C*-ALGEBRAS GENERATED BY GROUPS OF COMPOSITION OPERATORS

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Abstract. We compute the C*-algebra generated by a group of composition operators acting on certain reproducing kernel Hilbert spaces over the disk, where the symbols belong to a non-elementary Fuchsian group. We show that such a C*-algebra contains the compact operators, and its quotient is isomorphic to the crossed product C*-algebra determined by the action of the group on the boundary circle. In addition we show that the C*-algebras obtained from composition operators acting on a natural family of Hilbert spaces are in fact isomorphic, and also determine the same Ext-class, which can be related to known extensions of the crossed product.

1. Introduction

The purpose of this paper is to begin a line of investigation suggested by recent work of Moorhouse et al. [12, 8, 7]: to describe, in as much detail as possible, the C*-algebra generated by a set of composition operators acting on a Hilbert function space. In this paper we consider a class of examples which, while likely the simplest cases from the point of view of composition operators, nonetheless produces C*-algebras which are of great interest both intrinsically and for applications. In fact, the C*-algebras we obtain are objects of current interest among operator algebraists, with applications in the study of hyperbolic dynamics [5], noncommutative geometry [4], and even number theory [11].

Let $f \in H^2(\mathbb{D})$. For an analytic function $\gamma : \mathbb{D} \to \mathbb{D}$, the composition operator with symbol $\gamma$ is the linear operator defined by

$$(C_\gamma f)(z) = f(\gamma(z))$$

In this paper, we will be concerned with the C*-algebra

$$C_\Gamma = C^*(\{C_\gamma : \gamma \in \Gamma\})$$
where $\Gamma$ is a discrete group of (analytic) automorphisms of $\mathbb{D}$ (i.e. a Fuchsian group). For reasons to be described shortly, we will further restrict ourselves to non-elementary Fuchsian groups (i.e. groups $\Gamma$ for which the $\Gamma$-orbit of 0 in $\mathbb{D}$ accumulates at at least three points of the unit circle $\mathbb{T}$.) Our main theorem shows that $C_\Gamma$ contains the compact operators, and computes the quotient $C_\Gamma/K$:

**Theorem 1.1.** Let $\Gamma$ be a non-elementary Fuchsian group, and let $C_\Gamma$ denote the $C^*$-algebra generated by the set of composition operators on $H^2$ with symbols in $\Gamma$. Then there is an exact sequence

\[
0 \longrightarrow K \longrightarrow C_\Gamma \longrightarrow \pi : C(\mathbb{T}) \times \Gamma \longrightarrow 0
\]

Here $C(\mathbb{T}) \times \Gamma$ is the crossed product $C^*$-algebra obtained from the action $\alpha$ of $\Gamma$ on $C(\mathbb{T})$ given by

$$\alpha_\gamma(f)(z) = f(\gamma^{-1}(z))$$

(Since the action of $\Gamma$ on $\partial \mathbb{D}$ is amenable, the full and reduced crossed products coincide; we will discuss this further shortly.) We will recall the relevant definitions and facts we require in the next section.

There is a similar result for the $C^*$-algebras

$$C^n_\Gamma = C^* \left( \{ C_\gamma \in \mathcal{B}(A^2_n) | \gamma \in \Gamma \} \right),$$

acting on the family of reproducing kernel Hilbert spaces $A^2_n$ (defined below), namely there is an extension

\[
0 \longrightarrow K \longrightarrow C^n_\Gamma \longrightarrow \pi : C(\mathbb{T}) \times \Gamma \longrightarrow 0
\]

and we will show that each of these extensions represents the same element of the $Ext$ group $Ext(C(\partial \mathbb{D}) \times \Gamma, K)$, and we will also prove that $C_\Gamma$ and $C^n_\Gamma$ are isomorphic as $C^*$-algebras. In fact we obtain a stronger isomorphism result, namely that any unital $C^*$-extension of $C(\partial \mathbb{D}) \times \Gamma$ which defines the same $Ext$-class as $C_\Gamma$ is isomorphic to $C_\Gamma$:

**Theorem 1.2.** Let $x \in Ext(C(\partial \mathbb{D}) \times \Gamma)$ denote the class of the extension

\[
0 \longrightarrow K \longrightarrow C_\Gamma \longrightarrow \pi : C(\mathbb{T}) \times \Gamma \longrightarrow 0.
\]

If $e \in Ext(C(\partial \mathbb{D}) \times \Gamma)$ is a unital extension represented by

\[
0 \longrightarrow K \longrightarrow E \longrightarrow C(\partial \mathbb{D}) \times \Gamma \longrightarrow 0
\]

and $e = x$, then $E \cong C_\Gamma$ as $C^*$-algebras.

Finally, we will compare the extension determined by $C_\Gamma$ to two other recent constructions of extensions of $C(\partial \mathbb{D}) \times \Gamma$. We show that the $Ext$-class of $C_\Gamma$ coincides with the class of the $\Gamma$-equivariant Toeplitz extension of $C(\partial \mathbb{D})$ constructed by J. Lott \cite{10}, and differs from the
extension of crossed products by cocompact groups constructed by H. Emerson [5]. Finally we show that this extension in fact gives rise to a $\Gamma$-equivariant $KK_{1}$-cycle for $C(\partial \mathbb{D})$ which also accords with the construction in [10].

2. Preliminaries

We will consider $C^*$-algebras generated by composition operators which act on a family of reproducing kernel Hilbert spaces on the unit disk. Specifically we will consider the spaces of analytic functions $A_{n}^{2}$, where $A_{n}^{2}$ is the space with reproducing kernel

$$k_{n}(z, w) = (1 - zw)^{n}$$

When $n = 1$ this space is the Hardy space $H^{2}$, and its norm is given by

$$\|f\|^{2} = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{2} d\theta$$

For $n \geq 2$, the norm on $A_{n}^{2}$ is equivalent to

$$\|f\|^{2} = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^{2}(1 - |z|^{2})^{n-2} dA(z)$$

though the norms are equal only for $n = 2$ (the Bergman space).

An analytic function $\gamma : \mathbb{D} \to \mathbb{D}$ defines a composition operator $C_{\gamma}$ on $A_{n}^{2}$ by

$$(C_{\gamma}f)(z) = f(\gamma(z))$$

In this paper, we will only consider cases where $\gamma$ is a Möbius transformation; in these cases $C_{\gamma}$ is easily seen to be bounded, by changing variables in the integrals defining the norms. An elementary calculation shows that if $\gamma : \mathbb{D} \to \mathbb{D}$ is analytic, then

$$C_{\gamma}^{*}k_{w}(z) = k_{\gamma(w)}(z)$$

where $k$ is any of the reproducing kernels $k_{n}$.

We recall here the definitions of the full and reduced crossed product $C^*$-algebras; we refer to [13] for details. Let a group $\Gamma$ act by homeomorphisms on a compact Hausdorff space $X$. This induces an action of $\Gamma$ on the commutative $C^*$-algebra $C(X)$ via

$$(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x)$$

The algebraic crossed product $C(X) \times_{alg} \Gamma$ consists of formal finite sums $f = \sum_{\gamma \in \Gamma} f_{\gamma} [\gamma]$, where $f_{\gamma} \in C(\mathbb{T})$ and the $[\gamma]$ are formal symbols.
Multiplication is defined in \( C(X) \times_{\text{alg}} \Gamma \) by
\[
\left( \sum_{\gamma \in \Gamma} f_{\gamma} \right) \left( \sum_{\gamma' \in \Gamma} f'_{\gamma'} \right) = \sum_{\delta \in \Gamma} \sum_{\gamma \gamma' = \delta} f_{\gamma} (\gamma \cdot f'_{\gamma'}) [\delta]
\]
For \( f = \sum_{\gamma} f_{\gamma} \), define \( f^* \in C(X) \times_{\text{alg}} \Gamma \) by
\[
f^* = \sum_{\gamma \in \Gamma} (\gamma \cdot f_{\gamma^{-1}}) [\gamma]
\]
With this multiplication and involution, \( C(X) \times_{\text{alg}} \Gamma \) becomes a \(*\)-algebra, and we may construct a \( C^*\)-algebra by closing the algebraic crossed product with respect to a \( C^*\)-norm.

