SELF-ADJOINT INDEFINITE LAPLACIANS

By

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Abstract. Let $\Omega_-$ and $\Omega_+$ be two bounded smooth domains in $\mathbb{R}^n$, $n \geq 2$, separated by a hypersurface $\Sigma$. For $\mu > 0$, consider the function $h_{\mu} = 1_{\Omega_-} - \mu 1_{\Omega_+}$. We discuss self-adjoint realizations of the operator $L_{\mu} = -\nabla \cdot h_{\mu} \nabla$ in $L^2(\Omega_- \cup \Omega_+)$ with the Dirichlet condition at the exterior boundary. We show that $L_{\mu}$ is always essentially self-adjoint on the natural domain (corresponding to transmission-type boundary conditions at the interface $\Sigma$) and study some properties of its unique self-adjoint extension $L_{\mu} := L_{\mu}$. If $\mu \neq 1$, then $L_{\mu}$ simply coincides with $L_{\mu}$ and has compact resolvent. If $n = 2$, then $L_1$ has a non-empty essential spectrum, $\sigma_{\text{ess}}(L_1) = \{0\}$. If $n \geq 3$, the spectral properties of $L_1$ depend on the geometry of $\Sigma$. In particular, it has compact resolvent if $\Sigma$ is the union of disjoint strictly convex hypersurfaces, but can have a non-empty essential spectrum if a part of $\Sigma$ is flat. Our construction features the method of boundary triplets, and the problem is reduced to finding the self-adjoint extensions of a pseudodifferential operator on $\Sigma$. We discuss some links between the resulting self-adjoint operator $L_{\mu}$ and some effects observed in negative-index materials.

1 Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary $\partial \Omega$. Let $\Omega_-$ be a subset of $\Omega$ having a smooth boundary $\Sigma$ (called interface) and such that $\overline{\Omega_-} \subset \Omega$. In addition, we consider the open set $\Omega_+ := \Omega \setminus \overline{\Omega_-}$, whose boundary is $\partial \Omega_+ = \Sigma \cup \partial \Omega$, and denote by $N_\pm$ the unit normal on $\Sigma$ exterior with respect to $\Omega_\pm$. For $\mu > 0$, consider the function $h : \Omega \setminus \Sigma \to \mathbb{R}$,

$$h_{\mu}(x) = \begin{cases} 1, & x \in \Omega_-; \\ -\mu, & x \in \Omega_+. \end{cases}$$

The aim of the present work is to construct self-adjoint operators in $L^2(\Omega)$ corresponding to the formally symmetric differential expression $L_{\mu} = -\nabla \cdot h_{\mu} \nabla$. The operators of such a type appear, e.g., in the study of negative-index metamaterials, and we refer to the recent paper [30] for a survey and an extensive bibliography; we
remark that the parameter $\mu$ is usually called the \textbf{contrast}. A possible approach is to consider the sesquilinear form
\[
\ell_\mu(u, v) = \int_{\Omega} h_\mu \nabla u \cdot \nabla v \, dx, \quad u, v \in H^1_0(\Omega),
\]
and then to define $L_\mu$ as the operator generated by $\ell_\mu$; in particular, for all functions $v$ from the domain of $L_\mu$ one should then have
\[
(1) \quad \int_{\Omega} \pi L_\mu v dx = \ell_\mu(u, v), \quad u \in H^1_0(\Omega).
\]
But the form $\ell_\mu$ is not semibounded below, hence the operator obtained in this way can have exotic properties, in particular, its self-adjointness is not guaranteed. We refer to [20, 36, 37] for some available results in this direction.

In [4] a self-adjoint operator for the expression $-\nabla \cdot h \nabla$ was constructed for the very particular case of
\[
\Omega = (-1, 1) \times (0, 1), \quad \Omega_- = (-1, 0) \times (0, 1), \quad \Omega_+ = (0, 1) \times (0, 1), \quad h = \pm 1 \text{ in } \Omega_\pm,
\]
which is not covered by the assumptions of the present paper (the subdomains $\Omega_\pm$ are non-smooth and touch just along a part of the boundary), but it admits a separation of variables and has a number of symmetries. An interesting feature of the model is the possibility of a non-empty essential spectrum although the domain is bounded. Constructing self-adjoint operator realizations of $L_\mu$ for the general case is an open problem; see [24]. In the present note, we give a solution in the case of a smooth interface.

One should remark that the study of various boundary value problems involving differential expressions $\nabla \cdot h \nabla$ with sign-changing $h$ has a long history, and the most classical form involves unbounded domains with a suitable radiation condition at infinity; cf. [13, 18, 32]. In particular, the geometric conditions appearing in the main results below are very close to those of [29, 32] for the well-posedness of a transmission problem. The case of a non-smooth interface $\Sigma$, which was partially studied in [8, 9, 10], is not covered by our approach.

In fact, the problem of self-adjoint realizations in the non-critical case $\mu \neq 1$ was essentially settled in [8], while for the critical case $\mu = 1$ it was only studied for the above-mentioned example of [4], in [37, Chapter 8] for another similar situation (symmetric $\Omega_-$ and $\Omega_+$ separated by a finite portion of a hyperplane), and in [21] for the one-dimensional case. In a sense, in the present work we recast some techniques of the transmission problems and the pseudodifferential operators into the setting of self-adjoint extensions. Using the machinery of boundary triplets we reduce the problem first to finding self-adjoint extensions of a symmetric
differential operator and then to the analysis of the associated Weyl function acting on the interface $\Sigma$. Then one arrives at the study of the essential self-adjointness of a pseudodifferential operator on $\Sigma$, whose properties depend on the dimension. We hope that, in view of the recent progress in the theory of self-adjoint extensions of partial differential operators (see, e.g., [6, 7, 16]), such a direct reformulation could be a starting point for a further advance in the study of indefinite operators.

Similar to [4], our approach is based on the theory of self-adjoint extensions. Using the natural identification $L^2(\Omega) \simeq L^2(\Omega_-) \oplus L^2(\Omega_+)$, $u \simeq (u_-, u_+)$, we introduce the sets

$$D^\mu_\Sigma(\Omega) := \{ u = (u_-, u_+) \in H^s(\Omega_-) \oplus H^s(\Omega_+) : \Delta u_\pm \in L^2(\Omega_\pm), \ u_- = u_+ \text{ and } N_- \nabla u_- = \mu N_+ \nabla u_+ \text{ on } \Sigma, \ u_+ = 0 \text{ on } \partial \Omega \}, \ s \geq 0.$$  

Here and below, the values at the boundary are understood as suitable Sobolev traces; the exact definitions are given in Section 3. Let us recall that for $\frac{1}{2} < s < \frac{3}{2}$ and $u = (u_-, u_+) \in H^s(\Omega_-) \oplus H^s(\Omega_+)$ the conditions $u_- = u_+$ on $\Sigma$ and $u_+ = 0$ on $\partial \Omega$ entail $u \in H^s_0(\Omega)$; see, e.g., [1, Theorem 3.5.1]. In particular,

$$D^2_\mu(\Omega \setminus \Sigma) \subseteq H^{\frac{3}{2} - \epsilon}_0(\Omega) \text{ for } \epsilon > 0, \ D^1_\mu(\Omega \setminus \Sigma) \subseteq H^1_0(\Omega).$$

Consider the operator

$$L_\mu(u_-, u_+) = (-\Delta u_-, \mu \Delta u_+), \ \text{with } \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

acting on the domain

$$\text{dom } L_\mu = D^2_\mu(\Omega \setminus \Sigma).$$

We remark that $L_\mu$ satisfies (1) and it is a densely defined symmetric operator in $L^2(\Omega)$. Therefore, we use $L_\mu$ as a starting point and seek its self-adjoint extensions. Even if the case $\mu \neq 1$ was studied earlier, we include it into consideration as it does not imply any additional difficulties.

**Theorem 1 (Self-adjointness).** The operator $L_\mu$ is essentially self-adjoint, and we denote

$$\mathcal{L}_\mu := \overline{L_\mu}$$

its closure and unique self-adjoint extension. Furthermore, if $\mu \neq 1$, then $\mathcal{L}_\mu = L_\mu$, i.e., $L_\mu$ itself is self-adjoint, and has compact resolvent.
Now we consider in greater detail the critical case $\mu = 1$. The properties of $\mathcal{L}_1$ appear to depend on the dimension. In two dimensions, we have a complete result:

**Theorem 2** (Critical contrast in two dimensions). Let $\mu = 1$ and $n = 2$; then

$$\text{dom } \mathcal{L}_1 = \mathcal{D}_1^0(\Omega \setminus \Sigma), \quad \mathcal{L}_1(u_-, u_+) = (-\Delta u_-, \Delta u_+),$$

and the essential spectrum is non-empty, $\sigma_{\text{ess}}(\mathcal{L}_1) = \{0\}$.

Note (see Proposition 4 below) that 0 is not necessarily an eigenvalue of $\mathcal{L}_1$, contrary to the preceding examples given in [4] and [37, Chapter 8] for which the essential spectrum consisted of an infinitely degenerate zero eigenvalue.

In dimensions $n \geq 3$ the result appears to depend on the geometric properties of $\Sigma$:

**Theorem 3** (Critical contrast in dimensions $\geq 3$). Let $\mu = 1$ and $n \geq 3$; then $\mathcal{L}_1$ acts as $\mathcal{L}_1(u_-, u_+) = (-\Delta u_-, \Delta u_+)$, and its domain satisfies

$$\mathcal{D}_1^1(\Omega \setminus \Sigma) \subseteq \text{dom } \mathcal{L}_1.$$

Furthermore:

(a) If on each connected component of $\Sigma$ the principal curvatures are either all strictly positive or all strictly negative (in particular, if each maximal connected component of $\Sigma$ is strictly convex), then

$$\text{dom } \mathcal{L}_1 = \mathcal{D}_1^1(\Omega \setminus \Sigma)$$

and $\mathcal{L}_1$ has compact resolvent.

