Quantification of intrinsic quality of a principal dimension in correspondence analysis and taxicab correspondence analysis

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Abstract

Collins (2002, 2011) raised a number of issues with regards to correspondence analysis (CA), such as: qualitative information in a CA map versus quantitative information in the relevant contingency table; the interpretation of a CA map is difficult and its relation with the % of inertia (variance) explained. We tackle these issues by considering CA and taxicab CA (TCA) as a stepwise Hotelling/Tucker decomposition of the cross-covariance matrix of the row and column categories into four quadrants. The contents of this essay are: First, we review the notion of quality/quantity in multidimensional data analysis as discussed by Benzécri, who based his reflections on Aristotle. Second, we show the importance of unravelling the interrelated concepts of dependence/heterogeneity structure in a contingency table; and to picture them two maps are needed. Third, we distinguish between intrinsic and extrinsic quality of a principal dimension; the intrinsic quality is based on the signs of the residuals in the four quadrants, hence to the interpretability. Furthermore, we provide quantifications of the intrinsic quality and use them to uncover structure in particular in sparse contingency tables. Finally, we emphasize the importance of looking at the residual cross-covariance values at each iteration.

Key words: dependence/heterogeneity; correspondence analysis; contribution map; residual cross-covariance; intrinsic quality.

AMS 2010 subject classifications: 62H25, 62H30
1 Introduction

In this essay we comment on Collins’ (2002, 2011) statement "correspondence analysis makes you blind". Collins stated it twice as a reply to Whitlark and Smith (2001) and Bock (2011a), who analyzed two different brand image count data sets by correspondence analysis (CA). Both Whitlark and Smith (2002) and Bock (2011b) in their reply to Collins (2002, 2011) accepted without hesitation the insightful and original observations described by Collins. Beh and Lombardo (2014, p.131-132) provide a cursory report on these dialogues.

Table 1: WS brand-attribute count table.

| Attribute | Company | innovative | leader | solution | rapport | efficient | relevant | essential | trusted |
|-----------|---------|------------|--------|----------|---------|-----------|----------|-----------|---------|
| Oracle    | 155     | 157        | 109    | 133      | 151     | 96        | 35       | 170       |
| Nokia     | 375     | 350        | 274    | 318      | 351     | 284       | 91       | 408       |
| Fedex     | 476     | 675        | 550    | 669      | 748     | 627       | 307      | 754       |
| A         | 86      | 66         | 105    | 110      | 117     | 76        | 30       | 122       |
| B         | 30      | 21         | 25     | 37       | 40      | 20        | 9        | 43        |
| C         | 18      | 12         | 11     | 16       | 17      | 12        | 2        | 18        |
| D         | 25      | 23         | 33     | 36       | 34      | 28        | 12       | 35        |
| E         | 21      | 20         | 21     | 26       | 27      | 18        | 9        | 36        |
| F         | 190     | 307        | 305    | 332      | 355     | 309       | 131      | 392       |
| G         | 18      | 16         | 16     | 25       | 21      | 18        | 10       | 29        |
| H         | 408     | 549        | 467    | 551      | 613     | 523       | 239      | 624       |
| I         | 143     | 225        | 194    | 191      | 206     | 184       | 121      | 248       |

1.1 WS brand-attribute count data

Table 1, from Whitlark and Smith (2001), shows a brand-attribute count data. Whitlark and Smith (2001) analyzed it by CA resulting in a map, Figure 1, very similar to the Taxicab CA (TCA) map in Figure 2; they interpreted Figure 1 using the adjusted chi-square residuals by arguing that
it corrects "inaccuracies introduced by the dimensionality problem". TCA is a $l_1$ variant of CA, see Choulakian (2006). The interpretation of Figures 1 and 2 does not correspond to the simple seriated structure uncovered by Collins (2002) given in Table 2, where the rows and columns of Table 1 are permuted according to the row and column averages in descending order. Collins interpreted the seriated structure in Table 2 as:

"a relief map of a plain sloping steeply down from north ($FedEx$) to south ($Brand C$) and less steeply from west ($Trusted$) to east ($Essential$). On the plain would be a molehill ($Nokia$ is more innovative). But isn’t the table easy enough?"

Note that this simple structure uncovered by Collins is not found in Figures 1 and 2. Why? Lemma 1 provides a partial answer: The brands and the attributes are not independent, because of the existence of multiple ‘molehill’s, the most important molehill being ($Nokia$ is more innovative) as mentioned by Collins. The molehills (in bold in Table 2) represent positive associations, and can be analyzed and visualized in a complementary form in Figures (1 and 4) or in Figures (2 and 3). Figures 3 and 4 are TCA and CA contribution plots, which help us to interpret Figures 1 and 2. This is a summary of the paper.
1.2 Issues raised by Collins

Collins (2002) summarized his arguments against CA in Whitlark and Smith (2001) as:

"The popularity of CA and other techniques seems to arise from two feelings: The data are too complex to be handled by the human brain, and pictures communicate more than tables. In fact, the analysis of brand "image" data like that given by the authors is easy, and quantitative patterns are best communicated through simple tables supported by words. A picture like the correspondence "map" shown by W&S may say something qualitative about patterns in the data, but it says nothing quantitative or testable. Not even, for example, that the two dimensions of the map account for nearly 90% of the variance of the data, as reported in the text."

The highlighted parts are ours. There are two main points raised by Collins.

First, qualitative patterns in CA map Figure 1 do not reflect quantitative patterns in Table 2. Friendly and Kwan (2011) divided statisticians (data analysts) into two categories: graph-people and table-people. Clearly Collins belongs to the group of table-people. In this paper, we will show that both, tables and maps, are needed for global and local analysis. Additionally, in section 2 we will discuss the important notions of quality and quantity in data analysis, based on Benzéli’s reflections.

Second, the interpretation of a CA map is difficult and its relation with
the % of inertia (variance) explained. Here, we have to discuss two interrelated issues in CA. a) It concerns the interpretation of CA maps, which as we mentioned, was raised by Whitlark and Smith (2001); then clearly stated by Bock (2011b, pp. 587–588), and further discussed by Bock (2017) in a R-Blog titled “How to Interpret Correspondence Analysis Plots (It Probably Isn’t the Way You Think)” in the R-blog are discussed nine complex issues for the interpretation of CA maps. Beh and Lombardo (2014, p.132) summarize it as: in CA “the inability of the principal coordinate to provide a meaningful interpretation of the distance between a row and column point in these plots”. Greenacre (2013) also proposed contribution biplots to tackle this issue. b) In this paper, we will distinguish between the intrinsic quality and the extrinsic quality of a principal dimension, and introduce indices that quantify the intrinsic quality of a principal axis. In CA maps “variance accounted for” reflects the extrinsic quality. The intrinsic quality of a principal dimension examines the four quadrants of the residual cross-covariance matrix. We tackle these issues by considering CA and TCA as a stepwise Hotelling/Tucker decomposition of the residual cross-covariance matrix of the row and column categories into four quadrants.

1.3 Hotelling and Tucker decompositions

In mathematics, a data set $X = (x_{ij})$ for $i = 1,...,I$ and $j = 1,...,J$ can be interpreted as three kinds of mapping, see Benzécri (1973, p.56) and Choulakian (2016a). First, as a linear mapping: $X: \mathbb{R}^J \to \mathbb{R}^I$; second, as a linear mapping: $X^T: \mathbb{R}^I \to \mathbb{R}^J$; third, as a bilinear mapping $X: (\mathbb{R}^I, \mathbb{R}^J) \to \mathbb{R}$. Hotelling’s (1933) principal components analysis (PCA) is developed within the first two settings, while Hotelling’s (1936) canonical correlation analysis is developed within the third setting, where $X$ represents a cross-covariance matrix. Benzécri (1973) emphasized the development of CA within the first two settings as a weighted PCA method. In this paper, we shall emphasize the third setting.

It is well known that CA is a particular kind of Hotelling’s canonical correlation analysis, see for instance Goodman (1991), where the two sets of variables are the indicator sets of the categories of the two nominal variables. Another method, similar in perspective to canonical correlation analysis is Tucker’s (1958) interbattery analysis, which maximizes the covariance measure between the linear combination of the two indicator sets of quantitative variables. When CA did not produce interpretable maps of contingency
tables, Tenenhaus and Augendre (1996) proposed the Tucker interbattery analysis as an alternative to CA.

The parameters in Hotelling’s canonical correlation and Tucker’s covariance analyses are generally estimated by singular value decomposition (SVD). When we estimate the parameters by TaxicabSVD (TSVD) introduced by Choulakian (2006) in place of SVD, surprisingly we notice that these two analyses complement each other because they are linearly related, see equation (17). For further details, see Choulakian, Simonetti and Gia (2014).

