Graphs with maximum degree $\Delta \geq 17$ and maximum average degree less than 3 are list 2-distance $(\Delta + 2)$-colorable

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Abstract

For graphs of bounded maximum average degree, we consider the problem of 2-distance coloring. This is the problem of coloring the vertices while ensuring that two vertices that are adjacent or have a common neighbor receive different colors. It is already known that planar graphs of girth at least 6 and of maximum degree $\Delta$ are list 2-distance $(\Delta + 2)$-colorable when $\Delta \geq 24$ (Borodin and Ivanova (2009)) and 2-distance $(\Delta + 2)$-colorable when $\Delta \geq 18$ (Borodin and Ivanova (2009)). We prove here that $\Delta \geq 17$ suffices in both cases. More generally, we show that graphs with maximum average degree less than 3 and $\Delta \geq 17$ are list 2-distance $(\Delta + 2)$-colorable. The proof can be transposed to list injective $(\Delta + 1)$-coloring.

1 Introduction

In this paper, we consider only simple and finite graphs. A 2-distance $k$-coloring of a graph $G$ is a coloring of the vertices of $G$ with $k$ colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi_2^2(G)$ as the smallest $k$ such that $G$ admits a 2-distance $k$-coloring. This is equivalent to a proper vertex-coloring of the square of $G$, which is defined as a graph with the same set of vertices as $G$, where two vertices are adjacent if and only if they are adjacent or have a common neighbor in $G$. For example, the cycle of length 5 cannot be 2-distance colored with less than 5 colors as any two vertices are either adjacent or have a common neighbor: indeed, its square is the clique of size 5. An extension of the 2-distance $k$-coloring is the list 2-distance $k$-coloring, where instead of having the same set of $k$ colors for the whole graph, every vertex is assigned some set of $k$ colors and has to be colored from it. We define $\chi_2^2(G)$ as the smallest $k$ such that $G$ admits a list 2-distance $k$-coloring of $G$ for any list assignment. Obviously, 2-distance coloring is a sub-case of list 2-distance coloring (where the same color list is assigned to every vertex), so for any graph $G$, $\chi_2^2(G) \geq \chi^2(G)$. Kostochka and Woodall [19] even conjectured that it is actually an equality. The conjecture is still open.

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [21], and has been actively studied because of the conjecture below. The maximum degree of a graph $G$ is denoted $\Delta(G)$.

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Conjecture 1 (Wegner [21]). If $G$ is a planar graph, then:

- $\chi^2(G) \leq 7$ if $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ if $\Delta(G) \geq 8$

This conjecture remains open.

Note that any graph $G$ satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence at least $\Delta(G) + 1$ colors are needed for a 2-distance coloring of $G$. It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [20], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds.

A first measure of the sparseness of a planar graph is its girth. The girth of a graph $G$, denoted $g(G)$, is the length of a shortest cycle. Wang and Lih [20] conjectured that for any integer $k \geq 5$, there exists an integer $D(k)$ such that for every planar graph $G$ verifying $g(G) \geq k$ and $\Delta(G) \geq D(k)$, $\chi^2(G) = \Delta(G) + 1$. This was proved by Borodin, Ivanova and Neustroeva [11,13] to be true for $k \geq 7$, even in the case of list-coloring, and false for $k \in \{5,6\}$. So far, in the case of list coloring, it is known [3,18] that we can choose $D(7) = 16$, $D(8) = 10$, $D(9) = 8$, $D(10) = 6$, $D(12) = 5$. Borodin, Ivanova and Neustroeva [12] proved that the case $k = 6$ is true on a restricted class of graphs, i.e. for a planar graph $G$ with girth 6 where every edge is incident to a vertex of degree at most two and $\Delta(G) \geq 179$, we have $\chi^2(G) \leq \Delta(G) + 1$. Dvořák et al. [16] proved that the case $k = 6$ is true by allowing one more color, i.e. for a planar graph $G$ with girth 6 and $\Delta(G) \geq 8821$, we have $\chi^2(G) \leq \Delta(G) + 2$. They also conjectured that the same holds for a planar graph $G$ with girth 5 and sufficiently large $\Delta(G)$, but this remains open. Borodin and Ivanova improved [5] Dvořák et al.’s result and extended it to list-coloring [6,7] as follows.

Theorem 1 (Borodin and Ivanova [5]). Every planar graph $G$ with $\Delta(G) \geq 18$ and $g(G) \geq 6$ admits a 2-distance $(\Delta(G) + 2)$-coloring.

Theorem 2 (Borodin and Ivanova [7]). Every planar graph $G$ with $\Delta(G) \geq 24$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.

Theorems 1 and 2 are optimal with regards to the number of colors, as shown by the family of graphs presented by Borodin et al. [4], which are of increasing maximum degree, of girth 6 and are not 2-distance $(\Delta + 1)$-colorable. We improve Theorems 1 and 2 as follows.

Theorem 3. Every planar graph $G$ with $\Delta(G) \geq 17$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.

Another way to measure the sparseness of a graph is through its maximum average degree. The average degree of a graph $G$, denoted $\text{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The maximum average degree of a graph $G$, denoted $\text{mad}(G)$, is the maximum of $\text{ad}(H)$ over all subgraphs $H$ of $G$. Intuitively, this measures the sparseness of a graph because it states how great the concentration of edges in a same area can be. For example, stating that $\text{mad}(G)$ has to be smaller than 2 means that $G$ is a forest. Using this measure, we prove a more general theorem than Theorem 3.
Theorem 4. Every graph $G$ with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.

Euler’s formula links girth and maximum average degree in the case of planar graphs.

Lemma 1 (Folklore). For every planar graph $G$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$.

By Lemma 1 Theorem 4 implies Theorem 3.

An injective $k$-coloring [17] of $G$ is a (not necessarily proper) coloring of the vertices of $G$ with $k$ colors such that two vertices that have a common neighbor receive distinct colors. We define $\chi_i(G)$ as the smallest $k$ such that $G$ admits an injective $k$-coloring. A 2-distance $k$-coloring is an injective $k$-coloring, but the converse is not true. For example, the cycle of length 5 can be injective colored with 3 colors. The list version of this coloring is a list injective $k$-coloring of $G$, and $\chi_{i,\ell}(G)$ is the smallest $k$ such that $G$ admits a list injective $k$-coloring.

Some results on 2-distance coloring have their counterpart on injective coloring with one less color. This is the case of Theorems 1 and 2 [8, 9]. The proof of Theorem 4 also works with close to no alteration for list injective coloring, thus yielding a proof that every graph $G$ with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list injective $(\Delta(G) + 1)$-coloring.

In Sections 2 and 3, we introduce the method and terminology. In Sections 4 and 6, we prove Theorem 4 and its counterpart on injective coloring by a discharging method.

2 Method

The discharging method was introduced in the beginning of the 20th century. It has been used to prove the celebrated Four Color Theorem in [1, 2]. A discharging method is said to be local when the weight cannot travel arbitrarily far. Borodin, Ivanova and Kostochka introduced in [10] the notion of global discharging method, where the weight can travel arbitrarily far along the graph.

We prove for induction purposes a slightly stronger version of Theorem 4 by relaxing the constraint on the maximum degree. Namely, we relax it into “For any $k \geq 17$, every graph $G$ with $\Delta(G) \leq k$ and $\text{mad}(G) < 3$ verifies $\chi_{i,\ell}(G) \leq k + 2$” so that the property is closed under vertex- or edge-deletion. A graph is minimal for a property if it satisfies this property but none of its subgraphs does.

The first step is to consider a minimal counter-example $G$, and prove it cannot contain some configurations. To do so, we assume by contradiction that $G$ contains one of the configurations. We consider a particular subgraph $H$ of $G$, and color it by minimality (the maximum average degree of any subgraph of $G$ is bounded by the maximum average degree of $G$). We show how to extend the coloring of $H$ to $G$, a contradiction.

The second step is to prove that a graph that does not contain any of these configurations has a maximum average degree of at least 3. To that purpose, we assign to each vertex its degree as a weight. We apply discharging rules to redistribute weights along the graph with conservation of the total weight. As some configurations are forbidden, we can then prove that after application of the discharging rules, every vertex has a final weight of at least 3. This implies that the average degree of the graph is at least 3, hence the maximum average degree is at least 3. So a minimal counter-example cannot exist.

We finally explain how the same proof holds also for list injective $(\Delta + 1)$-coloring.
3 Terminology

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. White vertices may coincide with other vertices of the figure. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label \( i \) means "exactly \( i \) neighbors", the label \( i^+ \) (resp. \( i^- \)) means that it has at least (resp. at most) \( i \) neighbors.

