A Wideband Dual Function Radar Communication System With Sparse Array and OFDM Waveforms

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Abstract—A novel multiple-input multiple-output (MIMO) dual-function radar communication (DFRC) system is proposed. The system transmits wideband, orthogonal frequency division multiplexing (OFDM) waveforms using a small subset of the available antennas in each channel use. The proposed system assigns most carriers to antennas in a shared fashion, thus efficiently exploiting the available communication bandwidth, and a small set of subcarriers to active antennas in an exclusive fashion (private subcarriers). A novel target estimation approach is proposed to overcome the coupling of target parameters introduced by subcarrier sharing. The obtained parameters are further refined via an iterative approach, which formulates a sparse signal recovery problem based on the data of the private subcarriers. The system is endowed with beamforming capability, via waveform precoding and antenna selection. The precoding and antenna selection matrices are optimally co-designed to meet a joint sensing-communication system performance. The sparsity of the transmit array is exploited at the communication receiver to recover the transmitted information. The use of shared subcarriers enables high communication rate, while the sparse transmit array maintains low system hardware cost. The sensing problem is formulated by taking into account frequency selective fading, and a method is proposed to estimate the channel coefficients during the sensing process. The functionality of the proposed system is demonstrated via simulations.

Index Terms—DFRC system, MIMO radar, OFDM radar waveforms, subcarrier sharing, sparse array, antenna selection

I. INTRODUCTION

An emerging trend in next-generation wireless applications [1] is to allow unconstrained access to spectrum for radar and communication systems for the purpose of increasing spectral efficiency. This has given rise to a lot of interest in designing systems that can coexist in the spectrum while using different platforms [2]–[4], or to Dual Function Radar Communication (DFRC) systems that perform sensing and communication from a single platform [5]. The former class can work with existing systems but requires means for controlling the interference between the two systems, for example, via a control center [4]. On the other hand, DFRC systems require new signaling designs, but do not require interference control, and further they offer reduced cost, lighter hardware, and lower power consumption. For those reasons, DFRC systems are of great interest to vehicular networks, WLAN indoor positioning, unmanned aerial vehicle networks [5]–[9]. The contribution of this work falls along the lines of DFRC systems.

DFRC systems typically involve multi-antenna systems. There are various ways via which communication information can be embedded in the sensing process. It can be embedded directly in the radar waveforms [9]–[19], or in the way the waveforms are paired with transmit antennas [19]–[22], or in the phase of the sidelobes in the array beampattern [23], or in the antenna activation pattern [9], [16], [24].

Multi-carrier waveforms can enable high communication rate, and thus have been studied in DFRC systems. Multi-carrier waveforms with Frequency-Hopping (FH) were proposed in [19], [22], where the bandwidth is divided into sub-bands, and antennas are paired with subbands in an exclusive fashion, with the pairing changing over time. However, this type of subbands assignment uses only part of the available bandwidth, which reduces the attainable target range resolution. Orthogonal frequency division multiplexing (OFDM) waveforms for DFRC systems have been explored in [11], [12], [17], [18]. OFDM is a popular approach to achieve high communication rate and also deal with frequency selective fading. For those reasons, OFDM has been widely used in modern communication applications, such as wireless local area network (WLAN) [25], power line communication (PLC) [26] and 4G/5G mobile communications [27]. In radar, OFDM waveforms provide ability to flexibly occupy the available spectral resources [11], and can easily overcome frequency selective propagation effects. Typically [11], [12], [17], in OFDM radar, the subcarriers are assigned to antennas in an exclusive fashion, so that the transmissions of antennas are orthogonal, and a virtual array can be constructed at the receiver to provide high resolution angle estimation. As compared to FH methods, OFDM methods use all available bandwidth for sensing, which allows for a higher range resolution. However, due to the way carriers are assigned to antennas, the communication bandwidth is not used efficiently.

In this paper, we propose a novel DFRC system that uses the available bandwidth efficiently for both sensing and communication. The proposed system comprises a sparse MIMO radar that transmits wideband precoded OFDM waveforms. The transmit array contains a small number of radio frequency (RF) chains and a large number of antennas, and in each channel use, only a small number of antennas are connected to the available RF chains. Unlike other OFDM DFRC systems, where each subcarrier can only contain one data symbol [11], [12], [17], [18], our proposed system allows subcarrier sharing between the communication data symbols. In particular, the OFDM subcarriers are divided into two groups, i.e., the shared subcarriers group, where each subcarrier carries symbols from all active antennas, and the private subcarriers group, where each subcarrier is assigned to each active antenna in an exclusive fashion. Subcarrier sharing between RF chains exploits
the available bandwidth and thus achieves high communication rate. However, in prior works it has been avoided because it destroys waveform orthogonality, and results in coupling of radar target parameters. In this paper we propose a novel approach to resolve the coupling and estimate the target parameters.

**Sensing** - While Doppler information can be obtained based on one for the private subcarriers, range estimation cannot be obtained with good enough resolution based on the small number of private subcarriers used here. We propose to first obtain coarse angle estimates based on the receive array, and subsequently, within each angle bin, obtain range information based on the received and transmitted symbols. The obtained estimates are further refined via an iterative approach, which formulates a sparse signal recovery (SSR) problem based on the symbols transmitted and received on the private subcarriers.

**Communication** - The sparsity of the transmit array is exploited at the communication receiver to recover the transmitted data symbols on the shared subcarriers via a sparse signal recovery approach, which makes the transmission of multiple data symbols on each subcarrier possible.

**Co-Design** - The proposed system is endowed with beamforming capability via waveform precoding and antenna selection. The precoding and antenna selection matrices are optimally designed to maximize a weighted combination of the radar beampattern performance and the communication signal-to-noise (SNR) ratio. In general, selection of a small group of active antennas from the set of available antennas is an NP-hard problem. Several optimization schemes have been proposed for antenna selection [28]–[37]. Most formulations typically result in NP hard optimization problems, which require relaxation techniques and various assumptions in order to reach a solution. Machine Learning approaches for antenna selection have been proposed in [38]–[40], however they suffer from the combinatorial explosion problem, which renders them impractical in cases with large or even medium number of antennas. In this paper we optimally design the sparse antenna pattern and precoding matrix using the L2S framework of [41]. Instead of choosing the best sparse pattern out of all the combinations, L2S approximates the selection of each active antenna by a softmax neural network, thus avoiding the combinatorial explosion problem, while the softmax selection models can run in parallel to save time. However, the parallel softmax selection models lead to a selection matrix with a column permutation ambiguity, which would permute the pairing between RF chains and antennas, thus causing permutation ambiguity in data recovery at the communication system. By leveraging the private subcarriers, we show that the permutation ambiguity can be resolved.

Assuming that the targets and the communication receiver are in the same space, the sensing and communication signals experience similar channel conditions. Therefore, we formulate the sensing problem by taking into account frequency selective fading. While for the communication operation the channels can be estimated via pilots, for the sensing operation there is no known way to estimate the channels. We propose a novel method to periodically estimate the channel coefficients during the sensing process by exploiting private subcarriers.

The motivation behind using a small number of RF chains is cost reduction. RF chains are expensive as they involve analog-to-digital converters, and also consume a lot of power. On the other hand, antennas are inexpensive. The selection of active antennas provides degrees of freedom allowing to meet the beampattern required to have good sensing and communication performance.

Preliminary results of this work were reported in [9]. In addition to [9], we here consider multipath fading in the problem formulation, introduce precoding and antenna selection for beamforming purposes, and propose a novel co-design scheme that optimizes a combination of probing beampattern and communication SNR performances.

The remainder of this paper is organized as follows. In Section II, we describe the target estimation process using precoded OFDM radar with shared subcarriers in a frequency selective channel, while in Section III we introduce the use of private subcarriers in addition to shared subcarriers, for achieving better target detection. In Section IV, we present how to estimate the coefficients of the radar frequency selective channel. In Section V, we discuss the corresponding communication data symbol recovery. In Section VI we formulate the precoder design and antenna selection problems by jointly optimizing sensing and communication performance. We provide simulation results on the system performance in Section VII, and concluding remarks in Section VIII.

**Notation**: Throughout this paper, we use \( \mathbb{R} \) and \( \mathbb{C} \) to denote the sets of real and complex numbers, respectively. \((\cdot)^T\) stands for the transposition operator, \((\cdot)^*\) means the complex conjugate and \((\cdot)^H\) refers to the complex conjugate transpose. \(\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_F\) represent \(\ell_1, \ell_2\) and Frobenius norms, respectively. \(I_N\) denotes an identity matrix of size \(N \times N\).

**II. The Radar System Model**

Let us consider a collocated MIMO radar with \(N_x\) RF chains, which can be connected to a uniform linear array (ULA) transmit array with \(N_t\) transmit and \(N_r\) receive antennas, spaced by \(d_t\) and \(d_r\), respectively. In each channel use, \(N_x\) antennas out of the \(N_t\) available ones are selected to transmit. We will denote by \(\mathcal{N}\) the set of selected antenna indices.

