Shear-free, Irrotational, Geodesic, Anisotropic Fluid Cosmologies

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Abstract

General relativistic anisotropic fluid models whose fluid flow lines form a shear-free, irrotational, geodesic timelike congruence are examined. These models are of Petrov type D, and are assumed to have zero heat flux and an anisotropic stress tensor that possesses two distinct non-zero eigenvalues. Some general results concerning the form of the metric and the stress-tensor for these models are established. Furthermore, if the energy density and the isotropic pressure, as measured by a comoving observer, satisfy an equation of state of the form \( p = p(\mu) \), with \( \frac{dp}{d\mu} \neq -\frac{1}{3} \), then these spacetimes admit a foliation by spacelike hypersurfaces of constant Ricci scalar. In addition, models for which both the energy density and the anisotropic pressures only depend on time are investigated; both spatially homogeneous and spatially inhomogeneous models are found. A classification of these models is undertaken. Also, a particular class of anisotropic fluid models which are simple generalizations of the homogeneous isotropic cosmological models is studied.

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1. Introduction

In a recent paper [1], a comprehensive analysis of general relativistic spacetimes which admit a shear-free, irrotational and geodesic (SIG) timelike congruence was undertaken. This paper concentrates on a particular class of such spacetimes, namely, anisotropic fluid models whose fluid flow lines form a SIG timelike congruence (SIGA models). SIGA spacetimes are characterised by a single scale factor and a purely spatial three-metric [1]. If the additional assumption is made that the energy density and the isotropic pressure, as measured by a comoving observer, satisfy an equation of state of the form

\[ p = p(\mu), \]

with \( \frac{dp}{d\mu} \neq -\frac{1}{3} \), then the Ricci scalar associated with the spatial three-metric is constant; we note that these spacetimes are not the same as the perfect fluid Friedman-Robertson-Walker (FRW) models, where the three-space is a space of constant curvature. However, the FRW models are a subclass of the SIGA models. The SIGA models studied in this paper are assumed to have zero heat flux and a non-zero anisotropic stress tensor, \( \pi_{ab} \), as measured by a comoving observer. The usual phenomenological assumption \( \pi_{ab} = -\lambda\sigma_{ab} \) does not hold for these models since the fluid congruence has zero shear. However, there are a variety of physical situations that are described by such SIGA models, for example, the interaction of a perfect fluid and a pure magnetic field [1, 2, 3, 4]. Furthermore, since the anisotropic stress tensor is trace-free and three-dimensional in nature, it must either have three or two distinct non-zero eigenvalues. When there are only two distinct eigenvalues then the matter content of a SIGA spacetime is that of an anisotropic fluid; this paper is devoted to the study of such models.

The paper is organised as follows: In section 2, we derive some general results about the form of the anisotropic stress tensor. The anisotropic stress tensor is shown to only depend on a single function, \( P \), of the spatial coordinates, a three-dimensional vector that has unit norm with respect to the three-dimensional spatial metric and the spatial three-metric. The spacetime is also determined to be of Petrov type D. In sections 3 and 4, we specialise to the case where the three-dimensional hypersurfaces have constant Ricci scalar, \( ^3R \). In section 3, we use a 2+1 split of the three-geometry to investigate a particular class of anisotropic fluid solutions which are simple generalizations of the FRW models. In section 4, we consider the class of solutions that have constant \( P \) and \( ^3R \), and prove that both spatially homogeneous and spatially inhomogeneous solutions exist. We finish by deriving some specific spatially homogeneous solutions for the case where both \( P \) and \( ^3R \) are constant.

2. The Stress-Energy Tensor and the Metric

We consider general relativistic spacetimes with the following anisotropic fluid stress-energy tensor

\[ T_{ab} = \mu u_a u_b + p_\parallel n_a n_b + p_\perp (u_a u_b - n_a n_b + g_{ab}) , \]

where \( u^a \) is a unit timelike vector and \( n^a \) is a unit spacelike vector orthogonal to \( u^a \), \( u_a n^a = 0 \). The scalars \( p_\parallel \) and \( p_\perp \) are the pressures parallel and perpendicular to \( n^a \), respectively, and \( \mu \) is the energy-density as measured by an observer moving with four-velocity \( u^a \). Decomposing (2.1) with respect to \( u^a \) and \( n^a \), we find that the energy flux relative to \( u^a \) is zero, and that the isotropic pressure and anisotropic stress tensor (as measured by \( u^a \)) are given by

\[ p = \frac{1}{3}(p_\parallel + 2p_\perp) , \]

\[ \pi_{ab} = (p_\parallel - p_\perp)(n_a n_b - \frac{1}{3}(g_{ab} + u_a u_b)) . \]

*We shall follow the notation and conventions in Ellis [5] and in [1]; in particular, all kinematical quantities are defined therein. Also, Latin indices range from 0 to 3 and Greek indices from 1 to 3, and subscripts indicate differentiation with respect to the relevant spacetime coordinates.*
In particular, when the timelike congruence formed by $u_\alpha$ is geodesic, shear-free, and twist-free then the line element of the spacetime may be written as

$$\text{ds}^2 = -dt^2 + H^2(t) h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta .$$  \hfill (2.4)

The anisotropic stress-tensor (2.3) is now independent of $t$ and is defined entirely in terms of the Ricci tensor, $3R_{\alpha\beta}$, associated with the three-metric $h_{\alpha\beta}$:

$$\pi_{\alpha\beta} = 3R_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}3R .$$  \hfill (2.5)

Furthermore, $\pi_{\alpha\beta}$ must satisfy the conservation equation

$$\nabla^\alpha \pi_{\alpha\beta} = -\frac{1}{3}\nabla^\beta 3R ,$$  \hfill (2.6)

where $\nabla_\alpha$ is the covariant derivative with respect to the three-metric $h_{\alpha\beta}$.

