A note on stretched exponential decay of correlations for the Viana-Alves map

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1 Introduction

Let \( \varphi : S^1 \times \mathbb{R} \) to itself be given by

\[
\varphi(\omega, x) = (g(\omega), f(\omega, x)) = (d\omega, a_0 + \varepsilon \sin(2\pi \omega) - x^2)
\]

(1)

where \( a_0 \in (1, 2) \) is fixed such that \( x = 0 \) is a preperiodic point for the map \( h(x) = a_0 - x^2 \), and \( d \) is an integer, say \( \geq 16 \).

For small \( \varepsilon > 0 \), this map leaves invariant a set of the form \( S^1 \times I \) for some nonempty compact interval \( I \). It is known that this map has two positive Lyapunov exponents Lebesgue-almost everywhere (\([6]\)), that it has an ergodic SRB probability (\([1]\)), and that the decay of the correlations for this measure is faster than any polynomial (\([2]\)). The aim of this work is to show that the decay of the correlations is in fact at least \( O(e^{-c\sqrt{n}}) \).

The main difference between our method and the method of \([2]\) is that our construction is inductive. In their article, if a point has many hyperbolic times between 0 and \( N \) but has not yet been chosen, then it is not in contradiction with Pliss’ Lemma that this point does not have hyperbolic times between \( N \) and \( 2N \) for example. Thus, it is possible that the measure of points remaining at time \( 2N \) is quite large (and a careful study shows that, without new ideas, their method will not give a decay rate better than \( e^{-(\log n)^2} \)). In our inductive setting, everything restarts afresh after each iteration, so we do not have this kind of problem. This is made possible by a precise control of the geometry of the system (while the result of \([2]\) is valid in a much more general setting) – in particular, we need to use so-called hyperbolic returns to control the size of the sets given by the induction.

This note was written in December 2002, when our result was announced \([4]\). Since then, the second named author has found the “new ideas” needed to enhance the techniques in

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obtaining a general abstract result \( \mathbb{A} \) which gives as a particular case another proof of the present result. We nevertheless believe that the ad hoc proof in this rough note based on ideas from \( \mathbb{B} \), \( \mathbb{C} \) should be made publicly available, at least on arxiv.org.

2 Preliminary estimates

We introduce a partition of \( I \) (mod 0) into the following intervals:

\[
I_r = (\sqrt{\varepsilon} e^{-r}, \sqrt{\varepsilon} e^{-(r-1)}) \quad \text{for} \quad r \geq 1,
\]

\[
I_r = -I_{-r} \quad \text{for} \quad r \leq -1,
\]

\[
I_{0+} = I \cap [\sqrt{\varepsilon}, +\infty) \quad \text{and} \quad I_{0-} = I \cap (-\infty, -\sqrt{\varepsilon}]
\]

We also write \( I^+_r \) for the union of the three consecutive intervals centered on \( I_r \) (with the straightforward modifications for \( I_{0+} \) and \( I_{0-} \)).

Given \((\omega, x) \in S^1 \times I\), we define \((\omega_j, x_j) = \varphi_j(\omega, x)\). Following \( \mathbb{D} \), we take \( \eta \) a positive constant smaller than \( 1/3 \) depending only on the quadratic map \( h \). We have (\([6, \text{Lemma 2.1}]\))

**Lemma 2.1.** There are constants \( C_0, C_1 > 0 \) such that for every small \( \varepsilon > 0 \), we have an integer \( N(\varepsilon) \) satisfying

1. If \( |x| < 3\sqrt{\varepsilon} \) then \( \prod_{j=0}^{N(\varepsilon)-1} |\partial_x f(\omega_j, x_j)| \geq |x|e^{-1+\eta} \).

2. If \( |x| < 3\sqrt{\varepsilon} \) then \( |x_j| \geq \sqrt{\varepsilon} \) for \( j = 1, \ldots, N(\varepsilon) \).

**Lemma 2.2.** There are \( \sigma_2 > 1 \) and \( C_2 > 0 \) such that \( \prod_{j=0}^{k-1} \partial_x f(\omega_j, x_j) \geq C_2\sigma_2^k \) whenever \( |x_0|, \ldots, |x_{k-1}| \geq e^{-9}\sqrt{\varepsilon} \) and \( |x_k| \leq 2\sqrt{\varepsilon} \).

