Scaling Behavior of Level Statistics in Quantum Hall Regime

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The scaling property of level statistics in the quantum Hall regime, i.e. 2D disordered electron systems subject to strong magnetic fields, is analyzed numerically in the light of the random matrix theory. The energy dependences of the effective level repulsion parameter, the two level correlation, the GUE-GOE crossover parameter, and the rigidity (or $\Delta_3$-statistics) of the level distributions are investigated for different system sizes by unfolding the original data and by dividing the unfolded spectrum into small regions. It is shown that the critical exponent of the localization length as a function of energy can be determined through the energy dependence of the level statistics. The analyses are carried out not only for the lowest Landau band (LB) but also for the second lowest LB. Furthermore the effect of finite range of disordered potential is studied. The short-ranged potential case in the second lowest LB is found to be pathological as in other studies of critical behavior, and it is confirmed that this pathological behavior is improved in the case of disordered potential with finite ranges.

Keywords: level statistics, level spacing distribution, random matrix theory, GUE, localization, scaling behavior, GUE-GOE crossover, quantum Hall effect
1 Introduction

The study of electronic states in disordered systems is inevitable to understand electronic transport properties in conducting materials. Most directly the electronic states are described by wave functions. In recent researches, however, the importance of energy level distributions is pointed out. Since the wave functions and the energies are both obtained by solving the eigenvalue problem for the disordered Hamiltonian, it is quite natural to expect that the energy spectrum involves some informations on the eigenfunctions.

In disordered systems all the spatial symmetries which may exist in pure systems are lost, and only the fundamental symmetries under the operation of the time reversal can survive. The importance of such fundamental symmetries in the problem of the Anderson localization were first pointed out by Wegner[1] and Hikami, Larkin and Nagaoka, [2] in the treatment of weakly localized regime. The relation between the level statistics and the fundamental symmetry was originally proposed in nuclear physics in order to explain complicated energy spectra in heavy nuclei[3, 4, 5, 6] and then applied to the problems of metallic fine particles[7, 8] and chaos. [9, 10] These level statistics are treated within the random matrix theory(RMT), [7, 8] where the energy levels are mimicked by eigenvalues of random Hamiltonian matrices whose elements are chosen randomly. The RMT tells us that the level statistics have quite universal properties depending only on the fundamental symmetry of the system. If the system is time reversal invariant, there are two universality classes; orthogonal when there is no spin-orbit interaction or the spin of the particle is integer and symplectic when the spin of the particle is half odd-integer and there exists a spin-orbit interaction. When the time reversal symmetry is violated e.g. by the presence of an external magnetic field or magnetic impurities, the corresponding universality class is unitary.

The usual random matrices and the Hamiltonian matrices describing disordered electrons are different in the sense that the matrix elements in the former have no correlation while those in the latter have rather strong correlations. This will be clear if we remind ourselves of the Anderson model where the diagonal elements of the Hamiltonian matrix are uncorrelated random numbers but the off-diagonal elements takes 0 or a finite value equal to the nearest-neighbor transfer integral in a regular way. Because of this correlation, the Anderson model can yield both of localized and delocalized states. It is not a trivial task to connect the correlations in matrix elements to the localization-delocalization properties of electronic states, since the apparent correlations of matrix elements depend on the choice of basis functions. It is usually believed that the level statistics of extended states are well described by the RMT using Gaussian ensembles. [7, 8] The characteristic of the level statistics in the extended regime is the strong level repulsion, which can be expressed by the behavior $t^\beta$ of the level spacing distribution function $P(t)$ in the small $t$ (= spacing) region. It is well-known that the level repulsion parameter $\beta$ takes 1, 2 and 4 for the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE) and the Gaussian symplectic ensemble (GSE), respectively. [7, 8] This fact represents that the level correlation in the extended regime is rather strong. On the other hand the levels in the localized regime has no correlation reflecting negligible spatial overlap of wave functions of different localized states. The level statistics in this case become Poissonian. [7, 8]

