Quantum information transfer
from one system to another one

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Abstract

The topics of the paper are: a) Some anti-linear maps governing EPR tasks if no reference bases are distinguished. b) Imperfect quantum teleportation and the composition rule. The ancilla is supposed pure but otherwise arbitrary. c) Quantum teleportation with distributed measurements. d) Remarks on EPR with a mixed state, triggered by a Lüders measurement.

1 Introduction

The problem of transferring “quantum information” from one quantum system into another one has its roots in the 1935 paper \cite{EPR} of A. Einstein, B. Podolski, and N. Rosen. These authors posed a far reaching question, but they doubt the answer given by quantum theory. The latter, as was pointed out by them, asserts the possibility to create simultaneously and at different places exactly the same random events. The phenomena is often called “EPR effect” or, simply, “EPR”.

Early contributions to the EPR problem are due to Schrödinger, \cite{Schr}. Since then a wealth of papers had appeared on the subject, see \cite{Classical} and \cite{Rev} for a résumé. Even to-day some authors consider it more a “paradox” than a physical “effect”, because EPR touches the question, whether and how space and time can live with the very axioms of quantum physics, axioms which, possibly, are prior to space and time\footnote{Sometimes it seems helpful to think space-time a user interface of the quantum world.}.

Quantum information theory considers EPR as a map or as a “channel”, as an element of protocols transferring “quantum information” from one system to another one or supporting the transmission of classical information, \cite{QIT}, \cite{QI}, \cite{QIP}. One of our aims is to present a certain calculus for EPR and EPR-like processes or, more general, for processes triggered by measurements. We begin, therefore, with some selected fundamentals of quantum measurements.

The following treatment of EPR has its origin in the identification problem in comparing two or more quantum systems. It is by far not obvious how to identify two density operators, say $\rho^a$ and $\rho^b$, belonging to two different Hilbert spaces, $\mathcal{H}_a$ and $\mathcal{H}_b$. Often one fixes two bases, $\{\phi_i^a\}$ and $\{\phi_j^b\}$, and defines $\rho^a$ and $\rho^b$ to be “equal” one to another if they have the same matrix representation with respect to the reference bases. The more,
one needs a stable synchronization if several tasks have to be done in the course of time. It seems, therefore, worthwhile to postpone the selection of the reference bases as long as possible. If that can be done, it can be done using the s-maps, [12], of the EPR section. These maps are anti-linear. The anti-linearity in the EPR problem is usually masked by the reference bases: The bases provide conjugations which create, combined with the “natural” anti-linearity, the suggestion of an unrestricted linearity. An interesting, though quite different approach, [21], is founded on Ohy’a’s idea of compound states [20]. Also in [22] there is a side remark on anti-linearity.

For pure states in quantum systems with finitely many degrees of freedom, there is a duality between pure states and maximal properties in the sense of von Neumann and Birkhoff. In the section “inverse” EPR we show by an example the meaning of the mentioned duality.

We proceed with the beautiful quantum teleportation protocol of Bennett at al [18]. Here we prove a composition rule for imperfect (i.e. not faithful) quantum teleportation. Then we show its use in quantum teleportation with distributed measurements by an example with a 5-partite system, and an EPR example based on a 4-partite Hilbert space.

There is a short section on polar decompositions of the s-maps, including a quite elementary link to operator representations. From a physical point of view, *-representation theory provides a classification of the ways a quantum system can be embedded in a larger one. However, this topic is outside the realm of the present paper.

Finally we show how to handle, again by some anti-linear maps, an EPR task in a bi-partite system if its state is mixed and if a measurement is performed in one of its subsystems by a projection operator of any rank.

Remarks on notation: In this paper the Hermitian adjoint of a map or of an operator \( A \) is denoted by \( A^* \). The scalar product in Hilbert spaces is assumed linear in its second argument.

2 Preliminaries

The implementation independence in quantum information theory is guarantied by the use of Hilbert spaces, states (density operators), and operations between and on them. It is not said, what they physically describe in more concrete terms, whether we are dealing with spins, polarizations, energy levels, particle numbers, or whatever you can imagine. Because of this, the elements of quantum information theory, to which the EPR-effect belong, are of rather abstract nature.

Let a physical system be is described by an Hilbert space \( \mathcal{H} \). A quantum state of the system is then given by a density operator \( \omega \), a positive operator with finite trace, the latter normalized to be one. Thus every positive trace-class operator different from the zero operator uniquely defines a state. One only has to divide it by its trace. (If the Hilbert space is not of finite dimension, there are also so-called singular states. For our purposes we can safely ignore them.)

Every vector \( \psi \in \mathcal{H}, \psi \neq 0 \), defines a vector state, the density operator of which is the projection operator, say \( P_\psi \), onto the 1-dimensional subspace generated by \( \psi \). It is
common use to speak of “the state $\psi$” if the state can be described by $P\psi$. The vector states of our system are called pure if the properties of linear independent vectors do not coincide.

The quantum version of Boolean Logics is due to Birckhoff and J. von Neumann, [7]. According to them a property, a quantum state can have, is a subspace of $\mathcal{H}$ or, equivalently, a projection operator onto that subspace. Not every subspace may be considered a property. The point is, that there are no other properties a quantum state can have. This well established postulate excludes some hidden parameter dreams.

*Here we shall assume that every subspace defines a property, and that two different subspaces encode different properties.* It is another way to express the purity assumption for vector states.

Looking at these two concepts, states and properties, there is a certain “degeneracy”. A vector can denote a state or a property. A (properly) minimal projection operator represents either a maximal property or the density operator of a pure state. What applies depends on the context. The existence of maximal properties is a special feature of physical systems with a finite number of degrees of freedom.

Let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ denoting a property. A state, given by a density operator $\omega$, possesses property $\mathcal{H}_0$, if and only if its support is in $\mathcal{H}_0$. That is, $\omega$ must annihilate the orthogonal complement of $\mathcal{H}_0$. If $\omega = P\psi$ is a vector state, this is equivalent to $\psi \in \mathcal{H}_0$.

Let $P_0$ denote the ortho-projection onto $\mathcal{H}_0$. A test, whether $\omega$ has property $P_0$ results in one bit of information: Either the answer is YES or it is NO.

i) The probability of outputting the answer YES is $p := \text{Tr} P_0 \omega$.

ii) If $p$ is not zero, and if the answer is YES, then the test has prepared the new state $P_0\omega P_0$. Multiplying by $p^{-1}$ gives its density operator.

