The early stage of the interaction between a planar shock and a cylindrical droplet considering cavitation effects: theoretical analysis and numerical simulation

Sheng Xu¹, Wenqi Fan¹, Wangxia Wu², Wei Wang³, Bing Wang¹*

1 School of Aerospace Engineering, Tsinghua University, Beijing 100084, PR China
2 School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, PR China
3 National Clinical Research Center for Orthopedics, General Hospital of Chinese PLA, Beijing 100853, PR China

Abstract

The interaction between shock waves and droplets, involved the evolution of high transient unsteady wave structures coupled with a complex physical and chemical process, occurs widely in nature and industry. In this paper, a combination of theoretical analysis and high-resolution numerical simulation is employed to reveal the inherent characteristics for the interaction of a planar shock wave with a cylindrical droplet. A multi-component two-phase compressible flow model, coupled with the phase transition procedure, is used to demonstrate the evolution mechanism of wave structures and cavitation behaviours, including the inception, growth and collapse of cavitation. Based on the ray analysis, the evolution characteristics of wave structures can be equivalent to the motion of a series of rays whose emission angle is correlated to a dimensionless wave speed. Under the influence of cylindrical curvature, those rays, the same as the reflected expansion wave, focus inside the droplet. The position of the focusing point is determined by the dimensionless wave speed and is well consistent with numerical simulation results. Due to the significant difference in the properties between the upper and lower branches of the second reflected wave, a high-transient pressure region is observed, and its position is clarified detailly. Further research found that if the incident shock wave intensity is high enough, the focusing area of the reflected expansion wave can be identified as a cavitation zone. Furthermore, the numerical simulation indicates that the cavitation zone is enlarged, and the stronger collapsing waves are induced by increasing the incident shock wave intensity.
**Keywords:** shock wave, droplet, cavitation, ray analysis, numerical simulation

1. Introduction

The shock-droplet interaction is a fundamental and challenging two-phase flow problem ubiquitous in supersonic flows. In recent decades, due to its vital importance in industrial production and scientific research, the interaction between shock waves and droplets has been widely studied, such as supersonic combustion in two-phase rotating detonation combustor, supernova explosion, and shock wave lithotripsy.

As for the interaction of the shock wave with a droplet, the droplet deformation or breakup mechanism has been investigated for decades, summarised in works of Theofanous et al. (2004) and Theofanous & Li (2008). With the help of a high-speed camera, the evolution characteristics of wave structures in the droplet are progressively observed and investigated. Nevertheless, it is still hard to attain the flow patterns in a three-dimensional spherical droplet. Consequently, numerous studies have been carried out to reveal the flow characteristics inside the liquid column. Igra & Takayama (2001) studied the interaction of a planar shock wave with a water column based on many experiments and obtained the evolution characteristics of wave structures for the first time. The results provide a solid basis for numerical model and physical mechanism analysis. Igra & Takayama (2001) also investigated the interaction of a planar shock wave with two liquid columns and revealed the difference in deformation and dynamic characteristics of two liquid columns. Sembian et al. (2016) and Meng & Colonius (2015) reported the wave structures at the early stages of shock interaction with a cylindrical water column considering different incident shock wave intensities and cylinder diameters. Using high-speed photography technology, Liang et al. (2020) captured the deformation of a water droplet embedded with a vapour cavity and analysed the influence of the relative size and eccentricity of the vapour cavity on the mechanism of droplet deformation.

With the rapid development in computer technology and numerical algorithms, numerical simulation becomes more feasible to solve compressible multiphase flows, consuming enormous computation resources but demonstrating spatiotemporal details of flow structures. Hu et al. (2009) performed numerical simulations on the interaction of a shock wave with a water column to verify the effectiveness of the Harten–Lax–van
Leer Contact (HLLC) approximate Riemann solver for handling the two-phase flows. Meng & Colonius (2015) studied the interaction of a shock wave with a water column based on the third-order weighted essentially non-oscillatory (WENO) reconstruction scheme and the HLLC approximate Riemann solver, ignoring the surface tension, evaporation, and other factors, and described the deformation characteristics of the water column under different shock wave intensities. A fifth-order incremental stencil weighted essentially non-oscillatory (WENO-IS) reconstruction combined with multidimensional optimal order detection (MOOD)-type positivity preservation, as previously proposed by Wang et al. (2018). The WENO-IS scheme is more robust than the traditional WENO scheme and successfully applied to numerous numerical simulations of compressible multiphase flows. Based on experiments reported by Sembian et al. (2014), Xiang & Wang (2017) performed a numerical study on the interaction of a planar shock wave with a water column embedded with/without a cavity of different sizes at high Weber numbers. This work qualitatively and quantitatively analysed the morphological and dynamic characteristics of the water column. It revealed the mechanisms of droplet breakup and transverse jet formation under different incident shock wave intensities and different cavity sizes. Boyd & Jarrahbashi (2021) extended the shock-droplet interaction problem from subcritical conditions to supercritical conditions and studied the effects of temperature, pressure, and shock intensity on the interaction.

As illustrated by Obreschkow et al. (2006, 2011), the evidence of cavitation induced by confined shocks inside the liquid column was found in experiments. Sembian et al. (2016) observed the cavitation region when a water column was impinged by a high-intensity planar shock wave. Field, Dear & Ogren (1989), Field et al. (2012) observed cavitation bubbles caused by the convergence of reflected expansion waves when a high-speed droplet impacts a rigid wall. Following the work of Field et al. (1989, 2012), Kondo & Ando (2016) used the numerical method to verify the possibility of cavitation when a droplet impacts a deformable wall at high speed. The morphology and dynamic characteristics of the high-speed droplet impingement process are analysed qualitatively and quantitatively from the perspectives of detailed theoretical and numerical investigations, as reported by Wu et al. (2018). In their work,
mechanisms of wave structures evolution and the inception of a cavitation bubble are performed in detail. Subsequently, the high-speed droplet impingement on typically curved surfaces was numerically investigated to analyse the inherent complex wave structures and cavitation (Wu et al. 2021). It is found that a convex surface can reduce the possibility of homogenous cavitation during high-speed impingement. Poulanges & Rabii (2021) reported that whether cavitation occurs inside the liquid column, impacted by a shock wave, depends on the intensity of the incident shock wave and the value of cavitation threshold pressure, while the phase transition model was ignored. Based on numerical results, Xiang & Wang (2017) expounded the physical mechanism of the inception cavitation under the impaction of shock waves. However, the research failed to introduce a suitable cavitation model to capture cavitation behaviours inside the droplet and lacked an in-depth theoretical analysis on the propagation characteristic of wave structures.

Thanks to numerous numerical studies, the compressible two-phase flow field has developed rapidly in the past few decades. However, there is still a lack of theoretical analysis in this field. The ray analysis method is successfully used to theoretically analyse the high-speed impingement of droplets on a wall (Wu et al. 2018), while the analysis results are one-sided because only the initial impact point is considered. Poulanges & Rabii (2021) theoretically investigated the interaction of a planar shock wave with a liquid column with acoustic principle and ray analysis method and derived the concentration of rays with different reflection times then verified by numerical simulations. Nevertheless, the inception, growth, and collapse of cavitation and the propagation characteristic of wave structures were not revealed.

Even though many findings of shock wave interactions with a droplet were supported by experiments and numerical simulations, the theoretical analysis of wave structures and cavitation behaviours have not been well investigated in physics. In this study, we aim to investigate the flow characteristics in the interaction of a planar shock wave with a cylindrical droplet through a combination of theoretical analysis and numerical simulations.

This paper is organised as follows. In § 2, the governing equations, phase-transition model, and numerical treatments are presented. In § 3, the physical model of
the interaction of a planar shock wave with a cylindrical droplet and the grid sensitivity study is conducted. In § 4, the morphology and dynamical evolutions of wave structures are analysed qualitatively and quantitatively. In § 5, the cavitation behaviours inside the water column and the effect of incident shock wave intensities are investigated. Finally, the conclusions are presented in § 6.

2. Numerical methodology

2.1. Governing equations and equations of state (EOS)

In the 1980s, the seven-equation model was first presented to solve compressible granular flows in the heat flux relaxation conditions (Baer & Nunziato 1986), known as the Bear–Nunziato model. Saurel & Abgrall (1999) extended the Bear–Nunziato model and developed the Saurel–Abgrall model for compressible gas-liquid two-phase flows. The Saurel–Abgrall model was further developed into different forms, including the full non-equilibrium seven-equation model (Saurel & Abgrall 1999), six-equation single-velocity model with stiff mechanical relaxation (Pelanti & Shyue 2014), and the simplified five-equation single-velocity and single-pressure model with stiff mechanical and thermal relaxations (Johnsen & Colonius 2006; Coralic & Colonius 2014). Numerous studies have proven that the five-equation model has significant advantages such as high robustness and low system complexity and is widely used by researchers. In this study, the multi-component compressible two-phase model, proposed by Wu et al. (2018), is employed. Based on the traditional five-equation model, the vapour phase coupled with the phase transition model is introduced to capture the process of cavitation inception and cavity collapse accurately.

