Which semifields are exact?

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Abstract

Every (left) linear function on a subspace of a finite-dimensional vector space over a (skew) field can be extended to a (left) linear function on the whole space. This paper explores the extent to what this basic fact of linear algebra is applicable to more general structures. Semifields with a similar property imposed on linear functions are called (left) exact, and we present a complete description of such semifields. Namely, we show that a semifield $S$ is left exact if and only if $S$ is either a skew field or an idempotent semiring. In particular, our result is new even for the tropical semiring and gives a solution to the problem posed by Wilding. Also, we point out several problems that require further investigation.

Keywords: exact semiring, idempotent semiring, semifield

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1. Introduction

A set $S$ equipped with two binary operations $+$ and $\cdot$ is called a semiring if the following conditions are satisfied: (i) $(R, +)$ is a commutative monoid, (ii) $(R, \cdot)$ is a monoid, (iii) multiplication distributes over addition from both sides, and (iv) the additive identity 0 satisfies $0x = x0 = 0$, for any $x \in R$. In other words, semirings differ from rings by the fact that their elements are not required to have additive inverses. We denote the multiplicative identity by 1, and we assume that $0 \neq 1$. The set $S^n$ becomes a free left semimodule if we define the operations $(s_1, \ldots, s_n) \to (\lambda s_1, \ldots, \lambda s_n)$ for all $\lambda \in S$.

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A considerable amount of recent work \[6, 7, 8\] is devoted to the concept of so-called *exactness*, which gives a characterization of semirings that behave nicely with respect to basic linear algebraic properties. Namely, a semiring \(S\) is called *left exact* if, for every finitely generated left semimodule \(L \subseteq S^n\) and every left \(S\)-linear function \(\varphi: L \to S\), there is a left \(S\)-linear function \(\varphi_0: S^n \to S\) that coincides with \(\varphi\) on \(L\). This property becomes a standard result of linear algebra if \(S\) is a division ring, so we can conclude that division rings are left exact. The concept of right exactness can be defined dually, and the semirings that are both left and right exact are called simply *exact*. Therefore, the division rings are the first examples of exact semirings.

In this paper, we continue studying the semirings in which all the non-zero elements have multiplicative inverses. Such objects form an important class of semirings and are known as *semifields*. Various examples of semifields arise in different applications, and they include the division rings, the semiring of nonnegative reals \([9]\), the tropical semiring \([5]\), the binary Boolean algebra \([4]\), and many others. The aim of our paper is to give a complete characterization of those semifields that are exact.

**Theorem 1.1.** Let \(S\) be a semifield. Then \(S\) is left exact if and only if

1. \(S\) is a division ring, or
2. we have \(1 + 1 = 1\) in \(S\).

By symmetry, the conclusion of the theorem holds for right exactness as well. In particular, we get that a semifield is left exact if and only if it is right exact. As a corollary of Theorem 1.1, we get the exactness of the tropical semiring \(T = (\mathbb{R} \cup \{\infty\}, \min, +)\) because \(T\) is an idempotent semifield.

**Corollary 1.2.** The tropical semiring is exact.

We note that Corollary 1.2 gives a solution to the problem left open in the work \([8]\) by Wilding, Johnson, and Kambites. In fact, the authors developed the tools that allowed them to show the exactness of the related semiring \(\overline{T} = (\mathbb{R} \cup \{+\infty, -\infty\}, \min, +)\) but not of \(T\). Another property that is weaker than the exactness of \(T\) appears in Theorem 5.1 of \([2]\), as pointed out in \([7]\) by Wilding. However, the exactness of \(T\) remained an open problem until now, and it was stated explicitly in the above mentioned work by Wilding.

Our paper is structured as follows. In Section 2, we obtain a useful characterization of exactness resembling some of the results in \([8]\). We use this characterization (Theorem 2.1) to prove the 'only if' part of Theorem 1.1.
In Section 3, we get an improved version of Theorem 2.1 which is valid for semifields. In Section 4, we employ the developed technique and complete the proof of Theorem 1.1. In Section 5, we discuss the perspectives of further work and point out several intriguing open questions.

