Weak Limits of Consecutive Projections
and of Greedy Steps

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Abstract—Let $H$ be a Hilbert space. We investigate the properties of weak limit points of iterates of random projections onto $K \geq 2$ closed convex sets in $H$ and the parallel properties of weak limit points of the residuals of random greedy approximation with respect to $K$ dictionaries. In the case of convex sets these properties imply weak convergence in all the cases known so far. In particular, we give a short proof of the theorem of Amemiya and Ando on weak convergence when the convex sets are subspaces. The question of weak convergence in general remains open.

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1. INTRODUCTION

In what follows $H$ is a real Hilbert space with scalar product $\langle \cdot , \cdot \rangle$ and norm $|\cdot |$.

Let $A_1, \ldots , A_K$ be closed and convex sets in $H$, $K \geq 2$, such that $A_1 \cap \ldots \cap A_K = \{0\}$. Let $P_i$ denote the metric projection onto $A_i$. Let $i(n) \in \{1, \ldots , K\}$ be a fixed sequence containing each $k \in \{1, \ldots , K\}$ infinitely often. For $x_0 \in H$, we consider the sequence

$$x_n = P_{i(n)} x_{n-1}, \quad n = 1, 2, \ldots \quad (1.1)$$

In the case when $A_i$ are closed subspaces of $H$, the convergence properties of the sequence $\{x_n\}$ are well understood. If the sequence of the indices $\{i_n\}$ is periodic, then the sequence $\{x_n\}$ converges in norm [18, 10]. The rate of the convergence depending on the position of the subspaces and of the initial point is known [2–4, 7]. In this context a connection with the convergence properties of the greedy algorithm has been recently discovered [7]. If no extra information about the indices or the position of the subspaces is known, divergence might occur even for $K = 3$ (see [19, 13–15]). The sequence $\{x_n\}$, however, always converges weakly to zero in the case of subspaces [1].

In the absence of linearity, when the sets $A_i$ are just closed and convex, the situation is different. Even for $K = 2$ the sequence $\{x_n\}$ might diverge in norm, although the sequence of indices is inevitably periodic [11, 12, 16]. Weak convergence is known only under additional conditions: when $K \leq 3$ (see [9]), or when the indices $i(n)$ are periodic [8], or when the sets are “somewhat symmetric” [9].

**Problem 1.** Let $W = W(x_0)$ be the set of all partial weak limits of the sequence (1.1). Is it true that $W = \{0\}$?

We investigate the structure of the set $W$ and give new short proofs of the weak convergence in all of the cases mentioned above. In particular, we give a short proof of the theorem of Amemiya and Ando on weak convergence when the convex sets are subspaces. The general case remains, however, open.

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In the spirit of [7], we establish a connection with the weak convergence problem for the greedy approximation with respect to $K$ dictionaries. The structural properties of the set of weak partial limits of this greedy approximation turn out to be the same. We have hit the same bounds of knowledge while seeking weak convergence.

2. PROJECTIONS ONTO CONVEX SETS

Let $A_1, \ldots, A_K$ be closed and convex sets in $H$, $K \geq 2$, such that $A_1 \cap \ldots \cap A_K = \{0\}$. Let $i(n) \in \{1, \ldots, K\}$ be a fixed sequence containing each $k \in \{1, \ldots, K\}$ infinitely often, and let the sequence $\{x_n\}$ be defined by (1.1). We assume $i(n) \neq i(n+1)$ without loss of generality.

We study the structure of the set $W = W(x_0)$ of all partial weak limits of the sequence $\{x_n\}$.

Since the nearest point projection onto a convex set is a 1-Lipschitz mapping, the norms $|x_n|$ decrease and the set $W$ is always nonempty. We may assume that $|x_n| \langle K, R > 0$, as $R = 0$ implies the convergence in norm and hence $W = \{0\}$.

For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \ldots, K\}$ such that $w \in A_{J(w)}$. Here we use the notation $A_{J} = \bigcap_{j \in J} A_j$.

