THE ABELIAN PROJECTION VERSUS THE HITCHIN FIBRATION OF $K(D)$ PAIRS IN FOUR-DIMENSIONAL QCD

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ABSTRACT

We point out that the concept of Abelian projection gives us a physical interpretation of the role that the Hitchin fibration of parabolic $K(D)$ pairs plays in the large-$N$ limit of four-dimensional QCD.

This physical interpretation furnishes also a simple criterium for the confinement of electric fluxes in the large-$N$ limit of QCD.

There is also an alternative, compatible interpretation, based on the QCD string.

April 1999
1 Introduction

Some years ago ’t Hooft introduced the concept of Abelian projection [1] into non-Abelian gauge theories, in order to explain the confinement of quarks in four-dimensional $QCD$ as a dual Meissner effect in a dual superconductor [2, 3].

The Abelian projection allows us, by a careful choice of the gauge, to describe the physical variables of a non-Abelian $SU(N)$ gauge theory, without scalar matter fields, as a set of electric charges and magnetic monopoles interacting via a residual $U(1)^{N-1}$ Abelian gauge coupling.

The occurrence of magnetic monopoles into a non-Abelian gauge theory without matter fields is perhaps the most crucial feature of the Abelian projection, that furnishes a precise understanding of the structure of the phases of non-Abelian gauge theories, according to the following alternatives [4].

If there is a mass gap, either the electric charge condenses in the vacuum (Higgs phase) or the magnetic charge does (confinement phase). If there is no mass gap, the electric and magnetic fluxes coexist (Coulomb phase).

Recently, in an apparently unrelated development [5], some mathematical control was gained over the large-$N$ limit of four-dimensional $QCD$, mapping, by means of a chain of changes of variables, the function space of the $QCD$ functional integral into an elliptic fibration of Hitchin bundles.

Hitchin bundles [6] are themselves a fibration of $U(1)$ bundles over spectral branched covers of a Riemann surface, that, in the case of [6], is a torus.

In this paper, we point out that the map in [5] is a version, in a perhaps global algebraic-geometric setting, of the concept of Abelian projection [1].

In fact, the branching points of the spectral cover are identified with the magnetic monopoles of the Abelian projection, the parabolic points of the cover with (topological) electric charges and the $U(1)$ gauge group on the cover with a global version (on the cover) of the $U(1)^{N-1}$ gauge group of the Abelian projection.

The identifications that we have just outlined provide a physical interpretation of the mathematical construction in [5]. Indeed it is precisely this physical interpretation that explains naturally why the functional integral, once it is expressed as a functional measure supported over the collective field of the Hitchin fibration, is dominated by a saddle-point condition in the large-$N$ limit.
On the other side, we may think that the mathematical proof, that the variables of the Abelian projection really capture the physics of four-dimensional $QCD$ in the large-$N$ limit, relies on the fact that those variables may be employed to dominate the functional integral in the large-$N$ limit.

The only qualitative feature, in the treatment in [5], that was not already present in the concept of the Abelian projection, is the occurrence of Riemann surfaces and it is due to the global algebraic-geometric nature of the methods in [5]. This, however, makes contact, at least qualitatively, with another long-standing conjecture about the $QCD$ confinement, the occurrence of string world sheets [7] and the string program [8].

Our last concluding remark is that the electric/magnetic alternative [4] and the physical interpretation based on the Abelian projection, applied in the mathematical framework of [5], give us a simple qualitative criterium to characterize the confinement phase of $QCD$ in the large-$N$ limit: confinement is equivalent to magnetic condensation, in absence of electric (parabolic) singularities of the spectral covers.

An alternative, compatible interpretation, based on the idea that $QCD$ is equivalent, in the large-$N$ limit, to a theory of strings [7, 8] is outlined in the following section. The rest of the paper is devoted to a technical explanation of the correspondence between the Abelian projection and the Hitchin fibration in four-dimensional $QCD$.

