Propagating Profiles of a Chemotaxis Model with Degenerate Diffusion: Initial Shrinking, Eventual Smoothness and Expanding

Tianyuan Xu\textsuperscript{a}, Shanming Ji\textsuperscript{b}, Ming Mei\textsuperscript{c,d}, Jingxue Yin\textsuperscript{a},

\textsuperscript{a}School of Mathematical Sciences, South China Normal University
Guangzhou, Guangdong, 510631, P. R. China
\textsuperscript{b}School of Mathematics, South China University of Technology
Guangzhou, Guangdong, 510641, P. R. China
\textsuperscript{c}Department of Mathematics, Champlain College Saint-Lambert
Quebec, J4P 3P2, Canada, and
\textsuperscript{d}Department of Mathematics and Statistics, McGill University
Montreal, Quebec, H3A 2K6, Canada

Abstract

We investigate the propagating profiles of a degenerate chemotaxis model describing the bacteria chemotaxis and consumption of oxygen by aerobic bacteria, in particular, the effect of the initial attractant distribution on bacterial clustering. We prove that the compact support of solutions may shrink if the signal concentration satisfies a special structure, and show the finite speed propagating property without assuming the special structure on attractant concentration, and obtain an explicit formula of the population spreading speed in terms of model parameters. The presented results suggest that bacterial cluster formation can be affected by chemotactic attractants and density-dependent dispersal.

Keywords: Chemotaxis, degenerate diffusion, initial shrinking, propagating speed, eventual smoothness, eventual expanding.

\textsuperscript{*}Corresponding author, email:jism@scut.edu.cn
1 Introduction

We consider the following chemotaxis model with chemotactic consumption and porous media diffusion

\[
\begin{cases}
u_t = \nabla (\phi(u) \nabla u) - \chi \nabla \cdot (u \nabla v), \\
v_t = \Delta v - \alpha u v,
\end{cases}
\quad x \in \Omega, \ t > 0,
\]

(1)

where \( u \) represents the number per unit volume of aerobic bacteria cells, \( v \) denotes the oxygen concentration, \( \chi \) is the chemotactic coefficient, \( \alpha \) denotes the fractional rate of oxygen consumption per unit concentration of bacteria cell. The diffusion of species is considered to be degenerate in the form of \( \nabla (\phi(u) \nabla u) \) with \( \phi(u) = D u^m - 1 \) and \( m > 1 \), which is dependent of the population density due to the population pressure. This model can also describe other chemotaxis progress with nutrient consumption.

In many biological cases, the diffusion coefficient \( \phi(u) \) is not constant, which can be regarded as a consequence of the interaction between cells \([3, 42, 19, 27]\). It is worthy of mentioning that the porous medium type diffusion can represent “population pressure” in cell invasion models \([23]\), which initially arises from the ecology literature \([8, 9, 21, 40]\). In fact, experimental investigation has shown that the diffusion coefficient depends on the bacterial density \([33]\). In the bacterial experiments done by Ohgiwari et al. \([22]\), they recognized that cells located inside the bacterial colonies move actively, but cells became sluggish at the outermost front with apparently low cell density. This phenomenon indicates that bacteria become active as the cell density \( u \) increases. Thus, a natural choice of the bacterial diffusion coefficient is \( \phi(u) = u^{m-1} (m > 1) \), and this porous medium type bacterial diffusivity is based on the degenerate diffusion model proposed by Kawasaki et al. \([12]\). Recently, Leyva et al. \([16]\) incorporates a chemotactic term into the original model by Kawasaki et al., and explores the effects of chemotaxis on bacterial aggregation patterns.

Chemotaxis is the biased migration in the direction of a chemical stimulus concentration gradient \([13, 10]\). Bacteria can sense a large range of chemical signals, such as the concentrations of nutrients, toxins, oxygen, minerals, etc. A mathematical model for the process of aerobic motile bacteria toward oxygen which they consume was first proposed in \([24]\). When \( m = 1 \) in \((1)\), namely, the diffusion of bacteria cell are assumed to be random, Tao and Winkler \([29]\) proved that this model admits a global weak solution, and a more interesting fact is that, the weak solution will become smooth after some time. Recently, chemotaxis models featuring a density-dependent diffusion term have drawn great attention from many authors \([7, 11, 28, 35, 38, 13, 54]\). For this system with the porous medium diffusion (i.e. \( m > 1 \) in \((1)\)), it was shown that the weak solution is globally solvable in
two dimension for \( m > 1 \) \[30\]. In three dimensional space, the authors made great efforts to prove the global existence of weak solutions for this model for any \( m > 1 \). Winkler and Tao \[31, 37\] proved this problem admits a global weak solution for the case \( m \in (1, \frac{9}{4}) \).

The interaction between diffusion and chemotaxis contributes substantial influences on the behavior of solutions for chemotaxis model with degenerate diffusion. In \[4\], Burger, Di Francesco, and Dolak considered the Keller-Segel model of chemotaxis with volume filling effect, which is degenerate when bacteria densities approaching either 0 or 1, and they investigated the qualitative behavior of solutions, such as finite speed of propagation and asymptotic behavior of solutions. Kim and Yao \[14\] studied the qualitative properties of the Patlak-Keller-Segel model with porous medium type diffusion term by using maximum principle type arguments, and they proved the finite propagation property of the compactly supported solutions generated by this type of degeneracy of diffusivity. In \[6\], Fischer proved finite speed of support propagation for the parabolic-elliptic chemotaxis Keller-Segel system with porous medium type diffusive term and gave sufficient criteria for support shrinking, based on the integral estimates and the Stampacchia’s lemma.

The main feature of our model \( (1) \) lies in that the porous medium diffusive term and the chemotactic term are in competition. The dispersal term induces forward motion, whereas the chemotactic attraction may account for cohesive swarm and induce backward motion of the invasion boundary \[6\]. We explore the effect of density dependent diffusion and chemotactic attraction, which can account for cohesive, finite swarms with realistic density profiles.

To understand how changes in the initial conditions of chemotaxis can so dramatically alter the aggregation behavior of bacteria, we study the effects of attractant concentration on bacteria distribution. We will give (see Theorem \[2.1\]) a mathematical understanding of the collective behavior of bacteria chemotactic toward oxygen. We find that under certain initial conditions, the boundary of \( \text{supp} \ u(\cdot, t) \) moves backward in response to the gradient of attraction at early stage. This indicates that the size of the swarm is defined by a balance of chemotactic attraction and cell dispersal: the greater the attraction the smaller its size for a given total number of organisms. This is observed biologically: bacteria exhaust the local oxygen and then react to the attractant gradient they have created, producing a flux towards the region with more oxygen. Early in bioconvection, this process generated accumulations of cells, resulting in smaller size of cell collective region. This experiment was conducted on \textit{Bacillus subtilis} \[5\].

One of the intrinsic characteristics of porous medium diffusions is the population moves with a finite speed of propagation, which seems more reasonable than infinite speed in biological applications. To put it concisely, for any non-zero
initial data $u_0$, the solution of linear diffusion equation $u(x,t) > 0$ for $t > 0$ and any $x \in \mathbb{R}^N$, thus a linear diffusion process predicts an infinite propagation [32]. However, the spatial support of the solution to the degenerate diffusion equation remains bounded for all time $t > 0$ [6].

Bacteria are known to exhibit very diverse morphological aggregation patterns depending on a variety of environmental conditions [2, 22, 20, 33]. These experimental observations showed the bacterial envelop front propagate outward gradually over time and the velocity of front propagation is finite. In order to explain these phenomena, a variety of mathematical models have been proposed [12, 26, 25, 15]. The density-dependent degenerate diffusion model may capture more pattern features found experimentally and provides a better match to experimental cell density profiles. The difference between these diffusion types is that the porous medium type diffusion leads to distinct boundaries, and the population density decreases to zero at a finite point in space, rather than tends to zero asymptotically. It is therefore not surprising that the behavioral property of living organisms in these two models is different. The porous medium type models allow the cells aggregate rather than spread out. The non-physical diffusion is eliminated in this model.

Although the underlying dynamics of the chemotaxis model with degenerate mobility can be complicated, explicit description of bacteria invasion process can be given. The challenge in the mathematical analysis consists of the chemotactic term as well as the degeneracy of the diffusion term which generates compactly supported solutions. We prove several propagating properties of solutions, including the initial shrinking, finite propagation property, eventual smoothness and eventual expanding. The spreading speed is the rate at which the species with uniformly positive initial distribution over a large interval and zero distribution outside an interval expands its spatial range [15]. Theorem 2.3 below provides an explicit formula for the spreading speed in terms of model parameters. To the best of our knowledge, it is the first work that presents a precise description of the propagating speed for this model. These results provide important insight into the spatial patterns and rates of invading bacteria species in space.

The outline of this paper is as follows. In Section 2, we state our main results and some notations. We leave the comparison principle of the corresponding degenerate chemotaxis equation and its Hölder continuity into Section 3 as preliminaries. Section 4 is devoted to the study of the propagating properties of bacteria cells and the large time behavior of the weak solution.
2 Main results and Notations

We consider the following chemotaxis system (2) with degenerate diffusion

\[
\begin{align*}
\frac{\partial u^m}{\partial t} &= \Delta u^m - \chi \nabla \cdot (u \nabla v), & x \in \Omega, & t > 0, \\
v_t &= \Delta v - u v, & x \in \Omega, & t > 0, \\
\frac{\partial u^m}{\partial n} &= \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, & t > 0, \\
u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in \Omega,
\end{align*}
\]

where \( m > 1, \chi > 0, \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary and the spatial dimension \( N \in \{1, 2, 3\} \), \( u_0, v_0 \) are nonnegative functions, \( n \) is the unit outer normal vector.

Since degenerate diffusion equations may not have classical solutions in general, we need to formulate the following definition of generalized solutions for the initial boundary value problem (2).

**Definition 2.1.** Let \( T \in (0, \infty) \). A pair of \((u, v)\) is said to be a weak solution to the problem (2) in \( Q_T = \Omega \times (0, T) \) if

1. \( u \in L^\infty(Q_T), \nabla u^m \in L^2((0, T); L^2(\Omega)), \) and \( u^{m-1}u_t \in L^2((0, T); L^2(\Omega)) \);
2. \( v \in L^\infty(Q_T) \cap L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \);
3. the identities

\[
\int_0^T \int_\Omega u \psi_t \, dx dt + \int_\Omega u_0(x) \psi(x, 0) \, dx = \int_0^T \int_\Omega \nabla u^m \cdot \nabla \psi \, dx dt - \int_0^T \int_\Omega \chi u \nabla v \cdot \nabla \psi \, dx dt,
\]

\[
\int_0^T \int_\Omega v \varphi_t \, dx dt + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi \, dx dt = \int_0^T \int_\Omega w \varphi \, dx dt,
\]

hold for all \( \psi, \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \) with \( \psi(x, T) = 0 \) for \( x \in \Omega \);

4. \( v \) takes the value \( v_0 \) in the sense of trace at \( t = 0 \).

If \((u, v)\) is a weak solution of (2) in \( Q_T \) for any \( T \in (0, \infty) \), then we call it a global weak solution.

A pair of \((u, v)\) is said to be a globally bounded weak solution to the problem (2) if there exists a positive constant \( C \) such that

\[
\sup_{t \in \mathbb{R}^+} \left\{ \|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} \right\} \leq C.
\]
Throughout this paper we assume that the initial data satisfies
\[
  u_0 \in C(\overline{\Omega}), \quad \nabla u_0^m \in L^2(\Omega), \quad \frac{\partial v_0}{\partial n} = 0 \text{ on } \partial \Omega, \quad v_0 \in C^{2,\alpha_0}(\overline{\Omega}) \text{ for some } \alpha_0 \in (0,1).
\]

We are aiming at the propagating properties of the cell invasions. Let us first focus on the waiting time and the initial shrinking of the compact support caused by chemotaxis. Our approach is based on the comparison principle and the technique of self-similar weak lower and upper solutions with compact support.

**Theorem 2.1** (Initial shrinking caused by chemotaxis). Let \((u,v)\) be a globally bounded weak solution of (2) with (i) \(N = 1\); or (ii) \(\sup_{t \in (0,\infty)} \| u(\cdot, t) \|_{C^{1/(2m)}(\overline{\Omega})} \leq C\) for some constant \(C > 0\). Further we assume that
\[
  \text{supp } u_0 \subset \overline{B_{R_0}(x_0)} \subset \Omega, \quad u_0 \leq K_0(R_0^2 - |x - x_0|^2)^{d_0}, \quad x \in B_{R_0}(x_0),
\]
\[
  \nabla v_0 \cdot (x - x_0) \leq -\mu|x - x_0|^2, \quad x \in B_{R_0}(x_0),
\]
for some \(x_0 \in \Omega\) and positive constants \(d_0 \geq 1/(m-1)\), and \(R_0, K_0, \mu > 0\). such that \(\chi \mu > \frac{4m}{m-1} R_0^{m-1} \max \{1, R_0^{2(m-1)d_0 - 1}\}\). Then there exist a family of shrinking open sets \(\{A(t)\}_{t \in (0,t_0)}\) with \(t_0 > 0\) such that \(A(0) = B_{R_0}(x_0)\) and
\[
  \text{supp } u(\cdot, t) \subset \overline{A(t)} \subset \Omega, \quad t \in (0,t_0),
\]
and \(\partial A(t)\) has a finite negative derivative with respect to \(t\).

