A NOTE ON PRINCIPAL PARTS ON PROJECTIVE SPACE AND
LINEAR REPRESENTATIONS

HELGE MAAKESTAD

Abstract. Let $H$ be a closed subgroup of a linear algebraic group $G$ defined
over a field of characteristic zero. There is an equivalence of categories between
the category of linear finite-dimensional representations of $H$, and the category
of finite rank $G$-homogeneous vector bundles on $G/H$. In this paper we will
study this correspondence for the sheaves of principal parts on projective space,
and we describe the representation corresponding to the principal parts of a
line bundle on projective space.

1. Introduction

In this note we will study the vector bundles of principal parts $P^k(O(n))$ of a line
bundle on projective space over a field $F$ of characteristic zero from a representation
theoretic point of view. We consider projective $N$-space as a quotient $SL(V)/P$,
where $V$ is an $N+1$-dimensional vectorspace over $F$, and $P$ is the subgroup of
$SL(V)$ stabilizing a line $L$ in $V$. There is an equivalence of categories between the
category of finite rank $SL(V)$-homogeneous vector bundles on $SL(V)/P = P(V^*)$
and the category of linear finite-dimensional representations of $P$. The principal
parts $P^k(O(n))$ are $SL(V)$-homogeneous vector bundles on $P(V^*)$, and the novelty
of this note is that we describe the $P$-representation corresponding to the principal
parts. The main result is Theorem 2.4. The Theorem says the following: Let $L^*$
be the dual of the $P$-module $L$. Then for all $1 \leq k < n$, the $P$-representation
corresponding to $P^k(O(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$. As a corollary, we obtain the
splitting-type of $P^k(O(n))$ on $P(V^*)$ for all $1 \leq k < n$, and recover results obtained
in [5], [6], [7] and [8].

2. Principal parts on projective space

In this section we give the representation corresponding to $P^k(O(n))$ on $P(V^*)$
for all $1 \leq k < n$, where $V$ is an $F$-vectorspace of dimension $N+1$ and $F$ is a field
of characteristic $0$. A variety is an integral scheme of finite type over $F$. We will
consider closed points when we talk about points of a scheme. Let $V$ be a finite
dimensional vectorspace over $F$. We let $GL(V)$ denote the group of all invertible
linear transformations of $V$. It is an algebraic group in the sense of [2], chapt. 1.
A linear algebraic group is a closed subgroup of $GL(V)$. Let $SL(V)$ be the linear

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algebraic group of linear transformations of $V$ with determinant 1. Let $L$ in $V$
be a line, and $P$ the closed subgroup of $SL(V)$ stabilizing $L$. Then the quotient
$SL(V)/P$ (which exists by [2], Theorem 6.8) is isomorphic to $P(V^*)$, the projective
space of lines in $V$ (see [1], Section 4.2. This works over any field, not only the
complex numbers). There exists a natural left $SL(V)$-action on $P(V^*)$, making
it into a homogeneous space for $SL(V)$. And by [1], Chapt. 4, there exists an
equivalence of categories between the category of finite rank homogeneous vector
bundles on $SL(V)/P$ and the category of linear finite-dimensional representations
of $P$, and under this correspondence the dimension of the representation gives
the rank of the corresponding vector bundle. Hence any character of $P$ gives a
homogeneous line bundle on $P$. The line $L$ corresponds to a character of $P$,
and the bundle corresponding to the dual line $L^*$, is the line bundle $O(1)$ on $P(V^*)$
(see [1], Section 4.2). It is also a standard fact that any linear finite-dimensional
representation $\rho$ of $P$ lifting to a representation $\tilde{\rho}$ of $SL(V)$ corresponds to a trivial
abstract vector bundle on $P(V^*)$. There exists on any scheme an equivalence of
categories between the category of locally free finite rank sheaves and the category
of finite rank vector bundles, hence we will use these two notions interchangeably.

