BETTI NUMBERS OF MULTIGRADED MODULES OF GENERIC TYPE

HARA CHARALAMBOUS AND ALEXANDRE TCHERNEV

Abstract. Let \( R = \mathbb{k}[x_1, \ldots, x_m] \) be the polynomial ring over a field \( \mathbb{k} \) with the standard \( \mathbb{Z}^m \)-grading (multigrading), let \( L \) be a Noetherian multigraded \( R \)-module, let \( \beta_{i,\alpha}(L) \) the \( i \)th (multigraded) Betti number of \( L \) of multidegree \( \alpha \). We introduce the notion of a generic (relative to \( L \)) multidegree, and the notion of multigraded module of generic type. When the multidegree \( \alpha \) is generic (relative to \( L \)) we provide a Hochster-type formula for \( \beta_{i,\alpha}(L) \) as the dimension of the reduced homology of a certain simplicial complex associated with \( L \). This allows us to show that there is precisely one homological degree \( i \geq 1 \) in which \( \beta_{i,\alpha}(L) \) is non-zero and in this homological degree the Betti number is the \( \beta \)-invariant of a certain minor of a matroid associated to \( L \). In particular, this provides a precise combinatorial description of all multigraded Betti numbers of \( L \) when it is a multigraded module of generic type.

Introduction

Throughout this paper \( \mathbb{k} \) is a field, \( R = \mathbb{k}[x_1, \ldots, x_m] \) is the polynomial ring over \( \mathbb{k} \) with the standard \( \mathbb{Z}^m \)-grading (multigrading), \( L \) is a Noetherian multigraded \( R \)-module with minimal multihomogeneous free presentation

\[
E \xrightarrow{\Phi} G \xrightarrow{} L \xrightarrow{} 0,
\]

and \( S \) is a (multi)homogeneous basis of \( E \). An ongoing project of the second author is to use the combinatorial properties of the free multigraded resolution \( T(\Phi, S)_\bullet \) of \( L \) and the matroid \( M(\Phi, S) \) from [12] to study the homological properties of \( L \). In this paper we apply this technique to investigate the multigraded Betti numbers \( \beta_{i,\alpha}(L) \) in a generic situation. We introduce the notion of a multigraded module of generic type which generalizes the notion of genericity introduced previously by the authors in [7]. Our definition is new even in the case of monomial ideals, where it properly (and in a strong sense) subsumes the notion of generic ideals from [1], and differs in an essential way from the notion of genericity introduced in [10], see Examples 2.4. We also introduce the finer notion of a generic (relative to \( L \)) multidegree (the module \( L \) is then of generic type if each multidegree \( \alpha \in \mathbb{Z}^m \) is generic relative to \( L \)).

When \( I \) is a monomial ideal and \( R/I \) is of generic type, as a first result we show that the algebraic Scarf complex is a minimal resolution of \( R/I \), Corollary 2.6. Later on we show that if \( I \) is a monomial ideal and \( \alpha \) is a generic multidegree

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then the $i$th Betti number of $R/I$ is nonzero precisely when $\alpha$ belongs to the Scarf complex of $I$, Corollary 5.9.

The main result of this paper is the computation of the $i$th Betti number of $L$ when $\alpha$ is generic relative to $L$, Theorem 5.8. We provide a Hochster-type formula for the multigraded Betti numbers $\beta_{i,\alpha}(L)$ in terms of the relative homology of a certain simplicial complex associated with $\alpha$ and the matroid $M(\Phi, S)$. We analyze the properties of that simplicial complex to show that the Betti numbers $\beta_{i,\alpha}(L)$ are zero except in at most one degree $i$, and in that degree the Betti number equals the $\beta$-invariant of a certain minor $M_\alpha$ of the matroid $M(\Phi, S)$. In particular, this provides a detailed combinatorial description of the multigraded Betti numbers for the class of multigraded modules of generic type.

The material is organized as follows. In Section 2 we introduce the notion of a generic element in the LCM-lattice of $L$, the notion of a generic multidegree relative to $L$, and the notion of a multigraded module of generic type. We compare the new notions with the existing notions of genericity. We show that if $I$ is a generic in the sense of [1] monomial ideal then the module $R/I$ is automatically of generic type. We show that when $L$ is of generic type, then the proof of [7, Theorem 5.6] works and generalize the above theorem. In particular this implies that the algebraic Scarf complex is a minimal free resolution of $R/I$ when $R/I$ is a multigraded module of generic type.

In Section 3 we introduce the affine simplicial complexes of a matroid $M$ on the set $S$. We show their homology equals the $\beta$-invariant of the matroid. When a representation $\phi$ of the matroid is given, then for any ordering $\omega$ of $S$ we introduce a certain complex of vector spaces $V(\phi, \omega)$ with the same homology.

