Distribution of Angular Momentum in the Transverse Plane

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Abstract

Fourier transforms of GPDs describe the distribution of partons in the transverse plane. The 2nd moment of GPDs has been identified by X.Ji with the angular momentum (orbital plus spin) carried by the quarks - a fundamental result that is being widely utilized in the spin decomposition of a longitudinally polarized nucleon. However, I will demonstrate that, despite the above results, the Fourier transform of the 2nd moment of GPDs does not describe the distribution of angular momentum in the transverse plane for a longitudinally polarized target.

Keywords: GPDs, angular momentum

1. Introduction

The 2-dimensional Fourier transform of Generalized Parton Distribution (GPD) \( H(x, 0, t) \) yields the distribution of partons in the transverse plane for an unpolarized target [1].

\[
q(x, \vec{b}_\perp) = \int \frac{d^2 \vec{\Delta}_\perp}{(2\pi)^2} H(x, 0, -\vec{\Delta}_\perp^2) e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} \tag{1}
\]

As a corollary, one finds that the distribution of charge in the transverse plane is given by the 2-dimensional Fourier transform of the Dirac form factor \( F_1(t) \) [2].

GPDs can also be used to study the angular momentum carried by quarks of flavor \( q \) using the Ji-relation [3]

\[
J_q = \frac{1}{2} \int dx x [H_q(x, \xi, 0) + E_q(x, \xi, 0)] \tag{2}
\]

which requires GPDs extrapolated to momentum transfer \( t = 0 \). The observation that GPDs describe the distribution of partons in the transverse plane led to the conjecture [5] that the Fourier transform of

\[
J_q(t) \equiv \frac{1}{2} \int dx x [H_q(x, \xi, t) + E_q(x, \xi, t)] \tag{3}
\]

yields the distribution of angular momentum in position space. This suggested interpretation regarding the distribution of angular momentum is frequently used in the physics motivation for experiments as well as the 12 GeV upgrade at Jefferson Lab (see e.g. [6]).

In this note, we will investigate whether such an interpretation is justified. For this purpose, we consider the 2-dimensional Fourier transform of \( J_q(t) \). Although Ref. [5] originally suggested taking a 3-dimensional Fourier transform, most experimental papers that quote the idea that \( J_q(t) \) can be used to understand the distribution of angular momentum in the transverse plane consider a 2-dimensional Fourier transform. If the 3-dimensional Fourier transform yields information about the distribution in 3-dimensional space then by integrating over the \( z \) coordinate one reduces the distribution to the transverse plane. Hence, if one can demonstrate that the interpretation of the 2-dimensional Fourier transform as the distribution of angular momentum in the transverse plane is flawed, then the interpretation of the 3-dimensional Fourier transform must automatically be flawed as well.

Using a scalar diquark model, we will calculate the distribution of quark Orbital Angular Momentum (OAM) using two complementary approaches: in the
first approach, we take the 2-dimensional Fourier transform of \(J_q(t)\) calculated in this model. From that we subtract the spin-distribution in the transverse plane evaluated from the same light-cone wave functions that were used to calculate the GPDs. In the second approach we calculate the distribution of quark OAM as a function of the impact parameter also directly from the same light-cone wave functions used in the first approach.

We selected the scalar diquark model for this study not because we think it is a good approximation for QCD, but to make a point of principle for which that fact that it is straightforward to maintain Lorentz invariance in this model is very important. Furthermore, since it is not a gauge theory, no issues arise as to whether one should include the vector potential in the definition of OAM or in which gauge the calculation should be done, i.e. there is no difference between Ji’s OAM \([3]\) and that of Jaffe and Manohar \([4]\).

2. Distribution of Angular Momentum in the Transverse Plane

Following Ref. \([5]\), we define

\[
\rho_{\perp}(\vec{b}_\perp) = \frac{d^2 \vec{k}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{k}_\perp} J_q(-\vec{k}_\perp^2),
\]

where

\[
\begin{align*}
J_q(-\vec{k}_\perp^2) &\equiv \frac{1}{2} \int dx \chi H_q(x, \xi, -\vec{k}_\perp^2) + E_q(x, \xi, -\vec{k}_\perp^2) \\
&= \frac{1}{2}(A_q(-\vec{k}_\perp^2) + B_q(-\vec{k}_\perp^2)).
\end{align*}
\]

The main goal of this work is to investigate whether it is justified to interpret \(\rho_{\perp}(\vec{b}_\perp)\) as the distribution of angular momentum in the transverse plane.

