Iterated Peiffer pairings in the Moore complex
of a simplicial group

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Abstract

We introduce a pairing structure within the Moore complex \( \mathbf{N}G \) of a simplicial group \( G \) and use it to investigate generators for \( \mathbf{N}G_n \cap D_n \) where \( D_n \) is the subgroup generated by degenerate elements. This is applied to the study of algebraic models for homotopy types.

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Introduction

Simplicial groups and simplicial groupoids are valuable algebraic models for homotopy types. Much has been studied about the way the group structure interacts with the simplicial structure to yield homotopy information.

Recently the work of Wu, \[22\], \[23\], has shown that there is still progress that can be made in calculation of homotopy invariants such as homotopy groups from simplicial groups. Wu used techniques of combinatorial group theory, iterated commutators and properties related to the semidirect product decompositions of the individual \( G_n \) to give some insight into,
for instance, $\pi_{n+1}(\Sigma K(\pi, 1))$, the homotopy groups of the suspension of an Eilenberg-MacLane space.

Earlier Brown and Loday, [8], had used techniques derived from their generalised van Kampen theorem and Loday’s theory of cat$^n$-groups to give a complete description of the 3-type of $\Sigma K(\pi, 1)$. This raises the possibility of linking the results of Wu with crossed algebraic techniques and to combine the two techniques in order to give descriptions of, for instance, the $k$-type of $\Sigma K(\pi, 1)$ for $k = 4$ and 5. This is still out of our reach with the techniques of this paper, but other results suggest the way to develop tools for this sort of task.

Carrasco [10], and with Cegarra in [11], gave a complete description of the extra structure of the Moore complex, $NG$, of a simplicial group $G$ needed to reconstruct $G$ from $NG$, a sort of ultimate generalisation of the classical Dold-Kan theorem that links simplicial abelian groups with chain complexes. The controlled vanishing of this extra structure given necessary and sufficient conditions for the Moore complex to be a crossed complex or crossed chain complex. Further links between simplicial groups, their Moore complexes and crossed algebraic models for homotopy types have been given by Baues [2], [3] and [4] and also by the second author [19].

In this article we will develop a variant of the Carrasco - Cegarra pairing operators, that we will call Peiffer pairings, and will show that these pairings give products of commutators, and thus, by repeated application, iterated commutators that generate the Moore complex terms in those dimensions where additional non-degenerate generators are not present and in general, they generate $NG_n \cap D_n$ where $D_n$ is the subgroup of $G_n$ generated by the degenerate elements. So far it has not been possible to find a general form for the relations between these generators. This would seem to be an extremely hard problem in general. Some results in low dimensions and for free simplicial groups have been obtained, but as they are incomplete they
will not be included here. Some sample calculations of these generating elements will be given as will some fairly elementary examples of the use of this result.

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1 Simplicial groups, Moore complexes and Peiffer pairings

We refer the reader to Curtis’s survey article [13] or May’s book, [17], for most of the basic properties of simplicial sets, simplicial groups, etc. that we will be needing.

1.1 The Moore complex

If $G$ is a simplicial group, the Moore complex $(NG, \partial)$ of $G$ is the (non-abelian) chain complex defined by

$$NG_n = \bigcap_{i=0}^{n-1} \text{Ker}d_i$$

with $\partial_n : NG_n \rightarrow NG_{n-1}$ induced from $d_n$ by restriction. It is well known that the $n^{th}$ homotopy group $\pi_n(G)$ of $G$ is the $n^{th}$ homology of the Moore complex of $G$

$$\pi_n(G) \cong H_n(NG, \partial) = \bigcap_{i=0}^{n} \text{Ker}d_i^n / d_{n+1}^{n+1}(\bigcap_{i=0}^{n} \text{Ker}d_i^{n+1}).$$

Remark and Warning

There is a possibility of confusion as to the exact definition of $NG$ as two conventions are currently used, one as above takes the intersection of the
Kerd_i for i < n, the other the intersection of the Kerd_i for 0 < i ≤ n. (Curtis [13] uses this latter convention, whilst May, [17], uses the former.) The two theories run parallel and are essentially ‘dual’ to each other, however there is a necessity for checking, which convention is being used in any source as the actual form of any formula usually depends on the convention being used.

1.2 The poset of surjective maps

We recall the following notation and terminology referring the reader to the work of Conduché, [12], Carrasco and Cegarra [11] for more motivation and some related results.

For the ordered set \([n] = \{0 < 1 < \cdots < n\}\), let \(\alpha^n_i : [n+1] \to [n]\) be the increasing surjective map given by

\[
\alpha^n_i(j) = \begin{cases} 
  j & \text{if } j \leq i \\
  j - 1 & \text{if } j > i
\end{cases}
\]

Let \(S(n, n-l)\) be the set of all monotone increasing surjective maps from \([n]\) to \([n-l]\). This can be generated from the various \(\alpha^n_i\) by composition. The composition of these generating maps satisfies the rule \(\alpha_j \alpha_i = \alpha_{i-1} \alpha_j\) with \(j < i\). This implies that every element \(\alpha \in S(n, n-l)\) has a unique expression as \(\alpha = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_l}\) with \(0 \leq i_1 < i_2 < \cdots < i_l \leq n\), where the indices \(i_k\) are the elements of \([n]\) at which \(\{i_1, \ldots, i_l\} = \{i : \alpha(i) = \alpha(i+1)\}\). We thus can identify \(S(n, n-l)\) with the set \(\{(i_1, \ldots, i_l) : 0 \leq i_1 < i_2 < \cdots < i_l \leq n-1\}\).

In particular the single element of \(S(n, 0)\), defined by the identity map on \([n]\), corresponds to the empty 0-tuple \(()\) denoted by \(\emptyset_n\). Similarly the only element of \(S(n, 0)\) is \((n-1, n-2, \ldots, 0)\). For all \(n \geq 0\), let

\[S(n) = \bigcup_{0 \leq l \leq n} S(n, n-l)\]

We say that \(\alpha = (i_1, \ldots, i_1) > \beta = (j_1, \ldots, j_1)\) in \(S(n)\)

if \(i_1 = j_1, \cdots, i_k = j_k\) but \(i_{k+1} < j_{k+1}\) \((k \geq 0)\)
or

\[ i_1 = j_1, \ldots, i_m = j_m \text{ and } l > m. \]

This makes \( S(n) \) an ordered set. For instance, the orders of \( S(2) \) and \( S(3) \) and \( S(4) \) are respectively:

\[
\begin{align*}
S(2) &= \{\emptyset < (1) < (0) \}, \\
S(3) &= \{\emptyset < (2) < (1) < (2, 1) < (0) < (2, 0) < (1, 0) < (2, 1, 0) \}, \\
S(4) &= \{\emptyset < (3) < (2) < (3, 2) < (1) < (3, 1) < (2, 1) < (3, 2, 1) < (0) < (3, 0) < (2, 0) < (3, 2, 0) < (1, 0) < (3, 1, 0) < (2, 1, 0) < (3, 2, 1, 0) \}.
\end{align*}
\]

If \( \alpha, \beta \in S(n) \), we define \( \alpha \cap \beta \) to be the set of indices which belong to both \( \alpha \) and \( \beta \).

If \( \alpha = (i_1, \ldots, i_l) \), then we say \( \alpha \) has length \( l \) and will write \( \# \alpha = l \).

### 1.3 The semidirect decomposition of a simplicial group

The fundamental idea behind this can be found in Conduché [12]. A detailed investigation of this for the case of simplicial groups is given in Carrasco and Cegarra [11].

