SOLVING FRACTIONAL ADVECTION-DIFFUSION EQUATION USING GENOCCHI OPERATIONAL MATRIX BASED ON ATANGANA-BALEANU DERIVATIVE

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Abstract. In recent years, a new definition of fractional derivative which has a nonlocal and non-singular kernel has been proposed by Atangana and Baleanu. This new definition is called the Atangana-Baleanu derivative. In this paper, we present a new technique to obtain the numerical solution of advection-diffusion equation containing Atangana-Baleanu derivative. For this purpose, we use the operational matrix of fractional integral based on Genocchi polynomials. An error bound is given for the approximation of a bivariate function using Genocchi polynomials. Finally, some examples are given to illustrate the applicability and efficiency of the proposed method.

1. Introduction. Atangana and Baleanu have proposed a new fractional derivative defined by Mittag-Leffler function in 2016 [2]. More details and results can be seen in [1, 3, 4, 7, 15, 18, 22, 27, 25]. It is worthy to be noted here that the definition of Atangana-Baleanu derivative (AB-derivative) besides having all advantages of the Riemann-Liouville and Caputo derivatives, has a non-singular kernel. Some important properties of AB-derivative have been investigated in [4, 5].

Approximation of the solution using polynomials is known as a useful tool in solving many classes of equations, numerically, e.g., fractional differential equations (FDEs) [24], variable order differential equations [10, 11, 12, 13] and fractional partial differential equations [6, 14, 16, 17, 21, 26]. This technique reduces differential equations to a system of algebraic equations which greatly simplifies the problem.

In this article, we obtain the operational matrix for Atangana-Baleanu integral (AB-integral) based on Genocchi polynomials (GPs) and apply it for solving advection-diffusion equations of the form

\[ \text{ABC}_{t}^{\alpha} h(x, t) + \lambda \frac{\partial h(x, t)}{\partial x} = \mu \frac{\partial^2 h(x, t)}{\partial x^2}, \]

with the initial condition

\[ h(x, 0) = \omega(x), \]

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and the boundary conditions
\[ h(0,t) = v_0(t), \quad h(1,t) = v_1(t), \quad (3) \]
where \( \lambda \) is an arbitrary constant which denotes the speed of advection and \( \mu > 0 \) is the diffusion coefficient. Also, \( \omega(x) \), \( v_0(t) \) and \( v_1(t) \) are known functions.

Advection-diffusion equations consist an important class of differential equations with many applications in the modeling of physical systems, especially those involving fluid flow [9, 23]. Nevertheless, solving problem (1)–(3) using the analytical methods is very difficult. This is the reason why numerical solution of (31) is important.

The organization of this article is as follows. In Section 2, we recall some important properties of AB-derivative and integral. In Section 3, some properties of GPs are reviewed. Section 4 is devoted to introducing the operational matrix of AB-integral and its error bound. Section 5 is concerned with presenting a new numerical method for solving problem (1)–(3) based on the properties of the GPs. In Section 6, we solve some test examples using this new technique. In the last section, we conclude the paper.

2. AB–derivative and some properties. In this section, we bring definitions of the AB–derivative and integral operators. Moreover, some properties of AB-integral operator are recalled.

**Definition 2.1.** Let \( 0 \leq \alpha < 1 \) and \( h(t) \in H^1(a,b) \). The fractional order AB-derivative is defined as follows [2]:
\[ ABC D_0^\alpha h(t) = \frac{B(\alpha)}{(1-\alpha)} \int_a^t h'(x) E_\alpha[-\alpha \frac{(t-x)^\alpha}{1-\alpha}] dx, \]
where \( B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \) is the normalization function and \( E_\alpha \) is Mittag-Leffler function.

**Definition 2.2.** Let \( 0 \leq \alpha < 1 \) and \( h(t) \in H^1(a,b) \). The fractional AB–integral is defined as follows [2]:
\[ AB I_0^\alpha h(t) = \left( \frac{1-\alpha}{B(\alpha)} \right) h(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_a^t h(x) (t-x)^{\alpha-1} dx. \]

In particular, when \( \alpha = 0 \), we recover the initial function and when \( \alpha = 1 \), we will have the classical integral.

