On a class of compact perturbations of the special pole-free joint solution of KdV and $P_I^2$.

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Abstract

We consider perturbations of the special pole-free joint solution $U(x, t)$ of the Korteweg–de Vries equation $u_t + uu_x + \frac{1}{6} u_{xxx} = 0$ and $P_I^2$ equation $u_{xxxx} + 10u^2_x + 20uu_{xx} + 40(u^3 - 6tu + 6x) = 0$ under the action of the KdV flow. We show that if the perturbation is compact and of bounded variation, then the initial value problem for the KdV equation has a classical solution. Our method is the inverse scattering transform method in the form of the Riemann-Hilbert problem method. Namely, we construct the corresponding spectral functions $a(\lambda), r(\lambda)$, and give characterization of the compact perturbations in terms of $a(\lambda), r(\lambda)$.

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1 Introduction

We construct a class of classical smooth solutions of the Korteweg–de Vries equation (KdV), which are unbounded as \( x \to \pm \infty \).

There are several papers concerning unbounded classical solutions of KdV.

In Menikoff’72 the author showed that if an initial datum \( u_0(x) \) satisfies
\[
\frac{\partial^n}{\partial x^n} u_0(x) = \sigma(|x|^{1-n}) \quad \text{as} \quad x \to \pm \infty, \quad 0 \leq n \leq 7,
\]
where \( \sigma \) is the standard Bachmann–Landau little-o notation, then the initial value problem for KdV has a unique solution that exists for all times \( -\infty < t < +\infty \). Solutions which admit asymptotics with power growth at \( x \to \pm \infty \) were considered in Bondareva Shubin ‘82. Initial data with a polynomial growth as \( x \to \pm \infty \) were considered by Kenig Ponce Vega ‘97.

The Cauchy problem for KdV with an initial datum, which is a perturbation of different (bounded) quasi-periodic solutions as \( x \to \pm \infty \) was investigated by Egorova Teschl ‘11. We refer also to Bona Smith ‘75, Maspero Schaad ‘16, where KdV equation with vanishing datum is treated in different Sobolev spaces.

We develop formalism of direct and inverse scattering transform for the Schrödinger operator with a potential from a class of unbounded at \( x \to \pm \infty \) functions. Such a study for perturbations of finite-gap potentials (bounded at \( x = \pm \infty \)) was done in BET ‘08.

The case of an initial datum, which decays exponentially as \( x \to +\infty \), and might be unbounded as \( x \to -\infty \), was studied by Rybkin ‘11.

In Claeyss Vanlessen ‘07 the authors proved that one special solution \( U(x,t) \) to the Korteweg–de Vries equation
\[
u_t + u u_x + \frac{1}{12} u_{xxx} = 0 \tag{1}
\]
which behaves as \( \sqrt[3]{-6x} \) as \( x \to \pm \infty \), and is simultaneously a solution to the second member of the first Painlevé hierarchy (\( P_2 \))
\[
x = tu - \left( \frac{1}{6} u^3 + \frac{1}{24} (u_x^2 + 2uu_{xx}) + \frac{1}{240} u_{xxxx} \right) \tag{2}
\]
is pole free for all \((x,t) \in \mathbb{R}^2\). This is a part of the so-called Universality Conjecture Dubrovin ‘06, that states that the function \( U(x,t) \) describes the behavior of a generic solution to the general perturbed Hamiltonian equation
\[
u_t + a(u) u_x + \varepsilon^2 \left[ b_3(u) u_{xxx} + b_4(u) u_x u_{xx} + b_5(u) u_x^3 \right] + \ldots = 0
\]
near the point of gradient catastrophe of the unperturbed solution
\[
u_t + a(u) u_x = 0.
\]

In Claeyss Grava ‘09 this was proved for the equations from the KdV hierarchy with some restrictions on the initial data. It also was conjectured by Dubrovin ‘06 that \( U(x,t) \) is the unique real smooth for all \( x, t \in \mathbb{R} \) solution to (2). To the best of our knowledge, the part of the Dubrovin’s Universality Conjecture that there are no other real smooth for all \( x, t \in \mathbb{R} \) solutions to equation (2) remains open. Class of degenerate tritronquee solutions of (2) was studied in Grava Kapaev Klein ‘15.

Equation (2) for the particular value of the parameter \( t = 0 \) was studied by Kapaev ‘95, and it also appeared in the study of the double scaling limit for the matrix model with the multicritical index \( m = 3 \) BMP ‘99.

In Claeyss ‘10 the long-time asymptotics as \( t \to +\infty \) of \( U(x,t) \) were studied, and the complete asymptotic expansion for large \( x \) was obtained in Suleimanov ‘13. Further, similar common smooth solutions of KdV (1) and equations of the first Painlevé hierarchy were constructed in Claeyss ‘12. They behave like \( e^{\pm \sqrt{-x}} \) as \( x \to \pm \infty, m \geq 1 \).

In this paper we take a compactly supported perturbation of \( U(x,t_0) \) at a given time \( t_0 \), and study the Cauchy problem for KdV equation (1) with this initial datum.
To formulate our main results, we recall that the special solution $U(x, t; \lambda)$ of the equations (1) and (2) can be constructed in the following way [Claeys Vanlessen'07], [Grava Kapaev Klein'15], [Dubrovin'06], [Kapaev'95]; consider the Riemann-Hilbert problem (RHP) (see Figure 2):

**Riemann-Hilbert problem 1.** Find a $2 \times 2$ matrix-valued function $E(x, t; \lambda)$, which

1. is analytic in $\lambda \in \mathbb{C} \setminus \Sigma$ and continuous up to the boundary, where $\Sigma$ is the contour

$$
\Sigma = \mathbb{R} \cup \left( e^{6\pi i/7} \infty, 0 \right) \cup \left( e^{-6\pi i/7} \infty, 0 \right),
$$

with the orientation as is written (we denote by $(0, e^{i\alpha})$ the ray emanating from the origin and coming to infinity at an angle $\alpha$, and $(e^{i\alpha}, 0)$ means the ray with the opposite orientation);

2. satisfies the jump conditions $E_\pm$ on the contour $\Sigma$, where

$$
\begin{align*}
J_E &= \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, & \lambda \in \gamma_3 := \left( e^{6\pi i/7} \infty, 0 \right) \quad \text{and} \quad \lambda \in \gamma_{-3} := \left( e^{-6\pi i/7} \infty, 0 \right), \\
J_E &= \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, & \lambda \in \gamma_0 := (0, \infty), & J_E &= \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix}, & \lambda \in \rho := (-\infty, 0),
\end{align*}
$$

Here $E_\pm$ stands for the limiting values of $E$ on the contour $\Sigma$. The positive side of the contour is from the left, the negative one is from the right; these relations define segments $\gamma_{0,3,-3,\rho}$;

3. has the following asymptotics as $\lambda \to \infty$:

$$
E(\lambda) \sim \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (I + \frac{\tilde{b}}{\lambda}) \frac{\theta(\lambda^{-1})}{\lambda}, \quad \text{where} \quad \theta(x, t; \lambda) := \frac{1}{105} \lambda^{7/2} - \frac{t}{3} \lambda^{3/2} + x \lambda^{1/2},
$$

where $\tilde{b} = \tilde{b}(x, t)$ is a scalar (which is not fixed, but is introduced to fix the form of the asymptotics). We take the standard branch along $-\infty, 0$ for roots of $\lambda$. Here

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \lambda^{-\sigma_3/4} = \begin{pmatrix} \lambda^{-1/4} & 0 \\ 0 & \lambda^{1/4} \end{pmatrix}, \quad e^{\theta} = \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}
$$

are the matrix exponents.

**Reconstruction of $U(x, t)$ from $E(x, t; \lambda)$.** Having the solution $E(x, t; \lambda)$ of the RHP [1] the function $U(x, t)$ can be defined as

$$
U(x, t) := 2 \lim_{\lambda \to \infty} \frac{\lambda}{\sqrt{2}} \left( \sigma_3 + \sigma_1 \right) \lambda^{\sigma_3/4} E(x, t; \lambda) e^{-\theta},
$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the subscript $12$ denotes the element staying in the intersection of the first row and the second column.

**Jost solutions associated with $U(x, t)$.* Function $E(x, t; \lambda)$ satisfies the differential equation

$$
E_x(x, t; \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - 2U(x, t) & 0 \end{pmatrix} E(x, t; \lambda),
$$

where the subscript $x$ denotes the differentiation with respect to (w.r.t.) $x$. Hence, the elements of the matrix-valued function $E(x, t; \lambda)$ have the following structure in different parts of the complex $\lambda$ plane:

$$
\begin{align*}
\begin{pmatrix} E_{1u}(x, t; \lambda), E_{1r}(x, t; \lambda) \end{pmatrix} &= \begin{pmatrix} e_{1u}(x, t; \lambda) & e_{1r}(x, t; \lambda) \\ e_{1u,x}(x, t; \lambda) & e_{1r,x}(x, t; \lambda) \end{pmatrix}, & \lambda \in (0, \frac{6\pi}{7}), \\
\begin{pmatrix} E_{2d}(x, t; \lambda), E_{2r}(x, t; \lambda) \end{pmatrix} &= \begin{pmatrix} e_{2d}(x, t; \lambda) & e_{2r}(x, t; \lambda) \\ e_{2d,x}(x, t; \lambda) & e_{2r,x}(x, t; \lambda) \end{pmatrix}, & \lambda \in (-\frac{6\pi}{7}, 0), \\
\begin{pmatrix} E_{1u}(x, t; \lambda), -iE_{1d}(x, t; \lambda) \end{pmatrix} &= \begin{pmatrix} e_{1u}(x, t; \lambda) & -ie_{1d}(x, t; \lambda) \\ e_{1u,x}(x, t; \lambda) & -ie_{1d,x}(x, t; \lambda) \end{pmatrix}, & \lambda \in (\frac{6\pi}{7}, \pi), \\
\begin{pmatrix} E_{2d}(x, t; \lambda), iE_{2u}(x, t; \lambda) \end{pmatrix} &= \begin{pmatrix} e_{2d}(x, t; \lambda) & ie_{2u}(x, t; \lambda) \\ e_{2d,x}(x, t; \lambda) & ie_{2u,x}(x, t; \lambda) \end{pmatrix}, & \lambda \in (-\pi, -\frac{6\pi}{7}).
\end{align*}
$$

The above formula \([6]\) defines scalar functions \(e_{iu}, e_{id}, e_{r},\) and their vector counterparts \(E_{iu}, E_{id}, E_{r},\) where the subscripts \(r, iu, id\) mean “right”, “left, upper half-plane”, “left, lower half-plane”, and the subscript \(x\) means the differentiation w.r.t. \(x\). Further, out of functions \(E_{iu}, E_{id}\) we construct the piece-wise analytic functions \(E_{l}, e_{l}\)

\[
E_{l}(x, t; \lambda) = \begin{cases} 
E_{iu}(x, t; \lambda), & \exists \lambda > 0, \\
E_{id}(x, t; \lambda), & \exists \lambda < 0,
\end{cases}
\]

which are discontinuous across the real line \(\lambda \in \mathbb{R}\) and continuous up to the boundary. We call the functions \(E_{l}(x, t; \lambda), e_{l}(x, t; \lambda)\) the vector and scalar left Jost solutions, associated with \(U(x, t)\), respectively, and the functions \(E_{r}(x, t; \lambda), e_{r}(x, t; \lambda)\) the vector and scalar right Jost solutions, associated with \(U(x, t)\). Functions \(e_{r}, e_{l}\) are solutions to the associated spectral problem

\[
e_{xx}(x, t; \lambda) + 2U(x, t)e(x, t; \lambda) = \lambda e(x, t; \lambda), \quad \text{(7)}
\]

whence the definition (here the subscript \(xx\) denotes the second derivative w.r.t. to \(x\)).

Our first preliminary result describes the properties of the associated with \(U(x, t)\) Jost solutions:

**Lemma 1.1.** For any \(\lambda \in \mathbb{C}, t \in \mathbb{R}\), function \(E_{r}(x, t; \lambda)\) vanishes exponentially as \(x \to +\infty\). For any \(\lambda \in \mathbb{C} \setminus \mathbb{R}, t \in \mathbb{R}\), function \(E_{l}(x, t; \lambda)\) vanishes exponentially as \(x \to -\infty\). For \(\lambda \in \mathbb{R} \pm i0, t \in \mathbb{R}\), function \(E_{l}(x, t; \lambda)\) might have at most polynomial growth in \(x\) as \(x \to -\infty\).

This lemma justifies the names “left”, “right” in the definitions of the Jost solutions. More detailed behavior of Jost solutions is given in Lemma 2.7 below.

Define the following classes of functions:

**Definition 1.2.** Denote by \(BV_{loc} = BV_{loc}(\mathbb{R})\) the class of functions \(u : \mathbb{R} \mapsto \mathbb{R}\), which on every compact subset of \(\mathbb{R}\) are functions of bounded variation.

Denote by \(BV_{loc}^{(n)} = BV_{loc}^{(n)}(\mathbb{R}), n = 0, 1, 2, \ldots,\) the class of functions \(u : \mathbb{R} \mapsto \mathbb{R}\), which are \(n\) times differentiable, and the \(n\)th derivative \(u^{(n)}\) belongs to \(BV_{loc}\). We set \(BV_{loc}^{(0)} \equiv BV_{loc}\). Finally, the class \(BV_{loc}^{(\infty)}(\mathbb{R})\) of functions, which belong to \(BV_{loc}^{(n)}\) for any \(n \geq 0\), is \(C^{\infty}(\mathbb{R})\).

Now we are ready to formulate our first main result.

**Theorem 1.3.** (forward scattering) Let \(t_{0}, A < B\) are real, and let function \(u_{t_{0}}(x)\) be a function of the class \(BV_{loc}(\mathbb{R})\), which equals \(U(x, t_{0})\) for \(x \in (-\infty, A) \cup (B, +\infty)\). Then

1. The associated spectral problem

\[
\partial_{x}^{2}f_{0}(x; \lambda) + 2uf_{0}(x; \lambda)f_{0}(x; \lambda) = \lambda f_{0}(x; \lambda)
\]

has two solutions \(f_{r}, f_{l}\) (we call them Jost solutions associated with \(u_{t_{0}}(x)\)), which are determined by the conditions

\[
f_{r}(x, t_{0}; \lambda) = e_{r}(x, t_{0}; \lambda), \quad \forall x > B, \quad f_{l}(x, t_{0}; \lambda) = e_{l}(x, t_{0}; \lambda), \quad \forall x < A
\]

on their domains of definition, and which are analytic respectively in \(\lambda \in \mathbb{C}\) and \(\lambda \in \mathbb{C} \setminus \mathbb{R}\).

II. Define functions \(a(\lambda) \equiv a(\lambda; t_{0}), b(\lambda) \equiv b(\lambda; t_{0}), r(\lambda) \equiv r(\lambda; t_{0})\) (we call them the spectral functions associated with \(u_{t_{0}}(x)\)) as follows:

\[
a(\lambda; t_{0}) := \{f_{r}(x, t_{0}; \lambda), f_{l}(x, t_{0}; \lambda)\},
\]

where the bracket \(\{g, h\} \equiv W(g, h) = gh_{x} - g_{x}h\) denotes the Wronskian of two functions, and

\[
b(\lambda; t_{0}) := (f_{l}(x, t_{0}; \lambda) - a(\lambda; t_{0})e_{l}(x, t_{0}; \lambda))^{-1}, \quad x > B, \quad r(\lambda; t_{0}) = b(\lambda; t_{0})a^{-1}(\lambda; t_{0}),
\]

The spectral functions possess the following properties:

1. the restrictions of \(a(\lambda), b(\lambda), r(\lambda)\) to the upper half-plane \(\Re \lambda > 0\), we call them \(a_{u}(\lambda), b_{u}(\lambda), r_{u}(\lambda)\), can be extended analytically to \(\mathbb{C}\); the restrictions of \(a(\lambda), b(\lambda), r(\lambda)\) to the lower half-plane \(\Re \lambda < 0\), we call them \(a_{d}(\lambda), b_{d}(\lambda), r_{d}(\lambda)\), are related with \(a_{u}, b_{u}, r_{u}\) as \(a_{u}(\lambda) = a_{d}(\lambda), b_{u}(\lambda) = b_{d}(\lambda), r_{u}(\lambda) = r_{d}(\lambda)\);
2. \( \mathfrak{S}_u(s) = \mathcal{O}(|s|^{-1}) \) for \( s \in \mathbb{R}, s \to \pm \infty \) (here \( \mathcal{O} \) is the Bachmann–Landau big-\( O \) notation);
3. \( \mathfrak{S}_u(s) < \frac{1}{2} \) and \( \mathfrak{S}_d(s) > -\frac{1}{2} \) for \( s \in \mathbb{R} \);
4. \( r_u(\lambda) - r_u(\overline{\lambda}) \neq i \) for all \( \lambda \in \mathbb{C} \);
5. \( a(\lambda) = 1 + \frac{1}{2\pi i} \int_{\gamma} (U(x, t_0) - u_t(x))dx + \pi(\frac{1}{4\pi}) \) as \( \lambda \to \infty \) uniformly w.r.t. \( \arg \lambda \in [-\pi, \pi] \);
6. \( r_u(\lambda) = \mathcal{O}(\frac{1}{\lambda}) \cdot e^{2\theta(B,t_0;\lambda)} \) as \( \lambda \to \infty \) uniformly w.r.t. \( \arg \lambda \in [-\pi, \pi] \);
7. \( a_u(\lambda; t_0) - a_u(\overline{\lambda}; t_0) = \mathcal{O}(\frac{1}{\lambda}) \cdot e^{-2\theta(A,t_0;\lambda)} \) as \( \lambda \to \infty \) uniformly w.r.t. \( \arg \lambda \in [-\pi, \pi] \);
8. \( r_u(\lambda) - r_d(\lambda) = i \left( 1 - \frac{1}{a_u(\lambda)a_d(\lambda)} \right) \);
9. function \( a(\lambda) \) does not vanish nowhere, i.e. \( a_u(\lambda) \neq 0 \) for \( \Im \lambda \geq 0 \);
10. functions \( a_u, a_d \) can be expressed in terms of \( r_u \) in the following way:

\[
a_u(\lambda) = \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - 2\mathfrak{S}_u(s))ds \right\}, \quad \text{for } \Im \lambda > 0,
\]

\[
= \frac{1}{1 + i \left( r_u(\lambda) - r_u(\overline{\lambda}) \right)} \cdot \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - 2\mathfrak{S}_u(s))ds \right\}, \quad \text{for } \Im \lambda < 0,
\]

and

\[
a_d(\lambda) = \frac{1}{1 + i \left( r_u(\lambda) - r_u(\overline{\lambda}) \right)} \cdot \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - 2\mathfrak{S}_u(s))ds \right\}, \quad \text{for } \Im \lambda > 0,
\]

\[
= \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - 2\mathfrak{S}_u(s))ds \right\}, \quad \text{for } \Im \lambda < 0,
\]

and hence, not only \( a_u(\lambda) \neq 0 \) for \( \Im \lambda \geq 0 \), but \( a_u(\lambda) \neq 0 \) for all \( \lambda \in \mathbb{C} \).

Let us observe, that the property \([\text{[8]}\) of the part II of Theorem \([\text{[1]}\)] implies that there is no more than finite number of points \( \lambda \) in the sector \( \arg \lambda \in (\frac{\pi}{2}, \pi) \), at which \( r_u(\lambda) = i \); and the property \([\text{[3]}\)] implies that \( r_u(s) \neq i \) for real \( s \). Define the ray \( \gamma_3^\pm \) as follows: if there are no roots of \( r_u(\lambda) = i \) on \( \gamma_3 = (e^{i\pi/7}, \infty, 0) \), then \( \gamma_3^\pm \) := \( \gamma_3 \). If there are some roots of \( r_u(\lambda) = i \) on \( \gamma_3 \), then we move locally the ray \( \gamma_3 \) in such a way, that it would not contain roots of \( r_u(\lambda) = i \) anymore, and call the resulting contour by \( \gamma_3^\pm \). The contour \( \gamma_3^\pm \) is symmetric to \( \gamma_3^\pm \) w.r.t the real line, with orientation imposed by this symmetry. Let us call the domain included between \( \gamma_0 \) and \( \gamma_3^\pm \) by \( \Omega_I \), the one between \( \gamma_3^\pm \) and \( \rho \) by \( \Omega_{II} \), the one between \( \rho \) and \( \gamma_3^\pm \) by \( \Omega_{III} \), and the one between \( \gamma_3^\pm \) and \( \gamma_0 \) by \( \Omega_{IV} \). Denote by \( \lambda_j \), \( j = 1, \ldots, J \) the points in \( \Omega_{II} \) at which \( r_u(\lambda_j) = i \).

Given the spectral functions we can define the following RHP (see Figure \([\text{[1]}\)]:

**Riemann–Hilbert problem 2.** To find a \( 2 \times 2 \) matrix-valued function \( \hat{\mathbb{R}}(x, t; \lambda) \), which

1. is meromorphic in \( \lambda \in \mathbb{C} \setminus \Sigma_F \), with finite number of poles at \( \lambda_j \), \( \lambda_j^\pm \), \( j = i, \ldots, J \). Here \( \Sigma_F = \gamma_0 \cup \gamma_3^\pm \cup \gamma_3^\mp \cup \rho \).
2. has the following jump $\hat{F}_+ = \hat{F}_- \hat{J}_F$ across $\Sigma$:

$$J_F = \begin{cases}
\left( \begin{array}{cc}
1 - ir_u & 0 \\
0 & 1 + ir_u
\end{array} \right), & \lambda \in \rho, \\
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right), & \lambda \in \gamma_0,
\end{cases}$$

3. has the following pole conditions at the points $\lambda_j, j = 1, \ldots, J$:

$$\hat{F}_2(\lambda) + \frac{i}{1 + ir_u(\lambda)} \hat{F}_1(\lambda) = O(1) \quad \text{and} \quad \hat{F}_1(\lambda) = O(1) \quad \text{for} \quad \lambda \to \lambda_j,$$

$$\hat{F}_2(\lambda) - \frac{i}{1 - ir_d(\lambda)} \hat{F}_1(\lambda) = O(1) \quad \text{and} \quad \hat{F}_1(\lambda) = O(1) \quad \text{for} \quad \lambda \to \lambda_j,$$

4. has the following asymptotics as $\lambda \to \infty$, which are uniform w.r.t. $\arg \lambda \in [-\pi, \pi]$:

$$\hat{F}(x, t; \lambda) = (I + O(1)) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\theta \sigma_3},$$

Figure 1: RHP 2 for the function $F(x, t; \lambda)$.

**Remark 1.4.** Let us notice that the set of data of the RHP 2 is the function $r_u(\lambda)$, which is an entire function. Thus, it contains also all the information about the points $\lambda_j$, $\Im \lambda_j > -0$, where $r_u(\lambda_j) = i$.

In the theorem below we assume more regularity on the initial function $u_{t_0}(x)$, namely, we will take it from $C^\infty(\mathbb{R})$.

**Theorem 1.5.** Let $t_0, A < B$, $u_{t_0}(x)$ be a $C^\infty(\mathbb{R})$ function, which coincides with $U(x, t_0)$ for $x \in (-\infty, A) \cup (B, +\infty)$. Then RHP 2 has a unique solution $\hat{F}(x, t; \lambda)$, and $\hat{F}(x, t; \lambda)$ satisfies a stronger asymptotic condition

$$\hat{F}(x, t; \lambda) = \left( I + O\left( \frac{1}{\lambda} \right) \right) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\theta \sigma_3}, \lambda \to \infty, \quad \text{uniformly w.r.t.} \quad \arg \lambda \in [-\pi, \pi].$$
Furthermore, there exists a limit when \( \lambda \to \infty \) along non-transverse directions with the rays \( \gamma_3, \gamma_{-3}, \rho \)
\[
A(x, t) := \lim_{\lambda \to \infty} \left( \hat{F}(x, t; \lambda) e^{-\theta(x,t,\lambda)} \frac{1}{\sqrt{2}} (\sigma_3 + \sigma_1) \lambda^{\sigma_3/4} - I \right),
\]
and the function
\[
u(x, t) := 2A_{11}(x, t) - A_{12}^2(x, t)
\]
satisfies the following properties:
- \( \nu(x, t) \) is \( C^\infty \) smooth in \( x, t \) for all \( x, t \in \mathbb{R} \);
- \( \nu(x, t) \) satisfies KdV equation \( (1) \) for all \( t \in \mathbb{R} \);
- \( \nu(x, t_0) = \nu_u(x) \).

The following theorem is in some sense inverse to Theorems 1.3, 1.5:

**Theorem 1.6.** Let a function \( r_u(.) \) be any function, that satisfies the following set of conditions:

1. \( r_u(\lambda) \) is an entire function in \( \lambda \in \mathbb{C} \); define \( r_d(\lambda) = \overline{r_u(\lambda)} \);
2. properties 3, 4 of the part II of Theorem 1.3 are satisfied;
3. define functions \( a_u, a_d \) by formulas (8), (9). Then property 8 of the part II of Theorem 1.3 is satisfied;
4. there exist real \( t_0, A < B \) such that properties 6\(^{\infty} \), 7 of the part II of Theorem 1.3 are satisfied.

Then RHP 2 has a unique solution \( \hat{F}(x, t; \lambda) \), and \( \hat{F}(x, t; \lambda) \) satisfies a stronger asymptotic condition
\[
\hat{F}(x, t; \lambda) = (I + O(1/\lambda))^{\lambda^{\sigma_3/4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\theta \sigma_3} \lambda \to \infty, \text{ uniformly w.r.t. } \arg \lambda \in [-\pi, \pi].
\]

Furthermore, there exist limit when \( \lambda \to \infty \) along non-transverse directions with the segments \( \gamma_3, \gamma_{-3}, \rho \)
\[
A(x, t) := \lim_{\lambda \to \infty} \left( \hat{F}(x, t; \lambda) e^{-\theta(x,t,\lambda)} \frac{1}{\sqrt{2}} (\sigma_3 + \sigma_1) \lambda^{\sigma_3/4} - I \right),
\]
and the function
\[
u(x, t) := 2A_{11}(x, t) - A_{12}^2(x, t)
\]
satisfies the following properties:
- \( \nu(x, t) \) is \( C^\infty \) smooth in \( x, t \) for all \( x, t \in \mathbb{R} \);
- \( \nu(x, t) \) satisfies KdV equation \( (1) \) for all \( x, t \in \mathbb{R} \);
- \( \nu(x, t_0) = U(x, t_0) \) for any \( x \leq A \) and any \( x \geq B \).

Furthermore, functions \( r_u, r_d, a_u, a_d \) are the spectral functions associated with \( u(x, t_0) \).

