ON FIRST ORDER DEFORMATIONS OF HOMOGENEOUS FOLIATIONS

ARIEL MOLINUEVO AND BRUNO SCÁRDUA

Abstract. We study analytic deformations of holomorphic foliations given by homogeneous integrable one-forms in the complex affine space \( \mathbb{C}^n \). The deformation is supposed to be of first order (order one in the parameter). We also assume that the deformation is given by homogeneous polynomial one-forms. The deformation takes place in the affine space since we are not assuming that the foliations descent to the projective space. We describe the space of such deformations in three main situations: (1) the given foliation is given by the level hypersurfaces of a homogeneous polynomial. (2) the foliation is rational, i.e., has a first integral of type \( P^r/Q^s \) for some homogeneous polynomials \( P, Q \). (3) the foliation is logarithmic of a generic type. We prove that, for each class above, the first order homogeneous deformations of same degree are in the very same class. We also investigate the existence of such deformations with different degree.

Contents

1. Introduction and main results 1
2. Notation 4
3. First order deformations of a codimension one foliation 4
4. Rational and logarithmic foliations in \( \mathbb{P}^n \) 6
4.1. Rational foliations 6
4.2. Logarithmic foliations 6
5. Rational and logarithmic foliations in \( \mathbb{C}^n \) 7
5.1. Affine rational foliations defined by homogeneous one-forms 8
5.2. Affine logarithmic foliations defined by homogeneous one-forms 9
6. Affine deformations of affine rational and logarithmic foliations 9
7. Relative cohomology with poles: the equation \( d\left(\frac{\mathbf{1}}{\mathbf{H}}\right) \wedge \omega_0 = 0 \) 13
7.1. First order perturbations (solutions of degree \( = \partial(\omega_0) \)) 14
7.2. Solutions of degree \( \neq \partial(\omega_0) \) 16
8. Deformations of dicritical homogeneous one-forms 17
9. Stability of an exact differential form \( \omega = dP \) 19
References 20

1. Introduction and main results

Foliations are an important tool in the classification of manifolds, specially in low dimension. This refers initially to the study of smooth non-singular integrable structures on closed real manifolds. This is particularly evident in the case of codimension one foliations. Following this spirit the notion of holomorphic foliation with singularities was brought to
the scene. The subject has grown and spread to other areas as algebraic geometry. Indeed, the classification of compact complex surfaces is strongly related to the study of foliations in such objects. Much more has been done in this direction. Related to this is the structure of the space of foliations. As it is known, the space of codimension one holomorphic foliations in a complex manifold has the structure of an analytic variety. One may then ask for its irreducible components. This is a quite vast and rich topic. We will focus on a very specific case in this framework. More precisely, we will study codimension one holomorphic foliations in the complex affine space of dimension $\geq 3$. We shall restrict our study to the algebraic (polynomial) case. More precisely, in this paper we are concerned with the space of deformations of a polynomial homogeneous one-form satisfying the integrability condition of Frobenius. We consider deformations as perturbations of first order, also satisfying the integrability condition. Our starting one-form is assumed to admit an integrating factor which is reduced. This implies that the corresponding foliation is logarithmic in the sense of [CA94] and [CM82], i.e., given by a simple poles rational one-form. In a certain sense after those admitting rational first integral, this is the simplest class of foliations and correspond to a linear model. A number of authors have addressed the problem of finding the irreducible components of the space of codimension one holomorphic foliations in the complex projective space of dimension $n \geq 3$. This is treated by studying deformations by same degree homogeneous integrable one-forms of a given homogeneous integrable one-form $\omega$ in $n+1$ complex variables $(x_1, \ldots, x_{n+1})$. Since the corresponding foliation in $\mathbb{C}^{n+1} \setminus \{0\}$ descends to the projective space $\mathbb{P}^n$, we also have $i_R(\omega) = 0$ where $R = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$ is the radial vector field.

One of the very first results in this subject is the finding of the so called rational components ([GMLN91]). This corresponds to the stability of foliations given by the fibers of maps of the form $P^n/Q^s: \mathbb{P}^n \to \mathbb{P}^1$ for $n \geq 3$ and suitable homogeneous polynomials $P, Q$ where $r = \partial(Q)$ and $s = \partial(P)$, and where we are denoting with the symbol $\partial$ the degree of the given polynomial. The stability of such foliations is pretty much a consequence of the very special geometry of the projective space $\mathbb{P}^n$ (characteristic classes of line bundles) ([CA94]). In this paper we shall resume the study of deformations of rational foliations, but for a wider class. Indeed, we shall consider foliations with a rational homogeneous first integral, but not necessarily descending to the projective space. We shall refer to these as homogeneous affine rational foliations. Our main result for this class of foliations is (cf. Theorem [3]):

**Theorem 1.1.** Let $\omega_{R_0} = r f_1 df_2 - s f_2 df_2$ define a homogeneous affine rational foliation of generic type in $\mathbb{C}^n$, $n \geq 3$, where $f_1, f_2$ are homogeneous polynomials and $r, s \in \mathbb{C}$. All first order deformations of $\omega_{R_0}$ by homogeneous integrable one-forms of same degree, are obtained by perturbations of the polynomial parameters $f_1$ and $f_2$ or of the eigenvalues $(r, s)$.

The study of deformations of logarithmic foliations has started with the work of Calvo Andrade in [CA94] where the author proves their stability under some mild conditions, i.e., for generic elements. This result is based on a byproduct of a result of Hirsch about fixed points for central hyperbolic elements in a group of diffeomorphisms and a result of Nori for the fundamental group of the complement of a codimension one divisor with normal crossings in $\mathbb{P}^n$. Later on, other authors added more information to this subject.
and obtained more general results by considering more Algebraic geometry type arguments ([CSV06] and [CGM18]). In this paper we resume this subject with a slightly different standpoint. We consider perturbations of order one, i.e., of the form \( \omega_t = \omega + t\eta \) where \( t \) is a complex parameter such that \( t^2 = 0 \). We consider though the case where \( \omega \) is an affine logarithmic foliation. In short, this means that \( \omega \) admits a reduced integrating factor but not necessarily descends to the projective space, i.e., \( i_R(\omega) \neq 0 \). This situation has to be dealt with via different techniques resembling more to the local cases considered in [CS18]. Indeed, one rapidly reaches the connections with the relative cohomology introduced in [BC93]. Nevertheless, we are working with the meromorphic case, so we cannot apply the results in [BC93]. Roughly speaking we shall be concerned with the following equation
\[
\omega \wedge d\eta + d\omega \wedge \eta = 0,
\]
which parameterize, in \( \eta \), the perturbations of order one of the given foliation \( \omega \), see Section 3.

Let us make our framework more clear. We are considering one-forms \( \omega \) as follows:
\[
\omega = \left( \prod_{i=1}^{s} f_i \right) \left( \sum_{i=1}^{s} \lambda_i \frac{df_k}{f_k} \right)
\]
where \( \lambda_i \in \mathbb{C} \) are called eigenvalues of \( \omega \) and the homogeneous polynomials \( f_1, \ldots, f_s \) are the polynomial parameters of \( \omega \). We shall refer to \( \omega \) then as a homogeneous affine logarithmic foliation in \( \mathbb{C}^n \). We shall say that \( \omega \) is generic if it verifies the following conditions ([CGM18]):

1. the \( \{ f_i = 0 \} \) are smooth, irreducible \( \forall i = 1, \ldots, s \) and \( D = \{ f_1, \ldots, f_s = 0 \} \) is a divisor with normal crossings
2. \( \lambda_i \neq \lambda_j (\neq 0) \) for every \( i \neq j \).

We give solutions to the eq. (1) above in all cases. We shall also describe all the solutions of same degree of \( \omega \) for a generic element \( \omega \). As a result we are able to prove (see Theorem 6.4 for a complete statement) for dimension \( n \geq 3 \):

**Theorem 1.2.** All same degree polynomial first order deformations of a generic homogeneous affine logarithmic foliation, defined by integrable homogeneous one-forms of same degree, are obtained by perturbations of the polynomial parameters \( f_i \) or of the eigenvalues \( \lambda_i \).

Finally, in the last part of this work, we consider first order integrable perturbations of an exact homogeneous one-form \( \omega = dP \). This may be considered as a global version of the main result in [CS18]. We obtain a particular case of a result recently proved in [CS].

