Quantum Mechanics as a Classical Theory  
XIV: Connection with Stochastic Processes

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september, 27, 1996

Abstract

In this paper we are interested in unraveling the mathematical connections between the stochastic derivation of Schrödinger equation and ours. It will be shown that these connections are given by means of the time-energy dispersion relation and will allow us to interpret this relation on more sounded grounds. We also discuss the underlying epistemology.

1 Introduction

The search for a stochastic support of quantum mechanics is already known since the 50s[1] and was a fertile research field in the decades of 50 and 60[2]-[10].

It is still a sounded field for the investigation of the mathematical and epistemological foundations of quantum mechanics.

This approach can be seen as the mathematical demonstration that one may derive the quantum mechanical formalism (Schrödinger equation) using only the formal apparatus of classical statistic mechanics, together with the Brownian movement theory[11].

In this case the cinematic description of the Brownian movement is the one related with a movement with no friction—the same used in the Einstein-Smoluchovski theory[12, 13].
The picture drawn from this approach is such that a particle, submitted to an external force, remains in dynamic equilibrium because of the balance of this force with a stochastic force responsible for the random movement.

The important point here is that in such a theory, where \( x(t) \) is considered a stochastic process, it is not possible to define a total time derivative \( d/dt \), since the movement is discontinuous, and we have to search for substitutes of this operator that might be used to formulate another new ‘Newtonian’ theory, formally equivalent to the mathematical structure of quantum mechanics, as given by the Schrödinger equation. This is amply known in the literature and will be also shown in the present paper.

We have already shown in some previous papers[14]–[26] that it is possible to derive the quantum formalism (Schrödinger equation) from three classical postulates: the first one is the general validity of Newton’s laws for the individual systems, the second is the validity of Liouville’s equation for the ensembles and the last one is the possibility of connecting the joint probability density function on phase-space \( F(x, p; t) \) to a characteristic function \( \rho(x, \delta x; t) \) by means of an infinitesimal transformation

\[
\rho(x, \delta x; t) = \int e^{i p \cdot \delta x / \hbar} F(x, p; t) dp,
\]

where \( \delta x \) is considered an infinitesimal displacement.

With these two derivation methods, it would be of some relevance to take a look at their epistemological and mathematical connection. Indeed, from the fact that it was already shown that quantum mechanics may be understood as a stochastic process, it would be rather interesting to know in what point of our approach this stochastic character was effectively introduced and what is its ontological status, according to our approach.

This is the objective of the present paper.

To achieve this goal we will develop briefly, in the second section, the derivation of the Schrödinger equation from the stochastic point of view. We will follow the pioneering papers of Nelson[11], Kershaw[6] and De La Peña[27], and the revision paper of De La Peña[28] as our guides in the development of the related formalism and interpretation.

In the third section, we will show again and very schematically our derivation of the Schrödinger equation using the infinitesimal Wigner-Moyal transformation given by (1).
In the fourth section we will show how the stochastic derivation is connected with the one made in the third section. The last section is left to our final considerations.

2 Stochastic Derivation

As we have seen in the previous section, \( x(t) \) is a stochastic process and we cannot define a time derivative \( d/dt \) for it. This means that the velocity related with this process cannot be obtained by its direct derivation, for \( x(t) \) is not, in general, differentiable.

In this case we have to introduce a finite time interval \( \Delta t \), small compared with the characteristic times of the systematic movement (the one related with Newton’s equation), but large enough compared with the correlation time of the fluctuating force\(^2\).

Using this finite time interval we may define the forward time derivative\(^1\) as

\[
D_x(x(t)) = \lim_{\Delta t \to 0^+} E_t \frac{x(t + \Delta t) - x(t)}{\Delta t} = \langle \frac{\delta x(\Delta t)}{\Delta t} \rangle_t, \tag{2}
\]

where the mean \( E_t \) or \( \langle \rangle_t \) is taken over the \( \Delta t \) distribution, which means that it is the conditional mean in the interval \( \Delta t \) and reflects a statistical distribution of the displacements \( \delta x \).\(^3\) We may also define the backward derivative as

\[
D_x^*(x(t)) = \lim_{\Delta t \to 0^+} E_t \frac{x(t) - x(t - \Delta t)}{\Delta t} = \langle \frac{\delta x(-\Delta t)}{-\Delta t} \rangle_t, \tag{3}
\]

where, in general, \( D_x(x(t)) \neq D_x^*(x(t)) \).

