A Batalin-Vilkovisky Algebra structure on the Hochschild Cohomology of Truncated Polynomials

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April 23, 2010

Abstract

We calculate the Batalin-Vilkovisky structure of $\text{HH}^*(\mathbb{C}^*(\mathbb{K}P^n; \mathbb{R}); \mathbb{C}^*(\mathbb{K}P^n; \mathbb{R}))$ for $\mathbb{K} = \mathbb{C}$ and $\mathbb{H}$, and $R = \mathbb{Z}$ and any field; and show that in the special case when $M = \mathbb{C}P^1 = S^2$, and $R = \mathbb{Z}$, this structure can not be identified with the BV-structure of $\mathbb{H}_*(LS^2; \mathbb{Z})$ computed by Luc Menichi in [19]. However, the induced Gerstenhaber structures are still identified in this case. Moreover, according to the work of Y.Felix and J.Thomas [8], the main result of the present paper eventually calculates the BV-structure of the rational loop homology, $\mathbb{H}_*(L\mathbb{C}P^n; \mathbb{Q})$ and $\mathbb{H}_*(L\mathbb{H}P^n; \mathbb{Q})$, of projective spaces.

1 Introduction

Let $M$ be a connected, closed oriented manifold of dimension $d$, and $LM$ the free loop space of $M$. In [5], Theorem 5.4.4.7 and 6.1 Chas and Sullivan defined a Batalin-Vilkovisky structure on the loop homology $\mathbb{H}_*(LM)$ inducing a Gerstenhaber structure, and a Lie algebra structure on the string homology $H_\text{st}^S(LM)$. In [3], Theorem 3 Cohen and Jones suggested an identification of the loop homology $\mathbb{H}_*(LM)$ with the Hochschild cohomology $\text{HH}^*(\mathbb{C}^*(M); \mathbb{C}^*(M))$ as graded algebras. Similar results for rational and real coefficients were proved by Felix,Thomas and Vigue-Poirrier in [11] and Merkulove in [20] respectively. What is interesting is that there is also a way of defining a Gerstenhaber [12], and even a BV-structure on $H^*(\mathbb{C}^*(M); \mathbb{C}^*(M))$, see [18], Theorem 1.4 a and [23], Theorem 3.1, and these two Gerstenhaber, and BV-structures are expected to be identified.

Also, as a direct consequence of [14], Theorem A, there is a nature isomorphism of the string homology of $M$ and the negative cyclic homology of the chain complex of $M$. In [13], Theorem 1.4 b, Luc Menichi defined a Lie algebra structure on $HC_\text{st}^*(\mathbb{C}^*(M))$, at least when $M$ is formal; and it is natural to expect that this Lie algebra structure can be identified with the string bracket.

For the computational aspect of this theory, there are not so many results yet. According to the author’s knowledge, only for some really familiar family of manifolds, some partial results of the loop homology and the string homology are computed, i.e. $S^n$’s,
and $CP^n$'s. Precisely, for $M = S^n$, in [19], Luc Menichi calculates the BV-structures of $\mathbb{H}_*(LS^2; \mathbb{Z}_2)$ and $\mathbb{H}_*(LS^2; \mathbb{Z})$.

In [4], Cohen, Jones and Yan developed a spectral sequence to compute the loop homology of $M = CP^n$. However, their paper did not mention the Gerstenhaber and the BV structures.

For the string homology, in [10], Felix, Thomas and Vigue-Poirrier has constructed a model for the string bracket and computed this for $CP^n$'s with rational coefficients which turns out to be degenerated.

For the Hochschild cohomology, there are not so many results either. In [19], Luc Menichi calculates the BV-structure of $HH^*(C^*(S^n; \mathbb{Z}_2); C^*(S^n; \mathbb{Z}_2))$, and shows that this BV-structure can not be identified with the one on $\mathbb{H}_*(LS^2; \mathbb{Z}_2)$ that he computes, but the induced Gerstenhaber structures can be identified in this case. In [27] Corollary 4.2, C.Westerland calculates the BV-structure of $HH^*(C^*(CP^n; \mathbb{Z}_2); C^*(CP^n; \mathbb{Z}_2))$. Both of them are working with the $\mathbb{Z}_2$ coefficients.

The main result of the present paper describes the Hochschild cohomology of the cochain complex of $CP^n$, and also of $HP^n$, with coefficients in the integers $\mathbb{Z}$, and in any field $k$ with $\text{char}(k) = 0$ and $\mathbb{F}_p$ with $\text{char}(\mathbb{F}_p) = p$, an arbitrary prime number other than 2. Precisely,

**Main Theorem:** Let $A = R[x]/(x^{n+1})$, where $|x| = 2m$ and $R = \mathbb{Z}$ and $k$; and also $\mathbb{F}_p$, but in this case we require that $n \neq kp - 1$. Then as a Batalin-Vilkovisky algebra:

$$HH^*(A; A) = R[x, u, t]/(x^{n+1}, u^2, (n + 1)x^nt, ux^n),$$

where $|x| = -2m$, $|u| = -1$ and $|t| = 2mn + 2(m - 1)$, and

$$\Delta(t^k x^l) = 0$$
$$\Delta(t^k ux^l) = (- (k + 1)n - k + l)t^k x^l.$$

If $n = kp - 1$, we have:

**Theorem 4.7** Let $A = \mathbb{F}_p[x]/(x^{kp})$, where $|x| = 2m$, as a BV-algebra:

$$HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^{kp}, v^2),$$

where $|x| = -2m$, $|v| = 2m - 1$ and $|t| = 2mn + 2(m - 1)$, and

$$\Delta(t^k x^l) = 0$$
$$\Delta(t^k vx^l) = lt^k x^{l-1}.$$

As an immediate consequence, we have
Corollary: Let \( A = R[x]/(x^{n+1}) \), where \( |x| = 2m \) and \( R = \mathbb{Z} \) and \( k \); and also \( \mathbb{F}_p \), but in this case we require that \( n \neq kp - 1 \). The Gerstenhaber bracket on \( HH^*(A; A) \) is determined as follows:

\[
\{ t^{k_1}x^{l_1}, t^{k_2}x^{l_2} \} = 0
\]

\[
\{ t^{k_1}x^{l_1}, t^{k_2}ux^{l_2} \} = (-k_1n - k_1 + l_1)t^{k_1+k_2}x^{l_1+l_2}
\]

\[
\{ t^{k_1}ux^{l_1}, t^{k_2}ux^{l_2} \} = ((k_1 + k_2 + 2)n + k_1 + k_2 - l_1 - l_2)t^{k_1+k_2}ux^{l_1+l_2}.
\]

In the case that \( A = \mathbb{F}_p[x]/(x^{kp}) \), where \( |x| = 2m \), the Gerstenhaber bracket is determined as:

\[
\{ t^{k_1}x^{l_1}, t^{k_2}x^{l_2} \} = 0
\]

\[
\{ t^{k_1}x^{l_1}, t^{k_2}ux^{l_2} \} = l_1t^{k_1+k_2}x^{l_1+l_2-1}
\]

\[
\{ t^{k_1}ux^{l_1}, t^{k_2}ux^{l_2} \} = -(l_1 + l_2)t^{k_1+k_2}ux^{l_1+l_2-1}.
\]

As a consequence, since \( \mathbb{C}P^1 = S^2 \), there comes:

**Theorem 4.1** As Batalin-Vilkovisky algebras, \( \mathbb{H}_*(LS^2; \mathbb{Z}) \) is not isomorphic to \( HH^*(C^*(S^2; \mathbb{Z}); C^*(S^2; \mathbb{Z})) \), but the induced Gerstenhaber algebra structures are isomorphic.

