ONE-PARAMETER FAMILIES OF OPERATORS IN $\mathbb{C}$

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Abstract. We introduce classes of one-parameter families (OPF) of operators on $C^\infty_\sigma(\mathbb{C})$ which characterize the behavior of operators associated to the $\bar{\partial}$-problem in the weighted space $L^q(\mathbb{C}, e^{-2p})$ where $p$ is a subharmonic, nonharmonic polynomial. We prove that an order 0 OPF operator extends to a bounded operator from $L^q(\mathbb{C})$ to itself, $1 < q < \infty$, with a bound that depends on $q$ and the degree of $p$ but not on the parameter $\alpha$ or the coefficients of $p$. Last, we show that there is a one-to-one correspondence given by the partial Fourier transform in $\alpha$ between OPF operators of order $m \leq 2$ and nonisotropic smoothing (NIS) operators of order $m \leq 2$ on polynomial models in $\mathbb{C}^2$.

1. Introduction.

The goal of this paper is to introduce classes of one-parameter families (OPF) of operators on $\mathbb{C}$ which characterize the behavior of kernels associated to the weighted $\bar{\partial}$-problem in $\mathbb{C}$. The need for OPF operators stems from problems associated to the inhomogeneous $\bar{\partial}$-equation on polynomial models in $\mathbb{C}^2$ and the $\bar{\partial}$-problem in weighted $L^2$ spaces in $\mathbb{C}$. A polynomial model $M$ is the boundary of an unbounded weakly pseudoconvex domain of finite type of the form $\{(z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_2 > p(z_1)\}$ where $p$ is a subharmonic, nonharmonic polynomial. $M \cong \mathbb{C} \times \mathbb{R}$ and $\bar{\partial}_b$, defined on $M$, can be identified with the vector field $\bar{L} = \frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$. Under the partial Fourier transform in $\alpha$ the vector field $\bar{L}$ becomes

$$Z_{\tau p} = \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial \bar{z}}$$

which we regard as a one-parameter family of differential operators acting on functions defined on $\mathbb{C}$. OPF operators will be defined so that $Z_{\tau p}$ and $Z_{\tau p} = -Z^*_{\tau p} = \frac{\partial}{\partial z} - \tau \frac{\partial}{\partial \bar{z}}$ are the natural differential operators under whose action OPF operators behave well.

When $\tau = 1$, the differential operator $\bar{Z}_p = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$ has been well studied [Chr91, Ber96, Rai05, Rai06]. Christ [Chr91] and the author [Rai05, Rai06] respectively cite the study of $\bar{\partial}$ on polynomial models as motivation to study the $\bar{\partial}$-problem on weighted $L^2$ in $\mathbb{C}$. In Section 1.1 we review the equivalence of the $\bar{\partial}$-problem in $L^2(\mathbb{C}, e^{-2p})$ with the $\bar{Z}_p$-problem, $\bar{Z}_p u = f$, in $L^2(\mathbb{C})$. When $p$ is a subharmonic function satisfying mild hypotheses on $\Delta p$, Christ [Chr91] solves the equation $\bar{Z}_p u = f$ on $L^2(\mathbb{C})$ via the complex Green operator $G_p$ for $\bar{Z}_p$ which is a subharmonic function satisfying mild hypotheses on $\Delta p$. $G_p$, both $G_p$ and the relative fundamental solution $Z_p G_p$ are given as fractional integral operators. Also, Christ shows that if $Y^\alpha$ is a product of length 2 of operators of the form $Y = \bar{Z}_p$ of $Z_p$, then $Y^\alpha G_p$ is bounded on $L^q(\mathbb{C})$, $1 < q < \infty$. When $\tau = 1$, $G_p$ serves as a model for an order 2 OPF operator, while $Y^\alpha G_p$ serves as model for an order 0 OPF operator. Christ and the author [Rai05, Rai06] find pointwise estimates of the integral kernel of $G_p$ and its derivatives (Christ in the case $\tau = 1$ and the author for $\tau > 0$), and the author [Rai06] finds cancellation conditions for $G_p$ and its derivatives when $\tau > 0$. Similarly to the ordinary Laplace operator, $\Delta_p$ is a second order, nonnegative elliptic operator, and there is a strong analogy between $G_p$ and the Newtonian potential $N$ on $\mathbb{C}$. Both invert “Laplace” operators, and if $D^2$ is a second order derivative, $D^2 N$ is a Calderon-Zygmund operator and bounded on $L^q$, $1 < q < \infty$. In Theorem 2.1 we will see that order 0 OPF operators are bounded in $L^q$, $1 < q < \infty$.

1.1. Connection of $\bar{Z}_{\tau p}$ with $\bar{\partial} u = f$ on weighted $L^2$. Hörmander’s work [Har65] on solving the inhomogeneous Cauchy-Riemann equations on pseudoconvex domains in $\mathbb{C}^n$. Hörmander’s methods, now classical in the subject [Har65], rely on proving that if $\text{diam}(\Omega) \leq 1$, there is a solution to $\bar{\partial} u = f$ satisfying in $L^2(\Omega, e^{-2p})$ satisfying the estimate $\int_{\Omega} |u|^2 e^{-2p} d\Omega \leq \int_{\Omega} |f|^2 e^{-2p} d\Omega$. Using the techniques of Hörmander,

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Fornæss and Sibony [FS01] generalize the $L^2$ estimate to an $L^q$ estimate, $1 < q \leq 2$, and prove that $\bar{\partial}u = f$ has a solution satisfying: \[
\left( \frac{1}{|\Omega|} \int_{\Omega} |u|^q e^{-2p} \, dz \right)^{\frac{1}{q}} \leq C_{p,T} \left( \frac{1}{|\Omega|} \int_{\Omega} |f|^q e^{-2p} \, dz \right)^{\frac{1}{q}}.
\] They also show that the estimate fails if $q > 2$. Berndtsson builds on the work of Fornæss and Sibony and shows an $L^q$-$L^1$ result. He shows that if diam $\Omega < 1$ and $1 < q < 2$, then $\bar{\partial}u = f$ has a solution so that \[\|ue^{-p}\|_{L^q(\Omega)} \leq C_q \|e^{-p}\|_{L^1(\Omega)}\].
Berndtsson also proves a weighted $L^\infty$-$L^q$ estimate when $q > 2$.

In Christ [Chr91], Christ recognizes that it is possible to study the $\bar{\partial}$-problem in $L^2(\mathbb{C}, e^{-2p})$ by working with a related operator in the unweighted space $L^2(\mathbb{C})$. If $\bar{\partial}u = \tilde{f}$ and both $\tilde{u} = e^pu$ and $\tilde{f} = e^pf$ are in $L^2(\mathbb{C}, e^{-2p})$, then \[\frac{\partial \tilde{u}}{\partial \overline{z}} = \tilde{f} \iff e^{-p}\frac{\partial}{\partial \overline{z}} e^p u = f\]. However, \[\frac{\partial}{\partial \overline{z}} u = \partial u / \partial \overline{z}\]. Consequently, the $\bar{\partial}$-problem on $L^2(\mathbb{C}, e^{-2p})$ is equivalent to the $\bar{\partial}$-problem, $\bar{\partial}u = f$, on $L^2(\mathbb{C})$. Berndtsson solves $\bar{\partial}u = f$ on smoothly bounded domains in $\mathbb{C}$ and views $\bar{\partial}$ from the viewpoint of mathematical physics. He writes $\bar{\partial}$ as a magnetic Schrödinger operator with an electric potential and his estimates follow from Kato’s inequality, a result from mathematical physics.

The author [Raic05] solves the heat equation associated to $\bar{\partial}$ and uses techniques both from mathematical physics and the solution of the $\bar{\partial}$-heat equation on polynomial models in $\mathbb{C}^2$.

1.2. The relationship between NIS and OPF operators. For computations involving $\bar{\partial}b$ on both polynomial models in $\mathbb{C}^2$ and the boundaries of other weakly pseudoconvex domains of finite type in $\mathbb{C}^n$, nonisotropic smoothing operators (NIS) operators have played a critical role in the analysis of the relative fundamental solutions of $\bar{\partial}$, and related operators. Nagel and Stein use properties of NIS operators in their analysis of the heat kernel on polynomial models in $\mathbb{C}^2$ [NS01] and both the relative fundamental solution of $\bar{\partial}$, and the Szegő kernel on product domains and decoupled domains in $\mathbb{C}^n$ [NS04] [NS].

A motivation for developing NIS operators is that the class of NIS operators have invariances that individual operators do not. NIS operators are invariant under translations and dilations, derivatives of NIS operators are again an NIS operators, and order 0 NIS operators have desirable mapping properties, namely $L^p$-boundedness [NS04].

In [NS01], Nagel and Stein solve the $\bar{\partial}_b$-heat equation $\frac{\partial u}{\partial s} + \bar{\partial}_b u = 0$ with initial condition $u(0, \alpha) = f(\alpha)\,$ where $s \in (0, \infty)$ and $\alpha \in \mathbb{C} \times \mathbb{R}$. They write their solution using the heat semigroup $e^{-s\bar{\partial}_b}$ and in turn express $e^{-s\bar{\partial}_b} f$ as integration against a kernel called the heat kernel. NIS operators are one of the workhorses of their arguments because as a class of operators, NIS operators (1) commute with vector fields $L$ and $L^*$, (2) remain invariant under translations and scaling, and (3) change products of arbitrary compositions of $L$ and $L^*$ to a composition of a power of $\bar{\partial}$ with a well-controlled NIS operator. The analogy of NIS operators with Calderón-Zygmund operators is strong. For example, (3) is analogous to writing an arbitrary derivative as the composition of $\Delta^k$ for some $k$ with a Riesz transform.

A goal for OPF operators is to play the analogous role for objects associated to the operators $\bar{\partial}_p$ and $\bar{\partial}_p w$, as NIS operators do to objects related to $\bar{\partial}_b$ and $\bar{\partial} w$ defined on the boundaries of weakly pseudoconvex domains in $\mathbb{C}^2$. In [Raic05] [Raic06], the author solves the $\bar{\partial}_p$-heat equation for $\tau \in \mathbb{R},$ i.e. he solves the equation $\frac{\partial u}{\partial \tau} + \bar{\partial}_p u = 0$ with initial condition $u(0, z) = f(z)$. The solution is written as integration against a kernel called the heat kernel which is shown to be smooth off the diagonal $\{(s, z, w) : s = 0\}$ and $z = w\}$. Also, the author finds pointwise decay estimates for the heat kernel and its derivatives. OPF operators play a fundamental role in these articles. They are an essential tool in the regularity arguments and the derivative estimates. Also, the ability to scale an OPF operator and stay within the class of OPF operators is crucial in the time decay estimate of the heat kernel $e^{-s\bar{\partial}_p}$.\n
2. Main Results.