The radar transmitter is illustrated in Fig. 1. The radar transmits precoded OFDM waveforms, which are generated as follows. The binary source data are divided into \(N_x\) parallel streams, they are modulated via phase-shift keying (PSK) modulation, or quadrature amplitude modulation (QAM), and they are distributed to the OFDM subcarriers assigned to each RF chain. The outputs of all \(N_x\) RF chains are connected to a ULA of \(N_t >> N_x\) antennas, after they are processed by a precoding matrix, \(\mathbf{P}\), and an antenna selection matrix, \(\mathbf{S}\). The combined effect of the two aforementioned matrices, i.e., \(\mathbf{SP}\), is to select \(N_x\) antennas, and assign to the \(i\)-th subcarrier of each selected antenna a weighted sum of the symbols the RF fronts placed on subcarrier \(i\). The details on these matrices will
be provided in the following. Each selected antenna applies an inverse discrete Fourier transform (IDFT) on the data symbols assigned to it, pre-appends a cyclic prefix (CP), converts the samples into an analog multicarrier signal and transmits it with carrier frequency $f_c$. That signal will be referred to as OFDM symbol, and has duration $T_p$.

Let $\mathbf{D} \in \mathbb{C}^{N_s \times N_s}$ denote a matrix that contains the symbols to be transmitted during the $\mu$-th OFDM symbol, i.e.,

$$
\mathbf{D} = \begin{bmatrix}
    d(0, 0, \mu) & \cdots & d(0, N_s - 1, \mu) \\
    d(1, 0, \mu) & \cdots & d(1, N_s - 1, \mu) \\
    \vdots & \ddots & \vdots \\
    d(N_s - 1, 0, \mu) & \cdots & d(N_s - 1, N_s - 1, \mu)
\end{bmatrix} (1)
$$

where $N_s$ is the number of subcarriers, $d(n, i, \mu)$ denotes the symbol transmitted by the $n$-th antenna, on the $i$-th subcarrier, during the $\mu$-th OFDM symbol. The $i$-th column of $\mathbf{D}$, i.e., $\mathbf{d}_i$, contains the symbols transmitted by all antennas on subcarrier $i$, while the $j$-th row of $\mathbf{D}$ contains the symbols transmitted by the $j$-antenna on all subcarriers. Since only $N_x$ antennas are selected to transmit, the rows corresponding to not selected antennas will contain zeros. Unless otherwise indicated, the use of the symbol matrix will refer to one OFDM symbol, thus, for notational simplicity, the dependence on $\mu$ is not shown in the notation $\mathbf{D}$.

Let $\mathbf{P} \in \mathbb{C}^{N_s \times N_s}$ denote the precoding matrix and $\mathbf{Q} \in \mathbb{C}^{N_s \times N_s}$, the matrix containing the data symbols in each OFDM symbol before precoding. Then,

$$
\mathbf{D} = \mathbf{S}\mathbf{P}\mathbf{Q},
$$

where $\mathbf{S} \in \mathbb{R}^{N_s \times N_s}$ is the antenna selection matrix. Each column in $\mathbf{S}$ has exactly one nonzero element, equal to one, corresponding to the index of an active antenna. In order to avoid selecting the same antenna twice, each row of $\mathbf{S}$ contains at most one 1. For example, in Fig. 1, the selection matrix defined as

$$
\mathbf{S}^T = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} (3)
$$
distributes the precoded data to the first ($n = 0$), and fourth ($n = 3$) antennas of an array with a total of $N_t = 4$ antennas. The antenna selection matrix, $\mathbf{S}$, allows the RF fronts to connect to an equal number of antennas, which can be selected out of large number of possibilities. This introduces a large number of degrees of freedom in system design. The optimal design problem is addressed in Section VI. The precoding scheme endows the system with beamforming capability, and the precoding matrix, $\mathbf{P}$, will be optimally designed to meet a certain performance metric.

The complex envelope of the transmitted baseband signal on the $i$-th subcarrier due to the $n$-th antenna equals

$$
x(n, i, t) = \sum_{\mu=0}^{N_s-1} d(n, i, \mu) e^{j2\pi i \Delta t \text{rect}(t - \mu T_p / T_p)},
$$

where $\text{rect}(t/T_p)$ denotes a rectangular pulse of duration $T_p$ and $\Delta$ is the subcarrier spacing.

Suppose that there are $K$ point targets in the far field, each characterized by angle $\theta_k$, range $R_k$, and Doppler frequency $f_{dk}$. It holds that $f_{dk} = 2v_k / \lambda$, with $c$ denoting the speed of light, and $v_k$ the velocity of the $k$-th target. The baseband equivalent of the multicarrier signal reflected back and received by the $m$-th receive antenna on the $i$-th subcarrier is

$$
y(m, i, t) = \sum_{k=1}^{K} \sum_{n \in \mathcal{N}} \beta_{mnk}^i (n, i, t - \tau_{kmn}^i) e^{j2\pi f_{dk} t} + u_i(m, t)
$$

for $m = 0, \ldots, N_s - 1$, where $\beta_{mnk}^i$ is a complex coefficient accounting for multipath propagation and scattering process from the $n$-th transmit antenna to the $m$-th receive antenna of the $k$-th target on the $i$-th subcarrier.

$$
\tau_{kmn}^i = 2R_k / c + (nd_i + md_r) \sin \theta_k / \lambda_i,
$$
is the roundtrip delay of the $k$-th target, with $\lambda_i$ the wavelength of the $i$-th subcarrier, and $u_i(m, t)$ denoting noise or clutter. (5) represents the effect of a single tap delay channel, with the delay and channel coefficient depending on the transmit and receive antenna indices, the target and the subcarrier frequency, i.e., it is a frequency selective channel.

### A. Assumptions

For the rest of the paper, we make the following assumptions:

1. The length of the CP is larger than the maximum roundtrip delay to the target, so that inter-symbol interference can be avoided during demodulation.
2. We assume that the secondary reflections from background objects will be attenuated before they reach the targets of interest, and thus will not generate significant echoes to the receive antennas, but rather can be considered as noise.
3. The channel coefficients $\beta_{mnk}^i$ can be expressed as $\beta_{mnk}^i = \beta_{nk}^i \beta_{km}$, where the two multiplicative terms represent the effect of the channel from the $n$-th transmit antenna to the $k$-th target, and that of the channel from the $k$-th target to the $m$-th receive antennas. Due to

![Fig. 1. Beamforming with precoding on source data symbols](image-url)
the sparse transmit array structure, the spacing of the transmit antennas is in general larger than that of the receive antennas, hence, we assume that the term $\beta_{kn}$ is approximately the same for all receive antennas. Further, these coefficients are the same on all subcarriers. The latter assumption is justified based on the fact that for a large number of subcarriers the subcarrier frequencies are very closely spaced. Based on the above assumptions, we have that $\beta_{kn} \approx \beta_{nk}$.

4) Initially, we will assume that these coefficients are known. In Section IV we will discuss how they can be estimated.

5) The noise terms $u(m,t)$ are independent identically distributed (i.i.d.) white Gaussian noise processes with zero mean and variance $\sigma^2_r$ and do not depend on the frequency.

6) The OFDM signal bandwidth is much smaller than the carrier frequency, thus, within the same OFDM symbol, the phase shifts due to the Doppler effect are identical on all subcarriers.

7) The targets do not move much during the duration of the OFDM symbol, thus, we the target range is constant over the duration of the OFDM symbol and only changes between OFDM symbols.

8) The Doppler frequency of the target stays constant for $N_p$ OFDM symbols.

**B. Target Angle Coarse Estimation**

Due to the existence of CP, the radar receive antennas can recover the precoded symbols via an $N_r$-point discrete Fourier transform (DFT). For simplicity, the rectangular pulse function and CP will be omitted in the following formulations.

Based on the above assumptions, the received signal of (5) can be written as

$$y(m, i, t) \approx \sum_{n=1}^{K} \sum_{n \in N} \beta_{nk} x(n, i, t - \pi \tau_{kn}) e^{j2\pi f_{nk} t} + u_i(m, t)$$

Let us consider an observation interval equal to the duration of the OFDM symbol duration. Based on assumptions 2, 3, 6, 7 and 8, the received signal of (6) in the $\mu$-th OFDM symbol over all $N_s$ subcarriers equals

$$y(m, t) = \sum_{k=1}^{K} \sum_{n=1}^{N_s-1} e^{-j2\pi m d_{nk}} \sin \theta_n \frac{L+i\Delta}{c}$$

$$\times \sum_{n \in N} \beta_{nk} d(n, i, \mu) e^{-j2\pi n d_{nk} \sin \theta_n \frac{L+i\Delta}{c}}$$

$$\times e^{-j2\pi i \Delta 2n f_{nk}} e^{j2\pi i \Delta t} + u(m, t).$$

where $u(m, t)$ is the cumulative noise over all subcarriers.

The time during one OFDM symbol is usually referred to as fast time, while the time between OFDM symbols is referred to as slow time. Based on (7), the linear phase shift from the roundtrip propagation to the target and back changes in the fast time domain, while the linear phase shift from the Doppler effect changes only along the slow time domain. This makes it possible to measure range and Doppler independently.