In [8], Theorem 1 Y. Felix and J. Thomas proved that the BV-structures on \( \mathbb{H}_*(LM; k) \) and on \( HH^*(C^*(M; k); C^*(M; k)) \) can be naturally identified. Therefore, the main result of the present paper eventually calculates the BV-structure of the rational loop homology, \( \mathbb{H}_*(LCP^n; k) \) and \( \mathbb{H}_*(LHP^n; k) \), of projective spaces. Precisely, we have:

**Theorem 4.3** As BV-algebras:

\[
\mathbb{H}_*(LCP^n; k) = k[x, u, t]/(x^{n+1}, u^2, x^nt, ux^n),
\]

with \( |x| = -2, \ |u| = -1 \) and \( |t| = 2n \); and

\[
\mathbb{H}_*(LHP^n; k) = k[x, u, t]/(x^{n+1}, u^2, x^nt, ux^n),
\]

with \( |x| = -4, \ |u| = -1 \) and \( |t| = 4n + 2 \); and in both cases,

\[
\Delta(t^kx^l) = 0
\]

\[
\Delta(t^kux^l) = (-(k + 1)n - k + l)t^kx^l.
\]

With the aid of the main theorem, there also comes the calculation of the negative cyclic cohomology of cochain complex of \( \mathbb{C}P^n \) which was identified by Jones with the string homology of \( \mathbb{C}P^n \) as graded modules. The Lie structure is also calculated. When the coefficients are in the rationals, the Lie bracket turns out to be trivial as expected.

**Theorem 4.4** As a Lie algebra,

\[
HC^m_*(k[x]/(x^{n+1})) = k, \ m = 2q, \ q = 0, 1, 2, ...
\]

\[
HC^m_*(k[x]/(x^{n+1})) = 0, \ m = 2q + 1, \ q = 0, 1, 2, ...
\]

\[
HC^m_*(k[x]/(x^{n+1})) = 0, \ m = -2q, \ q = 1, 2, ...
\]

\[
HC^m_*(k[x]/(x^{n+1})) = k^n, \ m = -2q + 1, \ q = 1, 2, ...
\]
with the Lie bracket $[,] = 0$.

The method of proving the main theorem is purely homological algebra. For the graded module structure of $HH^*(A, A)$, it is from the derived functoriality of the Hochschild cohomology and a standard 2-periodic resolution of truncated polynomials

$$(P^*(A), d) : 0 \rightarrow A \xrightarrow{0} A \xrightarrow{(n+1)x^n} A \xrightarrow{0} A \xrightarrow{(n+1)x^n} \cdots,$$

which was already known by many authors, [15] [26].

For the algebra structure, there comes the key step of the argument, i.e., we carefully constructed a chain map

$$\varphi : P(A) \rightarrow C_{\text{bar}}^*(A)$$

from the periodic resolution to the bar resolution inducing the isomorphism of homology as graded modules. Therefore, the product can be traced by the explicit formula of the cup product on the bar complex via $\varphi^*$. The algebra structure was also known by [4] and the identification in [3], but the approach in the present paper is more direct and necessary for getting the BV-structure.

The BV-structure comes from the checking of a relatively manageable formula for the $\Delta$-operator. This formula for $\Delta$ makes use of the Poincare dual basis. The Gerstenhaber structure is a direct consequence of the BV-structure.

The paper is organized as follows. In section 2, the Hochschild cohomology of differential graded algebras is reviewed. The explicit formula for the cup product and the $\Delta$-operator is obtained. Section 3 concentrates on the proof of the main theorem. Section 4 consists of some consequences of the main theorem.

## 2 An explicit formula of the $\Delta$-operator

In this section an explicit formula for the $\Delta$-operator which will play a key role in the later calculation is obtained.

According to [13], Theorem 1.6 when $A$ has certain symmetry, such as the Poincaré duality, the Gerstenhaber structure can be extended to a BV-structure as in the non-DG case, at least for the special case when the differential $d$ is trivial. To this end, we should employ the Connes’ boundary-operator with a slight modification of the sign.

**Connes’ boundary-operator for DG-algebra:** Let $A$ be a DG-algebra, the Connes’ boundary operator $B : A \otimes (sA)^{\otimes n} \rightarrow A \otimes (sA)^{\otimes n+1}$ on the Hochschild chain complex of $A$ with coefficients in itself, $C_\bullet(A; A)$, is defined by

$$B(a_0 \otimes (a_1, ..., a_n)) = \sum_{i=0}^n (-1)^{\sum_{k=0}^{i-1} |s_{a_k}| \sum_{k=i}^n |s_{a_k}|} 1 \otimes (a_i, ..., a_n, a_0, a_1, ..., a_{i-1}).$$

To get a BV-structure on the Hochschild cohomology $HH^*(A)$, we should use a sequence of isomorphisms coming from the duality of the algebra $A$ and the adjunction between the tensor and $Hom$ functors. The duality gives an isomorphism $Hom(T(sA), A) \cong$
$\text{Hom}(T(sA), A^*)$ as chain complexes, and the adjunction gives $\text{Hom}(T(sA), A^*) \cong \text{Hom}(A \otimes T(sA), k)$, where the right hand side is the dual of the Hochschild chain complex. Therefore, we can put the dual of the Connes’ boundary-operator onto $\text{Hom}(T(sA), A^*)$, the Hochschild cochain complex, via this string of isomorphisms. Chasing each of the isomorphisms carefully, we have:

**Proposition 2.1** The operator $\Delta : \text{Hom}(sA \otimes n, A) \to \text{Hom}(sA \otimes n, A)$ is given by

$$
\Delta(f)((a_1, ..., a_n)) = \sum_{j=1}^{n} (-1)^{|f|+|a_j|} \sum_{k=0}^{n} (-1)^{(|a_0|+|a_k|) \sum_{k=1}^{n} |a_k|} < 1, f((a_i, ..., a_n, a_0, a_1, ..., a_{j-1})) > a_j^*,
$$

for all $f \in \text{Hom}(sA \otimes n+1, A)$.

where $a_0 = a_j$ in this formula. That is the explicit formula of $\Delta$-operator with which we can do some concrete calculation; and by Luc Menichi’s theorem [18, Theorem 1.6], this $\Delta$-operator induces a BV structure on $HH^*(A; A)$ which induces the Gerstenhaber structure of $HH^*(A; A)$.

3 The main theorem

In this section, we prove the main theorem of the present paper. For convenience, $A$ will always mean $\mathbb{Z}[[x]]/(x^{n+1})$ throughout this section.

**Main Theorem:** Let $A = \mathbb{Z}[x]/(x^{n+1})$, where $|x| = 2$, then as a Batalin-Vilkovisky algebra:

$$HH^*(A; A) = \mathbb{Z}[x, u, t]/(x^{n+1}, u^2, (n+1)x^nt, ux^n),$$

where $|x| = -2$, $|u| = -1$ and $|t| = 2n$, and

$$\Delta(t^k x^l) = 0$$
$$\Delta(t^k ux^l) = (-k+1)n - k + l) t^k x^l.$$

3.1 The graded $R$-module structure

By the $Tor$ interpretation of $HH^*(A; A)$, it is much more convenient to take a simple projective resolution other than the bar resolution. We have:

**Proposition 3.1** The following

$$P_*(A) : \ldots \to \Sigma^{2(n+1)}(A \otimes A)^{y^n+y^{n-1}z+\cdots+z^n} \to \Sigma^2(A \otimes A)^{y-z} A \otimes A \mu \to A \to 0$$

gives a 2-periodical resolution of $A$ as $A \otimes A$-module with $P_{2k}(A) = \Sigma^{2k(n+1)}A \otimes A$ with $d_{2k}(a \otimes b) = y^n a \otimes b + y^{n-1} a \otimes zb + \cdots + a \otimes z^n b$ and $P_{2k+1} = \Sigma^{2k(n+1)+2}A \otimes A$ with
\[ d_{2k+1}(a \otimes b) = ya \otimes b - a \otimes zb, \text{ where } \Sigma \text{ is the degree increasing operator which makes the } \\
\text{differential of degree 0 }, \text{ and } y, z \text{ are generators of the copies of } A \text{ in } A \otimes A \text{ respectively.} \]

