SHARP HEAT KERNEL ESTIMATES IN THE FOURIER-BESSEL SETTING 
FOR A CONTINUOUS RANGE OF THE TYPE PARAMETER

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Abstract. The heat kernel in the setting of classical Fourier-Bessel expansions is defined by an 
oscillatory series which cannot be computed explicitly. We prove qualitatively sharp estimates 
of this kernel when the associated type parameter is not less than $-1/2$. Our method relies on 
establishing a connection with a situation of expansions based on Jacobi polynomials and then 
transferring known sharp bounds for the related Jacobi heat kernel.

1. Introduction

Let $J_\nu$ denote the Bessel function of the first kind and order $\nu > -1$, and let $\{\lambda_{n,\nu} : n \geq 1\}$ 
be the sequence of successive positive zeros of $J_\nu$. The Fourier-Bessel heat kernel is given by the 
oscillating sum

$$(1) \quad G_t^\nu(x, y) = 2(xy)^{-\nu} \sum_{n=1}^{\infty} \exp \left( -t\lambda_{n,\nu}^2 \frac{J_\nu(\lambda_{n,\nu}x)}{J_{\nu+1}(\lambda_{n,\nu})^2} \right).$$

Our main result is the following (see the end of this section for the notation).

Theorem A. Assume that $\nu \geq -1/2$. Given any $T > 0$, we have

$$G_t^\nu(x, y) \asymp \left( t \lor xy \right)^{-\nu-1/2} \left[ \frac{(1-x)(1-y)}{t} \lor 1 \right] \frac{1}{\sqrt{t}} \exp \left( -c \frac{(x-y)^2}{t} \right),$$

uniformly in $x, y \in [0, 1]$ and $0 < t \leq T$. Moreover,

$$G_t^\nu(x, y) \asymp \exp \left( -t\lambda_{1,\nu}^2 \right) (1-x)(1-y),$$

uniformly in $x, y \in [0, 1]$ and $t \geq T$.

Thus we obtain a qualitatively sharp description of the kernel (1). The restriction on $\nu$ 
is imposed by a similar restriction in the Jacobi setting from which the heat kernel bounds are 
transferred. Nevertheless, it is natural to conjecture that the same estimates hold for all $\nu > -1$.

The main contents of Theorem A are of course the short time bounds. Proving them by a 
direct analysis of the heavily oscillating series in (1) is practically impossible. Note that the 
Bessel function $J_\nu$ is transcendental in general, and can be expressed by means of elementary 
functions only if the index $\nu$ is half-integer, i.e. $\nu = k/2$ for some integer $k$. Furthermore, the 
zeros of $J_\nu$ are known explicitly only when $\nu = \pm 1/2$. Note also that the order of magnitude 
of the numbers $\lambda_{n,\nu}^2$ appearing in the exponential factor is $n^2$ as $n \to \infty$; in particular, the 
asymptotic distribution of $\lambda_{n,\nu}^2$ is not linear.

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The behavior of the Fourier-Bessel heat kernel does not seem to have been studied before, except for our previous paper [3]. More precisely, in [3] Theorem 3.3 we derived qualitatively sharp estimates for $G_\nu^\nu(t, x, y)$ under the assumption that $\nu$ is a half-integer not less than $-1/2$ (actually, this restriction is essential only for the short time bounds, see [3] Theorem 3.7). We also conjectured analogous estimates for general $\nu > -1$, see [3, Conjecture 4.3], and Theorem A confirms this for all $\nu \geq -1/2$. However, the methods we use in the present paper are much different from those applied in [3]. Here the main argument is based on a relation we establish between the Fourier-Bessel setting and the framework related to Jacobi ‘functions’. This makes it possible to transfer qualitatively sharp estimates of the Jacobi heat kernel obtained recently by Nowak and Sjögren [4].