**Lemma 1.1** Let \( G \) be a simplicial group. Then \( G_n \) can be decomposed as a semidirect product:

\[
G_n \cong \text{Ker}d_n^0 \rtimes s_0^{n-1}(G_{n-1})
\]

**Proof:** The isomorphism can be defined as follows:

\[
\theta : G_n \to \text{Ker}d_n^0 \rtimes s_0^{n-1}(G_{n-1})
\]

\[
g \mapsto (gs_0d_0g^{-1}, s_0d_0g).
\]

Since we have the isomorphism \( G_n \cong \text{Ker}d_n \rtimes s_0G_{n-1} \), we can repeat this process as often as necessary to get each of the \( G_n \) as a multiple semidirect
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product of degeneracies of terms in the Moore complex. In fact, let $K$ be the simplicial group defined by

$$K_n = \ker d_0^{n+1}, \quad d^n_i = d_{i+1}^{n+1}|_{\ker d_0^{n+1}} \quad \text{and} \quad s^n_i = s_{i+1}^{n+1}|_{\ker d_0^{n+1}}.$$ 

Applying Lemma 1.1 above, to $G_{n-1}$ and to $K_{n-1}$, gives

$$G_n \cong \ker d_0 \times s_0 G_{n-1}$$
$$= \ker d_0 \times s_0(\ker d_0 \times s_0 G_{n-2})$$
$$= K_{n-1} \times (s_0 \ker d_0 \times s_0 s_0 G_{n-2}).$$

Since $K$ is a simplicial group, we have the following

$$\ker d_0 = K_{n-1} \cong \ker d_0^K \times s^K_0 K_{n-2}$$
$$= (\ker d_1 \cap \ker d_0) \times s_1 \ker d_0$$

and this enables us to write

$$G_n = ((\ker d_1^n \cap \ker d_0^n) \times s_1(\ker d_0^{n-1})) \times (s_0(\ker d_0^{n-1}) \times s_0 s_0 (G_{n-2})).$$

We can thus decompose $G_n$ as follows:

**Proposition 1.2** (cf. [13], p.158) If $G$ is a simplicial group, then for any $n \geq 0$

$$G_n \cong (\ldots (NG_n \times s_{n-1} NG_{n-1}) \times \ldots \times s_{n-2} \ldots s_1 NG_1) \times$$
$$\ldots (s_0 NG_{n-1} \times s_1 s_0 NG_{n-2}) \times \ldots \times s_{n-1} s_{n-2} \ldots s_0 NG_0).$$

The bracketing and the order of terms in this multiple semidirect product are generated by the sequence:

$$G_1 \cong NG_1 \times s_0 NG_0$$
$$G_2 \cong (NG_2 \times s_1 NG_1) \times (s_0 NG_1 \times s_1 s_0 NG_0)$$
$$G_3 \cong ((NG_3 \times s_2 NG_2) \times (s_1 NG_2 \times s_2 s_1 NG_1)) \times$$
$$(s_0 NG_2 \times s_2 s_0 NG_1) \times (s_1 s_0 NG_1 \times s_2 s_1 s_0 NG_0)).$$

and

$$G_4 \cong (((NG_4 \times s_3 NG_3) \times (s_2 NG_3 \times s_3 s_2 NG_2)) \times$$
$$((s_1 NG_3 \times s_3 s_1 NG_2) \times (s_2 s_1 NG_2 \times s_3 s_2 s_1 NG_1))) \times$$
$$s_0 \text{(decomposition of } G_3).$$
Note that the term corresponding to $\alpha = (i_l, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NG_{n-\#\alpha}) = s_{i_l} \cdots s_{i_1}(NG_{n-\#\alpha}) = s_{i_l} \cdots s_{i_1}(NG_{n-\#\alpha}),$$

where $\#\alpha = l$. Hence any element $x \in G_n$ can be written in the form

$$x = y \prod_{\alpha \in S(n)} s_\alpha(x_\alpha) \quad \text{with } y \in NG_n \text{ and } x_\alpha \in NG_{n-\#\alpha}.$$  

2 Peiffer pairings generate

In the following we will define a normal subgroup $N_n$ of $G_n$. First of all we adapt ideas from Carrasco \[10\] to get the construction of a useful family of natural pairings. We define a set $P(n)$ consisting of pairs of elements $(\alpha, \beta)$ from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_l, \ldots, i_1), \beta = (j_m, \ldots, j_1) \in S(n)$. The pairings that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \rightarrow NG_n : (\alpha, \beta) \in P(n), \ n \geq 0\}$$

are given as composites by the diagram

$$\begin{array}{ccc}
NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\
\downarrow{s_\alpha \times s_\beta} & & \downarrow{p} \\
G_n \times G_n & \xrightarrow{\mu} & G_n
\end{array}$$

where

$$s_\alpha = s_{i_l} \cdots s_{i_1} : NG_{n-\#\alpha} \rightarrow G_n, \ s_\beta = s_{j_m} \cdots s_{j_1} : NG_{n-\#\beta} \rightarrow G_n,$$

$p : G_n \rightarrow NG_n$ is defined by the composite projections $p(x) = p_{n-1} \cdots p_0(x)$, where

$$p_j(z) = zs_jd_j(z)^{-1} \quad \text{with } j = 0, 1, \ldots, n-1,$$

$\mu : G_n \times G_n \rightarrow G_n$ is given by the commutator map and $\#\alpha$ is the number of the elements in the set of $\alpha$, similarly for $\#\beta$. Thus

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = p\mu(s_\alpha \times s_\beta)(x_\alpha, y_\beta)$$

$$= p[s_\alpha x_\alpha, s_\beta y_\beta].$$
Definition Let $N_n$ or more exactly $N_n^G$ be the normal subgroup of $G_n$ generated by elements of the form
\[ F_{\alpha,\beta}(x_\alpha, y_\beta) \]
where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$.

This normal subgroup $N_n^G$ depends functorially on $G$, but we will usually abbreviate $N_n^G$ to $N_n$, when no change of group is involved.

We illustrate this subgroup for $n = 2$ and $n = 3$ to show what it looks like.

Example (a) : For $n = 2$, suppose $\alpha = (1), \beta = (0)$ and $x_1, y_1 \in NG_1 = \text{Kerd}_0$. It follows that
\[
F_{(0)(1)}(x_1, y_1) = p_1p_0[s_0x_1, s_1y_1] \\
= p_1[s_0x_1, s_1y_1] \\
= [s_0x_1, s_1y_1][s_1y_1, s_1x_1]
\]
is a generating element of the normal subgroup $N_2$.

