VACUUM INSTABILITY

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Abstract: Following fresh attempts to resolve the problem of the energy density of the vacuum, we reconsider the case where the cosmological constant is derived from a higher-dimensional version of general relativity, and interpret the gauge-dependence of \( \Lambda \) as a dynamical effect. This leads to a relation between the change in \( \Lambda \) and the line element (action) which is independent of gauge choices and fundamental constants: \( d\Lambda ds^2 = -6 \). This implies that the (classical) vacuum is unstable, with implications for particle production.

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1 Introduction

The cosmological-constant problem has a long history, and while there are many possible resolutions, none has gained widespread acceptance. In classical general relativity, the energy density and pressure of the vacuum obey

$$\rho c^2 = -p = \Lambda c^4 / 8\pi G$$

where $c$ is the speed of light and $G$ is the gravitational constant. The astrophysically-determined value of $\Lambda$, for the present epoch at least, is small. But in quantum theory, the vacuum (or zero-point) energies associated with particle interactions lead to a value of $\Lambda$ which is big. The discrepancy may be large as $10^{120}$. Padmanabhan has recently reviewed this problem, and outlined a resolution wherein the classical value of $\Lambda$ is essentially the statistical one “left over” from numerous stronger interactions described by quantum field theory [1, 2]. Mashhoon and Wesson have recently reconsidered the case where $\Lambda$ in four-dimensional general relativity is derived from a five-dimensional formalism, such as membrane or induced-matter theory, and found that the classical value can depend on the size of the extra coordinate [3, 4]. In what follows, we will present a short analysis that has something in common with both of the aforementioned approaches, and calculate the change in the 4D cosmological ”constant” due to a change in the size of the fifth coordinate. We will work with a specific gauge in order
to get an answer, but the latter will turn out to be independent of choices of gauge and fundamental constants. The equation concerned is $d\Lambda ds^2 = -6$.

This relates the change in the cosmological constant (or energy density of the vacuum) to the change in the interval (or action for a particle of unit mass). It implies that the classical vacuum is unstable. This invites application to a Dirac-like model, where fluctuations in a vacuum field are balanced by the production of massive particles.

The present account is brief and exploratory. But we believe that this approach is worth pursuing, since while it is somewhat phenomenological it has numerous applications, particularly to cosmology.

2 Relations for Vacuum Instability

In this section, we make use of technical results derived from 5D field theory. This in general describes the classical fields associated with the spin-2 graviton (Einstein gravity), the spin-1 photon (Maxwell electromagnetism) and a spin-0 scalaron (Higgs-type mass field). It is the basic extension of 4D general relativity, and is commonly regarded as the low-energy limit of 10, 11 and 26 D (etc.) theories which may lead to a grand unification of all
of the known interactions [5]. There are currently two versions of 5D field theory in vogue, namely membrane theory [6] and induced-matter theory [7]. Both make essential use of a non-compact extra dimension (which we label $x^4 = l$, where the spacetime coordinates are $x^\alpha$ with $\alpha = 0, 123$; we temporarily absorb $c$ and $G$ via a choice of units which renders them unity).

Physically, membrane theory allows gravity to propagate freely (into the “bulk”), whereas other interactions are confined to a singular hypersurface (the “brane”), thus giving insight into the hierarchy problem and the masses of particles. Alternatively, induced-matter (or space-time-matter) theory places no restrictions on the dynamics other than those which follow from solving the geodesic equation, using Campbell’s theorem to go from 5D to 4D and giving an account of matter in terms of pure geometry. Mathematically, the two theories are equivalent: (a) The field equations contain the same information (the non-linear terms associated with the brane are contained as vector components of the complete energy-momentum tensor). (b) Both theories involve an extra force associated with the extra dimension (the discontinuities across the brane balance, and reproduce the acceleration derived from the 5D geodesic). (c) In either approach a massless particle in 5D can be viewed as a massive particle in 4D (the photon is unique, being
confined to the hypersurface $l = 0$). For technical details on these three points, the reader is referred to [8], [9] and [10] respectively. We will use some of the relevant technical results below, but our starting point will be the recent analysis of the gauge-dependence of the cosmological constant $\Lambda$ referred to above [3]. This employs the “canonical” gauge of induced-matter theory, which via a quadratic in $l$ defines a coordinate system analogous to the synchronous one of standard cosmology, and leads to a ready comparison with the usual action and masses of particles. Alternatively, there is the “warp” metric of membrane theory, which via an exponential in $l$ defines a coordinate system similar to that used in deSitter cosmology, and weakens gravity away from the brane and leads to an explanation of why particles have masses less than the Plank value. These two gauges are both valid, but using the former we will obtain a result which is independent of either.

