Innerness of Derivations on Subalgebras of Measurable Operators

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Abstract

Given a von Neumann algebra $M$ with a faithful normal semi-finite trace $\tau$, let $L(M, \tau)$ be the algebra of all $\tau$-measurable operators affiliated with $M$. We prove that if $A$ is a locally convex reflexive complete metrizable solid $*$-subalgebra in $L(M, \tau)$, which can be embedded into a locally bounded weak Fréchet $M$-bimodule, then any derivation on $A$ is inner.

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1. Introduction

The structure of automorphisms and derivations of operator algebras is an important part of the theory of operator algebras and their applications in quantum dynamics.

Recall that a linear operator $D$ on an algebra $A$ is called a derivation if it satisfies the condition

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

Every (but fixed) element $a \in A$ generates a derivation $D_a$ on $A$, defined as $D_a(x) = ax - xa$, $x \in A$. Such derivations are said to be inner derivations.

Derivations on $C^*$-algebras and von Neumann algebras have been studied in the monographs of Sakai [13], [14]. It is well-known that every derivation on a $C^*$-algebra $A$ is norm continuous and if $C^*$-algebras $A$ is unital and simple or it is weakly closed (i.e. is a von Neumann algebra) then any derivation on $A$ is inner. For general Banach algebras similar problems were considered in the monograph [8].

Investigation of derivations on unbounded operator algebras and, in particular, on the algebra $L(M)$ of measurable operators affiliated with a von Neumann algebra $M$, was initiated in the papers [5], [6]. One of the main problems posed in these papers was: whether any derivation on $L(M)$ is inner. A negative answer to this problem in the general setting was given in the paper [7] (see also [10]). Namely, it was proved that if $M$ is a non atomic abelian von Neumann algebra (in particular $L^\infty(0; 1)$) then $L(M)$ (resp. $L^0(0; 1)$) admits a non trivial (and hence discontinuous, and non inner) derivation.

Further there were some positive results on this way. In [3] we have proved that if $M$ is type I von Neumann algebra with a faithful normal semi-finite trace $\tau$, then a derivation on the algebra $L(M, \tau)$ of all $\tau$-measurable operators affiliated with $M$ is inner if and only if it is $Z$-linear, or equivalently if it is identically zero on the center $Z$ of $M$. Recently [4] we gave a complete description of derivation on $L(M, \tau)$ and in particular proved that if $M$ is of type $I_\infty$ then any derivation on $L(M, \tau)$ is inner.

If $M$ is a general von Neumann algebra with a faithful normal semi-finite trace $\tau$, then the algebra $L(M, \tau)$ contains various subalgebras with different properties of derivations. One of the interesting classes of subalgebras in $L(M, \tau)$ are so called Arens algebras

$$L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$
In the paper [2] we gave a complete description of derivations on $L^\omega(M, \tau)$ and proved that any derivation on $L^\omega(M, \tau)$ is inner if and only if the trace $\tau$ is finite.

In this connection a natural question arises: which subalgebras in $L(M, \tau)$ admit only inner derivation?

In this paper we give a sufficient condition for subalgebras in $L(M, \tau)$ to have such a property. Namely, we prove that if $A$ is locally convex reflexive complete metrizable solid $*$-subalgebra in $L(M, \tau)$, which can be embedded into a locally bounded weak Fréchet $M$-bimodule, then any derivation on $A$ is inner.

2. Preliminaries

Let $H$ be a Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on $H$. Consider a von Neumann algebra $M \subset B(H)$ with a faithful normal semi-finite trace $\tau$, and denote by $\mathcal{P}(M)$ the lattice of (orthogonal) projections in $M$.

A linear subspace $\mathcal{D}$ in $H$ is affiliated with $M$ (denoted as $\mathcal{D} \eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for any unitary operator $u$ from the commutant $M' = \{y \in B(H) : xy = yx, \forall x \in M\}$ of the von Neumann algebra $M$.

A linear operator $x$ with the domain $\mathcal{D}(x) \subset H$ is said to be affiliated with $M$ (denoted as $x \eta M$) if $u(\mathcal{D}(x)) \subset \mathcal{D}(x)$ and $ux(\xi) = xu(\xi)$ for all $u \in M'$, $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in $H$ is called $\tau$-dense, if

1) $\mathcal{D} \eta M$;

2) given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(M)$ such that $p(H) \subset \mathcal{D}$ and $\tau(p^\perp) \leq \varepsilon$.