Figure 3, named TCov map, displays taxicab interbattery analysis map of WS data. Figures 2 and 3 (TCA and TCov maps) are different (similarly Figures 1 and 4 (CA and CA contribution maps) are different): thus they provide different information to us, sometimes confusing (for instance observe the positions of the brands B, E and F in Figures 2 and 3). One of the major novelties of this paper is that, we interpret Figure 3, the taxicab interbattery analysis TCov map, as TCA contribution map, and consequently we provide a new perspective on the interpretation of the associated TCA map Figure 2 via Lemma 6. For the interpretation of the row and column labels on the TCov map, we shall use a quantification of the intrinsic quality of a principal dimension, named quality of signs of residuals (QSR) index; which will be complimented by a look at the seriated residual covariance matrix. Then we extend the development of these ideas to CA; where we also discuss sparse contingency tables having the quasi-two blocks diagonal structure, which, according to Benzécri (1973, p.188-190), is quite common. Greenacre (2013) introduced and discussed CA contribution biplots, but did not relate them to CA maps; Lemma 6 accomplishes this task.

1.4 Organisation

This paper is organized as follows: Section 2 sketches Benzécri’s reflections on quality and quantity in data analysis. Section 3 presents an overview of taxicab singular value decomposition (TSVD); section 4 presents preliminaries. In section 5 we develop the main subject matter, the quantification of the intrinsic quality of a principal dimension in CA and TCA. Section 6 presents applications. Finally we conclude in section 7.
2 Quantity and Quality

Benzécri (1982, 1988) has two papers on quality and quantity; in the second he discussed the relationship between quantity and quality historically, starting with Aristotle and finishing it with his description within the philosophy of data analysis, aka CA framework. Here we quote from Benzécri (1988, section 1.7):

"Pour l’analyse des données, nous retenons d’abord, suivant Aristote, que "le caractère propre de la quantité qu’on peut lui attribuer l’égal et l’inégal", tandis que, "semblable ou dissemblable se dit uniquement des qualités". De ce point de vue, une description multidimensionnelle est toujours qualitative même si elle comporte que des variables numériques précises, parceque la multiplicité des descriptions possibles est telle qu’on rencontrera jamais d’égales, mais seulement de semblables”.

The following two definitions and the corollary provide a succinct summary of the quote.

**Definition 1 (Aristotle on quantity)**: X is a quantitative variable if, given two realizations \(x\) and \(y\) of X, then either \(x = y\) or \(x \neq y\).

**Definition 2 (Aristotle on quality)**: X is a qualitative variable if, given two realizations \(x\) and \(y\) of X, then either \(x\) is similar to \(y\) or \(x\) is dissimilar to \(y\).

**Corollary 3 (Benzécri)**: Any multidimensional description is always qualitative even though its components are precisely numerical.

Similar ideas also are expressed in a forward essay by Benzécri in Murtagh (2005).

Benzécri’s schematic conceptual formulation of data analysis is the following directed diagram

\[
\text{Quality(data)} \rightarrow \text{Quantity(factors)} \rightarrow \text{Quality(clusters)}. 
\]

The first step: Quality(data) → Quantity(factors) is done by dimension reduction. The nature of each factor (latent variable) is quantitative and there are almost always more than one factor. Even though each latent variable is quantitative, but its interpretation is qualitative: According to Benzécri (1988, section 1.3, in comments on Descartes): "Toute qualité n’est que l’expression d’un rapport de quantités"; that is, quality is the expression
of a ratio of quantities. For interpretation of a principal dimension, we apply Aristotles/Benzécri principle. Aristotle in his book PHYSICS defined "principles are contraries” and cited as examples taken from his predecessors "hot and cold”, "the rare and the dense” and “plenum and void” see Aristotle (1960, p.14). In CA, Benzécri (1973, p.227) following Aristotle based the interpretation of a principal dimension on contraries (dichotomies, oppositions) and gradations, where an opposition or a gradation represent a latent variable. In another context, Choulakian (2014, 2016b) used Euclid’s principle of contradiction for interpretation of the first principal dimension for the analysis of rank data.

The second step: Quantity(factors) \(\rightarrow\) Quality(clusters) is done by usual methods such as k-means.

Murtagh (2005, section 1.1), described Benzécri’s paradigm “a tale of three metrics”; which clearly characterizes the diagram where : the chi-squared and the Euclidean metrics are for the first step, and the ultrametrics for the second step. A similar description to the above diagram is also stated by De Leeuw (2005). This fact also is reflected in the first printed work, Benzécri (1973), titled DATA ANALYSIS; which is composed of two volumes: The first volume’s subtitle is La Taxonomie; the second volume’s subtitle is Analyse des Correspondances.

3 An overview of taxicab singular value decomposition

Consider a matrix \(X\) of size \(I \times J\) and \(rank(X) = k\). Taxicab singular value decomposition (TSVD) of \(X\) is a decomposition similar to SVD of \(X\); see Choulakian (2006, 2016a).

In TSVD the calculation of the dispersion measures (\(\delta_\alpha\)), principal axes (\(u_\alpha, v_\alpha\)) and principal scores (\(a_\alpha, b_\alpha\)) for \(\alpha = 1, ..., k\) is done in an stepwise manner. We put \(X_1 = X = (x_{ij})\) and \(X_\alpha\) be the residual matrix at the \(\alpha\)-th iteration.

The variational definitions of the TSVD at the \(\alpha\)-th iteration are
The $\alpha$-th principal axes are

$$\alpha = \max_{u \in \mathbb{R}^J} ||X_\alpha u||_1 = \max_{v \in \mathbb{R}^I} ||X'_\alpha v||_1 = \max_{u \in \mathbb{R}^J, v \in \mathbb{R}^I} \frac{v'X_\alpha u}{||u||_\infty ||v||_\infty},$$

subject to $u \in \{-1, +1\}^J$, $v \in \{-1, +1\}^I$.

The $\alpha$-th principal vectors are

$$a_\alpha = X_\alpha u_\alpha \quad \text{and} \quad b_\alpha = X'_\alpha v_\alpha.$$ (2)

Furthermore the following relations are also useful

$$u_\alpha = sgn(b_\alpha) \quad \text{and} \quad v_\alpha = sgn(a_\alpha),$$ (3)

where $sgn(.)$ is the coordinatewise sign function, $sgn(x) = 1$ if $x > 0$, and $sgn(x) = -1$ if $x \leq 0$.

The $\alpha$-th taxicab dispersion measure $\delta_\alpha$ can be represented in many different ways

$$\delta_\alpha = v'_\alpha X_\alpha u_\alpha = ||X_\alpha u_\alpha||_1 = ||a_\alpha||_1 = a'_\alpha v_\alpha,$$

$$= ||X'_\alpha v_\alpha||_1 = ||b_\alpha||_1 = b'_\alpha u_\alpha.$$ (4)

The $(\alpha + 1)$-th residual correspondence matrix is

$$X_{\alpha+1} = X_\alpha - a_\alpha b'_\alpha / \delta_\alpha.$$ (5)

An interpretation of the term $a_\alpha b'_\alpha / \delta_\alpha$ in (5) is that, it represents the best rank-1 approximation of the residual correspondence matrix $X_\alpha$, in the sense of taxicab norm.

Thus TSVD of $X$ corresponds to
\[ x_{ij} = \sum_{\alpha=1}^{k} a_{\alpha}(i)b_{\alpha}(j)/\delta_{\alpha}, \]  
\[ (6) \]
a decomposition similar to SVD, but where the vectors \((a_{\alpha}, b_{\alpha})\) for \(\alpha = 1, ..., k\) are conjugate, a weaker property than orthogonality. That is
\[ a_{\alpha}'sgn(a_{\beta}) = b_{\alpha}'sgn(b_{\beta}) = 0 \text{ for } \alpha > \beta. \]

In TSVD, the calculation of the principal component weights, \(u_{\alpha}\) and \(v_{\alpha}\), and the principal scores, \(a_{\alpha}\) and \(b_{\alpha}\), can be accomplished by two algorithms. The first one is based on complete enumeration based on equation (1). The second one is based on iterating the transition formulae (2,3). This is an ascent algorithm; that is, it increases the value of the objective function at each iteration, see Choulakian (2006, 2016a). The iterative algorithm could converge to a local maximum; so it should be restarted from several initial configurations. The rows or the columns of the data can be used as starting values.

