New Pieri type formulas for Jack polynomials, and
difference or Pieri formulas for interpolation Jack
polynomials

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Abstract
We obtain new Pieri type formulas for Jack polynomials. From these new Pieri formulas, we give a new derivation of higher order difference equations for interpolation Jack polynomials originally found by Knop-Sahi (1996). Similarly, we also obtain Pieri formulas for interpolation Jack polynomials and some intertwining relations for a kernel function for Jack polynomials.

1 Introduction

Interpolation Jack (shifted Jack) Polynomials (IJP) are a continuance deformation of shifted Schur polynomials [M1], [OO2], which are introduced by Sahi [Sa], Knop-Sahi [KS] and Okounkov-Olshanski [OO1]. Since IJP are regarded as a multivariate analogue of a usual falling factorial

\[ P_k^{ip}(x) := \begin{cases} x(x-1)\cdots(x-k+1) & (k \neq 0) \\ 1 & (k = 0) \end{cases}, \]

(1.1)
or binomial coefficient

\[ \binom{x}{k} := \frac{P_k^{ip}(x)}{k!}, \]

they appear in some explicit formulas (binomial type formulas) of multivariate hypergeometric functions or orthogonal functions (see [O1], [O2] and

*Dedicated to M. Noumi for his 65th birthday.*
Knop-Sahi investigated IJP, and in particular, derived its difference formulas. In this article, we give some new Pieri type formulas for Jack polynomials, which we call twisted Pieri formulas. As an application of our new Pieri formulas for Jack polynomials, we give another proof of difference formulas and obtain Pieri type formulas for IJP.

Let us describe our scheme in the one variable case more precisely. In the one variable case, IJP are uniquely defined by

\[(1)_{IP} \quad p_k^m(m) = 0, \quad \text{unless } k < m \in \mathbb{Z}_{\geq 0}\]
\[(2)_{IP} \quad p_m^m(z) = z^m + \text{(lower terms)}.

Difference and Pieri type formulas for IJP are the following:

\[k P_k^m(x) = x P_k^m(x) - x P_k^m(x-1), \quad (1.2)\]
\[x P_k^m(x) = P_{k+1}^m(x) + k P_k^m(x). \quad (1.3)\]

These formulas (1.2) and (1.3) are proved by explicit formula (1.1) immediately. However, in the multivariate case, IJP have no simple expressions like (1.1). Hence we consider the following complicated but generalizable proofs. For non-negative integer \(m\), we put

\[P_m(z) := z^m, \quad \Phi_m(z) := \frac{P_m(z)}{P_m(1)} = z^m,\]
\[\Psi_m(z) := \frac{P_m(z)}{P_m^m(m)} \quad \text{and} \quad \frac{\partial}{\partial z} := \frac{d}{dz}.

We list five key formulas to prove (1.2) and (1.3).

1. Sekiguchi (Euler) operator

\[\left( z \frac{\partial}{\partial z} \right) \Phi_m(z) = \Phi_m(z)m, \quad (1.4)\]
\[\left( z \frac{\partial}{\partial z} \right) \Psi_m(z) = \Psi_m(z)m. \quad (1.5)\]

2. Pieri formulas for Jack

\[\frac{\partial}{\partial z} \Phi_m(z) = \Phi_{m-1}(z)m, \quad (1.6)\]
\[\frac{\partial}{\partial z} \Psi_m(z) = \Psi_{m-1}(z), \quad (1.7)\]
\[z \Phi_m(z) = \Phi_{m+1}(z), \quad (1.8)\]
\[z \Psi_m(z) = \Psi_{m+1}(z)(m + 1). \quad (1.9)\]
3. Binomial formula

\[ \Phi_x(1 + z) = \sum_{0 \leq k \leq x} \binom{x}{k} \Phi_k(z) = \sum_{0 \leq k \leq x} \frac{P^p_k(x)}{P_k(1)} \Psi_k(z). \] (1.10)

4. Mysterious summation

\[(x + 1) - x = 1. \] (1.11)

5. Twisted Pieri formulas for Jack

\[
\begin{align*}
\left[ \frac{(\text{ad} \partial_z)^1}{1!} z\partial_z \right] \Phi_x(z) &= \Phi_{x-1}(z)x, \\
\left[ \frac{(\text{ad} \partial_z)^1}{1!} z\partial_z \right] \Psi_x(z) &= \Psi_{x-1}(z).
\end{align*}
\] (1.12) (1.13)

Here

\[(\text{ad} A)(B) := AB - BA.\]

Although all the above formulas are obvious, in the multivariate case these formulas are non-trivial results. Thus, as an introduction to the multivariate case, we give the proofs of (1.11) and (1.12).

From Pieri formulas (1.6) and (1.8),

\[ [\partial_z, z] \Phi_x(z) = \Phi_x(z)(x + 1) - \Phi_x(z)x = ((x + 1) - x)\Phi_x(z). \]

Since \([\partial_z, z] = 1\), we have

\[ [\partial_z, z] \Phi_x(z) = 1 \cdot \Phi_x(z). \]

Then by comparing the coefficients of \(\Phi_x(z)\), we obtain (1.11).

By (1.4), (1.5), (1.6) and (1.7), we have

\[
\begin{align*}
\left[ \frac{(\text{ad} \partial_z)^1}{1!} z\partial_z \right] \Phi_x(z) &= \partial_z((z\partial_z)\Phi_x(z)) - (z\partial_z)(\partial_z\Phi_x(z)) \\
&= (\partial_z\Phi_x(z))x - (z\partial_z)\Phi_{x-1}(z)x \\
&= (\Phi_{x-1}(z)x - \Phi_{x-1}(z)(x - 1))x \\
&= (x - (x - 1))\Phi_k(z)x.
\end{align*}
\]

Finally, by applying mysterious summation (1.11), we obtain the conclusion.
From the above preparations, we obtain (1.2) and (1.3). Since (1.2) and (1.3) are some relations for polynomials, it is enough to prove when variable \( x \in \mathbb{Z}_{\geq 0} \). To prove (1.2), we compute

\[(z\partial_z)\Phi_x(1 + z)\]

in two different ways. First, from the twisted Pieri (1.12) and binomial formula (1.10)

\[(z\partial_z)\Phi_x(1 + z) = e^{\partial_z}(z\partial_z)\Phi_x(z)\]

\[= e^{\partial_z} \sum_{p=0}^{1} \left[ (z\partial_z)^p \Phi_x(z) \right] = e^{\partial_z} \Phi_x(z) x - \Phi_{x-1}(z) x \]

\[= x\Phi_x(1 + z) - x\Phi_{x-1}(1 + z)\]

\[= \sum_{k=0}^{x} \left( \frac{P_{ip}^k(x)}{P_k(1)} - x \frac{P_{ip}^k(x-1)}{P_k(1)} \right) \Psi_k(z). \quad (1.14)\]

On the other hand, by (1.10) and (1.5)

\[(z\partial_z)\Phi_x(1 + z) = \sum_{k=0}^{x} \frac{P_{ip}^k(x)}{P_k(1)} (z\partial_z)\Phi_k(z) = \sum_{k=0}^{x} k \frac{P_{ip}^k(x)}{P_k(1)} \Psi_k(z). \quad (1.15)\]

By comparing the coefficients of \( \Psi_k(z) \) on (1.14) and (1.15), we have

\[k P_{ip}^k(x) = x P_{ip}^k(x) - x P_{ip}^k(x - 1).\]

