“FASTER THAN LIGHT” PHOTONS IN GRAVITATIONAL FIELDS

– CAUSALITY, ANOMALIES AND HORIZONS

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Abstract

A number of general issues relating to superluminal photon propagation in gravitational fields are explored. The possibility of superluminal, yet causal, photon propagation arises because of Equivalence Principle violating interactions induced by vacuum polarisation in QED in curved spacetime. Two general theorems are presented: first, a polarisation sum rule which relates the polarisation averaged velocity shift to the matter energy-momentum tensor and second, a ‘horizon theorem’ which ensures that the geometric event horizon for black hole spacetimes remains a true horizon for real photon propagation in QED. A comparison is made with the equivalent results for electromagnetic birefringence and possible connections between superluminal photon propagation, causality and the conformal anomaly are exposed.
1. Introduction

The possibility of superluminal photon propagation in gravitational fields is one of the most remarkable predictions of quantum field theory in curved spacetime. It appears that real photons propagating in a variety of background spacetimes may, depending on their direction and polarisation, travel with speeds exceeding the normal speed of light \( c \).

This phenomenon was discovered by Drummond and Hathrell in 1980 [1]. It is a quantum effect induced by vacuum polarisation and implies that the Principle of Equivalence is violated in interacting quantum field theories such as QED.

In their original paper [1], Drummond and Hathrell studied photon propagation in Schwarzschild, Robertson-Walker, gravitational wave and de Sitter backgrounds. In each case, except the totally isotropic de Sitter spacetime, it was possible to find directions and polarisations for which the photon velocity exceeds \( c \). In a subsequent paper with Daniels [2], we extended this analysis to the Reissner-Nordström spacetime describing a charged black hole with similar results. Previously, Ohkuwa [3] had generalised the Drummond-Hathrell result to massless neutrino propagation in a Robertson-Walker metric using the Weinberg-Salam model.

The effect is best described as a modification of the light cone in a local inertial frame (LIF) to \((\eta_{ab} + \alpha \sigma_{ab}(R))k^a k^b = 0\), where \( \eta_{ab} \) is the Minkowski metric, \( \alpha \) is the fine structure constant and \( \sigma_{ab}(R) \) depends on the Riemann curvature at the origin of the LIF. The correction arises from vacuum polarisation induced interactions in the QED effective action of the typical form \( \alpha \frac{1}{m^2} R_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \), where \( m \) is the electron mass. Such direct curvature couplings violate the Principle of Equivalence. As discussed in section 2, it is this fact which allows spacelike motion in a LIF without necessarily implying a violation of causality.

The physical origin of this effect may be understood in qualitative terms as follows. Vacuum polarisation allows the photon to exist as a virtual \( e^+ e^- \) pair and so at the quantum level it is characterised by a spacetime scale \( \lambda_c \), the Compton wavelength of the electron. In a gravitational field, the photon is therefore sensitive to an anisotropic spacetime curvature and its characteristics of propagation may become curvature dependent. The spatial anisotropy of the gravitational field results in a polarisation dependence of the effect (gravitational birefringence).

In this paper, we collect a number of new results and theorems on the basic interpretation of the superluminal effect and its realisation in black hole spacetimes.

The Principle of Equivalence would require that the only allowed kinetic term in the effective action for the electromagnetic field is simply \( F_{\mu\nu} F^{\mu\nu} \). In section 2, we review the geometric optics approximation for photon propagation and show how this requirement implies that photon trajectories are null geodesics. We then include the new vacuum polarisation induced interactions and derive the modified, curvature-dependent equations for the photon trajectories.

The violation of the Principle of Equivalence allows the possibility of superluminal propagation without causality violation and we discuss the conditions that must be satisfied to maintain causality. We also discuss the intrinsically quantum question of the propagation of photons whose polarisations do not satisfy the curvature dependent equa-
One of the main potential applications of superluminal propagation is to the physics of black holes. The case of the Schwarzschild black hole was discussed in ref.[1] and the extension to charged (Reissner-Nordström) black holes was given in our previous paper[2]. The analogous results for the Kerr metric describing a rotating black hole will be presented in ref.[4]. We again find a variety of directions and polarisations for which the photons have superluminal velocities. Unlike the Schwarzschild and Reissner-Nordström spacetimes, however, radially directed photons in the Kerr metric have velocities differing from \( c \), except at the event horizon. It turns out that this is a non-trivial special case of a general ‘horizon theorem’, which is stated precisely and proved here in section 3. The theorem states that in a general spacetime with an event horizon, the light cone for photons travelling normal to the horizon surface remains \( k^2 = 0 \) even in the presence of the vacuum polarisation induced interactions. The physical implications of this theorem merit further investigation, but the result seems to ensure that the geometric event horizons of the classical spacetime continue to be true horizons for real photon trajectories.

An amusing application of this theorem is to de Sitter spacetime. This has a cosmological event horizon, specific to each observer. In fact, every point in the spacetime lies on the horizon of some observer and thus, assuming the validity of the horizon theorem in the cosmological context, the velocity of light must be \( c \) everywhere. This is in agreement with the conclusion that the light cone is unchanged because the spacetime is totally isotropic.

