POSITIVE PERIODIC SOLUTION FOR GENERALIZED BASENER-ROSS MODEL

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Abstract. This paper is devoted to the existence of at least one positive periodic solution for generalized Basener-Ross model with time-dependent coefficients. Our proof is based on Manásevich-Mawhin continuation theorem, Leray-Schauder alternative principle, fixed point theorem in cones. Moreover, we obtain that there are at least two positive periodic solutions for this model.

1. Introduction. The Easter Island, which once flourished, and finally, due to the rapid population growth and excessive use of resources, the people on the island was almost extinct. According to records, Basener-Ross [2] in 2004 developed a model of resources and population on an isolated island as follows

\[
\begin{align*}
x'(t) &= cx(t) \left(1 - \frac{x(t)}{k}\right) - hy(t), \\
y'(t) &= ay(t) \left(1 - \frac{y(t)}{x(t)}\right),
\end{align*}
\]

where \(x\) and \(y\) represent the total amount of resources and the population of the island, respectively. \(k\) is the capacity of the island to carry resources, and \(c\) is the resource growth rate when the resources on the island are far less than the carrying capacity, \(a\) is the net growth rate of the population when resources are abundant, \(hy\) represent the amount of resources that human acquired, and \(h\) is the harvesting constant. It is obvious that \(a, c, h, k\) are positive constants.

According to model (1), residents will survive rely on the existing resources on the island [11, 13]. If resources are used excessively, humans will increase rapidly in a short time and eventually disappear. If resources are used reasonably, people and resources will always coexist. That is to say, the rate of human access to resources is not always greater than the rate of growth of resources, otherwise there will be a catastrophe of human. In other words, \(h \geq c\) does not always stand up.

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Of course, the Basener-Ross model is just an ideal model. In real life, both changes in the external environment and human living habits will lead to changes in parameters in the model. For example, the changes of climate conditions, the occurrence of natural disasters, the development and evolution of granite landforms, the changes of surface temperatures, and changes in the proportion of elements in air and soil all will lead to changes in resource carrying capacity and resource growth rate (see [3], [1], [8], [7]). In this paper, we consider the case that the parameters $c, k$ are functions about $t$. Then system (1) is transformed into the following form

\[
\begin{cases}
x'(t) = c(t)x(t) \left(1 - \frac{x(t)}{k(t)}\right) - hy(t), \\
y'(t) = ay(t) \left(1 - \frac{y(t)}{x(t)}\right),
\end{cases}
\]  

where $a$, $h$ are positive constants and $c, k \in C(\mathbb{R}, (0, +\infty))$ are $\omega$-periodic functions. It is meaningful to study the existence of positive $\omega$-periodic solutions for the Basener-Ross model (2). From the point of view of scientific development, it is close relationship between human development and resource utilization, as well as the important conditions for sustainable development.

By the second equation of system (2), take

\[x(t) = \frac{a y^2(t)}{ay(t) - y'(t)}.
\]

Substituting the above equation into the first equation of system (2), we get the following second order differential equation

\[y''(t) + p(t)y'(t) + q(t)y(t) = \alpha \frac{(y'(t))^2}{y(t)} + \beta(t)y^2(t), \]  

where $p(t) = a + c(t) - 2h$, $q(t) = (h - c(t))a$, $\alpha = 2 - \frac{h}{a}$, $\beta(t) = -\frac{ac(t)}{k(t)}$. It is clear that the derivable term is relatively complex, in order to get around the derivable term, the change of variable $y = u^\mu$, where $\mu = \frac{1}{1-\alpha}$. Due to definitions of $\mu$ and $\alpha$, it is easy to see that $\mu \neq 0$ and $\mu \neq -1$. With the change, converting equation (3) into the following form

\[u''(t) + p(t)u'(t) + \frac{q(t)}{\mu}u(t) = \frac{\beta(t)}{\mu}u^{\mu+1}(t).\]  

It is readily seen that the existence of positive $\omega$-periodic solutions to model (2) reduces to the existence of positive $\omega$-periodic solutions to equation (4).

The rest of this paper is organized as follows. In section 2, applying Manásevich-Mawhin continuation theorem, we prove that equation (4) has at least one positive $\omega$-periodic solution, which can be used the cases of a strong singularity and without singularity. In section 3, using Green function and Leray-Schauder alternative principle, we get second existence results for equation (4), the results are applicable to the case of a strong singularity, as well as the case of a weak singularity. In section 4, fixed point theorem in cones is applied to present third existence results for equation (4). Moreover, we prove that equation (4) has at least two positive $\omega$-periodic solutions.
2. Existence results (I). In this section, we investigate the existence of a positive \(\omega\)-periodic solution for equation (4) by using Manásevich-Mawhin continuation theorem. At first, embed equation (4) into the following equation family with a parameter \(\lambda \in (0, 1)\):
\[
 u''(t) + \lambda p(t)u'(t) + \lambda \frac{q(t)}{\mu} u(t) = \lambda \frac{\beta(t)}{\mu} u^{\mu+1}(t). \tag{5}
\]

By applications of Theorem 3.1 in [9], we give the following lemma which will be used to prove the existence of a positive \(\omega\)-periodic solution for equation (4).

Lemma 2.1. Assume that there exist constants \(E_1, E_2, E_3\) with \(E_2, E_3 > 0\) and \(E_1 < E_2\) such that the following conditions hold:

1. Each possible \(\omega\)-periodic solution \(v(t)\) to equation (3) such that \(E_1 < v(t) < E_2\) for all \(t \in [0, \omega]\) and \(\|v\| < E_3\), here \(\|v\| := \max_{t \in [0, \omega]} |v(t)|\).

2. Each possible solution \(C\) to equation
\[
 \int_0^\omega \frac{q(t)}{\mu} C dt - \int_0^\omega \frac{\beta(t)}{\mu} C^{\mu+1} dt = 0
\]
satisfies \(E_1 < C < E_2\).

3. It holds
\[
 \left( \int_0^\omega \frac{q(t)}{\mu} E_1 dt - \int_0^\omega \frac{\beta(t)}{\mu} E_1^{\mu+1} dt \right) \cdot \left( \int_0^\omega \frac{q(t)}{\mu} E_2 dt - \int_0^\omega \frac{\beta(t)}{\mu} E_2^{\mu+1} dt \right) < 0.
\]

Then equation (4) has at least one \(\omega\)-periodic solution.

Next, we investigate the existence of a positive \(\omega\)-periodic solution for equation (4) by applying Lemma 2.1. According to the value ranges of \(\mu\), we get two existence results of positive \(\omega\)-periodic solutions for equation (4) without a singularity (i.e. \(-1 < \mu < 0\)) and with a strong singularity (i.e. \(\mu \leq -2\)).

2.1. \(-1 < \mu < 0\).

Theorem 2.2. Assume that \(-1 < \mu < 0\) holds. Furthermore, suppose the following conditions hold:

(H1) There are constants \(D_1, D_2\) with \(D_1 < 0 < D_2\) such that \(\frac{\beta(t)}{\mu} u^{\mu+1}(t) - \frac{q(t)}{\mu} u(t) > 0\) for \((t, u) \in [0, \omega] \times (-\infty, D_1)\), and \(\frac{\beta(t)}{\mu} u^{\mu+1}(t) - \frac{q(t)}{\mu} u(t) < 0\) for \((t, u) \in [0, \omega] \times (D_2, +\infty)\).

(H2) \(\|p\| \omega + \|q\| \omega^2 \mu^2 < 1\).

If \(\mu = -\frac{\nu_2}{\nu_2} = -\frac{\text{odd}}{\text{even}}\), then equation (4) has at least one positive \(\omega\)-periodic solution, where \(\nu_1\) and \(\nu_2\) are inter-prime positive constants.