From these behavior of the level statistics it is clear that the localization-delocalization
properties of disordered systems can be analyzed in terms of level statistics. First attempt along this line was carried out by Shklovskii and co-workers. They have analyzed the level spacing distribution in 3D Anderson model by changing the system size and the strength of disorder, and have found that the level spacing distribution function satisfies a certain scaling property. From this scaling property they could derive the critical exponent of the localization length at the metal-insulator transition. The obtained value of the exponent is consistent with other studies such as the real space scaling analysis. A similar scaling behavior in the quantum Hall effect regime, i.e. in 2D disordered electron systems subject to strong magnetic fields has been found by Ono and Ohtsuki, who have found the localization length exponent for the lowest Landau band with short-ranged impurity potentials, which is also consistent with other studies. The level spacing distribution at the center of the lowest Landau band was studied by Huckestein and Schreiber, who found a distribution quite near to that of GUE, though they did not discuss the scaling behavior. The level statistics in 3D systems without time reversal symmetry have been investigated by Hofstetter and Schreiber. Particularly, they have paid attention to the level statistics at the transition, and found that, as far as the level statistics are concerned, the behavior at the transition seems not to be affected by the violation of the time reversal symmetry, namely that the orthogonal and unitary cases show the same level distribution at the transition point, at least in 3D. The conclusion that the two universality classes show the same critical level statistics is still controversial, since in the above analysis the time reversal symmetry is violated by introducing Aharonov-Bohm type phase factor which can affect only the delocalized states and have no effect on the localized states. In order to resolve this controversy, studies on the system with real magnetic fields will be necessary.

Level statistics are studied also analytically. The results by Kravtsov et al. are particularly interesting. According to their result, the behavior of the level spacing distribution for large spacing region at the transition is determined by the localization exponent. Therefore, by analyzing the large spacing tail structure of the spacing distribution function at the critical point, it is possible to derive the critical exponent of the localization length without changing the system size. It is confirmed that this is the case at least for 3D orthogonal systems. It will be discussed in this paper that the critical level spacing distribution has a size independent form, but that a similar analysis as suggested by Kravtsov et al. does not yield the correct exponent.

The existence of the critical level distribution is understood as follows. Let us assume the level distribution is characterized by a certain parameter $A$, which takes $A_{\text{loc}}$ for localized regime and $A_{\text{ext}}$ for extended regime. When the system size is infinite, this parameter will change abruptly from $A_{\text{loc}}$ to $A_{\text{ext}}$ as a function of energy or disorder strength at the critical point (see Fig. 1). When the system size is finite, however, the change will be gradual. It is quite natural that the size ($L$) dependence of this parameter is scaled by the characteristic length $\xi$ which diverges as a function of energy or disorder ($X$) at the transition point ($X_c$) with the critical exponent $\nu$. That is

$$A(L, X) = f(L/\xi(X)), \quad \xi(X) \sim |X - X_c|^{-\nu}. \quad (1)$$

At the critical point where $\xi$ is infinite, $A$ is independent of the system size as is seen in Fig. 1. This scale invariance at the transition will be applied for any parameter, as far
as it characterizes the difference of the level statistics between the localized and extended regimes. This means that there exists a critical scale independent distribution, as long as the above mentioned scaling hypothesis is valid. The validity of this hypothesis can be checked by analyzing numerical data.

In this paper we discuss the scaling behavior of level statistics in the quantum Hall regime. Not only the level spacing distribution but also the $\Delta_3$-statistics, two level correlation function and so forth are found to obey the same scaling behavior. We also study the effects of potential range on the level statistics. In previous works, mainly the lowest Landau band was studied, but in the present work we discuss also the level statistics in the second lowest Landau band. [24]

2 Model and Formulation

The system discussed in this paper is a two-dimensional non-interacting electron gas in a strong magnetic field and a random potential. The magnetic field is assumed to be so strong that the Landau band separation is much larger than the Landau band width due to scattering by the random potential. As for the random potential $V(\mathbf{r})$ we assume that of the Gaussian type with the following properties,

$$<V(\mathbf{r})> = 0,$$

$$<V(\mathbf{r} + \mathbf{a})V(\mathbf{r})> = \frac{u^2}{2\pi s^2} \exp(-a^2/2s^2),$$

where $s$ and $u$ represent the correlation length and the strength of the potential, respectively. They are related to the lowest Landau band width $\Gamma_{SCBA}^{(0)}$ calculated within the self-consistent Born approximation (SCBA), as follows, [25]