**Theorem 1.3.** Let \( \omega = dP \) be an exact differential form in \( \Omega^1_{\mathbb{C}^n}, n \geq 3 \), homogeneous of degree \( e \). Let us suppose also that the codimension of the singular locus of \( dP \) is \( \geq 3 \). Then all first order deformations of \( \omega \), defined by integrable homogeneous one-forms of the same degree, are exact of type
\[
\omega_\varepsilon = d(P + \varepsilon Q)
\]
where \( Q \) is a homogeneous polynomial of degree \( e \).

Theorem 1.3 is proved as Theorem 9.1 and may be seen as a global version of Malgrange’s “singular Frobenius” result ([Mal76]).
2. Notation

We will denote with $S_n = \mathbb{C}[x_1, \ldots, x_n]$ the ring of polynomial in $n$ complex variables. We would like to recall here that the global sections of the twisted sheaf of differential forms of $\Omega^1_{\mathbb{P}^n}(e)$ it is a finitely generated $S_{n+1}$-module defined by $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))$ if and only if $\omega$ can be written as

$$\omega = \sum_{i=1}^{n+1} A_i dx_i$$

where:

1. The $A_i$'s are homogeneous polynomials in $S_{n+1}$, of degree $e - 1$.
2. The one-form $\omega$ satisfies the condition of descent to the projective space: given the radial vector field $R = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$ we have

$$i_R(\omega) = \sum_{i=1}^{n+1} x_i A_i = 0.$$

Regarding foliations in the affine space $\mathbb{C}^n$, we are going to denote as $\Omega^1_{\mathbb{C}^n}$ the $S_n$-module of Kähler differentials. This module it is given by polynomial 1-differential forms in $n$-variables, i.e., it is generated by the differentials $(dx_1, \ldots, dx_n)$.

If $\eta \in \Omega^1_{\mathbb{C}^n}$ we will say that $\eta$ is \textit{homogeneous of degree} $e$ if $\eta$ is of the form

$$\eta = \sum_{i=1}^{n} H_i dx_i$$

where the $H_i$ are homogeneous polynomials of degree $e - 1$. We will also denote as $\partial(\eta)$ or $\partial(H_i)$ to the degree of $\eta$ and $H_i$, respectively.

We also by $\Omega^1_{\mathbb{C}^n,0}$ the space of germs of differential forms, i.e., an element $\omega \in \Omega^1_{\mathbb{C}^n,0}$ is of the form

$$\omega = \sum_{i=1}^{n} \tilde{A}_i dx_i$$

where the $\tilde{A} \in O_{\mathbb{C}^n,0}$ are germs of holomorphic functions at the origin.

3. First order deformations of a codimension one foliation

As it is well-known any holomorphic foliation of codimension one with singularities in the complex projective space $\mathbb{P}^n$ is given by a section $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))/\mathbb{C}$, for example see [CPV09], for some $e \in \mathbb{N}$. We will therefore denote as $\mathcal{F}^1(\mathbb{P}^n,e)$ the space of \textit{codimension one foliations} in $\mathbb{P}^n$ of degree $e - 2$, i.e.,

$$\mathcal{F}^1(\mathbb{P}^n,e) = \{ \omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))/\mathbb{C} : \omega \wedge d\omega = 0 \}.$$
The first order deformations of an integrable differential one form $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))$, are given by the $\eta \in H^0(\Omega^1_{\mathbb{P}^n}(e))$ such that

$$\omega_\varepsilon = \omega + \varepsilon \eta$$

is integrable (in the sense of Frobenius), where the parameter $\varepsilon$ is infinitesimal, in the sense that $\varepsilon^2 = 0$. The integrability condition means that $\omega_\varepsilon \wedge d\omega_\varepsilon = 0$. If we expand this equation we get $\omega_\varepsilon \wedge d\omega_\varepsilon = \omega \wedge d\omega + \varepsilon (\omega \wedge d\eta + d\omega \wedge \eta) = 0$. Since $\omega$ is already integrable, the integrability condition of $\omega_\varepsilon$ is then equivalent to the following equation:

$$\omega \wedge d\eta + d\omega \wedge \eta = 0.$$  

We may therefore parameterize first order deformations of $\omega \in \mathcal{F}^1(\mathbb{P}^n, e)$ as the space

$$D_{\mathbb{P}^n}(\omega) = \left\{ \eta \in H^0(\Omega^1_{\mathbb{P}^n}(e)) : \omega \wedge d\eta + d\omega \wedge \eta = 0 \right\} / \mathbb{C}.\omega$$

As expected, the vector space $D(\omega)$ can be identified with the tangent space at $\omega T_\omega \mathcal{F}^1(\mathbb{P}^n, e)$, see [CPV09, Section 2.1, pp. 709].

In the affine (algebraic) case, let us define the space of affine codimension one foliations as

$$\mathcal{F}^1(\mathbb{C}^n) = \{ \omega \in \Omega^1_{\mathbb{C}^n}/\mathbb{C} : \omega \wedge d\omega = 0 \},$$

where we are considering $\omega \in \mathcal{F}^1(\mathbb{C}^n)$ homogeneous of degree $e$.

Similarly to above the space of first order perturbations of $\omega$ is defined as

$$D_{\mathbb{C}^n}(\omega) = \left\{ \eta \in \Omega^1_{\mathbb{C}^n} : \omega \wedge d\eta + d\omega \wedge \eta = 0 \right\} / \mathbb{C}.\omega .$$

We shall only consider deformations preserving the degree of the given foliation. Thus, given $\omega$ of degree $e$ we define the space of deformations homogeneous of the same degree of $\omega$ as

$$D_{\mathbb{C}^n}(\omega, e) = \left\{ \eta \in \Omega^1_{\mathbb{C}^n} : \omega \wedge d\eta + d\omega \wedge \eta = 0, \text{ and } \eta \text{ of degree } e \right\} / \mathbb{C}.\omega .$$

4. **Rational and logarithmic foliations in $\mathbb{P}^n$**

Very basic examples of foliations in $\mathbb{P}^n$ are given by the classes of rational and logarithmic foliations. In this section we should review their definitions and basic results that we are going to use in the rest of the paper.

---

1. We point out that there are codimension one holomorphic foliations with singularities in the affine space $\mathbb{C}^n$ which are not given by polynomial one-forms. We shall be working with those which are given by polynomial one-forms.
4.1. **Rational foliations.** In the case of foliations in the projective space, the class of rational foliations corresponds to the pull-backs of the two dimensional model \( xdy - ydx = 0 \) by maps \( \sigma : \mathbb{P}^n \to \mathbb{P}^2 \) of the form \( \sigma = (P^n, Q^n) \) where \( P, Q \) are homogeneous polynomials in \( \mathbb{C}^{n+1} \) of degree \( \partial(P) = s \) and \( \partial(Q) = r \). More precisely we have:

**Definition 4.1.** A rational foliation of type \((d_1, d_2)\) in \( \mathcal{F}^1(\mathbb{P}^n, e) \), is defined by a global section \( \omega_{\mathcal{R}} \in H^0(\Omega^1_{\mathbb{P}^n}(e)) \) of the form

\[
\omega_{\mathcal{R}} = d_1 f_1 df_2 - d_2 f_2 df_1,
\]

where \( \partial(f_1) = d_1 \) and \( \partial(f_2) = d_2 \) and \( d_1 + d_2 = e \).

In the definition above, the \(-\) sign and the coefficients \( d_1 \) and \( d_2 \) are taken in order to guarantee the descent to projective space of the differential form \( \omega_{\mathcal{R}} \).

We will note \( \mathcal{R}(n, (d_1, d_2)) \) the space of rational foliations of this kind, and define the generic open set \( \mathcal{U}_{\mathcal{R}} \subset \mathcal{R}(n, (d_1, d_2)) \) as

\[
\mathcal{U}_{\mathcal{R}} = \{ \omega \in \mathcal{R}(n, (d_1, d_2)) : \text{codim}(\text{Sing}(d\omega)) \geq 3, \text{codim}(\text{Sing}(\omega)) \geq 2 \}.
\]

First order deformations of rational foliations are studied in the works [GMLN91] and [CPV09]. We recall from [CPV09] Proposition 2.4, p. 711] the following result.

