Abstract
For a half-translation surface $(S, q)$, the associated saddle connection complex $A(S, q)$ is the simplicial complex where vertices are the saddle connections on $(S, q)$, with simplices spanned by sets of pairwise disjoint saddle connections. This complex can be naturally regarded as an induced subcomplex of the arc complex. We prove that any simplicial isomorphism $\phi : A(S, q) \to A(S', q')$ between saddle connection complexes is induced by an affine diffeomorphism $F : (S, q) \to (S', q')$. In particular, this shows that the saddle connection complex is a complete invariant of affine equivalence classes of half-translation surfaces. Throughout our proof, we develop several combinatorial criteria of independent interest for detecting various geometric objects on a half-translation surface.

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1 | INTRODUCTION

For a closed topological surface $S$ of finite type with a non-empty finite set $Z$ of marked points, the associated arc complex $A(S, Z)$ has as vertices the essential arcs on $(S, Z)$ (considered up to proper homotopy), with simplices spanned by sets of arcs that can be realised pairwise disjointly. The (extended) mapping class group $\text{Mod}(S, Z)$ acts naturally on $A(S, Z)$ by simplicial isomorphisms. In [16], Irmak and McCarthy proved a combinatorial rigidity theorem for arc complexes: any simplicial isomorphism $\phi : A(S, Z) \to A(S', Z')$ between arc complexes is induced by a homeomorphism $F : (S, Z) \to (S', Z')$ between the associated surfaces. In particular, this implies that, except for finitely many cases, the automorphism group of $A(S, Z)$ is isomorphic to $\text{Mod}(S, Z)$.
In this paper, we explore an analogous combinatorial rigidity phenomenon for a complex associated to half-translation surfaces. Let \((S, q)\) be a half-translation surface, with \(Z\) taken to be the set of singularities.

The **saddle connection complex** \(\mathcal{A}(S, q)\) is the simplicial complex where vertices are saddle connections on \((S, q)\), with simplices spanned by sets of pairwise disjoint saddle connections. In particular, \(\mathcal{A}(S, q)\) can be regarded as the induced subcomplex of \(\mathcal{A}(S, Z)\) spanned by the arcs that can be realised as saddle connections on \((S, q)\). Any affine diffeomorphism between half-translation surfaces naturally induces a simplicial isomorphism between the respective saddle connection complexes. In particular, the **affine diffeomorphism group** \(\text{Aff}(S, q)\) acts naturally on \(\mathcal{A}(S, q)\) by simplicial automorphisms.

The main result of this paper is the converse, that is, any simplicial isomorphism between saddle connection complexes arises from an affine diffeomorphism. This gives a rigidity theorem analogous to that of Irmak–McCarthy for the arc complex.

**Theorem 1** (Rigidity of the saddle connection complex). Let \((S, q)\) and \((S', q')\) be half-translation surfaces, neither of which are flat tori with exactly one removable singularity. Suppose \(\phi : \mathcal{A}(S, q) \to \mathcal{A}(S', q')\) is a simplicial isomorphism. Then there exists a unique affine diffeomorphism \(F : (S, q) \to (S', q')\) inducing \(\phi\).

In the case where at least one of \((S, q)\) or \((S', q')\) is a flat torus with exactly one removable singularity, and \(\phi : \mathcal{A}(S, q) \to \mathcal{A}(S', q')\) is a simplicial isomorphism, there are exactly two affine diffeomorphisms \(F_1, F_2 : (S, q) \to (S', q')\) inducing \(\phi\). In particular, both \((S, q)\) and \((S', q')\) are flat tori, and the maps \(F_1\) and \(F_2\) differ by a hyperelliptic involution.

Consequently, the natural map \(\text{Aff}(S, q) \to \text{Aut}(\mathcal{A}(S, q))\) is an isomorphism (unless \((S, q)\) is a flat torus with one removable singularity, in which case the kernel is the order-2 subgroup generated by the hyperelliptic involution). In particular, if \([F] \in \text{Mod}(S, Z)\) is a mapping class that preserves the subcomplex \(\mathcal{A}(S, q) \subset \mathcal{A}(S, Z)\) (setwise), then \([F]\) has an affine representative \(F : (S, q) \to (S, q)\).

Theorem 1 shows that the combinatorial structure of the saddle connection complex completely governs the half-translation structure on the surface up to affine equivalence. In particular, the saddle connection complex is a complete invariant of \(\text{SL}(2, \mathbb{R})\)-orbits in the moduli space of quadratic differentials, taken over all surfaces. Consequently, there are uncountably many isomorphism classes of saddle connection complexes. Finding explicit relations between the combinatorial properties of saddle connection complexes and the dynamical aspects of half-translation surfaces and their \(\text{SL}(2, \mathbb{R})\)-orbit closures would provide an interesting direction for future research.

**Overview of the paper**

In Section 2, we review some results about arc complexes and flip graphs. While not directly necessary for proving our main theorem, we shall give proofs of some standard properties concerning arc complexes to provide context for the analogous results for the saddle connection complex. These also serve to point out the differences between the topological and the Euclidean settings. In Section 3, we give some basic definitions and properties of half-translation surfaces and saddle connection complexes. We then prove some elementary results regarding links in Section 4, followed by a detailed examination of simplices with infinite links in Section 5. In Sections 6, 7, and 8, we develop some technical results in order to establish the key combinatorial criteria: the Tri-
angle Test (Section 9) and the Orientation Test (Section 10). Finally, we give a proof of Theorem 1 in Section 11 using the following strategy.

(i) **Triangle Test (Corollary 9.5):** There is a combinatorial criterion that can detect the simplices in \( \mathcal{A}(S, q) \) that bound triangles on \( (S, q) \).

(ii) **Orientation Test (Proposition 10.1):** There is a combinatorial criterion that can detect whether two given oriented triangles on \( (S, q) \) are consistently oriented.

(iii) Using the previous two tests, we recover the gluing pattern of a triangulation and hence the underlying surface \( (S, \mathcal{Z}) \). By construction, we show that any simplicial isomorphism \( \phi : \mathcal{A}(S, q) \rightarrow \mathcal{A}(S', q') \) is induced by a piecewise affine diffeomorphism \( F : (S, q) \rightarrow (S', q') \).

(iv) **Cylinder Test (Corollary 5.5):** There is a combinatorial criterion that can detect the set of cylinder curves on \( (S, q) \).

(v) Finally, we apply the Cylinder Rigidity Theorem (Theorem 5) of Duchin, Leininger, and Rafi [8] to deduce that \( F \) is an affine diffeomorphism.

Throughout this paper, we develop several other combinatorial criteria to detect various features of a half-translation surface via its saddle connection complex. We expect these tools to be of independent interest, with potential applications to future work.

As readers might have background in combinatorial complexes or in (half-)translation surfaces but not both, we give some basics for both topics. These parts can be skipped by experts in the respective topics. A list of notation is given in Section B.

**Combinatorial rigidity results in the topological setting**

The combinatorics of simplicial complexes associated to a topological surface play an important role in several groundbreaking papers on the algebraic properties of the mapping class group in the 1980s. For instance, the arc complex was defined by Harer in his study of the cohomology of the mapping class group (see [10] and [11]); the Hatcher–Thurston complex was used to find the first explicit presentation of the mapping class group [13]; the curve complex was introduced by Harvey in order to study the action of the mapping class group on the boundary of Teichmüller space [12].

Rigidity phenomena for combinatorial complexes associated to surfaces were first studied by Ivanov in the case of the curve complex [17, 18]. Ivanov showed that, when the surface has genus at least 2, any automorphism of the curve complex is induced by a homeomorphism of the surface. He applied this result to study the isometry group of Teichmüller space, giving a new proof of a celebrated result of Royden [33]. Korkmaz then extended Ivanov’s result to the remaining low-genus cases [21]. In [23], Luo gave a new proof of the curve complex rigidity theorem, showing also that the isomorphism type of the curve complex is a complete invariant of the topological type of the surface.

Subsequently, *Ivanov-style rigidity* has been established for other complexes such as the arc complex (Irmak–McCarthy [16], Disarlo [4]), the pants complex (Margalit [24]), the Hatcher–Thurston complex (Irmak–Korkmaz [15]), the flip graph (Korkmaz–Papadopoulos [22]), the polygonalisation complex (Bell–Disarlo–Tang [2]), and many other complexes built from topological objects on surfaces (for a survey, see McCarthy–Papadopoulos [28]). In response to the multitude of rigidity results, Ivanov [19] formulated a *metaconjecture*, stating that every object naturally associated to a surface and having a sufficiently rich structure has the mapping class group as its group of automorphisms. At present, Ivanov’s metaconjecture remains unresolved, however, Brendle and Margalit in [3] have recently verified it for a large class of complexes by proving general results related to the complex of domains, introduced by McCarthy and Papadopoulos [28].
Relations to other work

In light of Ivanov’s meta-conjecture, Theorem 1 demonstrates that the saddle connection complex also enjoys combinatorial rigidity properties. In particular, its automorphism group is the affine diffeomorphism group of the half-translation structure on the surface. To the best of our knowledge, this is the first combinatorial complex known to exhibit Ivanov-style rigidity for a geometric structure on surfaces. (Here, we are using only the isomorphism type of $A(S, q)$, with no a priori information regarding the underlying surface $(S, \mathcal{Z})$ or arc complex $A(S, \mathcal{Z})$.) It would be interesting to see if other combinatorial complexes associated to half-translation structures also exhibit analogous rigidity properties.

An independent proof of Theorem 1 in the special case of translation surfaces was given simultaneously by Huiping Pan [32]. The overall strategy of Pan’s proof is similar to ours. He gives a direct proof that any simplicial isomorphism $\phi : A(S, q) \to A(S', q')$ must send triangles on $(S, q)$ to triangles on $(S', q')$, and that consistently oriented triangles remain consistently oriented. This shows that $\phi$ is induced by a piecewise affine homeomorphism. He then applies a standard argument of Duchin, Leininger, and Rafi [8] (see also [30]) to prove that the resulting homeomorphism is affine.

There have been several other recent works on combinatorial complexes associated to (half-)translation surfaces. Minsky and Taylor define the saddle connection complex in [29], as well as a subcomplex of the arc complex defined using Veering triangulations, and relate their geometry to that of an associated fibred hyperbolic 3-manifold. Nguyen defines a graph of (degenerate) cylinders for genus 2 translation surfaces as a subgraph of the curve complex, and proves that any mapping class that stabilises it must have an affine representative [30]. He also defines a graph of periodic directions in [31], and uses this to give an algorithm to compute a coarse fundamental domain for the associated Veech group. In [36], Tang and Webb consider the multi-arc graph and filling multi-arc graph associated to a half-translation surface, and use these to obtain a bounded geodesic image theorem for a ‘straightening operation’ (see Section 3.1) defined on the curve complex.

Another rigidity result for singular Euclidean surfaces was recently established by Duchin, Erlandsson, Leininger, and Sadanand in [7]. They show that the ‘bounce spectrum’ associated to a general polygonal billiard table determines the shape of the table up to affine equivalence (for right-angled tables) or up to similarity. Their techniques deal with flat surfaces with arbitrary cone angles, and so it would be interesting to see whether these methods can be adapted to obtain combinatorial rigidity results for complexes associated to this broader class of geometric structures.

2 | SURFACES AND ARCS

In this section, we review some background regarding topological surfaces and the arc complex, and also give proofs of some standard results. While they are not strictly necessary for the proof of Theorem 1, these results will provide context for the analogous results for the saddle connection complex.

Let $(S, \mathcal{Z})$ denote a compact surface $S$, possibly with boundary, equipped with a finite, non-empty set $\mathcal{Z} \subset S$ of marked points such that each boundary component contains at least one marked point. A surface homeomorphic to a closed disc with $n$ marked points on its boundary and having no interior marked points is called an $n$-gon, or simply a polygon. In particular, we call
an $n$-gon a monogon, bigon, triangle, or quadrilateral in the cases when $n = 1, 2, 3, 4$, respectively. A surface homeomorphic to a closed annulus with $m$ marked points on one boundary component, $n$ marked points on the other boundary component, and with no interior marked points is called an $(m, n)$-annulus.

2.1 Arc complexes

A simplicial complex is called flag if for every complete subgraph in the 1-skeleton, there exists a simplex in the complex which has the subgraph as its 1-skeleton. All simplicial complexes considered in this paper shall be flag, and so we may instead work with their 1-skeleta whenever convenient. Distances between vertices of the complex shall always be measured using the combinatorial metric in the 1-skeleton. Given a simplicial complex $\mathcal{K}$, we shall write $\sigma \in \mathcal{K}$ to refer to both a simplex of $\mathcal{K}$, and the vertex set of that simplex. We write $\# \mathcal{K}$ for the number of vertices of $\mathcal{K}$. The link $\text{lk}_{\mathcal{K}}(\sigma)$ of a simplex $\sigma \in \mathcal{K}$ is the subcomplex of $\mathcal{K}$ consisting of all simplices $\sigma' \in \mathcal{K}$ such that $\sigma \cap \sigma' = \emptyset$ and $\sigma \cup \sigma'$ is a simplex.

**Definition 2.1** (Essential arc). An arc on $(S, \mathcal{Z})$ is a map $\alpha : [0, 1] \to S$ such that $\{\alpha(0), \alpha(1)\} \subseteq \mathcal{Z}$ and the restriction $\alpha|_{(0, 1)}$ is an embedding into $S \setminus \mathcal{Z}$. An arc whose image lies in $\partial S$ is called a boundary arc. An arc is called essential if it is not homotopic (relative to $\mathcal{Z}$) to a point or a boundary arc.

All arcs considered in this paper will be essential unless stated otherwise. We usually consider arcs up to reversal of orientation and proper isotopy. Furthermore, we shall usually consider arcs up to proper homotopy, that is, homotopy relative to $\mathcal{Z}$. Two arcs $\alpha, \beta$ are said to be disjoint if they have disjoint interiors; otherwise we say they intersect or cross and write $\alpha \pitchfork \beta$.

A multi-arc on $(S, \mathcal{Z})$ is a non-empty set of pairwise disjoint and non-homotopic arcs (see Figure 2.1 for an example), and a triangulation of $(S, \mathcal{Z})$ is a maximal multi-arc. Indeed, cutting $S$ along a maximal multi-arc as described in the next section decomposes it into a finite union of triangles.

**Definition 2.2** (Arc complex). The arc complex $\mathcal{A}(S, \mathcal{Z})$ is the flag simplicial complex whose vertices are the essential arcs on $(S, \mathcal{Z})$ (considered up to homotopy), with simplices corresponding to multi-arcs on $(S, \mathcal{Z})$.

Every simplex in $\mathcal{A}(S, \mathcal{Z})$ can be extended to a maximal one, and every maximal simplex in $\mathcal{A}(S, \mathcal{Z})$ corresponds to a triangulation of $(S, \mathcal{Z})$. Moreover, $\mathcal{A}(S, \mathcal{Z})$ is finite-dimensional and every maximal simplex has the same dimension. In the case where $S$ is closed, connected, orientable, and has genus $g \geq 2$ then $\mathcal{A}(S, \mathcal{Z})$ is connected, locally infinite, has dimension $\kappa(S, \mathcal{Z}) = \# \mathcal{Z}$.
6g + 3|Z| − 7, and has infinite diameter. Masur and Schleimer [27] proved that \( A(S, Z) \) is Gromov hyperbolic, with constants depending on the topology of \((S, Z)\); a new proof yielding a hyperbolicity constant independent of the topology was later given in [14].

The results of [1, 4, 16] can be combined (as done in [2, Theorem 2.2]) to deduce the following:

**Theorem 2** (Rigidity of arccomplexes). Assume \((S, Z)\) and \((S', Z')\) are surfaces such that the associated arccomplexes are non-empty. Suppose \( \phi : A(S, Z) \to A(S', Z') \) is a simplicial isomorphism. Then there exists a homeomorphism \( F : (S, Z) \to (S', Z') \) inducing \( \phi \).

Here the homeomorphism must act bijectively on the sets of marked points, and is allowed to reverse orientation.

**Definition 2.3** (Mapping class group \( \text{Mod}(S, Z) \)). Let \( \text{Homeo}(S, Z) \) denote the group of homeomorphisms of \( S \) that fix \( Z \) setwise. The extended mapping class group \( \text{Mod}(S, Z) \) is the group of the isotopy classes of elements of \( \text{Homeo}(S, Z) \), where isotopies are required to fix \( Z \) pointwise. An element of \( \text{Mod}(S, Z) \) is called a mapping class. Mapping classes are allowed to reverse the orientation of \( S \).

Since mapping classes preserve the property of disjointness for arcs, it follows that \( \text{Mod}(S, Z) \) acts naturally on \( A(S, Z) \) by simplicial automorphisms.

In the remainder of this article, we restrict to the case where \( S \) is a closed surface.

**Theorem 3** (Automorphisms of the arc complex [16]). Assume \((S, Z)\) is not a sphere with at most 3 marked points, nor a torus with one marked point. Then the natural map \( \text{Mod}(S, Z) \to \text{Aut}(A(S, Z)) \) is an isomorphism.

### 2.1.1 Links, cutting, and gluing

Let \( \sigma \in A(S, Z) \) be a multi-arc. Then the link of \( \sigma \) is the induced subcomplex \( \text{lk}_{A(S, Z)}(\sigma) \) of \( A(S, Z) \) with vertex set

\[
\{ \alpha \in A(S, Z) \mid \alpha \notin \sigma \text{ and } \alpha \text{ is disjoint from all arcs in } \sigma \}.
\]

Define the surface \((S - \sigma, Z)\) obtained by cutting \( S \) along the multi-arc \( \sigma \in A(S, Z) \) as follows. Equip \( S \) with a Riemannian metric, and realise \( \sigma \) as a set of pairwise disjoint smooth arcs. For each complementary component \( R'_i \) of \( S\setminus\sigma \), let \( R_i \) be its metric completion with respect to the induced metric. The inclusion \( \text{int}(R_i) = R'_i \hookrightarrow S \) extends to a continuous map \( t_i : R_i \to S \). Then \( Z_i = t_i^{-1}(Z) \) is a non-empty set of marked points on \( R_i \). In particular, \((R_i, Z_i)\) is a surface with boundary, where each boundary component has at least one marked point. We then define \((S - \sigma, Z)\) to be the (possibly disconnected) surface \( \bigsqcup_i (R_i, Z_i) \). We call each \((R_i, Z_i)\) a region of \((S - \sigma, Z)\). When it is clear from context that \( Z \) is the set of marked points on a fixed ambient surface \((S, Z)\), then we may simply write \( S - \sigma \) for \((S - \sigma, Z)\) and \( R_i \) for \((R_i, Z_i)\). For brevity, we shall also refer to each \( R_i \) as a region of \( \sigma \).

The map \( \iota : (S - \sigma, Z) \to (S, Z) \) defined by \( \iota|_{R_i} = t_i \) is surjective, sends marked points to marked points, and restricts to an embedding on the interior of \( S - \sigma \). If \( \alpha \) is a boundary arc
of \( S - \sigma \) then \( \iota(\alpha) \) is either a boundary arc of \( S \) or an arc in \( \sigma \). (In fact, there are exactly two boundary arcs on \( S - \sigma \) mapping to each arc in \( \sigma \) via \( \iota \).) If \( \alpha \) is an essential arc on \( S - \sigma \) then \( \iota(\alpha) \) is an essential arc on \( S \) disjoint from \( \sigma \). Conversely, if \( \beta \) is an essential arc on \( S \) disjoint from \( \sigma \) then \( \iota^{-1}(\beta) \) is the disjoint union of an essential arc on \( S - \sigma \) with a (possibly empty) set of marked points. Therefore, \( \iota \) preserves disjointness of arcs, and so we deduce the following.

**Lemma 2.4** (Links and cutting). The map \( \iota : (S - \sigma, \mathcal{Z}) \to (S, \mathcal{Z}) \) as above induces a simplicial isomorphism \( \iota^* : \mathcal{A}(S - \sigma, \mathcal{Z}) \to \text{lk}_{\mathcal{A}(S, \mathcal{Z})}(\sigma) \).

Given a region \( R \) of \( \sigma \), we shall regard its boundary \( \partial R \subseteq \sigma \) as a multi-arc on \( S \), and hence a simplex in \( \mathcal{A}(S, \mathcal{Z}) \). Each boundary arc of \( R \) shall also be called a side of \( R \). If an arc \( \alpha \in \sigma \) appears as two sides of \( R \) then we shall refer to these sides as the two copies of \( \alpha \) on \( \partial R \). We also say a region \( R \) meets an arc \( \alpha \) if \( \alpha \) is a side of \( R \). Note that there are either one or two regions of \( \sigma \) meeting a given arc in \( \sigma \); and each arc appears at most twice as a side of any given region. In particular, a triangular region with an arc appearing twice as a side is called a **folded triangle** (see the triangle \( T \) in Figure 2.2.)

If \( R \) and \( R' \) are regions of \( \sigma \) meeting along an arc \( \alpha \) in \( \sigma \), then we may obtain a region of \( \sigma \setminus \{\alpha\} \) by gluing them along \( \alpha \): this is the unique region of \( \sigma \setminus \{\alpha\} \) containing the interiors of \( R \) and \( R' \). If \( \alpha \) appears twice as a side of \( R \), then we may glue \( R \) to itself along the two copies of \( \alpha \) to produce a region of \( \sigma \setminus \{\alpha\} \). Observe that marked points are mapped to marked points under any of these gluing operations.

Note that the regions obtained by the cutting operation (considered up to proper isotopy) do not depend on the choice of the Riemannian metric or on the representatives of the smooth arcs in \( \sigma \).

### 2.1.2 Join decompositions of links

Given a finite set of flag simplicial complexes \( A_1, \ldots, A_n \), we define their **join** \( A_1 \ast \ldots \ast A_n \) as follows: the vertex set is the disjoint union of each vertex set of each \( A_i \); the simplices are spanned by sets of vertices of the form \( \sigma_1 \sqcup \ldots \sqcup \sigma_n \), where each \( \sigma_i \) is a (possibly empty) simplex of \( A_i \).

**Lemma 2.5** (Uniqueness of minimal join decomposition). Let \( \mathcal{K} \) be a finite-dimensional flag simplicial complex. Then there exist subcomplexes \( A_1, \ldots, A_n \) of \( \mathcal{K} \), where \( n \geq 1 \), such that \( \mathcal{K} = A_1 \ast \ldots \ast A_n \) and each \( A_i \) cannot be decomposed as a non-trivial join. Moreover, the \( A_i \)'s are unique up to permutation.

**Proof.** Since \( \mathcal{K} \) is a flag complex, we may instead work with its 1-skeleton \( G \). Observe that the graph \( G \) cannot be decomposed as a non-trivial join if and only if the complement graph \( \overline{G} \) (that
is, the graph where vertices are adjacent if and only if they are not adjacent in \( G \) is connected. Since \( \mathcal{K} \) is finite-dimensional, there are finitely many connected components \( H_1, \ldots, H_n \) of \( \overline{G} \). Let \( V_i \) be the vertex set of \( H_i \), and \( A_i \) be the induced subgraph of \( V_i \) in \( G \). Then \( G = A_1 \ast \cdots \ast A_n \), and each \( A_i \) cannot be decomposed as a non-trivial join. Alternatively, we can take \( A_i \) to be the induced subcomplex of \( V_i \) in \( \mathcal{K} \) which yields the desired decomposition \( \mathcal{K} = A_1 \ast \cdots \ast A_n \). □

Note that \( S - \sigma \) cannot have any monogons or bigons as regions since \( \sigma \) is a multi-arc which, by definition, consists of essential and pairwise non-homotopic arcs. A region \( R \) of \( S - \sigma \) is a triangle if and only if \( \mathcal{B}(R) = \emptyset \); and \( R \) is a monogon with one interior marked point if and only if \( \mathcal{B}(R) \) is a single vertex.

**Proposition 2.6** (Decomposition of links in \( A(S, \mathcal{Z}) \)). Suppose \( \sigma \in A(S, \mathcal{Z}) \) is a simplex, and let \( \text{lk}_{A(S, \mathcal{Z})}(\sigma) = A_1 \ast \cdots \ast A_n \) be the minimal join decomposition of its link. Then

(i) \( S - \sigma \) has exactly \( n \) non-triangular regions \( R_1, \ldots, R_n \),

(ii) \( A_i = \mathcal{B}(R_i) \) (up to permutation), and

(iii) the topological type of \( R_i \) is determined by the isomorphism type of \( A_i \).

**Proof.** Let \( R_1, \ldots, R_m \) be the non-triangular regions of \( S - \sigma \). Let \( B_i = A(R_i) \) be the induced subcomplex in \( A(S, \mathcal{Z}) \) of the arcs contained in \( R_i \). Any arc \( \gamma \in \text{lk}_{A(S, \mathcal{Z})}(\sigma) \) is contained in exactly one non-triangular region of \( S - \sigma \), and hence is a vertex of precisely one of the \( B_i \). Moreover, if arcs \( \beta, \gamma \in \text{lk}_{A(S, \mathcal{Z})}(\sigma) \) are contained in distinct regions then they are disjoint. Therefore, \( \text{lk}_{A(S, \mathcal{Z})}(\sigma) = B_1 \ast \cdots \ast B_m \). We shall show that each \( B_i \) cannot be decomposed as a non-trivial join. Together with the above lemma, this will yield parts (i) and (ii). Part (iii) follows from the combinatorial rigidity of arc complexes (Theorem 2).

To simplify the exposition, we shall refer to each \( A_i \) as a ‘colour’. In particular, any pair of intersecting arcs in \( \text{lk}_{A(S, \mathcal{Z})}(\sigma) \) have the same colour. Our goal is to show that all vertices in \( B_i \) have the same colour. If \( B_i \) is a single vertex then we are done, so suppose it has at least two vertices. Let \( \mathcal{T} \) be a triangulation of \( (S, \mathcal{Z}) \) containing \( \sigma \). Note that \( R_i \) contains at least two triangles of \( \mathcal{T} \), for otherwise \( R_i \) is a once-marked monogon (as \( R_i \) is assumed to be non-triangular) and hence contains only one arc. Let \( \mathcal{T}_i = \mathcal{T} \cap B_i \) be the arcs of \( \mathcal{T} \) contained in \( R_i \). If we can show that every arc in \( \mathcal{T}_i \) has the same colour, then we are done since each \( \gamma \in B_i \) either intersects \( \mathcal{T}_i \) or is an arc in \( \mathcal{T}_i \).

**Claim.** If \( B_i \) has at least two vertices, then for every arc \( \gamma \in B_i \) there exists an arc \( \gamma' \in B_i \) intersecting \( \gamma \). If \( \gamma \) meets two triangles of \( \mathcal{T} \), then we let \( \gamma' \) be the other diagonal of the quadrilateral formed by the two triangles meeting \( \gamma \). If \( \gamma \) meets only one triangle \( T \) of \( \mathcal{T} \), it has to form two sides of the folded triangle \( T \) (see Figure 2.2). Let \( \beta \) be the other side of \( T \). Since \( R_i \) contains at least two triangles of \( \mathcal{T} \), there is a triangle \( T' \neq T \) of \( \mathcal{T} \) also meeting \( \beta \). Gluing \( T \) to itself along \( \gamma \) and to \( T' \) along \( \beta \) forms a once-marked bigon, with \( \gamma \) connecting the interior marked point to a boundary marked point of the bigon. We can take \( \gamma' \) to be the essential arc with both end points on the other boundary marked point. Therefore, there always exists an arc \( \gamma' \in B_i \) that intersects \( \gamma \).