Let \( u \) be a vertex. The neighborhood \( N(u) \) of \( u \) is the set of vertices that are adjacent to \( u \). Let \( d(u) = |N(u)| \) be the degree of \( u \). A p-link \( x - a_1 - ... - a_p - y \), \( p \geq 0 \), between \( x \) and \( y \) is a path between \( x \) and \( y \) such that \( d(a_1) = ... = d(a_p) = 2 \). When a p-link exists between two vertices \( x \) and \( y \), we say they are \( p \)-linked. If there is a \( p \)-link \( x - a_1 - ... - a_p - y \) between \( x \) and \( y \), we say \( x \) is \( p \)-linked through \( a_1 \) to \( y \). A partial 2-distance list coloring of \( G \) is a 2-distance list-coloring of a subgraph \( H \) of \( G \).

A vertex is weak when it is of degree 3 and is 1-linked to two vertices of degree at most 14, or twice 1-linked to a vertex of degree at most 14 (see Figure 1). A weak vertex is represented with a \( w \) label inside (\( w \) if it is not weak).

![Figure 1: A weak vertex x.](image1)

A vertex is support when it is either (see Figure 2):

Type \( (S_1) \): a vertex of degree 2 adjacent to another vertex of degree 2;

Type \( (S_2) \): a vertex of degree 2 that is adjacent to a vertex of degree 3 which is adjacent to a vertex of degree 2 and to a vertex of degree at most 7;

Type \( (S_3) \): a weak vertex 1-linked to another weak vertex.

A vertex is positive when it is of degree at least 4 and is adjacent to a support vertex. A vertex \( u \) is locked if it has two neighbors \( v_1 \) and \( v_2 \), where \( v_1 \) and \( v_2 \) are both 1-linked to the same two

![Figure 2: Support vertices x.](image2)
vertices \(w_1\) and \(w_2\) that have a common neighbor, and \(d(v_1) = d(v_2) = d(w_1) = d(w_2) = 3\) (see Figure 3). This configuration is called a \textit{lock}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lock.png}
\caption{A locked vertex \(u\).}
\end{figure}

4 Forbidden Configurations

In all the paper, \(k\) is a constant integer greater than 17 and \(G\) is a minimal graph such that \(\Delta(G) \leq k\) and \(G\) admits no 2-distance \((k + 2)\)-list-coloring.

We define configurations \((C_1)\) to \((C_{11})\) (see Figures 4, 5 and 6). Note that configurations similar to Configurations \((C_1), (C_2)\) and \((C_4)\) already existed in the literature, for example in [16].

- \((C_1)\) is a vertex \(u\) with \(d(u) \leq 1\)
- \((C_2)\) is a vertex \(u\) with \(d(u) = 2\) that has two neighbors \(v, w\) and \(u\) is 1-linked through \(v\) to a vertex of degree at most \(k - 1\).
- \((C_3)\) is a vertex \(u\) with \(d(u) = 3\) that has three neighbors \(v, w, x\) with \(d(w) + d(x) \leq k - 1\), and \(u\) is 1-linked through \(v\) to a vertex of degree at most \(k - 1\).
- \((C_4)\) is a vertex \(u\) with \(d(u) = 3\) that has three neighbors \(v, w, x\) with \(d(w) + d(x) \leq k - 1\), and \(v\) has exactly three neighbors \(u, y, z\) with \(d(z) \leq 7\) and \(d(y) = 2\).
- \((C_5)\) is a vertex \(u\) with \(d(u) = 3\) that has three neighbors \(v, w, x\) with \(d(x) \leq k - 1\) and \(u\) is 1-linked through \(v\) (resp. through \(w\)) to a vertex of degree at most 14. (Note that \(u\) is weak vertex.)
- \((C_6)\) is a vertex \(u\) with \(d(u) = 4\) that has four neighbors \(v, w, x, y\) with \(d(w) \leq 7, d(x) \leq 3, d(y) \leq 3\), and \(u\) is 1-linked through \(v\) to a vertex of degree at most 14.
- \((C_7)\) is a vertex \(u\) with \(d(u) = 4\) that has four neighbors \(v, w, x, y\) with \(d(x) + d(y) \leq k - 1\) and \(u\) is 1-linked through \(v\) (resp. through \(w\)) to a vertex of degree at most 14.
- \((C_8)\) is a vertex \(u\) with \(d(u) = 5\) that has five neighbors \(v, w, x, y, z\) with \(d(w) \leq 7, d(x) \leq 3, d(y) \leq 3, d(z) = 2\), and \(u\) is 1-linked through \(v\) to a vertex of degree at most 7.
- \((C_9)\) is a vertex \(u\) with \(d(u) = 6\) that has six neighbors \(v, w, x, y, z, t\) with \(d(w) \leq 7, d(x) \leq 3, d(y) \leq 3, d(z) = 2, d(t) = 2\), and \(u\) is 1-linked through \(v\) to a vertex of degree at most 7.
- \((C_{10})\) is a vertex \(u\) with \(d(u) = 7\) that has seven neighbors \(v, w_1, \ldots, w_6\) with \(d(v) \leq 7\) and \(u\) is 1-linked through \(w_i, 1 \leq i \leq 6\), to a vertex of degree at most 3.
Figure 4: Forbidden configurations (C₁) to (C₅).

- (C₁₁) is a vertex u with d(u) = k that has three neighbors v, w, x with x is a support vertex, v, w are both 1-linked to a same vertex y of degree 3, and v (resp. w) is 1-linked to a vertex of degree at most 14 distinct from y. (Note that v, w are weak vertices.)

Lemma 2. G does not contain Configurations (C₁) to (C₁₁).

Proof. Given a partial 2-distance list-coloring of G, a constraint of a vertex u is color appearing on a vertex at distance at most 2 from u in G.

Notation refers to Figures 4, 5 and 6.

Claim 1. G does not contain (C₁).

Proof. Suppose by contradiction that G contains (C₁). Using the minimality of G, we color G \ {u}. Since ∆(G) ≤ k, and d(u) ≤ 1, vertex u has at most k constraints (one for its neighbor and at most k − 1 for the vertices at distance 2 from u). There are k + 2 colors available in the list of u, so the coloring of G \ {u} can be extended to G, a contradiction.

Claim 2. G does not contain (C₂).

Proof. Suppose by contradiction that G contains (C₂). Using the minimality of G, we color G \ {u, v}. Vertex u has at most k + 1 constraints. Hence we can color u. Then v has at most k − 1 + 2 = k + 1 constraints. Hence we can color v. So we can extend the coloring to G, a contradiction.

Claim 3. G does not contain (C₃).

Proof. Suppose by contradiction that G contains (C₃). Using the minimality of G, we color G \ {v}. Because of u, vertices w and x have different colors. We discolor u. Vertex v has at most k − 1 + 2 = k + 1 constraints. Hence we can color v. Vertex u has at most d(w) + d(x) + 2 ≤ k + 1 constraints. Hence we can color u. So we can extend the coloring to G, a contradiction.

Claim 4. G does not contain (C₄).

Proof. Suppose by contradiction that G contains (C₄). Let e be the edge uv. Using the minimality of G, we color G \ {e}. We discolor u and v. Vertex u has at most d(w) + d(x) + 2 ≤ k + 1 constraints. Hence we can color u. Vertex v has at most 7 + 3 + 2 ≤ k + 1 constraints. Hence we can color v. So we can extend the coloring to G, a contradiction.
Claim 5. \( G \) does not contain \((C_5)\).

Proof. Suppose by contradiction that \( G \) contains \((C_5)\). Using the minimality of \( G \), we color \( G \setminus \{u, v, w\} \). Vertex \( u \) has at most \( k - 1 + 2 = k + 1 \) constraints. Hence we can color \( u \). Vertices \( v \) and \( w \) have at most \( 14 + 3 \leq k + 1 \) constraints respectively. Hence we can color \( v \) and \( w \). So we can extend the coloring to \( G \), a contradiction.

Claim 6. \( G \) does not contain \((C_6)\).

Proof. Suppose by contradiction that \( G \) contains \((C_6)\). Using the minimality of \( G \), we color \( G \setminus \{v\} \). We discolor \( u \). Vertex \( v \) has at most \( 14 + 3 \leq k + 1 \) constraints. Hence we can color \( v \). Vertex \( u \) has at most \( 14 + 3 + 7 \leq k + 1 \) constraints. Hence we can color \( u \). So we can extend the coloring to \( G \), a contradiction.

Claim 7. \( G \) does not contain \((C_7)\).

Proof. Suppose by contradiction that \( G \) contains \((C_7)\). Using the minimality of \( G \), we color \( G \setminus \{v, w\} \). We discolor \( u \). Vertex \( u \) has at most \( d(x) + d(y) + 2 \leq k + 1 \) constraints. Hence we can color \( u \). Vertices \( v \) and \( w \) have at most \( 14 + 4 \leq k + 1 \) constraints respectively. Hence we can color \( v \) and \( w \). So we can extend the coloring to \( G \), a contradiction.

Claim 8. \( G \) does not contain \((C_8)\).