$$\Gamma_{SCBA}^{(0)} = \sqrt{\frac{2u^2}{\pi(l^2 + s^2)}},$$

where $l$ is the magnetic length. The SCBA band width is independent of Landau band index in the case of short-ranged potentials ($s = 0$, i.e. $\delta$-correlated), but depends on it in the case of finite ranges ($s \neq 0$). [25] In the present treatment, however, such detailed dependences of the band width on various parameters are not important, since the characteristics of the unfolded levels are considered to be independent of the absolute value of the band width (see below).

In order to obtain the energy spectra, one must first calculate the Hamiltonian matrix elements by introducing appropriate basis functions. In the present situation a set of Landau functions is the most appropriate basis. As discussed in refs. [18] and [26], it is possible to generate random elements of the Hamiltonian matrix which reflect the essential features of the Landau functions as basis and of the random potential as described in eqs.
Details of preparing the matrix elements have been described in refs. \[18\] and \[26\], and therefore we do not repeat them here. As in ref. \[26\], the periodic boundary conditions are assumed for both directions in order to avoid edge effects. In the present work, we treat not only the lowest Landau band but also the second lowest one to study the effect of different basis sets. \[27\], \[28\] In fact the case of the short-ranged potential and the second lowest Landau band is pointed out to behave pathologically in the study of the localization length by the real space scaling analyses. \[17\], \[27\], \[29\], \[30\] The density of states for each Landau band takes bell shape in most cases. This means that the average level spacing changes as a function of energy. As is well-known, a procedure called “unfolding” maps the original energy levels \(\{E_i\}\) to a new set of levels \(\{x_i\}\) which has a constant average spacing independent of the level region. \[4\] If we represent the integrated density of states by \(N(E)\), then the unfolding is carried out by

\[
x_i = N(E_i).
\]

It is straightforward to show that the density of states and therefore the average spacing for the new levels are constant. \[26\] Throughout the paper, we discuss the statistics of the unfolded levels.

The Hamiltonian matrix size increases when we consider larger systems. If we denote the matrix size by \(N_{\text{mat}}\) and the system size by \(L \times L\) (i.e. the linear dimension, \(L\)), they are related to each other in the following form,

\[
L = \sqrt{2\pi N_{\text{mat}}} l.
\]

The matrix sizes treated here are \(N_{\text{mat}} = 200, 400, 600\) and \(800\). The number of samples for each size is chosen to be \(N_{\text{sample}} = 288,000/N_{\text{mat}}\) so that the total number of energy levels may be \(288,000\). The potential range discussed here covers 0 to \(2l\).

3 Level Spacing Distribution

As is clear from the definition of the unfolding procedure, the unfolded levels are distributed between 0 and \(N_{\text{mat}}\), and the average spacing is equal to unity. \[26\] It is well known that the truly delocalized states are possible only at the Landau band center. We shift the unfolded levels by \(N_{\text{mat}}/2\) in order to make \(x = 0\) correspond to the band center. As mentioned in § 1, the statistics of levels of delocalized states and those of localized ones are qualitatively different. In order to see the transition between localized and delocalized characteristics, we consider the energy dependent level statistics. In practice we take a region of \(x\) with a width of \(0.1N_{\text{mat}}\), and discuss the level statistics within that region. The region is represented by its central value. By changing this central value, we can discuss the level statistics dependent on the energy region. In the case of the quantum Hall regime, the Landau band center is always the critical point in the sense of the Anderson localization. It is, therefore, not possible to see the transition by changing the strength of disorder as in the case of the 3D orthogonal case. \[12\], \[13\], \[14\] This is the reason why we discuss the energy dependence.

The examples of level spacing distributions \(P(t)\) \((t\ the\ spacing\ of\ the\ unfolded\ levels)\) are given in Fig. 2 for the case of the lowest Landau band with \(\delta\)-correlated potential.
Figure 2: Examples of level spacing distributions for the lowest Landau band with \( \delta \)-correlated potential; (a) \( x = 0 \) and (b) \( x = -240 \), \( N_{\text{mat}} \) being 600 in both figures.

