Maximal Branching Processes in Random Environment

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Abstract

The work continues the author’s many-year research in theory of maximal branching processes, which are obtained from classical branching processes by replacing the summation of descendant numbers with taking the maximum. One can say that in each generation, descendants of only one particle survive, namely those of the particle that has the largest number of descendants. Earlier, the author generalized processes with integer values to processes with arbitrary nonnegative values, investigated their properties, and proved limit theorems. Then processes with several types of particles were introduced and studied. In the present paper we introduce the notion of maximal branching processes in random environment (with a single type of particles) and an important case of a “power-law” random environment. In the latter case, properties of maximal branching processes are studied and the ergodic theorem is proved. As applications, we consider gated infinite-server queues.

Key words: maximal branching processes; random environment; ergodic theorem; stable distributions; extreme value theory

1 Introduction

Classical objects of research in theory of stochastic processes are Galton–Watson branching processes (with a single type of particles and discrete time) [1]. Their extremal counterparts are referred to as maximal branching processes (MBPs). Namely, summation of the number of particle descendants (when finding the size of the next generation) is replaced with taking the maximum.

Let us recall the history of the question. MBPs were introduced and studied by J. Lamperti [2, 3] in 1970–1972, but were later completely abandoned by researchers (though mentioned in the survey [4]). A new stage of studying the MBPs was started by A.V. Lebedev in 2001. Processes with integer values were generalized to processes with arbitrary nonnegative values [5]. First, MBPs with particles of a single type were studied (see the survey [6]), and then those with several types of particles (multi-type MBPs) [7]. At present, the most complete overview of results and literature is given in [8] Chs. 4

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and 5]. Until recently, the author’s studies on this subject remained solitary. Only in 2012 they were unexpectedly pursued in the work of foreign researchers O. Aydogmus, A.P. Ghosh, S. Ghosh, and A. Roitershtein [9], who introduced colored maximal branching processes. Their difference from multi-type MBPs is that types (colors) of particles are determined after forming a generation, at random, and the type influences further fecundity. Another distinction from the author’s approach was considering solely processes that go to infinity.

Let us recall basic notions and properties of MBPs with a single particle type.

Consider random processes with values in \( \mathbb{Z}_+ \) defined stochastically by recurrence relations of the form

\[
Z_{n+1} = \bigvee_{m=1}^{Z_n} \xi_{m,n},
\]

where \( \bigvee \) stands for the maximum operation and where \( \xi_{m,n}, m \geq 1, n \geq 0 \), are independent random variables with a common distribution \( F \) on \( \mathbb{Z}_+ \). We assume (as in the case of summation) that the result of taking the maximum “zero times” (when \( Z_n = 0 \)) is zero.

One can say (by the analogy with Galton–Watson processes) that in each generation of a maximal branching process descendants of only one particle survive, namely of the particle that has the largest number of descendants. It is also clear that the set of possible values of an MBP (for \( n \geq 1 \)) coincides with the set of possible values of the number of descendants. It follows from (1) that the process is a homogeneous Markov chain on this set.

Another interpretation of an MBP can be proposed in queueing theory, by considering gated multi-server queues. These are queues with infinitely many servers where access of customers to service is regulated by a gate. The gate is assumed to be open only when all servers are free. Customers enter a queue with infinitely many waiting places, and servicing is performed in stages. At the beginning of a stage, when the gate opens, all customers in the queue instantly get access to servers and then are being served in parallel and independently until all servers become free. At the moment when all servers become free, the gate opens again for a new batch of customers (that have arrived during this time period) and the next stage.

Note that this queueing system is very easy to manage: there is no need to keep a permanent record of arriving and leaving customers, free and busy servers, etc. The allocation of customers to servers (which are all free at that moment) is made once and simultaneously at the beginning of each stage. Another advantage of a gated system may manifest itself in a situation where customers in the queue and servers are somehow separated from each other and establishment of communication requires certain costs. For example, it may be disadvantageous (or impossible) to keep the communication channel on all the time, and short connections from time to time may be preferable.

Of course, any infinite-linear system is only an approximation to the case where the real number of servers is large. On the other hand, it makes sense to study such systems to estimate different characteristics of the service quality (which for any queue with finitely many servers can only be worse).

