SUPERSYMMETRY AND THE
ATIYAH-SINGER INDEX THEOREM II:
The Scalar Curvature Factor in the
Schrödinger Equation

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Abstract

The quantization of the superclassical system used in the proof of the index theorem results in a factor of $\frac{\hbar^2}{8}R$ in the Hamiltonian. The path integral expression for the kernel is analyzed up to and including 2-loop order. The existence of the scalar curvature term is confirmed by comparing the linear term in the heat kernel expansion with the 2-loop order terms in the loop expansion.
1 Introduction

In [1], a supersymmetric proof of the twisted spin index theorem is presented. There, the Peierls bracket quantization is applied to the following supersymmetric Lagrangian:

\[ L = \left[ \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\epsilon\gamma} \psi^\epsilon \left( \dot{\psi}^\gamma + \dot{x}^\mu \Gamma^\gamma_{\mu\theta} \psi^\theta \right) \right] + \kappa \left[ i \eta^a \left( \dot{\eta}^a + \dot{x}^\sigma A_{a}^{\sigma\beta} \eta^\beta \right) + \frac{1}{2} F_{\sigma\gamma}^{ab} \psi^\sigma \psi^\gamma \eta^a \eta^b \right] + \frac{\bar{\hbar}}{2} \eta^a \eta^a. \]  

(1)

The classical “momenta” are defined by:

\[ p_\mu := g_{\mu\nu} \dot{x}^\nu. \]  

(2)

The Peierls bracket quantization leads to the quantization of the supersymmetric charge:

\[ Q = \frac{1}{\sqrt{\hbar}} \psi^\nu g^{\frac{1}{2}} p_\nu g^{-\frac{1}{2}}. \]  

(3)

The quantum mechanical Hamiltonian is then given by:

\[ H = Q^2 = \frac{1}{2} \hbar^{-\frac{3}{2}} p_\mu g^{\frac{1}{2}} g^{\mu\nu} p_\nu g^{-\frac{1}{2}} + \frac{\hbar^2}{8} R - \frac{\kappa}{2} F_{\sigma\gamma}^{ab} \psi^\sigma \psi^\gamma \eta^a \eta^b. \]  

(4)

The fact that \( Q \) is identified with the (twisted) Dirac operator,

\[ Q \equiv \bar{D}, \]  

(5)

defines \( H \) uniquely.

Reducing (1) to a purely bosonic system, i.e. setting \( \psi = \eta = \eta^* = 0 \), one arrives at the Lagrangian for a free particle moving on a Riemannian manifold [1]. Equation (4) is in complete agreement with the analysis of the reduced system presented in [2]. There, the Hamiltonian was defined by requiring a particular factor ordering, namely by the time ordering [2, §6.5]. Furthermore, it was shown in [2, §6.6] that the term \( \frac{\hbar^2}{8} R \) in the Hamiltonian contributed to the kernel:

\[ K(x'; t''; x'; t') := \langle x'' = x'; t''|x'; t' \rangle, \]  

(6)

\footnote{In this paper the case of closed and simply connected Riemannian manifolds is considered.}
a factor of

\[ -\frac{i\hbar^2}{24} R(x') (t'' - t'). \]  

This was obtained using the heat kernel expansion of (6). The quantity: (7) is the linear term in the heat kernel expansion. In [2, §6.6], the 2-loop terms were computed. It was shown that indeed the loop expansion validates the existence of $\frac{R}{8}\hbar^2$ term. The presence of the scalar curvature factor in the Schrödinger equation is discussed in [3]. For further review see, [4, 5] and references therein.

The present paper is devoted to the 2-loop analysis of the path integral formula for the kernel defined by the quantization of (1). It is shown that indeed the path integral used in the derivation of the index formula, [1], corresponds to the Hamiltonian given by (3). This serves as an important consistency check for the supersymmetric proof of the index theorem presented in [1].

In Section 2, the loop expansion is reviewed and the relevant 2-loop terms for the system of (1) are identified. In Sections 3 and 4, the 2-loop calculations are presented for the spin ($\kappa = \alpha = 0$) and twisted spin ($\kappa = 1$) cases, respectively.

To avoid redundancy, the results of the first part [1] are used freely. As in [1], the Latin indices label $\eta$'s, the indices from the first and the second halves of the Greek alphabet label $\psi$'s and $x$'s, respectively. The condensed notation of [2] is also employed. Finally, the following choices are made:

\[ \hbar = 1, \quad \beta := t'', \quad \text{and} \quad t' = 0. \]

2 The Loop Expansion

Let $\Phi^i$ denote the coordinate (field) variables of a superclassical system. Then, if the Lagrangian is quadratic in $\dot{\Phi}$'s, one has [2, §5]:

\[ K(\Phi'', t''|\Phi', t') := \langle \Phi'', t''|\Phi', t' \rangle = Z \int_{\Phi', t'}^{\Phi'', t''} e^{iS[\Phi]} (sdet G^+[\Phi])^{-\frac{1}{2}} D\Phi. \]

In the loop expansion of (8), one expands $\Phi$ around the classical paths $\Phi_0$. Defining $\phi$ by:

\[ \Phi(t) =: \Phi_0(t) + \phi(t), \]

\[^2\psi's \text{are labelled by } \gamma, \delta, \epsilon, \theta, \eta.\]
one obtains:

\[
K(\Phi'', t'' \mid \Phi', t') = Z (s \det G^+)^{-\frac{1}{2}} e^{iS_0} \int e^{\frac{i}{2} \phi_i^\dagger S_{ij} \phi^j} \{ 1 + \}
\]

+ \frac{i}{24} S_{0,ijkl} \phi^i \phi^k \phi^j \phi^l - \frac{1}{2} S_{0,ijkl} S_{0,lmn} \phi^m \phi^l \phi^k \phi^j \phi^i + \n

