Abstract: The problem of defining the average kinetic energy of statistical systems is addressed. The conditions of applicability for the formula, relating the average kinetic energy with the mass derivative of the internal energy, are analysed. It is shown that incorrectly using this formula, outside its region of validity, leads to paradoxes. An equation is found for a parametric derivative of the average for an arbitrary operator. A special attention is paid to the mass derivative of the internal energy, for which a general formula is derived, without invoking the adiabatic approximation and taking into account the mass dependence of the potential-energy operator. The results are illustrated by the case of a low-temperature dilute Bose gas.

Kinetic energy of Bose systems and variation of statistical averages

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1. Introduction

There exists a known formula by Landau and Lifshitz [1] (see § 15) connecting the average kinetic energy of a statistical system with the mass derivative of the internal energy. This formula has recently been used in several papers for evaluating the average kinetic energy of low-temperature dilute Bose gas. This gas is known to be well described by the Bogolubov theory [2–5]. However, there is an ambiguity in calculating the average kinetic energy in the dilute-gas approximation [6]. To avoid this ambiguity, several authors have applied the Landau-Lifshitz formula for evaluating the kinetic energy. Unfortunately, one usually forgets the conditions, under which this formula was derived, because of which the formula employed outside its region of applicability, leads to incorrect results.

To be concrete, let us recall this formula [1] together with the conditions of its validity. One considers a statistical system described by a Hamiltonian

\[ \hat{H} = \hat{K} + \hat{W} , \]

which is a sum of the kinetic-energy operator \( \hat{K} \) and of the potential-energy operator \( \hat{W} \). The kinetic-energy operator is assumed to have the standard form of an operator for nonrelativistic particles, when \( \hat{K} \propto 1/m \), so that

\[ \frac{\partial \hat{K}}{\partial m} = -\frac{1}{m} \hat{K}, \]

where \( m \) is the particle mass. The potential-energy operator is supposed to have no dependence on the mass, such that

\[ \frac{\partial \hat{W}}{\partial m} = 0 . \]

The total internal energy of the system is given by the statistical average

\[ E \equiv \langle \hat{H} \rangle \]

of the Hamiltonian (1). Let the mass be varied under the fixed temperature \( T \), volume \( V \), and entropy \( S \). Then one has the average kinetic energy

\[ \langle \hat{K} \rangle = -m \left( \frac{\partial E}{\partial m} \right)_S , \]

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which is the Landau-Lifshitz formula [1]. Here and in what follows all parametric derivatives are taken under fixed
temperature and mass. Hence, fixing the entropy means
the temperature becomes a function
of the given entropy and mass. Therefore, the derivative of the temperature becomes a function
\[ T = T(S, m) \]
of the temperature and mass. Hence, fixing the entropy means
that \( T = 0 \) and, hence, \( S = \text{const.} \). This is why formula
(5), in the general sense, corresponds to an adiabatic approximation, becoming exact solely for those cases, when 
both \( T \) and \( S \) can be made constant.

But even when \( T = 0 \), hence \( S = \text{const.} \), it is necessary to keep in mind other conditions, only under which formula (5) is valid. Going outside the region of applicability of this formula can yield principally wrong results. As an illustration, let us consider the uniform low-temperature dilute Bose gas described by the Bogolubov theory [2–5] (see also the recent review papers [7–9]). The particle interactions are characterized by a scattering length \( a_0 \). We assume that there are no bound states, so that \( a_0 \) does not depend on the mass. At zero temperature and in the first order with respect to the interaction intensity, the internal energy per particle is the ground-state energy \( E/N = \pi^2 a_0^2 / m \). \( T = 0 \), where \( E \equiv N/V \) is the average particle density. It is a trivial task to check (see the original works [2–5], review papers [7–9], or the textbook [10]) that the average potential energy for this case is \( \langle \hat{W} \rangle = E \). At the same time, invoking formula (5), one gets \( \langle \hat{K} \rangle = E \). By definition (4), one has \( E = \langle \hat{K} + \hat{W} \rangle \). Since \( \langle \hat{W} \rangle = \langle \hat{K} \rangle = E \), one comes to the equality \( E = 2E \), implying that \( 1 = 2 \), which is, of course, sense-
less. Such a paradoxical conclusion stems from the fact that Eq. (3), which is a necessary condition for the validity of formula (5), has been forgotten.