The structure of the paper is as follows. In Section 2 we study the properties of the special pole-free joint solution \( U(x, t) \), and associated with it Jost solutions of the associated Schrödinger equation. In Section 3 we study the properties of perturbed Jost solutions, associated with a perturbation of \( U(x, t_0) \) at a given time \( t_0 \). In Section 4 we introduce the spectral functions \( a, b, r \), associated with a perturbation of \( U(x, t_0) \), and study their properties. The statements of Theorem 1.3, i.e. the properties of the spectral functions, follow from the material given in Sections 3, 4, namely Lemmas 3.1, 3.2, 3.3, 3.5, 3.6, 4.1, 4.2, 4.4, 4.7, 4.11. In Sections 5, 6 we solve an inverse scattering problem, by constructing an appropriate Riemann-Hilbert problem (Section 5), and proving its solvability (Section 6). The latter also proves the solvability of the initial value problem for the KdV and give a way to link the solution of KdV with a solution of a corresponding
RHP. Theorems 1.4 follows from Theorems 6.3, 6.8, and Theorem 1.5 follows from Theorems 6.3, 6.8, 6.9.

Finally, in Appendix we prove a spectral decomposition of the Schrödinger operator with the potential $U(x,t)$. In other words we show that the corresponding Jost solutions are orthogonal to each other. Let us notice that for bounded potentials (finite-gap or vanishing) a spectral decomposition (which is orthogonality of the exponents, or Fourier inversion formula) plays a central role in constructing of the integral representation for the Jost solutions, and studying their properties. In our work, since our potential is a compactly supported perturbation of $U(x,t_0)$, we managed to study the properties of the Jost solutions by hands, without referring to a spectral theorem. However, should one try to construct Jost solutions for more general class of perturbations, she or he or they might need to use that theorem.

The properties at $x \to \pm \infty$ or $t \to \infty$ of the constructed solution $u(x,t)$ of the KdV equation will be studied somewhere else.

Schematically, we can represent the contents of Theorems 1.3, 1.5, 1.6 by the following diagrams:

Theorem 1.3 : $u_\sigma(x) \rightarrow \{a(\lambda), b(\lambda), \sigma(\lambda)\}$.

Theorem 1.5 : $u_\sigma(x) \rightarrow \{a(\lambda), b(\lambda), r(\lambda)\} \rightarrow \text{RHP 1}$ for $\hat{\mathcal{F}}(x;\lambda) \rightarrow u(x,t)$.

Theorem 1.6 : $r(\lambda) \rightarrow \{a(\lambda), b(\lambda)\} \rightarrow \text{RHP 2}$ for $\hat{\mathcal{F}}(x;\lambda) \rightarrow u(x,t) \rightarrow u_\sigma(x)$.

2 Special pole-free joint solution $U(x,t)$ of the KdV equation and the $P^2_t$ equation.

Here we list some further properties of the function $U(x,t)$ and the corresponding RHP [1]. Along with the RHP [1] we consider another RHP for a function $\hat{\mathcal{E}}(x;\lambda)$, with the same analyticity and jump conditions as in RHP [1] but with the condition at $\lambda \to \infty$ altered.

Riemann-Hilbert problem 3. Find a $2 \times 2$ matrix-valued function $\mathcal{E}(x,t;\lambda)$, which satisfies analyticity and jump conditions of RHP [1] and has the following asymptotics:

3a. asymptotics as $\lambda \to \infty$

$$\hat{\mathcal{E}}(\lambda) = (I + O(\lambda^{-1})) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{B_3}.$$  

Remark 2.1. Both conditions at infinity fix uniquely the solution of the RHP. This is obvious for the condition 3a. in RHP [3]. For the condition 3. in RHP [1], we first conclude that the ratio of any 2 solutions $\hat{\mathcal{E}}$ and $\mathcal{E}$ is constant in $\lambda$ and lower triangular:

$$\hat{\mathcal{E}} : \mathcal{E}^{-1} = \begin{pmatrix} 1 & 0 \\ e(x,t) & 1 \end{pmatrix},$$

and then multiplying the above ratio by $\mathcal{E}$ from the right and checking the asymptotics for $\hat{\mathcal{E}}$ at $\lambda \to \infty$, we see that this triangular matrix should be the identity.

There is a simple relation between the solutions of the RHPs [1] and [3] and each admits a full asymptotic expansion at infinity of the following form:

Theorem 2.2 ([Claey Vanlessen 07]). For any $x, t \in \mathbb{R}$ there exists a unique solution $\hat{\mathcal{E}}(x;\lambda)$ to the RH problem [3] [3] It is smooth (infinitely many times differentiable) w.r.t. $x, t$, and admits the following uniform asymptotic expansion as $\lambda \to \infty$: for any integer $J \geq 1$

$$\hat{\mathcal{E}}(x,t;\lambda) = \left(I + \sum_{j=1}^{J-1} \begin{pmatrix} a_j(x,t) \\ c_j(x,t) \end{pmatrix} \begin{pmatrix} b_j(x,t) \\ d_j(x,t) \end{pmatrix} \lambda^{-j} + O(\lambda^{-J}) \right) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\vartheta(x,t;\lambda)\sigma_3}. \quad (12)$$
Corollary 2.3. The solution \( E(x, t; \lambda) \) of the RHP \[ \square \] is related to \( \hat{E} \) as follows:

\[
E(x, t; \lambda) = \left( \begin{array}{cc} 1 & 0 \\ b_1(x, t) & 1 \end{array} \right) \hat{E}(x, t; \lambda),
\]

and hence admits the uniform asymptotic expansion

\[
E(x, t; \lambda) = \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( I + \frac{b_1(x, t) \sigma_3}{\sqrt{\lambda}} + \frac{a_j + d_j + b_1 b_j}{2 \lambda^{j+\frac{1}{2}}} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) + \frac{a_j - d_j - b_1 b_j}{2 \lambda^{j+\frac{1}{2}}} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) + \sum_{j \geq 1} \frac{b_1 a_j + c_j + b_j}{2 \lambda^{j+\frac{1}{2}}} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + \sum_{j \geq 1} \frac{b_1 a_j + c_j - b_j + 1}{2 \lambda^{j+\frac{1}{2}}} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right) e^{\theta(x, t; \lambda) \sigma_3} = \\
\frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \left( 1 + \frac{1}{\sqrt{\lambda}} \sum_{j \geq 1} \frac{b_j}{\lambda^{j+\frac{1}{2}}} + \sum_{j \geq 1} \frac{a_j}{\lambda^j} \right) e^{\theta(x, t; \lambda)}
\]

(14)

Remark 2.4. Symmetry. Let us notice that the jump matrix \( J_E \) satisfies the symmetry

\[
J_E(\lambda) = J_E^{-1}(\frac{1}{\lambda}),
\]

which implies

\[
E(x, t; \lambda) = \overline{E(x, t; \lambda)}, \quad \hat{E}(x, t; \lambda) = \overline{E(x, t; \lambda)},
\]

and hence all the coefficients \( a_j(x, t), b_j(x, t), c_j(x, t), d_j(x, t) \) in expansion \[ \square \] are real.

Furthermore, since the jumps of \( E \) are independent of \( x, t, \lambda \), we have

\[
E_x E^{-1} =: \Upsilon(x, t; \lambda), \quad E_t E^{-1} =: \Psi(x, t; \lambda), \quad E_\lambda E^{-1} =: \Omega(x, t; \lambda),
\]

(16)

where \( \Upsilon(x, t; \lambda), \Psi(x, t; \lambda), \Omega(x, t; \lambda) \) are polynomials in \( \lambda \). Substituting asymptotic series \[ \square \], \[ \square \] into \[ \square \], and taking into account that \( \det E = -1 \), one obtains

\[
\Upsilon(x, t; \lambda) =: \left( \lambda - 2U(x, t) \right) \left( \begin{array}{cc} 1 & 0 \\ -\lambda & 1 \end{array} \right), \quad \Psi(x, t; \lambda) =: \left( \frac{-\lambda}{3} + \frac{U x}{4} + \frac{4 U^2 + U_{xx}}{6} \right) \left( \begin{array}{cc} \frac{-U}{6} & 0 \\ 0 & -U \end{array} \right)
\]

(17)
\[\mathcal{W}(x, t; \lambda) = \left( \frac{-U_x \lambda}{60} - \frac{1}{200} \left( \frac{1}{20} U_x^2 + \frac{1}{20} U_{xxx} \right) \lambda \right) \frac{\lambda^2}{30} + \frac{-60t + 6U^2 + U_{xx}}{120}, \]

and some of the coefficients \(a_j, b_j, c_j, d_j\) in [12] are related to the function \(U = U(x, t)\) in the following way:

\[U(x, t) = 2a_1 - b_1^2 = -b_1 x, \quad U_x := 2(3a_1 b_1 - b_1^2 - b_2 + c_1), \]

\[x + \frac{U^3}{15} - \frac{U_{xx}}{60} = -2a_1^2 + a_2 + 4a_1 b_1^2 - b_1^2 - 2b_2 b_1 + b_1 c_1 - d_2 + 3b_1 t, \]

\[b_1 = \frac{1}{480} \left( 240U^2 - 20U^4 - 20U(24x + U_x^2) + U_x^2 - 2U_x U_{xxx} \right). \]

Hence, asymptotic series [12], [14] develop into

\[\hat{E}(x, t; \lambda) = \left( I + \frac{\lambda \lambda}{4 \sqrt{2}} \right) \frac{1}{1 - \lambda^2} + \mathcal{O}(\lambda^{-2}) \lambda^{-\sigma/4} \left( \frac{1}{1 - 1} \right) e^{\theta(x, t; \lambda)} \sigma_3, \]

\[E(x, t; \lambda) = \frac{\lambda^{-\sigma/4}}{\sqrt{2}} \left( \frac{1}{1 - 1} \right) \left( I + \frac{b_1(x, t) \sigma_3}{\sqrt{\lambda}} \right) + \frac{1}{2\lambda} \left( b_1^2 U U_x \right) + \mathcal{O}(\lambda^{-3/2}) \right) \cdot e^{\theta(x, t; \lambda)} \sigma_3, \]

which allows one to reconstruct \(U(x, t)\), once \(E\) or \(\hat{E}\) are known.

The consistency condition of the system

\[\begin{cases} E_x = \nabla E, \\ E_t = \mathcal{V} E, \end{cases} \]

i.e.

\[\mathcal{V}_t - \mathcal{V}_x + [\mathcal{V}, \mathcal{W}] = 0, \]

gives that \(U(x, t)\) satisfies the KdV equation [1], and the consistency condition of the system

\[\begin{cases} E_x = \nabla E, \\ E_\lambda = \mathcal{W} E, \end{cases} \]

i.e.

\[\mathcal{W}_\lambda - 2\mathcal{W}_x + [\mathcal{W}, \mathcal{W}] = 0, \]

gives that \(U(x, t)\) satisfies the PDE equation [2].

**Remark 2.5.** In the notations of [Grava Kapteev Klein’15], \(H_1 = -b_1, (H_1(x, t))_x = U(x, t), \)

\[E(\lambda) = \left( \begin{array}{cc} 1 & 0 \\ -H_1 & 1 \end{array} \right) \left( I + \frac{H_1^2 + U}{\lambda} \right) + \mathcal{O}(\lambda^{-2}) \left( \frac{1}{\lambda} \right)^{-\sigma} \left( \frac{1}{1 - 1} \right) e^{\theta(x, t; \lambda)} \sigma_3. \]

It was obtained in [Claeys Vanlessen’07] that

**Theorem 2.6.** [Claeys Vanlessen’07]

- \(U(x, t)\) is real-valued and pole-free for \(x, t \in \mathbb{R}\),

- for fixed \(t \in \mathbb{R}\), \(U(x, t)\) has the following asymptotic behavior:

\[U(x, t) = \frac{z_0(x, t)}{2} \sqrt{|x|} + \mathcal{O}(x^{-2}) = \sqrt{-6x} + \frac{2t}{\sqrt{-6x}} + \frac{8}{3(6x)^{3/2}} + \mathcal{O}(|x|^{-2}), \quad x \to \pm \infty, \]

where \(z_0(x, t)\) is the real solution of

\[z_0^3 = -48 \text{sgn}(x) + 24 \frac{z_0 t}{|x|^{2/3}}. \]
2.1 Jost solutions associated with $U(x, t)$.

It follows from the RHP \[1\] that the Jost solutions $E_t$, $E_r$ defined in \[1\] satisfy the relation

$$
E_{lu}(x, t; \lambda) = E_{ld}(x, t; \lambda) - i E_r(x, t; \lambda), \quad E_r(x, t; \lambda) = i E_{lu}(x, t; \lambda) - i E_{ld}(x, t; \lambda). \tag{24}
$$

Large $\lambda$ behavior of $E_t$, $E_r$ follows from formulas \[12\], \[13\], \[14\], \[22\], \[23\]. The following lemma, which refines Lemma \[1.1\], gives us large $x$ behavior of $E_t$, $E_r$.

**Lemma 2.7.** [Properties of $E_r$, $E_t$ as $x \to \pm \infty$]

1. Let $t \in \mathbb{R}$, $\lambda \in C$ be fixed, and $x \to +\infty$. Then

$$
(E_t, E_r) = \frac{1}{\sqrt{2}} \left( 1 + O(|x - i\lambda|) \right) \left( 1 + O(|x + i\lambda|) \right) (\lambda - \lambda_0)^{-2\lambda} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^{\sigma_3 x}.
$$

2. Let $t \in \mathbb{R}$, $\lambda \in (C \setminus \mathbb{R}) \cup (\mathbb{R} + i0) \cup (\mathbb{R} - i0)$ be fixed, and $x \to -\infty$. Then

$$
(E_t, -i \cdot \text{sgn} \Im \lambda \cdot E_t(\lambda)) = \left\{ \begin{array}{l} (E_{lu}, -i E_{ld}), \ \Im \lambda > 0, \\
(E_{ld}, i E_{lu}), \ \Im \lambda < 0 \end{array} \right\}
$$

$$
= \frac{1}{\sqrt{2}} \left( 1 + O(|x - i\lambda|) \right) \left( 1 + O(|x + i\lambda|) \right) (\lambda - \lambda_0)^{-2\lambda} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^{\sigma_3 x}.
$$

3. $E_t(x, t; \lambda) = E_t(x, t; \lambda)$; $E_r(x, t; \lambda) = E_r(x, t; \lambda)$.

4. $\det(E_t, E_r) = W \{e_t, e_r\} = -1, \quad \det(E_{lu}, E_{ld}) = W \{e_{lu}, e_{ld}\} = -i$.

Here

$$
g = g(x, t; \lambda) = \frac{1}{165} (\lambda - \lambda_0)^{7/2} + \frac{\lambda_0}{30} (\lambda - \lambda_0)^{5/2} + \frac{\lambda_0^2}{24} (\lambda - \lambda_0)^{3/2}, \tag{25}
$$

where $\lambda_0 = \lambda_0(x, t)$ is the solution of the equation

$$
\lambda_0^3 - 24 t \lambda_0 + 48 x = 0, \tag{26}
$$

which is fixed for $x \to \pm \infty$ by the condition that $\lambda_0(x, t)$ is real for real $x, t$.

**Remark 2.8.** It is convenient to expand $g(x, t; \lambda)$ for large $x \to \pm \infty$. We have:

1. as $x \to -\infty$, $\lambda_0(x, t) \to +\infty$, $g = \Re g + i\Im g$, where

$$
\Re g = -\text{sgn } \Im \lambda \left( \frac{1}{80} \Im(\lambda) \lambda_0^{5/2} + \frac{1}{192} \Im(\lambda) \lambda_0^{3/2} + \frac{1}{128} \Im(\lambda - 64 t \lambda) \lambda_0^{1/2} + O(\lambda_0^{-3/2}) \right),
$$

$$
\Im g = -\text{sgn } \Im \lambda \left( \frac{1}{56} \lambda_0^{7/2} - \frac{1}{80} \Re(\lambda) \lambda_0^{5/2} - \frac{1}{192} \Im(\lambda^2 + 64 t) \lambda_0^{3/2} + \frac{1}{128} \Re(64 t - \lambda) \lambda_0^{1/2} + O(\lambda_0^{-3/2}) \right),
$$

where we set $\text{sgn } \Im \lambda = \pm 1$ for $\lambda \in \mathbb{R} \setminus i0$, since $g$ has discontinuity across $\lambda \in (-\infty, \lambda_0]$;

2. as $x \to +\infty$, $\lambda_0(x, t) \to -\infty$ and

$$
g = \frac{(-\lambda_0)^2}{56} + \frac{\lambda}{80} (-\lambda_0)^{5/2} - \frac{t}{3} \left( -\lambda_0 \right)^2 + \frac{(\lambda^3 - 64 t \lambda)}{128} (-\lambda_0)^{1/2} + O(\lambda_0^{-3/2}).
$$
Lemma 2.7 implies Lemma 1.1 and shows that $E_r(x, t; \lambda)$ is rapidly vanishing for any $\lambda \in \mathbb{C}$ as $x \to +\infty$. On the other hand,

$$E_l = \begin{cases} E_{lu}, \Im \lambda > 0, \\ E_{ld}, \Im \lambda < 0 \end{cases}$$

is rapidly vanishing for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ as $x \to -\infty$, while for $\lambda \in \mathbb{R} \pm i0$, the function $E_l$ has oscillatory behavior of finite amplitude. This observation justifies referring to $E_l, E_r$ as Jost solutions, and the subscripts $r, t, lu, ld$.

Proof of Lemmas 3.1, 3.2, 3.3. Large $x$ behavior of $E_l, E_r$ can be obtained in the same manner, as large $x$ behavior of $U(x, t)$, see [Claeys Vanlessen’07], [Claeys’10], [Grava Kapaev Klein’15]. However, for convenience of the reader, we will sketch the derivation.

In order to study the asymptotics RHP 1’s solution for fixed $\lambda, t$, and $x \to \pm \infty$, first we shift the contours

$\gamma_3 = (e^{\frac{6\pi i}{7}} \infty, 0), \gamma_{-3} = (e^{-\frac{6\pi i}{7}} \infty, 0)$ to

$\gamma_3 + \lambda_0 = (e^{\frac{6\pi i}{7}} \infty, \lambda_0), \gamma_{-3} + \lambda_0 = (e^{-\frac{6\pi i}{7}} \infty, \lambda_0),$

where $\lambda_0$ is defined in (26). We call the solution of such a shifted RHP by $E_{\lambda_0}(x, t; \lambda)$.

Next we make a scaling change of the variables:

$$\lambda =: \zeta \sqrt{|x|}, \quad \lambda_0 =: \zeta_0 \sqrt{|x|}, \quad \zeta^3 - 24 \frac{t}{\sqrt{|x|^2}} \zeta_0 + 48 \text{sgn}(x) = 0,$$

then the function $g$ defined in (25) and the function $\theta$ can be written as

$$g(\lambda) = |x|^{7/6} \left( \frac{1}{105} (\zeta - \zeta_0)^{7/2} + \frac{\zeta_0}{30} (\zeta - \zeta_0)^{5/2} + \frac{\zeta_0^2 - \frac{8t}{12\pi^2|x|^2}}{24} (\zeta - \zeta_0)^{3/2} \right),$$

$$\theta(\lambda) = |x|^{7/2} \left( \frac{1}{105} \zeta^{3/2} - \frac{t}{3|\zeta|^2} \zeta^{3} + \text{sgn} x \cdot \zeta^{3/2} \right).$$

The signature table (distribution of signs) of the function $\Im g$ is shown in Figure 3.

Figure 3: Distribution of signs of $\Re g(\lambda)$: for $x \to -\infty$ (on the left), for $x \to +\infty$ (on the right). Blue dashed lines are separatrices of $\Re g = \text{const} \neq 0$. Red lines correspond to $\Re g = 0.$
We see that the position of \( \zeta_0 = \zeta_0(x,t) \) tends to fixed limits when \( x \to \pm \infty \).

The next step is to introduce a new local variable \( z(\zeta) \) in the vicinity of the point \( \zeta_0 \),

\[
\frac{2}{3} z^{3/2} = g(\lambda), \quad |\zeta - \zeta_0| < R, \quad \text{where } R > 0 \quad \text{is sufficiently small},
\]

which allows us to define an approximation for

\[
\chi E_{\lambda_0},
\]

where the matrix \( \chi \) is to be determined. Namely, define

\[
E_\infty(\zeta) = \begin{cases} 
(\zeta - \zeta_0)^{-\sigma_1/4} |x|^{\sigma_3/12 + \frac{3}{4} \sigma_1} e^{g(\lambda) \sigma_3}, & |\zeta - \zeta_0| > R, \\
B(z) Z(\zeta(\zeta)), & |\zeta - \zeta_0| < R,
\end{cases}
\]

where the matrix-valued function \( Z(z) \) is constructed as follows: denote

\[
v_1(z) = \sqrt{2\pi} Ai(z), \quad v_2(z) = \sqrt{2\pi} e^{\pi i/6} Ai(e^{\pi i/3} z), \quad v_3(z) = \sqrt{2\pi} e^{-\pi i/6} Ai(e^{-\pi i/3} z),
\]

then

\[
Z(z) = \begin{cases} 
v_3 v_1 \\
v_3 v_1 \\
v_2 v_1
\end{cases}
\]

arg \( z \in (0, 2\pi i/3) \), \( v_3 - iv_2 \), arg \( z \in (2\pi i/3, \pi i) \), \( v_3 + iv_2 \), arg \( z \in (-2\pi i/3, -\pi i) \). \quad (27)

The function \( Z \) has the following asymptotics as \( z \to \infty \):

\[
Z(z) = (1 + O(z^{-3}) \frac{O(z^{-2})}{O(z^{-1})}) z^{-\sigma_3/4} e^{\frac{\pi i}{6} \sigma_1 + \frac{1}{2} \sigma_3 + \frac{1}{8} \frac{1}{4} (1 + O(z^{-3/2}) e^{\frac{\pi i}{3} \sigma_3}) \sqrt{2},
\]

and \( Z \) has the same jumps as \( E \).

The function \( \chi \) is introduced in order to ensure identical asymptotics at infinity of

\[
E_{err}(\zeta) := \chi E_{\lambda_0}(\lambda(\zeta)) E_\infty^{-1}(\zeta),
\]

\[
E_{err} := \chi \left( \begin{array}{ccc} 1 & 0 \\
1 & 1 \\
\end{array} \right) (I + O(\lambda^{-1})) \lambda \frac{\sigma_3 + \sigma_1}{\sqrt{2}} e^{\frac{\pi i}{6} \sigma_1 + \frac{1}{2} \sigma_3 + \frac{1}{8} (1 + O(z^{-3/2}) e^{\frac{\pi i}{3} \sigma_3}) \sqrt{2},
\]

Hence, we take

\[
\chi := \left( \begin{array}{ccc} 1 & 0 \\
-1 & 1 \\
\end{array} \right) (I + O(\lambda^{-1})).
\]

(28)

where \( h_1 = h_1(x,t) \) is determined from the expansion at \( \lambda \to \infty \) of \( g(\lambda) - \theta(\lambda) = \frac{h_1}{\sqrt{\lambda}} + O(\lambda^{-3/2}) \), i.e.

\[
h_1 = \frac{\lambda_0^4}{128} - \frac{t \lambda_0^2}{8} - \frac{t \lambda_0^2}{16} - \frac{3 x \lambda_0}{8} = |x|^{\frac{1}{4}} \left( \frac{c_0^4}{128} - \frac{t}{8} |x|^{-\frac{1}{4}} \right) = |x| \left( \frac{t \lambda_0^2}{16} - \frac{3 x \lambda_0}{8} \right).
\]

The jump for \( E_{err}(\zeta) \) on \( |\zeta - \zeta_0| = R \) is \( E_{err,+} = E_{err,-} - J_{E_{err}} \), where

\[
J_{E_{err}} = (\lambda - \lambda_0) \frac{\sigma_3 + \sigma_1}{\sqrt{2}} e^{\frac{\pi i}{6} \sigma_1 \sigma_3} Z^{-1}(\zeta(\zeta)) B^{-1}(z) = \left( \frac{\lambda - \lambda_0}{z} \right)^{-\sigma_3} \left( 1 + O(z^{-3}) \frac{O(z^{-2})}{O(z^{-1})} \frac{1}{1 + O(z^{-3})} \right) B(z).
\]

Hence, in order to make this jump close to the identity matrix on the circle \( |\zeta - \zeta_0| = R \), we choose an analytic in \( |\zeta - \zeta_0| < R \) matrix \( B(z) \) as

\[
B(z) := \left( \frac{\lambda - \lambda_0}{z} \right)^{-\sigma_3}.
\]
Then the jump on \(|\zeta - \zeta_0| = R\) satisfies

\[
J_{E_{\text{err}}} = \begin{pmatrix}
1 + \mathcal{O}(z^{-3}) & \mathcal{O}(z^{-2}) \\
\mathcal{O}(z^{-1}) & 1 + \mathcal{O}(z^{-3})
\end{pmatrix}^{\frac{1}{2}}(\mathcal{E}_{\text{err}}) = \begin{pmatrix}
1 + \mathcal{O}(|x|^{-\frac{1}{3}}) & \mathcal{O}(|x|^{-\frac{2}{3}}) \\
\mathcal{O}(|x|^{-1}) & 1 + \mathcal{O}(|x|^{-\frac{2}{3}})
\end{pmatrix}.
\] (29)

The RHP for \(E_{\text{err}}\) is equivalent to the following singular integral equation (SIE):

\[
E_{\text{err},-} = I + C_- [E_{\text{err},-}(J_{E_{\text{err}}} - I)],
\] (30)

where \(C_- = C_{\Sigma_{E_{\text{err}}}}\) is the operator defined by

\[
[C_{\pm} f](\lambda) = \frac{1}{2\pi i} \lim_{\lambda^\prime \to \lambda, \lambda^\prime \in \pm \text{side} \Sigma_{E_{\text{err}}}} \int_{\Sigma_{E_{\text{err}}}} \frac{f(s)ds}{s - \lambda^\prime} = \frac{1}{2\pi i} \int_{\Sigma_{E_{\text{err}}}} \frac{f(s)ds}{s - \lambda} \mathcal{C}_{\pm} f = f,
\]

and

\[
\Sigma_{E_{\text{err}}} = (\zeta_0 + |x|^{-1/3}\Sigma) \cup \{ \zeta : |\zeta - \zeta_0| = R \} \setminus \{ \zeta : |\zeta - \zeta_0| < R \}
\]
is the contour for \(E_{\text{err}}(\zeta)\). Once the solution of the above SIE (30) is known, the solution to the RHP is given by the formula

\[
E_{\text{err}}(\zeta) = I + \frac{1}{2\pi i} \int_{\Sigma_{E_{\text{err}}}} \frac{E_{\text{err},-}(J_{E_{\text{err}}} (s) - I)ds}{s - \zeta} = I + [C_{\Sigma_{E_{\text{err}}}}(J_{E_{\text{err}}}(s) - I)](s).
\]

Analyzing the SIE (30), taking into account formula (29) for the jump \(J_{E_{\text{err}}}\) on the circle \(|\zeta - \zeta_0| = R\), and also that on the other parts of the contour \(\Sigma_{E_{\text{err}}}\), the jump matrix \(J_{E_{\text{err}}}\) is exponentially close to \(I\), we conclude that the entries for \(E_{\text{err}}\) have the following asymptotics as \(|x| \to \infty\), which are uniform w.r.t. \(\zeta \in \mathbb{C} \cup \{\infty\}:

\[
E_{\text{err}}(\zeta) \approx \begin{pmatrix}
1 + \mathcal{O}(|x|^{-\frac{1}{3}}) & \mathcal{O}(|x|^{-\frac{2}{3}}) \\
\mathcal{O}(|x|^{-1}) & 1 + \mathcal{O}(|x|^{-\frac{2}{3}})
\end{pmatrix}.
\]

Moreover, the entries in the large \(\zeta\) expansion of \(E_{\text{err}}\) are well controlled in \(x\). Now, to obtain the large \(x\) asymptotics for \(E_{\lambda_0}\), we recall that \(E\), and hence \(E_{\lambda_0}\), admit asymptotic expansion of the form

\[
E_{\lambda_0} = \begin{pmatrix}1 & 0 \\ b_1 & 1\end{pmatrix} \begin{pmatrix}I + \sum_{j \geq 1} \begin{pmatrix}a_j & b_j \\ c_j & d_j\end{pmatrix} \lambda^{-j} + \mathcal{O}(\lambda^{-\infty})\end{pmatrix} \lambda^{\frac{-\sigma_1}{\sqrt{2}}} \frac{\sigma_3 + \sigma_1 e^{\theta_{\sigma_3}}}{\sqrt{2}}.
\]

and hence

\[
E_{\text{err}} \cdot E_{\infty} = \chi E_{\lambda_0} = \begin{pmatrix}1 & 0 \\ h_1 & 1\end{pmatrix} \begin{pmatrix}I + \sum_{j \geq 1} \begin{pmatrix}a_j & b_j \\ c_j & d_j\end{pmatrix} \lambda^{-j} + \mathcal{O}(\lambda^{-\infty})\end{pmatrix} \lambda^{\frac{-\sigma_1}{\sqrt{2}}} \frac{\sigma_3 + \sigma_1 e^{\theta_{\sigma_3}}}{\sqrt{2}}.
\]

From here (after some computations) we first obtain

\[
b_1 = h_1 + E_{\text{err}}^{1,12}, \quad a_1 = h_1^2 + h_1 E_{\text{err}}^{1,12} + \lambda_0 + E_{\text{err}}^{1,11}, \quad \text{(hence \(U = 2a_1 - b_1^2 = \lambda_0^2 + 2E_{\text{err}}^{1,11} - (E_{\text{err}}^{1,12})^2\)}
\]

where we denoted

\[
E_{\text{err}} = I + \sum_{j \geq 1} \begin{pmatrix}E_j^{1,11} & E_j^{1,12} \\ E_j^{2,11} & E_j^{2,12}\end{pmatrix} \lambda^{-j} = I + \sum_{j \geq 1} \begin{pmatrix}E_j^{1,11} & E_j^{1,12} \\ E_j^{2,11} & E_j^{2,12}\end{pmatrix} |x|^{-j/3} \zeta^{-j},
\]

and thus, since

\[
E_{\text{err}}^{1,11} = \mathcal{O}(|x|^{-7/3}|x|^{1/3}) = \mathcal{O}(|x|^{-2}), \quad E_{\text{err}}^{1,12} = \mathcal{O}(|x|^{-4/3}|x|^{1/3}) = \mathcal{O}(|x|^{-1}),
\]
In this context, namely, in formula (7), bounded potentials, we suggest a way to develop some intuition for the properties of Jost solutions. We see that the right Jost solution is analytic in $\mathbb{C}$, and

$$E_{\lambda_0} = \chi^{-1} \cdot \text{err} \cdot E_{\infty} = \left( \frac{1}{\mathcal{O}(|x|^{-1})} \right)^{1} \cdot \left( \frac{1 + \mathcal{O}(|x|^{-\frac{7}{2}})}{\mathcal{O}(|x|^{-1})} \right)^{\frac{\mathcal{O}(|x|^{-\frac{7}{2}})}{1 + \mathcal{O}(|x|^{-\frac{7}{2}})} \left( \lambda - \lambda_0 \right)^{-\frac{\sigma_1}{\sqrt{2}} \cdot e^{\sigma_2}}.