**Remark 2.1.** The existence of globally bounded weak solutions of (2) is proved in [11]. We will prove in Lemma 3.5 that (i) implies (ii). The finite propagating speed (i.e. the derivative of \(\partial A(t)\) with respect to \(t\)) is interpreted as in the sense of Theorem 2.3.

We show the finite speed propagating property without the special structure (5) on signal concentration.

**Theorem 2.2** (Finite speed propagating). Let the assumptions in Theorem 2.1 be valid except for (5). Then there exist a family of open sets \(\{A(t)\}_{t \in (0,t_0)}\) with \(t_0 > 0\) such that \(A(0) = B_{R_0}(x_0)\) and
\[
  \text{supp } u(\cdot, t) \subset \overline{A(t)} \subset \Omega, \quad t \in (0,t_0),
\]
and \(\partial A(t)\) has a finite derivative with respect to \(t\).

**Remark 2.2.** Without the structure (5) on signal concentration, we do not know the shrinking or expanding of the cells. However, Theorem 2.2 shows the propagating speed is finite.
If the cell density and the signal concentration have special structure, we will present the exact propagating speed as follows.

**Theorem 2.3** (Exact propagating speed). Let \((u, v)\) be a globally bounded weak solution of (2) with (i) \(N = 1\); or (ii) \(\sup_{t \in (0, \infty)} ||u(\cdot, t)||_{C^{1/2}(\Omega)} \leq C\) for some constant \(C > 0\). Further we assume that the initial values satisfy

\[
\begin{cases}
u_0 = K_0[(R_0^2 - |x - x_0|^2)_+, \ x \in \Omega, \\
\nabla v_0 \cdot (x - x_0) = -\mu|x - x_0|^2, \ x \in B_{R_0}^e(x_0),
\end{cases}
\]

(6)

for some \(x_0 \in \Omega\) and positive constants \(d = 1/(m - 1)\), \(R_0, K_0, \mu, \delta > 0\) such that \(\overline{B}_{R_0}(x_0) \subset \Omega\) and \(B_{R_0}^0(x_0) = \{x \in B_{R_0}(x_0); \text{dist}(x, \partial B_{R_0}(x_0)) < \delta\}\). Here, \((R_0^2 - |x - x_0|^2)_+ = \max\{0, R_0^2 - |x - x_0|^2\}\). Then

\[
\text{supp } u(x, t) = \{(\theta, \rho(\theta, t)); \theta \in S^{N-1}\},
\]

where \((\theta, \rho)\) is the spherical coordinate centered at \(x_0\), \(\rho(\theta, 0) = R_0\) for all \(\theta \in S^{N-1}\), and the propagating speed

\[
\frac{\partial \rho(\theta, t)}{\partial t} igg|_{t=0} = R_0 \left( \frac{2m}{m-1} K_0^{m-1} - \chi \mu \right), \ \forall \theta \in S^{N-1}.
\]

With the signal being consumed as time grows, we show that the cells will eventually expand.

**Theorem 2.4** (Eventual expanding). Let \((u, v)\) be a globally bounded weak solution of (2) with (i) \(N = 1\); or (ii) \(\sup_{t \in (0, \infty)} ||u(\cdot, t)||_{C^{1/2}(\Omega)} \leq C\) for some constant \(C > 0\). Further we assume the initial data \(u_0 \geq 0\), \(u_0 \not\equiv 0\) and \(\Omega\) is convex. Then there exist \(\hat{T} > \hat{t} > 0\) and \(t_0 \in (\hat{t}, \hat{T})\), \(\varepsilon_0 > 0\), and a family of expanding open sets \(\{A(t)\}_{t \in (\hat{t}, \hat{T})}\) such that

\[
\text{A(t) } \subset \text{supp } u(x, t), \ \ t \in (\hat{t}, \hat{T}),
\]

and \(A(t) = \Omega\), \(u(x, t) \geq \varepsilon_0\) for all \(x \in \Omega\) and \(t \in [t_0, \hat{T}]\).

Theorem 2.4 implies that the cells will eventually expand to the whole domain. After that we can show the eventual smoothness and large time behavior.

**Theorem 2.5** (Eventual smoothness). Let the assumptions in Theorem 2.4 be valid. Then \(u(x, t) \geq \varepsilon_0\) for all \(x \in \Omega\) and \(t \geq t_0\) with \(t_0 > 0\) and \(\varepsilon_0 > 0\) in Theorem 2.4 \(u \in C^{2,1}(\overline{\Omega} \times [t_0, \infty))\) and there exist \(C > 0\) and \(c > 0\) such that

\[
||u(\cdot, t) - \overline{u}||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{W^{1,\infty}(\Omega)} \leq Ce^{-ct}, \ \ t > 0,
\]

where \(\overline{u} = \int_{\Omega} u_0 dx/|\Omega|\).
The main difficulty lies in the balance between the degenerate diffusion (expanding) and the possible aggregating effect (shrinking) caused by the chemotaxis. According to the exact propagating speed Theorem \[2,3\] it is clear that the profile near the boundary of its support competes with the gradient of the signal concentration. We first prove the comparison principle by the approximate Hohmgren’s approach, and then construct several kinds of lower and upper solutions. The self similar weak lower and upper solutions with shrinking or expanding support are comparable with the Barenblatt solution to the porous medium equation

\[
B(x,t) = (1 + t)^{-k} \left[ \left(1 - \frac{k(m-1)}{2mN} \frac{|x|^2}{(1 + t)^{2k/N}} \right)^{\frac{1}{m-1}} \right]
\]

with \(k = 1/(m-1+2/N)\) for \(m > 1\). After showing the eventual expanding property, we formulate the eventual smoothness and large time behavior.

3 Preliminaries: comparison principle and H"older continuity

3.1 Comparison principle of degenerate diffusion equations

We present the following comparison principle of degenerate diffusion equation in general form

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta A(u) - \nabla \cdot (B(u)\Phi(x,t)), \quad x \in \Omega, \; t > 0, \\
(\nabla A(u) - B(u)\Phi) \cdot n &= 0, \quad x \in \partial\Omega, \; t > 0, \\
u(x,0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \(A(s)\) is strictly increasing and locally Lipchitz continuous for \(s \in \mathbb{R}\), \(B(s)\) is locally Lipchitz continuous for \(s \in \mathbb{R}\), and \(\Phi : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N\) is bounded. Here the degenerate set \(\{s \in \mathbb{R}; A'(s) = 0\}\) has no interior point and the equation \(\Box s\) is weakly degenerate. The typical case is \(A(u) = u^m\) with \(m > 1\), \(B(u) = \chi u\) and the solution \(u\) is non-negative (otherwise, one may write \(A(u) = |u|^{m-1}u\)).

**Lemma 3.1** (Comparison principle). Let \(T > 0\) and the function space \(E = \{u \in L^\infty(Q_T); \nabla A(u) \in L^2(Q_T)\}\), \(u_1, u_2 \in E\), \(\Phi \in L^\infty(Q_T)\), and \(u_1, u_2\) satisfy the
following differential inequality

\[
\begin{aligned}
\begin{cases}
\frac{\partial u_1}{\partial t} - \Delta A(u_1) + \nabla \cdot (B(u_1)\Phi(x,t)), \\
\frac{\partial u_2}{\partial t} - \Delta A(u_2) + \nabla \cdot (B(u_2)\Phi(x,t)),
\end{cases}
\end{aligned}
\]

\[x \in \Omega, t \in (0, T),\]

\[
\begin{aligned}
\nabla A(u_1) - B(u_1)\Phi \cdot n \geq (\nabla A(u_2) - B(u_2)\Phi) \cdot n,
\end{aligned}
\]

\[x \in \partial\Omega, t \in (0, T),\]

\[u_1(x, 0) \geq u_2(x, 0),\]

\[x \in \Omega,\]

in the sense that the following inequality

\[
\begin{aligned}
\iint_{Q_T} u_1 \varphi_t \, dx \, dt + \int_{\Omega} u_{10}(x) \varphi(x, 0) \, dx - \iint_{Q_T} \nabla A(u_1) \cdot \nabla \varphi \, dx \, dt \\
+ \iint_{Q_T} B(u_1) \Phi(x, t) \cdot \nabla \varphi \, dx \, dt + \iint_{\partial \Omega \times (0, T)} (\nabla A(u_1) - B(u_1)\Phi) \cdot nd \, dx \\
\leq \iint_{Q_T} u_2 \varphi_t \, dx \, dt + \int_{\Omega} u_{20}(x) \varphi(x, 0) \, dx - \iint_{Q_T} \nabla A(u_2) \cdot \nabla \varphi \, dx \\
+ \iint_{Q_T} B(u_2) \Phi(x, t) \cdot \nabla \varphi \, dx \, dt + \iint_{\partial \Omega \times (0, T)} (\nabla A(u_2) - B(u_2)\Phi) \cdot nd \, dx,
\end{aligned}
\]

hold for some fixed \(u_{10}, u_{20} \in L^2(\Omega)\) such that \(u_{10} \geq u_{20}\) on \(\Omega\) and all test functions \(0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \) with \(\varphi(x, T) = 0\) on \(\Omega\). Then \(u_1(x, t) \geq u_2(x, t)\) almost everywhere in \(Q_T\).

**Proof.** The following inequality

\[
\begin{aligned}
\iint_{Q_T} (u_1 - u_2) \varphi_t \, dx \, dt \leq \iint_{Q_T} \nabla (A(u_1) - A(u_2)) \cdot \nabla \varphi \, dx \, dt \\
- \iint_{Q_T} (B(u_1) - B(u_2))\Phi(x, t) \cdot \nabla \varphi \, dx \, dt,
\end{aligned}
\]

holds for all \(0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega))\) with \(\varphi(x, T) = 0\). If we further assume that \(\frac{\partial \varphi}{\partial n} = 0\) for \(x \in \partial\Omega\) and \(t \in (0, T)\), then we have

\[
\begin{aligned}
\iint_{Q_T} (u_1 - u_2) \varphi_t \, dx \, dt \leq \iint_{Q_T} \left( - (A(u_1) - A(u_2)) \Delta \varphi - (B(u_1) - B(u_2))\Phi(x, t) \cdot \nabla \varphi \right) \, dx \, dt.
\end{aligned}
\]

(9)

Let

\[
\begin{aligned}
a(x, t) = \int_0^1 A'(su_1 + (1 - s)u_2) \, ds = \begin{cases}
\frac{A(u_1) - A(u_2)}{u_1 - u_2}, & u_1(x, t) \neq u_2(x, t), \\
A'(u_1), & u_1(x, t) = u_2(x, t),
\end{cases}
\end{aligned}
\]

9
the following approximated dual problem and Lemma 4.1 in [40]). For any smooth function \(0 \leq u \leq 39\), see also the comparison principle Lemma 3.4 in [41] on unbounded domain approach to complete this proof (see Theorem 6.5 in [32], Chapter 1.3 and 3.2 in ... line to line) is denoted by \(C\) and therefore \(\phi_{\|} = \phi \in C_{0}(\Omega)\) are locally Lipchitz continuous, there exists a constant \(C > 0\) such that \(|a|, |b|\) and \(|u_1|, |u_2| \leq C\). Henceforth, a generic positive constant (possibly changing from line to line) is denoted by \(C\). According to the strictly increasing property of \(A(s)\) and the boundedness of \(u_1, u_2\), there exists a constant \(L(\gamma) > 0\) such that

\[a(x, t) \geq L(\gamma), \quad \text{for all} \ (x, t) \in F_{\gamma},\]

and therefore

\[|c_{\phi}^{\eta}| \leq L(\delta)^{-\frac{1}{2}}|b| \leq L(\delta)^{-\frac{1}{2}}C =: K(\delta).\]

