Abstract. In this manuscript, we present several new results in finite and countable dimensional real Hilbert space phase retrieval and norm retrieval do norm retrieval. Also, we show that the families of norm retrievable frames \{f_i\}_{i=1}^m \in \mathbb{R}^n are not dense in the family of \(m \leq (2n - 2)\)-element sets of vectors in \(\mathbb{R}^n\) for every finite \(n\) and the families of vectors which do norm retrieval in \(\ell^2\) are not dense in the infinite families of vectors in \(\ell^2\). We also show that if a Riesz basis does norm retrieval in \(\ell^2\), then it is an orthogonal sequence. We provide numerous examples to show that our results are best possible.

1. Introduction

The concept of frames in a separable Hilbert space was originally introduced by Duffin and Schaeffer in the context of non-harmonic Fourier series \cite{15}. Frames have the redundancy property that make them more applicable than bases. Phase retrieval and norm retrieval are one of the most applied and studied areas of research today. Phase retrieval for Hilbert space frames was introduced in \cite{2} and quickly became an industry. Although much work has been done on the complex infinite dimensional case of phase retrieval, only a few papers exist on infinite dimensional real phase retrieval or norm retrieval, e.g., \cite{3}. In \cite{3}, some concepts such as “full spark” and “finitely full spark” were introduced and were generalized. We will present some examples for them.

Fusion frames are an emerging topic of frame theory, with applications to communications and distributed processing. Fusion frames were introduced by Casazza and Kutyniok in \cite{10} and further developed in their joint paper \cite{11} with Li. The theory for fusion frames is available in arbitrary separable Hilbert spaces (finite dimensional or not).

We first give the background material needed for the paper. Let \(\mathbb{H}\) be finite or infinite dimensional Real Hilbert space and \(B(\mathbb{H})\) be the class of all bounded linear operators defined on \(\mathbb{H}\). The natural numbers and real numbers are denoted by “\(\mathbb{N}\)” and “\(\mathbb{R}\)”, respectively. We use \([m]\) instead of the set \(\{1, 2, 3, \ldots, m\}\) and use \(\{f_i\}_{i \in I}\) instead of \(\text{span}\{f_i\}_{i \in I}\) where \(I\) is a finite or countable subset of \(\mathbb{N}\). We denote by \(\mathbb{R}^n\) a \(n\) dimensional real Hilbert space. We start with the definition of a real Hilbert space frame.
Definition 1. A family of vectors \( \{f_i\}_{i \in I} \) in a finite or infinite dimensional separable Hilbert space \( \mathbb{H} \) is a frame if there are constants \( 0 < A \leq B < \infty \) so that
\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad \text{for all } f \in \mathbb{H}.
\]
The constants \( A \) and \( B \) are called the lower and upper frame bounds for \( \{f_i\}_{i \in I} \), respectively. If only an upper frame bound exists, then \( \{f_i\}_{i \in I} \) is called a B-Bessel set or simply Bessel when the constant is implicit. If \( A = B \), it is called an A-tight frame and in case \( A = B = 1 \), it called a Parseval frame. The values \( \{\langle f, f_i \rangle\}_{i=1}^{\infty} \) are called the frame coefficients of the vector \( f \in \mathbb{H} \).

We will need to work with Riesz sequences.

Definition 2. A family \( \Phi = \{\phi_i\}_{i \in I} \) in a finite or infinite dimensional Hilbert space \( \mathbb{H} \) is a Riesz sequence if there are constants \( 0 < A \leq B < \infty \) satisfying
\[
A \sum_{i \in I} |c_i|^2 \leq \| \sum_{i \in I} c_i \phi_i \|^2 \leq B \sum_{i \in I} |c_i|^2
\]
for all sequences of scalars \( \{c_i\}_{i \in I} \). If it is complete in \( \mathbb{H} \), we call \( \Phi \) a Riesz basis.

It is well known that every finite dimensional real Hilbert space \( \mathbb{H} \) is isomorphic to \( \mathbb{R}^n \), for some \( n \), and every infinite dimensional real Hilbert space \( \mathbb{H} \) is isomorphic to \( \ell^2(\mathbb{R}) \) (countable real sequences with \( \ell^2 \)-norm). We will use \( \ell^2 \) instead of \( \ell^2(\mathbb{R}) \) for simplicity. Throughout the paper, \( \{e_i\}_{i=1}^{\infty} \) will be used to denote the canonical basis for the real space \( \ell^2 \), i.e., a basis for which
\[
\langle e_i, e_j \rangle = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

Definition 3. A family of vectors \( \{f_i\}_{i \in I} \) in a real Hilbert space \( \mathbb{H} \) does phase (norm) retrieval if whenever \( x, y \in \mathbb{H} \), satisfy
\[
|\langle x, f_i \rangle| = |\langle y, f_i \rangle| \quad \text{for all } i \in I,
\]
then \( x = \pm y (\|x\| = \|y\|) \).

Note that if \( \{f_i\}_{i \in I} \) does phase (norm) retrieval, then so does \( \{a_i f_i\}_{i \in I} \) for any \( 0 < a_i < \infty \) for all \( i \in I \). But in the case where \( |I| = \infty \), we have to be careful to maintain frame bounds. This always works if \( 0 < \inf_{i \in I} a_i \leq \sup_{i \in I} a_i < \infty \). But this is not necessary in general.

The complement property is an essential issue here. Since in the finite dimensional setting frames are equivalent with spanning sets, first we give the complement property in the finite case from [8].