To obtain the first two statements of the lemma we use the definition of $E_l$, $E_r$ (6), and track how $E_{\lambda_0}$ is related to them. As to the latter, for any fixed $\lambda \in \mathbb{C}, t \in \mathbb{R}$ and $x \to +\infty$ we have

$$E_{\lambda_0}(x, t; \lambda) = \left( E_l(x, t; \lambda), E_r(x, t; \lambda) \right), \quad x \to +\infty$$

(here we incorporated both cases $\Re \lambda \geq 0$, since $E_l$ has cut along $\mathbb{R}$). On the other hand, for any fixed $\lambda, t \in \mathbb{R}$ and $x \to -\infty$ we have

$$E_{\lambda_0}(x, t; \lambda) = \left( E_l(x, t; \lambda), -i \cdot \text{sgn} \Re \lambda \cdot \text{err}(x, t; \lambda) \right).$$

Hence, for any $\lambda$, as $x \to -\infty$, $\lambda_0 \to +\infty$, we have

$$e_l(x, t; \lambda) = \frac{1}{\sqrt{2}}(\lambda - \lambda_0)^{-1/4}(1 + \mathcal{O}(|x|^{-7/6}))e^{\sigma_2(x, \lambda)},$$

$$e_r(x, t; \lambda) = \frac{i}{\sqrt{2}}(\lambda - \lambda_0)^{-1/4}(1 + \mathcal{O}(|x|^{-7/6}))e^{\sigma_2(x, \lambda)} + \frac{1}{\sqrt{2}}(\lambda - \lambda_0)^{-1/4}(1 + \mathcal{O}(|x|^{-7/6}))e^{-\sigma_2(x, \lambda)},$$

and for $x \to +\infty$, $\lambda_0 \to -\infty$, we have

$$e_l(x, t; \lambda) = \frac{1}{\sqrt{2}}(\lambda - \lambda_0)^{-1/4}(1 + \mathcal{O}(|x|^{-7/6}))e^{\sigma_2(x, \lambda)},$$

$$e_r(x, t; \lambda) = \frac{1}{\sqrt{2}}(\lambda - \lambda_0)^{-1/4}(1 + \mathcal{O}(|x|^{-7/6}))e^{-\sigma_2(x, \lambda)}.$$}

This gives us the first and the second statements of the lemma. The 3rd statement follows from formula (15). The 4th statement follows from the fact that $\det E = -1$ and the definition (6). □

**Remark 2.9.** Since the properties of Jost solutions to the Sturm-Liouville equation with the potential $-2U(x, t)$ are different from the properties of Jost solutions associated with vanishing or bounded potentials, we suggest a way to develop some intuition for the properties of Jost solutions in this context. Namely, in formula (7), instead of the function $U$, take a function

$$u = \begin{cases} c_r, & x > 0, \\ c_l, & x < 0. \end{cases}$$

The corresponding Jost solutions are of the form

$$F_r = \begin{pmatrix} e^{-\sqrt{\lambda - 2c_r}x} \\ 2 \sqrt{\lambda - 2c_r} e^{-\sqrt{\lambda - 2c_r}x} \end{pmatrix}, \quad F_l = \begin{pmatrix} e^{\sqrt{\lambda - 2c_r}x} \\ 2 \sqrt{\lambda - 2c_r} e^{\sqrt{\lambda - 2c_r}x} \end{pmatrix}.$$}

We see that the right Jost solution $F_r$, which is vanishing for $x \to +\infty$ and $\lambda \in \mathbb{C} \setminus (-\infty, 2c_r]$, is analytic in $\mathbb{C} \setminus (-\infty, 2c_r]$. At the same time, the left Jost solution $F_l$, which is vanishing for $x \to -\infty$ and $\lambda \in \mathbb{C} \setminus (-\infty, 2c_l]$, is analytic in $\mathbb{C} \setminus (-\infty, 2c_l]$. 15
We know that $U(x,t) \sim \sqrt{-6x}$, $x \to \pm \infty$, hence we can simulate this behavior of the potential $U(x,t)$ by taking $c_l \to +\infty$ and $c_r \to -\infty$. We see that in the limit the cut for the domain of $F_r$ will shrink, and $F_r$ will become analytic in the whole complex plane, while the cut for $F_l$ will increase and in the limit $F_l$ will be analytic discontinuous across $\mathbb{R}$.

This analogy works only up to some extent. For example, the properties of the transmission and reflection coefficients are different. Indeed, the usual scattering relation

$$F_r = R(\lambda)F_l + T(\lambda)F_l(\bar{\lambda}),$$

where $R,T$ are the reflection and transmission coefficients, respectively, becomes

$$E_r = iE_{lu} - iE_{ld} = iE_{lu} - i\overline{E_{lu}(\bar{\lambda})},$$

and thus the transmission coefficient becomes $-i$, and the reflection coefficient becomes $i$.

### 2.2 Analogy with Airy functions

To gain some intuition, whenever it is possible we will use some similarity of the RHPs [3,4] with the following RHP, whose solution can be constructed explicitly in terms of Airy functions.

**Riemann-Hilbert problem 4.** Find a $2 \times 2$ matrix-valued function $E^{Ai}(x,t;\lambda)$, that

1. has the same analyticity and jump conditions as in RHP [4];
2. has the following asymptotics as $\lambda \to \infty$, uniformly w.r.t. arg $\lambda \in [-\pi, \pi]$:

$$E^{Ai}(x,t;\lambda) = \frac{1}{\sqrt{2}}\lambda^{-\sigma_3/4}(\sigma_3 + \sigma_1) \left( I + \frac{\tilde{b}^{Ai}}{\sqrt{\lambda}} + O(\lambda^{-1}) \right) e^{\theta^{Ai}(x,t)\sigma_3},$$

where

$$\theta^{Ai} = \theta^{Ai}(x,t;\lambda) := -\frac{t}{3}\lambda^2 + x\lambda^{\frac{1}{2}},$$

and a scalar $\tilde{b}^{Ai} = \tilde{b}^{Ai}(x,t)$ is not fixed, but is introduced in order to fix the structure of the asymptotics.

**The solution $U^{Ai}(x,t)$ of the KdV equation [1]** associated with RHP [4] can be constructed by formula [5] (in which we replaced $E$ with $E^{Ai}$).

The solution $E^{Ai}(x,t;\lambda)$ of RHP [4] can be constructed as follows:

$$E^{Ai}(x,t;\lambda) := \begin{cases} 
(e^{Ai}_{lu,x}, -ie^{Ai}_{id,x}), & \text{arg} \lambda \in \left(\frac{6\pi}{7}, \pi\right), \\
(e^{Ai}_{id,x}, ie^{Ai}_{lu,x}), & \text{arg} \lambda \in \left(0, \frac{6\pi}{7}\right), \\
(e^{Ai}_{id,x}, ie^{Ai}_{id,x}), & \text{arg} \lambda \in \left(-\pi, -\frac{6\pi}{7}\right), \\
(e^{Ai}_{id,x}, ie^{Ai}_{lu,x}), & \text{arg} \lambda \in \left(-\frac{6\pi}{7}, 0\right),
\end{cases}$$

where

$$e^{Ai}_{lu,x}(x,t;\lambda) := (-t)^{\frac{1}{3}}2^{\frac{1}{3}}\pi Ai \left( \frac{\sqrt{-t}}{2} \right)^{2/3} \left( \lambda - \frac{2x}{t} \right) = e^{-\lambda^{1/2} + \frac{t}{2}\lambda^{3/2}} \frac{1}{\sqrt{2\sqrt{\lambda}}} \left( 1 + \frac{x^2}{2\sqrt{\lambda}} + \frac{x^4 + 4xt}{8t^2\lambda} + \frac{x^6 + 20xt^3 + 10t^2}{48t^4} \lambda^{-3/2} + \ldots \right),$$

$$e^{Ai}_{id,x}(x,t;\lambda) = -e^{-\lambda^{1/2} + \frac{t}{2}\lambda^{3/2}} \frac{1}{\sqrt{2}} \left( 1 + \frac{x^2}{2\sqrt{\lambda}} + \frac{x^4 - 4xt}{8t^2\lambda} + \frac{x^6 - 4xt^3 - 14t^2}{48t^4} \lambda^{-3/2} + \ldots \right),$$

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The function $E_2$ and $U$ from here, by formulas (19), (23) we get a solution to the KdV equation (1):

\[ x, \lambda, x \]

Lemma 3.1. Let $A_i$ be Jost solutions associated with a perturbation $u_{t_0}(x)$, which is differentiable in $x$, and satisfies the $x$-equation (the subscript $x$ denotes the derivative w.r.t. $x$)

\[ F_l(x, t_0; \lambda) := \left( \begin{array}{c} \partial_x \frac{x^2}{2t} = x/t, \quad U_l^{A_i} + U_{t_1} U_x^{A_i} = U_t^{A_i} + U_{t_1} U_{x_2}^{A_i} + \frac{1}{12} U_{t_1} U_{x_2}^{A_i} = 0. \end{array} \right. \]

The function $U^{A_i}(x, t; \lambda)$ as $t \to -\infty$ behaves similarly in some sense with $U(x, t)$ as $t \to +\infty$.

3 Jost solutions associated with a perturbation $u_{t_0}(x)$ of $U(x, t_0)$.

3.1 Left Jost solution

Lemma 3.1. Let $t_0 \in \mathbb{R}$ and $u_{t_0}(x) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ be a locally integrable function such that

\[ \int_{-\infty}^{-1} |u_{t_0}(x) - U(x, t_0)| \frac{dx}{\sqrt{|x|}} < \infty. \]

Then there exists a unique $2 \times 1$ vector-valued function $F_l(x, t_0; \lambda)$ (which we call the left Jost solution), which is differentiable in $x$, and satisfies the $x$-equation (the subscript $x$ denotes the derivative w.r.t. $x$)

\[ F_{l,x} = \left( \begin{array}{c} \lambda - 2u_{t_0}(x) \end{array} \right) \]

such that

\[ F_l(x, t_0; \lambda) = \left( \begin{array}{c} f_l(x, t_0; \lambda) \end{array} \right) \]

(32)
1. Analyticity: $F_l(x, t_0; \lambda)$ is analytic in $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and continuous up to the boundary. Denote

$$F_l \equiv \begin{cases} F_{lu}, \Im \lambda > 0, \\ F_{ld}, \Im \lambda < 0. \end{cases}$$

2. Symmetry:

$$F_{lu}(x, t_0; \lambda) = F_{ld}(x, t_0; \lambda), \quad \text{i.e.} \quad F_l(x, t_0; \lambda) = F_l(x, t_0; \lambda). \quad (33)$$

3. Large $x \to -\infty$ asymptotics:

$$F_l(x, t_0; \lambda) = E_l(x, t_0; \lambda)(1 + O(\sigma_l(x))), \quad x \to -\infty,$$

where

$$\sigma_l(x) = \int_{-\infty}^{x} \frac{|u_{t_0}(y) - U(y, t_0)|}{\sqrt{|y|}} \, dy.$$

4. Determinant:

$$\det(F_{lu}, F_{ld}) = W\{f_{lu}, f_{ld}\} = -i.$$

5. Additional smoothness: if $u_{t_0}(x) \in C^n(\mathbb{R})$, then $F_l(x, t_0; \lambda) \in C^{n+2}(\mathbb{R})$.

If we strengthen the condition on the rate of convergence of $u$ to $U$, namely if require exponential fast convergence, then we can extend $F_{lu}, F_{ld}$ analytically to some strips:

**Lemma 3.2.** If in addition to conditions of Lemma 3.1

$$\int_{-\infty}^{-1} |u_{t_0}(y) - U(y, t_0)| \cdot |y|^{-1/2} \cdot e^{C|y|^5} \, dy < \infty, \quad C = \frac{48^{5/6}}{80} \cdot l > 0, \quad (34)$$

then

- $F_{lu}$ can be extended analytically to the strip $\Im \lambda > -l$,
- $F_{ld}$ can be extended analytically to the strip $\Im \lambda < l$.

In particular, if (34) is valid for all $C > 0$, then $F_{lu}, F_{ld}$ are entire functions in $\lambda$.

In order to study large $\lambda$ behavior of $F_l$ we need to further strengthen the decaying conditions on $u_{t_0}(x) - U(x, t_0)$:

**Lemma 3.3.** Suppose, in addition to conditions of Lemma 3.1, that for some $A \in \mathbb{R}$

$$u_{t_0}(x) = U(x, t_0), \quad x < A.$$

Then, for any fixed $x \in \mathbb{R}$, uniformly w.r.t. arg $\lambda \in [-\pi, 0] \cup [0, \pi]$,

$$F_l(x, t_0; \lambda) = E_l(x, t_0; \lambda) \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right), \quad \lambda \to \infty,$$

and the latter relation we understand in the sense that $f_l = e_l(1 + O(\frac{1}{\sqrt{\lambda}}))$, $f_{lx} = e_{lx}(1 + O(\frac{1}{\sqrt{\lambda}}))$.

**Proof of Lemmas 3.1, 3.2, 3.3.** It is enough to prove the statements for $F_{lu}$, because of symmetry (33). We will look for solution of the $x$–equation (32) as a solution to the integral equation (IE)

$$f_{lu}(x, t_0; \lambda) = e_{lu}(x, t_0; \lambda) + \int_{-\infty}^{x} i \cdot (e_{lu}(x)e_{ld}(y) - e_{ld}(x)e_{lu}(y)) \cdot 2(u_{t_0}(y) - U(y, t_0)) \cdot f_{lu}(y, t_0; \lambda) \, dy.$$

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For fixed $\lambda$ and big enough negative $x$, the function $e_{lu}(y) = e_{lu}(y, t_0; \lambda)$ does not vanish for $y < x$, so we can divide by it, getting

$$f_{lu}(x, t_0; \lambda) = \frac{1}{e_{lu}(x, t_0; \lambda)} \left( e_{lu}(y)e_{ld}(y) - \frac{e_{ld}(x)}{e_{lu}(x)} e_{lu}(y) \right) \cdot 2(u_{lu}(y) - U(y, t_0)) \cdot f_{lu}(y, t_0; \lambda) \ dy. \tag{35}$$

Lemma 2.7 yields, that for fixed $\lambda$, $\exists \lambda > 0$, the kernel

$$-i \left( e_{lu}(y)e_{ld}(y) - \frac{e_{ld}(x)}{e_{lu}(x)} e_{lu}(y) \right) = 1 + \mathcal{O}(|y|^{-1/6}) - \frac{\mathcal{O}(|y|^{-1/6})}{2\lambda - \lambda_0(y)}$$

and for $\exists \lambda < 0$

$$i \left( e_{lu}(y)e_{ld}(y) - \frac{e_{ld}(x)}{e_{lu}(x)} e_{lu}(y) \right) = 1 + \mathcal{O}(|y|^{-1/6}) - \frac{\mathcal{O}(|y|^{-1/6})}{2\lambda - \lambda_0(y)}$$

Hence, it is bounded by

$$\left| \frac{C}{\sqrt{\lambda - \lambda_0(y)}} \right| \leq C \left| y \right|^{1/6}, \ y \to -\infty \quad \text{for } \exists \lambda > 0,$$

and by

$$\left| C e^{2(g(x)-g(y)/\sqrt{\lambda - \lambda_0(y)}} \right| \leq C \exp \left\{ \frac{(1+\varepsilon)\lambda^{5/2}(y)\left| 2\lambda - \lambda_0(y) \right|}{80} \right\} \left| y \right|^{1/6}, \ y \to -\infty \quad \text{for } \exists \lambda < 0.$$

Solvability of the IE \tag{35} for sufficiently large negative $x$ now follows by the successive approximation method. Once the existence of $f_{lu}$ is established for sufficiently large negative $x$, we can extend it to all real $x$. The statement for the derivative $f_{lu}'$ (which is taken w.r.t. $x$) follows from the integral representation

$$f_{lu}'(x, t_0; \lambda) = 1 + \int_{-\infty}^{x} i \cdot \left( e_{lu}(y)e_{ld}(y) - \frac{e_{ld}(x)}{e_{lu}(x)} e_{lu}(y) \right) \cdot 2(u_{lu}(y) - U(y, t_0)) \cdot f_{lu}(y, t_0; \lambda) \ dy.$$

This proves statements 1.2.3 of Lemma 3.1 and Lemma 3.2. Statement 4 of Lemma 3.1 follows from the fact that the determinant does not depend on $x$, and then we obtain it by taking the limit $x \to -\infty$ and using property 4 of Lemma 2.7 of $E_{lu}, E_{ld}$.

To prove Lemma 3.3 we notice that since $(u - U)(y, t_0) = 0$ for $y < A$, the integral in \tag{35} is taken over a finite interval. When $y$ varies over a finite interval and $\lambda \to \infty$, the functions $e_{lu}(x, t_0; \lambda)$ do not vanish, and have large $\lambda$ asymptotics followed from \tag{4}, \tag{6}. Hence, the kernel of \tag{35} for $\exists \lambda > 0$ admit the estimate

$$-i \cdot \left( e_{lu}(y)e_{ld}(y) - \frac{e_{ld}(x)}{e_{lu}(x)} e_{lu}(y) \right) = 1 + \mathcal{O}(\lambda^{-1/2}) \frac{2\lambda - \lambda_0(y)}{2\lambda} - \frac{2\lambda - \lambda_0(y)}{2\lambda} \ e^{2(\theta(y) - \theta(x))},$$

where $\theta(y) - \theta(x) = (y - x)\sqrt{\lambda}$ is bounded for $y < x$ and $\exists \lambda \geq 0$, and by successive approximation method we obtain Lemma 3.3.

**Remark 3.4.** It is not trivial to extend the result of Lemma 3.3 beyond the case of compactly supported perturbation. This is due to the presence of the term $\frac{1}{\sqrt{\lambda - \lambda_0(y)}}$, in which both $\lambda$ and $\lambda_0(y)$ might be large, but their difference might be small.
3.2 Right Jost solution.

Lemma 3.5. Let $t_0 \in \mathbb{R}$ and $u_{t_0}(x) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ be a locally integrable function such that
\[
\int_{1}^{+\infty} \frac{|u_{t_0}(x) - U(x, t_0)| \, dx}{\sqrt{x}} < \infty.
\]
Then there exists a unique $2 \times 1$ vector-valued function $F_r(x, t_0; \lambda)$ (which we call the right Jost solution), which is differentiable in $x$, and satisfies the $x$-equation
\[
F_{r,x} = \begin{pmatrix} 0 & 1 \\ \lambda - 2u_{t_0}(x) & 0 \end{pmatrix} F_r, \quad F_r(x, t_0; \lambda) =: \begin{pmatrix} f_r(x, t_0; \lambda) \\ f_{r,x}(x, t_0; \lambda) \end{pmatrix}
\]
(36)
such that

1. Analyticity: $F_r(x, t_0; \lambda)$ is analytic in the whole complex plane $\lambda \in \mathbb{C}$.
2. Symmetry:
\[F_r(x, t_0; \lambda) = \overline{F_r(x, t_0; \bar{\lambda})}.
\]
3. Large $x \to +\infty$ asymptotics:
\[F_r(x, t_0; \lambda) = E_r(x, t_0; \lambda)(1 + \mathcal{O}(\sigma_r(x))), \quad x \to +\infty,
\]
where
\[
\sigma_r(x) = \int_{x}^{+\infty} \frac{|u_{t_0}(y) - U(y, t_0)| \, dy}{\sqrt{y}}.
\]
4. Additional smoothness: if $u_{t_0}(x) \in C^n(x \in \mathbb{R})$, then $F_r(x, t_0; \lambda) \in C^{n+2}(x \in \mathbb{R})$.

In order to study large $\lambda$ behavior of $F_1$ we need to strengthen more the decaying conditions on $u_{t_0}(x) - U(x, t_0)$:

Lemma 3.6. Assume in addition to conditions of Lemma 3.5 that for some $B \in \mathbb{R}$
\[u_{t_0}(x) = U(x, t_0), \quad x > B.
\]
Then for any fixed $x \in \mathbb{R}$ and small enough $\varepsilon > 0$ uniformly w.r.t. $\arg \lambda \in [-\pi + \varepsilon, \pi - \varepsilon]$
\[F_r(x, t_0; \lambda) = E_r(x, t_0; \lambda)\left(1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\right), \quad \lambda \to \infty,
\]
and the latter relation we understand in the sense that $f_r = e_r(1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}}))$, $f_{rx} = e_{rx}(1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}}))$.

Proof. Lemmas 3.5, 3.6. The proof is very similar to the case of the left Jost solution, but with slight differences, which we point out. We will look for solution of the $x$-equation (36) as a solution to the integral equation (IE)
\[f_r(x, t_0; \lambda) = e_r(x, t_0; \lambda) + \int_{x}^{+\infty} (e_r(x)e_l(y) - e_l(x)e_r(y)) \cdot 2(U(y, t_0) - u_{t_0}(y)) \cdot f_r(y, t_0; \lambda) \, dy.
\]
Here the kernel $(e_r(x)e_l(y) - e_l(x)e_r(y))$ does not have discontinuity across the real line $\lambda \in \mathbb{R}$, since in view of (24)
\[(e_r(x)e_{lu}(y) - e_{lu}(x)e_r(y)) = (e_r(x)e_{ld}(y) - e_{ld}(x)e_r(y)).
\]
For a fixed $\lambda$ and big enough positive $x$, the function $e_r(y) = e_r(y, t_0; \lambda)$ does not vanish for $y > x$, so we can divide by it.

$$
\frac{f_r(x, t_0; \lambda)}{e_r(x, t_0; \lambda)} = 1 + \int_x^{\infty} \left( \frac{e_l(y)e_r(y) - e_l(x)}{e_r(x) e_r^2(y)} \right) \cdot 2(U(y, t_0) - u_{t_0}(y)) \cdot \frac{f_r(y, t_0; \lambda)}{e_r(y, t_0; \lambda)} \ dy. \tag{37}
$$

Lemma 2.7 yields that for a fixed $\lambda$ the kernel equals

$$
\left( \frac{e_l(y)e_r(y) - e_l(x)}{e_r(x) e_r^2(y)} \right) = \frac{1 + O(|y|^{-1/6})}{2\sqrt{\lambda - \lambda_0(y)}} - \frac{(1 + O(|y|^{-1/6})) (1 + O(|y|^{-1/6})) e^{2(\theta(y) - \theta(y))}}{2\sqrt{\lambda - \lambda_0(y)}}.
$$

Hence, it is bounded by

$$
\left| \frac{C}{\sqrt{\lambda - \lambda_0(y)}} \right| \leq C |y|^{1/6}, \quad y \to +\infty.
$$

Solvability of the IE (37) for sufficiently large positive $x$ now follows by successive approximation method. Once existence of $f_r$ is established for sufficiently large positive $x$, we can extend it for all real $x$. The statement for the derivative $f'_r$ (which is taken w.r.t. $x$) follows from the integral representation

$$
\frac{f'_r(x, t_0; \lambda)}{e'_r(x, t_0; \lambda)} = 1 + \int_x^{\infty} \left( \frac{e_l(y)e_r(y) - e_l(x)}{e_r(x) e_r^2(y)} \right) \cdot 2(U(y, t_0) - u_{t_0}(y)) \cdot \frac{f_r(y, t_0; \lambda)}{e_r(y, t_0; \lambda)} \ dy.
$$

This proves Lemma 3.5.