An *executable measurement within* $\mathcal{H}$ is characterized by a finite orthogonal decomposition of $\mathcal{H}$ into subspaces. The subspaces are assumed to be properties. Denoting by $P_j$ the orthogonal projections onto the subspaces, the requirement reads

$$\sum_{j=1}^{m} P_j = 1, \quad P_i P_k = 0 \text{ if } i \neq k. \quad (1)$$

*Remark:* The phrase “executable” asserts the possible existence of an apparatus doing the measurement. A general observable can be approximated (weakly) by such devises. Important physical quantities like energy, momentum, and position in Schrödinger theory represent examples of observables, which can be approximated by executable ones without being executable themselves.

*Remark:* In saying that the measurement is “in” $\mathcal{H}$ we exclude measurements in an upper-system containing the system in question as a sub-system. Such a larger system allows for properties not present in the smaller one. In allowing such measurements we arrive at the so-called POVMs, “positive operator valued measurements”.

To be a measurement, the device testing the properties $P_j$ should output a definite signal $a_j$ if it decides to prepare the state $P_j \omega P_j$. Well, $a_1, \ldots, a_m$ constitute the letters of an alphabet. The device randomly decides what letter to choose. The probability of a decision in favor of the letter $a_j$ is $\text{Tr} P_j \omega$ with $\omega$ the density operator of the system’s
state. Thus, the classical information per probing the properties (1) is

$$H(p_1, \ldots, p_n) = - \sum p_j \log_2 p_j, \quad p_j = \operatorname{Tr} P_j \omega.$$  

A little more physics come into the game in assuming that the alphabet consists of \( m \) different complex numbers. Then the operator

$$A := \sum_{j=1}^{n} a_j P_j$$  

is an observable for the measurement of the properties (1). Clearly, the executable observables are normal operators, \( AA^* = A^*A \), and their spectra are finite sets.

One observes that information theory is not interested in the nature of the alphabet that distinguishes the outcomes of a measurement. It suffices for its purposes to discriminate the outcomes and to know the state that is prepared. Portability is gained that way.

It is standard that two properties can be checked simultaneously if and only if their ortho-projections commute. Otherwise one gets in trouble with the probability interpretation. Two observables, \( A \) and \( B \), can be measured (or approximated by such procedures) simultaneously provided they commute. Executing a set \( A_1, \ldots, A_n \) of mutually commuting observables will be called a distributed measurement.

Non-relativistically a distributed measurement may consist of several measuring devices, sitting on different (possibly overlapping) places in space, but being triggered at the same time.

Relativistically, every measurement is done in a certain space-time or “world” region. A particular case of a distributed measurement consists of devices doing their jobs in disjunct, mutually space-like world regions: Quantum theory does not enforce restrictions for measurements (or “interventions” a la A. Peres) for space-like separated world regions. EPR and quantum teleportation make use of it in an ingenious way.

Thinking in terms of the evolution of states in the course of time, these tasks update the initial conditions of the evolution. The choice of the new Cauchy data is done randomly and governed by transition probabilities.

In Minkowski space the problem is somehow delicate. According to Hellwig and Kraus [8] it is consistent to let take place the state change at the boundary of the past of the region. The past of the world region is the union of all backward light-cones terminating in one of the world points of the region the measurement is done. Finkelstein [9] has argued that it is also possible to allow the change at the light-like future of the world region in question. We, [11], think it even consistent to assume a slightly stronger rule: The state changes accompanied by a measurement in a space-time region takes place at the set of those points, which are neither in the past nor in the future of that region. The assumed region of influence is bounded to the past a la Hellwig and Kraus and to the future according to Finkelstein. The remarkable experiments of Zbinden et al [10] agree with it.

Finally, I mention some specialties in testing the properties of vector states: If \( \psi \) is a vector state, and \( A \) an observable (2), any vector, prepared by testing \( A \), is of the form \( P_i \psi \). It follows that the relative phase between \( \psi \) and a non-zero \( P_i \psi \) is real and positive. Hence, the Hilbert space distance between them equals their Fubini-Study distance: The state changes by measurements proceeds along Study-Fubini geodesic arcs.
Similar considerations with general ("mixed") states are more involved. These states allow for quite different "purifications", i.e. lifts to vector states living in larger quantum systems: One only gets inequalities for the distance. However, the case of the minimal possible distance is a distinguished one.

3 EPR

Let us consider a bi-partite quantum system composed of two Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ and one of its vectors

$$\mathcal{H} := \mathcal{H}_a \otimes \mathcal{H}_b, \quad \psi \in \mathcal{H}.$$ (3)

In such a bi-partite system $\mathcal{H}_a$ characterizes a subsystem, the a-system, which is embedded in the system of the Hilbert space $\mathcal{H}$. The same is with the b-system.

We assume the state of the composed system is the vector state defined by $\psi$. We are interested in what is happening if a property is checked in the a-system. A local subspace of $\mathcal{H}$ is a direct product of two subspaces, one of $\mathcal{H}_a$, the other one of $\mathcal{H}_b$. A local property of $\mathcal{H}$ is, therefore, a projection operator of the form $P_a \otimes P_b$. $P_a$ and $P_b$ are projectors from the subsystems. Similarly one proceeds in multi-partite systems.

If $P_a$ is a property of $\mathcal{H}_a$, the local property in the composed system that checks nothing in the b-system reads $P_a \otimes 1_b$. If so, and if the test of $P_a$ outputs YES, the newly prepared state is again a vector state. The state change is

$$\psi \mapsto (P_a \otimes 1_b)\psi.$$ (4)

Is something to be seen in the b-system by such a change? Posing and answering the question is an essentially part of the EPR problem. In pointing out the intrinsic anti-linearity in the EPR problem we follow [12] and [13].

Let us consider maximal properties of the a-system,

$$P_a = \frac{|\phi^a\rangle\langle\phi^a|}{\langle\phi^a, \phi^a\rangle}, \quad \phi^a \in \mathcal{H}_a.$$ (5)

Then the state prepared in (4) must be a product vector, the first factor being a multiple of $\phi^a$. Therefore, given $\psi$, there must be a map from $\mathcal{H}_a$ into $\mathcal{H}_b$ associating to any given $\phi^a$ its partner in the product state. Let us denote this map by

$$\mathcal{H}_a \ni \phi^a \mapsto \mathbf{s}_{\psi}^{ba} \phi^a \in \mathcal{H}_b.$$ (6)

It is defined by

$$\langle \phi^a \rangle (\phi^a \otimes 1_b) \psi = \phi^a \otimes \mathbf{s}_{\psi}^{ba} \phi^a, \quad \forall \phi^a \in \mathcal{H}_a.$$ (6)

We see: If in testing the property $\phi^a$ the answer is YES, the same is true with certainty if in the b-system one is asking for the property $\mathbf{s}_{\psi}^{ba} \phi^a$.

