DECOHERENCE-INSENSITIVE QUANTUM COMMUNICATION BY OPTIMAL $C^*$-ENCODING

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ABSTRACT. The central issue in this article is to transmit a quantum state in such a way that after some decoherence occurs, most of the information can be restored by a suitable decoding operation. For this purpose, we incorporate redundancy by mapping a given initial quantum state to a messenger state on a larger-dimensional Hilbert space via a $C^*$-algebra embedding. Our noise model for the transmission is a phase damping channel which admits a noiseless or decoherence-free subspace or subsystem. More precisely, the transmission channel is obtained from convex combinations of a set of lowest rank yes/no measurements that leave a component of the messenger state unchanged. The objective of our encoding is to distribute quantum information optimally across the noise-susceptible component of the transmission when the noiseless component is not large enough to contain all the quantum information to be transmitted. We derive simple geometric conditions for optimal encoding and construct examples.

1. Introduction

The use of quantum systems as a medium to carry information enables algorithms and communication techniques not possible in the purely classical setting. While there is considerable excitement about these new capabilities, their implementation poses severe theoretical and experimental challenges. One of the major impediments to large-scale experimental advances is the decoherence of quantum systems as they evolve in time. Indeed, finding efficient methods for “quantum error correction” is of central importance in the fields of quantum computation, cryptography and communication.

A fundamental, “passive” technique for quantum error correction grew out of the recognition that certain error models contain symmetries that can be used to hide encoded qubits from the overall noise of the model [1–7]. This scheme relies on the identification of what are now known as noiseless or decoherence-free subspaces or subsystems. (For convenience we will just use the noiseless terminology.) A number of experimental advances have been based on this approach [8–11]. Noiseless subsystems and subspaces have recently received wider attention, such as in quantum information processing.

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and cryptography [12–14], and in quantum cosmology and gravity [15–17].
Moreover, a structure theory is now emerging [18–23] which shows, among
other things, how such subsystems and subspaces can be identified [22, 23].

The basic problem we address in this paper is, given a certain class of
decoherence error models, how to distribute quantum information optimally
across the noise-susceptible component of the transmission when the noiseless
component is not large enough to encode all the information to be trans-
mited. Our analysis is motivated by recent work [24–28] on the construction
and use of frames to incorporate redundancy in classical signal transmission.

For the purposes of this paper, a quantum communication framework is
given by triples from a collection of quantum channels. Each triple contains
the maps for encoding, transmission and reconstruction of quantum states.

We exclusively consider encoding maps that are trace-preserving $\mathcal{C}^*$-al-
bracket{bra embeddings in order to preserve the structure of the set of encoded quan-
tum states. We call such maps $\mathcal{C}^*$-encodings. Consequently, the encoding
and the subsequent reconstruction are quantum channels, that is, completely
positive and trace-preserving (or at least non-increasing for the trace). The
encoding channel incorporates redundancy by mapping a quantum state on
a given Hilbert space to a “messenger” state on a larger-dimensional Hilbert
space.

It is common in the context of quantum error correction to model noise in
transmissions by a so-called interaction algebra $\mathcal{A}$ and search for states that
are in the commutant of $\mathcal{A}$. The interaction algebra we consider is given by
the commutative set of operators $\mathcal{A} = \{D \otimes I : D \text{ diagonal}\}$ on a Hilbert
space $\mathbb{C}^m \otimes \mathbb{C}^l$, admitting a noiseless subsystem in the second component.
However, we do not consider the entire commutative interaction algebra $\mathcal{A}$,
but rather focus on a “minimal” set $\mathcal{Q}^{(1)}$ of error models for the transmission.
This set contains all nontrivial lowest-rank projective measurements and
their convex combinations. Such errors model generalized phase dampin-
g, that is, decoherence due to minimal interaction with a macroscopic sys-
tem that performs with certain probabilities one among several possible yes/no
measurements on our system. The commutativity means that the outcomes
of the measurements obey the rules of classical Boolean logic. Apart from
naturally occurring decoherence due to interactions with an environment,
the motivation for considering this type of error is that such measurements
may be used to remove corrupted information in a quantum state after the
occurrence of a quantum error has been detected.

Minimal projections in the interaction algebra $\mathcal{A}$ are tensor products
of diagonal matrix units $\{E_{jj}\}_{j=1}^m$ and the identity, henceforth denoted as
$Q_j = E_{jj} \otimes I$. Accordingly, a transmission channel which performs a mini-
mal projective measurement of a messenger state $M$ is $\mathcal{E}_j : M \mapsto Q_jMQ_j +$
$Q^\perp_j M Q^\perp_j$, where $Q^\perp_j$ complements the orthogonal projection $Q_j$ to the identity, $Q_j + Q^\perp_j = I$. We denote the phase damping channels obtained by convex combinations of such measurements as

$$Q^{(1)} = \left\{ \sum_{j=1}^m p_j E_j(M) \mid \text{all } p_j \geq 0, \sum_{j=1}^m p_j = 1 \right\}.$$  

In the following, we shall use $B(\mathbb{C}^d)$ to denote the algebra to be encoded. This algebra is embedded via the encoding map $\Phi$ into the higher-dimensional algebra $B(\mathbb{C}^m \otimes \mathbb{C}^l) = B(\mathbb{C}^m) \otimes B(\mathbb{C}^l)$. The subsequent decoding is a “blind” reversal of the encoding operation.

For our purposes, an optimal $C^*$-encoding $\Phi$ minimizes $\sup_{E} e(\Phi, E)$, the supremum of the Hilbert-Schmidt norms for the reconstruction error occurring among all quantum states and all noisy transmission channels $E$ in the convex set $Q^{(1)}$. While other measures for distortion are also used in quantum information, we have chosen the Hilbert-Schmidt norm because it is most closely related to the geometry of trace-preserving $C^*$-algebra embeddings.

Our main result is that if $l < d$, a $C^*$-encoding $\Phi$ satisfies

$$\sup_{E \in Q^{(1)}} e(\Phi, E)/d \geq \begin{cases} 2 - 2d/ml, & d/ml \in [0, 1/2] ; \\ (8mdl - 2m^2l^2 - 4d^2)^{-1/2}, & d/ml \in (1/2, 1]. \end{cases}$$

In addition, we characterize the case of equality as that for which the encoding map is constructed from what we call a family of “uniformly weighted projections resolving the identity”. In many cases we show how such families can be constructed as an extension of recent work on classical signal encoding problems [25]. However, we leave the problem of determining the possible existence of these families in complete generality for investigation elsewhere. Finally, for all $C^*$-encodings that are optimal for $Q^{(1)}$, we derive an inequality for transmission channels with next-to-minimal decoherence and characterize the case of equality as that for which all pair sums of the underlying projections have equal operator norm. This geometric condition on the ranges of the projections generalizes the notion of equiangular lines.

