Hadamard matrices related to a certain series of ternary self-dual codes

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Abstract
In 2013, Nebe and Villar gave a series of ternary self-dual codes of length $2(p + 1)$ for a prime $p$ congruent to 5 modulo 8. As a consequence, the third ternary extremal self-dual code of length 60 was found. We show that the ternary self-dual code contains codewords which form a Hadamard matrix of order $2(p+1)$ when $p$ is congruent to 5 modulo 24. In addition, it is shown that the ternary self-dual code is generated by the rows of the Hadamard matrix. We also demonstrate that the third ternary extremal self-dual code of length 60 contains at least two inequivalent Hadamard matrices.

1 Introduction
Self-dual codes are one of the most interesting classes of codes. This interest is justified by many combinatorial objects and algebraic objects related to self-dual codes (see e.g., \cite{15}). A Hadamard matrix is a kind of orthogonal matrix appearing in many research areas of Mathematics and practical applications (see e.g., \cite{16} and \cite{17}). One of the interesting and successful applications

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of Hadamard matrices is their use as codes. In particular, a special class
of Hadamard matrices can give rise to self-dual codes as their row spaces.
In this paper, we are interested in Hadamard matrices related to ternary
self-dual codes found by Nebe and Villar [11].

A ternary self-dual code $C$ of length $n$ is an $[n, n/2]$ code over the finite
field of order 3 satisfying $C = C^\perp$, where $C^\perp$ is the dual code of $C$. A
ternary self-dual code of length $n$ exists if and only if $n$ is divisible by four.
It was shown in [10] that the minimum weight $d$ of a ternary self-dual code of
length $n$ is bounded by $d \leq 3\lfloor n/12 \rfloor + 3$. If $d = 3\lfloor n/12 \rfloor + 3$, then the code is
called extremal. For $n \in \{4, 8, 12, \ldots , 64\}$, it is known that there is a ternary
extremal self-dual code of length $n$ (see [7, Table 6]). The ternary extended
quadratic residue codes and the Pless symmetry codes are well known families
of ternary (self-dual) codes. It is known that the ternary extended quadratic
residue code $QR_{60}$ of length 60 and the Pless symmetry code $P_{60}$ of length
60 are ternary extremal self-dual codes (see [15, Table XII]). In 2013, Nebe
and Villar [11] gave a series of ternary self-dual codes of length $2(p + 1)$ for
all primes $p \equiv 5 \pmod{8}$. As a consequence, the third ternary extremal
self-dual code of length 60 was found.

A Hadamard matrix $H$ of order $n$ is an $n \times n$ matrix whose entries are from
$\{1, -1\}$ such that $HH^T = nI_n$, where $H^T$ is the transpose of $H$ and $I_n$ is the
identity matrix of order $n$. It is known that the order $n$ is necessarily 1, 2, or
a multiple of 4. Recently, Tonchev [19] studied Hadamard matrices of order $n$
formed by codewords of weight $n$ in ternary extremal self-dual codes of length
$n$, especially the extended quadratic residue codes and the Pless symmetry
codes. From the construction, the extended quadratic residue code contains a
type I Paley-Hadamard matrix. The Pless symmetry code contains a type II
Paley-Hadamard matrix [13]. Tonchev [19] showed that the Pless symmetry
code of length 36 contains exactly two inequivalent Hadamard matrices of
order 36. This motivates us to study the existence of Hadamard matrices of
order $n$ formed by codewords of weight $n$ in ternary self-dual codes found by
Nebe and Villar [11].