A new physical insight into the more general phenomenon of photon propagation in non-trivial vacua has been given by Latorre, Pascual and Tarrach[5]. Modifications to the velocity of light occur in a variety of situations apart from gravitational backgrounds, including propagation in electromagnetic background fields[6,2], Casimir-type regions with boundaries[7,8,9], finite temperature[5] or density, etc. Latorre et al. have identified an intriguing general formula covering all these cases, relating the polarisation (and, if necessary, direction) averaged velocity shift to the background energy density, with a universal numerical coefficient.* In cases where the background energy is positive, the average velocity is less than \( c \) whereas in negative energy situations such as Casimir regions, the average velocity is greater than \( c \).

Electromagnetic birefringence, described by Adler in ref.[6], was studied for an arbitrary anisotropic (but homogeneous) background electromagnetic field in our earlier paper[2]. In section 4, we make some further observations. First, motivated by Latorre et al., we rewrite our previous result for the photon velocities in such a way as to make clear that, for an arbitrary background, the direction and polarisation averaged photon velocity shift is indeed proportional to the electromagnetic field energy, with the required numerical coefficient.

Superluminal propagation in this Minkowski spacetime situation would violate causality and so the velocity shifts for both polarisations must be negative. This requires certain

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* Curiously, the coefficient involves the unusual number 11, hinting at a possible relation with the conformal anomaly. The numerical coefficient of the Euler term in the gravitational conformal anomaly is \( N_S + 11 N_F + 62 N_V \), where \( N_S \), \( N_F \) and \( N_V \) are the number of scalar, fermion and vector fields respectively. In the conformal anomaly, the factor 11 arises from a one-loop, background field calculation involving a single fermion, as does the effective action required to calculate corrections to photon propagation.
combinations of coefficients appearing in the effective action to be negative. Remarkably enough, we find that this is precisely the condition for the VEV of the trace of the energy-momentum tensor in the electromagnetic background to be negative, although the physical significance of such a requirement is not immediately clear. This is another piece of evidence suggesting a deeper connection between the conformal anomaly and photon propagation. In a sense, some such relation might be expected since the presence in the quantum theory of background interaction terms in the effective action necessarily introduces a mass scale, removing the conformal invariance of the classical photon action.

The explicit results for Ricci flat metrics, including Schwarzschild and Kerr, suggest a second theorem, which we prove in section 3. We show that in Ricci flat spacetimes, the velocity shifts are equal and opposite in sign for the two physical transverse polarisations. The polarisation averaged velocity shift is therefore zero in these vacuum solutions of the Einstein equation where the matter energy-momentum tensor vanishes. However, the fact that a superluminal birefringent effect exists even for Ricci flat spacetimes (with $T_{\mu\nu} = 0$) shows that vacuum energy density cannot provide the whole explanation for modified photon propagation. Birefringence associated with an explicit Riemann curvature interaction in a non-isotropic gravitational field is an alternative mechanism to modify the light cone.

The generalisation of this polarisation sum rule for non Ricci flat spacetimes is also given in section 3. This is particularly relevant for the Friedmann-Robertson-Walker and Reissner-Nordström spacetimes. In the general case, the polarisation average acquires a term proportional to $T_{\mu\nu} e^\mu e^\nu$, where $e^\mu = k^\mu / |k|$ and $T_{\mu\nu}$ is the matter energy-momentum tensor, in accord with the specific results of refs.[1] and [2]. This term is precisely the same as arises in the case of electromagnetic birefringence. The additional Riemann curvature term which induces birefringence in the gravitational case is identified as the Newman-Penrose scalar $\Psi_0$.

The major interpretational issues left open in previous work have concerned the question of dispersion and whether this superluminal propagation is observable in principle. These difficulties primarily concern the specific mechanism of generating explicit curvature couplings through vacuum polarisation proposed in ref.[1]. We address the problem of dispersion in the context of the effective action for QED in a background gravitational field in ref.[10]. It should be emphasised, however, that the interest in such Equivalence Principle violating interactions is broader than this. Once we have established that such interactions may exist without necessarily involving causality violation, it becomes an experimental question whether they do in fact exist and with what characteristic length scale. Such interactions could be searched for, for example, by studying polarisation dependence in gravitational lensing.\*

Of course, for weak fields the magnitude of the vacuum polarisation induced effect is tiny. For example, the modification to the angle of deflection of light by the sun is only a factor of $O(10^{-47})$ [1]. However, this is typical of all quantum field effects in macroscopic gravitational fields, an important example being Hawking radiation[11]. Just as for Hawking radiation, the superluminal effect should become large for curvatures comparable with the quantum scale, in this case $\lambda_c$. Such curvatures may arise either for microscopic black holes or in the very early universe.

\* This suggestion is due to I.T. Drummond (private communication).
2. Photon Propagation and Causality

We are concerned with photon propagation in QED in a fixed background curved spacetime. In this paper, we consider the properties of photon propagation implied by the effective action in the form derived by Drummond and Hathrell [1],

\[ \Gamma = \int dx \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{m^2} \left( a RF_{\mu\nu} F^{\mu\nu} + b R_{\mu\nu} F^{\mu\nu} F_{\lambda\mu} + c R_{\mu\nu\lambda\rho} F^{\mu\nu} F^\rho{}_{\lambda\mu} \right) \right] \quad (2.1) \]

Here, \( a = -\frac{1}{144} \frac{\alpha}{\pi}, b = \frac{13}{360} \frac{\alpha}{\pi}, \) and \( c = -\frac{1}{360} \frac{\alpha}{\pi}, \) where \( \alpha \) is the fine structure constant, and \( m \) is the electron mass. Further contributions to the effective action involving higher derivatives will be discussed in ref. [10], where we consider the question of dispersion. Alternatively, the action (2.1) could be regarded as a starting point in its own right, with no prior assumption on the magnitude of the constants \( a, b \) and \( c \) or the scale \( m^2 \).