This paper is organized as follows: In Section 2 we describe the concept of a quantum communication framework and examine the structure of this framework in the presence of a noiseless subsystem. In Section 3, we obtain precise estimates on the reconstruction error due to a noisy transmission and use this to characterize optimal encoding in terms of uniformly weighted projective resolutions of the identity. Section 4 treats optimality for secondary decoherence errors in the uniform encoding case. Finally, we comment on a generalization that includes the noiseless subspace case, with similar results on the reconstruction errors and optimal encoding schemes.
2. Quantum Communication Framework

We recall that for the purposes of this paper, a quantum communication framework is given by three quantum channels: encoding, transmission, and decoding. Hereby, a quantum channel is understood to be a completely positive, trace preserving (or at least trace non-increasing) linear map between $C^*$-algebras. We point the reader to [29–31] as entrance points into the literature on completely positive maps and their use in quantum computation and information.

2.1. $C^*$-Encoding and Decoding. Apart from being quantum channels, we require that these maps preserve the algebraic structure of the quantum states to be encoded.

**Definition 2.1.** Let $\mathcal{H}$ and $\mathcal{K}$ be finite-dimensional Hilbert spaces. We say that a map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is a $C^*$-encoding of $B(\mathcal{H})$ if $\Phi$ is a trace-preserving $*$-monomorphism. The associated decoding map is the Hilbert-Schmidt dual $\Phi^* : B(\mathcal{K}) \to B(\mathcal{H})$.

**Remarks 2.2.** Algebra embeddings of the type introduced here have also been used for a unified treatment of a hybrid quantum/classical memory [32].

In the context of our quantum communication framework, the most important property of the encoding is that it preserves the Hilbert-Schmidt inner product. Thus $\Phi^*$ is a left inverse of $\Phi$, and encoding followed by decoding is the identity map, $\Phi^* \Phi = \text{id}$. Moreover, the dual $\Phi^*$ is completely positive and non-increasing for the trace.

Another consequence of $\Phi$ being a Hilbert-Schmidt isometry is that, instead of performing a measurement after decoding a transmitted state, one may simply encode the measurement and apply it directly to the transmitted state.

Furthermore, the benefit of encoding with a $C^*$-algebra embedding is that a quantum algorithm performed on the initial state, implemented by conjugation with a unitary operator, could be performed either prior to encoding or afterwards.

2.2. Error Model for Transmission. The transmission of an encoded state $M \in B(\mathcal{K})$ is described by a quantum channel. When $\dim \mathcal{K} < \infty$, such a map has a representation of the form $\mathcal{E}(M) = \sum_a E_a ME_a^*$ for a finite set of operators $\{E_a\} \subset B(\mathcal{H})$. These operators $\{E_a\}$ are said to form an “error model” for the channel. It is the impact of such error operators that must be overcome to safely transmit quantum information through the channel $\mathcal{E}$.

As discussed in the introduction (see Eq. (1)), here we consider phase damping channels given by convex combinations of commuting projective measurements.
Definition 2.3. We call a quantum channel \( \mathcal{E} \) a generalized phase-damping channel on \( B(\mathcal{K}) \) if there is a commutative set of projections \( \{Q_j\}_{j=1}^m \subset B(\mathcal{K}) \) and a probability vector \( p \in \mathbb{R}_+^m, \sum_j p_j = 1 \) such that for \( M \in B(\mathcal{K}) \),
\[
\mathcal{E}(M) = \sum_{j=1}^m p_j (Q_j M Q_j + Q_j^\perp M Q_j^\perp),
\]
where each \( Q_j^\perp \) is the projection on the orthogonal complement of the range of \( Q_j \).

We define the overall reconstruction error of the framework as follows.

Definition 2.4. Let \( \Phi : B(\mathcal{H}) \to B(\mathcal{K}) \) be a \( C^* \)-encoding as in Definition 2.1, and let \( \mathcal{E} : B(\mathcal{K}) \to B(\mathcal{K}) \) be a completely positive, trace preserving map on the Hilbert space \( \mathcal{K} \). We then define the reconstruction error caused by \( \mathcal{E} \) in the course of transmitting a state \( W \in B(\mathcal{H}) \) as
\[
Y = W - (\Phi^* \circ \mathcal{E} \circ \Phi)(W)
\]
and the maximum of the Hilbert-Schmidt norm of \( Y \) over the set of all states \( \{W \geq 0, \text{tr}W = 1\} \) as the worst-case error norm
\[
e(\Phi, \mathcal{E}) = \max_W (\text{tr}[YY^*])^{1/2}.
\]

Remark 2.5. The Hilbert-Schmidt norm is not directly motivated by information-theoretic considerations. However, in the situation considered here, it is a natural way to quantify the impact of an error \( \mathcal{E} \) in the transmission with \( \Phi^* \circ \mathcal{E} \circ \Phi \). In fact, the Hilbert-Schmidt norm is non-increasing under any generalized phase-damping channel \( \mathcal{E} \). This follows from the convexity of the norm and from the case of a single projection \( Q \) and a state \( M \),
\[
\text{tr}[(QMQ + Q^\perp MQ^\perp)^2] = \text{tr}[QM^2Q + Q^\perp M(Q^\perp)^2 MQ^\perp] \\
\leq \text{tr}[QM^2Q + Q^\perp M^2Q^\perp] = \text{tr}[M^2].
\]
Since this norm is also non-increasing when \( \Phi^* \) is applied to \( \mathcal{E}(M) \), we can use it to measure the amount of corruption caused by the error \( \mathcal{E} \) and the subsequent decoding. When calculating the Hilbert-Schmidt norm of the maximal reconstruction error, we thus bound the maximal extent of corruption for any state.

2.3. Encoding in the Presence of Noiseless Subsystems. Consider a Hilbert space \( \mathcal{K} \) for a quantum system and a channel \( \mathcal{E} : B(\mathcal{K}) \to B(\mathcal{K}) \). Suppose we have a decomposition \( \mathcal{K} = (\mathcal{K}^{(1)} \otimes \mathcal{K}^{(2)}) \oplus \mathcal{V} \), so \( \mathcal{V} = (\mathcal{K}^{(1)} \otimes \mathcal{K}^{(2)})^\perp \), and for all states \( W^{(1)} \) and \( W^{(2)} \) on \( \mathcal{K}^{(1)} \) and \( \mathcal{K}^{(2)} \) there is a state \( S^{(1)} \) on \( \mathcal{K}^{(1)} \) with
\[
\mathcal{E}(W^{(1)} \otimes W^{(2)} \oplus 0) = S^{(1)} \otimes W^{(2)} \oplus 0.
\]
Then the second component of the tensor product is said to be a *noiseless subsystem* (or *noiseless subspace* in the case dim \( K^{(1)} = 1 \)) for \( \mathcal{E} \). The terminology “decoherence-free” subspace and subsystem is also quite common. This definition ensures that quantum information encoded in the second component of the tensor product is unaffected by the noise of the channel \( \mathcal{E} \).