It becomes clear by inspection of (6) that $\mathbf{s}_{\psi}^{ba}$ is an anti-linear map from $\mathcal{H}_a$ into $\mathcal{H}_b$ which depends linearly on $\psi$. We also may ask the same question starting from the b-system, resulting in an anti-linear map $\mathbf{s}_{\psi}^{ab}$ from $\mathcal{H}_b$ into $\mathcal{H}_a$,

$$(1_a \otimes |\phi^b\rangle\langle\phi^b|) \psi = \mathbf{s}_{\psi}^{ab} \phi^b \otimes \phi^b, \quad \forall \phi^b \in \mathcal{H}_b.$$ (7)
Let us go back to (6) and let us choose a vector $\phi^b$ in $\mathcal{H}_b$. Taking the scalar product (6) with $\phi^a \otimes \phi^b$, one easily finds
\[
\langle \phi^a \otimes \phi^b, \psi \rangle = \langle \phi^b, s_{\psi}^{ba} \phi^a \rangle.
\]
By symmetry, or by using (7) appropriately, one finally arrives at the identity
\[
\langle \phi^a \otimes \phi^b, \psi \rangle = \langle \phi^b, s_{\psi}^{ba} \phi^a \rangle = \langle \phi^a, s_{\psi}^{ab} \phi^b \rangle
\]
which is valid for all $\phi^a \in \mathcal{H}_a$ and $\phi^b \in \mathcal{H}_b$. Obviously, taking into account their anti-linearity, the two maps between the Hilbert spaces of the subsystems are Hermitian adjoints one from another.

\[
(s_{\psi}^{ab})^* = s_{\psi}^{ba}, \quad (s_{\psi}^{ba})^* = s_{\psi}^{ab}
\]
Finally, by the linearity of the $s$-maps with respect to $\psi \in \mathcal{H}$, one arrives at the following recipe for their construction:
\[
\psi = \sum a_{jk} \phi^a_j \otimes \phi^b_k \Rightarrow s_{\psi}^{ba} \phi^a = \sum a_{jk} \langle \phi^a_j, \phi^b_k \rangle \phi^b_k
\]
(9)
The exchange $a \leftrightarrow b$ of the letters $a$ and $b$ in (9) produces the adjoint $s$-map.

The $s$-maps obey some simple rules if local operations are applied to them. The most obvious is
\[
\phi = (A \otimes B) \psi \leftrightarrow s_{\psi}^{ab} = A s_{\psi}^{ab} B^*.
\]
(10)

Let us now escape from the formalism to a short discussion. We assume, as starting point, the bi-partite system in a pure state $\psi \in \mathcal{H}$. We can assume that $\psi$ and an arbitrarily chosen $\phi^a$ are unit vectors. $P_a$ denotes the projection operator of the 1-dimensional subspace generated in $\mathcal{H}_a$ by $\phi^a$.

What can be seen from $\psi$ in the subsystems? This is encoded in the reduced states, in the density operators $\varrho^a_\psi$ and $\varrho^b_\psi$, respectively. In more general terms: The state of a subsystem is given by the expectation values of the operators accessible within the subsystem. All what an owner, say Bob, can learn within his subsystem $\mathcal{H}_b$ without resources from outside, he has to learn from $\varrho^b_\psi$. Any belief, he could learn anything else from its quantum system alone, is nothing than a reanimation of the hidden parameter story.

The reduced density operators can be calculated by partial traces. In the case at hand a definition for the b-system is
\[
\langle \psi, (1_a \otimes B) \psi \rangle = \text{Tr} \varrho^b_\psi B, \quad \forall B \in \mathcal{B}(\mathcal{H}_b).
\]
The reduced density operators can also be expressed by the s-maps,
\[
\varrho^a_\psi = s_{\psi}^{ab} s_{\psi}^{ba}, \quad \varrho^b_\psi = s_{\psi}^{ba} s_{\psi}^{ab}
\]
(11)
The probability, $p$, for a successful test of $\phi^a$ is $\langle \phi^a, \varrho^a_\psi \phi^a \rangle$. The maximal possible probability appears if $\phi^a$ is an eigenvector to the largest eigenvalue of $\varrho^a_\psi$.

The square roots of the eigenvalues $p_i > 0$ of $\varrho^a_\psi$ are the Schmidt-coefficients of the Schmidt decomposition of $\psi$ and, according to (11), also the singular values of $s_{\psi}^{ab}$. 
Let \( \{\phi_j^a\} \) be the vectors of a basis and \( P_j^a \) the ortho-projection onto the space generated by \( \phi_j^a \). Let us now ask what is going on if we test the properties \( P_j^a \). We can use any operator

\[
A = \sum a_j P_j^a
\]

with mutually different numbers \( a_j \). The probability \( p_j^a \) of preparing \( \phi_j^a \) is \( \langle \phi_j^a, \varrho^a \phi_j^a \rangle \). It is well known, that the probability vector \( \{ p_j^a \} \) is majorized by the set of eigenvalues \( \{ p_j \} \) of \( \varrho^a \). Any probability vector, which is majorized by the vector of its eigenvalues, can be gained this way by the use of a suitable basis of \( \mathcal{H}_a \). Consequently, in measuring \( A \), one can produce a message with an entropy not less than the entropy of the eigenvalue distribution of \( \varrho^a \). If and only if the chosen basis is an eigen-basis of \( \varrho^a \), we get the minimally possible entropy.

Enhancing the entropy of Alice’s side is not useful for Bob. Though his system will definitely be in the state \( \phi_j^b = s^{ba} \phi_j^a \) if on Alice’s side the state \( \phi_j^a \) is prepared, he cannot always make too much use of it. While Alice is preparing states which must be mutually orthogonal, and hence distinguishable, the vector states on Bob’s side do not share this necessity. Indeed, Bob’s state are mutually orthogonal if and only if Alice had minimized the entropy, i.e., if she had chosen an eigen-basis of her density operator.