**Theorem 4.2.** Let \( \omega_{\mathcal{R}} \in \mathcal{U}_{\mathcal{R}} \) be a generic rational foliation. Then, the first order deformations of \( \omega_{\mathcal{R}} \), or the tangent space of \( \mathcal{F}^1(\mathbb{P}^n, e) \) at \( \omega_{\mathcal{R}} \), can be given by the perturbations of the parameters \( f_1 \) and \( f_2 \)

\[
T_{\omega_{\mathcal{R}}} \mathcal{F}^1(\mathbb{P}^n, e) = D_{\mathbb{P}^n}(\omega_{\mathcal{R}}) = \text{Span} \left\{ \eta \in \mathcal{R}(n, (d_1, d_2)) : \eta = d_1 f'_1 df_2 - d_2 f_2 df'_1 \text{ or } \eta = d_1 f_1 df_2' - d_2 f_2 df_1 \right\} / \mathcal{C}\omega_{\mathcal{R}}.
\]

4.2. **Logarithmic foliations.** Logarithmic foliations in the projective space \( \mathbb{P}^n \) are pullback of linear foliations in \( \mathbb{C}^s \), of the form \( \left( \prod_{i=1}^s x_i \right) \sum_{i=1}^s \lambda_i \frac{dx_i}{x_i} = 0 \) by maps \( \sigma : \mathbb{P}^n \to \mathbb{C}^s \). In a more formal way we have:

**Definition 4.3.** A logarithmic foliation of type \((d_1, \ldots, d_s)\) in \( \mathcal{F}^1(\mathbb{P}^n, e) \), is defined by a global section \( \omega_{\mathcal{L}} \in H^0(\Omega^1_{\mathbb{P}^n}(e)) \) of the form

\[
\omega_{\mathcal{L}} = \left( \prod_{i=1}^s f_i \right) \sum_{i=1}^s \lambda_i \frac{df_i}{f_i},
\]

where \( s \geq 3 \) and

1. \( (\lambda_1, \ldots, \lambda_s) \in \Lambda(s) := \{ (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s : \lambda_1 d_1 + \ldots + \lambda_s d_s = 0 \} \)

2. \( f_i \) is homogeneous of degree \( d_i \) and \( d_1 + \ldots + d_s = e \).

We will note \( \mathcal{L}(\mathbb{P}^n, \overline{d}) \) the space of logarithmic foliations of this kind and define the generic open set \( \mathcal{U}_{\mathcal{L}}(d) \subset \mathcal{L}(\mathbb{P}^n, \overline{d}) \) as

\[
\mathcal{U}_{\mathcal{L}}(d) := \{ \omega \in \mathcal{L}(\mathbb{P}^n, \overline{d}) : \omega \text{ verifies a) and b) below } \},
\]

writing \( \omega = (\prod_{i=1}^s f_i) \sum_{i=1}^s \lambda_i \frac{df_i}{f_i} \) we have the conditions:
a) the \( \{ f_i = 0 \} \) are smooth, irreducible \( \forall i = 1, \ldots, s \) and \( D = \{ f_1, \ldots, f_s = 0 \} \) is a divisor with normal crossings

b) \( \lambda_i \neq \lambda_j (\neq 0) \) for every \( i \neq j \).

Then \( U_L(d) \) is a Zariski dense open subset of \( \mathcal{L}(\mathbb{P}^n, (d)) \). We will usually note \( \overline{d}, \overline{x} \) and \( \overline{f} \) the \( s \)-uples involved in the expression of a logarithmic foliation. Noting \( F_i = \prod_{j \neq i} f_j \), we will frequently write \( \omega_L \) as

\[
\omega_L = \sum_{i=1}^{s} \lambda_i F_i \, df_i.
\]

We now give an example:

**Example 4.4.** Given homogeneous polynomials \( P_1, P_2, Q \) of same degree in \( n \) complex variables, we consider

\[
\omega_\varepsilon = (P_1 + \varepsilon Q)P_2 \left[ \frac{d(P_1 + \varepsilon Q)}{P_1 + \varepsilon Q} + \lambda_2 \frac{dP_2}{P_2} \right]
\]

Then \( \omega_\varepsilon = \omega + \varepsilon \eta \) where \( \omega = \lambda_1 P_2 dP_1 + \lambda_2 P_1 dP_2 \) and \( \eta = \lambda_1 P_2 dQ + \lambda_2 Q dP_2 \). Thus we have a first order deformation of a logarithmic foliation. We may choose the eigenvalues \( \lambda_1, \lambda_2 \) and polynomial parameters \( P_1, P_2 \) in such a way that the logarithmic form associated to \( \omega \) is generic. The deformation \( \omega_\varepsilon \) is given by logarithmic one-forms, obtained by first order perturbation in the polynomial parameter \( P_1 \).

Let us fix \( \omega_L \in \mathcal{L}(\mathbb{P}^n, \overline{d}) \) as before, as in eq. (7), and define the spaces of perturbation of parameters of \( \omega_L \) as

\[
D_{\mathbb{P}^n}(\omega_L, \overline{f}) = \text{Span} \left( \{ \eta_{g_i} \in \mathcal{L}(\mathbb{P}^n, \overline{d}) : \eta_{g_i} \text{ equals } \omega_L \text{ with } f_i \text{ replaced by } g_i \} \right) / \mathbb{C}.\omega_L
\]

\[
D_{\mathbb{P}^n}(\omega_L, \overline{x}) = \text{Span} \left( \{ \eta_{\overline{x}} \in \mathcal{L}(\mathbb{P}^n, \overline{d}) : \eta_{\overline{x}} \text{ equals } \omega_L \text{ with } \overline{x} \text{ replaced by } \overline{y} \} \right) / \mathbb{C}.\omega_L.
\]

By direct computation, it is straight forward to check that \( D_{\mathbb{P}^n}(\omega_L, \overline{f}) \) and \( D_{\mathbb{P}^n}(\omega_L, \overline{x}) \) are subspaces of \( D_{\mathbb{P}^n}(\omega_L) \).

We know from [CGM18, Theorem 25, pp. 14, and Remark 26, pp. 15] that the tangent space of \( \mathcal{F}^1(\mathbb{P}^n, e) \) at a generic point given by \( \omega_L \) it is defined by these perturbations:

**Theorem 4.5.** Let \( \omega_L \in U_L(d) \subset \mathcal{L}(\mathbb{P}^n, \overline{d}) \) be a generic logarithmic foliation. Then, the first order deformations of \( \omega_L \), or the tangent space of \( \mathcal{F}^1(\mathbb{P}^n, e) \) at \( \omega_L \), can be decomposed as

\[
T_{\omega_L} \mathcal{F}^1(\mathbb{P}^n, e) = D_{\mathbb{P}^n}(\omega_L) = D_{\mathbb{P}^n}(\omega_L, \overline{f}) \oplus D_{\mathbb{P}^n}(\omega_L, \overline{x})
\]

5. **Rational and Logarithmic Foliations in \( \mathbb{C}^n \)**

We can repeat the definitions of the preceeding section for rational and logarithmic foliations defined in the affine space \( \mathbb{C}^n \).

In this situation, what changes is condition (1) in definition (4.3) of logarithmic foliation, i.e., we do not need anymore that

\[
\sum_{i=1}^{s} \lambda_i d_i = 0.
\]
The same goes for rational foliations, now we can take every pair of coefficients \((r, s)\) in the definition of rational foliation.

5.1. Affine rational foliations defined by homogeneous one-forms. We shall now introduce an intermediate class between the class of rational foliations in the projective space and the class of foliations admitting a rational first integral in the affine space. For this sake we shall consider latter foliations which are defined by homogeneous one-forms. Indeed, we also allow some more generic models since we do not demand the eigenvalues to have a rational quotient. We define:

**Definition 5.1.** A homogeneous affine rational foliation of type \((d_1, d_2)\) and degree \(e\) in \(\mathcal{F}^1(\mathbb{C}^n)\), is defined by an element \(\omega_{R_0} \in \Omega^1_{\mathbb{C}^n}\) of the form

\[
(8) \quad \omega_{R_0} = rf_1 df_2 - sf_2 df_1,
\]

where \(f_1\) and \(f_2\) are homogeneous of degree \(d_1\) and \(d_2\) respectively, \(d_1 + d_2 = e\) and \(r, s \in \mathbb{C}\).

We will denote with \(\mathcal{F}\) the pair \((f_1, f_2)\) and with \(\mathcal{D}\) the pair \((d_1, d_2)\).

We shall refer to \(f_1, f_2\) as the *polynomial parameters* and to \(r, s\) as the *eigenvalues* of the foliation. If no confusion can arise we will call this foliations just affine rational foliations or rational foliations as well.