2. Another characterization of exactness

Let us begin with some notational conventions. We will denote matrices and vectors over a semiring $S$ by bold letters. We denote by $A_i$ and $A_j^j$ the $i$th row and $j$th column of a matrix $A$, and by $A^j_i$ the entry at the intersection of the $i$th row and $j$th column. By $E$ we denote the unit matrices, that is, square matrices with ones on the diagonal and zeros everywhere else. In particular, $E_i$, $E_j^i$ stand for the $i$th unit row and column vectors, respectively. Every $d \times n$ matrix induces the function $S^{1 \times d} \rightarrow S^{1 \times n}$ defined as $u \rightarrow uA$. We denote the image of this operator by left im $A$, and the kernel of this operator as left ker $A$. In other words, left ker $A$ is the set of all pairs $(u,v) \in S^{1 \times d} \times S^{1 \times d}$ such that $uA = vA$. The right image and right kernel of $A$ are defined dually in a natural way. In particular, we define right im $A$ as the set of all vectors in $S^{d \times 1}$ that can be written as $Aw$ with some $w \in S^{n \times 1}$. We proceed with a characterization of exactness that seems to be implicit in [8]. In particular, it looks very similar to Theorem 3.2 in [8], so we do not dare calling the following result 'new'.

Theorem 2.1. Let $S$ be a semiring. The following are equivalent:

(E1) $S$ is left exact;

(E2) for any $A \in S^{d \times n}$, $b \in S^{d \times 1}$, the condition left ker $A \subseteq$ left ker $b$ implies $b \in$ right im $A$.

Proof. Assume (E1) is true, and let $A \in S^{d \times n}$, $b \in S^{d \times 1}$ be such that

$$\text{left ker } A \subseteq \text{left ker } b. \quad (2.1)$$

We define the mapping $\varphi : \text{left im } A \rightarrow S$ by

$$\varphi \left( \sum_{i=1}^{d} \lambda_i A_i \right) = \sum_{i=1}^{d} \lambda_i b_i,$$

which is well defined because of (2.1). Since $\varphi$ is left $S$-linear, we can use the exactness of $S$ and obtain a left $S$-linear mapping $\psi : S^n \rightarrow S$ such that
\( \psi \mid_{\text{left im } A} = \varphi \). Denoting \( \alpha_i := \psi(E_i) \), we get

\[
b_i = \varphi(A_i) = \psi \left( \sum_{j=1}^{n} A_j^i E_i \right) = \sum_{j=1}^{n} A_j^i \alpha_j,
\]

which implies \( b \in \text{right im } A \) and proves \((E2)\).

Now we assume that \((E2)\) is true, and we consider a finitely generated left semimodule \( L \subseteq S^n \) and a left \( S \)-linear function \( \varphi : L \to S \). We can write \( L = \text{left im } A \) for some matrix \( A \), and we define the vector \( b \in S^{d \times 1} \) by the formula \( b_i = \varphi(A_i) \). (Here, the dimension \( d \) is the number of rows of \( A \), or, equivalently, the number of generators of \( L \).) The equation (2.1) is true because \( \varphi \) is well defined, so \((E2)\) implies \( b \in \text{right im } A \). Therefore, we have \( b = \sum_j A_j^i \alpha_j \) for some \( \alpha_1, \ldots, \alpha_n \in S \), and then \( \psi(x_1, \ldots, x_n) = x_1 \alpha_1 + \ldots + x_n \alpha_n \) is a mapping from \( S^n \) to \( S \) that coincides with \( \varphi \) on \( R \). We see that \( S \) is left exact, so \((E1)\) is true. \( \square \)

Let us present an application of Theorem 2.1. The following corollary presents a rather powerful condition that holds in all exact rings. This result seems to be new.

**Corollary 2.2.** Any left exact semiring contains an element \( e \) such that \( 1 + 1 + e = 1 \).

**Proof.** We consider the matrices

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 + 1 \\ 1 \end{pmatrix},
\]

and we apply Theorem 2.1. The condition \((E2)\) shows that either \( b \in \text{right im } A \) or \( \text{left ker } A \not\subseteq \text{left ker } b \). Let us treat these two cases separately.

