Electromagnetic fields in jets

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ABSTRACT
The magnetic fields and energy flows in an astronomical jet described by our earlier model are calculated in detail. Though the field distribution varies with the external pressure function \( p(z) \), it depends only weakly on the other boundary conditions. Individual field lines were plotted; the lines become nearly vertical at the bottom and are twisted at the top. An animation of a field line’s motion was made, which shows the line being wound up by the accretion disc’s differential rotation and rising as a result of this. The distribution of Poynting flux within the jet indicates that much of the energy flows up the jet from the inside of the accretion disc but a substantial fraction flows back down to the outside.

Key words: accretion, accretion discs – MHD – ISM: jets and outflows – galaxies: jets.

1 INTRODUCTION
Collimated radiating jets are a feature of many astronomical objects. They have been observed in radio galaxies (see Hargrave & Ryle 1974; Begelman, Blandford & Rees 1984; Carilli & Barthel 1996; Gizani & Leahy 2003), quasars and young stellar objects. They may also be present in gamma ray bursts (see Uzdensky & MacFadyen 2006), dying stars and microquasars (see Mirabel & Rodríguez 1999). In all these objects, the jet emerges along the axis of an accretion disc. As the form of the highly relativistic jets in active galactic nuclei and the much slower ones around stars is very similar, it is likely that relativistic effects are not essential for the formation of jets. The presence of strong magnetic fields in radio galaxies was deduced from their synchrotron radiation by Baade (1956), but the similarity of the observed structures suggests that magnetism plays a large role in the formation of all jets. In particular, recent observations of maser emission confirm that magnetic fields are important in collimating stellar jets (see Vlemmings, Diamond & Imai 2006).

Following earlier numerical work on the Grad–Shafranov equation by Barnes & Sturrock (1972) and Sturrock & Barnes (1972), Lovelace (1976) obtained jet-like magnetic structures by considering the winding up of a pre-existing uniform field. However, collimation by a pre-existing field side-steps the problem of why the field is collimated to start with. In Lynden-Bell (1996) (hereafter Paper II), the growing magnetic towers picture was developed in which a magnetic field twisted by an accretion disc was confined by an ambient coronal pressure. This provided a crude picture as to why the field was collimated. In later work (Lynden-Bell 2006—hereafter Paper IV) a non-relativistic model was developed which involved a force-free magnetic field being twisted up by a swirling accretion disc to give a jet-like collimated magnetic tower, now in a stratified atmosphere. Expressions for the magnetic fields were given; however, the fields were only illustrated in the idealized ‘dunce’s cap’ configuration for which the solutions were explicit and exact.

In this paper the more general implicit expressions will be numerically evaluated to depict in detail the fields and energy flows within the jet predicted by the model under various boundary conditions. Kato, Mineshinge & Shibata (2004) have computed the starting of such jets in a magnetic accretion disc. Li et al. (2006) extended these ideas to long jets that are not force-free initially but become so as they evolve. Nakamura, Li & Li (2006) performed numerical calculations of jets within an isothermally stratified atmosphere up to heights that are twice the diameter of the disc; our analytical solutions extend to far greater heights. Discussions of jet stability are given by Nakamura & Meier (2004) and Nakamura, Li & Li (2007). Lovelace et al. (2002) computed solutions in a toroidal box and found a strongly collimated Poynting flux as predicted by the growing towers picture. However, their calculations were limited by their box. Romanova et al. (1998) and Ustyugova et al. (1999) computed the evolution of magnetic fields above discs, with the fields not force-free. Though Ustyugova et al. (1999) considered fields initially of split monopole type and Romanova et al. (1998) treated a more complicated field structure, both obtained magnetically dominated collimated outflows with the matter flow everywhere away from the disc. As discussed later, we find that there is also a smaller Poynting flux back down on to the outer feet of the field lines, which is due to the field lines’ tension pulling on the disc. There are many simulations of highly relativistic black hole environments, such as Gammie, McKinney & Gábor (2003), McKinney (2005) and McKinney & Narayan (2007a,b).

