Two-parameter differential calculus on the $h$-superplane

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Abstract

We introduce a noncommutative differential calculus on the two-parameter $h$-superplane via a contraction of the $(p, q)$-superplane. We manifestly show that the differential calculus is covariant under $GL_{h_1, h_2}(1|1)$ transformations. We also give a two-parameter deformation of the $(1 + 1)$-dimensional phase space algebra.

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I. INTRODUCTION

Quantum groups are a generalization of the concept of classical groups. The theory of quantum groups has become an important branch of mathematical physics and a new branch of mathematics. An approach to obtain the quantum groups is to identify the elements of a quantum group with the linear transformations of a space with noncommuting coordinates. It is known, from the work of Woronowicz,\textsuperscript{1} that one can define a consistent differential calculus on the noncommutative space of a quantum group. Thus quantum group is a concrete example of noncommutative differential geometry.\textsuperscript{2}

During the past few years, Wess-Zumino\textsuperscript{3} have developed a differential calculus on the quantum (hyper)plane which is covariant under the action of the quantum group $GL_q(n)$. The natural extension of their scheme to superspace\textsuperscript{4} was given by Soni\textsuperscript{5} and the two-parameter differential calculus on the superplane has been worked out by Chung.\textsuperscript{6} A differential calculus on the $h$-plane was given by Karimipour\textsuperscript{7} and the two-parameter analogue was introduced by Aghamohammadi.\textsuperscript{8}

In this paper we construct a two-parameter differential calculus on the quantum $h$-superplane using the methods of Ref. 9. The paper is organized as follows: in Sec. II we obtain the $(h_1, h_2)$-superplanes via a contraction from the $(p, q)$-superplanes. We define derivatives and differentials on the $(h_1, h_2)$-superplane of noncommuting coordinates and give their commutation rules. In Sec. III we manifestly show that the differential calculus is covariant under the action of the quantum supergroup $GL_{h_1, h_2}(1|1)$ of Ref. 10. We give a two-parameter deformation of the $(1 + 1)$-dimensional phase space algebra in Sec. IV and in the following section we show that the $(p, q)$-deformed superoscillator algebra satisfies the undeformed superoscillator algebra when objects are transformed into new objects such that they are singular for certain values of the deformation parameters.

II. DIFFERENTIAL CALCULUS ON $h$-SUPERPLANE

In this work we denote $(p, q)$-deformed objects by primed quantities. Unprimed quantities represent transformed coordinates. As usual, we shall always assume that even (bosonic) objects commute with everything and odd (Grassmann) objects anti-commute among themselves. Before discussing the two-parameter differential calculus on $h$-superplane we give some notations and useful formulas in the following section. This first section closely follows the approach of Ref. 10.

A. Quantum $h$-superplane
Quantum superplane is an associative coordinate algebra $A_q$ equipped with a set $\{x', \theta'\}$ of generators $x'$, $\theta'$. The commutation relations of the generators is defined by

\begin{align}
  x' \theta' - q \theta' x' &= 0, \\
  \theta'^2 &= 0,
\end{align}

where $q$ is a nonzero complex deformation parameter. The coordinates neither commute nor anticommute unless $q \to \pm 1$, respectively. In this work we shall use the limits $p \to 1$, $q \to 1$ to make a contraction.

We now introduce new coordinates $x$ and $\theta$, in terms of $x'$ and $\theta'$, by

\begin{align}
  x &= x' - \frac{h_1}{p-1} \theta', \\
  \theta &= -\frac{h_2}{q-1} x' + \left(1 - \frac{h_1 h_2}{(p-1)(q-1)}\right) \theta'.
\end{align}

Using relation (1), it is easy to verify that

\begin{align}
  x \theta &= q \theta x + h_2 x^2,
\end{align}

where the new deformation parameter $h_2$ commutes with the coordinate $x$ and anticommutes with the coordinate $\theta$. Similarly, from (1a) one obtains

\begin{align}
  \theta^2 &= -h_2 \theta x,
\end{align}

where

\begin{align}
  h_1 h_2 &= -h_2 h_1, \\
  h_1^2 &= 0 = h_2^2.
\end{align}

That is, the new deformation parameters $h_1$ and $h_2$ are odd (Grassmann) numbers which anticommute. Note that although in the $p \to 1$, $q \to 1$ limits the transformation (2) is ill behaved, the resulting commutation relations are well defined.

