Simultaneous inhomogeneous Diophantine approximation on manifolds

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Dedicated to A.O. Gelfond on what would have been his 100th birthday

Abstract

In 1998, Kleinbock & Margulis [KM98] established a conjecture of V.G. Sprindzuk in metrical Diophantine approximation (and indeed the stronger Baker-Sprindzuk conjecture). In essence the conjecture stated that the simultaneous homogeneous Diophantine exponent \( w_0(x) = 1/n \) for almost every point \( x \) on a non-degenerate submanifold \( M \) of \( \mathbb{R}^n \). In this paper the simultaneous inhomogeneous analogue of Sprindzuk’s conjecture is established. More precisely, for any ‘inhomogeneous’ vector \( \theta \in \mathbb{R}^n \) we prove that the simultaneous inhomogeneous Diophantine exponent \( w_0(x, \theta) = 1/n \) for almost every point \( x \) on \( M \). The key result is an inhomogeneous transference principle which enables us to deduce that the homogeneous exponent \( w_0(x) = 1/n \) for almost all \( x \in M \) if and only if for any \( \theta \in \mathbb{R}^n \) the inhomogeneous exponent \( w_0(x, \theta) = 1/n \) for almost all \( x \in M \). The inhomogeneous transference principle introduced in this paper is an extremely simplified version of that recently discovered in [BV]. Nevertheless, it should be emphasised that the simplified version has the great advantage of bringing to the forefront the main ideas of [BV] while omitting the abstract and technical notions that come with describing the inhomogeneous transference principle in all its glory.

1 Introduction

The metrical theory of Diophantine approximation on manifolds dates back to 1932 with a conjecture of K. Mahler [Mah32] in transcendence theory. The conjecture was easily seen to be equivalent to a metrical Diophantine approximation problem restricted to Veronese curves. Mahler’s problem remained a key open problem in metric number theory for over 30 years and was eventually solved by Sprindzuk [Spr69] in 1964. Moreover, its solution eventually lead Sprindzuk [Spr80] to make an important general conjecture which we now describe. For a vector \( x \in \mathbb{R}^n \), let

\[
w_0(x) := \sup \{ w : \|qx\| < q^{-w} \text{ for i.m. } q \in \mathbb{N} \},
\]

and

\[
w_{n-1}(x) := \sup \{ w : \|q.x\| < |q|^{-w} \text{ for i.m. } q \in \mathbb{Z}^n \setminus \{0\} \}.
\]

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Here and elsewhere ‘i.m.’ is the abbreviation for ‘infinitely many’, \(|q| := \max\{|q_1|, \ldots, |q_n|\}\) is the supremum norm, \(q \cdot x := q_1 x_1 + \ldots + q_n x_n\) is the standard inner product and \(\|\cdot\|\) is the distance to the nearest integer. For obvious reasons, \(w_0(x)\) is referred to as the *simultaneous Diophantine exponent* and \(w_{n-1}(x)\) is referred to as the *dual Diophantine exponent*.

A trivial consequence of Dirichlet’s theorem, or simply the ‘pigeon-hole principle’, is that

\[
w_0(x) \geq \frac{1}{n} \quad \text{and} \quad w_{n-1}(x) \geq n \quad \text{for all } x \in \mathbb{R}^n.
\]  

(1)

The Diophantine exponents \(w_0(x)\) and \(w_{n-1}(x)\) can in principle be infinite. For example, this is the case when \(n = 1\) and \(x\) is a Liouville number. Nevertheless, a relatively easy consequence of the Borel-Cantelli lemma in probability theory is that the inequalities in (1) are reversed for almost all \(x \in \mathbb{R}^n\) with respect to Lebesgue measure on \(\mathbb{R}^n\). Thus,

\[
w_0(x) = \frac{1}{n} \quad \text{and} \quad w_{n-1}(x) = n \quad \text{for almost all } x \in \mathbb{R}^n.
\]

Sprindzuk conjectured that a similar statement holds for any non-degenerate submanifold \(\mathcal{M}\) in \(\mathbb{R}^n\) with respect to the Lebesgue measure induced on \(\mathcal{M}\). Essentially, these are smooth submanifolds of \(\mathbb{R}^n\) which are sufficiently curved so that they deviate from any hyperplane with a ‘power law’ (see [Ber02]). Formally, a differentiable manifold \(\mathcal{M}\) of dimension \(d\) embedded in \(\mathbb{R}^n\) is said to be non-degenerate if there exists an atlas \(\{\mathcal{M}_i, g_i\}_{i \in \mathbb{N}}\) such that each map \(g_i : \mathcal{M}_i \rightarrow U_i\), where \(U_i\) is an open subset of \(\mathbb{R}^d\), is a diffeomorphism and \(f_i := g_i^{-1}\) is non-degenerate. The map \(f : U \rightarrow \mathbb{R}^n : u \mapsto f(u) = (f_1(u), \ldots, f_n(u))\) is said to be non-degenerate at \(u \in U\) if there exists some \(l \in \mathbb{N}\) such that \(f\) is \(l\) times continuously differentiable on some sufficiently small ball centred at \(u\) and the partial derivatives of \(f\) at \(u\) of orders up to \(l\) span \(\mathbb{R}^n\). The map \(f\) is non-degenerate if it is non-degenerate at almost every (in terms of \(d\)-dimensional Lebesgue measure) point in \(U\). Any real, connected analytic manifold not contained in any hyperplane of \(\mathbb{R}^n\) is easily seen to be non-degenerate. For a planar curve, non-degeneracy is simply equivalent to the condition that the curvature is non-vanishing almost everywhere. In short, non-degeneracy is a natural generalisation of non-zero curvature and naturally excludes obvious counterexamples to Sprindzuk’s conjecture which we now formally state.

