NOTES ON VARIATION OF LEFSCHETZ STAR OPERATOR AND T-HODGE THEORY

XU WANG

ABSTRACT. These notes were written to serve as an easy reference for [34]. All the results in this presentation are well-known (or quasi-well-known) theorems in Hodge theory. Our main purpose was to give a unified approach based on a variation formula of the Lefschetz star operator, following [32]. It fits quite well with Timorin’s T-Hodge theory, i.e. the Hodge theory on the space of differential forms divided by T (i.e. forms like $T \wedge u$), where $T$ is a finite wedge product of Kähler forms.

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1. Preliminaries

1.1. Primitivity: linear setting. Let $V$ be an $N$-dimensional real vector space. Let $\omega$ be a bilinear form on $V$. We call $\omega$ a symplectic form if $\omega$ is non-degenerate and $\omega \in \wedge^2 V^*$, i.e. $\omega(u, v) = -\omega(v, u)$, $\forall u, v \in V$.

Proposition 1.1. Assume that there is a symplectic form $\omega$ on $V$. Then $N = 2n$ for some integer $n$ and there exists a base, say $\{e_1^*, f_1^*; \cdots; e_n^*, f_n^*\}$, of $V^*$ such that

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*.$$
Proof. Since \( \omega \) is non-degenerate, we know that \( N \geq 2 \). If \( N = 2 \) and \( \omega(e, f) = 1 \) then
\[
\omega = e^* \wedge f^*,
\]
where \( \{e^*, f^*\} \) is the dual base of \( \{e, f\} \). Assume that \( N \geq 3 \), consider
\[
V' := \{ u \in V : \omega(u, e) = \omega(u, f) = 0 \}.
\]
Then for every \( u \in V \), we have
\[
u' := u - \omega(u, f)e + \omega(u, e)f \in V',
\]
and
\[
ae + bf \in V' \text{ iff } a = b = 0,
\]
thus
\[
V = V' \oplus \text{Span}\{e, f\}.
\]
Since \( \omega \) is non-degenerate, we know that for every \( v \in V' \), there exists \( u \in V \) such that \( \omega(u, v) \neq 0 \). Thus
\[
\omega(u', v) = \omega(u, v) \neq 0,
\]
which implies that \( \omega|_{V'} \) is a symplectic form on \( V' \). Thus the theorem follows by induction on \( N \).

One may use \( \omega \) to define a bilinear form, say \( \omega^{-1} \), on \( V^* \) such that
\[
\omega^{-1}(f^*_j; e^*_k) = -\omega^{-1}(e^*_k; f^*_j) = \delta_{jk}, \ \omega^{-1}(f^*_j; f^*_k) = \omega^{-1}(e^*_j; e^*_k) = 0.
\]
Let \( T_\omega : V \rightarrow V^* \) be the linear isomorphism defined by
\[
T_\omega(u)(v) = \omega(v, u), \ \forall \ u, v \in V.
\]
Then we have
\[
T_{\omega^{-1}} = T_{\omega^{-1}},
\]
thus the definition of \( \omega^{-1} \) does not depend on the choice of bases in the above proposition. We shall also use \( \omega^{-1} \) to denote the following bilinear form on \( \wedge^p V^* \):
\[
\omega^{-1}(\mu, \nu) := \det(\omega^{-1}(\alpha_i, \beta_j)), \ \mu = \alpha_1 \wedge \cdots \wedge \alpha_p, \ \nu = \beta_1 \wedge \cdots \wedge \beta_p.
\]

Definition 1.1 (By Guillemin [16]). The symplectic star operator \( *_s : \wedge^p V^* \rightarrow \wedge^{2n-p} V^* \) is defined by
\[
(1.2) \quad \mu \wedge *_s \nu = \omega^{-1}(\mu, \nu) \frac{\omega^n}{n!}.
\]

The following theorem is the key to decode the structure of \( *_s \).

Theorem 1.2 (Hard Lefschetz theorem). For each \( 0 \leq k \leq n \),
\[
u \mapsto \omega^{n-k} \wedge u, \ u \in \wedge^k V^*,
\]
defines an isomorphism between \( \wedge^k V^* \) and \( \wedge^{2n-k} V^* \).

Proof. Notice that the theorem is true if \( n = 1 \) or \( k = 0, n \). Now assume that it is true for \( n \leq l, l \geq 1 \). We need to prove that it is true for \( n = l + 1, 1 \leq k \leq l \). Put
\[
\omega' = \sum_{j=1}^l e^*_j \wedge f^*_j.
\]
Then we have
\[
\omega^{l+1-k} = (\omega')^{l+1-k} + (l+1-k)(\omega')^{l-k} \wedge e^*_l \wedge f^*_l.
\]
Let us write \( u \in \wedge^k(V^*) \) as
\[
u = u^0 + u^1 \wedge e^*_{j+1} + u^2 \wedge f^*_{j+1} + u^3 \wedge e^*_{j+1} \wedge f^*_{j+1},
\]
where each \( u^j \) contains no \( e^*_{j+1} \) or \( f^*_{j+1} \) term. Then \( \omega^{l+1-k} \wedge u = 0 \) is equivalent to
\[
(\omega')^{l+1-k} \wedge u^0 = (\omega')^{l+1-k} \wedge u^1 = (\omega')^{l+1-k} \wedge u^2 = (\omega')^{l+1-k} \wedge u^3 + (l+1-k)(\omega')^{l-k} \wedge u^0 = 0,
\]
which implies \( u^1 = u^2 = 0 \) by our theorem for \( n = l \). Moreover, \( u^3 = 0 \) since
\[
(\omega')^{l+2-k} \wedge u^3 = (\omega')^{l+2-k} \wedge u^3 + (l+1-k)(\omega')^{l-k+1} \wedge u^0 = 0.
\]
Thus \((\omega')^{l-k} \wedge u^0 = 0\), which implies \( u^0 = 0 \). Now we know that \( u \mapsto \omega^{l+1-k} \wedge u \) is an injection, thus an isomorphism since \( \dim \wedge^k V^* = \dim \wedge^{2n-k} V^* \).
\( \square \)

The key notion in these notes is the following:

**Definition 1.2.** We call \( u \in \wedge^k V^* \) a primitive form if \( k \leq n \) and \( \omega^{n-k+1} \wedge u = 0 \).

The following Lefschetz decomposition theorem follows directly from Theorem 1.2 (see the proof of Theorem 2.1 in the next section).

**Theorem 1.3** (Lefschetz decomposition formula). Every \( u \in \wedge^k V^* \) has a unique decomposition as follows:

\[
u = \sum \omega_r \wedge u^r, \quad \omega_r := \frac{\omega^r}{r!},
\]

where each \( u^r \) is a primitive \((k-2r)\)-form.

By the above theorem, it is enough to study the symplectic star operator on \( \omega_r \wedge u \), where \( u \) is primitive.

**Theorem 1.4.** If \( u \) is a primitive \( k \)-form then \( \ast_s(\omega_r \wedge u) = (-1)^{k+\cdots+1} \omega_{n-k-r} \wedge u \).

The above theorem implies \( \ast_2^2 = 1 \). We shall use a symplectic analogy of the Berndtsson lemma (see Lemma 3.6.10 in [3]) to prove it.

**Definition 1.3.** \( u \in \wedge^k V^* \) is said to be an elementary form if there exists a base, say \( \{e^*_1, f^*_1; \cdots; e^*_n, f^*_n\} \), of \( V^* \) such that
\[
\omega = \sum_{j=1}^n e^*_j \wedge f^*_j, \quad u = e^*_1 \wedge \cdots \wedge e^*_k.
\]

