Lie-algebraic approach to the theory of polynomial solutions.

II. Differential equations in one real and one Grassmann variables
and 2x2 matrix differential equations

A. Turbiner

Theoretische Physik, ETH-Honggerberg, CH-8093 Zurich, Switzerland
and Centre de Physique Theorique, Marseille Cedex 9, F-13288, France
(submitted to Comm.Math.Phys.)

ABSTRACT

A classification theorem for linear differential equations in two variables (one real and one Grassmann) having polynomial solutions (the generalized Bochner problem) is given. The main result is based on the consideration of the eigenvalue problem for a polynomial element of the universal enveloping algebra of the algebra \( osp(2, 2) \) in the ”projectivized” representation (in differential operators of the first order) possessing an invariant subspace. A classification of 2 x 2 matrix differential equations in one real variable possessing polynomial solutions is described. Connection to the recently-discovered quasi-exactly-solvable problems is discussed.

\textsuperscript{1}This work was supported in part by the Swiss National Science Foundation
\textsuperscript{2}On leave of absence from: Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: turbir@cernvm or turbir@vxcern.cern.ch
Take the eigenvalue problem

\[ T \varphi = \epsilon \varphi \]  

(0)

where \( T \) is a linear differential operator of one real \( x \in \mathbb{R} \) and one Grassmann \( \theta \) variables and \( \epsilon \) is the spectral parameter.

**Definition.** Let us give the name *generalized Bochner problem* to the problem of classification of the differential equations (0) having \((2n + 1)\) eigenfunctions in the form of polynomials in \( x, \theta \) of a degree not higher than \( n \).

In Ref. [1] a general method has been formulated for generating eigenvalue problems for linear differential operators, linear matrix differential operators and linear finite-difference operators in one and several variables possessing polynomial solutions. The method is based on considering the eigenvalue problem for the representation of a polynomial element of the universal enveloping algebra of the Lie algebra in a finite-dimensional, 'projectivized' representation of this Lie algebra [1].

In a previous paper [2] it has been proven that in this approach consideration of the algebras \( sl_2(\mathbb{R}) \) and \( sl_2(\mathbb{R})_q \) in projectivized representations provides both necessary and sufficient conditions for existence of polynomial solutions in ordinary linear finite-order differential equations and in a certain class of finite-difference equations in one variable, respectively. Particularly, it manifested the classification theorems, which imply the solution of the Bochner problem (1929) posed for ordinary differential equations. In the present paper a similar classification theorem is given for finite-order linear differential equations in two variables: one real and one Grassmann, in connection to the algebra \( osp(2, 2) \). Also presented is a consideration of \( 2 \times 2 \) matrix differential equations in one real variable, which is closely connected to the previous problem of one real and one Grassmann variables.
1 Generalities

Define the following space of polynomials in $x, \theta$

$$\mathcal{P}_{N,M} = \langle x^0, x^1, \ldots, x^N, x^0\theta, x^1\theta, \ldots, x^M\theta \rangle$$

where $N, M$ are non-negative integers, $x \in \mathbb{R}$ and $\theta$ is Grassmann (anticommuting) variable.

The projectivized representation of the algebra $osp(2,2)$ is given as follows.

The algebra $osp(2,2)$ is characterized by four bosonic generators $T^\pm, 0, J$ and four fermionic generators $Q_{1,2}, \overline{Q}_{1,2}$ and given by the commutation and anti-commutation relations

$$[T^0, T^\pm] = \pm T^\pm , \quad [T^+, T^-] = -2T^0 , \quad [J, T^\alpha] = 0 , \quad \alpha = +, -, 0$$

$$\{Q_1, \overline{Q}_2\} = -T^- , \quad \{Q_2, \overline{Q}_1\} = T^+ ,$$

$$\frac{1}{2}(\overline{Q}_1, Q_1 + \overline{Q}_2, Q_2) = -J , \quad \frac{1}{2}(\overline{Q}_1, Q_1 - \overline{Q}_2, Q_2) = T^0 ,$$

$$[Q_1, T^+] = Q_2 , \quad [Q_2, T^+] = 0 , \quad [Q_1, T^-] = 0 , \quad [Q_2, T^-] = -Q_1 ,$$

$$[\overline{Q}_1, T^+] = 0 , \quad [\overline{Q}_2, T^+] = \overline{Q}_1 , \quad [\overline{Q}_1, T^-] = \overline{Q}_2 , \quad [\overline{Q}_2, T^-] = 0 ,$$

$$[Q_{1,2}, T^0] = \pm\frac{1}{2}Q_{1,2} , \quad [\overline{Q}_{1,2}, T^0] = \mp\frac{1}{2}\overline{Q}_{1,2}$$

$$[Q_{1,2}, J] = \pm\frac{1}{2}Q_{1,2} , \quad [\overline{Q}_{1,2}, J] = \mp\frac{1}{2}\overline{Q}_{1,2}$$

