Distributed Consensus of Nonlinear Multi-Agent Systems With Mismatched Uncertainties and Unknown High-Frequency Gains
(Extended Version)
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Abstract—This brief addresses the distributed consensus problem of nonlinear multi-agent systems under a general directed communication topology. Each agent is governed by higher-order dynamics with mismatched uncertainties, multiple completely unknown high-frequency gains, and external disturbances. The main contribution of this brief is to present a new distributed consensus algorithm, enabling the control input of each agent to require minimal information from its neighboring agents, that is, only their output information. To this end, a dynamic system is explicitly constructed for each agent to generate a reference output. Theoretical and simulation verifications of the proposed algorithm are rigorously studied to ensure that asymptotic consensus can be achieved and that all closed-loop signals remain bounded.

Index Terms—Consensus, multi-agent systems, higher-order systems, directed graphs, uncertain dynamics.

I. INTRODUCTION
Distributed control of multi-agent systems has attracted considerable attention over the last two decades, triggered mainly by its wide potential applications and theoretical challenges. As a fundamental problem within the scope of distributed control, designing consensus algorithms, according to which all agents strive to reach an agreement on certain states of interest using only local interactions, has become an active research branch [1]–[3]. In many applications (e.g., course control of marine vessels [4] and combustion control systems [5]), the high-frequency gain and even its sign may not be known a priori.

For individual systems with unknown high-frequency gains, the Nussbaum gain technique presented in [6] has been successfully applied to deal with the control problem [7], [8]. In case the boundaries of the gains are available, adaptive consensus algorithms have been proposed in [9] for first- and second-order agents with unknown high-frequency gains. In [10], the consensus output regulation problem has been addressed without any knowledge of high-frequency gains.

In order to deal with uncertain mismatched nonlinear multi-agent systems in the strict-feedback form, distributed adaptive backstepping control strategies have been presented in [11]–[13]. Within the framework of the prescribed performance technique, some research methods have emerged to solve the distributed leader-following control problem for nonlinear multi-agent systems with mismatched uncertainties [14], [15]. For higher-order nonlinear multi-agent systems with unknown control gains, adaptive consensus approaches have been developed by introducing novel Nussbaum functions in [16], [17]. Nevertheless, these algorithms [9], [10], [16], [17] require that the signs of high-frequency gains are unknown but identical. Recently, such a requirement has been removed in [18], [19], where a sub-Lyapunov function candidate was skillfully constructed for each agent to analyze the stability of the closed-loop system. Note that these aforementioned results [18], [19] can be applied only to a relatively simple class of nonlinear multi-agent systems, where each agent satisfies the matching condition and contains only one unknown high-frequency gain. However, various practical systems do not meet the condition and may involve multiple unknown high-frequency gains, as shown in [7], [8]. Another restrictive assumption typically made on higher-order nonlinear multi-agent environments is that all states of the neighboring agents must be available for use in the control law implementation of each agent. This assumption presents a formidable challenge when only the outputs of neighbors can be measured.

This brief investigates the consensus problem of nonlinear multi-agent systems with mismatched uncertainties and multiple unknown high-frequency gains under general directed graphs. The main differences between our work and the existing results can be emphasized as follows. (i) Compared with previous works on consensus with unknown high-frequency gains [9], [16]–[19], we consider a more general multi-agent system in which each agent is described by higher-order nonlinear dynamics with the mismatched condition, multiple unknown high-frequency gains, and unknown external disturbances. (ii) Contrary to [9], [16]–[19], the control input can be derived for each agent without requiring any additional information from its neighboring agents other than their outputs, thereby significantly alleviating the communication load in a multi-agent system. Furthermore, the communication graph in this brief is only assumed to have a directed spanning tree. This assumption is less stringent than the undirected
connected graph \cite{9}, \cite{17} and the strongly connected graph \cite{16}, \cite{18}. (iii) It should be noted that the solutions in \cite{12}, \cite{17} require each agent to know some prior information on the dynamics of its neighbors such that the adaptive updating laws can be explicitly designed to estimate the unknown dynamics parameters related to the neighbors. In contrast, no preliminary knowledge of the neighbors’ dynamics is needed in our work. Moreover, it is no longer necessary for the agent to account for the uncertainties associated with the dynamics of the neighbors.

Notation: Throughout the brief, we denote with \(1_m\) and \(0_m\), respectively, the \(m\)-vector of all ones and all zeros, and we let \(I_m\) denote the \(m\)-dimensional identity matrix. For a vector function \(u(t)\), it is said that \(u \in L_\infty[0, t_f]\), if \(\sup_{0 \leq t < t_f} ||u(t)|| < \infty\) and \(u \in L_p[0, t_f]\), if \((\int_0^{t_f} ||u(t)||^p dt)^{1/p} < \infty\), \(p = 1, 2\). Let \(y^{(n)}\) denote the \(n\)th derivative of \(y\).