3.1 Effective level repulsion parameter

As discussed in ref. [26], one of the parameters characterizing these distributions is the effective level repulsion parameter \( \beta \), which is thought to be 2 for the extended regime in the unitary symmetry and 0 for the localized regime. Several interpolation formulae are proposed, [31, 32] which describe intermediate distributions between those for the Gaussian ensembles,

\[
P_{\text{GOE}}(t) = \frac{\pi}{2} t \exp\left(-\frac{\pi}{4} t^2\right),
\]
for Gaussian orthogonal ensemble, \( \beta \leq 1 \) \( \tag{7} \)

\[
P_{\text{GUE}}(t) = \frac{32}{\pi^2} t^2 \exp\left(-\frac{4}{\pi} t^2\right),
\]
for Gaussian unitary ensemble, \( \beta \leq 1 \) \( \tag{8} \)

\[
P_{\text{GSE}}(t) = \frac{218}{3^6\pi^3} t^4 \exp\left(-\frac{64}{9\pi} t^2\right),
\]
for Gaussian symplectic ensemble, \( \beta > 1 \) \( \tag{9} \)

and the Poissonian,

\[
P(t) = e^{-t}
\]
\( \tag{10} \)

We apply the simplest interpolation formula proposed by Brody [31] and extended to cover the case \( \beta > 1 \). [26]

\[
P_{\beta}(t) = (1 + \beta) a(\beta) t^\beta \exp\left(-a(\beta) t^{1+\beta}\right),
\]
for \( 0 \leq \beta \leq 1 \) \( \tag{11} \)

\[
P_{\beta}(t) = c(\beta) t^\beta \exp\left(-b(\beta)^2 t^2\right),
\]
for \( \beta > 1 \) \( \tag{12} \)

where

\[
a(\beta) = \left[ \Gamma\left(\frac{2 + \beta}{1 + \beta}\right)\right]^{1+\beta},
\]
\( \tag{13} \)

\[
b(\beta) = \frac{\beta \Gamma\left(\frac{2}{2}\right)}{2\Gamma\left(\frac{1+\beta}{2}\right)},
\]
\( \tag{14} \)

\[
c(\beta) = \frac{2b(\beta)^{1+\beta}}{\Gamma\left(\frac{1+\beta}{2}\right)}.
\]
\( \tag{15} \)

Even if we use other more complicated formulae such as that proposed by Izrailev, [32] the final result is found not to be affected. Although there is some argument that the small \( t \) behavior of \( P(t) \) for delocalized states is described always by \( t, t^2 \), or \( t^4 \) depending on the basic symmetry of the system, we consider the effective level repulsion parameter \( \beta \) obtained by fitting the data to an interpolation formula like eqs. (11) and (12) can describe
Figure 3: The scaling behavior of the effective level repulsion parameter in the case of the lowest Landau band with two different correlation lengths of the random potential; (a) $s = 0$ and (b) $s = 2l$ ($l$: the magnetic length).

Figure 4: Scaling behavior of the effective level repulsion parameter in the second lowest Landau band with the correlation length of the potential (a) $s = 0$ and (b) $s = l$ ($l$ the magnetic length).

to some extent the deviation of the numerical data from the distribution for Gaussian ensemble, even if, precisely speaking, the fitting might be poor. In fact, Shklovskii et al. [12] chose an integral of $P(t)$ in a certain fixed region by noting that the intermediate distributions cross at a point near $t = 2$ in the 3D orthogonal case. Unfortunately there is no such common crossing point in the case of the quantum Hall regime.

Two examples of the scaling behavior of the effective level repulsion parameter, obtained as mentioned above, is shown in Fig. 3 for the case of the lowest Landau band with different correlation lengths of random potential. From this behavior we can deduce the critical exponent of the localization length $\nu$: Energy dependence of the localization length $\xi$ is assumed to be described by

$$\xi \propto |E - E_N|^{-\nu},$$

where $E_N$ represents the center of the $N$-th Landau band. Then $\beta$ as the function of $x$ and $N_{\text{mat}}$ is

$$\beta(x, N_{\text{mat}}) = f(|x/N_{\text{mat}}|^{-\nu} N_{\text{mat}}^{-1/2}) = \tilde{f}(x/N_{\text{mat}}^{1-1/2\nu}),$$

from eqs. (1) and (3). The obtained critical exponent $\nu = 2.4$ for the case of the lowest Landau band with $\delta$-correlated potential is consistent with that of different previous studies. [17, 18, 19, 27, 33]