To obtain a \( C^*\)-norm, one constructs \(*\)-representations of \( C(X) \times_{\text{alg}} \Gamma \) on Hilbert space. To do this, we first fix a faithful representation \( \pi \) of \( C(X) \) on a Hilbert space \( H \). We then construct a representation \( \sigma \) of the algebraic crossed product on \( H \otimes \ell^2(\Gamma) = \ell^2(\Gamma, H) \) as follows: define a representation \( \tilde{\pi} \) of \( C(X) \) by its action on vectors \( \xi \in \ell^2(\Gamma, H) \)
\[
(\tilde{\pi}(f))(\xi)(\gamma) = \pi(f \circ \gamma)\xi(\gamma)
\]
Represent \( \Gamma \) on \( \ell^2(\Gamma, H) \) by left translation:
\[
(U(\gamma))(\xi)(\eta) = \xi(\gamma^{-1}\eta)
\]
The representation \( \sigma \) is then given by
\[
\sigma \left( \sum_{\gamma} f_{\gamma} \right) = \sum \tilde{\pi}(f_{\gamma}) U(\gamma)
\]
The closure of \( C(X) \times_{\text{alg}} \Gamma \) with respect the norm induced by this representation is called the reduced crossed product of \( \Gamma \) and \( C(X) \), and is denoted \( C(X) \times_{r} \Gamma \). The full crossed product, denote \( C(X) \times \Gamma \), is obtained by taking the closure of the algebraic crossed product with respect to the maximal \( C^*\)-norm
\[
\|f\| = \sup_{\pi} \|\pi(f)\|
\]
where the supremum is taken over \textit{all} \(*\)-representations \( \pi \) of \( C(X) \times_{\text{alg}} \Gamma \) on Hilbert space. When \( \Gamma \) is discrete, \( C(X) \times \Gamma \) contains a canonical subalgebra isomorphic to \( C(X) \), and there is a natural surjective \(*\)-homomorphism \( \rho : C(X) \times \Gamma \rightarrow C(X) \times_{r} \Gamma \).

The full crossed product is important because of its universality with respect to covariant representations. A \textit{covariant representation} of the pair \((\Gamma, X)\) consists of a faithful representation \( \pi \) of \( C(X) \) on Hilbert space, together with a unitary representation \( u \) of \( \Gamma \) on the same space satisfying the covariance condition
\[
u(\gamma)\pi(f)u(\gamma)^* = f \circ \gamma^{-1}
\]
for all \( f \in C(X) \) and all \( \gamma \in \Gamma \). If \( \mathcal{A} \) denotes the C*-algebra generated by the images of \( \pi \) and \( u \), then there is a surjective \(*\)-homomorphism from \( C(X) \times \Gamma \) to \( \mathcal{A} \); equivalently, any C*-algebra generated by a covariant representation is isomorphic to a quotient of the full crossed product.

We now collect properties of group actions on topological spaces which we will require in what follows. Let a group \( \Gamma \) act by homeomorphisms on a locally compact Hausdorff space \( X \); The action of \( \Gamma \) on \( X \) is called \textit{minimal} if the set \( \{ \gamma \cdot x | \gamma \in \Gamma \} \) is dense in \( X \) for each \( x \in X \), and called \textit{topologically free} if, for each \( \gamma \in \Gamma \), the set of points fixed by \( \gamma \) has empty interior.

Suppose now \( \Gamma \) is discrete and \( X \) is compact. Let \( \text{Prob}(\Gamma) \) denote the set of finitely supported probability measures on \( \Gamma \). We say \( \Gamma \) acts \textit{amenably on} \( X \) if there exists a sequence of weak-* continuous maps \( b^n_x : X \to \text{Prob}(\Gamma) \) such that for every \( \gamma \in \Gamma \),

\[
\lim_{n \to \infty} \sup_{x \in X} \| \gamma \cdot b^n_x - b^n_{\gamma \cdot x} \|_1 = 0
\]

where \( \Gamma \) acts on the functions \( b^n_x \) via \( (\gamma \cdot b^n_x)(z) = b^n_x(\gamma^{-1} \cdot z) \), and \( \| \cdot \|_1 \) denotes the \( l^1 \)-norm on \( \Gamma \).

**Theorem 2.1.** [2] Let \( \Gamma \) be a discrete group acting on a compact Hausdorff space \( X \), and suppose that the action is topologically free. If \( \mathcal{J} \) is an ideal in \( C(X) \times \Gamma \) such that \( C(X) \cap \mathcal{J} = 0 \), then \( \mathcal{J} \subseteq \mathcal{J}_\lambda \), where \( \mathcal{J}_\lambda \) is the kernel of the projection of the full crossed product onto the reduced crossed product.

**Theorem 2.2.** [16, 18] If the \( \Gamma \) acts amenably on \( X \), then the full and reduced crossed product C*-algebras coincide, and this crossed product is nuclear.

In this paper we are concerned with the case \( X = \mathbb{T} \) and \( G = \Gamma \), a non-elementary Fuchsian group. The action of \( \Gamma \) on \( \mathbb{T} \) is always amenable, so by Theorem 2.2 the full and reduced crossed products coincide, \( C(\partial \mathbb{D}) \times \Gamma \) is nuclear.

We also require some of the basic terminology concerning Fuchsian groups. Fix a base point \( z_0 \in \mathbb{D} \). The \textit{limit set} of \( \Gamma \) is the set accumulation points of the orbit \( \{ \gamma(z_0) : \gamma \in \Gamma \} \) on the boundary circle; it is closed (and does not depend on the choice of \( z_0 \). The limit set can be one of three types: it is either finite (in which case it consists of at most two points), a totally disconnected perfect set (hence uncountable), or all of the circle. In the latter case we say \( \Gamma \) is of the \textit{first kind}, and of the \textit{second kind} otherwise. If the limit set is finite, \( \Gamma \) is called \textit{elementary}, and \textit{non-elementary} otherwise.
Though we do not require it in this paper, it is worth recording that if \( \Gamma \) is of the first kind, then \( C(\partial \mathbb{D}) \times \Gamma \) is simple, nuclear, purely infinite, and belongs to the bootstrap category \( \mathcal{N} \) (and is clearly separable and unital). Hence, it satisfies the hypotheses of the Kirchberg-Phillips classification theorem, and is classified up to isomorphism by its (unital) \( K \)-theory \([9,1]\).

Finally, we recall briefly the basic facts about extensions of \( C^* \)-algebras and the \( \text{Ext} \) functor. An exact sequence of \( C^* \)-algebras

\[
0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0
\]

is called an extension of \( A \) by \( B \). (In this paper we consider only extensions for which \( B = \mathcal{K} \), the \( C^* \)-algebra of compact operators on a separable Hilbert space.) Associated to an extension of \( A \) by \( B \) is a \( * \)-homomorphism \( \tau \) from \( A \) to the Calkin algebra \( Q(H) = \mathcal{B}(H)/\mathcal{K}(H) \); this \( \tau \) is called the Busby map associated to the extension. Conversely, given a map \( \tau : A \to Q(H) \) there is a unique extension having Busby map \( \tau \); we will thus speak of an extension and its Busby map interchangeably. Two extensions of \( A \) by \( B \) with Busby maps \( \tau_1 : A \to Q(H_1) \) and \( \tau_2 : A \to Q(H_2) \) are strongly unitarily equivalent if there is a unitary \( u : H_1 \to H_2 \) such that

\[
\pi(u)\tau_1(a)\pi(u)^* = \tau_2(a)
\]

for all \( a \in A \) (here \( \pi \) denotes the quotient map from \( \mathcal{B}(H_1) \) to \( Q(H_1) \)). We say \( \tau_1 \) and \( \tau_2 \) are unitarily equivalent, written \( \tau_1 \sim_u \tau_2 \), if (2.2) holds with \( u \) replaced by some \( v \) such that \( \pi(v) \) is unitary. An extension \( \tau \) is trivial if it lifts to a \( * \)-homomorphism, that is there exists a \( * \)-homomorphism \( \rho : A \to \mathcal{B}(\mathcal{H}) \) such that \( \tau(a) = \pi(\rho(a)) \) for all \( a \in A \). Two extensions \( \tau_1, \tau_2 \) are stably equivalent if there exist trivial extensions \( \sigma_1, \sigma_2 \) such that \( \tau_1 \oplus \sigma_1 \) and \( \tau_2 \oplus \sigma_2 \) are strongly unitarily equivalent; stable equivalence is an equivalence relation. If \( A \) is separable and nuclear (which will always be the case in this paper) then the stable equivalence classes of extensions of \( A \) by \( \mathcal{K} \) form an abelian group (where addition is given by direct sum of Busby maps) called \( \text{Ext}(A,\mathcal{K}) \), abbreviated \( \text{Ext}(A) \). Finally, each element of \( \text{Ext}(A) \) determines an index homomorphism \( \partial : K_1(A) \to \mathbb{Z} \) obtained by lifting a unitary in \( M_n(A) \) representing the \( K_1 \) class to a Fredholm operator in \( M_n(E) \) and taking its Fredholm index.