This algebra possesses the projectivized representation $\exists$

$$T^+ = x^2 \partial_x - nx + x\theta \partial_\theta ,$$

$$T^0 = x \partial_x - \frac{n}{2} + \frac{1}{2} \theta \partial_\theta ,$$

$$T^- = \partial_x .$$
\[ J = -\frac{n}{2} - \frac{1}{2} \theta \partial_\theta \]

for bosonic (even) generators and

\[
Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \partial_\theta \\ x \partial_\theta \end{bmatrix}, \quad \overline{Q} = \begin{bmatrix} \overline{Q}_1 \\ \overline{Q}_2 \end{bmatrix} = \begin{bmatrix} x \theta \partial_x - n \theta \\ -\theta \partial_x \end{bmatrix}, \quad (4)
\]

for fermionic (odd) generators, where \( x \) is a real variable and \( \theta \) is a Grassmann variable. Inspection of the generators shows that if \( n \) is a non-negative integer, the representation Eqs.(3),(4) is finite-dimensional representation of dimension \( (2n+1) \). The polynomial space \( P_{n,n-1} \) describes the corresponding invariant sub-space.

**Definition.** Let us name a linear differential operator of the \( k \)-th order, \( T_k(x, \theta) \), quasi-exactly-solvable if it preserves the space \( P_{n,n-1} \). Correspondingly, the operator \( E_k(x, \theta) \in T_k(x, \theta) \), which preserves the infinite flag \( P_{0,0} \subset P_{1,0} \subset P_{2,1} \subset \ldots \subset P_{n,n-1} \subset \ldots \) of spaces of all polynomials, is named exactly-solvable.

**LEMMA 1.** Take the space \( P_{n,n-1} \).

(i) Suppose \( n > (k-1) \). Any quasi-exactly-solvable operator of \( k \)-th order \( T_k(x, \theta) \), can be represented by a \( k \)-th degree polynomial of the operators (3),(4). If \( n \leq (k-1) \), the part of the quasi-exactly-solvable operator \( T_k(x, \theta) \) containing derivatives in \( x \) up to order \( n \) can be represented by an \( n \)-th degree polynomial in the generators (3), (4).

(ii) Inversely, any polynomial in (3),(4) is quasi-exactly solvable operator.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators \( E_k \subset T_k(x, \theta) \).

**Comment 1.** If we define the universal enveloping algebra \( U_g \) of a Lie algebra \( g \) as the algebra of all polynomials over the generators, then the meaning of the Lemma is the following: \( T_k(x, \theta) \) at \( k < n + 1 \) is simply an
element of the universal enveloping algebra $U_{osp(2,2)}$ of the algebra $osp(2,2)$ in the representation (3),(4). If $k \geq n + 1$, then $T_k(x, \theta)$ is represented as a polynomial in (3),(4) of degree $n$ plus $B \frac{\partial^{n+1}}{\partial x^{n+1} \partial \theta^{m}}$, where $m = 0, 1$ and $B$ is any linear differential operator of order not higher than $(k - n - 1)$.

**Proof.** The proof is technical and but a straightforward analogue of the proof of the similar lemma for the case of linear differential operators in one variable (see Lemma 1 in [2] and its proof).  

Let us introduce the grading of the bosonic generators (3)

\[ \text{deg}(T^+) = +1, \; \text{deg}(J, T^0) = 0, \; \text{deg}(J^-) = -1 \]

and fermionic generators (4)

\[ \text{deg}(Q_2, \overline{Q}_1) = +\frac{1}{2}, \; \text{deg}(Q_1, \overline{Q}_2) = -\frac{1}{2} \]

Hence the grading of monomials of the generators (3),(4) is equal to

\[ \text{deg}[(T^+)^{n_+}(T^0)^{n_0}J^+(T^-)^{n_-}Q_1^{m_1}Q_2^{m_2}\overline{Q}_1^{\overline{m}_1}\overline{Q}_2^{\overline{m}_2}] = \\
(n_+ - n_-) - (m_1 - m_2 - \overline{m}_1 + \overline{m}_2)/2 \]

The $n$'s can be arbitrary non-negative integers, while the $m$'s are equal either 0 or 1. The notion of grading allows us to classify the operators $T_k(x, \theta)$ in a Lie-algebraic sense.

**LEMMA 2.** A quasi-exactly-solvable operator $T_k(x, \theta) \subset U_{osp(2,2)}$ has no terms of positive grading other than monomials of grading $+1/2$ containing the generator $Q_1$ or $Q_2$, iff it is an exactly-solvable operator.