**Lemma 1.5** (Berndtsson lemma). The space of primitive forms is equal to the linear space spanned by elementary forms.

**Proof.** Since
\[
\omega_{n-k+1} = \sum_{j_1<\cdots<j_{n-k+1}} e^*_{j_1} \wedge f^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k+1}} \wedge f^*_{j_{n-k+1}},
\]
we know that \( \omega_{n-k+1} \wedge u = 0 \) if \( u \) is an elementary \( k \)-form. Thus every elementary form is primitive. Let us prove the other side by induction on \( n \). Notice that the lemma is true if \( n = 1 \). Assume that it is true for \( n \leq l, l \geq 1 \). We shall prove that it is also true for \( n = l + 1 \). With the notation in the proof of Theorem 1.2, \( \omega^{l-k+2} \wedge u = 0 \) is equivalent to
\[
(\omega')^{l+2-k} \wedge u^0 = (\omega')^{l+2-k} \wedge u^1 = (\omega')^{l+2-k} \wedge u^2 = (\omega')^{l+2-k} \wedge u^3 + (l+2-k)(\omega')^{l-k+1} \wedge u^0 = 0,
\]
which is equivalent to the \( \omega' \)-primitivity of \( u^1, u^2, u^3 \) and \( (l+2-k)u^0 + \omega' \wedge u^3 \). Now it suffices to show that
\[
u' := u^3 + ((l+2-k)e^*_{j+1} \wedge f^*_{j+1} - \omega') \]
is a linear combination of elementary forms. Since \( u^3 \) is \( \omega' \)-primitive, by the induction hypothesis, we can assume that
\[
u_3 = e_1^* \wedge \cdots \wedge e_{k-2}^*.
\]
Thus
\[
u' = \sum_{j=k-1}^{l} e_1^* \wedge \cdots \wedge e_{k-2}^* \wedge (e_l^* f_{l+1}^* - e_j^* f_j^*)
\]
Now it suffices to show that if \( n = 2 \) then \( e_1^* f_1^* - e_2^* f_2^* \) is a linear combination of elementary forms. Notice that
\[
e_1^* f_1^* - e_2^* f_2^* = (e_1^* + e_2^*) \wedge (f_1^* - f_2^*) + e_1^* f_2^* + f_1^* e_2^*.
\]
It is clear that \( e_1^* f_1^* \) and \( f_1^* e_2^* \) are elementary. \( (e_1^* + e_2^*) \wedge (f_1^* - f_2^*) \) is also elementary since we can write
\[
\omega = (e_1^* + e_2^*) \wedge f_1^* + e_2^* \wedge (f_2^* - f_1^*).
\]
The proof is complete. \( \square \)

We shall also use the following Lemma from [10].

**Lemma 1.6 (Guillemin Lemma).** Assume that \((V, \omega) = (V_1, \omega^{(1)}) \oplus (V_2, \omega^{(2)})\). Then
\[
s_* (u \wedge v) = (-1)^{k_1 k_2} s_* u \wedge s_* v, \ u \in \wedge^{k_1} V_1^*, \ v \in \wedge^{k_2} V_2^*,
\]
where \( s_* \) and \( s_*^2 \) are symplectic star operators on \( V_1 \) and \( V_2 \) respectively.

**Proof.** For every \( a \in \wedge^{k_1} V_1^*, \ b \in \wedge^{k_2} V_2^*, \) we have
\[
a \wedge b \wedge (-1)^{k_1 k_2} s_* u \wedge s_*^2 v = \omega^{-1} (a \wedge b \wedge v) \omega_n,
\]
which gives the lemma. \( \square \)

Now we are able to prove Theorem 1.4.

**Proof of Theorem 1.4.** By the Berndtsson lemma, we can assume that
\[
u = e_1^* \wedge \cdots \wedge e_k^*.
\]
Consider \( V = \text{Span}\{e_j^*, f_j^*\}_{1 \leq j \leq k} \oplus \text{Span}\{e_{k+1}^*, f_{k+1}^*\} \oplus \cdots \oplus \text{Span}\{e_n^*, f_n^*\} \) and write
\[
s_* = \epsilon^{k \leq k} \oplus s_*^{k+1} \oplus \cdots \oplus s_*^n,
\]
Since
\[
s_*^j (1) = e_j^* \wedge f_j^*, \ s_*^j (e_j^* \wedge f_j^*) = 1, \forall \ k + 1 \leq j \leq n,
\]
by the Guillemin lemma, we have
\[
s_* (e_{k+1}^* \wedge f_{k+1}^* \wedge \cdots \wedge e_{k+r}^* \wedge f_{k+r}^* \wedge u) = e_{k+r+1}^* \wedge f_{k+r+1}^* \wedge \cdots \wedge e_n^* \wedge f_n^* \wedge s_*^k u,
\]
which implies
\[
s_* (\omega_r \wedge u) = \omega_{n-k-r} \wedge s_*^k u.
\]
Since \( s_*^k = s_*^1 \oplus \cdots \oplus s_*^k \) and
\[
s_*^j e_j^* = -e_j^*, \ \forall \ 1 \leq j \leq k,
\]
the Guillemin lemma gives
\[
\ast s_*^k u = (-1)^{k-1} (-e_1^*) \wedge s_*^{(k-1)} (e_2^* \wedge \cdots \wedge e_k^*) = \cdots = (-1)^{k+\cdots+1} u,
\]
the proof is complete. \( \square \)
Definition 1.4. We call \( \{L, \Lambda, B\} \) the \( \mathfrak{sl}_2 \)-triple on \( \bigoplus_{0 \leq k \leq 2n} \bigwedge^k V^* \), where

\[
Lu := \omega \wedge u, \quad \Lambda := \ast_s^{-1} L \ast_s, \quad B := [L, \Lambda].
\]

We have

\[
\omega^{-1}(Lu, v) = \omega^{-1}(u, \Lambda v).
\]

Hence \( \Lambda \) is the adjoint of \( L \). Put

\[
L_r := \frac{L^r}{r!}, \quad L_0 := 1, \quad L_{-1} := 0.
\]

We have:

Proposition 1.7. If \( u \) is a primitive \( k \)-form then

\[
\Lambda(L_r u) = (n - k - r + 1)L_{r-1} u, \quad B(L_r u) = (k + 2r - n)L_r u,
\]

for every \( 0 \leq r \leq n - k + 1 \).

Proof. Put \( c = (-1)^{k+\cdots+1} \), then

\[
L \ast_s(L_r u) = cL(L_{n-k-r} u) = (n - k - r + 1)cL_{n-k-r+1} u = (n - k - r + 1) \ast_s(L_{r-1} u),
\]

which gives the first identity. The second follows directly from the first. \( \square \)

Now let us consider another structure on a linear space, which can be used to define an inner product structure on \( (V, \omega) \).

Definition 1.5. We call a linear map \( J : V \to V \) an almost complex structure on \( V \) if

\[
J(Ju) = -u \quad \text{for every} \quad u \in V.
\]

Definition 1.6. An almost complex structure \( J \) on \( (V, \omega) \) is said to be compatible with \( \omega \) if

\[
\omega(u, Ju) = \omega(v, Ju), \quad \forall \ u, v \in V,
\]

and \( \omega(u, Ju) > 0 \) if \( u \) is not zero.

If \( J \) is an almost complex structure on \( V \) then

\[
J(v)(u) := v(Ju), \quad \forall \ u \in V, \ v \in V^*;
\]

defines an almost complex structure on \( V^* \).

Definition 1.7. We call

\[
J(v_1 \wedge \cdots \wedge v_k) := J(v_1) \wedge \cdots \wedge J(v_k),
\]

the Weil operator on \( \bigoplus_{0 \leq k \leq 2n} \bigwedge^k V^* \).

Since the eigenvalues of \( J \) are \( \pm i \), its eigenvectors lie in \( \mathbb{C} \otimes V^* \). Put

\[
E_i := \{ u \in \mathbb{C} \otimes V^* : J(u) = iu \}, \quad E_{-i} := \{ u \in \mathbb{C} \otimes V^* : J(u) = -iu \},
\]

we know that

\[
E_i = \{ u - iJu : u \in V^* \}, \quad E_{-i} = \{ u + iJu : u \in V^* \}.
\]

and \( \mathbb{C} \otimes V^* = E_i \oplus E_{-i} \). Put

\[
\bigwedge^{p,q} V^* := (\bigwedge^p E_i) \wedge (\bigwedge^q E_{-i}).
\]

Then we have

\[
\mathbb{C} \otimes (\bigwedge^k V^*) = \bigwedge^k (\mathbb{C} \otimes V^*) = \bigoplus_{p+q=k} \bigwedge^{p,q} V^*;
\]

and

\[
J u = i^{p-q} u, \ \forall \ u \in \bigwedge^{p,q} V^*.
\]

We call \( \bigwedge^{p,q} V^* \) the space of \((p,q)\)-forms.
Proposition 1.8. An almost complex structure $J$ on $(V, \omega)$ is compatible with $\omega$ iff 
$$(\alpha, \beta) := \omega^{-1}(\alpha, J\bar{\beta}),$$
defines a Hermitian inner product structure on $\wedge^{p,q}V^*$, $0 \leq p, q \leq n$.

Proof. Assume that $J$ is compatible with $\omega$. Then 
$$T_\omega(Ju)(v) = \omega(v, Ju) = -\omega(Jv, u) = -J(T_\omega u)(v), \quad u, v \in V;$$
Thus $T_\omega \circ J = -J \circ T_\omega$. Now put 
$$a = T_\omega(u), \quad b = J(T_\omega(v)) = -T_\omega(Jv).$$
Then $\omega(v, Ju) = T_\omega(Ju)(v) = -(Ja)(v) = -a(Jv) = a(T_\omega^{-1}(b))$, thus 
$$\omega(v, Ju) = T_\omega^{-1}(b)(a) = \omega^{-1}(a, b) = \omega^{-1}(T_\omega u, J(T_\omega v)), $$
which gives the proposition. \hfill \Box

Definition 1.8. The Hodge star operator $\ast : \wedge^{p,q}V^* \to \wedge^{n-q, n-p}V^*$ is defined by 
$$u \wedge \ast \bar{v} = (u, v)\omega, \quad u \in \wedge^p V^*, \quad v \in \wedge^q V^*.$$ 

The above proposition gives 
$$\ast = \ast_s \circ J = J \circ \ast_s.$$ 

1.2. Application in complex geometry. Let $(X, \omega)$ be an $n$-dimensional complex manifold with a Hermitian form (smooth positive $(1, 1)$-form) $\omega$. Let $(E, h_E)$ be a holomorphic vector bundle over $X$ with a smooth Hermitian metric $h_E$. Let us denote by $V^k$ the space of $E$-valued $k$-forms with compact support on $X$. The following theorem is a direct consequence of Theorem 1.2.

