The Non-Local Massive Yang-Mills Action
as a
Gauged Sigma Model

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Abstract

We show that the massive Yang–Mills action having as a mass term the non-local operator introduced by Gubarov, Stodolsky, and Zakharov is classically equivalent to a principal gauged sigma model. The non-local mass corresponds to the topological term of the sigma model. The latter is obtained once the degrees of freedom implicitly generated in the non-local action are explicitly implemented as group elements. The non-local action is recovered by integrating out these group elements. In contrast to the usual gauge-fixed treatment, the sigma model point of view provides a safe framework in which calculations are tractable while keeping a full control of gauge-invariance. It shows that the non-local massive Yang–Mills action is naturally associated with the low-energy description of QCD in the Chiral Perturbation Theory approach. Moreover, the sigma model admits solutions called center vortices familiar in different (de)-confinement and chiral symmetry breaking scenarios. This suggests that the non-local operator introduced by Gubarov, Stodolsky, and Zakharov might be sensitive to center vortices configurations.

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1 Introduction

Non-locality appears in a wide range of areas from astrophysics and cosmology to string theories
and non-commutative theories [1, 2, 3, 4, 5, 6, 7]. However, it can frighten the basic principles of
a relativistic theory. The assumption of locality pervades most methods of quantum field theory
and is related to unitarity, causality and renormalizability [8, 9, 12, 10]. It is then far from obvious
that a given non-local theory can make sense physically.

In this letter we are interested in the non-local mass generation scheme developed recently for
Yang–Mills theories in the Landau gauge [11]. The interest in non-local mass generation scenarios
is mainly due to the absence of local gauge-invariant mass term in pure Yang–Mills theories in four
dimensions 1 . Consequently, a gauge-invariant massive Yang-Mills action requires an extension of
the number of fields, like in the Higgs mechanism, or the introduction of non-local quantities. The
Higgs mechanism is not appropriate when there is no room for new particles in the spectrum of the
theory. Therefore, although non-locality is a tricky concept in quantum field theory, it naturally
appears in mass generation without Higgs fields.

A non-local massive Yang–Mills model has emerged recently 2 following the introduction by
Gubarov, Stodolsky, and Zakharov [13] of a new gauge-invariant operator having the dimension
of a mass term. That operator is defined as the minimum along the gauge orbit of the square
potential \( \int A^2 \) and is usually denoted \( \|A\|_{\text{min}}^2 \) (‘\( A^2 \) minimum’). Gauge-invariance is ensured in
principle since the minimum is the same for any two points on the same gauge orbit. But the
price to pay is the loss of locality. A non-local massive action is obtained by adding \( \|A\|_{\text{min}}^2 \) to
the Yang–Mills action. \( \|A\|_{\text{min}}^2 \) is a Morse functional which contains topological information
on the gauge-orbit [14]. These type of functionals obtained from minimizing a given local quantities
along the gauge orbit reveals interesting properties of the structure of the configuration space of
Yang–Mills theories. The analysis of \( \|A\|_{\text{min}}^2 \) for example, requires a careful look at issues related to
Gribov copies [14, 15]. It also emphasises the role of the the center of the Lie group to understand
the structure of the configuration space, the distinction between reducible and irreducible gauge
potentials and the relevance of the stratification of the configuration space [14]. It is not very
surprising that a mass term is closely related to the topology. For example in three dimensions,
the Chern-Simons term provides important topological information. \( \|A\|_{\text{min}}^2 \) is supposed to give
as well information on the topological structures of gauge theories. This has been illustrated by
lattice simulations which seems to indicate that \( \|A\|_{\text{min}}^2 \) is sensitive to phase transitions in compact
QED [13]. It was also argued on the lattice that the expectation value of the square potential is
associated to instantons in the Landau gauge [16]. All these lattice results are not understood in
the continuous formulation of Yang–Mills theory.