We will note \(\mathcal{R}(\mathbb{C}^n, \mathcal{D})\) the space of affine rational foliations of this kind and define the generic open set \(\mathcal{U}_{\mathcal{R}_0} \subset \mathcal{R}(\mathbb{C}^n, \mathcal{D})\) as

\[
(9) \quad \mathcal{U}_{\mathcal{R}_0} = \{ \omega_{\mathcal{R}_0} \in \mathcal{R}(\mathbb{C}^n, \mathcal{D}) : \omega_{\mathcal{R}_0} \text{ verifies a) and b) below}\},
\]

writing \(\omega_{\mathcal{R}_0} = rf_1 df_2 - sf_2 df_1\) we have the conditions:

a) \(D = \{f_1, f_2 = 0\}\) is a normal crossing divisor

b) \(r \neq -s(\neq 0)\).

**Remark 5.2.** We stress the fact that, the eigenvalues \(r, s\) are allowed to be with non-rational quotient. Thus, our definition above of rational foliation in the affine space, includes the linear hyperbolic case \(xdy - \lambda ydx = 0, \lambda \in \mathbb{C} \setminus \mathbb{R}\) as well.

Let us now consider \(\omega_{\mathcal{R}_0} \in \mathcal{R}(\mathbb{C}^n, \mathcal{D})\) of the form of Definition (5.1), then we define the subspaces of \(D_{\mathbb{C}^n}(\omega_{\mathcal{R}_0}, e)\) as

\[
D_{\mathbb{C}^n}(\omega_{\mathcal{R}_0}, \mathcal{F}) = \text{Span} \left( \{ \eta \in \mathcal{R}(\mathbb{C}^n, \mathcal{D}) : \eta = rf_1 df_2 - sf_2 df_1 \text{ or} \right)
\eta = rf_1 df_2' - sf_2' df_1 ) / \mathbb{C}.\omega_{\mathcal{L}_0}
\]

\[
D_{\mathbb{C}^n}(\omega_{\mathcal{R}_0}, (r, s)) = \text{Span} \left( \{ \eta_{(r', s')} \in \mathcal{R}(\mathbb{C}^n, \mathcal{D}) : \eta_{(r', s')} = r' f_1 df_2 - s' f_2 df_1 \} / \mathbb{C}.\omega_{\mathcal{L}_0}\right)
\]

Later, in Theorem (6.3), we will see that these two spaces span all the first order deformations of the same degree of \(\omega_{\mathcal{R}_0}\). Notice that there is no equivalent space of \(D_{\mathbb{C}^n}(\omega_{\mathcal{R}_0}, (r, s))\) in the projective deformations case, this is because the condition of descent to projective space forces the coefficients to be such that there is no possible perturbations of them.
Remark 5.3. We would like to notice that we may also consider the space $D_{C^n}(\omega_{R^0},\overline{\jmath})^+$ defined as

$$D_{C^n}(\omega_{R^0},\overline{\jmath})^+ = \text{Span} \left( \{ \eta \in \mathcal{R}(C^n,\overline{d}) : \eta = r f'_1 df_2 - sf_2 df'_1 \text{ or} \right.$$

$$\eta = r f'_1 df'_2 - sf'_2 df_1 \} \} / \mathbb{C} \omega_{\mathcal{L}_0}$$

where $d'_i$ is the degree of the polynomial $f_i$ and/or $f'_i$, which can be different from the original degree $d_i$.

By direct computation, it is straight forward to check that $D_{C^n}(\omega_{R^0},\overline{\jmath})$ and $D_{C^n}(\omega_{R^0},(r,s))$ are subspaces of $D_{C^n}(\omega_{R^0},e)$ and that the space $D_{C^n}(\omega_{R^0},\overline{\jmath})^+$ is a subspace of $D_{C^n}(\omega_{R^0})$.

5.2. Affine logarithmic foliations defined by homogeneous one-forms. We shall now consider foliations of logarithmic type, but which are defined by homogeneous one-forms, though not necessarily satisfying the condition to descent to the projective space.

Definition 5.4. A homogeneous affine logarithmic foliation of type $(d_1,\ldots,d_s)$ and degree $e$ in $\mathcal{F}^1(C^n)$, is defined by an element $\omega_{\mathcal{L}_0} \in \Omega_{C^n}^1$ of the form

$$\omega_{\mathcal{L}_0} = \left( \prod_{i=1}^s f_i \right) \sum_{i=1}^s \lambda_i \frac{df_i}{f_i} = \sum_{i=1}^s \lambda_i F_i \, df_i,$$

where $s \geq 3$ and

1. $f_i$ is homogeneous of degree $d_i$ and $d_1 + \ldots + d_s = e$.

We shall refer to $f_1,\ldots,f_s$ as the polynomial parameters and to $\lambda_1,\ldots,\lambda_s$ as the eigenvalues of the foliation. If no confusion can arise we will call this foliations just affine logarithmic foliations or logarithmic foliations as well.

We will note $\mathcal{L}(C^n,\overline{d})$ the space of affine logarithmic foliations of this kind and define the generic open set $\mathcal{U}_{\mathcal{L}_0} \subset \mathcal{L}(C^n,\overline{d})$ as

$$\mathcal{U}_{\mathcal{L}_0} = \{ \omega \in \mathcal{L}(C^n,\overline{d}) : \omega \text{ verifies a) and b) below} \} ,$$

writing $\omega = \left( \prod_{i=1}^s f_i \right) \sum_{i=1}^s \lambda_i \frac{df_i}{f_i}$ we have the conditions:

a) $D = \{ f_1,\ldots,f_s = 0 \}$ is a normal crossing divisor

b) $\lambda_i \neq \lambda_j (\neq 0)$ for every $i \neq j$.

Let us now consider $\omega_{\mathcal{L}_0} \in \mathcal{L}(C^n,\overline{d})$ of the form of eq. (10), then we define the subspaces of $D_{C^n}(\omega_{\mathcal{L}_0},e)$ as

$$D_{C^n}(\omega_{\mathcal{L}_0},\overline{\jmath}) = \text{Span} \left( \{ \eta_{g_i} \in \mathcal{L}(C^n,\overline{d}) : \eta_{g_i} \text{ equals } \omega_{\mathcal{L}_0} \text{ with } f_i \text{ changed by } g_i \} \} / \mathbb{C} \omega_{\mathcal{L}_0}$$

$$D_{C^n}(\omega_{\mathcal{L}_0},\overline{x}) = \text{Span} \left( \{ \eta_{\overline{x}_i} \in \mathcal{L}(C^n,\overline{d}) : \eta_{\overline{x}_i} \text{ equals } \omega_{\mathcal{L}_0} \text{ with } \overline{x} \text{ changed by } \overline{x} \} \} / \mathbb{C} \omega_{\mathcal{L}_0}.$$

Later, in Theorem (6.3), we will see that, again, these two spaces span all the first order deformations of the same degree of $\omega_{\mathcal{L}_0}$.
Remark 5.5. As before, we would like to notice that we may also consider the space \( D_{\mathbb{C}^n}(\omega_{L_0}, \mathcal{F})^+ \) defined as
\[
D_{\mathbb{C}^n}(\omega_{L_0}, \mathcal{F})^+ = \text{Span} \left( \{ \eta g_i \in \mathcal{L}(\mathbb{C}^n, \mathcal{F}) : \eta g_i \text{ equals } \omega_{L_0} \text{ with } f_i \text{ changed by } g_i \} \right) / \mathbb{C}. \omega_{L_0}
\]
and \( \mathcal{F} \) is the s-uple defined as \((d_1, \ldots, d_{i-1}, d'_i, d_{i+1}, \ldots, d_s)\), where \( d'_i \) is the degree of the polynomial \( g_i \), which can be different from the original degree \( d_i \).

Again, by direct computation, it is straightforward to check that \( D_{\mathbb{C}^n}(\omega_{L_0}, \mathcal{F}) \) and \( D_{\mathbb{C}^n}(\omega_{L_0}, \lambda) \) are subspaces of \( D_{\mathbb{C}^n}(\omega_{L_0}, e) \) and that the space \( D_{\mathbb{C}^n}(\omega_{L_0}, \mathcal{F})^+ \) is a subspace of \( D_{\mathbb{C}^n}(\omega_{L_0}) \).

Remark 5.6. Notice that for \( s = 2 \) an affine logarithmic foliation is also an affine rational foliation.