For succinctness, we only treat the case \( V = \{ 0 \} \) here. Remark 4.5 sketches how the results generalize to the case dim \( V > 0 \).

For our transmission channels, the second component of the tensor product decomposition \( \mathbb{C}^N = \mathbb{C}^m \otimes \mathbb{C}^l \) used for the transmission is noiseless, whereas the first component may lose its off-diagonal entries. If \( l \geq d \), then we could simply encode \( W \) as \( M = \frac{1}{m} I \otimes W \), which is unaffected by any decoherence in the first component. So let us assume \( l < d \), which means the dimension of the second component is too small to contain all the information given by an arbitrary state \( W \) on \( \mathbb{C}^d \).

### 2.3.1. \( \mathcal{C}^* \)-encodings and Positive Operator Valued Measures.

Let us describe a useful concrete description of \( \mathcal{C}^* \)-encodings in terms of associated factorizations of POVMs.

**Lemma 2.6.** Given a \( \mathcal{C}^* \)-encoding \( \Phi : B(\mathbb{C}^d) \to B(\mathbb{C}^m \otimes \mathbb{C}^l) \), then there exists a set of so-called coordinate operators \( \{ V_j : \mathbb{C}^d \to \mathbb{C}^l \} \) resolving the identity \( \sum_{j=1}^m V_j^* V_j = I \) such that \( \Phi \) has the form

\[
\Phi : B(\mathbb{C}^d) \to B(\mathbb{C}^m \otimes \mathbb{C}^l) = B(\bigoplus_{j=1}^m \mathbb{C}^l)
\]

\[
F \mapsto G = (G_{ij})_{i,j=1}^m, G_{ij} = V_i W V_j^*.
\]

Moreover, the action of the decoding map is given by

\[
G \mapsto \Phi^*(G) = \sum_{i,j=1}^m V_i^* G_{ij} V_j.
\]

**Proof.** As \( \Phi \) is a trace-preserving \( \mathcal{C}^* \)-monomorphism of a simple, finite-dimensional \( \mathcal{C}^* \)-algebra \( B(\mathbb{C}^d) \), it follows from the representation theory for such algebras [33] that there is an isometry \( \tilde{V} : \mathbb{C}^d \to \mathbb{C}^m \otimes \mathbb{C}^l = \bigoplus_{j=1}^m \mathbb{C}^l \), \( \tilde{V}^* \tilde{V} = I \), which for all \( F \in B(\mathbb{C}^d) \) gives

\[
\Phi(F) = \tilde{V} F \tilde{V}^*.
\]

We may view \( \tilde{V} \) as a column matrix of coordinate operators \( V_j : \mathbb{C}^d \to \mathbb{C}^l \), \( 1 \leq j \leq m \),

\[
\tilde{V} = (V_1 \ V_2 \ \cdots \ V_m)^t : \mathbb{C}^d \longrightarrow \bigoplus_{j=1}^m \mathbb{C}^l ; \quad f \longmapsto (V_1 f \ V_2 f \ \cdots \ V_m f)^t.
\]

The linearity of \( \tilde{V} \) ensures that each \( V_j \) is linear. Also note that \( \text{rank}(V_j) \leq l \), \( 1 \leq j \leq m \).
Thus, the action of $\Phi$ is given by $G = \Phi(F) = \tilde{V}F\tilde{V}^*$. We may view $G$ as an operator on $\mathbb{C}^m \otimes \mathbb{C}^l$ expressed in block matrix form

$$G = (G_{ij})_{i,j=1}^m \text{ with } G_{ij} = V_i^*W_jV_j^*.$$  \hfill (5)

Now the Hilbert-Schmidt dual of $\Phi$ can be identified as $G \mapsto \Phi^*(G) = \tilde{V}^*G\tilde{V} = \sum_{i,j=1}^m V_i^*G_{ij}V_j$.

Remarks 2.7. The operators $\{V_j\}$ define a “positive operator valued measure” (POVM), $\{A_j\}$, where $A_j = V_j^*V_j$ and

$$I = \tilde{V}^*\tilde{V} = \sum_{j=1}^m A_j.$$  \hfill (6)

Conversely, if we start with a POVM on $\mathbb{C}^d$ given by $\{A_j\}$ such that $A_j \geq 0$, each $A_j$ has rank at most $l$, and $\sum_{j=1}^m A_j = I$, upon factoring $A_j = V_j^*V_j$ one can define $\tilde{V}$ as in Eq. (4) and a $C^*$-encoding as in Eq. (3).

Hence the $C^*$-encodings of Definition 2.1 are completely characterized by factorizations of POVMs in this way, and we shall use this concrete form in the analysis below. A special case is of central importance in this paper.

Definition 2.8. Let $\mathcal{H}$ be a $d$-dimensional complex Hilbert space and let $\{A_j\}_{j=1}^m \subset B(\mathcal{H})$ be a POVM on $\mathcal{H}$. If each operator is given by $A_j = k_jP_j$ with a non-negative coefficient $k_j$ and a projection operator $P_j = P_j^*P_j$, then we call $\{A_j\}$ a set of weighted projections resolving the identity. If $k_j = k = d/\sum_{j=1}^m \text{tr} P_j$ for all $j \in \{1, 2, \ldots, m\}$, we call $\{kP_j\}_{j=1}^m$ a set of uniformly weighted projections resolving the identity.

Remarks 2.9. (i) Suppose all operators in a POVM $\{A_j\}_{j=1}^m$ have a maximal rank $l$, and we denote $P_j$ the projection onto the range of each $A_j$, and abbreviate the operator norms $k_j = \|A_j\|$. Then from $k_jP_j \geq A_j$ and from taking traces on both sides of Eq. (6), we have

$$l \sum_{j=1}^m k_j \geq \sum_{j=1}^m \text{tr} A_j = d.$$  \hfill (7)

Equality holds in this inequality if and only if $A_j = k_jP_j$, and $P_j$ has rank $l$ for all $1 \leq j \leq m$.

(ii) Weighted projections giving a resolution of the identity and their robustness against perturbations have been investigated by Casazza and Kutyniok in the context of frames and distributed processing, where they are known as Parseval frames for subspaces [34] or fusion frames [35]. The $C^*$-encoding derived from this special case of POVMs is of central importance in this paper.
(iii) If $\sum_{j=1}^{m} \text{tr } P_j > d$, then the ranges of the $P_j$’s are not mutually orthogonal. However, this does not imply that the ranges of the $P_j$’s have to intersect nontrivially, see the example below.

(iv) Choosing orthonormal bases for the range of each $P_j$ and scaling the basis vectors spanning the range of each $P_j$ by a factor $\sqrt{k_j}$ would yield a Parseval frame of $N = \sum_{j=1}^{m} \text{tr } P_j$ vectors for the space $\mathbb{C}^d$. If the weights are uniform, so is the frame.

