**Atomic Filter: a Weak Form of Shift Operator for Graph Signals**

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**Abstract**

The shift operation plays a crucial role in the classical signal processing. It is the generator of all the filters and the basic operation for time-frequency analysis, such as windowed Fourier transform and wavelet transform. With the rapid development of internet technology and big data science, a large amount of data are expressed as signals defined on graphs. In order to establish the theory of filtering, windowed Fourier transform and wavelet transform in the setting of graph signals, we need to extend the shift operation of classical signals to graph signals.

It is a fundamental problem since the vertex set of a graph is usually not a vector space and the addition operation cannot be defined on the vertex set of the graph. In this paper, based on our understanding on the core role of shift operation in classical signal processing we propose the concept of atomic filters, which can be viewed as a weak form of the shift operator for graph signals. Then, we study the conditions such that an atomic filter is norm-preserving, periodic, or real-preserving. The property of real-preserving holds naturally in the classical signal processing, but no the research has been reported on this topic in the graph signal setting. With these conditions we propose the concept of normal atomic filters for graph signals, which degenerates into the classical shift operator under mild conditions if the graph is circulant. Typical examples of graphs that have or have not normal atomic filters are given. Finally, as an application, atomic filters are utilized to construct time-frequency atoms which constitute a frame of the graph signal space.

**Keywords:**  
Graph signal processing, graph shift operators, graph filters, atomic filters, windowed Fourier transform

**1. Introduction**

Graphs provide a natural representation for data in many applications, such as social networks, web information analysis, sensor networks and machine learning \cite{1,2,3}. Graph signals are functions defined on the vertices of graphs. To process such signals, one needs to extend the well-developed theory of classical signal processing to graph signals. There have been a lot of researches on graph signal processing, including graph shift operators \cite{4,5,6,7,8,9,10,11,12,13}, graph filters \cite{14,15,16,17,18}, graph Fourier transforms \cite{19,20,21}, windowed graph Fourier transforms \cite{22}, graph wavelets \cite{23,24,25,26,27}, graph signal sampling \cite{28,29,30,31,32,33,34,35,36,37}, multiscale analysis \cite{38,39,40}, and approximation theory for graph signals \cite{41,42,43}.

The shift operation play a crucial role in the classical signal processing. It is the generator of filters and the basic operation of defining windowed Fourier transform and wavelet transform \cite{44,45}. It is also used to define the moduli of smoothness of functions \cite{46}. However, the definition of the classical shift operator $S x(t) := x(t - h)$ cannot be formally extended to the graph signal setting since the vertex set of a graph is usually not a vector space, which makes it impossible to perform addition operation $x + h$ between the vertices of the graph. It is a fundamental problem of graph signal processing, both in theory and application. In recent years, from different perspectives of classical shift operation, several different definitions for graph shift operators have been proposed. D. I. Shuman \textit{et al.} presented a definition via the generalized convolution with a delta function centered at a given vertex \cite{4}. A. Sandryhaila \textit{et al.} utilized the adjacency matrix of the graph as the shift operator \cite{5}. In \cite{6,7,8}, B. Girault \textit{et al.} defined a norm-preserving graph shift operator depending on the spectrum of the graph Laplacian matrix. Later in \cite{9}, A. Gavili \textit{et al.} defined a set of norm-preserving graph shift operators flexible to accommodate desired properties. In \cite{10,11}, N. Grelier \textit{et al.} proposed a definition relying on neighborhood preserving properties. In \cite{12}, B. S. Dees \textit{et al.} employed the maximum entropy principle to define a general class of shift operators for random signals on a graph. To define the modulus of smoothness of graph functions, I. Z. Pesenson \textit{et al.} defined a family of shift operators by using the Schrödinger's semigroup of operators generated by the graph Laplacian $L$ in \cite{13}. Inspired by this idea of this definition, a similar definition was proposed by C. Huang \textit{et al.}

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in [13].

Understanding what role the shift operation plays in signal processing and what basic properties it must have will help us to establish the theory of shift for graph signals, which will be essential rather than formal. In this paper, we systematically examine the role of shift operation in the classical signal processing and summarize the following basic properties of the shift operator:

1. It is a permutation of the original signal;
2. It is norm-preserving;
3. It is smoothness-preserving;
4. It is periodic;
5. It is real-preserving;
6. It is a filter;
7. Any filter can be expressed as a polynomial of the shift operator.
8. It is time-invariant.

On this basis, we propose the concept of atomic filters in the setting of graph signals, which can be viewed as a weak form of the shift operator for graph signals. Then we study the characterization and implications of the above properties (1)–(8), and introduce the concept of the so-called normal atomic filters. It is shown that a normal atomic filter degenerates into the classical shift operator under mild conditions if the graph is circulant. Since not all graphs have normal atomic filters, typical examples of graphs that have or have not normal atomic filters are discussed. Finally, as an application, atomic filters are utilized to construct time-frequency atoms which constitute a frame of the graph signal space.

Throughout this paper we use the following notations and terminologies: Matrices and vectors are represented by uppercase and lowercase boldface letters, respectively. The entries of a matrix $A$ and lowercase boldface letters, respectively. The entries of a vector $a$ are denoted as $a(i,j)$, and the entries of a vector $x$ are denoted as $x(n)$ or $x$. Transpose and Hermitian (conjugate transpose) operations are represented by $(\cdot)^T$ and $(\cdot)^*$, respectively. For a vector $a \in \mathbb{C}^n$, the notation $\text{diag}(a)$ represents a diagonal matrix with entries of the vector $a$ along the main diagonal. For $x, y \in \mathbb{C}^n$, their inner product is defined as $(x,y) := y^*x$, and $\|x\|_2 := \sqrt{x^*x}$ is the Euclidean norm of $x$. For positive integers $m,n$, the notation $\delta_{m,n}$ is defined as $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$.

2. The Role of Shift Operator in the Classical Signal Processing

2.1. Shift Operator in the Classical Signal Processing

Let us first briefly review the concepts of shift operators in the classical signal processing. Without losing generality, we consider the discrete periodic signals. A discrete signal $\{x_n\}_{n \in \mathbb{Z}}$ is called periodic with period $N$ if $x_{n+N} = x_n$ holds for any $n \in \mathbb{Z}$. Then the shift operator $S$ is defined as

$$Sx_n := x_{n-1}, \quad n \in \mathbb{Z}. \quad (1)$$

It is easy to see that this signal can be expressed as an $N$-dimensional column vector $x := (x_1, \cdots, x_N)^T$ and the shift operator $S$ can be expressed as the following circulant matrix:

$$S = [e_2, \cdots, e_N, e_1]. \quad (2)$$

where $I_N = [e_1, \cdots, e_N]$ is the identity matrix of order $N$. For notational simplicity, we shall use in this paper the same notation $x$ for the signal $\{x_n\}_{n \in \mathbb{Z}}$ and the vector $x$; the notation $S$ for the shift operator (1) and the matrix (2); the context should make the distinction clear.