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Graph Theory

In this work, a weighted directed graph \(G = (\mathcal{V}, \mathcal{E})\) with the node set \(\mathcal{V} = \{1, \ldots, n\}\) and the edge set \(\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}\) is used to describe the communication topology among the \(n\) agents. An edge \((i, j) \in \mathcal{E}\) indicates that node \(j\) has access to the information of node \(i\), and node \(i\) is a neighbor of node \(j\). The set of all neighbors of node \(i\) is denoted by \(N_i\). A directed path from node \(i_1\) to node \(i_p\) is a sequence of ordered edges in the form of \((i_m, i_{m+1}), m = 1, \ldots, p - 1\). A directed graph is said to contain a directed spanning tree if there exists at least a node such that the node has directed paths to all other nodes in \(G\). The weighted adjacency matrix \(A_n = [a_{ij}] \in \mathbb{R}^{n \times n}\) associated with \(G\) is defined by \(a_{ij} > 0\) if \((j, i) \in \mathcal{E}\), and \(a_{ij} = 0\) otherwise.

The Laplacian matrix \(L_n \in \mathbb{R}^{n \times n}\) associated with \(G\) is defined as \(L_n = D_n - A_n\), where \(D_n = \text{diag}(d_1, \ldots, d_n)\) is the in-degree matrix with \(d_i = \sum_{j=1}^{n} a_{ij}\) being the weighted in-degree of node \(i\).

B. Problem Statement

Consider a multi-agent system with \(n\) agents, labeled as agents \(1\) to \(n\), under a directed interaction topology. The dynamics of the \(i\)th agent, \(i = 1, \ldots, n\), is described by

\[
\dot{x}_{i,\ell} = g_{i,\ell}x_{i,\ell+1} + \theta_{i,\ell}^T \varphi_{i,\ell}(\tilde{x}_{i,\ell}) + \tau_{i,\ell}(t)
\]

\[
\dot{\tilde{x}}_{i,m} = g_{i,m}u_{i} + \theta_{i,m}^T \varphi_{i,m}(\tilde{x}_{i,m}) + \tau_{i,m}(t)
\]

for \(\ell = 1, \ldots, m - 1\), \(\ell, m \in [1, \ldots, p]\) and \(p \in \mathbb{R}^p\) for \(p = 1, \ldots, m\), where \(u_i \in \mathbb{R}\) and \(x_{i,1} \in \mathbb{R}\) are, respectively, the control input and output of the \(i\)th agent, \(g_{i,p} \in \mathbb{R}\) are the high-frequency gains of the agent, \(\tau_{i,p}(t)\) denote uncertain time-varying disturbances evolving in \(R\), \(\theta_i \in \mathbb{R}^p\) is the constant vector of uncertain system parameters, and \(\varphi_{i,p}(\tilde{x}_{i,p}) : \mathbb{R}^p \rightarrow \mathbb{R}^p\) are known smooth nonlinear function vectors.

Remark 1: It is pointed out that the considered multi-agent system model \cite{11} is more general than the model in most of the currently available results on distributed consensus with unknown high-frequency gains \cite{9}, \cite{16}–\cite{19} in the following respects: (i) mismatched uncertainties, multiple unknown high-frequency gains, and uncertain disturbances exist simultaneously in agent dynamics; (ii) the signs of high-frequency gains are allowed to be completely unknown and non-identical.

Our control objective is to design a new consensus algorithm for agents \cite{11}, based only on their states and the output information of their neighbors, such that (i) all agents can reach asymptotic consensus on the output state, i.e.,

\[
\lim_{t \rightarrow \infty} (x_{i,1}(t) - x_{j,1}(t)) = 0
\]

for all \(i \neq j \leq n\), and (ii) all signals in the closed-loop system are bounded.

Assumption I: The high-frequency gains \(g_{i,\ell}, \ell = 1, \ldots, n\), \(i = 1, \ldots, m\) are unknown and nonzero constants. Besides, there exist unknown positive constants \(\tau_{i,\ell}^*\) such that the inequalities \(|\tau_{i,\ell}(t)| \leq \tau_{i,\ell}^*\) hold for all \(t \geq 0\).

Assumption \cite{11} is quite common in the consensus literature \cite{7}, \cite{15}. The following result can be obtained directly by applying elementary row operations.

Lemma 1: Consider a block matrix \(E = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n}\). If the matrix \(A \in \mathbb{R}^{\ell \times \ell}\) with \(1 \leq \ell \leq \min{m,n}\) is nonsingular, then \(\text{rank}(E) = \text{rank}(A) + \text{rank}(D - CA^{-1}B)\).

To deal with the unknown high-frequency gains of the agents, Nussbaum gain functions \(N(\cdot)\) that have the properties that \(\lim_{t \rightarrow \infty} \inf \frac{1}{\gamma} \int_0^t N(s) ds = -\infty\) and \(\lim_{t \rightarrow \infty} \sup \frac{1}{\gamma} \int_0^t N(s) ds = \infty\) are applied \cite{9}. In this brief, we utilize the Nussbaum gain function as \(N(k) = k^2 \cos(k)\).

III. CONSENSUS ALGORITHM DESIGN AND MAIN RESULT

Prior to beginning development, we separate the node set \(\mathcal{V}\) into two subsets as \(\mathcal{V}_1\) and \(\mathcal{V}_2\), where \(\mathcal{V}_1 = \{i \in \mathcal{V} | d_i \neq 0\}\) and \(\mathcal{V}_2 = \{i \in \mathcal{V} | d_i = 0\}\) with \(d_i\) being the weighted in-degree of the \(i\)th agent. Considering that only the output information of neighbors is available for each agent, we propose the following novel dynamic system to generate a reference output \(\xi_{i,1}\) for the \(i\)th agent as

\[
\dot{\xi}_{i,\ell} = \xi_{i,\ell+1} + \gamma_i \sum_{j=1}^{n} a_{ij} x_{j,1} - \sum_{p=1}^{m} \lambda_{i,p} \xi_{i,p}
\]