Theorem 2.1. If $T_\tau$ is an OPF operator of order 0, then $T_\tau, T_\tau^*$ are bounded operators from $L^q(\mathbb{C})$ to $L^q(\mathbb{C}),$ $1 < q < \infty,$ with a constant independent of $\tau$ but depending on $q.$

Also, the classes of OPF operators fulfill the promise of being an analog to NIS operators. We can use results about OPF operators to study NIS operators and vice versa. We have the theorem:

Theorem 2.2. Given a subharmonic, nonharmonic polynomial $p : \mathbb{C} \to \mathbb{R},$ there is a one-to-one correspondence between OPF operators of order $m \leq 2$ with respect to $p$ and NIS operators of order $m \leq 2$ on the polynomial model $M^p = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im} z_2 = p(z_1)\}.$ The correspondence is given by a partial Fourier transform in $\mathrm{Re} z_2.$
3. Notation and Definitions.

3.1. Notation for Operators on \( \mathbb{C} \). For the remainder of the article, let \( p \) be a subharmonic, nonharmonic polynomial. It will be important for us to expand \( p \) around an arbitrary point \( z \in \mathbb{C} \), and we set:

\[
a_{jk}^z = \frac{1}{j!k!} \frac{\partial^{j+k}p}{\partial z^j \partial \bar{z}^k}(z). \tag{3.1}
\]

We need the following two “size” functions to write down the size and cancellation conditions for both OPF operators and NIS operators. Let

\[
\Lambda(z, \delta) = \sum_{j,k \geq 1} |a_{jk}^z| \delta^{j+k}\]

and

\[
\mu(z, \delta) = \inf_{j,k \geq 1} \frac{|\delta|^{j+k}}{|a_{jk}^z|}. \tag{3.2}
\]

It follows \( \mu(z, \delta) \) is an approximate inverse to \( \Lambda(z, \delta) \). This means that if \( \delta > 0 \),

\[
\mu(z, \Lambda(z, \delta)) \sim \delta \quad \text{and} \quad \Lambda(z, \mu(z, \delta)) \sim \delta. \tag{3.3}
\]

We use the notation \( a \lesssim b \) if \( a \leq Cb \) where \( C \) is a constant that may depend on the dimension 2 and the degree of \( p \). We say that \( a \sim b \) if \( a \lesssim b \) and \( b \lesssim a \).

\( \Lambda(z, \delta) \) and \( \mu(z, \delta) \) are geometric objects from the Carnot-Carathéodory geometry developed by Nagel et al. [NSWS86, Nag88]. The functions also arise in the analysis of magnetic Schrödinger operators with electric potentials [She96, She99, Kur00, Rai05, Rai06].

Denote the “twist” at \( w \), centered as \( z \) by

\[
T(w, z) = -2 \text{Im} \left( \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^j p}{\partial z^j}(z)(w - z)^j \right) = i \left( \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^j p}{\partial z^j}(z)(w - z)^j - \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^j p}{\partial \bar{z}^j}(\bar{z})(\bar{w} - \bar{z})^j \right). \tag{3.4}
\]

Also associated to a polynomial \( p \) and the parameter \( \tau \in \mathbb{R} \) are the weighted differential operators

\[
\tilde{Z}_{\tau, p, z} = \frac{\partial}{\partial z} + \tau \frac{\partial p}{\partial z} = e^{-\tau p} \frac{\partial}{\partial z} e^{\tau p} \quad \text{and} \quad Z_{\tau, p, z} = \frac{\partial}{\partial z} - \tau \frac{\partial p}{\partial z} = e^{\tau p} \frac{\partial}{\partial z} e^{-\tau p}. \tag{3.5}
\]

We need to establish notation for adjoints. If \( T \) is an operator (either bounded or closed and densely defined) on a Hilbert space with inner product \((\cdot, \cdot)\), let \( T^* \) be the Hilbert space adjoint of \( T \). This means that if \( f \in \text{Dom}(T) \) and \( g \in \text{Dom}(T^*), \) then \( (Tf, g) = (f, T^*g) \). The Hilbert spaces that arise in this paper are \( L^2(\mathbb{C}) \) and \( L^2(\mathbb{C} \times \mathbb{R}) \). Since the \( L^2 \)-adjoints of \( \tilde{Z}_{\tau, p} \) and \( Z_{\tau, p} \) are different than their adjoints in the sense of distributions, for clarity we let \( \overline{W}_{\tau, p} \) and \( W_{\tau, p} \) be the negative of the distributional adjoints of \( \tilde{Z}_{\tau, p} \) and \( Z_{\tau, p} \), respectively. Thus,

\[
\overline{W}_{\tau, p, w} = \frac{\partial}{\partial w} - \tau \frac{\partial p}{\partial w} = e^{\tau p} \frac{\partial}{\partial w} e^{-\tau p} \quad \text{and} \quad W_{\tau, p, w} = \frac{\partial}{\partial w} + \tau \frac{\partial p}{\partial w} = e^{-\tau p} \frac{\partial}{\partial w} e^{\tau p}. \tag{3.6}
\]

We think of \( \tau \) as fixed and the operators \( \tilde{Z}_{\tau, p, z}, Z_{\tau, p, z}, \overline{W}_{\tau, p, w}, \) and \( W_{\tau, p, w} \) as acting on functions defined on \( \mathbb{C} \). Also, we will omit the variables \( z \) and \( w \) from subscripts when the application is unambiguous. Observe that \( (\overline{Z}_{\tau, p}) = \overline{W}_{\tau, p} \) and \( (Z_{\tau, p}) = W_{\tau, p} \). Finally, let

\[
M_{\tau} = \frac{1}{2\pi i} e^{\tau T(w,z)} \frac{\partial}{\partial \tau} e^{-\tau T(w,z)}. \tag{3.7}
\]
3.2. Definition of OPF Operators. Let \( p \) be a subharmonic, nonharmonic polynomial. We say that \( T_\tau \) is a one-parameter family (OPF) of operators of order \( m \) with respect to the polynomial \( p \) if the following conditions hold:

(a) There is a function \( K_\tau \in C^\infty \left( \left( \mathbb{C} \times \mathbb{C} \right) \setminus \{ z = w \} \right) \times (\mathbb{R} \setminus \{ 0 \}) \) so that for fixed \( \tau \), \( K_\tau \) is a distributional kernel, i.e. if \( \varphi, \psi \in C^\infty_\text{c}(\mathbb{C}) \) and supp \( \varphi \cap \text{supp} \psi = \emptyset \), then \( T_\tau[\varphi] \in (C^\infty_\text{c})'(\mathbb{C}) \) and
\[
\langle T_\tau[\varphi](\cdot), \psi \rangle_\mathbb{C} = \int_{\mathbb{C} \times \mathbb{C}} K_\tau(z, w) \varphi(w) \psi(z) \, dw \, dz.
\]

(b) There exists a family of functions \( K_{\tau, \epsilon}(z, w) \in C^\infty(\mathbb{C} \times \mathbb{C} \times \mathbb{R}) \) so that if \( \varphi \in C^\infty_\text{c}(\mathbb{C} \times \mathbb{R}) \),
\[
K_{\tau, \epsilon}[\varphi]_{\mathbb{C} \times \mathbb{R}}(z, \tau) = \int_{\mathbb{C} \times \mathbb{R}} \varphi(w, \tau) K_{\tau, \epsilon}(z, w) \, dw \, d\tau
\]
and
\[\lim_{\epsilon \to 0} K_{\tau, \epsilon}[\varphi]_{\mathbb{C} \times \mathbb{R}}(z) = K_\tau[\varphi]_{\mathbb{C} \times \mathbb{R}}(z) \text{ in } (C^\infty_\text{c})'(\mathbb{C} \times \mathbb{R}).\]

All of the additional conditions are assumed to apply to the kernels \( K_{\tau, \epsilon}(z, w) \) uniformly in \( \epsilon \).

(c) Size Estimates. If \( Y_{\tau p}^J \) is a product of \( |J| \) operators of the form \( Y_{\tau p}^J = Z_{\tau p, z}, \check{Z}_{\tau p, z}, W_{\tau p, w}, \check{W}_{\tau p, w} \), or \( M_{\tau p} \) where \( |J| = \ell + n \) and \( n = \# \{ j : Y_{\tau p}^j = M_{\tau p} \} \), for any \( k \geq 0 \) there exists a constant \( C_{\ell, n, k} \) so that
\[
\left| Y_{\tau p}^J K_{\tau, \epsilon}(z, w) \right| \leq C_{\ell, n, k} \frac{|z - w|^{m - 2 - \ell}}{|\tau|^{n + k}} \Lambda(z, |w - z|)^k \quad \text{if} \quad \begin{cases} m < 2 \\ m = 2, k \geq 1 \\ m = 2, |w - z| > \mu(z, \frac{1}{\tau}) \end{cases}
\]
for \( |J| = \ell + n \) and \( n = \# \{ j : Y_{\tau p}^j = M_{\tau p} \} \), for any \( k \geq 0 \) there exists a constant \( C_{\ell, n, k} \) and \( N_\ell \) so that for any \( \varphi \in C^\infty_\text{c}(D(z_0, \delta)) \),
\[
\sup_{z \in \mathbb{C}} \left| \int_{\mathbb{C}} Y_{\tau p}^J K_{\tau, \epsilon}(z, w) \varphi(w) \, dw \right| \leq C_{\ell, n, k} \left\{ \begin{array}{ll} \delta \left( \log \left( \frac{2\mu(z, \frac{1}{\tau})}{|w - z|} \right) \right) \| \varphi \|_{L^\infty(\mathbb{C})} + \sum_{1 \leq |l| \leq N_\ell} \delta |l| \| X_{\tau p}^l \varphi \|_{L^\infty(\mathbb{C})} & \text{if } \delta < \mu(z, \frac{1}{\tau}) \\ \delta |l| \| X_{\tau p}^l \varphi \|_{L^\infty(\mathbb{C})} & \text{otherwise} \end{array} \right.
\]

(d) Cancellation in \( w \). If \( Y_{\tau p}^J \) is a product of \( |J| \) operators of the form \( Y_{\tau p}^J = Z_{\tau p, z}, \check{Z}_{\tau p, z}, W_{\tau p, w}, \check{W}_{\tau p, w} \), or \( M_{\tau p} \) where \( |J| = \ell + n \) and \( n = \# \{ j : Y_{\tau p}^j = M_{\tau p} \} \), for any \( k \geq 0 \) there exists a constant \( C_{\ell, n, k} \) and \( N_\ell \) so that for any \( \varphi \in C^\infty_\text{c}(D(z_0, \delta)) \),
\[
\left| M_{\tau p} K_{\tau, \epsilon}(z, w) \right| \leq C_n \left\{ \begin{array}{ll} \log \left( \frac{2\mu(z, \frac{1}{\tau})}{|w - z|} \right) & n = 0 \\ \left| \tau \right|^{-n} \left( \frac{2\mu(z, \frac{1}{\tau})}{|w - z|} \right) & n \geq 1 \end{array} \right.
\]
for \( |J| = \ell + n \) and \( n = \# \{ j : Y_{\tau p}^j = M_{\tau p} \} \), for any \( k \geq 0 \) there exists a constant \( C_{\ell, n, k} \) and \( N_\ell \) so that for any \( \varphi \in C^\infty_\text{c}(D(z_0, \delta)) \),
\[
\sup_{z \in \mathbb{C}} \left| \int_{\mathbb{C}} Y_{\tau p}^J K_{\tau, \epsilon}(z, w) \varphi(w) \, dw \right| \leq C_{\ell, n, k} \left\{ \begin{array}{ll} \delta \left( \log \left( \frac{2\mu(z, \frac{1}{\tau})}{|w - z|} \right) \right) \| \varphi \|_{L^\infty(\mathbb{C})} + \sum_{1 \leq |l| \leq N_\ell} \delta |l| \| X_{\tau p}^l \varphi \|_{L^\infty(\mathbb{C})} & \text{if } \delta < \mu(z, \frac{1}{\tau}) \\ \delta |l| \| X_{\tau p}^l \varphi \|_{L^\infty(\mathbb{C})} & \text{otherwise} \end{array} \right.
\]
where \( X_{\tau p}^l \) is composed solely of \( Z_{\tau p} \) and \( \check{Z}_{\tau p} \).