In terms of the coordinates (2.4) the vector $n_\alpha = (0, n_\alpha)$ where $n_\alpha$ satisfies $h^{\alpha\beta}n_\alpha n_\beta = H^2(t)$. Thus, the only non-zero components of the anisotropic stress tensor are

$$\pi_{\alpha\beta} = (p_\parallel - p_\perp)(n_\alpha n_\beta - \frac{1}{3}H^2(t)h_{\alpha\beta}(x^\gamma)) .$$  \hfill (2.7)

The energy density $\mu$ and the isotropic pressure $p$ are both functions of $t$ only [3], and are related to $H$ by

$$\mu = \frac{1}{3} P^2 + \frac{3R}{2H^2} = \frac{6H^2 + 3R}{2H^2} ,$$

$$p = -\frac{1}{3}\dot{H} - \mu = -\frac{2\dot{H}}{H} - \frac{6H^2 + 3R}{6H^2} ,$$

where $\theta(t) = 3\dot{H}/H$ is the expansion of the timelike congruence.

**Theorem 1:** If the stress-energy tensor has the form (2.4) where the integral curves of $u^\alpha$ form a shear-free, irrotational, geodesic timelike congruence then there exists a coordinate system in which the metric has the form (2.4). In addition, the non-zero components of the anisotropic stress tensor (in these coordinates) are given by (2.7) where

(i) $p_\parallel - p_\perp = P(x^\alpha)/H^2(t) ,$

(ii) $n_\alpha = H(t)N_\alpha(x^\beta) ,$

such that $N_\alpha$ is a unit vector with respect to the spatial three-metric $h_{\alpha\beta}$.

**Proof:** Without loss of generality the three-metric can be diagonalised [4, 5]. We take the metric $h_{\alpha\beta}$ to have the form

$$h_{\alpha\beta} = \text{diag}(A^2, B^2, C^2) ,$$  \hfill (2.12)

where $A, B, C$ are independent of $t$. At least one component $n_\alpha$ is non-zero, say $n_1 \neq 0$. If $n_2 = 0$ (and/or $n_3 = 0$) then the component $\pi_{22}$ implies $(p_\parallel - p_\perp)H^2 = P(x^\gamma)$ where $P$ is as yet some undetermined function. If $n_2 \neq 0$ and $n_3 \neq 0$ then $\pi_{\alpha\beta}(x^\gamma) = (p_\parallel - p_\perp)n_\alpha n_\beta$ when $\alpha \neq \beta$. Therefore, there exist functions $f_\alpha(x^\gamma)$ such that $n_\alpha/n_\beta = f_\alpha^\beta$. Thus, the off-diagonal components of $\pi_{\alpha\beta}$ may be written as

$$\pi_{\alpha\beta} = (p_\parallel - p_\perp)(n_\alpha)^2f_\beta^\alpha , \quad (\alpha \neq \beta) .$$

(There is no summation over the repeated index $\alpha$ in the above equation.) Hence, there exists non-zero functions $G_\alpha(x^\gamma)$ such that

$$(p_\parallel - p_\perp)(n_\alpha)^2 = G_\alpha(x^\gamma) .$$  \hfill (2.14)
Two possibilities now exist. Either \( \pi_{11} \neq 0 \) or \( \pi_{11} = 0 \). In both cases, (2.7) together with (2.12) and (2.14) implies there exists a function \( P = P(x^\gamma) \) such that \( (p_\parallel - p_\perp)H^2 = P \).

Part (ii) of the theorem now follows directly from part (i) and expression (2.7) for \( \pi_{\alpha\beta} \). \( \square \)

**Corollary 1:** The anisotropic stress tensor may be written as

\[
\pi_{\alpha\beta} = P(x^\gamma)\{N_\alpha N_\beta - \frac{1}{3}h_{\alpha\beta}\} ,
\]

where \( N_\alpha \) and \( P \) satisfy \( h^{\alpha\beta}N_\alpha N_\beta = 1 \), and

\[
\frac{1}{6}\nabla_\alpha(3R) = \nabla_\beta(PN^\beta)N_\alpha - \frac{1}{3}\nabla_\alpha P + PN^\beta\nabla_\beta N_\alpha \quad \square
\]

The anisotropic stress tensor \( \pi_{\alpha\beta} \) has only two distinct eigenvalues which implies that the Weyl tensor for metric (2.4) is Petrov type D \( [10, 11] \). Conversely, if the metric is of the form (2.4) and \( \pi_{\alpha\beta} \) has two distinct eigenvalues then \( \pi_{\alpha\beta} \) must necessarily be of the form (2.12), and (2.4) can be interpreted as an anisotropic fluid, with four-velocity \( u^a = \delta^a_t \), whose flow lines are shear-free, twist-free, and geodesic.

