Asymptotic velocity of a position-dependent quantum walk

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Abstract

We consider a position-dependent coined quantum walk on \( \mathbb{Z} \) and assume that the coin operator \( C(x) \) satisfies

\[
\|C(x) - C_0\| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}
\]

with positive \( c_1 \) and \( \epsilon \) and \( C_0 \in U(2) \). We show that the Heisenberg operator \( \hat{x}(t) \) of the position operator converges to the asymptotic velocity operator \( \hat{v}_+ \) so that

\[
\text{s- lim } \lim_{t \to \infty} \exp \left( i\xi \frac{\hat{x}(t)}{t} \right) = \Pi_p(U) + \exp(i\xi \hat{v}_+) \Pi_{ac}(U)
\]

provided that \( U \) has no singular continuous spectrum. Here \( \Pi_p(U) \) (resp. \( \Pi_{ac}(U) \)) is the orthogonal projection onto the direct sum of all eigenspaces (resp. the subspace of absolute continuity) of \( U \). We also prove that for the random variable \( X_t \) denoting the position of a quantum walker at time \( t \in \mathbb{N} \), \( X_t/t \) converges in law to a random variable \( V \) with the probability distribution

\[
\mu_V = \|\Pi_p(U)\Psi_0\|^2 \delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{ac}(U)\Psi_0\|^2,
\]

where \( \Psi_0 \) is the initial state, \( \delta_0 \) the Dirac measure at zero, and \( E_{\hat{v}_+} \) the spectral measure of \( \hat{v}_+ \).

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1 Introduction

The weak limit theorems for discrete time quantum walks have been studied in various models (for reviews, see [7, 12]). In his papers [5, 6], Konno first proved the weak limit theorem for a position-independent quantum walk on \( \mathbb{Z} \). Grimmett et al [4] simplified the proof and extended the result to higher dimensions. For position-dependent quantum walks on \( \mathbb{Z} \), the weak limit theorems were obtained by Konno et al [9], Endo and Konno [2], and Endo et al [3].

We consider a position-dependent quantum walk on \( \mathbb{Z} \) given by a unitary evolution operator \( U \):

\[
(U \Psi)(x) = P(x + 1) \Psi(x + 1) + Q(x - 1) \Psi(x - 1), \quad x \in \mathbb{Z},
\]

where \( \Psi \) is a state vector in the Hilbert space \( \mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2) \) of states and

\[
P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.
\]

Let \( C(x) = P(x) + Q(x) \in U(2) \) and \( S \) be a shift operator such that \( U = SC \). Suppose that there exists a unitary matrix \( C_0 = P_0 + Q_0 \in U(2) \) such that

\[
\|C(x) - C_0\| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}
\]

with positive \( c_1 \) and \( \epsilon \) independent of \( x \). Here \( \|M\| \) stands for the operator norm of a matrix \( M \in M_2(\mathbb{C}) \). A typical example is the quantum walks with one defect [1, 8, 9, 13], which clearly satisfies (1.1). We note that the condition (1.1) allows not only finite but also infinite defects, whereas the models introduced in [2, 3] do not satisfy (1.1). The unitary operator \( U_0 = SC_0 \) also defines an evolution of a position-independent quantum walk on \( \mathbb{Z} \) and satisfies

\[
(U_0 \Psi)(x) = P_0 \Psi(x + 1) + Q_0 \Psi(x - 1), \quad x \in \mathbb{Z}
\]

with \( C_0 = P_0 + Q_0 \). Let \( \hat{x} \) be the position operator defined by \( (\hat{x} \Psi)(x) = x \Psi(x), \quad x \in \mathbb{Z} \), and \( \hat{x}_0(t) = U_0^{-t} \hat{x} U_0^t \) the Heisenberg operator of \( \hat{x} \) at time \( t \in \mathbb{N} \) with the evolution \( U_0 \). In [4], Grimmett et al essentially proved that the operator \( \hat{x}_0(t)/t \) weakly converges to the asymptotic velocity operator \( \hat{v}_0 \) so that

\[
\text{w- lim}_{t \to \infty} \exp \left( i \frac{\hat{x}_0(t)}{t} \right) = \exp (i \xi \hat{v}_0), \quad \xi \in \mathbb{R}.
\]

Let \( X^{(0)}_t \) be the random variable denoting the position of a quantum walker at time \( t \in \mathbb{N} \) with the evolution operator \( U_0 \). Then, the characteristic function
of $X_t^{(0)}/t$ is given by
\[ E(e^{i\xi X_t^{(0)}/t}) = \langle \Psi_0, e^{i\xi \hat{x}_0(t)/t} \Psi_0 \rangle, \quad \xi \in \mathbb{R}, \]
where $\Psi_0$ is the initial state of the quantum walker. Hence, (1.2) means that the random variable $X_t^{(0)}/t$ converges in law to a random variable $V_0$, which represents the linear spreading of the quantum walk: $X_t^{(0)} \sim tV_0$.