The paper is organized as follows. In Section 2, definitions, notations
and basic results are given. Especially, we review the construction of ternary
self-dual codes $NV(a)(p)$ in [11] of length $2(p + 1)$, where $p$ is a prime with
$p \equiv 5 \pmod{8}$ and $a \in \{1, -1\}$. In Section 3, we show that $NV(a)(p)$
contains $2(p+1)$ codewords of weight $2(p+1)$ which form a Hadamard matrix
$H_{NV(a)(p)}$ of order $2(p + 1)$ for any prime $p \equiv 5 \pmod{24}$ and $a \in \{1, -1\}$
(see Theorem 2, which is our main theorem of this paper). We also give
characterizations of the Hadamard matrices $H_{NV^{(a)}(p)}$ of order $2(p + 1)$. In particular, it is shown that the ternary self-dual code $NV^{(a)}(p)$ is generated by the rows of the Hadamard matrix $H_{NV^{(a)}(p)}$. This gives an alternative construction of the ternary self-dual code $NV^{(a)}(p)$. By Theorem 2, the third ternary extremal self-dual code $NV^{(1)}(29)$ of length 60, which was found in [11], contains a Hadamard matrix of order 60. In Section 4, our computer search shows that $NV^{(1)}(29)$ contains one more Hadamard matrix of order 60. Finally, in Section 5, we demonstrate that the currently known three ternary extremal self-dual codes of length 60 are constructed as four-negacirculant codes.

2 Preliminaries

In this section, we give definitions and some known results of ternary self-dual codes and Hadamard matrices used in this paper. Especially, we give details for the construction of ternary self-dual codes $NV^{(a)}(p)$ in [11] of length $2(p + 1)$, where $p$ is a prime with $p \equiv 5 \pmod{8}$ and $a \in \{1, -1\}$.

2.1 Ternary self-dual codes

Let $\mathbb{F}_3 = \{0, 1, 2\}$ denote the finite field of order 3. A ternary $[n, k]$ code $C$ is a $k$-dimensional vector subspace of $\mathbb{F}_3^n$. All codes in this paper are ternary. The parameter $n$ is called the length of $C$. A generator matrix of $C$ is a $k \times n$ matrix whose rows are a basis of $C$. The weight $wt(x)$ of a vector $x$ of $\mathbb{F}_3^n$ is the number of non-zero components of $x$. A vector of $C$ is called a codeword. The minimum non-zero weight of all codewords in $C$ is called the minimum weight of $C$. The weight enumerator of $C$ is given by $\sum_{c \in C} y^{wt(c)} \in \mathbb{Z}[y]$.

The dual code $C^\perp$ of a ternary code $C$ of length $n$ is defined as $C^\perp = \{x \in \mathbb{F}_3^n \mid x \cdot y = 0 \text{ for all } y \in C\}$, where $x \cdot y$ is the standard inner product. A ternary code $C$ is self-dual if $C = C^\perp$. A ternary self-dual code of length $n$ exists if and only if $n$ is divisible by four. Two ternary codes $C$ and $C'$ are equivalent if there is a monomial matrix $P$ over $\mathbb{F}_3$ with $C' = C \cdot P$, where $C \cdot P = \{xP \mid x \in C\}$. We denote two equivalent ternary codes $C$ and $D$ by $C \cong D$. All ternary self-dual codes were classified in [4], [6], [9] and [14] for lengths up to 24.
2.2 Ternary extremal self-dual codes

It was shown in [10] that the minimum weight \( d \) of a ternary self-dual code of length \( n \) is bounded by \( d \leq 3\lfloor n/12 \rfloor + 3 \). If \( d = 3\lfloor n/12 \rfloor + 3 \), then the code is called extremal. For \( n \in \{4, 8, 12, \ldots, 64\} \), it is known that there is a ternary extremal self-dual code of length \( n \) (see [7, Table 6]). By the Assmus–Mattson theorem [1], the supports of codewords of minimum weight in a ternary extremal self-dual code of length divisible by 12 form a 5-design. This is a reason for our interest in ternary extremal self-dual codes of length divisible by 12.

