Distributed Linear Equations over Random Networks *

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Abstract

Distributed linear algebraic equation over networks, where nodes hold a part of problem data and co-operatively solve the equation via node-to-node communications, is a basic distributed computation task receiving an increasing research attention. Communications over a network have a stochastic nature, with both temporal and spatial dependence due to link failures, packet dropouts or node recreation, etc. In this paper, we study the convergence and convergence rate of distributed linear equation protocols over a $\ast$-mixing random network, where the temporal and spatial dependencies between the node-to-node communications are allowed. When the network linear equation admits exact solutions, we prove the mean-squared exponential convergence rate of the distributed projection consensus algorithm, while the lower and upper bound estimations of the convergence rate are also given for independent and identically distributed (i.i.d.) random graphs. Motivated by the randomized Kaczmarz algorithm, we also propose a distributed randomized projection consensus algorithm, where each node randomly selects one row of local linear equations for projection per iteration, and establish an exponential convergence rate for this algorithm. When the network linear equation admits no exact solution, we prove that a distributed gradient-descent-like algorithm with diminishing step-sizes can drive all nodes’ states to a least-squares solution at a sublinear rate. These results collectively illustrate that distributed computations may overcome communication correlations if the prototype algorithms enjoy certain contractive properties or are designed with suitable parameters.

Keywords: distributed computation, multiagent systems, network linear equations, communication uncertainty, random graphs

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1 Introduction

1.1 Motivation

Distributed computation over networks emerges as an important and appealing research topic in engineering and machine learning, including average consensus [2–4], distributed optimization [5–7], distributed learning [8, 9], distributed estimation [10, 11], and distributed filtering [12]. The basic framework in distributed computation is that each node only has a part of overall problem data, while the nodes need to cooperatively accomplish a global computation task by manipulating local data and sharing information with neighbors over a network without relying on a center. In many cases, each node holds a dynamical state with locally preserving data, and can share its local dynamical state through node-to-node communications to facilitate all local dynamical states to converge to a consensual network level solution. Hence, distributed computation is attractive in large-scale networks due to its resilience, robustness and adaptivity. Moreover, distributed methods can keep the agent’s privacy, and remove the communication burden of data centralization.

In distributed computation, node-to-node communication over a network is essential for the nodes to cooperatively find the network level solution without accessing to the whole data, hence, how network topology and connectivity affects the convergence and convergence rate of distributed computation has been an important research topic, [29–32], and specifically, in distributed linear equations, [17, 33, 35]. Random graph models are extensively studied in distributed computation, since the practical communication networks are essentially uncertain and stochastic, due to the link failures, packet dropout and node sleeping, etc. For i.i.d. random graphs, distributed averaging consensus algorithms are analyzed with mean-square convergence rates in [2, 36, 37], and various distributed optimization algorithms are also analyzed with the almost sure convergence in [29], a convergence rate analysis in probability in [30] and a mean-squared convergence rate analysis in [31]. For distributed averaging consensus, the restrictive i.i.d. random graphs can be relaxed to Markovian switching graphs [38] and $*$-mixing graphs [39], partially due to the strict contractive property in consensus dynamics. To the best of our knowledge, the literature still lacks a study on the possibility and performance of more complex distributed computation schemes over random networks with node-to-node communication channel correlations. We aim to fill this gap in this paper.

1.2 Problem Definition

We consider the following system of linear equations

$$z = Hy$$

with respect to an unknown variable $y \in \mathbb{R}^m$, where $H \in \mathbb{R}^{l \times m}$ and $z \in \mathbb{R}^l$. Component-wise the equation (1) is a collection of $l$ linear equations. Let $l_1, \ldots, l_N$ be $N$ integers satisfying $\sum_{i=1}^N l_i = l$. The linear
equation (1) is distributed over a network with \( N \) nodes indexed in the set \( V \triangleq \{1, \ldots, N\} \) in the following way: Each node \( i \in V \) possesses the \( (\sum_{j=1}^{i-1} l_j + 1) \)th to \( (\sum_{j=1}^{i} l_j + 1) \)th row of \( H \), and \( (\sum_{j=1}^{i} l_j + 1) \)th to \( (\sum_{j=1}^{i} l_j) \)th elements of \( z \), i.e., node \( i \) holds a block of component-wise equations in (1). We have obtained a standard distributed decomposition of the linear equation, e.g., \( [23] \).

The discrete time is slotted at \( t = 0, 1, 2, \ldots \). The communication network among the nodes at the slotted time sequence is described by a random graph process \( G(t) = \{V, E(t)\} \), \( t = 0, 1, 2, \ldots \), where \( E(t) \) is a random variable taking its value from the set of all possible undirected edge sets among the nodes. Associated with the random graph process \( \langle G(t) \rangle_{t=0}^{\infty} \), we define a sequence of random vector \( I(t) = \text{col}(I_{ij}(t), i, j \in V), t = 0, 1, \ldots \), by \( I_{ij}(t) = 1 \) if \( \{i, j\} \in E(t) \) and \( I_{ij}(t) = 0 \) otherwise. We impose the following standing assumption of the paper.

**Standing Assumption.** The random process \( \langle I(t) \rangle_{t=0}^{\infty} \) is \( \ast \)-mixing \( [44] \), i.e., there exists a non-increasing sequence of real numbers \( \lambda_t, t = 0, 1, \ldots \) with \( \lim_{t \to \infty} \lambda_t = 0 \), such that for all \( A \in \mathcal{F}_0^m(I(t)) \) and \( B \in \mathcal{F}_m^s(I(t)) \) and for all \( m = 0, 1, 2, \ldots \), there holds \( |P(A \cap B) - P(A)P(B)| \leq \lambda_sP(A)P(B) \).

Solving linear equations is a fundamental and generic computation problem \( [13] \), and efficient numerical algorithms for linear equations have been a long standing research topic, such as the Kaczmarz algorithm and its randomized version \( [14, 15] \). Distributed methods for linear equations over networks receive an increasing research attention in recent years \( [16–21, 33] \), due to its applications in parameter estimation \( [11, 17] \), environmental monitoring \( [24] \), computerized tomography and image reconstruction \( [8, 25] \), etc. The \( \ast \)-mixing property describes the temporal dependence of the random process with the non-increasing correlation parameter sequence \( \lambda_t \). A strictly stationary Markovian random graph process is \( \ast \)-mixing if it is irreducible and aperiodic \( [44] \). In this paper, we are interested in the convergence and performance of the state-of-the-art distributed linear equation solvers over such a \( \ast \)-mixing random network.

### 1.3 Main Results

When the network linear equation admits exact solutions, we study the projection consensus algorithm motivated from \( [19, 35] \). Each node updates its state by projecting the weighted averaging with its neighbors’ states onto a local solution set specified by local data. We obtained the following results:

- We prove the exponential convergence rate of the mean-squared error even with \( \ast \)-mixing random graphs, ensuring the solvability of distributed linear equations under generic random networks. Specifically, when the random graph process is independent and identically distributed (i.i.d.), we give the lower and upper bound estimations of the mean-squared convergence rate with the spectrum theory of linear operators.
- Motivated by randomized Kaczmarz algorithms \( [14, 15] \), we further propose a distributed randomized projection consensus algorithm to solve the linear equation with exact solutions, where each node
only randomly selects one row of its local linear equation per iteration. We also prove its exponential convergence rate with \( *\)-mixing random graphs in a mean-squared sense when the linear equation admits a unique solution.

When the linear equation does not have exact solutions, we study a distributed algorithm to find a least-square solution over \( *\)-mixing random graphs, motivated by the distributed subgradient algorithms in distributed optimization [5] [39] [28]. We prove that with diminishing step-sizes, all nodes’ states converge to the unique least-squares solution at a sublinear rate.

1.4 Related Work

Existing works on distributed computation over random networks mostly assumed independence or Markovian property of the random graph process [3, 20, 29, 31, 38]. For averaging consensus over gossip communications, [2] showed that the second largest eigenvalue of the expected communication graph influences the convergence speed. Further, [39] proved almost sure convergence of averaging consensus algorithm for \( *\)-mixing random graphs. Distributed optimization with independent random graphs are studied in [29, 31], where an almost sure convergence analysis was provided in [29], and how the spectral gap of the expected graph influences the convergence rate was explicitly characterized in [30]. The role of communication networks has been investigated in distributed linear equation solvers with deterministically varying graphs. It was shown that convergence of distributed linear equation solvers over a time-varying communication structure, essentially depends on the ability for the union graph over a sequence of time intervals to maintain connectivity, e.g., [17, 19]. For network linear equations with randomly varying communication graphs, [20] considered the unreliable communication links modeled by independent Bernoulli processes. Recently, a distributed computing scheme for network linear equations was considered in [27] by a fixed-point iteration of random operators, which allowed temporal and spatial dependence. Additionally, the distributed randomized projection consensus algorithm of this work provides an extension of well-known randomized Kaczmarz algorithm [14] [15]. and a key technical obstacle in the convergence analysis lies in generalizing a key result [17] to a stochastic setting.

1.5 Paper Organization

Some preliminary results of the paper are scheduled to be presented at the IEEE Conference on Decision and Control in Dec. 2020 [1]. The remainder of the paper is organized as follows. Section 2 first investigates a distributed projection consensus algorithm along with the convergence rate analysis over \( *\)-mixing random graphs when the linear equation has exact solutions, and provides the convergence rate bound estimation over the i.i.d. random graphs as well. And then Section 2 designs a distributed randomized projection consensus algorithm and establishes the convergence results over \( *\)-mixing random graphs when the linear equation admits a unique exact solution. Section 3 studies a distributed algorithm to find
the least-squares solution to the linear equation, proves the almost sure convergence and the convergence rate over $*$-mixing random networks when the linear equation has a unique least-squares solution. Section 4 concludes the paper. All proofs are provided in the Appendix.

**Notation and Terminology.** All vectors are column vectors and are denoted by bold, lower case letters, i.e., $\mathbf{a}, \mathbf{b}, \mathbf{c}$, etc.; matrices are denoted with bold, upper case letters, i.e., $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc. The inner product between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^m$ is denoted as $\langle \mathbf{a}, \mathbf{b} \rangle$, and sometimes simply as $\mathbf{a}^T \mathbf{b}$. The Euclidean norm of a vector is denoted as $\| \cdot \|$. $\otimes$ denotes the Kronecker product. Denote the projection of $\mathbf{x}$ onto a closed convex set $\Omega$ as $\Pi_{\Omega}(\mathbf{x}) = \arg\min_{\mathbf{y} \in \Omega} \|\mathbf{x} - \mathbf{y}\|_2$. Denote by $\mathbf{I}_m$ the $m \times m$-dimensional identity matrix. Denote $\mathbf{1}_n$ ($\mathbf{0}_n$) as a vector of all ones (zeros) in $\mathbb{R}^n$. A nonnegative matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called row stochastic if $\mathbf{A}\mathbf{1}_n = \mathbf{1}_n$, and is called column stochastic if $\mathbf{1}_n^T \mathbf{A} = \mathbf{1}_n^T$. We denote range($\mathbf{A}$), kernel($\mathbf{A}$) and rank($\mathbf{A}$) as the range space, null space and rank of matrix $\mathbf{A}$. Denote $\text{sr}(\mathbf{A})$ as the spectral radius of a matrix $\mathbf{A}$ (linear operator): $\text{sr}(\mathbf{A}) = \max\{||\lambda||, \lambda$ is the eigenvalue of $\mathbf{A}\}$.

For a probability space $(\Xi, \mathcal{F}, \mathbb{P})$, $\Xi$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra and $\mathbb{P}$ is the probability measure. Let $(X)$ denote a random process with a family of random variables $X(0), X(1), X(2), \cdots$. Denote by $\mathcal{F}_t((X))$ the $\sigma$-algebra generated from the random variables $X(l), X(l + 1), \cdots, X(k)$ for any $k \geq l$. The expectation and variance of a random variable are denoted as $\mathbb{E}[\cdot]$ and $\forall a \mathbb{R}(\cdot)$, respectively.

An undirected graph, denoted by $\mathcal{G} = \{V, E\}$, is an ordered pair of two sets, where $V = \{1, \ldots, N\}$ is a finite set of vertices (nodes), and each element in $E$ is an unordered pair of two distinct nodes in $V$, called an edge. A path in $\mathcal{G}$ with length $p$ from $v_1$ to $v_{p+1}$ is a sequence of distinct nodes, $v_1 v_2 \ldots v_{p+1}$, such that $\{v_m, v_{m+1}\} \in E$, for all $m = 1, \ldots, p$. The graph $\mathcal{G}$ is termed connected if for any two distinct nodes $i, j \in V$, there is a path between them. The neighbor set of node $i$, denoted by $N_i$, is $N_i = \{j \in V : \{i, j\} \in \mathcal{E}\}$.

## 2 Projection Consensus Algorithm for Exact Solutions

In this section, we consider the case where the network linear equation \[1\] has exact solutions. We study the distributed projection consensus algorithm under $*$-mixing graphs.

### 2.1 Projection Consensus Algorithm

We define a mixing weight process $(\mathbf{W})$ according to $(\mathcal{G})$ such that for all $t$,

- $\mathbf{W}(t) \in \mathbb{R}^{N \times N}$ is $\mathcal{G}(t)$-measurable.
- There exists an $0 < \eta < 1$, such that $\mathbf{W}_{ii}(t) \geq \eta$ for all $i \in V$, $\mathbf{W}_{ij}(t) = \mathbf{W}_{ji}(t) \geq \eta$ if $\{i, j\} \in \mathcal{E}(t)$, and $\mathbf{W}_{ij}(t) = \mathbf{W}_{ji}(t) = 0$, otherwise.
- $\mathbf{W}(t)$ is row and column stochastic satisfying $\mathbf{W}(t)\mathbf{1}_N = \mathbf{1}_N$ and $\mathbf{1}_N^T \mathbf{W}(t) = \mathbf{1}_N^T$. 

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Let the rows of (1) held at node \(i\) form the local equation \(z_i = H_i y\). Let \(A_i = \{y \in \mathbb{R}^m : z_i = H_i y\}\) be a local solution space, and \(A^* = \{y \in \mathbb{R}^m : z = Hy\}\) be the solution space for linear equation (1). Obviously, both \(A_i\) and \(A^*\) are affine spaces, and \(A^* = \bigcap_{i \in V} A_i\). Denote \(\Pi_{A_i}\) as the projection operator over \(A_i\). Each node \(i\) at time \(t\) holds an estimate \(x_i(t) \in \mathbb{R}^m\) for the solution to equation (1). The projection consensus algorithm \([5, 19, 35]\) is defined by

\[
x_i(t + 1) = \Pi_{A_i} \left( \sum_{j=1}^{N} W_{ij}(t) x_j(t) \right), \quad i = 1, \ldots, N.
\]  

(2)

The iteration (2) takes the same form as the projection consensus algorithm in \([35]\) for distributedly finding a point in common at the intersection of convex sets that are held by each node over a network, while the projection in (2) is specified onto an affine set.

2.2 Main Convergence Result

We introduce the following definition.

**Definition 1** Given a random graph process \(G\) and a real number \(p \in (0, 1)\), we define its \(p\)-persistent graph as \(G_p(p) = (V, E_p(p))\) with \(E_p(p) = \{\{i, j\} : P(\{i, j\} \in E(t)) \geq p, \forall t \geq 0\}\).