According to (1) or (2), it is easy to see that $S$ is a downshift permutation operator which pushes the components of a vector down one notch with wraparound. Hence, it possesses the following basic properties:

1. $Sx$ is a permutation of $x$, as described above.
2. Norm-preserving property: $\|Sx\|_2 = \|x\|_2$;
3. Smoothness-preserving: $\sigma(Sx) = \sigma(x)$ holds for

$$\sigma(x) := \sum_{i=1}^N (|x_i - x_{i-1}|^2 + |x_i - x_{i+1}|^2).$$

4. Periodicity: $S^N x = x$;
5. Real-preserving property: If $x$ is real-valued, then $Sx$ is also real-valued.

Similarly, we have 2-dimensional shift operators:

$$(S_1x)_{n,m} := x_{n-1,m},$$
$$(S_2x)_{n,m} := x_{n,m-1}, \quad n, m \in \mathbb{Z}. \quad (3)$$

It is easy to verify that the three shifts satisfy the above conditions (1)–(5) for

$$\sigma(x) := \sum_{i,j=1}^N (|x_{i,j} - x_{i-1,j}|^2 + |x_{i,j} - x_{i+1,j}|^2 + |x_{i,j} - x_{i,j-1}|^2 + |x_{i,j} - x_{i,j+1}|^2).$$

The above properties or Definitions (1) and (3) are not convenient to be used to extend the shift operators to graph signals. As discussed in the previous section, to extend the concept of shift operator to graph signals, we need to examine systematically the crucial role of the shift operator in the classical signal processing.

2.2. As a Generator of Filters

In the classical signal processing, the discrete Fourier transform and the inverse discrete Fourier transform of $x$ are respectively defined as

$$\hat{x} := U^{-1}x, \quad x = U\hat{x},$$

where $U = (u_1, \cdots, u_N)$ is the matrix of discrete Fourier basis, whose $k$-th column is given by

$$u_k := \frac{1}{\sqrt{N}}(1, \omega^{k-1}, \omega^{2(k-1)}, \cdots, \omega^{(N-1)(k-1)})^T, \quad (4)$$

where $\omega := \exp(2\pi i/N)$.
Given a vector \(\mathbf{h} := (h_0, \cdots, h_{N-1})^T \in \mathbb{C}^N\), a filter corresponding to \(\mathbf{h}\) is defined by
\[
(\mathbf{S}\mathbf{x})(k) := \hat{x}(k)\mathbf{h}(k), \quad k = 1, \cdots, N,
\]
i.e.,
\[
(\mathbf{S}\mathbf{x})(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} h_k x_{n-k} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} h_k (\mathbf{S}^k \mathbf{x})(n),
\]
for \(n = 1, \cdots, N\), or equivalently,
\[
\mathbf{S} = \mathbf{E} \mathbf{H} \mathbf{E}^T,
\]
where \(\mathbf{S}\) is the shift operator defined by (2). Eq. (6) shows that the shift operator is a generator of any filters in the sense that \(S\).

A linear operator \(H\) is a filter if and only if it is commutative with \(\mathbf{S}\), i.e., \(\mathbf{HS} = \mathbf{SH}\). This equality is called the time-invariance of \(H\). It means that \(\mathbf{H}(t-t_0) = (\mathbf{H}(t) - t_0)\), in other words, the time-shift by \(t_0\) of the input signal creates the same time-shift by \(t_0\) at the output. As proved in [45, §2.1], the time-invariance is also a sufficient condition for a linear operator to be a filter. Thus, the shift operator serves as a descriptor of filters in the following sense:

(8) A linear operator \(H\) is a filter if and only if it is commutative with \(\mathbf{S}\), i.e., (7) holds.

3. Atomic Filters for Graph Signals

3.1. Graph Fourier Transforms and Graph Filters

In this paper, we consider connected, weighted, and undirected graphs. Let \(V = \{v_1, \cdots, v_N\}\) be the set of vertices of a graph \(\mathcal{G}\), and \(W \in \mathbb{R}^{N \times N}\) be the weighted adjacency matrix with its entry \(w_{ij}\) the nonnegative weight of the edge between the vertices \(v_i, v_j\). A graph signal is a function defined on \(V\) and can be expressed as a vector \(x \in \mathbb{C}^N\), whose \(n\)-th component represents the function value at the \(n\)-th vertex.

The Laplacian matrix is defined by \(\mathbf{L} := \mathbf{D} - \mathbf{W}\), where \(\mathbf{D}\) is the degree matrix \(\text{diag}(d_1, \cdots, d_N)\) with \(d_i := \sum_{j=1}^{N} w_{ij}\). By the spectral decomposition we have that \(\mathbf{L} = \mathbf{U} \Lambda \mathbf{U}^{-1}\), where \(\mathbf{U} := (u_1, \cdots, u_N)\), \(\Lambda := \text{diag}(\lambda_1, \cdots, \lambda_N)\), with \(0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N\) being all the eigenvalues of \(\mathbf{L}\) and \(u_1, \cdots, u_N\) being a set of eigenvectors, which constitute an orthonormal basis of \(\mathbb{C}^N\) and \(u_i = \mathbf{I}/\sqrt{N}\), where \(\mathbf{I} \in \mathbb{R}^{N}\) is an \(N\)-dimensional vector of ones. To accommodate desired properties for graph filters, we allow the eigenvectors \(u_2, \cdots, u_N\) to be complex vectors.

For a graph signal \(x \in \mathbb{C}^N\), its graph Fourier transform and inverse graph Fourier transform are respectively defined as
\[
\hat{x} := \mathbf{U}^{-1} x, \quad x = \mathbf{U} \hat{x}.
\]

Hereafter, we call \([u_k]_N^{N}\) the graph Fourier basis and \(\mathbf{U}\) the matrix of graph Fourier basis.

Using the graph Fourier transform, the graph filter can be defined in a way similar to Eq. (5). Given a vector \(\mathbf{h} \in \mathbb{C}^N\), the corresponding graph filter \(\mathbf{H}\) is defined as a linear operator satisfying [1][9]:
\[
(\mathbf{S}\mathbf{x})(k) := \hat{x}(k)\mathbf{h}(k), \quad k = 1, \cdots, N,
\]
for any graph signal \(x \in \mathbb{C}^N\).