Definition 4. A family of vectors \( \{f_k\}_{k=1}^{m} \) in \( \mathbb{R}^n \) has the complement property if for any subset \( I \subset [m] \),
\[
either \text{span}\{f_k\}_{k \in I} = \mathbb{R}^n \quad \text{or} \quad \text{span}\{f_k\}_{k \in I^c} = \mathbb{R}^n.
\]
This is generalized in [5].
Definition 5. A family of vectors \(\{f_k\}_{k=1}^\infty\) in \(\ell^2\) has the **complement property** if for any subset \(I \subset \mathbb{N}\),

\[
\text{either } \text{span}\{f_k\}_{k \in I} = \ell^2 \text{ or } \text{span}\{f_k\}_{k \in I^c} = \ell^2.
\]

The following result appeared in [5].

**Theorem 1.** A family of vectors \(\{f_i\}_{i=1}^\infty\) does phase retrieval for \(\ell^2\) if and only if it has the complement property.

The corresponding finite dimensional result first appeared in [7].

**Theorem 2.** A family of vectors \(\{f_i\}_{i=1}^m\) in \(\mathbb{R}^n\) does phase retrieval if and only if it has the complement property.

We recall

**Definition 6.** A family of vectors \(\{f_i\}_{i=1}^m\) in \(\mathbb{R}^n\) is **full spark** if for every \(I \subset [m]\) with \(|I| = n\), the set \(\{f_i\}_{i \in I}\) spans \(\mathbb{R}^n\).

**Corollary 1.** If \(\{f_i\}_{i=1}^m\) does phase retrieval in \(\mathbb{R}^n\), then \(m \geq 2n - 1\). If \(m \geq 2n - 1\) and the frame is full spark, then it does phase retrieval. If \(m = 2n - 1\), \(\{f_i\}_{i=1}^m\) does phase retrieval if and only if it is full spark.

For linearly independent sets there is a special case [6].

**Theorem 3.** If \(\{f_i\}_{i=1}^n\) in \(\mathbb{R}^n\) does norm retrieval, then the set is orthogonal.

It is clear that phase retrieval implies norm retrieval. The converse fails since an orthonormal basis does norm retrieval but fails phase retrieval since it fails complement property. Subsets of phase (norm) retrievable frames certainly may fail phase (norm) retrieval, since linearly independent subsets fail the complement property so fail phase retrieval and by Theorem 3 if every subset of a frame does norm retrieval then every two distinct vectors are orthogonal and so the frame is an orthogonal set plus possibly more vectors. But projections of these sets do still do phase (norm) retrieval.

### 2. Phase (norm) retrievable Fusion frames

In real life and some areas such as crystal twinning in X-ray crystallography [14], we need to project the signal onto higher than one dimensional subspaces and it has to be recovered from the norms of these projections. Throughout the paper, the term projection is used to describe an orthogonal projection onto subspaces. Norm retrieval is in fact the essential condition to pass phase retrievability of these projections to the corresponding orthogonal complements [9].

Fusion frames can be regarded as a generalization of conventional frame theory. It turns out that the fusion frame theory is in fact more delicate due to complicated relationships between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to sensitivity with respect to changes of the weights. Fusion frames were introduced by Casazza and Kutyniok in [10](under the name **frames of subspaces**) and further developed in their joint paper [11] with Li. Here \(\{W_i\}_{i \in I}\) is a family of closed subspaces of \(H\) and \(\{v_i\}_{i \in I}\) is a family of positive weights. Also we denote by \(P_i\) the orthogonal projection onto \(W_i\).
Definition 7. A family \( \{(W_i, v_i)\}_{i \in I} \) with \( W_i \) subspaces of \( \mathbb{H} \), \( v_i \) weights, and \( P_i \) the projection onto \( W_i \), is a **fusion frame** for \( \mathbb{H} \) if there exist constants \( A, B > 0 \) such that
\[
A\|f\|^2 \leq \sum_{i \in I} v_i^2\|P_i f\|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathbb{H}.
\]
The constants \( A \) and \( B \) are called the **fusion frames bounds**. We also refer to the fusion frames as \( \{P_i, v_i\}_{i \in I} \) or just \( \{P_i\}_{i \in I} \) if the weights are all one.

For more details on fusion frames, we recommend [10]. Improving and extending the notions of phase and norm retrievability, we present the definition of phase (norm) retrievable to fusion frames.

Definition 8. A family of projections \( \{P_i\}_{i \in I} \) in a real Hilbert space \( \mathbb{H} \) does **phase (norm) retrieval** if whenever \( x, y \in \mathbb{H} \), satisfy
\[
\|P_i x\| = \|P_i y\| \quad \text{for all } i \in I,
\]
then \( x = \pm y \) (\( \|x\| = \|y\| \)).

Definition 9. A fusion frame \( \{(W_i, v_i)\}_{i \in I} \) is phase (norm) retrievable for \( \mathbb{H} \) if and only if the family of projections \( \{P_i\}_{i \in I} \) is phase (norm) retrievable for \( \mathbb{H} \), where \( P_i = P_{W_i} \) is the orthogonal projection onto \( W_i \) \( (i \in I) \).

Remark 2.1. Note that \( \{W_i, v_i\}_{i=1}^m \) does phase (norm) retrieval if and only if \( \{W_i\}_{i=1}^m =: \{W_i, 1\}_{i=1}^m \) does phase (norm) retrieval.

It is well known that phase (norm) retrievable sets need not be frames (fusion frames). For example consider phase (norm) retrievable set \( \{e_i + e_j\}_{i < j} \) which does not satisfy the frame upper bound condition and therefore is not a (fusion) frame for \( \ell^2 \), but it does phase retrieval.