For highly compressible multi-component two-phase flow that involves phase transition, the governing equations, consisting of the Eulerian and scalar transportation equations for the volume of fraction, are as follows (Wu et al. 2018):
\[
\frac{\partial (\alpha_k \rho_k)}{\partial t} + \nabla \cdot (\alpha_k \rho_k u) = \dot{\mathcal{S}}_{\rho,k}, \quad k = 1, 2, \ldots, K,
\]

\[
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u + p I) = 0,
\]

\[
\frac{\partial E}{\partial t} + \nabla \left[ (E + p) u \right] = 0,
\]

\[
\frac{\partial (\alpha_k)}{\partial t} + u \cdot \nabla \alpha_k = \dot{\mathcal{S}}_{\alpha,k}, \quad k = 1, 2, \ldots, K - 1,
\]

where, \( \alpha_k, \rho_k \) and \( \alpha_k \rho_k \) represent the volume fraction, the density and the value of volume mass of components \( k \), respectively. \( \rho, \ u, \ p, \ I, \ E = (1/2) \rho u^2 + \rho e \) and \( e \) are the density, velocity, pressure, unit tensor, total energy and internal specific energy of the mixture, respectively. In this paper, we are interested in the three-component model, i.e., \( K = 3 \), with liquid water \( (k = 1) \), water vapour \( (k = 2) \) and air \( (k = 3) \). The saturation constraint of the volume of fraction yields: \( \alpha_k = 1 - \sum_{k=1}^{K-1} \alpha_k \).

The expression of the source terms, \( \dot{\mathcal{S}}_{\rho,k} \) and \( \dot{\mathcal{S}}_{\alpha,k} \), on the right-hand side of (2.1) are:

\[
\dot{\mathcal{S}}_{\rho,1} = -\dot{m} = \nu (\mu_2 - \mu_1), \quad \dot{\mathcal{S}}_{\rho,2} = \dot{m} = \nu (\mu_1 - \mu_2),
\]

\[
\dot{\mathcal{S}}_{\alpha,1} = -\dot{m} = \frac{\nu}{q_1} (\mu_2 - \mu_1), \quad \dot{\mathcal{S}}_{\alpha,2} = \dot{m} = \frac{\nu}{q_2} (\mu_1 - \mu_2).
\]

where, \( \mu_k \) is the chemical potential that participates in the phase change process. The values of \( \dot{\mathcal{S}}_{\rho,k} \) and \( \dot{\mathcal{S}}_{\alpha,k} \) are set as zero for the components that do not participate in the phase change. The specific parameter \( q_k \) can be found in Zein, Hantke & Warnecke (2010, 2013). The variable \( \nu (\geq 0) \) is the relaxation parameter for the chemical potential and can be regarded as the switch controlling the phase transition process. For details, please refer to the works of Wu et al. (2018, 2019, 2021, 2022).

In the present study, the stiffened gas equation of state (SG-EOS) is used to close the governing equations (Menikoff & Plohr 1989; Saurel et al. 2008), and the thermodynamic state of each component is described as follows:

\[
e_k (p, \rho_k) = \frac{p + \gamma_k p_{\alpha,k}}{\rho_k (\gamma_k - 1)} + q_k
\]

\[
\rho_k (p, T) = \frac{p + p_{\alpha,k}}{C_{v,k} T (\gamma_k - 1)}
\]
\[ h_k(T) = \gamma_k C_v k + q_k \]  
\[ c_k = \sqrt{\frac{(p + p_{\infty,k})\gamma_k}{\rho_k}} \]  
\[ g_k(p,T) = (C_{v,k} - q_k')T - C_{v,k} T \log \frac{T/\gamma_k}{(p + p_{\infty,k})\gamma_k} + q_k \]  
\[ \mu_2(T, g_2, \alpha_1, \alpha_2) = g_2 + (\gamma_2 - 1)C_{v,2} T \log \frac{\alpha_2}{1 - \alpha_1} \]

| Components | \( \gamma_k \) | \( p_{\infty,k} \) (Pa) | \( C_{v,k} \) (J kg\(^{-1}\) K\(^{-1}\)) | \( q_k \) (J kg\(^{-1}\)) | \( q_k' \) (J kg\(^{-1}\) K\(^{-1}\)) |
|-------------|----------------|-------------------------|----------------|----------------|----------------|
| Water (liquid) | 2.057 | 1.066\times10^9 | 3449 | -1.995\times10^6 | 3.578\times10^4 |
| Water (vapour) | 1.327 | 0 | 1200 | 1.995\times10^6 | 2.41\times10^3 |
| air | 1.4 | 0 | 717 | 0 | 0 |

**TABLE 1.** Parameters involved in the stiffened gas equation of state (Wu et al. 2018)

where, \( \gamma_k \) is the specific heat ratio, \( p_{\infty,k} \) is the material parameter with pressure dimension, \( C_{v,k} \) is the specific heat capacity at constant volume, \( q_k \) is the heat of formation, and \( q_k' \) is the entropy constant of component \( k \). The corresponding values of these parameters are referred from Wu et al. (2018), as shown in table 1.

Due to the numerical diffusion of the gas-liquid interface during the calculation, the mixed fluid variables in the diffusion region can be expressed as shown below (Wu et al. 2018, 2021, 2022):

\[ \rho = \sum_{k=1}^{K} \alpha_k \rho_k \]  
\[ \rho e = \sum_{k=1}^{K} \alpha_k \rho_k e_k \]  
\[ p = \frac{\rho e - \sum_{k=1}^{K} \alpha_k \rho_k q_k - \sum_{k=1}^{K} \alpha_k \gamma_k p_{\infty,k}}{\sum_{k=1}^{K} \alpha_k \gamma_k - 1} \]  
\[ c = \sqrt{\frac{\sum_{k=1}^{K} \alpha_k \rho_k c_k^2}{\rho}} \]
2.2. Numerical method

In the present study, the governing equation (2.1) is solved using the finite volume method. The splitting approach is applied; accordingly, the hyperbolic operator and the source terms related to the phase transition are solved separately. The WENO-IS scheme (Wang et al. 2018) is applied for spatial reconstructions to ensure computational stability. The HLLC approximate Riemann solver (Toro 2013) is used to solve the numerical flux at the edges of the cells. By referring to Han et al. (2017), the source terms on the right-hand side of the governing equation (2.1) are treated by a chemical relaxation procedure. The third-order total variation diminishing (TVD) Runge–Kutta scheme (Gottlieb & Shu 1998) is employed to time marching. In order to avoid an excessive numerical diffusion at the gas-liquid interface, the $\rho$-THINC algorithm (Garrick et al. 2017) is used to sharpen the interface.

3. Physical model and grid sensitivity analysis

3.1. Physical model

A schematic diagram of the shock wave interaction with a cylindrical droplet considering the effect of phase transition is shown in figure 1. We use $C$, LP, RP, LS, and RS to represent the nomenclature of liquid column center, left pole, right pole, upstream surface, and downstream surface, respectively. The schematic positions of LP, RP, LS, RS, and $C$ are also shown in figure 1. The diameter of the cylindrical droplet is chosen to be 4.8 mm, referring to the experiments of Igra & Takayama (2001a,b) and the numerical simulation of Xiang & Wang (2017).

![FIGURE 1. Schematic diagram of the interaction problem of a planar shock wave with a cylindrical droplet.](image)
In this paper, axisymmetric boundary conditions were used to improve the calculation efficiency. The position of the symmetry axis coincided with the horizontal axis of the droplet, and the non-reflective boundary conditions (Thompson 1987) are employed for the rest of the boundaries, as shown in figure 1. A uniform Cartesian grid system is used for all computations, and the Courant–Friedrich–Lewis (CFL) number is taken as 0.4 for all cases.