In the following Sec. 2, we derive a general equation for the derivative, with respect to any parameter, of a statistical average for an arbitrary operator. In this derivation, the adiabatic approximation is not involved. The obtained general formula is illustrated by considering the derivatives of the internal energy, including the case of the dilute Bose gas. In Sec. 3, we show how the average kinetic energy can be evaluated directly from the Bogolubov theory [2–5].

### 2. Parametric variation of statistical averages

Consider a statistical system characterized by a Hamiltonian \( \hat{H} \). In equilibrium, the average of an operator \( \hat{A} \) is given by the statistical average
\[ \langle \hat{A} \rangle \equiv \text{Tr} \rho \hat{A} , \]
with the Gibbs statistical operator
\[ \hat{\rho} = \frac{e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}} \quad (\beta \equiv \frac{1}{T}) . \]

Throughout the paper, the Heisenberg representation for the operators is kept in mind, and the system of units is used, where the Boltzmann and Planck constants are set to unity, \( \hbar = 1 \).

Let the Hamiltonian depend on a parameter \( \lambda \), such that \( \hat{H} = \hat{H}(\lambda) \), and the considered operator be also dependent on the same parameter, \( \hat{A} = \hat{A}(\lambda) \). Taking the parametric derivative of two traced operators, say \( \hat{A} \) and \( \hat{B} \), one should keep in mind the trace property
\[ \text{Tr} \hat{A} \hat{B} = \text{Tr} \hat{B} \hat{A} = \frac{1}{2} \text{Tr} \left[ \hat{A}, \hat{B} \right]_+ , \]
where the notation for the anticommutator
\[ \left[ \hat{A}, \hat{B} \right]_+ \equiv \hat{A} \hat{B} + \hat{B} \hat{A} \]
is used. That is, the parametric derivative
\[ \frac{\partial}{\partial \lambda} \text{Tr} \hat{A} \hat{B} = \frac{1}{2} \text{Tr} \left( \frac{\partial \hat{A}}{\partial \lambda} , \hat{B} \right)_+ + \left[ \hat{A}, \frac{\partial \hat{B}}{\partial \lambda} \right]_+ \]
has to always be represented in the symmetrized form.

For what follows, we shall also need the notation for the covariance of two operators,
\[ \text{cov} \left( \hat{A}, \hat{B} \right) \equiv \frac{1}{2} \left( \left[ \hat{A}, \hat{B} \right]_+ - \langle \hat{A} \rangle \langle \hat{B} \rangle \right) . \]

This is symmetric,
\[ \text{cov} \left( \hat{A}, \hat{B} \right) = \text{cov} \left( \hat{B}, \hat{A} \right) , \]

and possesses the property of additivity,
\[ \text{cov} \left( \hat{A} + \hat{B}, \hat{C} \right) = \text{cov} \left( \hat{A}, \hat{C} \right) + \text{cov} \left( \hat{B}, \hat{C} \right) . \]

The diagonal covariance defines the operator dispersion
\[ \Delta^2(\hat{A}) \equiv \text{cov} \left( \hat{A}, \hat{A} \right) = \langle \hat{A} \rangle^2 - \langle \hat{A} \rangle^2 . \]

Taking the parametric derivative of an operator average, we shall always keep the temperature and volume fixed. But no restrictions on the entropy will be imposed. Then
\[ \frac{\partial}{\partial \lambda} \langle \hat{A} \rangle = \left( \frac{\partial \hat{A}}{\partial \lambda} \right) - \beta \text{cov} \left( \hat{A}, \frac{\partial \hat{H}}{\partial \lambda} \right) . \]

Of particular importance is the case, when the operator \( \hat{A} \) is self-adjoint, representing an observable quantity, such that
\[ \langle \varphi_k, \hat{A} \varphi_p \rangle = \langle \hat{A} \varphi_k, \varphi_p \rangle . \]
Let also the operator $\hat{A}$ commute with $\hat{H}$, thus representing an integral of motion,

$$[\hat{A}, \hat{H}] = 0.$$  

Under these conditions, the operators $\hat{H}$ and $\hat{A}$ possess a common set of eigenfunctions,

$$\hat{H}\varphi_k = E_k\varphi_k, \quad \hat{A}\varphi_k = A_k\varphi_k.$$  

Owing to the Hamiltonian parametric dependence $\hat{H} = \hat{H}(\lambda)$, the eigenfunctions also depend on this parameter, $\varphi_k = \varphi_k(\lambda)$. For the normalized functions, one has

$$\frac{\partial}{\partial \lambda}(\varphi_k, \varphi_k) = 0, \quad (\varphi_k, \varphi_k) = 1.$$  

