The role of second viscosity in the velocity shear-induced heating mechanism

G Tsereteli and Z Osmanov

School of Physics, Free University of Tbilisi, 0183, Tbilisi, Georgia
E-mail: z.osmanov@freeuni.edu.ge

Received 25 May 2018, revised 13 September 2018
Accepted for publication 10 October 2018
Published 31 October 2018

Abstract
In the present paper we study the influence of second viscosity on a non-modally-induced heating mechanism. For this purpose we study the set of equations governing the hydrodynamic system. In particular, we consider the Navier–Stokes equation, the continuity equation and the equation of state, linearise them and analyse in the context of non-modal instabilities. Unlike previous studies using the Navier–Stokes equation we include the contribution of compressibility, thus the second viscosity. By analysing several typical cases we show that under certain conditions the second viscosity might significantly change the efficiency of the mechanism of heating.

Keywords: bulk viscosity, instability, shear flows

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that in many astrophysical scenarios the flows are characterised by complex kinematics with inhomogeneous velocity fields, the so-called shear flows (SFs). The typical examples are extragalactic jets [1, 2] and young stellar jets [3], which reveal tornado-like structures of kinematics. Such complexity is also observationally supported in solar spicules [4]. Therefore, it is clear that under certain conditions the SFs might somehow influence the plasma processes in the above-mentioned astrophysical situations.

It has been realised that even in relatively simple velocity fields the SFs lead to instabilities [5]. In the framework of the standard modal approach the physical system is governed by equations of the form \( \partial_t \xi + (V_0 \cdot \nabla) \xi = M \xi \). Here, \( M \) denotes a Hermitian differential operator, \( V_0 \) is the background velocity and \( \xi \) is the field of the Hilbert space. In this scheme a hydrodynamical picture is decomposed into a system of independent normal modes with certain time constants. For example, the linearised set of equations can be solved by using an anzatz, with a well known time dependence \( e^{-i\omega t} \), where \( \omega \) denotes the frequency of the corresponding oscillations. In the framework of this approach all physical quantities nontrivially depend on spacial coordinates and the SFs lead to a non-Hermitian operator, \( M \) [6]. This means that the non-modally excited waves are more complex and might be very interesting, which is the main reason why we will be examining this particular class of instabilities.

The corresponding mathematical model for non-modal waves has been developed by Lord Kelvin [7]. In the framework of this approach, if one uses a specially chosen anzatz, the physical quantities expanded in the linear approximation satisfy equations which are reduced to the ordinary differential equations with an initial value problem [8].

In [9] we studied the role of velocity shear-induced phenomena in magnetohydrodynamic flows. It has been shown that by means of non-modal instability the physical system exhibits an energy exchange between slow/fast magnetosonic and Alfvén waves. Electrostatic ion perturbations has been considered in [10]. Authors have found that depending on the shear parameters, the system undergoes two different kinds of instabilities. For the exponentially evolving wave vectors the electrostatic perturbations evolve exponentially as well. On the other hand, when wave vectors are limited in amplitude and the flow is mostly stable, for a certain set of parameters the physical system might exhibit the so-called parametric instability, which takes place only for relatively narrow ranges of physical parameters.

1 Author to whom any correspondence should be addressed.
In this paper we study another interesting consequence of the shear-induced instability. This problem was originally examined in [11] where the author considers the process of heating to be composed of three major steps: 1. excitation of waves; 2. their non-modal amplification and 3. dissipation in the form of heat by means of the viscosity. The role of viscosity is crucial, although not trivial. In particular, it is clear that dissipation is more efficient for more viscous fluids, but on the other hand, in a medium with high viscosity the waves cannot amplify significantly extracting energy from the mean flow. Therefore, only for relatively moderate values of viscosity will non-modal self-heating potentially be efficient.

Direct application of this mechanism to astrophysical problems has been presented in [12], where the problem was studied in the context of the well known chromospheric heating of solar-type stars. It has been shown that acoustic waves might undergo the non-modal SF instability, leading to damping via viscous dissipation. We have found that this mechanism might explain the unusually high temperature of solar-type stars. It has been shown that acoustic waves might undergo the non-modal SF instability, leading to heating to be composed of three major steps: 1. excitation of waves; 2. their non-modal amplification and 3. dissipation in the form of heat by means of the viscosity. The role of viscosity is crucial, although not trivial. In particular, it is clear that dissipation is more efficient for more viscous fluids, but on the other hand, in a medium with high viscosity the waves cannot amplify significantly extracting energy from the mean flow. Therefore, only for relatively moderate values of viscosity will non-modal self-heating potentially be efficient.