Since \( \Delta t \) is a very small time interval, we may write the following expansion

\[
\frac{1}{\Delta t} [f(x(t + \Delta t), t + \Delta t) - f(x(t), t)] \approx \left[ \frac{\partial}{\partial t} + \frac{1}{\Delta t} \sum_i [x_i(t + \Delta t) - x_i(t)] \frac{\partial}{\partial x_i} + \right. \\
+ \frac{1}{2\Delta t} \sum_{ij} [x_i(t + \Delta t) - x_i(t)][x_j(t + \Delta t) - x_j(t)] \frac{\partial^2}{\partial x_i x_j} \right]. \tag{4}
\]
Taking the mean \( \langle \rangle_t \) in the last expression and also taking the limit \( \Delta t \to 0 \) we have (up to second order)

\[
D f(x, t) = \left( \frac{\partial}{\partial t} + c \cdot \nabla + \nu \nabla^2 \right) f(x, t),
\]

where

\[
c = \lim_{\Delta t \to 0} E_t \left[ \frac{x(t + \Delta t) - x(t)}{\Delta t} \right] = D x(t),
\]

and

\[
\nu = \lim_{\Delta t \to 0} E_t \left[ \frac{(x(t + \Delta t) - x(t))^2}{2\Delta t} \right].
\]

We may now split the velocity \( c \) into two components: the systematic component \( \mathbf{v} \) and the stochastic one \( \mathbf{u} \)

\[
c = \mathbf{v} + \mathbf{u}.
\]

It is also interesting to define the time-inversion operator \( \hat{T} \), since it will be important to know the behavior of the processes \( \mathbf{v} \) and \( \mathbf{u} \) under its action when making the stochastic modification of Newton’s laws. Hence, we have

\[
c_s = \hat{T} c = \hat{T} D x(t) = D_s x(t),
\]

as in the definition \( \hat{T} \).

For a general function we have

\[
D_s f(x, t) = \left[ -\frac{\partial}{\partial t} + c_s \cdot \nabla + \nu_s \nabla^2 \right] f(x, t).
\]

The two derivative operators are given by

\[
\begin{cases} 
D &= \partial/\partial t + c \cdot \nabla + \nu \nabla^2, \\
D_s &= -\partial/\partial t + c_s \cdot \nabla + \nu_s \nabla^2.
\end{cases}
\]

We now define the two operators

\[
D_c = \frac{1}{2} (D - D_s), \quad D_s = \frac{1}{2} (D + D_s),
\]
named ‘systematic’ and ‘stochastic’ derivative, respectively. From relations (11) and (12) we have

\[
\begin{align*}
D_c &= \frac{\partial}{\partial t} + v \cdot \nabla - \nu_- \nabla^2, \\
D_s &= u \cdot \nabla + \nu_+ \nabla^2,
\end{align*}
\]  

(13)

where

\[
\nu_+ = \frac{1}{2}(\nu_s + \nu), \quad \nu_- = \frac{1}{2}(\nu_s - \nu),
\]  

(14)

and

\[
v = \frac{1}{2}(c + c_s), \quad u = \frac{1}{2}(c - c_s).
\]  

(15)

In the Newtonian limit \(\Delta t \to 0\) the operator \(D_s \equiv 0\) while the operator \(D_c\) becomes

\[
D_c \to \frac{\partial}{\partial t} + v \cdot \nabla = \frac{d}{dt},
\]  

(16)

and we may identify \(v\) with the systematic velocity. In this same limit \(u \to 0\).

