Decidability and Complexity for Quiescent Consistency and its Variations

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Abstract
Quiescent consistency is a notion of correctness for a concurrent object that gives meaning to the objects' behaviors in quiescent states, i.e., states in which none of the objects' operations are being executed. Correctness of an implementation object is defined in terms of a corresponding abstract specification. This gives rise to two important verification questions: membership (checking whether a behavior of the implementation is allowed by the specification) and correctness (checking whether all behaviors of the implementation are allowed by the specification). In this paper, we show that the membership problem for quiescent consistency is NP-complete and that the correctness problem is decidable, but coNP-hard and in EXPSPACE. For both problems, we consider restricted versions of quiescent consistency by assuming an upper limit on the number of events between two quiescent points. Here, we show that the membership problem is in PTIME, whereas correctness is in PSPACE.

Quiescent consistency does not guarantee sequential consistency, i.e., it allows operation calls by the same process to be reordered when mapping to an abstract specification. Therefore, we also consider quiescent sequential consistency, which strengthens quiescent consistency with an additional sequential consistency condition. We show that the unrestricted versions of membership and correctness are NP-complete and undecidable, respectively. When placing a limit on the number of events between two quiescent points, membership is in PTIME, while correctness is in PSPACE. Finally, we consider a version of quiescent sequential consistency that places an upper limit on the number of processes for every run of the implementation, and show that the membership problem for quiescent sequential consistency with this restriction is in PTIME.

1 Introduction

Due to the possibility of parallel executions, correctness of an operation of a concurrent object cannot be stated in terms of pre/post conditions. Instead, correctness is expressed in terms of a history of operation invocation/response events, capturing the interaction between a concurrent object and its client. There are many notions of correctness for safety [17, 10] — relaxed notions are more permissive, and hence, allow greater flexibility in an object’s design. Such flexibility is necessary in the presence of observations such as Amdahl’s Law and Gustafson’s Law [14], which show that sequential bottlenecks within an implementation must be reduced to improve performance [30].

This paper studies quiescent consistency [30, 8] a relaxed notion of correctness for concurrent objects, derived from a similar notion in replicated databases [11], that gives meaning to an object in its quiescent states, i.e., states in which none of its operations are currently executing. Here, correctness is defined by mapping a concurrent object’s history (with potentially overlapping operation calls) to a sequential history of its corresponding specification object (with no overlapping operation calls).

1. A history of a concurrent object is considered to be correct with respect to a correctness condition \( C \) iff the history can be mapped to a valid (sequential) history of the object’s specification and the
mapping satisfies \( C \).

2. A concurrent object satisfies \( C \) iff each of its histories is correct with respect to \( C \).

These two issues give rise to two distinct verification problems: the former gives rise to a membership problem, and the latter a correctness problem. In this paper, we extend the existing approach of Alur et al. \(^3\) and study the decidability and complexity of both membership and correctness for two correctness conditions: quiescent consistency and quiescent sequential consistency.

Informally speaking, quiescent consistency is defined as follows. A concurrent object is said to be in a quiescent state if none of its operations are being executed in that state. Quiescent consistency allows operations calls in a concurrent history between two consecutive quiescent states to be reordered when mapping to history of the sequential specification, but disallows events that are separated by a quiescent state from being reordered \(^10\) \(^8\) \(^30\). Compared to other conditions in the literature, quiescent consistency is more permissive. For example, unlike linearizability \(^18\) \(^17\) \(^10\), it allows the effects of operation calls to be reordered even if they do not overlap in a concurrent history. Unlike sequential consistency \(^24\) \(^17\) \(^10\), it allows the effects of operation calls by the same process to be reordered.

In the context of client-object systems, to guarantee observational refinement \(^15\) of the client, it turns out that it is necessary ensure process order is maintained \(^12\). Therefore, we additionally consider quiescent sequential consistency, a variation of quiescent consistency that adds a sequential consistency constraint \(^24\) to quiescent consistency, i.e., we are not allowed to reorder the events of the same process.

In this paper, we make the following main contributions.

1. We describe how quiescent consistency can be expressed using independence from Mazurkiewicz Trace Theory \(^25\) and encoded as finite automata.

2. Show that deciding membership for quiescent consistency is an NP-complete problem if the number of events between two quiescent states is unbounded, but deciding membership is polynomial (with respect to the size of the input run) if the number of events between two quiescent states has a fixed upper bound.

3. Show that correctness for quiescent consistency is decidable, coNP-hard, and in EXPSPACE, but correctness for quiescent consistency is in PSPACE if the number of events between two quiescent states has a fixed upper bound.

4. Show that deciding membership for quiescent sequential consistency is an NP-complete problem but can be solved in polynomial time (with respect to the size of the input run) if the number of events between two quiescent states has a fixed upper limit, or if the number of processes can be predetermined.

5. Show that correctness for quiescent sequential consistency is undecidable but is in PSPACE if the number of events between two quiescent states has a fixed upper bound.

This paper is organised as follows. In Section 2 we motivate the problem through an example, and describe the formal background of finite automata and independence used in the rest of the paper. Section 3 defines quiescent consistency, develops a finite automata encoding of quiescent consistency as well as the membership and correctness problems. Our results for the membership and correctness problems are given in Section 4 and Section 5 respectively. Section 6 describes quiescent sequential consistency, then Sections 7 and 8 present the results for membership and correctness for quiescent sequential consistency, respectively. A survey of related work and concluding remarks is given in Section 9.

### 2 Background

This section motivates quiescent consistency with a queue example (Section 2.1), then gives a finite automata formalisation for studying the problem (Section 2.2). We will use a notion of independence from Mazurkiewicz Trace Theory, which we describe in Section 2.3.
2.1 A quiescent consistent queue

We consider the quiescent consistent queue from [8] (see Figs. 1 and 2). The queue is based on the architecture of diffracting trees, which uses the following principle (adapted from counting networks [4]). Elements called balancers are arranged in a binary tree, which may have arbitrary depth. Each balancer contains one bit, which determines the direction in which the tree is traversed; a balancer value of 0 causes a traversal up and a value 1 causes a traversal down. The leaves of the tree point to a concurrent data structure. Operations on the tree (and hence data structures) start at the root of the tree and traverse the tree based on the balancer values. Each traversal is coupled with a bit flip, so that the next traversal occurs along the other branch. Upon reaching a leaf, the process performs a corresponding operation on the data structure at the leaf.

Our example consists of two 1-level balancers \( eb \) and \( db \) used by enqueue and dequeue operations, respectively. Both operations share the two queues at the leaves (see Fig. 1). Pseudocode for the queue is given in Fig. 2. Both operations are implemented using a non-blocking atomic CAS (Compare-And-Swap) operation that compares the stored local value \( old \) with the shared variable \( var \) and updates \( var \) to a new value \( new \) if the values of \( var \) and \( old \) are still equal:

\[
\text{CAS}(\text{var}, \text{old}, \text{new}) = \begin{cases} 
\text{atomic} \{ & \text{if var} = \text{old} \\
& \text{then var} := \text{new}; \text{return true} \\
& \text{else return false} 
\end{cases}
\]

Both operations read their corresponding bit and try to flip it using a CAS. If they succeed, they perform an enqueue \( \text{Enq} \) or dequeue \( \text{Deq} \) on the queue of their local bit. For simplicity, we assume that \( \text{Enq} \) and \( \text{Deq} \) are atomic operations (though they could be implemented by any linearizable operation). The queue only satisfies quiescent consistency if \( \text{Deq} \) is blocking, i.e., waits until an element is found in the queue. The diffracting queue is not quiescent consistent if \( \text{Deq} \) returns on empty (see [8] for details).

Example 1. The following is a possible history for the blocking concurrent queue implementation:

\[ h_1 = D_1 E_2(a) \hat{E}_2 E_3(b) \hat{E}_3 D_4 \hat{D}_4(b) D_5 \hat{D}_5(a) E_6(c) \hat{E}_6 \hat{D}_1(c) \]

where \( D_1 \) denotes a dequeue invocation by process 1, \( \hat{D}_1(c) \) denotes a dequeue by process 1 that returns \( c \), \( E_2(a) \) denotes an enqueue invocation by process 2 with input \( a \), and \( \hat{E}_2 \) denotes the corresponding return event. There is not much concurrency in \( h_1 \). Only the first dequeue is running concurrently with the rest of the operations. However, due to the first dequeue invocation, \( h_1 \) is only quiescent at the beginning and end.

History \( h_1 \) is not linearizable [15] because the dequeues by processes 4 and 5 violate the FIFO order of enqueues by processes 2 and 3, and linearizability does not allow non-overlapping operations to be reordered (see [8] for details). However, \( h_1 \) is quiescent consistent because quiescent consistency allows operations between two consecutive quiescent states to be reordered even if they do not overlap. This means that it may be matched with the following sequential history, which satisfies a specification of a sequential queue data structure.

\[ h_2 = E_3(b) \hat{E}_3 E_2(a) \hat{E}_2 D_4 \hat{D}_4(b) D_5 \hat{D}_5(a) E_6(c) \hat{E}_6 D_1 \hat{D}_1(c) \]

2.2 Problem representation

In this section, we present our formal framework. The behaviour of a system will be a sequence of events. Given a set \( A \) we will let \( A^* \) denote the set of finite sequences of elements of \( A \) and \( \varepsilon \notin A \) denote the empty
sequence. Like Alur et al. [3], the specification and implementation are both represented by finite automata, whose alphabet is a set of events recording the invocation/response of an operation.

**Definition 1.** A finite automaton (FA) is a tuple $(M, m_0, \Sigma, t, M_t)$ in which $M$ is the finite set of states, $m_0 \in M$ is the initial state, $\Sigma$ is the finite alphabet, $t : M \times \Sigma \rightarrow M$ is the transition relation and $M_t \subseteq M$ is the set of final states.

Given a finite automaton $M = (M, m_0, \Sigma, t, M_t)$, $m' \in t(m, e)$ is interpreted as “it is possible for $M$ to move from state $m$ to state $m'$ via event $e$” and this defines the transition $(m, e, m')$. A path of $M$ is a sequence $\rho = (m_1, e_1, m_2), (m_2, e_2, m_3), \ldots, (m_k, e_k, m_{k+1})$ of consecutive transitions. The path $\rho$ has starting state $\text{start}(\rho) = m_1$, ending state $\text{end}(\rho) = m_{k+1}$ and label $\text{label}(\rho) = e_1 e_2 \ldots e_k$. We let $\text{Paths}(M)$ denote the set of paths of $M$. The FA $M$ defines the regular language $L(M)$ of labels of paths that start in $m_0$ and end in final states. More formally,

$$L(M) = \{ \text{label}(\rho) \mid \rho \in \text{Paths}(M) \land \text{start}(\rho) = m_0 \land \text{end}(\rho) \in M_t \}$$

Given $\sigma \in L(M)$ we let $M(\sigma)$ denote the set of states of $M$ that are ending states of paths in $\text{Path}(M)$ that have label $\sigma$. Note that because quiescent consistency is a safety property, considering only finite runs is adequate; this restriction is also made by Alur et al. for linearizability [3].

Given an FA $M$ that represents either a specification or implementation, $\Sigma$ (the alphabet of $M$) is the set of events, and so, the language $L(M)$ denotes the possible sequences of events (called runs). In this setting, each $\sigma \in L(M)$ of an automaton representing an object is also a possible history of the object.

We will use $S = (S, s_0, \Sigma, t_S, S_1)$ to denote the FA that represents the specification and $Q = (Q, q_0, \Sigma, t_Q, Q_1)$ to denote the FA that represents the implementation. We will typically use $s_1, \ldots$ for the names of states of $S$ and $q_1, \ldots$ for the names of states of $Q$. If $S$ is the FA for a sequential queue object, it will generates runs like $h_2$ in Example 1 and if $Q$ is the FA for the implementation in Fig. 2 then it will generate runs such as $h_1$. In this paper we will be interested in two different problems.

