ON THE DIRICHLET FORM OF THREE-DIMENSIONAL BROWNIAN MOTION CONDITIONED TO HIT THE ORIGIN

PATRICK J. FITZSIMMONS AND LIPING LI

Abstract. Our concern in this paper is the energy form induced by an eigenfunction of a self-adjoint extension of the restriction of the Laplace operator to $C_\infty^\infty(R^3 \setminus \{0\})$. We will prove that this energy form is a regular Dirichlet form with core $C_\infty^\infty(R^3)$. The associated diffusion $X$ behaves like a 3-dimensional Brownian motion with a mild radial drift when far from 0, subject to an ever-stronger push toward 0 near that point. In particular $\{0\}$ is not a polar set with respect to $X$. The diffusion $X$ is rotation invariant, and admits a skew-product representation before hitting $\{0\}$: its radial part is a diffusion on $(0, \infty)$ and its angular part is a time-changed Brownian motion on the sphere $S^2$. The radial part of $X$ is a "reflected" extension of the radial part of $X^0$ (the part process of $X$ before hitting $\{0\}$). Moreover, $X$ is the unique reflecting extension of $X^0$, but $X$ is not a semi-martingale.

Contents

1. Introduction
2. The Dirichlet forms induced by eigenfunctions
3. Behavior near 0
4. Fukushima’s decomposition
References

1. Introduction

Our inspiration for this study is the work of Cranston, Koralov, Molchanov, and Vainberg ([13] and [14]) on a continuous model for homopolymers based on a "zero-range potential" perturbation of the 3-dimensional Laplacian. We start by describing probabilistically a process constructed in [14]. This process is symmetric (or reversible) with respect to a suitable measure on $R^3$, and our goal is to investigate it from the point of view of Dirichlet forms. Let $(Z_t)_{t \geq 0}$ be a standard Brownian motion in $R^3$, with law $Q^x$ when started at $x$. Fix $\gamma > 0$, and define $L_t^\epsilon := \epsilon \cdot \int_0^t 1_{\{|Z_s| \leq \epsilon\}} \, ds$, where $\epsilon := \frac{\pi^2}{8\cdot\epsilon^2} + \frac{\pi^2}{\epsilon^2}$. Now “tilt” the measure $Q_T^\epsilon := Q^x|_{F_T}$, where $F_T := \sigma\{Z_s : 0 \leq s \leq T\}$, as follows:

$$P_T^{\epsilon,\epsilon}(B) := \frac{\int_B \exp(L_T^\epsilon) \, dQ_T^\epsilon}{\int \exp(L_T^\epsilon) \, dQ_T^\epsilon}, \quad B \in F_T.$$ 

Thus paths that spend a lot of time near 0 are being heavily weighted under $P_T^{\epsilon,\epsilon}$. It is shown in [14] that as $\epsilon \downarrow 0$ and then $T \to \infty$, the probability measure $P_T^{\epsilon,\epsilon}$ converges weakly to the law $P^\epsilon$ of a certain “Brownian motion with singular drift”. It is this perturbed Brownian motion, which we label $X = (X_t)_{t \geq 0}$, that is the object of our interest. Roughly speaking, with the function $\psi_\gamma$ as defined below in (1.3), $X$ is the diffusion on $R^3$ with infinitesimal generator

2010 Mathematics Subject Classification. 31C25, 60J55, 60J60.

Key words and phrases. Dirichlet form, Reflected extension, Rotationally invariant process, Fukushima’s decomposition.
given, for smooth functions vanishing near 0, by
\[ Af = \frac{1}{2} \Delta f + \frac{\nabla \psi_\gamma \cdot \nabla f}{\psi_\gamma}. \]
(The operator \( L_\gamma u := \psi_\gamma A(\psi_\gamma^{-1} u) - (\gamma^2/2) u \) is a self-adjoint extension (on \( L^2(\mathbb{R}^3) \)) of \( \frac{1}{2} \Delta \) \( C_c^\infty(\mathbb{R}^3 \setminus \{0\}) \), and is viewed as the result of perturbing the Laplacian \( \frac{1}{2} \Delta \) by a zero-range potential.)

Because the drift \( \nabla \log \psi_\gamma(x) = -x(|x|^{-2} + \gamma |x|^{-1}), x \neq 0 \), blows up as \( x \to 0 \), the process \( X \) feels a strong push toward the origin. This push is such that the origin is a regular recurrent point for \( X \) (but all but one singleton subsets of \( \mathbb{R}^3 \) are polar for \( X \)). Away from the origin, \( X \) behaves like 3-dimensional Brownian motion with a moderate push toward the origin, but its behavior in time intervals when it visits the origin is so singular that \( X \) is not a semimartingale.

We now describe our main results in a little more detail. We work on the Hilbert space \( L^2(\mathbb{R}^3) \) and start with the unbounded operator
\[ L := \frac{1}{2} \Delta - \frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \]  
with domain
\[ D(L) = C_c^\infty(\mathbb{R}^3_0). \]
Here and in the sequel, \( \mathbb{R}^3_0 := \mathbb{R}^3 \setminus \{0\} \), and \( C_c^\infty(\mathbb{R}^3_0) \) denotes the class of all smooth functions with compact support in \( \mathbb{R}^3_0 \). Note that \( L \) is symmetric but not self-adjoint on \( L^2(\mathbb{R}^3) \). It is an interesting problem to describe the self-adjoint extensions of \( L \) acting on \( L^2(\mathbb{R}^3) \). At the same time, this is an important topic in the theory of quantum mechanics; see [5]. For the following complete characterization to the self-adjoint extensions of \( L \) see [5], [13], and [32].

**Lemma 1.1.** The self-adjoint extensions of \( L \), to an operator acting on \( L^2(\mathbb{R}^3) \), form a one-parameter family \( \mathcal{L}_\gamma, \gamma \in \mathbb{R} \). The spectrum of \( \mathcal{L}_\gamma \) is given by
\[ \text{spec}(\mathcal{L}_\gamma) = (-\infty, 0] \cup \{\gamma^2/2\}, \quad \gamma > 0, \]
\[ = (-\infty, 0], \quad \gamma \leq 0. \]
Moreover if \( \gamma > 0 \), then \( \gamma^2/2 \) is a simple eigenvalue of \( \mathcal{L}_\gamma \) with (normalized) eigenfunction
\[ \psi_\gamma(x) = \frac{\sqrt{\gamma}}{2\pi} e^{-\gamma |x|}. \]

In this paper we are concerned with the energy form induced by the eigenfunction \( \psi_\gamma \) for a fixed \( \gamma > 0 \). That is
\[ \mathcal{F} := \left\{ u \in L^2(\mathbb{R}^3, \psi_\gamma^2 dx) : \nabla u \in L^2(\mathbb{R}^3, \psi_\gamma^2 dx) \right\} \]
\[ \mathcal{E}(u, v) := \frac{1}{2} \int_{\mathbb{R}^3} \nabla u(x) \cdot \nabla v(x) \psi_\gamma(x)^2 dx, \quad u, v \in \mathcal{F}, \]  
where \( \nabla \) is the gradient of \( u \) in the sense of distributions. Note that \( \psi_\gamma \) is a smooth function on \( \mathbb{R}^3_0 \) but explodes at 0. Moreover, \( \psi_\gamma \in L^2(\mathbb{R}^3) \) but \( \nabla \psi_\gamma \notin L^1_{\text{loc}}(\mathbb{R}^3) \). The explosion of \( \psi_\gamma \) at 0 means that we cannot appeal to the classical results about energy forms similar to (1.4), as found in [1], [3], [23], [36], [37], [41]: these results all require \( \nabla \phi \in L^1_{\text{loc}}(\mathbb{R}^3) \).

However we will see in Theorem 2.1 that \( (\mathcal{E}, \mathcal{F}) \) is a regular Dirichlet form on \( L^2(\mathbb{R}^3, \psi_\gamma^2 dx) \) with the core \( C_c^\infty(\mathbb{R}^3) \), as a consequence of Lemma 1.1. As a corollary, the operator \(-\mathcal{L}_\gamma\), for each \( \gamma > 0 \), is lower bounded with parameter \( \gamma^2/2 \); see Corollary 2.2. Let \( X \) denote the diffusion process corresponding to \((\mathcal{E}, \mathcal{F})\). Then \( X \) is an \( m \)-symmetric recurrent diffusion on \( \mathbb{R}^3 \) without killing, where \( m(dx) := \psi_\gamma(x)^2 dx \). Moreover, \( X \) is recurrent, conservative, and irreducible; see Proposition 2.4. Outside \( \{0\} \) the diffusion \( X \) is similar to a 3-dimensional Brownian motion; especially, each singleton \( \{x\} \) is polar, for \( x \in \mathbb{R}^3_0 \). But as a result of the explosion of \( \psi_\gamma \) at 0, the 1-capacity of \( \{0\} \) with respect to \((\mathcal{E}, \mathcal{F})\) is positive; see Proposition 3.1. In addition \( \{0\} \) is regular for itself in the sense that \( P^0(T_0 = 0) = 1 \) where \( P^0 \) is the probability measure of \( X \) starting from 0, and \( T_0 \) is the hitting time of \( \{0\} \) with respect to \( X \); see Corollary 3.3.
Let \( X_0 \) be the part process of \( X \) on \( \mathbb{R}^3_0 \) and \( m^0 := m|_{\mathbb{R}^3_0} \). Then \( X^0 \) is an \( m^0 \)-symmetric diffusion with lifetime \( T_0 \) on \( \mathbb{R}^3_0 \), whose associated Dirichlet form is

\[
\mathcal{F}^0 = \{ u \in \mathcal{F} : \tilde{u}(0) = 0 \} \\
\mathcal{E}^0(u, v) = \mathcal{E}(u, v), \quad u, v \in \mathcal{F}^0.
\] (1.5)