Four incident shock waves with different intensities \( (Ms = 2.0, 2.4, 3.0 \text{ and } 3.6) \) are selected for the present simulations, referring to the experiments of Sembian et al. (2016) and the numerical simulation of Xiang & Wang (2017), and parameters are shown in table 2. Air 0 and Air 1 are pre-shock air and post-shock air, respectively. In the present study, liquid water is used to form the cylindrical droplet, and its dynamic viscosity, surface tension coefficient, and gravitational acceleration values are \( 2.98 \times 10^{-3} \text{ Pa} \cdot \text{s}, 72 \text{ mN/m} \) and \( 9.8 \text{ m/s}^2 \), respectively. In the present numerical simulations, the Weber numbers \( We \), Reynolds number \( Re \) and Froude number \( Fr \) are all over \( 10^3 \). Therefore, the viscous effect, surface tension, and gravity can be neglected (Meng & Colonius 2015).

| Gas/liquid | Air 0 | Air 1 | Air 1 | Water (liquid) |
|------------|-------|-------|-------|----------------|
| \( Ms \)   | /     | 2.0   | 2.4   | 3.0            | 3.6            |
| \( \rho \) (kg/m\(^3\)) | 1.21  | 3.21  | 3.87  | 4.65           | 5.22           | 998.07         |
| \( u \) (m/s) | 0.0   | 428.8 | 567.0 | 762.3          | 950.0          | 0.0            |
| \( p \) (bar) | 1.0   | 4.5   | 6.6   | 10.3           | 15.0           | 1.0            |

**TABLE 2. Initial thermodynamic state for the computation cases**

3.2. *Grid sensitivity analysis*

Before the formal numerical simulation, the grid sensitivity is analysed for the case of \( Ms = 2.4 \), and the numerical simulation results under three different grid resolutions are compared. The number of cells is 450,000 (Grid-I), 1.15 million (Grid-II) and 3.2 million (Grid-III), respectively. For the water column, the cell number across the column diameter is 1200, 1920 and 3200, respectively.

Figure 2 shows the schlieren (top) and pressure (bottom) contours under three grid resolutions at the fixed time \( t = 4.0 \mu\text{s} \). The numerical results obtained by three different
resolutions are overlapped. The flow field structures such as incident shock wave, reflected shock wave, Mach stem, reflected expansion wave, and gas-liquid interface could be obviously observed. As the grid resolution increases, the captured flow field structure becomes more apparent, as shown in figure 2. The pressure distributions along the horizontal axis of the droplet to compare the effects of different grid resolutions on the strength of wave structures are illustrated in figure 3. The three pressure curves with different grid resolutions are basically overlapped, and the pressure discontinuity caused by the reflected expansion wave is sharpened with the increase of grid resolution. Slight deviations from the pressure distribution are observed behind the reflected expansion wave in Grid-I due to the low grid resolution, while the pressure distributions overlap well for the other two higher grid resolutions. Grid-II is chosen in the present study to balance the computational efficiency and the resolution of the flow field.

**FIGURE 2.** The schlieren (top) and pressure (bottom) contours of the three different grid resolutions at $t = 4.0 \mu s$, $M_s = 2.4$.

**FIGURE 3.** Pressure distribution along the horizontal axis of the droplet under three
4. Theoretical analysis in evolution characteristics of wave structures

The evolution characteristics of wave structures in the early stage of the interaction of a planar shock wave with a cylindrical droplet, was reported in Xiang & Wang (2017). As presented in their work, when an incident shock wave impinges on the water column, it is reflected off the LS and transmitted into the liquid, and the reflection type of incident shock wave gradually changes from regular reflection to irregular Mach reflection with the increase of incident angle (the angle between the shock normal vector and the tangential vector of liquid-gas interface). As the speed of sound in water is much higher than the velocity of the shock wave in the air, the transmitted wave quickly detaches with the incident shock wave, forming a precursor transmitted shock wave. Since the curvature of the water column surface, the reflected expansion wave, reflected from the precursor transmitted shock wave at the inner interface, will gradually focus on the horizontal axis of the cylindrical droplet. Then, the expansion wave continues to propagate towards the LS after focusing and will be reflected inside the water column, forming the second reflected wave. The expansion wave and compression wave in the droplet propagate and reflect back and forth between the upstream and downstream, and the intensity decreases rapidly with the increase of reflecting times. Due to the limit of the grid resolution and the difference of research focus, the evolution characteristics of subsequent wave structures, such as the second reflection wave, were not clearly described in the work of Xiang & Wang (2017). Therefore, based on the numerical model introduced in § 2, the detailed evolution characteristics of wave structures in the early stage of shock wave and water column interaction are presented in appendix A.

Most previous studies focused on the deformation and breakup of the droplet and the kinematic characteristics of the incident shock wave affected by the gas-liquid interface. In this section, a detailed and in-depth theoretical analysis of the characteristics of the flow field inside the liquid column, impacted by a planar shock wave, is carried out. The theoretical analysis results are compared with the numerical simulation results at $Ms = 2.4$, neglecting the effects of phase transition or taking the maximum tensile stress of pure water (Caupin & Herbert 2006) as the threshold.
pressure of cavitation.

In order to better understand and analyse the physical mechanisms of the flow phenomena, the early stage of the interaction process, before the droplet deforms significantly, is divided into three stages, mainly according to the flow characteristics. In the first stage, the transmitted shock wave is generated and propagates inside the column (figure 17a–g). The second stage (figure 17h–k) begins when the transmitted shock wave touches the right pole of the column, and the reflected expansion wave propagates and focuses in this stage. The main features of the third stage (figure 17l–p) are the kinematic characteristics of the second reflected wave.

For the universality of the analysis, the dimensionless time \( t^* \) is used, which is the ratio of the physical time \( t \) over the characteristic time \( \tau \), namely, \( t^* = t/\tau \). The characteristic time \( \tau = 2R_0/V_s \) is when the transmitted shock wave in the droplet moves from LP to RP, where \( R_0 \) is the initial droplet radius and \( V_s \) is the velocity of the transmitted shock wave inside the liquid column. The velocity of the transmitted shock wave can be estimated by Rankine-Hugoniot relation (Gavrilyuk & Saurel 2007), and its expression is as follows:

\[
V_s = \frac{\gamma_i + 1}{4} \left( u_p + \sqrt{u_p^2 + \frac{1}{(\gamma_i + 1)^2 c_{l,0}^2}} \right)
\]

where, \( c_{l,0} \) represents the sound speed of the liquid in the initial state, and \( u_p \) represents the velocity of particles behind the transmitted shock wave. As the acoustic impedance of air is considerably smaller than that of liquid, the velocity variation of the liquid particles before and after the impaction of the incident shock wave is insignificant, namely, \( u_p \ll c_{l,0} \). Therefore, according to (4.1), the transmitted shock wave velocity \( V_s \) inside the droplet can be approximated by the sound speed of liquid \( c_{l,0} \) in the initial state, as written in terms of \( V_s \approx c_{l,0} \).

4.1. The first stage: the generation and propagation of the transmitted shock wave

The first stage starts at \( t_0 \) when the point \( O \) on the wavefront of incident shock touches the LP of the water column. The part of the liquid-gas interface, immersed in the incident shock wave, is denoted as \( PP' \), where \( P \) and \( P' \) are the upper and lower endpoints of the immersed region as shown in figure 4(a), respectively. Only the upper part of the water column is discussed for the following analysis because the flow is
When the incident shock wave contacts with the cylindrical droplet, the point P will move upward along the LS at the velocity as the interaction continues. Let \( \theta \) be the angle between the normal vector of the shock front and the tangential vector of the water column surface at \( P \), and \( \theta \) also is the central angle corresponding to \( OP \). Initially, \( P, \ P', \) and \( O \) all coincide with \( LP \), and \( \theta \) equals zero.

At a specific contact angle \( \theta \), the contact point velocity \( V_{p,j} \) is derived from the following expression:

\[
V_{p,j} = \frac{V_0}{\sin \theta}
\]

(4.2)

where, the velocity of the incident shock \( V_0 \) is the product of the air sound speed \( c_{g,0} \) at the initial state and the incident shock Mach number \( Ms \).

By the Huygens principle (Haller et al. 2003 and Wu et al. 2018, 2021), at each time instant \( t \), an individual compression wavelet will be emitted at \( P \) that will propagate inside the water column with the velocity \( V_s \), which can be approximately substituted by \( c_{f,0} \). From (4.2), the generation rate of the new compression wavelet is much larger than the propagation speed of the wavelet at the initial stage of interaction. In other words, the emitted compression wavelets cannot exceed the contact point \( P \).

