Consensus Control for Heterogeneous Multi-Robot Formation Systems with Time-Delays

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Abstract. Concerning the formation control problem of heterogeneous multi-robot systems with time-delays, a control algorithm based on the consensus theory is proposed. The formation control problem of heterogeneous multi-robot systems based upon leader-follower model is transformed into a stability problem by using the consensus theory. Then, by utilizing the Newton-Leibniz formula and the Lyapunov theorem, a sufficient condition of the formation control problem is presented. Finally, the correctness of the developed consensus control algorithm is verified by the simulation results.

Introduction

Recently, formation control has become a hot topic in the field of multi-robot collaborative control, and has been widely applied in satellite formation, unmanned aerial vehicle (UAV) formation, multi-arm systems, cluster robotic rescue and so on\cite{1-3}. Multi-robot formation control means that the multi-robot system forms and maintains the established formation by the communication protocol and the distributed control.

The consensus of multi-agent systems was firstly proved by use of the graph theory, laying the foundation for the study of the formation control problem based on the consensus theory in Ref\cite{4}. Most of the formation control consensus analysis was researched for the homogeneous system in which all individuals have the same dynamics model. But this hypothesis may not be true in many applications. Thus, many scholars have began to research on the consensus of heterogeneous multi-robot systems. Considering the consensus problem of continuous-time systems, the sufficient condition of the heterogeneous system consensus in undirected topological graph was obtained by the graph theory and the Lyapunov theory in Ref\cite{5}. For the formation control problem of continuous-time heterogeneous systems with or without the leader, Kim designed the consensus protocol based on the Lyapunov theory and proved its convergence in Ref\cite{6}. In order to be more practical, the consensus of discrete-time heterogeneous multi-agent systems was studied, and the sufficient condition for the consensus was obtained in Ref\cite{7}.

In above literatures, the authors studied the formation control problem of heterogeneous multi-robot systems from different aspects, and made some research results. However, if there are time-delays in the communication topology of these systems, it is difficult for the consensus analysis of different order robots in the system. Thus, in this paper, the formation control consensus of heterogeneous multi-robot systems with time-delays is discussed via leader-following model. And the consensus control protocol is proposed for the follower robots. It makes easier for the consensus analysis with different order robots.

The main work of this paper is that concerning the directed communication topology, the formation control problem of heterogeneous multi-robot systems with time-delays is studied. At first, based on the leader-follower model, a consensus protocol is proposed. Then, a Lyapunov-Razumikhin function is constructed. For ensuring the Lyapunov stability of the system, a sufficient condition is obtained by the graph theory and the matrix analysis. Finally, a simulation example is given to verify the correctness of the consensus control algorithm.
Model Description

Considering a continuous-time heterogeneous multi-robot system consisting of \( n \) followers and a leader, a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) is defined which can depict the system. \( \mathcal{V} = \{ v_0, v_1, ..., v_n \} \) is the set of vertices, where \( v_0 \) is the leader and \( v_k \) \((k = 1, ..., n)\) is the follower. \( \mathcal{E} \) is the set of directed edges. \( \mathcal{A} = [a_{ij}]_{n \times n} \) \((i, j = 1, ..., n)\) is the weighted adjacency matrix of followers. If there is a directed path among the leader \( v_0 \) and each of the other followers, then the leader \( v_0 \) is globally reachable. Furthermore, matrix \( B = \text{diag}(b_i) \) \((i = 1, 2, ..., n)\) is defined to describe the adjacency between followers and the leader.

If there is a direct communication path between the follower \( v_i \) and the leader \( v_0 \), then \( b_i > 0 \) otherwise \( b_i = 0 \). A graph is usually represented by a Laplacian matrix \( L \) such that:

\[
L = \begin{cases} 
-a_{ij} & \text{if } i \neq j \\
\text{diag} \left( \sum_{j \neq i} a_{ij} \right) & \text{if } i = j
\end{cases}
\]

(1)

Consider the following double integrator system of followers:

\[
\begin{align*}
\dot{p}_i(t) &= q_i(t) \\
\dot{q}_i(t) &= u_i(t), i = 1, ..., n
\end{align*}
\]

(2)

where \( p_i(t), q_i(t), u_i(t) \in \mathbb{R} \) denote the position (or angle), velocity and control input of \( n \) robots.

The dynamic model of the first-order leader can be described as follows:

\[
\dot{p}_0(t) = q_0
\]

(3)

where \( p_0(t), q_0(t) \in \mathbb{R} \) denote the position and desired constant velocity.

