Reaching the optomechanical strong-coupling regime with a single atom in a cavity

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A major goal within the field of optomechanics is to achieve the single-photon strong-coupling regime, wherein even a mechanical displacement as small as the zero-point uncertainty is enough to shift an optical cavity resonance by more than its linewidth. This goal is difficult, however, due to the small zero-point motion of conventional mechanical systems. Here, we show that an atom trapped in and coupled to a cavity constitutes an attractive platform for realizing this regime. In particular, while many experiments focus on achieving strong coupling between a photon and the atomic internal degree of freedom, this same resource also naturally enables one to observe optomechanical strong coupling, in combination with the low mass of an atom and the isolation of its motion from a thermal environment. As an example, we show that an optomechanically induced photon blockade can be realized in realistic setups, and provide signatures of how this effect can be distinguished from the conventional Jaynes-Cummings blockade associated with the two-level nature of the atomic transition.

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I. INTRODUCTION

Spectacular advances in optomechanics now allow quantum control over the interaction between photons and phonons [1]. In experiments thus far, a large classical field drives the system, around which the light-motion interaction can be linearized. This linear interaction enables many applications, ranging from laser cooling of the motion to its ground state [2,3], squeezed light generation [4,5], sensing [6,7], microwave-to-optical conversion [8,9], and nonreciprocal optical devices [10,11]. However, the interaction between photons and phonons is intrinsically nonlinear. These nonlinear effects can be observed in strongly driven systems (e.g., self-sustained oscillations [12–15]). A particularly interesting limit to consider is when this nonlinearity manifests itself at the level of single-photon inputs [16,17], leading to the ability to generate highly non-Gaussian states. To reach this regime, a zero-point mechanical displacement should shift the frequency of the optical resonator by an amount comparable to its linewidth, which is difficult due to the large mass of conventional mechanical elements and the implied small zero-point motion. While a number of schemes have been proposed to attain optomechanical strong coupling [18–20], this regime has yet to be experimentally demonstrated. Thus, finding a platform where this regime can be explored constitutes a major goal.

Separately, much work focuses on coupling single neutral atoms [21–24] or ions [25–31] to high-finesse cavities. Here, the primary motivation is to exploit the two-level atomic structure to generate nonclassical fields. However, in this article, we argue that such systems also constitute a natural platform to reach the strongly interacting limit of optomechanics. In particular, experiments [21,22,32,33] now routinely reach the strong-coupling regime of cavity QED, wherein an atom maximally coupled to the cavity (in an antinode) shifts the bare cavity frequency by more than a linewidth. Moving the atom by a quarter-wavelength to a node eliminates this shift. Thus, a zero-point motion on the order of a fractional wavelength is sufficient to attain optomechanical strong coupling, which is easily achievable given the light single-atom mass. In addition, the motion of the atom is effectively isolated from a thermal environment, providing the coherence times necessary to produce interesting quantum behavior. As a specific example, we show theoretically that one can observe an optomechanically induced photon blockade [16] in realistic cavity QED setups, where a nonclassical antibunched field is produced as the system is unable to transmit more than a single photon at a time. We also describe how this optomechanical behavior can be clearly distinguished from, and dominate over, the usual antibunching associated with the two-level nature of the atom. The explicit use of the strong-coupling regime of cavity QED to attain novel regimes of optomechanics, and the examination of the resulting nonclassical statistics of the outgoing field, distinguish the present work from previous experiments that explored optomechanical effects with atomic ensembles in cavities [32–34].

II. OPTOMECHANICAL PHOTON BLOCKADE

We begin by reviewing the phenomenon of photon blockade in a conventional optomechanical system. We focus on the system shown in Fig. 1(a), where a mechanical element such as a trapped particle [35–39] or membrane [40] can be positioned arbitrarily, and couples to a single standing-wave optical mode of a Fabry-Perot cavity. For small displacements of the mechanical degree of freedom around the equilibrium position $x_0$, the cavity frequency is given by
\[ \omega_c(x) \approx \omega_c(x_0) + \omega_1(x_0)(x - x_0). \]

