Some Theorems and Applications of the \((q, r)\)-Whitney Numbers

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Abstract

The \((q, r)\)-Whitney numbers were recently defined in terms of the \(q\)-Boson operators, and several combinatorial properties which appear to be \(q\)-analogues of similar properties were studied. In this paper, we obtain elementary and complete symmetric polynomial forms for the \((q, r)\)-Whitney numbers, and give combinatorial interpretations in the context of \(A\)-tableaux. We also obtain convolution-type identities using the combinatorics of \(A\)-tableaux. Lastly, we present applications and theorems related to discrete \(q\)-distributions.

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1 Introduction

In a recent paper, the author and Katriel \cite{21} introduced a new approach to generate \(q\)-analogues of Stirling and Whitney-type numbers. In this paper, the \((q, r)\)-Whitney numbers of the first and second kinds were defined as coefficients in

\[
m^n(a^\dagger)^n a^n = \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^\dagger a + r)^k
\]

and

\[
(ma^\dagger a + r)^n = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)(a^\dagger)^k a^k,
\]

respectively (cf. \cite{21}), by using as framework, the \(q\)-Boson operators \(a^\dagger\) and \(a\) of Arik and Coon \cite{2} which satisfy the commutation relation

\[
[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1.
\]
By convention, \(w_{m,r,q}(0,0) = W_{m,r,q}(0,0) = 1\) and \(w_{m,r,q}(n,k) = W_{m,r,q}(n,k) = 0\) for \(k > n\) and for \(k < 0\). Several combinatorial properties were already established, including the following triangular recurrence relations [21, Theorem 6]:

\[
w_{m,r,q}(n+1,k) = q^{-n}\left(w_{m,r,q}(n,k-1) - (m[n]_q + r)w_{m,r,q}(n,k)\right),
\]

with \([n]_q = \frac{q^n - 1}{q-1}\), the \(q\)-integer, and

\[
W_{m,r,q}(n+1,k) = q^{k-1}W_{m,r,q}(n,k-1) + (m[k]_q + r)W_{m,r,q}(n,k).
\]

From here, one readily obtains

\[
w_{m,r,q}(n,0) = (-1)^n q^{n-\binom{n}{2}} \prod_{i=0}^{n-1} (m[i]_q + r),
\]

\[
w_{m,r,q}(n,n) = q^{-\binom{n}{2}},
\]

\[
W_{m,r,q}(n,0) = r^n,
\]

and

\[
W_{m,r,q}(n,n) = q^{\binom{n}{2}}.
\]

The identities presented in Eqs. [4] and [5] can be used as tools to obtain further combinatorial identities for \(w_{m,r,q}(n,k)\) and \(W_{m,r,q}(n,k)\). For instance, with the aid of these recurrence relations, the vertical recurrence relations

\[
w_{m,r,q}(n+1,k+1) = \sum_{j=k}^{n} (-1)^{n-j} q^{\binom{n+1}{2} - \binom{j}{2}} w_{m,r,q}(j,k) \prod_{i=j+1}^{n} (m[i]_q + r),
\]

with \(\prod_{i=j+1}^{n} (m[i]_q + r) = 1\) when \(j = n\), and

\[
W_{m,r,q}(n+1,k+1) = q^k \sum_{j=k}^{n} (m[k+1]_q + r)^{n-j} W_{m,r,q}(j,k),
\]

can be proved by induction, as well as the rational generating function of the \((q,r)\)-Whitney numbers of the second kind given by

\[
\sum_{n=k}^{\infty} W_{m,r,q}(n,k)t^n = \frac{q^{\binom{r}{2}} t^k}{\prod_{i=0}^{k} (1 - (m[i]_q + r)t)}.
\]

On the other hand, the horizontal recurrence relations

\[
w_{m,r,q}(n,k) = q^n \sum_{j=0}^{n-k} (m[n]_q + r)^j w_{m,r,q}(n+1,k+j+1)
\]

with \([n]_q = \frac{q^{n+1} - 1}{q-1}\), the \(q\)-integer.
\[ W_{m,r,q}(n,k) = \sum_{j=0}^{n-k} (-1)^j q^{(k+j+1)} \frac{k+j}{j!} \prod_{i=0}^{j+1} (m[i]_q + r) W_{m,r,q}(n+1,k+j+1) \]  

(14)

can be verified by evaluating the right-hand sides using Eqs. (4) and (5). Before proceeding, we note that Eqs. (10) and (11) follow a behaviour similar to that of the Chu-Shi-Chieh’s identity (see [6]) for the classical binomial coefficients given by

\[ \binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k+1} + \cdots + \binom{n}{k} , \]

while Eqs. (13) and (14) are analogous with

\[ \binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + \cdots + (-1)^{n-k} \binom{n+1}{k} , \]

another known identity for the classical binomial coefficients.

The purpose of this paper is to express the \((q,r)-Whitney numbers of both kinds in symmetric polynomial forms. This proves to be useful in establishing combinatorial interpretations in terms of A-tableaux. In return, remarkable convolution-type identities are obtained and several other interesting theorems are also presented.

2 Explicit formulas in symmetric polynomial forms

2.1 \((q,r)-Whitney numbers of the first kind

Expanding the falling factorial \((x)_n = x(x-1)\cdots(x-n+1)\) in powers of \(x\), we obtain

\[ (x)_n = \sum_{k=0}^{n} (-1)^{n-k} x^k \prod_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} i_j , \]

which yield the well-known expression for the Stirling numbers of the first kind in terms of elementary symmetric functions. This relation can be generalized to the \(q\)-Stirling numbers as follows:

\[ [x]_q[x-1]_q \cdots [x-n+1]_q = [x]_q ([x]_q + q^x)([x]_q + q^x[2]_q) \cdots ([x]_q + q^x[n-1]_q) = \sum_{k=0}^{n} [x]_q^k \cdot q^{x(n-k)} \prod_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} [i_j]_q . \]

To further generalize this procedure to the \((q,r)-Whitney numbers of the first kind, recall that application of both sides of the defining relation in Eq. (11) on the \(q\)-boson number state \(|\ell\rangle\) gives

\[ m^n[\ell]_q[\ell-1]_q \cdots [\ell-n+1]_q = \sum_{k=0}^{n} w_{m,r,q}(n,k)(m[\ell]_q + r)^k . \]
Since both sides of this relation are finite polynomials in \( \ell \), and since the relation is valid for all integer \( \ell \), it remains valid when \( \ell \) is replaced by the real number \( x \), i.e.,

\[
m^n[x]_q[x-1]_q \cdots [x-n+1]_q = \sum_{k=0}^{n} w_{m,r,q}(n,k) \bigl(m[x]_q + r\bigr)^k.
\]

(15)

Now, defining \( y = [x]_q + \alpha \), where \( \alpha = \frac{x}{m} \), we note that \( [x-i]_q = q^{-i}(y - \alpha - [i]_q) \). Hence,

\[
m^n[y]_q[y-1]_q \cdots [y-n+1]_q = \sum_{k=0}^{n} (m[y]_q + r)^k q^{-\binom{n}{2}} (-1)^{n-k} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} (m[i_j]_q + r).
\]

(16)

The identity in the next theorem is obtained by comparing the right-hand-sides of Eqs. (15) and (16).