2 THEORETICAL BASIS
A brief summary of the theory in Paper IV will be provided, as the investigation of the fields which follows is based on the model developed in this paper. Cylindrical polar coordinates are used. The

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2.1 Introducing the problem to be solved

The field is located within a magnetically dominated cavity in the external medium. The displacement current is neglected, so that \( \nabla \times \mathbf{B} = 4\pi j \). Though the inside of the cavity has negligible gas pressure, it is assumed to be a perfect electrical conductor, so that the medium is frozen on to the magnetic field lines. Due to the negligible gas pressure the magnetic force has nothing to oppose it so the field in the cavity is force-free \((j \times B = 0)\). Hence a solution is defined by the following boundary conditions: the normal field \( B_n \) at the base, the twist on each field line \( \Phi \), and the gas pressure on the outside of the cavity \( p(z) \). As the equations for the magnetic field do not involve time derivatives, the time dependent solution can be found as a series of static solutions with different twists of the field lines at different times \( t \).

Within a radius \( R \) at \( z = 0 \), a total magnetic flux \( P(R, 0) \) rises out of the accretion disc. The field lines corresponding to this \( P \) turn over and return to the disc at an outer axis; radial symmetry means they define a surface of constant \( P(R, z) \) as shown in Fig. 4, where \( P(R, z) \) gives the flux through a circle of radius \( R \) at height \( z \). The disc’s rotation with angular velocity \( \Omega_d(R) \) produces a total twist on the field lines of \( \Phi(P) = [\Omega_d(R(R)) - \Omega_d(R_c(R))]t = \Omega(R)t \). Here \( R_c(P) \) and \( R_h(P) \) are the radii of the inner and outer foot points of the field line on the flux surface \( P(R, z) = P \) constant. At the boundary of the magnetic cavity, an external gas pressure \( p(z) \) is specified which must be balanced by the internal magnetic pressure. As \( \nabla \cdot \mathbf{B} = 0 \) the magnetic field may be written in terms of the flux function \( P(R, z) \) as (see Mestel 1999)

\[
\mathbf{B} = \frac{1}{2\pi} \nabla P \times \nabla \phi + B_\phi \hat{\phi},
\]

where \( \phi \) is the azimuthal coordinate. With the force-free condition

\[
j \times \mathbf{B} = 0 \implies \nabla \times \mathbf{B} = 4\pi j = \beta' \mathbf{B}.
\]

This implies that taking the poloidal component of the curl of (1) results in

\[
B_\phi = \frac{\beta(P)}{2\pi R}, \quad \beta' = \frac{d\beta}{dP},
\]

where \( \beta(P) \) is constant along each field line. The toroidal component leads to the differential equation for the flux function \( P \):

\[
R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial P}{\partial R} \right) + \frac{\partial^2 P}{\partial z^2} = -\beta(P)\beta'(P),
\]

where the magnetic pressure must be \( B^2/8\pi = p(z) \) on the surface of the magnetic cavity defined by \( R = R_m(z) \). This equation is non-linear and is to be solved within an unknown surface with an unknown function \( \beta(P) \) [which must result in the specified values for \( \Phi(P) \)].

2.2 Variational method and its results for the cavity shape

The easiest way to overcome these difficulties is to minimize the energy \( W \) given by

\[
W = \int \left[ \int \frac{B_\phi^2 + B_{\phi z}^2 + B_z^2}{8\pi} R \, dR \, d\phi + p(z)A \right] \, dz;
\]

using a trial function, apply the boundary conditions and solve the resulting variational equations as is done in Paper IV. As the radial flux is constant while the magnetic cavity grows to a great height, \( B_R \) is neglected. Using a trial function for the toroidal flux such that the twist on field lines is the geometric mean between the case with all the twist at the top and that with the twist uniformly distributed in height, the variational equation was solved generally and the following results were obtained:

\[
\pi R_m^2 = \frac{I_p}{\sqrt{2\pi p(z)}},
\]

\[
z^2 \frac{8\pi p(z)}{J^2} = \frac{\sqrt{2\pi p(z)}}{16\pi},
\]

where \( \sqrt{2\pi p(z)} = \Phi(P_m)/p_m \int \Phi(P) \, dP \). \( P_m(z) \) is the maximum flux \( P \) at a height \( z \) and \( F \) is the maximum \( P \) at \( z = 0 \). \( J \) and \( I \) are defined as \( J = (B_R^2)/B_z^2 \) and \( I = (B_z^2)/|B_\phi|^2 \) where \( | \cdot | \) indicates averaging over a horizontal plane, \( B_\phi = R_m^{-1} \int_0^{R_m} B_\phi dR \) and \( |B_\phi| = A^{-1} \int_0^a |B_\phi| 2\pi R dR \); however, we approximate \( J = I = 1 \) for our calculations (for exact values see Paper IV).