The simplest way to determine the characteristics of photon propagation starting from the effective action is to use geometric optics. In the leading geometric optics approximation, the electromagnetic field strength is written as the product of a slowly varying amplitude and a rapidly varying phase, i.e.

\[ F_{\mu\nu} = f_{\mu\nu} e^{i\theta} \quad (2.2) \]

and the wave vector is defined as \( k_{\mu} = \partial_{\mu} \theta. \) In the quantum interpretation in terms of photons, \( k_{\mu} \) is identified as the photon momentum. The Bianchi identity,

\[ D_{\mu} F_{\nu\lambda} + D_{\nu} F_{\lambda\mu} + D_{\lambda} F_{\mu\nu} = 0 \quad (2.3) \]

becomes

\[ k_{\mu} f_{\nu\lambda} + k_{\nu} f_{\lambda\mu} + k_{\lambda} f_{\mu\nu} = 0 \quad (2.4) \]

and constrains \( f_{\mu\nu} \) to be of the form

\[ f_{\mu\nu} = k_{\mu} a_{\nu} - k_{\nu} a_{\mu}. \quad (2.5) \]

The direction of \( a^\mu \) specifies the polarisation. Clearly we can assume \( k^\mu a_\mu = 0. \) Of the three remaining possibilities, only the two orthogonal to the photon momentum are physical in the quantum theory.

We illustrate this method first for the classical electromagnetic action. The photon equation of motion is simply

\[ D_{\mu} F^{\mu\nu} = 0 \quad (2.6) \]

i.e.

\[ k_{\mu} f^{\mu\nu} = 0 \quad (2.7) \]

It now follows from eqs. (2.4) and (2.7) that photon trajectories are null geodesics. To see this, first multiply (2.7) by \( k^\lambda \) and use the Bianchi identity. This gives

\[ 0 = k_{\mu} k^\lambda f^{\mu\nu} \]

\[ = k^2 f^{\lambda\nu} + k^\nu k_{\mu} f^{\mu\lambda} \]

\[ = k^2 f^{\lambda\nu} \quad (2.8) \]
from which we deduce \( k^2 = 0 \), i.e. \( k_\mu \) is a null vector. Next, it follows from the definition of \( k_\mu \) as a gradient that \( D_\mu k_\nu = D_\nu k_\mu \) and so

\[
k_\mu D_\mu k_\nu = k_\mu D_\nu k_\mu = \frac{1}{2} D^\nu k^2 = 0.
\]  (2.9)

Light rays (photon trajectories) are defined as the integral curves of the wave vector (photon momentum), i.e. the curves \( x_\mu(s) \) for which \( \frac{dx_\mu}{ds} = k_\mu \). Substituting into eq.(2.9) gives

\[
0 = k_\mu D_\mu k_\mu = \frac{d^2 x_\nu}{ds^2} + \Gamma^\nu_{\mu\lambda} \frac{dx^\mu}{ds} \frac{dx^\lambda}{ds}.
\]  (2.10)

This is the geodesic equation.

These familiar properties are no longer true when we consider the equations of motion derived from the effective action including the quantum corrections. The Bianchi identity remains unchanged, but the equation of motion becomes

\[
D_\mu F^{\mu\nu} + \frac{1}{m^2} \left[ 2b R^{\mu\lambda} D_\mu F^{\lambda\nu} + 4c R^{\mu\nu\lambda\rho} D_\mu F^{\lambda\rho} \right] = 0
\]  (2.11)

i.e.

\[
k_\mu f^{\mu\nu} + \frac{1}{m^2} \left[ 2b R^{\mu\lambda} k_\mu f^{\lambda\nu} + 4c R^{\mu\nu\lambda\rho} k_\mu f^{\lambda\rho} \right] = 0
\]  (2.12)

In writing eq.(2.11), we have neglected a number of terms of sub-leading order:

(i) Assuming the background gravitational field varies with the typical curvature scale \( L \), terms involving derivatives of the curvature are suppressed relative to the leading correction by \( O(\lambda/L) \), where \( \lambda \) is the photon wavelength.

(ii) The Ricci scalar term simply gives a correction to the \( D_\mu F^{\mu\nu} \) coefficient proportional to \( aR/m^2 \). This is suppressed by \( O(\lambda^2 c/L^2) \) in the weak field approximation where we neglect higher powers of the curvature in the effective action. A term involving the Ricci tensor multiplied by \( D_\mu F^{\lambda\mu} \) is neglected for a similar reason.

(iii) The standard geometric optics approximation is made where we neglect derivatives acting on \( f_{\mu\nu} \) relative to those acting on the phase factor to produce powers of momentum.