Since $|x_n - x_{n-1}|^2 \leq |x_{n-1}|^2 - |x_n|^2$, for $x_n \in A_{i(n)}$ we have

$$\text{dist}(x_n, A_{i(n)}) \to 0, \quad n \to \infty,$$

(2.1)

for any fixed $m$. Therefore $|J(w)| \geq 2$ for each $w \in W$, and $W$ is a weakly closed subset of $\bigcup_{|J| \geq 2} A_J \cap B(0, R)$, where $B(0, R)$ is the closed ball centered at 0 of radius $R$. If $w \neq 0$, then $|J(w)| < K$, since $\bigcap A_i = \{0\}$. It also follows from (2.1) that in the case $i(n) \equiv n (\text{mod} K)$ of alternating projections we have $J(w) = \{1, \ldots, K\}$ for each $w$, and hence $W = \{0\}$. In particular, if we have just two convex sets, then the sequence $\{x_n\}$ converges weakly to zero.

Next we show that if $W$ contains an element of maximal norm, then $W = \{0\}$.

**Theorem 1.** For each $w \in W$, $w \neq 0$, one can find another element $w' \in W$ with the following properties:

(i) $|J(w') \setminus J(w)| \geq 1$;

(ii) $|J(w') \cap J(w)| \geq 2$;

(iii) $|J(w')| \geq 3$;

(iv) $\langle w' - w, a \rangle \geq 0$ for every $a \in A_{J(w)}$.

In particular, $|w'| \geq |w|$ by property (i), since $\langle w' - w, w \rangle \geq 0$ and hence $|w'|^2 \geq |w|^2 + |w - w'|^2$.

**Proof.** 1. Let

$$x_{n_k} \to w, \quad i(n_k) \in J(w).$$

Taking a subsequence of indices $k$ if needed, we can choose $q \notin J(w)$ with the following property: for any $k$ there is a number $m_k \in (n_k, n_{k+1})$ with $i(m_k) = q$ such that for any $n \in [n_k, m_k]$ we have $i(n) \in J(w)$, and hence $i(n) \neq q$. Again taking a subsequence of indices $k$ if needed, we get $x_{m_k} \to w'$, and this is the definition of $w'$. Clearly $w' \in A_q$; hence $J(w') \supset q$ and property (i) holds.

2. We have $m_k - n_k \to \infty$ in view of (2.1). Therefore, the numbers $i(m_k - 1)$ and $i(m_k - 2)$ belong to $J(w)$ for all sufficiently large $k$. For each $k$, these numbers are different by assumption. We choose two different numbers $i, j \in J(w)$ such that $i(m_k - 1) = i$ and $i(m_k - 2) = j$ for infinitely many $k$. In view of (2.1) this implies $w' \in A_i \cap A_j$; hence $i, j \in J(w') \cap J(w)$ and property (ii) holds.

3. Property (iii) follows from properties (i) and (ii).

4. For any $a \in A_{J(w)}$, we have

$$\langle w' - w, a \rangle = \lim_{k \to \infty} \langle x_{m_k} - x_{n_k}, a \rangle = \lim_{k \to \infty} \sum_{n=n_k+1}^{m_k} \langle x_n - x_{n-1}, a \rangle = \lim_{k \to \infty} \sum_{n=n_k+1}^{m_k-1} \langle x_n - x_{n-1}, a \rangle$$

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be strictly larger than $A$ since every term in the sum is nonnegative and
\[
\lim_{k \to \infty} |x_{m_k} - 1| = \lim_{k \to \infty} |x_{n_k} - 1| = R. \quad \square
\]

Remark 1. The inequality in property (iv) of Theorem 1 holds for $a \in A_J(w, w')$, where $J(w, w') = \{ i(n) : n \in [n_k, m_k - 1], k = 1, 2, \ldots \}$. Since $J(w, w') \subset J(w)$, the set $A_J(w, w')$ can be strictly larger than $A_J(w)$.

The following corollary is a special case of [9, Theorem 2]; our proof is different.

Corollary 1. If $K \leq 3$, then $W = \{0\}$.

Proof. The case $K = 2$ was explained before Theorem 1. Assume that $K = 3$ and that there is $w \in W \setminus \{0\}$. By Theorem 1 there is $w' \in W$ with $|w'| > |w|$ and $J(w') = \{1, 2, 3\}$. Hence $w' = 0$, which is a contradiction. □

Assume all the convex sets $A_i$ are cones. Assume, moreover, that the intersection of any triple of these cones with the unit sphere has a positive distance to the intersection of any other triple. Then $W = \{0\}$ according to the next corollary.