Part of the importance of fusion frames is that it is both necessary and sufficient to be able to string together frames for each of the subspaces \( W_k \) (with uniformly bounded frame constants) to get a frame for \( \mathbb{H} \) which is proved in [7]:

Theorem 4. Let \( \{W_i\}_{i \in I} \) be subspaces of \( \mathbb{R}^n \). The following are equivalent:

1. \( \{W_i\}_{i \in I} \) is phase retrievable.
2. For every orthonormal basis \( \{f_{ij}\}_{j \in I_i} \) for \( W_i \), the family \( \{f_{ij}\}_{j \in I_i, i \in I} \) does phase retrieval.

We note that (2) of the theorem must hold for every orthonormal basis for the subspaces. For example, let \( \{\phi_i\}_{i=1}^3 \) and \( \{\psi_i\}_{i=1}^3 \) be orthonormal bases for \( \mathbb{R}^3 \) so that \( \{\phi_i\}_{i=1}^3 \cup \{\psi_i\}_{i=1}^3 \) is full spark. Let
\[
W_1 = [\phi_1] \quad W_2 = [\phi_2] \quad W_3 = [\phi_3] \quad W_4 = [\psi_1, \psi_2].
\]
Then \( \{W_i\}_{i=1}^4 \) is a fusion frame for \( \mathbb{R}^3 \) and \( \{\phi_1, \phi_2, \phi_3, \psi_1, \psi_2\} \) is full spark and so does phase retrieval for \( \mathbb{R}^3 \). But it is known that 4 subspaces of \( \mathbb{R}^3 \) cannot do phase retrieval [7].

The corresponding result for norm retrieval does not make sense, because every orthonormal basis for a subspace does norm retrieval.

We can strengthen this theorem.
Theorem 5. Let \( \{(W_k,v_k)\}_{k \in I} \) be a phase (norm) retrievable fusion frame for \( \mathbb{H} \) and \( \{f_{ij}\}_{j \in I} \) be a norm retrievable frame for \( W_i \) for \( i \in I \). Then \( \{v_if_{ij}\}_{j \in I, i \in I} \) is a phase (norm) retrievable frame for \( \mathbb{H} \).

Proof. Let \( f, g \in \mathbb{H} \), for any \( j \in I, i \in I \), we have
\[
|\langle f, v_if_{ij} \rangle| = |\langle g, v_if_{ij} \rangle| \Rightarrow |\langle f, v_if_{ij} \rangle| = |\langle g, v_if_{ij} \rangle|,
\]
\[
\Rightarrow v_i|\langle f, f_{ij} \rangle| = v_i|\langle g, f_{ij} \rangle|,
\]
since \( \{f_{ij}\} \) do norm retrieval \( \Rightarrow \|P_if\| = \|P_ig\|, \forall i \in I, \)
since \( \{P_i\}_{i \in I} \) do phase retrieval \( \Rightarrow f = \pm g. \)

The following theorem shows that the unitary operators preserve phase (norm) retrievability of fusion frames.

Theorem 6. Let \( \{(W_i,v_i)\}_{i \in I} \) be a phase (norm) retrievable fusion frame for \( \mathbb{H} \). If \( T \in B(\mathbb{H}) \) is a unitary operator, then \( \{(TW_i,v_i)\}_{i \in I} \) is also a phase (norm) retrievable fusion frame.

Proof. Let \( \{P_i\}_{i \in I} \) be the projections onto \( \{W_i\}_{i \in I} \). The projections onto \( TW_i \) are \( Q_i = TP_iT^* \). Assume \( f, g \in \mathbb{H} \) and
\[
\|Q_if\| = \|Q_ig\|, \text{ for all } i \in I.
\]
Then,
\[
\|TP_iT^*f\| = \|P_iT^*f\| = \|TP_iT^*g\| = \|P_iT^*g\|, \text{ for all } i \in I.
\]
If \( \{P_i\}_{i \in I} \) does norm retrieval, then \( \|T^*f\| = \|T^*g\| \), and so \( \|f\| = \|g\| \). If \( \{P_i\}_{i \in I} \) does phase retrieval, then \( T^*f = \pm T^*g \), and so \( f = \pm g. \)

Theorem 7. Let \( \{(W_i,v_i)\}_{i \in I} \) be a norm retrievable fusion frame for a Hilbert space \( \mathbb{H} \), with projections \( \{P_i\}_{i \in I} \). Let \( \{Q_i\}_{i \in I} \) be projections from \( W_i \) to \( W_i \), and let \( W_i' = Q_iW_i \) and \( W_i'' = (I - Q_i)W_i \) for all \( i \in I \). Then \( \{(W_i',v_i)\}_{i \in I} \cup \{(W_i'',v_i)\}_{i \in I} \) is a norm retrievable fusion frame for \( \mathbb{H} \).

Proof. If \( \|Q_ix\| = \|Q_iy\| \) and \( \|(I - Q_i)x\| = \|(I - Q_i)y\| \), for all \( i \in I \), then
\[
\begin{align*}
\|P_ix\|^2 &= \|Q_ix\|^2 + \|(I - Q_i)x\|^2 \\
&= \|Q_ix\|^2 + \|(I - Q_i)y\|^2 \\
&= \|P_iy\|^2
\end{align*}
\]
Since \( \{P_i\}_{i \in I} \) does norm retrieval, we have \( \|x\| = \|y\|. \)

The characterization of norm retrievable families of vectors first appeared in [10].