The total Hamiltonian of the system, including a coherent external driving field, is given in a frame rotating with the laser frequency \( \omega_L \) by

\[
H_{\text{op}} = \omega_{m}b^\dagger b - \left[ \omega_L - \omega_c(x_0) + i \frac{\kappa}{2} \right] a^\dagger a
+ g_m(b + b^\dagger)a^\dagger a + \sqrt{\frac{\kappa}{2}}E_0(a^\dagger + a). \tag{1}
\]

Here, \( \omega_m \) is the frequency of the vibrational mode, and \( a \) and \( b \) denote the photon and phonon annihilation operators, respectively. The quantity \( \omega_L - \omega_c(x_0) \) is the detuning between laser frequency \( \omega_L \) and the cavity frequency \( \omega_c(x_0) \) when the mechanical system lies at its equilibrium position. Each cavity mirror has a decay rate of \( \kappa/2 \) into outgoing radiation, while the left side also serves as the source of injection of a coherent state into the cavity with photon number flux \( E_0^2 \). The position-dependent cavity shift described previously has been rewritten in terms of phonon operators as \( \omega_c'(x_0)(x - x_0) = g_m(b + b^\dagger) \) where \( g_m = \omega_c(x_0)x_{\text{zp}} \) is the single-photon-phonon coupling strength and \( x_{\text{zp}} = \sqrt{\frac{\kappa}{2}}(\Delta\omega_L)^{1/2} \) is the zero-point motional uncertainty (\( \Delta\omega_L = \Delta\omega_{\text{eff}}/\omega_m \) being the effective mass). The cubic interaction term \( (b + b^\dagger)a^\dagger a \) gives rise to nonlinear equations of motion, but quantum signatures have not been observed, as the best ratio of coupling strength to linewidth so far is \( g_m/\kappa \approx 10^{-2} \) [19,41]. Thus, current experiments remain in the so-called optomechanical weak-coupling regime, where many photons inside the optical mode are required to see an appreciable interaction, and allowing for linearization around the strong classical cavity field. However, here we will focus on the regime where this linearization breaks down and the nonlinear nature of the optomechanical coupling manifests itself via photon coincidence measurements [16].

To quantify the optomechanical nonlinearity we change into a displaced oscillator representation, which diagonalizes \( H_{\text{op}} \) in the limit of weak driving [16]. The eigenvalues as \( E_0 \rightarrow 0 \) can then be written as \( E_{n,m} = m\omega_m + n\omega_1(x_0) - \frac{\kappa}{\omega_m}n^2 \omega_m^2 \) and correspond to the (displaced) eigenstates \( |n,m\rangle \). The spectrum is shown in Fig. 1(b). If the laser frequency is resonant with the transition \( |0,0\rangle \rightarrow |1,0\rangle \) [zero phonon line (ZPL)], then the transition for the second photon is off resonant from the transition \( |1,0\rangle \rightarrow |2,0\rangle \) by an amount \( E_{2,0} - 2E_{1,0} = -2g_m^2/\omega_m \). In order to have a substantial effect, this anharmonicity should be resolvable, \( g_m^2/\omega_m \gg \kappa \), and furthermore, one should operate in the sideband-resolved regime \( \omega_m \gg \kappa \) so that transitions to other motional states, e.g., the first phonon sideband \( |0,0\rangle \rightarrow |1,1\rangle \), are suppressed. These requirements for antibunching can also be observed in Fig. 1(c), where we have plotted the second-order correlation function \( g^{(2)}(0) \) of the transmitted field given a weak coherent state input for different values of \( \kappa \) and \( g_m \), taking the laser frequency \( \omega_L \) as being resonant with the ZPL (see Appendix A for details of the calculation). A value of \( g^{(2)}(0) < 1 \) indicates nonclassical antibunching, and a minimum value occurs around \( g_m \approx 0.5\omega_m \), which for well-resolved sidebands decreases as \( g^{(2)}(0) \approx 20k/(\omega_m)^2 \). One also sees that increasing the ratio \( g_m/\omega_m \) further does not improve the amount of antibunching, due to the possibility of resonantly coupling to other excited states. For example, at \( g_m/\omega_m \approx 1/\sqrt{2} \), the reduced antibunching arises as a second photon can resonantly excite the state \( |2,1\rangle \), since \( E_{2,0} - 2E_{1,0} = -\omega_m \).