After sampling the received signal of duration $T_p$ and applying an $N_p$-point DFT on the received samples, the symbol received by the $m$-th receive antennas equals

$$d_r(m, i, \mu) = \sum_{k=1}^{K} \sum_{n \in N} \beta_{nk} d(n, i, \mu) e^{-j2\pi (md_{nk} + nd_{nk}) \sin \theta_n \frac{L+i\Delta}{c}}$$

$$\times e^{-j2\pi i \Delta 2n f_{nk}} e^{j2\pi i \Delta t} + u(m, i, \mu).$$

where $U(m, i, \mu)$ denotes the $N_p$-point DFT of the noise during the $\mu$-th OFDM symbol. (8) can be viewed as

$$d_r(m, i, \mu) = \sum_{k=1}^{K} A(k, i, \mu) e^{j\omega(k, i)m} + U(m, i, \mu),$$

where

$$A(k, i, \mu) = \sum_{n \in N} \beta_{nk} d(n, i, \mu) e^{-j2\pi n d_{nk} \sin \theta_n \frac{L+i\Delta}{c}}$$

$$\times e^{-j2\pi i \Delta 2n f_{nk}} e^{j2\pi \mu f_{nk}}$$

and

$$\omega(k, i) = -d_i \sin \theta_n \frac{f_r + i\Delta}{c}$$

Assuming that $N_r > N_k$ and for a fixed $i$ and $\mu$, \{ $d_r(m, i, \mu), m = 0, ..., N_r - 1$ \} can be viewed as a sum of $K$ complex sinusoids with frequencies $\omega(k, i)$ and complex amplitudes $A(k, i, \mu)$. By applying an $N_r$-point DFT, we can get peaks at frequencies $\omega(k, i)$. The resolution of the peaks will depend on the number of receive antennas, $N_r$.

Once the $\omega(k, i)$’s are estimated, the target angles can be computed as

$$\theta_k = \arcsin \left( -\frac{\omega(k, i)c}{d_i(f_r + i\Delta)} \right)$$

In Section III-A we will discuss how to improve the angle resolution using some private subcarriers and formulating a virtual array.

**C. Range and Doppler Estimation**

The amplitudes corresponding to frequencies $\omega(k, i)$ contain known precoded symbols, estimated target angles, and unknown ranges and Doppler frequencies. Inside each angular bin, there may be multiple targets. Suppose that there are $N_k$ targets corresponding to the $k$-th estimated direction $\theta_k$. Then, the corresponding amplitude can be expressed as

$$A(k, i, \mu) = \sum_{n \in N} d(n, i, \mu) e^{-j2\pi n d_{nk} \sin \theta_n \frac{L+i\Delta}{c}}$$

$$\times e^{-j2\pi i \Delta 2n f_{nk}} e^{j2\pi \mu f_{nk}}$$

$$= \sum_{q=1}^{N_k} A'(k, i, q, \mu) e^{-j2\pi i \Delta 2n f_{nk} + j2\pi \mu f_{nk}}$$

where

$$A'(k, i, q, \mu) = \sum_{n \in N} \beta_{nq} d(n, i, \mu) e^{-j2\pi n d_{nk} \sin \theta_n \frac{L+i\Delta}{c}}$$

(14)

and

$$A'(k, i, q, \mu) = \sum_{n \in N} \beta_{nq} d(n, i, \mu) e^{-j2\pi n d_{nk} \sin \theta_n \frac{L+i\Delta}{c}}$$

(15)
(14) can be written as
\[ A(k, i, \mu) = \sum_{q=1}^{N_q} e^{j2\pi f_p T_p f_d q} A'(k, i, q, \mu) e^{j2\pi i \omega_r(q)} \]  
(16)
where
\[ \omega_r(q) = -\Delta \frac{2R_q}{c}. \]
Let \( A(k, \ell, \mu) \) and \( A'(k, \ell, q, \mu) \) denote respectively the \( N_q \)-point DFT of \( A(k, i, \mu) \) and \( A'(k, i, q, \mu) \) along the fast time \( i \). One can see that \( A(k, \ell, \mu) \) is a weighted sum of shifted versions of \( A'(k, \ell, q, \mu) \), where \( \omega_r(q) \) are the shifts and \( e^{j2\pi f_p T_p f_d q} \) the weights. Since \( A'(k, i, q, \mu) \) has already been estimated, the shifts \( \omega_r(q) \) can be measured based on the location of the peaks in the cross correlation of \( A'(k, \ell, q, \mu) \) and \( A(k, \ell, \mu) \). The peaks appear at indices
\[ l_q = \left\lfloor \frac{2N_q R_q \Delta}{c} \right\rfloor, \]
(17)
where \( \left\lfloor \cdot \right\rfloor \) denotes the floor function. The locations of the peaks can be used to obtain target range information. We should note here that for the cross correlation peaks to be resolvable, the spectrum of \( A'(k, \ell, q, \mu) \) should be narrow around zero. Based on our experience with simulations this holds reasonably well, (see Section VII). Similarly, by fixing \( k \) and \( i \), (15) can be written as
\[ A(k, i, \mu) = \sum_{q=1}^{N_q} e^{-j2\pi i \Delta \frac{2R_q}{c}} A''(k, i, q, \mu) e^{j2\pi i \omega_{rl}(q)} \]  
(18)
where \( \omega_{rl}(q) = T_p f_d q \). After cross correlation between the \( N_p \)-point IDFT of \( A(k, i, \mu) \) and \( A''(k, i, q, \mu) \), peaks will arise at
\[ p_q = \left\lfloor N_p T_p f_d q \right\rfloor = \left\lfloor \frac{2\nu_f f_c N_p T_p}{c} \right\rfloor, \]
(19)
and can be used to obtain the velocities of the targets.

However, there is a better and simpler way to obtain Doppler information, by applying the above described cross-correlation approach on the private subcarriers, as will be discussed in Sec.III-B.

For range estimation, the resolution is
\[ R_{res} = \frac{c}{2N_q \Delta} = \frac{c}{2B} \]
(20)
where \( B \) is the bandwidth of the OFDM waveforms. The maximum detectable range is determined by the spacing of subcarriers, i.e.,
\[ R_{max} = \frac{c}{2\Delta}. \]
(21)
Similarly, the resolution of the target velocity estimation and maximum detectable velocity are
\[ v_{res} = \frac{c}{2f_c N_p T_p} \]
(22)
\[ v_{max} = \frac{c}{2f_c T_p} \]
(23)
respectively. Note that the radial velocities could be both positive and negative thus the maximum unambiguous velocity is half of the detectable velocity.

III. ITERATIVE REFINING OF TARGET PARAMETERS USING PRIVATE SUBCARRIERS

Angle estimation via (13) relies on a physical array of aperture \((N_r - 1)c\)\(d_r\). As a result, target angle resolution is limited by the size of the receive array. If target angles are missed due to low resolution, there will be errors in the subsequent estimation of range and Doppler parameters. In order to address this problem, we propose an iterative estimation approach, which is capable of refining the angle estimates and then re-estimating the range/Doppler parameters by leveraging private subcarriers.

Let us reserve a subset of \( N_x \) subcarriers, denoted by \( I \), to be used as private subcarriers by the active antennas. Each private subcarrier \( i \) is uniquely assigned to an antenna \( n_i \in N \), and contains only one source data symbol from the corresponding data stream. Let there be no precoding on the private subcarriers, i.e., \( P = I \) for \( i \in I \), and the sole data symbol on the private subcarrier \( i \) be denoted by \( \tilde{Q}(n_i, i) \). Due to antenna selection, \( Q(n_i, i) \) will give rise to a transmitted symbol vector with only one non-zero element
\[ d_i = SP_{n_i}Q(n_i, i) = S_{n_i}Q(n_i, i), \]
(24)
on the \( i \)-th subcarrier, where \( P_{n_i} \) and \( S_{n_i} \) denote the \( n_i \)-th column of \( P \) and \( S \) respectively.

For a small number of RF fronts, \( N_x \) and a large number of antennas \( N_t \), the transmit snapshot along a private subcarrier will be \( 1\)-sparse, with the location of the non-zero entry corresponding to the transmit antenna. By controlling the pairing between antennas and subcarriers, the sparse pattern can be used to embed communication information [9] through the index modulation approach [20]. However, this is beyond the scope of our paper. For simplicity, here we let \( I = \{0, 1, \ldots, N_x - 1\} \) and set \( n_i = i \).

Let the transmit and receive steering vectors corresponding to the \( i \)-th subcarrier be
\[ a_t(\theta, i) = [1, e^{-j2\pi d_t \frac{\sin \theta}{\lambda_0}}, \ldots, e^{-j2\pi (N_t - 1)d_t \frac{\sin \theta}{\lambda_0}}]^T \]
(25)
\[ a_r(\theta, i) = [1, e^{-j2\pi d_r \frac{\sin \theta}{\lambda_0}}, \ldots, e^{-j2\pi (N_r - 1)d_r \frac{\sin \theta}{\lambda_0}}]^T \]
(26)
respectively, where \( \lambda_0 = \frac{L + i\Delta}{c} \). Assume that the private subcarriers are close to each other in frequency domain. Provided that the spacing between private subcarriers is much smaller as compared to \( f_c \), we can approximate \( \frac{L + i\Delta}{c} \approx \frac{L}{c} = \lambda_0 \). In the following, we will omit the subcarrier index in the symbol of the steering vector.