Similar scaling behaviors were investigated for different potential ranges and different Landau bands. Particularly it is confirmed that the case of the second lowest Landau band with the $\delta$-correlated potential is pathological (see Fig. 4) as already found in previous studies of localization length exponents. [17, 27, 29, 30] Furthermore, at least within the present analysis, the exponent depends on the potential range and universality was not confirmed, while the universality within the lowest Landau band has been confirmed in the real space scaling analysis. [34] The critical exponents obtained in this analysis are summarized in Table 1. It is found in general that the increase of the potential correlation length leads to smaller exponent. It is also found that the pathological behavior of the second lowest Landau band with short-ranged potential can be improved by introducing finite correlation length. [34]

The lack of universality within the present analysis may be partly due to the finite width of the energy window. In order to check the effect of the finiteness of the window, we might have to take many more samples and make the window width much smaller. We
Table 1: Critical exponents obtained from the scaling behavior of the effective level repulsion parameter $\beta$.

| Landau band index | $0$ | $1$ |
|-------------------|-----|-----|
| potential range $s$ | $0$ | $l$ | $2l$ | $0$ | $0.5l$ | $0.7l$ | $0.8l$ | $l$ | $2l$ |
| exponent $\nu$ | $2.4$ | $2.0$ | $1.5$ | $7.1$ | $5.0$ | $2.5$ | $2.4$ | $1.7$ | $1.3$ |

have made the window width half without increasing the sample number, but we have got no change in the scaling behavior of the effective level repulsion parameter, at least to this stage.

3.2 GUE–GOE crossover parameter

In ref. [26] it has been discussed that the level spacing distribution in the QH regime can be characterized by the GUE-GOE crossover parameter. According to the semiclassical argument by Argaman, Imry and Smilansky [35], a smaller level separation corresponds to a longer period of the semiclassical motion and therefore to a larger orbit. The larger the orbit, the stronger the effect of the magnetic field will be. From these consideration it is expected that the small $t$ behavior of $P(t)$ resembles that of GUE while the large $t$ behavior is similar to that of GOE. The crossover from GUE in small $t$ region to GOE in large $t$ region can be described by a model proposed by Pandey and Mehta [36] and utilized by Dupuis and Montambaux [37] to explain the level spacing distribution in a small metallic ring pierced by a magnetic flux.

In Pandey and Mehta’s model, [36] the random Hamiltonian matrix is expressed in a linear combination of symmetric and anti-symmetric matrices with a coefficient $\alpha$ ($\alpha$; real) for the latter. The case with $\alpha = 0$ corresponds to GOE and the case with $\alpha = 1$ to GUE. The level spacing distribution function in the intermediate case is obtained by considering $2\times2$ matrices. The resulting distribution function is given by [38]

$$P_\alpha(t) = \frac{t}{4v^2\sqrt{1-\alpha^2}}\exp(-t^2/8v^2)\text{Erf}(wt)$$

with

$$v = \sqrt{\frac{\pi}{8}}\left(\alpha + \frac{1}{\sqrt{1-\alpha^2}}\arctan\frac{\sqrt{1-\alpha^2}}{\alpha}\right)^{-1},$$

$$w = \sqrt{\frac{1-\alpha^2}{8\alpha^2v^2}}.$$ (20)

In fact, as far as the effective level repulsion parameter $\beta$ is larger than unity, the level spacing distribution data can be fitted to this interpolation formula and the crossover parameter is estimated as a function of energy. The energy and size dependence of $\alpha$ satisfies the scaling relation as is seen in Fig. 5. Note that data near the band edges cannot be used in this analysis.

It is worthwhile to note that this scaling relation is explained by the same critical exponent which is found for the effective level repulsion parameter. Unfortunately, however,
Figure 5: Scaling behavior of $\alpha$ in the lowest Landau band with $\delta$-correlated potential.