It is easy to see that Eq. (9) holds for any \(x \in \mathbb{C}^N\), if and only if \(\mathbf{H} = \mathbf{U} \text{diag}(\mathbf{h}) \mathbf{U}^{-1}\). Thus, any graph filter can be expressed as the following form
\[
\mathbf{H}_a := \mathbf{U} \text{diag}(\mathbf{a}) \mathbf{U}^{-1},
\]
where the vector \(\mathbf{a} \in \mathbb{C}^N\) is called the frequency response of the filter [2][9][13].

3.2. Definition of Atomic Filters for Graph Signals

As discussed in Section 2, the classical shift operator \(\mathbf{S}\) satisfies the properties (1)–(8). It will be shown that such operator that satisfies all the eight properties does not always exist for every graphs. Therefore, in order to extend the classical shift operator to graph signals, in this paper, we do not try to establish the shift operators with property (1)–(8), but use Property (6) and (7) to introduce the concept of “atomic filters” for general graphs, which can be viewed as a weak form of the shift operator for graph signals. Then we shall discuss the existence and conditions for atomic filters to satisfy the properties (1)–(8).

Definition 1. A matrix \(\mathbf{S} \in \mathbb{C}^{N \times N}\) is called an atomic filter if any graph filter can be expressed as a polynomial of \(\mathbf{S}\).

We first point out that an atomic filter is a filter, i.e., (6) holds. In fact, suppose any graph filter \(\mathbf{H}_b = \mathbf{U} \text{diag}(\mathbf{b}) \mathbf{U}^{-1}\) with \(\mathbf{b} \in \mathbb{C}^N\) can be expressed as a polynomial of \(\mathbf{S}\), then \(\mathbf{H}_b \mathbf{S} = \mathbf{S} \mathbf{H}_b\), i.e.,
\[
\text{diag}(\mathbf{b}) \mathbf{A} = \text{diag}(\mathbf{b})
\]
for \(\mathbf{A} := (a_{ij}) := \mathbf{U}^{-1} \mathbf{S} \mathbf{U}\). Consider the \((i, j)\)-th element on both sides of the above equation, we get
\[
b_j a_{ij} = a_j b_j, \quad i, j = 1, \cdots, N.
\]
Taking a vector \(\mathbf{b} \in \mathbb{C}^N\) with distinct components, we deduce that \(a_{ij} = 0\) for any \(i \neq j\), which implies that \(\mathbf{A} = \text{diag}(\mathbf{a})\) for some vector \(\mathbf{a} \in \mathbb{C}^N\) and consequently
\[
\mathbf{S} = \mathbf{U} \mathbf{A} \mathbf{U}^{-1} = \mathbf{U} \text{diag}(\mathbf{a}) \mathbf{U}^{-1} = \mathbf{H}_a
\]
is a filter.

It is a natural question to investigate the construction of atomic filters, that is, to make clear what $a \in \mathbb{C}^N$ makes $H_a$ satisfy the condition of Definition [1]. The following theorem gives a sufficient and necessary condition to this question.

**Theorem 3.1.** A graph filter $H_a$ is an atomic filter if and only if the vector $a \in \mathbb{C}^N$ has distinct components.

**Proof.** The filter $H_a$ is an atomic filter if and only if any graph filter $H_b = U\text{diag}(b)U^{-1}$ with $b \in \mathbb{C}^N$ can be expressed as

$$H_b = \sum_{k=0}^{N-1} c_k H_a^k$$

for some constants $c_0, c_1, \ldots, c_{N-1} \in \mathbb{C}$. Here, in the right hand of Eq. (10), the highest degree of the polynomial is $N-1$ due to the Cayley-Hamilton theorem [47] pp. 109]. It is easy to verify that Eq. (10) is equivalent to

$$\text{diag}(b) = \sum_{k=0}^{N-1} c_k \text{diag}(a)^k,$$

i.e.,

$$b_n = \sum_{k=0}^{N-1} c_k a_n^k, \quad n = 1, \ldots, N,$$

which is further equivalent to $Ac = b$ for the Vandermonde matrix $A$ determined by $a_1, a_2, \ldots, a_N$. Hence, $H_a$ is an atomic filter if and only if the matrix $A$ is invertible, namely, the vector $a \in \mathbb{C}^N$ has distinct components.

**Remark:** Theorem 3.1 shows that an atomic filter $S = H_a$ is a nonderogatory matrix. According to [47] p. 178, a linear operator $H$ is commutative with $S$ if and only if $H$ is a polynomial of $S$. This means an atomic filter satisfies the property (8).

### 3.3 Basic Properties of Atomic Filters

It has been already known that an atomic filter $H_a$ must satisfies the properties (6)–(8). In this section, we investigate whether or under what conditions an atomic filter satisfies the properties (1)–(5). Property (1) is obviously true in the classical signal processing. In graph signal processing, it is equivalent to the existence of a permutation matrix $P$ such that $U\text{diag}(a)U^{-1} = P$. It will be shown that this condition is usually not true in graph settings. Below, let us study Properties (2)–(5).

An atomic filter $H_a$ is called norm-preserving if $\|H_a x\|_2 = \|x\|_2$ holds for any $x \in \mathbb{C}^N$. It is smoothness-preserving if $\sigma(Sx) = \sigma(x)$ holds for

$$\sigma(x) := \text{trace}(Lx) = \sum_{k=1}^{N} \lambda_k |\hat{x}(k)|^2,$$

where $L$ is the Laplacian matrix and $0 = \lambda_1 \leq \cdots \leq \lambda_N$ are all the eigenvalues of $L$.

**Proposition 3.2.** Let $H_a$ be an atomic filter. Then

1. It is norm-preserving if and only if $|a_k| = 1$ for $k = 1, \ldots, N$.
2. If it is norm-preserving, then it is smoothness-preserving.
3. It is periodic, i.e., $H_a^N = I$, if and only if $a_k = e^{-i\theta_k}$, $k = 1, \ldots, N$, where

$$\{\theta_k | k = 1, \ldots, N\} = \left\{\frac{2\pi}{N} (k - 1) \big| k = 1, \ldots, N\right\}.$$

**Proof.** (1) By

$$\|\text{diag}(a)x\|_2 = \|U\text{diag}(a)U^{-1}Ux\|_2 = \|H_a Ux\|_2, \quad \|Ux\|_2 = \|x\|_2,$$

we have that $H_a$ is norm-preserving if and only if $\|H_a Ux\|_2 = \|Ux\|_2$, i.e.,

$$\|\text{diag}(a)x\|_2 = \|x\|_2, \quad \forall x \in \mathbb{C}^N.$$ 

It is easy to see that this equality is equivalent to $|a_k| = 1$ for $k = 1, \ldots, N$.