From (8), omitting the noise term and after performing element-wise division by the known data symbol that was transmitted on private subcarrier \( i \), the received symbol on the same subcarrier becomes
\[ d'_i(m, i, \mu) = \sum_{k=1}^{K} a_t^T(\theta_k) C(k) S_{n_i} e^{-j2\pi m d_r \frac{\sin \theta_k}{\lambda_0}} \times e^{-j2\pi i \Delta \frac{2R_u}{c}} e^{j2\pi i \mu f_d q_k}, \quad i \in I \]
(27)
where \( C(k) \) is a diagonal matrix with the channel coefficients of the \( k \)-th target on its diagonal
\[ C(k) \triangleq \text{diag}\{\beta_{0k}, \beta_{1k}, \ldots, \beta_{(N_r-1)k}\} \]
A. Refining angle estimation

Let us define \( \hat{\mathbf{a}}^T_k (\theta_k) \in \mathbb{C}^{1 \times N_s} \) as

\[
\hat{\mathbf{a}}^T_k (\theta_k) \triangleq \mathbf{a}^T_k (\theta_k) \mathbf{C}(k) \mathbf{S}
\]

(28)

By putting the symbols of (27) for all \( i \in \mathcal{I} \) in a column vector \( \mathbf{z}_m \in \mathbb{C}^{N_x \times 1} \) we have that

\[
\mathbf{z}_m = \sum_{k=1}^{N_k} e^{j2\pi \mu_m T_p f_{\lambda_0} e^{-j2\pi m d_r \sin \theta_k}} (\hat{\mathbf{a}}_i (\theta_k) \odot \mathbf{b}(R_k))
\]

(29)

where \( \odot \) denotes Hadamard product and

\[
\mathbf{b}(R_k) = \begin{bmatrix}
1 \\
\vdots \\
e^{-j2\pi(N_k-1)\Delta 2R_k}
\end{bmatrix}
\]

(30)

By stacking the resulted vectors from all the receive antennas into a long vector, we get

\[
\mathbf{z} = \sum_{k=1}^{N_k} e^{j2\pi \mu_m T_p f_{\lambda_0} \mathbf{a}_r (\theta_k) \odot (\hat{\mathbf{a}}_i (\theta_k) \odot \mathbf{b}(R_k))},
\]

(31)

where \( \otimes \) is the Kronecker product.

The above expression has a striking resemblance to a virtual array that can be formulated by a MIMO radar [42]. However, unlike the conventional virtual array that constructs a long ULA and thus achieves high target angle resolution, (31) does not correspond to a ULA, mainly due to the second term in the Kronecker product. Still, as target angle estimation for the conventional virtual array can be formulated as a sparse sensing recovery problem [43], (31) can also lead to targets angles via SSR methods, as explained below. In that sense, we will be referring to (31) as an effective virtual array. It turns out that the long effective virtual array results in high resolution angle estimation.

Let \( \{R_1, R_2, ..., R_N\} \) be the \( N \) previously estimated ranges. By discretizing the angle space on a grid of size \( N_a \), i.e., \( \{\hat{\theta}(1), ..., \hat{\theta}(N_a)\} \), (31) can be expressed as

\[
\begin{bmatrix}
\hat{\mathbf{z}}_{11} \\
\vdots \\
\hat{\mathbf{z}}_{N_a N_a}
\end{bmatrix} = \begin{bmatrix}
\hat{\beta}_{11} \\
\vdots \\
\hat{\beta}_{N_a N_a}
\end{bmatrix}
\]

(32)

where \( \hat{\beta}_{ij} \) is non zero if there is a target at range \( R_j \) and angle \( \hat{\theta}_i \), and

\[
\mathbf{z}_{ij} = \mathbf{a}_r (\hat{\theta}_i) \odot (\hat{\mathbf{a}}_i (\hat{\theta}_i) \odot \mathbf{b}(R_k))
\]

(33)

is the dictionary element for \( i = 1, 2, ..., N_a \) and \( j = 1, 2, ..., N \). The sparse vector \( \hat{\beta} \) can be estimated via \( \ell_1 \) norm minimization and its support will provide the target angle estimates, i.e., the angle space grid points that are the closest to the target angles.

B. Estimation of Doppler based on private subcarriers

Although Doppler estimation can be carried out via (18), an easier approach is to do the estimation based on the private subcarriers, which are free from coupling. Based on (27), the received symbols, viewed as function of \( \mu \), is a superposition of multiple complex sinusoids whose frequencies are functions of Doppler frequencies. After the same cross-correlation process, we find peaks appear at locations \( [N_p T_p f_{\lambda_0}] \), which reveal the target velocities.

C. Angle-range iterative estimation

Let us now revisit the range estimates of (17), which relied on angle estimate obtained based on the physical receive array. Now that we have obtained high resolution angle estimates, can we improve the range and Doppler estimates? As it will be demonstrated in the simulations section, indeed, we can. By evaluating (14)-(19) based on the high resolution angle estimates, we get better range and Doppler estimates. We can also use the private subcarriers to obtain the Doppler estimates as explained in the previous subsection.

The range estimate uses full bandwidth, and thus every range estimate has the maximum possible resolution. Also, based on the fact that the coarse angle estimation is conducted along the receive antenna index domain, while the range estimation along the subcarrier index domain, errors in the angle estimate do not affect the range resolution. However, if due to low angle resolution we end up missing targets, the corresponding ranges will also be missed. The improved angle estimates can reveal more angle bins occupied by targets, which in turn will lead to new range and Doppler estimates.

Based on the above observations, we propose an iterative angle-range estimation algorithm as summarized in Algorithm 1.

The iteration is initialized with the coarse angle estimates obtained from the physical array. Subsequently, in Step 2, the ranges are computed via (17). In step 3, using the private subcarriers and the previously obtained range estimates, an SSR problem is formulated based on (32). The solution of the SSR problem provides finer resolution angle estimates, based on which, refined range estimates can be obtained again via (17). Steps 3 and 4 are executed multiple times until no new targets are found.

We should note that the angle-range iterative estimation can be done using one OFDM symbol only, while \( N_p \) OFDM symbols are required to estimate the Doppler from the slow time domain. By the time the echoes of \( N_p \) OFDM symbols have been received, the occupied angle and range bins would have already been estimated. Then, we can estimate the Doppler frequencies within the occupied angle bins using the private subcarriers; in this way we avoid the coupling of channel coefficients from the multiple active antennas. If not all targets have been found after \( N_p \) OFDM symbols, due to the orthogonality between range and Doppler frequency, one can apply the iterative estimation algorithm on angle and Doppler to uncover the hidden targets.
Thus, taking an IDFT along complex sinusoids whose frequencies are functions of range. Subsequently, the coefficients of those sinusoids can be obtained by setting up a linear system of equations based on (34)

$$n_k, k = 1, \ldots, N_k.$$  

Revisiting (14), when antenna $n$ transmits on subcarrier $i$, and for the case in which there are multiple targets withing the estimated angles, we have that

$$\hat{A}(k, i, \mu) = \frac{A(k, i, \mu)}{|d(n, i, \mu)e^{-j2\pi nd_ki\sin \theta_k\Delta\mu}} = \sum_{q=1}^{N_x} \beta_{nq}e^{-j2\pi i\Delta\mu T_Tf_q}$$  

where $\beta_{nq}$ is the channel coefficients and $d(n, i, \mu)$ is the distance between the two systems. The corresponding coefficient is denoted as $\beta_{nq}^i$. The remaining tap delays are due to reflections of the radar transmissions and their contribution to the channel can be expressed as an additive term. The channel frequency response corresponding to the $i$-th subcarrier can be expressed as

$$\hat{H}_i = \mathbf{B}_i \odot (\mathbf{a}_i(\theta, i)\mathbf{a}_i^T(\phi, i)) + \sum_k c_k\mathbf{a}_k(\theta_k, i)\mathbf{a}_k^T(\phi_k, i)$$

where the $(\ell, n)$ element of $\mathbf{B}_i$ equals $\beta_{n\ell}^i e^{-j2\pi i\Delta R_c/c}$, and $\theta_k, \phi_k$ are angles of departure and incidence related to the various scatterers, and $c_k$ the corresponding coefficients.

Assuming that the channel spread is smaller than the CP length, and due to the narrow bandwidth of each subcarrier in the OFDM system, the effect from a frequency selective fading channel between transmit antenna and receive antenna can be mitigated.

Taking into account the fact that the transmit antennas share subcarriers, the symbols received on the $i$-th subcarrier across all receive antennas can be expressed as

$$r_i = \mathbf{H}_i\mathbf{d}_i + \mathbf{u}_i, \quad i = 0, \ldots, N_s - 1,$$

where $\mathbf{d}_i$ is the $i$-th column of $\mathbf{D}$ and $\mathbf{u}_i \in \mathbb{C}^{N_c \times 1}$ for $i = 0, \ldots, N_s - 1$ represents the measurement noise on the $i$-th carrier, which is assumed to be white, Gaussian with zero mean and covariance $\sigma^2 I$. Since only $N_x$ antennas are active, only $N_x$ entries in $\mathbf{d}_i$ are nonzero. In a given OFDM symbol, $\mu$, all $\mathbf{d}_i$ for $i = 0, \ldots, N_s - 1$ share the same sparsity pattern.