The shape of the magnetic cavity can be found by solving equations (6) and (7) for the cavity radius \( R_m(z) \). However, this only results in valid solutions when the pressure falls off more slowly than for \( n = 4 \), where \( p(z) = a(z + b)^{−4} \). (Paper IV also develops a ram pressure confined model which is valid for pressure falling faster than \( n = 4 \).) Plots of the resulting cavity shapes show that as the jet rises it becomes more collimated; constant pressure gives conical cavities whose cross-section decreases with height, decreasing pressure gives centrally bulged cavities.

The two models we will use for the relative angular velocity of the foot points \( \Omega(P) \) are

(i) the simple model \( \Omega(P) = \Omega_0(1 - P/F) \), where \( \Omega_0 \) is the maximum of \( \Omega \),

(ii) the more physically realistic dipole model \( \Omega(P) = \Omega_d[1 - (P/P)^{1/2}] \) which corresponds to a uniformly magnetized rotating body surrounded by an accretion disc. It arises from a flux distribution \( P(R, \nu) = FR^2/\nu \) for \( R < l \) (the ‘inside of the disc’) and \( P(R, \nu) = FL/\nu R \) for \( R > l \) (the ‘outside of the disc’), combined with a foot point angular velocity \( \Omega_d = \Omega_0 \) for \( R < l \) and \( \Omega_d = \Omega_d(l/R)^{−1/2} \) for \( R > l \).

2.3 Magnetic field in the cavity

The field within the cavity is found by writing

\[
P(R, z) = P_m(z) f(\lambda)
\]

for each height, where \( \lambda = R^2/R_m^2(z) \). The resulting conditions on \( f(\lambda) \) \([f(0) = f(1) = 0, f_{\text{max}} = 1]\) are very restrictive, so that in the tall tower approximation one can neglect variation of \( f \) with height. Additionally, neglecting second derivatives and squares of first derivatives of \( R_m(z) \) and \( P_m(z) \), equation (4) becomes

\[
\frac{d^2 P_m}{R_m^2} \frac{\lambda^2 f}{\lambda^2} = -\beta\beta',
\]

which gives \( \beta\beta' \) as a product of a function of \( \lambda \) and a function of \( \lambda \). However, it is also a function of \( P \), which is such a product. These facts imply that \( \beta\beta' \) is a power law in \( P \). Now \( B_\phi/B_R \) is zero on axis so \( \beta(P) = 0 \) when \( P \) is zero. It follows that since \( \beta\beta' \) is a power law, \( \beta \) is as well, so we can write \( \beta = C_1 P^C \). Inserting this in equation (4) and separating variables gives

\[
R_m = C_2 P_m^{l-1},
\]

\[
\lambda^2 f = -C_2 \nu f^{2l-1},
\]

with \( C_2 = C_1 C_2^l/4 \).
The second equation can be used to find an expression for \( v \). Multiplying it through by \(-f/\lambda\), integrating by parts, and multiplying by \( P_n/\lambda^2 \) it can be shown that this equation becomes

\[
\langle B_z^2 \rangle = v \langle B_\phi^2 \rangle. \tag{12}
\]

By expanding a slice through the jet as in Lynden-Bell (2003) with \( B_\phi = 0 \), a different expression for the ratio \( \langle B_z^2 \rangle/\langle B_\phi^2 \rangle \) can be derived:

\[
\frac{\langle B_z^2 \rangle}{\langle B_\phi^2 \rangle} = 1/(2-s), \tag{13}
\]

where at each height

\[
s(z) = -\frac{1}{A(z)p(z)} \int_0^{\top} A(z')(dp/dz')dz'. \tag{14}
\]

Thus \( v \) becomes a weak function of \( z \)

\[
v(z) = \frac{1}{2-s} \tag{15}
\]

so that the weak variation of the profile with height is determined, as a profile \( f(\lambda) \) for every height can be obtained from (11).