(b) If a subset of $\Sigma$ is isometric to a non-empty open subset of $\mathbb{R}^{n-1}$, then

$$\text{dom } \mathcal{L}_1 \neq \mathcal{D}_1^s(\Omega \setminus \Sigma) \quad \text{for any } s > 0,$$

the essential spectrum of $\mathcal{L}_1$ is non-empty, and $\{0\} \subseteq \sigma_{\text{ess}}(\mathcal{L}_1)$.

The proofs of the three theorems are given in Sections 2–4. In Section 2 we recall the tools of the machinery of boundary triplets for self-adjoint extensions of symmetric operators. In Section 3 we apply these tools to the operators $L_\mu$ and reduce the initial problem to finding self-adjoint extensions of a pseudodifferential operator $\Theta_\mu$ acting on $\Sigma$, which is essentially a linear combination of (suitably defined) Dirichlet-to-Neumann maps on $\Omega \pm$. The self-adjoint extensions of $\Theta_\mu$ are studied in Section 4 using a combination of some facts about Dirichlet-to-Neumann maps and pseudodifferential operators.
In addition, we use the definition of the operators $L_\mu$ to revisit some results concerning the so-called cloaking by negative materials; see, e.g., [30, 31]. For $\delta > 0$, consider the operator $T_{\mu, \delta}$ generated by the regularized sesquilinear form

$$t_{\mu, \delta}(u, v) := \int_{\Omega \setminus \Sigma} \nabla u \cdot (h_\mu + i\delta) \nabla v \, dx, \quad u, v \in H_0^1(\Omega).$$

By the Lax–Milgram theorem, the operator $T_{\mu, \delta} : L^2(\Omega) \supset H_0^1(\Omega) \supset \text{dom} T_{\mu, \delta} \to L^2(\Omega)$ has a bounded inverse, hence for $g \in L^2(\Omega)$ one may define $u_\delta := (T_{\mu, \delta})^{-1} g \in H_0^1(\Omega)$. It was observed in [31] that the limit properties of $u_\delta$ as $\delta$ tends to 0 can be quite irregular, in particular, the norm $\|u_\delta\|_{H^1(V)}$ may remain bounded for some subset $V \subset \Omega$ while $\|u_\delta\|_{H^1(\Omega \setminus V)}$ goes to infinity. The most prominent example is as follows: for $0 < r < R$ we denote

$$B_r := \{x \in \mathbb{R}^n : |x| < r\}, \quad B_{r,R} := \{x \in \mathbb{R}^n : r < |x| < R\},$$

$$S_r := \{x \in \mathbb{R}^n : |x| = r\},$$

pick three constants $0 < r_i < r_e < R$ and consider the above operator $T_{\mu, \delta}$ corresponding to

$$\Omega := B_R, \quad \Omega^- := B_{r_e,r_i},$$

and set $u_\delta := (T_{\mu, \delta})^{-1} g$ with $g$ supported in $B_{r_e,R}$. Then the norm $\|u_\delta\|_{H^1(\Omega)}$ remains bounded for $\delta$ approaching 0 provided $\mu \neq 0$ or $n \geq 3$. For $\mu = 1$ and $n = 2$ the situation appears to be different: if $g$ is supported outside the ball $B_a$ with $a = r_e^2/r_i$, then $\|u_\delta\|_{H^1(\Omega)}$ remains bounded, otherwise, for a generic $g$, the norm $\|u_\delta\|_{H^1(B_{r_e,R})}$ is bounded, while $\|u_\delta\|_{H^1(B_{r_e,r_i})}$ becomes infinite; see [31]. Such a non-uniform blow-up of the $H^1$ norm is often referred to as an anomalous localized resonance, and we refer to [2, 3, 11, 23, 28] for a discussion of other similar models and generalizations.

It is interesting to understand whether similar observations can be made based on the direct study of the operator $L_\mu$. In fact, instead of taking the limit of $u_\delta$ one may study directly the solutions $u$ of $L_\mu u = g$. If $\mu \neq 1$, then $u \in H_0^1(\Omega)$ by Theorem 1. Furthermore, due to Theorem 3(b) the same holds for $\mu = 1$ and $n \geq 3$ as the interface $\Sigma$ consists of two strictly convex hypersurfaces (the spheres $S_{r_i}$ and $S_{r_e}$), which is quite close to the discussion of [22]; we remark that a separation of variables shows that $L_1$ is injective and thus surjective in this case. The study of the case $\mu = 1$ and $n = 2$ is more involved, and the link to the anomalously localized resonance appears as follows (we assume a special form of the function $g$ to make the discussion more transparent):
Proposition 4. Let $\mu = 1$ and $n = 2$. Then the operator $L_1$ associated with (9) is injective, and a function $g \in L^2(B_R)$ of the form
\begin{equation}
 g(r \cos \theta, r \sin \theta) = 1_{(a,b)}(r) h(\theta), \quad h \in L^2(0, 2\pi), \quad r_e \leq a < b \leq R,
\end{equation}
belongs to $\text{ran} L_1$ if and only if
\begin{equation}
 \sum_{m \in \mathbb{Z} \setminus \{0\}} |h_m|^2 \left( \frac{r_e^2}{|m|^2} \right)^{2|m|} < \infty \quad \text{with} \quad h_m := \frac{1}{2\pi} \int_0^{2\pi} h(\theta) e^{-im\theta} d\theta.
\end{equation}
In particular, the condition (11) is satisfied for any $h$ if $a \geq r_e^2/r_i$, but fails generically for $a < r_e^2/r_i$.

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2 Preliminaries

2.1 Boundary triplets For a linear operator $B$ we denote $\text{dom} B$, $\text{ker} B$, $\text{ran} B$, $\sigma(B)$ and $\rho(B)$ its domain, kernel, range, spectrum and resolvent set respectively. For a self-adjoint operator $B$, by $\sigma_{\text{ess}}(B)$ and $\sigma_p(B)$ we denote respectively its essential spectrum and point spectrum (i.e. the set of the eigenvalues). The scalar product in a Hilbert space $H$ will be denoted as $\langle \cdot, \cdot \rangle_H$ or, if there is no ambiguity, simply as $\langle \cdot, \cdot \rangle$. By $B(\mathcal{H})$ we mean the Banach space of the bounded linear operators from a Hilbert space $\mathcal{H}$ to a Hilbert space $H$, and we set $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$.

Let us recall the key points of the method of boundary triplets for self-adjoint extensions [12, 14, 17]. Our presentation mostly follows the first chapters of [12]. Let $S$ be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$. A triplet $(\mathcal{H}, \Gamma_1, \Gamma_2)$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_1$ and $\Gamma_2$ are linear maps from the domain $\text{dom} S^*$ of the adjoint operator $S^*$ to $\mathcal{H}$, is called a boundary triplet for $S$ if the following three conditions are satisfied:

(a) the Green’s identity $\langle u, S^* v \rangle_{\mathcal{H}} - \langle S^* u, v \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_2 v \rangle_{\mathcal{H}} - \langle \Gamma_2 u, \Gamma_1 v \rangle_{\mathcal{H}}$ holds for all $u, v \in \text{dom} S^*$,

(b) the map $\text{dom} S^* \ni u \mapsto (\Gamma_1 u, \Gamma_2 u) \in \mathcal{H} \times \mathcal{H}$ is surjective,

(c) $\ker \Gamma_1 \cap \ker \Gamma_2 = \text{dom} S$.

It is known that a boundary triplet for $S$ exists if and only if $S$ admits self-adjoint extensions, i.e., if its deficiency indices are equal,
\[ \dim \ker (S^* - i) = \dim \ker (S^* + i) =: n(S). \]
A boundary triplet is not unique, but for any choice of a boundary triplet \((\mathcal{h}, \Gamma_1, \Gamma_2)\) for \(S\) one has \(\dim \mathcal{h} = n(S)\).

Let us assume from now on that the deficiency indices of \(S\) are equal and pick a boundary triplet \((\mathcal{h}, \Gamma_1, \Gamma_2)\); then the self-adjoint extensions of \(S\) are in a one-to-one correspondence with the self-adjoint linear relations in \(\mathcal{h}\) (multi-valued self-adjoint operators). In the present text we prefer to keep the operator language and reformulate this result as follows (cf. [34]): Let \(\Pi : \mathcal{h} \to \text{ran} \Pi \subseteq \mathcal{h}\) be an orthogonal projector in \(\mathcal{h}\) and \(\Theta\) be a linear operator in the Hilbert space \(\text{ran} \Pi\) with the induced scalar product. Denote by \(A_{\Pi, \Theta}\) the restriction of \(S\) to \(\text{dom} A_{\Pi, \Theta} = \{u \in \text{dom} S^*: \Gamma_1 u \in \text{dom} \Theta \text{ and } \Pi \Gamma_2 u = \Theta \Gamma_1 u\}\);

then \(A_{\Pi, \Theta}\) is symmetric/closed/self-adjoint iff \(\Theta\) possesses the respective property as an operator in \(\text{ran} \Pi\), and one has \(\overline{A_{\Pi, \Theta}} = A_{\Pi, \Theta}\), where as usual the bar means taking the closure. Futhermore, any self-adjoint extension of \(S\) is of the form \(A_{\Pi, \Theta}\).

