A simple, heuristic derivation of our ‘no backreaction’ results

Stephen R Green¹,³ and Robert M Wald²

¹ Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada
² Enrico Fermi Institute and Department of Physics, The University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637, USA

E-mail: sgreen@perimeterinstitute.ca and rmwa@uchicago.edu

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Abstract
We provide a simple discussion of our results on the backreaction effects of
density inhomogeneities in cosmology without mentioning one-parameter
families or weak limits. Emphasis is placed on the manner in which ‘aver-
ing’ is done and the fact that one is solving Einstein’s equation. The key
assumptions and results that we rigorously derived within our original math-
ematical framework are thereby explained in a heuristic way.

Keywords: general relativity, cosmology, backreaction

1. Introduction

In a series of papers [1–4], we provided a rigorous mathematical framework for analyzing the
effects of backreaction produced by small scale inhomogeneities in cosmology. We proved
results showing that no large backreaction effects can be produced by matter inhomogeneities,
provided that the energy density of the matter is positive in all frames. In particular, we
proved that at leading order in our approximation scheme, the effective stress-energy pro-
vided by the nonlinear terms in Einstein’s equation must be traceless and have positive energy
in all frames, corresponding to the backreaction effects of gravitational radiation.

Recently, our work has been criticized by Buchert et al [5]. We have responded to these
criticisms in [6] and see no need to further amplify our refutation of these criticisms here.
Nevertheless, it has become clear to us that it would be useful to provide a simple, heuristic
discussion of our results in order to make various aspects of our work clearer, including (i) the
nature of our assumptions, (ii) the relationship of our procedures to ‘averaging,’ (iii) the
manner in which we use Einstein’s equation, and (iv) the significance of our results. In this
way, the basic nature of our results can be seen without invoking technical mathematical processes, such as the taking of weak limits. The price paid for this, of course, is a loss of mathematical precision and rigor; for example, many of our equations below will involve the use of the relations \(\sim\) or \(\ll\)—which will not be given a precise meaning—and some terms in various equations will be dropped because they are ‘small.’ However, the reader desiring a more precise/rigorous treatment can simply re-read our original papers [1–4]. In section 3 below, we will provide a guide to relating the heuristic discussion of the present paper to the precise formulation (using one-parameter families and weak limits) given in our original papers.

To begin, the situation that we wish to treat is one where the spacetime metric, which solves the Einstein equation exactly on all scales, takes the form

\[
g_{ab} = g_{ab}^{(0)} + \gamma_{ab},
\]

where \(g_{ab}^{(0)}\) has ‘low curvature’ and \(\gamma_{ab}\) is ‘small,’ but derivatives of \(\gamma_{ab}\) may be large, so that the geodesics and the curvature (and, hence, the associated stress-energy distribution) of \(g_{ab}\) may differ significantly from that of \(g_{ab}^{(0)}\). In particular, it is not assumed that \(g_{ab}^{(0)}\) solves the Einstein equation. This should be an excellent description of the metric of our Universe except in the immediate vicinity of black holes and neutron stars. To make our assumptions about the form of the metric a bit more precise, it is convenient to introduce a Riemannian metric \(e_{ab}\) and use it to define norms on all tensors. We assume that

\[
|g_{ab}^{(0)}| \equiv [e^{ac}e^{bd}g_{ab}^{(0)}\gamma_{cd}]^{1/2} \sim 1,
\]

whereas

\[
|\gamma_{ab}| \ll 1.
\]

We denote the curvature length scale associated with \(g_{ab}^{(0)}\) by \(R\), i.e.

\[
|R^{(0)}_{abcd}| \sim 1/R^2.
\]

In cosmological applications, \(g_{ab}^{(0)}\) would be taken to be a metric with FLRW symmetry (but not assumed to satisfy the Friedmann equations) and \(R\) would be the Hubble radius, \(R = R_0 \sim 5\) Gpc (today), but our arguments apply to much more general situations. For our Universe, apart from the immediate vicinity of strong field objects, \(\gamma_{ab}\) would be largest near the centers of rich clusters of galaxies, where \(|\gamma_{ab}|\) can be as large as \(\sim 10^{-4}\). However, although \(\gamma_{ab}\) is required to be small, \(\gamma_{ab}\) is allowed to have large derivatives. We require that the first derivatives of \(\gamma_{ab}\) be constrained only by

\[
|\gamma_{ab} \nabla_{c} \gamma_{de}| \ll 1/R,
\]

where \(\nabla_{c}\) denotes the derivative operator associated with \(g_{ab}^{(0)}\). Second derivatives of \(\gamma_{ab}\) are entirely unconstrained, so locally we may have

\[
|\nabla_{c} \nabla_{d} \gamma_{ab}| \gg 1/R^2.
\]

Thus, the curvature of \(g_{ab}\) is allowed to be locally much greater than that of \(g_{ab}^{(0)}\), as is the case of main interest for cosmology. In this situation, ordinary perturbation theory about \(g_{ab}^{(0)}\) cannot be directly applied to Einstein’s equation for \(g_{ab}\), since even though \(\gamma_{ab}\) itself is small, the terms involving \(\gamma_{ab}\) that appear in Einstein’s equation are not small.