For $n = 3$, the possible pairings are the following
\[
F_{(1,0)(2)}, \quad F_{(2,0)(1)}, \quad F_{(0)(2)}, \quad F_{(1)(2)}, \quad F_{(0)(1)}.
\]

For all $x_1 \in NG_1, y_2 \in NG_2$, the corresponding generators of $N_3$ are:
\[
F_{(1,0)(2)}(x_1, y_2) = [s_1s_0x_1, s_2y_2][s_2y_2, s_2s_0x_1] \\
F_{(2,0)(1)}(x_1, y_2) = [s_2s_0x_1, s_1y_2][s_1y_2, s_2s_1x_1][s_2s_1x_1, s_2y_2][s_2y_2, s_2s_0x_1]
\]
and all $x_2 \in NG_2, y_1 \in NG_1,
\[
F_{(0)(2,1)}(x_2, y_1) = [s_0x_2, s_2s_1y_1][s_2s_1y_1, s_1x_2][s_2x_2, s_2s_1y_1]
\]
whilst for all $x_2, y_2 \in NG_2$,
\[
F_{(0)(1)}(x_2, y_2) = [s_0x_2, s_1y_2][s_1y_2, s_1x_2][s_2x_2, s_2y_2] \\
F_{(0)(2)}(x_2, y_2) = [s_0x_2, s_2y_2] \\
F_{(1)(2)}(x_2, y_2) = [s_1x_2, s_2y_2][s_2y_2, s_2x_2].
\]
Our aim in this paper is to prove that the images of these pairings generate $NG_n \cap D_n$. More precisely:

**Theorem 2.1 (Theorem A)** Let $G$ be a simplicial group and for $n > 1$, let $D_n$ the subgroup in $G_n$ generated by degenerate elements. Let $N_n^G$ be the normal subgroup generated by elements of the form

$$F_{\alpha, \beta}(x_\alpha, y_\beta) \quad \text{with} \ (\alpha, \beta) \in P(n)$$

where $x_\alpha \in NG_n - #\alpha$, $y_\beta \in NG_n - #\beta$. Then

$$NG_n \cap D_n = N_n^G \cap D_n.$$  

As a corollary we, of course, have that the image of $N_n^G \cap D_n$ is equal to the image of $NG_n \cap D_n$ i.e., $\partial_n(N_n \cap D_n) = \partial_n(NG_n \cap D_n)$.

The proof of 2.1 is given in the next section after some preparatory lemmas. Here we restrict to the case $n = 2$ by way of illustration. In their paper \cite{8}, Brown and Loday proved a lemma:

**Lemma 2.2** \cite{8} Let $G$ be a simplicial group such that $G_2 = D_2$ is generated by degenerate elements. Then in the Moore complex of $NG$ we have $\partial_2 NG_2 = \partial_2 N_2$ where $N_2$ is the normal subgroup of $G_2$ generated by elements of the form

$$F_{(0)(1)}(x_1, y_1) = [s_0 x_1, s_1 y_1] [s_1 y_1, s_1 x_1]$$

with $x_1, y_1 \in NG_1$.

This is, of course, a trivial consequence of Theorem A and their proof inspired that of the more general theorem given here.

**Remark:**

An unknown referee made the interesting observation that if $x$ is a Moore cycle, so $\partial x = 0$, then $F_{(0)(1)}(x, x)$ is one also. Thus $F_{(0)(1)}$ induces an
operation $\pi_*(G) \to \pi_*(G)$. Geometrically this operation can be described as the $\eta$-operation given by the composition $S^{m+1} \xrightarrow{\eta} S^m \xrightarrow{\varphi} G$, where $\eta$ is the (suspension of) the Hopf map. The geometric interpretation of the $F_{\alpha\beta(1)}$ in general would seem to be quite important but the authors have as yet little idea what it might be.

3 Elements of $N_n$ and properties of the pairings

In the following we analyse various types of elements in $N_n$ and show that products of them give elements that we want in giving an alternative description of $NG_n$.

Lemma 3.1 Given $x_\alpha \in NG_{n-\#\alpha}$, $y_\beta \in NG_{n-\#\beta}$ with $\alpha = (i_1, \ldots, i_l)$, $\beta = (j_m, \ldots, j_1) \in S(n)$. If $\alpha \cap \beta = \emptyset$ with $\beta < \alpha$ and $v = [s_\alpha x_\alpha, s_\beta y_\beta]$, then

(i) if $k \leq i_1$, then $p_k(v) = v$,

(ii) if $k > i_l + 1$ or $k > j_m + 1$, then $p_k(v) = v$,

(iii) if $k \in \{j_1, \ldots, j_m\}$ and $k = i_r + 1$ for some $r$, then for $\alpha' = (i_r + 1, i_r - 1, \ldots, i_1)$ and $\beta = (j_m, j_{m-1}, \ldots, j_1)$,

$$p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_{\alpha'} x_{\alpha'}, s_{\beta'} y_{\beta'}]^{-1},$$

(iv) if $k \in \{i_1, \ldots, i_l\}$ and $k = j_s + 1$ for some $s$, then for $\beta' = (j_m, \ldots, j_s + 1, j_s - 1, \ldots, j_1)$

$$p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_{\alpha'} x_{\alpha'}, s_{\beta'} y_{\beta'}]^{-1} = vv'$$

where $v' \in G_{n-1}$ and $0 \leq k \leq n-1$,

(v) if $k = j_m + 1$ (or $k = i_l + 1$) then

$$p_k(v) = v s_k(v_k)^{-1} = [s_\alpha x_\alpha, s_\beta y_\beta] s_k(v_k)^{-1}$$

where $s_k(v_k)^{-1} = [s_\beta y_\beta, s_\alpha x_\alpha]$ (or $s_k(v_k)^{-1} = [s_\beta y_\beta, s_{\alpha'} x_{\alpha'}]$) with respect to $k = j_m + 1$ (and $k = i_l + 1$ respectively) and for new strings $\alpha'$ and $\beta'$.
(vi) if \( k = j_1 + 1 \), then
\[
p_k(v) = vs_k(v_k) = [s_\alpha x_\alpha, s_\beta y_\beta]s_k(v_k)^{-1}
\]
where \( s_k(v_k)^{-1} = [s_\beta'y_\beta, s_\alpha x_\alpha] \)

(vii) if \( k \in \{j_1, \ldots, j_m, j_m+1\} \) and \( k = i_t + 1 \) for some \( t \), then
\[
p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_\beta'y_\beta, s_\alpha x_\alpha]
\]
where \( 0 \leq k \leq n - 1 \).

**Proof:** Assume \( \beta < \alpha \) and \( \alpha \cap \beta = \emptyset \) which implies \( i_1 < j_1 \). In the range \( 0 \leq k \leq i_1 \),
\[
p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_k d_k s_{j_1} \ldots s_{i_1} x, s_k d_k s_{j_m} \ldots s_{j_1} y]^{-1}
\]
\[
= [s_\alpha(x_\alpha), s_\beta(y_\beta)] [s_{j_m-1} \ldots s_{j_1-1} s_k d_k y, s_{i_m-1} \ldots s_{i_1-1} s_k d_k x]
\]
\[
= [s_\alpha x_\alpha, s_\beta y_\beta] \text{ since } d_k(x_\alpha) = 1 \text{ or } d_k(y_\beta) = 1.
\]
\[
= v.
\]

Similarly if \( k > i_1 + 1 \), then
\[
p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_j \ldots s_{j_1} s_{k-m} d_{k-m} x, s_{i_m} \ldots s_{i_1} s_{k-m} d_{k-m} y]
\]
\[
= [s_\alpha x_\alpha, s_\beta y_\beta] \text{ since } d_{k-m}(y_\beta) = 1 \text{ or } d_{k-1}(x_\alpha) = 1.
\]
\[
= v.
\]

Clearly the same sort of argument works if \( k > j_{m+1} \). If \( k \in \{j_1, \ldots, j_m, j_{m+1}\} \)
and \( k = j_t + 1 \) for some \( t \), then
\[
p_k(v) = [s_\alpha x_\alpha, s_\beta y_\beta][s_k d_k s_\beta y_\beta, s_k d_k s_\alpha x_\alpha]
\]
\[
= [s_\alpha x_\alpha, s_\beta y_\beta][s_\beta' x_\beta', s_\alpha' y_\alpha']
\]
\[
\square
\]

**Lemma 3.2** If \( \alpha \cap \beta = \emptyset \) and \( \beta < \alpha \), then
\[
p_l \ldots p_1 [s_\alpha x_\alpha, s_\beta y_\beta] = [s_\alpha x_\alpha, s_\beta y_\beta] \prod_{i=1}^{l} s_i(z_i)^{-1}
\]
where \( z_i \in \bigcap_{j=0}^{i-1} \text{Ker} d_j \subset G_{n-1} \) and \( l \in [n - 1] \).
Lemma 3.3 Let \( x_\alpha \in NG_{n-\#\alpha}, y_\beta \in NG_{n-\#\beta} \) with \( \alpha, \beta \in S(n) \), then

\[
s_\alpha x_\alpha s_\beta y_\beta s_\alpha(x_\alpha)^{-1} = s_{\alpha \cap \beta} z_{\alpha \cap \beta}
\]

where \( z_{\alpha \cap \beta} \) has the form \( s_{\bar{\alpha}} x_\alpha s_{\bar{\beta}} y_\beta s_{\bar{\alpha}}(x_\alpha)^{-1} \) and \( \bar{\alpha} \cap \bar{\beta} = \emptyset \).