Corollary 2. Suppose for every $r > 0$ there exists $\delta(r) > 0$ such that for any two different triples $\{i, j, k\}$ and $\{i, j, l\}$ and elements $u \in A_{\{i, j, k\}} \cap S(0, r)$ and $v \in A_{\{i, j, l\}} \cap S(0, r)$ we have $|u - v| > \delta(r)$. Then $W = \{0\}$.

Proof. Suppose $W \neq \{0\}$. Using Theorem 1, we construct a sequence $w_n \in W$ such that $w_1 \neq 0$, $w_{n+1} = w_n$, $|J(w_n)| \geq 3$, $J(w_n) \neq J(w_{n+1})$, and $|J(w_n) \cap J(w_{n+1})| \geq 2$ for each $n$. So we get $w_n \in A_{\{i, j, k\}}$ and $w_{n+1} \in A_{\{i, j, l\}}$ for some $i$, $j$, and $k \neq l$ depending on $n$. Since the sequence $|w_n|$ is bounded and increasing, let $r = \lim_{n \to \infty} |w_n|$. Hence, $u_n = r w_n/(|w_n|) \in A_{\{i, j, k\}}$ for all sufficiently large $n$. Defining $w_n = (1 + t_n)u_n$, $t_n > 0$, for those $n$ we have
\[
|w_{n+1}|^2 \geq |w_n|^2 + |w_{n+1} - w_n|^2 = |w_n|^2 + |(1 + t_n)u_{n+1} - (1 + t_n)u_n|^2
\]
\[
= |w_n|^2 + |u_{n+1} - u_n|^2 + |t_n u_{n+1} - t_n u_n|^2 + 2(t_n u_{n+1} - u_n, t_n u_{n+1} - t_n u_n)
\]
\[
\geq |w_n|^2 + |u_{n+1} - u_n|^2 + 2t_n |u_{n+1}|^2 + 2t_n |u_n|^2 - 2(t_n u_{n+1} + t_n)\langle u_{n+1}, u_n \rangle
\]
\[
\geq |w_n|^2 + |u_{n+1} - u_n|^2 + 2 \left( \frac{r}{2} \right)^2 (t_n + t_n - (t_{n+1} + t_n))
\]
\[
= |w_n|^2 + |u_{n+1} - u_n|^2 + |w_n|^2 + \delta \left( \frac{r}{2} \right)^2,
\]
This means, however, that $|w_n|$ is unbounded. □

Theorem 2. If $W \neq \{0\}$, then one can find two different elements $w, w' \in W$ such that
\[
w = \lim_{k \to \infty} x_{m_k} \quad \text{and} \quad w' = \lim_{k \to \infty} x_{m_k},
\]
where $n_1 < m_1 < n_2 < m_2 < \ldots$, and $i(n) \in J(w) \cap J(w')$ for any $n \in \bigcup_k (n_k, m_k)$. Consequently,
(i) $\langle w', w, a \rangle \geq 0$ for every $a \in A_J(w)$;
(ii) $\langle w', w, b \rangle \geq 0$ for every $b \in A_J(w')$.

Proof. Both inequalities (i) and (ii) follow from the first statement of the theorem. The proof repeats that of property (iv) in Theorem 1: all projections between $n_k$ and $m_k$ have indices in $J(w) \cap J(w')$. 
To prove the first statement, we take \( w \) and \( w' \) from Theorem 1. All indices in \( J = \{i(n) : n \in \bigcup_k(n_k, m_k)\} \) belong to \( J(w) \) by the proof of Theorem 1. If \( J \subset J(w') \), we are done. Otherwise we define \( v_k \) as the largest numbers \( n \in (n_k, m_k) \) such that \( i(n) \notin J(w') \). By taking a subsequence of indices \( k \) such that all these \( i(v_k) \) are the same, we get \( x_{v_k} \to v \neq w' \). Then we redefine \( w := v \) and \( n_k := v_k \). The renewed set \( J = \{i(n) : n \in \bigcup_k(n_k, m_k)\} \) is now a subset of \( J(w') \), and the number of elements in it has decreased by at least one. If this new \( J \) is also included in the new \( J(w) \), we stop. Otherwise this time we choose the numbers \( v_k \) as the least numbers \( n \in (n_k, m_k) \) such that \( i(n) \notin J(w) \). Then we redefine \( w' \) and the numbers \( m_k \). Since \( |J| \) is decreasing, this oscillation process stops in a finite number of steps: \( |J| \) cannot become less than 2. In case \( |J| = 2 \) we obviously have \( J \subset J(w) \cap J(w') \). 