2 The Hitchin fibration as the Abelian projection in the gauge in which the Higgs current is a triangular matrix

The Abelian projection, according to [1], is really the choice of a gauge-fixing in such a way that, after the gauge-fixing, the theory is no longer locally invariant under $SU(N)$ but only under its Cartan subgroup $U(1)^{N-1}$. The important point about this projection is that it is defined strictly locally, that is, the gauge rotation $\Omega$ performed at each point in space-time to implement the gauge-fixing condition, does not depend on the values of the physical fields in other points of space-time. This then guarantees that all observables in the new gauge frame are still locally observable. There are no
propagating ghosts. But $\Omega$ is not completely defined. There is a subgroup, $U(1)^{N-1}$, of gauge rotations that may still be performed. And this is why the theory, after the Abelian projection, looks like a local $U(1)^{N-1}$ gauge theory. If one now tries to gauge-fix this remaining gauge freedom, one discovers that it cannot be done locally, without encountering apparent difficulties. But local gauge-fixing is not needed, since the residual gauge symmetry is the one of a familiar Abelian theory.

There may be, however, isolated points, where the local gauge-fixing condition has coinciding eigenvalues, where the gauge symmetry is not $U(1)^{N-1}$ but a larger group. Here singularities appear, the magnetic monopoles. So we see that, topologically, the full theory can only be topologically equivalent to the $U(1)^{N-1}$ gauge theory if the latter is augmented with monopole singularities where the $U(1)$ conservation laws for the vortices are broken down into the (less restrictive) conservation laws of the $SU(N)$ vortices.

When we try to gauge-fix completely, we hit upon the Dirac strings, whose end points are the magnetic monopoles.

In addition to the magnetic monopoles, in the $QCD$ case, the gauge-fixed theory contains also gluon and quark fields, that are charged with respect to the residual $U(1)^{N-1}$.

Therefore we have a set of electric charges and magnetic monopoles interacting via a residual $U(1)^{N-1}$ Abelian gauge coupling.

We now compare this description with the one that arises in [5], for the pure gauge theory without quark matter fields.

The functional integral for $QCD$ in [5] is defined in terms of the variables $(A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})$, obtained by means of a partial duality transformation from $(A_z, A_{\bar{z}}, A_u, A_{\bar{u}})$, where $(z, \bar{z}, u, \bar{u})$ are the complex coordinates on the product of two two-dimensional tori, over which the theory is defined.

$(A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})$ define the coordinates of an elliptic fibration of $T^*A$, the cotangent bundle of unitary connections on the $(z, \bar{z})$ torus, whose base is the $(u, \bar{u})$ torus.

$\Psi_z$ transforms as a field strength by gauge transformations and it is a non-hermitian matrix.

Following Hitchin [6], the gauge is chosen in which $\Psi_z$ is a triangular matrix, for example lower triangular, that leaves a $U(1)^{N-1}$ residual gauge freedom as in the Abelian projection.

The points in space-time where $\Psi_z$ has a pair of coinciding eigenvalues, correspond to monopoles. In addition there are the charged components of
\((A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})\). We have thus a set of charges and monopoles with a residual \(U(1)^{N-1}\), according to the Abelian projection.

In \([5]\), however, it is found a dense set in the functional integral over (the elliptic fibration of) \(T^*A\), with the property that the quotient by the action of the gauge group exists as a Hausdorff (separable) manifold.

This dense set is defined in \([5]\) as the set of pairs \((A, \Psi)\) that are solutions of the following differential equations (elliptically fibered over the \((u, \bar{u})\) torus):

\[
F_A - i\Psi \wedge \Psi = \frac{1}{|D|} \sum_p \mu_p^0 \delta_p dz \wedge d\bar{z}
\]

\[
\partial_A \psi = \frac{1}{|D|} \sum_p \mu_p \delta_p dz \wedge d\bar{z}
\]

\[
\partial_A \bar{\psi} = \frac{1}{|D|} \sum_p \bar{\mu}_p \delta_p d\bar{z} \wedge dz
\]

where \(\delta_p\) is the two-dimensional delta-function localized at \(z_p\) and \((\mu_p^0, \mu_p, \bar{\mu}_p)\) are the set of levels for the moment maps. The moment maps are the three Hamiltonian densities generating gauge transformations on \(T^*A\) that appear in the left hand sides of Eq.(1) \([9]\).

\(\mu_p^0\) are hermitian traceless matrices, and \(\mu_p\) are matrices in the complexification of the Lie algebra of \(SU(N)\), that determine the residues of the poles the Higgs current \(\Psi. \psi\) and \(\bar{\psi}\) are the \(z\) and \(\bar{z}\) components of the one-form \(\Psi\).