A closed linear operator $x$ is called $\tau$-measurable with respect to the von Neumann algebra $M$, if $x \eta M$ and $\mathcal{D}(x)$ is $\tau$-dense in $H$.

Denote by $L(M, \tau)$ the set of all $\tau$-measurable operators affiliated with $M$. Consider the topology $t_\tau$ of convergence in measure on $L(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in L(M, \tau) : \exists e \in \mathcal{P}(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where $\| \cdot \|_\infty$ is the operator norm on $M$, and $\varepsilon, \delta$ are positive numbers.

It is well-known [12] that $L(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$-algebra.

Now let us recall the notion of a bimodule over a Banach algebra (see [8]).
Let $A$ be a complex algebra and let $E$ be a complex linear space. $E$ is called a left $A$-module (respectively right $A$-module) if a bilinear map $(a, x) \mapsto a \cdot x$ (respectively $(a, x) \mapsto x \cdot a$) from $A \times E$ into $E$ is defined, such that given any $a, b \in A$ and $x \in E$ one has

$$a \cdot (b \cdot x) = ab \cdot x \quad \text{(respectively } (x \cdot a) \cdot b = x \cdot ab),$$

$E$ is said to be $A$-bimodule if it is left and right $A$-module simultaneously, and

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b,$$

for all $a, b \in A, x \in E$.

Let $A$ be a Banach algebra and suppose that $E$ is a Fréchet space, i.e. a complete metric space with a shift invariant metric. If $E$ is an $A$-bimodule and the maps $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are continuous for each $a \in A$, then $E$ is called a weak Fréchet $A$-bimodule.

Interesting examples of weak Fréchet $A$-bimodules are given by non commutative $L^p$-spaces $L^p(M, \tau) \subset L(M, \tau)$, $p \geq 1$. Indeed, given any $a \in M$ and $x \in L^p(M, \tau)$ one has $ax \in L^p(M, \tau)$, $xa \in L^p(M, \tau)$ and $\|ax\|_p \leq \|a\|_\infty \|x\|_p$ which imply the above statement.

Let $E$ and $F$ be metrizable linear topological spaces and let $T : E \to F$ be a linear operator. The separating space of the linear map $T$, denoted by $S(T)$, is defined as

$$S(T) = \{y \in F : \text{there is } (x_n)_{n \in \mathbb{N}} \text{ in } E \text{ such that } x_n \to 0 \text{ and } T(x_n) \to y\}.$$

Recall that $S(T)$ is closed (see [8], Proposition 5.1.2) and that the closed graph theorem is valid for complete metrizable topological linear space. Therefore $T$ is continuous if and only if $S(T) = \{0\}$.

**Definition 2.1.** [8]. Let $A$ be an algebra, and let $E$ be a topological linear space which is a $A$-bimodule. Then $E$ is a separating module if, for each sequence $\{a_n\}$ in $A$, the nest $(a_1 \cdots a_n E)$ stabilizes, i.e. there is an $n_0 \in \mathbb{N}$ such that $(a_1 \cdots a_n E) = \overline{(a_1 \cdots a_{n+1} E)}$ for all $n > n_0$, where $\overline{F}$ – the closure of the set $F$.

A linear topological space $E$ is said to be locally bounded if there exists a bounded neighborhood of zero in $E$.

From [8, Theorem 5.2.15], we have the following

**Proposition 2.2.** Let $E$ be a weak Fréchet $A$-bimodule, and let $D : A \to E$ be a derivation. Then

1) The separating space $S(D)$ is a closed submodule of $E$;

2) Suppose that $E$ is locally bounded. Then $S(D)$ is a separating module.

Given a linear topological space $X$ with a topology $t_X$, let us denote by $x_n \overset{t_X}{\to} 0$ the convergence in the topology $t_X$. 
If \((A, t_A)\) and \((B, t_B)\) are linear topological spaces, with \(A \subseteq B \subseteq L(M, \tau)\) then we shall suppose that the topology \(t_A\) is stronger than \(t_B\), i.e. \((A, t_A)\) is topologically imbedded into \((B, t_B)\).