To prove Lemma 3.6, we notice that since $(u - U)(y, t_0) = 0$ for $y > B$, the integral in (37) is taken over a finite interval. When $y$ varies over a finite interval and $\lambda \to \infty$, $\arg \lambda \in [-\pi + \epsilon, \pi - \epsilon]$, function $e_r(x, t_0; \lambda)$ does not vanish, and has large $\lambda$ asymptotics followed by (4), (6). Hence, the kernel of (37) admits an estimate

$$
\left( \frac{e_l(y)e_r(y) - e_l(x)}{e_r(x) e_r^2(y)} \right) = \frac{1 + O(\lambda^{-1/2})}{2\sqrt{\lambda}} - \frac{(1 + O(\lambda^{-1/2})) e^{2(\theta(x) - \theta(y))}}{2\sqrt{\lambda}},
$$

where $\theta(x) - \theta(y) = (x - y)\sqrt{\lambda}$ is bounded for $y > x$, and by successive approximations method we obtain the statement of Lemma 3.6. \hfill \Box

Remark 3.7. It is not trivial to extend the result of Lemma 3.6 beyond the case of compactly supported perturbation. This is due to the presence of the term $\frac{1}{\sqrt{\lambda - \lambda_0(y)}}$, in which both $\lambda$ and $\lambda_0(y)$ might be large, but their difference might be small.

4 Spectral functions $a(\lambda)$ and $b(\lambda)$.

4.1 Function $a(\lambda)$.

Lemma 4.1. Scattering relation. Let $t_0 \in \mathbb{R}$, and $u_{t_0}(x) \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$
\int_{-\infty}^{+\infty} \frac{|u_{t_0}(x) - U(x, t_0)| \ dx}{1 + \sqrt[4]{|x|}} < \infty. \tag{38}
$$

By Lemmas 2.7, 3.5 there exist Jost solutions $F_l(x, t_0; \lambda)$, $F_r(x, t_0; \lambda)$. Define an analytic in $\lambda \in \mathbb{C} \setminus \mathbb{R}$ function $a(\lambda) = a(\lambda; t_0)$ by the formula

$$
a(\lambda) := \det(F_r, F_l), \quad a_u(\lambda) := \det(F_r, F_{l_0}), \quad a_d(\lambda) := \det(F_r, F_{d}), \tag{39}
$$
\[ a(\lambda) \equiv \begin{cases} a_u(\lambda), & 3\lambda > 0, \\ a_d(\lambda), & 3\lambda < 0. \end{cases} \] (40)

Then, for \( \lambda \in \mathbb{R} \),
\[ F_r(x, t_0; \lambda) = ia_d(\lambda)F_{iu}(x, t_0; \lambda) - ia_u(\lambda)F_{id}(x, t_0; \lambda) = ia(\lambda - i0)F_i(\lambda - i0) - ia(\lambda + i0)F_i(\lambda + i0). \] (41)

**Proof.** For any \( \lambda \in \mathbb{R} \), the functions \( F_r(\lambda), F_i(\lambda + i0) = F_{iu}(\lambda), F_i(\lambda - i0) = F_{id}(\lambda) \) are the solutions of the first equation in (16) \((x\text{-equation})\). Hence, there exist 4 functions \( a(\lambda + i0), a(\lambda - i0), d(\lambda + i0), d(\lambda - i0) \) such that for \( \lambda \in \mathbb{R} \)
\[ F_r(\lambda + i0) = -ia(\lambda + i0)F_i(\lambda - i0) + id(\lambda + i0)F_i(\lambda + i0), \]
\[ F_r(\lambda - i0) = ia(\lambda - i0)F_i(\lambda + i0) - id(\lambda - i0)F_i(\lambda - i0). \]

Due to symmetry properties of Lemmas 3.1, 3.5 and since \( F_r(\lambda + i0) = F_r(\lambda - i0) = F_r(\lambda) \), we conclude that
\[ a(\lambda + i0) = d(\lambda - i0), \quad d(\lambda + i0) = a(\lambda - i0), \quad a(\lambda - i0) = a(\lambda + i0), \quad \lambda \in \mathbb{R}. \]

Formula (39) for \( a \) can now be obtained from (41) using property 1 of Lemma 3.1. Formula (39) extends the domain of definition of \( a(\lambda) \) from \( \lambda \in \mathbb{R} \pm i0 \) to \( \lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (\mathbb{R} + i0) \cup (\mathbb{R} - i0) \).

**Lemma 4.2. Properties of \( a(\lambda) \).** Let \( t_0 \) and \( u_{n}(x) \) be as in Lemma 4.1, i.e. (38) holds. Then \( a(\lambda) = a(\lambda; t_0) \) satisfies the following properties:

1. **Symmetry:** \( \overline{a(\lambda)} = a(\lambda) \).
2. **Nonvanishing:** \( a(\lambda) \neq 0 \) for \( \lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (\mathbb{R} + i0) \cup (\mathbb{R} - i0) \).
3. **If in addition** \((U - U)(x, t_0) = 0 \) for \( x < A \) and \( x > B \),
   \[ a(\lambda) = 1 + \frac{1}{\sqrt{\lambda}} \int_{A}^{B} (U(x, t_0) - u_{n}(x))dx + \frac{1}{\sqrt{\lambda}} \] (42)
   as \( \lambda \to \infty \), uniformly in \( \arg \lambda \in [-\pi + \varepsilon, 0] \cup [0, \pi - \varepsilon] \), for any \( \varepsilon > 0 \).
4. **For compactly supported perturbation \( u_{n}(x) \) of \( U(x, t_0) \), the functions \( a_u(\lambda), a_d(\lambda) \) can be extended analytically to the whole complex plane.**

**Remark 4.3.** Later on, in Lemma 4.4 we will see that the asymptotics (42) are valid not only outside of a cone around the negative real axis, but uniformly in the whole complex plane.

**Proof.** Symmetry 1 follows from the symmetry properties of Lemmas 3.1, 3.5 and the definition of \( a(\lambda) \).

To prove that \( a(\lambda) \neq 0 \) everywhere, we suppose that, on the contrary, there exists \( \lambda^* \) such that \( a(\lambda^*) = 0 \). We have two possibilities: either \( \lambda^* \in \mathbb{R} \), or \( \exists \lambda^* \neq 0 \). If \( \Im(\lambda^*) = 0 \), then by symmetry 1 we have \( a(\lambda^* + i0) = a(\lambda^* - i0) = 0 \) and hence by (41) \( F_r(x, t_0; \lambda^*) = 0 \) for any \( x \), which contradicts the asymptotics of \( F_r(x, t_0; \lambda) \) for \( x \to +\infty \).

Suppose that \( \Im(\lambda^*) \in \mathbb{C} \setminus \mathbb{R} \), then we may assume \( \Im(\lambda^*) > 0 \), without loss of generality. In this case \( F_r(x, t_0; \lambda^*) = cF_i(x, t_0; \lambda^*) \) for some constant \( c \). Thus, \( F_r(x, t_0; \lambda^*) \) vanishes exponentially fast for both \( x \to \pm \infty \), therefore by the usual scheme
\[ \int_{-\infty}^{+\infty} |f_r|^2 + \int_{-\infty}^{+\infty} 2u|f_r|^2 = \lambda \int_{-\infty}^{+\infty} |f_r|^2 \]
and hence \( \lambda \) must be real.

Property 3 follows from the definition (39) of \( a(\lambda) \) and large \( \lambda \) asymptotics of \( F_r, F_i \) from Lemmas 3.3, 3.6. Property 4 follows from the corresponding property of \( F_i \) from Lemma 3.2. □
4.2 Function $b(\lambda)$.

Lemma 4.4. Form of Jost solutions for compactly supported perturbations. Let $t_0$ be real, $u_{t_0}(x)$ be a locally integrable function, and

\[ u_{t_0}(x) - U(x, t_0) = 0 \quad \text{for} \quad x < A \quad \text{and} \quad x > B \]

for some real $A < B$. Let the functions $h_1(x, t_0; \lambda)$, $h_2(x, t_0; \lambda)$ be solutions of (7), i.e.

\[ h_{xx} + 2u_{t_0}(x)h = \lambda h, \quad A < x < B, \]

and let their Wronskian

\[ W(\lambda) \equiv W(\lambda; t_0) = \{h_1, h_2\} \equiv h_1h_{2x} - h_2h_{1x} \]

not be identically 0.

Then the Jost solutions have the form

\[
    f_l(x, t_0, \lambda) = \begin{cases} 
    e_l(x, t_0, \lambda), & x < A, \\
    \frac{1}{W(\lambda)} \{ [e_r, h_2]_A h_1(x, t_0, \lambda) - [e_l, h_1]_A h_2(x, t_0, \lambda) \}, & A < x < B, \\
    b(\lambda; t_0)e_r(x, t_0, \lambda) + a(\lambda; t_0)e_l(x, t_0, \lambda), & x > B, 
    \end{cases} 
\]  

(43)

and

\[
    f_r(x, t_0, \lambda) = \begin{cases} 
    i(a_d - a_u)e_l(x, t_0, \lambda) + a(\lambda; t_0)e_r(x, t_0, \lambda), & x < A, \\
    \frac{1}{W(\lambda)} \{ [e_r, h_2]_B h_1(x, t_0, \lambda) - [e_l, h_1]_B h_2(x, t_0, \lambda) \}, & A < x < B, \\
    e_r(x, t_0, \lambda), & x > B. 
    \end{cases} 
\]  

(44)

Here the function $a(\lambda; t_0) \equiv a(\lambda)$ is defined in (39), and an analytic in $\mathbb{C} \setminus \mathbb{R}$ and continuous up to the boundary $\mathbb{R}$ function $b(\lambda; t_0) \equiv b(\lambda)$,

\[
    b(\lambda) \equiv \begin{cases} 
    b_u(\lambda), 3\lambda > 0, \\
    b_d(\lambda), 3\lambda < 0, 
    \end{cases} 
\]  

(45)

is determined by the representation (43). Furthermore, functions $b_u(\lambda)$, $b_d(\lambda)$, $a_u(\lambda)$, $a_d(\lambda)$ can be extended to entire functions, satisfying

\[
    \frac{b_u}{a_u} - \frac{b_d}{a_d} = \frac{i(a_u a_d - 1)}{a_u a_d}, \quad \text{or equivalently} \quad a_d b_u - a_u b_d = i(a_u a_d - 1), \quad \text{for} \quad \lambda \in \mathbb{C}. \]  

(46)

Proof. Let $u_{t_0}(x) - U(x, t_0) = 0$ for $x > B$ and $x < A$. Let $h_1(x, t_0, \lambda)$, $h_2(x, t_0, \lambda)$ be solutions of

\[ h_{xx} + 2u_{t_0}(x)h = \lambda h, \quad A < x < B \]

being normalized as above. Then $h_{1,2}$ are analytic in $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and

\[
    f_l(x, t_0, \lambda) = \begin{cases} 
    e_l(x, t_0, \lambda), & x < A, \\
    a_1(\lambda; t_0)h_1(x, t_0, \lambda) + a_2(\lambda; t_0)h_2(x, t_0, \lambda), & A < x < B, \\
    b(\lambda; t_0)e_r(x, t_0, \lambda) + \beta_2(\lambda; t_0)e_l(x, t_0, \lambda), & x > B, 
    \end{cases} 
\]

and

\[
    f_r(x, t_0, \lambda) = \begin{cases} 
    \delta_1(\lambda; t_0)e_l(x, t_0, \lambda) + \delta_2(\lambda; t_0)e_r(x, t_0, \lambda), & x < A, \\
    \gamma_1(\lambda; t_0)h_1(x, t_0, \lambda) + \gamma_2(\lambda; t_0)h_2(x, t_0, \lambda), & A < x < B, \\
    e_r(x, t_0, \lambda), & x > B, 
    \end{cases} 
\]

with some coefficients $a_{1,2}$, $\beta_2$, $\gamma_{1,2}$ and $\delta_{1,2}$ which are determined by the condition that the Jost solutions $f_{l,r}$ are continuously differentiable at the points $x = A, B$. 

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It follows from the scattering relation (41)

\[(F_r(\lambda) = i a_d(\lambda) F_{tu}(\lambda) - i a_u(\lambda) F_{td}(\lambda), \quad \det(F_{td}, F_{tu}) = i, \quad a(\lambda) = \det(F_r, F_t),)\]

that

\[\beta_2(\lambda) = \delta_2(\lambda) = a(\lambda), \quad \delta_1(\lambda; t_0) = i(a_d - a_u),\]

and that the function \(b(\lambda)\) (45) satisfies the following conjugation relation on the real axis:

\[a_d b_u - a_u b_d = i(a_u a_d - 1), \quad \frac{b_u}{a_u} - \frac{b_d}{a_d} = i\left(\frac{a_u a_d - 1}{a_u a_d}\right).\]

Since for a compactly supported perturbation \(u(x, t_0)\) the Jost solutions \(f_1, f_r\) are entire, the functions \(a_u, a_d, b_u, b_d\) are also entire functions, and the latter relation is valid for all complex \(\lambda\).

Finally, it is straightforward to express the coefficients \(\alpha_{1,2}, \gamma_{1,2}\) as in (43), (44).

\[\square\]

Remark 4.5. Later in Section 5 we will see that quantity \(b(\lambda)\) plays as fundamental role in the formulation of a Riemann-Hilbert problem as does \(a(\lambda)\). However, \(a(\lambda)\) can be defined by (39) for any perturbation, not necessarily compactly supported (we used compact support of the perturbation only to study large \(\lambda\) asymptotics of \(a(\lambda)\)), while \(b(\lambda)\) is so far defined only for compactly supported perturbations.

Remark 4.6. It follows from (46) that the function

\[E(\lambda) = \frac{b}{a} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1 - a_u(s) a_d(s)}{s - \lambda} ds + \mathcal{O}(\frac{1}{\lambda}),\]

is an entire function in the whole complex plane. It satisfies the symmetry condition

\[\overline{E(\lambda)} = E(\lambda),\]

and, if \(u_0(x) = c\) for \(A < x < B\), then it has the uniform w.r.t. \(\arg \lambda\) asymptotics as \(\lambda \to \infty\)

\[E(\lambda) = e^{2\pi(B - A + \gamma_1)} - e^{2\pi(B - A + \gamma_2)} + \mathcal{O}(\lambda^{-3/2}),\]

The existence of an entire function with such uniform asymptotics at infinity is quite remarkable.

Lemma 4.7. Let \(u_0(x)\) be as in Lemma 4.4. Denote \(r(\lambda) := \frac{b(\lambda)}{a(\lambda)}\),

\[r(\lambda) = \begin{cases} r_u(\lambda), & 3\lambda > 0, \\ r_d(\lambda), & 3\lambda < 0. \end{cases} \]

Then

1. Symmetry: \(\overline{b(\lambda)} = b(\lambda), \overline{r(\lambda)} = r(\lambda)\), i.e. \(\overline{b_d(\lambda)} = b_u(\lambda), \overline{r_d(\lambda)} = r_u(\lambda)\).
2. As \(\lambda \to \infty\), uniformly w.r.t. \(\arg \lambda \in [-\pi, 0] \cup [0, \pi]\),

\[a(\lambda) = 1 + \frac{1}{\sqrt{\lambda}} \int_A^B (U(x, t_0) - u_0(x)) dx + \mathcal{O}(\frac{1}{\sqrt{\lambda}}).\]

3. \(r(\lambda) = \mathcal{O}(\frac{1}{\sqrt{\lambda}}) e^{2\pi(\lambda_0; \lambda)}\).
4. \( a_d(\lambda) - a_u(\lambda) = C(\frac{1}{\lambda})e^{-2\theta(A,t_0;\lambda)} \).

5. Property 3 of the part II of Theorem 1.3 is satisfied, and hence \( r_u(s) \neq i \) and \( r_d(s) \neq -i \) for \( s \in \mathbb{R} \); the roots of the equation \( r_u(\lambda) = i \) in the upper half plane \( \Im \lambda \geq 0 \) can accumulate only along the rays \( \arg \lambda = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \); the roots of the equation \( r_d(\lambda) = -i \) in the lower half plane \( \Im \lambda \leq 0 \) are symmetric to the roots of \( r_u = i \), and can accumulate only along the rays \( \arg \lambda = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \).

6. For \( s \in \mathbb{R} \), \( 3r_u(s) = \frac{\lambda}{\sqrt{\beta}} \) as \( s \to \pm \infty \).

7. Property 7 of the part II of Theorem 1.3 is satisfied.

8. The functions \( a_u, a_d \) can be expressed in terms of \( r_u \) by formulas \([8], [9]\).

**Remark 4.8.** By Picard’s theorem, the set of values of a non-constant entire function is either the whole complex plane, or the complex plane minus a single point. Property 7 hence says that the set of values of \( r_u(\lambda) - \overline{r_u(\lambda)} \) is either a constant, or \( \mathbb{C} \setminus \{i\} \).

If it is a constant, it is \( 0 \), since for \( \lambda \in \mathbb{R} \) we have \( r_u(\lambda) - \overline{r_u(\lambda)} = 2\Im r_u(\lambda) \), which tends to \( 0 \) as \( \lambda \to \pm \infty \) by property 3. Furthermore, if \( r_u(\lambda) = r_d(\lambda) \) for all \( \lambda \in \mathbb{C} \), then by formula (16) \( u \lambda \) we have \( a_u(\lambda) a_d(\lambda) = a_u(\lambda) a_d(\lambda) \equiv 1 \). Hence, the function

\[
 f(\lambda) = \begin{cases} 
 a_u(\lambda), & \Im \lambda > 0, \\
 a_d^{-1}(\lambda), & \Im \lambda < 0 
\end{cases}
\]

satisfies the jump condition \( \frac{f(\lambda)}{f(\lambda)} = a_u a_d = 1 \) for \( \lambda \in \mathbb{R} \), and hence \( a_u(\lambda) \equiv a_d(\lambda) \equiv 1 \). Hence, \( b_u(\lambda) = b_d(\lambda) \).

**Remark 4.9.** Lemma 4.2 ensures that \( a_u(\lambda) \neq 0 \) for \( \Im \lambda \geq 0 \), and \( a_d(\lambda) \neq 0 \) for \( \Im \lambda \leq 0 \). Formulas \([8], [9]\) show that \( a_u(\lambda) \neq 0, a_d(\lambda) \neq 0 \) for all \( \lambda \in \mathbb{C} \).

**Proof of Lemma 4.3** The symmetry property follows from the corresponding symmetry for \( f_l, f_r \). Furthermore, it follows from the representations \([43], [44]\) that

\[
 a(\lambda) = \frac{1}{W(\lambda)} \left[ W\{e_r, h_1\}_B \cdot W\{e_l, h_2\}_A - W\{e_r, h_2\}_B \cdot W\{e_l, h_1\}_A \right],
\]

\[
 b(\lambda) = \frac{1}{W(\lambda)} \left[ -W\{e_l, h_2\}_A \cdot W\{e_r, h_1\}_B + W\{e_l, h_1\}_A \cdot W\{e_r, h_2\}_B \right],
\]

\[
 i(a_d - a_u) = \frac{1}{W(\lambda)} \left[ W\{e_r, h_2\}_B \cdot W\{e_r, h_1\}_A - W\{e_r, h_1\}_B \cdot W\{e_r, h_2\}_A \right].
\]

We can choose \( h_1, h_2 \) to be normalized in such a way that they admit the integral representations written below, and given enough smoothness of \( u_{t_0}(x) \), namely assuming that \( u_{t_0}(x) \) is \( N \) times differentiable (observe that the functions \( R(x,y), L(x,y) \) are one time more regular than function \( u_{t_0}(x) \)), we can develop asymptotic series of \( h_1, h_2 \) for large \( \lambda \) as follows:

1.

\[
 h_1(x, t_0; \lambda) = \frac{1}{\sqrt{2\sqrt{\lambda}}} \left( 1 - \int_{-\infty}^{\infty} R(x, y, t_0)e^{\sqrt{\lambda}(x-y)}dy \right) e^{-\theta(x, t_0; \lambda)}
\]

\[
 = \frac{e^{-\theta(x, t_0; \lambda)}}{\sqrt{2\sqrt{\lambda}}} \left( 1 - \frac{R(x, x, t_0)}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} R_y(x, y, t_0)e^{\sqrt{\lambda}(x-y)}dy \right)
\]

\[
 = \frac{e^{\pi \lambda^{3/2} - \pi \lambda^{1/2}}}{\sqrt{2\sqrt{\lambda}}} \left( e^{-x\sqrt{\lambda}} + \sum_{k=0}^{N} \frac{\partial^k R(x, y)}{\lambda^{k+1}} \cdot e^{-y\sqrt{\lambda}} \right)_{x}^{+\infty} - \frac{1}{\lambda^{3/2}} \int_{-\infty}^{+\infty} \partial_y^N R(x, y) \cdot e^{-y\sqrt{\lambda}}dy,
\]

(52)
Figure 4: Initial function $u_{t_0}(x)$ equals $U(x, t_0)$ outside of the interval $x \in [A, B]$, and equals some function $\tilde{u}_{t_0}(x)$ inside $x \in [A, B]$.

2.  
\[
h_{1x}(x, t_0; \lambda) = -\frac{\sqrt{\lambda}}{\sqrt{2}} \left( 1 - \frac{R(x, x)}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_{x}^{+\infty} R_x(x, y, t_0) e^{\sqrt{\lambda}(x-y)} dy \right) e^{-\theta(x, t_0; \lambda)} \\
= -\frac{\sqrt{\lambda} e^{\lambda^{3/2}/2 \lambda^{7/2}}}{\sqrt{2}} \left( e^{-x\sqrt{\lambda}} - \frac{R(x, x) e^{-x\sqrt{\lambda}}}{\sqrt{\lambda}} - \sum_{k=1}^{N} \frac{\partial_x \partial_{y}^k R(x, y) \cdot e^{-y\sqrt{\lambda}}}{\lambda^{k+1/2}} \right)_{x}^{+\infty} \\
+ \frac{1}{\lambda^{1/2}} \int_{x}^{+\infty} \partial_x \partial_{y}^N R(x, y) \cdot e^{-y\sqrt{\lambda}} dy, \tag{53}
\]

3.  
\[
h_{2x}(x, t_0; \lambda) = \frac{1}{\sqrt{2} \sqrt{\lambda}} \left( 1 + \int_{-\infty}^{x} L(x, y, t_0) e^{\sqrt{\lambda}(y-x)} dy \right) e^{\theta(x, t_0; \lambda)} \\
= \frac{e^{\theta(x, t_0; \lambda)}}{\sqrt{2} \sqrt{\lambda}} \left( 1 + \frac{L(x, x, t_0)}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{x} L_y(x, y, t_0) e^{\sqrt{\lambda}(y-x)} dy \right) = \frac{e^{-\frac{1}{2} \lambda^{3/2}/2 \lambda^{7/2}}}{\sqrt{2} \sqrt{\lambda}} \\
\cdot \left( e^{x\sqrt{\lambda}} + \sum_{k=0}^{N} \frac{(-1)^k \partial_y^k L(x, y) \cdot e^{y\sqrt{\lambda}}}{\lambda^{k+1}} \right)_{x}^{+\infty} + \frac{(-1)^{N+1}}{\lambda^{N+1}} \int_{-\infty}^{x} \partial_y^{N+1} L(x, y) \cdot e^{y\sqrt{\lambda}} dy, \tag{54}
\]
Then the formula of integration by parts takes the form

\[ \int_a^b f(x)\varphi'(x)\,dx = \sum_j \alpha_j \varphi(x) \bigg|_{x_j}^{x_{j+1}} + f_{ac}(x)\varphi(x) \bigg|_a^b - \int_a^b f_{ac}(x)\varphi(x)\,dx = - \int_{-\infty}^{+\infty} \bar{\varphi}(x)\,d\left(\chi_{[a,b]}(x)\tilde{f}(x)\right), \]

where \( \tilde{f} \) is an extension of \( f \) from \([a,b]\) to \( \mathbb{R} \), which itself is a function of bounded variation, and \( \bar{\varphi} \) is an extension of \( \varphi \) from \([a,b]\) to \( \mathbb{R} \), which itself is a differentiable function. Furthermore,
by \(d\left(\chi_{a,b}(x)\tilde{f}(x)\right)\) we denoted the signed measure corresponding to the function of bounded variation \(\chi_{a,b}(x)\tilde{f}(x)\).

With this in mind, formulas (52)-(55) make sense also for a function \(u_{t_0}(x)\) which is \(N-1\) time differentiable, with \(u^{(N-1)}(x,t_0)\) locally of bounded variation. In this case we need to interchange the last two terms of order \(\lambda^{\frac{N-1}{2}}\) in (52)-(55) with the terms

\[
\frac{1}{\lambda^{\frac{N}{2}}} \int_{-\infty}^{+\infty} e^{-y\sqrt{x}} d_y \left(\chi_{[x,\infty)}(x)\partial_y^N R(x,y)\right) - \lambda^{\frac{N}{2}} \int_{-\infty}^{+\infty} e^{-y\sqrt{x}} d_y \left(\chi_{[x,\infty)}(x)\partial_y^N R(x,y)\right),
\]

respectively, which are also of order \(O(\lambda^{-\frac{N+1}{2}})\).

Hence, uniformly w.r.t. \(\arg \lambda \in [-\pi, \pi]\), as \(\lambda \to \infty\),

\[
W(\lambda) \equiv W(t_0, \lambda) := W\{h_1, h_2\} \equiv \{h_1, h_2\} := h_1 h_{2x} - h_{1x} h_2 = 1 + O\left(\frac{1}{\sqrt{\lambda}}\right),
\]

provided that \(R, L\) are differentiable. Let us notice, that \(R_x, R_y, L_x, L_y\) have the same regularity w.r.t. \(x, y\) as \(u_{t_0}(x)\).

It was shown in Lemma 4.2 that asymptotics (42) are valid outside of a cone around the negative real axis. Hence, it is enough to study the behavior of \(a(\lambda)\) in the sector around \(\mathbb{R}_{-}\). To this end we rewrite the expression (49) for \(\arg \lambda \in \left[\frac{3\pi}{4} + \epsilon, \pi\right]\) using (24):

\[
a_u(\lambda) = \frac{1}{W(\lambda)} \left[ \frac{i\{e_{iu}, h_2\}_A}{e^{i\theta(\lambda)}(\frac{1}{\sqrt{\lambda}})} - 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right] \left[ \frac{i\{e_{iu}, h_1\}_A}{e^{i\theta(\lambda)}(\frac{1}{\sqrt{\lambda}})} - 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right]^{\frac{1}{\sqrt{\lambda}}}
\]

and hence, since for \(C \in \{A, B\}\),

\[
\{e_{iu}, h_2\}_C = e^{2\theta(C)\frac{i}{\sqrt{\lambda}}} - b(C) - \frac{b_1(C)}{\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda}}\right),
\]

and for \(\arg \lambda \in \left[\frac{\pi}{4} + \epsilon, \pi\right]\)

\[
\{e_{id}, h_2\}_C = e^{-2\theta(B)\frac{i}{\sqrt{\lambda}}} + \frac{L(C, C) - b_1(C)}{\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda}}\right),
\]

and the function

\[
e^{2\theta(A)\frac{i}{\sqrt{\lambda}}} - 2\theta(A)\frac{i}{\sqrt{\lambda}} = e^{2(A-B)\frac{i}{\sqrt{\lambda}}}
\]

is bounded, we obtain that

\[
a_u(\lambda)W(\lambda) = 1 + \frac{b_1(A) - b_1(B) + L(B, B) - R(A, A)}{\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda}}\right)
\]

for \(\arg \lambda \in \left[\frac{5\pi}{4} + \epsilon, \pi\right]\), and

\[
W(\lambda) = 1 + \frac{L(x, x) - R(x, x)}{\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{for} \quad \arg \lambda \in [-\pi, \pi],
\]

and hence, recalling property (19) \((\partial_x b_1 = -U)\) and equations (56), we obtain

\[
a(\lambda) = 1 + \frac{1}{\sqrt{\lambda}} \int_A^B (U(x, t_0) - u_{t_0}(x)) dx + O\left(\frac{1}{\sqrt{\lambda}}\right),
\]

and so on.
which proves the second statement of the lemma. Furthermore, from \ref{50}, the above asymptotics, and the second statement of the lemma we obtain the third statement of the lemma.