**Theorem 1.** The \((q,r)\)-Whitney numbers of the first kind satisfy the following explicit form

\[
w_{m,r,q}(n,k) = (-1)^{n-k} q^{-\binom{n}{2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} (r + [i_j]_q m).
\]

(17)

**Remark 2.** The sum within this theorem is the symmetric polynomial of degree \( n - k \) in the \( n \) variables \( \{(r + [i]_q m); i = 0, 1, \ldots, n - 1\} \). For \( r = 0 \) all the terms with \( i_1 = 0 \) vanish so the summation starts at 1, which is consistent with the expressions presented above for the Stirling and \( q \)-Stirling numbers of the first kind.

The above theorem can also be proved by induction as follows:

**Alternative proof of Theorem**[4]. The theorem readily yields \( w_{m,r,q}(0,0) = 1 \). Making the induction hypothesis that the theorem is true up to \( n \), for all \( k = 0, 1, \ldots, n \), we prove it for \( n + 1 \) and \( k = 0, 1, \ldots, n \), via the recurrence relation (4). Thus,

\[
w_{m,r,q}(n+1,k) = q^{-n} \left((-1)^{n+1-k} q^{-\binom{n+1}{2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n+1-k} \leq n-1} \prod_{j=1}^{n+1-k} (r + [i_j]_q m)\right.
\]

\[-(m[n]_q + r)(-1)^{n-k} q^{-\binom{n}{2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} (r + [i_j]_q m)\]

\[
= q^{-\binom{n+1}{2}} (-1)^{n+1-k} q^{-\binom{n}{2}} \left(\sum_{0 \leq i_1 < i_2 < \cdots < i_{n+1-k} \leq n-1} \prod_{j=1}^{n+1-k} (r + [i_j]_q m)\right)
\]

\[+(m[n]_q + r) \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} (r + [i_j]_q m).
\]

The first term within the large paretheses contains all products of \( n + 2 - k \) distinct factors out of \( \{(r + [i]_q m); i = 0, 1, \ldots, n - 1\} \), whereas the second term contains all products
of \( n + 2 - k \) distinct factors, one of which is \((r + m[n]_q)\) and the others chosen out of \(\{(r + [i]_q m); i = 0, 1, \ldots, n - 1\}\). Together, these sums yield
\[
\sum_{0 \leq i_1 < i_2 < \cdots < i_{n+1-k} \leq n} \prod_{j=0}^{n+1-k} (r + [i_j]_q m),
\]
thus establishing the theorem for the range of indices specified above. Finally, the theorem yields \( w_{m,r,q}(n+1,n+1) = q^{-{n \choose 2}}, \) in agreement with (7).

As \( q \to 1 \), the explicit formula (17) reduces to an expression for the \( r \)-Whitney numbers of the first kind given by
\[
w_{m,r}(n,k) = (-1)^{n-k} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} (r + i_j m).
\]

An equivalent of this identity was reported by Mangontarum et al. [18, Theorem 6]. For \( m = 1 \) and \( r = 0 \), (17) reduces to an explicit formula for a \( q \)-analogue of the Stirling numbers of the first kind, viz,
\[
{n \choose k}_q = (-1)^{n-k} q^{-n \choose 2} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} [i_j]_q,
\]
where \( \left[{n \atop k}\right]_q \) denote the \( q \)-Stirling numbers of the first kind defined by
\[
[x]_{q,n} = \sum_{k=1}^{n} (-1)^{n-k} \left[{n \atop k}\right]_q [x]_q^k,
\]
\( [x]_{q,n} = [x]_q [x-1]_q [x-2]_q \cdots [x-n+1]_q \) (cf. [4]). For any given set of \( n - k \) integers that satisfy \( 1 < i_2 < \cdots < i_{n-k} < n - 1 \), let
\[
\{\ell_1, \ell_2, \ldots, \ell_k\} \equiv \{1, 2, 3, \ldots, n - 1\} - \{i_1, i_2, \ldots, i_{n-k}\}
\]
be the complement with respect to \( \{1, 2, 3, \ldots, n - 1\} \). It follows that
\[
\prod_{j=0}^{n-k} [i_j]_q = \frac{[n-1]_q! \prod_{j=0}^{k} [\ell_j]_q}{[\ell_j]_q!}.
\]

This allows (19) to be written in the form
\[
{n \choose k}_q = q^{-n \choose 2} [n-1]_q! \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \frac{1}{\prod_{j=0}^{k} [\ell_j]_q}.
\]

As \( q \to 1 \), one recovers from (19) Comtet’s \([8\) identity given by
\[
{n \choose k} = (-1)^{n-k} \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \prod_{j=1}^{n-k} i_j,
\]
while (22) yields Adamchik’s \([1\) identity for the Stirling numbers of the first kind given by
\[
{n \choose k} = (n-1)! \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \frac{1}{\prod_{j=0}^{k} [\ell_j]_q}.
\]
2.2 \((q,r)\)-Whitney numbers of the second kind

**Theorem 3.** The \((q,r)\)-Whitney numbers of the second kind satisfy the following explicit form:

\[
W_{m,r,q}(n,k) = q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n-k}^k \prod_{j=0}^k (m[j]_q + r)^{c_j},
\]

(25)

where \(c_0, c_1, \ldots, c_k\) are non-negative integers.

*Proof.* We proceed by induction over \(n\). First, we note that the theorem is satisfied when \(n = k = 0\). That is, \(W_{m,r,q}(0,0) = 1\). Making the induction hypothesis that the theorem holds up to \(n\) (for all \(k = 0, 1, \ldots, n\)) we show, using the recurrence relation (5), that it holds for \(n + 1\) and \(k = 0, 1, \ldots, n\). Thus,

\[
W_{m,r,q}(n+1,k) = q^{k-1} q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n+1-k}^k \prod_{j=0}^{k-1} (m[j]_q + r)^{c_j}
\]

\[(+m[k]_q + r)q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n-k}^k \prod_{j=0}^k (m[j]_q + r)^{c_j}
\]

\[= q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n+1-k}^k \prod_{j=0}^{k-1} (m[j]_q + r)^{c_j}
\]

\[(+m[k]_q + r) \sum_{c_0+c_1+\ldots+c_k=n-k}^k \prod_{j=0}^k (m[j]_q + r)^{c_j}
\]

Now, the first term within the big parentheses is a sum of products of \(n+1-k\) factors, non of which contains \((m[k]_q + r)\). The second term is again a sum of \(n+1-k\) factors, each one of which containing \((m[k]_q + r)\) at least once. Thus,

\[
W_{m,r,q}(n+1,k) = q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n+1-k}^k \prod_{j=0}^k (m[j]_q + r)^{c_j}.
\]