3. The main results

The aim of the present section is to prove the following result.

**Theorem 3.1** Let \(M\) be a von Neumann algebra with a faithful normal semi-finite trace \(\tau\). Suppose that \(A\) is a complete metrizable solid \(*\)-subalgebra in \(L(M, \tau)\) and \(E\) is a locally bounded weak Fréchet \(M\)-bimodule in \(L(M, \tau)\). If

1. \(A\) is locally convex and reflexive;
2. \(M \subset A \subset E\) are topological imbedding,

then any derivation of the algebra \(A\) is inner.

Recall that a subalgebra \(A\) in \(L(M, \tau)\) is solid if \(x, y \in L(M, \tau), |y| \leq |x|\) implies \(y \in A\).

The proof of this theorem consists of several steps.

**Proposition 3.2.** Given an arbitrary von Neumann algebra \(M\), and a weak Fréchet \(M\)-bimodule \(E\), suppose that \(p\) is a projection in \(M\) and \(D : M \to E\) is a derivation, i.e. a linear map such that \(D(xy) = D(x)y + xD(y)\) for all \(x, y \in M\). Put \(D_\rho(x) = pD(x)p, x \in pMp\). Then

1. \(D_\rho : pMp \to pEp\) is a derivation;
2. \(pS(D)p \subseteq S(D_\rho)\).

Proof. 1) For \(x, y \in pMp\) we have \(x = pxp, y = pyp\). Therefore \(D_\rho(xy) = pD(pxyp)p = pD(pxpyp)p = pD(px)pyp + pxpD(pyp)p = D_\rho(x)y + xD_\rho(y)\), i.e. \(D_\rho\) is a derivation.

2) For \(y \in S(D)\) according the definition there exists a sequence \(\{x_n\}\) in \(M\) such that \(x_n \xrightarrow{\text{|||}} 0\) and \(D(x_n) \xrightarrow{t_E} y\) as \(n \to \infty\). But then \(px_n p \xrightarrow{\text{|||}} 0\) and \(D_\rho(px_n p) = pD(px_n p)p = pD(p)x_n p + pD(x_n)p + px_n D(p)p \xrightarrow{t_E} pyp, i.e. D_\rho(px_n p) \xrightarrow{t_E} pyp\), which means that \(pyp \in S(D_\rho)\) and therefore \(pS(D)p \subseteq S(D_\rho)\). The proof is complete. ■

**Proposition 3.3.** Let \(M\) be a von Neumann algebra with a faithful normal semi-finite trace \(\tau\) and let \(E\) be a locally bounded weak Fréchet \(M\)-bimodule in \(L(M, \tau)\). Then every derivation \(D : M \to E\) is automatically continuous.

Proof. Let us show that \(S(D) = \{0\}\). Suppose the opposite, i.e. \(S(D) \neq \{0\}\) and take a non zero \(y \in S(D)\). Chose a projection \(e\) in \(M\) such that \(ye \in M\) and \(ye \neq 0\). Since \(S(D)\) is a submodule in \(E\), we have that \(ye(ye)^* \in S(D)\). Thus without loss of the generality we may suppose that \(y \geq 0\). Take a projection \(p \in M\) such that \(pyp \neq 0\) and \(n^{-1}p \leq pyp \leq np\) for an appropriate \(n \in \mathbb{N}\). Then \(pyp\) is invertible in \(pMp\), i.e.
there exists an element \( z \in pMp \) such that \( pypz = p \). Since \( S(D) \) is an \( M \)-bimodule it follows that \( p \in S(D) \). Now consider two cases separately:

**The case 1.** \( pMp \) is finite dimensional. Observe the derivation \( D_p : pAp \to pEp \) defined by

\[
D_p(x) = pD(px)p, \quad x \in pAp.
\]

Since \( pMp \) is finite dimensional, the spaces \( pAp \) and \( pEp \) are also finite dimensional as subspaces of \( L(M, \tau) = pMp \). Therefore \( D_p \) is necessary continuous, i.e. \( S(D_p) = \{0\} \).

On the other hand Proposition 3.2 implies that \( pS(D)p \subset S(D_p) \), and from the consideration above we have that \( p \in S(D_p) \) and hence \( 0 \neq p \in pS(D)p \subset S(D_p) = \{0\} \). This contradiction implies that \( S(D) = \{0\} \).