3. The Extensions \( C_\Gamma \) and \( C^n_\Gamma \)

3.1. The Hardy space. This section is devoted to the proof of our main theorem, in the case of the Hardy space:
Theorem 3.1. Let $\Gamma$ be a non-elementary Fuchsian group, and let $C_\Gamma$ denote the $C^*$-algebra generated by the set of composition operators on $H^2$ with symbols in $\Gamma$. Then there is an exact sequence

$$0 \longrightarrow K_{\iota} \longrightarrow C_\Gamma \longrightarrow C(\mathbb{T}) \times \Gamma \longrightarrow 0 \quad (3.1)$$

While the proof in the general case of $A^2_n$ follows similar lines, we prove the $H^2$ case first since it is technically simpler and illustrates the main ideas. The proof splits into three parts: first, we prove that $C_\Gamma$ contains the unilateral shift $S$ and hence the compacts (Proposition 3.4). We then prove that the quotient $C_\Gamma/K$ is generated by a covariant representation of the topological dynamical system $(C(\mathbb{T}), \Gamma)$. Finally we prove that the $C^*$-algebra generated by this representation is all of the crossed product $C(\mathbb{T}) \times \Gamma$.

We first require two computational lemmas.

Lemma 3.2. Let $\gamma$ be an automorphism of $\mathbb{D}$ with $a = \gamma^{-1}(0)$ and let $f(z) = \frac{1 - \overline{a}z}{(1 - |a|^2)^{1/2}}$.

Then

$$C_\gamma C_\gamma^* = M_f M_f^*$$

where $M_f$ denotes the operator of multiplication by $f$.

Proof. It suffices to show

$$(3.2) \quad \langle C_\gamma C_\gamma^* k_w, k_z \rangle = \langle M_f M_f^* k_w, k_z \rangle$$

for all $z, w$ in $\mathbb{D}$. Since $C_\gamma^* k_\lambda = k_{\gamma(\lambda)}$ and $M_f^* k_\lambda = \overline{f(\lambda)} k_\lambda$, (3.2) reduces to the well-known identity

$$(3.3) \quad \frac{1}{1 - \gamma(z) \gamma(w)} = \frac{(1 - \overline{a}z)(1 - a\overline{w})}{(1 - |a|^2)(1 - z\overline{w})}.$$ 

Lemma 3.3. Let $\gamma(z)$ be an automorphism of $\mathbb{D}$ and let $S = M_z$. Then

$$M_\gamma C_\gamma = C_\gamma S$$

Proof. For all $g \in H^2$,

$$(C_\gamma S)(g)(z) = \gamma(z) g(\gamma(z)) = (M_\gamma C_\gamma)(g)(z)$$

Proposition 3.4. Let $\Gamma$ be a non-elementary Fuchsian group. Then $C_\Gamma$ contains the unilateral shift $S$. 


Proof. Let $\Lambda$ denote the limit set of $\Gamma$. Since $\Gamma$ is non-elementary, there exist three distinct points $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$. For each $\lambda_i$ there exists a sequence $\gamma_{n,i}$ such that $\gamma_{n,i}(0) \to \lambda_i$. Let $a_{n,i}^i = \gamma_{n,i}(0)$. By Lemma 3.2
\[
(1 - |a_{n,i}^i|^2)C_{\gamma_{n,i}}^{-1}C_{\gamma_{n,i}}^* = (1 - a_{n,i}^i)(1 - a_{n,i}^i)^*
\]
As $n \to \infty$, $a_{n,i}^i \to \lambda_i$ and the right-hand side converges to
\[
1 - \lambda S - \lambda S^* + SS^*
\]
in $C_{\Gamma}$. Taking differences of these operators for different values of $i$, we see that $\Re[\mu_1 S]$, $\Re[\mu_2 S] \in C_{\Gamma}$, with $\mu_1 = \lambda_1 - \lambda_2$ and $\mu_2 = \lambda_2 - \lambda_3$. Note that since $\lambda_1, \lambda_2, \lambda_3$ are distinct points on the circle, the complex numbers $\mu_1$ and $\mu_2$ are linearly independent over $\mathbb{R}$ when identified with vectors in $\mathbb{R}^2$. We now show there exist scalars $a_1, a_2$ such that
\[
a_1 \Re[\mu_1 S] + a_2 \Re[\mu_2 S] = S
\]
which proves the lemma. Such scalars must solve the linear system
\[
(3.4)
\begin{pmatrix}
\mu_1 & \mu_2 \\
\mu_1 & \mu_2
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]
Writing $\mu_1 = \alpha_1 + i\beta_1$, $\mu_2 = \alpha_2 + i\beta_2$, a short calculation shows that
\[
\det \begin{pmatrix}
\mu_1 & \mu_2 \\
\mu_1 & \mu_2
\end{pmatrix} = -2i \det \begin{pmatrix}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2
\end{pmatrix}
\]
The latter determinant is nonzero since $\mu_1, \mu_2$ are linearly independent over $\mathbb{R}$, and the system is solvable. □

The above argument does not depend on the discreteness of $\Gamma$; indeed it is a refinement of an argument due to J. Moorhouse [12] that the $C^*$-algebra generated by all Möbius transformations contains $S$. The argument is valid for any group which has an orbit with three accumulation points on $\mathbb{T}$; e.g. the conclusion holds for any dense subgroup of $PSU(1, 1)$.

Proof of Theorem 3.1 For an automorphism $\gamma$ of $\mathbb{D}$ set
\[
U_\gamma = (C_\gamma C_\gamma^*)^{-1/2}C_\gamma,
\]
the unitary appearing in the polar decomposition of $C_\gamma$. By Lemma C\gamma C_\gamma^* = T_f T_f^*, so we may write
\[
U_\gamma = T_{|f|^{-1}}C_\gamma + K
\]
for some compact $K$. Now by Lemma 3.3 if $p$ is any analytic polynomial,
\[ U_\gamma T_p = T_{|f|^{-1}}C_\gamma T_p + K' \]
\[ = T_{|f|^{-1}}T_{p \circ \gamma}C_\gamma + K' \]
\[ = T_{p \circ \gamma}T_{|f|^{-1}}C_\gamma + K'' \]
\[ = T_{p \circ \gamma}U_\gamma + K'' \]
where we have used the fact that $T_{|f|^{-1}}$ and $T_p$ commute modulo $K$.

Taking adjoints and sums shows that
\[ (3.5) \quad U_\gamma T_q U_\gamma^* = T_q \circ \gamma + K \]
for any trigonometric polynomial $q$. We next show that, for Möbius transformations $\gamma, \eta$,
\[ U_\gamma U_\eta = U_{\eta \circ \gamma} + K \]
for some compact $K$. To see this, write
\[ \gamma(z) = \frac{az + b}{cz + d}, \quad \eta(z) = \frac{ez + f}{gz + h} \]
Then
\[ U_\gamma = T_{|cz+d|^{-1}}C_\gamma + K_1, \quad U_\eta = T_{|gz+h|^{-1}}C_\eta + K_2. \]
Note that
\[ U_{\eta \circ \gamma} = T_{|g(a z + b) + h(c z + d)||^{-1}}C_{\eta \circ \gamma} + K_3. \]
Then
\[ U_\gamma U_\eta = T_{|cz+d|^{-1}}C_\gamma T_{|gz+h|^{-1}}C_\eta + K \]
\[ = T_{|cz+d|^{-1}}T_{|g(z) + h||^{-1}}C_\gamma C_\eta + K' \]
\[ = T_{|g(a z + b) + h(c z + d)||^{-1}}C_{\eta \circ \gamma} + K' \]
\[ = U_{\eta \circ \gamma} + K'' \]