Moreover, \( \prod_{j=0}^{k-1} \partial_x f(\omega_j, x_j) \geq C_2\varepsilon\sigma_2^k \) whenever \( |x_0|, \ldots, |x_{k-1}| \geq e^{-9}\sqrt{\varepsilon} \).

We say that the graph of a function \( X : J \subset S^1 \to I \) (where \( J \) is an interval) is an admissible curve if \( X \) is \( C^2 \) with \( |X'| \leq \varepsilon \) and \( |X''| \leq \varepsilon \). Then (\([6, \text{Lemma 2.1}]\))

**Proposition 2.3.** For small enough \( \varepsilon \), the image of an admissible curve defined on an interval of length \( < 1/d \) is still an admissible curve.

3 Construction of a Markov tower

3.1 Growing to a fixed size

A rectangle is a subset \( R \) of \( S^1 \times I \) bounded by two vertical lines, and two “horizontal” curves, i.e. graphs of functions from a subset of \( S^1 \) to \( I \). We shall write \( \text{left}(R) \) for the left
Lemma 3.1. There exist $q \in \mathbb{N}$ and $C > 0$ such that, for any admissible rectangle $R$ with basis $S^1$ and left boundary of size $\geq \varepsilon^{1-\frac{3}{4}q}$, there exists a partition $R_0, \ldots, R_s, R_0', \ldots, R'_k$ of $R$ and times $t_0, \ldots, t_s, t'_0, \ldots, t'_k \leq q$ such that:

1. For $0 \leq i \leq s$, $\varphi^{t_i}$ maps $R_i$ bijectively on $S^1 \times \Lambda$ for $\Lambda = I_1$ or $I_{-1}$, with distortion bounded by $C$.

2. For $0 \leq i \leq k$, the rectangle $\varphi^{t_i}(R'_i)$ is admissible and gentle, its left boundary is of size $\geq \varepsilon^{1-\frac{3}{4}q}/C$, and the distortion of $\varphi^{t_i}$ is bounded by $C$ on $R'_i$.

3. $\text{Leb}(\bigcup R_i) \geq \text{Leb}(R)/C$.

Proof. In this proof, every time we iterate the map, cut the rectangle vertically in $d$, and apply the following procedure independently to each part. Thus, at each step, the image of every rectangle will have $S^1$ as its basis. From this point on, we will only describe what happens in the $x$ direction.

Let $t$ be the first time such that $\varphi^t(R)$ meets $S^1 \times \{|x| < e^{-9} \sqrt{\varepsilon}\}$.

If $\varphi^t(R)$ also meets $S^1 \times \{|x| > 3 \sqrt{\varepsilon}\}$, then we can cut $\text{horz}(\varphi^t(R)) \times \Lambda$ as a part of $\varphi^t(R)$, and hence subdivide $\varphi^t(R)$ in three parts, for which the return time will be $t$. This gives the required construction: the number of iterates is bounded by a constant $C(\varepsilon)$ (according to the second part of Lemma 2.2), the distortion is bounded since in this finite number of iterates we have uniformly avoided the critical point, and the vertical size is $\geq (e^{-1} - e^{-2}) \sqrt{\varepsilon}$ at some point, whence it is $\geq (e^{-1} - e^{-2}) \sqrt{\varepsilon} - 2 \varepsilon$ on the left (because the rectangle is bounded by admissible curves). This is $\geq \varepsilon^{1-\frac{3}{4}q}$ if $\varepsilon$ is small enough. Finally, the upper part $U$ will contain $\text{horz}(U) \times (\sqrt{\varepsilon}, 2 \sqrt{\varepsilon})$, whence it is gentle, and the lower part $V$ contains $\text{horz}(V) \times (I_2 \cup \ldots \cup I_{-7})$, whence it is also gentle.