While mathematically the degree of antibunching is determined by the parameters \( g_m, \omega_m, \kappa \), it will also be helpful to “visualize” how the antibunching changes as the equilibrium position \( x_0 \) is scanned from a cavity antinode to node, to provide a useful comparison with atoms later. For a weak dielectric perturbation such as a thin membrane, intuitively one expects that the variation in the cavity frequency follows the intensity profile of the standing wave itself, \( \delta\omega_L(x) \propto \cos^2(k_x x) \) [42,43]. It follows then that \( g_m(x_0) = g_m(x_0) = g_m(x_0) \) vanishes at a node or antinode, and reaches the maximum possible value of \( g_m/\omega_m \) halfway between. In Fig. 1(d) we plot \( g^{(2)}(0) \) as a function of trapping position \( x_0 \) and detuning from the empty cavity \( \delta_0 = \omega_L - \omega_c \) for a mechanical system initially in its ground state. The dashed red (lower) line corresponds to a driving laser resonant with the ZPL, which requires the laser frequency to be tuned following the energy eigenvalue \( E_{1,0} \). In addition to the features along the ZPL,
antibunching can also be observed when a motional sideband 
$|1_x,m\rangle$ is resonantly driven, following the equation $\omega_L \approx E_{1,m}$ 
(see black dashed curve for $m = 1$). Below, we plot $g^{(2)}(0)$ fol-
lowing the ZPL (red, dashed). The oscillations in $g^{(2)}(0)$ along 
the ZPL versus $x_0$ occur as $g_m(x_0)$ sweeps into and away from 
the optimal values for antibunching [compare with Fig. 1(c)]. 
Here, we have chosen parameters of $g_m = 2\pi \times 0.16$ MHz,
$\kappa = 2\pi \times 0.02$ MHz, and $\omega_m = 2\pi \times 0.2$ MHz. These do not 
necessarily correspond to a practically realizable optome-
chanical system, but allow the interesting features to be observed.

III. CAVITY QED WITHOUT MOTION

We now consider an atom coupled to a cavity mode with 
amplitude $u(x) = \cos(kx)$ [see Fig. 2(a)], which is described 
by the Jaynes-Cummings (JC) Hamiltonian [44]. Due to the 
two-level nature of the atom, the spectrum of the JC Hamil-
tonian is nonlinear. We thus study the effect of this nonlinearity 
on $g^{(2)}(0)$ first without motion (i.e., the atom is infinitely tightly 
trapped), so that we can later clearly distinguish motional 
effects. The JC Hamiltonian, in an interaction picture rotating 
at $\omega_L$, is given by

$$H_{JC} = -\left(\delta_0 + i\frac{\gamma}{2}\right)\sigma_{ee} - \left(\delta_c + i\frac{\kappa}{2}\right)a^\dagger a$$

$$+ \sqrt{\frac{g_0}{2}}(a^\dagger + a)\sigma_{ee} + H.c..$$

The laser-atom detuning is $\delta_0 = \omega_L - \omega_0$, while $\omega_0$ being the resonance 
frequency of the atom, while $\sigma_{ee} = |\alpha\rangle\langle\beta|$, where $\alpha, \beta = e, g$ correspond to combinations of the atomic ground 
and excited states. As before, $\delta_c = \omega_c - \omega_c$ is the detuning 
relative to the bare cavity resonance. The atom-cavity coupling 
strength $g_0u(x_0)$ depends on the trapping position $x_0$, where $g_0$ is the magnitude of the vacuum Rabi splitting at the antinode 
at the cavity waist. The emission rate of an excited atom into 
free space is given by $\gamma$.