### 4 Preliminaries

Let \(P = N/n = (p_{ij})\) of size \(I \times J\) be the associated correspondence matrix of a contingency table \(N\), where \(n = \sum_{i=1}^{I} \sum_{j=1}^{J} N(i,j)\). We define as usual \(p_{is} = \sum_{j=1}^{J} p_{ij}\), \(p_{sj} = \sum_{i=1}^{I} p_{ij}\), the vector \(r = (p_{is}) \in \mathbb{R}^{I}\), the vector \(c = (p_{sj}) \in \mathbb{R}^{J}\), and \(D_I = Diag(r)\) the diagonal matrix having diagonal elements \(p_{is}\), and similarly \(D_J = Diag(c)\). We suppose that \(D_I\) and \(D_J\) are positive definite metric matrices of size \(I \times I\) and \(J \times J\), respectively; this means that the diagonal elements of \(D_I\) and \(D_J\) are strictly positive. Let

\[ P^{(1)} = (P - rc^\top) \]  
\[ = \text{Cov}(P) \]  
\[ (7) \]
or elementwise

\[ p_{ij}^{(1)} = p_{ij} - p_{is}p_{sj} \]  
\[ (8) \]
be the residual matrix with respect to the independence model. \(p_{ij}^{(1)}\) is the cross-covariance between the categories of the \(i\)th nominal row variable and the \(j\)th nominal column variable.
The independence assumption $p_{ij}^{(1)} = 0$ can also be interpreted in another way as

$$p_{ij} / p_{is}p_{sj} - 1 = 0,$$

which can be reexpressed as

$$\frac{1}{p_{is}p_{sj}}(p_{ij} - p_{is}) = 0$$

$$= \frac{1}{p_{is}p_{sj}}(p_{ij} - p_{sj});$$

this is the row and column homogeneity models. Benzécri (1973, p.31) named the vector $(p_{ij} / p_{is}p_{sj} \text{ for } i = 1, ..., I \text{ and } j \text{ fixed})$ the profile of the $j$th column; and the element $p_{ij} / p_{is}p_{sj}$ the density function of the probability measure $(p_{ij})$ with respect to the product measure $p_{is}p_{sj}$. The element $p_{ij} / p_{is}p_{sj}$ is named Pearson ratio in Goodman (1996) and Beh and Lombardo (2014, p.123).

4.1 Estimation of the parameters by SVD

Suppose the independence assumption $\text{Cov}(P) = P^{(1)} = 0$ is not true, then each of the two equivalent model formulations (8,10) can be generalized to explain the nonindependence by adding bilinear terms, where $k = \text{rank}(P^{(1)})$.

a) Cov (cross-covariance) decomposition:

$$p_{ij} - p_{is}p_{sj} = \sum_{\alpha=1}^{k} a_\alpha(i)b_\alpha(j)/\sigma_\alpha.$$  

(11)

This is an interbattery analysis proposed by Tucker (1958). Tenenhaus and Augendre (1996) estimated the parameters in (11) by singular value decomposition (SVD) of the matrix $\text{Cov}(P)$. The parameters in (11) satisfy the following equations

$$\sigma_\alpha^2 = \sum_{i=1}^{I} |a_\alpha(i)|^2 = \sum_{j=1}^{J} |b_\alpha(j)|^2 \text{ for } \alpha = 1, ..., k;$$

(12)
\[ 0 = \sum_{i=1}^{I} a_{\alpha}(i) a_{\beta}(i) = \sum_{j=1}^{J} b_{\alpha}(j) b_{\beta}(j) \text{ for } \alpha \neq \beta \quad (13) \]

\[ = \sum_{i=1}^{I} a_{\alpha}(i) = \sum_{j=1}^{J} b_{\alpha}(j) \text{ for } \alpha = 1, \ldots, k. \]

b) CA (correspondence analysis) decomposition

\[ \frac{p_{ij}}{p_i \ast p_j} - 1 = \sum_{a=1}^{k} \frac{f_{\alpha}(i)g_{\alpha}(j)}{\sigma_{\alpha}}. \quad (14) \]

This decomposition has many interpretations. Essentially, for data analysis purposes Benzécri (1973) interpreted it as weighted principal components analysis of row and column profiles. Another useful interpretation, comparable to Tucker interbattery analysis, is Hotelling’s correlation analysis, see Lancaster (1958) and Goodman (1991). The parameters in (14) satisfy the following equations

\[ \sigma_{\alpha}^2 = \sum_{i=1}^{I} |f_{\alpha}(i)|^2 p_{i*} = \sum_{j=1}^{J} |g_{\alpha}(j)|^2 p_{*j} \text{ for } \alpha = 1, \ldots, k; \quad (15) \]

\[ 0 = \sum_{i=1}^{I} f_{\alpha}(i) f_{\beta}(i)p_{i*} = \sum_{j=1}^{J} g_{\alpha}(j) g_{\beta}(j)p_{*j} \text{ for } \alpha \neq \beta \quad (16) \]

\[ = \sum_{i=1}^{I} f_{\alpha}(i)p_{i*} = \sum_{j=1}^{J} g_{\alpha}(j)p_{*j} \text{ for } \alpha = 1, \ldots, k. \]

The above two decompositions given in (11) and (14) are cross-covariance based. There are also association (log ratio) based decompositions see Goodman (1991, 1996) or Greenacre and Lewi (2009).

4.2 Estimation of the parameters by TSVD

First, we estimate the parameters \((a_{\alpha}(i), b_{\alpha}(j), \delta_{\alpha})\) in (11) by TSVD; then the parameters in (14) will be linearly related by

\[ a_{\alpha}(i) = p_{i*}f_{\alpha}(i) \quad , \quad b_{\alpha}(j) = p_{*j}g_{\alpha}(j) \quad \text{and} \quad \delta_{\alpha} = \sigma_{\alpha}. \quad (17) \]
The parameters $a_{\alpha}(i)$ and $b_{\alpha}(j)$ in (11) are the principal coordinates of the TCov decomposition and they satisfy

$$\delta_{\alpha} = \sum_{i=1}^{I} |a_{\alpha}(i)| = \sum_{j=1}^{J} |b_{\alpha}(j)| \quad \text{for} \quad \alpha = 1, \ldots, k; \quad (18)$$

$$0 = \sum_{i=1}^{I} a_{\alpha}(i) sgn(a_{\beta}(i)) = \sum_{j=1}^{J} b_{\alpha}(j) sgn(b_{\beta}(j)) \quad \text{for} \quad \alpha > \beta$$

$$= \sum_{i=1}^{I} a_{\alpha}(i) = \sum_{j=1}^{J} b_{\alpha}(j) \quad \text{for} \quad \alpha = 1, \ldots, k.$$

Similarly, the parameters $f_{\alpha}(i)$ and $g_{\alpha}(j)$ in (14) are the principal coordinates of the TCA decomposition and they satisfy

$$\delta_{\alpha} = \sum_{i=1}^{I} |f_{\alpha}(i)|p_{i*} = \sum_{j=1}^{J} |g_{\alpha}(j)|p_{*j} \quad \text{for} \quad \alpha = 1, \ldots, k; \quad (19)$$

$$0 = \sum_{i=1}^{I} f_{\alpha}(i)f_{\beta}(i)p_{i*} = \sum_{j=1}^{J} g_{\alpha}(j)g_{\beta}(j)p_{*j} \quad \text{for} \quad \alpha > \beta$$

$$= \sum_{i=1}^{I} f_{\alpha}(i)p_{i*} = \sum_{j=1}^{J} g_{\alpha}(j)p_{*j} \quad \text{for} \quad \alpha = 1, \ldots, k.$$

Let $P^{(m)} = (p_{ij}^{(m)})$ be the $m$th residual correspondence matrix, where

$$p_{ij}^{(m+1)} = p_{ij} - p_{i*}p_{*j} - \sum_{\alpha=1}^{m} a_{\alpha}(i)b_{\alpha}(j)/\delta_{\alpha} \quad \text{for} \quad m = 1, \ldots, k - 1. \quad (20)$$

Similarly, let $D^{(m)} = (d_{ij}^{(m)})$ be the $m$th residual density matrix, where

$$d_{ij}^{(m+1)} = \frac{p_{ij}}{p_{i*}p_{*j}} - 1 - \sum_{\alpha=1}^{m} f_{\alpha}(i)g_{\alpha}(j)/\delta_{\alpha} \quad \text{for} \quad m = 1, \ldots, k - 1. \quad (21)$$

Let $S \cup \overline{S} = I$ be an optimal binary partition of $I$, and $T \cup \overline{T} = J$ be an optimal binary partition of $J$, such that $S = \{ i : a_{\alpha}(i) \geq 0 \}$ and $T = \{ j : b_{\alpha}(j) \geq 0 \}$. 13
Besides (18), the taxicab dispersion $\delta_\alpha$ will additionally be related to the TCov principal coordinates $a_\alpha(i)$ and $b_\alpha(j)$ in (11) by the following useful equations:

\[
\frac{\delta_\alpha}{2} = \sum_{i \in S} a_\alpha(i) = - \sum_{i \in S} a_\alpha(i) \tag{22}
\]

\[
= \sum_{j \in T} b_\alpha(j) = - \sum_{j \in T} b_\alpha(j).
\]

\[
\frac{\delta_\alpha}{4} = \sum_{(i,j) \in S \times T} p^{(\alpha)}_{ij} = \sum_{(i,j) \in S \times T} p^{(\alpha)}_{ij} \tag{23}
\]

\[
= - \sum_{(i,j) \in S \times T} p^{(\alpha)}_{ij} = - \sum_{(i,j) \in S \times T} p^{(\alpha)}_{ij}.
\]

Equations (22, 23) follow from the fact that $P^{(\alpha)}$ for $\alpha = 1, \ldots, k$ is a double-centered matrix, see Choulakian and Abou-Samra (2020). The quantification of the intrinsic quality of a principal dimension is based on (23).