Proof. Suppose by contradiction that \( G \) contains \((C_8)\). Using the minimality of \( G \), we color \( G \setminus \{v\} \). We discolor \( u \). Vertex \( u \) has at most \( 7 + 3 + 3 + 2 \leq k + 1 \) constraints. Hence we can color \( u \). Vertex \( v \) has at most \( 7 + 5 \leq k + 1 \) constraints. Hence we can color \( v \). So we can extend the coloring to \( G \), a contradiction.

Claim 9. \( G \) does not contain \((C_9)\).

Proof. Suppose by contradiction that \( G \) contains \((C_9)\). Using the minimality of \( G \), we color \( G \setminus \{v\} \). We discolor \( u \). Vertex \( u \) has at most \( 7 + 3 + 3 + 2 + 2 + 1 \leq k + 1 \) constraints. Hence we can color \( u \). Vertex \( v \) has at most \( 7 + 6 \leq k + 1 \) constraints. Hence we can color \( v \). So we can extend the coloring to \( G \), a contradiction.
Claim 10. $G$ does not contain $(C_{10})$.

Proof. Suppose by contradiction that $G$ contains $(C_{10})$. Using the minimality of $G$, we color $G \setminus \{u, w_1, \ldots, w_6\}$. Vertex $u$ has at most $7 + 6 \leq k + 1$ constraints. Hence we can color $v$. Vertices $w_i$ have at most $3 + 7 \leq k + 1$ constraints. Hence we can color $w_1, \ldots, w_6$. So we can extend the coloring to $G$, a contradiction. \hfill \Box

Claim 11. $G$ does not contain $(C_{11})$.

Proof. Suppose by contradiction that $G$ contains $(C_{11})$. Since $x$ is a support vertex, and $u$ is of degree $k$, it is of type $(S_1)$, $(S_2)$ or $(S_3)$ of support vertices with the notation of Figure 2. Note that some vertices may coincide between Figure 2 and Figure 6.

We define a set of vertices $A$ as follows:

$$A = \begin{cases} 
\{a\} & \text{if } x \text{ is of Type } (S_1) \\
\{a, c\} & \text{if } x \text{ is of Type } (S_2) \\
\{a, c\} & \text{if } x \text{ is of Type } (S_3)
\end{cases}$$

Using the minimality of $G$, we color $G \setminus \{(v, w, x, y, z_1, \ldots, z_4) \cup A\}$. If $x$ is of Type $(S_1)$ (resp. $(S_2)$), $a$ (resp. $c$) has at most $k + 1$ constraints, hence we can color it. For the three types $(S_i)$, $x$ has at most $k - 3 + 1 + 2 = k$ constraints, thus it has at least 2 available colors. Vertex $y$ has at most $k$ constraints, thus it has at least 2 available colors. Both $v$ and $w$ have at most $k - 3 + 1 + 1 \leq k - 1$ constraints, so they have at least 3 available colors in their list.

We now explain how to color $v, w, x, y$ (other uncolored vertices will be colored after). Suppose $x$ and $y$ can be assigned the same color, then both $v$ and $w$ have at least 2 available colors and thus can be colored.

Suppose the lists of available colors of $x$ and $y$ are disjoint. We color $v$ with a color not appearing in the list of $x$. Then we color $y$ that has $k + 1$ constraints. (Vertex $x$ has still at least 2 available colors.) Then we color $w$ that has $k + 1$ constraints and finally $x$.

Now we assume that we cannot assign the same color to $x$ and $y$ and that their lists of available colors are not disjoint. This means that $x$ and $y$ are either adjacent or have a common neighbor. So some vertices coincide between Figure 2 and Figure 6. The different cases where $x$ and $y$ are either adjacent or have a common neighbor are the following:

$(S_1)$ \quad $b = y$
(S2) \(- b = y\)
- \(- a = y \) and \(w.l.o.g \ b = z_2, c = z_3 \) and \(d = w\).

(S3) \(- b = y\)
- \(- d = y, \) and \(w.l.o.g. \ f = z_2, \) \(g = v \) and \(e = z_3\).

In all these cases, \(y\) has at most 1 contstraint. So we can color \(x, v, w, y\), in this order as they all have at most \(k + 1\) constraints when they are colored.

If \(x\) is of Type \((S_2)\) (resp. \((S_3)\)), vertex \(a\) (resp vertices \(a, c\)) has at most 11 constraints (resp. 18, 6), so we can color them. The vertices \(z_i\) have at most \(17 \leq k + 1\), so we can color them. Thus the coloring have been extended to \(G\), a contradiction.

\(5\) Structure of support vertices

Let \(H(G)\) be the subgraph of \(G\) induced by the edges incident to at least a support vertex. We prove several properties of support vertices and of the graph \(H(G)\).

Lemma 3. Each positive vertex is of degree \(k\) and each support vertex is adjacent to exactly one positive vertex.

Proof. By Lemma 2 \(G\) does not contain Configurations \((C_2), (C_3)\) and \((C_5)\). So a support vertex is adjacent to a vertex of degree \(k\) (Configurations \((C_2)\), \((C_3)\) and \((C_5)\) correspond respectively to support vertices of Type \((S_1)\), \((S_2)\) and \((S_3)\)). By definition, a support vertex has at most one neighbor of degree at least 4, thus it is adjacent to exactly one vertex of degree at least 4 and this vertex has in fact degree \(k\). So all the positive vertices are of degree \(k\) and a support vertex is adjacent to exactly one positive vertex.

\(\square\)

Lemma 4. Each cycle of \(H(G)\) with an odd number of support vertices contains a subpath \(s_1v_1s_2v_2s_3\) where \(s_1, s_2, s_3\) are support vertices of type \((S_3)\) and \(v_1, v_2\) are vertices of degree 2.

Proof. Let \(C\) be cycle of \(H(G)\) with an odd number of support vertices. Cycle \(C\) does not contain just one support vertex, as all its edges have to be adjacent to a support vertex (there is no loop nor multiple edge in \(H(G)\)). So \(C\) contains at least three support vertices.

Suppose that \(C\) contains no positive vertices. Then it contains no support vertices of type \((S_1)\) or \((S_2)\) as such vertices are of degree 2, so all their neighbors would be on \(C\), and they are adjacent to a positive vertex by Lemma 2. \(C\) contains only support vertices of type \((S_3)\). Let \(s_1, s_2, s_3\) be three support vertices of \(C\) appearing consecutively along \(C\). A support vertex of Type \((S_3)\) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of \(s_1\) on \(C\) are vertices of degree 2 that are not support vertices. As \(H(G)\) contains only edges incident to support vertices, there exist \(v_1, v_2\) of degree 2 such that \(s_1v_1s_2v_2s_3\) is a subpath of \(C\).

Suppose now that \(C\) contains some positive vertices. Let \(p_1, \ldots, p_\ell\) be the set of positive vertices of \(C\) appearing in this order along \(C\) while walking in a chosen direction (subscript are understood modulo \(\ell\)). Let \(Q_i, 1 \leq i \leq \ell\), be the subpath of \(C\) between \(p_i\) and \(p_{i+1}\) (in the same chosen direction along \(C\)). (Note that if \(\ell = 1\), then \(Q_1 = C\) is not really a subpath.) As \(C\) contains an odd number of support vertices, there exists \(i\) such that \(Q_i\) contains an odd number of support
vertices. If $Q_i$ contains just one support vertex $v$, then $Q_i$ has length 2, since $H(G)$ contains only edges incident to support vertices. So $v$ is adjacent to two different positive vertices (or has a multiple edge if $\ell = 1$), a contradiction to Lemma 3. So $Q_i$ contains at least 3 support vertices. Let $s_1, s_2, s_3$ be three support vertices of $Q_i$ appearing consecutively along $Q_i$.

If one of the $s_i$ is of Type $(S_1)$, let $x$ be such a vertex. With the notation of Figure 2, vertex $x$ is of degree 2, so its two neighbors $u, a$ are on $C$, with $u$ a positive vertex and $a$ a support vertex of Type $(S_1)$. Then vertex $a$ is of degree 2 so its neighbor $b$ distinct from $x$ is also on $C$. Vertex $b$ is positive so $Q_i$ is the path $u, x, a, b$ and contains just two support vertices, a contradiction.

If one of the $s_i$ is of Type $(S_2)$, let $x$ be such a vertex. With the notation of Figure 2, vertex $x$ is of degree 2, so its two neighbors $u, a$ are on $C$, with $u$ a positive vertex and $a$ a vertex of degree 3. Vertex $a$ is not adjacent to vertices of degree $k$ so by Lemma 3 it is not a support vertex. Let $c'$ be the neighbor of $a$ on $C$ that is distinct from $x$. As all the edges of $H(G)$ are incident to support vertices, $c'$ is a support vertex. Since $c'$ is adjacent to a vertex of degree 3 it is a support vertex of Type $(S_2)$ and can play the role of $c$ of Figure 2. Then $c$ is of degree 2 and its neighbor on $C$ distinct from $a$ is a positive vertex $d$. So $Q_i$ is the path $u, x, a, c, d$ and contains just two support vertices, a contradiction.