(e) Cancellation in \( \tau \). If \( X_{\tau p}^J \) is a product of \( |J| \) operators of the form \( X_{\tau p}^J = Z_{\tau p, z}, \check{Z}_{\tau p, z}, W_{\tau p, w}, \check{W}_{\tau p, w} \) and \( |J| = n \), there exists a constant \( C_n \) so that
\[
\int_{\mathbb{R}} X_{\tau p}^J \left( e^{i\tau t} K_{\tau, \epsilon}(z, w) \right) \, d\tau \leq C_n \frac{\mu(z, t + T(w, z))^{m - n}}{\mu(z, t + T(w, z))^2 |t + T(w, z)|}
\]
for \( |J| = n \) and \( \mu(z, t + T(w, z))^2 |t + T(w, z)| \).

(f) Adjoint. Properties (a)-(e) also hold for the adjoint operator \( T_\tau^* \) whose distribution kernel is given by \( K_{\tau, \epsilon}(w, z) \).

Note that for the \( \tau \)-cancellation condition \( c \), we do not need to consider the case \( X_{\tau p} = M_{\tau p} \) since \( \int_{\mathbb{R}} \frac{d}{d\tau} \left( e^{i\tau (t + T(w, z))} K_{\tau, \epsilon}(z, w) \right) \, d\tau = 0. \)

In the size condition (c) and cancellation condition (d), the \( \tau^k \Lambda(z, |z - w|)^k \) and \( \tau^k \Lambda(z, \delta)^k \) terms indicate rapid decay. If OPF operators are to be partial Fourier transforms of NIS operators on polynomial models, rapid decay should not be surprising; it is consequence of being able to integrate parts from the Fourier transform formula. This will be seen explicitly in Lemma 6.3. Ignoring the rapid decay terms, the size and cancellation conditions of OPF operators are familiar. An order 2 OPF operator should “invert” two derivatives, like the Newtonian potential. In \( \mathbb{R}^2 \), the Newtonian potential has a logarithmic blowup on the
diagonal, just like an order 2 OPF operator. For an order 0 OPF operator, the blowup on the diagonal is the same as a Calderón-Zygmund kernel, and the decay of \( K_{\tau}(0, z) \) is \( |z|^{-2} \), the same as a Calderón-Zygmund kernel. For the cancellation conditions, if \( \varphi \) is “normalized” appropriately, the cancellation condition simplifies to

\[
\|Y_{tp}^j T_{\tau}^j [\varphi]\|_{L^\infty(\mathbb{C})} \lesssim \delta^j.
\]

This is reminiscent of cancellation of a Calderón-Zygmund operator or an NIS operator.

3.3. Notation for Carnot-Carathéodory geometry and Vector Fields on \( \mathbb{C} \times \mathbb{R} \). In order to write down the definition of an NIS operator on a polynomial model in \( \mathbb{C}^2 \), we need to establish notation for the Carnot-Carathéodory metric \( \rho \) and corresponding balls \( B_{NI}(\zeta, \delta) \). If \( M^p \) is a polynomial model in \( \mathbb{C}^2 \) given by \( M^p = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Im} z_2 = p(z_1)\} \), then \( M^p \cong \mathbb{C} \times \mathbb{R} \). Under the isomorphism, a representation of the Carnot-Carathéodory metric is the nonisotropic pseudodistance \( \rho((z, t), (w, s)) = |z - w| + \mu(z, t - s + T(w, z)) \) where \( (z, t), (w, s) \in \mathbb{C} \times \mathbb{R} \). Since \( \rho((z, t), (w, s)) \) is a function of \( z, w \), and \( t - s \), we define a new function

\[
d_{NI}(z, w, t) = |z - w| + \mu(z, t + T(w, z)).
\]

We will see that \( d_{NI}(z, w, t) \) is essentially symmetric in \( (z, w) \). The nonisotropic ball

\[
B_{NI}(\zeta, \delta) = \{(w, s) : d_{NI}(z, w, t - s) < \delta\}.
\]

We also define a volume function

\[
V_{NI}(\zeta, t, (w, s)) = |B_{NI}(\zeta, t, d_{NI}(z, w, t - s))| \sim d_{NI}(z, w, t - s)^2 \Lambda(z, d_{NI}(z, w, t - s)).
\]

That the volume function is comparable to \( d_{NI}(z, w, t - s)^2 \Lambda(z, d_{NI}(z, w, t - s)) \) follows from \( \text{[3.10]} \).

If \( \tau \) is the transform variable of \( t \), observe that under the partial Fourier transform in \( t \), \( Z_{\tau p} \) and \( Z_{\tau p} \) map to the vector fields

\[
\tilde{L}_z = \frac{\partial}{\partial z} - \frac{i}{\rho} \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \tau}
\]

while \( W_{\tau p} \) and \( W_{\tau p} \) map to the vector fields

\[
\tilde{L}_w = \frac{\partial}{\partial w} + \frac{i}{\rho} \frac{\partial}{\partial v} \frac{\partial}{\partial \tau}.
\]

As we know from Section \( \text{[1]} \), \( \tilde{b}_\partial \) (defined on \( M \)) becomes the operator \( \tilde{L}_z \) on \( \mathbb{C} \times \mathbb{R} \). It follows that \( -L_z \) is the Hilbert space adjoint to \( \tilde{L}_z \) in \( L^2(\mathbb{C} \times \mathbb{R}) \). The translation invariance in \( t \) causes many operators of interest to have a convolution structure in \( t \). A consequence is that if we have a function \( \tilde{f}((z, t), (w, s)) = f(z, w, t - s) \), we may study \( f(z, w, t) \). By the chain rule, \( \tilde{L}_w \) and \( \tilde{L}_w \) are the versions of \( L_z \) and \( L_z \) in the \( w \)-variable. Finally, let

\[
\mathcal{M} = -i(t + T(w, z)).
\]

3.4. NIS operators on polynomial models in \( \mathbb{C}^2 \). There are different notions of NIS operators (e.g. \( \text{[NRSW89]} \)). We use the definition from \( \text{[NRSW89]} \).

**Definition 3.1** (Nonisotropic Smoothing Operator of order \( m \)). Let

\[
T[f](z, t) = \int_{\mathbb{C} \times \mathbb{R}} T((z, t), (w, s)) f(w, s) \, dwds,
\]

where \( T((z, t), (w, s)) \) is a distribution which is \( C^\infty \) away from the diagonal. We shall say that \( T \) is a nonisotropic smoothing operator which is smoothing of order \( m \) if there exists a family

\[
T_\epsilon[f](z, t) = \int_{\mathbb{C} \times \mathbb{R}} T_\epsilon((z, t), (w, s)) f(w, s) \, dwds,
\]

so that:

(a) \( T_\epsilon[f] \rightarrow T[f] \) in \( C^\infty(\mathbb{C} \times \mathbb{R}) \) as \( \epsilon \rightarrow 0 \) whenever \( f \in C^\infty_c(\mathbb{C} \times \mathbb{R}) \);

(b) Each \( T_\epsilon((z, t), (w, s)) \in C^\infty((\mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \times \mathbb{R})) \);

The following two conditions hold uniformly in \( \epsilon \):
Proposition 4.1. \( \mathcal{X}^J T_\epsilon((z, t), (w, s)) \leq c_{|J|} \frac{d_{NI}(z, w, t - s)^{m - |J|}}{V((z, t), (w, s))}; \) \( |I| = \ell; \)

Corollary 4.2. \( \lim_{\epsilon \to 0} \int_{B_N} \frac{\mathcal{X}^J T_\epsilon((z, t), (w, s))}{V((z, t), (w, s))} = \frac{1}{V((z, t), (w, s))} \int_{B_N} \mathcal{X}^J T((w, s), (z, t)) \).

4. Properties of \( T(w, z). \)

To prove Theorem 4.1. and Theorem 4.2. we need to understand the “twist” \( T(w, z) \) and how it behaves under differentiation.

Proposition 4.1. \( T(w, z) = -T(z, w). \)

Proof. Since \( p(z) = \sum_{j, k} \frac{1}{j!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)(z - w)^j(z - w)^k \), we have

\[
\frac{\partial^\ell p}{\partial z^\ell}(z) = \sum_{j \geq 0} \frac{j!}{(j - \ell)!} \frac{1}{j! \ell!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)(z - w)^j(z - w)^k.
\]

Since \( p \) is \( \mathbb{R} \)-valued, the twist [Equation 3.51] \( T(w, z) = -2 \text{Im} \left( \sum_{\ell \geq 0} \frac{1}{\ell!} \frac{\partial^\ell p}{\partial z^\ell}(z)(w - z)^\ell \right) \), so

\[
\sum_{\ell \geq 0} \frac{1}{\ell!} \frac{\partial^\ell p}{\partial z^\ell}(z)(w - z)^\ell = \sum_{\ell \geq 0} \frac{1}{\ell!} \left( \sum_{j \geq 0} \frac{j!}{(j - \ell)!} \frac{1}{j! \ell!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)(z - w)^j(z - w)^k \right) (w - z)^\ell
\]

\[
= \sum_{\ell \geq 0} \frac{1}{\ell!} \left( \sum_{j \geq 0} \frac{j!}{(j - \ell)!} \frac{1}{j! \ell!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)(z - w)^j(z - w)^k \right) (w - z)^\ell
\]

\[
= \sum_{\ell \geq 0} \frac{1}{\ell!} \frac{\partial^\ell p}{\partial z^\ell}(w)(z - w)^\ell = \sum_{\ell \geq 0} \frac{1}{\ell!} \frac{\partial^\ell p}{\partial z^\ell}(w)(z - w)^\ell.
\]