It is convenient to think of the non-local massive Yang–Mills action as a (low-energy) effect-
tive action. The possible dangerous effects associated with non-locality [1, 2, 3, 4, 5] are then not
necessarily the result of fundamental violations of first principles. They can follow from the approx-

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1 This is a general feature of Yang–Mills theories in any number of dimensions. However, in three space-time
dimensions the Chern–Simons term provides a topological term.
We recall that [12] a local function is a polynomial in the fields and a finite number of their derivatives all evaluated
at the same space-time point. A local functional is the integral of a local function. This definition of locality is
valid for theory involving only point-wise objects and is very practical for calculations. It is based on properties of
distributions having support on a unique space-time point [8]. From this perspective, theories involving extended
objects are classified as non-local. But a more appropriate definition of locality exists to deal with extended objects
[10].

2 See [11] and references therein.
imative nature of the model \[3\]. In particular, unitarity and causality might be preserved under a certain cut-off and renormalizability is not an issue. Since Yang–Mills theories have a mass gap, an effective massive theory provides an accurate description of the physics at low energy. Another point of view is to treat the massive action as an effective action used for example to renormalize the operator \( \|A\|^{2}_{m}\).

The functional \( \|A\|^{2}_{m}\) is difficult to analyze in a gauge-invariant way because of its unconventional definition. Therefore, the associated non-local massive action is often defined by the trivial massive action \( \int F^{2} + m^{2}A^{2} \) considered as a gauge-fixed action in the Landau gauge. Indeed, the square potential \( \int A^{2} \) reaches its local minima when it is transverse and this corresponds to the Landau gauge condition \( (\partial \cdot A = 0) \) \[15\]. At first sight, the gauge-fixed action in the Landau gauge seems to be easier to handle because of its locality. However, this type of ‘locality’ is a pure gauge-fixed mirage that can lead to erroneous conclusions when the usual techniques of (local) quantum field theory are used without checking their relevance in a non-local context \[3\]. The topology of the configuration space is very important to understand low-energy physics of Yang–Mills theories. Therefore gauge-invariance should be kept manifest as far as possible since gauge-fixing conditions can constrain the value of boundary terms and therefore interfere with the topology. A better understanding of the non-local massive model based on the functional \( \|A\|^{2}_{m}\) calls for a tractable gauge-invariant formulation which takes into account the consequences of the non-local nature of the gauge-invariant theory. Providing such a description is the aim of this letter.

We show that the non-local massive Yang–Mills action associated with the operator introduced by Gubarev, Stodolsky, and Zakharov is classically equivalent to a Yang–Mills principal gauged sigma model also known as the Stueckelberg model \[17, 18\]. In this reformulation, the operator \( \|A\|^{2}_{m}\) corresponds to the so-called topological term of the sigma model. The non-local action contains implicitly new degrees of freedom. Locality emerges once these degrees of freedom are explicitly implemented in the action. They corresponds to the group elements in the sigma model action. Conversely, the non-local action is retrieved from the sigma model once the group elements are integrated out. A similar scheme is familiar in the Schwinger model in 1+1-dimensions: a non-local mass term appears once the fermionic field is integrated out. The sigma model formulation provides a local theory that encodes many effects coming from the non-locality of the original model in a gauge-invariant way. The local and non-local formulation are equivalent only at the classical level. We shall see that the local theory fulfilled the requirement of tree unitarity and is ‘renormalizable in the modern sense’. These two properties are lost in the non-local model. The sigma model formulation is valid under a certain ultra-violet cut-off which is determined so that the loop corrections are irrelevant in comparison to the tree level amplitudes. The classical reformulation helps to understand many features of the non-local theories in a gauge-invariant way. Moreover, it provides a safe framework in which calculation are tractable. We shall see that the non-local massive action although quite odd at first look, happens to be naturally related to two more conventional approaches of low-energy QCD. The sigma model is relevant in the chiral effective theory of low-energy QCD \[22\] which describes gluons and the lightest quarks at low-energy QCD. Interestingly enough, the Stueckelberg action admits a type of solutions which correspond in the usual Yang-Mills theory to configurations called center vortices \[23\]. The latter are familiar

\[3\] For example, the observables of the theory are usually identified with the BRST cohomology. But the gauge-fixed BRST cohomology has very different properties when it is computed in the space of local or non–local functionals \[21\]. This is well illustrated by the proof of the non-physical nature of some global symmetries observed in the gauge-fixed action of local Yang–Mills theory in the Curci-Ferarri gauge \[19\] and the difficulties to define a gauge-independent way to renormalize these non-local operators that take a local expression in particular gauge \[20\].
in different (de)-confinement and chiral symmetry breaking scenarios \cite{25,26,27,28}. We shall see that non-locality also constrains the topology.