So $s_1, s_2, s_3$ are all of Type $(S_3)$. A support vertex of Type $(S_3)$ is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of $s_2$ on $C$ are vertices $v_1, v_2$ of degree 2 that are not support vertices. As $H(G)$ contains only edges incident to support vertices, we can assume w.l.o.g. that $s_1v_1s_2v_2s_3$ is a subpath of $C$.

Lemma 5. $H(G)$ does not contain a 2-connected subgraph of size at least three with exactly two support vertices.

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph $C$ of size $\geq 3$ that has exactly two support vertices $S = \{s_1, s_2\}$. We color by minimality $G \setminus (S \cup \{v \in N_G(S) | d_G(v) \leq 3\})$. (Note that by Lemma 3 the set $\{v \in N_G(S) | d_G(v) \leq 3\}$ corresponds to vertex $a$ of Figure 2 if the support vertex is of Type $(S_1)$ or $(S_2)$ and to vertices $a, c$ if the support vertex is of Type $(S_3)$.)

We first show how to color $S$. For that purpose we consider three cases corresponding to the type of $s_1$.

- $s_1$ is of Type $(S_1)$. Then $s_1$ is of degree 2, has a positive neighbor $u$ and a support neighbor $a$ of Type $(S_1)$. As $s_1$ is of degree 2, both its neighbors are in $C$. So $a$ is a support vertex of $C$, thus $a = s_2$. Then $u$ is of degree $k$, has two neighbors $s_1, s_2$ that are not colored, so $s_1$ and $s_2$ have at most $k$ constraints, and we can color them.

- $s_1$ is of Type $(S_2)$. Then $s_1$ is of degree 2, has a positive neighbor $u$ and another neighbor $a$ of degree 3. Vertex $a$ is not a support vertex by Lemma 3 since it has no neighbor of degree $k$. As $s_1$ is of degree 2, all its neighbors are in $C$. Vertices $u$ and $a$ are in $C$ that is 2-connected so they have at least two neighbors in $C$. Since they are not support vertices, all their neighbors in $C$ are support vertices. So both $u$ and $a$ are adjacent to $s_2$. Vertex $s_2$ is support, it is adjacent to $a$ that is of degree 3, so $s_2$ is of Type $(S_2)$. Then $u$ is of degree $k$, has two neighbors $s_1, s_2$ that are not colored, so $s_1$ and $s_2$ have at most $k$ constraints, and we can color them.

- $s_1$ is of Type $(S_3)$. Then $s_1$ is of degree 3, has a positive neighbor $u$ and two other neighbors $w, w'$ of degree 2. Vertices $w, w'$ are not support vertices by Lemma 3 since they have no neighbor of degree $k$. As $s_1$ is of degree 3, two of $u, w, w'$ are in $C$. Let $Y$ be the neighbors of
Every vertex of \( \{v \in N_G(S) \mid d_G(v) \leq 3\} \) has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction.

**Lemma 6.** Every 2-connected subgraph of \( H(G) \) that contains exactly three support vertices is a cycle.

**Proof.** Suppose by contradiction that \( H(G) \) contains a 2-connected subgraph \( C \) of size \( \geq 3 \) that has exactly three support vertices \( S = \{s_1, s_2, s_3\} \) and that is not a cycle.

Suppose by contradiction that \( C \) contains no cycle \( C' \) with \( S \subseteq C' \subseteq C \). As \( C \) is 2-connected, by Menger’s Theorem there exist two internally vertex-disjoint paths \( Q, Q' \) between \( s_1, s_2 \). Let \( C'' \) be the cycle \( Q \cup Q' \). By assumption \( C'' \) does not contain \( s_3 \). So it contains just two support vertices, a contradiction to Lemma 5. So \( C \) contains a cycle \( C' \) with \( S \subseteq C' \subseteq C \).

By Lemma 4, cycle \( C' \) contains a subpath \( x_1v_1x_2v_2x_3 \) where \( x_1, x_2, x_3 \) are support vertices of Type (\( S_3 \)) and \( v_1, v_2 \) are vertices of degree 2. As \( C \) contains just three support vertices, we have \( S = \{x_1, x_2, x_3\} \). Vertices \( x_1, x_3 \) are support vertices of Type (\( S_3 \)), they are of degree 3 and only adjacent to positive vertices and to vertices of degree 2 so they are not adjacent. The graph \( H(G) \) contains only edges incident to support vertices, so there exists a vertex \( y \) of \( C' \) adjacent to \( x_1, x_3 \), and \( x_1v_1x_2v_2x_3y \) is the cycle \( C' \). If \( C' \) has some chords in \( H(G) \), then \( H(G) \) contains a cycle with two support vertices only, a contradiction to Lemma 5. So \( C' \) is an induced cycle of \( H(G) \) and so \( C' \) has strictly less vertices than \( C \). Let \( y' \) be a vertex of \( C \) distinct from \( x_1, v_1, x_2, v_2, x_3, y \). Vertex \( y' \) is not a support vertex, \( C \) is 2-connected and \( H(G) \) contains only edges incident to support vertices, so \( y' \) is adjacent to at least two vertices in \( S \). Then \( H(G) \) contains a cycle with two support vertices only, a contradiction to Lemma 5.

We need the following lemma from Brooks [14]:

**Lemma 7 ([14]).** If \( G \) is a 2-connected graph that is neither a clique nor an odd cycle, and \( L \) is a list assignment on the vertices of \( G \) such that \( \forall u \in V(G), |L(u)| \geq d(u) \), then \( G \) is \( L \)-colorable.

**Lemma 8.** Every 2-connected subgraph of \( H(G) \) of size at least three is either a cycle with an odd number of support vertices or a subgraph of a lock of \( H(G) \).

**Proof.** Suppose by contradiction that \( H(G) \) contains a 2-connected subgraph \( C \) of size \( \geq 3 \) that is not a cycle with an odd number of support vertices nor a subgraph of a lock of \( H(G) \). Let \( S = \{s_1, \ldots, s_p\} \) be the support vertices of \( C \). By Lemma 5, \( p \geq 3 \). Let \( S \) be the graph with \( V(S) = S \) where there is an edge between \( s_i \) and \( s_j \) if and only if they are adjacent or have a common neighbor in \( G \).

**Claim 12.** \( S \) is not a clique of size at least four.
Proof. Suppose, by contradiction that \( S \) is a clique with \( p \geq 4 \).

Given a support vertex \( x \), we say that a support vertex \( x' \), distinct from \( x \), satisfies the property \( P_x \) if it is either adjacent to \( x \) in \( G \) or has a non-positive common neighbor with \( x \) in \( G \). At most two vertices can satisfy \( P_x \) (vertex \( a \) of Figure 2). If \( x \) is of Type \( (S_1) \), vertices \( b, c \) if \( x \) is of Type \( (S_2) \), vertices \( b, d \) if \( x \) is of Type \( (S_3) \). Note that if \( x \) satisfies \( P_x \), then \( x \) satisfies \( P_x' \).

We claim that there exist two support vertices in \( S \) that do not have a positive common neighbor in \( G \). Suppose by contradiction that every pair of vertices of \( S \) has a positive common neighbor. By Lemma 4 every support vertex has at most one positive neighbor, so all the vertices of \( S \) are adjacent to the same positive vertex \( v \). As \( C \) is 2-connected, there is a path \( Q \) in \( C \) between \( s_1, s_2 \). Let \( s_i \) be the first support vertex, distinct from \( s_1 \), appearing along \( Q \) while starting from \( s_1 \) (maybe \( i = 2 \) if there is no support vertex in the interior of \( Q \)). Let \( Q' \) be the subpath of \( Q \) between \( s_1 \) and \( s_i \) (maybe \( Q = Q' \)). Then \( Q' \cup \{ v \} \) forms a 2-connected subgraph of size \( \geq 3 \) with exactly two support vertices, a contradiction to Lemma 5. So there exist two support vertices \( x, x' \) in \( S \) that do not have a positive common neighbor in \( G \). Since \( S \) is a clique, vertices \( x, x' \) are adjacent or have a common non-positive neighbor, so \( x \) satisfies \( P_x' \) (and \( x' \) satisfies \( P_x \)).