Now suppose arcs \( \beta, \gamma \in \mathcal{T}_i \) bound a common triangle \( T \) of \( \mathcal{T} \). Applying the above claim, there exist arcs \( \beta', \gamma' \in B_i \) intersecting \( \beta \) and \( \gamma \), respectively. If \( \beta' = \gamma', \beta' \pitchfork \gamma, \beta \pitchfork \gamma', \) or \( \beta' \pitchfork \gamma' \) then \( \beta \) and \( \gamma \) have the same colour. Otherwise, \( \beta' \) and \( \gamma' \) must both intersect the third side \( \alpha \) of \( T \),
implying that \( \alpha \) is an arc in \( T_i \) sharing the same colour with both \( \beta \) and \( \gamma \). Since \( R_i \) is connected, we deduce that every arc in \( T_i \) has the same colour as desired. \( \square \)

2.1.3 Infinite links

**Definition 2.7** (Infinite link simplices). Given a simplicial complex \( K \), define its set of infinite link simplices to be

\[
\text{IL}(K) = \{ \sigma \in K | \# \text{lk}_K(\sigma) = \infty \}.
\]

Let \( \text{MIL}(K) \subseteq \text{IL}(K) \) be the set of \( \sigma \in \text{IL}(K) \) for which \( \text{lk}(\sigma') \) is finite for all \( \sigma' \supseteq \sigma \). In other words, \( \text{MIL}(K) \) comprises the simplices which are maximal among those with infinite link.

A simple closed curve (or a curve) on \((S, Z)\) is an embedded loop on \( S \setminus Z \). Here, curves are considered up to proper homotopy. A curve on \((S, Z)\) is essential if it does not bound a disc with at most one marked point. Note that an essential curve of \( S \) contained in some region \( R \) could be peripheral on \( R \), that is, parallel to a boundary component of \( R \).

We also extend the notion of cutting a surface \( S \) along a simple closed curve \( \gamma \) to obtain a surface \( S - \gamma \). The two boundary components of \( S - \gamma \) obtained by cutting along \( \gamma \) will have no marked points.

**Remark 2.8** (Regions with finitely many arcs). Let \( \sigma \) be a multi-arc. A region \( R \) of \( \sigma \) contains an essential simple closed curve if and only if \( R \) is not a polygon with at most one interior marked point. For regions without simple essential closed curves, the arc complex is finite. In fact, these are the only possible regions with finitely many arcs: if \( R \) contains an essential simple closed curve \( \gamma \), there must exist some arc \( \alpha \) intersecting \( \gamma \) non-trivially; applying Dehn twists about \( \gamma \) yields infinitely many arcs.

**Proposition 2.9** (Infinite links in the arc complex). Suppose \( \sigma \in \mathcal{A}(S, Z) \) is a multi-arc. Then

(i) \( \sigma \in \text{IL}(\mathcal{A}(S, Z)) \) if and only if \( \sigma \) is disjoint from an essential simple closed curve on \((S, Z)\), and

(ii) \( \sigma \in \text{MIL}(\mathcal{A}(S, Z)) \) if and only if \( S - \sigma \) has a (1,1)-annulus as its unique non-triangular region.

**Proof.** Let \( R_1, \ldots, R_n \) be the non-triangular regions of \( S - \sigma \), and \( A_i = \mathcal{A}(R_i) \). By Proposition 2.6, we have the minimal join decomposition \( \text{lk}_{\mathcal{A}(S, Z)}(\sigma) = A_1 \ast \cdots \ast A_n \). Thus, \( \# \text{lk}_{K}(\sigma) = \infty \) if and only if \( \#A_i = \infty \) for some \( i \). By the above remark, this holds if and only if \( A_i \) contains an essential simple closed curve, yielding part (i).

Now, suppose \( S - \sigma \) has a (1,1)-annulus as its unique non-triangular region. The core curve of this annulus is an essential simple closed curve disjoint from \( \sigma \), and so \( \sigma \in \text{IL}(\mathcal{A}(S, Z)) \). However, any arc \( \alpha \in \text{lk}_{\mathcal{A}(S, Z)}(\sigma) \) must connect the marked points on opposite boundary components of the annulus. Thus, \( S - (\sigma \cup \{\alpha\}) \) has a quadrilateral as its unique non-triangular region. Therefore, \( \sigma \cup \{\alpha\} \) has finite link, and so it follows that \( \sigma \in \text{MIL}(\mathcal{A}(S, Z)) \).

Conversely, assume \( \sigma \in \text{MIL}(\mathcal{A}(S, Z)) \). Since \( \sigma \in \text{IL}(\mathcal{A}(S, Z)) \), there exists an essential simple closed curve on \((S, Z)\) disjoint from \( \sigma \). Let \( R \) be the region of \( S - \sigma \) containing such a curve. By the maximality assumption, every essential simple closed curve on \( R \) must intersect every essential
arc on $R$. This implies that $R$ is the only non-triangular region of $S - \sigma$. We want to show that $R$ is a (1,1)-annulus.

First, suppose there exists a non-separating curve $\gamma$ on $R$. Then there exists an arc $\alpha$ on $R$ that intersects $\gamma$ exactly once (in minimal position). Let $p$ and $p'$ be the end points of $\alpha$ (these could possibly coincide). Let $a, a'$ be the subsegments of $\alpha$ separated by the intersection point $x$ of $\alpha$ and $\gamma$, with an end point at $p, p'$, respectively. Let $\beta$ be the arc that begins at $p$, runs along $a$ to $x$, follows $\gamma$ exactly once, and then returns from $x$ to $p$ along $a$. Define $\beta'$ similarly using $a'$ instead (see Figure 2.3). Then, after a homotopy, $\beta$ and $\beta'$ bound a (1,1)-annulus $R' \subseteq R$, containing $\gamma$ as a core curve and $\alpha$ as an arc connecting opposite boundary components. By the maximality assumption, $\beta$ and $\beta'$ cannot be essential arcs on $R$, and so they must belong to $\partial R$. Therefore, $R = R'$ is a (1,1)-annulus.

Next, we consider the case where all curves on $R$ are separating; this occurs precisely when $R$ has genus 0. Let $\gamma$ be any simple closed curve on $R$, and suppose $R'$ and $R''$ are the two components obtained by cutting $R$ along $\gamma$. Our goal is to show that $R'$ and $R''$ are (0,1)-annuli.

Suppose, for a contradiction, that $R'$ has at least two marked points $p, p'$. If $p$ and $p'$ are not contained in the same boundary component, then any arc connecting them is essential. If $p$ and $p'$ lie on the same boundary component, then there exists an essential arc with both end points on $p$ which cuts off a polygon having $p'$ as one of its marked points. This is an arc that is also essential on $R$. This contradicts the assumption on maximality.

Thus, we have shown that $R'$ (and $R''$) has at most one marked point. In particular, $R'$ and $R''$ have at most two boundary components (including the one obtained by cutting along $\gamma$). Since $\gamma$ is essential on $R$, neither $R'$ or $R''$ can be a closed disc with at most one interior marked point. Therefore, $R'$ (and $R''$) has exactly one marked point, and this marked point must be a boundary marked point. It follows that both $R', R''$ are (0,1)-annuli. Gluing $R'$ and $R''$ along $\gamma$ yields a (1,1)-annulus $R$. This completes the proof of part (ii). □

2.2 Triangles and flip graphs

Recall that a triangulation $\mathcal{T}$ on $(S, \Sigma)$ is a maximal multi-arc. We refer to each (triangular) region of $S - \mathcal{T}$ as a triangle of $\mathcal{T}$. Note that two arcs $\alpha, \beta \in \mathcal{A}(S, \Sigma)$ bound a folded triangle $T$, with $\beta$ forming two sides of $T$, if and only if $\text{lk}_{\mathcal{A}(S, \Sigma)}(\alpha) = \{\beta\} * \mathcal{K}$ for some simplicial complex $\mathcal{K} \subset \mathcal{A}(S, \Sigma)$.

The following was proven in [16, Proposition 3.2].
Proposition 2.10 (Extending triangles to hexagons). Let $T$ be a non-folded triangle on $(S, \mathcal{Z})$. Then there exists a triangulation $\mathcal{T}$ of $(S, \mathcal{Z})$ containing $T$ such that $S - (\mathcal{T} \setminus \beta T)$ has a hexagon as its unique non-triangular region. In particular, each side of $T$ cuts the hexagon into a pentagon and a triangle.

This result can be used to give a purely combinatorial characterisation of non-folded triangles on $(S, \mathcal{Z})$.

Proposition 2.11 (Topological Triangle Test). Let $\tau \in \mathcal{A}(S, \mathcal{Z})$ be a 2-simplex. Then $\tau$ bounds a non-folded triangle on $(S, \mathcal{Z})$ if and only if there exists a maximal simplex $\mathcal{T} \in \mathcal{A}(S, \mathcal{Z})$ containing $\tau$ such that:

(i) $\text{lk}_{\mathcal{A}(S, \mathcal{Z})}(\mathcal{T} \setminus \tau)$ is isomorphic to the arc complex of the hexagon, and
(ii) $\text{lk}_{\mathcal{A}(S, \mathcal{Z})}(\mathcal{T} \setminus \sigma)$ is isomorphic to the arc complex of the pentagon for every 1-simplex $\sigma \subset \tau$.

Proof. The necessity of these conditions follows from the previous proposition.

To prove sufficiency, suppose $\mathcal{T}$ is a triangulation containing $\tau$ which satisfies the given conditions. Combining condition (i) and Proposition 2.6, we deduce that $S - (\mathcal{T} \setminus \tau)$ has one unique non-triangular region $R$, and $R$ is a hexagon. Moreover, $\tau$ forms three disjoint diagonals of $R$. This can occur in three possible ways: either $\tau$ bounds a triangle inside $R$, or $\tau$ contains a diagonal of $R$ that cuts $R$ into two quadrilaterals, leaving two possible choices for the other two diagonals (see Figure 2.4). Applying condition (ii) rules out the latter cases. $\square$

Suppose $\alpha$ is an arc of a triangulation $\mathcal{T}$ that meets two distinct triangles of $\mathcal{T}$. Then $S - (\mathcal{T} \setminus \{\alpha\})$ has a quadrilateral as its unique non-triangular region, with $\alpha$ as a diagonal; let $\beta \in \mathcal{A}(S, \mathcal{Z})$ be the other diagonal of this quadrilateral. The triangulation $\mathcal{T}'$ obtained by flipping $\alpha$ in $\mathcal{T}$ is given by $\mathcal{T}' = (\mathcal{T} \setminus \{\alpha\}) \cup \{\beta\}$ (see Figure 2.5). Note that

$$\text{lk}_{\mathcal{A}(S, \mathcal{Z})}(\mathcal{T} \setminus \{\alpha\}) = \{\alpha\} \cup \{\beta\} = \text{lk}_{\mathcal{A}(S, \mathcal{Z})}(\mathcal{T}' \setminus \{\beta\}).$$
We say that \( \alpha \) is \textit{flippable} in \( \mathcal{T} \) if the above operation can be performed; this holds precisely when \( \alpha \) meets two distinct triangles of \( \mathcal{T} \) or, equivalently, when \( \alpha \) does not form two sides of a folded triangle of \( \mathcal{T} \). The following is immediate.

**Lemma 2.12** (Sides in non-folded triangles are flippable). If \( T \) is a non-folded triangle on \((S, \mathcal{Z})\), then every side of \( T \) is flippable in any triangulation \( \mathcal{T} \) containing \( T \).

**Definition 2.13** (Flip graph). The flip graph \( \mathcal{F}(S, \mathcal{Z}) \) is the graph with the set of all triangulations of \((S, \mathcal{Z})\) as its vertex set, where two triangulations \( \mathcal{T}, \mathcal{T}' \) are joined by an edge if and only if they differ in precisely one flip.

The flip graph \( \mathcal{F}(S, \mathcal{Z}) \) is connected; see [1, 6] for a discussion about its geometric properties.

## 3 Half-Translation Surfaces and Saddle Connections

In this section, we discuss singular Euclidean metrics on \((S, \mathcal{Z})\) known as \textit{half-translation structures}. This class of Euclidean structures on \( S \) naturally arises through the study of \textit{holomorphic quadratic differentials} on \( S \). However, we will use a more geometric approach instead of working directly with quadratic differentials. For further details, refer to [34, 38, 39].

For the remainder of this paper, assume \((S, \mathcal{Z})\) is a closed, connected, orientable surface. A \textit{half-translation structure} on \((S, \mathcal{Z})\) consists of an atlas of charts from \( S \setminus \mathcal{Z} \) to \( \mathbb{C} \), with transition maps of the form \( z \mapsto \pm z + c \) for some \( c \in \mathbb{C} \). If all transition maps are of the form \( z \mapsto z + c \) for some \( c \in \mathbb{C} \) then the atlas defines a \textit{translation surface}. Pulling back the Euclidean metric on \( \mathbb{C} \) gives a locally Euclidean metric on \( S \setminus \mathcal{Z} \) whose metric completion endows each point of \( \mathcal{Z} \) with the structure of a Euclidean cone point with cone angle \( k\pi \) for some positive integer \( k \). Two half-translation structures on \((S, \mathcal{Z})\) are considered equivalent if they are related by an isometry isotopic to the identity (the isometries are not required to fix \( \mathcal{Z} \)).

We shall denote a half-translation surface by \((S, q)\). The set of marked points \( \mathcal{Z} \) is implicitly assumed to be the set of singularities of the half-translation structure \( q \). In particular, we allow for \textit{removable singularities} and \textit{simple poles}; these are, respectively, singularities with cone angle \( 2\pi \) and \( \pi \). Let \( QD(S) \) denote the space of all half-translation structures on \( S \) (or, equivalently, the space of holomorphic quadratic differentials on \( S \)).

Half-translation surfaces enjoy a consistent notion of slope. Given any slope \( \theta \in \mathbb{RP}^1 \), we can pull back the foliation of \( \mathbb{C} \) by straight lines with slope \( \theta \) to a singular foliation on \((S, q)\). Masur showed in [25, Theorem 2] that for every half-translation surface \((S, q)\), there exists a dense set of slopes in \( \mathbb{RP}^1 \) for which the corresponding foliation on \((S, q)\) possesses a closed (non-singular) leaf. Each such closed leaf on \((S, q)\) is contained in a (maximal) open Euclidean cylinder foliated by closed leaves parallel (and isotopic) to the given leaf; each of these closed leaves is called a \textit{core curve} of the cylinder. A simple closed curve on \( S \) isotopic to a core curve of a cylinder on \((S, q)\) is called a \textit{cylinder curve}; we denote the set of all cylinder curves on \((S, q)\) by \( \text{cyl}(q) \).

**Remark 3.1** (Isotopies of curves). We can consider simple closed curves on \((S, \mathcal{Z})\) up to two forms of isotopy: when isotopies cannot pass through \( \mathcal{Z} \), and when they can. Let \( C(S, \mathcal{Z}) \) and \( C(S) \) denote the respective sets of isotopy classes of simple closed curves. We want to consider every isotopy class in \( C(S, \mathcal{Z}) \) as an isotopy class in \( C(S) \). Strictly speaking, such a map can only be defined on curves in \( C(S, \mathcal{Z}) \) not bounding discs with marked points, but curves of this type cannot be
cylinder curves. Additionally, each isotopy class in \( C(S) \) arises from some isotopy class in \( C(S, Z) \) upon allowing isotopies to pass through \( Z \). Therefore, cyl\((q)\) can be equally regarded as a subset of \( C(S, Z) \) or of \( C(S) \). More precisely, the composition \( \text{cyl}(q) \hookrightarrow C(S, Z) \to C(S) \) is injective because if \( \gamma \in C(S) \) can be realised by a cylinder curve on \((S, q)\), then there is a unique Euclidean cylinder on \((S, q)\) with \( \gamma \) as its core curve.

### 3.1 Saddle connection complex

We now introduce the main objects of study in this paper.

**Definition 3.2** (Saddle connection). A **saddle connection** on \((S, q)\) is a locally isometric embedding \( a : [0, l] \to (S, q) \) such that \( \{a(0), a(l)\} \subseteq Z \) and the restriction \( a_{|_{(0,l)}} \) is an embedding into \( S \setminus Z \). We shall usually identify \( a \) with its image. In particular, we do not care about the orientation.

Saddle connections are Euclidean straight line segments, and thus have a well-defined slope on \((S, q)\). In particular, the boundary of a Euclidean cylinder on \((S, q)\) is a non-empty, finite set of saddle connections with the same slope. Therefore, the set of slopes of saddle connections on \((S, q)\) is also dense in \( \mathbb{RP}^1 \).

**Definition 3.3** (Saddle connection complex). The **saddle connection complex** \( \mathcal{A}(S, q) \) is the simplicial complex whose vertices are saddle connections on \((S, q)\), and whose simplices are sets of pairwise disjoint saddle connections.

Any saddle connection on \((S, q)\) is also (a representative of) an essential arc on \((S, Z)\); in fact, it is the unique geodesic representative in its proper homotopy class. Let \( i_q : \mathcal{A}(S, q) \to \mathcal{A}(S, Z) \) be the natural inclusion map. Minsky and Taylor introduced the **saddle connection graph**, the 1-skeleton of \( \mathcal{A}(S, q) \), in [29] and established some important results regarding geodesic representatives of arcs, which we shall recall.

Consider the universal cover of \((S, q) \setminus Z\), and take its metric completion. This metric completion is CAT(0), since \((S, q)\) is locally CAT(0), and so every path has a unique geodesic representative in its proper homotopy class. The action of the deck group extends to the set of completion points \( \tilde{Z} \); indeed, the set of orbits in \( \tilde{Z} \) is in natural bijection with \( Z \). Thus, every arc \( \alpha \in \mathcal{A}(S, q) \) lifts to an arc \( \tilde{\alpha} \) connecting a pair of distinct points in \( \tilde{Z} \). The unique geodesic representative of \( \tilde{\alpha} \) descends to a path \( \alpha_q \) on \((S, q)\) formed by concatenating a finite sequence of saddle connections. Note that this path does not depend on the choice of the lift \( \tilde{\alpha} \).

We would like to say that \( \alpha_q \) is the geodesic representative of \( \alpha \) on \((S, q)\), however, in order to do this we need to allow for homotopies to move arcs so that they touch points of \( Z \), but without being pushed past them. More precisely, we say that \( \alpha_1 \) is a representative of \( \alpha \in \mathcal{A}(S, q) \) if there is an embedded representative \( \alpha_0 \) of \( \alpha \) on \((S, Z)\), and a path homotopy \( \alpha_t : [0, 1] \to S \) such that for all \( 0 < s < 1 \) and \( 0 \leq t < 1 \) we have \( \alpha_t(s) \in S \setminus Z \); thus \( \alpha_t \) is permitted to pass through points of \( Z \) only when \( t = 1 \). Note that if \( \alpha_1 \) passes through some points of \( Z \) in its interior, then it does not necessarily determine a unique proper homotopy class of arcs; there is the ambiguity arising from the choice of side for ‘pushing’ \( \alpha_1 \) off a marked point. In the case where consecutive saddle connections of \( \alpha_q \) form an angle of less than \( \pi \) on one side, then \( \alpha_q \) cannot be pushed off on that side while remaining in the homotopy class of \( \alpha \).
Proposition 3.4 (Geodesic representatives of arcs [29]). Every arc \( \alpha \in \mathcal{A}(S, Z) \) has a unique geodesic representative on \((S, q)\) in its proper homotopy class. The geodesic representative \( \alpha_q \) of \( \alpha \) is a finite concatenation of saddle connections, possibly with repetition, where no two of these saddle connections have transverse intersections. Furthermore, if \( \alpha, \beta \in \mathcal{A}(S, Z) \) are disjoint arcs then no saddle connection appearing on \( \alpha_q \) crosses a saddle connection appearing on \( \beta_q \) (some saddle connections may appear on both, however).

Minsky and Taylor use this result in [29, Section 4.3] to define a straightening operation \( t_q : \mathcal{A}(S, Z) \to \mathcal{A}(S, q) \) by mapping a multi-arc \( \sigma \in \mathcal{A}(S, Z) \) to the simplex \( t_q(\sigma) \in \mathcal{A}(S, q) \) whose vertices correspond to the saddle connections on \((S, q)\) appearing on \( \alpha_q \) for some \( \alpha \in \sigma \). Note that \( t_q \circ i_q \) is the identity on \( \mathcal{A}(S, q) \), and so \( i_q(\mathcal{A}(S, q)) \) is an induced subcomplex of \( \mathcal{A}(S, Z) \). If \( \alpha_0, \ldots, \alpha_k \) is a path in \( \mathcal{A}(S, Z) \) with end points \( \alpha_0, \alpha_k \in \mathcal{A}(S, q) \), then choosing a vertex \( \beta_i \in t_q(\alpha_i) \) for each \( i \) yields a path \( \beta_0, \ldots, \beta_k \) (possibly with consecutive vertices coinciding) in \( \mathcal{A}(S, q) \) connecting \( \alpha_0 = \beta_0 \) to \( \alpha_k = \beta_k \). Thus, \( i_q \circ t_q \) defines a coarse retraction from \( \mathcal{A}(S, Z) \) onto \( i_q(\mathcal{A}(S, q)) \) that sends sets of diameter at most 1 to sets of diameter at most 1. A similar construction was also used in [36]. Using the connectedness and uniform hyperbolicity of arc complexes, we deduce the following.

Proposition 3.5 (Saddle connection complex is connected and hyperbolic). The map \( i_q : \mathcal{A}(S, q) \to \mathcal{A}(S, Z) \) is an isometric embedding. In particular, \( \mathcal{A}(S, q) \) is connected and uniformly Gromov hyperbolic.

To prove our main theorem for the saddle connection complex, we only use its isomorphism class as a simplicial complex. This means that we can only use the fact that \( \mathcal{A}(S, q) \) arises as a saddle connection complex of some half-translation surface without knowing what the underlying surface is. In particular, we do not know the labels of the vertices, that is, which topological arcs or saddle connections they represent. The topological information must be recovered from the combinatorics of the graph in the process of the proof.

We shall show, at the end of Section 5, that every vertex in the saddle connection complex has infinite valence, and so the saddle connection complex is also locally infinite. By following a standard method of Luo (which is adapted from an argument of Kobayashi [20] and appears in [26]), Pan shows that \( \mathcal{A}(S, q) \) has infinite diameter [32]. A consequence of Proposition 3.5 is that \( i_q \) induces an embedding of the Gromov boundary of \( \mathcal{A}(S, q) \) into that of \( \mathcal{A}(S, Z) \). The large-scale geometry of \( \mathcal{A}(S, q) \), including its Gromov boundary, has been described in the paper [5], which is a follow-up to this work.

We finish this subsection with an example of a saddle connection complex.

Example 3.6 (Saddle connection complex of the flat torus). If \((S, q)\) is a flat torus with exactly one marked point, then it can be formed by gluing two Euclidean triangles along three pairs of sides. It is well-known that its arc complex \( \mathcal{A}(S, Z) \) is isomorphic to the Farey tessellation of the hyperbolic plane. Moreover, every arc can be realised as a saddle connection and so the saddle connection complex \( \mathcal{A}(S, q) \) is also isomorphic to the Farey tessellation.

Any two flat tori with exactly one marked point are affine equivalent. Moreover, this is the only affine equivalence class of half-translation surfaces that admit triangulations by three saddle connections, and hence a saddle connection complex of dimension 2. Therefore, \((S, q)\) is a flat torus with one marked point if and only if \( \mathcal{A}(S, q) \) is isomorphic to the Farey tessellation.
This example shows that if \((S, q)\) or \((S', q')\) is a flat torus with one marked point then for every simplicial isomorphism \(\phi : \mathcal{A}(S, q) \to \mathcal{A}(S', q')\), there exists an affine diffeomorphism \(F : (S, q) \to (S', q')\) which induces \(\phi\). However, it is not true that this affine diffeomorphism is unique. This is because of the existence of a hyperelliptic involution that acts trivially on \(\mathcal{A}(S, q)\), but exchanges the two triangles in any given triangulation (see Irmak–McCarthy [16]). Composing \(F\) with this hyperelliptic involution yields the only other affine diffeomorphism inducing \(\phi\).

From now on, we consider only half-translation surfaces whose triangulations contain more than two triangles.

### 3.2 Regions on half-translation surfaces

Given a simplex \(\sigma \in \mathcal{A}(S, q)\), we can cut \((S, q)\) along all saddle connections in \(\sigma\) to obtain a surface with boundary in a similar manner as for topological multi-arcs. The resulting surface, which we shall denote by \((S - \sigma, q)\), has a locally Euclidean metric, with boundary formed by finitely many straight line segments.

We shall define the associated saddle connection complex \(\mathcal{A}(S - \sigma, q)\) as in the situation of the arc complex. Lemma 2.4 also holds for the saddle connection complex: the natural map \(i : (S - \sigma, q) \to (S, q)\) induces a simplicial isomorphism \(i_* : \mathcal{A}(S - \sigma, q) \to \text{lk}_{\mathcal{A}(S, q)}(\sigma)\).

Suppose \(R\) is a polygonal region of \((S - \sigma, q)\). We shall refer to (essential) topological arcs on polygons as diagonals. A diagonal of \(R\) is straight if it can be realised by a saddle connection, otherwise we call it broken (see Figure 3.1).

On a polygon \(R\), every diagonal is straight if and only if all corners of \(R\) have angle strictly less than \(\pi\); in this case we call \(R\) a strictly convex polygon.

**Lemma 3.7** (Convex polygons are embedded). Let \(\tilde{R}\) be a strictly convex polygon in the universal cover \((\tilde{S}, \tilde{q})\). Then its image \(R\) on \((S, q)\) has embedded interior. In particular, if \(R\) is a triangle then it is embedded, except for possibly at the corners.

**Proof.** Recall that a strictly convex polygon meets the set of (pre-images of) singularities exactly at its corners and hence contains no singularities in its interior. Suppose the interior of \(R\) is not embedded. Then there is some deck transformation \(g\) of \((\tilde{S}, \tilde{q})\) such that \(\tilde{R}\) and \(g(\tilde{R})\) have overlapping interiors. This implies that \(\tilde{R} \cap g(\tilde{R})\) is a strictly convex polygon with non-empty interior; in particular it has at least three sides. We may choose local co-ordinates such that \(R\) is identified with a Euclidean polygon in \(\mathbb{C}\), and such that \(g\) is given by \(w \mapsto \pm w + w_0\) for some \(w_0 \in \mathbb{C}\).
First, suppose \( g \) is a translation. Without loss of generality, we may apply a rotation to assume that \( w_0 \) is purely imaginary. Consider a tallest vertical line segment \( L \) contained in \( \tilde{R} \); this can be chosen so that at least one end point of \( L \) is a corner of \( \tilde{R} \) (see Figure 3.2). Since \( \tilde{R} \) and \( g(\tilde{R}) \) have overlapping interiors, we deduce that \( |w_0| < \text{length}(L) \). But this implies that there is a singularity in the interior of \( R \), or in the interior of one of its sides, contradicting the assumption that \( \tilde{R} \) is strictly convex.