\[\text{Equation (5) will be used to justify dropping various terms that arise in integrals over large regions, such as those that occur in going from equations (17) and (18) below. It is therefore fine if (5) fails to hold in highly localized regions, such as near the surface of a massive body.}\]
We assume that the matter in the Universe is described by a stress-energy tensor $T_{ab}$ that satisfies the weak energy condition,

$$T_{ab}u^au^b \geq 0,$$

for all timelike $u^a$. (Here ‘timelike’ means with respect to $g_{ab}$, i.e. $g_{ab}u^au^b < 0$, although it makes essentially no difference whether we use $g_{ab}$ or $g^{(0)}_{ab}$ since $|\gamma_{ab}| < 1$.) We assume further that $T_{ab}$ is (essentially) homogeneous on some scale $L$ with $L \ll R$. By this we mean that $T_{ab}$ can be written as

$$T_{ab} = T^{(0)}_{ab} + \Delta T_{ab},$$

where $|T^{(0)}_{ab}| \lesssim 1/R^2$ and $\Delta T_{ab}$ ‘averages’ to (nearly) zero on large scales compared with $L$ (even though $\Delta T_{ab}$ may locally be extremely large compared with $T^{(0)}_{ab}$). In the case of interest for cosmology, $T^{(0)}_{ab}$ would have FLRW symmetry.

The assumption that $\Delta T_{ab}$ averages to zero on large scales is a key assumption, so we should further explain both its meaning and our justification for making it. First, it is not obvious what one means by the ‘averaging’ of a tensor quantity such as $\Delta T_{ab}$. In a non-flat spacetime, parallel transport is path dependent so the values of tensors at different points cannot be meaningfully compared, as required to give any invariant meaning to an averaging procedure. Now, since $g^{(0)}_{ab}$ is locally flat in any region, $D$, of size $D \ll R$, averaging of tensor fields over such a region $D$ is well defined. However, we do not want to require $D \ll R$. Furthermore, even if we restricted the size of $D$ to $D \ll R$, we do not want to simply integrate quantities over such a region $D$ because the introduction of sharp boundaries for $D$ will produce artifacts that we wish to avoid. We will therefore do our ‘averaging’ in the following manner: We choose a region $D$ with $D > L$ and introduce a smooth tensor field $f^{ab}$ with support in $D$ such that $f^{ab}$ ‘varies as slowly as possible’ over $D$ compatible with its vanishing outside of $D$ and with the curvature of $g^{(0)}_{ab}$. Specifically, for any region $D$ with $L < D \lesssim R$, we require $f^{ab}$ to be chosen so that

$$\max |\nabla f^{ab}| \lesssim \max |f^{ab}|/D.$$  

A more precise statement of our homogeneity requirement on $T_{ab}$ is that $\Delta T_{ab}$ is such that for any region $D$ with $L < D \lesssim R$ and any such $f^{ab}$, we have

$$\left| \int f^{ab} \Delta T_{ab} \right| \ll \left| \int f^{ab} T^{(0)}_{ab} \right| \lesssim \frac{1}{R^2} \int |f^{ab}|.$$  

The integral appearing in equation (10) is a spacetime integral over the region $D$. In a general context—where significant amounts of gravitational radiation may be present and the motion of matter may be highly relativistic—both $\gamma_{ab}$ and $T_{ab}$ may vary rapidly in both space and time, and it is important that $D$ be sufficiently large in both space and time. However, for cosmological applications, the case of greatest interest is one in which there is rapid spatial variation on small scales compared with the Hubble radius, but time variations are negligibly small. In this case, it is important that the spatial extent, $D$, of $D$ be larger than the spatial homogeneity scale $L$, but the time extent of $D$ may be taken to be significantly smaller than $D/c$. For our Universe, the assumptions of the previous paragraph should hold for

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5 We refer to small fluctuations in $\Delta T_{ab}$ beyond the homogeneity scale as its ‘long-wavelength part’ [2]. These fluctuations can be described by linear perturbation theory, and will be neglected in this paper. For further discussion, see section III of [1], and [2].

6 If we chose $D > R$, we would have to replace the right side of this equation with $\max |f^{ab}|/R$. However, there is no reason to choose $D > R$. 

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$L \sim 100$ Mpc. (Beyond this scale, equation (10) does not preclude the presence of small fluctuations in $\Delta T_{ab}$ (see footnote 5).)

To summarize, there are three length scales that appear in our analysis. The first is the curvature length scale, $R$, of $g_{ab}^{(0)}$ (i.e. the Hubble radius). The second is the homogeneity length scale, $L$. It is an essential assumption that $L \ll R$. The third is the averaging length scale, $D$, which is up to us to choose. We must always choose $D > L$ if we wish to (very nearly) average out the stress-energy inhomogeneities. It is never useful to choose $D > R$, since we do not wish to average over the background structure. For some computations, it will be useful to choose $D \sim R$, and for others it will be more useful to choose $L < D \ll R$. In all cases, the averaging will be done over a region $D$ of size $D$ using a slowly varying test tensor field (see equation (9)).