Rewriting eq.(2.12) as an equation for the polarisation vector \( a^\mu \) (which from now on we take to be spacelike normalised, \( a^\mu a_\mu = -1 \)), we find

\[
k^2 a_\nu + \frac{2b}{m^2} \left[ R^{\mu\lambda} (k_\mu k^\lambda a_\nu - k_\mu k^\nu a^\lambda) \right] + \frac{8c}{m^2} \left[ R^{\mu\nu\lambda\rho} k_\mu k_\lambda a_\rho \right] = 0
\]  (2.13)

In what follows we will be concerned with the modifications to the light cone condition in a local inertial frame at each point in spacetime. Introducing local Lorentz components using the vierbeins \( e^a_\mu \) defined by \( g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \), we have

\[
k^2 a^b + \frac{2b}{m^2} \left[ R_{ac} (k^a k^c a^b - k^a k^b a^c) \right] + \frac{8c}{m^2} \left[ R_{abcd} k^a k^c a^d \right] = 0
\]  (2.14)
Provided we find a polarisation for which this equation is satisfied, the corresponding light cone condition is found by contracting with $a_b$, giving

$$k^2 + \frac{2b}{m^2} R_{ac} k^a k^c - \frac{8c}{m^2} R_{abcd} k^a k^b a^d = 0 \quad (2.15)$$

Eqs.(2.14) and (2.15) are manifestly local Lorentz invariant, provided the curvatures $R_{abcd}$ and $R_{ab}$ are appropriately transformed. However, the presence of the (position-dependent) curvature terms means that it does not reduce to the special relativistic equation at the origin of the LIF and is different for LIFs at different points in spacetime. This dynamical equation therefore violates the Principle of Equivalence, which asserts that all LIFs are equivalent.

At this point, we should perhaps digress a little on the rôle of the Principle of Equivalence in general relativity. Throughout the paper, we understand this to refer to the strong Principle of Equivalence. The so-called weak Principle of Equivalence states that at each point in spacetime there exists a local Minkowski frame. This is simply the requirement, fundamental to general relativity, that spacetime is (pseudo-)Riemannian. The strong Principle of Equivalence, however, goes on to assert the equivalence of the laws of physics in LIFs established at different points in spacetime, and furthermore that these laws take their special relativistic form at the origin of each LIF. The strong Principle of Equivalence is therefore an extra arbitrary dynamical condition added to the basic structure of general relativity and is essentially a specification of minimal coupling to the gravitational field in the effective action.* Its rôle is merely to exclude direct curvature couplings such as those appearing in eq.(2.1).

Viewed in this light, there is nothing fundamental about the Principle of Equivalence. Whether or not it is true is an experimental question. In the light of the Drummond-Hathrell result, it seems extremely unlikely that it should remain true, unless of course some unified theory incorporating QED and the standard model provided a mechanism for the systematic cancellation of the vacuum polarisation corrections identified in ref.[1]. In any case, it would be interesting experimentally to look for evidence of equivalence principle violating interactions such as those in eq.(2.1) without theoretical prejudice as to the controlling mass scale (in our case, the electron mass). In principle, such effects could be observed through, for example, polarisation dependence in gravitational lensing.

Of course, one apparent motivation for imposing the Principle of Equivalence is that by excluding curvature couplings a priori, photons are constrained to follow null geodesics and the question of possible causality violation does not arise. What Drummond and Hathrell have shown is that this is not possible in QED. We shall return to the rôle of the Principle of Equivalence in relation to causality below.

The photon trajectories corresponding to the new equation of motion are easily found by a straightforward generalisation of eq.(2.9), though the resulting equation appears too complicated to be useful in general. The trajectories satisfy

$$\frac{d^2 x^\nu}{ds^2} + \Gamma_{\lambda \rho}^\nu \frac{dx^\lambda}{ds} \frac{dx^\rho}{ds} + \frac{1}{m^2} D_\nu \left[ (b R_{\lambda \rho} - 4c R_{\lambda \sigma \rho \tau} a^\sigma a^\tau) \frac{dx^\lambda}{ds} \frac{dx^\rho}{ds} \right] = 0 \quad (2.16)$$

* Notice, however, that we do impose minimal coupling in the classical action in the path integral. The violation of the Principle of Equivalence occurs at the quantum level.
where $a^a$ are polarisations satisfying the equation of motion (2.14).

This brings us to the issue of possible violations of causality due to the modified trajectories. First, consider the situation in special relativity. It is certainly true that a spacelike signal from $A$ to $B$ necessarily corresponds to motion backwards in time in a class of inertial frames. However, it should be recognised that this in itself is not a problem with regard to causality[1]. A causal paradox only arises if a signal can then be sent back from $B$ to a point $C$ on the past world line of $A$. The point is that in special relativity, Poincaré invariance (the equivalence of the laws of physics in all inertial frames) assures us that such a signal is possible, given the possibility of $A \to B$. Two conditions are necessary to establish a causal paradox – spacelike motion and Poincaré invariance.

In the general relativistic context, we of course lose global Poincaré invariance. The nearest analogue to the second condition above is the Principle of Equivalence, i.e. the equivalence of the laws of physics in all local inertial frames. But this is precisely the condition which we have shown is violated in establishing the possibility of spacelike photon propagation. The allowed photon motion depends explicitly on the local curvature. Provided the Principle of Equivalence is violated, therefore, it does not follow that spacelike motion in general relativity necessarily implies a violation of causality.