Next, suppose \( g \) is of the form \( w \mapsto -w + w_0 \). Then \( g \) is a rotation about a point \( p \in \mathbb{C} \) through an angle of \( \pi \). Now \( p \) must lie in the interior of \( \tilde{R} \), for otherwise \( \tilde{R} \) and \( g(\tilde{R}) \) will have disjoint interiors (see Figure 3.2). Then \( p \) descends to a cone point of angle \( \pi \) on \((S, q)\), which again implies the existence of a singularity in the interior of \( R \), a contradiction. \( \square \)

### 3.3 Triangles and saddle flip graphs

In Section 10, we will consider oriented triangles; an oriented triangle is specified by a triple of sides \( \vec{T} = [a, b, c] \), considered up to cyclic permutation, where \( a, b, c \in \mathcal{A}(S, q) \). Any non-cyclic permutation of the sides determines the same triangle with the opposite orientation. We say that two oriented triangles are consistently oriented if they give the same orientation on \( S \).

**Lemma 3.8** (Maximal simplices in \( \mathcal{A}(S, q) \)). A maximal simplex of \( \mathcal{A}(S, q) \) corresponds to a triangulation of \((S, q)\) by saddle connections.

**Proof.** Let \( \sigma \) be a maximal simplex in \( \mathcal{A}(S, q) \). Cutting \((S, q)\) along \( \sigma \) gives rise to a disjoint union of Euclidean cone surfaces with piecewise geodesic boundary. It is a standard result that any such surface can be triangulated by Euclidean polygons (with corners at singularities). If some component \( R \) is triangulated by at least two triangles, then it must contain a saddle connection that is not contained in \( \partial R \). But this saddle connection would be disjoint from \( \sigma \), contradicting the maximality assumption. Therefore, \( \sigma \) cuts \((S, q)\) into a finite collection of Euclidean triangles. \( \square \)

We call such a triangulation a saddle triangulation of \((S, q)\). Whenever we speak of a triangulation of a half-translation surface, we shall implicitly mean a saddle triangulation. Given a simplex \( \sigma \in \mathcal{A}(S, q) \), we may also consider triangulations of \((S - \sigma, q)\); these are the maximal simplices of \( \mathcal{A}(S - \sigma, q) \).
A flip of a saddle triangulation is defined in the same way as for the (topological) flip graph. Given a triangulation $\mathcal{T}$ of $(S, q)$, a saddle connection $a \in \mathcal{T}$ is flippable in $\mathcal{T}$ if and only if the unique non-triangular region of $\mathcal{T} \setminus \{a\}$ is a strictly convex quadrilateral; in this case the quadrilateral has two straight diagonals, and so the flip is performed by replacing $a$ with the other diagonal. Note that $\mathcal{T}$ cannot have any folded triangles, since the three sides of a Euclidean triangle have distinct slopes. However, folded $n$-gons with $n \geq 4$ such as folded quadrilaterals can appear. These will be relevant in Lemma 4.6.

**Definition 3.9** (Saddle flip graph). The **saddle flip graph** $\mathcal{F}(S, q)$ is the graph with the set of all saddle triangulations on $(S, q)$ as its vertex set, where two triangulations are joined by an edge if and only if they differ in precisely one flip.

Given a simplex $\sigma \in \mathcal{A}(S, q)$, let $\mathcal{F}_\sigma(S, q)$ be the induced subgraph of $\mathcal{F}(S, q)$ whose vertices are $\{T \in \mathcal{F}(S, q) \mid \sigma \subseteq T\}$.

We shall write $\mathcal{F}(S - \sigma, q)$ for the saddle flip graph of $(S - \sigma, q)$. The simplicial isomorphism $\iota_* : \mathcal{A}(S - \sigma, q) \to \operatorname{lk}_{\mathcal{A}(S, q)}(\sigma)$ induces a natural graph isomorphism $\mathcal{F}(S - \sigma, q) \to \mathcal{F}_\sigma(S, q)$, given by $T \mapsto \iota_*(T) \cup \sigma$.

**Theorem 4** (Flip graph is connected [35]). For any simplex $\sigma \in \mathcal{A}(S, q)$, the graph $\mathcal{F}_\sigma(S, q)$ is connected. In particular, $\mathcal{F}(S, q)$ is connected.

Tahar’s proof works for a more general class of surfaces. He allows for singular Euclidean surfaces with arbitrary cone angles, possibly with boundary formed by finitely many straight line segments. The surface $(S - \sigma, q)$ falls into this class, and so $\mathcal{F}(S - \sigma, q)$ is connected.

Note that if $\sigma \in \mathcal{A}(S, \mathcal{Z})$ is a topological triangulation, then $t_q(\sigma)$ is not necessarily a saddle triangulation. However, all regions of $S - t_q(\sigma)$ will be polygons.

### 3.4 Affine diffeomorphisms

**Definition 3.10** (Affine diffeomorphism). Let $(S, q)$ and $(S', q')$ be half-translation surfaces, with respective set of singularities $\mathcal{Z}$ and $\mathcal{Z}'$. A homeomorphism $F : (S, q) \to (S', q')$ is called an **affine diffeomorphism** if it is locally affine on $S \setminus \mathcal{Z}$ with respect to the underlying half-translation structures.

We call two half-translation surfaces **affine equivalent** if there exists an affine diffeomorphism between them. The group of affine self-diffeomorphisms of $(S, q)$ shall be denoted $\operatorname{Aff}(S, q)$. We also allow orientation-reversing diffeomorphisms.

Note that if $F : (S, q) \to (S', q')$ is an affine diffeomorphism then $F^{-1}$ is also an affine diffeomorphism. Furthermore, $F$ maps saddle connections to saddle connections and preserves disjointness, and so it induces a simplicial isomorphism $\mathcal{A}(S, q) \to \mathcal{A}(S', q')$. The main goal of this paper is to prove that the converse also holds.

There is a natural $\operatorname{GL}(2, \mathbb{R})$-action on $\mathcal{QD}(S)$ defined as follows. Given $M \in \operatorname{GL}(2, \mathbb{R})$ and a half-translation structure $q \in \mathcal{QD}(S)$, define $M \cdot q \in \mathcal{QD}(S)$ to be the half-translation structure obtained by postcomposing the charts from $(S, q)$ to $\mathbb{C}$ (defined away from the set of singularities), with the $\mathbb{R}$-linear map on $\mathbb{C} \cong \mathbb{R}^2$ given by $z \mapsto M(z)$, where $z \in \mathbb{C}$ is a local co-ordinate. The metric can be extended to the set of singularities in the usual manner. Thus, the identity map
rigidity of the saddle connection complex

id$_S : (S, q) \to (S, M \cdot q)$ on the underlying surface $S$ is (isotopic to) an affine diffeomorphism with derivative $M$.

The following is a key ingredient for our proof of the main theorem.

**Theorem 5** (Cylinder Rigidity Theorem [8]). Let $q, q' \in QD(S)$ be half-translation structures on $S$. Then $\text{GL}(2, \mathbb{R}) \cdot q = \text{GL}(2, \mathbb{R}) \cdot q'$ if and only if $\text{cyl}(q) = \text{cyl}(q')$.

The statement in [8, Lemma 22] regards $\text{cyl}(q)$ as a subset of $C(S)$. However, by Remark 3.1, this theorem also holds when $\text{cyl}(q)$ is viewed as a subset of $C(S, \mathcal{Z})$. We thank Kasra Rafi for pointing this out.

## 4 | LINKS OF SIMPLICIES IN $\mathcal{A}(S, q)$

In this section, we discuss an important distinction between join decompositions of links in the saddle connection complex and in the arc complex. We introduce the notion of a **cordon** of a simplex, which shall be used throughout this paper. Furthermore, we give a classification of link types for low-codimensional simplices.

### 4.1 | Join decompositions

By Proposition 2.6, there is a natural correspondence between the factors of the minimal join decomposition of the link of a simplex $\sigma$ in the arc complex, and the non-triangular regions of $S - \sigma$. In contrast, the analogous correspondence does not necessarily hold for links in the saddle connection complex.

**Example 4.1** (Fish with cordon). Figure 4.1 shows a possible region $R$ of a collection of saddle connections on a half-translation surface $(S, q)$. Let $A$ be the induced subcomplex of $\mathcal{A}(S, q)$ whose vertices are all the saddle connections in $R$ (these are indicated with dashed lines). Let $A_1, A_2, \text{ and } A_3$ be the induced subcomplexes of $\mathcal{A}(S, q)$ spanned by the saddle connections in red, blue, and orange, respectively. The blue saddle connection cuts $R$ into two regions, each containing the saddle connections of a single colour. Therefore, no two saddle connections with distinct colours intersect, and so $A = A_1 \ast A_2 \ast A_3$ has a non-trivial join decomposition.
We shall see that the presence of ‘separating’ saddle connections, such as the blue one in the above example, is the only obstruction to having the desired correspondence between non-triangular regions and factors in the minimal join decomposition.

For the rest of this paper, \( \text{lk}(\sigma) \) shall always denote the link of \( \sigma \) in \( \mathcal{A}(S, q) \). We shall also write \( S - \sigma \) for the surface \((S - \sigma, q)\) when the underlying half-translation structure \( q \) is clear. Thus, each region of \( S - \sigma \) will be equipped with a half-translation structure.

**Definition 4.2** (Cordon). Let \( \sigma \in \mathcal{A}(S, q) \) be a simplex. Call a saddle connection \( \gamma \in \text{lk}(\sigma) \) a **cordon** of \( \sigma \) if any of the following equivalent conditions hold:

1. \( \gamma \) is disjoint from every \( \gamma' \in \text{lk}(\sigma) \setminus \{\gamma\} \).
2. \( \text{lk}(\sigma) = \{\gamma\} \ast A \) for some simplicial complex \( A \subseteq \mathcal{A}(S, q) \), or
3. \( \gamma \) belongs to every saddle triangulation containing \( \sigma \) (and hence is non-flippable).

In particular, the cordons of \( \sigma \) are precisely the single-vertex factors appearing in the minimal join decomposition of \( \text{lk}(\sigma) \). If we were working in the topological setting, then these arcs are precisely those contained in once-marked monogon regions of \( S - \sigma \). However, saddle connections on a half-translation surface cannot bound monogons with one singularity. Instead, cordons play an important role in our setting: after cutting along all cordons of \( \sigma \), we obtain our desired analogue of Proposition 2.6 for the saddle connection complex.

**Proposition 4.3** (Decomposition of links in \( \mathcal{A}(S, q) \)). Suppose \( \sigma \in \mathcal{A}(S, q) \) is a simplex. Let \( \text{lk}(\sigma) = c_1 \ast \ldots \ast c_k \ast A_1 \ast \ldots \ast A_n \) be the minimal join decomposition of its link, where the \( c_i \)'s are precisely the single-vertex factors (that is, the cordons of \( \sigma \) ). Then \( S - (\sigma \cup c_1 \cup \ldots \cup c_k) \) has exactly \( n \) non-triangular regions \( R_1, \ldots, R_n \). Moreover, up to permutation, \( A_i \) is the induced subcomplex of \( \mathcal{A}(S, q) \) whose vertices are the saddle connections contained in \( R_i \).

**Proof.** Let \( R_1, \ldots, R_m \) be the non-triangular regions of \( S - (\sigma \cup c_1 \cup \ldots \cup c_k) \), and \( B_i \) be the subcomplex of \( \mathcal{A}(S, q) \) spanned by the saddle connections contained in \( R_i \). We need to show that each \( B_i \) cannot be decomposed as a non-trivial join. The proof proceeds in exactly the same manner as in Proposition 2.6, taking \( T \) to be a saddle triangulation containing \( \sigma \), and using the following claim instead. (Saddle triangulations do not have any folded triangles, so we do not need to deal with that case.)

**Claim.** For every saddle connection \( \gamma \in B_i \), there exists a saddle connection \( \gamma' \in B_j \) that intersects \( \gamma \).

Since we have already cut along all cordons of \( \sigma \) to obtain the regions \( R_i \), it follows that \( \gamma \in B_i \) cannot be a cordon of \( \sigma \). Therefore, there is some saddle connection \( \gamma' \in \text{lk}(\sigma) \) intersecting \( \gamma \). Now, \( \gamma' \) cannot be cordon of \( \sigma \), and so it is contained in some region of \( S - (\sigma \cup c_1 \cup \ldots \cup c_k) \). But \( \gamma \) and \( \gamma' \) intersect, and so they must belong to the same region, namely \( R_j \). Thus, \( \gamma' \in B_j \), yielding the claim. \( \square \)

### 4.2 Classification of low-codimensional simplices

Recall from Proposition 2.6 that the isomorphism type of the link of a simplex in the arc complex completely determines the topological type of its non-triangular regions. For the saddle connect-
RIGIDITY OF THE SADDLE CONNECTION COMPLEX

A strictly convex quadrilateral and a quadrilateral that is not strictly convex, with their straight diagonals

tion complex, however, this does not hold. We shall nevertheless provide a partial classification for simplices corresponding to triangulations missing at most two saddle connections.

Recall that every maximal simplex in \( \mathcal{A}(S, q) \) corresponds to a triangulation, and all triangulations possess the same number of saddle connections. Thus, we may define the **codimension** of a simplex \( \sigma \in \mathcal{A}(S, q) \) to be \( \text{codim}(\sigma) = \text{dim}(\mathcal{A}(S, q)) - \text{dim}(\sigma) \); this counts the number of saddle connections that need to be added to \( \sigma \) in order to produce a triangulation. Observe that \( \text{dim}(\text{lk}(\sigma)) = \text{codim}(\sigma) - 1 \). Thus, for simplices \( \sigma \in \mathcal{A}(S, q) \) of codimension at most 2, the link \( \text{lk}(\sigma) \) is a simplicial graph.

Let \( C_k \) denote the cycle graph of length \( k \); \( P_k \) the path graph of length \( k \); and \( N_k \) the edgeless graph on \( k \) vertices. We shall write \( P_\infty \) for the bi-infinite path graph.

The classification of links for codimension-1 simplices is as follows (see Table A.1 in Appendix A).

**Lemma 4.4 (Codimension-1 simplices).** Let \( \sigma \in \mathcal{A}(S, q) \) be a codimension-1 simplex. Then \( S - \sigma \) has a quadrilateral as its unique non-triangular region which is:

(i) **strictly convex** \( \iff \text{lk}(\sigma) \cong N_2 \),

(ii) **not strictly convex** \( \iff \text{lk}(\sigma) \cong N_1 \).

**Proof.** Complete \( \sigma \) to a triangulation \( \mathcal{T} \), and let \( a \) be the unique saddle connection in \( \mathcal{T} \setminus \sigma \). Then \( a \) meets two distinct triangles \( T, T' \) of \( \mathcal{T} \). Gluing \( T \) and \( T' \) along \( a \) produces a quadrilateral \( Q \) with \( a \) as a diagonal. Now, \( Q \) is strictly convex if and only if the angles at each of its corners are strictly less than \( \pi \). This occurs precisely when there exists another diagonal \( b \) of \( Q \), obtained by flipping \( a \) in \( \mathcal{T} \), cutting \( Q \) into two triangles (see Figure 4.2). In this case, we have \( \text{lk}(\sigma) = \{a\} \cup \{b\} \cong N_2 \). On the other hand, if \( Q \) is not strictly convex then \( a \) is the only diagonal of \( Q \), and so \( \text{lk}(\sigma) = \{a\} \cong N_1 \). \( \Box \)

**Corollary 4.5 (Flippability condition).** Let \( \mathcal{T} \) be a triangulation. Then a saddle connection \( a \in \mathcal{T} \) is flippable in \( \mathcal{T} \) if and only if \( \text{lk}(\mathcal{T} \setminus \{a\}) \cong N_2 \). \( \Box \)

For codimension-2 simplices, there are six possibilities for the links and nine possibilities for the non-triangular regions (see Table A.2 in Appendix A).

**Lemma 4.6 (Codimension-2 simplices).** Let \( \sigma \in \mathcal{A}(S, q) \) be a codimension-2 simplex. Then the non-triangular region(s) of \( S - \sigma \) comprises:

(i) a \( (1,1) \)-annulus \( \iff \text{lk}(\sigma) \cong P_\infty \),

(ii) a strictly convex pentagon \( \iff \text{lk}(\sigma) \cong C_5 \),

(iii) a pentagon with one broken diagonal \( \iff \text{lk}(\sigma) \cong P_3 \),

(iv) two strictly convex quadrilaterals \( \iff \text{lk}(\sigma) \cong C_4 \).
(v) either a pentagon with two broken diagonals, or two quadrilaterals where exactly one is strictly convex $\iff \text{lk}(\sigma) \cong \mathbb{P}_2$,

(vi) either a pentagon with three broken diagonals, two quadrilaterals that are both not strictly convex, or a bigon containing a simple pole $\iff \text{lk}(\sigma) \cong \mathbb{P}_1$.

Moreover, the above list is exhaustive.

**Proof.** Complete $\sigma$ to a triangulation $\mathcal{T}$, and suppose $a_1, a_2$ are the saddle connections of $\mathcal{T} \setminus \sigma$. Let $T_i \neq T'_i$ be the triangles of $\mathcal{T}$ meeting $a_i$, for $i = 1, 2$, and $Q_i$ be the quadrilateral obtained by gluing $T_i$ and $T'_i$ along $a_i$. We shall go through the cases depending on how many of the triangles $T_1, T'_1, T_2, T'_2$ coincide.

First, suppose that the four triangles are distinct. Then $Q_1$ and $Q_2$ have disjoint interiors, and form the two non-triangular regions of $S - \sigma$. Therefore, $\text{lk}(\sigma) = A_1 \ast A_2$, where $A_i$ is the induced subcomplex of $\mathcal{A}(S, q)$ with the saddle connections contained in $Q_i$ as vertices. By the classification of links of codimension-1 simplices, we have

$$\text{lk}(\sigma) \cong \begin{cases} N_2 \ast N_2 \cong C_4 & \text{if both } Q_1, Q_2 \text{ are strictly convex,} \\ N_2 \ast N_1 \cong P_2 & \text{if exactly one of } Q_1, Q_2 \text{ is strictly convex, or} \\ N_1 \ast N_1 \cong P_1 & \text{if neither of } Q_1, Q_2 \text{ is strictly convex.} \end{cases}$$

Next, suppose $T_1 = T_2$, but $T'_1 \neq T'_2$. Then gluing $T := T_1 = T_2$ to $T'_1$ and $T'_2$ along $a_1$ and $a_2$ produces a pentagon $P$, with $a_1, a_2$ as non-intersecting straight diagonals. There are three more diagonals in $P$: the topological arcs $b_1, b_2$ obtained by, respectively, flipping $a_1, a_2$ in $\mathcal{T}$, and the arc $c$ that intersects both $a_1, a_2$ (see Figure 4.3). If $P$ is strictly convex, then all five diagonals are straight, and so $\text{lk}(\sigma)$ is a copy of $C_5$ as shown:

If $P$ is not strictly convex, then $\text{lk}(\sigma)$ is an induced subgraph of the above. We claim that if $c$ is a straight diagonal, then at least one of $b_1$ or $b_2$ is also straight. This will imply that $\text{lk}(\sigma)$ is isomorphic to a connected subgraph of $C_5$. Consequently, if $P$ has $1 \leq k \leq 3$ broken diagonals then $\text{lk}(\sigma) \cong P_{4-k}$.

To prove the claim, suppose $c$ is a straight diagonal. Cutting $P$ along $c$ produces a triangle $T'$ and a quadrilateral $Q$. By considering the angle sum of quadrilaterals, at least three corners of $Q$ have angle strictly less than $\pi$. Suppose this holds for the corner of $Q$ lying at an end point of $a_1$ (otherwise at an end point of $a_2$). Note that this corner is also a corner of $Q_1$. A small neighbourhood of
the corner of $Q_1$ at the other end point of $a_1$ lies strictly inside $T'$, and so it also has angle strictly less than $\pi$. The remaining two corners of $Q_1$ are themselves corners of $T$ and $T'_1$, respectively. Therefore, $Q_1$ is a strictly convex quadrilateral and so $b_1$ is a straight diagonal of $P$.

Finally, we consider the case where $T_1 = T_2$ and $T'_1 = T'_2$. Observe that $a_2$ appears as a side of both $T_1$ and $T'_1$. Therefore, $Q_1$ is either a parallelogram with $a_2$ appearing on opposite sides, or is a folded quadrilateral with a corner of angle $\pi$ formed by adjacent copies of $a_2$; see Figure 4.4. In the first case, gluing $Q_1$ along the two copies of $a_2$ produces a $(1,1)$-annulus $A$ forming the unique non-triangular region of $S - \sigma$. Every topological arc in $A$ is realisable as a saddle connection. Therefore, $\text{lk}(\sigma)$ is isomorphic to the arc complex of the $(1,1)$-annulus, the bi-infinite path graph $P_{\infty}$. In the second case, gluing $Q_1$ along the two copies of $a_2$ produces a bigon containing a simple pole as the unique triangular region of $S - \sigma$. Moreover, $a_1$ and $a_2$ are the only saddle connections in this bigon, so $\text{lk}(\sigma) \cong P_1$.

Observe that a region of a codimension-2 simplex that can be recognised by its link is either a $(1,1)$-annulus, a strictly convex pentagon, or a pentagon with at most one broken diagonal. We shall call a pentagon almond † if it has at most one broken diagonal. Also note that any $(1,1)$-annulus on a half-translation surface must be a cylinder; call any triangle contained in such an annulus a $(1,1)$-annular triangle. By examining cases (i)–(iii) in the above proposition, we deduce the following.

Corollary 4.7 (Detectable codimension-2 simplices). Let $a, b$ be saddle connections in a triangulation $\mathcal{T}$. Then $\text{lk}(\mathcal{T} \setminus \{a, b\})$ contains $P_3$ as an induced subgraph if and only if $\mathcal{T} \setminus \{a, b\}$ has either an almond pentagon or a $(1,1)$-annulus as its unique non-triangular region. In this situation, the region contains a triangle of $\mathcal{T}$ having $a$ and $b$ as two of its sides, and at least one of $a$ or $b$ is flippable in $\mathcal{T}$.

5 | CYLINDERS AND INFINITE LINKS

By Proposition 2.9, a simplex in the arc complex of $(S, \mathcal{Z})$ has infinite link if and only if it is disjoint from some simple closed curve on $(S, \mathcal{Z})$. For the saddle connection complex, this does not hold, as the following example shows.

† Almond stands for ‘at most one non-straight diagonal’ (where at most one letter in the acronym does not stand for a word).
Example 5.1 (Finite-link region with simple closed curve). Suppose $\sigma \in \mathcal{A}(S, q)$ is a simplex where every region is planar. Then each region contains only finitely many saddle connections, and so $\text{lk}(\sigma)$ is finite. But if some region of $S - \sigma$ is not a topological disc with at most one interior marked point, then $\sigma$ will be disjoint from some simple closed curve (see Figure 5.1 for such an example).

Suppose $C$ is a cylinder on $(S, q)$. A transverse arc of $C$ is an arc contained in $C$ whose end points lie on opposite boundary components of $C$. An arc contained in $C$ is realisable as a saddle connection if and only if it is a transverse arc. In particular, $C$ contains infinitely many saddle connections. Therefore, if a simplex $\sigma \in \mathcal{A}(S, q)$ is disjoint from (the interior of) some Euclidean cylinder $C$ on $(S, q)$, then $\text{lk}(\sigma)$ is infinite. We shall see that the converse is also true. Call a simplex $\sigma \in \mathcal{A}(S, q)$ a triangulation away from a cylinder if $\sigma$ has a Euclidean cylinder as its only non-triangular region on $(S, q)$.

Recall that for a simplicial complex $\mathcal{K}$, the set of infinite link simplices is

$$\text{IL}(\mathcal{K}) = \{ \sigma \in \mathcal{K} : \# \text{lk}_\mathcal{K}(\sigma) = \infty \},$$

and $\text{MIL}(\mathcal{K}) \subseteq \text{IL}(\mathcal{K})$ is the set of $\sigma \in \text{IL}(\mathcal{K})$ for which $\text{lk}(\sigma')$ is finite for all $\sigma' \supsetneq \sigma$. We now state analogues of Proposition 2.9 for the saddle connection complex.

Proposition 5.2 (Infinite link simplices). Let $\sigma \in \mathcal{A}(S, q)$ be a simplex. Then $\sigma \in \text{IL}(\mathcal{A}(S, q))$ if and only if $\sigma$ is disjoint from some cylinder curve on $(S, q)$.

The proof of the above proposition shall be given over the next two subsections. Before going into the details, let us give an immediate corollary.

Corollary 5.3 (Maximal infinite links simplices). A simplex $\sigma \in \mathcal{A}(S, q)$ is a triangulation away from a cylinder on $(S, q)$ if and only if $\sigma \in \text{MIL}(\mathcal{A}(S, q))$.

Proof. If $\sigma$ is a triangulation away from some cylinder $C$, then $\# \text{lk}(\sigma) = \infty$ by the above proposition. Any $\alpha \in \text{lk}(\sigma)$ must be a transverse arc of $C$, and so $\sigma \cup \{\alpha\}$ has a polygon as its unique non-triangular region. Therefore, $\text{lk}(\sigma \cup \{\alpha\})$ is finite and so $\sigma \in \text{MIL}(\mathcal{A}(S, q))$.

Conversely, if $\sigma \in \text{MIL}(\mathcal{A}(S, q))$ then there exists some cylinder $C$ disjoint from $\sigma$ by Proposition 5.2. Note that $\sigma$ cannot intersect any saddle connection in $\partial C$ transversely. By the maximality assumption, we deduce that $\partial C \subseteq \sigma$, and so $C$ is a non-triangular region of $S - \sigma$. If there exists
any $\alpha \in \text{lk}(\sigma)$ not contained in $C$, then $\text{lk}(\sigma \cup \{\alpha\})$ is infinite, violating the maximality assumption. Therefore, $\sigma$ must be a triangulation away from $C$. 

\[ \Box \]

\textbf{Remark 5.4 (Triangulations away from the same cylinder).} Observe that if $\sigma \in \text{MIL}(A(S,q))$ then the vertices of $\text{lk}(\sigma)$ are precisely the transverse arcs in the cylinder that is the unique region of $S - \sigma$.

Consequently, if $\sigma, \sigma' \in \text{MIL}(A(S,q))$ then $\text{lk}(\sigma) = \text{lk}(\sigma')$ if and only if they are triangulations away from the same cylinder.

To use the Cylinder Rigidity Theorem (Theorem 5) to prove our main theorem, we need to detect cylinder curves on $(S,q)$ using only combinatorial information from $A(S,q)$. By Remark 3.1, the set of cylinder curves $\text{cyl}(q)$ on $(S,q)$ can be regarded as a subset of either $C(S,Z)$ or $C(S)$; in other words, it does not matter whether or not we permit isotopies of curves to pass through $Z$. Corollary 5.3 can be rephrased as follows.