Theorem 1.9 (Hard Lefschetz theorem). For each $0 \leq k \leq n$, 
$$(1.4) \quad u \mapsto \omega^{n-k} \wedge u, \quad u \in V^k,$$ 
defines an isomorphism between $V^k$ and $V^{2n-k}$.

Definition 1.9. We call an $E$-valued $k$-form, say $u$, on $X$ a primitive form if $k \leq n$ and 
$$\omega^{n-k+1} \wedge u \equiv 0.$$ 

Now we have the following analogy of Theorem 1.3.

Theorem 1.10 (Lefschetz decomposition formula). Every $E$-valued $k$-form $u$ on $X$ has a unique decomposition as follows: 
$$(1.5) \quad u = \sum \omega_r \wedge u^r, \quad \omega_r := \frac{\omega^r}{r!},$$ 
where each $u^r$ is an $E$-valued primitive $(k - 2r)$-form.

Let $\{e_\alpha\}$ be a local holomorphic frame of $E$, then 
$$||u||^2 := \int_X \sum h_E(e_\alpha, e_\beta) u^\alpha \wedge \ast \bar{u}^\beta, \quad u := \sum u^\alpha \otimes e_\alpha.$$ 
defines a Hermitian inner product structure on $V^k$, we call it $(\omega, J, h_E)$-metric on $V^k$.

Definition 1.10. The Hodge star operator on $V^k$ is defined by 
$$\ast u = \sum (\ast u^\alpha) \otimes e_\alpha, \quad u := \sum u^\alpha \otimes e_\alpha.$$
Put
\[ \left\{ \sum u^\alpha \otimes e_\alpha, \sum u^\beta \otimes e_\beta \right\}_{h_E} := \sum h_E(e_\alpha, e_\beta)u^\alpha \wedge \bar{u}^\beta. \]
Then we have
\[ ||u||^2 = \int_X \{ u, \ast u \}_{h_E}. \]
The Hodge-Riemann bilinear relation is a direct consequence of Theorem 1.4 and \( \ast = J \circ \ast_s \).

**Theorem 1.11 (Hodge-Riemann bilinear relation).** If \( u \) is an \( E \)-valued primitive \( (p, q) \)-form then its \( (\omega, J, h_E) \)-norm satisfies
\[
||u||^2 = \int_X \{ u, \omega_{n-k} \wedge Iu \}_{h_E}, \quad Iu := (-1)^{k+\cdots+p-q}u, \quad k := p + q.
\]

### 2. Lefschetz bundle

**Definition 2.1.** Let \( V = \oplus_{k=0}^{2n} V^k \) be a direct sum of complex vector bundles over a smooth manifold \( M \). Let \( L \) be a smooth section of \( \text{End}(V) \). We call \( (V, L) \) a Lefschetz bundle if
\[ L(V^l) \subset V^{l+2}, \quad \forall 0 \leq l \leq 2(n-1), \quad L(V^{2n-1}) = L(V^{2n}) = 0, \]
and each \( L^k : V^{n-k} \to V^{n+k}, \quad 0 \leq k \leq n, \) is an isomorphism.

**Definition 2.2.** Let \( (V, L) \) be a Lefschetz bundle. \( u \in V^k \) is said to be primitive if \( k \leq n \) and \( L^{n-k+1}u = 0 \).

**Theorem 2.1.** Let \( (V, L) \) be a Lefschetz bundle. Then every \( u \in V^k \) has a unique decomposition as follows:
\[
(2.1) \quad u = \sum L_ru^r, \quad L_r := \frac{L}{r!}.
\]
where each \( u^r \) is a primitive form in \( V^{k-2r} \).

**Proof.** We can assume that \( k \leq n \) since we have the isomorphism \( L^k : V^{n-k} \to V^{n+k} \).
Notice that the theorem is trivial if \( k = 0, 1 \). Assume that \( 2 \leq k \leq n \). The isomorphism
\[ L^{n-k+2} : V^{k-2} \to V^{2n-k+2}, \]
gives \( \hat{u} \in V^{k-2} \) such that \( L^{n-k+2}\hat{u} = L^{n-k+1}u \). Put \( u^0 = u - L\hat{u} \), we know that \( u^0 \) is primitive and \( u = u^0 + L\hat{u} \). Consider \( \hat{u} \) instead of \( u \), we have \( \hat{u} = u^1 + L\hat{u} \), where \( u^1 \) is primitive. By induction, we know that \( u \) can be written as
\[ u = \sum L_ru^r, \]
where each \( u^r \in V^{k-2r} \) is primitive. For the uniqueness part, assume that
\[ 0 = \sum_{r=0}^{j} L_ru^r, \]
where each \( u^r \in V^{k-2r} \) is primitive. Then we have
\[ 0 = L_{n-k+j} \left( \sum_{r=0}^{j} L_ru^r \right) = L_{n-k+j}L_ju^j, \]
which gives \( u^j = 0 \). By induction on \( j \) we know that all \( u^r = 0 \). \qed
Definition 2.3. We call the following $\mathbb{C}$-linear map $*: V \to V$ defined by
\[ *_s(L_r u) := (-1)^{k+\ldots+1}L_{n-r-k}u, \]
where $u \in V^k$ is primitive, the Lefschetz star operator on $V$.

Notice that $*_s^2 = 1$. We know from the last section that the Lefschetz star operator is a generalization of the symplectic star operator.

Definition 2.4. Put $\Lambda = *^-1L*_s$, $B := [L, \Lambda]$. We call $(L, \Lambda, B)$ the $sl_2$-triple on $(V, L)$ (Proposition 1.7 is also true for general Lefschetz bundle).

3. Variation of Lefschetz Star Operator

3.1. Main theorem. Our main theorem is a generalization of the main result in [32].

Theorem 3.1. Let $D$ be a degree preserving connection on a Lefschetz bundle $(V, L)$. Put $\theta := [D, L]$. If $[L, \theta] = 0$ then $*_s^{-1}D*_s = D + [\Lambda, \theta]$.

Proof. By the Lefschetz decomposition theorem and $*_s^2 = 1$, it suffices to prove
\[ *_sD*_s(L_r u) = D(L_r u) + [\Lambda, \theta](L_r u), \]
where $u$ is a primitive $k$-form. Since $[L, \theta] = 0$, we have
\[ D*_s(L_r u) = cD(L_{n-r-k}u) = c(L_{n-r-k-1}u + L_{n-r-k}Du), \quad c := (-1)^{k+\ldots+1}. \]

Step 1: Since $u$ is primitive, we have
\[ L_{n-k+1}u = 0, \]
which implies that the primitive decomposition of $\theta u$ contains at most three terms. Thus we can write
\[ \theta u = a + Lb + L^2c, \]
where $a, b, c$ are primitive, which gives
\[ *_s D*_s(L_r u) = -L_{r-1}a + ML_r b - M(M + 1)L_{r+1}c + c*_s L_{n-r-k}Du, \quad M := n - r - k. \]

Step 2: Since
\[ \theta L_r u = L_r(a + Lb + L^2c), \]
Proposition 1.7 gives
\[ \Lambda \theta L_r u = (M - 1)L_{r-1}a + (r + 1)ML_r b + (r + 1)(r + 2)(M + 1)L_{r+1}c, \]
and
\[ \theta L_{r-1}u = (M + 1)\theta L_{r-1}u = (M + 1)L_{r-1}(a + Lb + L^2c). \]

Thus
\[ [\Lambda, \theta]L_r u = -2L_{r-1}a + (M - r)L_r b + (2r + 2)(M + 1)L_{r+1}c. \]

Step 3: Put
\[ A := *_s D*_s(L_r u) - [\Lambda, \theta]L_r u, \quad B := D(L_r u). \]
We have
\[ B = L_{r-1}u + L_r Du \]
\[ = L_{r-1}a + rL_r b + (r + 1)L_{r+1}c + L_r Du. \]
Since the first two steps gives
\[ A = L_{r-1}a + rL_r b - (M + 1)(M + 2r + 2)L_{r+1}c + c*_s L_{n-r-k}Du, \]
we have
\[ A - B = c \ast_s L_{n-r-k}Du - L_rDu - [(M+1)(M+2r+2)+r(r+1)]L_{r+1}c. \]

