Symmetry analysis of transport properties in helical superconductor junctions

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Abstract

We study the discrete symmetries satisfied by helical $p$-wave superconductors with the $d$-vectors $k_x \hat{x} \pm k_y \hat{y}$ or $k_x \hat{x} \pm k_y \hat{y}$ and the transformations brought by symmetry operations to ferromagnet and spin-singlet superconductors, which show intimate associations with the transport properties in heterojunctions, including helical superconductors. In particular, the partial symmetries of the Hamiltonian under spin-rotation and gauge-rotation operations are responsible for the novel invariances of the conductance in tunnel junctions and the new selection rules for the lowest current and peculiar phase diagrams in Josephson junctions, which were reported recently. The symmetries of constructed free energies for Josephson junctions are also analyzed, and are consistent with the results from the Hamiltonian.

Keywords: discrete symmetries, helical superconductor, heterojunction, transport properties, Ferromagnet

(Some figures may appear in colour only in the online journal)

1. Introduction

Symmetries play an important role in the classification of topological matter\textsuperscript{[1–6]} and the definition of topological invariants\textsuperscript{[7–12]}. Spin-triplet superconductors (TSs) with the $d$-vectors $k_x \hat{x} \pm k_y \hat{y}$ or $k_x \hat{x} \pm k_y \hat{y}$ belong to the symmetry class DIII according to the tenfold classification of topological insulators and superconductors\textsuperscript{[13–15]}, and satisfy time-reversal symmetry (TRS), particle-hole (or charge-conjugation) symmetry (PHS) and chiral symmetry (CS), but break SU(2) spin-rotation symmetry (SRS). Superconducting systems that possess the non-trivial $\mathbb{Z}_2$ topological invariant and support helical Majorana modes are analogous to the quantum spin Hall state\textsuperscript{[7, 16]}. Such superconductors are called helical superconductors. Helical superconducting states are candidates for pairing in Sr$_2$RuO$_4$\textsuperscript{[17–19]} and the triplet part of the order parameter in the non-centrosymmetric superconductor CePt$_3$Si\textsuperscript{[20]}. Among others, $k_x \hat{x} + k_y \hat{y}$ is the two-dimensional analog of the Balian–Werthamer state (B phase) in $^3$He\textsuperscript{[21]}. By contrast, chiral TSs with the $d$-vectors $(k_x \pm k_y) \hat{z}$ can be viewed as two copies of the spinless superconductor that belongs to the symmetry D class with the non-trivial $\mathbb{Z}$ topological number and a chiral Majorana mode\textsuperscript{[2, 3, 5]}. The chiral superconducting states satisfy PHS and SRS about the $\hat{z}$-axis, but break TRS, and are analogous to the quantum Hall state\textsuperscript{[22]}. Research on the transport properties of topological junctions can not only provide methods for the detection of...
Majorana fermion edge states [23–30], but it can also probe the symmetries satisfied by topological superconductors [31–33]. The charge conductance in the ferromagnet (F)chiral TS junction is invariant when the magnetization in F is rotated about the \( \hat{z} \)-axis due to the SU(2) SRS kept by TS [34]. The Josephson effects in spin-singlet superconductor (SS)Fchiral TS junctions show that the form of the lowest current (sin \( \phi \) or cos \( \phi \)) is strongly related to the symmetries of the Cooper pair functions in the SS [35]. The \( 0-\pi \) phase transition in purely chiral superconductor junctions with different chiralities is also clarified [36]. These results are very different from those for helical superconductor junctions, since the SU(2) rotation symmetry for the latter is completely broken. The conductance in Fchiral TS junctions exhibits higher anisotropy when the direction of the magnetization is changed [37]. Both sin \( \phi \)-type and cos \( \phi \)-type currents always exist in SSFhelical TS junctions; the selection rules for the current can be given by the orientation of the magnetization [38]. Purely helical Josephson junctions with different \( d \)-vectors can host more ground states, such as \( \pi \)-phase, \( \varphi \)-phase and \( \varphi_0 \)-phase, which can be tuned easily by the direction of the magnetization [39]. In particular, the \( \varphi_0 \)-phase has recently received increasing interest [24, 40–46].

However, the SU(2) symmetry breaking in helical superconductor junctions does not imply the absence of symmetries for transport quantities. In fact, both the charge conductance and the Josephson current possess invariances under the rotation of magnetization [37]. The invariances have important effects for classifying the SSSFhelical TS junctions into two types and forming the symmetric phase diagrams for the purely helical junctions [38, 39]. In this paper, we consider the transport properties of Fchiral TS, SSFhelical TS and helical TSFhelical TS junctions from the viewpoint of the symmetries of the Hamiltonian in order to clarify the origin of the invariances of the transport quantities. The main findings in this paper are as follows.

(a) The Hamiltonian for the helical superconductor is invariant under the joint operation of the SU(2) spin rotation and the U(1) gauge transformation, which will lead to the invariances of the conductance and Josephson current.

(b) The Hamiltonian satisfies partial symmetries under a particular spin-rotation operation or the joint operation of rotation and gauge transformation; the operations can transform the BdG Hamiltonian from \( H(k_x, k_y) \) into \( H(k_x, -k_y) \). The partial symmetries are responsible for the symmetries of the transport quantities and phase diagrams. In particular, if we take the different \( k_z \) parities of superconducting wave functions in SSs into account, the partial symmetries will produce different selection rules for the lowest order current for different SSSFhelical TS junctions, which will divide the junctions into two types.

(c) The TRS and PHS of the Hamiltonian bring about the sign reversal of the Josephson current and the symmetry of the current for junctions without the non-magnetic interfacial potential, respectively.

(d) The symmetries of current derived from the Hamiltonian are consistent with those from the constructed free energies for the helical Josephson junctions, which directly reflects the mechanism of the interaction between ferromagnetism and helical superconductivity.

2. BdG Hamiltonian

2.1. Superconducting system

The generic Hamiltonian for the superconductor is

\[
\mathcal{H} = \frac{1}{2} \sum_k \Psi_k^\dagger H(k) \Psi_k, 
\]

with \( \Psi_k = (c_k|, c_k^+|, c_{-k}^+|, c_{-k}^{}|)^T \) and the BdG Hamiltonian

\[
H(k) = \begin{pmatrix}
\hat{\tau}(k) + \hat{\Delta}(k) & -\hat{\Delta}^*(k) \\
-\hat{\Delta}^*(k) & \hat{\tau}^*(k) + \hat{\Delta}^*(k)
\end{pmatrix},
\]

where \( c_{-k}(c_{k}^\dagger) \) is the annihilation (creation) operator of an electron with momentum \( k = (k_x, k_y) \), spin \( \alpha \), \( \hat{\tau}(k) \) describes the normal dispersion of the electron and \( \hat{\Delta}(k) \) denotes the energy gap matrix. The normal state is invariant under the space inversion, i.e. \( \hat{\tau}(k) = \hat{\tau}^*(-k) \).

2.1.1. Spin-triplet system. For TS, the matrix \( \hat{\Delta}(k) = (d(k) \cdot \delta) i\hat{\sigma}_2 \) with \( \hat{\Delta}(-k) = -\hat{\Delta}(k) \), in which \( \delta_i \) (\( i = 1, 2, 3 \)) denote the usual Pauli matrices in spin space and \( d(k) \) denotes the \( d \)-vector.