To illustrate our method, we shall first treat the Abelian theory. We then generalize to the non-Abelian case. We consider the simple form of the action in the Landau gauge and we obtain a gauge-invariant formulation by computing its gauge-invariant extension \cite{20}. That is, we obtain a gauge-invariant action that reduces to $\int F^2 + m^2 A^2$ in the Landau gauge. Gauge-invariant extension is a general technique that has been recently applied to non-local Yang–Mills theories \cite{20} and effective field theories for massive gravitons \cite{29}.

2 Sigma model reformulation in the Abelian case

The action of QED: $S[A_\mu] = -\frac{1}{2e^2} \int F^2$ (where $F^2 = \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) is invariant under an irreducible Abelian gauge symmetry $\delta_\varepsilon A_\mu = \partial_\mu \varepsilon$ which is not compatible with the trivial mass term $\int A^2$. The gauge-invariant extension of the trivial mass term $\int A^2$ from the Landau gauge is the non-local functional $\int (A^2 + \partial \cdot A \frac{\partial A}{\partial A})$ \cite{20}. This expression reduces to $\int A^2$ in the Landau gauge ($\partial \cdot A = 0$) and corresponds exactly to the non-local mass term of the Schwinger model in 1 + 1-dimensions. The non-local massive action is

$$S[A_\mu] = -\frac{1}{2e^2} \int F^2 - m^2 \left( A^2 + \partial \cdot A \frac{\partial A}{\partial A} \right).$$ \hspace{1cm} (1)

This type of non-locality due to the inverse of differential operators (like $\Box^{-1}$) is called “derived non-locality” because it generically comes from integrating out some fields in a local theory \cite{2}. Therefore, one recovers locality by introducing a new field $\varphi$ in such a way that the new action $S[A_\mu, \varphi]$ reduces to the non-local one after $\varphi$ has been replaced by solving its equation of motion. The local-casting gives here a Stueckelberg model \cite{17}

$$S[A_\mu, \varphi] = -\frac{1}{2e^2} \int F^2 - m^2 (A^2 - 2 A^\mu \partial_\mu \varphi + \partial^\mu \varphi \partial_\mu \varphi)$$

$$= -\frac{1}{2e^2} \int F^2 - m^2 (A_\mu - \partial_\mu \varphi)^2. $$ \hspace{1cm} (2)

The gauge transformation of the scalar field $\varphi$ is a shift symmetry $\delta_\varepsilon \varphi = \varepsilon$. That symmetry has to be spontaneously broken in a quantum theory since it cannot be realized in a unitary way as a symmetry of the vacuum state. The Stueckelberg action can be written explicitly as a principal gauged sigma-model with the circle $S^1$ as target space

$$S[A_\mu, \phi] = -\frac{1}{2e^2} \int F^2 - m^2 D_\mu \phi (D^\mu \phi)^1,$$ \hspace{1cm} (3)

where $D_\mu X = \partial_\mu X - i A_\mu X$ and $\phi = e^{i\varphi}$. The shift symmetry becomes a shift of phase: $\delta_\varepsilon \phi = i \varepsilon \phi$.

An elegant way to understand the connection between the Proca action and the sigma model is to use the theory of constrained system. It is easy to see that the Proca action is a gauge fixed version of the Stueckelberg action. As usual, fixing the gauge replaces first-class constraints by second-class ones. Similarly, an Hamiltonian analysis of the Proca action reveals that it possesses second-class constraints. The Stueckelberg action is obtained by trading these second-class constraints for first class ones. This implies the addition of a compensating field. The gauge transformation of $\phi$ is generated by the first class constraints. Therefore it is often stated that “the Proca action possesses a hidden symmetry”. The replacement of a second class constrained system by an equivalent first class one is a general method nowadays known under the name of conversion. It is used for the quantification of theories with second class constraints See \cite{30} and reference therein.
3 Generalization to Yang–Mills theory