for \(\ell = 1, \ldots, m - 1\), \(\ell, m \in [1, \ldots, p]\) and \(p \in \mathbb{R}^p\) for \(p = 1, \ldots, m\), where \(\gamma_i, \lambda_{i,1}, \ldots, \lambda_{i,m}\) are positive constants and are selected such that the roots of the characteristic equation \(s^m + \lambda_{i,m}s^{m-1} + \cdots + \lambda_{i,1} = 0\) are negative real numbers, and \(\gamma_i\) is chosen as \(\gamma_i = \lambda_{i,1}/d_i\). Since the agents in \(\mathcal{V}_2\) cannot receive any information from other agents, the reference output \(\xi_{i,1}\) for the \(i\)th agent is developed as \(\xi_{i,1} = \gamma_i\) with \(\xi_{i,1}^{(1)} = \xi_{i,1}^{(1)} = 0\) for \(\ell = 1, \ldots, m\), where \(\gamma_i\) is a constant chosen arbitrarily by the designer.

In what follows, we design the control input \(u_i\) for the \(i\)th agent, \(i = 1, \ldots, n\) such that the output state \(x_{i,1}\) can converge towards the reference output \(\xi_{i,1}\). To cope with the higher-order dynamics of the agents in \cite{11}, the recursive design methodology using the backstepping technique \cite{20} is adopted. We define the tracking errors as follows:

\[
z_{i,1} = x_{i,1} - \xi_{i,1}, \quad z_{i,\ell} = x_{i,\ell} - \alpha_{i,\ell-1}, \quad \ell = 2, \ldots, m
\]

where \(\alpha_{i,\ell-1}\) are virtual control signals to be selected.
Step 1: From (1) and (2), the dynamics of $z_{i,1}$ can be obtained as follows:

$$
\dot{z}_{i,1} = g_i(z_{i,2} + \alpha_{i,1}) + \theta^T \varphi_{i,1}(\bar{x}_{i,1}) + \tau_{i,1}(t) - \xi_{i,2}.
$$

(4)

The virtual control $\alpha_{i,1}$ is designed as

$$
\alpha_{i,1} = N(k_{i,1})(z_{i,1}z_{i,1} + \tilde{\theta}_{i,1}^T \varphi_{i,1}(\bar{x}_{i,1}) + \Delta_{i,1} - \xi_{i,2})
$$

(5)

where $N(k_{i,1}) = k_{i,1}^2 + cos(\nu)(k_{i,1})$ is the Nussbaum gain, $\varphi_{i,1}$ is a smooth function, and $\tau_{i,1}$ is a positive constant, $\Delta_{i,1}(t)$ is a finite positive constant. $\theta_{i,1}$ and $\nu_{i,1}$ represent the estimates of the unknown parameters $\theta_i$ and $\nu_i$ with $\tilde{\tau}_{i,1} = \tau_{i,1} + 1$, respectively. The adaptive laws for $\theta_{i,1}$, $\nu_{i,1}$, and $\kappa_{i,1}$ are proposed as

$$
\dot{\theta}_{i,1} = \bar{\varphi}_{i,1}(\bar{x}_{i,1}) z_{i,1} \tau_{i,1} - \mu_{i,1} z_{i,1} \tan (z_{i,1}/\varepsilon_i(t))
$$

(6)

and

$$
\dot{\nu}_{i,1} = \gamma_{i,m}^x \frac{\partial \nu_{i,1}}{\partial \nu_{i,1}} \frac{\partial \nu_{i,1}}{\partial \nu_{i,1}} - \frac{1}{\mu_{i,1}} (\tau_{i,1} - \zeta_{i,1}) \tilde{\beta}_{i,1}.
$$

(7)

Then, substituting adaptive laws (6) into (7) results in

$$
\dot{\nu}_{i,1} \leq (\gamma_{i,m}^x \frac{\partial \nu_{i,1}}{\partial \nu_{i,1}} \frac{\partial \nu_{i,1}}{\partial \nu_{i,1}} - \frac{1}{\mu_{i,1}} (\tau_{i,1} - \zeta_{i,1}) \tilde{\beta}_{i,1}.
$$

(8)

where $|z_{i,1}| < z_{i,1}^* = z_{i,1}^* = \zeta_{i,1} \tau_{i,1} \tan (z_{i,1}/\varepsilon_i(t)) \leq \kappa_{i,1} \varepsilon_i$ and $\kappa_{i,1} = 0.2785 \varepsilon_i$. Young's inequality $g_{i,1} \tau_{i,1} \varepsilon_i^2 \leq \kappa_{i,1} \varepsilon_i$ has been used.

