Mechanism and Lineshapes of Sub-Fourier Resonances

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Subtle internal interference effects allow quantum-chaotic systems to display “sub-Fourier” resonances, i.e. to distinguish two neighboring driving frequencies in a time shorter than the inverse of the difference of the two frequencies. We report experiments on the atomic version of the kicked rotor showing the unusual properties of the sub-Fourier resonances, and develop a theoretical approach (based on the Floquet theorem) explaining these properties, and correctly predicting the widths and lineshapes.

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A major difference between classical and quantum systems is the existence in the latter of interferences between various paths. At long times, there are typically a large number of trajectories and one could expect that, in the average, the contributions of the various paths have uncorrelated phases, leading to vanishingly small interference terms; in this view, quantum and classical transport should be identical. This simple expectation is however too naive. Even for classically chaotic systems where trajectories are very complicated and proliferate rapidly, it is not true that the phases of the various contributions are uncorrelated. A dramatic example is the so-called dynamical localization (DL) taking place in a periodically kicked rotor, described by the following Hamiltonian:

$$H_0 = \frac{p^2}{2} + K \cos \theta \sum_n \delta(t-nT) \quad (1)$$

where $$\theta$$ is the $$2\pi$$-periodic position of the rotor, $$p$$ the conjugate momentum, $$T$$ the period and $$K$$ is proportional to the strength of the kicks). For $$K > 5$$, the classical dynamics is essentially chaotic. Although deterministic, the motion can be, on the average, described as a diffusion in momentum space, with the average kinetic energy growing linearly with time. The situation is completely different for the quantum evolution: after an initial diffusive growth, the average kinetic energy saturates after some break-time.

DL, leading to a complete freeze of the classical diffusion, is a non-trivial effect, as it means that the quantum interferences between classical diffusive trajectories are on the average completely destructive at long time. A fruitful insight is obtained by using the Floquet theorem. The so-called Floquet states $$|\varphi_k\rangle$$ are defined as the eigenstates of the unitary evolution operator $$U(T)$$ associated with $$H_0$$ over one period: $$U(T)|\varphi_k\rangle = \exp(-i\epsilon_k) |\varphi_k\rangle$$ where $$\epsilon_k$$ is the eigenphase. The temporal evolution of any state $$|\psi\rangle$$ after $$n$$ periods is:

$$|\psi(nT)\rangle = [U(T)]^n |\psi(0)\rangle = \sum_k c_k e^{-i\epsilon_k n} |\varphi_k\rangle \quad (2)$$

with $$c_k = \langle \varphi_k | \psi(0) \rangle$$. The evolution of any quantity can be calculated using the Floquet basis, for example:

$$\langle p^2(nT) \rangle = \sum_{k,k'} c_k c_{k'}^* e^{-i(\epsilon_k - \epsilon_{k'})} \langle \varphi_k | p^2 | \varphi_{k'} \rangle \quad (3)$$

The Floquet states $$|\varphi_k\rangle$$ of the chaotic kicked rotor are well known: they are on the average exponentially localized in momentum space around a most probable momentum $$p_k$$, with a characteristic localization length $$\ell$$ [1]. Such a localization – at the origin of DL – is far from obvious and is closely related to the Anderson localization in time-independent disordered one-dimensional systems [2]. As it is typical for classically chaotic systems, the Floquet spectrum is highly sensitive to changes of the parameters, and the momentum wavefunction $$\varphi_k(p)$$ displays large fluctuations.

If the initial state $$|\psi(0)\rangle$$ is well localized in momentum space around, say, zero momentum, Eq. (2) implies that only Floquet states with roughly $$|p_k| \lesssim \ell$$ (“important” Floquet states) will play a significant role in the dynamics. Eq. (3) is a coherent sum over Floquet states. However, as times goes on, non-diagonal interference terms accumulate larger and larger phases. In a typical chaotic system, these phases will be uncorrelated at long times, and interference terms will on the average cancel out, leading to an incoherent sum:

$$\langle p^2 \rangle \approx \sum_k |c_k|^2 \langle \varphi_k | p^2 | \varphi_k \rangle \quad (4)$$

This equation is valid when DL is established. How long does it take for the phases $$n(\epsilon_k - \epsilon_{k'})$$ to be of the order of $$2\pi$$? This can be simply estimated from the level spacing between important Floquet states and turns out to be roughly $$t_{\text{break}} \approx \ell T$$, while $$\langle p^2 \rangle$$ saturates to a value $$\propto \ell^2$$.

