Lattice QCD Spectroscopy with an Improved Wilson Fermion Action

Maria-Paola Lombardo
Department of Physics,
University of Illinois at Urbana-Champaign,
1110 West Green Street, Urbana, IL 61801, U.S.A.

Giorgio Parisi and Anastassios Vladikas
Dipartimento di Fisica,
Università di Roma Tor Vergata
Via E.Carnevale, 00173 Roma, Italy
and
Infn, Sezione di Roma Tor Vergata

1 on leave from Infn, Sezione di Pisa, Italy
Abstract

We study the hadronic spectrum in quenched lattice QCD using the improved Wilson fermion action proposed in [1] at \( \beta = 5.7 \) and \( \beta = 6.0 \). We find a systematic reduction of the finite spacing effects compared to the results obtained by using the standard Wilson action.
1 Introduction

Most of the recent results about quenched QCD spectroscopy exhibit the scaling behaviour expected near the continuum limit. However, deviations are still observed in the phenomenologically relevant heavy quark mass region, and in these cases progress using the standard Wilson action seems to be slow. Moreover, an important self-consistency check of lattice computations is the agreement between results coming from different discretizations.

For the above two reasons, we decided to study lattice spectroscopy by using an improved version of the Wilson fermion action, inspired by Symanzik’s perturbative procedure of eliminating finite-$a$ corrections. Other proposals along the same line are the ones of ref. and ref., which, by combining the improved action with a suitable modification of the fermion operators, eliminates all terms of order $a$ in the hadronic matrix elements. Some results obtained by using the actions proposed in refs. are already available and we remand to the references for their discussion.

There are two points to be considered, in relation to an improvement program based on perturbative arguments. The first one is the real source of the scaling violations, which can be completely non-perturbative, the other one is the practical applicability. Indeed, the success of an improved action depends on the interplay between the reduction of $O(a)$ effects, and the possibly induced new systematics, and it is not guaranteed a priori. To understand the real effectiveness of the action we study in this paper $\beta = 5.7$ and $\beta = 6.0$. From the results for the standard Wilson case we know that the finite spacing effects are strong at $\beta = 5.7$, while $\beta = 6.0$ seems to be the onset of scaling for many relevant quantities. We thus use the data at $\beta = 5.7$ to demonstrate the suppression of the most evident scaling violations which occur using the Wilson action at the same value of the coupling, and the ones at $\beta = 6.0$ to explore the possibility to actually improve the current Wilson results.

We introduce the action we use, and summarize its basic properties in the next section. Then, we describe the numerical simulation and the analysis procedure. Finally we discuss the results for the hadron spectrum and the meson decay constants. In the following we keep as a reference the results obtained with the standard Wilson action at $\beta = 5.7, \beta = 6.0, \beta = 6.3$ of ref., ref., ref. Our conclusions are based on this comparative analysis.
The action

We use the improved fermion action proposed by Hamber and Wu [1]. A next nearest neighbor term is added to the Wilson fermion action:

\[ S = S_G + S_W + S_{II} \]  (1)

\( S_G \) and \( S_W \) are the usual pure gauge and Wilson fermionic term, while the new term reads:

\[ S_{II} = \sum \bar{\psi}_n (C - D \gamma_\mu) U_{n,\mu} U_{n+\mu,\mu} \psi_{n+2\mu} + \text{h.c.} \]  (2)

The choice \( C = -\frac{1}{4kr} \) cancels at tree level the term \( O(p^2) \) in the inverse propagator \( S_F^{-1}(p) \), \( D \) is free. We choose \( D = -\frac{1}{8kr} \) which cancels also the terms \( O(p^3) \) in the inverse fermion propagator, and set \( r = 1 \). In this way the fermionic part of the action coincides with the Eguchi-Kawamoto one.

We refer to the original references [1], and to ref. [8] for a detailed discussion. It is however interesting pointing out here that this action exhibits positivity violations stronger that the ones of the standard Wilson case. The zero-momentum free quark propagator \( S_F(p_0, \vec{0}) \) has four poles for any \( k \) value, two of them complex, the other two turning from complex to real for \( k > \approx 0.14 \). (The critical value \( k_c \), defined by \( m_q(k_c) = 0 \), \( m_q \) being the quark mass, is \( 1/6 \).) So, the free propagator, computed by Fourier transforming \( S_F(p) \) to the coordinate space, deviates from a simple exponential behaviour. It shows a clear ripple at small \( k \)’s (i.e., in the region of four complex poles), which is more evident in small lattices. The effect progressively disappears by increasing the size of the lattice, and approaching \( k_c \), suggesting that no problem arises in the continuum limit. Indeed, in the continuum limit \( m_{PHYS} a = m_{LAT} \to 0 \), while \( k \to k_c \). So, in order to define properly the \( a \to 0 \) limit only one real pole (the one corresponding to the physical mass) should be zero as \( k \to k_c \), while the other real pole, and the real parts of the complex ones, should be different from zero. It is trivial to show that this is the case. Turning to the amplitudes, no problem should arise with the ones involving physical status: violations of positivity only affects the residuals of non-physical poles.