To complete the proof we need to show that the theorem holds for \(n + 1\) and \(k = n + 1\). For this case the theorem yields \(W_{m,r,q}(n+1,n+1) = q^{(n^2+1)}\), which is in agreement with (25). □

Apart from \(q^{(2)}\), (25) is a homogeneous complete symmetric polynomial of degree \(n-k\) in the variables \(\{(r + [j]_q m); j = 0, 1, 2, \ldots, k\}\). As \(q \to 1\), we obtain

\[
W_{m,r}(n,k) = \sum_{c_0+c_1+\ldots+c_k=n-k}^k \prod_{j=0}^k (r + mj)^{c_j},
\]

(26)

and for \(r = 0\), (25) reduces to an expression for the \(q\)-Stirling numbers of the second kind, viz,

\[
\binom{n}{k}_q = q^{k(2)} \sum_{c_0+c_1+\ldots+c_k=n-k}^k [1]_q^{c_1} [2]_q^{c_2} \cdots [k]_q^{c_k}.
\]

(27)
The $q$-Stirling numbers of the second kind were originally defined as

$$[x]_q^n = \sum_{k=1}^{n} \binom{n}{k} [x]_{q,k}$$

(cf. [4]). Moreover, when $q \to 1$, Eq. (27) yields an expression for the classical Stirling numbers of the second kind reported by Comtet [8].

Notice that from the inner product

$$\prod_{j=0}^{k}(m[j]_q + r)^{c_j} = (m[0]_q + r)^{c_0} (m[1]_q + r)^{c_1} (m[2]_q + r)^{c_2} \cdots (m[k]_q + r)^{c_k}$$

in the explicit formula in (25), we observe that there are exactly $n - k$ factors of $(m[j]_q + r)$ which is repeated $c_j$ times for each $j$. From here, we write

$$(m[0]_q + r)^{c_0} = (m[j_1]_q + r)(m[j_2]_q + r) \cdots (m[j_0]_q + r),$$

where $j_i = 0, i = 1, 2, \ldots, c_0$;

$$(m[1]_q + r)^{c_1} = (m[j_{c_0+1}]_q + r)(m[j_{c_0+2}]_q + r) \cdots (m[j_{c_0+c_1}]_q + r),$$

where $j_{c_0+i} = 1, i = 1, 2, \ldots, c_1$;

$$(m[2]_q + r)^{c_2} = (m[j_{c_0+c_1+1}]_q + r)(m[j_{c_0+c_1+2}]_q + r) \cdots (m[j_{c_0+c_1+c_2}]_q + r),$$

where $j_{c_0+c_1+i} = 2, i = 1, 2, \ldots, c_2$ and so on until

$$(m[k]_q + r)^{c_k} = (m[j_{c_0+c_1+\cdots+c_{k-1}+1}]_q + r)(m[j_{c_0+c_1+\cdots+c_{k-1}+2}]_q + r) \cdots (m[j_{c_0+c_1+\cdots+c_k}]_q + r),$$

where $j_{c_0+c_1+\cdots+c_{k-1}+i} = k, i = 1, 2, \ldots, c_k$ and $c_0 + c_1 + c_2 + \cdots + c_{k-1} + c_k = n - k$. Thus, 0 ≤ $j_1$ ≤ $j_2$ ≤ ··· ≤ $j_{n-k}$ ≤ $k$ and we have

$$W_{m,r,q}(n,k) = q^{(k)}_{m} \sum_{0 \leq j_1 \leq j_2 \leq \cdots \leq \sum_{i=1}^{n-k} j_i \leq k} \prod_{i=1}^{n-k} (m[j_i]_q + r).$$

We formally state this result in the next theorem.

**Theorem 4.** The $(q,r)$-Whitney numbers of the second kind satisfy the following explicit form:

$$W_{m,r,q}(n,k) = q^{(k)}_{m} \sum_{0 \leq j_1 \leq j_2 \leq \cdots \leq \sum_{i=1}^{n-k} j_i \leq k} \prod_{i=1}^{n-k} (m[j_i]_q + r).$$

Notice that when $q \to 1$, we obtain an identity similar to the result obtained by Mangontarum et al. [13, Theorem 11].
3 On the context of $A$-tableaux

De Medicis and Leroux [23] defined a 0-1 tableau to be a pair $\varphi = (\lambda, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of an integer $m$ and $f = (f_{ij})_{1 \leq j \leq \lambda_i}$ is a “filling” of the cells of the corresponding Ferrers diagram of shape $\lambda$ with 0’s and 1’s such that exactly one 1 in each column. For instance, Figure 1 represents the 0-1 tableau $\varphi = (\lambda, f)$, where $\lambda = (8, 7, 5, 4, 1)$ with $f_{14} = f_{16} = f_{18} = f_{22} = f_{25} = f_{27} = f_{33} = f_{41} = 1$ and $f_{ij} = 0$ elsewhere for $1 \leq j \leq \lambda_i$. In the same paper, an $A$-tableau is defined to be

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 &
\end{bmatrix}
\]

Figure 1: A 0-1 tableau $\varphi$

a list $\Phi$ of columns $c$ of a Ferrers diagram of a partition $\lambda$ (by decreasing order of length) such that the length $|c|$ is part of the sequence $A = (a_i)_{i \geq 0}$, a strictly increasing sequences of non-negative integers. Combinatorial interpretations of Stirling-type numbers in terms of $A$-tableaux are already done in the past. Similar works can be seen in [9, 12, 14, 17, 23] and some of the references therein. In particular, Corcino and Montero [14] defined a $q$-analogue of the Rucinski-Voigt numbers (an equivalent of the $r$-Whitney numbers of the second kind) and then presented a combinatorial interpretation using the theory of $A$-tableaux. The same type of interpretation was obtained by Mangontarum et al. [17] for the case of the translated Whitney numbers (see [20]) and their $q$-analogues. It is important to note that the $q$-analogues of these authors follow motivations which differ from that of the $(q, r)$-Whitney numbers. Furthermore, the numbers considered in the paper of Ramírez and Shattuck [26] belong to $p, q$-analogues, a natural extension of $q$-analogues.

Now, we let $\omega$ be a function from the set of non-negative integers $N$ to a ring $K$, and suppose that $\Phi$ is an $A$-tableau with $r$ columns of length $|c|$. Also, it is known that $\Phi$ might contain a finite number of columns whose lengths are zero since $0 \in A$ and if $\omega(0) \neq 0$ (cf. [23]). Before proceeding, we denote by $T^A(x, y)$ the set of $A$-tableaux with $A = \{0, 1, 2, \ldots, x\}$ and exactly $y$ columns (with some columns possibly of zero length), and by $T^A_d(x, y)$ the subset of $T^A(x, y)$ which contains all $A$-tableaux with columns of distinct lengths. The next theorem relates the $(q, r)$-Whitney numbers of both kinds to certain sets of $A$-tableaux.