For the heterogeneous multi-robot system, if the velocity of all followers converges to \( q_0(t) \), namely, \( q_i \to q_0, p_i \to p_0 \) as \( t \to \infty \), the system can achieve the expected formation. Due to the presence of time-delays, each robot in the heterogeneous multi-robot system cannot get information immediately from others. Thus, a consensus control protocol is proposed for followers of the system as follows:

\[
u_i = -\sum_{j \neq i} a_{ij} \left( p_j(t - \tau_{ij}) - p_j(t - \tau_{ji}) - r_j + r_i \right) - b_i \left( p_i(t - \tau_{ij}) - p_0(t - \tau_{ij}) - r_i \right) - k (q_i - q_0), i = 1, 2, ..., n
\]

(4)

where \( k > 0 \) denotes the control parameter; \( r_i \) denotes the relative distance between a follower and the leader in the desired formation; \( \tau_{ij} = \tau_{ji} \) denotes the communication time-delay between node \( i \) and node \( j \).

For convenience of description, let \( \tau_r = \{ \tau_{ij} : i, j = 1, 2, ..., n, i \neq j \} \), where \( r = 1, 2, ..., m \ (m \leq n(n-1)) \). We define the system communication topology sub-graph \( \mathcal{G}_r \) of the graph \( \mathcal{G} \), the adjacency matrix \( A_r \), the degree matrix \( D_r \), the Laplacian matrix \( L_r \), and the leader's adjacency matrix \( B_r \) [8]. Since sub-graphs are disjoint, we have \( A = \sum_{r=1}^{m} A_r, D = \sum_{r=1}^{m} D_r, L = \sum_{r=1}^{m} L_r, B = \sum_{r=1}^{m} B_r \).

Regarding the fixed communication topology, the consensus control protocol (4) and the double integrator system of followers (2) can be written in a matrix form as follows:
\[
\begin{align*}
\dot{p} &= q \\
q &= -\sum_{i=1}^{n} (L_i + B_i)(p(t - \tau_i) - r) + \sum_{i=1}^{m} B_i (p_0(t - \tau_i) \otimes I) - k(q - q_0 \otimes I)
\end{align*}
\]  

(5)

where \( p = [p_1, p_2, \ldots, p_n]^T \), \( q = [q_1, q_2, \ldots, q_n]^T \) and \( r = [r_1, r_2, \ldots, r_m]^T \).

Let \( x = p - p_0 \otimes I - r \), \( v = q - q_0 \otimes I \) and \( \varepsilon = \begin{pmatrix} x \\ v \end{pmatrix} \). The error model of the system (5) can be expressed as:

\[
\varepsilon = \mathbf{Y}_e \varepsilon(t) + \sum_{i=1}^{m} \mathbf{Y}_e \varepsilon(t - \tau_i)
\]

(6)

where \( \mathbf{Y}_0 = \begin{pmatrix} 0 & I \\ 0 & -kI \end{pmatrix} \), \( \mathbf{Y}_r = \begin{pmatrix} 0 & 0 \\ -H & 0 \end{pmatrix} \), \( H = L + B \).

The Stability Analysis

This section mainly analyzes the stability of the error system (5), and obtains the basic conditions for the system to realize formation control with time-delays.

**Lemma 1** [9]. Let \( Q = \begin{pmatrix} 0 & I_n \\ -F & -H_n \end{pmatrix} \). The control parameter \( k \) satisfies

\[
k^2 > \left( \frac{\max_{\rho(F)} \{\text{Im } \mu\}}{\min_{\rho(F)} \{\text{Re } \mu\}} \right)^2,
\]

where \( \text{Re} \) denotes the real part and \( \text{Im} \) denotes the imaginary part. In addition, \( \min_{\rho(F)} \{\text{Re } \mu\} > 0 \), where \( \rho(F) \) denotes the set of all eigenvalues of matrix \( F \). \( \max_{\rho(F)} \{\text{Re } \mu\} > 0 \) if and only if \( F \) is a positive stable matrix, that is, all eigenvalues of \( F \) have a positive real part.

**Lemma 2** [10]. The matrix \( H = L + B \) is positive stable if and only if node 0 is globally reachable in the graph \( G \).

**Theorem 1.** The system control parameter \( k \) of the control protocol (4) satisfies

\[
k^2 > \left( \frac{\max_{\rho(F)} \{\text{Im } \mu\}}{\min_{\rho(F)} \{\text{Re } \mu\}} \right)^2.
\]

Meanwhile, there is a constant \( \tau_0 = \frac{\|E\|}{\|W^T P W^{-1} P^T W \dot{e} + \alpha W\|} \) such that, when \( \tau < \tau_0 \), the error system (6) can be stabilized; namely,

\[
\lim_{t \to 0} \varepsilon(t) = 0
\]

(7)

if and only if node 0 is globally reachable in the communication topology \( G \).