**The case 2.** \( pMp \) is infinite dimensional. In this case there exists a strictly monotone decreasing sequence \( (p_n) \) of projections in \( M \) such that \( p_n \leq p \) for all \( n \in \mathbb{N} \). Since \( S(D) \) is a closed submodule in \( E \) (Proposition 2.2) \( p_nS(D) \) is also closed. Further \( p_n \neq p_{n+1} \leq p \) implies that

\[
\overline{p_nS(D)} \neq \overline{p_{n+1}S(D)}. \tag{1}
\]

On the other hand, since \( E \) is locally bounded Proposition 2.2 implies that the nest \( \{p_nS(D)\} \) stabilizes, i.e. there exist \( n_0 \in \mathbb{N} \) such that \( \overline{p_nS(D)} = \overline{p_{n+1}S(D)} \) for all \( n > n_0 \) in a contradiction with (1). Therefore \( S(D) = \{0\} \) and the proof is complete.

**Remark 1.** Proposition 3.3 implies the well-known fact that every derivation on the von Neumann algebra \( M \) is norm continuous. It is sufficient to put \( E = M \).

**Proposition 3.4.** Let \( M \) be a von Neumann algebra with a faithful normal semifinite trace \( \tau \). Suppose that \( E \) is a locally bounded weak Fréchet \( M \)-bimodule in \( L(M, \tau) \) and \( A \) is complete metrizable algebra such that \( A \subseteq E \). Then any derivation \( D : M \to A \) is continuous.

Proof. Let \( \{x_n\} \subset M, x_n \xrightarrow{\|\cdot\|} 0 \) and \( D(x_n) \xrightarrow{t_A} y \), which implies that \( D(x_n) \xrightarrow{t} y \). Let us show that \( y = 0 \). By Proposition 3.3 \( D : A \to E \) is continuous and thus \( D(x_n) \xrightarrow{t_E} 0 \) and hence \( D(x_n) \xrightarrow{t} 0 \). This implies that \( y = 0 \). The proof is complete.

If \( A \) is a locally convex metrizable space, then its topology can be generated by an increasing sequence of seminorms \( \{\rho_n, n \in \mathbb{N}\} \).

Denote by \( U \) the group of all unitaries of the von Neumann algebra \( M \).

**Proposition 3.5.** Let \( A \) be a locally convex weak Fréchet \( M \)-bimodule in \( L(M, \tau) \). Given any non zero element \( x \in A \) there exists a seminorm \( \rho_n \) such that

\[
\inf\{\rho_n(uxu^*) : u \in U\} \neq 0.
\]
Proof. Suppose that opposite, i.e. \( \inf \{ \rho_n(uxu^*) : u \in U \} = 0 \) for all \( n \in \mathbb{N} \).

Chose the unitaries \( u_n \in U \) such that \( \rho_n(u_n xu_n^*) \leq n^{-1} \). Since \( \rho_k \leq \rho_{k+1} \), we have that \( \rho_k(u_n xu_n^*) \to 0 \) as \( n \to \infty \) for each fixed \( k \in \mathbb{N} \). This means that \( u_n xu_n^* \xrightarrow{\tau} 0 \), and hence \( u_n xu_n^* \xrightarrow{\tau} 0 \). Since \( \|u_n\|_\infty = 1 \) for all \( n = 1, 2, \ldots \), we obtain that \( x = u_n^*(u_n xu_n^*)u_n \xrightarrow{\tau} 0 \), i.e. \( x = 0 \) a contradiction. The proof is complete. \( \blacksquare \)

**Proposition 3.6.** Let \( M, A \) and \( E \) be as in theorem 3.1. Then every derivation \( D : M \to A \) is spatial, i.e. \( D(x) = ax - xa \) for an appropriate \( a \in A \) and every \( x \in M \).

Proof. By Proposition 3.4 the derivation \( D : M \to A \) is continuous.

Let \( U \) be the group of all unitary elements in \( M \). Given any \( u \in U \) put

\[
T_u(x) = uxu^* + D(u)u^*, \quad x \in A.
\]

Since map \( x \mapsto uxu^* \) is continuous, the map \( T_u \) is \( \sigma(A, A^*) \)-continuous.