Observe now that $C_\Gamma$ is equal to the C*-algebra generated by $S$ and the unitaries $U_\gamma$. We have already shown that $C_\Gamma$ contains $S$ and each $U_\gamma$, for the converse we recall that $C_\gamma = (C_\gamma C_\gamma^*)^{1/2}U_\gamma$, and since $C_\gamma C_\gamma^*$ lies in $C^*(S)$ by Lemma 3.2 we see that $C_\gamma$ lies in the C*-algebra generated by $S$ and $U_\gamma$. Letting $\pi$ denote the quotient map $\pi : C_\Gamma \to C_\Gamma / K$, it follows that $C_\Gamma / K$ is generated as a C*-algebra by a copy of $C(T)$ and the unitaries $\pi(U_\gamma)$, and the map $\gamma \to \pi(U_{\gamma^{-1}})$ defines a unitary representation of $\Gamma$. Let $\alpha : \Gamma \to \text{Aut}(C(T))$ be given by
\[ \alpha_\gamma(f)(z) = f(\gamma^{-1}(z)) \]
Then by (3.6),
\[ (3.6) \quad \pi(U_{\gamma^{-1}})\pi(T_f)\pi(U_{\gamma^{-1}}^*) = \pi(T_{f \circ \gamma^{-1}}) = \pi(T_{\alpha_\gamma(f)}) \]
for all trigonometric polynomials \( f \), and hence for all \( f \in C(\mathbb{T}) \) by continuity. Thus, \( C^*_T/K \) is generated by \( C(\mathbb{T}) \) and a unitary representation of \( \Gamma \) satisfying the relation (3.0). Therefore there is a surjective *-homomorphism \( \rho : C(\mathbb{T}) \times \Gamma \to C^*_T/K \) satisfying

\[
\rho(f) = \pi(T_f), \quad \rho(u_\gamma) = \pi(U_{\gamma^{-1}})
\]

for all \( f \in C(\mathbb{T}) \) and all \( \gamma \in \Gamma \). Let \( J = \ker \rho \). The theorem will be proved once we show \( J = 0 \). To do this, we use Theorems 2.1 and 2.2. Indeed, it suffice to show that \( C^*_T \cap J = 0 \); then \( J \subset J_\lambda \) by Theorem 2.1, and \( J_\lambda = 0 \) by Theorem 2.2.

To see that \( C^*_T \cap J = 0 \), choose \( f \in C^*_T \cap J \). Then \( \pi(T_f) = \rho(f) = 0 \), which means that \( T_f \) is compact. But then \( f = 0 \), since nonzero Toeplitz operators are non-compact.

\[\square\]

3.2. Other Hilbert spaces. In this section we prove the analogue of Theorem 3.1 for composition operators acting on the reproducing kernel Hilbert spaces with kernel given by

\[ k(z, w) = \frac{1}{(1 - z \bar{w})^n} \]

for integers \( n \geq 2 \).

We let \( A^2_n \) denote the Hilbert function space on \( \mathbb{D} \) with kernel \( k(z, w) = (1 - z \bar{w})^{-n} \). For \( n \geq 2 \), this space consists of those analytic functions in \( \mathbb{D} \) for which

\[
\int_\mathbb{D} |f(w)|^2(1 - |w|^2)^{n-2}dA(w)
\]

is finite, and the square root of this quantity is a norm on \( A^2_n \) equivalent to the norm determined by the reproducing kernel \( k \) (they are equal for \( n = 2 \)). We fix \( n \geq 2 \) for the remainder of this section, and let \( T \) denote the operator of multiplication by \( z \) on \( A^2_n \). The operator \( T \) is a contractive weighted shift, and essentially normal. Moreover, if \( f \) is any bounded analytic function on \( \mathbb{D} \), then multiplication by \( f \) is bounded on \( A^2_n \) and is denoted \( M_f \). If \( \gamma \) is a Möbius transformation, then \( C_\gamma \) is bounded on \( A^2_n \). We let \( C^*_T \) denote the \( C^* \)-algebra generated by the operators \( \{C_\gamma : \gamma \in \Gamma\} \) acting on \( A^2_n \). In this section we prove:

**Theorem 3.5.** The \( C^* \)-algebra \( C^*_T \) contains \( C^*(T) \), and in particular the compact operators \( K(A^2_n) \). The map defined by \( \pi(C_\gamma) = |\gamma'|^{n/2}u_\gamma \) extends to a *-homomorphism from \( C^*_T \) onto \( C(\partial \mathbb{D}) \times \Gamma \), and there is an exact sequence

\[
0 \to K(A^2_n) \to C^*_T \to C(\partial \mathbb{D}) \times \Gamma \to 0
\]
We collect the following computations together in a single Lemma, the analogue for $A_n^2$ of the lemmas at the beginning of the previous section.

**Lemma 3.6.** For $\gamma \in \Gamma$, with $a = \gamma^{-1}(0)$ and $n$ fixed, let

$$f(z) = \frac{(1 - \pi z)^n}{(1 - |a|^2)^{n/2}}$$

Then $C_\gamma C_\gamma^* = M_fM_f^*$, and $M_\gamma C_\gamma = C_\gamma T$.

**Proof.** As in the Hardy space, for the first equation it suffices to prove that

$$\langle C_\gamma C_\gamma^* k_w, k_z \rangle = \langle T_f T_f^* k_w, k_z \rangle$$

for all $z,w$ in $\mathbb{D}$. This is the same as the identity

$$\left(\frac{1}{1 - \gamma(z)\gamma(w)}\right)^n = \frac{(1 - \overline{a}z)^n(1 - aw)^n}{(1 - |a|^2)^n(1 - zw)^n},$$

which is just (ref) raised to the power $n$. As before, the second equation follows immediately from the definitions. □

The proof of Theorem 3.5 follows the same lines as that for the Hardy space, except that it requires more work to show that $C_n^\Gamma$ contains $T$.

We first prove that $C_n^\Gamma$ is irreducible.

**Proposition 3.7.** The $C^*$-algebra $C_n^\Gamma$ is irreducible.

**Proof.** We claim that the operator $T^n$ lies in $C_n^\Gamma$; we first show how irreducibility follows from this and then prove the claim.

By [17], the only reducing subspaces of $T$ are direct sums of subspaces of the form

$$X_k = \text{span}\{z^{k+mn}|m = 0, 1, 2, \ldots\}$$

for $0 \leq k \leq n - 1$. Thus, since $T^n \in C_n^\Gamma$, these are the only possible reducing subspaces for $C_n^\Gamma$. Suppose, then, that $X$ is a nontrivial direct sum of distinct subspaces $X_k$ (i.e. $X \neq A_n^2$). Observe that, for each $k$, either $X$ contains $X_k$ as a summand, or $X$ is orthogonal to $X_k$. In the latter case (which we assume holds for some $k$), the $k$th Taylor coefficient of every function in $X$ vanishes.

It is now easy to see that $X$ cannot be reducing (or even invariant) for $C_n^\Gamma$. Indeed, if $X$ does not contain $X_0$ as a summand, then every function $f \in X$ vanishes at the origin, but if $\gamma \in \Gamma$ does not fix the origin then $C_\gamma f \notin X$. On the other hand, if $X$ contains the scalars, then consider the operator $F(\lambda) = (1 - \overline{\lambda}T)^n(1 - \lambda T^*)^n$, for $\lambda \in \Lambda$. Since $T^*$ annihilates the scalars, we have $F(\lambda)1 = (1 - \overline{\lambda}z)^n = p(z)$. 


But since $|\lambda| = 1$, the $k$th Taylor coefficient of $p$ does not vanish, so $p \notin X$. But since $F(\lambda)$ belongs to $C^n_\Gamma$, it follows that $X$ is not invariant for $C^n_\Gamma$ and hence $C^n_\Gamma$ is irreducible.

To prove that $T^n \in C^n_\Gamma$, we argue along the lines of Proposition 3.4, though the situation is somewhat more complicated. Using the same reproducing kernel argument as in that Proposition, we see that the operators

$$F(\lambda) = (1 - \bar{\lambda}T)^n(1 - \lambda T^*)^n \quad (3.9)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{(j+k)} \binom{n}{j} \binom{n}{k} \bar{\lambda}^j \lambda^k T^j T^*^k \quad (3.10)$$

$$= \sum_{d=0}^{n} \binom{n}{d}^2 T^d T^*^d + 2n \sum_{m=1}^{n} (-1)^m \bar{\lambda}^m \sum_{m \leq k \leq n} \binom{n}{k} \binom{n}{k-m} T^k T^*^{k-m} \quad (3.11)$$

$$= \sum_{d=0}^{n} \binom{n}{d}^2 T^d T^*^d + \sum_{m=1}^{n} \bar{\lambda}^m E_m + \lambda^m E_m^* \quad (3.12)$$

lie in $C^n_\Gamma$ for all $\lambda$ in the limit set $\Lambda$ of $\Gamma$. Here we have adopted the notation

$$E_m = (-1)^m \sum_{m \leq k \leq n} \binom{n}{k} \binom{n}{k-m} T^k T^*^{k-m}$$

