OBSTRUCTIONS FOR SUBGROUPS OF THOMPSON’S
GROUP V

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Abstract. Thompson’s group $V$ has a rich variety of subgroups, contain-
ing all finite groups, all finitely generated free groups and all fini-
tely generated abelian groups, the finitary permutation group of a countable
set, as well as many wreath products and other families of groups. Here,
we describe some obstructions for a given group to be a subgroup of $V$.

Thompson constructed a finitely presented group now known as $V$ as an
early example of a finitely presented infinite simple group. The group $V$
contains a remarkable variety of subgroups, such as the finitary infinite per-
mutation group $S_{\infty}$, and hence all (countable locally) finite groups, fini-
tely generated free groups, finitely generated abelian groups, Houghton’s groups,
copies of Thompson’s groups $F$, $T$ and $V$, and many of their generalizations,
such as the groups $G_{n,r}$ constructed by Higman [9]. Moreover, the class of
subgroups of $V$ is closed under direct products and restricted wreath products
with finite or infinite cyclic top group.

In this short survey, we summarize the development of properties of $V$
focusing on those which prohibit various groups from occurring as subgroups
of $V$.

Thompson’s group $V$ has many descriptions. Here, we simply recall that
$V$ is the group of right-continuous bijections from the unit interval $[0,1]$ to
itself, which map dyadic rational numbers to dyadic rational numbers, which
are differentiable except at finitely many dyadic rational numbers, and with
slopes, when defined, integer powers of 2. The elements of this group can
be described by reduced tree pair diagrams of the type $(S,T,\pi)$ where $\pi$ is a
bijection between the leaves of the two finite rooted binary trees $S$ and $T$.

Higman [9] gave a different description of $V$, which he denoted as $G_{2,1}$ in
a family of groups generalizing $V$.

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1. Obstructions

Higman [9] described several important properties of $V$ which can serve as obstructions to subgroups occurring in $V$.

**Theorem 1.1** ([9]). An element of infinite order in $V$ has only finitely many roots.

This prevents all Baumslag-Solitar groups $B_{m,n} = \langle a, b \mid a^n b = b a^m \rangle$ from occurring as subgroups of $V$, if $m$ properly divides $n$; see [13].

**Theorem 1.2** ([9]). Torsion free abelian subgroups of $V$ are free abelian, and their centralizers have finite index in their normalizers in $V$.

This prevents $GL_n(\mathbb{Z})$ from occurring as a subgroup of $V$ for $n \geq 2$.

A group is torsion locally finite if every torsion subgroup is locally finite. That is, if every finitely generated torsion subgroup is finite. Röver [12] showed

**Theorem 1.3** ([12]). Thompson’s group $V$ is torsion locally finite.

This rules out many branch groups from occurring as subgroups of $V$, including the Grigorchuk groups of intermediate growth [7] and the Gupta-Sidki groups [8]. It also rules out Burnside groups.

Holt and Röver [10] showed that $V$ has indexed co-word problem.

**Theorem 1.4** ([10]). The set of words (over an arbitrary but fixed finite generating set) which do not represent the identity in $V$ is an indexed language, and hence can be recognized by a nested-stack automaton.

This property is not easy to verify, however. But it is inherited by finitely generated subgroups (see [10]), and hence groups which do not have an indexed co-word problem cannot occur as a subgroup of $V$.

Lehnert and Schweitzer [11] improved this result.

**Theorem 1.5** ([11]). The set of words (over an arbitrary but fixed finite generating set) which do not represent the identity in $V$ is a context-free language, and hence can be recognized by a pushdown automaton.

Again, this property is inherited by finitely generated subgroups, but the condition is still not easy to verify.

More recently, Bleak and Salaza-Díaz [4] and subsequently Corwin [6], using similar techniques showed

**Theorem 1.6** ([4, 6]). Neither the free product $\mathbb{Z} * \mathbb{Z}^2$ nor the standard restricted wreath product $\mathbb{Z} \wr \mathbb{Z}^2$ with $\mathbb{Z}^2$ as top group are subgroups of $V$.

One theorem of Higman [9] together with a metric estimate of Birget [1] gives another obstruction.

**Theorem 1.7** ([9]). For any element $v$ of infinite order in $V$, there is a power $v^n$ such that for the reduced tree pair diagram $(S, T, \pi)$ for $v^n$, there is a leaf $i$ in the source tree $S$ which is paired with a leaf $j$ in the target tree $T$ so that $j$ is a child of $i$. 
Theorem 1.8 ([1]). For any finite generating set of $V$, there are constants $C$ and $C'$ such that word length $|v|$ of an element of $V$ with respect to that generating set satisfies $Cn \leq |v| \leq C'n \log n$ where $n$ is the size of the reduced tree pair diagram representing $v$.

Since the powers of $v^n$ will have length thus growing linearly, these two theorems give as a consequence the following.

Theorem 1.9. Cyclic subgroups of $V$ are undistorted.

We note that this argument applies as well to generalizations of $V$ where there is a linear lower bound on word length in terms of the number of carets, such as braided versions of $V$ [5].

This last theorem has an obvious corollary.

Corollary 1.10. If a group embeds in $V$, its cyclic subgroups must be undistorted.

The reason for this is that in a chain of subgroups $G \supset H \supset K$ the distortion of $K$ in $H$ cannot be larger than the distortion of $K$ in $G$.

We note that Bleak, Bowman, Gordon, Graham, Hughes, Matucci and J. Sapir [3] used Brin’s methods of revealing pairs for elements of $V$ to show that cyclic subgroups of $V$ are undistorted.

This result excludes all Baumslag-Solitar groups with $|n| \neq |m|$, as these have distorted cyclic subgroups. It also rules out nilpotent groups which are not virtually abelian. An alternative argument excluding the Baumslag-Solitar groups is due to Bleak, Matucci and Neunhöffer [2].

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