We list here some of the main properties of AB-integral [8]. If \( \alpha, \beta \in [0,1] \) and \( C, a, b \in \mathbb{R} \), then:

1) \[ AB I_t^\alpha C = \frac{C}{B(\alpha)} \left( 1 - \alpha + \frac{x^\alpha}{\Gamma(\alpha)} \right), \]
2) \[ AB I_t^\alpha t^\beta = \frac{t^\beta}{B(\alpha)} \left( 1 - \alpha + \frac{\alpha \Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^\alpha \right), \]
3) \[ AB I_t^\alpha (a f(t) + b g(t)) = a AB I_t^\alpha f(t) + b AB I_t^\alpha g(t). \]
3. Properties of GPs. In this section, we recall some basic properties of GPs and explain how to approximate a bivariate function in terms of these basis functions.

Genocchi polynomials and Genocchi numbers are respectively given by

\[
2^t \exp xt = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi, \tag{4}
\]

\[
2^t \exp t + 1 = \sum_{n=0}^{\infty} \frac{g_n t^n}{n!}, \quad |t| < \pi.
\]

The Genocchi polynomial \(G_n(x)\) defined in (4) has the following analytic form

\[
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} g_{n-k} x^k, \tag{5}
\]

where \(g_{n-k}\) is Genocchi number. Some important properties of GPs are listed below:

\[
\int_{0}^{1} G_n(x)G_m(x)dx = \frac{2(-1)^n n! m!}{(m + n)!} g_{m+n}, \quad m, n \geq 1,
\]

\[
\frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad 1 \leq n,
\]

\[
G_n(1) + G_n(0) = 0, \quad 1 < n.
\]

For two arbitrary functions \(f\) and \(g\) in \(L^2[0, 1]\), we consider the inner product and norm in the space \(L^2[0, 1]\) as follows:

\[
\langle f, g \rangle = \int_{0}^{1} f(x)g(x)dx,
\]

\[
\|f\| = \sqrt{\langle f, f \rangle}.
\]

Suppose that \(f \in L^2[0, 1]\). Then, the function \(f\) can be approximated using GPs as

\[
f(x) \simeq f_M(x) = \sum_{m=1}^{M+1} c_m G_m(x) = C^T \Phi(x), \tag{6}
\]

where

\[
C = [c_1, c_2, \cdots, c_{M+1}]^T,
\]

and

\[
\Phi(x) = [G_1(x), G_2(x), \cdots, G_{M+1}(x)]^T. \tag{7}
\]

From (6), we have

\[
\langle f(x), \Phi(x) \rangle = C^T \langle \Phi(x), \Phi(x) \rangle.
\]

By considering \(W = \langle \Phi(x), \Phi(x) \rangle\), we get

\[
C = W^{-1} \langle f(x), \Phi(x) \rangle.
\]

Similarly, a function \(h(x, t) \in L^2([0, 1] \times [0, 1])\) can be expanded in terms of the GPs as

\[
h(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} h_{i,j} G_i(x) G_j(t).
\]
By considering a finite number of terms in the above infinite series we obtain an approximation of \( h \) as follows:

\[
h(x, t) \simeq h_M(x, t) = \sum_{i=1}^{M+1} \sum_{j=1}^{M+1} h_{i,j} G_i(x) G_j(t) = \Phi^T(x)H\Phi(t), \tag{8}
\]

where \( H = [h_{i,j}]_{(M+1) \times (M+1)} \). From (8), we have

\[
\langle \Phi(x), \langle h(x, t), \Phi(t) \rangle \rangle = \langle \Phi(x), \Phi(t) \rangle H \langle \Phi(t), \Phi(t) \rangle.
\]

Therefore, we get

\[
H = W^{-1} \langle \Phi(x), \langle h(x, t), \Phi(t) \rangle \rangle W^{-1}.
\]

Let \( X = L^2([0, 1] \times [0, 1]) \) and \( X_M = \text{span} \{ G_1(x) G_1(t), \ldots, G_1(x) G_{M+1}(t), \ldots, G_{M+1}(x) G_1(t), \ldots, G_{M+1}(x) G_{M+1}(t) \} \), be the space of all bivariate polynomials of degree less than or equal to \( M \) on \( x \) and \( t \). Then, \( h_M(x, t) \) is the best approximation of the function \( h(x, t) \in X \) onto \( X_M \). We prove the following result for the error of approximation of a bivariate function using GPs.