1. Deciding whether a run $\sigma \in L(Q)$ of the implementation is allowed by the specification $S$. (membership)

2. Deciding whether all runs of $Q$ are allowed by the specification $S$ and thus whether $Q$ is a correct implementation of $S$. (correctness)

To model concurrent operations, we assume that an operation has separate invoke and return events. We will use natural numbers $\mathbb{N}$ to identify processes and make the following assumption, which is a common restriction used in the literature.

**Assumption 1.** The number of processes in the specification and implementation is bounded.

This assumption is implicitly met by the fact that we use FA $S$ and $Q$. Others have considered infinite-state systems in the context of linearizability [3]. Here, dropping Assumption 1 causes the correctness problem for linearizability to become undecidable, whereas correctness for linearizability with a bound on the number of processes is decidable [4]. To recover decidability in the infinite case, one must place restrictions on the algorithms under consideration, in particular, linearizability is EXPSPACE-complete for implementations with “fixed” linearization points [5] (see [17, 9] for examples of such implementations).

Each event in $\Sigma$ is associated with a process, an operation, and an input or output value. Like [3], our theory is data independent in the sense that the input and output values are ignored. We simply assume that the event sets of the specification and implementation are equal, and hence, every input/output that is possible for an event of the implementation is also possible for the specification. Given process $p$, $\Sigma(p)$ denotes the set of events associated with $p$. We write $e \rightarrow e'$ to denote that $e$ matches $e'$, i.e., $e$ is an invoke event and $e'$ the corresponding response, which holds whenever the process and operation corresponding to $e$ and $e'$ are the same. We let $\pi_p(\sigma)$ denote the run that restricts $\sigma$ to events of process $p$, which is defined by

$$\pi_p(\varepsilon) = \varepsilon \quad \pi_p(e \sigma) = \text{if } e \in \Sigma(p) \text{ then } e \pi_p(\sigma) \text{ else } \pi_p(\sigma)$$
The empty run $\varepsilon$ is *sequential*. A non-empty run $\sigma = e_0 \ldots e_k$ is *sequential* iff $e_0$ is an invoke event, for each even $i < k$, $e_i \rightarrow e_{i+1}$, and if $k$ is even, $e_k$ is an invoke event. $\sigma$ is *legal* iff for each process $p$, $\pi_p(\sigma)$ is sequential. Legality ensures that each process calls at most one operation at a time. Furthermore, legality is *prefix closed*, i.e., if $\sigma$ is legal, then all prefixes of $\sigma$ are legal.

As is common in the literature, we make the following assumption on each specification object, which essentially means that its operations are atomic.

**Assumption 2.** The specification $S$ is sequential.

Furthermore, as is common in the literature [3, 18, 17, 22], we ignore the behaviour of clients that use the concurrent object in question, but assume that each client process calls at most one operation of the object it uses at a time (although different client threads may call concurrent operations). This is captured by Assumption 3 below.

**Assumption 3.** All runs of specification $S$ and implementation $Q$ are legal.

**Example 2.** Consider the history $h_1$ from Example 1. We have that $D_1 \rightarrow \hat{D}_1(c), E_2(a) \rightarrow \hat{E}_2$, etc. Furthermore, $h_1$ is legal because $\pi_p(\tau)$ is sequential for each process $p$. □

### 2.3 Independence

In this paper, we study quiescent consistency and explore how it can be represented in terms of *independence* from Mazurkiewicz Trace Theory [25]. Here, a symmetric independence relation $I \subseteq \Sigma \times \Sigma$ is used to define equivalence classes of runs. If $(e, e') \in I$, then consecutive $e$ and $e'$ within a run can be swapped. The independence relation defines a partial commutation — some pairs of elements commute, but there may also be pairs that do not. This leads to an equivalence relation $\sim_I$, where $\sigma \sim_I \sigma'$ iff run $\sigma$ can be transformed into run $\sigma'$ via a sequence of rewrites of the form $\sigma_1 e e' \sigma_2 \rightarrow_I \sigma_1 e' e \sigma_2$ for each $(e, e') \in I$.

**Example 3.** For $h_1$ and $h_2$ in Example 1, if $I = \Sigma \times \Sigma$ then $h_1 \sim_I h_2$. □

Given a run $\sigma$ we will let $|\sigma|_I = \{\sigma' | \sigma \rightarrow_I \sigma'\}$ denote the set of runs that can be produced from $\sigma$ using zero or more applications of the rewrite rules defined by $I$. We will let $L_I(M) = \cup_{\sigma \in L(M)} |\sigma|_I$ denote the set of runs that can be formed from those in $M$ using rewrites based on $I$. We can now state membership and correctness as stated in Section 2.2 more precisely as follows.

1. Deciding whether $\sigma \in L_I(S)$ for a given $\sigma \in L(Q)$. (membership)
2. Deciding whether $L(Q) \subseteq L_I(S)$. (correctness)

N.B., the correctness problem is sometimes referred to as the *model checking* problem. In the next section we explore how problems associated with quiescent consistency can be expressed in this manner, and will see that this requires the FA that represent the specification and implementation to be slightly adapted.

### 3 Quiescent consistency

In this section we define quiescent consistency and explore its properties. In Section 3.1 we define quiescent runs and state a number of properties that will be used in the rest of the paper, then in Section 3.2 we present an adaptation of the FA from the previous section to enable reasoning about membership and correctness for quiescent consistency. In Section 3.3 we define quiescent consistency, state the membership and correctness problems in terms of the adapted FA. Sections 4 and 5 then explore these problems.
3.1 Quiescent runs

We first define quiescent runs and state some properties that we use in the rest of this paper. If \( \sigma = \sigma_1 e \sigma_2 \) and \( e \) is an invocation event, we say \( e \) is a pending invocation if for all \( e' \in \sigma_2, e \not\rightarrow e' \). A run \( \sigma \) is quiescent if it does not contain any pending invocations. Thus, if a legal run is quiescent then there is a one-to-one correspondence between invoke and response events. A path \( \rho = (q_0, e_1, q_1, (q_1, e_2, q_2), \ldots, (q_k-1, e_k, q_k) \) is quiescent if \( \text{label}(\rho) \) is quiescent.

Example 4. Run \( h_1 \) in Example 1 is quiescent, but the run \( h_1 E_1(x) D_3 \Xi_1 \) is not because the invocation \( D_3 \) is pending. Note that quiescence does not guarantee legality, e.g., runs \( \Xi_2(\xi) \) and \( D_1 D_1 \Xi_2(\xi) \Xi_2(\xi) \) are both quiescent, but neither is legal.

The following result links quiescence and legality.

Proposition 1. Suppose \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_k \) is a legal and quiescent run, such that each \( \sigma_i \) (for \( 1 \leq i \leq k \)) is a quiescent run. Then for all \( 1 \leq i \leq k, \sigma_i \) is legal.

Proof. Suppose \( \sigma \) is a legal quiescent run and \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_k \), where each \( \sigma_j \) is quiescent. If \( k = 1 \) then we are done so assume that \( k > 1 \). Let \( \sigma_i \) for \( 1 \leq i \leq k \) be the first subsequence that is not legal, i.e., there exists a process \( p \) such that \( \pi_p(\sigma_i) \) is non-empty and not sequential. Because legality is prefix closed, \( \sigma' = \sigma_1 \ldots \sigma_{i-1} \) must be legal. Moreover, for each process \( q, \pi_q(\sigma') \) is either empty, or a non-empty sequential run ending with a return event. Thus for \( p \), we have that \( \pi_p(\sigma' \sigma_i) \) is not sequential, which contradicts the assumption that \( \sigma \) is legal.

We say that a run \( \sigma \) is end-to-end quiescent iff it is quiescent and all non-empty proper prefixes of \( \sigma \) are not quiescent. We write \( \xi(\sigma) \) to denote \( \sigma \) being end-to-end quiescent. For example, the run \( h_1 \) in Example 1 is end-to-end quiescent, and \( h_2 \) is quiescent but not end-to-end quiescent. The next result states that any legal quiescent run can be expressed as the concatenation of legal end-to-end quiescent runs.

Proposition 2. Suppose \( \sigma \) is a legal quiescent run. Then \( \sigma \) can be written in the form \( \sigma_1 \sigma_2 \ldots \sigma_k \) such that each \( \sigma_i \) is a legal end-to-end quiescent run.

Proof. If \( \sigma \) is end-to-end quiescent, we are done. Otherwise, there must exist \( \sigma_1 \) and \( \sigma_2 \), where \( \sigma = \sigma_1 \sigma_2 \), such that \( \sigma_1 \) is legal and end-to-end quiescent, and \( \sigma_2 \) is legal and quiescent. Because \( \sigma_2 \) is quiescent, it is possible to inductively apply the construction above, which completes the proof.

3.2 Distinguishing quiescence

We now develop an extension to the FA in Section 2.2 to facilitate reasoning about quiescent consistency in an automata-theoretic setting. Quiescent consistency is defined in terms of quiescent runs and so we will consider the behaviour of the implementation/specification to be its quiescent runs. This assumption is stated formally in terms of FA as follows.

Assumption 4. A path of \( S \) (and \( Q \)) starting from the initial state of \( S \) (and \( Q \)) is quiescent iff it ends in a final state of \( S \) (and \( Q \)).

By Assumption 4, \( S \) is sequential, and hence, distinguishing its quiescent states is straightforward. The following proposition gives a sufficient condition under which it is possible to partition the state set of \( Q \) into quiescent states and non-quiescent states (and so it is straightforward to ensure that Assumption 4 holds).

Proposition 3. Suppose that every path of \( Q \) starting from \( q_0 \) is a prefix of a legal quiescent path of \( Q \). If \( \rho \) is a quiescent path of \( Q \) such that \( \text{start}(\rho) = q_0 \) and \( \text{end}(\rho) = q \), then all paths of \( Q \) starting from \( q_0 \) and ending in \( q \) are quiescent.
Proof. The proof is by contradiction. Assume that there exist \( \rho, \rho' \) and \( q \) such that paths \( \rho \) and \( \rho' \) end at \( q \), \( \rho \) is quiescent and \( \rho' \) is not quiescent. Since \( \rho' \) can be completed to form a quiescent path, there must be a path \( \rho'' \) from \( q \) such that \( \rho' \rho'' \) is quiescent. Further, \( \rho'' \) must contain more responses than invokes. Thus, since \( \rho \) is quiescent we can conclude that the path \( \rho \rho'' \) from the initial state of \( Q \) has more response than invokes. This provides a contradiction as required, since, by Assumption 3, all histories of \( Q \) are legal.

Note that in the proof of Proposition 3 it might be necessary to invoke a new operation in order to complete the non-quiescent path \( \rho \) under consideration and reach a quiescent state. For example, consider our diffraction queue in Section 2.1 where the dequeue operation that blocks when the queue is empty. Suppose we have a path \( \rho' \) such that

\[
\text{label}(\rho') = D_1 D_2 E_3(x) \hat{D}_2(x) \hat{E}_3 \]

It is not possible for \( \rho' \) to reach a quiescent state by only completing the pending invocations in \( \text{label}(\rho') \) — the only pending invocation \( D_1 \) is blocked because the queue is empty. However, it is possible to reach a quiescent state by following a path where a new enqueue operation is invoked (by some process), and this new operation along with the pending \( D_1 \) in \( \text{label}(\rho') \) is completed by adding matching returns. This observation does not invalidate our results, which only require that we identify the quiescent states.

We now work towards a definition of allowable behaviours for quiescent consistency (Section 3.3), stated in terms of an independence relation (Section 2.3). In particular, by using a special event \( \delta \notin \Sigma \) that signifies quiescence, we aim to use the universal independence relation:

\[
U = \Sigma \times \Sigma
\]

which defines a partial commutation that allows all events different from \( \delta \) in a run to commute. Thus, the alphabet of the FA we use is extended to \( \Sigma_\delta = \Sigma \cup \{ \delta \} \). Note that using \( U \) as the independence relation means that matching invocations and responses of the specification may also be reordered when checking both membership and correctness. However, as is standard in the literature, we have assumed that all runs of the implementation are legal (Assumption 3), and hence, do not generate runs such that a response precedes an invocation, i.e., commutations of a response that is followed by a matching invocation will never be used.