Proof: If \( \alpha \cap \beta = \emptyset \), then this is trivially true. Assume \( \#(\alpha \cap \beta) = t \), with \( t \in \mathbb{N} \). Take \( \alpha = (i_1, \ldots, i_k) \) and \( \beta = (j_1, \ldots, j_k) \) with \( \alpha \cap \beta = (k_1, \ldots, k_k) \),

\[
s_\alpha x_\alpha = s_{i_1} \cdots s_{i_k} x_\alpha \quad \text{and} \quad s_\beta y_\beta = s_{j_1} \cdots s_{j_k} y_\beta.
\]

Using repeatedly the simplicial axiom \( s_{a}s_b = s_{b}s_{a-1} \) for \( b < a \) until \( s_{k_1} \cdots s_{k_k} \) is at the beginning of the string, one gets the following

\[
s_\alpha x_\alpha = s_{k_1} \cdots s_{k_k} (s_{\bar{\alpha}} x_\alpha) \quad \text{and} \quad s_{\beta} y_\beta = s_{k_1} \cdots s_{k_k} (s_{\bar{\beta}} y_\beta).
\]

Multiplying these expressions together gives

\[
s_\alpha x_\alpha s_\beta y_\beta s_\alpha(x_\alpha)^{-1} = s_{k_1} \cdots s_{k_k} (s_{\bar{\alpha}} x_\alpha) s_{k_1} (s_{\bar{\beta}} y_\beta) s_{k_1} \cdots s_{k_k} (s_{\bar{\alpha}}(x_\alpha)^{-1})
\]

\[
= s_{k_1} \cdots s_{k_k} (s_{\bar{\alpha}} x_\alpha s_{\bar{\beta}} y_\beta s_{\bar{\alpha}}(x_\alpha)^{-1})
\]

\[
= s_{\alpha \cap \beta} (z_{\alpha \cap \beta}),
\]

where \( z_{\alpha \cap \beta} = s_{\bar{\alpha}} x_\alpha s_{\bar{\beta}} y_\beta s_{\bar{\alpha}}(x_\alpha)^{-1} \in NG_{n-\#(\alpha \cap \beta)} \) and where \( \bar{\alpha} = (i_1 - t, \ldots, k_t + 1 - t, \ldots, i_1) \) and \( \bar{\beta} = (j_m - t, \ldots, k_t' + 1 - t, \ldots, j_1) \). Hence \( \bar{\alpha} \cap \bar{\beta} = \emptyset_{n-\#(\alpha \cap \beta)} \). Moreover \( \bar{\alpha} < \alpha \) and \( \bar{\beta} < \beta \) as \( \#\bar{\alpha} < \#\alpha \) and \( \#\bar{\beta} < \#\beta \).

\[ \square \]

Suppose \( \alpha = (i_s, \ldots, i_1) \in S(m) \) and \( \gamma : [n] \rightarrow [m] \in S(n, m) \). Define \( \gamma_*(\alpha) \) by \( s_{\gamma_*(\alpha)} = s_\gamma s_\alpha \).

Corollary 3.4 Let \( \beta \leq \alpha \) and \( \gamma : [n] \rightarrow [m] \). Then \( \gamma_*(\beta) \leq \gamma_*(\alpha) \iff \beta \leq \alpha \), where \( \gamma_*(\alpha), \gamma_*(\beta) \in S(n) \).

\[ \square \]

The following lemma is proved similarly.
Lemma 3.5 For \( m \leq n \), suppose given in \( G_m \) an element

\[
g = \prod_{\beta \leq \gamma' \leq \alpha'} s_{\gamma'}(z_{\gamma'})
\]

and \( s_\delta : G_m \rightarrow G_n \). Then setting \( \alpha, \beta \in S(n) \) such that

\[
s_\delta s_{\alpha'} = s_\alpha, \quad s_\delta s_{\beta'} = s_\beta
\]

\[
s_\delta(g) = \prod_{\beta \leq \gamma \leq \alpha} s_{\gamma}(z_{\gamma})
\]

for some elements \( z_\gamma \in NG_{n-\#\gamma} \) and where \( s_\delta s_{\gamma'} = s_{\gamma} \). □

Proof of Theorem A :

From Proposition 1.2, \( G_n \) is isomorphic to

\[
NG_n \rtimes s_{n-1}NG_{n-1} \rtimes s_{n-2}NG_{n-1} \rtimes \ldots \rtimes s_{n-1}s_{n-2} \ldots s_0NG_0.
\]

Similarly \( D_n \) is isomorphic to

\[
(NG_n \cap D_n) \rtimes s_{n-1}NG_{n-1} \rtimes s_{n-2}NG_{n-1} \rtimes \ldots \rtimes s_{n-1}s_{n-2} \ldots s_0NG_0.
\]

Hence any element \( g \) in \( D_n \) can be written in the following form

\[
g = g_n s_{n-1}(y_{n-1}) s_{n-2}(y'_{n-1}) s_{n-1}s_{n-2}(y_{n-2}) \ldots s_{n-1}s_{n-2} \ldots s_0(y_0),
\]

with \( g_n \in NG_n \cap D_n, y_{n-1}, y'_{n-1} \in NG_{n-1}, y_{n-2} \in NG_{n-2}, y_0 \in NG_0 \) etc.

To simplify the notation a little, we will assume that \( G_n = D_n \), so that \( N_n \subset D_n \). The general case would replace \( NG_n \) by \( NG_n \cap D_n \) and similarly \( N_n \) by \( N_n \cap D_n \) from here on.

As it is easily checked that \( N_n \subset NG_n \cap D_n \), it is enough to prove that any element in \( D_n/N_n \) can be written in the form

\[
s_{n-1}(y_{n-1}) s_{n-2}(y'_{n-1}) s_{n-1}s_{n-2}(y_{n-2}) \ldots s_{n-1}s_{n-2} \ldots s_0(y_0) N_n
\]

that is, for any \( g \in D_n \),

\[
g N_n = s_{n-1}(y_{n-1}) s_{n-2}(y'_{n-1}) \ldots s_{n-1}s_{n-2} \ldots s_0(y_0) N_n.
\]

for some \( y_{n-1} \in NG_{n-1} \), etc. We refer to this as the standard form of \( g N_n \). If \( g \in D_n \), it is a product of degeneracies. If \( g \) is itself a degenerate element, it is obvious that it is a product of elements in the semidirect factors,
Assume therefore that provided an element $g$ can be written as a product of $k - 1$ degeneracies of this form, then it has the desired form modulo $N_n$. Now for an element $g$ which needs $k$ degenerate elements, we have

$$g = s_\alpha x_\alpha g' \quad \text{with } x_\alpha \in NG_{n-\#\alpha}$$

where $g'$ needs fewer than $k$ and so

$$gN_n = s_\alpha x_\alpha g'N_n = s_\alpha x_\alpha (s_{n-1}(y_{n-1})s_{n-2}(y'_{n-1})\ldots s_{n-1}s_{n-2}\ldots s_0(y_0))N_n.$$  

We prove that this can be rewritten in the desired form mod $N_n$ by using induction on $\alpha$ within the linearly ordered set $S(n) - \{\emptyset_n\}$.