We are now ready to state the almost sure convergence result as well as the convergence rate of mean-squared error for the projection consensus algorithm (2).

**Theorem 1** Assume the linear equation (1) admits at least one exact solutions. Suppose the considered random graph process \(G\) induces a connected \(p\)-persistent graph \(G_p(p)\). Then the following statements hold.

(i) For any fixed initial states \(x(0) = (x_1^T(0), \cdots, x_N^T(0))^T\), the projection consensus algorithm (2) has all local estimates converge almost surely to a consensual solution of the linear equation (1), i.e.,

\[
P\left( \lim_{t \to \infty} x_i(t) = \frac{1}{N} \sum_{i=1}^{N} \Pi_{A^*}(x_i(0)) \right) = 1, \quad \forall i \in V.
\]  

(3)

(ii) The algorithm (2) has the mean-squared error converge to zero at an exponential rate, i.e., there exists a \(0 < \mu < 1\) and a constant \(c > 0\) such that for any \(t \geq 0\),

\[
E\left[ \sum_{i \in V} \left\| x_i(t) - \frac{1}{N} \sum_{i=1}^{N} \Pi_{A^*}(x_i(0)) \right\|^2 \right] \leq c\mu^t.
\]  

(4)

The proof of Theorem 1 can be found in Appendix A while we give the intuitions behind the proofs as follows. Firstly, a projection invariance of the estimates generated by iteration (2) is given via the double stochasticity of \(W(t)\), implying that the convergent solution should be

\[
y^*(x(0)) = \frac{\sum_{i=1}^{N} \Pi_{A^*}(x_i(0))}{N}.
\]
We then rewrite the iterate error as a stochastic linear recursion, and show the monotonicity of the squared error $f(t) \triangleq \sum_{i=1}^{N} \|x_i(t) - y^*(x(0))\|^2$ since two-norms of the weight matrix $W(t)$ and the projection matrix is less or equal to one. Next, we deliberately construct a $*$−mixing events over a finite time interval such that the graphs are jointly connected, show that the product of the stochastic linear maps is contractive and that $f(t)$ is contractive conditioned on the $*$−mixing events. Finally, we apply the Borel-Cantelli lemma to show that the event happens infinitely times, which together with the monotonicity of $f(t)$ implies that the squared error converges almost surely to zero, and hence (3) follows by. Meanwhile, the exponential convergence of the mean-squared error (4) is obtained by the monotonicity of $f(t)$, the contraction property of $f(t)$ conditioned on the $*$−mixing events, and the fact that the event happens with a positive probability uniformly greater than zero.

From the proofs, we see that the exponential rate constant $\mu$ in Theorem 1 is influenced by the connectivity of the $p$−persistent graph, the mixing parameter, as well as the projection matrix of the linear equation. The challenge to establish Theorem 1 lies in the fact that the graphs can switch at an arbitrary order with both temporal and spatial dependence such that there does not exist a uniform time interval bound to ensure a jointly graph connectivity, which is necessary in the analysis of deterministically switching graphs [17,20,26]. The novel technical contribution is to provide a lower bound estimation of the probability for jointly graph connectivity by fully exploiting the $*$−mixing properties of the random graph process. The established probability estimation also ensures the exponential convergence in a mean-squared sense, hence, guarantees the fast convergence rate that has been provided in literature for distributed linear equations with fixed or uniformly jointly connected graphs [17,20,26]. The results demonstrate that distributed computation is still achievable with a similar performance even under $*$−mixing random graphs, if the prototype distributed algorithm fits the computation task with proper contractive properties.

The assumption that $G_P(p)$ is connected can be easily satisfied by the Erdős-Rényi random graph process and the Markovian graph process. For example, the $p$−persistent graph of an Erdős-Rényi random graph process is just its base graph if each edge is independently connected at a probability $p$, and the $p$−persistent connectivity is satisfied when the base graph is connected. Hence, the random graphs that have been used in average consensus [37,38] and distributed optimization [29,31] are all special cases of the random graph process with a connected $G_P(p)$. However, with $G_P(p)$ we only require the edges with a positive probability to constitute a connected graph, while neither spatial independence nor temporal independence is required. The recent work [27] also studied linear equations over random graphs that are even more general than the $*$−mixing random graph adopted in this paper. With the help of the $*$−mixing condition, we manage to bound the decaying of random events' dependence with the increasing of intervals separating the events for the convergence analysis. As a result, Theorem 1 further establishes an exponential rate of convergence in the mean-squared error, which is an improvement to the result in [27].
2.3 Independent Random Networks: Explicit Convergence Rate

Next, we give the upper and lower bound estimation for the convergence rate of the iteration (2) when (5) is an independent and identically distributed random graph sequence, e.g., \[2, 36\]. Denote

\[
\sup_{x(0)} \lim_{t \to \infty} \mathbb{E} \left[ \sum_{i=1}^{N} \left\| x_i(t) - y^*(x(0)) \right\|^2 \right]^{1/t},
\]

where \( y^*(x(0)) = \frac{\sum_{i=1}^{N} \Pi A^r(x_i(0))}{N} \). The following result characterizes the lower and upper bounds on the exponential rate \( r \). The analysis is motivated by \[37\] and can be found in Appendix B.

**Proposition 1** Assume the linear equation (11) admits at least one exact solutions. Suppose the random graph process (5) is an i.i.d. sequence with the corresponding mixing weight process (W) being an i.i.d. sequence of symmetric stochastic matrices. Define \( W \triangleq \mathbb{E}[W(t)] \). Then there holds for the exponential convergence rate \( r \) with \( P \triangleq \text{diag}\{P_1, \ldots, P_N\} \) that

\[
\text{sr}(PW \otimes I_m P)^2 \leq r \leq \text{sr}(PE[(W(0) \otimes I_m)P(W(0) \otimes I_m)]P).
\]

We have already shown in Theorem 1 that \( r \) must be bounded by some constant smaller than 1. For the unique solution case with i.i.d. random graphs, (3) provides a lower bound and upper bound estimate of the convergence rate, which explicitly shows its dependence on the graph properties and projection matrices. The bounds can be calculated numerically once the problem data is given. By \[26\], Proposition 1, the matrix \( PW \otimes I_m P \) is Schur stable, and hence \( \theta_1 \triangleq \text{sr}(PW \otimes I_m P)^2 < 1 \). It is easily seen from Theorem 1 that \( \theta_2 \triangleq \text{sr}(PE[(W(0) \otimes I_m)P(W(0) \otimes I_m)]P) < 1 \).

2.4 Exact Solutions with Randomized Projection

In practical problems, the local data \( H_i \in \mathbb{R}^{l_i \times m} \) can still have a large number of rows and a high dimension decision variable, that is, a large \( l_i \) and a large \( m \). Motivated by the randomized Kaczmarz algorithm, we propose a distributed iteration with a random sampling mechanism, where each node only selects one row of \( H_i \) at a certain positive probability at each iteration. For each \( i \in V \), we denote the rows of \( H_i \in \mathbb{R}^{l_i \times m} \) by \( H_i^{(1)}, \ldots, H_i^{(l_i)} \). Let \( H_i \) and \( z_i \) have atomic partitions as, respectively,

\[
H_i = \begin{pmatrix}
H_i^{(1)} \\
H_i^{(2)} \\
\vdots \\
H_i^{(l_i)}
\end{pmatrix}, \quad z_i = \begin{pmatrix}
z_i^{(1)} \\
z_i^{(2)} \\
\vdots \\
z_i^{(l_i)}
\end{pmatrix}.
\]

Independent from time, other nodes in \( V \), and the random graph process (5), at each time \( t \) each node \( i \) selects \( s_i(t) \) as an integer in \( \{1, 2, \ldots, l_i\} \) at random with probability \( \|H_i^{(s_i(t))}\|^2 / \|H_i\|^2 \). Let \( A_i^{s_i(t)} \) be the linear affine space

\[
A_i^{s_i(t)} = \{ y \in \mathbb{R}^m : z_i^{s_i(t)} = H_i^{s_i(t)} y \},
\]
where $z_i^{(s_i(t))}$ denotes the $s_i(t)$-th entry of $z_i$, and $H_i^{(s_i(t))}$ is the $s_i(t)$-th row of $H_i$. We present the following algorithm with a randomized projection as a generalization to the projection consensus algorithm (2):

$$x_i(t+1) = \Pi_{A_i^{s_i(t)}} \left( \sum_{j=1}^{N} W_{ij}(t)x_j(t) \right), \quad i = 1, \ldots, N. \quad (7)$$

In the algorithm (7), the cost for computing the local projections at each node is reduced compared to the algorithm (2) since $A_i^{s_i(t)}$ is much simplified than $A_i$. We present the following result.

**Theorem 2** Suppose the linear equation (1) has a unique solution $x^\ast$. Suppose the considered random graph process $\langle G \rangle$ induces a connected $p-$persistent graph $G^P(p)$. Then, the iteration (7) has all local estimates converge almost surely to the unique solution $x^\ast$. Moreover, the error with iteration (7) converges to zero at an exponential rate in the mean-squared sense, i.e., there exists a constant $0 < \mu < 1$ and a constant $c > 0$ such that for any $t \geq 0$,

$$E \left[ \sum_{i \in V} \| x_i(t) - x^\ast \|^2 \right] \leq c \mu^t.$$

The exponential rate constant $\mu$ is influenced by the connectivity of the $p-$persistent graph, the mixing parameter, the randomized projection selection rule $s_i(t), i \in V$, as well as the projection matrix of the linear equation. It might be of interests to explicitly characterize the exponential rate $\mu$ when $\langle G \rangle$ is an i.i.d. random graph sequence as a further work.

## 3 Distributed Gradient Descent for Least-square Solutions

In this section, we consider the case where the network linear equation (1) only has least-square solutions defined via the following optimization problem:

$$\min_{y \in \mathbb{R}^m} \| z - Hy \|^2. \quad (8)$$

### 3.1 The Algorithm

We study the following distributed algorithm where each node merely uses its local data $H_i, z_i$ and information from its neighboring agents $N_i(t) = \{ j : \{ i, j \} \in \mathcal{E}(t) \}$. The algorithm could be treated as an application of the distributed sub-gradient algorithm (Refer to [5, 28]) to linear equation over networks with $*$-mixing graphs.

Each node $i \in V$ at time $t + 1$ updates its estimate $x_i(k+1)$ as follows,

$$x_i(t+1) = x_i(t) - h \sum_{j \in N_i(t)} (x_i(t) - x_j(t)) - \alpha(t)H_i^T (H_i x_i(t) - z_i), \quad (9)$$

where $h > 0$ and $0 < \alpha(t) \leq h$ is the decreasing step-size. We impose the following condition on the step-size $\{\alpha(t)\}$. The iteration (9) can be treated as an application of the well-known distributed subgradient
algorithm in [5] to the least-squares optimization problem [5], where the consensus weight is constructed with the help of graph Laplacian matrix.

**Assumption 1** Let $\alpha(t) > 0$, $\alpha(t)$ be monotonically decreasing to 0, $\sum_{t=1}^{\infty} \alpha(t) = \infty$, and $\sum_{t=1}^{\infty} \alpha(t)^2 < \infty$. In addition, there exists a constant $\alpha \geq 0$ such that

$$\lim_{t \to \infty} \frac{1}{\alpha(t+1)} - \frac{1}{\alpha(t)} = \alpha.$$ 

We can take $\alpha(t) = \frac{1}{t+1}$ to satisfy Assumption [1] with $\alpha = 1$. We can also take $\alpha(t) = \frac{1}{(t+1)^{\delta_1}}$ with $\delta \in (\frac{1}{2}, 1)$ to satisfy Assumption [1] with $\alpha = 0$.

### 3.2 Main Result

In this part, we analyze the iteration (9) for the problem with a unique least-squares solution, denoted by $x^*_{LS} = (H^T H)^{-1} H^T z$. We present the convergence analysis for (9).

**Theorem 3** Suppose $\text{rank}(H) = m$, and Assumptions [1] holds. Suppose the considered random graph process $\{G_p\}$ induces a connected $p-$persistent graph $G_p(p)$. For any fixed initial state $x(0)$, the iteration (9) has all local estimates converge almost surely to the unique least-squares solution of (8), i.e.,

$$P \left( \lim_{t \to \infty} x_i(t) = x^*_{LS} \right) = 1, \forall i \in V. \tag{10}$$

Specially, when $\alpha(t) = \frac{1}{(t+1)^{\delta_1}}$ for some $\delta_1 \in (0, \frac{1}{2}]$, there exists some positive constant $\delta_2 \in (0, 2\delta_1)$ such that for each $i \in V$,

$$\|x_i(t) - x^*_{LS}\| = o((t+1)^{-\delta_2}), \text{ a.s.} \tag{11}$$

By setting $\alpha(t) = \frac{1}{t+1}$ in the distributed iteration (9), the result (11) implies that each $x_i(t)$ converges almost surely to the least-squares solution $x^*_{LS}$ with a sublinear rate $(t+1)^{-\delta_2}$ for some $\delta_2 \in (0, 1)$. The established rate is nearly tight for iteration (9) since the iterate $\{u(t)\}$ generated by the recursion $u_{t+1} = (1 - \alpha(t))u(t)$ with $u(0) > 0$ satisfies the following

$$u(t+1) \leq \exp(-\alpha(t))u(t) = \exp \left( -\sum_{p=0}^{t} \frac{1}{p+1} \right) u(0) \leq \exp(-\ln(t+1))u(0) = \frac{u(0)}{t+1}.$$ 

The proof of Theorem 3 is given in Appendix D. The proof applies stochastic approximation theories [47] to prove the almost sure convergence result and the resulting convergence rate. The analytical techniques in turn become quite different from Theorem 1:

(i) Theorem 1 applies the Borel-Cantelli lemma to the suitably defined $\ast-$mixing events to prove the almost sure convergence of (2), while Theorem 3 utilizes the convergence result of stochastic approximation to validate the almost sure convergence of (9) with decreasing step-sizes.
Theorem 1 shows the exponential convergence of the mean-squared error via the facts that the non-increasing squared error is contractive conditioned on the deliberately defined $*$-mixing events, and that each event happens with a positive probability.

While Theorem 3 establishes the sublinear rate in an almost sure sense according to the procedures that the average estimate can be rewritten as a stochastic linear recursion with decreasing step-sizes and the stochastic noise being a combination of consensus errors. Then the stochastic noise is shown to satisfy a specific summable condition utilizing properties of the $*$-mixing random graphs, and finally, the rate analysis of stochastic approximation is adapted to conclude the result.

4 Conclusions

Understanding how randomly switching communication topology with temporal correlations influences the performance of distributed computation can provide the theoretical guarantee for the applicability of various distributed algorithms in practical communication networks. This paper provided the analysis of distributed algorithms for solving linear equations over $*$-mixing random graphs, since linear equations is a basic problem in distributed computation and $*$-mixing random graphs cover a generic class of wired/wireless communication networks. Assuming the $p$-persistent connectivity of the random graphs, we showed the almost sure convergence of various distributed linear equation algorithms. When the linear equation admits exact solutions, we proved that the projection consensus algorithm enjoys the exponential convergence rate in term of the mean-squared error. We further gave an estimation of the mean-squared convergence rate bounds for the i.i.d. random graph when the linear equation has a unique solution. Extending the well-known randomized Kaczmarz method, we further designed a distributed randomized projection consensus algorithm, and showed its almost sure convergence and exponential convergence rate when the linear equation has a unique solution. In the last, we studied a distributed gradient-descent-like algorithm with decreasing step-sizes when the linear equation admits least-squares solutions, and proved that all nodes’ states converge almost surely to the unique least-squares solution at a sublinear rate. For future works, it would be interesting to investigate the exponentially convergent algorithm for least-squares, and extending the randomized Kaczmarz method to a distributed setting for least-squares. It is also promising to study other distributed computation tasks, such as distributed resource allocation, distributed optimization and distributed machine learning, over $*$-mixing random graphs.