The weight enumerator of a ternary extremal self-dual code of length \( n \) is uniquely determined for each \( n \) [10]. The number \( A_n \) of codewords of weight \( n \) in a ternary extremal self-dual code of length \( n \) is listed in Table 1 for \( n = 12, 24, 36, 48, 60 \) (see [19]). Note that \( A_n = 2n \) for \( n = 12, 24, 48 \).

| \( n \) | 12 | 24 | 36 | 48 | 60 |
|-------|----|----|----|----|----|
| \( A_n \) | 24 | 48 | 888 | 96 | 41184 |

The ternary extended quadratic residue codes and the Pless symmetry codes are well known families of ternary (self-dual) codes. More precisely, the extended quadratic residue code \( QR_{p+1} \) of length \( p + 1 \) is a ternary self-dual code when \( p \) is a prime such that \( p \equiv -1 \pmod{12} \) (see [8, Chapter 6]). The Pless symmetry code \( P_{2q+2} \) of length \( 2q + 2 \) is a ternary self-dual code when \( q \) is a prime power such that \( q \equiv -1 \pmod{6} \) [13] (see also [8, Chapter 10]). The extended quadratic residue codes \( QR_n \) and the Pless symmetry codes \( P_n \) yield ternary extremal self-dual codes when \( n \leq 60 \) (see [15]). More precisely, \( P_{36} \) is the currently known ternary extremal self-dual code of length 36, \( QR_{48} \) and \( P_{48} \) are the currently known ternary extremal self-dual codes of length 48. In addition, \( QR_{60} \) and \( P_{60} \) are ternary extremal self-dual codes of length 60.

2.3 Ternary self-dual codes given in [11]

In 2013, Nebe and Villar [11] gave a new series of ternary self-dual codes \( NV^{(a)}(p) \) of length \( 2(p + 1) \) for all primes \( p \equiv 5 \pmod{8} \) and \( a \in \{1, -1\} \) (see also [3, Section 4] for the details). Here, we review the construction of the ternary self-dual codes \( NV^{(a)}(p) \).
Suppose that \( p \equiv 5 \pmod{8} \). Let \( \mathbb{F}_p = \{0, 1, \ldots, p-1\} \) denote the finite field of order \( p \). Let \( \chi \) denote the quadratic character of \( \mathbb{F}_p \). Define two \( p \times p \) matrices \( R_X = (r_{X_{a,b}}) \) and \( R_Y = (r_{Y_{a,b}}) \) as follows

\[
r_{X_{a,b}} = \begin{cases} 0, & \text{if } a = b \text{ or } b - a \text{ is not a nonzero square in } \mathbb{F}_p, \\ \chi(c), & \text{if } b - a \text{ is a nonzero square } c^2 \text{ in } \mathbb{F}_p. \end{cases}
\]

\[
r_{Y_{a,b}} = \begin{cases} 0, & \text{if } a = b \text{ or } 2(b - a) \text{ is not a nonzero square in } \mathbb{F}_p, \\ \chi(c), & \text{if } 2(b - a) \text{ is a nonzero square } c^2 \text{ in } \mathbb{F}_p. \end{cases}
\]

where rows and columns of \( R_X \) and \( R_Y \) are indexed by the elements of \( \mathbb{F}_p \) with a fixed ordering. Then define two \( (p+1) \times (p+1) \) matrices \( X \) and \( Y \) as follows

\[
X = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & \ddots & R_X \\ \vdots & \ddots & \ddots & \ddots \\ -1 & \cdots & -1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \chi(c) \end{pmatrix}.
\]

In addition, define two \( 2(p+1) \times 2(p+1) \) matrices as follows

\[
B_w = \begin{pmatrix} X & Y \\ -Y^T & X^T \end{pmatrix} \quad \text{and} \quad B_{cw} = \begin{pmatrix} -Y^T & X^T \\ -X & -Y \end{pmatrix}.
\]

Throughout this paper, let \( I_n \) denote the identity matrix of order \( n \). For \( a = 1 \) and \( -1 \), let \( NV^{(a)}(p) \) denote the ternary code generated by the matrix \( M \), where

\[
M = \begin{cases} aI_{2(p+1)} + B_w, & \text{if } p \equiv 5 \pmod{24}, \\ aI_{2(p+1)} + B_w + B_{cw}, & \text{if } p \equiv 13 \pmod{24}. \end{cases}
\]

Then \( NV^{(a)}(p) \) is self-dual \[\text{[11]}\] (see also \[\text{[3, Theorem 8]}\]).

