We show that the Hilbert space spanned by a continuously parametrized wavefunction family—is dominated by a subspace, onto which all member states have close to unity projection weight. Its characteristic dimensionality \( D_P \) is much smaller than the full Hilbert space dimension, and is equivalent to a statistical complexity measure \( e^{S_2} \), where \( S_2 \) is the 2\(^\text{nd} \) Renyi entropy of the manifold. In the thermodynamic limit, \( D_P \) closely approximates the quantum geometric volume of the manifold under the Fubini-Study metric, revealing an intriguing connection between information and geometry. This connection persists in compact manifolds such as a twisted boundary phase, where the corresponding geometric circumference is lower bounded by a term proportional to its topological index, reminiscent of entanglement entropy.

Introduction—A ubiquitous notion in quantum physics is a state manifold, i.e., a continuously parametrized wavefunction family \( \Psi[M] \equiv \{ |\Psi(\lambda)\rangle \in \mathcal{H} | \lambda \in M \}. \) One example, typically encountered in the study of quantum phase transitions [1, 2], is the ground state of a Hamiltonian \( H(\lambda) \), where \( \lambda \) may represent external field, interaction strength, adiabatic pumping, etc. Another one is the Hilbert space trajectory generated by time evolution, e.g., after a quantum quench. Conventionally, \( \Psi[M] \) is analyzed through a few carefully designed characteristic quantities such as order parameters, topological indices, etc., which are highly system dependent. With the recent influx of machine learning techniques [3–10], a new paradigm is emerging, where the task of identifying such reductions is delegated to domain-agnostic algorithms, and the role of human becomes instead to understand is the fundamental information limit of \( \Psi[M] \), and \( \lambda \) can be restored at the end if necessary. Taking each \( |\Psi_m\rangle \equiv |\Psi(\lambda_m)\rangle \) as a \( D_H \)-dimensional column vector, the amplitude matrix \( A \) is constructed as

\[
A = \frac{1}{\sqrt{M}} (|\Psi_1\rangle \ |\Psi_2\rangle \cdots |\Psi_M\rangle).
\]

SVD of \( A \) then gives (subscripts denote matrix sizes)

\[
A_{D_H \times M} = U_{D_H \times K} \Lambda_{K \times K} V_{K \times M}^T,
\]

in which \( \Lambda_{K \times K} = \sqrt{w_1} \delta_{\lambda} \) is a positive diagonal matrix with \( 1 \geq w_1 \geq w_2 \geq \cdots w_K > 0 \), \( U \) and \( V \) are column unitary matrices, \( U^U = V^V = I_{K \times K} \), and \( K \leq \min(D_H, M) \) is the rank of \( A \), which is also the exact dimensionality of the Hilbert space spanned by the \( M \) sampled states. In a manybody setting, \( M \ll D_H \).
thus $K = M$ if we assume all sampled states are linearly independent. The eigen decomposition of $\hat{\rho}$ then reads

$$\hat{\rho} = AA^\dagger = \frac{1}{M} \sum_{m=1}^{M} |\Psi_m\rangle\langle \Psi_m| = \sum_{\kappa=1}^{K} w_\kappa |U_\kappa\rangle\langle U_\kappa|, \quad (4)$$

where $|U_\kappa\rangle$ is the $\kappa^{th}$ column of $U$. Note that $\text{Tr}(\hat{\rho}) = \sum_{\kappa} w_\kappa = 1$. The expansion coefficient $\langle U_\kappa |\Psi_m\rangle = (U^\dagger A)_{km} = \sqrt{M} w_\kappa V_{km}$ only involves $V$ and $\{w_\kappa\}$, and can be obtained alternatively via the eigen decomposition of the $M \times M$ overlap matrix $G$,

$$G_{mn} \equiv \frac{1}{M} \langle \Psi_m |\Psi_n\rangle = (A^\dagger A)_{mn}, \quad G = V^\dagger AV, \quad (5)$$

which will be useful in numerical schemes where overlaps can be efficiently computed. In statistical sciences, $\{|U_\kappa\rangle\}$ are known as principal vectors, and the above procedure amounts to the principal component analysis (PCA) of $\{|\Psi_m\rangle\}$. PCA of spin configuration data has recently been used to identify phase transitions [4, 19, 20].