The helical TS with the \( d \)-vector \( d^0_{uv}(k) = \Delta_0(k_0, \pm k_0) \) with \((u, v = x, y)\) is denoted by the \( p_{uv}^z \)-wave TS in this paper, which can be regarded as the superposition of the equal-spin wave functions with different orbital angular momentum,

\[
\begin{align*}
p_{xy}^+ & : -(k_x + i k_y) \uparrow\downarrow + (k_x - i k_y) \downarrow\uparrow, \\
p_{xy}^- & : -(k_x - i k_y) \uparrow\downarrow + (k_x + i k_y) \downarrow\uparrow, \\
p_{xy}^z & : i(k_x + i k_y) \uparrow\uparrow + i(k_x - i k_y) \downarrow\downarrow, \\
p_{xy}^- & : -i(k_x - i k_y) \uparrow\downarrow - i(k_x + i k_y) \downarrow\uparrow,
\end{align*}
\]

as shown in figure 1. Note, the phase before the orbital wave functions is different for \( p_{uv}^z \)-wave states. In fact, the Hamiltonian for the \( p_{uv}^z \)-wave states can be written in a block diagonal form in the basis \( \Psi = (c_{k|}, c_{-k}^+|, c_{k}^{}|, c_{-k}^+)^T \), which is

\[
H(k) = \begin{pmatrix}
\epsilon(k) & -d_x + id_y & 0 & 0 \\
-d_x - id_y & -\epsilon(k) & 0 & 0 \\
0 & 0 & \epsilon(k) & d_x + id_y \\
0 & 0 & d_x - id_y & -\epsilon(k)
\end{pmatrix}.
\]

The block diagonal Hamiltonian in the spin subspace is similar to the one discussed in [47], which describes a system consisting of two separate copies of spinless superconductors.
with opposite chirality. This is the reason for our use of the term ‘helical superconductor’ in this paper.

The helical superconductor possesses TRS. We introduce the time-reversal operator $T = \chi^T K$, where $\chi_T = \bar{\bar{\gamma}} \otimes \bar{\sigma}_0$ with $\bar{\bar{\gamma}}$ being the $2 \times 2$ unit matrix in particle-hole space and $K$ being the complex conjugation operator, which flips the momentum sign and the direction of spin. The actions of $T$ on the creation and the annihilation operators give $\hat{T} \epsilon^{\dagger}_{\sigma} a^\dagger_{\alpha \lambda} \Rightarrow \sum_{\sigma \alpha \lambda} i \hat{\sigma}_2 a^\dagger_{\alpha \lambda}^{\dagger}$ and $\hat{T} \epsilon_{\sigma} a_{\alpha \lambda} \Rightarrow \sum_{\sigma \alpha \lambda} \epsilon^{\dagger}_{\sigma} a^{\dagger}_{\alpha \lambda} \hat{\sigma}_2 a_{\alpha \lambda}$, respectively. The symmetry with $[T, \mathcal{H}] = 0$ requires

$$\chi_T H(k) \chi_T^{-1} = H^*(-k),$$

which is obviously satisfied by the BdG Hamiltonian. After the time-reversal operation, the Hamiltonian in equation (2) stays invariant. However, if the superconductor has a phase $\phi$, which is an important quantity in the Josephson effect, the operation $T$ will change the phase into $-\phi$.

The superconductor also obeys PHS. We define the charge conjugation operator $C = \chi_c K$ with $\chi_c = \bar{\bar{\gamma}} \otimes \bar{\sigma}_0$, where $\bar{\bar{\gamma}}$ is the Pauli matrices in particle-hole space and $\bar{\sigma}_0$ is the unit matrix in spin space. The action of $C$ on the electron state $|k\uparrow\rangle$ will produce a hole in the state $| -k\downarrow\rangle$. Consequently, the transformations of the creation and the annihilation operators are $\hat{C} \epsilon^{\dagger}_{\sigma} a^\dagger_{\alpha \lambda} = \epsilon^{\dagger}_{\sigma} a^\dagger_{\alpha \lambda}$ and $\hat{C} \epsilon_{\sigma} a_{\alpha \lambda} = \epsilon_{\sigma} a_{\alpha \lambda}$, respectively. The symmetry with $[C, \mathcal{H}] = 0$ requires

$$\chi_c H(k) \chi_c^{-1} = -H^*(-k),$$

which is also satisfied by the BdG Hamiltonian in equation (2). After the charge conjugation operation, the energy of the quasiparticles becomes negative. In other words, $\hat{\epsilon}(k)$ and $\hat{\Delta}(k)$ in equation (2) become $-\hat{\epsilon}(-k)$ and $\hat{\Delta}(-k)$, respectively. However, the phase $\phi$ keeps its value.

Since the system satisfies both TRS and PHS, we can define a CS operator $S = -i TC = \bar{\bar{\gamma}} \otimes \bar{\sigma}_2$, which gives $S \epsilon^{\dagger}_{\sigma} a^\dagger_{\alpha \lambda} S^{-1} = \sum_{\sigma \alpha \lambda} \epsilon^{\dagger}_{\sigma} a^\dagger_{\alpha \lambda}$ and $S \epsilon_{\sigma} a_{\alpha \lambda} S^{-1} = \sum_{\sigma \alpha \lambda} \epsilon_{\sigma} a_{\alpha \lambda}$. A Hamiltonian will possess CS, $[S, \mathcal{H}] = 0$, if

$$SH(k) S^{-1} = -H(k).$$

Evidently, the BdG Hamiltonian for the helical superconductor has this property. After the chiral symmetry operation, $\hat{\epsilon}(k), \hat{\Delta}(k)$ and $\phi$ in equation (2) will become $-\hat{\epsilon}(-k), \Delta(-k)$ and $-\phi$.

Finally, we consider the spin rotation operation. The rotation matrix in the particle-hole space is diagonal, i.e. $R(\xi, \eta) = \text{diag}(R(\xi), \eta, \hat{R}(\xi), \eta))$ with the element

$$\hat{R}(\xi, \eta) = \begin{pmatrix} \cos \frac{\xi \pm \eta}{2} & -\sin \frac{\xi \pm \eta}{2} \\ \sin \frac{\xi \pm \eta}{2} & \cos \frac{\xi \pm \eta}{2} \end{pmatrix},$$

in the spin space. The element $\hat{R}$ represents a $\xi$-angle rotation about the $y$-axis followed by a $\eta$-angle rotation about the $z$-axis. For a system with SRS, $[R, \mathcal{H}] = 0$, one has

$$R(\xi, \eta) H(k) R(\xi, \eta)^{-1} = H(k),$$

for all values of $\xi$ and $\eta$, which is not satisfied by the Hamiltonian with $d_m^d(k) = \Delta(k, \hat{x} \pm k, \hat{y})$. In fact, the helical superconductor breaks all the non-trivial rotation symmetries with the angles $\xi \neq 0$ and $\eta \neq 0$. Nevertheless, there are three special rotations, $R(0, \pi), R(\pi, 0)$ and $R(\pi, \pi)$, which can help bring the so-called partial symmetries.

For the $p_{xy}^\pm$-wave TS with the $d$-vector $d_m^d(k)$, we have

$$R(0, \pi) H(k) R(0, \pi)^{-1} = H(k),$$

$$R(\pi, 0) H(k) R(\pi, 0)^{-1} = H(k, -k_y),$$

$$R(\pi, \pi) H(k) R(\pi, \pi)^{-1} = H(k_x, -k_y),$$

where the operation $R(\xi, \eta) \equiv \text{diag}(R(\xi), \eta, U_{1, 2}^\pm, \hat{R}(\xi, \eta) U_{1, 2}^\pm)$ with the pure rotation $\hat{R}(\xi, \eta)$, and the $\mp U_1$ gauge transformation, which will be called ‘gauge-rotation’ operation for simplicity. Equation (13) shows that the Hamiltonian is invariant under the gauge-rotation $R(0, \pi)$; equations (14) and (15) indicate that the Hamiltonian obeys the partial symmetry under the gauge-operation $R(\pi, 0)$ or the pure rotation $R(\pi, \pi)$.