The free Yang-Mills action is

\[ S = \frac{1}{g^2} \mathrm{tr} F^2, \]

where \( g \) is the coupling constant. Our conventions are

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \]

with \( A_{\mu} = A_{\mu}^a T_a \) where \( [T_a, T_b] = f_{ab}^c T_c, \) \( \mathrm{tr}(T_a T_b) = -\frac{1}{2} \delta_{ab}, \) and \( T_a^\dagger = -T_a. \)

In the Abelian case, the Stueckelberg action \( S[A_{\mu}, \varphi] \) can be obtained from the Proca action

\[ \frac{1}{2g^2} \int F^2 - m^2 A^2 \]

by applying the Stuckelberg transformation \( A_{\mu} \rightarrow A_{\mu} - U(x)^{-1} \partial_\mu U(x) \), where the matrix \( U(x) \) is defined in terms of the Stueckelberg variables \( \varphi^a(x) \) as \( U(x) = \exp[\varphi^a(x) T_a] \). By applying the Stueckelberg transformation to the Proca action \( S = \frac{1}{g^2} \mathrm{tr} [F^2 - m^2 A^2] \) we get

\[ S[A_{\mu}, U] = \frac{1}{g^2} \int \mathrm{tr} \left[ F^2 - m^2 (A_{\mu} - U^{-1} \partial_\mu U)^2 \right]. \]  

This action is invariant under the finite gauge transformations \( A_{\mu} \rightarrow V^{-1}(A_{\mu} + \partial_\mu) V, U \rightarrow UV \), where \( V = \exp[\varepsilon^a(x) T_a] \). All the non-polynomial sectors of the action depend on \( U^{-1} \partial_\mu U(\varphi) \). Using the covariant derivative \( D_\mu U = \partial_\mu U - U A_\mu \), the previous action can be rewritten in the more familiar sigma model formulation

\[ S[A, U] = \frac{1}{g^2} \int \mathrm{tr} \left[ F^2 + m^2 D_\mu U(D^\mu U) \right]. \]  

The non-local operator \( ||A||^2_{\text{min}} \) corresponds here to the so-called topological term of the sigma model

\[ ||D U||^2 \equiv - \int \mathrm{tr} \left[ D U (D U)^\dagger \right] = - \int \mathrm{tr} (A_{\mu} - U^{-1} \partial_\mu U)^2. \]

The equations of motion of the sigma model are

\[ D^\mu F_{\nu\mu} + m^2 [A_{\mu} - U^{-1} \partial_\mu U] = 0, \]  

\[ m^2 D^\mu A_{\mu} - U^{-1} \partial_\mu U = 0, \]

where \( D_{\mu} X = \partial_{\mu} X - [A_{\mu}, X] \) is the (adjoint)-covariant derivative. These two equations are not independent as the second can be derived from the first one by acting with the adjoint-covariant derivative.

If the matrix \( U \) is expanded in terms of the field \( \varphi \), one can easily solve the equation of motion of \( \varphi \) in terms of the Yang-Mills field \( A_{\mu} \) order by order in \( \Phi \equiv \{A_{\mu}, \varphi\} \) using

\[ U(\varphi)^{-1} \partial_\mu U(\varphi) = \sum_{n=0}^\infty \frac{(-)^n}{(n+1)!} [\mathrm{ad} \varphi]^n \partial_\mu \varphi, \]

where \( \mathrm{ad} X(Y) = [X, Y] \). We obtain a non-local expression for \( \varphi \) which vanishes in the Landau gauge and starts like

\[ \varphi = \partial \cdot A - \frac{1}{2} \left[ \frac{1}{2} [\partial \cdot A, \frac{\partial \cdot A}{4}] + [A^\mu, \partial_\mu \frac{\partial \cdot A}{4}] \right] + O(A^3) \]

Using the equations and we get the non-local mass term as a functional of \( A_{\mu} \)

\[ \int \mathrm{tr} \left[ A^2 + \partial \cdot A \frac{\partial \cdot A}{4} + A^\mu \left( \partial_\mu \frac{\partial \cdot A}{4} \right) \right] + O(A^4). \]  

The Abelian result is recovered when the structure constants vanish.
4 Discussion and Conclusion

We shall present two points of view which come naturally from the reformulation of the effective action as a sigma model. The sigma model fits naturally in the description of gluons and the lightest quarks for low energy QCD in the spirit of the chiral effective theory. It can also be viewed as an effective action in the center vortices description of the features of low-energy QCD. We shall also discuss the shortcoming of our methods. This is mainly related to the existence of Gribov copies. We shall see that non-locality constrains the topology.