If $\alpha = (n - 1)$, then

$$gN_n = s_{n-1}(xy_{n-1})s_{n-2}(y'_{n-1})\ldots s_{n-1}s_{n-2}\ldots s_0(y_0)N_n$$

where $x \in NG_{n-1}$ and $(xy_{n-1}) \in NG_{n-1}$.

If $\alpha = (n - 2)$, then since

$$F_{(n-2)(n-1)}(x_{n-1}, y_{n-1}) = [s_{n-2}x_{n-1}, s_{n-1}y_{n-1}] [s_{n-1}y_{n-1}, s_{n-1}x_{n-1}]$$

and

$$s_{n-2}(x_{n-1})s_{n-1}(y_{n-1})s_{n-2}(x_{n-1})^{-1} \equiv s_{n-1}(x_{n-1}y_{n-1}x_{n-1}^{-1}) \mod N_n,$$

we have

$$gN_n = (s_{n-2}(x_{n-1})s_{n-1}(y_{n-1})s_{n-2}(x_{n-1})^{-1})s_{n-2}(x_{n-1})s_{n-2}(y'_{n-1})\ldots s_{n-1}s_{n-2}\ldots s_0(y_0)N_n$$

$$= s_{n-1}(xyx_{n-1}^{-1})s_{n-2}(xy'_{n-1})\ldots s_{n-1}s_{n-2}\ldots s_0(y_0)N_n$$

where $x_{n-1}, y_{n-1} \in NG_{n-1}$ so $(xyx_{n-1}^{-1}), (xy'_{n-1}) \in NG_{n-1}$.

In general we need to sort $s_\alpha x_\alpha$ into its correct place in the product but in so doing will conjugate earlier terms in the product as happened in the case
\( \alpha = (n - 2) \) above. Each of these terms must be shown to consist only of subterms of types we have already dealt with, that is further to the left in the standard form of the product. Explicitly we assume that we can do this sorting for any term \( s_{x_{\alpha}} \) with \( \gamma < \alpha \) and examine

\[
g_{N_n} = s_{x_{\alpha}}(s_{n-1}(y_{n-1})s_{n-2}(y'_{n-1}) \ldots s_{n-1}s_{n-2} \ldots s_0(y_0))N_n
\]

\[
= s_{x_{\alpha}} \prod_{\beta \in S(n) - \{\emptyset\}} s_{\beta}y_{\beta}N_n
\]

\[
g_{N_n} = \prod_{\alpha > \beta} s_{x_{\alpha}}s_{\beta}y_{\beta}s_{\alpha}(x_{\alpha})^{-1} \cdot s_{\beta}(x_{\beta}) \cdot \prod_{\beta > \alpha} s_{\beta}y_{\beta}N_n
\]

where \( \beta \in S(n) - \emptyset \) and \( \alpha > \beta \) with respect to the lexicographic ordering in \( S(n) \).

We look at products of the following type

\[
s_{x_{\alpha}}s_{\beta}y_{\beta}s_{\alpha}(x_{\alpha})^{-1} \quad (*)
\]

and we want to show that these can always be written in the form

\[
\prod_{\gamma \leq \beta} s_{\gamma}(z_{\gamma})
\]

for some \( z_{\gamma} \in NG_{n-\#\gamma} \). This will mean that we already know how to sort all the terms that arise since none occur ‘to the right of’ \( \beta \) in the lexicographic order in the product.

We check this product case by case as follows:

If \( \alpha \cap \beta = \emptyset \), then by Lemma 3.2,

\[
s_{x_{\alpha}}s_{\beta}y_{\beta}s_{\alpha}(x_{\alpha})^{-1} \equiv \prod_{k=l}^{i_1+1} s_k(z_k)s_{\beta}y_{\beta} \mod N_n,
\]

where \( \beta \in S(n) - \{\emptyset\} \). Now we need to show that each \( s_k(z_k) \) is made up of terms \( s_{\mu}(z_{\mu}) \) with \( (z_{\mu}) \in NG_{n-\#\mu}, \mu \leq \beta \). (We will use the notation of Lemma 3.1.) Since \( \alpha > \beta \) then \( i_1 \leq j_1 \). We have

\[
z_k = \prod_{\mu \leq (k-1)} s_{\mu}(z_{\mu}) \quad \text{since} \quad z_k \in \bigcap_{j=0}^{k-1} \text{Ker}_j
\]
so
\[ s_k(z_k) = \prod_{\mu \leq (k-1) \leq (i_1)} s_{k\mu}(z_\mu) \]
where we write \( \mu = (m_1, \ldots, m_r) \) so we have \( k - 1 \leq m_1 \) or \( k \leq m_1 + 1 \).

We compare \( k \) with \( m_1 \) and \( m_2 \): either

(a) \( k = m_1 + 1 < m_2 \);
(b) \( k = m_1 + 1 = m_2 \);
(c) \( k = m_1 = \begin{cases} m_2 = m_1 + 1, \text{or} \\ m_2 > m_1 + 1, \end{cases} \)

or

(d) \( k < m_1 \).

Thus
\[ s_k s_\mu = s_{m_1 + 1} \cdots s_{m_2 + 1} s_k s_{m_1} = \begin{cases} s_{m_1 + 1} \cdots s_{m_2 + 1} s_k s_{m_1} & \text{cases (a) and (b)}, \\ s_{m_1 + 1} \cdots s_{m_1 + 1} s_{m_1} & \text{case (c)}, \\ s_{m_1 + 1} \cdots s_{m_1 + 1} s_k & \text{case (d)}, \end{cases} \]

so in each case \( s_k s_\mu = s_\mu' \) where \( m_1' = \min\{k, m_1\} \). We compare \( \mu' \) with \( \beta \).

If \( m_1' > j_1 \), then \( \mu' \leq \beta \). If \( m_1' = j_1 \), then either \( k = j_1 \) or \( m_1 = j_1 \) then \( k = j_1 + 1 \). Thus we need to show

\[ w = s_{j+1} d_{j+1} [s_{\beta y_\beta}, \ s_{\alpha x_\alpha}] = \prod_{\vartheta \leq \beta} s_\vartheta(z_\vartheta). \]

There are two cases:

(i) If \( j_1 \in \alpha \), then

\[ w = [s_{\beta y_\beta}, \ s_{\alpha x_\alpha}] = \prod_{\vartheta \leq \beta \leq \beta'} s_\vartheta(z_\vartheta) \]

where \( \beta \leq \beta' \) and since \( j_1 + 1 = j_s \notin \beta, \) then \( \beta' = \{j_m, \ldots, j_{s+1} + 1, j_s, j_{s-1}, \ldots, j_1\} \).

(ii) If \( j_1 + 1 \in \beta \), then

\[ w = [s_{\beta y_\beta}, \ s_{\alpha' x_\alpha}] = \prod_{\vartheta \leq \beta} s_\vartheta(z_\vartheta) \]
where \( \alpha \leq \alpha' \) and since \( j_1 + 1 = i_r \notin \alpha \), then \( \alpha' = \{i_1, \ldots, i_{r+1} + 1, i_r, i_{r-1}, \ldots, i_1\} \).