![FIG. 2. Cavity QED without motion. (a) Schematic of an atom 
infinity tightly trapped inside a cavity mode at position $x_0$. The 
cavity and atomic excited state decay rates are $\kappa$ and $\gamma$, respectively. (b) Second-order correlation function $g^{(2)}(0)$.](https://example.com/fig2.png)

IV. FULL MODEL: CAVITY QED WITH MOTION

We now include atomic motion into the Jaynes-Cummings Hamiltonian $H = \omega_m b^\dagger b + H_{JC}$ by treating $x_0 \rightarrow x$ as a 
dynamical variable. We assume that the atom sees an internal-
state independent and harmonic trapping potential, which 
occurs naturally for trapped ions or using magic wavelength 
traps for neutral atoms [45]. In Fig. 3(a), we plot $g^{(2)}(0)$ as 
a function of laser-cavity detuning $\delta_c$ and the central position 
x_0 of the trap, for parameters $g_0 = 2\pi \times 10$ MHz, $\kappa = \gamma = 2\pi \times 0.02$ MHz, $\Delta = 3g_0$, and $\omega_m = 2\pi \times 0.5$ MHz. It can be 
seen that this figure captures a combination of the pure JC plot 
[Fig. 2(b)] and pure optomechanical plot [Fig. 1(d)], where 
the largest degree of antibunching occurs around the antinode 
$|x_0 = 0\rangle$ or in between the node and antinode, respectively. In 
particular, the presence of sideband features, and the extended 
antibunching away from the antinode are qualitative signatures 
of motional effects. Below we plot $g^{(2)}(0)$ following the ZPL 
(red, dashed). The region of negligible antibunching, $g^{(2)}(0) \approx$ 
1, at $k_c x_0 \approx \pm \pi/\delta$ originates from an exact cancellation 
of the nonlineairies induced by motion and the two-level nature.

To better understand the contribution from motion, under 
certain conditions one can effectively map the JC model to the 
optomechanical model. In particular, for large laser-
atom detunings $\delta_0 \gg g_0$, the atomic ground-state population 
is approximately one which allows for an effective elimination 
of the atomic excited state [46,47] using the Nakajima-
Zwanzig projection operator formalism [48,49]. The Lamb-
Dicke regime is given by $\hbar \kappa_{xc} = k_c x_0 \approx \sqrt{\omega_{xc}/\omega_m} \ll 1$, 
where the atomic recoil frequency $\omega_{xc} = \hbar k_c^2/(2m_{atom})$ relates 
the resonant wave vector with the atomic mass. In this
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FIG. 3. JC model including motion. (a) Top: $g^{(2)}(0)$ of the transmitted field versus trapping position $x_0$ and detuning from the empty cavity $\delta = \omega_c - \omega_s$, for detunings near the photonic eigenstate and for atom-cavity detuning $\Delta = 5\gamma_0$. Here, we use idealized parameters $g_0 = 2\pi \times 10$ MHz, $\kappa = \gamma = 2\pi \times 0.02$ MHz, and $\omega_m = 2\pi \times 0.5$ MHz so that all of the key features can be clearly observed. Below: $g^{(2)}(0)$ following the ZPL [red (lower), dashed]. (b) We plot the same as in Fig. 3(a), but using the parameters for a realistic cavity QED experiment given below. In this figure, we choose $\Delta = 12g_0$, and $\omega_m = 2\pi \times 0.1$ MHz. (c) $g^{(2)}(0)$ as a function of atom-cavity detuning $\Delta$ and trapping frequency $\omega_m$. (d) $g^{(2)}(0)$ as a function of trapping position $x_0$ and trapping frequency $\omega_m$, for $\Delta = 12g_0$. For Figs. 3(b)–3(d) we choose parameters $g_0 = 2\pi \times 1.4$ MHz, $\kappa = 2\pi \times 0.05$ MHz, $\gamma = 2\pi \times 1.1$ MHz, and $\omega_{at} = 2\pi \times 6.8$ kHz.

In conclusion, we have shown that cavity QED experiments approaching the strong-coupling regime are natural platforms to explore the single-photon, single-phonon strong-coupling regime of optomechanics, in the limit that the motional sidebands can be resolved. Since many of those experiments, which allow for the realization of motional nonlinear effects, already exist, we anticipate that such platforms will stimulate much theoretical and experimental work to further explore the generation of nonclassical light from motion and its consequences.