Suppose there exists a support vertex \( y \in S \) that does not satisfy \( P_x \) nor \( P_x' \). Since \( S \) is a clique, vertex \( y \) has a common positive neighbor \( z \) with \( x \) and \( y \) with \( x' \). Since \( x \) and \( x' \) have no positive common neighbor, \( z \) and \( x' \) are distinct. Thus \( y \) has two positive neighbors, a contradiction. So every vertex of \( S \setminus \{ x, x' \} \) satisfies either \( P_x \) or \( P_x' \). If two vertices \( y, y' \) of \( S \setminus \{ x, x' \} \) satisfy \( P_x \), then at least three vertices, \( x', y, y' \) verify \( P_x \), a contradiction. So there is at most one vertex of \( S \setminus \{ x, x' \} \) satisfying \( P_x \) and similarly at most one satisfying \( P_x' \). So \( p \leq 4 \) and we can assume, w.l.o.g., that \( S = \{ x, x', y, y' \} \), where vertex \( y \) satisfies \( P_x \) and not \( P_x' \) and vertex \( y' \) satisfies \( P_x' \) and not \( P_x \). Thus \( x \) has a common positive neighbor \( z \) with \( y \) and \( x' \) has a common positive neighbor \( z' \) with \( y \). Since \( x, x' \) do not have a common positive neighbor, \( z \) and \( z' \) are distinct. Vertices \( y, y' \) have at most one positive neighbor, thus, they do not have a common positive neighbor. Since \( S \) is a clique, \( y \) satisfies \( P_y' \). Let \((y_1, y_2, y_3, y_4) = (x, x', y', y)\) (subscript are understood modulo 4).

Suppose there exists \( i \in \{ 1, 2, 3, 4 \} \) such that \( y_i, y_{i+1} \) are adjacent in \( G \). Two support vertices can be adjacent only if they are of Type \( (S_1) \). So \( y_i, y_{i+1} \) are of Type \( (S_1) \) and of degree two. Then \( y_i \) is only adjacent to \( y_{i+1} \) and to a positive vertex in \( \{ z, z' \} \). If \( y_i \) is adjacent to \( y_{i-1} \), then \( y_{i-1} = y_{i+1} \), a contradiction. If \( y_i \) is not adjacent to \( y_{i-1} \), then \( y_{i+1} \) is a common neighbor of \( y_i \) and \( y_{i-1} \). Since \( y_{i+1} \) is of degree two and has a positive neighbor, \( y_{i-1} = y_{i+1} \), a contradiction. So \( y_i, y_{i+1} \) are not adjacent in \( G \) for any \( 1 \leq i \leq 4 \). Let \( w_i \) be a non-positive common neighbor of \( y_i, y_{i+1} \).

Suppose there exists \( i \in \{ 1, 2, 3, 4 \} \) such that \( d(y_i) = 2 \). Then \( w_i = w_{i-1} \). So \( \{ y_{i-1}, y_i, y_{i+1} \} \subseteq N(w_i) \), and \( w_i \) is not positive, so \( d(w_i) = 3 \). Two support vertices can have a common neighbor of degree 3 only if they are both of degree two (Type \( (S_2) \)). So \( d(y_{i-1}) = d(y_{i+1}) = 2 \). Since \( y_{i+1} \) is of degree two and has a positive neighbor, \( w_i = w_{i+1} \), so \( y_{i+2} \in N(w_i) \), a contradiction. So \( d(y_i) \geq 3 \) for any \( 1 \leq i \leq 4 \).

Then all the \( y_i \) are of Type \( (S_3) \), they are of degree three and their non positive neighbors are of degree two. Thus \( d(w_i) = 2 \) for any \( 1 \leq i \leq 4 \). So \( y_1, y_4, w_1, \ldots, w_4, z, z' \) induce a lock. So all the edges incident to \( S = \{ y_1, \ldots, y_4 \} = \{ s_1, \ldots, s_4 \} \) belong to a lock, contradicting the definition of \( C \).

By Lemma 5 the graph \( S \) is not an edge. If \( S \) is a triangle, then \( C \) contains exactly three support vertices and, by Lemma 6 it is a cycle with an odd number of support vertices, a contradiction. So \( S \) is not a triangle. By Claim 12 \( S \) is not a clique of size at least 4. So finally, \( S \) is not a clique.
Suppose, by contradiction, that \( S \) is an odd cycle with \( \geq 5 \) vertices. Then \( C \) is a 2-connected graph that is not a cycle, so it contains a vertex \( v \) with at least 3 neighbors in \( C \). If \( v \) is not a support vertex, then it has at least 3 support neighbors in \( C \) that form a triangle in \( S \), a contradiction. So \( v \) is a support vertex. Then either \( v \) has three neighbors in \( S \), a contradiction to \( S \) being a cycle, or \( C \) contains a cycle with two support vertices, a contradiction to Lemma 5. So \( S \) is not an odd cycle.

Suppose, by contradiction, that \( S \) is not 2-connected. Then there exist three support vertices \( s, s', s'' \) of \( S \) such that \( s', s'' \) appears in two different connected components of \( S \setminus \{s\} \). As \( C \) is 2-connected, there exists a path \( Q \) between \( s', s'' \) in \( C \setminus \{s\} \). This path \( Q \) is composed only of edges incident to support vertices so in \( S \setminus \{s\} \) it corresponds to a path between \( s', s'' \), a contradiction. So \( S \) is 2-connected.

We now consider the graph \( G \), we color by minimality \( G \setminus (S \cup \{v \in N_G(S)|d_G(v) \leq 3\}) \). We show how to color \( S \). In the three Types \((S_j)\), the number of constraints on a support vertex \( s_i \) of Type \((S_j)\) is at most \( k + 2 \) minus the number of its neighbors in \( S \). So the number of available colors of a support vertex is at least its degree in \( S \). Now Lemma 7 can be applied to \( S \) that is not a clique, not an odd cycle and 2-connected. So we can color \( S \). Every vertex of \( \{v \in N_G(S)|d_G(v) \leq 3\} \) has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction.

A **cactus** is a connected graph in which any two cycles have at most one vertex in common.

**Lemma 9.** Every connected component of \( H(G) \) is either a cactus where each cycle has an odd number of support vertices or a lock.

**Proof.** All the edges of a lock are incident to support vertices of type \((S_3)\) so all the edges of a lock of \( G \) appear in \( H(G) \). The only vertices of a lock that can have neighbors outside a lock are locked vertices (vertices \( u \) and \( x \) on Figure 3). By Lemma 2 graph \( G \) does not contain Configuration \((C_{11})\), so a locked vertex is incident to only two support vertices, the two support vertices of a lock. A lock is a connected component of \( H(G) \).

Let \( C \) be a connected component of \( H(G) \) that is not a lock. By Lemma 8 each 2-connected subgraph of \( C \) is a cycle with an odd number of support vertices. So \( C \) is a cactus where each cycle of \( C \) has an odd number of support vertices. \( \Box \)

### 6 Discharging rules

A **negative** vertex is a support vertex of type \((S_1)\) or \((S_2)\) or a vertex of degree 2 adjacent to two support vertices of type \((S_3)\). In this case we say that the negative vertex is of type \((N_1)\), \((N_2)\) or \((N_3)\) respectively.

We design discharging rules \( R_{1.1}, R_{1.2}, R_{1.3}, R_{1.4}, R_{1.5}, R_2, R_3, R_4 \) and \( R_y \) (see Figure 7): for any vertex \( x \) of degree at least 3,

- Rule \( R_1 \) is when \( 3 \leq d(x) \leq 7 \), and \( x \) is 1-linked (with a path \( x - a - y \)) to a vertex \( y \).
  - Rule \( R_{1.1} \) is when \( x \) is weak with \( d(y) \leq 7 \). Then \( x \) gives \( \frac{2}{5} \) to \( a \).
  - Rule \( R_{1.2} \) is when \( x \) is not weak and \( y \) is weak. Then \( x \) gives \( \frac{3}{5} \) to \( a \).
  - Rule \( R_{1.3} \) is when \( x \) and \( y \) are not weak, with \( d(y) \leq 7 \). Then \( x \) gives \( \frac{1}{2} \) to \( a \).
  - Rule \( R_{1.4} \) is when \( 8 \leq d(y) \leq 14 \). Then \( x \) gives \( \frac{2}{5} \) to \( a \).
  - Rule \( R_{1.5} \) is when \( 15 \leq d(y) \) and \( a \) is not negative. Then \( x \) gives \( \frac{1}{9} \) to \( a \).
Figure 7: Discharging rules $R_{1,i}, R_2, R_3,$ and $R_4$

- Rule $R_2$ is when $3 \leq d(x) \leq 7$ and $x$ is adjacent to a vertex $u$ of degree 3 that is adjacent to a vertex of degree 2 and a vertex of degree at most 7. Then $x$ gives $\frac{5}{14}$ to $u$.
- Rule $R_3$ is when $8 \leq d(x) \leq 14$. Then $x$ gives $\frac{5}{8}$ to each of its neighbors.
- Rule $R_4$ is when $15 \leq d(x)$. Then $x$ gives $\frac{4}{5}$ to each of its neighbors.
- Rule $R_g$ states that each positive vertex gives $\frac{2}{5}$ to a common pot, and that each negative vertex receives $\frac{1}{5}$ from the common pot.

**Lemma 10.** The common pot has non-negative value after applying $R_g$.

**Proof.** Given a set of vertices $X$, let $n(X)$ be its number of negative vertices and $p(X)$ its number of positive vertices. To prove that the common pot has positive value after applying $R_g$, we show that each connected component $C$ of $H(G)$ satisfies $p(C) \geq \left\lceil \frac{n(C)}{2} \right\rceil$.

