Sharp error estimate of variable time-step IMEX BDF2 scheme for parabolic integro-differential equations with initial singularity arising in finance

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Abstract

The recently developed technique of DOC kernels has been a great success in the stability and convergence analysis for BDF2 scheme with variable time steps. However, such an analysis technique seems not directly applicable to problems with initial singularity. In the numerical simulations of solutions with initial singularity, variable time-steps schemes like the graded mesh are always adopted to achieve the optimal convergence, whose first adjacent time-step ratio may become pretty large so that the acquired restriction is not satisfied. In this paper, we revisit the variable time-step implicit-explicit two-step backward differentiation formula (IMEX BDF2) scheme presented in [W. Wang, Y. Chen and H. Fang, SIAM J. Numer. Anal., 57 (2019), pp. 1289-1317] to compute the partial integro-differential equations (PIDEs) with initial singularity. We obtain the sharp error estimate under a mild restriction condition of adjacent time-step ratios $r_k := \tau_k / \tau_{k-1} \ (k \geq 3) < r_{\text{max}} = 4.8645$ and a much mild requirement on the first ratio, i.e., $r_2 > 0$. This leads to the validation of our analysis of the variable time-step IMEX BDF2 scheme when the initial singularity is dealt by a simple strategy, i.e., the graded mesh $t_k = T(k/N)^\gamma$. In this situation, the convergence of order $O(N^{-\min\{2, \gamma_{\text{opt}}\}})$ is achieved with $N$ and $\alpha$ respectively representing the total mesh points and indicating the regularity of the exact solution. This is, the optical convergence will be achieved by taking $\gamma_{\text{opt}} = 2/\alpha$. Numerical examples are provided to demonstrate our theoretical analysis.

Keywords: implicit-explicit method; two-step backward differentiation formula; the discrete orthogonal convolution kernels; the discrete complementary convolution kernels; error estimates; variable time-step;

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1 Introduction

In this paper, we consider the computation of parabolic integro-differential equations (PIDEs) which arise in option pricing theory when the underlying asset follows a jump diffusion process \cite{2,12,24,32,33}

\[
\begin{align*}
\partial_t u(x, t) - c_1 u_{xx}(x, t) + c_2 u(x, t) + c_3 u + \mathcal{J}(u(x, t)) &= f(x, t), \quad (x, t) \in \Omega \times (0, T], \\
u(x, t) &= u_b(x, t), \quad (x, t) \in \partial \Omega \times (0, T], \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

(1.1)

where \( \Omega = (x_l, x_r) \) with its boundary \( \partial \Omega \), the parameters \( c_1 > 0, c_2, c_3 \in \mathbb{R} \). Here \( \mathcal{J}(\cdot) \) represents the nonlocal integral operator and is defined by

\[
\mathcal{J}(u) := \int_{\Omega} u(z, t) \rho(x - z) \, dz,
\]

where \( \rho : \mathbb{R} \to \mathbb{R}^+ \) is a given function satisfying \( \| \rho(x) \|_{L^\infty} \leq C \) for some positive constant \( C \). In this situation, there exists a constant \( \tilde{C}_\rho \) (only depends on the given function \( \rho \) and \( \Omega \)) such that

\[
\| \mathcal{J}(u) \|_{L^2(\Omega)} \leq \tilde{C}_\rho \| u \|_{L^2(\Omega)}. \tag{1.2}
\]

The bound (1.2) is satisfied in many practical problems such as the finite activity jump diffusion model (e.g., Merton model and Kou model), CGMY and KoBoL with infinite activity and finite variation \cite{6,8}.

There are two main considerations in numerically solving PIDEs (1.1). The first one is the discretization of nonlocal integral operators. An undue discretization of the nonlocal integral operator with finite difference schemes, such as the fully implicit time stepping scheme, may increase the computational complexity since inversions will need solve the resulting systems with full matrices at each time step. To deal with the full matrix, many methods are designed such as iterative methods \cite{1,9,23,30}, FFT \cite{2,9} and the alternating direction implicit (ADI) method \cite{2}. An alternative approach is to avoid the inversion of a full matrix. To do so, an increasingly popular alternative is the implicit-explicit (IMEX) method \cite{12,14,26,32}, which typically treats the nonlocal integral term explicitly and the rest part implicitly. The IMEX method leads to a tridiagonal system and can be solved efficiently.

A feature of PIDEs (1.1) is the weak singularity near the initial time \( t = 0 \) arose from the nonsmooth initial data. For example, as noted in \cite{32}, with a nonsmooth payoff function in option models, the regularity of exact solution \( u(x, t) \) may have the form \( \| \partial_t^k u \|_{L^2} \leq C t^{\frac{k}{2} - k}, k = 1, 2, 3 \). To be more general, in this paper, we consider the regularity assumption as follows:

**Assumption 1.1.** There exists a constant \( \hat{C} \) such that the solution to (1.1) satisfies

\[
\| \partial_t^k u \| \leq \hat{C} t^{-k+\alpha}, \quad t \in (0, T], \quad k = 1, 2, 3, \tag{1.3}
\]

where \( \partial_t^k := \frac{\partial^k u}{\partial t^k} \) and the regularity parameter \( \alpha \) satisfies \( \frac{1}{2} \leq \alpha \leq 1 \).

For solving such initial singularity problems, another consideration is how to develop efficient and accurate algorithms. A heuristic method to improve efficiency without sacrificing accuracy is the adaptive time-step scheme. Thus, one can employ small time steps when the dynamics evolves rapidly, and use large time steps when the dynamics evolves slowly \cite{16,32,35}. Another method is to employ high-order schemes in time to have the same accuracy with a relatively large time-step, for example, the Runge–Kutta methods \cite{4}, Crank–Nicolson method \cite{9,27} and two-step backward differentiation.
formula (BDF2) [1, 12, 32]. Specifically, the BDF2 method has received much attention due to its strong stability (A-stable) [7, 16, 19, 20, 28, 32, 34, 35].

The numerical analysis of BDF2 scheme with variable time steps receives much attention. For instance, Becker [3] (also Thomée’s classical book [31, Lemma 10.6]) presents the stability and convergence with a factor \( \exp(C\Gamma_n) \) under the adjacent time-step ratio satisfying \( 0 < r_k \leq (2 + \sqrt{13})/3 \approx 1.868 \), where \( \Gamma_n := \sum_{k=2}^{n-2} \max\{0, r_k - r_{k+2}\} \). Emmrich [10] improves the Becker’s condition up to 1.91. By using a novel generalized discrete Grönwall-type inequality, Chen et al. [7] consider the Cahn-Hilliard equation, and present the energy stability without the factor \( \Gamma_n \) with \( 0 < r_k \leq 3.561 \). The works [21, 34] consider linear parabolic equations based on DOC kernels with \( 0 \leq r_k \leq 3.561 \) and \( 0 < r_k \leq 4.8645 \) [34], respectively. There are also a great progress on the error estimates for nonlinear equations, see [16, 19, 20, 35].

It is worthy to point out that the analysis in literature mentioned above is based on smooth solutions, and will fail for the initial singularity (1.3). In fact, an established technique to restore an optimal convergence rate is to employ a graded mesh

\[
t_k = T(k/N)^\gamma \quad \text{for } 0 \leq k \leq N,
\]

where the parameter \( \gamma \) is used to adapt to the strength of the singularity. Choosing \( \gamma = 1 \) will lead to a uniform mesh, and the larger the value of \( \gamma \) the more strongly the temporal meshes are concentrated at the initial time. For instance, such meshes have long been applied to numerically solve Fredholm integral equations, and time-fractional PDEs [17, 18, 22, 29].

Given the typical regularity (1.3) with \( \alpha = 1/2 \), Wang et al. [32] consider the sharp error estimate of a variable-time-step IMEX-BDF2 scheme for the problem (1.1). A half-order convergence is presented under \( 0 < r_k \leq 1.91 \), and the second-order convergence is achieved by introducing two kinds of nonuniform grids. For the graded mesh (1.4), the the maximum error for the fully BDF2 scheme will be order of

\[
N^{-\min\{2, \gamma\alpha\}},
\]

where we ignore the additional error due to the spatial discretization. From (1.5), if \( \gamma \geq 2/\alpha \) then the error is \( N^{-2} \). Specifically, for the case of \( \alpha = 1/2 \), one needs to take \( \gamma \geq 4 \) to have the optimal convergence of \( N^{-2} \). In this situation, the first-level adjacent time-step ratio defined in (2.6) satisfies

\[
r_2 := \tau_2/\tau_1 = 2^7 - 1 \geq 15,
\]

which significantly breaks the adjacent time-step ratio restriction given in [21, 32, 34] at the first-step ratio. This is to say, the analysis in [21, 32, 34] will not be valid any longer when the initial regularity is considered. It is numerically indicated in [25] that the error at the first step shall become larger as the first ratio \( r_2 \) increases. Thus, it is desired to develop a more general analytic framework and find out the effect of the first ratio on the error bound.

The aim of this paper is to revisit the IMEX-BDF2 scheme with variable time steps given in [32] for solving the PIDE (1.1), and achieve the sharp error estimate under the following adjacent time-step ratios condition

\[
A1 : \quad r_2 > 0 \quad \text{and} \quad 0 < r_k \leq r_{\max} - \delta, \quad 3 \leq k \leq N \quad \text{for any small constant } 0 < \delta < r_{\max}.
\]

In A1, we point out that \( r_{\max} = \frac{1}{6} \left( \sqrt[3]{1196 - 12\sqrt{177}} + \sqrt[3]{1196 + 12\sqrt{177}} \right) + \frac{4}{5} \approx 4.8645 \) is the root of \( x^3 = (2x + 1)^2 \), and the parameter \( \delta \) can be taken any small value in the practical adaptive time-step strategies. Our contributions can be listed as follows.