Similarly for Pieri formula (1.3), we calculate

\[e^{\text{ad}(\partial_z)}(z\partial_z)\Phi_x(1 + z)\]

in two different ways. First, we have

\[e^{\text{ad}(\partial_z)}(z\partial_z)\Phi_x(1 + z) = e^{\partial_z}(z\partial_z)e^{-\partial_z}\Phi_x(1 + z)\]

\[= e^{\partial_z}(z\partial_z)\Phi_x(z)\]

\[= e^{\partial_z}\Phi_x(z)x\]

\[= \Phi_x(1 + z)x.\]
Thus, from binomial formula (1.10)

\[ [e^{ad(\partial_z)}(z\partial_z)]\Phi_x(1 + z) = \Phi_x(1 + z)x = \sum_{k=0}^{x} x \frac{P_k^p(x)}{P_k(1)} \Psi_k(z). \] (1.16)

On the other hand, from the binomial (1.10) and twisted Pieri (1.13) formulas,

\[ [e^{ad(\partial_z)}(z\partial_z)]\Phi_x(1 + z) = \sum_{k=0}^{x} \frac{P_k^p(x)}{P_k(1)} \left[ e^{ad(\partial_z)(z\partial_z)} \right] \Psi_k(z) \]

\[ = \sum_{k=0}^{x} \frac{P_k^p(x)}{P_k(1)} \sum_{p=0}^{1} \frac{(ad(\partial_z))^p}{p!} (z\partial_z) \Psi_k(z) \]

\[ = \sum_{k=0}^{x} \frac{P_k^p(x)}{P_k(1)} (\Psi_k(z)k + \Psi_{k-1}(z)) \]

\[ = \sum_{k=0}^{x} \left( k \frac{P_k^p(x)}{P_k(1)} + \frac{P_{k+1}^p(x)}{P_{k+1}(1)} \right) \Psi_k(z). \] (1.17)

By comparing the coefficients of \( \Psi_k(z) \) of (1.16) and (1.17), we obtain the conclusion (1.3).

The purpose of this article is to provide a multivariate analogue of the above argument which has not been known even for one variable. For the purpose, we consider a multivariate analogue of the five key formulas;

1. Sekiguchi operator,
2. Pieri for Jack,
3. Binomial formula,
4. Mysterious summation,
5. Twisted Pieri for Jack.

Since 1., 2., 3. are known results, we give proofs of 4. and 5. which is the most difficult and complicated part in the article. By applying the above multivariate analogue 1.-5., we derive difference and Pieri formulas for IJP.

The content of this article is as follows. In Section 2, we introduce Jack and interpolation Jack polynomials. Section 3 is the main part of this article. We explain a multivariate analogue of (1.4) - (1.13). In particular, we obtain new Pieri type formulas for the ordinary Jack polynomials which we call twisted Pieri formulas. Twisted Pieri formulas are regarded as a multivariate analogue of (1.12) and (1.13). With these preparations, in Section 4, we give a new derivation of higher order difference equations for interpolation Jack polynomials originally found by Knop-Sahi (1996).
Similarly, we also obtain Pieri formulas for interpolation Jack polynomials. Finally, in Section 5, we give some intertwining relations for a kernel function for Jack polynomials, which is a multivariate analogue of
\[ \partial_z e^{zw} = e^{zw} \w. \]

2 Preliminaries

Refer to [Ka], [Ko], [L], [M2], [St], [VK] for the details in this section. Let \( r \in \mathbb{Z}_{\geq 1}, d \neq 0 \in \mathbb{C} \) and
\[
\mathcal{P} := \{ \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0 \},
\]
\[
\delta := (r - 1, r - 2, \ldots, 2, 1, 0) \in \mathcal{P},
\]
\[
e_{r,k}(z) := \sum_{1 \leq i_1 < \cdots < i_k \leq r} z_{i_1} \cdots z_{i_k} \quad (k = 1, \ldots, r), \quad e_{r,0}(z) := 1, \quad |z| := e_{r,1}(z),
\]
\[
E_k(z) := \sum_{j=1}^r z_j^k \partial z_j \quad (k \in \mathbb{Z}_{\geq 0}),
\]
\[
D_k(z) := \sum_{j=1}^r z_j^k \partial^2 z_j + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \partial z_j \quad (k \in \mathbb{Z}_{\geq 0}).
\]

For any partition \( \mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{P} \) and \( z = (z_1, \ldots, z_r) \in \mathbb{C}^r \), put
\[
m_{\mathbf{m}}(z) := \sum_{n \in \mathfrak{S}_r, m} z^n,
\]
where \( \mathfrak{S}_r \) is the symmetric group in \( r \) letters and
\[
\mathfrak{S}_r, \mathbf{m} := \{ \sigma \mathbf{m} := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(r)}) \mid \sigma \in \mathfrak{S}_r \} \subseteq \mathbb{Z}^{(r)}_{\geq 0},
\]
\[
z_n := z_1^{n_1} \cdots z_r^{n_r}.
\]

The dominance partial ordering \( \leq \) and the inclusion partial ordering \( \subseteq \) are defined by:
\[
k \leq \mathbf{m} \iff \sum_{l=1}^i k_l \leq \sum_{l=1}^i m_l \quad i = 1, \ldots, r,
\]
\[
k \subseteq \mathbf{m} \iff k_i \leq m_i \quad i = 1, \ldots, r.
\]
Jack polynomials $P_m(z; \frac{d}{2})$ are uniquely defined by the following two conditions.

\begin{align*}
(1) \quad & D_2(z)P_m\left(z; \frac{d}{2}\right) = P_m\left(z; \frac{d}{2}\right) \sum_{j=1}^{r} m_j (m_j - 1 - d(r - j)), \\
(2) \quad & P_m\left(z; \frac{d}{2}\right) = m_m(z) + \sum_{k < m} c_{mk} m_k(z).
\end{align*}

Here, $<$ is defined by $k < m \iff k \neq m, k \leq m$.

Similarly, interpolation Jack polynomials (IJP) $P_{ip}^m(z; \frac{d}{2})$ are uniquely defined by the following two conditions.

\begin{align*}
(1) & \quad ip_k P_{ip}^m\left(m + \frac{d}{2}; \frac{d}{2}\right) = 0, \quad \text{unless } k \subseteq m \in P \\
(2) & \quad ip_m P_{ip}^m\left(z; \frac{d}{2}\right) = m_m(z) + (\text{lower terms}).
\end{align*}

Further, we put

\begin{align*}
\Phi_m^{(d)}(z) := & \quad \frac{P_m(z; \frac{d}{2})}{P_m(1; \frac{d}{2})} \quad \text{(normalized Jack polynomials)}, \\
\Psi_m^{(d)}(z) := & \quad \frac{P_m(z; \frac{d}{2})}{P_{ip}^m(m + \frac{d}{2}; \frac{d}{2})} = \frac{P_m(1; \frac{d}{2})}{P_{ip}^m(m + \frac{d}{2}; \frac{d}{2})} \Phi_m^{(d)}(z)
\end{align*}

and

\begin{align*}
\binom{z}{k}^{(d)} := & \quad \frac{P_{ip}^m(z + \frac{d}{2}; \frac{d}{2})}{P_{ip}^m(k + \frac{d}{2}; \frac{d}{2})} \quad \text{(generalized (or Jack) binomial coefficients)}, \\
0\mathcal{F}_0^{(d)}(z; u) := & \quad \sum_{m \in \mathcal{P}} \psi_m^{(d)}(z) \phi_m^{(d)}(u) = \sum_{m \in \mathcal{P}} \Phi_m^{(d)}(z) \Psi_m^{(d)}(u).
\end{align*}