Both cases are covered by the induction hypothesis. Both cases can thus be written

\[
\prod_{\vartheta \leq \beta} s_{\vartheta}(z_{\vartheta}).
\]

We have \( z_{\vartheta} \in NG_{n-\#\vartheta} \) and for some \( r \) and \( s \) then it can be written for above cases

\[
s_k(z_k) = \prod_{0 \leq \gamma' \leq (k) \leq (i_1)} s_{\gamma'}(z_{\gamma'})
\]

where \( s_k s_{\gamma'} = s_{\gamma'} \), \( \gamma' \leq \beta \) and \( z_{\gamma'} \in NG_{n-\#\gamma'} \) so

\[
g_{\bar{\alpha} n} = \prod_{\gamma' \leq \beta} s_{\gamma'}(z_{\gamma'}) \cdot s_{\alpha}(x_{\alpha}) \cdot \prod_{\alpha < \beta} s_{\alpha}(x_{\alpha})
\]

as required.

If \( \alpha \cap \beta \neq \emptyset \), then one gets, from Lemma 3.3, the following

\[
s_{\bar{\alpha}} x_{\alpha} s_{\bar{\beta}} y_{\beta} s_{\alpha}(x_{\alpha})^{-1} = s_{\alpha \cap \beta}(s_{\bar{\alpha}} x_{\alpha} s_{\bar{\beta}} y_{\beta} s_{\bar{\alpha}}(x_{\alpha})^{-1})
\]

where \( \bar{\alpha} > \bar{\beta}, \alpha \cap \beta \in \emptyset_{n-\#(\alpha \cap \beta)} \). Using Lemma 3.5 in dimension \( n - \#(\alpha \cap \beta) \) and Corollary 3.4 then we have

\[
(s_{\bar{\alpha}} x_{\alpha} s_{\bar{\beta}} y_{\beta} s_{\alpha}(x_{\alpha})^{-1}) \equiv \prod_{\theta \in [\emptyset_n, \beta]} s_{\theta}(z_{\theta})
\]

hence

\[
s_{\alpha} x_{\alpha} s_{\beta} y_{\beta} s_{\alpha}(x_{\alpha})^{-1} = \prod_{\theta \in [\emptyset_n, \beta]} s_{\alpha \cap \beta \cap \theta}(z_{\theta})
\]

\[
= \prod_{\eta \in [\alpha \cap \beta, \beta]} s_{\eta}(z_{\eta})
\]

where \( z_{\eta} \in NG_{n-\#\eta} \),

\[
z_{\nu} = \begin{cases} 
  z_{\eta} & \text{if } \eta = \gamma_{s}(\eta) \\
  1 & \text{otherwise}
\end{cases}
\]

and \( s_{\alpha \cap \beta \cap \theta} = s_{\eta} \). Then \( g_{\bar{\alpha} n} \) can be written

\[
g_{\bar{\alpha} n} = \prod_{\nu \in [\alpha \cap \beta, \beta]} s_{\nu}(z_{\nu}) \cdot s_{\beta}(x_{\beta}) \cdot \prod_{\beta > \alpha} s_{\beta} y_{\beta} N_{\beta}.
\]
where \( s_\eta = s_\nu \). Thus we have shown that every product can be rewritten in the required form modulo \( N_n \), so in general, \( N_n \cap D_n = NG_n \cap D_n \). □

4 Applications and implications

Kan introduced the notation of a CW-basis for a free simplicial group and used this to proved that free simplicial groups model all connected homotopy types. The idea is that as one adds cells to the CW-complex one adds new generators to the free simplicial group, but one does this within the Moore complex so the new simplices have all but their last face at the identity element in the next dimension down. In homotopy types where there are few such non degenerate generators or where these generators are ‘generated’ in a simple way then the methods behind Theorem A raise the hope of finding a detailed presentation of the segments of the homotopy type between those dimensions in which there are non-degenerate generators. The means of presenting this information may vary with the context, but one set of fairly compact methods comes from the crossed algebraic techniques pioneered by J. H. C. Whitehead in [20]. (Modern references for this and for more recent developments can conveniently be found in the survey article by Baues [4].)

4.1 Crossed complexes

As an illustration we examine the impact of Theorem A on the links between simplicial groups and the homotopy systems of Whitehead, more exactly the connected crossed complexes of Brown and Higgins (cf. [5] and [6]) or the crossed chain complexes of Baues (cf. [3] and [4]) as no freeness assumptions will be made here.

Let \( G \) be a group, then a \( G \)-group is a group \( H \) together with a given action of \( G \) on \( H \), that is a homomorphism from \( G \) to \( \text{Aut}(H) \).
Definition 4.1 (cf. Baues, [4] p.22) A crossed complex $\rho$ is a sequence

$$d_4 \rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1$$

of homomorphisms between $\rho_1$-groups where $d_2$ is a crossed module and $\rho_n$, $n \geq 3$ is abelian and a $\pi_1$-module via the action of $\rho_1$, where $\pi_1 = \text{cokernel}(d_2)$. Moreover $d_{n-1}d_n = 0$ for $n \geq 3$.

It is known (cf. Ashley [1], Carrasco and Cegerra [11] or Ehlers and Porter [14] and the references therein) that crossed complexes correspond, via a nerve-type functor, to simplicial groups with a ‘thin’ structure. Each simplicial group is a Kan complex as a well known algorithm gives a filler for any horn. A Kan complex, $K$, is a $T$-complex if there is for each $n$ a subset $T_n$ of $K_n$, made up of so called ‘thin’ elements, such that any horn has a unique thin filler and two other more technical conditions hold (cf. Ashley [1]). A simplicial $T$-complex which is also a simplicial group is a group $T$-complex provided in each dimension $T_n$ is a subgroup of $K_n$. In this case one easily checks that $T_n$ must be $D_n$ the subgroup of $K_n$ generated by the degenerate elements.

Proposition 4.2 [1] A simplicial group $G$ has $NG$ a crossed complex if and only if for each $n \geq 1$, $NG_n \cap D_n$ is trivial. $\square$

The idea of the proof is that two $D_n$ fillers for the same horn must differ by an element of $NG_n \cap D_n$, so uniqueness corresponds to the simplicial group being a group $T$-complex. The final part uses Ashley’s equivalence between group $T$-complexes and crossed complexes.

Carrasco and Cegarra used this in [11] to prove (p. 215) that a simplicial group has Moore complex a crossed complex if and only if their pairings vanish (these are similarly defined to those used here but are based on products rather than commutators). Similarly we have:

Corollary 4.3 A necessary and sufficient condition that a simplicial group
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G has NG a crossed complex is that for all \( n \) and all \( \alpha, \beta \in P(n) \), \( F^n_{\alpha,\beta}(x, y) \) is trivial for all pairs \((x, y)\).

**Proof:** Since the \( F^n_{\alpha,\beta}(x, y) \) normally generate \( NG_n \cap D_n \), this is immediate. \( \square \)

The importance of this result is probably for the interpretation of the \( F^n_{\alpha,\beta} \) as their vanishing has a great simplifying effect on the Moore complex.