Pick a basis $e_0, \ldots, e_N$ for $V$. Let $x_0, \ldots, x_N$ be the dual basis, and let $L$ be
the line spanned by $e_0$. Having chosen a basis for $V$ it follows that $SL(V)$ may
be identified with the group of square rank $N + 1$ matrices with determinant equal
to 1. The group $P$ may be identified with the subgroup of $SL(V)$ consisting of
matrices $g$ of the form

$$
g = \begin{pmatrix}
a & * & \cdots & * \\
0 & a_{11} & \cdots & a_{1n} \\
& \vdots & \ddots & \vdots \\
0 & a_{n1} & \cdots & a_{nn}
\end{pmatrix}.
$$

The one dimensional representation $\chi : P \to GL(S^n(L^*))$ corresponding to the line
bundle $O(n)$ is given by $\chi(g) = a^{-n}$.

Let $X$ be a smooth variety of dimension $d$ and consider the diagonal $\Delta$ in $X \times X$.
Let $I$ be the sheaf of ideals of $O_{X \times X}$ defining the diagonal $\Delta$, and put $O_{\Delta^k}$ to be
$O_{X \times X}/I^{k+1}$.

**Definition 2.1.** Let $p, q$ be the projection maps from $X \times X$ to $X$, and let $E$ be an
$O_X$-module. Define $P^k(E) = p_\ast(O_{\Delta^k} \otimes q^\ast E)$ to be the $k$th order principal parts
of the module $E$. We put $P^k(O_X) = P^k$.

Note that by [5], if the rank of $E$ is $e$, $P^k(E)$ is a vector bundle of rank $e \binom{d+k}{d}$
on $X$. Assume that $G$ is an algebraic group, and that $X$ is a homogeneous space
for $G$. Assume furthermore that $E$ is a $G$-homogeneous vector bundle on $X$, then
it follows that $P^k(E)$ is again a $G$-homogeneous vector bundle on $X$. Consider the
line bundle $O(n)$ on $P(V^*) = SL(V)/P$, then $O(n)$ is an $SL(V)$-homogeneous line
bundle on $P(V^*)$ for all $n$, and we may consider the $SL(V)$-homogeneous vector
bundle $P^k(O(n))$. We want to compute the representation $\rho$ of $P$ corresponding
to the homogeneous vector bundle $P^k(O(n))$ for all $1 \leq k < n$ on $P(V^*)$. Let in
the following $X = P(V^*)$ and consider the projection maps $p, q$ from $X \times X$ to
$X$. Let $I$ in $O_{X \times X}$ be the ideal of the diagonal. We have an exact sequence of
$SL(V)$-homogeneous vector bundles on $X \times X$:

$$
0 \to I^{k+1} \to O_{X \times X} \to O_{\Delta^k} \to 0.
$$

(2.1.1)
Apply the functor \( p_* (\otimes q^* \mathcal{O}(n)) \) to the sequence (2.1.1) to get a long exact sequence (2.1.2)
\[
0 \to p_*(\mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n)) \to p_* q^* \mathcal{O}(n) \to \mathcal{P}^k(\mathcal{O}(n))
\]
\[
\to R^1 p_*(\mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n)) \to R^1 p_* q^* \mathcal{O}(n) \to R^1 p_*(\mathcal{O}_\Delta \otimes q^* \mathcal{O}(n)) \to \cdots
\]
of vector bundles. The sequence (2.1.2) is a sequence of vector bundles because all sheaves in the sequence are coherent, and it is a standard fact that a homogeneous coherent sheaf on a homogeneous space is locally free. Since the sequence (2.1.2) is a sequence of vector bundles, we get an exact sequence of \( P \)-representations when we pass to the fiber at \( \tau \).