Theorem 8. A family of vectors \( \{f_k\}_{k=1}^\infty \) does norm retrieval for \( \mathbb{H} \) if and only if for any subset \( I \subset \mathbb{N} \),
\[
\text{span}\{f_k\}_{k \in I}^\perp \perp \text{span}\{f_k\}_{k \in I^c}^\perp.
\]
One direction of this implication holds for fusion frames.

Theorem 9. Let \( \{W_i,v_i\}_{i \in I} \) be a fusion frame in \( \mathbb{R}^n \). If \( \{W_i,v_i\}_{i \in I} \) does norm retrieval, then whenever \( J \subset I \), and \( x \perp W_j \) for all \( j \in J \) and \( y \perp W_j \) for all \( j \in J^c \), then \( x \perp y. \)
Theorem 11. Let \( \{e_i\}_{i=1}^n \) be the canonical orthonormal basis for \( R^n \). Let \( \{I_i\}_{i=1}^m \) be subsets of \([n]\). Let
\[
W_i = \text{span}\{e_j\}_{j \in I_i}, \text{ for all } i = 1, 2, \ldots, m.
\]
Assume there exists a natural number K and \( \varepsilon_i = \pm 1 \) so that
\[
\sum_{i=1}^m \epsilon_i I_i = K(1, 1, \ldots, 1) \in R^n.
\]
Then \( \{W_i, v_i\}_{i=1}^m \) does norm retrieval for all \( 0 < v_i < \infty \).

Proof. Let \( \{P_i\}_{i \in I} \) be the projections onto \( \{W_i\}_{i \in I} \). For \( x \in R^n \) we have
\[
\sum_{i=1}^m \|P_i x\|^2 = \sum_{i=1}^m \epsilon_i \sum_{j \in I_i} a_j^2 = \sum_{j=1}^n \sum_{i=1}^m \{\epsilon_i a_j^2 : j \in I_i\} = \sum_{j=1}^n K a_j^2 = K\|x\|^2.
\]
It follows that the fusion frame does norm retrieval. \( \square \)
The converse to the above theorem fails. Let \( \{e_i\}_{i=1}^4 \) be a orthonormal basis for \( \mathbb{R}^4 \) and let
\[
W_1 = [e_1, e_4], \ W_2 = [e_2, e_4], \ W_3 = [e_3, e_4], \ W_4 = [e_4].
\]
This clearly does not retrieval. Also,
\[
I_1 = \{1, 4\}, \ I_2 = \{2, 4\}, \ I_3 = \{3, 4\}, \ I_4 = \{4\}.
\]
If \( I \subset \{4\} \), \( e_i = \pm 1 \) for \( i \in I \), and \( \sum_{i \in I} e_i I_i = K(1, 1, 1, 1) \), then \( K = 1 \) and \( e_i = 1 \) for \( i = 1, 2, 3 \). Since \( \sum_{i=1}^3 I_i = (1, 1, 1, 3) \), this set fails the assumption in the theorem.

3. Hyperplanes

The following appeared in [9].

**Theorem 12.** If \( \{W_i\}_{i=1}^m \) are hyperplanes doing norm retrieval in \( \mathbb{R}^n \) and \( \{W_i^\perp\}_{i=1}^m \) are linearly independent, then \( m \geq n \).

The following appeared in [9].

**Theorem 13.** Let \( \{W_i\}_{i=1}^m \) be subspaces of \( \mathbb{R}^n \) with projections \( \{P_i\}_{i=1}^m \). The following are equivalent:

1. \( \{W_i\}_{i=1}^m \) does norm retrieval.
2. For every \( 0 \neq x \in \mathbb{R}^n \), \( x \in \text{span}\{P_ix\}_{i=1}^m \).

The following example appears in [9]. We will give a new proof which generalizes to answer another problem.

**Example 2.** If \( \{W_i\}_{i=1}^n \) are hyperplanes doing norm retrieval, this does not imply that \( \{W_i^\perp\}_{i=1}^n \) are independent. Let \( \{e_i\}_{i=1}^3 \) be a orthonormal basis of \( \mathbb{R}^3 \) and let \( \phi_1 = e_1 \), \( \phi_2 = e_2 \), and \( \phi_3 = (e_1 - e_2)/\sqrt{2} \). The vectors are not independent but Note that
\[
\phi_1^\perp = W_1 = [e_2, e_3], \ \phi_2^\perp = W_2 = [e_1, e_3], \ \phi_3^\perp = W_3 = [(e_1 + e_2)/\sqrt{2}, e_3].
\]
The \( \{W_i^\perp\}_{i=1}^3 \) are hyperplanes doing norm retrieval. To see this let \( \{P_i\}_{i=1}^3 \) be the projections onto \( \{W_i\}_{i=1}^3 \) and we will check Theorem 13. Then if \( x = (a, b, c) \in \mathbb{R}^3 \),
\[
(3.1) \quad P_1 x = (0, b, c) \\
(3.2) \quad P_2 x = (a, 0, c) \\
(3.3) \quad P_3 x = \left(\frac{a + b}{2}, \frac{a + b}{2}, c\right)
\]
Then,
\[
(3.2) - (3.1) = (a, -b, 0), \ 2(3.3) = (a + b, a + b, 2c).
\]
Also,
\[
2(3.3) - (3.1) - (3.2) = (b, a, 0).
\]
We leave it to the reader to check the special cases where some of \( a, b, c \) are zero. For example, if \( a = 0 \neq b, c \) then \( P_2 x = (0, 0, c) \) and so \( e_3 \in \text{span}\{P_ix\}_{i=1}^3 \) and now by (3.2), \( e_2 \in \text{span}\{P_ix\}_{i=1}^3 \) and hence \( x \in \text{span}\{e_i\}_{i=2}^3 = \text{span}\{P_ix\}_{i=1}^3 \). So we assume \( a, b, c \neq 0 \). Since \( (a, -b, 0) \perp (b, a, 0) \) and both are in \( \text{span}\{P_ix\}_{i=1}^3 \), it follows that \( e_1, e_2 \in \text{span}\{P_ix\}_{i=1}^3 \). Combining this with (3.3) puts \( e_3 \in \text{span}\{P_ix\}_{i=1}^3 \). So \( x \in \mathbb{R}^3 = \text{span}\{P_ix\}_{i=1}^3 \).
Example 3. Theorem 12 may fail if the \( \{W_i^\perp\}_{i=1}^n \) are not linearly independent. Let \( \{e_i\}_{i=1}^4 \) be an orthonormal basis for \( \mathbb{R}^4 \) and define hyperplanes