Note in particular that $E_n = (-1)^n T^n$. Forming differences as $\lambda$ ranges over $\Lambda$ shows that $C^n_\Gamma$ contains all operators of the form

$$G(\lambda, \mu) = F(\lambda) - F(\mu) = \sum_{m=1}^{n} (\bar{\lambda}^m - \bar{\mu}^m) E_m + (\lambda^m - \mu^m) E_m^*$$

for all $\lambda, \mu \in \Lambda$. We wish to obtain $T^n$ as a linear combination of the $G(\lambda, \mu)$; since $E_n$ is a scalar multiple of $T^n$ it suffices to show that there exist $2n$ pairs $(\lambda_j, \mu_j) \in \Lambda \times \Lambda$ and $2n$ scalars $\alpha_j$ such that

$$E_n = \sum_{j=1}^{2n} \alpha_j G(\lambda_j, \mu_j)$$
Let $L$ be the $2n \times 2n$ matrix whose $j^{th}$ column is

\[
\begin{pmatrix}
\lambda_j - \mu_j \\
\lambda_j - \mu_j \\
\lambda_j^2 - \mu_j^2 \\
\vdots \\
\lambda_j^n - \mu_j^n \\
\lambda_j^n - \mu_j^n
\end{pmatrix}
\]

We must therefore solve the linear system

\[
L \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n-1} \\ \alpha_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}
\]

for which it suffices to show that the matrix $L$ is nonsingular for some choice of the $\lambda_j$ and $\mu_j$ in $\Lambda$. To do this, we fix $2n + 1$ distinct points $z_0, z_1, \ldots z_{2n}$ in $\Lambda$ and set $\lambda_j = z_0$ for all $j$ and $\mu_j = z_j$ for $j = 1, \ldots 2n$. The matrix $L$ then becomes the matrix whose $j^{th}$ column is

\[
\begin{pmatrix}
z_0 - z_j \\
z_0 - z_j \\
z_0^2 - z_j^2 \\
\vdots \\
z_0^n - z_j^n \\
z_0^n - z_j^n
\end{pmatrix}
\]

To prove that $L$ is nonsingular, we prove that its rows are linearly independent. To this end, let $c_j$ and $d_j$, $j = 1, \ldots n$ be scalars such that for each $k = 1, \ldots 2n$,

\[
(3.13) \quad \sum_{j=1}^{n} c_j(z_0^j - z_k^j) + \sum_{j=1}^{n} d_j(z_0^j - z_k^j) = 0
\]

To see that all of the $c_j$ and $d_j$ must be 0, consider the harmonic polynomial

\[
P(z,\overline{z}) = \sum_{j=1}^{n} c_j(z_0^j - z^j) + \sum_{j=1}^{n} d_j(\overline{z}_0^j - \overline{z}^j)
\]
By (3.13), $P$ has $2n + 1$ distinct zeroes on the unit circle, namely the points $z_0, \ldots, z_{2n}$. But this means that the rational function

$$Q(z) = \sum_{j=1}^{n} c_j (z_0^j - z^j) + \sum_{j=1}^{n} d_j (z_0^j - \frac{1}{z^j})$$

also has $2n + 1$ zeroes on the circle. But since the degree of $Q$ is at most $2n$, it follows that $Q$ must be the zero polynomial, and hence all the $c_j$ and $d_j$ are 0.

**Proposition 3.8.** $C^n_\Gamma$ contains $T$.

**Proof.** We have established that for each $\lambda$ in the limit set, the operators

$$(1 - \lambda T)^n (1 - \lambda T)^*$${}$^n$$

lie in $C^n_\Gamma$. Now, since $T$ is essentially normal, the difference

$$(1 - \lambda T)^n (1 - \lambda T)^* - [(1 - \lambda T)(1 - \lambda T)^*]^n$$

is compact. Given two positive operators which differ by a compact, their (unique) positive $n^{th}$ roots also differ by a compact. The positive square root of the left-hand term above is thus equal to

$$(1 - \lambda T)(1 - \lambda T)^* + K$$

for some compact operator $K$ (depending on $\lambda$), and lies in $C^n_\Gamma$. Forming linear combinations as in Proposition 3.4, we conclude that $T + K$ lies in $C^n_\Gamma$ for some compact $K$. Now, as $C^n_\Gamma$ is irreducible and contains a nonzero compact operator (namely, the self-commutator of $T + K$), it contains all the compacts, and hence $T$. □

We now prove the analogue of Theorem 3.1 for $C^n_\Gamma$:

**Theorem 3.9.** Let $\Gamma$ be a non-elementary Fuchsian group, and let $C^n_\Gamma$ denote the $C^*$-algebra generated by the set of composition operators on $A^2$ with symbols in $\Gamma$. Then there is an exact sequence

$$(3.14) \quad 0 \longrightarrow K \overset{i}{\longrightarrow} C^n_\Gamma \overset{\pi}{\longrightarrow} C(\mathbb{T}) \times \Gamma \longrightarrow 0$$

**Proof.** The proof is similar to that of Theorem 3.1; we define the unitary operators $U_\gamma$ in the same way (using the polar decomposition of $C_\gamma$), and check that he map that sends $\gamma$ to $U_{\gamma^{-1}}$ is a unitary representation of $\Gamma$ on $A^2_\Gamma$, modulo $K$. The computation to check the covariance condition is essentially the same. As $C^n_\Gamma$ is generated by the $T$ and the unitaries $U_\gamma$, the quotient $C^n_\Gamma/K$ is generated by a copy of $C(\partial \mathbb{D})$ (since $\partial \mathbb{D}$ is the essential spectrum of $T$) and a representation of $\Gamma$ which satisfy the covariance condition; the rest of the proof proceeds exactly as for $H^2$, up until the final step. At the final step in that proof, we have a Toeplitz operator $T_f$ which is compact; this implies that $f = 0$.
in the $H^2$ case but not for the spaces $A^2_n$. However the symbol of a compact Toeplitz operator on $A^2_n$ must vanish at the boundary of $\mathbb{D}$, so we may still conclude that $f = 0$ on $\partial \mathbb{D}$ and the proof is complete. $\square$

4. $K$-Homology of $C(\partial \mathbb{D}) \times \Gamma$

4.1. $Ext$ classes of $C_\Gamma$ and $C^n_\Gamma$. In this section we prove that the extensions of $C(\partial \mathbb{D}) \times \Gamma$ determined by $C_\Gamma$ and $C^n_\Gamma$ determine the same element of the group $\text{Ext}(C(\partial \mathbb{D}) \times \Gamma)$, and we prove that $C_\Gamma$ and $C^n_\Gamma$ are isomorphic as $C^*$-algebras. (In general neither of these statements implies the other.) We borrow several ideas from [Rørdam] concerning the classification of extensions by the associated six-term exact sequence in $K$-theory. The reader is referred to [14] for an introduction to $K$-theory and [3, Chapter 15] for the basic definitions and theorems concerning extensions of $C^*$-algebras and the $Ext$ group.

**Theorem 4.1.** The extensions of $C(\partial \mathbb{D}) \times \Gamma$ determined by $C_\Gamma$ and $C^n_\Gamma$ define the same element of the group $\text{Ext}(C(\partial \mathbb{D}) \times \Gamma)$.

We will show that the Busby maps of the extensions determined by $C_\Gamma$ and $C^n_\Gamma$ are strongly unitarily equivalent. To do this we introduce an integral operator $V \in B(A^2_n, A^2_{n-1})$ and prove the following lemma, which describes the properties of $V$ needed in the proof of the theorem.

**Lemma 4.2.** Consider the integral operator defined on analytic polynomials $f(w)$ by

$$(V f)(z) = \int_D \frac{f(w)}{(1 - wz)(1 - |w|^2)^{-1/2}} dA(w)$$

where $dA$ is normalized Lebesgue area measure on $\mathbb{D}$. Then:

1. For each $n \geq 0$, $V$ extends to a bounded diagonal operator from $A^2_{n+1}$ to $A^2_n$, and there exists a compact operator $K_n$ such that $V + K_n$ is unitary.
2. If $T_{n+1}$ and $T_n$ denote multiplication by $z$ on $A^2_{n+1}$ and $A^2_n$ respectively, then $VT_{n+1} - T_n V$ is compact.
3. If $g$ is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, then the integral operator

$$(V g f)(z) = \int_D \frac{f(w)g(w)}{1 - wz}(1 - |w|^2)^{-1/2} dA(w)$$

is bounded from $A^2_{n+1}$ to $A^2_n$, and is a compact perturbation of $VM^*_y$. 