Proof. Firstly, we claim that the set of all \(\omega\)-periodic solutions for equation (5) are bounded. Let
\[
 u \in X := \{u \in C(\mathbb{R}, \mathbb{R}) : u(t + \omega) - u(t) \equiv 0 \text{ for } t \in \mathbb{R}\} \tag{6}
\]
be an arbitrary \(\omega\)-periodic solution of equation (5). We claim that there is a point \(\eta_1 \in [0, \omega]\) such that
\[
 |u(\eta_1)| \leq D, \tag{7}
\]
where \(D := \max\{|D_1|, D_2\}\) and \(D_1, D_2\) are defined by condition (H1).
In fact, let \( t_* \), \( t^* \) be the global minimum, maximum points of \( u(t) \), respectively. Then we get \( u'(t_*) = u''(t^*) = 0 \), \( u''(t_*) \geq 0 \) and \( u''(t^*) \leq 0 \). Embed \( t_* \), \( t^* \) into equation (5), we get

\[
\frac{\beta(t_*)}{\mu} u^{\mu+1}(t_*) - \frac{q(t_*)}{\mu} u(t_*) \geq 0 \tag{8}
\]

and

\[
\frac{\beta(t^*)}{\mu} u^{\mu+1}(t^*) - \frac{q(t^*)}{\mu} u(t^*) \leq 0. \tag{9}
\]

From equations (8) and (9), condition \((H_1)\) implies that

\[ u(t^*) \geq D_1 \text{ and } u(t_*) \leq D_2. \]

Case I: If \( u(t^*) \in [D_1, D_2] \), let \( \eta_1 = t^* \), then, \( |u(\eta_1)| \leq D \).

Case II: If \( u(t^*) \in [D_2, +\infty) \), from continuity of \( u(t) \), we know that there will exist a point \( \eta_1 \in [0, \omega] \) such that \( |u(\eta_1)| = D_2 \leq D \).

Combining the above two cases, equation (7) holds. Therefore, it is clear that

\[
||u|| = \max_{t \in [0, \omega]} u(t) = \max_{t \in [0, \omega]} \frac{1}{2} (u(t) + u(t - \omega))
\]

\[
= \max_{t \in [0, \omega]} \frac{1}{2} \left( u(\eta_1) + \int_{\eta_1}^{t} u'(s)ds + u(\eta_1) - \int_{t-\omega}^{\eta_1} u'(s)ds \right)
\]

\[
\leq |u(\eta_1)| + \frac{1}{2} \left( \int_{\eta_1}^{t} |u'(s)|ds + \int_{t-\omega}^{\eta_1} |u'(s)|ds \right) \tag{10}
\]

\[
\leq D + \frac{1}{2} \int_{t-\omega}^{t} |u'(s)|ds
\]

\[
= D + \frac{1}{2} \int_{0}^{\omega} |u'(t)|dt.
\]

Multiplying both sides of equation (5) by \( u(t) \) and integrating over the interval \([0, \omega]\), we get

\[
\int_{0}^{\omega} u''(t)u(t)dt + \lambda \int_{0}^{\omega} p(t)u'(t)u(t)dt + \lambda \int_{0}^{\omega} \frac{q(t)}{\mu} u^{2}(t)dt = \lambda \int_{0}^{\omega} \frac{\beta(t)}{\mu} u^{\mu+2}(t)dt. \tag{11}
\]

Substituting \( \int_{0}^{\omega} u''(t)u(t)dt = -\int_{0}^{\omega} |u'(t)|^2dt \) into equation (11), we have

\[
\int_{0}^{\omega} |u'(t)|^2dt = \lambda \int_{0}^{\omega} p(t)u'(t)u(t)dt + \lambda \int_{0}^{\omega} \frac{q(t)}{\mu} u^{2}(t)dt - \lambda \int_{0}^{\omega} \frac{\beta(t)}{\mu} u^{\mu+2}(t)dt
\]

\[
\leq ||p||||u|| \int_{0}^{\omega} |u'(t)|dt + \frac{||q||}{\mu} ||u||^2 + \frac{||\beta||}{\mu} ||u||^{\mu+2}
\]

\[
\leq ||p|| \left( D + \frac{1}{2} \int_{0}^{\omega} |u'(t)|dt \right) \int_{0}^{\omega} |u'(t)|dt + \frac{||q||}{\mu} \left( \int_{0}^{\omega} |u'(t)|dt \right)^2 + \frac{||\beta||}{\mu} \left( D + \frac{1}{2} \int_{0}^{\omega} |u'(t)|dt \right)^{\mu+2}
\]

\[
= \left( \frac{1}{2} + \frac{||q||}{4\mu} \right) \left( \int_{0}^{\omega} |u'(t)|dt \right)^2 + \left( ||p||D + \frac{||q||D\omega}{\mu} \right) \int_{0}^{\omega} |u'(t)|dt + \frac{||q||D^2\omega}{2\mu} + \frac{||\beta||}{2\mu^{\mu+2}} \left( 1 + \frac{D}{2} \int_{0}^{\omega} |u'(t)|dt \right)^{\mu+2}, \tag{12}
\]
where \( p := \max_{t \in [0, \omega]} |p(t)|, q := \max_{t \in [0, \omega]} |q(t)|, \beta := \max_{t \in [0, \omega]} |\beta(t)|. \)

Next, we introduce a classical inequality, there exist constants \( \nu > 0 \) and \( m(\nu) > 0 \) which is only dependent on \( \nu \) such that
\[
(1 + u)^\nu \leq 1 + (1 + \nu)u, \; \text{for} \; u \in (0, m(\nu)).
\]

Then, we consider the following two cases.

Case I: If \( \frac{D}{\int_{0}^{\omega} |\omega'(t)|dt} \geq m(\mu + 2) \). Then, it is obvious that
\[
\int_{0}^{\omega} |u'(t)|^2 dt \leq \frac{2D}{m(\mu + 2)} := M_{11}.
\]

Case II: If \( \frac{D}{\int_{0}^{\omega} |\omega'(t)|dt} < m(\mu + 2) \). Since \( \mu + 2 > 1 \), using inequality (13) and the Hölder inequality, equation (12) turns into
\[
\int_{0}^{\omega} |u'(t)|^2 dt \leq \left( \frac{\|p\|}{2} + \frac{\|q\| \omega}{4|\mu|} \right) \left( \int_{0}^{\omega} |u'(t)|^2 dt \right)^{\frac{1}{2}} + \left( \frac{\|p\| D + \|q\| D\omega}{|\mu|} \right) \left( \int_{0}^{\omega} |u'(t)| dt \right) \left( \frac{\|p\| D + \|q\| D\omega}{|\mu|} \right)^{\frac{1}{2}}
\]
\[
+ \frac{\|\beta\| \omega^2}{2|\mu|} \left( \int_{0}^{\omega} |u'(t)|^2 dt \right)^{\frac{1}{2}} + \frac{\|\beta\| \omega^{\mu+3}}{2^{\mu+2}|\mu|} \left( \int_{0}^{\omega} |u'(t)|^2 dt \right)^{\frac{\mu+3}{2}}
\]
\[
+ \frac{(\mu + 3)\|\beta\| D\omega^{\mu+3}}{2^{\mu+1}|\mu|} \left( \int_{0}^{\omega} |u'(t)|^2 dt \right)^{\frac{\mu+3}{2}}.
\]

In view of \(-1 < \mu < 0 \), it is obvious that \( \frac{1}{2} < \frac{\mu+3}{2} < 1 \), from equation (15) and condition \((H_2)\), we can observe that \( \int_{0}^{\omega} |u'(t)|^2 dt \) is bounded, that is to say, there exists a constant \( M_{12} \) such that
\[
\int_{0}^{\omega} |u'(t)|^2 dt \leq M_{12}.
\]

Then, applying the Hölder inequality, we see that
\[
\int_{0}^{\omega} |u'(t)| dt \leq \omega^{\frac{1}{2}} \left( \int_{0}^{\omega} |u'(t)|^2 dt \right)^{\frac{1}{2}} \leq \omega^{\frac{1}{2}} (M_{12})^{\frac{1}{2}} := M_{12}.
\]

Take \( M' := \max\{M_{11}, M_{12}\} \), from equations (14) and (16), we have
\[
\int_{0}^{\omega} |u'(t)| dt \leq M'.
\]

From equation (10), we arrive at
\[
\|u\| \leq D + \frac{1}{2} \int_{0}^{\omega} |u'(t)| dt \leq D + \frac{1}{2} M' := M.
\]

On the other hand, in view of \( u(0) = u(\omega) \), there exists a point \( \eta_2 \in (0, \omega) \) such that
\[
u'(\eta_2) = 0.
\]
Hence, from equations (5), (10), (17), (18) and (19), the following inequality is given that

\[ \|u''\| \leq \frac{1}{2} \int_0^\omega |u'''(t)| dt \]
\[ = \frac{\lambda}{2} \int_0^\omega \left| -\mu(t)u'(t) - \frac{q(t)}{\mu} u(t) + \frac{\beta(t)}{\mu} u^{\mu+1}(t) \right| dt \]
\[ \leq \|p\| \int_0^\omega |u'(t)| dt + \|q\| \frac{\omega}{2|\mu|} \|u\| + \|\beta\| \frac{\omega}{2|\mu|} \|u\|^{\mu+1} \]
\[ \leq \frac{\|p\|}{2} M_1' + \frac{\|q\| \omega}{2|\mu|} M_1 + \frac{\|\beta\| \omega}{2|\mu|} M_1^{\mu+1} := M_2. \]  