Consider such a queue with discrete time, and assume that there is exactly one arrival at each time instant. Then, since the servers operate in parallel, the time required for servicing a current batch of arrivals (and hence, the number of arrivals in the next batch) equals the maximum of their service times. Thus, if we denote by \( Z_n \) the duration of the
nth stage and by $\xi_{m,n}$ the service times of arrivals within it, we obtain exactly (1).

Discrete-time gated infinite-server queues with Poisson arrival flows were analyzed in [10–13] (using other methods) and by the author in [14, 15]. In this case it is necessary to specify what happens if a gate opens while the queue is empty. It is most natural to assume that the queue waits for a new customer to arrive, from which the next stage begins.

Note that models with parallel data processing have become very popular in recent years due to the development of cloud computing technologies. At the same time, there arises a need to study maxima of random variables. As a recent work on this topic, note [16].

MBPs were introduced in [2] (in connection with long-range percolation models), where recurrence criteria for them were also obtained. Namely (assuming $F(0) = 0$):

$$\limsup_{x \to +\infty} x(1 - F(x)) < e^{-\gamma},$$

(2)

where $\gamma = 0, 577 \ldots$ is the Euler constant, the chain $\{Z_n\}$ is positively recurrent; and vice versa, if

$$\liminf_{x \to +\infty} x(1 - F(x)) > e^{-\gamma},$$

then $Z_n \to +\infty$, $n \to \infty$, almost surely (a.s.).

Then, in [3], the critical case $x(1 - F(x)) \to e^{-\gamma}$, $x \to +\infty$, was considered taking into account further terms of the tail expansion at infinity. It was shown that if $(e^{\gamma}x(1 - F(x)) - 1)\ln x \to d$, $x \to +\infty$, then the process is recurrent when $d < \pi^2/12$ and goes to infinity a.s. when $d > \pi^2/12$.

According to (1) and the assumption on the case of $Z_n = 0$, the process has transition probabilities

$$P(Z_{n+1} \leq j \mid Z_n = i) = F^i(j), \quad i, j \in \mathbb{Z}_+$$

(where we assume $0^0 = 1$), which suggested in [3] to consider Markov chains on an arbitrary measurable set $T \subset \mathbb{R}_+$ with transition probabilities

$$P(Z_{n+1} \leq y \mid Z_n = x) = F^x(y), \quad x, y \in T,$$

(3)

where $F$ is also supported on $T$.

Such processes can be considered both in their own right and as limit processes (in any sense) for MBPs on $\mathbb{Z}_+$ (normalized in a certain way). For instance, they can be used to describe the behavior of gated infinite-server queues, in particular, limit behavior under heavy traffic conditions, etc.

In particular, for a continuous-time gated infinite-server queue with Poisson arrivals, the sequence of stage durations over a busy period satisfies [3] with $F(x) = \exp\{-\lambda B(x)\}$, $x \geq 0$, where $\lambda$ is the arrival flow density, $B(x)$ is the distribution function for the service time of a single customer, and $\bar{B}(x) = 1 - B(x)$. Indeed, denote the duration of the $n$th stage by $Z_n$. Given that $Z_n = x$, the number of arrivals in this stage is Poissonian with parameter $\lambda x$, and $Z_{n+1}$ is the maximum of this (random) number of independent random variables with distribution $B$. Thus,

$$P(Z_{n+1} \leq y \mid Z_n = x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} B(y)^k = \exp\{-\lambda x \bar{B}(y)\} = F(y)^x.$$
To describe the system over the whole time horizon, one needs MBPs with immigration at the vanishing moment, etc.; see [8, Section 4.3].

Below we will speak about maximal branching processes on $T$ and denote them by $\text{MBP}(T)$. Note that a similar generalization for Galton–Watson processes are Jiřina processes [17] (with a continuous state set and discrete time); however, in this case $T$ can be any measurable subset of $\mathbb{R}_+$.

Equation (1) for such MBPs does not hold in the general case, but according to (3) they admit an equivalent representation by a stochastic recurrence sequence of the form

$$Z_{n+1} = \begin{cases} F^{-1}(U_{n+1}^{1/Z_n}), & Z_n > 0, \\ 0, & Z_n = 0, \end{cases} \quad n \geq 0, \tag{4}$$

where $F^{-1}(y) = \inf\{x : F(x) \geq y\}$; $U_n, n \geq 1$, are i.i.d. random variables on $(0,1)$; and $Z_0 \geq 0$ is independent of them. In this case the distribution $F$ is still called the distribution of the number of (direct) descendants.