To build a dynamic theory we have now to postulate the following relation between the stochastic acceleration and the stochastic derivative of \(c\)

\[
a = D_c v + D_s u + D_c u + D_s v.
\]  

(19)

If the forces involved do not depend upon the velocity, then these forces are \(\hat{T}\)-invariant

\[
\hat{T}f = f.
\]  

(20)

But the acceleration \(a\), as defined in (19), is not \(\hat{T}\)-invariant. Indeed

\[
D_c = -\hat{T}D_c, \quad D_s = +\hat{T}D_s;
\]  

(21)
Thus
\[ \mathbf{T} \mathbf{a} = D_c \mathbf{v} + D_s \mathbf{u} - D_c \mathbf{u} - D_s \mathbf{v}. \]  
(22)

To have \( \mathbf{a} = \mathbf{T} \mathbf{a} \) as required by (20) we must have
\[
\begin{cases}
D_c \mathbf{v} + D_s \mathbf{u} = \mathbf{a} \\
D_s \mathbf{v} + D_c \mathbf{u} = 0
\end{cases}
\]  
(23)

Writing
\[
\mathbf{a} = \mathbf{a}_c + \mathbf{a}_s = D_c \mathbf{v} + D_s \mathbf{u},
\]  
(24)

then
\[
\begin{cases}
\mathbf{a}_c = D_c \mathbf{v} = D_c^2 \mathbf{x}(t) = \mathbf{T} \mathbf{a}_c \\
\mathbf{a}_s = D_s \mathbf{u} = D_s^2 \mathbf{x}(t) = \mathbf{T} \mathbf{a}_s
\end{cases}
\]  
(25)
or else, by means of the operator \( D \) and \( D^* \),
\[
\begin{cases}
\mathbf{a}_c = \frac{1}{4}(D - D^*)^2 \mathbf{x}(t) \\
\mathbf{a}_s = \frac{1}{4}(D + D^*)^2 \mathbf{x}(t)
\end{cases}
\]  
(26)

Now let \( \mathbf{f}_0 \) be an external applied force. This force also has a stochastic component and may be equally written as a linear combination of systematic plus stochastic forces. In this case we may write
\[
\mathbf{f}_0 = m \lambda_1 \mathbf{a}_c + m \lambda_2 \mathbf{a}_s,
\]  
(27)

where \( \lambda_1 \) and \( \lambda_2 \) are constants.

In the newtonian limit we must have \( \mathbf{f}_0 = m \mathbf{a}_c \) which implies that \( \lambda = 1 \). Now putting \( \lambda_2 = -\lambda \) we get
\[
\mathbf{f}_0 = m(\mathbf{a}_c - \lambda \mathbf{a}_s).
\]  
(28)

Since the total force is
\[
\mathbf{f} = m(\mathbf{a}_c + \mathbf{a}_s),
\]  
(29)

we finally get
\[
\mathbf{f} = \mathbf{f}_0 + m(1 + \lambda) \mathbf{a}_s,
\]  
(30)
or
\[
\begin{cases}
\mathbf{f}_0 = D_c \mathbf{v} - \lambda D_s \mathbf{u} \\
D_s \mathbf{u} + D_c \mathbf{v} = 0
\end{cases}
\]  
(31)
which may be written explicitly as

\[
\begin{align*}
    \frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu_\nabla^2v - \lambda(u \cdot \nabla)u - \lambda\nu_+ \nabla^2u &= f_0, \\
    \frac{\partial u}{\partial t} + (v \cdot \nabla)v + u \cdot \nabla)v + \nu_\nabla^2v - \nu_- \nabla u &= 0
\end{align*}
\] (32)

In the newtonian limit[28] we have

\[
\nu_+ = \nu_- = 0 \rightarrow u = 0
\] (33)

thus giving

\[
m \frac{dv}{dt} = f_0,
\] (34)

as desired.