We illustrate the representational power of the principal vectors using the bilinear-biquadratic $S = 1$ spin chain [21], $H(\lambda) = \sum_{x=1}^{N} S_x \cdot S_{x+1} + \lambda(\sum_{x=1}^{N} S_x \cdot S_{x+1})^2$, where $N$ is chain length, and $S_{N+1} \equiv S_1$ (periodic boundary condition). For $N = 14$, we compute its ground state (exact diagonalization) at $M = 180$ points uniformly sampled between $-1 < \lambda < 0.8$, which covers most of the gapped Haldane phase [22, 23] [24]. Their PCA, shown in Fig. 1(a), has several interesting features:

(i) The $\kappa^{th}$ expansion coefficient $f_\kappa(\lambda) \equiv \langle U_\kappa |\Psi(\lambda)\rangle$ has $\kappa - 1$ nodes in the $\lambda$ space. This is reminiscent of the radial nodal structure in the hydrogen atom, which is attributed to the orthogonality between states of different principal quantum numbers. A similar orthogonality exists between $f_\kappa$ and $f_{\kappa'}$: $\int d\lambda f_\kappa(\lambda) f_{\kappa'}(\lambda)/\int d\lambda \to \frac{1}{M} \sum_{m=1}^{M} f_\kappa(\lambda_m) f_{\kappa'}(\lambda_m) = (AV^\dagger AV)_{\kappa\kappa'} = w_\kappa \delta_{\kappa\kappa'}$, which follows from Eq. 3 [25], and is suggestive of an “emergent quantum mechanics” in the $\lambda$ space.

(ii) The first few principal vectors already reproduce $\sim 1$ total weight. For the $N = 14$ chain, the leading weights ($w_\kappa$) are $\{0.71, 0.23, 0.052, 0.0067, \ldots\}$, thus a cutoff dimension at $D_{\text{cutoff}} = 3$, for example, preserves a total weight of $\sim 0.99$. Note that $D_{\text{cutoff}} \ll D_H \equiv 3^{14}$. Indeed, one can define a principal Hilbert space $\mathcal{H}_P$ as the span of the first $D_{\text{cutoff}}$ principal vectors; then as shown in Fig. 1(a), most ground states within the sampled region can be approximated to a fidelity of $\geq 0.99$ by its projection onto $\mathcal{H}_P$, i.e., a 3-term truncation $|\Psi(\lambda)\rangle \approx \sum_{\kappa=1}^{3} f_\kappa(\lambda)|U_\kappa\rangle$.

The compactness of $\mathcal{H}_P$ suggests that a sparse sampling of the $\lambda$ space at $M \sim D_{\text{cutoff}}$ points could gather enough information to generate as good a set of principal vectors as a dense sampling. In Fig. 1(b), we show that taking $M = 4$ samples is sufficient to approximate any unsampled state in the manifold. Thus the entire ground state manifold can be approximately reconstructed by diagonalizing a $4 \times 4$ Hamiltonian $\bar{H}_{\kappa\kappa'}(\lambda) = \langle U_\kappa |H(\lambda)| U_{\kappa'}\rangle$ for continuous $\lambda$, using $\{|U_\kappa\rangle\}$ generated from a 4-point sampling. Similarly, all physical operators are approximately $4 \times 4$ matrices.

**Principal dimensionality**—The impressive compactness of $\mathcal{H}_P$ prompts the question: what determines an appropriate cutoff dimension? Below we provide an estimation. Consider a state $|\Psi_m\rangle \in \mathcal{H}[M]$. Its average projection weight onto a random state in $\mathcal{H}[M]$ is

$$\bar{w}_m \equiv \frac{1}{M} \sum_{n=1}^{M} \langle \Psi_m |\Psi_n\rangle^2, \quad (6)$$