The above informal discussion, which leads to the conclusion that the spurious poles are harmless, concerns the free case. However, the results for the standard Wilson action suggest that what has been found in the free case
is general: for the Wilson action, the free propagator has a complex pole for \( K > 1/6 \), and a real one otherwise, and it has been shown \[12\] that the same pattern of violation of positivity is maintained also in the interacting case. In conclusion, a more detailed investigation of the positivity properties of the improved actions would be welcome, but at the moment we have no reason to think that their continuum limit is suspect.

Anyway — even if this may seem rather paradoxical — an action supposed to reduce finite size effects potentially leads to a peculiar finite spacing systematics, whose actual impact is not possible to estimate a priori. Happily enough, positivity violations affected the results only at \( \beta = 5.7 \), for the \( L = 1 \) mesons.

### 3 The Numerical Simulation and the Data Analysis

We used the Wilson pure-gauge action, and we have thermalized the gauge fields at \( \beta = 5.7 \) and \( \beta = 6.0 \). We used a standard Metropolis-5 hits algorithm to generate the background gauge field configurations. The onset of the scaling region for the standard Wilson action seems to be around \( \beta = 6.0 \) while the results at \( \beta = 5.7 \) are definitively affected by finite spacing effects. Thus, the values we choose are good to assess the effectiveness of the method.

We explored the heavy quark region, which is appropriate to test this action. We choose four equispaced \( K \), ranging from 0.186 to 0.198 at \( \beta = 5.7 \), and from 0.176 to 0.188 at \( \beta = 6.0 \). This gives the ratio \( \frac{\pi^2}{\rho^2} \) between 0.4 and 0.8 at \( \beta = 5.7 \) and between 0.6 and 0.9 at \( \beta = 6.0 \).

One configuration every 800 iterations was sampled for the propagator inversion. At \( \beta = 5.7 \) we have computed (140 \( \times \) 4 quark masses) propagators on a \( 24 \times 12^3 \) lattice, at \( \beta = 6.0 \) we have (160 \( \times \) 4) propagators on a \( 32 \times 12^3 \) lattice. We also collected (180 \( \times \) 3) propagators on a \( 18 \times 9^3 \) lattice (at the three bigger masses) to monitor the finite size effects.

All the measures were performed according to the methods introduced in \[10\] and reviewed in \[13\], which we briefly summarize in the following. We compute the fermionic propagators by a preconditioned minimal residue algorithm, as modified for parallel updating. We apply the incomplete LU
decomposition as a preconditioner, making use in this step of the standard Wilson operator. In practice the original equation for the improved propagator

\[ DX = B \]  

is replaced by

\[ U^{-1}L^{-1}DX = U^{-1}L^{-1}B \]  

where L and U are defined in the usual way \[ [10] \] in term of the Wilson fermionic operator. We have indeed found also with the improved action a considerable speed-up in the convergence rate by using this preconditioning.

We use smearing \[ [13] \ [10] \]: after the necessary gauge fixing we solve the Dirac equation with an extended source (a \( 3^3 \) spatial cube). The propagators computed in this way can be smeared on the final point too. From the quark propagators we form the contractions needed to evaluate the hadronic Green functions \( G_0 \) (smeared only at the origin) and \( G_x \) (at the origin and at the final point). We preferred the second ones (\( G_x \), smeared at the origin and in \( x \)) since \( G_0 \) sometimes tends to amplify the problems related to the lack of positivity of this action, but the difference is small and on the overall the results obtained from \( G_0 \) and \( G_x \) are fully consistent. A joint fit of \( G_0 \) and \( G_x \) does not further reduce the errors, which are always computed by a standard jacknife analysis.