For the $p_{xy}^\pm$-wave TS with the $d$-vector $d_m^d(k)$, equation (13) still holds. Equations (14) and (15) turn into

$$R(\pi, 0) H(k) R(\pi, 0)^{-1} = H(k_x, -k_y),$$

$$R(\pi, \pi) H(k) R(\pi, \pi)^{-1} = H(k_x, -k_y),$$

respectively. Similarly, the Hamiltonian satisfies the partial symmetries. In writing equations (13)–(17), we use the even parity of $\hat{\epsilon}(k)$ and the odd parity of $\hat{\Delta}(k)$. Obviously, after the rotation operations $R(\pi, 0), R(\pi, \pi)$ ($R(\pi, 0)$ and $R(\pi, \pi)$)
for the $p^\pm_\alpha(p^\pm_\beta)$-wave TS, $\tilde{\Delta}(k)$ in equation (2) will become

$$\Delta(k, -k).$$

### 2.12. Spin-singlet system.**

For SS, the gap matrix $\tilde{\Delta}(k) = \Delta(k)\sigma_2$ with the even parity $\tilde{\Delta}(k) = \tilde{\Delta}(-k)$ in equation (2). The energy gap function $\Delta(k)$ is $\Delta_0$ for the $s$-wave pairing, $\Delta(k) = \Delta_0(k_x^2 - k_y^2)$ for the $d_{x^2-y^2}$-wave pairing and $\Delta(k) = \Delta_0 2k_xk_y$ for the $d_{xy}$-wave pairing. The spin-singlet system not only preserves TRS, PHS and CS, but it also obeys SRS. In order to discuss the symmetries of transport quantities conveniently later, here we give the changes of $\tilde{\epsilon}(k)$ and $\tilde{\Delta}(k)$ in equation (2) after the operations $T$, $C$ and $S$,

$$T: \tilde{\epsilon}(k), \tilde{\Delta}(k), \phi \rightarrow \tilde{\epsilon}(k), \tilde{\Delta}(k), -\phi;$$

$$C: \tilde{\epsilon}(k), \tilde{\Delta}(k), \phi \rightarrow -\tilde{\epsilon}(-k), \tilde{\Delta}(-k)e^{i\pi}, \phi;$$

$$S: \tilde{\epsilon}(k), \tilde{\Delta}(k), \phi \rightarrow -\tilde{\epsilon}(-k), \tilde{\Delta}(-k)e^{i\pi}, -\phi.$$

The changes brought about by the rotation operations for SS are presented in Table 1. For later use, the transformed gap matrix has been expressed with $(k_x, -k_y)$ in order to remain consistent with equations (14)–(17). When we do this, there will be a $\pi$-phase difference between the gap matrix for the $s(d_{x^2-y^2})$-wave and that for the $d_{xy}$-wave. The difference originates from their opposite ‘parities’ under $k_y \rightarrow -k_y$, as shown in Figure 2, although they are both spin-singlet even parity superconductors. The subtle difference between the gap functions can bring about important physical results for Josephson effects, which will be seen in part 4. For simplicity, $\tilde{\epsilon}(k)$ is omitted in the table since it is invariant under the operations and it is an even function both for $k_x$ and $k_y$.

**Table 1. Changes of the gap matrix $\tilde{\Delta}(k)$ for SS after the rotation operations.**

| $R(0, \pi)$ | $R(\pi, 0)$ | $R(\pi, \pi)$ | $R(\pi, 0)$ | $R(\pi, \pi)$ |
|-------------|-------------|-------------|-------------|-------------|
| $s, d_{x^2-y^2}$ | $\tilde{\Delta}(k_x, -k_y)$ | $\tilde{\Delta}(k-x, -k_y)$ | $\tilde{\Delta}(k_x, -k_y)$ | $\tilde{\Delta}(k-x, -k_y)$ |
| $d_{xy}$ | $\tilde{\Delta}(k_x, -k_y)$ | $\tilde{\Delta}(k_x, -k_y)$ | $\tilde{\Delta}(k_x, -k_y)$ | $\tilde{\Delta}(k_x, -k_y)$ |

### 2.2. Ferromagnetic system

The BdG-type Hamiltonian in equation (1) for F is denoted by $H_F(k)$, which can be written as

$$H_F(k) = \begin{pmatrix} \tilde{\epsilon}(k) - M \cdot \sigma + \tilde{V}_0 & 0 \\ 0 & -\tilde{\epsilon}(-k) + M \cdot \sigma - \tilde{V}_0 \end{pmatrix},$$

(21)

where the magnetization $M = M\hat{n}(\theta_m, \phi_m)$ is specified by the direction $\hat{n} = (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m)$ with the polar angle $\theta_m$ and the azimuthal angle $\phi_m$. $\tilde{V}_0 = V_0 I_{2 \times 2}$ is the non-magnetic potential. The Hamiltonian $H_F(k)$ can describe the bulk F or the ferromagnetic interface in heterostructures. We take $V_0 = 0$ for the bulk and the purely ferromagnetic interface cases.

The ferromagnetic system breaks TRS, PHS and SRS, i.e. the Hamiltonian in equation (21) does not satisfy equations (8)–(10). For later use, we also present the changes of the Hamiltonian brought about by the $T$, $C$ and $S$ transformations as we did for SS,

$$T: \tilde{\epsilon}(k), \tilde{V}_0, \hat{n}(\theta_m, \phi_m) \rightarrow \tilde{\epsilon}(k), \tilde{V}_0, \hat{n}(\pi - \theta_m, \pi + \phi_m);$$

(22)

$$C: \tilde{\epsilon}(k), \tilde{V}_0, \hat{n}(\theta_m, \phi_m) \rightarrow -\tilde{\epsilon}(-k), -\tilde{V}_0, \hat{n}(\pi - \theta_m, \pi + \phi_m);$$

(23)

$$S: \tilde{\epsilon}(k), \tilde{V}_0, \hat{n}(\theta_m, \phi_m) \rightarrow -\tilde{\epsilon}(-k), -\tilde{V}_0, \hat{n}(\theta_m, \phi_m).$$

(24)

From the above equations, it is found that the direction of magnetization is ‘rotated’ after $T$ and $C$ operations; the energy $\tilde{\epsilon}(k)$ and $\tilde{V}_0$ become negative after the $S$ operation.

The actions of rotation operations are summarized in Table 2. A pure rotation and its corresponding gauge rotation lead to the same change of the ferromagnetic system due to the $U_1$ gauge symmetry obeyed by F. For simplicity, we do not show $\tilde{\epsilon}(k)$ and $\tilde{V}_0$ in the table since they are invariant under rotation operations.

**Table 2. The direction of magnetization after rotation operations.**

| $R(0, \pi)$ | $R(0, \pi)$ | $R(\pi, 0)$ | $R(\pi, 0)$ | $R(\pi, \pi)$ |
|-------------|-------------|-------------|-------------|-------------|
| $\tilde{\epsilon}(\theta_m, \pi + \phi_m)$ | $\tilde{\epsilon}(\theta_m, \pi - \phi_m)$ | $\tilde{\epsilon}(\theta_m, 2\pi - \phi_m)$ | $\tilde{\epsilon}(\theta_m, 2\pi - \phi_m)$ | $\tilde{\epsilon}(\theta_m, 2\pi - \phi_m)$ |

### 3. Symmetries of charge conductance

#### 3.1. Blonder–Tinkham–Klapwijk formalism

We consider the $F$-helical $p$-wave TS junctions as shown in Figure 3(a). The non-magnetic interface with $M = 0$ is located at $x = 0$ and along the $y$ axis. The ferromagnetic region with $x < 0$ is described by the Hamiltonian in equation (21) with $V_0 = 0$; the superconducting region with $x > 0$ is described by the Hamiltonian in equation (2). The wave functions of the quasiparticles in F and TS can be obtained through solving the BdG equations $H_F(k)\Psi = E(k)\Psi$ and $H_F(k)\Psi = -E(k)\Psi$, respectively.
under some unitary operation and simultaneously the operation rotates the direction of magnetization from \( \mathbf{n}(\theta_m, \phi_m) \) to \( \mathbf{n}(\theta_m', \phi_m') \), we will have \( \sigma(\theta_m, \phi_m) = \sigma(\theta_m', \phi_m') \). For SS, the spin of Cooper pairs is zero; the system obeys the full SRS. Consequently, the conductance is independent of the orientation of magnetization, i.e. \( \sigma(\theta_m, \phi_m) = \sigma(\theta_m', \phi_m') \) for all values of \( \theta_m, \phi_m, \theta_m', \phi_m' \). For the chiral p-wave superconductor with the d-vector \( \mathbf{d} = \Delta_0(k_x \pm ik_y) \mathbf{\hat{z}} \), which is parallel to the crystallographic c-axis, the spin of the Cooper pairs lies in the ab-plane. The system obeys the symmetry of the spin rotation about the z-axis. Consequently, the conductance is invariant under the magnetization rotation about the z-axis, i.e. \( \sigma(\theta_m, \phi_m) = \sigma(\theta_m, \phi_m') \) for all values of \( \phi_m \) and \( \phi_m' \). For the helical TS, the symmetry operation is the gauge-rotation \( \mathcal{R}(0, \pi) \), which ‘rotates’ the direction of magnetization from \( \mathbf{n}(\theta_m, \phi_m) \) to \( \mathbf{n}(\theta_m, \pi + \phi_m) \) as given in table 2. Thus, we obtain

\[
\sigma(\theta_m, \phi_m) = \sigma(\theta_m, \pi + \phi_m). \tag{26}
\]