**Proof.** By the Newton-Leibniz formula, we have

\[
\varepsilon(t - \tau_i) = \varepsilon(t) - \int_{t - \tau_i}^{t} \dot{\varepsilon}(s)ds
\]

\[
= \varepsilon(t) - \int_{t - \tau_i}^{t} \mathbf{Y}_e \varepsilon(s) + \sum_{j=1}^{m} \mathbf{Y}_e \varepsilon(s - \tau_j)ds
\]

\[
= \varepsilon(t) - \mathbf{Y}_r \int_{t - \tau_i}^{t} \varepsilon(s)ds - \sum_{j=1}^{m} \mathbf{Y}_r \int_{t - \tau_j}^{t} \varepsilon(s - \tau_j)ds
\]

Due to \( \mathbf{Y}_r \mathbf{Y}_0 = 0 \), \( \sum_{i=1}^{m} \mathbf{H}_i = \mathbf{H} \), and \( \mathbf{P}_r = \mathbf{Y}_r \mathbf{Y}_0 \), the system (6) can be rewritten as
\[ \dot{e} = E_\epsilon(t) + \sum_{i=1}^{n} Y_i e(t - \tau_i) \]

\[ = \sum_{i=1}^{n} Y_i e(t) + \sum_{i=1}^{n} Y_i e(t) - \sum_{i=1}^{n} Y_i \int_{t-\tau_i}^{t} e(s) ds \]

\[ = Q e(t) - \sum_{i=1}^{n} P_i \int_{t-\tau_i}^{t} e(s) ds \]

where \( Q = \begin{bmatrix} 0 & I \\ -H & -kI \end{bmatrix} \).

According to Lemma 1 and Lemma 2, it can be seen that \( \max_{\theta \in [0, \tau]} \Re \theta < 0 \). By the Lyapunov theorem[11], there is a positive definite matrix \( W \in \mathbb{R}^{n \times n} \) satisfying:

\[ Q^T W + W Q = -E \]

where \( E \) is a positive definite matrix.

Construct a Lyapunov-Razumikhin function as follows:

\[ V(\epsilon) = \epsilon^T W \epsilon \]

Then, using (8), the derivative of \( V(\epsilon) \) can be expressed as

\[ \dot{V}(\epsilon) = \epsilon^T W \dot{e} + \epsilon^T \dot{e} \]

\[ = \left( Q e(t) - \sum_{i=1}^{n} P_i \int_{t-\tau_i}^{t} e(s) ds \right)^T W e + \epsilon^T W \left( Q e(t) - \sum_{i=1}^{n} P_i \int_{t-\tau_i}^{t} e(s) ds \right) \]

\[ = \epsilon^T (Q^T W + W Q) - 2 \epsilon^T W \sum_{i=1}^{n} P_i \int_{t-\tau_i}^{t} e(s) ds \]

Since there is \( \pm 2a^T b \leq a^T \Phi a + b^T \Phi^{-1} b \) for any positive definite matrix \( \Phi \), column vectors \( a \) and \( b \), equation (11) can be rewritten as

\[ \dot{V}(\epsilon) = -\epsilon^T E \epsilon - 2 \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \left( P_i W e(t) \right)^T e(s) ds \]

\[ = -\epsilon^T E \epsilon - 2 \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \left( P_i W e(t) \right)^T e(s') ds' \]

\[ \leq -\epsilon^T E \epsilon + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \left( P_i W e(t) \right)^T e(s') ds' \]

\[ = -\epsilon^T E \epsilon + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \epsilon^T W^T P_i W e + \int_{s'-\tau_i}^{s'} \epsilon^T (s'+t) W e(s'+t) ds' \]

By the Lyapunov-Razumikhin theorem[12], when \( \tau \in [-\tau_0, 0] \), there is \( V(e(t+s)) < \varphi(e(t)) \), where \( \varphi(s) = \alpha s (\alpha > 1) \). We have

\[ \dot{V}(\epsilon) \leq -\epsilon^T E \epsilon + \sum_{i=1}^{n} \epsilon^T \left( W^T P_i W e + \alpha W \right) \epsilon \]

So that if we take

\[ \tau_i < \tau_0 = \frac{\|E\|}{\|W^T P_i W e + \alpha W\|} \]

433
Then $V(\varepsilon) < 0$. Therefore, the error system (6) can achieve stability, that is, the conclusion follows by Theorem 1.