Step 2 ($2 \leq \ell \leq m$): A similar procedure is employed recursively for each step. To obtain a recursive formula for $z_{i,\ell+1}$, the following steps are performed.

$$
\dot{z}_{i,\ell} = g_i(t) z_{i,\ell+1} + \alpha_{i,\ell} + \theta_{i,\ell}^T \varphi_{i,\ell}(\bar{x}_{i,\ell}) - \omega_{i,\ell} - \sum_{p=1}^{\ell-1} (\partial \theta_{i,p} / \partial x_i) \tau_{i,p} + \tau_{i,\ell} - \xi_{i,\ell+1}
$$

(9)

where $\theta_{i,\ell} = [-g_{i,1}, \ldots, -g_{i,\ell-1}, \theta_{i,\ell}]^T$ are uncertain parameters, $\bar{\varphi}_{i,\ell}(\bar{x}_{i,\ell}) = \frac{\partial}{\partial x_i} \varphi_{i,\ell}(\bar{x}_{i,\ell})$, $f_{i,\ell}(x_{i,\ell}) = \sum_{p=1}^{\ell-1} (\partial \theta_{i,p} / \partial x_i) \tau_{i,p} + (\partial \theta_{i,\ell} / \partial \nu_{i,\ell}) \tilde{\beta}_{i,\ell} + \zeta_{i,\ell} \tilde{\beta}_{i,\ell} + (\partial \theta_{i,\ell} / \partial \nu_{i,\ell}) \tilde{\beta}_{i,\ell} + (\partial \theta_{i,\ell} / \partial \nu_{i,\ell}) \tilde{\beta}_{i,\ell}

(10)

The virtual control $\alpha_{i,\ell}$ is designed as

$$
\alpha_{i,\ell} = N(k_{i,\ell})(z_{i,\ell} \tau_{i,\ell} + \tilde{\theta}_{i,\ell}^T \bar{\varphi}_{i,\ell}(\bar{x}_{i,\ell}) + \Delta_{i,\ell} - \xi_{i,\ell+1})
$$

(11)

where $N(k_{i,\ell}) = k_{i,\ell}^2 + cos(\nu)(k_{i,\ell})$ is the Nussbaum gain, $\Delta_{i,\ell}$ is the Nussbaum gain, $\theta_{i,\ell}$ and $\nu_{i,\ell}$ are, respectively, the estimates of the unknown parameters $\theta_i$ and $\nu_i$ with $\tilde{\tau}_{i,\ell} = \tau_{i,\ell} + 1$, respectively. The adaptive laws for $\theta_{i,\ell}$, $\nu_{i,\ell}$, and $\kappa_{i,\ell}$ are selected as

$$
\dot{\theta}_{i,\ell} = \bar{\varphi}_{i,\ell}(\bar{x}_{i,\ell}) z_{i,\ell} \tau_{i,\ell} - \mu_{i,\ell} z_{i,\ell} \tan (z_{i,\ell}/\varepsilon_i(t))
$$

(12)

and

$$
\dot{\nu}_{i,\ell} = \gamma_{i,m}^x \frac{\partial \nu_{i,\ell}}{\partial \nu_{i,\ell}} \frac{\partial \nu_{i,\ell}}{\partial \nu_{i,\ell}} - \frac{1}{\mu_{i,\ell}} (\tau_{i,\ell} - \zeta_{i,\ell}) \tilde{\beta}_{i,\ell}.
$$

(13)

where the fact that $(\tau_{i,\ell} - \sum_{p=1}^{\ell-1} (\partial \theta_{i,p} / \partial x_i) \tau_{i,p}) \zeta_{i,\ell} \leq \tilde{\tau}_{i,\ell} \| \theta_{i,\ell} \|_2 \zeta_{i,\ell}$ has been applied. Note that $\tilde{\tau}_{i,\ell} \| \theta_{i,\ell} \|_2 \zeta_{i,\ell}$ and $\tilde{\tau}_{i,\ell} \| \nu_{i,\ell} \|_2 \zeta_{i,\ell}$ are bounded. Then, substituting adaptive laws (11) into (12) leads to

$$
\dot{V}_{i,\ell} \leq (\gamma_{i,m}^x \frac{\partial \nu_{i,\ell}}{\partial \nu_{i,\ell}} \frac{\partial \nu_{i,\ell}}{\partial \nu_{i,\ell}} - \frac{1}{\mu_{i,\ell}} (\tau_{i,\ell} - \zeta_{i,\ell}) \tilde{\beta}_{i,\ell}.
$$

(14)

Summarizing the above discussion, we can now state the main result of this brief.

**Theorem 1:** Suppose that the directed graph $G$ contains a spanning tree. Consider a higher-order nonlinear multi-agent system of $n$ agents (1). Under Assumption 1, the proposed distributed control algorithm (10) with the reference output (2) and parameter update laws (6) and (11) ensures that (i) all agents can reach asymptotic consensus on the output state, i.e., $\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0$ for all $1 \leq i \neq j \leq n$, and (ii) all signals in the closed-loop system remain bounded.

Proof: In view of $z_{i,m+1} = 0$, we can obtain from (13) that $V_{i,m} \leq (\gamma_{i,m}^x \frac{\partial \nu_{i,m}}{\partial \nu_{i,m}} \frac{\partial \nu_{i,m}}{\partial \nu_{i,m}} - \frac{1}{\mu_{i,m}} (\tau_{i,m} - \zeta_{i,m}) \tilde{\beta}_{i,m}.$ Integrating this inequality between 0 and $\ell$ leads to

$$
V_{i,m}(t) \leq V_{i,m}(0) + \int_0^t (\gamma_{i,m}^x \frac{\partial \nu_{i,m}}{\partial \nu_{i,m}} \frac{\partial \nu_{i,m}}{\partial \nu_{i,m}} - \frac{1}{\mu_{i,m}} (\tau_{i,m} - \zeta_{i,m}) \tilde{\beta}_{i,m}.) d\sigma.
$$

(15)