**Theorem 3.1.** Let \( h \) be a sufficiently smooth function on \( \Omega := [0, 1] \times [0, 1] \) and \( h_M \) be its approximation using GPs given by (8). Then, we have

\[
||h(x, t) - h_M(x, t)||^2 = O \left( \frac{1}{(M + 1)!2^{M+1}} \right).
\]

**Proof.** Suppose that \( P_M \) is the interpolating polynomial to \( h \) at points \((x_i, t_j)\), where \( x_i, i = 1, \ldots, M+1 \), and \( t_j, j = 1, \ldots, M+1 \) are the roots of \( (M+1)\)–degree shifted Chebyshev polynomial on \([0, 1]\). Then [20]

\[
h(x, t) - P_M(x, t) = \frac{\partial^{M+1} h(\xi, t)}{\partial x^{M+1}} \prod_{i=1}^{M+1} (x - x_i) + \frac{\partial^{M+1} h(x, \eta)}{\partial t^{M+1}} \prod_{j=1}^{M+1} (t - t_j) \]

\[
- \frac{\partial^{2M+2} h(\xi', \eta')}{\partial x^{M+1} \partial t^{M+1}} \prod_{i=1}^{M+1} (x - x_i) \prod_{j=1}^{M+1} (t - t_j),
\]

where \( \xi, \xi', \eta, \eta' \in [0, 1] \). Therefore

\[
|h(x, t) - P_M(x, t)| \leq \max_{(x, t) \in \Omega} \left| \frac{\partial^{M+1} h(x, t)}{\partial x^{M+1}} \right| \prod_{i=1}^{M+1} |x - x_i| \frac{(M + 1)!}{(M + 1)!} \]

\[
+ \max_{(x, t) \in \Omega} \left| \frac{\partial^{M+1} h(x, t)}{\partial t^{M+1}} \right| \prod_{j=1}^{M+1} |t - t_j| \frac{(M + 1)!}{(M + 1)!} \]

\[
+ \max_{(x, t) \in \Omega} \left| \frac{\partial^{2M+2} h(x, t)}{\partial x^{M+1} \partial t^{M+1}} \right| \prod_{i=1}^{M+1} |x - x_i| \prod_{j=1}^{M+1} |t - t_j| \frac{(M + 1)!}{((M + 1)!)^2}.
\]

Since \( h \) is a sufficiently smooth function, there are real numbers \( K_1, K_2 \) and \( K_3 \), such that

\[
\max_{(x, t) \in \Omega} \left| \frac{\partial^{M+1} h(x, t)}{\partial x^{M+1}} \right| \leq K_1, \tag{10}
\]

\[
\max_{(x, t) \in \Omega} \left| \frac{\partial^{M+1} h(x, t)}{\partial t^{M+1}} \right| \leq K_2, \tag{11}
\]

\[
\max_{(x, t) \in \Omega} \left| \frac{\partial^{2M+2} h(x, t)}{\partial x^{M+1} \partial t^{M+1}} \right| \leq K_3. \tag{12}
\]
Using (10)–(12) in (9) and employing the estimates for Chebyshev interpolation nodes, we get

\[ |h(x, t) - P_M(x, t)| \leq \frac{K}{(M + 1)!2^{2M+1}} + \frac{K_3}{((M + 1)!)^2 2^{2(M+2)}}, \]

where \( K = K_1 + K_2 \). Since \( h_M \) is the best approximation of \( f \) in \( X_M \), we have

\[ \|h(x, t) - h_M(x, t)\|_2 \leq \|h(x, t) - f(x, t)\|_2, \]

where \( f(x, t) \) is any arbitrary polynomial in \( X_M \). In particular, we get

\[ \|h(x, t) - h_M(x, t)\|_2^2 = \int_0^1 \int_0^1 |h(x, t) - h_M(x, t)|^2 \, dx \, dt \]

\[ \leq \int_0^1 \int_0^1 |h(x, t) - P_M(x, t)|^2 \, dx \, dt. \]

Taking into consideration (13) and (14), we obtain

\[ \|h(x, t) - h_M(x, t)\|_2^2 \leq \int_0^1 \int_0^1 \left( \frac{K}{(M + 1)!2^{2M+1}} + \frac{K_3}{((M + 1)!)^2 2^{2(M+2)}} \right)^2 \, dx \, dt \]

\[ = \left( \frac{K}{(M + 1)!2^{2M+1}} + \frac{K_3}{((M + 1)!)^2 2^{2(M+2)}} \right)^2, \]

which gives the desired result by taking the squared root from the both sides. \( \square \)

4. Operational matrix of AB–integral. This section is devoted to introducing the operational matrix of AB–integral operator based on the GPs and its error. In order to introduce operational matrix for AB–integral, first we recall the following theorem from [8].