**Sprindzuk’s conjecture.** *Let \(\mathcal{M}\) be a non-degenerate submanifold of \(\mathbb{R}^n\). Then*

\[
w_0(x) = \frac{1}{n} \quad \text{and} \quad w_{n-1}(x) = n \quad \text{for almost all } x \in \mathcal{M}.
\]

(2)

In the case \(\mathcal{M} := \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}\), Sprindzuk’s conjecture reduces to Mahler’s problem. Manifolds that satisfy (2) are simply referred to as *extremal*. Thus, Sprindzuk’s conjecture simply states that any non-degenerate submanifold of \(\mathbb{R}^n\) is extremal. To be absolutely precise, the definition of non-degeneracy given in the form above was actually introduced by Kleinbock & Margulis and not Sprindzuk. Essentially, Sprindzuk considered the case of analytic manifolds.

**Remark.** It is worth stressing that the equalities in (2) concerning the simultaneous Diophantine exponent \(w_0(x)\) and the dual Diophantine exponent \(w_{n-1}(x)\) are intimately related via
a classical transference principle – see [2]. Thus, in order to establish the above conjecture it suffices to prove either of the two equalities. In other words, the notion of extremal is actually independent of the type of Diophantine exponent under consideration and we may freely work within either the simultaneous framework or the dual framework depending on what is most convenient. A priori, this is not the case when considering the inhomogeneous analogue of Sprindzuk’s conjecture.

By the time Sprindzuk made his conjecture, the case \( n = 2 \) (planar curves) had already been established by W.M. Schmidt [Sch64] many years earlier. Until 1998, the numerous contributions towards Sprindzuk’s conjecture had all been limited to special classes of manifolds – the most significant being that of Beresnevich & Bernik [BB96] who proved the conjecture in the case of curves in \( \mathbb{R}^3 \). In groundbreaking work, Kleinbock & Margulis [KM98] established Sprindzuk’s conjecture in full generality. Furthermore, they answered a more general question of A. Baker concerning a stronger notion of extremality. This is nowadays referred to as the Baker-Sprindzuk conjecture and will not be discussed within this paper however the inhomogeneous version of the Baker-Sprindzuk conjecture is addressed in [BV].

**Theorem A (Kleinbock & Margulis)** Let \( \mathcal{M} \) be a non-degenerate submanifold of \( \mathbb{R}^n \). Then \( \mathcal{M} \) is extremal.

Kleinbock & Margulis used ideas drawn from dynamical systems to prove their theorem, in particular the theory of flows on homogeneous spaces. Independently of their work, Beresnevich [Ber02] used classical methods to establish a convergence (Khintchine-Groshev type) criterion from which Theorem A readily follows. Recently, Kleinbock [Kle03] has extended Theorem A to incorporate non-degenerate submanifolds of linear subspaces of \( \mathbb{R}^n \). This naturally broadens the class of extremal manifolds beyond the notion of non-degeneracy. For example, affine subspaces of \( \mathbb{R}^n \) are degenerate and so the result that non-degenerate manifolds are extremal is not applicable.

**Theorem B (Kleinbock)** Let \( L \) be an affine subspace of \( \mathbb{R}^n \).

(a) If \( L \) is extremal and \( \mathcal{M} \) is a non-degenerate submanifold of \( L \), then \( \mathcal{M} \) is extremal.

(b) If \( L \) is not extremal, then no subset of \( L \) is extremal.

Since \( \mathbb{R}^n \) is itself extremal, Theorem A is obviously covered by part (a) of Theorem B.

The main substance of the present work is to establish the inhomogeneous analogue of Sprindzuk’s conjecture in the case of simultaneous approximation. Naturally, we begin by introducing the simultaneous inhomogeneous Diophantine exponent. For \( \theta \in \mathbb{R}^n \), let

\[
  w_0(x, \theta) := \sup \left\{ w \geq 0 : \|qx + \theta\| < |q|^{-w} \text{ for i.m. } q \in \mathbb{Z} \setminus \{0\} \right\}.
\]

A manifold \( \mathcal{M} \) is said to be \textit{simultaneously inhomogeneously extremal} (SIE for short) if for every \( \theta \in \mathbb{R}^n \),

\[
  w_0(x, \theta) = \frac{1}{n} \quad \text{for almost all } x \in \mathcal{M}.
\]

The main result of this paper is the following transference statement.
Theorem 1  Let $\mathcal{M}$ be a differentiable submanifold of $\mathbb{R}^n$. Then

$\mathcal{M}$ is extremal $\iff$ $\mathcal{M}$ is SIE.