(v) Given a set of weighted projections resolving the identity such that $\sum_{j=1}^{m} \text{tr } P_j > d$, we can replace each $P_j$ by $P_j' = I - P_j$ and each coefficient $k_j$ by $k_j' = k_j/(\sum_{n=1}^{m} k_n - 1)$ in order to arrive at a complementary resolution of the identity, $\sum_{j=1}^{m} k_j' P_j' = I$.

2.3.2. Examples. We sketch the construction of uniformly weighted projections resolving the identity.

Example 2.10. It is fairly straightforward to construct examples of weighted projections resolving the identity for which all $\{k_j P_j\}_{j=1}^{m}$ are rank-one and $m > d$. To this end, we take a rank-$d$ orthogonal projection $G = G^* G$ on $\mathbb{C}^m$ and factor it as $G = V V^*$, where $V$ is an isometry $V : \mathbb{C}^d \rightarrow \mathbb{C}^m$. Now denote the $m$ column vectors of the matrix representation for $V^*$ in the standard basis as $\{f_j\}_{j=1}^{m}$, choose each $P_j$ to be the projection on the one-dimensional subspace of $\mathbb{C}^d$ spanned by $f_j$, and let the set of weights be $\{k_j = \|f_j\|^2\}$. It is then straightforward to verify that $\sum_{j=1}^{m} k_j P_j = I$.

Holmes and Paulsen [25, Remark 1.1] show how to make the weights uniform by rotating amongst the frame vectors.

The next example describes how rank-one projections may be used to construct higher-rank weighted projections which resolve the identity.

Example 2.11. If the Hilbert space is a tensor product $\mathbb{C}^d = \mathbb{C}^l \otimes \mathbb{C}^q$, so $d = lq$, and there is a set of weighted rank-one projections $\{k_j \Pi_j\}_{j=1}^{m}$ resolving the identity on the second component $\mathbb{C}^q$, then $\{k_j P_j = k_j I \otimes \Pi_j\}$ is a set of weighted rank-$l$ projections resolving the identity on $\mathbb{C}^d$.

If all weights are uniform in the resolution of the identity for the second component, the same holds for the tensor product.

However, there are examples of weighted projections realized on tensor product spaces that are not of the type $\{I \otimes \Pi_j\}$.

Example 2.12. We now describe a set $\{kP_1, ..., kP_m\}$ of $m$ uniformly weighted projections resolving the identity in the case that $d = 2l$ is even.

This example has the property that $\text{rk}(P_j) = l$ for every $j$ and if $i \neq j$, then the ranges of $P_i$ and $P_j$ intersect only in the zero vector.
To establish the claimed properties, we identify in the usual way
\[ B(\mathbb{C}^d) = B(\mathbb{C}^l \oplus \mathbb{C}^l) = B(\mathbb{C}^2) \otimes B(\mathbb{C}^l); \]
that is, as \(2 \times 2\) block matrices, each of whose entries is an \(l \times l\) matrix. Let
\[
P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U_j = \begin{pmatrix} C_j & -S_j \\ S_j & C_j \end{pmatrix}
\]
for all \(1 \leq j \leq m\), where each \(C_j\) and \(S_j\) is an \(l \times l\) diagonal matrix of the form
\[
C_j = \sum_{i=1}^{l} \cos(\theta_{i,j})E_{ii}, \quad S_j = \sum_{i=1}^{l} \sin(\theta_{i,j})E_{ii},
\]
where \(\theta_{i,j} = \frac{\pi(i-j)}{m}, 1 \leq j \leq m, 1 \leq i \leq l\), and \(\{E_{ii}\}_{i=1}^{l}\) are the diagonal matrix units. Using these matrices, we specify our set of \(m\) projections of rank \(l\), by setting
\[
P_j = U_j^*PU_j = \begin{pmatrix} C_j^2 & -C_jS_j \\ -C_jS_j & S_j^2 \end{pmatrix},
\]
for all \(1 \leq j \leq m\), so that \(\text{rk}(P_j) = l\) for all \(j \in \{1, 2, \ldots m\}\).

Note that in this case, \(k = d/\sum_j \text{tr}(P_j) = d/(ml) = 2/l\). To verify that \(\sum_{j=1}^{m} P_j = \frac{m}{2}I_d\), observe that \(\sum_{j=1}^{m} \cos^2\left(\frac{\pi(i-j)}{m}\right) = \frac{m}{2}\), while \(\sum_{j=1}^{m} \cos(\frac{\pi(i-j)}{m})\sin(\frac{\pi(i-j)}{m}) = 0\). Finally, to verify that the ranges of a pair \(P_i\) and \(P_j\), \(i \neq j\), do not intersect nontrivially, we show \(\|P_i + P_j\| < 2\). Since the operator norm is invariant under conjugation with unitaries, we observe
\[
\|P_i + P_j\| = \|P + U_iU_j^*PU_jU_i^*\|.
\]
Notice that for all \(1 \leq i, j \leq m\), we have \(U_iU_j^* = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\), where
\[
C = \cos\left(\frac{\pi(i-j)}{m}\right)I \quad \text{and} \quad S = \sin\left(\frac{\pi(i-j)}{m}\right)I.
\]
Thus, it will be enough to verify that the matrix
\[
P + U_iU_j^*PU_jU_i^* = \begin{pmatrix} I + C^2 & -CS \\ -CS & S^2 \end{pmatrix}
\]
has norm strictly less than 2. However, this last matrix decomposes into a direct sum of \(2 \times 2\) Hermitian matrices of the same form and one can check that the eigenvalues of such a matrix are, \(1 \pm \cos(\theta)\), with \(\theta = \frac{\pi(i-j)}{m}\). Thus, for \(i \neq j, 1 \leq i, j \leq m\), the angle \(\theta\) is not an integer multiple of \(\pi\), so the eigenvalues of these Hermitian matrices, and hence their norms, will be bounded above by 2. Indeed, a tight upper bound on the norm is \(1 + \cos(\frac{\pi}{m})\).
3. Uniformly weighted projections as optimal encoders for minimal decoherence

The main theorem of this section characterizes optimal encoding in our communication framework. We prepare the result with three lemmas.

**Lemma 3.1.** Given a nonnegative operator $A$ with eigenvalues $\{\alpha_r\} \subseteq [0,1]$ and at least one nonzero $\alpha_r$, then a vector $\rho$ with $\rho_r \geq 0$, $\sum_r \rho_r = 1$ maximizes

$$\sum_{r,s} ((1 - \alpha_r)\alpha_s + (1 - \alpha_s)\alpha_r)^2 \rho_r \rho_s$$

if and only if at most two entries of $\rho$ are nonzero.

