Special functions and spectrum of Jacobi matrices

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Abstract

Several examples of Jacobi matrices with an explicitly solvable spectral problem are worked out in detail. In all discussed cases the spectrum is discrete and coincides with the set of zeros of a special function. Moreover, the components of corresponding eigenvectors are expressible in terms of special functions as well. Our approach is based on a recently developed formalism providing us with explicit expressions for the characteristic function and eigenvectors of Jacobi matrices. This is done under an assumption of a simple convergence condition on matrix entries. Among the treated special functions there are regular Coulomb wave functions, confluent hypergeometric functions, q-Bessel functions and q-confluent hypergeometric functions. In addition, in the case of q-Bessel functions, we derive several useful identities.

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1 Introduction

Special functions usually depend on a complex variable and an additional parameter called order. Typically, they obey a three-term recurrence relation with respect to the order. This is the basis of their relationship to Jacobi (tridiagonal) matrices. In more detail, the zeros of an appropriate special function are directly related to eigenvalues of a Jacobi matrix operator, and components of corresponding eigenvectors can be expressed in terms of special functions as well. One may also say that the characteristic function of the (infinite) matrix operator in question is written explicitly in terms of special functions. Particularly, Gard and Zakrajšek reported in [4] a matrix equation approach for numerical computation of the zeros of Bessel functions; on this point see
also [9]. In [8], Ikebe then showed that the same approach was applicable, too, for
determining the zeros of regular Coulomb wave functions. In practical computations,
an infinite tridiagonal matrix should be truncated which raises a question of error
estimates. Such an analysis has been carried out in [10, 12].

In [14], the authors initiated an approach to a class of Jacobian matrices with discrete
spectra. The basic tool is a function \( F \) depending on a countable number of variables.
In more detail, we define \( F : D \to \mathbb{C} \),

\[
F(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1} \cdots x_{k_m}x_{k_m+1},
\]

(1)

where the set \( D \) is formed by complex sequences \( x = \{x_k\}_{k=1}^{\infty} \) obeying

\[
\sum_{k=1}^{\infty} |x_kx_{k+1}| < \infty.
\]

(2)

For a finite number of variables we identify \( F(x_1, x_2, \ldots, x_n) \) with \( F(x) \) where \( x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) \). By convention, we put \( F(\emptyset) = 1 \) where \( \emptyset \) is the empty
sequence. Notice that the domain \( D \) is not a linear space though \( \ell^2(\mathbb{N}) \subseteq D \).

In the same paper, two examples are given of special functions expressed directly
in terms of \( F \). The first example is concerned with Bessel functions of the first kind.
For \( w, \nu \in \mathbb{C}, \nu \notin -\mathbb{N}, \) one has

\[
J_{\nu}(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \hat{F}\left(\left\{ \frac{w}{\nu+k} \right\}_{k=1}^{\infty}\right).
\]

Secondly, the formula

\[
\hat{F}\left(\left\{ t^{k-1}w \right\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4) \cdots (1-t^{2m})} = \phi_1(0; t^2, -tw^2)
\]

(4)

holds for \( t, w \in \mathbb{C}, |t| < 1 \). Here \( \phi_1 \) is the basic hypergeometric series (also called
q-hypergeometric series) being defined by

\[
\phi_1(0; b; q, z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q; q)_k (b; q)_k} z^k,
\]

and

\[
(a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j), \ k = 0, 1, 2, \ldots,
\]

is the q-Pochhammer symbol, see [3].

In [15], the approach is further developed and a construction in terms of \( \hat{F} \) of the
characteristic function of certain Jacobian matrices is established. As an application, a
series of examples of Jacobian matrices with explicitly expressible characteristic functions
is described. The method works well for Jacobian matrices obeying a simple convergence
condition imposed on the matrix entries which is in principle dictated by condition (2) characterizing the domain of $\mathfrak{F}$.

In the current paper, we present more interesting examples of Jacobi matrices whose spectrum coincides with the set of zeros of a particular special function. As a byproduct, we provide examples of sequences on which the function $\mathfrak{F}$ can be evaluated explicitly. The paper is organized as follows. In Section 2 we recall from [14, 15] some basic facts needed in the current paper. Section 3 is concerned with regular Coulomb wave functions. Here we reconsider the example due to Ikebe while using some basic facts needed in the current paper. Section 4 we deal with confluent hypergeometric functions. Here we go beyond the above mentioned convergence condition (see (14) below) which is violated in this example. Section 5 is concerned with $q$-Bessel functions. This example is particular in that respect that the constructed second order difference operator is bilateral, i.e. it acts in $\ell^2(\mathbb{Z})$ rather than in $\ell^2(\mathbb{N})$. We first derive several useful properties of $q$-Bessel functions and then we use this knowledge to solve the spectral problem for the bilateral difference operator fully explicitly. Finally, another interplay between special functions, namely $q$-confluent hypergeometric functions, and an appropriate Jacobi matrix is demonstrated in Section 6.

2 Preliminaries

Let us recall from [14, 15] some basic facts concerning the function $\mathfrak{F}$ and its properties and possible applications. First of all, quite crucial property of $\mathfrak{F}$ is the recurrence rule

$$\mathfrak{F}(\{x_k\}_{k=1}^\infty) = \mathfrak{F}(\{x_k\}_{k=2}^\infty) - x_1 x_2 \mathfrak{F}(\{x_k\}_{k=3}^\infty).$$

(5)

In addition, $\mathfrak{F}(x_1, x_2, \ldots, x_{k-1}, x_k) = \mathfrak{F}(x_k, x_{k-1}, \ldots, x_2, x_1)$. Furthermore, for $x \in D$, 

$$\lim_{n \to \infty} \mathfrak{F}(\{x_k\}_{k=n}^\infty) = 1 \quad \text{and} \quad \lim_{n \to \infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x).$$

(6)

Let us note that the definition of $\mathfrak{F}$ naturally extends to more general ranges of indices. For any sequence $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $N_1 \leq N_2 + 1$, (if $N_1 = N_2 + 1 \in \mathbb{Z}$ then the sequence is considered as empty) such that $\sum_{k=N_1}^{N_2-1} |x_k x_{k+1}| < \infty$ one defines

$$\mathfrak{F}(\{x_k\}_{k=N_1}^{N_2}) = 1 + \sum_{m=1}^\infty (-1)^m \sum_{k \in I(N_1, N_2, m)} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1},$$

where

$$I(N_1, N_2, m) = \{k \in \mathbb{Z}^m; k_j + 2 \leq k_{j+1} \text{ for } 1 \leq j \leq m - 1, \ N_1 \leq k_1, \ k_m < N_2\}.$$

With this definition, one has the generalized recurrence rule

$$\mathfrak{F}(\{x_k\}_{k=N_1}^{N_2}) = \mathfrak{F}(\{x_k\}_{k=N_1}^{n}) \mathfrak{F}(\{x_k\}_{k=n+1}^{N_2}) - x_n x_{n+1} \mathfrak{F}(\{x_k\}_{k=N_1}^{n-1}) \mathfrak{F}(\{x_k\}_{k=n+2}^{N_2})$$

(7)
provided \( n \in \mathbb{Z} \) satisfies \( N_1 \leq n < N_2 \).

Let us denote by \( J \) an infinite Jacobi matrix of the form

\[
J = \begin{pmatrix}
\lambda_1 & w_1 & & \\
w_1 & \lambda_2 & w_2 & \\
& w_2 & \lambda_3 & \\
& & \ddots & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
\]

(8)

where \( \{ w_n; \ n \in \mathbb{N} \} \subset \mathbb{C} \setminus \{ 0 \} \) and \( \{ \lambda_n; \ n \in \mathbb{N} \} \subset \mathbb{C} \). In all examples treated in the current paper, the matrix \( J \) determines in a natural way a unique closed operator in \( \ell^2(\mathbb{N}) \) (in other words, \( J_{\min} = J_{\max} \); see, for instance, [3]). If the matrix is real then the operator is self-adjoint. For the sake of simplicity of the notation the operator is again denoted by \( J \). One notes, too, that all eigenvalues of \( J \), if any, are simple since any solution \( \{ x_k \} \) of the formal eigenvalue equation

\[
\lambda_1 x_1 + w_1 x_2 = z x_1, \quad w_{k-1} x_{k-1} + \lambda_k x_k + w_k x_{k+1} = z x_k \quad \text{for} \ k \geq 2,
\]

(9)

with \( z \in \mathbb{C} \), is unambiguously determined by its first component \( x_1 \).

Let \( \{ \gamma_k \} \) be any sequence fulfilling \( \gamma_k \gamma_{k+1} = w_k, \ k \in \mathbb{N} \). If \( J_n \) is the principal \( n \times n \) submatrix of \( J \) then

\[
\det(J_n - zI_n) = \left( \prod_{k=1}^{n} (\lambda_k - z) \right) \mathfrak{F} \left( \frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \ldots, \frac{\gamma_n^2}{\lambda_n - z} \right).
\]

(10)

The function \( \mathfrak{F} \) can also be applied to bilateral difference equations. Suppose that sequences \( \{ w_n \}_{n=-\infty}^{\infty} \) and \( \{ \zeta_n \}_{n=-\infty}^{\infty} \) are such that \( w_n \neq 0, \zeta_n \neq 0 \) for all \( n \), and

\[
\sum_{k=-\infty}^{\infty} \left| \frac{w_k^2}{\zeta_k \zeta_{k+1}} \right| < \infty.
\]

Consider the difference equation

\[
w_n x_{n+1} - \zeta_n x_n + w_{n-1} x_{n-1} = 0, \ n \in \mathbb{Z}.
\]

(11)

Define the sequence \( \{ \mathcal{P}_n \}_{n \in \mathbb{Z}} \) by \( \mathcal{P}_0 = 1 \) and \( \mathcal{P}_{n+1} = (w_n/\zeta_{n+1}) \mathcal{P}_n \) for all \( n \). The sequence \( \{ \gamma_n \}_{n \in \mathbb{Z}} \) is again defined by the rule \( \gamma_n \gamma_{n+1} = w_n \) for all \( n \in \mathbb{Z} \), and any choice of \( \gamma_1 \neq 0 \). Then the sequences \( \{ f_n \}_{n \in \mathbb{Z}} \) and \( \{ g_n \}_{n \in \mathbb{Z}} \):

\[
f_n = \mathcal{P}_n \mathfrak{F} \left( \left\{ \frac{\gamma_k^2}{\zeta_k} \right\}_{k=1}^{\infty} \right), \quad g_n = \frac{1}{w_{n-1} \mathcal{P}_{n-1}} \mathfrak{F} \left( \left\{ \frac{\gamma_k^2}{\zeta_k} \right\}_{k=-\infty}^{n-1} \right),
\]

(12)

represent two solutions of the bilateral difference equation (11). With the usual definition of the Wronskian, \( \mathcal{W}(f, g) = w_n (f_n g_{n+1} - f_{n+1} g_n) \), one has

\[
\mathcal{W}(f, g) = \mathfrak{F} \left( \left\{ \frac{\gamma_n^2}{\zeta_n} \right\}_{n=-\infty}^{\infty} \right).
\]

(13)
For \( \lambda = \{ \lambda_n \}_{n=1}^{\infty} \) let us denote \( \mathbb{C}^\lambda_0 := \mathbb{C} \setminus \{ \lambda_n; \, n \in \mathbb{N} \} \), and let \( \text{der}(\lambda) \) stand for the set of all finite accumulation points of the sequence \( \lambda \). Further, for \( z \in \mathbb{C} \setminus \text{der}(\lambda) \), let \( r(z) \) be the number of members of the sequence \( \lambda \) coinciding with \( z \) (hence \( r(z) = 0 \) for \( z \in \mathbb{C}^\lambda_0 \)). We assume everywhere that \( \mathbb{C}^\lambda_0 \neq \emptyset \).

