Solution of the Linear Ordering Problem
(NP=P)

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We consider the following problem

\[
\begin{align*}
\text{max} & \quad \sum_{i=1, i \neq j}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s. t.} & \quad 0 \leq x_{ij} \leq 1, \\
& \quad x_{ij} + x_{ji} = 1, \\
& \quad 0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, i \neq j, i \neq k, j \neq k, i, j, k = 1, \ldots, n.
\end{align*}
\]

We denote the corresponding polytope by \( B_n \). The polytope \( B_n \) has integer vertices corresponding to feasible solutions of the linear ordering problem as well as non-integer vertices. We denote the polytope of integer vertices as \( P_n \).

Let us give an example of non-integer vertex in \( B_n \) and describe an exact facet cut. In what follows we will interested only in generating exact facet cuts.

Fig. 1 shows a graph interpretation of a non-integer vertex \([1]\),
where \( \{i_1, ..., i_m\} \), \( \{j_1, ..., j_m\} \subset \{1, ..., n\} \); \( i_1, ..., i_m \cap j_1, ..., j_m = \emptyset \); 
\( x_{ij} = 0, x_{qij} = 1, l \neq q, l, q = 1, ..., m \); the other variables that are not shown at the Figure 1 are equal to \( \frac{1}{2} \). This is the simplest non-integer vertex of the polytope \( B_n \). For this vertex all adjacent integer vertices can be written as:

\[ \alpha_k i_k j_k \beta_k, \quad \text{where } \alpha_k \text{ is any ordering from the set } \{j_1, ..., j_m\} \{j_k\}, \]

\[ \beta_k \text{ is any ordering from the set } \{i_1, ..., i_m\} \{i_k\}, \]

\[ \alpha_{kp} i_k j_k i_p j_p \beta_{kp}, \quad \text{where } \alpha_{kp} \text{ is any ordering from the set } \{j_1, ..., j_m\} \{j_k, j_p\}, \]

\[ \beta_{kp} \text{ is any ordering from the set } \{i_1, ..., i_m\} \{i_k, i_p\}, \]

\[ k \neq p, k, p = 1, ..., m. \]

All adjacent integer vertices, which number is equal to 
\[ m \left( (m - 1)! \right)^2 + \frac{m(m - 1)}{2} [(m - 2)!]^2 \]
lie in one hyperplane

\[ f(x) = 2 \sum_{l=1}^{m} x_{ij} - \sum_{l=1}^{m} \sum_{q=1}^{m} x_{qij} = 1. \]

This hyperplane for \( f(x) \leq 1 \) is a facet of the polytope \( P_n \).

Our aim is to determine exact facet cuts for any non-integer vertex of \( B_n \) (and not only for them) in an analogous fashion.

Figures 2 and 3 show non-integer vertices of the polytope \( B_n \):

Noninteger vertex at Figure 2 has an oriented chain 7583 of the length 3, and non-integer vertex at Figure 3 has an oriented 614 chain of the length
2. The oriented chain 758 at Figure 2 is independent, that is if we exchange the chain 758 with any other chain the rest of the graph does not change, while the chains 81 at Figure 2 and 614 at Figure 3 are dependent. We define corresponding dependent and independent oriented chains.

The following Theorems take place.

**Theorem 1** Let $x^0$ be a noninteger vertex in $B_n$ and assume that in graph interpretation there is a graph vertex $i$ which is the begin or the end of all adjacent arcs. Assume that the vertex $i$ can be repeated arbitrarily many times such that each of the new vertices has the same location with the other part of the graph as the vertex $i$. Then the new noninteger vertex, corresponding to the new graph, is a noninteger vertex of $B_n$, and in the new graph the vertices $i$ and new inserted vertices may be put in any order.

**Theorem 2** Let $x^0$ be a noninteger vertex in $B_n$. Then there does not exist corresponding dependent oriented chains of the length 4.

The polytope $B_n$ has noninteger vertices whose fractional components are equal to $\frac{l}{r}$, $r \geq 2$, $l < r$, as well.

For $r = 2$ after matrix transformation we get in all cases the following non unimodular minimal standard matrix:

$$
\begin{pmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{pmatrix}.
$$

For $r = 3$ after matrix transformation we get in all cases the known combination of two minimal standard matrices:

$$
\begin{pmatrix}
1 & -1 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
$$

For any $r$ after matrix transformation we get known combination of $r - 1$ minimal standard matrices.
Theorem 3 Let \( x^0 \) be a noninteger vertex of \( B_n \), which has fractional components \( \frac{l_1}{r_1} \), \( l_1 < r_1, r_1 \geq 3 \). Then we pass to an adjacent noninteger vertex with fractional components \( \frac{l_2}{r_2} \), \( r_2 < r_1, l_2 < r_2 \), by changing an equality in a basis (thus changing one or more minimal standard matrices).

Let \( I_1, I_2, \ldots, I_s \subset \{1, \ldots, n\} \), and assume that each set \( I_p, p = 1, \ldots, s \), corresponds to noninteger components of a vertex. Then for each \( I_p, p = 1, \ldots, s \), we can construct a facet cut. If \( s = 1 \) a noninteger vertex is called simple. A noninteger vertex is called complex if \( s \geq 2 \).