If the communication receive array has more antennas than the radar transmit array, i.e., $N_c > N_t$, $\mathbf{d}_i$ can be recovered by seeking the $\mathbf{d}$ with the minimum $\ell_1$-norm that gives rise to $r_i$, i.e.,

$$\min \|\mathbf{d}\|_1$$

s.t. $\|r_i - \mathbf{H}\mathbf{d}\|^2_2 \leq N_r\sigma^2_c$

The above assumes that the receiver has knowledge of the communication channel, which can be obtained via the transmission of pilots. Based on the support of the estimated $\mathbf{d}$, one can recover $\mathbf{S}$; this is because the location of active antennas determines the nonzero entries in $\mathbf{S}$. Noting that $\mathbf{S}^H\mathbf{S} = \mathbf{I}_{N_c}$, and based on (2) it holds that $\mathbf{S}^H\mathbf{D} = \mathbf{PQ}$.

### Algorithm 1: Angle-Range iterative estimation

**Step 1:** Obtain coarse angle estimates via (13)

**Step 2:** Obtain range estimates via (17) within the occupied angle bins found in Step 1

**do**

**Step 3:** Formulate SSR problem based on estimated ranges via (32), and refine angle estimates

**Step 4:** Refine range estimates corresponding to angle bins from Step 3

**until** no changes are made

### IV. ESTIMATION OF CHANNEL COEFFICIENTS

Here, we propose a scheme to estimate the channel coefficients $\beta_{nk}$. Suppose that periodically we allow each antenna to transmit on a subgroup of subcarriers in an exclusive fashion. Estimates of target angles can be obtained along the lines of Section II-B, by taking a DFT across receive antennas. This step does not require knowledge of channel coefficients. Also, the Doppler frequencies can be obtained based on the private subcarriers, as explained in Section III-B.

Revisiting (14), when antenna $n$ transmits on subcarrier $i$, and for the case in which there are multiple targets withing the estimated angles, we have that

$$\hat{A}(k, i, \mu) = \frac{A(k, i, \mu)}{|d(n, i, \mu)e^{-j2\pi nd_ki\sin \theta_k\Delta\mu}} = \sum_{q=1}^{N_x} \beta_{nq}e^{-j2\pi i\Delta\mu T_Tf_q}$$

Viewing $\hat{A}(k, i, \mu)$ as function of $i$, we have a sum of complex sinusoids whose frequencies are functions of range. Thus, taking an IDFT along $i$, we will see peaks at $\left[\frac{2N_c R_k\Delta}{c}\right]$. Subsequently, the coefficients of those sinusoids can be obtained by setting up a linear system of equations based on (34) for different $i$'s. Those coefficients will contain the Doppler terms and the channel coefficients, from where, given that the Doppler coefficients are known, we can compute the coefficients $\beta_{nk}, k = 1, \ldots, N_k$. Note that the coefficient estimation is based on the range estimation, and range’s resolution is only decided by the bandwidth (20). So, when we estimate the channel coefficients within a subgroup of subcarriers, the subcarriers should be assigned to each subgroup evenly such that the bandwidth of each subgroup is close to the whole available bandwidth. For instance, the indices of subcarriers in the first subgroup could be $\{0, N_x, 2N_x, \ldots\}$.  

### V. COMMUNICATION SYSTEM MODEL

Let us consider a communication receiver with $N_c$ antennas, spaced apart by $d_c^x$. The $i$-th subcarrier, between the $n$-th transmit antennas and the $\ell$-th receive antenna, undergoes an effect that can be modeled as propagation through a multiple tap delay channel. The first delay $\tau_{nk}^i$ is due to the direct path from the radar transmitters, i.e.,

$$\tau_{nk}^i = R_c/c + nd_c^x\sin \theta_i + nd_c^x\sin \phi_i/\lambda_i,$$  

where $\theta$ is the angle of the communication system from the point of view of the radar, $\phi$ is the angle of the radar from the point of view of the communication system, and $R_c$ is the distance between the two systems. The corresponding coefficient is denoted as $\beta_{nk}^i$. The remaining tap delays are due to reflections of the radar transmissions and their contribution to the channel can be expressed as an additive term. The channel frequency response corresponding to the $i$-th subcarrier can be expressed as

$$\hat{H}_i = \mathbf{B}_i \odot (\mathbf{a}_i(\theta, i)\mathbf{a}_i^T(\phi, i)) + \sum_k c_k\mathbf{a}_k(\theta_k, i)\mathbf{a}_k^T(\phi_k, i)$$

where the $(\ell, n)$ element of $\mathbf{B}_i$ equals $\beta_{n\ell}^i e^{-j2\pi i\Delta R_c/c}$, and $\theta_k, \phi_k$ are angles of departure and incidence related to the various scatterers, and $c_k$ the corresponding coefficients.

Assuming that the channel spread is smaller than the CP length, and due to the narrow bandwidth of each subcarrier in the OFDM system, the effect from a frequency selective fading channel between transmit antenna and receive antenna can be mitigated.

Taking into account the fact that the transmit antennas share subcarriers, the symbols received on the $i$-th subcarrier across all receive antennas can be expressed as

$$r_i = \mathbf{H}_i\mathbf{d}_i + \mathbf{u}_i, \quad i = 0, \ldots, N_s - 1,$$

where $\mathbf{d}_i$ is the $i$-th column of $\mathbf{D}$ and $\mathbf{u}_i \in \mathbb{C}^{N_c \times 1}$ for $i = 0, \ldots, N_s - 1$ represents the measurement noise on the $i$-th carrier, which is assumed to be white, Gaussian with zero mean and covariance $\sigma^2 I$. Since only $N_x$ antennas are active, only $N_x$ entries in $\mathbf{d}_i$ are nonzero. In a given OFDM symbol, $\mu$, all $\mathbf{d}_i$ for $i = 0, \ldots, N_s - 1$ share the same sparsity pattern.

If the communication receive array has more antennas than the radar transmit array, i.e., $N_c > N_t$, $\mathbf{d}_i$ can be recovered by seeking the $\mathbf{d}$ with the minimum $\ell_1$-norm that gives rise to $r_i$, i.e.,

$$\min \|\mathbf{d}\|_1$$

s.t. $\|r_i - \mathbf{H}\mathbf{d}\|^2_2 \leq N_r\sigma^2_c$

The above assumes that the receiver has knowledge of the communication channel, which can be obtained via the transmission of pilots. Based on the support of the estimated $\mathbf{d}$, one can recover $\mathbf{S}$; this is because the location of active antennas determines the nonzero entries in $\mathbf{S}$. Noting that $\mathbf{S}^H\mathbf{S} = \mathbf{I}_{N_c}$, and based on (2) it holds that $\mathbf{S}^H\mathbf{D} = \mathbf{PQ}$. Assuming that the communication receiver knows the precoding matrix and the constellation diagram used at the transmitter, the decoding of the estimated symbol vector can be achieved by looking for

| Name                        | Notation | Size       |
|-----------------------------|----------|------------|
| Data symbols                | $Q$      | $N_x \times N_x$ |
| Precoding matrix            | $P$      | $N_x \times N_c$ |
| Selection matrix            | $S$      | $N_x \times N_x$ |
| Transmitted symbols         | $D$      | $N_t \times N_c$ |
| Channel matrix              | $H$      | $N_c \times N_c \times N_t$ |
| Communication received symbols | $R$   | $N_c \times N_x$ |
precoded combination of data symbols that is the closests to \( S^H \hat{d} \), i.e.,

\[
\arg \min_{\mathbf{q}_s} \| S^H \hat{d} - \mathbf{P} \mathbf{q}_s \|^2_2
\]

subject to \( \mathbf{q}_s \in \mathcal{Q} \) 

where \( \mathcal{Q} \) denotes the set containing all the possible combinations of data symbol vectors. If \( \mathcal{Q} \) is the cardinality of \( \mathcal{Q} \), the complexity of the above step is \( O(2^{N_s}) \), since there are in total \( 2^{N_s} \) possible combinations.

After the decoding process, the original information can be extracted from the recovered data symbols. By applying the same process to every subcarrier and every OFDM symbol, all transmitted symbols can be recovered. Compared with an OFDM communication system with the same modulation scheme but without subcarrier sharing, the proposed scheme increases the number of information bits transmitted in one period by a factor of up to \( N_s \).