**Proof**: See [15] and [25]. □

After taking \( \text{Hom}_{A \otimes A}(A, A) \) of this periodical complex \( P_*(A) \), we get the following periodical cochain complex

\[ (P^*(A), d) : 0 \to A \xrightarrow{0} A \xrightarrow{(n+1)x^n} A \xrightarrow{0} A \xrightarrow{(n+1)x^n} \cdots \]

by chasing the differential. Therefore, a direct computation tells us:

**Proposition 3.2** As abelian groups,

\[
\begin{align*}
HH^0(A; A) &= A = \mathbb{Z}[x]/(x^{n+1}) \\
HH^{2q-1}(A; A) &= \ker (n+1)x^n = A \cong \mathbb{Z}^n \\
HH^{2q}(A; A) &= A/((n+1)x^n) \cong \mathbb{Z}^n \oplus \mathbb{Z}_{n+1}.
\end{align*}
\]

Where \( \mathbb{Z}_m \) stands for \( \mathbb{Z}/m\mathbb{Z} \) for typing convenience.

### 3.2 Key step: the construction of \( \varphi \)

To get the algebra structure of \( HH^*(A; A) \), we go back to \( \text{Hom}(TsA, A) \) on which the cup product is defined. To do this, we define \( A \otimes A \)-module maps

\[ \varphi_* : P_*(A) \to C^{\text{bar}}_*(A), \]

with

\[
\varphi_{2q} : \Sigma^{2q(n+1)} A \otimes A \to A \otimes (sA)^{\otimes 2q} \otimes A
\]

defined by

\[
\varphi_{2q}(1 \otimes 1) = \sum 1[x^{n-a_1}|x]x^{n-a_2}|x| \cdots |x^{n-a_q}|x]x^{\sum_{k=1}^q a_k}, \tag{1}
\]

and

\[
\varphi_{2q+1} : \Sigma^{2q(n+1)+2} A \otimes A \to A \otimes (sA)^{\otimes 2q+1} \otimes A
\]

defined by

\[
\varphi_{2q+1}(1 \otimes 1) = \sum 1[x|x^{n-a_1}|x|x^{n-a_2}|x| \cdots |x^{n-a_q}|x]x^{\sum_{k=1}^q a_k}, \tag{2}
\]

where the sum is taken over \( 0 \leq a_k < n, k = 1, 2, \ldots, q \) and \( \sum_{k=1}^q a_k \leq n \). We have the following key lemma:

**Lemma 3.3** \( \varphi^* = \text{Hom}_{A \otimes A}(\varphi, A) : C^*(A; A) \to P^*(A) \) is a cochain map.
Proof: Since $\Phi^* : (\text{Hom}_k(T(sA), A), \beta) \to (\text{Hom}_A(A \otimes T(sA) \otimes A, A), \beta')$ is an isomorphism of cochain complex, it suffices to show $d \circ (\varphi^* \circ \Phi^*) = (\varphi^* \circ \Phi^*) \circ \beta : \text{Hom}_k(T(sA), A) \to P^*(A)$, i.e. for any $f \in \text{Hom}_k(sA^\otimes m, A)$,

$$(d \circ \varphi_m^* \circ \Phi^*)(f)(1 \otimes 1) = (\varphi_{m+1}^* \circ \Phi^* \circ \beta)(f)(1 \otimes 1),$$

where $1 \otimes 1 \in P_m(A)$.

For $m = 2q$, we have on one side $(d_{2q} \circ \varphi_{2q}^*)(f) = 0$, since $d_{2q} = 0$. The calculation for the other side is more complicated. We have

$$\varphi_{2q+1}^* \circ \Phi^*(\beta(f))(1 \otimes 1) = \Phi^*(\beta(f))(\phi_{2q+1}(1 \otimes 1))$$

$$= \beta'(\Phi^*(f))(\sum_{0 \leq a_k < n} 1[x]x^{n-a_1}[x] \cdots [x^{n-a_q}][x]x\sum_{k=1}^q a_k)$$

$$= \sum_{0 \leq a_k < n} x\sum_{k=1}^q a_k \beta(f)(x, x^{n-a_1}, x, \cdots , x^{n-a_q}, x),$$

the last equality is from the commutativity of $A$ and that the degree of $x$ is even. Now as a typical term,

$$T = x\sum_{k=1}^q a_k \beta(f)(x, x^{n-a_1}, x, \cdots , x^{n-a_q}, x)$$

$$= x\sum_{k=1}^{q+1} a_k f(x^{n-a_1}, x, \cdots , x^{n-a_q}, x)$$

$$- \sum_{k=1}^q x\sum_{a_k=1}^{a_k} f(x, x^{n-a_1}, x, \cdots , x^{n-a_{a_k-1}}, x^{n-a_{a_k}+1}, x, \cdots , x^{n-a_q}, x)$$

$$+ \sum_{k=1}^q x\sum_{a_k=1}^{a_k} f(x, x^{n-a_1}, x, \cdots , x^{n-a_{a_k-1}}, x^{n-a_{a_k}+1}, x^{n-a_{a_k}+1}, \cdots , x^{n-a_q}, x)$$

$$- x\sum_{k=1}^q a_k f(x, x^{n-a_1}, \cdots , x, x^{n-a_q}).$$

Note that the other term

$$x\sum_{k=1}^q a_k \beta(f)(x, x^{n-a_1}, x, \cdots , x^{n-a_q}, x)$$

has a term

$$-x\sum_{a_k=1}^{a_k} f(x^{n-a_1}, x, \cdots , x^{n-a_q}, x)$$

as part of it, which cancels the first term

$$x\sum_{a_k=1}^{a_k} f(x^{n-a_1}, x, \cdots , x^{n-a_q}, x)$$

in

$$T = x\sum_{k=1}^q a_k \beta(f)(x, x^{n-a_1}, x, \cdots , x^{n-a_q}, x).$$
and
\[\sum_{k=1}^{q} a_k + 1 \beta(f)(x, x^{n-a_1}, \ldots, x^{n-a_q-1}, x)\]
has term
\[\sum_{k=1}^{q} a_k + 1 f(x, x^{n-a_1}, \ldots, x, x^{n-a_q})\]
which cancels the last term
\[-\sum_{k=1}^{q} a_k + 1 f(x, x^{n-a_1}, \ldots, x, x^{n-a_q})\]
in
\[T = \sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x, x^{n-a_q}, x);\]
also,
\[\sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x^{a_i+1}, x^{n-a_{i+1}+1}, \ldots, x^{n-a_q}, x)\]
have terms
\[-\sum_{k=1}^{q} a_k f(x, x^{n-a_1}, \ldots, x^{a_i+1}, x^{n-a_{i+1}+1}, \ldots, x^{n-a_q}, x)\]
which cancel terms
\[\sum_{k=1}^{q} a_k f(x, x^{n-a_1}, \ldots, x, x^{n-a_{i+1}}, \ldots, x^{n-a_q}, x)\]
in
\[T = \sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x, x^{n-a_q}, x);\]
and
\[\sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x^{n-a_i-1}, x, x^{n-a_i+1}, \ldots, x^{n-a_q}, x)\]
give rise to terms
\[\sum_{k=1}^{q} a_k f(x, x^{n-a_1}, \ldots, x^{n-a_i-1}, x^{n-a_i+1}, \ldots, x^{n-a_q}, x)\]
which cancel
\[-\sum_{k=1}^{q} a_k f(x, x^{n-a_1}, \ldots, x^{n-a_i-1}, x^{n-a_i+1}, x, \ldots, x^{n-a_q}, x)\]
in
\[T = \sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x^{n-a_q}, x).\]
Therefore, for each \(a_k, 0 < a_k < n - 1,\)
\[\sum_{k=1}^{q} a_k \beta(f)(x, x^{n-a_1}, \ldots, x^{n-a_q}, x)\]
is canceled by other terms in \(\varphi_{2q+1}(\beta(f))(1 \otimes 1).\)
Now, we only need to focus on the edge effects, i.e. those terms with \( a_k = n - 1 \) or 0, and those not get canceled in the way above. When \( a_1 = n - 1 \),

\[
x^{n-1} \beta(f)(x, x, x^n, \ldots, x^n, x)
\]
gives rise to one term

\[
x^n f(x, x^n, \ldots, x^n, x)
\]
which is unable to be canceled by the way above, but it gets canceled by a term that also could not be canceled in the way above coming from

\[
x^n \beta(f)(x, x, x^{n-1}, \ldots, x^n, x).
\]