Of course, this argument merely removes the most obvious objection to the possibility of superluminal propagation. It does not prove that causality violation does not occur – to prove this would require showing that trajectories satisfying (2.16) can never result in signals returning to the past world line of the emitter. However, since we have established that there is no necessity for causality violation, there seems no strong reason to doubt that, despite the modification to the light cone, QED in curved spacetime remains a causal theory.

Finally in this section, notice that the equation of motion (2.14) only admits solutions for certain choices of the polarisation vector $a^b$. Classically, other polarisations do not propagate – the spacetime is opaque to all but the selected polarisations. However, this clearly has to be reassessed in the quantum theory where any superposed state is permissible and where we must take account of the fact that particle trajectories in the above sense are not well-defined. Suppose, therefore, that we have a polarisation state which does not correspond to one of the classically allowed states. To determine how it propagates, we first reexpress it as a linear superposition of the two permitted polarisation states transverse to the photon momentum. These states propagate according to their respective light cone conditions with different $k^2$. The finally observed state is then in general a different linear superposition, in the usual way for elementary two-state quantum mechanical systems.
3. The Horizon Theorem and Polarisation Sum Rule

Our experience with the explicit examples of photon propagation in the different types of black hole spacetimes, Schwarzschild[1], Reissner-Nordström[2] and Kerr[4], suggests two general features. First, no matter what happens inside or outside the horizon, it seems to be a general result that the velocity of radially directed photons remains equal to $c$ exactly at the horizon. Second, for the Ricci flat spacetimes, the velocity shifts for the two transverse polarisations are always equal and opposite. This is no longer true for non-Ricci flat spacetimes such as FRW [1] and Reissner-Nordström[2]. In these cases, the polarisation averaged velocity shift is proportional to the matter energy-momentum tensor.

These theorems are most easily proved by using the Newman-Penrose formalism, which characterises spacetimes using a set of complex scalars found by contracting the Weyl tensor with elements of a null tetrad. (For a clear introduction to this formalism, see e.g. ref.[12].)

For our purposes, we choose the basis vectors of the null tetrad as follows[13]. Choose $\ell^\mu = k^\mu$, the photon momentum. Then, denote the two spacelike, normalised, transverse polarisation vectors by $a^\mu$ and $b^\mu$ and construct the null vectors $m^\mu = \frac{1}{\sqrt{2}}(a^\mu + ib^\mu)$ and $\bar{m}^\mu = \frac{1}{\sqrt{2}}(a^\mu - ib^\mu)$. Complete the tetrad with a further null vector $n^\mu$ orthogonal to $m^\mu$ and $\bar{m}^\mu$. We therefore have the usual Newman-Penrose conditions,

$$\ell.m = \ell.\bar{m} = n.m = n.\bar{m} = 0 \quad (3.1)$$

from orthogonality, and

$$\ell.\ell = n.n = m.m = \bar{m}.\bar{m} = 0 \quad (3.2)$$

since the basis vectors are null. In addition, we impose

$$\ell.n = 1 \quad \text{and} \quad m.\bar{m} = -1 \quad (3.3)$$

We denote components with respect to this tetrad frame by the indices $p,q,\ldots = 1,2,3,4$ corresponding to $\ell, n, m, \bar{m}$ respectively. (We follow the notation of ref.[12], sect. 1.8.) The metric takes the form

$$\eta^{pq} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix} \quad (3.4)$$

In this basis, the components of the Weyl tensor are given in terms of the Riemann and Ricci tensors by

$$C_{pqrs} = R_{pqrs} - \frac{1}{2}(\eta_{pq}R_{qs} - \eta_{qr}R_{ps} - \eta_{ps}R_{qr} + \eta_{qs}R_{pr}) + \frac{1}{6}(\eta_{pq}R_{qs} - \eta_{ps}R_{qr})R \quad (3.5)$$

where $R_{pr} = \eta^{qs}R_{pqrs}$ and $R = \eta^{pq}R_{pq}$. The Weyl tensor satisfies the trace-free condition

$$\eta^{ps}C_{pqrs} = 0 \quad (3.6)$$
together with cyclicity

\[ C_{1234} + C_{1342} + C_{1423} = 0 \] (3.7)

An important property of the Weyl tensor, which is relevant to our later discussion, is that it is invariant under conformal (Weyl) rescalings of the metric. The ten independent components of the Weyl tensor are denoted by the five complex Newman-Penrose scalars:

\[
\begin{align*}
Ψ_0 &= -C_{pqrs}\ell^p m^q \ell^r m^s \\
Ψ_1 &= -C_{pqrs}\ell^p n^q \ell^r m^s \\
Ψ_2 &= -C_{pqrs}\ell^p m^q \bar{m}^r n^s \\
Ψ_3 &= -C_{pqrs}\ell^p n^q \bar{m}^r n^s \\
Ψ_4 &= -C_{pqrs}\bar{r}^p m^q n^r \bar{m}^s
\end{align*}
\] (3.8)

The components of the Ricci tensor are given a similar representation[12]. In particular, we will use the notation \( \Phi_{00} = -\frac{1}{2}R_{11} = -\frac{1}{2}R_{pq}\ell^p \ell^q \).