We now consider how we should add \( \delta \) events to the FA \( S \) (representing the specification) and \( Q \) (representing the implementation) by extending their transition relations, which results in automata \( S_\delta \) and \( Q_\delta \).

First, consider the specification \( S \). One option is to insist that a \( \delta \) is included in a run whenever a quiescent state is reached. However, if we apply this approach to the specification, then the runs of \( S \) will all be of the form \( d e_1 \hat{e}_1 d e_2 \hat{e}_2 \delta \ldots \), i.e., a run \( \sigma \) of the implementation can only be equivalent to a run \( \sigma' \) of \( S \) under the partial commutation defined by \( U \) if \( \sigma = \sigma' \), which is not what is intended under quiescent consistency. This situation is a result of applying the restriction — that one can only reorder between instances of quiescence — to runs of the (sequential) specification; this restriction should only be applied to runs of the implementation. Thus, we should not require a \( \delta \) to appear in a run of \( S \) whenever a quiescent state is reached. Instead, we rewrite \( S \) to form an FA \( S_\delta \) so that if \( s \) is a quiescent state of \( S \) (i.e., after each return event) then there is a self-loop transition \((s, \delta, s)\) in \( S_\delta \). These are the only transitions of \( S_\delta \) that have label \( \delta \). Overall, we construct \( S_\delta \) such that we allow the inclusion of \( \delta \) whenever a run of \( S \) reaches a quiescent state.

Now consider the implementation \( Q \). Here, we must insist that there is a \( \delta \) in a run of \( Q \) whenever a quiescent state is reached, therefore we rewrite \( Q \) to form an FA \( Q_\delta \) such that if \( q \) is a quiescent state of \( Q \) then all transitions that leave \( q \) in \( Q_\delta \) have label \( \delta \). These are the only transitions of \( Q_\delta \) that have label \( \delta \). In particular, for each quiescent state \( q \) of \( Q \) we simply add a new state \( q_\delta \), make \( q_\delta \) the initial state of all transitions of \( Q \) that leave \( q \), and add the transition \((q, \delta, q_\delta)\). If \( q \) is a final state of \( Q \), i.e., \( q \in Q_1 \), we will make \( q_\delta \) a final state of \( Q_\delta \) instead of \( q \). Overall, we construct \( Q_\delta \) such that we require the inclusion of \( \delta \) when \( Q \) reaches a quiescent state.

The inclusion of \( \delta \) in runs of \( S \) allows us to compare runs of \( S \) and \( Q \) (once rewritten based on independence relation \( U \)).
Example 5. Returning to runs $h_1$ and $h_2$ in Example 1 there are many possible $\delta$ extensions of $h_2$ (which is a run of the specification), for example:

$$h_{1,1}^\delta = \delta E_3(b) \hat{E}_3 \delta E_2(a) \hat{E}_2 \delta D_4(b) \delta D_5 \hat{D}_5(a) \delta E_6(c) \hat{E}_6 \delta D_1 \hat{D}_1(c) \delta$$

$$h_{1,2}^\delta = \delta E_3(b) \hat{E}_3 \delta E_2(a) \hat{E}_2 \delta D_4(b) \delta D_5 \hat{D}_5(a) \delta E_6(c) \hat{E}_6 \delta D_1 \hat{D}_1(c) \delta$$

$$h_{1,3}^\delta = \delta E_3(b) \hat{E}_3 \delta E_2(a) \hat{E}_2 \delta D_4(b) \delta D_5 \hat{D}_5(a) \delta E_6(c) \hat{E}_6 \delta D_1 \hat{D}_1(c) \delta$$

In contrast, there is exactly one $\delta$ extension of $h_1$, namely $h_1 \delta$. If $h_2$ had been a run of the implementation, then the only $\delta$ extension of $h_2$ is $h_{1,1}^\delta$.

In addition to adding $\delta$ to the runs of $S$ and $Q$, we must also reason about runs with $\delta$ removed. To this end, we define the following projection

$$\pi_\Sigma(e) = e \quad \pi_\Sigma(e\sigma) = \text{if } e \in \Sigma \text{ then } e\pi_\Sigma(\sigma) \text{ else } \pi_\Sigma(\sigma)$$

Thus, for example $\pi_\Sigma(\delta h_1) = h_1$ and $\pi_\Sigma(h_{1,1}^\delta) = h_2$.

3.3 Allowable quiescent consistent behaviours

In this section we formalise what it means for a run of $Q$ to be allowed by $S$, and this is stated in terms of the extended automaton $Q_b$. Under quiescent consistency, runs $\sigma$ and $\sigma'$ are equivalent if they have the same (multi-)sets of events between two consecutive occurrences of quiescence. As a result, all elements in $\Sigma$ commute (we do not care about the relative order of these events) but nothing commutes with $\delta$.

Under quiescent consistency a quiescent run $\sigma$ is allowed by specification $S$ if $\sigma$ can be rewritten to form a run of $S$ by permuting events between consecutive quiescent points. We thus obtain the following definition.

Definition 2. Suppose $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$ is a legal quiescent run and each $\sigma_i$ is legal and end-to-end quiescent (N.B. it is always possible to write a quiescent $\sigma$ in such a form due to Proposition 1). Then $\sigma$ is allowed by $S$ under quiescent consistency if and only if there exists a permutation $\sigma_i' \sim_U \sigma_i$ for each $1 \leq i \leq k$ such that $\sigma_1' \sigma_2' \ldots \sigma_k' \in L(S)$.

We now define what it means for a run $\sigma \in L(Q_b)$ to be allowed by a specification $S$ under quiescent consistency.

Definition 3. Run $\sigma \in L(Q_b)$ is allowed by $S$ under quiescent consistency if $\pi_\Sigma(\sigma)$ is allowed by $S$ under quiescent consistency.

We say $\sigma'$ is a legal permutation of a legal run $\sigma$ if $\sigma \sim_U \sigma'$ and $\sigma'$ is legal.

Proposition 4. If $\sigma$ is legal and quiescent, then any legal permutation of $\sigma$ is quiescent.

We can now express the membership and correctness problems in terms of $Q_b$ and $S_b$, instead of between $Q$ and $S$ as done in Section 2.3.

Lemma 1 (Membership). Suppose $\sigma \in L(Q_b)$. Then $\sigma$ is allowed by $S$ under quiescent consistency iff $\sigma \in L(S_b)$.

Proof. Suppose $\sigma \in L(Q_b)$. By Proposition 1 and the construction of $Q_b$, we have $\sigma = \delta \sigma_1 \delta \ldots \delta \sigma_k \delta$ such that the $\sigma_i$ do not include $\delta$ (i.e., each $\sigma_i$ is end-to-end quiescent).

First assume that $\sigma$ is allowed by $S$ under quiescent consistency. By Definition 3 $\sigma_1 \sigma_2 \ldots \sigma_k$ is allowed by $S$, and hence, by Definition 2 we have that $S$ has a run $\sigma_1' \sigma_2' \ldots \sigma_k'$ such that $\sigma_i'$ is a legal permutation of $\sigma_i$ (for all $1 \leq i \leq k$). Furthermore, $S$ is initially quiescent and by Proposition 1 each $\sigma_i'$ is quiescent, therefore $S_b$ has the run $\sigma' = \delta \sigma_1' \delta \sigma_2' \delta \ldots \delta \sigma_k' \delta$. By definition, $\sigma \in L(S_b)$ as required.

Now assume $\sigma \in L(S_b)$. Then, $L(S_b)$ contains a run $\sigma' = \delta \sigma_1' \delta \sigma_2' \delta \ldots \delta \sigma_k' \delta$ for some $\sigma_1', \ldots, \sigma_k'$ such that $\sigma_i'$ is a permutation of $\sigma_i$ (all $1 \leq i \leq k$). We therefore have that $L(S)$ contains a run $\sigma_1' \ldots \sigma_k'$ such that $\sigma_i'$ is a permutation of $\sigma_i$ (all $1 \leq i \leq k$), and hence, have that $\sigma$ is allowed by $S$ as required. 

Lemma 2 (Correctness). Under quiescent consistency, \( Q \) is a correct implementation of \( S \) iff \( L(Q) \subseteq L_U(S_δ) \).

Proof. By Lemma 1 and the definition of quiescent consistency.

4 The Membership Problem

In this section we explore the following problem: given a specification \( S \), do we have that \( \sigma \in L_U(S_δ) \)? We show that this question is in general NP-complete (Section 4.1), but by assuming an upper bound between occurrences of two quiescent states, the question can be solved in polynomial time (Section 4.2).

4.1 Unrestricted quiescent consistency

We first establish that the membership problem for quiescent consistency is indeed in NP.

Lemma 3. The membership problem for quiescent consistency is in NP.

Proof. Given a run \( \sigma \in L(Q_δ) \) and a specification \( S \), a non-deterministic Turing machine can solve the membership problem of deciding whether \( \sigma \in L_U(S_δ) \) as follows. First, the Turing machine guesses a run \( \sigma' \) of \( S_δ \) with the same length as \( \sigma \). The Turing machine then guesses a permutation \( \sigma'' \) of \( \sigma \) that is consistent with the independence relation \( U \). Finally, the Turing machine checks whether \( \sigma' = \sigma'' \). This process takes polynomial time and hence, since a non-deterministic Turing machine can solve the membership problem in polynomial time, the problem is in NP.

We now prove that this problem is NP-hard by showing how instances of the one-in-three SAT problem can be reduced to it. An instance of the one-in-three SAT problem is defined by boolean variables \( v_1, \ldots, v_k \) and clauses \( C_1, \ldots, C_n \) where each clause is the disjunction of three literals (a literal is either a boolean variable or the negation of a boolean variable). The one-in-three SAT problem is to decide whether there is an assignment to the boolean variables such that each clause contains exactly one true literal and is known to be NP-complete [28].

The construction in the proof of the result below takes an instance of the one-in-three SAT problem and constructs a specification \( S \) that has \( k + 1 \) ‘main’ states \( s_0, \ldots, s_k \) and for boolean variable \( v_i \) it has two paths from \( s_{i-1} \) to \( s_i \): one path \( ρ_i^T \) has a matching invocation/response pair \( e_j, \bar{e}_j \) for every clause \( C_j \) that contains literal \( v_i \) and the other path \( ρ_i^F \) has a matching invocation/response pair \( e_j, \bar{e}_j \) for every clause \( C_j \) that contains literal \( \neg v_i \). The relative order of the pairs of events in \( ρ_i^T \) and \( ρ_i^F \) will not matter. A path from \( s_0 \) to \( s_{k+1} \) is of the form \( ρ_1^{B_1} ρ_2^{B_2} \cdots ρ_k^{B_k} \) for some \( B_1, \ldots, B_k \in \{ T, F \} \). Furthermore, the number of times that the events \( e_j \) and \( \bar{e}_j \) appear in the label of the path is the number of literals in clause \( C_j \) that evaluate to true under this assignment of values to \( v_1, \ldots, v_k \). As a result, such a path contains \( e_j \) and \( \bar{e}_j \) exactly once iff the assignment of \( B_i \) to \( v_i \) (all \( 1 \leq i \leq k \)) leads to exactly one literal in \( C_j \) evaluating to true. Thus, there is a path from \( q_0 \) to \( q_k \) that contains each \( e_j \) and \( \bar{e}_j \) exactly once iff there is a solution to this instance of the one-in-three SAT problem.

Example 6. Suppose we have four boolean variables \( v_1, \ldots, v_4 \) and clauses \( C_1 = v_1 \lor v_2 \lor \neg v_3, \quad C_2 = v_1 \lor \neg v_2 \lor v_4, \quad \text{and} \quad C_3 = v_2 \lor v_3 \lor \neg v_4. \) This leads to the FA shown in Figure 3. In this, for example, the label of \( ρ_1^T \) is \( e_1 \bar{e}_1 e_2 \bar{e}_2 \) because \( C_1 \) and \( C_2 \) both have literal \( v_1 \), and the label of \( ρ_2^F \) is \( e_2 \bar{e}_2 \) since \( C_2 \) is the only clause that contains literal \( \neg v_2 \). Consider now the path \( ρ_1^T ρ_2^F ρ_3^F ρ_4^F \), which has label \( e_1 \bar{e}_1 e_2 \bar{e}_2 e_3 \bar{e}_3 \). The label of this path tells us that if we assign true to \( v_1 \) and false to each of \( v_2, v_3, v_4 \) then clause \( C_1 \) contains two true literals (since \( v_1 \) appears twice), \( C_2 \) contains two true literals (since \( e_2 \) appears twice), and \( C_3 \) contains one true literal (since \( e_3 \) appears once). Thus, this assignment is not a solution to this instance of the one-in-three SAT problem because more than one clause of \( C_1 \) and \( C_2 \) evaluates to true.