When \( a_k = n - 1, \ 0 < k < q \), the terms

\[
-x^n f(x, x^n, x, \ldots, x, x^n, x, \ldots, x^n, x)
\]
coming from

\[
x^n \beta(f)(x, x^n, \ldots, x, x, x^{n-1}, \ldots, x^n, x)
\]
cancels those corresponding positive ones given by

\[
x^n \beta(f)(x, x^n, \ldots, x^{n-1}, x, x, \ldots, x^n, x).
\]

When \( a_q = n - 1 \),

\[
-x^n f(x, x^n, x, \ldots, x^n, x)
\]
given by

\[
x^{n-1} \beta(f)(x, x^n, \ldots, x, x)
\]
cancels the corresponding positive one given by

\[
x^n \beta(f)(x, x^n, x, \ldots, x^{n-1}, x, x, x, \ldots, x^n, x, x, x).
\]

Finally, when \( a_k = 0, \ 0 \leq k \leq q \), the terms which could not get canceled vanish automatically by the degree reason. Therefore, all the terms are mutually canceled, and \( \varphi_{2q}^+(\beta(f))(1 \otimes 1) \) vanishes as expected.

Now for \( m = 2q - 1 \), we have on one side

\[
(d_{2q-1} \circ \varphi_{2q-1}^* \circ \Phi^*)(f)(1 \otimes 1) = (n + 1)x^{n} \Phi^*(f)(\varphi(1 \otimes 1))
\]

\[
= (n + 1)x^n \hat{f} \left( \sum_{0 \leq a_k < n} 1[x|x^{n-a_1}|x| \ldots |x^{n-a_q-1}|x]x^{\sum_{k=1}^{q-1} a_k} \right)
\]

\[
= (n + 1)x^n \sum_{0 \leq a_k < n} f(x, x^{n-a_1}, x, \ldots, x^{n-a_q-1}, x)x^{\sum_{k=1}^{q-1} a_k}
\]

\[
= (n + 1)x^n f(x, x^n, x, \ldots, x^n, x).
\]

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For the other side, we have

\[
\varphi_{2q} \circ \Phi^*(\beta(f))(1 \otimes 1) = \beta'(\Phi^*(f))(\varphi_{2q}(1 \otimes 1)) = \sum_{0 \leq a_k < n} x^{\sum_{k=1}^q a_k} \beta(f)(x^{n-a_1}, x, \cdots, x^{n-a_q}, x),
\]

and

\[
x^{\sum_{k=1}^q a_k} \beta(f)(x^{n-a_1}, x, \cdots, x^{n-a_q}, x) = x^{n+\sum_{i=2}^q a_i} f(x, x^{n-a_2}, \cdots, x^{n-a_q}, x) \\
- \sum_{i=1}^q x^{\sum_{k=1}^q a_k} f(x^{n-a_1}, x, \cdots, x^{n-a_1+1}, x^{n-a_1+1}, x, \cdots, x^{n-a_q}, x) \\
+ \sum_{i=1}^{q-1} x^{\sum_{k=1}^q a_k} f(x^{n-a_1}, x, \cdots, x^{n-a_1}, x^{n-a_1+1}, x, \cdots, x^{n-a_q}, x) \\
+ x^{\sum_{k=1}^q a_k+1} f(x^{n-a_1}, x, \cdots, x, x^{n-a_q}).
\]

Similar to the previous case, most of the terms cancel each other, and the only exceptions are:

\[
x^n \beta(f)(x, x, x^{n-1}, \cdots, x^n, x)
\]
gives rise to one term

\[
x^n f(x, x^n, x, \cdots, x^n, x)
\]
that cannot be canceled; and each

\[
x^i \beta(f)(x^{n-i}, x, \cdots, x^n, x), \ i = 0, 1, \ldots, n-1,
\]
gives rise to the term

\[
x^n f(x, x^n, x, \cdots, x^n, x)
\]
that survives from canceling. Therefore,

\[
\varphi_{2q}(\beta(f))(1 \otimes 1) = (n+1)x^n f(x, x^n, x, \cdots, x^n, x)
\]
as expected. This completes the proof of the lemma. □

### 3.3 The graded commutative algebra structure

Note that, there are three distinguished elements \( \bar{x}, \bar{u} \) and \( \bar{t} \) in \( C^*(A; A) \), where

\[
\bar{x} \in \text{Hom}_k(k, A) \text{ with } \bar{x}(1) = x, \\
\bar{u} \in \text{Hom}_k(sA, A) \text{ with } \bar{u}(x^i) = ix^i, \text{ and} \\
\bar{t} \in \text{Hom}_k(sA \otimes sA, A) \text{ with } \bar{t}(x^i, x^j) = x^{i+j-(n+1)},
\]
with $|\bar{x}| = -2$, $|\bar{u}| = -1$ and $|\bar{t}| = 2n$. The significance of these elements is that they represent non-trivial cohomology classes, and moreover, give rise to the generators of $HH^*(A;A)$ as a graded commutative algebra.

**Lemma 3.4** The elements $\bar{x}, \bar{u}$ and $\bar{t}$ in $C^*(A;A)$ are cocycles but not coboundaries, hence present non-trivial cohomology classes in $HH^*(A;A)$.

**Proof:** Since

\[
\beta(\bar{x})(x^i) = x^i \bar{x}(1) - \bar{x}(1) x^i = x^i x - xx^i = 0, \quad 0 < i \leq n;
\]

and the cochain complex is non-negative, $\bar{x}$ represents a non-trivial class of $HH^0(A,A)$. Also, if $i + j \leq n$,

\[
\beta(\bar{u})(x^i \otimes x^j) = x^i \bar{u}(x^j) - \bar{u}(x^{i+j}) + \bar{u}(x^i)x^j = jx^{i+j} - (i+j)x^{i+j} + ix^{i+j} = 0;
\]

and if $i + j > n$,

\[
\beta(\bar{u})(x^i \otimes x^j) = x^i \bar{u}(x^j) + \bar{u}(x^i)x^j = 0, \quad \text{since} \ x^{i+j} = 0.
\]

Hence $\bar{u}$ is a cocycle. $\bar{u}$ is not a coboundary, since for any $g \in Hom_k(k,A)$,

\[
\beta(g)(x^i) = x^i g(1) - g(1)x^i = 0.
\]

Therefore, $\bar{u}$ represents a non-trivial class in $HH^1(A;A)$.