The results for the hadron masses quoted in Table 1 and in Table 3 are from a single mass fit (for \( t > 5 \)) of the Green functions smeared both at the origin and at the final points. We systematically controlled the stability of the results by performing two particle fits as well (in this case we included points starting form \( t = \approx 3 \)), and we cross checked with the local fits. As an example of the overall quality of the data we show in Fig. 1 the effective masses of the proton at \( \beta = 5.7 \) and in Figs. 2 the fits at \( \beta = 6.0 \) for the proton and the pion.

Using the data on the \( 9^3 \times 18 \) lattice we checked that finite size effects are small. The big symbols in Fig. 1 which perfectly circle the results obtained on the \( 12^3 \times 24 \) lattice are from the smaller one. The full set of results on the \( 9^3 \times 18 \) lattice is shown in Table 2. (We note that the plateau in the effective masses in this lattice is not as extended as in the bigger one, so in this case we had to rely on two particle fits.) The results reported in Table 2 agree with the ones of Table 1, thus demonstrating the good control we have over finite size effects. We are confident that, since the time extents
in physical units of the lattices at $\beta = 5.7$ and $\beta = 6.0$ are roughly the same, the masses evaluated at $\beta = 6.0$ do not suffer from finite size effects as well. (However, the number of points in the space directions is the same for the two lattices, so some problem at least with the smallest mass at $\beta = 6.0$ cannot be excluded.)

At $\beta = 5.7$ we do not quote the results for the $a1$ and $b1$ particles. These mesons deserve some separate comments, since their behaviour turned out to be quite peculiar. Their Green functions can even change sign, thus indicating violations of positivity (amplitudes of different signs in different channels, or manifestations of the spurious complex poles). As a consequence, the effective masses have a wiggle which becomes less and less apparent as $k \to k_c$ or $a \to 0$, in agreement with the qualitative discussion of Sect.2 above. In fact, the results at $\beta = 6.0$ for the fits of the $L = 1$ mesons are reasonable. (Of course in this case we do not have any hint about residual finite size effects, and it is clear that having amplitudes of comparable magnitude and

| Table 1: Hadron masses as a function of $k$ at $\beta = 5.7$ on the $24 \times 12^3$ lattice |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| Particle | 0.186 | 0.190 | 0.194 | 0.198 |
| $\pi$ | 1.0267(28) | 0.8661(29) | 0.6975(31) | 0.5069(40) |
| $\tilde{\pi}$ | 1.0239(39) | 0.8624(42) | 0.6925(50) | 0.4975(73) |
| $\rho$ | 1.1432(40) | 1.0109(44) | 0.8851(59) | 0.7676(132) |
| $\tilde{\rho}$ | 1.1451(44) | 1.0141(51) | 0.8905(73) | 0.7811(161) |
| $p$ | 1.7972(108) | 1.5811(116) | 1.3616(141) | 1.1186(277) |
| $\Delta$ | 1.8646(138) | 1.6665(150) | 1.4761(198) | 1.2990(557) |

| Table 2: Hadron masses as a function of $k$ at $\beta = 5.7$ on the $18 \times 9^3$ lattice |
|---------------------------------|-----------------|-----------------|-----------------|
| Particle | 0.186 | 0.190 | 0.194 |
| $\pi$ | 1.0199(30) | 0.8638(31) | 0.6970(37) |
| $\tilde{\pi}$ | 1.0207(40) | 0.8623(46) | 0.6714(59) |
| $\rho$ | 1.1326(43) | 1.0073(47) | 0.8863(57) |
| $\tilde{\rho}$ | 1.1327(48) | 1.0052(56) | 0.8814(75) |
| $p$ | 1.6807(182) | 1.5465(126) | 1.3505(100) |
| $\Delta$ | 1.6541(294) | 1.5798(219) | 1.4707(212) |
Table 3: Hadron masses as a function of $k$ at $\beta = 6.0$ on the $32 \times 12^3$ lattice

| Particle | $k$ | \hline
| --- | --- | --- | --- | --- |
|  | 0.176 | 0.180 | 0.184 | 0.188 |
| $\pi$ | 1.0014 (15) | 0.8179 (17) | 0.6287 (23) | 0.4204 (39) |
| $\bar{\pi}$ | 0.9976 (17) | 0.8141 (21) | 0.6252 (29) | 0.4170 (52) |
| $\rho$ | 1.0424 (19) | 0.8712 (24) | 0.7024 (33) | 0.5333 (63) |
| $\bar{\rho}$ | 1.0408 (20) | 0.8691 (25) | 0.6990 (36) | 0.5238 (76) |
| a1 | 1.2794 (142) | 1.1203 (187) | 0.9676 (285) | 0.8150 (570) |
| b1 | 1.2772 (140) | 1.1159 (177) | 0.9553 (262) | 0.7873 (557) |
| $p$ | 1.6356 (43) | 1.3734 (55) | 1.1126 (80) | 0.8458 (150) |
| $\Delta$ | 1.6602 (48) | 1.4061 (61) | 1.1584 (89) | 0.9120 (183) |

opposite signs is the worst possible situation from this point of view.) As a general comment, the orbitally excited hadrons are difficult to treat, also with the ordinary Wilson action, and their signal is intrinsically noisy: a recent discussion, together with new results, can be found in ref. [14].