Our principal motivation for investigating $G_\nu^\nu(t, x, y)$ comes from an interest in harmonic analysis related to Fourier-Bessel expansions, see the references in [3, Section 1]. Additional motivation emerges from the probabilistic interpretation of the Fourier-Bessel heat kernel, see [2, Section I]. Namely, it is a transition probability density for the time scaled Bessel process $X_{2t}^{\nu+1/2}$ on $(0, \infty)$ (with reflecting barrier at $x = 0$ when $-1 < \nu < 0$, see [1, Appendix I, Section 21]) killed upon leaving the interval $(0, 1)$. We also note that for $\nu = d/2 - 1$, $d \geq 1$, the kernel $G_\nu^\nu(x, y)$ provides a fundamental solution to the standard heat equation in the $d$-dimensional Euclidean unit ball with Dirichlet boundary condition and a radial initial condition.

There are two simple cases occurring already in Fourier’s works, at least implicitly. These are $G_t^{-1/2}(x, y)$ and $G_t^{1/2}(x, y)$, and the two kernels can be written by means of non-oscillating series, see [3, Section 4]. The argument is based on the periodized Gauss-Weierstrass kernel and simple boundary-value problems for the classical heat equation in an interval. No more elementary representations in these cases seem to be possible, and this suggests that a general closed explicit formula for $G_\nu^\nu(x, y)$ does not exist. The estimates in Theorem A are therefore a natural and desirable substitute for an exact expression.

When writing estimates, we use the notation $X \lesssim Y$ to indicate that $X \leq CY$ with a positive constant $C$ independent of significant quantities. We write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$. Further, we will also use the notation $X \simeq \sim Y \exp(-cZ)$ to indicate that

$$C^{-1}Y \exp(-c_1Z) \leq X \leq CY \exp(-c_2Z)$$

with positive constants $C, c_1$ and $c_2$ independent of significant quantities.

2. Fourier-Bessel and Jacobi settings

Given $\nu > -1$, consider the Bessel differential operator

$$L^\nu = -\frac{d^2}{dx^2} - \frac{2\nu + 1}{x} \frac{d}{dx}$$

on the interval $(0, 1)$. It is well known that $L^\nu$ is symmetric and nonnegative on $C^2_c(0, 1) \subset L^2(d\mu_\nu)$, where $\mu_\nu$ is a power measure in the interval $[0, 1]$ given by

$$d\mu_\nu(x) = x^{2\nu+1} dx.$$

A classical orthonormal basis in $L^2(d\mu_\nu)$ consisting of eigenfunctions of $L^\nu$ is the Fourier-Bessel system $\{\phi_n^\nu : n \geq 1\}$ (cf. [7, Chapter XVIII]) defined by

$$\phi_n^\nu(x) = \frac{\sqrt{2}}{|J_{\nu+1}(\lambda_n, \nu)|} x^{-\nu} J_\nu(\lambda_n, \nu x), \quad n \geq 1, \quad x \in [0, 1];$$
here the value $\phi_n^\nu(0)$ is understood in the limiting sense. We have

$$L^\nu \phi_n^\nu = \lambda_{n,\nu}^2 \phi_n^\nu, \quad n \geq 1.$$  

The self-adjoint extension of $L^\nu$ associated with the Fourier-Bessel system is defined naturally in $L^2(d\mu_\nu)$ via its spectral decomposition as

$$\tilde{L}^\nu f = \sum_{n=1}^{\infty} \frac{\lambda_{n,\nu}^2 \langle f, \phi_n^\nu \rangle d\mu_\nu \phi_n^\nu}{\lambda_{n,\nu}^2}$$

on the domain

$$\text{Dom } \tilde{L}^\nu = \left\{ f \in L^2(d\mu_\nu) : \sum_{n=1}^{\infty} \left| \frac{\lambda_{n,\nu}^2 \langle f, \phi_n^\nu \rangle d\mu_\nu \phi_n^\nu}{\lambda_{n,\nu}^2} \right|^2 < \infty \right\}.$$  

The semigroup generated by $-\tilde{L}^\nu$ has the integral representation

$$\exp (-t\tilde{L}^\nu) f(x) = \int_0^1 G_t^\nu(x,y) f(y) d\mu_\nu(y),$$

where the kernel is given by (1).