\textbf{Corollary 5.5 (Cylinder Test).} Let $\gamma$ be (an isotopy class of) a simple closed curve on $S$. Then $\gamma \in \text{cyl}(q)$ if and only if there exists some $\sigma \in \text{MIL}(A(S,q))$ disjoint from $\gamma$.

\subsection{A limiting geodesic}

Assume $\sigma \in \text{IL}(A(S,q))$. Since $\text{lk}(\sigma)$ is infinite, there exists a region $R$ of $\sigma$ containing infinitely many saddle connections. Our goal is to prove that $R$ contains a cylinder curve. As $R$ contains only finitely many singularities, we may choose a singularity $p \in R$ (possibly on the boundary $\partial R$) that forms an end point of infinitely many saddle connections in $R$. Since $\mathbb{RP}^1$ is compact, the slopes of these saddle connections ending at $p$ must accumulate; we shall rotate $(S,q)$ so that they accumulate on the horizontal slope.

Suppose $p$ has cone angle $k\pi$ and choose a sufficiently small $\epsilon > 0$ so that every saddle connection on $(S,q)$ has length greater than $2\epsilon$. Cutting the open $\epsilon$-neighbourhood of $p$ on $(S,q)$ along the $k$ horizontal line segments of length $\epsilon$ emanating from $p$ yields $k$ half-discs (with horizontal boundary) centred at $p$. Each half-disc has the form

$$\{z \in \mathbb{C} \mid |z| < \epsilon, \ 0 \leq \arg(z) \leq \pi\}$$

for some suitable choice of local co-ordinates, with $p$ identified with $0 \in \mathbb{C}$. Say a saddle connection $a : [0,1] \to (S,q)$ begins at a half-disc $H$ centred at $p$ if (up to possibly reversing the orientation) $a(0) = p$ and $a \cap H$ contains $a([0,\epsilon])$.

Consider an infinite sequence $a_i \in \text{lk}(\sigma)$ of saddle connections beginning at a common half-disc $H$, whose slopes $\theta_i \in \mathbb{RP}^1$ converge to the horizontal slope. Without loss of generality, we may assume that $0 < \theta_{i+1} < \theta_i < \pi/2$ for all $i$, that is, the slopes ‘decrease’ to the horizontal slope $\theta = 0$. (If this is not possible, then we can instead assume $\pi > \theta_{i+1} > \theta_i > \pi/2$ for all $i$ and argue similarly.) We shall define a ‘limiting’ geodesic $\beta$ of the $a_i$.

Let $\tilde{H}$ be a lift of $H$ to the universal cover $(\tilde{S},\tilde{q})$, with centre $\tilde{p}$ descending to $p$. Choose local co-ordinates so that $\tilde{H}$ is identified with

$$\{z \in \mathbb{C} \mid |z| < \epsilon, \ 0 \leq \arg(z) \leq \pi\} \subset \mathbb{C}.$$
For any $r > 0$, the open $r$-ball $B_r(\tilde{p})$ in $(\tilde{S}, \tilde{q})$ centred at $\tilde{p}$ contains finitely many singularities. Therefore, given $r > 0$ there exists some $\psi_r > 0$ such that the sector

$$U(r, \psi_r) := \{ z \in \mathbb{C} \mid |z| < r, \ 0 \leq \arg(z) \leq \psi_r \} \subset \mathbb{C}$$

isometrically embeds into $(\tilde{S}, \tilde{q})$, with the embedding agreeing with that of $\tilde{H}$ on $\tilde{H} \cap U(r, \psi_r)$ (compare Figure 5.2). In particular, the image of the interior of $U(r, \psi_r)$ in $(\tilde{S}, \tilde{q})$ contains no singularities. Therefore, any saddle connection starting at $\tilde{H}$ with slope $0 < \theta < \psi_r$ has length at least $r$. It follows that the lengths $|\alpha_i| \to \infty$ as $i \to \infty$.

Let $\tilde{\beta} : [0, \infty) \to (\tilde{S}, \tilde{q})$ be the horizontal unit-speed geodesic ray defined so that $\tilde{\beta}(t)$ coincides with the image of $t \in U(r, \psi_r)$ in $(\tilde{S}, \tilde{q})$ whenever $0 \leq t < r$. Let $\beta$ be its projection to $(S, q)$. Note that whenever $\beta$ passes through a singularity, it has an angle of $\pi$ to its left; in particular, if $\beta$ passes through a simple pole then it must immediately backtrack. The saddle connections $\alpha_i$ converge to $\beta$ in the following sense:

**Lemma 5.6.** Let $\tilde{\alpha}_i$ be the lift of $\alpha_i$ beginning at $\tilde{H}$. Then for all $r > 0$, the sequence $\tilde{\alpha}_i \cap B_r(\tilde{p})$ converges to $\tilde{\beta} \cap B_r(\tilde{p})$ as subsets of $(\tilde{S}, \tilde{q})$ under the Hausdorff topology.

**Proof.** Fix some $r > 0$. Then for all $i$ sufficiently large, we have $\theta_i < \psi_r$ and so $\tilde{\alpha}_i \cap B_r(\tilde{p})$ can be identified with the straight-line segment in $U(r, \psi_r)$ defined by $\arg(z) = \theta_i$ (together with the origin). Then the Hausdorff distance between $\tilde{\alpha}_i \cap B_r(\tilde{p})$ and $\tilde{\beta} \cap B_r(\tilde{p})$ is at most $r|\sin \theta_i|$. The result follows since $r|\sin \theta_i| \to 0$ as $i \to \infty$.

**Lemma 5.7.** Let $L$ be a non-horizontal line segment on $(S, q)$ that starts at a point on $\beta$, and leaves $\beta$ on its left. Then $L$ intersects some $a \in \text{lk}(\sigma)$ transversely. In particular, no saddle connection in $\sigma$ intersects $\beta$ transversely.

**Proof.** Suppose $L$ starts at $\tilde{\beta}(t)$ for some $t > 0$. Let $\tilde{L}$ be the lift of $L$ to $(\tilde{S}, \tilde{q})$ that starts at $\tilde{\beta}(t)$. For a fixed $r > t$, the pre-image of $\tilde{L}$ in $U(r, \psi_r)$ contains a straight line segment connecting $t \in \mathbb{C}$ to some point $z_0 \in \mathbb{C}$ satisfying $|z_0| < r$ and $0 < \arg(z_0) < \psi_r$; refer to Figure 5.2. For $i$ sufficiently large, the saddle connection $\alpha_i$ has slope $0 < \theta_i < \arg(z_0)$ and length at least $r$. Therefore, the lift $\tilde{\alpha}_i$ starting at $\tilde{H}$ must intersect $\tilde{L}$ transversely, and the first part of the statement follows.

In particular, the segment $L$ cannot be a segment of a saddle connection in $\sigma$.

### 5.2 Cylinders on minimal components

We shall use some results concerning measured foliations and their components. These can be defined for any foliation with constant slope on $(S, q)$ but, for simplicity, we shall assume we are always working with the horizontal foliation.
Every non-singular leaf is either closed or is dense in a subsurface, called a minimal component of the horizontal foliation. The horizontal foliation decomposes the surface $S$ into a finite union of maximal horizontal cylinders and minimal components. The boundary of each such component is formed by a set of horizontal saddle connections. Refer to [9] for more background.

Our goal is to prove that there is some cylinder curve on $(S, q)$ disjoint from $\sigma \in \text{IL}(A(S, q))$ from the previous section.

Recall that the limiting geodesic $\beta$ always has an angle of $\pi$ on its left whenever it passes through a singularity. This property, together with any segment $\beta([t_1, t_2])$ for any $0 \leq t_1 < t_2$, uniquely determines $\beta$ for all $t \geq t_2$. Therefore, if there exists some half-disc $H$ centred at a singularity $p$ such that $\beta$ passes through $p$ with $H$ on its left more than once, then it must eventually repeatedly run over some finite set of horizontal saddle connections. In this case, since $\beta$ always has an angle of $\pi$ to its left, there exists a closed leaf of the horizontal foliation on the left of $\beta$. Therefore, $\beta$ forms a boundary component of some horizontal cylinder $C$. If some saddle connection $\gamma \in \sigma$ intersects a core curve of $C$ transversely, then $\gamma \cap C$ contains a straight line segment starting at $\beta$, and leaving on its left. But this contradicts Lemma 5.7. Therefore, $\sigma$ is disjoint from a cylinder curve as desired.

Now assume $\beta$ does not pass through a singularity with a given half-disc on its left more than once. Since there are finitely many such half-discs on $(S, q)$, there is some largest value of $t_0 \geq 0$ such that $\beta(t_0)$ is a singularity. Then the geodesic ray $\beta' = \beta([t_0, \infty))$ is a non-compact leaf of the horizontal foliation on $(S, q)$. Let $X \subseteq S$ be the minimal component containing $\beta'$.

Lemma 5.8. If $\gamma \in \sigma$ is a non-horizontal saddle connection then it is disjoint from the interior of $X$.

**Proof.** Suppose $\gamma \in \sigma$ is not horizontal. If $\gamma$ meets the interior of $X$, then it must intersect $\beta'$ transversely since $\beta'$ is dense in $X$. But this contradicts Lemma 5.7. \hfill \square

The rest of this section is devoted to proving the following result. Together with the above lemma, this will complete the proof of Proposition 5.2.

**Proposition 5.9** (Subsurfaces with horizontal boundary contain cylinders). Let $\zeta \in A(S, q)$ be a set of horizontal saddle connections and $X$ be a connected component of $S - \zeta$. Then there exists a cylinder curve on $X$ that is disjoint from every horizontal saddle connection.

In particular, Proposition 5.9 implies that every vertex of $A(S, q)$ has infinite valence, as stated in Section 3.1: every saddle connection is disjoint from a maximal cylinder, and hence is disjoint from infinitely many saddle connections.

The proposition follows immediately if $X$ is a cylinder, so we assume otherwise. Our strategy is to adapt a theorem of Vorobets to the case of half-translation surfaces with horizontal boundary.

**Theorem 6** (Cylinders of definite width [37]). Let $\Sigma$ be a finite-area translation surface (without boundary), possibly with a finite set of removable singularities. Then there exists a Euclidean cylinder on $\Sigma$ with width at least $W \sqrt{\text{area}(\Sigma)}$, where $W$ is a constant depending only on the genus and the cone angles of singularities of $\Sigma$.

**Proof of Proposition 5.9.** First, double $X$ along its boundary to obtain a closed half-translation surface $X'$ (we regard any point arising from a boundary singularity with interior angle $\pi$ as a removable singularity on $X'$). The boundary arcs of $\partial X$ give rise to a set $\delta$ of disjoint horizon-
tal saddle connections on $X'$ which are fixed under the natural involution of $X'$. Next, take the canonical translation double branched cover of $X'$ to obtain a translation surface $\hat{X}$ (with simple poles lifting to removable singularities). Let $\hat{\delta}$ be the pre-image of $\delta$ on $\hat{X}$. Observe that any saddle connection on $\hat{X}$ that does not intersect $\hat{\delta}$ transversely descends to a saddle connection of the same length and slope on $X$. (Note that a saddle connection on $\hat{X}$ descends to a boundary saddle connection on $X$ if and only if it belongs to $\hat{\delta}$.) Conversely, every (non-boundary) saddle connection on $X$ has exactly four saddle connections in its pre-image on $\hat{X}$, also with the same slope and length.

By applying an $\text{SL}(2, \mathbb{R})$-deformation that shrinks in the horizontal direction, we can assume that all horizontal saddle connections on $\hat{X}$ have length strictly less than $W \sqrt{\text{area}(\hat{X})}$. Let $\hat{C}$ be a Euclidean cylinder on $\hat{X}$ as given by Theorem 6. Then any horizontal saddle connection on $\hat{X}$ must be disjoint from the interior of $\hat{C}$. Thus, any core curve $\hat{\eta}$ of $\hat{C}$ is disjoint from every horizontal saddle connection on $\hat{X}$. It follows that its image $\eta$ on $X$ is also disjoint from $\partial X$ and all horizontal saddle connections. But $\eta$ is a closed geodesic of constant slope, and so must be a cylinder curve on $X$, as desired. □

6 | FLOWING RESULTS

In this section, we shall establish some results concerning straight-line flows. A straight-line flow on $(S, q)$ in direction $\vartheta$ is an action of $\mathbb{R}_{>0}$ on $(S, q)$ such that for every $p \in S$, the map $t \mapsto t \cdot p$ is a unit-speed geodesic starting from $p$ with direction $\vartheta$. Strictly speaking, this action may not be defined for all times for some points as trajectories can hit singularities, however, this only occurs on a measure zero set. Thus, we will still call this action a flow as it is standard terminology in the context of (half-) translation surfaces.

For our purposes, we consider a flow $\varphi^t$ emanating from a given saddle connection $a \in \mathcal{A}(S, q)$ in some direction $\vartheta$ not parallel to $a$. To simplify the exposition, we shall assume that $a$ is vertical and the direction of the flow $\varphi^t$ is horizontal. However, the results in this section work in general by applying an appropriate $\text{SL}(2, \mathbb{R})$-deformation. The height of a saddle connection $a' \in \mathcal{A}(S, q)$ with respect to $\varphi^t$ is given by taking its length measured orthogonally to the flow direction. Equivalently, height($a'$) is the intersection number between $a'$ and the foliation on $(S, q)$ with slope $\vartheta$. Call $a \in \sigma$ a tallest saddle connection in $\sigma$ if it has maximal height among all saddle connections in $\sigma$. Analogously, we define the width of a saddle connection, measuring the length along the flow direction.

Throughout this section, we shall assume that $a$ is a vertical saddle connection in a simplex $\sigma \in \mathcal{A}(S, q)$, with height($a$) = $h > 0$. Equip $a$ with a unit-speed parameterisation and orientation $a : [0, h] \to (S, q)$ so that the flow $\varphi^t$ emanates from the right of $a$.

If the end points of $a$ coincide, then the flow $\varphi^t$ is not uniquely defined at the common end point. This will not cause us any problems; we shall work in the universal cover to define the two trajectories $\varphi^t(a(0))$ and $\varphi^t(a(h))$ in the next subsection.

A notion we shall use many times in the rest of this paper is that of visibility with respect to $\varphi^t$ and $\sigma$; a precise formulation is given in Section 6.2. Informally, a singularity $z$ is visible if some trajectory of $\varphi^t$ starting on the interior of $a$ hits $z$ at some time $t = t_0 > 0$ without intersecting any saddle connection of $\sigma$ transversely. (Under an additional condition, we also allow for trajectories to start at an end point of $a$.)

The following two propositions will be proven in the subsequent subsections.
Proposition 6.1 (Tallest saddle connections see singularities). Suppose $a$ is a tallest saddle connection in $\sigma$. Then there exists a visible singularity with respect to $\sigma$ and $\varphi^t$. In particular, if $a$ is strictly tallest in $\sigma$ and does not lie in a horizontal cylinder then there exists a visible singularity along some trajectory of $\varphi^t$ that starts in the interior of $a$.

Proposition 6.2 (Visible singularities yield visible triangles). Suppose that for some $y \in [0, h]$, the trajectory $\varphi^t(a(y))$ hits a visible singularity at $t = t_0 > 0$. Then there exists a triangle $T$ on $(S, q)$, with height($T$) = $h$, such that:

- $\partial T$ has no transverse intersections with $\sigma$, and
- $T$ has $a$ as one of its sides, and appears on the side of $a$ from which $\varphi^t$ emanates.

Furthermore, $T$ can be chosen so that the corner opposite of $a$ is $\varphi^{t_1}(a(y_1))$, where $t_1 \leq t_0$ and $y_1 \in (0, h) \cup \{y\}$. In particular, if $0 < y < h$ then $T$ has no horizontal sides.

In other words, if a singularity on $(S, q)$ is visible (not blocked by $\sigma$) along some trajectory of $\varphi^t$ starting from $a$, then there exists a simplex $\sigma' \supseteq \sigma$ such that $(S - \sigma', q)$ has a triangular region meeting $a$ on its right.

As many readers would be familiar with these flowing-style arguments for half-translationsurfaces, let us give an informal sketch of the main ideas; the formal proofs of the above propositions shall be given in the following two subsections.

To prove Proposition 6.1, suppose that the flow $\varphi^t$ emanating from $a$ does not see any visible singularities. Then for all $y \in [0, h]$, the flow $\varphi^t(a(y))$ will hit the interior of some saddle connection in $\sigma$ before any singularities. We then argue that there exists a saddle connection $a' \in \sigma$ that blocks every trajectory. But this implies that $a'$ is strictly taller than $a$.

For Proposition 6.2, suppose that $\varphi^t(y)$ hits a visible singularity at time $t = t_0$. Consider the two (oriented) topological arcs $\eta^+, \eta^-$ that start at an end point of $a$, run vertically along $a$ until $a(y)$, and then follow the horizontal flow trajectory until they hit $z$. Straighten $\eta^\pm$ to obtain its geodesic representative $t^*_q(\eta^\pm)$. Let $z^\pm_1$ be the terminal end point of the first saddle connection appearing along $t^*_q(\eta^\pm)$. Then at least one of $z^+_1$ or $z^-_1$ will be a corner of a triangle $T$ satisfying the desired properties.

6.1 Constructing a triangle

Choose local complex co-ordinates for $(S, q)$ so that $a(y)$ is given by $iy \in \mathbb{C}$. Thus, in these co-ordinates, $\varphi^t(a(y))$ is given by $t + iy$ for small $t \geq 0$. There are only countably many $y_i \in [0, h]$ such that the trajectory $\varphi^t(a(y_i))$ is singular: it (first) hits a singularity at some time $t = t_i > 0$ and so $\varphi^t$ cannot be defined for $t > t_i$. Let $Z \subset \mathbb{C}$ denote the countable set of points $z_i = t_i + iy_i$, arising in this manner. We claim that the set $Z$ is non-empty. By Poincaré recurrence, almost every trajectory starting in $[0, h]$ has to hit a copy of $a$. If some trajectory starting in $[0, h]$ hits a singularity without hitting a copy of $a$ then we are done, so let us assume otherwise. Given $y \in [0, h]$, let $a'$ be the first copy of $a$ hit by the trajectory starting at $y$. Consider the maximal subinterval $I \subseteq [0, h]$ containing $y$ such that every trajectory starting on $I$ hits $a'$ first among all copies of $a$. If the trajectory starting at some $y' \in I$ hits the interior of $a'$ then, by our assumption, so must every trajectory in an open neighbourhood of $y'$ in $[0, h]$. Therefore, at least one of the trajectories starting an end point of $I$ must hit an end point of $a'$, a singularity.
The trajectories \( \varphi^t(y) \) for all other \( y \in [0, h] \) can be defined for all \( t \geq 0 \). Therefore, the map \( \varphi \) with \( \varphi(t + iy) := \varphi^t(a(y)) \) is defined on

\[
Y := \{ z \in \mathbb{C} \mid \text{Re}(z) \geq 0, 0 \leq \text{Im}(z) \leq h \} \setminus \bigcup_{z_i \in Z} \{ z_i + s \mid s > 0 \} \subset \mathbb{C}.
\]

This domain is an infinite horizontal strip with countably many horizontal rays deleted (see Figure 6.1). The map \( \varphi : Y \to (S, q) \) is a locally isometric embedding (with respect to the induced Euclidean path metric on \( Y \)) compatible with the half-translation structure on \( (S, q) \). Furthermore, \( \varphi \) lifts to a map \( \tilde{\varphi} : Y \to (\tilde{S}, \tilde{q}) \). Since the set of singularities on the universal cover \( (\tilde{S}, \tilde{q}) \) is discrete, it follows that \( Z \subset \mathbb{C} \) is discrete also.

Let \( J_0 = a([0, h]) \subset Y \), that is \( J_0 \) is a vertical line segment, with \( J_0^- = 0 \) and \( J_0^+ = ih \) as its end points. Given \( z \in Z \), consider the finite sets

\[
Z^+(z) = \{ z' \in Z \mid \text{Im}(z') \geq \text{Im}(z), \text{Re}(z') \leq \text{Re}(z) \} \cup J_0^+ \quad \text{and} \\
Z^-(z) = \{ z' \in Z \mid \text{Im}(z') \leq \text{Im}(z), \text{Re}(z') \leq \text{Re}(z) \} \cup J_0^-.
\]

These are the points on the left and above \( z \), respectively, on the left and below \( z \). Let \( Y^\pm(z) \) be the convex hull of \( Z^\pm(z) \) in \( \mathbb{C} \). If \( Y^\pm(z) \) has non-empty interior, then there are two polygonal paths in \( \partial Y^\pm(z) \) from \( J_0^\pm \) to \( z \). Let \( \eta^\pm(z) \subseteq \partial Y^\pm(z) \) be the ‘left’ path: the one so that \( Y^\pm(z) \) lies entirely to the right (see Figure 6.1). If \( Y^\pm(z) \) is degenerate (which occurs precisely when the points in \( Z^\pm(z) \) are collinear), then it is a straight line segment from \( J_0^\pm \) to \( z \); we take \( \eta^\pm(z) \) to be this path in this situation. In either case, \( \eta^\pm(z) \) is a concatenation of straight line segments connecting consecutive points in some sequence

\[
J_0^\pm = z_0^\pm, z_1^\pm, \ldots, z_k^\pm = z,
\]

where each \( z_i^\pm \in Z^\pm(z) \) and \( k^\pm \geq 1 \).
Let $L_z$ be the horizontal line segment in $Y$ connecting $\text{Im}(z)$ to $z$. Note that $\eta^\pm(z)$ is path homotopic within $Y$ to the concatenation of the vertical path from $J_0^\pm$ to $\text{Im}(z) \in J_0$ with $L_z$. Moreover, $\varphi(\eta^\pm(z))$ is the unique geodesic path on $(S, q)$ from $\varphi(J_0^\pm)$ to $\varphi(z)$ in its path homotopy class. Note that $\varphi(\eta^\pm(z))$ is not necessarily simple on $(S, q)$, that is, homotopic to an arc with embedded interior. Observe that every point in $Z$ either lies on $\eta^+(z) \cup \eta^-(z)$, or to its right. Therefore, the polygonal region $R(z) \subset Y$ bounded by $J_0, \eta^+(z)$, and $\eta^-(z)$ contains no points of $Z$ in its interior, nor any points lying directly to the right of points in $Z$.

Let $T^\pm(z)$ be the triangle formed by taking the convex hull of $J_0 \cup z \pm 1$ in $\mathbb{C}$, and choose $T(z) \in \{T^+(z), T^-(z)\}$ to be a triangle with the lesser width. Then

$$R(z) \cap \{w \in \mathbb{C} \mid \text{Re}(w) \leq \text{width}(T(z))\}$$

is a trapezium containing $T(z)$, and so it follows that $T(z)$ intersects $Z$ precisely at its three corners. Applying Lemma 3.7, the restriction of $\varphi$ to $T(z)$ is injective, except for possibly at the corners. Note that the corner $z_1 \in \{z^+_1, z^-_1\}$ of $T(z)$ opposite $J_0$ satisfies $\text{Im}(z_1) = 0$ or $h$ only if $\text{Im}(z) = 0$ or $h$, respectively. We have thus constructed a triangle satisfying the following.

**Lemma 6.3** (Producing triangles). For any $z \in Z$, the triangle $T = \varphi(T(z))$ on $(S, q)$ has $a$ as one of its sides, appears on the side of $a$ from which $\varphi^i$ emanates, and satisfies $\text{height}(T) = h$ and $\text{width}(T) \leq \text{Re}(z)$. Moreover, the corner $z_1 \in Z$ of $T(z)$ opposite $J_0$ has imaginary part satisfying $\text{Im}(z_1) \in (0, h) \cup \{\text{Im}(z)\}$.

### 6.2 Visibility

We now wish to determine conditions so that $\varphi(T(z))$ has no sides intersecting any saddle connection in the simplex $\sigma \in A(S, q)$ transversely. Consider the pre-image $\varphi^{-1}(\sigma)$ in $Y$. This is a countable collection $J'(\sigma)$ of (maximal) straight line segments (or singletons contained in $Z$, but we may safely ignore these). Recall that in the construction of $Y$, the (open) horizontal ray starting from any point in $Z$ is deleted. Therefore, if there is a horizontal line segment in $J'(\sigma)$, it must have non-trivial intersection with $J_0$. Since $a$ is disjoint from any other saddle connection in $\sigma$, any horizontal line segments in $J'(\sigma)$ must have an end point at $J_0^\pm$. Let $J = J(\sigma)$ be the subset of $J'(\sigma) \setminus \{J_0\}$ consisting of all non-horizontal line segments (see Figure 6.1). Applying a Poincaré recurrence argument to the flow $\varphi^i$ on $(S, q)$ starting from the saddle connection $a$, we deduce that any trajectory that does not hit a singularity must eventually cross $a$ transversely. Therefore, the set $J$ is non-empty.

**Remark 6.4** (Types of line segments). Each line segment $J \in J$ is homeomorphic to either an open, half-open, or a closed interval. Any open end point of $J$ must lie on some deleted horizontal ray, and so is of the form $z + s$ for some $z \in Z$ and $s > 0$. Any closed end point of $J$ either belongs to $Z$, or has imaginary part equal to 0 or $h$. Note that $J \setminus Z$ maps into the interior of a saddle connection under $\varphi$.

Given a line segment $J \in J$, observe that $Y \setminus J$ has two connected components; let $Y(J) \subset Y$ be the component comprising all points lying directly to the right of some point on $J$. Define

$$V = V(\sigma) := Y \setminus \bigcup_{J \in J} Y(J).$$
Definition 6.5 (Visibility). Call a point \( p \in Y \) visible (with respect to \( J \) and \( \varphi' \)) if \( p \in V \). We also call its image \( \varphi(p) \) on \((S, q)\) visible with respect to \( \sigma \) and \( \varphi' \). More generally, we call any subset \( U \subseteq V \) and its image visible.

Let us point out here the difference of \( p \in Y \) being visible to the informal definition given at the beginning of the section. If \( \Im(p) \in (0, h) \), then the formal and informal definition are equivalent as the horizontal line segment from \( \Im(p) \) to \( p \) cannot cross any line segment \( J \in J \) transversely if \( p \in Y \). If \( \Im(p) \in \{0, h\} \), the characterisation by transverse crossings does not work. Indeed, suppose that \( J \) is a line segment in \( Y \) with positive slope, having an end point at \( J^-_0 \). Then the trajectory of \( \varphi' \) starting from \( J^-_0 \) does not cross \( J \) transversely for small \( t \), but the corresponding points lie in \( Y(J) \) and hence are not visible.

A point in \( Z \) having minimal real part is not necessarily visible (see the green point in Figure 6.1).

Lemma 6.6. If \( z \in Z \) is visible then the triangle \( T(z) \) is also visible.

Proof. We shall prove the contrapositive. Suppose that \( T(z) \) is not visible. Then \( T(z) \) contains some point lying directly to the right of some non-horizontal line segment \( J \in J \). It follows that \( J \) intersects the interior of \( T(z) \); see Figure 6.1. Since \( T(z) \subseteq R(z) \), the line segment \( J \) also intersects the interior of \( R(z) \). Therefore, \( J \cap R(z) \) must be a line segment connecting two points on \( \partial R(z) \). Note that \( J \) cannot intersect \( J_0 \), except possibly at the end points. Furthermore, the end points of \( J \cap R(z) \) cannot both lie on the same path \( \eta^\pm(z) \), for otherwise \( J \cap R(z) \subseteq Y^\pm(z) \), which is disjoint from the interior of \( R(z) \). Therefore, \( J \cap R(z) \) must connect some point on \( \eta^+(z) \setminus \{z\} \) to a point on \( \eta^-(z) \setminus \{z\} \). It follows that \( J \) must cross the line \( L_z \), and so \( z \) is not visible.