**Step 4:** Primitivity of \( u \) implies
\[ 0 = D(L_{n-k+1}u) = \theta L_{n-k}u + L_{n-k+1}Du. \]

Notice that
\[ \theta L_{n-k}u = L_{n-k}(a + Lb + L^2c) = L_{n-k}L^2c. \]
Thus
\[ L_{n-k+1}(Du + (n-k+1)Lc) = 0, \]
which implies the primitivity of
\[ (3.4) \quad v := Du + (n-k+1)Lc. \]

Now we have
\[ c \ast_s L_{n-r-k}Du = c \ast_s L_{n-r-k}(v - (n-k+1)Lc) = L_rv + (n-k+1)(M+1)L_{r+1}c. \]
and
\[ (3.5) \quad L_rDu = L_r(v - (n-k+1)Lc) = L_rv - (n-k+1)(r+1)L_{r+1}c. \]
Thus \( c \ast_s L_{n-r-k}Du - L_rDu \) can be written as
\[ (3.6) \quad [(n-k+1)(M+1) + (n-k+1)(r+1)]L_{r+1}c. \]

**Step 5:** Since our formula is equivalent to \( A = B \), by step 3 and 4, it is enough to prove
\[ (n-k+1)(M+1) + (n-k+1)(r+1) = (M+1)(M+2r+2) + r(r+1), \]
which is true (recall that \( M = n-r-k \)).
\[ \square \]

**Remark:** If we write
\[ D = \sum dt^i \otimes D_j, \]
where \( \{t^i\}_{1 \leq j \leq m} \) are smooth local coordinates. Then our main theorem is equivalent to
\[ (3.7) \quad s^{-1}D_j \ast_s u = D_ju + [\Lambda, \theta_j]u, \quad \theta_j := [D_j, L], \]
for every \( 1 \leq j \leq m \).

### 3.2 Corollary.

**Corollary 3.2.** With the same assumption in Theorem 3.1, we have
\[ (3.8) \quad s^{-1}\theta \ast_s = [\Lambda, D]. \]

**Proof.** The lemma below gives
\[ (3.9) \quad s^{-1}\theta \ast_s = -\frac{1}{2}[\Lambda, [\Lambda, \theta]]. \]
Since \( \Lambda = s^{-1}L \ast_s \) and \( \theta = [D, L] = DL - LD \), we have
\[ s^{-1}\theta \ast_s = [s^{-1}D, \Lambda]. \]
Our main theorem and (3.9) give
\[ s^{-1}\theta \ast_s = [D + [\Lambda, \theta], \Lambda] = [D, \Lambda] + 2s^{-1}\theta \ast_s, \]
which gives (3.8).
\[ \square \]
Lemma 3.3. \( \ast_s^{-1} \theta \ast_s = -\frac{1}{2} [\Lambda, [\Lambda, \theta]]. \)

Proof. For a primitive \( k \)-form \( u \), we have \( \theta \ast_s (L_r u) := cL_{n-r-k} \theta u \). By the proof of our main theorem, we can write \( \theta u = a + L b + L^2 c \), where \( a, b, c \) are primitive. Thus

\[ \ast_s \theta \ast_s (L_r u) = -L_{r-2} a + (M + 1) L_{r-1} b - (M + 1) (M + 2) L_r c, \quad M := n - r - k. \]

On the other hand, notice that

\[ -\frac{1}{2} [\Lambda, [\Lambda, \theta]] = \Lambda \theta \Lambda - \frac{1}{2} (\Lambda^2 \theta + \theta \Lambda^2). \]

Now

\[ \Lambda^2 \theta L_r u = \Lambda^2 L_r (a + L b + L^2 c). \]

thus Proposition 1.7 gives

\[ \Lambda^2 \theta L_r u = (M - 1) M L_{r-2} a + (r + 1) M (M + 1) L_{r-1} b + (r + 1) (r + 2) (M + 1) (M + 2) L_r c. \]

By a similar argument, we also get

\[ \theta \Lambda^2 L_r u = (M + 1) M L_{r-2} a + (r + 1) (M + 1) L_{r-1} b + r (r + 1) L_r c, \]

and

\[ \Lambda \theta \Lambda L_r u = (M + 1) M L_{r-2} a + r (M + 1) L_{r-1} b + r (r + 1) (M + 2) L_r c. \]

Thus \( -\frac{1}{2} [\Lambda, [\Lambda, \theta]](L_r u) \) equals

\[ -L_{r-2} a + (M + 1) L_{r-1} b - (M + 1) (M + 2) L_r c = \ast_s \theta \ast_s (L_r u). \]

The proof is complete. \( \square \)

The following proposition can be seen as a generalization of formula 1 in [26].

Proposition 3.4. Let \((V, L)\) be a Lefschetz bundle. Let \( \sigma \) be a smooth degree one section of \( \text{End}(V) \). If \([L, \sigma] = 0\) then \( \ast_s^{-1} \sigma \ast_s = (-1)^k [\Lambda, \sigma] \) on \( V^k \).

Proof. For a primitive \( k \)-form \( u \), we have \( \sigma \ast_s (L_r u) := cL_{n-r-k} \sigma u \). Since \( \sigma \) is degree one, we can write \( \sigma u = a + L b \), where \( a, b \) are primitive. Thus

\[ \ast_s \sigma \ast_s (L_r u) = (-1)^{k+1} (L_{r-1} a - (M + 1) L_r b), \quad M := n - r - k. \]

On the other hand, Proposition 1.7 gives

\[ \Lambda \sigma L_r u = M L_{r-1} a + (r + 1) (M + 1) L_r b. \]

and

\[ \sigma \Lambda L_r u = (M + 1) L_{r-1} a + r L_r b. \]

Thus

\[ (-1)^k [\Lambda, \sigma](L_r u) = \ast_s \sigma \ast_s (L_r u). \]

Since \( \ast_s^2 = 1 \), the proposition follows. \( \square \)
4. T-Hodge theory

4.1. Timorin’s theorem. Timorin’s theorem \[30\] is a mixed linear version of the Hodge-Riemann bilinear relation. Let \((V, \omega, J)\) be a 2n-dimensional real vector space with compatible pair \((\omega, J)\). Let \(\alpha_0, \alpha_1, \ldots, \alpha_n\) be \(J\)-compatible symplectic forms on \(V\). Put

\[
T_k := \alpha_k \wedge \cdots \wedge \alpha_n, \quad 0 \leq k \leq n, \quad T_{n+1} := 1.
\]

and

\[
V^k := \mathbb{C} \otimes \wedge^k V^*.
\]

Timorin introduced the following definition in \[30\].

**Definition 4.1.** We call \(u \in V^k\) a \(T_k\)-primitive form if \(k \leq n\) and \(u \wedge T_k = 0\).

**Theorem 4.1** (Timorin’s mixed Hodge-Riemann bilinear relation, MHR-n). Let \(u\) be a non-zero \(T_k\)-primitive form. Then \(Q(u) := (-1)^{k+\cdots+1} T_{k+1} \wedge u \wedge Ju > 0\).

**Proof.** We claim that MHR-n follows MHL-n and usual Hodge-Riemann bilinear relation. Notice that MHL-n below implies that the space, say \(P_k\), of \(T_k\)-primitive forms has constant dimension \(\dim V^k - \dim V^{k-2}\) and \(Q\) is non-degenerate on \(P_k\), consider

\[
\alpha_j' := (1-t)\alpha_j + t\omega, \quad 0 \leq t \leq 1,
\]

then the positivity of \(Q\) at \(t = 0\) follows from the positivity of \(Q\) at \(t = 1\) (the usual Hodge-Riemann bilinear relation). \(\square\)

**Theorem 4.2** (Timorin’s mixed hard-Lefschetz theorem, MHL-n). For every \(0 \leq k \leq n\),

\[
u \mapsto u \wedge T_{k+1},
\]

defines an isomorphism from \(V^k\) to \(V^{2n-k}\).

**Proof.** Since MHR-1 is true, it suffices to show MHR-(n-1) implies MHL-n. Assume that \(u \in V^k, k \leq n - 1\). If \(u \wedge T_{k+1} = 0\) then

\[
u_H \wedge T_{k+1} |_H = 0,
\]

for every hyperplane \(H\). Thus if \(u \wedge T_{k+1} = 0\) then \(u|_H\) is \(T_{k+1}|_H\)-primitive for every \(H\).