When the external pressure is constant, \( f \) can be found exactly (as \( v = 1/2 \)). Integrating equation (11) leads to the solution

\[
f(\lambda) = -e\lambda \log \lambda. \tag{16}
\]

The second equation can be used to approximate analytic solutions for \( f(\lambda) \) to increase calculation speed.

In the cases plotted here, the value of \( v(z) \) is mostly <1. A good approximation for \( v(z) < 1 \), which we will use for falling pressure throughout this paper, is \( f(\lambda) = g(\lambda)/g(\lambda_1) \) where

\[
g(\lambda) = \frac{\lambda(1-\lambda^v)}{\alpha(1+\alpha\lambda^v)} \tag{17}
\]

with \( \alpha_2 \) determined by the condition that \( f \) has a maximum at \( \lambda = \lambda_1 \)

\[
\alpha_2 = \frac{\lambda_1^z(1+\alpha) - 1}{\lambda_1^z[1(1-\alpha) - \lambda_1^v]} \tag{18}
\]

and with \( \alpha(z) = 2v(z) - 1 \) and \( \lambda_1(z) = [1 + 0.07073\alpha(z)]/e \). Comparing this solution to exact numerical computations shows that it is gives good results for \( v < 3 \).

### 3 ELECTROMAGNETIC FIELDS

#### 3.1 General method

The jet fields must be found numerically in all but the simplest cases, as the fields are described by both implicit and differential equations. The calculations were performed using the \textsc{Mathematica} program. The errors, unless otherwise indicated, are negligible compared with the magnitude of the results.

The different boundary conditions which define the model are specified by the parameters \( t, \Omega_0, F \) as well as, assuming a power pressure law, \( n, a, b \) where \( p(z) = a(z + b)^n \). As similar jets occur in a variety of objects with very different sizes and properties, we decided not to attempt to choose physically realistic values for the parameters; instead we use the following default values to investigate the fields. We choose units of time \( t \) such that \( 2\pi t \) corresponds to one turn at angular velocity \( \Omega_0 \) and measure magnetic flux in units of \( F \). The length-scale was chosen so that \( a = 1 \). With \( b = 0.1 \) this corresponds to a radius of the disc of order \( b \) for the \( n = 3 \) case. Most plots are either for \( n = 3 \) (which is a realistic value) or for constant pressure. Values of \( t = 30 \) and \( b = 0.1 \) were used; these give a collimated tall tower where the approximations are valid. We use the simple model \( \Omega(P) \) unless mentioned otherwise.

#### 3.2 Flux surfaces

As the field lines rise up from and return to the disc, they define a tube-like surface of constant flux \( P \). The equation giving \( P \) is

\[
P(R, z) = P_m(z)f(\lambda) = \frac{R_m^2(z)\sqrt{2\pi p(z)f(R^2/R_m^2(z))}}{I}. \tag{18}
\]

The cross-sections of the surfaces of constant \( P \) on which the lines of force lie are plotted in Fig. 1 for both constant pressure and for a pressure which falls as \( n = 3 \). The boundary of the magnetic cavity is also indicated in the diagrams.

#### 3.2.1 Results

For the \( n = 3 \) and \( n = 0 \) external pressure distributions the constant flux contours/poloidal fields conform to the shape of the corresponding magnetic cavity. At the centre of the jet, the contour lines are nearly vertical; they are closely spaced at the base.
3.2.2 Discussion

In the constant pressure case, it is remarkable how similar the form of the contours and the cavity are to those in the ‘dunce’s cap’ model evaluated analytically in Paper IV. While the pressure in the ‘dunce’s cap’ model is also constant, \( \Omega(P) \) on the disc is different to the simple model used here. This suggests that the field structure and the shape of the jet are determined mainly by the external pressure as the disc’s rotation profile only has a secondary influence on the jet’s structure.