Dye and Reich used in [9] the so-called weak internal points (WIPs) of a convex set to prove a nonlinear result that properly contains the original linear theorem of Amemiya and Ando: if all the \( K \) closed convex sets are linear subspaces, then the sequence \( \{x_n\} \) converges weakly [1]. In our version of the theorem we assume that zero is a WIP in each of the convex sets \( A_k \). Again, the result is a special case of Theorem 5 of [9]; our proof is different.

**Corollary 3.** Assume that zero is a weak internal point of each of the \( K \) convex sets \( A_k \): if \( a \in A_k \) then \( -\lambda a \in A_k \) for some \( \lambda = \lambda(a, k) > 0 \). Then \( W = \{0\} \). In particular, if all \( A_k \) are closed linear subspaces of \( H \), then the sequence (1.1) converges weakly to zero.

**Proof.** Assuming \( W \neq \{0\} \), we take the two different elements \( w, w' \in W \) from Theorem 2. Using Theorem 2(i) for \( a = w \) gives \( \langle w' - w, w \rangle \geq 0 \) and \( |w'|^2 \geq |w|^2 + |w - w'|^2 \). Using Theorem 2(ii) for \( b = -\lambda w' \) gives \( \langle w' - w, -\lambda w' \rangle \geq 0 \) and \( |w|^2 \geq |w'|^2 + |w - w'|^2 \). Hence \( w = w' \), which is a contradiction.  

### 3. PARALLELS BETWEEN PROJECTING ONTO CONVEX SETS AND GREEDY APPROXIMATION

A subset \( D \) of the unit sphere \( S(H) \) of the Hilbert space \( H \) is called a dictionary if its span is dense in \( H \). Assume, moreover, that \( D \) does not lie in a half-space: for any nonzero \( v \in H \), there exists \( g \in D \) such that \( \langle v, g \rangle > 0 \). The greedy approximation algorithm with respect to \( D \) then generates for any element \( x = x_0 \in H \) the sequence

\[
x_{n+1} = x_n - \langle x_n, g_{n+1} \rangle g_{n+1}, \quad n = 0, 1, \ldots, \tag{3.1}
\]

where the element \( g_{n+1} \in D \) is such that

\[
\langle x_n, g_{n+1} \rangle = \max \{ \langle x_n, g \rangle : g \in D \}.
\]

The existence of \( \max \{ \langle x, g \rangle : g \in D \} \) for every \( x \in H \) is an additional condition on \( D \). If the maximum is attained on several elements of \( D \), any of them is selected as \( g_{n+1} \). This algorithm is called the pure greedy algorithm, in contrast to other approximation algorithms whose names contain the word “greedy” (see [20]).

For any symmetric dictionary \( D \) the pure greedy algorithm converges in norm (see [20, Ch. 2]). That is, \( x_n \to 0 \) for any initial element \( x = x_0 \), and \( x \) is represented as a norm-convergent series \( \sum_{n=0}^{\infty} \langle x_n, g_{n+1} \rangle g_{n+1} \). If \( D \) is not symmetric, the greedy algorithm may diverge in norm [6], although it always converges weakly to zero [5].

Several details of the divergence construction in [6] occur to be similar to those of [12]. The “bridge” between these two seemingly different examples is the theorem of Moreau [17]:

\[
P_A(x) = x - P_{A^*}(x) \tag{3.2}
\]
for any \( x \in H \), any convex cone \( A \subset H \), and its polar cone
\[
A^* = \{ y \in H : \langle y, z \rangle \leq 0 \ \forall z \in A \}.
\]

Recall that both papers [11] and [12] provide examples of convex cones \( A_1, A_2 \subset H \) such that \( A_1 \cap A_2 = \{0\} \) and the alternating projections on these cones diverge in norm for certain starting elements. Formula (3.2) allows us to interpret this result as an example of a divergent greedy algorithm with respect to the dictionary \( D = (A_1^* \cup A_2^*) \cap S(H) \). Indeed, \( D \) does not lie in a half-space, as \( A_1 \cap A_2 = \{0\} \), and for any greedy residual \( x_n \) lying in, say, \( A_1 \), we have
\[
\max \{ \langle x_n, g \rangle : g \in D \} = \max \{ \langle x_n, g \rangle : g \in A_2^* \cap S(H) \},
\]
so that \( x_{n+1} = x_n - P_{A_2^*}(x_n) = P_{A_2}(x_n) \in A_2 \). Thus the author of [6] did not have to reinvent the wheel: [11] and [12] both provided the example he needed. However, the example in [6] is simpler than those of [11] and [12]: it uses a discrete dictionary without the extra care needed to build it of convex cones.