We conclude the description of the analysis procedure by discussing the computations of the amplitudes of the Green functions in the fundamental channels, which are important for the estimates of the decay constants. We are interested in the local amplitudes, which, as discussed in ref [11], can be recovered from the ratio of $G_0^2$ to $G_X$ in the hypothesis of long-distance factorization of the Green functions. We checked that the amplitude of the fit of the ratio has a smaller statistical error compared to the ratio of the amplitudes of the fits, typically by a factor 2. This is understandable, because by fitting the ratio we get rid of the coherent fluctuations. So the errors we quote are obtained in this way (the central values are basically the same).

At $\beta = 5.7$ we also computed 24 propagators with a point source, so we can evaluate directly the local amplitudes which turn out to be in complete agreement with the ones coming from the fit of the ratios. The summary of the results for the amplitudes is given in Tables 4 and 5.
| Particle | $0.186$ | $0.190$ | $0.194$ | $0.198$ |
|----------|---------|---------|---------|---------|
| $\pi^1$  | $2.140(57)$ | $1.950(53)$ | $1.834(56)$ | $1.876(78)$ |
| $\pi^2$  | $2.123(100)$ | $1.904(105)$ | $1.739(112)$ | $1.647(141)$ |
| $\tilde{\pi}^1$ | $0.308(14)$ | $0.202(10)$ | $0.119(7)$ | $0.058(5)$ |
| $\tilde{\pi}^2$ | $0.299(26)$ | $0.192(21)$ | $0.112(16)$ | $0.058(14)$ |
| $\rho^1$  | $0.977(38)$ | $0.779(33)$ | $0.611(33)$ | $0.484(53)$ |
| $\rho^2$  | $0.912(61)$ | $0.714(51)$ | $0.538(42)$ | $0.358(54)$ |

Table 4: Amplitudes in the fundamental channel as a function of $k$ at $\beta = 5.7$ on the $24 \times 12^3$ lattice, from the ratios’ fit$^1$ and from local operators$^2$.

| Particle | $0.176$ | $0.180$ | $0.184$ | $0.188$ |
|----------|---------|---------|---------|---------|
| $\pi$    | $0.675(15)$ | $0.540(14)$ | $0.427(13)$ | $0.342(17)$ |
| $\tilde{\pi}$ | $0.183(4)$ | $0.116(3)$ | $0.065(2)$ | $0.027(1)$ |
| $\rho$   | $0.340(8)$ | $0.242(7)$ | $0.160(6)$ | $0.092(6)$ |

Table 5: Amplitudes in the fundamental channel as a function of $k$ at $\beta = 6.0$.

### 4 Hadron masses, decay constants and scaling behaviour

We begin the discussion of our results with the chiral behaviour. Since we were particularly concerned with heavy masses, the chiral extrapolation, especially at $\beta = 6.0$, is somewhat delicate. For heavy flavors, the data should follow the predictions of the potential models for quarkonium, whose curvatures as a function of the quark mass are different from the ones in the chiral limit. Consequently, it is difficult to decide when a fit, even if satisfactory on statistical grounds, correctly describes the chiral behaviour. With this warning in mind, we fitted the squared hadron masses as second order polynomials in the quark masses, defined as $m_q = 2/3(1/k - 1/k_c)$. This turned out to be the most effective parametrization, in agreement with previous experiences with heavy masses.

The first step is the determination of $k_c$ which is obtained by demanding
Table 6: Coefficients of the extrapolating polynomial for the hadron masses at $\beta = 5.7$

$m_\pi(k_c) = 0$. We get at $\beta = 5.7$

$$m_\pi^2 = 2.687(49)m_q + 2.492(84)m_q^2$$

(5)

and $k_c = 0.20333(12)$ At $\beta = 6.0$ we have

$$m_\pi^2 = 2.032(40)m_q + 3.504(46)m_q^2$$

(6)

and $k_c = 0.19216(12)$

It is interesting to compare with the analytical predictions for $k_c$ which are poorly reproduced on the lattice in the standard Wilson case. We have a 9 percent deviation at $\beta = 6.0$ and 15 percent at $\beta = 5.7$ from the analytical result $k_c = 1/6(1 + 1/18g^2 + O(g^4))$ (second entry of ref. [1]), to be compared with 12 percent and 17 percent for the standard Wilson action at the corresponding $\beta$ values.