We employ the standard duality proof method or the approximate Hohmgren’s approach to complete this proof (see Theorem 6.5 in [32], Chapter 1.3 and 3.2 in [39], see also the comparison principle Lemma 3.4 in [41] on unbounded domain and Lemma 4.1 in [40]). For any smooth function \(0 \leq \psi(x, t) \in C^{2}_{0}(\Omega)\), consider the following approximated dual problem

\[
\begin{aligned}
-\varphi_t - (\eta + a_{\epsilon}(x, t))\Delta \varphi - c_{\epsilon,\phi}^{\eta}(x, t)(\eta + a_{\epsilon}(x, t))^\frac{1}{2} \cdot \nabla \psi = \psi, \quad (x, t) \in Q_T, \\
\frac{\partial \varphi}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
\varphi(x, T) = 0, \quad x \in \Omega,
\end{aligned}
\]
where \( \eta > 0, \delta > 0, \epsilon > 0, a_\epsilon \) is a smooth approximation of \( a \) in \( L^4(Q_T) \), \( a_\epsilon \geq a \), and \( c^{\eta}_{\delta,\epsilon}(x,t) \) is a smooth approximation of \( \psi \) in \( L^4(Q_T) \). Here we note that (11) is a standard parabolic problem as the initial data is imposed at the end time \( t = T \). Therefore, it has a smooth solution \( \varphi \geq 0 \). Maximum principle shows the boundedness of \( \varphi \) such that \( 0 \leq \varphi \leq C(\psi) \). Then we get from (10) and (11) the estimate

\[
\iint_{Q_T} (u_1 - u_2)\psi \, dx \, dt \geq - \iint_{Q_T} |u_1 - u_2|a_\epsilon|\Delta \varphi| \, dx \, dt \\
- \eta \iint_{Q_T} |u_1 - u_2| |\Delta \varphi| \, dx \, dt \\
- \int \int_{Q_T} |u_1 - u_2| |c^{\eta}_{\delta,\epsilon}(\eta + a_\epsilon)b_{\Delta \psi}| \, dx \, dt \\
=: - I_1 - I_2 - I_3. \tag{12}
\]

Next, we need the a priori estimate on \((\eta + a_\epsilon)|\Delta \varphi|^2\). We multiply the equation (11) by \(-\Delta \varphi\). Integrating over \( Q_T \) yields

\[
\iint_{Q_T} \varphi_t \Delta \varphi \, dx \, dt + \iint_{Q_T} (\eta + a_\epsilon)(\Delta \varphi)^2 \, dx \, dt \\
\leq \iint_{Q_T} |c^{\eta}_{\delta,\epsilon}(\eta + a_\epsilon)^{\frac{3}{2}}| \nabla \varphi||\Delta \varphi| \, dx \, dt + \iint_{Q_T} \psi \Delta \varphi \, dx \, dt \\
\leq \frac{1}{4} \iint_{Q_T} (\eta + a_\epsilon)(\Delta \varphi)^2 \, dx \, dt + \iint_{Q_T} |c^{\eta}_{\delta,\epsilon}|^2 |\nabla \varphi|^2 \, dx \, dt + \iint_{Q_T} |\Delta \psi| |\varphi| \, dx \, dt \\
\leq \frac{1}{4} \iint_{Q_T} (\eta + a_\epsilon)(\Delta \varphi)^2 \, dx \, dt + (K(\delta))^2 \iint_{Q_T} |\nabla \varphi|^2 \, dx \, dt + C(\psi).
\]

Using \( \varphi(x, T) = 0 \) and \( \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \), we have

\[
\iint_{Q_T} \varphi_i \Delta \varphi \, dx \, dt = - \iint_{Q_T} \nabla \varphi \cdot \nabla \varphi \, dx \, dt = - \frac{1}{2} \iint_{Q_T} \frac{\partial}{\partial t} |\nabla \varphi|^2 \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} |\nabla \varphi(x, 0)|^2 \, dx \geq 0,
\]

and

\[
\iint_{Q_T} |\nabla \varphi|^2 \, dx \, dt = \iint_{Q_T} \nabla \varphi \cdot \nabla \varphi \, dx \, dt = - \iint_{Q_T} \varphi \Delta \varphi \, dx \, dt \\
\leq \frac{1}{4(K(\delta))^2} \iint_{Q_T} (\eta + a_\epsilon)(\Delta \varphi)^2 \, dx \, dt + \eta^{-1} (K(\delta))^2 C(\psi).
\]
Therefore,

$$(K(\delta))^2 \iint_{Q_T} |\nabla \varphi|^2 dxdt + \iint_{Q_T} (\eta + a_\epsilon)(\Delta \varphi)^2 dxdt \leq C(\psi)(K(\delta))^4 \eta^{-1},$$

(13)

and

$$\|\Delta \varphi\|_{L^2(Q_T)} \leq (C(\psi)(K(\delta))^4 \eta^{-2})^{\frac{1}{2}} \leq C(\psi)(K(\delta))^2 \eta^{-1}.$$

It follows that

$$I_1 = \iint_{Q_T} |u_1 - u_2| |a - a_\epsilon| |\Delta \varphi| dxdt$$

$$\leq C\|\Delta \varphi\|_{L^2(Q_T)} \|a - a_\epsilon\|_{L^2(Q_T)} \leq C(\psi)(K(\delta))^2 \eta^{-1} |a - a_\epsilon|_{L^2(Q_T)},$$

which converges to zero if we let $\epsilon \to 0$. We can estimate $I_2$ as follows

$$I_2 = \eta \iint_{Q_T} |u_1 - u_2| |\Delta \varphi| dxdt$$

$$\leq \eta \iint_{G_\gamma} |u_1 - u_2| |\Delta \varphi| dxdt + \eta \iint_{F_\gamma} |u_1 - u_2| |\Delta \varphi| dxdt$$

$$\leq \gamma \iint_{G_\gamma} \eta |\Delta \varphi| dxdt + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} \iint_{F_\gamma} a_\epsilon^2 |\Delta \varphi| dxdt$$

$$\leq \gamma \iint_{G_\gamma} \eta |\Delta \varphi| dxdt + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} \iint_{F_\gamma} a_\epsilon^2 |\Delta \varphi| dxdt$$

$$\leq C\gamma \eta^{\frac{1}{2}} C(\psi)(K(\delta))^2 \eta^{\frac{1}{2}} + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} C(\psi)(K(\delta))^2 \eta^{-\frac{1}{2}}$$

$$= \gamma C(\psi)(K(\delta))^2 + \eta^{\frac{1}{2}} C(\psi)(K(\delta))^2 / L(\gamma)^{\frac{1}{2}}.$$
We note that

\[ c^{\eta}_{\delta, \varepsilon}(\eta + a) \to c^{\eta}_\delta(\eta) = \begin{cases} 0, & (x, t) \in G_\delta, \\ b(x, t), & (x, t) \in F_\delta, \end{cases} \]

almost everywhere and also in \( L^2(Q_T) \). It follows that

\[ \limsup_{\varepsilon \to 0} I_3 \leq C \delta \int_{G_\delta} |\nabla \varphi| dx dt. \]

We leave the uniform \( L^1 \) estimate of \( \|\nabla \varphi\|_{L^1(Q_T)} \leq C(\psi) \) to the next lemma (Lemma 3.2), and we combine the above estimates to find

\[ \limsup_{\varepsilon \to 0} (I_1 + I_2 + I_3) \leq \gamma C(\psi)(K(\delta))^2 + C(\psi)K(\delta)^2 / L(\gamma)^{\frac{1}{2}} + C(\psi)\delta. \]

Now we conclude according to (12) that

\[ \iint_{Q_T} (u_1 - u_2) \psi dx dt \geq -\gamma C(\psi)(K(\delta))^2 + C(\psi)K(\delta)^2 / L(\gamma)^{\frac{1}{2}} + C(\psi)\delta, \]

for any given \( \delta > 0, \eta > 0, \gamma > 0 \) and \( \psi \geq 0 \), which yields that

\[ \iint_{Q_T} (u_1 - u_2) \psi dx dt \geq 0, \]

by taking \( \eta \to 0 \), then \( \gamma \to 0 \), and at last \( \delta \to 0 \). Since \( 0 \leq \psi \in C^2_0(Q_T) \) is arbitrary selected, we see that \( u_1 \geq u_2 \) almost everywhere on \( Q_T \). \( \square \)

**Lemma 3.2.** Let \( \varphi \) be the solution of the approximated dual problem (11) in the proof of Lemma 3.1. Then there holds

\[ \sup_{t \in (0, T)} \int_{Q_T} |\nabla \varphi(x, t)| dx \leq \int_{Q_T} |\nabla \psi| dx dt. \]

**Proof.** Since \( \varphi \) is smooth enough, \( \varphi(x, T) = 0 \) on \( \Omega \) and \( \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \), we take the gradient of (11) and then multiply it by \( |\nabla \varphi|^{\beta - 1} \nabla \varphi \) with \( \beta \in (0, 1) \), integrate over \( Q_{t,T} = \Omega \times(t, T) \), to find

\[
\frac{1}{\beta + 1} \int_{\Omega} |\nabla \varphi(x, t)|^{\beta + 1} dx + \beta \int_{Q_{t,T}} (\eta + a) |\Delta \varphi|^2 |\nabla \varphi|^{\beta - 1} dx dt \\
= -\beta \iint_{Q_{t,T}} c^{\eta}_{\delta, \varepsilon}(\eta + a) \cdot \nabla \varphi |\nabla \varphi|^{\beta - 1} \Delta \varphi dx dt + \iint_{Q_{t,T}} \nabla \psi \cdot |\nabla \varphi|^{\beta - 1} \nabla \varphi dx dt
\]
\[ \leq \beta \int_{Q_T} (\eta + a_\epsilon) |\Delta \varphi|^2 |\nabla \varphi|^{\beta-1} dx dt + \beta \int_{Q_T} |\nabla v|^2 |\nabla \varphi|^\beta dx dt \]
\[ + \int_{Q_T} |\nabla \psi||\nabla \varphi|^\beta dx dt. \] (14)

According to (13), we see that
\[ \int_{Q_T} |\nabla \psi||\nabla \varphi|^\beta dx dt \leq \int_{Q_T} (K(\delta))^2 (1 + |\nabla \varphi|^2) dx dt \leq C(\psi)(K(\delta))^4 \eta^{-1}, \]
and
\[ \limsup_{\beta \to 0} \int_{Q_T} |\nabla \psi||\nabla \varphi|^\beta dx dt \leq \int_{Q_T} |\nabla \psi| dx dt, \]
by the dominated convergence theorem. Now we let \( \beta \) tends to zero, and (14) implies that
\[ \int_{\Omega} |\nabla \varphi(x, t)| dx \leq \int_{Q_T} |\nabla \psi| dx dt, \]
for all \( t \in (0, T) \). The proof is completed. \( \Box \)

The comparison principle together with specially constructed weak lower and upper solutions are used to show the propagating properties. Hence we define the following weak lower and upper solutions of the first equation in (2).

**Definition 3.1 (Weak lower and upper solutions).** A function \( g(x, t) \) is said to be a weak lower (or upper) solution of the first equation in (2) on \( Q_T \) corresponding to the initial value \( u_0 \) and a given function \( v \) such that \( \nabla v \in L^\infty(\Omega_T) \), if \( 0 \leq g \in L^\infty(Q_T) \), \( \nabla g^m \in L^2(Q_T) \), and it satisfies the following differential inequality

\[
\begin{cases}
\frac{\partial g}{\partial t} \leq (\geq) \Delta g^m - \nabla \cdot (g \nabla v), & x \in \Omega, t \in (0, T), \\
\frac{\partial g^m}{\partial n} - g \nabla v \cdot n \leq (\geq) 0, & x \in \partial \Omega, t \in (0, T), \\
g(x, 0) \geq 0, & g(x, 0) \leq (\geq) u_0(x), & x \in \Omega,
\end{cases}
\]

where the first two inequality is satisfied in the following sense
\[
\int_{Q_T} g \varphi dx dt + \int_{\Omega} g(x, 0) \varphi(x, 0) dx 
\geq (\leq) \int_{Q_T} \nabla g^m \cdot \nabla \varphi dx dt - \int_{Q_T} \nabla \psi \cdot \nabla \varphi dx dt,
\]
holds for all test functions \( 0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \) with \( \varphi(x, T) = 0 \) on \( \Omega \).
Lemma 3.3 (Comparison principle). Let \((u, v)\) be a globally bounded weak solution of (2). If \(g(x, t)\) is a weak lower (or upper) solution of the first equation in (2) on \(Q_T\), then
\[ u(x, t) \geq (\leq) g(x, t), \quad \forall (x, t) \in Q_T. \]

**Proof.** This is a simple corollary of comparison principle Lemma 3.1. 

3.2 Regularity of Hölder continuity

In order to show the propagation properties of the degenerate chemotaxis system (2), we need to know the existence, global boundedness, regularity and large time behavior of its solutions.

We recall the existence and the global boundedness of solutions to the degenerate chemotaxis model (2).

Lemma 3.4 (\cite{11}). Assume that \(u_0 \in L^\infty(\Omega), \nabla u_0^m \in L^2(\Omega), v_0 \in W^{2,\infty}(\Omega), u_0, v_0 \geq 0\) and \(m > 1\), the spacial dimension \(N = 3\). Then the problem (2) admits a nonnegative global bounded weak solution \((u, v)\) with
\[
\sup_{t \in (0, \infty)} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)}) \leq C,
\]
\[
\sup_{t \in (0, \infty)} \int_\Omega |\nabla u|^2 dx + \sup_{t \in (0, \infty)} \|u_0^m\|_{W^{2,1}_p(\Omega \times (t, t+1))} \leq C,
\]
\[
\sup_{t \in (0, \infty)} \|v\|_{W^{2,1}_p(\Omega \times (t, t+1))} \leq C(p), \quad \forall p > 1.
\]
Furthermore,
\[
\lim_{t \to \infty} \|v\|_{L^\infty(\Omega)} = 0, \quad \lim_{t \to \infty} \|u - \bar{u}\|_{L^p(\Omega)} = 0, \quad \forall p > 1,
\]
where \(\bar{u} = \frac{1}{|\Omega|} \int_\Omega u_0 dx > 0\).