Appendix A  Proof of Theorem 1

A.1 Preliminary Lemmas

The following lemma is from [19, Lemma 5].
Lemma 1 Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two affine spaces with $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subset \mathbb{R}^m$, and denote $\Pi_{\mathcal{K}_1}$ and $\Pi_{\mathcal{K}_2}$ as their projection operators. Then $\Pi_{\mathcal{K}_1}(y) = \Pi_{\mathcal{K}_2}(\Pi_{\mathcal{K}_2}(y))$ for any $y \in \mathbb{R}^m$.

We first show a projection invariance of the estimates generated by iteration (2).

Lemma 2 For any $t \geq 1$, $\sum_{i=1}^{N} \Pi_{A^*}(x_i(t)) = \sum_{i=1}^{N} \Pi_{A^*}(x_i(0))$ holds for the nodes' states generated by (2), irrespective of the random graph process (5).

Proof. With the iteration in (2), we have

$$\Pi_{A^*}(x_i(t+1)) = \Pi_{A^*} \left( \Pi_{A_i} \left( \sum_{j=1}^{N} W_{ij}(t) x_j(t) \right) \right)$$

$$= \Pi_{A^*} \left( \sum_{j=1}^{N} W_{ij}(t) x_j(t) \right) = \sum_{j=1}^{N} W_{ij}(t) \Pi_{A^*}(x_j(t)), \quad \forall t \geq 0. \quad (12)$$

where (i) is due to Lemma 1 and $A^* \subseteq A_i$, (ii) is due to $\Pi_{A^*}$ is an affine operator, and $\sum_{j=1}^{N} W_{ij}(t) = 1$ for each $i \in \mathcal{V}$. Then by using (12) and $\sum_{j=1}^{N} W_{ij}(t) = 1$ for each $j \in \mathcal{V}$, we obtain that

$$\sum_{i=1}^{N} \Pi_{A^*}(x_i(t)) = \sum_{i=1}^{N} \Pi_{A^*}(x_i(0))$$

Thus, $\sum_{i=1}^{N} \Pi_{A^*}(x_i(t)) = \sum_{i=1}^{N} \Pi_{A^*}(x_i(0))$ holds for any $t \geq 1$ for any realization of $\langle 5 \rangle$.

With Lemma 2 the iteration (2) should drive each node's state to $y^*(x(0)) = \sum_{i=1}^{N} \Pi_{A^*}(x_i(0))$ if all nodes' states converge to a consensual solution. Define $f(t) \triangleq \sum_{i=1}^{N} \|x_i(t) - y^*(x(0))\|^2$ for any $t \geq 0$. Next, we show the monotonicity of $f(t)$.

Lemma 3 For any $t \geq 0$, $f(t+1) \leq f(t)$ holds for the states with iteration (2), irrespective of (5).

Proof. Note that the projector $\Pi_{A_i}$ is affine. Therefore, we denote $\Pi_{A_i}(x) = P_i x + b_i$. Since $P_i \triangleq I_m - H_i^T (H_i^T)^\dagger$ is an orthogonal projector onto kernel($H_i$), it is both Hermitian ($P_i^T = P_i$) and idempotent ($P_i^2 = P_i$). Recall from [13, p.433] that a projection matrix $P_i$ has $\|P_i\|_2 = 1$.

With (2) and $y^*(x(0)) \in A^* \subseteq A_i$, we have

$$x_i(t+1) - y^*(x(0)) = P_i \left( \sum_{j=1}^{N} W_{ij}(t) x_j(t) \right) + b_i - \Pi_{A_i}(y^*(x(0)))$$

$$= P_i \left( \sum_{j=1}^{N} W_{ij}(t) x_j(t) - y^*(x(0)) \right) + P_i \left( \sum_{j=1}^{N} W_{ij}(t) (x_j(t) - y^*(x(0))) \right), \quad (14)$$

where the last equality holds by $\sum_{j=1}^{N} W_{ij}(t) = 1$. Note by $x_j(t) \in A_j$ and $y^*(x(0)) \in A^* \subseteq A_j$ that

$$P_j (x_j(t) - y^*(x(0))) = P_j (x_j(t) - y^*(x(0))) + b_j - b_j$$

$$= \Pi_{A_j}(x_j(t)) - \Pi_{A_j}(y^*(x(0))) = x_j(t) - y^*(x(0)). \quad (15)$$
This combined with (14) produces
\[ \mathbf{x}_i(t+1) - \mathbf{y}^*(\mathbf{x}(0)) = \mathbf{P}_i \sum_{j=1}^{N} \mathbf{W}_{ij}(t) \mathbf{P}_j (\mathbf{x}_j(t) - \mathbf{y}^*(\mathbf{x}(0))). \] (16)

Denote by \( \mathbf{e}_i(t) \triangleq \mathbf{x}_i(t) - \mathbf{y}^*(\mathbf{x}(0)) \), \( \mathbf{e}(t) \triangleq \text{col}\{\mathbf{e}_1(t), \cdots, \mathbf{e}_N(t)\} \), and \( \mathbf{P} \triangleq \text{diag}\{\mathbf{P}_1, \cdots, \mathbf{P}_N\} \). Then with (16), we get the following recursion for \( \mathbf{e}(t) \):
\[ \mathbf{e}(t+1) = \mathbf{P} (\mathbf{W}(t) \otimes \mathbf{I}_m) \mathbf{P} \mathbf{e}(t). \] (17)

Since \( \mathbf{W}(t) \) is row stochastic, the eigenvalues of \( \mathbf{W}(t) \) are less than or equal to 1 with the Geršgorin disks theorem, i.e., \( \| \mathbf{W}(t) \| \leq 1 \). Hence, with (17), we have
\[ \| \mathbf{e}(t+1) \| \leq \| \mathbf{P} \| \| \mathbf{W}(t) \otimes \mathbf{I}_m \| \| \mathbf{P} \| \| \mathbf{e}(t) \| \leq \| \mathbf{e}(t) \|. \]

By the definition \( f(t) = \| \mathbf{e}(t) \|^2 \), we obtain that \( f(t+1) \leq f(t) \) for any \( t \geq 0 \). \( \square \)

To analyze (17), we need to quantify the matrix product of the form \( \mathbf{P} \mathbf{W}(t) \otimes \mathbf{I}_m \cdots \mathbf{P} \mathbf{W}(0) \otimes \mathbf{I}_m \mathbf{P} \). For this, we introduce a special “mixed matrix norm” defined in [17]. Let write \( \mathbb{R}^{mN \times mN} \) for the vector space of all \( N \times N \) block matrices \( \mathbf{Q} = [\mathbf{Q}_{ij}] \) whose \( ij \)th entry is a matrix \( \mathbf{Q}_{ij} \in \mathbb{R}^{m \times m} \). We define the mixed matrix norm of \( \mathbf{Q} \in \mathbb{R}^{mN \times mN} \), denoted by \( \| \mathbf{Q} \|_M \), to be
\[ \| \mathbf{Q} \|_M = \| \langle \mathbf{Q} \rangle \|_\infty, \] (18)
where \( \langle \mathbf{Q} \rangle \in \mathbb{R}^{N \times N} \) with the \( ij \)th entry being \( \| \mathbf{Q}_{ij} \|_2 \). It has been shown in [17] Lemma 3] that \( \| \cdot \|_M \) is a sub-multiplicative norm, i.e., \( \| \mathbf{Q}_1 \mathbf{Q}_2 \|_M \leq \| \mathbf{Q}_1 \|_M \| \mathbf{Q}_2 \|_M \), \( \forall \mathbf{Q}_1, \mathbf{Q}_2 \in \mathbb{R}^{mN \times mN} \).

For any given symmetric stochastic matrix \( \mathbf{M} \) with positive diagonal elements, denote by \( \mathcal{S}(\mathbf{M}) \) the undirected graph with self-loop edge derived from \( \mathbf{M} \) so that \( \{i, j\} \) is an edge in the graph if \( \mathbf{M}_{ij} > 0 \). Denote by \( \mathcal{C} \) the set of \( N \) by \( N \) symmetric stochastic matrices with positive diagonal elements. Let \( r \) be a positive integer. Denote \( \mathcal{C}_r \) as the set of all sequences of symmetric stochastic matrices \( \mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_r \) with \( \mathbf{M}_i \in \mathcal{C} \) and the union graph \( \bigcup_{i=1}^r \mathcal{S}(\mathbf{M}_i) \) being connected.

In the following, we state two lemmas from [17] and [26] with adaptation of notations.

**Lemma 4 (Theorem 3 in [26])** Define \( \rho \triangleq (N - 1)N/2 \) and
\[ \theta \triangleq \left( \sup_{\mathcal{C}_r \in \mathcal{C}_r} \sup_{\mathcal{C}_{r-1} \in \mathcal{C}_r} \cdots \sup_{\mathcal{C}_1 \in \mathcal{C}_r} \| \mathbf{P}(\mathbf{M}_{pr} \otimes \mathbf{I}_m) \mathbf{P}(\mathbf{M}_{pr-1} \otimes \mathbf{I}_m) \cdots \mathbf{P}(\mathbf{M}_1 \otimes \mathbf{I}_m) \mathbf{P} \|_M \right), \]
where for each \( i \in \{1, \cdots, \rho\} \), \( \mathcal{C}_i \) is a sequence of stochastic matrices \( \mathbf{M}_{(i-1)r+1}, \mathbf{M}_{(i-1)r+2}, \cdots, \mathbf{M}_{ir} \) from \( \mathcal{C}_r \). Suppose that the LAE (1) has a unique solution, i.e., \( \bigcap_{i=1}^N \ker(\mathbf{H}_i) = 0 \). Then we have \( \theta < 1 \).

**Lemma 5 (Lemma 1 in [17])** Denote by \( \text{range}(\mathbf{P}_i) \) the space spanned with columns of \( \mathbf{P}_i \). Then \( \text{range}(\mathbf{P}_i) = \ker(\mathbf{H}_i) \). Suppose \( \bigcap_{i=1}^N \ker(\mathbf{H}_i) \neq \emptyset \), or equivalently, \( \ker(\mathbf{H}) \neq \emptyset \). Let \( \mathbf{Q}^T \) be a matrix with columns
forming an orthogonal basis for the space \( \text{range}(H^T) \). By defining \( \bar{P}_i = QP_iQ^T \) for each \( i \in \mathcal{V} \), the following statements are true.

(i) Each \( \bar{P}_i, i \in \mathcal{V} \) is an orthogonal projection matrix.

(ii) For each \( \bar{P}_i, i \in \mathcal{V} \), \( QP_i = \bar{P}_iQ \).

(iii) \( \cap_{i=1}^N \text{range}(\bar{P}_i) = 0 \).

We now ready to show the contractive property of the iteration (2) conditioned on specific events.

**Lemma 6** Suppose there are \( b \) edges in the \( p \)-persistent graph \( \mathcal{G}_P(p) \) denoted as \( \{i_1, j_1\}, \cdots, \{i_b, j_b\} \). Set \( \rho \triangleq N(N - 1)/2 \). Given a fixed integer \( \kappa \), we define an event for any \( s \geq 0 \):

\[
\omega(s) = \{(i_1, j_1) \in \mathcal{G}(s), (i_2, j_2) \in \mathcal{G}(s + \kappa), \cdots, (i_b, j_b) \in \mathcal{G}(s + (b-1)\kappa)\}. \tag{19}
\]

Then with (2), given any fixed integer \( s_0 \), there exists a constant \( 0 < \gamma < 1 \) such that

\[
P(f(s_0 + \rho \kappa) \leq \gamma f(s_0) | \{\omega(s_0), \omega(s_0 + \kappa), \cdots, \omega(s_0 + (\rho - 1)\kappa)\}) = 1. \tag{20}
\]

**Proof.** With (17) and the fact that \( \mathbf{P}^2 = \mathbf{P} \)

\[
e(s_0 + \rho \kappa) = \mathbf{P}(\mathbf{W}(s_0 + \rho \kappa - 1) \otimes \mathbf{I}_m)\mathbf{P} \cdots \mathbf{P}(\mathbf{W}(s_0) \otimes \mathbf{I}_m)\mathbf{P}e(s_0). \tag{21}
\]

Part (1): Firstly, suppose that \( \cap_{i=1}^N \ker\mathbf{H}_i = 0 \). Then for any \( l = 1, \cdots, \rho \), the sequence of stochastic matrices \( \mathbf{W}(s_0 + (l-1)b\kappa), \mathbf{W}(s_0 + (l-1)b\kappa+1), \cdots, \mathbf{W}(s_0 + lb\kappa - 1) \) has the union of their induced graphs \( \bigcup_{k=0}^{b\kappa - 1} \mathcal{G}((s_0 + (l-1)b\kappa + k)) \) being connected conditioned on the events \( \{\omega(s_0), \cdots, \omega(s_0 + (\rho - 1)b\kappa)\} \), since every edge of the connected \( \mathcal{G}_P(p) \) must appear at least once with the event \( \omega(s_0 + (l-1)b\kappa) \). In other words, the sequence of stochastic matrices \( \mathbf{W}(s_0 + (l-1)b\kappa), \mathbf{W}(s_0 + (l-1)b\kappa+1), \cdots, \mathbf{W}(s_0 + lb\kappa - 1) \) belongs to \( \mathcal{C}_{b\kappa} \) as defined before Lemma (4) conditioned on \( \{\omega(s_0), \omega(s_0 + b\kappa), \cdots, \omega(s_0 + (\rho - 1)b\kappa)\} \).