Suppose
\[
\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty
\]  
for at least one \( z_0 \in \mathbb{C}^\lambda_0 \). Then (14) is true for all \( z_0 \in \mathbb{C}^\lambda_0 \) [15]. In particular, the following definitions make good sense. For \( k \in \mathbb{Z}_+ \) (\( \mathbb{Z}_+ \) standing for nonnegative integers) and \( z \in \mathbb{C} \setminus \text{der}(\lambda) \) put
\[
\xi_k(z) := \lim_{u \to z} (u - z)^{r(z)} \left( \prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F} \left( \left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=k+1}^{\infty} \right).
\]  
(15)
Here one sets \( w_0 := 1 \). Particularly, for \( z \in \mathbb{C}^\lambda_0 \), one simply has
\[
\xi_k(z) = \left( \prod_{l=1}^{k} \frac{w_{l-1}}{z - \lambda_l} \right) \mathfrak{F} \left( \left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=k+1}^{\infty} \right)
\]  
(16)
(this is in fact nothing but the solution \( f_n \) from [12] restricted to nonnegative indices).

All functions \( \xi_k(z) \), \( k \in \mathbb{Z}_+ \), are holomorphic on \( \mathbb{C}^\lambda_0 \) and extend to meromorphic functions on \( \mathbb{C} \setminus \text{der}(\lambda) \), with poles at the points \( z = \lambda_n \), \( n \in \mathbb{N} \), and with orders of the poles not exceeding \( r(z) \). This justifies definition (15).

The sequence \( \{ \xi_k(z) \} \) solves the second-order difference equation
\[
w_{k-1}x_{k-1} + (\lambda_k - z)x_k + w_kx_{k+1} = 0 \quad \text{for} \quad k \geq 2.
\]  
(17)
In addition, \((\lambda_1 - z)\xi_1(z) + w_1\xi_2(z) = 0 \) provided \( \xi_0(z) = 0 \). Proceeding this way one can show [15] Section 3.3] that if \( \xi_0(z) \) does not vanish identically on \( \mathbb{C}^\lambda_0 \) then
\[
\text{spec}(J) \setminus \text{der}(\lambda) = \{ z \in \mathbb{C} \setminus \text{der}(\lambda) ; \, \xi_0(z) = 0 \}.
\]  
(18)
Moreover, if \( z \in \mathbb{C} \setminus \text{der}(\lambda) \) is an eigenvalue of \( J \) then \( \xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \ldots) \) is a corresponding eigenvector. If \( J \) is real and \( z \in \mathbb{R} \cap \mathbb{C}^\lambda_0 \) is an eigenvalue then \( \|\xi(z)\|^2 = \xi_0(z)\xi_1(z) \). Finally, let us remark that the Weyl m-function can be expressed as \( m(z) = \xi_1(z)/\xi_0(z) \).

**Lemma 1.** For \( p, r, \ell \in \mathbb{N}, 1 < p \leq r + 1 \leq \ell \), and any \( \ell \)-tuple of complex numbers \( x_j, 1 \leq j \leq \ell \), it holds true that
\[
\mathfrak{F} \left( \{ x_j \}_{j=1}^{r} \right) \mathfrak{F} \left( \{ x_j \}_{j=p}^{\ell} \right) - \mathfrak{F} \left( \{ x_j \}_{j=1}^{\ell} \right) \mathfrak{F} \left( \{ x_j \}_{j=r+2}^{\ell} \right) = \left( \prod_{j=p-1}^{r} x_jx_{j+1} \right) \mathfrak{F} \left( \{ x_j \}_{j=1}^{p-2} \right) \mathfrak{F} \left( \{ x_j \}_{j=r+2}^{\ell} \right),
\]
If \( p, r \in \mathbb{N}, 1 < p \leq r + 1 \), and a complex sequence \( \{x_j\}_{j=1}^{\infty} \) fulfills (2) then

\[
\mathfrak{F}(\{x_j\}_{j=1}^{r+1}) - \mathfrak{F}(\{x_j\}_{j=p}^{r}) \mathfrak{F}(\{x_j\}_{j=r+2}^{\infty}) = \left( \prod_{j=p-1}^{r} x_j x_{j+1} \right) \mathfrak{F}(\{x_j\}_{j=1}^{p-2}) \mathfrak{F}(\{x_j\}_{j=r+2}^{\infty}).
\]

Proof. Suppose \( \{z_j\}_{j=-\infty}^{\infty} \) is any nonvanishing bilateral complex sequence. In [15, Section 2] it is shown (under somewhat more general circumstances) that there exists an antisymmetric matrix \( \mathfrak{J}(m,n), m,n \in \mathbb{Z} \), such that

\[
\mathfrak{J}(m,n) = \left( \prod_{j=m+1}^{n-1} \frac{1}{z_j} \right) \mathfrak{F}(z_{m+1}, z_{m+2}, \ldots, z_{n-1})
\]

for \( m < n \), and \( \mathfrak{J}(m,k)\mathfrak{J}(n,\ell) - \mathfrak{J}(m,\ell)\mathfrak{J}(n,k) = \mathfrak{J}(m,n)\mathfrak{J}(k,\ell) \) for all \( m, n, k, \ell \in \mathbb{Z} \).

In particular, assuming that indices \( p, r, \ell \) obey the restrictions from the lemma,

\[
\mathfrak{J}(0,r+1)\mathfrak{J}(p-1,\ell+1) - \mathfrak{J}(0,\ell+1)\mathfrak{J}(p-1,r+1) = \mathfrak{J}(0,p-1)\mathfrak{J}(r+1,\ell+1).
\]

After obvious cancellations in this equation one can drop the assumption on nonvanishing sequences. The lemma readily follows.

Lemma 2. Let \( x = \{x_n\}_{n=1}^{\infty} \) be a nonvanishing complex sequence satisfying (2). Then

\[
F_n := \mathfrak{F}(\{x_k\}_{k=n}^{\infty}), \quad n \in \mathbb{N},
\]

is the unique solution of the second order difference equation

\[
F_n - F_{n+1} + x_n x_{n+1} F_{n+2} = 0, \quad n \in \mathbb{N},
\]

satisfying the boundary condition \( \lim_{n \to \infty} F_n = 1 \).

Proof. The sequence \( \{F_n\} \) defined in (19) fulfills all requirements, as stated in (5) and (6). It suffices to show that there exists another solution \( \{G_n\} \) of (20) such that \( \lim_{n \to \infty} G_n = \infty \). If \( F_1 = \mathfrak{F}(x) \neq 0 \) then \( \{G_n\} \) can be defined by \( G_1 = 0 \) and

\[
G_n = \left( \prod_{k=1}^{n-2} \frac{1}{x_k x_{k+1}} \right) \mathfrak{F}(\{x_k\}_{k=1}^{n-2}), \quad \text{for } n \geq 2.
\]

If \( F_1 = 0 \) then necessarily \( F_2 \neq 0 \) since otherwise (20) would imply \( F_n = 0 \) for all \( n \) which is impossible. Hence in that case one can shift the index by 1, i.e. one can put \( G_2 = 0 \),

\[
G_n = \left( \prod_{k=2}^{n-2} \frac{1}{x_k x_{k+1}} \right) \mathfrak{F}(\{x_k\}_{k=2}^{n-2}), \quad \text{for } n \geq 3,
\]

(and \( G_1 = -x_1 x_2 \)). In any case, \( F_n \) is the minimal solution of (20), see [6].
Remark 3. If $\mathfrak{F}(x) = 0$ then $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ tends to 0 as $n \to \infty$ quite rapidly, more precisely,

$$\mathfrak{F}(x_1, x_2, \ldots, x_{n+1}) = o\left(\prod_{k=1}^{n} x_k x_{k+1}\right), \text{ as } n \to \infty. \quad (22)$$

In fact, if $\mathfrak{F}(x) = 0$ then $F_2 \neq 0$ and the solutions $\{F_n\}$ and $\{G_n\}$ defined in (19) and (21), respectively, are linearly dependent, $F_n = F_2 G_n$, $\forall n$. Sending $n$ to infinity one gets

$$1 = F_2 \lim_{n \to \infty} \left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right) \mathfrak{F}(\{x_k\}_{k=1}^{n}) = \lim_{n \to \infty} \left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right) \mathfrak{F}(\{x_k\}_{k=1}^{n}) \mathfrak{F}(\{x_k\}_{k=2}^{n+1}).$$

Now, Lemma 1 provides us with the identity

$$\mathfrak{F}(\{x_k\}_{k=1}^{n}) \mathfrak{F}(\{x_k\}_{k=2}^{n+1}) - \mathfrak{F}(\{x_k\}_{k=1}^{n+1}) \mathfrak{F}(\{x_k\}_{k=2}^{n}) = \prod_{k=1}^{n} x_k x_{k+1},$$

and so one arrives at the equation

$$\lim_{n \to \infty} \left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right) \mathfrak{F}(\{x_k\}_{k=1}^{n+1}) \mathfrak{F}(\{x_k\}_{k=2}^{n}) = 0.$$

Since $\mathfrak{F}(\{x_k\}_{k=2}^{n}) \neq 0$ this shows (22).

### 3 Coulomb wave functions

For $x > 1, y \in \mathbb{R}$, put

$$\lambda(x, y) = \frac{y}{(x - 1)x}, \quad w(x, y) = \frac{1}{x} \sqrt{x^2 + y^2 - 1},$$

and

$$\gamma(x, y) = \frac{\Gamma\left(\frac{1}{2} x\right)}{\sqrt{2x - 1} \Gamma\left(\frac{1}{2}(x + 1)\right)} \left|\frac{\Gamma\left(\frac{1}{2}(x + iy + 1)\right)}{\Gamma\left(\frac{1}{2}(x + iy)\right)}\right|.$$

Then $\gamma(x, y)\gamma(x + 1, y) = w(x, y)$. For $\mu > 0, \nu \in \mathbb{R}$, consider the Jacobi matrix $J = J(\mu, \nu)$ of the form (8), with

$$\lambda_k = \lambda(\mu + k, \nu), \quad w_k = w(\mu + k, \nu), \quad k = 1, 2, 3, \ldots \quad (23)$$

Similarly, $\gamma_k = \gamma(\mu + k, \nu)$. Clearly, the matrix $J(\mu, \nu)$ represents a Hermitian Hilbert-Schmidt operator in $L^2(\mathbb{N})$. Moreover, the convergence condition (11) is satisfied for any $z_0 \in \mathbb{C} \setminus \{0\}$ such that $z_0 \neq \lambda_k, \forall k \in \mathbb{N}$.

Recall the definition of regular Coulomb wave functions [11, Eq. 14.1.3]

$$F_L(\eta, \rho) = 2^L e^{-\eta/2} \frac{\Gamma(L + 1 + i\eta)}{\Gamma(2L + 2)} \rho^{L+1} e^{-i\rho} 1_F(L + 1 - i\eta; 2L + 2; 2i\rho), \quad (24)$$
valid for \( L \in \mathbb{Z}_+, \eta \in \mathbb{R}, \rho > 0 \). Let us remark that, though not obvious from its form, the values of the regular Coulomb wave function in the indicated range are real. But nothing prevents us to extend, by analyticity, the Coulomb wave function to the values \( L > -1 \) and \( \rho \in \mathbb{C} \) (assuming that a proper branch of \( \rho^{L+1} \) has been chosen).