There is clearly a trivial part of Theorem 1 as any simultaneously inhomogeneously extremal manifold is extremal. This is simply due to the fact that $w_0(\mathbf{x}) = w_0(\mathbf{x}, 0)$. Thus the converse part of Theorem 1 constitutes the main substance. Indeed, it is rather surprising that a homogeneous statement ($\mathcal{M}$ is extremal) implies an inhomogeneous statement ($\mathcal{M}$ is SIE). The philosophy behind the above inhomogeneous transference principle is broadly comparable with the recently discovered mass transference principle [BV06, BV06a] in metric number theory.

The following results are simple consequences of Theorem 1 and represent the simultaneous inhomogeneous analogues of Theorems A and B.

**Corollary A**  Let $\mathcal{M}$ be a non-degenerate manifold of $\mathbb{R}^n$. Then $\mathcal{M}$ is SIE.

**Corollary B**  Let $\mathcal{L}$ be an extremal affine subspace of $\mathbb{R}^n$ and let $\mathcal{M}$ be a non-degenerate submanifold of $\mathcal{L}$. Then both $\mathcal{L}$ and $\mathcal{M}$ are SIE.

We stress that Corollary A alone does not establish the complete inhomogeneous analogue of Sprindzuk’s conjecture. For this we would also have to establish the analogue of Corollary A for the dual form of approximation. More precisely, for $\mathbf{\theta} \in \mathbb{R}^n$ let

$$w_{n-1}(\mathbf{x}, \mathbf{\theta}) := \sup\{w > 0 : \|\mathbf{q} \cdot \mathbf{x} + \mathbf{\theta}\| < |\mathbf{q}|^{-w} \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^n \setminus \{0\}\}.$$ 

A manifold $\mathcal{M}$ is said to be dually inhomogeneously extremal (DIE for short) if for every $\mathbf{\theta} \in \mathbb{R}^n$,

$$w_{n-1}(\mathbf{x}, \mathbf{\theta}) = n \quad \text{for almost all } \mathbf{x} \in \mathcal{M}.$$ 

Moreover, a manifold $\mathcal{M}$ is simply said to be inhomogeneously extremal if it is both SIE and DIE. The following statement represents the natural inhomogeneous analogue of Sprindzuk’s conjecture.

**Conjecture IE**  Let $\mathcal{M}$ be a non-degenerate submanifold of $\mathbb{R}^n$. Then $\mathcal{M}$ is inhomogeneously extremal.

The following corollary is a simple consequence of the general framework developed in [BV] and together with Corollary A establishes the above conjecture.

**Corollary A’**  Let $\mathcal{M}$ be a non-degenerate submanifold of $\mathbb{R}^n$. Then $\mathcal{M}$ is DIE.

Unlike in the homogeneous case, there is no classical transference principle that allows us to deduce Corollary A’ from Corollary A and vice versa. The upshot is that the two forms of inhomogeneous extremality, namely SIE and DIE a priori have to be treated separately. It turns out that establishing the dual form of inhomogeneous extremality is technically far more complicated than establishing the simultaneous form. The framework developed in [BV]...
naturally incorporates both forms of inhomogeneous extremality and indeed other stronger notions associated with the inhomogeneous analogue of the Baker-Sprindzuk conjecture. In particular, the general inhomogeneous transference principle of [BV] enables us to establish the following transference for non-degenerate manifolds:

\[ \mathcal{M} \text{ is extremal} \iff \mathcal{M} \text{ is inhomogeneously extremal}. \]

This together with Theorem A clearly establishes the inhomogeneous extremality conjecture and also enables us to deduce that:

\[ \mathcal{M} \text{ is SIE} \iff \mathcal{M} \text{ is DIE}. \]

In other words, a transference principle between the two forms of inhomogeneous extremality does exist at least for the class of non-degenerate manifolds.

As indicated above, the inhomogeneous transference principle introduced in this paper (Theorem 1) is an extremely simplified version of that in [BV]. Nevertheless and most importantly, the simplified version has the great advantage of bringing to the forefront the main ideas of [BV] and at the same time leads to a transparent and self-contained proof of the inhomogeneous analogue of Sprindzuk’s conjecture in the case of simultaneous approximation – Corollary A.

## 2 Diophantine exponents and transference inequalities

Transference inequalities in the theory of Diophantine approximation are often attributed to Khintchine who established the first set of inequalities relating the dual and simultaneous Diophantine exponents. Recently, Bugeaud & Laurent [BL05] have discovered new transference inequalities which we shall conveniently make use of in our proof of Theorem 1. In this section we give a brief overview of these new and classical transference results.