These compression wavelets form a shock envelope, which is also called the transmitted shock wave, denoted as \( T \), figure 4(a). As the interaction continues, the contact angle \( \theta \) increases and \( V_{p,j} \) decreases gradually. We define the time when \( V_{p,j} \) equals the compression wavelet propagation velocity as the critical time \( t_c \). The contact angle at \( t_c \) is called the critical contact angle \( \theta_c \). If \( t > t_c \), the transmitted shock wave will detach itself from the incident shock wave and the reflected shock wave, forming a precursor transmitted shock wave as shown in figure 4(b), and will propagate towards the RP of the water column. The expression of the critical contact angle is shown as:

\[
\theta_c = \arcsin \frac{V_0}{V_s} = \arcsin \left( Ms \cdot \frac{c_{g,0}}{c_{f,0}} \right)
\]

(4.3)

Initially, the air ahead of the incident shock wave and the cylindrical droplet are both in the environment of 101325 Pa and 293 K, and the sound speed of the water \( (c_{f,0}) \) and air \( (c_{g,0}) \) is 1482.3 m/s and 343.0 m/s, respectively. Consequently, the critical
contact angle $\theta_c$ calculated from (4.3) is $33.67^\circ$, at $Ms = 2.4$.

![FIGURE 4. Schematic diagram of the generation and propagation of transmitted shock and the ray analysis in the early stage of interaction: (a) the schematic diagram at critical time $t_c$; (b) the schematic diagram at the time instant, $t_1$, which is selected at the instant after the critical time.](image)

The path of the rays tracing the compression wavelets is demonstrated to understand the generation and propagation of transmitted shock. The motion of each compression wavelet generated from different contact points is represented by a series of rays emitted in different specified directions, figure 4(a). For one contact point, the rays represent the paths of the compression wavelet, and the length of the rays is equal to the propagation distance or the radius of the compression wavelet. The radius of the compression wavelet emitted from the contact angle $\theta$ at time $t$ is denoted as $r(\theta,t)$, and the centre of this compression wavelet is $O_\theta$. In addition, the time of $P$ moving from LP to $O_\theta$ is denoted as $t_\theta$, and its expression is as follows:

$$t_\theta = \frac{R_0 (1-\cos \theta)}{V_0}$$

(4.4)

Therefore, the radius of the compression wavelet can be written as:

$$r(\theta,t) = (t-t_0-t_\theta)V_S$$

(4.5)

As shown in figure 4(a) and figure 5(a), for the compression wavelet emitted from $O_\theta$, only rays in a specific direction will have a contribution to the envelope of the compression wavelets (the transmitted shock wave) at time $t$. Therefore, it is necessary
to understand the formation mechanism of the envelope of compression wavelets to analyse the evolution characteristics of the transmitted shock wave. The compression wavelets emitted by two different contact points $O_\theta$ and $O_{\theta+\Delta\theta}$ are selected for detailed discussion. The radius of two compression wavelets are $r_\theta(t)$ and $r(\theta+\Delta\theta,t)$, respectively. The intersection point of two compression wavelets is $G_\theta$, and the angle between $O_\theta G_\theta$ and $O_{\theta+\Delta\theta} G_\theta$ is denoted as $\alpha_\theta$, as shown in figure 5.

The expression of $\alpha_\theta$ can be derived from the law of cosines in the triangle $\triangle O_{\theta+\Delta\theta} O_\theta G_\theta$:

$$\cos \alpha_\theta = \frac{[r(\theta+\Delta\theta,t) + r(\theta,t)] [r(\theta,t) - r(\theta+\Delta\theta,t) + 2r_\theta \sin^2(\Delta\theta/2) + \frac{R_\theta \Delta\theta}{r(\theta,t) \Delta\theta}} \quad (4.6)$$

When $O_{\theta+\Delta\theta}$ is infinitely close to $O_\theta$ ($\Delta\theta \to 0^+$), and $\alpha_\theta$ is the angle between the vector of the emitted ray and the tangent vector of the water column at $O_\theta$. The endpoint of this ray is the unique contribution of the compression wavelet, emitted at $O_\theta$, on the transmitted shock wave (the envelope of compression wavelets). The expression of $\alpha_\theta$ is given by defining a dimensionless wave speed $\kappa$, the ratio of the velocity of the transmitted shock wave to that of the incident shock wave.

$$\cos \alpha_\theta = \lim_{\Delta\theta \to 0^+} \frac{[r(\theta,t) - r(\theta+\Delta\theta,t) + \frac{V_s \sin \theta}{V_0} = \kappa \sin \theta} \quad (4.7)$$

FIGURE 5. Schematic diagram of the transmitted shock wave generation and the ray analysis: (a) the schematic diagram at $t$ ($t^* = 0.136$); the enlarged view of the schematic diagram at $t$.

According to (4.3), it can be concluded that the emission angle $\alpha_{\theta_c}$ of the ray generated by the critical contact point $P_{\theta_c}$ is equal to zero. That is to verify that when
After the critical time $t_c$, the transmitted shock wave detaches itself from the incident shock and propagates inside the water column moving towards the RP. The reflected and transmitted waves are generated on the curved column surface, as shown in Figure 15(e). Since the acoustic impedance $Z_l$ in water is much larger than that in air, the reflected waves should be expansion waves, and the transmitted waves should be compression waves. The transmitted compression wave in the air is so weak that it is almost invisible in the numerical results.

In order to deeply understand the evolution characteristics of different types of waves, such as the transmitted shock wave and the reflected expansion wave inside the water column, the ray analysis method introduced above is adopted for further discussion. Only one particular ray is analysed for the infinite rays emitted from any compression wavelet, whose emission angle $\alpha_\theta$ satisfies (4.7). Meanwhile, the rays emitted from different contact points will be reflected symmetrically on the curved column surface for the convenience of the analysis.

![Diagram of ray analysis](image)

**FIGURE 6.** The schematic diagram of ray analysis: (a) the schematic diagram at $t (t^* =$...
After the critical time $t_c$, the rays emitted from the vicinity of the critical point $P_{oc}$ will be reflected on the column surface many times, as shown in figure 6(a). In contrast, the rays emitted from the vicinity of LP will not be reflected. According to (4.5), for a specific time instant $t$, the length of ray emitted from $O_\phi$ is $r(\theta,t)$. If the ray is reflected from the column surface, $r(\theta,t)$ will represent the total range of the ray before and after reflection. Furthermore, the ray may be reflected more than once at the column surface. For each ray, whether it has been reflected or it has been reflected for $N$ times, it can be divided by the emission angle $\alpha_\phi$ corresponding to the observation time $t$ and the contact angle $\theta$, and the specific expression is as follows:

For the ray emitted from $O_\phi$, if no reflection occurs at time $t$, the emission angle $\alpha_\phi$ will satisfy:

$$\alpha_\phi \geq \arcsin \frac{r(\theta,t)}{2R_0} = \arcsin \left[ \frac{t-t_0 - 1 - \cos \theta}{2V_0} V_S \right] = \alpha_\phi^{(1)}.$$

(4.8)

**FIGURE 7.** Numerical results at $Ms = 2.4$ and the schematic diagram of ray analysis in the first stage at time $t_2$ ($t^* = 0.845$): (a) the schematic diagram of ray analysis; (b) Numerical schlieren contour (top) and pressure contour (bottom); (c) the enlarged view of the region posted in figure 7(a,b) (the comparison between schematic diagram and numerical schlieren contour).

If the rays are reflected $N$ times ($N = 1, 2, 3, \ldots$), the angle $\alpha_\phi$ will satisfy:
\[ \alpha_\theta < \arcsin \frac{r(\theta,t)}{2NR_0} = \arcsin \left[ \frac{t-t_0}{2R_0} - \frac{1-\cos \theta}{2V_0} \right] \frac{V_s}{N} = \alpha_\theta^{(N)} \]

\[ \alpha_\theta \geq \arcsin \frac{r(\theta,t)}{2(N+1)R_0} = \arcsin \left[ \frac{t-t_0}{2R_0} - \frac{1-\cos \theta}{2V_0} \right] \frac{V_s}{N+1} = \alpha_\theta^{(N+1)} \]  \( (4.9) \)

Thus, the emission angle \( \alpha_\theta \) can be divided into different intervals according to reflection times \( N \), and the intervals of angle \( \theta \) can be obtained combining with \( (4.7) \).

The transmitted shock envelope is formed by the infinite rays emitted from different compression wavelets before the critical time \( t_c \). It is not difficult to prove that when \( t < t_c \), inequality \( r(\theta,t) < 2R_0 \cos \alpha_\theta \) is always authentic, that is, the transmitted shock wave does not reflect on the column surface. The reflection of these rays also represents the reflection of the wave structures on the column surface. For the rays whose emission angle \( \alpha_\theta \) within the interval \( [\alpha_\theta^{(2)}, \alpha_\theta^{(1)}] \), the one-time reflection will occur at \( t \), and the endpoints of these rays represent the head of the first reflection waves (the first reflected expansion wave), figure 6(b). Similarly, for \( \alpha_\theta \in [\alpha_\theta^{(3)}, \alpha_\theta^{(2)}] \), the rays will be reflected twice, and their endpoints represent the head of the second reflection wave. Thus, the position of the head of the \( N \)th reflection waves can be obtained according to the intervals of angle \( \alpha_\theta \).