(2) If $|a_k| = 1$ for $k = 1, \ldots, N$, then for any $x \in \mathbb{C}^N$ there holds

$$H_a x = U^{-1}H_a x = U^{-1}\text{diag}(a)U^{-1}Ux = \text{diag}(a)x,$$

which implies $\|H_a x(k)\| = |a_k| |x(k)| = |x(k)|$, $k = 1, \ldots, N$, and consequently $\sigma(H_a x) = \sigma(x)$.

(3) It is easy to see that $H_a^N = I$ is equivalent to $U\text{diag}(a)^N U^{-1} = I$, which implies $a_k^N = 1$, $k = 1, \ldots, N$.

Necessity: Let $H_a^N = I$. Then for $k = 1, \ldots, N$, we have $a_k^N = 1$, which implies $a_k := e^{-i\theta_k}$ with $0 \leq \theta_k < 2\pi$. Since $H_a$ is an atomic filter, $\{\theta_k | k = 1, \ldots, N\}$ are distinct numbers. By $a_k^N = 1$ we have $\theta_k = \frac{2\pi}{N} k \in \mathbb{Z}$. Thus $\{(\frac{2\pi}{N} k) | k = 1, \ldots, N\}$ are $N$ distinct integers in $[0, N)$. Since there are only $N$ distinct integers $\{0, 1, \ldots, N-1\}$ in $[0, N)$, we conclude that Eq. (11) holds.

Sufficiency: If Eq. (11) is satisfied, then $a_k^N = 1$, $k = 1, \ldots, N$, which is equivalent to $H_a^N = I$.

Let us consider the Real-Preserving Property (5). In the classical signal processing, it is known that the shift operator transforms a real-valued signal into a real-valued signal. A natural question is whether or under what conditions this property holds in the graph signal setting. There is no report on the research on this topic.

**Theorem 3.3.** An atomic filter $H_a$ is a real matrix if and only if there exist a rearrangement of $\{2, \ldots, N\}$, denoted by $\{p_2, \ldots, p_N\}$, and $c_2, \ldots, c_N \in \mathbb{C}$ with unit moduli, such that

$$a_1 \in \mathbb{R}, \quad a_k = e^{i\beta_{p_k}}, \quad u_k = c_k u_{p_k}, \quad k = 2, \ldots, N.$$ 

**Proof.** It is easy to see that $H_a$ is real-preserving if and only if $H_a$ is a real matrix: $H_a = H_a^T$, or equivalently $\text{diag}(a)U^T U = U^T \text{diag}(a)U$. The last equality can be rewritten as

$$a_j - \overline{a_j} u_j^T u_k = 0, \quad \forall j, k = 1, \ldots, N.$$

Necessity: Assume that Eq. (13) is satisfied. Since $u_1 = 1/\sqrt{N}$, where $1 \in \mathbb{R}^N$ is an $N$-dimensional vector of ones, then $u_j^T u_1 = 1$. By Eq. (13) we deduce $a_1 = \overline{a_1}$, i.e., $a_1 \in \mathbb{R}$.
Remark: If \( \{u_k\}_{k=1}^N \) are real-valued vectors, it is obvious that \( \{u_k\}_{k=1}^N \) satisfies the last equality of Eq. (12), i.e., \( u_k = \bar{u}_k \) for \( p_k = k \) and \( c_k = 1 \), \( k = 2, \ldots, N \).

Let us illustrate the result of Theorem 3.3 by experiments. Consider a ring graph, its graph Fourier basis \( \{u_k\}_{k=1}^N \) is the discrete Fourier basis defined in Eq. (4). It is easy to verify that \( \{u_k\}_{k=1}^N \) satisfy Eq. (12) for \( p_k = N + 2 - k \) and \( c_k = 1 \), \( k = 2, \ldots, N \). Thus, \( \mathbf{H}_a \) is real-preserving if and only if \( \{a_k\}_{k=1}^N \) are distinct numbers satisfying

\[
a_1 \in \mathbb{R} \quad \text{and} \quad a_k = \bar{a}_{N+2-k}, \quad k = 2, \ldots, N. \tag{14}
\]

Now we set \( \mathbf{a} := (1, e^{2\pi/N}, \ldots, e^{2\pi(N-1)/N})^T \). It is easy to see that Eq. (14) holds. Next, we disorder the components of the vector \( \mathbf{a} \), denoted by \( \mathbf{b} = (b_1, \ldots, b_N)^T \), such that \( \{b_k\}_{k=1}^N \) do not satisfy Eq. (14). It is easy to see that \( \mathbf{H}_a \) and \( \mathbf{H}_b \) are both periodic but \( \mathbf{H}_a \) is real-preserving and \( \mathbf{H}_b \) is not. Fig. 1 plots a Gaussian signal \( \mathbf{x} \) on a ring graph and its filtered versions \( \mathbf{H}_a^0 \mathbf{x} \) and \( \mathbf{H}_b^0 \mathbf{x} \). As is shown, \( \mathbf{H}_a^0 \mathbf{x} \) is real-valued but \( \mathbf{H}_b^0 \mathbf{x} \) is complex-valued.

The relationship between the properties (1)–(8) is illustrated in Fig. 2.

3.4. Normal Atomic Filters

The shift operator in the classical signal processing satisfy all the properties (1)–(8). For atomic filters in graph signal processing, these properties may not always be satisfied simultaneously. In fact, according to Theorem 3.3 an atomic filter is not real-preserving if the Fourier basis does not satisfy (12). In this section, we discuss under what conditions an atomic filter satisfies Properties (2), (4) and (5) simultaneously.

Fig. 2 reveals the logical relationship of Properties (1)–(8). It shows that both Property (2) (norm-preserving) and (5) (real-preserving) are crucial ones of atomic filters. In this section, we shall discuss the existence and construction of such atomic filters. For simplicity of expression, we introduce the concept of normal atomic filter in the following definition.

Definition 2. An atomic filter is called normal if it is both norm-preserving and real-preserving.