Let $C$ be a connected components of $H(G)$. By Lemma 9, $C$ is either a cactus where each cycle has an odd number of support vertices or a lock. If $C$ is a lock, then $n(C) = 4$ and $p(C) = 2$, so we are done. So we can assume that $C$ is a cactus where each cycle has an odd number of support vertices.

**Claim 13.** Every connected subgraph $C'$ of $C$, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in $C'$ and whose negative vertices of Type ($N_3$) are adjacent to their two neighbors in $C'$, satisfies $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

**Proof.** Suppose by contradiction that this is false. Let $C'$ be a connected subgraph of $C$ of minimum number of vertices, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in $C'$, and such that $p(C') < \left\lceil \frac{n(C')}{2} \right\rceil$. The graph $C'$ is a connected subgraph of a cactus so it is also a cactus.

Suppose first that $C'$ contains a pendant vertex $u$. Let $x$ be the neighbor of the positive vertex $u$ in $C'$. As $H(G)$ contains only edges incident to support vertices, $x$ is a support vertex. So it is not
positive and thus is not a pendant vertex of $C'$. So $x$ has at least two neighbors in $C'$. We consider different cases according to the Type of $x$ and its number of neighbors in $C'$.

- **$x$ is of Type ($S_1$).** Then let $a$ be the neighbor of $x$ distinct from $u$. We have $a \in C'$ and $a$ is a support vertex of Type ($S_1$). The positive neighbor $b$ of $a$ is in $C'$ by assumption. Let $C''$ be the graph $C' \setminus \{u, x, a\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph $C''$ is a connected subgraph of $C$ since $u, x, a$ is subpath of $C'$ where $u$ is pendant and $x, a$ are of degree 2. All the pendant vertices of $C''$ are positive since the only new possible pendant vertex is $b$. All the support vertices of $C''$ are adjacent to their positive neighbor in $C''$ since the only positive vertex that has been removed is $u$ and its support neighbor $x$ has also been removed. All the negative vertices of Type ($N_3$) are adjacent to their two neighbors in $C'$ as no support vertex of Type ($S_3$) has been removed. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') = p(C'') + 1 \geq \left\lceil \frac{n(C'' + 2)}{2} \right\rceil = \left\lceil \frac{n(C')}{2} \right\rceil$.

- **$x$ is of Type ($S_2$).** Then let $a$ be the neighbor of $x$ distinct from $u$. We have $a \in C'$ and $a$ is of degree 3. Let $b, c$ be the neighbors of $a$ distinct from $x$. Since $a$ is not positive, it is not a pendant vertex of $C'$, so at least one of $b, c$ is in $C'$. As $H(G)$ contains only edges incident to support vertices, vertex $c$ is a support vertex of Type ($S_2$). We consider two cases depending on whether $a$ has its three neighbors in $C'$ or not.

  If $b \in C'$, then let $C''$ be the graph $C' \setminus \{u, x\}$. We have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph $C''$ is a connected subgraph of $C$, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in $C''$ and all its negative vertices of Type ($N_3$) are adjacent to their two neighbors in $C'$. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C'' + 2)}{2} \right\rceil$.

  If $b \notin C'$, then let $C''$ be the graph $C' \setminus \{u, x, a\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph $C''$ is a connected subgraph of $C$, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in $C''$ and all its negative vertices of Type ($N_3$) are adjacent to their two neighbors in $C'$. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$.

- **$x$ is of Type ($S_3$) and has two neighbors in $C'$.** Then let $c$ be the neighbor of $x$ distinct from $u$ that is in $C'$. Vertex $c$ is of degree 2, it is not positive, so its neighbor $d$, distinct from $x$, is in $C'$. As $H(G)$ contains only edges incident to support vertices and $c$ is not a support vertex, vertex $d$ is a support vertex and so of Type ($S_3$). Let $e, f$ be the neighbors of $d$ distinct from $c$ where $e$ is a positive vertex and $f$ is a vertex of degree 2. Vertex $e$ is the positive neighbor of $d$ so it is in $C'$ by assumption. We consider two cases corresponding to whether $d$ has its three neighbors in $C'$ or not. If $f \in C'$, then let $C''$ be the graph $C' \setminus \{u, x, c\}$. If $f \notin C'$, then let $C''$ be the graph $C' \setminus \{u, x, c, d\}$. In both cases, we have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph $C''$ is a connected subgraph of $C$, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in $C''$ and all its negative vertices of Type ($N_3$) are adjacent to their two neighbors in $C'$. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$.

- **$x$ is of Type ($S_3$) and has three neighbors in $C'$.** Then let $a, c$ be the neighbors of $x$ distinct from $u$. We have $a, c$ in $C'$. Vertex $a$ (resp. $c$) is of degree 2, it is not positive, so its neighbor $b$ (resp. $d$)
is in $C'$. As $H(G)$ contains only edges incident to support vertices and $a$ and $c$ are not support vertices, vertices $b$ and $d$ are support vertices and so of Type $(S_3)$. The positive neighbor $h$ of $b$ (resp. $e$ of $d$) is in $C'$, by assumption. We consider several cases corresponding to whether $b$ and $d$ both have their three neighbors in $C'$ or not. If $b$ and $d$ both have their three neighbors in $C'$, then let $C''$ be the graph $C' \setminus \{u, x, c, a\}$. If $b$ has its three neighbors in $C'$ but not $d$, then let $C''$ be the graph $C' \setminus \{u, x, c, a, d\}$. If $d$ has its three neighbors in $C'$ but not $b$, then let $C''$ be the graph $C' \setminus \{u, x, c, a, b\}$. If none of $b$ and $d$ has its three neighbors in $C'$, then let $C''$ be the graph $C' \setminus \{u, x, c, a, b, d\}$. In the four cases we have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph $C''$ is not necessarily connected but it is composed of one or two connected subgraph of $C$ whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in $C''$ and all its negative vertices of Type $(N_3)$ are adjacent to their two neighbors in $C'$. So by minimality (on each component of $C''$), we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

Now we can assume that $C'$ contains no pendant vertex. Suppose that $C'$ is a single vertex $v$. Then $v$ is not support as all support vertices have their positive neighbor in $C'$ and $v$ is not negative of Type $(N_3)$ as negative vertices of Type $(N_3)$ have their two neighbors in $C'$. So $v$ is not negative and $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil = 0$. Now we can assume that $C'$ is not a single vertex. The graph $C'$ is a cactus, not a single vertex, contains no pendant vertex, so it contains a cycle $C''$, of size $\geq 3$, such that $C'' = C' \setminus C'$ is connected (note that we may have $C' = C''$ and $C''$ is empty). Cycle $C''$ is a cycle of $C$ so it has an odd number of support vertices by Lemma 9. Let $S$ be the set of support vertices of $C''$, with $s = |S|$. By Lemma 9 cycle $C''$ contains a subpath $s_1v_1s_2v_2s_3$ where $s_1, s_2, s_3$ are support vertices of Type $(S_3)$ and $v_1, v_2$ are vertices of degree 2. By assumption, the positive vertex $z$ that is adjacent to $s_2$ is in $C'$. It is not in $C''$ as there is no chord in $C''$. So the only vertex of $C''$ that has some neighbors in $C' \setminus C''$ is $s_2$. So all the positive vertices that are adjacent to $S \setminus \{s_2\}$ are vertices of $C'$ and thus of $C''$. A positive vertex of $C''$ has at most two support neighbors in $C''$ so $p(C'') \geq \left\lceil \frac{s+1}{2} \right\rceil$. A support vertex of Type $(S_1)$ or $(S_2)$ is a negative vertex of Type $(N_1)$ or $(N_2)$. A negative vertex of Type $(N_3)$ of $C''$ is of degree 2 and so has its two neighbors on $C''$ and this two neighbors are support vertices of Type $(S_3)$. So the number of negative vertices of $C''$ is less or equal to the number of support vertices of $C''$ and strictly less if $C''$ contains a vertex of Type $(N_3)$. Vertex $v_1$ is of Type $(N_3)$, so $s > n(C'')$ and so $p(C'') \geq \left\lceil \frac{s+1}{2} \right\rceil \geq \left\lceil \frac{n(C'')}{2} \right\rceil$. The graph $C'''$ is a connected subgraph of $C$ whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in $C''$ and all its negative vertices of Type $(N_3)$ are adjacent to their two neighbors in $C'$. So by minimality we have $p(C''') \geq \left\lceil \frac{n(C''')}{2} \right\rceil$. So finally, $p(C') = p(C'') + p(C''') \geq \left\lceil \frac{n(C'')}{2} \right\rceil + \left\lceil \frac{n(C''')}{2} \right\rceil \geq \left\lceil \frac{n(C'')}{2} \right\rceil \geq \frac{n(C')}{2}.$
negative vertex have been removed from \( C \) and \( n(C') = n(C) \). A pendant vertex of \( C \) that has been removed is not positive, not support, not negative but incident to a support, so it is necessarily a degree 2 vertex \( a \) incident to a support vertex \( x \) of Type \( (S_3) \) (with notations of Figure 2). When \( a \) is removed from \( C \), this does not create any new pendant vertex as \( x \) has degree 2 after the removal. All pendant vertices that are not positive are removed from \( C \), no new pendant vertices are created, thus in \( C \) all pendant vertices are positive. No positive vertex has been removed and each support vertex is adjacent to its positive neighbor in \( H(G) \), so support vertices of \( C' \) are adjacent to their positive neighbor in \( C' \). No support vertex have been removed and each negative vertex of Type \( (N_3) \) is adjacent to its support neighbors of Type \( (S_3) \) in \( H(G) \), so negative vertices of Type \( (N_3) \) of \( C' \) are adjacent to their two neighbors in \( C' \). By Claim 13 applied to \( C' \), we have \( p(C') \geq \lceil \frac{n(C')}{2} \rceil \). So \( p(C) = p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil = \left\lceil \frac{n(C)}{2} \right\rceil \) and we are done.