thus it takes approximately $D_m = [\bar{w}_m]$ random states in $\mathcal{H}[M]$ to supply a total weight of $\sim 1$. In the thermodynamic limit, these $D_m$ random states are themselves mutually orthogonal, and can serve as a minimal basis set to expand $|\Psi_m\rangle$. A further average over $m$ removes the $m$-dependence, and there are two natural choices: $1/\langle \bar{w}_m \rangle$ and $\langle 1/\bar{w}_m \rangle$. We adopt the first one as it puts the $m$ and $n$ dependence on an equal footing, but remark that the second option would eliminate a dominant correction $(\langle \sigma^2 \rangle)$ from Eq. 13 and therefore has its own merit [26]. This leads to our definition of principal dimensionality,

$$D_P \equiv \frac{1}{\langle \bar{w}_m \rangle} = \frac{M^2}{\sum_{m,n=1}^{M} \langle \Psi_m |\Psi_n\rangle^2} = \frac{1}{\text{Tr}(G^2)}, \quad (7)$$

where $\langle \cdots \rangle = \sum_{m}(\cdots)/M \overset{M \rightarrow \infty}{\longrightarrow} \int d\lambda (\cdots)/\int d\lambda.$
The last equality in Eq. 7 suggests an interesting connection with Renyi entropy. Recall that the $\alpha$th Renyi entropy of a trace-normalized hermitian $\hat{O}$ is $S_\alpha[O] = \frac{1}{1-\alpha} \log \text{Tr}(\hat{O}^\alpha)$, thus $D_\rho = e^{S_2[\hat{O}]}$. Since $S_\alpha[O]$ can be evaluated using the nonzero eigenvalues of $\hat{O}$, two operators with identical nonzero spectra—in this case $\hat{\rho} = AA^\dagger$ and $G = A^\dagger A$—must have the same Renyi entropy, hence

$$D_\rho = e^{S_2[\hat{O}]} = e^{S_2[\rho]} = \frac{1}{\sum_{k=1}^K w_k^2}.$$  \hspace{1cm} (8)

Our definition of $D_\rho$, based on considerations of wavefunction overlaps, is thus consistent with the intuitive correspondence between number of effective degrees of freedom and exponential of entropies \cite{27}. The physical meaning of $D_\rho$ is now more transparent: the last expression is an \textit{inverse participation ratio}; thus $D_\rho$ represents the number of principal vectors participating in the manifold $\Psi[M]$, and is therefore a natural measure of its effective Hilbert space dimension, and equivalently, its degree of \textit{Hilbert space localization} \cite{28}.

Note that $D_\rho$ thus defined is not an integer. For numerical comparison, we introduce an integer truncation dimension $D(W) \equiv \min_k \left[ \sum_{\kappa'=1}^k w_{\kappa'} \geq W \right]$, i.e., the smallest number of principal vectors needed to reach a total weight of $W$. We expect $D_\rho$ to be comparable to a $D(W)$ with $W \sim 1$.

Since exact diagonalization of interacting models, such as the bilinear-biquadratic spin chain used earlier, can only be implemented for small system size, we now switch to a free fermion model to illustrate $D_\rho$ at large $N$. The Su-Schrieffer-Heeger (SSH) model \cite{29} describes fermions hopping on a one-dimensional lattice with two alternating hopping amplitudes ($1$ and $\lambda$) on neighboring bonds. With $2N$ lattice sites, its Hamiltonian is $H(\lambda) = \sum_{x=1}^{2N-1} \epsilon_{x-1} \epsilon_{x} + \lambda \epsilon_{x-1} \epsilon_{x+1} + h.c.$, and we use periodic boundary condition $(\epsilon_{2N+1}, \epsilon_{2} = (\epsilon_1, \epsilon_2)$. Its ground state at half filling is $|\Psi(\lambda)\rangle = \prod_a \psi_{k_a}(\lambda)|0\rangle$, where $\psi_{k_a}(\lambda)$ creates a lower band eigenmode of momentum $k_a = 2\pi a/N$ for $a = 1, 2, \ldots, N$. The weights $\{w_\kappa\}$ can be obtained by diagonalizing the overlap matrix $G_{\lambda\lambda'} = \langle \psi(\lambda)|\psi'(\lambda')\rangle = \prod_a \langle 0|\psi_{k_a}(\lambda)\psi_{k_a}(\lambda')|0\rangle$.