Property (5), i.e., real-preserving, is a very strong condition in graph signal processing. If the graph Fourier basis \( \{u_k\}_{k=1}^N \) are real vectors, then the rearrangement \( \{p_2, \ldots, p_N\} \) of \( \{2, \ldots, N\} \)
in Theorem 3.3 is exactly \([2, \ldots, N]\) itself, that is, \(p_k = k\) for \(k \in \{2, \ldots, N\}\). This fact implies that \(a_{\theta} = \overline{a}_{\theta}\), or equivalently, \(|a_{\theta}| = 1\), is a normal atomic filter. The latter will be answered in the following theorem and the first will be discussed in detail in the next two sections.

**Theorem 3.4.** Given a undirected and weighted graph, there exist normal atomic filters if and only if one of the following conditions is satisfied:

1. The Laplacian matrix \(L\) has at most one nonzero eigenvalue of odd multiplicity;
2. There exist a family of orthonormal eigenvectors \(\{u_k\}_{k=1}^N\) of \(L\) satisfying

\[
u_1 = 1/\sqrt{N}, \quad u_k := \overline{u}_{N/2+k}, \quad k = 2, \ldots, [N/2] + 1,
\]

where \([N/2]\) denotes the largest integer not exceeding \(N/2\).

**Proof.** First, we suppose \(H_a\) is a normal atomic filter. Since \(H_a\) is real-preserving, according to Theorem 3.3 Eq. (12) holds. Since \(u_k = c_k \overline{u}_{p_k} = c_k \overline{u}_{p_k} u_{p_k}\), we have that \(p_{p_k} = k\), which means that \(p: k \rightarrow p_k\) is a bijection on \([2, \ldots, N]\) satisfying \(p^{-1} = p\).

Let \(K_0 := \{k | 2 \leq k \leq N, \ k \neq p_k\}\). It is easy to see that \(a_{\theta} \in \mathbb{R}\) if \(k \in K_0\). Since there are only two real numbers, 1 and 0, with absolute value of 1 and \(a_{\theta} \in \mathbb{R}\), \(K_0\) contains at most one element. Let \(0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N\) be all the eigenvalues of \(L\) satisfying \(L u_k = \lambda_k u_k, \ k = 1, \ldots, N\). If \(\lambda_j \notin \{\lambda_1, \lambda_2\} \in K_0\), then for any \(2 \leq j \leq N\) with \(\lambda_j = \lambda_j\), we have \(\lambda_j = \lambda_{p_j}\). Multiplying the both sides of \(u_j = c_j \overline{u}_{p_j}\) by \(L\) we deduce that \(\lambda_j = \lambda_{p_j}\). It follows that \(u_j, u_{p_j} \in V_{\lambda_j} := \{x | L x = \lambda_j x\}\), the eigenspace associated with the eigenvalue \(\lambda_j\). In conclusion, where

\[V_{\lambda_j} = \text{span}\left(\bigcup_{j \in K_1, \lambda_j = \lambda_k} \{u_k, u_{p_j}\}\right),\]

where \(K_1 := \{k | 2 \leq k \leq N, \ k \neq p_k\}\), which shows that the multiplicity of \(\lambda_k\), that equals the dimension of \(V_{\lambda_k}\), is an even number. Thus, the condition (1) is satisfied.

Then, we suppose the condition (1) is satisfied. Changing the order if necessary, we assume that all the nonzero eigenvalues \(\lambda_k\) of \(L\) satisfy

\[
\lambda_k = \lambda_{N/2+k}, \quad k = 2, \ldots, [N/2] + 1.
\]

By the spectral decomposition theorem, there exist a family of orthonormal real-valued vectors \(e_1, \ldots, e_N\) with \(e_1 = 1/\sqrt{N}\) such that \(L e_k = \lambda_k e_k\) for \(k = 1, \ldots, N\). Define \(u_1 := e_1, \ u_k := \frac{1}{2}(e_k + \overline{e}_{N-k}), \ u_{N/2+k} := \frac{1}{2}(e_k - \overline{e}_{N-k}), \ k = 2, \ldots, [N/2] + 1\), and \(u_{N/2+1} := \overline{e}_{N/2+1}\) if \(N\) is even. It is easy to verify that \(u_1, \ldots, u_N\) are orthonormal eigenvectors of \(L\) satisfying Eq. (15). Thus, the condition (2) is satisfied.

Finally, we suppose the condition (2) is satisfied. Choosing \(N\) distinct complex numbers on the unit circle of the complex plane satisfying

\[a_1 = 1, \quad a_k = \overline{a}_{N/2-k}, \ k = 2, \ldots, [N/2] + 1,\]

and \(a_{N/2+1} = -1\) if \(N\) is even, we know Eq. (12) holds and \(|a_k| = 1, \ k = 1, \ldots, N\). By Theorems 3.2 and 3.3 we know \(H_a\) is normal-preserving and real-preserving, that is, \(H_a\) is a normal atomic filter.

It is a natural question to ask what graphs meet this condition of Theorem 3.4 and how to choose a vector \(a \in \mathbb{C}^N\) such that \(H_a\) is a normal atomic filter. The latter will be answered in the following theorem and the first will be discussed in detail in the next two sections.

**Corollary 3.5.** Let \(\{u_k\}_{k=1}^N\) be a graph Fourier basis satisfying Eq. (15). Then for any \(a = (a_1, \ldots, a_N)^t\) with \(a_1 = 1\), \(H_a\) is a normal atomic filter if and only if \(a_k = e^{i\theta_k}, \ k = 1, \ldots, N\), for distinct numbers \(\{\theta_k\}_{k=1}^N\) in \([0, 2\pi]\) satisfying

\[
\theta_1 = 0, \quad \theta_{N/2-k} + \theta_k = 2\pi, \quad k = 2, \ldots, [N/2] + 1.
\]

Furthermore, \(H_a\) is a periodic if and only if Eq. (16) holds and

\[
\theta_1, \ldots, \theta_N = \left\{\frac{2\pi}{N}(k - 1) \bigg| k = 1, \ldots, N\right\}.
\]

**Proof.** From the proof of Theorem 3.4 we know that \(H_a\) is a normal atomic filter if and only if \(\{a_{\theta}\}_{\theta=0}^{2\pi}\) are distinct complex numbers on the unit circle of the complex plane satisfying \(a_1 = 1\) and \(a_{\theta} = \overline{a}_{\theta}\), \(k = 2, \ldots, [N/2] + 1\). Write \(a_k = e^{i\theta_k}\) with \(\theta_k \in [0, 2\pi]\). It is easy to see that \(a_1 = 1\) is equivalent to \(\theta_1 = 0\) and \(a_k = \overline{a}_{N/2-k}\) is equivalent to \(\theta_{N/2-k} + \theta_k = 2\pi\). Thus, \(H_a\) is a normal atomic filter if and only if Eq. (16) holds.