The proof is essentially as follows: let \( e^{(A)}_\alpha \) be the two linearly independent eigenvectors corresponding to the same eigenvalue, \( \lambda \). Then the eigenvalue of the third eigenvector, \( e^{(1)}_\alpha \), is \( -2\lambda \) since \( \pi_{\alpha\beta} \) is traceless. Furthermore, \( e^{(1)}_\alpha \) is orthogonal to \( e^{(A)}_\alpha \). If \( e^{(1)}_\alpha \) is chosen to have unit norm then \( \pi_{\alpha\beta} \) can be decomposed as

\[
\pi^{\alpha\beta} = \lambda\{-2e^{(1)}_\alpha e^{(1)}_\beta + g^{AB}e^{(A)}_\alpha e^{(B)}_\beta\} ,
\]

where \( g^{AB}g_{BC} = \delta^A_C \) and \( g_{AB} = e^{(A)}_\alpha e^{(B)}_\beta \). The desired result follows from the fact that the three-metric, \( h_{\alpha\beta} \), has the decomposition

\[
h_{\alpha\beta} = e^{(1)}_\alpha e^{(1)}_\beta + g^{AB}e^{(A)}_\alpha e^{(B)}_\beta .
\]

### 3. A Particular Class of Solutions

We wish to solve the three-dimensional field equations (2.5) and (2.15). As a general prescription, we write the field equations in terms of a 2+1 split. We suppose that we can choose coordinates \((x, x^A)\) such that \( N_\beta \propto \partial_\beta x \). (This will always be possible if \( N_\alpha \) is hypersurface orthogonal.) We define base vectors \( e^{(A)}_\alpha = \partial_A x^\alpha \) and a lapse function \( M \) and a shift vector \( M^A \) via

\[
\frac{\partial x^\alpha}{\partial x} = MN^\alpha + MA\cdot e^{(A)}_\alpha .
\]

The extrinsic curvature and the intrinsic metric for the surface \( x = \text{const} \) are \( K_{AB} \) and \( g_{AB} \), respectively. They are related by

\[
\partial_x g_{AB} = 2MK_{AB} + 2\nabla_{(A}M_{B)} .
\]

It is unlikely that a general solution can be found. Hence, we impose some further physical constraints so that the matter content of the spacetimes under consideration is physically relevant. Specifically, we consider the case where the energy density and the isotropic pressure satisfy an equation of state of the form

\[
p = p(\mu).
\]

Thus, either (i) \( 3R = \text{const} \) or (ii) \( \frac{dp}{d\mu} = -\frac{1}{3} \) (see reference [1]). We shall only consider the former case here, \( 3R = R_0 = \text{const} \). (For a discussion of the latter case see [1].) In addition, we shall only look for solutions that have unit lapse and zero shift.
Using the field equations (2.5) and (2.15), and decomposing the Ricci tensor in terms of a 2+1 split [12, 13], we obtain the following equations:

\[ 3R_{AB} \equiv 2R_{AB} + 2K^C_{AB}K_{BC} - K_{AB} - \partial_c K_{AB} = (3R - P)g_{AB}/3 \quad , \tag{3.3} \]
\[ 3R_{AN} \equiv \nabla_B K_{AC} - \partial_A K = 0 \quad , \tag{3.4} \]
\[ 3R_{NN} \equiv -K_{AB}K^{AB} - \partial_x K = (2P + 3R)/3 \quad , \tag{3.5} \]

where \( K \) is the trace of the extrinsic curvature. The Ricci scalar is given by

\[ 3R = 2R - K^2 - K_{AB}K^{AB} - 2\partial_x K \quad . \tag{3.6} \]

(Note that the Ricci tensor for any two metric is \( 2R_{AB} = 2R_{gAB}/2 \).)

Under the conditions we have assumed here, equation (3.2) can be integrated to yield

\[ g_{AB} = \Phi(x, x^A)h_{AB}(x^C) \quad \tag{3.7} \]

with \( 2\lambda = \partial_x \Phi/\Phi \). Equation (3.4) implies that \( \lambda = \lambda(x) \) which in turn yields \( \Phi = \psi(x)\phi(x^a) \). We can absorb \( \phi \) into \( h_{AB} \). Thus, without loss of generality, the two-metric (3.7) can be expressed as

\[ g_{AB} = \psi(x)\phi(x^C)\delta_{AB} \quad . \tag{3.8} \]

Now the function \( 2\lambda = \psi'/\psi \). (We have introduced the notation \( \psi' = \partial_x \psi \).) For the remainder of the paper, we shall take \( x^A = (y, z) \).

The equation for the Ricci scalar (3.6) reduces to

\[ R_0 = 2R - 6\lambda^2 - 4\lambda' \quad . \tag{3.9} \]

The Ricci scalar for the metric (3.8) is easily calculated since the metric is conformally flat [17], and is given by

\[ 2R = -\frac{\nabla^2 \ln \phi}{\psi} \quad \tag{3.10} \]

(\( \nabla^2 \) is simply the flat space Laplacian). But (3.9) implies that \( 2R = 2R(x) \). Therefore, \( \phi \) must satisfy

\[ \nabla^2 \ln \phi + k\phi = 0 \quad \tag{3.11} \]

where \( k \) is some constant. Solving (3.11) is equivalent to finding the conformal factor for a two-space of constant curvature. A rescaling of the \( y \) and \( z \) coordinates enables us to write the general solution as [16]

\[ \phi = [1 + \frac{\kappa}{4}(y^2 + z^2)]^{-2} \quad , \tag{3.12} \]

where the constant \( \kappa \) has been normalised to take values \( \pm 1 \) and 0.