4.2 \( \sum K(\pi, 1) \)

As mentioned earlier Brown and Loday used their generalised van Kampen theorem to calculate \( \pi_3 \sum K(\pi, 1) \) as \( \text{Ker}(\pi \otimes \pi \rightarrow \pi) \), the kernel of the commutator map. Jie Wu ([23] Theorem 5.9) proves that for any group, \( \pi \), and set of generators \( \{x^\alpha \mid \alpha \in J\} \) for \( \pi \), then for \( n \neq 1 \), \( \pi_{n+2}(\sum K(\pi, 1)) \) is isomorphic to the center of the quotient group of the free product

\[
\prod_{0 \leq j \leq n} (\pi)
\]

modulo the relation

\[
[y_1^{(\alpha_1)\varepsilon_1}, y_2^{(\alpha_2)\varepsilon_2}, \ldots, y_t^{(\alpha_t)\varepsilon_t}]
\]

where \( \{i_1, \ldots, i_t\} = \{-1, 0, \ldots, n\} \) as sets, where \( (\pi)_j \) is a copy of \( \pi \) with generators \( \{x_j^{(\alpha)} \mid \alpha \in J\} \), \( \varepsilon_j = \pm 1 \), \( y_{-1}^{(\alpha)} = x_0^{(\alpha)^{-1}} \), \( y_j^{(\alpha)} = x_j^{(\alpha)}x_{j+1}^{(\alpha)^{-1}} \) for \( 1 \leq i \leq n - 1 \) and \( y_n^{(\alpha)} = x_n^{(\alpha)} \), and finally the commutator bracket \([\ldots]\) runs over all the commutator bracket arrangements of weight \( t \) for each \( t \).

Wu’s methods rely on using a construction he attributes to Carlsson [9]. This gives a simplicial group \( F^\pi(S^1) \) that has \( \pi_{n+2} \sum K(\pi, 1) \cong \Omega \sum K(\pi, 1) \cong \pi_{n+1}F^G(S^1) \). Our Theorem A above provides a link between Wu’s methods and the Brown-Loday result. We will explore this link to some extent but cannot as yet retrieve the Brown-Loday result by purely algebraic methods. Potentially however this might yield a tensor-like description in dimension 4 and higher, but we will not explore that here.
Although Carlsson introduced the construction $F^G(X)$ in 1984, the construction is essentially the same as the tensoring operation used by Quillen and others. Working in the simplicially enriched category of simplicial groups, there is a tensor operation defined as follows: let $K$ be a simplicial set and $G_1$, $G_2$ simplicial groups. The simplicial group $G_1 \tilde{\otimes} K$ has the universal property given by the natural isomorphism

$$S(K, SGp(G_1, G_2)) \cong SGp(G_1 \tilde{\otimes} K, G_2).$$

The category of simplicial groups is also enriched over $S_*$, the category of pointed simplicial sets. We define $G_1 \tilde{\wedge} K$ by

$$S_* (K, SGp(G_1, G_2)) \cong SGp(G_1 \tilde{\wedge} K, G_2).$$

There is an isomorphism $F^G(K) \cong G \tilde{\wedge} K$. The advantage of this approach is that it makes it clear that $\tilde{\wedge}$ generalises $\wedge$ just as $\tilde{\otimes}$ generalises $\times$.

**Lemma 4.4** If $f : G \to H$ is a morphism of simplicial groups, it induces $f \tilde{\wedge} K : G \tilde{\wedge} K \to H \tilde{\wedge} K$, moreover if $f$ is a weak homotopy equivalence, so is $f \tilde{\wedge} K$.

**Proof:** As Carlsson noted, $(G \tilde{\wedge} K)_n$ is

$$\prod_{x \in K_n} (G_n)_x / (G_n)_*$$

and is thus the diagonal of a bisimplicial group having $\coprod \{(G_m)_x \mid x \in K_n \setminus \{\ast\}\}$ in its $(m,n)$-position. A simple spectral sequence argument, or direct manipulation, completes the proof. □

**Proposition 4.5** There is a natural weak homotopy equivalence

$$\Omega \sum K(\pi, 1) \simeq K(\pi, 0) \tilde{\wedge} S^1$$

where $K(\pi, 0)$ is the constant simplicial group with value, $\pi$, $S^1$ is the simplicial 1-sphere and $\sum$ is reduced suspension.
Proof: As Kan’s loop group functor models $\Omega$ and $K \wedge S^1$ the suspension,

$$\Omega(\sum K(\pi, 1)) \simeq G(K(\pi, 1) \wedge S^1)$$

then the adjunction between $G$ and the classifying space functor $\bar{W}$ gives for $K, L$, arbitrary pointed simplicial sets, and $H$ an arbitrary simplicial group, the natural isomorphisms

$$SGp(G(K) \wedge L, H) \cong S_*(L, SGp(G(K), H))$$

$$\cong S_*(L, S_*(K, \bar{W}H))$$

$$\cong S(K \wedge L, \bar{W}H)$$

$$\cong SGp(G(K \wedge L), H)$$

thus $G(K) \wedge L \cong G(K \wedge L)$. As Curtis notes (p. 137) $\bar{W}K(\pi, 0)$ is a minimal complex for $K(\pi, 1)$ so taking $K = \bar{W}K(\pi, 0)$ we get $\Omega \sum K(\pi, 1)$ has as model $G(\bar{W}(K(\pi, 0)) \wedge S^1)$ and hence $G(\bar{W}(K(\pi, 0))) \wedge S^1$. By Lemma, 4.4 given the weak homotopy equivalence $G(\bar{W}(K(\pi, 0))) \to K(\pi, 0)$, the result follows. □

This implies that, like Jie Wu [23], we can take a simple model for $\Omega \sum K(\pi, 1)$. First we introduce notation for $S^1$. We write $S^1_0 = *, S^1_1 = \{\sigma, *\}$, $S^1_2 = \{x_0, x_1, *\}$ where $x_0 = s_1\sigma$, $x_1 = s_0\sigma$ and in general $S^1_{n+1} = \{x_0, \ldots, x_n, *\}$ where $x_i = s_n \ldots s_{i+1}s_{i-1} \ldots s_0\sigma$, $0 \leq i \leq n$.

For simplicity we write $G = K(\pi, 0)$ and make no distinction between simplicies in different dimensions unless confusion might arise. This then gives

$$(G \wedge S^1)_0 = 1, \quad \text{the trivial group}$$

$$(G \wedge S^1)_1 \cong \pi,$$

$$(G \wedge S^1)_2 \cong \pi \ast \pi, \quad \text{the free product of two copies of } \pi$$

$$(G \wedge S^1)_3 \cong \pi \ast \pi \ast \pi \quad \text{and so on.}$$

We write $g\wedge x$ for the $x$-indexed copy of $g \in \pi$ in the coproduct $\coprod \{ (\pi)_x : x \in S^1_n \setminus \{ * \} \}$ so the only relations we have are of the form

$$(gg'\wedge x) = (g\wedge x)(g'\wedge x).$$
We next analyse \( N(G\bar{\wedge}S^1) \) in low dimensions. For simplicity we will write \( H \) instead of \( G\bar{\wedge}S^1 \). Of course \( NH_0 = 1, \ NH_1 = \pi \).

By Theorem A, \( NH_2 \) is generated by all \( F_{(0)(1)}(g\bar{\wedge}\sigma, h\bar{\wedge}\sigma), g, h \in \pi : \)

\[
F_{(0)(1)}(g\bar{\wedge}\sigma, h\bar{\wedge}\sigma) = (g\bar{\wedge}x_1)(h\bar{\wedge}x_0)(g^{-1}\bar{\wedge}x_1)(gh^{-1}\bar{\wedge}x_0).
\]

In fact although Theorem A gives these as normal generators, it is clear that these are generators since conjugates of them are expressible as product of other terms of the same form. For instance

\[
k\bar{\wedge}x_1 F_{(0)(1)}(g\bar{\wedge}\sigma, h\bar{\wedge}\sigma) = F_{(0)(1)}(kg\bar{\wedge}\sigma, h\bar{\wedge}\sigma)F_{(0)(1)}(kg\bar{\wedge}\sigma, ghg^{-1}\bar{\wedge}\sigma)
\]

and a similar expression can be found for conjugation by \( k\bar{\wedge}x_0 \).