Figure 6: Critical level spacing distribution at the center of the second lowest Landau band with the potential correlation length $s = 2l$. The solid curve is the best fit to eq. (3.17) with $\beta = 0.66$ and $\gamma = -0.58$.

the crossover fitting is not possible when the maximum value of the effective level repulsion parameter $\beta$ is less than unity, which occurs for long-ranged potentials.

### 3.3 Critical distribution

The behavior of the scale invariant critical distribution function of the level spacing at the center of the lowest Landau band with $\delta$-correlated potential has been discussed in ref. [23]. Therefore we mention here that the critical distribution is well fitted over all the region of $t$ by the expression,

$$ P_{\text{crit}}(t) = At^2 \exp(-Bt^{2-\gamma}), \quad (21) $$

with $\gamma$ the fitting parameter. The coefficients $A$ and $B$ are determined by two normalization conditions, i.e. $<1 > = <t> = 1$. According to the analytic study, [22] $\gamma$ is related to the dimension $d$ of the system and the critical exponent $\nu$ as

$$ \gamma = 1 - \frac{1}{\nu d}. \quad (22) $$

The best fit value of $\gamma$ in the case of the lowest Landau band with short-ranged potential is about 0.35, which yields the critical exponent smaller than unity in disagreement with other studies.

We have studied the critical distributions for different values of the potential range and also for the second lowest Landau bands. It was confirmed that the critical distributions are scale invariant and can be fitted to eq. (21) quite well as long as the potential is short-ranged. For longer-ranged potentials the single parameter fitting to eq. (21) is not successful, and we have to use a two parameter fitting to

$$ P'_{\text{crit}}(t) = A't^\beta \exp(-B't^{2-\gamma}). \quad (23) $$

Here again the parameters $A'$ and $B'$ are related to $\beta$ and $\gamma$ due to the two normalization condition. In order to show how the potential range affects the critical distribution, we give the data of the critical level spacing distribution for the case of the second lowest Landau band with the potential correlation length $s = 2l$ in Fig. 6 along with the fitting curve to eq. (23). The values of $\beta$ and $\gamma$ obtained by the best fit to eq. (23) are summarized in Table 2.

It will be worthwhile to point out that the fitting to eq. (23) and that to eq. (18) are almost indistinguishable as far as the latter fitting is possible.
Table 2: Values of $\beta$ and $\gamma$ obtained by fitting the critical level spacing distribution at the band center to eq. (3.17) for different cases.

| Landau band index | 0 | 1 |
|-------------------|---|---|
| potential range   | 0 | $l$ | $2l$ | 0 | 0.5$l$ | 0.7$l$ | 0.8$l$ | $l$ | 2$l$ |
| $\beta$           | 2.19 | 1.95 | 1.03 | 2.04 | 2.01 | 2.02 | 2.03 | 2.01 | 0.66 |
| $\gamma$          | 0.47 | 0.48 | 0.02 | 0.17 | 0.16 | 0.25 | 0.31 | 0.43 | −0.58 |

Figure 7: Examples of the $\Delta_3$-statistics in the lowest Landau band with $\delta$-correlated potential. Two values of $x_0$ are chosen; $x_0 = 0$ (the band center) and $x_0 = 0.2N_{\text{mat}}$. The solid line is the analytic form for the GUE.

4 $\Delta_3$-Statistics

In the random matrix theory, the rigidity of the level distribution is studied in terms of the $\Delta_3$-function. It is defined by

$$
\Delta_3(K, x_0) = \min_{A, B} \frac{1}{K} \int_{x_0-K/2}^{x_0+K/2} dx \ [M(x) - Ax - B]^2
$$

$$
= \langle M^2 \rangle - \langle M \rangle^2 - \frac{12}{K^2} \langle (x-x_0) \rangle < M >^2,
$$

(24)

where $M(x)$ is the integrated density of states for the unfolded levels

$$
M(x) = \int_0^x W(x) dx \quad (W(x); \text{the density of states}),
$$

(25)

and $\langle \cdots \rangle$ means

$$
\langle \cdots \rangle = \frac{1}{K} \int_{x_0-K/2}^{x_0+K/2} dx \cdots.
$$

(26)

If $\Delta_3$ is a constant independent of $K$, the levels must be equidistant. If the levels are uncorrelated and obey the Poisson distribution, it can be shown that $\Delta_3$ is proportional to $K$. The exact $K$ dependences of $\Delta_3$ for GOE, GUE, GSE and the Poissonian are known.