We may interpret $g_{ab}^{(0)}$ as the ‘averaged metric’ (although no actual averaging need be done since $|\gamma_{ab}| \ll 1$), whereas $T_{ab}^{(0)}$ represents the large-scale average of $T_{ab}$. The issue at hand is whether the small scale inhomogeneities of $g_{ab}$ and $T_{ab}$ can contribute nontrivially to the dynamics of $g_{ab}^{(0)}$. A priori, this is possible because even though $\gamma_{ab}$ is assumed to be small, Einstein’s equation for $g_{ab}$ contains derivatives of $\gamma_{ab}$, which need not be small. Consequently, the average of the Einstein tensor, $G_{ab}$, of $g_{ab}$ need not be close to the Einstein tensor, $G_{ab}^{(0)}$, of $g_{ab}^{(0)}$. Thus, although $g_{ab}$ is assumed to be an exact solution of Einstein’s equation (with cosmological constant, $\Lambda$) with stress-energy source $T_{ab}$, it is possible that $g_{ab}^{(0)}$ may not be close to a solution to Einstein’s equation with source $T_{ab}^{(0)}$. If we have

$$G_{ab}^{(0)} + \Lambda g_{ab}^{(0)} - 8\pi T_{ab}^{(0)} \ll 1/R^2,$$

then we say that there is a negligible backreaction effect of the small scale inhomogeneities on the effective dynamics of $g_{ab}^{(0)}$. Conversely, if

$$G_{ab}^{(0)} + \Lambda g_{ab}^{(0)} - 8\pi T_{ab}^{(0)} \sim 1/R^2,$$

then the backreaction effects are large. Our aim is to determine whether the backreaction effects can be large and, if so, to determine the properties of the averaged effective stress-energy tensor of backreaction, defined by

$$8\pi T_{ab}^{(0)} \equiv G_{ab}^{(0)} + \Lambda g_{ab}^{(0)} - 8\pi T_{ab}^{(0)}.$$

The most interesting possibility would be to have large backreaction effects with $T_{ab}^{(0)}$ of the form $-CG_{ab}^{(0)}$ with $C \sim 1/R^2$, in which case the backreaction effects of small scale inhomogeneities would mimic that of a cosmological constant, and the observed acceleration of our Universe could be attributed to these backreaction effects, without the need to postulate the presence of a true cosmological constant, $\Lambda$, in Einstein’s equation. However, we will show that this is not possible.

Our strategy, now, is simply the following. We write down the exact Einstein equation,

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}.$$  

We then take suitable averages of this equation in the manner described above to obtain an expression for the effective stress-energy of backreaction, $T_{ab}^{(0)}$, and determine its properties, using only our assumptions (3), (5), (7), and (10). In order to implement our strategy, it is extremely useful to write the exact Einstein equation (14) in the form

$$G_{ab}^{(0)} + \Lambda g_{ab}^{(0)} - 8\pi T_{ab}^{(0)} = 8\pi \Delta T_{ab} - \Lambda \gamma_{ab} - G_{ab}^{(1)} - G_{ab}^{(2)} - G_{ab}^{(3+)}.$$  

Here $G_{ab}^{(1)}$ denotes the terms in the exact Einstein tensor $G_{ab}$ that are linear in $\gamma_{ab}$; $G_{ab}^{(2)}$ denotes the terms in $G_{ab}$ that are quadratic in $\gamma_{ab}$; and $G_{ab}^{(3+)}$ denotes the terms in $G_{ab}$ that are cubic and higher order in $\gamma_{ab}$. 


Before proceeding with our analysis, we comment that if, following [7], one simply inserts Newtonian estimates for \( \gamma_{ab} \) associated with the various density inhomogeneities found in our Universe (clusters of galaxies, galaxies, stars, etc.), one can easily see that the backreaction effects in our Universe are negligible. Our aim here is to significantly improve upon such ‘back of the envelope estimates’ by showing that the backreaction associated with density inhomogeneities can never be large if (3), (5), (7), (10), and (14) hold.

2. Determination of the averaged effective stress-energy

We now analyze the contributions of the various terms on the right side of equation (15). Our first claim is that, under our assumptions, the contribution of \( G_{ab}^{(3+)} \) is negligible compared with \( G_{ab}^{(2)} \). This is because no term in Einstein’s equation contains more than a total of two derivatives. Thus, a term that is cubic or higher order in \( \gamma_{ab} \) must contain at least one factor of \( \gamma_{ab} \) that is undifferentiated. Since \( \gamma_{ab} \ll 1 \), all such terms will be much smaller than corresponding terms in \( G_{ab}^{(2)} \). Thus, we neglect \( G_{ab}^{(3+)} \) in equation (15). As far as we are aware, this conclusion is in agreement with all other approaches to backreaction, i.e. we are not aware of any approach to backreaction that claims that the dominant effects are produced by cubic or higher order terms in Einstein’s equation.