**Theorem 4.1.** [8] Let \( 0 \leq \alpha < 1 \). For \( k \in \mathbb{N} \), the AB–integral operator of order \( \alpha \neq [\alpha] \) of \((t-a)^k\) is given as:

\[ AB I_t^\alpha (t-a)^k = \frac{(t-a)^k}{B(\alpha)} \left[ 1 - \alpha + \frac{\alpha(t-a)^\alpha}{\Gamma(\alpha)} \sum_{r=0}^k \frac{(-1)^r k!}{r! (\alpha)^r} (1 - \frac{a}{t})^r \right]. \]

In the following theorem, we describe the process of obtaining GPs operational matrix of AB–integral operator.

**Theorem 4.2.** Suppose that \( \Phi(t) \) be GPs vector defined by (7) and \( 0 \leq \alpha \leq 1 \), then:

\[ AB I_t^\alpha \Phi(t) \approx \Psi^\alpha \Phi(t), \]

where \( \Psi^\alpha \) is the \((M+1) \times (M+1)\) operational matrix of AB-integral operator of order \( \alpha \) which is given by:

\[
\Psi^\alpha = \begin{bmatrix}
\Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,M+1} \\
\sum_{k=1}^{M+1} \Omega_{2,1,k} & \sum_{k=1}^{M+1} \Omega_{2,2,k} & \cdots & \sum_{k=1}^{M+1} \Omega_{2,M+1,k} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{M+1} \Omega_{i,1,k} & \sum_{k=1}^{M+1} \Omega_{i,2,k} & \cdots & \sum_{k=1}^{M+1} \Omega_{i,M+1,k} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{M+1} \Omega_{M+1,1,k} & \sum_{k=1}^{M+1} \Omega_{M+1,2,k} & \cdots & \sum_{k=1}^{M+1} \Omega_{M+1,M+1,k}
\end{bmatrix},
\]
with:

\[ \Omega_{i,j,k} = \frac{(i)!g_{i-k}}{(i-k)!(k)!} \left[ \frac{1-\alpha}{B(\alpha)}d_{j,k} + \frac{\alpha e_{j,k}}{B(\alpha)\Gamma(\alpha)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \right]. \]

**Proof.** By using equations (5) and (15), for \( i = 1, 2, 3, \ldots, M + 1 \) we have

\[ AB I_0^\alpha G_i(t) = \sum_{k=1}^{i} \frac{(i)!g_{i-k}}{(i-k)!(k)!} AB I_0^\alpha t^k, \]

\[ = \sum_{k=1}^{i} \frac{(i)!g_{i-k}}{(i-k)!(k)!} t^k \left[ \frac{1-\alpha}{B(\alpha)}d_{j,k} + \frac{\alpha e_{j,k}}{B(\alpha)\Gamma(\alpha)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \right] \]

(18)

\[ = \sum_{k=1}^{i} \frac{(i)!g_{i-k}}{(i-k)!(k)!} \left[ \frac{1-\alpha}{B(\alpha)}t^k + \frac{\alpha t^{k+\alpha}}{B(\alpha)\Gamma(\alpha)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \right]. \]

Let \( g_1(t) = t^k, \ g_2(t) = t^{k+\alpha} \), then we approximate the functions \( g_1(t) \) and \( g_2(t) \) using the GPs as follows:

\[ g_1(t) = \sum_{j=1}^{M+1} d_{j,k} G_j(t), \quad (19) \]

\[ g_2(t) = \sum_{j=1}^{M+1} e_{j,k} G_j(t). \quad (20) \]

By substituting equations (19) and (20) into (18), we get

\[ AB I_0^\alpha G_i(t) \simeq \sum_{j=1}^{M+1} \left( \sum_{k=1}^{i} \frac{(i)!g_{i-k}}{(i-k)!(k)!} \left[ \frac{1-\alpha}{B(\alpha)}d_{j,k} + \frac{\alpha e_{j,k}}{B(\alpha)\Gamma(\alpha)} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \right] \right) G_j(t) \]

\[ = \sum_{j=1}^{M+1} \left( \sum_{k=1}^{i} \Omega_{i,j,k} \right) G_j(t), \]

which completes the proof. \( \square \)

In this part, we prepare an error bound for operational matrix of AB–integral. To do this, we give the following theorem.