The second situation which can lead to the invariance of conductance is partial symmetries of the BdG Hamiltonian. The conductance \( \sigma(\theta_m, \phi_m) \) is an average value of the angle-resolved conductance on \( k_y \), which indicates that the junctions with \( k_y \) and those with \( -k_y \) correspond to the same conductance. For the \( p^+ \)-wave and the \( p^- \)-wave TSs, the partial symmetry operations are \( \mathcal{R}(\pi, 0), \mathcal{R}(\pi, \pi) \), and \( \mathcal{R}(\pi, 0) \), \( \mathcal{R}(\pi, \pi) \), respectively, which transform the Hamiltonian \( H(k) \) into \( H(k_y - k_x) \). They change the direction of magnetization from \( \mathbf{n}(\theta_m, \phi_m) \) to \( \mathbf{n}(\pi - \theta_m, \pi - \phi_m) \) and \( \mathbf{n}(\pi - \theta_m, 2\pi - \phi_m) \). Consequently, we have

\[
\sigma(\theta_m, \phi_m) = \sigma(\pi - \theta_m, \pi - \phi_m), \tag{27}
\]

\[
\sigma(\theta_m, \phi_m) = \sigma(\pi - \theta_m, 2\pi - \phi_m). \tag{28}
\]

The joint operations \( \mathcal{R}(0, \pi) \mathcal{R}(\pi, 0) \) and the pure operation \( \mathcal{R}(\pi, \pi) \) give the same symmetry relation of conductance due to the \( U_1 \) gauge symmetry satisfied by F.

The magnetization with \( \theta_m = 0 \) or \( \pi \) is a special case which means the magnetization is parallel to the spin quantization axis of the helical TS. In the coordinate of spin space in F, the TS are in purely equal-spin pairing states. When an electron is injected from F, the normal reflected electron, the Andreev reflected hole and the transmitted quasiparticles all possess the same spin as the injected electron. As a result, the eight lines denoting the scattering processes presented in figure 3 will degenerate into four. The conductance in this situation is independent of the azimuthal angle \( \phi_m \). The equations (27) and (28) will give \( \sigma(\theta_m = 0) = \sigma(\theta_m = \pi) \).

The results obtained in this part demonstrate that the conductance, as an observable quantity, possesses a higher symmetry than the system itself. The SRS breaking of the Hamiltonian favors strong anisotropy of the conductance when the direction of the magnetization is changed. However, the remaining partial symmetries of the Hamiltonian help to maintain the invariance of the conductance. In addition, the symmetry relations in equations (26)–(28) were proved in [37], where the detailed derivation and the numerical calculations are presented. Here, for a convenient comparison between the

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**Figure 3.** (a) Schematic illustration of the Fhelical TS junctions. The current is flowing along the x-axis, which is defined by the crystallographic a-axis. (b) The scattering process for an electron-like quasiparticle (EQ) injected from F. The solid lines represent the normally reflected and the transmitted EQs. The dashed lines represent the Andreev reflected and the transmitted hole-like quasiparticles (HQs). The spin for the injected EQ is assumed along \( \mathbf{n}(\theta_m, \phi_m) \); quasiparticles denoted by lines of the same color possess the same spin.

For the spin of quasiparticles in F, we chose the direction \( \mathbf{\hat{n}}(\theta_m, \phi_m) \) as the quantization axis. When an electron with wavevector \( k_y \) is injected from F, there will be four physical processes, as shown in figure 3(b). The electron is normally reflected as electrons and Andreev reflected as holes; the electron transmits into TS as electron-like quasiparticles and hole-like quasiparticles. The wave function in F (TS) is the superposition of the reflected (transmitted) quasiparticles. The reflection and transmission coefficients of the quasiparticles can be derived from boundary conditions at the interface, which are functions of \( M, V_0, \mathbf{n}(\theta_m, \phi_m), k_y \) and the bias V. According to the Blonder–Tinkham–Klapwijk formalism [48], the angle-resolved conductance \( \sigma' \) of the junctions can be expressed as the composition of the coefficients, which is a function of \( \theta_m, \phi_m \) and \( k_y \) when \( M, V_0 \) and V are fixed. As a result, the angle-averaged conductance \( \sigma \) can be given by

\[
\sigma(\theta_m, \phi_m) = C_0 \sum_{k_y} \sigma'(\theta_m, \phi_m, k_y), \tag{25}
\]

where \( C_0 \) is a constant independent of \( \theta_m, \phi_m \) and \( k_y \). Generally speaking, the expression of the angle-averaged conductance is very complex. A more detailed description for the derivation of conductance can be found in the appendix.

**3.2. Fhelical TS junctions**

Now, we clarify the symmetries of the conductance. We want to know which orientations of magnetization can lead to the same conductance. First, if the superconductor is invariant

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4. Symmetries of Josephson current

4.1. Quasiclassical Green’s function formalism

The theory of the quasiclassical Green’s function is a powerful tool to study the Josephson effect in superconducting junctions, in which the Josephson current can be expressed by the retarded Green’s function. One method of obtaining the retarded Green’s function is to solve the Eilenberger equation [49]. Riccati parametrization of the function and diagrammatic representation of the boundary conditions have been developed for the method [50]. Another effective method of obtaining the retarded Green’s function is to construct it by composing wave functions derived from the BdG equation \( H(\mathbf{k})\Psi = E(\mathbf{k})\Psi \); this was developed by McMillan [51] and then extended to the case of anisotropic superconductors by Tanaka et al [52]. In this paper, the latter method is adopted.

We consider two-dimensional Josephson junctions with a ferromagnetic interface located at \( x = 0 \) and along the \( y \)-axis. The interface is described by the Hamiltonian in equation (21). For a given \( k_y \) and fixed spin, there are four types of quasiparticle injection processes: the left or right injection of the electron-like quasiparticle and the hole-like quasiparticle. For each injection, there are four scattering processes, as shown in figure 5. The wave function in the superconductor is the superposition of the scattered quasiparticles. For the given \( k_y \), there are four types of conjugated injection processes with the wavevector \( -k_y \). The wave function in the superconductor for the conjugated process can also be obtained from the superposition of the scattered quasiparticles. The retarded Green’s function is constructed from a linear combination of the product of wave functions and the transpose of the conjugated wave functions. The combination coefficients can be solved by using the boundary conditions at the ferromagnetic interface.

After an analytical continuation, the obtained Green’s function \( G(x, x', k_y, \omega_n) \) is not only dependent on the spatial position, the wavevector \( k_y \) and the Matsubara frequency \( \omega_n \), but also a function of \( M, \hat{n}(\theta_m, \phi_m), \tilde{V}_0 \) and the superconducting phase difference \( \Delta \phi \). The Josephson current for fixed \( M \) and \( \tilde{V}_0 \) at the interface can be expressed as

\[
I_j(\theta_m, \phi_m, \Delta \phi) = C_1 \lim_{\omega_n \to 0} \left( \frac{\partial}{\partial x'} - \frac{\partial}{\partial x} \right) \sum_{\omega_n, k_y} Tr[G(x, x', k_y, \omega_n)] |_{x = 0} (29)
\]

\[
= C_2 \sum_{\omega_n, k_y} F(\omega_n, k_y), \tag{30}
\]

where \( C_1 \) and \( C_2 \) are constants that are irrelevant to the symmetries of the Josephson current. For simplicity, in the following we will assume that the superconducting phase of the left superconductor is \( \phi \) and that the right one’s is zero. Under this assumption, we have \( I_j(\theta_m, \phi_m^*, \Delta \phi) = I_j(\theta_m, \phi_m^*, \phi) \). A concise discussion about how the symmetries of Hamiltonian affect the Josephson current is given in the appendix. In the following, we use equation (30) as the starting point for analyzing the properties of the current.