Let us repeat it from another perspective. Let Alice perform some measurements using the observable \( A \). Assume that just before any measurement, the state of the bi-partite system is the vector state \( \psi \). Then, whenever the device answers “\( a_j \)”, the state of the \( a \)-system changes to \( \phi_j^a \). The state of the \( b \)-system becomes \( \phi_j^b = s^{ba} \phi_j^a \). Bob knows this state iff he knows \( \psi \) and which of the values \( a_j \) the measuring device has given to Alice. Now, if Alice uses an eigen-basis of the density operator \( \varrho^a \), then Bob himself is able to measure which state he get and, therefore, which \( a_j \) Alice has obtained. On the contrary, if Alice does not use a basis of eigenvectors, Bob’s possible states are not orthogonal and he cannot distinguish exactly between them. Therefore, the gain in entropy in the \( a \)-system by using a measurement basis distinct from the eigenvector basis is compensated by a loss of Bob’s possibility to distinguish between the states he gets.

One can prove the assertion by calculating

\[
\langle \phi_j^b, \phi_k^b \rangle = \langle s^{ba} \phi_j^a, s^{ba} \phi_k^a \rangle = \langle \phi_j^a, s^{ba} s^{ba} \phi_k^a \rangle
\]

or, by (11),

\[
\langle \phi_j^b, \phi_k^b \rangle = \langle \phi_j^a, \varrho^a \phi_k^a \rangle. \tag{12}
\]

If all von Neumann measurements of Alice are on equal footing, and Bob can always discover the state prepared by Alice within his system to any precision, the EPR settings is “perfect” or “tight”. In the tight case the reduced density operator \( \varrho^b \) of Alice is equal to \( (\dim \mathcal{H}_a)^{-1}1_a \), i.e. to the unique tracial state of her system. This state is like “white quantum paper”, there is no quantum information at all in it. The “more white” Alice’s “quantum paper” \( \varrho^a \) is, the better EPR is working. That somewhat fabulous language can be made precise substituting “more mixed” or “less pure” for “more white”.

A further remark should be added to our short and incomplete account of the EPR mechanism. It is a well known theorem that \( \mathcal{H}_a \otimes \mathcal{H}_b \) is canonically isomorphic to the space \( \mathcal{L}^2(\mathcal{H}_a, \mathcal{H}_b^a) \) of Hilbert-Schmidt mappings form \( \mathcal{H}_a \) into \( \mathcal{H}_b^a \). On the other hand, \( \mathcal{H}_b^a \) is canonically anti-linearly isomorphic to \( \mathcal{H}_b \), a fact used by P. A. Dirac to establish his bra-ket correspondence \( \langle | \leftrightarrow \langle | \) Composing both maps one immediately see the
isomorphism between $\mathcal{H}_a \otimes \mathcal{H}_b$ and the space of anti-linear Hilbert-Schmidt maps $L^2(\mathcal{H}_a, \mathcal{H}_b)_{\text{anti}}$. The isomorphism is an isometry expressed by

$$\langle \varphi, \psi \rangle = \text{Tr} s^b_{\varphi} s^a_{\rho} = \text{Tr} s^a_{\rho} s^b_{\varphi}$$

with $\psi$ and $\varphi$ from $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$.

4 “Inverse” EPR

In the preceding section we have considered three vectors: $\psi$ from the composite Hilbert space $\mathcal{H}$ and $\phi^a$, $\phi^b$ from its constituents. In the EPR setting $\psi$ is a given pure state which is to test whether it enjoys the local properties defined either by $\phi^a$, by $\phi^b$, or by both. In the “dual” or “inverse” EPR setting their roles are just reversed: $\psi$ appears as a non-local property which is to check. $\phi^a \otimes \phi^b$ is the state to be tested for the property $\psi$. Because transition probabilities are symmetric in their arguments, one can enrol the EPR setting backwards. The trick has been clearly seen and used by C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters in their famous quantum teleportation paper [18], see also [23].

To demonstrate what is going on, let us consider a simple but instructive example. Here $\mathcal{H}$ is of dimension four, and its two factors 2-dimensional. Dirac’s bra-ket notation is used, but in scalar products anti-linear maps should be applied to kets only! In the example we choose the vectors

$$\psi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad \phi^a = |x\rangle, \quad \phi^b = |0\rangle$$

with $x = 0, 1$. Alice is trying to send a bit-encoded message to Bob by choosing $|x\rangle$ accordingly one after the other. Bob’s input is always $|0\rangle$. By doing so, they enforce the bi-partite system into the state

$$|x0\rangle = |x\rangle \otimes |0\rangle, \quad x = 0, 1$$

Then it is checked whether it has the property $\psi$. If $x = 1$, the measuring apparatus will necessarily answer the question with NO because the state is orthogonal to $\psi$. If, however, $x = 0$, the answer is YES with probability 0.5 and NO with the same probability. The input state $|00\rangle$ of the bi-partite system now has changed as follows:

$$\text{YES} \mapsto \psi, \quad \text{NO} \mapsto \psi' = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Let now $q$ be the probability of an input $x = 0$. Then the input ensemble is transformed by the measurement in the following way:

$$\{|00\rangle, |10\rangle; q, 1 - q\} \to \{\psi, \psi'; |10\rangle; \frac{q}{2}, \frac{q}{2}, 1 - q\}$$

The classical information encoded in the input state is not lost. It could be regained by measuring the property

$$|\psi\rangle\langle\psi| + |\psi'\rangle\langle\psi'|,$$
a task which does not change the states involved. Now, next, Bob and Alice perform local measurements by testing the properties

\[ P_b := |0\rangle\langle 0|_b, \quad P_a := |0\rangle\langle 0|_a. \]

If \( \psi \) or \( \psi' \) is the state of \( \mathcal{H} \), the states of the local parts will be \((1/2)1_a\) and \((1/2)1_b\) respectively. The answer is either YES or NO with equal property \( 1/2 \) as seen from

\[
(\langle 0 | a \otimes 1_b) \psi = \frac{1}{\sqrt{2}} (00) = (1_a \otimes |0\rangle \langle 0|_b) \psi
\]

and from the similar relation with \( \psi' \). There is a strong correlation: Either both devices return YES or both say NO. Therefore, if the input of Alice is \(|0\rangle\), the output is either YES for Alice as well as for Bob, or it is NO for both. If, however, \(|1\rangle\) is the input of Alice, then \(|10\rangle\) becomes the state of \( \mathcal{H} \). It follows that Alice gets necessarily NO and Bob YES.