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\[ E_{n,m} \approx m\omega_m + \left( \omega_c - \frac{g_0^2}{\Delta} u^2(\chi_0) \right) n + \left( \frac{g_0^2}{\Delta^3} u^4(\chi_0) - \frac{\kappa_{eff}}{\omega_m} \right) n^2. \]
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**APPENDIX A: CALCULATION OF SECOND-ORDER PHOTON CORRELATIONS $g^{(2)}(0)$**

Here, we discuss how to calculate the second-order correlation function $g^{(2)}(0)$ of the transmitted field given a weak coherent state input, such as plotted in Figs. 1(c), 1(d), 2(b), and 3 of the main text. Formally, the quantum properties of the transmitted field are encoded in the input-output relation function. For simplicity, we only consider a single direction of spontaneous emission.

The full master equation corresponding to the Hamiltonian $\sigma_{ge}$ [Eq. (2) of the main text], where we treat $x_0 \rightarrow x$ as a dynamical variable is given by

$$\dot{\rho} = -i(H_{JC}\rho - \rho H_{JC}^\dagger) + \sigma_{ge} e^{-ikx} \rho e^{ikx} \sigma_{eg} + \kappa a \rho a \equiv L\rho.$$ 

(B1)

The term $\sigma_{ge} e^{-ikx} \rho e^{ikx} \sigma_{eg}$ physically describes quantum jumps corresponding to atomic spontaneous emission, accompanied by a momentum recoil kick $e^{-ikx}$ acting on the atomic motion. For simplicity, we only consider a single direction of spontaneous emission.

In the limit where the cavity is driven near resonantly and the atom is far-detuned, the atomic excited state can be eliminated to yield an effective optomechanical system involving just the atomic motion and the cavity mode. We will now use the Nakajima-Zwanzig projection operator formalism to eliminate the atomic excited state. We define a set of operators $P, Q$, which project the entire system density matrix

$$\rho = |g\rangle\langle g| \rho_{gg} + |e\rangle\langle e| \rho_{ee} + |e\rangle\langle e| \rho_{eg},$$

(B2)

into the subspace spanned by $|g\rangle\langle g|$ (which we want to project the dynamics into), and its orthogonal $1 - |g\rangle\langle g|$. Here $\rho_{ij} = \langle i | \rho | j \rangle$ are the reduced density matrices for the reduced Hilbert space, which still contain all other existing degrees of freedom. Thus, we define a projection operator $P$:

$$P\rho = |g\rangle\langle g| \rho_{gg}$$

(B3)

and its complementary

$$Q\rho = |g\rangle\langle e| \rho_{ge} + |e\rangle\langle g| \rho_{eg} + |e\rangle\langle e| \rho_{ee}.$$ 

(B4)

It is straightforward to show $P^2 = P$, $Q^2 = Q$, $QP = 0$, $P + Q = 1$. We will now divide the superoperator $L$ into parts according to the way they act on the Hilbert space describing the internal degrees of freedom of the atom:

$$L = L_0 + L_a + L_I + J.$$ 

(B5)

Here, $L_0 = L_m + L_c$ is composed of terms that do not act on the internal degrees of freedom, with $L_m$ and $L_c$ describing, respectively, the trapped atomic motion and the bare dynamics of the driven cavity mode:

$$L_m \rho = -i[\omega_m b \dagger b, \rho],$$

(B6)

$$L_c \rho = i\delta_{\alpha}(a, \rho) - i\sqrt{\kappa} E_0 (a + a \dagger),$$

(B7)

The superoperator

$$L_0 \rho = i\delta [\sigma_{ee}, \rho] - \frac{\gamma}{2} [\sigma_{ee}, \rho]$$

(B8)

acts on $|e\rangle\langle e|$ and $|g\rangle\langle g|$ and just multiplies those terms by a constant. It describes evolution and damping of the excited internal state of the atom.

$$L_I \rho = -i[g(x)(\sigma_{ee} + \alpha e^{\dagger}), \rho]$$

(B9)

acts on all the states and all Hilbert spaces, describing the interaction of the atom with the cavity field and

$$J\rho = \gamma \sigma_{ge} e^{-ikx} \rho e^{i k x} \sigma_{eg}$$

(B10)

describes the spontaneous jump of the excited state of the atom into its ground state accompanied by a momentum recoil. We define $v = P\rho$ and $w = Q\rho$ and insert $P + Q = 1$ into Eq. (B1):

$$\dot{v} = P\dot{\rho} = PL\rho = PLP\rho + PLQ\rho.$$ 

(B11)