The spectral analysis of the self-adjoint extensions can be carried out using the associated Weyl functions. Namely, let \(A\) be the restriction of \(S^*\) to \(\ker \Gamma_1\), i.e., corresponding to \((\Pi, \Theta, 0, 0)\) in the above notation, which is a self-adjoint operator. For \(z \in \rho(A)\) the restriction \(\Gamma_1 : \ker (S^* - z) \to \mathcal{h}\) is a bijection, and we denote its inverse by \(G_z\). The map \(z \mapsto G_z\), called the associated \(\gamma\)-field, is then a holomorphic map from \(\rho(A) \to \mathcal{B}(\mathcal{h})\) with

\[
G_z - G_w = (z - w)\left(A - z\right)^{-1}G_w, \quad z, w \in \rho(A).
\]

The Weyl function associated with the boundary triplet is then the holomorphic map

\[
\rho(A) \ni z \mapsto M_z := \Gamma_2 G_z \in \mathcal{B}(\mathcal{h}).
\]

To describe the spectral properties of the self-adjoint operators \(A_{\Pi, \Theta}\) let us consider first the case \(\Pi = 1\); then \(\Theta\) is a self-adjoint operator in \(\mathcal{H}\), and the following holds:

**Proposition 5.** For any \(z \in \rho(A) \cap \rho(A_{1, \Theta})\) one has \(0 \in \rho(\Theta - M_z)\) and the resolvent formula

\[
(A_{1, \Theta} - z)^{-1} = (A - z)^{-1} + G_z(\Theta - M_z)^{-1}G_z^*
\]

holds. In addition, for any \(z \in \rho(A)\) one has the equivalences:

(a) \(z \in \sigma(A_{1, \Theta})\) iff \(0 \in \sigma(\Theta - M_z)\),
(b) \(z \in \sigma_{\text{ess}}(A_{1, \Theta})\) iff \(0 \in \sigma_{\text{ess}}(\Theta - M_z)\),
(c) \(z \in \sigma_p(A_{1, \Theta})\) iff \(0 \in \sigma_p(\Theta - M_z)\) with \(G_z\) being an isomorphism of the eigensubspaces,
(d) if \( f \in \mathcal{H} \), then \( f \in \text{ran}(A_{1,\Theta} - z) \) iff \( G_z^* f \in \text{ran}(\Theta - M_z) \); if \( \Theta - M_z \) is injective, the resolvent formula (13) still holds on such \( f \).

**Proof.** The points (a), (b) and (c) are contained in Theorems 1.29 and Theorem 3.3 of [12], and the point (d) is proved in Theorem 6.16 of [5]. □

Now let \( A_{\Pi,\Theta} \) be an arbitrary self-adjoint extension. Denote by \( S_{\Pi} \) the restriction of \( S^* \) to
\[
\text{dom} S_{\Pi} = \{ u \in \text{dom} S^* : \Gamma_1 u = \Pi \Gamma_2 u = 0 \},
\]
which is a closed densely defined symmetric operator whose adjoint \( S^*_{\Pi} \) is the restriction of \( S^* \) to
\[
\text{dom} S^*_{\Pi} = \{ u \in \text{dom} S^* : \Gamma_1 u \in \text{ran} \Pi, \Gamma_2 u = 0 \},
\]
then \( (\text{ran} \Pi, \Gamma_1^\Pi, \Gamma_2^\Pi) \), with \( \Gamma_j^\Pi := \Pi \Gamma_j \), is a boundary triplet for \( S_{\Pi} \), and the restriction of \( S^*_{\Pi} \) to \( \ker \Gamma_1^\Pi \) is the same operator \( A \) as previously. The associated \( \gamma \)-field and Weyl function take the form
\[
z \mapsto G_z^\Pi := G_z^\Pi, \quad z \mapsto M_z^\Pi := \Pi M_z^\Pi,
\]
and \( \text{dom} A_{\Pi,\Theta} := \{ u \in \text{dom} S^*_{\Pi} : \Gamma_2^\Pi u = 0 \} \); see [12, Remark 1.30]. A direct application of Proposition 5 gives

**Corollary 6.** For any \( z \in \rho(A) \cap \rho(A_{\Pi,\Theta}) \) one has \( 0 \in \rho(\Theta - M_z^\Pi) \) and the resolvent formula
\[
(A_{\Pi,\Theta} - z)^{-1} = (A - z)^{-1} + G_z^\Pi (\Theta - M_z^\Pi)^{-1} (G_z^\Pi)^*
\]
holds, and, in addition, for any \( z \in \rho(A) \) one has

(a) \( z \in \sigma(A_{\Pi,\Theta}) \) iff \( 0 \in \sigma(\Theta - M_z^\Pi) \),
(b) \( z \in \sigma_{\text{ess}}(A_{\Pi,\Theta}) \) iff \( 0 \in \sigma_{\text{ess}}(\Theta - M_z^\Pi) \),
(c) \( z \in \sigma_{p}(A_{\Pi,\Theta}) \) iff \( 0 \in \sigma_{p}(\Theta - M_z^\Pi) \) with \( G_z^\Pi \) being an isomorphism of the eigensubspaces,
(d) if \( f \in \mathcal{H} \), then \( f \in \text{ran}(A_{\Pi,\Theta} - z) \) iff \( (G_z^\Pi)^* f \in \text{ran}(\Theta - M_z^\Pi) \); if \( \Theta - M_z^\Pi \) is injective, the resolvent formula (14) still holds on such \( f \).

### 2.2 Singular perturbations.
In this section let us recall a special approach to the construction of boundary triplets as presented in [33] and [34] or in [12, Section 1.4.2]. Let \( A \) be a self-adjoint operator in a Hilbert space \( \mathcal{H} \); then we denote by \( \mathcal{H}_A \) the Hilbert space given by the linear space \( \text{dom} A \) endowed with the scalar product \( \langle u, v \rangle_A = \langle u, v \rangle_H + \langle Au, Av \rangle_H \). Let \( \mathfrak{h} \) be an auxiliary Hilbert space
and $\tau : \mathcal{H}_A \to \mathfrak{h}$ be a bounded linear map which is surjective and whose kernel $\ker \tau$ is dense in $\mathcal{H}$; then the restriction $S$ of $A$ to $\ker \tau$ is a closed densely defined symmetric operator in $\mathcal{H}$. To simplify the formulas we assume additionally that

$$0 \in \rho(A),$$

which always holds in the subsequent applications. For $z \in \rho(A)$ consider the maps

$$G_z := (\tau(A - z)^{-1})^* \in \mathcal{B}(\mathfrak{h}, \mathcal{H}), \quad M_z := \tau(G_z - G_0) \equiv z\tau A^{-1}G_z \in \mathcal{B}(\mathfrak{h}).$$

**Proposition 7.** The adjoint $S^*$ is given by

$$\text{dom } S^* := \{ u = u_0 + G_0 f_u : u_0 \in \text{dom } A \text{ and } f_u \in \mathfrak{h} \}, \quad S^* u = Au_0.$$

Furthermore, the triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ with $\Gamma_1 u := f_u$ and $\Gamma_2 u := \tau u_0$ is a boundary triplet for $S$, and the associated $\gamma$-field $G_z$ and Weyl function $M_z$ are given by (15).

**Example 8.** Let $A_{\pm}$ be self-adjoint operators in Hilbert spaces $\mathcal{H}_{\pm}$ with $0 \in \rho(A_{\pm})$, and let $\mathfrak{h}_{\pm}$, $\tau_{\pm}$, $S_{\pm}$, $G^\pm_z$, $M^\pm_z$, $\Gamma_{\pm}$ be the spaces and maps defined as above and associated with $A_{\pm}$. For $\nu \in \mathbb{R}\setminus\{0\}$ consider the operator $A := A_- \oplus \nu A_+$ acting in the Hilbert space $\mathcal{H} := \mathcal{H}_- \oplus \mathcal{H}_+$. Set $\tau = \tau_- \oplus \nu \tau_+$. Then the restriction $S$ of $A$ to $\ker \tau$ has again the structure of a direct sum, $S = S_- \oplus \nu S_+$, with $\gamma$-field and Weyl function given by

$$G_z = G^+_z \oplus G^-_z, \quad M_z = M^-_z \oplus \nu M^+_z.$$

Thus, by the preceding considerations, the adjoint $S^*$ acts on the domain

$$\text{dom } S^* = \{ u = (u_-, u_+) : u_{\pm} = u^{\pm}_0 + G^{\pm}_0 \phi_{\pm}, \quad u^{\pm}_0 \in \text{dom } A_{\pm}, \quad \phi_{\pm} \in \mathfrak{h}_{\pm} \}$$

by $S^*(u_-, u_+) = A(u^-_0, u^+_0)$, and one can take $(\mathfrak{h}_- \oplus \mathfrak{h}_+, \Gamma_1, \Gamma_2)$ as a boundary triplet for $S$,

$$\Gamma_1 u = (\phi_-, \phi_+), \quad \Gamma_2 u = (\tau_- u^-_0, \nu \tau_+ u^+_0).$$