It is straightforward to check that

$$\frac{\partial}{\partial \lambda}(\varphi_k, \hat{A}\varphi_k) = \left(\varphi_k, \frac{\partial \hat{A}}{\partial \lambda} \varphi_k\right).$$  

Comparing Eqs. (7) and (8), and recalling that

$$\lim_{T \to 0} \frac{\partial}{\partial \lambda}(\hat{A}) = \frac{\partial}{\partial \lambda} \left(\varphi_0, \hat{A}\varphi_0\right),$$

$$\lim_{T \to 0} \frac{\partial \hat{A}}{\partial \lambda} = \left(\varphi_0, \frac{\partial \hat{A}}{\partial \lambda} \varphi_0\right),$$

with $\varphi_0$ being the ground state, we see that for a self-adjoint integral of motion $\hat{A}$, one finds the limit

$$\lim_{\beta \to \infty} \beta \text{ cov} \left(\hat{A}, \frac{\partial \hat{H}}{\partial \lambda}\right) = 0. \quad (9)$$

In particular, when $\hat{A}$ is $\hat{H}$, then for the ground-state energy

$$E_0 \equiv \langle \varphi_0, \hat{H}\varphi_0 \rangle$$

one gets

$$\frac{\partial E_0}{\partial \lambda} = \left(\varphi_0, \frac{\partial \hat{H}}{\partial \lambda} \varphi_0\right) \quad (T = 0)$$

and the limit

$$\lim_{\beta \to \infty} \beta \text{ cov} \left(\hat{H}, \frac{\partial \hat{H}}{\partial \lambda}\right) = 0.$$  

The first term in the right-hand side of Eq. (10) can be expressed through the derivative

$$\frac{\partial F}{\partial \lambda} = \left(\frac{\partial \hat{H}}{\partial \lambda}\right) \quad (11)$$

of the free energy

$$F = -T \ln \text{Tr} e^{-\beta \hat{H}}.$$  

Hence, Eq. (10) can be rewritten as

$$\frac{\partial F}{\partial \lambda} = \frac{\partial E}{\partial \lambda} - T \frac{\partial S}{\partial \lambda}. \quad (12)$$

Due to the equality $F = E - TS$, one has

$$\frac{\partial E}{\partial \lambda} = \frac{\partial E}{\partial \lambda} - T \frac{\partial S}{\partial \lambda}.$$  

Then another form of Eq. (10) is

$$\lim_{T \to 0} T \frac{\partial S}{\partial \lambda} = \beta \text{ cov} \left(\hat{H}, \frac{\partial \hat{H}}{\partial \lambda}\right).$$  

From the comparison of Eqs. (12) and (13) it follows that

$$\lim_{T \to 0} T \frac{\partial S}{\partial \lambda} = -\beta^2 \text{ cov} \left(\hat{H}, \frac{\partial \hat{H}}{\partial \lambda}\right).$$  

The same expression (14) can be derived from the direct differentiation of the definition for the entropy $S \equiv -\text{Tr} \ln \hat{\rho}$, for which we have

$$\lim_{\beta \to \infty} \beta \text{ cov} \left(\hat{H}, \frac{\partial \hat{H}}{\partial \lambda}\right).$$  

Substituting here

$$\ln \hat{\rho} = \beta (F - \hat{H}),$$

we return to Eq. (14). According to the limit (9), one gets

$$\lim_{T \to 0} T \frac{\partial S}{\partial \lambda} = 0.$$  

Let us emphasize that the second terms in the right-hand sides of Eqs. (10), (12), and (13), generally, are not zero for finite temperatures. It is only in the adiabatic approximation, when one has

$$\left(\frac{\partial E}{\partial \lambda}\right)_S = \left(\frac{\partial F}{\partial \lambda}\right)_S = \left(\frac{\partial \hat{H}}{\partial \lambda}\right)_S \quad \left(\frac{\partial S}{\partial \lambda} = 0\right).$$  