As observed in [3], the eigenvalue equation for \( J(\mu, \nu) \) may be written in the form \( F_{\mu-1}(-\nu, z^{-1}) = 0 \). Moreover, if \( z \neq 0 \) is an eigenvalue of \( J(\mu, \nu) \) then the components \( v_n(z), n \in \mathbb{N} \), of a corresponding eigenvector \( v(z) \) are proportional to \( \sqrt{2 \mu + 2n - 1} F_{\mu} - 1(\mu + 1; 2n; 2i\nu) \). Thus, using definition (24), one can write

\[
\text{spec}(J(\mu, \nu)) \setminus \{0\} = \left\{ \zeta^{-1}; e^{-i\zeta} \right\}
\]

and

\[
v_n(\zeta^{-1}) = \sqrt{2 \mu + 2n - 1} \frac{\Gamma(\mu + n + i\nu)}{\Gamma(2 \mu + 2n)} (2 \zeta)^{n-1} e^{-i\zeta} F_1(\mu + n + i\nu; 2 \mu + 2n; 2i\nu).
\]

Here we wish to shortly reconsider this example while using our formalism.

**Proposition 4.** Under the above assumptions (see (23)),

\[
\tilde{\mathcal{F}} \left( \left\{ \frac{\gamma_k^2}{\lambda_k - \zeta^{-1}} \right\}_{k=1}^\infty \right) = \frac{\Gamma(\mu + 1 \sqrt{1 + 4 \nu \zeta})}{\Gamma(\mu) \Gamma(\mu + 1)} e^{-i\zeta} F_1(\mu + \nu; 2 \mu; 2i\nu).
\]

**Proof.** Observe that the convergence condition (14) is satisfied in this example. For \( n \in \mathbb{N} \) put

\[
f_{1,n} = \tilde{\mathcal{F}} \left( \left\{ \frac{\gamma_k^2}{\lambda_k - \zeta^{-1}} \right\}_{k=n}^\infty \right),
\]

and let \( f_{2,n} \) be equal to the RHS of (27) where we replace \( \mu \) by \( \mu + n - 1 \). According to (3), the sequence \( \{f_{1,n}\} \) obeys the recurrence rule

\[
f_{1,n} - f_{1,n+1} + X(\mu + n) f_{1,n+2} = 0, \quad n \in \mathbb{N},
\]

where

\[
x(\mu) = \frac{w(x, \nu)^2}{(\lambda(x, \nu) - \zeta^{-1}) (\lambda(x + 1, \nu) - \zeta^{-1})}
\]

\[
= \frac{(x^2 - 1)(x^2 + \nu^2) \zeta^2}{((x - 1)x - \nu \zeta) (x(x + 1) - \nu \zeta)} \quad \text{for} \quad x > 1.
\]

Next one can apply the identity

\[
\frac{1}{b^2} F_1(a - 1; b - 2; z) - \frac{b^2 - 2b + (2a - b)z}{(b - 2)b} F_1(a; b; z) - \frac{a(b - a)z^2}{(b^2 - 1)b^2} F_1(a + 1; b + 2; z) = 0,
\]

(29)
as it follows from [1, §13.4], to verify that the sequence \( \{f_{2,n}\} \) obeys (28) as well. Notice that, if rewritten in terms of Coulomb wave functions, (29) amounts to the recurrence rule [1, Eq. 14.2.3]

\[
L \sqrt{(L+1)^2 + \eta^2} u_{L+1} - (2L+1) \left( \eta + \frac{L(L+1)}{\rho} \right) u_L + (L+1) \sqrt{L^2 + \eta^2} u_{L-1} = 0,
\]

where \( u_L = F_L(\eta, \rho) \).

To evaluate the limit of \( f_{2,n} \), as \( n \to \infty \), one may notice that

\[
\lim_{n \to \infty} F_1(a_n, b + \kappa n, z) = e^{z/\kappa}
\]

for \( \kappa \neq 0 \), and apply the Stirling formula. Alternatively, avoiding the Stirling formula, the limit is also obvious from the identity [7, Eq. 8.325(1)]

\[
\prod_{k=0}^{\infty} \left( 1 + \frac{z}{(y+k)(y+k+1)} \right) = \frac{\Gamma(y) \Gamma(y+1)}{\Gamma\left( \frac{1}{2} + y - \frac{1}{2} \sqrt{1 - 4z} \right) \Gamma\left( \frac{1}{2} + y + \frac{1}{2} \sqrt{1 - 4z} \right)} .
\]

In any case, \( \lim_{n \to \infty} f_{2,n} = 1 \) and so, in virtue of Lemma 2, \( f_{1,n} = f_{2,n} \), \( \forall n \). In particular, for \( n = 1 \) one gets (27).

**Proof of formulas (25) and (26).** As recalled in Section 2 (see (18)), \( z = \zeta - 1 \neq 0 \) is an eigenvalue of \( J(\mu, \nu) \) if and only if \( \xi_0(z) = 0 \) which means nothing but (25). In that case the components \( \xi_n(z), n \in \mathbb{N} \), of a corresponding eigenvector can be chosen as described in (15). Note that

\[
\prod_{k=1}^{n-1} w_k = 2^{n-1} \sqrt{(2\mu + 1)(2\mu + 2n - 1)} \left| \frac{\Gamma(\mu + n + iv)}{\Gamma(\mu + 1 + iv)} \right| \frac{\Gamma(2\mu + 1)}{\Gamma(2\mu + 2n)}
\]

and that (30) means in fact the equality

\[
\prod_{k=1}^{\infty} \frac{1}{1 - \lambda(\mu + k, \nu) z} = \frac{\Gamma\left( \frac{1}{2} + \mu - \frac{1}{2} \sqrt{1 + 4\nu z} \right) \Gamma\left( \frac{1}{2} + \mu + \frac{1}{2} \sqrt{1 + 4\nu z} \right)}{\Gamma(\mu) \Gamma(\mu + 1)} .
\]

Using these equations and omitting a constant factor one finally arrives at formula (26).

**□**

4 Confluent hypergeometric functions

First, let us show an identity.

**Proposition 5.** The equation

\[
\frac{\Gamma(x + \gamma + n)}{\Gamma(x + \gamma)} e^{x^2} \mathcal{F} \left( \left\{ \frac{\sqrt{2x} \Gamma\left( \frac{1}{2}(\gamma - \alpha + k + 1) \right)}{(x + \gamma + k - 1) \Gamma\left( \frac{1}{2}(\gamma - \alpha + k) \right)} \right\}_{k=1}^{n} \right)
\]

\[
= \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \genfrac{[}{]}{0pt}{}{\alpha}{\gamma} - n 1\genfrac{[}{]}{0pt}{}{\alpha - \gamma - n}{1 - \gamma - n} x \genfrac{[}{]}{0pt}{}{\alpha - \gamma + 1}{\gamma + n + 1} x^{n+1} \genfrac{[}{]}{0pt}{}{\alpha - \gamma + 1; 2 - \gamma}{\gamma + n + 1} x,
\]

\[
(31)
\]
is valid for \( \alpha, \gamma, x \in \mathbb{C} \) and \( n \in \mathbb{Z}_+ \) (if considering both sides as meromorphic functions).

**Remark 6.** For instance, as a particular case of (31) one gets, for \( n = 0 \),
\[
\text{1}_1 F_1(\alpha; \gamma; x) \times \text{1}_1 F_1(\alpha - \gamma) \times (\gamma - x) = e^x.
\]

**Proof.** For \( \alpha, \gamma \) and \( x \) fixed and \( n \in \mathbb{Z} \), put
\[
\varphi_n = \frac{1}{\Gamma(n + \gamma)} \times \text{1}_1 F_1(\alpha; n + \gamma; x), \quad \psi_n = \frac{1}{\Gamma(n + \gamma - \alpha)} \times U(\alpha, n + \gamma, x).
\]

Then \( \{\varphi_n\} \) and \( \{\psi_n\} \) obey the second-order difference equation [1, Eqs. 13.4.2, 13.4.16]
\[
(n + \gamma - \alpha) x u_{n+1} - (n + \gamma + x - 1) u_n + u_{n-1} = 0, \quad n \in \mathbb{Z}.
\]

Note also that
\[
(\alpha - \gamma) \times \text{1}_1 F_1(\alpha; \gamma + 1; x) \times U(\alpha, \gamma, x) + \gamma \times \text{1}_1 F_1(\alpha; \gamma; x) \times U(\alpha, \gamma + 1, x) = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha)} x^{-\gamma} e^x
\]
(as it follows, for example, from equations 13.4.12 and 13.4.25 combined with 13.1.22 in [1]). Whence
\[
\varphi_0 \psi_1 - \varphi_1 \psi_0 = \frac{1}{\Gamma(\alpha) \Gamma(1 - \alpha + \gamma)} x^{-\gamma} e^x,
\]
and so the solutions \( \varphi_n, \psi_n \) are linearly independent except of the cases \( -\alpha \in \mathbb{Z}_+ \) and \( \alpha - \gamma \in \mathbb{N} \).

The difference equation (33) can be symmetrized using the substitution
\[
u_n = \frac{x^{-n}}{\Gamma(\gamma - \alpha + n)} v_n.
\]
Then
\[
w_n v_{n+1} - \zeta_n v_n + w_{n-1} v_{n-1} = 0
\]
where
\[
w_n = \frac{x^{-n}}{\Gamma(\gamma - \alpha + n)}, \quad \zeta_n = \frac{(x + \gamma + n - 1) x^{-n}}{\Gamma(\gamma - \alpha + n)}.
\]

For a solution of the equation \( \gamma_n \gamma_{n+1} = w_n, \forall n \), one can take
\[
\gamma_n = 2^{\frac{1}{2}} x^{-\frac{1}{2} + \frac{n}{4}} \sqrt{\frac{\Gamma(\frac{1}{2}(\gamma - \alpha + n + 1))}{\Gamma(\gamma - \alpha + n) \Gamma(\frac{1}{2}(\gamma - \alpha + n))}}.
\]

Referring to another solution, namely
\[
g_n = \frac{1}{w_n} \times \text{P}_{n-1} \times \left( \left\{ \frac{\gamma_k^2}{\zeta_k} \right\}_{k=-\infty}^{n-1} \right), \quad \text{with} \ n \in \mathbb{N},
\]
using otherwise the same notation as in (12), one concludes that there exist constants
$A$ and $B$ such that
\[
\frac{\Gamma(x + \gamma + n)}{\Gamma(\gamma - \alpha + n + 1)} x^{-n} \mathfrak{F}\left(\left\{ \frac{\sqrt{2x} \Gamma(\frac{1}{2}(\gamma - \alpha + k + 1))}{(x + \gamma + k - 1) \Gamma(\frac{1}{2}(\gamma - \alpha + k))} \right\}_{k=1}^{n} \right) = A\varphi_{n+1} + B\psi_{n+1}
\]
for all $n \in \mathbb{Z}_+$. $A$ and $B$ can be determined from the values for $n = -1, 0$ (putting
$\mathfrak{F}\left(\{x\}_{k=1}^{n}\right) = 0$, as dictated by the recurrence rule (7) provided the admissible values
are extended to $N_1 = N_2 = 1, n = 0$). After some manipulations one gets
\[
\frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma - \alpha + n + 1)} \, {}_1F_1(\alpha; \gamma; x)U(\alpha, \gamma + n + 1, x)
\]
\[- \frac{\Gamma(\gamma)}{\Gamma(\gamma + n + 1)} U(\alpha, \gamma, x) \, {}_1F_1(\alpha; \gamma + n + 1; x)
\]
\[
= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha)\Gamma(x + \gamma + n)}{\Gamma(\alpha)\Gamma(\gamma - \alpha + n + 1)\Gamma(x + \gamma)} \, x^{-\gamma-n} e^x \mathfrak{F}\left(\left\{ \frac{\sqrt{2x} \Gamma(\frac{1}{2}(\gamma - \alpha + k + 1))}{(x + \gamma + k - 1) \Gamma(\frac{1}{2}(\gamma - \alpha + k))} \right\}_{k=1}^{n} \right).
\]
Recall that
\[
U(a, b, x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \, {}_1F_1(a; b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} \, x^{1-b} \, {}_1F_1(a - b + 1; 2 - b; x), \quad (34)
\]
whence (31).