6. Affine deformations of affine rational and logarithmic foliations

Along this section we are going to prove that an affine first order deformation of a homogeneous differential form \( \omega \) can be projectivized and it still defines a first order deformation of the projectivization of \( \omega \), given that they are homogeneous and have the same degree, see Lemma (6.1) below. This lemma is used for proving Theorem (6.3) and Theorem (6.4) which classify first order perturbations of affine rational and logarithmic foliations.

Let us consider a differential form \( \eta \in \Omega^1_{\mathbb{C}^n} \), homogeneous, of degree \( e \). If \( \eta = \sum h_i dx_i \), then its projectivization is given by
\[
\tilde{\eta} = z\eta - i_R(\eta)dz = z\eta - \left( \sum x_i h_i \right) dz.
\]
And we also have that
\[
d\tilde{\eta} = -2\eta \wedge dz - \left( \sum x_i dh_i \right) \wedge dz + zd\eta.
\]
We have the following equality, having \( d\eta = \sum dh_i \wedge dx_i \) we get that, since the degree of \( \eta \) is equal to \( e \), and following Euler’s formula \( i_R(dh_i) = \partial(h_i)h_i \),
\[
i_R(d\eta) = \sum (e-1)h_idx_i - \sum x_i dh_i = (e-1)\eta - \sum x_i dh_i.
\]

Lemma 6.1. Let us consider \( \omega \) and \( \eta \) a degree \( e \) homogeneous, 1-differential forms such that
\[
\omega \wedge d\omega = 0, \omega \wedge d\eta + d\omega \wedge \eta = 0.
\]
Now, consider the projectivization of these two differential forms in the sense of eq. (12), let us name them \( \tilde{\omega} \) and \( \tilde{\eta} \), respectively.

Then, we have that
\[
\tilde{\omega} \wedge d\tilde{\eta} + d\tilde{\omega} \wedge \tilde{\eta} = 0.
\]
Proof. From the above equations we have the following equalities, writing \( \omega \) as

\[
\omega = \sum_{i=1}^{n} f_i dx_i,
\]

then to compute \( \tilde{\omega} \wedge d\tilde{\eta} + d\tilde{\omega} \wedge \tilde{\eta} \) we proceed as follows:

\[
\tilde{\omega} \wedge d\tilde{\eta} = -z \left( \sum x_i f_i \right) dz \wedge d\eta + z \omega \wedge \left[ -2\eta \wedge dz - \left( \sum x_i dh_i \right) \wedge dz \right] + z^2 \omega \wedge d\eta =
\]

\[
= -z \left( \sum x_i f_i \right) dz \wedge d\eta - 2z \omega \wedge \eta \wedge dz - z \omega \wedge \left( \sum x_i dh_i \right) \wedge dz +
\]

\[
\quad + z^2 \omega \wedge d\eta =
\]

\[
\tilde{\eta} = z \eta - \left( \sum x_i h_i \right) dz,
\]

then we write

\[
d\tilde{\eta} = -2\eta \wedge dz - \left( \sum x_i dh_i \right) \wedge dz + zd\eta.
\]

So, we finally get:

\[
\tilde{\omega} \wedge d\tilde{\eta} + d\tilde{\omega} \wedge \tilde{\eta} = -z \left( \sum x_i f_i \right) d\eta \wedge dz - z \omega \wedge \left( \sum x_i dh_i \right) \wedge dz +
\]

\[
\quad - z \left( \sum x_i h_i \right) d\omega \wedge dz + z \left( \sum x_i df_i \right) \wedge \eta \wedge dz =
\]

\[
= z \left[ - \left( \sum x_i f_i \right) d\eta - \omega \wedge \left( \sum x_i dh_i \right) - \left( \sum x_i h_i \right) d\omega +
\]

\[
\quad + \left( \sum x_i df_i \right) \wedge \eta \right] \wedge dz
\]

This way, we would like to see the annihilation of the following equation

\[
(14) \quad - \left( \sum x_i f_i \right) d\eta - \omega \wedge \left( \sum x_i dh_i \right) - \left( \sum x_i h_i \right) d\omega + \left( \sum x_i df_i \right) \wedge \eta = 0
\]

This can be seen by contracting the following equation

\[
\omega \wedge d\eta + d\omega \wedge \eta = 0
\]
with the radial vector field \( R \). Then we get that, following eq. (13),
\[
(\sum x_if_i)\,d\eta - \omega \wedge [(e-1)\eta - (\sum x_idh_i)] + [(e-1)\omega - (\sum x_idf_i)] \wedge \eta + \\
+ \left(\sum x_ih_i\right)\,d\omega = \\
= (\sum x_if_i)\,d\eta + \omega \wedge (\sum x_idh_i) - (\sum x_idf_i) \wedge \eta + \\
+ \left(\sum x_ih_i\right)\,d\omega = 0
\]
showing that eq. (14) is zero, as expected. □

**Remark 6.2.** The converse of the above lemma is clear: if \( \tilde{\omega} \) and \( \tilde{\eta} \) are such that \( \tilde{\omega} \wedge d\tilde{\eta} + d\tilde{\omega} \wedge \tilde{\eta} = 0 \) then \( \omega \wedge d\eta + d\omega \wedge \eta = 0 \).

Now, let us see that the projectivization of an affine rational and logarithmic foliation given by \( \omega_{R_0} \in \mathcal{R}(\mathbb{C}^n, \overline{d}) \) and \( \omega_{L_0} \in \mathcal{L}(\mathbb{C}^n, \overline{d}) \), respectively, is logarithmic. For that, let us write as \( \omega_0 \) to both of our foliations, and since \( i_R(\omega_{R_0}) = i_R(\omega_{L_0}) = \mu F \), where \( \mu = d_1 \gamma - d_2 s \) for the rational case and \( \mu = \sum_{i=1}^s \lambda_i d_i \) for the logarithmic case, we just need to see that
\[
\tilde{\omega}_0 = z\omega_0 - \left(\sum_{i=1}^n x_i i_{\frac{\partial}{\partial z_i}}(\omega_0)\right)\,dz = z\omega_0 - i_R(\omega_0)dz = \\
= z\omega_0 - \mu F\,dz
\]
which has effectively the form of a logarithmic foliation defined in \( \mathbb{P}^m \), with parameters given by
\[
f_1, f_2 \text{ and } z \text{ and } d_1, d_2 \text{ and } (-\mu) \text{ for } \omega_0 \text{ rational} \\
f_1, \ldots, f_s \text{ and } z \text{ and } \lambda_1, \ldots, \lambda_s \text{ and } (-\mu) \text{ for } \omega_0 \text{ logarithmic.}
\]

Now, by [CGM18, Theorem 25, pp. 14, and Remark 26, pp. 15] we know that the first order (same degree projective homogeneous) deformations (of a projective homogeneous logarithmic foliation) are given by perturbing the polynomial parameters \( f_i \) and \( z \) and the eigenvalues \( \lambda_i \) (or the \( d_1, d_2 \)) and \( \mu \).

After dehomogenization we get that the only perturbations of the same degree of \( \omega_0 \), i.e. of degree \( e \), are those given by the perturbations of the polynomial parameters \( f_i \) and eigenvalues \( \lambda_i \) (or the \( d_1, d_2 \)), since the perturbation given in the direction of the infinite hyperplane \( z \), after dehomogenization, gives a differential form of degree \( e+1 \), and the perturbation of \( \mu \), after dehomogenization, gives the trivial deformation. Using the fact that the first order (degree one in the parameter) restricts the perturbation to either the polynomial parameters or the eigenvalues we obtain:

**Theorem 6.3.** Let \( \omega_{R_0} \in \mathcal{U}_{R_0} \) be a generic affine rational foliation in \( \mathcal{R}(\mathbb{C}^n, \overline{d}) \), defined by homogeneous polynomials \( f_1, f_2 \)
\[
\omega_{R_0} = r\,f_1df_2 - s\,f_2df_2.
\]
Then all first order perturbations of \( \omega_{R_0} \), of the same degree of \( \omega_{R_0} \), are the perturbations of the polynomial parameters \( f_1 \) and \( f_2 \) or of the eigenvalues \( (r, s) \), i.e., we have that
\[
D_{\mathbb{C}^n}(\omega_{R_0}, e) = D_{\mathbb{C}^n}(\omega_{R_0}, \overline{f}) \oplus D_{\mathbb{C}^n}(\omega_{R_0}, (r, s))
\]
Theorem 6.4. Let $\omega_{L_0} \in \mathcal{U}_L(d)$ be a generic affine logarithmic foliation in $L(\mathbb{C}^n, \overline{d})$, defined by homogeneous polynomials $f_1, \ldots, f_s$:

$$\omega_{L_0} = \left( \prod_{k=1}^{s} f_k \right) \left( \sum_{k=1}^{s} \lambda_k \frac{df_k}{f_k} \right) = \sum_{k=1}^{s} \lambda_k F_k df_k.$$