As the precursory transmitted shock wave propagates forwards, the reflected expansion wave can be observed gradually, and a specific time, \( t^* = 0.845 \) (\( t_2 \)), is chosen for the following analysis. The numerical schlieren and the pressure contour, neglecting the effect of phase transition, together with the schematic of the emitted rays, are shown in figure 7. The position and shape of the reflected expansion wave are obtained from the analysis of the emitted rays.

The emitted rays, which can be divided into five regimes based on the intervals of angle \( \alpha_\theta \), corresponding to the non-reflected transmitted shock wave, the first reflected expansion wave, and the second reflected wave, are shown in figure 7(a). The five regimes consist: (I): \( \alpha_\theta^{(1)} \leq \alpha_\theta \leq \pi/2 \); (II): \( \alpha_\theta^{(i*)} \leq \alpha_\theta < \alpha_\theta^{(1)} \), (III): \( \alpha_\theta^{(2)} \leq \alpha_\theta < \alpha_\theta^{(i*)} \), (IV): \( \alpha_\theta^{(2*)} \leq \alpha_\theta < \alpha_\theta^{(2)} \) and (V): \( \alpha_\theta^{(3)} \leq \alpha_\theta < \alpha_\theta^{(2*)} \). In regime (I), the rays are non-reflected, and the endpoints of these rays correspond to the transmitted shock wave \( (T_1) \).

Meanwhile, the low limit of the emission angle interval can be obtained as \( \alpha_\theta^{(i)} = \arcsin[(t-t_0)/(\tau - \kappa(1-\cos \theta))] \), based on \( (4.8) \). The wave head of the first
reflected expansion wave, made up of the endpoints of the rays whose emission angle $\alpha_\phi$ lies in regime (II) and (III), is divided into the upper branch (Re-EW$_U$) and the lower branch (Re-EW$_L$), corresponding to the regime (II) and (III), respectively. Similarly, the wave head of the second reflected wave, made up of the endpoints of the rays whose emission angle $\alpha_\phi$ lies in regime (IV) and (V), is divided into the upper branch (Re$^2$-EW) and the lower branch (Re$^2$-CW), corresponding to the regime (IV) and (V), respectively. Moreover, for the regime (III) and (V), the low limit of the emission angle interval can be obtained as $\alpha_\phi^{(3)} = \arcsin\left\{\frac{(t-t_0)/\tau - \kappa(1-\cos\theta)}{2}\right\}$ and $\alpha_\phi^{(3)} = \arcsin\left\{\frac{(t-t_0)/\tau - \kappa(1-\cos\theta)}{3}\right\}$, respectively.

The analytical shape and position of the first reflected expansion waves (Re-EW$_U$ and Re-EW$_L$) overlap well with those visualised by the numerical schlieren results, figure 7(c). The second reflected waves (Re$^2$-EW and Re$^2$-CW) are too weak to be observed from the numerical schlieren results. Ideally, the emitted rays can be reflected infinite times, while the other reflected waves are weaker except for the first and second reflected waves. Therefore, they cannot be clearly recognised in numerical images.

In addition, a local minimum pressure regime can be observed, in figure 7(b), due to the superposition of the first reflected expansion wave, and a detailed analysis was discussed in Wu et al. (2018). Meanwhile, the pressure behind the lower branch of the first reflected expansion waves recovers swiftly because of the subsequent emitting compression wavelets catching up with the expansion wave.

4.2. The second stage: the propagation and focusing of the reflected expansion wave

The second stage occurs at the instant $t^* = 1.0$ ($t_3$) when the transmitted shock wave ($T_t$) moves to the RP of the water column and is entirely reflected by the column surface. The schematic of ray analysis, demonstrating the evolution characteristics of the reflected expansion wave, is presented in figure 8, and the numerical schlieren contour at $t_3$ is shown in figure 8(b). The rays emitted from the contact points whose contact angle is less than the critical angle $\theta_c$ are drawn, figure 8(a), and the first reflected rays are considered only to obtain the envelope of these one-time reflected rays. The schematic of Re-EW$_L$ and Re-EW$_U$ at four different time instants shows that the intersection points of Re-EW$_L$ and Re-EW$_U$, which always lie on the envelope of the one-time reflected rays, moves along the envelope from time $t_2$ to $t_3$. The two branches
of Re-EW\textsubscript{L}, from the top and bottom sides, meet at the horizontal axis of the cylindrical droplet and form a continuous contracting arched Re-EW\textsubscript{L} with the two Re-EW\textsubscript{U} following on either side. The arched Re-EW\textsubscript{L} gradually focuses on the centerline of the water column under the influence of the surface curvature. Figure 8 \textit{(b)} shows the evolution diagram of Re-EW\textsubscript{L} and Re-EW\textsubscript{U} at five different time instants, from \( t_3 \) to \( t_4 \) \( (t^* = 1.256) \) in the focusing of Re-EW\textsubscript{L}. The curvature radius of the arched Re-EW\textsubscript{L} gradually decreases as the reflected expansion wave propagates and disappears at the moment of complete focus. Meanwhile, the two following Re-EW\textsubscript{U} become wider as they propagate forwards. The two symmetrical Re-EW\textsubscript{U} merge when the Re-EW\textsubscript{L} focuses completely, forming a continuous emanative reflected expansion wave (Re-EW\textsubscript{C}), and the evolution diagram of Re-EW\textsubscript{C} at five different time instants from \( t_4 \) to \( t_5 \) \( (t^* = 1.58) \) is shown in figure 8\textit{(c)}. Moreover, the shape and position of Re-EW\textsubscript{C}, at \( t_5 \) and \( t_6 \) \( (t^* = 1.88) \), are shown in the schematic diagram of ray analysis, figure 8\textit{(d)}. Besides the first reflected expansion wave (Re-EW\textsubscript{C}), two other waves are also observed, which are the second reflected wave and the third reflected wave.
FIGURE 8. Computational results at $Ms = 2.4$ and schematic diagram of ray analysis: (a) All the first reflected rays emitted from different contact points with the contact angle of the interval $[0, \theta_c)$, and the schematic diagram of the first reflected expansion wave propagation from $t_2$ ($t^* = 0.845$) to $t_3$ ($t^* = 1.0$). (b) The comparison between schematic diagram and numerical schlieren contour at time $t_3$, and the schematic diagram of the first reflected expansion wave propagation from $t_3$ to $t_4$ ($t^* = 1.256$). (c) the schematic diagram of the first reflected expansion wave propagation from $t_4$ to $t_5$ ($t^* = 1.58$). (d) All of the emitted rays from different contact points with one-, two- and three-time reflection with the contact angle of the interval $[0, \theta_c)$, and schematic diagram of the reflected waves’ propagation from $t_5$ to $t_6$ ($t^* = 1.88$).

Like the first reflected expansion wave, the second and third reflected waves also have upper and lower branches. If only the difference of acoustic impedance across the liquid-gas interface is considered, the second reflected wave will be the compression wave with a single property. However, in the numerical simulation results (figure 15(l)) and the experiment images (Sembian et al. 2016), it is found that the upper and lower branches of the second reflected wave have entirely different properties, namely, the compression wave ($Re^2-CW$) and the expansion wave ($Re^2-EW$) respectively. In the
present study, a speculative explanation is given: under the joint influence of the expansion angle of the first reflected expansion wave and the curvature of the liquid-gas interface, the second reflected wave has two branches of different properties. Three envelopes representing the trajectories of the intersection points of the upper and lower branches of the first, second, and third reflected waves, respectively, are presented in figure 8(d).