In Fig. 2, we uniformly sample $M = 1000$ ground states within $-0.5 < \lambda < 0.5$, and plot $D_\rho$ and $D(W)$ for several $W$s, as functions of system size $N$. As shown, $D_\rho > D(0.9)$, i.e., the first $[D_\rho]$ principal vectors represent over 90% total weight of the state manifold. In addition, $D(W)W$ are linearly related to $D_\rho$ (see inset). Thus $D_\rho$ accurately captures the Hilbert space size of the SSH ground state manifold.

\textbf{Relation with quantum geometry—}The geometric content of quantum state manifolds has been intensively studied in the past decade, particularly in relation with quantum phases and phase transitions \cite{30–41}. Since two wavefunctions that differ by an infinitesimal $\delta \lambda$ are in general not orthogonal, their overlap amplitude can be interpreted as a Hilbert space distance, $d(\Psi, \Phi) = \cos^{-1} \langle \Psi|\Phi\rangle$, a measure of their maximal experimental distinguishability \cite{42}. This endows $\Psi[M]$ with a metric \cite{33}, $F_{\mu\nu}(\lambda) \equiv \langle \partial_{\lambda_\mu} \Psi(\lambda)|\partial_{\lambda_\nu} \Psi(\lambda)\rangle - \langle \partial_{\lambda_\mu} \Psi(\lambda)|\Psi(\lambda)\rangle\langle \Psi(\lambda)|\partial_{\lambda_\nu} \Psi(\lambda)\rangle$, which turns out to encode surprisingly rich physics. Its anti-symmetric and symmetric parts are, respectively, the Berry curvature $\Omega_{\mu\nu} = -iF_{\mu\nu} - F_{\nu\mu}$ \cite{43}, and the Fubini-Study metric $g_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + F_{\nu\mu})$ \cite{11}. The former has long been known to play a fundamental role in systems with nontrivial topology \cite{43, 44}. Study of the latter in the context of quantum many-body physics initiated only more recently with Refs. \cite{33, 35}, which showed that $F_{\mu\nu}$ exhibits universal scaling near phase transitions, and becomes singular at quantum critical points. The more global aspects of $F_{\mu\nu}$ have since been systematically investigated \cite{37, 38, 40, 41}. Note that the Fubini-Study metric can be viewed as a quantum generalization of the Fisher-Rao information metric \cite{45}, and the divergence of the latter has been shown to signify \textit{thermal} phase transitions \cite{46}, \textit{vis-a-vis} its quantum counterpart.

The appearance of wavefunction overlaps in Eq. 7 suggests a potential geometric interpretation of $D_\rho$ and $S_\alpha$. Interestingly, Hall \cite{47} has argued from general grounds that $e^{S_1}$—where $S_1$ is the von Neumann entropy—can be viewed as a geometric volume measure for any statistical ensemble, although Renyi entropies $S_\alpha$ with $\alpha \neq 0, 1$ were explicitly excluded. We now establish a connection between $D_\rho$ and the Fubini-Study volume.

For clarity, we will consider a scalar parameter $\lambda$. Straightforward Taylor expansion gives $\left| \langle \Psi(\lambda)|\Psi(\lambda + \Delta \lambda)\rangle \right|^2 = 1 - g_\lambda \Delta \lambda^2 + O(\Delta \lambda^3)$, where $g_\lambda$ is the Fubini-Study metric,

$$g(\lambda) = \langle \partial_\lambda \Psi|\partial_\lambda \Psi\rangle - \langle \partial_\lambda \Psi|\Psi\rangle\langle \Psi|\partial_\lambda \Psi\rangle ,$$  \hspace{1cm} (9)

note that its length element reproduces the Hilbert space distance, $d\lambda = \sqrt{g(\lambda)} \Delta \lambda = \cos^{-1} \langle \Psi|\Psi(\lambda + \Delta \lambda)\rangle$, and is-
variant under reparametrization. In the thermodynamic limit, the overlap quickly drops to zero as \(\Delta \lambda\) increases, thus one can write
\[
|\langle \Psi(\lambda) | \Psi(\lambda + \Delta \lambda) \rangle|^2 \simeq e^{-g(\lambda) \Delta \lambda^2}.
\] (10)