Thus, inserting the results (3.12) and (3.10) into (3.9), we obtain the following differential equation for \( \psi \):

\[ \psi \psi'' - \frac{1}{4}(\psi')^2 + \frac{R_0}{2}\psi^2 = \kappa \psi \quad . \tag{3.13} \]

We introduce a new function \( S(x) \) by letting \( \psi = S^2 \). The function \( S \) then satisfies the differential equation

\[ 2SS'' + (S')^2 + \frac{R_0}{2}S^2 = \kappa \quad . \tag{3.14} \]
The first integral of the above equation is

\[(S')^2 = \kappa + \frac{2\alpha}{S} - \frac{R_0}{6} S^2, \quad (3.15)\]

where \(\alpha\) is an integration constant. Equation (3.15) is very reminiscent of the Friedmann equation. Indeed an equation similar to (3.15) has been studied in the context of homogeneous and isotropic cosmological models [17, 18].

An expression for \(P\) is found by subtracting (3.3) from (3.5):

\[P = -\frac{S''}{S} + \left(\frac{S'}{S}\right)^2 - \frac{\kappa}{S^2}. \quad (3.16)\]

Equations (3.14) and (3.15) then imply

\[P = \frac{3\alpha}{S^3}, \quad (3.17)\]

which is in complete agreement with the conservation equation (2.16).

Thus, the metric

\[ds^2 = -dt^2 + H^2(t) \left\{ dx^2 + S^2(x) \left[ \frac{dy^2 + dz^2}{1 + \frac{\kappa}{3} (y^2 + z^2)} \right] \right\} \quad (3.18)\]

can be interpreted as an anisotropic fluid, with four-velocity \(u^a = \delta^a_t\), when \(\kappa = \pm 1\) or \(\kappa = 0\), and \(S(x)\) satisfies (3.15). The energy density, \(\mu\) and the isotropic pressure, \(p\), as measured by a comoving observer are given by (2.8) and (2.9), respectively. The pressures \(p_\perp\) and \(p_\parallel\) are

\[p_\perp = p - \frac{\alpha}{H^2 S^3}, \quad (3.19)\]

\[p_\parallel = p + \frac{2\alpha}{H^2 S^3}. \quad (3.20)\]

We note that if \(\alpha\) equals zero then the model does not represent an anisotropic fluid model since \(P = 0\). This implies that \(p_\parallel = p_\perp\), and thus, the stress-energy tensor (2.1) corresponds to a perfect fluid and the spacetime is consequently a perfect fluid FRW model. Therefore, we shall only consider the case of non-zero \(\alpha\).

A close examination of (3.15) reveals that it is remarkably similar to the zero pressure Friedmann equation. However, there is one notable exception; the constant \(\alpha\) must be strictly non-negative for the analogue to be complete since \(\alpha\) basically corresponds to the energy density, \(\rho\), in the Friedmann equation. In our analysis, \(\alpha\) simply represents an arbitrary integration constant, and thus may take on negative values.

Equation (3.15) is difficult to analyse in general. However, a qualitative analysis can easily be undertaken. First, we shall consider solutions of (3.17) that allow \(S(x)\) to tend to zero as unphysical. In other words, there exists a number \(S_{\min}\) such that \(0 < S_{\min} \leq S(x)\) for all values of \(x\). In addition, if \(R_0 > 0\) then \(S(x)\) must also be bounded from above, that is, there exists a number \(S_{\max} < \infty\) such that \(S(x) \leq S_{\max}\).

In terms of the Harrison-Robertson classification scheme [17, 18], there are four basic types of solutions for which \(S(x)\) is non-zero everywhere. These solutions are referred to as: (i) static \(S_1\) (unstable) and \(S_2\) (stable), where \(S'(x) = S_s = \text{const}\) and \(S'' = S''' = 0\); (ii) asymptotic \(A_2\), \(S_s \leq S \leq \infty\) where \(S_s\) is a static point, that is, \(S'(S = S_s) = S''(S = S_s) = 0\); (iii) monotonic \(M_2\), \(S_{\min} \leq S \leq \infty\) with
The static models are by far the simplest models. Equations (3.14) and (3.15) imply that \( S^2 = 2\kappa/R^0 \) (provided \( R^0 \neq 0 \) and \( \kappa \neq 0 \)) and \( \alpha = \kappa R^0/3 \). Thus, \( \text{sign}(R^0) = \kappa \). Equation (3.16) then gives \( P = -R^0/2 = \text{const.} \) Thus, the metric for the static models \( (\kappa = \pm 1) \) is
\[
\text{ds}^2 = -dt^2 + H^2(t) \left\{ dx^2 + \frac{dy^2 + dz^2}{1 + \frac{2}{3}(y^2 + z^2)^2} \right\} . \tag{3.21}
\]
(The coordinates have been rescaled to normalise the Ricci scalar, \( R^0 \), to \( 2\kappa \).) The above metric is a Kantowski-Sachs metric \[19\] with equal scale factors. The energy density and the isotropic pressure are given by (2.8) and (2.9), respectively, viz.
\[
\mu = 3\dot{H}^2 + \kappa , \tag{3.22}
\]
\[
p = -6\dot{H} + 3\dot{H}^2 + \kappa \quad \text{and} \quad \frac{3\dot{H}^2}{3H^2} . \tag{3.23}
\]
The anisotropic pressures associated with the metric (3.21) are
\[
p_\parallel = -\frac{2\dot{H}H + \dot{H}^2}{H^2} , \tag{3.24}
\]
\[
p_\perp = -\frac{2\dot{H}H + \dot{H}^2 + \kappa}{H^2} . \tag{3.25}
\]
The following anisotropic fluid with planar symmetry is another particular example:
\[
\text{ds}^2 = -dt^2 + H^2(t)\{ dx^2 + S^2(x)(dy^2 + dz^2) \} \tag{3.26}
\]
and corresponds to taking \( \kappa = 0 \). The only non-singular solution occurs when \( R^0 < 0 \) and \( \alpha < 0 \), and then
\[
S(x) = \left( \frac{12\alpha}{R^0} \right)^\frac{1}{2} \left[ \cosh \left\{ \sqrt{-3R^0/8}x \right\} \right]^{\frac{1}{2}} . \tag{3.27}
\]