Thus we have given a general description of the polytope \( B_n \).

Theorem 4 Let \( x^0 \) be a simple noninteger vertex of the polytope \( B_n \) with fractional components \( \frac{1}{2} \), assume further that there does not exist dependent oriented chains with the length 3. Then all adjacent integer vertices lie in one hyperplane, this hyperplane is a facet of the polytope \( P_n \), and it can be constructed by a polynomial algorithm.

Now we describe the principle for constructing facets.

Consider a noninteger vertex \( x^0 \). It can be defined as the solution of the following system of the linear basic equalities

\[
\begin{align*}
  x_{i,l} & = 0, \quad l = 1, \ldots, q; \quad (1) \\
  x_{i,l} + x_{j,k,l} - x_{i,k,l} & = 0, \quad l = q + 1, \ldots, \frac{n^2 - n}{2}.
\end{align*}
\]

We introduce artificial variables \( x_{j,n} = 0, x_{i,n} = 0 \), into the first \( q \) equalities of the system (1):

\[
x_{i,j,l} + x_{j,n+1} - x_{i,n+1} = 0, \quad l = 1, \ldots, q.
\]

With the help of the notation

\[
x_{i,j,l} + x_{j,k,l} - x_{i,k,l} := x(i_l, j_l, k_l), \quad l = 1, \ldots, \frac{n^2 - n}{2},
\]

we rewrite the system (1) as follows:

\[
x(i_l, j_l, k_l) = 0, \quad l = 1, \ldots, \frac{n^2 - n}{2}.
\]
We can determine \( \frac{n^2 - n}{2} \) linear independent adjacent integer vertices

\[
x^q(i_s, j_s, k_s) = \delta^q(i_s, j_s, k_s), \quad s, q = 1, \ldots, \frac{n^2 - n}{2},
\]

where \( \delta^q \) is either 1 or 0. We can prove that all adjacent integer vertices lie in the hyperplane:

\[
f(x) = \begin{vmatrix}
  x(i_1, j_1, k_1) & \cdots & x(i_m, j_m, k_m) & 1 \\
  \delta^1(i_1, j_1, k_1) & \cdots & \delta^1(i_m, j_m, k_m) & 1 \\
  \vdots & & \vdots & \vdots \\
  \delta^m(i_1, j_1, k_1) & \cdots & \delta^m(i_m, j_m, k_m) & 1
\end{vmatrix} = 0
\]

where \( m = \frac{n^2 - n}{2} \).

**Theorem 5** Let \( x^0 \) be a simple noninteger vertex of the polytope \( B_n \) with fractional components \( \frac{1}{2} \), assume further that there exist \( \tau \) dependent oriented chains with the length 3. Then all adjacent integer vertices lie in \( 2^\tau \) hyperplanes, each of them is a facet of the polytope \( P_n \), and can be constructed by a polynomial algorithm.

**Theorem 6** Let \( x^0 \) be a simple noninteger vertex of the polytope \( B_n \) with fractional components \( \frac{1}{2}, r \geq 3, l < r \). Then we can construct all minimal standard matrices and corresponding noninteger vertices with fractional components \( \frac{1}{2} \). For every such vertex we can construct facet cuts.

**Theorem 7** Let \( x^0 \) be a complex noninteger vertex of the polytope \( B_n \), and \( I_1, I_2, \ldots, I_s \) correspond to noninteger values. Then we can construct facet cuts for each set \( I_p, p = 1, \ldots, s \).

Assume we have generated facet cuts. Solving the problem again we get the noninteger vertex \( x^1 \) of the polytope

\[
0 \leq x_{ij} \leq 1, \\
x_{ij} + x_{ji} = 1, \\
0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, \quad i \neq j, \quad i \neq k, \quad j \neq k, \quad i, j, k = 1, \ldots, n, \\
f^1_s \leq f_s(x) \leq f^2_s, \quad s = 1, \ldots, q.
\]
Without loss of generality we may assume that the noninteger vertex $x^1$ satisfies the following linear system:

\[ x(i_s, j_s, k_s) = 1, \quad i = 1, \ldots, p, \]
\[ f_s(x) = f_s^2, \quad s = 1, \ldots, q. \]

Now we find all adjacent integer vertices. If all of them lie in one hyperplane and this hyperplane is a facet of $P_n$ then we generate this facet and re-solve the problem with a new facet. In case we cannot determine the facet we solve the auxiliary problem:

\[
\max \left( \sum_{s=1}^{p} x(i_s, j_s, k_s) + \sum_{s=1}^{q} \frac{f_s(x) - f_s^1}{f_s^2 - f_s^1} \right),
\]
\[
0 \leq x_{ij} \leq 1,
\]
\[
x_{ij} + x_{ji} = 1,
\]
\[
0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, \quad i \neq j, \quad i \neq k, \quad j \neq k, \quad i, j, k = 1, \ldots, n.
\]

With the solution of this problem we can construct the facet of the polytope $P_n$. In the case of theorems 5 and 6 we can construct the necessary facets by means of a polynomial algorithm.

References

[1] Bolotashvili G., Kovalev M., Girlich E. New Facets of the linear ordering Polytope. SIAM J. Discrete Mathematics 12(3):326-336, 1999.