The results for the pion at $\beta = 5.7$ and $\beta = 6.0$ with the results fits superimposed are shown in Fig. 3. In the same figure we also show the results for the $\bar{\pi}$’s, which turn out to be in full agreement with the ones for the $\pi$.

Once $k_c$ is determined, we go ahead with fits in other channels. As already said, they are completely satisfactory on statistical grounds, and do not deserve any special comments, so we simply quote in Table 6 and 7 the coefficients of the extrapolating polynomials,

$$M_{hadron}^2 = M_0^2 + s_1m_q + s_2m_q^2$$

(7)

for the two $\beta$'s.
We remark again the agreement between the results for the π and the \( \tilde{\pi} \) particles, and for the \( \rho \) and the \( \tilde{\rho} \) particles.

We note that at \( \beta = 5.7 \) the result \( m_\rho(0) \) for the extrapolated \( \rho \) mass lies in between the Kogut-Susskind [15] result and the Wilson one, supporting the intuitive picture of an “effective spacing” between \( a \) and \( 2a \) for the improved action:

\[
m_\rho(0)(W) = 0.185(18) < m_\rho(0)(WI) = 0.387(38) < m_\rho(0)(KS) = 0.76(5)
\]

At \( \beta = 6.0 \) the results for Wilson and Wilson improved are mutually consistent, while the result for Kogut-Susskind is getting closer to them (the ratio of the K-S results to the Wilson one is 1.14(7) at \( \beta = 6.0 \)):

\[
m_\rho(0)(W) = 0.111(5) \simeq m_\rho(0)(WI) = 0.119(10) < m_\rho(0)(KS) = 0.144(27)
\]

As already noticed in [16], \( \beta = 6.0 \) seems the onset of the region in which the details of lattice discretization are forgotten.

The comparison of the results obtained with the Wilson action and with the improved one is better done by using adimensional quantities: for this purpose we give in Table 8 the values of some relevant ratios. For the string tension at \( \beta = 5.7 \) we use the result of ref [17]: \( \sigma a^2 = 0.056(2) \). At \( \beta = 6.0 \) and \( \beta = 6.3 \) we get, by analyzing the same configurations we were using in [11], [3], \( \sigma a^2 = 0.0471(35) \) and \( \sigma a^2 = 0.01704(8) \), respectively [18]. (Let us remind that the experimental values to be compared with are \( p/\rho = 1.22 \), \( p/\sigma = 2.24 \), \( \rho/\sigma = 1.83 \), assuming the accepted estimate for \( \sigma \) of 420 MeV.)

| Particle | \( m_0^2 \) | \( s_1 \) | \( s_2 \) |
|----------|-------------|----------|----------|
| \( \pi \)   | 0           | 2.032(40) | 3.504(46) |
| \( \tilde{\pi} \) | 0.000(2)    | 1.996(31) | 3.541(85) |
| \( \rho \)   | 0.119(10)   | 1.872(63) | 3.658(108) |
| \( \tilde{\rho} \) | 0.103(12)   | 1.971(78) | 3.474(136) |
| \( a_1 \)    | 0.436(146)  | 2.742(942) | 3.217(1698) |
| \( b_1 \)    | 0.362(136)  | 3.180(864) | 2.520(1548) |
| \( p \)      | 0.290(36)   | 4.926(198) | 8.047(323) |
| \( \Delta \) | 0.422(51)   | 4.718(316) | 8.194(541) |

Table 7: Coefficients of the extrapolating polynomial for the hadron masses at \( \beta = 6.0 \).
Table 8: Relationship between the extrapolated values for $m_q = 0$ of the $\rho$ and proton masses and the string tension

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & 6.0 & 5.7 & 6.3 W & 6.0 W & 5.7 W \\
\hline
p/\rho & 1.561(117) & 1.193(136) & 1.247(95) & 1.298(53) & 1.451(75) \\
p/\sigma & 2.481(179) & 2.124(220) & 2.220(166) & 1.993(101) & 2.217(93) \\
\rho/\sigma & 1.590(89) & 1.780(88) & 1.780(64) & 1.535(66) & 1.528(49) \\
\hline
\end{array}
\]

It is remarkable the full agreement of the improved results at $\beta = 5.7$ with the ones obtained with the standard Wilson action at $\beta = 6.3$, which in turn match the experimental results. At $\beta = 6.0$ the situation is less clear (the $\rho$ is the same as the normal Wilson case at the same $\beta$, the proton is closer to the one at $\beta = 6.3$, but as previously discussed we do not have a safe estimate of the systematic errors induced by the chiral extrapolation at $\beta = 6.0$).