Consider the diagram

\[
\begin{array}{ccc}
\text{Spec}(\kappa(\tau)) \times X & \xrightarrow{j} & X \times X \\
\downarrow \pi & & \downarrow p \\
\text{Spec}(\kappa(\tau)) & \xrightarrow{i} & X
\end{array}
\]

then by [4], Chapt. III, Sect.12 we get maps
\[
\phi^i : R^i p_*(\mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n))(\tau) \to R^i p_*(j^*(\mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n)))
\]
of \( \mathcal{O}_X \)-modules. Put for any \( \mathcal{O}_{X \times X} \)-module \( \mathcal{E} \)
\[
h^i(y, E) = \dim_{\kappa(y)} H^i(X_y, \mathcal{E}_y),
\]
where \( X_y \) is the fiber \( p^{-1}(y) \) and \( \mathcal{E}_y \) is the restriction of \( \mathcal{E} \) to \( X_y \). We see that
\[
h^i(y, \mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n)) = \dim_{\kappa(y)} H^i(X, m_y^{k+1} \otimes \mathcal{O}(n))
\]
is a constant function of \( y \) for \( i = 0, 1, \ldots \) for the following reason: Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(\kappa(y)) \times X & \xrightarrow{\bar{g}} & \text{Spec}(\kappa(gv)) \times X \\
\downarrow j & & \downarrow k \\
X \times X & \xrightarrow{g} & X \times X
\end{array}
\]
where the action of \( SL(V) \) on \( X \times X \) is given by \( g(x, y) = (gx, gy) \). In general if \( G \times Y \to \sigma Y \) is an algebraic group acting on a scheme \( Y \), and \( \mathcal{E} \) is a \( G \)-linearized sheaf on \( Y \), then there exists an isomorphism \( I : \sigma^* \mathcal{E} \to p^* \mathcal{E} \) where \( p : G \times Y \to Y \) is the projection map. It follows that for all \( g \in G \), we get an isomorphism \( g^* \mathcal{E} \cong \mathcal{E} \) of sheaves. Then since \( \mathcal{O}(n) \) and \( \mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n) \) are \( SL(V) \)-homogeneous sheaves, we have an isomorphism
\[
\tilde{g}^* (m_y^{k+1} \otimes \mathcal{O}(n)) = j^* g^* (\mathcal{I}^{k+1} \otimes q^* \mathcal{O}(n)) = m_y^{k+1} \otimes \mathcal{O}(n)
\]
hence since \( \tilde{g} \) is an isomorphism, we see that we have an isomorphism
\[
m_y^{k+1} \otimes \mathcal{O}(n) \cong m_y^{k+1} \otimes \mathcal{O}(n)
\]
of sheaves for all \( g \) in \( SL(V) \). It follows that
\[
\dim_{\kappa(y)} H^i(X, m_y^{k+1} \otimes \mathcal{O}(n)) = \dim_{\kappa(gy)} H^i(X, m_y^{k+1} \otimes \mathcal{O}(n))
\]
for all \( g \) in \( SL(V) \), hence by \([4]\), chapter III Corr. 12.9, it follows that the maps \( \phi^i \) are isomorphisms for \( i = 0, 1, \ldots \). Here \( m_y \) is the sheaf of ideals corresponding to the point \( y \) in \( X \). We get an exact sequence

\[
(2.1.3) \quad 0 \to H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) \to H^0(X, \mathcal{O}(n)) \to P^k(\mathcal{O}(n))(_\pi) \to H^1(X, \mathcal{O}(n)) \to \cdots
\]

of \( P \)-representations.

**Lemma 2.2.** For all \( 1 \leq k < n \) we have that \( H^1(X, \mathcal{O}(n) \otimes m_y^{k+1}) = 0 \).

**Proof.** Consider the exact sequence \((2.1.3)\). We prove that

\[
\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) = \dim_F P^k(\mathcal{O}(n))(_\pi),
\]

and then the result follows by counting dimensions. We have that \( H^0(X, \mathcal{O}(n)) \) equals \( S^n(V^*) \), where \( V^* \) is the \( F \)-vector space on the basis \( x_0, \ldots, x_N \). We also see that \( H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) \) equals \( m_y^{k+1} S^{n-(k+1)}(V^*) \) considered as a subspace of \( S^n(V^*) \). Here \( m \) is the \( F \)-vector space on the basis \( x_1, \ldots, x_N \) and \( m^{k+1} S^{n-(k+1)}(V^*) \) is the image of the natural map

\[
S^{k+1}(m) \otimes S^{n-(k+1)}(V^*) \to S^n(V^*).
\]