In this paper, we derive the asymptotic velocity $\hat{v}_+$ for the Heisenberg operator $\hat{x}(t) = U^{-t}\hat{x}U^t$ with the evolution $U$ of the position-dependent quantum walk. The decaying condition (1.1) implies that $U - U_0$ is a trace class operator and allows us to prove the existence and completeness of the wave operator
\[ W_+ = \text{s- lim}_{t \to \infty} U^{-t}U_0^\dagger \Pi_{ac}(U_0) \]
using a discrete analogue of the Kato–Rosenblum Theorem (See [11] for details), where $\Pi_{ac}(U_0)$ is the orthogonal projection onto the subspace of absolute continuity of $U_0$. We also prove that
\[ \text{s- lim}_{t \to \infty} \exp \left( i\xi \frac{\hat{x}_0(t)}{t} \right) = \exp (i\xi \hat{v}_0), \quad \xi \in \mathbb{R} \]
under a reasonable condition, which is essentially the same as that of [4]. Furthermore, we assume that $U$ has no singular continuous spectrum. Then, we prove that
\[ \text{s- lim}_{t \to \infty} \exp \left( i\xi \frac{\hat{x}(t)}{t} \right) = \Pi_p(U) + \exp(i\xi \hat{v}_+) \Pi_{ac}(U), \quad (1.3) \]
where $\Pi_p(U)$ is the orthogonal projection onto the direct sum of all eigenspaces of $U$ and $\hat{v}_+ = W_+ \hat{v}_0 W_+^*$. We believe that the absence of a singular continuous spectrum can be checked with a concrete example such as the one-defect model. As a consequence of (1.3), we have the following weak limit theorem. Let $X_t$ be the random variable denoting the position of a quantum walker at time $t \in \mathbb{N}$ with the evolution operator $U$ and the initial state $\Psi_0$. We prove that $X_t/t$ converges in law to a random variable $V$ with a probability distribution
\[ \mu_V = \| \Pi_p(U) \Psi_0 \|^2 \delta_0 + \| E_{\hat{v}_+}(\cdot) \Pi_{ac}(U) \Psi_0 \|^2, \]
where $\delta_0$ is the Dirac measure at zero and $E_{\hat{v}_+}$ the spectral measure of $\hat{v}_+$.

The remainder of this paper is organized as follows. In Section 2, we present the precise definition of the model and our results. Section 3 is devoted to the proof of the existence and completeness of the wave operator. In Section 4, we construct the asymptotic velocity.
2 Definition of the model

Let $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$ be the Hilbert space of the square-summable functions $\Psi : \mathbb{Z} \to \mathbb{C}^2$. We define a shift operator $S$ and a coin operator $C$ on $\mathcal{H}$ as follows. For a vector $\Psi = \begin{pmatrix} \Psi(0) \\ \Psi(1) \end{pmatrix} \in \mathcal{H}$, $S\Psi$ is given by

$$(S\Psi)(x) = \begin{pmatrix} \Psi(0)(x+1) \\ \Psi(1)(x-1) \end{pmatrix}, \ x \in \mathbb{Z}.$$ 

Let $\{C(x)\}_{x \in \mathbb{Z}} \subset U(2)$ be a family of unitary matrices with

$$C(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$ 

$C\Psi$ is given by

$$(C\Psi)(x) = C(x)\Psi(x), \ x \in \mathbb{Z}.$$ 

We define an evolution operator as $U = SC$. $U$ satisfies

$$(U\Psi)(x) = P(x+1)\Psi(x+1) + Q(x-1)\Psi(x-1), \ x \in \mathbb{Z}$$

with

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \ Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$ 

For a matrix $M \in M(2, \mathbb{C})$, we use $\|M\|$ to denote the operator norm in $\mathbb{C}^2$: $\|M\| = \sup_{\|x\|_{\mathbb{C}^2} = 1} \|Mx\|_{\mathbb{C}^2}$. We suppose that:

(A.1) There exists a unitary matrix $C_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in U(2)$ such that

$$\|C(x) - C_0\| \leq c_1|x|^{-1-\epsilon}, \ x \in \mathbb{Z} \setminus \{0\}$$

with some positive $c_1$ and $\epsilon$ independent of $x$.

We denote by $\mathcal{T}_1$ the set of trace class operators.

**Lemma 2.1.** Let $U$ satisfy (A.1) and set $U_0 = SC_0$. Then, $U - U_0 \in \mathcal{T}_1$.

**Proof.** Let $T = U - U_0$ and $T(x) = C(x) - C_0$. Then

$$T^*T = (C - C_0)^*(C - C_0)$$

is the multiplication operator by the matrix-valued function $T(x)^*T(x)$. Let $t_i(x)$ ($i = 1, 2$) be the eigenvalues of the Hermitian matrix $T(x)^*T(x) \in \mathbb{R}$. We have

$$T^*T = \begin{pmatrix} t_1(x) & t_2(x) \\ t_2(x) & t_1(x) \end{pmatrix}$$

with

$$t_i(x) = \begin{pmatrix} a_i(x) & b_i(x) \\ c_i(x) & d_i(x) \end{pmatrix}$$

and

$$\|T(x)^*T(x)\| \leq c_2|x|^{-1}, \ x \in \mathbb{Z} \setminus \{0\}$$

with some positive $c_2$. Therefore, $T(x)$ is a trace class operator for all $x \in \mathbb{Z} \setminus \{0\}$.
$M(2, \mathbb{C})$ and take an orthonormal basis (ONB) $\{\tau_i(x)\}_{i=1,2}$ of corresponding eigenvectors for all $x \in \mathbb{Z}$. We use $|\xi\rangle\langle\eta|$ to denote the operator on $\mathcal{H}$ defined by $|\xi\rangle\langle\eta|\Psi = \langle\eta, \Psi\rangle|\xi\rangle$. Then, we have

$$T^*T = \sum_{i=1,2} \sum_{x \in \mathbb{Z}} t_i(x)|\tau_{i,x}\rangle\langle\tau_{i,x}|,$$

where $\{\tau_{i,x}\}$ is the ONB given by

$$\tau_{i,x}(y) = \delta_{xy}\tau_i(x), \quad y \in \mathbb{Z}.$$

Since $T^*(x)T(x) \geq 0$, we have $t_i(x) \geq 0$. By (A.1), we know that

$$\max_{i=1,2} t_i(x) \leq c_1^2|x|^{-2}\epsilon.$$

Hence, we have

$$\text{Tr}|T| = \sum_{x \in \mathbb{Z}} \sum_{i=1,2} t_i(x)^{1/2} \leq 2c_1 \sum_{x \in \mathbb{Z}} |x|^{-1-\epsilon} < \infty,$$

which means that $T \in \mathcal{B}$. Since $\mathcal{B}$ is an ideal, $U - U_0 = ST \in \mathcal{B}$.