**Case 1.** If \( b \in \text{right im } A \), then there are \( x_1, x_2 \in S \) such that \( x_2 = 1 + 1 \) and \( x_1 + x_2 = 1 \). We get \( x_1 + 1 + 1 = x_1 + x_2 = 1 \), which implies the desired conclusion.

**Case 2.** If \( \text{left ker } A \not\subseteq \text{left ker } b \), then there are vectors \( u, v \in S^{1 \times 2} \) such that \( uA = vA \) and

\[
u b \neq v b. \tag{2.2}
\]

The former condition shows that \( u^2 = v^2, u^1 + u^2 = v^1 + v^2 \), so we get

\[
u^1 + u^1 + u^2 = u^1 + v^1 + v^2 = u^1 + v^1 + u^2 = v^1 + v^1 + v^2, \tag{2.3}
\]
which is a contradiction. In fact, the left-hand side of (2.2) coincides with the left-hand side of (2.3), and the right-hand side of (2.2) coincides with the right-hand side of (2.3). Therefore, Case 2 is not an option, and the proof is complete.

**Corollary 2.3.** Any left exact semifield is either a ring or satisfies $1+1 = 1$.

**Proof.** Let $e$ be the element as in Corollary 2.2. We get

$$(1 + e)^2 = 1 + e + e + e^2 = 1 + e(1 + 1 + e) = 1 + e,$$

so that $1 + e = 0$ or $1 + e = 1$. The former condition would imply that we have a ring, and the latter one shows that $1 + 1 + e = 1 + 1$ or $1 + 1 = 1$ again by Corollary 2.2.

3. A semifield version of Theorem 2.1

In this section we sharpen the condition (E2) in Theorem 2.1 under the additional assumption that $S$ is a semifield. Recall that a matrix $C \in S^{n \times n}$ is invertible if there exists a matrix $C^{-1}$ such that $CC^{-1} = C^{-1}C = E$.

**Observation 3.1.** Let $C \in S^{d \times d}$, $D \in S^{n \times n}$ be invertible matrices, and let $A \in S^{d \times n}$, $b \in S^{d \times 1}$ be arbitrary. Then

1. left ker $A \subseteq$ left ker $b$ if and only if left ker $CAD \subseteq$ left ker $Cb$,
2. $b \in$ right im $A$ if and only if $Cb \in$ right im $CAD$.

**Proof.** Let us assume $(u, v) \in$ left ker $A \setminus$ left ker $b$, which means that $uA = vA$, $ub \neq vb$. We define $u' = uC^{-1}$, $v' = vC^{-1}$, and we get

$$u'CAD = uC^{-1}CAD = uAD = vAD = vC^{-1}CAD = v'CAD,$$

$$u'Cb = uC^{-1}Cb = ub \neq vb = vC^{-1}Cb = v'Cb,$$

which means that $(u', v') \in$ left ker $CAD \setminus$ left ker $Cb$. This proves the 'only if' direction of (1), and the 'if' direction follows as well by symmetry.

To prove (2), we note that a vector $w$ satisfies $Aw = b$ if and only if the vector $w' = D^{-1}w$ satisfies $CADw' = Cb$.

Let us say that a matrix is column-stochastic if the sum of elements in every column equals one. A semiring $S$ is called zero-sum free if $a + b = 0$ implies $a = b = 0$ for all $a, b \in S$. We are ready to show that, in the case of zero-sum-free semifields, Theorem 2.1 remains true if we restrict the possible choices of $A$ by column-stochastic matrices and the choices of $b$ by vectors whose coordinates are zeros and ones.
Corollary 3.2. Let $S$ be a zero-sum free semifield. Then the condition (E2) in Theorem 2.1 is equivalent to the following: (E2') for any column-stochastic matrix $A \in S^{d \times n}$ and any vector $b \in \{0, 1\}^{d \times 1}$, the condition $\text{left ker } A \subseteq \text{left ker } b$ implies $b \in \text{right im } A$.