Let us write

\[
\alpha_{k+1} = \sum_{j=1}^{n} i\sigma_j \wedge \bar{\sigma}_j, \quad H_j := \ker \sigma_j.
\]

Then MHR-(n-1) gives

\[
Q_j(u) := (-1)^{k+\cdots+1} T_{k+2}|_H \wedge u|_H \wedge \overline{Ju}|_H \geq 0.
\]

If \(u \wedge T_{k+1} = 0\) then

\[
0 = (-1)^{k+\cdots+1} T_{k+1} \wedge u \wedge \overline{Ju} = \sum Q_j(u) \wedge (i\sigma_j \wedge \bar{\sigma}_j),
\]

which implies each \(Q_j(u) = 0\). Thus \(u|_{H_j} = 0\) for every \(1 \leq j \leq n\), which gives \(u \wedge \alpha_{k+1} = 0\), thus \(u = 0\) since \(\deg u \leq n - 1\). \(\square\)
4.2. **Hodge star operator on** $V_T$. Let $(E, h_E)$ be a holomorphic vector bundle over an $n$-dimensional complex manifold $(X, \omega)$. Denote by $V^{p,q}$ the space of smooth $E$-valued $(p,q)$-forms with compact support on $X$. Put 

$$V := \oplus_{0 \leq p,q \leq n} V^{p,q}, \quad V^k := \oplus_{p+q=k} V^{p,q}.$$ 

Fix $0 \leq m \leq n$ and smooth positive $(1,1)$-forms $\alpha_{m+1}, \ldots, \alpha_n$ on $X$. Consider 

$$f_T : u \mapsto T \wedge u, \quad u \in V.$$ 

where 

$$T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n, \quad T := 1, \quad \text{if } m = n.$$ 

We call the Hodge theory on 

$$\text{Im} f_T := \{ T \wedge u : u \in V \},$$ 

the $T$-Hodge theory. Put 

$$V_T^{p,q} := f_T(V^{p,q}), \quad V_T := \oplus_{0 \leq p,q \leq n} V_T^{p,q}, \quad V^k_T := \oplus_{p+q=k} V_T^{p,q}.$$ 

We have 

$$V_T = \oplus_{k=0}^{2m} V^k = \oplus_{0 \leq p,q \leq m} V_T^{p,q},$$ 

and 

$$L : u \mapsto \omega \wedge u, \quad u \in V_T,$$ 

maps $V_T^{p,q}$ to $V_T^{p+1,q+1}$. Timorin’s mixed hard-Lefschetz theorem gives: 

**Theorem 4.3.** For every $0 \leq k \leq m$, 

$$L^{m-k} : u \mapsto u \wedge \omega^{m-k},$$ 

defines an isomorphism from $V^k_T$ to $V_T^{2m-k}$. 

**Proof.** By Timorin’s theorem, we know that 

$$u \mapsto T \wedge u, \quad u \in V^k$$ 

is injective. Thus $f_T$ defines an isomorphism from $V^k$ to $V^k_T$. Again by Timorin’s theorem, we have the following isomorphism 

$$A : u \mapsto u \wedge T \wedge \omega^{m-k},$$ 

from $V^k$ to $V_T^{2n-k}$. Thus $L^{m-k} = A \circ f_T^{-1}$ is an isomorphism. \hfill $\square$ 

**Definition 4.2.** We call $u \in V^k_T$ a primitive $k$-form if $k \leq m$ and $L^{m-k+1}u = 0$. 

The proof of Theorem 2.1 implies: 

**Theorem 4.4.** Every $u \in V^k_T$ has a unique decomposition as follows: 

\begin{equation}
(4.1) \quad u = \sum L_r u^r, \quad L_r := \frac{L_r}{r!}.
\end{equation}

where each $u^r$ is a primitive form in $V^k_T$. 

**Definition 4.3.** We call the following $\mathbb{C}$-linear map $*_s : V_T \rightarrow V_T$ defined by 

$$*_s(L_r u) := (-1)^{k+\cdots+1} L_{m-r-k} u,$$ 

where $u \in V^k_T$ is primitive, the Lefschetz star operator on $V_T$. 

In case $m = n$, the Lefschetz star operator above is just the symplectic star operator. 

**Definition 4.4.** Put $\Lambda = *^{-1}_s L *_s, \quad B := [L, \Lambda]$. We call $(L, \Lambda, B)$ the $sl_2$-triple on $V_T$. 

Since $J$ commutes with $f_T$, the Weil-operator is also well defined on $V_T$, we shall also denote it by $J$.

**Definition 4.5.** We call $\ast := \ast_s \circ J$ the Hodge star operator on $V_T$.

Timorin’s mixed Hodge-Riemann bilinear relation gives:

**Theorem 4.5.** Put

$$ (u,v) := \int_X \{ f_T^{-1}(u), \ast v \} h_E, \quad u, v \in V_T^k, \quad 0 \leq k \leq m, $$

and

$$ (u,v) := \int_X \{ u, f_T^{-1}(\ast v) \} h_E, \quad u, v \in V_T^k, \quad m \leq k \leq 2m. $$

Then $(u,v)$ is a Hermitian inner product on $V_T$.

**Definition 4.6.** Let us define $(u,v)_T$ such that

$$ \{ f_T^{-1}(u), \ast v \} h_E = (u,v)_T \omega_m \wedge T, \quad u, v \in V_T^k, \quad 0 \leq k \leq m, $$

and

$$ \{ u, f_T^{-1}(\ast v) \} h_E = (u,v)_T \omega_m \wedge T, \quad u, v \in V_T^k, \quad m \leq k \leq 2m. $$

We call $(u,v)_T$ the pointwise Hermitian inner product of $u,v$ in $V_T$.

**Remark:** If $m = n$ then $T = 1$ and $(u,v)_T$ is just the pointwise $(\omega, J, h_E)$-inner product. Moreover, our Hermitian metric in Theorem 4.5 is compatible with the current norm on $V_T^p$ defined by Berndtsson-Sibony in [8].

4.3. **Kähler identity in $T$-Hodge theory.** Let

$$ D := \overline{\partial} + \partial^E, $$

be the Chern connection on $(E,h_E)$. Assume that $T$ is $d$-closed, then $Du \in V_T$ if $u \in V_T$. Let $D^*, \overline{\partial}, (\partial^E)^*$ be the adjoint of

$$ D, \quad \overline{\partial}, \quad \partial^E : V_T \to V_T, $$

respectively. We shall use Theorem 3.1 and Proposition 3.4 to prove the following $T$-geometry generalization of the Demailly-Griffiths-Kähler identity (see page 307 in [11]).

**Theorem 4.6.** If $T$ is $d$-closed then $[\overline{\partial}, L] = i(\partial^E + [\Lambda, [\partial^E, L]])$ on $V_T$.

**Proof.** Since $\ast = \ast_s \circ J = J \circ \ast_s$ and $\ast_s^2 = 1$, we have

$$ \overline{\partial}^* = - \ast \partial^E \ast = (-1)^{k+1} \ast_s \partial^E \ast_s = (-1)^{k+1} \sum_{j=1}^n (\ast_s^{-1} \partial^E \ast_s)(\ast_s^{-1} \sigma_j \ast_s), $$

on $V_T^k$, where $\sigma_j := dz^j \wedge$. Thus

$$ [\overline{\partial}^*, L] = (-1)^{k+1} \sum_{j=1}^n (\ast_s^{-1} \partial^E \ast_s)(\ast_s^{-1} \sigma_j \ast_s)L - L(\ast_s^{-1} \partial^E \ast_s)(\ast_s^{-1} \sigma_j \ast_s). $$
Now Proposition 3.4 implies $[\ast_{s}^{-1}\sigma_{j}, L] = (-1)^{k+1}\sigma_{j}$, thus
\[
[\bar{\partial}, L] = (-1)^{k+1}i\sum_{j=1}^{n}((s_{s}^{-1}\partial_{j}E_{s})((-1)^{k+1}\sigma_{j} + L\ast_{s}^{-1}\sigma_{j} - L(s_{s}^{-1}\partial_{j}E_{s})(\ast_{s}^{-1}\sigma_{j})).
\]
(3.8) gives $[\ast_{s}^{-1}\partial_{j}E_{s}, L] = -\theta_{j}$, where $\theta_{j} := [\partial_{j}E, L]$, thus our main theorem gives
\[
[\bar{\partial}, L] = i\partial_{E} + i\sum_{j=1}^{n}[\Lambda, \theta_{j}]\sigma_{j} + (-1)^{k}\theta_{j}(\ast_{s}^{-1}\sigma_{j}).
\]
Now it suffices to show
(4.2) \[
\sum_{j=1}^{n}[\Lambda, \theta_{j}]\sigma_{j} + (-1)^{k}\theta_{j}(\ast_{s}^{-1}\sigma_{j}) = [\Lambda, [\partial_{E}, L]].
\]
Since $\sum_{j=1}^{n}_j \sigma_{j} = [\partial_{E}, L]$, we have
(4.3) \[
\sum_{j=1}^{n}[\Lambda, \theta_{j}]\sigma_{j} = \Lambda[\partial_{E}, L] - \sum_{j=1}^{n}\theta_{j}\Lambda\sigma_{j}.
\]
Proposition 3.4 gives
(4.4) \[
(-1)^{k}\theta_{j}(\ast_{s}^{-1}\sigma_{j}) = \theta_{j}\Lambda\sigma_{j} - \theta_{j}\sigma_{j}\Lambda.
\]
Thus the left hand side of (4.2) can be written as
(4.5) \[
\Lambda[\partial_{E}, L] - [\partial_{E}, L]\Lambda,
\]
which equals the right hand side of (4.2). \qed

**Theorem 4.7.** Assume that both $T$ and $\omega$ are d-closed, then
\[
D^{*} = [\Lambda, D^{c}], \quad (D^{c})^{*} = [D, \Lambda],
\]
on $V_{T}$, where $D^{c} := \bar{i}\partial_{c} - i\partial^{E}$.