Eq.(1) defines a dense stratification of the functional integral over \(T^*A\) because the set of levels is dense everywhere in function space, in the sense of the distributions, as the divisor \(D\) gets larger and larger.

Eq.(1) defines the data of parabolic \(K(D)\) pairs \([10]\) on a torus valued in the Lie algebra of the complexification of \(SU(N)\): a holomorphic connection \(\partial_A\) of a holomorphic bundle, \(E\), with a parabolic structure and a parabolic morphism \(\psi\) of the parabolic bundle. The parabolic structure at a point \(p\) \([11, 10]\) consists in the choice of a set of ordered weights, that are positive real numbers modulo 1, and a flag structure, that is a collection of nested subspaces \(\mathcal{F}_\infty \subset \mathcal{F}_\epsilon \subset \ldots \mathcal{F}_j\) labelled by the weights \(\alpha_1 \geq \alpha_2 \geq \ldots \alpha_k\), with the associated multiplicities defined as: \(m_{i+1} = \dim \mathcal{F}_{i+\infty} - \dim \mathcal{F}_i\). A parabolic morphism, \(\phi\), is a holomorphic map between parabolic bundles, \(E^1, E^2\), that preserves the parabolic flag structure at each parabolic point \(p\) in the sense that \(\alpha^1_i > \alpha^2_j\) implies \(\phi(\mathcal{F}_i^\infty) \subset \phi(\mathcal{F}_j^\epsilon)\). We should now explain
how a parabolic structure arises from Eq.(1) and how it follows that $\psi$ is a parabolic morphism with respect to the given parabolic structure. Though we are going to choose the gauge in which $\psi$ is a lower triangular matrix in most of this paper, we start at an intermediate stage with a gauge in which $\mu_0^p$ is diagonal. The eigenvalues of $\mu_0^p$ modulo $2\pi$ and divided by $2\pi$ define the parabolic weights. Their multiplicities will turn out to be the multiplicities of the yet to be defined flag structure.

Fixed $\mu_0^p$ and $\mu_p$ in Eq.(1), let $(e_k)$ be an orthonormal basis of the eigenvectors of $\mu_0^p$ in decreasing order. This basis is not necessarily unique if the eigenvalues have non-trivial multiplicities. However the corresponding flag structure will not be affected by this lack of uniqueness. Let $g$ be the gauge transformation that puts $\mu$ and $\psi$ into lower triangular form. Let $(ge_k)$ be the transformed basis and let $F$ be the flag obtained by taking the unions of subspaces generated by the vectors in the transformed basis that are the images of eigenvectors of the ordered eigenvalues with the given multiplicity in such a way that the multiplicities of the resulting flag are the same as the multiplicities of the eigenvalues. In addition, by construction, $\psi$ is a parabolic morphism with respect to the flag since it is holomorphic and lower triangular in the basis $(ge_k)$.

We have thus the data of a parabolic $K(D)$ pair from Eq.(1).

There is also a representation theoretic interpretation of Eq.(1).

The three equations for the moment maps are equivalent to a vanishing curvature condition for the non-hermitian connection one-form $B = A + i \Psi$ plus a harmonic condition for $\psi$ away from the parabolic divisor [12].

Therefore the set of solution of Eq.(1) can be figured out essentially as a collection of monodromies around the points of the divisor with values in the complexified gauge group, that form a representation of the fundamental group of the torus with the points of the parabolic divisor deleted.

’t Hooft description of the Abelian projection previously outlined, applies to $T^*A$ and to its dense subset defined by Eq.(1) a fortiori. In addition, we have just shown that there is an embedding of the solutions of Eq.(1) into the parabolic $K(D)$ pairs.

However, on the parabolic $K(D)$ pairs, ’t Hooft concept of Abelian projection can be carried to its extreme consequences.

Indeed, in the global algebraic-geometric framework of the Hitchin fibration [3, 10] of parabolic $K(D)$ pairs, it is preferable to concentrate ourselves on the first eigenvalue and the first eigenstate of the lower triangular matrix.
Ψ_z, since all the information of the original parabolic bundle, up to gauge equivalence, can be reconstructed from these only data [3].