With Lemma (4) there exists \( \theta < 1 \) such that the following inequality is a sure event conditioned on \( \{\omega(s_0), \omega(s_0 + b\kappa), \cdots, \omega(s_0 + (\rho - 1)b\kappa)\} \):

\[
||\mathbf{P}(\mathbf{W}(s_0 + \rho \kappa - 1) \otimes \mathbf{I}_m)\mathbf{P} \cdots \mathbf{P}(\mathbf{W}(s_0) \otimes \mathbf{I}_m)\mathbf{P}||_M \leq \theta < 1.
\]

This incorporating with (21) implies that

\[
P(||e(s_0 + \rho \kappa)|| \leq \theta ||e(s_0)|| | \{\omega(s_0), \omega(s_0 + b\kappa), \cdots, \omega(s_0 + (\rho - 1)b\kappa)\}) = 1.
\]

Part (2): Secondly, suppose that \( \cap_{i=1}^N \ker\mathbf{H}_i \neq 0 \). Then we denote \( \mathbf{Q}^T \) as a matrix with columns forming an orthogonal basis for \( \text{range}(H^T) \). With Lemma (5) and Example 5.13.3 in [13], we know the projection matrix onto \( \ker\mathbf{H} \) is \( \mathbf{P}_{\ker\mathbf{H}} = \mathbf{I} - \mathbf{Q}^T(\mathbf{QQ}^T)^{-1}\mathbf{Q} = \mathbf{I} - \mathbf{Q}^T\mathbf{Q} \) and the projection matrix onto \( \text{range}(H^T) \) is \( \mathbf{P}_{\text{range}(H^T)} = \mathbf{Q}^T\mathbf{Q} \). Moreover, \( \ker\mathbf{H} \) and \( \text{range}(H^T) \) forms an orthogonal
decomposition of $\mathbb{R}^m$. Therefore, we decompose each $e_i, i \in \mathcal{V}$ along the spaces $\text{kernel}(H)$ and $\text{range}(H^T)$. Define a transformation $e_i^\dagger = Qe_i$ and $e_i^\perp = e_i - Q^T e_i^\dagger$. Then we decompose $e_i$ as

$$e_i = P_{\text{range}(H^T)}(e_i) + P_{\text{kernel}(H)}(e_i) = Q^T e_i^\dagger + e_i - Q^T e_i^\dagger = Q^T e_i^\dagger + e_i^\perp.$$ 

By using (16), $QP_i P_j = \check{P}_i QP_j = \check{P}_i \check{P}_j Q$, and Lemma 5 (ii), we have the following for $e_i^\dagger$:

$$e_i^\dagger(t + 1) = QP_i \sum_{j=1}^N W_{ij}(k) P_j e_j(t) = \check{P}_i \sum_{j=1}^N W_{ij}(k) QP_j e_j(t) = \check{P}_i \sum_{j=1}^N W_{ij}(k) \check{P}_j Q e_j(t) = \check{P}_i \sum_{j=1}^N W_{ij}(k) \check{P}_j e_j^\dagger(t).$$

Moreover, with Lemma 5 $\bigcap_{i=1}^N \text{range}(\check{P}_i) = 0$. The iteration of $e_i^\perp(t)$ can be treated as an error system for solving a LAE with a unique solution. With Part (1) of the proof, there exists a $\theta^2 < 1$ such that

$$\mathbb{P}(||e_i^\dagger(s_0 + \rho bk)|| \leq \theta^2 ||e_i^\dagger(s_0)|| |\{\omega(s_0), \omega(s_0 + bk), \ldots, \omega(s_0 + (\rho - 1)bk)\}) = 1.$$

We denote $A^* = v + \text{kernel}(H)$ with $v \in A^*$, hence by [Example 5.13.5, [13]] there holds

$$y^*(x(0)) = \frac{1}{N} \sum_{i=1}^N \Pi_{A^*} (x_i(0)) = \frac{1}{N} \sum_{i=1}^N (I - P_{\text{kernel}(H)}) v + P_{\text{kernel}(H)} x_i(0))$$

$$= (I - P_{\text{kernel}(H)}) v + P_{\text{kernel}(H)} \sum_{i=1}^N x_i(0)/N.$$

For $e_i^\perp = P_{\text{kernel}(H)} e_i$, by using $P_{\text{kernel}(H)}^2 = P_{\text{kernel}(H)}$, we have that

$$e_i^\perp = P_{\text{kernel}(H)} (x_i(t) - y^*(x(0)))$$

$$= P_{\text{kernel}(H)} x_i(t) - P_{\text{kernel}(H)} (I - P_{\text{kernel}(H)}) v - P_{\text{kernel}(H)} \sum_{i=1}^N x_i(0)/N$$

$$= P_{\text{kernel}(H)} \left( x_i(t) - \frac{1}{N} \sum_{i=1}^N x_i(0)/N \right).$$

Denote $A_i = v_i + \text{kernel}(H_i)$ with $v_i \in A_i$. Then

$$e_i^\perp(t + 1) = P_{\text{kernel}(H)} \left( \Pi_{A_i} \left( \sum_{j=1}^N W_{ij}(t) x_j(t) \right) - \frac{1}{N} \sum_{i=1}^N x_i(0) \right)$$

$$= P_{\text{kernel}(H)} \left( (I - P_{\text{kernel}(H_i)}) v_i + P_{\text{kernel}(H_i)} \left( \sum_{j=1}^N W_{ij}(t) x_j(t) - \frac{1}{N} \sum_{i=1}^N x_i(0) \right) \right)$$

$$\overset{(i)}{=} P_{\text{kernel}(H)} \left( \sum_{j=1}^N W_{ij}(t) x_j(t) - \frac{1}{N} \sum_{i=1}^N x_i(0) \right)$$

$$\overset{(ii)}{=} \sum_{j=1}^N W_{ij}(t) P_{\text{kernel}(H)} (x_j(t) - \frac{1}{N} \sum_{i=1}^N x_i(0)) = \sum_{j=1}^N W_{ij}(t) e_j^\perp(t),$$

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where (i) is due to kernel($H$) $\subset$ kernel($H_i$) and Lemma 11 and (ii) is due to the row stochasticity of $W(t)$.

Note that $e_i(t + 1) = \sum_{j=1}^N W_{ij}(t)e_j^i(t)$ is the well studied consensus algorithm over random graphs. When $W_{ij}(t)$ is defined from uniformly jointly connected graphs, each $e_i(t + 1)$ should converge to the same $\frac{1}{N} \sum_{i=1}^N e_i^i(0)$ exponentially fast. Moreover,

$$\frac{1}{N} \sum_{i=1}^N e_i^i(0) = \frac{1}{N} \sum_{i=1}^N P_{\text{kernel}(H)}(X_i(0) - \frac{1}{N} \sum_{i=1}^N X_i(0)) = 0.$$

Conditioned on $\{\omega(s), \omega(s + \kappa), \ldots, \omega(s + (\rho - 1)\kappa)\}$, the sequence of $W(s + \rho\kappa - 1), \ldots, W(s)$ have the union of their induced graphs $\cup_{t=0}^{\rho-1} \cup_{\kappa=0}^{b-1} \mathcal{G}(W(s + (l - 1)\kappa + \kappa))$ being connected. Thereby, there exists a constant $\theta^1 < 1$ such that

$$P(||e_i^1(s) + \rho\kappa|| \leq \theta^1 ||e_i(s)||) = 1.$$

Moreover, $\theta^1$ is uniformly upper bounded irrespective with any realization of $\mathcal{G}$ since all edges in $\mathcal{G}(P)$ must appears at least once conditioned on $\{\omega(s), \omega(s + \kappa), \ldots, \omega(s + (\rho - 1)\kappa)\}$. Note that $||e_i||^2 = ||Q^T e_i^1 + e_i^1||^2 = ||e_i^1||^2 + ||e_i||^2$ with kernel($H$) and range($H^T$) being orthogonal and $QQ^T = 1$. Now, conditioned on $\{\omega(s), \omega(s + \kappa), \ldots, \omega(s + (\rho - 1)\kappa)\}$ we have

$$f(s + \rho\kappa) = ||Q^T e_i^1(s + \rho\kappa) + e_i^1(s + \rho\kappa)||^2 \leq \max\{(\theta^2)^2, (\theta^1)^2\} ||Q^T e_i^1(s) + e_i^1(s)||^2.$$

Combining the proofs in Part (1) and Part (2), we can always find a $0 < \gamma < 1$ such that (20) holds. \square

The following Borel-Cantelli lemma for $\ast-$mixing events will be used to prove Theorem 11.

**Lemma 7** (Lemma 6] Let $\{A\}$ be a sequence of $\ast-$mixing events. Then $\sum_{k=0}^\infty P(A(k)) = \infty$ implies

$$P\left(\limsup_{k\to\infty} A(k)\right) = 1.$$

**A.2 Proof of the Theorem**

With the $\ast-$mixing property on the random graph process $\mathcal{G}$, given a $0 < \bar{\lambda} < 1$, there exists a large enough integer $\kappa$ possibly depending on $\bar{\lambda}$ such that, for any $t \geq 0$, $A \in \mathcal{T}_0(\{\mathbb{I}\})$ and $B \in \mathcal{T}_t(\{\mathbb{I}\})$, we have that

$$|P(A \cap B) - P(A)P(B)| \leq \bar{\lambda}P(A)P(B). \quad (22)$$

We first give a lower bound on the probability of the event $\{\omega(s_0), \omega(s_0 + \kappa), \ldots, \omega(s_0 + (\rho - 1)\kappa)\}$ defined as in Lemma 6. Here, $s_0$ is any time index, $\rho = N(N - 1)/2$, and $\kappa$ is taken such that (22) holds with $\bar{\lambda}$. Note by (19) that

$$\omega(s_0) = \{i_1, j_1\} \in \mathcal{G}(s_0), \{i_2, j_2\} \in \mathcal{G}(s_0 + \kappa), \ldots, \{i_b, j_b\} \in \mathcal{G}(s_0 + (b - 1)\kappa)\}.$$
We denote two events $A(s_0) \triangleq \{(i_1, j_1) \in G(s_0)\}$ and $B(s_0 + \kappa) \triangleq \{(i_2, j_2) \in G(s_0 + \kappa)\}$. Since the events $\{(i_1, j_1) \in G(s_0)\}$ and $\{(i_2, j_2) \in G(s_0 + \kappa)\}$ are two indicator variables of $I(s_0)$ and $I(s_0 + \kappa)$, we have $A(s_0) \in \mathcal{F}_{s_0}^2(I)$ and $B(s_0 + \kappa) \in \mathcal{F}_{s_0+\kappa}^\infty(I)$. Then with $\kappa$ chosen for (22) we have that

$$|\mathbb{P}(A(s_0) \cap B(s_0 + \kappa)) - \mathbb{P}(A(s_0))\mathbb{P}(B(s_0 + \kappa))| \leq \bar{\lambda}\mathbb{P}(A(s_0))\mathbb{P}(B(s_0 + \kappa)).$$

Therefore,

$$\mathbb{P}(A(s_0) \cap B(s_0 + \kappa)) \geq (1 - \bar{\lambda})\mathbb{P}(A(s_0))\mathbb{P}(B(s_0 + \kappa)).$$

Since both $\{i_1, j_1\}$ and $\{i_2, j_2\}$ belong to the $p$–persistent graph $G_P(p)$, $\mathbb{P}(A(s_0)) \geq p$ and $\mathbb{P}(B(s_0 + \kappa)) > p$. Hence,

$$\mathbb{P}(A(s_0) \cap B(s_0 + \kappa)) \geq (1 - \bar{\lambda})p^2. \tag{23}$$

Next, we denote two events $A(s_0 + \kappa) \triangleq \{A(s_0) \cap B(s_0 + \kappa)\} \in \mathcal{F}_{s_0}^{2+\kappa}(I)$ and $B(s_0 + 2\kappa) \triangleq \{(i_3, j_3) \in G(s_0 + 2\kappa)\} \in \mathcal{F}_{s_0+2\kappa}^\infty(I)$. Similarly to (23), we have

$$\mathbb{P}(A(s_0 + \kappa) \cap B(s_0 + 2\kappa)) \geq (1 - \bar{\lambda})\mathbb{P}(A(s_0 + \kappa))\mathbb{P}(B(s_0 + 2\kappa)) \geq (1 - \bar{\lambda})^2p^3.$$

We repeat the procedure by defining proper events $A(s_0 + (l - 1)\kappa), B(s_0 + l\kappa), l = 3, \ldots, b - 1$, and have

$$\mathbb{P}(\omega(s_0)) \geq (1 - \bar{\lambda})^{b-1}p^b. \tag{24}$$

Note that

$$\omega(s_0 + b\kappa) = \{i_1, j_1\} \in G(s_0 + b\kappa), \{i_2, j_2\} \in G(s_0 + b\kappa + \kappa), \ldots, \{i_b, j_b\} \in G(s_0 + 2b\kappa - \kappa)\}.$$

Denote $A(s_0 + (b - 1)\kappa) \triangleq \{\omega(s_0)\} \in \mathcal{F}_{s_0}^{2+(b-1)\kappa}(I)$ and $B(s_0 + b\kappa) \triangleq \{(i_1, j_1) \in G(s_0 + b\kappa)\} \in \mathcal{F}_{s_0+b\kappa}^\infty(I)$. Similarly to (23), we obtain that

$$\mathbb{P}(A(s_0 + (b - 1)\kappa) \cap B(s_0 + b\kappa)) \geq (1 - \bar{\lambda})\mathbb{P}(A(s_0 + (b - 1)\kappa))\mathbb{P}(B(s_0 + b\kappa)) \geq (1 - \bar{\lambda})^bp^{b+1}.$$

Therefore, we can repeat the above recursion by defining property events through out $\{\omega(s_0), \omega(s_0 + b\kappa), \ldots, \omega(s_0 + (\rho - 1)b\kappa)\}$, and obtain

$$\mathbb{P}(\{\omega(s_0), \omega(s_0 + b\kappa), \ldots, \omega(s_0 + (\rho - 1)b\kappa)\}) \geq (1 - \bar{\lambda})^{\rho b-1}p^\rho. \tag{25}$$

This combined with Lemma 4 implies that there exists a constant $0 < \gamma < 1$, possibly depending on $\bar{\lambda}$ and $\kappa$, such that for any time index $s_0$:

$$\mathbb{P}(f(s_0 + \rho b\kappa) \leq \gamma f(s_0)) \geq (1 - \bar{\lambda})^{\rho b-1}p^\rho. \tag{26}$$

Next, we denote a sequence of events

$$D(l) \triangleq \{f((l + 1)\rho b\kappa) \leq \gamma f(l\rho b\kappa)\}, \quad \forall l \geq 0.$$
Then from (26) it follows that for any \( l = 0, 1, \cdots \):

\[
\mathbb{P}(\mathcal{D}(l)) \geq (1 - \bar{\lambda})^{\rho b - 1} p^{\rho b} > 0. \tag{27}
\]

Since the initial node states are fixed and the stochasticity in node states only comes from the random graph process \( \langle G \rangle \), the sequence of events \( \mathcal{D}(l) \) is also \( \ast \)-mixing. From (27), we have

\[
\sum_{l=0}^{\infty} \mathbb{P}(\mathcal{D}(l)) = \infty.
\]

This combined with Lemma 7 implies the following

\[ \mathbb{P}\left( \limsup_{l \to \infty} \mathcal{D}(l) \right) = 1. \]

Note that

\[
\limsup_{l \to \infty} \mathcal{D}(l) = \{ \omega : \omega \in \mathcal{D}(l) \text{ for infinitely many } l \}. \tag{28}
\]

Therefore, the probability that the event \( \{ f((l + 1)\rho b \kappa) \leq \gamma f(l\rho b \kappa) \} \) happens for infinitely many times is 1. Moreover, \( f(t + 1) \leq f(t) \) is always true for any time \( t \) with Lemma 3 Therefore, \( f(t) \) decreases to 0 with probability one, implying (3).