After identifying all vanishing terms, we obtain

$$\dot{v} = L_a v + \{J + L_I\} w.$$ 

(B12)
\[ \dot{w} = Q L_I v + Q(L_o + L_a + L_I)w. \]  

(13)

Since \( w \) describes the evolution of the fluctuations out of the subspace of interest, the term \( L_o w = (L_m + L_c)w \) describes the free evolution of motion and of the cavity mode during one of these fluctuations. As the timescale of these fluctuations is set by \( \delta_0 \) and \( \gamma \) and we assume that either \( \delta_0 \) or \( \gamma \) is much larger than both \( \omega_m \) and \( \kappa \), we can neglect the time evolution of motion and cavity mode during one of these fluctuations by approximating \( L_o w \approx 0 \) in Eq. (B13). Then the general solution to this equation reads

\[ w(t) = \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} Q L_I w(\tau) \]

\[ + \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} Q L_I v(\tau), \]

(14)

where we set \( w(0) = 0 \) as the initial condition. Now we plug this equation twice into Eq. (B12) (iterative) in order to catch a term of the order \( JL_I^2 \) and include the process of spontaneous emission:

\[ \dot{v}(t) = L_o v + P(J + L_I) \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} Q L_I v(\tau) \]

\[ + P(J + L_I) \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} Q L_I v(\tau) \]

\[ \times \int_0^\tau d\tau' e^{Q(L_o + L_c)(\tau'-\tau)} Q L_I v(\tau'), \]

(15)

where we neglected the term proportional to \( w(\tau') \) since it produces only terms \( \propto L_I^3 \) or higher. After identifying vanishing terms, we are left with

\[ \dot{v}(t) = L_o v + PL_I \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} L_I v(\tau) \]

\[ + P J \int_0^t d\tau e^{Q(L_o + L_c)(t-\tau)} L_I \int_0^\tau d\tau' e^{Q(L_o + L_c)(\tau'-\tau)} L_I v(\tau'), \]

(16)

After extending the lower integral borders to \(-\infty \) (Markov approximation) and evaluating the integrals, we obtain the effective optomechanical master equation:

\[ \dot{\rho} = -i[H_{\text{opt}}, \rho] + L_{\text{opt}} \rho, \]

(17)

with an effective optomechanical Hamiltonian

\[ H_{\text{opt}} = \omega_m b^\dagger b - \Delta_c(x)a^\dagger a + \sqrt{\kappa/2}E_0(a + a^\dagger). \]

(18)

The position-dependent cavity-laser detuning is given by

\[ \Delta_c(x) = \delta_0 - \frac{\delta_0^2 \delta_0}{\delta_0^2 + \frac{\gamma^2}{4}} u^2(x). \]

(19)

Expanding \( \Delta_c(x) \) around \( x_0 \) to linear order and replacing \( x \) with phonon operators \( b \) and \( b^\dagger \) yields \( \Delta_c(x) \approx \Delta_c(x_0) - g_{\text{eff}}(b + b^\dagger) \) with \( g_{\text{eff}} \) of the main text. The system losses are given by the effective Liouvillian

\[ L_{\text{opt}} \rho = -\frac{i}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a\rho a^\dagger) \]

\[ - \frac{\gamma}{2} \frac{\delta_0^2}{\delta_0^2 + \frac{\gamma^2}{4}} [\frac{1}{2} (u^2(x) a^\dagger a + \rho a^\dagger a^\dagger u^2(x))] \]

\[ + \frac{\gamma}{2} \frac{\delta_0^2}{\delta_0^2 + \frac{\gamma^2}{4}} \frac{1}{2} [2a\rho u^2(x) e^{-i\kappa x} \rho e^{i\kappa x} u(x) a^\dagger], \]

(20)

which describes the broadening of the cavity linewidth due to atomic spontaneous emission,

\[ \kappa(x) = \kappa + \gamma \frac{\delta_0^2}{\delta_0^2 + \frac{\gamma^2}{4}} u^2(x). \]

(21)

Averaging with the atomic wave function located at \( x_0 \) yields \( \kappa_{\text{eff}} \) of the main text.