In [5], a number of MBP properties (similarity condition, association, monotonicity in parameters, degeneration condition) were obtained and an ergodic theorem was proved.

Note that zero is always an absorbing state for an MBP. Thus, for $\text{MBP}(\mathbb{Z}_+)$ under condition (2) and $F(0) > 0$, this leads to degeneration a.s. [2]. For an $\text{MBP}(T)$, if $F(0) = 0$, zero can simply be excluded from the state set by considering a process with nonzero initial condition. However, if zero is a limit point of $T$, the possibility of asymptotic convergence to it as $n \to \infty$ remains. The following theorem provides sufficient conditions to eliminate the possibilities for the process to go to either zero or infinity and to make it ergodic. Here and in what follows we assume Harris ergodicity [18, ch. 1].

**Theorem A** [5, Theorem 1]. If for an $\text{MBP}(T)$, $T \subset (0, +\infty)$, the conditions (2) and

$$\lim \inf_{x \to 0} x(-\ln F(x)) > e^{-\gamma}$$

are fulfilled, then the process is ergodic.

Now we define MBPs in random environment (MBPREs).

In applications, “random environment” may describe various factors of natural, technical, or social nature randomly varying in time and affecting the system (for instance, speeding up or slowing down the operation of a queue).

Let us be given a sequence $F_l, l \geq 1$, of distributions on $\mathbb{Z}_+$; a collection of independent random variables $\xi_{m,n,l}, m \geq 1, n \geq 0, l \geq 1$, with distributions $F_l, l \geq 1$; and a sequence of independent random variables $\nu_n, n \geq 0$, with common distribution $G$ on $\mathbb{N}$ and independent of the $\xi_{m,n,l}, m \geq 1, n \geq 0, l \geq 1$. Then an MBPRE($\mathbb{Z}_+$) can be defined with the help of the stochastic recurrence relation of the form

$$Z_{n+1} = \bigvee_{m=1}^{Z_n} \xi_{m,n,\nu_n}. \tag{5}$$

The random environment is reflected here by the random variables $\nu_n, n \geq 1$, on which the choice of a distribution $F_l, l \geq 1$, of the number of particle descendants in each step depends.

According to (5) and the assumption on the case of $Z_n = 0$, the process has transition probabilities

$$P(Z_{n+1} \leq j \mid Z_n = i) = E(F^{i(j)}_{\nu}(j)), \quad i, j \in \mathbb{Z}_+,$$
where $\nu$ has distribution $G$, which suggests to consider a process on an arbitrary measurable set $T \subset \mathbb{R}_+$ with transition probabilities

$$P(Z_{n+1} \leq y \mid Z_n = x) = E(F_{\nu}^x(y)), \quad x, y \in T, \quad (6)$$

where a family of distributions $F_s, s > 0$, on $T$ and a random variable $\nu$ with distribution $G$ on $(0, +\infty)$ are assumed. In this way, we define MBPRE($T$).

Such processes can be considered both in their own right and as limit processes (in any sense) for the MBPRE($\mathbb{Z}_+$) processes (normalized in a certain way).

Below we will study a class of processes with a power-law property

$$\text{MBPPLRE}(\lambda T) \quad (\text{or } \text{MBPPLRE}(\lambda T))$$

for any number $\lambda > 0$. As an MBPPLRE, we also have a constructive representation, which follows from (7), namely

$$Z_{n+1} = \begin{cases} F^{-1}(\exp\{-\varphi^{-1}(U_{n+1})/Z_n\}), & Z_n > 0, \\ 0, & Z_n = 0, \end{cases} \quad n \geq 0, \quad (8)$$

where $U_n, n \geq 1$, are i.i.d. random variables on $(0, 1)$ and where $Z_0 \geq 0$ does not depend on them.

We prove a series of MBPPLRE properties by analogy with [5] as the following propositions.

**Proposition 1.** If $\{Z_n\}$ is an MBPPLRE($T$) with $F(x)$, then $\{\lambda Z_n\}$ for any $\lambda > 0$ is an MBPPLRE($\lambda T$) with $F^{1/\lambda}(x/\lambda)$.