**Theorem 5.** Let $\Omega : N \rightarrow K$ and $\omega : N \rightarrow K$ be functions from the set of non-negative integers $N$ to a ring $K$ (column weights according to length) defined by

$\Omega(|c|) = m[|c|]_q + r$

and

$\omega(|c|) = m[|\bar{c}|]_q + r$,

8
where \( m \) and \( r \) are complex numbers, \(|c|\) is the length of column \( c \) of an \( A \)-tableau in \( T^A_d(n-1,n-k) \), and \(|\bar{c}|\) is the length of column \( c \) of an \( A \)-tableau in \( T^A(k,n-k) \). Then

\[
(-1)^{n-k} q^\binom{n}{2} w_{m,r,q}(n,k) = \sum_{\Phi \in T^A_d(n-1,n-k)} \prod_{c \in \Phi} \Omega(|c|) \tag{32}
\]

and

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\phi \in T^A(k,n-k)} \prod_{\bar{c} \in \phi} \omega(|\bar{c}|). \tag{33}
\]

Proof. Let \( \Phi \in T^A_d(n-1,n-k) \). This means that \( \Phi \) has exactly \( n-k \) columns, say \( c_1, c_2, \ldots, c_{n-k} \) whose lengths are \( j_1, j_2, \ldots, j_{n-k} \), respectively. Now, for each column \( c_i \in \Phi \), \( i = 1, 2, 3, \ldots, n-k \), we have \(|c_i| = j_i\) and

\[
\Omega(|c_i|) = m[j_i]_q + r.
\]

Thus,

\[
\prod_{c \in \Phi} \Omega(|c|) = \prod_{i=1}^{n-k} \Omega(|c_i|)
\]

\[
= \prod_{i=1}^{n-k} (m[j_i]_q + r).
\]

Since \( \Phi \in T^A_d(n-1,n-k) \), then

\[
\sum_{\Phi \in T^A_d(n-1,n-k)} \prod_{c \in \Phi} \Omega(|c|) = \sum_{0 \leq j_1 < j_2 < \cdots < j_{n-k} \leq n-1} \prod_{c \in \Phi} \Omega(|c|)
\]

\[
= \sum_{0 \leq j_1 < j_2 < \cdots < j_{n-k} \leq n-1} \prod_{i=1}^{n-k} (m[j_i]_q + r)
\]

\[
= (-1)^{n-k} q^\binom{n}{2} w_{m,r,q}(n,k).
\]

The second result is obtained similarly. \( \square \)

3.1 Combinatorics of \( A \)-tableaux

In the following theorem, we will demonstrate the simple combinatorics of \( A \)-tableaux. To start, note that Eqs. (32) and (33) are equivalent to

\[
(-1)^{n-k} q^\binom{n}{2} w_{m,r,q}(n,k) = \sum_{\Phi \in T^A_d(n-1,n-k)} \Omega_A(\Phi) \tag{34}
\]

and

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\phi \in T^A(k,n-k)} \omega_A(\phi), \tag{35}
\]
respectively, where
\[ \Omega(A(\Phi)) = \prod_{c \in \Phi} \Omega(|c|) = \prod_{c \in \Phi} (m[|c|]_q + r), \quad |c| \in \{0, 1, 2, \ldots, n - 1\} \]  
(36)

and
\[ \omega(A(\phi)) = \prod_{\bar{c} \in \phi} \omega(|\bar{c}|) = \prod_{\bar{c} \in \phi} (m[|\bar{c}|]_q + r), \quad |\bar{c}| \in \{0, 1, 2, \ldots, k\}. \]  
(37)

**Theorem 6.** For nonnegative integers \(n\) and \(k\), and complex numbers \(m\) and \(r\), the following identities hold:
\[ w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} (-r_2)^{j-k} w_{m,r_1,q}(n,j) \]  
(38)

\[ W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} W_{m,r_1,q}(j,k), \]  
(39)

where \(r_1 + r_2 = r\).

**Proof.** Let \(\Phi \in T_d^A(n-1)\). Substituting \(j_i = |c|\) in Eq. (36) gives
\[ \Omega(A(\Phi)) = \prod_{i=1}^{n-k} (m[j_i]_q + r), \]
where \(j_i \in \{0, 1, 2, \ldots, n - 1\}\). Suppose that for some numbers \(r_1\) and \(r_2\), \(r = r_1 + r_2\). Then, with \(\Omega^*(j_i) = m[j_i]_q + r_1\), we may write
\[ \Omega(A(\Phi)) = \prod_{i=1}^{n-k} [(m[j_i]_q + r_1) + r_2] \]
\[ = \prod_{i=1}^{n-k} (\Omega^*(j_i) + r_2) \]
\[ = \sum_{\ell=0}^{n-k} r_2^{n-k-\ell} \sum_{j_1 \leq q_2 < \cdots < q_\ell \leq j_{n-k}} \prod_{i=1}^{\ell} \Omega^*(q_i). \]

Let \(B_\Phi\) be the set of all \(A\)-tableaux corresponding to \(\Phi\) such that for each \(\psi \in B_\Phi\), one of the following is true:

\(\psi\) has no column whose weight is \(r_2\);

\(\psi\) has one column whose weight is \(r_2\);

\(\psi\) has two columns whose weight are \(r_2\);

\(\vdots\)

\(\psi\) has \(n-k\) columns whose weight are \(r_2\).
Then,

\[ \Omega_A(\Phi) = \sum_{\psi \in B_\Phi} \Omega_A(\psi). \]

Now, if \( \ell \) columns in \( \psi \) have weights other than \( r_2 \), then

\[ \Omega_A(\psi) = \prod_{c \in \psi} \Omega^*(|c|) = r_2^{n-k\ell} \prod_{i=1}^\ell \Omega^*(q_i), \]

where \( q_1, q_2, q_3, \ldots, q_\ell \in \{j_1, j_2, j_3, \ldots, j_{n-k}\} \). Hence, Eq. (34) may be written as

\[ (-1)^{n-k} q^{(n)} w_{m,r,q}(n,k) = \sum_{\Phi \in T_d^A(n-1,n-k)} \Omega_A(\Phi) = \sum_{\Phi \in T_d^A(n-1,n-k)} \sum_{\psi \in B_\Phi} \Omega_A(\psi). \]

For each \( \ell \), it is known that there correspond \( \binom{n-k}{\ell} \) tableaux with \( \ell \) distinct columns with weights \( \Omega^*(q_i) \), \( q_i \in \{j_1, j_2, j_3, \ldots, j_{n-k}\} \). Since \( T_d^A(n-1,n-k) \) contains \( \binom{n}{k} \) tableaux, then for each \( \Phi \in T_d^A(n-1,n-k) \), the total number of \( A \)-tableaux \( \psi \) corresponding to \( \Phi \) is

\[ \frac{\binom{n}{k} \binom{n-k}{\ell}}{\binom{n}{\ell}} = \binom{n - \ell}{k}. \]