For a parameter space \(\lambda \in (\lambda_a, \lambda_b)\), this allows us to rewrite Eq. 6, in the continuum limit, as
\[
\bar{w}(\lambda) = \int_{\lambda_a}^{\lambda_b} d\lambda' \frac{e^{-g(\lambda') \lambda^2}}{\lambda_b - \lambda_a} = \sqrt{\frac{\pi}{g(\lambda)}} \frac{1 - \xi(\lambda)}{\lambda_b - \lambda_a},
\] (11)

where \(\xi(\lambda)\) accounts for the effect of finite integration limits, \(\xi(\lambda) = \frac{1}{2} \text{erfc} \left( \sqrt{g(\lambda)}(\lambda - \lambda_a) \right) + \frac{1}{2} \text{erfc} \left( \sqrt{g(\lambda)}(\lambda_b - \lambda) \right)\), and \(\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt e^{-t^2}\). \(\xi(\lambda) \simeq \frac{1}{2}\) at the boundaries \(\lambda_a, \lambda_b\), but rapidly approaches zero away from them [26]. As discussed before, \(\bar{w}(\lambda)\) counts the number of random states in \(\Psi[M]\) needed to span \(\langle \Psi(\lambda) \rangle\), thus away from the boundaries \(\lambda_a, \lambda_b\), \(\sqrt{g}\) is a “local density of dimensions”,
\[
\sqrt{g(\lambda)} \simeq \bar{w}(\lambda)^{-1} \quad \text{Density of principal dimension}.
\] (12)

One can now anticipate that the quantum geometric length \(L = \int d\lambda \sqrt{g(\lambda)}\), an integrated “dimension density”, should recover the principal dimensionality. Indeed, we have [26]
\[
D_P = \frac{L}{\sqrt{\pi}} (1 + c), \quad L \equiv \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{g(\lambda)},
\] (13)

where \(c = \frac{1}{1 - \xi(\lambda)} - 1 = \langle \xi(\lambda) \rangle - \langle \sigma(\lambda)^2 \rangle + \cdots\), \(\sigma(\lambda) = \frac{\sqrt{g(\lambda)}}{\langle \sqrt{g(\lambda)} \rangle - 1}\), and \(\langle \cdots \rangle = \int_{\lambda_a}^{\lambda_b} d\lambda \frac{\cdots}{\sqrt{g(\lambda)}}\). \(\langle \sigma^2 \rangle\) is the relative standard deviation of \(\sqrt{g}\) independent of system size \(N\). \(\xi\) is an “edge” effect around \(\lambda_a, \lambda_b\), and decreases with increasing \(N\). Thus for a smooth enough \(\Psi[M]\), \(c\) should be negligible. In Fig. 2, we show that for the SSH ground state manifold, \(L/\sqrt{\pi}\) almost perfectly tracks \(D_P\). The Gaussian integral Eq. 11 can easily accommodate \(d_M\)-dimensional parameters, with which the dominant contribution in Eq. 13 becomes \(D_P = \sqrt{\frac{\pi}{N}}\text{det}(g_{\mu\nu}(\lambda))\) in the Fubini-Study volume. Note that for gapped ground state manifolds, \(g_{\mu\nu} \propto N\) where \(N\) is system size [35], thus \(D_P \propto N^{d_M/2}\).

**Compact manifold and topology**—Compact manifolds may host topologically protected degeneracies, which can be interpreted as a lower bound to Hilbert space dimension. This in turn implies topological lower bounds to \(S_2\) and \(\mathcal{L}\). We briefly discuss this point using again the SSH model, but on the compact space of a twisted boundary phase \(\theta \in [0, 2\pi]\) [48, 49], \(c_{2N+i} = e^{i\theta} c_i\), \(i = 1, 2\). The ground state at half filling becomes \(|\Psi(\lambda, \theta)\rangle = \prod_i \psi_{k_i,\theta}(\lambda)|\theta\rangle\), where \(k_i(\theta) = \frac{2\pi n + \theta}{N}, n = 1, 2, \cdots, N\).