4.3 An observation

The TCov principal coordinates, $a_\alpha(i)$ and $b_\alpha(j)$, are uniformly weighted, see equation (18); meanwhile TCA principal coordinates, $f_\alpha(i)$ and $g_\alpha(j)$, are marginally weighted, see equation (19). What is the consequence to this? The answer to this question is: Benzécri’s principle of distributional equivalence, which states that CA (and TCA) results are not changed if two proportional columns or rows are merged into one. This has the practical consequence that the effective size of sparse and large data sets can be smaller than the observed size; for further details concerning sparse contingency tables see Choulakian (2017).

5 Main developments

Let $L$ be a permutation matrix such that the coordinates of $r_L = Lr$ are in decreasing order, $r_L(i) \geq r_L(i + 1)$ for $i = 1, \ldots, I - 1$. Similarly, $M$ be a permutation matrix such that the coordinates of $c_M = Mc$ are in decreasing order, $c_M(j) \geq c_M(j + 1)$ for $j = 1, \ldots, J - 1$. 

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We consider the matrix
\[ LP^{(1)} M^\top = L(P - rc^\top)M^\top. \] (24)

We have the following easily proved result

**Lemma 4**: Let \( S = LPM^\top \). A necessary condition for the independence model, \( LP^{(1)} M^\top = 0 \) or \( p_{ij} - p_{i*}p_{*j} = 0 \), is that
\[
S(i, j) \geq S(i, j + 1) \quad \text{for} \quad j = 1, ..., J - 1 \quad \text{and} \quad i \text{ fixed} \quad (25)
\]
and
\[
S(i, j) \geq S(i + 1, j) \quad \text{for} \quad i = 1, ..., I - 1 \quad \text{and} \quad j \text{ fixed}. \quad (26)
\]

**Remark**: Relations (25 and 26) characterize Robinson matrices used for seriation of artifacts or sites in archeology. That is why we named \( S \), see Table 2, seriated contingency table following its seriated row \( r_L \) and column \( c_M \) marginals.

**Lemma 5**: TSVD of \( P \) is equivalent to TCov(\( P \)) = TSVD of \( P^{(1)} \).

Lemma 6 states that the \( \alpha \)th row TCA (or CA) principal factor score \( f_\alpha(i) \) is the weighted covariance of the \( \alpha \)th residual density function \( D^{(\alpha)}(i,:) \) with the \( \alpha \)-th principal axis \( u_\alpha \); where \( D^{(\alpha)}(i,:) \) is the ith row of \( D^{(\alpha)} \) and \( D^{(\alpha)}(:,j) \) is the jth column of \( D^{(\alpha)} \).

**Lemma 6**: In CA and TCA
\[
f_\alpha(i) = D^{(\alpha)}(i,:D_J u_\alpha \quad (27)
\]
and
\[
g_\alpha(j) = v_\alpha^\top D_I D^{(\alpha)}(:,j) \quad (28)
\]
where \( u_\alpha \) represents the \( \alpha \)-th standardized principal axis in each method.

In CA, \( u_\alpha = \frac{f_\alpha}{\sigma_\alpha} \) and \( v_\alpha = \frac{f_\alpha}{\sigma_\alpha} \). In TCA, \( u_\alpha = sign(b_\alpha) = sign(g_\alpha) \) and \( v_\alpha = sign(a_\alpha) = sign(f_\alpha) \) for \( \alpha = 1, ..., k \), see equations (3 and 17).
Proof: Here, we provide a proof for TCA. We use the transition formula (2) for $m = 1, \ldots, k - 1$,

\[ a_{m+1}(i) = \sum_j p_{ij}^{(m+1)} u_{m+1}(j) \]

\[ = \sum_j \left[ p_{ij} - p_i^* p_j^* - \sum_{\alpha=1}^m a_\alpha(i) b_\alpha(j)/\delta_\alpha \right] u_{m+1}(j) \]

\[ = \sum_j \left[ p_{ij} - p_i^* p_j^* - \sum_{\alpha=1}^m a_\alpha(i) b_\alpha(j)/\delta_\alpha \right] \frac{p_i^* p_j^*}{p_{i^* p_j^*}} u_{m+1}(j) \]

\[ = \sum_j \left[ \frac{p_{ij}}{p_i^* p_j^*} - 1 - \sum_{\alpha=1}^m f_\alpha(i) g_\alpha(j)/\delta_\alpha \right] p_{i^* j^*} u_{m+1}(j), \text{ by (17)}, \]

\[ f_{m+1}(i) = \sum_j \left[ \frac{p_{ij}}{p_i^* p_j^*} - 1 - \sum_{\alpha=1}^m f_\alpha(i) g_\alpha(j)/\delta_\alpha \right] p_{i^* j^*} u_{m+1}(j) \]

\[ = \sum_j d_{ij}^{(m+1)} p_{i^* j^*} u_{m+1}(j), \]

which is the required result (27).

**Remark**: In CA, due to (16), equation (27), similarly (28), can further be simplified to

\[ f_{m+1}(i) = \sum_j \frac{p_{ij}}{p_i^* p_j^*} p_{i^* j^*} u_{m+1}(j) \]

\[ = \text{cov}(D(i,:), u_{m+1}), \]

a well known result in Bastin et. al. (1980, p.157) or Goodman(1991, p. 1105, eq. A.1.3).

### 5.1 Quantifying the intrinsic quality of a taxicab principal axis

Within the Euclidean framework a measure of the quality of a principal dimension is the proportion (or percentage) of the residual variance explained (or inertia in the case of CA)

\[ \%\text{(explained residual variance by dimension } \alpha) = \frac{100 \sigma^2_\alpha}{\sum_{\beta=1}^k \sigma^2_\beta}, \]
This is an extrinsic measure of quality, because it compares the dispersion of a principal axis \( \sigma_\alpha^2 \) with the residual dispersion \( \sum_{\beta=1}^{k} \sigma_\beta^2 \). In the above equation replacing the \( l_2 \) terms by the corresponding \( l_1 \) terms, we obtain the measure of intrinsic quality \( QSR_\alpha \) expressed in Definition 7.

Let \( S \cup \overline{S} = I \) be an optimal binary partition of \( I \), and similarly \( T \cup \overline{T} = J \) be an optimal binary partition of \( J \) for the \( \alpha \)th principal dimension. Thus the data set is divided into four quadrants. We define a new index showing the quality of signs of the residuals (QSR) in each quadrant of the \( \alpha \)th residual cross-covariance matrix \( P^{(\alpha)} \) for \( \alpha = 1, \ldots, k \) in (20).

**Definition 7:** For \( \alpha = 1, \ldots, k \), the measure of the quality of signs of the residuals in the quadrant \( E \times F \subseteq I \times J \) is

\[
QSR_\alpha(E, F) = \frac{\sum_{(i,j)\in E \times F} p_{ij}^{(\alpha)}}{\sum_{(i,j)\in E \times F} |p_{ij}^{(\alpha)}|}, \quad \text{and by (23)}
\]

\[
= \frac{\delta_\alpha/4}{\sum_{(i,j)\in E \times F} |p_{ij}^{(\alpha)}|} \quad \text{for } (E, F) = (S, T) \text{ and } (\overline{S}, \overline{T})
\]

\[
= -\frac{\delta_\alpha/4}{\sum_{(i,j)\in E \times F} |p_{ij}^{(\alpha)}|} \quad \text{for } (E, F) = (\overline{S}, T) \text{ and } (S, \overline{T}).
\]

Similarly, a quantification of the quality of signs of the optimal cut of dimension \( \alpha \) is

\[
QSR_\alpha = \frac{\delta_\alpha}{\sum_{(i,j)} |p_{ij}^{(\alpha)}|}.
\]

**Remark:** The computation of the elements of \( QSR_\alpha(E, F) \) are done easily in the following way. We note that the \( \alpha \)th principal axis can be written as

\[
u_\alpha = u_{\alpha+} + u_{\alpha-}\]

where \( u_{\alpha+} = (u_\alpha + 1_J)/2 \in \{0,1\}^J \) and \( u_{\alpha-} = (u_\alpha - 1_J)/2 \in \{-1,0\}^J \); similarly

\[
u_\alpha = v_{\alpha+} + v_{\alpha-}\]

where \( v_{\alpha+} = (v_\alpha + 1_I)/2 \in \{0,1\}^I \) and \( v_{\alpha-} = (v_\alpha - 1_I)/2 \in \{-1,0\}^I \), and \( 1_I \) designates a column vector of 1’s of size \( I \). So

\[
QSR_\alpha(S, T) = QSR_\alpha(v_{\alpha+}, u_{\alpha+})
\]

\[
= \frac{\delta_\alpha/4}{v_{\alpha+} \text{abs}(X_\alpha)u_{\alpha+}} > 0.
\]
\[ QSR_\alpha(S, T) = QSR_\alpha(v_\alpha-, u_\alpha-) = \frac{\delta_{\alpha}/4}{v'_\alpha \cdot \text{abs}(X_\alpha) u_\alpha-} > 0, \]
\[ QSR_\alpha(S, \overline{T}) = QSR_\alpha(v_\alpha+, u_\alpha-) = \frac{\delta_{\alpha}/4}{v'_\alpha \cdot \text{abs}(X_\alpha) u_\alpha+} < 0, \]
\[ QSR_\alpha(\overline{S}, T) = QSR_\alpha(v_\alpha-, u_\alpha+) = \frac{\delta_{\alpha}/4}{v'_\alpha \cdot \text{abs}(X_\alpha) u_\alpha-} < 0, \]

where \( \text{abs}(X_\alpha) = (|X_\alpha(i, j)|) \).