Proof. It is trivial that (E2) implies (E2'). To prove the opposite direction, assume that (E2) is not true. Then there are $A \in S^{d \times n}, b \in S^{d \times 1}$ such that $\text{left ker } A \subseteq \text{left ker } b$ and $b \notin \text{right im } A$. The removal of zero columns of $A$ does not change these properties, so we can assume that every column of $A$ contains at least one non-zero entry. We define $\beta_i = b_i$ if $b_i \neq 0$ and $\beta_i = 1$ otherwise, and we set $C$ to be the diagonal matrix with $\beta_1, \ldots, \beta_d$ on the diagonal. Further, we define $\alpha_j$ as the sum of the entries of the $j$th column of $C^{-1}A$. Since the semifield is zero-sum-free, the $\alpha_j$’s are non-zero, so we get an invertible matrix $D$ if we put $\alpha_1, \ldots, \alpha_n$ on the diagonal and zeros everywhere else. Now we see that the matrix $C^{-1}AD^{-1}$ is stochastic, the vector $C^{-1}b$ consists of zeros and ones, and the conditions $\text{left ker } C^{-1}AD^{-1} \subseteq \text{left ker } C^{-1}b$ and $C^{-1}b \notin \text{right im } C^{-1}AD^{-1}$ hold by Observation 3.1. This shows that (E2') is not true. □

4. Idempotent semifields are exact

In this section we complete the proof of Theorem 1.1. Namely, we show that any semifield satisfying $1 + 1 = 1$ is necessarily left exact. In general, a semiring in which $1 + 1 = 1$ (or, equivalently, $x + x = x$ for all $x$) is called idempotent. A natural (and very well known) ordering on an idempotent semiring is given as $x \succeq y$ if and only if $x + y = x$. It is easy to see that the relation $\succeq$ is a partial order compatible with the operations. In other words, the following result is true, see [1] for details.

Observation 4.1. Let $S$ be an idempotent semiring and $p, q, r, s \in S$. Then

$(1) \quad p \succeq p$;
$(2) \quad$ If $p \succeq q, q \succeq r$, then $p \succeq r$;
$(3) \quad$ If $p \succeq q, q \succeq p$, then $p = q$;
$(4) \quad$ If $p \succeq r, q \succeq s$, then $p + q \succeq r + s$ and $pq \succeq rs$.

We will write $p > q$ if $p \succeq q$ and $p \neq q$. If $p \neq p + q$, then we write $q \nleq p$. The set of matrices or vectors over $S$ is still an idempotent semigroup with respect to addition, so these relations are applicable to matrices and vectors. Another obvious property of idempotent semirings is that they are zero-sum free.
Observation 4.2. Let $S$ be an idempotent semiring and $p, q \in S$. If $p + q = 0$, then $p = q = 0$.

Observation 4.3. Let $S$ be an idempotent semifield containing at least three elements. Then there is an element $\lambda$ such that $\lambda \neq 1$.

Proof. Choose an arbitrary $a \not\in \{0, 1\}$. If $a \neq 1$, then we are done, and otherwise we have $1 + a = 1$. This implies $a^{-1} + 1 = a^{-1}$, so we can take $\lambda = a^{-1}$. \qed

Before we proceed, we recall that a matrix $A$ is called column-stochastic if the sum of elements in every column of $A$ equals one. A matrix is row-stochastic if its transpose is column-stochastic.

Lemma 4.4. Let $S$ be an idempotent semifield containing at least three elements. Let $A \in S^{d \times n}$ be a column-stochastic matrix that is not row-stochastic. Then there is a vector $\Lambda \neq (1, \ldots, 1) \in S^{1 \times d}$ such that $\Lambda A = (1, \ldots, 1)$.