**Example 2.1** (one-defect model). Let $C_0, C'_0 \in U(2)$ be unitary matrices with $C_0 \neq C'_0$ and set

$$C(x) = \begin{cases} C'_0, & x = 0 \\ C_0, & x \neq 0. \end{cases}$$

$U = SC$ satisfies (A.1), because $C(x) - C_0 = 0$ if $x \neq 0$.

**Example 2.2.** Let $C_0 \in U(2)$ be a unitary matrix and $\{C(x)\} \subset U(2)$ a family of unitary matrices. Assume that

$$\max_{i,j} |(C(x) - C_0)_{ij}| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\},$$

where $M_{ij}$ denotes the $ij$-component of a matrix $M$. Then, $U = SC$ satisfies (A.1), because all norm on a finite dimensional vector space are equivalent.

We prove the following theorem in Section 3 using a discrete analogue of the Kato–Rosenblum theorem.

**Theorem 2.1.** Let $U$ and $U_0$ be as above and assume that (A.1) holds. Then

$$W_+ = \text{s-\lim}_{t \to \infty} U^{-t}U_0^t\Pi_{ac}(U_0)$$

exists and is complete.
In what follows, we introduce the asymptotic velocity $\hat{v}_0$, obtained first in [4], of the quantum walk with the evolution $U_0$ as follows. Let

$$
\hat{U}_0(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} C_0, \quad k \in [0, 2\pi).
$$

Since $\hat{U}_0(k) \in U(2)$, $\hat{U}_0(k)$ is represented as

$$
\hat{U}_0(k) = \sum_{i=1,2} \lambda_i(k) |u_j(k)\rangle \langle u_j(k)|,
$$

where $\lambda_j(k)$ is an eigenvalue of $\hat{U}_0(k)$ and $u_j(k)$ is the corresponding eigenvector with $\|u_j(k)\| = 1$. The function $k \mapsto e^{ik}$ is analytic, and so is $\lambda_j(k)$.

We need the following assumption on $u_j(k)$:

(A.2) The functions $k \mapsto u_j(k)$ are continuously differentiable in $k$ with

$$
\sup_{k \in [0, 2\pi]} \left\| \frac{d}{dk} u_j(k) \right\|_{C^2} < \infty.
$$

Let $K$ be the Hilbert space of square integrable functions $f : [0, 2\pi) \to \mathbb{C}^2$ with norm

$$
\| f \|_K = \left( \int_{[0,2\pi)} \|f(k)\|^2_{\mathbb{C}^2} \right)^{1/2}.
$$

Let $\mathcal{F} : \mathcal{H} \to K$ be the discrete Fourier transform given by

$$
(\mathcal{F} \Psi)(k) = \sum_{x \in \mathbb{Z}} e^{-ik \cdot x} \Psi(x), \quad \Psi \in \mathcal{H}.
$$

We also use $\hat{\Psi}(k) = \begin{pmatrix} \hat{\Psi}^{(0)}(k) \\ \hat{\Psi}^{(1)}(k) \end{pmatrix}$ to denote the Fourier transform of $\Psi$. The asymptotic velocity $\hat{v}_0$ is the self-adjoint operator defined by

$$
\hat{v}_0 = \mathcal{F}^{-1} \left( \int_{[0,2\pi)} \frac{dk}{2\pi} \sum_{j=1,2} \left( \frac{i\lambda_j(k)}{\lambda_j(k)} \right) |u_j(k)\rangle \langle u_j(k)| \right) \mathcal{F}
$$

The position operator $\hat{x}$ is a self-adjoint operator defined by

$$
(\hat{x} \Psi)(x) = x \Psi(x), \quad x \in \mathbb{Z}
$$

with domain

$$
D(\hat{x}) = \left\{ \Psi \in \mathcal{H} \mid \sum_{x \in \mathbb{Z}} |x|^2 \| \Psi(x) \|^2_{\mathbb{C}^2} < \infty \right\}.
$$

Let $\hat{x}_0(t) = U_0^{-t} \hat{x} U_0^t$ be the Heisenberg operator of $\hat{x}$ for the evolution $U_0$. 
Theorem 2.2. Let \( \hat{v}_0 \) and \( \hat{x}_0 \) be as above. Suppose that (A.2) holds. Then,

\[
s-\lim_{t \to \infty} \exp \left( i \xi \frac{\hat{x}_0(t)}{t} \right) = \exp(i \xi \hat{v}_0), \quad \xi \in \mathbb{R}. \tag{2.3}
\]

Proof. By [10, Theorem VIII.21], (2.3) holds if and only if

\[
s-\lim_{t \to \infty} \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

which is proved in Subsection 4.1. \( \Box \)

Example 2.3. (i) Let \( C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then, \( \hat{U}_0(k) \) has eigenvalues 1 and \(-1\), which are independent of \( k \). By definition, \( \hat{v}_0 = 0 \). Hence, the random variable \( X_t^{(0)}/t \) converges in law to a random variable \( V_0 \) with a probability distribution \( \delta_0 \).