4. Solutions with constant \( P \)

In this section, we shall look for solutions that have both \( 3R \) and \( P \) constant. First, we note that this assumption greatly simplifies the conservation equations (2.16). However, if \( P \) is not constant then the resulting models are inhomogeneous; consequently a more pertinent reason for choosing this ansatz is the fact that we would like to determine the spatially homogeneous SIGA models.

We shall assume that the matter associated with a spatially homogeneous model inherits its symmetries \[20\]; in particular, any scalars constructed from the stress-energy tensor should be independent of the spatial coordinates. For an anisotropic fluid this would imply that \( P \) is constant since \( P = P(x^\alpha) \). The vector \( N_\alpha \) must be both geodesic, \( N^\beta\nabla_\beta N_\alpha = 0 \), and expansion-free, \( \nabla^\alpha N_\alpha = 0 \), with respect to the three-metric \( h_{\alpha\beta} \) (if \( P = \text{const} \) and \( 3R = R^0 \)). We already know that there exist spatially homogeneous SIGA models with constant \( P \) and \( 3R \) (see \[21\]). Thus, we now ask the following two questions:(1) are the Mimoso and Crawford solutions \[21\] the only spatially homogeneous SIGA models, and (2) are there any spatially inhomogeneous models with \( P \) and \( 3R \) constant?
Thus, suppose that the Ricci tensor has the form
\[ 3R_{\alpha\beta} = (\alpha - \beta) N_\alpha N_\beta + \beta h_{\alpha\beta} \]
where the eigenvalues, \( \alpha \) and \( \beta \), are constant and \( N_\alpha \) is a unit vector. The contracted Bianchi identities imply that \( N_\alpha \) can only have twist and shear, and hence
\[ \nabla_\beta N_\alpha = \omega_{\alpha\beta} + \sigma_{\alpha\beta} \]
where \( \omega_{\alpha\beta} = \nabla_\beta N_\alpha \) is the twist tensor and \( \sigma_{\alpha\beta} = \nabla_\beta N_\alpha \) is the shear tensor. If the shear tensor is zero then a theorem due to Bona and Coll \([22, 23]\) implies that the three-metric \( h_{\alpha\beta} \) admits a \( G_3 \) as the maximal isometry group. If the shear is non-zero then we can use its unit orthogonal eigenvectors to form an orthogonal triad \( \{ N_\alpha, E_{(1)\alpha}, E_{(2)\alpha} \} \). The shear and twist can then be decomposed as
\[ \sigma_{\alpha\beta} = \sigma \left[ E_{(1)\alpha} E_{(1)\beta} - E_{(2)\alpha} E_{(2)\beta} \right] \]
\[ \omega_{\alpha\beta} = \omega \left[ E_{(1)\alpha} E_{(2)\beta} - E_{(2)\alpha} E_{(1)\beta} \right] \]
where the vorticity and shear scalars are given by \( 2 \omega^2 = \omega_{\alpha\beta} \omega^{\alpha\beta} \) and \( 2 \sigma^2 = \sigma_{\alpha\beta} \sigma^{\alpha\beta} \neq 0 \), respectively. The integrability conditions for \( N_\alpha \),
\[ \nabla_\gamma \nabla_\beta N_\alpha - \nabla_\beta \nabla_\gamma N_\alpha = N_\delta R_{\delta\alpha\beta\gamma} \]
can be employed to show that
\[ \omega' = \sigma' = (E_{(1)\alpha})' = (E_{(2)\alpha})' = 0 \]
where \( \cdot \cdot \cdot' \equiv N_\alpha \nabla_\alpha \cdot \cdot \cdot \) is the covariant derivative in the direction of \( N_\alpha \). Decomposing the covariant derivatives of \( E_{(A)\alpha} \) in terms of the triad, we find
\[ \nabla_\beta E_{(1)\alpha} = -\theta_2 E_{(2)\alpha} E_{(1)\beta} + \theta_1 E_{(2)\alpha} E_{(2)\beta} - \sigma N_\alpha E_{(1)\beta} - \omega N_\alpha E_{(2)\beta} \]
\[ \nabla_\beta E_{(2)\alpha} = -\theta_1 E_{(1)\alpha} E_{(2)\beta} + \theta_2 E_{(1)\alpha} E_{(1)\beta} + \omega N_\alpha E_{(1)\beta} + \sigma N_\alpha E_{(2)\beta} \]
where \( \theta_1 \) and \( \theta_2 \) are the expansions of \( E_{(1)} \) and \( E_{(2)} \), respectively. Equation (4.5) also yields three further equations:
\[ \alpha = 2(\omega^2 - \sigma^2) \]
\[ \sigma_1 - \omega_2 + 2 \sigma (\theta_1) = 0 \]
\[ \sigma_2 - \omega_2 + 2 \sigma (\theta_2) = 0 \]
where \( \sigma_1 = E_{(1)\alpha} \nabla_\alpha \sigma, \sigma_2 = E_{(2)\alpha} \nabla_\alpha \sigma \), etc.