According to the analysis in § 4.1 and referring to the studies of Obreschkow et al. (2011) and Wu et al. (2018), the motion characteristics of the one-time reflected ray are the same as that of the first reflected expansion wave. Therefore, these one-time reflected rays can be used to analyse the position of the focusing point of Re-EW₁, denoted as \( F_{\text{Re-EW}} \). Some rays emitted by the contact point whose contact angle is less than the critical contact angle \( \theta_c \) are presented in figure 9(a). In order to ensure that the one-time reflected rays can intersect with the horizontal central axis, the emission angle \( \alpha_\theta \) and the contact angle \( \theta \) will satisfy: \( 4\alpha_\theta + \theta \geq 180^\circ \). Select an arbitrary ray (the black solid lines) for analysis in figure 9(a), the intersection points of the one-time reflected part of this ray with the column surface and the horizontal central axis are \( E_\theta \) and \( F_\theta \), respectively, and the distance \( \lambda_\theta \) between the intersection \( F_\theta \) and the RP can be expressed as follows:

FIGURE 9. Computational results at \( Ms = 2.4 \) and schematic diagram of ray analysis: (a) the schematic diagram of ray analysis for the focus of the one-time reflected rays; (b) the comparison between schematic diagram and numerical results at \( t' = 1.283 \). The bottom side presents the numerical pressure contour, while the top side compares the schematic diagram and numerical schlieren contour.
\[
\lambda_\theta = R_0 - \frac{\sin \beta_\theta}{\sin \phi_\theta} R_0
\]  

(4.10)

where, the angles \( \beta_\theta \) and \( \phi_\theta \) are shown in figure 9(a), and the distance \( \lambda_\theta \) can be rewritten as expressions for \( \theta \):

\[
\lambda_\theta = R_0 \left[ 1 - \frac{\kappa}{\sqrt{1 - \kappa^2 \sin^2 \theta \cdot (4\kappa^2 \sin^2 \theta - 1) + \cos \theta \cdot (3\kappa - 4\kappa^3 \sin^2 \theta)}} \right]
\]  

(4.11)

A maximum limiting value of \( \lambda_\theta \) exists. Accordingly, the position of \( R_{Re-EW} \), as well as the focus point of the one-time reflected rays, is obtained, which has the following expression (the detailed derivation can be found in appendix B):

\[
\lambda_{\text{max}} = \lim_{\theta \to 0} \left[ R_0 - \frac{\kappa R_0}{\sqrt{1 - \kappa^2 \sin^2 \theta \cdot (4\kappa^2 \sin^2 \theta - 1) + \cos \theta \cdot (3\kappa - 4\kappa^3 \sin^2 \theta)}} \right]
\]  

(4.12)

\[
= R_0 - \frac{\kappa R_0}{3\kappa - 1} = \frac{2\kappa - 1}{3\kappa - 1} R_0
\]

Figure 9 (b) shows the wave structures and pressure contour inside the water column, obtained from the ray analysis and numerical simulation when \( R_{Re-EW} \) focuses entirely \( (r^* = 1.283) \). It can be found that when the phase transition effect is not taken into account, the liquid near the focus point bears an extreme negative pressure, which is more than -90 times the initial pressure. Moreover, the position of \( R_{Re-EW} \) obtained by ray analysis almost coincides with the numerical result. Wu et al. (2018) observed that the reflected expansion waves focus at a position 2/3 of the droplet radius away from the top pole of the water column in the study of the high-speed impingement of droplets on a rigid wall. In fact, since the droplet impact velocity \( (110 \text{ m/s}) \) is much smaller than the transmitted shock wave propagation velocity \( (1500 \text{ m/s}) \), the \( \kappa \) will be significantly greater than 1.0. The focusing point obtained from (4.12) also degenerates to the same position. Hence, the position of the focusing point obtained in the present study is more general and extendable.

Since the switch of phase transition is shut down in this section, no phase transition phenomenon is captured. However, Poulanges & Rabii (2021) and Xiang & Wang (2017) expounded that the focusing area of the reflected expansion wave has a high possibility for cavitation, and it was verified through the experimental observations in
Sembian et al. (2016). Therefore, the phase transition effect will be considered in § 5, and the cavitation behaviours inside the water column due to the focus of the reflected expansion waves will be deeply investigated. When Re-EW_C moves to the LP and is entirely reflected by the column surface, the second stage of interaction between shock wave and water column ends, and the third stage begins.

4.3. The third stage: the propagation and focusing of the second reflected wave

The third stage begins when Re-EW_C reaches the LP of the water column at the time instant \( t^* = 2.0 \) \((t_7)\). Similar to the second-stage analysis in § 4.2, the evolution characteristics of Re^2-CW and Re^2-EW at four instants from \( t_7 \) to \( t_8 \) \((t^* = 2.36)\) are shown in figure 10(a). Figure 10(b) compares the schematic diagram and numerical simulation results at \( t_8 \).

![Figure 10](image)

**FIGURE 10.** Computational results at \( Ms = 2.4 \) and schematic diagram of ray analysis: (a) Schematic diagram of the second reflected wave propagation from \( t_7 \) \((t^* = 2.0)\) to \( t_8 \) \((t^* = 2.36)\), and schematic diagram of the reflected waves’ propagation at time \( t_8 \) \((a,b)\); (b) Numerical schlieren contour (top) and pressure contour (bottom) at time \( t_8 \).

Under the effect of the surface curvature, the one branch of the second reflected wave (Re^2-CW) will focus on the horizontal central axis near the upstream stagnation point, causing a high-pressure region whose maximum pressure reaches more than 60 times the initial pressure, as shown in figure 10(b). At the same time, the two symmetrical reflected expansion waves (Re^2-EW), located on the upper and lower sides of Re^2-CW, merge, causing a rapid decrease of the local liquid pressure. The pressure distribution curves on the centerline of the water column at six different time instants, before and after the focusing of Re^2-CW, are shown in figure 11. Obviously, the maximum pressure in the high-pressure region increases rapidly as the focus process
continues, before the complete focus of Re\textsuperscript{2}-CW. After that, a significant negative pressure region appears inside the water column, and the maximum pressure in the high-pressure region decreases rapidly and disappears.

Similar to the transmitted shock wave, the first reflected wave, and the second reflected wave, the third reflection compression wave will also reflect on the column surface, forming the quaternary reflected wave. Due to the curvature of the column surface, the quaternary reflected wave has two branches with entirely different properties, namely, the compression wave (Re\textsuperscript{4}-CW) and the expansion wave (Re\textsuperscript{4}-EW), respectively. As the interaction continues, the high-pressure region and the low-pressure region appear alternately at different positions on the centerline, and the analysis is similar to the focus of the first reflected expansion wave and the second reflected compression wave. Nonetheless, we will not elaborate further on this in the present study.

![Figure 11](image.png)

**FIGURE 11.** Dimensionless pressure value along the horizontal centerline of the water column for \( Ms = 2.4 \) at six different time instants.

Due to the different properties of the two branches of the second reflected wave, the liquid, near the focusing point of Re\textsuperscript{2}-CW, undergoes a violent pressure oscillation in a short time. The high-transient characteristics of liquid pressure have great significance in the biomedical field, which can remove diseased tissue successfully.

Similarly, the specific position of the Re\textsuperscript{2}-CW focus point can be obtained by ray analysis. A schematic diagram of rays emitted from some contact points when Re\textsuperscript{2}-CW focuses is shown in figure 10(c), and select an arbitrary ray (the black lines) for further
analysis. Denote the intersection points of the two-time reflected part of this ray with the column surface and the horizontal central axis as $E_2$ and $F_2$, respectively, and the distance between $F_2$ and $F$ is $\delta_0$:

$$\delta_0 = \frac{\sin(2\beta_0)}{\sin(\phi_0 - 2\beta_0)} \left( 2\cos \beta_0 - \frac{\sin(\pi - \beta_0 - \phi_0)}{\sin \phi_0} \right) R_0$$

(4.13)

where, the angles $\beta_0$ and $\phi_0$ are shown in figure 9(a), and the minimum distance of $\delta_0$ can be obtained as $\delta_{\text{min}}$, which has the following expression:

$$
\delta_{\text{min}} = \lim_{\theta \to 0} \delta_0 = \lim_{\theta \to 0} \left[ \frac{\sin(2\beta_0)}{\sin(\phi_0 - 2\beta_0)} \left( 2\cos \beta_0 - \frac{\sin(\pi - \beta_0 - \phi_0)}{\sin \phi_0} \right) R_0 \right]
$$

(4.14)

The physical meaning of $\delta_{\text{min}}$ can be recognised as the distance from the focus point of Re-EW$_L$ to that of RE$^2$-CW. The detailed derivation is presented in appendix B.

When the Re$^2$-EW touches the RP, the second reflected wave is wholly reflected to form the third reflected wave, and the third stage ends accordingly. In the subsequent interactions, new reflected waves are constantly generated and disappear. However, the intensity of these waves has been dramatically reduced compared with the intensity of the transmitted shock wave, the first reflected wave, and the second reflected wave, and the analysis methods are similar.