The situation here is the same as is mentioned in the Introduction. At finite temperatures, the entropy is a function $S = S(T, \lambda)$. Hence fixing the entropy requires that the temperature be a function $T = T(S, \lambda)$ of the parameter $\lambda$, so that $(\partial T/\partial \lambda)_S \neq 0$. In general, it is impossible to fix simultaneously both the temperature and entropy. This could be done either in some marginal cases or approximately.
Let us now take the particle mass \( m \) as a parameter \( \lambda \). Then, with the derivative
\[
\frac{\partial \hat{H}}{\partial m} = -\frac{1}{m} \hat{K} + \frac{\partial \hat{W}}{\partial m} ,
\]
from Eq. (10) we find
\[
\frac{\partial E}{\partial m} = -\frac{1}{m} \langle \hat{K} \rangle + \left( \frac{\partial \hat{W}}{\partial m} \right) + T \frac{\partial S}{\partial m} .
\]
This formula is a generalization of the Landau-Lifshitz equation (5). At zero temperature, this becomes
\[
\frac{\partial E}{\partial m} = -\frac{1}{m} \langle \hat{K} \rangle + \left( \frac{\partial \hat{W}}{\partial m} \right) \quad (T = 0) .
\]
The second term in the right-hand side of Eq. (18), generally, is not zero. It is only under the supposition (3), when Eq. (18) reduces to the Landau-Lifshitz formula (5).

An obvious case, when the condition (3) is not valid, is exactly the case of the dilute Bose gas, whose interaction energy is proportional to the interaction intensity
\[
\Phi_0 = 4\pi \frac{a_s}{m} .
\]
The potential of pair interactions for the dilute gas is proportional to \( \Phi_0 \), whether the Fermi contact potential or the Huang effective potential are considered [10]. Therefore for the dilute gas, one has
\[
\frac{\partial \hat{W}}{\partial m} = \left( \frac{\partial \hat{W}}{\partial m} \right) .
\]
Note that relation (19) is valid for many systems, but not solely for the dilute gas. Together with Eq. (2), this gives
\[
\frac{\partial \hat{H}}{\partial m} = -\frac{1}{m} \hat{H} .
\]
From Eq. (14), we find
\[
\frac{\partial S}{\partial m} = -\beta^2 \text{cov} \left( \hat{H}, \frac{\partial \hat{H}}{\partial m} \right) = \frac{\beta^2}{m} \Delta^2(\hat{H}) .
\]
Remembering the definition of the specific heat
\[
C_V \equiv \frac{1}{N} \left( \frac{\partial E}{\partial T} \right)_V = \frac{\Delta^2(\hat{H})}{NT^2} ,
\]
we get
\[
\frac{\partial S}{\partial m} = \frac{1}{m} NC_V .
\]
The latter equation emphasizes again that the adiabatic approximation, in the sense of the independence of the entropy from the mass, can become asymptotically exact solely for \( T \to 0 \), when \( C_V \to 0 \). But for finite temperatures, when the specific heat can be rather large, one cannot neglect the dependence of \( S \) on \( m \), so that \( \partial S/\partial m \neq 0 \).

Finally, from Eq. (17) we find
\[
\frac{\partial E}{\partial m} = \frac{1}{m} (NTC_V - E) .
\]
At zero temperature, we have
\[
\frac{\partial E}{\partial m} = -\frac{E}{m} \quad (T = 0) ,
\]
where the limit
\[
\lim_{T \to 0} T C_V = 0
\]
is taken into account.

It is formula (24), instead of Eq. (5), which is to be used, when no adiabatic approximation is valid and the potential-energy operator depends on the mass according to Eq. (19). At zero temperature, Eq. (24) reduces to Eq. (25). For the low-temperature dilute Bose gas, for which \( E/N = 2\pi \rho a_b/m \), Eq. (25) becomes a simple identity. In this way, taking the mass derivative correctly, without forgetting the dependence on the mass of the potential-energy term, results just in an identity, but by no means this defines the kinetic energy.