Taking \( \{ N_\alpha dx^\alpha, E_{(1)\alpha} dx^\alpha, E_{(2)\alpha} dx^\alpha \} \) as an orthogonal basis of 1-forms for the metric and using Cartan’s structural equations, we obtain the extra integrability conditions:
\[ (\theta_2)' + (\theta_2)\sigma + (\theta_1)\omega = 0 \]
\[ (\theta_1)' - (\theta_2)\omega - (\theta_1)\sigma = 0 \]
\[ (\theta_2)_2 + (\theta_1)_1 + (\theta_2)^2 + (\theta_1)^2 + \beta = 0 \]
We now notice that (4.12) implies \( (\theta_2)' = -\frac{d}{dx} (\theta_2) \). Introducing coordinates \( x^\alpha = (x, y^A) \) such that \( x \) is the affine distance along the integral curves of \( N_\alpha \), (4.12) and (4.13) can then be integrated. At this stage the analysis splits into two parts: (i) \( \alpha \neq 0 \) and (ii) \( \alpha = 0 \).

(i): If \( \alpha \neq 0 \) then (4.14) can be employed to show that the only valid solution is \( \theta_1 = \theta_2 = 0 \), whence \( \beta = 0 \). Equations (4.3)–(4.11) then imply that both \( \omega \) and \( \sigma \) are constants. Hence, the metric must admit a \( G_3 \).
since all the eigenvalues and spin coefficients are constant (see [24] for a complete classification of all positive definite three-metrics that satisfy (4.1) and admit either a $G_3$ or a $G_4$ as the maximal isometry group). Thus, we have proved the following result:

**Theorem 2:** If $h_{\alpha \beta}$ is a positive definite Riemannian three-metric whose Ricci tensor has exactly two distinct constant eigenvalues then the metric is homogeneous if either (i) the shear of the principal Ricci direction corresponding to the simple eigenvalue is zero or (ii) the degenerate eigenvalue is zero.

We note that the Killing vectors of the three-space are also Killing vectors of the four-dimensional spacetime and hence, the resulting SIGA models are spatially homogeneous.

(ii) If $\alpha = 0$ then we can always choose $\omega = \sigma$, and consequently (4.13) implies $\theta_2 = -\theta_1 = \theta(y^A)$. Equations (4.9)–(4.11) yield

$$\theta_{,1} - \theta_{,2} = 2\theta^2 + \beta$$ \hspace{1cm} (4.15)
$$\sigma_{,1} - \sigma_{,2} = 2\sigma \theta$$ \hspace{1cm} (4.16)

We note that $N^\alpha$ is not hypersurface orthogonal. However, we can pick a new orthogonal triad $\{N^\alpha, \xi^{(+)\alpha}, \xi^{(-)\alpha}\}$, where

$$\xi^{(\pm)\alpha} = \left[ E^\alpha_{(1)} \pm E^\alpha_{(2)} \right] / \sqrt{2}$$ \hspace{1cm} (4.17)

such that $\xi^{(+)\alpha}$ is hypersurface orthogonal (we will label these surfaces by $z = \text{const}$) and such that $\xi^{(-)\alpha}$ is geodesic. If we let $y$ be the affine distance along the integral curves of $\xi^{(-)\alpha}$ then $(x,y)$ will be intrinsic coordinates for the surfaces $z = \text{const}$ and a 2+1 split can be employed to show that the three-metric can then be written as

$$3ds^2 = dx^2 + dy^2 + 2M_1 dx dz + 2M_2 dy dz + [M^2 + (M_1)^2 + (M_2)^2] dz^2$$ \hspace{1cm} (4.18)

where

$$M_1 = -y + g(z) \quad \text{and} \quad M_2 = x + h(z)$$ \hspace{1cm} (4.19)

and

$$M = \begin{cases} A(z) e^{\sqrt{-\beta}y} + B(z) e^{-\sqrt{-\beta}y} & \text{if } \beta < 0 \\ A(z) \cos(\sqrt{\beta}y) + B(z) \sin(\sqrt{\beta}y) & \text{if } \beta > 0 \end{cases}$$ \hspace{1cm} (4.20)

where $A, B, g$ and $h$ are arbitrary functions.

**Theorem 3:** If $h_{\alpha \beta}$ is a positive definite inhomogeneous Riemannian three-metric whose Ricci tensor has exactly two distinct constant eigenvalues then (1) the simple eigenvalue of the Ricci tensor is zero and (2) there exist coordinates $(x,y,z)$ such that the inhomogeneous metric is given by (4.18)–(4.20).

From theorems 2 and 3, we have the following corollary:

**Corollary 2:** If $h_{\alpha \beta}$ is a positive definite Riemannian three-metric whose Ricci tensor has exactly two distinct constant eigenvalues then $h_{\alpha \beta}$ is not homogeneous if and only if both the simple eigenvalue of the Ricci tensor is zero and the shear of the principal Ricci direction corresponding to the simple eigenvalue is nonzero.