With these preliminaries, we are now ready to state our theorems:-

**(a) Polarisation Sum Rule**

For Ricci flat spacetimes, the sum over the two physical polarisations of the velocity shift is zero, i.e.

\[ \sum_{\text{pol}} \delta v = 0 \] (3.9)

For non Ricci flat spacetimes satisfying the Einstein field equations, we have

\[ \sum_{\text{pol}} \delta v = -\frac{8\pi}{m^2}(2b + 4c) T_{\mu\nu} e^\mu e^\nu \] (3.10)

where \( T_{\mu\nu} \) is the energy-momentum tensor and \( e^\mu = k^\mu/|k| \) specifies the photon direction.

If we substitute the vacuum polarisation induced values for the constants \( b \) and \( c \) into this equation, we find

\[ \sum_{\text{pol}} \delta v = -\frac{22}{45} \frac{\alpha}{m^2} T_{\mu\nu} e^\mu e^\nu \] (3.11)

Notice the occurrence of the universal coefficient[5] involving the factor 11. It is straightforward to check that this formula is consistent with the explicit results for the non Ricci flat FRW [1] and Reissner-Nordström[2] spacetimes.

We present the proof for the Ricci flat case first. From the modified light cone condition (2.15), eq.(3.9) follows immediately if we can show

\[ \sum_{\text{pol}} R_{abcd}k^a k^c a^b a^d = 0 \] (3.12)
i.e. in the Newman-Penrose tetrad basis,

\[ C_{pqrs}^{\ell p \ell q} \left( m^q \bar{m}^s + \bar{m}^q m^s \right) = 0 \]  

(3.13)

Notice that it is legitimate to take \( k^a \) to be null in eq.(3.12). This is consistent with the perturbation expansion in powers of \( O(\lambda^2/L^2) \) used in the starting point (2.15).

Now,

\[ C_{pqrs}^{\ell p m^q \ell r \bar{m}^s} = C_{1314} = 0 \]  

(3.14)

from the trace-free condition on the Weyl tensor. To see this, write out eq.(3.6) explicitly,

\[ C_{1qr2} + C_{2qr1} - C_{3qr4} - C_{4qr3} = 0, \]  

(3.15)

set \( q = r = 1 \), and use the symmetries of the Weyl tensor. This establishes (3.13).

Now consider the non-Ricci flat case. Here,

\[ \sum_{\text{pol}} R_{abcd} k^a k^c a^b a^d = C_{pqrs}^{\ell p \ell r} \left( m^q \bar{m}^s + \bar{m}^q m^s \right) - R_{pr}^{\ell p \ell r} \]  

(3.16)

From eq.(2.15) we therefore have

\[ \sum_{\text{pol}} k^2 = -\frac{1}{m^2} \left( 4b + 8c \right) R_{ac} k^a k^c \]  

(3.17)

For spacetimes satisfying the Einstein field equations we can simply replace \( R_{ac} \) by \( 8\pi T_{ac} \) (since \( k^a \) is null) and eq.(3.10) follows.

This brings us to the second theorem, which applies specifically to black hole spacetimes with an event horizon.

**(b) Horizon Theorem**

At the event horizon, photons with momentum directed normal to the horizon have velocity equal to \( c \), i.e. the light cone remains \( k^2 = 0 \), independent of their polarisation.\(^*\)

We prove this in general for Ricci flat or non Ricci flat spacetimes. The null tetrad is chosen as above, so that the physical, spacelike, polarisation vectors \( a^\mu \) and \( b^\mu \) lie in the event horizon 2-surface while \( k^\mu \) is the null vector normal to them. From eq.(2.15) we have

\[ k^2 = -\frac{2b}{m^2} R_{ac} k^a k^c + \frac{8c}{m^2} R_{abcd} k^a k^c a^b a^d \]

\[ = -\frac{1}{m^2} \left( 2b + 4c \right) R_{pr}^{\ell p \ell r} + \frac{4c}{m^2} C_{pqrs}^{\ell p \ell r} \left( m + \bar{m} \right)^q \left( m + \bar{m} \right)^s \]  

(3.18)

\(^*\) This theorem has been formulated and proved independently by G.W. Gibbons and by M.J. Perry (private communication)[16].
in the tetrad basis. Using eq.(3.13), and assuming the Ricci tensor is given in terms of the energy-momentum tensor by the Einstein field equations, this reduces to

\begin{equation}
    k^2 = -\frac{8\pi}{m^2} (2b + 4c) T_{\mu\nu} k^\mu k^\nu + \frac{4c}{m^2} (C_{pqrs} \ell^p m^q \ell^r m^s + \text{c.c.})
\end{equation}

(3.19)

In general, this is non-zero. However, at the event horizon itself, both the Ricci tensor term \( \Phi_{00} = -4\pi T_{\mu\nu} k^\mu k^\nu \) and the Newman-Penrose scalar \( \Psi_0 = C_{pqrs} \ell^p m^q \ell^r m^s \) are zero for stationary spacetimes.*

The proof of these assertions is not simple and may be found in lectures by Hawking[13] (see also ref.[14]). They follow from consideration of the convergence and shear of the generators of the event horizon. The physical interpretation, on the other hand, is clear – the Ricci term represents the flow of matter across the event horizon while the Weyl term represents the flow of gravitational radiation across the event horizon. Both are zero in the classical theory.