1Note that the one-in-three SAT problem differs slightly from the more well-known 3SAT problem.
Note that we are not asking whether the clauses can be satisfied, but whether they can be satisfied in a way that makes exactly one literal of each true. This is equivalent to asking whether we can make the clauses true if they are stated in terms of isolating disjunction \( \lor \), where

\[
\lor(v_1, v_2, \ldots, v_n) = \bigvee_{1 \leq i \leq n} (v_i \land \neg v_j)
\]

**Example 7.** Checking the assignment in Example 6 is equivalent to checking for a satisfying assignment to the clauses \( \tilde{C}_1 = \lor(v_1, v_2, \neg v_3), \tilde{C}_2 = \lor(v_1, \neg v_2, v_4), \) and \( \tilde{C}_3 = \lor(v_2, v_3, \neg v_4) \), where, for example,

\[
\hat{C}_1 = (v_1 \land \neg v_2 \land v_3) \lor (\neg v_1 \land v_2 \land v_3) \lor (\neg v_1 \land \neg v_2 \land \neg v_3)
\]

We now prove NP-hardness of the membership problem. The proof essentially uses the finite automaton described above and the run \( \sigma = e_1 \hat{e}_1 \ldots e_n \hat{e}_n \). Additional events are included to allow these events to be reordered. In particular, we add an initial invocation \( e_0 \) and a final response \( \hat{e}_0 \) in the run and so the implementation is only quiescent in its initial state and at the end of the run. This allows the events of the run to be reordered; without the initial invocation and final response we could only compare \( \sigma \) with runs of the specification in which the pairs \( e_i, \hat{e}_i \) are met in the order found in \( \sigma \).

**Lemma 4.** The membership problem for quiescent consistency is NP-hard.

**Proof.** Assume that we are given an instance of the one-in-three SAT problem defined by boolean variables \( v_1, \ldots, v_k \) and clauses \( C_1, \ldots, C_n \). We define a specification with invocation events \( e_0, e_1, \ldots, e_n, e \) and corresponding return events \( \hat{e}_0, \hat{e}_1, \ldots, \hat{e}_n, \hat{e} \). Define a finite automaton specification \( S \) as follows. The state set of \( S \) includes states \( s_0, s_1, \ldots, s_k \) and \( s, s' \) with \( s \) being the initial state. For all \( 1 \leq i \leq k \) there are two paths from \( s_{i-1} \) to \( s_i \): path \( \rho^i \) has invocation/response pair \( e_j, \hat{e}_j \) for every clause \( C_j \) that contains literal \( v_i \); and path \( \rho^i \) has invocation/response pair \( e_j, \hat{e}_j \) for every clause \( C_j \) that contains literal \( \neg v_i \). Thus, a path from \( s_0 \) to \( s_k \) is of the form \( \rho^1 \rho^2 \ldots \rho^k \) for some \( B_1, \ldots, B_k \in \{ T, F \} \). From the initial state \( s \) the path to \( s_0 \) has run \( e_0 \hat{e}_0 \) and from \( s_k \) the path to the final state \( s' \) has run \( e \hat{e} \).

Consider the run \( \sigma = e_0 e_1 \hat{e}_1 \ldots e_n \hat{e}_n e_0 \). We prove that \( \sigma \) is in \( L_U(S_b) \) iff there is a solution to the instance of the one-in-three SAT problem defined by \( v_1, \ldots, v_k \) and \( C_1, \ldots, C_n \). First note that if \( \sigma \in L_U(S_b) \) then the corresponding run of \( S_b \) must end at state \( s' \) since \( \sigma \) contains the events \( e \) and \( \hat{e} \). In addition, \( \sigma \) is end-to-end quiescent and so we simply require that some permutation of \( \sigma \) is in \( L(S_b) \). Thus, \( \sigma \in L_U(S_b) \) iff \( S_b \) has a path from \( s_0 \) to \( s_k \) whose label \( \sigma \) contains each \( e_i \) and \( \hat{e}_i \) exactly once for \( 1 \leq i \leq n \). Furthermore, \( \sigma_1 \) must be the label of a path of \( S_b \) that is of the form \( \rho = \rho^1 \rho^2 \ldots \rho^k \). Thus, \( \sigma \in L_U(S_b) \) iff there is an assignment \( v_1 = B_1, \ldots, v_k = B_k \) such that each clause \( C_1, \ldots, C_n \) contains exactly one true literal. This is the case iff there is a solution to this instance of the one-in-three SAT problem. The result now follows from the one-in-three SAT problem being NP-complete and the construction of \( S_b \) and \( \sigma \) taking polynomial time.

The following brings together these results.

**Theorem 1.** The membership problem for quiescent consistency is NP-complete.
4.2 Upper bound for restricted quiescent consistency

We now consider a restricted version of quiescent consistency that assumes an upper limit on the number of events between two quiescent states. It turns out that the membership problem under this assumption is polynomial with respect to the size of the specification \( S \) and the length of \( \sigma \). To prove this, we convert the membership problem into the problem of deciding whether two finite automata define a common word, which is a problem that can be solved in polynomial time. In particular, for a given run \( \sigma \in L(Q) \), we construct a finite automaton \( M[\sigma] \) (see Definition 4) such that \( \sigma \in L(U(S_b)) \) iff \( L(M[\sigma]) \cap L(S) \) is non-empty.

The proof (in particular, the construction of \( M[\sigma] \) in Definition 4) requires that it is possible to distinguish each event within \( \sigma \). Therefore, we introduce the following assumption.

**Assumption 5.** For any \( \sigma \in Q \), the events within \( \sigma \) are unique, i.e., if \( \sigma = e_1 \ldots e_k \), for all \( 1 \leq i < j \leq k \), we have \( e_i \neq e_j \).

This assumption can be trivially ensured, for example, by labelling the events.

For any run \( \sigma \) whose events are unique, we define a machine \( M[\sigma] \) that is a finite automaton for \( \sigma \) that accepts any permutation of the events in \( \sigma \).

**Definition 4.** Suppose \( \sigma = e_1 \ldots e_k \) is a run whose events are unique. We let \( M[\sigma] \) be the finite automaton \((Q^n, \emptyset, \Sigma, t, \{\Sigma\})\) where \( \Sigma = \{e_1, \ldots, e_k\} \) and for all \( T, T' \in 2^\Sigma \), we have \((T, e, T') \in \delta \) iff \( e \notin T \) and \( T' = T \cup \{e\} \).

Note that the construction \( M[\sigma] \) is generic, but we only use it in situations where \( \sigma \) is legal and end-to-end quiescent.

Next, we define \( M[\sigma] \) for runs \( \sigma \in L(Q_b) \). We use \( L_1 \cdot L_2 \) to denote the language product of languages \( L_1 \) and \( L_2 \) and for FA \( A \) and \( B \), we let \( A \cdot B \) be the FA such that \( L(A \cdot B) = L(A) \cdot L(B) \). In what follows, \( A \) only has one final state (\( A = M[\sigma] \) for some \( \sigma \)), and hence, we can construct \( A \cdot B \) by adding an empty transition from the final state of \( A \) to the initial state of \( B \).

**Definition 5.** For run \( \sigma = \delta \sigma_1 \delta \sigma_2 \ldots \delta \sigma_k \delta \) such that \( \xi(\sigma_i) \) for each \( 1 \leq i \leq k \), we let \( M[\sigma] = M[\sigma_1] \cdot M[\sigma_2] \cdot \ldots \cdot M[\sigma_k] \).

The next result uses the automata construction in Definition 5 to convert the membership problem into a problem of deciding whether two automata accept a common word. Its proof is clear from the definitions.

**Proposition 5.** For any \( \sigma \in L(Q_b) \), we have \( \sigma \in L(U(S_b)) \) iff \( L(M[\sigma]) \cap L(S) \neq \emptyset \).

We now arrive at our main result for this section.

**Theorem 2.** Suppose that there exists an upper limit \( b \in \mathbb{N} \), such that for each \( \sigma \in L(Q_b) \) there are at most \( b \) events between two occurrences of \( \delta \) in \( \sigma \). Then the membership problem for quiescent consistency is in PTIME.

**Proof.** By Assumption 5 \( \sigma \) is quiescent, and by Proposition 3 and the definition of \( Q_b \), \( \sigma \) can be written as \( \sigma = \delta \sigma_1 \delta \sigma_2 \delta \ldots \delta \sigma_k \delta \), where each \( \sigma_i \) is legal and end-to-end quiescent.

For each \( \sigma_i \), the size of \( M[\sigma_i] \) is exponential in terms of the length of \( \sigma_i \). If we place an upper limit \( b \) on the number of events between two occurrences of quiescence then the size of \( M[\sigma_i] \) is polynomial (it is exponential in terms of \( b \)). Therefore, \( M[\sigma] \) is of polynomial size (the sum of the sizes of the \( M[\sigma_i] \)) and the result follows from it being possible to decide whether \( L(M[\sigma]) \cap L(S) \neq \emptyset \) in time that is polynomial in terms of the sizes of \( S \) and \( M[\sigma] \).

5 The correctness problem

For the correctness problem, we might directly compare \( L(Q) \) and \( L(S) \), i.e., require that \( L(Q) \subseteq L(S) \). However, this limits the potential for concurrency — \( Q \) would essentially be sequential. The effect of using
a relaxed notions of correctness (such as quiescent consistency) is that it allows \( L(Q) \) to be compared with \( L(S) \) using some notion of observational equivalence. Therefore, for quiescent consistency, we explore the following problem: given an implementation \( Q \) and specification \( S \), do we have that \( L(Q) \subseteq L_U(S) \)? We show that this question is decidable, coNP-hard and in EXPSPACE.

A language is a rational trace language if it is defined by a finite automaton and a symmetric independence relation. Decidability of the correctness problem is proved by using the following result from trace theory [1].

Lemma 5. Suppose \( A \) and \( B \) are FA with set of events \( \Sigma \) and \( I \subseteq \Sigma \times \Sigma \) is a symmetric independence relation. Then, the inclusion \( L_I(A) \subseteq L_I(B) \) is decidable iff \( I \) is transitive.

The following is an immediate consequence.

Theorem 3. \( L(Q) \subseteq L_U(S) \) is decidable.

Proof. The independence relation \( U = \Sigma \times \Sigma \) is transitive. This result thus follows from Lemma 5 and the fact that \( L(Q) \subseteq L_U(S) \) iff \( L_U(Q) \subseteq L_U(S) \).

We now explore the complexity of the correctness problem, which is equivalent to the complexity of deciding whether the inclusion \( L_U(Q) \subseteq L_U(S) \) holds. We show that this problem is coNP-hard by considering the problem of deciding inclusion of the set of Parikh images of regular languages. For the rest of this section we assume that \( A \) and \( B \) are FA.

5.1 Lower bound for unrestricted quiescent consistency

Given alphabet \( \Sigma = \{e_1, \ldots, e_k\} \) and \( \sigma \in \Sigma^* \), the Parikh image of \( \sigma \) is the tuple \( (n_1, \ldots, n_k) \) such that \( \sigma \) contains exactly \( n_i \) instances of \( e_i \) (all \( 1 \leq i \leq k \)). We use \( PI(A) \) to denote the set of Parikh images of the runs in \( L(A) \) and the inclusion problem for Parikh images is to decide whether \( PI(A) \subseteq PI(B) \). Deciding inclusion for the Parikh images of regular languages is known to be coNP-hard (even if the size of the alphabets of both \( A \) and \( B \) are fixed) [23].