**Proof.** Make a substitution in (7). Indeed,

$$P(\lambda Z_{n+1} \leq y \mid \lambda Z_n = x) = P(Z_{n+1} \leq y/\lambda \mid \lambda Z_n = x/\lambda)$$

$$= \varphi(-(x/\lambda) \ln F(y/\lambda))$$

$$= \varphi(-x \ln F^{1/\lambda}(y/\lambda)).$$

This property implies closeness of the class MBPPLRE($\mathbb{R}_+$) with respect to multiplication by $\lambda > 0$ and closeness of MBPPLRE($\mathbb{Z}_+$) for $\lambda \in \mathbb{N}$.

**Lemma 1.** For any numbers $Z_0' \leq Z_0''$ and $U_1' \leq U_1''$, $n \geq 1$, with $Z_0', Z_0'' \geq 0$ and $U_1', U_1'' \in (0, 1)$, number sequences $\{Z_n'\}$ and $\{Z_n''\}$ constructed according to (8) satisfy the condition $Z_n' \leq Z_n''$ for all $n \geq 0$.

**Proof.** By the condition, we have $Z_0' \leq Z_0''$. Assume that $Z_n' \leq Z_n''$ for some $n \geq 0$. Then, since $F^{-1}$ is a nondecreasing function, from $U_{n+1}' \leq U_{n+1}''$ and equation (8) we obtain $Z_{n+1}' \leq Z_{n+1}''$. Now the claim of the lemma holds by the induction principle.
Recall the notion of association of random variables \[19, 20\]. A multivariable function \( f(x) \) with \( x = (x_1, \ldots, x_n) \) is said to be monotonically nondecreasing if \( x_i' \leq x_i'' \), \( 1 \leq i \leq n \), implies \( f(x') \leq f(x'') \). Random variables of a tuple \( \zeta = (\zeta_1, \ldots, \zeta_n) \) are said to be associated if \( \text{cov}(f(\zeta), g(\zeta)) \geq 0 \) holds for all monotonically nondecreasing \( f \) and \( g \) such that this covariance exists. A random process or a field \( \{\zeta(t) : t \in T\} \) are said to be associated if their values \( \zeta(t_1), \ldots, \zeta(t_n) \) are associated for any finite set \( \{t_1, \ldots, t_n\} \subset T \).

According to \[19, 20\] Theorem 1.8, independent random variables are always associated; monotonically nondecreasing functions of associated random variables also possess this property.

**Proposition 2.** Any MBPPLRE is associated.

*Proof.* It suffices to note that, by Lemma 1, any \( Z_i, \ldots, Z_{i_n} \) are nondecreasing functions of the independent random variables \( Z_0 \) and \( U_n, 1 \leq n \leq \max\{i_1, \ldots, i_m\} \). \( \square \)

To establish the monotonicity in parameters, introduce a (partial) ordering relation between distributions: \( F_1 \prec F_2 \) if \( F_1(x) \geq F_2(x), \forall x \). Note that \( F_1 \prec F_2 \) implies \( F_1^{-1}(y) \leq F_2^{-1}(y), \forall y \in (0, 1) \).

Denote by \( Z = Z(F, G, H) \) an MBPPLRE with \( Z_0 \) having distribution \( H \).

**Proposition 3.** If \( F' \prec F'', G' \prec G'', H' \prec H'' \), then one can construct processes \( Z' = Z(F', G', H') \) and \( Z'' = Z(F'', G'', H'') \) on the same probability space so that \( Z'_n \leq Z''_n \) for all \( n \geq 0 \) a.s.

*Proof.* Let \( U_0 \) be the random variable uniformly distributed on \((0, 1)\) and independent of \( U_n, n \geq 1 \). Letting \( Z'_0 = (H')^{-1}(U_0) \) and \( Z''_0 = (H'')^{-1}(U_0) \), we obtain \( Z'_0 \leq Z''_0 \). Assume that \( Z'_n \leq Z''_n \) for some \( n \geq 0 \). \( G' \prec G'' \) implies \( \varphi'(u) \geq \varphi''(u), \forall u > 0 \), and \( (\varphi'(u))^{-1} \geq (\varphi''(u))^{-1}, \forall u \in (0, 1) \). \( F' \prec F'' \) implies \( (F')^{-1}(y) \leq (F'')^{-1}(y), \forall y \in (0, 1) \). By equation (9) we obtain \( Z'_{n+1} \leq Z''_{n+1} \). Proposition 3 is proved by the induction principle. \( \square \)

Denote the limit distribution of \( Z_n \) as \( n \to \infty \) (if exists) by \( \Psi \).