(ii) Let \( C_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). \( \hat{U}_0(k) \) has eigenvalues \( e^{ik} \) and \( -e^{-ik} \). Hence, \( \hat{v}_0 \) has eigenvalues \(-1\) and \( 1\). The random variable \( X_t^{(0)}/t \) converges in law to a random variable \( V_0 \) with a probability distribution \( \|\Psi^{(0)}\|_2^2 \delta_{-1} + \|\Psi^{(1)}\|_2^2 \delta_1 \).

(iii) Let \( C_0 \) be the Hadamard matrix. The eigenvalues of \( \hat{U}_0(k) \) are given by \( \lambda_j(k) = ((-1)^j w(k) + i \sin k)/\sqrt{2} \) \((j = 1, 2)\), where \( w(k) = \sqrt{1 + \cos^2 k} \). Hence, \( \hat{v}_0 \) has no eigenvalue. The corresponding eigenvectors

\[
u_j(k) = \sqrt{w(k) + (-1)^j \cos k} \left( \frac{e^{ik}}{(-1)^j w(k) - \cos k} \right)
\]

form an ONB of \( \mathbb{C}^2 \) and satisfy (A.2). The random variable \( X_t^{(0)}/t \) converges in law to a random variable \( V_0 \) with a probability distribution \( \|E_{\hat{v}_0} \cdot \Psi_0\|^2 \), where \( E_{\hat{v}_0} \) is the spectral measure of \( \hat{v}_0 \). Let us consider the Hadamard walk starting from the origin. Let the initial state \( \Psi_0 \) satisfy \( \Psi_0(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) \((|\alpha|^2 + |\beta|^2 = 1)\) and \( \Psi(x) = 0 \) if \( x \neq 0 \). Then,

\[
d\|E_{\hat{v}_0}(v)\Psi_0\|^2 = (1 - c_{\alpha,\beta}v)f_K \left( v; \frac{1}{\sqrt{2}} \right) dv,
\]

where \( c_{\alpha,\beta} = |\alpha|^2 - |\beta|^2 + \alpha\bar{\beta} + \bar{\alpha}\beta \),

\[
f_K(v; r) = \frac{\sqrt{1 - r^2}}{\pi(1 - v^2)\sqrt{r^2 - v^2}} I_{(-r, r)}(v)
\]

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is the Konno function, and \( I_A \) is the indicator function of a set \( A \). For more details, the reader can consult [4, 7].

Let \( \hat{x}(t) = U^{-1} \hat{x}U \) be the Heisenberg operator of \( \hat{x} \) and define the asymptotic velocity \( \hat{v}_+ \) for the evolution \( U \) by

\[
\hat{v}_+ = W_+ \hat{v}_0 W_+^*.
\]

We need the following assumption:

(A.3) The singular continuous spectrum of \( U \) is empty.

We are now in a position to state our main result, which is proved in Subsection 4.2.

**Theorem 2.3.** Let \( \hat{x}(t) \) and \( \hat{v}_+ \) be as above. Suppose that (A.1) - (A.3) hold. Then,

\[
\lim_{t \to \infty} \exp \left( i \xi \frac{\hat{x}(t)}{t} \right) = \Pi_p(U) + \exp (i \xi \hat{v}_+) \Pi_{ac}(U), \quad \xi \in \mathbb{R}.
\]

Let \( X_t \) be the random variable denoting the position of the walker at time \( t \in \mathbb{N} \) with the initial state \( \Psi_0 \). We use \( \Pi_p(U) \) to denote the orthogonal projection onto the direct sum of all eigenspaces of \( U \) and \( E_A \) to denote the spectral projection of a self-adjoint operator \( A \).

**Corollary 2.4.** Let \( X_t \) be as above. Suppose that (A.1) - (A.3) hold. Then, \( X_t/t \) converges in law to a random variable \( V \) with a probability distribution

\[
\mu_V = \| \Pi_p(U) \Psi_0 \|^2 \delta_0 + \| E_{\hat{v}_+}(\cdot) \Pi_{ac}(U) \Psi_0 \|^2,
\]

where \( \delta_0 \) is the Dirac measure at zero.

**Proof.** From Theorem 2.1 \( \lim_{t \to \infty} U^{-1} U^t \Pi_{ac}(U) \) exists and is equal to \( W_+^* \). Then, \( W_+ \) is unitary from \( \text{Ran} W_+^* = \text{Ran} \Pi_{ac}(U_0) \) to \( \text{Ran} W_+ = \text{Ran} \Pi_{ac}(U) \). Since, by Lemma 4.1, \( U_0 \) is strongly commuting with \( \hat{v}_0 \), we know, from the intertwining property \( U W_+ = W_+ U_0 \), that \( U \) is also strongly commuting with \( \hat{v}_+ \). Hence, \( \hat{v}_+ \) is strongly commuting with \( \Pi_{ac}(U) \) and \( e^{i \xi \hat{v}_+} \Pi_{ac}(U) = \Pi_{ac}(U) e^{i \xi \hat{v}_+} \). Hence, by Theorem 2.3, \( \exp(i \xi \hat{x}(t)/t) \Psi_0 \) converges strongly to \( \Pi_p(U) \Psi_0 + e^{i \xi \hat{v}_+} \Pi_{ac}(U) \Psi_0 \) and