Assuming that \(\nu_+ \) and \(\nu_- \) will depend only upon the time, the velocities are rotational-free and that the external forces are derivable from a potential \(V\), we may rewrite the system of equations (32) as

\[
\begin{align*}
    \frac{\partial v}{\partial t} + \nabla[v^2/2 - \nu_\nabla \cdot v - \lambda u^2/2 - \lambda\nu_+ \nabla \cdot u] &= -\nabla V/m, \\
    \frac{\partial u}{\partial t} + \nabla[v \cdot u + \nu_\nabla \cdot v - \nu_- \nabla \cdot u] &= 0
\end{align*}
\] (35)

To obtain the Schrödinger equation from equation (35) we have only to make the ansatz

\[
v = 2D_0 \nabla S \quad \text{and} \quad u = 2D_0 \nabla R,
\] (36)

with \(\nu_+ = D_0\), \(\nu_- = 0\) and \(\psi_\pm = \exp(R \pm iS/\sqrt{-\lambda})\), where \(R\) and \(S\) are functions depending upon \(x(t)\) and \(t\). After some algebra we get

\[
\mp 2i mD_0 \sqrt{-\lambda} \frac{\partial \psi_\pm}{\partial t} = -2m\lambda D_0^2 \nabla^2 \psi_\pm + V \psi_\pm,
\] (37)

where \(V\), as was said above, is the potential function related with the external force \(f_0\).

Since the parameters \(\lambda\) and \(D_0\) appear in equation (37) only through the product \(D_0 \sqrt{-\lambda}\), it is clear that we may adjust the scale through \(D_0\) and take \(|\lambda| = 1\)[28].

If \(\lambda = -1\) we get the equation

\[
\mp 2mD_0 \partial \psi_\pm /\partial t = 2mD_0^2 \nabla^2 \psi_\pm + V \psi_\pm
\] (38)

having as its solution

\[
\psi_\pm = e^{R \pm S} = \rho^{1/2} e^{\pm S} \quad \text{onde} \quad \rho = \psi_+ \psi_-.
\] (39)
This equation is of the parabolic type \[28\] and describes the irreversible time evolution of the (real) amplitudes $\psi_-$ and $\psi_+$. If $\lambda = +1$, then equation (37) becomes

$$\mp 2imD_0 \partial \psi_+/\partial t = -2mD_0^2 \nabla^2 \psi_+ + V \psi_+$$  \hspace{1cm} (40)$$

having as its solution

$$\psi_\pm = e^{R \pm iS} = \rho^{1/2} e^{\mp iS} \quad \text{onde} \quad \rho = \psi_+ \psi_-.$$  \hspace{1cm} (41)$$

This is an hyperbolic type equation \[28\] and describes the reversible evolution of the (complex) amplitudes $\psi_- = \psi$ and $\psi_+ = \psi^\dagger$.

If we put into equation (40)

$$D_0 = \frac{\hbar}{2m},$$  \hspace{1cm} (42)$$

where $\hbar$ is Planck’s constant, we finally get the Schrödinger equation.

With the definition

$$\rho = \psi^\dagger \psi = e^{2R},$$  \hspace{1cm} (43)$$

we have, because of the second relation in (10),

$$u = D_0 \nabla \frac{\rho}{\rho}, \quad D_0 = E_t \left[\frac{(\delta x(\Delta t))^2}{2\Delta t}\right] = \frac{\hbar}{2m}. \hspace{1cm} (44)$$

In the same way

$$f = f_0 + 2ma_s.$$  \hspace{1cm} (45)$$

The results (44) and (45) will be very important for the comparison between this derivation and the one based upon the infinitesimal Wigner-Moyal transformation, to be presented in the next section.

The process of derivation of the equation related with Brownian movement and the one related with the quantum formalism doesn’t leave any doubt about the irreducibility of one type of phenomenon into the other. Indeed, since the very beginning, we have said that the quantum mechanical process has to be understood as one where there is no room for friction, which distinguishes it from the usual Brownian process.
3 Second Derivation

Now we will present again the mathematical demonstration of the Schrödinger equation from the Liouville equation and Newton’s laws using the infinitesimal Wigner-Moyal transformation, which has been our approach since the first paper of this series. This will be done for the commodity of references to be made latter on.