**Proposition 4.** If for two MBPPLRE \((T)\) we have \( F' \prec F'' \) and \( G' \prec G'' \), then \( \Psi' \prec \Psi'' \).

*Proof.* Take arbitrary \( H' = H'' \) on \( T \) and construct processes on the same probability space according to Proposition 3. Then \( Z'_n \leq Z''_n \) a.s. implies \( \mathbf{P}(Z'_n \leq x) \geq \mathbf{P}(Z''_n \leq x), n \geq 1 \), whence as \( n \to \infty \) we obtain \( \Psi'(x) \geq \Psi''(x), x > 0 \). \( \square \)

### 3 Ergodic Theorem

Recall the Gumbel distribution function \( \Lambda(x) = \exp\{-e^{-x}\} \), which plays an important role in stochastic extreme value theory.

If a distribution function \( F(x) \) is continuous and strictly increasing and \( F(0) = 0 \), then equation (3) reduces by the transformation \( \zeta_n = \Lambda^{-1}(F(Z_n)) \) to the form of the general (nonlinear) first-order autoregression

\[ \zeta_{n+1} = f(\zeta_n) + \eta_{n+1}, \quad n \geq 0, \quad (9) \]
where $f(u) = \ln F^{-1}(\Lambda(u))$ and where independent random variables $\eta_n = -\ln \varphi^{-1}(U_n)$, $n \geq 1$, have distribution function $\varphi(e^{-x})$.

Ergodicity of such models has been studied, e.g., in [18, Section 8.4; 21].

Denote $\delta = \mathbb{E}\eta_1$ and assume that this mean value does exist. We have

$$
\varphi(e^{-x}) = \mathbb{E}\exp\{-\nu e^{-x}\} = \mathbb{E}\Lambda(x - \ln \nu),
$$

which implies

$$
\delta = \gamma + \mathbb{E}\ln \nu, \quad (10)
$$

since the Gumbel distribution $\Lambda$ has mathematical expectation $\gamma$.

Note that $\delta$, as the mathematical expectation under the distribution function $\phi(e^{-x})$, can also be represented as

$$
\delta = \int_0^{+\infty} (1 - \varphi(e^{-x})) \, dx - \int_{-\infty}^0 \varphi(e^{-x}) \, dx = \int_{-\infty}^\infty (1 - \varphi(e^{-x})) \, dx - \int_0^\infty \varphi(e^x) \, dx \quad (11)
$$

when all the integrals converge.

**Example 1.** Let $F(x) = \exp\{-(x/c)^{-\beta}\}$, $x, c, \beta > 0$ (Fréchet distribution). Then the MBPPLRE admits a constructive representation

$$
Z_{n+1} = W_{n+1} Z_n^{1/\beta}, \quad (12)
$$

where $W_n$, $n \geq 1$, are independent and have distribution function $\varphi((x/c)^{-\beta})$, and (9) can be rewritten in the linear autoregression form

$$
\zeta_{n+1} = \zeta_n / \beta + \ln c + \eta_{n+1}, \quad n \geq 0. \quad (13)
$$

For $\beta < 1$, the process $\{\zeta_n\}$ goes to $\pm\infty$ depending on the sign of the initial condition $\zeta_0$. For $\beta > 1$, the process $\{\zeta_n\}$ is ergodic. For $\beta = 1$, we have a simple random walk going to $+\infty$ when $c > e^{-\delta}$, to $-\infty$ when $c < e^{-\delta}$, and oscillating between $\pm\infty$ when $c = e^{-\delta}$.

From this, one can easily obtain results for $\{Z_n\}$ taking into account that $Z_n = F^{-1}(\Lambda(\zeta_n))$, whence $Z_n \to 0$ for $\zeta_n \to -\infty$ and $Z_n \to +\infty$ for $\zeta_n \to +\infty$, and ergodicity is preserved.

This example suggests that the following theorem holds.