Lastly, for $\bar{t}$, we have

\[
\beta(\bar{t})(x^k, x^i, x^j) = x^k \bar{t}(x^i, x^j) - \bar{t}(x^{k+i}, x^j) + \bar{t}(x^k, x^{i+j}) - \bar{t}(x^k, x^i)x^j.
\]

At this point, we do a case by case discussion. When $k + i < n + 1$, $i + j < n + 1$,

\[
\beta(\bar{t})(x^k, x^i, x^j) = -\bar{t}(x^{k+i}, x^j) + \bar{t}(x^k, x^{i+j}) = -x^{k+i+j-(n+1)} + x^{k+i+j-(n+1)} = 0;
\]

when $k + i < n + 1$, $i + j \geq n + 1$ (similarly, $k + i \geq n + 1$, $i + j < n + 1$), $x^{i+j} = 0$ so

\[
\beta(\bar{t})(x^k, x^i, x^j) = x^k \bar{t}(x^i, x^j) - \bar{t}(x^{k+i}, x^j) = x^{k+i+j-(n+1)} - x^{k+i+j-(n+1)} = 0;
\]

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and when $k + i \geq n + 1, i + j \geq n + 1,\[
\beta(\bar{t})(x^k, x^i, x^j) = x^k \bar{t}(x^i, x^j) - \bar{t}(x^k, x^i)x^j \\
= x^{k+i+j-(n+1)} - x^{k+i+j-(n+1)} = 0.
\]

Therefore, $\bar{t}$ is a cocycle.

To see that it is not a coboundary, we have, $\forall g \in \text{Hom}_k(sA, A),\[
\beta(g)(x^n, x^n) = x^n g(x^n) + g(x^n)x^n = 2x^n g(x^n)
\]
which cannot be equal to $\bar{t}(x^n, x^n) = x^{n-1}$ for dimensional reasons. Therefore, $\bar{t}$ is not a coboundary, hence represents a non-trivial cohomology class in $HH^2(A; A)$. \hfill \Box

Lemma 3.5

1. As a commutative graded algebra, $HH^*(A; A)$ is generated by $\bar{x}$, $\bar{u}$ and $\bar{t}$.

2. $\varphi^*$ induces an isomorphism on homology.

Proof: By (1) and (2), $\varphi^*(\bar{x})(1 \otimes 1) = \bar{x}(1) = x$, so $\varphi^*(\bar{x}) = x \in A \cong HH^0(A; A);\[
\varphi^*(\bar{u})(1 \otimes 1) = \bar{u}(x) = x, \text{ so } \varphi^*(\bar{u}) = x \in \ker(n+1)x^n \cong HH^1(A; A); \text{ and } \varphi^*(\bar{t})(1 \otimes 1) = \sum_{k=0}^{n-1} \bar{t}(x^{n-k}, x)x^k = 1, \text{ hence } \varphi^*(\bar{t}) = 1 \in A/(n+1)x^n \cong HH^2(A; A).\]

Moreover, for $\bar{t}^q x^l \in \text{Hom}_k(sA^{2q}, A),$ we have

$$\varphi^*(\bar{t}^q x^l)(1 \otimes 1) = \sum_{0 \leq a_k < n} \bar{t}^q x^l(x^{n-a_1}, x, x^{n-a_2}, x, \ldots, x^{n-a_q}, x)x^\sum_{k=1}^q a_k$$

$$= \bar{t}^q x^l(x^n, x, \ldots, x^n, x)$$

$$= x^l,$$

hence

$$\varphi^*(\bar{t}^q x^l) = x^l \in A/(n+1)x^n \cong HH^{2q}(A; A);$$

and for $\bar{t}^q \bar{u} x^l \in \text{Hom}_k(sA^{2q+1}, A),$ we have

$$\varphi^*(\bar{t}^q \bar{u} x^l)(1 \otimes 1) = \sum_{0 \leq a_k < n} \bar{t}^q \bar{u} x^l(x, x^{n-a_1}, x, x^{n-a_2}, x, \ldots, x^{n-a_q}, x)x^\sum_{k=1}^q a_k$$

$$= \bar{t}^q \bar{u} x^l(x, x^n, x, \ldots, x^n, x)$$

$$= x^{l+1},$$

hence

$$\varphi^*(\bar{t}^q \bar{u} x^l) = x^{l+1} \in \ker(n+1)x^n \cong HH^{2q+1}(A; A).$$

Therefore, $\varphi^*$ induces a surjection on homology. Since we are computing a value of $Tor$ which is finitely generated using two projective resolutions, $\varphi^*$ induces an isomorphism on homology. Therefore, $\bar{x}$, $\bar{u}$ and $\bar{t}$ generate $HH^*(A; A)$ as an algebra. \hfill \Box

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Proposition 3.6  As a commutative graded algebra,

$$HH^*(A; A) = \mathbb{Z}[x, u, t]/(x^{n+1}, u^2, (n+1)x^n t, ux^n).$$

Proof: Let $x$, $u$ and $t$ denote the cohomology classes of $\bar{x}$, $\bar{u}$ and $\bar{t}$ respectively. The relation between them is straightforward on $P^*(A)$.

Precisely, consider $x = \varphi^*(\bar{x}) \in P^0(A) = Hom_{A\otimes A}(A \otimes A, A)$. We have $x^{n+1}(1 \otimes 1) = (x(1 \otimes 1))^{n+1} = x^{n+1} = 0 \in A$, so $x^{n+1} = 0 \in P^0(A)$. Therefore, $\bar{x}^{n+1} = 0 \in HH^*(A; A)$. The class $\bar{u}x^n = 0$ for the same reason. $\bar{u}^2 = 0$, since $\bar{u}^2(1 \otimes 1) = \varphi^2(\bar{u}^2)(1 \otimes 1) = \varphi^2(\varphi_2(1 \otimes 1) = \sum_{k=0}^{n-1} \bar{u}^2(x^{n-k}, x)x^k = 0 \in A$. The element $(n+1)x^n t = \varphi^*((n+1)\bar{x}^n \bar{t}) \in P^2(A)$ is a coboundary, i.e. the image of $1 \in P^1(A)$, hence vanishes in the homology $HH^*(A; A)$. All the other relations are generated by those four above. □

The algebra structure was also known to [4] and the identification in [3]; but the approach in the present paper is more direct and necessary for getting the BV-structure.

3.4 The BV-structure

We are now left to find the BV-structure. Since in a BV-algebra, we have equation [13]:

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|}\Delta(bc) + (-1)^{|a|-1|b|}b\Delta(ac)$$

$$- \Delta(a)bc - (-1)^{|a|}a(\Delta(b))c - (-1)^{|a|+|b|}ab\Delta(c),$$

it suffice to find the value of $\Delta(x), \Delta(x^2), \Delta(u), \Delta(t), \Delta(t^2), \Delta(tx), \Delta(tu)$ and $\Delta(ux)$; and these values determine $\Delta$ via the equation. This leads us to the following calculation:

$\Delta(x^k) = 0$ by the degree reason. By proposition [21] we take $\{x^k\}_{0 \leq k \leq n}$ as basis (hence $\{x^{n-k}\}_{0 \leq k \leq n}$ as dual basis), and have

$$\Delta(u)(1) = \sum_{k=0}^{n} - <1, u(x^k) > x^{n-k} = - <1, nx^n > 1 = -n,$$

hence $\Delta(u) = -n$.

$$\Delta(ux)(1) = \sum_{k=0}^{n} - <1, ux(x^k) > x^{n-k} = - <1, (n-1)x^n > x = -(n-1)x,$$

hence $\Delta(ux) = -(n-1)x$.

$$\Delta(t)(x^i) = \sum_{k=0}^{n} <1, t(x^k, x^i) > x^{n-k} - \sum_{k=0}^{n} <1, t(x^i, x^k) > x^{n-k} = 0,$$
since \(t(x^i, x^k) = t(x^k, x^i)\); hence \(\Delta(t) = 0\).

\[
\Delta(t^2)(x^i, x^j, x^h) = \sum_{k=0}^{n} <1, t^2(x^k, x^j, x^h) > x^{n-k}
- \sum_{k=0}^{n} <1, t^2(x^i, x^j, x^h) > x^{n-k}
+ \sum_{k=0}^{n} <1, t^2(x^j, x^h, x^k, x^i) > x^{n-k}
- \sum_{k=0}^{n} <1, t^2(x^h, x^i, x^j) > x^{n-k}.
\]