\left[ -\frac{1}{27} \left( \epsilon_1 S_{0,ijkl} S_{0,lmn} G^{+nm}_{0} + \epsilon_2 S_{0,nmkl} G^{+nm}_{0} S_{0,jkl} \right) \phi^i \phi^k \phi^j \phi^l + \frac{1}{8} \epsilon_3 S_{0,ijkl} G^{+kl}_{0} S_{0,lmn} G^{+nm}_{0} \phi^i \phi^j \phi^k \phi^l \phi^m \phi^n \phi^p \phi^q \phi^r \phi^s \right] \cdots \}
\]

where,

\[
\epsilon_1 := (-1)^{m(l+1)+ln}, \ \epsilon_2 := (-1)^{m(i+1)+ln}, \ \text{and} \ \epsilon_3 := (-1)^{j(i+1)+m(l+1)+k+ln} .
\]

In (9), \(G^+\) is the advanced Green’s function for the Jacobi operator:

\[
(i, S_{ij}) := \left( \frac{\delta}{\delta \Phi^i(t)} S[\Phi] \frac{\delta}{\delta \Phi^j(t')} \right) .
\]

The subscript “0” indicates that the corresponding quantity is evaluated at the classical path, e.g. \(G^{+ij}_{0} := G^{+ij}[\Phi_0]\). Finally, “\(\cdots\)” are 3 and higher loop order terms.

To evaluate the right hand side of (9), one needs to perform the following functional Gaussian integrals:

\[
\int e^{\frac{i}{2} \phi_i^\dagger S_{ij} \phi^j} D\phi = c (s \det G)^{\frac{1}{2}}
\]

\[
\int e^{\frac{i}{2} \phi_i^\dagger S_{ij} \phi^j} \phi^k \phi^l D\phi = -ic (s \det G)^{\frac{1}{2}} G^{kl}
\]

\[
\int e^{\frac{i}{2} \phi_i^\dagger S_{ij} \phi^j} \phi^k \phi^l \phi^m \phi^n D\phi = (-i)^2 c (s \det G)^{\frac{1}{2}} \left( G^{kl} G^{mn} \pm \text{perm.} \right)
\]

\[
\int e^{\frac{i}{2} \phi_i^\dagger S_{ij} \phi^j} \phi^k \phi^l \phi^m \phi^n \phi^p \phi^q D\phi = (-i)^3 c (s \det G)^{\frac{1}{2}} \left( G^{kl} G^{mn} G^{pq} \pm \text{perm.} \right) .
\]

In (11), “\(G\)” is the Feynman propagator, “perm.” are terms obtained by some permutations of the indices of the previous term, “\pm” depends on the “parity” of \(\phi\)’s appearing on the left hand side, and “\(c\)” is a possibly infinite constant of functional integration. The functional integrals in (8) and (11) are taken over all paths with fixed end points:

\[
\phi'' := \phi(t'') = 0 \text{ and } \phi' := \phi(t') = 0 .
\]
This justifies the appearance of $G$ in (11).

One must realize that the terms in square bracket in (9) originate from the expansion of $(s\text{det} G^+)^{-\frac{1}{2}}$ in (8). For the system under consideration in (1), it was shown in [1, Sec. 5] that

$$s\text{det}(G^+) = 1 \quad \text{for: } x'' = x'.$$

This simplifies the computations of Sections 3 and 4 considerably. In view of (12), the square bracket in (9) drops and (9) reduces to:

$$K = K_{\text{WKB}} \left\{ 1 - i \frac{1}{12} S_{0,ijkl} (G^{ik} G^{ji} \pm \text{perm.}) + i \frac{1}{72} S_{0,ijk} S_{0,lmn} (G^{mn} G^{ik} G^{ji} \pm \text{perm.}) + \cdots \right\} ,$$

where,

$$K_{\text{WKB}} := Z c e^{i S_0} (s\text{det} G)^{\frac{1}{2}}$$

is the “WKB” approximation of the kernel. In Sections 3 and 4, the terms:

$$I := S_{0,ijkl} (G^{ik} G^{ji} \pm \text{perm.})$$
$$J := S_{0,ijk} S_{0,lmn} (G^{mn} G^{ik} G^{ji} \pm \text{perm.})$$

are computed explicitly. They correspond to the following Feynman diagrams:

$I \equiv \bigcirc \bigcirc$ and $J \equiv \bigcirc$.

### 3 2-Loop Calculations for the Case: $\kappa = \alpha = 0$

For $\kappa = \alpha = 0$, the dynamical equations [1, eq. 28] are solved by [1, eq. 90]:

$$x_0(t) = x_0 , \quad \psi_0(t) = \psi_0 .$$

As in [1], all the computations will be performed in a normal coordinate system centered at $x_0$. Since $K$ and $K_{\text{WKB}}$ in (13) have the same tensorial properties, the curly bracket in (13) must be a scalar. This justifies the use of the normal coordinates.
The Feynman propagator is given by [1, Sec. 6]:

\[
G^{\pi \xi'} = g_0^{\pi \mu} \left[ \theta(t - t') \left( \frac{e^{R(t - \beta)} - 1}{R(1 - e^{-R\beta})} \right) + \theta(t' - t) \left( \frac{e^{Rt} - 1}{R(e^{R\beta} - 1)} \right) \right]^{\xi}_{\mu} 
\]

\[G^{\pi \delta'} = G^{* \xi'} = 0 \]

\[G^{\gamma \delta'} = i g_0^{\gamma \delta} [\theta(t - t') - \theta(t' - t)] \]

where,

\[
R = (R^\mu_\tau) = (g^{\rho \nu} R_{\rho \tau \nu \sigma} \psi_0^\sigma \psi_0^\delta) \ .
\]