By technical reasons, we also need to consider a closely related setting emerging from incorporating the measure $\mu_\nu$ into the system $\{\phi_n^\nu\}$. In this way we derive the Fourier-Bessel system $\{\psi_n^\nu : n \geq 1\}$,

$$\psi_n^\nu(x) = x^{\nu+1/2} \phi_n^\nu(x), \quad n \geq 1, \quad x \in [0,1],$$

which for each $\nu > -1$ is an orthonormal basis in $L^2(dx)$; here and elsewhere $dx$ stands for Lebesgue measure in the interval $[0,1]$. This system consists of eigenfunctions of the differential operator

$$L^\nu = -\frac{d^2}{dx^2} - \frac{1/4 - \nu^2}{x^2};$$

more precisely, we have

$$L^\nu \psi_n^\nu = \lambda_{n,\nu}^2 \psi_n^\nu, \quad n \geq 1.$$  

The operator $L^\nu$ is symmetric and nonnegative on $C_c^2(0,1) \subset L^2(dx)$. Similarly as in the previous setting, there is a natural self-adjoint extension of $L^\nu$ in $L^2(dx)$, denoted by $\tilde{L}^\nu$, whose spectral decomposition is given by the $\psi_n^\nu$. The associated heat semigroup $\exp(-t\tilde{L}^\nu)$ possesses an integral representation (with integration against $dx$), and the relevant kernel is

$$K_t^\nu(x,y) = (xy)^{\nu+1/2} G_t^\nu(x,y).$$

Finally, we introduce the Jacobi setting based on Jacobi trigonometric ‘functions’, see [4, Section 2], scaled to the interval $[0,1]$. Let $P_k^{\alpha,\beta}, \ k = 0, 1, 2, \ldots$, be the classical Jacobi polynomials with type parameters $\alpha, \beta > -1$, as defined in Szegö’s monograph [6]. Define

$$\Phi_k^{\alpha,\beta}(x) = c_k^{\alpha,\beta} \left( \sin \frac{\pi x}{2} \right)^{\alpha+1/2} \left( \cos \frac{\pi x}{2} \right)^{\beta+1/2} P_k^{\alpha,\beta}(\cos \pi x), \quad k \geq 0, \quad x \in [0,1],$$

with

$$c_k^{\alpha,\beta} = \left( \frac{\pi(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \right)^{1/2}.$$
where for \( k = 0 \) and \( \alpha + \beta = -1 \) the product \( (2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1) \) must be replaced by \( \Gamma(\alpha + \beta + 2) \). Then the system \( \{\Phi_k^{\alpha,\beta} : k \geq 0\} \) is an orthonormal basis in \( L^2(dx) \). Moreover, each \( \Phi_k^{\alpha,\beta} \) is an eigenfunction of the differential operator

\[
\mathcal{J}^{\alpha,\beta} = -\frac{d^2}{dx^2} - \frac{\pi^2(1/4 - \alpha^2)}{4\sin^2(\pi x/2)} - \frac{\pi^2(1/4 - \beta^2)}{4\cos^2(\pi x/2)},
\]

and we have

\[
\mathcal{J}^{\alpha,\beta} \Phi_k^{\alpha,\beta} = \Lambda_k^{\alpha,\beta} \Phi_k^{\alpha,\beta}, \quad k \geq 0,
\]

where \( \Lambda_k^{\alpha,\beta} = \pi^2(k + \frac{\alpha + \beta + 1}{2})^2 \).