**Theorem 4.3.** The error \( |\Delta_M| = |AB I_0^\alpha h(t) - AB I_0^\alpha h_M(t)| \) in approximating \( AB I_0^\alpha h(t) \) with operational matrix of fractional integral is bounded as follows:

\[ |\Delta_M| \leq \sum_{i=M+2}^{\infty} |c_i| \sum_{j=1}^{M+1} |\psi_{i,j}| \sum_{k=1}^{j} \binom{j}{k} |g_{j-k}|, \]

where \( c_i, i = 1, \ldots, M + 1, \) are the coefficients of the GPs approximation of the function \( h(t) \), and

\[ \psi_{i,j} = \sum_{k=1}^{i} \Omega_{i,j,k}, \quad i = 1, \ldots, M + 1, \quad j = 1, \ldots, M + 1. \]

**Proof.** We have

\[ h(t) = \sum_{i=1}^{\infty} c_i G_i(t). \]
Using (16), we get

\[ AB I_t^\alpha h(t) = \sum_{i=1}^{\infty} c_i \sum_{j=1}^{M+1} \psi_{i,j} G_j(t). \]

By considering only the first \( M + 1 \) terms of the above infinite series, we obtain

\[ AB I_t^\alpha h(t) - \sum_{i=1}^{M+1} c_i \sum_{j=1}^{M+1} \psi_{i,j} G_j(t) = \sum_{i=M+2}^{\infty} c_i \sum_{j=1}^{M+1} \psi_{i,j} G_j(t), \quad (21) \]

then, using (6) and (16), we show (21) in a matrix form as follow:

\[ AB I_t^\alpha h(t) - C^T \Psi^\alpha \Phi(t) = \sum_{i=M+2}^{\infty} c_i \sum_{j=1}^{M+1} \psi_{i,j} G_j(t). \]

So, we have

\[ |AB I_t^\alpha h(t) - C^T \Psi^\alpha \Phi(t)| = \left| \sum_{i=M+2}^{\infty} c_i \sum_{j=1}^{M+1} \psi_{i,j} G_j(t) \right| \leq \sum_{i=M+2}^{\infty} |c_i| \sum_{j=1}^{M+1} |\psi_{i,j}| |G_j(t)|. \quad (22) \]

We get an upper bound for GPs as follows:

\[ |G_j(t)| = \left| \sum_{k=1}^{j} \binom{j}{k} g_{j-k} t^k \right| \leq \sum_{k=1}^{j} \binom{j}{k} |g_{j-k}| |t^k| \leq \sum_{k=1}^{j} \binom{j}{k} |g_{j-k}|. \quad (23) \]

Therefore by substituting (23) into (22), we will have

\[ |AB I_t^\alpha h(t) - C^T \Psi^\alpha \Phi(t)| \leq \sum_{i=M+2}^{\infty} |c_i| \sum_{j=1}^{M+1} \psi_{i,j} \sum_{k=1}^{j} \binom{j}{k} |g_{j-k}|. \]

So, we have

\[ |AB I_t^\alpha h(t) - AB I_t^\alpha h_M(t)| \leq \sum_{i=M+2}^{\infty} |c_i| \sum_{j=1}^{M+1} \psi_{i,j} \sum_{k=1}^{j} \binom{j}{k} |g_{j-k}|. \]

Therefore, the proof is complete. \( \square \)

5. **Numerical method.** In this section, we introduce a numerical method based on the GPs for solving (1) with initial condition (2) and boundary conditions (3).