The relations (3) define a new deformation,\textsuperscript{11} which we called the $h_2$-deformation of the algebra of coordinate functions on the Manin superplane generated by $x$ and $\theta$ in the limit $q \to 1$, and will be denoted by $A_{h_2}$.

Differential calculus on the quantum superplane $A_{h_2}$ requires the introduction of differentials $dx$, $d\theta$. The complete framework also includes the commutation relations of these differentials with the coordinates and derivatives.

\textbf{B. Relations between coordinates and differentials}
To establish a noncommutative differential calculus on the quantum superplane $A_{h2}$, we assume that the commutation relations between the coordinates and their differentials have the following form:

$$x'dx' = A dx'x', \quad x'd\theta' = C_{11} d\theta'x' + C_{12} dx'\theta', \quad (5)$$

$$\theta'dx' = C_{21} dx'\theta' + C_{22} d\theta'x', \quad \theta'd\theta' = B d\theta'\theta'.$$

Now we would like to transform these relations to unprimed quantities to determine the coefficients $A$, $B$ and $C_{ij}$. We first introduce the exterior differential $d$.

The exterior differential $d$ is an operator which gives the mapping from the coordinates to the differentials

$$d : Z^i \longrightarrow dZ^i, \quad (6)$$

where $Z^1 = x$, $Z^2 = \theta$, $dZ^1 = dx$ and $dZ^2 = d\theta$. We demand that the exterior differential $d$ has to satisfy two properties: the nilpotency

$$d^2 = 0, \quad (7)$$

and the graded Leibniz rule

$$d(FG) = (dF)G + (-1)^{\hat{F}} F(dG), \quad (8)$$

where $\hat{F}$ is the Grassmann degree of $F$, that is, $\hat{F} = 0$ for even variables and $\hat{F} = 1$ for odd variables. We wish to substitute into (5) the differentials $dx'$ and $d\theta'$ together with the coordinates $x'$ and $\theta'$. The deformation parameters $h_1$ and $h_2$ are both odd numbers and the exterior differential $d$ is also odd. Therefore the action of the exterior differential $d$ on $\alpha u$ is defined by

$$d(\alpha u) = (-1)^{\hat{\alpha}} \alpha du, \quad (9)$$

where $\alpha$ is a number (even or odd) and $u$ is a coordinate of superplane. So we can write from (2)

$$dx = dx' + \frac{h_1}{p-1}d\theta', \quad \quad d\theta = \frac{h_2}{q-1}dx' + \left(1 - \frac{h_1 h_2}{(p-1)(q-1)}\right)d\theta'. \quad (10)$$

Note that if we consider $x$ and $\theta$ as functions of two variables (say $x'$ and $\theta'$) and differentiate (2), as usual, then we do not obtain the expressions in (10). To obtain (10) one must take the differential from the left in Eq. (2). In the Appendix, we explain this in detail.
We now substitute (2) and (10) into (5) which are not explicitly written here. It will be calculated the coefficients $A$, $B$ and $C_{ij}$. We first assume that
\[ \frac{dx'd\theta'}{d\theta'dx'} = p^{-1}d\theta'dx', \quad (dx')^2 = 0. \] (11)

Then we have
\[ d\theta dx = p dx d\theta - h_1 (d\theta)^2, \] (12a)
and
\[ (dx)^2 = h_1 dx d\theta. \] (12b)

Consequently, the coefficients are determined as follows:
\[ A \text{ undetermined, } B = 1, \]
\[ C_{11} = q, \quad C_{12} = pq - 1, \quad C_{21} = 0, \quad C_{22} = -p. \] (13)

Here we shall choose $A$ equal to $pq$ since the relations are then well defined.