Thus \( \mathcal{J}^{\alpha,\beta} \) has a natural self-adjoint extension in \( L^2(dx) \) given by

\[
\tilde{\mathcal{J}}^{\alpha,\beta} f = \sum_{k=0}^{\infty} \Lambda_k^{\alpha,\beta} \langle f, \Phi_k^{\alpha,\beta} \rangle \Phi_k^{\alpha,\beta}
\]
on the domain

\[
\text{Dom} \tilde{\mathcal{J}}^{\alpha,\beta} = \left\{ f \in L^2(dx) : \sum_{k=0}^{\infty} |\Lambda_k^{\alpha,\beta} \langle f, \Phi_k^{\alpha,\beta} \rangle|^2 < \infty \right\}.
\]

The semigroup generated by \( -\tilde{\mathcal{J}}^{\alpha,\beta} \) in \( L^2(dx) \) has the integral representation

\[
\exp(-t\tilde{\mathcal{J}}^{\alpha,\beta}) f(x) = \int_0^1 G_t^{\alpha,\beta}(x,y) f(y) \, dy,
\]

where the Jacobi heat kernel is defined by the oscillating series

\[
G_t^{\alpha,\beta}(x,y) = \sum_{k=0}^{\infty} \exp\left(-t\Lambda_k^{\alpha,\beta}\right) \Phi_k^{\alpha,\beta}(x) \Phi_k^{\alpha,\beta}(y).
\]

Recently, qualitatively sharp estimates of \( G_t^{\alpha,\beta}(x,y) \) were obtained in [4], under the restriction \( \alpha,\beta \geq -1/2 \). The result below is a consequence of [4, Theorem A], the relation between various Jacobi heat kernels [4, (3)] and a simple scaling argument.

**Theorem 2.1** ([4]). Assume that \( \alpha,\beta \geq -1/2 \). Given any \( T > 0 \), we have

\[
G_t^{\alpha,\beta}(x,y) \asymp \left[ \frac{xy}{t} \land 1 \right]^{\alpha+1/2} \left[ \frac{(1-x)(1-y)}{t} \land 1 \right]^{\beta+1/2} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{t}\right),
\]

uniformly in \( x, y \in [0,1] \) and \( 0 < t \leq T \).

These estimates are crucial for our reasoning proving Theorem [A]

### 3. Proof of Theorem [A]

We first focus on showing the short time bounds, which constitute the main part of Theorem [A]. Our argument relies on relating the Fourier-Bessel heat kernel \( K_t^\nu(x,y) \) with the Jacobi heat kernel \( G_t^{\alpha,\beta}(x,y) \) and then transferring the estimates of Theorem [2.1] This will be achieved by proving that the generator of the Fourier-Bessel semigroup \( \{\exp(-tK_t^\nu)\} \) is a ‘slight’ perturbation of the Jacobi generator with the parameters of type \( \alpha = \nu \) and \( \beta = 1/2 \). Then the desired relation will follow by the so-called Trotter product formula. Note that this transference method works for all \( \nu > -1 \), and the only reason for the restriction on \( \nu \) in Theorem [A] is the analogous restriction on \( \alpha \) in Theorem [2.1].
A close connection between the generators is suggested by a relation at the level of differential operators. Observe that \( L^\nu \) and \( \mathbb{J}^{\nu,1/2} \) differ only by a zero order term,

\[
L^\nu - \mathbb{J}^{\nu,1/2} = \left( \frac{1}{4} - \nu^2 \right) \left[ \frac{\pi^2}{4 \sin^2 \frac{x}{2}} - \frac{1}{x^2} \right] := H^\nu(x).
\]

Moreover, the difference function \( H^\nu(x) \) is continuous and bounded on \([0, 1]\), with the value at \( x = 0 \) understood in the limiting sense. Indeed, as easily verified, we have

\[
|H^\nu(x)| \leq |H^\nu(1)| = \left| \frac{1}{4} - \nu^2 \right| \left( \frac{\pi^2}{4} - 1 \right), \quad x \in [0, 1].
\]

In the sequel this bound will be used repeatedly, sometimes implicitly.