For $m \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{C}$, we put

\begin{align*}
(x)_m := \begin{cases} 
  x(x + 1) \cdots (x + m - 1) & (m \neq 0) \\
  1 & (m = 0)
\end{cases}.
\end{align*}
From [M2] VI (6.14), (10.20) and [Ko] (4.8), we have
\[
P_m \left( 1; \frac{d}{2} \right) = \prod_{(i,j) \in \mathbf{m}} \frac{j - 1 + \frac{d}{2}(r - i + 1)}{m_i - j + \frac{d}{2}(m_j' - i + 1)}
\]
\[
= \prod_{1 \leq i < j \leq r} \frac{\left( \frac{d}{2}(j - i + 1) \right)_{m_i - m_j}}{\left( \frac{d}{2}(j - i) \right)_{m_i - m_j}}. \tag{2.1}
\]

Here a partition \( m \) is identified with a Young diagram, also notated by \( m \), which consists of boxes \((i, j)\) with \( i = 1, \ldots, r \) and \( j = 1, \ldots, m_i \) for a given \( i \)
\[
m = \{ s = (i, j) \mid 1 \leq i \leq r, 1 \leq j \leq m_i \},
\]
and \( m_j' \) is the \( j \)-th entry of \( m' \) which is the transpose of \( m \). Further, by [Ko] (7.4) and (7.5)
\[
P_{m}^{ip} \left( \mathbf{m} + \frac{d}{2}; \frac{d}{2} \right) = \prod_{(i,j) \in \mathbf{m}} \left( m_i - j + 1 + \frac{d}{2}(m_j' - i) \right)
\]
\[
= \prod_{j=1}^{r} \left( \frac{d}{2}(r - j) + 1 \right)_{m_j}
\cdot \prod_{1 \leq i < j \leq r} \frac{\left( \frac{d}{2}(j - i - 1) + 1 \right)_{m_i - m_j}}{\left( \frac{d}{2}(j - i) + 1 \right)_{m_i - m_j}}. \tag{2.2}
\]

Although these multivariate special functions are very complicated, we write down these functions explicitly in \( r = 1, r = 2 \) and \( d = 2 \).

**The \( r = 1 \) case** For non positive integer \( m \) and \( z \in \mathbb{C} \),
\[
P_m \left( z; \frac{d}{2} \right) = z^m, \quad P_{m}^{ip} \left( z; \frac{d}{2} \right) = \begin{cases} z(z - 1) \cdots (z - m + 1) & (m \neq 0) \\ 1 & (m = 0) \end{cases}.
\]

Further,
\[
P_m \left( 1; \frac{d}{2} \right) = 1, \quad P_{m}^{ip} \left( m; \frac{d}{2} \right) = m!, \quad \Phi_{m}^{(d)}(z) = z^m, \quad \Psi_{m}^{(d)}(z) = \frac{z^m}{m!},
\]
\[
\binom{z}{k}^{(d)}(d) = \begin{cases} \frac{z(z-1)\cdots(z-k+1)}{k!} & (k \neq 0) \\ 1 & (k = 0) \end{cases},
\]
\[
0 \mathcal{F}_0^{(d)}(z, u) = \sum_{m \geq 0} \frac{z^m}{m!} u^m = \sum_{m \geq 0} \frac{z^m}{m!} u^m = e^{zu}. \tag{2.3}
\]

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The \( r = 2 \) case (see [Ko] 10.3, [VK] 3.2.1) For any partition \( \mathbf{m} = (m_1, m_2) \in \mathcal{P} \) and \( \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \),

\[
P^m_\mathbf{m}(\mathbf{z}; \frac{d}{2}) = z_1^{m_1}z_2^{m_2} \binom{-m_1 + m_2, \frac{d}{2}; \frac{z_2}{z_1}}{1 - m_1 + m_2 - \frac{d}{2}; \frac{z_2}{z_1}}
\]

\[
P^{ip}_\mathbf{m}(\mathbf{z}; \frac{d}{2}) = (-1)^{m_1 + m_2} (-z_1)^{m_2} (-z_2)^{m_1} \cdot \binom{-m_1 + m_2, \frac{d}{2}, -m_1 + 1 - \frac{d}{2} + z_1, 1}{1 - m_1 + m_2 - \frac{d}{2}, -m_1 + 1 + z_2, 1},
\]

where \( n+1F_n \) is the generalized hypergeometric function;

\[
n+1F_n(a_1, \ldots, a_{n+1}; b_1, \ldots, b_n; x) := \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_{n+1})_k}{(b_1)_k \cdots (b_n)_k} x^k.
\]

Further,

\[
P^m_\mathbf{m}(1; \frac{d}{2}) = \frac{(d)_{m_1 - m_2}}{(\frac{d}{2})_{m_1 - m_2}}
\]

\[
P^{ip}_\mathbf{m}(\mathbf{m} + \frac{d}{2}; \frac{d}{2}) = \frac{(\frac{d}{2} + 1)_{m_1} m_2! (m_1 - m_2)!}{(\frac{d}{2} + 1)_{m_1 - m_2}},
\]

\[
\Phi^{(d)}_\mathbf{m}(\mathbf{z}) = \frac{(\frac{d}{2})_{m_1 - m_2} z_1^{m_1} z_2^{m_2} \binom{-m_1 + m_2, \frac{d}{2}; \frac{z_2}{z_1}}{1 - m_1 + m_2 - \frac{d}{2}; \frac{z_2}{z_1}}}{(\frac{d}{2} + 1)_{m_1 - m_2}},
\]

\[
\Psi^{(d)}_\mathbf{m}(\mathbf{z}) = \frac{(\frac{d}{2} + 1)_{m_1 - m_2} z_1^{m_1} z_2^{m_2} \cdot \binom{-m_1 + m_2, \frac{d}{2}; \frac{z_2}{z_1}}{1 - m_1 + m_2 - \frac{d}{2}; \frac{z_2}{z_1}}}{(\frac{d}{2} + 1)_{m_1 - m_2}},
\]

\[
\left(\begin{array}{c}
\mathbf{z} \\
\mathbf{k}
\end{array}\right)^{(d)} = \frac{(\frac{d}{2} + 1)_{k_1 - k_2}}{(\frac{d}{2} + 1)_{k_1} (k_1 - k_2)! k_2!} (-1)^{k_1 + k_2} \binom{-z_1 - \frac{d}{2}; k_2}{-z_2 - k_2} \cdot \binom{-k_1 + k_2, \frac{d}{2}; k_1 + 1 + z_1, 1}{1 - k_1 + m_2 - \frac{d}{2}, -k_1 + 1 + z_2; 1},
\]

\[
0F_0^{(d)}(\mathbf{z}; \mathbf{u}) = e^{z_1 u_1 + z_2 u_2} \binom{\frac{d}{2}; (z_1 + z_2)(u_1 + u_2)}{1 - z_1 - z_2}(u_1 - u_2).
\]

The \( d = 2 \) case In this case, \( P^m_\mathbf{m}(\mathbf{z}; 1) \) and \( P^{ip}_\mathbf{m}(\mathbf{z}; 1) \) are Schur polynomials
and shifted Schur polynomials respectively \[\text{[OO2]}\].