We conclude that $V_{i,m} \in L_{\infty}(0, t_f]$; furthermore, $z_i(t) \in L_2(0, t_f)$ and $z_i(t) \in L_2(0, t_f)$ then follows from the fact that $\theta_{i,m}$ and $\tilde{\tau}_{i,m}$ are constants. Employing (7) Lem. 1) recursively $(m - 2)$ times, it can be shown from the aforementioned design procedures that...
\( V_{i,\ell}, k_{i,\ell}, z_{i,\ell}, \bar{\theta}_{i,\ell}, \xi_{i,\ell} \in \mathcal{L}_0[0, tf) \) and \( z_{i,\ell} \in \mathcal{L}_2[0, tf) \) for all \( \ell = 2, \ldots, m - 1 \). Since \( z_{i,2} \in \mathcal{L}_2[0, tf) \), there exists a positive constant \( \bar{b}_i \) such that \( \int_0^{tf} \frac{z_{i,2}^2}{2} \, \sigma \, ds \leq \bar{b}_i \) for all \( t \in [0, tf) \). Integrating (\ref{eq:4}) between 0 and \( t \), \( \forall t \in (0, tf) \), gives \( V_{i,1}(t) \leq V_{i,1}(0) + \int_0^t \bar{b}_i - \frac{f_i^T \xi_{i,1}(\sigma) + 1}{\lambda_i} \xi_{i,1}^T(\sigma) \cdot \xi_{i,1}(\sigma) \, \sigma \, d\sigma \). Notice (\ref{eq:3}), the dynamics of the reference output can be rewritten as \( \xi_{i,1} = A_i \xi_{i,1} + B_i \xi_i, \) where \( i \in \mathcal{V}_i, \xi_{i,1} = [\xi_{i,1}, \ldots, \xi_{i,m}]^T, A_i = [0_{m-1,1}]^T, \) and \( B_i = \begin{bmatrix} 0_{m-1,1} & \cdots & 0_{m-1,1} \end{bmatrix} \). Let the eigenvalues of \( B_i \) be denoted, without a particular order, by \( \delta_{i,\ell} < 0 \) for \( \ell = 1, \ldots, m \). For analysis purposes, we introduce a nonsingular transformation matrix \( J_i \) for the \( \ell \)th agent as \( J_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \delta_{i,1} & 0 & \cdots & 0 \\ \delta_{i,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i,m} & 0 & \cdots & 0 \end{bmatrix} \), where \( j_{i,p} = j_{i,(\ell-1)p} - \delta_{i,\ell-1} j_{i,\ell-1} \) for \( \ell = 2, \ldots, m, 2 \leq p \leq \ell \). Then, it can be verified that \( J_i B_i = \Lambda_i J_i \) and \( j_{i,m,m} = -\delta_{i,m} / \lambda_i \), with \( \Lambda_i = \begin{bmatrix} \delta_{i,1} & 0 & \cdots & 0 \\ 0 & \delta_{i,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{i,m} \end{bmatrix} \). Taking the state transformation \( q_i = [q_{i,1}, \ldots, q_{i,m}]^T = J_i \xi_{i,1} \) for \( i \in \mathcal{V}_i \), and noting (\ref{eq:3}), we have

\[
q_{i,\ell} = \delta_{i,\ell} q_{i,\ell-1} - \delta_{i,\ell-1} q_{i,\ell+1} + \xi_{i,\ell-1}, \quad \ell = 1, \ldots, m - 1
\]

\[
q_{i,m} = \delta_{i,m} q_{i,1} - \delta_{i,m-1} q_{i,2} + (\sum_{j=1}^n a_{ij} z_{j,1}^i)_{\ell = 1, \ldots, m}
\]

Since the directed graph \( \mathcal{G} \) contains a spanning tree, there is at most one agent with no neighbors. We consider two cases: (C1) each agent can obtain information from at least one other agent, i.e., \( \mathcal{V}_i = \mathcal{V} \) and (C2) there exists one agent that cannot receive any information from any other agent.

**C1:** Define the column vectors \( \tilde{q}_i = [q_{i,1}, \ldots, q_{i,m}]^T \), \( z = [0_{(m-1)n}, \tilde{z}_{i,1}, \ldots, \tilde{z}_{i,m}]^T \), and \( q = [\tilde{q}_1^T, \ldots, \tilde{q}_n^T]^T \), where \( \tilde{z}_{i,1} = -(\delta_{i,m} / d_i) \sum_{j=1}^n a_{ij} z_{j,1} \) for \( i = 1, \ldots, n \), \( \ell = 1, \ldots, m \). It follows from (\ref{eq:17}) that

\[
\tilde{q}(t) = -\tilde{L} q(t) + z(t)
\]

where

\[
\tilde{L} = \begin{bmatrix} -\delta_{i,1} & \delta_{i,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\delta_{m-1} & \delta_{m-1} \\ 0 & 0 & \cdots & 0 & -\delta_{m} \end{bmatrix}
\]
and $q = [\dot{q}_1^T, \ldots, \dot{q}_{n-1}^T]^T$, where $i = 2, \ldots, n$, $\ell = 2, \ldots, m$, and $\tilde{z}_{i,1} = -\left(\delta_{i,m}/d_i\right) \sum_{j=2}^{n} a_{ij} z_{j,1}$. Noting (15) and the fact that $\xi_{1,1}$ is a constant, we have

$$\dot{q}(t) = -Lq(t) + z(t)$$

where

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ h & L_{(n-1)m} \end{bmatrix}$$

with $h = [0_{(n-1)(m-1)}, h^T]^T \in \mathbb{R}^{(n-1)m}$.