To use the coNP-hard result for Parikh images, we construct FA \( A' \) and \( B' \) from \( A \) and \( B \) such that \( PI(A) \subseteq PI(B) \) iff \( L_U(A'_\delta) \subseteq L_U(B'_\delta) \), where \( A'_\delta \) (and \( B'_\delta \)) extends \( A' \) (resp. \( B' \)) with \( \delta \) events and transitions as defined in Section 5.2. Suppose \( \Sigma \) is the alphabet of both \( A \) and \( B \). For each \( x \in \Sigma \) we define an invoke event \( e_x \) and corresponding response event \( \hat{e}_x \). We also include an additional invoke event \( e \) and corresponding response \( \hat{e} \) that do not correspond to any \( x \in \Sigma \) and hence, the resulting event set is:

\[
\Gamma = \{e, \hat{e}\} \cup \{e_x \mid x \in \Sigma\} \cup \{\hat{e}_x \mid x \in \Sigma\}
\]

To construct FA \( A' \), we initialise the state set of \( A' \) to the state set of \( A \) and the event set of \( A' \) to \( \Gamma \). We then modify \( A' \) and construct the initial state, transitions, and final states of \( A' \) as follows.

1. For the initial state \( q_0 \) of \( A \), add a new state \( q'_0 \notin A \) to \( A' \), make \( q'_0 \) the initial state of \( A' \), and add the transition \((q'_0, e, q_0) \to A'\).
2. For each transition \( t = (q, x, q') \) in \( A \), add transitions \((q, e_x, q_t) \) and \((q_t, \hat{e}_x, q') \) in \( A' \), where \( q_t \notin A \), then add \( q_t \) to \( A' \).
3. Add a state \( q_F \notin A \) to \( A' \), make this the only final state, and from every final state \( q \) of \( A \), add the transition \((q, \hat{e}, q_F) \).

We have the following relationship between \( L(A) \) and \( L(A') \).

Proposition 6. \( x_1 x_2 \ldots x_k \in L(A) \) iff \( e e_{x_1} \hat{e}_{x_1} e_{x_2} \hat{e}_{x_2} \ldots e_{x_k} \hat{e}_{x_k} \hat{e} \in L(A') \).
We have the following relationship between the specification.

Proof.

Lemma 6. By Proposition 6 there is some proposition \( \sigma \) to \( \Gamma \), then set the initial state of \( \delta \sigma \) and derive an upper bound on its running time.

2. Add new states \( q' \) and \( q \) to \( B' \), then for every final state \( q \) of \( B \) add transitions \((q, e, q')\) and \((q', \hat{e}, q_F)\) to \( B' \). Finally, make \( q_F \) the only final state of \( B' \).

We have the following relationship between \( L(B) \) and \( L(B') \).

Proposition 7. \( x_1x_2 \ldots x_k \in L(B) \) iff \( e_1e_2e_3 \ldots e_{k-1}e_{k} \) \( \in L(B') \).

The next lemma links inclusion of Parikh images for \( A \) and \( B \) to inclusion of the languages of \( A'_0 \) and \( B'_0 \) under independence relation \( U \).

Lemma 6. \( PI(A) \subseteq PI(B) \) iff \( L_U(A'_0) \subseteq L_U(B'_0) \).

Proof. First assume \( PI(A) \subseteq PI(B) \). Suppose that \( \sigma \in L_U(A'_0) \); it is sufficient to prove that \( \sigma \in L_U(B'_0) \). By Proposition \( P \) there is some \( x_1x_2 \ldots x_k \in L(A) \) such that \( \sigma \sim_U \delta \sigma' \delta \), where \( \sigma' = e_{x_1}e_{x_2}e_{x_3} \ldots e_{x_k} \). Since \( PI(A) \subseteq PI(B) \), \( L(B) \) contains a permutation \( y_1 \ldots y_k \) of \( x_1 \ldots x_k \). By Proposition \( Q \) \( y_1y_2y_3 \ldots y_{k-1}y_k \) \( \in L(B') \) and we also have that \( \delta \sigma'' \delta \in L_U(B'_0) \) where \( \sigma'' = e_{y_1}e_{y_2}e_{y_3} \ldots e_{y_k} \). As \( y_1 \ldots y_k \) is a permutation of \( x_1 \ldots x_k \), \( \sigma'' \sim_U \sigma' \). Since \( \delta \sigma'' \delta \in L_U(B'_0) \) and \( \sigma'' \sim_U \sigma' \) we have that \( \delta \sigma' \delta \in L_U(B'_0) \). Thus, since \( \sigma = \delta \sigma' \delta \), we have that \( \sigma \in L_U(B'_0) \) as required.

Now assume \( L_U(A'_0) \subseteq L_U(B'_0) \). Suppose that \( \gamma \in PI(A) \) and so there is some \( \sigma' = x_1 \ldots x_k \in L(A) \) with Parikh Image \( \gamma \). By Proposition \( P \) \( \in L(A') \). Thus, \( \delta e_{x_1}e_{x_2}e_{x_3} \ldots e_{x_k} \) \( \in L(U') \) for some permutation \( y_1 \ldots y_k \) of \( x_1 \ldots x_k \). By Proposition \( Q \) we therefore know that \( \sigma' = y_1 \ldots y_k \in L(B) \). Finally, since \( y_1 \ldots y_k \) and \( x_1 \ldots x_k \) are permutations of one another they have the same Parikh Image and so \( \gamma \in PI(B) \) as required.

We therefore have the following result.

Theorem 4. The correctness problem for Quiescent Consistency is \( coNP \)-hard.

Proof. By Lemma \( L \) and inclusion of Parikh images being \( coNP \)-hard.

5.2 Upper bound for unrestricted quiescent consistency

We now investigate the upper bounds on the complexity of deciding correctness of quiescent consistency and show that the problem is in \( EXPSPACE \). This proof is much more involved than the lower bound result as it is necessary to first derive an algorithm for checking correctness quiescent consistency (see Algorithm \( I \)) and derive an upper bound on its running time.

We start by introducing some new notation. For \( m \in \in \) and \( FA \ M = (M, m_0, \Sigma, t, M_t) \), we let \( m \triangleleft M \) denote the \( FA \ (M, m, \Sigma, t, M_t) \) formed by replacing the initial state of \( M \) by \( m \). Furthermore, for \( M' \subseteq M \) (recalling that \( \xi(\sigma) \) denotes that \( \sigma \) is end-to-end quiescent), we define:

\[
Z_M(m) = \{ \sigma \in L_U(m \triangleleft M) \mid \xi(\sigma) \} \\
Z_M(M') = \bigcup_{m \in M'} Z_M(m)
\]

Thus, \( Z_M(m) \) is the set of end-to-end quiescent runs that start in state \( m \in M \). The following is immediate from this definition.
Proposition 8. If \( Q \) is a correct implementation of \( S \) with respect to quiescent consistency and \( q_0 \) and \( s_0 \) are the initial states of \( Q \) and \( S \) respectively then \( Z_Q(q_0) \subseteq Z_S(s_0) \).

We will use an implicit powerset construction when reasoning about quiescent consistency. Given states \( m, m' \in M \) of \( M \), sets of states \( M_1, M_2 \subseteq M \) and run \( \sigma \), we define some further notation:

\[
\begin{align*}
    m \xrightarrow{\sigma} m' & \iff \exists \rho \in \text{Paths}(M). \text{start}(\rho) = m \land \text{end}(\rho) = m' \land \text{label}(\rho) \in |\sigma|_U \\
    M_1 \xrightarrow{\sigma} M_2 & \iff \forall \rho \in \text{Paths}(M). \text{start}(\rho) \in M_1 \land \text{label}(\rho) \in |\sigma|_U \Rightarrow \text{end}(\rho) \in M_2
\end{align*}
\]

Thus, \( m \xrightarrow{\sigma} m' \) holds iff there is some path in \( M \) with labels in \( |\sigma|_U \) from state \( m \) to state \( m' \). Furthermore, \( M_1 \xrightarrow{\sigma} M_2 \) holds iff every path of \( M \) starting from a state in \( M_1 \) with label in \( |\sigma|_U \) ends in a state of \( M_2 \).

If \( Q \) is not a correct implementation of \( S \) with respect to quiescent consistency then there must be a quiescent run \( \sigma \) that demonstrates this. We will use the following result, which shows that if there is a counterexample to quiescent consistency then there is one of the form \( \sigma = \sigma_1 \ldots \sigma_{k+1} \) (where \( \xi(\sigma_i) \)) such that \( \sigma_{k+1} \) is the portion of \( \sigma \) that is in \( Q \) but not in \( S \) (under independence relation \( U \)) and \( k \) is bounded by \( |Q| \cdot 2^{|S|} \).

Proposition 9. \( Q \) is not a correct implementation of \( S \) under quiescent consistency iff there exists some run \( \sigma = \sigma_1 \ldots \sigma_{k+1} \) for end-to-end quiescent \( \sigma_1, \ldots, \sigma_{k+1} \) and corresponding pairs \( (q_0, S_0), (q_1, S_1), \ldots, (q_k, S_k) \in Q \times 2^S \) such that \( S_0 = \{ s_0 \} \), \( q_i \xrightarrow{\sigma_i} q_i \), and \( S_{i-1} \xrightarrow{\sigma} S_i \) (all \( 1 \leq i \leq k \)) such that:

1. \( \sigma_{k+1} \notin Z_Q(q_k) \) and \( \sigma_{k+1} \notin Z_S(S_k) \), and
2. \( k \leq |Q| \cdot 2^{|S|} \).

Proof. The existence of such a \( \sigma \) demonstrates that \( Q \) is not a correct implementation of \( S \) under quiescent consistency and so it is sufficient to prove the left-to-right direction. We therefore assume that \( Q \) is not a correct implementation of \( S \) under quiescent consistency. Thus, there exists a quiescent run \( \sigma \) that is in \( L(Q) \) but not in \( L_U(S) \). Assume that we have a shortest such run \( \sigma, \sigma = \sigma_1 \ldots \sigma_{k+1} \) for end-to-end quiescent \( \sigma_1, \ldots, \sigma_{k+1} \). Since \( \sigma \) is in \( L(Q) \) but not in \( L_U(S) \), by the minimality of \( \sigma \) we must have that \( \sigma_{k+1} \in Z_Q(q_k) \) and \( \sigma_{k+1} \notin Z_S(S_k) \) and so the first condition holds. Further, by the minimality of \( \sigma \) we must have that \( (q_i, S_i) \neq (q_j, S_j) \), all \( 0 \leq i < j \leq k \); otherwise we can remove \( \sigma_i \ldots \sigma_{j-1} \) from \( \sigma \) to obtain a shorter run that is in \( L_U(Q) \) but not in \( L_U(S) \). But, there are \( |Q| \cdot 2^{|S|} \) possible pairs and so the second condition, \( k < |Q| \cdot 2^{|S|} \), must hold.

Using Proposition 9, we develop Algorithm 7 which defines a non-deterministic Turing Machine that solves the problem of deciding correctness. At each iteration, the non-deterministic Turing Machine first checks whether \( Z_Q(q_i) \not\subseteq Z_S(S_i) \); if not, it has demonstrated that \( Q \) is not a correct implementation of \( S \) (the first condition of Proposition 9). If this condition holds then the non-deterministic Turing Machine increments the counter \( c \) and guesses a next pair \( (q_c, S_c) \). It then checks that there is some \( \sigma_c \) such that \( q_{c-1} \xrightarrow{\sigma_c} q_c \) and \( S_{c-1} \xrightarrow{\sigma_c} S_c \). If there is such a \( \sigma_c \) then the process can continue, otherwise the result is inconclusive. The bound on \( c \) ensures that the algorithm terminates as long as we can decide the conditions contained in the if statements (we explore this below).

If a non-deterministic Turing Machine operates as above then it will return Fail if there is some sequence of choices that leads to Fail being returned. The following is thus immediate from Proposition 9.

Proposition 10. If a non-deterministic Turing Machine applies Algorithm 7 to \( Q \) and \( S \) then it returns Fail iff \( Q \) is not a correct implementation of \( S \) with respect to quiescent consistency.