The first eigenvalue defines a spectral covering, that is a branched cover of the two-torus. The eigenspace defines a section of a line bundle, that determines a $U(1)$ connection on the cover of the torus, instead of the $U(1)^{N-1}$ bundle on the torus of the Abelian projection.

The $U(1)$ connection on the cover, $a$, and the eigenvalue, $\lambda$, of the Higgs current can be considered as coordinates of the cotangent bundle of unitary $U(1)$ connections on the cover, or as parabolic $K(D)$ pairs $(a, \lambda)$ on the cover, valued in the complexification of the Lie algebra of $U(1)$.

The system is now completely abelianized. Correspondingly, not only the magnetic charges, but also the electric ones can occur only as gauge invariant topological configurations.

The points in space-time where $\Psi_z$ has a pair of coinciding eigenvalues, that in the Abelian projection correspond to monopoles, are here, according to Hitchin, simple branching points of the spectral covers, defined by means of the characteristic equation:

$$\text{Det}(\lambda I - \Psi_z) = 0,$$

in which the coordinates $(u, \bar{u})$ are kept fixed.

All the other branching points can be obtained by collision of these simple branching points, in the same way monopoles can in the Abelian projection. The branching points are the end points of string cuts on the Riemann surfaces, the Dirac strings of the Abelian projection.

These Riemann surfaces, the only additional global ingredient with respect to the Abelian projection, are interpreted as the world sheets of strings made by electric flux lines.

A closed string of electric flux is represented by a Wilson loop of the $U(1)$ connection $a$ on the cover, along a non-trivial generator of the fundamental group of the surface.

In addition, the Riemann surfaces, defined by the spectral equation, may posses parabolic points, associated to poles of the eigenvalues of the Higgs current $\Psi_z$, whose origin is in the parabolic singularities of the original $su_c(N)$-valued $K(D)$ pair, which may be reflected into a parabolic structure for the $u_c(1)$-valued $K(D)$ pair on the cover.

These poles, together with the ones of the $U(1)$ connection, are interpreted as electric charges. Indeed it is not difficult to see that they are electric sources,
that appear where a boundary-electric loop shrinks to a point. Therefore, the electric charges occur here as topological objects associated to the parabolic degree \([11]\) of the \(u_c(1)\)-valued \(K(D)\) pair. On the other side, magnetic topological quantum numbers are associated, as usual, to the ordinary degree of the \(U(1)\) bundle.

We should mention however that a subtlety arises in our interpretation of the Hitchin fibration in terms of the Abelian projection. As we mentioned in the first part of this section, in the Abelian projection the gauge-fixing condition leaves a residual non-Abelian gauge symmetry where a magnetic monopole occurs. This is essentially due to the fact that ‘t Hooft chooses to diagonalize a hermitian functional of the fields. On the contrary, in the case of the dense set defined by Eq.(1), since \(\psi\) is a non-hermitian matrix, it can only be put in triangular form. This gauge-fixing does not leave in general a residual compact non-Abelian gauge symmetry even when the eigenvalues coincide. However this difficulty can be resolved in the following way, anticipating somehow some of the conclusions of this paper and the result of \([13]\).

Let us require for the moment that the levels of the non-hermitian moment maps be nilpotent. Since these are only \(N\) conditions at each parabolic point they do not modify essentially the entropy of the functional integration in the large-\(N\) limit. The true physical meaning of this choice has to do with confinement and it is explained in \([13]\). If the residues of the Higgs field are nilpotent, Eq.(1) can be interpreted as the vanishing condition for the moment maps of the action of the compact \(SU(N)\) gauge group on the pair \((A, \Psi)\) and on the cotangent space of coadjoint orbits \([14]\):

\[
F_A - i \Psi \wedge \Psi - \frac{1}{|D|} \sum_p \mu_p^0 \delta_p i dz \wedge d\bar{z} = 0
\]

\[
\bar{\partial}_A \psi - \frac{1}{|D|} \sum_p n_p \delta_p dz \wedge d\bar{z} = 0
\]

\[
\partial_A \bar{\psi} - \frac{1}{|D|} \sum_p \bar{n}_p \delta_p d\bar{z} \wedge dz = 0
\]