An essential point of the whole method is the coincidence of domains stated below.

**Theorem 3.1.** For each \( \nu > -1 \),

\[
\text{Dom} \, \mathcal{\bar{L}}^\nu = \text{Dom} \, \mathbb{J}^{\nu,1/2}.
\]

To prove this we will need a preparatory result.

**Lemma 3.2.** Let \( \nu > -1 \). Then

\[
\langle \mathcal{\bar{L}}^\nu \psi_n, \Phi_k^{\nu,1/2} \rangle = \langle \psi_n, L^\nu \Phi_k^{\nu,1/2} \rangle, \quad n \geq 1, \quad k \geq 0,
\]

where \( L^\nu \) is the differential operator given by (2).

**Proof.** We shall use the divergence form of \( L^\nu \),

\[
L^\nu f(x) = -x^{-\nu-1/2} D \left( x^{2\nu+1} D \left( x^{-\nu-1/2} f(x) \right) \right),
\]

and integrate by parts. Denote

\[
\mathcal{I} = \langle \mathcal{\bar{L}}^\nu \psi_n, \Phi_k^{\nu,1/2} \rangle = \int_0^1 L^\nu \psi_n(x) \Phi_k^{\nu,1/2}(x) \, dx.
\]

Clearly, the integral here converges since \( L^\nu \psi_n \) and \( \Phi_k^{\nu,1/2} \) are in \( L^2(dx) \). We have

\[
\mathcal{I} = - \int_0^1 x^{-\nu-1/2} D \left( x^{2\nu+1} D \left( x^{-\nu-1/2} \psi_n(x) \right) \right) \Phi_k^{\nu,1/2}(x) \, dx
\]

\[
= -x^{\nu+1/2} D \left( x^{-\nu-1/2} \psi_n(x) \right) \Phi_k^{\nu,1/2}(x) \bigg|_0^1 + \int_0^1 D \left( x^{-\nu-1/2} \psi_n(x) \right) D \left( x^{-\nu-1/2} \Phi_k^{\nu,1/2}(x) \right) x^{2\nu+1} \, dx
\]

\[
= -x^{\nu+1/2} D \left( x^{-\nu-1/2} \psi_n(x) \right) \Phi_k^{\nu,1/2}(x) \bigg|_0^1 + x^{\nu+1/2} \psi_n(x) D \left( x^{-\nu-1/2} \Phi_k^{\nu,1/2}(x) \right) \bigg|_0^1
\]

\[
- \int_0^1 \psi_n(x) x^{-\nu-1/2} D \left( x^{2\nu+1} \left( x^{-\nu-1/2} \Phi_k^{\nu,1/2}(x) \right) \right) \, dx
\]

\[
\equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\]

Since \( \mathcal{I}_3 = \langle \psi_n, L^\nu \Phi_k^{\nu,1/2} \rangle \), it remains to ensure that \( \mathcal{I}_1 = \mathcal{I}_2 = 0 \).

From the definition of \( \Phi_k^{\nu,1/2} \) it is clear that

\[
\Phi_k^{\nu,1/2}(x) = O(x^{\nu+1/2}), \quad x \to 0^+, \quad \text{and} \quad \Phi_k^{\nu,1/2}(x) = O(1-x), \quad x \to 1^-.
\]