We now consider the two problems encoded in the conditions of Algorithm 7: deciding whether \( Z_Q(q_c) \not\subseteq Z_S(S_c) \); and deciding whether there exists a run \( \sigma_c \) such that \( q_{c-1} \xrightarrow{\sigma_c} q_c \) and \( S_{c-1} \xrightarrow{\sigma_c} S_c \).

We start with problem of deciding whether \( Z_Q(q_c) \subseteq Z_S(S_c) \). This involves checking whether the Parikh Image of one regular language is contained in the Parikh Image of another regular language. It is known that this problem can be solved in non-deterministic exponential time (NEXPTIME) [20].
Proposition 11. It is possible to decide whether \( Z_M(q_e) \subseteq Z_S(S_c) \) in NEXPTIME.

The remaining problem we need to decide, for states \( q_{c-1}, q_c \) of \( Q \) and sets \( S_{c-1}, S_c \) of states of \( S \), whether there exists some \( \sigma_c \) that can

(i) take \( Q \) from \( q_{c-1} \) to \( q_c \) and

(ii) take \( S \) from the set \( S_{c-1} \) of states to the set \( S_c \) of states.

We introduce some further notation. For \( m \in M \) and \( M' \subseteq M \), we let \( m \triangleleft M' \triangleright M' \) denote the FA \((M, m, \Sigma, \delta, M')\) formed by making \( m \) the initial state of \( M \) and \( M' \) the final states. We introduce the following (assuming all states in \( M' \) and \( M'' \) are quiescent).

\[
Z_M(m, M') = \{ \sigma \in L_U(m \triangleleft M' \triangleright M') \mid \xi(\sigma) \}
Z_M(M', M'') = \bigcup_{m \in M'} Z_M(m, M'')
\]

That is, \( Z_M(m, M'') \) is the set of end-to-end quiescent runs of the FA that start in state \( m \) and end at a state in \( M'' \). We use shorthand \( Z_M(m, m') \) for \( Z_M(m, \{m'\}) \) (similarly \( Z_M(M', m') \)).

Using this notation, condition (i) above may be formalised as the predicate \( \sigma_c \in Z_M(q_{c-1}, q_c) \). Condition (ii) above requires that \( \sigma_c \) can take \( S \) to all states in \( S_c \) (and so that \( \sigma_c \in \bigcap_{s \in S_c} Z_S(S_{c-1}, s) \)) and cannot take \( S \) from \( S_{c-1} \) to any state outside of \( S_c \) (and so that \( \sigma_c \notin \bigcup_{s \in (S \setminus S_c)} Z_S(S_{c-1}, s) \)). The negation of the overall condition thus reduces to the following.

\[
\exists \sigma_c \in (A \setminus B) \cap C \tag{1}
\]

where

\[
A = \bigcap_{s \in S_c} Z_S(S_{c-1}, s) \quad B = \bigcup_{s \in (S \setminus S_c)} Z_S(S_{c-1}, s) \quad C = Z_Q(q_{c-1}, q_c)
\]

Using some straightforward set manipulation, (1) is equal to \( A \cap C \subseteq B \).

Thus, the problem is reduced to deciding whether the intersection of a set of Parikh Images of regular languages is contained within the Parikh Image of another regular language. We also note that if we use \( \overline{T} \) to represent the complement of a language \( L \) then \( B \subseteq C \iff B \subseteq C \cup \overline{A} \), and by de Morgan’s Law \( \bigcap_i \overline{A}_i \) is equivalent to \( \bigcup_i A_i \). The condition therefore becomes

\[
Z_Q(q_{c-1}, q_c) \subseteq \left( \bigcup_{s \in (S \setminus S_c)} Z_S(S_{c-1}, s) \right) \cup \left( \bigcup_{s \in S_c} \overline{Z_S(S_{c-1}, s)} \right)
\]

The Parikh Image of a regular language can be represented by a semi-linear set that contains exponentially many terms \([23]\). In addition, the complement of a semi-linear set can be represented by polynomially many terms \([19]\). Thus, all of \( Z_Q(q_{c-1}, q_c), \bigcup_{s \in (S \setminus S_c)} Z_S(S_{c-1}, s), \) and \( \bigcup_{s \in S_c} \overline{Z_S(S_{c-1}, s)} \) can be represented using

---

**Algorithm 1 Deciding correctness for quiescent consistency**

\[
c = 0, S_0 = \{s_0\}, Q_0 = \{q_0\}
\]

while \( c \leq |Q| \cdot 2^{|S|} \)

if \( Z_Q(q_e) \not\subseteq Z_S(S_c) \) then

Return Fail

end if

\( c = c+1 \)

Choose some \((q_e, S_c) \in Q \times 2^S\)

if \( \not\exists \sigma_c \) such that \( q_{c-1} \overset{\sigma_c}{\rightarrow} q_e \) and \( S_{c-1} \overset{\sigma}{\Rightarrow} S_c \) then

Return Ok

end if

end while
exponentially many terms (linear sets). Further, the problem of deciding whether one semi-linear set is contained in another is in $\Sigma^p_2$ [21] and so is in PSPACE. The overall problem is thus in EXPSPACE (since there are exponentially many terms).

**Proposition 12.** It is possible to decide whether there exists run $\sigma_c$ such that $q_{c-1} \xrightarrow{\sigma_c} Q q_c$ and $S_{c-1} \xrightarrow{\sigma_c} S S_c$ in EXPSPACE.

We can now bring these results together.

**Theorem 5.** The correctness problem for quiescent consistency is in EXPSPACE.

*Proof. We know that a non-deterministic Turing Machine can use Algorithm 1 to solve the problem. Further, by Propositions 11 and 12 the conditions of the if statements can be solved in NEXPTIME and EXPSPACE. Observe also that the storage required for the algorithm, beyond determining the conditions in the if statements, is polynomial since the algorithm only has to store the current values of $q_c$, $S_c$ and $c$, the latter taking $\log(|Q|^2|S|)$ space. Since NEXPTIME is contained in EXPSPACE, we therefore have that a non-deterministic Turing Machine can solve the problem in nondeterministic EXPSPACE (NEXPSPACE).

The result now follows from Savitch’s theorem [27], which implies that NEXPSPACE = EXPSPACE. □

### 5.3 Upper bound for restricted quiescent consistency

We now consider the case where there is a limit $b$ on the lengths of subsequences of runs of $Q$ between two occurrences of quiescence.

**Proposition 13.** If there is a bound on the length of end-to-end quiescent runs in $Q$ and $S$, then it is possible to decide whether $Z_Q(q_c) \subseteq Z_S(S_c)$ in PSPACE.

*Proof. A nondeterministic Turing Machine can solve this problem in PSPACE as follows. First, it guesses a run $\sigma$ whose length is at most the upper bound. It then checks that $\sigma$ is end-to-end quiescent. It then checks whether $\sigma \in Z_Q(q_c)$ and whether $\sigma \in Z_S(S_c)$; we know that these checks can be performed in polynomial time since this is an instance of the restricted membership problem. Finally, it returns failure if and only if $\sigma \in Z_Q(q_c)$ and $\sigma \notin Z_S(S_c)$. □

**Proposition 14.** Let us suppose that there is a bound on the length of end-to-end quiescent runs in $Q$ and $S$. It is possible to decide whether there exists run $\sigma_c$ such that $q_{c-1} \xrightarrow{\sigma_c} Q q_c$ and $S_{c-1} \xrightarrow{\sigma_c} S S_c$ in PSPACE.

*Proof. A nondeterministic Turing Machine can solve this problem in PSPACE as follows. First, it guesses a run $\sigma$ whose length is at most the upper bound and checks that $\sigma$ is end-to-end quiescent. It then checks whether $\sigma \in Z_Q(q_{c-1}, q_c)$ and $\sigma \in Z_S(S_{c-1}, S_c) \setminus Z_S(S_{c-1}, S \setminus S_c)$.

We know that the first check (solving the membership problem for bounded quiescent consistency) can be performed in polynomial time. The second check can be solved by deciding whether $\sigma \in Z_S(S_{c-1}, S_c)$ and whether $\sigma \in Z_S(S_{c-1}, S \setminus S_c)$ and, again, these checks can be performed in polynomial time. The nondeterministic Turing Machine returns True if it finds that $\sigma \in Z_Q(q_{c-1}, q_c)$, $\sigma \in \sigma \in Z_S(S_{c-1}, S_c)$, and $\sigma \notin Z_S(S_{c-1}, S \setminus S_c)$. □

**Theorem 6.** The correctness problem for bounded quiescent consistency is in PSPACE.

*Proof. From Propositions 13 and 14 we know that the two conditions in Algorithm 1 can be decided in PSPACE. Thus, a nondeterministic Turing Machine can apply Algorithm 1 using polynomial space. The result thus follows. □

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Cited in [20].
6 Quiescent sequential consistency

In this section, we consider quiescent sequential consistency, which adds a sequential consistency constraint \cite{24} to quiescent consistency, i.e., we are not allowed to reorder the events of the same process. For concurrent objects, this means that the order of effects of operation calls by the same process will take place in program order: if operation calls identified by events $e$, $\hat{e}$ and $e'$, $\hat{e}'$ all have the same process, and a concrete implementation has a run where $\hat{e}$ occurs before $e'$, then such a trace cannot be justified by a sequential run where $e'$ occurs before $\hat{e}$.

In the context of client-object systems, sequential consistency has been shown to be equivalent to observational refinement \cite{15} provided that the client threads are independent (i.e., do not share data) \cite{12}. Observational refinement provides the conditions necessary for replacing a specification object within a client program by an implementation. The sorts of guarantees that quiescent consistency provides a client is still a subject of further study; as Shavit says, exploiting concurrency in the multiprocessor age requires a rethinking of traditional notions of correctness \cite{30}.

We now present some background for quiescent sequential consistency in preparation for the membership and correctness problems. In order to formally define quiescent sequential consistency, we define a projection function that also preserves $\delta$ states in the projection.

**Definition 6.** Given run $\sigma \in \sum_3$, event $e \in \sum_{\delta}$ and process $p$, $\pi_p^\delta(\sigma)$ is defined by the following:

$$
\pi_p^\delta(\epsilon) = \epsilon \\
\pi_p^\delta(e) = e \text{ if } e \in \sum(p) \cup \{\delta\} \text{ then } e\pi_p^\delta(\sigma) \text{ else } \pi_p^\delta(\sigma)
$$

For $\sigma, \sigma' \in \sum_3$, we write $\sigma \approx \sigma'$ iff $\pi_p^\delta(\sigma) = \pi_p^\delta(\sigma')$ for every process $p$.

We can now define quiescent sequential consistency in a similar manner to quiescent consistency, except that we include the constraint that events on a process are ordered.

**Definition 7.** Suppose $\sigma = \sigma_1\sigma_2 \ldots \sigma_m \in \sum^*$ is a quiescent run and each $\sigma_i$ is end-to-end quiescent. Then $\sigma$ is allowed by specification $\mathcal{S}$ under quiescent sequential consistency if there exists a permutation $\sigma'_i$ for each $1 \leq i \leq m$ such that $\sigma_i \approx \sigma'_i$ and $\sigma'_1\sigma'_2 \ldots \sigma'_m$ is a run of $\mathcal{S}$.

Since we cannot reorder events on a process we obtain the following independence relation.

$$
R = \{(a, b) \mid \exists p, p'. p \neq p' \land a \in \sum(p) \land b \in \sum(p')\}
$$

The essential idea is that quiescence ($\delta$) does not commute with anything, as with quiescent consistency, and that two events from $\sum$ are independent if and only if they are on different processes.

Given an FA $M$ with alphabet $\sum_{\delta}$, we will use $\mathcal{L}_R(M)$ to denote $\mathcal{L}_I(M)$ in which the independence relation $I$ is $R$.

We will use the FA $Q_{\delta}$ and $S_{\delta}$, which are derived from the implementation $Q$ and specification $S$, respectively, via the construction described in Section \ref{3.2}. If we consider a quiescent run $\sigma$ of $Q_\delta$ we have that $\delta$ is included whenever $\sigma$ is quiescent. We can define what it means for a run that includes $\delta$ to be allowed by $S$.