**Remark 3.1.** The same global boundedness and asymptotic behavior results hold for the lower spatial dimensional case \(N = 1, 2\).

**Remark 3.2.** We note that the boundedness of \(\|u\|_{L^\infty(Q_T)}\) and \(\|v\|_{L^\infty(Q_T)}\) is insufficient for the boundedness of \(\|\Delta v\|_{L^\infty(Q_T)}\) according to the strong theory of the second equation in (2). Hence the \(W^{2,1}_p\) estimate for \(p = \infty\) is not obtained in the above Lemma.

**Remark 3.3.** One of the basic features for the degenerate diffusion equations, such as the porous medium equation, is the property of finite speed of propagation. Therefore, the first component \(u\) may not have positive minimum for some time.
$t > 0$. For the large time behavior, it is proved in Lemma 3.4 that $u(x, t)$ converges to $\bar{u}$ in $L^p(\Omega)$ for $p < \infty$, while the $L^\infty(\Omega)$ and some other more regular convergence are not deduced.

In a special case that $v_0 \equiv 0$, we see that $v(x, t) \equiv 0$ and $u$ satisfies the porous medium equation. The Barenblatt solution (7) of the porous medium equation shows that the best regularity of the first equation in (2) is no better than Hölder continuous $C^{1/2}((Q_T))$ (for $m > 2$) even for the one spatial dimensional case $N = 1$.

In what follows, we will show the Hölder continuous of $u$. That is, the uniform continuity $C^{1/2}$ shows that the best regularity of the first equation in (2) is no better than Hölder continuous $C^{1/2}((Q_T))$.

Lemma 3.5. Let $N = 1$ and $(u, v)$ be the globally bounded weak solution of (2). Then there exists a constant $C > 0$ such that

$$\sup_{t \in (0, \infty)} \left( \|u^m(\cdot, t)\|_{C^{1/2}(\Omega)} + \|v(\cdot, t)\|_{C^{1/2m}(\Omega)} \right) \leq C.$$  

Proof. According to Lemma 3.4 $\|\nabla u^m(\cdot, t)\|_{L^\infty(\Omega)}$ is uniformly bounded. The Sobolev embedding theorem for one dimensional case implies the uniform boundedness of $\|u^m(\cdot, t)\|_{C^{1/2m}(\Omega)}$.

We assert that for $m > 1$,

$$|a - b|^m \leq C(M)|a^m - b^m|, \quad \forall a, b \in [0, M].$$

This is a simple result of calculus. Actually, we can choose $C(M) = 1$. Therefore,

$$\left( \frac{|u(x_1, t) - u(x_2, t)|^m}{|x_1 - x_2|^{1/(2m)}} \right) \leq C(\sup_{t \in (0, \infty)} \|u\|_{L^\infty(\Omega)}) \left( \frac{|u^m(x_1, t) - u^m(x_2, t)|}{|x_1 - x_2|^{1/2}} \right), \quad x_1 \neq x_2.$$  

That is, the uniform $C^{1/2}$ regularity of $u^m(\cdot, t)$ implies the uniform $C^{1/(2m)}$ regularity of $u(\cdot, t)$.

The following continuity of $\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)}$ and boundedness of $\|\Delta v(\cdot, t)\|_{L^\infty(\Omega)}$ will be used to formulate varies types of upper and lower solutions in the next section.

Lemma 3.6. Let $(u, v)$ be the globally bounded weak solution of (2) such that

$$\sup_{t \in (0, \infty)} \|u(\cdot, t)\|_{C^{1/2m}(\Omega)} \leq C,$$

and $v_0 \in C^{2, \alpha_0}(\Omega)$ for some $\alpha_0 \in (0, 1)$, $\frac{\partial v_0}{\partial n} = 0$ on $\partial \Omega$. Then $\nabla v(\cdot, t)$ is continuous in the $\|\cdot\|_{L^\infty(\Omega)}$ norm with respect to time and there exist $\alpha \in (0, 1)$ and a constant $C(T, \delta) > 0$ such that

$$\|\Delta v(x, t)\|_{C^{\alpha}(\overline{\Omega} \times [0, T])} \leq C(T, \delta),$$

where $\Omega_\delta = \{x \in \Omega; \text{dist}(x, \partial \Omega) > \delta\}$.  

16
Proof. Since $\|v\|_{W^{1,1}_p(\Omega \times (t_0, t_1))}$ is uniformly bounded for $p > 1$ in Lemma 3.4, we see that $\sup_{t \in (0, \infty)} \|v(\cdot, t)\|_{C^\alpha(\Omega)} \leq C$ for some $\beta \in (0, 1)$. Therefore,

$$\sup_{t \in (0, \infty)} \|uv(\cdot, t)\|_{C^\alpha(\Omega)} \leq C$$

for some $\alpha \in (0, 1)$. Indeed, we can choose $\alpha = \min\{1/(2m), \beta\}$. The Schauder theory via Campanato space theory in [17] implies the interior Hölder continuity of $\Delta v$ with respect to space and time, and the Hölder continuity of $v_t$ with respect to space (the Hölder continuity of $v_t$ with respect to time is insufficient). \qed

For large time behavior, we present the following regularity.

Lemma 3.7. Let $(u, v)$ be the globally bounded weak solution of (2). Then

$$\lim_{t \to \infty} \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} = 0.$$ 

Proof. Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in $\Omega$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary condition. Then the solution $v$ can be expressed as follows

$$v(x, t) = e^{t\Delta}v_0(x) - \int_0^t e^{(t-s)\Delta}(uv)(x, s)ds, \quad t \geq t_0 \geq 0.$$ 

According to the $L^p - L^q$ estimates for the Neumann heat semigroup (see for example [36]),

$$\|\nabla v(x, t)\|_{L^\infty(\Omega)} \leq \|\nabla e^{t\Delta}v_0(x)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}(uv)(x, s)\|_{L^\infty(\Omega)}ds$$

$$\leq C(1 + t^{-\frac{1}{2}})e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + \int_0^t C(1 + (t - s)^{-\frac{1}{2}})e^{-\lambda_1 (t-s)}\|uv(\cdot, s)\|_{L^\infty(\Omega)}ds$$

$$\leq C(1 + t^{-\frac{1}{2}})e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + C\int_0^t e^{-\lambda_1 (t-s)}ds$$

$$+ C\int_{t-1}^t (1 + (t - s)^{-\frac{1}{2}})ds \sup_{\tau \in (t-1, t)} \|v(\cdot, \tau)\|_{L^\infty(\Omega)},$$

which tends to zero since $\|u(\cdot, t)\|_{L^\infty(\Omega)}$, $\|v(\cdot, t)\|_{L^\infty(\Omega)}$ are uniformly bounded and $\|v(\cdot, t)\|_{L^\infty(\Omega)}$ tends to zeros as $t \to \infty$ from Lemma 3.4. \qed

Lemma 3.8. Let the conditions in Lemma 3.6 be valid. Then

$$\lim_{t \to \infty} \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} = 0.$$
Proof. We rewrite \( v = v_1 + w \) such that
\[
\begin{align*}
\begin{cases}
    v_t = \Delta v_1, & x \in \Omega, \ t > 0, \\
    v_1(x, 0) = v_0(x), & x \in \Omega, \\
    \frac{\partial v_1}{\partial n} = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
    w_t = \Delta w - uv, & x \in \Omega, \ t > 0, \\
    w(x, 0) = 0, & x \in \Omega, \\
    \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
\end{align*}
\]

The Neumann heat semigroup theory shows \( \lim_{t \to \infty} \| \Delta v_1(\cdot, t) \|_{L^\infty(\Omega)} = 0 \). We note that
\[
\|(uv)(\cdot, t)\|_{C^\alpha(\Omega)} \leq \|u(\cdot, t)\|_{L^\infty(\Omega)}\|v(\cdot, t)\|_{C^\alpha(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\|u(\cdot, t)\|_{C^\alpha(\Omega)} \to 0,
\]
as \( t \) tends to infinity since \( \lim_{t \to \infty} \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} = 0 \) according to Lemma 3.7 and \( \|u(\cdot, t)\|_{C^\alpha(\Omega)} \) are uniformly bounded in Lemma 3.6 for some \( \alpha \in (0, 1) \). The Schauder theory in [17] shows the Hölder continuity
\[
\|\Delta w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 \sup_{s \in [t/2, t]} \|(uv)(\cdot, s)\|_{C^\alpha(\Omega)} + C_2(t) \sup_{s \in (0, \infty)} \|(uv)(\cdot, s)\|_{C^\alpha(\Omega)},
\]
where \( C_1 > 0 \) is a constant and \( C_2(t) \) decays to zeros as \( t \) tends to infinity. \( \square \)

4 Propagation properties: shrinking versus expanding

This section is devoted to the study of the propagating properties of bacteria cells and the large time behavior of the weak solution \((u, v)\) to the problem (2). In contrast with the heat equation, it is known that the porous medium equation has the property of finite speed of propagation. Therefore, the first component \( u \) may not have positive minimum for some time \( t > 0 \). We use the comparison principle together with weak lower solutions.

Our interest lies in the propagating properties of the cell invasions. Let us first focus on the waiting time and initial shrinking of the compact support. Our approach is the combination of the comparison principle Lemma 5.3 and weak lower and upper solutions with compact support.

4.1 Initial shrinking caused by the chemotaxis

The Barenblatt solution [7] of the classical porous medium equation indicates the slow diffusion with finite speed of expanding support; while the chemotaxis may
cause backward diffusion, i.e. the aggregation, which in competition with the slow diffusion results in an initial shrinking of the support provided specified structures of the signal concentration.

We consider a typical situation in which the cells are concentrated in a compact support and the signal concentration has the aggregation effect. Specifically speaking, assume that

\[
\begin{align*}
\text{supp } u_0 \subset B_{R_0}(x_0) \subset \Omega, \quad u_0 & \leq K_0(R_0^2 - |x - x_0|^2)^{d_0}, \quad x \in B_{R_0}(x_0), \\
\nabla v_0 \cdot (x - x_0) & \leq -\mu |x - x_0|^2, \quad x \in B_{R_0}(x_0),
\end{align*}
\]

(15)

for some \( x_0 \in \Omega \) and positive constants \( d_0 \geq 1/(m - 1) \), and \( R_0, K_0, \mu > 0 \).

We construct self similar upper and lower solution with compact support to show the propagating property. We note that for the degenerate porous medium type equation and the self similar function of the form \( g = [(1 - |x|^2)_+]^d \) with \( md > 1 \), we can check that \( \nabla g^m \) is continuous and \( \Delta g^m \in L^q(\Omega) \) for some \( q > 1 \). This shows that the differential inequality for an upper (or lower) solution only need to be valid almost everywhere, without the possible Radon measures on the boundary of its support, which is completely different from the uniform parabolic cases.

**Lemma 4.1.** Let the conditions in Lemma 3.6 be valid with the initial values satisfying (15) and \( \chi \mu > \frac{4m}{m-1} K_0^{m-1} \max\{1, R_0^{2(m-1)d_0-1}\} \). Define a function

\[
g(x, t) = \epsilon (\tau + t)\eta \left[ \left( \eta^2 - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)_+ \right]^d, \quad x \in \Omega, \quad t \geq 0,
\]

where \( d = 1/(m - 1) \), \( \beta, \sigma \in \mathbb{R}, \epsilon > 0, \eta > 0, \tau > 0 \). Then by appropriately selecting \( \beta < 0, \sigma > 0, \epsilon, \eta \) and \( \tau \), the support of \( g(x, t) \) is contained in \( \Omega \) and shrinks for \( t \in (0, t_0) \) with some \( t_0 > 0 \) and the function \( g(x, t) \) is an upper solution of the first equation in (2) on \( \Omega \times (0, t_0) \) corresponding to \( v(x, t) \) and the initial date \( u_0 \). Therefore, \( u(x, t) \leq g(x, t) \) and there exist a family of shrinking open sets \( \{A(t)\}_{t \in (0, t_0)} \) such that

\[
\text{supp } u(\cdot, t) \subset \overline{A(t)} \subset \Omega, \quad t \in (0, t_0),
\]

and \( \partial A(t) \) has a finite derivative with respect to \( t \).