We begin with the three basic axioms

• Newtonian mechanics is valid for all the particles composing the systems of a given ensemble;

• For an ensemble of isolated systems, the joint probability density function is a conserved quantity

\[
\frac{dF(x, p; t)}{dt} = 0; \quad (46)
\]

• The infinitesimal Wigner-Moyal transformation, defined as

\[
\rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) = \int F(x, p; t) \exp \left( \frac{i p \cdot \delta x}{\hbar} \right) dp, \quad (47)
\]

may be applied to the problem.

We may write the Liouville equation as

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{p}{m} \cdot \frac{\partial F}{\partial x} - \frac{\partial V}{\partial x} \cdot \frac{\partial F}{\partial p} = 0,
\]

where we have already used Newton’s equation, valid according to the first postulate.

With the characteristic function given by (47) we may multiply the Liouville equation by the exponential factor there defined and integrate term by term in the momenta to get the equation

\[
-i \hbar \frac{\partial \rho}{\partial t} - \frac{\hbar^2}{m} \frac{\partial^2 \rho}{\partial x \delta x} + \delta x \cdot \frac{\partial V}{\partial x} \rho = 0. \quad (49)
\]

Now, making the ansatz

\[
\rho(x, \delta x; t) = \psi^\dagger(x - \delta x/2; t)\psi(x + \delta x/2; t), \quad (50)
\]
using the infinitesimal character of the displacement $\delta x$ and writing the amplitudes as $\psi = \exp(R + iS)$, with $R$ and $S$ as in the previous section, we finally get the amplitude equation given by
\[
-i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = 0, \tag{51}
\]
which is nothing but Schrödinger equation for the amplitude.

It is interesting to stress again that, as was shown in paper 1 of this series [14], the equation (49) has no dispersion associated to it in a necessary manner, whereas for equation (51) the Heisenberg relations are necessarily valid. This result will be of some relevance in what follows.

4 Comparison Between the Derivations

Now that both derivations were presented it remains the question about what connection they maintain with each other. The unraveling of such a connection may be of great importance to clarify some of the significances of the quantities involved in the formalism, and also their place in the underlying epistemology.

To find this connection, consider the transformation given in (1)
\[
\rho(x, \delta x; t) = \int_{-\infty}^{+\infty} e^{i \mathbf{p} \cdot \delta \mathbf{x}/\hbar} F(x, \mathbf{p}; t) d\mathbf{p}, \tag{52}
\]
where $F(x, \mathbf{p}; t)$ is the joint probability density defined over phase-space and $\rho(x, \delta x; t)$ is the corresponding characteristic function.

We may expand the exponential with respect to the infinitesimal displacement $\delta x$ and integrate in the momentum to get
\[
\rho(x, \delta x; t) = \rho(x) \left[ 1 + \frac{i < \mathbf{p} >_F \cdot \delta \mathbf{x}}{\hbar} - \frac{< p^2 >_F (\delta x)^2}{2\hbar^2} + O[(\delta x)^3] \right], \tag{53}
\]
where $\langle \rangle_F$ represents the mean taken using the function $F(x, \mathbf{p}; t)$.

Inserting this result into the equation for the characteristic function (49) and equating to zero the real and imaginary parts (or the terms in zeroth and first order in the infinitesimal displacement) we get the set of equations
\[
\begin{cases}
\partial \rho_0/\partial t + \frac{< \mathbf{p} >_F}{m} \cdot \nabla \rho_0 = 0 \\
< \mathbf{p} >_F \partial \rho_0/\partial t + \frac{< p^2 >_F}{m} \nabla \rho_0 + \rho_0 \nabla V = 0
\end{cases}, \tag{54}
\]
with \( \rho_0 = \rho(x, t) \) as in the second section, equation (43).