\[
\begin{align*}
P_m(z; 1) &= s_m(z) = \frac{\det \left( z_i^{m_j + r - j} \right)_{1 \leq i, j \leq r}}{\Delta(z)}, \\
P_{ip}^m(z; 1) &= \frac{\det \left( P_{m_j + r - j}^{ip} \left( z_i + r - i; 1 \right) \right)_{1 \leq i, j \leq r}}{\Delta(z)}, \\
\Phi_m^{(2)}(z) &= \prod_{1 \leq i < j \leq r} \frac{(j - i + 1)_{m_i - m_j} (j - i)_{m_i - m_j}}{(j - i)_{m_i - m_j}}, \\
\Psi_m^{(2)}(z) &= \prod_{1 \leq i < j \leq r} \frac{(j - i + 1)_{m_i - m_j}}{(j - i)_{m_i - m_j}} s_m(z).
\end{align*}
\]

where \(\Delta(z) := \prod_{1 \leq i < j \leq r} (u_i - u_j)\). Further,

\[
P_m(1; 1) = s_m(1) = \prod_{1 \leq i < j \leq r} \frac{(j - i + 1)_{m_i - m_j}}{(j - i)_{m_i - m_j}}.
\]

Remark 2.1. We remark normalization of various Jack polynomials. First, we list some notations of Jack polynomials and their special values at \(z = 1\) (see Table 1). In this article, our notations are based on \[\text{[FK]}\]. In particular,

\[
\Psi^{(d)}_m(z) = d_m \frac{1}{\binom{n}{m}} \Phi^{(d)}_m(z),
\]

where

\[
n := r + \frac{d}{2} r (r - 1), \\
(\alpha)_m := \prod_{j=1}^r \left( \alpha - \frac{d}{2} (j - 1) \right)_{m_j}, \\
d_m := \prod_{1 \leq i < j \leq r} \frac{m_i - m_j + \frac{d}{2} (j - i)}{\frac{d}{2} (j - i)} \frac{(\frac{d}{2} (j - i + 1))_{m_i - m_j}}{(\frac{d}{2} (j - i - 1) + 1)_{m_i - m_j}} \quad (\text{[FK], p315}).
\]

From special values of Jack polynomials \(P_m(z; \frac{d}{2})\) \[\text{(2.1)}\] and interpolation Jack polynomials \(P_{ip}^k (z + \frac{d}{2} \delta; \frac{d}{2})\) \[\text{(2.2)}\], we have

\[
d_m \frac{1}{\binom{n}{m}} = \frac{P_m(1; \frac{d}{2})}{P_{ip}^m (m + \frac{d}{2} \delta; \frac{d}{2})}.
\]
Next, we remark the relationship between Stanley style $J_m^{(\frac{d}{2})}(z)$ and MacDonald style $P_m\left(z; \frac{d}{2}\right)$

\[
J_m^{(\frac{d}{2})}(z) = \left(\frac{2}{d}\right)^{|m|} \prod_{(i,j) \in m} \left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right) P_m\left(z; \frac{d}{2}\right)
\]

(M2) \text{M (10.22)).}

Hence we have

\[
\Psi_m^{(d)}(z) = \frac{P_m\left(z; \frac{d}{2}\right)}{P_m^{ip}\left(m + \frac{d}{2}\delta; \frac{d}{2}\right)} = \left(\frac{d}{2}\right)^{|m|} \prod_{(i,j) \in m} \left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right) \frac{1}{P_m^{ip}\left(m + \frac{d}{2}\delta; \frac{d}{2}\right)} J_m^{(\frac{d}{2})}(z).
\]

The relationship between $C_m^{(\frac{d}{2})}(1)$ (Kaneko style) and $\Psi_m^{(d)}(1)$ (our style)

\[
C_m^{(\frac{d}{2})}(1) = |m|! \prod_{(i,j) \in m} \frac{(j - 1 + \frac{d}{2}(r - i + 1))}{\left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right) \left(m_i - j + 1 + \frac{d}{2}(m'_j - i)\right)}
\]

\[
= |m|! \frac{P_m\left(1; \frac{d}{2}\right)}{P_m^{ip}\left(m + \frac{d}{2}\delta; \frac{d}{2}\right)}.
\]

It follows from (2.1), (2.2) and [Ka] (18). Thus, we have

\[
\Psi_m^{(d)}(z) = \frac{1}{|m|!} C_m^{(\frac{d}{2})}(z).
\]

To summarize the above results, we obtain

\[
\Psi_m^{(d)}(z) = d_m \frac{1}{|m|!} \Phi_m^{(d)}(z)
\]

\[
= \frac{1}{P_m^{ip}\left(m + \frac{d}{2}\delta; \frac{d}{2}\right)} P_m\left(z; \frac{d}{2}\right)
\]

\[
= \frac{1}{|m|!} C_m^{(\frac{d}{2})}(z).
\]
Table 1: Notations and normalizations of Jack polynomials

3 The five key formulas

In this section, we give a multivariate analogue of the five key formulas.

1. Sekiguchi operator (Sekiguchi [Se], Debiard [D], et.al.)

Let

\[
H^{(d)}_{r,p}(z) := \sum_{l=0}^{p} \left(\frac{2}{d}\right)^{p-l} \sum_{I \subseteq [r], |I| = l} \frac{1}{\Delta(z)} \left(\prod_{i \in I} z_i \partial z_i \right) \Delta(z) \sum_{J \subseteq [r] \setminus I, |J| = p-l} \left(\prod_{j \in J} z_j \partial z_j \right),
\]

\[
S^{(d)}_{r}(u;z) := \sum_{p=0}^{r} H^{(d)}_{r,p}(z) u^{r-p},
\]

where \([r] := \{1, 2, \ldots, r\}\) and \(\Delta(z) := \prod_{1 \leq i<j \leq r} (z_i - z_j)\). We have

\[
S^{(d)}_{r}(u;z) P_{\mathbf{m}} \left( z; \frac{d}{2} \right) = P_{\mathbf{m}} \left( z; \frac{d}{2} \right) I^{(d)}_{r}(u; \mathbf{m}), \tag{3.1}
\]

\[
H^{(d)}_{r,p}(z) P_{\mathbf{m}} \left( z; \frac{d}{2} \right) = P_{\mathbf{m}} \left( z; \frac{d}{2} \right) e_{r,k} \left( \mathbf{m} + \frac{d}{2} \delta \right), \tag{3.2}
\]

where

\[
I^{(d)}_{r}(u; \mathbf{m}) := \prod_{k=1}^{r} \left( u + r - k + \frac{2}{d} m_k \right) = \left(\frac{2}{d}\right)^{r} \prod_{k=1}^{r} \left( m_k + \frac{d}{2} (u + r - k) \right).
\]

2. Pieri type formulas for Jack polynomials (Lassalle [L], et.al.)
Let
\[ e_{r,1}(z) := \sum_{j=1}^r z_j, \quad E_0(z) := \sum_{j=1}^r \partial_{z_j}, \quad A_{\pm,i}^{(d)}(x) := \prod_{1 \leq k \neq i \leq r} \frac{x_i - x_k - \frac{d}{2}(i-k) \pm \frac{d}{2}}{x_i - x_k - \frac{d}{2}(i-k)}, \]
\[ \epsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r. \]