Since Bloch states with different \(\theta\) are no longer orthogonal, the overlap matrix elements become Slater determinants, \(G_{ab}(\lambda) = \langle \Psi(\lambda, \theta) | \Psi(\lambda, \theta') \rangle = \det g(\lambda), \text{where } G_{ab}(\lambda) = \langle \emptyset | \psi_{k_{a,\theta}}(\lambda) \psi_{k_{b,\theta'}}(\lambda) | \emptyset \rangle\). The system undergoes a topological phase transition at \(|\lambda_c| = 1\), where its Berry phase \(\gamma = -\int_0^{2\pi} d\theta \text{det} (\psi(\theta)|\partial_\theta \psi(\theta))\) changes discretely from 0 (\(0 < |\lambda| < |\lambda_c|\)) to \(2\pi\) (\(|\lambda| > |\lambda_c|\)).

In Fig. 3, we take a \(2N = 400\)-site chain and compute \(\{w_a\}, D_P\), and \(\mathcal{L}\) for its ground state ensemble \(\{ |\Psi(\lambda, \theta)\rangle \}\) at 100 \(\theta\)s evenly sampled between 0 and 2\(\pi\), and plot them as functions of \(\lambda \in [0, 2]\). In the non-topological phase, a single weight dominates, with two degenerate subleading ones appreciating near the critical \(\lambda_c = 1\); \(D_P\) stays close to 1 and only exhibits a sharp upturn near \(\lambda_c\). The topological phase is dominated by two degenerate weights, thus \(D_P\) stabilizes toward 2. The cusp at \(\lambda_c\) reflects the topological phase transition.

The behavior of \(\mathcal{L}\) mirrors that of \(D_P\), but does not quantitatively satisfy \(D_P \simeq \mathcal{L}/\sqrt{\pi}\). This is expected, because the deviation of \(\langle \Psi(\theta) | \Psi(\theta + \Delta \theta) \rangle\) from 1 is driven solely by the single bond on which the twisted boundary phase is applied, which thus spoils the many-body character of Eq. 10 and hence its quantitative accuracy. To acquire some analytical understanding, we truncate to a two-dimensional subspace, justified because \(D_P \leq 2\) in both phases except near the critical point. One can then write (dropping \(\lambda\) dependence) \(|\Psi(\theta)\rangle = \frac{1}{\sqrt{2}} \left[ |\tilde{V}_1\rangle + e^{i\phi(\theta)} |\tilde{V}_2\rangle \right]\), which is the most general form of a two-component wavefunction that can host a robust winding number, \(\nu = \int_0^{2\pi} \frac{d\theta}{2\pi} \partial_\theta \phi\), and \(\gamma = \nu \pi\) is the Berry phase. From Eq. 9, \(g(\theta) = \frac{1}{4} (\partial_\theta \phi)^2\), thus \(\mathcal{L} = \frac{1}{2} \int_0^{2\pi} d\theta |\partial_\theta \phi|^2 \geq \frac{1}{4} \int_0^{2\pi} d\theta |\partial_\theta \phi|^2 = |\nu| \pi\). Indeed, in Fig. 3(b), \(\mathcal{L}\) stabilizes to \(\pi\) for \(\lambda > \lambda_c\) and touches 0 at \(\lambda = 0\). The geometric length of a compact manifold is
thus lower bounded by its topological index, reminiscent of the behavior of entanglement entropy [50, 51].

Conclusion and discussions—We have shown that the effective Hilbert space dimension $D_P$ of a quantum state manifold, its complexity $e^{S_2}$, and its Fubini-Study geometric volume $V$ are quantitatively related, $D_P = e^{S_2} \simeq V/\sqrt{\pi^d D}$. On a compact manifold, there is a topological obstruction to their trivialization, with a lower bound determined by the topological index. The interpretation of geometric volume as a Hilbert space dimension is suggestive of an approximate “geometric quantization”, in unit of $\sqrt{\pi^d D}$, of parameter spaces (which may coincide with real space and time). This is analogous to the quantization of classical phase space in unit of $(2\pi)^d$. In this sense, $D_P$ is a natural quantum generalization of the notion of phase space volume. For a $\lambda$ that parametrizes a ground state, $D_P$ represents the amount of quantum fluctuation driven by $\lambda$. Divergence of $g_{\mu\nu}$ at critical point [35] then implies that as one approaches quantum criticality, increasingly more states are “pulled down” from higher energies to span the effective Hilbert space. For a manifold generated from unitary time evolution (i.e., $\lambda = \text{time}$), $[D_P]$ is roughly the number of Hilbert space dimensions “activated” over the course of time, and can be measured as the integrated energy fluctuation [52]. In the context of adiabatic quantum computation, this becomes the number of orthogonal “machine states” needed to carry out an algorithm, and is therefore a kind of computational complexity.