4.1 The Perturbative Chiral Effective Action Approach.

The sigma model approach shows that the massive Yang–Mills action associated with the non-local operator $\|A\|^2_{\text{min}}$ is essentially the same as the model introduced as an alternative to the Higgs model by Delbourgo and Thompson [32]. However, it has been found not to be satisfactory as a fundamental theory because of problems with unitarity [33]. This explains why we considered the non-local massive action as an effective action for low-energy QCD. This point of view makes sense as unitarity problem can be disregarded once an appropriate cut-off is considered. In the spirit of effective action [31] it is interesting to look at all the possible local terms that are consistent with the symmetries of the action. Indeed, they might be used as corrections to the action. For the principal gauged sigma model this question has been carefully analyzed by Henneaux and Wilch [18]: in spite of the usual curvature terms coming from pure Yang-Mills theory there are also ‘winding number terms’ that depend on the group elements and cannot be eliminated when the topology is non-trivial.

The Stueckelberg fields $\varphi^a$, play the role of unphysical Goldstone-boson fields since they decouple in the so-called unitary gauge ($\varphi = 0$ $\iff$ $U = I$). The unitary gauge is equivalent to the Landau gauge thanks to equation (5). If we are not in the Landau gauge (which corresponds here to the unitary gauge $\varphi \neq 0$), equation (9) shows clearly that they are an infinite number of non-renormalizable vertices involving $(ad \varphi)^n \partial_\mu \varphi$. However, if we were working only in the Landau gauge, we would have the impression that the action is power counting renormalizable. The gauge invariant analysis shows that renormalizability in the Landau gauge is a gauge artifact. However, although the sigma model is not power counting renormalizable in four spacetime dimensions it is renormalizable in the modern sense of Gomis and Weinberg [31] as shown in [18]. That is, all the infinities can be regulated by counterterms respecting the symmetries of the original action. But we might need an infinite number of them. Renormalizability in the modern sense is specially well appropriate in the analysis of effective actions [31].

It is interesting to write equation (5) at first order in the fields $\Phi \in \{ A_\mu, \varphi \}$

\[ \partial \cdot A = \Box \varphi + O(\Phi^2) \implies \partial_\mu \varphi^a = \partial_\mu \frac{\partial \cdot A^a}{\Box} + O(\Phi^2). \]  

It follows that the asymptotic field associated to the longitudinal mode of the $A_\mu$ is just equivalent to $\partial_\mu \varphi$. Therefore, the latter can validly replace the former when computing $S$-matrix elements at tree level on the mass-shell in the limit in which the energies are all much greater than the vector boson mass. This is a direct application of the Equivalence Theorem [34].

The principal gauged sigma model is the starting block of the chiral perturbative theory of QCD. The latter provides an effective action of gluons and lightest hadronics fields. It describes the chiral symmetry properties of QCD in the light of effective field theory. The central phenomena is here the spontaneous chiral symmetry breaking which admits the quark condensate as an order
parameter. The effective theory breaks down at the scale $\Lambda$ such that the loop correction becomes relevant. The naive dimensional analysis (NDA) gives $\Lambda \sim 4\pi f$ where $f = \frac{m_g}{g}$. In the spirit of large $N$-limit calculation, the cut-off can be improved to $\Lambda_N \sim \frac{4\pi f}{\sqrt{N}}$ by taking into account the number of colors ($N$) in coefficients of loop corrections [22]. The cut-off is chosen so that loop corrections are negligible compare to tree-level amplitudes. Therefore, it also protects the model from unitarity violation since the latter are only seen at loop-levels. It is amusing that from a non-local action which involves only Yang–Mills field we end up with an action that aims to describe hadronic matter as well.