We note that these spatially inhomogeneous models may be of some interest in the study of anisotropic fluid cosmologies. We also note that the three-metric, (4.18), possesses at most one Killing vector field (none in general) and that there exist examples of these spatially inhomogeneous models satisfying all of the relevant energy conditions.

9
4.1 Spatially Homogeneous Solutions

Let us explicitly construct some of the spatially homogeneous three-metrics. Now, since all of the spatially homogeneous cosmological models have at least one translational Killing vector field, say $\xi = \frac{\partial}{\partial x}$, we shall consider the following class of three-metrics as an illustrative example:

$$3ds^2 = A^2(y, z) dx^2 + B^2(y, z)(dy^2 + dz^2) . \quad (4.21)$$

The Ricci tensor components $R_{yy}$ and $R_{xx}$ for the above metric are identically zero, which implies that $\pi_{xy} = \pi_{xz} = 0$. Therefore, we must have $P N_1 N_2 = P N_1 N_3 = 0$. There are three possible solutions:

(i) $N_1 \neq 0$ and $N_2 = N_3 = 0$ with $P \neq 0$, (ii) $N_1 = 0$ and $P \neq 0$, and (iii) $P = 0$. We can neglect the third possibility since it would imply that the stress-energy tensor represents a perfect fluid, as was noted earlier.

$$\text{(i) } N_1 \neq 0; \quad N_2 = N_3 = 0$$

The expansion is $\nabla^\alpha N_\alpha \equiv A^{-2} N_{1,x} = 0$, and the acceleration equations are $\nabla^\alpha N_1 \equiv 0$, $\nabla^\alpha N_2 \equiv -N_x^2 A^{-3} A_y = 0$, $\nabla^\alpha N_3 \equiv -N_3^2 A^{-3} A_z = 0$. Hence, we can take $N_1 = A = 1$ since $N_\alpha$ is a unit vector. Thus, the metric belongs to the class of metrics discussed in section 3. Now, the metric must be the static homogeneous cosmological models have at least one translational Killing vector field, say $\xi$.

$$\text{(ii) } N_1 = 0$$

Now, $N_1 = N^1 = 0$ implies that there exists a coordinate transformation $(y, z) \to (v, w)$ such that $w$ is the affine-distance along the integral curves of $N^\alpha$; that is, $N^\alpha = \delta^\alpha_w$. Thus, since $N^\alpha$ is geodesic the three-metric (4.21) may be written as

$$3ds^2 = A^2(v, w) dx^2 + \alpha^2(v, w) dv^2 + 2 \beta(v) dw + dw^2 \ . \quad (4.22)$$

The coordinate transformations $w' = w + \int \beta(v) dv$, $v' = \int \beta(v) dv$ reduce the above metric to the form

$$ds^2 = A^2(v, w) dx^2 + B^2(v, w) dv^2 + dw^2 \quad (4.23)$$

(the primes have been dropped) and maintains the form of $N^\alpha$, $N^\alpha = \delta^\alpha_w$.

The expansion of $N_\alpha$ is $\nabla^\alpha N_\alpha \equiv A^{-1} A_w + B^{-1} B_w = 0$, which yields the equation $\partial_w (AB) = 0$. Also, the field equations imply that $R_{uw} \equiv -A^{-1} A_{uw} + A^{-1} B^{-1} A_w B_w = 0$, which implies that $\partial_w (A_v / B) = 0$. Hence, we find

$$A^2 = F(v) + G(w) \quad , \quad B^2 = \frac{g^2(v)}{F(v) + G(w)} \ , \quad (4.24)$$

where $F, G$, and $g$ are arbitrary functions (we can set $g = 1$ without loss of generality). Next, we can reduce the field equation $R_{uw} \equiv -B^{-1} B_{uw} - A^{-1} A_{uw} = (2P + R_0)/3$ to

$$\left( \frac{G_w}{F + G} \right)^2 = -\frac{2}{3}(2P + R_0) \ . \quad (4.26)$$

Therefore, we need only examine the two cases: (a) $G = 0$ and (b) $F = 0$.

(a) $G = 0$

Here, $A = A(v) = B^{-1}$ and equation (4.26) gives $2P + R_0 = 0$. Thus, $B$ can be set equal to 1. The field equations $R_{xx} = (R_0 - P) A^2/3$ and $R_{uv} = (R_0 - P) B^2/3$ both reduce to the single ordinary differential equation

$$A_{uv} = -\frac{1}{2} R_0 A \ . \quad (4.27)$$
We can rescale the \((x, v, w)\) coordinates so that \(R_0 = 2\kappa\). Then, the solution to (4.27) is
\[
A = \begin{cases} 
    a \sin(v) + b \cos(v) & \text{if } \kappa = 1 \\
    a \exp(v) + b \exp(-v) & \text{if } \kappa = -1 
\end{cases},
\]
where \(a\) and \(b\) are arbitrary constants. (When \(\kappa = 0\) the metric is the flat FRW model.) If \(\kappa = -1\) then there are essentially three distinct choices for \(A\)
\[
A = \begin{cases} 
    (i) & e^v \\
    (ii) & \cosh(v) \\
    (iii) & \sinh(v)
\end{cases}.
\]
(4.29)

However, in each of the above cases, the two-surface \(w = \text{constant}\) is a surface of constant negative curvature. Thus the three metrics are diffeomorphic. The metric admits a \(G_4\) with a simply-transitive \(G_3\) subgroup of Bianchi type III and a \(G_3\) subgroup of Bianchi type VIII acting on the two-dimensional space \(w = \text{constant}\). If \(\kappa = 1\) then the metric is a Kantowski-Sachs [19] metric
\[
ds^2 = -dt^2 + H^2(t)(\sin^2(v)dx^2 + dv^2 + dw^2) .
\]
(4.30)

Again, the three-metric is homogeneous and admits a \(G_4\).