Furthermore, if \(H_a\) is also periodic, then by Proposition 3.2 there exist

\[
\{\phi_k | k = 1, \ldots, N\} = \left\{\frac{2\pi}{N}(k - 1) \bigg| k = 1, \ldots, N\right\}
\]

such that \(a_k = e^{i\phi_k}, \ k = 1, \ldots, N\). Since \(\theta_k, \phi_k \in [0, 2\pi]\) and \(e^{i\theta_k} = e^{i\phi_k}\), we obtain that \(\theta_k = \phi_k\), which implies Eq. (17) immediately.

### 4. Typical graphs with normal atomic filters

In this section, we give some typical graphs that have normal atomic filters.

#### 4.1. Circulant Graphs

The graphs that have normal atomic filters that we want to discuss first are the so-called circulant graphs. An undirected weighted graph is called circulant if its adjacency matrix \(W = (w_{ij})\) is a symmetric and circulant matrix, that is, there exists a vector \(e = (c_0, c_1, \cdots, c_{N-1})^t\) satisfying \(c_0 = 0\) and \(c_{N-k} = c_k\) for \(k = 1, \ldots, N - 1\), such that

\[w_{ij} = c_{j - i (\text{mod } N)}, \quad i, j = 1, \ldots, N,\]
The vector \( c \in \mathbb{C}^N \) is called the generating vector of the circulant matrix \( W \). It is easy to see that the generating vector \( c \in \mathbb{C}^N \) is completely determined by its first half components \( c_1, \cdots, c_{N/2} \).

It is easy to verify that the Laplacian matrix \( L \) of a circulant graph is also a circulant matrix, whose generating vector is given by \((\alpha, -c_1, \cdots, -c_{N-1})^T\), where \( \alpha := \sum_{k=1}^{N-1} c_k \). According to the spectral decomposition of circulant matrices [47], we have \( L = U A U^{-1} \), where \( U := (u_1, \cdots, u_N) \) is the matrix of discrete Fourier basis defined by Eq. (4). It is easy to verify that \( \{u_i\}_{k=1}^{N} \) satisfy the condition (15), which implies the existence of normal atomic filters. Particularly, if \( a_k = e^{-i \theta_k}, k = 1, \cdots, N \), with \( \{\theta_k\}_{k=1}^{N} \) satisfying Eq. (16) and Eq. (17), then by Corollary 3.5 \( H_a \) is also periodic. If we choose
\[
\theta_k = \frac{2\pi}{N}(k-1), \quad k = 1, \cdots, N,
\]
a simple calculation shows that \( H_a \) is exactly the classical shift operator: \( H_a x(n) = x(n-1) \).

A typical example of circulant graphs is the ring graph with adjacency matrix the circulant matrix generated by \((0, 1, 0, \cdots, 0)^T\). Another example is the fully connected graph with adjacency matrix the circulant matrix generated by \((0, 1, \cdots, 1)^T\). For the periodic normal atomic filter \( H_a \) with \( a = (1, e^{\frac{i\pi}{N}}, e^{\frac{2i\pi}{N}}, \cdots, e^{\frac{(N-1)i\pi}{N}})^T \), Fig. 4 in Section 2 plots \( H_a x \) for a Gaussian signal \( x \) defined on a ring graph, Fig. 5 plots \( H_a x \) and \( H_b x \) for a pulse signal \( x \) defined on a fully connected graph.

### 4.2. Complete Bipartite Graphs

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. The adjacent matrix of a complete bipartite graph with \( N \) vertices has the form
\[
W = \begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}
\]
where \( p, q \) are respectively the cardinalities of \( V_1 \) and \( V_2 \), and \( N = p + q \). It can be shown that the eigenvalues of its Laplacian matrix \( L \) are [48]:
\[
0, \ p, \ \cdots, \ p, \ q, \ \cdots, \ q, \ N.
\]
Fig 5: Left: a sinusoidal signal $x$ defined on a path graph; Middle: the real part of $H_a x$; Right: the imaginary part of $H_a x$. The atomic filter $H_a$ is given by $a = (1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \cdots, e^{i\frac{2\pi(N-1)}{N}})^T$, it is norm-preserving and periodic but not real-preserving.

Fig 6: Left: a Gaussian signal $x$ defined on a sensor graph with $N = 500$ vertices; Middle: the real part of $H_a x$; Right: the imaginary part of $H_a x$. The atomic filter $H_a$ is given by $a = (1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \cdots, e^{i\frac{2\pi(N-1)}{N}})^T$, it is norm-preserving and periodic but not real-preserving.

According to Theorem 3.4, there exist atomic filters that are both norm-preserving and real-preserving if and only if both $p$ and $q$ are odd numbers. Fig. 4 plots $H_a x$ and $H_3 a x$ for a pulse signal $x$ defined on a complete bipartite graph with $p = 5$ and $q = 3$, where $H_a$ is a periodic normal atomic filter given by $a = (1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \cdots, e^{i\frac{2\pi(N-1)}{N}})^T$. Since $H_a$ is norm-preserving and periodic but not real-preserving, part of the energy of $H_a x$ is transferred to the imaginary part.

**4.3. Graphs without Normal Atomic Filters**

In this section, we give examples of graphs that have no normal atomic filters.

**4.3.1. Path Graphs**

A path graph is a graph in which the first and last vertices are only connected to one adjacent vertex, and every other vertex is only connected to two adjacent vertices. The adjacent matrix of a path graph is given by

$$W = \begin{pmatrix}
0 & 1 & & \\
1 & 0 & 1 & \\
& 1 & 0 & 1 \\
& & \ddots & \ddots & \ddots \\
& & & 1 & 0 & 1 \\
& & & & 1 & 0
\end{pmatrix}$$

The eigenvalues of the Laplacian matrix $L$ of the path graph with $N$ vertices are [48]:

$$\lambda_k = 2 - 2 \cos \frac{k\pi}{N}, \quad k = 0, 1, \cdots, N - 1.$$

Since these $N$ eigenvalues are distinct from each other, the multiplicity of each eigenvalue is 1. According to Theorem 3.4, there exists no normal atomic filter on the path graph if $N \geq 3$. Nevertheless, there exist atomic filters, which are norm-preserving and periodic but not real-preserving, on this kind of graph. Fig. 5 plots $H_a x$ for a sinusoidal signal $x$ defined on a path graph, where the atomic filter $H_a$ is given by $a = (1, e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \cdots, e^{i\frac{2\pi(N-1)}{N}})^T$. Since $H_a$ is norm-preserving and periodic but not real-preserving, part of the energy of $H_a x$ is transferred to the imaginary part.