\[
\lim_{t \to \infty} \mathbb{E}(e^{i \xi X_t/t}) = \langle \Psi_0, \Pi_p(U) \Psi_0 + e^{i \xi \hat{v}_+} \Pi_{ac}(U) \Psi_0 \rangle
\]

\[
= \| \Pi_p(U) \Psi_0 \|^2 + \int_{-\infty}^{\infty} e^{i \xi v} \| E_{\hat{v}_+}(v) \Pi_{ac}(U) \Psi_0 \|^2 dv
\]

\[
= \int_{-\infty}^{\infty} e^{i \xi v} d\mu_V(v),
\]

which proves the corollary. \( \square \)
Example 2.4. Let $C_0$ be the Hadamard matrix and $C(x)$ satisfy (A.1). As seen in Example 2.3 (iii), (A.2) is satisfied and the spectrum of $U_0$ is purely absolutely continuous. Let $\Psi_+ \in \mathcal{H}$ satisfy $\Psi_+(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ($|\alpha|^2 + |\beta|^2 = 1$) and $\Psi_+(x) = 0$ if $x \neq 0$. By Example 2.3,

$$d\|E_\psi(v)\Pi_{ac}(U)W_+\Psi_+\|^2 = d\|E_\psi(v)\Psi_+\|^2 = (1 - c_{\alpha,\beta}v)f_K \left(v; \frac{1}{\sqrt{2}}\right) dv.$$ 

Let $\Psi_p \in \text{Ran}\Pi_p(U_0)$ be a unit vector and take the initial state $\Psi_0 = C_1 \Psi_p + C_2 W_+ \Psi_+ (|C_1|^2 + |C_2|^2 = 1)$. Suppose that $U = SC$ satisfies (A.3). By Corollary 2.4, $X_t/t$ converges in law to $V$ with a probability distribution $\mu_V$ and

$$\mu_V(dv) = |C_1|^2 \delta_0 (dv) + |C_2|^2 (1 - c_{\alpha,\beta}v)f_K \left(v; \frac{1}{\sqrt{2}}\right) dv.$$ 

3 Wave operator

To prove Theorem 2.1, we use the following general proposition:

**Proposition 3.1.** Let $U$ and $U_0$ be unitary operators on a Hilbert space $\mathcal{H}$ and suppose that $U - U_0 \in \mathcal{T}_1$. The following limit exists:

$$W_+ = \lim_{t \to \infty} U_0^{-t}U^t\Pi_{ac}(U_0)$$

**Proof of Theorem 2.1.** Since, by Lemma 2.3, $U - U_0 \in \mathcal{T}_1$, the wave operator $W_+$ exists. If we interchange the roles of $U$ and $U_0$, then the proposition says that the limit $\lim_{t \to \infty} U_0^{-t}U^t\Pi_{ac}(U)$ also exists, which implies that $W_+$ is complete. This completes the proof. 

In the remainder of this section, we suppose that $U - U_0 \in \mathcal{T}_1$ and prove Proposition 3.1. This is done by a discrete analogue of [11, Theorem 6.2]. We use $\mathcal{H}_{ac}$ and $\mathcal{H}_p$ to denote the subspaces of absolute continuity and the direct sum of all eigenspaces of $U_0$. Let $E_0$ be the spectral measure of $U_0$ with $E_0([0, 2\pi]) = I$. Let

$$\mathcal{H}_{ac,0} = \{ \psi \in \mathcal{H}_{ac} \mid d\|E_0(\lambda)\psi\|^2 = G_\psi(\lambda)^2 d\lambda \text{ and } G_\psi \in L^2 \cap L^\infty \},$$

where $L^2 = L^2([0, 2\pi])$ and $L^\infty = L^\infty([0, 2\pi])$. Although the following lemma may be well known, we give proofs for completeness.

**Lemma 3.1.** $\mathcal{H}_{ac,0}$ is dense in $\mathcal{H}_{ac}$. 

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Lemma 3.2. Let \( \phi \in H \) and \( \psi \in H_{ac,0} \). Then,
\[
\sum_{t \in \mathbb{Z}} |\langle \phi, U_t^\dagger \psi \rangle|^2 \leq 2\pi \| \phi \|^2 \sup_{\lambda} G_\psi(\lambda)^2.
\]

Proof. Let \( \psi \in H_{ac,0} \) and \( L = L^2([0,2\pi), G_\psi^2(\lambda)d\lambda) \). Let \( H_0 \) be the self-adjoint operator defined by \( \langle \xi, H_0 \eta \rangle = \int_0^{2\pi} \lambda d\langle \xi, E_0(\lambda) \eta \rangle \) \( (\xi, \eta \in H) \). Let \( \mathcal{U} : L \to H \) be an injection defined by \( \mathcal{U} f = f(H_0) \psi \) \( (f \in L) \). Then \( \mathcal{U} 1 = \psi \) and \( \mathcal{U} e^{it\lambda} = U_t^\dagger \psi \) \( (t \in \mathbb{N}) \). We use \( P \) to denote the orthogonal projection onto \( UL \). Let \( \phi \in H \) and \( F = \mathcal{U}^{-1} P \phi \in L \). Then we have
\[
\langle \phi, U_t^\dagger \psi \rangle = \int_0^{2\pi} e^{it\lambda} \bar{F}(\lambda) G_\psi(\lambda)^2 d\lambda = 2\pi \bar{F} G_\psi^2(t).
\]
This completes the proof.