Note by \( f(t + 1) \leq f(t), \forall t \geq 0 \) that

\[
\mathbb{E}[f(l\rho b \kappa)] = \mathbb{E}[f(l\rho b \kappa)I_{f(l\rho b \kappa) \leq f((l-1)\rho b \kappa)}] + \mathbb{E}[f(l\rho b \kappa)I_{f(l\rho b \kappa) > f((l-1)\rho b \kappa)}] \\
\leq \gamma \mathbb{E}[f((l-1)\rho b \kappa)] \mathbb{P}(\mathcal{D}(l-1)) + \mathbb{E}[f((l-1)\rho b \kappa)](1 - \mathbb{P}(\mathcal{D}(l-1))) \\
= (1 - (1 - \gamma)\mathbb{P}(\mathcal{D}(l-1))) \mathbb{E}[f((l-1)\rho b \kappa)] \tag{29}
\]

\[
\leq (1 - (1 - \gamma)(1 - \bar{\lambda})^{\rho b - 1} p^{\rho b}) \mathbb{E}[f((l-1)\rho b \kappa)] \\
\leq (1 - (1 - \gamma)(1 - \bar{\lambda})^{\rho b - 1} p^{\rho b})^l \mathbb{E}[f(0)] \triangleq c_0 \nu^l.
\]

For any \( t > \rho b \kappa \), it could be written as \( t = l \rho b \kappa + q \) with \( l, q \) being positive integers. Then

\[
\mathbb{E}[f(t)] \leq \mathbb{E}[f(l\rho b \kappa)] \leq c_0 \nu^l = c_0 \nu^{\frac{t}{\rho b \kappa}} = c_0 (\nu^{\frac{1}{\rho b \kappa}})^q (\nu^{\frac{1}{\rho b \kappa}})^l \leq c_0 (\nu^{\frac{1}{\rho b \kappa}})^{-\rho b \kappa + 1}(\nu^{\frac{1}{\rho b \kappa}})^t.
\]

Thus, by the definition of \( f(t) \), we obtain the exponential convergence rate of the mean-squared error. □

### Appendix B Proof of Proposition 1

Define \( \mathcal{F}_t = \sigma\{\mathbf{W}(0), \cdots, \mathbf{W}(t - 1)\} \). Then \( e(t) \) is adapted to \( \mathcal{F}_t \) by the iteration (2), and \( \mathbf{W}(t) \) is independent of \( \mathcal{F}_t \). Then by (17), we obtain that

\[
\mathbb{E}[e(t + 1)|\mathcal{F}_t] = \mathbb{E}[\mathbf{P}\mathbf{W}(t) \otimes I_m \mathbf{P}|\mathcal{F}_t]e(t) = \mathbf{P}\mathbf{W} \otimes I_m \mathbf{P}e(t).
\]

This implies that

\[
\mathbb{E}[e(t + 1)] = \mathbf{P}\mathbf{W} \otimes I_m \mathbb{E}[e(t)] = (\mathbf{P}\mathbf{W} \otimes I_m \mathbf{P})^{t+1}e(0).
\]
Then by the Jensen’s inequality, we obtain that
\[
E[\|e(t)\|^2] \geq \|E[e(t)]\|^2 = (P \bar{W} \otimes I_mP)^t e(0)^2.
\]
Hence, we obtain the left side of (6).

Since e(t) is adapted to \(\mathcal{F}_t\) and \(W(t)\) is independent of \(\mathcal{F}_t\), from (7) it follows that
\[
E[\|e(t + 1)\|^2|\mathcal{F}_t] = e(t)^T E[P(W(t) \otimes I_m)P(W(t) \otimes I_m)P]e(t)
\]
\[
\leq \|e(t)\|^2 \|E[P(W(t) \otimes I_m)P(W(t) \otimes I_m)P]\|.
\]
Then by a simple recursion, we obtain that
\[
E[\|e(t)\|^2] \leq \|e(0)\|^2 \|E[P(W(t) \otimes I_m)P(W(t) \otimes I_m)P]\|^t.
\]
Therefore, the right side of (6) is proved. \(\square\)

Appendix C  

Proof of Theorem 2

C.1 Preliminary Lemmas

For (7), we denote \(v_i(t) = \sum_{j=1}^{N} W_{ij}(t)x_j(t)\). Let \(x^*\) be a solution to the LAE (1). Then \(z^{(s_i(t))}_i = H^{(s_i(t))}_i x^*\). By defining
\[
P^{(s_i(t))}_i \triangleq I_m - \frac{(H^{(s_i(t))}_i)^T H^{(s_i(t))}_i}{\|H^{(s_i(t))}_i\|^2},
\]
from (7) it follows that
\[
x_i(t + 1) - x^* = v_i(t) - x^* - \left(H^{(s_i(t))}_i\right)^T \frac{H^{(s_i(t))}_i (v_i(t) - x^*)}{\|H^{(s_i(t))}_i\|^2} = P^{(s_i(t))}_i (v_i(t) - x^*).
\]
(30)

Observe that \(P^{(s_i(t))}_i\) is symmetric and \(P^{(s_i(t))}_i F^{(s_i(t))}_i = F^{(s_i(t))}_i\), i.e., \(P^{(s_i(t))}_i\) is a projection matrix. Define \(e_i(t) \triangleq x_i(t) - x^*\). Note by \(\sum_{j=1}^{N} W_{ij}(t) = 1\) that
\[
v_i(t) - x^* = \sum_{j=1}^{N} W_{ij}(t)(x_j(t) - x^*) = \sum_{j=1}^{N} W_{ij}(t)e_j(t).
\]
Define \(e(t) \triangleq \text{col}\{e_1(t), \ldots, e_N(t)\}\), \(S(t) \triangleq \text{diag}\{s_1(t), \ldots, s_N(t)\}\), and \(P^{(S(t))} \triangleq \text{diag}\{P^{(s_1(t))}_1, \ldots, P^{(s_N(t))}_N\} \in \mathbb{R}^{mN \times mN}\) with the ith diagonal matrix being \(P^{(s_i(t))}_i \in \mathbb{R}^{m \times m}\). Then by (30), we have the following
\[
e(t + 1) = P^{(S(t))} W(t) \otimes I_m e(t). W(0) \otimes I_m P e(0).
\]
(31)

Note that \(s_i(t) \in \{1, 2, \ldots, l_i\}\) for each \(i \in V\). Then \(S(t)\) belongs to a finite set of diagonal matrices with the \(i\)-th diagonal entry taking value from the set \(\{1, 2, \ldots, l_i\}\), for which the cardinality is \(\prod_{i=1}^{N} l_i\).
We consider the case where \( z = H y \) has a unique solution, this indeed implies that

\[
\bigcap_{i=1}^{N} \bigcap_{s=1}^{l_i} \text{kernel}(H_i^{(s)}) = \emptyset.
\]

Since \( \text{kernel}(H_i^{(s)}) = \mathcal{P}_i^{(s)} \), where \( \mathcal{P}_i^{(s)} \) denotes the column span of the projection matrix \( P_i^{(s)} = I_m - \left( H_i^{(s)} \right)^T H_i^{(s)} \). Thus, the uniqueness assumption is equivalent to the condition \( \bigcap_{i=1}^{N} \bigcap_{s=1}^{l_i} \mathcal{P}_i^{(s)} = \emptyset \). Then from [17, Lemma 2] it follows that

\[
\left\| \bigcap_{i=1}^{N} \bigcap_{s=1}^{l_i} P_i^{(s)} \right\| < 1.
\]

A route over a given sequence of undirected graphs \( \mathcal{G}_1 = \{ V, \mathcal{E}_1 \}, \cdots, \mathcal{G}_q = \{ V, \mathcal{E}_q \} \) is meant a sequence of vertices \( i_0, i_1, \cdots, i_q \) such that \( \{ i_{k-1}, i_k \} \in \mathcal{E}_k \) for all \( k \in \{ 1, \cdots, q \} \). For each \( k \geq 0 \), let \( S_k \) be a positive diagonal matrix and \( \mathbf{P}(S_k) = \text{diag}\{ P_i^{(S_k)} \} \in \mathbb{R}^{N \times N} \) with \( S_k, i \in \{ 1, \cdots, l_i \} \) denoting the \( i \)th diagonal entry of \( S_k \). Then similarly to [17, Lemma 4], we obtain the following result.

**Lemma 8** Let \( M_1, M_2, \cdots, M_q \) be a sequence of symmetric stochastic matrices with positive diagonal elements. If \( j = i_0, i_1, \cdots, i_q = i \) is a route over the graph sequence \( \mathcal{G}(M_1), \cdots, \mathcal{G}(M_q) \), then the matrix product \( \mathbf{P}(S_{q,i_q}) \cdots \mathbf{P}(S_{i,i}) \mathbf{P}(S_{i_0}) \) is a component of the \( i \)th block entry of

\[
\Phi = \mathbf{P}(S_0) M_q \otimes I_m \cdots M_1 \otimes I_m \mathbf{P}(S_0).
\]

To proceed, we call matrices of the form

\[
\mu \left( P_i^{(1)}, \cdots, P_i^{(l_i)}, i \in V \right) = \sum_{k=1}^{d} \lambda_k P_i^{(f_{i,k,1})} P_i^{(f_{i,k,2})} \cdots P_i^{(f_{i,k,q})}
\]

the projection matrix polynomials, where \( q_k \) and \( d \) are positive integers, \( \lambda_k \) is a real positive number, and for each \( j \in \{ 1, 2, \cdots, q_k \}, h_{k,j} \in \{ 1, \cdots, N \} \) and \( f_{i,j} \in \{ 1, \cdots, l_{h_{k,j}} \} \). We say that a nonzero matrix polynomial \( \mu( P_i^{(1)}, \cdots, P_i^{(l_i)}, i \in V \) is complete if it has a component \( P_i^{(f_{i,k,1})} P_i^{(f_{i,k,2})} \cdots P_i^{(f_{i,k,q})} \) within which each of the projection matrices \( P_i^{(s)}, i \in \{ 1, \cdots, N \}, s \in \{ 1, \cdots, l_i \} \) appears at least once. Let \( N \) be a positive integer. Denote by \( \mathcal{S}_n \) the set of all sequences of projection matrices \( \mathbf{P}(S_1), \cdots, \mathbf{P}(S_n) \), where for every \( i \in \{ 1, \cdots, N \} \), each of the projection matrices \( P_i^{(s)}, s \in \{ 1, \cdots, l_i \} \) appears at least once the \( i \)th diagonal entry \( \prod_{k=1}^{n} P_i^{(S_{k,i})} \) of the matrix product \( \prod_{k=1}^{n} P_i^{(S_{k,i})} \).

We then give Lemma [9] for which the proof is modified based on that of [17, Proposition 2]. For proving the lemma, we introduce the graph composition. Let \( \mathcal{G}_p = \{ V, \mathcal{E}_p \} \) and \( \mathcal{G}_q = \{ V, \mathcal{E}_q \} \) be two undirected graphs. The composition of \( \mathcal{G}_p \) with \( \mathcal{G}_q \), denoted by \( \mathcal{G}_p \circ \mathcal{G}_q \), is meant that undirected graph over the node set \( V \) with the edge set defined so that \( \{ i, j \} \) is an edge in the composition \( \mathcal{G}_p \circ \mathcal{G}_q \) whenever there is a vertex \( k \) such that \( \{ i, k \} \in \mathcal{E}_p \) and \( \{ k, j \} \in \mathcal{E}_q \). By the definition of graph composition, it is seen that for any pair of \( N \times N \) stochastic matrices \( M_1 \) and \( M_2 \), there holds \( \mathcal{G}(M_1 M_2) = \mathcal{G}(M_1) \circ \mathcal{G}(M_2) \).
Lemma 9 Suppose \( \tau \) has a unique solution. Let \( \rho \) be the least common multiple of \( n \) and \( \tau \). Define 
\[ \rho \triangleq N^2 - 1, \quad \rho_1 = \rho \tau / r \quad \text{and} \quad \rho_2 = \rho / n. \]
Then the matrix \( \Phi \equiv \prod_{l=1}^{n} (M_{l\rho_l} \otimes I_{m_l}) \prod_{l=1}^{n} (M_{l\rho_l-1} \otimes I_{m_l}) \) is a contraction in the mixed matrix norm defined by [18], where for each \( i \in \{1, \ldots, \rho_1\} \), the sequence of stochastic matrices \( M_{(i-1)r+1}, M_{(i-1)r+2}, \ldots, M_{ir} \) is from \( \mathcal{C}_r \), and for each \( j \in \{1, \ldots, \rho_2\} \), the sequence of projection matrices \( \prod_{i=1}^{n} (S_{(j-1)n+1}), \prod_{i=1}^{n} (S_{(j-1)n+2}), \ldots, \prod_{i=1}^{n} (S_{jn}) \) is from \( \mathcal{S}_n \).

Proof. We partition the sequence \( \mathcal{S}(M_1), \ldots, \mathcal{S}(M_{\rho_2}) \) into \( N - 1 \) subsequences, where for each \( k = 1, \ldots, N - 1 \) : 
\[ G_k = \{ \mathcal{S}(M_{(k-1)r(N+1)+1}), \mathcal{S}(M_{(k-1)r(N+1)+2}), \ldots, \mathcal{S}(M_{k\tau(N+1)}) \}. \]
For each \( k = 1, \ldots, N - 1 \), we further partition \( G_k \) into three subsequences as 
\[ \begin{align*}
G_k^1 & = \{ \mathcal{S}(M_{(k-1)r(N+1)+1}), \ldots, \mathcal{S}(M_{(k-1)r(N+1)+\tau}) \}, \\
G_k^2 & = \{ \mathcal{S}(M_{(k-1)r(N+1)+\tau+1}), \ldots, \mathcal{S}(M_{(k-1)r(N+1)+n\tau}) \}, \\
G_k^3 & = \{ \mathcal{S}(M_{(k-1)r(N+1)+n\tau+1}), \ldots, \mathcal{S}(M_{k\tau(N+1)}) \},
\end{align*} \]
and define the composite graph \( \mathbb{H}_k = \mathcal{S}(M_{(k-1)r(N+1)+\tau+1}) \circ \ldots \circ \mathcal{S}(M_{(k-1)r(N+1)+n\tau}) \). Since \( \tau \) is divisible by \( r \), \( \mathbb{H}_k \) can be written as the composition of \((N - 1)\tau / r \) connected graphs. It has been shown in [22], Proposition 4] that the composition of any sequence of \( N - 1 \) or more connected graph is a complete graph. Thus, the graph \( \mathbb{H}_k \), \( k = 1, \ldots, N - 1 \) is a complete graph. Hence for each pair \( i, j \in V \) and each \( k \in \{1, \ldots, N - 1\} \), there must be a route over the sequence \( G_k^2 \) from \( i \) to \( j \).