### 3 Boundary triplets for indefinite Laplacians

We start with some constructions for the closed symmetric operator

$$S = (-\Delta^-_{\min}) \oplus \mu \Delta^+_{\min}, \quad \Delta^\pm_{\min} : L^2(\Omega_{\pm}) \supset H^2_0(\Omega_{\pm}) \to L^2(\Omega_{\pm}),$$

where

$$H^2_0(\Omega_-) := \{ u_- \in H^2(\Omega_-) : (\gamma^-_0 u_-, \gamma^-_1 u_-) = (0, 0) \},$$

$$H^2_0(\Omega_+) := \{ u_+ \in H^2(\Omega_+) : (\gamma^+_0 u_+, \gamma^+_1 u_+) = (0, 0) \}.$$
Here and later on, \( H^m(\Omega_\pm) \) denotes the usual Sobolev–Hilbert space of the square-integrable functions on \( \Omega_\pm \) with square-integrable partial (distributional) derivatives of any order \( k \leq m \), and the linear operators

\[
\begin{align*}
\gamma_0^\pm : H^2(\Omega_\pm) &\to H^{\frac{3}{2}}(\Sigma), \\
\gamma_1^\pm : H^2(\Omega_\pm) &\to H^{\frac{1}{2}}(\Sigma), \\
\gamma_0^\delta : H^2(\Omega_+) &\to H^{\frac{3}{2}}(\partial\Omega), \\
\gamma_1^\delta : H^2(\Omega_+) &\to H^{\frac{1}{2}}(\partial\Omega),
\end{align*}
\]

are the usual trace maps first defined on \( u_\pm \in C^\infty(\overline{\Omega}_\pm) \) by

\[
\begin{align*}
\gamma_0^\pm u_\pm(x) &:= u_\pm(x), & \gamma_1^\pm u_\pm(x) &:= N_\pm(x) \cdot \nabla u_\pm(x), & x &\in \Sigma, \\
\gamma_0^\delta u_+(x) &:= u_+(x), & \gamma_1^\delta u_+(x) &:= N_\delta(x) \cdot \nabla u_+(x), & x &\in \partial\Omega,
\end{align*}
\]

with \( N_\delta \) being the outer unit normal on \( \partial\Omega \), and then extended by continuity. It is well-known (see, e.g., [27, Chapter 1, Section 8.2]) that the maps

\[
\begin{align*}
H^2(\Omega_-) &\ni u_- \mapsto (\gamma_0^- u_-, \gamma_1^- u_-) \in H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\Sigma), \\
H^2(\Omega_+) &\ni u_+ \mapsto (\gamma_0^+ u_+, \gamma_1^+ u_+, \gamma_0^\delta u_+, \gamma_1^\delta u_+) \\
&\quad \in H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{3}{2}}(\partial\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega)
\end{align*}
\]

are bounded and surjective.

We remark that both \( \Sigma \) and \( \partial\Omega \) can be made smooth compact Riemannian manifolds. For \( \Xi = \Sigma \) or \( \Xi = \partial\Omega \), the fractional order Sobolev–Hilbert spaces \( H^s(\Xi) \) with \( s \in \mathbb{R} \), are defined in the standard way as the completions of \( C^\infty(\Xi) \) with respect to the scalar products

\[
\langle \phi_1, \phi_2 \rangle_{H^s(\Xi)} := \langle \phi_1, (-\Delta_\Xi + 1)^s \phi_2 \rangle_{L^2(\Xi)},
\]

where \( \Delta_\Xi \) is the (negative) Laplace–Beltrami operator in \( L^2(\Xi) \) (see, e.g., [27, Remark 7.6, Chapter 1, Section 7.3]) and then \((-\Delta_\Xi + 1)^{\frac{1}{2}}\) extends to a unitary map from \( H^r(\Xi) \) to \( H^{r-\frac{1}{2}}(\Xi) \). In what follows we denote for brevity

\[
\Lambda := \sqrt{-\Delta_\Sigma + 1}, \quad \Lambda_\delta := \sqrt{-\Delta_{\partial\Omega} + 1}.
\]

By Green’s formula, the linear operators \( \gamma_0^\pm, \gamma_1^\pm, \gamma_0^\delta \) and \( \gamma_1^\delta \) can be then extended to continuous (with respect to the graph norm) maps

\[
(19) \quad \begin{align*}
\gamma_0^\pm &: \text{dom } \Delta_\pm^{\text{max}} \to H^{-\frac{1}{2}}(\Sigma), & \gamma_1^\pm &: \text{dom } \Delta_\pm^{\text{max}} \to H^{-\frac{1}{2}}(\Sigma), \\
\gamma_0^\delta &: \text{dom } \Delta_+^{\text{max}} \to H^{-\frac{3}{2}}(\partial\Omega), & \gamma_1^\delta &: \text{dom } \Delta_+^{\text{max}} \to H^{-\frac{1}{2}}(\partial\Omega),
\end{align*}
\]

where \( \Delta_\pm^{\text{max}} := (\Delta_\pm^{\text{min}})^* \) acts as the distributional Laplacian on the domain

\[
\text{dom } \Delta_\pm^{\text{max}} := \{ u_\pm \in L^2(\Omega_\pm) : \Delta u_\pm \in L^2(\Omega_\pm) \};
\]
see [27, Chapter 2, Section 6.5]. Now consider the operator

\[ A = (−Δ^D) ⊕ μ Δ^D_+ \]

acting in \( L^2(Ω) \equiv L^2(Ω_-) ⊕ L^2(Ω_+) \), where \( Δ^D_± \) are the Dirichlet Laplacians in \( L^2(Ω_±) \), i.e.

\[
\text{dom } Δ^D_- = \{ u_- ∈ H^2(Ω_-) : γ^0_- u_- = 0 \}, \\
\text{dom } Δ^D_+ = \{ u_+ ∈ H^2(Ω_+) : (γ^0_+ u_+, γ^0_0 u_+) = (0, 0) \}.
\]

As both \( Δ^D_± \) are self-adjoint with compact resolvents, the same applies to \( A \). The maps

\[
τ_- : \text{dom } Δ^D_- \to H^\frac{1}{2}(Σ), \quad τ_- u_- := γ^1_- u_-,
\]

\[
τ_+ : \text{dom } Δ^D_+ \to H^\frac{1}{2}(Σ) ⊕ H^\frac{1}{2}(∂Ω), \quad τ_+ u_+ := (γ^1_+ u_+, γ^0_0 u_+),
\]

are linear, continuous, surjective, and their kernels are dense in \( L^2(Ω_±) \). Moreover, \( Δ^\text{min}_± \) is exactly the restriction of \( Δ^D_± \) to \( \ker τ_± \). Therefore, we may use the construction of Example 8 with \( ν = −μ \) to obtain a description of the self-adjoint extensions of \( S \) from (18). To this end, an expression for the associated operators \( G^±_z \) and \( M^±_z \) is needed. These were already obtained in [34, Example 5.5], and we recall the final result. The Poisson operators

\[
P^-_z : H^s(Σ) → \text{dom } Δ^\text{max}_-, \quad P^+_z : H^s(Σ) ⊕ H^s(∂Ω) → \text{dom } Δ^\text{max}_+, \quad s ≥ −\frac{1}{2},
\]

are defined through the solutions of the respective boundary value problems,

\[
P^-_z φ = f \text{ iff } \begin{cases} -Δ^\text{max}_- f = zf, \\ γ^0_- f = φ, \end{cases} z ∈ \rho(−Δ^D_-),
\]

\[
P^+_z(φ, ψ) = g \text{ iff } \begin{cases} -Δ^\text{max}_+ g = zg, \\ γ^0_+ g = φ, \end{cases} z ∈ \rho(−Δ^D_+),
\]

and the associated (energy-dependent) Dirichlet-to-Neumann operators are given by

\[
D^-_z : H^s(Σ) → H^{s−1}(Σ), \quad D^-_z := γ^1_- P^-_z, \quad s ≥ −\frac{1}{2},
\]

\[
D^+_z : H^s(Σ) ⊕ H^s(∂Ω) → H^{s−1}(Σ) ⊕ H^{s−1}(∂Ω), \quad s ≥ −\frac{1}{2},
\]

\[
D^+_z(φ, ψ) := (γ^1_+ P^+_z(φ, ψ), γ^0_0 P^+_z(φ, ψ)).
\]
\[ G^-_z = -P^-_z \Lambda, \quad M^-_z = (D^-_0 - D^-_z)\Lambda, \]
\[ G^+_z = P^+_z (\Lambda \oplus \Lambda_\partial), \quad M^+_z = (D^+_0 - D^+_z)(\Lambda \oplus \Lambda_\partial). \]

Thus, by Remark 8, the adjoint \( S^* \) acts as \( S^*u = (-\Delta^\text{max}_- u_-, \mu \Delta^\text{max}_+ u_+) \) on the domain \( \text{dom}(\Lambda^\text{max}_- \oplus \Lambda^\text{max}_+) \), and using (16) and (17) one obtains the boundary triplet \((h, \Gamma_1, \Gamma_2)\) for \( S \) with \( h = H^\frac{1}{2}(\Sigma) \oplus H^\frac{1}{2}(\Sigma) \oplus H^\frac{1}{2}(\partial\Omega) \) and
\[ \Gamma_1 u = - \begin{pmatrix} \Lambda^{-1} \gamma_0^- u_- \\ \Lambda^{-1} \gamma_0^+ u_+ \\ \Lambda_\partial^{-1} \gamma_0^\partial u_+ \end{pmatrix}, \quad \Gamma_2 u = \begin{pmatrix} \gamma_1^- (u_- - P^-_0 \gamma_0^- u_-) \\ -\mu \gamma_1^+ (u_+ - P^+_0 (\gamma_0^+ u_+, \gamma_0^\partial u_+)) \\ -\mu \gamma_1^\partial (u_+ - P^+_0 (\gamma_0^+ u_+, \gamma_0^\partial u_+)) \end{pmatrix}. \]