(20)

Let \( E_1 < -M_1, E_2 > M_1, E_3 > M_2 \) are constants, then it is clear that \( \omega \)-periodic solution \( u(t) \) to equation (5) satisfies

\[ E_1 < u(t) < E_2, \quad \|u''\| < E_3. \]

Then condition (1) of Lemma 2.1 is satisfied. For possible solution \( C \) to equation

\[ \int_0^\omega \frac{q(t)}{\mu} C dt - \int_0^\omega \frac{\beta(t)}{\mu} C^{\mu+1} dt = 0 \]

satisfies \( E_1 < C < E_2 \). Therefore, condition (2) of Lemma 2.1 holds. Finally, we consider the condition (3) of Lemma 2.1 is also satisfied. In fact, from condition \((H_1)\), we can observe that

\[ \int_0^\omega \frac{q(t)}{\mu} E_1 dt - \int_0^\omega \frac{\beta(t)}{\mu} E_1^{\mu+1} dt < 0, \]

and

\[ \int_0^\omega \frac{q(t)}{\mu} E_2 dt - \int_0^\omega \frac{\beta(t)}{\mu} E_2^{\mu+1} dt > 0. \]

So condition (3) of Lemma 2.1 is also satisfied. By application of Lemma 2.1, we get that equation (4) has at least one \( \omega \)-periodic solution.

In the following, we prove that \( \omega \)-periodic solution for equation (4) must be positive \( \omega \)-periodic solution. From equation (8) and the bounds for \( \mu \), we get

\[ \beta(t_*) u^{\mu+1}(t_*) \leq q(t_*) u(t_*). \]

(21)

If \( \mu = -\frac{\mu_3}{\mu_2} = -\frac{\text{odd}_{even}}{\text{even}}, \mu + 1 = \frac{\mu_2 - \mu_1}{\mu_2} = \frac{\text{odd}_{even}}{\text{even}}. \) Only if \( u(t_*) \geq 0 \) can \( u^{\mu+1}(t_*) \) be meaningful. From equation (3) and the relation between \( y \) and \( u \), it is clear that \( u(t) \neq 0 \), then \( u(t_*) > 0 \). Thus,

\[ u(t) \geq \min_{t \in [0, \omega]} u(t) = u(t_*) > 0, \]

which implies \( \omega \)-periodic solution for equation (4) must be positive \( \omega \)-periodic solution. \( \square \)

**Remark 1.** If \( \mu = -\frac{\mu_3}{\mu_4} = -\frac{\text{even}_{odd}}{\text{odd}}, \) where \( \mu_3 \) and \( \mu_4 \) are inter-prime positive constants. Then \( \mu + 1 = \frac{\mu_4 - \mu_3}{\mu_4} = \frac{\text{odd}_{odd}}{\text{odd}}. \) Next, we prove that the minimum value of \( \omega \)-periodic solution for equation (4) is positive \( \omega \)-periodic solution. Assume \( u(t_*) < 0 \), then equation (21) can be turned into

\[ \beta(t_*) \geq q(t_*) u^{-\mu}(t_*). \]

(22)

We know \( \beta(t) = -\frac{a(t)}{k(t)} < 0 \) and \( u^{-\mu}(t_*) > 0. \) If \( q(t) \geq 0 \), which is contradictory with equation (22). Therefore, \( u(t_*) \geq 0 \). The above statement indicates that \( q(t) \geq 0 \) is a necessary condition for \( u(t_*) \geq 0 \). From the fact that \( h \geq c(t) \) (i.e., \( q(t) \geq 0 \))
0) does not always stand up, \( \omega \)-periodic solution of equation (4) can not find a positive lower bound by this method.

**Remark 2.** If \( \mu = -\frac{\mu_5}{\mu_6} = -\text{odd} \) \( \text{odd} \), where \( \mu_5 \) and \( \mu_6 \) are inter-prime positive constants. Under the circumstances, \( \mu + 1 = \frac{\mu_5 - \mu_6}{\mu_6} = \text{even} \) \( \text{odd} \), and it is obvious that \( u^{\mu+1}(t) > 0 \), then equation (21) can be turned into

\[
\beta(t_*) \leq q(t_*) u^{-\mu}(t_*).
\]

Since \( \beta(t_*) < 0 \), we can not determine the sign of \( u^{-\mu}(t_*) \) (i.e., the sign of \( u(t_*) \)) by adding conditions to \( q(t_*) \).

**Remark 3.** If \( \mu > 0 \), the nonlinear term \( \frac{\beta(t)}{\mu} u^{\mu+1}(t) \) satisfies super-linearity condition. By the above method, equations (12)-(15) is not available for a estimating a prior bounds of \( \omega \)-periodic solution for equation (4).

2.2. \( \mu \leq -2 \).

**Theorem 2.3.** Assume that \( \mu \leq -2 \) holds. Furthermore, suppose the following condition holds:

(H3) There are constants \( D_3 \), \( D_4 \) with \( 0 < D_3 < D_4 \) such that \( \frac{\beta(t)}{\mu} u^{\mu+1}(t) - \frac{q(t)}{\mu} u(t) > 0 \) for \( (t, u) \in [0, \omega] \times (0, D_3) \), and \( \frac{\beta(t)}{\mu} u^{\mu+1}(t) - \frac{q(t)}{\mu} u(t) < 0 \) for \( (t, u) \in [0, \omega] \times (D_4, +\infty) \).

If \( \|p\|\omega + \|q\|\omega^2 < 1 \), then equation (4) has at least one positive \( \omega \)-periodic solution.

**Proof.** Similar to the proof procedure of Theorem 2.2, we get equations (8) and (9). From these two equations, condition (H3) implies that

\[ u(t_*) \geq D_3 \text{ and } u(t_*) \leq D_4. \]

Case I: If \( u(t^*) \in [D_3, D_4] \), let \( \eta_3 = t^* \), then, \( D_3 \leq u(\eta_3) < D_4 \).

Case II: If \( u(t^*) \in [0, \omega] \), from continuity of \( u(t) \), we know that there will exist a point \( \eta_3 \in [0, \omega] \) such that \( u(\eta_3) = D_4 \).

Combining the above two cases, there is a point \( \eta_3 \in [0, \omega] \) such that

\[ D_3 \leq u(\eta_3) \leq D_4. \] (23)

It follows from equation (10) that

\[
\|u\| \leq D_4 + \frac{1}{2} \int_0^\omega |u'(t)|dt. \quad (24)
\]

Multiplying both sides of equation (5) by \( u(t) \) and integrating over the interval \([0, \omega]\). Similar to equations (11) and (12), we obtain

\[
\int_0^\omega |u'(t)|^2 dt \leq \|p\||u\| \int_0^\omega |u'(t)|dt + \|q\|\omega^2 + \|u\| \int_0^\omega \left| \frac{\beta(t)}{\mu} u^{\mu+1}(t) \right| dt. \quad (25)
\]

Integrating equation (5) over the interval \([0, \omega]\), since \( \beta(t) < 0 \) and \( \mu \leq -2 \), we arrive at