The second to last line uses the identity \( \sum_{j=0}^{\ell} \binom{j}{\ell} (-1)^j = \delta_0(\ell) \). The result follows easily. \( \square \)

Corollary 4.2. \( d_{NI}(z, w, t) \sim d_{NI}(w, z, t). \)

Proof. This is a well known fact (NagS5, NagS6), but we are in a situation where the computations can be explicit. We sketch a proof. If \( r = |t + T(w, z)| \), it follows from Proposition 4.1. that it is enough to show that

\[
|z - w| + \mu(z, r) \sim |z - w| + \mu(w, r).
\]

If \( \mu(z, r) < |z - w| \) and \( \mu(w, r) < |z - w| \), there is nothing to prove, so (without loss of generality) assume that \( \mu(z, r) > |z - w| \). By expanding \( p(z) \) around \( w \) and \( p(w) \) around \( z \), it can be shown that \( \Lambda(z, \delta) \sim \Lambda(w, \delta) \) if \( \delta > |w - z| \). Thus, we see

\[
\Lambda(w, \mu(z, r)) \sim \Lambda(z, \mu(z, r)) \sim r,
\]

and it follows that \( \mu(z, r) \sim \mu(w, r). \) \( \square \)

The next proposition contains two useful, though simple, computations.
Proposition 4.3.

\[ \frac{\partial T}{\partial z}(w, z) = -i \frac{\partial p}{\partial z}(z) - i \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(z)(w - \bar{z})^j \]

and

\[ \frac{\partial T}{\partial z}(w, z) = i \frac{\partial p}{\partial z}(z) + i \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(z)(w - \bar{z})^j. \]

Proof. The proof is a short computation.

\[
\frac{\partial T}{\partial z}(w, z) = i \left( \sum_{j=1}^{\deg(p)-1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(z)(w - \bar{z})^j - \sum_{j=1}^{\deg(p)} \frac{1}{(j-1)!} \frac{\partial^j p}{\partial z^j}(z)(w - \bar{z})^{j-1} - \sum_{j=1}^{\deg(p)-1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(z)(w - \bar{z})^j \right)
\]

since the first sum cancels all but the first term of the second sum. Since \( T \) is \( \mathbb{R} \)-valued, \( \frac{\partial T}{\partial z}(w, z) = \frac{\partial T}{\partial z}(w, z) \) which gives the result for the second sum.

A useful consequence of these calculations is

Proposition 4.4. Let \( Y^j \) be a product of \(|J| \) operators of the form \( Y^j = L_z, \bar{L}_z, L_w, \bar{L}_w \). Then

\[ |Y^j(t + T(w, z))| \leq C_{|J|} \Lambda(z, d_{N1}(z, w, t)) \frac{\Lambda(z, d_{N1}(z, w, t))}{d_{N1}(z, w, t)^{|J|}}. \]

Before we prove the Proposition 4.4, we note that the result would be false if we replaced \( t + T(w, z) \) with \( t \) or \( T(w, z) \). Without both terms, there would be uncontrolled derivatives of \( p \) remaining after applying \( Y^j \).

Proof. We have \( L_z(t + T(w, z)) = \frac{\partial T}{\partial z}(w, z) + i \frac{\partial p}{\partial z}(z) = -i \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(z)(w - \bar{z})^j \). Similarly, \( \bar{L}_z(t + T(w, z)) = i \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(w - \bar{z})^j \). Analogous equalities (with \( z \) and \( w \) interchanged and the sign switched) hold for \( L_w(t + T(w, z)) \) and \( \bar{L}_w(t + T(w, z)) \) since

\[
L_w(t + T(w, z)) = \left( \frac{\partial}{\partial w} - \frac{\partial p}{\partial w} \frac{\partial}{\partial t} \right)(t - T(z, w)) = -i \frac{\partial p}{\partial w}(w) - \frac{\partial T}{\partial w}(z, w)
\]

and \( \bar{L}_w(t + T(w, z)) = -\bar{L}_w(t + T(z, w)) \). But

\[
\left| \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z^j \partial \bar{z}}(w - \bar{z})^j \right| \leq c_1 \Lambda(z, d_{N1}(z, w, t)) \frac{\Lambda(z, d_{N1}(z, w, t))}{d_{N1}(z, w, t)^{|J|}}.
\]

Higher order derivatives are easier. As we just showed, the result of applying \( Y^j \) to \( t + T(w, z) \) leaves a polynomial that is a sum of derivatives of \( \Delta p \) (and hence well controlled). There are no \( t \) terms remaining, so if \( j \geq 2 \), applying \( Y^j \) is a matter of applying one of \( \frac{\partial}{\partial w}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{w}} \). Hence, the computation is simpler, and it can be done naively, i.e. there is no need to find any cancelling terms (which in general are absent). □

5. \( L^q \) Boundedness of Order 0 Operators.

We are now ready to begin the proof Theorem 2.1. The idea is to show that \( e^{-i \tau T(w, z)} K_{\tau, e} \) satisfies the bounds of a Calderon-Zygmund kernel and the operator \( S_\tau \) with kernel \( e^{-i \tau T(w, z)} K_{\tau, e} \) is restrictively bounded. These two facts, proven in Lemma 4.1 and Lemma 5.2 respectively, show \( S_\tau \) satisfy the hypotheses of \( T(1) \) theorem [Ste83]. Consequently, \( S_\tau \) is a bounded operator on \( L^q(\mathbb{C}) \). A result by Ricci and Stein [RSS7] applies to pass from \( L^q(\mathbb{C}) \) boundedness of \( S_\tau \) to \( L^q(\mathbb{C}) \) boundedness of \( T_\tau \).
Lemma 5.1. Let $T_\tau$ be an OPF operator of order $m \leq 2$ with a family of kernel approximating functions $K_{\tau,\epsilon}$. For $k \geq 0$, there exists $C_k$ independent of $\tau$ so that $K_{\tau,\epsilon}(z, w)$ satisfies:

(a)  
$$\left| \nabla_{z,w} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right| \leq C_k \frac{|w-z|^{m-3}}{|\tau|^k \Lambda(z, |w-z|)^k}$$  
(5.1)

(b) If $2|w-w'| \leq |w-z|$, then
$$\left| e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) - e^{-i\eta T(w',z)} K_{\tau,\epsilon}(z, w') \right| \leq C_k \frac{|w-w'|}{|w-z|^{3-m} |\tau|^k \Lambda(z, |w-z|)^k}$$  
(5.2)

(c) If $2|z-z'| \leq |w-z|$, then
$$\left| e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) - e^{-i\eta T(w',z')} K_{\tau,\epsilon}(z', w) \right| \leq C_k \frac{|z-z'|}{|w-z|^{3-m} |\tau|^k \Lambda(z, |w-z|)^k}$$  
(5.3)

Also, the constants are uniform in $\epsilon$.

Proof. It is immediate from the Mean Value Theorem that (5.1) implies (5.2) and (5.3). To prove (5.1), we use Proposition 4.3 and compute:

$$e^{i\eta T(w,z)} \frac{\partial}{\partial z} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) = -i\eta \frac{\partial T}{\partial z}(w, z) K_{\tau,\epsilon}(z, w) + \frac{\partial K_{\tau,\epsilon}}{\partial z}(z, w)$$
$$= \frac{\partial K_{\tau,\epsilon}}{\partial z}(z, w) - \frac{\partial}{\partial z}(\tau \frac{\partial}{\partial z}(z) K_{\tau,\epsilon}(z, w) - \tau \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1}}{\partial z^j}(w-z)^j K_{\tau,\epsilon}(z, w).$$

Using the size estimate (5.4),

$$\left| \frac{\partial}{\partial z} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right| \leq Z_\tau K_{\tau,\epsilon}(z, w) + \frac{\tau \Lambda(z, |w-z|)}{|w-z|} K_{\tau,\epsilon}(z, w)$$
$$\leq C_k \frac{|w-z|^{m-3}}{|\tau|^k \Lambda(z, |w-z|)^k}.$$  

A virtually identical calculation shows

$$\left| \frac{\partial}{\partial z} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right| \leq C_k \frac{|w-z|^{m-3}}{|\tau|^k \Lambda(z, |w-z|)^k}$$

which proves $\left| \frac{\partial}{\partial w} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right|$ satisfies the bound in (5.1). The bounds for the $w$ and $w'$ derivatives, $\left| \frac{\partial}{\partial w} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right|$ and $\left| \frac{\partial}{\partial w'} \left( e^{-i\eta T(w,z)} K_{\tau,\epsilon}(z, w) \right) \right|$, use a repetition of the calculations just performed and the identity $e^{-i\eta T(w,z)} = e^{i\eta T(z,w)}$ (which follows from Proposition 4.1).

We now restrict ourselves to the case $m = 0$. Given an operator $T_\tau$, order $0$, define a related family of operators $S_\tau$ so that if $K_\tau(z, w)$ is the kernel of $T_\tau$, the kernel of $S_\tau$ is given by $e^{-i\eta T(w,z)} K_\tau(z, w)$. We have the following:

Lemma 5.2. $S_\tau$ and $S_\tau^*$ are restrictly bounded uniformly in $\tau$, i.e. if $\varphi \in C^\infty_c(D(0,1))$, $\|\varphi\|_{C^{N_0}} \leq 1$ where $N_0$ is the constant from the cancellation condition (5.8) and $\varphi^{R,z_0}(z) = \varphi(z-z_0/R)$, then

$$\|S_\tau(\varphi^{R,z_0})\|_{L^2(\mathbb{C})} \leq AR, \quad \|(S_\tau)^*(\varphi^{R,z_0})\|_{L^2(\mathbb{C})} \leq AR$$

(5.4)

with the constant $A$ independent of $\tau$. 

The final ingredient we need to prove Theorem 2.1 is a result by Ricci and Stein [RS87].