Let \( i_1, i_2, \ldots, i_N \) be any reordering of the node sequence \( \{1, 2, \ldots, N\} \). Based on the discussions in the aforementioned paragraph, it is clear that for each \( k \in \{1, \ldots, N - 1\} \), there exist a route 
\[ j_{(k-1)r(N+1)+\tau} = i_k, j_{(k-1)r(N+1)+\tau+1}, \ldots, j_{(k-1)r(N+1)+n\tau} = i_{k+1} \]
over \( G_k^2 \) from \( i_k \) to \( i_{k+1} \). Since each symmetric stochastic matrix \( M_{\rho_l}, \rho \geq 1 \) has positive diagonal elements, the undirected graph \( \mathcal{G}(M_{\rho_l}) \) have the self-loop edge \( \{i, i\} \) for each \( i \in V \). Then for each \( k \in \{1, \ldots, N - 1\} \), there exists a route 
\[ j_{(k-1)r(N+1)} = i_k, j_{(k-1)r(N+1)+1} = \cdots = j_{(k-1)r(N+1)+\tau} = i_k \]
over \( G_k^1 \) from \( i_k \) to \( i_k \), and also exists a route 
\[ j_{(k-1)r(N+1)+n\tau} = \cdots = j_{k\tau(N+1)} = i_{k+1} \]
over \( G_k^3 \) from \( i_{k+1} \) to \( i_{k+1} \). In view of Lemma \( i_{N+1}^{th} \) block entry of \( \Phi \) contains the following matrix product as a component
\[ \prod_{k_1=0}^{\tau} P_{i_{k_1}^{(s_1)}} \cdots \prod_{k_2=\tau N}^{\tau (N+2)} P_{i_{k_2}^{(s_2)}} \cdots \prod_{k_{N-1}=(N^2-N+1)\tau}^{(N^2-N+1)\tau} P_{i_{k_{N-1}}^{(s_{N-1})}} \cdots \prod_{k_{N}=(N^2-N+2)\tau}^{(N^2-1)\tau} P_{i_{k_{N}}^{(s_{N})}}. \]
By recalling that \( \tau \) is divisible by \( n \) and the definition of \( \mathcal{S}_n \), we see that each of the projection matrices \( P_{i_{k_1}^{(s)}} \), \( i \in \{1, \ldots, N\} \), \( s \in \{1, \ldots, l_i\} \) appears at least once in the above matrix product. Therefore, the \( i_{N+1}^{th} \) block entry of \( \Phi \) is complete.

Since the above procedure applies for any sequence of \( N \) distinct node labels \( i_1, i_2, \ldots, i_N \) of the set \( \{1, 2, \ldots, N\} \), every block entry of \( \Phi \) except for the diagonal blocks must be a complete projection matrix polynomial. By recalling that the LAE \( \mathcal{L} \) has a unique solution, it follows from [17], Proposition 1] that \( \Phi \) is a contraction in the mixed matrix norm defined in [18].
Then based on Lemma \[9\] we conclude that

\[
\psi \triangleq \sup_{C_{r_1} \in \mathcal{C}_r, \ldots, C_{r_n} \in \mathcal{C}_r} \sup_{s_{r_1} \in \mathcal{S}_{r_1}, \ldots, s_{r_n} \in \mathcal{S}_{r_n}} \left\| P^{(S_{r_1})} (M_{r_1} \otimes I_m) \cdots P^{(S_{r_n})} (M_{r_n} \otimes I_m) P^{(S_{0})} \right\|_M < 1, \tag{33}
\]

where for each \( i \in \{1, \ldots, p_1\} \), \( C_i \) is a sequence of stochastic matrices \( M_{(i-1)r_1+1}, \ldots, M_{r_1} \) from \( \mathcal{C}_r \), and for each \( j \in \{1, \ldots, p_2\} \), \( S_j \) is a sequence of projection matrices \( P^{(S_{(j-1)n_1})}, \ldots, P^{(S_{jn})} \) from \( S_n \).

**C.2 Proof of the Theorem**

Similarly to (15), we have that

\[
\forall \kappa \in \mathcal{C}_r, \exists \kappa \in \mathcal{C}_r, \text{ such that for any } t \geq 0 : \quad \phi(t) = P(t) \mathcal{W}(t) \otimes I_m \cdots P(0) \mathcal{W}(0) \otimes I_m \mathcal{P}(t). \tag{34}
\]

Let \( n \) be the least common multiplier of integers \( l_i, i \in \mathcal{V} \). We define the following event for \( t \geq 0 \):

\[
\varphi(t) \triangleq \left\{ P^{(S_{i(t)})} \cdots P^{(S_{(t+n-1)})} \right\} \text{ with each of the projection matrices } P^{(s)}_{i}, s \in \{1, \ldots, l_i\} \text{ appearing at least once in the } i \text{th diagonal entry for each } i \in \mathcal{V} \right\}. \tag{35}
\]

For each \( i \in \mathcal{V} \) and \( s = 1, \ldots, l_i \), define \( p_{i,s} \triangleq P(s_i(t) = s) = \left\| H_{i}^{(s)} \right\|^2/\left\| H_{i} \right\|^2_F \). Since the sequences \( \{s_i(t), i \in \mathcal{V} \} \) are mutually independent, by (35) it is seen that

\[
\mathbb{P}(\varphi(t)) = \prod_{i=1}^{N} \mathbb{P}(\varphi_i(t)), \quad \forall t \geq 0.
\]

Based on the above definition, we know that

\[
\mathbb{P}(\varphi(t)) = \mathbb{P} \left( \text{each element of } \{1, \ldots, l_i\} \text{ appears at least once in the sequence } \{s_i(p)\}_{p=t}^{t+n-1} \right) \geq C(n, l_i) \prod_{s=1}^{l_i} p_{i,s} \quad \text{ with } C(n, l_i) = \frac{n!}{l_i!(n-l_i)!},
\]

where the above inequality follows from the fact that for each \( i \in \mathcal{V} \), \( s_i(t), t \geq 0 \) are independent variables. Then by \( \mathbb{P}(\varphi(t)) = \prod_{i=1}^{N} \mathbb{P}(\varphi_i(t)), \) we conclude that there exits a positive constant \( \bar{p} > 0 \) such that

\[
\mathbb{P}(\varphi(t)) \geq \bar{p}, \quad \forall t \geq 0.
\tag{36}
\]

For a given \( 0 < \bar{\lambda} < 1 \), there exists a large integer \( \kappa \) possibly depending on \( \bar{\lambda} \) such that for any \( t \geq 0 \): (22) holds for all \( A \in \mathcal{F}_0(t) \) and \( B \in \mathcal{F}_\kappa(t) \). Suppose there are \( b \) edges in the \( p \)-persistent graph \( G_P(p) \) denoted as \( \{i_1, j_1\}, \ldots, \{i_b, j_b\} \). Define \( r \triangleq \kappa r \), and let \( \tau \) be the least common multiplier of \( n \) and \( r \). Define \( \rho \triangleq N^2 - 1 \), \( \rho_1 = \rho r / r \) and \( \rho_2 = \rho r / n \). From (24) it is seen that \( \omega(s) \) defined by (19) satisfies the following

\[
\mathbb{P}(\{\omega(t), \omega(t+r), \cdots, \omega(t+(\rho_1-1)r)\}) \geq (1 - \bar{\lambda})^{\rho_1b} \mathbb{P}^{\rho_1b}, \quad t \geq 0.
\tag{37}
\]
Note by the definition (35) that the events $\varphi(t), \varphi(t+n), \cdots, \varphi(t+(\rho_2-1)n)$ are mutually independent. Then by (36), we obtain that for any $t \geq 1$:

$$\mathbb{P}\left(\{\varphi(t), \varphi(t+n), \cdots, \varphi(t+(\rho_2-1)n)\}\right) = \mathbb{P}(\varphi(t))\mathbb{P}(\varphi(t+n))\cdots\mathbb{P}(\varphi(t+(\rho_2-1)n)) \geq \bar{p}^{\rho_2}. \quad (38)$$

With (37), we have that

$$e(t+\rho \tau) = P(S(t+\rho \tau-1))W(t+\rho \tau - 1) \otimes I_m \cdots P(S(t))W(t) \otimes I_m Pe(t). \quad (39)$$

Note that for each $l = 1, \cdots, \rho_2$ and $i \in V$, each of the projection matrices $P_i^{(s)}$, $s \in \{1, \cdots, l_i\}$ appears in the $i$th diagonal entry of $P(S(t+(l-1)n)) \cdots P(S(t+ln-1))$ at least once conditioned on the events $\{\varphi(t), \cdots, \varphi(t+(\rho_2-1)n)\}$. In other words, the sequence of projection matrices $P(S(t+(l-1)n)), \cdots, P(S(t+ln-1))$ belongs to $S_r$ as conditioned on $\{\varphi(t), \cdots, \varphi(t+(\rho_2-1)n)\}$.

In addition, note by the definition (35) that for each $l = 1, \cdots, \rho_2$ and $i \in V$, each of the projection matrices $P_i^{(s)}$, $s \in \{1, \cdots, l_i\}$ appears in the $i$th diagonal entry of $P(S(t+(l-1)n)) \cdots P(S(t+ln-1))$ at least once conditioned on the events $\{\varphi(t), \cdots, \varphi(t+(\rho_2-1)n)\}$. In other words, the sequence of projection matrices $P(S(t+(l-1)n)), \cdots, P(S(t+ln-1))$ belongs to $S_r$ as conditioned on $\{\varphi(t), \cdots, \varphi(t+(\rho_2-1)n)\}$.

With (37), the following inequality holds conditioned on $\{\varphi(t), \varphi(t+n), \cdots, \varphi(t+(\rho_2-1)n)\}$:

$$||P(S(t+\rho \tau-1))W(t+\rho \tau - 1) \otimes I_m \cdots P(S(t))W(t) \otimes I_m P||_M \leq \vartheta < 1.$$  

Hence, from (39) it follows that for any $t \geq 0$:

$$\mathbb{P}(||e(t+\rho \tau)||^2 \leq \vartheta^2 ||e(t)||^2 | \{\varphi(t), \cdots, \varphi(t+(\rho_2-1)n)\}) = 1.$$  

Since the events $\{\varphi(t), \varphi(t+n), \cdots, \varphi(t+(\rho_2-1)n)\}$ are independent, by (37) and (38) we obtain that

$$\mathbb{P}(||e(t+\rho \tau)||^2 \leq \vartheta^2 ||e(t)||^2) \geq (1 - \lambda)p^{\rho_1 b - 1}p^{\rho_1 b} \bar{p}^{\rho_2}. \quad (40)$$

We define a sequence of events $D(l) \triangleq \{||e(t+\rho \tau)||^2 \leq \vartheta^2 ||e(t)||^2\}$ for any $l \geq 0$. Then from (40) it follows that for any $l \geq 0$,

$$\mathbb{P}(D(l)) \geq (1 - \lambda)p^{\rho_1 b - 1}p^{\rho_1 b} \bar{p}^{\rho_2} > 0.$$  

Since the stochasticity in node states only come from the random graph process (9) and the randomized projection selection $\{S^{(l)}\}_{l \geq 0}$, the sequence of events $D(l), l \geq 0$ is also $*$–mixing. Note that $\sum_{l=0}^{\infty} \mathbb{P}(D(l)) = \infty$. This combined with Lemma 7 produces $\mathbb{P} \left( \lim \sup_{l \rightarrow \infty} D(l) \right) = 1$. Hence by the definition (28), the probability that the event $\{||e(t+\rho \tau)||^2 \leq \vartheta^2 ||e(t)||^2\}$ happens for infinitely many times
is 1. Moreover, \( \|e(t + \rho \tau)\|^2 \leq \|e(t)\|^2 \) holds for any \( t \) by (44), \( |P^{(S(t))}|^2 \leq 1 \), and \( |W(t)| \leq 1 \). Therefore, \( \|e(t)\|^2 \) decreases to 0 with probability one, proving the theorem.

The proof for mean-squared convergence rate is similar to that of Theorem 1. By (40) and \( \|e(t+1)\|^2 \leq \|e(t)\|^2 \), we have

\[
E[\|e((l+1)\rho \tau)\|^2] = E[\|e(l\rho \tau)\|^2] + E[\|e((l+1)\rho \tau)\|^2 - \|e(l\rho \tau)\|^2] \\
\leq \vartheta^2 E[\|e(l\rho \tau)\|^2] \mathbb{P}(\mathcal{D}(l) \cap \mathbb{P}(\mathcal{D}(l))) \\
= (1 - (1 - \vartheta^2)\mathbb{P}(\mathcal{D}(l))) E[\|e(l\rho \tau)\|^2] \\
\leq (1 - (1 - \vartheta^2)(1 - \lambda)^\rho \tau - 1) \rho \tau + 1 \mathbb{E}[\|e(0)\|^2] = c_0 \nu^{l+1}.
\]

For any \( t > \rho \tau \), it could be written as \( t = l \rho \tau + q \) with \( l, q \) being positive integers. Then

\[
E[\|e(t)\|^2] \leq E[\|e(l\rho \tau)\|^2] \leq c_0 \nu^l = c_0 \frac{1}{\nu^{\rho \tau}} = c_0 (\nu^{\rho \tau})^{-q} (\nu^{\rho \tau})^l \leq c_0 (\nu^{\rho \tau})^{-\rho \tau + 1} (\nu^{\rho \tau})^l.
\]

Thus, by the definition of \( e(t) \), we obtain the exponential convergence rate of the mean-squared error. \( \square \)

### Appendix D Proof of Theorem 3

#### D.1 Preliminary Lemmas

We introduce a result from [47] Lemma 3.1.1] about the convergence of a linear recursion corrupted by noises, which will be used to establish the almost sure convergence and convergence rate of the iteration (1).

**Lemma 10** Let \( \{F(t)\} \) and \( F \) be \( m \times m \)-matrices. Suppose Assumption 1 holds, \( F \) is a stable matrix, and \( \lim_{t \to \infty} F(t) = F \). If the \( m \)-dimensional vector \( \nu(t) \) and \( \zeta(t) \) satisfy \( \sum_{t=0}^{\infty} \alpha(t)\nu(t) < \infty \) and \( \lim_{t \to \infty} \zeta(t) = 0 \). Then \( \{u(t)\} \) generated by the following recursion with arbitrary initial value \( u(0) \) tends to zero:

\[
u(t + 1) = u(t) + \alpha(t)F(t)u(t) + \alpha(t)(\varepsilon(t) + \zeta(t)).
\]

Denote by \( L(t) \) the Laplacian matrix of the graph \( G(t) \), where \( [L(t)]_{ij} = -1 \) if \( \{i, j\} \in \mathcal{E}(t) \), \( [L]_{ii}(t) = |N_i(t)| \), and \( [L]_{ij}(t) = 0 \), otherwise. Here and thereafter, \( |\cdot| \) stands for the cardinality of a set. Define

\[
z_H \triangleq \begin{pmatrix} z_1^T \mathbf{H}_1, \ldots, z_N^T \mathbf{H}_N \end{pmatrix}^T \in \mathbb{R}^{mN}, \quad \mathbf{H}_d \triangleq \text{diag} \left\{ \mathbf{H}_1^T \mathbf{H}_1, \ldots, \mathbf{H}_N^T \mathbf{H}_N \right\} \in \mathbb{R}^{mN \times mN},
\]

\[
\Gamma(t) \triangleq I_{mN} - hL(t) \otimes I_m - \alpha(t)\mathbf{H}_d,
\]

\[
\Phi(t, t + 1) \triangleq I_{mN} \quad \text{and} \quad \Phi(t_1, t_2) \triangleq \Gamma(t_1) \cdots \Gamma(t_2), \quad \forall t_1 \geq t_2 \geq 0.
\]

Define \( x(t) \triangleq (x_1^T(t), \cdots, x_N^T(t))^T \). Then (41) can be rewritten in the following compact form:

\[
x(t + 1) = (I_{mN} - hL(t) \otimes I_m)x(t) - \alpha(t)(H_d x(t) - z_H)
\] (41)
\[ = \Gamma(t)x(t) + \alpha(t)z_H = \Phi(t,0)x(0) + \sum_{s=0}^{t} \alpha(s)\Phi(t,s+1)z_H. \] (42)

The following lemma shows that the iterate \( \{x(t)\} \) generated by the iteration (9) is almost surely bounded.