3.3 Magnetic field components

Substituting equation (8) into the expressions for the fields given by equation (4) leads to

\[
B_\lambda(R, z) = \frac{P_m(z)}{\pi R_{m}^3} f'(\lambda),
\]

(19)

\[
B_\phi(R, z) = \frac{C_3}{2\pi R} [P_m(z) f(\lambda)]^{\lambda z},
\]

(20)

with \( P_m(z) = \pi R_{m}^2(z) \sqrt{2\pi p(z)} / I \) and \( \lambda = R^2 / R_{m}^2(z) \).

\(|B_\lambda|\) and \(|B_\phi|\) were calculated for the \( n = 3 \) pressure law and are shown in Fig. 2. \( B^2 = B_\lambda^2 + B_\phi^2 \) and the twist \( |B_\phi|/|B| \) were plotted for both constant pressure and the \( n = 3 \) pressure law in Figs 2 and 3, respectively (\( n = 0 \) plots were made at \( t = 50 \) so that both cavities have nearly equal height). The plots are linear; to preserve detail we cut them off at large field strengths (\( n = 3 \) cut-offs 270, 10, 15 for \( B^2, |B_\phi|, |B_\lambda| \)). To investigate the dependence of the fields on \( \Omega(P) \), \( B^2 \) was also found for the dipole model \( \Omega(P) \). It is depicted in Fig. 2.

It is also useful to determine how the fields change when the values for \( t, \Omega_0, F, a \) and \( b \) are varied. For the \( n = 3 \) case the parameters were individually changed from their default values and the results noted.

3.3.1 Results

When the pressure falls as \( n = 3 \) it can be seen that \( B_\lambda \) is largest at the base and core of the jet. At a radius \( R_1(z) \) which corresponds to \( \lambda_1 \) at each height, \( B_\lambda \) goes to zero. \( B_\phi \) is large at \( R_1 \) and at the base; there is almost no azimuthal field at the centre and at the edges of the cavity. The twist is highest at \( R_1 \), and is negligible on axis and near the cavity boundary, \( B^2 \) is maximal at the bottom of the jet and on the upper axis.

The constant pressure jet has the highest \( B^2 \) field strength on axis. The field’s twist is zero near the axis and the boundary.

The plot of \( B^2 \) for the dipole rotation model is very similar to the simple model plot; the most noticeable difference is that the dipole jet is taller.

The height and collimation of the magnetic cavities increases with time \( t \), though the distribution of fields does not change much. Varying the other parameters \( \Omega_0, F \) and \( a \) changes the height of the cavity \( h_{\text{cav}} \) at a given time (increasing \( \Omega_0 \) or \( F \) increases the height, increasing \( a \) decreases it); however, this only affects the field distribution substantially if the cavity is thus in a different pressure regime. So for \( p(z) = a(z + b)^{-3} \), if \( h_{\text{cav}} \) becomes < \( b \), the fields become as in the \( n = 0 \) case, if \( h_{\text{cav}} \approx b \), the fields are an intermediate case, and when \( h_{\text{cav}} > b \) we retain the fields as in Fig. 2.

3.3.2 Discussion

One clearly expects strong magnetic fields at the base because all the field lines rise up from and return to the disc. When the outside pressure is approximately constant, theory predicts pinching of the
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The behaviour of $B_r, B_\phi$ and the twist at $R_1$ is due to the field lines turning over at this radius with a large proportion of their total twist at their tops.

The observation that the form of the fields does not depend very strongly on $\Omega(P)$ or the other boundary conditions lets us deduce that the plots made are quite generally valid. Ram pressure confined jet fields (see Paper IV) are probably like the $n=3$ fields, as the cavity shapes are similar. The increased cavity height for the dipole model at the same $t$ is due to the larger differential rotation of the dipole model for all values of $P$.