However, only \( \binom{n}{\ell} \) tableaux in \( B_\Phi \) with \( \ell \) distinct columns of weights other than \( r_2 \) are distinct. It then follows that every distinct tableau \( \psi \) appears

\[ \frac{\binom{n}{k} \binom{n-k}{\ell}}{\binom{n}{\ell}} = \binom{n - \ell}{k} \]

times in the collection (cf. [12]). Thus, we consequently obtain

\[ (-1)^{n-k} q^{(n)} w_{m,r,q}(n,k) = \sum_{\ell=0}^{n-k} \binom{n - \ell}{k} r_2^{n-k-\ell} \sum_{\psi \in B_\ell} \prod_{c \in \psi} \Omega^*(|c|), \]

where \( B_\ell \) denotes the set of all tableaux \( \psi \) having \( \ell \) distinct columns whose lengths are in the set \( \{0, 1, 2, \ldots, n-1\} \). Reindexing the double sum yields

\[ (-1)^{n-k} q^{(n)} w_{m,r,q}(n,k) = \sum_{j=k}^n \binom{j}{k} r_2^{j-k} \sum_{\psi \in B_{n-j}} \prod_{c \in \psi} \Omega^*(|c|). \tag{40} \]

Since \( B_{n-j} = T_d^A(n-1,n-j) \), then

\[ \sum_{\psi \in B_{n-j}} \prod_{c \in \psi} \Omega^*(|c|) = (-1)^{n-j} q^{(n)} w_{m,r_1,q}(n,j). \tag{41} \]
Combining Eqs. (40) and (41) gives

\[ (-1)^{n-k} q(\ell) w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{j}{k} r^{j-k}(n-j) q^{(\ell)} w_{m,r,q}(n,j) \]  

which is equivalent to the desired result in Eq. (38). Similarly, if \( \phi \in T^A(n-1) \), then substituting \( j_i = |\vec{c}| \) in Eq. (37) gives

\[ \omega_A(\phi) = \prod_{i=1}^{n-k} (m[j_i]q + r), \]

where \( j_i \in \{0,1,2,\ldots,k\} \). If for some numbers \( r_1 \) and \( r_2 \), \( r = r_1 + r_2 \), then

\[ \omega_A(\phi) = \prod_{i=1}^{n-k} [(m[j_i]q + r_1) + r_2] = \prod_{i=1}^{n-k} (\omega^*(j_i) + r_2), \omega^*(j_i) = m[j_i]q + r_1 = \sum_{\ell=0}^{n-k} r_2^{n-k-\ell} \sum_{j_1 \leq q_1 \leq q_2 \leq \ldots \leq q_{n-k}} \prod_{i=1}^{\ell} \omega^*(q_i). \]

Suppose \( B_\phi \) is the set of all \( A \)-tableaux corresponding to \( \phi \) such that for each \( \zeta \in B_\phi \), one of the following is true:

- \( \zeta \) has no column whose weight is \( r_2 \);
- \( \zeta \) has one column whose weight is \( r_2 \);
- \( \zeta \) has two columns whose weight are \( r_2 \);
- : 
- \( \zeta \) has \( n-k \) columns whose weight are \( r_2 \).

Then, we may write

\[ \omega_A(\phi) = \sum_{\zeta \in B_\phi} \omega_A(\zeta). \]

If there are \( \ell \) columns in \( \zeta \) with weights other than \( r_2 \), then we have

\[ \omega_A(\zeta) = \prod_{c \in \zeta} \omega^*(|\vec{c}|) = r_2^{n-k \ell} \prod_{i=1}^{\ell} \omega^*(q_i), \]
where \( q_1, q_2, q_3, \ldots, q_\ell \in \{ j_1, j_2, j_3, \ldots, j_{n-k} \} \). It then follows that Eq. (35) may be expressed as

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\phi \in T^A(k,n-k)} \omega_A(\phi) = \sum_{\phi \in T^A(k,n-k)} \sum_{\zeta \in B_\phi} \omega_A(\zeta).
\]

Like in the previous, for each \( \ell \), there correspond \( \binom{n-k}{\ell} \) tableaux with \( \ell \) columns having weights \( \omega^*(q_i), q_i \in \{ j_1, j_2, \ldots, j_{n-k} \} \). Since the set \( T^A(k,n-k) \) contains \( \binom{n}{k} \) tableaux, then for each \( \phi \in T^A(k,n-k) \), there are \( \binom{n}{k} \binom{n-k}{\ell} \) \( A \)-tableaux corresponding to \( \phi \). But only \( \binom{\ell+k}{\ell} \) of these tableaux are distinct. Hence, every distinct tableau \( \zeta \) with \( \ell \) columns of weights other than \( r_2 \) appears

\[
\binom{n}{k} \binom{n-k}{\ell} = \binom{n}{\ell+k}
\]
times in the collection (cf. [9]). It implies that

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\ell=0}^{n-k} \binom{n}{\ell+k} r_2^{n-k-\ell} \sum_{\zeta \in B_\ell} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|),
\]

where \( B_\ell \) is the set of all tableaux \( \zeta \) having \( \ell \) columns of weights \( \omega^*(j_i) \). Reindexing the sums yield

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} \sum_{\zeta \in B_{j-k}} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|). \tag{43}
\]

Since \( B_{n-j} = T^A(k,n-j) \), then

\[
\sum_{\zeta \in B_{j-k}} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|) = q^{-\binom{j}{2}} W_{m,r_1,q}(j,k). \tag{44}
\]

Moreover, by Eqs. (43) and (44), we obtain

\[
q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} q^{-\binom{j}{2}} W_{m,r_1,q}(j,k) \tag{45}
\]

which is equivalent to the second desired result.
Let \( r_1 = r - 1 \) and \( r_2 = 1 \) in Eqs. (38) and (39). Then,

\[
w_{m,r,q}(n, k) = \sum_{j=k}^{n} \binom{n}{j} (-1)^{j-k} w_{m,r-1,q}(n, j)
\]

(46)

and

\[
W_{m,r,q}(n, k) = \sum_{j=k}^{n} \binom{n}{j} W_{m,r-1,q}(j, k).
\]

(47)

These identities were first seen in [21, Theorem 9]. Now, using Eq. (39), the \((q,r)\)-Dowling numbers \( D_{m,r,q}(n) \) [21] may be expressed as

\[
D_{m,r,q}(n) = \sum_{k=0}^{n} W_{m,r,q}(n, k)
\]

\[
= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} W_{m,r-1,q}(j, k)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} W_{m,r-1,q}(j, k)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} D_{m,r-1,q}(j).
\]

Moreover, by applying the binomial inversion formula \([8]\)

\[
f_n = \sum_{j=0}^{n} \binom{n}{j} g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f_j
\]

to this identity gives

\[
D_{m,r-1,q}(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} D_{m,r,q}(j).
\]

These results are formally stated in the following corollary:

**Corollary 7.** The \((q,r)\)-Dowling numbers satisfy the recurrence relations with respect to \( r \) given by

\[
D_{m,r+1,q}(n) = \sum_{j=0}^{n} \binom{n}{j} D_{m,r,q}(j)
\]

(48)

and

\[
D_{m,r,q}(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} D_{m,r+1,q}(j).
\]

(49)
Remark 8. When \( q \to 1 \) and \( m = \beta \), we obtain the following identities by Corcino and Corcino [11]:

\[
G_{n,\beta,r+1} = \sum_{j=0}^{n} \binom{n}{j} G_{j,\beta,r}
\]

\[
G_{n,\beta,r} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} G_{j,\beta,r+1},
\]

where \( G_{n,\beta,r} := D_{\beta,r,1}(n) \) is the generalized Bell numbers in [10, 11]. These identities were used to identify the Hankel transform of \( G_{n,\beta,r} \).