We see that Bob and Alice would have the full information of the message, Alice had encoded in her system, if both parties could communicate their measurement results – even if Alice has forgotten her original message. No information is lost, but it is non-locally distributed after testing the property \( \psi \).

A particular interesting case is the transmission of information from Alice to Bob, who knows neither the result of testing the property \( \psi \) nor has he obtained any information from Alice. He knows, which property has been checked, but does not know the result.

Though there is no classical information transfer, Bob gets some information from Alice by testing in his system property \( P_b \). Considering all intermediate state changes as done by a quantum black box, the process is stepwise described by

\[
\{|0\rangle\langle 0|_a, |1\rangle\langle 1|_a\} \mapsto \{\frac{1}{2}1_b, |0\rangle\langle 0|_b\}
\]

and can be represented as an application of the stochastic cp-map

\[
\left( \begin{array}{cc} \omega_{00} & \omega_{01} \\ \omega_{10} & \omega_{11} \end{array} \right)_a \mapsto \frac{1}{2} \left( \begin{array}{cc} \omega_{00} + 2\omega_{11} & 0 \\ 0 & \omega_{00} \end{array} \right)_b.
\]

A message encoded by Alice with probabilities \( q \) or \( 1 - q \) per letters 0 or 1 carries an information \( H(q, 1 - q) \). The Holevo bound for the quantum message Bob obtains by measuring \( P_b \) can be calculated to be

\[
H(1 - \frac{1}{2}q, \frac{1}{2}q) - q.
\]

Its maximum is reached at \( q = (2/5) \). In that case Bob receives approximately 0.322 bit per letter, while Alice has encoded her message with 0.962 bit per letter.

What we have just discussed is a slight variation of protocols invented independently by Aharonov and Albert, [14], and by R. D. Sorkin [15]. The latter claimed it to be an example of a measurement “forbidden by Einstein causality”. More recently Beckman et al [16], adding an interesting collection of similar measurements, have extended and sharpened Sorkin’s assertion. On the other hand, Vaidman [17] presented teleportation protocols of non-local measurements. We, B. Crell and me, [11], think the causality considerations of Sorkin and Beckmann et al not conclusive: While a measurement
allows for instantaneous changes of states, the output of an apparatus includes classical information processing which has to go on in the world region the device is working. To detect the output of the signal can only be possible in the intersection of all future cones originating in world points of the measuring region. Bob can detect Alice’s message not before his world lines have crossed all the future light cones originating from the world points at which the measuring process is going on. Hence, though the state change has taken place, Bob can be informed only after a time delay of the order “radius of the measuring device / velocity of light”. Before that time has elapsed, the state change is hidden to Bob – as required by causality.

More accurate [11], the rule with which quantum theory outlines the defect of being not causal, is as follows. Let $A$ and $B$ be two non-commuting observables which we like to measure sequentially, say $A$ before $B$. Let $G_A$ and $G_B$ denote the world region at which the measurements should take place. Then $G_B$ must be in the complete future of $G_A$, that is $G_B$ must be in the intersection of all forward cones originating in the world points of $G_A$.

The return to the general case of inverse EPR with $\psi$ an arbitrary vector of a bi-partite system with Hilbert space $\mathcal{H}$ is formally straightforward: Checking the property $\psi$ if the system is in a product state $\phi^a_1 \otimes \phi^b_1$ one comes across

$$|\psi\rangle\langle\psi| \phi^a_1 \otimes \phi^b_1 = \langle\psi, \phi^a_1 \otimes \phi^b_1 \rangle \psi.$$ 

If Alice and Bob can communicate, and they can check with which probability their states enjoy the property $\phi^a_2 \otimes \phi^b_2$. The transition amplitude for an affirmative answer can be expressed, according to (8), by

$$\langle\psi, \phi^a_1 \otimes \phi^b_1 \rangle \langle \phi^a_2 \otimes \phi^b_2, \psi \rangle = \langle \phi^a_1, s_{ab}^a \phi^b_1 \rangle^* \langle \phi^a_2, s_{ab}^a \phi^b_2 \rangle.$$ 

5 Imperfect quantum teleportation

Quantum teleportation has been invented by Bennett et al [18]. “Perfect” or faithful quantum teleportation starts within a product of three Hilbert spaces of equal finite dimension and with a maximal entangled vector in the last two. It is triggered by a von Neumann measurement in the first two spaces using a basis of maximally entangled vectors. The measurement randomly chooses one of several quantum channels. The information, which quantum channel has been activated, is carried by the classical channel. It serves to reconstruct, by a unitary move, the desired state at the destination.

All those possible “perfect” or “tight” schemes, together with their dense coding counterparts, have been reviewed by R. F. Werner [19].

Following [18] and analyzing their computations, one can decompose the chosen quantum channel into two parts, an inverse EPR and an EPR setting. As one can identify two particular s-maps with them, one is tempted to use two general s-maps. In doing so one can treat a more general setup. But even in “perfect” circumstances the explicit use of the mentioned decomposition may be of some interest.

Let $\mathcal{H}$ be a tri-partite Hilbert space

$$\mathcal{H}_{abc} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c.$$ 

(14)
There is no restriction on the dimensions of the factor spaces. The input is a vector \( \phi^a \in \mathcal{H}_a \), possibly unknown, and a known vector \( \varphi^{bc} \), the “ancilla”, out of \( \mathcal{H}_b \otimes \mathcal{H}_c \). The teleportation protocol is to start with the initial vector

\[
\varphi^{abc} := \phi^a \otimes \varphi^{bc} \in \mathcal{H}_{abc}.
\]

Now one performs a measurement on \( \mathcal{H}_a \otimes \mathcal{H}_b \). Instead of a complete von Neumann measurement we ask just whether a property, given by a vector \( \psi^{ab} \), is present or not. In doing “nothing” on the c-system, one is checking a local property of the abc-system. If the check runs affirmative, the vector state \( \psi^{ab} \) is prepared in \( \mathcal{H}_{ab} \), inducing a state change in the larger abc-system:

\[
(|\psi^{ab}\rangle\langle\psi^{ab}| \otimes \mathbf{1}_c)(\phi^a \otimes \varphi^{bc}) = \psi^{ab} \otimes \phi^c,
\]

with a vector \( \phi^c \in \mathcal{H}_c \) yet to be determined. Indeed,

\[
\phi^a \mapsto \phi^c
\]

represents the teleportation channel which is triggered by an affirmative check of the property defined by \( \psi^{ab} \). Letting \( \phi^a \) as a free variable, we introduce the teleportation map \( t^{ca} \) by

\[
t^{ca}\phi^a \equiv t^{ca}_{\psi,\varphi}\phi^a = \phi^c, \quad \psi \equiv \psi^{ab}, \quad \varphi \equiv \varphi^{bc}.
\]

The teleportation map \( t^{ca} \) is governed by the composition rule, \[12\],

\[
t^{ca}_{\psi,\varphi} = s^b_{\psi} s^{ca}_{\varphi}.
\]

The s-maps being Hilbert-Schmidt, the t-maps must be of trace class and linear. Indeed, every trace class map from \( \mathcal{H}_a \) into \( \mathcal{H}_c \) can be gained as a t-map, provided its rank does not exceed the dimension of \( \mathcal{H}_b \). Of course, this fact can be obtained also directly, without relying on the decomposition rule, \[25, 26, 24, 27, 28, 29\], where also cases with a mixed ancilla have been studied.