**Proposition 1** (Nebe and Villar \[\text{[11]}\]). \( NV^{(1)}(29) \cong NV^{(-1)}(29) \) and \( QR_{60} \not\cong NV^{(1)}(29) \not\cong P_{60} \).

The above proposition means that \( NV^{(1)}(29) \) is the third ternary extremal self-dual code of length 60. In this paper, we denote the code \( NV^{(1)}(29) \) by \( NV_{60} \).
2.4 Hadamard matrices and results in [19]

A Hadamard matrix $H$ of order $n$ is an $n \times n$ matrix whose entries are from \{1, -1\} such that $HH^T = nI_n$, where $H^T$ is the transpose of $H$. It is known that the order $n$ is necessarily 1, 2, or a multiple of 4. A Hadamard matrix $H$ of order $n$ is called skew if $H = A + I_n$, where $A = -A^T$. Two Hadamard matrices $H$ and $K$ are said to be equivalent if there is $(1, -1, 0)$-monomial matrices $P$ and $Q$ with $K = PHQ$. An automorphism of a Hadamard matrix $H$ is an equivalence of $H$ to itself, i.e., a pair $(P, Q)$ of monomial matrices $P$ and $Q$ such that $H = PHQ$. The set of all automorphisms of $H$ forms a group, called the automorphism group of $H$, under the component-wise product: $(P_1, Q_1)(P_2, Q_2) = (P_1P_2, Q_1Q_2)$.

Recently, Tonchev [19] studied Hadamard matrices of order $n$ formed by codewords of weight $n$ in ternary extremal self-dual codes of length $n$, especially the extended quadratic residue codes and the Pless symmetry codes. In the context of Hadamard matrices, we consider the element 0, 1, 2 of $\mathbb{F}_3$ as 0, 1, -1 of $\mathbb{Z}$, throughout this paper. It is trivial that $n \equiv 0 \pmod{12}$ if a ternary (extremal) self-dual code of length $n$ contains a Hadamard matrix formed by codewords of weight $n$. This is another reason for our interest in ternary extremal self-dual codes of length divisible by 12.

From the construction, the extended quadratic residue code contains a type I Paley-Hadamard matrix. The Pless symmetry code contains a type II Paley-Hadamard matrix [13]. Tonchev [19] showed that $P_{36}$ contains exactly two inequivalent Hadamard matrices of order 36. In addition, Tonchev [19] gave a natural question, namely, is there any other ternary extremal self-dual code of length 36, 48, or 60 which contains a Hadamard matrix? This motivates us to study the existence of Hadamard matrices of order $2(p + 1)$ formed by codewords of weight $2(p+1)$ in the ternary self-dual codes $NV^{(a)}(p)$ found by Nebe and Villar [11].

3 Hadamard matrices related to $NV^{(a)}(p)$

Throughout this section, suppose that $p$ is a prime with $p \equiv 5 \pmod{24}$. In this section, we show that $NV^{(a)}(p)$ contains $2(p + 1)$ codewords of weight $2(p + 1)$ which form a Hadamard matrix $H_{NV^{(a)}(p)}$ of order $2(p + 1)$ for $a \in \{1, -1\}$. We also give characterizations of the Hadamard matrices $H_{NV^{(a)}(p)}$ of order $2(p + 1)$. 