**4.3.2. Sensor Graphs**

A sensor graph is a graph whose vertices are placed randomly in the unit square, and edges are placed between any vertices within a fixed radius of each other. The edge weights are assigned via a thresholded Gaussian kernel. Using the graph signal processing toolbox GSPBOX [49], we find that many eigenvalues of the Laplacian matrix of the sensor graph have a multiplicity of 1. According to Theorem 3.4, there exists no normal atomic filter on the sensor graph. Fig. 6 plots
Hₜₓ for a Gaussian signal x defined on a sensor graph with N = 500 vertices, where the atomic filter Hₜ is given by a = (1, e^(2πi/3), e^(4πi/3), ..., e^(2π(N-1)/N))^T. Since Hₜ is norm-preserving and periodic but not real-preserving, part of the energy of Hₜₓ is transferred to the imaginary part.

5. Relations to Existing Works in the Literature

In this section, we make a discussion of the relations between our work and those in the literature.

In [4], the graph shift Tₜ is defined via generalized convolution with a pulse signal located at vertex vₖ. It is actually a graph filter, that is, for k = 1, ..., N,

\[ Tₜ = \text{Udiag}(a)\text{U}^{-1}, \text{ where } a = \sqrt{N}(u₁(k), ..., uₙ(k))^T. \]

where U is the matrix of graph Fourier basis. For each k, if the k-th row of U has distinct components, Tₜₖ is an atomic filter. Generally, it is not norm-preserving since |uₙ(k)]= 1/\sqrt{N} does not hold. Accordingly, it is not a normal atomic filter. Moreover, the composability Tₜₖ = TₜₖTₜₖ usually does not hold.

In [5], the authors regarded the adjacency matrix W as a graph as the graph shift. In the special case of directed ring graphs, the normal atomic filter does not hold.

In [6], the authors defined a set of graph shifts by

\[ \phiₜ := \text{Udiag}(a)\text{U}^{-1}, \text{ where } a = (e^{i\varphi₁}, ..., e^{i\varphiₙ})^T, \]

where \( \varphi₁, ..., \varphiₙ \in [0, 2\pi) \) are distinct numbers. It is a norm-preserving atomic filter. By Proposition [5], if \( |\varphi_k| = 1, k = 1, ..., N \), then it is periodic. The real-preserving property is not taken into account in [9].

In [7], the authors studied the shift-enabled condition that the characteristic and minimum polynomials of the matrix H are identical. They prove that H is shift-enabled if and only if every linear shift-invariant operator is a polynomial of H. Since an atomic filter Hₜ is shift-enabled, it satisfies the shift-enabled condition. However, it is easy to understand that a shift-enabled matrix H is not necessarily an atomic filter since it may even not be a graph filter.

In [8], by using the Schrödinger’s semigroup of operators generated by the graph Laplacian L, I. Z. Pesenson et al. defined a family of shift operators

\[ Tₜₖ := \text{Udiag}(aₖ)\text{U}^{-1}, \]

where aₖ = (e^{i\varphiₖ₁}, ..., e^{i\varphiₖₙ}) and h ∈ \mathbb{R}. Inspired by the idea of this definition, a similar definition was proposed in [13] as

\[ \hat{Tₜₖ} := e^{ih\sqrt{N}} = \text{Udiag}(\hat{aₖ})\text{U}^{-1}, \]

where \( \hat{aₖ} := (e^{i\varphiₖ₁\sqrt{N}}, ..., e^{i\varphiₖₙ\sqrt{N}})^T \) and h ∈ \mathbb{R}. It is easy to see that both Tₜₖ and \( \hat{Tₜₖ} \) are atomic filter if and only if L has distinct eigenvalues. If they are atomic filters, they must be norm-preserving. However, whether they are real-preserving depends on whether they satisfy (12).

6. Windowed Fourier Time-frequency Atom for Graph Signals

The windowed Fourier transform is a powerful tool in time-frequency analysis in the classical signal processing. Its windowed Fourier time-frequency atom is defined as the product of the window function and the shift. It is usually a compactly supported function and the shift controls the center of the locality. It is obviously seen that the shift operator play a crucial role. In order to extend the theory to graph signal processing, we use normal atomic filters as an alternative to shift operator and define the windowed Fourier time-frequency atom of the graph setting as follows:

\[ \text{gₜₖ} := (Hₜₖg) ∪ uₖ, \text{ where } g ∈ \mathbb{C}^N, \]

where g ∈ \mathbb{C}^N is a graph signal (window function), Hₜₖ is an atomic filter, ∪ represents the Hadamard product of two vectors. In the following theorem, we prove that under proper conditions \( \{\text{gₜₖ}\}_{1≤j≤L, 1≤k≤N} \) constitute a frame for graph signals, and any graph signal f ∈ \mathbb{C}^N can be completely reconstructed from \( \{f, \text{gₜₖ}\}_{1≤j≤L, 1≤k≤N} \). It extends the results in [4].
Theorem 6.1. Let \(|u_k|^N\) be a graph Fourier basis, \(g \in \mathbb{C}^N\) and \(A := (a_1, \ldots, a_j) \in \mathbb{C}^{N \times J}\). If the row vectors of \(A\) are orthonormal and
\[
C_n := \sum_{j=1}^{J} |g_j|^2 |u_j(n)|^2 > 0, \quad n = 1, \ldots, N, \quad (18)
\]
then for any graph signal \(f \in \mathbb{C}^N\), the following reconstruction formula holds:
\[
f(n) = \frac{1}{C} \sum_{j=1}^{J} \sum_{k=1}^{N} \langle f, g_{a_j,k} \rangle g_{a_j,k}(n), \quad n = 1, \ldots, N, \quad (19)
\]
and
\[
\alpha \|f\|_2^2 \leq \sum_{j=1}^{J} \sum_{k=1}^{N} |\langle f, g_{a_j,k} \rangle|^2 \leq \beta \|f\|_2^2, \quad (20)
\]
where \(\alpha := \min_{1 \leq n \leq N} C_n\), \(\beta := \max_{1 \leq n \leq N} C_n\).

Proof. Since the row vectors of \(A\) are orthonormal, i.e.
\[
\sum_{j=1}^{J} \sum_{k=1}^{N} \langle f, g_{a_j,k} \rangle g_{a_j,k}(n) = \delta_{n,k},
\]
which yields Eq. (19) immediately. Multiplying the both sides of the above equality by \(f(n)\) and summing over \(n\) from 1 to \(N\), we obtain that
\[
\sum_{j=1}^{J} \sum_{k=1}^{N} |f(n)|^2 = \sum_{k=1}^{K} C_n |f(n)|^2.
\]
Using \(\alpha \leq C_n \leq \beta\) in the above equality, we deduce Eq. (20).