- An alternative representation of A1 is given in [15] by \( 0 < r_k \leq r_{\user} < r_{\max} \), where \( r_{\user} \) can be considered as \( r_{\user} = r_{\max} - \delta \) in this paper. We introduce \( \delta \) aimed to clearly indicate how do the stability and convergence estimates depend on the gap \( \delta = r_{\max} - r_{\user} \).
Comparing the recent work [32], we extend to the adjacent time-step ratios condition from $0 < r_k \leq 1.91$ to $0 < r_k \leq 4.86$. More importantly, our ratio condition in A1 has no restriction on the first ratio $r_2$. It implies that our analysis will remain valid to deal with the initial singularity by using the graded mesh for any $\gamma \leq 2/\alpha$. This is, an optimal convergence of order $O(N^{-2})$ is achieved under the graded mesh $t_k = T(k/N)^{\gamma_{opt}}$ with an optimal parameter $\gamma_{opt} = 2/\alpha$.

Besides, our error estimates clarify that the error increases polynomially with respect to the first ratio $r_2$, which is consistent with the numerical experiments in [25].

Our stability analysis and sharp error estimate for the IMEX-BDF2 scheme are based on the recently developed DOC [21] and DCC [34] kernels by developing their new properties, which is used to overcome the difficulties arising from the initial singularity, including a strictly positive definiteness of the BDF2 kernels $b_{n-k}^{(m)}$ (see Lemma 2.3) and a sharper estimate of DCC kernels $p_{n-k}^{(m)}$ (see Proposition 2.1). In addition, the finite difference method is used for spatial discretization. Differing from the finite element method and spectral method, the finite difference method may break the discrete integration by parts formula, which may bring extra difficulties in the analysis. A novel convolution-type Young’s inequality (see Lemma 3.3) is developed to handle the spatial difficulties.

The remainder is organized as follows. Section 2 presents IMEX BDF2 scheme, and several properties of DOC and DCC kernels including some new properties such as the strictly positive definiteness of the BDF2 kernels $b_{n-k}^{(m)}$ and a sharper estimate of the kernels. The stability and half-order convergence under the ratio restriction A1 are presented in Section 3 and the optimal second-order convergence are also achieved under a graded mesh with an optimal parameter in Section 4. Numerical experiments are provided to demonstrate our theoretical analysis in Section 5.

2 IMEX BDF2 scheme, DOC and DOC kernels

In this paper, we consider an IMEX-BDF2 scheme with variable time steps combining with the finite difference method in space. We first take the generally variable time grids by $0 = t_0 < t_1 < \cdots < t_N = T$, and denote the $k$th time-step size by $\tau_k := t_k - t_{k-1}$, the maximum time step size by $\tau := \max_{1 \leq k \leq N} \tau_k$, and the adjacent time-step ratio by

$$r_k = \frac{\tau_k}{\tau_{k-1}}, \quad 2 \leq k \leq N, \quad r_1 = 0. \quad (2.6)$$

Denote $u^n(x)$ by the approximation of $u(x,t_n)$, and $\nabla_\tau u^n = u^n - u^{n-1}$ by the difference operator. The BDF1 and BDF2 with variable time steps are respectively defined by

$$D_1 u^n = \frac{1}{\tau_n} \nabla_\tau u^n, \quad D_2 u^n = \frac{1 + 2r_n}{\tau_n(1 + r_n)} \nabla_\tau u^n - \frac{r_n^2}{\tau_n(1 + r_n)} \nabla_\tau u^{n-1}, \quad \text{for} \quad 2 \leq n \leq N.$$

Since the BDF2 needs two starting values, the BDF1 is used to compute the first-level solution $u^1$ and the BDF2 is used when $n \geq 2$. If we introduce the notations, i.e., $b_0^{(1)} := 1/\tau_1$ and

$$b_0^{(n)} = \frac{1 + 2r_n}{\tau_n(1 + r_n)}, \quad b_1^{(n)} = -\frac{r_n^2}{\tau_n(1 + r_n)} \quad \text{and} \quad b_j^{(n)} = 0, \quad \text{for} \quad 2 \leq j \leq n - 1, \quad (2.7)$$
BDF2 (using BDF1 to compute $u^1$) can be written in the following discrete convolution form

$$D_2u^n := \sum_{k=1}^{n} b_{n-k}^{(n)} \nabla u^k, \quad n \geq 1. \quad (2.8)$$

The finite difference method is considered for the spatial discretization. We denote the spatial length $h = (x_r - x_l)/M$ for a positive integer $M$, the discrete grid $\bar{\Omega} = \{x_i + ih | 0 \leq i \leq M\}$, $\Omega_h = \Omega \cap \bar{\Omega}$, and $\partial \Omega_h = \partial \Omega \cup \partial \bar{\Omega}$. For any grid function $v_h = \{v_i | v_i = v(x_i), x_i \in \Omega_h\}$, let $\Delta_h v_i$ be the standard second-order approximation of $v_{xx}(x_i)$ and $\nabla_h v_i := \frac{v_{i+1} - v_{i-1}}{2h}$ be the approximation of $v_x(x_i)$. Denote the space of grid functions by $V_h = \{v_h | v_h \text{ vanishes on } \partial \Omega_h\}$. The discrete $L^2$ inner product and associated discrete $L^2$-norm are defined respectively by $\langle u, v \rangle := \sum_{i=1}^{M} h_i u_i v_i$, $\|u\| := \sqrt{\langle u, u \rangle}$, $\forall u, v \in V_h$. The discrete $H^1$ semi-norm and $H^1$-norm are defined respectively by $|u|_1 = \sqrt{\langle -\Delta h u, u \rangle}$, $\|u\|_1 = \sqrt{\|u\|^2 + |u|_1^2}$, $\forall u \in V_h$.

The integral operator in (1.1) is approximated by the composite trapezoidal rule

$$J(u^n_i) = \int_{\Omega} u(z, t_n) \rho(z-x_i) \, dz \approx \frac{h}{2} \left( u^n_0 \rho^n_{i,0} + 2 \sum_{j=1}^{M-1} u^n_{j} \rho^n_{i,j} + u^n_M \rho^n_{i,M} \right) := J_h(u^n_i),$$

where $\rho^n_{i,j} = \rho(x_j - x_i, t_n)$. Similar to (1.2), there exists constant $C_\rho$ such that

$$\|J_h(u^n_i)\| \leq C_J \|u^n_i\|. \quad (2.9)$$

Thus, a fully discrete IMEX BDF2 scheme for solving the PIDE (1.1) is given by

$$D_2u^n_i - c_1 \Delta_h u^n_i + c_2 \nabla_h u^n_i + c_3 u^n_i + J_h(Eu^{n-1}_i) = f^n_i, \quad x_i \in \Omega_h, 1 \leq n \leq N,$$
$$u^n_0 = u_0(x_i, t_n), \quad x_i \in \partial \Omega_h, 1 \leq n \leq N,$$
$$u^n_i = u_0(x_i), \quad x_i \in \Omega_h, \quad \forall n \geq 2 \text{ and } Eu^0 = u^0. \quad (2.10)$$

### 2.1 The definitions and properties of DCC and DOC kernels

The techniques of the DCC and DOC kernels will play a key role in the following stability and convergence analysis. Here we first introduce the DCC and DOC kernels [21, 34, 35], and also develop a new positive definiteness of the DOC kernels and a sharp estimate of the DCC kernels.

The discrete complementary convolution (DCC) kernels $p_{n-j}^{(n)}$ [34] is defined by

$$\sum_{j=k}^{n} p_{n-j}^{(n)} b_{j-k}^{(j)} = 1, \quad \forall 1 \leq k \leq n, 1 \leq n \leq N, \quad (2.11)$$

which has the property of

$$\sum_{j=1}^{n} p_{n-j}^{(n)} D_2 u^j = \sum_{j=1}^{n} p_{n-j}^{(n)} \sum_{l=1}^{j} b_{j-l}^{(j)} \nabla \nabla u^l = \sum_{l=1}^{n} \nabla \nabla u^l \sum_{j=l}^{n} p_{n-j}^{(n)} b_{j-l}^{(j)} = u^n - u^0, \quad \forall n \geq 1. \quad (2.12)$$
Denote by \( \delta_{nk} \) the Kronecker delta symbol with \( \delta_{nk} = 1 \) if \( n = k \) and \( \delta_{nk} = 0 \) if \( n \neq k \). The discrete orthogonal convolution (DOC) kernels [21] is defined by

\[
\sum_{j=k}^{n} \theta_{n-j}^{(n)} b_{j-k} = \delta_{nk}, \quad \forall 1 \leq k \leq n, \tag{2.13}
\]

which has the property of

\[
\sum_{j=1}^{k} \theta_{k-j}^{(n)} D_2 u^j = \sum_{l=1}^{k} \nabla \tau u^l \sum_{j=l}^{k} \theta_{k-j}^{(n)} b_{j-l} = u^k - u^{k-1}, \quad 1 \leq k \leq N. \tag{2.14}
\]

As noted in [34, Proposition 2.1], the DCC and DOC kernels have the following relations

\[
\theta_0^{(n)} = p_0^{(n)}, \quad \theta_{n-k}^{(n)} = p_{n-k}^{(n)} - p_{n-k-1}^{(n-1)} (1 \leq k \leq n - 1) \quad \text{and} \quad p_{n-j}^{(n)} = \sum_{k=j}^{n} \theta_{k-j}^{(n)} (1 \leq j \leq n). \tag{2.15}
\]

The DOC and DCC kernels may be calculated explicitly in the following two lemmas.