Otherwise, we set $(S_0, t_0) = (\varphi^t(R), t)$. Note that $S_0 \subset S^1 \times \{|x| < 3 \sqrt{\varepsilon}\}$. By Lemma 2.2, $|\text{left}(S_0)| \geq C_2 |\text{left}(R)| \geq C_2 \varepsilon^{1-\frac{3}{4}q}$. We construct inductively $(S_i, t_i)$ such that $S_i$ is a subset of $S^1 \times \{|x| < 3 \sqrt{\varepsilon}\}$, and with $|\text{left}(S_{i+1})| \geq C \varepsilon^{-\eta/2} |\text{left}(S_i)|$. This will imply that, if $\varepsilon$ is small enough, the process will stop after a finite number $C(\varepsilon)$ of iterates. Note that, with the process of vertical cutting, $S_i$ will be replaced by a smaller $S'_i$, but with
Assume \((S_i, t_i)\) is constructed. If \(S_i\) meets \(S^1 \times \{|x| < \varepsilon^{1-\frac{3}{2}\eta}/(5C_2)\}\), we can cut a part of \(S_i\) with a horizontal line at height \(\pm \varepsilon^{1-\frac{3}{2}\eta}/(5C_2)\) (recall that \(|\text{left}(S_i)| \geq \varepsilon^{1-\frac{3}{2}\eta}/C_2\)) and put it as a \(\varphi^i_j(R'_j)\) (for \(t'_j = t_i\)), such that the remaining part \(S\) satisfies \(|\text{left}(S)| \geq |\text{left}(S_i)|/2\) and \(S \subset S^1 \times \{|x| > \varepsilon^{1-\frac{3}{2}\eta}/(5C_2)\}\). Note that \(\varphi^i_j(R'_j)\) will contain \(\text{horz}(\varphi^i_j(R'_j)) \times \left(\varepsilon^{1-\frac{3}{2}\eta}/(10e^8C_2), \varepsilon^{1-\frac{3}{2}\eta}/(10C_2)\right)\) (there is a small loss due to the fact that the boundaries are not straight lines). The ratio of \(e^8\) ensures that this interval contains at least 6 consecutive \(I_r\), and proves the gentleness of \(\varphi^i_j(R'_j)\). Moreover, it will satisfy \(|\text{left}(\varphi^i_j(R'_j))| \geq \varepsilon^{1-\frac{3}{2}\eta}/(20C_2)\), which gives the claim on its size.

Let \(t\) be the first time such that \(\varphi^t(S)\) meets \(S^1 \times \{|x| < e^{-9}\sqrt{\varepsilon}\}\). If \(\varphi^t(S)\) also meets \(S^1 \times \{|x| > 3\sqrt{\varepsilon}\}\), we cut it in three pieces as at the beginning of the proof, and we stop the construction. Otherwise, we set \((S_{i+1}, t_{i+1}) = (\varphi^t(S), t_i + t)\). By Lemma 2.1, we will have \(t \geq N(\varepsilon)\), and during the first \(N(\varepsilon)\) iterates we will have an expansion \(\geq |x|e^{-1+\eta} \geq C\varepsilon^{\eta/2}\) (since \(S \subset S^1 \times \{|x| > \varepsilon^{1-\frac{3}{2}\eta}/5\}\)). During the next \(t - N(\varepsilon)\) iterates, we will have an expansion \(\geq C_2\sigma_2^{t-N(\varepsilon)}\) according to Lemma 2.2, which implies that \(t \leq C(\varepsilon)\), and that globally the expansion will be at least \(C_2C\varepsilon^{-\eta/2}\). This proves the claim \(|\text{left}(S_{i+1})| \geq C\varepsilon^{-\eta/2}|\text{left}(S_i)|\), and concludes the construction.

We check that the desired properties are satisfied: the claims on the size of the images come from the construction. The number of steps in the construction is bounded, since at each step we have an expansion of \(C\varepsilon^{-\eta/2} > 1\). In each step, the number of iterates is bounded by \(C(\varepsilon)\), thus the global number of iterates is bounded. Finally, we iterate the map only outside of the set \(\{|x| < \varepsilon^{1-\frac{3}{2}\eta}/(5C_2)\}\), which implies that the distortion will be bounded. Finally, the claim on \(\text{Leb}(\bigcup R_i)\) comes from the bounded distortion and the fact that the number of rectangles will be bounded by \((2d)^q\).

\[\square\]

### 3.2 Construction of the partition associated to an admissible rectangle

We fix \(p_0 = p_0(\varepsilon)\) such that the expansion during a time \(p_0\) more than compensates for the distortion and the possible contraction during the \(q\) iterates of Lemma 3.1.

