Numerical Studies on the Effect of Boundary Conditions for a Simply-Supported Plate

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Abstract. In this paper we analyze a simply-supported plate under the action of transverse uniform pressure. The Mindlin theory of moderately thick plates is used for analysis. Different boundary conditions are considered at the edges of the plate and their effect on the solution, in particular, shear forces, is studied. Only finite element results are presented.

1. Introduction
The objective of this work is the analysis of a moderately thick plate which is simply supported and subjected to the action of uniform transverse pressure. The theory of thick plates, in its simplest form, is described, for example, in the books by Wang et al. [1], Ferreira [2], Khennane [3]. More recent developments of the theory of thick plates are presented in the paper by Ma et al. [4].

We recall that in the thick plate theory transverse normal may rotate without remaining perpendicular to the mid-plane and thus the shear deformations are not zero. By contrast, the thin plate theory (Kirchhoff plate theory) does not include shear deformations and a vertical element of the plate remains perpendicular to the mid-plane after bending.

Analytical results for the simply-supported laminated plates and doubly-curved general panel are presented in the papers by Kabir [5,6] and Chaudhuri [7]. Analytical results for the thick plates are more difficult to obtain than for the thin plates but finite element implementation of the thick plate element is easier. In this paper only numerical (finite element) results will be presented for the analysis of thick plates. The finite element code written by Ferreira [2] in Matlab will be used for obtaining the numerical results.

It is natural to expect that in the shear-flexible plates there is a broader range of boundary conditions that one can prescribe at the edges. In fact, Kabir [5,6] notes that for the thick simply-supported plates there are four different types of boundary conditions denoted by SS1, SS2, SS3 and SS4. But it appears that this classification is mainly related to variations in prescribing in-plane displacements $u$ and $v$. The objective of this paper is to consider how prescribing or not prescribing zero rotations or twists $\phi_x$, $\phi_y$ in the planes of edges of simply-supported plate affects the solution.

2. Description of the problem and its solution
Consider a simply supported plate subjected to transverse uniformly-distributed pressure load $q$. The plate has a square planform in the $x$-$y$ plane and the side of the plate is $a$. The thickness of the plate is $h$ (figure 1).
Figure 1. Geometry of the square plate

Let $u$, $v$ and $w$ be the displacements of the plate midplane along $x$, $y$ and $z$ axis respectively. Thus, $w$ is the transverse displacement and it is assumed independent of the $z$-coordinate. Also, let $\phi_x$ and $\phi_y$ signify rotations of the normal to the midplane about $y$ and $x$-axis respectively. We also denote the bending moment about $y$ and $x$-axis by $M_x$ and $M_y$ respectively, the twisting moment by $M_{xy}$, the shear force acting in the cross-section $x = \text{const}$ by $Q_x$ and the shear force in the cross-section $y = \text{const}$ by $Q_y$. We note that different notation may be used in commercial finite element software (e.g., $\phi_x$ is the rotation about $x$-axis) but the notation adopted in this paper is used more commonly in plate theory.

Consider two types of simply-supported boundary conditions for this plate. For both types of boundary conditions, we prescribe zero transverse displacement and appropriate bending moments on the edges of the plate as follows

$$
\begin{align}
w &= 0 & x = 0, a, & y = 0, a \\
M_x &= 0 & x = 0, a \\
M_y &= 0 & y = 0, a.
\end{align}
$$

In the first type of boundary conditions, denoted here by SS-A, we prescribe zero rotations-twists on the edges of the plate as follows

$$
\phi_x = 0 & x = 0, a \\
\phi_y = 0 & y = 0, a.
$$

In the second type of boundary conditions, denoted here by SS-B, zero twisting moment is prescribed on the edges of the plate as follows

$$
M_{xy} = 0 & y = 0, a.
$$
Due to the symmetry of the plate and the absence of the in-plane loads, the displacements of the plate midplane along \( x \) and \( y \) axes, denoted here by \( u \) and \( v \), will be equal to zero.

The solution of the present problem will be based on Mindlin plate theory or theory suitable for moderately thick plates. Typically, in most research work, the first type of boundary conditions, SS-A, is considered only and the obtained results, at least for the symmetric homogeneous plate, are very similar to the solution based on the thin plate theory. In this work, however, we also attempt to obtain numerical solution for the plate with SS-B boundary conditions (typically not available) and compare two sets of results. Note again that for the SS-B type of boundary conditions, zero rotations are not prescribed on the edges of the plate. It is also useful to note that in terminology of Wang et al. [1] SS-A and SS-B types of boundary conditions are denoted by SS and SS* respectively.

The present problem is solved with the help of finite element method. For that, the plate is subdivided by equal number of elements along \( x \) and \( y \) axes (say, 40 by 40 elements). Four node quadrilateral finite elements are used. Finite element program written in Matlab by Ferreira [2] with some adjustments was used to obtain the numerical results.

In what follows we mainly concentrate on shear force distribution in the plates considered. This is due to the fact that the deflections \( w \) turned out to be very similar for the plates with these two types of boundary conditions. Also, the plots of deflections are usually presented in all relevant textbooks but the shear forces are typically not shown or discussed.

We observe here that the results for the shear forces are less accurate in finite element analysis and thus it is desirable to use finer meshes to obtain more reliable results. The code provided by Ferreira [2], however, could hardly be used for finer meshes (say, 200 by 200 elements) due to the fact that the \textit{total} stiffness matrix (will all its zero terms) is stored in computer memory. Thus, the code was adjusted to take advantage of the sparse nature of the stiffness matrix and reduce the memory requirements. Use of Matlab sparse solver for solving this problem efficiently will be addressed in the next paper of the author.