\[
W_1 = [e_2, e_3, e_4], \quad W_2 = [e_1, e_3, e_4], \quad W_3 = [e_1 + e_2, e_3, e_4],
\]

and let \( \{P_i\}_{i=1}^3 \) be the corresponding projections. Then

\[
W_i^\perp = [e_1], \quad W_2^\perp = [e_2], \quad W_3^\perp = [\frac{e_1 - e_2}{\sqrt{2}}],
\]

and \( \{W_i^\perp\}_{i=1}^3 \) are not independent. We will show that \( \{W_i^\perp\}_{i=1}^3 \) does norm retrieval in \( \mathbb{R}^4 \). We mimic Example 2. Let \( x = (a, b, c, d) \) and we check Theorem 13. Again we leave it to the reader to check the simple cases where one or more of the \( a, b, c, d \) are zero. Using exactly the same argument as Example 2 we discover that \( e_1, e_2 \in \text{span}\{P_i x\}_{i=1}^3 \) so \( (a, 0, 0, 0) \in \text{span}\{P_i x\}_{i=1}^3 \). Also, \( P_1 x = (0, b, c, d) \in \text{span}\{P_i x\}_{i=1}^3 \). It follows that \( x \in \text{span}\{P_i x\}_{i=1}^3 \).

An examination of the above two examples shows a general result.

Theorem 14. Let \( \{e_i\}_{i=1}^n \) be an orthonormal basis for \( \mathbb{R}^n \). If \( \{W_i\}_{i=1}^m \) does norm retrieval for \( \{e_1, e_2, \ldots, e_k\}, \quad 1 \leq k < n \), then \( \{W_i \cup \{e_{k+1}, e_{k+2}, \ldots, e_n\}\}_{i=1}^m \) does norm retrieval in \( \mathbb{R}^n \). If the \( W_i \) are hyperplanes in \( \{e_1, \ldots, e_k\} \) then the new sets are hyperplanes in \( \mathbb{R}^n \).

4. Full spark, finitely full spark and norm retrievability

Spark is also an essential issue here.

Definition 10. A family of vectors \( \{f_i\}_{i=1}^m \) in \( \mathbb{R}^n \) \((m \geq n)\) has \textbf{spark} \( k \) if for every \( I \subset [m] \) with \( |I| = k-1 \), \( \{f_i\}_{i \in I} \) is linearly independent. It is full spark if \( k = n+1 \) and hence every \( n \)-element subset spans \( \mathbb{R}^n \).

It is proved in [3] that finitely full spark frames are dense in all frames in both the finite and infinite dimensional case. By theorem 2 the families of vectors \( \{f_k\}_{k=1}^m \) which do phase retrieval in \( \mathbb{R}^n \) are dense in the family of \( m \geq (2n-1) \)-element sets of vectors in \( \mathbb{R}^n \) for every finite \( n \). Since every phase retrievable frame in \( \mathbb{R}^n \) is a norm retrievable frame we have: the families of vectors \( \{f_k\}_{k=1}^m \) which do norm retrieval in \( \mathbb{R}^n \) are dense in the family of \( m \geq (2n-1) \)-element sets of vectors in \( \mathbb{R}^n \) for every finite \( n \). Now we focus on the case when \( m \leq (2n-2) \). We will show that the families of norm retrievable frames \( \{f_k\}_{k=1}^m \) in \( \mathbb{R}^n \) are not dense in the family of \( m \leq (2n-2) \)-element sets of vectors in \( \mathbb{R}^n \) for every finite \( n \). This will require some preliminary results.

Theorem 15. Let \( X \) and \( Y \) be subspaces of \( \mathbb{R}^n \) and \( T : X \rightarrow Y \) be an operator with \( \|I - T\| < \epsilon \). Let \( P \) be the projection onto \( X \). Define \( S : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by:

\[
S(Px + (I - P)x) = TPx + (I - P)x.
\]

Then

\[
\begin{align*}
(1) \quad & \|I - S\| < \epsilon \quad \text{and} \quad \|S\| < 1 + \epsilon \\
(2) \quad & \|S^{-1}\| < \frac{1}{1 - \epsilon} \quad \text{and} \quad \|I - S^{-1}\| < \frac{\epsilon}{1 - \epsilon} \\
(3) \quad & Q = SPS^{-1} \quad \text{is a projection onto} \ Y.
\end{align*}
\]
(4) \[ \| P - Q \| < \frac{\epsilon^2}{2} \]

(5) We have for \( x \in X \), \( \|Qx\| < \epsilon \|x\| \).

(6) For \( x \perp X \), \( \|(I - Q)x\| \geq (1 - \epsilon)\|x\| \).