We start by recalling the classical transference principle of Khintchine. For any \( x \in \mathbb{R}^n \), Khintchine’s transference principle relates the Diophantine exponents \( w_0(x) \) and \( w_{n-1}(x) \) in the following way:

\[
\frac{w_{n-1}(x)}{(n-1)w_{n-1}(x) + n} \leq w_0(x) \leq \frac{w_{n-1}(x) - n + 1}{n}.
\]

These inequalities readily imply that

\[ w_0(x) = \frac{1}{n} \iff w_{n-1}(x) = n. \]

A particular implication for us is that in order to establish Sprindzuk’s conjecture it suffices to prove either of the above equalities. Thus, within the homogeneous setting there is no need to differentiate between the two implicit forms (dual and simultaneous) of extremality because one naturally implies the other. As already mentioned, this is far from the situation within the inhomogeneous setting. Nevertheless, there are various transference inequalities between homogeneous and inhomogeneous Diophantine exponents (see [Cas57]) which we are able to utilise. The recent inequalities discovered by Bugeaud & Laurent that relate the
above Diophantine exponents with their uniform counterparts are particularly relevant to establishing Theorem 1.

Uniform Diophantine exponents are defined as follows. Let $\mathbf{x}, \theta \in \mathbb{R}^n$. The simultaneous uniform inhomogeneous exponent $\hat{w}_0(\mathbf{x}, \theta)$ is defined to be the supremum of real numbers $\hat{w}$, such that for any sufficiently large integer $Q$ there exists an integer $q$ so that

$$
\|q \mathbf{x} + \theta\| \leq Q^{-\hat{w}} \quad \text{and} \quad 0 < |q| \leq Q.
$$

Let $\mathbf{x} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$. The dual uniform inhomogeneous exponent $\hat{w}_{n-1}(\mathbf{x}, \theta)$ is defined to be the supremum of real numbers $\hat{w}'$, such that for any sufficiently large integer $Q$ there exists an integer point $q$ so that

$$
\|q \mathbf{x} + \theta\| \leq Q^{-\hat{w}'} \quad \text{and} \quad 0 < |q| \leq Q.
$$

If $\theta = 0 \pmod{1}$, the above exponents are naturally referred to as the homogeneous uniform Diophantine exponents and we write $\hat{w}_0(\mathbf{x})$ for $\hat{w}_0(\mathbf{x}, \theta)$ and $\hat{w}_{n-1}(\mathbf{x})$ for $\hat{w}_{n-1}(\mathbf{x}, 0)$.

A trivial consequence of Dirichlet’s theorem is that $\hat{w}_0(\mathbf{x}) \geq \frac{1}{n}$ and $\hat{w}_{n-1}(\mathbf{x}) \geq n$ for all $\mathbf{x} \in \mathbb{R}^n$. This is not at all the case for the inhomogeneous exponents. Indeed, if $\mathbf{x} \in \mathbb{R}$ is a Liouville number then $\hat{w}_0(\mathbf{x}, \theta)$ vanishes for almost all $\theta \in \mathbb{R}$ – see [BL05] or [Cas57] for further details. A simplified version of the main result in [BL05] is as follows.

**Theorem C (Bugeaud & Laurent)** Let $\mathbf{x}, \theta \in \mathbb{R}^n$. Then

$$
w_0(\mathbf{x}, \theta) \geq \frac{1}{\hat{w}_{n-1}(\mathbf{x})} \quad \text{and} \quad \hat{w}_0(\mathbf{x}, \theta) \geq \frac{1}{w_{n-1}(\mathbf{x})},
$$

with equalities in (6) for almost every $\theta \in \mathbb{R}^n$.

The following statement is a consequence of Theorem C.

**Corollary 1** Let $\mathcal{M}$ be an extremal differentiable submanifold of $\mathbb{R}^n$. Then for every $\theta \in \mathbb{R}^n$ we have that

$$
w_0(\mathbf{x}, \theta) \geq \frac{1}{n} \quad \text{for almost all } \mathbf{x} \in \mathcal{M}.
$$

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Proof. By definition, for an extremal manifold \( M \) we have that \( w_{n-1}(x) = n \) for almost all \( x \in M \). This together with (4) and (5), implies that \( \hat{w}_{n-1}(x) = n \) for almost all \( x \in M \). Thus, for every \( \theta \in \mathbb{R}^n \),

\[
0(x, \theta) \geq \frac{1}{\hat{w}_{n-1}(x)} = \frac{1}{n} \quad \text{for almost all } x \in M .
\]

\( \Box \)

In view of Corollary 1 and the fact that \( w_0(x) = w_0(x, 0) \), the proof of Theorem 1 is reduced to establishing the following statement.