The functional derivatives of the action which appear in (13) are listed below:

\[
S_{0,\pi \rho''} = R_{\pi \tau \rho} \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \delta(t - t''') + R_{\rho \tau \pi} \delta(t - t') \left[ \frac{\partial}{\partial t} \delta(t - t'') \right] \\
S_{0,\pi \epsilon''} = i R_{0,\pi \gamma \epsilon} \psi_0^\gamma \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \delta(t - t''') + i \Gamma_{0,\pi \gamma \tau, \pi} \psi_0^\gamma \delta(t - t') \left[ \frac{\partial}{\partial t} \delta(t - t'') \right] \\
S_{0,\tau' \epsilon''} = S_{0,\epsilon'' \gamma'} = 0 \\
S_{0,\tau' \rho'' \kappa} = \left\{ -g_{0,\tau' \rho'', \kappa} \left[ \frac{\partial^2}{\partial t^2} \delta(t - t') \right] \delta(t - t''') + \delta(t - t'') \left[ \frac{\partial^2}{\partial t^2} \delta(t - t''') \right] \delta(t - t') + \delta(t - t'') \left[ \frac{\partial^2}{\partial t^2} \delta(t - t''') \right] \delta(t - t') \right\}_1 + \\
\left\{ -2 \Gamma_{0,\tau' \rho', \kappa} \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \left[ \frac{\partial}{\partial t} \delta(t - t''') \right] \delta(t - t'') + \right\}_2 + \\
\left\{ R_{\tau' \rho', \kappa} \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \delta(t - t''') + \right\}_3 + \\
\left\{ i (\Gamma_{0,\mu \tau} \Gamma_0^\mu_{\epsilon''}, \rho', \pi) \psi_0^\epsilon \psi_0^\rho \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \delta(t - t''') \delta(t - t'') + \right\}_4
\]
\[ S_{0, \tau \pi' \nu' \gamma''} = i R_{0 \pi \gamma} \left( \frac{\partial}{\partial t} \delta(t - t') \right) \delta(t - t'') \delta(t - t''') + \]
\[ + i \Gamma_{0 \pi' \tau \pi} \left( \frac{\partial}{\partial t} \delta(t - t') \right) \delta(t - t''') + \]
\[ - i \Gamma_{0 \pi' \tau \pi} \delta(t - t') \delta(t - t'') \left( \frac{\partial}{\partial t} \delta(t - t''') \right) \]
\[ S_{0, \epsilon' \eta' \delta''} = 0 \]

where the indices are placed on some of the curly brackets for identification purposes. \( S_{0, \tau \pi' \nu' \gamma''} \) and \( S_{0, \tau \epsilon' \nu' \eta''} \) are omitted because, as will be seen shortly, they do not contribute to (15).

In view of (18), one needs to consider only the following terms:

\[ I_1 := S_{0, \tau \pi' \rho' \kappa''} \left[ G^{\tau \pi' \rho'} G^{\rho' \kappa''} + G^{\tau \rho'} G^{\rho' \kappa''} + G^{\tau \kappa''} G^{\pi' \rho'} \right] \]
\[ = 3 S_{0, \tau \pi' \rho' \kappa''} G^{\tau \pi' \rho'} G^{\rho' \kappa''} \]
\[ I_2 := S_{0, \tau \pi' \epsilon' \nu''} G^{\tau \pi' \epsilon'} G^{\nu'' \gamma''} \] (22)
\[ J_1 := S_{0, \tau \pi' \rho' \sigma''} S_{0, \mu' \nu' \rho'' \sigma''} G^{\tau \pi' \rho'} G^{\rho' \kappa''} G^{\mu'' \sigma''} \]
\[ J_2 := S_{0, \tau \pi' \epsilon''} S_{0, \mu' \nu' \rho'' \gamma''} G^{\tau \pi' \epsilon''} G^{\nu'' \kappa''} G^{\mu'' \sigma''} , \]

where,

\[ I = I_1 + 6 I_2 . \] (23)

Here, 6 is a combinatorial factor and \( J \) is a linear combination of \( J_1 \) and \( J_2 \).

**Calculation of \( I_1 \) and \( I_2 \)**

Let us denote by \( I_{1, \alpha} \) the terms in \( I_1 \) which correspond to \{ \}_\alpha in \( S_{0, \tau \pi' \rho' \kappa''} \), with \( \alpha = 1, 2, 3, 4 \). The computation of \( I_{1, \alpha} \) is in order. One has:

\[ I_{1,1} := 3 \int_0^\beta dt \int_0^\beta dt' \left( -g_{0 \pi \pi', \rho} \left[ \frac{\partial^2}{\partial t^2} \delta(t - t') \right] G^{\tau \pi' \rho} \right) + \]
\[ 3 \int_0^\beta dt \int_0^\beta dt'' \left( -g_{0 \pi \rho, \pi} \left[ \frac{\partial^2}{\partial t^2} \delta(t - t'') \right] G^{\tau \pi \rho' \kappa} \right) + \]
\[ 3 \int_0^\beta dt \int_0^\beta dt''' \left( -g_{0 \rho', \pi \pi} \left[ \frac{\partial^2}{\partial t^2} \delta(t - t''') \right] G^{\tau \pi \rho'' \kappa} \right) \]
\[ = -3 \int_0^\beta dt \left( [g_{0 \pi \pi', \rho} + 2 g_{0 \pi \rho, \pi} \] \( G^{\rho \kappa} I^{\tau \pi} \) \right) \]
\[ = 3 \int_0^\beta dt \left( R_{0 \pi \pi', \rho} G^{\rho \kappa} I^{\tau \pi} \right) \]
\[ = 0 . \] (24)
In (24),
\[ I^{\tau\pi} := \int_0^\beta dt' \left[ \frac{\partial^2}{\partial t'^2} \delta(t - t') G^{\tau\pi'} \right], \] (25)
and the third and fourth equalities are established using [4, p. 56]:
\[ g_{0\mu\nu,\sigma\tau} = -\frac{1}{3} (R_{0\mu\nu\tau} + R_{0\nu\sigma\mu}) , \] (26)
and
\[ G^{\rho\kappa} = G^{\kappa\rho} . \] (27)