Now using the first of these equations into the second, we find

\[
\frac{(\Delta_F p)^2}{m} \nabla \rho_0 + \rho_0 \nabla V = 0,
\]

where

\[
(\Delta_F p)^2 = \langle p^2 \rangle_F - \langle p \rangle_F^2.
\]

If we make the identification

\[
p = m \frac{\delta x}{\Delta t},
\]

then equation (55) may be rewritten as

\[
m \frac{(\Delta_t (\delta x))^2}{\Delta t} \nabla \rho_0 + \rho_0 \nabla V \Delta t = 0,
\]

which may be understood, when we compare it with the results (44) and (45), as representing the equilibrium between the external force and the stochastic component given by

\[
f_s = 2ma_s = 2m \frac{u}{\Delta t} = 2m \frac{(\Delta_t (\delta x))^2}{2(\Delta t)^2} \nabla \rho_0,
\]

and coming from the fluctuations, where

\[
(\Delta_t (\delta x))^2 = \langle (\delta x)^2 \rangle_t - \langle (\delta x) \rangle_t^2,
\]

with the mean \( <\cdot> \), defined as in (2).

This is precisely the result one would expect from a stochastic approach to the problem, as we have seen in the second section, and allows us to make the connection between the two apparently distinct manners of deriving quantum mechanics. This, in turn, gives us the possibility of studying their respective ontological values.

The key to such a connection are the equations (58) and (57), where the symbols of one approach (\( p \) and \( \delta x \)) are written in terms of the symbols of the other (\( \delta x \) and \( \Delta t \)).

In the following subsection the relations between these symbols will be finally made clear.
4.1 Theoretical Connections

We begin this section by first noting that the connection between the momenta dispersion and the one related with the infinitesimal displacements implies in fixing a minimum value for the time variation $\Delta t$. Indeed,

$$\left(\Delta_F p\right)^2 = m^2 \frac{\left(\Delta_t (\delta x)\right)^2}{\left(\Delta t\right)^2},$$  \hspace{1cm} (61)

where, using the expression

$$\nu = D_0 = \frac{\left(\Delta_t (\delta x)\right)^2}{2(\Delta t)} = \frac{\hbar}{2m},$$  \hspace{1cm} (62)

we get

$$\Delta t = \frac{m\hbar}{\left(\Delta_F p\right)^2}.$$  \hspace{1cm} (63)

This last relation may be cast into a more familiar format. To show this we write (63) as

$$\Delta t = \frac{\hbar}{2} \frac{2m}{\left(\Delta_F p\right)^2},$$  \hspace{1cm} (64)

and note that the last term in the right-hand side may be identified with a dispersion in the kinetic energy ($\Delta_F E_k$), then we get

$$\left(\Delta t\right)\left(\Delta F E_k\right) = \frac{\hbar}{2},$$  \hspace{1cm} (65)

showing that the limitation in the time variation obeys a dispersion relation (by its minimum value) with the kinetic energy. Since we must have for the dispersion in the potential energy ($\Delta_F V \geq 0$), the dispersion in the total energy may be written as

$$\left(\Delta_F E\right) = \left(\Delta_F E_k\right) + \left(\Delta_F V\right),$$  \hspace{1cm} (66)

and we find, immediately

$$\left(\Delta t\right)\left(\Delta F E\right) \geq \frac{\hbar}{2},$$  \hspace{1cm} (67)

as expected.
This result may, in fact, be considered as a demonstration of the validity of the time-energy dispersion relation, by means of the concepts apprehended from the stochastic approach. Note, however, that, while \((\Delta F/E)\) is a dispersion, \(\Delta t\) is not, being simply a variation. This was expected since, as we know, the time \(t\) is only a parameter in the quantum formalism (has no operator associated to it) and is not subjected to dispersions.

What the stochastic approach shows is that, even not being related to dispersions, its variation is fixed by a minimum value.

Since the time variation \(\Delta t\) has a minimum value distinct from zero the evolution of the ensemble in the interval \([t-\Delta t, t]\) may be distinct from its evolution in the next time interval \([t, t+\Delta t]\) [28]. This fact is the very reason for the stochastic behavior to appear in the process \(\delta x\).

This result establishes the relation between the two formal derivations of the Schrödinger equation. We may now go to the conclusions.

5 Conclusions

Returning to equation (55), we may argue that, while using the characteristic function \(\rho(x, \delta x; t)\) we had not imposed upon the dispersions a value necessarily distinct from zero [14]. In this case, we still have

\[(\Delta_F p) = 0,\]

and the minimum value for the dispersion is introduced into the theory when we write the characteristic function as the product

\[\rho(x, \delta x; t) = \psi^\dagger(x - \delta x/2; t)\psi(x + \delta x/2; t),\]

and find Schrödinger equation for the amplitudes, as was already demonstrated.