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Let $X$ and $Y$ be the $(p+1) \times (p+1)$ matrices as defined in Section 2.3. As described there, for $a = 1$ and $-1$, the ternary code $NV^{(a)}(p)$ generated by the following matrix

$$aI_{2(p+1)} + \begin{pmatrix} X & Y \\ -Y^T & X^T \end{pmatrix}$$

is a ternary self-dual code [11] (see also [3, Theorem 8]).

The following is our main theorem of this paper.

**Theorem 2.** Suppose that $p \equiv 5 \pmod{24}$ and $a \in \{1, -1\}$. Then the ternary self-dual code $NV^{(a)}(p)$ of length $2(p+1)$ contains $2(p+1)$ codewords which form a Hadamard matrix of order $2(p+1)$.

**Proof.** Since $NV^{(a)}(p)$ is generated by the following matrix

$$\begin{pmatrix} X + aI_{p+1} & Y \\ -Y^T & X^T + aI_{p+1} \end{pmatrix},$$

the rows of the following two matrices

$$\begin{pmatrix} X - Y^T + aI_{p+1} & Y + X^T + aI_{p+1} \\ -Y^T - X - aI_{p+1} & X^T - Y + aI_{p+1} \end{pmatrix}$$

are codewords of $NV^{(a)}(p)$. From the definition of $X$ and $Y$, the $2(p+1)$ codewords has weight $2(p+1)$. In addition, we regard the following matrix as a $\mathbb{Z}$-matrix

$$H_{NV^{(a)}(p)} = \begin{pmatrix} X - Y^T + aI_{p+1} & Y + X^T + aI_{p+1} \\ -Y^T - X - aI_{p+1} & X^T - Y + aI_{p+1} \end{pmatrix}. \quad (1)$$

Since it is known [3] that

$$X^T = -X, Y^T = -Y, XY = YX \text{ and } X^2 + Y^2 = -pI_{p+1},$$

$H_{NV^{(a)}(p)}$ is a Hadamard matrix of order $2(p+1)$.

Now we give characterizations of Hadamard matrices $H_{NV^{(a)}(p)}$ of order $2(p+1)$.

**Proposition 3.** Let $H_{NV^{(a)}(p)}$ denote the Hadamard matrix given in (1). Then $aH_{NV^{(a)}(p)}$ is a skew Hadamard matrix for $a = 1$ and $-1$. 

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Proof. The claim follows from that \( H_{NV(a)(p)} + H_{NV(a)(p)}^T = 2aI_2(p+1) \). □

Note that \( H_{NV(a)(p)} \) has the form \( H_{NV(a)(p)} = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix} \), where

\[
A = X - Y^T + aI_{p+1} = \begin{pmatrix} a & 1 & \cdots & 1 \\ -1 & \ddots & A' \\ -1 & & \ddots & \ddots \\ & & -1 & \cdots & 1 \end{pmatrix}
\]

and

\[
B = Y + X^T + aI_{p+1} = \begin{pmatrix} a & -1 & \cdots & -1 \\ 1 & \ddots & \ddots & \ddots \\ \vdots & 1 & \ddots & \ddots \\ & \vdots & \ddots & -1 & \cdots & -1 \end{pmatrix}
\]

for some \( p \times p \) matrices \( A' \) and \( B' \). Let \( \omega \) be a fixed primitive element of \( \mathbb{F}_p \). Define

\[
C_i = \omega^i(\omega^4), i = 0, 1, 2, 3,
\]

which are the cosets of the multiplicative subgroup of index 4 of \( \mathbb{F}_p \). Note that \( C_1 \) and \( C_3 \) are interchanged if we choose \( \omega^{-1} \) as a primitive element instead of \( \omega \). Since \( p \equiv 5 \pmod{8} \), we have \( -1 \in C_2 \) and \( 2 \in NQ = C_1 \cup C_3 \), where \( NQ \) denotes the set of nonsquares in \( \mathbb{F}_p \). Hence, \( -2 \in C_1 \) or \( -2 \in C_3 \), depending on the choice of \( \omega \). We denote by \( C_\epsilon \) the coset containing \(-2\), that is,

\[
-2 \in C_\epsilon.
\]