**Lemma 2.1.** [21, Lemma 2.3] The DOC kernels \( \theta_{n-j}^{(n)} \) have the following properties:

\[
\theta_{n-j}^{(n)} > 0, \quad \forall 1 \geq j \geq n \quad \text{and} \quad \sum_{j=1}^{n} \theta_{n-j}^{(n)} = \tau_n, \quad \text{for} \ n \geq 1. \tag{2.16}
\]

**Proposition 2.1.** [34, Proposition 2.2] The DCC kernels \( p_{n-k}^{(n)} \) defined in (2.11) satisfy

\[
p_{n-j}^{(n)} = \sum_{k=j}^{n} \tau_k (1 + r_j) \prod_{i=j+1}^{k} \frac{r_i}{1 + 2 r_i} \leq 2 \tau, \quad 1 \leq j \leq n \quad \text{and} \quad \sum_{j=1}^{n} p_{n-j}^{(n)} = t_n, \tag{2.17}
\]

where \( \prod_{i=j+1}^{k} = 1 \) for \( j \geq k \) is defined.

We point out that the BDF2 kernels \( p_{n-k}^{(n)} (k \geq 2) \) still keep positively defined for any ratio \( r_2 > 0 \) in A1. The similar result for the version of \( r_2 < 4.8645 \) is presented in [15].

**Lemma 2.2.** Assume the time step ratio \( r_k \) satisfies A1. For any real sequence \( \{ \omega_k \}_{k=1}^{n} \), it holds that

\[
2 \omega_k \sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j \geq \frac{r_{k+1} \sqrt{r_{\max}} \omega_{k+1}^2}{1 + r_{k+1}} - \frac{r_k \sqrt{r_{\max}} \omega_k^2}{1 + r_k} + C_r \delta \frac{\omega_k^2}{\tau_k}, \quad k \geq 3, \tag{2.18}
\]

\[
2 \sum_{k=2}^{n} \omega_k \sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j \geq \sum_{k=2}^{n} C_r \delta \frac{\omega_k^2}{\tau_k}, \quad n \geq 2, \tag{2.19}
\]

where \( C_r = \sqrt{r_{\max} / (1 + r_{\max})}^2 \) and \( \delta \) is any small constant satisfying \( 0 < \delta < r_{\max} \) (see A1).
Proof. The inequality (2.18) can be obtained by [35, Lemma 3.2]. Summing (2.18) from 2 to \(n\) yields
\[
2 \sum_{k=2}^{n} \omega_k \sum_{j=2}^{k} b_{k-j}^{(k)} \omega_j \geq 2 \sum_{k=2}^{n} \omega_k \sum_{j=2}^{k} r_{n+1} \sqrt{r_{max}} \omega_n \frac{2}{\tau_2} + \frac{r_{n+1} \sqrt{r_{max}} \omega_n}{1 + r_{n+1}} \frac{\omega_n}{\tau_2} + \frac{r_3 \sqrt{r_{max}} \omega_3}{1 + r_3} + \sum_{k=3}^{n} \delta \sqrt{r_{max}} \omega_k^2 \left(1 + r_{max}\right)^2 \tau_k
\]
\[
\geq \left(2 - \frac{r_{max}^2}{1 + r_{max}}\right) \frac{\omega_2^2}{\tau_2} + \sum_{k=2}^{n} \frac{\delta \sqrt{r_{max}} \omega_k^2}{(1 + r_{max})^2 \tau_k}
\]
\[
\geq \frac{\omega_2^2}{(1 + r_{max}) \tau_2} + \sum_{k=2}^{n} \delta \sqrt{r_{max}} \omega_k^2 \left(1 + r_{max}\right)^2 \tau_k
\]
where one uses the facts \(r_{max}^2 = 1 + 2r_{max}\) and \(0 < \delta < r_{max}\). The proof is completed.

2.2 The new properties of DOC kernels and DCC kernels

Lemma 2.3. Assume the condition A1 holds. Then the DOC kernels \(\theta_{n-k}^{(n)}(n \geq 2)\) defined in (2.13) are positive definite. Moreover, for any real sequences \(\{\omega_j\}_{j=1}^{n}\), it holds that
\[
2 \sum_{k=2}^{n} \omega_k \sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j \geq C_r \delta \sum_{k=2}^{n} \tau_{k}^{-1} \left(\sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j\right)^2, \quad \forall n \geq 2,
\]
where \(C_r = \sqrt{r_{max}}/(1 + r_{max})^2\).

Proof. For any \(\omega = (\omega_2, \ldots, \omega_n) \in \mathbb{R}^n\), we construct the vector \(v = (v_2, \ldots, v_n) \in \mathbb{R}^n\) by
\[
\omega_j = \sum_{l=2}^{j} b_{j-l}^{(j)} v_l, \quad 2 \leq j \leq n.
\]
It is easy to check \(v\) is uniquely determined by \(\omega\). Multiplying \(\theta_{k-j}^{(k)}\) on both sides of (2.21) and summing \(j\) from 2 to \(k\), one finds
\[
\sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j = \sum_{j=2}^{k} \theta_{k-j}^{(k)} \sum_{l=2}^{j} b_{j-l}^{(j)} v_l = \sum_{l=2}^{k} v_l \sum_{j=l}^{k} \theta_{k-j}^{(k)} b_{j-l}^{(j)} = v_k,
\]
where one exchanges the order of summations and uses the definition (2.13). Again, multiplying (2.22) by \(\omega_k\) and taking summation from 2 to \(n\), we find
\[
2 \sum_{k=2}^{n} \omega_k \sum_{j=2}^{k} \theta_{k-j}^{(k)} \omega_j = 2 \sum_{k=2}^{n} \omega_k v_k = 2 \sum_{k=2}^{n} v_k \sum_{j=2}^{k} b_{k-j}^{(k)} v_j.
\]
Thus, the proof is completed by Lemma 2.2 and (2.22).

The following proposition gives a sharp estimate of the DCC kernels, which play a key role in the stability and convergence analysis.
Proposition 2.2. Assume A1 holds and \( c_r = r_{\text{max}}^{5/2} \). The DCC kernels \( p_{n-k}^{(n)} \) defined in (2.11) satisfy
\[
\begin{align*}
    p_{n-1}^{(n)} &\leq c_r \delta^{-1} \sqrt{\tau} \sqrt{r}, \quad 2 \leq j \leq n, \\
p_{n-2}^{(n)} &\leq \tau_1 + c_r \delta^{-1} \sqrt{\tau_2} \sqrt{r}.
\end{align*}
\] (2.23) (2.24)

Proof. From Proposition 2.1 and the condition A1, the DCC kernels \( p_{n-j}^{(n)} \) can be bounded by
\[
p_{n-j}^{(n)} = \sum_{k=j}^{n} \tau_k \frac{r_i}{1 + 2r_j} \prod_{i=j+1}^{k} \frac{r_i}{1 + 2r_i} = \sqrt{\tau_j} \sum_{k=j}^{n} \sqrt{\tau_k} \frac{r_i}{1 + 2r_i} \prod_{i=j+1}^{k} \frac{r_i}{1 + 2r_i}
\]
\[
\leq \sqrt{\tau_j} \sum_{k=j}^{n} \sqrt{\tau_k} \prod_{i=j+1}^{k} \frac{(r_{\text{max}} - \delta)^{3/2}}{1 + 2(r_{\text{max}} - \delta)}
\]
\[
\leq \sqrt{\tau_j} \sum_{k=j}^{n} \sqrt{\tau_k} \left( \frac{(r_{\text{max}} - \delta)^{3/2}}{1 + 2(r_{\text{max}} - \delta)} \right)^{k-j}
\]
\[
\leq \sqrt{\tau_j} \left( 1 - \frac{(r_{\text{max}} - \delta)^{3/2}}{1 + 2(r_{\text{max}} - \delta)} \right)^{-1}, \quad \text{for } 2 \leq j \leq n.
\] (2.25)

Noting that \((r_{\text{max}})^{3/2} = 1 + 2r_{\text{max}}\), we get
\[
1 - \frac{(r_{\text{max}} - \delta)^{3/2}}{1 + 2(r_{\text{max}} - \delta)} = \frac{(r_{\text{max}})^{3/2}}{1 + 2r_{\text{max}}} - \frac{(r_{\text{max}} - \delta)^{3/2}}{1 + 2(r_{\text{max}} - \delta)} \geq \frac{\sqrt{r_{\text{max}}}}{(1 + 2r_{\text{max}})^2} \delta = r_{\text{max}}^{-5/2} \delta.
\] (2.26)

Thus, by inserting (2.26) into (2.25), we obtain (2.23).

For \( j = 1 \), the estimate is different because of \( r_2 \in \mathbb{R}^+ \). In this situation, from the definition we have the following estimate
\[
p_{n-1}^{(n)} = \sum_{k=1}^{n} \tau_k \prod_{i=2}^{k} \frac{r_i}{1 + 2r_i} = \tau_1 + \frac{r_2}{1 + 2r_2} \sum_{k=2}^{n} \tau_k (1 + r_2) \prod_{i=3}^{k} \frac{r_i}{1 + 2r_i} = \tau_1 + \frac{r_2}{1 + 2r_2} p_{n-2}^{(n)}.
\] (2.27)

The proof is completed by inserting the estimate (2.23) into (2.27). \(\square\)

3 Stability and convergence analysis for IMEX BDF2 scheme

We now consider the stability and convergence analysis for the IMEX BDF2 scheme (2.10), which requires a discrete Grönwall inequality given as follows.