C. Relations of derivatives and coordinates

In this section we shall define the derivatives and find the commutation relations of derivatives with coordinates and the commutation relations between derivatives. We first introduce the matrix
\[ g = \begin{pmatrix} 1 + h_1 h_2/(p - 1)(q - 1) & h_1/(p - 1) \\ h_2/(q - 1) & 1 \end{pmatrix}. \] (14)

It is easy to verify that the matrix $g$ is a supermatrix. Thus we can write the transformation in (2) of the form
\[ Z' = gZ, \quad Z' = \begin{pmatrix} x' \\ \theta' \end{pmatrix}. \] (15)

Let us denote the partial derivatives with respect to $x'$ and $\theta'$ by
\[ \partial_x' = \frac{\partial}{\partial x'}, \quad \partial_{\theta'} = \frac{\partial}{\partial \theta'}. \] (19)
respectively. The transformation law of the partial derivatives is then defined by
\[ \partial' = (g^{st})^{-1} \partial, \quad \partial = \begin{pmatrix} \partial_x \\ \partial_{\theta} \end{pmatrix} \] (16)
where $g^{st}$ denotes the supertranspose of $g$. Explicitly
\[ \partial_x' = \partial_x - \frac{h_2}{q - 1} \partial_{\theta}, \quad \partial_{\theta'} = \frac{h_1}{p - 1} \partial_x + \left(1 - \frac{h_1 h_2}{(p - 1)(q - 1)}\right) \partial_{\theta}. \] (17)
Note that, when one demands the validity of the chain rule, to obtain the expressions in (17) it must be assumed that the derivatives act from the left on the transformed variables. This case will also be explained in detail in the Appendix.

We know that the exterior differential $d$ is defined by
\[ d = dx' \partial_{x'} + d\theta' \partial_{\theta'}. \]

Substituting (10) and (17) into (18a) one obtains
\[ d = d x \partial_{x} + d \theta \partial_{\theta}, \]
that is, $d$ preserves its form. So, since
\[ dF(x, \theta) = dx\partial_{x}F + d\theta\partial_{\theta}F, \]
for any function $F$, replacing $F$ with $xF$ and $\theta F$ we get the following relations:
\[ \partial_{x}x = 1 + px\partial_{x} + h_{1}\partial_{x}x + h_{2}(x\partial_{x} + \theta\partial_{\theta}) + (pq - 1)\theta\partial_{\theta}, \]
\[ \partial_{x}\theta = p\theta\partial_{x} - ph_{2}(x\partial_{x} + \theta\partial_{\theta}), \]
\[ \partial_{\theta}x = qx\partial_{\theta} - qh_{1}(x\partial_{x} + \theta\partial_{\theta}), \]
\[ \partial_{\theta}\theta = 1 - \theta\partial_{\theta} + h_{1}\partial_{x}x + h_{2}(x\partial_{x} + \theta\partial_{\theta}). \]

We now find the commutation rules between derivatives. These rules can be easily obtained by using the nilpotency of the exterior differential. Thus we write
\[ 0 = d^{2} = dx\theta p\partial_{x}\partial_{\theta} - \partial_{\theta}x\partial_{\theta} + h_{1}\partial_{x}^{2} + (d\theta)^{2}(\partial_{\theta}^{2} - h_{1}\partial_{x}\partial_{\theta}) \]
which says that
\[ \partial_{\theta}\partial_{x} = p\partial_{x}\partial_{\theta} + h_{1}\partial_{x}^{2}, \quad \partial_{\theta}^{2} = h_{1}\partial_{x}\partial_{\theta}. \]

The complete framework of the differential calculus requires commutation relations of the differentials with derivatives.