(Here \( \tilde{u} \) is a quasi-continuous \( m \)-version of \( u \).) It follows from Theorem 3.3.9 of \([11]\) that \( (\mathcal{E}^0, \mathcal{F}^0) \) is regular on \( L^2(\mathbb{R}^3_0, m^0) \) with core \( C_c^{\infty}(\mathbb{R}^3_0) \). Because \( \text{Cap}(\{0\}) > 0 \), the part process \( X^0 \) is different from \( X \), but the behavior of \( X^0 \) is easier to understand. Roughly speaking, \( X^0 \) moves as a diffusion satisfying

\[
X^0_t - X^0_0 = B_t - \int_0^t \frac{\gamma|X^0_s| + 1}{|X^0_s|^2} \cdot X^0_s \, ds
\]

before hitting \( \{0\} \), where \( (B_t) \) is a three-dimensional standard Brownian motion. We can reconstruct \( X \) from \( X^0 \) by stringing together excursions of \( X^0 \). The requisite entrance law \( \{\nu_t : t > 0\} \) is uniquely determined by

\[
\int_0^{\infty} \nu_t \, dt = m.
\]

On the other hand, \( X \) and \( X^0 \) are both rotationally-invariant diffusions. In Proposition 3.7 we will write down the skew product presentation of \( X^0 \). Its radial part, which dies upon hitting \( \{0\} \), is an absorbing diffusion on \( (0, \infty) \) and the angular part is a time-changed spherical Brownian motion on \( S^2 \). In addition the set of limit points of the angular part of \( X^0 \), as time approaches its lifetime, coincides a.s. with the entire sphere \( S^2 \). Unfortunately \( X \) does not have an analogous skew product presentation because the spinning about 0 of the paths of \( X \) at the beginning (and end) of its excursions from \( \{0\} \) is too violent. However the radial part of \( X \) which is a diffusion on \( [0, \infty) \) is simply the reflection of the radial part of \( X^0 \). In other words setting \( r_t := |X^0_t| \) and \( \hat{r}_t := |X_t| \), we have

\[
\begin{align*}
  r_t - r_0 &= \beta_t - \gamma t, & t < T_0, \\
  \hat{r}_t - \hat{r}_0 &= \beta_t - \gamma t + \pi \gamma \cdot l_0^0, & t \geq 0
\end{align*}
\]

where \( \beta \) is a 1-dimensional standard Brownian motion and \( (t^0_t)_{t \geq 0} \) is the local time of \( (\hat{r}_t)_{t \geq 0} \) at \( \{0\} \). We will also examine the Fukushima decomposition of \((\mathcal{E}, \mathcal{F})\) with respect to the coordinate functions:

\[
X_t - X_0 = B_t + N_t, \quad t > 0
\]

where \( B \) is a 3-dimensional standard Brownian motion and \( N \) is the zero energy part of \( X \). It will be shown in Theorem 4.2 that \( N \) is not of bounded variation. As a corollary, \( X \) is not a semi-martingale. By moderating the pole of \( \psi_r \) at 0 we can generate a sequence of nice semi-martingales that approximate \( X \) in a certain sense; see Proposition 4.4.

**Notation.** For a domain \( E \subset \mathbb{R}^d \), the function classes \( C(E), C^1(E), \) and \( C^\infty(E) \) are the continuous functions, continuously differentiable differentiable functions, and smooth functions on \( E \), respectively. For a Radon measure \( \mu \) on \( E \), the Hilbert space of all \( \mu \)-square-integrable functions on \( E \) is denoted by \( L^2(E, \mu) \) or \( L^2(\mu) \), and its norm and inner product will be written as \( \| \cdot \|_\mu \) and \( \langle \cdot, \cdot \rangle_\mu \). If \( \mu \) is the Lebesgue measure, the preceding notation will abbreviated to \( L^2(E) \) or \( L^2 \). Similar notation will be used for integrable functions. For any function class \( \Theta \), the subclass of all the functions locally in \( \Theta \) (resp. with compact support, bounded) will denoted by \( \Theta_{\text{loc}} \) (resp. \( \Theta_c, \Theta_0 \)).

If \((\mathcal{E}, \mathcal{F})\) is a Dirichlet form on \( L^2(E, \mu) \) and \( F \) is a subset of \( E \), then the subclass \( \mathcal{F}_F \) of \( \mathcal{F} \) is defined by

\[
\mathcal{F}_F := \{ u \in \mathcal{F} : u = 0, \ \mu\text{-a.e. on } F^c \}.
\]

The \( \mathcal{E}_1^{1/2} \)-norm of \( u \in \mathcal{F} \) is

\[
||u||_{\mathcal{E}_1^{1/2}} := [\mathcal{E}(u, u) + (u, u)_\mu]^{1/2}.
\]
The 1-capacity of \((E,F)\), as defined in [27], will always be denoted by \(\text{Cap}\). For basic concepts to do with Dirichlet forms and associated potential theoretic notions, especially polar set, nest, generalized nest, quasi-continuous function, quasi everywhere (q.e. in abbreviation), we refer the reader to [27]. The quasi-continuous version of function \(u\) is always denoted by \(\tilde{u}\).

The diffusion process associated with \((E,F)\) is denoted by \(X = (X_t)_{t \geq 0}\). The law of \(X\) started at \(x \in \mathbb{R}^3\) is \(P^x\), and the transition semigroup of \(X\) is \((P_t)_{t \geq 0}\) defined by \(P_tf(x) := E^x[f(X_t)]\) for \(t \geq 0\) and \(f : \mathbb{R}^3 \to \mathbb{R}\) bounded and measurable. Here \(E^x\) is the expectation with respect to \(P^x\).

2. The Dirichlet forms induced by eigenfunctions

The general energy form may be defined by the expression

\[
\int u(x)v(x)\phi(x)^2\,dx \tag{2.1}
\]

for \(u,v \in C_c^\infty(\mathbb{R}^3)\), with respect to some specific function \(\phi\) and dimension \(d \geq 1\). The mildest condition on \(\phi\) that we know ensuring the closability of \((E,C_c^\infty(\mathbb{R}^3))\) on \(L^2(\mathbb{R}^3,\phi^2\,dx)\) is this: \(\phi \in L^2_{\text{loc}}(\mathbb{R}^3)\) and there is a closed set \(N\) of Lebesgue measure zero such that the distribution \(\nabla \phi\) is in \(L^2_{\text{loc}}(\mathbb{R}^3 \setminus N)\); see Theorem 2.4 of [1]. In particular \((E,C_c^\infty(\mathbb{R}^3))\) is closable on \(L^2(\mathbb{R}^3,\psi_2^2\,dx)\) if we choose \(N = \{0\}\) in this condition. Note that \(\nabla \psi_0\) is locally bounded on \(\mathbb{R}^3_0\) but is not in \(L^1_{\text{loc}}(\mathbb{R}^3)\). (Certain other properties of the diffusion associated with the energy form \((2.1)\), such as the semi-martingale property, require the additional property \(\nabla \phi \in L^1_{\text{loc}}(\mathbb{R}^3)\), which \(\nabla \psi_0\) does not satisfy.) We begin with a proof of the regularity of \((E,F)\).

**Theorem 2.1.** Fix \(\gamma > 0\) and let \(\psi_\gamma\) be the eigenfunction given by \([1.5]\). Then the form \((1.4)\) is a regular Dirichlet form on \(L^2(\mathbb{R}^3,\psi_\gamma^2(x)\,dx)\) with core \(C_c^\infty(\mathbb{R}^3)\).

**Proof.** Since \(\psi_\gamma\) is smooth and strictly positive on \(\mathbb{R}^3_0\), it follows that \((E,F)\) is a Dirichlet form (not regular) on \(L^2(\mathbb{R}^3_0,m^0)\). Hence it is also a Dirichlet form on \(L^2(\mathbb{R}^3,m)\). On the other hand clearly

\[
C_c^\infty(\mathbb{R}^3) \subset F.
\]

Denote the closure of \((E,C_c^\infty(\mathbb{R}^3))\) in \((E,F)\) by \((\tilde{E},\tilde{F})\). Then \((\tilde{E},\tilde{F})\) is a regular Dirichlet form on \(L^2(\mathbb{R}^3,m)\). Let \(A\) and \(\tilde{A}\) be the associated generators of the Dirichlet forms \((E,F)\) and \((\tilde{E},\tilde{F})\) respectively. We only need to prove \(A = \tilde{A}\).