In addition the quotient under the action of the compact gauge group is hyper-Kahler \([10]\). By a general result of Hitchin, Karlhede, Lindström and Roček \([15]\), the hyper-Kahler quotient under the action of the compact gauge group in Eq.(3) is the same as the quotient defined by the non-hermitian
moment maps:

$$\bar{\partial}_A \psi - \frac{1}{|D|} \sum_p n_p \delta_p d\bar{z} \wedge dz = 0$$

$$\partial_A \bar{\psi} - \frac{1}{|D|} \sum_p \bar{n}_p \delta_p dz \wedge d\bar{z} = 0$$  (4)

under the action of the complexification of the gauge group. We can therefore impose a gauge condition compatible with the compact action in Eq.(3) or a gauge condition compatible with the action of the complexified group in Eq.(4) getting the same moduli space. In the second case we choose the gauge in which $\psi$ is diagonal. This condition becomes singular where two or more eigenvalues coincide. In fact it cannot be extended continuously to the points where the eigenvalues coincide. There it can only be required that $\Psi_z$ be a triangular matrix. However this condition leaves now a residual non-Abelian gauge symmetry in the complexification of the gauge group: the freedom of making triangular gauge transformations, thus confirming our analogy with 't Hooft definition of magnetic monopoles.

To summarize, the ingredients of the Hitchin fibration of the $\mathfrak{su}_c(N)$-valued $K(D)$ pairs, are the branching points, that are interpreted as magnetic monopoles, and the $U(1)$ monodromies around closed loops, that are interpreted as electric lines. In addition, the ordinary degree of the $U(1)$ bundle is interpreted as a (topological) magnetic charge, while the parabolic degree of the $U(1)$ bundle is interpreted as a (topological) electric charge.

The difference here, with the letter but not with the spirit of the Abelian projection, is that the system has been completely abelianized, so that both the magnetic and the electric charges are topological. We are thus given a set of charges and monopoles with a $U(1)$ gauge group on the covering, in analogy with the Abelian projection.

We call this description a complete Abelian projection.

The string interpretation is as follows. The spectral covers are the world sheets of strings, made by the electric flux lines. The confinement condition is equivalent to requiring that only closed string world sheet occur, since confinement requires that the flux lines can never break in absence of quarks.

If the spectral covers posses parabolic points, the same as electric charges in the complete Abelian projection, they are, topologically, Riemann surfaces with boundaries at infinity.
For example a sphere with two parabolic points is a topological cylinder. But a cylinder can occur in vacuum string world sheets (we are describing the contributions to the partition function, the vacuum to vacuum amplitude indeed) only if open strings propagate. In fact, a closed string that propagates through the torus breaks into an open one at the parabolic points, since the parabolic points do not belong to the world sheet.

On the contrary, when a closed string meets a branching point, for example in a once-branched double cover of a torus, the closed string is pinched into another closed string with the form of a double loop intersecting at the (simple) branching point. Notice also that the branching points do belong to the world sheet. Thus, the string picture is consistent with the interpretation of branching points as magnetic charges, where the string electric line can self-intersect but not break, and of parabolic points as electric charges, where closed string break into open strings with the parabolic points as boundaries.

3 Conclusions

Our conclusion is that the concept of Abelian projection in \([1]\) furnishes a physical interpretation of the structures that appear in the Hitchin fibration of \(K(D)\) pairs, as it is embedded in the \(QCD\) functional integral in \([3]\). In addition, there is a complementary consistent string interpretation. The most relevant consequence of these interpretations is a criterium for electric confinement in the framework of \([3]\), that is the usual criterium of magnetic condensation of \([1]\). Therefore, if \(QCD\) confines the electric charge, the functional measure must be localized, in the large-\(N\) limit, on those parabolic \(K(D)\) pairs, whose image through the Hitchin map, contains monopoles but no charges, that is, in geometric language, on those spectral covers that are arbitrarily branched, but that do not posses a parabolic divisor. In turn, this is equivalent to the condition that only spectral covers spanned by closed strings occur as configurations in the vacuum to vacuum amplitude. It is amusing to notice that this condition is satisfied by the string of two-dimensional \(QCD\) in the large-\(N\) limit \([4]\).
4 Acknowledgements

We would like to thank Gerard ’t Hooft for several clarifying remarks on the Abelian projection.

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