$L_{(n-1)m} = \begin{bmatrix} -\delta_1 & \delta_1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\delta_{m-1} & \delta_{m-1} & \cdots & \cdots & 0 & -\delta_m \end{bmatrix}$

$\delta_\ell = \text{diag}(\delta_{2,\ell}, \ldots, \delta_{n,\ell})$ for $\ell = 1, \ldots, m$, and $\tilde{A}_{n-1} = D^{-1}_{n-1} \tilde{A}_{n-1}$. Notice that the matrix $\tilde{L}$ takes the form of a Laplacian matrix, as $L_{1(n-1)m+1} = 0_{(n-1)m+1}$ and all the off-diagonal entries of $\tilde{L}$ are non-positive. The system (18) can be regarded as a system consisting of $(n-1)m+1$ agents that are interconnected according to the augmented directed graph $\tilde{G} = (\tilde{V}, \tilde{E})$, where $\tilde{V} = \{1, \ldots, (n-1)m+1\}$. $\tilde{L}$ is the associated Laplacian matrix, and the edge set $\tilde{E}$ can be determined from (19). Since $\text{rank}(\delta_\ell) = n - 1$ and $\text{rank}(\delta_m(I_{n-1} - \tilde{A}_{n-1})) = \text{rank}(L_{(n-1)m}) = n - 1$, it follows from Lemma [1] that $\text{rank}(L_{(n-1)m}) = (n-1)m$. By (19), we have $\text{rank}(L) = (n-1)m$. Now proceeding in a manner similar to the proof of C1, we can conclude that all signals in the closed-loop system are bounded and that $\lim_{t \to \infty} (\xi_{i,1}(t) - \xi_{\ell,1}(t)) = 0$, $\lim_{t \to \infty} (x_{i,1}(t) - \xi_{\ell,1}(t)) = 0$, and $\lim_{t \to \infty} (x_{i,1}(t) - x_{j,1}(t)) = 0$ for all $i, j = 1, \ldots, n$. Furthermore, since $\xi_{1,1}$ is a constant for $i \in \mathcal{V}_2$ in this case, we can also conclude that $\lim_{t \to \infty} x_{j,1}(t) = \gamma_i$ for all $j = 1, \ldots, n$, which completes the proof.

IV. SIMULATION STUDY

Example 1: An example application with five marine vessels is considered to clarify and verify the theoretical findings of our work. The Norrbin model for the $i$th, $i = 1, \ldots, 5$, vessel (agent) can be described by

$$T_i \ddot{\theta}_i + \dot{\theta}_i + W_i \ddot{\psi}_i = M_i \dot{\psi}_i + \rho_i(t)$$

where $\theta_i$ denotes the actual course of the $i$th vessel, $\psi_i$ is the rudder angle, $M_i$ is the gain constant, $T_i$ is the time constant, $W_i$ is the Norrbin coefficient, and $\rho_i(t)$ denotes the time-varying disturbance term. The control objective is to design the distributed rudder angle $\psi_i$ for the $i$th vessel by utilizing only its states $\theta_i$, $\dot{\theta}_i$ and the course $\dot{\psi}_j$ ($j \in \mathcal{N}_i$) of its neighbors so that the course of all vessels can achieve asymptotic consensus. Note that since only the course (output information) of the neighbors is available, the results in [9], [16], [17], [19] cannot be applied to realize this goal. We define $x_{i,1} = \theta_i$, $x_{i,2} = \dot{\theta}_i$, and $u_i = \psi_i$. Then, the dynamics of the vessel can be equivalently written as

$$\dot{x}_{i,1} = x_{i,2}, \dot{x}_{i,2} = g_i u_i + \theta^T \varphi_i(x_{i,2}) + \tau_i(t),$$

where $x_{i,2} = [x_{i,1}, x_{i,2}]^T$, $g_i = M_i / T_i$, $\varphi_i(x_{i,2}) = [x_{i,2}, x_{i,2}]^T$, $\dot{\theta}_i = [-1/T_i, -W_i / T_i]^T$, and $\tau_i(t) = \rho_i(t) / T_i$. The simulation parameters are set as follows: $W_i = 0.4 - 0.05i$, $M_i = 20 + 0.5i$, $T_i = (1)^i + 0.02i$, and $\rho_i = 2i \cos(t)$. The initial postures of vessels are $x_{1,2}(0) = [\pi/2, 0]^T$, $x_{2,2}(0) = [-\pi/4, 0]^T$, $x_{3,2}(0) = [-\pi/3, 0]^T$, $x_{4,2}(0) = [\pi/5, 0]^T$, and $x_{5,2}(0) = [-\pi/2, 0]^T$. We consider two cases: (S1) each vessel can obtain information from at least one other vessel, and (S2) there exists one vessel that cannot receive any information from any other vessel.

S1: The communication topology among the vessels is shown in Fig. 1. Since $g_{i,1} = 1$ and $\varphi_i(x_{i,1}) = 0$ for $i = 1, \ldots, 5$, we choose the virtual control $\alpha_{i,1}$ as $\alpha_i = -c_{i,2} z_{i,1} - \Delta_i z_{i,2}$. The control input $u_i$ is designed as $u_i = N(k_i) (c_{i,2} z_{i,2} + \hat{\theta}_i^T \varphi_i(x_{i,2}) + \Delta_i z_{i,2} - \hat{\alpha}_i)$.