First we have

\[
C_c^\infty(\mathbb{R}^3_0) \subset D(A) \tag{2.2}
\]

and

\[
Au = \frac{1}{2} \Delta u + \frac{\nabla \psi_\gamma}{\psi_\gamma} \cdot \nabla u \tag{2.3}
\]

for all \(u \in C_c^\infty(\mathbb{R}^3_0)\). Indeed for a given \(u \in C_c^\infty(\mathbb{R}^3_0)\), we can deduce that \(u \in F\) and moreover that for all \(f \in F\),

\[
E(u,f) = \frac{1}{2} \sum_{i=1}^3 \int \frac{\partial u}{\partial x_i}(x) \frac{\partial f}{\partial x_i}(x) \psi_\gamma(x)^2\,dx
\]

\[
= -\frac{1}{2} \sum_{i=1}^3 \int f(x) \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i}(x) \psi_\gamma^2(x) \right)\,dx
\]

\[
= -\frac{1}{2} \sum_{i=1}^3 \int f(x) \left( \frac{\partial^2 u}{\partial x_i^2}(x) + \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \frac{1}{\psi_\gamma(x)} \frac{\partial \psi_\gamma}{\partial x_i}(x) \right) \psi_\gamma^2(x)\,dx
\]

\[
= \left( \frac{1}{2} \Delta u - \frac{\nabla \psi_\gamma}{\psi_\gamma} \cdot \nabla u, f \right)_m.
\]

Thus \(u \in D(A)\) and \(2.3\) holds. In the same way we deduce that

\[
1 \in D(A), \quad A1 = 0. \tag{2.4}
\]
Now define an operator $H$ on $L^2(\mathbb{R}^3)$ by
\[
\mathcal{D}(H) = \{ u \in L^2(\mathbb{R}^3) : \psi_\gamma^{-1} \cdot u \in \mathcal{D}(A) \},
\]
\[
Hu = \psi_\gamma \cdot A(\psi_\gamma^{-1} \cdot u), \quad u \in \mathcal{D}(H).
\] (2.5)

Then $H$ is a self-adjoint operator on $L^2(\mathbb{R}^3)$. In fact for each $u \in \mathcal{D}(H)$ and $v \in \mathcal{D}(H^*)$, where $H^*$ is the adjoint operator of $H$, we have
\[
(u, H^* v)_{L^2(\mathbb{R}^3)} = (Hu, v)_{L^2(\mathbb{R}^3)} = (A(\psi_\gamma^{-1} \cdot u), \psi_\gamma^{-1} \cdot v)_m.
\]

It follows that
\[
(\psi_\gamma^{-1} \cdot u, \psi_\gamma^{-1} \cdot H^* v)_m = (A(\psi_\gamma^{-1} \cdot u), \psi_\gamma^{-1} \cdot v)_m.
\]

Hence $\psi_\gamma^{-1} \cdot v \in \mathcal{D}(A)$ and $H^* v = \psi_\gamma \cdot A(\psi_\gamma^{-1} \cdot v)$. In other words, $\mathcal{D}(H^*) \subset \mathcal{D}(H)$ and $Hu = H^* u$ for all $u \in \mathcal{D}(H^*)$. On the other hand for all $u, v \in \mathcal{D}(H),

\[
(Hu, v)_{L^2(\mathbb{R}^3)} = (A(\psi_\gamma^{-1} \cdot u), \psi_\gamma^{-1} \cdot v)_m = (u, \psi_\gamma \cdot A(\psi_\gamma^{-1} \cdot v))_{L^2(\mathbb{R}^3)} = (u, Hv).
\]

Hence $\mathcal{D}(H) \subset \mathcal{D}(H^*)$ and $H^* v = Hv$ for all $v \in \mathcal{D}(H)$. Therefore $H$ is self-adjoint on $L^2(\mathbb{R}^3)$. It follows from (2.2) and (2.5) that
\[
C^\infty_c(\mathbb{R}^3_0) \subset \mathcal{D}(H)
\]
and for all $u \in C^\infty_c(\mathbb{R}^3_0)$,
\[
Hu = \psi_\gamma \cdot A(\psi_\gamma^{-1} \cdot u) = \frac{1}{2} \Delta u - \frac{\gamma^2}{2} u.
\]

Let $H_\gamma := H + \frac{\gamma^2}{2}$. Then $H_\gamma$ is a self-adjoint extension of the operator $L$ defined by (1.1) and (1.2). On the other hand from (2.4) we see that
\[
H_\gamma \psi_\gamma = H \psi_\gamma + \frac{\gamma^2}{2} \psi_\gamma = \frac{\gamma^2}{2} \psi_\gamma.
\]

Hence it follows from Lemma 1.1 that $H_\gamma = \mathcal{L}_\gamma$.

Similarly, we can show that (2.2), (2.4), and (2.5) hold for the generator $\tilde{A}$, and then define an operator $\tilde{H}$ on $L^2(\mathbb{R}^3)$ as in (2.5). As before, $\tilde{H}_\gamma := H + \gamma^2/2$ is a self-adjoint extension of $L$ and similarly
\[
\tilde{H}_\gamma = \mathcal{L}_\gamma = H_\gamma.
\]

It follows that $A = \tilde{A}$. \qed

Remark 2.2. For a given $\gamma > 0$ let $\mathcal{L}_\gamma$ be the self-adjoint extension of $L$ as in Lemma 1.1. It follows from the above proof that the generator $A$ of $(\mathcal{E}, \mathcal{F})$ is characterized by
\[
\mathcal{D}(A) = \{ u \in L^2(\mathbb{R}^3, m) : u \cdot \psi_\gamma \in \mathcal{D}(\mathcal{L}_\gamma) \}
\]
\[
Au = \psi_\gamma^{-1} \cdot \mathcal{L}_\gamma(\psi_\gamma \cdot u) - \frac{\gamma^2}{2} u, \quad u \in \mathcal{D}(A).
\] (2.6)

The operator $A$ is actually a $\psi_\gamma$-transform of $\mathcal{L}_\gamma$. In other words, let $(P_t)_{t \geq 0}$ and $(Q_t)_{t \geq 0}$ be the semigroups generated by $A$ and $\mathcal{L}_\gamma$ respectively. Then $(P_t)$ and $(Q_t)$ are symmetric with respect to $m$ and Lebesgue measure, respectively. And for all $u \in L^2(\mathbb{R}^3, m)$, it follows that
\[
P_t u = e^{tA} u = e^{-t^2/2} \psi_\gamma^{-1} \cdot e^{t\mathcal{L}_\gamma}(\psi_\gamma \cdot u) = e^{-t^2/2} \psi_\gamma^{-1} \cdot Q_t(\psi_\gamma \cdot u).
\] (2.7)

Since $Q_t$ has a (continuous) density function $q_t(x, y)$ (that is, $Q_t(x, dy) = q_t(x, y) dy$; see [5], [13], or [32]), it follows that
\[
P_t(x, dy) = p_t(x, y)m(dy)
\] (2.8)
where
\[
p_t(x, y) = \frac{e^{-\gamma^2 t/2}}{\psi_\gamma(x) \psi_\gamma(y)} q_t(x, y)
\] (2.9)
for \(x, y \neq 0\). Recall that \([31]\) has provided a characterization of \(h\)-transforms of symmetric Markov processes. The difference here is that neither \(\mathcal{L}_\gamma\) nor \(\mathcal{L}_\gamma - \gamma^2/2\) is the generator of Markov process because neither is Markovian, whereas the transformed operator \(A\) is the generator of the Dirichlet form \((\mathcal{E}, \mathcal{F})\).

**Corollary 2.3.** Let \(\gamma > 0\) and the operator \(\mathcal{L}_\gamma\) be in Lemma\([4]\). Then \(-\mathcal{L}_\gamma\) is lower bounded with parameter \(\gamma^2/2\), i.e.

\[
(-\mathcal{L}_\gamma u, u)_{L^2(\mathbb{R}^3)} + \frac{\gamma^2}{2} (u, u)_{L^2(\mathbb{R}^3)} \geq 0
\]

for all \(u \in \mathcal{D}(\mathcal{L}_\gamma)\). Consequently, the semigroup \((Q_t)\) of \(\mathcal{L}_\gamma\) is bounded by \(\exp\{\gamma^2 \cdot t/2\}\); i.e.

\[
\|Q_t\|_{L^2(\mathbb{R}^3)} \leq \exp\{\gamma^2 t/2\}
\]

for all \(t \geq 0\).

We now record several global properties of the Dirichlet form \((\mathcal{E}, \mathcal{F})\) and the associated diffusion \(X\).

**Proposition 2.4.** The Dirichlet form \((\mathcal{E}, \mathcal{F})\) is recurrent, conservative, and irreducible. Consequently, the symmetry measure \(m\) is an invariant distribution for \(X\).