4.2 The center vortices point of view.

The equations of motion of the Stuckelberg action admit non-trivial topological solutions as was emphasized particularly by Cornwall [23]. Their non-trivial topological features can already be seen by noticing that the second equation of motion is just the Landau background gauge condition, which is well known not to have a unique solution when physical boundary conditions are considered. These Gribov ambiguities are a manifestation of the non-trivial topology of the configuration space of non-Abelian gauge theories and also occur in QED when the gauge group is compact and spacetime has a finite volume. One can see that static solitonic excitation are possible by using a familiar rescaling argument in the static action. After a rescaling $x \rightarrow \lambda x$, the dynamical term $\int dx^3 F^2$ scales like $\lambda$ whereas the topological term $\int dx^3 \| D_\mu \|^2$ scales like $1/\lambda$. Therefore the sum of the two terms always has a minimum with $\lambda \sim m$ [24].

An important type of solutions of the equations of motion of the Stueckelberg action are the so-called center vortices [23, 24]. They are localized gauge field configurations which carry flux concentrated on a closed hypersurfaces of co-dimension two (sheet in 4D, loops in 3D, point in 2D) and many of their properties are encoded in the topological features of the hypersurfaces like the intersection number and the linking number [27, 28, 24]. Center vortices have the nice property of being described and analysed in a gauge-invariant way [24, 27]. Center vortices have attracted a lot of attention following famous lattice simulations which indicate that their removal from the grand ensemble destroys confinement and chiral symmetry breaking and eliminates all the topological non-trivial field configurations [25, 26]. However, the analytic confirmation or refutation of these scenarios is still missing in the continuous formulation of QCD.

$\| A \|^2_{\text{min}}$ was first introduced to study topological structures in gauge theories. This was illustrated by a lattice simulation which indicates that the expectation value of $\| A \|^2_{\text{min}}$ is sensitive to the phase transition of compact QED [13]. It was also argued on the lattice that the expectation value of the square potential is associated to instantons in the Landau gauge [16].

The expectation value of $\| A \|^2_{\text{min}}$ can also be formulated in terms of a principal gauged sigma model. Indeed, the expectation value of an operator $\mathcal{O}$ in a model defined by an action $S(\Phi)$ is defined by $\langle \mathcal{O} \rangle = \int d\Phi \mathcal{O}(\Phi) \exp \left[ S(\Phi) \right]$. It is useful to rewrite it in the following way

$$\langle \mathcal{O} \rangle = \frac{\partial}{\partial \lambda} \int d\Phi \exp \left[ S(\Phi) + \lambda \mathcal{O}(\Phi) \right] \bigg|_{\lambda=0}$$

(13)

where the functional $S(\Phi) + \lambda \mathcal{O}(\Phi)$ is called the effective action for the operator $\mathcal{O}(\Phi)$. This is a very practical concept. For example, the correlation functions $\langle \phi(y_1) \cdots \phi(y_r) \mathcal{O}(z) \rangle$ are just the first order $\lambda$-coefficients in the usual Schwinger functions for the action $S(\Phi) + \lambda \mathcal{O}(\Phi)$. The effective action for the operator $\| A \|^2_{\text{min}}$ is the non-local massive Yang-Mills action $F^2 + \lambda \| A \|^2_{\text{min}}$, which can be reformulated as a principal gauged sigma model of equations (14) and (5). In view of the relation
between the sigma model and center vortices [23], it is natural to ask if \( \|A\|_{min}^2 \) is sensitive to center vortex configurations. It would be possible to study this possibility with a lattice simulation.

### 4.3 Local vs non-local formulation.

The non-local massive Yang–Mills theory having as a mass term the minimum of the square potential along the gauge potential is equivalent to a gauged-principal sigma model in the following sense. The non-local theory is obtained from the sigma model by integration out the group elements present in the sigma model action. In the other direction, the sigma model emerges once the degrees of freedom implicitly generated by non-locality are explicitly included in the action.