**Proof.** For simplicity, we let

\[
h(x, t) = \left( \eta^2 - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)_+, \quad x \in \Omega, \quad t \geq 0.
\]
Straightforward computation shows that $g_t = \sigma \varepsilon (\tau + t)^{\sigma-1} h^d + \varepsilon (\tau + t)^{\sigma} dh^{d-1} \frac{\beta |x|^2}{(\tau + t)^{\beta+1}}$, 

$\nabla g = - \varepsilon (\tau + t)^{\sigma} dh^{d-1} \frac{2x}{(\tau + t)^\beta}$,

$\nabla g^m = - \varepsilon^m (\tau + t)^{m\sigma} mdh^{m-1} \frac{2x}{(\tau + t)^\beta}$,

$\Delta g^m = \varepsilon^m (\tau + t)^{m\sigma} md(m - 1)h^{m-2} \frac{4|x|^2}{(\tau + t)^{2\beta}} - \varepsilon^m (\tau + t)^{m\sigma} mdh^{m-1} \frac{2N}{(\tau + t)^\beta}$,

for all $x \in A(t)$ and $t > 0$. According to the initial condition (15) and the regularity result Lemma 3.5 we see that at the initial time

$\nabla g(x, 0) \cdot \nabla v(x, 0) = \nabla g(x, 0) \cdot \nabla v_0(x)$

$= - \varepsilon \tau^\sigma dh^{d-1} \frac{2x}{\tau^\beta} \cdot \nabla v_0(x) \geq \varepsilon \tau^\sigma dh^{d-1} 2\mu |x|^2$,

and there exists a $\hat{t} > 0$ by the continuity such that

$\nabla v(x, t) \cdot x \leq \frac{\mu}{2} |x|^2$, \quad $x \in B_{R_0} \setminus B_{R_0/2}$, \quad $t \in [0, \hat{t}]$,

$\nabla v(x, t) \cdot x \leq \frac{\mu}{2} R_0^2$, \quad $x \in B_{R_0/2}$, \quad $t \in [0, \hat{t}]$.

Therefore,

$\nabla g(x, t) \cdot \nabla v(x, t) = - \varepsilon (\tau + t)^{\sigma} dh^{d-1} \frac{2x}{(\tau + t)^\beta} \cdot \nabla v(x, t)$

$\geq \varepsilon (\tau + t)^{\sigma-\beta} dh^{d-1} \mu |x|^2$, \quad $x \in B_{R_0} \setminus B_{R_0/2}$, \quad $t \in [0, \hat{t}]$. \quad (16)

Let $\tau > 0$ to be determined and

$\eta^2 = \frac{R_0^2}{\tau^\beta}$, \quad $t_0 = \min(\tau, \hat{t})$. \quad (17)

According to the definition of $g(x, t)$, we see that $A(0) = B_{R_0}(0)$, supp $u_0 \subset A(0) \subset \Omega$, and $A(t) \subset B_{R_0}(0) \subset \Omega$ for $t \in [0, t_0]$. Therefore, $\frac{\partial g}{\partial n} = 0$ and $\frac{\partial g}{\partial n} = 0$ on $\partial \Omega$ for all $t \in (0, t_0)$, and

$g(x, 0) = \varepsilon \tau^\sigma \left( |x|^2 - \frac{|x|^2}{\tau^\beta} \right)^d = \varepsilon \tau^\sigma \left( \frac{R_0^2}{\tau^\beta} - \frac{|x|^2}{\tau^\beta} \right)^d \cdot 1_{B_{R_0}(0)}$
provided that
\[ \varepsilon \tau^{\sigma-d}\beta \geq K_0 \max\{1, R_0^{2(\delta-d)}\}. \] (18)

In order to find a weak upper solution \( g \), we only need to check the following differential inequality on \( A(t) \)
\[ \frac{\partial g}{\partial t} \geq \Delta g - \chi \nabla \cdot (g\nabla v) = \Delta g - \chi \nabla g \cdot \nabla v - \chi g \Delta v, \quad x \in A(t), \ t \in (0, t_0). \] (19)

We denote \( C_1 = ||\nabla v||_{L^\infty(\Omega \times [0,1])} \) and \( C_2 = ||\Delta v||_{L^\infty(\Omega \times [0,1])} \) for convenience, since they are bounded according to Lemma 3.6. A sufficient condition of inequality (19) is
\[
\sigma \varepsilon (\tau + t)^{\sigma-1}h^d + \varepsilon (\tau + t)^{\sigma}dh^{d-1}\left[ \frac{\beta|x|^2}{(\tau + t)^{\beta+1}} + \frac{2N}{(\tau + t)^{\beta}} \chi \nabla g \cdot \nabla v \right] \\
\geq \varepsilon (\tau + t)^{\sigma}mdh^{md-1}\left[ \frac{4|x|^2}{(\tau + t)^{2\beta}} + C_2 \chi \varepsilon (\tau + t)^{\sigma}h \right].
\] (20)

for all \( x \in A(t), \ t \in (0, t_0). \) As we have chosen \( d = 1/(m-1) \), we rewrite (20) into
\[
\sigma \varepsilon (\tau + t)^{\sigma-1}h + \frac{\varepsilon \beta}{m-1}(\tau + t)^{\sigma}h^{d-1}\frac{|x|^2}{(\tau + t)^{\beta+1}} \\
+ 2N \frac{m}{m-1} \varepsilon (\tau + t)^{\sigma}h^{d-1}\frac{|x|^2}{(\tau + t)^{\beta}} + \chi \varepsilon (\tau + t)^{\sigma}h^{d-1}\chi \nabla g \cdot \nabla v \\
\geq \frac{m}{(m-1)^2} \varepsilon (\tau + t)^{\sigma}h^{d-1}\frac{4|x|^2}{(\tau + t)^{\beta+1}} + C_2 \chi \varepsilon (\tau + t)^{\sigma}h \] (21)

for all \( x \in A(t), \ t \in (0, t_0). \) For simplicity, we denote (21) by \( LHS \geq RHS. \)

Now, we give sufficient conditions of (21) to be valid on \( B_{R_0/2} \) and \( B_{R_0} \) respectively (Note that \( A(t) \subset B_{R_0}(0) \) for \( t \in (0, t_0) \) as \( \beta < 0 \). For \( x \in (B_{R_0} \setminus B_{R_0/2}) \cap A(t) \) and \( t \in (t_0, t_0) \) we have according to the estimate (16) that
\[
LHS - RHS \geq \sigma \varepsilon (\tau + t)^{\sigma-1}h + \frac{\varepsilon \beta}{m-1}(\tau + t)^{\sigma}h^{d-1}\frac{|x|^2}{(\tau + t)^{\beta+1}} \\
+ 2N \frac{m}{m-1} \varepsilon (\tau + t)^{\sigma}h^{d-1}\frac{|x|^2}{(\tau + t)^{\beta}} + \chi \varepsilon (\tau + t)^{\sigma-\beta}d\mu|x|^2 \\
- \frac{m}{(m-1)^2} \varepsilon (\tau + t)^{\sigma}h^{d-1}\frac{4|x|^2}{(\tau + t)^{\beta+1}} - C_2 \chi \varepsilon (\tau + t)^{\sigma}h
\]
\begin{align*}
\geq & \left(\sigma + 2N \frac{m}{m-1} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} + C_2 \chi (\tau + t) \right) e(\tau + t)^{\sigma-1} h \\
& + \left( d \chi \mu + \frac{\beta}{m-1} (\tau + t)^{1-1} - \frac{4m}{(m-1)^2} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} \right) e(\tau + t)^{\sigma-1} |x|^2 \\
\geq & \left(\sigma + 2N \frac{m}{m-1} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} + C_2 \chi (\tau + t) \right) e(\tau + t)^{\sigma-1} h \\
& + \left( d \chi \mu + \frac{\beta}{m-1} (\tau + t)^{1-1} - \frac{4m}{(m-1)^2} e^{m-1} (2\tau)^{(m-1)\sigma-\beta} \right) e(\tau + t)^{\sigma-1} |x|^2.
\end{align*}

(22)

For $x \in (B_0/2) \cap A(t)$ and $t \in (t_0, t_0)$, we also have

\begin{align*}
LHS - RHS \geq & \left(\sigma + 2N \frac{m}{m-1} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} + C_2 \chi (\tau + t) \right) e(\tau + t)^{\sigma-1} h \\
& + \left( d \chi \mu + \frac{\beta}{m-1} (\tau + t)^{1-1} - \frac{4m}{(m-1)^2} e^{m-1} (2\tau)^{(m-1)\sigma-\beta} \right) e(\tau + t)^{\sigma-1} |x|^2 \\
& - d \chi \mu e(\tau + t)^{\sigma-1} R_0^2.
\end{align*}

(23)

Let $\beta \in [-2 \ln(4/3)/\ln 2, 0)$, i.e. $2^{\beta/2} \in [3/4, 1)$. For $t \in (0, t_0)$, we see that

$A(t) = B_{\eta(t+\eta/\beta)}(0) = B_{R_0(1+t/\eta^{\beta/2})}(0) \supset B_{2^{\beta/2}R_0}(0) \supset B_{3R_0/4}(0).$

Further if $x \in (B_0/2) \cap A(t) = B_{R_0/2}$ and $t \in (0, t_0) \subset (0, \tau)$,

$h(x, t) = \left(\eta^2 - \frac{|x|^2}{(\tau + t)^{\beta}}\right) = \frac{R_0^2}{\tau^\beta} - \frac{|x|^2}{(\tau + t)^{\beta}} \geq \frac{R_0^2}{\tau^\beta} - \frac{(R_0/2)^2}{(2\tau)^{\beta}} \geq \frac{5}{9} \frac{R_0^2}{\tau^\beta}.

Then (23) reads

\begin{align*}
LHS - RHS \geq & \left(\sigma + 2N \frac{m}{m-1} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} + C_2 \chi (\tau + t) \right) e(\tau + t)^{\sigma-1} h \\
& + \left( d \chi \mu + \frac{\beta}{m-1} (\tau + t)^{1-1} - \frac{4m}{(m-1)^2} e^{m-1} (2\tau)^{(m-1)\sigma-\beta} \right) e(\tau + t)^{\sigma-1} |x|^2 \\
& \geq \left[(\sigma + 2N \frac{m}{m-1} e^{m-1} (\tau + t)^{(m-1)\sigma-\beta} + C_2 \chi (\tau + t)\right] \frac{5}{9} \frac{R_0^2}{\tau^\beta} \\
& + \left( d \chi \mu + \frac{\beta}{m-1} (\tau + t)^{1-1} - \frac{4m}{(m-1)^2} e^{m-1} (2\tau)^{(m-1)\sigma-\beta} \right) \frac{(\tau + t)^{1-\beta}}{4} e(\tau + t)^{\sigma-1} R_0^2.
\end{align*}

(24)

Let $\varepsilon > 0, \beta \in [-2 \ln(4/3)/\ln 2, 0), \sigma > 0, \tau > 0, \eta > 0$ and $t_0 > 0$ be chosen
such that (17), (18) are valid and
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\sigma + 2N \frac{m}{m-1} e^{m-1} (m-1) & (m-1) \chi_{\mu} - 2C_2 \chi \tau \geq 0, \\
d\chi_{\mu} + \frac{\beta}{m-1} - \frac{4m}{(m-1)^2} e^{m-1} (2 \tau)^{(m-1) \chi_{\mu}} \geq 0, \\
(\tau + 2N \frac{m}{m-1} e^{m-1} (m-1) \chi_{\mu} - 2C_2 \chi \tau) \frac{5}{9 \tau^{\beta}} \\
\beta \frac{\tau - 2N \frac{m}{m-1} e^{m-1} (m-1) \chi_{\mu} - 2C_2 \chi \tau - 4d \chi_{\mu}}{4 (2 \tau)^{1-\beta}} \geq 0.
\end{array} \right.
\end{aligned}
\tag{25}
\]

We can fix \( \tau = 1, \eta \) and \( t_0 \) to be determined by (17), \( \varepsilon = K_0 \max \{1, R_0^{2(d_0 - d)} \} \) as (18) is valid, and \( \beta < 0 \) with \( |\beta| \) being sufficiently small such that the second inequality in (25) is true since \( \chi_{\mu} > \frac{4m}{m+K_0} \max \{1, R_0^{2(m-1)d_0-1} \} \), and at last we choose \( \sigma > 0 \) to be sufficiently large such that the first and the third inequalities are satisfied.

Now, (25) is valid for those parameters. Then according to the inequalities (22), (23), (24), we find that
\[
LHS - RHS \geq 0, \quad x \in B_{R_0} \cap A(t) = A(t), \quad t \in (0, t_0),
\]
which yields (19), (21), and then \( g(x, t) \) is an upper solution.

The comparison principle Lemma 3.3 implies that \( u(x, t) \leq g(x, t) \) for all \( x \in \Omega \) and \( t \in (0, t_0) \). Thus,
\[
\text{supp } u(\cdot, t) \subset \overline{A}(t) = \{ x \in \Omega; |x - x_0|^2 < \eta^2 (\tau + t)^{\beta} \}, \quad t \in (0, t_0),
\]
and
\[
\partial A(t) = \{ x \in \Omega; |x - x_0| = \eta (\tau + t)^{\beta} \}, \quad t \in (0, t_0),
\]
which has finite derivative with respect to \( t \). The family of sets \( \{ A(t) \}_{t \in (0, t_0)} \) is shrinking with respect to \( t \) since \( \beta < 0 \).