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SUPPLEMENTAL MATERIALS

In this note, we provide derivation details and some numerical justifications for Eq. (13) in the text, reproduced below,

\[ D_P = \frac{L}{\sqrt{\pi}}(1 + c), \quad L = \int d\lambda \sqrt{g(\lambda)}, \quad c = \left. \left( \frac{1 - \xi(\lambda)}{1 + \sigma(\lambda)} \right)^{-1} \right|_{\lambda = \lambda_m}. \tag{14} \]

We also discuss an alternative definition of characteristic dimensionality, \( D_{alt} \equiv \langle 1/\bar{w}_m \rangle \) (whereas \( D_P \equiv \langle 1/\bar{w}_m \rangle \)).

Consider a one parameter wavefunction \( |\Psi(\lambda)\rangle \), and discretize the \( \lambda \) space into a regular grid,

\[ \lambda_m = m\delta\lambda, \quad m = 1, 2, \ldots, M, \quad |\Psi_m\rangle \equiv |\Psi(\lambda_m)\rangle. \tag{15} \]

Our task is to compute \( D_P \),

\[ D_P = \sum_{m,n} \frac{M^2}{|\langle \Psi_m | \Psi_n \rangle|^2}. \tag{16} \]

We use the approximation

\[ C_{mn} \equiv |\langle \Psi_m | \Psi_n \rangle|^2 \approx \exp[-g_m(n - m)^2\delta\lambda^2], \tag{17} \]

where \( g_m \) is the Fubini-Study metric,

\[ g_m \equiv g(\lambda_m) = \langle \partial_{\lambda} \Psi(\lambda_m) | \partial_{\lambda} \Psi(\lambda_m) \rangle - \langle \partial_{\lambda} \Psi(\lambda_m) | \Psi(\lambda_m) \rangle \langle \Psi(\lambda_m) | \partial_{\lambda} \Psi(\lambda_m) \rangle. \tag{18} \]

Then

\[ \sum_{n=1}^{M} C_{mn} = \sum_{x=1-m}^{M-m} e^{-g_mx^2\delta\lambda^2} \rightarrow \int_{1-m}^{M-m} dx e^{-g_mx^2\delta\lambda^2} \]

\[ = \int_{-\infty}^{1-m} - \int_{-\infty}^{M-m} \cdots \frac{\sqrt{\pi}}{\sqrt{g_m}\delta\lambda} [1 - \xi_m], \tag{19} \]

where the prefactor in the last expression results from the infinite-limit Gaussian integral, and \( \xi_m \) accounts for the second and third integration ranges on the second line, i.e., the effect of finite integration limits,

\[ \xi_m = \xi(\lambda_m) = \frac{1}{2} \text{erfc} \left[ \sqrt{g_m} \delta\lambda (m - 1) \right] + \frac{1}{2} \text{erfc} \left[ \sqrt{g_m} \delta\lambda (M - m) \right], \tag{20} \]

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} dt e^{-t^2}. \tag{21} \]
Note that the arguments in the erfcs can be interpreted as “quantum distances” from \( \lambda_m \) to the two boundaries \( \lambda_1 \) and \( \lambda_M \), respectively, as propagated by the metric at \( \lambda_m \). \( \xi(\lambda) \approx \frac{1}{2} \) at the boundaries \( \lambda_{a,b} \), but quickly approaches zero away from them, increasingly so with larger system size. Fig. 4 shows its behavior in the SSH model with three different system sizes.

Upon normalization, this gives the weight

\[
\tilde{w}_m = \frac{1}{M} \sum_{n=1}^{M} C_{mn} = \sqrt{\frac{\pi}{g_m}} \frac{1 - \xi_m}{\lambda_M - \lambda_1} \frac{M - 1}{M}.
\]

(23)

In the limit \( M \to \infty \), the last fraction goes to 1, and we recover the continuum limit obtained in the main text.