Then all first order perturbations of $\omega_{L_0}$, of the same degree of $\omega_{L_0}$, are the perturbations of the polynomial parameters $f_i$ or of the eigenvalues $\lambda_i$, i.e., we have that

$$D_{\mathbb{C}^n}(\omega_{L_0}, e) = D_{\mathbb{C}^n}(\omega_{L_0}, \vec{f}) \oplus D_{\mathbb{C}^n}(\omega_{L_0}, \vec{\lambda}).$$

7. Relative cohomology with poles: the equation $d \left( \frac{\eta}{f} \right) \wedge \omega_0 = 0$

The problem of relative cohomology for holomorphic differential forms has been studied by Cerveau and Berthier. We recall that given a one-form $\omega$ and a one-form $\eta$ both defined in the same domain, we say that $\eta$ is closed relatively to $\omega$ if $d\eta \wedge \omega = 0$. We also say that $\eta$ is exact relatively to $\omega$ if $\eta = dh + a \omega$ for some holomorphic functions $a$ and $h$ in the same domain of definition as $\omega$ and $\eta$. The basic question is whether a relatively closed one-form $\eta$ with respect to $\omega$ is also exact with respect to $\omega$. Assume that the form $\omega$ is integrable, i.e., $\omega \wedge d\omega = 0$. In this case $\omega = 0$ defines a holomorphic foliation of codimension one and with singular set given by $\text{sing}(\omega)$. In this case the condition $d\eta \wedge \omega = 0$ means that the restriction of $\eta$ to the leaves of $\omega$ is a closed one-form. This indicates that the topology of the leaves of $\omega$ may be an ingredient in the solution to be the above question. In the case of germs of one-forms, Cerveau and Berthier have proved (see [BC93, Théorème 4.1.1, pp. 422]) that for a generic logarithmic one-form $\omega_{L_0}$ with some additional diophantine conditions in the coefficients $\lambda_j$, the equation

$$d\eta \wedge \omega_{L_0} = 0$$

is equivalent to the fact that $\eta$ is of the form

$$\eta = a \omega_{L_0} + dh$$

for some $a, h \in \mathcal{O}_{\mathbb{C}^n, 0}$, i.e., $a, h$ germs of holomorphic functions in $n$-variables around $0 \in \mathbb{C}^n$.

Nevertheless, we shall address a different situation. Since we are interested in deformations of an affine rational or logarithmic foliation, we must study the following equation

$$d \left( \frac{\eta}{f} \right) \wedge \omega_0 = 0$$

where $\omega_0 \in \mathcal{R}(\mathbb{C}^n, \overline{d})$ or $\omega_0 \in \mathcal{L}(\mathbb{C}^n, \overline{d})$ is an affine rational or logarithmic foliation of type

$$\omega_0 = r f_1 df_2 - s f_2 df_1$$

in the rational case, or

$$\omega_0 = \left( \prod_{k=1}^{s} f_k \right) \left( \sum_{k=1}^{s} \lambda_k \frac{df_k}{f_k} \right) = \sum_{k=1}^{s} \lambda_k F_k df_k.$$
where \( F_k = \prod_{j \neq k} f_j \), \( \lambda_k \in \mathbb{C} \) and \( F = \prod_{k=1}^{s} f_k \), in the logarithmic case. In other words, we would like to know what happens when we divide \( \eta \) by the polynomial \( F \), the integrating factor of the affine rational or logarithmic differential form \( \omega_0 \).

In any case, we are going to work with the following equivalent equation to eq. (15) which is

\[
(F d\eta - dF \wedge \eta) \wedge \omega_0 = 0 ,
\]

and we will still denote with \( F \) the product \( f_1.f_2 \), of the polynomials involved in the definition of the affine rational foliation.

7.1. **First order perturbations (solutions of degree \( \partial(\omega_0) \)).**

Along this section we are going to see the equivalence between the equation that defines a first order deformation of a foliation defined by an affine rational or logarithmic form \( \omega_0 \), which says that it is given by the differential forms \( \eta \) such that, see eq. (2),

\[
\omega_0 \wedge d\eta + d\omega_0 \wedge \eta = 0 ,
\]

and between the equation defining the relative cohomology of \( \omega_0 \), with poles in the integrating factor defined by \( \omega_0 \), see eq. (18),

\[
d\left( \frac{\eta}{F} \right) \wedge \omega_0 = 0 ,
\]

see Corollary (7.3) below for a complete statement of the result.

We begin with the following proposition:

**Proposition 7.1.** If \( \omega_0 \in \mathcal{R}(\mathbb{C}^n, \overline{\mathbb{C}}) \) or \( \omega_0 \in \mathcal{L}(\mathbb{C}^n, \overline{\mathbb{C}}) \) then if \( \eta \in \Omega^1_{\mathbb{C}^n} \) then we have that

\[
\omega_0 \wedge d\eta + d\omega_0 \wedge \eta = 0 \Rightarrow (F d\eta - dF \wedge \eta) \wedge \omega_0 = 0
\]

**Proof.** For this we make use of the following well-known fact. For \( \omega_0 \) rational, as in Definition (5.1), or logarithmic, as in Definition (5.4), the following equation holds, since \( F \), defined as above, is an integrating factor of \( \omega_0 \) then

\[
(F d\omega_0 = dF \wedge \omega_0 .
\]

Then, by multiplying by \( F \) the eq. (19) we get

\[
F \omega_0 \wedge d\eta + F d\omega_0 \wedge \eta = 0 \\
F \omega_0 \wedge d\eta + dF \wedge \omega_0 \wedge \eta = 0 \\
F d\eta \wedge \omega_0 - dF \wedge \eta \wedge \omega_0 = 0 \\
(F d\eta - dF \wedge \eta) \wedge \omega_0 = 0
\]

concluding our first result.

**Proposition 7.2.** If \( \omega_0 \in \mathcal{R}(\mathbb{C}^n, \overline{\mathbb{C}}) \) or \( \omega_0 \in \mathcal{L}(\mathbb{C}^n, \overline{\mathbb{C}}) \) and if \( \eta \in \Omega^1_{\mathbb{C}^n} \) is homogeneous and has the same degree of \( \omega_0 \) then we have that

\[
\omega_0 \wedge d\eta + d\omega_0 \wedge \eta = 0 \Leftrightarrow (F d\eta - dF \wedge \eta) \wedge \omega_0 = 0
\]
Proof. To see this, let us proceed as follows.

First we apply the exterior differential to eq. (18), getting

\[ 2dF \wedge d\eta \wedge \omega_0 + (F \ d\eta - dF \wedge \eta) \wedge \omega_0 = 0, \]

now, we multiply the above equation by \( F \), and using eq. (20) above, we get

\[ 2F \ dF \wedge d\eta \wedge \omega_0 + F \ (F \ d\eta - dF \wedge \eta) \wedge d\omega_0 = 0 \]

\[ 2F \ dF \wedge d\eta \wedge \omega_0 + F \ d\eta \wedge dF \wedge \omega_0 = 0 \]

\[ 2F \ dF \wedge d\eta \wedge \omega_0 + F \ d\eta \wedge \omega_0 = 0 \]

\[ 3F \ dF \wedge d\eta \wedge \omega_0 = 0 \]

\[ dF \wedge d\eta \wedge \omega_0 = 0 \]

\[ d\eta \wedge dF \wedge \omega_0 = 0 \]

\[ F \ d\eta \wedge d\omega_0 = 0 \]

\[ d\eta \wedge d\omega_0 = 0 \]

Now, we use the contraction with the radial vector field \( R \) applied to the last equation, and using Cartan’s formula

\[ L_R(\omega_0) = e\omega_0 = d_iR(\omega_0) + iR(d\omega_0), \]

we get

\[ i_R(d\eta) \wedge d\omega_0 + d\eta \wedge i_R(d\omega_0) = 0 \]

\[ e\eta \wedge d\omega_0 - di_R(\eta) \wedge d\omega_0 + e\eta \wedge \omega_0 - d\eta \wedge di_R(\omega_0) = 0 \]

which can be written as

\[ \omega_0 \wedge d\eta + d\omega_0 \wedge \eta = \frac{1}{e} \left[ d_iR(\eta) \wedge d\omega_0 + d\eta \wedge di_R(\omega_0) \right] \]

where we are assuming that \( \partial(\omega_0) = \partial(\eta) = e \).