**Remark 4.1.** We compare the self similar weak upper solution \( g(x, t) \) in the proof of Lemma 4.4 to the Barenblatt solution of porous medium equation
\[
B(x, t) = (1 + t)^{-k} \left[ \left( 1 - \frac{k(m-1)}{2mN} \frac{|x|^2}{(1 + t)^{2k/N}} \right) \right]^{1/m},
\]
with \( k = 1/(m - 1 + 2/N) \). The Barenblatt solution \( B(x, t) \) is decaying at the rate \( (1 + t)^{-1/(m-1+2/N)} \) in \( L^\infty(\mathbb{R}^N) \) and the support is expanding at the rate \( (1 + t)^{k/N} \). Here, the upper solution is increasing at the rate \((\tau + t)^\sigma\) and its support is shrinking at the rate \((\tau + t)^{\beta/2} \). The increasing of \( g(x, t) \) makes it possible to be an upper solution, which is crucial in the proof.
4.2 Finite speed propagating and the exact propagating speed

We have proved that the compact support may shrink if the signal concentration satisfies a special structure such as (15). Now, we will show the finite speed propagating property without assuming the special structure on signal concentration. Assume that

\[ \text{supp} \ u_0 \subset B_{R_0}(x_0) \subset \Omega, \quad u_0 \leq K_0(R_0^2 - |x - x_0|^2)^{d_0}, \quad x \in B_{R_0}(x_0), \] (26)

for some \( x_0 \in \Omega \) and positive constants \( d_0 \geq 1/(m-1) \) and \( R_0, K_0 > 0 \).

**Lemma 4.2.** Let the conditions in Lemma 3.6 be valid with the initial values satisfying (26). Define a function

\[ g(x, t) = \varepsilon(\tau + t)^\sigma \left( \eta^2 - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)^d, \quad x \in \Omega, \ t \geq 0, \]

where \( d = 1/(m-1) \), \( \beta, \sigma \in \mathbb{R}, \varepsilon > 0, \eta > 0, \tau > 0 \). Then by appropriately selecting \( \beta > 0, \sigma > 0, \varepsilon, \eta, \text{ and } \tau \), the support of \( g(x, t) \) is contained in \( \Omega \) for \( t \in (0, t_0) \) with some \( t_0 > 0 \) and the function \( g(x, t) \) is an upper solution of the first equation in (2) on \( \Omega \times (0, t_0) \) corresponding to \( v(x, t) \) and the initial data \( u_0 \).

Therefore, \( u(x, t) \leq g(x, t) \) and there exist a family of open sets \( \{A(t)\}_{t \in (0, t_0)} \) such that

\[ \text{supp } u(\cdot, t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0, t_0), \]

and \( \partial A(t) \) has a finite derivative with respect to \( t \).

**Proof.** This proof is similar to the proof of Lemma 4.1 except there is no structure condition (16) and we need minor modifications. We still define \( h(x, t) \) and \( A(t) \) as in the proof of Lemma 4.1 and we assume \( x_0 = 0 \) for simplicity. Let

\[ \eta^2 = \frac{R_0^2}{\tau^\beta}, \quad \varepsilon \tau^{\sigma-d_0} \geq K_0 \max\{1, R_0^{2(d_0-d)}\}, \] (27)

and \( C_1, C_2 \) be defined as in the proof of Lemma 4.1. We need to check the differential inequality (19) (i.e., (20)). A sufficient condition of (20) is

\[
\begin{align*}
\sigma \varepsilon(\tau + t)^{\sigma-1}h + \frac{\varepsilon \beta}{m-1}(\tau + t)^\sigma \frac{|x|^2}{(\tau + t)^{\beta+1}} + 2N \frac{m}{m-1} \varepsilon^m(\tau + t)^{m \sigma} \frac{h}{(\tau + t)^\beta} - C_1 \varepsilon(\tau + t)^{\sigma-\beta} \frac{2|x|}{m-1} \\
\geq \frac{m}{(m-1)^2} \varepsilon^m(\tau + t)^{m \sigma} \frac{4|x|^2}{(\tau + t)^{2 \beta}} + C_2 \varepsilon(\tau + t)^{\sigma} h,
\end{align*}
\] (28)
for all \( x \in A(t), t \in (0, t_0) \). For simplicity, we denote (28) by LHS \( \geq \) RHS.

According to (29), \( B_{R_0}(0) \subset \Omega \), there exists a \( R > R_0 \) such that \( B_{R_0}(0) \subset B_R(0) \subset \subset \Omega \). Let \( \delta t > 0 \) depending on \( \beta \) and \( \tau \) such that

\[
\left( 1 + \frac{\delta t}{\tau} \right)^\beta \leq \frac{R^2}{R_0^2}.
\]

Let \( t_0 = \min(\tau, \delta t) \). We see that for \( t \in (0, t_0) \),

\[
\sup \sigma g(x, t) = \sup |t_0(x, t)| = \sup B_{(t_0 + t)^\beta/2} \subset \sup B_{R_0/2} \subset \sup B_R \subset \sup \Omega = B_R \subset \sup \Omega.
\]

Then \( \frac{\partial \sigma}{\partial m} = 0 \) and \( \frac{\partial \sigma}{\partial m} = 0 \) on \( \partial \Omega \) for all \( t \in (0, t_0) \). For \( x \in A(t) \setminus B_{R_0/2} \) and \( t \in (0, t_0) \), we have

\[
\begin{align*}
LHS - RHS & \geq \sigma e(t + t)^{\beta - 1} h + \frac{e \beta}{m - 1} (t + t)^{\tau} \frac{|x|^2}{(t + t)^{\beta + 1}} + 2N \frac{m}{\lambda - 1} e^{m - 1} (t + t)^{\lambda - 1} h \\
& \quad - C_1 |\chi e(t + t)^{\sigma - \beta} h |^2 (t + t)^{\beta + 1} \frac{1}{(m - 1)R_0} \\
& \quad - C_2 |\chi e(t + t)^{\sigma} h |^2 \frac{1}{(m - 1)R_0} \\
& \quad - 4C_1 |\chi e(t + t)^{\sigma - \beta} h |^2 \\
& \quad - 4C_2 |\chi e(t + t)^{\sigma} h |^2 \\
& \quad - 4C_1 |\chi e(t + t)^{\sigma - \beta} h |^2 \\
& \quad - 4C_2 |\chi e(t + t)^{\sigma} h |^2.
\end{align*}
\]

We note that \( B_{R_0/2} \subset B_{R_0} \subset A(t) \) for \( t \in (0, t_0) \) since \( \beta > 0 \). For \( x \in (B_{R_0/2}) \cap A(t) \) and \( t \in (t, t_0) \), we find that

\[
\frac{h(x, t)}{(t + t)^{\beta}} = \frac{R^2 - 0}{\lambda - 1} \geq \frac{R_0^2}{\lambda - 1} \geq \frac{R^2}{\lambda - 1} \geq \frac{(R_0/2)^2}{\lambda - 1} \geq \frac{3R^2}{4\lambda - 1},
\]

then we also have

\[
\begin{align*}
LHS - RHS & \geq \sigma e(t + t)^{\beta - 1} h + \frac{e \beta}{m - 1} (t + t)^{\tau} \frac{|x|^2}{(t + t)^{\beta + 1}} \\
& \quad + 2N \frac{m}{\lambda - 1} e^{m - 1} (t + t)^{\lambda - 1} h \\
& \quad - C_1 |\chi e(t + t)^{\sigma - \beta} h |^2 \frac{1}{(m - 1)R_0} \\
& \quad - C_2 |\chi e(t + t)^{\sigma} h |^2 \frac{1}{(m - 1)R_0} \\
& \quad - 4C_1 |\chi e(t + t)^{\sigma - \beta} h |^2 \\
& \quad - 4C_2 |\chi e(t + t)^{\sigma} h |^2.
\end{align*}
\]
provided that \( \sigma \geq 2C_2 \chi \tau \).

Let \( \tau = 1, \eta = R_0, \varepsilon = K_0 \max\{1, R_0^{2(d_0-d)}\} \), and \( \beta = (m-1)\sigma \) with \( \sigma > 0 \) being sufficiently large such that

\[
\begin{align*}
\left( \frac{\beta}{2(m-1)} - \frac{4m}{(m-1)^2} \varepsilon^{m-1} - \frac{4C_1 \chi}{(m-1)R_0} \right) & \geq 0, \\
(\sigma - 2C_2 \chi \tau) \left( \frac{3R_0}{4 \tau^\beta} \min\{1, 2^{\beta-1}\} - \frac{C_1 \chi}{m-1} \right) & \geq 0.
\end{align*}
\]

Then (31) tells us \( \text{LHS} \geq \text{RHS} \) for all \( x \in A(t) \) and \( t \in (0, t_0) \). It follows that \( g(x, t) \) is an upper solution. The comparison principle Lemma 3.3 completes the proof. \( \square \)

Lemma 4.3 implies the finite speed propagating property of the degenerate diffusion equation. We will present the exact propagating speed for a special structure initial data.

**Lemma 4.3 (Exact propagating speed).** Let the conditions in Lemma 3.6 be valid with the initial values satisfying

\[
\begin{align*}
\left\{ \begin{array}{l}
\varepsilon_0 = K_0 \left[ (R_0^2 - |x - x_0|^2) \right]^{\frac{\beta}{2}}, \quad x \in \Omega, \\
\nabla \varepsilon_{00} \cdot (x - x_0) = -\mu |x - x_0|^2, \quad x \in B_{R_0}(x_0),
\end{array} \right.
\end{align*}
\]

for some \( x_0 \in \Omega \) and positive constants \( d = 1/(m-1), R_0, K_0, \mu, \delta > 0 \) such that \( \Omega \) and \( B_{R_0}^{\delta}(x_0) := \{ x \in B_{R_0}(x_0); \text{dist}(x, \partial B_{R_0}(x_0)) < \delta \} \). Then

\[
\text{supp} \ \varepsilon(x, t) = \{(\theta, \rho(\theta, t)); \theta \in S^{N-1}\},
\]

where \( (\theta, \rho) \) is the spherical coordinate centered at \( x_0, \rho(\theta, 0) = R_0 \) for all \( \theta \in S^{N-1} \), and the propagating speed

\[
\frac{\partial \rho(\theta, t)}{\partial t} \bigg|_{t=0} = R_0 \left( \frac{2m}{m-1} K_0^{m-1} - \chi \mu \right), \quad \forall \theta \in S^{N-1}.
\]

26
**Proof.** Define

\[ g_\pm(x, t) = \varepsilon(t + t)^{\sigma_\pm} \left( \eta^2 - \frac{|x - x_0|^2}{(t + t)^{\beta_\pm}} \right)^d, \quad x \in \Omega, \ t \geq 0, \]

with \( \varepsilon = K_0, \ \tau = 1, \ \eta = R_0, \ \sigma_\pm \in \mathbb{R}, \ \beta_\pm \in \mathbb{R} \) are to be determined. We have

\[ g_\pm(x, 0) = K_0[(R_0^2 - |x - x_0|^2)]^d = u_0, \quad x \in \Omega, \]

and \( \frac{\partial g_\pm}{\partial n} = 0, \ \frac{\partial g_\pm}{\partial n} = 0 \) on \( \partial \Omega \) at least for a small time interval since \( \overline{B}_{R_0} \subset \Omega \). Here we only aim to find the exact propagating speed and we only need to construct upper and lower solutions on a small time interval. We note that

\[ \nabla g_\pm(x, 0) \cdot \nabla v_0 = -\varepsilon(t + t)^{\sigma_\pm - \beta} d^{d-1} 2(x - x_0) \cdot \nabla v_0 = 2\mu \varepsilon(t + t)^{\sigma_\pm - \beta} d^{d-1}|x - x_0|^2, \]

for \( x \in B^\delta_{R_0}(x_0) \). Let

\[ \beta = \frac{4m}{m - 1} K_0^{m-1} - 2\chi \mu, \]

and \( \beta_\pm \) approach \( \beta \) from above and below. Take \( \sigma_+ > 0 \) sufficiently large and \( \sigma_- < 0 \) with \( |\sigma_-| \) being sufficiently large, we can check as in the proof of Lemma 4.1 and next Lemma 4.4 that \( g_\pm(x, t) \) are upper and lower solutions for a small time interval \((0, T_\pm)\), where \( T_\pm > 0 \) depend on \( |\beta_\pm - \beta| \). Here we omit the details. Then the comparison principle Lemma \[3.3 \] implies that there exists \( A_{\beta_\pm}(t) \) such that

\[ A_{\beta_\pm}(t) = B_{R_0(1 + t)^{1/2}}(x_0), \quad t \in (0, T_\pm), \]

and

\[ A_{\beta_-}(t) \subset \text{supp} \ u(x, t) \subset A_{\beta_+}(t), \quad t \in (0, T_\pm). \]