(b) \(F = 0\)

In this case, \(A = A(w) = B^{-1}\). The field equations reduce to:
\[
\begin{align*}
A^{-2}R_{xx} &= -A^{-1}A_{ww} + A^{-2}A^2_{ww} = (R_0 - P)/3 , \\
B^{-2}R_{vv} &= -A^{-1}A_{ww} - A^{-2}A^2_{ww} = (R_0 - P)/3 , \\
R_{ww} &= -2A^{-2}A^2_{ww} = (2P + R_0)/3 .
\end{align*}
\]
(4.31)

We observe that \(P = R_0\), and that a solution only exists for \(R_0 < 0\). Without loss of generality, we can take \(R_0 = -2\), and then we have the solution \(A = \exp(\pm w)\). The solution is the Bianchi type VI\(_0\) metric
\[
ds^2 = -dt^2 + H^2(t)(e^{2w}dx^2 + e^{-2w}dv^2 + dw^2).
\]
(4.32)

Mimoso and Crawford [21] discussed Bianchi types I, III and Kantowski-Sachs metrics in the context of anisotropic fluid models. However, these are not the only SIGA models that can be interpreted as SIGA models. There are also Bianchi type II, VI, VIII and IX examples of SIGA models [24].

5. Conclusion

We have demonstrated that there exist coordinates such that the metric of a SIGA spacetime has the form (2.4), where \(3R_{\mu\nu}(h_{\gamma\delta}) = (\alpha - \beta)N_{\mu}N_{\nu} + \beta h_{\mu\nu}\). In particular, we examined models in which both of the eigenvalues of \(3R_{\alpha\beta}\) are constant; In this case, if either \(\beta\) or the shear of \(N_{\mu}\) is zero then the three-metric, \(h_{\mu\nu}\), is homogeneous and hence, the associated SIGA models are spatially homogeneous. A necessary condition for the three-metric to be inhomogeneous is \(\alpha = 0\). However, this condition is not sufficient. The static models discussed in section 3 are examples of spatially homogeneous models with \(\alpha = 0\).

In this paper and our recent paper [1], we have only been interested in the study of spacetimes which admit a SIG timelike congruences. However, it is also of interest to study spacetimes which admit a
SIG congruence which is either null or spacelike. We shall investigate the null case in a future paper. In the spacelike case, perhaps the most natural (and most simple) setting in which such a congruence may take on a physical role is that in which the spacelike congruence is related to the preferred spacelike direction associated with an anisotropic fluid source [see (2.1)]. Let us now briefly describe this case since the analysis involved is similar to that employed in this paper.

Thus, we study anisotropic fluid spacetimes with an energy-momentum tensor given by (2.1) in which the preferred spacelike vector \( u^a \) is shear-free, irrotational and geodesic. In analogy with (2.4), there exist coordinates in which \( n^a = \delta^a_x \) and the metric is given by

\[
ds^2 = dx^2 + L^2(x) l_{\xi \zeta}(x^\psi) dx^\xi dx^\zeta ,
\]

(5.1)

where \( \xi, \zeta, \psi = 0, 2, 3 \) \( x^\psi = (t, y, z) \) and \( l_{\xi \zeta}(x^\psi) \) are the components of a Lorentzian three-metric with signature +1 (and coordinates can always be chosen such that \( l_{\xi \xi} \) is diagonal). Exact solutions of the Einstein field equations can now be sought in a similar manner to those obtained earlier in this paper, particularly in the case in which the adapted coordinates \( (x, y, z) \) are also comoving coordinates (that is, in the case in which the anisotropic fluid four-velocity \( u^a \) can be aligned along the timelike coordinate axis). However, we note that these exact solutions will be somewhat different in form to those obtained earlier due to the Lorentzian signature of \( l_{\xi \xi} \). We also note that in the special perfect fluid subcase these solutions are related to the solutions of Stephani [25].

In particular, the results of Bona and Coll [26] on Lorentzian three-metrics can be utilized in the special case in which the three-dimensional Ricci tensor, \( ^3R_{\xi \zeta} \), associated with the metric \( l_{\xi \xi} \) has exactly two distinct constant eigenvalues. The results of section 4 can be generalized to the Lorentzian case and hypersurface homogeneous models can be obtained.

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Table 1:

| κ  | α  | R₀     | Type |
|----|----|--------|------|
| 0  | < 0| ≤ 0    | M₂   |
| 1  | < 0| < 0    | M₂   |
|    | > 0|        | O₂, S₂|
| -1 | < 0| < 0    | M₂   |
|    | > 0| < 0    | S₁, A₂, M₂ |

Table caption: Harrison-Robertson classification [17, 18] for anisotropic fluid solutions with \( S(x) \geq S_{\text{min}} > 0 \). (The definitions of the constants \( \kappa, \alpha, R₀ \) and the spatial types can be found in the text.)