**Theorem 1** Let \( M \) be a differentiable submanifold of \( \mathbb{R}^n \). If \( M \) is extremal, then for every \( \theta \in \mathbb{R}^n \) we have that

\[
w_0(x, \theta) \leq \frac{1}{n} \quad \text{for almost all } x \in M .
\]

**An important remark.** It is worth pointing out that Corollary 1, which allows us to reduce Theorem 1 to Theorem 1`, can in fact be established without appealing to Theorem C. Indeed, a proof can be given which only makes use of classical transference inequalities; namely Theorem VI of Chapter 5 in [Cas57]. Thus, the proof of Theorem 1 is not actually dependent on the recent developments regarding transference inequalities. However, given the existence of Theorem C it would be rather absurd to make no use of it. Furthermore, Theorem C actually gives us information beyond inequality (7) that supports our main result (Theorem 1). More precisely, it enables us to deduce that inequality (7) is in fact an equality for almost all \( \theta \). The real significance of Theorem 1 is therefore in establishing (3) for all \( \theta \) rather than for almost all \( \theta \).

3 Proof of the Theorem 1*

Throughout, let \( \mu \) denote the induced Lebesgue measure on the differentiable submanifold \( M \) in \( \mathbb{R}^n \). For \( \varepsilon > 0 \), let

\[
S_0^\theta(\varepsilon) := \{ y \in \mathbb{R}^n : \| qy + \theta \| < |q|^{-\frac{1}{n}} \varepsilon \text{ for i.m. } q \in \mathbb{Z} \setminus \{0\} \}
\]

and

\[
S_\varepsilon(\varepsilon) := S_0^\theta(\varepsilon).
\]

Theorem 1* will follow on establishing that

\[
\mu(S_\varepsilon(\varepsilon) \cap M) = 0 \quad \text{for any } \theta \in \mathbb{R}^n \text{and any } \varepsilon > 0 ,
\]

under the hypothesis that \( M \) is extremal; i.e.

\[
\mu(S_\varepsilon(\varepsilon) \cap M) = 0 \quad \text{for any } \varepsilon > 0 .
\]
3.1 Slicing into extremal curves

In this section we show that it is sufficient to prove Theorem 1 and therefore Theorem 1* for extremal differentiable curves.

Let $\mathcal{M}$ be an extremal differentiable submanifold of $\mathbb{R}^n$ of dimension $d > 1$. Consider the local parameterisation of $\mathcal{M}$ given by

$$f : U \rightarrow \mathbb{R}^n : x = (x_1, \ldots, x_d) \mapsto f(x) \in \mathcal{M}$$

where $U$ is a ball in $\mathbb{R}^d$ and $f$ is a diffeomorphism. Since $\mathcal{M}$ is extremal and the fact that sets of full and zero measure are invariant under diffeomorphisms, the set

$$\mathcal{E} := \{x \in U : w_0(f(x)) = 1/n\}$$

has full Lebesgue measure in $U$. Now for every $x' = (x_2', \ldots, x_d') \in \mathbb{R}^{d-1}$ consider the line $L_{x'}$ in $\mathbb{R}^d$ given by

$$L_{x'} := \{x \in \mathbb{R}^d : x_2 = x_2', \ldots, x_d = x_d'\}.$$  

Also define $\mathcal{E}_{x'} = \mathcal{E} \cap L_{x'}$ and $U_{x'} = U \cap L_{x'}$. Clearly, $U_{x'}$ is either an interval or is empty and that $\mathcal{E}_{x'} \subset U_{x'}$. For obvious reasons, we only consider the case when $U_{x'} \neq \emptyset$. Since $\mathcal{E}$ has full measure in $U$, it follows from Fubini’s theorem that for almost every $x' \in \mathbb{R}^{d-1}$ the set $\mathcal{E}_{x'}$ has full measure in $U_{x'}$. Now let $f_{x'}$ denote the map $f$ restricted to $U_{x'}$. Clearly, $f_{x'}$ is a diffeomorphism from $U_{x'}$ onto $\mathcal{M}_{x'} = f(U_{x'})$. Since $\mathcal{E}_{x'}$ has full measure in $U_{x'}$ and $f_{x'}$ is a diffeomorphism, $\mathcal{M}_{x'}$ is extremal for almost all $x' \in \mathbb{R}^{d-1}$. It follows, under the assumption that Theorem 1 is true for curves, that $\mathcal{M}_{x'}$ is SIE; i.e. for every fixed $\theta \in \mathbb{R}^n$ the set $\mathcal{E}^\theta := \{x \in U : w_0(f(x), \theta) = 1/n\}$ has full Lebesgue measure in $U_{x'}$ for almost all $x' \in \mathbb{R}^{d-1}$. On applying Fubini’s theorem, we conclude that $\mathcal{E}^\theta$ has full Lebesgue measure in $U$. Since the latter holds for every $\theta \in \mathbb{R}^n$, we have established that $\mathcal{M}$ is SIE. The upshot of this is that we only need to establish Theorem 1* in the case that $\mathcal{M}$ is an extremal differentiable curve.

