trap potential. We use the coordinate system with the axes directed along the principal directions of the trap potential.

The transformation to the rotating frame preserves the separability of the center of mass motion from the internal motion. Therefore, the spectrum of the many-body Hamiltonian is a Cartesian product of the center of mass spectrum and the spectrum of the internal motion. Each eigenvalue of the Hamiltonian $\hat{H}_I$ gives rise to the whole ladder made of the levels of the center of mass Hamiltonian $\hat{H}_{CM}$. These levels can be exactly calculated regardless of the form of the mutual interaction. Since for a fixed number of particles the commutation relations between $\hat{R}$ and $\hat{P}$ are canonical (up to a numerical factor
of $n$, the center of mass dynamics is that of a three-
dimensional anisotropic harmonic oscillator in a rotating
frame. The characteristic frequencies of this oscillator
can be calculated from the following classical equations
of motion determined by the Hamiltonian $\hat{H}$ (the equa-
tions of motion and the characteristic frequencies given
in Ref. [1] are incorrect)

$$\frac{d^2 \mathbf{R}(t)}{dt^2} = -\hat{A} \cdot \mathbf{R}(t) - \Omega \times \left(2 \frac{d\mathbf{R}(t)}{dt} + \Omega \times \mathbf{R}(t)\right).$$

One may solve this third order equation for $\omega^2$ but the
formulas are quite complicated. We give here the explicit
solution only in the simple case, when the angular rota-
tion vector is directed along one of the principal axes, say
the $z$ axis, of the potential ellipsoid and the axes $x$ and
$y$ are directed along the two remaining principal direc-
tions. In this case the frequency of oscillations along the
$z$ direction is not modified by the rotation, $\omega_z = \sqrt{\omega_{z,0}}$. The frequencies of the oscillations in the perpendicular
directions become equal to

$$\omega_{\pm} = \sqrt{2\Omega^2 + a_0 \pm \sqrt{a_0^2 + 8\Omega^2 a_0}},$$

where $a_\pm = a_x \pm a_y$. The spectrum of $\hat{H}_{\text{CM}}$ is discrete
when the frequencies $\omega_\pm$ are real. This is true (cf. [10]) for
slow rotations ($\Omega < \sqrt{\sigma}$) or for fast rotations ($\Omega > \sqrt{\sigma}$),
where we have assumed for definiteness that $a_x < a_y$. In
the intermediate region, the classical oscillations are un-
stable. This means that for the intermediate values of the
rotational frequency the spectrum of $\hat{H}_{\text{CM}}$ is continuous
and the condensate as a whole falls out of the trap. The
counterintuitive result that for fast rotations the oscillations
are stable is due to the action of the Coriolis force.
The stabilizing effect of the Coriolis force is well known in
other physical situations. It plays the crucial role in the
Paul [7] trap and also in the dynamics of nonscattering
electronic wave packets in Rydberg states, called Trojan
states [8, 9].

The appearance of an unstable region of rotational ve-
obilities between the stable regions of slow and fast rota-
tions is not limited to the special case discussed above,
but is a general property of the anisotropic rotating os-
cillator. This can be seen from the properties of the
free term in the characteristic equation (37). This term
(taken with the minus sign) is always equal to the pro-
duct of the squares of the three eigenfrequencies. In the
absence of rotation this term is equal to $a_x a_y a_z$. In turn,
for fast rotations this term is again positive since it is
dominated by $\Omega^2 \cdot \hat{A} \cdot \mathbf{R}$. The signature of an unstabile
behavior is the change of sign that indicates that one of
the characteristic frequencies becomes imaginary. In
the Appendix we show that for an anisotropic trap this
change of sign always takes place. Therefore there always
exists an intermediate region of rotational velocities for
which the center of mass escapes from the rotating trap.

V. CONCLUSIONS

We have made a full use of the fact that for a many-
body system of interacting atoms in a general harmonic
potential the dynamics of the center of mass decouples
from the internal dynamics. In the time-dependent de-
scription, this leads to the appearance of families of states
generated by the action of the displacement operator.
The members of each family are labeled by the solutions
of the equations of motion for a classical particle in the
trap. Each family is built on an internal state of the
system. The internal states are not affected by the dis-
placement operator.