Calculating the relevant GPDs is straightforward using the light-cone wave functions \([7]\) for the scalar diquark model

\[
\begin{align*}
\psi_+^{\pm}(x, \vec{k}_\perp) &= (M + \frac{m}{x}) \tilde{\phi}(x, \vec{k}_\perp^2), \\
\psi_-^{\pm}(x, \vec{k}_\perp) &= -\frac{k^1 + i k^2}{x} \tilde{\phi}(x, \vec{k}_\perp^2), \\
\psi_+^{-}(x, \vec{k}_\perp) &= \frac{k^1 + i k^2}{x} \tilde{\phi}(x, \vec{k}_\perp^2), \\
\psi_-^{-}(x, \vec{k}_\perp) &= (M + \frac{m}{x}) \tilde{\phi}(x, \vec{k}_\perp^2)
\end{align*}
\]

with \(\phi(x, \vec{k}_\perp^2) = \frac{g/\sqrt{\pi}}{m^2 - \frac{\vec{k}_\perp^2}{2} + \frac{x^2}{2}}\). Here \(g\) is the Yukawa coupling and \(M/m/\lambda\) are the masses of the ‘nucleon’/‘quark’/diquark respectively. Furthermore \(x\) is the momentum fraction carried by the quark and \(\vec{k}_\perp \equiv \vec{k}_{\perp x} - \vec{k}_{\perp y}\) represents the relative \(\perp\) momentum. The upper wave function index \(\uparrow\) refers to the helicity of the ‘nucleon’ and the lower index to that of the quark.

For the generalized form factors needed to evaluate \([5]\) one finds \([7]\)

\[
A_q(-\vec{k}_\perp^2) = \int dx \chi H_q(x, 0, -\vec{k}_\perp^2) = \int \frac{d^2 \vec{k}_\perp}{(2\pi)^2} e^{-i \vec{b}_\perp \cdot \vec{k}_\perp} J_q(-\vec{k}_\perp^2)
\]

where

\[
H_q(x, 0, -\vec{k}_\perp^2) = \int \frac{d^2 \vec{k}_\perp}{16\pi^2} \left[ \psi_+^{\uparrow}(x, \vec{k}_\perp) \psi_+^{\downarrow}(x, \vec{k}_\perp) + \psi_-^{\uparrow}(x, \vec{k}_\perp) \psi_-^{\downarrow}(x, \vec{k}_\perp) \right]
\]

From these GPDs one can determine the OAM as obtained from GPDs through the Ji relation \([7]\) as

\[
\Delta q(x) = \frac{1}{2} \int_0^1 dx \left[ x H_q(x, 0, 0) + x E(x, 0, 0) - \Delta q(x) \right]
\]

Since some of the above \(\vec{k}_\perp\)-integrals diverge, a manifestly Lorentz invariant Pauli-Villars regularization (subtraction with heavy scalar \(\lambda^2 \to \Lambda^2\)) is always understood.

To evaluate relation \([4]\), we simplify and rewrite \([8]\) and \([10]\) as:

\[
\begin{align*}
H_q(x, 0, -\vec{k}_\perp^2) &= \frac{g^2}{16\pi^3} \int \frac{d^2 \vec{k}_\perp}{16\pi^3} \left[ \psi_+^{\uparrow}(x, \vec{k}_\perp) \psi_+^{\downarrow}(x, \vec{k}_\perp) + \psi_-^{\uparrow}(x, \vec{k}_\perp) \psi_-^{\downarrow}(x, \vec{k}_\perp) \right] \\
&= \int_0^1 \frac{dx}{(\vec{k}_\perp^2 + x^2)^3} \\
&= -\int_0^1 \frac{dx}{(\vec{k}_\perp^2 + x^2)^3}
\end{align*}
\]
where $u = x^2 - 2x + 1 + x\lambda^2$ and $F = (1 - x)^2(A^2 - \alpha(1 - \alpha) + x^2 - 2x + 1 + x\lambda^2)$

Similarly,

$$E(x, 0, -\lambda^2) = \frac{Mg^2}{8\pi^2} \int_0^1 dx \frac{1}{\lambda} (m + xM)$$