Now, let us see that the right side of this last equation is zero. So, we want to see that

\[ \omega_0 \wedge d\eta + d\omega_0 \wedge \eta = 0 \]

(21)

For that, we are going to apply the contraction with the radial vector field to eq. (18), and we will also write \( i_R(\omega_0) = \mu F \), for a \( \mu \in \mathbb{C} \). We get:

\[ [Fi_R(d\eta) - eF\eta + i_R(\eta)dF] \wedge \omega_0 + \mu F (F d\eta - dF \wedge \eta) = 0 \]

\[ F \left[ i_R(d\eta) \wedge \omega_0 - e\eta \wedge \omega_0 + i_R(\eta)d\omega_0 + \mu (F d\eta - dF \wedge \eta) \right] = 0 \]

\[ F [\omega_0 \wedge di_R(\eta) + i_R(\eta)d\omega_0 + \mu (F d\eta - dF \wedge \eta)] = 0 \]

This last equation, can be rewritten as

\[ \omega_0 \wedge di_R(\eta) + i_R(\eta)d\omega_0 = -\mu (F d\eta - dF \wedge \eta) . \]
Applying the exterior differential to this last equation we get
\begin{equation}
2 \ di_R(\eta) \wedge d\omega_0 = -2\mu \ dF \wedge d\eta
\end{equation}
\begin{equation}
di_R(\eta) \wedge d\omega_0 = -\mu \ dF \wedge d\eta
\end{equation}

If, we now take eq. (21) and we use the equality \( i_R(\omega_0) = \mu F \) then we have
\[ di_R(\eta) \wedge d\omega_0 + \mu \ d\eta \wedge dF , \]
and using eq. (22) we finally get that
\[ -\mu \ dF \wedge d\eta + \mu \ d\eta \wedge dF = 0 \]
as we wanted to see.

\[ \square \]

**Corollary 7.3.** If \( \omega_0 \in \mathcal{R}(\mathbb{C}^n, \mathcal{d}) \) or \( \omega_0 \in \mathcal{L}(\mathbb{C}^n, \mathcal{d}) \) then if \( \eta \in \Omega_{\mathbb{C}^n}^1 \) is homogeneous and has the same degree of \( \omega_0 \), let us assume that \( \partial(\omega_0) = e \), then
\[ \eta \in D(\omega_0, e) \iff d\left( \frac{\eta}{F} \right) \wedge \omega_0 = 0 \]

**Remark 7.4.** With this result, we have that if \( \eta \) is a perturbation of \( \omega_0 \) given by changing one of the parameters \( f_i \) or one of the \( \lambda_i \) (or one of \( d_1, d_2 \)), then it verifies eq. (18).

Following [CGM18] and the computations of section 6, see Theorem 6.3 and Theorem 6.4, we have that this are all possible solutions for an affine rational or logarithmic foliation, if the \( \omega_0 \) is generic and we consider only solutions of the same degree of \( \omega_0 \).

### 7.2. Solutions of degree \( \neq \partial(\omega_0) \).

In this section we show some examples of solutions of the equation of relative cohomology, see eq. (18), by using Proposition (7.1), when considering degrees of \( \eta \) such that \( \partial(\eta) \neq \partial(\omega_0) \).

By following Remark 6.3, considering \( D_{\mathbb{C}^n}(\omega_0, \mathcal{d})^+ \) where \( \omega_0 \in \mathcal{R}(\mathbb{C}^n, \mathcal{d}) \) or \( \omega_0 \in \mathcal{L}(\mathbb{C}^n, \mathcal{d}) \), we have that as a corollary of Proposition 7.1 above, we can also get solutions of the eq. (18) of different degrees to the one given by \( \omega_0 \). In contrast to the case when the degree is the same as of the original \( \omega_0 \), as we show in the preceeding section, we do not know whether these are all the possible solutions.

In particular we can give, as an example for the case when \( \omega_0 \in \mathcal{R}(\mathbb{C}^n, \mathcal{d}) \) or \( \omega_0 \in \mathcal{L}(\mathbb{C}^n, \mathcal{d}) \), the extreme case where the polynomial \( f_i \) is changed by the constant polynomial equal to 1, and then we get that the differential form, in the logarithmic case,
\[ \eta = \sum_{j \in J} \lambda_j \overline{F}_j df_j \]
is a solution of eq. (18), for \( J \subset [1, \ldots, s] \), such that \( \#(J) = s - 1 \) and \( \overline{F}_j = \prod_{i \neq j} f_i \).
In the rational case, the situation is much simpler, since we should consider \( \eta \) such that
\[
\eta = df_1 \quad \text{or} \quad \eta = df_2 .
\]

8. Deformations of dicritical homogeneous one-forms

Let \( \omega \) be a homogeneous one-form in \( \mathbb{C}^n, n \geq 3 \) satisfying the integrability condition \( \omega \wedge d\omega = 0 \). According to [CMS82, Part 4, Chap. I pp. 86-95] we have that either \( F = i_R(\omega) \equiv 0 \) or \( F \) is an integrating factor for \( \omega \). In the non-dicritical case, \( \text{i.e.} \), for \( F \not\equiv 0 \), we can write
\[
\omega = \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} + (1 + \varepsilon) \left( \sum_{i=1}^{s} \frac{g}{\prod_{i=1}^{s} f_i^{n_i-1}} \right)
\]
for some \( \lambda_i \in \mathbb{C}, n_i \geq 1 \) and some homogeneous polynomials \( f_i, g \). Clearly the \( f_i \) are factors of \( F \) so that we must have \( F = \prod_{i=1}^{s} f_i^{n_i} \). Put \( f = f_1 \ldots f_s \). We may then rewrite
\[
\omega = \sum_{i=1}^{s} \lambda_i \frac{F}{f_i} df_i + f dg - g \sum_{i=1}^{s} (n_i - 1) \frac{f_i}{f_i} df_i
\]

We may deform \( \omega \) as follows:
\[
\omega_\varepsilon = F \left[ \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} + (1 + \varepsilon)d \left( \frac{g}{\prod_{i=1}^{s} f_i^{n_i-1}} \right) \right]
\]
Notice that each \( \omega_\varepsilon \) admits \( F \) as integrating factor and therefore it is integrable. We have \( \omega_\varepsilon = \omega + \varepsilon \eta \) where \( \eta = F d \left( \frac{g}{\prod_{i=1}^{s} f_i^{n_i-1}} \right) \). It is interesting to observe that \( F \) is also an integrating factor for \( \eta \), \( \text{i.e.} \), \( \frac{F}{f_i} \eta \) is closed. The deformations of non-dicritical homogeneous integrable one-forms (by same degree homogeneous integrable one-forms) are described in [CMS82, Part 4, Chap. I pp. 86-95]. Now we turn our attention to the dicritical case, \( \text{i.e.} \), when \( i_R(\omega) \equiv 0 \). We have:

**Proposition 8.1.** Let \( \omega \in \mathcal{F}^1(\mathbb{P}^n, e) \) be a dicritical homogeneous one-form, \( \text{i.e.} \), such that \( i_R(\omega) = 0 \). Let \( \eta \) be a solution of
\[
(Fd\eta - dF \wedge \eta) \wedge \omega = 0
\]

homogeneous of the same degree of \( \omega \). Then either \( i_R(\eta) = 0 \) or it is an integrating factor of \( \omega \) and \( \eta \).