Next, we compute:
\[ I_{1.2} := -6\Gamma_{0\tau\pi,\rho,k} \int_0^\beta dt \int_0^\beta dt' \int_0^\beta dt'' \left[ \frac{\partial}{\partial t'} \delta(t - t') \right] \frac{\partial}{\partial t''} \left[ \delta(t - t'') \right] G^{\tau\pi'} G^{\rho''\kappa} \]
\[ -6\Gamma_{0\tau\rho,k,\pi} \int_0^\beta dt \int_0^\beta dt'' \int_0^\beta dt''' \left[ \frac{\partial}{\partial t''} \delta(t - t') \right] \frac{\partial}{\partial t'''} \left[ \delta(t - t''') \right] G^{\tau\pi'} G^{\rho''\kappa''} \]
\[ -6\Gamma_{0\tau\kappa,\rho,p} \int_0^\beta dt \int_0^\beta dt' \int_0^\beta dt''' \left[ \frac{\partial}{\partial t'} \delta(t - t') \right] \frac{\partial}{\partial t''} \left[ \delta(t - t''') \right] G^{\tau\pi'} G^{\rho''\kappa'''} \]
\[ = -12\Gamma_{0\tau\pi,\rho,k} \int_0^\beta dt \left( \frac{\partial}{\partial t'} G^{\tau\pi'} \right) \left|_{t' = t} \frac{\partial}{\partial t''} G^{\rho''\kappa'} \right|_{t'' = t} + \]
\[ -6\Gamma_{0\tau\rho,k,\pi} \int_0^\beta dt \left( G^{\tau\pi} \frac{\partial^2}{\partial t'' \partial t'''} G^{\rho''\kappa''} \right) \left|_{t'' = t'''} = t \right) . \] (28)

To evaluate the right hand side of (28), one needs the following relations:
\[ \frac{\partial}{\partial t''} G^{\rho''\kappa'} \left|_{t'' = t} \frac{\partial}{\partial t'} G^{\rho'} \left|_{t' = t} = \frac{1}{2} g_{0\mu} \left[ 1 + e^{R_{B^\mu}} - 2 e^{R_{B^t}} \right]_{\kappa} \right) \] (29)
\[ \frac{\partial^2}{\partial t' \partial t} G^{\rho''\kappa'} \left|_{t' = t} = -\frac{1}{2} g_{0\mu} \left[ \frac{\mathcal{R}(1 + e^{R_{B^\mu}})}{1 - e^{R_{B^t}}} \right]_{\rho} \right) \] (30)

and [4, p. 55]:
\[ \Gamma_{0\tau\pi,\rho,k} = -\frac{1}{3} (R_{0\tau\pi\rho} + R_{0\pi\rho\kappa}) . \] (31)

Equations (29) and (30) are obtained by differentiating (17), using symmetries of $\mathcal{R}$ and:
\[ \theta(0) := \frac{1}{2} . \] (32)
Combining equations (28)-(31) and using (17), one obtains:

\[ I_{1.2} = (R_0 \tau_\pi \rho_\kappa + R_0 \tau_\rho \pi_\kappa) \times \int_0^\beta dt \left( g_0^{\pi \mu} \left[ \frac{1 + e^{R_\beta} - 2e^{R_\beta t}}{1 - e^{R_\beta}} \right] \right)^\gamma g_0^{\rho \nu} \left[ \frac{1 + e^{R_\beta} - 2e^{R_\beta t}}{1 - e^{R_\beta}} \right]^\kappa + (33) \]

\[ - (R_0 \rho_\kappa \pi + R_0 \tau_\kappa \rho_\pi) \times \int_0^\beta dt \left( g_0^{\tau \nu} \left[ e^{R_\beta} - e^{R_\beta t} - e^{-R(t-\beta)} + 1 \right]^{\pi \kappa} g_0^{\kappa \mu} \left[ \frac{R(1 + e^{R_\beta})}{1 - e^{R_\beta}} \right]^{\nu \mu} \right) \]

To identify the terms in (33) which are linear in \( \beta \), one may recall that for every integral

\[ I(\beta) := \int_0^\beta dt \ f(\beta, t), \]

with an analytic integrand in both \( t \) and \( \beta \), the linear term in \( \beta \) is given by:

\[ \left. \frac{\partial}{\partial \beta} \right|_{\beta=0} I(\beta) = f(\beta = 0, t = 0) \beta. \quad (34) \]

Thus, one needs to examine the integrands in (33). This leads to

\[ I_{1.2} = (R_0 \tau_\pi \rho_\kappa + R_0 \tau_\rho \pi_\kappa) \ g_0^{\pi \tau} g_0^{\rho \kappa} \beta + O(\beta^2) = R_0 \beta + O(\beta^2), \quad (35) \]

where, \( R_0 \) is the Ricci scalar curvature evaluated at \( x_0 \).

Next step is to compute:

\[ I_{1.3} := 3 \left( \mathcal{R}_{\tau_\pi \rho_\kappa} \int_0^\beta dt \ G^{\rho \kappa} \frac{\partial}{\partial t'} G^{\tau \pi'} \bigg|_{t'=t} + \mathcal{R}_{\tau_\rho \pi_\kappa} \int_0^\beta dt \ G^{\tau \pi} \frac{\partial}{\partial t'} G^{\rho \kappa'} \bigg|_{t'=t} \right) + \mathcal{R}_{\tau_\kappa \pi_\rho} \int_0^\beta dt \ G^{\tau \pi} \frac{\partial}{\partial t'} G^{\rho \kappa'} \bigg|_{t'=t} \right) \]

This is done by using (29) and (34). The result is:

\[ I_{1.3} = O(\beta^2). \quad (36) \]

The computation of \( I_{1.4} \) is similar. Again, one obtains:

\[ I_{1.4} = O(\beta^2). \quad (37) \]
This completes the calculation of $I_1$. Combining (24), (33), (36), and (37), one has:

$$I_1 = R_0 \beta + O(\beta^2) \, .$$  \hfill (38)