This establishes \( k^2 = 0 \) at the level of weak field perturbation theory at which we are working. It is tempting to speculate that the result is more general and would hold also for strong gravitational fields, where we do not have an explicit expression for the modified light cone. The theorem ensures that the geometric event horizon remains a true horizon for real photons in QED.

Before leaving this section, we make a final comment on the light cone condition for photons propagating in a weak gravitational field. We have shown that for an individual polarisation state,

\begin{align*}
    k^2 &= \frac{1}{m^2} (4b + 8c) \Phi_{00} \pm \frac{4c}{m^2} (\Psi_0 + \Psi_0^*) \\
    &= -\frac{8\pi}{m^2} (2b + 4c) T_{\mu\nu} k^\mu k^\nu \pm \frac{4c}{m^2} (\Psi_0 + \Psi_0^*)
\end{align*}

(3.20)

while the polarisation sum removes the Weyl term, leaving

\begin{equation}
    \sum_{\text{pol}} k^2 = -\frac{8\pi}{m^2} (4b + 8c) T_{\mu\nu} k^\mu k^\nu
\end{equation}

(3.21)

The weak energy condition[14] in gravitation theory implies \( T_{\mu\nu} k^\mu k^\nu \geq 0 \) for any null vector \( k^\mu \). Assuming this to be true, we always have \( \sum_{\text{pol}} \delta v \leq 0 \). The relation between the polarisation summed velocity shift and the matter energy-momentum tensor is consistent with, but more general than, the observation of Latorre et al.[5] and should be compared with the corresponding result for photon propagation in a background electromagnetic field in section 4. The specifically gravitational birefringent shift in the photon velocity dependent on the Weyl curvature shows up in the second contribution to eq.(3.20) proportional to the Newman-Penrose scalar \( \Psi_0 \).

* See ref.[13] for a careful discussion, including the distinction for non-stationary spacetimes between the event horizon and the apparent horizon.
4. Electromagnetic Birefringence

In an earlier paper[2], we calculated the modification to the velocity of light in an arbitrary anisotropic (but homogeneous) electromagnetic field in flat spacetime. In this section, we discuss these results a little further, first to make contact with the formula of Latorre et al.[5] and the gravitational formulae in section 3, and second to discuss the relation with the conformal anomaly.

The starting point is the Euler-Heisenberg effective action,

\[ \Gamma = \int dx \left[ -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{m^4} \left( z \left( F_{ab} F^{ab} \right)^2 + y F_{ab} F_{cd} F_{ac} F_{bd} \right) \right] \] (4.1)

where \( z = -\frac{1}{36} \alpha^2 \) and \( y = \frac{7}{90} \alpha^2 \), from which we derive the equation of motion

\[ D_a F^{ab} - \frac{16}{m^4} z m^4 F_{ab} F_{cd} D_a F^{cd} - \frac{8y}{m^4} \left( F_{ac} F_{cd} D_a F^{bd} + F_{ac} F^{bd} D_a F_{cd} \right) = 0 \] (4.2)

These expressions are the analogues of eqs.(2.1) and (2.11) in the gravitational case and are derived using similar approximations. In the geometric optics approximation, the equation of motion becomes[2]

\[ k_a f^{ab} - \frac{16z}{m^4} F_{ab} F_{cd} k_a f^{cd} - \frac{8y}{m^4} \left( F_{ac} F_{cd} k_a f^{bd} + F_{ac} F^{bd} k_a f_{cd} \right) = 0 \] (4.3)

where \( F_{ab} \) is the background electromagnetic field strength. Using the Bianchi identity to set \( f_{ab} = k_a a_b - k_b a_a \), and rewriting in terms of the polarisation vector \( a^a \), we find

\[ k^2 a^b - \frac{8}{m^4} (4z + y) F_{ab} F_{cd} k^a k^c a^d - \frac{8y}{m^4} F_a c F_{cd} \left( k^a k^b a^d - k^a k^d a^b \right) = 0 \] (4.4)

Provided we have a polarisation satisfying this equation, the corresponding modified light cone condition is

\[ k^2 - \frac{8}{m^4} (4z + y) F_{ab} F_{cd} k^a k^c a^d + \frac{8y}{m^4} F_a d F_{cd} k^a k^c = 0 \] (4.5)

It is easy to see that a polarisation satisfying \( F_{ab} a^b = 0 \) solves eq.(4.4).\(^*\) The light cone condition is then

\[ k^2 = \frac{-8y}{m^4} F_a d F_{cd} k^a k^c \] (4.6)

Recalling the form of the classical electromagnetic energy-momentum tensor,

\[ T_{ac} = \left( F_a d F_{cd} - \frac{1}{4} \eta_{ac} F_{bd} F^{bd} \right) \] (4.7)

\(^*\) For the case of a pure magnetic field[6], the first and second polarisation states considered here correspond to polarisations respectively orthogonal to and coplanar with the plane spanned by the photon momentum and magnetic field directions.
and remembering that consistent with the weak field perturbative expansion we can take \( k^a \) to be null on the r.h.s. of eq.(4.6), we have

\[
k^2 = -\frac{8y}{m^4} T_{ac} k^a k^c \tag{4.8}
\]

corresponding to a velocity shift for this polarisation state of

\[
\delta v = -\frac{4y}{m^4} T_{ae} e^a e^c \tag{4.9}
\]

where as before \( e^a = k^a/|k| \).