To make \(<1, \_\_\_\_ > \) non-zero, \(i+j+k+h - 2(n+1)\) should be equal to \(n\), i.e. \(i+j+k+h = 3n+2\). Therefore, all of \(\{k+i, i+j, j+h, h+k\}\) should be \(\geq n+1\), so

\[
\Delta(t^2)(x^i, x^j, x^h) = x^{i+j+h-2(n+1)} - x^{i+j+h-2(n+1)} + x^{i+j+h-2(n+1)} - x^{i+j+h-2(n+1)} = 0,
\]

hence \(\Delta(t^2) = 0\).

\[
\Delta(tx)(x^i) = \sum_{k=0}^{n} <1, tx(x^k, x^i) > x^{n-k} - \sum_{k=0}^{n} <1, tx(x^i, x^k) > x^{n-k}
= 0,
\]

hence \(\Delta(tx) = 0\).

\[
\Delta(tu)(x^i, x^j) = - \sum_{k=0}^{n} <1, tu(x^k, x^i, x^j) > x^{n-k}
- \sum_{k=0}^{n} <1, tu(x^j, x^k, x^i) > x^{n-k}.
\]

When \(i+j < n+1\), \(\forall k, <1, \_\_\_\_ >\) will vanish; hence \(\Delta(tu)(x^i, x^j) = 0\) in this case. When \(i+j \geq n+1\), to make \(<1, \_\_\_\_ >\) non-vanished, \(k\) should be equal to \(k_0 = 2n+1 - (i+j)\), then \(k_0 + i = 2n+1 - (i+j) + i = 2n+1 - j \geq n+1\), and similarly, \(j + k_0 \geq n+1\). Therefore, in this case,

\[
\Delta(tu)(x^i, x^j)
= - <1, tu(x^{k_0}, x^i, x^j) > x^{n-k_0} - <1, tu(x^i, x^j, x^{k_0}) > x^{n-k_0} - <1, tu(x^j, x^{k_0}, x^i) > x^{n-k_0}
= - jx^{(i+j)-(n+1)} - (2n - (i+j))x^{(i+j)-(n+1)} - ix^{(i+j)-(n+1)}
= -(2n+1)x^{(i+j)-(n+1)}.
\]
Therefore, $\Delta(tu) = -(2n + 1)t$.

Now we have all the data we need to determine $\Delta$. An induction on the powers of $x$ and $t$ tells us that:

$$\Delta(t^k x^l) = 0$$
$$\Delta(t^k ux^l) = (-(k + 1)n - k + l)t^k x^l.$$

That completes the proof of the main theorem. □

Note that there is no essential difference between $|x| = 2$ and $|x| = 2m$. In the later case, we go through the same argument word by word, and have

**Theorem 3.7** Let $A = \mathbb{Z}[x]/(x^{n+1})$, where $|x| = 2m$, then as a Batalin-Vilkovisky algebra:

$$HH^*(A; A) = \mathbb{Z}[x, u, t]/(x^{n+1}, u^2, (n + 1)x^n t, ux^n),$$

where $|x| = -2m$, $|u| = -1$ and $|t| = 2mn + 2(m - 1)$, and

$$\Delta(t^k x^l) = 0$$
$$\Delta(t^k ux^l) = (-(k + 1)n - k + l)t^k x^l.$$

4 Consequences of the main theorem

In this section, we give some consequences of the main theorem.

4.1 Consequence 1

**Theorem 4.1** As Batalin-Vilkovisky algebras, $\mathbb{H}_*(LS^2; \mathbb{Z}) \not\cong HH^*(C^*(S^2; \mathbb{Z}); (C^*(S^2; \mathbb{Z})))$, but the induced Gerstenhaber algebra structures are isomorphic.

**Proof:** In the special case when $n = 1$, $A = \mathbb{Z}[x]/(x^2)$, the main theorem gives: as a BV-algebra,

$$HH^*(A) = \mathbb{Z}[x, u, t]/(x^2, u^2, 2xt, ux),$$
$$\Delta(t^k x) = 0, \quad \Delta(t^k u) = -(2k + 1)t^k.$$

Recall Luc Menichi’s result in [19, Theorem 25], as a BV-algebra,

$$\mathbb{H}_*(LS^2; \mathbb{Z}) = \mathbb{Z}[a, b, v]/(a^2, b^2, ab, 2av),$$
$$\Delta(v^k a) = 0, \quad \Delta(v^k b) = (2k + 1)v^k + av^{k+1}.$$

This is not isomorphic to $HH^*(\mathbb{Z}[x]/(x^2); \mathbb{Z}[x]/(x^2))$ as BV-algebras, since by dimensional reason, any isomorphisms of algebras

$$\Phi : HH^*(\mathbb{Z}[x]/(x^2); \mathbb{Z}[x]/(x^2)) \rightarrow \mathbb{H}_*(LS^2)$$

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must maps \( x \) to \( \pm a \), \( u \) to \( \pm b \) and \( t \) to \( \pm v \) or \( \pm v + av^2 \), but none of them could be a BV-isomorphism. Since if \( \Phi(t) = \pm v \), then
\[
\Phi \circ \Delta(ut^k) = \pm(2k + 1)v^k,
\]
but
\[
\Delta \circ \Phi(ut^k) = \pm(2k + 1)v^k + av^{k+1} \neq \Phi \circ \Delta(ut^k).
\]
If \( \Phi(t) = \pm v + av^2 \), then by induction
\[
\Phi \circ \Delta(ut^k) = \Phi(-\pm(2k + 1)t^k) = \pm(2k + 1)v^k + kav^{k+1},
\]
but
\[
\Delta \circ \Phi(ut^k) = \Delta(\pm bv^k) = \pm(2k + 1)v^k + av^{k+1} \neq \Phi \circ \Delta(ut^k)
\]
when \( k \) is even. So there is no possibility for \( \Phi \) to be a BV-isomorphism. However, if we let
\[
\Phi : HH^*(\mathbb{Z}[x]/(x^2); \mathbb{Z}[x]/(x^2)) \to \mathbb{H}_*(L^2)
\]
given by
\[
\Phi(x) = a, \quad \Phi(u) = -b \quad \text{and} \quad \Phi(t) = v;
\]
checking the formula
\[
\{x, y\} = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}(\Delta x)y - a(\Delta b)
\]
shows that \( \Phi \) preserves the Gerstenhaber structure. \( \Box \)

4.2 Consequence 2

The whole argument also works for \( \mathbb{Q} \), or any field \( k \) with \( \mathbb{Q} \subset k \), coefficients, and we have:

**Theorem 4.2** Let \( A = \mathbb{k}[x]/(x^{n+1}) \), where \( |x| = 2m \), then as a BV-algebra:
\[
HH^*(A; A) = \mathbb{k}[x, u, t]/(x^{n+1}, u^2, x^n, u x^n),
\]
where \( |x| = -2m, \ |u| = -1 \) and \( |t| = 2mn + 2(m - 1) \), and
\[
\Delta(t^k x^l) = 0, \quad \Delta(t^k u x^l) = (-k + 1)n - k + l)t^k x^l.
\]

In [8], Y.Felix and J.Thomas proved that the BV-structures on \( \mathbb{H}_*(LM; k) \) and on \( HH^*(C^*(M; k); C^*(M; k)) \) can be naturally identified. Precisely, they showed:

**Theorem 8.1** Theorem 1: If \( M \) is 1-connected and the field of coefficients has characteristic zero then there exists a BV-structure on \( HH^*(C^*(M; k); C^*(M; k)) \) and an isomorphism of BV-algebras \( \mathbb{H}_*(LM) \cong HH^*(C^*(M; k); C^*(M)) \).