**Lemma 11** Suppose the considered random graph process (5) induces a connected \( p \)-persistent graph \( \mathcal{G}_P(p) \). Suppose \( \text{rank}(H) = m \) and Assumption 1 holds. Let the iterate \( \{x(t)\} \) be generated by (9), then \( \{x(t)\} \) is bounded almost surely.

**Proof.** For a given \( 0 < \bar{\lambda} < 1 \), there exists a large enough integer \( \kappa \) possibly depending on \( \bar{\lambda} \) such that (22) holds for any \( t \geq 0 \), all \( A \in \mathcal{F}_0^\infty(I) \) and \( B \in \mathcal{F}_t^\infty(I) \). Suppose there are \( b \) edges in the \( p \)-persistent graph \( \mathcal{G}_P(p) \) denoted as \( \{i_1,j_1\}, \ldots, \{i_b,j_b\} \). Let \( \omega(s_0) \) be defined by (19). Then by (24), we have that \( \mathbb{P}(\omega(s_0)) \geq (1 - \bar{\lambda})^{b-1}p^{b} \) for any \( s_0 \geq 0 \). Then conditioned on the event \( \omega(s_0) \), the union graph \( \bigcup_{k=0}^{kb-1} \mathcal{G}(s_0+k) \) is connected and undirected, and the matrix \( G \mathcal{L}(s_0+k) \), \( k^* \equiv b\kappa \) is a corresponding Laplacian matrix. Hence by (19), Lemma 9 and \( \text{rank}(H) = m \), we conclude that the matrix

\[ F_d(s_0) \triangleq \frac{1}{k^*} \sum_{k=0}^{k^*-1} \mathcal{L}(s_0+k) \otimes I_m + H_d \] (43)

is positive definite. Since the value space of \( \mathcal{L}(t) \) has finite elements, there exists a constant \( \mu_0 > 0 \) such that for any \( s \geq 0 \), the smallest eigenvalue of \( F_d(s_0) \) is greater than \( \mu_0 \).

Note that for each \( t \geq 0 \) and any \( x \in \mathbb{R}^{mN} \):

\[
\min\{h, \alpha(t)\}x^T (\mathcal{L}(t) \otimes I_m + H_d) x \leq x^T (h \mathcal{L}(t) \otimes I_m + \alpha(t)H_d)(t)x
\]

\[
\leq \max\{h, \alpha(t)\}x^T (\mathcal{L}(t) \otimes I_m + H_d)x.
\]

Recall by \( 0 < \alpha(t) \leq h \) that \( \min\{h, \alpha(t)\} = \alpha(t) \) and \( \max\{h, \alpha(t)\} = h \). Thus, for each \( t \geq 0 \), the smallest eigenvalue of \( h\mathcal{L}(t) \otimes I_m + \alpha(t)H_d \) is greater than or equal to \( \alpha(t)\lambda_{\text{min}}(\mathcal{L}(t) \otimes I_m + H_d) \), while the largest eigenvalue is smaller than or equal to \( h\lambda_{\text{max}}(\mathcal{L}(t) \otimes I_m + H_d) \). Then the eigenvalues of \( \Gamma(t) \) can be sorted in an ascending order as \( 1 - h\lambda_{\text{max}}(\mathcal{L}(t) \otimes I_m + H_d) \leq \cdots \leq 1 - \alpha(t)\lambda_{\text{min}}(\mathcal{L}(t) \otimes I_m + H_d) \). Thus, for any \( t \geq 0 \) and small \( h > 0 \), the matrix \( \Gamma(t) \) is a positive semidefinite with \( \| \Gamma(t) \| \leq 1 \). A sufficient selection of \( h \) is \( h \in (0, \frac{1}{N}) \), which guarantees that \( I_{mN} - h\mathcal{L}(t) \otimes I_m \) is a symmetric stochastic matrix.

Note by Assumption 1 that

\[
\frac{\alpha(t-1)}{\alpha(t)} - 1 = \alpha(t-1) \left( \frac{1}{\alpha(t)} - \frac{1}{\alpha(t-1)} \right) = O(\alpha(t)).
\]

We can recursively show that for any \( s = 1, \ldots, k^* \):

\[
\frac{\alpha(t + k^* + s)}{\alpha(t + 2k^*)} - 1 = O(\alpha(t + 2k^*)).
\]

Hence

\[
\alpha(t + k^* + s) - \alpha(t + 2k^*) = O(\alpha^2(t + 2k^*)). \tag{44}
\]
Note by $\alpha(t) \leq h$ and the definition of $\Gamma(t)$ in (42) that

$$\Gamma(t) \leq \mathbf{I}_{mN} - (h\mathbf{L}(t) \otimes \mathbf{I}_m + \alpha(t)\mathbf{H}_d) \leq \mathbf{I}_{mN} - \alpha(t)(\mathbf{L}(t) \otimes \mathbf{I}_m + \mathbf{H}_d).$$

Since $\mathbf{L}(t) \otimes \mathbf{I}_m + \mathbf{H}_d$ is positive semidefinite and $\alpha(t)$ is a decreasing sequence, by using (44), we have that for any $t \geq 0$:

$$\Phi(t + 2k^*, t + 1) \leq (I_{mN} - \alpha(t + 2k^*) (L(t + 2k^*) \otimes I_m + H_d)) \ldots$$

$$(I_{mN} - \alpha(t + k^* + 1) (L(t + k^* + 1) \otimes I_m + H_d)) \Phi(t + k^*, t + 1)$$

$$= \left( I_{mN} - \sum_{s=1}^{k^*} (\alpha(t + k^* + s) (L(t + k^* + s) \otimes I_m + H_d) + o(\alpha(t + 2k^*)) \right) \Phi(t + k^*, t + 1)$$

$$= \left( I_{mN} - \alpha(t + 2k^*) \sum_{s=1}^{k^*} (L(t + k^* + s) \otimes I_m + H_d) \right) \Phi(t + k^*, t + 1)$$

$$- \left( \sum_{s=1}^{k^*} (\alpha(t + k^* + s) - \alpha(t + 2k^*)) (L(t + k^* + s) \otimes I_m + H_d) + o(\alpha(t + 2k^*)) \right) \Phi(t + k^*, t + 1)$$

$$\leq \left( I_{mN} - \alpha(t + 2k^*) \sum_{s=1}^{k^*} (L(t + k^* + s) \otimes I_m + H_d) + o(\alpha(t + 2k^*)) \right) \Phi(t + k^*, t + 1).$$

Then conditioned on $\omega(s_0 + k^*)$, by (43) and $\lambda_{\min}(F_d(s_0 + k^*)) \geq \mu_0$, we obtain that for any $s_0 \geq 0$:

$$\|\Phi(s_0 + 2k^* - 1, s_0)\|$$

$$\leq (\alpha(s_0 + 2k^* - 1) + I_{mN} - k^* \alpha(s_0 + 2k^* - 1) F_d(s_0 + k^*)) \|\Phi(s_0 + k^* - 1, s_0)\|$$

$$\leq (\alpha(s_0 + 2k^* - 1) + 1 - \mu_0 k^* \alpha(s_0 + 2k^* - 1)) \|\Phi(s_0 + k^* - 1, s_0)\|$$

$$\leq (\alpha(s_0 + 2k^* - 1)) + 1 - \mu_0 \sum_{k=0}^{k^*-1} \alpha(s_0 + k)) \|\Phi(s_0 + k^* - 1, s_0)\|. $$

This combined with $\|\Phi(s_0 + 2k^* - 1, s_0)\| \leq 1$ and $P(\omega(s_0 + k^*)) \geq (1 - \bar{\lambda})^{b-1} p^b$ (by (24)) implies that for sufficiently large $s_0$, there exists some positive constant $c_0$ such that

$$E[\|\Phi(s_0 + 2k^* - 1, s_0)\|]$$

$$\leq \left( \alpha(s_0 + 2k^* - 1) + 1 - (1 - \bar{\lambda})^{b-1} p^b \mu_0 \sum_{k=0}^{k^*-1} \alpha(s_0 + k^*) \right) E[\|\Phi(s_0 + k^* - 1, s_0)\|]$$

$$\leq \exp \left( -c_0 \sum_{k=s_0+k^*-1}^{s_0+2k^*} \alpha(k) \right) E[\|\Phi(s_0 + k^* - 1, s_0)\|],$$

where the last inequality holds by $1 - x \leq \exp(-x)$, $\forall x \in (0, 1)$. We can recursively show that for sufficiently large $s_0$ and any positive integer $\rho \geq 1$:

$$E[\|\Phi(s_0 + \rho k^* - 1, s_0)\|] \leq \exp \left( -c_0 \sum_{k=s_0}^{s_0+\rho k^*-1} \alpha(k) \right).$$
For any $t \geq s \geq s_0$, it could be written as $t = slk^* + q$ with $l, q$ being positive integers. Hence by

$$
\| \Phi(t,s) \| \leq \| \Phi(t,s+lk^*) || \Phi(s+lk^*-1,s) \| \leq \| \Phi(s+lk^*-1,s) \|,
$$

we have that

$$
\mathbb{E} [\| \Phi(t,s) \|] \leq \mathbb{E} [\| \Phi(s+lk^*-1,s) \|] \leq \exp \left( -c_0 \sum_{k=s}^{s+lk^*-1} \alpha(k) \right)
$$

Paying attention to that $\| \Phi(s_0 - 1,s) \| \leq \prod_{t=s}^{s_0-1} (1 + h||L(t)|| + \alpha(t)||H_d||)$ and $||L(t)|| \leq N$, from the above inequality, we have that for any $t \geq s_0 \geq s$:

$$
\mathbb{E} [\| \Phi(t,s) \|] \leq \| \Phi(s_0 - 1,s) \| \mathbb{E} [\| \Phi(t,s_0) \|]
$$

$$
\leq \| \Phi(s_0 - 1,s) \| \exp \left( c_0 \sum_{k=s_0+lk^*}^{t} \alpha(k) \right) \exp \left( -c_0 \sum_{k=s_0}^{t} \alpha(k) \right)
$$

$$
\leq \prod_{t=s}^{s_0-1} (1 + hN + \alpha(t)||H_d||) \exp \left( c_0 \sum_{k=s_0+lk^*}^{t} \alpha(k) + c_0 \sum_{k=s}^{s_0-1} \alpha(k) \right) \exp \left( -c_0 \sum_{k=s}^{t} \alpha(k) \right).
$$

Therefore, we conclude that there exists some $c_1 > 0$ such that

$$
\mathbb{E} [\| \Phi(t,s) \|] \leq c_1 \exp \left( -c_0 \sum_{k=s}^{t} \alpha(k) \right), \quad \forall t \geq s \geq 0.
$$

(46)

Based on (42), we obtain that

$$
\| x(t + 1) \| \leq \| \Phi(t,0) \| \| x(0) \| + \sum_{s=0}^{t} \alpha(s) \| \Phi(t,s+1) \| \| z_H \|,
$$

Hence by taking unconditional expectations on both sides of the above inequality, there holds

$$
\mathbb{E} [\| x(t + 1) \|] \leq \mathbb{E} [\| \Phi(t,0) \|] \| x(0) \| + \sum_{s=0}^{t} \alpha(s) \mathbb{E} [\| \Phi(t,s+1) \|] \| z_H \|
$$

$$
\leq c_1 \exp \left( -c_0 \sum_{k=0}^{t} \alpha(k) \right) \| x(0) \| + c_1 \| z_H \| \sum_{s=0}^{t} \exp \left( -c_0 \sum_{k=s+1}^{t} \alpha(k) \right) \alpha(s).
$$

(47)

Since $\alpha(t)$ is a decreasing sequence, there exists $k_1 \geq 1$ such that $c_0 \alpha(s) \leq 1$ for any $s \geq k_1$. Then

$$
c_0 \alpha(s) \leq 2 \left( c_0 \alpha(s) - c_0^2 \alpha(s)^2/2 \right), \quad \forall s \geq k_1.
$$

Now observe that for any $x \in (0,1)$, $x - x^2/2 < 1 - \exp(-x)$. Then we have the following inequalities:

$$
\sum_{s=k_1}^{t} c_0 \exp \left( -c_0 \sum_{k=s+1}^{t} \alpha(k) \right) \alpha(s)
$$

$$
\leq \sum_{s=k_1}^{t} \left( c_0 \alpha(s) - c_0^2 \alpha(s)^2/2 \right) \exp \left( -c_0 \sum_{k=s+1}^{t} \alpha(k) \right)
$$

$$
\leq \sum_{s=k_1}^{t} \left( 1 - \exp(-c_0 \alpha(s)) \right) \exp \left( -c_0 \sum_{k=s+1}^{t} \alpha(k) \right)
$$

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This incorporating with (47) and \( \alpha(s) > 0 \) produces

\[
\mathbb{E}[\|x(t + 1)\|] \leq c_1\|x(0)\| + c_1\|z_H\| \left( \frac{2}{c_0} + \sum_{s=0}^{k_1-1} \alpha(s) \right) \triangleq c_2. \tag{48}
\]

Therefore, we conclude that the sequence \( \{x(t)\} \) is almost surely bounded. \( \square \)

Next, we give a lemma to characterize the convergence properties of the consensus error.