Figure 4. The conductance as a function of \( \theta_m \) and \( \phi_m \). The symmetry of the figure expresses the relations in equations (26)–(28). The other parameters are taken as \( 2\Delta_0 = 1, \frac{M}{\Delta_0} = 0.9, \frac{\Delta_1}{\Delta_0} = 0.3 \) and \( T = 0 \), with \( k_F \) being the Fermi wavevector, \( E_F \) being the Fermi energy, \( V \) being the voltage and \( T \) being the temperature. The conductance is normalized by the normal value [37].

Figure 5. (a) Schematic illustration of the Josephson junctions considered in this paper. The \( x \)-axis is defined by the crystallographic \( a \)-axis. The magnetization in the ferromagnetic interface is assumed along \( \hat{n}(\theta_m, \phi_m) \). (b) The scattering process for an EQ injected from the left superconductor. The lines have the same meaning as in figure 3. The spin for the injected EQ is assumed along the \( z \)-axis defined by the crystallographic \( c \)-axis; quasiparticles denoted by lines of the same color possess the same spin.

| SS or \( P_{xy}^+ \) -wave TS | \( P_{uv}^\pm \) -wave TS |
|---------------------------|---------------------------|
| EQ                       | EQ                       |
| HQ                       | HQ                       |

Figure 3. The conductance as a function of \( \theta_m \) and \( \phi_m \). The symmetry of the figure expresses the relations in equations (26)–(28). The other parameters are taken as \( 2\Delta_0 = 1, \frac{M}{\Delta_0} = 0.9, \frac{\Delta_1}{\Delta_0} = 0.3 \) and \( T = 0 \), with \( k_F \) being the Fermi wavevector, \( E_F \) being the Fermi energy, \( V \) being the voltage and \( T \) being the temperature. The conductance is normalized by the normal value [37].
4.2. SSF helical TS Josephson junctions

We consider the Josephson junctions as shown in figure 5(a). The SS with phase $\phi$ is located in the region with $x < 0$ and the TS with zero phase is located in the region with $x > 0$. They are described by the Hamiltonian in equations (1) and (2). The ferromagnetic interface is described by the Hamiltonian in equations (1) and (21).

Firstly, we discuss the general results for the Josephson current of SSF helical TS junctions. For one thing, the time-reversal operation will change the superconducting phase of SS from $\phi$ to $-\phi$ according to equation (18). The direction of the Josephson current is reversed accordingly, i.e. $I_1 \rightarrow -I_1$. For another thing, the time-reversal ‘rotates’ the magnetization in F from the direction $\hat{n}(\theta_m, \phi_m)$ to $\hat{n}(\pi - \theta_m, \pi + \phi_m)$. As a result, one obtains

$$I_I(\theta_m, \phi_m, \phi) = -I_I(\pi - \theta_m, \pi + \phi_m, 2\pi - \phi). \quad (31)$$

The $C$ operator is another symmetry operation that can lead to a general result. The energy of quasiparticles becomes negative, while the phase $\phi$ remains unchanged under the operation. Accordingly, the quantity $F(\omega_n, k_x)$ will become $F(-\omega_n, k_x)$ in equation (30). However, the Josephson current $I_I$, as the sum of $F(\omega_n, k_x)$, will keep its value. In fact, after the particle–hole transformation, the helical superconductor can be viewed as a new electron system with the normal dispersion $-\epsilon(-k)$ and the energy gap matrix $\Delta(-k)$. The SS can be viewed as a new electron system with the normal dispersion $-\epsilon(-k)$ and the energy gap matrix $\Delta(-k)e^{i\pi}$ according to equation (19). The ferromagnetic interface becomes a new electron system with the normal dispersion $-\epsilon(-k)$, the non-magnetic potential $-V_0$ and the magnetization along $\hat{n}(\pi - \theta_m, \pi + \phi_m)$ according to equation (23). As a result, we have

$$I_I(\theta_m, \phi_m, \phi, V_0) = I_I(\pi - \theta_m, \pi + \phi_m, \pi + \phi, -V_0). \quad (32)$$

Since an actual interface possesses non-negative potential, i.e. $V_0 \geq 0$, only the case of $V_0 = 0$ for equation (32) makes sense. Then, we have

$$I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, \pi + \phi_m, \pi + \phi), \quad (33)$$

with $V_0 = 0$. Combining equations (31) and (33), we obtain

$$I_I(\theta_m, \phi_m, \phi) = -I_I(\theta_m, \phi_m, \pi - \phi), \quad (34)$$

which is just the result brought about by the chiral symmetry operation.

The operator $\mathcal{R}(0, \pi)$ can also bring about a general result. Under the operation, the helical TS is invariant, as given in equation (13), while the SS will acquire a $\pi$ phase, as shown in table 1. Meanwhile, the operation changes the direction of the magnetization from $\hat{n}(\theta_m, \phi_m)$ to $\hat{n}(\theta_m, \pi + \phi_m)$. As a unitary transformation, $\mathcal{R}(0, \pi)$ will not change the Josephson current. As a result, one obtains

$$I_I(\theta_m, \phi_m, \phi) = I_I(\theta_m, \pi + \phi_m, \pi + \phi). \quad (35)$$

Secondly, we discuss other symmetry operations that will give different results for junctions with different pairing symmetries of the superconducting wave functions. Now, we take $\mathcal{R}(\pi, 0)$ as an example to analyze the symmetry of the Josephson current. Under the operation, the junctions with $k_i$ become those with $-k_i$ due to the partial symmetry of the $p_x$-wave superconductor. Simultaneously, the $s(d_{x^2-y^2})$-wave SS acquires a phase of $\pi$ according to table 1; the direction of the magnetization turns into $\hat{n}(\pi - \theta_m, \pi - \phi_m)$ according to table 2. However, as an average quantity of $k_x$, the Josephson current in equation (30) will not change its value. Thus, we have

$$I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, \pi - \phi_m, \pi + \phi) \quad (36)$$

for the $s(d_{x^2-y^2})$-wave SSF $p_x^\pm$-wave TS junctions. Similarly, from the operation $\mathcal{R}(\pi, \pi)$ we can derive

$$I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, 2\pi - \phi_m, \phi), \quad (37)$$

for the $d_{x^2-y^2}$-wave SSF $p_x^\pm$-wave TS junctions. From the operations $\mathcal{R}(\pi, 0)$ and $\mathcal{R}(\pi, \pi)$, we can also derive

$$I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, \pi - \phi_m, \phi) \quad (38)$$

and

$$I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, 2\pi - \phi_m, \pi + \phi), \quad (39)$$

respectively, for the $d_{xy}$-wave SSF $p_x^\pm$-wave TS junctions.

For the SSF $p_x^\pm$-wave TS junctions, there are similar symmetries of the Josephson current. The $s(d_{x^2-y^2})$-wave SSF $p_x^\pm$-wave TS junctions satisfy equations (38) and (39), which originate from the operations $\mathcal{R}(\pi, 0)$ and $\mathcal{R}(\pi, \pi)$, respectively. The $d_{xy}$-wave SSF $p_x^\pm$-wave TS junctions satisfy equations (36) and (37), which also originate from the operations $\mathcal{R}(\pi, 0)$ and $\mathcal{R}(\pi, \pi)$, respectively. From the results, we come to the following conclusions.

(a) The same spin-rotation or gauge-rotation operation can bring about different symmetries of the Josephson current for the $s(d_{x^2-y^2})$-wave SS and the $d_{xy}$-wave SSW when the gap function in the helical superconductor is fixed. The difference results from the partial symmetry obeyed by the helical superconductor and the different $k_x$-parities of the gap functions in SS, as discussed in section 2.1.2.