Lemma 3.1. Assume that \( \lambda > 0 \) and the sequences \( \{v_j\}_{j=1}^{N} \) and \( \{\eta_j\}_{j=1}^{N} \) are nonnegative. If
\[
v_n \leq \lambda \sum_{j=1}^{n-1} \tau_j v_j + \sum_{j=0}^{n} \eta_j, \quad \text{for } 1 \leq n \leq N,
\]
then it holds that
\[
v_n \leq \exp(\lambda t_{n-1}) \sum_{j=0}^{n} \eta_j, \quad \text{for } 1 \leq n \leq N.
\]

The proof of Lemma 3.1 can be derived by the standard induction hypothesis and we omit it here.
3.1 Stability

Let \( u^n_h \) and \( \tilde{u}^n_h \) be solutions of the IMEX BDF2 scheme (2.10) with initial values \( u^0_h, \tilde{u}^0_h \) and source terms \( f^0_h, \tilde{f}^0_h \), respectively. Let \( \phi^n_h := u^n_h - \tilde{u}^n_h, \zeta^n_h := f^n_h - \tilde{f}^n_h (0 \leq n \leq N) \) be the perturbations. Then \( \phi^n_h \) solves

\[
D_2\phi^n_h - c_1 \Delta_h \phi^n_h + c_2 \nabla_h \phi^n_h + c_3 \phi^n_h + \mathcal{J}(E\phi^n_h^{-1}) = \zeta^n_h, \quad 1 \leq n \leq N,
\]

with the initial value \( \phi^0_h(x) \) and homogeneous Dirichlet boundary condition. Different from the spectral method and finite element method, the finite difference method may bring extra difficulties in the analysis. For example, the discrete integration by parts formula that \((-\Delta_h u_h, v_h) = (-\nabla_h u_h, \nabla_h v_h)\) is not satisfied due to the incompatible discretizations of \( u_{xx} \) and \( u_x \). The following lemmas will play a key role in overcoming this difficulty.

Lemma 3.2. It holds for any \( \epsilon > 0 \) and \( u_h, v_h \in V_h \) that

\[
2\langle -\Delta_h u_h, u_h - v_h \rangle \geq |u_h|_1^2 - |v_h|_1^2, \quad \text{(3.29)}
\]

\[
|u_h|_1^2 + \epsilon^2 \|u_h\|^2 \geq 2\epsilon \langle \nabla_h u_h, u_h \rangle. \quad \text{(3.30)}
\]

\[
|u_h|_1^2 \geq |u_h|_2^2 - |v_h|_1^2. \quad \text{(3.31)}
\]

**Proof.** The first claim (3.29) holds by the inequality \( 2a(a - b) \geq a^2 - b^2 \), i.e.,

\[
2\langle -\Delta_h u_h, u_h - v_h \rangle = \frac{2}{h} \sum_{i=1}^M (u_i - u_{i-1}) ((u_i - u_{i-1}) - (u_i - v_{i-1}))
\]

\[
\geq \frac{1}{h} \sum_{i=1}^M (u_i - u_{i-1})^2 - (v_i - v_{i-1})^2 = |u_h|_1^2 - |v_h|_1^2.
\]

And the third claim (3.31) can be immediately derived by the second claim (3.30) via the Young’s inequality. Hence, we only focus on proving (3.30). From the definition of discrete semi-norm, we have

\[
|u_h|_1^2 = \frac{1}{h} \sum_{i=1}^{M-1} (2u_i - u_{i+1} - u_{i-1})u_i
\]

\[
= \frac{1}{2h} \sum_{i=1}^{M-1} ( (u_{i+1} - u_i)^2 + (u_i - u_{i-1})^2 ) + \frac{u_i^2 + u_{M-1}^2}{2h}
\]

\[
= \frac{1}{4h} \sum_{i=1}^{M-1} ( (u_i - u_{i-1})^2 + (u_{i+1} - 2u_i + u_{i-1})^2 ) + \frac{u_i^2 + u_{M-1}^2}{2h}
\]

\[
= \| \nabla_h u_h \|^2 + \frac{h^2}{4} \| \Delta_h u_h \|^2 + \frac{u_i^2 + u_{M-1}^2}{2h}.
\]

where the identity \( 2a^2 + 2b^2 = (a+b)^2 + (a-b)^2 \) is used. The proof is completed. \( \Box \)

Lemma 3.3. Assume A1 holds. Then it holds for any \( \epsilon > 0 \) and \( u^k_h \in V_h (1 \leq k \leq n) \) that

\[
2 \sum_{k=2}^n \sum_{j=2}^k \theta_{k-j}^{(j)} \langle -\Delta_h u^j_h, u^k_h \rangle + \frac{\epsilon^2}{C_r^2} \sum_{k=2}^n \tau_k \|u^k\|^2 \geq 2\epsilon \sum_{k=2}^n \sum_{j=2}^k \theta_{k-j}^{(j)} \langle \nabla_h u^j_h, u^k_h \rangle, \quad \text{(3.32)}
\]

where \( C_r \) is defined in (2.20).
Proof. On the one hand, from the definition of discrete Laplacian operator $\Delta_h$, we have
\[
\langle -\Delta_h u^k_h, u^k_h \rangle = \frac{1}{h} \sum_{i=1}^{M-1} (2u^j_i - u^j_{i+1} - u^j_{i-1})u^k_i
\]
\[
= \frac{1}{2h} \sum_{i=1}^{M-1} \left( (u^j_{i+1} - u^j_i)(u^k_{i+1} - u^k_i) + (u^j_i - u^j_{i-1})(u^k_i - u^k_{i-1}) \right) + \frac{u^j_i u^k_i + u^{M-1}k_{M-1}}{2h}.
\]
From Lemma 2.3, one further has
\[
2 \sum_{k=2}^{n} \sum_{j=2}^{k} \theta^{(k)}_{k-j} \langle -\Delta_h u^j_h, u^k_h \rangle
\geq \frac{C_r \delta}{2h} \sum_{i=1}^{M-1} \sum_{k=2}^{n} \frac{1}{\tau_k} \left( \left( \sum_{j=2}^{k} \theta^{(k)}_{k-j} (u^j_{i+1} - u^j_i) \right)^2 + \left( \sum_{j=2}^{k} \theta^{(k)}_{k-j} (u^j_i - u^j_{i-1}) \right)^2 \right).
\]
(3.33)
On the other hand,
\[
2 \epsilon \sum_{k=2}^{n} \sum_{j=2}^{k} \theta^{(k)}_{k-j} \langle \nabla_h u^j, u^k \rangle
\]
\[
= \epsilon \sum_{k=2}^{n} \sum_{j=2}^{k} \theta^{(k)}_{k-j} \sum_{i=1}^{M-1} \left( (u^j_{i+1} - u^j_i) + (u^j_i - u^j_{i-1}) \right) u^k_i
\]
\[
= \epsilon \sum_{i=1}^{M-1} \sum_{k=2}^{n} \left( \sum_{j=2}^{k} \theta^{(k)}_{k-j} \left( (u^j_{i+1} - u^j_i) + (u^j_i - u^j_{i-1}) \right) \right) u^k_i
\]
\[
\leq \frac{C_r \delta}{2h} \sum_{i=1}^{M-1} \sum_{k=2}^{n} \frac{1}{\tau_k} \left( \left( \sum_{j=2}^{k} \theta^{(k)}_{k-j} (u^j_{i+1} - u^j_i) \right)^2 + \left( \sum_{j=2}^{k} \theta^{(k)}_{k-j} (u^j_i - u^j_{i-1}) \right)^2 \right)
\]
\[
+ \frac{\epsilon^2}{C_r \delta} \sum_{i=1}^{M-1} \sum_{k=2}^{n} \tau_k(u^k)\).
\]
(3.35)
Thus, combining (3.33) with (3.35), we have (3.32). The inequality (3.31) can be proven similarly and is omitted here. The proof is completed.

We now present the result of stability.