The above parallels between projecting onto convex sets and greedy approximation have already been noticed in [7] in the special case of subspaces. In the context of this paper, these parallels bring up the question of weak divergence of random greedy steps with respect to several dictionaries. This problem is considered in the next section. It turns out to have the same “bounds of knowledge” as the problem of the weak divergence of random projections onto several convex sets.

4. GREEDY APPROXIMATION WITH RESPECT TO SEVERAL DICTIONARIES

Let \( K \geq 2 \) and \( D_1, \ldots, D_K \) be subsets of \( S(H) \) such that their union \( \bigcup_{i=1}^{K} D_i \) is contained in no half-space: for any nonzero \( v \in H \), there exists \( g \in \bigcup_{i=1}^{K} D_i \) such that \( \langle v, g \rangle > 0 \). This implies that the set \( \bigcup_{i=1}^{K} D_i \) is spanning; we will call here the sets \( D_i \) dictionaries.

Assume that for every \( x \in H \) and each \( i \in \{1, \ldots, K\} \) the following condition holds: if \( \sup_{g \in D_i} \langle x, g \rangle > 0 \), then the supremum is attained at some element \( g_i(x) \in D_i \). If it is attained at several elements of \( D_i \), then we denote by \( g_i(x) \) any one of them. If \( \sup_{g \in D_i} \langle x, g \rangle \leq 0 \), we put \( g_i(x) = 0 \).

Clearly, our assumption means that the set \( \Lambda(D_i) = \{ \lambda g : \lambda \geq 0, g \in D_i \} \) is proximal and the element \( \langle x, g_i(x) \rangle g_i(x) \) belongs to the metric projection \( P_{\Lambda(D_i)}(x) \).

Let \( G_i \) denote the mapping corresponding to one step of the greedy algorithm with respect to the dictionary \( D_i \):
\[
G_i(x) = x - \langle x, g_i(x) \rangle g_i(x).
\]

Note that
\[
|G_i(x)|^2 = |x|^2 - |x - G_i(x)|^2. \tag{4.1}
\]

Let \( i(n) \in \{1, \ldots, K\} \) be a fixed sequence containing each \( k \in \{1, \ldots, K\} \) infinitely often and such that \( i(n) \neq i(n+1) \) for all \( n \in \mathbb{N} \). For \( x_0 \in H \), we consider the sequence
\[
x_n = G_{i(n)}x_{n-1}, \quad n = 1, 2, \ldots. \tag{4.2}
\]

As we have already mentioned above, this sequence may diverge in norm even in the case of one dictionary. Both examples in [11] and [12] can be interpreted as norm divergence examples of the residuals \( x_n \) for alternating greedy steps with respect to two dictionaries. So we are interested in weak convergence, just as in the case of projections.

**Problem 2.** Let \( \tilde{W} = \tilde{W}(x_0) \) be the set of all partial weak limits of the sequence (4.2). Is it true that \( \tilde{W} = \{0\} \)?
Note that our assumption that \( i(n) \neq i(n + 1) \) for all \( n \in \mathbb{N} \), which is very natural for successive projections onto convex sets, no longer seems so natural in the setting of several dictionaries. Successive greedy steps with respect to the same dictionary may well be nontrivial. That is, they can change the next remainder \( x_n \) each time. However, in the proofs below it is essential that the sequence of indices \( i(n) \) is not locally constant.

According to (4.1),
\[
|x_{n+1}|^2 = |x_n|^2 - |x_n - x_{n+1}|^2,
\]
so the norms \( |x_n| \) are decreasing. We may assume that \( |x_n| \gg R > 0 \), since \( R = 0 \) implies \( \tilde{W} = \{0\} \).