Proof. If \( f(w) = w^k \), then the integrand is absolutely convergent for each \( z \in \mathbb{D} \), and we calculate

\[
(Vf)(z) = \int_{\mathbb{D}} \frac{w^k}{1 - wz} (1 - |w|^2)^{-1/2} \, dA(w) \tag{4.1}
\]
\[
= \sum_{j=0}^{\infty} \int_{\mathbb{D}} w^k|w|^j z^j (1 - |w|^2)^{-1/2} \, dA(w) \tag{4.2}
\]
\[
= \left( \int_{\mathbb{D}} |w|^{2k} (1 - |w|^2)^{-1/2} \, dA(w) \right) z^k \tag{4.3}
\]
\[
= \alpha_k z^k \tag{4.4}
\]

It is known that the sequence \((\alpha_k)\) defined by the above integral is asymptotically \((k + 1)^{-1/2}\). Each space \(A^2_n\) has an orthonormal basis of the form \(z^k\), where the sequence \(\beta_k\) is asymptotically \((k + 1)^{(n-1)/2}\).

To prove the last statement, first suppose \(g(w) = w\), and let \(z^k\) be an orthonormal basis for \(A^2_{n+1}\). Then a direct computation shows that \(V\) is a weighted backward shift with weight sequence \(\alpha_{j+1}/\alpha_j\), \(j = 0, 1, \ldots\). Thus with respect to this basis, \(V^*V\) is a weighted shift with weight sequence \(\alpha_{j+1}/\alpha_j\). Since \(\lim_{j \to \infty} \alpha_{j+1}/\alpha_j = 1\), it follows that \(V^*V\) is a compact perturbation of \(M_z^*\). By linearity, the lemma holds for polynomial \(g\). Finally, if \(p_n\) is a sequence of polynomials such that \(M_{p_n}\) converges to \(M_g\) in the operator norm (which is equal to the supremum norm of the symbol), then \(M_{p_n}\) converges to \(M_g\) in the essential norm, and \(p_n\) converges to \(g\) uniformly, so that \(V_{p_n} \to V_g\) in norm and the result follows.

Proof of Theorem 4.1

We will prove that the Busby maps of the extensions determined by \(C_\Gamma\) and \(C^n_\Gamma\) are strongly unitarily equivalent. It suffices to prove, for each fixed \(n\), the equivalence between \(C^n_\Gamma\) and \(C^{n+1}_\Gamma\). We first establish some notation: for a function \(g\) in the disk algebra, we let \(M_g\) denote the multiplication operator with symbol \(g\) acting on \(A^2_{n+1}\), and \(\tilde{M}_g\) the corresponding operator acting on \(A^2_n\). Similarly, for the composition operators \(C_\gamma\) and the unitaries \(U_\gamma\) on \(A^2_{n+1}\), a tilde denotes the corresponding operator on \(A^2_n\). Now, since each of the algebras \(C^n_\Gamma, C^{n+1}_\Gamma\) is generated by the operators \(M_z\) (resp. \(\tilde{M}_z\)) and the unitaries \(U_\gamma\) (resp.
on the respective Hilbert spaces, it suffices to produce a unitary $U$ (or indeed a compact perturbation of a unitary) from $A^2_{n+1}$ to $A^2_n$ such that the operators $UM_z - M_z U$ and $UU_\gamma - U_\gamma U$ are compact for all $\gamma \in \Gamma$. In fact we will prove that the operator $V$ of the previous lemma does the job.

To prove this, we will first calculate the operator $\tilde{C}_\gamma V C_{\gamma^{-1}}$. Written as an integral operator,

$$(\tilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(\gamma^{-1}(w))}{1 - \overline{w}\gamma(z)} (1 - |w|^2)^{-1/2} dA(w)$$

Applying the change of variables $w \to \gamma(w)$ to this integral, we obtain

$$(\tilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \gamma(w)\gamma(z)} (1 - |\gamma(w)|^2)^{-1/2} |\gamma'(w)|^2 dA(w)$$

A little algebra shows that

$$1 - |\gamma(w)|^2 = (1 - |w|^2)|\gamma'(w)|$$

Furthermore, multiplying and dividing the identity (3.3) by $1 - aw$, we obtain

$$\frac{1}{1 - \gamma(w)\gamma(z)} = \frac{1 - \overline{a}z}{(1 - \overline{aw})(1 - \overline{w}z)} |\gamma'(w)|^{-1}$$

Thus the integral may be transformed into

$$(\tilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \overline{w}z} \frac{1 - \overline{aw}}{1 - \overline{aw}z} |\gamma'(w)|^{1/2}(1 - |w|^2)^{-1/2} dA(w)$$

Since $\gamma'$ is analytic and non-vanishing in a neighborhood of the closed disk, there exists a function $\psi$ in the disk algebra such that $|\psi|^2 = |\gamma'|^{1/2}$ (choose $\psi$ to be a branch of $(\gamma')^{1/4}$). Thus we may rewrite this integral as

$$(4.5) \quad (\tilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \overline{w}z} \frac{1 - \overline{aw}}{1 - \overline{aw}z} \psi(w)\overline{\psi(w)} (1 - |w|^2)^{-1/2} dA(w)$$

Now, applying the lemma to $V\psi$ and using the fact that $V$ intertwines the multiplier algebras of $A^2_{n+1}$ and $A^2_n$ modulo the compacts, we have

$$(4.6) \quad (\tilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = M_{1-\pi_z} V\psi M_{(1-\pi_z)^{-1}\psi(z)}$$

Now, applying the lemma to $V\psi$ and using the fact that $V$ intertwines the multiplier algebras of $A^2_{n+1}$ and $A^2_n$ modulo the compacts, we have

$$(4.7) \quad \tilde{C}_\gamma V C_{\gamma^{-1}} = M_{1-\pi_z} V\psi M_{(1-\pi_z)^{-1}\psi(z)}$$

$$(4.8) \quad = M_{1-\pi_z} VM_\psi M_{(1-\pi_z)^{-1}} + K$$

$$(4.9) \quad = VM_\psi M_\psi + K$$
modulo the compacts. Multiplying on the right by $C_{\gamma-1}$ and on the left by $V^*$ we get

\[(4.10) \quad V^*\tilde{C}_\gamma V = M^*_\psi M_\psi C_{\gamma} + K\]

Since $\psi^2(z) = (1 - |a|^2)^{1/2}(1 - az)^{-1}$, the calculations in the proof of 3.1 show that

\[
(C_{\gamma}C^*_\gamma)^{-1/2} = M^*_\psi M_\psi + K \quad \text{and} \quad (\tilde{C}_\gamma\tilde{C}^*_\gamma)^{-1/2} = \tilde{M}^*_\psi \tilde{M}_\psi + K
\]

Using the fact that $V$ intertwines the $C^*$-algebras generated by $Mz$ and $\tilde{M}z$ modulo $K$, and that these algebras are commutative modulo $K$, we conclude after multiplying both sides of (4.10) on the left by $M^*_\psi M_\psi$ that

\[
V^*\tilde{U}_\gamma V = V^*\tilde{M}_\psi \tilde{M}^*_\psi \tilde{C}_\gamma V + K = M^*_\psi M_\psi C_{\gamma} + K = U_\gamma + K
\]

which completes the proof.

4.2. $C^*$-algebras isomorphic to $C_\Gamma$. In this section we prove that all unital extensions of $C(\partial \D) \times \Gamma$ which determine the same $Ext$ class as $C_\Gamma$ have isomorphic pull-backs, i.e. the middle term in the exact sequence is isomorphic to $C_\Gamma$. To prove this, we apply Voiculescu’s theorem to obtain a stable isomorphism and use an analysis of the $K_0$-classes of finite projections in $C_\Gamma$ to obtain a unital isomorphism.

We recall that if $A$ is a unital, nuclear $C^*$-algebra, an extension of $A$ by $K$ is called unital if the Busby map $\tau : A \rightarrow \mathcal{Q}(\mathcal{H})$ is unital. A trivial extension is called strongly unital $\tau$ lifts to a unital homomorphism $\rho : A \rightarrow \mathcal{B}(\mathcal{H})$. (Not all unital trivial extensions are strongly unital.)

We begin with the following lemma, which is a special case of [15, Proposition 5.1].