**THEOREM 1.** Let $n$ is a non-negative integer. Take the eigenvalue problem for a linear differential operator in one real and one Grassmann variables

\[ T_k(x, \theta)\varphi = \varepsilon \varphi \]
where $T_k$ is symmetric in a certain manner. In general, the problem (8) has $(2n + 1)$ linear independent eigenfunctions in the form of polynomials in variable $x, \theta$ of order not higher than $n$, iff $T_k$ is quasi-exactly-solvable. The problem (8) has an infinite sequence of polynomial eigenfunctions, iff the operator is exactly-solvable.

This theorem gives a general classification of differential equations

$$
\sum_{i,j=0}^{i=k,j=1} a_{i,j}(x, \theta) \varphi_{x,\theta}^{(i,j)}(x, \theta) = \varepsilon \varphi(x, \theta),
$$

where the notation $\varphi_{x,\theta}^{(i,j)}$ means the $i-$th order derivative with respect to $x$ and $j-$th order derivative with respect to $\theta$, having at least one polynomial solution in $x, \theta$, thus resolving the generalized Bochner problem. Suppose that $k > 0$, then the coefficient functions $a_{i,j}(x, \theta)$ should have the form

$$
a_{i,0}(x, \theta) = \sum_{p=0}^{k+i} a_{i,0,p} x^p + \theta \sum_{p=0}^{k+i-1} \pi_{i,0,p} x^p
$$

$$
a_{i,1}(x, \theta) = \sum_{p=0}^{k+i-1} a_{i,1,p} x^p + \theta \sum_{p=0}^{k+i-1} \pi_{i,1,p} x^p
$$

The explicit expressions (10) are obtained by substituting (3),(4) into a general, the $k$-th order, polynomial element of the universal enveloping algebra $U_{osp(2,2)}$. Thus the coefficients $a_{i,j,p}$ can be expressed through the coefficients of the $k$-th order polynomial element of universal enveloping algebra $U_{osp(2,2)}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing general, a $k$-th order polynomial element of the universal enveloping algebra $U_{osp(2,2)}$. However, in counting parameters a certain ordering of the generators should be introduced to avoid double counting due to commutation and anti-commutation relations. Also some relations between generators should be taken into account, specific for the given representation (3),(4), like
between quadratic expressions in generators and the double-sided ideals generated by them. Straightforward analysis leads to the following expression for the number of free parameters of a quasi-exactly-solvable operator $T_k(x, \theta)$

$$par(T_k(x, \theta)) = 4k(k + 1) + 1$$
where we denoted the number of free parameter of the operator $T$ through the symbol $\text{par}(T)$ . Note, that for the second-order quasi-exactly solvable operator $\text{par}(T_2) = 25$ \footnote{Recall, that for the case of the second-order differential operator in one real variable, the number of free parameter was equal to 9 \cite{2}.}. For the case of an exactly-solvable operator (an infinite sequence of polynomial solutions of Eq. (9)), the Eq. (10) simplifies and reduces to

$$a_{i,0}(x, \theta) = \sum_{p=0}^{i} a_{i,0,p} x^p + \theta \sum_{p=0}^{i-1} \bar{a}_{i,0,p} x^p$$

$$a_{i,1}(x, \theta) = \sum_{p=0}^{i} a_{i,1,p} x^p + \theta \sum_{p=0}^{i-1} \bar{a}_{i,1,p} x^p$$

(13)

Correspondingly, the number of free parameters reduces to

$$\text{par}(E_k(x, \theta)) = 2k(k + 2) + 1$$

(14)

For the second-order exactly solvable operator $\text{par}(E_2(x, \theta)) = 17$ \footnote{Recall, that for the case of the second-order differential operator in one real variable, the number of free parameter was equal to 6 \cite{2}.}. Hence, Eq.(9) with coefficient functions (13) gives a general form of eigenvalue problem for the operator $T_k$, which can lead to an infinite family of orthogonal polynomials as eigenfunctions. If we put formally all coefficients in (13), $\bar{a}_{i,0,p}$ and $a_{i,1}(x, \theta)$ equal to zero, we reproduce the eigenvalue problem for the differential operators in one real variable, which gives rise to all known families of orthogonal polynomials in one real variable (see \cite{2}).