We claim $R^{[I]} |Y_{\tau p}^I (e^{i\tau T(z,w)} \varphi(\frac{w-z_0}{R}))| \leq C_{|\tau|} \max \{1, |\tau|^I \Lambda(z,R)^{[I]} \}$. To see this, we first do the case $Y_{\tau p}^I = Z_{\tau p,w}$. It follows from Proposition 4.1 and Proposition 4.3 that

$$Z_{\tau p,w} (e^{i\tau T(z,w)} \varphi(\frac{w-z_0}{R})) = e^{i\tau T(z,w)} \frac{\partial \varphi(\frac{w-z_0}{R})}{\partial w} + \tau e^{i\tau T(z,w)} \sum_{j \geq 1} \frac{1}{j!} \frac{\partial^{j+1} \varphi(\frac{w-z_0}{R})}{\partial w^{j}} (w-z) \varphi(\frac{w-z_0}{R}).$$

Hence, $|Z_{\tau p,w} (e^{i\tau T(z,w)} \varphi(\frac{w-z_0}{R}))| \leq C (1 + \tau \Lambda(z,R))$. Iterating this argument proves the claim. Thus, for $|z-z_0| \leq 2R$,

$$\left| \int_{\mathbb{C}} K_{\tau,e}(z,w) e^{-i\tau T(w,z)} \varphi(\frac{w-z_0}{R}) \, dw \right| \leq C,$$

and

$$I \leq C \left( \int_{|z-z_0|<2R} \frac{1}{|z-z_0|^4} \left( \int_{\mathbb{C}} \left| \varphi(\frac{w-z_0}{R}) \right| \, dw \right)^2 \, dz \right)^{\frac{1}{2}} \leq AR.$$  

When $|z-z_0| \geq 2R$, $|z-z_0| \sim |z-w|$ for $w \in \text{supp} \varphi(\frac{w-z_0}{R})$, so

$$II \leq C \left( \int_{|z-z_0|\geq2R} \frac{1}{|z-z_0|^4} \left( \int_{\mathbb{C}} \left| \varphi(\frac{w-z_0}{R}) \right| \, dw \right)^2 \, dz \right)^{\frac{1}{2}} \leq CR^2 \left( \int_{r>R} \frac{1}{r^3} \, dr \right)^{\frac{1}{2}} \leq AR.$$

The final ingredient we need to prove Theorem 2.1 is a result by Ricci and Stein [RS87].

**Theorem 5.3** (Ricci-Stein). In $\mathbb{R}^n \times \mathbb{R}^n$, let $K(\cdot, \cdot)$ satisfy the following:

(a) $K(\cdot, \cdot)$ is a $C^1$ function away from the diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y \}$,

(b) $|\nabla K(x,y)| \leq A|x-y|^{-n-1}$ for some $A \geq 0$,

(c) the operator $f \mapsto \int_{\mathbb{R}^n} K(x,y) f(y) \, dy$ initially defined on $C_0^\infty(\mathbb{R}^n)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$.

If $P : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial, then the operator $T$ defined by

$$T[f](x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) \, dy$$
can be extended to a bounded operator from $L^q(\mathbb{R}^n)$ to itself, with $1 < q < \infty$. The bound of this operator may depend on $K$, $q$, $n$ and the degree $d$ of $P$ but is otherwise independent of the coefficients of $P$.

**Proof of Theorem 2.7.** The first step of the proof is to use the T(1) Theorem (p. 294 in [Ste93]) on $S_n$. The T(1) Theorem says that if $S$ is a continuous linear mapping from $S$ to $S'$ satisfying (5.2) and (5.3) (when $k = 0$) and $S$ and $S'$ are restrictively bounded in the sense of (5.4), then $S$ extends to a bounded linear operator from $L_2$ to itself. In our case, this means $S_n$ extends to a bounded linear operator. However, since all of the constants in Lemma 5.1 and Lemma 5.2 are independent of $\tau$, it follows that $S_n$ is a bounded linear operator from $L_2$ to itself with constants independent in $\tau$.

Next, $S_n$ satisfies the hypotheses of Theorem 5.3, so $T_n$ is a bounded linear operator from $L^q$ to itself for $1 < q < \infty$ with a constant independent of $\tau$ but possibly depending on the $L^q$ constant of $S_n$ and the degree of $\tau T$ (which is $\leq \deg p$), both of which are independent of $\tau$. □

6. Equivalence with NIS operators.

We now generate an OPF operator $T_\tau$ from an NIS operator $\tilde{T}$ on a polynomial model $M^p$. Let $\tilde{k}(p,q)$ be the kernel of an NIS operator $\tilde{T}$. On $\mathbb{C} \times \mathbb{R}$, each kernel $\tilde{k}$ can be associated with a kernel $k$ by setting
\[ k(z,w,t-s) = \tilde{k}((z,t),(w,s)). \]

The convolution structure in $t$ follows from the property that a polynomial model is translation invariant in $t = \Re z_2$. Thus we have (for appropriate $\varphi$),
\[ \tilde{T}[\varphi](z,t) = \int_{\mathbb{C} \times \mathbb{R}} \tilde{k}((z,t),(w,s)) \varphi(w,s) \, dw \, ds = \int_{\mathbb{C} \times \mathbb{R}} k(z,w,t-s) \varphi(w,s) \, dw \, ds. \]

We set
\[ K_\tau(z,w) = \int_{\mathbb{R}} e^{-it\tau} k(z,w,t) \, dt \]
and observe we also have
\[ k(z,w,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} K_\tau(z,w) \, dt. \]

The integrals representing $K_\tau(z,w)$ and $k(z,w,t)$ do not necessarily converge. For a tempered distribution $T$ and a Schwartz function $\varphi$, we know that if $F$ represents the partial Fourier transform in $t$, by definition, $(FT, \varphi) = (T,F\varphi)$. As an integral, this corresponds to:
\[ (FT, \varphi) = \int_{\mathbb{C} \times \mathbb{R}} k(z,w,t) \int_{\mathbb{R}} e^{-it\tau} \varphi(w,\tau) \, dw \, d\tau \, dt = \int_{\mathbb{C} \times \mathbb{R}} \int_{\mathbb{R}} k(z,w,t) e^{-it\tau} \varphi(w,\tau) \, dw \, d\tau \, dt. \]

We make sense of (6.1) by the string of equalities in (6.2), and we say the integral $\int_{\mathbb{R}} k(z,w,t) e^{-it\tau} \, dt$ is defined in the sense of Schwartz distributions. We similarly justify writing $k(z,w,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} K_\tau(z,w) \, d\tau$. If one of (or both of) the kernels is actually in $L^1(\mathbb{R})$ (in $t$ or $\tau$), then the integral defined in the sense of Schwartz distributions agrees with the standard definition.

### 6.1. An NIS Operator on $\mathbb{C} \times \mathbb{R}$ generates an OPF operator $T_\tau$ on $\mathbb{C}$.

**Theorem 6.1.** An NIS operator $\tilde{T}$ of order $m \leq 2$ on a polynomial model $M^p = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_2 = p(z_1)\}$ generates an OPF operator $T_\tau$ of order $m$ with respect to the polynomial $p$.

**Remark 6.2.** The approximation conditions, (b) in the definition of OPF operators and (a) in the definition of NIS operators, imply one another since a partial Fourier transform is a continuous operator on the space of Schwartz distributions. Also, the adjoint conditions (f) from OPF operators and (e) from NIS operators, allow us to focus only $k$ and $K_\tau$ as the computations will automatically apply to $k^*$ and $K_\tau^*$.

Theorem 6.1 is proved in a series of lemmas. We first show that if $\tilde{k}$ is an NIS operator of order $m \leq 2$, then $K_\tau$ is the kernel for a family $T_\tau$ of operators on $\mathbb{C}$.

The proof that $K_\tau, \epsilon$ satisfies the size conditions (3.6) and (3.8) is broken into two lemmas. We handle the $m \leq 1$ case and the $m = 2$ case.

**Lemma 6.3.** If $m \leq 1$, the kernel $K_\tau, \epsilon$ satisfies the size condition (3.6).
Lemma 6.4. If \(|J| = n\), let \(\eta \in C^\infty_\mathbb{C}(\mathbb{R})\) so that \(\eta \equiv 1\) on \([-1, 1]\), \(0 \leq \eta \leq 1\), and \(|\eta^{(n)}| \leq c_n\). Also, let \(\eta_A(t) = \eta(t/A)\). We will estimate
\[
\frac{\partial^n}{\partial t^n} \int_\mathbb{R} e^{-i\tau(t + T(w, z))} k(z, w, t) \eta_A(t) \, dt,
\]
and (3.6) will follow by sending \(A \to \infty\). The integral is compactly supported and the integrand is smooth, so we can apply the derivatives inside of the integral. Integrating by parts \((n + k)\) times shows
\[
c_n \left| \int_\mathbb{R} e^{-i\tau(t + T(w, z))} \left( t + T(w, z) \right)^n k(z, w, t) \eta_A(t + T(w, z)) \, dt \right|
= c_{n+k} \left| \int_\mathbb{R} e^{-i\tau(t + T(w, z))} \frac{\partial^{n+k}}{\partial t^{n+k}} \left( t + T(w, z) \right)^n k(z, w, t) \eta_A(t + T(w, z)) \, dt \right|
= c_{n+k} \left| \int_\mathbb{R} e^{-i\tau(t + T(w, z))} \sum_{j=0}^{n+k} \frac{\partial^j}{\partial t^j} \left( t + T(w, z) \right)^n k(z, w, t) \eta_A^{(n+k-j)}(t + T(w, z)) \, dt \right|
\leq c_{n+k} \left| \sum_{j=1}^{n+k} \left( \int_{t - t + T(w, z) \leq A(z, w - z)} \right)^{n-1-j} \mu(z, \left| t + T(w, z) \right|) \left| \frac{1}{A^{n+k-j}} \right| \, dt \right|
+ \int_{t - t + T(w, z) \leq 2A} \left| \frac{1}{A^{n+k-j}} \right| \, dt.
\]
If \(j = n + k\), then
\[
1 \left| \frac{1}{|t - t + T(w, z)|^{n-1-(n+k)}} \int_{t - t + T(w, z) \leq A(z, w - z)} \right| \frac{1}{A^{n+k-j}} \left( \frac{t + T(w, z)}{A} \right) \, dt
\]
\[
\leq c_{n+k} \left| \frac{1}{A^{n+k-j}} \int_{t - t + T(w, z) \leq 2A} \left| \frac{1}{A^{n+k-j}} \right| \, dt \right|
\]
Using the substitution \(s = \mu(z, \left| t + T(w, z) \right|)^{-1}, \frac{ds}{dt} \sim \frac{1}{\mu(z, \left| t + T(w, z) \right|)} \), so
\[
1 \left| \frac{1}{|t - t + T(w, z)|^{n-1-k}} \int_{t - t + T(w, z) \leq 2A} \right| \frac{1}{A^{n+k-j}} \, dt
\]
\[
\sim \left| \frac{1}{|t - t + T(w, z)|^{n-1-k}} \int_{|s| \leq \frac{1}{A^{n+k-j}}} \right| \frac{1}{A^{n+k-j}} \, ds \leq c_{n+k} \left| \frac{1}{|t - t + T(w, z)|^{n-1-k}} \right|
\]
If \(j < n + k\), then using the support condition of \(\eta_A^{(j)}(t + T(w, z))\) that \(|t + T(w, z)| \sim A\), the estimate simplifies to
\[
1 \left| \frac{1}{|t - t + T(w, z)|^{n-1-j}} \int_{t - t + T(w, z) \leq A(z, w - z)} \right| \frac{1}{A^{n+k-j}} \, dt
\]
\[
+ \frac{1}{|t - t + T(w, z)|^{n-1-j}} \mu(z, \left| t + T(w, z) \right|) \frac{1}{A^{n+k-j}} \left( \frac{t + T(w, z)}{A} \right) \, dt
\]
\[
\leq c_{n+k} \left| \frac{1}{A^{n+k-j}} \right| \mu(z, \left| t + T(w, z) \right|) \frac{1}{A^{n+k-j}} \left( \frac{t + T(w, z)}{A} \right) \, dt
\]
This complete the proof for \(m \leq 1\).

\[\square\]

Lemma 6.4. If \(m = 2\), the kernel \(K_{r, \epsilon}\) satisfies the size conditions (3.6) and (3.7).