To obtain the fourth statement, for $3\lambda \in [-\pi + \varepsilon , 0] \cup [0, \pi - \varepsilon ]$, we use formula \ref{51} together with the asymptotics

$$\{e_r , h_2 \} = 1 + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \{e_r , h_1 \} = O\left(\frac{1}{\sqrt{\lambda}}\right) e^{-2\theta(C,t_0,\lambda)}, \quad 3\lambda \in [-\pi + \varepsilon , 0] \cup [0, \pi - \varepsilon],$$

which give

$$(a_d - a_u) = O\left(\frac{1}{\sqrt{\lambda}}\right) e^{-2\theta(A,t_0,\lambda)} + O\left(\frac{1}{\sqrt{\lambda}}\right) e^{-2\theta(B,t_0,\lambda)} = O\left(\frac{1}{\sqrt{\lambda}}\right) e^{-2\theta(A,t_0,\lambda)}.$$ 

For $\arg \lambda \in \left[\frac{5\pi}{7} + \varepsilon , \pi\right]$ we substitute $e_r = i\varepsilon u - i\varepsilon d$ in formula \ref{51}, which gives

$$i(a_d - a_u) = O\left(\frac{1}{\sqrt{\lambda}}\right) e^{2\theta(A)} O\left(\frac{1}{\sqrt{\lambda}}\right) e^{2\theta(B)} O\left(\frac{1}{\sqrt{\lambda}}\right) +$$

$$+ e^{2(A-B)\sqrt{\lambda}} O\left(\frac{1}{\lambda}\right) + e^{2(B-A)\sqrt{\lambda}} O\left(\frac{1}{\lambda}\right);$$

and

$$i(a_d - a_u) e^{2\theta(A,t_0,\lambda)} = e^{2\theta(A)} O\left(\frac{1}{\sqrt{\lambda}}\right) e^{2\theta(B)} O\left(\frac{1}{\sqrt{\lambda}}\right) +$$

$$+ e^{2(A-B)\sqrt{\lambda}} O\left(\frac{1}{\lambda}\right) + e^{2(B-A)\sqrt{\lambda}} O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{\sqrt{\lambda}}\right)$$

for $\arg \lambda \in \left[\frac{5\pi}{7} + \varepsilon , \pi\right]$. The fifth statement that $r_u(\lambda) \neq i$ for $\lambda \in \mathbb{R}$ follows from relation \ref{46}

$$r_u(\lambda) - r_d(\lambda) = i(1 - \frac{1}{a_u a_d}).$$

Indeed, by Lemma \ref{4.2}

$$a_u(\lambda) \neq 0 \quad \text{for} \quad 3\lambda \geq 0 \quad \text{and} \quad a_d(\lambda) \neq 0 \quad \text{for} \quad 3\lambda \leq 0.$$ 

Hence, assuming that $r_u(\lambda^*) = i$ for some real $\lambda^*$, and hence, by symmetry $\overline{r_d(\lambda)} = r_u(\lambda)$ also $r_d(\lambda^*) = -i$, one obtains

$$2 = 1 - \frac{1}{a_u(\lambda^*) a_d(\lambda^*)},$$

which cannot be true since $a_u(\lambda^*) a_d(\lambda^*) = |a_u(\lambda^*)|^2 \geq 0$. The remaining part of property 5 follows from the asymptotics of $r(\lambda)$ described in property 3.

Properties 6, 7, 8 follow from \ref{46}, applied to a real $\lambda = s \in \mathbb{R}$. Indeed, we have

$$r_u(s) - r_d(s) = i\left(1 - \frac{1}{a_u(s) a_d(s)}\right) \implies r_u(s) - r_d(s) = i\left(1 - \frac{1}{a_u(s) a_u(s)}\right) \implies 2\Im r_u(s) = 1 - \frac{1}{|a_u(s)|^2}.$$ 

Hence, together with the asymptotics of $a(\lambda)$ from property 2, this gives us properties 6, 7, 8. To obtain property 9, we consider the scalar conjugation problem for the function

$$f(\lambda) = \begin{cases} a_u(\lambda), & 3\lambda > 0, \\ \frac{1}{\overline{a_u(\lambda)}}, & 3\lambda < 0, \end{cases}, \quad \frac{f^+}{f^-} = \overline{|a_u(s)|^2} = \frac{1}{1 - 2\Im r_u(s)},$$

and then use the Sokhotsky-Plemelj formula. This finishes the proof of Lemma \ref{4.7}.

We think of $r(\lambda) \equiv \frac{b(\lambda)}{a(\lambda)}$ as of a reflection coefficient, and hence we would expect that the rate of vanishing of $r(\lambda)$ as $\lambda \to \infty$ is related to the smoothness of the initial function $u_{t_0}(x)$. The following lemma shows that this is indeed the case.
Lemma 4.10. Refined decay of $r(\lambda)$. Let $t_0 \in \mathbb{R}$ and $u_{t_0}(x)$ be equal to $U(x,t_0)$ outside of an interval $x \in [A, B]$. Then if

1. $u_{t_0}(x) - U(x,t_0) \in \text{BV}_{\text{loc}}$ is a function of bounded variation, then

   \[ a(\lambda) = 1 + \frac{1}{\sqrt{\lambda}} \int_A^B (U(x,t_0) - u_{t_0}(x)) \, dx + O\left(\frac{1}{\lambda}\right), \quad a_d - a_u = O\left(\frac{1}{\lambda}e^{-20(A)}, \quad \lambda \to \infty, \right. \]

   \[ r(\lambda) = O\left(\frac{1}{\lambda}\right)e^{2\theta(B,t_0;\lambda)}, \quad \lambda \to \infty, \quad 3r_u(s) = O\left(\frac{1}{s}\right), \quad s \to \pm \infty. \]

2. $u_{t_0}(x)$ is $N$ times differentiable ($N = 0, 1, 2, \ldots$), and $u^{(N)}(x,t_0) - U^{(N)}(x,t_0)$ is a function of bounded variation, then

   \[ r(\lambda) = O\left(\lambda^{-\frac{N}{2} - 1}\right)e^{2\theta(B,t_0;\lambda)}, \quad \lambda \to \infty. \]

Proof of Lemma 4.10. For the parts of function $u_{t_0}(x)$ outside of the interval $x \in [A, B]$, where it equals $U(x,t_0)$, we have Jost solutions $E$, $E_l$, properties of which are described in Lemma 2.7, Section 2.1. In order to use the machinery of Jost solutions also in the interval $x \in [A, B]$, we take a compactly supported function $u_{t_0}(x)$, which on the interval $x \in [A, B]$ coincides with $u_{t_0}(x)$ (see Figure 3).

Function $b(\lambda)$ has representation (50). Now we need to develop an asymptotic series for all the ingredients in the above formula. For the functions $h_1$, $h_{1x}$, $h_2$, $h_2x$ we use formulas (52)-(55), and for $c_1$, $c_{tx}$ formula (14).

For $u_{t_0}(x) \in \text{BV}_{\text{loc}}$ we can take $N = 1$ in formulas (52)-(55) with remainder terms (58), (59), to obtain

\[ \{c_1, h_1\}_C = -1 - b_1(C) - C(R(C, C)) + O\left(\frac{1}{\lambda}\right), \]

with $C = A$ or $C = B$. Let us develop asymptotics for the term $\{c_1, h_2\}_C$. For the sake of clarity we precede the general case $N$ by the cases $N = 0, 1, 2$. We have

\[
\{c_1, h_2\}_C \cdot 2e^{-2C\sqrt{\lambda} + \frac{3L}{2}\lambda^{3/2} - \frac{2L}{\lambda^{3/2}} - \frac{3b_1}{2\lambda^{3/2}}} = \left(1 + \frac{L}{\sqrt{\lambda}} + \frac{Lx}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right)\right) \left(1 + \frac{b_1}{\lambda} + \frac{a_1}{\lambda^2} + \frac{b_2}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right)\right) \]

\[
- \left(1 + \frac{L}{\sqrt{\lambda}} + \frac{L_y}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right)\right) \left(1 + \frac{b_1}{\lambda} + \frac{b_2}{\lambda^{3/2}} + \frac{a_1}{\lambda^2} + \frac{b_1a_1 + c_1}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right)\right) \bigg|_{y=x=C}
\]

\[
= \frac{a_1 - b_1^2 - d_1 + L_x + L_y}{\lambda} + \frac{-L_{xy} - L_{yy} + b_1(L_x + L_y) + (a_1 - b_1^2)L + b_2 - b_2b_1 - c_1}{\lambda^{3/2}} \bigg|_{y=x=C} + O(\lambda^{-2}),
\]

where we assumed that $u_{t_0}(x)$ is twice differentiable and $u^{(2)}(x,t_0) \in \text{BV}_{\text{loc}}$. If the function $u_{t_0}(x)$ is only 1 time differentiable with $u'(x,t_0) \in \text{BV}_{\text{loc}}$, then, in formula (60), we would have to reduce the expansion by the last element, and if $u_{t_0}(x)$ is just locally a function of bounded variation, we reduce it by the last two elements.

Hence, for $u_{t_0}(x) \in \text{BV}_{\text{loc}}$ we already have that

\[ \{c_1, h_2\}_C \cdot 2e^{-2C\sqrt{\lambda} + \frac{3L}{2}\lambda^{3/2} - \frac{2L}{\lambda^{3/2}} - \frac{3b_1}{2\lambda^{3/2}}} = O\left(\frac{1}{\lambda}\right), \]

and hence

\[ b(\lambda) = O\left(\frac{1}{\lambda}\right)e^{2B\sqrt{\lambda} - \frac{2L}{\lambda^{3/2}} + \frac{3L}{\lambda^{3/2}}} + \frac{3b_1}{2\lambda^{3/2}}. \]

For $u_{t_0}(x)$ differentiable with $u'(x,t_0) \in \text{BV}_{\text{loc}}$ we need to check that

\[ (a_1 - b_1^2 - d_1 + L_x + L_y) \bigg|_{y=x=C} = 0, \]

\[ (a_1 - b_1^2 - d_1 + L_x + L_y) \bigg|_{y=x=C} = 0, \]

(61)
and for \( u_{t_0}(x) \) \( 2 \) times differentiable with \( u''(x, t_0) \in BV_{loc} \) we need to check that

\[
(-L_{xy} - L_{yy} + b_1(L_x + L_y) + (a_1 - b_2 \lambda^2) L + b_1 a_1 - c_1) \bigg|_{y=x=C} = 0.
\]

(62)

From \eqref{56}, setting \( L(x, y) =: H(x, y) \), and differentiating, and then coming back from \( H \) to \( L \), we obtain

\[
L(x, x) = - \int_{-\infty}^{x} \tilde{u}(s, t_0) ds, \quad \text{for } u \in L_{loc},
\]

\[
L_x + L_y \bigg|_{y=x} = - \tilde{u}_{t_0}(x), \quad \text{for } u \in C,
\]

\[
L_{xy} + L_{yy} \bigg|_{y=x} = - \frac{1}{2} \tilde{u}'(x, t_0) - \tilde{u}_{t_0}(x) \int_{-\infty}^{x} \tilde{u}(s, t_0) ds, \quad \text{for } u \in C^1.
\]

On the other hand, from \eqref{19} and \( d_1 = -a_1 \) we obtain that

\[
2a_1 - b_1^2 = U, \quad b_2 - b_1 a_1 - c_1 = -\frac{U_x}{2} + U b_1,
\]

and since \( \tilde{u}(C, t_0) = U(C, t_0) \) for \( C = A, B \) and continuous \( u \), and \( \tilde{u}'(C, t_0) = U'(C, t_0) \) for \( C = A, B \) and continuously differentiable \( u \), we see that \eqref{61} and \eqref{62} holds, and hence we obtain the statements of the lemma.

The general case can be proven as follows: denote

\[
f(x) = h_1(-x, t_0; \lambda) \cdot \sqrt{2} \cdot \sqrt{\lambda} \cdot e^{-\frac{\lambda}{4} x^{2}} + \frac{1}{\sqrt{\pi}} x^{3/2}, \quad g(x) = e_t(-x, t_0; \lambda) \cdot \sqrt{2} \cdot \sqrt{\lambda} \cdot e^{-\frac{\lambda}{4} x^{2}} + \frac{1}{\sqrt{\pi}} x^{3/2}.
\]

If \( u_{t_0}(x) \) is \( N \) times differentiable with \( u^{(N)}(x, t_0) \in BV_{loc} \), then

\[
f(x) = e^{\sqrt{\lambda} x} \left( \sum_{j=0}^{N} \frac{\alpha_j}{\lambda^{j/2}} + O(\lambda^{-N/2-1}) \right), \quad f'(x) = \sqrt{\lambda} e^{\sqrt{\lambda} x} \left( \sum_{j=0}^{N} \frac{\alpha_j + \alpha_j-1,x}{\lambda^{j/2}} + O(\lambda^{-N/2-1}) \right),
\]

\[
f''(x) = \lambda e^{\sqrt{\lambda} x} \left( \sum_{j=0}^{N} \frac{\alpha_j + 2\alpha_j-1,x + \alpha_j-2,x,x}{\lambda^{j/2}} + O(\lambda^{-N/2-1}) \right),
\]

and similar formulas, with \( \alpha_j(x) \) substituted by \( \beta_j(x) \), hold for \( g(x) \). We put here \( \alpha_0 = \beta_0 = 1, \alpha_{-j} = \beta_{-j} = 0, j \geq 1 \). Substituting the above expansions into

\[
f_{xx} + 2\tilde{u}(x)f = \lambda f, \quad g_{xx} + 2U(x)g = \lambda g,
\]

one obtains

\[
\alpha_{j,x} = -u\alpha_{j-1} - \frac{1}{2} \alpha_{j-1,x}, \quad \beta_{j,x} = -u\beta_{j-1} - \frac{1}{2} \beta_{j-1,x}, \quad j = 1, \ldots, N.
\]

(63)

Now, the term of order \( e^{2x \sqrt{\lambda}} x^{-k/2} \) in the Wronskian

\[
\sqrt{\lambda} (f(x)g'(x) - f'(x)g(x))
\]

is equal to

\[
\sum_{j=0}^{k} \alpha_{k-j} (\beta_j + \beta_{j-1,x}) - \sum_{j=0}^{k} \beta_{k-j} (\alpha_j + \alpha_{j-1,x}),
\]

and substituting subsequently expressions \( \eqref{63} \) instead of \( \alpha_{j,x}, \beta_{j,x} \), we find that the above term is equal to 0, since the function \( \tilde{u}(x) - U(x) \) and its first \( N - 1 \) derivatives vanish at the point \( x = A \).
4.3 Example: compactly supported perturbation with a constant in the middle.

For the initial function

\[ u(x, t_0) = \begin{cases} 
  c, & A < x < B, \\
  U(x, t_0), & x < A, x > B, 
\end{cases} \]

where \( c \in \mathbb{R} \) is a constant, the functions \( h_1, h_2 \) have the explicit form

\[ h_1(x) = a \tilde{h}_1(x) + \beta \tilde{h}_2(x), \quad h_2(x) = \gamma \tilde{h}_1(x) + \delta \tilde{h}_2(x), \]

where

\[
\tilde{h}_1 = \frac{1}{\sqrt{2} \sqrt{\lambda - 2c}} e^{-x \sqrt{\lambda - 2c} + \frac{1}{2} x \lambda^{3/2} - \frac{1}{6} \lambda^{7/2}}, \quad \tilde{h}_2 = \frac{1}{\sqrt{2} \sqrt{\lambda - 2c}} e^{x \sqrt{\lambda - 2c} - \frac{1}{2} x \lambda^{3/2} + \frac{1}{6} \lambda^{7/2}},
\]

and

\[
\begin{align*}
\alpha &= \frac{1}{2} \left( \sqrt{\frac{\lambda - 2c}{\lambda}} + \frac{\lambda}{\lambda - 2c} \right) e^{B(\sqrt{\lambda - 2c} - \sqrt{\lambda})}, \\
\beta &= \frac{1}{2} \left( \sqrt{\frac{\lambda - 2c}{\lambda}} - \frac{\lambda}{\lambda - 2c} \right) e^{-B(\sqrt{\lambda - 2c} + \sqrt{\lambda})},
\end{align*}
\]

\[
\begin{align*}
\gamma &= \frac{1}{2} \left( \frac{\lambda - 2c}{\lambda} - \frac{\lambda}{\lambda - 2c} \right) e^{A(\sqrt{\lambda - 2c} + \sqrt{\lambda})}, \\
\delta &= \frac{1}{2} \left( \frac{\lambda - 2c}{\lambda} + \frac{\lambda}{\lambda - 2c} \right) e^{-A(\sqrt{\lambda - 2c} - \sqrt{\lambda})}.
\end{align*}
\]

Furthermore, in the formulas for \( a(\lambda), b(\lambda) \) in Section 4.2 we can replace \( h_1, h_2 \) everywhere with \( \tilde{h}_1, \tilde{h}_2 \), and we can write the large \( \lambda \) asymptotics of \( a, b \) in a more explicit way. Namely, \( (a_1, b_1) \) below are defined in asymptotic expansion \[\{12\}],

\[ a(\lambda) = 1 + \frac{(c(A - B) + b_1(A) - b_1(B))}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}), \]

\[
\arg \lambda \in (\varepsilon, 0) \cup (-\varepsilon, 0),
\]

\[ a(\lambda) = 1 + \frac{(c(A - B) + b_1(A) - b_1(B))}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) +
\]

\[
\begin{align*}
&+ i \left\{ e^{2\theta(A)} \left( \frac{[c - 2a_1(A) + b_1^2(A)]}{2\lambda} + O(\lambda^{-3/2}) \right) - e^{2\theta(B)} \left( \frac{[c - 2a_1(B) + b_1^2(B)]}{2\lambda} + O(\lambda^{-3/2}) \right) \right\},
\end{align*}
\]

exponentially small term, which becomes oscillatory of order \( O(\lambda^{-1}) \) at \( \arg \lambda = \pi = 0 \)

\[ \arg \lambda \in [\pi, \varepsilon, \pi - 0], \]

\[ a(\lambda) = 1 + \frac{(c(A - B) + b_1(A) - b_1(B))}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) -
\]

\[
\begin{align*}
&- i \left\{ e^{2\theta(A)} \left( \frac{[c - 2a_1(A) + b_1^2(A)]}{2\lambda} + O(\lambda^{-3/2}) \right) - e^{2\theta(B)} \left( \frac{[c - 2a_1(B) + b_1^2(B)]}{2\lambda} + O(\lambda^{-3/2}) \right) \right\},
\end{align*}
\]

exponentially small term, which becomes oscillatory of order \( O(\lambda^{-1}) \) at \( \arg \lambda = -\pi + 0 \)

\[ \arg \lambda \in [-\pi + 0, -\pi + \varepsilon], \]

\[ a_u(\lambda) = 1 - \frac{(A - B)c + b_1(A) - b_1(B)}{\sqrt{\lambda}} + \frac{((A - B)c + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) +
\]

\[
\begin{align*}
&- i \left( -e^{2\theta(B)}(c - 2a_1(B) + b_1^2(B) + O(\lambda^{-3/2})) - e^{-2\theta(A)}(c - 2a_1(A) + b_1^2(A) + O(\lambda^{-3/2})) \right),
\end{align*}
\]

leading term

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\[ a_u(\lambda) = \begin{cases} 
\frac{c(A - B) + b_1(A) - b_1(B)}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}), & \text{arg } \lambda \in [-\pi + 0, -\frac{5\pi}{7} - \varepsilon] \ni 0, \pi - \varepsilon], \\
\frac{c(A - B) + b_1(A) - b_1(B)}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) + \frac{-i}{2\lambda} \left[ e^{-2\theta(B)} \left( c - 2a_1(B) + b_1^2(B) + O(\lambda^{-1/2}) \right) - e^{-2\theta(A)} \left( c - 2a_1(A) + b_1^2(A) + O(\lambda^{-1/2}) \right) \right], & \text{arg } \lambda \in [-\pi + \varepsilon, 0], \\
\frac{c(A - B) + b_1(A) - b_1(B)}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) + \frac{-i}{2\lambda} \left[ e^{-2\theta(B)} \left( c - 2a_1(B) + b_1^2(B) + O(\lambda^{-1/2}) \right) - e^{-2\theta(A)} \left( c - 2a_1(A) + b_1^2(A) + O(\lambda^{-1/2}) \right) \right], & \text{arg } \lambda \in [\frac{5\pi}{7} + \varepsilon, \pi], \\
\frac{c(A - B) + b_1(A) - b_1(B)}{\sqrt{\lambda}} + \frac{(c(A - B) + b_1(A) - b_1(B))^2}{2\lambda} + O(\lambda^{-3/2}) + \frac{-i}{2\lambda} \left[ e^{-2\theta(B)} \left( c - 2a_1(B) + b_1^2(B) - e^{-2\theta(A)} \left( c - 2a_1(A) + b_1^2(A) \right) \right), & \text{arg } \lambda \in [-\pi, -\frac{5\pi}{7} - \varepsilon]. 
\end{cases}\]

Asymptotics for \( a_u(\lambda) \) in all the sectors of the complex plane \( \lambda \) follows from the asymptotics of \( a_u(\lambda) \) by the formula \( a_u(\lambda) = a_u(\overline{\lambda}) \). Furthermore,

\[ b(\lambda) = \left( -\frac{c}{2\lambda} + \frac{2a_1(A) - b_1^2(A)}{2\lambda} + O(\lambda^{-3/2}) \right) e^{2\theta(A)} + \left( \frac{c - 2a_1(B) + b_1^2(B)}{2\lambda} + O(\lambda^{-3/2}) \right) e^{2\theta(B)} \]

\[ = e^{2\theta(B)} \left( \frac{c - 2a_1(B) + b_1^2(B)}{2\lambda} + O(\lambda^{-3/2}) - e^{2\theta(A)} \sqrt{\frac{c - 2a_1(A) + b_1^2(A)}{2\lambda} + O(\lambda^{-3/2})} \right) \]

exponentially small term which becomes oscillatory for \( \text{arg } \lambda = \pm \pi \)

\[ \text{arg } \lambda \in [-\pi + 0, -0] \cup [0, \pi - 0], \]

\[ b_u(\lambda) = \frac{e^{-2\theta(B)} \left( c - 2a_1(B) + b_1^2(B) + O(\lambda^{-1/2}) \right) - e^{-2\theta(A)} \left( c - 2a_1(A) + b_1^2(A) + O(\lambda^{-1/2}) \right)}{2\lambda} \]

\[ \text{arg } \lambda \in [-\pi + 0, -\frac{5\pi}{7} - \varepsilon], \]

\[ b_d(\lambda) = \frac{e^{-2\theta(B)} \left( c - 2a_1(B) + b_1^2(B) + O(\lambda^{-1/2}) \right) - e^{-2\theta(A)} \left( c - 2a_1(A) + b_1^2(A) + O(\lambda^{-1/2}) \right)}{2\lambda} \]

\[ \text{arg } \lambda \in [\frac{5\pi}{7} + \varepsilon, \pi - 0]. \]
Remark 4.11. The functions $a_u(\lambda), a_d(\lambda)$ are entire, i.e. they do not have any jump on the real axis. This is not obvious since their representation involves $h_{1,2}$, which do have jumps across some part of the real axis.

Remark 4.12. Consider the KdV equation \([1]\) (with reverse time, i.e. instead of $t \geq t_0$ we take $t \leq t_0 < 0$)
\[
u_t(x,t) + u(x,t)\nu_x(x,t) + u_{xxx}(x,t) = 0, \quad t \leq t_0 < 0
\]
with the initial datum of the form
\[
\nu_{t_0}(x) = \begin{cases} 
\frac{u_0}{t_0}, & x < A, \ x > B, \\
\frac{c}{t_0}, & c, A < x < B,
\end{cases}
\]
where $A < B, c, t_0 < 0$ are real constants. Based on the unperturbed Jost solutions $e^{h_{1,2}}$ associated with the function $\frac{x}{t}$ and constructed in subsection 2.2, one can construct the Jost solutions $f^{A_i}_{\lambda, r}$ associated with the function $\nu_{t_0}(x)$, and then the corresponding spectral functions $a^{A_i}(\lambda)$, $b^{A_i}(\lambda)$, $r^{A_i}(\lambda)$. Then the function $a^{A_i}(\lambda)$ does not vanish nowhere, but if we take the constant $c$ big enough, then $a^{A_i}(\lambda)$ takes some values which are very close to 0 (“quasi-spectrum”).

5 Construction of Riemann-Hilbert problems

In order to construct a solution $\nu(x,t)$ to the KdV equation, which at the time $t = t_0$ is equal to the given initial function $\nu(x,t_0)$, our strategy is to construct a solution to a Riemann-Hilbert problem out of the Jost solutions $F_l(x,t_0; \lambda)$, $F_r(x,t_0; \lambda)$, in such a way that this RH problem makes sense also for $t \neq t_0$. There are several ways to do this. In this section the initial function $\nu_{t_0}(x)$ is a compactly supported perturbation of $U(x,t_0)$, i.e. $\nu(x,t_0)$ satisfies the conditions of Lemma 4.4. The functions $a(\lambda), b(\lambda), r(\lambda)$ are the spectral functions associated with the initial function $\nu_{t_0}(x)$.