2 Second-order differential equations in $x, \theta$

Now let us consider in more detail the second order differential equation Eq.(9), which can possess polynomial solutions. As follows from The-
orem 1, the corresponding differential operator $T_2(x, \theta)$ should be quasi-exactly-solvable. Hence, this operator can be expressed in terms of $osp(2,2)$ generators taking into account the relations (11)

$$
T_2 = c_{++} T^+ T^+ + c_{+0} T^+ T^0 + c_{+-} T^+ T^- + c_{0-} T^0 T^- + c_{--} T^- T^- +
$$

$$
c_{++} T^+ J + c_{00} T^0 J + c_{--} T^- J +
$$

$$
c_{++} T^+ Q_1 + c_{+2} T^+ Q_2 + c_{+1} T^+ Q_1 + c_{+2} T^+ Q_2 + c_{01} T^0 Q_1 +
$$

$$
c_{00} T^0 Q_2 + c_{-1} T^- Q_1 + c_{-2} T^- Q_2 +
$$

$$
c_{+} T^+ + c_{0} T^0 + c_{-} T^- + c_{J} Q_1 + c_{2} Q^2 + c_{1} Q_1 + c_{2} Q_2 + c
$$

(15)

where $c_{\alpha\beta}, c_{\alpha}, c$ are parameters. Following Lemma 2, under the conditions

$$
c_{++} = c_{+0} = c_{+} = c_{+2} = c_{+} = c_{+} = c_{+} = 0,
$$

(16)

the operator $T_2(x, \theta)$ in representation (13) becomes exactly-solvable.

Now we proceed to the detailed analysis of the quasi-exactly-solvable operator $T_2(x, \theta)$. Set

$$
c_{++} = 0
$$

(17)

in Eq.(15). The remainder will possess an exceptionally rich structure. The whole situation can be subdivided into three cases

$$
c_{+2} \neq 0, c_{+} = 0 \text{ (case I)}
$$

(18)

$$
c_{+2} = 0, c_{+} \neq 0 \text{ (case II)}
$$

(19)

$$
c_{+2} = 0, c_{+} = 0 \text{ (case III)}
$$

(20)

We emphasize that we keep the parameter $n$ in the representation (3),(4) as a fixed, non-negative integer.
Case I. The conditions (17) and (18) are fulfilled (see Fig. I).

Case I.1.1. If

\[(n + 2)c_{+0} + n c_{+J} + 2c_+ = 0 ,\]

\[c_{+2} = c_T = 0 ,\]

\[(n + 1)c_{0+} + 2c_T = 0 ,\]

\[(21)\]

then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{n+1,n-1}\).

Case I.1.2. If

\[(n + 4 + 2m)c_{+0} + n c_{+J} + 2c_+ = 0 ,\]

\[c_{+2} = c_T = 0 ,\]

\[c_{0+} = c_T = c_{-2} = 0 ,\]

\[(22)\]

at a certain integer \(m \geq 0\), then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{n+2+m,n-1}\).

If \(m\) is non-integer, then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{\infty,n-1}\).

Case I.1.3. If

\[(n + 1)c_{+J} + 2c_+ = 0 ,\]

\[c_{+0} = 0 ,\]

\[c_{+2} = c_T = 0 ,\]

\[c_{0+} = c_T = c_{-2} = 0 ,\]

\[(23)\]

then \(T_2(x, \theta)\) preserves the infinite flag of polynomial spaces the \(P_{n+m,n-1}\), \(m = 0, 1, 2, \ldots\).

Case I.2.1. If

\[(n - 3)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0 ,\]

\[(n - 1)c_{+2} = c_T ,\]
\[(n-1)c_{02} + 2c_{-2} = 0, \quad (24)\]

then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{n,n-2}\).

**Case I.2.2.** If

\[
3c_{+0} - c_{+J} = 0,
\]

\[
(2k + 2n + 4)c_{+0} + 2c_+ = 0,
\]

\[
c_{+\pi} = c_{\pi} = 0,
\]

\[
(2k - n + 3)c_{0\pi} + 2c_{-2} = 0, \quad (25)
\]

at a certain integer \(k \geq 0\), then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{k+2,k}\).

**Case I.2.3.** If

\[
c_{+0} = c_{+J} = c_+ = 0,
\]

\[
c_{+\pi} = c_{\pi} = 0,
\]

\[
c_{0\pi} = c_{-2} = 0, \quad (26)
\]

then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and the infinite flag of the polynomial spaces \(P_{k+2,k}, \ k = 0, 1, 2, \ldots\). Note in general for this case \(c_{-2} \neq 0\).

**Case I.3.1.** If

\[(n - 5 - 2m)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0, \quad (27)\]

at a certain integer \(0 \leq m \leq (n - 3)\), then \(T_2(x, \theta)\) preserves \(P_{n,n-1}\) and \(P_{n,n-3-m}\).

**Case I.3.2.** If

\[
c_{+0} = 0,
\]

\[
(n + 1)c_{+J} + 2c_+ = 0,
\]
\[ c_+ \varpi = c_\Gamma = 0 \, , \]
\[ c_0 \varpi = c_\tau = c_- \varpi = 0 \, , \]
\[ (28) \]
then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the sequence of the polynomial spaces \( P_{n,n-3-m}, m = 0, 1, 2, \ldots, (n-3) \).