For solving (1), we present an approximation of \( \frac{\partial^2 h(x,t)}{\partial x^2} \) using the GPs as follows:

\[ \frac{\partial^2 h(x,t)}{\partial x^2} \simeq \sum_{i=1}^{M+1} \sum_{j=1}^{M+1} h_{i,j} G_i(x) G_j(t) = \Phi^T(x) H \Phi(t). \quad (24) \]

By integrating of the both sides of (24), we get

\[ \frac{\partial h(x,t)}{\partial x} = \frac{\partial h(x,t)}{\partial x} \bigg|_{x=0} + \Phi^T(x) \rho^T H \Phi(t), \quad (25) \]

where \( \rho = \Psi^4 \) which is given by (17) with \( \alpha = 1 \). Also, by integrating of (25) and using the boundary condition given by (2), we have

\[ h(x,t) = v_0(t) + x \frac{\partial h(x,t)}{\partial x} \bigg|_{x=0} + \Phi^T(x) (\rho^T)^2 H \Phi(t). \quad (26) \]
Let $x = 1$, then from (26) we get
\[ \frac{\partial h(x,t)}{\partial x} \bigg|_{x=0} = v_1(t) - v_0(t) - \Phi^T(1)(\rho^T)^2H\Phi(t). \] (27)

By substituting (27) into (25), we get
\[ \frac{\partial h(x,t)}{\partial x} = v_1(t) - v_0(t) - \Phi^T(1)(\rho^T)^2H\Phi(t) + \Phi^T(x)\rho^T H\Phi(t) = \Phi^T(x)uQ_1^T\Phi(t) - \Phi^T(x)uQ_0^T\Phi(t) - \Phi^T(x)u\Phi^T(1)(\rho^T)^2H\Phi(t) + \Phi^T(x)\rho^T H\Phi(t) = \Phi^T(x)\left[uQ_1^T - uQ_0^T - u\Phi^T(1)(\rho^T)^2H + \rho^T H\right] \Phi(t) = \Phi^T(x)\Delta_1 \Phi(t), \] (28)
where
\[ u = [1,0,\cdots,0], \]
\[ v_0(t) = Q_0^T\Phi(t), \]
\[ v_1(t) = Q_1^T\Phi(t), \]
\[ \Delta_1 = uQ_1^T - uQ_0^T - u\Phi^T(1)(\rho^T)^2H + \rho^T H. \]

Furthermore, by substituting (27) into (26), we have
\[ h(x,t) = v_0(t) + x [v_1(t) - v_0(t) - \Phi^T(1)(\rho^T)^2H\Phi(t)] + \Phi^T(x)\rho^T H\Phi(t) = \Phi^T(x)uQ_0^T\Phi(t) + \Phi^T(x)XQ^T_1\Phi(t) - \Phi^T(x)XQ^T_0\Phi(t) - \Phi^T(x)H\Phi(t) + \Phi^T(x)(\rho^T)^2H\Phi(t) = \Phi^T(x)\left[uQ_0^T + XQ^T_1 - XQ^T_0 - X\Phi^T(1)(\rho^T)^2H + (\rho^T)^2H\right] \Phi(t) = \Phi^T(x)\Delta_2 \Phi(t), \] (29)
where we have used
\[ x \simeq X^T \Phi(x), \]
\[ \Delta_2 = uQ_0^T + XQ^T_1 - XQ^T_0 - X\Phi^T(1)(\rho^T)^2H + (\rho^T)^2H. \]

Now by applying $\Lambda^B I_t^\alpha$ to both sides of (1), we have
\[ h(x,t) = \omega(x) + \Lambda^B I_t^\alpha \left[ \mu \frac{\partial^2 h(x,t)}{\partial x^2} - \lambda \frac{\partial h(x,t)}{\partial x} \right] = \omega(x) + \mu^\Lambda^B I_t^\alpha \left( \frac{\partial^2 h(x,t)}{\partial x^2} \right) - \lambda^\Lambda^B I_t^\alpha \left( \frac{\partial h(x,t)}{\partial x} \right). \] (30)

We set
\[ \omega(x) \simeq J^T \Phi(x), \]
then, by substituting equations (24), (28) and (29) into (30), we obtain
\[ \Phi^T(x)\Delta_2 \Phi(t) = \Phi^T(x)Ju\Phi(t) + \mu^\Lambda^B I_t^\alpha (\Phi^T(x)H\Phi(t)) - \lambda^\Lambda^B I_t^\alpha \left( \Phi^T(x)\Delta_1 \Phi(t) \right) = \Phi^T(x)Ju\Phi(t) + \mu^\Phi^T(x)H\Psi^\alpha \Phi(t) - \lambda^\Phi^T(x)\Delta_1 \Psi^\alpha \Phi(t) = \Phi^T(x)J[Ju + \mu H\Psi^\alpha - \lambda \Delta_1 \Psi^\alpha] \Phi(t). \]

So, we get
\[ \Phi^T(x)\left[ \Delta_2 - Ju = \mu H\Psi^\alpha + \lambda \Delta_1 \Psi^\alpha \right] \Phi(t) = 0. \]
Therefore, the unknown parameters of $H$ can be given by solving the following system

$$\Delta_2 - Ju - \mu H\Psi^\alpha + \lambda \Delta_1 \Psi^\alpha = 0.$$ 

Finally, by substituting the obtained result for $H$ into (29), an approximation of the unknown function $h(x,t)$ is given.

