On the spherical derivative of a rational function

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Abstract

For a rational function $f$ we consider the norm of the derivative with respect to the spherical metric and denote by $K(f)$ the supremum of this norm. We give estimates of this quantity $K(f)$ both for an individual function and for sequences of iterates.

Keywords: rational function, spherical derivative, characteristic exponent.

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A rational function is a holomorphic map from the Riemann sphere into itself. We equip the Riemann sphere with the usual spherical metric whose length and area elements are

$$ds = \frac{|dz|}{1 + |z|^2} \quad \text{and} \quad dA = \frac{dxdy}{(1 + |z|^2)^2}.$$ 

So the norm of the derivative with respect to the spherical metric is

$$\|f'(z)\| := |f'(z)|(1 + |z|^2)/(1 + |f(z)|^2).$$

In this paper we study the quantity $K(f) = \max_{C} \|f'\|$. d’Ambra and Gromov [1] proposed to study the rate of growth of sup $\|(f^n)'\|$ as $n \to \infty$ for the iterates $f^n$ of smooth maps of Riemannian manifolds, especially those maps in a given class for which this growth rate is the smallest possible.

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Such maps are called “slow”. Slow maps of an interval and slow Hamiltonian
diffeomorphisms of a 2-torus have been investigated in [3, 16] and [2].

Let \( f \) be a rational function of degree \( d \). As the map \( f : \mathbb{C} \rightarrow \mathbb{C} \) is \( d \)-to 1, we conclude that

\[
\int \int_{\mathbb{C}} ||f'||^2 dA = d(f) \int \int_{\mathbb{C}} dA.
\]

This implies that

\[
K(f) \geq \sqrt{d(f)} \tag{1}
\]

We ask how small can \( K(f) \) be for a function of given degree.

1. It is known that \( K(f) \geq 2 \) for all rational functions of degree at least 2.
   In fact this holds for all smooth maps of the sphere into itself which satisfy
   \( \text{deg}(f) \not\in \{0, 1, -1\} \) [10]. It is not known whether \( K(f) = 2 \) can hold for rational functions of degrees 3 or 4.

2. An interesting question is whether (1) is best possible in certain sense. We have

Theorem 1. There exists an absolute constant \( C \) with the following property. For every \( d \geq 2 \) there exists a rational function of degree exactly \( d \) such that

\[
K(f) \leq C \sqrt{d}.
\]

An analogous result was obtained by Gromov [10, Ch. 2D] for smooth maps of spheres of arbitrary dimension.

Littlewood [14] and Hayman [12] studied the quantity

\[
\phi(d) = \sup_{\text{deg } f = d} \sup_{R > 0} \frac{1}{R} \int \int_{|z| \leq R} \frac{|f'(z)|}{1 + |f(z)|^2} \, dxdy,
\]

where the sup is taken over all rational functions of degree \( d \). For polynomials \( \phi(d) \) was also studied in [9, 6, 13]. It is easy to see that \( \phi(d) \leq \pi \sqrt{d} \) and Hayman obtained \( \phi(d) \geq c_1 \sqrt{d} \) using a rational approximation of elliptic functions. Our Theorem 1 implies this with a more elementary proof. Indeed, by a change of the independent variable,

\[
\phi(d) = \sup_{\text{deg } f = d} \int \int_{|z| \leq 1} \frac{||f'|| \, dxdy}{1 + |z|^2}.
\]
Denote
\[ f^\# = |f'|/(1 + |f|^2). \] (2)

Let \( f \) be the function from Theorem 1. By rotating the sphere of the independent variable we may achieve that
\[ \int \int_{|z| \leq 1} (f^\#)^2 dxdy \geq \pi d/2, \]
because the spherical area of the image sphere is \( \pi \) and it is covered \( d \) times. Let \( M = \max_{|z| \leq 1} f^\# \). Theorem 1 implies that \( M \leq C \sqrt{d} \), so
\[
\phi(d) \geq \int \int_{|z| \leq 1} f^\# dxdy \geq M^{-1} \int \int_{|z| \leq 1} (f^\#)^2 dxdy
\geq C^{-1} d^{-1/2} \pi d/2 = \pi C \sqrt{d}/2.