**Lemma 4.3.** Let $A$ be a unital $C^*$-algebra, and let $\tau_0 = \pi \circ \alpha$, $\tau_1 : A \rightarrow \mathcal{Q}(\mathcal{H})$ be unital extensions, with $\tau_0$ trivial. Then there is an isometry $v \in \mathcal{B}(\mathcal{H})$ such that $\alpha(1_A) = vv^*$. Set $\alpha'(a) = v^*\alpha(a)v$ and set $\tau'_0 = \pi \circ \alpha'$, so that $\tau'_0$ is a strongly unital trivial extension. Let

\[
e : \quad 0 \longrightarrow K \longrightarrow E \quad \psi \longrightarrow A \longrightarrow 0
\]

\[
e' : \quad 0 \longrightarrow K \longrightarrow E' \quad \psi' \longrightarrow A \longrightarrow 0
\]

be the extensions with Busby maps $\tau_1 \oplus \tau_0$, respectively, $\tau_1 \oplus \tau'_0$. Let $s_1, s_2$ be isometries in $\mathcal{B}(\mathcal{H})$ with $s_1 s_1^* + s_2 s_2^* = 1$ and set $w = s_1 s_1^* + s_2 v s_2^*$, $p = ww^*$. Then $\beta(b) = w^*bw$ and $\eta(x) = w^*xw$ define an isomorphism
Theorem 4.4. Let \( x \in \text{Ext}(C(\partial \mathbb{D}) \times \Gamma) \) denote the class of the extension
\[
0 \longrightarrow \mathcal{K} \longrightarrow C_{\Gamma} \longrightarrow C(\partial \mathbb{D}) \times \Gamma \longrightarrow 0.
\]
If \( e \in \text{Ext}(C(\partial \mathbb{D}) \times \Gamma) \) is a unital extension represented by
\[
0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow C(\partial \mathbb{D}) \times \Gamma \longrightarrow 0
\]
and \( e = x \), then \( E \cong C_{\Gamma} \) as \( C^* \)-algebras.

Proof. We first prove that the index homomorphism \( \partial \) from \( K_1(C(\partial \mathbb{D}) \times \Gamma) \) to \( \mathbb{Z} \) determined by the extension \( x \) is surjective (and hence so is the homomorphism determined by \( e \)). First, consider the function \( f(z) = z \); this function is a unitary in \( C(\partial \mathbb{D}) \times \Gamma \) lying in the canonical subalgebra isomorphic to \( C(\partial \mathbb{D}) \). By construction, this unitary lifts to the unilateral shift \( S \) on \( \mathcal{H} \), which is a Fredholm isometry of index \(-1\). Thus \( \partial[z]_1 = -1 \), and since \(-1\) generates \( \mathbb{Z} \), \( \partial \) is surjective.

Since \( \partial \) is surjective, the six-term exact sequence in \( K \)-theory associated to \( E \)
\[
0 \longrightarrow K_1(E) \longrightarrow K_1(C(\partial \mathbb{D}) \times \Gamma) \rightarrow \mathbb{Z} \]
(4.11)

splits into the two sequences
\[
0 \longrightarrow K_1(E) \longrightarrow K_1(C(\partial \mathbb{D}) \times \Gamma) \longrightarrow \mathbb{Z} \longrightarrow 0
\]
(4.12)

Thus \( K_0(E) \cong K_0(C(\partial \mathbb{D}) \times \Gamma) \), and as the isomorphism is induced by the quotient map which annihilates the compacts, it follows that \([p]_0 = 0\) for any finite projection \( p \in E \).

Now let \( \tau_1 \) and \( \tau_2 \) denote the Busby maps associated to \( C_{\Gamma} \) and \( E \) respectively. Since \( C_{\Gamma} \) and \( E \) determine the same element in \( \text{Ext}(C(\partial \mathbb{D}) \times \Gamma) \), there exist unital trivial extensions \( \sigma_1 \) and \( \sigma_2 \) such that \( \tau_1 \oplus \sigma_1 \cong_u \tau_2 \oplus \sigma_2 \). Let \( E_j \) denote the extension with Busby map \( \tau_j \oplus \sigma_j \); we have \( E_1 \cong E_2 \). Using \( \sigma_1 \) and \( \sigma_2 \), we may choose isometries \( v_1 \) and \( v_2 \) in \( \mathcal{B}(\mathcal{H}) \)
as in the lemma to obtain strongly unital extensions $\sigma'_1$ and $\sigma'_2$. Let $E'_j$ be the extension of $C(\partial \mathbb{D}) \times \Gamma$ with Busby map $\tau_j \circ \sigma'_j$, $j = 1, 2$. By Voiculescu’s theorem, $\tau_j \sim u \tau_j \circ \sigma'_j$, and it follows (since unitarily equivalent Busby maps determine isomorphic extensions) that $E'_1 \cong C_\Gamma$ and $E'_2 \cong E$. Applying the lemma to $E_1$ and $E_2$, we obtain projections $p_j \in E_j$ such that $p_j E_j p_j \cong E'_j$. We claim that $[p_j]_0 = [1_{E_j}]_0$; from this the theorem follows, since we then have $p_j E_j p_j \cong E_j$.

To prove the claim, observe that for the isometries $v_j$, the projections $1 - v_j v_j^*$ are finite, and hence so are the projections $1 - p_j$:

$$1 - p = 1 - uu^* = 1 - [s_1 s_1^* + s_2 v v^* s_2^*] = s_2 s_2^* - s_2 v v^* s_2^* = s_2 [1 - vv^*] s_2^*$$

which is finite. Thus $1 - p_j$ lies in $E_j$ (since $E_j$ contains the compacts) and $p_j \in E_j$ since $E_j$ is unital. Finally, $[1 - p_j]_0 = 0$, and $[p_j]_0 = [1_{E_j}]_0$.

\[\square\]

4.3. Emerson’s construction. When $\Gamma$ is cocompact, there is another extension of $C(\partial \mathbb{D}) \times \Gamma$ which was constructed by Emerson [5], motivated by work of Kaminker and Putnam [6]; we will show that the $\text{Ext}$-class of $C_\Gamma$ differs from the $\text{Ext}$-class of this extension.

The extension is constructed as follows: since $\Gamma$ is cocompact, we may identify $\mathbb{T}$ with the Gromov boundary $\partial \Gamma$. Let $f$ be a continuous function on $\partial \Gamma$, extend it arbitrarily to $\Gamma$ by the Tietze extension theorem, denote this extended function by $\tilde{f}$. Let $e_x, x \in \Gamma$ be the standard orthonormal basis for $\ell^2(\Gamma)$, and let $u_\gamma, \gamma \in \Gamma$ denote the unitary operator of left translation on $\Gamma$. Define a map $\tau : C(\partial \mathbb{D}) \times \Gamma \to \mathcal{Q}(\ell^2(\Gamma))$ by

$$\tau(f)e_x = \tilde{f}(x)e_x$$

and

$$\tau(\gamma)e_x = u_\gamma e_x = e_{\gamma x},$$

it can be shown that these expressions are well defined modulo the compact operators and determine a $\ast$-homomorphism from $C(\partial \mathbb{D}) \times \Gamma$ to the Calkin algebra of $\ell^2(\Gamma)$, i.e. an extension of $C(\partial \mathbb{D}) \times \Gamma$ by the compacts. Let $\pi$ denote the quotient map $\pi : \mathcal{B}(\ell^2(\Gamma)) \to \mathcal{Q}(\ell^2(\Gamma))$, and consider the pull-back $C^*$-algebra $E$:

\[
\begin{array}{ccc}
E & \xrightarrow{\tilde{\tau}} & C(\partial \mathbb{D}) \times \Gamma \\
\downarrow & & \downarrow \tau \\
\mathcal{B}(\ell^2(\Gamma)) & \xrightarrow{\pi} & \mathcal{Q}(\ell^2(\Gamma))
\end{array}
\]
We will show this extension is distinct from \( \mathcal{C}_\Gamma \) by showing that it induces a different homomorphism from \( K_1(C(\partial \mathbb{D}) \times \Gamma) \) to \( \mathbb{Z} \). Indeed, consider the class \([z]_1\) of the unitary \( f(z) = z \) in \( K_1(C(\partial \mathbb{D}) \times \Gamma)\); the extension belonging to \( \mathcal{C}_\Gamma \) sends this class to \(-1\). On the other hand, for the extension \( \tau \) we claim the function \( z \) lifts to a unitary in \( E \); it follows that \( \partial([z]_1) = 0 \) in this case. To verify the claim, note that \( z \) lifts to a diagonal operator on \( \ell^2(\Gamma) \), and if \((x_n)\) is a subsequence in \( \Gamma \) tending to the boundary point \( \lambda \in \mathbb{T} \), then \( \tilde{f}(x_n) \to f(\lambda) = \lambda \). We may thus choose all the \( \tilde{f}(x) \) nonzero, and by dividing \( \tilde{f}(x) \) by its modulus, we may assume they are all unimodular. Thus \( z \) lifts to the unitary \( \text{diag}(\tilde{f}(x)) \oplus z \) in \( E \).