Remark: When \(C_n > 0\) for all \(n = 1, \ldots, N\), \(g_{a_j,k}\) constitute a tight frame with frame bound \(C\) and the reconstruction Eq. (19) can be rewritten as
\[
f = \frac{1}{C} \sum_{j=1}^{J} \sum_{k=1}^{N} \langle f, g_{a_j,k} \rangle g_{a_j,k}.
\]

A special example of the case is when the graph is circulant. In this case, according to Eq. (4) we have that \(|u_k(n)| = 1/\sqrt{N}\). Thus
\[
C_n = \frac{1}{N} \sum_{k=1}^{N} |g_k|^2 = \frac{1}{N} \cdot 1 = \frac{1}{N}, \quad n = 1, \ldots, N. \quad (21)
\]
Let us consider the following special family of time-frequency atoms:
\[
g_{jk} := \frac{1}{\sqrt{N}} (H^{-1}_a g) \odot u_k, \quad j, k = 1, \ldots, N, \quad (22)
\]
where \(a := (a_1, \ldots, a_N)^T \in \mathbb{C}^N\). It is easy to see that
\[
\frac{1}{\sqrt{N}} H^{-1}_a = H_a,
\]
where
\[
a_j := \frac{1}{\sqrt{N}} (a_1^{j-1}, \ldots, a_N^{j-1})^T, \quad j = 1, \ldots, N.
\]
A simple calculation shows that
\[
A := (a_1, \ldots, a_N) = \frac{1}{\sqrt{N}} \begin{bmatrix}
a_1 & a_2^2 & \cdots & a_N^{N-1} \\
a_2 & a_1^2 & \cdots & a_N^{N-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_N^2 & \cdots & a_1^{N-1}
\end{bmatrix}. \quad (23)
\]

Lemma 6.2. For distinct numbers \(a_1, \ldots, a_N \in \mathbb{C}\), the matrix \(A\) defined by Eq. (23) is a unitary matrix if and only if there exist a permutation matrix \(P\) and a diagonal matrix \(C := \text{diag}(1, c, \ldots, c^{N-1})\) with \(|c| = 1\) such that \(A = PUC\), where \(U\) is the matrix of discrete Fourier basis defined in Eq. (4).

Proof. The sufficiency holds obviously since \(PP^* = UU^* = CC^* = I_N\). To prove the necessity we assume that \(A\) is a unitary matrix. For \(k = 1, 2, \ldots, N\), we have that \(1 + |a_k|^2 + |a_k|^4 + \cdots |a_k|^{2N-1} = N\), which implies that \(|a_k| = 1\). Write \(a_k = e^{i\omega_k}\). Multiplying a permutation matrix \(P\) on the left of \(A\) we can assume that \(0 \leq \omega_1 < \omega_2 < \cdots < \omega_N < 2\pi\).

For \(k = 2, \ldots, N\), we have
\[
1 + \sum_{l=1}^{k-1} a_l^2 + a_l^4 + \cdots + a_l^{2N-1} = 0,
\]
i.e.,
\[
1 + e^{i(\omega_k - \omega_1)} + e^{i2(\omega_k - \omega_2)} + e^{i(2N-1)(\omega_k - \omega_N)} = 0.
\]
It follows that \(e^{N(\omega_k - \omega_1)} = 1\) and consequently \(N\) \((\omega_k - \omega_1) \in \mathbb{Z}\).

Since
\[
\left(\frac{N}{2\pi} (\omega_k - \omega_1) \right) |k = 1, \ldots, N| \in \mathbb{N},
\]
and there are exactly \(N\) integers in \([0, N]\), we deduce that \(N\) \((\omega_k - \omega_1) \in \mathbb{N}\), which implies that \(|a_k| = 1\). Hence
\[
a_k = e^{i\omega_k}, \quad k = 1, \ldots, N,
\]
which is equivalent to \(QA = UC\), where \(C := \text{diag}(1, c, \ldots, c^{N-1})\). Then we obtain \(A = PUC\), where \(P := Q^T\) is a permutation matrix. The necessity is proved.

Lemma 6.2 shows that the matrix \(A\) defined in Eq. (23) satisfies the condition of Theorem 6.1 and only if \(A\) (possibly after row permutation) is equivalent to the classical discrete Fourier basis matrix up to a permutation. Based on this fact we have the following corollary.
Corollary 6.3. For \( \{u_k\}_{k=1}^N; g, C, \alpha, \beta \) stated in Theorem 6.1 If \( C_n > 0 \) for \( n = 1, \ldots, N \), then for \( a := (1, \omega, \ldots, \omega^{N-1}) \) with \( \omega := e^{\pi i/N} \), the time-frequency atoms

\[
g_{jk} := \frac{1}{\sqrt{N}} (H_{\alpha^{-1}} g) \otimes u_k, \quad j, k = 1, \ldots, N,
\]

constitute a frame of \( \mathbb{C}^N \) with bounds \( \alpha \) and \( \beta \), that is,

\[
\alpha \|f\|_2^2 \leq \sum_{j=1}^{N} \sum_{k=1}^{N} |\langle f, g_{jk} \rangle|^2 \leq \beta \|f\|_2^2, \quad \forall f \in \mathbb{C}^N,
\]

and the reconstruction formula

\[
f(n) = \sum_{j=1}^{N} \sum_{k=1}^{N} \langle f, g_{jk} \rangle g_{jk}(n), \quad n = 1, \ldots, N.
\]

holds for any \( f \in \mathbb{C}^N \).

7. Conclusion

The definition of shift operation for signals defined on graphs is a fundamental problem since the vertex set of a graph is usually not a vector space. This paper provides a system research on this topic. A weak form, called atomic filter, is presented. The contributions of this paper are summarized as follows:

- Based on the action of shift operator in the classical signal processing, we introduce the concept of atom filters. The basic properties, including the norm-preserving, smooth-preserving, periodic, and real-preserving are studied.

- The property of real-preserving holds naturally in the classical signal processing, but no report on the researches on this topic. This paper shows that real-preserving may not always hold for any graph. The sufficient and necessary conditions are presented.

- The concept of normal atomic filters is introduced. Typical examples of graphs that have or have not normal atomic filters are given.

- Finally, as an application, atomic filters are utilized to construct time-frequency atoms which constitute a frame of the graph signal space.

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