For \( u, v \in U \) we have

\[
T_u(T_v(x)) = T_u(vxv^* + D(v)v^*) = u(vxv^* + D(v)v^*)u^* + D(u)u^* = uvxv^*u^* + uD(v)v^*u^* + D(u)u^* = (uvx + D(u)v + uD(v))u^* = uvxu^* + D(uv)(uv)^* = T_{uv}(x),
\]
i.e.

\[
T_u T_v = T_{uv}, \quad u, v \in U.
\]  

(2)

Further we have \( T_u(0) = D(u)u^* \) and the continuity of \( D \) implies that the set \( K_D = \{ T_v(0) : v \in U \} = \{ D(v)v^* : v \in U \} \) is bounded in \( A \). Moreover, the set \( K = \text{cl}(\text{co}(K_D)) \) – the closure of the convex hull of \( K_D \), is a closed convex bounded subset in \( A \). The reflexivity of the space \( A \) then implies that \( K \) is a non-void \( \sigma(A, A^*) \)-compact convex set. From (2) it follows that \( T_u(K_D) \subseteq K_D \) for all \( u \in U \). Since \( T_u \) is an affine homeomorphism we have \( T_u(\text{cl}(\text{co}(K_D))) = \text{cl}(\text{co}(T_u(K_D)))) \subseteq \text{cl}(\text{co}(K_D)) \), i.e. \( T_u(K) \subseteq K \) for all \( u \in U \).

According Proposition 3.6, given any \( x, y \in A, x \neq y \) there exists seminorm \( \rho_n \) such that

\[
\inf \{ \rho_n(T_u(x) - T_u(y)) : u \in U \} = \inf \{ \rho_n(u(x - y)u^*) : u \in U \} \neq 0.
\]

Therefore \( \{ T_u : u \in U \} \) is a non-contracting (in the sense of [11]) semigroup of \( \sigma(A, A^*) \)-continuous affine mappings of a \( \sigma(A, A^*) \)-compact convex set \( K \). By Ryll-Nardzewski’s fixed point theorem [11], there exists \( a \in K \) such that \( T_u(a) = a \) for all \( u \in U \). This means that \( uau^* + D(u)u^* = a \), i.e. \( D(u) = au - ua \) for all \( u \in U \). Since every element of \( M \) is a linear combination of unitaries from \( M \), we have \( D(x) = ax - xa \) for all \( x \in M \), i.e. \( D \) is a spatial derivation on \( M \) with values in \( A \). The proof is complete. \( \blacksquare \)
Proof of Theorem 3.1. According to Proposition 3.6 there exists element $a \in A$ such that $D(x) = ax - xa$ for all $x \in M$. We shall proof that this is true for all $x \in A$.

First suppose that $x \in A$, $x \geq 0$. In this case the element $1 + x \in A \subset L(M, \tau)$ is invertible and moreover $(1 + x)^{-1} \in M$.

For an invertible element $b \geq 0$ in $A$ one has

$$0 = D(1) = D(bb^{-1}) = D(b)b^{-1} + bD(b^{-1}),$$

i.e. $D(b) = -bD(b^{-1})b$.

Therefore

$$D(x) = D(1 + x) = -(1 + x)D((1 + x)^{-1})(1 + x).$$

On the other hand, since $(1 + x)^{-1} \in M$ we have from the above

$$D((1 + x)^{-1}) = a(1 + x)^{-1} - (1 + x)^{-1}a.$$

Therefore,

$$-(1 + x)D((1 + x)^{-1})(1 + x) = -(1 + x)[a(1 + x)^{-1} -
-(1 + x)^{-1}a](1 + x) = -(1 + x)a + a(1 + x) = ax - xa,$$

i.e.

$$D(x) = -(1 + x)D((1 + x)^{-1})(1 + x) = ax - xa.$$ 

Therefore $D(x) = ax - xa$ for every $x \in A$, $x \geq 0$. Since $A$ is a solid $*$-subalgebra in $L(M, \tau)$, every element from $A$ is a linear combination of positive elements of $A$. Thus $D(x) = ax - xa$ for all $x \in A$. The proof is complete. $\blacksquare$

Remark 2. The condition on $A$ to be solid is used only at the end of proof of Theorem 3.1. In fact this condition may be replaced by the condition $Lin(A_+) = A$, where $A_+$ is the positive cone of $A$ and $Lin(A_+)$ is the linear span of $A_+$ (i.e. the positive cone $A_+$ is hereditary in $A$).