**Proof.** This is a standard Lagrange-multiplier argument. Suppose $\rho$ is a maximizing vector and there is an index $r$ such that $\rho_r = 1$, then the assertion is true. On the other hand, assume all $\rho_r < 1$. Then there exists a $\lambda \in \mathbb{R}$ such that for any $r$ with $\rho_r \neq 0$,

$$\sum_s ((1 - \alpha_r)\alpha_s + (1 - \alpha_s)\alpha_r)^2 \rho_s = \lambda .$$

Since the left hand side is quadratic in $\alpha_r$, there can be at most two different values for $\alpha_r$ that satisfy this equation. Consequently, $\rho_r$ can be nonzero for at most two $r$. \qed

**Lemma 3.2.** Let $A$ be a nonnegative operator with eigenvalues $\{\alpha_r\} \subseteq [0,1]$, the maximum of which is given by $\kappa = \|A\|$. Let the function $g : [0, \kappa]^2 \to \mathbb{R}$ be defined by $g(x, y) = ((1 - kx)y + (1 - ky)x)^2$. Then the maximum of

$$G(p, x, y) = p^2 g(x, x) + 2p(1 - p)g(x, y) + (1 - p)^2 g(y, y)$$

over $p \in [0,1]$ and $\{(x, y) : \kappa \geq x \geq y \geq 0\}$ is achieved for some $p$ when $x = \kappa$ and $y = 0$.

**Proof.** We note that the auxiliary function

$$h(x, y) = x + y - 2xy$$

is harmonic in $\mathbb{D} = [0, \kappa]^2$ and piecewise linear when restricted to the boundary $\partial \mathbb{D}$. Denote

$$H(p, x, y) = p^2 h(x, x) + 2p(1 - p)h(x, y) + (1 - p)^2 h(y, y) .$$

Let $\eta_{x,y}$ be the so-called harmonic measure on $\partial \mathbb{D}$ for the point $(x, y)$. That is, for any harmonic function $u$ that is continuous on $\mathbb{D}$, it gives $u(x, y) = \int_{\partial \mathbb{D}} u \, d\eta_{x,y}$. Using a suitable convex combination of harmonic measures for the points $(x, x), (x, y), (y, x)$ and $(y, y)$, we construct $\mu_{p,x,y}$ supported on the boundary $\partial \mathbb{D}$ such that $\int_{\partial \mathbb{D}} h \, d\mu_{p,x,y} = H(p, x, y)$. Since $h$ is piecewise linear
on ∂D, this measure can be replaced by weights \( w_{11}, w_{12}, w_{21}, w_{22} \geq 0 \) at the corners. These weights give
\[
\begin{align*}
  w_{11} h(\kappa, \kappa) + w_{12} h(\kappa, 0) + w_{21} h(0, \kappa) + w_{22} h(0, 0) &= H(p, x, y) \\
  w_{11} &= \frac{1}{\kappa^2} (px + (1 - p)y)^2, \\
  w_{22} &= \frac{1}{\kappa^2} (p(\kappa - x) + (1 - p)(\kappa - y))^2,
\end{align*}
\]
and can be determined as
\[
\begin{align*}
  w_{12} &= w_{21} = \frac{1}{\kappa^2} \left[ p^2 x(\kappa - x) + p(1 - p)(y(\kappa - x) + x(\kappa - y)) \\
  &\quad + (1 - p)^2 y(\kappa - y) \right].
\end{align*}
\]
It can be verified that \( w_{12} = \sqrt{w_{11} w_{22}} \) and that \( w_{11} + 2w_{12} + w_{22} = 1 \).

We observe that \( g = h^2 \) is sub-harmonic. Consequently, Jensen’s inequality yields
\[
G(p, x, y) \leq \int_{\partial D} g \, d\mu_{p,x,y}
\]
and since the restriction of \( g \) to each side of \( \partial D \) is convex, applying Jensen’s inequality again gives us
\[
G(p, x, y) \leq w_{11} g(\kappa, \kappa) + w_{12} g(\kappa, 0) + w_{21} g(0, \kappa) + w_{22} g(0, 0) = G(\sqrt{w_{11}}, \kappa, 0).
\]
Thus, \( G \) assumes its maximum when the point \((x, y)\) is on the boundary of the square \( D \). \( \square \)

**Lemma 3.3.** Let \( \Phi : B(\mathbb{C}^d) \to B(\mathbb{C}^m) \otimes B(\mathbb{C}^l) \) be a \( C^* \)-encoding and let \( \{V_j\}_{j=1}^m \) be the associated coordinate operators of rank \( l < d \) as defined in Lemma 2.6. Let \( Q_n = E_{nn} \otimes I \), for \( 1 \leq n \leq m \), be the orthogonal projection onto the \( n \)th copy of \( \mathbb{C}^l \) in \( \mathbb{C}^m \otimes \mathbb{C}^l \), and denote \( E_n \) the projective measurement applied to any state \( M \in B(\bigoplus_{j=1}^m \mathbb{C}^l) \),
\[
E_n(M) = Q_n M Q_n + Q_n^\perp M Q_n^\perp.
\]
Then the maximum of the Hilbert-Schmidt norm \( (\text{tr}[YY^*])^{1/2} \) of the reconstruction error \( Y = W - \Phi^*(E_n(\Phi(W))) \) over all states \( W \) only depends on the operator norm \( k_n = \|V_n^*V_n\| \). Specifically, we have
\[
(14) \quad e(\Phi, E_n) = \max_W (\text{tr}[YY^*])^{1/2} = 2k_n(1 - k_n) \quad \text{when} \quad k_n \leq 1/2,
\]
and
\[
(15) \quad e(\Phi, E_n) = \frac{k_n}{\sqrt{-2 + 8k_n - 4k_n^2}} \quad \text{when} \quad k_n > 1/2.
\]
Proof. We fix $n$ and abbreviate $A = A_n = V_n^* V_n$. After decoding, the resulting error in the reconstructed state is
\[ Y = (I - A)WA + AW(I - A). \]
The square of the Hilbert-Schmidt norm is
\[ \text{tr}[Y Y^*] = \text{tr}[(I - A)WA + AW(I - A)]^2 \]
\[ = 2 \text{tr}[(I - A)AW]^2 + (I - A)^2 WA^2 W]. \]
In the eigenbasis of the operator $A$, we can express this as
\[ \text{tr}[Y Y^*] = 2 \sum_{r,s} ((1 - \alpha_r)\alpha_r(1 - \alpha_s)\alpha_s|W_{r,s}|^2 + (1 - \alpha_r)^2 \alpha_s^2|W_{r,s}|^2), \]
where the matrix with entries $(W_{r,s})$ represents the state $W$ in the eigenbasis of $A$ and $\{\alpha_r\}$ are the corresponding eigenvalues of $A$.

We notice that if we assume $\alpha_r \leq 1$ for all $r$, the expression for $\text{tr}[Y Y^*]$ is just the square of a weighted $\ell^2$-norm on the set of all states, and thus convex in $W$. Moreover, the set of all states is convex, and therefore, a state that maximizes the Hilbert-Schmidt norm of $Y$ must be a pure state, $W_{r,s} = \phi_r \phi_s^*$, with $\sum_r |\phi_r|^2 = 1$. Since $|W_{r,s}|$ is symmetric with respect to exchanging $r$ and $s$, we can symmetrize the term $(1 - \alpha_r)^2 \alpha_s^2$ in the above sum.