In the following we will discuss finite mass data, using occasionally the extrapolated values for the $\rho$ mass to set a common scale for the results obtained at different $\beta$ values.

An interesting quantity to look at is the splitting between pseudoscalar and vector mesons with the same flavour content. Experimentally, $M_V^2 - M_{PS}^2$ is known to satisfy $(\rho^2 - \pi^2) > (K^{*2} - K^2) \simeq (D^{*2} - D^2) \simeq (B^{*2} - B^2)$. The available lattice data fail to reproduce the approximate plateau exhibited at large quark masses. Since we have to reproduce a $\vec{\mu}_1 \vec{\mu}_2 \delta(\vec{r}_{12})$ interaction, the hyperfine splitting is a natural candidate for the improvement.

To show the trend in the splitting, we compute the average derivative $< d(m_\rho^2 - m_\pi^2)/dm_\pi^2 >$ for the various cases of interest. This is done via the linear fit

\[
(m_\rho^2 - m_\pi^2) = Sm_\pi^2 + Cm_\rho^2(0)
\]

(note that both $S$ and $C$ are dimensionless) The summary of these results is given in Table 9. $C$ should be consistent with 1 when both $m_\pi^2$ and $m_\rho^2$ are linear in $m_q$. In this respect, it is interesting to note that $C$ is roughly consistent with 1 also when we the quark masses are large. The slope decreases with $\beta$ both for the standard and for the improved Wilson actions. Again, the results at $\beta = 5.7$, improved, are consistent with the ones at $\beta = 6.3$, standard Wilson, while the improved data at $\beta = 6.0$ have definitively the smaller slope, closer to the experimental results (however, still inconsis-
tent with them). Anyway, since the slope is a (slowly varying) function of the quark mass, a more detailed comparison is done by simply superimposing the results in units of the extrapolated \( \rho \) mass. This is done in Fig. 5. We show in Fig.5a the improved data alone, and the collection of results in Fig.5b.

The results for the Wilson action at \( \beta = 5.7 \) are more steeper than the other ones in the overlapping region of \( \pi \) masses \( (0.2 \lesssim m_\pi^2 \lesssim 2.0) \), while at \( \beta = 6.0 \) and 6.3 the residual difference in the local slope is very small (from the plot it is clear that the difference in the average slopes comes mostly from the contribution at large masses at \( \beta = 6.3 \), and from the one at small masses at \( \beta = 6.0 \)). Turning to the improved action, the results at \( \beta = 5.7 \) closely follow the ones at \( \beta = 6.3 \), Wilson. The slope of the data at \( \beta = 6.0 \), which can be compared to the one of \( \beta = 6.3 \), Wilson up to \( m_\pi^2 \simeq 4 \), is clearly more flat, so closer to the experimental data.

The standard Wilson results seem to have a very slow evolution with \( \beta \). In addition the results at \( \beta = 6.2 \) reported in [19] which explore up to \((\pi/\rho(0))^2 \simeq 30\) do not show any reversal in the decreasing trend, while the UKQCD collaboration reported no improvement in the hyperfine splitting at the same \( \beta \) value with the Clover action [8] (this in the small mass range). On the contrary, the data for the improved action from \( \beta = 5.7 \) to \( \beta = 6.0 \) are still appreciably moving in the right direction, and it is not unreasonable to expect good results at a feasible \( \beta \) value (i.e. 6.3 – 6.5). Interesting results about the splittings are the ones obtained by the FNAL group [9], which use the action \([5]\). They observe that the spin splittings are very sensitive to the parameter \( c \) in the \( O(a) \) correction term, \(-i/2c\bar{\psi}\Sigma_{\mu\nu}F_{\mu\nu}\psi\) which contributes mainly an additional magnetic interaction to the quarks.