6. **Numerical examples.** In this section, we solve some examples of the fractional advection–diffusion equations with AB–derivative using the method introduced in the previous section.

**Example 6.1.** Consider the fractional advection–diffusion equation [26]:

$$ABC \, D_t^\alpha h(x,t) + \lambda \frac{\partial h(x,t)}{\partial x} = \mu \frac{\partial^2 h(x,t)}{\partial x^2},$$

with the initial condition

$$h(x,0) = \exp \left(-\frac{(x - \lambda)^2}{4\mu}\right),$$

and the boundary conditions

$$h(0,t) = \frac{1}{\sqrt{1+t}} \exp \left(-\frac{((1+t)\lambda)^2}{4\mu(1+t)}\right),$$

$$h(1,t) = \frac{1}{\sqrt{1+t}} \exp \left(-\frac{(1-(1+t)\lambda)^2}{4\mu(1+t)}\right).$$

The exact solution of the above problem in the case $\alpha = 1$ is

$$h(x,t) = \frac{1}{\sqrt{1+t}} \exp \left(-\frac{(x -(1+t)\lambda)^2}{4\mu(1+t)}\right).$$

By setting $M = 4$, we have solved this problem with $\alpha = 0.99, \lambda = 0.25$ and $\mu = 0.1$ and reported the numerical results in Figure 1. Moreover, the numerical results for the absolute error obtained by $M = 3, 6$ and $\alpha = 0.99$, are displayed in Table 1. In Figure 2, we compare the approximate solutions at $t = 0.5$ given for $\alpha = 0.9, 0.95, 0.99$ and $M = 5$ with the exact solution for $\alpha = 1$. As it can be seen from this figure, the numerical solution converges to the exact solution when $\alpha \rightarrow 1$.

| Table 1. Numerical results of the absolute error when $\alpha = 0.99$ and $t = 1$ for Example 6.1. |
|---|---|---|
| $x$ | $M=3$ | $M=6$ |
| 0.0 | $4.39878e-3$ | $7.19623e-4$ |
| 0.1 | $2.83179e-3$ | $6.56846e-4$ |
| 0.2 | $3.61572e-3$ | $1.03531e-3$ |
| 0.3 | $1.44208e-3$ | $8.15792e-4$ |
| 0.4 | $8.62515e-4$ | $2.26231e-4$ |
| 0.5 | $1.46612e-3$ | $5.53976e-4$ |
| 0.6 | $2.43959e-4$ | $1.33992e-3$ |
| 0.7 | $3.58188e-3$ | $1.91911e-3$ |
| 0.8 | $6.64254e-3$ | $2.04788e-3$ |
| 0.9 | $6.52617e-3$ | $1.47985e-3$ |
| 1.0 | $3.29430e-4$ | $6.37627e-5$ |
Example 6.2. Consider the fractional advection–diffusion equation [19, 28]:

\[ ABC D^\alpha_t h(x,t) + \lambda \frac{\partial h(x,t)}{\partial x} = \mu \frac{\partial^2 h(x,t)}{\partial x^2}, \]

with the initial condition

\[ h(x,0) = a \exp(-cx), \]

and the boundary conditions

\[ h(0,t) = a \exp(bt), \]
\[ h(1,t) = a \exp(bt-c). \]

For \( \alpha = 1 \), the exact solution is given by

\[ h(x,t) = a \exp(bt-c), \quad c = \frac{-\lambda + \sqrt{\lambda^2 + 4\mu b}}{2\mu}. \]

By solving this problem using the proposed method with \( \alpha = 0.99, \lambda = 0.3, \mu = 0.1, a = 1, b = 0.2 \) and \( M = 4 \), the numerical results are displayed in Figure 3.
Figure 2. (Example 6.1) The exact and approximate solutions with different values of $\alpha$ and $M = 5$ at $t = 0.5$.