The associated \( \gamma \)-field \( G_\gamma \) and \( M_\gamma \) are given by
\[ G_\gamma \begin{pmatrix} \varphi_- \\ \varphi_+ \\ \varphi_\partial \end{pmatrix} = - \begin{pmatrix} P^-_z \Lambda \varphi_- \\ P^+_z \Lambda \varphi_+ \end{pmatrix}, \quad M_\gamma \begin{pmatrix} \varphi_- \\ \varphi_+ \\ \varphi_\partial \end{pmatrix} = \begin{pmatrix} (D^-_0 - D^-_z) \Lambda \varphi_- \\ -\mu (D^+_0 - D^+_\partial) (\Lambda \varphi_+ \Lambda \varphi_\partial) \end{pmatrix}. \]

Let us represent the operator \( L_\mu \) given by (3) and (4) in the form \( A_{11,0} \). Remark first that, in view of the elliptic regularity (see, e.g., [19, Proposition III.5.2]) we have
\[ H^\frac{1}{2}(\Omega_-) = \{ u_- \in L^2(\Omega_-) : \Lambda^\text{max}_- u_- \in L^2(\Omega_-), \gamma_0^\partial u_- \in H^\frac{1}{2}(\Sigma) \}, \]
\[ H^\frac{1}{2}(\Omega_+) = \{ u_+ \in L^2(\Omega_+) : \Lambda^\text{max}_+ u_+ \in L^2(\Omega_+), (\gamma_0^\partial u_+, \gamma_0^\partial u_+) \in H^\frac{1}{2}(\Sigma) \oplus H^\frac{1}{2}(\partial\Omega) \}. \]

Therefore, \( L_\mu \) is exactly the restriction of \( S^* \) to the functions \( u = (u_-, u_+) \) with
\[ \gamma^-_0 u_- = \gamma^+_0 u_+ =: \gamma_0 u, \quad \gamma^-_0 u_+ = 0, \quad \gamma_0 u \in H^\frac{1}{2}(\Sigma), \quad \gamma^-_1 u_- = \mu \gamma^+_1 u_+. \]
The first two conditions can be rewritten as \( \Gamma_1 u \in \text{ran} \Pi \), where \( \Pi \) is the orthogonal projector in \( h \) given by
\[ \Pi(\varphi_-, \varphi_+, \varphi_\partial) = \frac{1}{2}(\varphi_- + \varphi_+, \varphi_- + \varphi_+, 0). \]
For the subsequent computations it is useful to introduce the unitary operator
\[ U : \text{ran} \Pi \to H^\frac{1}{2}(\Sigma), \quad U(\varphi, \varphi, 0) = \sqrt{2} \varphi; \]
then
\[ U \Pi \Gamma_2 u = \frac{1}{\sqrt{2}} [\gamma_1^-(u_- - P_0^- \gamma_0^- u_-) - \mu \gamma_1^+(u_+ - P_0^+ (\gamma_0^+ u_+, \gamma_0^+ u_+))], \]
and the third and the fourth conditions in (20) can be rewritten respectively as
\[ \Gamma_1 u \in U^* \text{ dom } \Theta_\mu, \quad U \Pi \Gamma_2 u = \Theta_\mu U \Gamma_1 u, \]
where \( \Theta_\mu \) is the symmetric operator in \( H^2(\Sigma) \) given by
\[ \Theta_\mu := \frac{1}{2} (D_0^- - \mu \tilde{D}_0^+), \quad \text{ dom } \Theta_\mu = H^2(\Sigma), \]
with
\[ \tilde{D}_z^+ := \gamma_z^+ \tilde{P}_z^+, \quad \tilde{P}_z^+ := P_z^+(\cdot, 0). \]
Therefore, one has the representation \( L_\mu = A_{\Pi, U^* \Theta_\mu U}, \) and, due to the unitarity of \( U \) and to the discussion of Section 2, the operator \( L_\mu \) is self-adjoint/essentially self-adjoint in \( L^2(\Omega) \) if and only if \( \Theta_\mu \) has the respective property as an operator in \( H^2(\Sigma) \). Remark that for the associated maps \( G_z^{\Pi} := G_z \Pi^* \) and \( M_z^{\Pi} := \Pi M_z \Pi^* \) (see Subsection 2.1) one has
\[ (21) \quad G_z^{\Pi} U^* \phi = -\frac{1}{\sqrt{2}} \left( \frac{P_z^- \Lambda \phi}{\tilde{P}_z^+ \Lambda \phi} \right), \quad U M_z^{\Pi} U^* = \frac{1}{2} \left( (D_0^- - D_z^-) - \mu (\tilde{D}_0^+ - \tilde{D}_z^+) \right) \Lambda. \]

4 Proofs of main results

With the above preparations, the proofs will be based on an application of the theory of pseudodifferential operators; see, e.g., [38] and [39]. At first we recall some known results adapted to our setting.

If \( \Psi \in \mathcal{B}(H^s(\Sigma), H^{s-k}(\Sigma)) \) is a symmetric pseudodifferential operator of order \( k \), we set \( k_0 := \max(k, 0) \) and denote by \( \Psi^{\text{min}} \) and \( \Psi^0 \) the symmetric operators in \( L^2(\Sigma) \) given by the restriction of \( \Psi \) to \( \text{dom } \Psi^{\text{min}} = C^\infty(\Sigma) \) and \( \text{dom } \Psi^0 = H^{k_0}(\Sigma) \), respectively; then \( \Psi^0 \subseteq \Psi^{\text{min}} \subseteq \Psi^0 \). Furthermore, if \( \Psi \) is elliptic, then \( \Psi^0 \) is closed and, hence, \( \Psi^{\text{min}} = \Psi^0 \). Since \( \text{dom}(\Psi^{\text{min}}) = \{ f \in L^2(\Sigma) : \Psi f \in L^2(\Sigma) \} \), for elliptic \( \Psi \) one has \( \text{dom}(\Psi^{\text{min}})^* \subseteq H^{k_0}(\Sigma) = \text{dom } \Psi^{\text{min}}, \) and so \( \Psi^{\text{min}} \) is essentially self-adjoint and \( \Psi^0 \) is self-adjoint. It is important to recall that for \( k = 1 \) one does not need the ellipticity:

**Lemma 9.** If \( \Psi \) is a symmetric first order pseudodifferential operator, then \( \Psi^{\text{min}}, \) and then also \( \Psi^0, \) is essentially self-adjoint in \( L^2(\Sigma). \)
Proof. By [38, Proposition 7.4], for any \( f \in L^2(\Sigma) \) with \( \Psi f \in L^2(\Sigma) \) there exist \( (f_j) \subset C^\infty(\Sigma) \) such that \( f_j \to f \) and \( \Psi f_j \to \Psi f \) in \( L^2(\Sigma) \), which literally means that \( \text{dom}(\Psi f_j^*) \subseteq \text{dom}(\Psi f^*) \). □

In what follows, instead of studying \( \Theta_\mu \) in \( H^{\frac{1}{2}}(\Sigma) \) we prefer to deal with the unitarily equivalent operator \( \Phi_\mu := \Lambda^{\frac{1}{2}}\Theta_\mu \Lambda^{-\frac{1}{2}} \) acting in \( L^2(\Sigma) \). Set

\[
\Psi_\mu := \frac{1}{2}\Lambda^{\frac{1}{2}}(D_0^- - \mu \tilde{D}_0^+)\Lambda^{\frac{1}{2}};
\]

then \( \Phi_\mu \) is the restriction of \( \Psi_\mu \) to \( \text{dom}(\Phi_\mu) = H^2(\Sigma) \). Furthermore, denote by \( \Psi_\mu^\text{min} \) and \( \Psi_\mu^0 \) the symmetric operators in \( L^2(\Sigma) \) given respectively by the restrictions of \( \Psi_\mu \) to \( \text{dom}(\Psi_\mu^\text{min}) = C^\infty(\Sigma) \) and to \( \text{dom}(\Psi_\mu^0) = H^{k_0}(\Sigma) \), where \( k \) is the order of \( \Psi_\mu \) and \( k_0 = \max(k, 0) \). We remark that we always have \( k \leq 2 \), hence \( \Psi_\mu^\text{min} \subseteq \Phi_\mu \subseteq \Psi_\mu^0 \).

Proof of Theorem 1. Assume first that \( \mu \neq 1 \). Let us show that the operator \( \Theta_\mu \) is self-adjoint in \( H^{\frac{1}{2}}(\Sigma) \); then this will imply the self-adjointness of \( L_\mu \) in \( L^2(\Omega) \). It is a classical result that \( D_0^\pm \) are first order pseudodifferential operators with the principal symbol \( |\xi|^2 \) (see, e.g., [39, Chapter 7, Section 11]) and, in view of the definition, the same applies then to \( \tilde{D}_0^\pm \). Then \( \Psi_\mu \) is a pseudodifferential operator with principal symbol \( \frac{1}{2}\mu|\xi|^2 \). As such a principal symbol is non-vanishing, \( \Psi_\mu \) is a second order elliptic pseudodifferential operator and, by the results recalled at the beginning of the section, \( \Phi_\mu \equiv \Psi_\mu^0 \) is self-adjoint on the domain \( H^2(\Sigma) \). Hence, since \( \Lambda^{\frac{1}{2}}: H^{\frac{1}{2}}(\Sigma) \to L^2(\Sigma) \) is unitary, the operator \( \Theta_\mu = \Lambda^{-\frac{1}{2}}\Phi_\mu \Lambda^{\frac{1}{2}} \) is self-adjoint on the initial domain \( H^{\frac{3}{2}}(\Sigma) \), which implies the self-adjointness of \( L_\mu \) on the initial domain \( D^{\frac{3}{2}}_\mu(\Omega \setminus \Sigma) \). Due to (2) we have \( \text{dom}(L_\mu) \subseteq H^1_0(\Omega) \), and the compact embedding of \( H^1_0(\Omega) \) into \( L^2(\Omega) \) proves that the resolvent of \( L_\mu \) is a compact operator.