**Theorem 3.1.** Assume the condition **A1** holds. Then the IMEX BDF2 scheme (2.10) is unconditionally stable in the $L^2$-norm. This is, if the maximum time-step size satisfies
\[
\tau \leq \min \left\{ \frac{1}{2C_1}, \frac{1}{2(c_2^2/c_1 + 4|c_3| + 2C_d)}, \frac{1}{4(5C_d + 4|c_3|)} \right\},
\]
(3.36)
it holds for \( n \geq 2 \) that

\[
\|\phi_h^n\| \leq 2 \exp(2C_1 t_{n-1}) \left( (3 + r_2) \|\phi_h^0\| + 7 \sum_{k=1}^n p_{n-k}^{(n)} \|\zeta_k^1\| + 6p_{n-2}^{(n)} C_J (r_1 + r_2) \|\zeta_k^1\| + C_2 \sqrt{r} \|\phi_h^0\|_1 \right)
\]

(3.37)

\[
\leq 2 \exp(2C_1 t_{n-1}) \left( (3 + r_2) \|\phi_h^0\| + 7t_n \max_{1 \leq k \leq n} \|\zeta_k^1\| + 6p_{n-2}^{(n)} C_J (r_1 + r_2) \|\zeta_k^1\| + C_2 \sqrt{r} \|\phi_h^0\|_1 \right),
\]

(3.38)

\[
\|\phi_h^n\| \leq 2\|\phi_h^0\| + 3t_1 \|\zeta_1^1\|.
\]

(3.39)

where

\[
C_1 := \frac{c_2}{c_1 C_r \delta} + 2|c_3| + 2C_J (1 + 2r_{\max}), \quad C_2 := \frac{4c_r}{\delta \sqrt{c_1 r_2}}.
\]

Proof. For \( n \geq 2 \), it follows from (3.28) and (2.14) that

\[
\sum_{j=2}^k \theta_{k-j}^{(k)} D_2 \phi_h^j = \sum_{j=1}^k \theta_{k-j}^{(k)} D_2 \phi_h^j - \theta_{k-j}^{(k)} b_0^{(1)} \nabla_t \phi_h^1 = \nabla_t \phi_h^k - \theta_{k-j}^{(k)} b_0^{(1)} \nabla_t \phi_h^1.
\]

(3.40)

Applying the identity (3.40) to the perturbed equation (3.28), one has

\[
\nabla_t \phi_h^k - \theta_{k-1}^{(k)} b_0^{(1)} \nabla_t \phi_h^1 = c_1 \sum_{j=2}^k \theta_{k-j}^{(k)} (c_2 \nabla_h \phi_h^j + c_3 \phi_h^j + \mathcal{J}(E \phi_h^{j-1}) - \zeta_h^j).
\]

(3.41)

Taking inner products with \( 2\phi_h^k \) on both sides of (3.41) and summing the resulting from 2 to \( n \), we get

\[
2 \sum_{k=2}^n \langle \nabla_t \phi_h^k, \phi_h^k \rangle - 2c_1 \sum_{k=2}^n \sum_{j=2}^k \theta_{k-j}^{(k)} (\Delta_h \phi_h^j, \phi_h^k)
\]

\[
= -2 \sum_{k=2}^n \sum_{j=2}^k \theta_{k-j}^{(k)} (c_2 \nabla_h \phi_h^j + c_3 \phi_h^j + \mathcal{J}(E \phi_h^{j-1}) - \zeta_h^j, \phi_h^k) + 2 \sum_{k=2}^n \theta_{k-1}^{(k)} (b_0^{(1)} \nabla_t \phi_h^1, \phi_h^k).
\]

(3.42)

Taking \( \epsilon = c_2/c_1 \) in (3.32) and inserting the resulting into (3.42), one has

\[
2 \sum_{k=2}^n \langle \nabla_t \phi_h^k, \phi_h^k \rangle \leq \frac{c_2}{c_1 C_1 \delta} \sum_{k=2}^n T_k \|\phi_h^k\|^2 + 2 \sum_{k=2}^n \langle \theta_{k-1}^{(k)} b_0^{(1)} \nabla_t \phi_h^1, \phi_h^k \rangle
\]

\[
+ 2 \sum_{k=2}^n \sum_{j=2}^k \theta_{k-j}^{(k)} \left( c_3 \phi_h^j + \mathcal{J}(E \phi_h^{j-1}) - \zeta_h^j, \phi_h^k \right).
\]

(3.43)
Applying $2(a - b) \geq a^2 - b^2$, (2.9) and the Cauchy-Schwarz inequality to (3.43), one has

\[
\|\phi_h^k\|^2 - \|\phi_h^l\|^2 \leq \frac{c_2}{c_1 C^r \delta} \sum_{k=2}^{n} \tau_k \|\phi^k_h\|^2 + 2|c_3| \sum_{k=2}^{n} \sum_{j=2}^{k} \theta_{k-j}^{(k)} \|\phi_j^h\| \|\phi_k^h\| + 2 \sum_{k=2}^{n} \|\phi_k^h\| \sum_{j=2}^{k} \theta_{k-j}^{(k)} \|\phi^j_h\| \|\phi_k^h\|
\]

\[
+ 2C_J \sum_{k=3}^{n} \sum_{j=3}^{k} \theta_{k-j}^{(k)} \left( (1 + r_{\text{max}})\|\phi_{k-1}^l\| + r_{\text{max}}\|\phi_{k-2}^l\| \right) \|\phi_k^h\|
\]

\[
+ 2C_J \sum_{k=2}^{n} \sum_{j=2}^{k} \theta_{k-j}^{(k)} \left( (1 + r_2)\|\phi_{k}^h\| + r_2\|\phi_0^h\| \right) \|\phi_k^h\| + \frac{2}{\tau_1} \sum_{k=2}^{n} \theta_{k-1}^{(k)} \|\nabla \phi_{k}^h\|. \tag{3.44}
\]

Selecting an integer $n_0$ ($0 \leq n_0 \leq n$) such that $\|\phi_{n_0}^h\| = \max_{0 \leq k \leq n} \|\phi^k_h\|$, then we have

\[
\|\phi_{h}^{n_0}\|^2 \leq \|\phi_1^h\|^2 \|\phi_{n_0}^h\| + \frac{c_2}{c_1 C^r \delta} \sum_{k=2}^{n_0} \tau_k \|\phi^k_h\|^2 + 2|c_3| \sum_{k=2}^{n_0} \|\phi^h_k\| \sum_{j=2}^{k} \theta_{k-j}^{(k)} \|\phi^j_h\|
\]

\[
+ 2C_J \sum_{k=2}^{n_0} \sum_{j=2}^{k} \theta_{k-j}^{(k)} \left( (1 + r_2)\|\phi_{k}^h\| + r_2\|\phi_0^h\| \right) + \frac{2}{\tau_1} \|\phi_{n_0}^h\| \sum_{k=2}^{n_0} \theta_{k-1}^{(k)} \|\nabla \phi_{k}^h\|. \tag{3.45}
\]

Eliminating a $\|\phi_{n_0}^h\|$ from both sides, and setting $C_1 := \frac{c_2}{c_1 C^r \delta} + 2|c_3| + 2C_J(1 + 2r_{\text{max}})$, one arrives at

\[
\|\phi_h^k\|^2 \leq \|\phi_1^h\|^2 + C_1 \sum_{k=2}^{n} \tau_k \|\phi^k_h\| + 2 \sum_{k=2}^{n} \|p^{(n)}_{n-k}\| \|\phi^k_h\|
\]

\[
+ 2C_J \sum_{k=2}^{n} \theta_{k-1}^{(k)} \left( (1 + r_2)\|\phi_{k}^h\| + r_2\|\phi_0^h\| \right) + \frac{2}{\tau_1} \|p^{(n)}_{n-1}\| \|\nabla \phi_{k}^h\|. \tag{3.46}
\]

where one exchanges the order of summation, uses the property (2.15) and the facts $\phi_{n_0}^h \geq \phi_k^h (0 \leq k \leq n)$ and $n_0 \leq n$. It is necessary to bound the terms $\|\phi_k^h\|$ and $\|\nabla \phi_{k}^h\|$. For the first term $\|\phi_k^h\|$, setting $n = 1$ in (3.28) and taking inner products with $2\phi_{k}^h$ on both sides, one has

\[
2 \langle \nabla \phi_{1}^h, \phi_{k}^h \rangle + 2c_1 \tau_1 \langle -\Delta \phi_{1}^h, \phi_{k}^h \rangle = -2c_2 \tau_1 \langle \nabla \phi_{1}^h, \phi_{1}^h \rangle - 2|c_3| \tau_1 \|\phi_{1}^h\|^2 - 2\tau_1 \langle J(\phi_{1}^0) + \zeta_{1}, \phi_{1}^1 \rangle. \tag{3.47}
\]

Applying the property (2.9), the identity $2(a - b) = a^2 - b^2 + (a - b)^2$, the inequality (3.31) and the Cauchy-Schwarz inequality to (3.47), we obtain

\[
\|\phi_k^h\|^2 - \|\phi_0^h\|^2 + \|\nabla \phi_{k}^h\|^2 \leq \left( \frac{c_2^2}{2c_1^2} + 2|c_3|\tau_1 \right) \|\phi_{1}^h\|^2 + 2C_J \tau_1 \|\phi_{1}^0\| \|\phi_{k}^h\| + 2\tau_1 \|\zeta_{1}\| \|\phi_{1}^h\|. \tag{3.48}
\]

Applying the Young’s inequality again to (3.48), one has

\[
\left( \frac{3}{4} - \frac{c_2^2}{2c_1^2} + 2|c_3| + C_J \tau_1 \right) \|\phi_{1}^h\|^2 \leq (1 + C_J \tau_1) \|\phi_{0}^h\|^2 + 4\tau_1^2 \|\zeta_{1}\|^2. \tag{3.49}
\]
Taking $\tau \leq 1/(2c_2^2/c_1 + 8|c_3| + 4C_J)$, we finally get
\[
\|\phi_h^n\| \leq \frac{5}{3}\|\phi_h^0\|^2 + 8\tau_1^2\|\zeta_h^1\|^2.
\] (3.50)

Noting that $\sqrt{a^2 + b^2} \leq |a| + |b|$, we further have
\[
\|\phi_h^n\| \leq 2\|\phi_h^0\| + 3\tau_1\|\zeta_h^1\|.
\] (3.51)

We now estimate the second term $\frac{2\|\nabla_\tau \phi_h^1\|^2}{\tau_1}$. Setting $n = 1$ in (3.28) and taking inner products on both sides with $2\nabla_\tau \phi_h^1$, one has
\[
\frac{2\|\nabla_\tau \phi_h^1\|^2}{\tau_1} - 2c_1 \langle \Delta_h \phi_h^1, \nabla_\tau \phi_h^1 \rangle = -2c_2 \langle \nabla_h \phi_h^1, \nabla_\tau \phi_h^1 \rangle - 2c_3 \langle \phi_h^1, \nabla_\tau \phi_h^1 \rangle - 2\langle J(\phi_h^0), \nabla_\tau \phi_h^1 \rangle - 2\langle \zeta_h^1, \nabla_\tau \phi_h^1 \rangle.
\] (3.52)