Therefore,

\[ \frac{\partial \rho(\theta, t)}{\partial t} \bigg|_{t=0} \in [R_0\beta_-/2, R_0\beta_+/2]. \]

Since \( \beta_\pm \) approach \( \beta \), we have \( \frac{\partial \rho(t)}{\partial t} = R_0\beta/2. \)

\[ \Box \]

### 4.3 Eventual smoothness and expanding

The large time behavior in Lemma \[3.4 \] and Lemma \[3.7 \] shows that \( \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \) tends to zero as time grows. This indicates that the chemotaxis effect decays and the support will expand to the whole domain. Now we construct a self similar weak lower solution with expanding support.
Lemma 4.4. Let the conditions in Lemma 3.6 be valid with the initial data \( u_0 \geq 0, u_0 \neq 0 \) and \( \Omega \) is convex. Define a function

\[
g(x, t) = \varepsilon(\tau + t)^\gamma \left( \eta^2 - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)^d, \quad x \in \Omega, \ t > -\tau,
\]

where \( d = 1/(m - 1), \beta > 0, \sigma < 0, \varepsilon, \eta > 0, \tau \in \mathbb{R} \) and \( x_0 \in \Omega \). Then by appropriately selecting \( \beta, \varepsilon, \tau, \sigma, \eta \) and \( x_0 \), the function \( g(x, t) \) is a weak lower solution of the first equation in (2) on \( \Omega \times (\hat{t}, \hat{T}) \) corresponding to \( v(x, t) \) and \( u_0 \) for some \( \hat{T} > \hat{t} > 0 \). Therefore, \( u(x, t) \geq g(x, t) \) and there exist \( t_0 \in (\hat{t}, \hat{T}), \varepsilon_0 > 0 \), and a family of expanding open sets \( \{A(t)\}_{\hat{t} \leq t \leq \hat{T}} \), such that

\[
A(t) \subset \text{supp} \ u(x, t), \quad t \in (\hat{t}, \hat{T}),
\]

and \( A(t) = \Omega, \ u(x, t) \geq \varepsilon_0 \) for all \( x \in \Omega \) and \( t \in [t_0, \hat{T}] \).

**Proof.** Since \( u_0 \geq 0, u_0 \neq 0 \) and \( u_0 \in C(\overline{\Omega}) \), the first equation in (2) shows that

\[
\int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) > 0, \quad t > 0.
\]

For any \( t > 0 \), there exists a \( x_0(t) \in \Omega \) such that \( u(x_0(t), t) \geq \bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u_0(x) > 0 \). According to the uniform Hölder continuity of \( u(\cdot, t) \), we find that there exists a \( R_0 > 0 \) independent of \( t \) such that

\[
u(x, t) \geq \frac{\bar{u}}{2} =: \varepsilon_1, \quad \forall x \in B_{R_0}(x_0(t)). \tag{33}
\]

We denote \( C_1(t) = ||\nabla v(\cdot, t)||_{L^\infty(\Omega)} \) and \( C_2(t) = ||\Delta v(\cdot, t)||_{L^\infty(\Omega)} \) for convenience. According to Lemma 3.7 and Lemma 3.8 \( C_1(t) \) and \( C_2(t) \) tend to zero. For fixed \( \delta > 0 \) to be determined, let \( \hat{t} > 0 \) depend on \( \delta \) such that

\[
C_1(t) \leq \delta, \quad C_2(t) \leq \delta, \quad \forall t \geq \hat{t}. \tag{34}
\]

Note that \( u(x, \hat{t}) \geq \varepsilon_1 \) on \( B_{R_0}(x_0(\hat{t})) \). Without loss of generality, we may assume that \( B_{R_0} = B_{R_0}(x_0(\hat{t})) \subset \Omega \) and \( x_0 = x_0(\hat{t}) = 0 \).

Similar to the proof of Lemma 4.1, we let

\[
h(x, t) = \left( \eta^2 - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)^d, \quad x \in \Omega, \ t \geq 0,
\]

and

\[
A(t) = \left\{ x \in \Omega; \frac{|x - x_0|^2}{(\tau + t)^\beta} < \eta^2 \right\}, \quad t \geq 0.
\]

\[28\]
According to the definition of \( g \), we see that \( \frac{\partial g}{\partial n} \leq 0 \) and \( \frac{\partial g^m}{\partial n} \leq 0 \) on \( \partial \Omega \) since \( \Omega \) is convex, and for \( \tau = 1 - \hat{t} \) we have

\[
g(x, t) = e[(\eta^2 - |x|^2)^d] \leq e_1 1_{B_{R_0}(x_0)} \leq u_0(x), \quad x \in \Omega,
\]

provided that

\[
\eta \leq R_0, \quad e\eta^2 \leq e_1. \tag{35}
\]

In order to find a weak lower solution \( g \), we only need to check the following differential inequality on \( A(t) \)

\[
\frac{\partial g}{\partial t} \leq \Delta g^m - \chi \nabla \cdot (g \nabla v) = \Delta g^m - \chi \nabla g \cdot \nabla v - \chi g \Delta v, \quad x \in A(t), \ t \in (\hat{t}, \hat{T}), \tag{36}
\]

for some \( \hat{T} > \hat{t} \) to be determined.

A sufficient condition of inequality (36) is

\[
\sigma e(\tau + t)^{\sigma - 1} + \frac{e\beta}{m - 1}(\tau + t)^{\sigma - 1} \leq \frac{|x|^2}{(\tau + t)^{\beta + 1}}
\]

\[
+ 2N - m - 1 \epsilon^m(\tau + t)^{m\sigma} \frac{h}{(\tau + t)^{\sigma - \beta + 1} + C_1(t) \sqrt{e}(\tau + t)^{\sigma - \beta}} + C_2(t) \sqrt{e}(\tau + t)^{\sigma - h}, \tag{37}
\]

for all \( x \in A(t), \ t \in (\hat{t}, \hat{T}) \). For simplicity, we denote (37) by \( \text{LHS} \leq \text{RHS} \). The estimates on the above inequality is quit similar to (28) in the proof of Lemma 4.2 except some terms are with inverse signs. Here, (30) and (31) are changed into

\[
\text{LHS} - \text{RHS} \leq \sigma e(\tau + t)^{\sigma - 1} + \frac{e\beta}{m - 1}(\tau + t)^{\sigma - 1} \leq \frac{|x|^2}{(\tau + t)^{\beta + 1}}
\]

\[
+ 2N - m - 1 \epsilon^m(\tau + t)^{m\sigma} \frac{h}{(\tau + t)^{\sigma - \beta + 1} + C_1(t) \sqrt{e}(\tau + t)^{\sigma - \beta}} + C_2(t) \sqrt{e}(\tau + t)^{\sigma - h}
\]

\[
\leq \left[ \sigma + 2N - m - 1 \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + C_2(t) \sqrt{e}(\tau + t)^{\sigma - h} \right]
\]

\[
+ \left( \frac{\beta}{m - 1} - \frac{4m}{(m - 1)^2} \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + \frac{4C_1(t) \sqrt{e}(\tau + t)}{(m - 1)R_0} \right) \epsilon(\tau + t)^{\sigma - \beta - 1} |x|^2
\]

\[
\leq \left[ \sigma + 2N - m - 1 \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + C_2(t) \sqrt{e}(\tau + t)^{\sigma - h} \right]
\]

\[
+ \left( \frac{\beta}{m - 1} - \frac{4m}{(m - 1)^2} \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + \frac{4C_1(t) \sqrt{e}(\tau + t)}{(m - 1)R_0} \right) \epsilon(\tau + t)^{\sigma - \beta - 1} |x|^2
\]

\[
+ \left( \frac{\beta}{m - 1} - \frac{4m}{(m - 1)^2} \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + \frac{4C_1(t) \sqrt{e}(\tau + t)}{(m - 1)R_0} \right) \epsilon(\tau + t)^{\sigma - \beta - 1} |x|^2
\]

\[
+ \left( \frac{\beta}{m - 1} - \frac{4m}{(m - 1)^2} \epsilon^m(\tau + t)^{(m - 1)\sigma - \beta + 1} + \frac{4C_1(t) \sqrt{e}(\tau + t)}{(m - 1)R_0} \right) \epsilon(\tau + t)^{\sigma - \beta - 1} |x|^2,
\]

29
For this \( \varepsilon > 0 \), \( \sigma < 0 \), the above seven inequalities can be satisfied simultaneously in the following way.

\[
LHS - RHS \leq \sigma \epsilon (\tau + t)^{\sigma-1} h + \frac{\epsilon \beta}{m-1} \frac{|x|^2}{(\tau + t)^{\beta+1}} + 2N \frac{m}{m-1} \epsilon^m (\tau + t)^{m\sigma} \frac{h}{(\tau + t)^{\beta}} + C_1(t) \epsilon (\tau + t)^{\sigma-\beta} \frac{R_0}{m-1} - \frac{m}{(m-1)^2} \epsilon^m (\tau + t)^{m\sigma} \frac{4|x|^2}{(\tau + t)^{2\beta}} + C_2(t) \epsilon (\tau + t)^{\sigma} h \\
\leq (\sigma + 2N \frac{m}{m-1} \epsilon^{m-1} (\tau + t)^{(m-1)\sigma - \beta + 1} + C_2(t) \epsilon (\tau + t)) \epsilon (\tau + t)^{\sigma-1} h \\
+ \left( \frac{\beta}{m-1} - \frac{4m}{(m-1)^2} \epsilon^{m-1} (\tau + t)^{(m-1)\sigma - \beta + 1} \right) \epsilon (\tau + t)^{\sigma-\beta} - \frac{R_0}{m} \right) \\
+ C_1(t) \epsilon (\tau + t)^{\sigma-\beta} \frac{R_0}{m-1} \\
\leq (2N \frac{m}{m-1} \epsilon^{m-1} \max\{1, (\tau + \hat{T})^{(m-1)\sigma - \beta + 1} + C_2(t) \epsilon (\tau + \hat{T})\} \epsilon (\tau + t)^{\sigma-1} \eta^2 \\
+ \left( \frac{\beta}{m-1} - \frac{4m}{(m-1)^2} \epsilon^{m-1} \min\{1, (\tau + \hat{T})^{(m-1)\sigma - \beta + 1}\} \epsilon (\tau + t)^{\sigma-\beta - 1} |x|^2 \right) \\
+ \sigma \epsilon (\tau + t)^{\sigma-1} \frac{3}{4} \eta^2 + C_1(t) \epsilon (\tau + t)^{\sigma-\beta} \frac{R_0}{m-1}, \quad x \in (B_{R_0/2} \cap A(t), \quad t \in (\hat{t}, \hat{T}).
\]

(39)

Since \( \Omega \) is bounded, there exists \( R > R_0 \) such that \( \Omega \subset B_R(x_0) \). Let \( \eta = R_0 \), \( \epsilon > 0 \), \( \beta \in (0, 1) \), \( \tau = 1 - \hat{t}, \hat{T} > \hat{t} \) and \( \sigma = \frac{1-\beta}{m-1} < 0 \) be chosen such that

\[
\begin{align*}
\epsilon \eta^{2d} &\leq \epsilon_1, \\
2N \frac{m}{m-1} \epsilon^{m-1} &\leq -\sigma/4, \quad \beta \leq \frac{2m}{m-1} \epsilon^{m-1}, \\
\delta \chi(\hat{T} - \hat{t} + 1) &\leq -\sigma/4, \quad 4\delta \chi(\hat{T} - \hat{t} + 1) \leq \frac{2m}{m-1} \epsilon^{m-1} R_0, \\
\delta \chi(\hat{T} - \hat{t} + 1)^{1-\beta} \frac{R_0}{m-1} &\leq -\sigma \eta^2/4, \quad (\hat{T} - \hat{t} + 1)^{\beta/2} \geq 2R/R_0.
\end{align*}
\]

(40)

The above seven inequalities can be satisfied simultaneously in the following way. We first fix \( \beta \in (0, 1) \) sufficiently small such that \( N \beta \leq (1-\beta)/(4(m-1)) \). Then we set \( \epsilon = \epsilon(\beta) > 0 \) such that \( \frac{2m}{m-1} \epsilon^{m-1} = \beta \). Now we can modify \( \beta \) to be smaller such that \( \epsilon \eta^{2d} \leq \epsilon_1 \). The first three inequalities are valid. Let \( L = e^{\frac{1}{2} \ln \frac{R_0}{4} - 1} \) and

\[
\delta = \min\{-\sigma/(4\chi(L+1)), \frac{2m}{m-1} \epsilon^{m-1} R_0/(4\chi(L+1)), -\sigma \eta^2 (m-1)/(4\chi(L+1)^{1-\beta} R_0)\}.
\]