5. Cavitation inside the water column

As seen in the preceding section, a local low-pressure region is generated at the intersection of the upper and lower branches of the first reflected wave. The degree of negative pressure will increase rapidly with the Re-EW$_L$ focusing. When the incident shock wave intensity is chosen as $M_s = 2.4$, the negative pressure generated at the complete focusing reaches more than -90 times the initial pressure. It is well known that when the liquid is subjected to tensile stress exceeding a critical value depending on the nature of the liquid and its purity, the liquid ruptures, and cavitation occurs. For pure liquids, cavitation arises from microscopic bubbles caused by random thermal motions of molecules (Balibar & Caupin 2002), i.e., homogeneous nucleation cavitation. In contrast, when the liquid contains impurities, the maximum tensile stress they can withstand drastically drops termed as heterogeneous nucleation cavitation.
because of the expansion of submicroscopic gas pockets trapped at the two-phase interface on column surface or particles present in the liquid. Water, in particular, has a wide range of tensile stress limits. The maximum tensile stress that pure water can withstand is about 140 MPa at 300 K, which is expected by classical nucleation theory (Fisher 1948), and its reliability in predicting homogeneous cavitation has been verified by experiments (Azouzi et al. 2012) and numerical simulations (Menzl et al. 2016). However, for unpurified water, the maximum tensile stress is several orders of magnitude lower than pure water, about 0.1-1.0 MPa provided by Caupin & Herbert (2006). Given such a disparity in tensile limits for water, the choice of the liquid cavitation threshold pressure will directly affect the cavitation behaviours within the liquid column. In order to realise a relatively apparent cavitation phenomenon, the saturated vapour pressure as the cavitation threshold pressure is used in the present study. The saturated vapour pressure of water at room temperature is about 2,337 Pa (Wu et al. 2019).

FIGURE 12. Numerical schlieren contours (top) and pressure contours (bottom) at different time intervals for $M_s = 2.4$, considering the effect of phase transition: (a) $t = 3.0 \mu s$; (b) $t = 5.68 \mu s$; (c) $t = 10.10 \mu s$; (d) $t = 10.46 \mu s$. 
Figure 12 presents the evolution of cavitation behaviour inside the water column, impacted by a planar shock wave, at \( M_s = 2.4 \). The reflected expansion wave is formed when the transmitted shock wave is reflected at the RS. A violent phase transition occurs under the focusing of the reflected expansion wave and once the local fluid pressure is lower than the cavitation threshold pressure, as shown in figure 12(a). As the continued focus of the reflected expansion wave, the cavitation degree inside the droplet will gradually increase, eventually forming a large-scale cavitation zone, figure 12(c). Because of the enormous energy conversion in the phase transition process, the thermodynamic state of the liquid near the cavitation zone restores gradually. After which, the cavitation degree gradually decreases and entirely collapses at a particular position on the centerline of the water column, forming a collapsing shock wave, figure 12(e). The collapsing centre of the cavitation zone is denoted as \( F_{\text{collapse}} \). The collapsing shock wave propagates from the collapsing centre to the surroundings, as shown in figure 12(f). Furthermore, an expansion wave will be formed due to the reflections of the collapsing shock on the column surface, which can induce the second cavitation near the liquid surface. However, the cavitation behaviours after the collapse of the first cavitation zone are weak and short, so the present study will not discuss and analyse them in detail.

![Figure 13](image)

**FIGURE 13.** The characteristic cavitation parameters related to different incident shock wave intensities: (a) the maximum cavitation amount \( c_{\text{max}} \); (b) the maximum collapsing pressure \( p_{c,\text{max}} \).

Combined with the analysis in § 4.4, it is observed that the collapse centre of the cavitation zone is not consistent with the focus point of the first reflected expansion wave \( (F_{\text{Re-EW}}) \), and the main reason is the choice of the cavitation threshold pressure.
The minimum pressure that the liquid can withstand and the minimum pressure generated when the first reflected expansion wave focuses entirely are recorded as $p_{\text{threshold}}$ and $p_{\text{fmax}}$, respectively. When $p_{\text{threshold}}$ is less than $p_{\text{fmax}}$, no cavitation phenomenon occurs inside the water column. If the value of $p_{\text{threshold}}$ is increased to be precisely equal to that of $p_{\text{fmax}}$, only the liquid at $F_{\text{Re-EW}}$ will enter the critical state. The cavitation occurs, and $F_{\text{collapse}}$ coincidences with $F_{\text{Re-EW}}$. Moreover, with the continual increase of $p_{\text{threshold}}$, the area of cavitation zone inside the water column grows gradually, and the collapse centre $F_{\text{collapse}}$ gradually deviates from the focus point $F_{\text{Re-EW}}$. In the present study, the saturated vapour pressure, the maximum limit of $p_{\text{threshold}}$, is chosen for numerical simulations, and the distance between $F_{\text{collapse}}$ and $F_{\text{Re-EW}}$ reaches the maximum limit.

Poulanges & Rabii (2021) found that with the increase of the incident shock wave intensity, the stronger the negative pressure effect caused by the first reflected expansion wave, the greater the possibility of cavitation. In this section, different incident shock wave intensities, $Ms = 2.0, 2.4, 3.0$ and $3.6$, are selected to investigate the influence of the shock wave intensity on the cavitation characteristic parameters. The maximum cavitation amount $c_{\text{max}}$ (the ratio of the vapour phase volume to the initial volume of the liquid water) significantly increases with the increase of $Ms$, and the value of $c_{\text{max}}$ at $Ms=3.6$ is about 25 times that at $Ms=2.0$, figure 13(a). In addition, from figure 13(b), the maximum collapsing pressure $p_{c,\text{max}}$, generated by the collapsing of cavitation zone, is approximately linearly related to the incident shock wave intensity $Ms$, and $p_{c,\text{max}}$ reaches about 1000 times the initial pressure $p_i$, at $Ms = 3.6$. Namely, the collapsing shock wave is much stronger than the intensity of the transmitted shock wave. Therefore, the collapse of vapour bubbles can be used to realise high energy convergence and peak pressure amplification, as Wu et al. (2022) reported.
The dimensionless parameter $l/R_0$, the distance from the RP to the focus point of $F_{Re-EW}$ and the collapsing centre $F_{\text{collapse}}$, versus the incident shock wave intensity $Ms$, varies from 1.6 to 4.0.

The relations between the dimensionless parameter $l/R_0$, the distance from the RP to the focusing point $F_{Re-EW}$ and the collapsing centre $F_{\text{collapse}}$, and the incident shock wave intensity $Ms$ are presented in figure 14. Obviously, with the increase of the incident shock wave intensity, the focusing point of the first reflected expansion wave and the collapse centre of the first cavitation zone both move towards the RP along the horizontal centerline of the column. Moreover, other than that, the theoretical result of $F_{Re-EW}$ is always well coincided with that obtained from numerical simulations. Of course, the position of $F_{\text{collapse}}$ calculated in the present study is based on the saturated vapour pressure as the cavitation threshold pressure. When the cavitation pressure threshold changes, the specific position of $F_{\text{collapse}}$ will also change.

6. Conclusion

In this paper, we have examined the early stage of the interaction of a planar shock wave with a cylindrical droplet. The analysis was conducted using the ray analysis, which has not been presented in previous work, and the analytical results were compared with and complemented by numerical simulations. The promoted two-phase multi-component compressible fluid model, coupled with a phase-transition model describing the homogeneous cavitation process, was used to analyse the physical
mechanism, such as the generation and propagation of complex wave structures and the inception collapsing of the cavitation zone.

A detailed analysis showing the evolution and interaction of the complicated waves inside the column has been presented. The transmitted shock wave is generated immediately after the impaction of a planar shock on the liquid column, and it propagates inside the droplet. When the contact angle $\theta$ is larger than its critical value $\theta_c$, determined by the incident shock wave intensity and the sound speed ratio between gas and liquid, the transmitted shock wave will detach from the incident shock wave. Furthermore, the precursive transmitted shock wave continually propagates towards the RP. Meanwhile, the envelope of all compression wavelets emitted from different contact points, whose contact angle $\theta$ is less than the critical value, corresponds to the transmitted shock wave. Furthermore, only one particular ray contributes to the envelope of compression wavelets for the infinite rays emitted from any compression wavelet. The transmitted shock wave will be reflected by the surface of the liquid column and generate a series of expansion waves. The convergence of the reflected expansion waves can induce low-pressure regions, which directly causes cavitation inside the droplet. And, those expansion waves gradually focus at the horizontal axis, the position of which is wholly determined by the dimensionless wave speed $\kappa$. In addition, a high-transient pressure region was observed near the focusing area of the second reflected wave, whose two branches have opposite properties, which have not been discovered and analysed in previous studies.