Let \( \mu = 1 \); then \( \Psi_1 \) is a first order pseudodifferential operator and \( \Psi_1^\text{min} \) is essentially self-adjoint due to Lemma 9. Then \( \Phi_1 \) is also essentially self-adjoint being a symmetric extension of \( \Psi_1^\text{min} \). The unitarity of \( \Lambda^{\frac{1}{2}}: H^{\frac{1}{2}}(\Sigma) \to L^2(\Sigma) \) implies the essential self-adjointness of \( \Theta_1 \) in \( H^{\frac{3}{2}}(\Sigma) \) and, in turn, that of \( L_1 \) in \( L^2(\Omega) \).

Recall that in what follows we denote by \( \mathcal{L}_1 \) the unique self-adjoint extension of \( L_1 \). In view of the discussion of Section 3 one has \( \mathcal{L}_1 = A_{\Pi, U, \Theta U} \) with \( \Theta = \bar{\Theta}_1 \) being the closure (and the unique self-adjoint extension) of \( \Theta_1 \) in \( H^{\frac{1}{2}}(\Sigma) \).

Proof of Theorem 2. Assume that \( n = 2 \) and \( \mu = 1 \); then

\[
\Psi_1 = \frac{1}{2}\Lambda^{\frac{1}{2}}(D_0^- - \tilde{D}_0^+)\Lambda^{\frac{1}{2}}.
\]

It is well-known that the classical Dirichlet-to-Neumann map (at \( z = 0 \)) on a smooth bounded domain in \( \mathbb{R}^2 \) is represented as \( \sqrt{-\Delta_{\partial}} + K \) with \( -\Delta_{\partial} \) and \( K \) being
respectively the Laplace–Beltrami operator on the boundary and a pseudodifferential operator of order \((-\infty)\); see [15, Proposition 1] for a direct proof or [26, Section 1] for an iterative computation of the symbol. (Informally, one often says that the full symbol with respect to the arclength is equal to \(|\xi|\), see [15].) It follows that \(D_0^{-} = \sqrt{-\Delta} + K^{-}\) and \(\tilde{D}_0^{+} = \sqrt{-\Delta} + K^{+}\), where \(K^{\pm}\) are pseudodifferential operators of order \((-\infty)\). Therefore, the symmetric pseudodifferential operator \(\Psi_1\) is of order \((-\infty)\), and \(\Psi_0^{-} : L^2(\Sigma) \to L^2(\Sigma)\) is bounded, self-adjoint and compact. As \(\Phi_1\) is densely defined, it follows that \(\Phi_1^{-} = \Psi_0^{-}\). Since \(\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \to L^2(\Sigma)\) is unitary, the closure of \(\Theta_1\) is given by \(\Theta = \Lambda^{\frac{1}{2}} \Phi_1 \Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)\), and it is a compact self-adjoint operator in \(H^{\frac{1}{2}}(\Omega)\). As \(\text{dom} \Theta = H^{\frac{1}{2}}(\Sigma)\), the boundary condition \(\Gamma_1 u \in U^* \text{dom} \Theta\) takes the form \(\gamma_0^{-} u_- = \gamma_0^{+} u_+ \in H^{-\frac{1}{2}}(\Sigma)\), \(\gamma_0^{+} u_+ = 0\), and, in view of (19), the domain of \(\mathcal{L}_1 = A_{\Pi, U^*} C\) is given by (5).

Let us study the spectral properties of \(\mathcal{L}_1\) using Corollary 6. As

\[
U^* \Theta U - M_{0}^{\Pi} \equiv U^* \Theta U
\]

is compact, one has \(0 \in \sigma_{\text{ess}}(U^* \Theta U - M_{0}^{\Pi})\) implying \(0 \in \sigma_{\text{ess}}(\mathcal{L}_1)\). To prove the reverse inclusion \(\sigma_{\text{ess}}(\mathcal{L}_1) \subseteq \{0\}\) we note first that the operators \(U^* \Theta U - M_{z}^{\Pi}\) are unitarily equivalent to \(\Theta - U M_{z}^{\Pi} U^*\) and, hence, have the same spectra. Furthermore, the principal symbol of \(D_0^{\pm} - D_0^{\pm}\) is \(\sqrt{|\xi|}\) for any \(\lambda \in \rho(-\Delta_{\pm}^{D})\); see [25, Lemma 1.1]. As the principal symbol of \(\Lambda\) is \(|\xi|\), it follows that, for any \(z \in \rho(A)\), the operators \((D_0^{\pm} - D_0^{\pm})\Lambda\) and \((\tilde{D}_0^{\pm} - \tilde{D}_0^{\pm})\Lambda\) are bounded in \(H^{\frac{1}{2}}(\Sigma)\) being pseudodifferential operators of order zero, and their principal symbols are \(\tilde{\frac{1}{2}}\) and \((-\tilde{\frac{1}{2}})\) respectively. By Eq. (21) it follows that the principal symbol of \(\Theta - U M_{z}^{\Pi} U^*\) is simply \(\tilde{\frac{1}{2}}\), and one can represent \(\Theta - U M_{z}^{\Pi} U^* = \tilde{\frac{1}{2}} + K_z\), where \(K_z\) are compact operators depending holomorphically on \(z \in \rho(A)\). As the operator \(A\) has compact resolvent, it follows by (12) that the only possible singularities of \(z \mapsto K_z\) at the points of \(\sigma(A)\) are simple poles with finite-dimensional residues. Therefore, the operator function

\[
z \mapsto U^* \Theta U - M_{z}^{\Pi} := U^* (\Theta - U M_{z}^{\Pi} U^*) U
\]

satisfies the assumptions of the meromorphic Fredholm alternative on \(\mathbb{C}_0 := \mathbb{C} \setminus \{0\}\) (see [35, Theorem XIII.13]) and either (a) \(0 \in \sigma(U^* \Theta U - M_{z}^{\Pi})\) for all \(z \in \mathbb{C}_0 \setminus \rho(A)\), or (b) there exists a subset \(B \subset \mathbb{C}_0\), without accumulation points in \(\mathbb{C}_0\), such that the inverse \((U^* \Theta U - M_{z}^{\Pi})^{-1}\) exists and is bounded for \(z \in \mathbb{C}_0 \setminus (B \cup \sigma(A))\) and extends to a meromorphic function in \(\mathbb{C}_0 \setminus B\) such that the coefficients in the Laurent series at the points of \(B\) are finite-dimensional operators. The case (a) can be excluded: By Corollary 6 this would imply the presence of a non-empty non-real spectrum for \(\mathcal{L}_1\), which is not possible due to the self-adjointness. Therefore, we are in the
case (b), and the resolvent formula (14) for $\mathcal{L}_1 \equiv A_{\Pi, U^* \theta U}$ implies that the set $\mathbb{C}_0 \cap \sigma(\mathcal{L}_1) \cap \rho(A) \subseteq B$ has no accumulation points in $\mathbb{C}_0$, and each point of this set is a discrete eigenvalue of $\mathcal{L}_1$. Furthermore, by (12) the maps $z \mapsto G_z^\Pi$ can have at most simple poles with finite-dimensional residues at the points of $\sigma(A)$, and it is seen again from the resolvent formula (14) that the only possible singularities of $\mathcal{L}_1 - z)^{-1}$ at the points of $\sigma(A)$ are poles with finite-dimensional residues. It follows that each point of $\sigma(A)$ is either not in the spectrum of $\mathcal{L}_1$ or is its discrete eigenvalue. Therefore, $\mathcal{L}_1$ has no essential spectrum in $\mathbb{C} \setminus \{0\}$, and the only possible accumulation points for the discrete eigenvalues are 0 and $\infty$. \hfill \Box

**Proof of Theorem 3.** Assume $n \geq 3$ and $\mu = 1$; then again

$$
\Psi_1 = \frac{1}{2} \Lambda^{\frac{1}{2}} (D_0^- - \tilde{D}_0^+) \Lambda^{\frac{1}{2}}.
$$

By [39, Chapter 12, Proposition C.1], we have

$$
D_0^- = \sqrt{-\Delta} + B^- + C^-, \quad \tilde{D}_0^+ = \sqrt{-\Delta} + B^+ + C^+,
$$

where $C^\pm$ are pseudodifferential operators of order $(-1)$ and $B^\pm$ are pseudodifferential operator of order 0 whose principal symbols are $\pm b_0(x, \xi)$, with

$$
b_0(x, \xi) = \frac{1}{2} \left( \text{tr} W_x - \frac{\langle \xi, W_x^* \xi \rangle_{T_x \Sigma}}{\langle \xi, \xi \rangle_{T_x \Sigma}} \right)
$$

and $W_x := dN_-(x) : T_x \Sigma \to T_x \Sigma$ being the Weingarten map and $W_x^*$ its adjoint. Therefore, $\Psi_1$ is a pseudodifferential operator of order 1 whose principal symbol is $\frac{1}{2} b_0(x, \xi) |\xi|^2$. As already seen, $\Psi_1^{\\text{min}}$ is then essentially self-adjoint by Lemma 9, and, as before, $L_1$ is essentially self-adjoint and its self-adjoint closure is $A_{\Pi, U^* \theta U}$, where $\Theta := \overline{\Theta}_1$. As $\Theta_1$ is a first order operator, one has $H^{\frac{1}{2}}(\Sigma) \subseteq \text{dom} \Theta$. In particular, the boundary condition $\Gamma_1 u \in U^* H^{\frac{1}{2}}(\Sigma)$ entails $\gamma_0^- u_\pm = \gamma_0^0 u_\pm \in H^{\frac{1}{2}}(\Sigma)$ and $\gamma_0^0 u_\pm = 0$. Due to the elliptic regularity (see, e.g., [27, Chapter 2, Section 7.3]) this can be rewritten as $u \in H^{\frac{1}{2}}(\Omega)$ and gives the inclusion (6).