**Remark 7.** Let us point out two particular cases of (31). Putting $\alpha = 0$ one gets the identity
\[
\mathfrak{F}\left(\left\{ \frac{\sqrt{2x} \Gamma(\frac{1}{2}(\gamma + k + 1))}{(x + \gamma + k - 1) \Gamma(\frac{1}{2}(\gamma + k))} \right\}_{k=1}^{n} \right) = \frac{\Gamma(x + \gamma)}{\Gamma(x + \gamma + n)} \sum_{j=0}^{n} \frac{\Gamma(\gamma + n - j)}{\Gamma(\gamma)} \, x^j, \quad (35)
\]
and for $\alpha = -1$ one obtains
\[
\mathfrak{F}\left(\left\{ \frac{\sqrt{2x} \Gamma(\frac{1}{2}(\gamma + k + 2))}{(x + \gamma + k - 1) \Gamma(\frac{1}{2}(\gamma + k + 1))} \right\}_{k=1}^{n} \right)
\]
\[
= \frac{\Gamma(x + \gamma)}{\Gamma(x + \gamma + n)} \sum_{j=0}^{n} \frac{\Gamma(\gamma + n - j)}{\Gamma(\gamma + 1)} \, (\gamma - j(n - j)) \, x^j. \quad (36)
\]

Let us sketch a derivation of (36), equation (35) is simpler. Substitute $-1$ for $\alpha$
and $\gamma + n$ for $\gamma$ in (32), and put
\[
B_n = \frac{\Gamma(\gamma + n - 1)}{\gamma - x + n} x^{-n} \, {}_1F_1(-\gamma - n; 2 - \gamma; x).
\]

Then
\[
B_{n+1} - B_n = \frac{\Gamma(\gamma + n + 1)}{(\gamma - x + n)(\gamma - x + n + 1)} x^{-n-1} e^x.
\]
Whence
\[ B_{n+1} = B_0 + e^x \sum_{j=1}^{n+1} \frac{\Gamma(\gamma+j)}{(\gamma-x+j-1)(\gamma-x+j)} x^{-j} \]
which means nothing but
\[
\Gamma(\gamma+n)(\gamma-x)_{1F1}(-\gamma-n-1; 1-\gamma-n; x) \\
- \Gamma(\gamma-1)(\gamma-x+n+1) x^{n+1} _{1F1}(-\gamma; 2-\gamma; x) \\
= (\gamma-x+n+1)(\gamma-x) e^x \sum_{j=1}^{n+1} \frac{\Gamma(\gamma+j)}{(\gamma-x+j-1)(\gamma-x+j)} x^{n+1-j}.
\]

Set \( \alpha = -1 \) in (31) and notice that \( _{1F1}(-1; b; x) = 1 - x/b \). After some simplifications, a combination of thus obtained identity with (37) gives (36).

As an application of (31) consider the Jacobi matrix operator \( J(\alpha, \beta, \gamma) \) depending on parameters \( \alpha, \beta, \gamma \), with \( \beta > 0, \gamma > 0 \) and \( \alpha + \beta > 0 \), as introduced in (8) where we put
\[
\lambda_k = \gamma k, \quad w_k = \sqrt{\alpha + \beta k}, \quad k = 1, 2, 3, \ldots.
\]

For the sequence \( \gamma_k \) (fulfilling \( \gamma_k \gamma_{k+1} = w_k \)) one can take
\[
\gamma_k^2 = \sqrt{2\beta} \Gamma \left( \frac{1}{2} \left( \frac{\alpha}{\beta} + k + 1 \right) \right) \Gamma \left( \frac{1}{2} \left( \frac{\alpha}{\beta} + k \right) \right).
\]

Regarding the diagonal of \( J(\alpha, \beta, \gamma) \) as an unperturbed part and the off-diagonal elements as a perturbation one immediately realizes that the matrix \( J(\alpha, \beta, \gamma) \) determines a unique semibounded self-adjoint operator in \( \ell^2(\mathbb{N}) \). Moreover, the Weyl theorem about invariance of the essential spectrum tells us that its spectrum is discrete and simple. Our goal here is to show that one can explicitly construct a “characteristic” function of this operator in terms of confluent hypergeometric functions.

**Proposition 8.** The spectrum of \( J(\alpha, \beta, \gamma) \) defined in (8) and (38) coincides with the set of zeros of the function
\[
_{1F1}(1-\frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; 1-\frac{\beta}{\gamma^2} - \frac{z}{\gamma}; \frac{\beta}{\gamma^2})/\Gamma \left( 1-\frac{\beta}{\gamma^2} - \frac{z}{\gamma} \right).
\]

Moreover, if \( z \) is an eigenvalue then the components of a corresponding eigenvector \( v \) can be chosen as
\[
v_k = (-1)^k \beta^{k/2} \gamma^{-k} \frac{\Gamma \left( \frac{\alpha}{\beta} + k \right)^{1/2}}{\Gamma \left( 1-\frac{\beta}{\gamma^2} - \frac{z}{\gamma} + k \right)} _{1F1} \left( 1-\frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; 1-\frac{\beta}{\gamma^2} - \frac{z}{\gamma} + k; \frac{\beta}{\gamma^2} \right),
\]
k \in \mathbb{N}.
Remark 9. (i) In principle it would be sufficient to consider the case $\gamma = 1$; observe that

$$F_J(\alpha, \beta, \gamma; z) = F_J\left(\frac{\alpha}{\gamma^2}, \frac{\beta}{\gamma}, 1; \frac{z}{\gamma}\right).$$

Thus for $\gamma = 1$ we get a simpler expression,

$$F_J(\alpha, \beta, 1; z) = \frac{F_1\left(1, 1 - \alpha; 1 - \beta - z; \beta \right)}{\Gamma(1 - \beta - z)}.$$

(ii) Notice that the convergence condition (14) is violated in this example.

Before the proof we consider analogous results for finite matrices. Let $J_n(\alpha, \beta, \gamma)$ be the principal $n \times n$ submatrix of $J(\alpha, \beta, \gamma)$. The characteristic polynomial $F_{J_n}(z)$ of $J_n(\alpha, \beta, \gamma)$ can be expressed in terms of confluent hypergeometric functions, too. According to (10),

$$F_{J_n}(\alpha, \beta, \gamma; z) = \frac{\Gamma\left(1 - \frac{z}{\gamma} + n\right)}{\Gamma\left(1 - \frac{z}{\gamma}\right)} \mathfrak{F} \left(\left\{\frac{2}{\gamma} \left(\frac{n}{\beta} + k + 1\right)\right\}_{k=1}^n\right).$$

Applying (31) one arrives at the expression

$$F_{J_n}(\alpha, \beta, \gamma; z) = \gamma^n e^{-\frac{z}{\gamma}} \frac{\Gamma\left(n + 1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right)}{\Gamma\left(1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right)} F_1\left(1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; 1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; \beta \right)$$

$$\times \frac{\Gamma\left(n - \frac{\alpha}{\beta} + 1\right)}{\Gamma\left(\frac{\alpha}{\beta}\right)} F_1\left(1 - \frac{\alpha}{\beta} + \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; n + 2 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; \beta \right)$$

$$- \left(\frac{\beta}{\gamma^2}\right)^{n+1} \frac{\Gamma\left(n + 1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right)}{\Gamma\left(\frac{\alpha}{\beta}\right)} F_1\left(1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; n + 2 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; \beta \right) \right).$$

Eigenvectors can be explicitly expressed as well. If $z$ is an eigenvalue of $J_n(\alpha, \beta, \gamma)$ then formula (16) admits adaptation to this situation giving the expression for the components of a corresponding eigenvector,

$$\xi_k^{(n)} = (-1)^{k-1} \left(\prod_{j=1}^{k-1} w_j\right) \left(\prod_{j=k+1}^{n} (\lambda_j - z)\right) \mathfrak{F} \left(\left\{\frac{\gamma^2 j^2}{\lambda_j - z}\right\}_{j=k+1}^n\right), \quad k = 1, 2, \ldots, n.$$
computation at the formula for an eigenvector \( v^{(n)} \) of \( J_n(\alpha, \beta, \gamma) \):

\[
v_k^{(n)} = (-1)^k \beta^k / 2 \gamma^{-k} \frac{1}{\Gamma(\alpha/\beta + k)} \left( \frac{\Gamma(\alpha/\beta + k)}{\Gamma(1 - \beta/\gamma + k)} \right) \Gamma \left( 1 - \frac{\beta}{\gamma} - \frac{z}{\gamma} + n \right) \times_1 F_1 \left( 1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma} - \frac{z}{\gamma} + k; 1 - \frac{\beta}{\gamma} - \frac{z}{\gamma} + k; \frac{\beta}{\gamma} \right) \times_1 F_1 \left( 1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma} - \frac{z}{\gamma}; 1 + \frac{\beta}{\gamma} + \frac{z}{\gamma} - k; \frac{\beta}{\gamma} \right) - \left( \frac{\beta}{\gamma^2} \right)^{n-k+1} \frac{1}{\Gamma(1 + \alpha/\beta + n)} \frac{\Gamma(1 + \alpha/\beta + n)}{\Gamma(2 - \frac{\beta}{\gamma} - \frac{z}{\gamma} + n)} \times_1 F_1 \left( 1 - \frac{\alpha}{\beta} - k; 1 + \frac{\beta}{\gamma} + \frac{z}{\gamma} - k; \frac{\beta}{\gamma} \right) \times_1 F_1 \left( 1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma} - \frac{z}{\gamma}; 2 - \frac{\beta}{\gamma} - \frac{z}{\gamma} + n; \frac{\beta}{\gamma} \right), \quad 1 \leq k \leq n.
\]

Remark 10. Formula (10) can be derived informally using a limit procedure. Suppose \( z \) is an eigenvalue of the infinite Jacobi matrix \( J(\alpha, \beta, \gamma) \). For \( k \in \mathbb{N} \) fixed, considering the asymptotic behavior of \( v_k^{(n)} \) as \( n \to \infty \) one expects that the leading term may give the component \( v_k \) of an eigenvector corresponding to the eigenvalue \( z \). Omitting some constant factors one actually arrives in this way at (10). But having in hand the explicit expressions (39) and (40) it is straightforward to verify directly that the former one represents a characteristic function while the latter one describes an eigenvector.