Hence, applying (2.9), (2.29) and the Cauchy-Schwarz inequality to (3.52), we obtain
\[
\frac{2\|\nabla_\tau \phi_h^1\|^2}{\tau_1} + c_1\|\phi_h^1\|^2 \leq c_1\|\phi_h^0\|^2 + 2(2c_2\|\nabla_h \phi_h^1\| + |c_3|\|\phi_h^1\| + C_J\|\phi_h^0\| + \|\zeta_h\|)\|\nabla_\tau \phi_h^1\|.
\] (3.53)

Then applying the inequality (3.30) and the Young’s inequality to (3.53), one yields
\[
\frac{\|\nabla_\tau \phi_h^1\|^2}{\tau_1} \leq c_1\|\phi_h^0\| + \frac{c_1^2}{\tau_1}\|\nabla_\tau \phi_h^1\|^2 + \tau_1(3c_3^2\|\phi_h^1\|^2 + 3C_J^2\|\phi_h^0\|^2 + 3\|\zeta_h\|^2).
\] (3.54)

Taking $\tau \leq 1/(2c_2^2/c_1 + 4|c_3| + 2C_J) \leq c_1/(2c_2^2)$, we arrive at
\[
\frac{\|\nabla_\tau \phi_h^1\|^2}{\tau_1} \leq 2c_1\|\phi_h^0\| + 2\tau_1(3c_3^2\|\phi_h^1\|^2 + 3C_J^2\|\phi_h^0\|^2 + 3\|\zeta_h^1\|^2).
\] (3.55)

Inserting the inequality (3.50) into (3.55), one has
\[
\|\nabla_\tau \phi_h^1\|^2 \leq 2c_1\tau_1\|\phi_h^0\|^2 + 3\tau_1^2(5c_3^2 + 2C_J^2)\|\phi_h^0\|^2 + 6\tau_1^2(8c_3^2\tau_1^2 + 1)\|\zeta_h^1\|^2,
\]
which together with the inequality $\sqrt{a^2 + b^2} \leq |a| + |b|$ implies
\[
\|\nabla_\tau \phi_h^1\| \leq 2\sqrt{c_1\tau_1}\|\phi_h^0\| + \tau_1(4|c_3| + 3C_J)\|\phi_h^0\| + \tau_1(7|c_3|\tau_1 + 3)\|\zeta_h^1\|.
\] (3.56)

Inserting the estimates (3.51) and (3.56) into (3.46), one produces
\[
\|\phi_h^n\| \leq 2\|\phi_h^0\| + 3\tau_1\|\zeta_h^1\| + 2C_1\sum_{k=2}^n \tau_k\|\phi_h^k\| + 2\sum_{k=2}^n p_{n-k}^{(n)}\|\zeta_h^k\|
\]
\[
+ 2\left(2p_{n-2}^{(n)}C_J(\tau_1 + \tau_2) + (p_{n-1}^{(n)} - \tau_1)(7|c_3|\tau_1 + 3)\right)\|\zeta_h^1\|
\]
\[
+ 2p_{n-2}^{(n)}(5 + 3\tau_2)C_J + 4|c_3|)\|\phi_h^0\| + \frac{C_J}{\tau_1}\|\phi_h^0\|_1.
\]

where one uses Lemma 2.1 and Proposition 2.2 and identity (3.33). It follows from (3.36) that
\[
\|\phi_h^n\| \leq (6 + 2\tau_2)\|\phi_h^0\| + 2C_1\sum_{k=2}^n \tau_k\|\phi_h^k\| + 14\sum_{k=1}^n p_{n-k}^{(n)}\|\zeta_h^k\|
\]
\[
+ 12p_{n-2}^{(n)}C_J(\tau_1 + \tau_2)\|\zeta_h^1\| + 2C_2\sqrt{\tau}|\phi_h^0|_1.
\] (3.57)
where Proposition 2.2 and identity (2.27) are used. Hence, it follows from Lemma 3.1 that
\[
\|\phi_h^n\| \leq 2 \exp(2C_1 t_{n-1}) \bigg((3 + r_2)\|\phi_h^n\| + 7 \sum_{k=1}^{n} p_{n-k}^{(n)}\|\xi_h^k\| + 6p_{n-2}^{(n)}C_J(t_1 + \tau_2)\|\xi_h^k\| + C_2\sqrt{\tau}\|\phi_h^n\|_1\bigg).
\] (3.58)

According to (2.17), we finally arrive at
\[
\|\phi_h^n\| \leq 2 \exp(2C_1 t_{n-1}) \bigg((3 + r_2)\|\phi_h^n\| + 7 \sum_{k=1}^{n} p_{n-k}^{(n)}\|\xi_h^k\| + 6p_{n-2}^{(n)}C_J(t_1 + \tau_2)\|\xi_h^k\| + C_2\sqrt{\tau}\|\phi_h^n\|_1\bigg)
\leq 2 \exp(2C_1 t_{n-1}) \bigg((3 + r_2)\|\phi_h^n\| + 7t_n \max_{1 \leq k \leq n} \|\xi_h^k\| + 6p_{n-2}^{(n)}C_J(t_1 + \tau_2)\|\xi_h^k\| + C_2\sqrt{\tau}\|\phi_h^n\|_1\bigg).
\]
The proof is completed.

Remark 3.1. Although the stability estimate (3.37) relies on the $H^1$ semi-norm of the initial perturbation, i.e., $|\phi_h^0|_1$, its weight shall tends to zero as $\tau \to 0$. In this sense, the $H^1$ semi-norm $|\phi_h^0|_1$ has mild effect on stability of the IMEX-BDF2 scheme (2.10). Actually, the effect of $H^1$ semi-norm $|\phi_h^0|_1$ can be completely removed if $r_k < 1 + \sqrt{2}$ for all $3 \leq k \leq N$. More specifically, under such a severe restriction, the DCC kernels may be further estimated as
\[
p_{n-j}^{(n)} = \frac{(1 + r_j)\tau_j}{1 + 2r_j} \sum_{k=j}^{n} \prod_{i=j+1}^{k} \frac{r_i^2}{1 + 2r_i} \leq C\tau_j, \quad 2 \leq j \leq n,
\] (3.59)
where $C$ is a constant independent of ratio $r_2$ and mesh sizes $\tau, h$. Hence, the last term in (3.46) can be bounded by $2r_2(\|\phi_h^n\| + \|\phi_h^0\|)$ and the $H^1$ semi-norm $|\phi_h^0|_1$ is removed.

3.2 Consistence and convergence

Set $e_h^n = u(t_n, x_h) - u_h^n$, $x_h \in \Omega_h$. From (1.1), the error function satisfies the governing equation
\[
D_2 e_h^n - c_1 \Delta_h e_h^n + c_2 \nabla_h e_h^n + c_3 e_h^n + \mathcal{J}_h(Ee_h^{n-1}) = \xi_h^n + \eta_h^n + \nu_h^n, \quad \text{for} \quad 1 \leq n \leq N,
\] (3.60)
where $\xi_h^n := \mathcal{J}(Eu(t_{n-1}, x_h)) - \mathcal{J}(u(t_n, x_h))$, $\eta_h^n := D_2u(t_n, x_h) - \partial_h u(t_n, x_h)$ with $x_h \in \mathcal{V}_h$, and $\nu_h^n$ represents the spatial truncation error. For the uniform spatial mesh, it is known the spatial truncation error has second-order accuracy. This is, there exists a constant $C_s$ such that $\|\nu_h^n\| \leq C_s h^2$.

Lemma 3.4. Under the assumption 1.1, it holds that
\[
\|\xi_h^n\| \leq C_J \mathcal{C}_2 \tau_2^{\alpha - 2} + \tau_1 \tau_2^{-1} t_2^{\alpha - 2}), \quad j \geq 3,
\] (3.61)
\[
\|\xi_h^n\| \leq C_J \mathcal{C}_2 \tau_2^{\alpha} (r_2 + 1/\alpha),
\] (3.62)
\[
\|\xi_h^n\| \leq C_J \mathcal{C}_2 \tau_2^{\alpha}/\alpha,
\] (3.63)

Proof. By using the Taylor expansion, it is easy to check that the integro error $\xi_h^n := \mathcal{J}(Eu(t_{j-1}, x_h)) - \mathcal{J}(u(t_j, x_h))(x_h \in \mathcal{V}_h, 1 \leq j \leq N)$ can be expressed as
\[
\xi_h^n := \mathcal{J}(R_h^n),
\] (3.64)
where

\[ R_j^l := \int_{t_j-1}^{t_j} (t - t_j)\partial_t u(t, x_h) \, dt + r_j \int_{t_j-2}^{t_j-1} (t_j-2 - t)\partial_t u(t, x_h) \, dt, \quad 2 \leq j \leq N, \]
\[ R_1^l := u(t_0, x_h) - u(t_1, x_h) = -\int_0^{t_1} \partial_t u(t, x_h) \, dt. \]  \hspace{1cm} (3.65)

Combining (2.9) and assumption 1.1, one immediately has (3.61)–(3.63). The proof is completed. \( \square \)