(a) Recall that the principal curvatures $k_1(x), \ldots, k_{n-1}(x)$ of $\Sigma$ at a point $x$ are the eigenvalues of $W_x$, hence,

$$
\frac{1}{2} (k_1(x) + \cdots + k_{n-1}(x) - \max_j k_j(x)) \leq b_0(x, \xi) \leq \frac{1}{2} (k_1(x) + \cdots + k_{n-1}(x) - \min_j k_j(x)).
$$

Let $\Sigma'$ be a maximal connected component of $\Sigma$. If all $k_j$ are either all strictly positive or all strictly negative on $\Sigma'$, one can estimate $a_1 \leq |b_0(x, \xi)| \leq a_2$ for all $x \in \Sigma'$ with some $a_1 > 0$ and $a_2 > 0$. Therefore, in this case $\Psi_1$ is a first order elliptic pseudodifferential operator and so, by the results recalled at the beginning.
of this section, $\Psi_1^0$ is self-adjoint. This implies that $\text{dom } \Theta \equiv \text{dom } \overline{\Theta}_1 = H_1^+(\Sigma)$. As before, the boundary condition $\Gamma_1 u \in U^* \text{dom } \Theta$ for $u \in \text{dom } \mathcal{L}_1$ entails $u \in H_0^1(\Omega)$, and one arrives at the equality (7). The inclusion $\text{dom } \mathcal{L}_1 \subseteq H_0^1(\Omega)$ and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ imply that $\mathcal{L}_1$ has compact resolvent.

(b) As $M_0^\Pi = 0$, by Corollary 6(b) and by the unitarity of $U$, to get $0 \in \sigma_{\text{ess}}(\mathcal{L}_1)$ it suffices to show that $0 \in \sigma_{\text{ess}}(\Theta)$. As $\Lambda^\pm : H_1^+(\Sigma) \rightarrow L^2(\Sigma)$ is a unitary operator, it is sufficient to show $0 \in \sigma_{\text{ess}}(\overline{\Phi}_1)$ for the unitarily equivalent operator $\overline{\Phi}_1 \equiv \Lambda^\pm \Theta \Lambda^\pm$ in $L^2(\Sigma)$, which will be done by constructing a singular Weyl sequence, i.e. a sequence of non-zero functions $(u_j) \subset \text{dom } \overline{\Phi}_1$ weakly converging to $0$ in $L^2(\Sigma)$ and such that the ratio $\|\Psi_1 u_j\|_{L^2(\Sigma)}/\|u_j\|_{L^2(\Sigma)}$ tends to $0$. While the domain of $\overline{\Phi}_1$ is not known explicitly, we know already that it contains $H^1(\Sigma)$.

Without loss of generality, we assume that $\Sigma_\varepsilon := \{(\varepsilon', 0) : \varepsilon' \in B_\varepsilon\} \subset \Sigma$, where $B_\varepsilon$ is the ball in $\mathbb{R}^{n-1}$ centered at the origin and of radius $\varepsilon > 0$. The iterative procedure of [26, Section 1] shows that the full symbols of $D_0^-$ and $D_0^+$ on $\Sigma_\varepsilon$ in the local coordinates $x'$ equal to $[x']$, and it follows that the full symbol of $\Psi_1$ vanishes on $\Sigma_\varepsilon$. Hence, there exists a smoothing operator $K$ and $\delta \in (0, \varepsilon)$ such that $\Psi_1 \tilde{u} = K \tilde{u}$ for all $u \in C_\varepsilon(\partial B_\delta)$, where $\tilde{u}$ is the extension of $u$ by zero to the whole of $\Sigma$. Take an orthonormal sequence $(u_j) \subset L^2(B_\delta)$ with $u_j \in C_\varepsilon(\partial B_\delta)$; then the sequence $(\tilde{u}_j) \subset H^1(\Sigma)$ is orthonormal and weakly converging to $0$ in $L^2(\Sigma)$. Due to the compactness of $K$ in $L^2(\Sigma)$ there exists a subsequence $(\Psi_1 \tilde{u}_{j_k})$ strongly converging to zero in $L^2(\Sigma)$. Therefore, the sequence $\nu_k := \tilde{u}_{j_k}$ is a sought singular Weyl sequence for $\overline{\Phi}_1$, which gives the result.

Suppose now $\text{dom } \mathcal{L}_1 = D_1^+(\Omega \setminus \Sigma) \subseteq H^s(\Omega_+) \oplus H^s(\Omega_-)$ for some $s > 0$. As the set on the right-hand side is compactly embedded in $L^2(\Omega)$ (see, e.g., [1, Theorem 14.3.1]) this implies the compactness of the resolvent of $\mathcal{L}_1$ and the equality $\sigma_{\text{ess}}(\mathcal{L}_1) = \emptyset$, which contradicts the previously proved relation $0 \in \sigma_{\text{ess}}(\mathcal{L}_1)$. 

**Remark 10.** After some simple cancellations, the resolvent formula of Corollary 6 for $\mathcal{L}_\mu$ takes the following form:

$$(\mathcal{L}_\mu - z)^{-1}\begin{pmatrix} u_- \\ u_+ \end{pmatrix} = \begin{pmatrix} (-\Delta_+ - z)^{-1} u_- \\ (\mu \Delta_+ - z)^{-1} u_+ \end{pmatrix} - \begin{pmatrix} R^-_\mu(u_-, u_+) \\ R^+_\mu(u_-, u_+) \end{pmatrix},$$

with

$$R^-_\mu(u_-, u_+) = P^{-\mu}_\varepsilon(D^-_\varepsilon - \mu \hat{D}_\mu^-)^{-1}(\sqrt{1}(-\Delta_+ - z)^{-1} u_- - \mu \gamma^1_\varepsilon(\mu \Delta_+ - z)^{-1} u_+),$$
$$R^+_\mu(u_-, u_+) = P^+\mu(D^-_\varepsilon - \mu \hat{D}_\mu^-)^{-1}(\sqrt{1}(-\Delta_+ - z)^{-1} u_- - \mu \gamma^1_\varepsilon(\mu \Delta_+ - z)^{-1} u_+).$$
5 Proof of Proposition 4

We continue using the conventions and notation introduced just before Theorem 4. In addition to (9) we have

\[ \Omega_+ = B_{r_1} \cup B_{R_r}, \quad \Sigma = S_{r_1} \cup S_r, \quad \partial \Omega = S_R, \]

and for the subsequent computations we use the identification

\[ L^2(S_\rho) \simeq L^2((0, 2\pi), \rho d\theta); \]

then \( L^2(\Sigma) \simeq L^2((0, 2\pi), r_1 d\theta) \oplus L^2((0, 2\pi), r_e d\theta), \) and similar identifications hold for the Sobolev spaces.

In view of Corollary 6 and of the expressions (21), the injectivity of \( L_1 \) is equivalent to the injectivity of the map

\[ D := D_0^- - \tilde{D}_0^+ : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma), \]

and then the condition \( g = (0, g_+) \in \text{ran} L_1 \) is equivalent to

\[ (G_0^{\Pi_1})^* g = -\gamma_1^+(-\Delta_+^D)^{-1} g_+ \in \text{ran} D, \]

or, as \( (\Delta_+^D)^{-1} : L^2(\Omega_+) \to H^2(\Omega) \) and \( \gamma_+^1 : H^2(\Omega) \to H^{\frac{1}{2}}(\Sigma), \) to

\[ (22) \quad D^{-1} \gamma_1^+(-\Delta_+^D)^{-1} g_+ \in H^{-\frac{1}{2}}(\Sigma). \]

The condition will be checked using an explicit computation of the Dirichlet-to-Neumann maps \( D_0^\pm \) and of the inverse of \( \Delta_+^D . \)

For a function \( f \) defined in \( \Omega_\pm, \) define its Fourier coefficients with respect to the polar angle by

\[ f_m(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, r \sin \theta)e^{-im\theta} d\theta, \quad m \in \mathbb{Z}; \]

then \( f \) is reconstructed by \( f(r \cos \theta, r \sin \theta) = \sum_{m \in \mathbb{Z}} f_m(r)e^{im\theta}. \) Furthermore, the separation of variables shows that a function \( f \) is harmonic iff \( f_m \) satisfy the Euler equations

\[ f_m''(r) + r^{-1} f_m'(r) - m^2 r^{-2} f_m(r) = 0, \]

whose linearly independent solutions are 1 and \( \ln r \) for \( m = 0 \) and \( r^{\pm m} \) for \( m \neq 0. \) This shows that for

\[ (\phi_i, \phi_\epsilon) \in H^s(\Sigma), \quad \phi_{i/e}(\theta) = \sum_{m \in \mathbb{Z}} \phi_{i/e,m} e^{im\theta}, \quad \phi_{i/e,m} := \frac{1}{2\pi} \int_0^{2\pi} \phi_{i/e}(\theta)e^{-im\theta} d\theta, \]
one has the following expressions for the Poisson operators:

\[
P_0^- \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) (r \cos \theta, r \sin \theta) = \frac{\ln \frac{r_0}{r_i}}{\ln \frac{r_i}{r_e}} \phi_{i,0} + \frac{\ln \frac{r_0}{r_e}}{\ln \frac{r_i}{r_e}} \phi_{e,0} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{[(\frac{r_0}{r_e})^{|m|} - (\frac{r_0}{r_i})^{|m|}]\phi_{i,m} + [(\frac{r_0}{r_e})^{|m|} - (\frac{r_0}{r_i})^{|m|}]\phi_{e,m}}{e^{i|m|}} e^{im\theta},
\]