Proof of (18). Let us abbreviate the left hand side of (16) by \( \psi^{abc} \). Choosing in \( \mathcal{H}_b \) an ortho-normal basis \( \{ \phi^b_j \} \) gives the opportunity to write

\[
\varphi^{bc} = \sum \phi^b_j \otimes s^c_{\varphi} \phi^b_j
\]

and hence

\[
\psi^{abc} = \psi^{ab} \otimes \sum_j \langle \psi^{ab}, \phi^a \otimes \phi^b_j \rangle s^c_{\psi} \phi^b_j.
\]

We choose in \( \mathcal{H}_a \) an ortho-normal basis, \( \{ \phi^a_k \} \), to resolve the scalar product in the last equation:

\[
\psi^{abc} = \psi^{ab} \otimes \sum_{jk} \langle \phi^a_k, \psi^{ab} \rangle \langle \phi^b_j, s^c_{\psi} \phi^b_j \rangle s^b_{\psi} \phi^b_j.
\]

Using anti-linearity,

\[
\psi^{abc} = \psi^{ab} \otimes s^c_{\psi} \sum_k \langle \phi^a_k, \psi^{ab} \rangle \langle \phi^b_j, s^b_{\psi} \phi^a_k \rangle s^b_{\psi} \phi^b_j.
\]

The summation over \( j \) results in \( s^b_{\psi} \phi^a_k \). Next, again by anti-linearity, the sum over \( k \) comes down to

\[
s^b_{\psi} \sum_k \langle \phi^a_k, \psi^{ab} \rangle \phi^a_k = s^b_{\psi} \psi^{ab}.
\]
and we get finally
\[ \psi^{abc} = \psi^{ab} \otimes s^b \varphi^c \phi^a \]
and the composition rule is proved.

**Distributed measurements**

The next aim is to present an extension of the composition rule to multi-partite systems. In a multi-partite system one can distribute the measurements and the entanglement resources over some pairs of subsystems. With an odd number of subsystems we get *distributed teleportation*, with an even number something like *distributed EPR*.

At first let us see, as an example, distributed teleportation with five subsystems.

\[ \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_d \otimes \mathcal{H}_e. \]  

(19)

The input is an unknown vector \( \phi^a \in \mathcal{H}_a \), the ancillarian vectors are selected from the \( bc \)- and the \( de \)-system,

\[ \varphi^{bc} \in \mathcal{H}_{bc} = \mathcal{H}_b \otimes \mathcal{H}_c, \quad \varphi^{de} \in \mathcal{H}_{de} = \mathcal{H}_d \otimes \mathcal{H}_e, \]  

(20)

and the vector of the total system we are starting with is

\[ \varphi^{abcde} = \phi^a \otimes \varphi^{bc} \otimes \varphi^{de}. \]  

(21)

The channel is triggered by measurements in the \( ab \)- and in the \( cd \)-system. Suppose these measurements are successful and they prepare the vector states

\[ \psi^{ab} \in \mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b, \quad \psi^{cd} \in \mathcal{H}_{cd} = \mathcal{H}_c \otimes \mathcal{H}_d. \]  

(22)

Then we get the relation

\[ (|\psi^{ab}\rangle \langle \psi^{ab}| \otimes |\psi^{cd}\rangle \langle \psi^{cd}| \otimes 1^e) \varphi^{abcde} = \psi^{ab} \otimes \psi^{cd} \otimes \phi^e \]  

(23)

and the vector \( \phi^a \) is mapped onto \( \phi^e \). Introducing the s-maps corresponding to the vectors

\[ \psi^{ab} \to s^{ba}, \quad \varphi^{bc} \to s^{cb}, \quad \psi^{cd} \to s^{dc}, \ldots, \]

the *factorization rule* becomes

\[ \phi^e = t^{ea} \phi^a, \quad t^{ea} = s^{ed} s^{dc} s^{cb} s^{ba}. \]  

(24)

Next we consider a setting with *four* Hilbert spaces, \( \mathcal{H}_b \) to \( \mathcal{H}_e \). The input state is

\[ \varphi^{bcde} = \varphi^{bc} \otimes \varphi^{de} \]

and we perform a test to check whether the property \( \psi^{cd} \) is present or not. Let the answer be YES. Then the subsystems \( bc \) and \( de \) become disentangled. The \( cd \) system gets \( \psi^{cd} \) and, hence, the entanglement of this vector state. The previously unentangled systems \( \mathcal{H}_b \) and \( \mathcal{H}_e \) will now be entangled.

The newly prepared state is

\[ \chi^{bcde} := (1_b \otimes |\psi^{cd}\rangle \langle \psi^{cd}| \otimes 1_e) \varphi^{bcde}. \]  

(25)
With
\[ \psi^{cd} = \sum \lambda_j \phi^c_j \otimes \phi^d_j \]
we obtain
\[ \chi^{bcde} = \sum \lambda_j \lambda_k \left[ (|\phi^c_j\rangle \langle \phi^c_j|) \otimes \left( (|\phi^d_j\rangle \langle \phi^d_j| \otimes |1_e\rangle \phi^{de} \right) \right]. \]
Let us denote just by \( s^{bc} \) and \( s^{de} \) the s-maps of \( \varphi^{bc} \) and \( \varphi^{de} \) respectively. They allow to rewrite \( \chi^{bcde} \) as
\[ \chi^{bcde} = \sum \lambda_j \lambda_k \left( (s^{bc})_{\phi^c_k} \otimes \phi^c_j \right) \otimes \left( \phi^d_j \otimes (s^{ed})_{\phi^d_k} \right) \]
which is equal to
\[ \chi^{bcde} = \sum \lambda_k \left( (s^{bc})_{\phi^c_k} \right) \otimes \psi^{cd} \otimes \left( (s^{ed})_{\phi^d_k} \right). \]
(26)
The Hilbert space \( H^c \otimes H^d \) is decoupled from \( H^b \) and \( H^e \). The vector state of the latter can be characterized by a map from \( H^c \otimes H^d \) into \( H^b \otimes H^e \).
\[ \varphi^{be} := (s^{bc} \otimes s^{ed}) \psi^{cd} \]
(27)
is indicating how the entanglement within the be-system is arising, and how the three vectors involved come together to achieve it.