The situation diversifies when, in addition to the interaction energy, the potential energy term contains an external potential depending on the particle mass. For instance, the external potential can be a confining potential acting on atoms in a trap. Let the potential-energy operator consist of two terms,
\[
\hat{W} = \hat{V} + \hat{U} ,
\]
of which the first term corresponds to particle interactions, with the mass dependence according to Eq. (19), while the second term describes a power-law trapping potential, proportional to \( m^{n/2} \). For a parabolic trap, \( n = 2 \). Thus, we have
\[
\frac{\partial \hat{V}}{\partial m} = \frac{1}{m} \hat{V} , \quad \frac{\partial \hat{U}}{\partial m} = \frac{n}{2m} \hat{U} .
\]
Consequently, instead of Eq. (19), we get
\[
\frac{\partial \hat{W}}{\partial m} = -\frac{1}{m} \hat{W} + \frac{1}{m} \left( 1 + \frac{n}{2} \right) \hat{U} .
\]
Then relation (17) transforms to
\[
\frac{\partial E}{\partial m} = -\frac{1}{m} E + \frac{1}{m} \left( 1 + \frac{n}{2} \right) \langle \hat{U} \rangle + T \frac{\partial S}{\partial m} .
\]
For a harmonic trap at zero temperature, we obtain
\[
\frac{\partial E}{\partial m} = -\frac{1}{m} E + \frac{2}{m} \langle \hat{U} \rangle \quad (T = 0) .
\]
This again does not allow one to get in a simple way the average kinetic energy.

Thus, in any case, considering whether homogeneous systems or trapped atoms, one must accurately take into account the mass dependence of the potential-energy term. Then no paradoxes appear. While blindly applying Eq. (5)
outside of the region of its validity yields principally wrong results.

It is worth emphasizing that Eq. (25) is an identity, for the
dilute Bose gas at zero temperature, for all known or-
ders of the perturbation theory in powers of the dimension-
less coupling \( \alpha \equiv \rho a_s^2 \ll 1 \). The ground-state energy, as
an expansion with respect to this coupling, is

\[
E = 2\pi\rho a_s \frac{2}{m} \left( 1 + b_1 \alpha^{1/2} + b_2 \alpha + b_3 \alpha \ln \alpha \right).
\]

The coefficients in this expansion, found in [11–14] and
summarized in the review article [15], are

\[
b_1 \approx 4.814, \quad b_2 \approx 71.337, \quad b_3 \approx 19.654.
\]

As is evident, Eq. (25) is an identity for the ground-state
energy of the dilute Bose gas in all orders with respect to
\( \alpha \). Moreover, the above expansion can be extrapolated, by
invoking resummation techniques, to arbitrary values of
\( \alpha \in [0, \infty) \). Consequently, relation (25) seems to be valid,
not merely for the dilute Bose gas in the weak-coupling
limit, but for this gas with arbitrarily strong couplings.

### 3. Kinetic energy of Bose-condensed systems

The direct definition of the kinetic energy per particle, for a
homogeneous Bose-Einstein condensed system, is

\[
\mathcal{K} = \frac{\langle K \rangle}{N} = \frac{1}{N} \sum_{k \neq 0} \frac{k^2}{2m} n_k, \tag{31}
\]

where \( n_k \) is the momentum distribution of atoms. The contribution
of condensed particles is, as is evident, zero because
of the zero momenta of particles in the condensate.
For a dilute Bose gas, the number of noncondensed parti-
cles, with a momentum \( k \), is much smaller than the number
of condensed particles, \( n_k \ll N_0 \approx N \). The potential energy is mainly due to the interaction of the condensed
atoms, so that \( \langle W \rangle \approx 2\pi\rho a_s/m \). But the kinetic energy
is caused by the noncondensed particles, because of which it
must be of higher order with respect to the interaction
intensity \( \Phi_0 = 4\pi a_s/m \), hence of the higher order in the
scattering length \( a_s \). From here, even with no calcula-
tions, it is obvious that the average kinetic energy \( \langle K \rangle \)
cannot be of the same order in \( a_s \) as the ground-state energy
\( E \approx \langle W \rangle \). Recall that incorrectly employing Eq. (5)
one would get \( \langle K \rangle \approx E \approx \langle W \rangle \), which is in an explicit
contradiction with the direct definition (31).