**Case I.3.3.** If
\[ (2k + 1 - n)c_+0 + (n + 1)c_+J + 2c_+ = 0 \, , \]
\[ (2m + 5)c_+0 - c_+J = 0 \, , \]
\[ c_+ \varpi = c_\Gamma = 0 \, , \]
\[ c_0 \varpi = c_\tau = c_- \varpi = 0 \, , \]
\[ (29) \]
at certain integers \( k, m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{k+3+m,k} \).

**Case I.3.4.** If
\[ c_+0 = c_+J = c_+ = 0 \, , \]
\[ c_+ \varpi = c_\Gamma = 0 \, , \]
\[ c_0 \varpi = c_\tau = c_- \varpi = 0 \, , \]
\[ (30) \]
then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of polynomial spaces \( P_{k+3+m,k} \), \( k, m = 0, 1, 2, \ldots \) (cf. **Cases I.1.3 and I.2.3**).

**Case II.** The conditions (17) and (19) are fulfilled (see Fig.II).

**Case II.1.1.** If
\[ (n + 1)c_+0 + (n + 1)c_+J + 2c_+ = 0 \, , \]
\[ c_2 = 0 \, , \]
\[ (31) \]
then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n} \).
Case II.1.2. If

\[(n + 3)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0 ,\]

\[(n + 2)c_{01} + 2c_1 = 0 ,\]

\[c_{+1} = c_2 = 0 ,\]

then \(T_2(x, \theta)\) preserves \(\mathcal{P}_{n,n-1}\) and \(\mathcal{P}_{n,n+1}\).

Case II.1.3. If

\[(2k + 5 + n)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0 ,\]

\[c_{01} = c_1 = 0 ,\]

\[c_{+1} = c_2 = 0 ,\]

\[c_{-1} = 0 ,\]

at a certain integer \(k \geq 0\), then \(T_2(x, \theta)\) preserves \(\mathcal{P}_{n,n-1}\) and \(\mathcal{P}_{n,n+2+k}\).

Case II.1.4. If

\[c_{+0} = 0,\]

\[(n + 1)c_{+J} + 2c_+ = 0 ,\]

\[c_{01} = c_1 = 0 ,\]

\[c_{+1} = c_2 = 0 ,\]

\[c_{-1} = 0 ,\]

then \(T_2(x, \theta)\) preserves \(\mathcal{P}_{n,n-1}\) and the infinite flag of polynomial spaces \(\mathcal{P}_{n,n+k}, k = 0, 1, 2, \ldots\) (cf. Cases I.1.3, I.2.3 and I.3.4).

Case II.2.1. If

\[(n - 2)c_{+0} + nc_{+J} + 2c_+ = 0 ,\]

\[c_{+1} = c_2 ,\]

\[(35)\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n-1}$.

**Case II.2.2.** If

\[
\begin{align*}
(n+1)c_0 - c_+ &= 0, \\
3c_0 + c_+J &= 0, \\
nc_0 + 2c_1 &= 0, \\
c_+ &= c_2 = 0,
\end{align*}
\]

then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n}$.

**Case II.2.3.** If

\[
\begin{align*}
(n-2)c_0 + nc_+J + 2c_+ &= 0, \\
(2k+1)c_0 + c_+J &= 0, \\
c_+ &= c_2 = 0, \\
c_0 &= c_1 = c_{-1} = 0,
\end{align*}
\]

at a certain integer $k \geq 0$, then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n+k+1}$.

**Case II.2.4.** If

\[
\begin{align*}
c_0 &= c_+ = c_+ = 0, \\
c_+ &= c_2 = 0, \\
c_0 &= c_1 = c_{-1} = 0,
\end{align*}
\]

then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and the infinite flag of the polynomial spaces $P_{n-1,n+k}, k = 0, 1, 2, \ldots$ (cf. Cases I.1.3, I.2.3, I.3.4 and II.1.4).

**Case II.3.1.** If

\[
\begin{align*}
(n-4)c_0 + nc_+J + 2c_+ &= 0, \\
(n-2)c_0 + 2c_1 &= 0, \\
c_+ &= c_2 = 0,
\end{align*}
\]

13
then $T_2(x, \theta)$ preserves $\mathcal{P}_{n,n-1}$ and $\mathcal{P}_{n-2,n-1}$.