We now complete the proof of Proposition 6.2. Suppose that for some \( 0 \leq y \leq h \) and \( t_0 > 0 \), the point \( \varphi^{t_0}(a(h)) \) on \((S, q)\) is a visible singularity with respect to \( \varphi' \) and \( \sigma \). Then \( z = t_0 + iy \in Z \) is a visible point descending to this singularity. By the above lemma and Lemma 6.3, taking \( T = \varphi(T(z)) \) gives a triangle on \((S, q)\) satisfying the required properties. This completes the proof.

6.3 Visible singularities exist

Throughout the rest of this section, we assume that \( a \) is a tallest saddle connection of \( \sigma \). Under this assumption, we want to show that some point of \( Z \) is visible. We shall prove some stronger results.

For each \( J \in J \), observe that \( Y(J) \subset Y \) is open under the subspace topology. Therefore, \( \Im(J \setminus Z) = \Im(Y(J)) \) is a connected open subset of \([0, h] \subset \mathbb{R} \) under the subspace topology. We shall always work this topology in this section. Given distinct \( J, J' \in J \), observe that \( Y(J) \) and \( Y(J') \) are either nested or disjoint, and so the intervals \( \Im(J \setminus Z) \) and \( \Im(J' \setminus Z) \) are either nested or disjoint. In particular, if some point on \( J' \) lies strictly to the right of a point on \( J \setminus Z \), then \( Y(J') \subset Y(J) \). We deduce the following.

Lemma 6.7 (Visibility and nesting). Let \( J \in J \). If some point on \( J \setminus Z \) is visible, then \( J \) is visible. Furthermore, \( J \) is visible if and only if for all \( J' \in J \) such that \( J' \neq J \), either \( Y(J') \cap Y(J) = \emptyset \) or \( Y(J') \subset Y(J) \) holds.
Write $\mathring{J} \subseteq J$ for the set of non-horizontal visible line segments. This is precisely the set of $J \in J$ for which $Y(J)$ is maximal with respect to inclusion. In particular, $\mathring{J}$ is non-empty.

For every $y \in [0, h]$, the trajectory $\varphi'(a(y))$ terminates at some point in $Z$, or eventually crosses some $J \in J$ by Poincaré recurrence. (In the former case, the trajectory could possibly cross some $J \in J$ before terminating.) By the above lemma, the first line segment in $J$ crossed by the trajectory (if it exists) must be visible. Since $Z$ is countable, we deduce that the disjoint union

$$\bigsqcup_{J \in J} \operatorname{Im}(J \setminus Z)$$

is dense in $[0, h]$. Our goal is to show that this union is not equal to $[0, h]$.

**Lemma 6.8** (Tall saddle connections). Let $J \in J$ be a straight line segment. If $\operatorname{Im}(J \setminus Z) \supseteq [0, h)$ then one end point of $J$ is at $J_0^- = 0$, and the other has imaginary part $h$. If $\operatorname{Im}(J \setminus Z) \supseteq (0, h]$ then one end point of $J$ is at $J_0^+ = ih$, and the other is on the real axis. In addition, $\operatorname{Im}(J \setminus Z) \neq [0, h]$.

**Proof.** Assume that $\operatorname{Im}(J \setminus Z) \supseteq [0, h)$. Then $J$ is a line segment that connects a point $p$ with $\operatorname{Im}(p) = 0$ to a point $p'$ with $\operatorname{Im}(p') = h$. Note that $p \in J$, while $p'$ either lies in $J$, or is an open end point of $J$. If $p \neq 0$, then $\varphi$ maps $J$ into some saddle connection $a' \in \sigma$ on $(S, q)$, with $\varphi(p)$ contained in the interior of $a'$. But this implies that $\operatorname{height}(a') > \operatorname{height}(J) = h$, contradicting the assumption that $a$ is tallest in $\sigma$, and so $p = 0$. By a similar argument, we deduce that if $\operatorname{Im}(J \setminus Z) \supseteq (0, h]$ then $ih$ is an end point of $J$. Finally, if $\operatorname{Im}(J \setminus Z) = [0, h]$ then the end points of $J$ are 0 and $ih$ and hence $J$ must coincide with $J_0 \not\in J$, a contradiction. 

It follows that for every $J \in \mathring{J}$, the interval $\operatorname{Im}(J \setminus Z)$ has some open end point $y \in [0, h]$. In fact, every point of $[0, h] \setminus \bigsqcup_{J \in J} \operatorname{Im}(J \setminus Z)$ arises in this manner. Note that if $\operatorname{Im}(J \setminus Z) = (0, h)$, then the closure of $\varphi(J)$ in $(S, q)$ is a tallest saddle connection of $\sigma$. In particular, if $a$ is strictly tallest then any such $J$ must map onto $a$ under $\varphi$, in which case $a$ is contained in a horizontal cylinder. Therefore, if $a$ is strictly tallest and not contained in a horizontal cylinder then $\operatorname{Im}(J \setminus Z)$ has an open end point $y \in (0, h)$ for all $J \in \mathring{J}$.

By Remark 6.4, there exists some $z \in Z$ satisfying $\operatorname{Im}(z) = y$. Note that $z$ must be visible, for otherwise $z \in Y(J)$ for some $J \in J$, and so $y = \operatorname{Im}(z) \in \operatorname{Im}(Y(J)) = \operatorname{Im}(J \setminus Z)$, a contradiction. The results in this section can be summarised as follows.

**Proposition 6.9** (Structure of the set of visible line segments). The set $\{\operatorname{Im}(J \setminus Z) \mid J \in \mathring{J}\}$ is a collection of pairwise disjoint connected open sets in $[0, h]$, whose union is dense in $[0, h]$. Furthermore, the complement of this union is non-empty, and comprises precisely of all $y \in [0, h]$ for which there exists a visible $z \in Z$ satisfying $\operatorname{Im}(z) = y$. In particular, if $a$ is strictly tallest and not contained in a horizontal cylinder, then there exists a visible $z \in Z$ where $0 < \operatorname{Im}(z) < h$.

Proposition 6.1 now follows from the above proposition. We conclude this section with one final result.

**Lemma 6.10** (Visibility after extending simplices). Suppose $\sigma \subseteq \sigma'$ are simplices in which $a$ is a tallest saddle connection. Assume $J \in \mathring{J}(\sigma)$ is visible. Then either $J \in \mathring{J}(\sigma')$, or there exists some $J' \in \mathring{J}(\sigma')$ such that $\operatorname{Im}(J' \setminus Z) \supseteq \operatorname{Im}(J \setminus Z)$. In the latter case, $J'$ maps into some saddle connection in $\sigma' \setminus \sigma$ under $\varphi$. 
**FIGURE 7.1** All horizontal and vertical saddle connections have the same length. There is a heptagon $R$ on this half-translation surface bounded by the saddle connections indicated in thick lines. The saddle connections $a', b', c'$ must intersect one another transversely inside $R$. (The gluings are indicated by the number of arrows, not by colour)

**Proof.** Suppose $J$ is not visible with respect to $J(\sigma')$. Then there exists some $J' \in J(\sigma')$ such that $Y(J') \supset Y(J)$, and so $\text{Im}(J' \setminus Z) \supset \text{Im}(J \setminus Z)$. Observe that $J' \notin J(\sigma)$, for otherwise $J$ would not be visible with respect to $J(\sigma)$. Therefore, $\varphi(J')$ is contained in some saddle connection in $\sigma' \setminus \sigma$. □

# 7 | EXTENDING TRIANGLES

Recall from Lemma 2.12 that any non-folded (topological) triangle $T$ on $(S, Z)$ has all three sides flippable in any triangulation $T' \in \mathcal{F}(S, Z)$ containing $T$. In contrast, this is far from true in the case of triangulations on half-translation surfaces (see Example 7.2). Nevertheless, knowing that some sides of a triangle are flippable will help us to detect neighbouring triangles. Let us begin with an observation that will be relevant in the next example.

**Remark 7.1 (Flipping within a pentagon).** Let $T$ be a triangle on $(S, q)$ with sides $a, b, c \in A(S, q)$. Assume $T$ is not (1,1)-annular. Suppose $T$ is a triangulation containing $T$ in which both $a$ and $b$ are flippable. Appealing to Lemma 4.6, we deduce that $T - \{a, b\}$ has an almond pentagon as its unique non-triangular region, in which $a$ and $b$ form non-intersecting diagonals. Consequently, the saddle connections $a'$ and $b'$ obtained by, respectively, flipping $a$ and $b$ in $T$ can only intersect in the interior of $T$.

**Example 7.2 (Triangle with flipping difficulties).** Let $T$ be the triangle with sides $a, b, c$ as shown in Figure 7.1. Observe that $T$ cannot be (1,1)-annular.

Let $R$ be the heptagonal region bounded by the thick saddle connections. The boundary $\partial R$ is partitioned into six subintervals: as one follows the boundary, these intervals cycle through the colours red, orange, and blue, while also alternating between being open and closed. Any saddle connection $a'$ obtained by flipping $a$ in any triangulation containing $T$ must contain a line segment in $R$ connecting the two red intervals. Similarly, any $b'$ or $c'$ obtained by flipping $b$ or $c$
will contain a line segment, respectively, connecting two blue intervals or two orange intervals. Then \(a', b', c'\) must pairwise intersect in the interior of \(R\). By the above remark, we deduce that \(T\) has at most one flippable side in any triangulation containing \(T\).

The goal of this section is to prove the following. In light of the above example, this is the strongest result one could hope for in general. Recall that a pentagon is almond if it has at most one broken diagonal, that is, at most one topological diagonal that cannot be realised by a saddle connection.

**Proposition 7.3** (Extending triangles). Suppose \(T\) is a triangle on \((S, q)\) with sides \(a, b, c \in \mathcal{A}(S, q)\). Then there exists a triangulation \(\mathcal{T} \supseteq \partial T\) in which \(a\) is flippable, and such that \(\text{lk}(\mathcal{T} \setminus \{a, b\})\) contains \(P_3\) as an induced subgraph.

If \(b\) is also flippable in \(\mathcal{T}\), then the non-triangular region of \(\mathcal{T} \setminus \{a, b\}\) is either a (1,1)-annulus or an almond pentagon and \(\text{lk}(\mathcal{T} \setminus \{a, b\})\) contains \(\bullet a \bullet b \bullet\) as a subgraph; otherwise \(b\) is a non-flippable diagonal of an almond pentagon of \(\mathcal{T} \setminus \{a, b\}\), in which case \(\text{lk}(\mathcal{T} \setminus \{a, b\})\) is equal to \(\bullet a \bullet b \bullet\).

We can also obtain stronger extension results for certain triangles.

**Proposition 7.4** (Extending major triangles). Suppose \(T\) is a triangle on \((S, q)\) with sides \(a, b, c \in \mathcal{A}(S, q)\). Assume \(Q \supseteq T\) is a strictly convex quadrilateral that has \(c\) as a diagonal, and satisfying \(\text{area}(T) \geq \frac{1}{2}\text{area}(Q)\). Then the triangulation \(\mathcal{T} \supseteq \partial T\) in Proposition 7.3 can be chosen so that both \(a\) and \(c\) are flippable. Furthermore, if \(T\) is not (1,1)-annular, then \(\mathcal{T}\) can be chosen so that \(\partial Q \subseteq \mathcal{T}\).

**Definition 7.5** (Major triangle). Call \(T\) a major triangle if there exists a strictly convex quadrilateral \(Q \supseteq T\) such that \(\text{area}(T) \geq \frac{1}{2}\text{area}(Q)\); if this holds, call the side of \(T\) that forms a diagonal of \(Q\) a base of \(T\). A major triangle may have more than one base. Any (1,1)-annular triangle is major.

The proofs of Propositions 7.3 and 7.4 are given in Subsections 7.1 and 7.2, respectively. In these subsections, we shall continue using the following notation as defined in the previous section: the collection of line segments \(J_i\), the segment \(J_0\) and its end points \(J_0^\pm\), the singularities \(Z \subset \mathbb{C}\), the domain \(Y \subset \mathbb{C}\), and the locally isometric embedding \(\varphi : Y \to (S, q)\).

### 7.1 Extending triangles to almond pentagons or (1,1)-annuli

Let \(T\) be a triangle on \((S, q)\) with sides \(a, b, c \in \mathcal{A}(S, q)\). Apply an \(\text{SL}(2, \mathbb{R})\)-deformation to \((S, q)\) to make \(a\) vertical and \(c\) horizontal. This ensures that \(a\) and \(b\) are both tallest saddle connections in the simplex \(\tau = \{a, b, c\} \in \mathcal{A}(S, q)\), with height \(h = \text{height}(T) > 0\). Let \(\varphi'\) be the horizontal unit-speed flow emanating from \(a\), and flowing away from \(T\). Equip \(a\) with a unit-speed parameterisation \(a : [0, h] \to (S, q)\) so that \(\varphi'\) emanates from its right. There is a unique isometric identification of \(T\) with a Euclidean triangle in \(\mathbb{C}\) so that \(a(y)\) maps to \(iy\) for \(0 \leq y \leq h\). Since \(T\) appears to the left of \(a\), the map \(\varphi : Y \to (S, q)\) can be extended to a locally isometric embedding defined on \(T \cup Y \subset \mathbb{C}\).

Without loss of generality, we shall assume that the sides \(a, b, c\) appear in anticlockwise order around \(T\) in \(\mathbb{C}\), and so \(c\) is a line segment lying on the negative real axis. Our goal is to extend \(\tau\)
FIGURE 7.2 The line segment $J$ is an awning for $a$ with respect to $J$ and the flow $\phi'$. The unique visible singularity $z \in Z$ may possibly lie on an end point of $J$.
The two cases corresponding to when \( \text{Im}(z') > 0 \) or \( \text{Im}(z') = 0 \). In either case, the pentagon \( P \) formed by gluing \( T', T'', \) and \( T''' \) along \( a \) and \( b \) is almond.

**Definition 7.7 (Awning).** An awning for \( a \) (with respect to \( \mathcal{J} \) and the flow \( \varphi' \)) is a visible non-horizontal line segment \( J \in \mathcal{J}(\sigma) \), with one end point at \( J_0^+ = ih \in Y \) and the other on the real axis (as in condition (iii) in the above lemma). If \( J \) is an awning for \( a \), then \( \varphi(J) \) maps into a tallest saddle connection \( a' \) of \( \sigma \), which we shall also refer to as an awning.

Note that any awning \( a' \) for \( a \) cannot be parallel to \( a \), so \( a' \neq a \). Furthermore, \( a \) and \( a' \) must have the same height since \( a \) is tallest in \( \sigma \).

Let us now return our attention to Proposition 7.3. For this situation, we choose \( \sigma = \tau \). We will construct a triangulation that contains \( \tau \) and in which \( a \) is flippable. Any awning \( a' \in \tau \) is a tallest saddle connection with negative slope. But the tallest saddle connections in \( \tau = \{a, b, c\} \) are \( a \) and \( b \), neither of which have negative slope, and so \( a \) does not have any awnings. By the above lemma, there exists a visible singularity \( z \in Z \) such that \( \text{Im}(z) > 0 \). This gives a visible triangle \( T' = T(z) \) which can be glued to \( T \) along \( a \) to form a strictly convex quadrilateral \( Q \). Moreover, \( \text{height}(T') = \text{height}(a) \). Therefore, \( a \) and \( b \) remain tallest in the simplex \( \tau' = \partial \tau \cup \partial T' \supset \tau \).

If \( b \) is also a side of \( T' \), then gluing \( Q \) along the two copies of \( b \) yields a \((1,1)\)-annulus in which \( a \) and \( b \) are both transverse arcs. This gives a link as desired.

Now suppose \( b \) is not a side of \( T' \). Consider the horizontal unit-speed flow emanating from \( b \) away from \( T \). By Propositions 6.1 and 6.2, there exists a triangle \( T'' \) meeting \( Q \) along \( b \), of \( \text{height}(T'') = h \), and such that \( \partial T'' \) has no transverse intersections with \( \tau \) (see Figure 7.3). The isometric embedding \( T \cup Y \to \mathbb{C} \) can be uniquely extended to an isometric embedding \( T'' \cup T \cup Y \to \mathbb{C} \). Let \( z' \in \mathbb{C} \) be the corner of \( T'' \) opposite \( b \). Note that \( \text{Re}(z') < 0 \) and \( 0 \leq \text{Im}(z') \leq h \).

Let \( P \) be the pentagon formed by gluing \( Q \) and \( T'' \) along \( b \). Note that \( a, b, \) and the diagonal of \( Q \) obtained by flipping \( a \) are all diagonals of \( P \). If \( \text{Im}(z') > 0 \), then \( b \) is also flippable in \( P \) which implies that \( P \) is an almond pentagon. Therefore, \( \text{lk}(\mathcal{T} \setminus \{a, b\}) \) is either \( C_5 \) (if \( P \) is strictly convex) or \( \bullet a \bullet b \bullet \) (if \( P \) has a broken diagonal, occurring precisely when \( \text{Im}(z') = h \)). If \( \text{Im}(z') = 0 \), the straight line segment in \( \mathbb{C} \) connecting \( z' \) to \( z \) gives a fourth diagonal of \( P \) and so \( P \) is an almond pentagon. In this case, \( \text{lk}(\mathcal{T} \setminus \{a, b\}) \) is equal to \( a \bullet b \bullet \bullet \bullet \).

This completes the proof of Proposition 7.3.

**Remark 7.8 (Specific link after extending triangles).** From the above proof, we see that \( \text{lk}(\mathcal{T} \setminus \{a, b\}) \) contains \( a \bullet b \bullet \bullet \bullet \) unless \( \text{Im}(z') = h \). In this case, if \( z' \) is the only visible singularity (seen from \( b \)), then \( b \) is contained in a horizontal cylinder \( C \) of height \( h \). The lower boundary of \( C \) can only contain one singularity and, moreover, \( C \) contains at least three disjoint saddle connections of height \( h \). Hence, \( T \) is a \((m, 1)\)-annular triangle for some \( m \geq 2 \).
FIGURE 7.4 The two cases in the proof of Lemma 7.9. Left: \( a' \) and \( J \) are related by a rotation by \( \pi \) about \( p \). Right: \( a' \) and \( J \) are related by a translation \( g \). There may be singularities on the horizontal boundary of \( R \) (black dotted line).

7.2 Extending major triangles

In this section, we assume that \( T \) is a major triangle with sides \( a, b, c \in A(S, q) \), having \( c \) as a base. Thus, there exists another triangle \( T' \) having \( c \) as a side such that gluing \( T \) and \( T' \) along \( c \) yields a strictly convex quadrilateral \( Q \). Apply an \( \text{SL}(2, \mathbb{R}) \)-deformation to make \( c \) horizontal and \( a \) vertical. Thus, \( \text{height}(T) \geq \text{height}(T') \) and so \( a \) and \( b \) are tallest saddle connections in \( \partial Q \cup \{c\} \). Note that \( c \) is flippable in any triangulation containing \( \partial Q \cup \{c\} \).

As in the previous subsection, we isometrically identify \( T \) with a triangle in \( \mathbb{C} \) so that \( a \) lies on the positive imaginary axis, \( c \) on the negative real axis, with their common corner at the origin. The triangle \( T' \) can be uniquely isometrically identified with a Euclidean triangle in the third quadrant of \( \mathbb{C} \), so that it glues to the copy of \( T \) in \( \mathbb{C} \) along \( c \) with the correct orientation. The map \( \varphi : Y \to (S, q) \) extends to a locally isometric embedding on \( \varphi : T \cup T' \cup Y \to (S, q) \), which has a lift \( \tilde{\varphi} \) to the universal cover.

Our goal is to prove the existence of a triangulation containing \( \partial T \) in which both \( a \) and \( c \) are flippable.

Lemma 7.9 (Major triangles and flippability). Suppose \( a \) is non-flippable in every triangulation \( T \supseteq \partial Q \cup \{c\} \). Then there exists a \((1,1)\)-annulus \( A \) in which \( a \) and \( c \) are transverse arcs, and \( b \) is contained in \( \partial A \). In particular, \( a \) and \( c \) are flippable in any triangulation containing \( \partial A \cup \{a, c\} \). In this case, we also have \( \text{height}(T) = \text{height}(T') \).

Proof. By Lemma 7.6, there exists an awning \( J \in J \) for \( a \) which descends to a tallest saddle connection \( a' \in \sigma = \partial Q \cup \{c\} \) via \( \varphi \). Since \( a' \) has negative slope, it cannot coincide with \( a, b, \) nor \( c \). Furthermore, the side \( b' \) of \( Q \) opposite \( b \) has positive slope since \( Q \) is strictly convex, and so does not coincide with \( a' \) neither. Therefore, \( a' \) must be the side of \( Q \) opposite \( a \). We shall identify \( a' \) with its copy in \( \partial T' \subset \mathbb{C} \), lying in the third quadrant. Furthermore, we have \( \text{height}(T') = \text{height}(a') = \text{height}(T) \).

Since \( a' \) and \( J \) have the same image on \( (S, q) \), there is a deck transformation \( \tilde{g} \) of the universal cover sending \( \tilde{\varphi}(a) \) to \( \tilde{\varphi}(J) \). We can realise \( \tilde{g} \) in local co-ordinates: there exists a map \( g : \mathbb{C} \to \mathbb{C} \), of the form \( g(w) = \pm w + w_0 \) for some \( w_0 \in \mathbb{C} \), such that \( (\tilde{g} \circ \tilde{\varphi})(w) = (\varphi \circ g)(w) \) whenever \( w, g(w) \in T \cup T' \cup Y \). Refer to Figure 7.4.

We claim that \( g \) must be a translation; in which case we have \( w_0 = \text{length}(c) + i \text{length}(a) \). Suppose otherwise, for a contradiction. The unique half-translation of \( \mathbb{C} \) mapping \( a' \) to \( J \) is a rotation by \( \pi \), with centre at the intersection point \( p \) of the two diagonals of \( Q = T \cup T' \) in \( \mathbb{C} \).
Let $R \subset \mathbb{C}$ be the region bounded by $J_0, J$, and the real axis. Since the awning $J$ is visible, the interior of $R$ is also visible and hence contains no singularities. Now consider the triangle $g(T') \subset \mathbb{C}$. This triangle has $J$ as one of its sides, and the point $w_0$ as the opposite corner. Furthermore, $R$ and $g(T')$ can be glued along $J$ to form a strictly convex quadrilateral $Q' \subset \mathbb{C}$ with no interior singularities. Note that $Q'$ is not necessarily a subset of $Y$, since the unique visible singularity $z \in Z$ could lie strictly to the left of the endpoint of $J$ on the real axis. However, the points in $Q'$ strictly to the right of $z$ are the only points in $Q \setminus Y$. Therefore, the line segment $J'$ connecting $0$ to $w_0$ in $\mathbb{C}$ lies in $Y$. The triangle $T'' \subset Q' \cap Y$ with sides $a$, $g(c)$, and $J'$ descends to a triangle on $(S, q)$ with $a$ and $c$ as two of its sides. This triangle cannot coincide with $T$ on $(S, q)$, since its interior overlaps that of $T'$. Gluing the triangles $T$ and $T''$ along $a$ and $c$ gives the desired $(1,1)$-annulus $A$ on $(S, q)$. □

This completes the proof of Proposition 7.4. We conclude this section with a lemma which will be useful in Sections 8 and 9.

**Lemma 7.10** (Annular triangles and flippability). Let $T$ be a triangle on $(S, q)$ with sides $a, b, c \in A(S, q)$. Suppose that $A$ is an $(m, 1)$-annulus that contains $b$ and $c$ as transverse arcs, where $m \geq 1$. Let $\sigma = \partial A \cup \{ b, c \}$. Then there exists a triangulation $T \supseteq \sigma$ in which $a, b,$ and $c$ are flippable. If $A$ is a $(1,1)$-annulus, then $T$ can be chosen such that every saddle connection in $\sigma$ is flippable.

**Proof.** Observe that $b$ and $c$ are flippable in any triangulation $T \supseteq \sigma$. As before, assume $a$ is vertical and $c$ is horizontal. Without loss of generality, suppose $b$ has positive slope. Note that $a$ and $b$ are tallest in $\sigma$.

Consider the horizontal straight-line flow emanating from $a$ away from $A$. By Lemma 7.6, the only obstruction for the flippability of $a$ in every triangulation $T \supseteq \sigma$ is the presence of an awning $J \in J$ for $a$, which maps to a tallest saddle connection of $\sigma$ via $\varphi$. Such an awning must have negative slope, but this does not hold for any saddle connection of $\sigma$. Therefore, $a$ does not have an awning. Applying Proposition 7.3, there exists a triangle $T''$ of height$(T'') = \text{height}(a)$ that can be glued to $A$ along $a$ so that $a$ is flippable in any triangulation containing $\sigma' = \sigma \cup \partial T''$.

Assume now that $A$ is a $(1,1)$-annulus and let $a$ and $d$ be the saddle connections forming $\partial A$. Then $d$ is also a tallest saddle connection in $\sigma$. As $a$ is flippable in $\sigma'$, $\partial T''$ cannot contain any saddle connection of negative slope that is tallest in $\sigma'$. Consider now the horizontal straight-line flow emanating from $d$ away from $A$. As before, an awning for $d$ must have negative slope. But such an awning cannot exist in $\sigma'$. Therefore, there exists a triangulation $T \supseteq \sigma'$ in which $d$ is also flippable. □

### 8 FLIP PAIRS AND CONVEX QUADRILATERAL BOUNDARIES

In this section, we attempt to detect the boundary of a strictly convex quadrilateral $Q$ given its diagonals $c$ and $d$, using only the combinatorial properties of $A(S, q)$. First, we define the sets of **barriers** $B(c, d)$ and of **flippable barriers** $FB(c, d)$ in $A(S, q)$ satisfying $FB(c, d) \subseteq \partial Q \subseteq B(c, d)$. However, these inclusions may be strict as we will show in Example 8.6. The main result of this section is the careful definition of a set $KFB(c, d) \subseteq \partial Q$, which we call **kite-or-flippable barriers**, that
is guaranteed to contain at least three sides of \( Q \) (see Corollary 8.14). This will play an important role in our proof of the Triangle Test in the following section.

### 8.1 Barriers

Recall from Lemma 4.4 that a simplex \( \sigma \in A(S, q) \) has link \( \text{lk}(\sigma) \cong \mathbb{N}_2 \) if and only if the unique non-triangular region of \( S - \sigma \) is a strictly convex quadrilateral. Moreover, the two diagonals of the quadrilateral are precisely the vertices of \( \text{lk}(\sigma) \).