(7) For \( x \perp X \), \( \|x\| = 1 \), \( \|x - (I - Q)x\| \leq \epsilon \|x\| \).

It follows that given \( \delta > 0 \) there is an \( \epsilon > 0 \) so that given the assumptions of the theorem, if \( x \perp X \) with \( \|x\| = 1 \), there is a \( y \perp Y \) with \( \|y\| = 1 \) and \( \|x - y\| < \delta \).

**Proof.** (1) We compute

\[
\|(I - S)x\| = \|x - Sx\|
\]
\[
= \|Px + (I - P)x - S(Px + (I - P)x)\|
\]
\[
= \|Px + (I - P)x - TPx - (I - P)x\|
\]
\[
= \|Px - TPx\|
\]
\[
= \|(I - T)Px\|
\]
\[
\leq \epsilon \|Px\| \leq \epsilon \|x\|.
\]

Also,

\[
\|Sx\| \leq \|x\| + \|x - Sx\| \leq \|x\| + \epsilon \|x\| = (1 + \epsilon)\|x\|.
\]

(2) By the Neuman series,

\[
S^{-1} = (I - (I - S))^{-1} = \sum_{i=0}^{\infty} (I - S)^i,
\]

and so

\[
\|S^{-1}\| \leq \sum_{i=0}^{\infty} \|(I - S)^i\| = 1 + \sum_{i=1}^{\infty} \epsilon^i = 1 + \frac{\epsilon}{1 - \epsilon} = \frac{1}{1 - \epsilon}.
\]

Also,

\[
\|I - S^{-1}\| = \|\sum_{i=1}^{\infty} (I - S)^i\| \leq \sum_{i=1}^{\infty} \|(I - S)^i\| \leq \frac{\epsilon}{1 - \epsilon}.
\]

(3) Since \( SPS^{-1}SPS^{-1} = SP^2S^{-1} = SPS^{-1} \), this is a projection. If \( y \in Y \), \( S^{-1}y \in X \) and so \( PS^{-1}y = S^{-1}y \) and hence \( SPS^{-1}SPS^{-1}y = SS^{-1}y = y \).

(4) We compute

\[
\|P - Q\| = \|P - SPS^{-1}\|
\]
\[
= \|P - SP + SP - SPS^{-1}\|
\]
\[
\leq \|(I - S)P\| + \|SP(I - S^{-1})\|
\]
\[
\leq \|I - S\||P| + \|SP||I - S^{-1}||
\]
\[
\leq \epsilon + \frac{\epsilon}{1 - \epsilon} \|S\| \|P\|
\]
\[
\leq 1 + \frac{\epsilon(1 + \epsilon)}{1 - \epsilon} = \frac{1 + \epsilon^2}{1 - \epsilon}.
\]

(5) We compute for \( x \perp X \),

\[
\|Qx\| \leq \|(P - Q)x\| + \|Px\| \leq (\epsilon + 0)\|x\|
\]
(6) We compute for $x \perp X$
$$\|(I - Q)x\| \geq \|x\| - \|Qx\| \geq \|x\| - \|Qx\| \geq (1 - \epsilon)\|x\|.$$ 
(7) We compute for $x \perp X$, $\|x\| = 1$ using (5) and (6),
$$1 - \epsilon \leq \|(I - Q)x\| \leq \|x\| + \|Qx\| \leq 1 + \epsilon.$$ 
So
$$\frac{1}{1 + \epsilon} \leq \frac{1}{\|(I - Q)x\|} \leq \frac{1}{1 - \epsilon},$$
so
$$-1 \leq \frac{1}{\|(I - Q)x\|} \leq 1 - \epsilon,$$
and
$$1 - \frac{1}{1 + \epsilon} = \frac{\epsilon}{1 + \epsilon} \geq 1 - \frac{1}{\|(I - Q)x\|} \geq 1 - \frac{1}{1 - \epsilon} = \frac{-\epsilon}{1 - \epsilon}.$$ 
Hence,
$$\left| 1 - \frac{1}{\|(I - Q)x\|} \right| \leq \frac{\epsilon}{1 - \epsilon}.$$ 

Now,
$$\left\| x - \frac{(I - Q)x}{\|(I - Q)x\|} \right\| \leq \left\| x - (I - Q)x \right\| + \left\| (I - Q)x - \frac{(I - Q)x}{\|(I - Q)x\|} \right\|$$
$$= \|Qx\| + \|(I - Q)x\| \left| 1 - \frac{1}{\|(I - Q)x\|} \right|$$
$$\leq \epsilon + (1 + \epsilon) \frac{\epsilon}{1 - \epsilon} = \epsilon \left( 1 + \frac{1}{1 - \epsilon} + \frac{\epsilon}{1 - \epsilon} \right) = \frac{2\epsilon}{1 - \epsilon}.\)

We can now prove the main result.

**Theorem 16.** Let $\{x_i\}_{i=1}^m$, $m \leq 2n - 2$ be a frame in $\mathbb{R}^n$ which fails norm retrieval. Then there is an $\epsilon > 0$ so that whenever $\{y_i\}_{i=1}^m$ are vectors satisfying
$$\sum_{i=1}^m \|x_i - y_i\| < \epsilon,$$
then $\{y_i\}_{i=1}^m$ also fails norm retrieval.