Then \( A' = (a'_{s,t}) \) and \( B' = (b'_{s,t}) \) can be written as

\[
a'_{s,t} = \begin{cases} 
  a, & \text{if } t = s, \\
  1, & \text{if } t - s \in C_0 \cup C_\epsilon, \\
  -1, & \text{if } t - s \in C_2 \cup C_{\epsilon+2}, 
\end{cases}
\]

and

\[
b'_{s,t} = \begin{cases} 
  a, & \text{if } t = s, \\
  1, & \text{if } t - s \in C_2 \cup C_\epsilon, \\
  -1, & \text{if } t - s \in C_0 \cup C_{\epsilon+2}.
\end{cases}
\]

Throughout this section, we reduce the subscript of \( C_i \) modulo 4.

Let \( G \) be an additively written abelian group of order \( v \). Two subsets \( D_1 \) and \( D_2 \) of \( G \) with \( k = |D_1| = |D_2| \) are called \((v, k, \lambda)\) supplementary
difference sets if the list of differences $x - y$, $x, y \in D_i$, $i = 1, 2$, represents every nonzero element of $G$ exactly $\lambda$ times. Fixing an ordering for the elements of $G$, we define a matrix $M = (m_{i,j})$ by

$$m_{i,j} = \begin{cases} 1, & \text{if } j - i \in X, \\ -1, & \text{if } j - i \notin X, \end{cases}$$

for $X \subset G$. The matrix $M$ is called a type-1 matrix of $X$.

The following construction of Hadamard matrices easily follows from [16, Corollary 4.5 (i) and Lemma 4.8].

**Lemma 4.** Let $D_i$, $i = 1, 2$, be $(2m + 1, m, m - 1)$ supplementary difference sets in an abelian group $G$ of order $v = 2m + 1$. Furthermore, let $M_1$ (resp. $M_2$) be the type-1 matrix of $D_1$ (resp. $D_2$). Then

$$H(D_1, D_2) = \begin{pmatrix} 1 & 1 & 1_v & -1_v \\ -1 & 1 & -1_v & -1_v \\ -1_v^T & 1_v^T & -M_1 & -M_2 \\ 1_v^T & -M_1^T & M_2 & -M_2^T \end{pmatrix}$$

is a Hadamard matrix of order $4(m + 1)$, where $1_v$ denotes the all-one vector of length $v$.

It is known [18] that for any prime $p \equiv 5 \pmod{24}$ and any $i = 0, 1, 2, 3$, the sets $C_i \cup C_{i+1}$ and $C_i \cup C_{i+2}$ are $(p, (p - 1)/2, (p - 3)/2)$ supplementary difference sets in the additive group of $\mathbb{F}_p$, where $C_i$’s are defined as in [2] (more generally, the same claim holds for any prime power $p \equiv 5 \pmod{8}$). By Lemma 4, $H(C_i \cup C_{i+1}, C_i \cup C_{i+2})$ is a Hadamard matrix of order $2(p+1)$. Furthermore, from (4), the following theorem holds.

**Theorem 5.** Let $H_{NV(a)}(p)$ denote the Hadamard matrix defined as in (1). Then $H_{NV(0)}(p)$ and $H_{NV(-1)}(p)$ are equivalent to $H(C_2 \cup C_{2+\epsilon}, C_0 \cup C_{\epsilon+2})$ and $H(C_0 \cup C_{\epsilon}, C_2 \cup C_{\epsilon})$, respectively, where $C_{\epsilon}$ is defined as in (3).

Although the proof of the following proposition is somewhat trivial, we give it for the sake of completeness.