(b) The SSF $p_x^\pm$-wave TS junctions can be classified into two types, depending on the symmetries satisfied by their current. The $s(d_{x^2-y^2})$-wave SSF $p_x^\pm$-wave TS junctions and the $d_{xy}$-wave SSF $p_x^\pm$-wave TS junctions satisfy equations (36) and (37); the $s(d_{x^2-y^2})$-wave SSF $p_x^\pm$-wave TS junctions and the $d_{xy}$-wave SSF $p_x^\pm$-wave TS junctions satisfy equations (38) and (39). This classification is consistent with the numerical results in [38]. In particular, the junctions of the same type possess the same selection rules of the lowest order Josephson current.

Finally, we show a brief explanation for the relation of the results in this paper to those in [38]. The combination of equations here can bring equations in [38]. For example, equations (33) and (35) will give $I_I(\theta_m, \phi_m, \phi) = I_I(\pi - \theta_m, \phi_m, \phi)$; equations (33)
and (36) will give $I_l(\theta_m, \phi_m, \phi) = I_l(\theta_m, 2\pi - \phi_m, \phi)$; and equations (33) and (39) will give $I_l(\theta_m, \phi_m, \phi) = I_l(\theta_m, \pi - \phi_m, \phi)$. The three equalities are just equations (8), (12) and (14) in [38], respectively. As pointed out in [38], the three equations will not hold when the non-magnetic potential $V_0$ is non-zero (i.e. $Z = 0$ there). This is because the equations are all based on equation (33), which only holds when the non-magnetic potential $V_0 = 0$. In order to compare our results with experiments easily, we present in figure 6 the Josephson current for the SSF $p_x^1$-wave TS junction in the orientation space of the magnetization and in the space spanned by the azimuthal angle $\phi_m$ and the phase $\phi$, which can explicitly express the symmetry relations.

### 4.3. Helical TS Josephson junctions

We consider purely helical Josephson junctions, as shown in figure 5(a). The $p_x^1$-wave TS on the left-hand side has the superconducting phase $\phi$. The right-hand side can be the $p_y$-wave or the $p_x$-wave TS with the zero phase. The magnetization in the interface is assumed to be along $\mathbf{m}(\theta_m, \phi_m)$.

Now we discuss the symmetries of the Josephson current. The time-reversal and the charge conjugation lead to

$$I_l(\theta_m, \phi_m, \phi) = -I_l(\pi - \theta_m, \pi + \phi_m, 2\pi - \phi),$$  \hspace{1cm} (40)

$$I_l(\theta_m, \phi_m, \phi, V_0) = I_l(\theta_m, \pi - \phi_m, \pi + \phi, -V_0),$$  \hspace{1cm} (41)

respectively, which are the common results for purely helical junctions. The latter equality is different from equation (32). The helical superconductor will not acquire a phase $\pi$ under the charge conjugation.

The rotation operations can bring about different results for different junctions. For the $p_x^1$-wave TS $p_x^1$-wave TS junction, $R(0, \pi)$ and $R(\pi, 0)$ give

$$I_l(\theta_m, \phi_m, \phi) = I_l(\theta_m, \pi + \phi_m, \phi)$$  \hspace{1cm} (42)

and

$$I_l(\theta_m, \phi_m, \phi) = I_l(\pi - \theta_m, \pi - \phi_m, \phi),$$  \hspace{1cm} (43)

respectively, which are related to the transformations in equations (13) and (14). The result for $R(\pi, \pi)$, as the joint operation of $R(0, \pi)$ and $R(\pi, 0)$, is the combination of equations (42) and (43), i.e. $I_l(\theta_m, \phi_m, \phi) = I_l(\pi - \theta_m, 2\pi - \phi_m, \pi + \phi)$. There is a $\pi$-phase difference between equation (43) and (44), since the $p_x^1$-wave TS is partially symmetric under $R(\pi, 0)$, while $R(\pi, 0)$ will bring an extra phase $\pi$ to the superconductor.

Helical Josephson junctions can host rich ground states and these can be classified into $0, \pi, 0 + \pi, \varphi_0$ and $\varphi$ phases [39]. The phase transition can be tuned by rotating the magnetization in the ferromagnetic interface. The phase diagram formed in the orientation space of the magnetization is an important aspect of research on Josephson junctions and also possesses some symmetries as a result of symmetries of the current.

For the $p_y$-wave $p_x^1$-wave TS $p_x^1$-wave TS junction, the diagram in the orientation space $0 \leq \theta_m < \pi$ and $0 \leq \phi < 2\pi$ is symmetric about both $\theta_m = \pi/2$ and $\phi_m = \pi$. It is the result of equations (42), (43) and

$$I_l(\theta_m, \phi_m, \phi) = I_l(\theta_m, \pi - \phi_m, \phi).$$  \hspace{1cm} (46)

Equation (46) originates from the close relation between $p_x^1$-wave TSs. When one changes the $y$-axis into $-y$ in the

Figure 6. (a) The current for $\phi = 0.3\pi$. (b) The current for $\phi = 1.7\pi$. The symmetry of (a) expresses equation (37). The connection between the two panels reflects the symmetry relations in equation (31). (c) The current for $\theta_m = 0.3\pi$. The symmetry of (c) expresses equation (35). The other parameters for all the panels are taken as $V_{xy} \equiv 1$, $M = 0.9$ and $T = 0.3T_C$, with $T_C$ being the critical temperature. The current is expressed by the resistance in the normal state [38].
three-dimensional coordinate space, the TSs exchange with each other; the direction of the magnetization in the new coordinate system becomes \( \mathbf{n}(\theta_m, 2\pi - \phi_m, \phi) \) accordingly. The relation can be reflected directly from the interaction of helical superconductivity and ferromagnetism, as can be seen in part 5.

For the \( p_\Delta \)-wave TS|TS\(^\pm\) wave TS junctions, the phase diagram for \( V_0 = 0 \) is symmetric about \( \theta_m = \pi/2 \) and invariant under the translation \( \phi_m \rightarrow \phi_m + \pi \). It is the result of equations (44) and (41) with \( V_0 = 0 \). In addition, we have another equality,

\[
I_0(\theta_m, \phi_m, \phi) = I_0(\theta_m, \pi/2 - \phi_m, \phi),
\]

which originates from the connection between the TSs. For example, for the \( p_\Delta \)-wave TS|TS\(^\pm\)wave TS junction, when one changes the x- and y-axes into y and x, respectively, the TSs exchange with each other; the direction of the magnetization becomes \( \mathbf{n}(\theta, \pi/2 - \phi_m, \phi) \) in the new coordinate space. The connection can also be reflected directly from the interaction of helical superconductivity and ferromagnetism, as can be seen in part 5.

As discussed in part 3.2 for the conductance, the magnetization with \( \theta_m = 0 \) or \( \pi \) is also a special case for the Josephson current. The purely equal-spin pairing will simplify the scattering process in figure 5. The current will become independent of the azimuthal angle \( \phi_m \). For the SS|FTS and TS|FTS junctions, we will have \( I_0(\theta_m = 0, \phi) = -I_0(\theta_m = \pi, 2\pi - \phi) \) from equations (31) and (40). This relation can also be found in the Josephson junction formed along the edges of quantum spin-Hall insulators with a Zeeman field along the spin quantization [45].

Another special case is the perpendicular magnetization with \( \theta_m = \pi/2 \), which is important for a helical superconductor realized in a topological insulator with the \( s \)-wave superconductor [53, 54]. The symmetries of the transport quantities in these references are different from those in this paper. For example, in [54], the transmission probability of the normal-metal–superconductor junction with a single ferromagnetic barrier is independent of the direction of the in-plane magnetization. For the normal-metal–superconductor junction with double barriers, the probability is dependent on the relative angle of the in-plane magnetization in the barriers. However, in our junctions with \( \theta_m = \pi/2 \), the conductance and the current is strongly dependent on the azimuthal angle of the magnetization due to the symmetry breaking of the spin-rotation.