Write \(r_j(\omega, x) = |r|\) if \(r_j \in I_r\) with \(|r| \geq 1\), 0 otherwise. Consider \(G_n(\omega, x) = \{1 \leq i \leq n - 1 \mid r_i(\omega, x) \geq \left(\frac{1}{2} - 2\eta\right) \log \frac{1}{r}\}\). Take \(c > 0\) small enough, and \(c' > c\) very close to \(c\). We say that \(n\) is a hyperbolic return for \((\omega, x)\) if for every \(0 < k < n\), we have

\[
\sum_{i \in G_n(\omega, x)} r_i(\omega, x) \leq c'(n - k)
\]

and

\[
r_n(\omega, x) \geq 1.
\]
Write \( H^*_n = \{(\omega, x) \mid n \text{ is the first hyperbolic time} \geq p_0\} \). Then there exists \( \gamma(\varepsilon) > 0 \) and \( C(\varepsilon) > 0 \) such that
\[
\forall n \in \mathbb{N}, \ \text{Leb}((S^1 \times I) - H^*_p \cup \ldots \cup H^*_n) \leq Ce^{-\gamma\sqrt{\varepsilon}}. \tag{2}
\]

**Proposition 3.2.** Let \( R \) be a gentle admissible rectangle. Then there exists a partition \( \mathcal{R}(R) = \bigcup_{n \geq p_0} \mathcal{R}_n(R) \) such that

1. \( H^*_n \cap R \subset \bigcup_{k \leq n} \bigcup_{S \in \mathcal{R}_k(R)} S \).
2. \( \forall S \in \mathcal{R}_n(R), \) the rectangle \( f^n(S) \) has basis \( S^1 \), satisfies \( |\text{left}(f^n(S))| \geq \varepsilon^{1-\frac{3}{2}\eta} \), and \( f^n \) is uniformly expanding and has uniformly bounded distortion on \( S \).

**Construction of the initial partition** \( \mathcal{Q}(R) \)

Since he wants a partition of the whole space, Alves starts from the partition \( \{I_r \times S^1\} \). However, we start from an admissible rectangle, whose boundary can have a slope \( \varepsilon \), and in particular this boundary may cross \( S^1 \times \{0\} \). Thus, we have to construct a more complicated initial partition.

This partition \( \mathcal{Q}(R) = \{Q_i\} \) will have the following properties:

1. Each \( Q_i \) is an admissible rectangle, contained in a set \( S^1 \times I^+_r \), and its horizontal size is of the form \( 1/d^s \) for some \( s \in \mathbb{N} \).
2. \( Q_i \) contains a set of the form \( \text{horz}(Q_i) \times I_r \).
3. If \( Q_i \cap H^*_n \neq \emptyset \), then the horizontal size of \( Q_i \) is at least \( 1/d^n \).

The last property is important because, if \( Q_i \) intersects \( H^*_n \), we will try to iterate \( Q_i \) exactly \( n \) times, and we need to recover a rectangle with basis \( S^1 \).

If the gentle rectangle \( R \) is contained in \( S^1 \times \{x > \sqrt{\varepsilon}\} \) and contains \( S^1 \times \{\sqrt{\varepsilon} < x < 2\sqrt{\varepsilon}\} \), it suffices to take \( Q_0 = R \). So, we can assume that \( R \) contains \( \text{horz}(R) \times (I_a \cup \ldots \cup I_{a+5}) \) for some \( a \).

To construct \( \mathcal{Q}(R) \), we start from the partition \( \mathcal{Q}' \) in sets \( S^1 \times I_r \) for \( r \leq \left(\frac{1}{2} - 2\eta\right) \log(1/\varepsilon) + 2 \) and \( \left[\frac{k}{d_r}, \frac{k+1}{d_r}\right] \times I_r \) for \( r > \left(\frac{1}{2} - 2\eta\right) \log(1/\varepsilon) + 2 \).

Note that if the horizontal boundary of \( R \) intersects a \( Q' \), then it can intersect at most one rectangle of the same horizontal size above or below (because of the bound \( \varepsilon \) on the slope and the smallness of the horizontal size). Otherwise, for \( r \leq \left(\frac{1}{2} - 2\eta\right) \log(1/\varepsilon) + 2 \), we would have \( \varepsilon \geq |I_{r+1}| \geq C\varepsilon^{1-2\eta} \), which is a contradiction for \( \varepsilon \) small enough, and for \( r > \left(\frac{1}{2} - 2\eta\right) \log(1/\varepsilon) + 2 \) we would have \( |\text{horz}(Q')| \varepsilon \geq |I_{r+1}| \), which implies \( \varepsilon d^{-r} \geq C\sqrt{\varepsilon}e^{-r} \) and is again a contradiction for \( \varepsilon \) small enough. Thus, if we form a block of three rectangles, one of them will be included in \( R \), and the intersection of this block with \( R \) will give a valid \( Q_i \). This deals with the boundaries of \( R \); in its interior, simply put the
remaining \( Q' \) as \( Q_j \). Note that the gentleness of \( R \) ensures that there will be no bad interaction between the lower and upper boundaries of \( R \).