Proof. As in Lemma 6.3, we can assume that
\[
Y^{r, \epsilon} = M^n = e^{i\tau T(w, z)} \frac{\partial^n}{\partial t^n} e^{-i\tau T(w, z)}
\]
where $|J| = n$.

We first show the case $\mu(z, \frac{1}{r}) \geq |w - z|$ and assume $n = 0$. From the definition of NIS operators, $|k_r(z, w, t)| \leq \frac{c_0}{A(z, |w - z|)^{1/2} + |t + T(w, z)|}$ and $\frac{\partial k_r}{\partial t}(z, w, t) \leq \frac{c_2}{A(z, |w - z|)^{1/2} + |t + T(w, z)|}$. Since $k_r$ is not integrable on $\mathbb{R}$, we need to integrate by parts to obtain an estimate on $K_{\tau, r}$. However, since $|w - z|$ is small, we need to be careful to integrate by parts as few times as possible and then only for large $t$. Let $A$ be a large number.

\[
\begin{align*}
\left| \int_{|t + T(w, z)| \leq \frac{1}{|\tau|}} e^{-i\tau t} k_r(z, w, t) dt \right| & \leq \left| \int_{|t + T(w, z)| \leq \Lambda(z, |w - z|)} e^{-i\tau t} k_r(z, w, t) dt \right| \\
+ \left| \int_{\Lambda(z, |w - z|) \leq |t + T(w, z)| \leq \frac{1}{|\tau|}} e^{-i\tau t} k_r(z, w, t) dt \right| & \leq 1 + \left| \int_{\Lambda(z, |w - z|) \leq |t + T(w, z)| \leq \frac{1}{|\tau|}} \frac{1}{|t + T(w, z)|} dt \right|
\end{align*}
\]

This is actually the estimate we are looking for since $\log \left( \frac{1/|\tau|}{\Lambda(z, |w - z|)} \right) \sim \log \left( \frac{\mu(z, \frac{1}{r})}{|w - z|} \right)$. Also, the estimate is independent of $A$, so we can let $A \to \infty$.

Now assume $k \geq 1$. Let $\eta \in C^\infty_c(\mathbb{R})$, $0 \leq \eta \leq 1$, supp $\eta(\cdot + T(w, z)) \subset [-2, 2]$, $\eta(t + T(w, z)) = 1$ if $|t| \leq 1$, and $\eta^{(k)}(t + T(w, z)) \leq c_k$. We show the case $|w - z| \geq \mu(z, \frac{1}{r})$. Let $A \in \mathbb{R}$ be large. Integration by parts $n + k$ times shows:

\[
\begin{align*}
\left| \frac{\partial^n}{\partial \tau^n} \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} e^{-i\tau t} k_r(z, w, t) dt \right| & = \frac{1}{A} + \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} \left| \frac{\partial^n}{\partial \tau^n} \left( t + T(w, z) \right)^n k_r(z, w, t) \right| dt \\
& \leq \frac{1}{A} + \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} \left| \frac{\partial^n}{\partial \tau^n} \left( t + T(w, z) \right)^n k_r(z, w, t) \right| dt \\
& \leq \frac{1}{A} + \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} \left| \frac{\partial^n}{\partial \tau^n} \left( t + T(w, z) \right)^n k_r(z, w, t) \right| dt
\end{align*}
\]

Sending $A \to \infty$ yields the desired estimate.

We have one estimate left to compute: the case $|w - z| < \mu(z, \frac{1}{r})$ and $n \geq 1$. Let $A$ be a large number. Let $0 \leq \psi_1, \psi_2 \leq 1$ so that $1 = \psi_1 + \psi_2^3$ on $[-A, A]$. Let supp $\psi_1 \subset [-2, 2]$ and supp $\psi_2^3 \subset \{ t : |t| \leq \frac{A^3}{2} \}$, and assume $|\frac{\partial^n}{\partial \tau^n} \psi_2^3| \leq c_0 |t|^{1/2}$ if $|t| \geq \frac{A^3}{2}$ and $|\frac{\partial^n}{\partial \tau^n} \psi_2^3| \leq c_0$ if $|t| \leq 2$. Since $|z - w| \leq \mu(z, \frac{1}{r})$, $\Lambda(z, |z - w|) \leq \frac{1}{r}$.

\[
\begin{align*}
\left| \frac{\partial^n}{\partial \tau^n} \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} e^{-i\tau t} k_r(z, w, t) dt \right| & \leq \frac{c_n}{A} + \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} \left| \frac{\partial^n}{\partial \tau^n} \left( t + T(w, z) \right)^n k_r(z, w, t) \right| dt \\
+ \sum_{j=0}^{n} c_j \int_{[t + T(w, z)] \leq \Lambda(z, |w - z|)} \left| \frac{\partial^n}{\partial \tau^n} \left( t + T(w, z) \right)^n k_r(z, w, t) \right| dt
\end{align*}
\]
Picking an arbitrary term and integrating by parts \((n+2)\) times, we have
\[
\left| \int_{\mathbb{R}} (t + T(w,z)) n \frac{\partial^j \psi^A_k(t + T(w,z))}{\partial \tau^j} k_e(z,w,t)e^{-i\tau(t+T(w,z))} \, dt \right|
\leq c_{n+2} \sum_{k=0}^{n+2} \int_{\mathbb{R}} \frac{1}{(t + T(w,z))^{n+2}} \frac{\partial^k}{\partial t^k} \left( (t + T(w,z))^n k_e(z,w,t) \right) \frac{\partial^{n+2+j-k} \psi^A_k(t + T(w,z))}{\partial \tau^{j} \partial \tau^{n+2-k}} \, dt
\]
If \(n + 2 + j - k \geq 1\), the term in the sum has support near \(\frac{1}{|\tau|}\) and \(\frac{1}{|\eta|}\), so it is bounded by
\[
\leq c_n |\tau|^{n+2-k} |\eta|^{n+2-k} \frac{1}{|\tau|^{n+2-k} |\eta|^{n+2-k}} \frac{A^{n-1-k}}{A^{n+2-k+j}} \frac{A \rightarrow \infty}{c_n}
\]
Finally, if \(n + 2 + j - k = 0\), then \(j = 0\) and \(k = n + 2\) and we have the estimate
\[
\int_{\mathbb{R}} \frac{1}{|\tau|^{n+2-k} |\eta|^{n+2-k}} \frac{1}{|\tau|^{n+2-k} |\eta|^{n+2-k}} \frac{A^{n-1-k}}{A^{n+2-k+j}} \frac{A \rightarrow \infty}{c_n}
\]
\[\square\]

**Lemma 6.5.** The operator \(T_r\) has the \(w\)-cancellation condition \(\text{(6.3)}\).

**Proof.** Let \(Y^j_{\tau r} = Z_{\tau r}, \tilde{Z}_{\tau r}, M_{\tau r}\) where \(|J| = \ell + n\) and \(n = \# \{ j : Y^j_{\tau r} = M_{\tau r} \}\) and \(\varphi \in C^\infty(D(z_0,\delta))\). We have
\[
K_{\tau r}(z,w) = \int_{\mathbb{R}} e^{-i\tau t} k_e(z,w,t) \, dt,
\]
so that integration by parts yields
\[
Z_{\tau r}K_{\tau r}(z,w) = Z_{\tau r} \int_{\mathbb{R}} e^{-i\tau t} k_e(z,w,t) \, dt
\]
\[
= \frac{\partial}{\partial z} \int_{\mathbb{R}} e^{-i\tau t} k_e(z,w,t) \, dt - \int_{\mathbb{R}} \frac{\partial}{\partial z} e^{-i\tau t} k_e(z,w,t) \, dt
\]
\[
= e^{-i\tau t} Lk_e(z,w,t) \, dt.
\]
Similarly, \(\tilde{Z}_{\tau r}K_{\tau r}(z,w) = \int_{\mathbb{R}} e^{-i\tau t} L\tilde{k}_e(z,w,t) \, dt\). Also, recalling that \(Mf(z,w) = -i(t + T(w,z))f(z,w)\), we have \(M_{\tau r}K_{\tau r}(z,w) = \int_{\mathbb{R}} e^{-i\tau t}(t + T(w,z))Mk_e(z,w,t) \, dt\). Thus,
\[
\int_{C} Y^j_{\tau r} K_{\tau r}(z,w)\varphi(w) \, dw = \int_{C} \int_{\mathbb{R}} e^{-i\tau t} Y^j k(z,w,t)\varphi(w) \, dt \, dw,
\]
with the correspondence that if \(Y^j_{\tau r} = Z_{\tau r}, \tilde{Z}_{\tau r}, M_{\tau r}\), then \(Y^j = L, \tilde{L}, M\) respectively. Integrating \((n+k)\) times gives us:
\[
\int_{C} Y^j_{\tau r} K_{\tau r}(z,w)\varphi(w) \, dw = \int_{C} \int_{\mathbb{R}} (Y^j k_e)(z,w,t)e^{-i\tau t} \varphi(w) \, dt \, dw
\]
\[
= \frac{c_{n+k}}{\tau^{n+k}} \int_{C} \int_{\mathbb{R}} \left( \frac{\partial^{n+k} Y^j}{\partial \tau^{n+k}} \right) k_e(z,w,t)e^{-i\tau t} \varphi(w) \eta(w,t) \, dt \, dw
\]
\[
+ \frac{c_{n+k}}{\tau^{n+k}} \int_{C} \int_{\mathbb{R}} \left( \frac{\partial^{n+k} Y^j}{\partial \tau^{n+k}} \right) k_e(z,w,t)e^{-i\tau t} \varphi(w)(1 - \eta(w,t)) \, dt \, dw
\]
\[(6.4)\]
where \(\eta \in C^\infty_c(C \times \mathbb{R})\) is a bump function on \(B_{N1}(z_0,\delta)\). To estimate the integrals in \((6.4)\), the strategy is to expand \(\left( \frac{\partial^{n+k} Y^j}{\partial \tau^{n+k}} \right) k_e(z,w,t)\) and estimate an arbitrary term. It is important to remember that in \(Y^j\), \(n\) of the terms are \(M\) and an \(L\) or \(\tilde{L}\) can hit either an \(M\) term or \(k_e(z,w,t)\).
Expanding \( \frac{\partial^{n+k}}{\partial t^{n+k}} X^J \) \( k_c(z, w, t) \), we see
\[
\frac{\partial^{n+k}}{\partial t^{n+k}} X^J k_c(z, w, t) = \frac{\partial^{n+k}}{\partial t^{n+k}} \left[ \sum_{|J_0| + \cdots + |J_n| = \ell} c_{|J_0|, \ldots, |J_n|} X^{J_0} k_c(z, w, t) \prod_{j=1}^n (-i) X^{J_j} (t + T(w, z)) \right]
\]
\[
= \sum_{|J_0| + \cdots + |J_n| = \ell} c_{|J_0|, \ldots, |J_n|} \frac{\partial^{f_0}}{\partial t^{f_0}} X^{J_0} k_c(z, w, t) \prod_{j=1}^n \frac{\partial^{f_j}}{\partial t^{f_j}} X^{J_j} (t + T(w, z)), \tag{6.5}
\]
where \( X^{J_j} \) is an operator composed only of \( X^J = L \) and \( L \). We pick an arbitrary term from the sum and show that it has the desired bound. Taking an arbitrary term from (6.5), we estimate the integrals from (6.4) which reduce to the following two integrals:
\[
I = \left| \frac{1}{\tau^{n+k}} \int \int_{C \times R} \frac{\partial^{f_0}}{\partial t^{f_0}} X^{J_0} k_c(z, w, t) \prod_{j=1}^n \frac{\partial^{f_j}}{\partial t^{f_j}} X^{J_j} (t + T(w, z)) e^{-ir t} \varphi(w) \eta(w, t) dtdw \right|
\]
and
\[
II = \left| \frac{1}{\tau^{n+k}} \int \int_{C \times R} \frac{\partial^{f_0}}{\partial t^{f_0}} X^{J_0} k_c(z, w, t) \prod_{j=1}^n \frac{\partial^{f_j}}{\partial t^{f_j}} X^{J_j} (t + T(w, z)) e^{-ir t} \varphi(w)(1 - \eta(w, t)) dtdw \right|
\]
where \( |J_0| + \cdots + |J_\ell| = \ell \) and \( f_0 + \cdots + f_n = n + k \). Using Proposition 4.3 and the cancellation condition (3.1), \( I \) has the estimate:
\[
I \leq \frac{c_{|J_0|, f_0}}{|\tau^{n+k}} \Lambda(z, \delta)^{m - |J_0|} \sup_{(w, t)} \sum_{|J_0| + \cdots + |J_n| = \ell} \delta^{\ell} \left| X^I (e^{-ir t} \varphi(w) \prod_{j=1}^n \left( \frac{\partial^{f_j}}{\partial t^{f_j}} X^{J_j} (t + T(w, z)) \right) \eta(w, t)) \right|
\]
\[
\leq \frac{c_{|J_0|, f_0}}{|\tau^{n+k}} \Lambda(z, \delta)^{m - |J_0|} \sup_{(w, t)} \sum_{|J_0| + \cdots + |J_n| = \ell} \delta^{\ell} \left| X^I \left( e^{-ir t} \varphi(w) \prod_{j=1}^n \left( \frac{\partial^{f_j}}{\partial t^{f_j}} X^{J_j} (t + T(w, z)) \right) \right) \right|
\]
\[
\leq \frac{c_{n, \ell, \delta}}{|\tau^{n+k}} \Lambda(z, \delta)^{-k \delta m - \ell} \sup_{(w, t)} \sum_{|J_0| \leq N_{|J_0|}, f_0} \delta^{\ell} \left| X^I (e^{-ir t} \varphi(w)) \right|
\]