In order to describe distributions in impact parameter space, we introduce wave functions in impact parameter space as [9]

$$\psi_\lambda(x, \vec{b}_\perp) \equiv \frac{1}{2\pi(1 - x)} \int d^2\vec{k}_\perp e^{i\vec{k}_\perp\cdot\vec{0}} \psi(x, \vec{k}_\perp)$$

where calculating suitable prefactor $\frac{1}{2\pi(1 - x)}$ is straightforward using the following relation:

$$\int |\psi_\lambda(x, \vec{b}_\perp)|^2 d^2\vec{b}_\perp = \int |\psi(x, \vec{k}_\perp)|^2 d^2\vec{k}_\perp.$$  \hspace{1cm} (16)

Note the factor $\frac{1}{2\pi(1 - x)}$ in the exponent which accounts for the fact that the variable $\vec{k}_\perp$ is conjugate to the displacement between the active quark and the spectator, while the impact parameter $\vec{b}_\perp$ represents the displacement of the active quark from the center of momentum of the entire hadron. Using these wave functions, it is straightforward to evaluate the quark spin distribution in the transverse plane for a longitudinally polarized 'nucleon' as

$$\rho_\lambda(\vec{b}_\perp) = \int dx \left[ |\psi_\lambda^\dagger(x, \vec{b}_\perp)|^2 - |\psi_{-\lambda}(x, \vec{b}_\perp)|^2 \right].$$  \hspace{1cm} (17)

where

$$|\psi_\lambda^\dagger(x, \vec{b}_\perp)|^2 = \frac{g^2}{16\pi^3(1 - x)} \int_0^\infty dk_\perp J_0((\xi - \lambda)^2)$$

and

$$|\psi_{-\lambda}(x, \vec{b}_\perp)|^2 = \frac{g^2}{16\pi^3(1 - x)} \int_0^\infty dk_\perp J_0((\xi + \lambda)^2)$$

If [4] can be interpreted as the angular momentum density then the difference

$$L_\lambda(\vec{b}_\perp) \equiv \rho_\lambda(\vec{b}_\perp) - \rho_\lambda(\vec{b}_\perp)$$

represents the orbital angular momentum density. In the following section, we will investigate if that is the case.

3. Impact Parameter Space Distribution Directly from Light Front Wave Functions

With the light-cone wave functions available [9], it is also straightforward to compute the orbital angular momentum $L_\lambda$ of the 'quark' for a 'nucleon' polarized in the $\hat{z}$ direction directly as [8]

$$L_\lambda = \int_0^1 dx \int d^2\vec{k}_\perp \frac{1}{16\pi^3(1 - x)} |\psi_\lambda(x, \vec{k}_\perp)|^2.$$  \hspace{1cm} (21)

Evaluating the above integrals is tedious, but straightforward, and one finds [8]

$$L_\lambda = \frac{L_\perp}{4}$$

as was expected since $L_\perp$ in the scalar diquark model does not contain a vector potential and therefore no gauge related issues arise (in QED for an electron $L_\perp \neq L_\perp$ [8]).

Likewise, one can define the orbital angular momentum density directly using light-cone wave functions [15] as $L_\lambda$ and $b \equiv |\vec{b}_\perp|$ can be simultaneously measured. For a nucleon with spin up, only the wave function component $\psi_\lambda$ has one unit or orbital angular momentum shared between the active quark (weight factor $1 - x$) and the spectator (weight factor $x$) [8] and therefore

$$L_\lambda = \int dx (1 - x) |\psi_\lambda(x, \vec{b}_\perp)|^2$$

represents the orbital angular momentum density for the active quark as a function of the distance from the center of momentum in a 'nucleon' that is polarized in the $+\hat{z}$ direction.

Evaluating integrals available [20] and [23] are tedious but straight forward. Use of manifestly Lorentz invariant Pauli-Villars regularization (subtraction with heavy scalar $\lambda^2 \rightarrow \lambda^2$) is easily understood to isolate the divergence piece for some of $k_\perp$ integrals. Both $L_\lambda(b)$ and $L_\perp(b)$ are shown in Fig. 1 and it is clear that the area under the curve is the only feature that these two distributions have in common.

$$\int_0^\infty db \ L_\lambda(b) = \int_0^\infty db \ L_\perp(b)$$

With the relations available [21], [11], and [20], it is also straightforward to show

$$\int d^2\vec{b}_\perp L_\lambda(\vec{b}_\perp) = L_\lambda = L_\perp = 2\pi \int_0^\infty db \ L_\lambda(b).$$  \hspace{1cm} (25)

This result clearly demonstrates that $L_\lambda(\vec{b}_\perp)$ does not represent the distribution of angular momentum for a
longitudinally polarized target, since \( L_q(b) \) already has that interpretation. As a corollary, we also conclude that the Fourier transform of \( J_\parallel^q(t) \) does not represent the distribution of angular momentum either - regardless whether the Fourier transform is two- or three-dimensional. These observations represent the main result of this work.