Let us review computation of the shear force in the finite element analysis based on the thick plate theory. Let \( \mathbf{d}^e \) be the element displacement vector

\[
\mathbf{d}^e = [w_1, \phi_{x1}, \phi_{y1}, \ldots, w_4, \phi_{x4}, \phi_{y4}]^T.
\]

This vector has 12 components for a four-node finite element. Then the shear strains \( \mathbf{\varepsilon}_c = [\gamma_{xz}, \gamma_{yz}]^T \) can be computed using the strain-displacement matrix \( \mathbf{B}_c \), i.e.,

\[
\mathbf{\varepsilon}_c = \mathbf{B}_c \mathbf{d}^e.
\]

Here the strain-displacement matrix \( \mathbf{B}_c \) is given by

\[
\mathbf{B}_c = \begin{bmatrix} \frac{\partial N_1}{\partial x} & N_1 & 0 & \ldots & \frac{\partial N_4}{\partial x} & N_4 & 0 \\ \frac{\partial N_1}{\partial y} & 0 & N_1 & \ldots & \frac{\partial N_4}{\partial y} & 0 & N_4 \end{bmatrix}.
\]

This matrix can be computed at any point within the element, \(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1\), where \( \xi \) and \( \eta \) are the natural coordinates of the point. In particular, to obtain distribution of the shear strains
or stresses along the side of the element, say $\eta = -1$, it is sufficient to compute the $B_\eta$ matrix at two points with natural coordinates $\xi = -1, \eta = -1$ and $\xi = 1, \eta = -1$ and then use linear interpolation. This is exactly the way that we use for evaluation of the shear forces acting along the edge $y = 0$ (see next section).

If we multiply the shear strains (5) by the shear modulus $G$ and thickness of the plate $h$ we obtain the shear force.

3. Results and discussions

As an example we consider a square plate of side $a = 1$. The magnitude of the uniform transverse pressure acting on the plate is $q = 1$. The modulus of elasticity is $E = 10920$ and Poisson's ratio is $\nu = 0.3$. The thickness of the plate is taken as $h = 0.1$. This data is taken from the book by Ferreira [2].

In what follows, we show distribution of the shear forces along the edge $y = 0$ for which $0 \leq x \leq 1$ (due to the symmetry of the plate the results for the opposite edge $y = 1$ are the same). The shear force was computed as described in the previous section.

In figure 2 we show the distribution of the shear force $Q_y$, and in figure 3 the distribution of the shear force $Q_x$ is shown for the same edge. Both sets of results are obtained for the finite element mesh with 40 by 40 elements. The results are shown for the plates with two types of boundary conditions described above, i.e., SS-A and SS-B.

In figure 2 we clearly see the difference between the distributions of the shear force $Q_y$ for these two types of boundary conditions. The shear force distribution $Q_y$ for SS-A plate is very similar to the thin plate solution. Indeed, we see the maximum of the shear force at the center of the edge, at $x = 1/2$, and zero shear force at the corners $x = 0$ and $x = 1$. The maximum of the shear force is approximately equal to 0.314 and this is smaller than the support reaction for the simply-supported beam: $qa / 2 = 0.5$. For the SS-B plate we see somewhat larger value of the shear force at the center of the edge, $x = 1/2$, ($\approx 0.374$) but we also see large forces of opposite sign at the corners of the plate ($\approx 0.724$). It is important to note here that the values for the shear force are converging slowly, especially for the SS-B plate, and thus for better accuracy finer meshes are desired. However, due to the reasons mentioned above, the finer meshes can be used only after considerable improvement of the code performance.

In figure 3 the shear force distribution $Q_x$ is shown for the same edge $y = 0$. For the SS-A plate, this shear force is zero along this edge and this is also true for the thin plate solution. But for the SS-B plate, this force is large near the corners of the plate ($\approx 0.724$) and has a maximum ($\approx 0.949$) at some distance away from the corners. Such solution can obviously not be obtained using the thin plate theory, which is insensitive to whether zero rotations are prescribed at the edges of the plate or zero twisting moment. We also note that due to the symmetry, $Q_x = Q_y$ at the corners of the plate as expected.
Figure 2. Shear force at the edges of the square plate subjected to uniform transverse pressure of magnitude 1. Two types of boundary conditions, SS-A and SS-B, are considered.

Figure 3. Shear force at the edges of the square plate subjected to uniform transverse pressure of magnitude 1. Two types of boundary conditions, SS-A and SS-B, are considered.

4. Conclusions
In this paper we analyzed the simply-supported plate under the action of transverse uniform pressure. Two types of boundary conditions were considered for this plate: SS-A type with zero rotations-twists in the plane of each edge, and SS-B type with zero twisting moment in the plane of each edge.
Distribution of the shear forces was examined for the plates with these two types of boundary conditions and it was shown that the shear force distributions for the SS-A and SS-B plates are considerably different. In particular, larger values of the shear forces are obtained for SS-B plate. Moreover, large non-zero values of the shear force are obtained for the SS-B plate at those points on the plate boundary where the corresponding shear force for the SS-A plate is equal zero.

Thus, when using the theory of thick plates or the thick plate finite element it is important to adequately prescribe boundary conditions at the edges of the plate. Different choices of boundary conditions for the simply-supported thick plate may lead to drastically different solutions, in particular, different distributions of shear forces.

References

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