Proof. We have
\[
\sum_{i=1}^{d} \sum_{j=1}^{n} A_{ij} = 1 + \ldots + 1 = 1,
\]
so that $\alpha_i := \sum_{j=1}^{n} A_{ij} \leq 1$. If $\alpha_i = 0$ for some $i$, then the $i$th row of $A$ consists of zeros by Observation 4.2. In this case, we define $\Lambda$ as the vector whose coordinates are ones except the $i$th coordinate which is equal to the element $\lambda$ as in Observation 4.3. We have $\Lambda \neq (1, \ldots, 1)$ and $\Lambda A = (1, \ldots, 1) A$; since $A$ is column-stochastic, we get $(1, \ldots, 1) A = (1, \ldots, 1)$ and complete the proof in our special case.

Now we assume that all of the $\alpha_i$’s are non-zero, and we define $\Lambda \in S^{1 \times d}$ with $\Lambda^i$ being the inverse of $\alpha_i$. As said above, $\alpha_i \leq 1$, so that $\Lambda^i \geq 1$ by the item (4) of Observation 4.1. Also, the assumption of the theorem states that $A$ is not row-stochastic, which implies $\Lambda > (1, \ldots, 1)$. Since $A$ is column stochastic, we have $(1, \ldots, 1) A = (1, \ldots, 1)$, and using the item (4) of Observation 4.1, we get
\[
\Lambda A \geq (1, \ldots, 1) A = (1, \ldots, 1).
\] (4.1)

Further, we get
\[
\sum_{i=1}^{d} \sum_{j=1}^{n} \Lambda^i A_{ij} = \sum_{i=1}^{d} \Lambda^i \alpha_i = 1 + \ldots + 1 = 1,
\]
which shows that
\[\sum_{i=1}^{d} \Lambda^i A_i^j \leq 1 \quad (4.2)\]
for all \(j\). Putting the inequalities (4.1) and (4.2) together and using the item (3) of Observation 4.1, we get \(\Lambda A = (1, \ldots, 1)\). In other words, \(\Lambda A\) is a column-stochastic matrix.

**Lemma 4.5.** Let \(S\) be an idempotent semifield containing at least three elements. Let \(A \in S^{d \times n}\) be a column-stochastic matrix and \(b \in \{0, 1\}^{d \times 1}\) be a vector outside \(\text{right im } A\). Then there are vectors \(u, v\) such that \(uA = vA\) and \(ub \neq vb\).

**Proof.** Assume \(b\) contains \(k\) ones. We can assume without loss of generality that the first \(k\) coordinates of \(b\) are ones, and we write
\[
A = \begin{pmatrix}
P_{k \times (n-m)} & Q_{k \times m} \\
R_{(d-k) \times (n-m)} & O_{(d-k) \times m}
\end{pmatrix}, \quad b = \begin{pmatrix}
J_{k \times 1} \\
O_{(d-k) \times 1}
\end{pmatrix},
\]
where \(J\) is a \(k \times 1\) vector of ones, the \(O\)’s are zero matrices of relevant sizes, and \(R\) has no zero column. If \(k = d\), then the result follows from Lemma 4.4 and we also have \(k \neq 0\) because otherwise \(b \in \text{right im } A\). Therefore, we can assume that both the upper and lower blocks of \(A\) and \(b\) are non-empty. The possibilities \(m = 0, m = n\) are not a priori prohibited, so we will treat them too.

By Lemma 4.4, there is a vector \(\Lambda \notin (1, \ldots, 1) \in S^{1 \times k}\) such that \(\Lambda Q = (1, \ldots, 1)\). (If \(m = 0\), that is, \(Q\) is empty, then we choose a vector \(\Lambda \notin (1, \ldots, 1)\) arbitrarily.) Let \(p \in S\) be the sum of all entries of the matrices \(P\) and \(\Lambda P\) plus one; let \(r\) equal one plus the sum of the inverses of all non-zero entries of \(R\). We set \(M = (p, \ldots, p) \in S^{1 \times (d-k)}\), and we get \(MR \geq (p, \ldots, p)\) whenever the left blocks of \(A\) are non-empty, that is, when \(m \neq n\). This means that \(MR\) is greater than or equal to any row of \(P\) and \(\Lambda P\), so we get
\[
(1, \ldots, 1|M)A = (MR|1, \ldots, 1) = (\Lambda|M)A,
\]
\[
(1, \ldots, 1|M)b = 1 \neq \Lambda^1 + \ldots + \Lambda^k = (\Lambda|M)b,
\]
which completes the proof. \(\square\)