In the aforementioned articles self-heating was studied for incompressible fluids and therefore the role of second viscosity (which is crucial for compressible fluids) was not studied. On the other hand, it is natural to generalise the previous study by including in the Navier–Stokes equation the terms corresponding to compressibility and considering the efficiency of self-heating.

The paper is organised in the following way: in section 2, a theoretical model of a shear-induced self-heating mechanism is presented, in section 3 we discuss the obtained numerical results and in section 4 we summarise them.

2. Main consideration

To conduct a detailed analysis of the viscous self-heating of non-modally generated waves and study the influence of second viscosity on the system’s behaviour, we consider the standard set of hydrodynamic equations. The mass conservation:

\[ D_t \rho + \rho \nabla \cdot \mathbf{V} = 0, \]  

the momentum conservation, taking into account the second viscosity

\[ D_t \mathbf{V} = -1/\rho \nabla P + \eta \Delta \mathbf{V} + (\eta/3 + \zeta) \nabla (\nabla \cdot \mathbf{V}) \]  

and the polytropic equation of state

\[ P = C\rho^n, \]  

where \( D_t \equiv \partial_t + (\mathbf{V} \cdot \nabla) \) is the notation for the convective derivative, \( \rho \) is the density, \( \mathbf{V} \) is the velocity, \( n \) is the polytropic index and \( \eta \) and \( \zeta \) are respectively the coefficients of kinematic shear viscosity and kinematic compression viscosity. After expanding all physical quantities up to the first order terms

\[ \rho \equiv \rho_0 + \rho', \]  
\[ \mathbf{V} \equiv \mathbf{V}_0 + \mathbf{V}', \]  
\[ P \equiv P_0 + P', \]  

where \( \rho_0, \mathbf{V}_0 \) and \( P_0 \) are the zeroth order quantities and \( \rho', \mathbf{V}' \) and \( P' \) are the corresponding linear terms, which satisfy the following linearised set of equations:

\[ D_t \rho' + \rho_0 (\nabla \cdot \mathbf{V}') = 0, \]  

\[ D_t \mathbf{V}' + (\mathbf{V}' \cdot \nabla) \mathbf{V}_0 = -C_2^2/\rho_0 \nabla \rho' + \eta \Delta \mathbf{V}' + (\eta/3 + \zeta) \nabla (\nabla \cdot \mathbf{V}'). \]  

Here \( D_t \equiv \partial_t + (\mathbf{V}_0 \cdot \nabla) \) and \( C_\eta \equiv \sqrt{dP_0/d\rho_0} \) are the sound speed. Next we follow the approach originally developed in [14] and we expand the velocity by the Taylor series in the vicinity of a point \( \mathbf{A}(x_0, y_0, z_0) \) preserving only the linear terms.

\[ \mathbf{V} = \mathbf{V}(\mathbf{A}) + \sum_{i=1}^{3} \frac{\partial \mathbf{V}(\mathbf{A})}{\partial x_i} (x_i - x_{0i}) \]  

In [14] it was shown that the set of linearised partial differential equations can be transformed into ordinary differential equations by employing the following ansatz.

\[ \mathbf{F}(x, y, z, t) \equiv \hat{\mathbf{F}}(t) e^{i(\phi_1 - \phi_2)} \]  
\[ \phi_1 \equiv \sum_{i=1}^{3} K_i(t) x_i \]  
\[ \phi_2 \equiv \sum_{i=1}^{3} V_i(A) \int K_i(t) dt \]  

where \( V_i(A) \) is the unperturbed velocity component and \( K_i(t) \) are the components of wave vectors. By following the standard approach developed in [14] it is straightforward to show that the convective derivative of a physical quantity \( \mathbf{F}(x, y, z, t) \)

\[ D_t \mathbf{F} = e^{i(\phi_1 - \phi_2)} D_t \hat{\mathbf{F}}(t) \]

\[ + i x (K_1^{(1)} + a_1 K_x + b_1 K_y + c_1 K_z) F \]
\[ + i y (K_2^{(1)} + a_2 K_x + b_2 K_y + c_2 K_z) F \]
\[ + i z (K_3^{(1)} + a_3 K_x + b_3 K_y + c_3 K_z) F, \]  