\[
\int_0^\omega p(t)u'(t)dt + \int_0^\omega \frac{q(t)}{\mu} u(t)dt = \int_0^\omega \frac{\beta(t)}{\mu} u^{\mu+1}(t)dt = \int_0^\omega \left| \frac{\beta(t)}{\mu} u^{\mu+1}(t) \right| dt. \quad (26)
\]
Using the Hölder inequality, we have
\[
\int_0^\omega |u'(t)|^2 dt \leq \|p\| \|u\| \int_0^\omega |u'(t)| dt + \frac{\|q\|\|\omega\|}{|\mu|} \|u\|^2
\]
\[
+ \|u\| \left( \int_0^\omega p(t)u'(t) dt + \int_0^\omega \frac{q(t)}{\mu} u(t) dt \right)
\]
\[
\leq 2\|p\| \|u\| \int_0^\omega |u'(t)| dt + \frac{2\|q\|\|\omega\|}{|\mu|} \|u\|^2
\]
\[
\leq 2\|p\| \left( D_4 + \frac{1}{2} \int_0^\omega |u'(t)| dt \right) \int_0^\omega |u'(t)| dt
\]
\[
+ \frac{2\|q\|\|\omega\|}{|\mu|} \left( D_4 + \frac{1}{2} \int_0^\omega |u'(t)| dt \right)^2
\]
\[
= \left( \|p\| + \frac{\|q\|\|\omega\|^2}{2|\mu|} \right) \left( \int_0^\omega |u'(t)| dt \right)^2 + \left( 2\|p\| D_4 + \frac{2\|q\| D_4 \omega^\frac{3}{2}}{|\mu|} \right)
\]
\[
\cdot \int_0^\omega |u'(t)| dt + \frac{2\|q\| D_4 \omega^\frac{3}{2}}{|\mu|}
\]
\[
\leq \left( \|p\| + \frac{\|q\|\|\omega\|^2}{2|\mu|} \right) \int_0^\omega |u'(t)|^2 dt + \left( 2\|p\| D_4 \omega^\frac{3}{2} + \frac{2\|q\| D_4 \omega^\frac{3}{2}}{|\mu|} \right)
\]
\[
\cdot \left( \int_0^\omega |u'(t)|^2 dt \right)^\frac{1}{2} + \frac{2\|q\| D_4 \omega^\frac{3}{2}}{|\mu|}.
\]
From the above equation, it is obvious that \( \int_0^\omega |u'(t)|^2 dt \) is bounded if \( \|p\| + \frac{\|q\|\|\omega\|^2}{2|\mu|} < 1 \), hence, there exists a positive constant \( M''_3 \) such that
\[
\int_0^\omega |u'(t)|^2 dt \leq M''_3.
\]
Using the Hölder inequality,
\[
\int_0^\omega |u'(t)| dt \leq \omega^\frac{1}{2} \left( \int_0^\omega |u'(t)|^2 dt \right)^\frac{1}{2} \leq \omega^\frac{1}{2} (M''_3)^\frac{1}{2} := M'_3.
\]
From equations (24) and (27), it is clear that
\[
\|u\| \leq D_4 + \frac{1}{2} \int_0^\omega |u'(t)| dt \leq D_4 + \frac{1}{2} M'_3 := M_3.
\]
From equations (20), (26), (27) and (28), the following inequality is given that
\[
\|u''\| \leq \frac{1}{2} \int_0^\omega |u''(t)| dt
\]
\[
\leq \frac{1}{2} \left( \int_0^\omega |p(t)u'(t)| dt + \int_0^\omega \left| \frac{q(t)}{\mu} u(t) \right| dt + \int_0^\omega \left| \frac{\beta(t)}{\mu} u^\mu + 1 (t) \right| dt \right)
\]
\[
\leq \int_0^\omega |p(t)u'(t)| dt + \int_0^\omega \left| \frac{q(t)}{\mu} u(t) \right| dt
\]
\[
\leq \|p\| \int_0^\omega |u'(t)| dt + \frac{\|q\|\|\omega\|}{|\mu|} \|u\|
\]
\[
\leq \|p\| M'_3 + \frac{\|q\|\|\omega\|}{|\mu|} M_3 := M_4.
\]
On the other hand, we prove that \( u(t) \) has a positive lower bound. Form \( \beta(t) \neq 0 \), equation (3) can be turned into
\[
\frac{\mu u''(t)}{\beta(t)} + \lambda \frac{\mu p(t)u'(t)}{\beta(t)} + \lambda \frac{q(t)u(t)}{\beta(t)} = \lambda u^{\mu + 1}(t).
\]

Consider the following interval in \([\eta_3, \tau] \subset [0, \omega]\) and \( u(\eta_3) \geq D_3 \) be as in equation (23). Multiplying both sides of equation (30) by \( u'(t) \) and integrating on \([\eta_3, \tau]\), we get
\[
\int_{\eta_3}^{\tau} \frac{\mu u''(s)u'(s)}{\beta(s)} ds + \lambda \int_{\eta_3}^{\tau} \frac{\mu p(s)u'(s)}{\beta(s)} ds + \lambda \int_{\eta_3}^{\tau} \frac{q(s)u(s)u'(s)}{\beta(s)} ds = \lambda \int_{\eta_3}^{\tau} u^{\mu + 1}(s)u'(s) ds.
\]

Furthermore, we have
\[
\left\| \lambda \int_{u(\eta_3)}^{u(t)} u^{\mu + 1} du \right\| = \lambda \int_{\eta_3}^{\tau} u^{\mu + 1}(s)u'(s) ds
\]

\[\leq \int_{\eta_3}^{\tau} \frac{\mu u''(s)u'(s)}{\beta(s)} ds + \lambda \int_{\eta_3}^{\tau} \frac{\mu p(s)u'(s)}{\beta(s)} ds + \lambda \int_{\eta_3}^{\tau} \frac{q(s)u(s)u'(s)}{\beta(s)} ds.\]

By equations (26), (28) and (29), we obtain
\[
\left| \int_{\eta_3}^{\tau} \frac{\mu u''(s)u'(s)}{\beta(s)} ds \right| \leq \frac{|\mu||u'|}{|\beta|_0} \int_0^\omega |u''(t)| dt
\]
\[\leq \frac{2|\mu||u'|}{|\beta|_0} \left( \int_0^\omega |p(t)u'(t)| dt + \int_0^\omega |q(t)\mu u(t)| dt \right)\]
\[\leq \frac{2|\mu||M_4|}{|\beta|_0} \left( \int_0^\omega |p(t)u'(t)| dt + \int_0^\omega |q(t)\mu u(t)| dt \right)\]
\[\leq \frac{2|\mu||M_4|}{|\beta|_0} \left( ||p||M_4 + ||q||\omega M_3 \right),\]
\[\lambda \int_{\eta_3}^{\tau} \frac{\mu p(s)u'(s)}{\beta(s)} ds \leq \lambda \int_0^\omega \frac{\mu p(t)u'(t)}{\beta(t)} dt \leq \frac{|\mu||p||\omega M_4^2}{|\beta|_0},\]
\[\lambda \int_{\eta_3}^{\tau} \frac{q(s)u(s)u'(s)}{\beta(s)} ds \leq \lambda \int_0^\omega \frac{q(t)u(t)u'(t)}{\beta(t)} dt \leq \frac{|\mu||q||\omega M_3 M_4}{|\beta|_0},\]
where \( |\beta|_0 := \min_{t \in [0, \omega]} |\beta(t)|. \)

From these inequalities, we derive from equation (31)
\[
\left| \int_{u(\eta_3)}^{u(t)} u^{\mu + 1} du \right| \leq \frac{3|\mu|M_4}{|\beta|_0} \left( ||p||M_4 + ||q||M_3 \right) := M_5.\]
In view of $\mu \leq -2$, i.e., $\mu + 1 \leq -1$ and $u(\eta_3) \geq D_3$, then there exists a constant $M_6 \in (0, D_3)$ such that
\[ \int_{M_6}^{D_3} u^{\mu+1} du > M_5. \] (33)
Thus, if there exists a point $\eta_4 \in [\eta_3, t]$ such that $u(\eta_4) \leq M_6$, then
\[ \left| \int_{u(\eta_4)}^{u(\eta_3)} u^{\mu+1} du \right| \geq \left| \int_{M_6}^{D_3} u^{\mu+1} du \right| > M_5, \]
which contradicts with equation (32). Therefore, we can obtain that $u(t) > M_6$ for all $t \in [\eta_3, \omega]$. For the case $t \in [0, \eta_3]$, we can handle similarly.

The proof left is as same as Theorem 2.2.

Remark 4. It is worth mentioning that method of Theorem 2.3 is no longer applicable to the proof of existence of an $\omega$-periodic solution for equation (4) with weak singularity (i.e. $-2 < \mu < -1$). Due to $-2 < \mu < -1$, we can not obtain that equation (33) holds. Hence, we do not obtain that equation (32) and equation (33) are contradictory.

From Theorems 2.2 and 2.3, we only consider the existence of a positive $\omega$-periodic solution for equation (4) in the cases that $-1 < \mu < 0$ and $\mu < -2$ by applications of Manásevich-Mawhin continuation theorem. Therefore, we need to find other methods to consider the existence of a positive $\omega$-periodic solution for equation (4) in the cases that $\mu > 0$ and $-2 < \mu < -1$.

3. Existence results (II). In this section, we investigate the existence of a positive $\omega$-periodic solution for equation (4) in the cases that $\mu > 0$, $-1 < \mu < 0$ and $\mu < -1$ by using Leray-Schauder alternative principle. At first, we discuss positivity and negativity of Green function of the linear equation corresponding to equation (4).