The above Kähler identity gives the following Bochner-Kodaira-Nakano identity in $T$-geometry:

**Theorem 4.8.** Assume that both $T$ and $\omega$ are d-closed. Then
\[
\square_{\bar{\tau}} = \square_{\partial_{E}} + [i\Theta(E, h_{E}), \Lambda],
\]
on $V_{T}$, where $\Theta(E, h_{E}) := D^{2}$, $\square_{\bar{\tau}} := \partial \bar{\partial} + \bar{\partial}\partial$, $\square_{\partial_{E}} := \partial^{E}(\partial^{E})^{*} + (\partial^{E})^{*}\partial^{E}$; moreover
\[
\square_{D^{c}} = \square_{D} = \square_{\bar{\tau}} + \square_{\partial_{E}},
\]
on $V_{T}$, where $\square_{D} := DD^{*} + D^{*}D$, $\square_{D^{c}} := D^{c}(D^{c})^{*} + (D^{c})^{*}D^{c}$.

**Remark:** If $1 \leq m < n$ then $\square_{D}$ is elliptic on $V^{k,0}_{T}$ and $V^{0,k}_{T}$ for $0 \leq k \leq m - 1$, but it is not elliptic on $V^{m,0}_{T}$ and $V^{0,m}_{T}$. In general, we don’t have
\[
(\partial \bar{\partial})^{*}(\partial \bar{\partial}) + (\partial \bar{\partial})(\partial \bar{\partial})^{*} = |\partial \bar{\partial}|^{2},
\]
but still we have the following theorem:

**Theorem 4.9.** $\square_{\bar{\tau}}, \square_{\partial_{E}}$ are elliptic on $V^{k}_{T}$ for every $k \neq m$. 
Proof. Recall that a differential operator of order $l$ is said to be elliptic if $\sigma_l(D)(x, \xi)$ is invertible for every $x \in M$ and every non-zero $\xi \in T_x M$, where

$$\sigma_l(D)(x, \xi) u := \lim_{t \to \infty} t^{-l} e^{-itf} D(e^{itf} u)(x),$$

and $f$ is a smooth function near $x$ such that $df(x) = \xi$. We have

$$\sigma_2(\square_\partial)(x, \xi) = (\partial f)^* (\overline{\partial f}) + (\overline{\partial f})^*(\partial f), \quad \sigma_2(\square_{\partial E})(x, \xi) = (\partial f)^*(\partial f) + (\partial f)(\partial f)^*.$$

Theorem 4.8 gives

$$\sigma_2(\square_\partial)(x, \xi) = \sigma_2(\square_{\partial E})(x, \xi).$$

Thus it suffices to prove that if $u \in V^p_k$, $k \neq m$, satisfies

$$(\overline{\partial f}) \wedge u = (\partial f) \wedge u = (\overline{\partial f})^* u = (\partial f)^* u = 0,$$

then $u(x) = 0$. It is clear that we can assume that $u \in V^p_k$, $p + q = k$. Consider $*u$ if $k > m$, one may assume further that $k < m$. Moreover, by a C-linear change of local coordinate, one may assume that $\overline{\partial f} = d\bar{z}_1$, $\partial f = dz_1$. Let

$$u = \sum_{r=0}^j \omega_r \wedge u^r,$$

be the Lefschetz decomposition of $u$. Then

$$*u = (-1)^{k+p+1} v^{p-q} \sum_{r=0}^j (-1)^r \omega_m^{-k+r} \wedge u^r.$$

The lemma below gives

$$u^r \wedge d\bar{z}_1 = u^r \wedge d\bar{z}_1 = 0, \quad \forall 0 \leq r \leq j,$$

hence

$$f_T^{-1}(u^r) \wedge d\bar{z}_1 = f_T^{-1}(u^r) \wedge d\bar{z}_1 = 0,$$

since the degree of each $u^r$ is no bigger than $m - 1$. Thus each $f_T^{-1}(u^r)$ can be written as

$$d\bar{z}_1 \wedge u^r.$$

Timorin’s Hodge-Riemann bilinear relation implies that if $f_T^{-1}(u^r) \neq 0$ then $f_T^{-1}(u^r) \wedge f_T^{-1}(u^r) \neq 0$. But obviously $(d\bar{z}_1 \wedge d\bar{z}_1)^2 = 0$ gives $f_T^{-1}(u^r) \wedge f_T^{-1}(u^r) = 0$. Thus $f_T^{-1}(u^r) = 0$ and $u = 0$. The proof is complete. \hfill \Box

Lemma 4.10. $u^r \wedge d\bar{z}_1 = u^r \wedge d\bar{z}_1 = 0, \quad \forall 0 \leq r \leq j$.

Proof. $dz_1 \wedge u = dz_1 \wedge *u = 0$ gives

$$A := \sum_{r=0}^j \omega_r \wedge (dz_1 \wedge u^r) = 0, \quad B := \sum_{r=0}^j (-1)^r \omega_m^{-k+r} \wedge (dz_1 \wedge u^r) = 0.$$

The lemma is true if $j = 0$. Assume the lemma is true for $0 \leq j \leq l$. We claim that it is true for $j = l + 1$. In fact, $\omega_m^{-k} \wedge A + (-1)^{l+1} C_m^{-k+l} B = 0$ gives

$$C := (\omega_m^{-k} \wedge \omega_{l+1} + C_m^{-k+l} \omega_m^{-k+l+1}) \wedge (dz_1 \wedge u^{l+1}) + \sum_{r=0}^{l-1} C_r \omega_m^{-k+r} \wedge (dz_1 \wedge u^r) = 0.$$

Since each $u^r$ is primitive, we know that

$$\omega_l \wedge C = \omega_l \wedge (\omega_m^{-k} \wedge \omega_{l+1} + C_m^{-k+l} \omega_m^{-k+l+1}) \wedge (dz_1 \wedge u^{l+1}),$$
which gives \( dz_1 \wedge u^{l+1} = 0 \) by Timorin’s hard Lefschetz theorem. Thus the lemma is also true for \( j = l + 1 \) by the induction hypothesis. \( \square \)

5. Applications

5.1. Alexandrov-Fenchel inequality.

**Definition 5.1.** A Hermitian manifold \((X, \hat{\omega})\) is said to be complete if there exists a smooth function, say

\[ \phi : X \to [0, \infty), \]

such that \( \phi^{-1}([0, c]) \) is compact for every \( c > 0 \) and

\[ |d\phi(\omega)(x)| \leq 1, \quad \forall \ x \in X. \]

**Theorem 5.1.** Let \((X, \hat{\omega})\) be an \( n \)-dimensional complete Kähler manifold with finite volume. Let \( \alpha_1, \ldots, \alpha_n \) be smooth \( d \)-closed semi-positive \((1, 1)\)-forms such that \( \alpha_j \leq \hat{\omega} \) on \( X \) for every \( 1 \leq j \leq n \). Assume that \( n \geq 2 \). Put

\[ T := \alpha_3 \wedge \cdots \wedge \alpha_n, \quad T := 1, \text{ if } n = 2. \]

Then

\[ \left( \int_X \alpha_1 \wedge \alpha_2 \wedge T \right)^2 \geq \left( \int_X \alpha_1^2 \wedge T \right) \left( \int_X \alpha_2^2 \wedge T \right). \]

**Remark:** In case \((X, \hat{\omega})\) is compact Kähler, the above theorem is just the Khovanskiĭ-Teissier inequality. The classical Alexandrov-Fenchel inequality follows from the above theorem with \( X = \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n \), see [34] for the proof.

**Proof of Theorem 5.1.** We shall follow the method in [34]. Consider \( \alpha_j + \epsilon \hat{\omega} \) instead of \( \omega \), one may assume that

\[ \frac{\hat{\omega}}{C} \leq \alpha_j \leq C \hat{\omega}, \]

for every \( 1 \leq j \leq n \), where \( C \) is a fixed positive constant.

**Step 1:** By Proposition 2.6 in [34], it suffices to show that

\[ \psi : t \mapsto -\log \int_X \omega_2 \wedge T, \quad \omega := t \alpha_1 + (1 - t) \alpha_2, \]

is convex on \((0, 1)\).