To determine a maximizer, we use a variational principle. We replace the discrete spectrum of $A$ with a continuous variable and examine the symmetrized quantity
\[ g(x, y) = xy(1 - x)(1 - y) + \frac{1}{2}(1 - x)^2 y^2 + \frac{1}{2}(1 - y)^2 x^2 \]
\[ = \frac{1}{2}((1 - x)y + (1 - y)x)^2. \]
A critical point of $g$ satisfies
\[ (1 - 2y)(y(1 - x) + x(1 - y)) = 0 \]
and the same equation with $x$ and $y$ exchanged. Thus, either $x = y = 0$ or $x = y = \frac{1}{2}$.

We denote $\kappa = \|A\|$.

Case I: $\kappa \leq 1/2$. Considering the boundaries, we have $g(x, 0) \leq g(\kappa, 0) = \frac{1}{2}\kappa^2$ and $g(x, \kappa) = x(1 - x)\kappa(1 - \kappa) + \frac{1}{2}(1 - x)^2 \kappa^2 + \frac{1}{2}(1 - \kappa)^2 x^2$ for $0 \leq x \leq \kappa$, which is convex if $1 - 2\kappa \neq 0$ and linear otherwise. In both cases the maximum on the boundary is either $g(0, \kappa) = \frac{1}{2}\kappa^2$ or $g(\kappa, \kappa) = 2(1 - \kappa)^2 \kappa^2$. When $\kappa \leq \frac{1}{2}$, the latter expression is the maximum. Moreover, the critical point is not contained in the open square $(0, \kappa)^2$. 
This means that the state which maximizes
\[
\text{tr}[Y Y^*] = 2 \sum_{r,s} ((1 - \alpha_r)\alpha_r(1 - \alpha_s)\alpha_s + (1 - \alpha_r)^2\alpha_s^2)|\phi_r|^2|\phi_s|^2
\]
is given by \(\phi\) being an eigenstate of \(A\) with highest eigenvalue \(\kappa\), and
\[
\max_W \text{tr}[Y Y^*] = 4(1 - \kappa)^2\kappa^2.
\]

**Case II:** \(1/2 < \kappa \leq 1\). Since the rank of \(A\) is not maximal, we know it has an eigenvalue zero. From the preceding lemmas, we know that the probability vector \(\rho = |\phi|^2\) which maximizes
\[
\text{tr}[Y Y^*] = \sum_{r,s} ((1 - \alpha_r)\alpha_s + (1 - \alpha_s)\alpha_r)^2 \rho_r \rho_s
\]
can have at most two nonzero entries, which correspond to the maximal eigenvalue \(\kappa\) and an eigenvalue zero.

Choosing \(\rho\) accordingly, and optimizing
\[
\text{tr}[Y Y^*] = (4p^2(1 - \kappa)^2\kappa^2 + 2p(1 - p)\kappa^2)
\]
gives
\[
p = \frac{1}{2 - 4(1 - \kappa)^2}.
\]
Inserting this, we obtain
\[
\max_W \text{tr}[Y Y^*] = \frac{\kappa^2}{2 - 4(1 - \kappa)^2}.
\]

After inspecting both cases for \(\kappa\), we conclude that
\[
\max_W \text{tr}[Y Y^*] = \begin{cases} 
4(1 - \kappa)^2\kappa^2, & \kappa \in [0, 1/2], \\
\frac{\kappa^2}{2 - 4(1 - \kappa)^2}, & \kappa \in (1/2, 1].
\end{cases}
\]

\[\square\]

**Theorem 3.4.** Consider the Hilbert space \(\mathbb{C}^m \otimes \mathbb{C}^l\) and let the set of generalized phase-damping channels \(Q^{(1)}\) be the convex hull of the channels \(\{E_n\}_{n=1}^m\) defined with the set of projections \(\{Q_n = E_{nn} \otimes I\}_{n=1}^m\). For any given \(C^*\)-encoding \(\Phi : B(\mathbb{C}^d) \to B(\mathbb{C}^m \otimes \mathbb{C}^l)\), \(l < d\), the worst case error is bounded below by

\[
\sup_{\mathcal{E} \in Q^{(1)}} e(\Phi, \mathcal{E}) \geq \begin{cases} 
2(d - \frac{d^2}{d_m}), & \frac{d}{d_m} \in [0, 1/2], \\
\frac{d}{8mld - 2m^2l^2 - 4d^2}, & \frac{d}{d_m} \in (1/2, 1],
\end{cases}
\]

and equality holds if and only if all \(\{V_j^* V_j\}\) are uniformly weighted projections; that is, for all \(j \in \{1, 2, \ldots, m\}\), we have \(V_j^* V_j = kP_j\), with \(P_j = P_j^* P_j\) and \(k = \frac{d}{d_m}\).
Proof. Given a $C^*$-encoding $\Phi$, we optimize among all convex combinations of $E_n$ as defined in the preceding lemma.

We observe that for any probability vector $p \in \mathbb{R}^m_+$, by Minkowski’s inequality for the Hilbert-Schmidt norm
\[ e(\Phi, \sum_j p_j E_j) \leq \sum_j p_j e(\Phi, E_j). \]
The right-hand side is maximized by the probability vector which is nonzero only in the index $j$ with the largest $e(\Phi, E_j)$. However, choosing the worst-case state for this channel and the maximizing probability vector on the left-hand side gives equality in the above inequality.

Therefore, optimal encoding suppresses the maximum among all $e(\Phi, E_n)$.

The function $e(\Phi, E_n)$ is increasing in $k_n = \|A_n\|$. Since we want to choose $\{A_j\}$ such that the maximum of $e(\Phi, E_n)$ is minimized among all choices of $n$, we want to minimize $\max_{1 \leq j \leq m} \|A_j\|$. The minimum is achieved when equality holds in Inequality (7). This requires that the POVM $\{A_j\}$ consists of uniformly weighted projections. \qed

Remark 3.5. Recall that in Example 2.10 we saw how to construct uniformly weighted rank-one projections resolving the identity. Then in Example 2.11 we indicated how to generalize this construction to higher-rank projections in the case that $l$ divides $d$. We are aware of some other cases for which such families can be constructed, but here we leave the general case as an open problem that warrants further investigation.

4. Optimality for next to minimal decoherence

Next, we investigate what happens if we choose $\Phi$ optimal for minimal decoherence, and consider convex combinations of measurements with next to minimal impact. By the results of the preceding section, we can assume that the coordinate operators $\{V_j\}_{j=1}^m$ of $\Phi$ factor a set of uniformly weighted projection operators, $V_j^* V_j = k P_j$, with $k = d/ml$ and $k \sum_{j=1}^m P_j = I$.