We have
\[ E_0(z) \Phi_x^{(d)}(z) = \sum_{i=1}^r \Phi_{x-\epsilon_i}^{(d)}(z) \left( x_i + \frac{d}{2}(r-i) \right) A_{-i}^{(d)}(x), \quad (3.3) \]
\[ E_0(z) \Psi_x^{(d)}(z) = \sum_{1 \leq i \leq r, x-\epsilon_i \in \mathcal{P}} \Psi_{x-\epsilon_i}^{(d)}(z) A_{+i}^{(d)}(x_i), \quad (3.4) \]
\[ e_{r,1}(z) \Phi_x^{(d)}(z) = \sum_{i=1}^r \Phi_{x+\epsilon_i}^{(d)}(z) A_{+i}^{(d)}(x), \quad (3.5) \]
\[ e_{r,1}(z) \Psi_x^{(d)}(z) = \sum_{1 \leq i \leq r, x+\epsilon_i \in \mathcal{P}} \Psi_{x+\epsilon_i}^{(d)}(z) \left( x_i + 1 + \frac{d}{2}(r-i) \right) A_{-i}^{(d)}(x), \quad (3.6) \]

We remark if \( x - \epsilon_i \notin \mathcal{P} \) (resp. \( x + \epsilon_i \notin \mathcal{P} \)) then \( A_{-i}^{(d)}(x) = 0 \) (resp. \( A_{+i}^{(d)}(x) = 0 \)).

3. **Binomial formula** (Knop-Sahi [KS], Okounkov-Olshanski [OO1], et.al.)
   For any partition \( x \),
   \[ \Phi_x^{(d)}(1 + z) = \sum_{k \subseteq x} \binom{x}{k}^{(d)} \Phi_k^{(d)}(z) = \sum_{k \subseteq x} \frac{P_{k}^{(d)}(x + \frac{d}{2} \delta; \frac{d}{2})}{P_{k}^{(d)}(1; \frac{d}{2})} \Psi_k^{(d)}(z). \quad (3.7) \]
   Since the above formulas are well known, we omit their proofs.

4. **Mysterious summation**

   **Lemma 3.1.** For any \( I \subseteq [r] \) and \( x = (x_1, \ldots, x_r) \in \mathbb{C}^r \), we have
   \[ \sum_{i \in I} \left( x_i + 1 + \frac{d}{2}(r-i) \right) A_{-i,I \setminus i}^{(d)}(x + \epsilon_i) A_{+i,I \setminus i}^{(d)}(x) \]
   \[ - \sum_{i \in I} \left( x_i + \frac{d}{2}(r-i) \right) A_{+i,I \setminus i}^{(d)}(x - \epsilon_i) A_{-i,I \setminus i}^{(d)}(x) = |I|, \quad (3.8) \]
where
\[ A^{(d)}_{\pm,i,R,j}(x) := \prod_{j \in I \setminus i} \frac{x_i - x_j - \frac{d}{2}(i - j) \pm \frac{d}{2}}{x_i - x_j - \frac{d}{2}(i - j)}. \]

**proof.** For convenience, we put
\[ s_j := x_j + \frac{d}{2}(r - j). \]

By Pieri type formulas for the Jack polynomials (3.3) and (3.5), we have
\[ [E_0(z), e_{r,1}(z)] \Phi^{(d)}_x(z) = \sum_{J \subseteq \{r\}, |J| = l} \Phi^{(d)}_{x-\epsilon, J}(z) A^{(d)}_{-+, J}(x) - \sum_{J \subseteq \{r\}, |J| = l} e_{r,1}(z) \Phi^{(d)}_{x-\epsilon, J}(z) s_j A^{(d)}_{-+, j}(x) \]
\[ = \Phi^{(d)}_x(z) \]
\[ \cdot \sum_{i=1}^{r} \left( (s_i + \delta_{j,i}) A^{(d)}_{-i, J}(x) A^{(d)}_{+, i}(x) - s_j A^{(d)}_{-i, J}(x) A^{(d)}_{+, i}(x - \epsilon_j) \right). \]

On the other hand,
\[ [E_0(z), e_{r,1}(z)] \Phi^{(d)}_x(z) = \Phi^{(d)}_x(z) r. \]
Then, we obtain (3.8). \(\square\)

5. Twisted Pieri formulas for Jack polynomials

**Theorem 3.2.** For \( l = 0, 1, \ldots, r \), we have
\[ \left[ \frac{(\text{ad} E_0(z))^l}{l!} S^{(d)}_{\ell}(u; z) \right] \Phi^{(d)}_x(z) = \sum_{J \subseteq \{r\}, |J| = l} \Phi^{(d)}_{x-\epsilon, J}(z) I^{(d)}_{J, c}(u; x) A^{(d)}_{-+, J}(x) \prod_{j \in J} \left( x_j + \frac{d}{2}(r - j) \right), \]
\[ \left[ \frac{(\text{ad} E_0(z))^l}{l!} S^{(d)}_{\ell}(u; z) \right] \Phi^{(d)}_x(z) = \sum_{J \subseteq \{r\}, |J| = l} \Psi^{(d)}_{x-\epsilon, J}(z) I^{(d)}_{J, c}(u; x) A^{(d)}_{+, J}(x - \epsilon J). \]
\[ (3.10) \]
Here $J^c := [r] \setminus J$, $\epsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, $\epsilon_J = \sum_{j \in J} \epsilon_j$,

$$A^{(d)}_{\pm,J}(x) := \prod_{j \in J, l \in J^c} \frac{x_j - x_l - \frac{d}{2}(j - l) \pm \frac{d}{2}}{x_j - x_l - \frac{d}{2}(j - l)},$$

$$I_{j, r}^{(d)}(u; x) := \left(\frac{2}{d}\right)^r \prod_{l \in J^c} \left(x_l + \frac{d}{2}(u + r - l)\right).$$

**proof.** Since (3.9) and (3.10) can be similarly proved, we only prove (3.9). These formulas are proved by induction on $l$.

The case of $l = 0$ is

$$S_r^{(d)}(u; z)\Phi^{(d)}_x(z) = \Phi^{(d)}_x(z)I_r^{(d)}(u; x). \quad (3.11)$$

This is (3.1).

$$S_r^{(d)}(u; z)P_m\left(z; \frac{d}{2}\right) = P_m\left(z; \frac{d}{2}\right)I_r^{(d)}(u; m)$$

exactly.

If $l = 1$, then

$$[\text{ad } E_0(z)S_r^{(d)}(u; z)]\Phi^{(d)}_x(z)$$

$$= E_0(z)\Phi^{(d)}_x(z)I_r^{(d)}(u; x) - S_r^{(d)}(u; z)\sum_{i=1}^r \Phi^{(d)}_{x - \epsilon_i}(z) \left(x_i + \frac{d}{2}(r - i)\right) A^{(d)}_{-i}(x)$$

$$= \sum_{i=1}^r \Phi^{(d)}_{x - \epsilon_i}(z) A^{(d)}_{-i}(x)(I_r^{(d)}(u; x) - I_r^{(d)}(u; x - \epsilon_i))$$

$$= \sum_{J \subseteq [r], |J| = 1} \Phi^{(d)}_{x - \epsilon_J}(z) A^{(d)}_{-J}(x) s_i.$$
Assume the \( n = l \) case holds. Hence,