**D. Relations of differentials with derivatives**

Finally we shall find the commutation relations between differentials and derivatives. We assume that they have the following form in terms of primed quantities:
\[ \partial_{x'}dx' = A_{11}dx'\partial_{x'} + A_{12}d\theta'\partial_{\theta'}, \]
\[ \partial_{x'}d\theta' = A_{21}d\theta'\partial_{x'} + A_{22}dx'\partial_{\theta'}. \]
\[ \partial_{\psi} d\psi' = B_{11} d\psi' \partial_{\psi} + B_{12} d\theta' \partial_{\psi}, \]
\[ \partial_{\psi} d\theta' = B_{21} d\theta' \partial_{\psi} + B_{22} d\theta' \partial_{\psi}. \]

Substituting (10) and (17) into (22) and using
\[ d(d \psi) = -d(\psi) d, \quad d(d \theta) = (d \theta) d, \quad (23a) \]
and the relation
\[ \partial_i(X^j dX^k) = \delta^i_j \delta_i^k dX^k, \quad (23b) \]
where \( \partial_1 = \partial_x \) and \( \partial_2 = \partial_{\theta} \), we determine the coefficients \( A_{ij} \) and \( B_{ij} \). So one has
\[ \partial_x d\psi = pq d\psi + h_1 d\theta \partial_x + h_2 d\psi \partial_{\theta} + h_1 h_2 (d\psi \partial_x + d\theta \partial_{\theta}) + (pq - 1) d\theta \partial_{\theta}, \]
\[ \partial_x d\theta = pd \theta \partial_x + ph_2 (d\psi \partial_x + d\theta \partial_{\theta}), \]
\[ \partial \theta d\psi = -q d\psi \partial_{\theta} - q h_1 (d\psi \partial_x + d\theta \partial_{\theta}), \quad (24) \]
\[ \partial_{\theta} d\theta = d\theta \partial_{\theta} - h_1 d\theta \partial_x + h_2 d\psi \partial_{\theta} + h_1 h_2 (d\psi \partial_x + d\theta). \]

E. Algebra of one-forms

In this section we shall define two one-forms using the generators of \( \mathcal{A} \) and find the commutation relations of one-forms.

If we call them \( w \) and \( u \) then one can define them as follows:
\[ w = d\psi x^{-1}, \quad u = d\theta x^{-1} - d\psi^{-1} \theta x^{-1}. \quad (25) \]
We denote the algebra of one-forms generated by two elements \( w \) and \( u \) by \( \Omega \). The generators of the algebra \( \Omega \) with the generators of \( \mathcal{A} \) satisfy the following relations:
\[ xw = wx - h_1 ux, \quad \theta w = -w \theta + h_1 u \theta, \]
\[ xu = ux, \quad \theta u = u \theta - h_2 (w \theta + ux). \quad (26) \]

The commutation rules of the generators of \( \Omega \) are
\[ w^2 = 0, \quad wu = uw. \quad (27) \]

Using (18b) and (25), if we define the operators \( T \) and \( \nabla \) as
\[ T = x \partial_x + \theta \partial_{\theta}, \quad \nabla = x \partial_{\theta}, \quad (28) \]
then we have
\[ T \nabla = \nabla T, \quad \nabla^2 = 0, \quad (29) \]
as a subalgebra of $\mathfrak{gl}(1|1)$.

The action of $T$ and $\nabla$ on the generators $x$ and $\theta$ is

\begin{align*}
T x &= x + xT, & \nabla x &= x \nabla - h_1 xT, \\
T \theta &= \theta + \theta T, & \nabla \theta &= x - \theta \nabla + h_1 \theta T.
\end{align*}

(30)
III. THE SUPERGROUP \(GL_{h_1,h_2}(1|1)\) AND COVARIANCE

It is well known that the quantum supergroup \(GL_{p,q}(1|1)\) acts as a linear transformation on the quantum superplane, preserves (1) and the dual relations

\[
\varphi'^2 = 0, \quad \varphi'y' - p^{-1}y'\varphi' = 0. \tag{31}
\]

In extending this property of covariance under the coaction of \(GL_{p,q}(1|1)\), from the superplane to its calculus, it will be assumed that the deformed group structure implies and is implied by invariance of the intermediary relations (5) under linear transformations of the quantum superplane. In the present work, this will be applied to the \((h_1, h_2)\)-deformed superplane.

In this section we would like to discuss the meaning of covariance in a graded version of noncommutative differential calculus of Wess-Zumino\(^3\) for the two-parameter case. Before proceeding, we define the dual quantum \(h\)-superplane.