**Case II.4.1.** If

$$
\begin{align*}
(m - 2n)c_{+0} + c_+ &= 0, \\
3c_{+0} + c_{+J} &= 0, \\
(2m + 2 - n)c_{01} + 2c_1 &= 0, \\
c_{+1} &= c_2 = 0,
\end{align*}
$$

(40)

at a certain integer $m \geq 0$, then $T_2(x, \theta)$ preserves $\mathcal{P}_{n,n-1}$ and $\mathcal{P}_{m,m+1}$.

**Case II.4.2.** If

$$
\begin{align*}
(2m - n)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
(2k + 5)c_{+0} + 2c_{+J} &= 0, \\
c_{+1} &= c_2 = 0, \\
c_{01} &= c_1 = c_{-1} = 0
\end{align*}
$$

(41)

at certain integers $k \geq 0$, $m \geq 0$, then $T_2(x, \theta)$ preserves $\mathcal{P}_{n,n-1}$ and $\mathcal{P}_{m,m+2+k}$.

**Case II.4.3.** If

$$
\begin{align*}
c_{+0} &= c_{+J} = c_+ = 0, \\
c_{+1} &= c_2 = 0, \\
c_{01} &= c_1 = c_{-1} = 0
\end{align*}
$$

(42)

then $T_2(x, \theta)$ preserves $\mathcal{P}_{n,n-1}$ and the infinite flag of the polynomial spaces $\mathcal{P}_{m,m+1+k}$, $m, k = 0, 1, 2, \ldots$ (cf. Cases I.1.3, I.2.3, I.3.4, II.1.4 and II.2.4).

**Case III.** The conditions (17) and (20) are fulfilled (see Fig.III).

**Case III.1.1.** If

$$
(2m - n)c_{+0} + nc_{+J} + 2c_+ = 0,
$$
\[ c_{+0} + c_{+J} = 0, \]
\[ (m-n)c_{+1} + c_2 = 0, \tag{43} \]

at a certain integer \( m \geq 0 \), then \( T_2(x, \theta) \) preserves \( \mathcal{P}_{n,n-1} \) and \( \mathcal{P}_{m,m} \).

**Case III.1.2.** If
\[ c_{+0} = c_{+J} = c_+ = 0, \]
\[ c_{+1} = c_2 = 0, \tag{44} \]
then \( T_2(x, \theta) \) preserves \( \mathcal{P}_{n,n-1} \) and the infinite flag of the polynomial spaces \( \mathcal{P}_{m,m}, m = 0, 1, 2, \ldots \) (cf. Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4 and II.4.3).

**Case III.2.1.** If
\[ (2m-n)c_{+0} + nc_{+J} + 2c_+ = 0, \]
\[ c_{+0} - c_{+J} = 0, \]
\[ mc_{+0} - c_T = 0, \tag{45} \]

at a certain integer \( m \geq 0 \), then \( T_2(x, \theta) \) preserves \( \mathcal{P}_{n,n-1} \) and \( \mathcal{P}_{m,m-1} \).

**Case III.2.2.** If
\[ c_{+0} = c_{+J} = c_+ = 0, \]
\[ c_{+0} = c_T = 0 \tag{46} \]
then \( T_2(x, \theta) \) preserves \( \mathcal{P}_{n,n-1} \) and the infinite flag of polynomial spaces \( \mathcal{P}_{m,m-1}, m = 0, 1, 2, \ldots \) (cf. Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4, II.4.3 and III.1.2). This case corresponds to exactly-solvable operators \( E_k \).

In \[4\] it has been shown that under a certain condition some quasi-exactly-solvable operators \( T_2(x) \) in one real variable can preserve two polynomial spaces of different dimensions \( n \) and \( m \). It has been shown that those quasi-exactly-solvable operators \( T_2(x) \) can be represented through the generators
of $sl_2(\mathbb{R})$ in projectivized representation characterized either by the mark $n$ or by the mark $m$. The above analysis shows that the quasi-exactly-solvable operators $T_2(x,\theta)$ in two variables (one real and one Grassmann) possess an extremely rich variety of internal properties. They are characterized by different structures of invariant sub-spaces. However, generically the quasi-exactly-solvable operators $T_2(x,\theta)$ can preserve either one, or two, or infinitely many polynomial spaces. For the latter, those operators become 'exactly-solvable' (see Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4, II.4.3 and III.1.2) giving rise to the eigenvalue problems (8) possessing infinite sequences of polynomial eigenfunctions. In general, for the two latter cases the interpretation of $T_2(x,\theta)$ in term of $osp(2,2)$ generators characterized by different marks does not exist unlike the case of quasi-exactly-solvable operators in one real variable. The only exceptions are given by the Case III.2.1 and Case III.2.2.