Finally, we discuss Josephson current in the non-magnetic case, i.e. \( M = 0 \). In this case, we have \( I_n(\phi) = I_n(\pi + \phi) \) for the SS|TS junctions, which means that the Josephson current is \( \pi \)-periodic. This result is consistent with that for the junction between singlet and triplet superconductors in the static thermodynamic limit [55]. For the \( p_\Delta \)-wave TS|TS\(^\pm\)-wave TS junctions, we also have \( I_n(\phi) = I_n(\pi + \phi) \) with \( \pi \) periodicity, which also indicates the absence of \( \sin \phi \)-type current. This result is also consistent with that for the triplet junction in which the two \( d \)-vectors are perpendicular [55]. For the helical states with \( k \)-dependent \( d \)-vectors in this paper, the perpendicular vectors possess the relation \( \langle \mathbf{d}_{\mathbf{y}, \mathbf{k}} \cdot \mathbf{d}^\dagger_{\mathbf{x}, \mathbf{k}} \rangle_k = 0 \) in which \( \langle \cdot \cdot \rangle_k \) denotes the average over the momentum parallel to the interface [39].

5. Symmetries of free energy

In the above sections we analyzed the symmetries of the current in Josephson junctions from the viewpoint of symmetries of the Hamiltonian. In fact, the current-phase relation has a more direct relationship with the free energy of junctions, which is derivative of the free energy with respect to \( \phi \). The free energy of the Josephson junctions was constructed in [38] and [39] on the basis of numerical results, and directly reflects the interaction of helical superconductivity and ferromagnetism. In the following sections we will show that the symmetries of the current derived from the Hamiltonian are consistent with those derived from the free energy.
5.1. SSSF/helical TS Josephson junctions

The free energy for the \( s(d_{x^2-y^2}) \)-wave SSSF/\( p_x^\pm \)-wave TS and the \( d_{x^2-y^2} \)-wave SSSF/\( p_y^\pm \)-wave TS junctions, denoted by \( F_{ST1} \), is given by

\[
F_{ST1} \propto \sin \theta_m \cos \phi_m \sin \phi
\]

for \( V_0 = 0 \) and

\[
F_{ST1} \propto \sin \theta_m \cos \phi_m \sin \phi \quad \& \quad \sin \theta_m \cos \theta_m \sin \phi_m \cos \phi
\]

for \( V_0 \neq 0 \). The proportional coefficient before each term in the expressions is generally a complex function of parameters such as \( M \), \( V_0 \) and \( \Delta_m \), irrespective of the symmetries of the free energy. We omitted them for simplicity. The symbol \& suggests that there are two terms contributing to the current when \( V_0 = 0 \). The Josephson current is accordingly given by

\[
I_J \propto \sin \theta_m \cos \phi_m \cos \phi
\]

for \( V_0 = 0 \) and

\[
I_J \propto \sin \theta_m \cos \phi_m \cos \phi \quad \& \quad \sin \theta_m \cos \theta_m \sin \phi_m \sin \phi
\]

for \( V_0 \neq 0 \). It is easy to verify that the current for \( V_0 = 0 \) satisfies equations (31) and (33)–(37). However, equations (33) and (34) do not hold for the current with \( V_0 \neq 0 \) due to the presence of the second term in equation (51), which is consistent with the true condition of equations (33) and (34).

The free energy for the \( s(d_{x^2-y^2}) \)-wave SSSF/\( p_x^\pm \)-wave TS and the \( d_{x^2-y^2} \)-wave SSSF/\( p_y^\pm \)-wave TS junctions, denoted by \( F_{ST2} \), is given by

\[
F_{ST2} \propto \sin \theta_m \sin \phi_m \sin \phi
\]

for \( V_0 = 0 \) and

\[
F_{ST2} \propto \sin \theta_m \sin \phi_m \sin \phi \quad \& \quad \sin \theta_m \cos \theta_m \cos \phi_m \sin \phi
\]

for \( V_0 = 0 \). The Josephson current is accordingly given by

\[
I_J \propto \sin \theta_m \sin \phi_m \cos \phi
\]

for \( V_0 = 0 \) and

\[
I_J \propto \sin \theta_m \sin \phi_m \cos \phi \quad \& \quad \sin \theta_m \cos \theta_m \sin \phi_m \sin \phi
\]

for \( V_0 \neq 0 \). It is easy to verify that the current for \( V_0 = 0 \) satisfies equations (31), (33)–(35), (38) and (39). However, equations (33) and (34) do not hold for the current with \( V_0 \neq 0 \) due to the presence of the second term in equation (55), which is consistent with the true condition of equations (33) and (34).

5.2. Helical TS/helical TS Josephson junctions

The free energy for the \( p_y^\pm \)-wave TS\( \mid p_y^\pm \)-wave TS junction, denoted by \( F_{TT1} \), is given by

\[
F_{TT1} \propto \cos^2 \theta_m \cos \phi \quad \& \quad \sin^2 \theta_m \cos 2\phi_m \cos \phi.
\]

The Josephson current is accordingly given as

\[
I_J \propto \cos^2 \theta_m \sin \phi \quad \& \quad \sin^2 \theta_m \cos 2\phi_m \sin \phi.
\]

It can easily be demonstrated that the current satisfies equations (40)–(43). In addition, equation (46) can be derived directly from equation (57), which is not obeyed by the SSSF/helical \( p \)-wave TS junctions due to the absence of the close relation discussed in part 4.3.

The free energy for the \( p_y^\pm \)-wave TS\( \mid p_y^\pm \)-wave TS junctions, denoted by \( F_{TT2} \), is given by

\[
F_{TT2} \propto \sin^2 \theta_m \sin 2\phi_m \cos \phi
\]

for \( V_0 = 0 \) and

\[
F_{TT2} \propto \sin^2 \theta_m \sin 2\phi_m \cos \phi \quad \& \quad \cos \theta_m \sin \phi
\]

for \( V_0 \neq 0 \). The Josephson current is accordingly given as

\[
I_J \propto \sin^2 \theta_m \sin 2\phi_m \sin \phi
\]

for \( V_0 = 0 \) and

\[
I_J \propto \sin^2 \theta_m \sin 2\phi_m \sin \phi \quad \& \quad \cos \theta_m \cos \phi
\]

for \( V_0 \neq 0 \). It can be demonstrated that the current satisfies equations (40), (41), (44) and (45). In addition, equations (46) and (47) can be derived directly from equations (60) and (61), which are not satisfied by the SSSF/helical \( p \)-wave TS junctions due to the absence of the connection discussed in part 4.3.

6. Conclusions

We establish links between the symmetries of the Hamiltonian for topological superconducting systems and the invariance of the transport quantities for the corresponding junctions. As observables, transport quantities exhibit higher symmetries than the systems themselves. We reveal the important role that partial symmetries play in the invariance of the conductance and Josephson current. Our analysis explains the numerical results reported recently, including the selection rules for the lowest order current and the rich phase diagrams in Josephson junctions. The analytical method not only provides a profound understanding of topological junctions, but it also helps to provide useful information about transport quantities before complex numerical calculations are carried out. In addition, the symmetry analysis of transport properties in this paper is general and applicable to other topological junctions, such as chiral ones. However, these studies do not include the \( 4\pi \)-periodic Josephson effect caused by the transmission of unpaired electrons [56]. This novel effect is related to the fermion-parity anomaly in the superconducting ground state.

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Appendix

Taking the $H_{p^+}$-wave TS junction in figure 3(a) as an example, we show in detail how the symmetries of the conductance are obtained from the symmetries of the Hamiltonian for the helical superconductor.