**Addendum:** A rearrangement lemma.
The starting point is a collection of bi-partite spaces and vectors,
\[ \psi_j \in H^{ij}_{ab}, \quad H^{ij}_{ab} = H^i_a \otimes H^j_b, \quad j = 1, \ldots, m \]
(28)
from which we build
\[ H_{ab} = H^1_{ab} \otimes \ldots \otimes H^m_{ab}, \quad \psi = \psi_1 \otimes \ldots \otimes \psi_m. \]
(29)
We abbreviate the s-maps accordingly,
\[ \psi_j \leftrightarrow s_{ab}^{ij} \leftrightarrow s_{ba}^{ij} \]
(30)
We now change to the rearranged Hilbert space
\[ H_{AB} = H_A \otimes H_B = (H^1_a \otimes \ldots \otimes H^m_a) \otimes (H^1_b \otimes \ldots \otimes H^m_b) \]
(31)
The Hilbert spaces (28) and (31) are unitarily equivalent in a canonical way:
\[ V : H_{ab} \mapsto H_{AB} \]
(32)
is defined to be the linear map satisfying
\[ V (\phi^a_1 \otimes \phi^b_1 \otimes \ldots \otimes \phi^a_m \otimes \phi^b_m) = (\phi^a_1 \otimes \ldots \otimes \phi^a_m) \otimes (\phi^b_1 \otimes \ldots \otimes \phi^b_m) \]
(33)
This is a unitary map, \( V^{-1} = V^* \).
Assume we need the s-maps of
\[ \varphi := V \psi \]
(34)
with \( \psi \) given by (29). The rearrangement lemma we have in mind reads
\[ s_{\varphi}^{AB} = V (s_1^{ab} \otimes \ldots \otimes s_m^{ab}) V^{-1}. \]
(35)
The proof uses the fact that both sides are multi-linear in the vectors \( \psi_j \). Therefore, it suffices to establish the assertion in the case, the \( \psi_j \) are product vectors. But then the proof consists of some lengthy but easy to handle identities.
6 Polar decompositions

Let us come back to the s-maps. It is worthwhile to study their polar decompositions. As we already know (11) it is evident that we should have

\[ s^ba_\psi = (\varrho^b_\psi)^{1/2} j^ba_\psi = j^ba_\psi (\varrho^a_\psi)^{1/2}, \]

\[ s^{ab}_\psi = (\varrho^a_\psi)^{1/2} j^{ab}_\psi = j^{ab}_\psi (\varrho^b_\psi)^{1/2}. \]

The j-maps are anti-linear partial isometries with left (right) supports equal to the support of their left (right) positive factor. From Alice’s point of view, who can know her reduced density operator but not the state from which it is reduced, \( j^{ab}_\psi \) is a non-commutative phase. It is in discussion whether and how relative phases of this kind can be detected experimentally.

One outcome of the polar decomposition is a unique labelling of purifications. If \( \varrho^a \) denotes a density operator on \( \mathcal{H}_a \), then all its purifications can be gained by the chain

\[ \varrho^a \mapsto j^{ba}(\varrho^a)^{1/2} = s^{ba}_\psi \mapsto \psi \]

where \( j^{ba} \) runs through all those anti-linear isometries from a to b whose right supports are equal to the support of \( \varrho^a \).

The uniqueness of the polar decomposition and (11) yields

\[ (j^{ba}_\psi)^* = j^{ab}_\psi, \quad \varrho^b_\psi = j^{ba}_\psi \varrho^a_\psi j^{ab}_\psi. \]  

(37)

Now we can relate the expectation values of the reduced density operators: Assume the bounded operators \( A \) and \( B \) on \( \mathcal{H}_a \) and \( \mathcal{H}_b \) are such that

\[ B^* j^{ba}_\psi = j^{ba}_\psi A. \]  

(38)

Then one gets, as a little exercise in anti-linearity,

\[ \text{Tr} \varrho^a_\psi A = \text{Tr} \varrho^b_\psi B. \]  

(39)

It is possible to express the condition (38) for the validity of (39) by an anti-linear operator \( J_\psi \) acting on \( \mathcal{H}_a \otimes \mathcal{H}_b \). To this end we define \( J_\psi \) as the anti-linear extension of

\[ J_\psi(\varrho^a \otimes \varrho^b) = j^{ab}_\psi \varrho^b \otimes j^{ba}_\psi \varrho^a. \]  

(40)

With this definition it is to be seen that (38) is as strong as

\[ J_\psi (A \otimes B) = (A \otimes B)^* J_\psi. \]  

(41)

(40) is a crossed tensor product, \( \hat{\otimes} \). With every pair of maps, one from \( \mathcal{H}_a \) to \( \mathcal{H}_b \) and one in the opposite direction, and both either linear or anti-linear, one can build the crossed tensor product \( \hat{\otimes} \). An important example is (40), where the two factors are j-maps. We may formally write

\[ J_\psi = j^{ab}_\psi \hat{\otimes} j^{ba}_\psi \]

for the just defined anti-linear operator acting on \( \mathcal{H}_{ab} \).
Now let the factors of $\mathcal{H}_{ab}$ be of equal dimension and $\psi$ “completely entangled”. In a more mathematical language $\psi$ is called a cyclic and separating vector, a so-called GNS-vector\(^2\) or a “GNS vacuum”, for the representation

$$A \mapsto A \otimes 1_b$$

of the algebra $\mathcal{B}(\mathcal{H}_a)$. In this context, $J_\psi$ is an elementary example of Tomita-Takeski’s modular conjugation. That $\psi$ is completely entangled can be expressed also in terms of $s$-maps: $s^{ab}_\psi$ must be invertible. (Its inverse, if it exists, must be unbounded for infinite dimensional Hilbert spaces.)