**Proof.** Since $\{x_i\}_{i=1}^m$ fails norm retrieval, there is some $I \subset [m]$ so that there are vectors $\|x\| = \|y\| = 1$, $x \perp x_i$ for all $i \in I$, $y \perp x_i$ for all $i \in I^c$ and $x$ is not orthogonal to $y$. That is, there is a $\delta > 0$ so that $\|x - y\| \leq \sqrt{2} - \delta$. Choose $\epsilon > 0$ so that
$$\frac{2\epsilon}{1 - \epsilon} < \frac{\delta}{3}.$$ 

Let $J_1 \subset I$ and $J_2 \subset I^c$ with $\{x_i\}_{i \in J_1}$ linearly independent for $k = 1, 2$ and
$$X_1 = \text{span}\{x_i\}_{i \in J_1} = \text{span}\{x_i\}_{i \in I} \quad \text{and} \quad X_2 = \text{span}\{x_i\}_{i \in J_2} = \text{span}\{x_i\}_{i \in I^c}.$$ 

Let $Y_1 = \text{span}\{y_i\}_{i \in J_1}$ and $Y_2 = \text{span}\{y_i\}_{i \in J_2}$ and define operators $T_k : X_k \rightarrow Y_k$ for $k = 1, 2$ by $T_kx_i = y_i$ for $i \in J_k$. By Theorem 15, there are vectors $\|z\| = \|w\| = 1$ with $z \perp y_i$ for all $i \in I$, $w \perp y_i$ for all $i \in I^c$ and
$$\|x - z\| < \frac{\delta}{3} \quad \text{and} \quad \|y - w\| < \frac{\delta}{3}.\)
Now
\[ \| z - w \| \leq \| z - x \| + \| x - y \| + \| y - w \| < \frac{\delta}{3} + \sqrt{2} - \delta + \frac{\delta}{3} = \sqrt{2} - \frac{\delta}{3}. \]

It follows that \( z \) and \( w \) are not orthogonal and so by Theorem 8, \( \{ y_i \}_{i=1}^m \) fails norm retrieval. \( \square \)

On the surface, it looks like the above argument works equally well for \( m \geq 2n-1 \). The problem is that in this case, if the vectors are full spark then whenever we divide them into two sets, one will span the whole space. So the only vector orthogonal to this set is the zero vector and this vector is orthogonal to all vectors. I.e. This set does norm retrieval.

Now, we will consider the infinite dimensional Hilbert space \( \ell^2 \). In \( \ell^2 \), every phase (norm) retrievable family need not to be full spark. For example if we write a norm retrievable set twice, the latter is a norm retrievable set again, but it is not full spark. The corresponding definition of a full spark family for \( \ell^2 \) first appeared in [3].

**Definition 11.** A family of vectors \( \{ f_k \}_{k=1}^\infty \) in \( \ell^2 \) is **full spark** if every infinite subset spans \( \ell^2 \).

If a set is full spark in \( \ell^2 \), we can drop infinitely many vectors and as long as there are infinitely many left, it still spans \( \ell^2 \).

The concept of “finitely full spark vectors for \( \ell^2 \)” first was introduced in [3].

**Definition 12.** A set of vectors \( \{ f_k \}_{k=1}^\infty \) in \( \ell^2 \) is **finitely full spark** if for every \( I \subset \mathbb{N} \) with \( |I| = n \), \( \{ P_I f_k \}_{k=1}^\infty \) is full spark (i.e., spark \( n + 1 \)), where \( P_I \) is the orthogonal projection onto \( \operatorname{span} \{ e_k \}_{k \in I} \).

**Example 4.** The set \( \{ 1/2^i e_1 + 1/\{2^i + 1\} e_2 + \cdots + 1/\{2^i + (n-1)\} e_n \}_{i=1}^\infty \) is finitely linear independent and so is finitely full spark for \( \ell^2 \) for any arbitrary \( n \in \mathbb{N} \).

It is shown in [3] that the families of vectors which do phase retrieval in \( \ell^2 \) are not dense in the infinite families of vectors in \( \ell^2 \). We will show a similar result for the families of vectors which do norm retrieval in \( \ell^2 \). By [6] we know if \( \{ \phi_i \}_{i=1}^n \) does norm retrieval in \( \mathbb{R}^n \), then the vectors of the frame are orthogonal. This is true in \( \ell^2 \) also.

**Proposition 1.** Let \( \{ \phi_i \}_{i=1}^\infty \) be a Riesz basis doing norm retrieval in \( \ell^2 \), then the vectors \( \{ \phi_i \}_{i=1}^\infty \) are orthogonal.

**Proof.** Without loss of generality, assume \( \| \phi_i \| = 1 \) and there is some \( j \in I \) with \( \phi_j \) not orthogonal to \( \operatorname{span} \{ \phi_i \}_{i \neq j} \). Choose a unit vector \( x \perp \phi_i \) for all \( i \neq j \). So that \( x \neq \phi_j \). Let \( y = x - \langle x, \phi_j \rangle \phi_j \). Now \( \langle \phi_j, y \rangle = \langle \phi_j, x \rangle - \langle x, \phi_j \rangle \langle \phi_j, \phi_j \rangle = 0 \). Let \( I = \{ i : i \neq j \} \). Then \( x \perp \operatorname{span} \{ \phi_i \}_{i \in I} \) and \( y \perp \phi_j \), but \( \langle x, y \rangle = \langle x, x \rangle - \langle x, \phi_j \rangle \langle x, \phi_j \rangle = 1 - |\langle x, \phi_j \rangle|^2 \neq 0 \), contradicting Theorem 8. \( \square \)