To interpret the above indices, we recall from elementary probability theory the definition of association between two events by defining an index of association \( \text{ass}(i, j) = p_{ij} - p_i^* p_j^* \) for \( i = 1, ..., I \) and \( j = 1, ..., J \).

a) When \( \text{ass}(i, j) = 0 \), then the \( i \)th category of the row variable and the \( j \)th category of the column variable are not associated (independent).

b) When \( \text{ass}(i, j) > 0 \), then the \( i \)th category of the row variable and the \( j \)th category of the column variable are attractively or positively associated; that is, the event \((i, j)\) occurs more than by chance.

c) When \( \text{ass}(i, j) < 0 \), then the \( i \)th category of the row variable and the \( j \)th category of the column variable are repulsively or negatively associated; that is, the event \((i, j)\) occurs less than by chance.

Based on these, the interpretation of the indices becomes evident: for instance, \( QSR_\alpha(S, T) > 0 \) measures the intensity of the attractive association between the subsets \( S \) and \( T \); while \( QSR_\alpha(\overline{S}, T) < 0 \) measures the intensity of the repulsive association between the subsets \( \overline{S} \) and \( T \).

Allard et al. (2020) used the QSR index to choose between two competing methods of data analysis, TCA and taxicab log-ratio analysis of contingency tables and compositional data.

**Notation:**
\[ QSR_\alpha(+) = \{QSR_\alpha(u_\alpha+, v_\alpha+), QSR_\alpha(u_\alpha-, v_\alpha-)\} \]
\[ QSR_\alpha(-) = \{QSR_\alpha(u_\alpha+, v_\alpha-), QSR_\alpha(u_\alpha-, v_\alpha+)\} \]

We have the following easily proved result
Lemma 8: a) For $\alpha = 1, \ldots, k$, $QSR_\alpha = 1$ if and only if $QSR_\alpha(S, T) = QSR_\alpha(S, T) = QSR_\alpha(S, T) = QSR_\alpha(S, T) = 1$.

b) For $\alpha = k$, $QSR_\alpha = 1$.

c) $(QSR_\alpha(S, T) + |QSR_\alpha(S, T)| + |QSR_\alpha(S, T)| + QSR_\alpha(S, T))/4 \geq QSR_\alpha$.

The proof of part c, is based on the arithmetic-harmonic means inequality which states that for four strictly positive real numbers $a, b, c$ and $d$
\[
\frac{a + b + c + d}{4} \geq \frac{4}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}};
\]
equality is attained when $a = b = c = d$.

5.2 Quantifying the intrinsic quality of a principal axis in CA

Let $D^{(m)} = (d^{(m)}_{ij})$ be the $m$th residual density matrix in CA,

\[
d^{(m+1)}_{ij} = \frac{p_{ij}}{p_i p_j} - 1 - \sum_{\alpha=1}^{m} f_\alpha(i) g_\alpha(j)/\sigma_\alpha
\]

\[
= \frac{p_{ij}}{p_i p_j} - 1 - \sum_{\alpha=1}^{m} \sigma_\alpha v_\alpha(i) u_\alpha(j) \quad \text{for } m = 1, \ldots, k - 1,
\]

where $u_\alpha = \frac{f_\alpha}{\sigma_\alpha}$ and $v_\alpha = \frac{g_\alpha}{\sigma_\alpha}$ represent the $\alpha$th standardized principal axis coordinates in CA.

Let $Q^{(m)} = (q^{(m)}_{ij})$ be the $m$th residual cross-covariance matrix in CA obtained from (29),

\[
q^{(m+1)}_{ij} = p_i p_j d^{(m+1)}_{ij}
\]

\[
= p_{ij} - p_i p_j - \sum_{\alpha=1}^{m} \sigma_\alpha p_i p_j v_\alpha(i) u_\alpha(j) \quad \text{for } m = 1, \ldots, k - 1.
\]

5.2.1 CA_QSR indices

Let $S \cup \overline{S} = I$ be an optimal binary principal axis partition of $I$, and similarly $T \cup \overline{T} = J$ be an optimal principal axis partition of $J$ by CA. Thus the residual covariance matrix is divided into four quadrants: $S = \{i : v_\alpha(i) \geq 0\}$ and $T = \{j : u_\alpha(j) \geq 0\}$. Based on the observation that both $q_{ij}^{(m+1)}$ and $p_{ij}^{(m+1)}$
are double centered, we can quantify the intrinsic quality of CA principal dimension by replacing $p_{ij}^{(m+1)}$ by $q_{ij}^{(m+1)}$ in subsection 5.1, and obtain CA_QSR measures.

**Definition 9**: For $\alpha = 1, ..., k$, the CA measure of the quality of signs of the residuals in the quadrant $E \times F \subseteq I \times J$ is

$$CA_{QSR_\alpha}(E, F) = \sum_{(i,j) \in E \times F} q^{(\alpha)}_{ij} \frac{\varpi_\alpha / 4}{\sum_{(i,j) \in E \times F} |q^{(\alpha)}_{ij}|}$$

for $(E, F) = (S, T)$ and $(S, T)$.

for $E = S$ and $\overline{S}$, and, $F = T$ and $\overline{T}$. Similarly the CA measure of the quality of signs of principal dimension $\alpha$ is

$$CA_{QSR_\alpha} = \frac{\varpi_\alpha}{\sum_{(i,j)} |q^{(\alpha)}_{ij}|},$$

where

$$\varpi_\alpha = sign(f'_{\alpha}) Q^{(\alpha)} sign(g_{\alpha})$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} sign(f_{\alpha}(i)) sign(g_{\alpha}(j)) q^{(\alpha)}_{ij}$$

$$= 4 \sum_{(i,j) \in E \times F} sign(f_{\alpha}(i)) sign(g_{\alpha}(j)) q^{(\alpha)}_{ij},$$

for $E = S$ and $\overline{S}$, and, $F = T$ and $\overline{T}$.

Note that equations (31) and (23) are similar, and they follow from the important observation that both residual cross-covariance matrices $q_{ij}^{(m)}$ and $p_{ij}^{(m)}$ for $m = 1, ..., k$ are double centered. Furthermore, the CA_QSR indices satisfy the three properties in Lemma 8.

We have the following

**Lemma 10**: a) When both CA and TCA produce the same binary partition (cut) of the set of column and the set of row variables of the cross-covariance matrix $\mathbf{P}^{(1)}$, then $CA_{QSR_1} = QSR_1$ and $CA_{QSR_1}(E, F) = QSR_1(E, F)$ for $E = S$ and $\overline{S}$, and, $F = T$ and $\overline{T}$. Thus $\delta_1 = \varpi_1$. 

20
The proof is evident based on the fact that \( p_{ij}^{(1)} = q_{ij}^{(1)} = p_{ij} - p_{i*}p_{*j} \). Note that for \( \alpha \neq 1 \), \( p_{ij}^{(\alpha)} \neq q_{ij}^{(\alpha)} \).

b) When CA and TCA produce different binary partitions of the set of column or the set of row variables of the cross-covariance matrix, then \( \varpi_1 < \delta_1 \).

The proof is evident, for TCA maximizes \( \delta_1 \).

5.2.2 Absolute and relative contributions of a quadrant in CA

A better known decomposition, similar to (31), where CA is interpreted as Hotelling’s canonical correlation analysis, is:

\[
\sigma_\alpha = v'_\alpha P u_\alpha = v'_\alpha Q^{(\alpha)} u_\alpha \quad \text{by (16)},
\]

\[
= \sum_{i=1}^{I} \sum_{j=1}^{J} v_\alpha(i)u_\alpha(j)q_{ij}^{(\alpha)}
\]

\[
= \sum_{F=T,\overline{T}} \sum_{E=S,\overline{S}} \sum_{(i,j) \in S \times T} v_\alpha(i)u_\alpha(j)q_{ij}^{(\alpha)}. \tag{32}
\]

Equation (32) shows that, the dispersion measure \( \sigma_\alpha = corr(v_\alpha, u_\alpha) \) is a weighted correlation measure, and is decomposed into four positive parts, each part representing the absolute contribution of a quadrant \( E \times F \) to \( \sigma_\alpha \). From the four positive parts in (32), we define the signed absolute (respectively relative) contribution of the quadrant \( sACQ \) (respectively \( sRCQ \)) to \( \sigma_\alpha \)

\[
sACQ_\alpha(E, F) = \sum_{(i,j) \in E \times F} \text{sign}(v_\alpha(i))\text{sign}(u_\alpha(j))v_\alpha(i)u_\alpha(j)q_{ij}^{(\alpha)},
\]

and

\[
sRCQ_\alpha(E, F) = sACQ_\alpha(E, F)/\sigma_\alpha,
\]

for \( E = S \) and \( \overline{S} \), and, \( F = T \) and \( \overline{T} \).