Proof of Proposition 3. Observe first that for \( k = 0 \) the RHS of (10) equals, up to a constant factor, to the announced characteristic function (39). If \( z \) solves the equation \( v_0 = 0 \) then one can make use of the identity [1], Eq. 13.4.2

\[
b(b - 1) \frac{1}{2} F_1(a; b - 1; x) + b(1 - b - x) \frac{1}{2} F_1(a; b; x) + (b - a) x_1 F_1(a; b + 1; x) = 0
\]

to verify that \( v \in \ell^2(\mathbb{N}) \) actually fulfills the eigenvalue equation (9). Note that the Stirling formula tells us that

\[
v_k = \left( \frac{-1}{2\pi} \right)^{1/4} k^{-3/2 + \alpha/\beta + \beta/\gamma + z} \left( \frac{\beta e}{\gamma^2 k} \right)^{k/2} \left( 1 + O \left( \frac{1}{k} \right) \right) \quad \text{as} \ k \to \infty.
\]

On the other hand, whatever the complex number \( z \) is, the sequence \( v_k, k \in \mathbb{N} \), solves the second-order difference equation (17), and in that case it is even true that

\[
w_0 v_0 + (\lambda_1 - z) v_1 + w_1 v_2 = 0.
\]

Let \( g_k, k \in \mathbb{N} \), be any other independent solution of (17). Since the Wronskian

\[
w_k (v_k g_{k+1} - v_{k+1} g_k) = \text{const} \neq 0
\]
does not depend on \( k \), and clearly \( \lim_{k \to \infty} w_k v_k = \lim_{k \to \infty} w_k v_{k+1} = 0 \), the sequence \( g_k \) cannot be bounded in any neighborhood of infinity. Hence, up to a multiplier, \( \{v_k\} \) is the only square summable solution of (17). One concludes that \( z \) is an eigenvalue of \( J(\alpha, \beta, \gamma) \) if and only if \( w_0 v_0 = 0 \) (which covers also the case \( \alpha = 0 \)).
Remark 11. A second independent solution of (17) can be found explicitly. For example, this is the sequence

\[ g_k = (-1)^k \beta^{k/2} \gamma^{-k} \Gamma\left(\frac{\alpha}{\beta} + k\right)^{-1/2} U\left(1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}, 1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma} + k, \frac{\beta}{\gamma^2}\right), \quad k \in \mathbb{N}, \]

as it follows from the identity [1, Eq. 13.4.16]

\[(b - a - 1)U(a, b - 1, x) + (1 - b - x)U(a, b, x) + xU(a, b + 1, x) = 0.\]

But using once more relation (34) one may find as a more convenient the solution

\[ g_k = \left(\frac{\beta}{\gamma^2}\right)^{\frac{a}{2} + \frac{z - 1}{2}} \frac{1}{\sqrt{\Gamma\left(\frac{a}{2} + k\right) \Gamma\left(1 + \frac{\beta}{\gamma^2} + \frac{z}{\gamma} - k\right)}} \times _1F_1\left(1 - \frac{\alpha}{\beta} - k; 1 + \frac{\beta}{\gamma^2} + \frac{z}{\gamma} - k; \frac{\beta}{\gamma^2}\right), \quad k \in \mathbb{N}. \]

Remark 12. Let us point out that for \(\alpha = 0\) one gets a nontrivial example of an unbounded Jacobi matrix operator whose spectrum is known fully explicitly. In that case

\[ \lambda_k = \gamma k, \quad w_k = \sqrt{\beta k}, \quad k = 1, 2, 3, \ldots, \]

and

\[ F_J(0, \beta, \gamma; z) = e^{\beta/\gamma^2} \left/ \Gamma\left(1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right) \right. \]

Hence

\[ \text{spec} \ J(0, \beta, \gamma) = \left\{ -\frac{\beta}{\gamma} + \gamma j; \quad j = 1, 2, 3, \ldots \right\}. \]

Remark 13. Finally we remark that another particular case of interest is achieved in the formal limit \(\beta \to 0\). Set \(\alpha = w^2\) for some \(w > 0\). Since [1, Eq. 13.3.2]

\[ \lim_{a \to \infty} _1F_1\left(a; b; -\frac{z}{a}\right) = z^{(1-b)/2} \Gamma(b) J_{b-1}(2\sqrt{z}) \]

one finds that

\[ \lim_{\beta \to 0} F_J(w^2, \beta, \gamma; z) = \left(\frac{w}{\gamma}\right)^{z/\gamma} J_{-z/\gamma}\left(\frac{2w}{\gamma}\right). \]

It is known for quite a long time [4, 9] that actually

\[ \text{spec} \ J(w^2, 0, \gamma) = \left\{ z \in \mathbb{C}; \quad J_{-z/\gamma}\left(\frac{2w}{\gamma}\right) = 0 \right\}. \]
5 Q-Bessel functions

5.1 Some properties of q-Bessel functions

Here we aim to explore a q-analogue to the following well known property of Bessel functions. Consider the eigenvalue problem

\[ w x_{k-1} - k x_k + w x_{k+1} = \nu x_k, \quad k \in \mathbb{Z}, \]

for a second order difference operator acting in \( \ell^2(\mathbb{Z}) \) and depending on a parameter \( w > 0 \). If \( \nu \not\in \mathbb{Z} \) then one can take \( \{J_{\nu+k}(2w)\} \) and \( \{(-1)^k J_{-\nu-k}(2w)\} \) for two independent solutions of the formal eigenvalue equation while for \( \nu \in \mathbb{Z} \) this may be the couple \( \{J_{\nu+k}(2w)\} \) and \( \{Y_{\nu+k}(2w)\} \). Taking into account the asymptotic behavior of Bessel functions for large orders (see [1, Eqs. 9.3.1, 9.3.2]) one finds that a square summable solution exists if and only if \( \nu \in \mathbb{Z} \). Then \( x_k = J_{\nu+k}(2w) \), \( k \in \mathbb{Z} \), is such a solution and is unique up to a constant multiplier. Since

\[ \sum_{k=-\infty}^{\infty} J_k(z)^2 = 1, \]

(41)

thus obtained eigenbasis \( v_\nu = \{v_{\nu,k}\}_{k=-\infty}^{\infty}, \nu \in \mathbb{Z}, \) with \( v_{\nu,k} = J_{\nu+k}(2w) \), is even orthonormal. One observes that the spectrum of the difference operator is stable and equals \( \mathbb{Z} \) independently of the parameter \( w \).

Hereafter we assume \( 0 < q < 1 \). Recall the second definition of the q-Bessel function introduced by Jackson [11] (for some basic information and references one can also consult [3]),

\[ J_{\nu}^{(2)}(x; q) = (q^{\nu+1}; q)_{\infty} \frac{x}{2} \phi_1(\frac{x}{2}; q^{\nu+1}; q, -q^{\nu+1}x^2/4). \]

Here we prefer a slight modification of the second q-Bessel function, obtained just by some rescaling, and define

\[ j_\nu(x; q) := q^{\nu^2/4} J_{\nu}^{(2)}(q^{1/4}x; q) = q^{\nu(\nu+1)/4} \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \frac{x}{2} \phi_1(\frac{x}{2}; q^{\nu+1}; q, -q^{\nu+3/2}x^2/4). \]

(42)

With our definition we have the following property.

**Lemma 14.** For every \( n \in \mathbb{N} \),

\[ j_{-n}(x; q) = (-1)^n j_n(x; q). \]

(43)

**Proof.** One can readily verify that

\[
\lim_{\nu \to -n} (1 - q^{\nu+n}) \phi_1(\nu+1; q, q^{\nu+3/2}w^2) = -q^{n^2/2}w^{2n} \frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{n+1}(q; q)_n} \phi_1(n+1; q, q^{n+3/2}w^2)
\]

and

\[
\lim_{\nu \to -n} \frac{(q^{\nu+1}; q)_{\infty}}{1 - q^{\nu+n}} = (-1)^{n-1} q^{-n(n-1)/2} (q; q)_{n-1}(q; q)_n.
\]

The lemma is an immediate consequence. \( \square \)
Proposition 15. For $0 < q < 1$, $w, \nu \in \mathbb{C}$, $q^{-\nu} \notin q^{\mathbb{Z}^+}$, one has
\[
\mathfrak{S}\left(\left\{\frac{w}{q^{-\nu k/2} - q^{\nu + k/2}}\right\}_{k=0}^{\infty}\right) = 0\phi_1(\nu; q, q, -q^{-1/2}w^2). \tag{44}
\]

Remark 16. If rewritten in terms of q-Bessel functions, (44) becomes a q-analogue of (3). Explicitly,
\[
\mathfrak{S}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right) = q^{\nu/4} \Gamma_q(\nu + 1)w^{-\nu}J^{(2)}_\nu(2q^{-1/4}(1 - q)w; q)
\]
where
\[
[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}, \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{-x}.
\]

Lemma 17. For $\nu \in \mathbb{C}$, $q^{-\nu} \notin q^{\mathbb{Z}^+}$, and all $s \in \mathbb{N},$
\[
\sum_{k=0}^{\infty} \frac{q^{sk}}{(q^{\nu+k}; q)_s+1} = \frac{1}{(1 - q^s)(q^\nu; q)_s}. \tag{45}
\]

Proof. One can proceed by mathematical induction in $s$. The identity
\[
\frac{q^{sk}}{(q^{\nu+k}; q)_s+1} = \frac{q^{s-1k}}{q^\nu(1 - q^s)} \left( \frac{1}{(q^{\nu+k}; q)_s} - \frac{1}{(q^{\nu+k+1}; q)_s} \right)
\]
can be used to verify both the case $s = 1$ and the induction step $s \to s + 1$. \hfill \Box

Proof of Proposition [13]. One possibility how to prove (44) is based on Lemma [2]. The proof presented below relies, however, on explicit evaluation of the involved sums. For $\nu \in \mathbb{C}$, $q^{-\nu} \notin q^{\mathbb{Z}^+}$, $k \in \mathbb{Z}$, put
\[
\rho_k = \frac{q^{(\nu+k)/2}}{1 - q^{\nu+k}}.
\]

Then (45) immediately implies that, for $n \in \mathbb{Z}$ and $s \in \mathbb{N},$
\[
\sum_{k=n}^{\infty} q^{(s-1)(\nu+k)/2} \rho_k \rho_{k+1} \cdots \rho_{k+s} = \frac{q^{s(\nu+n+1)/2}}{1 - q^s} \rho_n \rho_{n+1} \cdots \rho_{n+s-1}.
\]

This equation in turn can be used in the induction step on $m$ to show that, for $m \in \mathbb{N}$, $n \in \mathbb{Z},$
\[
\sum_{k_1=n}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \rho_{k_1} \rho_{k_1+1} \rho_{k_2+1} \cdots \rho_{k_m} \rho_{k_m+1}
\]
\[
= \frac{q^{m(3m+1)/4}}{(q; q)_m} q^{m(\nu+n-1)/2} \rho_n \rho_{n+1} \cdots \rho_{n+m-1}.
\]

In particular, for $n = 1$ one gets
\[
\sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \rho_{k_1} \rho_{k_1+1} \rho_{k_2+1} \cdots \rho_{k_m} \rho_{k_m+1} = \frac{q^{m(2m+1)/2+\nu m}}{(q; q)_m (q^\nu; q)_m}, \quad m \in \mathbb{N}.
\]