3.4 Individual field lines

In order to see the shapes of individual field lines, 3D plots were necessary. The field lines are given by $dR/BR = R d\phi/B_\phi = dz/B_z$, or using the expressions for the components of $B$:

$$\frac{dR}{dP/\beta} = R \frac{d\phi}{\beta(P)} = \frac{dz}{dP/\partial R}.$$  \hspace{1cm} (21)

Thus a differential equation for the field line is

$$\frac{dz}{d\phi} = R \frac{\partial P}{\beta(P)} = \frac{P_{\lambda}^{1-\nu}}{C_{\nu} f(\lambda)}. \hspace{1cm} (22)$$

The rightmost expression is a function only of $z$ for a given field line, that is, a given constant value of $P$. The expression is double valued: the field line rises from and then returns to the disc, so that $\lambda(z)$ is double valued. Two different functions for $\lambda(z)$ were obtained by solving $f(\lambda, z) = P/P_m(z)$ for $\lambda(z)$ subject to either $\lambda_{in}(z) < \lambda_1$ or $\lambda_{out}(z) > \lambda_1$. The function $z(\phi)$ for all points on the field line was found by solving equation (22) numerically using $\lambda_{in}(z)$ up until the point where $z$ reached a value corresponding to the top of the field line, then integrating back down to the base using $\lambda_{out}(z)$. The corresponding values of $R(\phi)$ were found using $R^2 = R_m^2(z(\phi)) \lambda(z(\phi))$.

3D plots of the $P = 0.3F$ field line for $n=3$ and simple model $\Omega(P)$ were thus constructed at three different times. They are shown in Fig. 4 along with the constant $P$ surface and an outline of the magnetic cavity.

Figure 3. $B^2$ and $|B_\phi|/|B|$ for $p = \text{constant at } t = 90$ for the simple model.

Figure 4. Individual field line ($P = 0.3F$) at $t = 30, 50, 90$ for simple model and $p \propto (z + 0.1)^{-3}$. 

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In order to be able to visualize the time development of the field lines, a 3D animation of the motion of the $P = 0.3\Omega$ field line was made for $30 < t < 80$. The outer foot point of the field line was set to rotate with the small angular velocity $\Omega_b = 0.07$. The animation is provided on the website www.ast.cam.ac.uk/research/repository/bfieldline.html.

### 3.4.1 Results

The field lines are twisted most at the top; the outer part of the field line has less twist on it than the inner part.

The video depicts the field line being wound up and rising rapidly. At the bottom of the cavity the field line’s twist decreases, so that eventually the line is nearly vertical.

### 3.4.2 Discussion

The increasing twist near the top is due to the trial function used. In the video it can be seen that the field line is dragged around by the disc at the inside foot point and held back by the outer foot point. The field line’s tension thus applies a torque and transfers energy from the inner to the outer part of the disc.

The straightening of the field lines near the base means that the fields become nearly antiparallel, which could lead to magnetic reconnection if the perfect conductivity breaks down.

### 3.5 Poynting flux

As there is no $E$ along $B$ due to the assumption of perfect conductivity, and as the velocity of a field line $u$ is the $cE \times B / B^2$ drift velocity, the electric field is given by $E = -u \times B / c$ and $u \cdot B = 0$. The azimuthal angle of an intersection of the field lines with a $z = \text{constant}$ plane is shown in Paper IV to be

$$\phi = \Psi + \Omega_d\phi_0 + \Omega(P)t\psi(z, P),$$

where $\Omega_d\phi_0$ is the angular velocity of the outer foot point of the field line, $\Psi$ is the initial azimuth at which the field line intersected the disc at the outer foot point, and $q = \int_0^t (\lambda d\log z / \lambda d\log f) dz / \int_1^P \log \lambda / \log f dz$ is the fraction of the total twist on the field line $P$ which occurs by height $z$ (integrating up from the outer foot point). In the same paper, the velocity of the intersection is derived in terms of the components of the field line’s velocity $u$:

$$R \dot{\phi} = u_{\phi} \left( 1 + \frac{B_z^2}{B_0^2} \right) + u_{B} \left( \frac{B_z B_0}{B_0^2} \right).$$

Applying Faraday’s law to the flow through a circle about the axis in the $z = \text{constant}$ plane gives

$$2\pi R E_{\phi} = -\frac{\dot{P}}{c} = -\frac{u_{B} B_0 - u_{B_R} B_0}{c}.$$  \hspace{1cm} (25)

We solve these two equations and $u \cdot B = 0$ for $u$; neglecting $B_R$ we obtain

$$u_{\phi} = -\frac{B_0 B_{\phi}}{|B|^2}; \quad u_{B} = -\frac{u_{B_z} B_0}{B_0}; \quad u_{B_R} = -\frac{\dot{P}}{B_{\phi}}.$$  \hspace{1cm} (26)