Consider then a 5D line element which contains the 4D one and depends on a constant length ($L$):

$$dS^2 = \frac{l^2}{L^2} ds^2 - dl^2$$

$$ds^2 \equiv g_{\alpha\beta}(x^\gamma, l) dx^\alpha dx^\beta .$$

Here the metric tensor can in principal depend on $x^4 = l$, in which case (1) is still general, because it uses the 5 available degrees of coordinate freedom.
to remove the electromagnetic potentials \( (g_{04} = 0) \) and flatten the scalar potential \( (g_{44} = -1) \), but imposes no further constraints. For (1), numerous solutions are known of the apparently-empty field equations, which in terms of the 5D Ricci tensor are

\[
R_{AB} = 0 \quad (A, B = 0, 123, 4) .
\]  

(3)

Using Campbell’s theorem [11], it can be shown that these equations always contain the equations of 4D general relativity, which in terms of the Einstein tensor and an effective energy-momentum tensor are

\[
G_{\alpha\beta} = 8\pi T_{\alpha\beta} (\alpha, \beta = 0, 123) .
\]  

(4)

In these, the energy density of the vacuum is nowadays frequently taken to be implicit in \( T_{\alpha\beta} \). But if it is taken to be explicit and measured by \( \Lambda \), then in the absence of ordinary matter the field equations in terms of the 4D Ricci tensor are just

\[
R_{\alpha\beta} = \Lambda g_{\alpha\beta} (\alpha, \beta = 0, 123) .
\]  

(5)

These equations for metric (1) identify the length scale in the latter via

\[
\Lambda = 3/L^2 .
\]  

(6)

This and the preceding results are by now well known [7]. However, it was shown recently [3] that the gauge transformation \( l \to (l - l_0) \) in metrics of
type (1) leads to a change in $\Lambda$ of (6) to

$$\Lambda = \frac{3}{L^2} \left( \frac{l}{l-l_0} \right)^2.$$  

(7)

This result is mathematically simple but physically profound. It indicates that the 4D cosmological “constant”, determined by (5), can diverge as one approaches a 5D state ($l \to l_0$) determined by (7). The latter equation was arrived at by tedious algebra, and holds when the 4D part of the metric (1) has the conformally-flat or deSitter form $g_{\alpha\beta}(x^\gamma, l) = f(x^\gamma, l) \eta_{\alpha\beta}$ [3]. This is a special case of the general situation, that in non-compact 5D field theory the form of 4D quantities can change under coordinate transformations that depend on $x^4 = l$. An alternative and instructive way to appreciate this kind of behaviour is as follows:

A corollary of Campbell’s theorem is that any solution of the source-free 5D field equations $R_{\alpha\beta} = 0$ with metric (1) can be written as a solution of the empty 4D field equations $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ with $\Lambda = 3/L^2$. Therefore, since the equations are covariant, the same must hold for any gauge transformation which leaves the form of the metric intact. For (1) with $l \to (l - l_0)$, the fifth part of (1) is unchanged, while the prefactor on the 4D part changes from $l^2/L^2$ to $(l - l_0)^2 / L^2 = (l^2 / L^2) [(l - l_0) / l]^2$. Let us write $\tilde{g}_{\alpha\beta} = [(l - l_0) / l]^2 g_{\alpha\beta}$. Then the field equations hold with $\tilde{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta} \tilde{g}_{\alpha\beta}$.
and $\Lambda = (3/L^2) l^2 / (l - l_0)^2$. This is the same as (7). Put another way: a translation along the fifth dimension necessarily changes the four-dimensional cosmological “constant”.

We now proceed to analyse the instability inherent in (7) by adding a series of mathematical and physical conditions to the problem.

Firstly, let us take derivatives of (7), to obtain
\[
\frac{d\Lambda}{dl} = -\frac{6}{L^2}(l - l_0)^{-3}l_0 dl.
\]
We are mainly interested in the region near $l = l_0$, where the energy density $\Lambda = \Lambda(l)$ is changing rapidly but smoothly (the change in the opposite regime leads to the same relation, but less by a factor of 2). Putting $dl = l - l_0$ for the change in the extra coordinate, we obtain
\[
\frac{d\Lambda}{dl} = -\frac{6l^2}{L^2}.
\] (8)

This is an alternative form of the instability inherent in (7) near to its divergence.