becomes a simple time derivative multiplied by \( e^{i(\phi_1 - \phi_2)} \) if the following set of equations is satisfied

\[ \partial_t \mathbf{K} + S^T \cdot \mathbf{K} = 0, \]

where

\[ S \equiv \begin{pmatrix} V_{x,xx} & V_{x,xy} & V_{x,xz} \\ V_{y,xx} & V_{y,xy} & V_{y,xz} \\ V_{z,xx} & V_{z,xy} & V_{z,xz} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \]

is the so-called shear matrix and \( S^T \) is its transposed form. In dimensionless quantities the complete set of equations
writes as
\[ d^{(1)} + k \cdot u = 0 \]
\[ u^{(1)} + s \cdot u = k d - \nu_1 k^2 u - \nu_2 (k \cdot u) \]
\[ k^{(1)} + s^T \cdot k = 0. \]

Notations are as follows: \( d = -i \rho' / \rho_0 \), \( u = \frac{V'}{c_0} \), \( k = \frac{k_t}{k_{t(0)}} \), \( \nu_{1,2} = \nu_{1,2} \frac{k_t}{c_0} \), \( \nu_1 \equiv p_1 \), \( \nu_2 \equiv \eta/3 + \zeta \), \( s \equiv S/(k_t(0) C_t) \). By \( \Psi^{(1)} \) we denote the derivative with respect to dimensionless time \( \tau \equiv K_t(0) C_t t \).

In order to analyse the process of heating by means of energy we define the perturbation energy density \([15]\)
\[ E_{tot} = \frac{u^2}{2} + \frac{d^2}{2}, \]
where the first and second terms are the so-called kinetic and compressional terms, respectively. It is straightforward to show that the time derivative of \( E_{tot} \) is given by
\[ E^{(1)} = -u(s \cdot u) - \nu_1 k^2 u^2 - \nu_2 (k \cdot u)^2. \]

As is evident from the above equation, the last two terms are responsible for heating, therefore the efficiency of the process might be defined as follows \([11]\)
\[ \psi(\tau) = \frac{1}{E_{tot}(0)} \int_0^\tau \left( \nu_1 k^2 (\tau') u^2(\tau') + \nu_2 (k (\tau') \cdot u(\tau'))^2 \right) d\tau', \]
where \( E_{tot}(0) \) is the initial energy of perturbation. It is worth noting that although the full picture of the heating process is purely non-linear, we study the process of dissipation of the first order terms, which show the tendency of the process and the role of second viscosity.

3. Discussion

In this section we are going to numerically demonstrate the effect of compression on the overall heating for arbitrary parameters in reasonable scopes. We start with the set of equations for different types of SFs. For the incompressible case the shear matrix is relatively trivial and is traceless. In particular, from equation (1) it is evident that from \( \rho = const \) follows the condition \( \nabla \cdot V = 0 \), which if rewritten for the matrix components means \( a_1 + b_2 + c_3 = 0 \). In all of the briefly discussed articles in the introduction the problem of
heating has been examined for incompressible fluids. One can straightforwardly show that the same condition for compressible flows is not satisfied at all and therefore, it is important to consider this particular case as well.

3.1. Density perturbation

As a first example we consider a case when initially only the density field is perturbed, thus an acoustic wave is excited. In figure 1 we demonstrate the behaviour of \( d(\tau) \) and \( u(\tau) \). The set of parameters is \( a_1 = -0.1, a_2 = 0.2 \) and the rest of the shear matrix elements are zero and \( k_x = 0.1, k_y = 0.2, k_z = 0, u_x = 0.3, u_y = -0.2, u_z = 0, d = 0, \eta = 0.001, \zeta = 0.003 \). Without going into detail we consider the case \( \zeta/\eta = 3 \), which is a reasonable value for gases [16].

As is clear from the plots, the acoustic waves excite the velocity components and the amplitude of corresponding oscillations initially increases but in due course the induced

Figure 3. Here we demonstrate the efficiency of heating versus time. The set of parameters is the same as described in the previous figure: left panel corresponds to \( \zeta = 0 \) and the right panel— to \( \zeta = 0.003 \).