Proof of Theorem 1. For every positive integer \( n \), consider the function
\[ f_n(z) = \prod_{k=-n}^{n} \tanh(z + 2k). \]

We first show that
\[
\frac{|f_n|}{1 + |f_n|^2} \leq C,
\] (3)
where \( C \) is independent of \( n \). We have
\[
|| \tanh z - 1 || \leq 4e^{-2x} < 1, \quad x = |\Re z| \geq 1.
\] (4)

Now fix \( z \in \mathbb{C} \) and let \( m \) be an integer such that \( |2m - \Re z| \leq 1 \). For \( |w - z| \leq 1/2 \) put
\[
g(w) = \begin{cases} \tanh(w - 2m), & |m| \leq n, \\ 1, & \text{otherwise} \end{cases}
\]

Then evidently
\[
\frac{|g'(w)|}{1 + |g(w)|^2} \leq \frac{|\tanh'(w)|}{1 + |\tanh(w)|^2} \leq C_0.
\] (5)

We write \( f_n = gh, f'_n = g'h + gh' \), and estimate \( h \) using (4):
\[
|h(w)| \leq \prod_{k=0}^{\infty} \left(1 + 4e^{-2(2k+1)/2}\right)^2 =: C_1.
\]
Now \( h \) is holomorphic in \( |w - z| \leq 1/2 \), so by Cauchy’s theorem,

\[
|h'(z)| \leq 4C_1.
\]

Next we estimate \( h(z) \) from below using (4) again:

\[
|h(z)| \geq \prod_{k=0}^{\infty} \left( 1 - 4e^{-2(2k+1)} \right)^2 =: C_2.
\]

Combining all these estimates we obtain

\[
\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq \frac{C_1|g'(z)| + 4C_1|g(z)|}{1 + C_2^2|g(z)|^2} \leq C_2^{-2}(C_1C_0 + 4C_1).
\]

This proves (3)

When \( |\Re z| \geq n + 1 \), we will obtain better estimates. Using

\[
|\tanh'(z)| = |\cosh(z)^2| \leq 16e^{-2|x|}, \quad |x| = |\Re z| \geq 1,
\]

we obtain for \( z > 2n \) and \( \xi = \Re z - 2n \):

\[
|f'_n(z - 2n)| \leq C_1 \sum_{k=0}^{\infty} |\cosh(\xi + 2k)|^2 \leq 16C_1 \sum_{k=0}^{\infty} e^{-2(\xi+2k)} = 16C_1C_3e^{-2\xi},
\]

which gives

\[
\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq Ce^{-2(|x| - n)} \leq Ce^{-2|x|/n}
\]

if \( |x| \geq n + 2 \). Combining this with (3) we obtain

\[
\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq Ce^{-2|x|/n}
\]

for all \( z \). As \( f_n \) has period \( \pi \), there exists a rational function \( R_n \) such that \( f_n(nz) = R_n(e^{2z}) \). This rational function has degree \( 2n^2 \) and the derivative satisfies \( \|R'_n\| \leq Cn \). This completes the proof of Theorem 1.

**Theorem 2.** There exists an absolute constant \( c > 1 \) with the property that

\[
K(f) \geq c\sqrt{d}
\]

for all rational functions of degree \( d \geq 2 \).
This can be considered as an analog of a result of Tsukamoto [18]. He studied spherical derivatives (2) of meromorphic functions $F : \Delta \to \mathbb{C}$, where $\Delta$ is the unit disc with the Euclidean metric and proved that there exists an absolute constant $c_1 < 1$ with the property that $\omega(F(\Delta)) \leq c_1 \pi$ for all meromorphic functions $F$ satisfying $F^\# \leq 1$, where

$$\omega(F(\Delta)) = \int_{|z| < 1} (F^\#(z))^2 \, dx \, dy.$$

We derive Theorem 2 from this result. In fact we show that Theorem 2 holds with $c = 1/\sqrt{c_1}$.