Definition 4.1. Let $K$ be a subset of two elements from the index set $J = \{1, 2, \ldots m\}$. Define $Q_K = \sum_{j \in K} E_{jj} \otimes I$, and let for $M \in B(\oplus_{j=1}^m C_l)$,
\[ \mathcal{E}_K : M \mapsto Q_K MQ_K + Q_K^\perp MQ_K^\perp. \]

Let $Q^{(2)}$ be the convex hull of all transmission errors $\mathcal{E}_K$ indexed by two-element subsets $K$ of $J$.

The following lemma reduces to the so-called Welch bound in the special case that all projections $\{P_j\}$ are of rank $l = 1$. 
Lemma 4.2. Let \( \{P_i\}_{i=1}^m \) be a set of rank-\( l \) projections forming a uniformly weighted resolution of the identity \( \frac{d}{m} \sum_{j=1}^m P_j = I \) on a Hilbert space of dimension \( d > l \). Then (a) if \( d < 2l \), we have that for all pairs \( i, j \in \{1, 2, \ldots, m\} \), \( \|P_i + P_j\| = 2 \); (b) if \( d \geq 2l \), on the other hand, then

\[
\max_{i \neq j} \|P_i + P_j\| \geq 1 + \sqrt{\frac{lm - d}{d(m - 1)}}
\]

and equality holds in this inequality if and only if for all \( i, j \in \{1, 2, \ldots, m\} \),

\[
\|P_i + P_j\| = 1 + \sqrt{\frac{lm - d}{d(m - 1)}}.
\]

Proof. We can assume that the Hilbert space is \( \mathbb{C}^d \), equipped with the canonical basis, and identify the projections with \( d \times d \) matrices.

(a) We first show that in case \( d < 2l \), the sum of two of the projections has norm \( \|P_i + P_j\| = 2 \) for all \( i, j \in J \). To see this, we write the projections as \( 2 \times 2 \) block matrices with the \((1,1)\)-block being a \( l \times l \) matrix, and choose a basis such that this block is the identity for \( P_i = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \). Now when we compress \( P_j \) to this first \( l \times l \) block, we remove \( r = d - \ell < l \) rows and columns and so by eigenvalue interlacing, the \( l \times l \) block still has eigenvalue 1 of multiplicity \( l - r > 0 \). Consequently, \( P_i + P_j \) has 2 as an eigenvalue.

(b) We now consider the case \( d \geq 2l \).

Splitting \( \mathbb{C}^d = \mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}^r \), \( r \geq 0 \), we write \( d \times d \) matrices as \( 3 \times 3 \) block matrices. By a suitable choice of basis, we may assume that

\[
P_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

In the same decomposition, each \( P_j \) is written as a block matrix, and we call \( B_j \) the \((1,1)\)-block entry, so \( B_j \) is an \( l \times l \) matrix.

Now summing these matrices we have that \( I + \sum_{j=2}^m B_j = \frac{lm}{d} I \) by the resolution of the identity and hence \( \sum_{j=2}^m \text{tr}(B_j) = \frac{lm}{d} - l \). Thus, selecting for each \( B_j \) its largest eigenvalue \( \lambda_j \) gives the chain of inequalities

\[
(17) \quad \max_{j \geq 2} \lambda_j \geq \frac{1}{m-1} \sum_{j=2}^m \lambda_j \geq \frac{lm - d}{d(m - 1)}.
\]

In the remaining part of the proof, we show that

\[
\|P_1 + P_j\| = 1 + \sqrt{\lambda_j}.
\]
Once this is established, we have by the monotonicity of the square root that

\[ \max_{j \geq 2} \|P_1 + P_j\| \geq 1 + \sqrt{\frac{lm - d}{d(m - 1)}} \]

and equality implies that equality holds in the chain of Inequalities (17), which in turn gives that all \( \lambda_j \) are equal, as well as all \( \|P_1 + P_j\| \). Since the choice of \( P_1 \) is arbitrary, this argument shows that all pairs \( \{P_i + P_j\}_{i \neq j} \) have equal norm.

For simplicity of notation, we continue the remaining part of the proof with \( i = 1 \) and \( j = 2 \). We first show that up to unitary equivalence \( P_2 = \begin{pmatrix} D & R \\ R & R_{22} & R_{23} \\ 0 & R_{32} & R_{33} \end{pmatrix} \), with diagonal blocks \( D, R \), and as yet unspecified parts \( R_{22}, R_{23}, R_{32}, \) and \( R_{33} \). To see this, first we write \( P_2 \) in 2 \( \times \) 2 block form with the (1, 1)-block an \( l \times l \) matrix and conjugate by a unitary to diagonalize \( B_2 \) as \( D \).

Now we polar decompose the (1, 2)-block as \( RV^* \) where \( R \geq 0 \) is \( l \times l \) and \( V^* \) is \( l \times (d - l) \), and by assumption \( d - l \geq l \). Let \( U \) be a \( (d - l) \times (d - l) \) unitary so that \( V^*U = (I_0) \). Conjugating by \( \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} \) we have that up to these two unitary transformations, \( P_2 = \begin{pmatrix} D & R \\ R & R_{22} & R_{23} \\ 0 & R_{32} & R_{33} \end{pmatrix} \). But since \( P_2 \) is a projection, \( D^2 + R^2 = D \), and we see that \( R \) is diagonal as well.

If \( R \) has entries that are zero, then the corresponding entries in \( D \) must vanish also, since by assumption \( \|P_1 + P_2\| < 2 \) which yields \( \|D\| < 1 \). Again using that \( P_2 \) is a projection, \( DR + RR_{22} = R \) and hence \( R_{22} \) is also diagonal in the rows and columns where \( R \) is non-zero. In addition, since \( 0 = RR_{23}, R_{33} \) vanishes in those rows.

We now observe, similarly as in Example 2.12, that \( P_2 \) is a direct sum of 2 \( \times \) 2 matrices and a piece which contains the blocks that are not diagonal.

If the eigenvector of \( P_1 + P_2 \) corresponding to the largest eigenvalue had nonzero entries in a row in which \( D \) and \( R \) vanish, then that eigenvalue would be one. Indeed, this is the case if \( B_2 = D = R = 0 \), since then the ranges of \( P_1 \) and \( P_2 \) are orthogonal, and \( \|P_1 + P_2\| = 1 \).