The sample averages are taken for fixed $K$ and $x_0$. An example of $\Delta_3(K, x_0)$ calculated from numerical data is shown in Fig. 7 for the case of the lowest Landau band with short-ranged disordered potential. It is confirmed that at the band centers the $K$-dependence of $\Delta_3$ is size independent. This means that $\Delta_3$ shows a kind of critical behavior at the band centers. Note that it is difficult to fix the energy region in this type of analysis since the finite value of $K$ is necessary by definition.

In order to parametrize the $\Delta_3$-function, we introduce its integral in the form,

$$
D_3(x_0) = \int_0^{K_0} dK \Delta_3(K, x_0),
$$

(27)

choosing $K_0$ as a sufficiently large constant. Then we consider the energy ($x_0$) and the size ($N_{\text{mat}}$) dependences of this integral. It is easily confirmed that this energy and size...
Figure 8: An example of the scaling behavior of the parameter $D_3$; the lowest Landau band with $\delta$-correlated potential. For the upper limit of the integral $K_0$, two values are chosen.

Figure 9: Examples of two level correlation function for the case with $N_{\text{mat}} = 400$ and $x_0 = 0$ and $-180$: (a) the lowest Landau band and (b) the second lowest Landau band. In both cases, the disordered potential is $\delta$-correlated. The dashed line represents $Y_{\text{GUE}}(\omega)$.

dependences follow the same scaling relation as other parameters discussed above (see Fig. 8).

The choice of $K_0$ is found not to be crucial for the final scaling argument (see Fig. 8). It is confirmed that different choices of $K_0$ gives the same scaling exponent though the absolute value of $D_3$ depends on $K_0$.

5 Two Level Correlation Function

It has been pointed out that the two level correlation function plays an important role to determine the behaviors of physical quantity such as response functions. [39]

By using the density of unfolded levels before sample average $W(x)$, the two level correlation function $Y$ is defined as

$$Y(\omega, x_0, N_{\text{mat}}) = \overline{W(x)W(x+\omega)},$$
where the overline means the averages over $x$ and over samples under the condition that both levels $x$ and $x+\omega$ are within the region $(x_0-N_{\text{mat}}/20, x_0+N_{\text{mat}}/20)$. [40] Examples of the two level correlation function in the cases of the two different Landau bands with short-ranged disordered potential and with the matrix size $N_{\text{mat}} = 400$ are shown in Fig. 9 for $x_0 = 0$ and $-180$. For large level separation $\omega$ it tends to unity irrespectively of the extent of the localization. This is because the densities of states at sufficiently separated energies are uncorrelated and the average value of the density of unfolded levels is unity by definition. [26] On the other hand, when the separation $\omega$ is small, the two level correlation function behaves in much different ways depending whether the corresponding states are delocalized or strongly localized. The exact functional forms of $Y(\omega)$ are known for GOE, GUE, and GSE cases, [6] which are believed to describe the level statistics in the metallic regime. For example, that for GUE is given by

$$Y_{\text{GUE}}(\omega) = 1 - \left( \frac{\sin \pi \omega \omega}{\pi \omega} \right)^2$$

In Fig. 9 we show $Y_{\text{GUE}}(\omega)$ by a thin dashed curve.

In order to parametrize the energy and size dependence of the two level correlation function, we introduce the following integral,

$$z(x_0, N_{\text{mat}}) = \int_0^2 [1 - Y(\omega, x_0, N_{\text{mat}})]d\omega.$$
Figure 10: Scaling behavior of the parameter \( z(x_0, N) \) obtained from the two level correlation function \( Y(\omega, x_0, N) \) in the lowest and the second lowest Landau bands with short-ranged random potential.

Here the upper bound of the integral has no special meaning; any value can be chosen as far as it is in the region where \( Y(\omega) \) is almost constant for any value of \( x_0 \). It is not difficult to confirm that this parameter \( z(x_0, N) \) obeys the same scaling relation as discussed for other parameters characterizing the level statistics in previous sections (see Fig. 10).