**Step 2:** Consider the trivial line bundle \( \ker d := U \times \mathbb{C} \) over

\[ U := \{ t + is : 0 < t < 1, \ s \in \mathbb{R} \}, \]

with metric

\[ h(1, 1)(t + is) := e^{-\psi(t)} = \int_X \omega_2 \wedge T, \]

Then \( \psi \) is convex iff the curvature of \((\ker d, h)\) is positive.

**Step 3:** Look at \( \ker d \) as a holomorphic subbundle of

\[ \mathcal{A} := U \times A, \]

where

\[ A := \{ f \in C^\infty(X, \mathbb{C}) : \int_X |f|^2 \hat{\omega}_n < \infty \}. \]
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$h$ extends to a metric on $\mathcal{A}$ as follows:

$$h(f, g)(t + is) := \int_X f \bar{g} \omega_T \wedge T = \int_X \{f, *g\}, \ \forall \ f, g \in \mathcal{A}.$$ 

Thus the Chern curvature operator of $(\mathcal{A}, h)$ can be written as

$$\Theta_A^t := [\Lambda, \partial/\partial t] = [\Theta_A^t, \partial/\partial t].$$

Our main theorem gives

$$\Lambda = \partial/\partial t \cdot \partial/\partial t = \{\partial/\partial t, \omega\} = \alpha_1 - \alpha_2,$$

thus we have

$$\Theta_A^t = [\Lambda, \partial/\partial t] = [\theta, \partial/\partial t],$$

by (3.8).

**Step 4**: Denote by $\Theta_K^t$ the Chern curvature operator of $(\ker d, h)$. By the subbundle curvature formula, we have

$$\psi_t = \frac{h(\Theta_K^t 1, 1)}{h(1, 1)} = \frac{h(\Theta_A^t 1, 1)}{h(1, 1)} - \frac{h((\Lambda \theta)\perp, (\Lambda \theta)\perp)}{h(1, 1)},$$

where $\Lambda \theta = [\Lambda, \theta] 1$ and

$$(\Lambda \theta)\perp := (\Lambda \theta) - \frac{h(\Lambda \theta, 1)}{h(1, 1)}.$$

is the $L^2$-minimal solution of

$$d(\cdot) = d(\Lambda \theta).$$

**Step 5**: Theorem 4.7 gives

$$d(\Lambda \theta) = (d^c)^* \theta.$$ 

If $u$ is a smooth one-form with compact support on $X$ then

$$|((d^c)^* \theta, u)|^2 = |(\theta, d^c u)|^2 \leq ||\theta||^2 ||d^c u||^2,$$

moreover, theorem 4.8 gives

$$||d^c u||^2 \leq ||du||^2 + ||d^* u||^2,$$

thus Hörmander's $L^2$-theory (here we use (5.1) and the completeness of $(X, \bar{\omega})$, see the proof of Lemma 5.2 in [34] for the details) implies

$$h((\Lambda \theta)\perp, (\Lambda \theta)\perp) = ||(\Lambda \theta)\perp||^2 \leq ||\theta||^2 = h(\Theta_A^t 1, 1),$$

where the last identity follows from $\Theta_A^t = [\theta, \theta]$. Thus $\psi_t \geq 0$. □

5.2. **Dinh-Nguyễn’s theorem.** Let $(X, \omega)$ be an $n$-dimensional compact Kähler manifold. Let $\alpha_1, \cdots, \alpha_n$ be smooth Kähler forms on $X$. Let $\mathcal{A}^{p,q}$ be the space of smooth $(p, q)$-forms on $X$ and $\mathcal{A}^k$ be the space of real-valued smooth $k$-forms on $X$. We have the Dolbeault cohomology group (a $\mathbb{C}$-vector space in fact)

$$H^{p,q}(X, \mathbb{C}) := \frac{\mathcal{A}^{p,q} \cap \ker \bar{\partial}}{\bar{\partial} \mathcal{A}^{p,q-1}},$$

and the de Rham cohomology group (an $\mathbb{R}$-vector space)

$$H^k(X, \mathbb{R}) := \frac{\mathcal{A}^k \cap \ker d}{d \mathcal{A}^{k-1}}.$$ 

The following theorem depends on the theory of elliptic operators, see [17], and Theorem 4.8 (when $(E, h_E)$ is trivial and $T = 1$).
Theorem 5.2 (Hodge-Dolbeault-de Rham theorem). Let $(X,\omega)$ be an $n$-dimensional compact Kähler manifold. Let $\Box_{\omega}$ be the $\bar{\partial}$-Laplacian with respect to the $(\omega,J)$-metric. Then each $H^{p,q}(X,\mathbb{C})$ is $\mathbb{C}$-linear isomorphic to

$$H^{p,q} := A^{p,q} \cap \ker \Box_{\omega}$$

which is finite dimensional. Let $\Box_d$ be the $d$-Laplacian with respect to the $(\omega,J)$-metric. Then each $H^k(X,\mathbb{R})$ is $\mathbb{R}$-linear isomorphic to

$$H^k := A^k \cap \ker \Box_d.$$

Moreover,

$$H^k + iH^k = \bigoplus_{p+q=k} H^{p,q}.$$

The above theorem implies that every class in $H^{p,q}(X,\mathbb{C})$ has a $d$-closed representative. Fix $0 \leq m \leq n$, put

$$T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n, \quad T := 1 \quad \text{if} \quad m = n.$$

Definition 5.2. We call a class $[u]$ in $H^{p,q}(X,\mathbb{C})$, $p+q = m$, a $T$-primitive class if

$$[u \wedge T \wedge \omega] = 0 \quad \text{in} \quad H^{p+n-m+1,q+n-m+1}(X,\mathbb{C}).$$

Theorem 5.3 (Dinh-Nguyễn’s theorem [12]). Assume that $[u] \in H^{p,q}(X,\mathbb{C})$, $p+q = m$, is a non-zero $T$-primitive class then

$$\int_X (-1)^{m+\cdots+1} u \wedge \overline{J} u \wedge T > 0,$$

where $u$ is a $d$-closed representative of $[u]$, $J u = i^{p-q} u$.

Proof. The case $m = 0$ is trivial. Assume that $m \geq 1$. Let $u$ be a $d$-closed representative of $[u]$. Since $[u]$ is $T$-primitive, there exists $v \in A^{p+n-m+1,q+n-m}$ such that

$$u \wedge T \wedge \omega = \overline{\partial} v.$$

Let us look at $v$ as an element in $V_T^{p+1,q}$. Timorin’s mixed hard Lefschetz theorem gives $v' \in V_T^{p,q-1}$ such that

$$v' \wedge \omega = v.$$

By Theorem [4.9] $\Box_{\omega}$ are elliptic on $V_T^k$ for every $k \neq m$. Thus the elliptic operator theory (see [17]) gives

$$v' = v'_h + \Box_{\partial} f', \quad v'_h \in V_T^{m+1} \cap \ker \Box_{\partial}, \quad f' \in V_T^{m-1}.$$

The Kähler identity in $T$-Hodge theory implies that $\Box_{\partial}$ commutes with $L$ and $\Lambda$, thus

$$v_h := v'_h \wedge \omega \in V_T^{m+1} \cap \ker \Box_{\partial},$$

and

$$v = v_h + \Box_{\partial} f, \quad f := \omega \wedge f'.$$

Since $\overline{\partial} v$ is $\partial$-closed, we have

$$0 = \partial \overline{\partial} v = \partial \overline{\partial}^* \partial f = 0.$$

Thus

$$\|\partial^* \overline{\partial} f\|^2 = (\partial \overline{\partial}^* \partial f, \overline{\partial} \partial f) = 0,$$

which gives $\partial^* \overline{\partial} f = \overline{\partial} \partial f = 0$. The Kähler identity in $T$-Hodge theory implies

$$[\partial^* \overline{\partial}, L] = 0.$$

Thus we have

$$\omega \wedge \overline{\partial}^* \partial f' = 0,$$
which gives
\[ \partial v = \omega \wedge \partial v' = \omega \wedge \partial \partial^* f'. \]

Let us write \( \partial^* f' = T \wedge g \), thus
\[ (u + \partial \partial g) \wedge T \wedge \omega = 0. \]

Thus Timorin’s mixed Hodge-Riemann bilinear relation gives
\[ \int_X (-1)^{m+\cdots+1} (u + \partial \partial g) \wedge J(u + \partial \partial g) \wedge T \geq 0, \]
where the equality holds if \( u + \partial \partial g \equiv 0 \). Stokes’ theorem implies
\[ \int_X (-1)^{m+\cdots+1} u \wedge Jt u \wedge T > 0 = \int_X (-1)^{m+\cdots+1} (u + \partial \partial g) \wedge J(u + \partial \partial g) \wedge T, \]
thus the theorem follows. \( \square \)

5.3. **Curvature of higher direct images.** We shall use the following setup:

1. \( \pi : \mathcal{X} \to B \) is a proper holomorphic submersion from a complex manifold \( \mathcal{X} \) to another complex manifold \( B \), each fiber \( X_t := \mathcal{X}|_{X_t} \);
2. \( E \) is a holomorphic vector bundle over \( \mathcal{X} \), \( E_t := E|_{X_t} \);
3. \( \omega \) is a smooth \((1,1)\)-form on \( \mathcal{X} \) that is positive on each fiber, \( \omega' := \omega|_{X_t} \);
4. \( h_E \) is a smooth Hermitian metric on \( E \), \( h_{E_t} := h_E|_{E_t} \).