DL has been observed in experiments with cold atoms exposed to kicks of a time-periodic, far detuned, standing laser wave [3, 4, 5]. Our experimental setup is described in [6]. Basically, cold cesium atoms are produced in a standard magneto-optical trap. The trap is turned off,
and pulses of a far-detuned (9.2 GHz $\sim$ 1700 $\Gamma$) standing wave (around 90 mW in each direction) are applied. At the end of the pulse series, counter-propagating phase-coherent Raman beams perform velocity-selective Raman stimulated transitions between the hyperfine ground state sublevels $F_g = 4$ and $F_g = 3$. A resonant probe beam is used to estimate the fraction of transfered atoms, thus measuring the population of a velocity class. Repeated measurements allow to reconstruct the atomic momentum distribution $P(p)$.

When the system is exposed to a two-frequency non-periodic driving, no DL is expected. Consider the Hamiltonian:

$$H(r, \lambda) = \frac{p^2}{2} + \frac{K}{T} \cos \theta \sum_n [\delta(t - nT) + \delta(t - nrT - \lambda T)],$$

where $r$ is the ratio of periods and $\lambda$ the initial phase between the two kick sequences. If $r$ is rational, the system is strictly time periodic and DL takes place, but DL is rapidly destroyed around any rational number, as experimentally observed in Fig. 4. Fig. 1 shows the experimentally measured population in the zero-momentum class $P(p = 0)$ after 20 and 100 double kicks as a function $r$. It displays a spectacular peak at $r = 1$ and a sharp decrease on both sides, due to the destruction of DL. There are two surprising features: (i) The resonance is very narrow: after $N$ kicks, it could be argued that the two quasi-periods can be distinguished only if they differ by $1/N$ (in relative value). This would predict a width of the order of $\Delta r = 1/N = 0.01$ for 100 kicks, whereas we experimentally observe 0.0018. (ii) This “sub-Fourier resonance” is not smooth, but has a marked cusp at the maximum. The aim of this paper is to discuss the physical mechanism responsible for the destruction of DL and to give an explanation for these two unexpected features.

Let us now consider the system evolution. Fig. 2(a) shows $(p^2)$ as a function of time, for various $r$ values. Note that we measure $P(p = 0)$ and then deduce $(p^2)$ assuming $(p^2) \propto 1/[P(p = 0)]^2$. For $r = 1$, DL is visible after a break-time of the order of 5-10 periods, as $(p^2)$ saturates to a finite value. For the other values of $r$, the short-time dynamics is similar, but, for $t > t_{\text{break}}$, $(p^2)$ increases slowly, linearly with time. In other words, in the vicinity of the resonance, the quantum transport is not completely frozen, a residual quantum diffusive process takes place, with a diffusion constant much smaller than the classical constant observed at short times. Fig. 2(b) shows the quantum diffusion constant, as a function of $r$. It displays a sharp minimum at $r = 1$ with a characteristic triangular shape. Due to residual spontaneous emission by the kicked atoms, the quantum diffusion constant at $r = 1$ is not strictly zero.

The key point to understand the dynamics of the sys-
tem is to realize that, near the threshold. Note that Floquet states rapidly change when very similar to a periodic doubly-kicked rotor except that the phase $\lambda$ drifts. The evolution operator $U(nT)$ is thus required to significantly modify the Floquet states: $U(r;\lambda_0; nT \rightarrow (n+1)T) = U(\lambda_0 + n(r-1))$, where $\lambda_0$ denotes the initial phase between the two pulse sequences. The total evolution operator from $t = 0$ to $(N-1)T$ can thus be written as the product $\prod_{n=0}^{N-1} U(\lambda_0 + n(r-1))$. For small enough $|r-1|$, the adiabatic approximation can be used, which states that, if the system is initially prepared in a Floquet eigenstate of $U(\lambda_0)$ it remains in an eigenstate of the instantaneous evolution operator $U(\lambda)$, following the continuous deformation of the Floquet eigenstate of the periodic doubly-kicked rotor. This is illustrated in Fig. 3 which shows the Floquet spectrum of the periodic doubly-kicked rotor, obtained by numerically diagonalizing $U(\lambda)$. When $\lambda$ is varied, the energy levels evolve in a complex manner. The complicated “spaghetti” behaviour, characteristic of quantum-chaotic systems, with a large number of avoided crossings (ACs) is directly responsible for the sub-Fourier character of the resonance lines, as we show below.