**Proof.** It follows from \([2,4]\) that \(1 \in \mathcal{F}\) and \(\mathcal{E}(1, 1) = 0\). Thus \((\mathcal{E}, \mathcal{F})\) is recurrent, hence also conservative (that is, \(P_{t1} = 1\), \(m\)-a.e., for all \(t > 0\)). On the other hand for all \(u \in \mathcal{F}\) satisfying \(\mathcal{E}(u, u) = 0\), it follows that \(\nabla u = 0\), \(m\)-a.e. hence a.e. because \(m\) is equivalent to the Lebesgue measure. Thus we can deduce that \(u\) is constant \(m\)-a.e. Because \(m(\mathbb{R}^3) < \infty\) it follows from Theorem 2.1.11 of \([11]\) that \((\mathcal{E}, \mathcal{F})\) is irreducible. The final assertion follows because \(X\) is conservative, and

\[
m(\mathbb{R}^3) = \int_{\mathbb{R}^3} \psi_1^2(x)dx = 1.
\]

□ □

### 3. Behavior near 0

The process induced by the energy form \([2,4]\) is sometimes called a **distorted Brownian motion**. In particular, the potential theoretic properties of \((\mathcal{E}, \mathcal{F})\) on a given relatively compact open subset \(G\) of \(\mathbb{R}_0^3\) are equivalent to those of the Brownian motion because the \(\mathcal{E}^{1/2}_{G,1}\)-norm of the part Dirichlet form \((\mathcal{E}_G, \mathcal{F}_G)\) is equivalent to that of \((\frac{1}{2}D_G, H^1_0(G))\). Here \((\frac{1}{2}D_G, H^1_0(G))\) is the Dirichlet form of the absorbing Brownian motion on \(G\). However, because of the singularity of \(\psi_\gamma\) at the origin, the process \(X\) behaves quite differently from Brownian motion near the state 0. It is well known that singletons are polar sets with respect to 3-dimensional Brownian motion. But \(\{0\}\) is not polar with respect to \((\mathcal{E}, \mathcal{F})\). Actually \(\{0\}\) is the only non-polar singleton with respect to \((\mathcal{E}, \mathcal{F})\).

**Proposition 3.1.** The 1-capacity of the set \(\{0\}\) with respect to \((\mathcal{E}, \mathcal{F})\) is positive.

**Proof.** Let \(B_\epsilon := \{x \in \mathbb{R}^3 : |x| < \epsilon\}\) for \(\epsilon > 0\). Note that the 1-capacity \(\text{Cap}\) satisfies

\[
\text{Cap}(\{0\}) = \inf_{\epsilon > 0} \text{Cap}(B_\epsilon).
\]

Hence we need only compute the capacity of \(B_\epsilon\) for each \(\epsilon > 0\).

Fix \(\epsilon > 0\), define \(f : \mathbb{R}^3 \to [0, \infty)\) by

\[
f_\epsilon(x) := \begin{cases} 
1, & 0 \leq |x| \leq \epsilon, \\
\exp\{c(|x| - \epsilon)\}, & |x| > \epsilon,
\end{cases}
\]

where \(c = \gamma - \sqrt{\gamma^2 + 2} < 0\). We shall demonstrate that \(f_\epsilon\) is the 1-equilibrium potential of the set \(B_\epsilon\); see \([27]\). Things being so we have

\[
\text{Cap}(B_\epsilon) = \mathcal{E}_1(f_\epsilon, f_\epsilon).
\]
ON THE DIRICHLET FORM OF THREE-DIMENSIONAL BROWNIAN MOTION CONDITIONED TO HIT THE ORIGIN

Firstly, $f_\varepsilon \in F$. In fact,

$$\int f_\varepsilon(x)^2 \psi_\gamma(x)^2 dx = 4\pi \gamma^2 \int_0^\infty \hat{f}_\varepsilon(r)^2 e^{-2\gamma r} r^2 dr$$

$$= 2\gamma \int_0^\varepsilon e^{-2\gamma r} dr + 2\gamma \int_\varepsilon^\infty e^{2(c-\gamma)r-2\varepsilon r} dr$$

$$= 1 + \frac{c}{\gamma-c} e^{-2\gamma\varepsilon}$$

where $\hat{f}_\varepsilon$ is the radial part of $f_\varepsilon$. And

$$\int |\nabla f_\varepsilon|^2(x) \psi_\gamma(x)^2 dx = 4\pi \gamma \int_0^\infty e^2 \hat{f}_\varepsilon(r)^2 e^{-2\gamma r} r^2 dr$$

$$= \frac{\gamma c^2}{\gamma-c} e^{-2\gamma\varepsilon}.$$ 

This shows that $f_\varepsilon \in F$ and

$$\mathcal{E}_1(f_\varepsilon, f_\varepsilon) = 1 + \frac{c + \gamma c^2}{\gamma-c} e^{-2\gamma\varepsilon}.$$ 

In particular,

$$\inf_{\varepsilon>0} \mathcal{E}_1(f_\varepsilon, f_\varepsilon) = \lim_{\varepsilon \downarrow 0} \mathcal{E}_1(f_\varepsilon, f_\varepsilon) = \frac{\gamma + \gamma c^2}{\gamma-c} > 0.$$ 

Next it follows from Theorem 2.1.5 of [27] that to show that $f_\varepsilon$ is 1-excessive we need to show that

$$\mathcal{E}_1(f_\varepsilon, v) \geq 0$$

for all $v \in F$ with $\hat{v} \geq 0$ on $B_\varepsilon$. Note that

$$\hat{f}_\varepsilon''(r) - 2\gamma \hat{f}_\varepsilon'(r) - 2 \hat{f}_\varepsilon(r) = 0, \quad r > \varepsilon. \quad (3.1)$$

We first assume $v$ has a compact support, say $K$. Clearly the weak distribution derivative satisfies

$$\frac{\partial}{\partial x_i} \left( v \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right) = \psi_\gamma^2 \frac{\partial v}{\partial x_i} \frac{\partial f_\varepsilon}{\partial x_i} + v \frac{\partial}{\partial x_i} \left( \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right).$$

Thus

$$\mathcal{E}(f_\varepsilon, v) = \frac{1}{2} \sum_{i=1}^3 \int \frac{\partial}{\partial x_i} \left( v \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right) dx - \frac{1}{2} \sum_{i=1}^3 \int v \frac{\partial}{\partial x_i} \left( \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right) dx.$$

Choose a function $g \in C_c^\infty(\mathbb{R}^3)$ such that $g \equiv 1$ on $K$. Because the support of $u$ is contained in $K$,

$$\int \frac{\partial}{\partial x_i} \left( v \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right) dx = \int g(x) \frac{\partial}{\partial x_i} \left( v \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} \right)(x) dx$$

$$= - \int \frac{\partial g}{\partial x_i} v \psi_\gamma^2 \frac{\partial f_\varepsilon}{\partial x_i} dx$$

$$= 0.$$
Hence from (3.1) we can deduce that
\[ E_1(f, v) = \int_{B_r} f(x)v(x)\gamma(x)^2\,dx \]
\[ + \int_{B_r^c} v(x)[f(x)\gamma(x)]^2 - \frac{1}{2} \sum_{i=1}^3 \frac{\partial}{\partial x_i} (\gamma^2 \frac{\partial f}{\partial x_i})(x)\,dx \]
\[ = \int_{B_r} f(x)v(x)\gamma(x)^2\,dx \]
\[ + \int_{B_r^c} v(x)\gamma(x)[\tilde{f}_r(|x|) + \gamma \tilde{f}'(|x|) - \frac{1}{2} \tilde{f}''(|x|)]\,dx \]
\[ = \int_{B_r} v(x)\gamma(x)^2\,dx \geq 0. \]

Now let \( v \) be an arbitrary element of \( \mathcal{F} \) with \( \tilde{v} \geq 0 \) on \( B_r \). Choose a sequence of functions \( \{g_n\} \subset C_c^\infty(\mathbb{R}^3) \) such that \( g_n \uparrow 1 \) pointwise and in the norm \( \| \cdot \|_{E_1/2} \). Then \( g_n \cdot v \to v \) in the norm \( \| \cdot \|_{E_1/2} \), while
\[ E_1(f, g_n \cdot v) \geq 0 \]
by the preceding discussion. By letting \( n \to \infty \) we deduce that
\[ E_1(f, v) \geq 0. \]

We have now shown that \( f_\epsilon \) is the 1-equilibrium potential of \( B_\epsilon \), and so
\[ \text{Cap}(\{0\}) = \frac{\gamma + \gamma c^2}{\gamma - c} > 0 \]
where \( c = \gamma - \sqrt{\gamma^2 + 2} \). In particular, \( \{0\} \) is not \( m \)-polar. \( \square \)

The part process \( X^0 \) on \( \mathbb{R}^3_0 \) (that is, \( X \) killed on first hitting \( 0 \)) is an \( m^0 \)-symmetric Markov process whose lifetime is the hitting time \( T_0 \) of \( \{0\} \) with respect to \( X \). The associated Dirichlet form \((\mathcal{E}_0, \mathcal{F}_0)\) is given by (1.5). In particular \((\mathcal{E}_0, \mathcal{F}_0)\) is regular with core \( C_c^\infty(\mathbb{R}^3_0) \). Clearly \( \mathcal{F}_0 \) is a proper subset of \( \mathcal{F} \) and we will see in Corollary 3.11 below that \((\mathcal{E}_0, \mathcal{F}_0)\) is irreducible and transient. For the following definition we refer the reader to Definition 7.2.6 of [11].

**Definition 3.2.** A symmetric Hunt process \( Y \) is said to be a reflecting extension of a symmetric standard process \( Y^0 \) if the following hold:

1. **(RE.1):** \( E \) is a locally compact separable metric space, \( m \) is an everywhere dense positive Radon measure on \( E \) and \( Y \) is an \( m \)-symmetric Hunt process on \( E \) whose Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(E, m) \) is regular.
2. **(RE.2):** \( Y^0 \) is the part process of \( Y \) on a non-\( \mathcal{E} \)-polar, \( \mathcal{E} \)-quasi-open subset \( E^0 \) of \( E \) whose Dirichlet form \((\mathcal{E}_0, \mathcal{F}_0)\) on \( L^2(E_0, m|_{E_0}) \) is irreducible.
3. **(RE.3):** \( m(F) = 0 \) where \( F = E \setminus E_0 \).
4. **(RE.4):** The active reflected Dirichlet form \((\mathcal{E}^0, \mathcal{F}^0, \mathcal{R}_0)\) of \((\mathcal{E}_0, \mathcal{F}_0)\) coincides with \((\mathcal{E}, \mathcal{F})\).