For rotating traps — that are often used in BEC ex-
periments — one may employ the time-independent de-
scription by using the rotating frame. In that case one
may study the spectrum characterizing stationary states
of the system. The decoupling of the center of mass
Hamiltonian leads to the splitting of the spectrum. The
spectrum of the full Hamiltonian is a Cartesian prod-
uct of the center of mass spectrum and the spectrum of the
internal motion. The spectrum of the center of mass
Hamiltonian may be discrete or continuous depending on
the value of the rotational velocity. For sufficiently slow
rotations the spectrum is discrete because the centrifugal
force is not strong enough to overcome the trap attrac-
tion. For sufficiently fast rotations the spectrum is again
discrete because the Coriolis force stabilizes the center
of mass motion. In the two stable regions the eigenvalue
spectrum forms a triplet of ladders with equally spaced rungs built on each eigenvalue of the internal Hamiltonian. The spacings are given by the three frequencies, solutions of the characteristic equation \( \Delta \). For intermediate values of the rotational velocity the spectrum is continuous that means that the center of mass motion is unbounded — the condensate escapes from the trap.

The spectrum of the center of mass Hamiltonian is not influenced by the internal dynamics but the inverse is not true. The eigenvalue spectrum of the internal Hamiltonian does depend on the properties of the trap. For example, in an exactly soluble model with harmonic mutual interactions studied previously \([6]\), the trap potential modifies the frequency of the quadrupole oscillations. In the present paper we have considered the case of arbitrary two-body mutual interactions but our results are also valid in the general case of \( n \)-body interactions provided the interaction potential is a function of coordinate differences only.

The decoupling of the center of mass motion may have direct observational consequences for small condensates, i.e. for condensates whose physical dimensions are small as compared to the extension of the center of mass wave function. Such condensates may be produced, for example, when interatomic forces are attractive (as in lithium). Then, all many-particle correlation functions are highly peaked at small distances between the particles. In this case, the center of mass wave function contains significant (probabilistic) information about the position of the whole condensate. In particular, the vortex lines embedded in this wave function will have experimentally testable consequences; the probability of finding the atoms of the condensate close to the vortex line is small. Since the center of mass wave function obeys the one-particle Schrödinger equation in a trap, the explicit solutions of this equation exhibiting various vortex structures that were studied in detail in Refs. \([10]\) and \([11]\) will become relevant.

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**APPENDIX A**

In this Appendix we prove that in a rotating anisotropic trap there always exists a range of rotational velocities \( \Omega \) where the center of mass motion is unstable. In such a region one of the characteristic frequencies will be imaginary. As mentioned in the main text, this proof is based on the analysis of the characteristic equation \( \Delta \). The (minus) free term in this equation is equal to the product of the three characteristic frequencies squared. As seen from \( \Delta \), it is a biquadratic polynomial in the absolute value of \( \Omega \)

\[
\omega_1^2 \omega_2^2 \omega_3^2 = \Omega^4 \mathbf{n} \cdot \hat{A} \cdot \mathbf{n}
- \Omega^2 \left( \text{Tr} \{ A \} \mathbf{n} \cdot \hat{A} \cdot \mathbf{n} - \mathbf{n} \cdot \hat{A}^2 \cdot \mathbf{n} \right) + \text{Det} \{ A \}, \quad (A1)
\]

where \( \mathbf{n} \) is the unit vector in the direction of \( \Omega \). The signature of the transition to an unstable region is the change of sign of this polynomial occurring at its zeros. The polynomial \( (A1) \) has zeros if the discriminant \( \Delta \) is positive

\[
\Delta = \left( \text{Tr} \{ A \} \mathbf{n} \cdot \hat{A} \cdot \mathbf{n} - \mathbf{n} \cdot \hat{A}^2 \cdot \mathbf{n} \right)^2
- 4 \text{Det} \{ A \} \mathbf{n} \cdot \hat{A} \cdot \mathbf{n} \cdot (A2)
\]

Without loss of generality, we may assume that \( a_x < a_y < a_z \). Then, by rearranging the terms, we can easily obtain the form of \( \Delta \) that exhibits its positivity

\[
\Delta = \left( n_x^2 a_x (a_z - a_y) + n_y^2 a_y (a_z - a_x) + n_z^2 a_z (a_x - a_y) \right)^2
+ 4n_x^2 n_y^2 n_z^2 a_x a_y (a_z - a_x) (a_y - a_x) \geq 0. \quad (A3)
\]

The vanishing of \( \Delta \) means that there is one double root for \( \Omega^2 \). In that case there is no region of instability. This occurs only when the trap is not fully anisotropic but has a symmetry axis and rotates around this direction.

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