\( D_p \) is defined as \( 1/\langle \tilde{w}_m \rangle \). To proceed, we introduce the length element and its relative fluctuation,

\[
\delta \ell_m \equiv \sqrt{g_m} \delta \lambda = \langle \delta \ell_m \rangle (1 + \sigma_i), \quad \sigma_m \equiv \frac{\delta \ell_m - \langle \delta \ell_m \rangle}{\langle \delta \ell_m \rangle} = \frac{M \delta \ell_m}{L} - 1,
\]

(24)

where \( L = \sum_{m=1}^{M} \delta \ell_m = M \langle \delta \ell_m \rangle \). Then

\[
\langle \tilde{w}_m \rangle = \frac{\sqrt{\pi}}{M} \frac{1 - \xi_m}{\langle \delta \ell_m \rangle} = \frac{\sqrt{\pi}}{L} \frac{1 - \xi_m}{1 + \sigma_m}.
\]

(25)

\[
= \frac{\sqrt{\pi}}{L} \sum_{a=0}^{\infty} (-)^a \langle (1 - \xi_m) \sigma_m^a \rangle
\]

(26)

\[
= \frac{\sqrt{\pi}}{L} \left[ 1 - \langle \xi_m \rangle + \langle \sigma_m^2 \rangle + \langle \xi_m \sigma_m \rangle - \langle \xi_m \sigma_m^2 \rangle + O((1 - \xi)\sigma^3) \right].
\]

(27)

For a smooth curve, \( \sigma_m^2 \ll 1 \). Thus one can truncate at the \( \sigma_m^2 \) order,

\[
D_p = \frac{1}{\langle \tilde{w}_m \rangle} = \frac{L}{\sqrt{\pi}} \frac{1 - \xi_m}{1 + \sigma_m}^{-1} \approx \frac{L}{\sqrt{\pi}} (1 + \langle \xi_m \rangle - \langle \sigma_m^2 \rangle + \cdots).
\]

(28)

As discussed before, with increasing system size, \( \langle \xi_m \rangle \) decreases, thus in the thermodynamic limit, the leading order correction should be \( \langle \sigma_m^2 \rangle \).

Fig. 5 shows \( \sigma_m \) in the SSH model with three different system sizes, note that it is size independent. Numerical values of the expansion terms in Eq. 27 with \( N = 50000 \) unit cells are

\[
\begin{array}{cccc}
\langle \xi_m \rangle & \langle \sigma_m^2 \rangle & \langle \xi_m \sigma_m \rangle & \langle \xi_m \sigma_m^2 \rangle \\
0.0067 & 0.0018 & 0.00067 & 0.000066
\end{array}
\]

(29)

**Alternative definition of characteristic dimensionality**

As discussed in the text, \( 1/\tilde{w}_m \) counts the number of random states in \( \{ |\Psi_n \rangle \} \) needed to represent \( |\Psi_m \rangle \), and to remove the \( m \) dependence, there are two natural choices, \( 1/\langle \tilde{w}_m \rangle \) and \( 1/\langle \tilde{w}_m \rangle \), where \( \langle \cdots \rangle = \sum_{m=1}^{M} (\cdots)/M \). We have

![Graph showing ξ correction in SSH model](image-url)
FIG. 5. $\sigma$ correction in the SSH model with periodic boundary condition. Using 1000 ground states evenly sampled between $-0.5 < \lambda < 0.5$. Note that $\sigma$ is size independent ($2N =$ number of lattice sites).

defined $D_P = 1/\langle \tilde{w}_m \rangle$ in the text, but the second choice has its own merit, as it naturally evaluates to the geometric length. In the thermodynamic limit, one can ignore the edge correction $\xi$ (see Fig. 4), then

$$D_{alt} \equiv \left\langle \frac{1}{\tilde{w}_m} \right\rangle = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{\tilde{w}_m} \xrightarrow{\xi_m = 0} \frac{1}{\sqrt{\pi}} \sum_{m=1}^{M} \sqrt{g_m} \delta \lambda = \frac{L}{\sqrt{\pi}} .$$  (30)