Consider the following nonhomogeneous second order linear differential equation
\[ \begin{cases} u''(t) + p(t)u'(t) + q(t)\mu u(t) = b(t), \\ u(0) = u(\omega), \ u'(0) = u'(\omega), \end{cases} \] (34)
where $b \in C(\mathbb{R}, \mathbb{R})$ is an $\omega$-periodic function. Then equation (34) have unique $\omega$-periodic solution which can be written as
\[ u(t) = \int_0^\omega G(t, s)b(s)ds, \]
where $G(t, s)$ is the Green function of equation (34) [4]. Throughout these sections 3 and 4, assume one of the following conditions is satisfied:

($F_1$) The Green function $G(t, s)$ of equation (34) is positive for all $(t, s) \in [0, \omega] \times [0, \omega]$.

($F_2$) The Green function $G(t, s)$ of equation (34) is negative for all $(t, s) \in [0, \omega] \times [0, \omega]$.

In the following, we study conditions ($F_1$) and ($F_2$). Let $X$ with norm $\|u\| := \max_{t \in [0, \omega]} |u(t)|$ and $X$ be a Banach space, where $X$ is defined by equation (6). On the basis of [12], Cheng and Ren [5] in 2018 proved the Green function is positive for all $(t, s) \in [0, \omega] \times [0, \omega]$ if the following condition is satisfied:
There are continuous $\omega$-periodic functions $a_1(t)$ and $a_2(t)$ such that
\[ \int_0^\omega a_1(t)dt > 0, \int_0^\omega a_2(t)dt > 0 \]
and
\[ a_1(t) + a_2(t) = p(t), \quad a_1'(t) + a_2(t)a_2(t) = \frac{q(t)}{\mu}, \quad \text{for } t \in \mathbb{R}. \]

Applying the methods of [12] and [5], we get that the Green function is negative for all $(t,s) \in [0,\omega] \times [0,\omega]$ if one of the following conditions is satisfied:

(A) There are continuous $\omega$-periodic functions $a_3(t)$ and $a_4(t)$ such that
\[ \int_0^\omega a_3(t)dt > 0, \int_0^\omega a_4(t)dt < 0 \]
and
\[ a_3(t) + a_4(t) = p(t), \quad a_3'(t) + a_4(t)a_4(t) = \frac{q(t)}{\mu}, \quad \text{for } t \in \mathbb{R}. \]

(B) There are continuous $\omega$-periodic functions $a_5(t)$ and $a_6(t)$ such that
\[ \int_0^\omega a_5(t)dt < 0, \int_0^\omega a_6(t)dt > 0 \]
and
\[ a_5(t) + a_6(t) = p(t), \quad a_5'(t) + a_6(t)a_6(t) = \frac{q(t)}{\mu}, \quad \text{for } t \in \mathbb{R}. \]

Next, we state and prove the existence of a positive $\omega$-periodic solution for equation (4). Our proofs are based on the following Leray-Schauder alternative principle.

**Lemma 3.1.** (see [6]) Assume $B_1$ is a relatively compact subset of a convex set $K$ in a normed space $X$, and $0 \in B_1$. Let $\overline{\Sigma} : \overline{B}_1 \to K$ be a compact map, then one of the following two conclusions holds:

(I) $\overline{\Sigma}$ has at least one fixed point in $\overline{B}_1$.

(II) There exist $u \in \partial B_1$ and $0 < \sigma < 1$ such that $u = \sigma \overline{\Sigma}u$.

In 1997, O’Regan improved the existing result by applying Leray-Schauder alternative principle to singular differential equations, namely the following lemma.

**Lemma 3.2.** (see [10]) Assume $B_2$ is a relatively compact subset of a convex set $K$ in a normed space $X$, and $\rho \in B_2$. Let $\overline{\Xi} : \overline{B}_2 \to K$ be a compact map, then one of the following two conclusions holds:

(I) $\overline{\Xi}$ has at least one fixed point in $\overline{B}_2$.

(II) There exist $u \in \partial B_2$ and $0 < \varpi < 1$ such that $u = \varpi \overline{\Xi}u + (1 - \varpi)\rho$.

From conditions (F1) and (F2), we denote
\[ l := \min_{s,t \in [0,\omega]} G(t,s), \quad L := \max_{s,t \in [0,\omega]} G(t,s), \quad \delta_1 := \frac{l}{L}, \quad \delta_2 := \frac{L}{l} \]

It is clear that $0 < l \leq L$ if condition (F1) holds; $l \leq L < 0$ if condition (F2) holds.

And $0 < \delta_i \leq 1, \quad i = 1, 2$. Define set
\[ K_i = \{ u \in X : u(t) \geq 0 \text{ for all } t \in \mathbb{R} \text{ and } \min_{t \in \mathbb{R}} u(t) \geq \delta_i \| u \|\}, \quad i = 1, 2. \]

It is easy to verify that $K_i$ is a cone in $X$.

In the following, according to different ranges of $\mu$, we study the existence of a positive $\omega$-periodic solution for equation (4) without singularity (i.e. $\mu > 0$ and $-1 < \mu < 0$) and with a singularity (i.e. $\mu < -1$).
3.1. \( \mu > 0 \). In this subsection, \( \mu > 0 \) is the precondition. Since \( \beta(t) < 0 \), in order to guarantee the \( \omega \)-periodic solution \( u(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \) of equation (4) is positive \( \omega \)-periodic solution, it is a requirement that \( G(t, s) < 0 \), for this purpose, we use condition \((F_2)\).

**Theorem 3.3.** Assume that \( \mu > 0 \) and condition \((F_2)\) hold. Furthermore, suppose the following condition holds:

\[(H_4)\] There exists a positive real constant \( r_1 \) such that

\[r_1 \leq \left( \frac{\mu}{\Lambda_\ast \beta_\ast} \right)^\frac{1}{\mu},\]

where \( \Lambda_\ast := \min_{t \in [0, \omega]} \int_0^\omega G(t, s) ds, \beta_\ast := \min_{t \in [0, \omega]} \beta(t). \)

Then equation (4) has at least one positive \( \omega \)-periodic solution \( u(t) \) with \( \|u\| \in (0, r_1) \).

**Proof.** Consider the family of equations

\[u''(t) + p(t)u'(t) + \frac{q(t)}{\mu} u(t) = \sigma_1 \frac{\beta(t)}{\mu} u^{\mu+1}(t),\]  

where \( \sigma_1 \in (0, 1) \). An \( \omega \)-periodic solution of equation (36) is just a fixed point of the operator equation

\[u = \sigma_1 \Xi_1 u, \]

where \( \Xi_1 \) is a continuous operator defined by

\[(\Xi_1 u)(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds.\]

Let \( B_1^1 := \{ u \in X : \|u\| < r_1 \} \), we claim that \( \Xi_1(B_1^1) \subset K_2 \). For any \( u \in \overline{B_1^1} \), from equation (35), we arrive at

\[\min_{t \in \mathbb{R}}(\Xi_1 u)(t) = \min_{t \in \mathbb{R}} \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \]
\[\geq L \int_0^\omega \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \]
\[= \delta_2 l \int_0^\omega \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \]
\[\geq \delta_2 \max_{t \in \mathbb{R}} \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \]
\[= \delta_2 \| \Xi_1 u \|,\]

which shows that \( \Xi_1(B_1^1) \subset K_2 \). In addition, from the continuity of \( G(t, s) \) and \( \frac{\beta(t)}{\mu} u^{\mu+1}(t) \), we deduce that \( \Xi_1 : \overline{B_1^1} \to K_2 \) is a continuous and completely continuous operator.

We claim that for any \( \sigma_1 \in (0, 1) \), there does not exist \( u \in \partial B_1^1 \) such that equation (37) holds. Otherwise, assume that for some \( \sigma_1 \in (0, 1) \), there is a fixed point \( u \) of
equation (37) such that $u \in \partial B_1^1$ (i.e., $\|u\| = r_1$). From condition $(H_4)$, we deduce

$$u(t) = \sigma_1(T_1 u)(t)$$

$$= \sigma_1 \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds$$

$$\leq \frac{\Lambda \beta_s}{\mu} \|u\|^{\mu+1}$$

$$= \frac{\Lambda \beta_s}{\mu} r_1^{\mu+1}$$

$$\leq r_1,$$

thus, $r_1 = \|u\| < r_1$, this is a contraction. From this claim and Lemma 3.1, we know that $u = T_1 u$ has a positive fixed point $u(t)$ with $\|u\| < r_1$. Therefore, equation (4) has a positive $\omega$-periodic solution $u(t)$ satisfies $\|u\| \in (0, r_1)$.