Now, in order to evaluate $\mathfrak{S}(\{w\rho_k\}_{k=1}^{\infty})$, it suffices to apply the very definition (11). \hfill \Box
The \( q \)-hypergeometric function is readily seen to satisfy the recurrence rule
\[
0\phi_1(; q^\nu; q, z) - 0\phi_1(; q^{\nu+1}; q, qz) - \frac{z}{(1 - q^\nu)(1 - q^{\nu+1})} 0\phi_1(; q^{\nu+2}; q, q^2z) = 0.
\]
Consequently,
\[
w j_\nu(2w; q) - (q^{-(\nu+1)/2} - q^{(\nu+1)/2}) j_{\nu+1}(2w; q) + w j_{\nu+2}(2w; q) = 0.
\]
This is in agreement with (12) if applied to the bilateral second order difference equation
\[
w x_{n-1} - (q^{-(\nu+n)/2} - q^{(\nu+n)/2}) x_n + w x_{n+1} = 0, \quad n \in \mathbb{Z}.
\]
Suppose \( q^\nu \notin q^2 \). Then the two solutions described in (12) in this case give
\[
f_n = q^{-\nu(\nu+1)/4} \frac{(q; q)_\infty}{(q^{\nu+1}; q)_\infty} w^{-\nu} j_{\nu+n}(2w; q),
\]
\[
g_n = (-1)^{n+1} q^{-\nu(\nu+1)/4} \frac{(q; q)_\infty}{(q^{-\nu}; q)_\infty} w^{\nu} j_{-\nu-n}(2w; q), \quad n \in \mathbb{Z}.
\]
Let us show that they are generically independent. For the proof we need the identity \[5, \S 1.3]\]
\[
\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} z^k = (z; q)_\infty.
\]

**Lemma 18.** For \( \nu \in \mathbb{C}, q^\nu \notin q^2 \), the Wronskian of the solutions of (46), \( \{f_n\} \) and \( \{g_n\} \) defined in (47) and (48), respectively, fulfills
\[
\mathcal{W}(f, g) = \mathfrak{F}\left( \left\{ \frac{w q^{(\nu+k)/2}}{1 - q^{\nu+k}} \right\}_{k=-\infty}^{\infty} \right) = (-q^{1/2} w^2; q)_\infty.
\]

**Proof.** The first equality in (49) is nothing but (13). Further, in virtue of (44), the second member in (49) equals
\[
\lim_{N \to \infty} 0\phi_1(; q^{\nu-N}; q, -q^{\nu-N+1/2} w^2) = \lim_{M \to \infty} \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q; q)_k (q^-M; q)_k} (-q^{-M} q^{1/2} w^2)^k
\]
\[= \sum_{k=0}^{\infty} \frac{q^{k^2/2}}{(q; q)_k} w^{2k} = (-q^{1/2} w^2; q)_\infty.
\]
The lemma follows.

At the same time, \( \mathcal{W}(f, g) \) equals
\[
\frac{q^{-\nu(\nu+1)/2} (q; q)_\infty^2 w}{(q^{\nu+1}; q)_\infty (q^{-\nu}; q)_\infty} (j_\nu(2w; q) j_{-\nu-1}(2w; q) + j_{\nu+1}(2w; q) j_{-\nu}(2w; q)).
\]
This implies the following result.
Proposition 19. For \( w \in \mathbb{C} \) one has
\[
j_\nu(2w; q) j_{\nu-1}(2w; q) + j_{\nu+1}(2w; q) j_{-\nu}(2w; q) = 
\frac{q^{\nu(\nu+1)/2} (q^{\nu+1}; q)_\infty (q^{-\nu}; q)_\infty (-q^{1/2}w^2; q)_\infty}{(q; q)_\infty^2 w}
\]
and, rewriting (50) in terms of \( q \)-hypergeometric functions,
\[
q^\nu z \left( q^\nu z \right)^{-1} \left( 1 - q^\nu \right) (1 - q^\nu z) = 
\frac{q^\nu z}{(1 - q^\nu)(1 - q^\nu z)} q^\nu z 
\frac{q^\nu (1; q^\nu z; q, -q^{-\nu} z)}{q^\nu (1; q^\nu z; q, -q^{-\nu} z)} q^\nu z 
\frac{q^\nu (1; q^\nu z; q, -q^{-\nu} z)}{q^\nu (1; q^\nu z; q, -q^{-\nu} z)} q^\nu z
\]

Remark 20. Let us examine the limit \( q \to 1^- \) applied to (50) while replacing \( w \) by \( (1 - q)w \). One knows that
\[
\lim_{q \to 1^-} j_\nu((1 - q)z; q) = J_\nu(z), \quad \lim_{q \to 1^-} (1 - q)^{1-z} \frac{(q; q)_\infty}{(q^2; q)_\infty} = \Gamma(x).
\]
Thus one finds that the limiting equation coincides with the well known identity
\[
J_\nu(2w)J_{-\nu-1}(2w) + J_{\nu+1}(2w)J_{-\nu}(2w) = \frac{1}{w\Gamma(\nu + 1)\Gamma(-\nu)} = \frac{-\sin(\pi\nu)}{\pi w}.
\]
It is desirable to have some basic information about the asymptotic behavior of \( q \)-Bessel functions for large orders. It is straightforward to see that
\[
J_\nu(2w)J_{-\nu-1}(2w) + J_{\nu+1}(2w)J_{-\nu}(2w) = \frac{1}{w\Gamma(\nu + 1)\Gamma(-\nu)} = \frac{-\sin(\pi\nu)}{\pi w}.
\]
The asymptotic behavior at \(-\infty\) is described as follows.

Lemma 21. For \( \sigma, w \in \mathbb{C} \), \( q^\sigma \notin q^2 \), one has
\[
\lim_{|\nu| \to \infty, \nu \in -\sigma - \mathbb{N}} \sin(\pi\nu) q^{\nu(\nu+1)/4} w^{-\nu} j_\nu(2w; q) = -\sin(\pi\sigma) q^{-\sigma(\sigma+1)/2} \frac{(q^\sigma; q)_\infty (q^{1+\sigma}; q)_\infty (-q^{1/2}w^2; q)_\infty}{(q; q)_\infty}.
\]
Proof. Put \( \nu = -\sigma - n \) where \( n \in \mathbb{N} \). Using (11) and (17) one can write
\[
0\phi_1(q^{-\sigma - n}; q, -q^{-\sigma - n + 1/2}w^2) = 
\mathfrak{F} \left( \left\{ \frac{w}{q^{(\sigma+k)/2} - q^{-(\sigma+k)/2}} \right\}_{k=0}^n \right) \mathfrak{F} \left( \left\{ \frac{w}{q^{(\sigma-k)/2} - q^{-(\sigma-k)/2}} \right\}_{k=1}^\infty \right) + 
\frac{w^2}{(q^{\sigma/2} - q^{-\sigma/2})(q^{1-\sigma/2} - q^{-(1-\sigma)/2})}
\times \mathfrak{F} \left( \left\{ \frac{w}{q^{(\sigma+k)/2} - q^{-(\sigma+k)/2}} \right\}_{k=1}^n \right) \mathfrak{F} \left( \left\{ \frac{w}{q^{(\sigma-k)/2} - q^{-(\sigma-k)/2}} \right\}_{k=2}^\infty \right).
\]
Applying the limit $n \to \infty$ one obtains
\[
\lim_{n \to \infty} \phi_1(q^{-\sigma-n}; q, -q^{-\sigma-n+1/2}w^2) = \phi_1(q^\sigma; q, -q^{\sigma+1/2}w^2) \phi_1(q^{1-\sigma}; q, -q^{-\sigma+3/2}w^2) \\
+ \frac{(q^{\sigma/2} - q^{-\sigma/2})(q^{1-\sigma/2} - q^{-(1-\sigma)/2})}{w^2} \phi_1(q^{1+\sigma}; q, -q^{\sigma+3/2}w^2) \phi_1(q^{2-\sigma}; q, -q^{-\sigma+5/2}w^2) \\
= (-q^{1/2}w^2; q)_{\infty}.
\]

To get the last equality we have used (51). Notice also that
\[
\lim_{n \to \infty} (-1)^n q^{-n(-\sigma-n+1/2)} (q^{-\sigma-n+1}; q)_{\infty} = q^{\sigma(\sigma-1)/2} (q^\sigma; q)_{\infty}(q^{1-\sigma}; q)_{\infty}.
\]
The limit (53) then readily follows.

Finally we establish an identity which can be viewed as a q-analogue to (41).

**Proposition 22.** For $0 < q < 1$ and $w \in \mathbb{C}$ one has
\[
\sum_{k=-\infty}^{\infty} q^{-k/2} j_k(2w; q)^2 = j_0(2w; q)^2 + \sum_{k=1}^{\infty} (q^{k/2} + q^{-k/2}) j_k(2w; q)^2 = (-q^{1/2}w^2; q)_{\infty}.
\]

Equivalently, if rewritten in terms of q-Bessel functions,
\[
J_0^{(2)}(2w; q)^2 + \sum_{k=1}^{\infty} (q^{k/2} + q^{-k/2}) q^{k^2/2} J_k^{(2)}(2w; q)^2 = (-w^2; q)_{\infty}.
\]

**Proof.** In [13 (1.20)] it is shown that
\[
\frac{J_\nu^{(2)}(2w; q)^2}{(-w^2; q)_{\infty}} = \left( \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \right)^2 w^{2\nu} 3\phi_2(q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q^{\nu+1}, q^{2\nu+1}, q, -w^2),
\]
and this can be rewritten as
\[
\phi_1(q^{\nu+1}; q, -q^{\nu+1}x)^2 = (-x; q)_{\infty} 3\phi_2(q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q^{\nu+1}, q^{2\nu+1}, q, -x).
\]
Hence (54) is equivalent to
\[
3\phi_2(q^{1/2}, -q^{1/2}; q, q; q, -x) \\
+ \sum_{k=1}^{\infty} \frac{(q^{-k/2} + q^{k/2}) q^{k^2/2}}{(q; q)_{k^2}} 3\phi_2(q^{k+1/2}, -q^{k+1/2}, -q^{k+1}; q^{k+1}, q^{2k+1}, q, -x) x^k = 1.
\]
Looking at the power expansion in $x$ one gets, equivalently, a countable system of equations, for $n = 1, 2, 3, \ldots$,

\[
\frac{(q^{1/2}; q)_n (-q^{1/2}; q)_n (-q; q)_n}{(q; q)_n^3} + \sum_{k=1}^{n} (-1)^k \frac{(q^{k-1/2}; q^{k-1/2}) q^{k^2/2} (q^{k+1/2}; q)_{n-k} (-q^{k+1/2}; q)_{n-k} (-q^{k+1}; q)_{n-k}^3}{(q; q)_{n-k} (q^{k+1}; q)_{n-k}} = 0.
\]

The equations can be brought to the form

\[
\frac{1}{(q; q)_n^2} + \sum_{k=1}^{n} (-1)^k \frac{q^{k(k-1)/2} (1 + q^k)}{(q; q)_{n+k}^2} = 0
\]

or, more conveniently,

\[
\sum_{j=0}^{2n} (-1)^j \frac{q^{-j(2n-j+1)/2}}{(q; q)_{2n-j}^2} = 0.
\]

This is true indeed since, for any $m \in \mathbb{Z}_+$,

\[
\sum_{j=0}^{m} (-1)^j \frac{(q; q)_m}{(q; q)_{m-j}^2} q^{-j(m-j)/2} x^j = (q^{-m-1/2} x; q)_m = \prod_{k=0}^{m-1} \left(1 - q^{-(m-1)/2+k} x \right).
\]

This concludes the proof.

\[\square\]

### 5.2 A bilateral second order difference equation

We know that the sequence $u_n = j_{\nu+n}(2w; q)$ obeys (46). Applying the substitution $q^{-\nu-1} = z, w = q^\nu + \frac{1}{2} \beta$, one finds that the sequence

\[
v_n = q^{-n/4} u_n = q^{-n/4} j_{\nu+n} (2q^{(2\nu+1)/4} \beta; q)
\]

\[
= q^{-\left(\nu^2 + 2\nu + 3\right)/4} q^{(n-1)(n-2)/4} \frac{(q^n z^{-1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{\beta}{z}\right)^{\nu+n} 0\phi_1 (z; q^n z^{-1}; q, -q^n z^{-2} \beta^2),
\]

fulfills

\[
q^{(n-1)/2} \beta v_n + (q^n - z) v_{n+1} + q^{n/2} \beta v_{n+2} = 0, \quad n \in \mathbb{Z}.
\]