\[
\frac{(ad E_0(z))^{l+1}}{(l+1)!} S_r^{(d)} (t; z) \Phi_x^{(d)} (z) \\
= \frac{1}{l+1} E_0(z) \left[ \frac{(ad E_0(z))^l}{l!} S_r^{(d)} (t; z) \right] \Phi_x^{(d)} (z) \\
- \left[ \frac{(ad E_0(z))^l}{l!} S_r^{(d)} (t; z) \right] \frac{1}{l+1} E_0(z) \Phi_x^{(d)} (z) \\
= \frac{1}{l+1} \sum_{j \in [r]} \sum_{|J|=l} \frac{\Phi_x^{(d)} (z) A^{(d)}_{-\nu, \nu} (x - \epsilon_j) (s_{\nu} - \delta_{\nu, j}) I_j^{(d)} (u; x) A^{(d)}_{-\nu, \nu} (x) \prod_{j \in J} s_j}{l!} \\
- \frac{1}{l+1} \sum_{\nu=1}^r \sum_{|J|=l} \frac{\Phi_x^{(d)} (z) I_j^{(d)} (u; x - \epsilon_{\nu}) A^{(d)}_{-\nu, \nu} (x - \epsilon_{\nu}) A^{(d)}_{-\nu, \nu} (x) s_{\nu} \prod_{j \in J} (s_j - \delta_{j, \nu})}{l!},
\]

where

\[ \delta_{i, J} := \begin{cases} 1 & (i \in J) \\ 0 & (i \not\in J) \end{cases}. \]

From a simple calculation, we have

\[
\frac{(ad E_0(z))^{l+1}}{(l+1)!} S_r^{(d)} (t; z) \Phi_x^{(d)} (z) \\
= \sum_{\substack{J \subseteq [r], J \neq \emptyset, \ |J|=l+1 \ \ |J|=l+1}} \frac{1}{l+1} \Phi_x^{(d)} (z) I_j^{(d)} (u; x) A^{(d)}_{-\nu, \nu} (x) \prod_{j \in J} s_j \\
\cdot \sum_{i \in I} \left\{ \frac{s_i + \frac{d}{2} u}{2} A^{(d)}_{-\nu, \nu} (x + \epsilon_i) A^{(d)}_{+\nu, \nu} (x) \\
- \frac{s_i - 1 + \frac{d}{2} u}{2} A^{(d)}_{-\nu, \nu} (x) A^{(d)}_{+\nu, \nu} (x) \right\}. \]

Here, the summation

\[
\sum_{i \in I} [s_i A^{(d)}_{-\nu, \nu} (x + \epsilon_i) A^{(d)}_{+\nu, \nu} (x) - (s_i - 1) A^{(d)}_{-\nu, \nu} (x) A^{(d)}_{+\nu, \nu} (x - \epsilon_i)] = l + 1
\]

is our mysterious summation (3.8) exactly.
By comparing the coefficients for $u^{-l}$ of the twisted Pieri (3.10) and (3.9), we obtain the following twisted Pieri type formulas.

**Corollary 3.3.** For any $z \in \mathbb{C}^r$ and $l = 0, 1, \ldots, r$,

\[
\left( \frac{d}{2} \right)^{r-l} \left[ \frac{(\text{ad} E_0(z))^l}{l!} H_{r,l}(z) \right] \Phi_x^{(d)}(z) = \sum_{J \subseteq [r], |J| = l, x - \epsilon J \in \mathcal{P}} \frac{\Phi_{x-\epsilon J}^{(d)}(z)}{l!} \prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right),
\]

(3.12)

\[
\left( \frac{d}{2} \right)^{r-l} \left[ \frac{(\text{ad} E_0(z))^l}{l!} H_{r,l}(z) \right] \Psi_x^{(d)}(z) = \sum_{J \subseteq [r], |J| = l, x - \epsilon J \in \mathcal{P}} \frac{\Psi_{x-\epsilon J}^{(d)}(z)}{l!} \prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right).
\]

(3.13)

To summarize the above results, we obtain the following dictionary.

0. **Jack polynomials**

\[
\Phi_m^{(d)}(z) := z^m \Rightarrow \Phi_m^{(d)}(z) := \frac{P_m(z; \frac{d}{2})}{P_m(1; \frac{d}{2})},
\]

\[
\Psi_m^{(d)}(z) := \frac{z^m}{m!} \Rightarrow \Psi_m^{(d)}(z) := \frac{P_m(1; \frac{d}{2}) \Phi_m^{(d)}(z)}{P_m^{\text{sp}}(m + \frac{d}{2}; \frac{d}{2})} = \frac{P_m(z; \frac{d}{2})}{P_m^{\text{sp}}(m + \frac{d}{2}; \frac{d}{2})}.
\]

1. **Sekiguchi operator**

\[
(z \partial_z) \Phi_m(z) = \Phi_m(z)m \Rightarrow S_r^{(d)}(u; z) \Phi_k^{(d)}(z) = \Phi_k^{(d)}(z) I_r^{(d)}(u; m),
\]

\[
(z \partial_z) \Psi_m(z) = \Psi_m(z)m \Rightarrow S_r^{(d)}(u; z) \Psi_k^{(d)}(z) = \Psi_k^{(d)}(z) I_r^{(d)}(u; m).
\]
2. Pieri type formulas for Jack polynomials

\[ \partial_z \Phi_m(z) = \Phi_{m-1}(z) m, \quad \Rightarrow \quad E_0(z) \Phi^{(d)}_x(z) = \sum_{i=1}^{r} \Phi^{(d)}_{x-\epsilon_i}(z) \left( x_i + \frac{d}{2} (r - i) \right) A^{(d)}_{-i}(x), \]

\[ \partial_z \Psi_m(z) = \Psi_{m-1}(z), \quad \Rightarrow \quad E_0(z) \Psi^{(d)}_x(z) = \sum_{1 \leq i \leq r, \ x-\epsilon_i \in \mathcal{P}} \Psi^{(d)}_{x-\epsilon_i}(z) A^{(d)}_{+i}(x - \epsilon_i), \]

\[ z \Phi_m(z) = \Phi_{m+1}(z), \quad \Rightarrow \quad e_{r,1}(z) \Phi^{(d)}_x(z) = \sum_{i=1}^{r} \Phi^{(d)}_{x+\epsilon_i}(z) A^{(d)}_{+i}(x), \]

\[ z \Psi_m(z) = \Psi_{m+1}(z)(m+1), \quad \Rightarrow \quad e_{r,1}(z) \Psi^{(d)}_x(z) = \sum_{1 \leq i \leq r, \ x+\epsilon_i \in \mathcal{P}} \Psi^{(d)}_{x+\epsilon_i}(z) \left( x_i + 1 + \frac{d}{2} (r - i) \right) A^{(d)}_{-i}(x + \epsilon_i). \]

3. Binomial formula

\[ \Phi_x(1 + z) = \sum_{k \geq 0} \frac{P^{ip}(x)}{P_k(1)} \Psi_k(z) \quad \Rightarrow \quad \Phi^{(d)}_x(1 + z) = \sum_{k \in \mathcal{P}} \frac{P^{ip}(x + \frac{d}{2}; \frac{d}{2})}{P_k(1; \frac{d}{2})} \Psi^{(d)}_k(z). \]