Keeping in mind a low-temperature dilute Bose gas,
satisfying the Bogolubov theory [2–5], we have

\[
n_k = \frac{\omega_k}{2\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) - \frac{1}{2}, \tag{32}
\]

Here the notation

\[
\omega_k = \frac{k^2}{2m} + mc^2
\]

is used; the Bogolubov spectrum

\[
\varepsilon_k = \sqrt{(ck)^2 + \frac{k^2}{2m}}^2,
\]

and the sound velocity is

\[
c = \frac{\sqrt{\pi\rho a_s}}{m}.
\]

The kinetic energy per particle (31) can be represented as

\[
\mathcal{K} = \frac{1}{\rho} \int \frac{k^2}{2m} n_k \frac{dk}{(2\pi)^3}. \tag{33}
\]

For the local interactions \( \Phi(r) = \Phi_0 \delta(r) \), the integrand
in \( \mathcal{K} \) behaves as \( k^4 n_k \propto \Phi_0^2 \) at large \( k \rightarrow \infty \), hence,
the above integral linearly diverges. However, this by no
means implies the failure of the Bogolubov theory, but
simply reflects a defect in the dilute-gas approximation.
The Bogolubov theory [2–5] has been formulated for ar-
bitrary interaction potentials \( \Phi(r) \), but not solely for the
local potentials. For a realistic interaction potential \( \Phi(r) \),
its Fourier transform \( \Phi_k \) steeply diminishes with increas-
ing \( k \rightarrow \infty \). Therefore, the integrand in \( \mathcal{K} \) behaves as
\( k^4 n_k \propto \Phi_0^2 \), which is a rapidly diminishing function of
\( k \rightarrow \infty \). Thus, for realistic interaction potentials, no
infrared divergence arises in calculating \( \mathcal{K} \). Instead of re-
sorting to realistic interaction potentials, which would es-
entially complicate calculations, one can model the fast
decay of \( \Phi_k \) at large \( k \) by the step function

\[
\Phi_k = \Phi_0 \Theta(k_0 - k), \tag{34}
\]

with \( \Theta(k) \) being the unit step function. The boundary
wave vector \( k_0 \) can be chosen as the vector, where the
phonon behaviour of the Bogolubov spectrum \( \varepsilon_k \) changes to the single-particle type dependence. This definition gives \( k_0 = 2mc \).

Our goal here is to evaluate the mean kinetic energy
(33) by its order of magnitude, in order to demonstrate its
nonlinear dependence on \( a_s \), which makes it principally
different from the ground-state energy, whose main term
is linear in \( a_s \). To this end, we resort to the simple form
(34), assuming that, by the order of magnitude, the in-
tegral (33) can be evaluated by cutting off its upper limit
with the boundary wave vector. Let us also consider the
zero-temperature case, when the particle distribution (32)
comes

\[
n_k = \frac{\omega_k - \varepsilon_k}{2\varepsilon_k}, \quad (T = 0) .
\]

Then the integral (33) gives

\[
\mathcal{K} = C \rho_s \sqrt{\rho a_s^3}, \tag{35}
\]

with the constant

\[
C = \frac{64}{5} \left( 3\sqrt{2} - 4 \right) \sqrt{\pi} \approx 5.505 .
\]
As is expected, $K \propto a_s^{5/2}$, making it principally different from $E \propto a_s$.

In this way, the Landau-Lifshitz equation (5) is an adiabatic approximation, valid under conditions (1)–(3), together with the assumption $\partial S/\partial m = 0$. Incorrectly using this relation, outside its region of applicability, leads to wrong paradoxical results. For instance, condition (3) does not hold true for the dilute Bose gas, whose interaction potentials do depend on particle mass.

Generally, a parametric differentiation for the statistical average of an operator is given by formula (7). If the operator is a self-adjoint integral of motion, it satisfies the zero-temperature limit (9).

The parametric differentiation of the internal energy is represented by the general expressions (10), (12), and (13). The mass differentiation of the internal energy is given by formula (17). For the dilute Bose gas, the latter takes the form (24), which at zero temperature reduces to relation (25). In the case of trapped atoms, the mass variation of the internal energy leads to Eq. (29). For the homogeneous Bose gas, the mean kinetic energy per particle can be evaluated as in Eq. (35). The dependence on the scattering length of the kinetic energy $K \propto a_s^{5/2}$ and of the internal energy $E \propto a_s$ is fundamentally different.

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