VI. THE DESIGN PROBLEM

A. Beampattern

Let \( \mathbf{F} \in \mathbb{C}^{N_s \times N_s} \) denote the inverse Fourier transform matrix. The baseband transmitted signal corresponding to the \( \mu \)-th OFDM symbol can be expressed in matrix form as

\[
\mathbf{X} = \mathbf{DF} = \mathbf{SPDF}. \quad (40)
\]

The \( j \)-th column of the baseband signal matrix, or otherwise, the \( j \)-th snapshot, equals

\[
x_j = \sum_{i=0}^{N_s-1} \mathbf{SPd}_i \mathbf{F}(i, j). \quad (41)
\]

where \( \mathbf{F}(i, j) \) is the element on the \( i \)-th row and \( j \)-th column of \( \mathbf{F} \). The sparse array output for direction \( \theta \) at the \( j \)-th snapshot is

\[
y_j(\theta) = \sum_{i=0}^{N_s-1} \mathbf{a}^T(\theta, i) \mathbf{SPd}_i \mathbf{F}(i, j). \quad (42)
\]

The transmitted power of the sparse array at direction \( \theta \) equals

\[
\hat{p}(\theta) = \mathbb{E}\{y_j(\theta)\bar{y}_j(\theta)^*\} = \frac{1}{N_s} \sum_{j=0}^{N_s-1} y_j(\theta)\bar{y}_j(\theta)^* = \frac{1}{N_s} \sum_{i=0}^{N_s-1} \sum_{j=0}^{N_s-1} \mathbf{a}^T(\theta, i) \mathbf{SPd}_i \mathbf{F}(i, j) \mathbf{F}^H(i, j) \mathbf{d}^H_i \mathbf{P}^H \mathbf{S}^H \mathbf{R}_i \mathbf{a}^*_i(\theta, i) \quad (43)
\]

where \( \mathbf{a}^T(\theta, i) \) is the transmit steering vector corresponding to the \( i \)-th subcarrier, \( \sum_{j=0}^{N_s-1} \mathbf{F}(i, j) \mathbf{F}^H(i, j) = 1 \) for \( i = 0, 1, \ldots, N_s - 1 \) and \( \mathbf{R}_i = \mathbf{q}_i \mathbf{q}^H_i \) can be viewed as the covariance matrix of the \( i \)-th original symbol vector \( \mathbf{q}_i \) (the \( i \)-th column of matrix \( \mathcal{Q} \)), which is white.

B. SNR at the communication receiver

From (37), the power of received signal from all \( N_s \) subcarriers in the \( \mu \)-th OFDM symbol can be expressed as

\[
\mathbf{P}_r = E\{\mathbf{tr}[\mathbf{r}_\mu \mathbf{r}_\mu^H]\} = \frac{1}{N_s} \sum_{i=0}^{N_s-1} \mathbf{tr}[\mathbf{H}_i \mathbf{d}_i \mathbf{d}_i^H \mathbf{H}_i^H], \quad (44)
\]

\[
= \frac{1}{N_s} \sum_{i=0}^{N_s-1} \mathbf{tr}[\mathbf{H}_i \mathbf{SPR}_i \mathbf{P}^H \mathbf{S}^H \mathbf{H}_i^H]. \quad (45)
\]

where \( \mathbf{tr}[\cdot] \) refers to the trace of a matrix. Recall that the power of communication noise is \( \sigma_r^2 \); thus, the SNR at the communication receiver equals

\[
\text{SNR} = \frac{\sum_{i=0}^{N_s-1} \mathbf{tr}[\mathbf{H}_i \mathbf{SPR}_i \mathbf{P}^H \mathbf{S}^H \mathbf{H}_i^H]}{N_s \sigma_r^2}. \quad (46)
\]

The selection matrix, by selecting transmit antennas, effectively selects the channels corresponding to those antennas, and thus plays an important role in the communication receiver SNR.

C. The sensing-communications co-design problem

Let \( p(\theta) \) be the desired power of the transmitted signal towards direction \( \theta \). The beampattern error with respect to the desired beampattern \( \hat{p}(\theta) \), equals

\[
\sum_{i=1}^{G} \| p(\theta_i) - \hat{p}(\theta_i) \|^2 = \| \mathbf{p} - \sum_{i=0}^{N_s-1} \text{diag}\{ \mathbf{A}^T_i \mathbf{SPR}_i \mathbf{P}^H \mathbf{S}^H \mathbf{A}^*_i \} \|_2^2 \quad (47)
\]

where \( \theta_i \) are discretized angles in \([-\pi/2, \pi/2]\) on a grid of \( G \) points. \( \mathbf{A}_i = [\mathbf{a}_i(\theta_1, i), \ldots, \mathbf{a}_i(\theta_G, i)] \in \mathbb{C}^{N_s \times G} \) is the transmit steering matrix on the \( i \)-th subcarrier.

Due to fading, the signal that will be received by the communication receiver depends on the channels, which in turn depend on the radar transmit antennas. In the following, we propose a scheme for selecting \( \mathbf{S} \) and \( \mathbf{P} \) so that we optimize a weighted combination of the array beampattern error and the SNR at the communication receiver. The selection method is based on the softmax learning approach of [41], referred to as Learn-to-Select (L2S), applied to the following loss function:

\[
\mathcal{L}(\mathbf{S}, \mathbf{P}) \triangleq \alpha_b \sum_{i=1}^{G} \gamma_i |p(\theta_i) - \hat{p}(\theta_i)|^2 + \alpha_s \| \mathbf{S}^T \mathbf{S} - \mathbf{I} \|_F^2
\]

\[
+ \alpha_{\text{snr}} \log_{10} \left( \frac{N_s \sigma_r^2}{\sum_{i=0}^{N_s-1} \mathbf{tr}[\mathbf{H}_i \mathbf{SPR}_i \mathbf{P}^H \mathbf{S}^H \mathbf{H}_i^H]} \right). \quad (48)
\]

where \( \alpha_b, \alpha_s \) and \( \alpha_{\text{snr}} \) are cost parameters, respectively reflecting the relative importance of the beampattern error (first term), a constraint required for \( \mathbf{S} \) to be a selection matrix (second term), and the inverse SNR (third term); \( \gamma_i \) are weights that control the importance of beampattern error from the \( i \)-th angle.
Each column of selection matrix $S$ is modeled by an independent softmax neural network [44]. The outputs of the $j$-th network equals

$$s_{ij} = \frac{\exp(w_{ij}^T x + b_j)}{\sum_{k=1}^{N_t} \exp(w_{kj}^T x + b_k)}, \quad i = 1, \ldots, N_t$$

(49)

where $w_{ij}$, $b_j$ are respectively weights and biases, and $x$ is the input. Note that $0 \leq s_{ij} \leq 1$ and $\sum_{i=1}^{N_t} s_{ij} = 1$, $\forall j \in \{1, \ldots, N_x\}$, essentially, $s_{ij}$ represents the probability that antenna $i$ will be our $j$-th selected antenna. The flow graph of L2S is shown in Fig. 2.

Since the selection matrix does not depend on time $t$, the input $x$ should be constant, and thus, the constant value $b'_j = w_{ij}^T x$ can be merged into the biases term $b_j$. Without loss of generality, such a model is equivalent to a softmax model with $x = 0$ where the only trainable parameters are the biases.

The approximated selection matrix is formed based on the outputs $s_j = [s_{1j}, \ldots, s_{N_tj}]^T$ of all the softmax models as its columns. Clearly, this will be a soft selection matrix since the values $s_{ij}$ range between 0 and 1. By the end of the training, the selection matrix should converge very close to hard binary values, i.e., the entries of it are either 0 or 1 so to approximate the defined selection matrix $S$. In order to achieve such a solution, the following constraint is added to enforce the softmax models to produce hard binary values:

$$\sum_{i=1}^{N_t} s_{ij}^2 = 1, \quad \forall j \in \{1, \ldots, N_x\}$$

We also impose another constraint on so that the same antenna cannot be selected more than once, i.e.,

$$s_{ij} = 1 \Rightarrow s_{ik} = 0, \quad \forall j \neq k$$

Combining both aforementioned constraints it follows that $S^T S$ must be equal to the identity matrix $I_{N_x}$, which explains the second term in the loss function of (48).

In total, there are two sets of parameters to be trained: (i) the biases $b_i$ to approximate the selection on the $i$-th antenna and (ii) the precoding matrix $P$. Along the lines of [41], use a two-stage alternating optimization approach to first optimize over $b_i$ while fixing $P$ and then fixing $Q$ while optimizing over $b_i$. The algorithm runs for $N_{\text{epoch}}$ learning epochs and each alternating stage runs for a small number of steps $N_{\text{step}}$. The L2S iteration is shown in Algorithm 2. In order to improve speed of convergence, other optimization algorithms instead of standard gradient descent (GD) can be used, for example, the Adam optimizer [45] is used in our simulations which has superior performance as compared to GD.

**Algorithm 2:** Learn to select.

```
\textbf{for} epoch=1 to $N_{\text{epochs}}$ \textbf{do}
\hspace{1cm} Fix $P$ and optimize $\mathcal{L}(S, P)$ w.r.t. $b_i$:
\hspace{1cm} \textbf{for} step=1 to $N_{\text{steps}}$ \textbf{do}
\hspace{1.5cm} Update $b_i$, $i = 1, \ldots, N_x$
\hspace{1cm} Fix $b_i$ and optimize $\mathcal{L}(S, P)$ w.r.t. $P$:
\hspace{1cm} \textbf{for} step=1 to $N_{\text{steps}}$ \textbf{do}
\hspace{2.5cm} Update $P$
\textbf{end}
```

In comparison to other machine learning based antennas selection methods [38]–[40], L2S is flexible enough to accommodate any loss function, different constraints, and offers significant computational savings.