6 Summary and Discussion

In this paper we have discussed the scaling behavior of the energy and system size dependences of the level statistics in 2D disordered electron systems subject to strong magnetic fields (the quantum Hall regime) from various points of view in the light of the RMT. Not only the level statistics in the lowest Landau band but also those in the second lowest Landau band are considered. Furthermore the correlation length of the disordered potential has been set to various values.

In treating the level spacing distribution, we have discussed the effective level repulsion parameter and the GUE-GOE crossover parameter. Both parameters are found to satisfy the same scaling relation. From this scaling behavior the critical exponent of the localization length can be estimated. In the case of the lowest Landau band with short-ranged potential, the exponent is consistent with that obtained by other studies. It is also confirmed that the second lowest Landau band with short-ranged potential is pathological. The longer correlation length of the disordered potential leads to the smaller exponent within the present treatment. Namely the universality was not confirmed even within the lowest Landau band in contrast to the conclusion of the real space scaling analysis.

Experimental situation about the universality is a bit delicate. In AlGaAs/GaAs systems, Koch et al. obtained the critical exponent \( \nu = 2.3 \) for three lowest Landau bands and claimed to have confirmed the universality. On the other hand, in the case of Si-MOS systems, magnetic field and sample dependent exponents are reported. Although one of the reasons of the lack of universality, if any, may be the mutual interaction of electrons, the possibility of the potential range dependence of the exponent should be reconsidered. In experiments, the energy of the carrier cannot be precisely fixed, and therefore the observed quantities may be coarse-grained. Even if the rigorous exponent might have a universality, the exponent estimated from coarse-grained measurements may not apparently satisfy the universality. The present analysis of the critical behavior in terms of level statistics will correspond to the measurements coarse-grained in the energy region.

The pathological behavior of the second lowest Landau band is improved by increasing the correlation length of the potential near to magnetic length. If we increase the correlation length up to two times the magnetic length, the maximum value of the effective level repulsion parameter gets very small. This would not be inconsistent with the notion that
in the limit of slowly varying potential (so-called percolation limit) the electronic states correspond to semiclassical orbits along equipotential lines. In this limit the level repulsion will become very small.

The same scaling behaviors have been found for the $\Delta_3$-statistics and the two level correlation function. The fact that various parameters characterizing the level statistics satisfy the same scaling relation means the existence of the characteristic length determining the size dependence of the level statistics. In the present problem it is most natural to consider that this length is the localization length.

The pathological behavior of the second lowest Landau band might be understood by introducing a very long irrelevant length as suggested by Huckestein. [30] However the physical meaning of the irrelevant field is not clear. Liu and Das Sarma [27] suggested that the spatial symmetry of the basis function of each Landau band might play some role in this pathological behavior. In order to see whether this is the case, we should investigate higher Landau bands.

One of the most important results in the present study is the existence of the scale invariant critical statistics. For the critical level spacing distribution, the form eq. (21) with eq. (22) is suggested in the analytic study by Kravtsov et al. [22] Our numerical data in the case of short-ranged potentials can be fitted to it, but cannot give a correct exponent. Furthermore we have to use two-parameters fitting in the case of long-ranged potentials. In the case of the symplectic symmetry, a similar problem has been pointed out by several authors. [23, 43, 16] In 3D orthogonal case, the critical level spacing distribution corresponding to eq. (21) could be successfully applied [17] to some extent. Even in this case, however, the existence of the deviation from that expression in the large separation region has been pointed out. [48] More elaborated analytical studies will be necessary in order to understand truly the critical level statistics at the Anderson transition. Present numerical work on the quantum Hall regime encourages certainly such studies.

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In fact, a detailed observation of Fig. 7 tells us that the data for $x = 0$ show a slight size dependence in the large $K$ region. This is thought to be due to the fact that the smaller the system size, the larger the original energy is if the value of $K$ is fixed; namely the effect of localization appears more strongly for smaller system sizes. This consideration is consistent with the direction of the size dependence of the data for $x = 0$ which is seen near $K \sim 30$ in Fig. 7.