For the definition of the active Dirichlet form see [9] (and also [11]).

**Theorem 3.3.** The process \( X \) on \( \mathbb{R}^3 \) corresponding to \((\mathcal{E}, \mathcal{F})\) given by (1.4) is a reflecting extension of \( X^0 \) on \( \mathbb{R}^3_0 \).

We only need to verify (RE.4) in the above definition.

**Lemma 3.4.** The active reflected Dirichlet form of \((\mathcal{E}_0, \mathcal{F}_0)\) is equal to \((\mathcal{E}, \mathcal{F})\). Here \((\mathcal{E}, \mathcal{F})\) and \((\mathcal{E}_0, \mathcal{F}_0)\) are given by (1.4) and (1.7) respectively.
Proof. Since $\psi_s$ is smooth and strictly positive on $\mathbb{R}^3_0$, it follows from Theorems 3.3.1 and 3.3.2 of [27] that the self-adjoint operator $A$ corresponding to $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^3_0, m^0)$ is the maximum element of the class of Silverstein extensions of the form $\mathcal{E}_S$ defined by
\[
\mathcal{D}(\mathcal{E}_S) := C^\infty_c(\mathbb{R}^3_0),
\]
\[
\mathcal{E}_S(u, v) := \mathcal{E}(u, v), \quad u, v \in \mathcal{D}(\mathcal{E}_S).
\]
(For the definitions of Silverstein extension and the related order, we refer the reader to § 3.3 and (3.3.3) of [27] (or Definition 6.6.1 and 6.6.8 of [11]).) It follows from Theorem 6.6.9 of [11] that
\[
(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_S^{0, \text{ref}}, (\mathcal{F}_S^0)^{\text{ref}}).
\]
\[\square \square\]

With Theorem 3.3 in hand we can appeal to Theorems 6.4.2 and 6.6.10 of [11] to deduce the following corollary.

Corollary 3.5. The extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ is equal to the reflected Dirichlet space $\mathcal{F}_S^{0, \text{ref}}$ of $(\mathcal{E}_S^{0, \text{ref}}, \mathcal{F}_S^0)$:
\[
\mathcal{F}_S = \mathcal{F}_S^{0, \text{ref}} = \{ u \in \mathcal{F}_0^{0, \text{loc}} : \nabla u \in L^2(\mathbb{R}^3, \psi^2(x)dx) \},
\]
(3.2)
where $\mathcal{F}_S^{0, \text{loc}}$ is the class of all the functions $u$ for which there is an increasing sequence of $\mathcal{E}_S^{0, \text{loc}}$-quasi-open sets $\{D_n\}$ with $\bigcup_{n=1}^\infty D_n = \mathbb{R}^3_0$ $\mathcal{E}_S^0$-q.e. and a sequence $\{u_n\} \subset \mathcal{F}_S^0$ such that $u = u_n$ a.e. on $D_n$.

Remark 3.6. Note that collection described in (3.2) contains all $u \in L^2_{\text{loc}}(\mathbb{R}^3, \psi^2(x)dx)$ such that $\nabla u \in L^2(\mathbb{R}^3, \psi^2(x)dx)$.

To get at the singular behavior of $X$ near $\{0\}$, we first examine the skew-product decomposition of the part process $X^0$.

Proposition 3.7. The process $X^0$ admits a skew-product representation
\[
(r_t \theta_{A_t})_{t \geq 0}
\]
(3.3)
where $(r_t)_{t \geq 0}$ is a symmetric diffusion on $(0, \infty)$, killed at $\{0\}$, whose speed measure $l$ and scale function $s$ are
\[
l(dx) = 2\gamma e^{-\gamma x}dx,
\]
\[
s(x) = \frac{1}{4\gamma^2} e^{2\gamma x};
\]
$(A_t)_{t \geq 0}$ is the PCAF of $(r_t)_{t \geq 0}$ with Revuz measure is
\[
\mu_A(dx) = \frac{l(dx)}{x^2};
\]
and $\theta$ is a spherical Brownian motion on $S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \}$, and is independent of $(r_t)_{t \geq 0}$.

Proof. Let $(r_t)_{t \geq 0}$ be the diffusion on $(0, \infty)$ described in the statement of the theorem, with speed measure $l$ and scale function $s$. Then $(r_t)_{t \geq 0}$ is $l$-symmetric, and it follows from Theorem 3 of [21] that the associated Dirichlet form on $L^2((0, \infty), l)$ is the closure of
\[
\mathcal{D}(\mathcal{E}^{s,l}) = C_c^\infty((0, \infty)),
\]
\[
\mathcal{E}^{s,l}(u, v) = \frac{1}{2} \int u'(x)v'(x)l(dx), \quad u, v \in \mathcal{D}(\mathcal{E}^{s,l}).
\]
Note that $\mu_A$ is a positive Radon measure on $(0, \infty)$ with full support. Let $\sigma$ be the normalized surface measure on $S^2$, so that $\sigma(S^2) = 1$. Then $\theta$ is $\sigma$-symmetric, and it follows from Theorem 1.1 of [24] that the skew product (3.3) is an $m := l \otimes \sigma$-symmetric diffusion on $\mathbb{R}^3_0$.
Dirichlet form can be expressed as the closure of
\[
D(\tilde{\mathcal{E}}) = C_c^\infty(\mathbb{R}^3_0),
\]
\[
\tilde{\mathcal{E}}(u,v) = \frac{1}{2} \int_{S^2} \int_0^\infty \frac{\partial u}{\partial r}(r,y) \frac{\partial v}{\partial r}(r,y) l(\sigma(dy))
+ \int_0^\infty \frac{1}{2}(-\Delta_{S^2} u(r,\cdot), v(r,\cdot))_\sigma \mu_A(dr)
\]
for \(u,v \in C_c^\infty(\mathbb{R}^3_0),\) where \(\Delta_{S^2}\) is the Laplace-Beltrami operator on \(S^2.\) It follows from (2.3) and
\[
\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^2} f, \quad f \in C_c^\infty(\mathbb{R}^3)
\]
that
\[
\tilde{\mathcal{E}}(u,v) = \mathcal{E}(u,v), \quad u,v \in C_c^\infty(\mathbb{R}^3_0).
\]
Hence the closure of \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\) coincides with \((\mathcal{E}^0, \mathcal{F}^0).\) \(\square \quad \square \)

Remark 3.8. In fact \(X^0\) is rotation invariant in the sense that for any orthogonal transformation \(T : \mathbb{R}^3 \to \mathbb{R}^3, T(X^0)\) has the same distribution as \(X^0.\) Moreover \((r_t)_{t \geq 0}\) is the radial part of \(X^0,\) i.e. \(r_t = |X_t^0|\) for all \(t \geq 0.\) Since \(s(0+) = -\infty\) and \(l\) is a bounded measure, the boundary point \(\{0\}\) is a regular boundary point for \((r_t)_{t \geq 0};\) see [11]. In particular \(\{0\}\) is not \(l\)-polar, hence non-polar, with respect to \((r_t)_{t \geq 0}.\) In other words,
\[
Q^x_r(\tau_0 < \infty) > 0 \tag{3.4}
\]
for each \(x \in (0, \infty),\) where \(Q^x_r\) is the law of \((r_t)_{t \geq 0}\) starting from \(x,\) and \(\tau_0 := \inf\{t > 0 : r_t = 0\}.
\]
Note that \(\tau_0\) is the lifetime of \((r_t)_{t \geq 0}.\) Since \(X^0\) is rotation invariant, we deduce that
\[
\phi(x) := P^x(0 < \infty) = Q^{|x|}_0(\tau_0 < \infty) > 0, \tag{3.5}
\]
for all \(x \in \mathbb{R}^3_0,\) where \(P^x\) is the law of \(X^0\) starting from \(x,\) and \(T_0 := \inf\{t > 0 : X_t = 0\}.\) On the other hand, note that the PCAF \(A\) satisfies
\[
A_t = \int_0^t \frac{1}{r_s^2} ds, \quad t < \tau_0.
\]
Because \((r_t)_{t \geq 0}\) and \(\theta\) are independent, it follows that the spherical part \(S_t := \theta A_t,\) of \(X^0\) is a diffusion on \(S^2\) and satisfies the SDE
\[
dS_t = \frac{1}{r_t^2} d\theta_t, \quad t < \tau_0.
\]
From an analogue of Theorem 2.12 of [14], we can deduce that
\[
\int_0^{\tau_0} \frac{1}{r_s^2} ds = \infty, \quad Q^x_r\text{-a.s.}
\]
for all \(x \in (0, \infty).\) Since \((r_t)_{t \geq 0}\) and \(\theta\) are independent, the set of limit points of \(S_t\) as \(t \uparrow \tau\) coincides with the entire sphere \(S^2,\) a.s.: cf. [17]. As noted by Erickson, this behavior of \(X^0\) at its lifetime is reflected in the fact that the excursions of \(X\) away from 0 end (and by symmetry begin) with the angular part of \(X\) oscillating so violently that each neighborhood of each point of the unit sphere is visited infinitely often. This behavior is the root cause of the fact that \(X\) is not a semi-martingale, as is shown in the next section.