Therefore, theorem 4.2 eventually calculates the BV-structure of the rational loop homology, \( \mathbb{H}_*(L\mathbb{C}P^n; k) \) and \( \mathbb{H}_*(L\mathbb{H}P^n; k) \), of projective spaces. We have:
**Theorem 4.3** Let $k$ be any field containing $\mathbb{Q}$, then as BV-algebras:

$$\mathbb{H}_*(L \mathbb{C}P^n; k) = k[x, u, t]/(x^{n+1}, u^2, x^n t, u x^n),$$

with $|x| = -2$, $|u| = -1$ and $|t| = 2n$; and

$$\mathbb{H}_*(L \mathbb{H}P^n; k) = k[x, u, t]/(x^{n+1}, u^2, x^n t, u x^n),$$

with $|x| = -4$, $|u| = -1$ and $|t| = 4n + 2$; and in both cases,

$$\Delta(t^k x^l) = 0$$

$$\Delta(t^k u x^l) = (-k + 1)n - k + l t^k x^l.$$

**Proof:** $K^{P^n}$’s are formal, i.e., $C^*(K^{P^n})$ is quasi-isomorphic to $H^*(K^{P^n})$. Therefore, by the naturality of the Hochschild cohomology with respect to quasi-isomorphisms [11], 3 and [8], Theorem 1:

$$\mathbb{H}_*(L K^{P^n}; k) \cong HH^*(C^*(K^{P^n}; k); C^*(K^{P^n}; k)) \cong HH^*(H^*(K^{P^n}; k); H^*(K^{P^n}; k))$$
as BV-algebras. Theorem 4.2 completes the proof, when $m = 1$ and 2. \hfill $\square$

As another consequence of the main theorem, we have:

**Theorem 4.4** Let $A = k[x]/(x^{n+1})$, then the Lie bracket on the negative cyclic cohomology $HC^*_-(A)$ is trivial.

It is a consequence of the following calculations and propositions.

Since $\phi^*: C^*(A; A) \to P^*(A)$ is a chain map, it takes the $\Delta$-operator of $C^*(A; A)$ onto the periodic resolution $P^*(A)$ making it into a mixed complex $(P^*(A), d, B)$. Moreover, $(P^*(A), d, B)$ is quasi-isomorphic to $(C^*(A; A), \beta, \Delta)$ as mixed complexes, since $\phi^*$ is a quasi-isomorphism. Now we can use this simpler mixed complex to calculate the negative cyclic cohomology of $A$. Precisely, as a mixed complex, $B^{2q} = 0$, $B^{2q+1} = B(q) \oplus 0$, where

$$B(q): \overline{A} \to \overline{A}, \quad B(q)(x^k) = (-q(n + 1) - n + k)x^k.$$

Therefore, by tracing the defining double complex $BC^*_{**}(A)$ of the negative cyclic cohomology of $A$ given by the mixed complex $(P^*(A), d, B)$, we have:

**Proposition 4.5** Let $k$ be any field of characteristic 0, then

$$HC^0_m(k[x]/(x^{n+1})) = k, \quad m = 2q \quad q = 0, 1, 2, ...$$

$$HC^0_m(k[x]/(x^{n+1})) = 0, \quad m = 2q + 1 \quad q = 0, 1, 2, ...$$

$$HC^0_m(k[x]/(x^{n+1})) = 0, \quad m = -2q \quad q = 1, 2, ...$$

$$HC^0_m(k[x]/(x^{n+1})) = k^n, \quad m = -2q + 1 \quad q = 1, 2, ...$$
Since in $k$, all elements are invertible hence $B(q)$'s are surjective.

Now it is time to compute the Lie bracket on $HC_+^*(A)$ defined in [18]. First consider the Connes’ long exact sequence for negative cyclic cohomology

$$
\cdots \to HH^n(A) \xrightarrow{I} HC^n(A) \to HC^{n+2}_-(A) \xrightarrow{\partial} HH^{n+1} \to \cdots.
$$

By checking the definition of the connecting morphism carefully, we have

$$\partial : HC^{n+2}_-(A) \to HH^{n+1}(A); \quad \partial(a_{n+2}, a_{n+4}, \ldots) = B(a_{n+2}).$$

For $a_i \in HC^{m_i}_-(A)$, $i = 1, 2$, $[a_1, a_2]$ is defined to be $I(\partial(a_1) \cup \partial(a_2))$. Therefore, if $a \in HC^{2q}_-(A)$, then $\partial(a) = B(a_{2q+2}) = 0$, hence $[a, \_] = [\_, a] = 0$. If $a_i \in HC^{2q+1}_-(A)$, then $[a_1, a_2] \in HC^{2(q+1)+1}_-(A) = 0$, hence trivial. Therefore, the Lie bracket is trivial. □

### 4.3 Consequence 3

Also, we have the similar result for $\mathbb{Z}_p$, and any field $\mathbb{F}_p$ of characteristic $p$, coefficients, where $p$ is an arbitrary prime number other than 2.

When $n \neq kp - 1$, all the argument in the $\mathbb{Z}$ coefficients case still works; and we have:

**Theorem 4.6** Let $A = \mathbb{F}_p[x]/(x^{n+1})$, where $|x| = 2m$ and $n \neq kp - 1$, then as a BV-algebra:

$$
HH^*(A; A) = \mathbb{F}_p[x, u, t]/(x^{n+1}, u^2, x^n t, ux^n),
$$

where $|x| = -2m$, $|u| = -1$ and $|t| = 2mn + 2(m - 1)$, and

$$
\Delta(t^k x^l) = 0,
\Delta(t^k u x^l) = -(k + 1)n - k + l)t^k x^l.
$$

When $n = kp - 1$, we have:

**Theorem 4.7** Let $A = \mathbb{F}_p[x]/(x^{kp})$, where $|x| = 2m$, as a BV-algebra:

$$
HH^*(A; A) = \mathbb{F}_p[x, v, t]/(x^{kp}, v^2),
$$

where $|x| = -2m$, $|v| = 2m - 1$ and $|t| = 2mn + 2(m - 1)$, and

$$
\Delta(t^k x^l) = 0,
\Delta(t^k v x^l) = lt^k x^{l-1}.
$$

**Proof:** We still use the 2-periodical resolution and the bridge $\varphi^*$ between $C^*(A; A)$ and $P^*(A)$. In this case, $P^*(A)$ turns out to be

$$
0 \to A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A \to \cdots.
$$
Let’s take a close look at the following elements in this case.

\[ x \in P^0(A) = A, \quad x(1) = x, \]
\[ v \in P^1(A) = A, \quad v(1) = 1, \text{ and} \]
\[ t \in P^2(A) = A, \quad t(1) = 1. \]

We have \(|x| = -2m, |v| = 2m - 1, \text{ and } |t| = 2mn + 2(m - 1)|; \text{ and these elements correspond via } \varphi^* \text{ to} \]
\[ \bar{x} \in \text{Hom}(\mathbb{F}_p, A), \quad \bar{x}(1) = x, \]
\[ \bar{v} \in \text{Hom}(sA, A), \quad \bar{v}(x^j) = ix^{i-1}, \text{ and} \]
\[ \bar{t} \in \text{Hom}(sA^{\otimes 2}, A), \quad \bar{t}(x^i, x^j) = x^{i+j-(n+1)}. \]

\( \bar{v} \) is a cocycle because, if \( i + j < n \),
\[ \beta(\bar{v})(x^i, x^j) = x^i\bar{v}(x^j) - \bar{v}(x^{i+j}) + \bar{v}(x^i)x^j = ix^{i+j-1} - (i + j)x^{i+j-1} + jx^{i+j-1} = 0; \]
if \( i + j = n + 1 \),
\[ \beta(\bar{v})(x^i, x^j) = x^i\bar{v}(x^j) + \bar{v}(x^i)x^j = ix^{i+j-1} + jx^{i+j-1} = (n+1)x^n = kpx^n = 0 \in \mathbb{F}_p[x]/(x^{kp}); \]
and if \( i + j > n + 1 \), every term vanishes, so \( \beta(\bar{v}) = 0 \).