The estimate \( |\phi_k^{\nu}(x)| \lesssim 1 - x \), see the proof of [3, Theorem 3.7], holds uniformly in \( x \in (0,1) \) and implies

\[
psi_n^{\nu}(x) = O(x^{\nu+1/2}), \quad x \to 0^+, \quad \text{and} \quad psi_n^{\nu}(x) = O(1-x), \quad x \to 1^-.
\]
Further, using $D(z^{-\nu} J_{\nu}(z)) = -z^{-\nu} J_{\nu+1}(z)$ (cf. [7] Chapter III, Section 3.2)), we see that

$$D(x^{-\nu-1/2} \psi_n^{\nu}(x)) = c x^{-\nu} J_{\nu+1}(\lambda_n, \nu x) = \begin{cases} O(x), & x \to 0^+ \\ O(1), & x \to 1^- \end{cases},$$

where $c$ does not depend on $x$. Moreover, by the differentiation rule for Jacobi polynomials (cf. [6] (4.21.7)),

$$\frac{d}{dz} P_k^{\alpha, \beta}(z) = \frac{1}{2}(k + \alpha + \beta + 1) P_{k-1}^{\alpha+1, \beta+1}(z), \quad k \geq 0,$$

(here we use the convention that $P_{-1}^{\alpha, \beta} = 0$) it follows that

$$D(x^{-\nu-1/2} \Phi_k^{\nu, 1/2}(x)) = c_1 \sin(\pi x) P_{k-1}^{\nu+1, 3/2}(\cos(\pi x)) \left( \frac{\sin \pi x}{x} \right)^{\nu+1/2} \cos \frac{\pi x}{2}$$

$$+ c_2 P_k^{\nu, 1/2}(\cos(\pi x)) \left( \frac{\sin \pi x}{x} \right)^{\nu+1/2} \frac{\sin \pi x}{2}$$

$$+ c_3 P_k^{\nu, 1/2}(\cos(\pi x)) \left( \frac{\pi x}{2} \cos \frac{\pi x}{x} - \sin \frac{\pi x}{x} \right) \left( \frac{\sin \pi x}{x} \right)^{\nu-1/2} \cos \frac{\pi x}{2},$$

with $c_1, c_2, c_3$ independent of $x$. Therefore

$$D(x^{-\nu-1/2} \Phi_k^{\nu, 1/2}(x)) = \begin{cases} O(x), & x \to 0^+ \\ O(1), & x \to 1^- \end{cases}.$$

Altogether, these facts imply $\mathcal{I}_1 = \mathcal{I}_2 = 0$. The lemma follows. \hfill \Box

**Proof of Theorem 3.1.** We demonstrate that $\operatorname{Dom} \tilde{\mathcal{J}}^{\nu, 1/2} \subset \operatorname{Dom} \tilde{\mathcal{L}}^{\nu}$. Then the other inclusion follows by self-adjointness of the two operators.

Assume that $f \in \operatorname{Dom} \tilde{\mathcal{J}}^{\nu, 1/2}$ and let

$$S := \sum_{n=1}^{\infty} |\lambda_n, \nu \langle f, \psi_n^{\nu} \rangle|^2.$$

We must show that $S$ is finite. We have

$$S = \sum_{n=1}^{\infty} |\langle f, \tilde{\mathcal{L}}^{\nu} \psi_n^{\nu} \rangle|^2 = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} |\langle f, \Phi_k^{\nu, 1/2} \rangle \langle \Phi_k^{\nu, 1/2}, \tilde{\mathcal{L}}^{\nu} \psi_n^{\nu} \rangle|^2.$$

Applying now Lemma 3.2 we can write

$$S = \sum_{n=1}^{\infty} \left| \sum_{k=0}^{\infty} \langle f, \Phi_k^{\nu, 1/2} \rangle \langle \tilde{\mathcal{J}}^{\nu, 1/2} + H^{\nu} \rangle \Phi_k^{\nu, 1/2}, \psi_n^{\nu} \rangle \right|^2$$

$$= \sum_{n=1}^{\infty} \left| \sum_{k=0}^{\infty} \Lambda_k^{\nu, 1/2} \langle f, \Phi_k^{\nu, 1/2} \rangle \Phi_k^{\nu, 1/2}, \psi_n^{\nu} \rangle + \langle f, \Phi_k^{\nu, 1/2} \rangle \Phi_k^{\nu, 1/2}, \psi_n^{\nu} H^{\nu} \rangle \right|^2$$

$$= \sum_{n=1}^{\infty} \left| \tilde{\mathcal{J}}^{\nu, 1/2} f, \psi_n^{\nu} \right|^2 + \langle f H^{\nu}, \psi_n^{\nu} \rangle^2.$$