**Definition 8.** Run $\sigma$ of $Q_\delta$ is allowed by $S$ under quiescent sequential consistency if the run $\pi_\Sigma(\sigma)$ formed from $\sigma$ by removing all instances of $\delta$ is allowed by $S$ under quiescent sequential consistency.

Recall also that all processes observe quiescence. As a result, we have the following property.

**Lemma 7.** Suppose $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ is a quiescent run and each $\sigma_i$ is end-to-end quiescent. Given run $\sigma' \in \sum^*$ we have that $\sigma' \approx \sigma$ iff $\sigma' = \sigma'_1\sigma'_2 \ldots \sigma'_m$ for some $\sigma'_1, \ldots, \sigma'_m$ with $\sigma_j \approx \sigma'_j$ (all $1 \leq j \leq m$).

Based on Definition \ref{8} this leads directly to the following simplified ways of expressing when a run is allowed under quiescent sequential consistency.

**Proposition 15.** Suppose $\sigma \in L(Q_\delta)$ is a quiescent run. Then the following statements are equivalent:
1. $\sigma$ is allowed by $S$ under quiescent sequential consistency.

2. There exists a $\sigma' \in L(S)_{\delta}$ such that $\sigma' \approx \sigma$.

3. $\sigma \in L_R(S)_{\delta}$.

The following lemma links quiescent consistency and quiescent sequential consistency.

**Lemma 8.** If $\sigma \in L(Q_{\delta})$ is allowed by $S$ under quiescent sequential consistency then $\sigma$ is allowed by $S$ under quiescent consistency, but not vice-versa.

**Proof.** The first part follows from the independence relation $R$ for quiescent sequential consistency being a subset of the independence relation $U$ for quiescent consistency. To prove the second part it is sufficient to obtain a run $\sigma$ and specification $S$ such that $\sigma$ is allowed by $S$ for quiescent consistency but not for quiescent sequential consistency. Suppose $Q$ allows run $\sigma = e_1e_2\hat{e}_1\hat{e}_2\hat{e}$ where events $e_1, e_2, \hat{e}_1$ and $\hat{e}_2$ are on the same process $p$ and $S$ allows the run $\sigma' = e\hat{e}e_1\hat{e}_1e_2\hat{e}_2$ but no other permutation of $\sigma$. Then $\sigma'$ is allowable under quiescent consistency, but not under quiescent sequential consistency.

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### 7 Membership for quiescent sequential consistency

In this section we consider the membership problem for quiescent sequential consistency. Some of the results are similar to those for quiescent consistency, and hence, the proofs for these results are elided. The structure of this section is similar to Section 4 — we first present the unrestricted case (Section 7.1), then present the upper bounds for the restricted cases (Section 7.2).

#### 7.1 Unrestricted quiescent sequential consistency

The unrestricted version of quiescent sequential consistency is NP-complete. First, we show that the problem is in NP, then show that the problem is NP-hard.

**Lemma 9.** The membership problem for quiescent sequential consistency is in NP.

**Proof.** Given run $\sigma$ and specification $S$, a non-deterministic Turing machine can solve the membership problem, of deciding whether $\sigma \in L_R(S)_{\delta}$, as follows. First, the Turing machine guesses a run $\sigma'$ of $S$ with the same length as $\sigma$. The Turing machine then guesses a permutation $\sigma''$ of $\sigma$ that is consistent with the independence relation $R$. Finally, the Turing machine checks whether $\sigma'' = \sigma'$. This process takes polynomial time and so, since a non-deterministic Turing machine can solve the membership problem in polynomial time, the problem is in NP.

We can adapt the proof, that the membership problem for quiescent consistency is NP-hard (Lemma 4), by simply having a separate process for each invoke/response pair. We therefore have the following.

**Lemma 10.** The membership problem for quiescent sequential consistency is NP-hard.

**Theorem 7.** The membership problem for quiescent sequential consistency is NP-complete.

#### 7.2 Restricted quiescent sequential consistency

We now adapt the approach developed for quiescent consistency, that showed that the membership problem can be solved in polynomial time if we either have have an upper limit on the number of events between any two instances of quiescence, or on the number of processes in the system.
Upper limit on number of events between quiescence  Like Definition 4, we construct a finite automaton that accepts any permutation of run $\sigma$ that preserves the order of events within a single process. For a run $\sigma = e_1, \ldots, e_k$ of distinct elements, $1 \leq i \leq k$ and process $p$, we let

$$\text{pre}_p(\sigma, i) = \{e_j | 1 \leq j < i \land e_j \in \Sigma(p)\}$$

be the set of elements of $\sigma$ with index smaller than $i$ that are part of process $p$.

**Definition 9.** Suppose run $\sigma_i = e_1 \ldots e_k$ is such that for all $i, j \in \{1, \ldots, k\}$, if $i \neq j$ then $e_i \neq e_j$. We let $\mathcal{M}_R[\sigma_i]$ be the finite automaton $(2^\Sigma, \emptyset, \Sigma, t, \{\Sigma\})$ such that:

- $\Sigma = \{e_1, \ldots, e_k\}$ and,
- for all $T, T' \in 2^\Sigma$ and $e_i \in \Sigma$, we have $(T, e_i, T') \in t$ iff $e_i \notin T$, $T' = T \cup \{e_i\}$ and $\text{pre}_p(\sigma, i) \subseteq T$.

Using this definition, we obtain a new FA $\mathcal{M}'[\sigma]$.

**Definition 10.** Given run $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k \in \Sigma^*$ such that each $\sigma_i$ is end-to-end quiescent,

$$\mathcal{M}'[\sigma] = \mathcal{M}_R[\sigma_1] \cdot \mathcal{M}_R[\sigma_2] \ldots \mathcal{M}_R[\sigma_k].$$

The following is clear from the definition and from Proposition 15.

**Lemma 11.** Given run $\sigma$ and specification $S$, $\sigma \in L_R(S_k)$ iff $L(\mathcal{M}'[\sigma]) \cap L(S) \neq \emptyset$.

As before, we have the following result.

**Theorem 8.** If $b$ is an upper limit on the number of events between two occurrences of quiescence in each run of $Q$, then the membership problem for quiescent sequential consistency is in PTIME.

Upper limit on number of processes  We now consider the membership problem for the case in which there is a fixed upper limit on the number of processes. Note that this notion is not covered by Assumption 1, which states that the number of processes for any particular implementation or specification is bounded. The results here state that if we place an upper bound on the number of processes, with that bound being applied to all specifications and implementations being considered, then the set of membership problems that satisfy this bound can be solved in polynomial time.

As before, we start by defining an FA whose language is $|\sigma|_R$. Given some $\sigma_i$, the basic idea is that the state of the FA will be a tuple that, for each process $p$, records the most recent event on $p$. Thus, a state $q$ will be represented by a tuple of events (the most recent events observed on each process) and an event $a$ on process $p$ will only be possible in state $q$ if the event that immediately precedes $a$ on $p$ is in this tuple.

**Definition 11.** Suppose that run $\sigma_i = e_1 \ldots e_k$, in which each $e_i$ is distinct, has projection $\pi_p(\sigma_i) = e_{i_1}^p \ldots e_{i_k}^p$ on process $p$ ($1 \leq p \leq n$). We let $\mathcal{M}_V[\sigma_i]$ be the FA $(T, q_0, \Sigma, t, F)$ such that

- $\Sigma = \{e_1, \ldots, e_k\}$,
- $T = \{e_0^1, e_1^1, \ldots, e_{k_1}^1, \varepsilon\} \times \{e_0^2, e_1^2, \ldots, e_{k_2}^2, \varepsilon\} \times \ldots \times \{e_0^n, e_1^n, \ldots, e_{k_n}^n, \varepsilon\}$,
- $q_0 = (e_0^1, \ldots, e_0^n)$,
- $F = \{(e_1^1, e_2^1, \ldots, e_{k_a}^1)\}$, and
- $(T, a, T') \in t$ for $a \in \Sigma(p)$ if and only if the following hold: $T = (e_1^1, e_2^1, \ldots, e_{j_p}^1)$, $j_p < k_p$, $a = e_{j_p+1}^p$, and $T' = (e_1^1, e_2^1, \ldots, e_{j_p}^1, \varepsilon)$. 

The following defines $\mathcal{M}''[\sigma]$. 

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Definition 12. Given run $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ such that each $\sigma_i$ is end-to-end quiescent,
\[ \mathcal{M}^n[\sigma] = \mathcal{M}_V[\sigma_1] \cdot \mathcal{M}_V[\sigma_2] \cdot \ldots \cdot \mathcal{M}_V[\sigma_k]. \]

Lemma 12. Given run $\sigma$ and specification $S$, $\sigma \in \mathcal{L}_R(S)$ iff $L(\mathcal{M}^n[\sigma]) \cap L(S) \neq \emptyset$.

The important point now is that the state set of $\mathcal{M}_V[\sigma_i]$ has size that is exponential in terms of the number of processes but if the number of processes is bounded then the size is polynomial in terms of the length of $\sigma$. In particular, if there is an upper bound $b$ on the number of processes then $\mathcal{M}_V[\sigma_i]$ has at most $|\sigma_i|^b$ states. We therefore obtain the following result.

Theorem 9. If $b$ is an upper limit on the number of processes for each run of $Q$ then the membership problem for quiescent sequential consistency is in PTIME.

8 Correctness for quiescent sequential consistency

This section now presents decidability and complexity results for quiescent sequential consistency. Following the pattern of the previous sections, we present unrestricted quiescent sequential consistency (Section 8.1), and then restricted quiescent sequential consistency (Section 8.2).

8.1 Unrestricted quiescent sequential consistency

If we consider a system with two processes 1 and 2 such that $e_1, e_2 \in \Sigma(1)$ and $e_3 \in \Sigma(2)$ then we have that: $(e_1, e_2) \in R, (e_2, e'_1) \in R$ but $(e_1, e'_2) \notin R$. Therefore, the independence relation is not transitive and so we expect correctness to be undecidable (Lemma 13). It could, however, be argued that we might have changed the nature of the problem by placing restrictions on the structure of the specification. In this section we therefore prove that, as expected, the correctness problem is undecidable for quiescent sequential consistency. The proof will be based on showing how an instance of Post’s Correspondence Problem can be reduced to an instance of the correctness problem for quiescent sequential consistency.

Definition 13. Given alphabet $\Gamma$ and sequences $\alpha_1, \ldots, \alpha_n \in \Gamma^*$ and $\beta_1, \ldots, \beta_n \in \Gamma^*$, Post’s Correspondence Problem (PCP) is to decide whether there is a non-empty sequence $i_1 \ldots i_k \in [1, n]$ of indices such that $\alpha_{i_1} \ldots \alpha_{i_k} = \beta_{i_1} \ldots \beta_{i_k}$.

Post’s Correspondence Problem is known to be undecidable [20].

First we explain how the proof operates. Given an instance of the PCP defined by sequences $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, we will construct FA $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$ that will act as the implementation. The quiescent runs of this FA will have a particular form: a quiescent run $\sigma$ will be defined by a sequence $i_1 \ldots i_k \in [1, n]$ of indices, the projection of $\sigma$ on process 1 will correspond to $\alpha_{i_1} \ldots \alpha_{i_k}$ and the projection of $\sigma$ on process 2 will correspond to $\beta_{i_1} \ldots \beta_{i_k}$. Thus, there is a solution to this instance of the PCP if and only if $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$ has a non-empty quiescent trace $\sigma$ such that the projections on the two processes define the same sequences in $\Gamma^*$. We then define a specification $S$ that allows the set of runs in which the projections on the two processes differ (plus the empty sequence). When brought together, $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$ is not a correct implementation of $S$ under quiescent sequential consistency if and only if $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$ has a quiescent run whose projections on the two processes define the same sequences in $\Gamma^*$ and this is the case if and only if there is a solution to this instance of the PCP. As a result, correctness under quiescent sequential consistency being undecidable follows from the PCP being undecidable.