Example 3.8. An example of algebras, satisfying the conditions of Theorem 3.1 is given by a non commutative Arens algebra $L^\omega(M, \tau)$ in the case of a finite trace $\tau$ (see [2]).

Given be a von Neumann algebra $M$ with a faithful normal semi-finite trace $\tau M$ and $p \geq 1$, put $L^p(M, \tau) = \{x \in L(M, \tau) : \tau(|x|^p) < \infty\}$. It is known [12] that $L^p(M, \tau)$ is a Banach space with respect to the norm

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(M, \tau).$$

Consider the space

$$L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$
It is known [1], [9] that $L^\omega(M, \tau)$ is a locally convex metrizable $*$-algebra with the topology $t$ generated by the sequence of norms 
\[\|x\|'_n = \max\{\|x\|_1, \|x\|_n\}, n \in \mathbb{N}.\]

The algebra $L^\omega(M, \tau)$ is called a (non commutative) Arens algebra. The dual space for $(L^\omega(M, \tau), t)$ was described in [1], where it has been proved that $(L^\omega(M, \tau), t)$ is reflexive if and only if trace $\tau$ is finite.

Therefore Theorem 3.1 implies that if the trace $\tau$ is finite, then every derivation on the algebra $L^\omega(M, \tau)$ is inner.

It should be noted also that a complete description of derivations on general $L^\omega(M, \tau)$ was obtained in [2]. Namely, it has been proved that every derivation on $L^\omega(M, \tau)$ is spatial and has the form 
\[D(x) = ax - xa, \; x \in L^\omega(M, \tau)\]
for an appropriate $a \in M + L^\infty_2(M, \tau)$, where $L^\infty_2(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau)$.

Now let us consider an example of an algebra $A$ satisfying all conditions of Theorem 3.1 except $M \subset A$, which admits non-inner derivations.

**Example 3.9.** Put 
\[A = L^\omega_2(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau)\]
and consider on $A$ the topology generated by the system of norms $\{\|\cdot\|_{p \geq 2}\}$. Then $A$ is a metrizable locally convex $*$-algebra [2] and 
\[L^\omega_2(M, \tau) = \bigcap_{n=2}^{\infty} L^n(M, \tau).\]

As the intersection of countable family of reflexive Banach space, $A$ is also reflexive. If the trace $\tau$ is semi-finite but not finite, $M$ is not contained in $A = L^\omega_2(M, \tau)$. Every derivation of $L^\omega_2(M, \tau)$ has the form 
\[D_a(x) = ax - xa, \; x \in L^\omega_2(M, \tau)\]
for an appropriate $a \in M + L^\infty_2(M, \tau)$ (see [2]).

Now if $M$ is non commutative and $a$ is a non central element from $M \setminus L^\infty_2(M, \tau)$ the spatial derivation $D_a$ on $L^\omega_2(M, \tau)$ is not inner.

Let us consider an example of an algebra $A$ which shows that the reflexivity of $A$ is not a necessary condition for the statement of Theorem 3.1.

**Example 3.10.** Let $M$ be the $C^*$-product of von Neumann algebra $M_n$, i.e.
\[M = \bigoplus_{n=1}^{\infty} M_n = \{\{x_n\} : x_n \in M_n, \sup \|x\|_{M_n} < \infty\},\]
where \( \| \cdot \|_{M_n} \) is the \( C^* \)-norm on \( M_n \).

Put
\[
A = \{ \{ x_n \} : x_n \in M_n, \sum_{n=1}^{\infty} \frac{1}{2^n} \| x_n \|_{M_n}^p < \infty, 1 \leq p < \infty \}.
\]

Then it is clear that \( \{ x_n \} \in A \) if and only if \( \{ \| x_n \|_{M_n} \} \in l^\omega \), where \( l^\omega = \bigcap_{p \geq 1} l^p \) – the Arens algebra associated with abelian von Neumann algebra \( l^\infty \) of all bounded complex sequences with the trace
\[
\nu(\{ \lambda_n \}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n, \{ \lambda_n \} \in l^\infty.
\]

Consider the topology \( t \) on \( A \), generated by the family of norms
\[
\| x \|_{A,p} = \left( \sum_{n=1}^{\infty} \frac{1}{2^n} \| x_n \|_{M_n}^p \right)^{\frac{1}{p}}.
\]

With the coordinatewise algebraic operations and involution \( (A, t) \) becomes a locally convex complete metrizable \(*\)-algebra. If at least one of the algebras \( M_n \) is infinite dimensional, the \( A \) is not isomorphic to any Arens algebra.