Lemma 3.3. Let \( t, s \in \mathbb{N} \) \( (s \neq t) \). Then, \( s^{-} \lim_{r \to \infty} (W_t - W_s) U_0^\dagger P_{ac}(U_0) = 0 \).

Proof. For \( t, s \in \mathbb{N} \) \( (t > s) \), we have \( W_t = \sum_{k=s+1}^{t} (W_k - W_{k-1}) + W_s \) and
\[
W_k - W_{k-1} = U^{-k}(-T) U_0^{k-1},
\]
where \( T = U - U_0 \in \mathcal{T}_1 \). Since \( \mathcal{T}_1 \) is an ideal, we know that
\[
W_t - W_s = \sum_{k=s+1}^{t} U^{-k}(-T) U_0^{k-1} \in \mathcal{T}_1.
\]
In particular, \( W_t - W_s \) is compact. Let \( H_0 \) be the self-adjoint operator defined in the proof of Lemma 3.2. Since \( w^{-} \lim_{r \to \infty} e^{irH_0} P_{ac}(H_0) = 0 \), we have
\[
s^{-} \lim_{r \to \infty} (W_t - W_s) U_0^\dagger P_{ac}(U_0) = s^{-} \lim_{r \to \infty} (W_t - W_s) e^{irH_0} P_{ac}(H_0) = 0.
\]
This completes the proof.
Proof of Proposition 3.1. By Lemma 3.1 it suffices to prove that, for \( \psi \in H_{ac,0} \),
\[
\| (W_t - W_s) \psi \| \to 0, \quad t, s \to \infty.
\]
Because
\[
\| (W_t - W_s) \psi \| = \langle \psi, W^*_t (W_t - W_s) \psi \rangle - \langle \psi, W^*_s (W_t - W_s) \psi \rangle,
\]
we need only to prove that
\[
\langle \psi, W^*_t (W_t - W_s) \psi \rangle \to 0, \quad t, s \to \infty.
\]
By direct calculation, we have, for \( r > 1 \),
\[
W^*_t (W_t - W_s) - U_0^{-r} W^*_t (W_t - W_s) U_0^r
= \sum_{k=0}^{r-1} (U_0^{-k-1} W^*_t W_s U_0^{k+1} - U_0^{-k} W^*_t W_s U_0^k).
\]
Since
\[
U_0^{-k-1} W^*_t W_s U_0^{k+1} - U_0^{-k} W^*_t W_s U_0^k = U_0^{-k-t-1} (TU^{t-s} - U^{t-s} T) U_0^{s+k},
\]
we obtain
\[
W^*_t (W_t - W_s) - U_0^{-r} W^*_t (W_t - W_s) U_0^r
= \sum_{k=0}^{r-1} U_0^{-k-t-1} (TU^{t-s} - U^{t-s} T) U_0^{s+k}.
\]
Since, by Lemma 3.3 \( s^{-\lim}_{r \to \infty} U_0^{-r} W^*_t (W_t - W_s) U_0^r \psi = 0 \), we have
\[
W^*_t (W_t - W_s) \psi = \sum_{k=0}^{\infty} U_0^{-k-t-1} (TU^{t-s} - U^{t-s} T) U_0^{s+k} \psi
= Z_{t,s}((U_0 T) U^{t-s} - (U_0 U^{t-s}) T) \psi,
\]
where
\[
Z_{t,s}(A) = \sum_{k=0}^{\infty} U_0^{-k-t} A U_0^{k+s}.
\]
By Lemma 3.4 below, we know that
\[
| \langle \psi, W^*_t (W_t - W_s) \psi \rangle | \leq | \langle \psi, Z_{t,s}((U_0 T) U^{t-s}) \psi \rangle | + | \langle \psi, Z_{t,s}(U_0 U^{t-s}) T) \psi \rangle | \to 0, \quad t, s \to \infty.
\]
This completes the proof.
Lemma 3.4. Let $Y \in \mathcal{F}_1$ and $\{Q(t, s)\}$ be a family of bounded operators with $\sup_{t, s} \|Q(t, s)\| < \infty$. Then, for all $\psi \in \mathcal{H}_{ac,0}$,

1. $\lim_{t, s \to \infty} \langle \psi, Z_{t,s}(YQ(t, s))\psi \rangle = 0$;
2. $\lim_{t, s \to \infty} \langle \psi, Z_{t,s}(Q(t, s)Y)\psi \rangle = 0$.

Proof. Let $Y = \sum_{n=1}^{\infty} \lambda_n |\psi_n\rangle \langle \phi_n|$ be the canonical expansion of the compact operator $Y$. Since $Y \in \mathcal{F}_1$, $\sum_n \lambda_n < \infty$. Then, by the Cauchy-Schwartz inequality, we have

$$| \langle \psi, Z_{t,s}(YQ(t, s))\psi \rangle | \leq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n \left| \langle U_0^{k+t}\psi, \psi_n \rangle \langle \phi_n, Q(t, s)U_0^{k+s}\psi \rangle \right|$$

$$\leq I_1(t, s)^{1/2} \times I_2(t, s)^{1/2},$$

where

$$I_1(t) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n \left| \langle \psi_n, U_0^{k+t}\psi \rangle \right|^2,$$

$$I_2(t, s) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n \left| \langle Q(t, s)^* \phi_n, U_0^{k+s}\psi \rangle \right|^2.$$

By Lemma 3.2, we have

$$I_2(t, s) \leq 2\pi \sup_{\lambda} G_{\psi}(\lambda) \sup_{t, s} \|Q(t, s)\| \sum_n \lambda_n < \infty,$$

where we have used the fact that $\phi_n$ is a normalized vector. Let $u_k = \sum_{n=1}^{\infty} \lambda_n \left| \langle \psi_n, U_0^k\psi \rangle \right|^2$. Then, similarly to the above, we observe that $\{u_k\} \in \ell^1(\mathbb{Z})$. Hence, we have

$$\lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} \sum_{k=t}^{\infty} u_k = 0.$$

This proves (i). The same proof works for (ii).