To estimate \( II \), we use size estimates and the support size of \( \varphi \).
\[
II \leq \frac{c_{n, \ell}}{|\tau^{n+k}} \| \varphi \|_{L^\infty} \int_{|w - z_0| \leq \delta} \int_{|t + T(w, z)| \geq \Lambda(z, \delta)} \frac{dN_1(z, w, t)^{m - |J_0|}}{\Lambda(z, dN_1(z, w, t))^{f_0}} \frac{1}{|t + T(w, z)|^{n + k - \ell}} dtdw
\]
\[
\leq \frac{c_{n, \ell}}{|\tau^{n+k}} \| \varphi \|_{L^\infty} \frac{1}{|t + T(w, z)|^{n + k - \ell}} \int_{|w - z_0| \leq \delta} \int_{|t + T(w, z)| \geq \Lambda(z, \delta)} \frac{1}{|t + T(w, z)|^{n + k - \ell}} dtdw. \tag{6.6}
\]
If \( m \leq 2 \) or \( m = 2 \) and \( \ell \geq 1 \), then we use the substitution \( s = (z, t + T(w, z))^{-1} \), so \( \left| \frac{ds}{dt} \right| \sim |t + T(w, z)|^{-1} \) and (6.6) becomes
\[
II \leq \frac{c_{n, \ell}}{|\tau^{n+k}} \| \varphi \|_{L^\infty} \Lambda(z, \delta)^{-k \delta m - \ell} \int_{|z| \leq \delta} s^{-m - \ell} ds \leq \frac{c_{n, \ell}}{|\tau^{n+k}} \| \varphi \|_{L^\infty} \Lambda(z, \delta)^{-k \delta m - \ell}.
\]
If \( m = 2, \ell = 0 \), and \( k \geq 1 \), then a straightforward integration shows that \( II \leq \frac{c_{n, \ell}}{|\tau^{n+k}} \| \varphi \|_{L^\infty} \Lambda(z, \delta)^{-k \delta 2} \). The integral in (6.6) diverges if \( m = 2 \) and \( \ell = k = 0 \), so we must estimate the tail term more carefully in
this case. With \( m = 2, \ell = 0, \) and \( k = 0 \), (6.6) simplifies to
\[
II \leq \frac{1}{|\tau|^n} \left| \int \int_{\mathbb{C}} e^{-it\tau} \frac{\partial^{\ell_0}}{\partial t^{\ell_0}} k_\tau(z, w, t) \left( t + T(w, z) \right)^{-n-(\ell_0-\ell)} \varphi(w) \left( 1 - \eta(w, t) \right) dt dw \right|.
\]
The key to this estimate is to recognize that \( \frac{\partial^{\ell_0}}{\partial t^{\ell_0}} k_\tau(z, w, t) \left( t + T(w, z) \right)^{\ell_0} \) satisfies the estimates of an order 2 NIS operator. To integrate in \( t \), we use the argument of (6.3) with \( \delta \) replacing \( |z-w| \) and see that
\[
|II| \leq c_{n,0} \left| \frac{1}{|\tau|^n} \int_{\mathbb{C}} |\varphi(w)| \log \left( \frac{1}{|z-w|} \right) dw \right| \leq c_{n,0} \delta^2 \log \left( \frac{1}{\delta} \right) \| \varphi \|_{L^\infty(\mathbb{C})}.
\]
Note that \( \log \left( \frac{1}{\|A(z, \delta)\|} \right) \sim \log \left( \frac{1}{\delta} \right) \). While this estimate is true for all \( \tau \) and \( \delta \), the previous estimate of \( II \) shows that we only have to consider the case when \( \delta \leq \mu(z, \frac{1}{\delta}) \) or equivalently, \( \tau \Lambda(z, \delta) \leq 1 \). \( \square \)

**Lemma 6.6.** The kernel \( K_{\tau, \epsilon} \) satisfies the \( \tau \)-cancellation condition (53).

**Proof.** Since \( \mathcal{F}^{-1} \mathcal{F} = I \) in the sense of Schwartz distributions,
\[
|X^J k(z, w, t)| \leq C_{|J|} \frac{\mu(z, t + T(w, z))^{m-|J|}}{\sqrt{\mu(z, t + T(w, z))}}
\]
implies \( \frac{1}{2\pi} \int_{\mathbb{R}} X^J \left( e^{i\tau \tau} K_{\tau, \epsilon}(z, w) \right) d\tau = X^J k(z, w, t) \) satisfies the same estimates. \( \square \)

The proof of Theorem 6.7 is complete.

6.2. An OPF operator \( T_{\tau} \) on \( \mathbb{C} \) generates an NIS operator \( \tilde{k} \) on \( \mathbb{C} \times \mathbb{R} \).

**Theorem 6.7.** A OPF operator \( T_{\tau} \) of order \( m \leq 2 \) with respect to the subharmonic, nonharmonic polynomial \( p \) generates an NIS operator \( \tilde{k} \) of order \( m \leq 2 \) on the polynomial model \( M^p = \{ (z_1, z_2) \in \mathbb{C}^2 : \text{Im} z_2 = p(z_1) \} \).

We prove Theorem 6.7 in the same manner as Theorem 6.1. Remark 6.2 applies to Theorem 6.7 as well.

**Lemma 6.8.** The operator \( \tilde{k} \) satisfies the \( \tilde{N} \) cancellations conditions (53).  

**Proof.** Let \( \varphi \in C^\infty_c (B((z, \tau), \delta)) \). Also, let \( \hat{\varphi}(z, \tau) = \int_{\mathbb{R}} e^{-it\tau} \varphi(z, t) dt \) be the partial Fourier transform in \( t \) of \( \varphi(z, t) \). Let \( \eta \in C^\infty_c (\mathbb{R}) \) with \( \text{supp } \eta \subset \left[ -\frac{1}{\Lambda(z, \delta)}, \frac{1}{\Lambda(z, \delta)} \right] \) and \( \eta(\tau) = 1 \) when \( \tau \in \left[ -\frac{1}{\Lambda(z, \delta)}, \frac{1}{\Lambda(z, \delta)} \right] \). Let \( \tilde{X}^J \) be a product of \( |J| \) operators of the form of \( \tilde{X}^J = \tilde{L}_z \) and \( \tilde{L}_w \). Then
\[
\tilde{X}^J \int_{\mathbb{C} \times \mathbb{R}} k_\tau(z, w, t-s) \varphi(w, s) dw ds = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} \int_{\mathbb{R}} \tilde{X}^J \left( e^{i\tau \tau} K_{\tau, \epsilon}(z, w) \right) \varphi(w, s) d\tau ds dw
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{C}} \int_{\mathbb{R}} \tilde{X}^J \left( e^{i\tau \tau} K_{\tau, \epsilon}(z, w) \right) \varphi(w, t) d\tau dt ds dw = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{X}^J \left( e^{i\tau \tau} K_{\tau, \epsilon}(z, w) \right) \varphi(w, \tau) d\tau dt ds dw
\]
\[
+ \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{X}^J \left( K_{\tau, \epsilon}(z, w) \right) \varphi(w, \tau) d\tau dt ds dw \left( 1 - \eta(\tau) \right) d\tau = I + II.
\]