We now average Einstein’s equation (15) (with \( G_{ab}^{(3+)} \) discarded) in the manner described in the previous section: We choose a region \( D \) of size \( D \sim R \), choose a slowly varying \( f^{ab} \) with support in \( D \) (see equation (9)), multiply equation (15) by \( f^{ab} \), and integrate. The term \( \int f^{ab} \Delta T_{ab} \) may be neglected by equation (10). We estimate \( \int f^{ab} G_{ab}^{(1)} \) by integrating by parts to remove all derivatives\(^7\) from \( \gamma_{ab} \). We obtain

\[
\left| \int f^{ab} G_{ab}^{(1)} \right| \lesssim \int |\nabla_v \nabla f^{ab}| |\gamma_{ab}| \lesssim \frac{1}{R^2} \max |\gamma_{ab}| \int |f^{ab}| \lesssim \frac{1}{R^2} \int |f^{ab}|. \tag{16}
\]

Therefore, the contribution of \( G_{ab}^{(1)} \) to the averaged Einstein equation may be neglected. Similarly, the contribution from \( A \int f^{ab} \gamma_{ab} \) may be neglected, and to our level of approximation we obtain

\[
\int f^{ab} \gamma_{ab} \approx -\frac{1}{8\pi} \int f^{ab} G_{ab}^{(2)}. \tag{17}
\]

Furthermore, the terms in \( G_{ab}^{(2)} \) can be divided into two types: (i) terms quadratic in first derivatives of \( \gamma_{ab} \), i.e. of the form \( (\nabla \gamma)(\nabla \gamma) \), and (ii) terms of the form \( \gamma \nabla \nabla \gamma \). For the terms in category (ii), we integrate by parts on one of the derivatives of \( \gamma \) to eliminate the second derivative terms. This derivative then will either act on the other \( \gamma \) factor—thereby converting it to a term of type (i)—or it will act on \( f^{ab} \)—in which case it can be neglected on account of equations (5) and (9). Consequently, we may replace \( G_{ab}^{(2)} \) in equation (17) by an expression, \( \tilde{G}_{ab}^{(2)} \), that is quadratic in first derivatives of \( \gamma_{ab} \), and we may rewrite equation (17) as

\[
\int f^{ab} \gamma_{ab} \approx -\frac{1}{8\pi} \int f^{ab} \tilde{G}_{ab}^{(2)}. \tag{18}
\]

for all \( f^{ab} \) that are ‘slowing varying’ in the sense discussed above. In order to make contact with commonly used terminology and notation (at least in discussions of gravitational radiation), it is useful to note that equation (18) can be rewritten as

\[7\] If time variations are slow, we may choose the time extent of \( D \) to be smaller than \( R/c \).

\[8\] If the time derivatives of \( \gamma_{ab} \) are negligibly small, there is no need to integrate by parts on the time derivatives, which is why the time extent of \( D \) may be chosen to be smaller than its spatial extent.
\[ t_{ab}^{(f)}(0) = -\frac{1}{8\pi} \langle G_{ab}^{(2)} \rangle, \]  

where \( \langle G_{ab}^{(2)} \rangle \) denotes the ‘Isaacson average’ [8, 9] of \( G_{ab}^{(2)} \), i.e. the quantity obtained by replacing \( G_{ab}^{(2)} \) by \( \tilde{G}_{ab}^{(2)} \) and ‘averaging’ over a region of size \( D > L \). The meaning of equation (19) is, of course, simply that equation \( (18) \) holds for all \( f^{\mu\nu} \) that are ‘slowing varying’ in the sense discussed above.

At this stage, we have ‘averaged’ the Einstein equation subject to our assumptions on the sizes of various terms. The average of the Einstein equation, however, contains much less information than the full Einstein equation, which holds at each spacetime point. This ignorance is encapsulated in an effective backreaction stress-energy tensor \( \tilde{\rho}_{ab} \) that is completely unconstrained, except to be expressed as an average of quadratic terms in the first derivatives of \( \tilde{\gamma}_{ab} \). Further constraints on \( t_{ab}^{(f)} \) can, however, be obtained by using the fact that the Einstein equation must hold at each spacetime point.

Our main results [1] on \( t_{ab}^{(f)} \) are that it is traceless and that it satisfies the weak energy condition. The proof of these results requires some complicated calculations that were done in [1]. Rather than repeat these calculations here, we will simply outline the logic of our arguments within the heuristic framework of the present paper, referring the reader to [1] for the details of the various calculations.

The quantity \( \tilde{G}_{ab}^{(2)} \) is given by a rather complicated expression, and in order to make progress on determining its properties, we need additional information about \( \tilde{\gamma}_{ab} \). However, the only available information about \( \tilde{\gamma}_{ab} \) comes from Einstein’s equation \( (15) \). On examination of this equation, one sees that, locally, the potentially largest terms are \( 8\pi \Delta T_{ab} \) and \( G_{ab}^{(1)} \). Therefore, it might be tempting to set these potentially largest terms to zero by themselves, i.e. to postulate that the equation \( G_{ab}^{(1)} = 8\pi \Delta T_{ab} \) holds. Indeed, equations along these lines (with certain gauge choices) were imposed in [8–10]. However, as we explained in section III of [1], this equation is not justified. Indeed, if \( G_{ab}^{(0)} + \tilde{A} g_{ab}^{(0)} = 8\pi T_{ab} \) this equation is not even gauge invariant.