Looking at the previous corollary, we see that the sequence \( (D_{m,r+1,q}(n)) \) is the binomial transform of the sequence \( (D_{m,r,q}(n)) \), for \( r = 0, 1, 2, \ldots \). Using “Layman’s Theorem” [16], \( (D_{m,0,q}(n)), (D_{m,1,q}(n)), (D_{m,2,q}(n)), \ldots, (D_{m,r,q}(n)) \ldots \) have the same Hankel transform. This directs our attention to the following open problem:

**Problem 9.** Is it possible to identify the Hankel transform of \( D_{m,r,q}(n) \) using a method parallel to what is being done in [11] for \( G_{n,\beta,r} \)?

### 3.2 Convolution-type identities

Recall that for any two sequences \( a_n \) and \( b_n \), we call the sequence \( c_n \) as convolution sequence if

\[
c_n = \sum_{k=1}^{n} a_k b_{n-k}, \quad n = 0, 1, 2, \ldots \tag{52}
\]

One of the most famous convolution-type identity is the Vandermonde’s formula [6, 8] given by

\[
\binom{a + b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}. \tag{53}
\]

The following theorem contains convolution-type identities for the \( (q,r) \)-Whitney numbers of the first kind which will be proved using the combinatorics of \( A \)-tableaux:

**Theorem 10.** The \( (q,r) \)-Whitney numbers of the first kind have convolution-type identities given by

\[
w_{m,r,q}(p + j, n) = q^{-pj} \sum_{k=0}^{n} w_{m,r,q}(p,k) w_{\tilde{m},\tilde{r},q}(j,n-k) \tag{54}
\]

and

\[
w_{m,r,q}(n + 1, j + p + 1) = \sum_{k=0}^{n} q^{k^2-nk-n} w_{m,r,q}(k,p) w_{\tilde{m},\tilde{r},q}(n-k,j), \tag{55}
\]

where \( \tilde{m} = mq^p \) and \( \tilde{r} = m[p]_q + r \).

**Proof.** For \( A_1 = \{0, 1, 2, \ldots, p - 1\} \) and \( A_2 = \{p, p + 1, p + 2, \ldots, p + j - 1\} \), let \( \Phi_1 \in T_{d}^{A_1}(p-1,p-k) \) and \( \Phi_2 \in T_{d}^{A_2}(j-1,j-n+k) \). Note that by joining the columns of the tableaux \( \Phi_1 \) and \( \Phi_2 \), we may generate an \( A \)-tableau \( \Phi \) with \( p + j - n \) distinct columns whose
lengths are in the set \( A = \{0, 1, 2, \ldots, p + j - 1\} \). That is, \( \Phi \in T_d^A(p + j - 1, p + j - n) \). Hence,

\[
\sum_{\Phi \in T_d^A(p+j-1,p+j-n)} \Omega_A(\Phi) = \sum_{k=0}^{n} \left\{ \sum_{\Phi_1 \in T_d^{A_1}(p-1,p-k)} \Omega_{A_1}(\Phi_1) \right\} \left\{ \sum_{\Phi_2 \in T_d^{A_2}(j-1,j-n+k)} \Omega_{A_2}(\Phi_2) \right\}.
\]

Note that in the right-hand side, we get

\[
\sum_{\Phi_2 \in T_d^{A_2}(j-1,j-n+k)} \Omega_{A_2}(\Phi_2) = \sum_{p \leq g_1 < g_2 < \cdots < g_{j-n+k} \leq p+j-1} \prod_{i=1}^{j-n+k} (m[g_i]q + r)
\]

\[
= \sum_{0 \leq g_1 < g_2 < \cdots < g_{j-n+k} \leq j-1} \prod_{i=1}^{j-n+k} (m[p + g_i]q + r)
\]

\[
= \sum_{0 \leq g_1 < g_2 < \cdots < g_{j-n+k} \leq j-1} \prod_{i=1}^{j-n+k} (mq^p[g_i]q + ([p]q + r))
\]

\[
= (-1)^{j-n+k}q^{(j)}w_{\bar{m},r,q}(j, n - k),
\]

where \( \bar{m} = mq^p \) and \( \bar{r} = m[p]q + r \). Also, using Eq. \( (34) \),

\[
\sum_{\Phi_1 \in T_d^{A_1}(p-1,p-k)} \Omega_{A_1}(\Phi_1) = (-1)^{p-k}q^{(p)}w_{m,r,q}(p, k)
\]

and

\[
\sum_{\Phi \in T_d^A(p+j-1,p+j-n)} \Omega_A(\Phi) = (-1)^{p+j-n}q^{(p+j)}w_{m,r,q}(p + j, n).
\]

Hence, by simplification, we obtain the convolution identity \( (34) \). Similarly, we let \( \Phi_1 \) be a tableau with \( k - p \) columns whose lengths are in \( B_1 = \{0, 1, 2, \ldots, k - 1\} \) and \( \Phi_2 \) be a tableau with \( n - k - j \) columns whose lengths are in \( B_2 = \{k + 1, k + 2, \ldots, n\} \) so that \( \Phi \in T_d^{B_1}(k-1,k-p) \) and \( \Phi \in T_d^{B_2}(n-k-1,n-k-j) \). Note that we may generate an \( A \)-tableau \( \Phi \) by joining the columns of \( \Phi_1 \) and \( \Phi_2 \) whose lengths are in \( A = \{0, 1, 2, \ldots, n\} \). Hence, we have

\[
\sum_{\Phi \in T_d^A(n,n-j-p)} \Omega_A(\Phi) = \sum_{k=0}^{n} \left\{ \sum_{\Phi_1 \in T_d^{B_1}(k-1,k-p)} \Omega_{B_1}(\Phi_1) \right\} \left\{ \sum_{\Phi_2 \in T_d^{B_2}(n-k-1,n-k-j)} \Omega_{B_2}(\Phi_2) \right\}.
\]