In our case, the initial state is a linear combination of Floquet eigenstates. In the adiabatic approximation, it remains a linear combination of the instantaneous Floquet states with the same weights (the phases change, but the squared moduli remain constant). As discussed above, the coherences between Floquet states play a role only at short time, and interference terms average to zero after the break-time, Eq. (1). This implies that Eq. (1) remains valid close to the resonance, provided one uses the instantaneous Floquet eigenbasis:

$$\langle p^2(\lambda) \rangle \approx \sum_k |\langle \psi(0)|\varphi_k(\lambda_0)\rangle|^2 \langle \varphi_k(\lambda)|p^2|\varphi_k(\lambda)\rangle. \quad (6)$$

The weights $|\langle \psi(0)|\varphi_k(\lambda_0)\rangle|^2$ are estimated at the initial time ($\lambda = \lambda_0$), while the average values $\langle \varphi_k(\lambda)|p^2|\varphi_k(\lambda)\rangle$ are calculated at the final value $\lambda = \lambda_0 + N(r-1)$. Eq. (4) explains the experimental observations. The fundamental point is that Eq. (4) describes a correlation between the localization properties of the Floquet eigenstates for two values of $\lambda$. The $|\langle \psi(0)|\varphi_k(\lambda_0)\rangle|^2$ term gives important weights to the Floquet states localized near zero momentum. As $\lambda$ moves away from $\lambda_0$, the important Floquet states evolve and, on the average, their center moves away from zero momentum. Thus, one expects $\langle p^2(\lambda) \rangle$ to have a minimum at $\lambda = \lambda_0$ and to rapidly increase on both sides of $\lambda_0$. Since $\lambda - \lambda_0$ is proportional to $r-1$, $\langle p^2 \rangle$ shall present a sharp minimum at $r = 1$; simultaneously, the population in the zero-momentum class shall present a sharp maximum at $r = 1$, which is experimentally observed in Fig. 1.

What determines the linewidth of the resonances? A state $|\varphi_k(\lambda)\rangle$ evolves because of its ACs with other Floquet eigenstates. Tiny ACs may be crossed diabatically, leading to a large modification of the state. On the average, the scale $\Delta \lambda = \lambda - \lambda_0$ on which $|\varphi_k(\lambda)\rangle$ loses the localization property of $|\varphi_k(\lambda_0)\rangle$ is the distance to the next AC. We immediately deduce that the width $\Delta f$ of the sub-Fourier line is such that:

$$NT\Delta f = 2\Delta \lambda. \quad (7)$$

Eq. (7) also explains why chaos is necessary to observe sub-Fourier resonances. If the classical dynamics is regular, the Floquet eigenstates evolve smoothly with the parameter $\lambda$: a change in $\lambda$ of the order of one is thus required to significantly modify the Floquet states: $2\Delta \lambda \approx 1$, which is the Fourier limit. On the contrary, in a classically chaotic system, the level dynamics is known to be complex with plenty of ACs (Fig. 3). $\Delta \lambda$ is then much smaller than unity, leading to highly sub-Fourier resonances (up to a factor 37 observed in Fig. 4). Eq. (7) also predicts the linewidth to be inversely proportional to the temporal length of the kick sequence beyond the break-time (i.e. the sub-Fourier character is independent of $N$), as numerically observed in Fig. 3. The value of $\Delta \lambda$ depends on the detailed dynamics of the system. It can be roughly estimated by visual inspection of the quasi-energy level dynamics, Fig. 3. The level spacing between important Floquet states is the inverse of the number of such states, itself roughly equal to the localization length, of the order of 5 for the present set of parameters. The typical relative slope of neighbor levels in Fig. 4 is of the order of 4, thus predicting $\Delta \lambda$ of the order of 0.05 and a “factor 10”
sub-Fourier line, about twice the experimentally observed factor (the difference is due to the transverse profile of the laser mode leading to spatial inhomogeneities in $K_i$).