**Proof.** Assume that \( i_R(\eta) \not\equiv 0 \). The one-forms \( \omega_\varepsilon = \omega + \varepsilon \eta \) are integrable and homogeneous. Moreover, \( i_R(\omega_\varepsilon) = i_R(\omega) + \varepsilon i_R(\eta) = \varepsilon i_R(\eta) \not\equiv 0, \forall \varepsilon \not\equiv 0 \). According to [CMS82] as
mentioned above, the one-form \( \frac{1}{i_{R}(\omega_{\varepsilon})}\omega_{\varepsilon} \) is closed for all \( \varepsilon \neq 0 \). Let \( F = i_{R}(\eta) \). We have

\[
\frac{1}{i_{R}(\omega_{\varepsilon})}\omega_{\varepsilon} = \frac{1}{\varepsilon F}(\omega + \varepsilon \eta) = (\varepsilon)^{-1}\frac{1}{F}\omega + \frac{1}{F}\eta.
\]

This implies that \( \frac{1}{F}\omega \) and \( \frac{1}{F}\eta \) are closed.

\[\Box\]

**Corollary 8.2.** Let \( \omega \in F^{1}(\mathbb{P}^{n}, e) \) be a dicritical homogeneous one-form, i.e., such that \( i_{R}(\omega) = 0 \). Given a first order deformation \( \omega_{\varepsilon} = \omega + \varepsilon \eta \) of \( \omega \) by integrable homogeneous one-forms we have the following possibilities:

1. \( \omega_{\varepsilon} \) descends to the projective space \( \mathbb{P}^{n} \).
2. \( \omega_{\varepsilon} \) is of the form

\[
\omega_{\varepsilon} = \left(\prod_{i=1}^{s} f_{i}^{n_{i}}\right) \left[ \sum_{i=1}^{s} (\lambda_{i} + \varepsilon \mu_{i}) \frac{df_{i}}{f_{i}} + d \left( \frac{g + \varepsilon h}{\prod_{i=1}^{s} f_{i}^{n_{i}-1}} \right) \right]
\]

where \( f_{i}, g, h \) are homogeneous polynomials, \( \lambda_{i}, \mu_{i} \in \mathbb{C} \).

**Proof.** Let \( F = i_{R}(\eta) \). If \( F = 0 \) then \( i_{R}(\omega_{\varepsilon}) = 0 \) and therefore the deformation descends to the projective space \( \mathbb{P}^{n} \). Assume now that \( F \neq 0 \). From Proposition 8.1 we know that \( \frac{1}{F}\omega \) and \( \frac{1}{F}\eta \) are closed. Put \( F = \prod_{i=1}^{s} f_{i}^{n_{i}} \) in irreducible homogeneous distinct factors. Then we can apply the Integration lemma from \[\text{CMS82}\] in order to write

\[
\omega = F \left[ \sum_{i=1}^{s} \lambda_{i} \frac{df_{i}}{f_{i}} + d \left( \frac{g}{\prod_{i=1}^{s} f_{i}^{n_{i}-1}} \right) \right]
\]

and

\[
\eta = F \left[ \sum_{i=1}^{s} \mu_{i} \frac{df_{i}}{f_{i}} + d \left( \frac{h}{\prod_{i=1}^{s} f_{i}^{n_{i}-1}} \right) \right]
\]

Then the result follows.

\[\Box\]

**Remark 8.3.** In case (1) the deformation can be viewed in the projective space \( \mathbb{P}^{n} \) and then we can apply the above discussion and corollary once again.
9. Stability of an exact differential form $\omega = dP$

Along this section we would like to study the stability under perturbations of an exact differential form of type $\omega = dP$, where $P \in S_n$ is a polynomial of degree $e$. As in the former sections we are considering first order deformations. These are given by one-forms $\omega_\varepsilon = \omega + \varepsilon \eta$, where $\eta$ is homogeneous of degree $e$ and each $\omega_\varepsilon$ is integrable $\omega_\varepsilon \wedge d\omega_\varepsilon = 0$.

In [CS] the authors prove a more general result than the one in Theorem (9.1). Anyway, we are writing this result here because of its similarity with the previous method of demonstration. Moreover, we highlight its strictly algebraic character, unlike the techniques used in [CS].

Let us consider $\omega \in \Omega^1_{\mathbb{C}^n}$ of the form

$$\omega = dP$$

for a homogeneous polynomial $P \in S_n$ of degree $e$. As before, we are going to consider only deformations of the same degree of $\omega$.

We would like to prove the following statement:

**Theorem 9.1.** Let $\omega = dP$ be an exact differential form in $\Omega^1_{\mathbb{C}^n}$, homogeneous of degree $e$. Let us suppose also that the codimension of the singular locus of $dP$ is $\geq 3$. Then all first order deformations of $dP$, of the same degree, are of type

$$\omega_\varepsilon = d(P + \varepsilon Q)$$

where $Q$ is a homogeneous polynomial of degree $e$.

**Proof.** Recalling Lemma (6.1), we consider the projectivization of such a differential form $\omega$. This way we get, by using Euler’s formula,

$$(23) \quad \tilde{\omega} = z \omega - i_R(\omega)dz = zdP - \left( \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} dP \right) dz = zdP - eP dz$$

which is a rational foliation of type $(1, e)$.

And, by the hypothesis on $P$ we have that $\tilde{\omega} \in U_R$, then using Theorem (4.2) we know that the first order deformations of such a foliation are the deformations of its polynomial parameters. Then we have that the space of first order deformations of $\tilde{\omega}$ are given by

$$\tilde{\eta}_1 = zdQ - eQ dz \quad \text{and} \quad \tilde{\eta}_2 = ldP - eP dl$$

where $Q$ is a homogeneous polynomial of degree $e$, and $l$ is an homogeneous polynomial of degree 1.

Now, after de-homogenisation, we get in the first case

$$\eta_1 = dQ .$$

In the second case, after dehomogenization we get a differential form degree $e + 1$, which we are not allowed to consider. Then we conclude that the deformation $\omega_\varepsilon = \omega + \varepsilon \eta$ is exact of the form $\omega = dP_\varepsilon$ where $P_\varepsilon$ is homogeneous and such that $P_0 = P$. The fact that the deformation is of order one implies that $P_\varepsilon = P + \varepsilon Q$ as stated.
Remark 9.2. We would like to clarify the meaning of the term *generic* on the polynomial $P$. The codimension of the singular locus of $\overline{\omega}$ being $\geq 2$ means nothing since $P$ and $z$ are always transversal, but the condition of codimension of the singular locus of $dP \wedge dz$ being $\geq 3$, means, in particular that $P$ has to be reduced, irreducible and smooth in codimension 2.

REFERENCES

[BC93] Michel Berthier and Dominique Cerveau, *Quelques calculs de cohomologie relative*, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 3, 403–424. MR 1222279

[CA94] Omegar Calvo-Andrade, *Irreducible components of the space of holomorphic foliations*, Math. Ann. 299 (1994), no. 4, 751–767. MR 1286897 (95i:32039)

[CGM18] Fernando Cukierman, Javier Gargiulo Acea, and César Massri, *Stability of logarithmic differential one-forms*, to appear in Transactions of the American Mathematical Society. Available at [http://arxiv.org/abs/1706.06534](http://arxiv.org/abs/1706.06534) 2018.

[CM82] Dominique Cerveau and Jean-François Mattei, *Formes intégrables holomorphes singulières*, Astérisque, vol. 97, Société Mathématique de France, Paris, 1982. With an English summary. MR 704017

[CPV09] Fernando Cukierman, Jorge Vitorio Pereira, and Israel Vainsencher, *Stability of foliations induced by rational maps*, Ann. Fac. Sci. Toulouse Math. (6) 18 (2009), no. 4, 685–715. MR 2590385 (2011d:32049)

[CS] Dominique Cerveau and Bruno Scárdua, *Integrable deformations of foliations: cycles and persistence of first integrals*, to appear.

[CS18] , *Integrable deformations of local analytic fibrations with singularities*, Ark. Mat. 56 (2018), no. 1, 33–44. MR 3800457

[CSV06] Fernando Cukierman, Marcio G. Soares, and Israel Vainsencher, *Singularities of logarithmic foliations*, Compos. Math. 142 (2006), no. 1, 131–142. MR 2197406

[GMLN91] Xavier Gómez-Mont and Alcides Lins Neto, *Structural stability of singular holomorphic foliations having a meromorphic first integral*, Topology 30 (1991), no. 3, 315–334. MR 1113681

[Mal76] B. Malgrange, *Frobenius avec singularités. I. Codimension un*, Inst. Hautes Études Sci. Publ. Math. (1976), no. 46, 163–173. MR 0508169

Ariel Molinuevo  
Instituto de Matemática  
Universidade Federal do Rio de Janeiro  
Caixa Postal 68530  
CEP. 21945-970 Rio de Janeiro - RJ  
BRASIL

Bruno Scárdua  
Instituto de Matemática  
Universidade Federal do Rio de Janeiro  
Caixa Postal 68530  
CEP. 21945-970 Rio de Janeiro - RJ  
BRASIL