**Theorem 1.** If MBPPLRE$(T)$, $T \subset (0, +\infty)$, satisfies the conditions

$$
\liminf_{x \to +\infty} x(-\ln F(x)) < e^{-\delta} \quad (14)
$$

and

$$
\liminf_{x \to 0} x(-\ln F(x)) > e^{-\delta}, \quad (15)
$$

where $\delta$ in (10) exists, then the process is ergodic.

*Proof.* Without loss of generality we may assume that $T = (0, +\infty)$. First of all, note that conditions (14) and (15) are equivalent to the claim that there exist $0 < x_1 < x_2$ and $0 < c_2 < e^{-\delta} < c_1$ such that $F(x) \leq e^{-c_1/x}$ for $x \leq x_1$ and $F(x) \geq e^{-c_2/x}$ for $x \geq x_2$. 

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As a Lyapunov function, consider \( g(x) = (\ln(x/x_2))_+ + (\ln(x_1/x))_+ \) with \( y_+ = \max\{0, y\} \). Denote

\[
\mu(x) = \mathbb{E}(g(Z_{n+1}) \mid Z_n = x) - g(x).
\]

Then

\[
\mu(x) = \mathbb{E}\{\ln(Z_{n+1}/x_2) \mid Z_n = x\} + \mathbb{E}\{\ln(x_1/Z_{n+1}) \mid Z_n = x\} - g(x)
\]

\[
= \int_0^\infty \mathbb{P}(\ln(Z_{n+1}) > y \mid Z_n = x) \, dy + \int_0^\infty \mathbb{P}(\ln(x_1/Z_{n+1}) > y \mid Z_n = x) \, dy - g(x)
\]

\[
= \int_0^\infty (1 - \varphi(-x \ln F(x_2 e^y))) \, dy + \int_0^\infty \varphi(-x \ln F(x_1 e^y))) \, dy - g(x)
\]

\[
\leq \int_0^\infty (1 - \varphi((c_2 x/x_2) e^y)) \, dy + \int_0^\infty \varphi((c_1 x/x_1) e^y)) \, dy - g(x)
\]

\[
= \int_{-\ln(c_2 x/x_2)}^\infty (1 - \varphi(e^{-z})) \, dz + \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz - g(x)
\]

\[
= \int_0^\infty (1 - \varphi(e^{-z})) \, dz + \int_0^\infty \varphi(e^z) \, dz + \int_{-\ln(c_2 x/x_2)}^0 (1 - \varphi(e^{-z})) \, dz
\]

\[
- \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz - ((\ln(x/x_2))_+ + (\ln(x_1/x))_+).
\]

Denote the obtained upper estimate for \( \mu(x) \) by \( \mu^*(x) \), and let

\[
\bar{\mu}(x) = \int_{-\ln(c_2 x/x_2)}^0 (1 - \varphi(e^{-z})) \, dz - \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz - ((\ln(x/x_2))_+ + (\ln(x_1/x))_+).
\]

Then

\[
\mu(x) \leq \mu^*(x) = \int_0^\infty (1 - \varphi(e^{-z})) \, dz + \int_0^\infty \varphi(e^z) \, dz + \bar{\mu}(x).
\]

(16)

For \( x \geq x_2 \), we have

\[
\bar{\mu}(x) = \int_{-\ln(c_2 x/x_2)}^0 (1 - \varphi(e^{-z})) \, dz - \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz - \ln(c_2 x/x_2) + \ln c_2
\]

\[
= - \int_{-\ln(c_2 x/x_2)}^0 \varphi(e^{-z}) \, dz - \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz + \ln c_2
\]

\[
\rightarrow -2 \int_0^\infty \varphi(e^z) \, dz + \ln c_2, \quad x \rightarrow +\infty,
\]

whence by (11) and (16) we obtain \( \mu(x) \leq \mu^*(x) \rightarrow \delta + \ln c_2 = \ln(c_2/e^{-\delta}) < 0, x \rightarrow +\infty. \)