For each \( t \in B \), let us denote by \( \mathcal{A}^{p,q}(E_t) \) the space of smooth \( E_t \)-valued \((p,q)\)-forms on \( X_t \). Let us recall the following definition in [7]:

**Definition 5.3.** Let \( V := \{V_t\}_{t \in B} \) be a family of \( \mathbb{C} \)-vector spaces over \( B \). Let \( \Gamma \) be a \( C^\infty (B) \)-submodule of the space of all sections of \( V \). We call \( \Gamma \) a smooth quasi-vector bundle structure on \( V \) if each vector of the fiber \( V_t \) extends to a section in \( \Gamma \) locally near \( t \).

Consider
\[ \mathcal{A}^{p,q} := \{ \mathcal{A}^{p,q}(E_t) \}_{t \in B}. \]
Denote by \( \mathcal{A}^{p,q}(E) \) the space of smooth \( E \)-valued \((p,q)\)-forms on \( \mathcal{X} \). Let us define
\[ \Gamma^{p,q} := \{ u : t \mapsto u^t \in \mathcal{A}^{p,q}(E_t) : \exists u \in \mathcal{A}^{p,q}(E), u|_{X_t} = u^t, \forall t \in B \}. \]
We call \( u \) above a smooth representative of \( u \in \Gamma^{p,q} \). We know that each \( \Gamma^{p,q} \) defines a quasi-vector bundle structure on \( \mathcal{A}^{p,q} \).

**Definition 5.4.** Let \( (V, \Gamma) = \oplus_{k=0}^{2n}(V^k, \Gamma^k) \) be a direct sum of quasi vector bundles over a smooth manifold \( B \). Let \( L \) be a section of \( \text{End}(V) \). We call \( (V, \Gamma, L) \) a Lefschetz quasi vector bundle if
\[ L(\Gamma^l) \subset \Gamma^{l+2}, \forall 0 \leq l \leq 2(n-1), \quad L(\Gamma^{2n-1}) = L(\Gamma^{2n}) = 0, \]
and each \( L^k : \Gamma^{n-k} \to \Gamma^{n+k}, 0 \leq k \leq n, \) is an isomorphism.

Same as before, one may define the Lefschetz star operator and the \( \text{sl}_2 \)-triple on a general Lefschetz quasi vector bundle. Consider
\[ (\mathcal{A}, \Gamma) := \oplus_{k=0}^{2n}(\mathcal{A}^k, \Gamma^k), \quad (\mathcal{A}^k, \Gamma^k) := \oplus_{p+q=k}(\mathcal{A}^{p,q}, \Gamma^{p,q}) \]
and define \( L \in \text{End}(\mathcal{A}) \) such that
\[ Lu(t) = \omega^t \wedge u^t, \forall u \in \Gamma. \]
Then the hard Lefschetz theorem implies that \( (\mathcal{A}, \Gamma, L) \) is a Lefschetz quasi vector bundle. One may also define the notion of connection on a general quasi vector bundle, see [7].
Thus our main theorem is still true for general Lefschetz quasi vector bundles. We shall use the following connection on $(A, \Gamma)$.

**Definition 5.5.** The Lie-derivative connection, say $\nabla^A$, on $(A, \Gamma)$ is defined as follows:

$$\nabla^A u := \sum dt^j \otimes [d^E, \delta V_j] u + \sum d\bar{t}^\bar{j} \otimes [d^E, \delta \bar{V}_j] u, \quad u \in \Gamma,$$

where $d^E := \partial + \partial^E$ denotes the Chern connection on $(E, h_E)$ and each $V_j$ is the horizontal lift of $\partial/\partial t^j$ with respect to $\omega$.

Our main theorem implies:

**Theorem 5.4.** If $d\omega \equiv 0$ then the Lie-derivative connection $\nabla^A$ commutes with the Lefschetz star operator $\ast_s$ on the Lefschetz quasi vector bundle $(A, \Gamma, L)$.

**Proof.** By our main theorem, it suffices to prove $[\nabla^A, L] = 0$, i.e. $[d^E, \delta V_j] = 0$. Notice that $d^E = \partial + \partial^E$ and $
abla^A - D^A$ satisfies

$$D^A u := \sum dt^j \otimes [\partial^E, \delta V_j] u + \sum d\bar{t}^\bar{j} \otimes [\bar{\partial}, \delta \bar{V}_j] u, \quad u \in \Gamma^{p,q}.$$

Thus

$$(\nabla^A - D^A) u = \sum dt^j \otimes [\bar{\partial}, \delta V_j] u + \sum d\bar{t}^\bar{j} \otimes [\partial^E, \delta \bar{V}_j] u.$$

Put

$$\kappa_j u := [\bar{\partial}, \delta V_j] u, \quad \kappa_{\bar{j}} u := [\partial^E, \delta \bar{V}_j] u.$$

Then

$$\nabla^A - D^A = \sum dt^j \otimes \kappa_j + \sum d\bar{t}^\bar{j} \otimes \kappa_{\bar{j}}.$$

**Definition 5.6.** We call

$$\kappa_j u := [\bar{\partial}, \delta V_j] u,$$

the non-cohomological Kodaira-Spencer action of $\kappa_j$ on $u \in \Gamma$.

The fiberwise Hodge star operator $\ast$ equals $\ast_s \circ J$, recall that $J$ is the Weil-operator

$$Ju = i^{p-q}u, \quad u \in \Gamma^{p,q}.$$

Thus Theorem 5.4 implies:

**Proposition 5.5.** If $d\omega \equiv 0$ then $D^A \ast = \ast D^A$, $\kappa_j \ast = - \ast \kappa_j$ and $\kappa_{\bar{j}} \ast = - \ast \kappa_{\bar{j}}$.

**Theorem 5.6.** If $d\omega \equiv 0$ then $D^A$ defines a Chern connection on each $(A^{p,q}, \Gamma^{p,q})$ and each $\kappa_j$ is the adjoint of $\kappa_j$.

**Proof.** First part: Since the metric on $A^{p,q}$ is defined by

$$(u, v) = \int_{X_t} \{u, *v\},$$

thus the above proposition implies that $D^A$ preserves the metric. The fact that the square of the $(0,1)$-part of $D^A$ vanishes follows from the usual Lie derivative identity, see [7] for the details. Thus $D^A$ is a Chern connection.
Second part: Let \( u \in \Gamma^{p,q}, \ v \in \Gamma^{p-1,q+1} \), then for bidegree reason, we have
\[
0 = \partial/\partial t^j(u,v) = (\kappa_j u, v) + (u, \ast^{-1} \kappa_j \ast v),
\]
which gives \( (\kappa_j u, v) = (u, \kappa_j v) \) by the above proposition. □

Remark: One may also prove the above theorem by a direct computation without using the Hodge star operator, see [25]. For other related results on the Lie-derivative connection, see [6], [14], [20], [21], [23], [24], [28], [29], [31].

The curvature of the Lie-derivative connection is
\[
(D^A)^2 = \sum \left[ \left[ dE, \delta V_j \right], \left[ dE, \delta V_k \right] \right] dt^j \wedge d\bar{t}^k.
\]
For bidegree reason, we have
\[
(5.2) \quad (D^A)^2 = (\nabla^A)^2 - \sum [\kappa_j, \kappa_k] dt^j \wedge d\bar{t}^k.
\]
One may get a curvature formula of the higher direct image bundles using the above formula and the sub-bundle-quotient-bundle curvature formula, see [7] for the details.

Definition 5.7. We call \( u \in \Gamma^{p,q} \) a holomorphic section of \( A^{p,q} \) if each
\[
[\overline{\partial}, \delta V_j] u = 0,
\]
on fibers, i.e. the \((0,1)\)-part of \( D^A \) vanishes on \( u \).

Theorem 5.6 and (5.2) give:

Proposition 5.7. If \( d\omega = 0 \) and \( u \) is a holomorphic section of \( A^{p,q} \) then
\[
 i\partial\overline{\partial}|u|^2 \geq -i((\nabla^A)^2 u, u) + i \sum ((\kappa_j^+ u, \kappa_j^- u) - (\kappa_j u, \kappa_j u)) dt^j \wedge d\bar{t}^k.
\]
We shall show how to use the above proposition in a future publication [33].

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