Brown and Loday \cite{8} calculated \( \pi_3(\sum K(\pi, 1)) \) using a van Kampen theorem for \( \text{cat}^2 \)-groups. This led to an expression for this group as being isomorphic to

\[
J_2(\pi) = \text{Ker}(\kappa : \pi \otimes \pi \rightarrow \pi).
\]

We refer to the paper \cite{7} Brown, Johnson and Robertson for some details on the non-abelian tensor product of groups and to Ellis \cite{15} for more on the representation of homotopy types by crossed squares and \( \text{cat}^2 \)-groups. Here it will suffice to say that if \( G, H \) are groups that act on themselves by conjugation and on each other in such a way that

\[
g' h' = ghg^{-1}g' \quad \quad \quad \quad \quad \quad \quad h' = hgh^{-1}h'
\]

(see \cite{7}), the tensor product \( G \otimes H \) is the group generated by symbols \( g \otimes h \) with relations

\[
\begin{align*}
    gg' \otimes h &= (g g' \otimes g h)(g \otimes h) \\
g \otimes h h' &= (g \otimes h) \otimes (h g \otimes h' h)
\end{align*}
\]

for all \( g, g' \in G, h, h' \in H \). We will need this only when \( G = H \), then there is a map

\[
\kappa : G \otimes G \rightarrow G
\]
given by \( \kappa(g \otimes h) = [g, h] \). This gives a homomorphism since the above relations are abstract versions of the usual commutator identities.

Using a combination of Ellis’s work in [15] and the second author’s description of the crossed \( n \)-cube associated to a simplicial group in [19], it is clear that the expression for the tensor product should be closely linked to one for \( NH_2/d_3NH_3 \) and with that in mind we set for \( g, h \in \pi \)

\[
g \bar{\otimes} h = \left[ g \tilde{x}_0, (h \tilde{x}_0)(h^{-1} \tilde{x}_1) \right] d_3NH_3.
\]

This ‘mysterious’ formula will be more fully explained in another paper where the relationship with crossed squares and Ellis’s work will be given in detail. For the moment the reader is asked merely to accept the left hand side as a shorthand for the right hand side.

**Proposition 4.6** The symbols \( g \bar{\otimes} h \) generate \( NH_2/d_3NH_3 \) and satisfy the tensor product relations above. The boundary homomorphism

\[
d'_2: \frac{NH_2}{d_3NH_3} \rightarrow NH_1
\]

sends \( g \bar{\otimes} h \) to \( [g, h] \bar{\otimes} \sigma \) in \( NH_1 \).

**Proof:** Direct calculation shows

\[
g \bar{\otimes} h = F_{(0)(1)}(h \bar{\otimes} \sigma, g \bar{\otimes} \sigma) d_3NH_3
\]

which clearly generate \( NH_2/d_3NH_3 \) by Theorem A and the commutator above. The tensor product relations are again directly verifiable (eg. using the description of the \( h \)-map given in [19]), and it is immediate that

\[
d_2[g \bar{\otimes} x_0, (h \bar{\otimes} x_0)(h^{-1} \bar{\otimes} x_1)] = [g, h].
\]

\[\square\]

The next term in the Moore complex \( NH_3 \) has 6 different types of generator and so there are 6 known types of relation in \( NH_2/d_3NH_3 \). Presumably it is possible to give a direct proof that

\[
\frac{NH_2}{d_3NH_3} \cong \pi \otimes \pi
\]
using these relation, but we have so far not managed to find one. A closer analysis of the relationship of the Peiffer pairings with the structure of crossed squares gives this isomorphism by a universal argument, but requires other techniques and so will be postponed to a later paper. Knowledge of the 6 types of generator for $NH_3$ hopefully will allows a detailed calculation of $NH_3/d_4NH_4$ in a similar manner.

References

[1] N. Ashley, Simplicial T-Complexes: a non abelian version of a theorem of Dold-Kan, Dissertations Math. 165, (1988), 11-58. Ph.D. Thesis, University of Wales, Bangor, (1978).

[2] H. J. Baues, Algebraic Homotopy, Cambridge Studies in Advanced Mathematics, 15, Cambridge Univ. Press., (1989), 450 pages.

[3] H. J. Baues, Combinatorial homotopy and 4-dimensional complexes, Walter de Gruyter, (1991).

[4] H. J. Baues, Homotopy Types, Handbook of Algebraic Topology, Edited by I. M. James, (1995), 1-72.

[5] R. Brown and P. J. Higgins, Colimit-theorems for relative homotopy groups, Jour. Pure Appl. Algebra, 22, (1981), 11-41.

[6] R. Brown and P. J. Higgins, Crossed Complexes and non-abelian extensions, Lecture Notes in Math., Springer, 962, (1982), 39-50.

[7] R. Brown, D. L. Johnson and E. F. Robertson, Some computations of non-abelian tensor products of groups, J. Algebra, 111, (1987), 177-202.

[8] R. Brown and J.-L. Loday, Van Kampen Theorems for Diagram of Spaces, Topology, 26, (1987), 311-335.
[9] G. Carlsson, A Simplicial Group Construction for Balanced Products, *Topology*, **23**, (1984), 85-89.

[10] P. Carrasco, Complejos Hipercruzados, Cohomologia y Extensiones. *Ph.D. Thesis*, Universidad de Granada, (1987).

[11] P. Carrasco and A. M. Cegarra, Group-theoretic Algebraic Models for Homotopy Types, *Jour. Pure Appl. Algebra*, **75**, (1991), 195-235.

[12] D. Conduché, Modules Croisés Généralisés de Longueur 2. *Jour. Pure Appl. Algebra*, **34**, (1984), 155-178.

[13] E. B. Curtis, Simplicial Homotopy Theory, *Advances in Math.*, **6**, (1971), 107-209.

[14] P. J. Ehlers and T. Porter, Varieties of Simplicial Groupoids, I: Crossed Complexes. *Jour. Pure Appl. Algebra*, **120**, (1997), 221-233.

[15] G. J. Ellis, Crossed Squares and Combinatorial Homotopy, *Math. Z.*, **214**, (1993), 93-110.

[16] J.-L. Loday, Spaces having finitely many non-trivial homotopy groups, *Jour. Pure Appl. Algebra*, **24**, (1982), 179-202.

[17] J. P. May, Simplicial Objects in Algebraic Topology, *Van Nostrand, Math. Studies*, **11**, (1967).

[18] A. Mutlu, Peiffer Pairings in the Moore Complex of a Simplicial Group *Ph.D. Thesis*, University of Wales, Bangor, (1997); *Bangor Preprint* 97.11.

[19] T. Porter, n-Types of simplicial groups and crossed n-cubes, *Topology*, **32**, (1993), 5-24.
[20] J. H. C. Whitehead, Combinatorial Homotopy I and II, *Bull. Amer. Math. Soc.*, 55, (1949), 231-245 and 453-496.

[21] J. H. C. Whitehead, A Certain Exact Sequence, *Annals. of Math.*, 52, (1950), 51-110.

[22] J. Wu, On Combinatorial Descriptions of Homotopy Groups and the Homotopy Theory of mod 2 Moore Spaces, *Ph.D. Thesis*, University of Rochester, Rochester, New York, USA, (1995).

[23] J. Wu, On combinatorial descriptions of $\pi_*(\Sigma K(\pi, 1))$, *MSRI preprint 069*, (1995).

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