Consider a derivation \( D : A \to A \). Let \( q_n \) be central projection in \( M \) such that \( q_n M = M_n \). Then we have
\[
D(q_n x) = q_n D(x), \quad x \in A,
\]
and therefore \( D(M_n) \subseteq M_n \) and the restriction
\[
D_n(x) = q_n D(x), \quad x \in M_n,
\]
gives a derivation \( D_n : M_n \to M_n \). The classical theorem of Sakai implies the existence of an appropriate \( a_n \in M_n \) such that \( D_n(x) = a_n x - x a_n \) for all \( x \in M_n \), moreover one can assume that \( \| a_n \| \leq \| D_n \| \) (see [13, Theorem 4.1.6])

Let us show that \( a = \{ a_n \} \in A \) and that \( D(x) = ax - xa \) for all \( x \in A \).

Take an arbitrary \( \varepsilon > 0 \). For each \( n \in \mathbb{N} \) there exists \( x_n \in M_n, \| x_n \|_{M_n} \leq 1 \) such that \( \| D_n \| \leq \| D_n(x_n) \| + \varepsilon \). Then \( x = \{ x_n \} \in M \) and \( D(x) = \{ D_n(x_n) \} \in A \). Therefore \( \{ \| D(x_n) \|_{M_n} \} \in l^\omega \). The inequalities
\[
\sum_{n=1}^{\infty} \frac{1}{2^n} \| a_n \|_{M_n}^p \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \| D_n \|^p \leq \sum_{n=1}^{\infty} \frac{1}{2^n} (\| D_n(x_n) \| + \varepsilon)^p \leq 2^{p-1} \sum_{n=1}^{\infty} \frac{1}{2^n} (\| D_n(x_n) \|^p + \varepsilon^p)
\]
imply that $\{\|a_n\|_{M_n}\} \in l^\omega$, i.e. $a \in A$. Further since

$$q_n D(x) = D_n (q_n x) = a_n q_n x - q_n x a_n = q_n (ax - xa)$$

for all $n = 1, 2, \ldots$, taking the sum over all $q_n$ we obtain $D(x) = ax - xa$ for all $x \in A$. The proof is complete. □

It well-known that every abelian von Neumann algebra is isomorphic to an algebra $L^\infty(\Omega)$ of all essentially bounded measurable complex functions on a measure space $(\Omega, \Sigma, \mu)$. In this case the algebra $\tau$-measurable operators $L(M, \tau)$ is isomorphic with the algebra $L^0(\Omega)$ of all measurable functions on $\Omega$.

**Proposition 3.11.** Let $A$ be a $*$-subalgebra in $L^0(\Omega)$ and let $E$ be a locally bounded weak Fréchet $L^\infty(\Omega)$-bimodule such that $L^\infty(\Omega) \subseteq A \subseteq E$. Then every derivation on $A$ is identically zero.

Proof. Since $L^\infty(\Omega)$ is abelian, every derivation $D$ is equal to zero on idempotents (projections) from $L^\infty(\Omega)$. Therefore $D(x) = 0$ on each step function $x \in A$. The space of step functions is dense in $L^\infty(\Omega)$ and by Proposition 3.4 the derivation $D : L^\infty(\Omega) \rightarrow A$ is continuous, therefore $D(x) = 0$ for all $x \in L^\infty(\Omega)$.

For $x \in A$, take a sequence of idempotents $e_n, n \in \mathbb{N}$ from $L^\infty(\Omega)$ such that $e_n x \in L^\infty(\Omega)$ and $e_n \uparrow 1$. Then $e_n D(x) = D(e_n x) - D(e_n) x = 0$, i.e. $e_n D(x) = 0$ for all $n \in \mathbb{N}$. Since $e_n \uparrow 1$ this implies that $D(x) = 0$. The proof is complete. □

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