4. Mysterious summation

\[ (x + 1) - x = 1 \quad \Rightarrow \quad \sum_{i \in I} \left( x_i + 1 + \frac{d}{2} (r - i) \right) A^{(d)}_{-i, I \setminus i}(x + \epsilon_i) A^{(d)}_{+i, I \setminus i}(x) \]
\[ - \sum_{i \in I} \left( x_i + \frac{d}{2} (r - i) \right) A^{(d)}_{+i, I \setminus i}(x - \epsilon_i) A^{(d)}_{-i, I \setminus i}(x) = |I|. \]
5. Twisted Pieri formulas for Jack polynomials

\[
\left( \frac{\text{ad} \partial_z}{1!} z \partial_z \right) \Phi_x(z) = \Phi_{x-1}(z)x \Rightarrow \left( \frac{\text{ad} E_0(z)}{l!} S^{(d)}_r(u; z) \right) \Phi^{(d)}_x(z) = \sum_{J \subseteq \mathbb{P}, |J| = l} \Phi^{(d)}_{x-\epsilon_j}(z) I^{(d)}_{j_c}(u; x) A^{(d)}_{-, j}(x) \cdot \prod_{j \in J} \left( x_j + \frac{d}{2} (r-j) \right),
\]

\[
\left( \frac{\text{ad} \partial_z}{1!} z \partial_z \right) \Psi_x(z) = \Psi_{x-1}(z) \Rightarrow \left( \frac{\text{ad} E_0(z)}{l!} S^{(d)}_r(u; z) \right) \Psi^{(d)}_x(z) = \sum_{J \subseteq \mathbb{P}, |J| = l} \Psi^{(d)}_{x-\epsilon_j}(z) I^{(d)}_{j_c}(u; x) A^{(d)}_{+, j}(x - \epsilon_j). \]

4 Difference and Pieri formulas for interpolation Jack polynomials

**Theorem 4.1** (Difference formula for IJP, Knop-Sahi). Let

\[ T_{x,f}(x) := f(x - \epsilon_j), \quad T^J_x := \prod_{j \in J} T_{x,j}. \]

Put

\[ D^{(d)}_{r, \text{ip}}(u; x) := \sum_{J \subseteq [r]} (-1)^{|J|} I^{(d)}_{j_c}(u; x) A^{(d)}_{-, j}(x) \prod_{j \in J} \left( x_j + \frac{d}{2} (r-j) \right) T^J_x. \]

We have

\[ D^{(d)}_{r, \text{ip}}(u; x) P^{\text{ip}}_k \left( x + \frac{d}{2}; \frac{d}{2} \right) = P^{\text{ip}}_k \left( x + \frac{d}{2}; \frac{d}{2} \right) I^{(d)}_r(u; k). \quad (4.1) \]

**Proof.** Since the difference formula (4.1) is a relation for rational function of \((x_1, \ldots, x_r)\), it is enough to prove when variable \(x \in \mathbb{P}\). To prove (4.1), we compute

\[ S^{(d)}_r(u; z) \Phi^{(d)}_x(1 + z) \]
in two different ways. From the twisted Pieri (3.9) and the binomial (3.7),

\[ S_r^{(d)}(u; z) \Phi_x^{(d)}(1 + z) \]

\[ = S_r^{(d)}(u; z) e^{E_0(z)} \Phi_x^{(d)}(z) \]

\[ = e^{E_0(z)} \left[ e^{-ad E_0(z) S_r^{(d)}(u; z)} \right] \Phi_x^{(d)}(z) \]

\[ = e^{E_0(z)} \sum_{l=0}^r \left[ \frac{(-ad (E_0(z)))^l}{l!} S_r^{(d)}(u; z) \right] \Phi_x^{(d)}(z) \]

\[ = \sum_{l=0}^r (-1)^l \sum_{J \subseteq [r], |J|=l} e^{E_0(z)} \Phi_x^{(d)}(1 + z) I_{j_e}^{(d)}(u; x) A^{(d)}_{-J}(x) \prod_{j \in J} \left( x_j + \frac{d}{2}(r - j) \right) \]

\[ = \sum_{J \subseteq [r]} (-1)^{|J|} \Phi_x^{(d)}(1 + z) I_{j_e}^{(d)}(u; x) A^{(d)}_{-J}(x) \prod_{j \in J} \left( x_j + \frac{d}{2}(r - j) \right) \]

\[ = \sum_{J \subseteq [r]} \sum_{k \in \mathcal{P}} \Psi_k^{(d)}(z) \prod_{J \subseteq [r]} (-1)^{|J|} \frac{P_{k}^{ip} \left( x - \epsilon_J + \frac{d \delta_j}{2} \right)}{P_k \left( 1; \frac{d}{2} \right)} I_{j_e}^{(d)}(u; x) A^{(d)}_{-J}(x) \prod_{j \in J} \left( x_j + \frac{d}{2}(r - j) \right) . \]

On the other hand, from the binomial formula (3.7) and (3.1), we have

\[ S_r^{(d)}(u; z) \Phi_x^{(d)}(1 + z) = \sum_{k \in \mathcal{X}} \frac{P_k^{ip} \left( x + \frac{d \delta_j}{2} \right)}{P_k \left( 1; \frac{d}{2} \right)} S_r^{(d)}(u; z) \Phi_k^{(d)}(z) \]

\[ = \sum_{k \in \mathcal{X}} \frac{P_k^{ip} \left( x + \frac{d \delta_j}{2} \right)}{P_k \left( 1; \frac{d}{2} \right)} \Phi_k^{(d)}(z) I_{r}^{(d)}(u; x) \]

\[ = \sum_{k \in \mathcal{P}} \frac{\Psi_k^{(d)}(z) I_{r}^{(d)}(u; x)}{P_k \left( 1; \frac{d}{2} \right)} . \]

Comparing coefficients for \( u^{r-l} \) in (4.1), we obtain higher order difference formulas for IJP.
Corollary 4.2. For any \( x \in \mathbb{C}^r \), \( k \in \mathcal{P} \) and \( l = 0, 1, \ldots, r \), we have

\[
e_{r,l} \left( k + \frac{d}{2} \delta \right) P_k^{\text{ip}} \left( x + \frac{d}{2} \delta; \frac{d}{2} \right) = \sum_{\substack{J \subseteq [r], \ 0 \leq |J| \leq l \ \text{and} \ \epsilon \in \mathbb{C} \ \text{and} \ \epsilon \in \mathbb{C}}} (-1)^{|J|} P_k^{\text{ip}} \left( x - \epsilon J + \frac{d}{2} \delta; \frac{d}{2} \right) e_{r-|J|,|J|} \left( \left( x + \frac{d}{2} \delta \right) \right)_J A_{-\epsilon,J}(x) \\
\prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right), \tag{4.2}
\]

where \( \left( x + \frac{d}{2} \delta \right)_J := \left( x_{i_1} + \frac{d}{2} (r - i_1), \ldots, x_{i_r-l} + \frac{d}{2} (r - i_{r-l}) \right)_{i_1, \ldots, i_{r-l} \in J^c}. \)

Originally, Theorem 4.1 or Corollary 4.2 were proved by Knop-Sahi [KS]. Knop-Sahi’s proof shows that \( D_r^{(d)}(u; x) P_k^{\text{ip}} \left( x + \frac{d}{2} \delta; \frac{d}{2} \right) \) satisfy the IJP’s conditions (1) and (2) up to a constant \( c(k) \) for any \( k \in \mathcal{P} \), and determine \( c(k) := I_r^{(d)}(u; k) \) explicitly. Knop-Sahi’s proof requires that the difference formulas for IJP are known in ad hoc, whereas our proof does not require it.