**Definition 8.1 (Flip pair).** The set of *flip pairs* of \( A(S, q) \) is

\[
\text{FP}(A(S, q)) := \{ \text{lk}(\sigma) \mid \sigma \in A(S, q), \text{lk}(\sigma) \cong \mathbb{N}_2 \}.
\]

Given a flip pair \( \{c, d\} \), there exists a (unique) strictly convex quadrilateral \( Q = Q(c, d) \) in which they form the diagonals. We would like to recover the simplex \( \partial Q \in A(S, q) \) of boundary saddle connections purely combinatorially from \( \{c, d\} \). However, this is not so straightforward. As a first approximation, we define the following.

**Definition 8.2 (Barrier).** The set of *barriers* of a flip pair \( \{c, d\} \in \text{FP}(A(S, q)) \) is

\[
B(c, d) := \{ \gamma \in A(S, q) \mid \gamma \neq c, d \text{ and } \forall \delta \in A(S, q) \text{ with } \delta \cap \gamma \text{ we have } \delta \cap c \text{ or } \delta \cap d \}.
\]

Observe that any saddle connection \( \gamma \) intersecting \( \partial Q \) transversely must necessarily intersect at least one of its diagonals, and so \( \partial Q \subseteq B(c, d) \). However, the barriers do not always coincide with \( \partial Q \); see Example 8.6. This disparity can be characterised as follows.

**Lemma 8.3 (Cordons appear as barriers).** The saddle connections in \( B(c, d) \setminus \partial Q \) are precisely the cordons of \( \partial Q \). Consequently, any triangulation \( \mathcal{T} \) containing \( \partial Q \) also contains \( B(c, d) \).

**Proof.** Suppose \( \gamma \notin \partial Q \) is a saddle connection. Recall from Definition 4.2 that \( \gamma \) is a cordon of \( \partial Q \) if and only if \( \gamma \in \text{lk}(\partial Q) \) and \( \gamma \) does not intersect any saddle connection in \( \text{lk}(\partial Q) \). This is equivalent to saying that whenever some \( \delta \in A(S, q) \) intersects \( \gamma \), then \( \delta \) also intersects some saddle connection of \( \partial Q \), and hence at least one of \( c \) or \( d \). By definition, this holds precisely when \( \gamma \in B(c, d) \setminus \partial Q \).

To exclude the cordons of \( \partial Q \), we use the fact that they cannot be flipped in any triangulation containing \( \partial Q \). This motivates a second approximation.

**Definition 8.4 (Flippable barrier).** The set of *flippable barriers* of a flip pair \( \{c, d\} \) is

\[
\text{FB}(c, d) = \{ \gamma \in B(c, d) \mid \exists \text{ a triangulation } \mathcal{T} \supseteq B(c, d) \text{ in which } \gamma \text{ is flippable} \}.
\]

This set can be characterised purely combinatorially, as the flippability condition can be restated as \( \text{lk}(\mathcal{T} \setminus \{\gamma\}) \cong \mathbb{N}_2 \). Since a cordon for \( \partial Q \) cannot be flipped in any triangulation containing \( \partial Q \), we deduce that \( \text{FB}(c, d) \subseteq \partial Q \).
RIGIDITY OF THE SADDLE CONNECTION COMPLEX

FIGURE 8.1 The flip pair \( \{c, d\} \) forms the diagonals of a strictly convex quadrilateral \( Q(c, d) \), whose sides are indicated by thick line segments. The saddle connections in \( \text{FB}(c, d) \) are given as dashed orange lines, while those in \( \text{B}(c, d) \setminus \text{FB}(c, d) \) are in blue.

Lemma 8.5 (Over–under). Given any flip pair \( \{c, d\} \), we have \( \text{FB}(c, d) \subseteq \partial Q(c, d) \subseteq \text{B}(c, d) \).

Unfortunately, these inclusions may be strict, as the following example demonstrates.

Example 8.6 (Non-flippable barriers). Figure 8.1 shows a genus 2 translation surface formed by gluing opposite sides of a regular octagon. The flip pair \( \{c, d\} \) forms the diagonals of a strictly convex quadrilateral \( Q(c, d) \). The flippable barriers are indicated as dashed orange lines, while the non-flippable barriers are given in blue. There are three cordons of \( \partial Q(c, d) \); these are the non-horizontal blue saddle connections. In this example, we have the strict inclusions \( \text{FB}(c, d) \subsetneq \partial Q(c, d) \subsetneq \text{B}(c, d) \).

8.2 Cylindrical kites

We now show that it is possible for a flip pair to only have two flippable barriers, and prove that this happens precisely when the flip pair satisfies the following conditions.

Definition 8.7 (Cylindrical kite). Suppose \( Q \) is a strictly convex quadrilateral, with diagonals \( c \) and \( d \), that can be made into a (non-rhombus) Euclidean kite under \( \text{SL}(2, \mathbb{R}) \)-deformations; this occurs if and only if exactly one of its diagonals bisect the other. Without loss of generality, suppose that \( c \) is horizontal and \( d \) is vertical, and that \( c \) bisects \( d \) (see Figure 8.2). Let \( T \) and \( T' \) be the triangles obtained by cutting \( Q \) along \( c \). We say \( Q \) is a cylindrical kite if there exists a horizontal Euclidean cylinder \( C \) such that:

\[ \bullet \ C \text{ contains } T \text{ and } T', \text{ and thus all saddle connections of } \partial Q \text{ are transverse arcs of } C, \]
\[ \bullet \ C - \partial Q \text{ has exactly four regions, among them } T \text{ and } T', \text{ and} \]
\[ \bullet \text{ for each such region } R \text{ with } T \neq R \neq T', \text{ we require the two non-horizontal sides to be adjacent in } R \text{ and opposite in } Q. \]

We also call a flip pair \( \{c, d\} \) a cylindrical kite pair if the quadrilateral \( Q(c, d) \) is a cylindrical kite.

Remark 8.8. If \( \{c, d\} \) is a cylindrical kite pair with a cylinder \( C \) as above, then exactly one of \( c \) or \( d \) belongs to \( \partial C \).
FIGURE 8.2 A cylindrical kite $Q(c, d)$ with sides $a, b, e,$ and $f$. The triangles $T$ and $T'$ are obtained by cutting $Q(c, d)$ along $c$, and are contained in a horizontal cylinder $C$. There may be singularities on the horizontal dotted lines lying on the boundary of $C$. The saddle connections $b'$ (in red, dashed) and $a'$ (in blue, dotted) appear in the proof of Proposition 8.11.

**Lemma 8.9** (At most two flippable barriers in cylindrical kites). If $\{c, d\} \in \text{FP}(A(S, q))$ is a cylindrical kite pair then $\# \text{FB}(c, d) \leq 2$.

*Proof.* Let $Q = Q(c, d)$ be a cylindrical kite, and assume that $c$ is horizontal, $d$ is vertical, and that $c$ bisects $d$. Label the sides of $Q$ by $a, b, e, f$ in cyclic order so that $a, b, c$ bounds a triangle $T$, the side $a$ has negative slope, and $\text{width}(a) < \text{width}(b)$ (see Figure 8.2). Let $T'$ be the triangle bounded by $c, e, f$. Let $C$ be a horizontal cylinder which fulfills the conditions for $Q$ to be a cylindrical kite. The triangles $T$ and $T'$ form exactly two of the regions of $C - \partial Q$.

Let $R$ be the region of $C - \partial Q$ meeting $T$ along $a$. Then $a$ and $e$ form the adjacent non-horizontal sides of $R$. It follows that all other sides of $R$ lie on one boundary component of $C$. Since each region of $C - \partial Q$ is a Euclidean planar polygon, $R$ is isometric to a Euclidean triangle, with its horizontal ‘side’ possibly subdivided into several saddle connections.

Now consider the horizontal straight-line flow emanating from $a$ away from $T$. Any trajectory starting from the interior of $a$ will cross $R$ and then hit the interior of $e$. Appealing to Proposition 6.9, we deduce that $e$ is the only visible line segment with respect to the flow and $\partial Q$. Since $e$ shares an end point with $a$ at the corner of $T$ opposite $c$, it follows that $e$ is an awning for $a$. Therefore, by Lemma 7.6, $a$ cannot be flippable in any triangulation $T \supset \partial Q$, hence $a \notin \text{FB}(c, d)$. Using a similar argument, we deduce that $f \notin \text{FB}(c, d)$, and so $\# \text{FB}(c, d) \leq 2$. \hfill $\Box$

However, there will always be at least two flippable barriers, and at least three if we are not in the cylindrical kite case.

**Lemma 8.10** (Flip pair boundaries). Let $\{c, d\} \in \text{FP}(A(S, q))$ be a flip pair. Then $\# \text{FB}(c, d) \geq 2$. Furthermore, equality occurs if and only if $\{c, d\}$ is a cylindrical kite pair.

*Proof.* Apply an $\text{SL}(2, \mathbb{R})$-deformation to make $c$ horizontal and $d$ vertical. If $Q$ has a pair of opposite sides identified, then it forms a $(1,1)$-annulus by gluing those sides. Applying Lemma 7.10, there exists a triangulation containing $\partial Q$ in which every saddle connection of $\partial Q$ is flippable. In this case, we have $\text{FB}(c, d) = \partial Q$. Note that $\# \partial Q = 3$, as identifying the second pair of opposite sides would yield a torus which we do not consider here.

Let us now assume $Q$ has four distinct sides $a, b, e, f$ appearing in the given cyclic order, with $a, b, c$ bounding a triangle $T$ in $Q$. Suppose that $b, e, d$ bound a triangle $T''$ in $Q$. Without loss of
First, we show that $b$ must be a flippable barrier. If $b \notin FB(c, d)$, then by Lemma 7.6 there exists an awning for $b$ with respect to $\partial Q$ and the horizontal flow away from $T$. This awning starts at the corner where $a$ and $b$ meet and hence has positive slope, which means it must be $f$. It follows that $\text{width}(f) > \text{width}(b)$. But this implies that $T''$ takes up less than half the area of $Q$, a contradiction.

Next, we prove that at least one of $a$ or $e$ is also a flippable barrier. Suppose $a \notin FB(c, d)$. By considering slopes, we deduce that $e$ is an awning for $a$ with respect to $\partial Q$ and the horizontal flow direction. It follows that $\text{width}(e) > \text{width}(a)$, and so $T''$ is a strictly major triangle. Now consider the vertical flow emanating from $e$ away from $T''$. Any awning for $e$ with respect to $\partial Q$ and the vertical flow must start at the corner where $b$ and $e$ meet. Hence, it must be a widest saddle connection of negative slope (and taller than $e$). But such a saddle connection cannot exist, since $\partial Q$ has $b$ and $e$ as its widest saddle connections. Therefore, by Lemma 7.6, $e$ is flippable in some triangulation containing $\partial Q$ and so $e \in FB(c, d)$. Note that in this case, the heights of $a, b, e, f$ are all equal, and $Q$ is a kite.

Finally, we characterise when there are only two flippable barriers. Assume that we are in the same setting as in the previous paragraph and suppose $a, f \notin FB(c, d)$. Let $T'$ be the triangle bounded by $c, e, f$. Then $b$ is an awning for $f$ with respect to $\partial Q$ and the horizontal flow away from $T'$. Note that $Q$ cannot be a rhombus as $b$ and $f$ have different slopes. Each trajectory starting at an interior point of $f$ will hit $b$ and continue across $T$ until it hits an interior point of $a$. The trajectory will then coincide with some trajectory starting from $a$ as described above (since it now flows away from $T$); these in turn will hit $e$ and cross $T'$, until they reach the initial starting point on $f$ (see Figure 8.3). Since this holds for every trajectory starting from the interior of $f$, the triangles $T$ and $T'$ are contained in a common horizontal cylinder $C$. Moreover, as $e$ is an awning for $a$, they share a common corner in some region $R \neq T, T'$ of $C - \partial Q$ and hence are the non-horizontal adjacent sides of $R$. Similarly, $b$ and $f$ are adjacent sides of the fourth region of $C - \partial Q$. It follows that $Q$ is a cylindrical kite.

\[\square\]

We have thus shown that $\# FB(c, d) = 2$ if and only if $\{c, d\}$ is a cylindrical kite pair. Moreover, when $c$ and $d$ are perpendicular, then $FB(c, d)$ are the two longer sides of $\partial Q(c, d)$. Cylindrical kites only arise in a very particular set of circumstances which we can use to our advantage. We propose the following purely combinatorial procedure for recovering cylindrical kite boundaries. As the two diagonals $c$ and $d$ of a cylindrical kite play different roles in the definition, we write
instead of \{c, d\} and then, in the process of recovering the kite, determine which element of \(\kappa\) plays the role of \(c\) (or \(d\)).

**Proposition 8.11** (Preparations to define kite barriers). Let \(\kappa \in \text{FP}(\mathcal{A}(S, q))\) be a cylindrical kite pair and let \(\{b, e\} = \text{FB}(\kappa)\) be the flippable barriers of \(\kappa\).

(i) There exists \(\sigma \in \text{MIL}(\mathcal{A}(S, q))\) satisfying \(\text{FB}(\kappa) \subset \text{lk}(\sigma)\). Furthermore, \(\kappa \cap \sigma\) consists of exactly one saddle connection. Call this saddle connection \(c\) and call \(d\) the other saddle connection in \(\kappa\).

(ii) There exists \(\sigma' \in \text{MIL}(\mathcal{A}(S, q))\) satisfying \(\sigma' \supset (\sigma \cup \{b\}) \setminus \{c\}\). Furthermore, there is a unique saddle connection in \(\text{lk}(\sigma' \cup \{c\})\) disjoint from \(d\). Call this saddle connection \(a\).

(iii) There exists \(\sigma'' \in \text{MIL}(\mathcal{A}(S, q))\) satisfying \(\sigma'' \supset (\sigma \cup \{e\}) \setminus \{c\}\). Furthermore, there is a unique saddle connection in \(\text{lk}(\sigma'' \cup \{c\})\) disjoint from \(d\). Call this saddle connection \(f\).

(iv) There exists a quadrilateral \(Q \coloneqq Q(c, d)\) with diagonals \(c\) and \(d\). Moreover, the set \(\{a, b, e, f\}\) is equal to \(\partial Q(c, d)\). In particular, \(\{a, b, e, f\}\) does not depend on the choices of \(\sigma, \sigma',\) and \(\sigma''\).

**Proof.** By definition, there exists a quadrilateral \(Q\) associated to \(\kappa\). Let us assume, without loss of generality, that the diagonals of \(Q\) are perpendicular, with the horizontal one bisecting the vertical one. The reader should refer to Figure 8.2 as a guide throughout this proof, but imagine that there are initially no labels on the diagram. We recover \(\text{FB}(\kappa) = \{b, e\}\), yielding the two long sides of \(Q\). For the sake of concreteness, choose \(b\) to have positive slope, and \(e\) negative.

By definition, there exists a cylinder \(C\) containing \(\text{FB}(\kappa) = \{b, e\}\) as transverse arcs. We claim that \(C\) is the unique cylinder with this property and, furthermore, that \(C\) is horizontal.

If there exists a cylinder with slope \(\theta \in \mathbb{RP}^1\) having \(b, e\) as transverse arcs, then every straight-line trajectory with slope \(\theta\) emanating from the interior of \(b\) and \(e\) will form closed (non-singular) loops. Consider the straight-line flows emanating from \(b\) and \(e\) and pointing into the kite \(Q\). For positive slopes, some trajectory emanating from the interior of \(e\) will hit a singularity which is an end point of the opposite side to \(e\) in \(Q\). Similarly, for negative slopes, some trajectory starting from the interior of \(b\) will hit an end point of its opposite side. Finally, for the vertical slope, there is a downward trajectory starting from an interior point of \(b\) that crosses \(e\), and then hits the corner of \(Q\) opposite the common corner of \(b\) and \(e\). Therefore, such a cylinder exists only for the horizontal slope, yielding the claim.

Now we can choose \(\sigma \in \text{MIL}(\mathcal{A}(S, q))\) to be any triangulation away from \(C\). By definition, it satisfies \(\text{FB}(\kappa) \subset \text{lk}(\sigma)\). By Remark 8.8, exactly one of the saddle connections in \(\kappa\) belongs to \(\partial C \subseteq \sigma\). Thus, we have proven statement (i) and can assign \(c\) to be the horizontal diagonal of \(Q\), and \(d\) the vertical.

Cutting \(C\) along \(b\) produces a polygonal region \(R\). Exactly one of the two copies of \(b\) on \(\partial R\) has the two copies of \(c\) as its adjacent sides (in Figure 8.2, this is the left copy of \(b\)). Let \(b'\) be the topological arc in \(R\) connecting the end points of the two copies of \(c\) that are not end points of \(b\).

Since \(b'\) is a transverse arc of \(C\), it can be realised as a saddle connection (it is shown in red in Figure 8.2). Thus, there is a quadrilateral \(Q' \subset R\) bounded by \(b, c, b', c\). Gluing \(Q'\) along the two copies of \(c\) yields a \((1,1)\)-annulus \(C'\), which must be a cylinder.

We can choose \(\sigma' \in \text{MIL}(\mathcal{A}(S, q))\) to be any triangulation away from \(C'\). Note that it satisfies \(\sigma' \supset (\sigma \cup \{b\}) \setminus \{c\}\). The unique non-triangular region of \(\sigma' \cup \{c\}\) is the parallelogram \(Q'\). Therefore, \(\text{lk}(\sigma' \cup \{c\})\) has exactly two vertices: one that intersects \(d\), and another that does not. The latter is the side of \(Q\) opposite \(e\). Therefore, statement (ii) recovers the saddle connection \(a\) as in Figure 8.2 (the other diagonal of this parallelogram appears as \(a'\)).
Using a similar argument with \( e \) taking the role of \( b \), we can define a cylinder \( C'' \) and deduce that statement (iii) correctly recovers the saddle connection \( f \) as given in Figure 8.2.

Finally, the cylinders \( C, C', \) and \( C'' \) are uniquely defined for \( \kappa \). Therefore, by Remark 5.4, any choice of \( \sigma, \sigma' \), and \( \sigma \) satisfying statements (i)–(iii) will produce the same set \{\( a, b, e, f \)\}. \( \square \)

**Definition 8.12** (Kite barriers). Let \( \kappa \in \text{FP}(\mathcal{A}(S, q)) \) be a cylindrical kite pair and choose \( a, b, c, d, e, f \) as in Proposition 8.11. We then define the kite barriers of \( \kappa \) to be \( \text{KB}(\kappa) = \text{KB}(c, d) := \{a, b, e, f\} \).

**Definition 8.13** (Kite-or-flippable barriers). Given a flip pair \{\( c, d \)\}, define its set of kite-or-flippable barriers to be \( \text{KFB}(c, d) := \text{KB}(c, d) \) if \(\{c, d\}\) is a cylindrical kite pair, and \( \text{KFB}(c, d) := \text{FB}(c, d) \) otherwise.

The results of this section can be summarised as follows.

**Corollary 8.14** (Three out of four ain’t bad). If \(\{c, d\}\) is a flip pair then \( \# \text{KFB}(c, d) \geq 3 \) and \( \text{KFB}(c, d) \subseteq \partial Q(c, d) \).

9 | DETECTING TRIANGLES

Throughout this section, assume \( \tau = \{a_1, a_2, a_3\} \in \mathcal{A}(S, q) \) is a 2-simplex. Our goal is to give a purely combinatorial criterion to detect whether \( \tau \) bounds a triangle on \((S, q)\). This will provide an analogue of the Topological Triangle Test (Proposition 2.11) for the saddle connection complex.

The Triangle Test comprises three subtests: the First Triangle Test, the Bigon Test, and the Second Triangle Test. Each test enlarges the pool of triangles that can be conclusively detected; moreover, this procedure also simplifies the case analysis in each of the tests.

**Proposition 9.1** (First Triangle Test). Assume \( \tau = \{a_1, a_2, a_3\} \in \mathcal{A}(S, q) \) is a 2-simplex satisfying the following two conditions.

(T1) There is a triangulation \( \mathcal{T} \supseteq \tau \) such that

(i) \( \text{lk}(\mathcal{T} \setminus \{a_k\}) = \{a_k\} \cup \{b\} \) for some \( b \in \mathcal{A}(S, q) \),

(ii) \( a_i, a_j \in \text{KFB}(a_k, b) \), and

(iii) \( \text{lk}(\mathcal{T} \setminus \{a_i, a_j\}) \) contains \( a_i \rightarrow a_j \) as an induced subgraph, for some choice of distinct \( i, j, k \in \{1, 2, 3\} \), and

(iv) if \( \text{lk}(\mathcal{T}' \setminus \tau) \) is infinite for some triangulation \( \mathcal{T}' \supseteq \tau \), then \( \text{lk}(\mathcal{T}' \setminus \{a_i, a_j\}) \) is also infinite for some choice of distinct \( i, j \in \{1, 2, 3\} \).

Then \( \tau \) bounds a triangle on \((S, q)\).

As a partial converse, if \( T \) is a major triangle then \( \partial T \) satisfies conditions (T1) and (T2).

Note that if \( \tau \) satisfies the First Triangle Test, then it bounds a triangle \( T \) contained in a triangulation in which at least two sides of \( T \) are flippable. Example 7.2 shows a triangle where at most one side is flippable, and so not all triangles can be detected by this test. Thus, more general tests are needed.
Proposition 9.2 (Bigon Test). Assume $\tau = \{a_1, a_2, a_3\} \in \mathcal{A}(S, q)$ is a 2-simplex that does not satisfy the First Triangle Test. Assume furthermore that there exists a choice of distinct $i, j, k \in \{1, 2, 3\}$ and a triangulation $\mathcal{T} \supseteq \tau$ satisfying the following five conditions:

- (B1) $\text{lk}(\mathcal{T} \setminus \{a_k\}) = \{a_k\} \cup \{b\}$ for some $b \in \mathcal{A}(S, q)$,
- (B2) $\text{lk}(\mathcal{T} \setminus \{a_i, a_j\}) = \{a_i\} \cup \{a_j\} \cup \{b\}$,
- (B3) $\text{lk}((\mathcal{T} \setminus \tau) \cup \{b\}) = \{a_i\} \cup \{a_j\} \cup \{a_k\} \cup \{b\}$,
- (B4) $\text{lk}(\mathcal{T} \setminus \tau) = \{a_i\} \cup \{a_j\} \cup \{a_k\}$, and
- (B5) there does not exist a 2-simplex $\tau' \subseteq \mathcal{T}$ that satisfies the First Triangle Test, and contains $\{a_i, a_k\}$ or $\{a_j, a_k\}$.

Then $\tau$ bounds a triangle contained in a bigon with a simple pole in the interior.

Conversely, if $T$ is a triangle contained in a bigon then $\partial T$ satisfies the First Triangle Test or conditions (B1)–(B5). In the latter case, the interior arcs of the bigon are $a_i$ and $a_j$.

Before stating the Second Triangle Test, we need the following definition.

Definition 9.3 (Compatible flip partner). Define the $\tau$-compatible flip partners of a saddle connection $a_i \in \tau$ to be

$$F_\tau(a_i) := \{b \in \mathcal{A}(S, q) \mid \{a_i, b\} \in \text{FP}(\mathcal{A}(S, q)), b \text{ is disjoint from both } a_j, a_k\}.$$ 

This is precisely the set of saddle connections that can be obtained by flipping $a_i$ in some triangulation containing $\tau$.

Proposition 9.4 (Second Triangle Test). Assume $\tau = \{a_1, a_2, a_3\} \in \mathcal{A}(S, q)$ is a 2-simplex that neither satisfies the conditions of the First Triangle Test nor of the Bigon Test. Then $\tau$ bounds a triangle on $(S, q)$ if and only if the following three conditions hold.

- (T3) If $\tau' \in \mathcal{A}(S, q)$ is a 2-simplex such that $\#(\tau \cap \tau') \geq 2$, then $\tau'$ neither satisfies the conditions of the First Triangle Test nor of the Bigon Test.
- (T4) For all distinct $i, j, k \in \{1, 2, 3\}$, there exists a triangulation $\mathcal{T}_i \supseteq \tau$ such that $\text{lk}(\mathcal{T}_i \setminus \{a_j, a_k\})$ contains $P_3$ as an induced subgraph.
- (T5) For all distinct $i, j, k \in \{1, 2, 3\}$ and $b \in F_\tau(a_i)$, we have $\text{KFB}(a_i, b) \cap \{a_j, a_k\} \neq \emptyset$.

Corollary 9.5 (Triangle Test). A 2-simplex $\tau \in \mathcal{A}(S, q)$ bounds a triangle on $(S, q)$ if and only if it satisfies the First Triangle Test, the Bigon Test, or the Second Triangle Test.

The proofs of necessity for the three tests essentially combine results from the preceding sections. The main content of this section is in the proof of sufficiency. We show that the First Triangle Test can detect many triangles, including major triangles. The Bigon Test, as the name suggests, detects triangles contained in bigons such as in Figure 4.4 on the right. Our general strategy for the Second Triangle Test is as follows. By Corollary 4.7, condition (T4) is equivalent to saying that for all distinct $i, j, k$, there exists a triangulation $\mathcal{T}_k \supseteq \tau$ such that $\text{lk}(\mathcal{T}_k \setminus \{a_i, a_j\})$ either has an almond pentagon or a (1,1)-annulus as its unique non-triangular region, in which $a_i$ and $a_j$ are disjoint.
saddle connections. This implies that there exists a triangle $T_k$ of $\mathcal{T}_k$ having $a_i$ and $a_j$ as two sides. The goal is to use the other properties to prove that $T_i$, $T_j$, and $T_k$ are indeed the same triangle. The fact that major triangles and triangles in bigons can already be detected by the First Triangle Test and the Bigon Test plays a key role in simplifying our case analysis.

9.1 Proof of the First Triangle Test

9.1.1 Necessity for major triangles

Assume $\tau = \{a_1, a_2, a_3\}$ bounds a major triangle $T$ with base $a_3$. Depending on whether $T$ is annular or not, we choose a triangulation $\mathcal{T} \supseteq \tau$ to satisfy condition (T1) as follows.

If $T$ is an $(m, 1)$-annular triangle with transverse arcs $a_1$ and $a_2$ for some $m \geq 1$, then by Lemma 7.10, we may take $\mathcal{T}$ to be a triangulation in which $a_1, a_2, a_3$ are all flippable. This implies that (T1)(i) is fulfilled for every choice of $(i, j, k)$. If $m = 1$, we choose $(i, j, k) = (1, 2, 3)$ and (T1)(iii) is fulfilled by Lemma 4.6. If $m \geq 2$, at least one of the unique non-triangular region of $\mathcal{T} - \{a_1, a_2\}$ and of $\mathcal{T} - \{a_2, a_3\}$ is a strictly convex pentagon. Therefore, (T1)(iii) is fulfilled for the corresponding choice of $(i, j, k)$. Let $Q$ be the unique non-triangular region of $\mathcal{T} - \{a_k\}$.