To define the dual quantum superplane, we interpret the differentials \(dx\) and \(d\theta\), as the coordinates of the dual superplane, as follows

\[
dx = \varphi, \quad d\theta = y. \tag{32}
\]

Now the quantum dual \(h\)-superplane generated by \(y, \varphi\) with the relations (12) in the limit \(p \rightarrow 1\) will be denoted by \(dA_{h_1}\). If we assume that \(A_{h_2}\) and \(dA_{h_1}\) have to be covariant under the coaction

\[
\delta(x) = a \otimes x + \beta \otimes \theta, \quad \delta(\theta) = \gamma \otimes x + d \otimes \theta, \tag{33a}
\]

\[
\delta(dZ) = (\tau \otimes d)\delta(Z), \quad \tau(u) = (-1)^u u \tag{33b}
\]

and that \(\beta, \gamma\) anti-commute with \(\theta, \varphi, h_1\) and \(h_2\) we get the corresponding \((h_1, h_2)\)-deformation of the supergroup \(GL(1|1)\) as a quantum matrix supergroup \(GL_{h_1,h_2}(1|1)\) generated by \(a, \beta, \gamma, d\) with the relations\(^{10}\)

\[
a\beta = \beta a - h_1(a^2 - \beta\gamma - da), \quad d\beta = \beta d + h_1(d^2 + \beta\gamma - da),
\]

\[
a\gamma = \gamma a + h_2(a^2 + \beta\gamma - ad), \quad d\gamma = \gamma d - h_2(d^2 - \gamma\beta - da),
\]

\[
\beta^2 = h_1\beta(a - d), \quad \gamma^2 = h_2\gamma(d - a),
\]

\[
\beta\gamma = -\gamma\beta + (h_1\gamma - h_2\beta)(a - d), \tag{34}
\]

\[
ad = da + h_1(a - d)\gamma + h_2\beta(a - d),
\]

where

\[
D = ad^{-1} - \beta d^{-1}\gamma d^{-1} = d^{-1}a - d^{-1}\beta d^{-1}\gamma.
\]
The two-parameter differential calculus on the quantum superplane is explicitly as follows:

The commutation relations of variables and their differentials are
\[
x\theta = \theta x + h_2 x^2, \quad \theta^2 = -h_2 \theta x,
\]
\[
\varphi y = y \varphi + h_1 y^2, \quad \varphi^2 = h_1 \varphi y.
\]
Note that the last two relations of (35) are obtained from (14) and (15). However, they can also be obtained from (31) with the limits \( p \to 1, q \to 1 \).

The commutation relations between variables and derivatives are
\[
\partial_x x = 1 + x \partial_x - h_1 \theta \partial_x + h_2 x \partial_\theta + h_1 h_2 (x \partial_x + \theta \partial_\theta),
\]
\[
\partial_x \theta = \theta \partial_x - h_2 (x \partial_x + \theta \partial_\theta),
\]
\[
\partial_\theta x = x \partial_\theta - h_1 (x \partial_x + \theta \partial_\theta),
\]
\[
\partial_\theta \theta = 1 - \theta \partial_\theta - h_1 \theta \partial_x + h_2 x \partial_\theta + h_1 h_2 (x \partial_x + \theta \partial_\theta),
\]
and those among the derivatives are
\[
\partial_x \partial_\theta = \partial_\theta \partial_x - h_1 \partial_x^2, \quad \partial_\theta^2 = h_1 \partial_\theta \partial_x.
\]

The commutation relations of variables with their differentials are
\[
x \varphi = \varphi x + h_1 (\varphi \theta - xy) + h_1 h_2 \varphi x,
\]
\[
xy = xy - h_1 y \theta - h_2 \varphi x + h_1 h_2 \varphi \theta,
\]
\[
\theta \varphi = -\varphi \theta + h_1 y \theta - h_2 \varphi x - h_1 h_2 xy,
\]
\[
\theta y = y \theta - h_2 (\varphi \theta + xy) - h_1 h_2 y \theta.
\]