Applying Eq. \( (34) \) gives

\[
\sum_{\Phi \in T_d^A(n,n-j-p)} \Omega_A(\Phi) = (-1)^{n-j-p}q^{(n+1)}w_{m,r,q}(n + 1, j + p + 1)
\]
and
\[ \sum_{\Phi_1 \in T_{d_1}(k-1,k-p)} \Omega_{B_1}(\Phi_1) = (-1)^{k-p}q^{\binom{k}{2}}w_{m,r,q}(k,p). \]
Also, in the right-hand side, we get
\[ \sum_{\Phi_2 \in T_{d_2}(n-k-11,n-k-j)} \Omega_{B_2}(\Phi_2) = \sum_{0 \leq g_1 < g_2 < \ldots < g_{n-k-j} \leq p} \prod_{i=1}^{n-k-j} (m[g_i]_q + r) \]
\[ = \sum_{0 \leq g_1 < g_2 < \ldots < g_{n-k-j} \leq n-k-1} \prod_{i=1}^{n-k-j} (m[p + g_i]_q + r) \]
\[ = \sum_{0 \leq g_1 < g_2 < \ldots < g_{n-k-j} \leq n-k-1} \prod_{i=1}^{n-k-j} (mq[g_i]_q + ([p]_q + r)) \]
\[ = (-1)^{n-k-j}q^{\binom{n-k}{2}}w_{\bar{m},\bar{r},q}(n-k,j), \]
where \( \bar{m} = mq^p \) and \( \bar{r} = m[p]_q + r \). This completes the proof.

The next theorem can be proved similarly.

**Theorem 11.** The \((q,r)\)-Whitney numbers of the second kind have convolution-type identities given by

\[ W_{m,r,q}(n+1,j+p+1) = \sum_{k=0}^{n} q^{p+j}W_{m,r,q}(k,p)W_{\bar{m},\bar{r},q}(n-k,j), \]  
(56)

and

\[ W_{m,r,q}(p+j,n) = \sum_{k=0}^{n} q^{n-k}W_{m,r,q}(p,k)W_{\bar{m},\bar{r},q}(j,n-k), \]  
(57)

where \( \bar{m} = mq^{p+1} \) and \( \bar{r} = m[p+1]_q + r \).

As \( q \to 1 \), we recover from Theorems 10 and 11 the results recently obtained by Xu and Zhou [27, Theorems 2.1 and 2.4].

### 4 On Heine and Euler distributions

Consider the Poisson distribution
\[ f_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \]  
(58)

for \( x = 0, 1, 2, \ldots \). The factorial moment of a Poisson random variable is readily evaluated, i.e.,

\[ E[(X)_n] = \lambda^n \]  
(59)
the mean, \( E[X] = \lambda \), being the special case \( n = 1 \). Expanding \( x^n \) in terms of falling factorials (using the Stirling numbers of the second kind), we obtain the \( n \)-th moment of \( X \) given by

\[
E[X^n] = B_n(\lambda),
\]

(60)

where \( B_n(\lambda) \) are the Bell polynomials. The \( q \)-analogues of the Poisson distribution introduced by Kemp [15], and Benkherouf and Bather in [3] are given by

\[
f_Y(y) = e_q(-\lambda) \frac{\lambda^y}{[y]_q!}, \quad y = 0, 1, 2, \ldots
\]

(61)

and

\[
f_Z(z) = \tilde{e}_q(-\lambda) \frac{\lambda^z}{[z]_q!}, \quad z = 0, 1, 2, \ldots
\]

(62)

These are called Heine and Euler distributions, respectively, where

\[
e_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!}
\]

(63)

and

\[
\tilde{e}_q(t) = \sum_{k=0}^{\infty} q^{(2)}(k) \frac{t^k}{[k]_q!}
\]

(64)

In line with this, Charalambides and Papadatos [5] obtained the following important results:

\[
E[[Y]_{r,q}] = \frac{q^{(r)} \lambda^r}{\prod_{i=1}^{r}(1 + \lambda(1-q)q^{i-1})},
\]

(65)

\[
E[[Z]_{r,q}] = \lambda^r,
\]

(66)

where \([x]_{r,q} = [x][x-1]_q[x-2]_q \cdots [x-r+1]_q\) is the \( q \)-falling factorial of \( x \) of order \( r \). Considering these, we now state the following theorem:

**Theorem 12.** If \( Y \) and \( Z \) are random variables with Heine and Euler distributions, respectively, and if the mean of \( Y \) is \( \phi = \frac{\lambda}{1+\lambda(1-q)} \) and the mean of \( Z \) is \( \lambda \), then

\[
E_\phi[(m[Y]_q + r)^n] = \sum_{\ell=0}^{n} \sum_{i=0}^{n} (-\lambda)^i q^{-(i-r)} \frac{\lambda^\ell}{[\ell]_q! [r]_q!} \prod_{j=1}^{\ell+i}(1 + \lambda(1-q)q^{j-1}),
\]

(67)

\[
E_\lambda[(m[Z]_q + r)^n] = \tilde{e}_q(-\lambda) \sum_{\ell=0}^{n} \frac{\lambda^\ell}{[\ell]_q!} (m[\ell]_q + r)^n.
\]

(68)

**Proof.** From the defining relation in (11) and the result in (65),

\[
E_\lambda[(m[Y]_q + r)^n] = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k) \frac{q^{(n)} \lambda^k}{\prod_{j=1}^{k}(1 + \lambda(1-q)q^{j-1})}.
\]

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Using the explicit formula for the \((q,r)\)-Whitney numbers of the second kind [21, Theorem 16] given by
\[
W_{m,r,q}(n,k) = \frac{1}{m^k [k]_q !} \sum_{\ell = 0}^{k} (-1)^{k-\ell} q^{(k-\ell)} \binom{k}{\ell}_q (m[\ell]_q + r)^n, \tag{69}
\]
we obtain
\[
E_\lambda [(m[Z]_q + r)^n] = \sum_{k=0}^{n} \frac{1}{[k]_q !} \sum_{\ell = 0}^{k} (-1)^{k-\ell} q^{(k-\ell)} \binom{k}{\ell}_q (m[\ell]_q + r)^n \frac{\lambda^\ell}{[\ell]_q ! [k-\ell]_q !} \prod_{j=1}^{k} (1 + \lambda(1 - q)q^{j-1}).
\]
Reindexing the second sum yields (67). Eq. (68) may be shown similarly.