It becomes clear thus that the stochastic behavior of the theory is fixed only at this latter stage, for if we have \((\Delta_F p)\), we still keep ourselves within a purely newtonian-liouvillian description.

Therefore, the stochastic behavior is introduced by the characteristic function decomposition as the amplitude product described above. We have to remember that, according to paper 11 of this series [24], this decomposition of the characteristic function is possible only if we may decompose the
probability density $F$ as

$$F(x, p; t) = \int \phi^\dagger(x, 2p - p'; t)\phi(x, p'; t)dp'$$

(70)

when the amplitudes $\phi$, defined upon phase-space, are related with the $\psi$ amplitudes by

$$\psi(x + \delta x/2; t) = \int e^{i p \cdot \delta x/\hbar} \phi(x, p; t)dp,$$

(71)

as was already shown[24].

Such a decomposition in the probability density $F(x, p; t)$ necessarily implies its non positive-definite character for the excited states. Indeed, for these states, the $\psi$ amplitudes, and in general the $\phi$ amplitudes, have nodes (are ‘wave functions’) and so, assume negative values over some regions; being $F$ an autocorrelation function of these amplitudes (and $\rho$ its convolution) it becomes clear that $F$ will assume, in general, negative values for these states. Note, however, that for the ground state there will be no nodes in the amplitudes, and these amplitudes will be positive-definite. In this cases the function $F$ will also be positive-definite.

This last observation may be a clue to understand the impossibility of having the joint probability density function $F$ positive-definite for the excited states. Indeed, these states are not, strictly speaking, stationary, since they have a finite mean life-time associated to them. Only the ground state is to be considered as really stationary. In this case, since the stationary property of the states is related with the possibility of having an equilibrium situation between the external force and the stochastic one—equation (55)—we may expect that, for these states, this force balance will be no longer valid, and equation (55) will have only an approximate validity.

In this sense, we may make the conjecture that the negative behavior of the probability density for the excited states is reflecting exactly this fact. The mathematical demonstration of this property, however, will be left to another paper.

Now it remains to make some comments about the ontological aspects revealed by the formal developments we have dealt with. To begin with, it is important to stress again that the stochastic character of the theory appears as a by-product of the momentum dispersion.
In a general problem we have the Heisenberg relation given by expression (67). In the specific harmonic oscillator problem, for example, we have

\[ (\Delta_F E) = \frac{\hbar \omega}{2}, \]

where \( \omega \) is the frequency of oscillation characteristic of the problem. For this problem, relation (67) gives

\[ \Delta t \geq \frac{1}{\omega} = \frac{T}{2\pi}, \]

where \( T \) is the characteristic period of the problem. The minimum time interval to be considered for the stochastic formalism to be still applicable is of the order of the movement period. Hence, it is not unexpected that the energy dispersion makes it impossible to take for granted in which specific state the oscillator is\[22\], for

\[ E = (n + 1/2)\hbar \omega \pm (\Delta_F E) = (n + 1/2)\hbar \omega \pm \hbar \omega/2. \] (74)

It is also interesting that, for this specific example, the minimum time interval \( \Delta t \) is not really small compared with the characteristic time of the system, as required in the development (2). In fact, this interval is of the same order or greater to this characteristic time. This can be considered another argument to explain the result (74).

It is important to stress that, being \( t \) a parameter, we may, from the experimental point of view, work with times much smaller than those allowed by the equation (73). In this case, we expect that the quantum formalism, now seen as a stochastic theory, will be no longer adequate to describe the problem.

Finally, we would like to stress that, as with the Brownian movement, the fluctuations are a mathematical artifice introduced to simulate the particle interactions with the environment, since the exact description of the interaction may not be possible. Hence, in the Brownian movement, for example, the fluctuations are caused by the collisions that, rigorously speaking, obey Newton’s equation, but whose exact consideration is despairingly prohibited.

This is exactly our point of view.
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