**Proposition 6.** The ternary self-dual code $NV(a)(p)$ is generated by the rows of the Hadamard matrix $H_{NV(a)}(p)$ defined as in (1).
Proof. It is sufficient to show that \( \text{rank}_3(H_{NV(a)}(p)) = p + 1 \). Since \( H_{NV(a)}(p) + H^T_{NV(a)}(p) = 2aI_{2(p+1)} \), we have

\[
2(p + 1) = \text{rank}_3(2aI_{2(p+1)}) = \text{rank}_3(H_{NV(a)}(p) + H^T_{NV(a)}(p)) \\
\leq 2 \text{rank}_3(H_{NV(a)}(p)),
\]

i.e., \( p + 1 \leq \text{rank}_3(H_{NV(a)}(p)) \). On the other hand, since \( p + 1 \equiv 0 \pmod{3} \), \( H_{NV(a)}(p)H^T_{NV(a)}(p) \equiv O \pmod{3} \), where \( O \) denotes the \( 2(p+1) \times 2(p+1) \) zero matrix. This implies that \( \text{rank}_3(H_{NV(a)}(p)) \leq p + 1 \). This completes the proof.

The above proposition gives an alternative construction of the ternary self-dual code \( NV(a)(p) \).

4 Hadamard matrices related to \( NV_{60} \)

By Theorem 2, the third ternary extremal self-dual code \( NV_{60} \) of length 60, which was found in [11], contains a Hadamard matrix of order 60. In this section, our computer search found one more Hadamard matrix of order 60 in \( NV_{60} \). All computer calculations in this section were done by programs in the language C and programs in Magma [2].

Any ternary extremal self-dual code of length 60 contains 41184 codewords of weight 60 (see Table 1). Let \( W_{60} \) be the set of 41184 codewords of weight 60 in \( NV_{60} \). It is trivial that there is a set \( W^+_{60} \) consisting of 20592 codewords of weight 60 such that

\[
W_{60} = W^+_{60} \cup \{2x \mid x \in W^+_{60}\}.
\]

Let \( \rho \) be a map from \( \mathbb{F}_3 \) to \( \mathbb{Z} \) sending 0, 1, 2 to 0, 1, -1, respectively. Define the following set

\[
W^Z_{60} = \{(\rho(x_1), \rho(x_2), \ldots, \rho(x_{60})) \mid (x_1, x_2, \ldots, x_{60}) \in W^+_{60}\} \subset \mathbb{Z}^{60}.
\]

Then we define the simple undirected graph \( \Gamma \), whose set of vertices is the set \( W^Z_{60} \) and two vertices \( x \) and \( y \) are adjacent if \( x \) and \( y \) are orthogonal, noting that \( x, y \in \mathbb{Z}^{60} \). Clearly, a 60-clique in \( \Gamma \) gives a Hadamard matrix. In addition, in order to find a Hadamard matrix, it is sufficient to consider only \( W^Z_{60} \) as the set of vertices of \( \Gamma \). Due to the computational complexity,
by the above approach, our computer search was able to find two 60-cliques, which imply two inequivalent Hadamard matrices $H_{NV,1}$ and $H_{NV,2}$. The computation for finding cliques was performed using the clique finding algorithm CLIQUER [12]. The computation for verifying the inequivalence of $H_{NV,1}$ and $H_{NV,2}$ was done by the Magma function IsHadamardEquivalent. Therefore, we have the following proposition.

**Proposition 7.** The third ternary extremal self-dual code $NV_{60}$ of length 60 contains at least two inequivalent Hadamard matrices of order 60 having as rows codewords of weight 60.

We verified by MAGMA that $H_{NV,1}$ and $H_{NV(1)(29)}$ are equivalent, and $H_{NV,1}$ and $H_{NV,2}$ have automorphism groups of orders 24360 and 812, respectively. These were done by the MAGMA functions IsHadamardEquivalent and HadamardAutomorphismGroup, respectively.