APPENDIX C: BEYOND THE LAMB-DICKE REGIME: INCLUDING QUADRATIC-ORDER TERMS IN DISPLACEMENT

In order to show that the strong-coupling regime of optomechanics can already be observed by an existing experiment, we plotted \( g^{(2)}(0) \) as a function of \( x_0 \) in Fig. 3(b) in the main text. In

![Fig. 4. JC model with motion expanding u(x) until quadratic order.](Image)

(a) \( g^{(2)}(0) \) of the transmitted field versus trapping position \( x_0 \) and detuning from the empty cavity \( \delta_0 = \omega_c - \omega_e \), for detunings near the photonic eigenstate and by using the parameters for a realistic cavity QED experiment given below. In this figure, we choose an atom-cavity detuning \( \Delta = 10 \gamma_0 \) and atomic trap frequency \( \omega_m = 2 \pi \times 0.09 \text{MHz} \), which produces the minimum possible \( g^{(2)}(0) \) including quadratic order corrections. (b) Following the ZPL of (a) (red, dashed). We compare \( g^{(2)}(0) \) calculated with only linear displacements (red, dashed) in Hamiltonian Eq. (2) of the main text with \( g^{(2)}(0) \) calculated by also including terms of quadratic order (blue, solid). (c) \( g^{(2)}(0) \) as a function of atom-cavity detuning \( \Delta \) and trapping frequency \( \omega_m \) including terms of quadratic order. Here, the atomic position is fixed at \( k \_x = 1.15 \). (d) \( g^{(2)}(0) \) as a function of trapping position \( x_0 \) and trapping frequency \( \omega_m \) for \( \Delta = 10 \gamma_0 \) including terms of quadratic order. As in the main text, we choose parameters of an existing cavity QED experiment with trapped \( ^{40}\text{Ca}^+ \) ions: \( \gamma_0 = 2 \pi \times 1.4 \text{MHz}, \kappa = 2 \pi \times 0.05 \text{MHz}, \gamma = 2 \pi \times 11 \text{MHz}, \text{and recoil frequency } \omega_{\text{rec}} = 2 \pi \times 6.8 \text{kHz}. \)
this calculation, we linearized the cavity mode profile $u(x)$ in Hamiltonian Eq. (2) of the main text around the trapping position $x_0$: $u(x) \approx u(x_0) + u'(x_0) k \cdot (x - x_0)$, which is strictly only valid in the Lamb-Dicke regime $\eta_{LD} = k \cdot x_0 \approx \sqrt{\omega_{rec}/\omega_m} \ll 1$.

However, in order to produce Fig. 3 of the main text, we used a trapping frequency of $\omega_m = 2\pi \times 0.1$ MHz. With the recoil frequency of $^{40}$Ca$^+$ ions this corresponds to $\eta_{LD} \approx 0.26$.

To ensure that the results are not significantly affected by this relatively large Lamb-Dicke parameter, we will now include the next order term $u(x) \approx u(x_0) + u'(x_0) k \cdot (x - x_0) + (1/2)u''(x_0)(x-x_0)^2$. In Fig. 4(a), we plot the adjusted $g^2(\Delta)$ as a function of atom position $x_0$ and detuning $\delta_c$. Here we choose $\Delta = 10 g_0$ and $\omega_m = 2\pi \times 0.09$ MHz in order to minimize $g^2(\Delta)$ including quadratic order corrections. Figure 4(b) shows $g^2(\Delta)$ as a function of atom position $x_0$ following the ZPL of (a) (blue, solid). In red (dashed) we plot $g^2(x_0)$, where $u(x)$ has only been expanded until linear order for the same parameters. We observe a reasonable match and conclude that linearizing motion on the Hamiltonian level at least qualitatively fully captures the relevant physics even for relatively large $\eta_{LD}$. For completeness, we plot $g^2(\Delta)$ as a function of $\omega_m$ and $\Delta$ in Fig. 4(c) for a fixed atomic position $k \cdot x_0 = 1.15$, and in Fig. 4(d) we plot $g^2(\Delta)$ as a function of trapping position $x_0$ and trap frequency $\omega_m$ for $\Delta = 10 g_0$.

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