Secondly, let us assume that the instability has a dynamical origin, and that the $l$-path involved is part of a null 5D geodesic as in other work [10]. Then by (1) with $dS^2 = 0$, we have $l = l_0 e^{\pm s/L}$. We take the upper sign as elsewhere [3], which means that the path drifts slowly away from the $l = l_0$ hypersurface. [The constant $L$ is large because the current astrophysical value of $\Lambda$ is small, the two being inversely related by (6). It should be
noted that if we reverse the sign of the last term in (1), \( \Lambda \) changes sign and the path oscillates around \( l_0 \).] The noted path implies \( dl/l = ds/L \), which in (8) yields

\[
d\Lambda ds^2 = -6.
\]

(9)

This is remarkable, in that it contains no reference to \( x^4 = l \) and is homogeneous in its physical dimensions (units), with no reference to fundamental constants. That is, it is gauge and scale invariant. [An alternative derivation of (9) may be made by using the expression for \( \Lambda = \Lambda(s) \) found in ref. 3, equation (24) and noting that \( s \) is measured from where \( \Lambda \) diverges at the big bang.] Again (9) confirms the instability, since \( d\Lambda \to \infty \) for \( ds \to 0 \).

This behaviour can be put into better physical perspective by recalling that the action for a particle of rest mass \( m \) in 4D dynamics is usually defined as \( I = \int mds \). So (9) can be interpreted as a change in the energy density of the vacuum for a particle of unit mass which changes its action.

Thirdly, let us assume that the action is quantized. In most higher-dimensional theories, the rest mass of a particle can change as it pursues its 4D path, so \( m = m(s) \) [9]. But irrespective of this, we have with units restored that \( dI = mcds = h \) where \( h \) is Planck’s constant. Then (9) gives

\[
d\Lambda = -6 \left( \frac{mc}{h} \right)^2.
\]

(10)
This says that a change in the energy density of the vacuum is related to the square of the mass of a particle. The implication is clearly that the vacuum (with an energy density proportional to \( \Lambda \)) gives up energy which corresponds to a particle (with rest mass \( m \)). The precise fashion in which this occurs cannot be investigated using the phenomenological relation (10). However, the situation is similar to the old Dirac theory, in which a positron is regarded as a hole created in an underlying sea of energy. Another way of interpreting (10) involves geometry. Globally, the vacuum is a sea of energy which curves spacetime, the gravitationally-defined “radius of curvature” being related to \( L = \sqrt{3/\Lambda} \) (see above). Locally, a perturbation in the vacuum corresponds to a change in the curvature; and (10) in this picture says that the change is related to the Compton wavelength \((h/mc)\) of the particle. It is interesting to note that relations like (10) have appeared previously in the literature [12]. Their rationale is to give a semi-classical account of the origin of mass, a problem whose analog in quantum theory involves the Higgs mechanism. At present, we are not sure how to incorporate the symmetries manifested by particles into higher-dimensional field theory. But (10) is a simple relation which is compatible with other more detailed approaches.
3 Conclusion

When the cosmological “constant” $\Lambda$ as measured in a 4D spacetime with proper time (action) $s$ is derived from a higher-dimensional model, dynamical changes in these parameters are related by (9): $d\Lambda ds^2 = -6$. This is free of fundamental constants and other parameters involving the choice of higher-dimensional gauge. It indicates that the (classical) vacuum is unstable to spacetime changes. If the latter are quantized, we obtain (10): this connects a change in the energy density of the vacuum (as measured by $\Lambda$) to particle mass, in a way reminiscent of the Dirac theory of the positron. Relations (9) and (10) are phenomenological, insofar as they are derived in the context of 5D (membrane and induced-matter) theory, without high-energy corrections from other dimensions or input from models of particle interactions. In this sense, they are like the relations of classical thermodynamics, which however provide reasonable approximations without knowledge of the underlying atomic physics. Both relations are compatible with recent research on the cosmological-“constant” problem and the nature of mass [1-4, 7-12]. We are of the opinion that they provide a basis for more detailed work, notably in the areas of particle production and cosmology.
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