Furthermore, by employing the method with $M = 3, 6$ and $\alpha = 0.99$, the numerical results of the absolute error are given in Table 2. Finally, Figure 4 shows the numerical solutions given with $\alpha = 0.3, 0.5, 0.9$ and $M = 5$ together with the exact solution of $\alpha = 1$ at $t = 0.01$.

Table 2. Numerical results of the absolute error when $\alpha = 0.99$, $t = 1$ for Example 6.2.

| $x$  | $M = 3$     | $M = 6$     |
|------|-------------|-------------|
| 0.0  | $2.08503e-2$| $3.99527e-5$|
| 0.1  | $1.05799e-2$| $7.02578e-5$|
| 0.2  | $1.21467e-2$| $2.89318e-5$|
| 0.3  | $4.94776e-3$| $1.02674e-4$|
| 0.4  | $2.35280e-4$| $1.53367e-4$|
| 0.5  | $2.36604e-3$| $1.16798e-4$|
| 0.6  | $1.08676e-2$| $5.12801e-5$|
| 0.7  | $2.18511e-2$| $2.11059e-5$|
| 0.8  | $2.91950e-2$| $3.51774e-7$|
| 0.9  | $2.49148e-2$| $7.30602e-5$|
| 1.0  | $3.02368e-8$| $2.05050e-6$|

Example 6.3. Consider the fractional advection–diffusion equation [19]:

$$\frac{ABC}{D^\alpha_1}h(x, t) + \lambda \frac{\partial h(x, t)}{\partial x} = \mu \frac{\partial^2 h(x, t)}{\partial x^2},$$

with the initial condition

$$h(x, 0) = \sin (\pi x),$$

and the boundary conditions

$$h(0, t) = h(1, t) = 0.$$

For $\alpha = 1$, the exact solution is given by

$$h(x, t) = \exp (-t) \sin (\pi x).$$
By setting $\lambda = 0$, $\mu = \frac{1}{\pi^2}$, we have solved this problem with $M = 4$, $\alpha = 0.99$ and reported the numerical results in Figure 5. Moreover, the numerical results of the...
absolute error obtained by $M = 3, 6$ and $\alpha = 0.98$, are reported in Table 3. Also, by considering $\alpha = 0.9, 0.95, 0.99$ and $M = 5$ at the point $t = 0.5$, the numerical results are plotted in Figure 6.

![Figure 5. (Example 6.2) The exact solution with $\alpha = 1$ and numerical solution with $\alpha = 0.99$.](image)

Table 3. Numerical results of the absolute error when $\alpha = 0.98$ and $t = 1$ for Example 6.3.

| $x$ | $M=3$            | $M=6$            |
|-----|------------------|------------------|
| 0.0 | $9.58422e-3$     | $0.00000$        |
| 0.1 | $9.47883e-3$     | $1.32252e-3$     |
| 0.2 | $1.1959e-2$      | $2.51929e-3$     |
| 0.3 | $7.30440e-3$     | $3.47139e-3$     |
| 0.4 | $2.89100e-3$     | $4.07566e-3$     |
| 0.5 | $3.24315e-3$     | $4.28296e-3$     |
| 0.6 | $9.53273e-3$     | $4.07566e-3$     |
| 0.7 | $1.94066e-2$     | $3.47139e-3$     |
| 0.8 | $2.71593e-2$     | $2.51929e-3$     |
| 0.9 | $2.42335e-2$     | $1.32252e-3$     |
| 1.0 | $4.19753e-16$    | $4.50522e-17$    |
\( \alpha = 0.9 \)
\( \alpha = 0.95 \)
\( \alpha = 0.99 \)
\( \alpha = 1.0 \)

\[ x \]

**Figure 6.** (Example 6.3) The exact and approximate solutions for \( M = 5 \) at \( t = 0.5 \).

7. **Conclusion.** In this work, the operational matrix of Atangana-Baleanu integral based on the Genocchi polynomials has been employed for the first time for solving the fractional advection-diffusion equations. Error bounds have been introduced for the approximation of a bivariate function based on the Genocchi polynomials and the operational matrix of fractional integral. Some properties of Genocchi polynomials help us to reduce the problem to a system of linear algebraic equations. Some examples have been included and solved to illustrate the efficiency and accuracy of the proposed technique.

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