Before turning to the baryons, we discuss the results for the meson decay constants which are given in Table 10. As explained in [11], we obtain \( f_\pi \) from the local amplitudes in the \( \bar{\pi} \) channel whose computation has been described in Sect. 3 above. The results are also plotted in Fig.5a, and again we compare with the Wilson ones in Fig. 5b.

|       | 6.0   | 5.7   | 6.3 W | 6.0 W | 5.7 W |
|-------|-------|-------|-------|-------|-------|
| \( S \) | -0.027(9) | -0.088(23) | -0.078(11) | -0.173(18) | -0.207(17) |
| \( K \) | 0.92(6)  | 0.88(5)  | 0.91(2)  | 1.01(2)  | 0.97(2)  |

Table 9: Results of the fits for \( (m_\rho^2 - m_\pi^2) = Sm_\pi^2 + Cm_\rho^2(0) \)
\[ \beta = 5.7 \]

| \( k \) | 0.186 | 0.190 | 0.194 | 0.198 |
|---|---|---|---|---|
| \( F_\pi (MeV) \) | 243.5(55) | 219.3(54) | 191.5(56) | 160.1(69) |
| \( \frac{f_\pi}{f_\pi^{(\pi)}} \) | 0.712(16) | 0.708(17) | 0.721(21) | 0.759(32) |
| \( f_\rho^{-1} \) | 0.281(11) | 0.309(13) | 0.341(19) | 0.383(47) |

\[ \beta = 6.0 \]

| \( k \) | 0.176 | 0.180 | 0.184 | 0.188 |
|---|---|---|---|---|
| \( F_\pi (MeV) \) | 325.9(35) | 293.6(38) | 256.3(39) | 206.4(39) |
| \( \frac{f_\pi}{f_\pi^{(\pi)}} \) | 0.669(7) | 0.614(7) | 0.561(8) | 0.521(9) |
| \( f_\rho^{-1} \) | 0.182(4) | 0.205(6) | 0.235(9) | 0.276(20) |

Table 10: Results for the pion and \( \rho \) decay constants.

The results obtained by making use of the Wilson action at \( \beta = 6.0 \) and \( \beta = 6.3 \) are in moderate disagreement, while we observe a perfect coincidence between the results of the improved action at \( \beta = 5.7 \) and \( \beta = 6.0 \) with the ones at \( \beta = 6.3 \), Wilson. This suggests that the data at \( \beta = 6.3 \) Wilson or, equivalently, 5.7, 6.0 improved, are the asymptotic quenched ones for \( f_\pi \).

Note also that we convert to MeV by using the extrapolated value for the \( \rho \) mass. The \( f_\pi \) data nicely extrapolate at 132 MeV thus demonstrating the consistence between the two scales induced by the \( \rho \) mass and \( f_\pi \). (We have to say that since we rely on the perturbative evaluation of \( Z_A \) \[20\], it is possible, even if unlikely, that this good result is a coincidence. It is clear that a non-perturbative computation of the multiplicative renormalization constant would be welcome.)

We can get an estimate for \( f_\pi \) also from the data in the \( \pi \) channel, modulo a constant. We denote this estimate \( (f_\pi^{(\pi)}) \), and we quote its ratio with the true \( f_\pi \), which is remarkably stable.

\( f_\rho \) is more delicate since with Wilson action there is a strong discrepancy between the perturbative and lattice evaluation for \( Z_V \) \[21\]. Even if in this case the situation is supposed to be better, the perturbative results for \( Z_V \) \[20\], hence for the lattice estimate of \( f_\rho \), should be considered with some extra care.

The results for baryons are shown in Figs.6 in the form of Ape invariant
mass plot. The plots we believe are self-explanatory, and demonstrate the consistency of the results for the improved action at $\beta = 5.7$ and $\beta = 6.0$ with the ones for the Wilson action at $\beta = 6.3$ and $\beta = 6.0$, while the Wilson results at $\beta = 5.7$ exhibit clear scaling violations.

A final comment concerns the Proton-$\Delta$ splitting. In this case we can compare data from different lattices by normalizing with the running value of the $\rho$ mass, like in the Ape plot, thus avoiding the systematics connected with the chiral extrapolation. Looking at Fig. 7, we see that the $\Delta$ and the proton masses are clearly well resolved, and the results, in the largish errors, agree with the ones obtained on the other lattices. (As discussed in ref. [3] the data at $\beta = 6.3$ are possibly biased from the opposite parity partner.) All the other data are interpolating almost linearly between the 0 and the infinite mass limit.

**Acknowledgements**

The numerical simulations were performed using the Ape computers in Roma and Pisa. The updating code, the bulk of the inversion and the analysis software were the same used in refs. [3], [10], [11]: we are especially grateful to Enzo Marinari for this crucial help.