**Remark 23.** One can as well consider the unilateral second order difference equation

\[
(1 - z) v_1 + \beta v_2 = 0, \quad q^{(n-1)/2} \beta v_n + (q^n - z) v_{n+1} + q^{n/2} \beta v_{n+2} = 0, \quad n = 1, 2, 3, \ldots
\]

From (52) it can be seen that the sequence $\{v_n\}$ given in (55) is square summable over $\mathbb{N}$. Considering the Wronskian one also concludes that any other linearly independent solution of (55) cannot be bounded on any neighborhood of $+\infty$. Hence the sequence $v_n, n \in \mathbb{N}$, solves the eigenvalue problem in $l^2(\mathbb{N})$ iff $v_0 = 0$, i.e. iff $j_{\nu}(2w; q) = 0$. In terms of the new parameters $\beta, z$ this condition becomes the characteristic equation for an eigenvalue $z$,

\[
(z^{-1}; q)_\infty 0\phi_1 (z^{-1}; q, -z^{-2} \beta^2) = 0.
\]

This example has already been treated in [13 Sec. 4.1].
For the bilateral equation it may be more convenient shifting the index by 1 in (56). This is to say that we are going to solve the equation
\[ q^{(n-1)/2} \beta v_{n-1} + (q^n - z) v_n + q^{n/2} \beta v_{n+1} = 0, \quad n \in \mathbb{Z}, \] (57)
rather than (56). Denote again by \( J = J(\beta, q) \), with \( \beta \in \mathbb{R} \) and \( 0 < q < 1 \), the corresponding matrix operator in \( \ell^2(\mathbb{Z}) \). One knows, however, that \( J(-\beta, q) \) and \( J(\beta, q) \) are unitarily equivalent and so, if convenient, one can consider just the values \( \beta \geq 0 \). In equation (57), \( z \) is playing the role of a spectral parameter. Using a notation analogous to (58) (now for the bilateral case), this means that
\[ w_n = q^{n/2} \beta, \quad \lambda_n = q^n, \quad \text{and} \quad \zeta_n := z - \lambda_n = z - q^n, \quad n \in \mathbb{Z}. \] (58)
Notice that for a sequence \( \{ \gamma_n \} \) obeying \( \gamma_n \gamma_{n+1} = w_n, \quad \forall n \in \mathbb{Z} \), one can take
\[ \gamma_{2k-1}^2 = q_k^{-1}, \quad \gamma_{2k}^2 = q_k^{2\beta}. \]

Since the sequence \( \{ w_n/(\lambda_n + 1) \} \) is summable over \( \mathbb{Z} \), the Weyl theorem tells us that the essential spectrum of the self-adjoint operator \( J(\beta, q) \) contains just one point, namely 0. Hence all nonzero spectral points are eigenvalues.

**Proposition 24.** For \( 0 < q < 1 \) and \( \beta > 0 \), the spectrum of the Jacobi matrix operator \( J(\beta, q) \) in \( \ell^2(\mathbb{Z}) \), as introduced above (see (58)), is pure point, all eigenvalues are simple and
\[ \text{spec}_{\text{p}} J(\beta, q) = (-\beta^2 q^2)^+ \cup q^2 \mathbb{Z}. \]

*Eigenvectors* \( v_m^{(+)} \) corresponding to the eigenvalues \( q^m, \quad m \in \mathbb{Z} \), can be chosen as \( v_m^{(+)} = \left\{ v_{m,k}^{(+)} \right\}_{k=-\infty}^{\infty}, \quad \text{with} \quad v_{m,k}^{(+)} = q^{(m-k)/4} j_{m+k}(2q^{-(2m+1)/4(\beta \cdot q)}). \)

They are normalized as follows:
\[ \| v_m^{(+)} \|^2 = \sum_{k=-\infty}^{\infty} q^{-k/2} j_{k}(2q^{-(2m+1)/4(\beta \cdot q)})^2 = (-q^{-m \beta^2} q_{\infty}, \quad \forall m \in \mathbb{Z}. \]

*Eigenvector* \( v_m^{(-)} \) corresponding to the eigenvalues \(-\beta^2 q^m, \quad m \in \mathbb{Z}^+ \), can be chosen as \( v_m^{(-)} = \left\{ v_{m,k}^{(-)} \right\}_{k=-\infty}^{\infty}, \quad \text{with} \quad v_{m,k}^{(-)} = \frac{(-1)^k q^{k(4m-1)/4}}{(q; q)_{\infty}} \beta^{-k} (-q^{-m+k+1 \beta^2}; q_{\infty}) \times \phi_1 (\ldots; -q^{-m+k+1 \beta^2}; q; -q^{-2m+k+1 \beta^2}). \] (59)

**Remark 25.** An expression for the norms of vectors \( v_m^{(-)} \) can be found, too,
\[ \| v_m^{(-)} \|^2 = (-1)^m q^{-m(3m+1)/2} \frac{(q^{-\beta^2}; q_{\infty})}{(q^{m+1}; q_{\infty})} \times \frac{\phi_1 (\ldots; -\beta^2; q; -q^{-m+1 \beta^2})}{\phi_1 (\ldots; q^{2 \beta^2}; q; -q^{-m+1 \beta^2})}, \quad \forall m \in \mathbb{Z}^+. \]
But the formula is rather cumbersome and its derivation somewhat lengthy and this is why we did not include it in the proposition and omit its proof.

Proof. We use the substitution \( z = q^{-\nu} \) where \( \nu \) is in general complex. The right hand sides in (12) can be evaluated using (44) and (42). Applying some easy simplifications one gets two solutions of (57):

\[
v_n = q^{-\nu(n+1)/4} j_{\nu+n}(2q^{(2\nu-1)/4} \beta; q), \quad \tilde{v}_n = (-1)^n q^{-\nu(n+1)/4} j_{-\nu-n}(2q^{(2\nu-1)/4} \beta; q), \quad n \in \mathbb{Z}.
\]

One can argue that in the bilateral case, too, all eigenvalues of \( J(\beta, q) \) are simple. In fact, the solution \( \{v_n\} \) asymptotically behaves as

\[
v_n = \frac{1}{(q; q)^{\infty}} q^{\frac{1}{2}(n^2+(4\nu-1)n+(3\nu-1)\nu)} \beta^{\nu+n} (1 + O(q^n)) \quad \text{as} \quad n \to +\infty.
\]

For any other independent solution \( \{y_n\} \) of (57), \( q^{n/2} (y_n v_{n+1} - y_{n+1} v_n) \) is a nonzero constant. Obviously, such a sequence \( \{y_n\} \) cannot be bounded on any neighborhood of \( +\infty \). A similar argument applies to the solution \( \{\tilde{v}_n\} \) for \( n \) large but negative. In particular, one concludes that \( z = q^{-\nu} \) is an eigenvalue of \( J(\beta, q) \) if and only if \( \{v_n\} \) and \( \{\tilde{v}_n\} \) are linearly dependent.

Using (50) one can derive a formula for the Wronskian,

\[
\mathcal{W}(v, \tilde{v}) = q^{k/2} \beta (v_k \tilde{v}_{k+1} - v_{k+1} \tilde{v}_k) = (-1)^{k+1} \beta q^{(2\nu+1)/4} \left( j_{\nu+k}(2q^{(2\nu-1)/4} \beta; q) j_{-\nu-k}(2q^{(2\nu-1)/4} \beta; q) + j_{\nu+k+1}(2q^{(2\nu-1)/4} \beta; q) j_{-\nu-k}(2q^{(2\nu-1)/4} \beta; q) \right)
\]

\[
= q^{\nu(\nu-3)/2} (q^\nu; q)_{\infty} (q^{1-\nu}; q)_{\infty} (-q^\nu \beta^2; q)_{\infty} \quad \frac{(q; q)^{\infty^2}}{(q; q)_{\infty^2}}.
\]

Thus \( z \) is an eigenvalue if and only if either \( (z^{-1}; q)_{\infty} (qz; q)_{\infty} = 0 \) or \( (-z^{-1} \beta^2; q)_{\infty} = 0 \). In the former case \( z \in q\mathbb{Z} \), in the latter case \( -z \in \beta^2 q\mathbb{Z} \).

Thus in the case of positive eigenvalues one can put \( \nu = -m \), with \( m \in \mathbb{Z} \). With this choice, \( \{v_k\} \) coincides with \( \{v_m^{(+)\infty}\} \). Notice that then the linear dependence of the sequences \( \{v_k\} \) and \( \{\tilde{v}_k\} \) is also obvious from (43). Normalization of the eigenvectors \( v_m^{(+)\infty} \) is a consequence of (54).

As far as the negative spectrum is concerned, one can put, for example, \( \tau = -(i\pi + \log \beta^2)/\log q \) and \( \nu = \tau - m \), \( m \in \mathbb{Z}_+ \). Then the sequence

\[
v_k = q^{-\tau(m+k)/4} j_{\tau-k}(2q^{-(2m+1)/4} \beta; q), \quad k \in \mathbb{Z},
\]

represents an eigenvector corresponding to the eigenvalue \( -\beta^2 q^m \). But it is readily seen to be proportional to the RHS of (57) whose advantage is to be manifestly real.

Finally let is show that 0 can never be an eigenvalue of \( J(\beta, q) \). We still assume \( \beta > 0 \). For \( z = 0 \), one can find two mutually complex conjugate solutions of (57) explicitly. Let us call them \( v_{\pm,n}, n \in \mathbb{Z}, \) where

\[
v_{\pm,n} = i^{\pm n} q^{-n/4} \phi_1 \left( 0; -q^{1/2}; q^{1/2}, \pm \frac{i q^{(2n+3)/4}}{\beta} \right) = i^{\pm n} q^{-n/4} \sum_{k=0}^{\infty} \frac{q^{k(k+2)/4}}{(q; q)_{k}} \left( \pm \frac{i q^{n/2}}{\beta} \right)^k.
\]
Clearly,

\[ v_{z,n} = i^{\pm n} q^{-n/4} \left( 1 + O \left( q^{n/2} \right) \right) \text{ as } n \to +\infty. \]

Using the asymptotic expansion one can evaluate the Wronskian getting

\[ W(v_+, v_-) = q^{n/2} \beta (v_{+,n} v_{-,n+1} - v_{+,n+1} v_{-,n}) = -2i q^{-1/4} \beta. \]

Hence the two solutions are linearly independent. It is also obvious from the asymptotic expansion that no nontrivial linear combination of these solutions can be square summable. Hence 0 cannot be an eigenvalue of \( J(\beta, q) \) whatever \( \beta \) is, and this concludes the proof.

So one observes that the positive part of the spectrum of \( J(\beta, q) \) is stable and does not depend on the parameter \( \beta \). This behavior is very similar to what one knows from the non-deformed case. On the other hand, there is an essentially new feature in the \( q \)-case when a negative part of the spectrum emerges for \( \beta \neq 0 \), and it is even infinite-dimensional though it shrinks to zero with the rate \( \beta^2 \) as \( \beta \) tends to 0.