There are two further operators, particularly tied to the modular conjugation. The first is introduced by

$$(A \otimes 1_b) \psi = S_\psi (A^* \otimes 1_b) \psi. \quad (42)$$

$S_\psi$ can also be gained by the help of the twisted cross product

$$S_\psi = (s^{ba}_\psi)^{-1} \hat{\otimes} s^{ba}_\psi. \quad (43)$$

It is standard to write the polar decomposition of the anti-linear $S$-operator

$$S_\psi = J_\psi \sqrt{\Delta_\psi}. \quad (44)$$

$\Delta_\psi$ is called the Tomita-Takesaki modular operator. The distinguished role of these and similar “modular objects” becomes apparent in the theory of general von Neumann algebras where they play an exposed and quite natural role. From them I borrowed the notations for the $s$- and the $j$-maps. In the elementary case we are dealing with, one has

$$\Delta_\psi = \varrho^a_a \otimes (\varrho^b_b)^{-1}. \quad (45)$$

See \[30\] for a physically motivated introduction. Further relations between the $s$- and $j$-maps and to modular objects can be found in \[12\] and \[13\].

7 From vectors to states

With $\varrho \equiv \varrho^{ab}$ we may write similar to \[6\],

$$\langle \phi^a \rangle (\phi^a \otimes 1_b) \varrho^{\phi^a} (|\phi^a\rangle \langle \phi^a| \otimes 1_b) = |\phi^a\rangle \langle \phi^a| \otimes \Phi^{\phi^a}_{\varrho}(|\phi^a\rangle \langle \phi^a|), \quad \forall \phi^a \in \mathcal{H}_a \quad (46)$$

For every decomposition

$$\varrho^{ab} = \sum c_{jk} |\psi_j\rangle \langle \psi_k|, \quad \mathcal{H}_{ab} \ni \psi_j \leftrightarrow s^{ab}_j \quad (46)$$

there is a representation

$$\Phi^{ba}_{\varrho}(|\phi^a\rangle \langle \phi^a|) = \sum c_{jk} s^{ba}_j |\phi^a\rangle \langle \phi^a| s^{ab}_k. \quad (47)$$

Similarly one defines $\Phi^{ab}_{\varrho}$. The maps are linear in $\varrho^{ab}$ and can be defined for every trace class operator $\varrho$. Moreover, their domain of definition can be extended to the bounded

\(^2\)GNS stands for I. M. Gelfand, M. A. Naimark, I. E. Segal
operators of the subsystems: Let $X$ and $Y$ denote bounded operators on $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively, then

$$X \mapsto \Phi^{ba}_\psi(X), \quad Y \mapsto \Phi^{ab}_\psi(Y)$$

are well defined and anti-linear in $X$ or $Y$. The equation

$$\text{Tr} \ X \tilde{\Phi}^{ab}_\psi(Y^*) = \text{Tr} \ Y \tilde{\Phi}^{ba}_\psi(X^*) = \text{Tr} \ \varrho (X \otimes Y)$$

is valid. Proving them at first for finite linear combinations of rank one operators, one finds the maps (48) mapping the bounded operators of one subsystem into the trace class operators of the other one. Indeed, the finite version of (49) provides us with estimates like

$$\| \Phi^{ba}_\psi(X^*) \|_1 \leq \| X \|_{\infty} \| \varrho \|_1$$

(50).

We now have a one-to-one correspondence

$$\tilde{\Phi}^{ab}_\psi \leftrightarrow \tilde{\Phi}^{ba}_\psi \leftrightarrow \varrho$$

(51).

That we have a map from the bounded operators of $\mathcal{H}_a$ into the trace class operators of $\mathcal{H}_b$ is physically quite nice. It is an opportunity to reflect on testing a property $P_a$ of $\mathcal{H}_a$ once more, but under the condition that $\varrho \equiv \varrho^{ab}$ is in any (normal) state. The rank of $P_a$ is not necessarily finite. The rule of Lüders, [31], says that the prepared state is

$$\omega^a := P_a \varrho P_a$$

if one finds the property $P_a$ valid and $\varrho^a$ is the reduced density matrix of $\varrho$ in the a-system before the test. The EPR channel asks for $\omega^b$, the density operator of the b-system after an affirmative checking of the property $P_a$. This density operator is given by a $\Phi$-map:

If $\varrho$ is the density operator of $\mathcal{H}_{ab}$ and if a local measurement establishes property $P_a$, then the state $\omega^b$ of the b-system is given by

$$\omega^b = \Phi^{ba}_\psi(P_a).$$

(52)

The proof is by looking at the effect in the bi-partite system resulting from a local measurement. Let $\phi_j^a$ be a basis of the support space of $P_a$. One obtains

$$(P_a \otimes 1_b) |\psi\rangle \langle \psi| (P_a \otimes 1_b) = \sum |\phi_j^a\rangle \langle \phi_k^a| \otimes s^{ba}_\psi|\phi_j^a\rangle \langle \phi_k^a| s^{ab}_\psi$$

and this is, up to normalization, the state prepared by the local measurement. Next we sandwich the equation between $1_a \otimes B$ and take the trace. At the left hand we get $\text{Tr} \ \omega^b B$. On the right we obtain $\Phi^{ba}_\psi(P_a)$. Now we have seen from (49) that (52) is correct for pure states. By linearity and (50) we get the assertion.

It may be worthwhile to compare (49) with the now well known “duality” between super-operators $T$ of $\mathcal{H}_a$ and operators on $\mathcal{H}_a \otimes \mathcal{H}_b$. Here the Hilbert spaces are of equal finite dimension. One selects a maximally entangled vector $\psi$ and defines

$$\rho := (T \otimes \text{id}_b)(|\psi\rangle \langle \psi|)$$

(53)

to express the structure of $T$ by that of $\rho$. This trick is due to A. Jamiolkowski, [32], and is now refined and much in use after the papers of B. Terhal [34] and of Horodecki et al [33]. Comparing (47) and (48), one can connect both approaches as follows:

From $\rho$ we get a map $\Phi^{ab}_\psi$. From a maximally entangled $\psi$ we get an anti-linear map $s^{ab}_\psi$, enabling the correspondence (53) to be expressed by

$$\varrho \leftrightarrow T, \quad T(X) = s^{ab}_\psi \Phi^{ba}_\psi(X) s^{ba}_\psi$$

(54)

In a certain way, anti-linearity is the prize for eliminating the reference state $\psi$ in Jamiolkowski’s approach.
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