The computation of $I_2$ is straightforward. Substituting (19) and (21) in (22), one has:

$$I_2 = \int_0^\beta dt \int_0^\beta dt' \int_0^\beta dt'' \int_0^\beta dt''' \left[ \mathcal{X}_{\tau\pi\gamma\epsilon}(t, t', t'', t''') \mathcal{Y}^{\tau\pi\gamma\epsilon}(t, t', t'', t''') \right] \, ,$$

where,

$$\mathcal{X}_{\tau\pi\gamma\epsilon} := -iR_{0\tau\pi\gamma\epsilon} \left[ \frac{\partial}{\partial t} \delta(t-t') \right] \delta(t-t'') \delta(t-t''') + \right.$$

$$+ i\Gamma_{0\mu\pi,\sigma} \delta(t-t') \left[ \frac{\partial}{\partial t} \delta(t-t'') \right] \delta(t-t''') + \right.$$

$$- i\Gamma_{0\gamma\epsilon,\tau,\pi} \delta(t-t') \delta(t-t'') \left[ \frac{\partial}{\partial t} \delta(t-t''') \right] \right.$$  \hfill (39)

Since $\mathcal{X}$ is antisymmetric in $(\gamma \leftrightarrow \epsilon)$ and $\mathcal{Y}$ is symmetric in $(\gamma \leftrightarrow \epsilon)$, $I_2$ vanishes. This together with (23) and (38) lead to:

$$I = R_0 \beta + O(\beta^2) \, .$$

### Calculation of $J_1$ and $J_2$

Substituting (21) in (22) and performing the integrations which involve $\delta$-functions, one obtains:

$$J_1 = \int_0^\beta dt \int_0^\beta dt'' \left( R_{\tau\pi,\rho} R_{\mu\nu,\sigma} \frac{\partial}{\partial t'} G^{\tau\pi}_{\mu\nu} \bigg|_{t'=t} \frac{\partial}{\partial t''} G^{\mu\nu}_{\sigma\pi} \bigg|_{t''=t''} \right) G^{\rho\mu''} + \right.$$  \hfill (40)

$$+ R_{\tau\pi,\rho} R_{\sigma,\mu,\nu} \bigg|_{t'=t} \frac{\partial}{\partial t'} G^{\tau\pi}_{\mu\nu} \bigg|_{t''=t''} G^{\rho\mu''} + \right.$$  \hfill (40)

$$+ R_{\rho,\tau,\mu} R_{\pi,\nu,\sigma} \bigg|_{t'=t} \frac{\partial}{\partial t''} G^{\rho\mu}_{\mu''} \bigg|_{t''=t''} G^{\tau\pi} + \right.$$  \hfill (40)

$$+ R_{\rho,\tau,\mu} R_{\pi,\nu,\sigma} \bigg|_{t''=t''} \frac{\partial}{\partial t'} G^{\rho\mu}_{\mu''} \bigg|_{t''=t''} G^{\tau\pi} \right) \right.$$  \hfill (40)

$$= \int_0^\beta dt \int_0^\beta dt'' \left( R_{\tau\pi,\rho} (R_{\mu\nu,\sigma} + R_{\nu,\mu,\sigma}) \right) \left[ \cdots + R_{\rho,\tau,\pi} (R_{\mu\nu,\sigma} + R_{\nu,\mu,\sigma}) \right] \left[ \cdots \right]$$  \hfill (40)

$$= 0 \, .$$
In (40), the second equality is obtained by rearranging the indices and using (29). The terms \[ \cdots \] involve \( G \)'s and their time derivatives. The last equality is established using the antisymmetry of \( R \):

\[
R_{\mu\nu} = -R_{\nu\mu}.
\]

(41)

The computation of \( J_2 \) is a little more involved. Carrying out the integrations involving \( \delta \)-functions, one can write \( J_2 \) in the following form:

\[
J_2 = \sum_{\alpha} J_{2,\alpha},
\]

(42)

where,

\[
J_{2,1} := -R_{0\tau\pi\delta\epsilon} R_{0\mu\nu\gamma} \psi^\delta_0 \psi^\eta_0 \times \int_0^\beta dt \int_0^\beta dt'' \left( \frac{\partial}{\partial t''} G^\gamma \right)^{\tau\pi}_{\mu\nu} \left| \frac{\partial}{\partial t'} G^{\epsilon\mu\nu} \right|_{t'=t, t''=t'''} G^{\gamma'''}
\]

(43)

\[
J_{2,2} := \frac{1}{4} R_{0\tau\pi\delta\epsilon} (R_{0\gamma\mu\nu} + R_{0\gamma\mu\nu}) \psi^\delta_0 \psi^\eta_0 \times \int_0^\beta dt \int_0^\beta dt'' \left( \frac{\partial}{\partial t''} G^\gamma \right)^{\tau\pi}_{\mu\nu} \left| \frac{\partial}{\partial t'} G^{\epsilon\mu\nu} \right|_{t'=t, t''=t'''} G^{\gamma'''}
\]

(44)

\[
J_{2,3} := \frac{1}{3} R_{0\mu\nu\eta\gamma} (R_{0\delta\tau\pi} + R_{0\delta\tau\pi}) \psi^\delta_0 \psi^\eta_0 \times \int_0^\beta dt \int_0^\beta dt'' \left( \frac{\partial}{\partial t''} G^{\epsilon\gamma'''} \right)^{\tau\pi}_{\mu\nu} \left| \frac{\partial}{\partial t'} G^{\epsilon\mu\nu} \right|_{t'=t, t''=t'''} G^{\gamma'''}
\]

(45)

\[
J_{2,4} := -\frac{1}{9} \left[ (R_{0\delta\tau\pi} + R_{0\delta\tau\pi}) (R_{0\gamma\mu\nu} + R_{0\gamma\mu\nu}) \right] \psi^\delta_0 \psi^\eta_0 \times \int_0^\beta dt \int_0^\beta dt'' \left( G^{\epsilon\gamma'''} \right)^{\tau\pi}_{\mu\nu} \frac{\partial^2}{\partial t' t''} G^{\epsilon\mu\nu} \left| t''=t, t''=t''' \right).
\]

(46)

In (44)-(46), use has been made of (31).