3 2 x 2 matrix differential equations in $x$

It is well-known that anti-commuting variables can be represented by matrices. In our case the matrix representation is as follows: substitute $\theta$ and $\partial_\theta$ in the generators (3),(4) by the Pauli matrices $\sigma^+$ and $\sigma^-$, respectively, acting on two-component spinors. In fact, all main notations are preserved like quasi-exactly-solvable and exactly-solvable operator, grading etc.

In the explicit form the fermionic generators (4) in matrix representation are written as follows:

$$Q = \begin{bmatrix} \sigma^- \\ x\sigma^- \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} x\sigma^+ \partial_x - n\sigma^+ \\ -\sigma^- \partial_x \end{bmatrix}. \quad (47)$$

The representation (47) implies that in the spectral problem (8) an eigenfunction $\varphi(x)$ is treated as a two-component spinor

$$\varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}. \quad (48)$$
In the matrix formalism, the polynomial space (1) has the form:

\[ \mathcal{P}_{N,M} = \left\{ x^0, x^1, \ldots, x^M \right\} \]

(49)

where the terms of zero degree in \( \theta \) come in as the lower component and the terms of first degree in \( \theta \) come in as the upper component. The operator \( T_k(x, \theta) \) becomes a 2x2 matrix differential operator \( T_k(x) \) having derivatives in \( x \) up to \( k \)-th order. In order to distinguish the matrix operator in \( x \) from the operator in \( x, \theta \), we will denote the former as \( T_k(x) \). Finally, as a consequence of Theorem 1, we arrive at the eigenvalue problem for a 2x2 matrix quasi-exactly-solvable differential operator \( T_k(x) \), possessing in general \((2n + 1)\) polynomial solutions of the form \( \mathcal{P}_{N,N-1} \). This eigenvalue problem can be written in the form (cf.Eq.(9))

\[ \sum_{i=0}^{k} a_{k,i}(x) \varphi^{(i)}(x) = \varepsilon \varphi(x) , \]

(50)

where the notation \( \varphi^{(i)}_x \) means the \( i \)-th order derivative with respect to \( x \) of each component of the spinor \( \varphi(x) \) (see Eq.(48)). The coefficient functions \( a_{k,i}(x) \) are 2x2 matrices and generically for the \( k \)-th order quasi-exactly-solvable operator their matrix elements are polynomials. Suppose that \( k > 0 \). Then the matrix elements are given by the following expressions

\[ a_{k,i}(x) = \begin{pmatrix} A^{[k+i]}_{k,i} & B^{[k+i-1]}_{k,i} \\ C^{[k+i+1]}_{k,i} & D^{[k+i]}_{k,i} \end{pmatrix} \]

(51)

at \( k > 0 \), where the superscript in square brackets displays the order of the corresponding polynomial.

It is easy to calculate the number of free parameters of a quasi-exactly-solvable operator \( T_k(x) \)

\[ \text{par}(T_k(x)) = 4(k + 1)^2 \]

(52)
For the case of exactly-solvable problems, the matrix elements (53) of the coefficient functions will be modified

\[ a_{k,i}(x) = \begin{pmatrix} A_k^{[i]} & B_k^{[-1]} \\ C_k^{[i+1]} & D_k^{[i]} \end{pmatrix} \]  

(53)

where \( k > 0 \). An infinite family of orthogonal polynomials as eigenfunctions of Eq.(50), if they exist, will occur, if and only if the coefficients functions have the form (53). The number of free parameters of an exactly-solvable operator \( E_k(x) \) and, correspondingly, the maximal number of free parameters of the 2 x 2 matrix orthogonal polynomials in one real variable is equal to

\[ \text{par}(E_k(x)) = 2k(k + 3) + 3 \]  

(54)

(cf.Eq.(14)).

The increase in the number of free parameters for the 2 x 2 matrix operators with respect to the case of the operators in \( x, \theta \) is connected to the occurrence of extra monomials of degree \((k+1)\) in generators of \( osp(2,2) \) (see Eqs.(3),(4),(47)), leading to the \( k \)-th order differential operators in \( x \).

Thus, the above formulas describe the coefficient functions of matrix differential equations (50), which can possess polynomials in \( x \) as solutions, resolving the analogue of the generalized Bochner problem Eq.(0) for the case of 2 x 2 matrix differential equations in one real variable.

Now let us take the quasi-exactly-solvable matrix operator \( T_2(x) \) and try to reduce Eq.(8) to the Schroedinger equation

\[ [-\frac{1}{2} \frac{d^2}{dy^2} + \hat{V}(y)] \Psi(y) = E \Psi(y) \]  

(55)

where \( \hat{V}(y) \) is a two-by-two hermitian matrix, by making a change of variable \( x \rightarrow y \) and "gauge" transformation

\[ \Psi = U \varphi \]  

(56)
where $U$ is an arbitrary two-by-two matrix depending on the variable $y$. In order to get some "reasonable" Schrödinger equation one should fulfill two requirements: (i) the potential $\hat{V}(y)$ must be hermitian and (ii) the eigenfunctions $\Psi(y)$ must belong to a certain Hilbert space.