We check the third claim on the hyperbolic returns: assume that \( Q_i \cap H^*_n \neq \emptyset \). If \(|\text{horz}(Q_i)| = S^1\), there is nothing to prove. Otherwise, \( Q_i = [\frac{k}{d^n}, \frac{k+1}{d^n}] \times I_r \) (or it comes from a block containing this), which implies that its height is at most \( e^{-r+2} \). Thus, for \((\omega, x) \in Q_i\), \( r(\omega, x) \geq r - 2 \) (because \( r - 2 \geq (1/2 - 2\eta) \log(1/\varepsilon) \), we have \( 0 \in G_n(\omega, x) \)), whence \( \sum_{j \in G_n(\omega, x)} r(\omega_j, x_j) \geq r - 2 \). Since \( n \) is a hyperbolic time for \((\omega, x)\), we obtain \( r - 2 \leq c'n \), whence \( n \geq r \) (as soon as \( n > c'n + 2 \), which will be true for \( \varepsilon \) small enough since \( n > (r - 2)/c' \geq \frac{1}{C}(1/2 - 2\eta) \log(1/\varepsilon) \)). Then \(|\text{horz}(Q_i)| = 1/d^r \geq 1/d^n \).

**Construction of the partition \( \mathcal{R}(R) \)**

Let \( R \) be an admissible rectangle. An admissible subrectangle \( S \) of \( R \) is \( n \)-good if for every \( 0 \leq j \leq n \), there exists \( r \) such that \( \varphi^j(\text{left}(S)) \subset I^+_r \), and there exists \( j \leq n \) such that \( \varphi^j(\text{left}(S)) \supset I_r \), and \( S \cap H^*_n \neq \emptyset \).

Then there exists a partition \( \mathcal{R}(R) = \bigcup_{n \geq p} \mathcal{R}_n(R) \) such that

1. If \( S \in \mathcal{R}_n(R) \), then \( \text{horz}(S) \) is of the form \( \left[ \frac{k}{d^n}, \frac{k+1}{d^n} \right] \) for some \( 0 \leq k \leq d^n - 1 \).
2. \( H^*_n \cap R \subset \bigcup_{k \leq n} \bigcup_{S \in \mathcal{R}_{k}(R)} S \).
3. For every \( 0 \leq j \leq n \) and \( S \in \mathcal{R}_n(R) \), there exists an \( I_{r_j} \) such that \( \varphi^j(\text{left}(S)) \subset I^+_r \).
4. For every \( S \in \mathcal{R}_n(R) \), either \( S \) is \( n \)-good, or there exists a \( j \)-good rectangle \( T \) for some \( j \leq n \), such that \( S \) is subordinate to \( T \). (Our definition of subordinate is adapted from [1]: \( S \) is subordinate to \( T \) if, on the one hand there are \( \ell \leq j - 1 \) and \( I_{r_\ell} \subset I_{r_j} \), and on the other hand \( S \) is a subrectangle of \( T \) with \( \text{horz} \tilde{T} = \text{horz}T \) and either the top admissible curve or the bottom admissible curve of \( \tilde{T} \) coincides with that of \( T \), and either \( I_{r_{j+1}} \) or \( I_{r_{j-1}} \) is included in \( \varphi^j(\text{left} \tilde{T}) \).

In fact, it is sufficient to construct such a partition for each \( Q \in \mathcal{Q}(R) \). And in this case, we can use more or less directly the construction of Alves, up to checking that we have enough control on horizontal sizes.

**Proof of proposition 3.2**

This is done in Alves, with minor modifications due to the fact that our boundaries are not straight line but admissible curves.

The expansion with the hyperbolic returns, done in [3], shows that the size at the end will be \( \geq \varepsilon^{1-2\eta}/C \) for a constant \( C \) independent of \( \varepsilon \). If \( \varepsilon \) is small enough, this will be \( \geq \varepsilon^{1-\frac{2\eta}{3}} \).