Nevertheless, we can obtain very useful information about \( \langle G_{ab}^{(2)} \rangle \) from Einstein’s equation in a completely reliable way by the following procedure due to Burnett [11]: We multiply equation \( (15) \) by \( \gamma_{cd} \) and ‘average’ the resulting equation, i.e. we multiply the resulting 4-index tensor equation by a slowing varying tensor \( f_{cdab} \) with support in region \( D \) (with \( D \sim R \) as above) and integrate. Since \( |\gamma_{ab}| \ll 1 \), the only terms in the resulting equation that are not \( a \) priori negligible are the ones arising from \( 8\pi \Delta T_{ab} \) and \( G_{ab}^{(1)} \). We therefore obtain

\[ \int f_{cdab} \gamma_{cd} G_{ab}^{(1)} = 8\pi \int f_{cdab} \gamma_{cd} \Delta T_{ab}. \]  

We now show that the right side of equation (20) is negligibly small as a consequence of the weak energy condition on \( T_{ab} \). To see this, we choose \( f_{cdab} \) to be of the form \( f_{cdab} = f_{cd} \gamma^{ab} \), where \( f_{cd} \) and \( t^{a} \) are slowly varying and \( t^{a} \) is unit timelike. For such an \( f_{cdab} \), the right side of equation (20) becomes

\[ \int f_{cd} \gamma_{cd} t^{a} T_{ab} = \int f_{cd} \gamma_{cd} [\rho - \rho^{(0)}], \]  

where \( \rho \equiv T_{ab} t^{a} t^{b} \) and \( \rho^{(0)} \equiv T_{ab}^{(0)} t^{a} t^{b} \). Since \( |T_{ab}^{(0)}| \lesssim 1/R^2 \) and \( |\gamma_{ab}| \ll 1 \), the contribution of \( \rho^{(0)} \) term is negligible. However, since locally we can have \( \rho \gg 1/R^2 \), it is conceivable that the \( \rho \) term could make a large contribution. The key point is that the positivity of \( \rho \) precludes this possibility because
\[ \left| \int f^{cd} \gamma_{cd} \rho \right| \lesssim \max |\gamma_{cd}| \int |f^{cd}| \rho \sim \max |\gamma_{cd}| \int |f^{cd}| \rho^{(0)} \]
\[ \sim \frac{\max |\gamma_{cd}|}{R^2} \int |f^{cd}| \ll \frac{1}{R^2} \int |f^{cd}|. \]  
(22)

Here the positivity of \( \rho \) was used to omit an absolute value sign on \( \rho \) in the first inequality; we were then able to replace \( \rho \) by \( \rho^{(0)} \) in the next (approximate) equality because \( |f^{cd}| \) is slowly varying. By contrast, if \( \rho \) were allowed to have large fluctuations of both positive and negative type, then we would not be able to get an estimate similar to (22). Basically, for \( \rho \geq 0 \), although we may have arbitrarily large positive density fluctuations in localized regions, we must compensate for these by having large voids where the density fluctuations are only mildly negative. The net contribution to equation (21) is then negligible.

Thus, we have shown that the right side of equation (20) may be neglected when \( f^{cdab} \) is of the form \( f^{cd} \rho^{(b)} \) for any (slowly varying) \( f^{ab} \) and any (slowing varying) unit timelike \( t^a \). But any slowly varying \( f^{cdab} \) can be approximated by a linear combination of terms of the form \( f^{cd} \rho^{(b)} \) (with different choices of \( t^a \) as well as \( f^{ab} \)). It follows that the right side of equation (20) is negligible for all slowly varying \( f^{cdab} \), as we desired to show, and hence

\[ \int f^{cdab} \gamma_{cd} G^{(1)}_{ab} = 0. \]  
(23)

The expression for \( G^{(1)} \) is of the form \( \nabla \nabla \gamma \). We can again integrate by parts in equation (23) to remove one of these derivatives from \( \gamma \). This derivative will then either act on the other \( \gamma \) factor or it will act on \( f^{abcd} \), in which case it can be neglected. Thus, we may rewrite equation (23) as

\[ \langle \gamma_{cd} G^{(1)}_{ab} \rangle = 0, \]  
(24)

where the ‘Issacson average’ again denotes the average of the quantity quadratic in first derivatives of \( \gamma \) obtained by integration by parts in equation (23).

In view of equation (19), the tracelessness of \( t^{ab}_{(0)} \) is then an immediate consequence of equation (24) together with the mathematical fact that

\[ \langle G^{(2)}_{ab} \rangle = \frac{1}{2} \langle \gamma^{(1)}{}_{ab} \rangle, \]  
(25)

as can be seen by direct inspection of the explicit formulas for both sides of this equation. We refer the reader to [1] or [11] for the details.

The demonstration that \( t^{(0)}_{ab} \) satisfies the weak energy condition—i.e. that \( t^{(0)}_{ab} t^{ab} \geq 0 \)—is considerably more difficult. To show this, it is convenient to now work in an ‘averaging region’ \( \mathcal{D} \) with \( L < D \ll R \). We choose a point \( P \in \mathcal{D} \) and a unit timelike vector \( t^a \) at \( P \) and construct Riemannian normal coordinates (with respect to \( g^{(0)}_{ab} \)) starting from 0. Since \( D < R \), these coordinates will cover \( \mathcal{D} \) and the components of \( g^{(0)}_{ab} \) will take a nearly Minkowskian form in \( \mathcal{D} \). We choose a positive function \( f \) with support in \( \mathcal{D} \) that is slowly varying in the sense used above (see equation (9)). Our aim is to show that for any such \( f \) we have

\[ \int f^{(0)}_{ab} t^{ab} \geq 0, \]  
(26)

where \( t^a \) has been extended to \( \mathcal{D} \) via our Riemannian normal coordinates, i.e. \( t^a = (\partial / \partial y)^a \).