The commutation relations between derivatives and differentials are
\[
\partial_x \varphi = \varphi \partial_x + h_1 y \partial_x - h_2 \varphi \partial_\theta + h_1 h_2 (\varphi \partial_x + y \partial_\theta),
\]
\[
\partial_x y = y \partial_x + h_2 (\varphi \partial_x + y \partial_\theta),
\]
\[
\partial_\theta \varphi = -\varphi \partial_\theta - h_1 (\varphi \partial_x + y \partial_\theta),
\]
\[
\partial_\theta y = y \partial_\theta - h_1 y \partial_x + h_2 \varphi \partial_\theta + h_1 h_2 (\varphi \partial_x + y \partial_\theta).
\]
Note that this calculus goes back to those of Ref. 9 when \( h_1 = 0 \) and \( h_2 = h \). This calculus is slightly different from Ref. 10. The reason for this difference is the use of
commutation relations of the dual exterior superplane in Ref. 10 instead of the dual
superplane in this work.

We now discuss the covariance of the differential calculus. The covariance here
means that all the relations between coordinates $x, \theta$, differentials $dx, d\theta$ and deriva-
tives $\partial_x, \partial_\theta$, etc. must preserve their form when one changes the coordinates by

$$x \rightarrow ax + \beta \theta, \quad \theta \rightarrow \gamma x + d\theta,$$

(40)

where the matrix $T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ is an element of the quantum supergroup GL$_{h_1,h_2}(1|1)$
acting on the quantum superplane. We must change the differentials by

$$dx \rightarrow adx - \beta d\theta, \quad d\theta \rightarrow -\gamma dx + dd\theta,$$

(41)

since the odd objects anti-commute among themselves. Covariance can be maintained
if one defines the transformation law of the partial derivatives as folllows

$$\partial_x \rightarrow (a^{-1} - a^{-1}\gamma d^{-1}\beta a^{-1})\partial_x - a^{-1}\gamma d^{-1}\partial_\theta,$$

$$\partial_\theta \rightarrow (d^{-1} - d^{-1}\beta a^{-1}\gamma d^{-1})\partial_\theta + d^{-1}\beta a^{-1}\partial_x.$$

(42)

IV. A TWO-PARAMETER DEFORMATION OF CLASSICAL PHASE
SPACE

We shall now give a two-parameter deformation of the (1 + 1)-dimensional classical
phase space. We denote the algebra (35) – (37) generated by coordinates $x, \theta$ and
the derivatives $\partial_x$ and $\partial_\theta$ by $\mathcal{B}_{h_1,h_2}$. It is interesting to note that simply identifying
$\partial_x$ and $\partial_\theta$ with $ip_x$ and $p_\theta$ is not compatible with the hermiticity of coordinates and
momenta. To identify $\partial_x$ and $\partial_\theta$ with the momenta $i p_x$ and $p_\theta$, one must take care
of the hermiticity of the coordinates and momenta. To this end, we first define the
hermitean conjugation of the coordinates $x$ and $\theta$, respectively, as

$$x^+ = (1 + 2h_1h_2)x + 2h_1\theta, \quad \theta^+ = (1 - 2h_1h_2)\theta + 2h_2x.$$  

(43)

It is then easy to see that the hermiticity of $x^+$ and $\theta^+$ impose some condition on
the deformation parameters, i.e., $h_1$ is a real parameter and $h_2$ is a pure imaginary
parameter:

$$\bar{h}_1 = h_1, \quad \bar{h}_2 = -h_2,$$

(44)

where the bar denotes complex conjugation. In this case, the hermitean conjugation
of the derivatives $\partial_x$ and $\partial_\theta$ are

$$\partial_x^+ = -(1 + 2h_1h_2)\partial_x + 2h_2\partial_\theta, \quad \partial_\theta^+ = (1 - 2h_1h_2)\partial_\theta + 2h_1\partial_x.$$

(45)
In the $h_1 \to 0, h_2 \to 0$ limits the definitions (43) and (45) go back to those of the classical case.