From this point onwards, $\mathcal{M}$ is an extremal differentiable curve in $\mathbb{R}^n$ and $\theta \in \mathbb{R}^n$ is fixed. Let $f = (f_1, \ldots, f_n) : I \rightarrow \mathbb{R}^n$ be a diffeomorphic parameterisation of $\mathcal{M}$, where $I$ is a finite interval and $f(I) \subset \mathcal{M}$. Note the fact that $f(I)$ is not necessarily the whole of $\mathcal{M}$ is not an issue since establishing Theorem 1* for every patch of $\mathcal{M}$ suffices. Since $f$ is a diffeomorphism, the Implicit Function Theorem enables us to change variables so that

$$f_1(x) = x. \quad (10)$$

This is the standard Monge parameterisation. Also, since we are able to work locally (i.e. on patches of $\mathcal{M}$), we can assume that

$$C := \sup_{x \in I} |f'(x)| < \infty, \quad (11)$$

as otherwise we can restrict $f$ to an interval $J$ such that the closure of $J$ is contained in $I$. For the same reason, there is no loss of generality in assuming that the curve $\mathcal{M}$ is bounded.
3.2 Auxiliary lemmas

Given \( p \in \mathbb{Z}^n, q \in \mathbb{Z} \setminus \{0\}, \theta \in \mathbb{R}^n \) and \( \varepsilon > 0 \) define the ball

\[
B_{p,q}^\theta(\varepsilon) := \{ y \in \mathbb{R}^n : |qy + p + \theta| < |q|^{-\frac{1}{n}} \}.
\]

In general, \( B = B(x,r) \) is the ball centred at \( x \) and of radius \( r > 0 \). For any \( \lambda > 0 \), we denote by \( \lambda B \) the ball \( B \) scaled by a factor \( \lambda \); i.e. \( \lambda B := B(x,\lambda r) \). In this notation,

\[
B_{p,q}^\theta(\varepsilon) = B((p + \theta)/q, |q|^{-\frac{1}{n}} - \varepsilon).
\]

**Lemma 1** Let \( M = \{ f(x) : x \in I \} \) be a curve satisfying (11) and (12). Then for any choice of \( p \in \mathbb{Z}^n, q \in \mathbb{Z} \setminus \{0\}, \varepsilon > 0 \) and \( \theta \in \mathbb{R}^n \) we have that

\[
\mu(B_{p,q}^\theta(\varepsilon) \cap M) \leq 2 n C |q|^{-1 - \frac{1}{n} - \varepsilon}.
\]

Furthermore, if \( \frac{1}{2} B_{p,q}^\theta(\varepsilon) \cap M \neq \emptyset \) then

\[
\mu(B_{p,q}^\theta(\varepsilon) \cap M) \geq \frac{1}{2} \min \{ |C|^{-1}, |I| \} |q|^{-1 - \frac{1}{n} - \varepsilon}.
\]

**Proof.** Let \( x, x' \in I \) be such that \( f(x), f(x') \in B_{p,q}^\theta(\varepsilon) \). Then, by the definition of \( B_{p,q}^\theta(\varepsilon) \) together with (11) and (12), we have that \( |qx + p_1 + \theta_1| < |q|^{-1 - \varepsilon} \) and \( |qx' + p_1 + \theta_1| < |q|^{-\frac{1}{n} - \varepsilon} \). On taking the obvious difference we find that \( |x - x'| < 2 |q|^{-\frac{1}{n} - \varepsilon} \). Hence

\[
|x - x'| < 2 |q|^{-\frac{1}{n} - \varepsilon}.
\]

Let \( J \) denote the smallest interval that contains all \( x \) such that \( f(x) \in B_{p,q}^\theta(\varepsilon) \). Then, in view of (14) it follows that \( |J| \leq 2 |q|^{-1 - \frac{1}{n} - \varepsilon} \). Therefore

\[
\mu(B_{p,q}^\theta(\varepsilon) \cap M) \leq \int_I |f'(x)| \, dx \leq n C \int_I n C \, dx \leq 2 n C |I| \leq 2 n C |q|^{-1 - \frac{1}{n} - \varepsilon},
\]

which is precisely (12). In order to prove (13), fix as we may a point \( x \in I \) such that \( f(x) \in \frac{1}{2} B_{p,q}^\theta(\varepsilon) \). Equivalently,

\[
|qf(x) + p + \theta| < \frac{1}{2} |q|^{-\frac{1}{n} - \varepsilon}.
\]

Now take any \( x' \in I \) such that \( |x - x'| < \delta |q|^{-1 - \frac{1}{n} - \varepsilon} \), where \( \delta := \frac{1}{2} \min \{ |C|^{-1}, |I| \} \). By the Mean Value Theorem and (11), we have that

\[
|f(x') - f(x)| \leq C \delta |q|^{-1 - \frac{1}{n} - \varepsilon} \leq \frac{1}{2} |q|^{-1 - \frac{1}{n} - \varepsilon}.
\]

On combining (15) and (16), we obtain that

\[
|qf(x') + p + \theta| = |(qf(x) + p + \theta) - q(f(x) - f(x'))| < \frac{1}{2} |q|^{-\frac{1}{n} - \varepsilon} + \frac{1}{2} |q|^{-\frac{1}{n} - \varepsilon} = |q|^{-\frac{1}{n} - \varepsilon}.
\]