To deduce the light cone condition for the second polarisation state, we use the identity

\[
\sum_{pol} a^b a^d = - (\eta^{bd} - \bar{e}^b \bar{e}^d) \tag{4.10}
\]

(where \( \bar{e}^a \) is simply \( e^a \) with the sign of the spacelike components reversed) to show that the two terms in eq.(4.5) are equal. So, for the second polarisation, we immediately find

\[
k^2 = -\frac{8}{m^4} (4z + 2y) T_{ac} k^a k^c \tag{4.11}
\]

that is,

\[
\delta v = -\frac{4}{m^4} (4z + 2y) T_{ae} e^a e^c \tag{4.12}
\]

The dependence of the velocity shifts on the energy-momentum tensor is therefore exactly the same as in the gravitational case, except that here the polarisation dependence enters already in determining the coefficient of this term. Unlike the gravitational case, therefore, birefringence occurs here already at the level of the energy-momentum tensor dependence. If we now take the polarisation sum, and substitute the explicit values for the coefficients \( y \) and \( z \), we find

\[
\sum_{pol} \delta v = -\frac{4}{m^4} (4z + 3y) T_{ae} e^a e^c = \frac{22}{45} \frac{\alpha^2}{m^4} T_{ae} e^a e^c \tag{4.13}
\]

Again notice the appearance of the universal coefficient involving the number 11, just as in the gravitational case.

Writing out the energy-momentum tensor in terms of \( E \) and \( B \) fields, we find

\[
T_{ae} e^a e^c = E^2 + B^2 - (E \cdot n)^2 - (B \cdot n)^2 - 2(E \times B) \cdot n \tag{4.14}
\]

where \( n \) is the direction of the photon momentum. In this form, we recover the results derived by a less direct method in ref.[2] (in the notation used there, \( X = T_{ac} k^a k^c \)). It was pointed out there that since the coefficients in both eqs.(4.9) and (4.12) are negative and \( T_{ae} e^a e^c \geq 0 \) (the r.h.s. of eq.(4.14) is a positive definite quantity), both polarisations have velocities less than \( c \), as required by causality in this special relativistic context.
As a final embellishment, we can follow ref.[5] and consider the direction averaged velocity shift. It is clear from the expression (4.14) that if we average over the photon direction, the $\mathbf{E}.\mathbf{n}$, $\mathbf{B}.\mathbf{n}$ and Poynting vector $(\mathbf{E} \times \mathbf{B}).\mathbf{n}$ terms disappear. The remainder is just the energy density of the electromagnetic field. So we have

$$\sum_{\text{pol}} \langle \delta v \rangle = -\frac{4}{m^2} (4z + 3y)(\mathbf{E}^2 + \mathbf{B}^2)$$

(4.15)

where $\langle \rangle$ denotes direction averaging. This is in agreement with the observation of ref.[5].

We now present an intriguing connection between these results and the conformal anomaly. The Euler-Heisenberg action can be rewritten as

$$\Gamma = \int \! dx \left[ -\mathcal{F} + \frac{1}{m^4} \left( 8(2z + y)\mathcal{F}^2 + 4y\mathcal{G}^2 \right) \right]$$

(4.16)

in terms of the Lorentz invariants $\mathcal{F} = \frac{1}{4} \text{tr} F^2 = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ and $\mathcal{G} = \frac{1}{4} \text{tr} FF^* = -\mathbf{E}.\mathbf{B}$.

The vacuum expectation value of the energy-momentum tensor for QED in a background electromagnetic field is given to one loop in terms of this effective Lagrangian $\mathcal{L}$ by[15]

$$\langle T_{ab} \rangle = \left( F_{ac} F_{b}^c - \frac{1}{4} \eta_{ab} F^2 \right) \frac{\partial \mathcal{L}}{\partial F} + \eta_{ab} \left( \mathcal{L} - \mathcal{F} \frac{\partial \mathcal{L}}{\partial \mathcal{F}} - \mathcal{G} \frac{\partial \mathcal{L}}{\partial \mathcal{G}} \right)$$

(4.17)

and so the conformal anomaly is

$$\langle T^a_a \rangle = 4 \left( \mathcal{L} - \mathcal{F} \frac{\partial \mathcal{L}}{\partial \mathcal{F}} - \mathcal{G} \frac{\partial \mathcal{L}}{\partial \mathcal{G}} \right)$$

$$= -\frac{16}{m^4} \left( (4z + 2y)\mathcal{F}^2 + y\mathcal{G}^2 \right)$$

(4.18)

Now compare with the velocity shift formulae (4.9) and (4.12). Notice that the coefficients of the $\mathcal{F}^2$ and $\mathcal{G}^2$ terms in the conformal anomaly are precisely those appearing in the velocity shifts for the two polarisations.

The requirement that both velocity shifts are negative is therefore equivalent to the requirement that the VEV of the trace of the energy-momentum tensor in the background electromagnetic field is negative. The sign of the conformal anomaly is therefore linked to causality in photon propagation. This is certainly a curious result, but beyond noting that it is one more hint of a deeper connection between photon propagation with a modified light cone and the conformal anomaly, we have no real physical understanding of why it should be true.

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