Consider the integrals \( \int f \int \) in (43). \( f \int \) is symmetric under the exchange of the pairs \((\tau, \pi) \leftrightarrow (\mu, \nu)\). The term \( \psi^\delta_0 \psi^\eta_0 \) is antisymmetric in \( \delta \leftrightarrow \eta \). Thus, the term \((R.R)\) is antisymmetrized in \( \epsilon \leftrightarrow \gamma \). However, according to (19), \( G^{\epsilon\gamma'''} \) involves \( g^{\epsilon\gamma}_{00} \) which is symmetric in \( \epsilon \leftrightarrow \gamma \). This makes \( J_{2,1} \) vanish. Furthermore, using (34) one finds out that \( J_{2,2} \) and \( J_{2,3} \) are at least of order \( \beta^2 \). \( f \int \) in (44) is symmetric under \( \tau \leftrightarrow \pi, \mu \leftrightarrow \nu, \) and \((\mu, \nu) \leftrightarrow (\tau, \pi)\). This allows only the term \( R_{0\epsilon\tau\delta\pi} R_{0\gamma\mu\nu} \) to survive in the square bracket in (44). Moreover, this term is symmetrized in \((\mu, \nu) \leftrightarrow (\tau, \pi)\), or alternatively in
\((\epsilon, \delta) \leftrightarrow (\gamma, \eta)\). Since \(\psi_0^\dagger \psi_0^*\) is antisymmetric in \(\delta \leftrightarrow \eta\), the surviving term which is multiplied by \(\int \int\) can be antisymmetrized in \(\epsilon \leftrightarrow \gamma\). However, due to (19) \(\int \int\) is symmetric in these indices. Hence, \(J_{2,4}\) vanishes too.

This concludes the computation of the 2-loop terms in the case \(\kappa = \alpha = 0\). Combinging (13), (15), (23), (39), (40), and (42), one finally obtains:

\[
K = K_{\text{WKB}} \left\{ 1 - \frac{i}{2\pi} R_0 \beta + O(\beta^2) \right\} .
\]

(47)

4 2-Loop Calculations for the Case: \(\kappa = 1\)

First, the following special case will be considered:

\[
\tilde{K} := \langle x, \psi, \eta^* = 0 | x, \psi, \eta = 0 \rangle .
\]

(48)

The dynamical equations, [1, eq. 28], are solved by [1, eq.’s 128]:

\[
\begin{align*}
x_0(t) &= x_0 \\
\psi_0(t) &= \psi_0 = \frac{1}{\sqrt{\beta}} \tilde{\psi}_0 \\
\eta_0(t) &= 0 \\
\eta_0^* &= 0.
\end{align*}
\]

(49)

Following [1, Sec. 7], one chooses a normal coordinate system centered at \(x_0\).

The Feynman propagator is given by:

\[
(G^{ij}) = \begin{pmatrix}
G^{\pi \xi'} & 0 & 0 & 0 \\
0 & G^{\gamma \delta'} & 0 & 0 \\
0 & 0 & G^{ac'} & 0 \\
0 & 0 & 0 & G^{a \ast c'}
\end{pmatrix},
\]

(50)

where \(G^{\pi \xi'}\) and \(G^{\gamma \delta'}\) are given by (17) and (19), respectively. Moreover, one has [1, eq.’s 135 and 136]:

\[
\begin{align*}
G^{ac'} &= \left[ \frac{i}{2} e^{i(F + \frac{\alpha}{2})} \right]^{ac} [\theta(t - t') - \theta(t' - t)] \\
G^{a \ast c'} &= \left[ \frac{i}{2} e^{i(F + \frac{\alpha}{2})} \right]^{ac} [\theta(t - t') - \theta(t' - t)] .
\end{align*}
\]

(51)
Here,
\[
\mathcal{F} = (\mathcal{F}^{ab}) := \left( \frac{1}{2} F_{\alpha \gamma}^{ab} \psi^\alpha_0 \psi^\gamma_0 \right),
\]
and \(I_{n \times n}\) is the \(n \times n\) unit matrix.

The functional derivatives of the action which enter into the computation of \(I\) and \(J\), (15), are listed below:

\[
\begin{align*}
\tilde{S}_{0, \tau \pi' \xi''} &= S_{0, \tau \pi' \xi''} |_{\kappa = \alpha = 0} \\
\tilde{S}_{0, \tau \pi' \epsilon''} &= S_{0, \tau \pi' \epsilon''} |_{\kappa = \alpha = 0} \\
\tilde{S}_{0, \epsilon' \theta''} &= 0 = \tilde{S}_{0, \epsilon' \theta''} \\
\tilde{S}_{0, \tau \pi' \xi'' \rho''} &= S_{0, \tau \pi' \xi'' \rho''} |_{\kappa = \alpha = 0} \\
\tilde{S}_{0, \tau \pi' \gamma'' \epsilon''} &= S_{0, \tau \pi' \gamma'' \epsilon''} |_{\kappa = \alpha = 0} \\
\tilde{S}_{\tau \alpha' \epsilon''} &= 0 = S_{\tau \alpha' \epsilon''} \\
\tilde{S}_{0, \epsilon' \epsilon''} &= \mathcal{F}^{ca}_{\tau \tau} \delta(t - t') \delta(t - t'') \\
\tilde{S}_{0, \epsilon' \epsilon''} &= 0 = \tilde{S}_{0, \epsilon' \epsilon''} \\
S_{\epsilon' \epsilon''} &= 0 = S_{\epsilon' \epsilon''} \\
\tilde{S}_{0, \epsilon' \epsilon''} &= F^{dc}_{\epsilon \epsilon'} \psi^d_0 \delta(t - t') \delta(t - t'') \\
\tilde{S}_{0, \tau \pi' \xi'' \epsilon''} &= \tilde{S}_{0, \tau \pi' \xi'' \epsilon''} \\
\tilde{S}_{0, \tau \pi' \gamma'' \epsilon''} &= \tilde{S}_{0, \tau \pi' \gamma'' \epsilon''} \\
S_{\tau \alpha' \epsilon''} &= 0 = S_{\tau \alpha' \epsilon''} \\
\tilde{S}_{0, \epsilon' \epsilon''} &= i F^{dc}_{\epsilon \epsilon'} \left[ \frac{\partial}{\partial t} \delta(t - t') \right] \delta(t - t'') \delta(t - t'') + \\
&\quad - i A^{dc}_{\epsilon \epsilon'} \delta(t - t') \frac{\partial}{\partial t} [\delta(t - t') \delta(t - t'')] + F^{dc}_{\epsilon \epsilon'} \delta(t - t') \delta(t - t'') \delta(t - t'') \\
\tilde{S}_{0, \tau \pi' \epsilon''} &= 0 = \tilde{S}_{0, \tau \pi' \epsilon''} \\
\tilde{S}_{0, \tau \pi' \epsilon''} &= F^{dc}_{\epsilon \epsilon'} \delta(t - t') \delta(t - t'') \delta(t - t'') \\
\tilde{S}_{0, \epsilon' \epsilon''} &= F^{dc}_{\epsilon \epsilon'} \delta(t - t') \delta(t - t'') \delta(t - t'') .
\end{align*}
\]