Corollary 3.9. The process \(X\) is the unique one-point extension of \(X^0\) in the sense that \(X\) is \(m\)-symmetric, admits no killing on \(\{0\},\) and the part process of \(X\) on \(\mathbb{R}^3_0\) is \(X^0.\) Consequently, \(\{0\}\) is regular for itself in the sense that \(P^0(T_0 = 0) = 1\) where \(P^0\) is the law of \(X\) starting from 0, and \(T_0 := \inf\{t > 0 : X_t = 0\}.\)

Proof. This is clear from Theorem 7.5.4 of [14] and (3.3). \(\square \quad \square \)

Corollary 3.10. The Dirichlet form \((\mathcal{E}^0, \mathcal{F}^0)\) is irreducible and transient.
Proof. Clearly \((\bar{r}_t)_{t \geq 0}\) and \(\theta\) are both irreducible. It follows from Theorem 7.2 of [24] and Proposition 19.10 that \((\mathcal{E}^0, \mathcal{F}^0)\) is also irreducible. Hence it is transient because \(1 \notin \mathcal{F}^0\). \(\square\ \square\)

Unsurprisingly, \((\mathcal{E}, \mathcal{F})\) is also rotation invariant. In fact let \(T\) be an orthogonal transformation from \(\mathbb{R}^3\) to \(\mathbb{R}^3\). Denote the probability measures, the semigroup and Dirichlet form of \(\hat{X} := T(X)\) by \((\hat{P}^x)_{x \in \mathbb{R}^3}\), \((\hat{P}_t)_{t \geq 0}\) and \((\hat{\mathcal{E}}, \hat{\mathcal{F}})\). Clearly \(\hat{X}\) is also \(m\)-symmetric and for any Borel subset \(B \subset \mathbb{R}^3\),
\[
\hat{P}_t 1_B(x) = \hat{P}^x(\hat{X}_t \in B) = P^{T^{-1}x}(X_t \in T^{-1}B) = P_t(1_B \circ T)(T^{-1}x).
\]
It follows that for \(f \in L^2(\mathbb{R}^3, m)\),
\[
\hat{P}_t f(x) = P_t(f \circ T)(T^{-1}x).
\]

Then
\[
(f - \hat{P}_t f, f)_m = \int (f \circ T - P_t(f \circ T))(T^{-1}x) f \circ T(T^{-1}x) \psi^2(x) dx
\]
\[
= \int (f \circ T - P_t(f \circ T))(y) f \circ T(y) \psi^2(y) dTy
\]
\[
= (f \circ T - P_t(f \circ T), f \circ T)_m.
\]

Thus \(f \in \hat{\mathcal{F}}\) if and only if \(f \circ T \in \mathcal{F}\). Moreover
\[
\hat{\mathcal{E}}(f,g) = \mathcal{E}(f \circ T, g \circ T), \quad f,g \in \hat{\mathcal{F}}.
\]
From the expression 14 of \((\mathcal{E}, \mathcal{F})\) we can easily deduce that \((\hat{\mathcal{E}}, \hat{\mathcal{F}}) = (\mathcal{E}, \mathcal{F})\). Hence \(\hat{X}\) and \(X\) have the same distribution.

Let \((\bar{r}_t)_{t \geq 0}\) be the radial part of \(X\). Then \((\bar{r}_t)_{t \geq 0}\) is a diffusion on \([0, \infty)\). Denote its semigroup by \((\bar{q}_t)_{t \geq 0}\). Clearly for any positive function \(f\) on \([0, \infty)\) and \(r \in [0, \infty)\),
\[
\bar{q}_t f(r) = P_t(f \otimes 1_{[0,r]})(x)
\]
for all \(x \in \mathbb{R}^3\) such that \(|x| = r\). It follows that \((\bar{r}_t)_{t \geq 0}\) is \(l\)-symmetric and its Dirichlet form is
\[
\mathcal{F}^{s,l} = \{u \in L^2([0, \infty), l) : u' \in L^2([0, \infty), l)\}
\]
\[
\mathcal{E}^{s,l}(u,v) = \frac{1}{2} \int_0^\infty u'(x) v'(x) l(dx), \quad u,v \in \mathcal{F}^{s,l}.
\]

Hence \((\bar{r}_t)_{t \geq 0}\) is a reflecting diffusion on \([0, \infty)\) which is reflected at the boundary \([0)\) and acts as \((r_t)_{t \geq 0}\) on \((0, \infty)\). On the other hand it follows from Corollary 1 and Theorem 5 of [21] that
\[
\bar{r}_t - \bar{r}_0 = B_t - \gamma t, \quad t < \tau_0,
\]
\[
\bar{r}_t - \bar{r}_0 = B_t - \gamma t + \pi \gamma \cdot \bar{t}_t
\]
(3.6)
where \((B_t)_{t \geq 0}\) is a one-dimensional standard Brownian motion and \((\bar{t}_t)_{t \geq 0}\) is the local time of \(\bar{r}\) at \([0)\); i.e., the PCAF of \((\bar{r}_t)_{t \geq 0}\) with smooth measure \(\delta_{[0]}\). In particular, it follows from (3.6) that the following corollary holds. Recall that we already have \(P^0(T_0 = 0) = 1\) in Corollary 14.

Corollary 3.11. For each \(x \in \mathbb{R}^3\),
\[
\phi(x) = \mathbb{P}^x(T_0 < \infty) = 1.
\]

Now we can reconstruct the diffusion \(X\) by “stringing together” its excursions away from \([0)\). The associated Itô excursion law \(\mathbf{n}\) is determined by \(X^0\) and a certain \(X^0\)-entrance law. A system \(\{\nu_t : t > 0\}\) of \(\sigma\)-finite measures on \(\mathbb{R}_3^0\) is said to be an \(X^0\)-entrance law if

\[
\nu_t \hat{P}_t^0 = \nu_{s+t}
\]
for every \(t,s > 0\) where \((\hat{P}_t^0)_{t \geq 0}\) is the semigroup of \(X^0\). For the details of constructing a process from excursions via a suitable entrance law, we refer the reader to [30], [40], [39], [25], [26], and [41]. Since \(X\) admits no killing inside its state space, it follows that the unique \(X^1\)-entrance law \(\{\nu_t\}\) needed to construct \(\mathbf{n}\) is characterized by the formula
\[
\int_0^\infty \nu_t dt = m.
\]
Here is a “skew-product” description of $n$ that parallels the earlier skew-product decomposition of $X^0$. On a suitable measure space prepare three independent random objects:

(a): A stationary Brownian motion in the unit sphere $S^2$, $(\Theta_t)_{-\infty < t < \infty}$;
(b): An excursion $(\rho_t)_{0 \leq t \leq \zeta}$ of the process $(\bar{r}_t)_{0 \leq t \leq \infty}$ found in (3.0);
(c): A random variable $U$ uniformly distributed on $(0, 1)$.

Now form the time change

\[
A(t) := \begin{cases} 
\int_t^\zeta \rho_s^{-2} \, ds, & U\zeta \leq t < \zeta; \\
- \int_t^\zeta \rho_s^{-2} \, ds, & 0 < t \leq U\zeta,
\end{cases}
\]

noticing that $A_{0+} = -\infty$ and $A_{1-} = +\infty$, almost surely. Then the “distribution” of the process $(\rho_t \Theta_{A(t)})_{0 < t < \zeta}$ is proportional to the excursion law $n$. This is consistent with Erickson’s observation that the angular part of the path of our process must “go wild” when approaching (or departing) the origin.

Remark 3.12. Although we have limited our discussion to 3-dimensional Brownian motion, a similar development can be made in dimension $d = 2$. However, there are natural obstructions to our story when $d \geq 4$. Analytically, it is known that the Laplacian restricted to $C_c^\infty (R^d_0)$ admits a unique self-adjoint extension to $L^2(R^d)$, namely the usual Laplacian on $L^2(R^d)$. Thus, the eigenfunction approach taken here appears to be unavailable for $d \geq 4$. Probabilistically, the function $h(x) := |x|^{2-d}$ is harmonic (on $R^d_0$) for the $d$-dimensional Brownian motion. (This corresponds to the limiting case $\gamma = 0$ of our construction.) Let $X^* = (X^*_t)_{t \geq 0}$ be the $h$-transform of $d$-dimensional Brownian motion. This is a diffusion on $R^d_0$ with infinitesimal generator $\frac{1}{2} \Delta f(x) - \frac{(d-2)}{|x|^2} x \cdot \nabla f(x)$. The push toward the origin represented by the drift term in this generator is strong enough that $X^*$ hits the origin with probability 1 if started away from 0. But the push is too strong for the process to be able to escape (continuously) from the origin. In fact, if we start $X^*$ uniformly at random on the sphere of radius $\epsilon$ centered at the origin, normalize by dividing by $\epsilon^{d-2}$, and then send $\epsilon$ to 0, we obtain a putative excursion law. The resulting measure, call it $n$ would be the Itô excursion measure for the recurrent extension of $X^*$, if there were one. But $n$ satisfies

$$n[\zeta \in dt] = C_d \cdot t^{-d/2}, \quad t > 0,$$

where $\zeta$ is the excursion lifetime. In order that it be possible to string together such excursions to obtain a recurrent extension of $X^*$, it is necessary that $\int_0^\infty \min(t, 1) n[\zeta \in dt] < \infty$. This latter condition fails for $d \geq 4$. This would seem to indicate that at least in the radially symmetric case, a recurrent distorted Brownian motion that hits the origin is impossible for $d \geq 4$. We hope to explore possible connections between the analytic and probabilistic obstructions in the future.