3.2. $-1 < \mu < 0$. In this subsection, $-1 < \mu < 0$ is the precondition. Since $\beta(t) < 0$, in order to guarantee the $\omega$-periodic solution $u(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds$ of equation (4) is positive $\omega$-periodic solution, it is a requirement that $G(t, s) > 0$, for this purpose, we use condition $(F_1)$.

Theorem 3.4. Assume that $-1 < \mu < 0$ and condition $(F_1)$ hold. Furthermore, suppose the following condition holds:

$$(H_5)$$ There exists a positive real constant $r_2$ such that

$$r_2 \geq \left( \frac{\Lambda \beta_s}{\mu} \right)^{-\frac{1}{\mu}},$$

where $\Lambda^* := \max_{t \in [0, \omega]} \int_0^\omega G(t, s) ds$.

Then equation (4) has at least one positive $\omega$-periodic solution $u(t)$ with $\|u\| \in (0, r_2)$.

Proof. Similar to the proof procedure of Theorem 3.3, we consider the family of equations

$$u''(t) + p(t)u'(t) + \frac{q(t)}{\mu} u(t) = \sigma_2 \frac{\beta(t)}{\mu} u^{\mu+1}(t),$$

where $\sigma_2 \in (0, 1)$. An $\omega$-periodic solution of equation (39) is just a fixed point of the following operator equation

$$u = \sigma_2 T_2 u,$$
which shows that $\mathcal{T}_2(B_1^2) \subset K_1$ and $\mathcal{T}_2 : B_1^2 \to K_1$ is a continuous and completely continuous operator.

Following, we claim that for any $\sigma_2 \in (0, 1)$, there does not exist $u \in \partial B_1^2$ such that equation (40) holds. Otherwise, assume that for some $\sigma_2 \in (0, 1)$, there is a fixed point $u$ of equation (40) such that $u \in \partial B_1^2$ (i.e., $\|u\| = r_2$). From condition $(H_5)$, we get

$$u(t) = \sigma_2(\mathcal{T}_2 u)(t) = \sigma_2 \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds$$

$$< \frac{\Lambda^* \beta}{\mu} \sigma_2^\mu r_2^{\mu+1} \leq r_2,$$

thus, $r_2 = \|u\| < r_2$, this is a contraction. From this claim and Lemma 3.1, we know that $u = \mathcal{T}_2 u$ has a positive fixed point $u(t)$ with $\|u\| < r_2$. Therefore, equation (4) has a positive $\omega$-periodic solution $u(t)$ satisfies $\|u\| \in (0, r_2)$.

**Remark 5.** If $-1 < \mu < 0$ and $\mu > 0$, the nonlinear term $\frac{\beta(t)}{\mu} u^{\mu+1}(t)$ has not singular, we discuss the existence of a positive $\omega$-periodic solution for equation (4) by using Lemma 3.1. However, if $\mu < -1$, equation (4) has a singularity of repulsive type at the origin, we verify the existence of a positive $\omega$-periodic solution for equation (4) by applications of Lemma 3.2.

3.3. $\mu < -1$. In this subsection, $\mu < -1$ is the precondition. Since $\beta(t) < 0$, in order to guarantee the $\omega$-periodic solution $u(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}$ of equation (4) is positive $\omega$-periodic solution, it is a requirement that $G(t, s) > 0$, for this purpose, we use condition $(F_1)$.

**Theorem 3.5.** Assume that $\mu < -1$ and condition $(F_1)$ hold. Furthermore, suppose the following condition holds:

$(H_6)$ There exists a positive real constant $r_3$ such that

$$r_3 > \left( \frac{\Lambda^* \beta}{\mu} \right)^{-\frac{1}{\mu}}.$$

If $\|p\| \omega < 1$, then equation (4) has at least one positive $\omega$-periodic solution $u(t)$ with $\|u\| \in (0, r_3)$.

**Proof.** Let $f(t, u(t)) = \frac{\beta(t)}{\mu} u^{\mu+1}(t)$. And consider the family of equations

$$u''(t) + p(t)u'(t) + \frac{q(t)}{\mu} u(t) = \tilde{\sigma} f_n(t, u(t)) + \frac{q(t)}{n\mu},$$

where $\tilde{\sigma} \in (0, 1)$, and

$$f_n(t, u) = \begin{cases} f(t, u) & \text{if } u \geq \frac{1}{n}, \\ f(t, \frac{1}{n}) & \text{if } u < \frac{1}{n}. \end{cases}$$

An $\omega$-periodic solution of equation (42) is just a fixed point of the operator equation

$$u = \tilde{\sigma} \mathcal{T}_n u + (1 - \tilde{\sigma}) \rho,$$
where \( \rho = \frac{1}{n} \) and \( \tilde{T}_n \) is a continuous operator defined by

\[
(\tilde{T}_n u)(t) = \int_0^\omega G(t,s)f_n(s,u(s))ds + \frac{1}{n}.
\]

Let \( B_2 := \{ u \in X : \|u\| < r_3 \} \), we claim that \( \tilde{T}_n(B_2) \subset K_1 \). For any \( u \in \overline{B}_2 \), from equation (41), we arrive at

\[
\min_{t \in \mathbb{R}}(\tilde{T}_n u)(t) - \frac{1}{n} = \min_{t \in \mathbb{R}} \int_0^\omega G(t,s)f_n(s,u(s))ds
\]

\[
\geq \delta_1 \| \tilde{T}_n u - \frac{1}{n} \|.
\]

From the equation above, it is clear that

\[
\min_{t \in \mathbb{R}}(\tilde{T}_n u)(t) \geq \delta_1 \| \tilde{T}_n u - \frac{1}{n} \| + \frac{1}{n}
\]

\[
\geq \delta_1 \| \tilde{T}_n u \| - \frac{\delta}{n} + \frac{1}{n}.
\]

\[
\geq \delta_1 \| \tilde{T}_n u \|
\]

where we used the fact \( 0 < \delta_1 \leq 1 \). This shows that \( \tilde{T}_n(B_2) \subset K_1 \). In addition, from the continuity of \( G(t,s) \) and \( f \), we deduce that \( \tilde{T}_n : B_2 \to K_1 \) is a continuous and completely continuous operator.

Next, we claim that for any \( \tilde{\sigma} \in (0,1) \), there does not exist \( u \in \partial B_2 \) such that equation (43) holds. Otherwise, assume that for some \( \tilde{\sigma} \in (0,1) \), there is a fixed point \( u \) of equation (43) such that \( u \in \partial B_2 \) (i.e., \( \|u\| = r_3 \)).

We claim that \( u(t) > \frac{1}{n} \). In fact, from

\[
u = \tilde{\sigma} \tilde{T}_n u + (1 - \tilde{\sigma})\rho = \tilde{\sigma} \int_0^\omega G(t,s)f_n(s,u(s))ds + \frac{1}{n}
\]

and \( \int_0^\omega G(t,s)f_n(s,u(s))ds > 0 \), we can see \( u(t) > \frac{1}{n} \) holds. Note that

\[
u(t) = \tilde{\sigma}(\tilde{T}_n u)(t) + (1 - \tilde{\sigma})\rho
\]

\[
= \tilde{\sigma} \int_0^\omega G(t,s)f_n(s,u(s))ds + \frac{\tilde{\sigma}}{n} + \frac{1 - \tilde{\sigma}}{n}
\]

\[
= \tilde{\sigma} \int_0^\omega G(t,s)f(s,u(s))ds + \frac{1}{n}
\]

\[
= \tilde{\sigma} \int_0^\omega G(t,s)\frac{\beta'(s)}{\mu}u^{\mu+1}(s)ds + \frac{1}{n}
\]

\[
< \frac{\Lambda^* \beta_s}{\mu} (\delta_1 r_3)^{\mu+1} + \frac{1}{n}.
\]

From condition \((H_6)\), we can choose \( n_0 \in \{1, 2, \ldots \} \) such that

\[
\frac{\Lambda^* \beta_s}{\mu} (\delta_1 r_3)^{\mu+1} + \frac{1}{n_0} < r_3.
\]

Let \( N_0 = \{n_0, n_0 + 1, \ldots \} \) and \( n \in N_0 \) and transform equation (44) into

\[
u(t) < \frac{\Lambda^* \beta_s}{\mu} (\delta_1 r_3)^{\mu+1} + \frac{1}{n_0} < r_3,
\]
thus, \( r_3 = \|u\| < r_3 \), this is a contraction. From this claim and Lemma 3.2, we know that
\[
\bar{u} = \bar{\bar{u}} = \bar{\bar{u}}
\]
has a positive fixed point in \( B_2 \), denoted by \( u_n \). Then, we get
\[
\beta(t) u'(t) + p(t) u(t) + \frac{q(t)}{\mu} u(t) = f_n(t, u(t)) + \frac{q(t)}{n\mu}
\]
\[
(45)
\]
has a positive \( \omega \)-periodic solution \( u_n \) with \( \|u_n\| \leq r_3 \).