S2: In this case, the communication topology among the vessels is shown in Fig. 2. Since the vessel indexed by 1 cannot receive any information from any other vessel, its reference output $\xi_{1,1}$ is designed as $\xi_{1,1} = \gamma_1$, where $\gamma_1$ is a randomly selected constant in the range of $[-\pi/2, \pi/2]$. Control input design, the rest of the reference output and parameter settings are the same as in S1. The profiles of each vessel’s angle and velocity are exhibited in Fig. 3.

Example 2: Consider that there is a group of five second-
order nonlinear agents with the dynamics
\[
\begin{align*}
\dot{x}_{i,1} &= g_{i,1}x_{i,2} + \varphi_{i,1}^T(x_{i,1}, t) + \tau_{i,1}(t) \\
\dot{x}_{i,2} &= g_{i,2}u_i + \varphi_{i,2}^T(x_{i,2}, t) + \tau_{i,2}(t)
\end{align*}
\]
where \(i = 1, \ldots, 5\), \(\varphi_{i,1}(x_{i,1}) = \cos(x_{i,1})\), \(\varphi_{i,2}(x_{i,2}) = x_{i,1}\sin(x_{i,2})\), \(\delta_{i,1} = 1.1 - 0.1i\), \(\delta_{i,2} = (-1)^i(0.5 + 0.1i)\), \(\theta_i = 1 - 0.1i\), \(\tau_{i,1}(t) = (0.6 - 0.1i)\sin(t)\), and \(\tau_{i,2}(t) = 0.1i\cos(t)\).

The communication topology among the agents is shown in Fig. 1. The control goal is to design consensus algorithms for agents based only on their states and the output information of their neighbors so that all agents can reach asymptotic consensus on the output state, i.e., \(\lim_{t \to \infty} x_{i,1}(t) - x_{j,1}(t) = 0\) for all \(i \leq 1 \neq j \leq 5\).

In the design procedure, the error variables for each agent are defined as \(e_{i,1} = x_{i,1} - \xi_{i,1}\) and \(e_{i,2} = x_{i,2} - \alpha_{i,1}\), where \(\xi_{i,1}\) is selected by (2). By (5), we design the virtual control \(\alpha_{i,1}\) as \(\alpha_{i,1} = N(k_{i,1})(c_{i,1}z_{i,1} + \delta_{i,1}^T \varphi_{i,1}(x_{i,1}, t) + \Delta_{i,1} - \xi_{i,2})\).

According to (10), the control input \(u_i\) is designed as \(u_i = N(k_{i,1})c_{i,2}z_{i,2} + \delta_{i,2}^T \varphi_{i,2}(x_{i,2}, t) + \Delta_{i,2} - w_{i,1}\), where \(w_{i,1} = (\partial\alpha_{i,1}/\partial k_{i,1}) \hat{k}_{i,1} + (\partial\alpha_{i,1}/\partial \hat{\theta}_{i,1}) \hat{\theta}_{i,1} + (\partial\alpha_{i,1}/\partial \hat{\xi}_{i,1}) \hat{\xi}_{i,1} + (\partial\alpha_{i,1}/\partial \hat{\psi}_{i,1}) \hat{\psi}_{i,1} + \sum_{p=1}^3 (\partial\alpha_{i,1}/\partial \hat{\epsilon}_{i,1}) \hat{\epsilon}_{i,p}\).

Parameter update laws are designed as
\[
\begin{align*}
\dot{\hat{\theta}}_{i,1} &= \rho_{i,2}\varphi_{i,2}^T(x_{i,1}, t)z_{i,1}, \\
\dot{\hat{\theta}}_{i,2} &= \rho_{i,2}\varphi_{i,2}^T(x_{i,2}, t)z_{i,2}
\end{align*}
\]
where \(\eta_{i,2} = \sqrt{1 + (\partial\alpha_{i,1}/\partial \Delta_{i,1})^2}\). The initial conditions are given by \(\hat{x}_{i,2}(0) = [\pi, -\pi/2]^T\), \(\hat{\theta}_{i,2}(0) = [-\pi/5, -\pi/3]^T\), \(\hat{\xi}_{i,1}(0) = [0, 0]^T\), \(\hat{\xi}_{i,2}(0) = [0, 0]^T\), \(\hat{\epsilon}_{i,1}(0) = [0, 0]^T\), \(\hat{\epsilon}_{i,2}(0) = [0, 0]^T\), \(\hat{\psi}_{i,1}(0) = [0, 0]^T\), and \(\hat{\varphi}_{i,1}(0) = [0, 0]^T\).

**V. Conclusion**

In this brief, a distributed control algorithm for nonlinear multi-agent systems with mismatched uncertainties and unknown high-frequency gains under a directed graph was proposed. One salient feature of the presented algorithm is that it requires minimal information from neighboring agents, namely, their output measurements, such that asymptotic consensus among agents can be reached with lower communication costs. This overcomes a typical problem of the existing consensus schemes on higher-order nonlinear agents, in which both the states of the neighbors and their preliminary dynamics knowledge are needed for each agent to design the control input. Simulation results on marine vessels validated the theoretical finding. Following the metaheuristic algorithm presented in [25]–[27], future work will be devoted to handling the consensus problem of redundant robotic manipulators.