4. **FUKUISHIMA’S DECOMPOSITION**

Fukushima’s decomposition for symmetric Markov processes is may be thought of an extension of the familiar semi-martingale decomposition, is valid even processes of the form $u(X_t) (u \in F)$ that are not semi-martingales. Note that the coordinate functions

$$f^i(x) := x_i, \quad x \in R^3, i = 1, 2, 3,$$

are in both $F$ and $F^0$. It follows from Proposition 6 of [21] that the Fukushima decomposition of these coordinate function relative to $(E^0, F^0)$ can be written as

$$X^0_t - X^0_0 = B_t - \int_0^t \gamma [X^0_s] + 1 |X^0_s| \cdot X^0_s ds, \quad t < T_0$$
where \( B \) is a 3-dimensional standard Brownian motion. Similarly, for the Dirichlet form \((\mathcal{E}, \mathcal{F})\) and the diffusion \( X \) there exists a unique additive functional of zero energy \( N^i \) for each \( i = 1, 2, 3 \), and a 3-dimensional standard Brownian motion \( B \) such that

\[
X_t - X_0 = B_t + N_t, \quad t \geq 0,
\]

(4.1)

where \( N_i = (N^1_i, N^2_i, N^3_i) \); see ?? in [27]. It follows from Lemma 5.4.4 of [27] that

**Lemma 4.1.** If \( t < T_0 \), then

\[
N_t = - \int_0^t \frac{\gamma|X_s| + 1}{|X_s|^2} X_s \, ds.
\]

(4.2)

On the other hand, recall that the radial parts of \( X \) and \( X^0 \) have the decompositions (8.30). It is a beautiful reflection from \((r_t)_{t \geq 0} \to (\bar{r}_t)_{t \geq 0}\). So the natural question is this: Is there an analogous expression relating \( X \) and \( X^0 \)? In other words does \( N \) have an expression similar to (4.1), obtained by adding another term with built out of the local time \((L^i_t)_{t \geq 0}\) of \( X \) at \( \{0\}\)?

An additive functional \( A \) is said to be of bounded variation if \( A_t(\omega) \) is of bounded variation in \( t \) on each compact subinterval of \([0, \zeta(\omega)]\) for every fixed \( \omega \) in the defining set of \( A \), where \( \zeta \) is the lifetime of \( X \). A continuous AF (CAF in abbreviation) \( A \) is of bounded variation if and only if \( A \) can be expressed as a difference of two PCAFs:

\[
A_t(\omega) = A^1_t(\omega) - A^2_t(\omega), \quad t < \zeta(\omega), A^1, A^2 \in \mathbb{A}^+,
\]

where \( \mathbb{A}^+ \) is the space of all the PCAFs. Let \( \mu_1 \) and \( \mu_2 \) be the Revuz measures of \( A^1 \) and \( A^2 \), then

\[
\mu_\Lambda := \mu_1 - \mu_2
\]

is the signed smooth measure associated with \( A \). For more details, see §5.4 of [27]. We say \( N \) in (4.1) is of bounded variation if \( N^i \) is of bounded variation for \( i = 1, 2, 3 \).

**Theorem 4.2.** The zero energy part \( N \) in (4.1) is not of bounded variation.

**Proof.** Arguing by contradiction, suppose that \( N \) is of bounded variation. It follows from Theorem 5.5.4 of [27] that for each \( i \) the signed smooth measure \( \mu_i \) of \( N^i \) satisfies

\[
\mathcal{E}(f^i, u) = -\langle \mu_i, u \rangle
\]

for all \( u \in \mathcal{F}_{b,F_k} \) where \( \{F_k\} \) is a generalized nest associated with \( \mu_i \); i.e., \( \mu_i(F_k) < \infty \) for all \( k \geq 1 \). Let \( F^i_k := F_k \cap \{ x : 1/n \leq |x| \leq n \} \) for each \( n \geq 1 \). Then \( F^i_k \) is compact. For all \( u \in C^\infty(R^3) \) with \( \text{supp} \, u \subset F^i_k \), clearly \( u \in \mathcal{F}_{b,F_k} \), and

\[
\langle \mu_i, u \rangle = -\mathcal{E}(f^i, u) = -\frac{1}{2} \int \frac{\partial u}{\partial x_i} \psi_\gamma^2(x) \, dx
\]

\[
= \int u(x) \frac{\partial \psi_\gamma^2}{\partial x_i}(x) \, dx
\]

\[
= -\int u(x) \frac{\gamma|x| + 1}{|x|} \frac{x_i}{|x|} \, m(dx).
\]

From this we deduce that

\[
\mu_i(dx) = -\frac{\gamma|x| + 1}{|x|} \frac{x_i}{|x|} m(dx)
\]

(4.3)

on each \( F^i_k \). It follows that (4.3) holds on \((\cup_{k \geq 1} F_k) \cap \{ x : |x| > 0 \}\). On the other hand since \((\cup_{k \geq 1} F_k)^c\) is \( \mathcal{E}\)-polar, we have \( \mu_i((\cup_{k \geq 1} F_k)^c) = m((\cup_{k \geq 1} F_k)^c) = 0 \). Thus (4.3) holds on \( \{ x : |x| > 0 \} \). Therefore there is a constant \( c_i \) such that

\[
\mu_i(dx) = -\frac{\gamma|x| + 1}{|x|} \frac{x_i}{|x|} m(dx) + c_i \delta_{\{0\}}.
\]

(4.4)
In particular \( -\frac{\gamma|x| + 1}{|x|} m(dx) \) is a signed smooth measure. Consequently,
\[
\frac{\gamma|x| + 1}{|x|} m(dx)
\]
is smooth. Since \(|x| \leq |x_1| + |x_2| + |x_3|\), it follows that
\[
\frac{\gamma|x| + 1}{|x|} m(dx)
\]
is also smooth. Then there exists a quasi-continuous and q.e. strictly positive function \(g\) such that
\[
\int g(x) \frac{\gamma|x| + 1}{|x|} m(dx) < \infty; \tag{4.5}
\]
see Thm. 4.22 in [20]. In particular, it follows from Proposition 3.1 that \(g(0) > 0\). Moreover, because \(\{0\}\) is not polar, \(g\) is finely continuous at 0. Thus, if we let \(B_\varepsilon := \{x : |x| \leq \varepsilon\}\) and \(T_\varepsilon\) the hitting time of \(B_\varepsilon^c\) by \(X\), then (as noted in [33])
\[
E^0(g(X_{T_\varepsilon})) \to g(0)
\]
as \(\varepsilon \to 0\), whereas \(X_{T_\varepsilon}\) is uniformly distributed on \(\partial B_\varepsilon := \{x : |x| = \varepsilon\}\) since \(X\) is rotation invariant. Thus
\[
\int_{S^2} g(\varepsilon u) \sigma(du) \to g(0)
\]
as \(\varepsilon \to 0\). Thus there is a constant \(\delta > 0\) such that when \(\varepsilon < \delta\),
\[
\int_{S^2} g(\varepsilon u) \sigma(du) > \frac{1}{2} g(0).
\]
It follows that
\[
\int g(x) \frac{\gamma|x| + 1}{|x|} m(dx) = 2\pi\gamma \int_0^\infty \frac{r}{r} e^{-2r} dr \int_{S^2} g(ru) \sigma(du)
\]
\[
\geq 2\pi\gamma \int_0^{\delta} \frac{r}{r} e^{-2r} dr \cdot \frac{1}{2} g(0)
\]
which is infinite, in violation of (4.5). \(\square\)

**Corollary 4.3.** The diffusion \(X\) associated with \((\mathcal{E}, \mathcal{F})\) is not a semi-martingale.

An interesting fact is that the first term in (4.4) is a signed smooth measure with respect to \((\mathcal{E}^0, \mathcal{F}^0)\) but not with respect to \((\mathcal{E}, \mathcal{F})\). The key to this phenomena is of course that \(\{0\}\) is not polar because \(\psi_{\gamma}\) explodes at 0. But by modifying \(\psi_{\gamma}\) near 0 we can obtain a sequence of nice semi-martingale distorted Brownian motions that are semi-martingales and that approximate \(X\) in a suitable sense.