Theorem 4.3 (Pieri formula for IJP). For any \( x \in \mathbb{C}^r \) and \( k \in \mathcal{P} \), we have

\[
I_j^{(d)}(u; x) P_k^{\text{ip}} \left( x + \frac{d}{2} \delta; \frac{d}{2} \right) P_k \left( 1; \frac{d}{2} \right) = \sum_{\substack{J \subseteq [r], \ k + \epsilon J \in \mathcal{P} \ \text{and} \ \epsilon \in \mathbb{C} \ \text{and} \ \epsilon \in \mathbb{C}}} \frac{P_k^{\text{ip}} \left( x + \frac{d}{2} \delta; \frac{d}{2} \right) I_{j^c}^{(d)}(u; k + \epsilon J) A_{+\epsilon,J}(k)}{P_k \left( 1; \frac{d}{2} \right)}.
\tag{4.3}
\]

proof. To prove the Pieri formula (4.3), it is enough to prove when variable \( x \in \mathcal{P} \). We compute

\[
[e^{\text{ad} E_0(z)} S_\tau^{(d)}(u; z)] \Phi_x^{(d)}(1 + z)
\]
in two different ways. From (3.1) and the binomial formula (3.7),

\[ e^{ad E_0(z)} S^{(d)}_r (u; z) \Phi^{(d)}_x (1 + z) = e^{E_0(z)} S^{(d)}_r (u; z) e^{-E_0(z)} \Phi^{(d)}_x (1 + z) \]

\[ = e^{E_0(z)} S^{(d)}_r (u; z) \Phi^{(d)}_x (z) \]

\[ = e^{E_0(z)} \Phi^{(d)}_x (z) I^{(d)}_r (u; x) \]

\[ = \Phi^{(d)}_x (1 + z) I^{(d)}_r (u; x) \]

\[ = \sum_{k \subseteq \chi} P^d_k \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right) \Psi^{(d)}_k (z) I^{(d)}_r (u; x) \]

\[ = \sum_{k \in \mathcal{P}} \Psi^{(d)}_k (z) I^{(d)}_r (u; x) \frac{P^d_k \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right)}{P_k \left( 1; \frac{d}{2} \right)}. \]

On the other hand, from the binomial (3.7) and the twisted Pieri (3.9)

\[ [e^{ad E_0(z)} S^{(d)}_r (u; z)] \Phi^{(d)}_x (1 + z) \]

\[ = \sum_{k \subseteq \chi} P^d_k \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right) \sum_{l=0}^r \left[ \frac{(ad (E_0(z)))^l}{l!} S^{(d)}_r (u; z) \right] \Psi^{(d)}_k (z) \]

\[ = \sum_{k \subseteq \chi} P^d_k \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right) \sum_{l=0}^r \sum_{J \subseteq [r], |J| = l, k_{-\epsilon_J} \in \mathcal{P}} \Psi^{(d)}_k (z) I^{(d)}_{\epsilon_J} (u; k) \hat{h}^{(d)}_+ (k - \epsilon_J) \]

\[ = \sum_{k \in \mathcal{P}} \Psi^{(d)}_k (z) \sum_{J \subseteq [r], k_{+\epsilon_J} \in \mathcal{P}} \frac{P^d_{k_{+\epsilon_J}} \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right)}{P_{k_{+\epsilon_J}} \left( 1; \frac{d}{2} \right)} I^{(d)}_{\epsilon_J} (u; k + \epsilon_J) A^{(d)}_+ \epsilon (k). \]

\[ \square \]

By comparing coefficients for \( u^{r-l} \) in (4.3), we obtain the Pieri type formulas for the interpolation Jack polynomials.

**Corollary 4.4.** For any \( x \in \mathbb{C}^r, k \in \mathcal{P} \) and \( l = 0, 1, \ldots, r, \)

\[ e_{r,l} \left( \frac{x + \frac{d}{2} \delta}{\frac{d}{2}} \right) \frac{P^d_k \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right)}{P_k \left( 1; \frac{d}{2} \right)} \]

\[ = \sum_{J \subseteq [r], |J| = l, 0 \leq k_{+\epsilon_J} \in \mathcal{P}} \frac{P^d_{k_{+\epsilon_J}} \left( \frac{x + \frac{d}{2} \delta; \frac{d}{2}}{\frac{d}{2}} \right)}{P_{k_{+\epsilon_J}} \left( 1; \frac{d}{2} \right)} e_{r-[J],l-[J]} \left( \left( k + \epsilon_J + \frac{d}{2} \delta \right) \right)_J A^{(d)}_+ \epsilon (k). \]

(4.4)
5 Some applications

As some applications of our twisted Pieri formulas for Jack and Pieri formulas for IJP, we obtain the following results. First, by comparing the coefficients for the highest degree term of \(4.4\) and the definition of IJP \((2)^{\text{IP}}\), we obtain Pieri formulas for the ordinary Jack polynomials.

**Corollary 5.1.** For \(l = 0, 1, \ldots, r\),
\[
e_{r,l}(z) \Phi_k^{(d)}(z) = \sum_{J \subseteq [r], |J| = l, k+\epsilon_j \in \mathcal{P}} \Phi_{k+\epsilon_j}^{(d)}(z) A_{\frac{k}{\epsilon_j} J}(k).
\] (5.1)

Next, we obtain the following intertwining relation for \(0 F_0^{(d)}\).

**Theorem 5.2.** For any \(l = 0, 1, \ldots, r\),
\[
\left( \frac{d}{2} \right)^{r-l} \left[ \frac{(ad E_0(z))^l}{l!} H_{r,l}^{(d)}(z) \right] 0 F_0^{(d)} (z, w) = 0 F_0^{(d)} (z, w) e_{r,l}(w),
\]

where
\[
0 F_0^{(d)} (z, w) := \sum_{m \in \mathcal{P}} \Psi_m^{(d)}(z) \Phi_m^{(d)}(w) = \sum_{m \in \mathcal{P}} \Phi_m^{(d)}(z) \Psi_m^{(d)}(w).
\]

It is a multivariate analogue of
\[
\partial_z 0 F_0^{(d)} (z, w) = 0 F_0^{(d)} (z, w) w,
\]
where
\[
0 F_0^{(d)} (z, w) := \sum_{m=0}^{\infty} \frac{1}{m!} z^m w^m = \sum_{m=0}^{\infty} \Psi_m(z) \Phi_m(w) = \sum_{m=0}^{\infty} \Phi_m(z) \Psi_m(w) = e^{zw}.
\]
**proof.** From two Pieri type formulas (3.13) and (5.1), we have

\[
\left( \frac{d}{2} \right)^{r-l} \left[ \frac{(ad E_0(z))^{l}}{l!} H_{r,l}^{(d)}(z) \right] 0\mathcal{J}_0^{(d)}(z, w) \\
= \sum_{m \in \mathcal{P}} \left( \frac{d}{2} \right)^{r-l} \left[ \frac{(ad E_0(z))^{l}}{l!} H_{r,l}^{(d)}(z) \right] \Psi_m^{(d)}(z) \Phi_m^{(d)}(w) \\
= \sum_{m \in \mathcal{P}} \sum_{J \subseteq [r], |J|=l, \ m-\epsilon_J \in \mathcal{P}} \Psi_m^{(d)}(z) A_{+J}^{(d)}(m-\epsilon_J) \Phi_m^{(d)}(w) \\
= \sum_{m \in \mathcal{P}} \Psi_m^{(d)}(z) \varepsilon_{r,l}(w) \Phi_m^{(d)}(w) \\
= 0\mathcal{J}_0^{(d)}(z, w) e_{r,l}(w).
\]

\[\square\]

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