The cavitation behaviours inside the droplet under the impaction of a planar shock wave, with different intensities, are investigated by choosing the saturated vapour pressure as the cavitation pressure threshold. It is found that the cavitation degree inside the water column gradually intensifies and forms a remarkable cavitation zone as the convergence of the reflected expansion wave continues. After complete focusing, the cavitation zone will gradually shrink until it collapses, generating a collapsing shock wave of extremely high intensity. The location of the collapsing centre $F_{\text{collapse}}$ is closely related to intensities of the incident shock and selections of the cavitation threshold pressure. The numerical results showed that the intensity of incident shock significantly influences the strength of wave structures, which increases with the Mach number of
the incident shock wave. Moreover, as the intensity of incident shock increases, the degree of cavitation inside the droplet increases rapidly, and the intensity of collapsing shock also strengthens nearly linearly.

Based on the above analysis, the evolution mechanism and essential influencing factors of wave structures and cavitation behaviour are explained in detail, guiding practical applications. The future work will further analyse the influence on cavitation of different purities of water and the shapes of incident shock waves.

7. Appendix A

This appendix obtains the dynamic evolution characteristics of flow field structures in the early stage of the interaction between a planar shock and a cylindrical droplet at $Ms = 2.4$ by the numerical models mentioned in § 2, as shown in figure 15. At the same time, the phase transition effects are neglected. Due to the large acoustic impedance mismatch ($\delta Z \sim 1.5 \times 10^6[\text{kg/(m}^2\cdot\text{s})]$), a strong shock is reflected, and a weak shock is transmitted within the droplet. As the refracted shock reaches the downstream interface, a weak expansion wave is internally reflected, and no significant shock is transmitted into the air. Due to the unique curvature of the droplet surface, a low-pressure area generated by the convergence of reflected expansion waves and a high-pressure area generated by the convergence of reflected compression waves will appear successively inside the droplet. In addition, as the contact angle between the incident shock and the cylindrical droplet increases, the reflection types of the incident shock on the droplet surface will change from the regular reflection to the irregular Mach reflection, forming the Mach stem, the slip line and the triple point.
FIGURE 15. Numerical schlieren contours (top) and pressure contours (bottom) at different time intervals for $Ms = 2.4$, neglecting the phase transition effect.

8. Appendix B

This appendix outlines the derivation of the positions of the focus point of the first reflected expansion wave and the second reflected wave, detailly. For the focus point of the reflected expansion wave, as sketched in figure 10(a), the distance $\lambda_0$ between the intersection $F_\theta$ and the right pole (RP) of the column surface can be expressed as follows:
\[ \lambda_\theta = \left(1 - \frac{\sin \beta_\theta}{\sin \varphi_\theta}\right) R_0 \]  

(4.15)

where, the angles \( \beta_\theta \) and \( \varphi_\theta \) satisfy: \( \beta_\theta = 90^\circ - \alpha_\theta \) and \( \varphi_\theta = 3\alpha_\theta + \theta - 90^\circ \), respectively. Hence, the distance \( \lambda_\theta \) can be rewritten as follows:

\[ \lambda_\theta = \left[1 - \frac{\cos \alpha_\theta}{\cos (3\alpha_\theta + \theta)}\right] R_0 \]  

(4.16)

The denominator \( \cos (3\alpha_\theta + \theta) \) can be expressed to:

\[ \cos (3\alpha_\theta + \theta) = \sin \theta \cdot \left(4\sin^3 \alpha_\theta - 3\sin \alpha_\theta\right) + \cos \theta \cdot \left(4\cos^3 \alpha_\theta - 3\cos \alpha_\theta\right) \]  

(4.17)

According to (4.7) and (4.17), eliminating the angle \( \alpha_\theta \) and the distance \( \lambda_\theta \) can be reformed only about \( \theta \):

\[ \lambda_\theta = \left[1 - \frac{\kappa}{\sqrt{1 - \kappa^2 \sin^2 \theta \cdot (4\kappa^2 \sin^2 \theta - 1) + \cos \theta \cdot (3\kappa - 4\kappa^3 \sin^2 \theta)}\right] R_0 \]  

(4.18)

Taking the limit of \( \theta \) in (4.18), and obtaining the maximum limiting value of \( \lambda_\theta \):

\[ \lambda_{\text{max}} = \lim_{\theta \to 0} \left[1 - \frac{\kappa}{\sqrt{1 - \kappa^2 \sin^2 \theta \cdot (4\kappa^2 \sin^2 \theta - 1) + \cos \theta \cdot (3\kappa - 4\kappa^3 \sin^2 \theta)}\right] R_0 \]

\[ = \left(1 - \frac{\kappa}{3\kappa - 1}\right) R_0 = \frac{2\kappa - 1}{3\kappa - 1} R_0 \]  

(4.19)

For the focus point of the second reflected wave, as sketched in figure 13, the distance \( \delta_\theta \) between the intersections \( F_\theta \) and \( F_\theta' \) can be expressed as follows:

\[ \delta_\theta = \frac{\sin(2\beta_\theta)}{\sin(\varphi_\theta - 2\beta_\theta)} \cdot \left[2\cos \beta_\theta - \frac{\sin(\pi - \beta_\theta - \varphi_\theta)}{\sin \varphi_\theta}\right] R_0 \]  

(4.20)

Substituting the relations of \( \beta_\theta \) and \( \varphi_\theta \) for \( \alpha_\theta \) and \( \theta \) into (4.20) yields

\[ \delta_\theta = \frac{\sin(2\alpha_\theta)}{\cos(5\alpha_\theta + \theta)} \left[2\sin \alpha_\theta + \frac{\sin(2\alpha_\theta + \theta)}{\cos (3\alpha_\theta + \theta)}\right] R_0 \]  

(4.21)

Denoting \( \cos(5\alpha_\theta + \theta)/\sin(2\alpha_\theta) \) and \( 2\sin \alpha_\theta + \sin(2\alpha_\theta + \theta)/\cos (3\alpha_\theta + \theta) \) as \( \delta_{\text{01}} \) and \( \delta_{\text{02}} \), respectively. Similarly, eliminating the angle \( \alpha_\theta \) and \( \delta_{\text{01}} \) can be reformed as follows:
\[
\delta_{\theta_1} = \frac{(16\cos^4 \alpha_\theta - 20\cos^2 \alpha_\theta + 5)\cos \theta}{2\sin \alpha_\theta} - \frac{(16\sin^4 \alpha_\theta - 20\sin^2 \alpha_\theta + 5)\sin \theta}{2\cos \alpha_\theta}
\]
\[
= \frac{\cos \theta (16\kappa^4 \sin^4 \theta - 20\kappa^2 \sin^2 \theta + 5)}{2\sqrt{1 - \kappa^2 \sin^2 \theta}} - \frac{16\kappa^4 \sin^4 \theta - 12\kappa^2 \sin^2 \theta + 1}{2\kappa}
\]

Similarly, according to (4.7), eliminating the angle \(\alpha_\theta\) and \(\delta_{\theta_2}\) can be reformed as follows:

\[
\delta_{\theta_2} = 2\sin \alpha_\theta + \frac{2\cos \theta \sin \alpha_\theta \cos \alpha_\theta + \sin \theta \left(2\cos^2 \alpha_\theta - 1\right)}{\sin \theta \sin \alpha_\theta \left(1 - 4\cos^2 \alpha_\theta\right) + \cos \theta \cos \alpha_\theta \left(4\cos^2 \alpha_\theta - 3\right)}
\]
\[
= \frac{2\kappa \cos \theta \sqrt{1 - \kappa^2 \sin^2 \theta} + \left(2\kappa^2 \sin^2 \theta - 1\right)}{\sqrt{1 - \kappa^2 \sin^2 \theta} \left(1 - 4\kappa^2 \sin^2 \theta\right) + \cos \theta \left(4\kappa^3 \sin^2 \theta - 3\kappa\right)}
\]

Synthesise the above analysis, and the distance \(\delta_{\theta}\) can be regarded as:

\[
\delta_{\theta} = \left(\frac{\delta_{\theta_2}}{\delta_{\theta_1}}\right)R_0
\]

Taking the limit of \(\theta\) in (4.22) and (4.23), and obtaining the maximum limiting value of \(\delta_{\theta}\):

\[
\delta_{\theta_{\text{min}}} = \lim_{\theta \to 0} \left(\frac{\delta_{\theta_2}}{\delta_{\theta_1}}\right)R_0 = \frac{\lim_{\theta \to 0} \delta_{\theta_2}}{\lim_{\theta \to 0} \delta_{\theta_1}}R_0 = \frac{(4\kappa - 1)/(3\kappa - 1)}{(5\kappa - 1)/2\kappa} = \frac{2\kappa(4\kappa - 1)}{(5\kappa - 1)(3\kappa - 1)}
\]