5.1 RH problem appropriate for $t > t_0, x \in \mathbb{R}$ and $t = t_0, x > B.$

Let us notice that relation \([46]\)
\[
\frac{b_u}{a_u} - \frac{b_d}{a_d} = i\frac{(a_u a_d - 1)}{a_u a_d}
\]

(together with scattering relation \([41]\))
\[
\frac{F_{lu}}{a_u} - \frac{F_{ld}}{a_d} = -\frac{i}{a_u a_d} F_r
\]

imply
\[
\frac{b_u}{a_u} F_r - \frac{b_d}{a_d} F_r = i\frac{(a_u a_d - 1)}{a_u a_d} F_r.
\]

Hence, substracting the two latter formulas and multiplying them by $i$, we get
\[
\frac{i}{a_u} (F_{lu} - b_u F_r) - \frac{i}{a_d} (F_{ld} - b_d F_r) = F_r.
\]

Now we are ready to define the piece-wise analytic in $\lambda$ matrix-valued function
\[
P(x,t_0; \lambda) := \begin{cases} 
\left( \frac{F_{lu} - b_u F_r e^{-\theta}}{a_u}, \left( F_r - \frac{i(F_{lu} - b_u F_r)}{a_u} e^{\theta} \right), \Omega_{II}, \frac{1}{a_u} F_{lu} e^{-\theta}, F_r e^{\theta} \right), \Omega_I, \\
\left( \frac{F_{ld} - b_d F_r e^{-\theta}}{a_d}, \left( F_r + \frac{i(F_{ld} - b_d F_r)}{a_d} e^{\theta} \right), \Omega_{II}, \frac{1}{a_d} F_{ld} e^{-\theta}, F_r e^{\theta} \right), \Omega_{IV},
\end{cases}
\]

where we denoted
\[
\Omega_{II} = \left\{ \lambda : \ \arg \lambda \in \left(\frac{6\pi}{T}, \pi\right) \right\}, \quad \Omega_I = \left\{ \lambda : \ \arg \lambda \in \left(0, \frac{6\pi}{T}\right) \right\}.
\]
\[ \Omega_{IIV} = \left\{ \lambda : \arg \lambda \in (-\pi, -\frac{6\pi}{7}) \right\}, \quad \Omega_{IV} = \left\{ \lambda : \arg \lambda \in (-\frac{6\pi}{7}, 0) \right\}, \]

This matrix-valued function \( P(x, t_0, \lambda) \) satisfies the following RH problem at the time \( t = t_0 \) for \( x > B \):

**Riemann-Hilbert problem 5. (appropriate for \( t > t_0, x \in \mathbb{R} \) or \( t = t_0 \) and \( x > B \).)** To find a 2 x 2 matrix-valued function \( P(x, t; \lambda) \), which

- is analytic in \( \lambda \in \mathbb{C} \setminus \Sigma \), where \( \Sigma \) is as in (3),
- has the following jump for \( \lambda \rightarrow \infty \), which is uniform w.r.t. \( \arg \lambda \in [-\pi, \pi] \):

\[
J_P = \begin{pmatrix}
\frac{b_u e^{-2\theta}}{a_u} & 1 \\
1 + \frac{ib_u}{a_u} & \exp \text{ decay due to } b_u
\end{pmatrix}, \quad \gamma_0, \quad J_P = \begin{pmatrix}
0 & -i \\
-1 & 0
\end{pmatrix}, \quad \rho,
\]

where we denoted

\[
\gamma_0 = (0, +\infty), \quad \gamma_3 = (e^{6\pi i/7} \infty, 0), \quad \gamma_{-3} = (e^{-6\pi i/7} \infty, 0), \quad \rho = (-\infty, 0),
\]

- has the following asymptotics as \( \lambda \rightarrow \infty \), which is uniform w.r.t. \( \arg \lambda \in [-\pi, \pi] \):

\[
P = \frac{1}{\sqrt{2}} \lambda^{-\sigma_{3/4}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \left( I + b_3 \frac{1}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1}) \right),
\]

where \( b = b(x, t) \) is some scalar (which is not fixed, but is introduced in order to fix the form of the asymptotics).

\[
P_+ = P_- J_P :
\]

\[
\begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{F_{1u} - b_u F_r}{a_u} & F_r e^{\theta} - \frac{ib_u}{a_u} - iF_{1u} e^{\theta} \\
F_{1d} - b_d F_r & F_r \frac{1 - ib_d}{a_d} + iF_{1d} e^{\theta}
\end{pmatrix}
\]

For this RH problem to be meaningful the jumps must vanish as \( \lambda \rightarrow \infty \). Notice that, by Lemma 4.7,

\[
r_u e^{-2\theta(x,t;\lambda)} = O\left( \frac{1}{\sqrt{\lambda}} \right), \quad \text{exp decay due to } b_u
\]

\[ \text{exp decay due to } b_u \]
and hence this is indeed the case for \( t > t_0, x \in \mathbb{R} \) and for \( t = t_0, x > B \).

Together with RHP 5 we can consider another one, for the function \( \hat{P}(x,t;\lambda) \), with the same analyticity and jump condition, but with different asymptotics as \( \lambda \to \infty \):

**Riemann-Hilbert problem 6.** Find a function \( \hat{P}(x,t;\lambda) \), with analyticity and jump as in RH 5 and with the asymptotic condition altered:

- asymptotics as \( \lambda \to \infty \)

\[
\hat{P}(x,t;\lambda) = \left( I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

### 5.2 RH problem appropriate for \( t < t_0, x \in \mathbb{R} \) and \( t = t_0, x < A \)

Another way to construct a RH problem is as follows. Define a piece-wise analytic matrix-valued function

\[
N = \begin{cases}
\left( \frac{F_1 x e^{-\theta}}{a_u}, (F_r - ia_d F_1 x) e^\theta \right), \Omega_{III} , & \left( \frac{1}{a_u} F_1 x e^{-\theta}, F_r e^\theta \right), \Omega_I , \\
\left( \frac{F_1 x e^{-\theta}}{a_d}, (F_r + ia_u F_1 x) e^\theta \right), \Omega_{IV} , & \left( \frac{1}{a_d} F_1 x e^{-\theta}, F_r e^\theta \right), \Omega_{IV} .
\end{cases}
\]

This matrix-valued function \( N(\lambda) \) satisfies the following RH problem at the time \( t = t_0 \) for \( x < A \):

**Riemann-Hilbert problem 7.** (appropriate for \( t < t_0, x \in \mathbb{R} \) or \( t = t_0 \) and \( x < A \)) To find a \( 2 \times 2 \) matrix-valued function \( N(x,t;\lambda) \), which

- is analytic in \( \lambda \in \mathbb{C} \setminus \Sigma \), where \( \Sigma \) is as in (3),
- has the following jump \( N_+ = N_- J_N \) across \( \Sigma \):

\[
J_N = \begin{pmatrix}
1 & 0 \\
-\frac{i e^{-2\theta}}{a_u a_d} & 1
\end{pmatrix}, \gamma_0 , \\
J_N = \begin{pmatrix}
0 & -\frac{i a_u a_d}{e^{-2\theta}} \\
\frac{1}{a_u a_d} & 0
\end{pmatrix}, \rho
\]

- has the following asymptics as \( \lambda \to \infty \), which is uniform w.r.t. \( \arg \lambda \in [-\pi, \pi] \):

\[
N = \frac{1}{\sqrt{2}} \lambda^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

A condition for this RH problem 5 to be meaningful is that the jumps vanish as \( \lambda \to \infty \). Notice that by Lemma 4.7

\[
(a_d - a_u) e^{2\theta(x,t;\lambda)} = \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) e^{\frac{2i(b(x) - t)}{\sqrt{\lambda}}} \lambda^{3/2} + (x-A) \lambda^{3/2},
\]

and hence this RH problem 5 is good for \( t < t_0 \) and for \( t = t_0, x < A \).

Together with the RHP 4 we can consider another one, for the function \( \hat{N}(x,t;\lambda) \), with the same analyticity and jump condition, but with different asymptotics as \( \lambda \to \infty \):

**Riemann-Hilbert problem 8.** To find a function \( \hat{N}(x,t;\lambda) \), with analyticity and jump conditions as in RH 7 and with the asymptotic condition altered:

- asymptotics as \( \lambda \to \infty \)

\[
\hat{N}(x,t;\lambda) = \left( I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]
5.3 **RH problem appropriate for all** $t \in \mathbb{R}$, $x \in \mathbb{R}$.

A third way to construct a RH problem is to define a piece-wise meromorphic in $\lambda$ matrix-valued function

$$
\mathcal{F}(x,t_0;\lambda) = \begin{cases}
\begin{pmatrix}
\frac{F_{II}}{a_u}, & F_r - \frac{i}{a_u+ib_u}F_{II} \\
\frac{F_{III}}{a_u}, & F_r + \frac{i}{a_u-ib_u}F_{III}
\end{pmatrix}, & \lambda \in \rho,

\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \lambda \in \gamma_0,
\end{cases}
\begin{cases}
\begin{pmatrix}
\frac{F_{II}}{a_u}, & F_r - \frac{i}{a_u+ib_u}F_{II} \\
\frac{F_{III}}{a_u}, & F_r + \frac{i}{a_u-ib_u}F_{III}
\end{pmatrix}, & \lambda \in \gamma_3,

\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \lambda \in \gamma_{-3},
\end{cases}
\end{cases}
$$

By Lemma 4.7 there is at most a finite number of zeros of $a_u + ib_u$ in the region $\Omega_{II}$, moreover, those zeros do not lie on the real axis. In case if some zeros fall on the border $\gamma_3$ between $\Omega_{II}$ and $\Omega_I$, we will locally deform a bit the line $\gamma_3$, so that $\gamma_3$ would be free of zeros of $a_u + ib_u$. Symmetrically, we will move $\gamma_{-3}$. We keep the same notations for the deformed rays $\gamma_3, \gamma_{-3}$.

Function $\mathcal{F}(x,t_0;\lambda)$ solves the following RHP [9] at the time $t = t_0$:

**Riemann-Hilbert problem 9.** *(Appropriate for all real $t$ and $x$.)* To find a $2 \times 2$ matrix-valued function $\mathcal{F}(x,t;\lambda)$, which

1. is analytic in $\lambda \in \mathbb{C} \setminus \Sigma$, where $\Sigma$ is as in [3],

2. has the following jump $\mathcal{F}_+ = \mathcal{F}_-$ across $\Sigma$ :

3. has the following pole conditions at the roots of $r_u = i, r_d = -i$:

   for $\lambda^* \in \Pi$, $\exists \lambda^* > 0$ such that $r_u(\lambda^*) = i$,

   $$
   \mathcal{F}[2](\lambda) + \frac{i}{1 + ir_u(\lambda)} \mathcal{F}[1] = O(1) \quad \text{and} \quad \mathcal{F}[1] = O(1) \quad \text{for} \quad \lambda \to \lambda^*,
   $$

   $$
   \mathcal{F}[2](\lambda) - \frac{i}{1 - ir_d(\lambda)} \mathcal{F}[1] = O(1) \quad \text{and} \quad \mathcal{F}[1] = O(1) \quad \text{for} \quad \lambda \to \overline{\lambda^*},
   $$

4. has the following asymptotics as $\lambda \to \infty$, which is uniform w.r.t. $\arg \lambda \in [-\pi, 0] \cup [0, \pi]$:

   $$
   \mathcal{F}(x,t;\lambda) = \frac{1}{\sqrt{2}} \lambda^{-\sigma_3/4} \left( \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \right) \left( I + \overline{b}(x,t) \frac{1}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \right) e^{\theta(x,t,\lambda)\sigma_3},
   $$

   where $\overline{b} = \overline{b}(x,t)$ is some scalar, which is not fixed, but introduced in order to fix the form of the asymptotics.
Together with RH problem \[9\] for the function \(F(x, t; \lambda)\) we will also consider another one for the function \(\tilde{F}(x, t; \lambda)\), which has the same analyticity, pole, jump conditions as \(F(x, t; \lambda)\), but asymptotic condition as \(\lambda \to \infty\) is replaced with

\[4a.\] Asymptotics as \(\lambda \to \infty\):

\[
\tilde{F}(\lambda) = (I + \mathcal{O}(1)) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{i \theta}.
\]

\[M = \tilde{F} e^{-\theta} \sigma_3, \quad M_+ = M_-J_M:\]

\[
\begin{align*}
&\begin{pmatrix}
\frac{\mathcal{O}}{7} & \mathcal{O} \\
\mathcal{O} & \mathcal{O}
\end{pmatrix} \begin{pmatrix}
1 & i v^{20} \\
0 & 1
\end{pmatrix}
\end{align*}
\]

Let us mention that there are 2 ways to rewrite meromorphic RHP \[9\] into a regular one.

**Regular RH problem.** The first one, is to redefine \(F\) in small neighborhoods of the points \(\lambda^* \in \Omega_{II}\) with \(r_u(\lambda^*) = i\), and points \(\overline{\lambda^*} \in \Omega_{III}\) with \(r_d(\lambda^*) = -i\):

\[
F_{\text{reg}}(x, t; \lambda) = \begin{pmatrix} F_{[1]}(\lambda), & F_{[2]}(\lambda) + \frac{i}{1 + i r_u(\lambda)} F_{[1]}(\lambda) \end{pmatrix}, |\lambda - \lambda^*| < \varepsilon, \lambda^* \in \Omega_{II}, r_u(\lambda^*) = i,
\]

\[
F_{\text{reg}}(x, t; \lambda) = \begin{pmatrix} F_{[1]}(\lambda), & F_{[2]}(\lambda) - \frac{i}{1 + i r_u(\lambda)} F_{[1]}(\lambda) \end{pmatrix}, |\lambda - \overline{\lambda^*}| < \varepsilon, \lambda^* \in \Omega_{II}, r_u(\lambda^*) = i,
\]

\[
F_{\text{reg}}(x, t; \lambda) = F(x, t; \lambda)\] elsewhere.

The function \(F_{\text{reg}}(x, t; \lambda)\) is regular at \(\lambda^*, \overline{\lambda^*}\), and solves RH problem \[9\] with pole conditions replaced by additional jumps across circles \(C_j, \overline{C_j}\) around the points \(\lambda^*, \overline{\lambda^*}\), oriented counter-clock-wise:

\[3a.\] \(F_{\text{reg}+} = F_{\text{reg}-} \begin{pmatrix} 1 & i v^{20} \\
0 & 1 + i r_u(\lambda) \end{pmatrix} C_j, \quad F_{\text{reg}+} = F_{\text{reg}-} \begin{pmatrix} 1 & i v^{20} \\
0 & 1 - i r_d(\lambda) \end{pmatrix} \overline{C_j}.
\]

RHP for \(\tilde{F}_{\text{reg}}\) is the same as for \(F_{\text{reg}}\), but with the asymptotic condition replaced with

\[4a.\] Asymptotics as \(\lambda \to \infty\):

\[
\tilde{F}_{\text{reg}}(\lambda) = (I + \mathcal{O}(1)) \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{i \theta}.
\]

**Shifted RH problem.** Another way is to move the intersection point of the contour \(\Sigma\) from \(\lambda = 0\) to some point \(\lambda = \lambda_0 << 0\). Indeed, as follows from Lemma 4.7 roots of \(r_u(\lambda) = i\) can accumulate only along the rays \(\arg \lambda = \frac{5\pi}{7}, \frac{3\pi}{7}, \frac{\pi}{7}\), and hence, if we move the intersection point of the contour \(\Sigma\) from 0 to some \(\lambda_0 << 0\), and denote such a contour by \(\lambda_0 + \Sigma\), corresponding domains by \(\lambda_0 + \Sigma_{I,II,III,IV}\), and rays by \(\lambda_0 + \gamma_{0,3,-3}, \lambda + \rho\), then for large enough negative \(\lambda_0\) the region \(\lambda_0 + \Omega_{II}\) will not contain any roots of \(r_u = i\).

We call \(F_{\lambda_0}\) the function, obtained from \(F\) by such a shift. It solves RHP \[9\] with contour \(\Sigma\) changed to \(\lambda_0 + \Sigma\), and asymptotics
Remark 5.1. The function \( \theta_{\lambda_0}(x, t; \lambda) \) is connected with \( \theta(x, t; \lambda) \) in the following way:

\[
\theta_{\lambda_0}(x, t; \lambda) = \theta(x, t; \lambda) + \frac{h_1}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-3/2}),
\]

where

\[
g_1 = -\frac{\lambda_0^3}{384} + \frac{t\lambda_0^2}{8} - \frac{x\lambda_0}{2}.
\]

Furthermore, quantities \( \tilde{b} \) and \( \hat{b} \) in the asymptotics for \( \mathcal{F} \) and \( \mathcal{F}_{\lambda_0} \) are related as

\[
\tilde{b} = \hat{b} - g_1.
\]

The RHP for \( \mathcal{F}_{\lambda_0}(x, t; \lambda) \) is the same as for \( \mathcal{F}_{\lambda_0} \), but with the asymptotic condition 4. replaced with

\[4a. \ \text{Asymptotics as } \lambda \to \infty :\]

\[
\mathcal{F}_{\lambda_0}(\lambda) = (I + \mathcal{O}(1)) \left( \frac{\lambda - \lambda_0}{\sqrt{2}} \right)^{-\gamma_3/4} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) e^{\theta_{\lambda_0} \sigma_3}.
\]

The functions \( \mathcal{F}_{\lambda_0} \) and \( \mathcal{F} \) are related in the following way: for those \( \lambda \) not lying between \( \gamma_{\pm 3} \) and \( \gamma_{\pm 3} + \lambda_0 \),

\[
\mathcal{F}_{\lambda_0}(x, t; \lambda) = \left( \begin{array}{cc} 1 & 0 \\ g_1 & 1 \end{array} \right) \mathcal{F}(x, t; \lambda).
\]

6 Existence of solution to the RH problems

RHPs constructed in Section 5 make sense not only for spectral functions \( a(\lambda), b(\lambda), r(\lambda) \) associated with a compactly supported perturbation \( u_{a_0}(x) \) of \( U(x, t_0) \), but also for a wider range of functions \( a(\lambda), b(\lambda), r(\lambda) \). We list below the properties of functions \( r(\lambda), b(\lambda), a(\lambda) \), which are sufficient to consider the RH problems from Section 5 and in what follows we do not associate \( r(\lambda), b(\lambda), a(\lambda) \) with initial function \( u_{a_0}(x) \), but only assume that they are three arbitrary functions, that satisfy properties 6.1 (the decay property for \( r(\lambda) \), listed in Properties 6.1, corresponds to the initial function \( u_{a_0}(x) \), which is locally a function of bounded variation, and not just locally integrable).

Properties 6.1. of spectral functions \( a(\lambda), r(\lambda), b(\lambda) = a(\lambda)r(\lambda) \).

Define the set \( \mathcal{M} \) of functions \( r_u : \mathbb{R} \to \mathbb{C} \) that satisfy the following properties:

1. \( r_u \) can be extended to an entire function,
2. \( \Im r_u(s) = \mathcal{O}(s^{-1}) \) for \( s \in \mathbb{R}, \ s \to \pm \infty \),
3. \( \Re r_u(s) < \frac{1}{2} \) for \( s \in \mathbb{R}, \)
4. \( r_u(\lambda) - \overline{r_u(\lambda)} \neq i \) for all \( \lambda \in \mathbb{C} \).
Define \( r_d(\lambda) := r_u(\overline{\lambda}) \), and \( a_u(\lambda), a_d(\lambda) \) by formulas (8), (9), set
\[
b_u(\lambda) = a_u(\lambda)r_u(\lambda), \quad b_d(\lambda) = a_d(\lambda)r_d(\lambda),
\]
and define functions \( a(\lambda), b(\lambda), r(\lambda) \) by formulas (40), (45), (48). Then obviously, by the Sokhotsky-Plemelji formula, \( a_u(\lambda), a_d(\lambda), b_u(\lambda), b_d(\lambda) \) are entire functions satisfying the symmetry condition
\[
a_u(\lambda) = a_d(\overline{\lambda}), \quad b_u(\lambda) = b_d(\overline{\lambda}), \quad r_u(\lambda) = r_d(\overline{\lambda}),
\]
and multiplying (8) and (9) we see that relation (46) holds.

Furthermore, the defining properties of the set \( M \) are: there exist real \( t_0, A < B \) such that
\[
5. \quad r_u(\lambda) = O(\frac{1}{\lambda})e^{2\Theta(B,t_0,\lambda)} \quad \text{uniformly w.r.t. } \arg \lambda \in [0, \pi],
\]
\[
6. \quad a_d - a_u = O(\frac{1}{\lambda})e^{-2\Theta(A,t_0;\lambda)} \quad \text{uniformly w.r.t. } \arg \lambda \in [-\pi, 0] \cup [0, \pi],
\]

**Remark 6.2.** Lemmas 4.2, 4.7, 4.10 show, that the spectral function \( \gamma \) with asymptotic condition \( \lambda \to \infty \) does not converge exponentially fast to the identity as \( \lambda \to \infty \). Indeed, for quite general RHPs, the scheme how to prove their solvability was introduced by [Zhou’89]. For some particular cases, the scheme was realised with all the necessary details by [DKMVZ’99] Theorems 5.3, 5.6, p.1387–1406, steps 1, 2, 3, for RHP [2] in the particular case \( r \equiv 0, a \equiv 1 \) by [Claeys Vanlessen’07], and in other situation by [Its Kuijlaars Ostensson’08] section 2.3, p.18]. The main distinction of our case is that the jump matrix \( J_L \) (defined in Preparatory Step 3 below) does not converge exponentially fast to the identity as \( \lambda \to \infty \). Indeed, \( J_L - I \) is exponentially small as \( \mu \to \infty \) along \( \gamma_0, \gamma_3, \gamma_3 \), but not across \( \rho \).

The general scheme from [Zhou’89] consists of 3 steps: after reformulating the RHP as a singular integral equation (SIE), Step 1 is to prove that the corresponding singular integral operator is Fredholm. Step 2 is to show that the index of that operator is 0. Step 3 is to show that the kernel of the corresponding operator is 0. This proves the invertibility of the operator. Before applying this scheme, we need to make 3 more preparatory steps: Preparatory Step 1 is to get rid of poles, Preparatory Step 2 is to make identity asymptotics at infinity, and Preparatory Step 3 is to make jump equals \( I \) at all the junctions of the contour.

**Preparatory Step 1: Get rid of poles.** To this end we consider the shifted RHP for the \( \tilde{F}_\lambda \) with large enough negative \( \lambda_0 \). Denote
\[
\mu = \lambda - \lambda_0.
\]

RHP [2] for \( \tilde{F} \) and the one for \( \tilde{F}_\lambda \) are equivalent to each other. Indeed, for \( \lambda \) not lying between \( \gamma_{\pm 3} \) and \( \gamma_{\pm 3} + \lambda_0 \), they are related by formula (66).

**Preparatory Step 2: To make identity asymptotics at infinity.** To this end we use a slight modification of the function \( Z(\cdot) \) defined in (27), in which we change the rays from \( \arg \ z = \pm \frac{2\pi}{3} \) to \( \arg \mu = \pm \frac{3\pi}{2} \), but keep the same notation \( Z(\cdot) \).

Now define
\[
Y(\mu) = Z(\mu)e^{-\frac{z^2}{z^2}}s, \quad \text{hence} \quad Y(\mu) = \left(1 + O(\mu^{-3}) \quad O(\mu^{-2}) \quad O(\mu^{-1}) \quad 1 + O(\mu^{-2}) \quad 1 + O(\mu^{-3})\right)\mu^{-3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \mu \to \infty,
\]
and define
\[
M(x,t;\mu) = \tilde{F}_\lambda(x,t;\mu)e^{-\theta_\lambda(x,t;\mu)}Y^{-1}(\mu).
\]
The function \( M(\mu) = M(x,t;\mu) \) solves the following RHP:

**Riemann-Hilbert problem 10.** To find a \( 2 \times 2 \) matrix-valued function \( M(x,t;\mu) \), which
\begin{enumerate}
\item is analytic in \( \mu \in \mathbb{C} \setminus \Sigma \), where \( \Sigma = \mathbb{R} \cup \gamma_3 \cup \gamma_{-3} \);
\end{enumerate}
2. has the following jump $M_+(\mu) = M_-(\mu) J_M(\mu)$ across $\Sigma$: $J_M = \begin{cases} Y_-(1 + r_u(1 - r_u)) \frac{r_u e^{-2\Phi_0}}{1 - r_u} Y_+^{-1}, & \mu \in \rho, \\ Y_-(1 + r_u(1 - r_u)) \frac{1}{a_u a_d} Y_+^{-1}, & \mu \in \gamma_0, \end{cases}$

Indeed, pick up 4 points due to the identity product $L$ has the following asymptotics as $\mu \to \infty$, which is uniform w.r.t. arg $\mu \in [-\pi, 0] \cup [0, \pi]:$

$$M(x,t; \mu) = I + \rho(1).$$

Preparatory Step 3: make the jump at the junction $\mu = 0$ equals $I$. This can be done due to the identity product

$$J_M|_{\gamma_3}(0) \cdot J_M|_{\rho}(0) \cdot J_M|_{\gamma_0}(0) \cdot J_M^{-1}|_{\gamma_0}(0) = I.$$

Indeed, pick up 4 points $\mu_j \notin \Omega_j$, $j = I, II, III, IV$, and define

$$L(x,t; \mu) = M(x,t; \mu) G, \quad G = G_j, \mu \in \Omega_j, \quad G_j = I + \frac{-\mu_j}{\mu - \mu_j} (B_j(x,t) - I),$$

where the matrices $B_j(x,t)$ are to be determined. The jumps for the function $L$ are: $L_+ = L - J_L$, where $J_L = G^{-1} J_M G_+$, and

$$J_L|_{\gamma_3} = G^{-1}_{III} J_M|_{\gamma_3} G_{II}, \quad J_L|_{\gamma_3} = G^{-1}_{IV} J_M|_{\gamma_3} G_{III}, \quad J_L|_{\gamma_0} = G^{-1}_{IV} J_M|_{\gamma_0} G_{II}, \quad J_L|_{\rho} = G^{-1}_{III} J_M|_{\rho} G_{II}.$$

We can take

$$B_I = I, \quad B_{II} = J_M|_{\gamma_3}(0), \quad B_{III} = J_M|_{\rho}(0) J_M|_{\gamma_3}(0), \quad B_{IV} = J_M|_{\gamma_3}(0) \cdot J_M|_{\rho}(0) \cdot J_M|_{\gamma_0}(0),$$

and the jump matrix $L_I$ equals $I$ at the origin $\mu = 0$ on every ray $\gamma_{\pm 3}, \gamma_0, \rho$.

Reformulation of RHP for $L$ as a SIE

The RH problem for $L$ is equivalent to the following singular integral equation (SIE)

$$L_+ = I + C_-(L_-(J_L - I)), \quad [I - C_-(J_L - I)] \circ (L_+ - I) = C_-((J_L - I)).$$

where $C_\pm$ are the Cauchy operators acting in $L_p(\Sigma), p > 1$, which for a H"older continuous function $f$ act as

$$C_\pm f(\lambda) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s) ds}{(s - \lambda)^\pm} = \pm \frac{f(\lambda)}{2} + \frac{1}{2\pi i} p.v. \int_{\Sigma} \frac{f(s) ds}{s - \lambda}.$$

Since $J_L - I \in L_\infty(\Sigma)$, then $C_{JI} := C_-((J_L - I))$ is an operator acting in $L_p(\Sigma)$. Define also the (Hilbert) operator

$$Hf(\lambda) := \frac{1}{2\pi i} p.v. \int_{\Sigma} \frac{f(s) ds}{s - \lambda}.$$

By Sokhotsky–Plemelj formula,

$$C_+ = \frac{1}{2} Id + H, \quad C_- = -\frac{1}{2} Id + H.$$

For the fact that $C_\pm$ are operators acting in $L_2(\Sigma)$ or $L_p(\Sigma)$ for contours with self-intersections we refer to [B"ottcher Karlovich'97, section 4.4, p.137, Theorem 4.15], see also [Lenells'18 section 2.2].
If \( L_- \in I + L_p(\Sigma) \) (i.e. \( L_- - I \in L_p(\Sigma) \)) is the solution of \([68]\), then the function

\[
L = I + C(L_-(J_L - I)), \quad \text{where } (Cf)(\mu) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - \mu} \, ds,
\]

is the solution of RHP \([10]\). Indeed, using the Sokhotsky-Plemelj formula \( C_+ - C_- = I \), one obtains

\[
L_+ = I + C_+(L_-(J_L - I)) = I + L_-(J_L - I) + C_-(L_-(J_L - I)) = L_-(J_L - I) + L_- = L_- J_L.
\]

This solution \( L \) is to be understood in the \( L_p \) sense (see [DKMVZ'99, p. 1388] for more details). However, using local analyticity of the jump matrix \( J_L \), one can obtain that \( L \) is a solution of the RHP also in the sense of continuous boundary values.