Figure 4. The plots of \( d(\tau) \) and \( u(\tau) \) are shown. The set of parameters is \( a_1 = -0.1, a_2 = 0.2 \) and the rest of the shear matrix elements are zero and \( k_x = 0.1, k_y = 0.2, k_z = 0, u_x = 0.3, u_y = -0.2, u_z = 0, d = 0, \eta = 0.001, \zeta = 0.003 \).

Figure 5. We show the plots of \( \psi \) versus time for \( \zeta = 0 \) (left panel) and \( \zeta = 0.003 \) (right panel), respectively. The rest of the parameters are the same as in figure 4.
3.2. Velocity perturbation

In figure 2 we show how the sensitive behaviour of excited waves might be able to change the second viscosity. In particular, if the instability is almost completely damped at $\tau \approx 55$ for $\zeta = 0$ (left panel), the corresponding oscillations for $\zeta = 0.003$ (right panel) dissipate at $\tau \approx 45$. Therefore, the second viscosity has a visible effect on the propagation of the acoustic waves and will inevitably affect the nature of the flow. In figure 3 we show the time dependence of $\psi$ on time. It is clear that the heating rate is a continuously increasing function of time with a relatively high initial increment. This is a direct consequence of the fact that initially the waves are efficiently amplifying, which in turn results in the efficient energy transform to heat. In due course, by means of viscous effects the oscillations start damping and consequently a value of $\psi$ saturates. As is clear from the plots, for the previously mentioned physical parameters, the efficiency of heating is almost two times smaller for $\zeta = 0.003$ than for $\zeta = 0$.

4. Conclusion

We have considered compressible fluids and studied the role of second viscosity in the process of heating. In particular, we have examined the Navier–Stokes equation, continuity equation and equation of state, linearised them and investigated the generation of non-modal SF instabilities.

It has been shown that initially generated acoustic waves perturb velocity components as well and as a result non-modal waves amplify the extracting energy from the mean flow. The already amplified waves are continuously damped by means of viscosity.

For relatively simple examples we have shown that the heating rate might be significantly changed by means of the second viscosity coefficient. In particular, it has been shown that initially driven acoustic waves finally dissipate and the corresponding efficiency is higher for smaller values of second viscosity. A similar result is valid for a regime, when initially only velocity components are perturbed.

It is worth noting that in the current manuscript we have generally examined the role of second viscosity and did not study the self-heating mechanism for physical processes implying realistic values of $\eta$ and $\zeta$. This particular problem is something that will be considered in forthcoming papers.

Acknowledgments

The research of GT was supported by the Knowledge Foundation at the Free University of Tbilisi. The research of ZO was supported by the Shota Rustaveli National Science Foundation grant (FR17-587) and partially by the grant (DI-2016-14).

References

[1] Broderick A E and Loeb A 2009 ApJ 703 104L
[2] Kharb P, Gabuzda D C, O’Dea C P, Shastri P and Baum S A 2009 ApJ 694 1485
[3] Chrysostomou A, Bacciotti F, Nisini B, Ray T P, Eislöffel J, Davis C J and Takami M 2008 Astron. Astrophys. 482 575
[4] Pike C D and Mason H E 1998 Sol. Phys. 182 333
[5] Nold A and Oberlack M 2013 Phys. Fluids 25 104101
[6] Tatsuno T, Volponti F and Yoshida Z 2001 Phys. Plasmas 8 399
[7] Lord Kelvin (W Thomson) 1887 Phil. Mag. 24 188 Ser. 5
[8] Trefethen L N, Trefethen A E, Reddy S C and Driscoll T A 1993 Sience 261 578
[9] Rogava A D, Bodo G, Massaglia S and Osmanov Z 2003 Astron. Astrophys. 408 401
[10] Osmanov Z, Rogava A D and Poedts S 2015 NJP 17 043019
[11] Rogava A D 2004 Astrophys. Space Sci. 293 189
[12] Rogava A D, Osmanov Z and Poedts S 2010 MNRAS 404 224
[13] Osmanov Z, Rogava A D and Poedts S 2012 Phys. Plasmas 19 012901
[14] Mahajan S and Rogava A D 1999 ApJ 518 814
[15] Landau L D and Lifshitz E M 2010 Fluid Mechanics (Oxford: Pergamon Press)
[16] Cramer M S 2012 Phys. Fluids 24 066102