Write \( V^* \) as the direct sum \( Fx_0 \oplus m \). Then it follows that

\[
m^{k+1} S^{n-(k+1)}(V^*) = x_0^{n-(k+1)} m^{k+1} + \cdots + x_0 m + m^{n},
\]

hence we see that the dimension of \( m^{k+1} S^{n-(k+1)}(V^*) \) equals \( \sum_{i=k+1}^n \binom{i+N-1}{N-1} \). We also see that the dimension of \( S^n(V^*) \) equals \( \sum_{i=0}^n \binom{i+N-1}{N-1} \), and it follows that

\[
\dim_F S^n(V^*) - \dim_F m^{k+1} S^{n-(k+1)}(V^*) = \sum_{i=0}^k \binom{i+N-1}{N-1}.
\]

It follows that

\[
\sum_{i=0}^k \binom{i+N-1}{N-1} = \binom{k+N}{N} = \dim_F P^k(\mathcal{O}(n))(_\pi),
\]

and we have proved that

\[
\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) = \dim_F P^k(\mathcal{O}(n))(_\pi),
\]

and the result follows from the fact that the sequence \((2.1.3)\) is exact and that \( H^1(X, \mathcal{O}(n)) = 0 \) for \( n \geq 1 \). \( \square \)

Note that by Lemma 2.2 and the sequence \((2.1.3)\) there exists for all \( 1 \leq k < n \) an exact sequence of \( P \)-representations

\[
0 \to H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) \to H^0(X, \mathcal{O}(n)) \to P^k(\mathcal{O}(n))(_\pi) \to 0.
\]

Since the representation \( H^0(X, \mathcal{O}(n) \otimes m_y^{k+1}) \) equals \( m_y^{k+1} S^{n-(k+1)}(V^*) \) as subrepresentation of \( H^0(X, \mathcal{O}(n)) = S^n(V^*) \), it follows that we have an exact sequence of \( P \)-representations

\[
(2.2.1) \quad 0 \to m^{k+1} S^{n-(k+1)}(V^*) \to S^n(V^*) \to P^k(\mathcal{O}(n))(_\pi) \to 0.
\]

From the exact sequence

\[
0 \to m \to V^* \to V^*/m \to 0,
\]
where \( m \) is the \( F \)-vectorspace on \( x_1, \ldots, x_N \), we see that the representation \( V^*/m \) is the representation corresponding to the module \( L^* \) of \( P \), giving the line bundle \( \mathcal{O}(1) \) on \( X = SL(V)/P \).

**Lemma 2.3.** For all \( 1 \leq k < n \) there exists a surjective map of \( P \)-representations

\[
\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*).
\]

*Proof.* Recall that we have chosen a basis \( e_0, \ldots, e_N \) for \( V \), with the property that \( \pi_0 \) is a basis for \( L^* \). The \( P \)-representation \( m \) with basis \( x_1, \ldots, x_N \) gives an exact sequence

\[
0 \to m \to V^* \to L^* \to 0
\]

of \( P \)-representations. Define a map

\[
\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*)
\]

as follows: \( \phi(f) = \pi_0^{-k} \otimes \partial_0^{n-k}(f) \), where \( \partial_0^{n-k} \) is \( n-k \) times partial derivative with respect to the \( x_0 \)-variable. Let \( g \) be an element of \( P \). Then by induction on the degree of the differential-operator \( \partial_0^{n-k} \) and applying the chain-rule for derivation, it follows that

\[
\phi(gf) = \pi_0^{-k} \otimes \partial_0^{n-k}(gf) = \pi_0^{-k} \otimes a^{-(n-k)}g(\partial_0^{n-k}f) = a^{-(n-k)}\pi_0^{-k} \otimes g(\partial_0^{n-k}f) = g(\pi_0^{-(n-k)} \otimes \partial_0^{n-k}f) = g\phi(f),
\]

and we see that \( \phi \) is \( P \)-linear. It is clearly surjective, and the lemma follows. \( \square \)