Now we display the Hadamard matrix $H_{NV,2}$. Here, instead of this matrix, we display its binary Hadamard matrix $B_{NV,2} = (H_{NV,2} + J)/2$, where $J$ is the $60 \times 60$ all-one matrix. Let $r_i$ denote the $i$-th row of $B_{NV,2}$. To save space, the vectors $r_1, r_2, \ldots, r_{60}$ are written in octal using $0 = (0,0,0)$, $1 = (0,0,1), \ldots, 7 = (1,1,1)$ in Figure 1. For example, the first row of $H_{NV,2}$

$$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$$

$$(−1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$$

corresponds to 777777777377777777777777.

It is worthwhile to determine whether $C$ contains a Hadamard matrix which is not equivalent to a Paley-Hadamard matrix for $C = QR_{60}$ and $P_{60}$.

## 5 Four-negacirculant codes of length 60

In this section, we demonstrate that the currently known ternary extremal self-dual codes of length 60 are constructed as four-negacirculant codes. All computer calculations in this section were done by programs in MAGMA [2].

An $n \times n$ negacirculant matrix has the following form

$$
\begin{pmatrix}
    r_0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\
    2r_{n-1} & r_0 & r_1 & \cdots & r_{n-3} & r_{n-2} \\
    2r_{n-2} & 2r_{n-1} & r_0 & \cdots & r_{n-4} & r_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    2r_1 & 2r_2 & 2r_3 & \cdots & 2r_{n-1} & r_0
\end{pmatrix}
$$
Let $A$ and $B$ be $n \times n$ negacirculant matrices. A ternary $[4n, 2n]$ code having the following generator matrix

$$
\begin{pmatrix}
I_{2n} & A & B \\
2B^T & B^T & A^T
\end{pmatrix}
$$

is called a \textit{four-negacirculant} code. Many ternary extremal self-dual four-negacirculant codes are known (see e.g., [5]).

Let $C_1, C_2$ and $C_3$ be the ternary four-negacirculant codes of length 60, having generator matrices of form \((5)\), where the pairs $(r_A, r_B)$ of the first
rows $r_A$ and $r_B$ of the negacirculant matrices $A$ and $B$ are as follows

$\begin{align*}
((1, 1, 0, 2, 1, 1, 2, 2, 2, 0, 1, 0, 0, 2),
(2, 0, 0, 2, 1, 0, 1, 2, 2, 0, 1, 0, 2, 2)), \\
((1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2),
(2, 2, 1, 2, 2, 0, 2, 1, 2, 2, 2, 1, 1)), \\
((1, 0, 0, 1, 2, 2, 0, 2, 1, 1, 0, 0, 0, 2),
(1, 2, 0, 0, 2, 1, 1, 0, 0, 0, 2, 2, 0))
\end{align*}$

respectively. We verified by MAGMA that $C_1 \cong QR_{60}$, $C_2 \cong P_{60}$ and $C_3 \cong NV_{60}$. This was done by the MAGMA function \texttt{IsIsomorphic}. Hence, we have the following proposition.

**Proposition 8.** For each $C$ of the codes $QR_{60}$, $P_{60}$ and $NV_{60}$, there is a four-negacirculant code $D$ such that $C \sim D$.

It is worthwhile to determine whether there is a new ternary extremal four-negacirculant self-dual code of length $60$.

**Remark 9.** Two ternary extremal self-dual codes $D_{60,1}$ and $D_{60,2}$ of length $60$ were constructed in [5]. We verified by MAGMA that $D_{60,1} \cong QR_{60}$ and $D_{60,2} \cong P_{60}$. This was also done by the MAGMA function \texttt{IsIsomorphic}.

**Remark 10.** Recently, it has been shown in [3] that there are exactly three inequivalent ternary extremal self-dual codes of length $60$ having an automorphism of order $29$. On the other hand, since each of $QR_{60}$, $P_{60}$ and $NV_{60}$ has an automorphism of order $29$, the three codes found in [3] are $QR_{60}$, $P_{60}$ and $NV_{60}$.

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