Although the local and non-local formulation are equivalent in the sense explain above, they do have important differences. For example, the Stueckelberg model satisfies the tree unitarity conditions [35]4 and is renormalizable in the modern sense of Gomis and Weinberg [31]5 as shown in [15]. This two properties disappear in the non-local theories [33, 20]. This can be understood from the fact that the transformation from one model to the other is highly non-local whereas tree unitarity and renormalizability in the modern sense are mostly valid for (perturbatively) local theories. Quantum correction can be seen as part of the measure of the path-integral. The latter is known to be invariant under field redefinitions. However, when non-locality is introduced, the physics can be altered. This is well illustrated by the re-investigation of the Equivalence Theorem by Tuytin [36]. It is shown there that two field theories related by a field redefinition can have different physical content even when the field redefinitions are local or perturbatively local.

### 4.4 The gauged-principal sigma model as a collective description of the local minima of the square-potential.

The sigma model that we present in this letter is the gauge-invariant extension of the trivial massive YM theory from the Landau gauge. Therefore its equivalence to the gauge-invariant definition of the \( \|A\|_{min}^2 \)-massive YM action (which requires the computation of the absolute minimum along the gauge orbit) relies on the ability of the Landau gauge to detect the absolute minimum. However, the Landau gauge condition does not single out a unique value of the potential \( A \), as it processes Gribov copies. That is, there exists different gauge fields related by a gauge transformation but satisfying all the Landau gauge condition. The Landau condition characterizes local minima of the square potential along the gauge orbit. The absolute minimum among all possible Gribov copies is never selected. Since the sigma model is obtained from the Landau gauge description, it inherits the Landau gauge ambiguities: the sigma model encodes only information associated with local minima of the square potential and does not identify the global minimum. Still it is the gauge-invariant formulation of the ‘Landau gauge dynamical mass generation’ as discuss in the literature. Indeed, the latter is only defines by the Landau gauge-fixed action and therefore suffer as well of the Gribov ambiguities. The difficulties of identifying the absolute minima of the square potential is a common feature of all the methods used to evaluate \( \|A\|_{min}^2 \). The problem of Gribov copies in the definition of \( \|A\|_{min}^2 \) is carefully analyzed in [15] to which we refer for more details. The ambiguities in the definition of \( \|A\|_{min}^2 \) are not only a technical problem. They are related to miscellaneous properties of the configuration space of Yang–Mills theory. They depend on particular features

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4 Tree unitary: The \( n \)-particle \( S \)-matrix elements do not grow more rapidly than \( E^{4-N} \) in the limit \( E \to \infty \). See [35] for more details.

5 That is, all the infinities can be eliminate by using local counter-terms that respects the symmetries of the original action, however, one might needs an infinite number of them.
of the gauge potential that we start with. In particular, they are unavoidable when we deal with reducible gauge potential \[14\].

We would like as well to point that topological consideration are crucial to make sense of the non-local mass term \([11, 11]\). Taking the inverse of a differential operator makes sense only if the latter has no non-trivial zero modes. The possible boundary conditions are therefore constrained. For consistency the set of gauge transformations has to be limited to those that preserve these boundary conditions. In the Hamiltonian analysis of the operator \( \|A\|^2_{\text{min}} \), restrictions are imposed in the Landau gauge to avoid Gribov copies \[15\].

We do not think that a model that determines the exact absolute minimum can be defined in a tractable way. When the connection field \( A_\mu \) is reducible, the spectrum of absolute minima is degenerated \[14\]. In this respect, the sigma model reformulation is a fair alternative. It is local, manifestly gauge-invariant, renormalizable in the modern sense, and fulfills the tree unitarity condition. It provides a gauge-invariant description in which calculations are possible while the effects of non-locality are taken into account through the additional fields. The gauged principal sigma model is also interesting in its own without any link with the \( \|A\|^2_{\text{min}} \) operator. At low energy the gauged principal sigma model can be used as an effective action for gluons and lightest hadrons in the spirit of perturbative chiral effective theory. In the spirit of the center vortices pictures of low energy QCD an effective action relevant to understand confinement and chiral symmetry breaking. In view of these features of the gauged principal sigma model, it would be interesting to study the sensitivity of the expectation value of \( \|A\|^2_{\text{min}} \) to center vortices configurations.

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