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Thus, for any $x' \in J' := \{ x' \in J : |x - x'| < \delta |q|^{-\frac{1}{n} - \varepsilon} \}$ we have that $f(x') \in B_{p,q}^\theta(\varepsilon)$. Since $\delta < |I|/2$, we have that $|J'| \geq \delta q^{-\frac{1}{n} - \varepsilon}$. Therefore,

$$\mu(B_{p,q}^\theta(\varepsilon) \cap M) \geq \int_{J'} |f'(x)|_2 \, dx \geq \int_{J'} \, dx = |J'| \geq \delta |q|^{-\frac{1}{n} - \varepsilon}.$$ 

This is precisely (12) and completes the proof of the lemma.

Theorem 1*. As we shall see, it is extremely simple yet very effective.

The following statement is an immediate consequence of the Lemma 1.

**Lemma 2** Let $M = \{ f(x) : x \in I \}$ be a curve satisfying (10) and (11). Then for any choice of $p \in \mathbb{Z}^n$, $q \in \mathbb{Z}$, $\varepsilon > 0$ and $\theta \in \mathbb{R}^n$ such that

$$|q| > \frac{2}{\varepsilon}$$

and $B_{p,q}^\theta(\varepsilon) \cap M \neq \emptyset$, we have

$$\mu(B_{p,q}^\theta(\varepsilon) \cap M) \leq \frac{4nC}{\min\{C^{-1}, |I|\}} |q|^{-\varepsilon/2} \mu(B_{p,q}^\theta(\varepsilon/2) \cap M).$$

**Proof.** Simply note that (17) implies that $B_{p,q}^\theta(\varepsilon) \subset \frac{1}{2} B_{p,q}^\theta(\varepsilon/2)$ and apply Lemma 1.

3.3 The disjoint and non-disjoint balls

The following decomposition of $S_n^\theta(\varepsilon) \cap M$ represents the key component in establishing Theorem 1*. As we shall see, it is extremely simple yet very effective.

Fix an $\varepsilon > 0$. The set $S_n^\theta(\varepsilon) \cap M$ can be written in the following manner to bring to the forefront its limsup nature:

$$\Lambda_n^\theta(\varepsilon) := \bigcap_{s=1}^\infty \bigcup_{q \geq s} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\theta(\varepsilon) \cap M. \quad (18)$$

Since $M$ is bounded, for each $q$ the above union over $p \in \mathbb{Z}^n$ is in fact finite. We now make a crucial distinction between two natural types of balls appearing in (18).

Fix $p \in \mathbb{Z}^n$ and $q \in \mathbb{Z} \setminus \{0\}$. Clearly, there exists a unique integer $t = t(q)$ such that $2^t \leq |q| < 2^{t+1}$. The ball $B_{p,q}^\theta(\varepsilon)$ is said to be disjoint if for every $q' \in \mathbb{Z}$ with $2^t \leq |q'| < 2^{t+1}$ and every $p' \in \mathbb{Z}^n$

$$B_{p,q}^\theta(\varepsilon/2) \cap B_{p',q'}^\theta(\varepsilon/2) \cap M = \emptyset.$$

Otherwise, the ball $B_{p,q}^\theta(\varepsilon)$ is said to be non-disjoint.

Naturally, the notion of disjoint and non-disjoint balls enables us to decompose the set $\Lambda_n^\theta(\varepsilon)$ into two limsup subsets:

$$D_n^\theta(\varepsilon) := \bigcap_{s=0}^\infty \bigcup_{t \geq s} \bigcup_{2^t \leq |q| < 2^{t+1}} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\theta(\varepsilon) \cap M,$$

$$B_{p,q}^\theta(\varepsilon) \text{ is disjoint} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\theta(\varepsilon) \cap M.$$
and
\[ \mathcal{N}_n^\theta(\varepsilon) = \bigcap_{s=0}^\infty \bigcup_{t \geq s} \bigcup_{2^t \leq |q| < 2^{t+1}} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\theta(\varepsilon) \cap \mathcal{M}. \]

Formally,
\[ S_\theta^\varepsilon \cap \mathcal{M} = \Lambda_\theta^\varepsilon = \mathcal{D}_n^\theta(\varepsilon) \cup \mathcal{N}_n^\theta(\varepsilon). \]

### 3.4 The finale

Our aim is to establish (8). In view of the above decomposition of \( \Lambda_\theta^\varepsilon \), this will clearly follow on showing that
\[ \mu(\mathcal{N}_\theta^\varepsilon) = \mu(\mathcal{D}_\theta^\varepsilon) = 0. \]

Naturally, we deal with the disjoint and non-disjoint sets separately.