Next, we prove that \( u_n(t) \) have a uniform positive lower bound \( \theta > 0 \) such that
\[
\min_{t \in [0, \omega]} u_n(t) \geq \theta
\]
for all \( n \in N_0 \). The following formula is true,
\[
\begin{align*}
\bar{u}_n(t) &= (\bar{\bar{\bar{u}}}_n)(t) \\
&= \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u_n^\omega(s) \, ds + \frac{1}{n}
\end{align*}
\]
\[
> A_3 \beta^* \rho_n^\omega := \theta,
\]
where \( \beta^* := \max_{t \in [0, \omega]} \beta(t) \).

In the end, we need to illustrate the solution \( u_n \) of equation (45) is that of the original equation (4), for this purpose, we prove \( \{u_n\} \in N_0 \) and \( \{u'_n\} \in N_0 \) are compact.

From equations (19), (20) and (45), it is clear that
\[
\begin{align*}
\|u'_n\| &\leq \frac{1}{2} \int_0^\omega |u''_n(t)| \, dt \\
&\leq \frac{1}{2} \left( \int_0^\omega |p(t)u'_n(t)| \, dt + \int_0^\omega \left| \frac{q(t)}{\mu} u_n + \frac{\beta(t)}{\mu} u_n^\omega + \frac{q(t)}{n\mu} \right| \, dt \right)
\end{align*}
\]
\[
(46)
\]
for all \( (t, u) \in [0, \omega] \times (0, r_3) \). Integrating equation (45) over the interval \( [0, \omega] \), since \( \beta(t) < 0 \) and \( \mu < -1 \), we arrive at
\[
\begin{align*}
\int_0^\omega p(t) u'(t) \, dt + \int_0^\omega \frac{q(t)}{\mu} u(t) \, dt - \int_0^\omega \frac{q(t)}{n\mu} \, dt = \int_0^\omega \frac{\beta(t)}{\mu} u_n^\omega(t) \, dt \\
= \int_0^\omega \left| \frac{\beta(t)}{\mu} u_n^\omega(t) \right| \, dt.
\end{align*}
\]
\[
(47)
\]
Substituting equation (47) into equation (46), we have
\[
\|u'_n\| \leq \|p\| \|u\| + \|q\| \|u_3\| + \|q\| \|u\| \leq \frac{\|q\| \|u_3\|}{|\mu|} + \|q\| \|u\|
\]
for all \( (t, u) \in [0, \omega] \times (0, r_3) \). We see from the above equation that \( \|u'_n\| \) is bounded if \( \|p\| \omega < 1 \), i.e.,
\[
\|u'_n\| \leq \frac{\|q\| \omega (1 + r_3)}{|\mu| (1 - \|p\| \omega)}.
\]

In consequence, sequences \( \{u_n\} \in N_0 \) and \( \{u'_n\} \in N_0 \) are bounded and equi-continuous family in \( C^1_{\omega} := \{u \in C(\mathbb{R}, \mathbb{R}) : u(t + \omega) \equiv u(t), u'(t + \omega) = u'(t) \} \) for \( t \in \mathbb{R} \). Using the Arzela-Ascoli theorem, it is clear that \( \{u_n\} \in N_0 \) has a subsequence \( \{u_{n_i}\} \in N_0 \), converging uniformly on \( \mathbb{R} \) to a function \( u \in X \). It follows
from $\|u_n\| \leq r_3$ and $u_n(t) \geq \theta$ that $u$ satisfies $\theta \leq u(t) \leq r_3$ for all $t \in \mathbb{R}$. Moreover, $u_n$, satisfies the integral equation

$$u_n(t) = \int_0^\omega G(t,s)f(s,u_n(s))ds + \frac{1}{n}.$$ 

Letting $i \to \infty$, we get

$$u(t) = \int_0^\omega G(t,s)f(s,u(s))ds.$$ 

Therefore, $u$ is a positive $\omega$-periodic solution for equation (4) and satisfies $\|u\| \in (0,r_3]$. Furthermore, we can show $\|u\| < r_3$ by noting $\|u\| = r_3$, the argument similar to the proof of the first claim and we can yield a contradiction. 

**Remark 6.** Comparing Theorem 2.3 with Theorem 3.5, it is clear that Theorem 3.5 embraces both a strong singularity and a weak singularity, while Theorem 2.3 just can be used to consider a strong singularity.

### 4. Existence results (III).

In the section, we consider the existence of positive $\omega$-periodic solutions for equation (4) by the following fixed point theorem in cones, which can be found in [10].

**Lemma 4.1.** Let $X$ be a Banach space and $K$ be a cone in $X$. Assume that $\Omega_1$, $\Omega_2$ are open subsets of $X$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let

$$Q : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) $\|Qu\| \leq \|u\|$ for $u \in K \cap \partial \Omega_2$; and

(ii) there exists $u_0 \in K \setminus \{0\}$ such that $u \neq Qu + \xi u_0$ for $u \in K \cap \partial \Omega_1$ and $0 < \xi \leq 1$. 

Then $Q$ has a fixed point in $K \cap (\overline{\Omega_2 \setminus \Omega_1})$.

In the following, we study the existence of positive $\omega$-periodic solutions for equation (4) by different values of $\mu$, and the division method is the same as section 3.

#### 4.1. $\mu > 0$.

**Theorem 4.2.** Assume that $\mu > 0$ and condition $(F_2)$ hold. Furthermore, suppose the following conditions hold:

(H7) There exists a positive real constant $r_4$ such that

$$r_4 \leq \left(\frac{\mu}{\lambda_i \beta_a}\right)^\frac{1}{\beta}.$$ 

(H8) There exists a positive real constant $R_1$ such that

$$R_1 \geq \left(\frac{\mu}{\lambda^* \beta^* \delta_2^{\mu+1}}\right)^\frac{1}{\beta}.$$ 

Then equation (4) has at least one positive $\omega$-periodic solution $u(t)$ with $\|u\| \in [r_4,R_1]$. 

Proof. It is easy to see that an ω-periodic solution of equation (4) is just a fixed point of the operator equation \( u = Q_1 u \), where the expression of \( Q_1 \) is the same as \( T_1 \).

Define two open sets
\[
\Omega_1^1 := \{ u \in X : \| u \| < r_4 \}, \quad \Omega_2^1 := \{ u \in X : \| u \| < R_1 \}.
\]
Similar to equation (38), we can verify that \( Q_1(K_2 \cap \Omega_2^1 \setminus \Omega_1^1) \subset K_2 \) and \( Q_1 : K_2 \cap \Omega_2^1 \rightarrow K_2 \) is a continuous and completely continuous operator.

Next, we prove that
\[
\| Q_1 u \| \leq \| u \|, \quad \text{for} \quad u \in K_2 \cap \partial \Omega_1^1.
\]
In fact, for any \( u \in K_2 \cap \partial \Omega_1^1 \), we have
\[
\delta_2 r_4 \leq u(t) \leq r_4.
\]
Then, from condition \((H_7)\), it is clear that
\[
(Q_1 u)(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds 
\leq \frac{\Lambda^* \beta^*}{\mu} r_4 \mu
\leq r_4 = \| u \|.
\]
Therefore, \( \| Q_1 u \| \leq \| u \| \), equation (48) holds.

Let \( u_0 = 1 \), \quad \text{(49)}
then \( u_0 \in K_2 \setminus \{0\} \). Now we prove that
\[
u \neq Q_1 u + \xi u_0, \quad \text{for} \quad u \in K_2 \cap \partial \Omega_2^1 \quad \text{and} \quad 0 < \xi \leq 1.
\]
In fact, if not, there would exist \( u_1 \in K_2 \cap \partial \Omega_2^1 \) and \( \xi_1 > 0 \) such that
\[
u_1 = Q_1 u_1 + \xi_1 u_0.
\]
Since \( u_1 \in K_2 \cap \partial \Omega_2^1 \), we have
\[
\delta_2 R_1 \leq u_1(t) \leq R_1.
\]
Then, from condition \((H_8)\), we arrive at
\[
u_1 = Q_1 u_1 + \xi_1 u_0
\leq \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u_1^{\mu+1}(s) ds + \xi_1
\geq \frac{\Lambda^* \beta^*}{\mu} (\delta_2 R_1)^{\mu+1}
\geq R_1,
\]
which contradicts with \( u_1 \in K_2 \cap \partial \Omega_2^1 \), i.e., equation (50) holds.