The BdG Hamiltonian for the ferromagnetic region ($x < 0$) is

$$H_F(k) = \left( \hat{\varepsilon}(k) - M \cdot \hat{\sigma} - i \varepsilon'(k) + M \cdot \hat{\sigma} \right), \quad (A.1)$$

with $\hat{\varepsilon}(k) = \left( \frac{\hbar^2 k^2}{2m} - E_F \right) \hat{1}$ and $M = M \hat{n}(\theta_m, \phi_m)$ specified by the direction $\hat{n} = (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m)$ with the polar angle $\theta_m$ and the azimuthal angle $\phi_m$. For the spin of electrons, we chose the direction of $M$ as the quantization axis. Let us consider that an electron with majority spin (spin-up) is injected from $F$. The wave functions can be given by

$$\Psi_F(x < 0) = \psi_F \eta e^{ik_1x} + \psi_F^\ast \bar{\eta} e^{-ik_1x},$$

where $\psi_F(= (\chi_1, \chi_2, 0, 0)^T)$, $\psi_F^\ast = (-\chi_2, \chi_1, 0, 0)^T$, $\eta = (0, 0, \chi_1, \chi_2)^T$, and $\bar{\eta} = \sin \frac{\theta_m}{2} e^{i\phi_m}$; the wavevectors $k_1 = k_h = \left\sqrt{\frac{2m}{\hbar} (E_F - M - \frac{k^2}{2})} \right.$ and $k_{e1} = k_{h1} = \left\sqrt{\frac{2m}{\hbar} (E_F - M + \frac{k^2}{2})} \right.$. The coefficients $b_1(b_1)$ and $a_1(a_1)$ represent the normal reflection to the majority (minority) spin subband and the Andreev reflection to the majority (minority) spin subband, respectively, which correspond to the scattering process shown in figure 3(b).

The BdG Hamiltonian for the superconducting region ($x > 0$) is

$$H_S(k) = \left( \hat{\Delta}(k) - \hat{\Delta}^\ast(-k) \right), \quad (A.3)$$

where $\hat{\Delta}(k) = (d_{k+}^\dagger(k) \cdot \hat{\sigma} \cdot d_{k-}^\dagger(k) \cdot \hat{\sigma}) = \Delta_0(k_x \hat{k} + k_y \hat{\gamma})$. The wave function is given by

$$\Psi_S(x > 0) = c_{\downarrow} \psi_S \eta e^{ik_2x} + c_{\downarrow} \psi_S^\ast \bar{\eta} e^{-ik_2x} + d_{\uparrow} \psi_S \eta e^{ik_2x},$$

where $\psi_S = (u, 0, -v y(k_x), 0)^T$, $\psi_S^\ast = (0, u, 0, v y(k_x))^T$, $\psi_{S_+} = (-v y'(k_x), 0, 0, 0)^T$, $\psi_{S_+}^\ast = (0, v y'(k_x), 0, 0)^T$ with $u(v) = \sqrt{\frac{E_F + co0127m}{co0127E - co0127\Delta}}$ and the phase factor $\eta(k_x) = \frac{\hbar}{\frac{2m}{\hbar}co0127k_x} + k_x = \frac{k^2_x - k^2_y}{k_y}$.

The coefficients $c_{\downarrow}$, $c_{\downarrow}$, $d_{\uparrow}$, and $d_{\uparrow}$ represent the transmission of the electron-like quasiparticle and the hole-like quasiparticle, respectively.

All the coefficients in the wave functions can be determined under the boundary conditions:

$$\Psi_F(x = 0^-) = \Psi_S(x = 0^+), \quad (A.5)$$

$$\Psi_F'(x = 0^-) - \Psi_S'(x = 0^+) = \frac{2mV_0}{\hbar^2} \Psi_F(x = 0). \quad (A.6)$$

Obviously, the obtained coefficients will be functions of $\theta_m$, $\phi_m$, $k_x$, and $V_0$. The coefficients for an injection electron with minority spin can be obtained in a similar way. According to the Blonder–Tinkham–Klapwijk formalism, the conductance for a given $V_0$ can be written as

$$\sigma(\theta_m, \phi_m) = \sum_{k} \sigma(\theta_m, \phi_m, k_y),$$

where

$$\sigma(\theta_m, \phi_m, k_y) = \sigma_1 + \sigma_0,$$

and

$$\sigma_1 = \frac{1}{2} \left( 1 + \frac{k_{h2}}{k_{e1}} \right) |a_1(\theta_m, \phi_m, k_y)|^2 + \frac{k_{h1}}{k_{e1}} |a_1(\theta_m, \phi_m, k_y)|^2$$

$$- \frac{k_{h2}}{k_{e1}} |b_1(\theta_m, \phi_m, k_y)|^2 - \frac{k_{h1}}{k_{e1}} |b_1(\theta_m, \phi_m, k_y)|^2, \quad (A.8)$$

with $X = M \frac{k}{k_0}$.

In order to derive the symmetry relations in equations (26)–(28), we firstly transform the Hamiltonian for the superconductor in equation (A.3) with the gauge-rotation operation $\mathcal{R}(\theta, \phi)$. The transformation does not change $H(k)$; the wave function in the superconducting region remain unchanged. The same operation can change the Hamiltonian $H_F(k)$ with $\hat{n}(\theta_m, \phi_m)$ in equation (A.1) into $H_F(k)$ with $\hat{n}(\theta_m + \pi, \phi_m)$; the wave function in the ferromagnetic region can correspondently be obtained through the substitution of $\pi + \phi_m$ for $\phi_m$ in $\Psi_F$. Under the boundary conditions in equations (A.5) and (A.6), the coefficients for the transformed system can be determined, which are functions of $\theta_m$, $\pi + \phi_m$, and $k_x$ for a given $V_0$. The conductance can be obtained by the substitution of $\pi + \phi_m$ for $\phi_m$ in equations (A.7)–(A.9), i.e., $\sigma(\theta_m, \pi + \phi_m) = \sigma(\theta_m, \phi_m)$, as a unitary operator, $\mathcal{R}(\theta, \phi)$ will keep the conductance invariant under the transformation. Then we have the relation $\sigma(\theta_m, \phi_m) = \sigma(\theta_m, \pi + \phi_m)$.

Secondly, we consider the operations $\mathcal{R}(\pi, 0)$ and $\mathcal{R}(\pi, \pi)$. They not only transform the direction $\hat{n}(\theta_m, \phi_m)$ in $H_F(k)$ into $\hat{n}(\pi - \theta_m, \pi - \phi_m)$ and $\hat{n}(\pi - \theta_m, 2\pi - \phi_m)$, respectively, but they also change the superconducting Hamiltonian $H_F(k)$ into $H(k_x, -k_y)$. The wave functions for the transformed Hamiltonian in the ferromagnetic region can be obtained by similar substitutions for $\theta_m$ and $\phi_m$ in $\Psi_F$ as discussed; the wave function in the superconducting region can be obtained by replacing $k_y$ with $-k_y$ in $\Psi_S$. Under the
boundary conditions, the coefficients and thus the con-ductance for the transformed system can be determined, i.e. \( \sigma(\pi - \theta_m, \pi - \phi_m) = C_0 \sum_k \sigma(\pi - \theta_m, \pi - \phi_m - k_y) = C_0 \sum_k \sigma(\pi - \theta_m, \pi - \phi_m, k_y) \) and \( \sigma(\pi - \theta_m, 2\pi - \phi_m - k_y) = C_0 \sum_k \sigma(\pi - \theta_m, 2\pi - \phi_m, k_y) \). Then we have the relations in equations (27) and (28) due to the unitarity of \( \mathcal{R}(\pi, 0) \) and \( \mathcal{R}(\pi, \pi) \). For other \( Y \) helical TS junctions, one can derive the symmetry relations in a similar way.

Finally, we discuss how to derive symmetries of the Josephson current from the symmetries satisfied by the Hamiltonian. In order to express the current, we first need to obtain the retarded Green’s function, which can be constructed with the scattering wave function in superconductors \cite{51, 52}. The wave function and the scattering coefficients can be obtained by solving the BdG equation under boundary conditions. The solving process is the same as that for the conductance case. Then, the changes of the Hamiltonian under the symmetry transformations will enter into the wave function and the coefficients. In fact, the final expression of the current can be written as a combination of the Andreev reflection coefficients \cite{57}, i.e. \( F(\omega_m, k_y) \) in equation (30) is proportional to \( a(\theta_m, \phi_m, \omega_m, k_y, \phi) \). The effects of symmetry transformations on the Andreev coefficient have been discussed carefully. The symmetries of the current can be derived in a similar way.

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