If $T$ is not $(m, 1)$-annular with transverse arcs $a_1$ and $a_2$, consider a strictly convex quadrilateral $Q$ containing $T$, with $a_3$ as a diagonal, in which $T$ takes up at least half the area. By Proposition 7.4, we may choose a triangulation $\mathcal{T} \supseteq \partial Q \cup \{a_3\}$ such that $a_1$ and $a_3$ are flippable, and $\text{lk}(\mathcal{T} \setminus \{a_1, a_3\})$ contains $P_3$ as an induced subgraph. Recall from Remark 7.8 that the subgraph is in fact “$\begin{array}{c} a_1 \\ \bullet \\ a_2 \\ \bullet \\ \bullet \\ \bullet \end{array}$” in our situation. Hence, (T1)(i) and (iii) are fulfilled for $(i, j, k) = (1, 2, 3)$. Another application of Proposition 7.4 shows that there exists some triangulation (not necessarily equal to $\mathcal{T}$) containing $\partial Q \cup \{a_3\}$ in which $a_2$ and $a_3$ are flippable.

We show that (T1)(ii) is fulfilled for all of the previously discussed cases at once. Let $b$ be the other diagonal of $Q$. If $\{a_k, b\}$ is a cylindrical kite pair then

$$a_i, a_j \in \partial Q = \text{KB}(a_k, b) = \text{KFB}(a_k, b)$$

by Proposition 8.11. Otherwise, we have that $a_i$ and $a_j$ are flippable (in some triangulation containing $\partial Q \cup \{a_k\}$) and hence by definition

$$a_i, a_j \in \text{FB}(a_k, b) = \text{KFB}(a_k, b).$$

To verify the necessity of condition (T2), let us assume that $\text{lk}(\mathcal{T}' \setminus \{a_i, a_j\})$ is finite for all distinct $i, j \in \{1, 2, 3\}$. Let $T_i$ be the triangle of $\mathcal{T}'$ meeting $T$ along $a_i$. Then $\mathcal{T}' \setminus \{a_i, a_j\}$ has a pentagon, formed by gluing $T$ to $T_i$ and $T_j$ along $a_i$ and $a_j$, as its unique non-triangular region. In particular, the triangles $T_1, T_2, T_3$ are distinct. Therefore, $\mathcal{T}' \setminus \tau$ has a hexagon formed by gluing $T$ to $T_1, T_2, T_3$ along $a_1, a_2, a_3$ as its unique non-triangular region. This hexagon contains only finitely many diagonals, and so $\text{lk}(\mathcal{T}' \setminus \tau)$ is finite.

9.1.2 Sufficiency

Assume $\tau$ satisfies conditions (T1) and (T2). Without loss of generality, assume condition (T1) holds for $(i, j, k) = (1, 2, 3)$. 
FIGURE 9.1 Left to right: an almond pentagon formed by gluing $T$ to $T'$ and $T''$ along $a_1$ and $a_2$; the case where $e$ is a straight diagonal; the case where $e$ is a broken diagonal. In either case, there is a cylinder curve $\gamma$ (in orange)

Let $Q_i$ be the unique quadrilateral region of $\mathcal{T} \setminus \{a_i\}$ for $i = 1, 2, 3$. By condition (T1)(i) and Lemma 4.4, $Q_3$ is strictly convex, with diagonals $a_3$ and $b$. By condition (T1)(ii), we have $a_1, a_2 \in \text{KFB}(a_3, b)$ and hence $a_1, a_2 \in \partial Q_3$ by Corollary 8.14.

If $\text{lk}(\mathcal{T} \setminus \{a_i, a_3\})$ is infinite for some $i = 1, 2$, then by Lemma 4.6, the unique non-triangular region of $\mathcal{T} \setminus \{a_i, a_3\}$ is a (1,1)-annulus. This annulus is obtained by gluing $Q_3$ along two copies of $a_i$, and so $a_i$ must form two opposite sides of $Q_3$. It follows that $\tau$ bounds a triangular region of $Q_3 - a_3$ and we are done. We may henceforth assume $\text{lk}(\mathcal{T} \setminus \{a_i, a_3\})$ is finite for $i = 1, 2$.

By condition (T1)(iii) and Corollary 4.7, the unique non-triangular region $R$ of $\mathcal{T} \setminus \{a_1, a_2\}$ is either an almond pentagon or a (1,1)-annulus; this region is, respectively, cut into three or two triangles by the saddle connections $a_1, a_2$. In particular, these are the only triangles of $\mathcal{T}$ meeting at least one of $a_1$ or $a_2$.

Let us consider the case where $R$ is a (1,1)-annulus; this occurs precisely when $\text{lk}(\mathcal{T} \setminus \{a_1, a_2\})$ is infinite. Since $a_1 \in \partial Q_3$, there is a triangle of $\mathcal{T}$ obtained by cutting $Q_3$ along $a_3$ having $a_1$ and $a_3$ as two of its sides. The only triangles of $\mathcal{T}$ that meet $a_1$ are the two triangles obtained by cutting $R$ along $a_1$ and $a_2$, and so $a_3$ must form a side of $R$. It follows that $\tau$ bounds a (1,1)-annular triangle as desired.

We shall henceforth assume that $R$ is an almond pentagon. In particular, $\text{lk}(\mathcal{T} \setminus \{a_i, a_j\})$ is finite for all $i \neq j$, and so $\text{lk}(\mathcal{T} \setminus \tau)$ is also finite by condition (T2). Let $T$ be the triangle in $R$ meeting both $a_1$ and $a_2$; and let $T'$ and $T''$ be the other triangles of $R$ that meet only $a_1$ or $a_2$, respectively. The third side of $T$ is a saddle connection $c \in \partial R$; see Figure 9.1. Our goal is to prove that $c = a_3$. If $T$ is contained in $Q_3$ then we are done, since $a_3$ will form a side of $T$.

Let us suppose otherwise for a contradiction (this is equivalent to assuming $c \neq a_3$). Since $a_1, a_2 \in \partial Q_3$, and $T', T''$ are the only triangles of $\mathcal{T}$ other than $T$ meeting $a_1$ or $a_2$, it follows that $T'$ and $T''$ are the two triangular regions of $Q_3 - a_3$. Therefore, $a_3$ must form two sides of the pentagon $R$, one meeting $T'$ and one meeting $T''$. We rule out the case where these two sides are adjacent on the pentagon $R$. If this were so, the angle between these two sides would be either $\pi$ or $2\pi$. In both cases, the diagonal that would cross both $a_1$ and $a_2$ is a broken diagonal. This implies that the link of $\mathcal{T} \setminus \{a_1, a_2\}$ is an induced subgraph of “$\bullet \longrightarrow \bullet \longrightarrow \bullet$” which contradicts (T1)(iii).

Thus, $a_3$ forms two non-adjacent sides of $R$ and $\mathcal{T} \setminus \tau$ has a topological (2,1)-annulus as its unique non-triangular region; it is formed by gluing $R$ along the two copies of $a_3$. We claim that there is a cylinder curve contained in this annulus. This will imply that $\text{lk}(\mathcal{T} \setminus \tau)$ is infinite, by Proposition 5.2, contradicting condition (T2).

Let $d \in \partial R$ be the side of $R$ adjacent to both copies of $a_3$. To avoid potential confusion, we shall label them $a_3$ and $a_3'$, respectively. Let $p$ and $p'$, respectively, be the corners of $R$ that lie on an end
point of $a_3$ and $a'_3$, but are not end points of $d$. Note that $p$ and $p'$ are not adjacent corners of $R$. Let $e$ be the diagonal of $R$ connecting $p$ and $p'$.

If $e$ is a straight diagonal, then there is a quadrilateral $Q \subset R$ bounded by $a_3, d, a'_3, e$. Gluing $Q$ to itself along the two copies of $a_3$ yields a (1,1)-annulus, which must be a cylinder. The core curve $\gamma$ of this cylinder is shown in Figure 9.1.

We are left with the case where $e$ is a broken diagonal. Since $R$ is an almond pentagon, $e$ must be the only broken diagonal. Therefore, there exists a triangle contained in $R$ with $a_3$ and $d$ as two of its sides, and so $\angle(d, a_3) < \pi$. Similarly, we deduce $\angle(d, a'_3) < \pi$. Since $a_3$ and $a'_3$ are parallel, a small regular neighbourhood of $d$ within $R$ is a Euclidean parallelogram. This parallelogram forms a cylinder upon gluing the two copies of $a_3$ in $\partial R$; thus there is a cylinder curve $\gamma$ that is parallel to $d$ and contained in $R$ (see Figure 9.1).

This completes the proof of the First Triangle Test.

9.2 | Proof of the bigon Test

9.2.1 | Necessity for triangles contained in bigons

We shall begin with a useful lemma for triangles sharing a pair of sides.

**Lemma 9.6** (Triangles with two common sides). Let $T$ and $T'$ be distinct triangles with two sides in common. Then $T$ and $T'$ are contained in a common bigon, or $\partial T$ and $\partial T'$ both satisfy the First Triangle Test.

**Proof.** Suppose $a, b$ are the common sides of $T$ and $T'$. By applying an $\text{SL}(2, \mathbb{R})$-deformation, we may assume that $a$ and $b$ are, respectively, vertical and horizontal, and both have length $h > 0$. Then $T$ and $T'$ are both right-angled isosceles triangles. There are two possible configurations (up to swapping the roles of $a$ and $b$) depending on the sides of $a$ and $b$ on which the triangles $T$ and $T'$ appear.

(i) $T$ and $T'$ appear on opposite sides of $a$, and on opposite sides of $b$.
(ii) $T$ and $T'$ appear on the same side of $a$, and on opposite sides of $b$.

Note that $T$ and $T'$ cannot appear on the same sides of both $a$ and $b$, for otherwise they would coincide. The different cases appear as shown in Figure 9.2 (up to reflection).

In case (i), the triangles $T$ and $T'$ can be glued along $a$ and $b$ to form either a (1,1)-annulus or a bigon. In either situation, the desired result holds.
FIGURE 9.3  The situation for the bigon before and after flowing. In the right one, all diagonals of $T \setminus \tau$ are drawn (dashed or full).

For case (ii), let $c$ and $c'$, respectively, be the third side of $T$ and $T'$. Let $\varphi'$ be the horizontal unit-speed flow emanating from $c$ away from $T$ (strictly speaking, we are working in the universal cover). Consider the trajectory that begins at the common corner of $a$ and $c$ in $T$. This trajectory runs along $b$ until it hits the other end point at time $t = h$. This end point is a visible singularity with respect to $\varphi'$ and $\partial T$. Applying Proposition 6.2, there exists a triangle $T''$ meeting $c$ on the side from which the flow emanates, of height($T'') = h$, with a visible singularity $z = \varphi^{h'}(p)$ as its corner opposite $c$, for some $0 < h' \leq h$ and a point $p \in c$. Then $T$ and $T''$ can be glued along $c$ to form a strictly convex quadrilateral in which $T$ takes up at least half the area. Thus, $T$ is a major triangle. Using a similar flowing argument with $c'$, we prove that $T'$ is also major. In particular, by Proposition 9.1, $\partial T$ and $\partial T'$ both satisfy the First Triangle Test.

Assume that $\tau = \{a_i, a_j, a_k\}$ bounds a triangle $T$ contained in a bigon with a simple pole in its interior and $a_i$ and $a_j$ as interior arcs. We may assume that $\tau$ does not satisfy the First Triangle Test. Then $a_k$ forms a side of this bigon; call the other side $c_k$. By applying an $SL(2, \mathbb{R})$-deformation, we can assume that $a_k$ is vertical and $c_k$ is horizontal as in Figure 9.3. Observe that $a_k$ is strictly tallest in $\sigma := \tau \cup \{c_k\}$ and is not contained in a horizontal cylinder. Consider the horizontal flow emanating from $a_k$ away from $T$. By Propositions 6.1 and 6.2, there exists an acute-angled triangle $T'$ with $a_k$ as one of its sides such that $\partial T' \cup \sigma$ spans a simplex in $\mathcal{A}(S, q)$; let $\mathcal{T}$ be any triangulation containing $\partial T' \cup \sigma$. The unique non-triangular region of $T \setminus \{a_i, a_j\}$ is a bigon containing a simple pole, and so condition (B2) holds by Lemma 4.6. Gluing $T$ to $T'$ along $a_k$ forms a strictly convex quadrilateral $Q$ with $a_i$ and $a_j$ appearing as adjacent sides. Setting $b$ to be the other diagonal of $Q$, we see that $\mathcal{T}$ satisfies condition (B1).

Let $\mathcal{T}'$ be the triangulation obtained from $\mathcal{T}$ by flipping $a_k$. Then $(\mathcal{T} \setminus \tau) \cup \{b\} = \mathcal{T'} \setminus \{a_i, a_j\}$ has an almond pentagon with one corner having angle $\pi$ as its unique non-triangular region. Since the unique broken diagonal is the topological arc intersecting $a_i$ and $a_j$, it follows that condition (B3) also holds. Observe that $\mathcal{T} \setminus \tau$ has a triangle containing a simple pole as its unique non-triangular region. All saddle connections belonging to $\text{lk}(\mathcal{T} \setminus \tau)$ appear in Figure 9.3 and it can be checked there that $\text{lk}(\mathcal{T} \setminus \tau)$ is as in condition (B4).

Finally, to check condition (B5), note that there are exactly two triangles in $\mathcal{T}$ that contain $a_i$ and exactly two triangles that contain $a_j$. These are precisely the triangles bounded by $\{a_i, a_j, a_k\}$ and $\{a_i, a_j, c_k\}$, respectively. Hence, condition (B5) has only to be checked for $\{a_i, a_j, a_k\}$ for which it is true by assumption.

Thus, all conditions hold and we have proven the statement.

9.2.2  Sufficiency

Assume $\tau = \{a_1, a_2, a_3\}$ is a 2-simplex that does not satisfy the First Triangle Test. Let $i, j, k$ be a permutation of 1, 2, 3 and $\mathcal{T} \supseteq \tau$ be a triangulation satisfying conditions (B1)–(B5). Condition (B1)
and Corollary 4.5 imply that $a_k$ can be flipped in $\mathcal{T}$, with flip partner $b$, to obtain a new triangulation $\mathcal{T}'$. Condition (B3) and Lemma 4.6 imply that $\mathcal{T}' \setminus \{a_i, a_j\} = (\mathcal{T} \setminus \tau) \cup \{b\}$ has a pentagon $P$ with one broken diagonal as its unique non-triangular region. Moreover, the broken diagonal of $P$ is the arc intersecting both $a_i$ and $a_j$ (which are themselves straight diagonals of $P$). Let $T'$ be the triangle contained in $P$ with $a_i$ and $a_j$ as two of its sides. Condition (B4) implies that $\mathcal{T}' \setminus \{a_i, a_j, b\} = \mathcal{T} \setminus \tau$ has exactly one non-triangular region $R$ and hence $b$ must appear as at least one side of $P$. We claim that $b$ appears as a pair of adjacent sides of $P$ forming an angle of $\pi$.

First, we rule out the case where $b$ appears as only one side. Suppose first that $b$ appears exactly as the third side of $T'$. Then $a_k$ and $b$ form the diagonals of a strictly convex quadrilateral having $a_i$ and $a_j$ as adjacent sides. This quadrilateral contains a major triangle $T''$ having $a_k$ and one of $a_i$ or $a_j$ among its sides. But then $\partial T'' \subseteq \mathcal{T}$ satisfies the First Triangle Test and contains $\{a_i, a_k\}$ or $\{a_j, a_k\}$, contradicting condition (B5). Without loss of generality, we may thus assume that $b$ appears exactly as a side of the triangle obtained by cutting $P$ along $a_i$. Then $a_k$ does not intersect the interior of the strictly convex quadrilateral obtained by cutting $P$ along $a_i$. It follows that the non-triangular region of $\mathcal{T} \setminus \{a_i, a_j\}$ contains the flip partner of $a_j$ in $\mathcal{T}'$ as straight diagonal. This contradicts condition (B2).

Suppose now that $b$ forms two non-adjacent parallel sides of $P$. If they are identified by a translation, then we can find a cylinder curve in $R$ parallel to the side of $P$ adjacent to both copies of $b$. This contradicts condition (B4) combined with Proposition 5.2. If they are identified by a half-translation, then $P$ will have at least two broken diagonals as can be seen in the left of Figure 9.4, a contradiction.

Thus, $b$ forms two adjacent parallel sides of $P$. Since $P$ is an almond pentagon, the corner $p$ formed by these two sides has angle $\pi$. Therefore, the broken diagonal of $P$ cuts off a (topological) triangle with $b$ as two of its sides. Since $a_i$ and $a_j$ intersect the broken diagonal, they both have an end point at $p$. As $a_k$ is the flip partner of $b$, it is contained in $R$. It follows that $a_i, a_j, a_k$ form three sides of a Euclidean triangle $T$ (see Figure 9.4). Gluing $T$ to $T'$ along $a_i$ and $a_j$ yields a bigon having $a_i$ and $a_j$ as interior arcs as desired.

This completes the proof of the Bigon Test.

9.3 Proof of the Second Triangle Test

9.3.1 Necessity

Assume $\tau = \{a_1, a_2, a_3\}$ bounds a triangle on $(S, q)$ that neither satisfies the First Triangle Test nor the Bigon Test. Condition (T4) is an immediate consequence of Proposition 7.3.
To verify condition (T3), suppose \( \tau' \in \mathcal{A}(S, q) \) is a 2-simplex sharing at least two vertices with \( \tau \). If \( \tau' \) satisfies the First Triangle Test or the Bigon Test, it must bound a triangle. Then by Lemma 9.6, \( \tau \) must also satisfy the First Triangle Test or the Bigon Test which is a contradiction.

To verify condition (T5), suppose \( b \) is obtained by flipping \( a_i \) in some triangulation \( T \supseteq \tau \). Then \( a_j, a_k \) form two sides of the strictly convex quadrilateral \( Q(a_i, b) \) with diagonals \( a_i \) and \( b \). By Corollary 8.14, KFB\((a_i, b)\) contains at least three sides of \( Q(a_i, b) \). Hence, at least one of \( a_j \) and \( a_k \) must belong to KFB\((a_i, b)\).

### 9.3.2 Sufficiency

Assume \( \tau = \{a_1, a_2, a_3\} \) satisfies the Second Triangle Test, but neither the First Triangle Test nor the Bigon Test. We shall use the indices \( i, j, k \) to stand for a fixed permutation of 1,2,3.

**Lemma 9.7** ((T3) and (T4) give three almond pentagons). For all distinct \( i, j, k \in \{1, 2, 3\} \), \( \mathcal{T}_i \setminus \{a_j, a_k\} \) has an almond pentagon \( P_i \) as its unique non-triangular region.

**Proof.** By condition (T4) and Corollary 4.7, \( \mathcal{T}_i \setminus \{a_j, a_k\} \) has either a (1,1)-annulus or an almond pentagon as its unique non-triangular region. If this region is a (1,1)-annulus, then \( a_j \) and \( a_k \) form two sides of some (1,1)-annular triangle \( T' \). But then \( \partial T' \) is a 2-simplex sharing at least two vertices with \( \tau \), and satisfying the First Triangle Test. This contradicts condition (T3). \( \square \)

The pentagon \( P_i \) contains a triangle \( T_i \) with \( a_j \) and \( a_k \) as two of its sides; let \( c_i \in \partial P_i \) denote the third side of \( T_i \). Our goal is to prove that \( c_i = a_i \) for some \( i \). Note that \( c_i \in \mathcal{T}_i \) and so it cannot intersect any of \( a_i, a_j, a_k \) transversely.

**Lemma 9.8** (Properties of \( T_i \)). The triangle \( T_i \) is not major. Moreover, \( T_i \) is the only triangle on \((S, q)\), having \( a_j \) and \( a_k \) as two of its sides.

**Proof.** Since \( \partial T_i \) shares at least two vertices with \( \tau \), it does not satisfy the First Triangle Test by condition (T3). Therefore, \( T_i \) cannot be major by Proposition 9.1. If \( T' \neq T_i \) is another triangle having \( a_j \) and \( a_k \) as its sides, then, by Lemma 9.6, \( \partial T' \) and \( \partial T_i \) both satisfy the First Triangle Test or the Bigon Test, a contradiction. \( \square \)

Next, orient the triangles \( T_i \) by choosing the cyclic order of the subscripts to agree with (1,2,3). To be explicit, set \( \vec{T}_1 = [c_1, a_2, a_3] \), \( \vec{T}_2 = [a_1, c_2, a_3] \), and \( \vec{T}_3 = [a_1, a_2, c_3] \). Our goal is to show that these triangles are consistently oriented on \((S, q)\). Before doing so, let us introduce some terminology. In each \( T_i \), direct the sides \( a_j \) and \( a_k \) so that they point towards their common corner. Say that \( T_i \) and \( T_j \) meet concordantly along \( a_k \) if the directions on \( a_k \) coming from \( T_i \) and \( T_j \) agree, otherwise we say that they meet discordantly; see Figure 9.5. Write \( \angle_{T_i}(a_j, a_k) \) for the angle between \( a_j \) and \( a_k \), measured inside the triangle \( T_i \).

**Lemma 9.9** (Consistent orientation). The triangles \( \vec{T}_1, \vec{T}_2, \) and \( \vec{T}_3 \) are consistently oriented on \((S, q)\).

**Proof.** Suppose \( \vec{T}_i \) and \( \vec{T}_j \) are not consistently oriented. We consider two cases, depending on whether they meet concordantly or discordantly along \( a_k \).
First, suppose $T_i$ and $T_j$ meet concordantly. Without loss of generality, we may suppose that $\angle T_i (a_j, a_k) < \angle T_j (a_k, a_i)$. (The angles cannot be equal, for otherwise $a_i = a_j$.) But then $a_j$ intersects $c_j$, or $T_j$ is strictly contained in $T_i$, which is both impossible.

Next, suppose $T_i$ and $T_j$ meet discordantly. Consider the quadrilateral $Q$ formed by $T_i$ and $T_j$, glued along $a_k$. (Strictly speaking, this does not always result in a quadrilateral on $(S, q)$ with embedded interior, so we should really work in the universal cover.) If $Q$ is strictly convex, then it has embedded interior on $(S, q)$ by Lemma 3.7. In this situation, at least one of $T_i$ or $T_j$ is major, which is ruled out by Lemma 9.8. Thus, $Q$ is not strictly convex. Assume, without loss of generality, that $\angle T_i (a_k, c_i) + \angle T_j (a_i, a_k) \geq \pi$, as shown in the fourth case of Figure 9.5. Recall that $P_j$ is a pentagon with at most one broken diagonal. Hence, there exists at least one straight diagonal $b$ of $P_j$ that intersects $a_k$. Note that $b$ cannot intersect $c_j$, so it has to intersect $a_j$. But $a_j$ is disjoint from the interior of $P_j$ and hence from $b$, a contradiction. □

Next we provide a criterion for $\tau$ bounding a triangle in the case when we have at least one discordant gluing.

**Lemma 9.10** (Parallel sides coincide). Assume $T_i$ and $T_j$ meet discordantly along $a_k$. If $a_i$ and $c_i$ are parallel, they must coincide. Consequently, $\tau$ bounds a triangle.

**Proof.** Observe that $\angle T_i (a_k, c_i) \leq \angle T_j (a_k, a_i)$, for otherwise either $a_i$ intersects $a_j$ transversely or $T_j$ is strictly contained in $T_i$, neither of which is possible. Therefore, $T_j \cap c_i$ contains a segment of $c_i$ starting from the common corner of $a_i$ and $a_k$ in $T_j$. Since this segment and $a_i$ are contained in a common triangle, we deduce that $\angle T_j (a_i, c_i) < \pi$. Hence, if $a_i$ and $c_i$ are parallel, we have $\angle T_j (a_i, c_i) = 0$. It follows that $a_i$ and $c_i$ are the same saddle connection. □

For the rest of this section, we shall assume for a contradiction that $\tau$ does not bound a triangle. Our strategy is to prove that the triangles $T_i$, $T_j$, and $T_k$ can be placed into one of three standard configurations as depicted in Figure 9.6, with $a_i$ strictly taller than $a_j$ and $a_k$, by applying $\text{SL}(2, \mathbb{R})$-deformations, reflections, and permutations of the indices.

Let us first consider the case where all gluings are concordant.

**Lemma 9.11** (When all gluings are concordant). If all three gluings among $T_i$, $T_j$, and $T_k$ are concordant, then they glue together to form a triangle containing exactly one removable singularity which, moreover, forms a common end point of $a_i$, $a_j$, and $a_k$. Furthermore, no two saddle connections in $\{a_1, a_2, a_3, c_1, c_2, c_3\}$ are parallel.
Proof. Suppose all gluings are concordant. Then $a_i$, $a_j$, and $a_k$ will be directed towards a common singularity. Examining the cyclic gluing pattern of $T_i$, $T_j$, and $T_k$ along the sides $a_k$, $a_i$, and $a_j$, we see that a small neighbourhood of the common singularity is formed by gluing together one corner from each of $T_i$, $T_j$, $T_k$. By summing these corner angles, it follows that this singularity has cone angle strictly less than $3\pi$.

Suppose this cone angle equals $\pi$. Gluing $T_i$, $T_j$, $T_k$ along $a_k$, $a_i$, and $a_j$ forms a triangle that contains a simple pole. By the Gauss–Bonnet theorem, the sum of the interior angles of this triangle is $2\pi$. In particular, there exists a corner of angle strictly less than $\pi$. Let $a_k$ be the saddle connection that connects this corner to the simple pole. Then $T_i$ and $T_j$ form a strictly convex quadrilateral which means that at least one of $T_i$ or $T_j$ is a major triangle. This contradicts Lemma 9.8.

Hence, the cone angle is $2\pi$. Gluing $T_i$, $T_j$, $T_k$ along $a_k$, $a_i$, and $a_j$ forms a Euclidean triangle that contains a removable singularity. The sides of this triangle are $c_1$, $c_2$, and $c_3$ whereas $a_1$, $a_2$, and $a_3$ connect the corners of the triangle to the removable singularity. In particular, no two of these six saddle connections can be parallel. \(\square\)

We now turn to the second case, that is, where there is at least one discordant gluing. Without loss of generality, suppose that $T_i$ and $T_j$ meet discordantly along $a_k$. As $T_i$ and $T_j$ are consistently oriented and cannot be contained in one another, $c_i$ and $c_j$ must intersect. Apply an SL(2, $\mathbb{R}$)-deformation to make $c_i$ horizontal and $c_j$ vertical. Note that $c_i$ and $c_j$ have a unique intersection point, for otherwise $T_i \cap c_j$ contains at least two distinct vertical line segments, and so $c_j$ would intersect $a_j$ or $a_k$ transversely. By Lemma 9.10 (and the assumption that $c$ does not bound a triangle), it follows that none of $a_i$, $a_j$, $a_k$ can be horizontal nor vertical. Therefore, the triangles $T_i$ and $T_j$ appear as in Figure 9.7 (up to horizontal or vertical reflection).

**Lemma 9.12** (Heights and widths). In the described situation, we have $\text{height}(T_i) < \text{height}(a_i)$ and $\text{width}(T_j) < \text{width}(a_j)$. Consequently, we have $\angle T_j(a_i, a_k) + \angle T_i(a_k, a_j) > \pi$. 

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**FIGURE 9.6** Three configurations for the triangles $T_i$, $T_j$, and $T_k$. From left to right: All triangles meet concordantly; $T_i$ and $T_j$ meet discordantly along $a_k$ and $T_j$ and $T_k$ meet concordantly along $a_i$; $T_i$ and $T_j$ meet discordantly along $a_k$ and $T_j$ and $T_k$ meet discordantly along $a_i$. Note that $\text{height}(a_i) > \text{height}(a_j) = \text{height}(a_k)$ in all cases.
FIGURE 9.7 The triangles $T_i$ and $T_j$ as in the proof of Lemma 9.12. The trajectory $\varphi'(p)$ (in red) runs along $c_i$ from $p$ to $p'$. Note that $p$ is not a singularity.