We now use the discharging rules to prove the following:

Lemma 11. mad\((G) \geq 3\).

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules \( R_1, R_2, R_3, R_4 \) and \( R_g \). The common pot is empty at the beginning and, by Lemma 10, it has non-negative value after applying \( R_g \). We show that all the vertices have a weight of at least 3 at the end.

Let \( u \) be a vertex of \( G \). By Lemma 2, graph \( G \) does not contain Configurations \((C_1)\) to \((C_{11})\). According to Configuration \((C_1)\), we have \( d(u) \geq 2 \). We now consider different cases corresponding to the value of \( d(u) \).

1. \( d(u) = 2 \).

So \( u \) has an initial weight of 2 and gives nothing. We show that it receives at least 1, so it has a final weight of at least 3.

(a) Assume \( u \) is adjacent to a vertex \( u_2 \) of degree 2.

Then \( u \) is a negative vertex of Type \((N_1)\) and receives \( \frac{1}{5} \) from the common pot by \( R_g \). According to Configuration \((C_2)\), vertex \( u \) is adjacent to a vertex \( v \) with \( d(v) = k \). Since \( k \geq 17 \), according to \( R_4 \), vertex \( v \) gives \( \frac{4}{17} \) to \( u \).

(b) Assume both neighbors \( v_1 \) and \( v_2 \) of \( u \) are of degree at least 3.

Vertex \( u \) is not a negative vertex of Type \((N_1)\) since it has no neighbor of degree 2.

i. \( u \) has two weak neighbors

Then \( u \) is a negative vertex of Type \((N_3)\). It receives \( \frac{1}{5} \) from the common pot by \( R_g \) and \( \frac{2}{9} \) from each of its two neighbors by \( R_{1.1} \).

ii. \( u \) has one weak neighbor \( w \) and one non-weak neighbor \( v \)

A. \( 3 \leq d(v) \leq 7 \)

Vertex \( u \) receives \( \frac{3}{9} \) from \( v \) by \( R_{1.2} \) and \( \frac{2}{9} \) from \( w \) by \( R_{1.1} \).

B. \( 8 \leq d(v) \leq 14 \)

Vertex \( u \) receives \( \frac{5}{9} \) from \( v \) by \( R_3 \) and \( \frac{2}{9} \) from \( w \) by \( R_{1.4} \).

C. \( 15 \leq d(v) \)

Vertex \( w \) is weak and \( v \) has degree at least 15, so one can check that \( u \) is not negative of Type \((N_1)\) or \((N_3)\). According to Configuration \((C_3)\), it not negative.
of Type \((N_2)\). So \(u\) is not negative and it receives \(\frac{1}{5}\) from \(w\) by \(R_{1.5}\) and \(\frac{1}{5}\) from \(v\) by \(R_4\).

iii. \(u\) has two non-weak neighbors \(v, v'\)

A. \(3 \leq d(v) \leq 7\) and \(3 \leq d(v') \leq 7\)
   Vertex \(u\) receives \(\frac{1}{7}\) from each neighbor by \(R_{1.3}\).

B. \(3 \leq d(v) \leq 7\) and \(8 \leq d(v') \leq 14\)
   Vertex \(u\) receives \(\frac{2}{7}\) from \(v'\) by \(R_3\) and \(\frac{3}{7}\) from \(v\) by \(R_{1.4}\).

C. \(3 \leq d(v) \leq 7\) and \(15 \leq d(v')\)
   If \(u\) is negative, it receives \(\frac{1}{7}\) from the common pot by \(R_g\). If \(u\) is non-negative, it receives \(\frac{1}{5}\) from \(v\) by \(R_{1.5}\). In both cases, it receives \(\frac{4}{5}\) from \(v'\) by \(R_4\).

D. \(8 \leq d(v)\) and \(8 \leq d(v')\)
   Vertex \(u\) receives at least \(\frac{5}{8}\) from each neighbor by \(R_3\) or \(R_4\).

2. \(d(u) = 3\).
   So \(u\) has an initial weight of 3. We show that it has a final weight of at least 3.

(a) Assume \(u\) has three neighbors \(y_1, y_2, y_3\) of degree 2.
   Let \(z_i, 1 \leq i \leq 3\), be the neighbors of \(y_i\) distinct from \(u\). According to Configuration \((C_3)\), \(d(z_1) = d(z_2) = d(z_3) = k\). So \(y_1, y_2, y_3\) are negative vertices of Type \((N_2)\). So no rule applies to \(u\).

(b) Assume \(u\) has exactly two neighbors \(y_1, y_2\) of degree 2.
   Let \(z_i, 1 \leq i \leq 2\), be the neighbors of \(y_i\) distinct from \(u\). Let \(x\) the third neighbor of \(u\), \(d(x) \geq 3\). According to Configuration \((C_3)\), we are in one of the two following cases:
   
   i. \(d(x) \geq k - 2\).
      Vertex \(x\) gives \(\frac{1}{5}\) to \(u\) by \(R_4\) and \(u\) gives nothing to \(x\).
      
      A. Assume vertex \(u\) is weak.
         Since \(u\) is weak, \(d(y_i) \leq 14\), so vertex \(u\) gives at most \(\frac{2}{5}\) to each of \(y_1, y_2\) by \(R_{1.1}\) or \(R_{1.4}\).
      
      B. Assume vertex \(u\) is not weak.
         Then, \text{w.l.o.g.}, \(d(z_1) \geq 15\). So vertex \(u\) gives at most \(\frac{1}{5}\) to \(y_1\) by \(R_{1.5}\). Vertex \(u\) gives at most \(\frac{4}{5}\) to \(y_2\) by \(R_{1.2}, R_{1.3}, R_{1.4}\) or \(R_{1.5}\).
   
   ii. \(d(z_1) = d(z_2) = k\).
      
      A. \(d(x) \leq 7\).
         According to Configuration \((C_4)\), vertex \(u\) gives nothing to \(x\) by \(R_2\). Vertices \(y_1\) and \(y_2\) are negative (of Type \((N_2)\)) and \(u\) gives nothing to \(y_1, y_2\).
      
      B. \(d(x) \geq 8\).
         Vertex \(u\) gives \(\frac{1}{5}\) to \(y_1\) and \(y_2\) by \(R_{1.5}\). Vertex \(x\) gives at least \(\frac{5}{8}\) to \(u\) by \(R_3\) or \(R_4\).

(c) Assume \(u\) has exactly one neighbor \(y\) of degree 2
   Let \(z\) be the neighbor of \(y\) distinct from \(u\). Let \(w\) and \(x\) the other neighbors of \(u\), where \(d(w) \geq d(x) \geq 3\). We consider three cases according to the value of \(d(w)\).
   
   i. \(15 \leq d(w)\).
      Then, vertex \(u\) gives at most \(\frac{1}{5}\) to \(y\) by \(R_{1.i}, 1 \leq i \leq 5\). Vertex \(u\) gives at most \(\frac{1}{10}\) to \(x\) by \(R_2\). Vertex \(w\) gives \(\frac{1}{5}\) to \(u\) by \(R_4\).
ii. $8 \leq d(w) \leq 14.$

According to Configuration ($C_4$), vertex $u$ gives nothing to $x$ by $R_2$. Vertex $u$ gives at most $\frac{3}{5}$ to $y$ by $R_{1,i}$, $1 \leq i \leq 5$. Vertex $w$ gives $\frac{5}{8}$ to $u$ by $R_3$.

iii. $d(w) \leq 7.$

According to Configuration ($C_4$), vertex $u$ gives nothing to $x$ and $w$ by $R_2$. According to Configuration ($C_3$), we have $d(z) = k$. Vertex $u$ gives $\frac{1}{3}$ to $y$ by $R_{1.5}$. Both $w$ and $x$ give $\frac{1}{10}$ to $u$ by $R_2$.