Similarly we define the signed residual contribution \( sRES_\alpha \) to be

\[
sRES_\alpha = \sum_{F=T,\overline{T}} \sum_{E=S,\overline{S}} sACQ_\alpha(E, F),
\]

which can be interpreted as a global index of attractive or repulsive association of a CA principal dimension.
5.2.3 Two blocks diagonal contingency tables

Here we discuss the particular case of contingency tables which have two blocks diagonal structure; that is, for binary partitions \( I = I_1 \cup I_2 \) and \( J = J_1 \cup J_2 \), \( p_{ij} = 0 \) for \( (i,j) \in I_1 \times J_2 \) and \( (i,j) \in I_2 \times J_1 \). Then a well-known result due to Benzécri (1973, p. 188-190) or Bastin et. al.(1980, pp. 174-179) is

**Theorem 10:** (Benzécri): A contingency table has two blocks diagonal structure if and only if \( \sigma_1 = 1 \); moreover, \( f_1(i) = g_1(j) = c_1 \) for \( (i,j) \in I_1 \times J_1 \) and \( f_1(i) = g_1(j) = c_2 \) for \( (i,j) \in I_2 \times J_2 \), where \( c_1 \) and \( c_2 \) are constants.

**Corollary 11:** If \( \sigma_1 = 1 \), then
a) \( CA_{QSR_1}(v_{1+}, u_{1-}) = CA_{QSR_1}(v_{1-}, u_{1+}) = -1 \);
b) \( sACQ_1(v_{1+}, u_{1-}) = sACQ_1(v_{1-}, u_{1+}) \).

**Remark:**
a) Benzécri (1973, p. 188-190) generalized the result of Theorem 10 to \( k \)-blocks diagonal contingency tables.
b) Benzécri (1973, p.189-190) observed that it is rare to have \( \sigma_1 = 1 \), but not uncommon to have \( \sigma_1^2 \geq 0.7 \); then in these cases the structure of the contingency table may be either quasi-two blocks diagonal, where few cells will be nonzero in the quadrants \( I_1 \times J_2 \) and \( I_2 \times J_1 \); or not, see Benzécri (1973, p.246) and Choulakian and de Tibeiro (2013).

Rodent species abundance data set, discussed in the next section provides an example of a sparse contingency table having quasi-two blocks diagonal structure; furthermore, it shows that \( \sigma_1 = 1 \) is a sufficient condition but not necessary to have \( minCA_{QSR_1}(-) = -1 \).

5.3 Two new unified formulas

Choulakian (2006) showed that both CA and TCA satisfy few fundamental identical formulas -such as data reconstruction formula (14) and the transition formulas (27, 28) of Lemma 6, even though mathematically they are different. This paper extends the similarity of both methods by showing that the dispersion measures also can be represented in a common form. For \( \alpha = 1, \ldots, k \)

\[
\varpi_\alpha \text{ and } \delta_\alpha = \sum_{(i,j) \in I \times J} sign(f_\alpha(i))sign(g_\alpha(j))p_{i*}p_{*j}d_{ij}^{(\alpha)},
\]
\[ \sigma_\alpha \text{ and } \delta_\alpha = \sum_{(i,j) \in I \times J} v_\alpha(i)u_\alpha(j)p_{i*}p_{*j}d_{ij}^{(\alpha)}. \]

### 5.4 Uncomparability of CA and TCA contribution maps

CA and TCA maps, Figures 1 and 2, are comparable because they are based on the same data reconstruction formula (14)

\[
\frac{p_{ij}}{p_{i*}p_{*j}} - 1 = \sum_{\alpha=1}^{k} f_\alpha(i)g_\alpha(j)/\gamma_\alpha,
\]

where \( \gamma_\alpha^2 = \sigma_\alpha^2 = \sum_{i=1}^{I} p_{i*}f_\alpha(i)^2 = \sum_{j=1}^{J} p_{*j}g_\alpha(j)^2 \) in CA and \( \gamma_\alpha = \delta_\alpha = \sum_{i=1}^{I} p_{i*}|f_\alpha(i)| = \sum_{j=1}^{J} p_{*j}|g_\alpha(j)| \) in TCA. Figures 1 and 2 are obtained by plotting the coordinates \((f_1(i), f_2(i))\) and \((g_1(j), g_2(j))\).

Figures 3 and 4 representing TCA and CA contribution maps are not comparable, because they do not represent the same object.

TCA contribution (TCov) is the factoring of the cross-covariance matrix by TSVD:

\[
p_{ij} - p_{i*}p_{*j} = \sum_{\alpha=1}^{k} a_\alpha(i)b_\alpha(j)/\delta_\alpha,
\]

where \( \delta_\alpha = \sum_{i=1}^{I} |a_\alpha(i)| = \sum_{j=1}^{J} |b_\alpha(j)| \) for \( \alpha = 1, \ldots, k \).

CA contribution is the factoring of the chi-square residuals by SVD:

\[
\frac{p_{ij}}{\sqrt{p_{i*}p_{*j}}} - 1 = \sum_{\alpha=1}^{k} a_\alpha(i)b_\alpha(j)/\sigma_\alpha,
\]

where \( \sigma_\alpha^2 = \sum_{i=1}^{I} |a_\alpha(i)|^2 = \sum_{j=1}^{J} |b_\alpha(j)|^2 \) for \( \alpha = 1, \ldots, k \); this is symmetric scaling, different from the one used by Greenacre (2013).

Figures 3 and 4 are obtained by plotting the coordinates \((a_1(i), a_2(i))\) and \((b_1(j), b_2(j))\). We observe that TCov map really represents TCA contribution plot, while CA contribution map represents only the orderings of the points according to their contributions. A really representative CA contribution plot should be based on the coordinates \(a_\alpha(i)^2 \text{sign}(a_\alpha(i))\) or \(b_\alpha(j)^2 \text{sign}(b_\alpha(j))\).
6 Applications

In this section First we revisit WS data set in detail; then consider briefly two other data sets.

6.1 WS brand-attribute count data

Table 3 displays the quality of signs of residuals ($QSR$) measures in % of WS data: We use it to choose the number of principal dimensions. Our interest focuses on $maxQSR_\alpha(\cdot)$, because maps essentially reflect attractive association between two optimal subsets of the row and column categories. The first two with values of 100% and above 89.29% are significant compared to the 3rd with value of 63.55%. QSR values, $CA_{QSR_2} = 75.6\% > QSR_2 = 72.99\%$, show that the CA map is slightly preferable to the TCA map. Given that CA and TCA maps Figures 1 and 2 are very similar, we use TCov-TCA framework, introduced in this essay, to interpret this data set; for there is a lot of literature on the interpretation of CA maps, starting with the pioneering work of Benzécri (1973, Vol.2, chapter 2) and the few references that we cited in the introduction.

Table 3: QSR values (in %) of WS data for the first 4 dimensions.

| $\alpha$ | $QSR_\alpha(\cdot)$ | $QSR_\alpha(-\cdot)$ | $QSR_\alpha$ | $\delta_\alpha$ |
|---------|---------------------|----------------------|--------------|----------------|
| 1       | (100, 52.05)        | (-52.65, -87.07)     | 67.01        | 0.0476         |
| 2       | (65.62, 89.29)      | (-84.01, -60.77)     | 72.99        | 0.0318         |
| 3       | (62.50, 63.55)      | (-78.88, -62.65)     | 66.25        | 0.0203         |
| 4       | (58.13, 64.54)      | (-46.90, -80.23)     | 60.17        | 0.0130         |

| $\alpha$ | $CA_{QSR_\alpha}(\cdot)$ | $CA_{QSR_\alpha}(-\cdot)CA_\cdot$ | $CA_{QSR_\alpha}$ | $\varphi_\alpha$ |
|---------|------------------------|----------------------------------|-------------------|----------------|
| 1       | (100, 52.05)           | (-52.65, -87.07)                 | 67.01             | 0.0476         |
| 2       | (67.95, 89.80)         | (-90.24, -62.60)                 | 75.60             | 0.0312         |
| 3       | (58.20, 44.56)         | (-49.68, -59.46)                 | 52.24             | 0.0155         |
| 4       | (88.57, 48.59)         | (-57.37, -69.26)                 | 62.76             | 0.0130         |

Figure 3 represents taxicab interbattery analysis TCov map, which can also be interpreted as TCA contribution map. In Figure 3, brand $i$ is represented by the principal coordinates $(a_1(i), a_2(i))$, and attribute $j$ is represented by the principal coordinates $(b_1(j), b_2(j))$. 
6.1.1 Interpretation of the first principal dimension of TCov map

Table 4 displays the cross-covariance matrix seriated along its first TCov principal coordinates; where we clearly also observe the four principal quadrants for \( \alpha = 1 \).