## 6 Q-confluent hypergeometric functions

In this section we deal with the q-confluent hypergeometric function

\[ \phi_1(a; b, q, q^2 z) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} \frac{(a; q)_k}{(b)_k (q; q)_k} z^k. \]

It can readily be checked to obey the recurrence rules

\[ - \frac{q^{\alpha+\gamma}(1-q^{-\alpha+1})}{(1-q^\gamma)(1-q^{\gamma+1})} z \phi_1(q^\alpha; q^{\gamma+2}; q, q^{\gamma+2} z) - \left( 1 - \frac{q^\gamma}{1-q^\gamma} \right) \phi_1(q^\alpha; q^{\gamma+1}; q, q^{\gamma+1} z) \]

\[ + \phi_1(q^\alpha; q^\gamma; q, q^\gamma z) = 0 \]

and

\[ \phi_1(q^{\alpha-\gamma+1}; q^{2-\gamma}; q, z) + \frac{q(q - q^\gamma - q^{1-\gamma} + 1)}{q^\gamma - q^\alpha} \phi_1(q^{\alpha-\gamma-1}; q^{-\gamma}; q, z) \]

\[ - \left( \frac{q(q - q^\gamma - q^{1-\gamma} + 1)}{q^\gamma - q^\alpha} + \frac{q - q^\gamma}{q^\gamma - q^\alpha} \right) \phi_1(q^{\alpha-\gamma}; q^{1-\gamma}; q, z) = 0. \]

Put, for \( n \in \mathbb{Z} \),

\[ \varphi_n = (q^{n+\gamma}; q)_\infty \phi_1(q^\alpha; q^{n+\gamma}; q, -q^{n+\gamma} z), \]

\[ \psi_n = q^{-\alpha(n+\gamma)-(n+\gamma-1)(n+\gamma-2)/2} \frac{(q^{n+\gamma-\alpha}; q)_\infty}{(q^{n+\gamma-1}; q)_\infty} z^{1-n-\gamma} \phi_1(q^{\alpha-n-\gamma+1}; q^{2-n-\gamma}; q, -q z). \]

Here \( z, \alpha, \gamma \in \mathbb{C}, q^\gamma \notin q^\mathbb{Z} \). The recurrence rules imply that both \( \{ \varphi_n \} \) and \( \{ \psi_n \} \) solve the three-term difference equation

\[ q^{\alpha+\gamma+n-1} \left( 1 - q^{-\alpha+n} \right) z u_{n+1} - (1 - q^{\gamma+n-1} + q^{\gamma+n-1} z) u_n + u_{n-1} = 0, \quad n \in \mathbb{Z}. \]
Lemma 26. The sequences \( \{ \varphi_n \} \) and \( \{ \psi_n \} \) defined in (61) and (62), respectively, fulfill
\[
\varphi_0 \psi_1 - \varphi_1 \psi_0 = q^{-\alpha(\gamma+1)-\frac{1}{2} \gamma(\gamma-1)}(q^{-\alpha+1};q)_\infty (-q^\alpha z;q)_\infty z^{-\gamma}.
\] (64)

Alternatively, (64) can be rewritten as
\[
1 \phi_1(q^\alpha;q^\gamma; q, q^{-\alpha-z}) 1 \phi_1(q^{\alpha-\gamma}; q^{1-\gamma}; q, q^{1-\alpha} z)
+ \frac{q^{\gamma-1}(1-q^{-\alpha}) z}{(1-q^{-\gamma})(1-q^\gamma)} 1 \phi_1(q^\alpha; q^{\gamma+1}; q, q^{\gamma-\alpha+1} z) 1 \phi_1(q^{\alpha-\gamma+1}; q^{2-\gamma}; q, q^{1-\alpha} z) = (z;q)_\infty.
\]

Proof. Checking the Wronskian of the solutions \( \varphi_n \) and \( \psi_n \) one finds that
\[
q^{\frac{1}{2} n(n-1)+\alpha \gamma} n(q^{\gamma-\alpha+1};q)_n z^n (\varphi_n \psi_{n+1} - \varphi_{n+1} \psi_n) = C
\] (65)
is a constant independent of \( n \). In particular, \( \varphi_0 \psi_1 - \varphi_1 \psi_0 = C \). It is straightforward to examine the asymptotic behavior for large \( n \) of the solutions in question getting \( \varphi_n = 1 + O(q^n) \) and
\[
\psi_n = q^{\frac{1}{2} n(n-1)-(\alpha+\gamma)n-\frac{1}{2} \gamma(\gamma-1)\alpha} z^{1-\gamma-n} (-q^\alpha z;q)_\infty (1 + O(q^n)).
\]
Sending \( n \) to infinity in (65) one finds that \( C \) equals the RHS of (64). \( \square \)

Proposition 27. For \( \alpha, \gamma, z \in \mathbb{C} \),
\[
\tilde{\mathcal{F}}^0 \left( \left\{ \frac{q^{\frac{1}{2} (\alpha+\gamma+k)-\frac{1}{2} \gamma}}{(q^{\gamma-\alpha+k+1};q^2)_\infty (1-(1-z)q^{\gamma+k+1})} \right\}_k \right)^\infty = \frac{(q^\gamma;q)_\infty}{((1-z)q^\gamma;q)_\infty} 1 \phi_1(q^\alpha; q^\gamma; q, -q^\gamma z).
\] (66)

Proof. The both sides of the identity are regarded as meromorphic functions in \( z \).
Setting \( \text{Im} \gamma \) to a constant, the both sides tend to 1 as \( \text{Re} \gamma \) tends to \( +\infty \). In virtue of Lemma 2 it suffices to verify that the sequence
\[
F_n = \frac{(q^{\gamma+n-1};q)_\infty}{((1-z)q^{\gamma+n-1};q)_\infty} 1 \phi_1(q^\alpha; q^{\gamma+n-1}; q, -q^{\gamma+n-1} z), \; n \in \mathbb{N},
\]
satisfies the three-term recurrence relation \( F_n - F_{n+1} + s_n z F_{n+2} = 0, \; n \in \mathbb{N}, \) where
\[
s_n = \frac{q^{\alpha+\gamma+n-1} (1-q^{\gamma-\alpha+n})}{(1-(1-z)q^{\gamma+n-1})(1-(1-z)q^{\gamma+n})}.
\]
Since \( \gamma \) here is arbitrary one can consider just the equality for \( n = 1 \). But then the three-term recurrence coincides with (60) (provided \( z \) is replaced by \( -z \)). \( \square \)

Let us now focus on equation (63). One can extract from it a solvable eigenvalue problem for a Jacobi matrix obeying the convergence condition (14).
Proposition 28. For $\sigma \in \mathbb{R}$ and $\gamma > -1$, let $J = J(\sigma, \gamma)$ be the Jacobi matrix operator in $\ell^2(\mathbb{N})$ defined by (63) and

$$w_n = \frac{1}{2} \sinh(\sigma) q^{(n-\gamma-1)/2} \sqrt{1 - q^{n+\gamma}}, \quad \lambda_n = q^{n-1}. \quad (67)$$

Then $z \neq 0$ is an eigenvalue of $J(\sigma, \gamma)$ if and only if

$$(\cosh^2(\sigma/2) z^{-1}; q) \propto 1 \phi_1 (q^{-\gamma} \cosh^2(\sigma/2) z^{-1}; \cosh^2(\sigma/2) z^{-1}; q, -\sinh^2(\sigma/2) z^{-1}) = 0.$$  

Moreover, if $z \neq 0$ solves this characteristic equation then the sequence $\{v_n\}_{n=1}^\infty$, with

$$v_n = q^{-\gamma n + \frac{1}{2} n(n-3)} \frac{\sinh^n(\sigma) (2z)^{-n}}{(q^{\gamma+n}; q)_{\infty}} \left( q^n \cosh^2\left(\frac{\sigma}{2}\right) z^{-1}; q \right)_{\infty}$$

$$\times 1 \phi_1 \left( q^{-\gamma} \cosh^2\left(\frac{\sigma}{2}\right) z^{-1}; q^n \cosh^2\left(\frac{\sigma}{2}\right) z^{-1}; q, -q^n \sinh^2\left(\frac{\sigma}{2}\right) z^{-1} \right), \quad (68)$$

is a corresponding eigenvector.

Remark 29. Notice that the matrix operator $J(\sigma, \gamma)$ is compact (even trace class).

Proof. First, apply in (63) the substitution

$$\gamma = \tilde{\gamma} + \alpha, \quad z = q^\beta, \quad u_n = q^{-\alpha n} \tilde{u}_n,$$

and then forget about the tilde over $\gamma$ and $u$. Next use the substitution

$$q^{\beta/2} = \tanh\left(\frac{\sigma}{2}\right), \quad q^\alpha = q^{-\gamma} \cosh^2\left(\frac{\sigma}{2}\right) z^{-1}, \quad u_n = \phi_n \tilde{u}_n,$$

where $\{\phi_n\}$ is a sequence obeying

$$\frac{\phi_n}{\phi_{n+1}} = q^{(\beta+\gamma+n-1)/2} \sqrt{1 - q^{\gamma+n}}.$$  

Up to a constant multiplier, $\phi_n^2 = q^{-\beta n - \gamma n + \frac{1}{2} n(n-3)} (q^{\gamma+n}; q)_{\infty}$. We again forget about the tildes over $z$ and $u$, and restrict the values of the index $n$ to natural numbers. If $u_0 = 0$ then the transformed sequence $\{u_k\}_{k=1}^\infty$ solves the Jacobi eigenvalue problem (9) with $w_n$ and $\lambda_n$ given in (67).

Further apply the same sequence of transformations to the solution $\varphi_n$ in (61). Let us call the resulting sequence $\{v_n\}$. A straightforward computation yields (68).

Clearly, the sequence $\{v_k; \ k \geq 1\}$ is square summable. On general grounds, since $J(\sigma, \gamma)$ falls into the limit point case, any other linearly independent solution of the recurrence in question, (17), cannot be square summable. Hence the characteristic equation for this eigenvalue problem reads $v_0 = 0$. This shows the proposition. \qed

Remark 30. In the particular case $\gamma = 0$ the characteristic equation simplifies to the form

$$(\cosh^2(\sigma/2) z^{-1}; q) \propto (-\sinh^2(\sigma/2) z^{-1}; q)_{\infty} = 0.$$  

Hence in that case, apart of $z = 0$, one knows the point spectrum fully explicitly,

$$\text{spec} J(\sigma, 0) \setminus \{0\} = \{ q^k \cosh^2(\sigma/2); \ k = 0, 1, 2, \ldots \} \cup \{-q^k \sinh^2(\sigma/2); \ k = 0, 1, 2, \ldots \}. $$
Remark 31. Of course, Proposition 28 can be as well derived using formulas (16), (18), while knowing that (14) is fulfilled. To evaluate $\xi_n(z)$ one can make use of (66). Applying the same series of substitutions as above to equation (66) one gets

$$\tilde{\xi} \left( \left\{ \frac{q^{\frac{1}{2} (k-\gamma)} \frac{3}{2} \sinh(\sigma) \left( q^{\gamma+k}; q^2 \right) \infty}{2 \left( q^{\gamma+k+1}; q^2 \right) \infty (q^{k-1} - z)} \right\}_{k=1}^\infty \right)$$

$$= \frac{(\cosh^2(\sigma/2) z^{-1}; q) \infty}{(z^{-1}; q) \infty} \, \phi_1 \left( q^{-\gamma} \cosh^2 \left( \frac{\sigma}{2} \right) z^{-1}; \cosh^2 \left( \frac{\sigma}{2} \right) z^{-1}; q, -\sinh^2 \left( \frac{\sigma}{2} \right) z^{-1} \right).$$

Then a straightforward computation yields

$$\xi_n(z) = 2 \frac{q^{(\gamma+1)/2} \sqrt{(q^{\gamma+1}; q) \infty}}{\sinh(\sigma) (z^{-1}; q) \infty} \, v_n, \quad n = 0, 1, 2, \ldots,$$

with $v_n$ being given in (68).

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