Define \(\tilde{\psi}_{\gamma}^n(x) := \psi_{\gamma}(x)\) if \(|x| \geq 1/n\) and \(\tilde{\psi}_{\gamma}(1/n)\) if \(|x| < 1/n\). Then \(\tilde{\psi}_{\gamma}^n\) is a bounded function on \(\mathbb{R}^3\) and \(\nabla \tilde{\psi}_{\gamma}^n \in L^2(\mathbb{R}^3)\). Let
\[
\mathcal{F}^n := \{u \in L^2(\mathbb{R}^3, \psi_{\gamma}^n(x)^2 dx) : \nabla u \in L^2(\mathbb{R}^3, \psi_{\gamma}^n(x)^2 dx)\},
\]
\[
\mathcal{E}^n(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} \nabla u(x) \cdot \nabla v(x) \psi_{\gamma}^n(x)^2 dx, \quad u, v \in \mathcal{F}^n.
\]
It follows from Theorem 3.1 of [37] (see also [35]) that \((\mathcal{E}^n, \mathcal{F}^n)\) is a regular Dirichlet form on \(L^2(\mathbb{R}^3, \psi_{\gamma}^n(x)^2 dx)\) with core \(C_c^\infty(\mathbb{R}^3)\). Note that the associated diffusion \(X^n\) of \((\mathcal{E}^n, \mathcal{F}^n)\) has the following Fukushima decomposition with respect to the coordinate functions:
\[
X^n_{t} - X^n_0 = B_t - \int_0^t \frac{\gamma |X^n_s| + 1}{|X^n_s|^2} \cdot X^n_s \cdot 1_{(|X^n_s| \geq {\delta})} ds.
\]
Notice that \((\mathcal{E}^n, \mathcal{F}^n)\) and \((\mathcal{E}, \mathcal{F})\) are defined on different Hilbert spaces. The natural way to relate them is by \(h\)-transforms. Recall that \((P_t)_{t \geq 0}\), \((Q_t)_{t \geq 0}\) are the semigroups associated with \((\mathcal{E}, \mathcal{F})\)
and \( \mathcal{L}_\gamma \) respectively, and are related by (2.7). Denote the semigroup of \((\mathcal{E}^n, \mathcal{F}^n)\) by \((Q^n_t)_{t \geq 0}\).

Define a semigroup \((Q^n_t)_{t \geq 0}\) on \(L^2(\mathbb{R}^3)\) by

\[
Q^n_t u := e^{-\frac{t}{n^2}} \psi^n_0 \cdot P^n_t (u \cdot (\psi^n_0)^{-1}), \quad u \in L^2(\mathbb{R}^3), t \geq 0.
\]

Then \((\mathcal{E}^n, \mathcal{F}^n)\) is convergent to \((\mathcal{E}, \mathcal{F})\) in the following sense:

**Proposition 4.4.** There is a subsequence \(\{n_k\}\) such that \(Q^{n_k}_t\) is strongly convergent to \(Q_t\) on \(L^2(\mathbb{R}^3)\) as \(k \to \infty\) for all \(t \geq 0\).

The above proposition follows from Theorem 2.3 of [1].

References

[1] Albeverio, S., Høegh-Krohn, R., Streit, L.: Energy forms, Hamiltonians, and distorted Brownian paths, *J. Math. Physics*, 18 (1977) 907–917.
[2] Albeverio, S., Høegh-Krohn, R., Streit, L.: Regularization of Hamiltonians and processes, *J. Math. Physics*, 21 (1980) 1636–1642.
[3] Albeverio, S., Gesztesy, F., Karwowski, W., Streit, L.: On the connection between Schrödinger and Dirichlet forms, *Mathematics + physics*, Vol. 1, pp. 65–97, World Scientific, Singapore, 1985.
[4] Albeverio, S., Høegh-Krohn, R., Streit, L.: On the connection between Schrödinger and Dirichlet forms, *Z. Wahrsch. Verw. Gebiete*, 63 (1983) 230–258.
[5] Albeverio, S., Kusuoka, S., Streit, L.: Energy forms, Hamiltonians, and distorted Brownian paths, *J. Math. Physics*, 26 (1985) 135–162.
[6] Albeverio, S., Kusuoka, S., Streit, L.: Convergence of Dirichlet forms and associated Schrödinger operators, *J. Funct. Anal.*, 68 (1986) 130–148.
[7] Albeverio, S., Getoor, R.K.: On construction of Markov processes, *Ann. Probab.*, 125 (1984) 1368–1393.
[8] Albeverio, S., Gesztesy, F., Karwowski, W., Streit, L.: On the connection between Schrödinger and Dirichlet forms, *Z. Wahrsch. Verw. Gebiete*, 63 (1983) 907–917.
[9] Albeverio, S., Kusuoka, S., Streit, L.: Convergence of Dirichlet forms and associated Schrödinger operators, *J. Funct. Anal.*, 68 (1986) 130–148.
[10] Chen, Z.Q.: On reflected Dirichlet spaces, *Probab. Th. Rel. Fields*, 91 (1992) 135–162.
[11] Chen, Z.Q., Fukushima, M.: Symmetric Markov Processes, Time Change, and Boundary Theory, Princeton University Press, 2011.
[12] Chen, Z.Q., Fukushima, M.: One-point Reflection, *Stochastic Process. Appl.*, 125 (2015) 1368–1393.
[13] Cranston, M., Koralov, L., Molchanov, S., Vainberg, B.: A solvable model for homopolymers and self-similarity near the critical point, *Random Operators and Stochastic Equations*, 18 (2010) 73–95.
[14] Cranston, M., Koralov, L., Molchanov, S., Vainberg, B.: Continuous model for homopolymers, *J. Funct. Anal.*, 256 (2009) 2656–2696.
[15] Dabrowski, L., Grosse, H.: On nonlocal point interactions in one, two, and three dimensions, *J. Math. Physics*, 26 (1985) 2777–2780.
[16] Doob, J.L.: *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, 1984.
[17] Erickson, K.B.: Continuous extensions of skew product diffusions, *Probab. Th. Rel. Fields*, 85 (1990) 73–89.
[18] Fitzsimmons, P.J.: On the quasi-regularity of semi-Dirichlet forms, *Potential Anal.*, 15 (2001) 151–185.
[19] Fitzsimmons, P.J.: Absolute continuity of symmetric diffusions, *Ann. Probab.*, 20 (1992) 1484–1497.
[20] Fitzsimmons, P.J.: On the quasi-regularity of semi-Dirichlet forms, *Potential Anal.*, 15 (2001) 151–185.
[21] Fitzsimmons, P.J., Li, L.: On Fukushima’s decompositions of symmetric diffusions, *Potential Anal.*, 15 (2001) 151–185.
[22] Fukushima, M.: On absolute continuity of multidimensional symmetric diffusions. *In Functional Analysis in Markov Processes*, (pp. 146–176), Lect. Notes in Math. 923, Springer Berlin-Heidelberg, 1982.
[23] Fukushima, M.: Energy forms and diffusion processes, *Mathematics + physics*, Vol. 1, pp. 65–97, World Scientific, Singapore, 1985.
[24] Fukushima, M., Oshima, Y.: On the skew product of symmetric diffusion processes, *Forum Math.*, 1 (1989) 103–142.
[25] Fukushima, M., Tanaka, H.: Poisson point processes attached to symmetric diffusions, *Ann. Inst. H. Poincaré Probab. Statist.*, 41 (2005) 419–459.
[31] Ying, J.: Remarks on $h$-transform and drift, *Chinese Ann. Math.*, **19** (1998) 473–478.
[32] Lin, P.: Non-Friedrichs Self-adjoint extensions of the Laplacian in $\mathbb{R}^d$, [arXiv:1103.6089](https://arxiv.org/abs/1103.6089), 2011.
[33] Meyer, P.-A.: Une remarque sur la topologie fine, In *Séminaire de Probabilités XIX*, Lect. Notes in Math. **1123**, p. 176, Springer Berlin-Heidelberg, 1985.
[34] Mijatovic, A., Urusov, M.: Convergence of integral functionals of one-dimensional diffusions, *Electronic Comm. Probab.* **17** (2012), no. 61, 13 pages.
[35] Pardoux, E., Williams, R.J.: Symmetric reflected diffusions, *Ann. Inst. H. Poincaré Probab. Statist.*, **30** (1994) 13–62.
[36] Röckner, M., Zhang, T.-S.: Uniqueness of generalized Schrödinger operators and applications, *J. Funct. Anal.*, **105** (1992) 187–231.
[37] Rockner, M., Zhang, T.-S.: Uniqueness of Generalized Schrödinger Operators, II, *J. Funct. Anal.*, **119** (1994) 455–467.
[38] Rogers, L.C.G., Williams, D.: *Diffusions, Markov Processes and Martingales: Volume 2, Itô calculus*, Cambridge Univ. Press, 2000.
[39] Salisbury, T.S.: On the Itô excursion process, *Probab. Th. Rel. Fields*, **73** (1986) 319–350.
[40] Salisbury, T.S.: Construction of right processes from excursions, *Probab. Th. Rel. Fields*, **73** (1986) 351–367.
[41] Streit, L.: Energy forms: Schroedinger theory, processes, *Physics Rep.*, **77** (1981) 363–375.
[42] Takemura, T., Tomisaki, M.: Recurrence/transience criteria for skew product diffusion processes, *Proc. Japan Acad. Ser. A Math.Sci.*, **87** (2011) 119–122.
[43] Vuolle-Apiala, J.: Excursion theory for rotation invariant Markov processes, *Probab. Th. Rel. Fields*, **93** (1992) 153–158.

Department of Mathematics, University of California, San Diego, La Jolla, California 92093-0112

*E-mail address*: pfitzsim@ucsd.edu

Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.

*E-mail address*: liliping@amss.ac.cn