For this \( \delta > 0 \), let \( \hat{t} \) be chosen such that (34) is fulfilled and \( \hat{T} = \hat{t} + L \).
For those parameters, we see that \([40]\) is valid and \([38], [39]\) tells us \(LHS \leq RHS\) for all \(x \in A(t)\) and \(t \in (i, \hat{T})\), i.e. \([37]\). It follows that \(g(x, t)\) is a lower solution. The comparison principle Lemma 3.3 shows that

\[
A(\hat{t}) = B_{g(\tau - \hat{T})^{\frac{\beta}{2}}}(x_0) \cap \Omega = B_{R_0}(x_0),
\]

and

\[
A(\hat{T}) = B_{g(\tau + \hat{T})^{\beta/2}}(x_0) \cap \Omega = B_{R_0(\hat{T} - i + 1)^{\beta/2}}(x_0) \cap \Omega \supset B_{2R}(x_0) \cap \Omega = \Omega,
\]

since \((\hat{T} - i + 1)^{\beta/2} \geq 2R/R_0\) in \([40]\) and \(\Omega \subset B_R(x_0)\). There exists a \(t_1 \in (i, \hat{T})\) such that

\[
\eta^2 - \frac{|x - x_0|^2}{\tau + t} \geq 0, \quad \forall x \in \Omega, \ t \in (t_1, \hat{T}),
\]

which means \(A(t) = \Omega\) for \(t \in (t_1, \hat{T})\). And there exists a \(t_0 \in (t_1, \hat{T})\) such that

\[
\eta^2 - \frac{|x - x_0|^2}{\tau + t} \geq \frac{\eta^2}{2}, \quad \forall x \in \Omega, \ t \in (t_0, \hat{T}),
\]

and thus

\[
u(x, t) \geq g(x, t) \geq e(\hat{T} - i + 1)^{\beta/2} \left(\frac{\eta^2}{2}\right)^d =: \varepsilon_0, \quad \forall x \in \Omega, \ t \in (t_0, \hat{T}).
\]

The proof is completed. \(\square\)

**Remark 4.2.** It is interesting to compare the self similar weak lower solution \(g(x, t)\) in the proof of Lemma 4.4 to the Barenblatt solution of porous medium equation

\[
B(x, t) = (1 + t)^{-k}\left[1 - \frac{k(m - 1)}{2mN}\left(\frac{|x|^2}{(1 + t)^{2k/N}}\right)\right]^{\frac{1}{m-1}},
\]

with \(k = 1/(m-1+2/N)\). The Barenblatt solution \(B(x, t)\) is decaying at the rate \((1 + t)^{-1/(m-1+2/N)}\) in \(L^\infty(\mathbb{R}^N)\) and the support is expanding at the rate \((1 + t)^{k/N}\). While the self similar weak lower solution \(g(x, t)\) is decaying at the rate \((1 + t)^{-\beta/(m-1)}\) and its support is expanding at the rate \((1 + t)^{\beta/2}\). Here in the proof we have selected \(\beta > 0\) sufficiently small, which means the support of \(g\) is expanding with a much slower rate and the maximum of \(g\) is decaying at a slightly faster rate.
Now that we have proved the lower bound of \( u(x, t) \) on \( \Omega \times (t_0, \hat{T}) \), we will show the globally lower bound at large time, as well as the non-degeneracy, regularity for large time behavior.

**Lemma 4.5 (Eventual smoothness).** Let the conditions in Lemma 4.4 be valid. Then \( u(x, t) \geq \varepsilon_0 \) for all \( x \in \Omega \) and \( t \geq t_0 \) with \( t_0 > 0 \) and \( \varepsilon_0 > 0 \) being defined as in the proof of Lemma 4.4 \( u \in C^{2,1}((\Omega \times [t_0, \infty)) \) and there exist \( C > 0 \) and \( c > 0 \) such that

\[
\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq Ce^{-ct}, \quad t > 0,
\]

where \( \overline{u} = \int_{\Omega} u_0 dx / |\Omega| \).

**Proof.** We point out that

\[
\varepsilon_0 = \varepsilon(\hat{T} - \hat{\theta} + 1)^\sigma \left( \frac{\eta^2}{2} \right)^d = \varepsilon(L + 1)^\sigma \left( \frac{\eta^2}{2} \right)^d
\]

is independent of \( \delta \) and \( \hat{\theta} \) therein, since \( L \) only depends on \( \beta, R_0 \) and \( R \) (note that \( \beta, \sigma, \varepsilon \) depend only on \( \varepsilon_1 \) and \( \varepsilon_1 = \overline{u}/2 \) is fixed). Therefore, we can take \( \hat{T} \) larger to be \( \hat{T} + \theta \) with any \( \theta > 0 \) such that (34) is also valid. Lemma 4.4 shows that \( u(x, t) \geq \varepsilon_0 \) for all \( x \in \Omega \) and \( t \in [t_0 + \theta, \hat{T} + \theta] \). Since \( \varepsilon_0 > 0 \) is fixed and \( \theta > 0 \) is arbitrary, we have \( u(x, t) \geq \varepsilon_0 \) for all \( x \in \Omega \) and \( t \geq t_0 \). It follows that the first equation in (2) is non-degenerate and uniform parabolic. The Hölder regularity and exponential decay can be verified similar to the proof of Theorem 1.3 in [40]. \( \square \)

**Acknowledgement**

The research of S. Ji is supported by NSFC Grant No. 11701184. The research of M. Mei was supported in part by NSERC Grant RGPIN 354724-2016, and FRQNT Grant No. 2019-CO-256440. The research of J. Yin was supported in part by NSFC Grant No. 11771156.

**References**

[1] N. Bellomo and M. Winkler. A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up. *Commun. Partial Diff. Equa.*, 42(3):436–473, 2016.

[2] E. Ben-Jacob, O. Schochet, A. Tenenbaum, I. Cohen, A. Czirók, and T. Vicsek. Generic modelling of cooperative growth patterns in bacterial colonies. *Nature*, 368(6466):46, 1994.
[3] S. Bian and J.G. Liu. Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$. *Commun. Mathem. Phys.*, 323(3):1017–1070, 2013.

[4] M. Burger, M. Di Francesco, and Y. Dolak-Struss. The Keller-Segel model for chemotaxis with prevention of overcrowding: Linear vs. nonlinear diffusion. *SIAM J. Math. Anal.*, 38(4):1288–1315, 2006.

[5] L. Cisneros. The organized melee: Emergence of collective behavior in concentrated suspensions of swimming bacteria and associated phenomena. 2008.

[6] J. Fischer. Advection-driven support shrinking in a chemotaxis model with degenerate mobility. *SIAM J. Math. Anal.*, 45(3):1585–1615, 2013.

[7] M. Di Francesco and J. Rosado. Fully parabolic Keller-Segel model for chemotaxis with prevention of overcrowding. *Nonlinearity*, 21(21):2715–2730(16), 2008.

[8] W.S. Gurney and R.M Nisbet. The regulation of inhomogeneous populations. *J. Theor. Biol.*, 52(2):441–457, 1975.

[9] M.E. Gurtin and R.C. Maccamy. On the diffusion of biological populations. *Mathe. Biosci.*, 33(1):35–49, 1977.

[10] T. Hillen and K.J. Painter. A user’s guide to pde models for chemotaxis. *J. Math. Biol.*, 58:183, 2009.

[11] C.H. Jin, Y.F. Wang, and J.X. Yin. Global solvability and stability to a nutrient-taxis model with porous medium slow diffusion. *arXiv:1804.03964*.

[12] K. Kawasaki, A. Mochizuki, M. Matsushita, T. Umeda, and N. Shigesada. Modeling spatio-temporal patterns generated by bacillus subtilis. *J. Theor. Biol.*, 188(2):177–185, 1997.

[13] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.

[14] I. Kim and Y. Yao. The Patlak-Keller-Segel model and its variations: Properties of solutions via maximum principle. *SIAM J. Math. Anal.*, 44(2):568–602, 2012.

[15] M.A. Lewis and B. Li. Spreading speed, traveling waves, and minimal domain size in impulsive reaction-diffusion models. *Bull. Math. Biol.*, 74(10):2383–2402, 2012.
[16] J.F. Leyva, C. Málaga, and R.G. Plaza. The effects of nutrient chemotaxis on bacterial aggregation patterns with non-linear degenerate cross diffusion. *Phys. A Stat. Mech. Appl.*, 392(22):5644–5662, 2013.

[17] G.M. Lieberman. Intermediate schauder theory for second order parabolic equations iv: time irregularity and regularity. *Diff. Integ. Equa.*, 5:1219–1236, 1992.

[18] M. Ma and Z.A. Wang. Global bifurcation and stability of steady states for a reaction-diffusion-chemotaxis model with volume-filling effect. *Nonlinearity*, 28(8):2639–2660, 2015.

[19] M. Mansour. Traveling wave solutions of a nonlinear reaction-diffusion-chemotaxis model for bacterial pattern formation. *Appl. Math. Model.*, 32(2):240–247, 2008.

[20] T. Matsuyama and M. Matsushita. Fractal morphogenesis by a bacterial cell population. *CRC Critical Reviews in Microbiology*, 19(2):117–135, 1993.

[21] J.D. Murry. *Mathematical biology I: an introduction*. Springer, New York, USA, 2002.

[22] M. Ohgiwari, M. Matsushita, and T. Matsuyama. Morphological changes in growth phenomena of bacterial colony patterns. *J. Phys. Soc. Japan*, 61(3):816–822, 1992.

[23] K.J. Painter and J.A. Sherratt. Modelling the movement of interacting cell populations. *J. Theor. Biol.*, 225:327–339, 2003.

[24] G. Rosen. Steady-state distribution of bacteria chemotactic toward oxygen. *Bull. Math. Biol.*, 40(5):671–674, 1978.

[25] R.A. Satnoianu, P.K. Maini, F.S. Garduno, and J.P. Armitage. Travelling waves in a nonlinear degenerate diffusion model for bacterial pattern formation. *Disc. Cont. Dyna. Syst. -B*, 1(3):339–362, 2012.

[26] J.A. Sherratt. On the form of smooth-front travelling waves in a reaction-diffusion equation with degenerate nonlinear diffusion. *Math. Model. Natur. Phen.*, 5(5):64–79, 2010.

[27] J.A. Sherratt and B.P. Marchant. Nonsharp travelling wave fronts in the fisher equation with degenerate nonlinear diffusion. *Appl. Math. Lett.*, 9(5):33–38, 1996.
[28] Y. Sugiyama. Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis. *Diff. Integ. Equa.*, 20(2):133–180, 2007.

[29] Y.S. Tao and M. Winkler. Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant. *J. Differential Equations*, 252(3):2520–2543, 2012.

[30] Y.S. Tao and M. Winkler. Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion. *Disc. Cont. Dyna. Syst. - Series A*, 32(5):1901–1914, 2013.

[31] Y.S. Tao and M. Winkler. Locally bounded global solutions in a three-dimensional chemotaxis-stokes system with nonlinear diffusion. *Annal. Instit. Henri Poincaré*, 30(1):157–178, 2013.

[32] J.L. Vázquez. *The Porous Medium Equation: Mathematical Theory*. Oxford Univ. Press, 2006.

[33] J. Wakita, K. Komatsu, A. Nakahara, T. Matsuyama, and M. Matsushita. Experimental investigation on the validity of population dynamics approach to bacterial colony formation. *J. Phys. Soc. Japan*, 63(3):1205–1211, 1994.

[34] Z.A. Wang and T. Hillen. Classical solutions and pattern formation for a volume filling chemotaxis model. *Chaos*, 17(3):299–304, 2007.

[35] Z.A. Wang, M. Winkler, and D. Wrzosek. Global regularity vs. infinite-time singularity formation in a chemotaxis model with volume filling effect and degenerate diffusion. *SIAM J. Math. Anal.*, 44(44):3502–3525, 2012.

[36] M. Winkler. Aggregation vs. global diffusive behavior in the higher dimensional Keller-Segel model. *J. Differential Equations*, 248:2889–2905, 2010.

[37] M. Winkler. Boundedness and large time behavior in a three-dimensional chemotaxis-stokes system with nonlinear diffusion and general sensitivity. *Cal. Vari. Partial Diff. Equa.*, 54(4):3789–3828, 2015.

[38] M. Winkler. Global existence and stabilization in a degenerate chemotaxis-stokes system with mildly strong diffusion enhancement. *J. Differential Equations*, 264: 6109–6151, 2018.

[39] Z. Wu, J. Zhao, J. Yin, and H. Li. *Nonlinear diffusion equations*. World Scientific Publishing Co. Pvt. Ltd., 2001.
[40] T.Y. Xu, S.M. Ji, C.H. Jin, M. Mei, and J.X. Yin. Early and late stage profiles for a chemotaxis model with density-dependent jump probability. *Math. Biosci. Engin.*, 15: 1345–1385, 2018.

[41] T.Y. Xu, S.M. Ji, M. Mei, and J.X. Yin. Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion. *J. Differential Equations*, 265: 4442–4485, 2018.

[42] Y. Yao and A.L. Bertozzi. Blow-up dynamics for the aggregation equation with degenerate diffusion. *Phys. D Nonl. Phen.*, 260(5):77–89, 2013.