**Proof of Theorem 2.** Proving by contradiction, we suppose that there exists a sequence $f_m$ of rational functions of degrees $m$, such that

$$K(f_m)/\sqrt{m} \to b < 1/\sqrt{c_1}. \quad (6)$$

Let $\omega$ be the spherical area measure, so that

$$\int_C d\omega = \pi,$$

and $\omega_m = f_m^* \omega$ the pull back of $\omega$ by $f_m$. Then

$$\int_C d\omega_m = \pi m. \quad (7)$$

It is easy to see that we can find discs $D_m = D(a_m, r_m) \subset \overline{C}$ (with respect to the spherical metric) of radii $r_m$ such that

$$\int_{D_m} d\omega = \pi/(mb^2) \quad \text{and} \quad \int_{D_m} d\omega_m \geq \pi/b^2. \quad (8)$$

To show this, choose $r_m$ so that the first equation is satisfied and then integrate

$$F(a) := \int_{D(a,r_m)} d\omega_m(a)$$

with respect to $a$. Evidently $r_m \sim 1/b \sqrt{\pi m}, m \to \infty$.

Let $a'_m$ be the point diametrically opposite to $a_m$, and let $\phi_m : \mathbb{C} \to \mathbb{C}\{a'_m\}$ be the conformal map (inverse to a stereographic projection) such that $\phi_m(0) = a, \phi_m(\Delta) = D_m$, then

$$\phi_m^\#(z) \leq r_m(1 + o(1)), m \to \infty. \quad (9)$$
uniformly with respect to $z$. Then $F_m = f_n \circ \phi_m$ is a normal family. Let $F = \lim F_m$. From (6) follows that then $F^\# \leq 1$, but the area of $F(\Delta) \geq \pi/b^2$ in view of (8), contradicting the result of Tsukamoto.

Theorems 1 and 2 have analogs for maps $\mathbf{P}^1 \to \mathbf{P}^n$ which are stated and proved in the same way as for $n = 1$, using the Fubini–Study metric for the norm of the derivative. Constants $C$ and $c$ will of course depend on $n$.

Now we consider dynamical questions. By $f^n$ we denote the $n$-th iterate, and our standing assumption is that $d(f) \geq 2$. We define

$$k_\infty(f) = \lim_{n \to \infty} \frac{1}{n} \log K(f^n).$$

The limit always exists because the sequence $a_n = \log K(f^n)$ is subadditive, $a_{m+n} \leq a_m + a_n$ and for every such positive sequence the limit $\lim_{n \to \infty} a_n/n$ exists and is equal to $\inf a_n/n$ (see, for example, Lemma 1.16 in [10]).

It follows that $k_\infty(f^m) = mk_\infty(f)$.

Notice that $k_\infty$ is independent of the choice of a smooth Riemannian metric on the sphere, and is invariant under conjugation by conformal automorphisms. Obviously, $k_\infty(f) \leq \log K(f)$.

3. What is the smallest value of $k_\infty(f)$ for rational functions of given degree?

The trivial lower estimate of $K(f)$ gives

$$k_\infty(f) \geq (1/2) \log d(f).$$

We will see that equality never happens, and that the Latté functions are not extremal for minimizing $k_\infty$. For functions $f_d$ of degree $d$ from Theorem 1 we have

$$k_\infty(f_d)/\log d \to 1/2, \quad d \to \infty,$$

so the $(1/2)$ in (10) cannot be replaced with a larger constant.

In [7] these quantities were studied for polynomials, in particular, the inequality $k_\infty(f) \geq \log d(f)$ was established for polynomials, with equality only if $f$ is conjugate to $z^d$.

Let us consider a slightly different quantity, the maximum characteristic exponent

$$\chi_m(f) = \sup_z \limsup_{n \to \infty} \frac{1}{n} \log \| (f^n)'(z) \| \leq k_\infty(f).$$

\[11\]
The difference in the definitions of $k_\infty$ and $\chi_m$ is in the order of $\max_z$ and $\lim_{n \to \infty}$. Przytycki proved in 1998 (reproduced in [17]) that the same quantity $\chi_m(f)$ can be obtained by taking the sup over periodic points $z$ of $f$, in which case the lim sup in (11) can be of course replaced by the ordinary limit. Moreover, he proved the following:

**Theorem P.** For every $\epsilon > 0$ there exists a periodic point $z$ such that

$$\frac{1}{m} \log \| (f^m)'(z) \| \geq k_\infty - \epsilon,$$

(12)

where $m$ is a period of $z$.

In particular, $k_\infty = \chi_m$, and one can replace $\sup_{z \in C}$ in (11) by sup over all periodic points.