We define the closed convex cones
\[
A_i = \{ y \in H : \langle y, g \rangle \leq 0 \ \forall g \in D_i \}, \quad i = 1, \ldots, K.
\]
Notice that \( A_i \) is the polar cone of \( \text{conv} \Lambda(D_i) \). We also note that \( \bigcap_{i=1}^K A_i = \{0\} \), since the union \( \bigcup_{i=1}^K D_i \) of the dictionaries is not contained in any half-space by assumption. As in Section 2, for \( w \in \tilde{W} \), we denote by \( J(w) \) the maximal subset of \( \{1, \ldots, K\} \) such that \( w \in A_{J(w)} \), and again use the notation \( A_J = \bigcap_{j \in J} A_j \). Let us nevertheless stress that the set \( \tilde{W} \) is the result of greedy approximation with respect to the dictionaries \( D_1, \ldots, D_K \).

Let us prove that \( |J(w)| \geq 2 \) for each \( w \in \tilde{W} \). The convergence \( x_{n_j} \to w \) implies the convergence \( x_{n_j+m} \to w \) for any fixed \( m \), since \( \lim_{i \to \infty} |x_i - x_{i+m}| = 0 \) by (4.3). Suppose the sequence \( i(n_j + m) \) contains some \( k \) infinitely often. If \( w \notin A_k \), then \( \langle w, g \rangle > \delta > 0 \) for some \( g \in D_k \), which yields \( \langle x_{n_j+m}, g \rangle > \delta \) for all sufficiently large \( j \), so that \( |x_{n_j+m+1}|^2 \leq |x_{n_j+m}|^2 - \delta^2 \) for such \( j \) with \( i(n_j + m) = k \), and we arrive at a contradiction with \( |x_n| \gg R > 0 \). So we get \( w \in A_k \), and since one can find at least two such \( k \) using different \( m \), we conclude that \( |J(w)| \geq 2 \).

The same argument shows that in the case \( i(n) \equiv n \pmod{K} \) of alternating greedy algorithm we have \( J(w) = \{1, \ldots, K\} \) for each \( w \), and hence \( \tilde{W} = \{0\} \).

Thus, \( \tilde{W} \) is a weakly closed subset of \( \bigcup_{2 \leq |J|} A_J \cap B(0,R) \).

**Theorem 3.** For each \( w \in \tilde{W} \), \( w \neq 0 \), one can find another element \( w' \in \tilde{W} \) with the following properties:

(i) \( |J(w') \setminus J(w)| \geq 1 \);
(ii) \( |J(w') \cap J(w)| \geq 2 \);
(iii) \( |J(w')| \geq 3 \);
(iv) \( \langle w' - w, a \rangle \geq 0 \) for every \( a \in A_{J(w)} \).

In particular, \( |w'| > |w| \) by property (i), since \( \langle w' - w, w \rangle \geq 0 \) and hence \( |w'|^2 \geq |w|^2 + |w - w'|^2 \).

**Proof.** Theorem 3 is formally identical to Theorem 1, and the proofs of properties (i)–(iii) follow the same reasoning.

The proof of property (iv) is slightly different. As in the proof of Theorem 1, we have two alternating sequences \( n_1 < m_1 < n_2 < m_2 < \ldots \) such that
\[
x_{n_k} \to w, \quad x_{m_k} \to w',
\]
and \( i(n) \in J(w) \) for all \( n \in \bigcup_k [n_k, m_k) \).

For any \( a \in A_{J(w)} \), we have
\[
\langle w' - w, a \rangle = \lim_{k \to \infty} \langle x_{m_k} - x_{n_k}, a \rangle = \lim_{k \to \infty} \sum_{n=n_k+1}^{m_k} \langle x_n - x_{n-1}, a \rangle = \lim_{k \to \infty} \sum_{n=n_k+1}^{m_k-1} \langle x_n - x_{n-1}, a \rangle
\]
\[
= \lim_{k \to \infty} \sum_{n=n_k+1}^{m_k-1} (-1)^{\langle x_{n-1}, g_{i(n)}(x_{n-1}) \rangle} \langle g_{i(n)}(x_{n-1}), a \rangle \geq 0.
\]
The last inequality holds since each of the summands is nonnegative: $\langle x, g_i(x) \rangle \geq 0$ for any $x$ and $i$ by the definition of $g_i$, and $\langle g_i(n)(x_{n-1}), a \rangle \leq 0$ since $i(n) \in J(w)$ and $a \in A_{J(w)}$. \hfill \Box

**Remark 2.** The inequality in property (iv) of Theorem 3 holds for $a \in A_{J(w',w''')}$, where $J(w,w') = \{i(n) : n \in [n_k, m_k - 1], k = 1, 2, \ldots \}$. Since $J(w,w') \subset J(w)$, the set $A_{J(w,w''')}$ can be strictly larger than $A_{J(w)}$.