**Theorem 2.4.** For all \( 1 \leq k < n \), the representation corresponding to \( \mathcal{P}^k(\mathcal{O}(n)) \) is \( S^{n-k}(L^*) \otimes S^k(V^*) \).

*Proof.* By Lemma 2.3 there exists a surjective map of \( P \)-representations

\[
\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*).
\]

We claim that \( m^{k+1}S^{n-(k+1)}(V^*) \) equals \( \ker \phi \): We first prove the inclusion

\[
m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi.
\]

Pick a monomial \( x_0^{p_0}x_1^{p_1} \cdots x_N^{p_N} \) in \( m^{k+1}S^{n-(k+1)}(V^*) \), hence \( p_0 + \cdots + p_N = n \) and \( p_0 < n-k \). These monomials form a basis for \( m^{k+1}S^{n-(k+1)}(V^*) \). We see that \( \partial_0^{n-k}(x_0^{p_0} \cdots x_N^{p_N}) \) is zero, hence since \( \phi \) is a linear map, it follows that we have an inclusion

\[
m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi
\]

of vectorspaces. The reverse inclusion follows from counting dimensions and the fact that \( \phi \) is surjective: We have that

\[
\dim_F \ker \phi = \dim_F S^n(V^*) - \dim_F S^{n-k}(V^*) \otimes S^k(V^*)
\]

\[
= \sum_{i=0}^{n} \binom{i + N - 1}{N - 1} - \sum_{i=0}^{k} \binom{i + N - 1}{N - 1} = \sum_{i=k+1}^{n} \binom{i + N - 1}{N - 1},
\]

and we see that \( \dim_F \ker \phi = \dim_F m^{k+1}S^{n-(k+1)}(V^*) \) and it follows that

\[
m^{k+1}S^{n-(k+1)}(V^*) = \ker \phi,
\]

hence we have an exact sequence of \( P \)-representations

\[
0 \to m^{k+1}S^{n-(k+1)}(V^*) \to S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*) \to 0.
\]
Using sequence \( \text{2.2.1} \) we get isomorphisms
\[
\mathcal{P}^k(\mathcal{O}(n))(\tau) \cong H^0(X, \mathcal{O}(n))/H^0(X, \mathcal{O}(n) \otimes m^{k+1}) \cong \\
S^n(V^*)/m^{k+1}S^{n-(k+1)}(V^*) \cong S^{n-k}(V^*) \otimes S^n(V^*),
\]
and it follows that \( \mathcal{P}^k(\mathcal{O}(n))(\tau) \) is isomorphic to \( S^{n-k}(L^*) \otimes S^k(V^*) \) as representation. \( \square \)

Note that the result in Theorem \( \text{2.4} \) is true if \( \text{char}(F) > n \).

**Corollary 2.5.** For all \( 1 \leq k < n \), \( \mathcal{P}^k(\mathcal{O}(n)) \) splits as abstract vector bundle as \( \oplus \binom{N+k}{N} \mathcal{O}(n-k) \).

**Proof.** Since \( S^k(V^*) \) corresponds to the trivial rank \( \binom{N+k}{N} \) abstract vector bundle on \( \mathbf{P}(V^*) \), and \( S^{n-k}(L^*) \) corresponds to the line bundle \( \mathcal{O}(n-k) \), the assertion is proved. \( \square \)

We see that we recover results on the splitting-type of the principal parts obtained in \( \text{[5], [6], [7] and [8]} \).

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Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel

Email address: makesth@macs.biu.ac.il