**Simulation Results**

In this section, the analytical results of this paper are verified by Matlab simulation results. The formation control of heterogeneous multi-robot systems based on the leader-follower mode is achieved.

**Parameter Initialization**

A heterogeneous system consisting of six robots is built, where $a_0$ is the first-order leader and the remaining $a_i$ ($i = 1, 2, 3, 4$) are the second-order followers. The position is denoted by $p_i = [p_{ix}, p_{iy}]^T$, and the velocity is denoted by $q_i = [q_{ix}, q_{iy}]^T$, where $p_{ix}, p_{iy}, q_{ix}, q_{iy}$ represent the x-displacement and the y-displacement of the robot $a_i$ in the global coordinate system, and $q_{ix}, q_{iy}$ represent the x-velocity and the y-velocity of the robot $a_i$ in the global coordinate system. In this paper, we assume that each robot in the system is a particle movement, so the rotation angle and the rotational angular velocity are not considered.

| Number $i$ | Desired relative position $p_{s_i}$ | Initial position $p_{s_0}$ | Initial velocity $q_{s_0}$ |
|------------|-----------------------------------|--------------------------|---------------------------|
| 0          | $[0,0,0]^T$                       | $[0,0,0]^T$              | $[0.1,0,1.0]^T$           |
| 1          | $[0,1,0]^T$                       | $[1.2,0,0]^T$            | $[-0.15,0,0]^T$           |
| 2          | $[-\sin(\frac{\pi}{2})\cos(\frac{\pi}{2}),0]^T$ | $[-2,-1.0]^T$           | $[0,0.1,0]^T$             |
| 3          | $[-\sin(\frac{\pi}{6}),-\cos(\frac{\pi}{6}),0]^T$ | $[-1,0,0.7]^T$           | $[0,0.1,0]^T$             |
| 4          | $[\sin(\frac{\pi}{2}),-\cos(\frac{\pi}{2}),0]^T$ | $[-1.9,0,0]^T$           | $[-2,0,0]^T$              |
| 5          | $[\sin(\frac{\pi}{4}),\cos(\frac{\pi}{4}),0]^T$ | $[3.2,-1.4,0]^T$         | $[-0.6,0.5,0]^T$          |

The desired formation is set a square, and the initial position and velocity are shown in Table 1 (random settings). Communication topology of the heterogeneous multi-robot system is shown as:

![Communication topology](image)

Then, using (1), the Laplacian matrix $L$ of followers can be expressed as

$$L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}$$
Formation simulation

With parameters set in Section 4.1, numerical simulation is executed in Matlab, when the control parameter $k=1$. The results are as follows:

Figure 2. The trajectory of the formation system.

Figure 3. The x-direction and the y-direction displacement.

Figure 4. The x-direction and the y-direction velocity.

According to Theorem 1, we can see that the upper bound of time-delays is $\tau_0 = 0.356$. Let
\( \tau_{10} = \tau_{30} = \tau_{25} = \tau_1 = 0.1 \), \( \tau_{12} = \tau_{43} = \tau_{34} = \tau_2 = 0.2 \), \( \tau_{25} = \tau_3 = 0.3 \). Node 0 is globally reachable in the heterogeneous multi-robot system, and the control parameter is \( \xi = 1 \). In this case, the heterogeneous multi-robot system can finally achieve formation control. In the simulation example, the initial states of each robot in the heterogeneous system are given by the legend. Fig.2 shows the trajectory of the formation system. When robots move for more than 30s and the displacement of robots exceeds 3.5m, each robot arrives at the desired relative position shown in Table 1, so the square formation is formed gradually and lastly. Fig.3 shows the x-displacement and the y-displacement of the formation system. Fig.4 shows the x-velocity and the y-velocity of the formation system. From Fig.4, we found that the velocity of each robot changes violently in early time, but it converges to 0.1m/s in the x and y-directions after 30s.

Summary

In this paper, we studied the formation control problem of heterogeneous multi-robot systems with time-delays. Firstly, in order to make the formation control consensus analysis of different order robots at the same time easier, a linear consensus control protocol is proposed based upon the leader-follower mode. Then, the Lyapunov-Razumikhin function is constructed, and the sufficient condition for the formation control problem of the system with time-delays is obtained by the matrix analysis theory. Finally, the theoretical results of analytical model are verified by simulation results through Matlab tools.

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