**REFERENCES**

[1] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. New York: Springer, 2008.

[2] J. Fu, G. Wen, W. Yu, and Z. Ding, “Finite-time consensus for second-order multi-agent systems with input saturation,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 65, no. 11, pp. 1758–1762, 2017.

[3] G. Wen, G. Hu, W. Yu, and G. Chen, “Distributed H∞ consensus of higher order multiagent systems with switching topologies,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 61, no. 5, pp. 359–363, 2014.

[4] T. I. Fossen, *Handbook of marine craft hydrodynamics and motion control*. Chichester: Wiley, 2011.
[5] S. Illingworth and A. Morgans, “Adaptive control of combustion instabilities for unknown sign of the high frequency gain,” in Proc. 38th Fluid Dynamics Conf. and Exhibit, Seattle, Washington, 2008, p. 4383.
[6] R. D. Nussbaum, “Some remarks on a conjecture in parameter adaptive control,” Syst. Control Lett., vol. 3, no. 5, pp. 243–246, 1983.
[7] X. Ye and J. Jiang, “Adaptive nonlinear design without a priori knowledge of control directions,” IEEE Trans. Autom. Control, vol. 43, no. 11, pp. 1617–1621, 1998.
[8] S. S. Ge, F. Hong, and T. H. Lee, “Adaptive neural control of nonlinear time-delay systems with unknown virtual control coefficients,” IEEE Trans. Syst., Man, Cybern. B, vol. 34, no. 1, pp. 499–516, 2004.
[9] W. Chen, X. Li, W. Ren, and C. Wen, “Adaptive consensus of multi-agent systems with unknown identical control directions based on a novel Nussbaum-type function,” IEEE Trans. Autom. Control, vol. 59, no. 7, pp. 1887–1892, 2014.
[10] Z. Ding, “Adaptive consensus output regulation of a class of nonlinear systems with unknown high-frequency gain,” Automatica, vol. 51, pp. 348–355, 2015.
[11] J. Peng and X. Ye, “Distributed adaptive controller for the output-synchronization of networked systems in semi-strict feedback form,” Journal of the Franklin Institute, vol. 351, no. 1, pp. 412–428, 2014.
[12] W. Wang, J. Huang, H. Fan, and W. Jiang, “Decentralized adaptive consensus control of uncertain nonlinear systems under directed topologies,” in Proc. 34th Chin. Control Conf. (CCC), China, 2015, pp. 7090–7095.
[13] J. Huang, W. Wang, C. Wen, J. Zhou, and G. Li, “Distributed adaptive leader-follower and leaderless consensus control of a class of strict-feedback nonlinear systems: A unified approach,” Automatica, vol. 118, p. 109021, 2020.
[14] G. Wang, C. Wang, and L. Li, “Fully distributed low-complexity control for nonlinear strict-feedback multiagent systems with unknown dead-zone inputs,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 50, no. 2, pp. 421–431, 2020.
[15] W. Wang, D. Wang, Z. Peng, and T. Li, “Prescribed performance consensus of uncertain nonlinear strict-feedback systems with unknown control directions,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 46, no. 9, pp. 1279–1286, 2016.
[16] C. Chen, C. Wen, Z. Liu, K. Xie, Y. Zhang, and C. P. Chen, “Adaptive consensus of nonlinear multi-agent systems with non-identical partially unknown control directions and bounded modelling errors,” IEEE Trans. Autom. Control, vol. 62, no. 9, pp. 4654–4659, 2017.
[17] J. Peng, C. Li, and X. Ye, “Cooperative control of high-order nonlinear systems with unknown control directions,” Syst. Control Lett., vol. 113, pp. 101–108, 2018.
[18] Q. Wang and C. Sun, “Adaptive consensus of multiagent systems with unknown high-frequency gain signs under directed graphs,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 50, no. 6, pp. 2181–2186, 2020.
[19] G. Wang, “Distributed control of higher-order nonlinear multi-agent systems with unknown non-identical control directions under general directed graphs,” Automatica, vol. 110, p. 108559, 2019.
[20] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, Nonlinear and adaptive control design. New York: Wiley, 1995.
[21] M. M. Polycarpou, “Stable adaptive neural control scheme for nonlinear systems,” IEEE Trans. Autom. Control, vol. 41, no. 3, pp. 447–451, 1996.
[22] A. Abdessameud and A. Tayebi, “Distributed consensus algorithms for a class of high-order multi-agent systems on directed graphs,” IEEE Trans. Autom. Control, vol. 63, no. 10, pp. 3464–3470, 2018.
[23] C.-T. Chen, Linear system theory and design. New York: Oxford University Press, 1999.
[24] P. J. Antsaklis and A. N. Michel, A linear systems primer. Boston: Birkhäuser, 2007.
[25] A. H. Khan, S. Li, D. Chen, and L. Liao, “Tracking control of redundant mobile manipulator: an RNN based metaheuristic approach,” Neurocomputing, 2020, in press.
[26] A. H. Khan, S. Li, and X. Cao, “Tracking control of redundant manipulator under active remote center of motion constraints: an RNN-based metaheuristic approach,” Sci. China Inf. Sci., 2020, in press.
[27] A. H. Khan, S. Li, and X. Luo, “Obstacle avoidance and tracking control of redundant robotic manipulator: an RNN based metaheuristic approach,” IEEE Trans. Ind. Informat., vol. 16, no. 7, pp. 4670–4680, 2020.