For \( x \leq x_1 \), we have

\[
\bar{\mu}(x) = \int_{-\ln(c_2 x/x_2)}^0 (1 - \varphi(e^{-z})) \, dz - \int_{\ln(c_1 x/x_1)}^\infty \varphi(e^z) \, dz + \ln(c_1 x/x_1) - \ln c_1
\]

\[
= - \int_{-\ln(c_2 x/x_2)}^0 \varphi(e^{-z}) \, dz - \int_{\ln(c_1 x/x_1)}^\infty (1 - \varphi(e^{-z})) \, dz - \ln c_1
\]

\[
\rightarrow -2 \int_0^\infty (1 - \varphi(e^{-z})) \, dz - \ln c_2, \quad x \rightarrow 0,
\]
whence by (11) and (16) we obtain
\[ \mu(x) \leq \mu^*(x) \rightarrow -\delta - \ln c_1 = -\ln(c_1/e^{-\delta}) < 0, \ x \rightarrow 0. \]
Hence, there exist \( \varepsilon > 0 \) and \( 0 < v_1 \leq v_2 \) such that \( \mu(x) \leq -\varepsilon \) for \( x \notin V = [v_1, v_2]. \)
Furthermore, \( \sup_{x \in V} E(g(Z_{n+1}) \mid Z_n = x) < \infty. \) Thus, the Lyapunov conditions [18, Section 4.2] are satisfied.

Now let us check the mixing condition.

Let \( 0 < u_1 \leq u \leq u_2, \ 0 < a < b. \) Since \( \varphi(s) \) is convex and decreasing, we have
\[ \varphi(-u \ln F(b)) - \varphi(u \ln F(a)) \geq \frac{u_1}{u_2} (\varphi(-u_2 \ln F(b)) - \varphi(-u_2 \ln F(a)), \]
which implies that for any measurable \( B \) we have
\[ P(Z_{n+1} \in B \mid Z_n = x) \geq \frac{v_1}{v_2} P(Z_{n+1} \in B \mid Z_n = v_2), \ \forall x \in V. \]  \hspace{1cm} (17)

Furthermore, any MBPPLRE is irreducible and aperiodic (in other words, from any state \( x \in T \) one can get into any set \( B \subset T \) and, moreover, in one step). Now the Lyapunov conditions and (17) imply the ergodicity of the MBPPLRE according to [18, Section 2, Theorem 2].

Note that if \( \nu = 1 \) a.s., Theorem 1 reduces to Theorem A, since in this case \( \delta = \gamma. \)

In some cases it is more convenient to compute \( \delta \) by the definition than by equation (10).

**Example 2.** Let \( \nu \) have exponential distribution with mean \( \theta. \) Then
\[ \varphi(u) = (1 + \theta u)^{-1} \]
and
\[ \varphi(e^{-x}) = \frac{1}{1 + \theta e^{-x}} = \frac{1}{1 + e^{-(x - \ln \theta)}}; \]
i.e., we obtain a logistic distribution with shift parameter \( \ln \theta. \) Hence, \( \delta = \ln \theta. \)

**Example 3.** Let \( \nu \) have a strictly stable distribution with \( \varphi(u) = e^{-cu^\alpha}, \ c > 0, \ 0 < \alpha < 1. \) Then
\[ \varphi(e^{-x}) = \exp\{-ce^{-\alpha x}\} = \Lambda(\alpha x - \ln c), \]
whence
\[ \delta = \frac{\gamma + \ln c}{\alpha}. \]

This, by the way, provides a convenient method for computing the mean logarithm of a stable random variable. Using (11), we obtain
\[ E \ln \nu = \frac{\gamma(1 - \alpha) + \ln c}{\alpha}. \]

For an ergodic MBPPLRE\((T),\) denote the random variable with the limit distribution by \( \tilde{Z}. \) In some cases, we can find numerical characteristics of this distribution.

**Example 4.** Let \( \nu \) have a strictly stable distribution with \( \varphi(u) = e^{-u^\alpha}, \ 0 < \alpha < 1, \) and \( F(x) = \exp\{-x^{-\beta}\}, \ x, \beta > 0. \) Then we obtain representation (12), where the \( W_n, \ n \geq 1, \) have the Fréchet distribution function \( \exp\{-x^{-\alpha \beta}\}, \ x > 0, \) and
\[ E W_1^s = \Gamma \left( 1 - \frac{s}{\alpha \beta} \right), \ 0 < s < \alpha \beta. \]

9
Representation (12) and ergodicity of the process imply that the limit distribution is the distribution of the following infinite product, which converges a.s.:

\[ \tilde{Z} \overset{d}{=} \prod_{n=0}^{\infty} W_{n}^{1/\beta n}, \]

whence

\[ \mathbb{E} \tilde{Z}^s = \prod_{n=1}^{\infty} \Gamma \left( 1 - \frac{s}{\alpha \beta^n} \right), \quad 0 < s < \alpha \beta. \]

Clearly, Proposition 4 also holds in the cases where one or both of the limit distributions is/are concentrated at zero. From this, in particular, one can obtain the degeneracy condition for processes with \( F(0) > 0 \). Here, by the degeneracy we mean vanishing of the process starting from some (random) moment.