However, if \( D \) has rows with non-zero entries, then we find the largest eigenvalue among the 2 \( \times \) 2 matrices. To this end, pick a non-zero entry of \( D \), say \( \lambda \). Note that since \( P_2 \) is a projection, the corresponding diagonal entry of \( R \) is then \( \sqrt{\lambda - \lambda^2} \) and that of \( R_{22} \) is \( 1 - \lambda \). Thus, in \( P_1 + P_2 \) we have a 2 \( \times \) 2 matrix of the form \( \begin{pmatrix} 1 + \lambda & \sqrt{\lambda - \lambda^2} \\ \sqrt{\lambda - \lambda^2} & 1 - \lambda \end{pmatrix} \). One finds that the norm
of this matrix is $1 + \sqrt{\lambda}$. Consequently, the largest entry $\lambda_2$ in $D$ maximizes this norm, which shows that $\|P_1 + P_2\| = 1 + \|\sqrt{D_2}\|$. \hfill \Box

\textbf{Theorem 4.3.} Assume the $C^*$-encoding $\Phi : B(\mathbb{C}^d) \to B(\mathbb{C}^m \otimes \mathbb{C}^l)$ has coordinate operators $\{V_j\}_{j=1}^m$ with rank $l < d$, such that each $V_j^*V_j = kP_j$ with a projection operator $P_j$ and $k = d/ml$. Denote

$$\kappa = \begin{cases} \frac{2d}{ml}, & d < 2l \\ \frac{d}{ml} + \sqrt{\frac{d(lm-d)}{m^2[l(m-1)]}}, & d \geq 2l, \end{cases}$$

then the worst case error under the $C^*$-encoding $\Phi$ and transmission channels from $Q^{(2)}$ is bounded below by

$$\max_{\mathcal{E} \in Q^{(2)}} e(\Phi, \mathcal{E}) \geq \begin{cases} 2\kappa(1 - \kappa), & \kappa \in [0, 1/2] \\ \kappa \sqrt{-2 + 8\kappa - 4\kappa^2}, & \kappa \in (1/2, 1]. \end{cases}$$

If $d \geq 2l$, then the value $\max_{\mathcal{E} \in Q^{(2)}} e(\Phi, \mathcal{E})$ attains the lower bound if and only if for all $i, j \in \{1, 2, \ldots m\}$,

$$\|P_i + P_j\| = 1 + \sqrt{\frac{lm-d}{d(m-1)}}.$$

\textbf{Proof.} Denote $A = k\sum_{j \in K} V_j^*V_j$. After decoding, the resulting error in the reconstructed state is

$$Y = (I - A)WA + AW(I - A).$$

In the eigenbasis of the operator $A$, the square of the Hilbert-Schmidt norm of $Y$ is

$$\text{tr}[YY^*] = 2 \sum_{r,s} \left((1 - \alpha_r)\alpha_r(1 - \alpha_s)\alpha_s|W_{r,s}|^2 + (1 - \alpha_r)^2\alpha_s^2|W_{r,s}|^2\right),$$

where again the matrix with entries $(W_{r,s})$ represents the state $W$ in the eigenbasis of $A$ and $\{\alpha_r\}$ are the corresponding eigenvalues of $A$.

We notice that finding the state $W$ which maximizes the worst-case error norm is exactly the same problem as in Lemma 3.3 of the preceding section. By the monotonicity of the resulting expression in $\|A\|$ and the convexity of the Hilbert-Schmidt norm, we can conclude analogously that the optimal encoding minimizes the largest of the eigenvalues of all pairs $\{P_i + P_j\}_{i \neq j}$. The preceding lemma states that this is the case if and only if the norms of all these pairs are equal. The common value of their norms then yields the claimed error bound. \hfill \Box

We describe a class of examples for which the lower bound for the worst-case error is attained.
Example 4.4. Let $C^d = C^q \otimes C^l$, and $\{P_j = \Pi_j \otimes I\}_{j=1}^m$ with rank-one projections $\{\Pi_j\}$ giving a uniformly weighted resolution of the identity on the first component $C^q$. We follow ideas of Holmes and Paulsen [25] to characterize optimal uniformly weighted projections of this type.

Let $\{f_j\}_{j=1}^m$ be a set of non-zero vectors, such that each $f_j$ is contained in the span of the corresponding rank-one projection $\Pi_j$, with the normalization $\|f_j\| = \sqrt{q/m}$. One can check that this set is a Parseval frame.

The largest eigenvalue of $P_i + P_j$ is $\|P_i + P_j\| = \|(\Pi_i + \Pi_j) \otimes I\| = \|\Pi_i + \Pi_j\|$. From identities for the trace of the Grammian $G$ of the Parseval frame $\{f_j\}$, we have $q = \text{tr} G = \sum_{j=1}^m \|f_j\|^2$ and $q = \text{tr} G^*G = \sum_{i,j} |\langle f_i, f_j \rangle|^2$, and we can infer the usual form of the Welch bound [25, 27, 36]

$$\max_{i \neq j} \|\Pi_i + \Pi_j\| = \max_{i \neq j} \left\{1 + \frac{m}{q} |\langle f_i, f_j \rangle|\right\} \geq 1 + \sqrt{\frac{m - q}{q(m - 1)}}$$

and equality holds if and only if $|\langle f_i, f_j \rangle| = \sqrt{q(m - q)/(m^2(m - 1))}$ for all $i \neq j$.

Therefore, an optimal set of projections of the form $\{\Pi_j \otimes I\}$ with rank-one projections $\{\Pi_j\}$ is given by a so-called two-uniform frame $\{f_j\}$ with all inner products between two different frame vectors having the same absolute value.

Remark 4.5. Now we briefly remark how to generalize to the case when $\dim V = s > 0$. This includes the case $m = 1$ of so-called decoherence-free subspaces. The minimal error model is the convex hull of transmission channels given by projections of both types $\{Q_j = (E_{jj} \otimes I) \oplus 0\}_{j=1}^m$ or $\{Q_{m+n} = (0 \otimes 0) \oplus E_{nn}\}_{n=1}^s$.

Suppose we have a $C^*$-encoding $\Phi : B(C^d) \to B((C^m \otimes C^l) \oplus V)$. Then we can view the tensor product as a direct sum, $(C^m \otimes C^l) \oplus V = (\oplus_{j=1}^m C^l) \oplus V$ and split the matrix $\tilde{V}$ associated with $\Phi$ similarly as in the proof of Lemma 2.6 into coordinate operators $\{V_j\}_{j=1}^m$ of rank $l$ and $\{V_j\}_{j=m+1}^{m+s}$ of rank one.

We note that the coordinate operators give rise to a POVM $\{A_j\}_{j=1}^{m+s}$ with $m$ rank-$l$ operators $\{A_j\}_{j=1}^m$ and $s$ rank-one operators $\{A_j\}_{j=m+1}^{m+s}$.

In analogy with Theorem 3.4, we can then derive a modified version of the lower bound for the worst case error, which amounts to replacing $ml$ with $ml + s$ in Ineq. (16). Again, equality would hold for $\Phi$ with coordinate operators that factor uniformly weighted projections. However, apart from the case when $s$ divides $l$ it is not clear under which general conditions an optimal $\Phi$ can be constructed.

Similarly, to find the best $\Phi$ for the next to minimal error model, we require the maximal eigenvalue among all $\{P_i + P_j\}_{i \neq j}$ to be minimized, but
at this point it is not clear whether there are enough interesting examples of this type.

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