**Theorem 4.6.** Every idempotent semifield is left exact.
Proof. Lemma 4.5 shows that an idempotent semifield $S$ possesses the property (E2’) as in Corollary 3.2 whenever $S$ has at least three elements. Therefore, we can apply Theorem 2.1 to see that every such $S$ is left exact.

If $S$ contains 0 and 1 only, then the definitions allow us to identify $S$ uniquely as the binary Boolean semiring $\mathbb{B}$. The exactness of $\mathbb{B}$ can be proved by a routine application of Theorem 2.1. Alternatively, one can get this result by applying a deeper one, Theorem 6.5 in [8].

The proof of Theorem 1.1 is now complete. In fact, Corollary 2.3 proves the 'only if' direction, and the 'if' direction follows from Theorem 4.6 and basic results of linear algebra which imply that any division ring is exact.

5. Further work

We gave the complete characterization of semifields that are exact. We do not know how to generalize this result to the case of arbitrary semirings, and we propose here several directions towards this goal. First of all, we do not know if Corollary 2.3 can be generalized to arbitrary semirings.

Problem 5.1. Does there exist a left exact semiring which is neither a ring nor an idempotent semiring?

If the answer to Problem 5.1 was negative, then exact semirings could arise only from the two well-studied classes. Wilding points out in [11] that the exactness of a ring is equivalent to the property known as $FP$-injectivity, and there different known and conjectured characterizations of this property, see [3]. Much less is known about exactness in the class of idempotent semirings. A relevant restriction of this class are the selective semirings, that is, those in which $a + b \in \{a, b\}$ for all $a, b$.

Problem 5.2. Give a characterization of selective left exact semirings. In particular, is there a selective left exact semiring which is not a semifield?

Another question left open in this work is the coincidence of left and right exactness. We know that they differ in the class of rings (see Example 5.46 in [3]), but is the same true in the idempotent case?

Problem 5.3. Provide an example of a left exact semiring which is neither a ring nor a right exact semiring.
Several interesting examples of idempotent semirings were examined in [7] by Wilding. Given a monoid \( M \), he defines the semiring \( \mathbb{B}M = (2^M, \cup, \cdot) \), where two subsets \( A, B \subseteq M \) are multiplied as \( AB = \{ab | a \in A, b \in B\} \). Wilding proves that \( \mathbb{B}M \) is exact if \( M \) is a group, and asks if the converse of this statement is true. Our approach is not sufficient to answer this question, and we believe that its resolution would lead to a significant step towards the classification of exactness in the idempotent case.

References

[1] J. S. Golan, Semirings and their applications. Springer Science & Business Media, 2013.

[2] C. Hollings, M. Kambites, Tropical matrix duality and Green's \( \mathcal{D} \) relation, J. London Math. Soc. (2015) jds015.

[3] W. K. Nicholson, M. F. Yousif, Quasi-Frobenius Rings, volume 158 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2003.

[4] H. Rasiowa, R. Sikorski, The mathematics of metamathematics, Panstwowe Wydawnictwo Naukowe, Warsaw, 1963.

[5] D. Speyer, B. Sturmfels, Tropical mathematics, Mathematics Magazine 82 (2009) 163–173.

[6] Y. Shitov, Group rings that are exact, J. Algebra 403 (2014) 179–184.

[7] D. Wilding, Linear Algebra Over Semirings, PhD dissertation, The University of Manchester, 2015.

[8] D. Wilding, M. Johnson, M. Kambites, Exact rings and semirings, J. Algebra 388 (2013): 324–337.

[9] M. Yannakakis, Expressing combinatorial optimization problems by linear programs, Comput. System Sci. 43 (1991) 441–466.