$E$ follows directly from $c E = B \times u$, giving

$$c E_{R} = -R B_{\phi}; \quad c E_{\phi} = -\dot{P}; \quad c E_{B} = -\frac{\dot{P} B_{\phi}}{B_0}.$$  \hspace{1cm} (27)

It is interesting to investigate the distribution of the Poynting flux component $S_z$. As $S = \frac{1}{4\pi} E \times B$

$$S \propto |B|^2 u$$

so that

$$S_z \propto -B_0 B_R R \dot{\phi}.$$  \hspace{1cm} (28)

$\dot{q}$ was approximated as $\Delta q / \Delta t$ by calculating $\Delta q$ at times separated by $\Delta t = 0.03$ (using either $\lambda_{in}$ or $\lambda_{out}$ to overcome the double-valuedness of $q$). Values of $r = 50$ and $b = 4$ were used to calculate $S, \Omega(P)$ was given by the dipole model as it has a well defined $\Omega_d$. It was verified that the changes did not affect the field distributions substantially. As calculations of $R \dot{\phi}$ were very time consuming, $B_R \dot{\phi}$ was evaluated on a $70 \times 70$ evenly spaced grid across the width and height of the jet; cubic interpolation was then applied to obtain values at other points. The vertical Poynting flux was thus plotted in Fig. 5.

### 3.5.1 Results

The $z$ component of the Poynting flux is large and positive just above the inside of the accretion disc. Above the outer part of the disc there is a large downward flow.

### 3.5.2 Discussion

Energy flows up from the inside of the disc because the disc drags the field lines around and thus does work on the field. The flow spreads to the top and sides of the jet, where it enables the expansion against the external pressure. Some of the energy returns to the outer part of the disc (as the fields pull on the disc there); the fraction this constitutes was calculated as $\int_B^{B_{\phi}} R S_z dR / \int_B^{B_{\phi}} R S_z dR = 0.47$. It thus appears that jets are quite an efficient mechanism for energy transfer from the inside to the outside of the accretion disc.
The energy flow into the jet can also be calculated by considering torques on the disc. The couple $GdP = -\frac{\mu_0}{4\pi} 2\pi RdP$ carried by the tube of flux between $P$ and $P + dP$ is constant along the tube. This couple inevitably does work on the rotating outer feet of the field lines, giving a power flow $G(P)dP \Omega(R_{i}(P))$ down from the jet into the outside of the disc. This differs from the results of many simulations (Ustyugova et al. 1999; Gammie et al. 2003; McKinney 2005; McKinney & Narayan 2007a,b; and Narayan, McKinney & Farmer (2007) on winds) because they have no magnetic flux returning to the disc and so do not have the downward Poynting flux which results from this. Romanova et al. (1998) do not have a force-free magnetosphere. The disc does work on the field lines at the inner feet, giving a power input of $G(P)dP \Omega(R_{i}(P))$. The difference between the power input and output goes into expanding the magnetic cavity against the pressure and increasing the magnetic energy. Integrating over all flux tubes, the net power put into the expanding jet is $\int_{0}^{P} G(P)\Omega(P)dP$.

As $-\overrightarrow{P}/c = E_{\phi}$ is small compared to the other $E$ components but $E \perp B$, $E$ points close to normally out of the $P$ surface at each point.

4 CONCLUSIONS

(i) The $B$ field is strongest at the base of the magnetic cavity and on axis. The field structure is only weakly affected by the detailed profile of the shear in the accretion disc’s rotation.

(ii) The field lines are twisted more at their tops and at smaller radii. At the cavity base the field lines become straight and vertical.

(iii) In the case investigated the energy rises from the inside of the disc; somewhat less than half returns to the outer part of the disc, the remainder flows up the jet.

The published numerical calculations of others referred to above do not extend to great heights due to the difficulties with numerically induced resistivity. Although in our work only the ‘dunce’s cap’ solution is truly exact, the solutions in this paper are still basically analytic and do not suffer this inconvenience. In the few cases such as Nakamura et al. (2006) where similar problems are solved, the results agree with the start of our solutions.

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