Let \( T = \{ \text{innovative, trusted} \} \) and \( S = \{ \text{Nokia, Oracle, A, B, C, G} \} \) for the first principal dimension. In Table 4 we observe that the covariance values in \( S \times T \) quadrant are all positive; and that is the reason that \( \max_{QSR} QSR_1(+) = 100\% \), the highest attainable value. Similarly one observes that for the first two principal dimensions \( \min_{QSR} QSR_\alpha(-) \) are \(-87.07\%\) and \(-84.04\%\), relatively significant values compared to the lower bound \(-100\%\).

The first taxicab dispersion measure, displayed in Table 3, is \( \delta_1 = 0.0476 = \sum_{i=1}^{12} |a_1(i)| = \sum_{j=1}^{8} |b_1(j)| \) by (18). The last column and the last row of Table 4 display the signed absolute contributions \( a_1(i) \) and \( b_1(j) \) to \( \delta_1 \), that according to Benzécri (1973, p.47) assist in the interpretation of the first factor. So the relative contribution (RC) of \text{innovative} \ to the first factor is \( RC_1(\text{innovative}) = 194/476 = 0.41 \leq 0.5 \), a very high value indeed. Similarly the \( RC_1(\text{relevant or essential}) = (98 + 86)/476 = 0.39 \leq 0.5 \), a very high value. In TCov(\( P \)), the attainable upper bound of a RC of a coordinate \( a_\alpha(i) \) or \( b_\alpha(j) \) is 0.5 by equation (22). So the first principal dimension represents the factor opposing (\text{innovative} associated with \text{Nokia, and} \ with lesser degree with \text{Oracle}) to (\text{relevant-essential} associated with the brands \text{Fedex, H and F}). Furthermore, the cross-covariance matrix in Table 4 informs us more about this opposition: \( \text{cov(Nokia, innovative)} = 60 \) and \( \text{cov(Oracle, innovative)} = 25 \), while \( \text{cov(F, innovative)} = -37 \), \( \text{cov(Fedex, innovative)} = -30 \), and \( \text{cov(H, innovative)} = -17 \). Note that in particular the intensity of the negative covariances, representing the three major repulsive associations, can not be assessed in Figure 2. So, to assess quantitatively an
association between a row and a column, one has to follow Collins advice and look at the value in a table of numbers.

Furthermore, examining the seriated Table 2, we see that the last three brands FedEx, H and F do not have any molehills: they satisfy equation (25) of Lemma 4, and they have quite large marginal weights.

### 6.1.2 Interpretation of the second principal dimension of TCov map

Table 5 displays the residual cross-covariance matrix $P^{(2)}$ seriated along its 2nd TCov principal coordinates: It shows that the second principal or latent variable is based on the opposition between (leader-innovative associated essentially with Nokia) and (solution-rapport-trusted associated with brands F and A).

Figure 2 shows that each of the brands \{C, G, D, E, B\} have very small almost insignificant contributions either to the first or to the second principal dimensions; and this is also evident in Tables 4 and 5.

| Company | leader | innovative | essential | relevant | efficient | trusted | rapport | solution |
|---------|--------|------------|-----------|----------|-----------|---------|---------|-----------|
| Nokia  | 14     | 7          | -9        | 12       | -9        | -7      | -11     | -6       | 65        |
| fedex  | 5      | 8          | -1        | +0       | -9        | -8      | 2       | -17      | 29        |
| I      | 15     | -7         | -7        | 12       | 3         | -10     | 7       | 28        |
| H      | 8      | 4          | 0         | 5        | -8        | 0       | -7      | 19        |
| Oracle | 9      | -3         | 15        | -7       | -14       | 3       | -10     | 7         |
| C      | -1     | 0          | -1        | 1        | 1         | 0       | 1       | -1       | -1        |
| G      | -3     | 2          | 1         | -1       | 2         | 2       | 1       | -1       | -4        |
| E      | -2     | -3         | 1         | -1       | 0         | 3       | 1       | 0         | -4        |
| D      | -5     | 1          | -1        | 0        | 0         | -1      | 3       | 3         | -10       |
| B      | -5     | -2         | 1         | -2       | 4         | 2       | 3       | -1       | -18       |
| A      | -17    | -2         | -3        | -4       | 5         | 2       | 7       | 12       | -52       |
| F      | -9     | -13        | -12       | 1        | 0         | 13      | 6       | 14       | -66       |

| $b_j(i)$ | 83 | 42 | 24 | 10 | -18 | -42 | -45 | -54 |

### 6.1.3 TCA map

Equation (17) shows that the TCA map, Figure 2, is a change of scale of the TCov map, Figure 3. In Figure 2, brand $i$ is represented by the principal coordinates $(f_1(i), f_2(i))$, and attribute $j$ is represented by the principal coordinates $(g_1(j), g_2(j))$. The first row and column principal coordinates are given in Table 6 in decreasing order, and accordingly the $(density - 1 = \frac{p_{ij}}{p_{ii}p_{jj}} - 1)$ values of the attribute and brand categories are displayed in Table 6. However, the relative position of some brands and attributes in Figure 2 are completely different in Figure 3. For instance, in TCA map Figure 2 on dimension 1 the brand C seems to be much more important than the brands Oracle or Nokia: By Lemma 6, the covariance of brand C with the first factor
is $f_1(C) = 13/100$, which is much larger than $f_1(Nokia) = 9.3/100$; while in TCov map Figure 3 it is the opposite, the contributions are $a_1(C) = 8/10000$ and $a_1(Nokia) = 129/10000$. This aspect is the cause of confusion and difficulty in the interpretation of CA or TCA maps. Lemma 6 helps us to explain this fact: The brand $C$ is strongly associated with the latent variable innovative, but it does not contribute to the construction of this latent variable. While Nokia is moderately associated with the latent variable innovative, even though it constructs this latent variable; because it also constructs the 2nd principal dimension. That is, the TCov map helps us to identify rows or columns that essentially contribute to the formation of a latent variable, while the TCA map shows us the rows and the columns which are highly associated with the latent variables defined by the TCov decomposition.

| Company | innovative | trusted | rapport | efficient | solution | leader | relevant | essential | $f_1(i)$ |
|---------|------------|---------|---------|-----------|----------|--------|----------|----------|---------|
| C       | 54         | 4       | 9       | 6         | $-18$    | $-17$  | $-9$     | $-67$    | 13      |
| Oracle  | 40         | 4       | $-4$    | $-1$     | $-9$     | 14     | $-23$    | $-38$    | 10      |
| B       | 21         | 17      | 19      | 17        | $-7$     | $-32$  | $-28$    | $-29$    | 10      |
| E       | 7          | 24      | 6       | 0         | $-1$     | $-18$  | $-19$    | $-10$    | 9.4     |
| Nokia   | 39         | 2       | $-6$    | $-6$     | $-6$     | 4      | $-7$     | $-34$    | 9.3     |
| G       | 7          | 16      | 18      | $-10$     | $-12$    | $-24$  | $-5$     | 16       | 6.8     |
| A       | 10         | 5       | 12      | 8         | 24       | $-32$  | $-14$    | $-25$    | 3.8     |
| D       | 1          | $-3$    | 16      | $-1$     | 22       | $-26$  | 0        | $-6$     | $-1.5$  |
| H       | $-7$       | $-4$    | 0       | 2         | $-2$     | 1      | 6        | 7        | $-2.7$  |
| I       | $-14$      | 1       | $-9$    | $-10$    | 7        | 9      | $-2$     | 42       | $-2.9$  |
| Fedex   | $-10$      | $-4$    | 1       | 3         | $-4$     | 3      | 5        | 13       | $-3.4$  |
| F       | $-25$      | 4       | 3       | 10        | $-3$     | 7      | 0        | $-4.5$   |         |

$g_1(j)$ | 17.6 | 2.7 | $-0.3$ | $-0.7$ | $-1.5$ | $-1.6$ | $-6.9$ | $-17.3$ |

### 6.1.4 WS data set by CA

Table 3 displays the $CA_{QSR}$ values for the WS data set: $CA_{QSR_1}$ and $QSR_1$ values are identical by Lemma 10; $CA_{QSR_2}$ values being a little bit better than the corresponding $QSR_2$ values. This is the main reason that both CA and TCA maps, Figures 1 and 2, are very similar.

Table 7 represents the $sACQ$ and $sRSQ$ values for the WS data set, which reflect, somewhat in a different way, the $CA_{QSR}$ values. For the first principal dimension, there is a positively associated quadrant which contributes 44.15% to $\sigma_1$, and a negatively associated quadrant whose contribution to $\sigma_1$ is 27.89%. Furthermore, we note that in the first principal dimension globally the excess attractive association is quite small $100 sRES_1/\sigma_1 = 9.9\%$. For the second principal dimension, there is a negatively associated quadrant which contributes 48.43% to $\sigma_2$; a positively associated quadrant whose contribution to $\sigma_2$ is 28.43%; and a small excess repulsive association $100 sRES_2/\sigma_2 = -12.14\%$. 