**Lemma 12** Suppose the considered random graph process \( \mathcal{G} \) induces a connected \( p \)-persistent graph \( \mathcal{G}_P(p) \). Suppose \( \text{rank}(H) = m \), and Assumption 7 holds. Let the iterate \( \{x(t)\} \) be generated by the iteration (9). Define \( \bar{x}(t) = \sum_{i=1}^{N} x_i(t)/N \). Then for each \( i \in \mathcal{V} \):

\[
\sum_{t=0}^{\infty} \alpha(t)\|\bar{x}(t) - x_i(t)\| < \infty, \quad \text{a.s.}
\]

**Proof.** Define

\[ \eta(t) \triangleq (D \otimes I_m) x(t) \text{ with } D \triangleq I_N - \frac{1_N 1_N^T}{N}. \tag{49} \]

Then by multiplying both sides of (41) from the left with \( D \otimes I_m \), using \( D^2 = D \) and \( DL(t) = L(t)D \), we obtain that

\[ \eta(t + 1) = D(I_N - hL(t)) \otimes I_m \eta(t) + \alpha(t)D \otimes I_m (z_H - H_d x(t)). \]

Define \( H(t) \triangleq I_N - hL(t) \). Then based on the above recursion, we obtain that

\[ \eta(t + 1) = \prod_{k=0}^{t} DH(k) \otimes I_m \eta(0) + \sum_{k=0}^{t} \alpha(t - k) \prod_{p=t-k+1}^{t} DH(p) D \otimes I_m (z_H - H_d x(t - k)). \tag{50} \]

Note by the definition of \( L(t) \) that \( [H(t)]_{ij} = h \) if \( \{i, j\} \in \mathcal{E}(t) \), \( [H(t)]_{ii} = 1 - h|N_i(t)| \), and \( [H(t)]_{ij} = 0 \), otherwise. Suppose \( h \leq 1/N \), then \( [H(t)]_{ii} \geq 1 - h(N-1) \). Thus, \( H(t) \) is a symmetric and stochastic matrix. By the definition of \( \omega(s_0) \) in (19), we see that \( \bigcup_{k=0}^{b\kappa-1} \mathcal{G}(s_0 + k) \) is connected and \( [H(s_0 + (p-1)\kappa)]_{ij} = h \) for each \( p = 1, \ldots, b \). Then conditioned on the event \( \omega(s_0) \), \( [\Pi_{k=0}^{b\kappa-1} H(s_0 + k)]_{ij} \geq h \left( 1 - h(N-1) \right)^{b\kappa-1} \) for each \( p = 1, \ldots, b \). Hence the graph derived from the matrix \( \Pi_{k=0}^{b\kappa-1} H(s_0 + k) \) is connected, and \( \|\Pi_{k=0}^{b\kappa-1} H(s_0 + k) - \frac{1_N 1_N^T}{N}\| < 1 \). Denote \( \mathcal{C}_k, \kappa_* \triangleq b\kappa \) as the set of all sequences of symmetric stochastic matrices \( M_1, \ldots, M_{\kappa_*} \) with \( \bigcup_{k=1}^{\kappa_*} \mathcal{G}(M_k) \) being connected. Define

\[ \theta_0 \triangleq \sup_{s \in \mathcal{C}_k} \left\| M_{\kappa_*} M_{\kappa_*-1} \cdots M_1 - \frac{1_N 1_N^T}{N} \right\| , \]

where \( \mathcal{S} \) is a sequence of symmetric stochastic matrices \( M_1, \ldots, M_1 \) from \( \mathcal{C}_k \). Then \( \theta_0 < 1 \). Hence

\[ \|\Pi_{k=0}^{b\kappa-1} H(s_0 + k) - \frac{1_N 1_N^T}{N}\| \leq \theta_0 \text{ conditioned on the event } \omega(s_0). \]
Note that
\[\prod_{k=s_0}^{s_0+2k^* -1} \mathbf{D}(k) = \left( \prod_{k=0}^{k^*-1} \mathbf{H}(s_0 + k^* + k) - \frac{1_N 1_N^T}{N} \right) \prod_{k=s_0}^{s_0+k^*-1} \mathbf{D}(k).\]

This combined with \(\|\sum_{k=s_0}^{s_0+2k^*-1} \mathbf{D}(k)\| \leq 1\) and \(\mathcal{P}(\omega(s_0 + k^*)) \geq (1 - \bar{\lambda})^{-b -1} p^b\) implies that
\[
\mathbb{E} \left[ \left\| \sum_{k=s_0}^{s_0+k^*-1} \mathbf{D}(k) \right\| \right] \leq \left( 1 - (1 - \theta_0)(1 - \bar{\lambda})^{-b -1} p^b \right) \mathbb{E} \left[ \left\| \sum_{k=s_0}^{s_0+k^*-1} \mathbf{D}(k) \right\| \right].
\]

We can recursively show that for any positive integer \(\rho \geq 1\):
\[
\mathbb{E} \left[ \left\| \sum_{k=s_0}^{s_0+\rho^l -1} \mathbf{D}(k) \right\| \right] \leq \nu^\rho.
\]

For any given \(t \geq s\), it could be written as \(t - s = lk^* + q\) for some \(l, q\) with \(l \geq 0\) and \(q < k^*\). Therefore,
\[
\mathbb{E} \left[ \left\| \sum_{k=s}^{t} \mathbf{D}(k) \right\| \right] \leq \mathbb{E} \left[ \left\| \sum_{k=s}^{s+lk^*} \mathbf{D}(k) \right\| \right] \left\| \sum_{k=s}^{t} \mathbf{D}(k) \right\| \leq \mathbb{E} \left[ \left\| \sum_{k=s}^{s+lk^* -1} \mathbf{D}(k) \right\| \right]
\]
\[
\leq \nu^l = \nu^{(t-s-q)/k^*} \leq \nu^{-(k^*/k^*)\nu^l(t-s)/k^*} \leq c_3 \nu_{1-s+1}^l.
\]

(51)

for some constant \(c_3 > 0\) and \(\nu_1 \in (0, 1)\). By taking two norms of (50), we obtain that
\[
\|\eta(t+1)\| \leq \sum_{k=0}^{t} \|\mathbf{D}(k)\| \|\eta(0)\| + \sum_{k=0}^{t} \alpha(t-k) (\|\mathbf{z}_H\| + \|\mathbf{H}_d\| + \|\mathbf{x}(t-k)\|) \prod_{p=t-k+1}^{t} \|\mathbf{D}(p)\|.
\]

By taking unconditional expectation on both sides of the above inequality, using (48) and (51), we have
\[
\mathbb{E}[\|\eta(t+1)\|] \leq c_3 \nu_{1}^{t+1}\|\eta(0)\| + c_3 (\|\mathbf{z}_H\| + c_2\|\mathbf{H}_d\|) \sum_{k=0}^{t} \alpha(t-k)\nu_{1}^{k}.
\]

Therefore,
\[
\sum_{t=0}^{\infty} \alpha(t) \mathbb{E}[\|\eta(t)\|] \leq c_3\|\eta(0)\| \sum_{t=0}^{\infty} \alpha(t)\nu_{1}^{t} + c_3 (\|\mathbf{z}_H\| + c_2\|\mathbf{H}_d\|) \sum_{t=0}^{\infty} \alpha(t) \sum_{k=0}^{t-1} \alpha(t-1-k)\nu_{1}^{k}.
\]

(52)

Since \(\alpha(t) \leq \alpha(t-1-k)\), we have that for any positive integer \(T\):
\[
\sum_{t=0}^{K} \alpha(t) \sum_{k=0}^{t-1} \alpha(t-1-k)\nu_{1}^{k} \leq \sum_{t=0}^{K} \sum_{k=0}^{t-1} \alpha(t-1-k)^2\nu_{1}^{k}
\]
\[
= \sum_{t=0}^{K} \sum_{s=0}^{t-1} \alpha(s)^2\nu_{1}^{t-s-1} = \sum_{s=0}^{K-1} \alpha(s)^2 \sum_{t=0}^{K-s} \nu_{1}^{t-1} \leq \sum_{s=0}^{K-1} \alpha(s)^2 \frac{1}{1 - \nu_{1}}.
\]

This combined with the assumption \(\sum_{s=0}^{\infty} \alpha(s)^2 < \infty\) produces
\[
\sum_{t=0}^{\infty} \alpha(t) \sum_{k=0}^{t-1} \alpha(t-1-k)\nu_{1}^{k} < \infty.
\]

(53)

Since \(\{\alpha(t)\}\) is a decreasing sequence, we have
\[
\sum_{t=0}^{\infty} \alpha(t)\nu_{1}^{t} \leq \alpha(0) \sum_{t=0}^{\infty} \nu_{1}^{t} = \frac{\alpha(0)}{1 - \nu_{1}}.
\]

(54)
Then by combining (52), (53), and (54), we obtain that
\[ \sum_{t=0}^{\infty} \alpha(t) \mathbb{E}[\|\eta(t)\|] < \infty. \]
This implies that
\[ \sum_{t=0}^{\infty} \alpha(t) \|\eta(t)\| < \infty, \text{ a.s. .} \]
Hence the lemma follows by the definition (19) that \( \|\eta(t)\| = \sqrt{\sum_{i=1}^{N} \|x_i(t) - \bar{x}(t)\|^2} \). \( \square \)

D.2 Proof of the Theorem

From (11), \( \bar{x}(t) = \mathbf{1}_N^T \otimes I_m \bar{x}(t)/N \), and \( \mathbf{1}_N^T \mathbf{L}(t) = \mathbf{0}_N^T \) it follows that
\[
\bar{x}(t + 1) = \bar{x}(t) - h \mathbf{1}_N^T \mathbf{L}(t) \otimes I_m \bar{x}(t)/N - \alpha(t) \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i (\bar{x}(t) - \mathbf{x}(t)) \equiv \bar{x}(t) - \alpha(t) \sum_{i=1}^{N} \mathbf{H}_i^T (\bar{x}(t) - \mathbf{x}(t)).
\]
Then by recalling that \( \mathbf{x}_{LS} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{z} \), we obtain that
\[
\bar{x}(t + 1) - \mathbf{x}_{LS} = \bar{x}(t) - \mathbf{x}_{LS} - \alpha(t) \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i (\bar{x}(t) - \mathbf{x}_{LS}) - \alpha(t) \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i (\mathbf{x}(t) - \bar{x}(t)).
\]
Define \( \mathbf{u}(t) \triangleq \bar{x}(t) - \mathbf{x}_{LS}^* \), \( \mathbf{F} \triangleq -\frac{1}{N} \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i \), and \( \varepsilon(t) \triangleq -\frac{1}{N} \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i (\mathbf{x}(t) - \bar{x}(t)) \). Then
\[
\mathbf{u}(t + 1) = \mathbf{u}(t) + \alpha(t) \mathbf{F} \mathbf{u}(t) + \alpha(t) \varepsilon(t). \tag{55}
\]
By rank(\( \mathbf{H} \)) = \( m \) it is seen that \( \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{H}_i = \mathbf{H}^T \mathbf{H} \) is positive definite. Hence \( \mathbf{F} \) is a stable matrix. In addition, by setting \( \mathbf{F}(t) \equiv \mathbf{F} \) and \( \zeta(t) = 0 \), we obtain from Lemma[12] that
\[ \sum_{t=0}^{\infty} \alpha(t) \varepsilon(t) < \infty. \]
Therefore, by using Lemma[10] we conclude that \( \lim_{t \to \infty} \mathbf{u}(t) = 0 \). Thus, \( \lim_{t \to \infty} \bar{x}(t) = \mathbf{x}_{LS}^* \), which together with \( \mathbf{x}(t) - \bar{x}(t) \to 0 \) implies (10).

Specially, by \( \alpha(t) = \frac{1}{(t + 1)^{1+\delta_1}} \) with \( \delta_1 \in (0, 0.5] \), Assumption[1] holds. Therefore, the results of Lemma[11], Lemma[12], and Theorem[8] hold. Similarly to (52), we can show that
\[
\sum_{t=0}^{\infty} \alpha(t)(t + 1)^{\delta_2} \mathbb{E}[\|\eta(t)\|] \leq c_3 \|\eta(0)\| \sum_{t=0}^{\infty} \alpha(t)(t + 1)^{\delta_2} \nu_1^t
\]
\[
+ c_3 (\|\mathbf{z}_H\| + c_2 \|\mathbf{H}_A\|) \sum_{t=0}^{\infty} \alpha(t)(t + 1)^{\delta_2} \sum_{k=0}^{t-1} \alpha(t - k) \nu_1^k. \tag{56}
\]
Since \( \alpha(t) = \frac{1}{(t + 1)^{1+\delta_1}} \) with \( \delta_1 \in (0, 1/2] \), by \( \delta_2 \in (0, 2\delta_1) \) we have \( \delta_2 - (1/2 + \delta_1) < 0 \) and
\[
\sum_{t=0}^{\infty} \alpha(t)(t + 1)^{\delta_2} \nu_1^t \leq \sum_{t=0}^{\infty} (t + 1)^{\delta_2 - \delta_1 - 1/2} \nu_1^t \leq \sum_{t=0}^{\infty} \nu_1^t = \frac{1}{1 - \nu_1}. \tag{57}
\]
In addition, note by $\delta_2 - (1/2 + \delta_1) < 0$ and that for any positive integer $K \geq 1$:

$$
\sum_{t=0}^{K} \alpha(t)(t+1)^{\delta_2} \sum_{k=0}^{t-1} \alpha(t-1-k)\mu_1^k = \sum_{t=0}^{K} (t+1)^{\delta_2-\delta_1-1/2} \sum_{k=0}^{t-1} (t-k)^{-\delta_1-1/2} \nu_1^k
$$

$$
\leq \sum_{t=0}^{K} \sum_{k=0}^{t-1} (t-k)^{\delta_2-2\delta_1-1} \nu_1^k \leq \sum_{s=0}^{K-1} s^{\delta_2-2\delta_1-1} \sum_{t=0}^{K-s} \nu_1^t \leq \frac{1}{1-\nu_1} \sum_{s=0}^{K-1} s^{\delta_2-2\delta_1-1} < \infty.
$$

Hence

$$
\sum_{t=0}^{\infty} \alpha(t)(t+1)^{\delta_2} \sum_{k=0}^{t-1} \alpha(t-1-k)\mu_1^k < \infty.
$$

This combined with (56) and (57) produces $\sum_{t=0}^{\infty} \alpha(t)(t+1)^{\delta_2} \mathbb{E}[||\eta(t)||] < \infty$, which implies that

$$
\sum_{t=0}^{\infty} \alpha(t)(t+1)^{\delta_2} ||\eta(t)|| < \infty, \quad \text{a.s.} . \tag{58}
$$

By recalling that $||\eta(t)|| = \sqrt{\sum_{i=1}^{N} ||x_i(t) - \bar{x}(t)||^2}$ and $\varepsilon(t) = -\frac{1}{N} \sum_{i=1}^{N} H_i^T H_i (x_i(t) - \bar{x}(t))$, we conclude from (58) that

$$
\sum_{t=0}^{\infty} \alpha(t)(t+1)^{\delta_2} \varepsilon(t) < \infty, \quad \text{a.s.} . \tag{59}
$$

By multiplying both sides of (59) with $(t+2)^{\delta_2}$, using the definitions of $u(t), F, \nu(t)$, we obtain that

$$
(t+2)^{\delta_2} u(t+1) = \left(1 + \frac{\delta_2}{t+1}\right)^{\delta_2} \left((I_m + \alpha(t)F)(t+1)^{\delta_2} u(t) + \alpha(t)(t+1)^{\delta_2} \varepsilon(t)\right).
$$

Define $z(t) \triangleq (t+1)^{\delta_2} u(t)$. Then by noting that $\left(1 + \frac{\delta_2}{t+1}\right)^{\delta_2} = 1 + \frac{\delta_2}{t+1} + O(1/(t+1)^2)$ and $\alpha(t) = \frac{1}{(t+1)^{1/2+\delta_1}}$, (55) can be rewritten as

$$
z(t+1) = \left(1 + \frac{\delta_2}{t+1} + O((t+1)^{-2})\right) \left(z(t) + \alpha(t)F z(t) + \alpha(t)(t+1)^{\delta_2} \varepsilon(t)\right)
$$

$$
= z(t) + \alpha(t) \left(F + \frac{\delta_2}{t+1}\right) I_m + O\left(\frac{1}{(t+1)^{3/2-\delta_1}}\right) I_m z(t)
$$

$$
\triangleq F(t) \begin{cases} 
\xi(t) = \alpha(t)(t+1)^{\delta_2} \varepsilon(t) + \alpha(t) \left(\frac{\delta_2}{t+1} + O((t+1)^{-2})\right)(t+1)^{\delta_2} \varepsilon(t).
\end{cases}
$$

From (59) and $\sum_{t=0}^{\infty} \alpha(t) = \infty$ it is easily seen that $(t+1)^{\delta_2} \varepsilon(t) = o(1)$. Hence $\lim_{t \to \infty} \xi(t) = 0$. Since Assumption 1 holds and $\lim_{t \to \infty} F(t) = F$ with $F$ being a stable matrix, we obtain from (59) and Lemma 10 that $\lim_{t \to \infty} z(t) = 0$. Then the result follows by the definition of $z(t)$.

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