**D. Resolving the permutation ambiguity in the estimated antenna selection matrix estimate**

Recall that the selection matrix is approximated by a stack of softmax neural networks, and each neural network runs in parallel. Therefore, the selection matrix $S$ is found within a column permutation ambiguity. Such column-permutated selection matrix will change the order of symbols. For example, if we exchange the first and second column of the selection matrix in (3), then the first active antenna will transmit the data from the last RF chain and the second active antenna will connect with the first RF chain. Thus, via SSR on (38), we estimate the row-permuted transmitted symbols. In the subsequent decoding process, the permutation will be an obstacle.

Here, we show how to address this issue by leveraging the private subcarriers. Assume that the communication receiver has prior knowledge on the indices of private subcarriers, i.e., $I$. Then, the location of the non-zero element in the recovered transmitted data symbol vector from (38) is exactly the location of the active antenna paired to that private subcarrier and (24) becomes $d_i = s_{ni} Q(n_i, i)$. By going through all the private subcarriers, we can recover the indices of all the active antennas. After eliminating the data symbol $Q(n_i, i)$ and stacking the remaining vectors in a long vector, the correct order in the columns of $S$ can be redetermined by matching the indices of private subcarriers and the known set $\mathcal{N}$. Note that the change of an identity precoding matrix on private subcarriers will not disturb the radar processing proposed before since the target estimation is only based on the transmitted symbol and received symbols.

While the use of private subcarriers enables higher angular resolution for sensing, and also helps the communication system to reconstruct the selection matrix, it comes at a cost of reduced number of shared subcarriers. The corresponding loss in the communication bit rate is

$$\text{Loss} = \frac{N_x - 1}{N_x} \times 100\%.$$ 

(50)
If \( N_s >> N_x \), the loss from private subcarriers is small as compared to the aforementioned advantages.

VII. NUMERICAL RESULTS

In this section, we demonstrate the sensing and communication performance of the proposed DFRC system via simulations. The data symbols were modulated by quadrature phase shift keying (QPSK). The system parameters are shown in Table II. Similar parameters were used in several studies [11], [17], [18]. The channels were simulated based on (36), and the coefficients, \( \beta_{nk} \) were taken to be complex random numbers with 0.1 mean and variance equal to 0.01.

| Parameter                      | Symbol | Value          |
|--------------------------------|--------|----------------|
| Center frequency               | \( f_c \) | 24GHz          |
| Subcarrier spacing             | \( \Delta \) | 0.25MHz        |
| Duration of OFDM symbol        | \( T_p \) | 5\( \mu \)s    |
| Number of subcarriers          | \( N_s \) | 512            |
| Number of OFDM symbols         | \( N_f \) | 256            |
| Total number of transmit antennas | \( N_t \) | 32             |
| Number of activated antennas   | \( N_a \) | 16             |
| Number of radar receive antennas | \( N_r \) | 64             |
| Number of communication receive antennas | \( N_c \) | 64             |
| Receive antenna spacing distance | \( d_r \) | 0.5\( \lambda \) |
| Transmit antenna spacing distance | \( d_t \) | 0.5\( \lambda \) |

A. Beampattern design

L2S was applied to iteratively optimize the total loss \( \mathcal{L} \) w.r.t to the selection matrix \( S \) and the precoding matrix \( P \) (see Algorithm 2). In order to improve convergence, the Adam stochastic optimization procedure was used, with different learning rates, starting at 0.02, and \( N_{epoch} = 400 \) epochs of training. A large number of epochs were used to ensure a stable and balanced result. In each epoch \( N_{step} = 5 \) steps were executed. First, we selected 16 out of 32 antennas and optimized with respect to the precoding matrix, so that the resulting beampattern approximated the desired one, and the SNR at the communication receiver was maximized. The weights in the total loss function of (48) were take as \( \alpha_b = 1 \), \( \alpha_s = 4 \times 10^6 \) and \( \alpha_{snr} = 1 \times 10^5 \). The biases of the softmax neural networks, \( b_i \) were initialized to 1/32, so that all the antennas had the same probability to be chosen.

In the training process, the desirable beam power profile was 0 everywhere except over the angle range \([-52, -37] \) degrees, corresponding to the region of interest for the radar, and over the angle range \([29, 31] \) degrees, corresponding to the region where the communication receiver is; over those angles ranges the beam profile was set to 1.

Fig. 3, shows the designed beampattern when selecting \( N_x = 16 \) out of \( N_t = 32 \) antennas, and Fig. 4 shows the locations of selected antennas. Fig. 5 shows the result of the softmax selection neural networks after convergence, where one can see that the probabilities converge to hard binary values, and thus only one antenna in each model has high probability to be selected.

Fig. 6 shows the changes of SNR during the iteration with respect to the initial value. One can see that the SNR gain first
Fig. 7. The learning curve of L2S.

grows rapidly in the first 10 epochs, then increases relative gently to the peak level. Next, the SNR gain drops slowly and reaches convergence. The rapid increase in SNR gain is due to the large weights, $\alpha_{\text{snr}}$, in the loss term. The subsequent drop in SNR is due to the optimization with respect to the precoding and selection matrices, which aim to reduce the beampattern error. In general, when the beampattern error dominates in the total loss, the model will optimize with respect to $P$ and $S$ to reduce it. But once the beampattern error becomes smaller than the loss due to SNR, the model chooses to optimize $P$ and $S$ to improve the communication performance at the cost of increasing the beampattern error. The corresponding learning curve is shown in Fig. 7. The jump seen at 20 epochs is due to the model trying to maintain a balance between the beampattern error and SNR.

L2S was used here because of its simplicity as compared to other classification-based machine learning methods [38]–[40]. To get an idea of the computational savings achieved with using L2S, choosing 16 out of 32 antennas via a classification-based method would result in selecting one from $6 \times 10^6$ combinations, and for each choice we would need to optimize with respect to the precoding matrix. This would take an unacceptably long time to compute. On the other hand, on a 2019 Apple Macbook Pro with a 2.3 GHz 8-Core Intel Core i9 processor, training an L2S model to solve the same problem took 6 minutes only.

Table III

| Angles | Ranges | Velocities |
|--------|--------|------------|
| $-42^\circ$ | 50m | 20m/s |
| $-42^\circ$ | 80m | 8m/s |
| $-45^\circ$ | 100m | 13m/s |
| $-48^\circ$ | 45m | $-10$ m/s |

B. Sensing performance

Let us considered 4 point targets in the far field of the radar array, each characterized by its angle, range, velocity, with values as shown in Table III. The targets were all closely placed in the angle space, while two of them had the same angle but different range and velocity parameters. The SNR in the sensing experiments was set to 15dB. The coefficients of targets $\beta$ were assumed to be known.

Following Algorithm 1 of Section III-C, we first obtained a coarse estimate of the target angles, based on the physical

Algorithm 3: Learn to select (updated).

```python
for epoch=1 to $N_{\text{epochs}}$ do
    if the reduction of $\mathcal{L}(S, P)$ > threshold 1
        Fix $P$ and optimize $\mathcal{L}$ w.r.t. $b_i$:  
        if the reduction of $\mathcal{L}_s >$ threshold 2
            for step=1 to $N_{\text{steps}}$ do
                Update $b_i$, $i = 1, \ldots, N_x$
            else
                break
        Fix $b_i$ and optimize $\mathcal{L}$ w.r.t. $P$:
        if the reduction of $\mathcal{L}_s >$ threshold 3
            for step=1 to $N_{\text{steps}}$ do
                Update $P$
        else
            break
    else
        break
```

Fig. 8. Angle estimation based on shared subcarriers (blue), and refined estimation via the SSR method (red).
array of \( N_r = 64 \) antennas. As shown in Fig. 8 (blue line), the estimate had low angular resolution and returned one peak at \(-45.95^\circ\). Based on the location of that peak, and via (15)-(17), we obtained one range peak within that angle bin, which equals to \(99.61m\) (see Fig. 9).

Via (33), the estimated range was used to construct an effective virtual array, which led to target estimation via SSR methods. Fig. 10 shows how the result of \(\ell_1\)-norm minimization refines the angular resolution, returning 2 angles at \(-42^\circ\) and \(-45^\circ\).

By repeating the range estimation process with the refined angles, we could recover more targets (see Fig. 11). The re-estimated results were \((-42^\circ, 50.39m)\), \((-42^\circ, 79.69m)\), \((-45^\circ, 79.69m)\), and \((-45^\circ, 99.61m)\). Those results were very close to the ground truth, but we could continue using the estimated ranges to re-estimate angles and then re-estimate ranges until no more targets were revealed, or no changes were made. After a total of six stages of estimation, the final result obtained is shown in Fig. 12, where the initially missed target is now at the correct position of \((-48^\circ, 44.53m)\).

The proposed angle-range alternating estimation algorithm is very useful in dealing with the coupling of parameters and works well even with a coarse estimate initialization.