\((r, \theta) \in (r_i, r_e) \times (0, 2\pi),\)

and

\[
\tilde{P}_0^+ \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) (r \cos \theta, r \sin \theta) = \begin{cases} \sum_{m \in \mathbb{Z}} (\frac{r_0}{r_i})^{|m|} \phi_{i,m} e^{im\theta}, & (r, \theta) \in (r_i, r_e) \times (0, 2\pi), \\ \frac{\ln \frac{r_0}{r_i}}{\ln \frac{r_i}{r_e}} \phi_{e,0} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(\frac{r_0}{r_i})^{|m|} - (\frac{r_0}{r_i})^{|m|}}{e^{i|m|}} \phi_{e,m} e^{im\theta}, & (r, \theta) \in (r_e, R) \times (0, 2\pi). \end{cases}
\]

It follows that

\[
\begin{align*}
D_0^- \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) &= \sum_{m \in \mathbb{Z}} B_m \left( \begin{array}{c} \phi_{i,m} \\ \phi_{e,m} \end{array} \right) e^{im\theta}, & \tilde{D}_0^+ \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) &= \sum_{m \in \mathbb{Z}} C_m \left( \begin{array}{c} \phi_{i,m} \\ \phi_{e,m} \end{array} \right) e^{im\theta}, \\
D \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) &= \sum_{m \in \mathbb{Z}} D_m \left( \begin{array}{c} \phi_{i,m} \\ \phi_{e,m} \end{array} \right) e^{im\theta}, & D_m := B_m - C_m,
\end{align*}
\]

with

\[
B_0 = \begin{pmatrix}
-\frac{1}{r_i} \ln \frac{r_0}{r_i} & -\frac{1}{r_i} \ln \frac{r_0}{r_e} \\
-\frac{1}{r_i} \ln \frac{r_0}{r_e} & -\frac{1}{r_i} \ln \frac{r_0}{r_e}
\end{pmatrix}, & C_0 = \begin{pmatrix}
0 & 0 \\
0 & 1 \frac{1}{r_i} \ln \frac{r_0}{r_e}
\end{pmatrix},
\]

\[
B_m = |m| \begin{pmatrix}
\frac{1}{r_i} \left( \frac{r_0}{r_i} \right)^{|m|} + \frac{1}{r_e} \left( \frac{r_0}{r_e} \right)^{|m|} \\
-\frac{2}{r_e} \left( \frac{r_0}{r_e} \right)^{|m|} - \frac{1}{r_i} \left( \frac{r_0}{r_i} \right)^{|m|} \\
\frac{1}{r_e} \left( \frac{r_0}{r_i} \right)^{|m|} + \frac{1}{r_e} \left( \frac{r_0}{r_e} \right)^{|m|}
\end{pmatrix}, & m \neq 0,
\]

\[
C_m = |m| \begin{pmatrix}
\frac{1}{r_i} & 0 \\
0 & \frac{1}{r_e} \left( \frac{r_0}{r_i} \right)^{|m|} + \frac{1}{r_e} \left( \frac{r_0}{r_e} \right)^{|m|}
\end{pmatrix}, & m \neq 0.
\]

Therefore,

\[
D_0 = \begin{pmatrix}
-\frac{1}{r_i} \ln \frac{r_0}{r_i} & -\frac{1}{r_i} \ln \frac{r_0}{r_e} \\
-\frac{1}{r_i} \ln \frac{r_0}{r_e} & -\frac{1}{r_i} \ln \frac{r_0}{r_e}
\end{pmatrix},
\]

\[
D_m = 2|m| \begin{pmatrix}
\frac{1}{r_i} \left( \frac{r_0}{r_i} \right)^{|m|} - 1 & -\frac{1}{r_i} \left( \frac{r_0}{r_i} \right)^{|m|} \\
-\frac{1}{r_i} \left( \frac{r_0}{r_i} \right)^{|m|} - 1 & \frac{1}{r_e} \left( \frac{r_0}{r_i} \right)^{|m|} - \frac{1}{r_e} \left( \frac{r_0}{r_i} \right)^{|m|} - 1
\end{pmatrix}, & m \neq 0,
\]
hence, all $D_m$ are invertible, and then $D$ is injective with the inverse

$$D^{-1} \left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) = \sum_{m \in \mathbb{Z}} D_{m}^{-1} \left( \begin{array}{c} \phi_{i,m} \\ \phi_{e,m} \end{array} \right) e^{im\theta},$$

which shows the injectivity of $\mathcal{L}_1$. Furthermore, for $m \neq 0$ we have

$$D_{m}^{-1} = -\frac{1}{2|m|} \left( \begin{array}{cc} r_i(1 - (\frac{r_i}{R})^2|m|) & r_e(\frac{r_e}{R})^{|m|}(1 - (\frac{r_e}{R})^2|m|) \\ r_i(\frac{r_i}{R})^{|m|}(1 - (\frac{r_i}{R})^2|m|) & r_e(1 - (\frac{r_e}{R})^2|m|) \end{array} \right),$$

and we conclude that a function $(\phi_i, \phi_e) \in H^{\frac{1}{2}}(\Sigma)$ belongs to ran $D$ iff $D^{-1}(\phi_i, \phi_e) \in H^{-\frac{1}{2}}(\Sigma)$, i.e., iff

$$\sum_{m \neq 0} \frac{1}{|m|} \left\| D_{m}^{-1} \left( \begin{array}{c} \phi_{i,m} \\ \phi_{e,m} \end{array} \right) \right\|^2_{\mathbb{C}^2} < \infty.$$

Therefore, the condition (22) is equivalent to (26) for

$$\phi_i, \phi_e := \gamma^+_f, \quad f := (\Delta_+^D)^{-1}g_+.$$

Note first that $f$ vanishes in $B_{r_i}$, hence, $\phi_i = 0$ and $f_m(r) = 0$ for $r < r_i$ and $m \in \mathbb{Z}$. To study the problem in $B_{r_e,R}$, let us pass to the Fourier coefficients; then we arrive at the system of equations

$$f''_m(r) + r^{-1}f'_m(r) - m^2r^{-2}f_m(r) = h_m1_{(a,b)}(r), \quad r_e < r < R, \quad f_m(r_e) = f_m(R) = 0,$$

and we have

$$\left( \begin{array}{c} \phi_i \\ \phi_e \end{array} \right) = -\sum_{m \in \mathbb{Z}} \left( \begin{array}{c} 0 \\ f'_m(r_e) \end{array} \right) e^{im\theta}.$$ 

One solves (28) using the variation of constants, and for $m \neq 0$ the solutions are

$$f_m(r) = a_m r^m + \beta_m r^{-m} + \frac{h_m r^m}{2m} \int_{r_e}^r s^{1-m} 1_{(a,b)}(s)ds - \frac{h_m r^{-m}}{2m} \int_{r_e}^r s^{1+m} 1_{(a,b)}(s)ds,$$

$$a_m = -\frac{h_m}{2mr^m} \frac{1}{(\frac{r_e}{R})^m} \int_{a}^{b} \left( \left( \frac{R}{s} \right)^m - \left( \frac{s}{R} \right)^m \right) sds, \quad \beta_m = -r_e^{-m} a_m,$$

and

$$f'_m(r_e) = \frac{mh_m}{r_e} (a_m r^m - \beta_m r^{-m}) = -\frac{1}{r_e} \frac{f'_b(\frac{R}{s})^{|m|} - (\frac{s}{R})^{|m|}}{(\frac{R}{r_e})^{|m|} - (\frac{s}{R})^{|m|}}.$$ 

Then for large $m$ one has

$$f'_m(r_e) = -\frac{(a^2 + o(1))h_m}{r_e|m|} \left( \frac{r_e}{a} \right)^{|m|}, \quad D_{m}^{-1} \left( \begin{array}{c} 0 \\ -f'_m(r_e) \end{array} \right) = h_m \left( \frac{a^2 + o(1)}{2m} \frac{r_e^2}{(\frac{r_e}{a})^{|m|}} \right),$$
and the condition (26) for the function (27) takes the form (11), which finishes the proof.

One should note that the condition (11) can still hold for \( a < r_e^2/r_i \) if the Fourier coefficients \( h_m \) of \( h \) are very fast decaying for large \( m \), i.e., if \( h \) extends to an analytic function in a suitable complex neighborhood of the unit circle.

**Remark 11.** Finally, we note that, in view of the injectivity of \( \mathcal{L}_1 \), the expression for its inverse given in Remark 10 can be extended naturally to a linear map \( \mathcal{L}^{-1} : L^2(\Omega) \to \mathcal{D}'(\Omega) \). As \( \text{ran}(\Delta_{\pm}^D)^{-1} = H^2(\Omega_{\pm}) \cap H^1_0(\Omega_{\pm}) \), the finiteness of the norms \( \|\mathcal{L}_1^{-1}g\|_{H^s(V)} \), \( V \subseteq \Omega, 0 \leq s \leq 1 \), is equivalent to the finiteness of \( \|v\|_{H^s(V)} \) for \( v := (R_0^g, R_0^+g) \). The direct substitution of the values of (25) and (29) into (23) and (24) shows that one always has \( v \in H^1(B_{r_e, r}) \), while the condition \( v \in L^2(B_{r_e, r}) \) appears to be equivalent to (11), so it holds for any \( h \) for \( a \geq r_e^2/r_i \) as before, otherwise a very strong regularity of \( h \) is required.

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