**Corollary 1.** If an MBPPLRE(\( \mathbb{R}_+ \)) satisfies (14) and \( F(0) > 0 \), then the process degenerates a.s.

**Proof.** Note that for any \( C > 0 \) and \( n \geq 0 \) by equation (7) we have

\[ \mathbb{P}(Z_{n+1} = 0) = \mathbb{E} \varphi(-Z_n \ln F(0)) \geq \mathbb{P}(Z_n = 0) + \varphi(-C \ln F(0)) \mathbb{P}(0 < Z_n \leq C). \]

The sequence \( \mathbb{P}(Z_n = 0) \) is monotone nondecreasing and bounded, and therefore tends to some limit \( p_0 \in (0, 1] \). We obtain

\[ \sum_{n=0}^{\infty} \mathbb{P}(0 < Z_n \leq C) \leq p_0 / \varphi(-C \ln F(0)) < \infty, \]

so by the Borel–Cantelli lemma \( Z_n \) gets into \((0, C]\) finitely many times a.s. for any \( C > 0 \), which may mean either degeneracy or going to infinity as \( n \to \infty \).

Let \( F^*(x) = F(x) \exp\{ -x^{-2} \} \mathbb{I}(x > 0) \); then \( F \prec F^* \). By Proposition 3, one can construct an MBPPLRE(\( \mathbb{R}_+ \)) with \( F^* \) such that \( Z_n \leq Z_n^* \) a.s. Note that \( F^* \) satisfies the conditions of Theorem 1, so that \( \mathbb{P}(Z_n^* \to +\infty) = 0 \), and hence \( \mathbb{P}(Z_n \to +\infty) = 0 \) too. Thus, the process degenerates a.s. \( \square \)

Up to now, we assumed that \( \delta \) is finite. However, a limit distribution may in some cases exist as well for \( \delta = +\infty \), which corresponds to super-heavy tails \( G \).

**Example 5.** Let \( F(x) = \exp\{ -x^{-\beta} \}, \ x > 0, \ \beta > 1, \) and \( G(x) = 1 - 1/\ln x, \ x \geq e \). Then by the Tauber theorem we have \( 1 - \varphi(u) \sim -1/\ln u, \ u \to 0, \) whence

\[ \mathbb{P}(\eta_1 > x) = 1 - \varphi(e^{-x}) \sim 1/x, \quad x \to +\infty. \quad (18) \]

On the other hand, since \( \nu \geq e \) a.s., we have \( \varphi(u) \leq e^{-eu}, \ u \geq 0, \) whence

\[ \mathbb{P}(\eta_1 < -x) = \varphi(e^x) \leq \exp\{ -e^{x+1} \}, \quad x > 0. \quad (19) \]

It follows from (13) that existence of a limit distribution is equivalent to convergence of the following random series a.s.:

\[ \tilde{\zeta} \overset{d}{=} \sum_{n=0}^{\infty} \beta^{-n} \eta_n, \quad (20) \]
where $\tilde{\zeta} = \Lambda^{-1}(F(\tilde{Z}))$. For any $1/\beta < \varepsilon < 1$ by virtue of (18) and (19) we obtain
\[
\sum_{n=0}^{\infty} P(|\eta_n| > (\varepsilon \beta)^n) < \infty,
\]
whence it follows by the Borel–Cantelli lemma that the events $A_n = \{\beta^{-n}|\eta_n| > \varepsilon^n\}$ occur at most finitely many times; hence, the series (20) converges a.s.

4 Conclusion

We have introduced maximal branching processes in random environment (with a single particle type). We have examined the case of “power-law” random environment; for this case we have studied a number of properties, proved an ergodic theorem, and considered examples. We have noted a relation between maximal branching process and infinite-server queues. Further research can address the analysis of a wider class of random environments and processes with immigration.

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