Remark 13. When \(m = 1\) and \(r = 0\) in the previous theorem, we have
\[
E_\phi [Y]^n_q = \sum_{\ell=0}^{n} \sum_{i=0}^{n} (-\lambda)^i q^{-\ell} \binom{\ell}{-\ell_i} \frac{\lambda^\ell}{[\ell]_q ! [\ell-i]_q !} \prod_{j=1}^{\ell-i} (1 + \lambda(1 - q)q^{j-1}),
\]
and
\[
E_\lambda [Z]^n_q = \tilde{e}_q(-\lambda) \sum_{\ell=0}^{n} \lambda^\ell \frac{\lambda^\ell}{[\ell]_q !} \equiv B_{n,q}(\lambda),
\]
where \(B_{n,q}(\lambda)\) is the \(q\)-Bell polynomials. On the other hand, if the mean is \(\lambda = \frac{x}{m}\),
\[
E_{x/m} [(m[Z]_q + r)^n] = \tilde{e}_q \left( -\frac{x}{m} \right) \sum_{\ell=0}^{n} \frac{x^\ell}{m^\ell} \frac{(m[\ell]_q + r)^n}{[\ell]_q !}.
\]
This explicit formula is due to Mangontarum and Katriel [21]. Thus
\[
E_{x/m} [(m[Z]_q + r)^n] = D_{m,r,q}(n, x),
\]
where
\[
D_{m,r,q}(n, x) = \sum_{k=0}^{n} W_{m,r,q}(n, k) x^k \tag{72}
\]
is the \((q, r)\)-Dowling polynomials.

It is worth mentioning that Mangontarum and Corcino [19] obtained the following pair of \(n\)-th order generalized factorial moments
\[
E_\lambda [(\beta X + \gamma|\alpha)_n] = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(i\beta + \gamma|\alpha)_n}{i!} \lambda^i \tag{73}
\]
$E_\lambda[(\alpha X - \gamma|\beta)_n] = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(i\alpha - \gamma|\beta)_n}{i!} \lambda^i,$  \hfill (74)

where $X$ is a Poisson random variable with mean $\lambda$ and $\alpha$, $\beta$ and $\gamma$ may be real or complex numbers. Here,

$$(t|\alpha)_n = t(t-\alpha)(t-2\alpha) \cdots (t-n\alpha + \alpha),$$  \hfill (75)

with initial conditions $(t|\alpha)_n = 0$ when $n \leq 0$ and $(t|\alpha)_0 = 1$. Notice that (73) unifies the factorial moment in (59) and the $n$-th moment in (60). More precisely,

- when $\beta = 1$, $\gamma = 0$ and $\alpha = 0$,
  $$E_\lambda[(\beta X + \gamma|\alpha)_n] = E_\lambda[X^n];$$
- when $\beta = 1$, $\gamma = 0$ and $\alpha = 1$,
  $$E_\lambda[(\beta X + \gamma|\alpha)_n] = E_\lambda[(X)_n].$$

Other known “Bell-type” and “Dowling-type” polynomials (see [7, 10, 18, 22, 24, 25]) can be shown to be particular cases of Eqs. (73) and (74). Furthermore, Corcino and Mangontarum [13] obtained the generalized $q$-factorial moments

$$E_\phi[[\beta Y]_q + [\gamma]_q[[\alpha]_q]_{n,q}] = \sum_{j=0}^{\infty} \hat{e}_{q^3,j}(-\lambda) \frac{(q^3 \lambda)^j [[\beta j]_q + [\gamma]_q[[\alpha]_q]_{n,q}}{[j]_q^a! \prod_{i=1}^n (1 + \lambda(1 - q^3)q^{3(i-1)})},$$  \hfill (76)

and

$$E_\lambda[[\beta Z]_q + [\gamma]_q[[\alpha]_q]_{n,q}] = \sum_{j=0}^{\infty} \hat{e}_{q^3,j}(-\lambda) \frac{[\beta j]_q + [\gamma]_q[[\alpha]_q]_{n,q}}{[j]_q^a!} \frac{\lambda^j}{[j]_q^a!},$$  \hfill (77)

where $Y$ is a random variable with Heine distribution and mean $\phi = \frac{\lambda}{1+\lambda(1-q^3)}$, and $Z$ is a random variable with an Euler distribution and mean $\lambda$. The notations

$$[[\beta Z]_q + [\gamma]_q[[\alpha]_q]_{n,q} = \prod_{j=0}^{n-1} ([\beta j]_q + [\gamma]_q - [\alpha j]_q)$$  \hfill (78)

and

$$\hat{e}_{q^3,j}(-\lambda) = \sum_{t=0}^{\infty} \left[ [t]_q^a! \prod_{i=1}^n (q^{3(i-1)} + \lambda(1 - q^3)q^{3(j-1)}) \right]$$  \hfill (79)

are used. (76) and (77) are found to be $q$-analogues of (73). By thoroughly investigating (68), it is obvious that this result is not generalized by (76) and (77).

Privault [25] defined an extension of the classical Bell numbers as

$$e^{ty - \lambda(e^t - 1)} = \sum_{k=0}^{\infty} B_n(y, \lambda) \frac{t^k}{k!}.$$
Moreover, he obtained the following $n$-th moment of a Poisson random variable

$$E_{\lambda}[(X + y - \lambda)^n] = B_n(y, -\lambda),$$

where

$$B_n(y, -\lambda) = \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} \sum_{j=0}^{k} \binom{k}{j} \lambda^j,$$  \hspace{1cm}  (81)

Corcino and Corcino [10] showed that the $(r, \beta)$-Bell polynomials satisfy

$$D_{m,r,q}(n, x) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} \beta^{k-j} \binom{k}{j} x^j.$$  \hspace{1cm}  (82)

It then follows that

$$G_{n,1,y-\lambda}(\lambda) = B_n(y, -\lambda).$$

The next theorem is analogous to these identities.

**Theorem 14.** The $(q,r)$-Dowling polynomials satisfy the identity

$$D_{m,r,q}(n, x) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} \beta^{k-j} \binom{k}{j} q^j x^j.$$  \hspace{1cm}  (83)

**Proof.** Using the binomial theorem, we have

$$E_{x/m}[m[Z]_q + r]^n] = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k E_{x/m}[[Z]_q]^k]$$

$$= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k B_{n,q}(\frac{x}{m})$$

$$= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k \sum_{j=0}^{k} \binom{k}{j} q^j (\frac{x}{m})^j.$$  \hspace{1cm}  (83)

The desired result follows from the fact that $E_{x/m}[m[Z]_q + r]^n] = D_{m,r,q}(n, x)$.  \hspace{1cm}  \qed

**Remark 15.** As $q \rightarrow 1$, we obtain the $(r, \beta)$-Bell polynomial identity in Eq. (82). If the mean is replaced with $\lambda$, then for an Euler random variable $Z$,

$$E_{\lambda}[(m[Z]_q + r)^n] = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} m^k \binom{k}{j} \lambda^j.$$  \hspace{1cm}  (84)

As $q \rightarrow 1$, we get [19] Eq. 34]

$$E_{\lambda}[(mX + r)^n] = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} m^k \binom{k}{j} \lambda^j.$$  \hspace{1cm}  (84)

When $m = 1$ and $r = y - \lambda$,

$$D_{1,y-\lambda,q}(n, x) = \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} \sum_{j=0}^{k} \binom{k}{j} q^j x^j.$$  \hspace{1cm}  (84)

This is a $q$-analogue of Privault’s identity since (84) $\rightarrow$ (81) as $q \rightarrow 1$.  \hspace{1cm}  \qed

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