**Lemma 3.5.** Under the assumption 1.1, for the truncation error \( \eta^j \) it holds that

\[ \left\| \eta_h^1 \right\| \leq 2\bar{C} \tau_j^2 t_{j-1}^{\alpha-3} + \frac{1}{2}\bar{C} \tau_{j-1}^2 t_{j-2}^{\alpha-3}, \quad j \geq 2, \]  \hspace{1cm} (3.66)
\[ \left\| \eta_h^2 \right\| \leq (2r_j^2 + 1/(2\alpha))\bar{C} \tau_j^{\alpha-1}, \]  \hspace{1cm} (3.67)
\[ \left\| \eta_h^1 \right\| \leq \frac{\bar{C}}{\alpha} \tau_1^{\alpha-1}. \]  \hspace{1cm} (3.68)

**Proof.** By using the Taylor expansion [21,31,34], the truncation error \( \eta_h^j \) may be expressed by

\[ \eta_h^j = -\frac{1 + 2r_j}{(2 + 2r_j)\tau_j} \int_{t_j-1}^{t_j} (t - t_j-1)^2\partial_t u(t, x_h) \, dt + \frac{r_j^2}{(2 + 2r_j)\tau_j} \int_{t_j-2}^{t_j-1} (t - t_j-2)^2\partial_t u(t, x_h) \, dt \]
\[ + \frac{r_j}{(2 + 2r_j)} \int_{t_j-1}^{t_j} (2(t - t_j-1) + \tau_j-1)\partial_t u(t, x_h) \, dt, \quad j \geq 2, \]
\[ \eta_h^1 = -\frac{1}{\tau_1} \int_0^{t_1} \partial_t u(t, x_h) \, dt. \]

Noting the weak singularity of solutions in the assumption 1.1, we have

\[ \left\| \eta_h^2 \right\| \leq \frac{1 + 2r_j}{(1 + r_j)} \bar{C} \tau_j^2 t_{j-1}^{\alpha-3} + \frac{r_j}{(2 + 2r_j)} \bar{C} \tau_{j-1}^2 t_{j-2}^{\alpha-3} \leq 2\bar{C} \tau_j^2 t_{j-1}^{\alpha-3} + \frac{1}{2}\bar{C} \tau_{j-1}^2 t_{j-2}^{\alpha-3}, \quad j \geq 2, \]  \hspace{1cm} (3.69)
\[ \left\| \eta_h^2 \right\| \leq \frac{1 + 2r_j}{(1 + r_j)} \bar{C} \tau_j^2 \tau_1^{\alpha-3} + \frac{r_j}{(2 + 2r_j)\tau_1} \bar{C} \int_0^{t_1} \bar{t}^{\alpha-1} \, dt \leq (2r_j^2 + 1/(2\alpha))\bar{C} \tau_1^{\alpha-1}, \]  \hspace{1cm} (3.70)
\[ \left\| \eta_h^1 \right\| \leq \frac{\bar{C}}{\alpha} \int_0^{t_1} \bar{t}\partial_t u \, dt \leq \frac{\bar{C}}{\alpha} \tau_1^{\alpha-1}. \]  \hspace{1cm} (3.71)

The proof is completed. \( \square \)

In the remainder of this paper, any subcripted \( C \) and \( C_u \), denotes positive constants, not necessarily the same at different occurrences, which is always dependent on the given data and the solution, but independent of the ratio \( r_2 \) and mesh sizes \( \tau \) and \( h \).

**Theorem 3.2.** Let \( u(t, x) \) be the solution to problem (1.1) and assume \( A1 \) holds, then the solution \( u_h^n \) to BDF2 scheme (2.10) is convergent in the \( L^2 \)-norm. This is, if the maximum time-step sizes satisfy
(3.36), it holds for $3 \leq n \leq N$ that
\[
\|e^N_h\| \leq C_u \exp(2C_1 t_{n-1}) \left( (1 + r_2)\|e^0_h\| + C_2 \sqrt{\tau} \|e^0_h\|_1 + \sum_{k=3}^{n} p^{(n)}_{n-k} (\tau^2 k^{-3} + \tau^{-2} t_{k-2}^{n-3}) + (1 + r_2)^2 (p^{(n)}_{n-2} + p^{(n)}_{n-1}) + h^2 \right),
\]
\[
\leq C_u \exp(2C_1 t_{n-1}) \left( (1 + r_2)\|e^0_h\| + C_2 \sqrt{\tau} \|e^0_h\|_1 + \delta^{-1} \sum_{k=3}^{n} \tau^{\alpha} (\tau^3 k^{-3} + \tau^{-2} t_{k-2}^{n-3}) + (1 + r_2)^2 (r_2^{1-\alpha} + 1)\tau^{\alpha} + h^2 \right),
\]
where $C_u$ is a constant independent of ratio $r_2$ and mesh sizes $\tau$ and $h$.

Proof. For $n \geq 2$, it follows from (3.39) and (3.58) that
\[
\|e^N_h\| \leq 2 \exp(2C_1 t_{n-1}) \left( (3 + r_2)\|e^0_h\| + C_2 \sqrt{\tau} \|e^0_h\|_1 + 14 \sum_{k=1}^{n} p^{(n)}_{n-k} \|\xi_k^h + \eta_k^h + \nu_k^h\| + 6 p^{(n)}_{n-2} C_J (\tau_1 + \tau_2) \|\xi_k^h + \eta_k^h + \nu_k^h\| \right),
\]
\[
\leq C_u \left( (1 + r_2)^{3} \tau^{\alpha} + t_1 h^2 \right),
\]
Combining with (3.76), one has
\[
\|e^1_h\| \leq 2\|e^0_h\| + 3\tilde{C}(C_J \tau_1 + 1)\tau^{\alpha} + 3C_u \tau_1 h^2 \leq C_u (\|e^0_h\| + \tau^{\alpha} + t_1 h^2).
\]

For $n = 2$, it follows from Lemmas 3.4 and 3.5 and Proposition 2.1 that
\[
7 \sum_{k=1}^{2} p^{(2)}_{2-k} \|\xi_k^h + \eta_k^h + \nu_k^h\| + 6 C_J p^{(2)}_{0} (\tau_1 + \tau_2) \|\xi_k^h + \eta_k^h + \nu_k^h\| \leq C_u \left( (1 + r_2)^{3} \tau^{\alpha} + t_2 h^2 \right),
\]
Combining with (3.76), one has
\[
\|e^2_h\| \leq C_u \exp(2C_1 t_1) \left( (1 + r_2)\|e^0_h\| + \tau^{\alpha} + t_2 h^2 + C_2 \sqrt{\tau} \|e^0_h\|_1 \right).
\]

For $n \geq 3$, from Lemmas 3.4 and 3.5, one has
\[
\sum_{k=1}^{n} p^{(n)}_{n-k} \|\eta_k^h\| \leq 2\tilde{C} \left( \sum_{k=3}^{n} p^{(n)}_{n-k} (\tau^2 k^{-3} + \tau^{-2} t_{k-2}^{n-3}) + (p^{(n)}_{n-2} (r_2^{1} + 1) + p^{(n)}_{n-1}) \tau^{-1} \right),
\]
\[
\sum_{k=1}^{n} p^{(n)}_{n-k} \|\xi_k^h\| \leq C_J \tilde{C} \left( \sum_{k=3}^{n} p^{(n)}_{n-k} (\tau^2 t_{k-1}^{-2} + \tau_2 \tau_{k-1} t_{k-2}^{n-2}) + (p^{(n)}_{n-2} r_2 + 2) + 2p^{(n)}_{n-1} \right),
\]
\[
p^{(n)}_{n-2} (\tau_1 + \tau_2) \|\xi_k^h + \eta_k^h\| \leq 2\tilde{C}(C_J \tau_1 + 1)(\tau_1 + \tau_2) p^{(n)}_{n-2} \tau^{\alpha-1}.
\]
where $\alpha \geq 1/2$ is used. Hence, (3.72) holds by inserting (3.80) and (3.81) into (3.76) and using the condition $\tau \leq 1$. From Proposition 2.2, we find

$$
(p_{n-2}^{(n)} + p_{n-1}^{(n)}) r_1^{\alpha - 1} \leq c_r \delta^{-1} (2r_1^{1-\alpha} + 1) \tau^\alpha,
$$

which yields (3.73). The proof is completed. \hfill \Box

**Remark 3.2.** Theorem 3.2 indicates that the error increases polynomially with respect to the first ratio $r_2$, which is consistent with the numerical experiments in [25].

### 4 Graded mesh for time grid

An important feature of the considered problem has the weak regularity of the solution arose from the nonsmooth initial data. A fundamental flaw of numerical scheme with uniform time step will bring the loss of accuracy. Theorem 3.2 indicates that taking smaller time steps near the initial time $t = 0$ may be an effective approach to improve the global accuracy. In this section, we consider the graded time mesh $t_k = T(k/N)^\gamma$ to achieve the second-order convergence, where the grading parameter $\gamma > 1$.

We claim that the graded time mesh is consistent with the ratio restriction $A1$. Actually, it is easy to verify the time-step ratio $r_k = \frac{\tau_k}{\tau_{k-1}} = \frac{k^\gamma - (k-1)^\gamma}{(k-1)^\gamma - (k-2)^\gamma}$ for $2 \leq k \leq N$. Note that the function $h(x) = x^\gamma - (x-1)^\gamma (x-1)^\gamma - (x-2)^\gamma$ is monotone decreasing. One can verify $r_3 \leq r_{\text{max}} \approx 4.8645$ for $\gamma \leq 4.2529$, which implies the ratio restriction $A2$ holds for the graded time mesh.