Unlike the case of quasi-exactly-solvable differential operators in one real variable (see [4]), this problem has no complete solution so far. Therefore it seems instructive to display a particular example [3].

Consider the quasi-exactly-solvable operator

$$T_2 = -2T^0 T^- + 2T^- J - i\beta T^0 Q_1 +$$

$$\alpha T^0 - (2n + 1) T^- - \frac{i\beta}{2} (3n + 1) Q^1 + \frac{i}{2} \alpha \beta Q^2 - i\beta \bar{Q}_1 , \quad (57)$$

where $\alpha$ and $\beta$ are parameters. Upon introducing a new variable $y = x^2$ and after straightforward calculations one finds the following expression for the matrix $U$ in Eq.(56)

$$U = \exp \left( -\frac{\alpha y^2}{4} + i\beta y^2 \sigma_1 \right) \quad (58)$$

and for the potential $\hat{V}$ in Eq.(56)

$$\hat{V}(y) = \frac{1}{8} (\alpha^2 - \beta^2) y^2 + \sigma_2 \left[ -(n + \frac{1}{4}) \beta + \frac{\alpha \beta}{4} y^2 - \frac{\alpha}{4} \tan \frac{\beta y^2}{2} \cos \frac{\beta y^2}{2} \right]$$

$$\sigma_3 \left[ -(n + \frac{1}{4}) \beta + \frac{\alpha \beta}{4} y^2 - \frac{\alpha}{4} \cot \frac{\beta y^2}{2} \sin \frac{\beta y^2}{2} \right] \quad (59)$$

It is easy to see that the potential $\hat{V}$ is hermitian; $(2n + 1)$ eigenfunctions have the form of polynomials multiplied by the exponential factor $U$ and they are obviously normalizable.

At closing, I am very indebted to V.Arnold for numerous useful discussions and also to J.Frohlich and M.Shubin for interest in the subject and valuable suggestions. Also I am very grateful to the Centre de Physique Theorique, where this work was initiated, and to the Institute of Theoretical Physics, ETH-Honggerberg, where this work was completed, for their kind hospitality extended to me.
References

[1] "On polynomial solutions of differential equations"
   A.V.Turbiner, Preprint CPT-91/P.2628; J.Math.Phys.(in print)

[2] "Lie-algebraic approach to the theory of polynomial solutions.
   I. Ordinary differential equations and finite-difference equations in one
   variable"
   A.V.Turbiner, Preprint CPT-91/P.2679 (submitted to
   Comm.Math.Phys.)

[3] "Quantal problems with partial algebraization of the spectrum”
   M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. 120 347-365
   (1989)

[4] "Normalizability properties of one-dimensional quasi-exactly-solvable
   Schroedinger operators ”
   A. Gonzales-Lopez, N. Kamran and P.J. Olver. Preprint of the School
   of Math., Univ. Minnesota (Invited talk at the Session of AMS No.873,
   March 20-21, Springfield, USA)

[5] "The theory of groups and Quantum mechanics”
   H.Weyl, Dover Publications, Inc. (1931)
FIGURE CAPTIONS

Figs. 0–III.

Newton diagrams describing invariant subspaces \( \mathcal{P}_{N,M} \) of the second-order polynomials in the generators of \( osp(2,2) \). The lower line corresponds to the part of the space of zero degree in \( \theta \) and the upper line of first degree in \( \theta \). The letters without brackets indicate the maximal degree of the polynomial in \( x \). The letters in brackets indicate the maximal (or minimal) possible degree, if the degree can be varied. The *thin* line displays schematically the length of polynomial in \( x \) (the number of monomials). The *thick* line shows, that the length of polynomial can not be more (or less) than that size. The *dashed* line means, that the length of polynomial can take any size on this line. If the dashed line is unbounded, it means that the degree of polynomial can be arbitrary up to infinity. Numbering the figures I-III corresponds to the cases, which satisfy the conditions (17),(18) (*Case I*); (17),(19) (*Case II*) and (17),(20) (*Case III*).
Fig. 0. Basic subspace

Fig. I.1. Subspaces for the Case I.1

Fig. I.2. Subspaces for the Case I.2

Fig. I.3. Subspaces for the Case I.3

Fig. II.1. Subspaces for the Case II.1
Fig. II.2. Subspaces for the Case II.2

Fig. II.3. Subspaces for the Case II.3

Fig. II.4. Subspaces for the Case II.4

Fig. III.1. Subspaces for the Case III.1

Fig. III.2. Subspaces for the Case III.2