4 Asymptotic velocity

4.1 Proof of Theorem 2.2

Let

$$\mathcal{H}_0 = \bigcup_{m=0}^{\infty} \{\Psi \in \mathcal{H} | \Psi(x) = 0, \ |x| \geq m\}.$$
We use $D$ to denote a subspace of vectors $\Psi \in H$ whose Fourier transform $\hat{\Psi}$ are differentiable in $k$ with
\[
\sup_{k \in [0,2\pi)} \left\| \frac{d}{dk} \hat{\Psi}(k) \right\| < \infty.
\]
Note that $H_0$ is a core for $\hat{x}$, and so is $D$. Let $D = \mathcal{F} \hat{x} \mathcal{F}^{-1}$. Then, by direct calculation, we know that $(D\hat{\Psi})(k) = i \frac{d}{dk} \hat{\Psi}(k)$ for $\Psi \in D$. We prove the following theorem:

**Theorem 4.1.** Suppose that (A.2) holds. Then,
\[
s\lim_{t \to \infty} \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{4.1}
\]

**Proof.** For all $\Psi \in H$ and $\epsilon > 0$, there exists a vector $\Psi_\epsilon \in D$ such that $\|\Psi - \Psi_\epsilon\| \leq \epsilon$. Because, by the second resolvent identity,
\[
\left\| \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi - (\hat{v}_0 - z)^{-1} \Psi_\epsilon \right\| \leq \frac{2\epsilon}{|\text{Im}z|} + \left\| \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi_\epsilon - (\hat{v}_0 - z)^{-1} \Psi_\epsilon \right\|
\]
\[
\leq \frac{2\epsilon}{|\text{Im}z|} + \frac{1}{|\text{Im}z|} \left\| \frac{\hat{v}_0 - \hat{x}_0(t)}{t} \right\| (\hat{v}_0 - z)^{-1} \Psi_\epsilon
\]

it suffices to prove that
\[
\lim_{t \to \infty} \left\| \left( \frac{\hat{v}_0 - \hat{x}_0(t)}{t} \right) (\hat{v}_0 - z)^{-1} \Psi \right\| = 0, \quad \Psi \in D.
\]

Note that
\[
(\hat{v}_0 - z)^{-1} = \mathcal{F}^{-1} \left( \int_{[0,2\pi)} dk \sum_{j=1,2} \left( i \frac{\lambda_j'(k)}{\lambda_j(k)} - z \right)^{-1} |u_j(k)\rangle \langle u_j(k)| \right) \mathcal{F}.
\]

Since $\lambda_j(k)$ is analytic and $|\lambda_j(k)| = 1$, we observe from (A.2) that $(\hat{v}_0 - z)^{-1}$ leaves $D$ invariant. Hence, we only need to prove that
\[
\lim_{t \to \infty} \left\| \left( \frac{\hat{v}_0 - \hat{x}_0(t)}{t} \right) \Psi \right\| = 0, \quad \Psi \in D.
\]
By direct calculation, we have

\[
\left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\|^2 = \int_0^{2\pi} dk \left\| \sum_{j=1,2} \left( \frac{i\lambda_j(k)}{\lambda_j(k)} \right) \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) - \hat{U}(k) \right\|^2 = \int_0^{2\pi} \frac{dk}{t^2} \left\| \sum_{j=1,2} \lambda_j(k) \hat{U}(k)^{-1} \left( i \frac{d}{dk} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\|^2.
\]

By the definition of \( D \) and (A.2), we know that

\[
\sup_{k \in [0,2\pi]} \left\| \left( i \frac{d}{dk} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\| < \infty.
\]

Hence, we have

\[
\left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\| = O(t^{-1}),
\]

which completes the proof. \( \square \)

### 4.2 Proof of Theorem 2.3

The proof falls naturally into two parts:

**Theorem 4.2.** Let \( U \) be a unitary operator on \( \mathcal{H} \). \( \hat{x}(t) = U^{-t} \hat{x} U^t \) satisfies

\[
s\lim_{t \to \infty} \exp \left( i \frac{\hat{x}(t)}{t} \right) \Pi_p(U) = \Pi_p(U), \quad \xi \in \mathbb{R}.
\]

**Theorem 4.3.** Let \( U = SC \) and \( U_0 = SC_0 \) satisfy (A.1) and (A.2). Then,

\[
s\lim_{t \to \infty} \exp \left( i \frac{\hat{x}(t)}{t} \right) \Pi_{ac}(U) = \exp(i\xi\hat{v}_+)\Pi_{ac}(U), \quad \xi \in \mathbb{R}.
\]

**Proof of Theorem 2.3** By (A.3), we have

\[
s\lim_{t \to \infty} \exp \left( i \frac{\hat{x}(t)}{t} \right) = s\lim_{t \to \infty} \exp \left( i \frac{\hat{x}(t)}{t} \right) \left( \Pi_p(U) + \Pi_p(U) \right) = \Pi_p(U) + \exp(i\xi\hat{v}_+)\Pi_{ac}(U).
\]

This prove the theorem. \( \square \)
It remains to prove Theorems 4.2 and 4.3.