It follows from Lemma 4.1 that \( Q_1 \) has a fixed point \( u \in K_2 \cap \Omega_2^1 \). Clearly, the fixed point is an ω-periodic solution for equation (4) satisfying \( \| u \| \in [r_4, R_1] \).

**Corollary 1.** Assume that \( \mu > 0 \) and condition \((F_2)\) hold. Furthermore, suppose \( r_1 \leq r_4 \) holds. Then equation (4) has at least two positive ω-periodic solutions in the interval \((0, R_1]\). □
4.2. $−1 < \mu < 0$.

**Theorem 4.3.** Assume that $−1 < \mu < 0$ and condition $(F_1)$ hold. Furthermore, suppose the following conditions hold:

$(H_9)$ There exists a positive real constant $r_5$ such that

$$r_5 \leq \left( \frac{\Lambda^* \beta^* \delta^*_1 + 1}{\mu} \right)^{-\frac{1}{\mu}}.$$

$(H_{10})$ There exists a positive real constant $R_2$ such that

$$R_2 \geq \left( \frac{\Lambda^* \beta^*}{\mu} \right)^{-\frac{1}{\mu}}.$$

Then equation (4) has at least one positive $\omega$-periodic solution $u(t)$ with $\|u\| \in [r_5, R_2]$.

**Proof.** It is easy to see that an $\omega$-periodic solution of equation (4) is just a fixed point of the operator equation $u = Q_2u$, where the expression of $Q_2$ is the same as $T_1$.

Define

$$\Omega_2 := \{u \in X : \|u\| < r_5\}, \quad \Omega_2^2 := \{u \in X : \|u\| < R_2\}.$$

Similar to equation (38), we know that $Q_2(K_1 \cap (\Omega_2 \backslash \Omega_1)) \subset K_1$ and $Q_2 : K_1 \cap (\Omega_2 \backslash \Omega_1) \rightarrow K_1$ is a continuous and completely continuous operator.

Now we prove that

$$\|Q_2u\| \leq \|u\|, \text{ for } u \in K_1 \cap \partial \Omega_2^2. \quad (51)$$

For any $u \in K_1 \cap \partial \Omega_2^2$, we have

$$\delta_1 R_2 \leq u(t) \leq R_2.$$

To make use of condition $(H_{10})$, it is clear that

$$Q_2u(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s)ds \leq \frac{\Lambda^* \beta^*}{\mu} R_2^{\mu+1} \leq R_2 = \|u\|.$$

Therefore, $\|Q_2u\| \leq \|u\|$, equation (51) holds.

On the other hand, we prove that

$$u \neq Q_2u + \xi u_0, \text{ for } u \in K_1 \cap \partial \Omega_1^2, \quad (52)$$

where $u_0$ and $\xi$ are same as equations (49) and (50). In fact, if not, there would exist $u_2 \in K_1 \cap \partial \Omega_1^2$ and $\xi_2 > 0$ such that

$$u_2 = Q_2u_2 + \xi_2 u_0.$$

Since $u_2 \in K_1 \cap \partial \Omega_2^2$, we have

$$\delta_1 r_5 \leq u_2(t) \leq r_5.$$
To make use of condition \((H_9)\), we arrive at

\[
\begin{align*}
    u_2 &= Q_2u_2 + \xi_2u_0 \\
    &= \int_0^\omega G(t,s) \int_0^1 \beta(s) \mu u_2^{\mu+1}(s) ds + \xi_2 \\
    &> \frac{\Lambda^* \beta^*}{\mu} (\delta_1 r_5)^{\mu+1} \\
    &\geq r_5,
\end{align*}
\]

which contradicts with \(u_2 \in K_1 \cap \partial \Omega_2^3\), i.e., equation (52) holds.

It follows from Lemma 4.1 that \(Q_2\) has a fixed point \(u \in K_1 \cap \partial \Omega_2^3\). Clearly, the fixed point is an \(\omega\)-periodic solution for equation (4) satisfying \(\|u\| \in [r_5, R_2]\).

\[\Box\]

\textbf{Remark 7.} If \(-1 < \mu < 0\), compare Theorem 3.4 with Theorem 4.3. Since \(r_5 > 0\), we get that if \(R_2 < r_2\), the range of the \(\omega\)-periodic solution proved by Theorem 4.3 is smaller.

4.3. \(\mu < -1\).

\textbf{Theorem 4.4.} Assume that \(\mu < -1\) and condition \((F_1)\) hold. Furthermore, suppose the following conditions hold:

\((H_{11})\) There exists a positive real constant \(r_6\) such that

\[
    r_6 \leq \left( \frac{\Lambda^* \beta^*}{\mu} \right)^{-\frac{1}{\mu}}.
\]

\((H_{12})\) There exists a positive real constant \(R_3\) such that

\[
    R_3 \geq \left( \frac{\Lambda^* \beta^* \delta_1^{\mu+1}}{\mu} \right)^{-\frac{1}{\mu}}.
\]

Then equation (4) has at least one positive \(\omega\)-periodic solution \(u(t)\) with \(\|u\| \in [r_6, R_3]\).

\textbf{Proof.} It is easy to see that an \(\omega\)-periodic solution of equation (4) is just a fixed point of the operator equation \(u = Q_3u\), where the expression of \(Q_3\) is the same as \(T_1\).

Define sets

\[
    \Omega_1^3 := \{ u \in X : \|u\| < r_6 \}, \quad \Omega_2^3 := \{ u \in X : \|u\| < R_3 \}.
\]

Similar to equation (41), we know that \(Q_3(K_1 \cap (\Omega_2^3 \setminus \Omega_1^3)) \subset K_1\) and \(Q_3 : K_1 \cap (\Omega_2^3 \setminus \Omega_1^3) \to K_1\) is a continuous and completely continuous operator.

Next, we prove that

\[
    \|Q_3u\| \leq \|u\|, \text{ for } u \in K_1 \cap \partial \Omega_2^3.
\]

(53)

For any \(u \in K_1 \cap \partial \Omega_2^3\), we have

\[
    \delta_1 R_3 \leq u(t) \leq R_3.
\]
Then, from condition $(H_{12})$, we get that
\[ (Q_3u)(t) = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u^{\mu+1}(s) ds \]
\[ \leq \frac{\Lambda^* \beta^* (\delta_1 R_3)^{\mu+1}}{\mu} \]
\[ \leq R_3 = ||u||. \]
Therefore, $||Q_3u|| \leq ||u||$, equation (53) holds.

Now we prove that
\[ u \neq Q_3u + \xi u_0, \quad \text{for} \quad u \in K_1 \cap \partial \Omega_3 \]
where $u_0$ and $\xi$ are same as equations (49) and (50). In fact, if not, there would exist $u_3 \in K_1 \cap \partial \Omega_3$ and $\xi_3 > 0$ such that
\[ u_3 = Q_3u_3 + \xi_3u_0. \]
It follows from the fact $u_3 \in K_1 \cap \partial \Omega_3$ that
\[ \delta_1 r_6 \leq u_3(t) \leq r_6. \]
Then, to make use of condition $(H_{11})$, we have
\[ u_3 = Q_3u_3 + \xi_3u_0 \]
\[ = \int_0^\omega G(t, s) \frac{\beta(s)}{\mu} u_3^{\mu+1}(s) ds + \xi_3 \]
\[ > \frac{\Lambda^* \beta^*}{\mu} r_6^{\mu+1} \]
\[ \geq r_6, \]
which contradicts with $u_3 \in K_1 \cap \partial \Omega_3$, i.e., equation (54) holds.

It follows from Lemma 4.1 that $Q_3$ has a fixed point $u \in K_1 \cap (\Omega_2 \setminus \Omega_3)$. Clearly, the fixed point is an $\omega$-periodic solution for equation (4) satisfying $||u|| \in [r_6, R_3]$.

\textbf{Remark 8.} If $\mu < -1$, the requirements for the coefficients of equation (4) in Theorem 4.4 is less than Theorem 3.5. Furthermore, since $r_6 > 0$, we get that if $R_3 < r_3$, the range of the $\omega$-periodic solution proved by Theorem 4.4 is smaller.

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