Here, “’s are placed to indicate that the special case of (15) is under consideration.

\[^3\text{The other terms are obtained from this list using the rules of changing the order of differentiation.}\]
Equations (52)-(56), indicate that the 2-loop contributions due to the terms which involve only the Greek indices are the same as the case of \( \kappa = \alpha = 0 \), i.e. these terms contribute a factor of \(-\frac{i}{24}R_0\beta\) to the kernel. Consequently, it is sufficient to show that the remaining 2-loop terms vanish.

**Computation of the Terms of Type I**

In view of (50) and (52)-(70), the following terms must be considered:

\[
\tilde{I}_1 := \tilde{S}_{0,\tau\pi} G^{\tau\pi} G^{c''d'}
\]

\[
\tilde{I}_2 := \tilde{S}_{0,\kappa\gamma} G^{\kappa\gamma} G^{c''d'}
\]

Performing the integration overs \( \delta \)-functions, one has:

\[
\tilde{I}_1 = \int_0^\beta dt \left( i F_{0\tau\pi} \frac{\partial}{\partial t'} G^{\tau\pi} \bigg|_{t=t'} G^{cd} + i A_{\alpha\tau,\pi}^{dc} G^{\tau\pi} \left[ \frac{\partial}{\partial t''} G^{c''d'} \bigg|_{t''=t} + \frac{\partial}{\partial t''} G^{cd''} \bigg|_{t''=t} \right] + F_{\kappa\pi}^{dc} G^{\tau\pi} G^{cd'} \right).
\]

The first and the last terms in the integrand of (73) vanish because according to (51):

\[
G^{cd} := G^{cd'} \bigg|_{t=t'} = 0.
\]

Moreover, one has:

\[
\frac{\partial}{\partial t'} G^{c'd'} \bigg|_{t'=t} = i \delta(0) \delta^{cd} = -\frac{\partial}{\partial t'} G^{c'd'} \bigg|_{t'=t}.
\]

Thus, the remaining terms cancel and one obtains:

\[\tilde{I}_1 = 0.\]

The computation of \( \tilde{I}_2 \) is quite simple. Substituting (19), (51), (70) in (72) and using (74), one finds:

\[
\tilde{I}_2 = \int_0^\beta dt \left( F^{dc} G^{c} G^{cd'} \right) = 0.
\]

The other terms of type I which involve Latin indices are proportional to \( \tilde{I}_1 \) or \( \tilde{I}_2 \) and hence vanish.
Computation of the Terms of Type $J$

There are six different terms of type $J$ which must be considered. These are:

\[
\begin{align*}
\tilde{J}_1 & := \tilde{S}_{0,\pi'}\tilde{S}_{0,\kappa''} G^{\tau\pi'} G^{\kappa''} G^{e^*d^v} \\
\tilde{J}_2 & := \tilde{S}_{0,\pi'}\tilde{S}_{0,\gamma''} G^{\tau\pi'} G^{\gamma''} G^{e^*d^v} \\
\tilde{J}_3 & := \tilde{S}_{0,\gamma'd''} \tilde{S}_{0,\pi''} G^{\tau\pi'} G^{a'b''} G^{e^*d^v} \\
\tilde{J}_4 & := \tilde{S}_{0,\tau ab'} \tilde{S}_{0,\pi''} G^{\tau\pi'} G^{a'd''} G^{e^*b''} \\
\tilde{J}_5 & := \tilde{S}_{0,\epsilon a'b''} \tilde{S}_{0,\gamma''} G^{\epsilon\gamma''} G^{a'b''} G^{e^*d^v} \\
\tilde{J}_6 & := \tilde{S}_{0,\epsilon a'b''} \tilde{S}_{0,\gamma''} G^{\epsilon\gamma''} G^{a'd''} G^{e^*b''}.
\end{align*}
\]

The following relations are useful in the computation of $\tilde{J}$'s. Using (74), one has:

\[
\begin{align*}
\tilde{S}_{0,\kappa''} G^{e^*d^v} & = \mathcal{F}_{\pi c} G^{cd} = 0 \quad \text{(77)} \\
\tilde{S}_{0,\gamma''} G^{e^*d^v} & = \mathcal{F}_{0,\psi'\delta} G^{cd} = 0. \quad \text{(78)}
\end{align*}
\]

Equations (77) and (78) lead immediately to:

\[
\tilde{J}_\alpha = 0 \quad \text{for} \quad \alpha = 1, 2, 3, 5. \quad \text{(79)}
\]

It remains to calculate $\tilde{J}_4$ and $\tilde{J}_6$. In view of (60) and (63), one has:

\[
\begin{align*}
\tilde{J}_4 & = \mathcal{F}_{\tau\pi} \mathcal{F}_{\pi c} \int_0^\beta dt \int_0^\beta \int_0^\beta \psi'^\delta \psi'^\theta G^{e^*d^v} \quad \text{(80)} \\
\tilde{J}_6 & = \mathcal{F}_{0,\delta} \mathcal{F}_{0,\psi'\delta} \psi'^\theta \psi'^\theta \int_0^\beta dt \int_0^\beta \int_0^\beta \psi'^\delta \psi'^\theta G^{e^*d^v} \quad \text{(81)}
\end{align*}
\]