We now construct the FA $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$. There will be two processes and for each $p \in \{1, 2\}$ and letter $a$ in the alphabet $\Gamma$ used, we will create an invoke event $e^p_a$ and a response event $\hat{e}^p_a$. We will add two additional events: a matching invoke $e$ and response $\hat{e}$ for process 1. (This choice of process 1 for $e$ and $\hat{e}$ is arbitrary, i.e., we could have chosen $e$ and $\hat{e}$ to be events of process 2 as well). Thus, we have that:

\[ \Sigma = \{ e^p_a \mid a \in \Gamma \land p \in \{1, 2\} \} \cup \{ \hat{e}^p_a \mid a \in \Gamma \land p \in \{1, 2\} \} \cup \{ e, \hat{e} \} \]
We define a mapping from a sequence in $\Gamma^*$ and process number $p$ to a sequence in $\Sigma^*$ as follows (in which $a \in \Gamma$ and $\gamma \in \Gamma^*$).

$$t_{\Sigma}(\varepsilon, p) = \varepsilon$$

$$t_{\Sigma}(a\gamma, p) = e_a^p \hat{e}_a^p \cdot t_{\Sigma}(\gamma, p)$$

Then, we define an equivalence relation on sequences in $\Sigma^*$ that essentially 'ignores' the process number.

$$\varepsilon \equiv \varepsilon$$

$$e_a^p \sigma_1 \equiv e_b^q \sigma_2 \text{ if and only if } a = b \text{ and } \sigma_1 \equiv \sigma_2$$

$$\hat{e}_a^p \sigma_1 \equiv \hat{e}_b^q \sigma_2 \text{ if and only if } a = b \text{ and } \sigma_1 \equiv \sigma_2$$

From the initial state $q_0$ of $Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]$ there is a transition with label $e$ to state $q$. For all $1 \leq i \leq n$ there is a path from $q$ to state $q'$ with the following label:

$$t_{\Sigma}(\alpha_1, 1) t_{\Sigma}(\beta_1, 2)$$

Similarly, for all $1 \leq i \leq n$ there is a path from $q'$ to state $q'$ with label $t_{\Sigma}(\alpha_1, 1) t_{\Sigma}(\beta_1, 2)$. Finally, there is a transition from $q'$ with label $\hat{e}$ to the unique final state $q_F$.

Now consider a path from $q$ or $q'$ that ends in $q'$. Such a path must have a corresponding sequence $i_1 \ldots i_k \in [1, n]$ of indices and the projections of the label of this path on processes 1 and 2 are $\alpha_{i_1} \ldots \alpha_{i_k}$ and $\beta_{i_1} \ldots \beta_{i_k}$ respectively. We therefore have the following key property.

**Lemma 13.** $L(Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n])$ contains a run $e \sigma \hat{e}$ such that $\pi_1(\sigma) \equiv \pi_2(\sigma)$ iff there is a solution to the instance of the PCP defined by $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$.

Another important property is that runs in $L(Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n])$ are end-to-end quiescent. If we define a FA $S$ such that $L_R(S)$ is the language of all runs of the form $\sigma e \hat{e}$ such that run $\sigma$ does not have the same projections at processes 1 and 2 then we will have that $L_R(Q[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n]) \subseteq L_R(S)$ if and only if there is no solution to this instance of the PCP. The following shows how we can construct such a specification $S$.

$S$ has initial state $s_0$ and for all $a \in \Gamma$ there is a cycle that has label $e_a^1 \hat{e}_a^1 e_a^2 \hat{e}_a^2$. This cycle models the case where the projections are identical. We add the following to $S$ to represent the ways in which a first difference in projections, on processes 1 and 2, can occur.

1. For all $a, b \in \Gamma$ with $a \neq b$ there is a path from $s_0$ to state $s_1$ with label $e_a^1 \hat{e}_a^1 e_b^2 \hat{e}_b^2$. In $s_1$ there are separate cycles with labels $e_a^1 \hat{e}_a^1$ and $e_a^2 \hat{e}_a^2$ for all $a \in \Gamma$.

2. For all $a \in \Gamma$ there is a path from $s_0$ to state $s_2$ with label $e_a^1 \hat{e}_a^1$. In $s_2$ there is a cycle with label $e_a^1 \hat{e}_a^1$ for all $a \in \Gamma$.

3. For all $a \in \Gamma$ there is a path from $s_0$ to state $s_3$ with label $e_a^2 \hat{e}_a^2$. In $s_3$ there is a cycle with label $e_a^2 \hat{e}_a^2$ for all $a \in \Gamma$.

State $s_1$ represents the case where after some common prefix we have $e_a^1 \hat{e}_a^1$ at process 1 and $e_b^2 \hat{e}_b^2$ at process 2 for some $a \neq b$. States $s_2$ and $s_3$ model the cases where one projection is a proper prefix of the other ($s_2$ models the case where the projection on process 1 is longer and $s_3$ models the case where the projection on process 2 is longer). We add a final state $s_F$ and paths from $s_1, s_2, s_3$ to $s_F$ with label $e \hat{e}$. The following is immediate from the construction.

**Lemma 14.** Run $\sigma \in \Sigma^*$ is in $L(S)$ if and only if $\sigma = \sigma' e \hat{e}$ for some $\sigma'$ such that $\pi_1(\sigma') \neq \pi_2(\sigma')$.

We therefore obtain the following result.

**Theorem 10.** The correctness problem for quiescent sequential consistency is undecidable.

**Proof.** This follows from Lemmas 13 and 14 and the PCP being undecidable. \( \square \)
8.2 Restricted quiescent sequential consistency

Our results for restricted quiescent sequential consistency extend the constructions used in Section 5.2. Given FA $M = (M, m_0, \Sigma, t, M_f)$ and state $m \in M$, we let $Y_M(q)$ denote the set of runs that are equivalent to end-to-end quiescent runs that label paths that start at $m$, i.e.,

$$Y_M(q) = \{ \sigma \in \mathcal{L}(m \triangleleft M) \mid \xi(\sigma) \}$$

We now consider the case where there is a fixed bound $b$ on the number of events that can occur in an end-to-end quiescent trace. For the unbounded case we could not use Algorithm 1 since this would require us to decide whether $Y_Q(q_c) \subseteq Y_S(S_c)$ and we know that this is undecidable. However, it is straightforward to see that in the presence of bound $b$ the conditions controlling the loop and If statement of Algorithm 1 involve reasoning about finite languages and are decidable. We can therefore apply Algorithm 1 and will now show that correctness is in PSPACE.

**Proposition 16.** Let us suppose that there is a limit $b$ on the length of end-to-end quiescent runs in $Q$ and $S$. It is possible to decide whether $Y_Q(q_c) \subseteq Y_S(S_c)$ in NP.

**Proof.** A non-deterministic Turing Machine can solve this problem as follows. First, it guesses a run $\sigma$ whose length is at most the upper limit. The Turing Machine then checks that $\sigma$ is end-to-end quiescent and whether $\sigma \in Y_Q(q_c)$ and $\sigma \in Y_S(S_c)$; from Theorem 8 we know that these checks can be performed in time that is polynomial in the lengths of $Q$ and $S$. It then returns failure if and only if $\sigma \notin Y_Q(q_c)$ and $\sigma \notin Y_S(S_c)$. Thus, a non-deterministic Turing Machine can solve this problem in polynomial time and so the problem is in NP.

**Proposition 17.** Let us suppose that there is a limit $b$ on the length of end-to-end quiescent runs in $Q$ and $S$. Given states $q_{c-1}$ and $q_c$ of $Q$ and sets $S_{c-1}$ and $S_c$ of states of $S$, it is possible to decide whether there exists run $\sigma_c$ such that $q_{c-1} \xrightarrow{\sigma_c} Q q_c$ and $S_{c-1} \xrightarrow{\sigma_c} S_c$ in NP.

**Proof.** A non-deterministic Turing Machine can solve this as follows. It guesses a run $\sigma$ whose length is at most $b$, requiring constant, and checks that $\sigma$ is end-to-end quiescent. It then determines whether $\sigma \in Y_Q(q_c)$ and $\sigma \in Y_S(S_c \setminus S_{c-1})$. The non-deterministic Turing Machine returns True if and only if it finds that $\sigma \in Y_Q(q_c)$, $\sigma \in Y_S(S_c)$, and $\sigma \notin Y_S(S \setminus S_c)$. From Theorem 8 we know that these steps can be performed in time that is polynomial in the sizes of $Q$ and $S$, and so in polynomial time. Thus, a non-deterministic Turing Machine can solve this problem in polynomial time and so the problem is in NP.

**Theorem 11.** The correctness problem for restricted quiescent consistency is in PSPACE.

**Proof.** From Propositions 16 and 17 we know that the two conditions in Algorithm 1 can be decided in NP. Thus, a non-deterministic Turing Machine can apply Algorithm 1 using polynomial space.

9 Conclusions

Concurrent objects (such as the queue example in Section 2.1) form an important class of objects, managing thread synchronisation on behalf of a programmer. The safety properties that a concurrent object satisfies can be understood in terms of the correctness conditions such as sequential consistency, linearizability and quiescent consistency [17].

Decidability and complexity for checking membership and correctness for these conditions have been widely studied. These generally extend Alur et al.’s methods [3], which in turn is based on the notions of independence from Mazurkiewicz Trace Theory. A summary of results from the literature is given in Table 11. The bounded and unbounded versions of linearizability that have been studied refer to the number of processes that a system is assumed to have — the unbounded version does not assume an upper limit on
| Correctness condition | Membership | Correctness |
|-----------------------|------------|-------------|
|                       | Unrestricted | Restricted | Unrestricted | Restricted |
| Sequential consistency| NP-complete | —           | Undecidable  | —           |
| Linearizability       | NP-complete(a) | PTIME(b)   | EXPSPACE(c) | EXPSPACE-complete(d) |
| Serializability       | —           | —           | PSPACE      | —           |
| Conflict serializability| —           | —           | EXPSPACE-complete | — |
| Quiescent consistency * | NP-complete | PTIME (b)  | coNP-hard, EXPSPACE | PSPACE (b) |
| Quiescent sequential consistency * | NP-complete | PTIME (b) or (c) | Undecidable | PSPACE (b) |

Table 1: Summary of decidability and complexity results

(a) Finite number of processes
(b) Predetermined upper limit on number of processes
(c) Potentially infinite number of processes
(d) Implementations with fixed linearization points, but potentially infinite number of processes
(e) Upper limit on number of events between two quiescent states
* Our results

the number of processes. Our results on quiescent consistency and quiescent sequential consistency adds to this existing body of work.

The notion of quiescent consistency we have considered is based on the definition by Derrick et al. [8], which is a formalisation of the definition by Shavit [30]. This definition allows operation calls by the same process to be reordered, i.e., sequential consistency is not necessarily preserved. However, for concurrent objects, sequential consistency is known to be necessary for observational refinement [12], which in turn guarantees substitutability on behalf of a programmer. Therefore, we also study a stronger version quiescent sequential consistency that disallows commutations of events corresponding to the same process.

There are further variations of quiescent consistency in the literature. Jagadeesan and Riely have developed a quantitative version of quiescent consistency [22] that only allows reordering if there is adequate contention in the system; here adequate contention is judged in terms of the number of open method calls in the system. It is straightforward to extend the approach used in this paper to show that the membership problem for this quantitative version is NP-complete, but decidability of correctness is not yet known. Versions of quiescent consistency suited to relaxed-memory architectures have also been developed [31, 10], where the notion of a quiescent state incorporates pending write operations stored in local buffers. Consideration of decidability and complexity for membership and correctness of these different variations is a task for future work. In particular, Jagadeesan and Riely’s condition forms forms a class of quantitative correctness conditions, which includes quantitative relaxations of linearizability [2, 16] and sequential consistency [29].

Bouajjani et al. have developed characterisations of algorithm designs that enable reduction of the linearizability problem to a (simpler) state reachability problem [6]. Other work [7] has considered (under) approximations of history inclusion with the aim of solving the observational refinement problem for concurrent objects [15, 12] directly. Linking quiescent consistency and quiescent sequential consistency to the state reachability problem and under approximations for observational refinement are both topics for future work.

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