4. Discussion

We have demonstrated within the context of a scalar Yukawa diquark model that although \( J_\parallel^q(t) \) yields, in the limit \( t \to 0 \), the \( \hat{z} \) component of the quark angular momentum for a target polarized in the \( +\hat{z} \) direction, the 2-dimensional Fourier transform of its \( t \)-dependence does not yield the distribution of angular momentum in impact parameter space.

This result is best understood by recalling that Lorentz/rotational invariance is heavily used when a relation between the quark angular momentum operator, which is not only leading twist, and twist-2 GPDs. The use of Lorentz invariance appears implicitly in the original paper [3], where it imposes constraints on the allowed tensor structure. In Ref. [10], Eq. (2) was rederived by considering the transverse deformation of parton distributions in a transversely polarized target. In this approach, the momentum density in the \( \hat{z} \) direction was correlated with the distribution in the transverse direction for a transversely polarized target (see also Ref. [11]). While \( T^{\parallel T_3}x \) comprises only half the angular momentum tensor \( T^{\parallel T_3}x - T^{\parallel T_3}z \), the two terms in the latter turn out to yield identical contributions - provided the target is invariant under rotations about the \( \hat{y} \) axis. Therefore, as long as one considers a target with rotational symmetry about the \( \hat{y} \) axis, one can identify the angular momentum in the \( \hat{y} \) direction with the expectation value of \( 2T^{\parallel T_3}x \), which in turn can be identified with off forward matrix elements of the twist two operator \( T^{\parallel T_3} \). Finally, as long as considering the However, as rotational invariance has been heavily used in this process, the resulting relation [3] should hold for any component of the quark angular momentum for a nucleon polarized in the corresponding direction. Hence one can relate the quark angular momentum in the \( \hat{z} \) direction, although it is not \( a \ priori \) twist-2, to matrix elements of twist-2 operators.

Our explicit calculation has shown that the Fourier transform of \( J_\parallel^q(t) \) does not yield the distribution of angular momentum in the transverse plane for a longitudinally polarized target. However, from the discussion above it should also be clear that it cannot be interpreted as the distribution of transverse angular momentum in a transversely polarized target: the Fourier transform of \( J_\parallel^q(t) \) yields the distribution of \( xT^{\parallel T_3} \). Using rotational symmetry arguments, that are applicable only after integration over the position, that can be related to the matrix element of \( xT^{\parallel T_3} \) and hence also of \( -T^{\parallel T_3} \). However, this is not possible for the local (unintegrated) densities.

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Appendix A. Different types of integrals used

\[
\int d^2\vec{k}_\perp \frac{1}{k_1^2 + u(\lambda^2)} = \pi \log \left( \frac{u(\lambda^2 = 10)}{u(\lambda^2 = 1)} \right) \quad (A.1)
\]

\[
\int d^2\vec{k}_\perp \frac{1}{(k_1^2 + F(\lambda^2))^2} = \pi \left( \frac{1}{F(\lambda^2 = 1)} - \frac{1}{F(\lambda^2 = 10)} \right) \quad (A.2)
\]

\[
J_0(|\vec{k}_\perp \cdot \vec{b}_\perp |) = \frac{1}{2\pi} \int d\phi \, e^{i \phi} |\vec{k}_\perp \cdot \vec{b}_\perp | \quad (A.3)
\]

\[
\int d^2\vec{b}_\perp e^{i \phi} \frac{d\phi}{\sin^{1/2}(\phi)} = (2\pi)^2 (1 - x)^2 \delta^2(|\vec{k}_\perp - \vec{k}_\perp '|) \quad (A.4)
\]
Appendix B. Part of calculation for relation (25)

\[
\int d^2 \vec{r}_\perp L_q(\vec{r}_\perp) = \int d^2 b \vec{\rho}_J(b_\perp) - \int d^2 b \vec{\rho}_S(b_\perp)(B.1)
\]

\[
\int d^2 \vec{r}_\perp \vec{\rho}_J(b_\perp) \equiv J_q(0) = \frac{1}{2} [A_q(0) + B_q(0)] \quad (B.2)
\]

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