### 4.4. Lott’s construction.

In this section we relate the extension \( \tau \) given by \( \mathcal{C}_\Gamma \) to an extension recently constructed by J. Lott [10]. In particular it will follow that (up to tensoring with \( \mathbb{Q} \)) \( \tau \) lies in the range of the Baum-Douglas-Taylor boundary map

\[
\partial : K^0(C(\overline{\mathbb{D}}) \times \Gamma, C(\partial \mathbb{D}) \times \Gamma) \to K^1(C(\partial \mathbb{D}) \times \Gamma)
\]

We first describe a construction of the extension \( \sigma_+ \) of [10]. Let \( \mathcal{D} \) denote the Hilbert space of analytic functions on \( \mathbb{D} \) with finite Dirichlet integral

\[
D(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z)
\]

equipped with the norm

\[
\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) = |f(0)|^2 + D(f)
\]

If \( f \) is represented in \( \mathbb{D} \) by the Taylor series \( \sum_{n=0}^\infty a_n z^n \), then this norm is given by

\[
\|f\|^2 = |a_0|^2 + \sum_{n=1}^\infty n|a_n|^2
\]

The operator of multiplication by \( z \) on \( \mathcal{D} \), denoted \( M_z \), is a weighted shift with weight sequence asymptotic to 1, and hence is unitarily equivalent to a compact perturbation of the unilateral shift on \( H^2 \). It follows that there is a \( * \)-homomorphism \( \rho : C(\partial \mathbb{D}) \to \mathbb{Q}(\mathcal{D}) \) with \( \rho(z) = \pi(M_z) \). Now, by changing variables one checks that if \( \gamma \) is a M"obius transformation, \( D(f \circ \gamma) = D(f) \). Let \( \mathcal{D}_0 \) denote the subspace of \( \mathcal{D} \) consisting of those functions which vanish at the origin. It then follows from the definition of the norm in \( \mathcal{D} \) that the operators

\[
u_\gamma(f)(z) = f(\gamma(z)) - f(\gamma(0))
\]
are unitary on $D_0$, and form a unitary representation of $\Gamma$. We extend this representation to all of $D$ by letting $\Gamma$ act trivially on the scalars. Moreover, it is simple to verify (by noting that $u_\gamma$ is a compact perturbation of the composition operator $C_\gamma$) that for all $\gamma \in \Gamma$,

$$u_\gamma M_z u_\gamma^* = M_{\gamma(z)}$$

modulo compact operators. Arguing as in the proof of Theorem 3.1, we conclude that the pair $(\rho(f), \pi(u_\gamma))$ determines an injective *-homomorphism from $C(\partial D) \times \Gamma$ to $Q(D)$, which is unitarily equivalent to the Busby map $\sigma_+$ of [10].

We may now state the main theorem of this subsection:

**Theorem 4.5.** The Busby maps $\tau$ and $\sigma_+$ are unitarily equivalent.

**Proof.** We first show that the Busby map $\tau : C_\Gamma \to Q(H^2)$ lifts to a completely positive map $\eta : C(\partial D) \times \Gamma \to B(H^2)$. Define a unitary representation of $\Gamma$ on $L^2(\partial D)$ by

$$U(\gamma^{-1}) = M_{|\gamma|^{1/2}} C_\gamma$$

Together with the usual representation of $C(\partial D)$ as multiplication operators on $L^2$, we obtain a covariant representation of $(\Gamma, \partial D)$ which in turn determines a representation $\rho : C(\partial D) \times \Gamma \to B(L^2)$. Letting $P$ denote the Riesz projection $P : L^2 \to H^2$, we next claim that the commutator $[\rho(a), P]$ is compact for all $a \in C(\partial D) \times \Gamma$, and so the pair $(\rho, P)$ is an abstract Toeplitz extension of $C(\partial D) \times \Gamma$ by $K$. To see this, it suffices to prove the compactness of the commutators

$$[\pi(f), P] \quad \text{and} \quad [U(\gamma), P].$$

Now, it is well known that $[M_f, P]$ is compact, and as $U(\gamma)$ has the form $M_{|\gamma|^{1/2}} C_{\gamma^{-1}}$ it suffices to check that $[C_{\gamma}, P]$ is compact. It is easily checked that this latter commutator is rank one. Indeed, the range of $P$ is invariant for $C_{\gamma}$, so $[C_{\gamma}, P] = PC_{\gamma}P^\perp$. If we expand $h \in L^2(\partial D)$ in a Fourier series

$$h \sim \sum_{n \in \mathbb{Z}} \hat{h}(n) e^{in\theta}$$

then a short calculation shows that

$$PC_{\gamma}P^\perp h \sim (\sum_{n<0} \hat{h}(n) \gamma(0)^{|n|}) \cdot 1$$

so $PC_{\gamma}P^\perp$ is rank one.

Thus, we have established that the completely positive map $\eta : C(\partial D) \times \Gamma \to B(H^2)$ given by

$$\eta(a) = P \rho(a) P$$
is a homomorphism modulo compacts, and the calculations in the proof of Theorem 3.1 show that the map
\[ \tau(a) = \pi(P\rho(a)P) \]
coincides with the Busby map associated to \( C_\Gamma \). Thus, \( \eta \) is a completely positive lifting of \( \tau \) as claimed.

With the lifting \( \eta \) in hand, to prove the unitary equivalence of \( \tau \) and \( \sigma_+ \) it will therefore suffice to exhibit an operator \( V : H^2 \to \mathcal{D} \) such that \( V \) is a compact perturbation of a unitary, and such that
\[ \pi(V\eta(a)V^*) = \sigma_+(a) \]
for all \( a \in C(\partial \mathbb{D}) \times \Gamma \). Since the crossed product is generated by the function \( f(z) = z \) and the formal symbols \([\gamma]\), it suffices to establish the above equality on these generators. In fact, we may use the operator \( V \) of Lemma 4.2. Since the map \( z^n \to n^{-1/2}z^n \) is essentially unitary from \( H^2 \) to \( \mathcal{D} \), the proofs of statements (1) and (2) of Lemma 4.2 are still valid. The conclusion of statement (3) holds provided the hypothesis on \( g \) is strengthened, by requiring that \( g \) be analytic in a neighborhood of \( \mathbb{D} \). Moreover, the arguments of Theorem 4.1 still apply, since the proof applies statement (3) of Lemma 4.2 only to the function \( \psi \), which is indeed analytic across the boundary of \( \mathbb{D} \). Thus, the arguments in the proof of Theorem 4.1 prove that \( V \) intertwines (modulo compacts) multiplication by \( z \) on \( H^2 \) and \( \mathcal{H} \), and also intertwines (modulo compacts) \( C_\gamma \) acting on \( \mathcal{H} \) with \( U_\gamma \) on \( H^2 \). Since \( u_\gamma \) is a compact perturbation of \( C_\gamma \) on \( \mathcal{H} \), it follows that \( V \) intertwines \( U_\gamma \) and \( u_\gamma \).

Now, the Busby map \( \sigma_+ \) takes the function \( z \) to the image of \( M_z \) in the Calkin algebra \( \mathcal{Q}(\mathcal{H}) \). Since \( \eta(z) = M_z \in \mathcal{B}(H^2) \), the intertwining property of \( V \) (modulo compacts) may be written as
\[ \pi(V\eta(z)V^*) = \sigma_+(z) \]
Similarly, since \( \eta \) applied to the formal symbol \([\gamma]\) (viewed as a generator of \( C(\partial \mathbb{D}) \times \Gamma \)) is \( U_\gamma \), the intertwining property for \( V \) with respect to \( U_\gamma \) and \( u_\gamma \) reads
\[ \pi(V\eta([\gamma])V^*) = \sigma_+([\gamma]) \]
Thus the equivalence of \( \tau \) and \( \sigma_+ \) on generators is established, which proves the theorem. \( \square \)

We conclude by observing that the covariant representation on \( L^2 \) described in the proof of the previous theorem gives rise to an equivariant \( KK_1 \)-cycle for \( C(\partial \mathbb{D}) \). Indeed, such a cycle consists of a triple \((U, \pi, F)\) where \((U, \pi)\) is a covariant representation on a Hilbert space...
$\mathcal{H}$ and $F$ is a bounded operator on $\mathcal{H}$ such that the operators

$$F^2 - I, \ F - F^*, \ [U(\gamma), F], \ \text{and} \ [\pi(f), F]$$

are compact for all $\gamma \in \Gamma$ and $f \in C(\partial \mathbb{D})$. The computations in the previous proof show that the triple $(U, \pi, 2P - I)$ satisfies all of these conditions, and essentially the same unitary equivalence argument shows that this cycle represents (up to a scalar multiple) the class of [10] Section 9.1.

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