Proof. Suppose $\text{width}(a_j) \leq \text{width}(T_j)$ for a contradiction. The saddle connection $c_j$ is vertical, and is tallest in the simplex $\tau \cup \{c_j\}$. Consider the horizontal unit-speed flow $\varphi^t$ emanating from $c_j$, away from $T_j$. Let $p$ be the intersection point of $c_i$ and $c_j$. The trajectory $\varphi^t(p)$ runs along $c_i$ until it hits the corner $p'$ of $T_i$ formed by $c_i$ and $a_j$ at time $t = \text{width}(a_j)$; see Figure 9.7. Since $c_i$ is disjoint from $\tau$, and $p$ is the only intersection point between $c_i$ and $c_j$, it follows that $p'$ is visible with respect to $\varphi^t$ and $\tau \cup \{c_j\}$. Applying Proposition 6.2, there exists a visible triangle $T$ with $c_j$ as one of its sides and satisfying

$$\text{height}(T) = \text{height}(T_j) \quad \text{and} \quad \text{width}(T) \leq \text{width}(a_j) \leq \text{width}(T_j).$$

But then $T_j$ is a major triangle, since it has at least half the area of the strictly convex quadrilateral formed by gluing $T_j$ and $T$ along $c_j$. This contradicts Lemma 9.8. The proof of the other inequality follows in a similar manner.

By considering the three right-angled triangles obtained by cutting $T_i \cup T_j$ along $c_i$ and $c_j$ (in the universal cover), together with the inequalities established above, we deduce the following.

$$\angle_{T_j}(a_i, a_k) + \angle_{T_i}(a_k, a_j) = \angle_{T_j}(a_i, c_i) + \angle_{T_j}(c_i, a_k) + \angle_{T_i}(a_k, c_j) + \angle_{T_i}(c_j, a_j)$$

$$> 2 \left( \angle_{T_j}(c_i, a_k) + \angle_{T_i}(a_k, c_j) \right) = 2 \cdot \frac{\pi}{2} = \pi.$$

This completes the proof. \qed

We continue in the previously described situation (as in Figure 9.7). By applying a horizontal shear to $(S, q)$, we now make $a_i$ vertical and $c_i$ horizontal, while maintaining the height of every saddle connection. Observe that the inequality $\angle_{T_j}(a_i, a_k) + \angle_{T_i}(a_k, a_j) > \pi$ persists under $\text{SL}(2, \mathbb{R})$-deformations. This implies that exactly one of $a_j$ or $a_k$ has positive slope, with the other having negative slope. Therefore, the triangles $T_i, T_j, T_k$ appear as in the configuration in Figure 9.6 in the middle or on the right (up to horizontal or vertical reflections), depending on whether $T_j$ and $T_k$ are glued concordantly or discordantly along $a_i$.

Summarising the discussion, we may thus assume that the triangles $T_i, T_j, T_k$ appear as one of the three possible configurations in Figure 9.6 up to horizontal or vertical reflections. For concreteness, let us assume that they appear exactly as shown, so that we may refer to top, bottom, left, and right. Observe that $\text{height}(T_j) = \text{height}(T_k) = \text{height}(T_i) + \text{height}(a_i)$. 
Figure 9.8 A strictly convex quadrilateral $Q$ (in red), formed by gluing triangles $T$ and $T'$ along $a_i$. The diagonals of $Q$ are $a_i$ and $b$ (in blue, dashed).

**Lemma 9.13.** There exists a strictly convex quadrilateral $Q$ with $a_i$ as one of its diagonals, such that height$(Q) = \text{height}(a_i)$, and neither $a_j$ nor $a_k$ intersect $\partial Q$ transversely.

**Proof.** Consider the leftwards unit-speed flow emanating from $a_i$. Since $a_i$ is strictly taller than both $a_j$ and $a_k$, we may construct a triangle $T$ meeting $a_i$ that is visible with respect to $\tau$ (see Figure 9.8). Note that height$(T) = \text{height}(a_i)$. Now let $\sigma = \tau \cup \partial T$, and consider the rightwards unit-speed flow emanating from $a_i$. Since $a_i$ is also tallest in $\sigma$, we can construct a triangle $T'$ meeting $a_i$ that is visible with respect to $\sigma$. We need to show that $T'$ can be chosen so that the quadrilateral $Q$ obtained by gluing $T$ and $T'$ along $a_i$ is strictly convex. If no sides of $T$ are horizontal, then this is immediate.

Suppose the side of $T$ meeting the bottom end point of $a_i$ is horizontal, as in Figure 9.8. By Lemma 7.6, the only obstruction for the existence of a triangle $T'$ so that $Q$ is strictly convex is the presence of an awning for $a_i$. Such an awning must be tallest in $\sigma$ and have negative slope. But this cannot occur, since the tallest saddle connections in $\sigma$ are the two non-horizontal sides of $T$, neither of which have negative slope. Arguing similarly for the case where a horizontal side of $T$ meets the top end point of $a_i$ completes the proof.

Let $Q$ be as in the above lemma, and $b$ be its diagonal obtained by flipping $a_i$. Since $\partial Q \cup \tau$ forms a simplex, it can be extended to a triangulation, and so $b \in F_\tau(a_i)$. By condition (T5), at least one of $a_j$ or $a_k$ belongs to KFB($a_i$, $b$). Recall from Corollary 8.14 that KFB($a_i$, $b$) $\subseteq \partial Q$. Therefore, one of the triangles obtained by cutting $Q$ along $a_i$ must also have at least one of $a_j$ or $a_k$ among its sides. By Lemma 9.8, this triangle must coincide with $T_j$ or $T_k$. But this is impossible, since

$$\text{height}(T_j) = \text{height}(T_k) = \text{height}(T_i) + \text{height}(a_i) > \text{height}(Q).$$

Hence, we have a contradiction to the assumption that $\tau$ does not bound a triangle.

This completes the proof of the Second Triangle Test.

**10 | ORIENTING TRIANGLES**

To recover the gluing pattern of a triangulation, we need to know not only the triangles themselves but also their orientations on $(S, q)$. Recall that an oriented triangle is given by a triple of sides...
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\[\vec{T} = [a, b, c],\] considered up to cyclic permutation. The main goal of this section is to develop a purely combinatorial test to detect whether two oriented triangles are consistently oriented, that is, when they determine the same orientation on \(S\).

**Proposition 10.1** (Orientation Test). Let \(\vec{T}\) and \(\vec{T}'\) be oriented triangles on \((S, q)\). Then \(\vec{T}\) and \(\vec{T}'\) are consistently oriented if and only if there exists a sequence of distinct triangles

\[\vec{T} = \vec{T}_0, \ldots, \vec{T}_k = \vec{T}'\]

such that for each \(0 \leq i < k\), the triangles \(\vec{T}_i\) and \(\vec{T}_{i+1}\) are consistently oriented and contained in a common strictly convex quadrilateral. Moreover, the property that \(\vec{T}\) and \(\vec{T}'\) are consistently oriented can be detected purely combinatorially in \(\mathcal{A}(S, q)\).

Certainly, if such a sequence exists then \(\vec{T}\) and \(\vec{T}'\) are consistently oriented. Our goal is to prove that there always exists such a sequence between two consistently oriented triangles.

**Lemma 10.2** (Detecting convex quadrilateral boundaries). Let \(\{c, d\}\) be a flip pair, and \(Q(c, d)\) be the strictly convex quadrilateral they span. Then \(\partial Q(c, d)\) can be detected using only the combinatorial structure of \(\mathcal{A}(S, q)\).

**Proof.** Recall from Section 8 that the set of barriers \(B(c, d)\) contains \(\partial Q(c, d)\), and can be defined purely combinatorially in terms of the pair of vertices \(\{c, d\}\). Let \(\mathcal{T}\) be any triangulation containing \(B(c, d) \cup \{c\}\). As \(B(c, d) \setminus \partial Q(c, d)\) consists exactly of the cordons of \(Q(c, d)\) (see Lemma 8.3), we deduce that \(Q(c, d)\) is the unique non-triangular region of \(\mathcal{T} \setminus \{c\}\). Applying the Triangle Test, we can detect the two triangles \(T\) and \(T'\) of \(\mathcal{T}\) that meet \(c\). Then \(\partial Q(c, d) = (\partial T \cup \partial T') \setminus \{c\}\). \(\square\)

Using the above lemma and the Triangle Test, we can consistently orient the four triangles contained in a strictly convex quadrilateral as follows. Let \(\{c, d\}\) be a flip pair.

(i) Use Lemma 10.2 to detect the sides of \(Q(c, d)\).

(ii) Observe that two sides of \(Q(c, d)\) (which may be the same saddle connection) are opposite one another if and only if they do not form a triangle together with neither \(c\) nor \(d\). Use this observation along with the Triangle Test to cyclically order the sides of \(Q(c, d)\).

(iii) Suppose that \(a, b, e, f\) form the sides of \(Q(c, d)\) in the given cyclic order, and so that \(\{a, b, c\}\) forms a triangle. Then \([a, b, c], [e, f, c], [b, e, d]\), and \([f, a, d]\) are triangles contained in \(Q(c, d)\) that all have the same orientation.

We now define an auxiliary graph \(G(S, q)\) as follows: The vertices of \(G(S, q)\) are the triangles on \((S, q)\) (in some triangulation), and two triangles \(T, T'\) are connected by an edge if and only if they are contained in a common strictly convex quadrilateral. (The triangles \(T\) and \(T'\) can be any two of the four triangles in the strictly convex quadrilateral, in particular, they are allowed to overlap.) Adjacent triangles in \(G(S, q)\) can be detected purely combinatorially using the definition of flip pairs together with the above procedure. Moreover, if \(T\) and \(T'\) are adjacent then we can also assign them consistent orientations using only combinatorial data. Thus, it suffices to show that \(G(S, q)\) is connected in order to prove Proposition 10.1.

Let us first focus on pairs of triangles that can be glued along a common side to form a quadrilateral. These triangles can be detected purely combinatorially as follows.
Lemma 10.3 (Gluable triangles). Let \( \tau, \tau' \in \mathcal{A}(S, q) \) be 2-simplices bounding triangles \( T, T' \). Then there exists a quadrilateral formed by gluing \( T \) and \( T' \) along a common side if and only if \( \tau \cup \tau' \) is a simplex in \( \mathcal{A}(S, q) \) and \( 1 \leq \#(\tau \cap \tau') \leq 2 \).

Proof. Observe that \( \tau \cup \tau' \) is a simplex in \( \mathcal{A}(S, q) \) if and only if no saddle connections of \( \tau \) intersect any saddle connection of \( \tau' \). The condition \( 1 \leq \#(\tau \cap \tau') \leq 2 \) is equivalent to saying that \( T \) and \( T' \) are distinct triangles that meet along at least one common side. \( \square \)

Lemma 10.4 (Connectedness of \( \mathcal{G}(S, q) \)). The graph \( \mathcal{G}(S, q) \) is connected.

Proof. Suppose first that \( \mathcal{T} \) is a triangulation of \( (S, q) \), and suppose \( T, T' \) are triangles of \( \mathcal{T} \) that share a common side. We show that there exists a path in \( \mathcal{G}(S, q) \) connecting \( T \) to \( T' \). Let \( c \in \partial T \) be the common side. By Proposition 7.3, there exists a triangulation \( \mathcal{T}' \supseteq \partial T \) in which \( c \) is flippable. Thus, the two triangles \( T, T'' \) that meet \( c \) in \( \mathcal{T}' \) can be glued along \( c \) to form a strictly convex quadrilateral. In particular, \( T \) and \( T'' \) are adjacent in \( \mathcal{G}(S, q) \).

By Theorem 4, \( \mathcal{F}(S, q) \) is connected, so there exists a sequence of triangulations

\[
\mathcal{T} = \mathcal{T}_0, \ldots, \mathcal{T}_j = \mathcal{T}'
\]

each containing \( \tau \), where consecutive triangulations are related by a single flip. Let \( T_i \neq T \) be the triangle of \( \mathcal{T}_i \) with \( c \) as one of its sides. Note that \( T_0 = T' \) and \( T_j = T'' \). If \( T_i \neq T_{i+1} \) then \( \mathcal{T}_i \) and \( \mathcal{T}_{i+1} \) are related by a flip in a strictly convex quadrilateral that contains both \( T_i \) and \( T_{i+1} \). Thus, for each \( 0 \leq i < j \), the triangles \( T_i \) and \( T_{i+1} \) either coincide or are adjacent in \( \mathcal{G}(S, q) \). Therefore, there exists a path in \( \mathcal{G}(S, q) \) connecting \( T' \) to \( T'' \), and hence to \( T \).

Consequently, by connectedness of \( (S, q) \), there exists a path in \( \mathcal{G}(S, q) \) connecting any two triangles of a given triangulation. If two triangulations \( \mathcal{T}, \mathcal{T}' \) differ by a single flip then they contain at least one triangle in common (since we are assuming that \( S \) is not a flat torus with exactly one marked point and hence contains at least three triangles in every triangulation). By Theorem 4, \( \mathcal{F}(S, q) \) is connected and so it follows that \( \mathcal{G}(S, q) \) is connected. \( \square \)

Note that all the steps and objects used in the previous proof can be stated using only the combinatorial structure of \( \mathcal{A}(S, q) \). This completes the proof of Proposition 10.1.

11 | RIGIDITY

We are now ready to prove our main theorem. Let \( (S, q) \) and \( (S', q') \) be half-translation surfaces. By the Triangle Test (Corollary 9.5) and Orientation Test (Proposition 10.1), the combinatorial structure of \( \mathcal{A}(S, q) \) can be used to recover the gluing pattern of any triangulation \( \mathcal{T} \) on \( (S, q) \), and hence the underlying topological surface \( (S, \mathcal{Z}) \). We can thus regard \( \mathcal{A}(S, q) \) as a subcomplex of \( \mathcal{A}(S, \mathcal{Z}) \). Similarly, \( \mathcal{A}(S', q') \) can be regarded as a subcomplex of \( \mathcal{A}(S', \mathcal{Z}') \), where \( (S', \mathcal{Z}') \) is the underlying topological surface of \( (S', q') \).

Given a homeomorphism \( F : (S, \mathcal{Z}) \to (S', \mathcal{Z}') \), write \( F^\# : \mathcal{A}(S, \mathcal{Z}) \to \mathcal{A}(S', \mathcal{Z}') \) for the induced map on the arc complexes. If \( F \) is isotopic to an affine diffeomorphism from \( (S, q) \) to \( (S', q') \), then the restriction \( F^\#|_{\mathcal{A}(S, q)} : \mathcal{A}(S, q) \to \mathcal{A}(S', q') \) is a simplicial isomorphism. Our goal is to prove the converse.
Theorem 1 (Rigidity of the saddle connection complex). Let \((S, q)\) and \((S', q')\) be half-translation surfaces, neither of which are flat tori with exactly one removable singularity. Suppose \(\phi : \mathcal{A}(S, q) \rightarrow \mathcal{A}(S', q')\) is a simplicial isomorphism. Then there exists a unique affine diffeomorphism \(F : (S, q) \rightarrow (S', q')\) inducing \(\phi\).

Recall that in the case where at least one of \((S, q)\) or \((S', q')\) is a flat torus with one removable singularity, we have exactly two such affine diffeomorphisms as described after Example 3.6. As we are not in this case, we can use the fact that triangles are uniquely determined by their sides in the following arguments.

Our strategy is to first define affine maps on individual triangles. These maps can be used to define a piecewise affine diffeomorphisms on \((S, q)\) associated to a given triangulation. We then show that all triangulations give rise to the same piecewise affine diffeomorphism \(F : (S, q) \rightarrow (S', q')\) yielding the map \(\phi\) on the saddle connection complexes. Finally, we use the Cylinder Rigidity Theorem (see Theorem 5) to show that \(F\) is affine.

The Triangle Test (Corollary 9.5) uses only the combinatorial structure of \(\mathcal{A}(S, q)\). Therefore, a 2-simplex \(\tau \in \mathcal{A}(S, q)\) bounds a triangle on \((S, q)\) if and only if \(\phi(\tau) \in \mathcal{A}(S', q')\) bounds a triangle on \((S', q')\). If \(T\) is a triangle on \((S, q)\) with sides \(\tau = \{a, b, c\}\) then there is a unique affine map \(F_T : T \rightarrow (S', q')\) such that \(F_T(a) = \phi(a), F_T(b) = \phi(b), \) and \(F_T(c) = \phi(c)\). In particular, the image of \(F_T\) is the unique triangle with sides \(\phi(\tau)\).

Suppose \(\vec{T} = [a, b, c]\) and \(\vec{T}' = [a', b', c']\) are oriented triangles on \((S, q)\). Since the Orientation Test (Proposition 10.1) only uses the combinatorial structure of \(\mathcal{A}(S, q)\), it follows that the oriented triangles \([\phi(a), \phi(b), \phi(c)]\) and \([\phi(a'), \phi(b'), \phi(c')]\) on \((S', q')\) are consistently oriented if and only if \(\vec{T}, \vec{T}'\) are consistently oriented.

We now want to define a piecewise affine diffeomorphism \(F_T : (S, q) \rightarrow (S', q')\) for a given triangulation \(T\) of \((S, q)\).

Lemma 11.1 (Candidate homeomorphisms). Let \(T\) be a triangulation of \((S, q)\) and define \(F_T : (S, q) \rightarrow (S', q')\) by declaring \(F_T|_T = F_T\) for every triangle \(T\) of \(T\). Then \(F_T\) is a well-defined, piecewise affine diffeomorphism.

Proof. Observe that \(F_T\) is well-defined on the interior of each triangle \(T\) of \(T\). We need to check that it is well-defined on the edges and vertices of \(T\). Suppose \(T, T'\) are distinct triangles of \(T\). Since \(\phi\) preserves disjointness of saddle connections, \(\partial F_T(T)\) and \(\partial F_T(T')\) have no transverse intersections. It follows that \(F_T(T), F_T(T')\) have disjoint interiors. By Proposition 10.1 and Lemma 10.3, if \(T, T'\) meet along a common side \(a \in T\), then \(F_T(T), F_T(T')\) meet along \(\phi(a)\) with the same orientation. Thus, \(F_T\) is well-defined on the complement of the singularities \(\mathcal{Z}\) of \((S, q)\).

Next, we check that \(F_T\) is well-defined on \(\mathcal{Z}\). Note that \(S\) and \(S'\) are, respectively, the metric completions of \(S \setminus \mathcal{Z}\) and \(S' \setminus \mathcal{Z}'\). Observe that each \(F_T\) is a Lipschitz map. Since \(T\) has finitely many triangles, \(F_T\) is also a Lipschitz map. As Lipschitz maps send Cauchy sequences to Cauchy sequences, it follows that \(F_T\) extends to a unique map between the respective metric completions. It follows that \(F_T : (S, q) \rightarrow (S', q')\) is well-defined.

Finally, we show that \(F_T\) is a homeomorphism. By construction, \(F_T\) is continuous and restricts to an affine map on each triangle \(T\) of \(T\). We can analogously define a piecewise affine map \(G_{\phi(T)} : (S', q') \rightarrow (S, q)\) using \(\phi^{-1}\) and the triangulation \(\phi(T)\) on \((S', q')\). By construction, \(G_{\phi(T)} \circ F_T\) restricts to the identity map on each triangle \(T\) of \(T\). Therefore, \(F_T\) and \(G_{\phi(T)}\) are inverses of one another. It follows that \(F_T\) is a piecewise affine diffeomorphism. \(\square\)
Observe that $F_T$ acts as a bijection between the singularities of $(S, q)$ and $(S', q')$. In fact, this bijection does not depend on the choice of the triangulation $T$.

As $F_T$ is in particular a homeomorphism from $(S, Z)$ to $(S', Z')$, it induces a simplicial isomorphism $F_T^#: A(S, Z) \to A(S', Z')$. We show first that $F_T^#$ is independent of the triangulation and then that it coincides with $\phi$ on the proper isotopy classes of arcs in $A(S, Z)$ realisable as saddle connections on $(S, q)$.

**Lemma 11.2** (Candidates are isotopic). For two triangulations $T, T'$ of $(S, q)$, we have $F_T^# = F_{T'}^#$. 

**Proof.** By Theorem 4, $F(S, q)$ is connected, and so there is a finite sequence of flips turning $T$ into $T'$. So, we may assume without loss of generality that $T$ and $T'$ differ by precisely one flip. Then $T \cap T'$ is a triangulation away from a strictly convex quadrilateral $Q$ on $(S, q)$. Then $F_T$ and $F_{T'}$ coincide outside the interior of $Q$. By Alexander’s Trick, any two homeomorphisms defined on a disc that agree on the boundary must be isotopic. Therefore, $F_T$ and $F_{T'}$ are isotopic, through isotopies fixing the singularities. Hence, $F_T(a) = F_{T'}(a)$ for every arc $a \in A(S, Z)$. □

**Lemma 11.3** (Candidates induce the correct isomorphism on $A(S, q)$). Let $T$ be a triangulation of $(S, q)$. Then the restricted simplicial map $F_T^#|_{A(S, q)}$ coincides with $\phi$.

**Proof.** Let $a \in A(S, q)$ be a saddle connection and let $T'$ be a triangulation with $a \in T'$. By definition, $F_T(a)$ is the unique geodesic representative of $\phi(a)$ in its isotopy class. But this implies $F_T^#(a) = F_{T'}^#(a) = \phi(a)$. □

It remains to show that $F_T$ is an affine diffeomorphism. For this, let us fix a choice of triangulation $T$ and write $F = F_T$. Let $m = F^*(q')$ be the half-translation structure on $S$ obtained by pulling back $q'$ via $F$. Note that the singularities of $(S, m)$ are naturally identified with $Z$. Now, consider the map $F' := F \circ \text{id}_S : (S, m) \to (S', q')$ where $\text{id}_S : (S, m) \to (S, q)$ is the identity map on the underlying surface $(S, Z)$. By definition of $m$, $F'$ maps the charts corresponding to the half-translation structure $m$ to the charts corresponding to $q'$ and hence $F'$ is an isometry. Therefore, a topological arc $\alpha \in A(S, Z)$ is realisable as a saddle connection on $(S, m)$ if and only if $F'(\alpha) \in A(S', Z')$ is realisable as a saddle connection on $(S', q')$. By the above corollary, this occurs precisely when $F^{-1} \circ F'(\alpha) = \text{id}_S(\alpha) = \alpha$ is realisable as a saddle connection on $(S, q)$. We have thus shown the following.

**Lemma 11.4** (Subcomplexes coincide). The saddle connection complexes $A(S, q)$ and $A(S, m)$ coincide as subcomplexes of the arc complex $A(S, Z)$. □

It follows that $\text{MIL}(A(S, q)) = \text{MIL}(A(S, m))$ as sets of simplices in $A(S, Z)$. Therefore, by Corollary 5.5, the sets of cylinder curves $\text{cyl}(q)$ and $\text{cyl}(m)$ coincide as sets of simple closed curves on $S$. Applying the Cylinder Rigidity Theorem (Theorem 5), we deduce that $m \in \text{GL}(2, \mathbb{R}) \cdot q$. Therefore, $\text{id}_S$ is an affine diffeomorphism. Since $F'$ is an isometry, it follows that $F = F' \circ \text{id}_S^{-1}$ is also an affine diffeomorphism as desired.

Finally, we check that $F$ is the unique affine diffeomorphism with $F_T^#|_{A(S, q)} = \phi$. Suppose $G : (S, q) \to (S', q')$ is an affine diffeomorphism inducing the isomorphism $\phi : A(S, q) \to A(S', q')$. Then $G$ maps a triangle $T$ on $(S, q)$ with sides $a, b, c$ to the triangle $G(T)$ on $(S', q')$ with sides $\phi(a) = G(a), \phi(b) = G(b)$, and $\phi(c) = G(c)$. Since $F_T : T \to G(T)$ is the unique affine map from $T$ to $G(T)$ which behaves correctly on the sides of $T$, it follows that $G|_T = F_T$. As this holds
for all triangles on \((S, q)\), we deduce that \(G = F_T\) for every triangulation \(T\) of \((S, q)\). (In particular, \(F_T\) does not depend on the choice of triangulation.) Therefore, \(F = G\) and we are done.

This completes the proof of Theorem 1.

**APPENDIX A: CLASSIFYING LINKS OF SIMPLICSES OF CODIMENSION 1 OR 2**

In the tables below, we classify the possible link types of a simplex \(\sigma \in \mathcal{A}(S, q)\) of codimension 1 or 2, together with the corresponding non-triangular regions of \(S - \sigma\). In the cases where \(\text{lk}(\sigma)\) decomposes as a non-trivial join, we indicate the factors by colouring the vertices red or blue.

**TABLE A.1** Classification of links of codimension-1 simplices

| \(\text{lk}(\sigma)\) | Non-triangular regions of \(S - \sigma\) |
|----------------------|----------------------------------------|
| ![Diagram](https://example.com/diagram1.png) | ![Diagram](https://example.com/diagram2.png) |

**TABLE A.2** Classification of links of codimension-2 simplices

| \(\text{lk}(\sigma)\) | Non-triangular regions of \(S - \sigma\) |
|----------------------|----------------------------------------|
| ![Diagram](https://example.com/diagram3.png) | ![Diagram](https://example.com/diagram4.png) |

(bi-infinite path graph)
APPENDIX B: LIST OF NOTATIONS

Let $\mathcal{K}$ be a simplicial complex. The following sets of simplices in $\mathcal{K}$ are used in this paper.

- $\text{IL}(\mathcal{K})$: Simplices $\sigma \in \mathcal{K}$ with infinite link (Definition 2.7)
- $\text{MIL}(\mathcal{K})$: Simplices $\sigma \in \text{IL}(\mathcal{K})$ that are maximal among those in $\text{IL}(\mathcal{K})$ (Definition 2.7)
- $\text{FP}(\mathcal{K})$: Flip pairs $\kappa \cong \mathbb{N}_2$ arising as links of codimension-1 simplices in $\mathcal{K}$ (Definition 8.1)
- $\text{B}(\kappa)$: Barriers of a flip pair $\kappa$ (Definition 8.2)
- $\text{FB}(\kappa)$: Flippable barriers of a flip pair $\kappa$ (Definition 8.4)
- $\text{KB}(\kappa)$: Kite barriers of a cylindrical kite pair $\kappa$ (Definition 8.12)
- $\text{KFB}(\kappa)$: Kite-or-flippable barriers of a flip pair $\kappa$ (Definition 8.13)
- $\text{F}_{\tau}(a)$: $\tau$-Compatible flip partners of a saddle connection $a$ (Definition 9.3)

The following notations for graphs are used in the paper.

- $\mathbb{C}_k$: The cycle graph of length $k$
- $\mathbb{N}_k$: The edgeless graph on $k$ vertices
- $\mathbb{P}_k$: The path graph of length $k$
- $\mathbb{P}_\infty$: The bi-infinite path graph

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