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Table 7: sACQ and sRCQ of WS data for the first 4 dimensions of CA.

|   | 100sACQ\(_\alpha\)(+) | 100sACQ\(_\alpha\)(-) | 100\(sRES\)\(_\alpha\) | 100\(\sigma\)\(_\alpha\) |
|---|-----------------------|-----------------------|------------------------|------------------------|
| 1 | (4.02, 0.98)          | (-1.56, -2.54)        | 0.90                   | 9.10                   |
| 2 | (0.76, 1.39)          | (-2.38, -0.37)        | -0.59                  | 4.90                   |
| 3 | (1.13, 0.29)          | (-0.29, -1.02)        | 0.11                   | 2.72                   |
| 4 | (0.48, 0.42)          | (-0.52, -0.73)        | -0.35                  | 2.15                   |

|   | 100sRCQ\(_\alpha\)(+) | 100sRCQ\(_\alpha\)(-) | 100\(sRES\)\(_\alpha\)/\(\sigma\)\(_\alpha\) | 100\(\sigma\)\(_\alpha\)/\(\sigma\)\(_\alpha\) |
|---|-----------------------|-----------------------|-----------------------------------------------|-----------------------------------------------|
| 1 | (44.15, 10.80)        | (-17.16, -27.89)      | 9.90                                          | 100                                           |
| 2 | (15.50, 28.43)        | (-48.43, -7.46)       | -12.14                                        | 100                                           |
| 3 | (41.48, 10.50)        | (-10.69, -37.34)      | 3.95                                          | 100                                           |
| 4 | (22.39, 19.38)        | (-24.25, -33.98)      | -16.47                                        | 100                                           |

Here, we also want to remind the reader that CA and TCA of two other data sets of marketing research, discussed in Bock (2011a) and Bendixen (1996), produced similar results to the analysis of WS data set by both methods.

### 6.2 Faust data set

Faust (2005) analyzed by CA a two-mode affiliation network, (0-1) matrix \(Z = (z_{ij})\) of size 22 \(\times\) 15; where the 22 rows represent 22 countries and the 15 columns the regional trade and treaty organizations in the American continent. The country \(i\) is a member of the organization \(j\) if \(z_{ij} = 1\); and \(z_{ij} = 0\) means the country \(i\) is not a member of the organization \(j\). Choulakian and Abou-Samra (2020) compared the CA and TCA maps, where they found that the TCA map is much more interpretable than the corresponding CA map because of the existence of some influential columns and rows (outliers?) that dominated the second CA principal dimension. Table 8 displays the QSR measures for both methods. \(QSR_1 = 63.67\%\) is of comparable value to \(CA_{QSR_1} = 61.25\%\). However the corresponding values for the second dimension are completely different: \(QSR_2 = 57.96\%\) and \(CA_{QSR_2} = 33.41\%\), a significant difference of 24.55\%. This is also reflected in the \(sRCQ_2\) values reported in Table 9, where a positively associated quadrant contributes 78.56\% to \(\sigma_2 = 0.5331\); while in TCA this value is constant and equals 25\%. Furthermore, \(100sRES_2/\sigma_2 = 63.41\%\) represents a significant excess of attractive association.
Table 8: QSR values (in %) of Faust data for the first 4 dimensions.

| α | QSR_α(+) | QSR_α(−) | QSR_α | δ_α |
|---|-----------|-----------|--------|------|
| 1 | (56.52, 53.79) | (-67.43, -85.43) | 63.67 | 0.4266 |
| 2 | (43.78, 70.80) | (-68.79, -57.12) | 57.96 | 0.2751 |
| 3 | (65.51, 44.38) | (-78.00, -82.57) | 63.77 | 0.2659 |
| 4 | (61.45, 55.04) | (-67.63, -44.33) | 55.72 | 0.1782 |

| α | CA_QSR_α(+) | CA_QSR_α(−) | CA_QSR_α | ω_α |
|---|-------------|-------------|-----------|------|
| 1 | (69.71, 44.86) | (-59.20, -84.94) | 61.25 | 0.4104 |
| 2 | (17.51, 63.72) | (-49.87, -37.22) | 33.41 | 0.1570 |
| 3 | (60.29, 43.54) | (-81.95, -45.06) | 54.09 | 0.2195 |
| 4 | (33.26, 55.77) | (-46.57, -54.61) | 45.57 | 0.1636 |

Table 9: sACQ and sRCQ of Faust data for the first 4 dimensions of CA.

| α | 100sACQ_α(+) | 100sACQ_α(−) | 100sRES_α | 100σ_α |
|---|--------------|--------------|------------|--------|
| 1 | (22.56, 10.21) | (-11.13, -15.90) | 5.74 | 59.80 |
| 2 | (1.68, 41.89) | (-5.75, -4.00) | 33.81 | 53.31 |
| 3 | (4.55, 25.96) | (-8.26, -6.22) | 16.02 | 44.99 |
| 4 | (8.32, 11.90) | (-11.00, -8.17) | 1.06 | 39.40 |

| α | 100sRCQ_α(+) | 100sRCQ_α(−) | 100sRES_α/σ_α | 100σ_α/σ_α |
|---|--------------|--------------|----------------|-------------|
| 1 | (37.72, 17.08) | (-18.62, -26.58) | 9.60 | 100 |
| 2 | (3.14, 78.56) | (-10.79, -7.50) | 63.41 | 100 |
| 3 | (10.11, 57.70) | (-18.36, -13.83) | 35.62 | 100 |
| 4 | (21.13, 30.21) | (-27.92, -20.74) | 2.68 | 100 |

6.3 Rodent species abundance data

Table 10 displays the seriated Rodent species abundance data of size 28 × 9, where 9 species of rodents have been counted at each of 28 sites in California. The data set is from Quinn and Keough (2002), but is available in the R package TaxicabCA. Choulakian (2017) presented a detailed analysis of this data set by CA and TCA. Given that, σ² ≥ 0.7465, a value greater than 0.7 marked by Benzécri, we clearly observe the quasi-two blocks diagonal structure in Table 10.
Furthermore,

a) $CA_QSR_1(v_{1+}, u_{1+}) = 0.965$, $CA_QSR_1(v_{1-}, u_{1-}) = 0.190$, $CA_QSR_1(v_{1-}, u_{1+}) = -0.943$, $CA_QSR_1(v_{1+}, u_{1-}) = -1$.

This shows that the quality of signs in the 4th quadrant of the $Cov(P)$ (not shown), is very poor, 0.190 ; and the result in Corollary 11a is approximately satisfied, -1 and -0.943.

b) $sACQ_1(v_{a+}, u_{a+}) = 0.674$, $sACQ_1(v_{1-}, u_{1-}) = 0.014$, $sACQ_1(v_{1-}, u_{1+}) = -0.084$, $sACQ_1(v_{1+}, u_{1-}) = -0.092$.

This shows that the result in Corollary 11b is approximately satisfied.

c) $sRCQ_1(v_{a+}, u_{a+}) = 0.7798$, $sRCQ_1(v_{1-}, u_{1-}) = 0.0158$, $sRCQ_1(v_{1-}, u_{1+}) = -0.0974$, $sRCQ_1(v_{1+}, u_{1-}) = -0.1070$.

This shows that 78% of the contribution to $\sigma_1 = 0.864$ comes from the 2nd quadrant of Table 10; a very small proportion 1.58% comes from the 4th quadrant of Table 10; and about 20% comes from the quasi-sparse blocks.

7 Conclusion

A crucial first step in data analysis of multivariate tables is the preprocessing step: centering and/or scaling of the data. In the case of CA of contingency tables, the row and column marginals are intricate part of the centering and scaling of the method via the chi-square residuals defined for the row and
column profiles. In a pioneering work, Goodman (1991) and his discussants compared the effects of marginal weighted scores and uniform weighted scores in association and correlation models in the analysis of contingency tables. A parallel to this problem is: Should we decompose the covariance matrix (do Tucker interbattery analysis) or the correlation matrix (do Hotelling canonical correlation analysis)?

This essay attempted to clarify mainly the following issues:

First, we showed that the aims of CA and TCA are different, but interrelated. The aim of CA is to explain the heterogeneity of the row and column profiles (row and column conditional distributions), from which as a byproduct we get a view of the dependence structure of the cross-covariance matrix. While, the aim of TCA is to explain the dependence structure of the cross-covariance matrix, then as a by product obtain a view of the heterogeneity of the row and column profiles. Empirical data have shown that the cross-covariance matrix \((p_{ij} - p_i^* p_j^*)\) is much more robust than the chi-square residual matrix \(\frac{(p_{ij} - p_i^* p_j^*)}{\sqrt{p_i^* p_j^*}}\), a fact first observed by Tenenhaus and Augendre (1996).

Second, two maps are needed to fully picture the dependence/heterogeneity structure in a contingency table; and Lemma 6 explains in a simple way the relationship between the two maps.

Third, for a principal dimension we introduced the new concept of the intrinsic quality and distinguished it from the often used extrinsic quality; and related the intrinsic quality to the quality of signs of the residuals in the four quadrants. Furthermore, we provided quantifications of the intrinsic quality by introducing \(QSR\) and \(sACQ\) indices.

Fourth, we emphasized the importance of looking at the residual covariance values at each iteration, a general procedure exemplified by Tukey (1977) in exploratory data analysis.

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