The relations (35) – (37) are now invariant under the transformations (43) and (45). The above involution allows us to define the hermitean operators

$$\hat{x} = (1 + h_1 h_2)x + h_1 \theta, \quad \hat{\theta} = (1 - h_1 h_2)\theta + h_2 x,$$

and, as bosonic and fermionic momenta,

$$\hat{p}_x = i[(1 + h_1 h_2)\partial_x - h_2 \partial_\theta], \quad \hat{p}_\theta = (1 - h_1 h_2)\partial_\theta + h_1 \partial_x.$$

The final form of the $(h_1, h_2)$-deformed phase space algebra is

$$\hat{x} \hat{\theta} = \hat{\theta} \hat{x} + h_2 \hat{x}^2, \quad \hat{\theta}^2 = -h_2 \hat{x},$$

$$\hat{p}_x \hat{p}_\theta = \hat{p}_\theta \hat{p}_x + i h_1 \hat{p}_x^2, \quad \hat{p}_\theta^2 = -i h_1 \hat{p}_x \hat{p}_\theta,$$

$$\hat{p}_x \hat{x} = i + \hat{x} \hat{p}_x + i h_2 \hat{x} \hat{p}_\theta - h_1 \hat{\theta} \hat{p}_x + h_1 h_2 (1 + \hat{x} \hat{p}_x + i \hat{\theta} \hat{p}_\theta),$$

$$\hat{p}_x \hat{\theta} = \hat{\theta} \hat{p}_x - h_2 (\hat{x} \hat{p}_x + i \hat{\theta} \hat{p}_\theta),$$

$$\hat{p}_\theta \hat{\theta} = 1 - \hat{\theta} \hat{p}_\theta + h_2 \hat{x} \hat{p}_\theta + i h_1 \hat{\theta} \hat{p}_x - h_1 h_2 (1 + i \hat{x} \hat{p}_x - \hat{\theta} \hat{p}_\theta).$$

This gives a $(h_1, h_2)$-deformed phase space algebra which may be used to study the $(1 + 1)$-dimensional quantum phase space.

Note that we can derive a deformed super-Clifford algebra from the phase space algebra as follows: suppose that we define gamma matrices

$$\gamma^1 \equiv \hat{p}_\theta, \quad \gamma^2 \equiv \hat{\theta}, \quad c^1 \equiv \hat{p}_x, \quad c^2 \equiv \hat{x}.$$  

Then, they satisfy super-Clifford algebra

$$c^1 c^2 = c^2 c^1 - h_1 \gamma^2 c^1 + i(1 + h_2 c^2 \gamma^1) + h_1 h_2 (1 + \gamma^2 \gamma^1 + c^2 c^1),$$

$$c^1 \gamma^2 = \gamma^2 c^1 - h_2 (c^2 c^1 + i \gamma^2 \gamma^1),$$

$$\gamma^1 c^1 = c^1 \gamma^1 - i h_1 (c^1)^2,$$

$$\gamma^1 c^2 = c^2 \gamma^1 - h_1 (\gamma^2 \gamma^1 - i c^2 c^1),$$

$$\gamma^1 \gamma^2 = 1 - \gamma^2 \gamma^1 + i h_1 \gamma^2 c^1 + h_2 c^2 \gamma^1 - h_1 h_2 (1 + c^2 c^1 - \gamma^2 \gamma^1),$$

$$\gamma^1 c^1 = i h_1 \gamma^1 c^1, \quad \gamma^2 c^2 = -h_2 \gamma^2 c^2,$$

$$\gamma^1 c^2 = c^2 \gamma^1 - h_2 c^2.$$
V. A COMMENT ON SUPEROSCILLATORS

We know that introducing one 'bosonic' and one 'fermionic' oscillator, \( A \) and \( B \), respectively, and making the usual identification

\[
x' \leftrightarrow A^+, \quad \theta' \leftrightarrow B^+,
\]

\[
\partial_{x'} \leftrightarrow A, \quad \partial_{\theta'} \leftrightarrow B,
\]

one constructs the quantum super-oscillator algebra which is covariant under the quantum supergroup \( \text{GL}_{p,q}(1|1) \). Under identification (2) and (17) one has