4. Let $\chi_a(f)$ be the average value of

$$\log \| (f^n)'(z) \|$$

over the measure $\mu$ of maximal entropy. According to the multiplicative ergodic theorem, we have

$$\chi_a(f) = \lim_{n \to \infty} \frac{1}{n} \log \| (f^n)'(z) \|$$

almost everywhere with respect to $\mu$.

**Theorem 4.** $\chi_a(f) \geq (1/2) \log d$, this is best possible and equality holds only for Lattés examples.

**Proof.** The estimate follows from the formula

$$\chi_a(f) = h(f) / \dim \mu,$$

where $h(f) = \log d$ is the topological entropy, and $\dim \mu$ is the Hausdorff dimension of the maximal measure, see, for example [8] and references therein. Obviously $\dim \mu \leq 2$ so we obtain our inequality. On the other hand, a theorem of A. Zdunik [20] says that $\dim \mu = 2$ can happen only for Lattés examples. This completes the proof of the theorem.

Now we consider the Lattés functions [15].

**Proposition 1.** If $L$ is a Lattés function then $\chi_m(L) \geq \log d(L)$. 

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Proof. A Lattés function can be defined by a functional equation

\[ F(\lambda z) = L \circ F(z), \quad d(L) = \lambda^2, \]

where \( F \) is an elliptic function with a critical point at 0. Assuming without loss of generality that \( F(0) = 0 \) we conclude that 0 is a fixed point of \( L \). Now the derivative at this fixed point is

\[ \lambda \lim_{z \to 0} \frac{F'(\lambda z)}{F'(z)} = \lambda^2 = d(L), \]

from which follows that \( \chi_m(L) \geq d(L) \).

To summarize, we have 4 quantities satisfying inequalities

\[ \frac{1}{2} \log d \leq \chi_a(f) \leq \chi_m(f) = k_\infty(f) \leq \log K(f). \]  \hfill (13)

For Latteés functions we have

\[ (1/2) \log d = \chi_a(L) < \chi_m(L) = \log d. \]

In the first inequality equality holds only for Lattés functions. For functions \( f_d \) constructed in Theorem 1, we have \( \log K(f_d) \leq (1/2) \log d + O(1) \).

Which of the rest of these inequalities \hfill (13) are strict and what are the conditions of equality? According to a private communication of Przytycki, the method of Zdunik can be used to show that \( \chi_a(f) = \chi_m(f) \) is only possible when \( f \) is conjugate to \( z^{\pm d} \) in which case both quantities are equal to \( \log d \).

Proposition. Let \( M = \{ z : \|f'(z)\| = K(f) \} \). Then \( k_\infty = \log K(f) \) if and only if \( M \) contains a cycle.

Proof. If \( z \) is a point whose trajectory is in \( M \), then \( \|f^m(z)\| = \|f'(z)\|^m \) so \( k_\infty(f) = \log K(f) \).

Suppose now that \( k_\infty = \log K(f) \). We claim that

\[ \bigcap_{j=0}^{\infty} f^j(M) \neq \emptyset. \]

Indeed, otherwise for some \( m \) we have

\[ \bigcap_{j=0}^{m} f^j(M) = \emptyset, \]
and if this holds, then
\[ mk_\infty(f) = k_\infty(f^m) \leq \log K(f^m) = \log \max_z \prod_{j=0}^{m-1} \| f'(f^j(z)) \| < m \log K(f), \]
contrary to our assumption. This proves the claim.

The set \( M_\infty \) is forward invariant. All sets
\[ \bigcap_{j=0}^{m} f^j(M) \]
are real algebraic subsets of \( \mathbb{C} \), and thus the set \( M_\infty \) is also real algebraic. If \( M_\infty \) is finite it must contain a cycle.

Suppose that \( M \) is infinite. Let \( M_1, \ldots, M_q \) be the list of connected components of \( M \). Then there must be a cycle of components. Let \( M_1, \ldots, M_k \) be this cycle, so that \( f(M_j) \subset M_{j+1}, f(M_k) = M_1 \). If any of these components contains singular points, then all singular points in these components form a finite invariant set, and this set must contain a cycle. If all \( M_j, 1 \leq j \leq k \) are smooth, then \( f^k : M_1 \to M_1 \) is an expanding map of a circle, so it must have a fixed point. Again we obtain a cycle in \( M \). This completes the proof.

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