Following [1], we start with formula (19) for \( t^{(0)}_{ab} \). By some relatively nontrivial manipulations using equation (24), it turns out that it is possible to rewrite \( \langle G^{(2)}_{ab} \rangle t^{ab} \) entirely in terms of spatial derivatives of spatial components of \( \gamma_{ab} \). The desired formula, derived in [1], is
\[
\int f_{ab}^{(0)} \varepsilon^{b} = \frac{1}{32\pi} \int d^{4}x \left[ \partial_{i} \gamma_{jk} \partial^{i} \gamma^{jk} - 2 \partial_{j} \gamma_{ik} \partial^{j} \gamma^{ik} + 2 \partial_{j} \gamma_{ik} \partial^{i} \gamma^{jk} - \partial_{j} \gamma_{ik} \partial^{i} \gamma^{jk} \right],
\]  
(27)

where \( i, j, k \) run over the spatial indices of the Riemannian normal coordinates, \( \partial_{i} \) denotes the partial derivative operator in these coordinates, and the raising and lowering of indices is done using the (essentially flat) background metric. We now define

\[
\psi_{ij} = \sqrt{\gamma} \gamma_{ij}.
\]  
(28)

Since \( f \) is ‘slowly varying,’ to a good approximation, we have

\[
\int f_{ab}^{(0)} \varepsilon^{b} = \frac{1}{32\pi} \int d^{4}x \left[ \partial_{i} \psi_{jk} \partial^{i} \psi^{jk} - 2 \partial_{j} \psi_{ik} \partial^{j} \psi^{ik} + 2 \partial_{j} \psi_{ik} \partial^{i} \psi^{jk} - \partial_{j} \psi_{ik} \partial^{i} \psi^{jk} \right].
\]  
(29)

Even though our Riemannian normal coordinates are not globally well defined on the actual spacetime, since \( \psi_{ij} \) has support in \( \Delta \), we can pretend that the coordinates \( x^{i} \) range from \( -\infty \) to \( +\infty \). Let \( \tilde{\psi}_{ij} \) denote the spatial Fourier transform of \( \psi_{ij} \), i.e.

\[
\tilde{\psi}_{ij}(t, k) = \frac{1}{(2\pi)^{3/2}} \int d^{3}x \exp(-ik_{1}x^{1}) \psi_{ij}(t, x).
\]  
(30)

We decompose \( \tilde{\psi}_{ij} \) into its scalar, vector, and tensor parts via

\[
\tilde{\psi}_{ij} = \partial_{k}k_{j} - 2\tilde{\omega}q_{ij} + 2k_{i}\tilde{\varepsilon}_{j} + \tilde{s}_{ij},
\]  
(31)

where \( q_{ij} \) is the projection orthogonal to \( k^{i} \) of the Euclidean metric on Fourier transform space and \( \tilde{\varepsilon}_{i}k^{i} = \tilde{s}_{ij}k^{i} = \tilde{s}_{ij} = 0 \). With this substitution, our formula for the effective energy density of backreaction becomes

\[
\int f_{ab}^{(0)} \varepsilon^{b} = \frac{1}{32\pi} \int d^{3}k k^{i}k^{j} [\tilde{s}_{ij}]^{2} - 8|\tilde{\omega}|^{2}.
\]  
(32)

The term involving \( |\tilde{s}_{ij}|^{2} \) arising from the ‘tensor part’ of \( \psi_{ij} \) is positive definite and corresponds to the usual formula for the effective energy density of short wavelength gravitational radiation [10]. This term can be ‘large,’ corresponding to the well known fact that gravitational radiation can produce large backreaction effects. In a cosmological context, this will contribute effects equivalent to that of a \( P = \rho / 3 \) fluid. The term of potentially much greater interest for the backreaction effects in cosmology associated with density inhomogeneities is the one involving \( |\tilde{\omega}|^{2} \), which arises from the ‘scalar part’ of \( \psi_{ij} \). This term is negative definite. The final—and most difficult—step of the proof is to show that, in fact, this term is negligibly small.

In position space, the term of interest takes the form

\[
E_{\phi} = -\frac{1}{4\pi} \int d^{4}x \partial_{i} \phi \partial^{i} \phi,
\]  
(33)

where \( \phi \) is the inverse Fourier transform of \( \tilde{\phi} \). This is of the form of Newtonian potential energy. Furthermore, it follows from Einstein’s equation (14) that \( \phi \) satisfies a Poisson-like equation. To illustrate the basic idea of our demonstration that \( E_{\phi} \) is negligible, suppose that \( \phi \) exactly satisfied the Poisson equation

\[
\partial^{i} \partial_{i} \phi = 4\pi \sqrt{f} \rho
\]  
(34)