In order to prove that equation \([68]\) has a solution \( L_- \in I + L_p(\Sigma) \), i.e. \( L_- - I \in L_p(\Sigma) \), following the approach from Zhou [89], [DKMVZ'99, pp.1387–1395], it suffices to show that the operator

\[
C_{J_L} = C_- (J_L - I)
\]

is invertible in \( L_p(\Sigma) \). This consists of 3 steps: step 1 is to show that \( I - C_{J_L} \) is a Fredholm operator, step 2 is to show that the index of \( I - C_{J_L} \) is 0, and step 3 is to show that the kernel of \( I - C_{J_L} \) is \( \{0\} \).

**Step 1.** To show that \( I - C_{J_L} \) is a Fredholm operator, one can show that it has a pseudo-inverse, i.e. there exists an operator \( O \) such that \( O(I - C_{J_L}) - I \) and \((I - C_{J_L})O - I \) are compact operators (see, for instance, [Pedersen'89, Prop 3.3.11, p.109–110]). Let us take

\[
O = I - C_{J_L}^{-1} := I - C_- (J_L^{-1} - I).
\]

Then similarly as in [DKMVZ'99] Step 1, p.1389] one can show that

\[
(I - C_{J_L}) (I - C_{J_L}^{-1}) f = f + C_- [C_+ (f \tilde{w}) w], \quad \text{where } w = J_L - I, \quad \tilde{w} = J_L^{-1} - I
\]

and

\[
(I - C_{J_L}^{-1}) (I - C_{J_L}) f = f + C_- [C_+ (fw) \tilde{w}].
\]

Since both the operators \((I - C_{J_L}^{-1}) (I - C_{J_L})\), \((I - C_{J_L}) (I - C_{J_L}^{-1})\) are of the same form, it suffices to prove compactness for one of it. To prove that the operator

\[
K : f \mapsto C_- [C_+ (f \tilde{w}) w]
\]

is compact, we follow the approach of [DKMVZ'99, p. 1400-1401], i.e. approximate the continuous function \( w \) (it is continuous at the origin because of our Preparatory Step 3) by rational functions \( w_\varepsilon \),

\[
\|w_L - w_\varepsilon\|_{L^\infty} < \varepsilon
\]

for any positive \( \varepsilon \). Then

\[
K_\varepsilon \to K \quad \text{as } \varepsilon \to 0,
\]

where \( K_\varepsilon \) is defined by almost the same formula as \( K \), but with \( w \) replaced with \( w_\varepsilon \). Hence, it is enough to show that \( K_\varepsilon \) is compact for every \( \varepsilon \). If

\[
w_\varepsilon(\mu) = \sum_{\nu} \frac{\alpha_\nu}{\mu - \mu_\nu},
\]

then in the same way as in [DKMVZ'99, p. 1401] one shows that for a weakly convergent to 0 sequence \( f_n \in L_2(\Sigma_L) \), the sequence

\[
(K_\varepsilon f_n)(\mu) = \sum_{\nu} \alpha_\nu \frac{C,f_n \tilde{w}_L)(\mu_\nu)}{\mu_\nu - \mu}
\]

strongly converges to 0.
Step 2. The proof that the index of $I - \mathcal{C}_{J_M}$ is 0 is the same as in [DKMVZ'99, p.1390], and is based on the fact that $I - s\mathcal{C}_{J_M}$ is Fredholm for all scalar $s$, which can be proved in the same way as in Step 1, and then using continuity of index and the fact that the identity operator has index 0.

Step 3. To prove that $\ker(I - \mathcal{C}_{J_M}) = 0$, we take any element from the kernel, i.e. $L_{0,-} \in L_2(\Sigma)$ such that

$$L_{0,-} - \mathcal{C}_-L_{0,-}(J_L - I) = 0,$$

and we will show that $L_{0,-} = 0$. Indeed, define the function $L_0(\mu) = C[L_0,-(J_L - I)](\mu) = \frac{1}{2\pi i} \int_{\Sigma_L} L_{0,-}(s)(J_L(s) - I) \frac{ds}{s - \mu}$.

It satisfies the following RH problem:

1. $L_0(\mu)$ is analytic in $\mu \in \mathbb{C} \setminus \Sigma_L$;
2. $L_{0,+} = L_{0,-}(\mu)J_L(\mu), \mu \in \Sigma$;
3. $L_0(\mu) = \mathcal{O}(\mu^{-1}), \mu \to \infty$.

The function $L_0(\mu)$ satisfies the above RH problem both in $L^2$ sense and, using local analyticity of the jump matrices, as in [DKMVZ'99, p. 1402, Proposition 5.7], in continuous sense.

So, $L_0$ solves the RH problem with the same jumps as $L$, but with zero asymptotics at infinity. We need to show that $L_0 \equiv 0$.

Making all the transformations which led us from $\hat{F}_{\lambda_0}(x,t;\mu)$ to $L$ in the reverse order, starting from $L_0$, we come to the following RH problem for a function $\hat{F}_{\lambda_0,0}(x,t;\mu)$:

Riemann-Hilbert problem 11. 1, 2. Analyticity, jumps are as in RHP for $\hat{F}_{\lambda_0}(x,t;\mu)$.

3b. Asymptotics at infinity:

$$\hat{F}_{\lambda_0,0}(\mu) = \mathcal{O}(\frac{1}{\mu})\mu^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\lambda_0 \sigma_3}. $$

We thus need to show that $\hat{F}_{\lambda_0,0}(\mu) \equiv 0$.

To prove the latter, define the matrix

$$A(\mu) = \hat{F}_{\lambda_0,0} e^{-\lambda_0 \sigma_3} = \begin{pmatrix} I, & \arg(\mu) \in (-\frac{6\pi}{7}, 0), \\ 1, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{4}} \\ -e^{-i\frac{\pi}{4}} \end{pmatrix}, & \arg(\mu) \in (-\pi, -\frac{6\pi}{7}), \\ 1, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} \end{pmatrix}, & \arg(\mu) \in (\frac{6\pi}{7}, \pi), \\ 0 & \arg(\mu) \in (0, \frac{6\pi}{7}, 0) \end{pmatrix}.$$

Furthermore, the function $A(\mu)$ has asymptotics $A(\mu) = \mathcal{O}(\mu^{-3/4})$ as $\mu \to \infty$, and jumps only on the real axis, reading

$$A_+(\mu) = A_-(\mu)J_A(\mu), \quad J_A(\mu) = \begin{pmatrix} 0, & 0 \\ ie^{\lambda_0 \sigma_3 - \theta_{\lambda_0} - \theta_{\lambda_0}^{-1}} & 1 \\ ie^{-\lambda_0 \sigma_3 - \theta_{\lambda_0} - \theta_{\lambda_0}^{-1}} & 0 \\ i & \frac{1}{a_u a_d} \end{pmatrix}, \mu < 0,$$

$$A_+(\mu) = A_-(\mu)J_A(\mu), \quad J_A(\mu) = \begin{pmatrix} 0, & 0 \\ ie^{\lambda_0 \sigma_3 - \theta_{\lambda_0} - \theta_{\lambda_0}^{-1}} & 1 \\ ie^{-\lambda_0 \sigma_3 - \theta_{\lambda_0} - \theta_{\lambda_0}^{-1}} & 0 \\ i & \frac{1}{a_u a_d} \end{pmatrix}, \mu > 0.$$

Now, integrating $A(\mu)A^T(\mu)$ over $\mathbb{R} + i0$, and adding the result to its Hermite conjugate, we obtain

$$0 = \int_{\mathbb{R}^+} A_+(\mu)A^T_+(\mu)d\mu = \int_{\mathbb{R}^+} A(\mu - i0)J_A(\mu)A^T(\mu - i0)d\mu,$$

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Proposition 6.4. An entire function is entire and has uniform asymptotics (compare with the function \(E\) and the new function \(\tilde{\mu}\)).

\[ A_1^+ = iA_{11}^\theta, \quad A_1^- = iA_{11}e^{\theta z_0} - e^{\theta z_0}, \quad \mu \in (-\infty, 0). \]

and \(A_{12}^+ = O(\mu^{-3/4}), A_{12}^- = O(\mu^{-3/4}).\) Hence, the function

\[ f(\mu) = O(\mu^{-3/4})e^{\theta z_0(\mu)}, \quad \mu \to \infty. \] (69)

Proposition 6.4. An entire function \(f(\mu)\), which has the asymptotics uniformly w.r.t. \(\arg \mu \in [-\pi, \pi]\), equals 0 identically.

Proof of Proposition 6.4. The proof is very similar to the one in [Its Kuijlaars Östensson '08, p.18], [DKMVZ '99, p.1395], but for the convenience of the reader we give it here. First we recall the following Carlson theorem, which is a variant of the maximum modulus principle:

Theorem 6.5. [Reed Simon '78, p.236] Let \(b(z)\) be a function, holomorphic in \(\mathbb{Rz} > 0\) and continuous up to \(\mathbb{Rz} \geq 0\). Let \(|b(z)| \leq Me^{|z|}\) for \(\Re z \geq 0\) and \(|b(z)| < Me^{-B|y|}\) for \(y \in \mathbb{R}\). Then \(b(z) \equiv 0\).

Now we define the function \(h(\mu) = f(\mu)e^{-\theta z_0(\mu)}\), which is discontinuous across the half-line \(\mu \in (-\infty, 0]\), where it has the jump

\[ h(\mu + i0) = h(\mu - i0)e^{2\theta z_0(\mu - i0)}, \] (70)

and define the new variable

\[ \zeta = \sqrt{\mu}, \]

and the new function

\[ \tilde{h}(\zeta) = \begin{cases} h(\zeta^2), & \Re \zeta > 0, \\ h(\zeta^2)e^{-\theta z_0(\zeta^2)}, & \Re \zeta < 0. \end{cases} \]

Despite the uniform definition for \(\Re \zeta < 0\), the function \(\tilde{h}(\zeta)\) is discontinuous across the half-line \(\zeta \in (-\infty, 0]\), but because of the jump (70), it is continuous across the imaginary line \(\zeta \in \mathbb{R}\). Now we introduce the variable

\[ z = \zeta^{7/8}, \quad \zeta = z^{8/7}, \]

with the standard cut across \(\zeta \in (-\infty, 0], z \in (-\infty, 0]\), and consider the function

\[ \tilde{h}(z) = \tilde{h}(z^{8/7}). \]

The function \(\tilde{h}(z)\) is continuous and bounded in \(\mathbb{Rz} \geq 0\), analytic in \(\mathbb{Rz} > 0\), and for \(z \in i\mathbb{R}\) it has the super exponential decay (with some positive \(c, C > 0\))

\[ |h(z)| \leq Ce^{-cz}. \]

Hence, by the Carlson’s Theorem 6.5, \(h(z) \equiv 0\) for \(\mathbb{Rz} \geq 0\), and hence \(f(\mu) \equiv 0\) for any \(\mu \in \mathbb{C}\). ☐
This finishes the proof that \( \hat{F}_{\lambda_0,0}(\mu) \) is identically 0, and hence that \( L_0(\mu) \) is identically 0, and thus that RHP 5 for the function \( \hat{F}(x,t;\lambda) \) with asymptotic condition [4] replaced with [4a] is solvable in the \( L_2 \) sense. Then using analyticity of the jump matrices, we can argue that this solution is indeed a solution in the continuous sense. \( \square \)

**Corollary 6.6.** If a function \( r_u(s) \) vanishes as \( \mathcal{O}([s]^{-1-N/2}), s \to -\infty \), then the function \( F(x,t;\lambda) \) is \( N \) times differentiable in \( x \), and \( \lfloor N/3 \rfloor \) times differentiable in \( t \).

**Proof.** Every time that we differentiate the jump matrix \( J_M \) w.r.t. \( x \), we gain a factor \( \sqrt{\lambda} \) under the symbol of the integral. Every time that we differentiate \( J_M \) w.r.t. \( t \), we gain \( \lambda^{3/2} \). Thus, in order to have convergent integrals in (68), we need to have stronger vanishing of \( r_u(s), s \to -\infty \).

Let us consider differentiation w.r.t. \( x \) and \( N \geq 1 \). Then the derivative w.r.t. \( x \) of the jump matrix \( J_x \) is still in \( L_2(\Sigma_M) \), and differentiating equation (68) w.r.t. \( x \), we obtain

\[
M_{-x} = C_-(M_{-x}(J_M - I)) + C_-(M_{-J_M, x}), \quad \text{or} \quad [\text{Id} - C_-(J_M - I)] \circ (M_{-x}) = C_-(M_{-J_M, x})
\]

and this equation is of the same kind as (68), just with a different right-hand-side. Since the operator \( \text{Id} - C_-(J_M - I) \) in (68) is invertible, the same is true for (71). \( \square \)

**Corollary 6.7.**

1. RHP 3 has a unique solution \( \hat{P}(x,t;\lambda) \) for \( t > t_0, x \in \mathbb{R} \) and for \( t = t_0, x \geq B \).

2. RHP 2 has a unique solution \( \hat{N}(x,t;\lambda) \) for \( t < t_0, x \in \mathbb{R} \) and for \( t = t_0, x \leq A \).

3. Further, for \( t > t_0, x \in \mathbb{R} \) and for \( t = t_0, x > B \) the function \( \hat{P}(x,t;\lambda) \) admits the following full asymptotic expansion:

\[
\hat{P}(x,t;\lambda) = \frac{1}{\sqrt{2}} \left( I + \sum_{j=1}^{\infty} P_j(x,t)\lambda^{-j} + \mathcal{O}(\lambda^{-\infty}) \right) \lambda^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

and for \( t < t_0, x \in \mathbb{R} \) and for \( t = t_0, x < A \) the function \( \hat{N}(x,t;\lambda) \) admits the following full asymptotic expansion:

\[
\hat{N}(x,t;\lambda) F^{\sigma_3}(\lambda) = \frac{1}{\sqrt{2}} \left( I + \sum_{j=1}^{\infty} N_j(x,t)\lambda^{-j} + \mathcal{O}(\lambda^{-\infty}) \right) \lambda^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

where the scalar function \( F \) is given by

\[
F(\lambda) = \exp \left\{ \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{0} \frac{\ln |a_u(s)|^2 \, ds}{(s-\lambda) \sqrt{|s|}} \right\}. \quad (72)
\]

4. \( \hat{P}(x,t;\lambda), \hat{N}(x,t;\lambda) \) are smooth (infinitely many times differentiable) in \( x,t \).

5. the functions

\[
P(x,t;\lambda) = \begin{pmatrix} 1 & 0 \\ (P_1(x,t;\lambda))_{12} & 1 \end{pmatrix} \hat{P}(x,t;\lambda),
\]

\[
N(x,t;\lambda) = \begin{pmatrix} 1 & 0 \\ (N_1(x,t;\lambda))_{12} & 1 \end{pmatrix} \hat{N}(x,t;\lambda)
\]

solve RHPs 5, 7, respectively.
Proof. [1., 2.] Let us define the functions \( \hat{P}, \hat{N} \) by the following formulas:

\[
\hat{P}(x, t; \lambda) = \hat{\mathbf{e}} e^{-\theta \sigma_3} \begin{cases}
\begin{pmatrix}
\frac{1}{1 + ir_u} & 0 \\
-r_u e^{-2\theta} & 1 + ir_u
\end{pmatrix}, & \Omega_{III}, \\
\begin{pmatrix}
\frac{1}{1 - ir_d} & 0 \\
-r_d e^{-2\theta} & 1 - ir_d
\end{pmatrix}, & \Omega_{III}, \\
I, & \text{elsewhere},
\end{cases}
\]  

\[
\hat{N}(x, t; \lambda) = \hat{\mathbf{e}} e^{-\theta \sigma_3} \begin{cases}
\begin{pmatrix}
a_u & b_u e^{2\theta} \\
\frac{1}{1 + ir_u} & a_u
\end{pmatrix}, & \Omega_{III}, \\
\begin{pmatrix}
a_d & b_d e^{2\theta} \\
\frac{1}{1 - ir_d} & a_d
\end{pmatrix}, & \Omega_{III}, \\
I, & \text{elsewhere}.
\end{cases}
\]  

It is straightforward to check that the function \( \hat{P} \) [74] satisfies RHP [5] and the function \( \hat{N} \) satisfies RHP [2] and that all the possible poles of \( \hat{F} \) disappear for \( \hat{P}, \hat{N} \). One needs to be careful however that the asymptotics of \( \hat{P} \) as \( \lambda \to \infty \) are not spoiled by the term

\[
r_u(\lambda)e^{-\theta(x,t;\lambda)} = O\left(\frac{1}{\lambda}\right)e^{2\theta(B,t_0;\lambda)-2\theta(x,t;\lambda)} = O\left(\frac{1}{\lambda}\right)e^{2\theta(1-\lambda)+2(x-B)\lambda^{\frac{1}{2}}},
\]

which is bounded in \( \Omega_{II} \) for \( t > t_0, x \in \mathbb{R} \) and for \( t = t_0, x \geq B \), and that the asymptotics of \( \hat{N} \) as \( \lambda \to \infty \) are not spoiled by the term

\[
\frac{b_d e^{2\theta}}{1 + ir_u} = \left(-i a_d + \frac{i}{a_u + i b_u}\right) e^{2\theta} = O\left(\frac{1}{\lambda}\right)e^{2\theta(x,t;\lambda)-2\theta(A,t_0;\lambda)} = O\left(\frac{1}{\lambda}\right)e^{2\theta(1-\lambda)+2(x-A)\lambda^{\frac{1}{2}}},
\]

which is bounded for \( \lambda \in \Omega_{II} \) for \( t < t_0, x \in \mathbb{R} \) and for \( t = t_0, x < A \).

[3] The statement concerning the full asymptotic expansion is similar to [Claeys Vanlessen ’07, lemma 2.3, (ii), p. 1168]. Now we proceed to the details. Let us observe that the function

\[
M_P(x, t; \lambda) = \hat{P}(x, t; \lambda)Y^{-1}(\lambda),
\]

where \( Y(\lambda) \) is defined in [67], solves the following RH problem:

**Riemann-Hilbert problem 12.** To find a \( 2 \times 2 \) matrix-valued function \( M_P(x, t; \lambda) \), which

1. is analytic in \( \lambda \in \mathbb{C} \setminus \Sigma, \Sigma = \gamma_0 \cup \gamma_r \cup \gamma_- \cup \rho \),

2. has the following jump

\[
J_{M_P} = \begin{cases}
I, & \lambda \in \rho, \\
\begin{pmatrix}
1 & 0 \\
-r_u & 1
\end{pmatrix}, & \lambda \in \gamma_0, \\
\begin{pmatrix}
1 + ir_u & ie^{2\theta} \\
r_u e^{-2\theta} & 1
\end{pmatrix}, & \lambda \in \gamma_3, \\
\begin{pmatrix}
1 & -ie^{2\theta} \\
-r_d e^{-2\theta} & 1 - ir_d
\end{pmatrix}, & \lambda \in \gamma_-.
\end{cases}
\]

3. has the following asymptotics as \( \lambda \to \infty \), which is uniform w.r.t. \( \arg \lambda \in [-\pi, \pi] \):

\[
M_P(x, t; \lambda) = I + \mathcal{O}(1).
\]

We see that for \( t > t_0, x \in \mathbb{R} \) and for \( t = t_0, x > B \) the jumps for \( M_P \) are exponentially close to \( I \) on the infinite parts of the contour \( \Sigma \), and hence from the SIE of the type [68], which now reads as

\[
M_{P-} = I + C_{\Sigma -} (M_{P-} (J_{M_P} - I)),
\]

and from the representation of \( M_P \) in terms of \( M_{P-} \)

\[
M_P = I + C_{\Sigma} (M_{P-} (J_{M_P} - I))
\]
we obtain that $M_P$ possesses the full asymptotic expansion

$$M_P = I + \sum_{j=1}^{\infty} M_{P,j} \lambda^{-j} + O(\lambda^{-\infty}).$$

Coming to $\tilde{N}(x,t;\lambda)$, we observe that the jump matrix for $\tilde{N}$ on $\lambda \in \rho$ is not exactly the same as the one of $Y$ \([67]\), and we hence first need to transform it to such a jump. In order to do this, we make a transformation

$$\tilde{N}^{(1)}(x,t;\lambda) = N(x,t;\lambda)F^{\sigma_3}(\lambda), \quad \tilde{N}^{(1)}_+ = \tilde{N}^{(1)}_N, \quad J_N^{(1)} = F^{-\sigma_3}(\lambda)J_N F^{\sigma_3},$$

where the scalar function $F$ is analytic in $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ solves the scalar conjugation problem

$$F_+ F_- = a_w a_d, \lambda \in (-\infty, 0),$$

has asymptotics $F \to 1$ as $\lambda \to \infty$, and hence can be found explicitly by formula \([72]\). Let us observe that $F(\lambda) = \exp\left\{ \frac{1}{2\pi \sqrt{\lambda}} \left( \int_{-\infty}^{0} \frac{\ln|a_w(s)|^2 ds}{\sqrt{|s|}} - \int_{-\infty}^{0} \frac{s \ln|a_w(s)|^2 ds}{(s-\lambda)\sqrt{|s|}} \right) \right\} = \exp\left\{ \frac{1}{2\pi \sqrt{\lambda}} \left( \int_{-\infty}^{0} \frac{\ln|a_w(s)|^2 ds}{\sqrt{|s|}} + \sigma(1) \right) \right\}.

The jump matrix for $\tilde{N}^{(1)}$ on $\rho$ now equals

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

and now we observe that the function

$$M_N(x,t;\lambda) = \tilde{N}(x,t;\lambda)F^{\sigma_3}(\lambda)Y^{-1}(\lambda)$$

(with $Y(\lambda)$ defined by \([67]\)) solves the following RH problem:

**Riemann-Hilbert problem 13.** To find a $2 \times 2$ matrix-valued function $M_N(x,t;\lambda)$, which

1. is analytic in $\lambda \in \mathbb{C} \setminus \Sigma, \Sigma = \gamma_0 \cup \gamma_r \cup \gamma_{-3} \cup \rho$,
2. has the following jump $M_{N,+} = M_{N,-} J_{M,N}$ accross $\Sigma$:
   $$J_{M,N} = \begin{cases} I, & \lambda \in \rho, \\
   Y_+ \begin{pmatrix} 1 & 0 \\ a_w a_d e^{-2\theta} \frac{F^{2}(\lambda)}{F^{2}(\lambda)} + i e^{-\frac{i}{3} \lambda^{3/2}} & 1 \end{pmatrix} Y_+^{-1}, & \lambda \in \gamma_0,
   \end{cases}$$

   $$Y_+ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} Y_+^{-1}, & \lambda \in \gamma_3, \quad Y_- \begin{pmatrix} 1 & 0 \\ a_w a_d e^{2\theta} \frac{F^{-2}(\lambda)}{F^{-2}(\lambda)} - i e^{\frac{i}{3} \lambda^{3/2}} & 1 \end{pmatrix} Y_-^{-1}, & \lambda \in \gamma_{-3},$$

3. has the following asymptotics as $\lambda \to \infty$, which is uniform w.r.t. $\arg \lambda \in [-\pi, 0] \cup [0, \pi]$:
   $$M_N(x,t;\lambda) = I + \mathfrak{M}(1).$$

We see that the jumps for $M_N$ are exponentially close to $I$ on the infinite parts of the contour $\Sigma$, and hence from the SIE of the type \([68]\), which now reads as

$$M_{N,-} = I + \mathfrak{C} \Sigma_- (M_{N,-} (J_{M,N} - I)),$$

and from the representation of $M_N$ in terms of $M_{N,-}$

$$M_N = I + \mathfrak{C} \Sigma (M_{N,-} (J_{M,N} - I))$$
we obtain that $M_N$ possesses the full asymptotic expansion

$$M_N = I + \sum_{j=1}^{\infty} M_{N,j} \lambda^{-j} + O(\lambda^{-\infty}).$$

[4.] To show that $\hat{P}$, $\hat{N}$ are infinitely many times differentiable with respect to $x, t$, one proceeds in the same way as in the proof of Theorem 6.3. Since the jumps of $M_P, M_N$ are exponentially close to $I$ in the infinite parts of the contour $\Sigma$, we can differentiate the corresponding SIE as many times as we wish. Hence, the arguments of Corollary 6.6 can be repeated infinitely many times.

[5.] We need only to check the fact that the functions $P(x, t; \lambda), N(x, t; \lambda)$ indeed have asymptotics prescribed by RHPs [4, 5] and this can be done by direct computations. \qed

6.1 Reconstruction of $u(x, t)$ in terms of $\hat{F}$.

**Theorem 6.8.** Let $a_1^\pm$, $b_1^\pm$ be the corresponding scalar coefficients in the expansion

$$\hat{F}(x, t; \lambda) = \left( I + \left( a_1^\pm(x, t) \frac{1}{\sigma(1)} + \sigma \left( \frac{1}{\lambda} \right) \quad b_1^\pm(x, t) \frac{1}{\sigma(1)} + \sigma \left( \frac{1}{\lambda} \right) \right) \right) \frac{1}{\sqrt{2}} \lambda^{-\sigma_3/4} \theta_{\sigma_3}.$$

Denote

$$u(x, t) := 2a_1^\pm - b_1^\pm.$$

Then $u(x, t)$ solves the KdV equation [1].