**The disjoint case:** By the definition of disjoint balls, for every fixed \( t \) we have that
\[ \sum_{2^t \leq |q| < 2^{t+1}} \sum_{p \in \mathbb{Z}^n} \mu(B_{p,q}^\varepsilon(\varepsilon/2) \cap \mathcal{M}) = \mu \left( \bigcup_{2^t \leq |q| < 2^{t+1}} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\varepsilon(\varepsilon/2) \cap \mathcal{M} \right) \leq \mu(\mathcal{M}) < \infty. \]

This together with Lemma 2 implies that for \( 2^t > 2/\varepsilon \)
\[ \sum_{2^t \leq |q| < 2^{t+1}} \sum_{p \in \mathbb{Z}^n} \mu(B_{p,q}^\varepsilon(\varepsilon) \cap \mathcal{M}) \ll 2^{-\varepsilon t/2}. \]

The implied constant in the Vinogradov symbol \( \ll \) does not depend on \( t \). Therefore,
\[ \sum_{t \geq s} \sum_{2^t \leq |q| < 2^{t+1}} \sum_{p \in \mathbb{Z}^n} \mu(B_{p,q}^\varepsilon(\varepsilon) \cap \mathcal{M}) \ll \sum_{t \geq s} 2^{-\varepsilon t/2} \to 0 \text{ as } s \to \infty. \quad (19) \]

By definition,
\[ \mathcal{D}_n^\theta(\varepsilon) \subset \bigcup_{t \geq s} \bigcup_{2^t \leq |q| < 2^{t+1}} \bigcup_{p \in \mathbb{Z}^n} B_{p,q}^\theta(\varepsilon) \cap \mathcal{M} \quad (20) \]

for arbitrary \( s \) and by (19) the measure of the right hand side of (20) tends to 0 as \( s \to \infty \). Therefore, the left hand side of (20) must have zero measure; i.e.
\[ \mu(\mathcal{D}_n^\theta(\varepsilon)) = 0. \]

**The non-disjoint case:** Let \( B_{p,q}^\theta(\varepsilon) \) be a non-disjoint ball and let \( t = t(q) \) be as above. Clearly
\[ B_{p,q}^\theta(\varepsilon) \subset B_{p,q}^\theta(\varepsilon/2). \]
By the definition of non-disjoint balls, there is another ball $B^\theta_{p',q'}(\varepsilon/2)$ with $2^t \leq |q'| < 2^{t+1}$ such that

$$B^\theta_{p,q}(\varepsilon/2) \cap B^\theta_{p',q'}(\varepsilon/2) \cap \mathcal{M} \neq \emptyset.$$ (21)

It is easily seen that $q' \neq q$, as otherwise we would have that $B^\theta_{p,q}(\varepsilon/2) \cap B^\theta_{p',q'}(\varepsilon/2) = \emptyset$.

Take any point $y$ in the non-empty set appearing in (21). By the definition of $B^\theta_{p,q}(\varepsilon/2)$ and $B^\theta_{p',q'}(\varepsilon/2)$, it follows that

$$|qy + p + \theta| < |q|^{-1-\frac{\varepsilon}{2}} \leq 2^{t\left(-\frac{1}{n} - \frac{\varepsilon}{2}\right)}$$

and

$$|q'y + p' + \theta| < |q'|^{-1-\frac{\varepsilon}{2}} \leq 2^{t\left(-\frac{1}{n} - \frac{\varepsilon}{2}\right)}.$$

On combining these inequalities in the obvious manner, we deduce that

$$\left|\left(\frac{q - q'}{q''}, \frac{p - p'}{p''}\right)\right| < 2 \cdot 2^{t\left(-\frac{1}{n} - \frac{\varepsilon}{2}\right)} < 2^{t+2}\left(-\frac{1}{n} - \frac{\varepsilon}{2}\right)$$ (22)

for all $t$ sufficiently large. Furthermore, $0 < |q''| \leq 2^{t+2}$ which together with (22) yields that

$$|q''y + p''| < |q''|^{-\frac{1}{2} - \frac{\varepsilon}{2}}.$$

If the latter inequality holds for infinitely many different $q'' \in \mathbb{Z}$, then $y \in \mathcal{S}_n(\varepsilon/3) \cap \mathcal{M}$. Otherwise, there is a fixed pair $(p'', q'') \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}$ such that (22) is satisfied for infinitely many $t$. Thus, we must have that $q''y + p'' = 0$ and so $y$ is a rational point. The upshot of the non-disjoint case is that

$$\mathcal{N}^\theta_n(\varepsilon) \subset (\mathcal{S}_n(\varepsilon/3) \cap \mathcal{M}) \cup (\mathbb{Q}^n \cap \mathcal{M}),$$

where $\mathbb{Q}^n$ is the set of rational points in $\mathbb{R}^n$. In view of (19) and since $\mathbb{Q}^n$ is countable, it follows that

$$\mu(\mathcal{N}^\theta_n(\varepsilon)) = 0.$$

This together with the analogous statement for $\mathcal{D}^\theta_n(\varepsilon)$ establishes (17) and thereby completes the proof of Theorem 1.

\[ \square \]

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