(with \( f \) as in equation (28)) and suppose it were known that \( |\phi| \ll \sqrt{f} \) (as would be expected since \( |\psi_{ij}| = |\sqrt{f} \gamma_{ij}| \ll \sqrt{f} \)). In that case, by integrating equation (33) by parts, we obtain

\[ \text{Note that we have not made any Newtonian approximations. Note also that this formula is off by a factor of two from the standard formula for Newtonian gravitational energy.} \]
\[ E_\phi = \frac{1}{4\pi} \int d^4x \phi \partial_i \phi = \int d^4x \phi \sqrt{f} \rho. \] (35)

and hence

\[ |E_\phi| \lesssim \int d^4x |\phi| |\sqrt{f} \rho| \ll \int d^4x \rho \sim \int d^4x \rho^{(0)} \sim \frac{1}{R^2} \int d^4x f, \] (36)

which shows that \( E_\phi \) indeed contributes negligibly to the effective energy density. Here, as in equation (22), the positivity of \( \rho \) was used to omit the absolute value sign on \( \rho \) in the first inequality, and the slowly varying character of \( f \) was then used to replace \( \rho \) by \( \rho^{(0)} \).

The actual proof that \( E_\phi \) is negligible is much more difficult than as just sketched above because (i) \( \phi \) does not satisfy the simple Poisson equation (34) but rather an equation that contains many other terms that, \textit{a priori}, are not negligibly small and (ii) since \( \phi \) is nonlocally related to \( \psi_i \), it is not obvious that \( \phi \) is ‘small’ in the required sense, i.e. this must be shown. The reader wishing to see the details of how these difficulties are overcome should read section II of our original paper [1]. However, the key element of the proof is the argument sketched in the previous paragraph.

The above results show that, assuming only that (3), (5), (7), and (10) hold, then in the absence of gravitational radiation, we must have \( |\rho^{(0)}| \ll 1/R^2 \), i.e. the backreaction effects of density inhomogeneities must be ‘small.’ However, in the present era of precision cosmology, it is of interest to know more precisely how ‘small’ the backreaction effects are. In particular, what are the size and nature of the various ‘small corrections’ that we neglected in our analysis above to the expansion rate and acceleration of the Universe? Corrections as small as, say, 1% would be of significant observational interest.

In order to analyze this, it is necessary to make further assumptions about the nature of the stress-energy, \( T_{ab} \), of matter and the perturbed metric \( g_{ab} \). We assume that \( T_{ab} \) takes the form of a pressureless fluid, \( T_{ab} = \rho u_a u_b \), and that appropriate quasi-Newtonian behavior holds for both \( T_{ab} \) and \( \gamma_{ab} \). With these assumptions, it is possible to solve Einstein’s equation to the accuracy required to compute the dominant contributions to the terms that were neglected in the above calculations. These calculations are quite involved, and we refer the reader to [2] for all details (see, particularly, appendix B of that reference). The upshot of these calculations is that backreaction effectively modifies the matter stress-energy by adding in the effects of the kinetic motion of the matter as well as its Newtonian potential energy and stresses.

Consequently, for a quasi-Newtonian Universe, as ours appears to be, the backreaction effects of small scale density inhomogeneities are extremely small (far smaller than 1%), mainly involving only a small ‘renormalization’ of the mass density to take account of the kinetic and Newtonian potential energy of matter. This result is in complete agreement with the analysis of [12], which was done prior to our work [2].

3. Relationship to our mathematically precise formulation

How can one make the arguments of the previous two sections more mathematically precise and rigorous, so that the results can be stated as mathematical theorems rather than heuristic estimates? Our approximations will become exact in the limit that both \( \gamma_{ab} \to 0 \) and \( L \to 0 \), where \( L \) denotes the homogeneity length introduced in section 1. Thus, if we wish to try to make these arguments mathematically precise, we are led to consider a one-parameter family of metrics \( g_{ab}(\lambda) \) and stress-energy tensors \( T_{ab}(\lambda) \) such that, as \( \lambda \to 0 \), we have

10 Note that for virialized systems, the kinetic motion and Newtonian potential contributions to stress cancel, so a Universe filled with virialized systems behaves like a dust-filled Universe [12].
\( g_{ab}(\lambda) \to g_{ab}^{(0)} \) (say, uniformly on compact sets) and such that the homogeneity length \( L \to 0 \). Now, as \( L \to 0 \) there is no longer any need for the ‘averaging field,’ \( f^{ab} \), of equation (10) to be ‘slowly varying,’ since everything is ‘slowly varying’ as compared with an arbitrarily small \( L \). Thus, as \( \lambda \to 0 \), equation (10) becomes the statement that for any smooth tensor field \( f^{ab} \) of compact support we have

\[
\int f^{ab} \Delta T_{ab} \to 0. \tag{37}
\]

Mathematically, equation (37) is precisely the statement that the weak limit as \( \lambda \to 0 \) of \( \Delta T_{ab}(\lambda) \) vanishes. However, note that we definitely do not want to require that \( \Delta T_{ab} \to 0 \) in a pointwise or uniform sense or we would be ‘throwing out the baby with the bathwater;’ we must allow the small scale inhomogeneities to remain present as \( \lambda \to 0 \) so that we can see their possible backreaction effects.