**3.3 Construction of the global partition**

We construct a partition \( T \) of \( X = S^1 \times \Lambda_\pm \) for which the induced map will be Markov.
We start from the sets $K_{\pm} = S^1 \times \Lambda_{\pm}$. For each set $S \in \mathcal{R}_n(K_{\pm})$, we apply Lemma 3.1 to $\varphi^n(S)$, giving $R_0, \ldots, R_s, R'_0, \ldots, R'_k$ and times $t_0, \ldots, t_s, t'_0, \ldots, t'_k$. Put the $n$-th preimage of each $R_i$ in the partition $T_i$, with return time $n + t_i$. For $0 \leq i \leq k$ apply inductively the construction process to $\varphi^i(R'_i)$: decompose it as $R(\varphi^i R'_i)$, then use Lemma 3.1 on each image, and go on.

Since, at each step, the process covers at least a proportion $\geq 1/C$ of the remaining space (using Lemma 3.1 and bounded distortion in between), this will cover the whole space mod 0. We write $R(\omega, x)$ for the return time of $(\omega, x)$ — it is defined almost everywhere.

**Theorem 3.3.** There exists a constant $C$ such that $\text{Leb}\{ (\omega, x) \in X \mid R(\omega, x) \geq n \} \leq C e^{-\sqrt{n}/C}$.

In the proof, we will use stopping time ideas, as introduced by Young in [7], but we will have to use slightly different technical ideas, since the arguments of Young would only give an estimate $Ce^{-n^v}$ for every $v < 1/2$ (this is the decay rate she obtains when the estimate on the return times is $e^{-\sqrt{n}}$). Note that the same technical idea can be used to enhance her result, and we will indeed deduce from this estimate on return times that the decorrelation rate is $O(e^{-\sqrt{n}})$.

**Proof.** In the proof, we shall write $T_0(\omega, x) \leq T_1(\omega, x) \leq \ldots T_{k_{\text{max}}(\omega, x)}(\omega, x)$ for the successive return times of $(\omega, x)$, i.e. the times given by the use of Lemma 3.1. While $(\omega, x)$ does not fall in a set $[k/d^q, (k+1)/d^q] \times \Lambda_{\pm}$ at time $T_i$, then Proposition 3.2 and Lemma 3.1 give a next time $T_{i+1}$, and the process stops only when the point returns to a set $[k/d^q, (k+1)/d^q] \times \Lambda_{\pm}$, the return time being then $T_{k_{\text{max}}(\omega, x)}(\omega, x) = R(\omega, x)$.

Fix some $\delta > 0$ very small. Since, at each step, a proportion $1/C$ of the points returns, we have for each $n \in \mathbb{N}$

$$\text{Leb}\{ (\omega, x) \mid k_{\text{max}}(\omega, x) \geq \delta \sqrt{n} \} \leq K e^{-C(\delta)\sqrt{n}}.$$  

(3)

Let $\tau_1 < \ldots < \tau_i$ be fixed return time, and consider

$$A(\tau_1, \ldots, \tau_i) = \{ (\omega, x) \mid k_{\text{max}} \geq i, T_1 = \tau_1, T_2 = \tau_2, \ldots, T_i = \tau_i \}.$$

Let $R$ be a rectangle on which $T_1 = \tau_1, \ldots, T_{i-1} = \tau_{i-1}$, and write $S = T_{\tau_{i-1}}(R)$. Then, by bounded distortion of $f^{T_{i-1}}$ on $R$,

$$\frac{\text{Leb}\{ (\omega, x) \in R \mid T_i = \tau_i \}}{\text{Leb}(R)} \leq C \frac{\text{Leb}\{ (\omega, x) \in S \mid T_1 = \tau_i - \tau_{i-1} \}}{\text{Leb}(S)}.$$

Lemma 3.1 gives that $\text{Leb}(S) \geq C(\varepsilon)$, and

$$\{ (\omega, x) \in S \mid T_1 = \tau_i - \tau_{i-1} \} \subset S^1 \times I - (H^{*}_{p_0} \cup \ldots \cup H^{*}_{\tau_1 - \tau_{i-1} - 1})$$

whence by Equation (2),

$$\text{Leb}\{ (\omega, x) \in S \mid T_1 = \tau_i - \tau_{i-1} \} \leq C e^{-\gamma \sqrt{\tau_i - \tau_{i-1}}}.$$
Summing these equations on all rectangles $R_i$, we obtain
\[ \text{Leb}(A(\tau_1, \ldots, \tau_i)) \leq C \text{Leb}(A(\tau_1, \ldots, \tau_{i-1})) e^{-\gamma \sqrt{n} - \tau_{i-1}}. \]