Using (34) and examining the integrands in (80) and (81), one finally arrives at:

\[
\tilde{J}_\alpha = O(\beta^2) \quad \text{for} \quad \alpha = 4, 6. \quad \text{(82)}
\]

This concludes the 2-loop calculations for the special case of (19). For this case the kernel is given by:

\[
\tilde{K} = \tilde{K}_{\text{WKB}} \left( 1 - \frac{i}{24} R_0 \beta + O(\beta^2) \right). \quad \text{(83)}
\]
In the rest of this section, it is shown that the same conclusion is reached even for the general case where \( \eta \neq 0 \neq \eta^* \), i.e. for

\[
K := \langle x, \psi, \eta^* | x, \psi, \eta \rangle.
\]

In the general case, the dynamical equations \([1, \text{eq. 28}]\) are solved by \([1, \text{eq.'s 123}, 153, \& 154]\):

\[
\begin{align*}
x_0(t) &= x_0 + O(\beta) \\
\psi_0(t) &= \frac{1}{\sqrt{\beta}} [\tilde{\psi}_0 + O(\beta)] \\
\eta_0^a(t) &= O(1) \\
\eta_0^a(t) &= O(1).
\end{align*}
\]

(84)

It is easy to check that the terms in \( S_0 \)'s which involve \( \eta \)'s or \( \eta^* \)'s, and thus survive in the general case, are all of higher order in \( \beta \) than the terms considered above. Therefore, the contribution of these terms are of order \( \beta^2 \) or higher. For instance,

\[
S_{0, \tau \pi \rho''} = \tilde{S}_{0, \tau \pi \rho''} + \Sigma_{\tau \pi \rho}
\]

where,

\[
\tilde{S}_{0, \tau \pi \rho''} = O(\beta^{-4})
\]

is given by (52), and

\[
\Sigma_{\tau \pi \rho} := \left[ i(A_{0\tau, \pi}^{ab} - A_{0r, \pi}^{ab}) \pi \eta_0^a \eta_0^b + iA_{0r, \pi \rho}^{ab}(\dot{\eta}_0^a \eta_0^b + \eta_0^c \dot{\eta}_0^d) + i(A_{0\sigma, \tau}^{cd} - A_{0r, \sigma}^{cd}) \pi \rho \dot{\eta}_0^c \eta_0^d + \mathcal{F}_{\tau \pi \rho} \eta_0^c \eta_0^d \right] \delta(t - t') \delta(t - t'') + i(A_{0\rho, \tau} - A_{0r, \rho}) \pi \delta(t - t') \frac{\partial}{\partial t} [\delta(t - t'')]
\]

\[
= O(\beta^{-3})
\]

To determine the order of the terms in \( \beta \), one proceeds as in \([1, \text{Sec. 7}]\), namely one makes the following change of time variable:

\[
s := \frac{t}{\beta} \in [0, 1].
\]

(85)
For example, one has:
\[
\frac{\partial}{\partial t} \delta(t-t') = \beta^{-2} \frac{\partial}{\partial s} \delta(s-s') = O(\beta^{-2}) .
\]

The terms for which the above argument might not apply are those that vanish identically in the special case of (49) but survive otherwise. These are:

\[
J_7 := S_{0,\tau\pi'c''} S_{0,\xi''} G^{\tau\pi'} G^{\xi''} G^{c''} d^{c'} \\
J_8 := S_{0,\tau\pi'c''} S_{0,\xi''} G^{\tau\pi'} G^{\xi''} G^{c''} d^{c'} \\
J_9 := S_{0,\tau\gamma'c''} S_{0,\xi''} G^{\tau\gamma'} G^{\xi''} G^{c''} d^{c'} \\
J_{10} := S_{0,\epsilon\gamma'c''} S_{0,\xi''} G^{\epsilon\gamma'} G^{\xi''} G^{c''} d^{c'} \\
J_{11} := S_{0,\epsilon\gamma'c''} S_{0,\xi''} G^{\epsilon\gamma'} G^{\xi''} G^{c''} d^{c'} .
\]

However, one can easily show that the contribution of these terms to the kernel is of the order \(\beta^2\). The following relations are useful:

\[
S_{0,\tau\pi'c''} = O(\beta^{-3}) = S_{0,\tau\pi'c''} \\
S_{0,\tau\gamma'c''} = O(\beta^{-2}) \\
S_{0,\epsilon\gamma'c''} = O(\beta^{-2}) = S_{0,\epsilon\gamma'c''} \\
G^{\tau\pi'} = O(\beta) \\
G^{\epsilon\gamma'} = O(1) = G^{c''} d^{c'} .
\]

Furthermore, each integral:
\[
\int_0^\beta dt \cdots = \beta \int_0^1 ds \cdots
\]
contributes a factor of \(\beta\). Putting all this together, one finds out that \(J_7, \ldots, J_{11}\) are all of order \(\beta^2\).

This concludes the 2-loop calculations for the general case of (84). The final result is:
\[
K = K_{\text{WKB}} \left[ 1 - \frac{i}{2\pi} R_0 \beta + O(\beta^2) \right] .
\]

Equation (86) verifies the existence of the scalar curvature factor in the Hamiltonian and provides a consistency check for the supersymmetric proof of the Atiyah-Singer index theorem presented in [1].

It must be emphasized that the 2-loop term in (86) does not contribute to the index formula [1, eq. 147]. This is simply because the \(\psi\)-integrations in [1, eq. 87] eliminate such a term.
5 Conclusion

The scalar curvature factor in the Schrödinger equation yields a factor of $-\frac{i}{24} R\beta$ in the heat kernel expansion of the path integral. The loop expansion provides an independent test of the validity of this assertion. In particular, this factor is obtained in the 2-loop order. Thus, it is shown that indeed the path integral used in the derivation of the index formula corresponds to the Hamiltonian defined by the twisted Dirac operator.

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