(d) Assume all the neighbors of $u$ have degree at least 3 and at most 7.

According to Configuration ($C_4$), vertex $u$ gives nothing to its neighbors by $R_2$.

(e) Assume $u$ has no neighbor of degree 2 and at least a neighbor $v$ of degree at least 8.

Vertex $v$ gives at least $\frac{5}{8}$ to $u$ by $R_3$ or $R_4$. Vertex $u$ gives at most $\frac{1}{10}$ to each of its other neighbors by $R_2$.

3. $d(u) = 4.$

So $u$ has an initial weight of 4. We show that it has a final weight of at least 3.

(a) Assume $u$ has at least three neighbors $y_1$, $y_2$ and $y_3$ of degree 2.

Let $z_i$ be the neighbors of $y_i$ distinct from $u$. We assume that $d(z_1) \geq d(z_2) \geq d(z_3)$. Let $x$ be the neighbor of $u$ distinct from $y_1$, $y_2$ and $y_3$. We consider three cases depending on $d(z_2)$ and $d(z_3)$.

i. $d(z_2) \leq 14.$

According to Configuration ($C_7$), we have $d(x) \geq k - 2$. Vertex $u$ gives at most $3 \times \frac{3}{5}$ by $R_{1,i}$, $1 \leq i \leq 5$. Vertex $x$ gives $\frac{1}{2}$ to $u$ by $R_4$.

ii. $d(z_2) \geq 15$ and $d(z_3) \leq 14.$

According to Configuration ($C_6$), we have $d(x) \geq 8$. Vertex $u$ gives at most $\frac{1}{5}$ to each of $y_1$, $y_2$ by $R_{1.5}$. Vertex $u$ gives at most $\frac{2}{5}$ to $y_3$ by $R_{1,i}$. 

iii. $d(z_3) \geq 15.$

Vertex $u$ gives at most $\frac{1}{5}$ to each of its neighbors by $R_{1.5}$.

(b) Assume $u$ has exactly two neighbors $y_1$ and $y_2$ of degree 2.

Let $z_i$ be the neighbors of $y_i$ distinct from $u$. We assume that $d(z_1) \geq d(z_2)$. Let $w$ and $x$ the neighbors of $u$ distinct from $y_1$, $y_2$. We assume that $d(w) \geq d(x) \geq 3$. We consider two cases depending on $d(z_1)$.

i. $d(z_1) \leq 14.$

According to Configuration ($C_7$), we have $d(w) \geq 9$. Vertex $u$ gives at most $\frac{3}{5}$ to each of $y_1$, $y_2$ by $R_{1,i}$, and at most $\frac{1}{10}$ to $x$ by $R_2$. Vertex $x$ gives at least $\frac{3}{8}$ to $u$ by $R_3$ or $R_4$.

ii. $d(z_1) \geq 15.$

Vertex $u$ gives at most $\frac{1}{3}$ to $y_1$ by $R_{1.6}$, at most $\frac{3}{5}$ to $y_2$ by $R_{1.6}$, and at most $\frac{1}{10}$ to each of $w$, $x$ by $R_2$.

(c) Assume $u$ has at most one neighbor of degree 2.

Vertex $u$ gives at most $3 \times \frac{1}{10}$ by $R_2$, and at most $\frac{2}{5}$ by $R_{1,i}$.

4. $d(u) = 5.$

So $u$ has an initial weight of 5. We show that it has a final weight of at least 3.
(a) Assume $u$ has at least four neighbors $y_1, y_2, y_3$ and $y_4$ of degree 2
Let $z_i$ be the neighbors of $y_i$ distinct from $u$. We assume that $d(z_1) \geq d(z_2) \geq d(z_3) \geq d(z_4)$. Let $x$ be the neighbor of $u$ distinct from the $y_i$'s. We consider two cases depending on $d(z_4)$.

i. $d(z_4) \leq 7$.
According to Configuration $(C_8)$, we have $d(x) \geq 8$. Vertex $u$ gives at most $\frac{3}{8}$ to each of $y_i$ by $R_{1,i}$. Vertex $x$ gives at least $\frac{5}{8}$ to $u$ by $R_3$ or $R_4$.

ii. $d(z_4) \geq 8$.
Vertex $u$ gives at most $5 \times \frac{3}{8}$ to each of $y_i$ and $x$ by $R_{1,4}$ or $R_{1,5}$.

(b) Assume $u$ has at most three neighbors of degree 2.
Vertex $u$ gives at most $3 \times \frac{5}{8}$ by $R_{1,i}$, and at most $2 \times \frac{1}{10}$ by $R_2$.

5. $d(u) = 6$.
So $u$ has an initial weight of 6. We show that it has a final weight of at least 3.

(a) Assume $u$ has at least five neighbors $y_1, \ldots, y_5$, of degree 2
Let $z_i$ be the neighbors of $y_i$ distinct from $u$. We assume that $d(z_1) \geq \cdots \geq d(z_5)$. Let $x$ be the neighbors of $u$ distinct from the $y_i$'s. According to Configuration $(C_9)$, we are in one of the following two cases.

i. $d(z_5) \geq 8$.
Vertex $u$ gives at most $6 \times \frac{2}{8}$ to each of its neighbors by $R_{1,4}$ or $R_{1,5}$.

ii. $d(x) \geq 8$.
Vertex $u$ gives at most $5 \times \frac{3}{8}$ to each of $y_i$.

(b) Assume $u$ has at most four neighbors of degree 2.
Vertex $u$ gives at most $4 \times \frac{5}{8}$ by $R_{1,i}$, and at most $2 \times \frac{1}{10}$ by $R_2$.

6. $d(u) = 7$.
So $u$ has an initial weight of 7. We show that it has a final weight of at least 3.

(a) Assume $u$ has at least six neighbors of degree 2 adjacent to vertices of degree at most 3.
According to Configuration $(C_{10})$, vertex $u$ has a neighbor $v$ of degree at least 8. Vertex $u$ gives at most $6 \times \frac{2}{8}$ by $R_{1,i}$.

(b) Assume $u$ has at most five neighbors of degree 2 adjacent to vertices of degree at most 3.
Vertex $u$ gives at most $5 \times \frac{3}{8}$ by $R_{1,i}$, and at most $2 \times \frac{1}{2}$.

7. $8 \leq d(u) \leq 14$.
Then Rule $R_3$ applies to every neighbor of $u$, and $d(u) - (d(u) \times \frac{5}{8}) \geq 3$.

8. $15 \leq d(u) < k$.
Then Rule $R_4$ applies to every neighbor of $u$, and $d(u) - (d(u) \times \frac{4}{9}) \geq 3$.

9. $d(u) = k$.
Then Rule $R_4$ applies to every neighbor of $u$ and $R_g$ applies to $u$. We have $k \geq 17$ so $k - (k \times \frac{4}{9} + \frac{2}{8}) \geq 3$.  

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Consequently, after application of the discharging rules, every vertex $v$ of $G$ has a weight of at least $3$, meaning that $\sum_{v \in G} d(v) \geq 3|V|$. Therefore, $\text{mad}(G) \geq 3$. \hfill \square

Finally, $k$ is a constant integer greater than $17$ and $G$ is a minimal graph such that $\Delta(G) \leq k$ and $G$ admits no $2$-distance $(k + 2)$-list-coloring. By Lemma\textsuperscript{[1]} we have $\text{mad}(G) \geq 3$. So Theorem\textsuperscript{4} is true.

7 Conclusion

We proved that graphs with $\Delta(G) \geq 17$ and maximum average degree less than $3$ are list $2$-distance $(\Delta(G) + 2)$-colorable. The key idea in the proof is to use Brooks’ lemma (Lemma\textsuperscript{7}) instead of the usual special case of an even cycle being $2$-choosable. Thus we can prove stronger structural properties, which results in a global arborescent structure that is a cactus. As far as we know, Brooks’ lemma has not been used in a global discharging proof before, and it might be useful for other problems. One remaining question would be to determine the maximum $\Delta(G)$ of a graph $G$ with $\text{mad}(G) < 3$ that is not $2$-distance $(\Delta(G) + 2)$-colorable. By Theorem\textsuperscript{4} it cannot be more than $16$.

Note that these proofs can be effortlessly transposed to list injective $(\Delta(G) + 1)$-coloring. Indeed, every vertex we color has a neighbor that is already colored. This means that in the case of list injective coloring, every vertex we color has at least one constraint less than in the case of list $2$-distance coloring. Consequently, $\Delta(G) + 1$ colors are enough in the case of list injective coloring, as mentioned in the introduction.

In contrast to Theorem\textsuperscript{4}, other results have been obtained on the $2$-distance coloring of planar graphs of girth at least $6$ when more colors are allowed. For example, Bu and Zhu\textsuperscript{[15]} proved that every planar graph $G$ of girth at least $6$ was $2$-distance $(\Delta(G) + 5)$-colorable.

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