These elements represent the generators of \( HH^*(A; A) \) as an associative algebra. \( v^2 = 0 \) because \( v^2(1 \otimes 1) = \bar{v}^2(\varphi_2(1 \otimes 1)) = -\frac{n(n+1)}{2}x^{n-1} = -\frac{(kp-1)kp}{2}x^{n-1}; \text{ if } k = 2l, \text{ then} \)
\[ -\frac{(kp-1)kp}{2} = -(kp-1)lp = 0 \in \mathbb{F}_p, \text{ and if } k = 2l + 1, \text{ then } kp - 1 = 2l' \text{ since } p \text{ is odd, so} \]
\[ -\frac{(kp-1)kp}{2} = l'kp = 0 \in \mathbb{F}_p. \] Then by checking the formula of \( \Delta \), we get the result. \( \square \)

In [19], Luc Menichi conjectured that for any prime \( p \), the free loop space modulo \( p \) of the complex projective space \( \mathbb{H}_n(LCP^{p-1}; \mathbb{Z}_p) \) is not isomorphic as BV-algebras to the Hochschild cohomology \( HH^*((\mathbb{C}P^{p-1}; \mathbb{Z}_p); H^*(\mathbb{C}P^{p-1}; \mathbb{Z}_p)) \). He also pointed that: in [1], M.Bökstedt and I.Ottosen have announced the computation of BV-structure of \( \mathbb{H}_n(LCP^n; \mathbb{Z}_p) \). Therefore, combining with theorem 4.7 this will give a complete answer of Menichi’s conjecture.

There is no essential difference between \( p = 2 \) and \( p = \) other prime numbers, so we have:

**Theorem 4.8** Let \( A = \mathbb{F}_2[x]/(x^{n+1}) \), where \(|x| = 2m \). If \( n \) is even, as a BV-algebra:

\[ HH^*(A; A) = \mathbb{F}_2[x, u, t]/(x^{n+1}, u^2, x^nt, ux^n), \]
where \(|x| = -2m, |u| = -1 \) and \(|t| = 2mn + 2(m - 1), \text{ and} \)
\[ \Delta(t^kx^l) = 0 \]
\[ \Delta(t^kux^l) = (-k + l)t^kx^l; \]

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if $n$ is odd, as a BV-algebra:

$$HH^*(A; A) = \mathbb{F}_2[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1}),$$

where $|x| = -2m$, $|v| = 2m - 1$ and $|t| = 2mn + 2(m - 1)$, and

$$\Delta(t^k x^l) = 0,$$
$$\Delta(t^k vx^l) = lt^k x^{l-1},$$

especially when $n = 1$, as a BV-algebra:

$$HH^*(A; A) = \mathbb{F}_2[x, v, t]/(x^2, v^2 - t) \cong \Lambda[x] \otimes \mathbb{F}_2[v],$$

where $|x| = -2m$ and $|v| = 2m - 1$ and

$$\Delta(v^k) = 0,$$
$$\Delta(v^k x) = kv^{k-1}.$$

Proof: Every step is exactly the same as the $\mathbb{F}_p$ coefficients case, except that when $n$ is odd, $v^2(1 \otimes 1) = \frac{n(n+1)}{2}x^{n-1} = \frac{n+1}{2}x^{n-1} = \frac{n+1}{2}tx^{n-1}(1 \otimes 1).$ 

This recalculates the results in [19], [26] and [27].

Acknowledgement

The author would like to thank Don Stanley for directing his attention to this problem. It was also Don who suggests him to construct the chain map $\varphi$ in the key step. He would also like to thank the referee for the careful revision and providing several instructive suggestions on improving this work, and J. Stasheff for showing interest to this work and warm encouragement to the author.

References

[1] M.Bökstedt and I.Ottosen, The homology of the free loop space on a projective space, talk at the first Copenhagen Topology Conference, Sept 1-3, [http://www.math.ku.dk/conf/CTC2006/2006](http://www.math.ku.dk/conf/CTC2006/2006).

[2] R-O. Buchweitz, E. Green, N, Snashall and O. Solberg, Multiplicative structures for Koszul algebras, Preprint, 2005, [arXiv:math.RA/02508117](http://arxiv.org/abs/math.RA/02508117).

[3] R.L. Cohen and J.D.S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (2002), no.4, 773-798.
[4] R.L. Cohen, J.D.S. Jones and J. Yan, *The loop homology algebra of spheres and projective spaces*, Preprint, October 2002, arXiv:math.AT/0210353 v1.

[5] M. Chas and D. Sullivan, *String topology*, Preprint, CUNY, November 1999, math.GT/9911159. To appear in Ann. of Math.

[6] X. Chen, *On a general chain model of the free loop space and string topology*, Preprint, 2007, arXiv:0708.1197.

[7] R.L. Cohen and A.A. Voronov *Notes on string topology*, Preprint, March 2005, arXiv:math.GT/0503625 v1.

[8] Y. Félix and J. Thomas, *Rational BV-algebra in string topology*, Preprint, May 2007, arXiv:math.AT/0705.4194 v1.

[9] Y. Félix, J. Thomas and M. Vigué-Poirrier, *Loop homology algebra of a closed manifold*, Preprint, June 2003, arXiv:math.AT/0203137 v2.

[10] Y. Félix, J. Thomas and M. Vigué-Poirrier, *Rational string topology*, *J. Eur. Math. Soc. (JEMS)* 9 (2007), no. 1, 123–156.

[11] Y. Félix, J. Thomas and M. Vigué-Poirrier, *Gerstenhaber duality in Hochschild cohomology*, *Journal of pure and applied algebra* 199 (2005), 43-59.

[12] M. Gerstenhaber, *The cohomology structure of an associative ring*, *Ann. of Math.* 78 (1963), no.2, 267-288.

[13] E.Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, *Comm. Math. Phys.* 159 (1994), no.2, 265-285.

[14] J.D.S. Jones, *Cyclic homology and equivariant homology*, *Invent.Math.* 87 (1987) 403-423.

[15] J.L. Loday, *Cyclic homology*, Berlin, Heidelberg, New York: Springer-Verlag, 1971.

[16] P. Lambrechts and D. Stanley, *The rational homotopy type of configuration spaces of two points*, *Ann. Inst. Fourier, Grenoble.* 54 (2004), no.4 1029-1052.

[17] P. Lambrechts and D. Stanley, *Poincaré Duality and commutative differential graded algebras*, Preprint, 2006, arXiv:math.AT/0701309 v1.

[18] L. Menichi, *Batalin-Vilkovisky algebra and cyclic cohomology of Hopf algebras*, *K-Theory*. 32 (2004), 231-251.

[19] L. Menichi, *String topology for spheres*, Preprint, September 2006, arXiv:math.AT/0609304 v1.

[20] S. Merkulov, *De Rham model for string topology*, *Int. Math. Res. Not.* (2004) no.55, 2955-2981.
[21] S. Siegel and S. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. 79 (1999) no.1, 131-157.

[22] T. Tradler, *Infinity-inner-product on A-infinity algebras*, Preprint, January 2002, arXiv:math.AT/0108027 v2.

[23] T. Tradler, *The BV algebra on hochschild cohomology induced by infinity inner products*, Preprint, October 2002, arXiv:math.QA/0210150 v1.

[24] T. Tradler and M, Zeinalian, *On the cyclic Deligne conjecture*, J. Pure Appl. Algebra. 204 2006) no.2, 280-299.

[25] C.A. Weibel, *An introduction to homological algebra*, Cambridge: Cambridge University Press, 1994.

[26] C. Westerland, *Dyer-Lashof operations in the string topology of spheres and projective spaces*, Math.Z. 250 (2005) 711-727.

[27] C. Westerland, *String homology of spheres and projective spaces*, AGT, 7 (2007) 309-325.

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