**Theorem 4.1.** Assume the conditions in Theorem 3.2 hold and take the graded mesh $t_k = T(k/N)^\gamma$. Then the global error $e^n$ has the following convergence rate:

$$
\| e^n \| \leq \begin{cases} 
CN^{-\alpha \gamma}, & \gamma < 2/\alpha \\
CN^{-2} \log N, & \gamma = 2/\alpha \\
CN^{-2}, & \gamma > 2/\alpha,
\end{cases}
$$

where $C$ is a constant independent with $N$.

**Proof.** For the case of $n \leq 2$, it is easy to verify that (4.84) holds. We only prove the case of $n \geq 3$.

It follows from Theorem 3.2 that

$$
\| e^n \| \leq C \left( \sum_{k=3}^{n} p_{n-k}^{(n)} (r_k^{2} t_k^{\alpha-3} + r_k^{2} t_k^{\alpha-3}) + (p_{n-2}^{(n)} + p_{n-1}^{(n)}) r_1^{\alpha - 1} + h^2) \right).
$$

Due to the decreasing property of function $h(x)$, there exists a positive integer $N_0(\gamma)$ such that $r_k < 1 + \sqrt{2}$ when $k \geq N_0(\gamma)$ (For example, $N_0(\gamma) = 5$ for $\gamma = 4$). Combining with proposition 2.1, i.e., the DCC kernels satisfy

$$
p_{n-j}^{(n)} = \sum_{k=j}^{n} \tau_k (1 + r_j) \prod_{i=j+1}^{k} \frac{r_i}{1 + 2r_i} \leq \tau_j \sum_{k=j}^{n} \prod_{i=j+1}^{k} \frac{r_i^2}{1 + 2r_i},
$$

(4.86)
which implies that there exists a constant $C$ such that $p^{(n)}_{n-j} \leq C\tau_j$. Hence, we have
\[
\|e^n\| \leq C\left(\sum_{k=3}^{n}(\tau_k^3 t_{k-1}^{\alpha-3} + \tau_k^3 t_{k-2}^{\alpha-3}) + \tau_1^\alpha + h^2\right). \tag{4.87}
\]

It is easy to verify $\tau_k \leq T\gamma N^{-1}(k/N)^{\gamma-1}$ for $1 \leq k \leq N$, then we have
\[
\sum_{k=3}^{n} \tau_k^3 t_{k-1}^{\alpha-3} \leq T^\alpha \gamma^3 \sum_{k=3}^{n} \frac{k^{\alpha\gamma-3}}{N^{\alpha\gamma}}. \tag{4.88}
\]

For $\alpha\gamma < 2$, obviously $\sum_{k=3}^{n} k^{\alpha\gamma-3}$ is summable. Hence, (4.88) reduces to
\[
\sum_{k=3}^{n} \tau_k^3 t_{k-1}^{\alpha-3} \leq C N^{-\alpha\gamma}. \tag{4.89}
\]

For $\alpha\gamma = 2$, it is easy to verify $\sum_{k=3}^{n} k^{-1} \leq C \ln N$, then we have
\[
\sum_{k=3}^{n} \tau_k^3 t_{k-1}^{\alpha-3} \leq C N^{-2} \ln N. \tag{4.90}
\]

For $\alpha\gamma > 2$, note that $\sum_{k=3}^{n} \frac{1}{N} (\frac{k}{N})^{\alpha\gamma-3} \leq \int_0^1 x^{\alpha\gamma-3} dx \leq \frac{1}{\alpha\gamma-2}$, we arrive at
\[
\sum_{k=3}^{n} \tau_k^3 t_{k-1}^{\alpha-3} \leq C N^{-2}. \tag{4.91}
\]

The proof is completed by inserting (4.89), (4.90) and (4.91) into (4.87). \qed

5 Numerical experiments

Two numerical examples are reported here to demonstrate our theory.

Example 1 (Construct an exact solution for abstract PIDE).

The IMEX BDF2 scheme (2.10) runs for solving the problem (1.1) in the spatial domain $(0, \pi)$ and time interval $(0, 1]$. We take $c_1 = c_2 = c_3 = 1$, $u_0(x, t) = 0$ and $\rho(x) \equiv 1$. The source term $f$ is chosen such that the continuous problem (1.1) has an exact solution $u = (1 + t^\alpha) \sin(x)$ with some parameters $0.5 \leq \alpha \leq 1$.

In our simulations, we uniformly divide the spatial domain $\Omega$ into $M$ subintervals and the time interval $[0, 1]$ by a graded mesh with $N$ points. Since the spatial error $O(h^2)$ is standard, we only investigate the temporal error. In each run, fixing $M = 8192$, then the discrete $L^2$-error is recorded as $e(N) = \|u(t_n, \cdot) - u_n^h\|$ and the temporal convergence rate (listed as “Order” in the tables) is calculated by $\text{Order} = \log_2(e(N)/e(2N))$. To test the sharpness of our error estimate (4.84), we list the $L^2$-error and convergence order in Table 1, in which different parameters $\alpha, \gamma$ are chosen. As shown in Table 1, the numerical results support the predicted time accuracy in Theorem 4.1 on the smoothly graded mesh $t_k = T(k/N)^\gamma$.  

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The nonuniform meshes evidently improve the numerical precision and convergence order of solution. More specifically, one can observe an accuracy of $O(N^{-\alpha \gamma})$ for $\gamma < 2/\alpha$, and also observe the accuracy of $O(N^{-1})$ for $\gamma > 2/\alpha$. Results in Table 1 suggest that $\gamma = 2/\alpha$ is the optimal choice to have the optimal convergence order, which agrees with our theoretical error estimate (4.84).

**Example 2** (European call option under Merton’s model).

To simulate a European call option under Merton’s model, we only consider the case that $\mu_M, \sigma_M, \sigma$ and $\lambda$ are constants for simplicity and investigate the convergence orders of IMEX BDF2 scheme with different time grids. By Merton [24], the first six terms can obtain the accurate price and the reference values for European call option are 0.52763802 at $S = 90$, 4.39124569 at $S = 100$, and 12.64340583 at $S = 110$.

By choosing the parameters $\sigma = 0.15, r_I = 0.05, \mu_M = -0.9, \sigma_M = 0.45, \lambda = 0.1, c_1 = \sigma^2/2, c_2 = -(r_I - \sigma^2/2 - \lambda \kappa), c_3 = r_I + \lambda, T = 0.25, K = 100, x_l = -1.5, x_r = 1.5$, we demonstrate the errors at the reference points and convergence orders of IMEX BDF2 scheme with different time grids in Table 2. Table 2 shows that the IMEX BDF2 scheme is convergent with quadratic rate, which is consistent with our theoretical analysis.
Table 2: The errors and convergence orders on graded time mesh with $\alpha = 1/2$ and $\gamma = 4$ for European call option (Merton’s)

| $S=90$ | $S=100$ | $S=110$ |
|-------|---------|---------|
| $M$   | $N$     | Error   | Order | Error   | Order | Error   | Order |
| 256   | 256     | 4.2388e-04 | –     | 9.0000e-03 | –     | 2.0000e-03 | –     |
| 512   | 512     | 1.1181e-04 | 1.92  | 2.2000e-03 | 2.01  | 5.0890e-04 | 1.99  |
| 1024  | 1024    | 2.8316e-05 | 1.98  | 5.5722e-04 | 2.00  | 1.2743e-04 | 2.00  |
| 2048  | 2048    | 7.1017e-06 | 2.00  | 1.3927e-04 | 2.00  | 3.1868e-05 | 2.00  |

6 Conclusion

The partial integrl-differential equations (PIDEs), which arises from option pricing theory when the underlying asset follows a jump diffusion process, may suffer from the weak regularity near $t = 0$ due to nonsmoothness of the initial value. In this paper, we revisit an implicit-explicit (IMEX) BDF2 (2.10) with variable time steps for solving the PIDEs with an initial singularity. To handle the initial singularity, nonuniform time steps like graded mesh are an efficient choice. If the graded mesh is used to deal with the initial singularity (1.3), one needs to $r_2 = 2^\gamma - 1 > 15$ when taking $\alpha = 0.5$, which significantly breaks the adjacent time-step ratio restriction $r_2 < r_{\text{max}}(\approx 4.86)$ in [21,32,34].

To fill the gap, several novel properties based on the DOC and DCC kernels are developed, including a strictly positive definiteness of the BDF2 kernels $b_{n-k}^{(n)}$ (see Lemma 2.3) and a sharper estimate of DCC kernels $p_{n-k}^{(n)}$ (see Proposition 2.1) for handling the initial singularity, and a novel convolution-type Young’s inequality (see Lemma 3.3) for dealing with the difficulty arising from spatial discretization. We have presented that the IMEX-BDF2 scheme is unconditionally stable and achieves a $\alpha$-order temporal convergence under mild assumptions on the ratio of adjacent time steps A1, where $\alpha$ is the regularity of the exact solution (see 1.3). Compared with the previous ratio restrictions in [21,32,34], our ratio restriction has no restriction on the ratio $r_2$. Thus, an optimal convergence of order $\mathcal{O}(N^{-2})$ is achieved under the graded mesh $t_k = T(k/N)^{\gamma_{\text{opt}}}$ with an optimal parameter $\gamma_{\text{opt}} = 2/\alpha$.

As far as we know, this is the first rigorous proof of second-order convergence for an IMEX-BDF2 scheme for initial singularity problems without severe restrictions on the ratio of adjacent time-steps especially no any restriction on the first ratio $r_2$.

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