This together with Parseval’s identity implies

$$S \leq 2 \left\| \tilde{\mathcal{J}}^{\nu, 1/2} f \right\|_{L^2(dx)}^2 + 2 \left\| f H^{\nu} \right\|_{L^2(dx)}^2 < \infty$$

and consequently $f \in \operatorname{Dom} \tilde{\mathcal{L}}^{\nu}$. \hfill \Box
We now recall the Trotter product formula, see [5, Chapter VIII, Section 8]. Let $A$ and $B$ be (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$. If $A + B$ is essentially self-adjoint on $\text{Dom } A \cap \text{Dom } B$, and $A$ and $B$ are bounded from below, then

$$
\exp \left( -t(A + B) \right) h = \lim_{m \to \infty} \left( \exp(-tA/m) \exp(-tB/m) \right)^m h, \quad h \in \mathcal{H}, \quad t \geq 0.
$$

Specifying this general situation to our context we take $\mathcal{H} = L^2(dx)$, $A = J^{\nu,1/2}$ and choose $B$ to be the multiplication operator by $H^\nu$. Then, in view of Theorem 3.1, $A + B = J^\nu$. Moreover, the Trotter product formula applies and since $|H^\nu(x)| \leq |H^\nu(1)|$, $x \in [0,1]$, it implies

$$
e^{-t|H^\nu(1)|} \exp \left( -tJ^{\nu,1/2} \right) f \leq \exp \left( -tJ^\nu \right) \exp \left( -tJ^{\nu,1/2} \right) f, \quad 0 \leq f \in L^2(dx),
$$

for each $t \geq 0$. Taking now into account that the Jacobi and the Fourier-Bessel heat kernels are continuous functions of their arguments (see [4, Section 2], [3, Section 2]), we infer that

$$e^{-t|H^\nu(1)|} \mathbb{K}^{\nu,1/2}_t(x,y) \leq \mathbb{K}^\nu_t(x,y) \leq e^{-t|H^\nu(1)|} \mathbb{G}^{\nu,1/2}_t(x,y), \quad x,y \in [0,1], \quad t > 0.
$$

This combined with Theorem 2.1 and [3, Theorem 3.7], under the assumption that the threshold $T$ is sufficiently large. However, given any $0 < T_0 < T$, the already proved short time bounds imply

$$G^\nu_t(x,y) \simeq (1 - x)(1 - y), \quad x,y \in [0,1], \quad T_0 \leq t \leq T,
$$

and the desired conclusion follows.

The proof of Theorem A is complete.

**Remark 3.3.** A slightly more detailed analysis reveals that for $\nu \in [-1/2,1/2]$

$$e^{-t|H^\nu(1)|} \mathbb{G}^{\nu,1/2}_t(x,y) \leq \mathbb{K}^\nu_t(x,y) \leq e^{-t|H^\nu(0)|} \mathbb{G}^{\nu,1/2}_t(x,y) \leq \mathbb{G}^{\nu,1/2}_t(x,y), \quad x,y \in [0,1], \quad t > 0,
$$

and for $\nu \notin [-1/2,1/2]$

$$\mathbb{G}^{\nu,1/2}_t(x,y) \leq e^{-t|H^\nu(0)|} \mathbb{K}_t^{\nu,1/2}(x,y) \leq \mathbb{K}_t^\nu(x,y) \leq e^{-t|H^\nu(1)|} \mathbb{G}^{\nu,1/2}_t(x,y), \quad x,y \in [0,1], \quad t > 0.
$$

Notice that $\mathbb{K}^{\pm,1/2}_t(x,y) = \mathbb{G}^{\pm,1/2}_t(x,y)$.

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