Proof of Theorem 4.2. Let $\mathcal{H}_p(U)$ be the direct sum of all eigenspaces of $U$. It suffices to prove that, for $\Psi \in \mathcal{H}_p(U)$,

$$\text{s- lim}_{t \to \infty} \exp \left( i \frac{\hat{x}(t)}{t} \right) \Psi = \Psi.$$ 

Let $\lambda_n$ be the eigenvalues of $U$ and take an ONB $\{ \eta_n \}_{n=1}^\infty$ of $\mathcal{H}_p$ such that $U \eta_n = \lambda_n \eta_n$. We have $\Pi_p(U) = \sum_n |\eta_n\rangle \langle \eta_n|$. Let $\epsilon > 0$. For sufficiently large $N$, $\Psi_N = \sum_{n=1}^N \langle \eta_n, \Psi \rangle \eta_n$ satisfies $\| \Psi - \Psi_N \| \leq \epsilon$. Then,

$$\left\| \exp \left( i \frac{\hat{x}(t)}{t} \right) \Psi - \Psi \right\| \leq 2\epsilon + \left\| \exp \left( i \frac{\hat{x}(t)}{t} \right) \Psi_N - \Psi_N \right\|.$$ 

By direct calculation, we have

$$\left\| \exp \left( i \frac{\hat{x}(t)}{t} \right) \Psi_N - \Psi_N \right\| = \left\| \left( \exp \left( i \frac{\hat{x}(t)}{t} \right) - 1 \right) U^t \Psi_N \right\|$$

$$= \left\| \sum_{n=1}^N \lambda_n^t \langle \eta_n, \Psi \rangle \left( \exp \left( i \frac{\hat{x}(t)}{t} \right) - 1 \right) \eta_n \right\|$$

$$\leq \sum_{n=1}^N |\langle \eta_n, \Psi \rangle| \left\| \left( \exp \left( i \frac{\hat{x}(t)}{t} \right) - 1 \right) \eta_n \right\|.$$ (4.2)

Since $\lim_{t \to \infty} |1 - e^{ix_0/t}| = 0$, $|1 - e^{ix_0/t}| \leq 2$ and $\sum_x \| \eta_n(x) \|_2^2 = \| \eta_n \|_2^2 < \infty$, we have

$$\lim_{t \to \infty} \left\| \left( \exp \left( i \frac{\hat{x}(t)}{t} \right) - 1 \right) \eta_n \right\|^2 = \lim_{t \to \infty} \sum_{x \in \mathbb{Z}} |e^{ix_0/t} - 1|^2 \| \eta_n(x) \|_2^2 = 0,$$

which, combined with (4.2), completes the proof. \qed

Lemma 4.1. $[U_0, \exp(i\xi \hat{v}_0)] = 0$.

Proof. By direct calculation, we have

$$[U_0, \exp(i\xi \hat{v}_0)] = \text{s- lim}_{t \to \infty} \left[ U_0, \exp \left( i \frac{\hat{x}_0(t)}{t} \right) \right]$$

$$= \text{s- lim}_{t \to \infty} U_0 \left\{ \exp \left( i \frac{\hat{x}_0(t)}{t} \right) - \exp \left( i \frac{\hat{x}_0(t+1)}{t} \right) \right\} = 0.$$ \qed
**Proof of Theorem 4.3.** By (A.1) and (A.2), Theorems 2.1 and 2.2 hold. Then, $W_+$ is a unitary operator from $H_{ac}(U_0)$ to $H_{ac}(U)$. Hence, we have

$$\exp(i\xi\hat{v}_+)\Pi_{ac}(U) = W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{ac}(U).$$

By direct calculation, we observe that

$$I(t) := \exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Pi_{ac}(U) - \exp(i\xi\hat{v}_+)\Pi_{ac}(U)$$

$$= W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)W_t^*\Pi_{ac}(U) - W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{ac}(U)$$

$$=: \sum_{j=1}^3 I_j(t),$$

where

$$I_1(t) = W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)(W_t^* - W_+^*)\Pi_{ac}(U),$$

$$I_2(t) = W_t\left(\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right) - \exp(i\xi\hat{v}_0)\right)W_+^*\Pi_{ac}(U),$$

$$I_3(t) = (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{ac}(U).$$

Because $W_t$ and $\exp(i\xi\hat{x}_0(t)/t)$ are uniformly bounded, we know from Theorems 2.1 and 2.2 that $\text{s-lim}_{t \to \infty} I_1(t) = \text{s-lim}_{t \to \infty} I_2(t) = 0$. Hence, we have

$$I(t) = (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{ac}(U) + o(1)$$

$$= (W_t - W_+)\Pi_{ac}(U_0)\exp(i\xi\hat{v}_0)W_+^*\Pi_{ac}(U)$$

$$+ (W_t - W_+) [\exp(i\xi\hat{v}_0), \Pi_{ac}(U_0)]W_+^*\Pi_{ac}(U) + o(1),$$

where we have used the fact that $\text{Ran}W_+^* = H_{ac}(U_0)$. Since, by Lemma 4.1 $[\exp(i\xi\hat{v}_0), \Pi_{ac}(U_0)] = 0$, we obtain from Theorem 2.1 that $\text{s-lim}_{t \to \infty} I(t) = 0$. This completes the proof. \qed

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