Let us write $a_n = Ce^{-\gamma \sqrt{n}}$. We get
\[ \text{Leb}(A(\tau_1, \ldots, \tau_i)) \leq a_{\tau_1} a_{\tau_2 - \tau_1} \cdots a_{\tau_i - \tau_{i-1}}. \]

Summing finally on all possible sequences $\tau_1 < \ldots < \tau_i$ with $\tau_i \geq n$ and $i \leq \delta \sqrt{n}$, we get
\[
\text{Leb}\{(\omega, x) \mid k_{\max}(\omega, x) < \delta \sqrt{n}, R(\omega, x) \geq n\} \leq \sum_{i \leq \delta \sqrt{n}} \sum_{\tau_1 < \ldots < \tau_i \geq n} a_{\tau_1} a_{\tau_2 - \tau_1} \cdots a_{\tau_i - \tau_{i-1}} = \sum_{i \leq \delta \sqrt{n}} \sum_{\tau_1 + \ldots + \tau_i \geq n} a_{\tau_1} \cdots a_{\tau_i}.
\]  

(4)

Lemma 3.4. Let $a_n$ be a sequence with $a_n = O(e^{-\gamma \sqrt{n}})$. Then there exists $D > 0$ such that $b_n = D^{-1}a_n$ satisfies
\[ u_n := \sum_{i=0}^{n} \sum_{j_1 + \ldots + j_i = n} b_{j_1} \cdots b_{j_i} = O(n^2 e^{-\gamma \sqrt{n}}). \]

Let $D$ be given by the lemma. Then Equation (4) implies that
\[
\text{Leb}\{(\omega, x) \mid k_{\max}(\omega, x) < \delta \sqrt{n}, R(\omega, x) \geq n\} \leq D^{\delta \sqrt{n}} \sum_{i \leq \delta \sqrt{n}} \sum_{p = n}^{\infty} \sum_{j_1 + \ldots + j_{i} = p \geq \tau_i \geq n} b_{j_1} \cdots b_{j_i} 
\]
\[ \leq D^{\delta \sqrt{n}} \sum_{p=n}^{\infty} u_p \leq D^{\delta \sqrt{n}} En^3 e^{-\gamma \sqrt{n}}. \]

We now choose $\delta$ small enough so that $D^\delta e^{-\gamma} < 1$ (note that $D$ does not depend on $\delta$), and Equation (4) gives a bound of the form $Se^{-\gamma \sqrt{n}}$. Add finally Equation (3), to get
\[ \text{Leb}\{(\omega, x) \mid R(\omega, x) \geq n\} \leq Te^{-\min(\delta, \gamma) \sqrt{n}} \]

which is the conclusion of the theorem. 

\[ \square \]

Proof of Lemma 3.4. Writing $s * t$ for the convolution of the sequences $s_n$ and $t_n$, i.e. $(s * t)_n = \sum_{i=0}^{n} s_i t_{n-i}$, then $u = \sum_{j=0}^{\infty} b^j$.

Write $w_n$ for a sequence equal to $e^{\gamma \sqrt{n}} / n^2$ for $n$ large enough, and satisfying $w_{n+p} \leq w_n w_p$. Let $\|s\| = \sum w_n s_n$ for a sequence $s_n$ such that this sum is finite. Then $\|s \ast t\| \leq \|s\| \|t\|$.

In particular, $\|a\| < \infty$, whence for $D$ large enough, $\|b\| = \|a\| / D < 1$. Then $\|b^j\| \leq \|b\|^j$, and $\|u\| \leq \sum \|b\|^j = 1 / \|b\| < \infty$. In particular, $w_n u_n$ is bounded, i.e. $u_n = O(n^2 e^{-\gamma \sqrt{n}})$. \[ \square \]
4 Decay of correlations

It is not difficult to obtain an aperiodic tower. Then, the rate of decay of correlations may be obtained by using the coupling argument in [7] combined with our technical lemma.

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