We estimate \( I \) and \( II \) separately. We first do the case \( m \leq 1 \) or \( m = 2 \) and \( |J| \geq 1 \). By (6.8),
\[
|I| \leq c_{|J|} \delta^{m-|J|} \int_{\mathbb{R}} \sup_w \left| \sum_{|I| \leq N_J} \delta^{|I|} \tilde{X}^J \left( \hat{\varphi}(w, \tau) \eta(\tau) \right) \right| d\tau
\]
\[
\leq c_{|J|} \delta^{m-|J|} \int_{\mathbb{R}} |\eta(\tau)| \sup_w \left| \sum_{|I| \leq N_J} \delta^{|I|} \tilde{X}^J \varphi \right|_{L^1(\tau)} d\tau
\]
\[
\leq c_{|J|} \delta^{m-|J|} \frac{1}{\Lambda(z, \delta)} \sum_{|I| \leq N_J} \delta^{|I|} \tilde{X}^J \varphi \right|_{L^\infty(\mathbb{C} \times \mathbb{R})} \Lambda(z, \delta).
\]
The last line follows from Hölder’s inequality and the size of \( \text{supp } \varphi \). The only difference between the case \( m = 2, J = 0 \) case and the previous estimate is the logarithm term in (3.8). The term to estimate is
\[
\int_{\mathbb{R}} |\eta(\tau)\delta^2 \log \left( \frac{1}{\|A(z, \delta)\|} \right) \sup_w |\hat{\varphi}(w, \tau)| d\tau \tag{6.7}
\]
However, integration shows that \( \int_0^{\infty} \log(\frac{1}{\Lambda(z,\delta)}) \, d\tau = \frac{1}{\Lambda(z,\delta)} \), so (6.17) simplifies to
\[
\delta^2 \| \hat{\varphi} \|_{L^\infty(C \times R)} \int_0^{\infty} \log(\frac{1}{\Lambda(z,\delta)}) \, d\tau = \delta^2 \| \hat{\varphi} \|_{L^\infty(C \times R)} \frac{1}{\Lambda(z,\delta)} \lesssim \delta^2 \| \varphi \|_{L^\infty(C \times R)}.
\]

We estimate \( II \) in a similar fashion. We first cover the case when \( m \leq 1 \) or \( m = 2 \) and \( |J| \geq 1 \).

\[
|II| = \frac{1}{2\pi} \int_R e^{i\tau t} \left( 1 - \eta(\tau) \right) \frac{1}{\tau^2} \left( X^t \int_C \tau^2 K_{\tau,\epsilon}(z,w) \hat{\varphi}(w,\tau) \, dw \right) \, d\tau
\]

\[
\leq c |J| \int_{|\tau| > \frac{1}{\Lambda(z,\delta)}} |\tau|^{-2} \delta^m |J| \sum_{|I| \leq |J|} \delta^{|I|} \| \tau^2 X^t \hat{\varphi}(w,\tau) \|_{L^\infty(w)} \, d\tau.
\]

The terms in the sum can be rewritten the more useful way:

\[
\| \tau^2 X^t \hat{\varphi}(w,\tau) \|_{L^\infty(w)} = \sup_w \left| \frac{1}{2\pi} \int_R \tau^2 X^t_{\tau,\epsilon} e^{i\tau t} \hat{\varphi}(w,\tau) \, dt \right|
\]

\[
= c \sup_w \left| \frac{1}{2\pi} \int_R e^{i\tau t} \left( \frac{\partial^2}{\partial t^2} \chi^t \varphi(w,\tau) \right) \, dt \right| \leq c_2 \Lambda(z,\delta) \left\| \frac{\partial^2}{\partial t^2} \chi^t \varphi \right\|_{L^\infty(C \times R)}.
\]

Using the estimate from (6.9) in (6.10),

\[
|II| \leq c_1 |J| \delta^m |J| \int_{|\tau| > \frac{1}{\Lambda(z,\delta)}} |\tau|^{-2} \sum_{|I| \leq |J|} \delta^{|I|} \left\| \left( \frac{\partial^2}{\partial t^2} \chi^t \right) \varphi(w,\tau) \right\|_{L^\infty(C \times R)} \, d\tau
\]

\[
\leq c_1 |J| \delta^m |J| \sum_{|I| \leq |J|} \delta^{|I|} \left\| \Lambda(z,\delta)^2 \left( \frac{\partial^2}{\partial t^2} \chi^t \varphi(w,\tau) \right) \right\|_{L^\infty(C \times R)}.
\]

In the final estimate, we used the fact that \( \Lambda(z,\delta) \frac{\partial}{\partial \tau} \) can be generated by commutators of \( \delta \chi \) terms. As in \( I \), the difference between the \( m = 2 \), \( J = 0 \) and the case already estimated is the logarithm term in (6.8).

However, \( \int_0^\infty \left( \log\left( \frac{1}{\Lambda(z,\delta)} \right) \right) \, d\tau = \Lambda(z,\delta) \), so we can repeat the estimate in (6.10) replacing \( |\tau|^{-2} \) with \( \left( \frac{1}{\log\left( \frac{1}{\Lambda(z,\delta)} \right)} \right) \) and achieve the same conclusion. \( \square \)

Lemma 6.9. The operator \( \hat{k} \) has the NIS size conditions (6.11).

Proof. It is enough to find the estimate on \( |k_\epsilon(z,w,t)| \). We handle the \( m = 2 \) separately. First assume \( m \leq 1 \). If \( d_{N1}(z,w,t) = |z-w| \), then we break the integral in two pieces and estimate each piece separately.

\[
\int_R e^{i\tau t} K_{\tau,\epsilon}(z,w) \, d\tau = \frac{1}{2\pi} \int_{|\tau| \leq \frac{1}{\Lambda(z,w-\epsilon)} \frac{1}{|w-z|^2 \Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w) \, d\tau + \frac{1}{2\pi} \int_{|\tau| \geq \frac{1}{\Lambda(z,w-\epsilon)} \frac{1}{|w-z|^2 \Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w) \, d\tau.
\]

Estimating the first integral gives us:

\[
\left| \int_{|\tau| \leq \frac{1}{\Lambda(z,w-\epsilon)} \frac{1}{|w-z|^2 \Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w) \, d\tau \right| \leq c_0 \frac{|w-z|^m}{|w-z|^2 \Lambda(z,|w-z|)} = c_0 \frac{d_{N1}(z,w,t)^m}{V(z,d_{N1}(z,w,t))}.
\]

The tail term is no harder: by (6.6) with \( \ell = n = 0 \) and \( k = 2 \),

\[
\left| \int_{|\tau| \geq \frac{1}{\Lambda(z,w-\epsilon)} \frac{1}{|w-z|^2 \Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w) \, d\tau \right| \leq c_2 \frac{|w-z|^m}{|w-z|^2 \Lambda(z,|w-z|)} \int_{|\tau| \geq \frac{1}{\Lambda(z,w-\epsilon)} \frac{1}{|w-z|^2 \Lambda(z,|w-z|)}} \frac{1}{\tau^2} \, d\tau
\]

\[
\leq c_2 \frac{|w-z|^m}{|w-z|^2 \Lambda(z,|w-z|)}.
\]

The case \( d_{N1}(z,w,t) = \mu(z,t + T(w,z)) \) is the \( \tau \)-cancellation condition (6.9).

Now assume \( m = 2 \). The estimate to prove is

\[
|k_\epsilon(z,w,t)| \leq \frac{d_{N1}(z,w,t)^2}{V(z,d_{N1}(z,w,t))} = \frac{1}{\Lambda(z,d_{N1}(z,w,t))}.
\]
Let $\eta \in C^\infty_c(\mathbb{R})$ where $\text{supp} \eta \subset [-2, 2]$, $\eta(\tau) = 1$ if $|\tau| \leq 1$, $0 \leq \eta \leq 1$, and $|\frac{\partial \eta}{\partial \tau'}(\tau)| \leq C_k$. Let $\Lambda = \Lambda(z, d_N(z, w, t))$. We have

$$k_\epsilon(z, w, t) = \int_{\mathbb{R}} e^{i\tau t} K_{\tau, \epsilon}(z, w) \eta(\tau \Lambda) \, d\tau + \int_{\mathbb{R}} e^{i\tau t} K_{\tau, \epsilon}(z, w)(1 - \eta(\tau \Lambda)) \, d\tau = I + II.$$  

Before we estimate $I$, observe $\int_{\mathbb{R}} \log s \, ds = -k \log s + \log s + 1 = \frac{1}{s^2}$ for $s \to \infty$. Also, with the change of variables $s = \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}$, $|\frac{\partial s}{\partial \tau}| \sim \frac{2\mu(z, \frac{1}{w-z})}{|w-z|^2}$. Also, with the change of variables $s = \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}$, $|\frac{\partial s}{\partial \tau}| \sim \frac{2\mu(z, \frac{1}{w-z})}{|w-z|^2}$, so

$$I \lesssim \int_{|\tau| \leq \frac{1}{2}} \log \left( \frac{2\mu(z, \frac{1}{w-z})}{|w-z|} \right) \frac{1}{s} \log s \, d\tau \sim \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{s} \log s \, d\tau \sim \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{s} \log s \, d\tau \sim \frac{1}{s^2} \frac{1}{|w-z|^2} \Lambda \sim \frac{1}{\Lambda}.$$

To estimate $II$, we need to separate the cases $\Lambda = \Lambda(z, w)$ and $\Lambda = |t + T(w, z)|$. We first do the case $\Lambda = \Lambda(z, w)$. By (3.6) with $k = 2$ and $\ell = n = 0$,

$$II \lesssim \int_{\frac{1}{2}}^\infty \frac{1}{\tau^3 \Lambda^2} \, d\tau \sim \frac{1}{\Lambda}.$$

Now assume $\Lambda = |t + T(w, z)|$. Then

$$II \lesssim \frac{1}{(t + T(w, z))^2} \left| \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{\tau^2} \left( e^{-i\tau T(w, z)} K_{\tau, \epsilon}(z, w)(1 - \eta(\tau|t + T(w, z)|)) \right) \, d\tau \right|.$$

If both $\tau$-derivatives are applied to $K_{\tau, \epsilon}$,

$$\frac{1}{(t + T(w, z))^2} \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{\tau^2} \left( e^{-i\tau T(w, z)} K_{\tau, \epsilon}(z, w) \right) \left( 1 - \eta(\tau|t + T(w, z)|) \right) \, d\tau \sim \frac{1}{(t + T(w, z))^2} \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{\tau^2} \, d\tau \sim \frac{1}{(t + T(w, z))^2}.$$

Next, if one $\tau$-derivative is applied to $K_{\tau, \epsilon}$ and one to $\eta$, then

$$\frac{1}{(t + T(w, z))} \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{\tau} \left( e^{-i\tau T(w, z)} K_{\tau, \epsilon}(z, w) \right) \eta'(|\tau| t + T(w, z)) \, d\tau \sim \frac{1}{(t + T(w, z))} \int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} \frac{1}{\tau} \, d\tau \sim \frac{1}{(t + T(w, z))^2}.$$

Finally, if $\eta$ receives both $\tau$-derivatives,

$$\int_{|\tau| \geq \frac{2\mu(z, \frac{1}{w-z})}{|w-z|}} |K_{\tau, \epsilon}(z, w) \eta''(|\tau| t + T(w, z))| \, d\tau \sim \frac{1}{(t + T(w, z))^2}.$$

Proving Theorem 6.1 and Theorem 6.7 proves Theorem 2.2.

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