We owe special thanks to Carlotta Pittori for innumerable valuable discussions about improved actions and renormalization constants. Finally, we wish also thank Herbert Hamber, Enzo Marinari, Guido Martinelli, Federico Rapuano, Gian Carlo Rossi, Gaetano Salina and Raffaele Tripiccione for interesting conversations and useful comments.

The work of MPL is supported by the National Science Foundation, NSF-PHY 92-00148.

**References**

[1] T. Eguchi and N. Kawamoto, Nucl. Phys. B237 (1984) 609.; H. Hamber and C. M. Wu, Phys. Lett. 133B (1983) 351.

[2] T. A. DeGrand, Nucl. Phys. B20 (Proc. Suppl.) (1991) 353.
[3] M. Guagnelli, M.-P. Lombardo, E. Marinari, G. Parisi and G. Salina, 
Quenched Mass Spectrum of Lattice QCD on a 1 Gigaflops Computer, 
Nucl. Phys. B, in press

[4] K. Symanzik, Nucl. Phys. B226 1983 187, 205.

[5] B. Sheikholeslami and R. Wohlert, Nucl.Phys. B310 572.

[6] G. Heatlie et al., Nucl. Phys. B352 (1991) 266

[7] G. Martinelli,C. T. Sachrajda,G. Salina and A. Vladikas, An Exploratory 
Study of Meson Spectroscopy and Matrix Elements with an Improved 
Wilson action at $\beta = 6.0$, Nucl. Phys. B, in press

[8] UKQCD Collaboration, C.R. Allton et al., Quenched Hadrons using Wil- 
son and O(a)-Improved Fermion actions at $\beta = 6.2$, Edinburgh preprint 
92/506, SHEP 91/92-15, 9205016/hep-lat

[9] Talks given by P. B. Mackenzie and A. X. El-Khadra at Lattice ’91, 
Fermilab preprints CONF-92/09-T and CONF-92/10-T, and references 
therein.

[10] The Ape Collaboration, P. Bacilieri et al., Nucl. Phys. B317 (1989) 509.

[11] The Ape Collaboration, S. Cabasino et al., Phys. Lett. 258B (1991) 195.

[12] P. Menotti and A. Pelissetto, Comm.Math.Phys. 113 (1987) 369, and 
references therein.

[13] E. Marinari, Nucl. Phys. B9 (Proc. Suppl.) (1989) 209.

[14] T. A. DeGrand and M. W. Hecht, More about Orbitally Excited Hadrons 
from Lattice QCD, Colorado preprint, COLO-HEP-282.

[15] The Ape Collaboration, S. Cabasino et al., Nucl.Phys. B343 (1990) 228.

[16] The Ape Collaboration, S. Cabasino et al., Nucl.Phys. 258B (1991) 202.

[17] The Ape Collaboration, P. Bacilieri et al., Phys.Lett 214B (1988) 115.

[18] M. Guagnelli, M.-P. Lombardo, E. Marinari, G. Parisi and G. Salina, in 
preparation.
[19] M. Bochicchio et al., Nucl. Phys. B372 (1992) 903.

[20] H. Hamber and C. M. Wu, Phys. Lett. B136B (1984) 255.

[21] L. Maiani and G. Martinelli, Phys. Lett. B178B (1986) 265, and references therein.
Figure Captions.

1 Effective proton mass estimator as a function of the time distance at $\beta = 5.7$ for the different quark masses. $24^3 \times 32$, squares. $9^3 \times 18$, circles.

2 a) Pion Green functions for the four different quark masses at $\beta = 6.0$. The results of two (one) particle fits are shown as dashed (dotted) lines. b) as a), for the proton.

3 $\pi$ (squares, dashed lines) and $\tilde{\pi}$ (diamonds, dotted lines) chiral extrapolations at $\beta = 5.7$ (left) and $\beta = 6.0$ (right). The lines are the results of the fits.

4 a) The splitting $\rho^2 - \pi^2$ plotted against $\pi^2$. b) as a), with included the Wilson results. Everything in unit of the extrapolated $\rho$ mass.

5 a) $f_\pi$ in MeV versus $\pi^2$ in unit of the extrapolated $\rho$ mass. b) as a), with included the Wilson results.

6 a) Ape plot : $m_p/m_\rho$ vs $m_\pi^2/m_\rho^2$ b) as a), with Wilson results.

7 $(m_\Delta - m_p)/m_\rho$ vs $m_\pi^2/m_\rho^2$ for the improved and standard Wilson action.