From Eq. 6 and the previous analysis, it is expected that $\langle p^2 \rangle$ significantly increases from its minimum value at $\lambda = \lambda_0$ as $\lambda$ varies to $\lambda_0 + \Delta \lambda$. It seems reasonable to assume that this increase is initially quadratic in $\lambda - \lambda_0$ [10]. The situation here is more complicated because Floquet states are localized in momentum space. ACs between Floquet states localized around the same momentum space are typically much smaller and scale like $\exp(-L/\ell)$. There is thus a very broad distribution of AC widths, with a large number of tiny ACs. A tiny AC typically extends over a small $\lambda$ interval and thus tends to produce small values of $\Delta \lambda$. The increase of $\langle p^2 \rangle$ with $\lambda$ thus depends on the number of small ACs encountered. In the presence of exponential localization, the AC density scales with $L/\ell$, which is itself a linear function of time and $|r - 1|$. Thus, our model correctly predicts two non-trivial properties: $\langle p^2 \rangle$ increases linearly with time and the corresponding diffusion constant is proportional to $|r - 1|$.

There will always be some degree of nonadiabaticity. Whatever small $|r - 1|$ is, tiny enough ACs will be crossed diabatically. This puts a lower bound on the size of the ACs effectively participating in the quantum dynamics of the system and produces a rounding of the top of the sub-Fourier resonance line, which is too small to be seen in the experiment after 100 kicks, but is easily visible after 20 kicks, Fig. 1.

Our approach concentrates on the immediate vicinity of the resonance. What happens in the wings of the sub-Fourier line? Eq. (6) indicates that this depends on the residual correlation between $|\varphi_k(\lambda_0)|$ and $|\varphi_k(\lambda)|$ for $|\lambda - \lambda_0| > \Delta \lambda$. A quantitative answer to this question cannot be given, as there is no typical AC width. However, it seems clear that the quantum diffusion constant does not exceed the classical one, which corresponds to the killing of all interference terms. Random Matrix Theory tells us that this type of parametric correlation usually decays algebraically with $\lambda - \lambda_0$. We thus propose the following ansatz:

$$\langle p^2(nT) \rangle = \langle p^2 \rangle_{DL} + D_{cl} \frac{|r - 1| + \Delta \lambda / t_{break}}{|r - 1| + \Delta \lambda / t_{break}} nT$$

where $D_{cl}$ is the classical diffusion constant and $\langle p^2 \rangle_{DL}$ the saturation value of $p^2$ due to DL. This equation fits (using $P(p = 0) \propto \langle p^2 \rangle^{-1/2}$) very well the experimental curves in Fig. 1 reproducing the linear behaviour at the center of the resonance and the classical diffusion in the wings. Nevertheless, because of the laser inhomogeneities, the parameters are fitted, not extracted from the previous analysis.

In summary, we have developed a theoretical approach for the mechanism of sub-Fourier resonances, which correctly predicts the unexpected observed features. Deviations from exact periodicity are treated in the framework of the adiabatic approximation for the Floquet spectrum of the system, which thus goes beyond a perturbative approach. The dynamics is governed by instantaneous Floquet eigenstates, which are non-trivial objects, as they need to have very well defined internal (i.e. between the various parts of the wavefunction for each Floquet state) phase coherence to be stationary states of the strictly periodic system, but at the same time, the inter-state coherences do not play any role (beyond the break-time, Floquet states effectively add incoherently). We are not aware of any other quantum system where the dynamics is dominated by such an incoherent sum of internally extremely coherent states. It shows that the role of interferences in quantum mechanics is far from obvious, and can produce unexpected behaviors.

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* URL: [http://www.phlam.univ-lille1.fr/atfr/cq](http://www.phlam.univ-lille1.fr/atfr/cq)

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