**Proof.** The proof goes along the well-known scheme by Zakharov–Shabat. Let $\hat{F}$ admit an asymptotic expansion of the form (12), with coefficients $a_j, b_j, c_j, d_j$, changed with $a_j^\pm, b_j^\pm, c_j^\pm, d_j^\pm$, and $J=3$. Suppose that we can differentiate this expansion 1 time w.r.t. $x, t$. Define

$$\hat{F}(x, t; \lambda) := \begin{pmatrix} 1 & 0 \\ b_1^\pm(x, t) & 1 \end{pmatrix} \hat{F}(x, t; \lambda).$$

Notice that the jumps for the function $\hat{F}$ do not depend on $x, t$, hence the ratio

$$\left( \hat{F} \right)_+ \left( \hat{F} \right)^{-1}, \quad \left( \hat{F} \right)_- \left( \hat{F} \right)^{-1}$$

are analytic functions (do not have jumps). By the Liouville theorem, from the asymptotics at $\lambda \to \infty$ we find that

$$\hat{F}_{P,x} \hat{F}_P^{-1} = \left( \begin{array}{cc} 0 & 1 \\ \lambda - 2u_P(x, t) & 0 \end{array} \right), \quad \hat{F}_{P,t} \hat{F}_P^{-1} = \left( \begin{array}{cc} \frac{u_{P,x}}{6} & \frac{\lambda + u_P(x, t)}{-3} \\ \frac{\lambda^2}{3} + \frac{u_{P,x}}{3} & -\frac{u_{P,xx}}{6} \end{array} \right).$$

where

$$u_P(x, t) = 2a_1^\pm(x, t) - (b_1^\pm)^2(x, t) = -\partial_x (b_1^\pm(x, t)).$$

The consistency condition for the two above differential equations gives us that $u_P(x, t)$ satisfies the KdV equation [1].

The only delicate moment here is the possibility of expansion of $\hat{F}$ at $\lambda \to \infty$, and the differentiability of $\hat{F}$ and its expansion. To address this issue, let us consider the RHP [10]. Its solution can be obtained by the formula

$$M = 1 + \mathcal{C}(M_\cdot (J_M - I)),$$

where $M_\cdot - 1 \in L_2$ is the solution of the singular integral equation

$$[Id - \mathcal{C}_\cdot (J_M - 1)] \circ (M_\cdot - 1) = \mathcal{C}_\cdot (J_M - I).$$
Furthermore, we can write the derivative of $M$ as

$$M_x = C(M_{-x} \cdot (J_M - 1)) + C(M_- \cdot J_{M,x}),$$

where $M_{-x}$ is the solution of the singular integral equation

$$[Id - C_- \cdot (J_M - 1)] \circ M_{-x} = C_- \cdot (M_- \cdot J_{M,x}).$$

We see that SIE for $M_-$ and for $M_{-x}$ have the same operator, and hence the derivative $M_{-x} \in L_2$.

Theorem 6.9. The function $\Lambda$ possesses the following property:

$$\Lambda = F \cdot e^{-\theta \sigma_3}.$$

Remark 6.10. It is not a complete characterization, since we do not show that $u(x,t)$ obtained from the solution of the RH problem $\frac{\partial}{\partial x}$ is a function of bounded variation for $t = t_0$, $A \leq x \leq B$.

Proof. Let us consider RHP 9 with functions $u_r, r_d, \ f_0, \ d_0$ satisfying Properties 6.8. First we divide $F$ by $e^{\theta \sigma_3}$:

$$\Lambda = \frac{\frac{1}{1+i\theta u_a}}{1+i\theta u_d}, \Omega_{II}, \frac{1}{1-i\theta u_d}, \Omega_{II}, \frac{1}{1-i\theta u_a}, \Omega_{II}, \frac{1}{1+i\theta u_a}, \Omega_{II}.$$

Function $\Lambda^{(1)}$ is regular at the points $\Lambda^* \in \Omega_{II}$, $\ r_u(\Lambda^*) = i$, and the jumps of $\Lambda^{(1)}$ are exactly the same as those of $E$ (RHP 3). The only issue is the asymptotics as $\lambda \to \infty$. Since

$$r_u(\lambda) e^{-2\theta(x,t,\lambda)} = \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \cdot e^{\frac{2(1-t_0)}{\lambda} \lambda^{3/2} + 2(B-x)\lambda^{1/2}},$$

the asymptotics for $\Lambda^{(1)}(x,t_0; \lambda)e^{\theta(x,t_0;\lambda)\sigma_3} \to 0$ only when $t = t_0, \ x > B$, and hence

$$\Lambda^{(1)}(x,t_0; \lambda)e^{\theta(x,t_0;\lambda)\sigma_3} = \mathcal{E}(x,t_0; \lambda), \ x > B.$$

Case $t = t_0, \ x < A$.

In this case we apply the transformation

$$\Lambda^{(2)} = \Lambda \cdot \left\{ \begin{array}{l}
\begin{pmatrix}
a_u & \frac{i}{a_u-ib_u} \\
1 & \frac{i}{a_u+ib_u}
\end{pmatrix} \Omega_{II}, & \begin{pmatrix} a_u & -i(a_d-a_u)e^{2\theta} \\
0 & \frac{1}{a_u}
\end{pmatrix} \Omega_{II}, \\
\begin{pmatrix}
a_d & \frac{i}{a_d-ib_d} \\
0 & \frac{i}{a_d+ib_d}
\end{pmatrix} \Omega_{II}, & \begin{pmatrix} a_d & -i(a_d-a_u)e^{2\theta} \\
0 & \frac{1}{a_d}
\end{pmatrix} \Omega_{II}.
\end{array} \right\}.$$
Function \( \Lambda^{(2)} \) is regular at the points \( \lambda^* \in \Omega_{II} \) with \( r_u(\lambda^*) = i \), and the jumps for \( \Lambda^{(2)} \) are again exactly the same as for \( E \) (RHP [1]). As to the asymptotics of \( \Lambda^{(2)} \) as \( \lambda \to \infty \), we notice that

\[
(ad(\lambda) - a_u(\lambda))e^{2\theta(x,t;\lambda)} = O\left(\frac{1}{\sqrt{\lambda}}\right)\quad e^{\frac{2(t-\tau)}{\lambda} + 2 \frac{1}{\lambda} + 2 \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda}}.
\]

and hence we do not spoil the asymptotics when \( t = t_0 \), \( x < A \), and hence

\[
\Lambda^{(2)}(x, t_0; \lambda)e^{\theta(x,t_0;\lambda) \sigma_3} = \mathbb{E}(x, t_0; \lambda) \quad \text{for} \quad x < A.
\]

\[\square\]

### 6.3 Uniqueness of solution of the Cauchy problem for KdV equation

**Lemma 6.11.** Suppose that \( v_1(x, t) \) and \( v_2(x, t) \) are two solutions of the KdV equation

\[
v_1 + v_1x + \frac{1}{12} v_1xx = 0
\]

with the same initial data

\[v_1(x, t_0) = v_2(x, t_0).\]

Assume that \( v_j(x, t) \) is 3 times differentiable in \( x \) and 1 time differentiable in \( t \) for \( t > t_0 \), and that \( v_j(x, t_0) \) is 3 times differentiable in \( x \). Suppose also that, writing \( \omega(x, t) = v_1(x, t) - v_2(x, t) \),

- \( \forall t \geq t_0 \) \( \omega(x, t) \to 0 \), \( \omega_{xx}(x, t) \to 0 \), \( \omega_x(x, t) \to 0 \) as \( x \to \pm \infty \),

- \( \forall t \geq t_0 \) \( \exists \int_{-\infty}^{+\infty} \omega^2(x, t) \, dx < \infty \),

- \( \forall T > t_0 \) \( \exists \sup_{T \geq t \geq t_0} \sup_{x \in \mathbb{R}} |v_{1,x}(x, t)| = M(T) < \infty \).

Then \( v_1(x, t) \equiv v_2(x, t) \) for all \( t \geq t_0 \).

**Proof.** The proof mimicks the one from [Jakovleva]. Suppose that \( v_1(x, t) \) and \( v_2(x, t) \) are 2 solutions of KdV. Then their difference \( \omega(x, t) = v_1(x, t) - v_2(x, t) \) satisfies the following equation:

\[
\omega_t + \frac{1}{12} \omega_{xxx} - \omega_\omega x + 2\omega_1x = 0.
\]

Multiplying the above expression by \( \omega \) and integrating it from \( A \) to \( B \), we obtain

\[
\frac{d}{dt} \int_A^B \frac{\omega^2}{2} \, dx + \int_A^B 2\omega^2 v_{1x} \, dx + \left( \frac{1}{12} \omega_{xx} \omega - \frac{1}{24} \omega_x^2 - \frac{1}{3} \omega^3 \right) \bigg|_A^B = 0.
\]

Now we take the limit \( A \to -\infty, B \to +\infty \), which is possible due to our assumptions on \( \omega, v_1 \), and we get

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \frac{\omega^2}{2} \, dx + \int_{-\infty}^{\infty} 2\omega^2 v_{1x} \, dx = 0.
\]

Denote

\[
q(t) = \int_{-\infty}^{\infty} \frac{\omega^2}{2} \, dx, \quad M(t) = 4 \sup_{x \in \mathbb{R}} |v_{1,x}(x, t)| \leq M(T),
\]

then

\[
q(t) \leq M(T) q(t), \quad \Rightarrow \frac{d}{dt} \left( e^{-M(T) t} q(t) \right) \leq 0 \quad \Rightarrow q(t) \leq q(t_0) e^{M(T) (t-t_0)},
\]

and since \( q(t_0) = 0 \), then \( q(t) = 0 \) for all \( t \geq t_0 \), hence \( \omega(x, t) = 0 \) for all \( x \in \mathbb{R}, t \geq t_0 \). \[\square\]

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Remark 6.12. We do not make any assumptions on the decay at \( x \to \pm \infty \) of \( v_1(x, t) \). For example, we may take \( v_1(x, t) = U(x, t) \) or \( v_1(x, t) = \frac{1}{t^2} \). In the first case the derivative \( v_{1,x} \) is decaying, in the second case it is bounded.

Remark 6.13. The discontinuous initial datum

\[
v_1(x,t_0) = \begin{cases} 
U(x,t), & x < A, \\
e^{-cA}, & A < x < B, \\
U(x,t), & x > B,
\end{cases}
\]

or

\[
v_1(x,t_0) = \begin{cases} 
-x, & x < A, \\
e^{-cA}, & A < x < B, \\
-x, & x > B,
\end{cases}
\]

where \( c \) is a constant, is not within the scope of the lemma.

Appendices

A Derivation of the inverse Fourier Transform and the Parseval identity.

Theorem A.1. For any compactly supported locally integrable function \( f(y) \), the following inverse Fourier transform relation is hold:

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^r(x, t; \lambda) \int_{-\infty}^{+\infty} f(y)e^r(y, t; \lambda)d\lambda = f(x).
\]

Proof. We follow the well-known ideas from [Titchmarsh(1960)], [Levitan(1987)]. For a given fixed \( t \), denote

\[
\varphi(x, \lambda) = e^r(x, t; \lambda), \quad \psi(x, \lambda) = e_t(x, t; \lambda).
\]

Furthermore, for a given function \( f(x) \), define the function

\[
\Phi(x, \lambda) = \varphi(x, \lambda) \int_{-\infty}^{0} f(y)\psi(y, \lambda)dy + \psi(x, \lambda) \int_{0}^{+\infty} f(y)\varphi(y, \lambda)dy.
\]

We will integrate it over some contour in the variable \( k = i\sqrt{\lambda} \) from \(-N+iM\) to \( N+iM \), \( 0 < M < N \), in two different ways: in the first case the contour \( C \) consists of intervals \([-N+iM, -N+iN), [-N+iN, N+iN], [N+iN, N+iM] \). The second contour \( C_2 \) is the union of the intervals \(-\rho_{N,M} := [-N+iM, -0], \gamma_{0,N} := [0, iN-0], \gamma_{0,N} := [iN+0, +0], \rho_{N,M} := [+0, N+iM] \).

We take \( N, M \) to be large, with the constant relation \( \frac{M}{N} < 1 \). We notice that

\[
d\lambda = -2dk,
\]

\[
\varphi(x, \lambda)\psi(y, \lambda) = \frac{e^{\sqrt{\lambda}(x+y)}}{2\sqrt{\lambda}}(1 + o(1)) = \frac{e^{ik(x+y)}}{-2ik}(1 + o(1)), \quad (75)
\]

\[
\psi(x, \lambda)\varphi(y, \lambda) = \frac{e^{\sqrt{\lambda}(x-y)}}{2\sqrt{\lambda}}(1 + o(1)) = \frac{e^{-ik(x-y)}}{-2ik}(1 + o(1)), \quad (76)
\]

\[
\int_{C} \Phi(x, \lambda)d\lambda = \int_{C} \Phi(x, \lambda)d\lambda = \int_{C} \left( \varphi(x, \lambda) \int_{-\infty}^{x} f(y)\psi(y, \lambda)dy + \psi(x, \lambda) \int_{x}^{+\infty} f(y)\varphi(y, \lambda)dy \right) d\lambda,
\]

Here we can switch the order of integration (for example, we can take \( f \) with a compact support), and then we can use asymptotics (75)-(76) the issue whether those asymptotics are uniform w.r.t.
$y$ does not arise here, since again we can take $f$ with compact support). Then the integral becomes

$$
\int_{-\infty}^{\infty} dyf(y) \int_{C} \varphi(x, \lambda) \psi(y, \lambda) d\lambda + \int_{x}^{+\infty} dyf(y) \int_{C} \psi(x, \lambda) \varphi(y, \lambda) d\lambda =
$$

$$
= \int_{-\infty}^{\infty} dyf(y) \int_{C} (-i) e^{ikx} (1 + o(1)) dk + \int_{x}^{+\infty} dyf(y) \int_{C} (-i) e^{ik(x-y)} (1 + o(1)) d\lambda =
$$

$$
= \int_{-\infty}^{+\infty} (-i)f(y) \frac{e^{i|x-y|(N+iM)} - e^{i|x-y|(-N+iM)}}{|x-y|} (1 + o(1)) dy =
$$

$$
= \int_{-\infty}^{+\infty} (-i)f(y) \frac{e^{-M|x-y||y|}}{|x-y|} (1 + o(1)) dy =
$$

$$
= -2i \int_{-\infty}^{+\infty} f(y) \frac{e^{-2M|x-y||y|}}{|x-y|} (1 + o(1)) dy.
$$

Now we make the change of variable $y = x + \frac{\tilde{y}}{N}$. Then the integral becomes

$$
= -2i \int_{-\infty}^{+\infty} f(x + \frac{\tilde{y}}{N}) \frac{e^{-\frac{M}{N} |\tilde{y}| \sin(|\tilde{y}|)}}{|\tilde{y}|} (1 + o(1)) d\tilde{y}
$$

When $N \to \infty$, $\frac{M}{N} = \text{const}$, the above integral tends to

$$
= -2i f(x) \int_{-\infty}^{+\infty} \frac{e^{-\frac{M}{N} |\tilde{y}| \sin(|\tilde{y}|)}}{|\tilde{y}|} d\tilde{y} = -4i \arctan \frac{N}{M} f(x). \quad (77)
$$

**Integrating over $C_2$.** Next we integrate over $C_2$. We split the contribution of integration over $\gamma_{0,N}^+, -\gamma_{0,N}^-$, and over $\rho_{N,M}^+, -\rho_{N,M}^-$. The integral over $\gamma_{0,N}^+, -\gamma_{0,N}^-$ is

$$
\int_{\gamma_{0,N}^+} \varphi(x, \lambda) \psi(y, \lambda) d\lambda + \int_{-\gamma_{0,N}^-} \varphi(x, \lambda) \psi(y, \lambda) d\lambda = \int_{\gamma_{0,N}^+} (\varphi(x, \lambda) \psi_+(y, \lambda) - \varphi(x, \lambda) \psi_-(y, \lambda)) d\lambda =
$$

$$
= \int_{\gamma_{0,N}^-} (\varphi(x, \lambda) \psi_+(y, \lambda) - \varphi(x, \lambda) \psi_-(y, \lambda)) d\lambda = -i \int_{\gamma_{0,N}^-} \varphi(x, \lambda) \varphi(y, \lambda) d\lambda \xrightarrow{N \to \infty} -i \int_{\gamma_0} \varphi(x, \lambda) \varphi(y, \lambda) d\lambda.
$$

Hence, the contribution from the integrals over $\gamma_{0,N}^+, -\gamma_{0,N}^-$ is

$$
- i \int_{\gamma_{0,N}^+} + \int_{-\gamma_{0,N}^-} f(y) \varphi(y, \lambda) d\lambda.
$$

The integrals over $\rho_{N,M}^+/\rho_{N,M}^-$ are

$$
\int_{-\rho_{N,M}^-}^{\rho_{N,M}^+} dyf(y) \int \varphi(x, \lambda) \psi(y, \lambda) d\lambda + \int_{-\rho_{N,M}^-}^{\rho_{N,M}^+} dyf(y) \int \varphi(x, \lambda) \psi(y, \lambda) d\lambda +
$$

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\[ + \int_{x}^{+\infty} df(y) \int_{-\rho_{N,M}}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda + \int_{x}^{+\infty} df(y) \int_{\rho_{N,M}}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda. \]

Since the asymptotics for \( \varphi \) is not uniform up to \( \rho^\pm \), we will use the identities

\[ \varphi(\lambda) = i(\psi(\lambda) - \overline{\psi(\lambda)}), \quad \Im \lambda > 0, \quad \varphi(\lambda) = -i(\psi(\lambda) - \overline{\psi(\lambda)}), \quad \Im \lambda < 0. \]

Hence the above sum of integrals can be written in the form

\[
\int_{x}^{+\infty} df(y) \left[ \int_{-\rho_{N,M}}^{+\infty} i(\psi(x,\lambda) - \overline{\psi(x,\lambda)})\varphi(y,\lambda) d\lambda + \int_{\rho_{N,M}}^{+\infty} (-i)(\psi(x,\lambda) - \overline{\psi(x,\lambda)})\varphi(y,\lambda) d\lambda \right] + \int_{x}^{+\infty} \left[ \int_{-\rho_{N,M}}^{+\infty} i(\psi(x,\lambda) - \overline{\psi(x,\lambda)})\varphi(y,\lambda) d\lambda + \int_{\rho_{N,M}}^{+\infty} (-i)(\psi(x,\lambda) - \overline{\psi(x,\lambda)})\varphi(y,\lambda) d\lambda \right].
\]

Now we notice that due to the decay properties at infinity, we have

\[
\int_{-\rho_{N,M}}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda = \int_{-\rho}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda, \quad \int_{\rho_{N,M}}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda = \int_{\rho^+} \psi(x,\lambda)\varphi(y,\lambda) d\lambda.
\]

Then the input from the terms containing \( \psi(x,\lambda)\varphi(y,\lambda) \) is

\[
\int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} i\psi(x,\lambda)\varphi(y,\lambda) d\lambda + \int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} i\psi(x,\lambda)\varphi(y,\lambda) d\lambda + \int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} -i\psi(x,\lambda)\varphi(y,\lambda) d\lambda + \int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} -i\psi(x,\lambda)\varphi(y,\lambda) d\lambda =
\]

\[
= \int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} (-i)\psi(x,\lambda)\varphi(y,\lambda) d\lambda + \int_{-\infty}^{x} df(y) \int_{-\rho}^{+\infty} (-i)\psi(x,\lambda)\varphi(y,\lambda) d\lambda.
\]

Now, since \( \psi(x,\lambda - i0) = \overline{\psi(x,\lambda + i0)} \) for \( \lambda \in \mathbb{R} \), the above expression becomes

\[
= -i \int_{-\infty}^{x} df(y) \int (\psi(x,\lambda)\varphi(y,\lambda) + \overline{\psi(x,\lambda)\varphi(y,\lambda)}) d\lambda.
\]

Integrals in \([78]\), which contain \( \overline{\psi(x,\lambda)}\varphi(y,\lambda) \), \( \overline{\psi(y,\lambda)}\psi(x,\lambda) \), can be treated in the following way. Consider, for example,

\[
- i \int_{-\infty}^{x} df(y) \int_{-\rho_{N,M}}^{+\infty} \psi(x,\lambda)\varphi(y,\lambda) d\lambda = -i \int_{-\infty}^{x} df(y) \left( \int_{-\rho_{N,O}}^{\rho_{N,M}} \psi(x,\lambda)\varphi(y,\lambda) d\lambda - \int_{-\rho_{N,O}}^{-N+iM} \int_{\rho_{N,O}}^{N} \psi(x,\lambda)\varphi(y,\lambda) d\lambda \right).
\]

Here the first integral is already in the form which is OK for us, and to compute the second integral, we use the asymptotics

\[
\psi(x,\lambda) = \exp \left\{ \frac{1}{105} \lambda^{7/2} - \frac{1}{4} \lambda^{3/2} + x \lambda^{1/2} \right\} \frac{1}{\sqrt{2\pi \lambda}} (1 + \omega(1)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

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\[ \overline{\psi(x, \lambda)} = i \text{sign}(3\lambda) \exp \left\{ \frac{-1}{105} \lambda^{7/2} + \frac{4}{9} \lambda^{3/2} - x \lambda^{1/2} \right\}, \quad \text{arg } \lambda \in (\pi - \varepsilon, \pi - 0) \cup [-\pi + 0, -\pi + \varepsilon), \]

Hence, for \( x > y \) we have

\[
\int_{-N}^{-N+iM} \frac{\psi(x, \lambda) \psi(y, \lambda) d\lambda}{-N+iM} = \int_{-N}^{-N+iM} \frac{i e^{(y-x)\sqrt{\lambda}} (1 + o(1)) d\lambda}{2 \sqrt{\lambda}} = \int_{-N}^{-N+iM} \frac{i e^{-i(y-x)k} (-2k) d\lambda}{-2ik} =
\]

\[
= \int_{-N}^{-N+iM} e^{ik(x-y)} d\lambda = e^{-i(x-y)N} \left( e^{-N}(x-y) - 1 \right) (1 + o(1)),
\]

then the second sum in the integral (79) becomes

\[
i \int_{-\infty}^{x} dy f(y) \frac{e^{-i(x-y)N} \left( e^{-M(x-y)} - 1 \right)}{i(x-y)},
\]

and after change of variables \( y = x + \frac{y}{M} \) we get

\[
- \int_{-\infty}^{0} d\tilde{y} f \left( x + \frac{\tilde{y}}{M} \right) \frac{e^{i\tilde{y} \left( \frac{N}{M} - 1 \right)}}{i} \rightarrow -f(x) \int_{-\infty}^{0} e^{i\tilde{y} \left( \frac{N}{M} - 1 \right)} d\tilde{y} =
\]

\[
= f(x) \left[ i \left( \frac{\pi}{2} - \arctan \frac{N}{M} \right) - \frac{1}{2} \ln \left( \frac{M^2}{N^2} + 1 \right) \right]
\]

Hence, that term tends to

\[
-i \int_{-\infty}^{x} dy f(y) \int_{-\rho_{N,M}}^{\rho_{N,M}} \psi(x, \lambda) \psi(y, \lambda) d\lambda 
\]

\[
= f(x) \left[ i \left( \frac{\pi}{2} - \arctan \frac{N}{M} \right) - \frac{1}{2} \ln \left( \frac{M^2}{N^2} + 1 \right) \right]
\]

Similarly,

\[
\int_{x}^{+\infty} dy f(y) \int_{-\rho_{N,M}}^{\rho_{N,M}} (-i) \overline{\psi(x, \lambda)} \psi(y, \lambda) d\lambda 
\]

\[
= f(x) \left[ i \left( \frac{\pi}{2} - \arctan \frac{N}{M} \right) + \frac{1}{2} \ln \left( \frac{M^2}{N^2} + 1 \right) \right],
\]

\[
\int_{-\infty}^{x} dy f(y) \int_{-\rho_{N,M}}^{\rho_{N,M}} i \overline{\psi(x, \lambda)} \psi(y, \lambda) d\lambda 
\]

\[
= f(x) \left[ i \left( \frac{\pi}{2} - \arctan \frac{N}{M} \right) + \frac{1}{2} \ln \left( \frac{M^2}{N^2} + 1 \right) \right],
\]

\[
\int_{x}^{+\infty} dy f(y) \int_{-\rho_{N,M}}^{\rho_{N,M}} i \overline{\psi(y, \lambda)} \psi(x, \lambda) d\lambda 
\]

\[
= f(x) \left[ i \left( \frac{\pi}{2} - \arctan \frac{N}{M} \right) + \frac{1}{2} \ln \left( \frac{M^2}{N^2} + 1 \right) \right].
\]
Summing up the last four expressions, and taking into account the property $\psi(x, \lambda - i0) = \psi(x, \lambda + i0)$, $\lambda \in \mathbb{R}$, we can combine the integrals in the 2\textsuperscript{nd} and the 3\textsuperscript{rd} lines, and in the first and the fourth lines. We get

$$-i \int_{-\infty}^{\infty} dy f(y) \int_{-\rho,0}^{\infty} \left( \overline{\psi(y, \lambda)} \psi(y, \lambda) + \overline{\psi(y, \lambda)} \psi(y, \lambda) \right) d\lambda + f(x)i \left( 2\pi - 4 \arctan \frac{N}{M} \right).$$

Summing up, the integral $\int_{C_2} \Phi(x, t; \lambda) d\lambda$ equals

$$f(x)i \left( 2\pi - 4 \arctan \frac{N}{M} \right) - i \int_{\gamma_0}^{+\infty} \phi(x, \lambda) f(y) \psi(y, \lambda) d\lambda - i \int_{-\infty}^{+\infty} f(y) \int_{\rho^-}^{+\infty} \phi(x, \lambda) d\lambda d\lambda + i \int_{-\infty}^{+\infty} f(y) \int_{\rho^-}^{+\infty} \phi(x, \lambda) f(y, \lambda) d\lambda,$

and, using the relation $\phi(\lambda) = i(\psi(\lambda) - \overline{\psi(\lambda)})$, $\lambda \in \rho_-$, we get

$$f(x)i \left( 2\pi - 4 \arctan \frac{N}{M} \right) - i \int_{\gamma_0}^{+\infty} \phi(x, \lambda) f(y) \psi(y, \lambda) d\lambda + i \int_{-\infty}^{+\infty} f(y) \int_{\rho^-}^{+\infty} \phi(x, \lambda) f(y, \lambda) d\lambda,$$

and, since $\rho^-$ has the direction from 0 to $-\infty$, the last expression is equal to

$$f(x)i \left( 2\pi - 4 \arctan \frac{N}{M} \right) - i \int_{-\infty}^{\infty} \phi(x, \lambda) f(y) \psi(y, \lambda) d\lambda. \quad (80)$$

Comparing $\int_{C} \Phi(x, t; \lambda) d\lambda$ and $\int_{C_2} \Phi(x, t; \lambda) d\lambda$ (80), we come to conclusion that the following inverse Fourier transform formula holds true:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x, \lambda) f(y) \psi(y, \lambda) dy d\lambda. \quad (81)$$

**Remark A.2.** Let us notice, that when we wrote integrals of $\psi(y, \lambda)$, we must take $f(y) = 0$, $y \to \infty$, but in the final formula (81) we don’t have this restriction. In the symbolic form (81) can be written as

$$\delta(x - y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x, \lambda) \psi(y, \lambda) d\lambda.$$

Multiplying (81) by a function $g(x)$ and integrating over $\mathbb{R} dx$, we come to the Parseval identity

$$\int_{-\infty}^{+\infty} f(x)g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) \phi(x, \lambda) d\lambda \int_{-\infty}^{+\infty} f(y) \psi(y, \lambda) dy. \quad (82)$$

**Remark A.3.** The analogy with (81) with the potential $x$ instead of $U(x, t)$ is

$$\delta(x - y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sqrt{2\pi} Ai(x + \lambda) \sqrt{2\pi} Ai(y + \lambda) d\lambda.$$
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