If we take the weak limit as \( \lambda \to 0 \) of Einstein’s equation (15) under the assumptions that \( \gamma_{ab} \to 0 \) uniformly and \( \Delta T_{ab} \to 0 \) weakly, and if we also assume that \( |\nabla \gamma_{ab}| \) remains bounded as \( \lambda \to 0 \), we find that

\[
t_{ab}^{(0)} = -\frac{1}{8\pi} w\text{-lim}_{\lambda \to 0} g^{(2)}_{ab}, \tag{38}
\]

which effectively replaces equation (19). Thus, for the one-parameter families that we wish to consider in order to give a precise mathematical formulation of our results, the weak limit of the particular quadratic expression in \( \nabla \gamma_{ab} \) that appears on the right side of (38) must exist. Note that it is essential for our analysis that this quantity be allowed to be nonzero, since, otherwise, we would preclude backreaction. It is mathematically convenient to slightly further restrict the one-parameter families we consider in order to require that all quadratic expressions in \( \nabla \gamma_{ab} \) have a well defined weak limit.

The above considerations lead us to the following framework [11] for stating our results in a mathematically precise form: We consider a one-parameter family of metrics \( g_{ab}(\lambda) \) and stress-energy tensors \( T_{ab}(\lambda) \) such that the following conditions hold: (i) Einstein’s equation (14) holds with \( T_{ab} \) satisfying the weak energy condition. (ii) \( |\gamma_{ab}(\lambda)| \leq \lambda C_1(x) \) (where \( \gamma_{ab}(\lambda) \equiv g_{ab}(\lambda) - g_{ab}^{(0)} \)) for some positive function \( C_1(x) \), so, in particular, \( g_{ab}(\lambda) \to g_{ab}^{(0)} \) uniformly on compact sets as \( \lambda \to 0 \). (iii) \( |\nabla \gamma_{ab}(\lambda)| \leq C_2(x) \) for some positive function \( C_2(x) \), so derivatives of \( \gamma_{ab} \) remain bounded as \( \lambda \to 0 \). (iv) The weak limit of \( \nabla \gamma_{ab} \) exists as \( \lambda \to 0 \). This is precisely the mathematical framework of our original papers [1–4]. (In [3] we constructed explicit nontrivial examples of such one-parameter families.) It can be readily seen that condition (i) is precisely equations (7) and (14), condition (ii) is a precise version of (3), condition (iii) is a slightly strengthened version of (5), and condition (iv) corresponds (under Einstein’s equation) to a slightly strengthened version of (10). With the replacement of (3), (5), (7), (10), and (14) by conditions (i)–(iv), our heuristic arguments of the previous two sections concerning the properties of \( t_{ab}^{(0)} \) can be transformed into mathematically precise theorems.

4. Further implications

We have discussed above the application of our work to the analysis of backreaction effects in cosmology. However, we believe that our work provides an indication of aspects of Einstein’s equation that may underlie fundamental stability properties of its solutions.

We have derived, in a very general context, what may be viewed as the ‘long wavelength effective equations of motion’ for the metric in the presence of ‘short wavelength
disturbances.’ The key point is that the long wavelength effective stress-energy tensor, $\tau_{ab}^{(0)}$, associated with the short wavelength disturbances, always has positive energy properties\footnote{We proved that $\tau_{ab}^{(0)}$ satisfies the weak energy condition. We conjecture that $\tau_{ab}^{(0)}$ satisfies the dominant energy condition.}, provided only that the matter itself has positive energy. But positivity of energy together with local conservation of total (i.e. real plus effective) stress-energy at long wavelengths suggests that there cannot be a rapid, uncontrolled growth in solutions at long wavelengths arising from the short wavelength behavior. In other words, the nonlinear effects resulting from the insertion of a perturbation at short wavelengths should not be able to locally\footnote{In asymptotically anti-de Sitter spacetimes, reflections off of $\mathcal{I}$ can lead to inverse cascades [13].} trigger a catastrophic ‘inverse cascade’ that has a large effect on the long wavelength behavior. Although it is normally taken for granted that ‘unphysical behavior’ of this sort does not occur, it is a nontrivial feature for long wavelength behavior to be ‘protected’ in this manner from dynamical effects occurring at short wavelengths. Einstein’s equation appears to have this property. It is far from obvious that, e.g. various modified theories of gravity will share this property.

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References

[1] Green S R and Wald R M 2011 Phys. Rev. D 83 084020
[2] Green S R and Wald R M 2012 Phys. Rev. D 85 063512
[3] Green S R and Wald R M 2013 Phys. Rev. D 87 124037
[4] Green S R and Wald R M 2014 Class. Quant. Grav. 31 234003
[5] Buchert T et al 2015 Class. Quant. Grav. 32 215021
[6] Green S R and Wald R M 2015 in preparation (arXiv:1506.06452)
[7] Ishibashi A and Wald R M 2006 Class. Quant. Grav. 23 235–50
[8] Isaacson R A 1968 Phys. Rev. 166 1263–71
[9] Isaacson R A 1968 Phys. Rev. 166 1272–9
[10] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[11] Burnett G A 1989 J. Math. Phys. 30 90–6
[12] Baumann D, Nicolis A, Senatore L and Zaldarriaga M 2012 J. Cosmol. Astropart. Phys. 1207 JCAP07(2012)051
[13] Carrasco F, Lehner L, Myers R C, Reula O and Singh A 2012 Phys. Rev. D 86 126006