\[
x \leftrightarrow A^+ - \frac{h_1}{p-1}B^+, \quad \partial_{x} \leftrightarrow \left(1 + \frac{h_1 h_2}{(p-1)(q-1)}\right) A + \frac{h_2}{q-1} B, \]

\[
\theta \leftrightarrow \left(1 - \frac{h_1 h_2}{(p-1)(q-1)}\right) B^+ - \frac{h_2}{q-1} A^+, \quad \partial_{\theta} \leftrightarrow B - \frac{h_1}{p-1} A, \quad (52)
\]

where

\[
p = q. \quad (53)
\]

Substituting (52) into (20) and (3), surprisingly all \((h_1, h_2)\)-dependence cancels and one obtains the usual \((p, q)\)-deformed super-oscillator algebra\(^{12}\)

\[
AA^+ = 1 + pqA^+ A + (pq - 1)B^+ B, \\
BB^+ = 1 - B^+ B, \quad B^2 = 0 = B^{+2}, \\
AB^+ = pB^+ A, \quad AB = p^{-1}BA, \\
A^+ B = q^{-1}BA^+, \quad A^+ B^+ = qB^+ A^+, \quad (54)
\]

In the \( p \rightarrow 1, q \rightarrow 1 \) limits, we get undeformed super-oscillator algebra.

VI. APPENDIX

In this Appendix, we show that a two-parameter covariant differential calculus on the quantum \( h \)-superplane can be constructed only if the derivatives and differentials act from the left.

Consider the change of coordinates which is given by (2)

\[
x' = \left(1 + \frac{h_1 h_2}{(p-1)(q-1)}\right) x + \frac{h_1}{p-1}\theta, \quad \theta' = \theta + \frac{h_2}{q-1} x. \quad (A1)
\]

If we interpret the symbols \( dx \) and \( d\theta \) as differentials acting from the right and demanding the validity of the chain rule, we have

\[
dx = dx' \frac{\partial x}{\partial x'} + d\theta' \frac{\partial x}{\partial \theta'} = dx' + d\theta' \left(-\frac{h_1}{p-1}\right) = dx' - \frac{h_1}{p-1} d\theta' \quad (A2)
\]
and
\[ d\theta = \frac{h_2}{q-1} dx' + \left( 1 + \frac{h_1 h_2}{(p-1)(q-1)} \right) d\theta'. \] (A3)

Therefore, for example,
\[ dx' = \left( 1 + \frac{h_1 h_2}{(p-1)(q-1)} \right) dx + \frac{h_1}{p-1} d\theta \] (A4)

so that
\[
\text{RHS of (A4) } = \left( 1 + \frac{h_1 h_2}{(p-1)(q-1)} \right) dx' - \frac{h_1}{p-1} d\theta' + \frac{h_1}{p-1} d\theta' + \frac{h_1 h_2}{(p-1)(q-1)} dx' \neq dx'.
\]

Similarly, if we write, from the chain rule,
\[ \partial_x = \left( 1 + \frac{h_1 h_2}{(p-1)(q-1)} \right) \partial_{x'} + \frac{h_2}{q-1} \partial_{\theta'}, \] (A5)

and
\[ \partial_\theta = \frac{h_1}{p-1} \partial_{x'} + \partial_{\theta'}, \] (A6)

then
\[ \partial_{x'} = \partial_x - \frac{h_2}{q-1} \partial_\theta, \quad \partial_{\theta'} = -\frac{h_1}{p-1} \partial_x + \left( 1 - \frac{h_1 h_2}{(p-1)(q-1)} \right) \partial_\theta, \] (A7)

so that, for example
\[
\text{RHS of (A6) } = \left( 1 - 2 \frac{h_1 h_2}{(p-1)(q-1)} \right) \partial_\theta \neq \partial_\theta.
\]

This asymmetry between right and left derivative and differential for transformed variables stems from the matrix \( g \) in (14) which off diagonal elements are odd. That is, \( g \) is a supermatrix and so the supertranspose must be used.

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