We should note that the above angle and range estimation and refinement process required one OFDM symbol only, while the estimation of velocity requires multiple OFDM symbols. According to the system parameters, the duration of each OFDM symbol is \(T_p = 5\mu s\) and the duration of in total \(N_p = 256\) OFDM symbol is \(1.28ms\), which is a short period of time, and with a high probability, the targets will remain in the same angle and range bin during that time. Based on (19), we can estimate the velocities of targets which are \(21.36m/s\), \(9.16m/s\), \(12.21m/s\) and \(-9.16m/s\) on the private subcarriers.

1) Cross-correlation based estimation of range and Doppler parameters: From (16) and (18), one can see that the use of cross-correlation is the same on the estimation of range and Doppler, thus we only take the cross-correlation based range estimation as an example. Note that, since the ranges are all positive numbers, only the non-negative \(x\)-axis is shown in the figures. In the above estimation process, after the first SSR estimation, two angles \(-42^\circ\) and \(-45^\circ\) were revealed. Fig. 13 shows the cross correlation result (see (17)) in the angle bin corresponding to \(-41.01^\circ\) and containing two targets. One can clearly see two narrow peaks at \(50.39m\) and \(79.69m\) and the ground truth is \(50m\) and \(80m\).

In (16) both the private subcarriers and shared ones are used in range estimation to enable a maximum range resolution.

2) The effect of channel coefficients: coarsely We should note that channel coefficients do not play a role for coarse angle estimation based on (13). Here, we examined the effect of channel coefficients on the cross-correlation based range estimation and SSR based refined angle estimation.

First, we show how the coefficients affect the target parameter estimation if they were ignored during the estimation process. This means that the data were synthesized based on channels that were random with mean equal to 0.1 mean and
variance equal to 0.01, as in the previous experiments, but during the estimation we assume that they are equal to 1.

In Fig. 14 we show the cross-correlation based range estimation results in the angle bin corresponding to $-45.95^\circ$. The SNR is again 15dB. In this case, the ground truth for the target range is 100m. From Fig. 14, one can see that the cross-correlation method still works; the highest peak location is in agreement with the ground truth, and the level of sidelobes is not significant.

Similarly, in Fig. 15, the cross-correlation based method succeeds in finding the target velocity, but a significant side-lobe shows up at $-9.16$ m/s. We should note that the velocity estimation in Fig. 15 is done based on one private subcarrier. One can find that the highest sidelobe both in Fig. 14 and Fig. 15 are from one of the targets, which is located at angle $-48^\circ$, range 45m and has velocity $-10$m/s. In theory, the range and Doppler estimation is completed within each angle bin and targets in different angle bins do not interfere each other. However here the information of other target distorts the velocity estimation result and leading one to infer that there are two targets with different velocities in that angle bin. This leakage of target information is due to the low resolution of DFT on receive array since the cross-correlation based estimation is based on the coefficients of the analyzed frequencies (see Section.II-C). One can increase the number of receive antennas to provide a higher resolution and finer angle bins, but this would require more cost. The SSR estimation stage, however, can address this leakage problem since it repairs the estimated ranges/velocities with the angles, thus the wrong estimated targets would be eliminated.

The SSR based angle estimation still works when we ignore the coefficients, as shown in Fig. 16, but the corresponding relative peak levels are changed by the unknown coefficients. In this separate SSR estimation, three previously estimated ranges 50.39m, 79.68m and 99.61m are used to formulate the basis matrix in (32). We should note that while the range of the fourth target, 44.53m, is not used, the SSR still finds the correct angle of that target.

3) Channel estimation: We also tested the estimation of the channel coefficients with Monte Carol simulations, based
on the method in Section IV. Since the system should be functional in both sensing and communication, the SNR range was taken from 0dB to 20dB.

The mean squared error (MSE) over all channels involved in all antennas, under different noise level is shown in Fig. 17, where one can see that the MSE of the estimated channel coefficients first drops as the noise level decreases, and then stabilizes after 10dB. This is due to the fact that the estimation of channel coefficients is based on the target estimation results, which contributes to the major part of the MSE.

Next, we used the estimated the channel coefficients to estimate the target parameters for SNR equal to 10dB. The corresponding range estimation result within the $-45 \leq \theta \leq 95$ degree bin, shown in Fig. 18, yields a very similar result as that of Fig. 14, where the channel has been ignored, except that the highest sidelobe level is reduced. As the angles do not depend on the channels, when using those range estimates in the angle-range iterative estimation algorithm, the obtained results are similar those when we ignore the channel coefficients.

Given that the channel estimation would require all the subcarriers to periodically work in private fashion, thus sacrificing bandwidth and bit rate, ignoring the channel during sensing is probably better than trying to estimate it.

C. Communication performance

We considered a communication receiver with $N_c = 64$ receive antennas. The channels are simulated following (36), where the communication receiver is at $30^\circ$ from the point of view of radar and the radar is at $-45^\circ$ to the communication receiver and the channel coefficients follow the same distribution as in the radar channel. The departure and incidence angles of scatters are random values ranging from $-90^\circ$ to $90^\circ$ and the corresponding coefficients $c_k$ are random complex numbers with 0.01 mean and variance equal to $1 \times 10^{-4}$. As mentioned in Section VI-D, the output of softmax selection models will permutate the columns of selection matrix $S$. Thus the communication receiver needs to first recover $S$ with the help of private subcarriers. We computed the probability of successful recovery of $S$ under different levels of SNR with Monte-Carlo simulations, and the results are given in Fig. 19. One can observe that the reconstruction of the column-permutated selection matrix is more robust to noise when fewer transmit antennas are active. The robustness to noise supports the subsequent recovery on source data symbols. Indeed, the use of private subcarriers not only enables the construction of a virtual array and the powerful angle-range alternating estimation algorithm for the radar system, but it also helps the communication system to resolved the permutation ambiguity of L2S.

Assuming that the channel is known at the communication receiver, and after the selection matrix $S$ has been estimated, the data symbols can be recovered and mapped back to binary bits via the QPSK demodulation.

In (39), we need to search all combinations of precoded symbols to decode the data, which involves high complexity. If the precoding matrix is full rank, this decoding process can be simplified by multiplying $S^H \hat{d}$ with $P^{-1}$. In the following results, we have checked and all the precoding matrices were full rank, and thus, we applied the easier decoding approach. In our future work, we will study how to guarantee a full rank precoding matrix from L2S.

The bit error rate (BER) of the proposed DFRC system

![Fig. 17. MSE of estimated channel coefficients versus SNR.](image1)

![Fig. 18. Cross-correlation based range estimation result with estimated channel coefficients.](image2)

![Fig. 19. Probability of reconstructing the selection matrix versus SNR.](image3)
under different SNRs with different number of active antennas is shown in Fig. 20. Although \( N_x = 16 \) will enable better beampattern performance, it will result in higher BER. Although the BER is almost the same when \( N_x = 8 \) and \( N_x = 12 \), the former selection has much higher probability for good reconstruction of the selection matrix and thus is a better choice in terms of the overall communication performance.

The SNR gain for different number of active antennas is plotted in Fig. 21. One can see that a sparser selection pattern gives a bigger SNR gain, since in that case there is more freedom to choose antennas that correspond to better communication channels.

Under the configuration provided in the table II, the maximum bit rate of the system is 1.638 Gigabits per second, while the loss on bit rate from the use of private subcarrier is 0.096 Gigabits per second. Given the advantages from the private subcarriers, such cost is acceptable.

**VIII. CONCLUSION**

We have proposed a wideband OFDM-MIMO DFRC system with sparse transmit array, where subcarrier sharing is employed to increase the communication bit rate. Our formulation has taken into account a frequency selective fading channel model both for the sensing and communication environments. Regarding the sensing functionality, it has been shown that a coarse angle estimate can be first obtained based on one snapshot of the receive array, and based on that, range information can be obtained within each occupied angle bin with maximum resolution via cross-correlation operations. Doppler information can be obtained based on private subcarriers. We have also shown that the obtained angles can be refined by exploiting an effective virtual array, synthesized based on a set of private subcarriers. With the refined angles, the range estimation can be further improved, which in return can further improve the angle estimation on private subcarriers. Based on that idea, we have proposed an iterative algorithm to improve both the angle and range estimation, which has been shown to work well, pairing successfully estimated ranges and angles. After receiving several OFDM symbols, the Doppler frequencies can be estimated on the private subcarriers.

Our formulation took into account frequency selective fading during sensing as well as communication operations, and we have proposed a method to estimate the channel coefficients that appear during the target estimation. However, we have shown that the channel does not significantly impact target estimation and can be ignored.

Regarding the communication functionality, we have shown that the sparsity of the transmit array can be exploited to facilitate communication data symbol recovery via \( \ell_1 \) minimization on the shared subcarriers. The permutation ambiguity problem has been addressed with the use of private subcarriers.

The sparse transmit array and the baseband transmit precoding matrix have been co-designed via a learning approach, aiming to approximate a desired probing pattern and maximize the SNR at the communication receiver simultaneously.

The functionality of our proposed DFRC system has been demonstrate via simulations. It has been shown that the proposed system has good sensing performance and sustains high communication rate.

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