**Corollary 4.** If $K \leq 3$, then $\tilde{W} = \{0\}$.

**Proof.** If $K = 2$, we have an alternating greedy algorithm and hence weak convergence to zero, as explained before Theorem 3.

Assume that $K = 3$ and that there is $w \in \tilde{W} \setminus \{0\}$. By Theorem 3 there is $w' \in \tilde{W}$ with $|w'| > |w|$ and $J(w') = \{1, 2, 3\}$ Hence $w' = 0$, which is a contradiction. \hfill \Box

**Corollary 5.** Suppose for any four indices $i,j,k,l \in \{1, \ldots, K\}$ the inequality

$$\inf_{s \in S(H)} \sup_{g \in D_{ij} \cup D_{jk} \cup D_{kl} \cup D_{il}} \langle s, g \rangle > 0 \quad (4.4)$$

holds. Then $\tilde{W} = \{0\}$.

**Proof.** Inequalities (4.4) provide $\delta > 0$ such that for any distinct $i,j,k,l$ and $u \in A_{\{i,j,k\}} \cap S(H)$ there exists $g \in D_l$ with $\langle u, g \rangle > \delta$. Hence, for any two different triples $\{i,j,k\}$ and $\{i,j,l\}$ and unit elements $u \in A_{\{i,j,k\}}$ and $v \in A_{\{i,j,l\}}$ we have $|u - v| > \delta$:

$$|u - v| \geq \langle u - v, g \rangle \geq \langle u, g \rangle > \delta.$$

Further, we repeat the proof of Corollary 2. Suppose $\tilde{W} \neq \{0\}$. By Theorem 3, we can produce a sequence $w_n \in \tilde{W}$ such that $w_{n+1} = w'_n$, $|J(w_n)| \geq 3$, $J(w_n) \neq J(w_{n+1})$, and $|J(w_n) \cap J(w_{n+1})| \geq 2$ for each $n$. So we get $w_n \in A_{\{i,j,k\}}$ and $w_{n+1} \in A_{\{i,j,l\}}$ for some $i,j$, and $k \neq l$ depending on $n$. Therefore, using the fact that the sets $A_{\{i,j,k\}}$ and $A_{\{i,j,l\}}$ are cones, we can refine the inequality from Theorem 3:

$$|w_{n+1}|^2 \geq |w_n|^2 + |w_{n+1} - w_n|^2 \geq |w_n|^2 + \delta^2 |w_n|^2.$$ 

This, however, means that $|w_n|$ is unbounded. \hfill \Box

**Theorem 4.** If $\tilde{W} \neq \{0\}$, then one can find two different elements $w,w' \in \tilde{W}$ such that

$$w = \text{weak lim}_{k \to \infty} x_{n_k} \quad \text{and} \quad w' = \text{weak lim}_{k \to \infty} x_{m_k},$$

where $n_1 < m_1 < n_2 < m_2 < \ldots$, and $i(n) \in J(w) \cap J(w')$ for any $n \in \bigcup_k (n_k, m_k)$. Consequently,

(i) $\langle w', w, a \rangle \geq 0$ for every $a \in A_{J(w)}$;

(ii) $\langle w', w, b \rangle \geq 0$ for every $b \in A_{J(w')}$. 

**Proof.** We repeat the proof of Theorem 2; it is purely combinatorial. The inequalities follow from the first statement as in the proof of property (iv) in Theorem 3. \hfill \Box

**Corollary 6.** If all $D_k$ are symmetric, then $\tilde{W} = \{0\}$.

**Proof.** Assume that $\tilde{W} \neq \{0\}$. We take the two different elements $w,w' \in \tilde{W}$ from Theorem 4. Using Theorem 4(i) for $a = w$ gives $\langle w' - w, w \rangle \geq 0$, and hence $|w'|^2 \geq |w|^2 + |w - w'|^2$. Using Theorem 4(ii) for $b = -w'$ gives $\langle w' - w, -w' \rangle \geq 0$, and hence $|w|^2 \geq |w'|^2 + |w - w'|^2$. Thus we get $w = w'$, which is a contradiction. \hfill \Box

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