**Lemma 1.** Let \( \{ \phi_i \}_{i=1}^\infty \) and \( \{ \psi_i \}_{i=1}^\infty \) be Riesz bases for \( \ell^2 \). Given \( \epsilon > 0 \) arbitrary, there exists \( \delta > 0 \) such that
\[
\text{if } \sum_{i=1}^\infty \| \phi_i - \psi_i \| \leq \delta \quad \text{then} \quad \sum_{i=1}^\infty \left\| \frac{\phi_i}{\| \phi_i \|} - \frac{\psi_i}{\| \psi_i \|} \right\| \leq \epsilon .
\]
Proof. Since \( \{ \phi_i \}_{i=1}^{\infty} \) and \( \{ \psi_i \}_{i=1}^{\infty} \) are Riesz bases, there are constants \( 0 < A \leq B < \infty \) satisfying \( A \leq \| \phi_i \| \leq B \) and \( A \leq \| \psi_i \| \leq B \), for all \( i \in \mathbb{N} \). Assume \( \sum_{i=1}^{\infty} \| \phi_i - \psi_i \| \leq \delta \) with \( \frac{2B}{A} \delta = \epsilon \), then we have

\[
\sum_{i=1}^{\infty} \left\| \frac{\phi_i}{\| \phi_i \|} - \frac{\psi_i}{\| \psi_i \|} \right\| = \sum_{i=1}^{\infty} \frac{1}{\| \phi_i \| \| \psi_i \|} \left( \| \psi_i \| \| \phi_i \| - \| \phi_i \| \| \psi_i \| \right)
\leq \frac{1}{A^2} \sum_{i=1}^{\infty} \left( \| \psi_i \| \| \phi_i \| - \| \phi_i \| \| \psi_i \| \right)
\leq \frac{1}{A^2} \sum_{i=1}^{\infty} \left( \| \psi_i \| \| \phi_i \| - \| \phi_i \| \| \psi_i \| \right)
\leq \frac{1}{A^2} \sum_{i=1}^{\infty} \| \psi_i \| - \| \phi_i \| \| \psi_i \| + \frac{1}{A^2} \sum_{i=1}^{\infty} \| \phi_i \| \| \phi_i - \psi_i \| \nleq \frac{B}{A^2} \sum_{i=1}^{\infty} \| \psi_i - \phi_i \| + \frac{B}{A^2} \sum_{i=1}^{\infty} \| \psi_i - \phi_i \|
= \frac{2B}{A^2} \delta = \epsilon
\]

\( \Box \)

**Theorem 17.** The families of vectors which do norm retrieval in \( l^2 \) are not dense in the infinite families of vectors in \( l^2 \).

**Proof.** We prove the Theorem in three steps.

Step 1: By Proposition [1] if \( \{ \phi_i \}_{i=1}^{\infty} \) be a Riesz basis doing norm retrieval in \( l^2 \), then the vectors of \( \{ \phi_i \}_{i=1}^{\infty} \) are orthogonal.

Step 2: It is known that if \( \{ \phi_i \}_{i=1}^{\infty} \) is a Riesz basis and \( \{ \psi_i \}_{i=1}^{\infty} \) is close enough to it, then \( \{ \psi_i \}_{i=1}^{\infty} \) must be a Riesz basis.

Step 3: If \( \{ \phi_i \}_{i=1}^{\infty} \) is a Riesz basis and \( \{ \psi_i \}_{i=1}^{\infty} \) is an orthogonal set of vectors arbitrary close enough to \( \{ \phi_i \}_{i=1}^{\infty} \), then \( \{ \phi_i \}_{i=1}^{\infty} \) must be orthogonal. To prove it: We know by the parallelogram law if \( \| x \| = \| y \| = 1 \), then \( \| x - y \| = \sqrt{2} \) if and only if \( x \perp y \). In the above problem we may assume that for simplicity we only have two vectors \( x_1, x_2 \subset \{ \phi_i \}_{i=1}^{\infty} \) with \( \| x_1 \| = \| x_2 \| = 1 \) and given \( \epsilon > 0 \) by Lemma [2] there are vectors \( y_1, y_2 \subset \{ \psi_i \}_{i=1}^{\infty} \) such that \( \| y_1 \| = \| y_2 \| = 1 \), \( y_1 \perp y_2 \) and \( \| x_i - y_i \| < \epsilon \) then

\[
\sqrt{2} - 2 \epsilon = \| y_1 - y_2 \| - 2 \epsilon \leq \| x_1 - x_2 \| \leq \| y_1 - y_2 \| + 2 \epsilon = \sqrt{2} + 2 \epsilon.
\]

Since \( \epsilon \) was arbitrary, it follows that \( \sqrt{2} = \| x_1 - x_2 \| \) and \( \{ \phi_i \}_{i=1}^{\infty} \) is an orthogonal sequence. Therefore if \( \{ \phi_i \}_{i=1}^{\infty} \) is a non-orthogonal Riesz basis, there is no norm retrievable sequence \( \{ \psi_i \}_{i=1}^{\infty} \) that is orthogonal and close enough to \( \{ \phi_i \}_{i=1}^{\infty} \). Thus the families of vectors which do norm retrieval in \( l^2 \) are not dense in the infinite families of vectors in \( l^2 \).

\( \Box \)

Of course by Theorem 8, we can construct many none orthogonal Riesz basic sequences that are close to an orthogonal Riesz basic sequence.
Example 5. Let $0 < \epsilon < 1$ be given, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\ell^2$, and choose any vectors $\{x_i\}_{i=1}^{\infty}$ so that
\[
\sum_{i=1}^{\infty} \|x_i\| < \epsilon.
\]
Then $\{e_i + x_i\}_{i=1}^{\infty}$ is a Riesz basis failing norm retrieval and it is $\epsilon$-close to an orthonormal basis.

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