In Section 4 when $\alpha$ is a generic element we examine certain minors $M_\alpha$ and $M^\alpha$ of the matroid $M(\Phi, S)$ of $L$. We introduce a new complex of vector spaces $V(\alpha, \phi, \omega)$ whose homology is nonzero at precisely one position and equals the $\beta$-invariant of $M_\alpha$.

Finally, in Section 5 we use the free multigraded resolution $T_{\bullet}(\Phi, S)$ of [12] to construct a certain double complex. According to the first filtration of this complex we recover the complex $V(\alpha, \phi, \omega)$, while according to the second filtration we can compute the $\alpha$-graded Betti numbers of $L$. Thus we prove that when $\alpha$ is generic relative to $L$, the reduced homology of the affine simplicial complexes of $M_\alpha$ determines the $\alpha$-Betti numbers of $L$. Moreover the $\alpha$-Betti number of $L$ is nonzero in exactly one position and equals the $\beta$-invariant of a minor of $M_\alpha$.

1. Preliminaries

For the rest of this paper $L$ is a Noetherian multigraded $R$-module, $\Phi: E \rightarrow G$ is a minimal free multigraded presentation of $L$, and $S$ a homogeneous basis of $E$. All vector spaces, homomorphisms and unadorned tensor operations are over $k$.

1.1. Complexes of vector spaces. Let $C_{\bullet} = (C_i, \varphi_i)$ be a complex of vector spaces. The dual complex of $C_{\bullet}$ is the complex

$$C^*_\bullet = (C^*_i, \varphi^*_i) = (\text{Hom}_k(C_{-i}, k), \text{Hom}_k(\varphi_{-i+1}, k)),$$

its shift by an integer $k$ is the complex

$$C[k]_{\bullet} = (C[k]_i, \varphi[k]_i) = (C_{i+k}, (-1)^k \varphi_{i+k}).$$
its \emph{shift in homological degrees} by \( k \) is the complex
\[ C^i[k] = (C^i, \varphi^i[k]) = (C^i_{i+k}, \varphi^{i+k}), \]
and its \emph{truncation} at \( k \) is the complex
\[ C^i_{\geq k} = (C^i_{\geq k}, \varphi^i_{\geq k}) = (\tau^k_iC_i, \tau^k_i\varphi_i) \text{ where } \tau^k_i = \begin{cases} \text{id} & \text{if } i \geq k; \\ 0 & \text{otherwise.} \end{cases} \]
We call \( C^i \) \emph{acyclic} if \( H_i(C^i) = 0 \) for \( i \neq 0 \), and \emph{exact} if also \( H_0(C^i) = 0 \).

1.2. Maps and vector spaces. We recall here and in the following subsections some of the notation introduced in [12]. Let \( U_S \) be the vector space with basis the set of symbols \( \{e_a \mid a \in S\} \). For each subset \( A \subseteq S \) we denote by \( U_A \) the subspace of \( U_S \) spanned by the set \( \{e_a \mid a \in A\} \). Whenever a map of vector spaces \( \phi : U_S \to W \) is given and \( A \subseteq S \) we denote by \( V_A \) the subspace of \( W \) spanned by the set \( \{\phi(e_a) \mid a \in A\} \); thus \( V_A = \phi(U_A) \). We denote by \( \phi_A : U_A \to W \) the restriction of \( \phi \) to \( U_A \).

Next, we consider \( k \) as an \( R \)-module via the canonical \( k \)-algebra map \( R \to k \) that sends each variable \( x_i \) to \( 1 \in k \). We denote by \( W(\Phi) \) the \( k \)-vector space \( G \otimes_R k \).
Since the set \( \{a \otimes 1 \mid a \in S\} \) forms a basis of the space \( E \otimes_R k \) we can canonically identify it with \( U_S \) by identifying \( e_a \) with \( a \otimes 1 \) for each \( a \in S \). We denote by \( \phi(\Phi) \) the \( k \)-linear map \( \Phi \otimes_R k \); thus we have \( \phi(\Phi) : U_S \to W(\Phi) \).

For \( A \subseteq S \) we let \( E_A \) denote the free direct summand of \( E \) generated over \( R \) by the set \( \{a \mid a \in A\} \), and let \( \Phi_A \) denote the restriction of \( \Phi \) to \( E_A \). It is straightforward that \( \phi(\Phi_A) = \phi(\Phi)_A \).

1.3. Matroids. For more details on the basic properties of matroids we refer the reader to [14, 11]. For a quick summary with notation in the spirit of this paper see [12, Section 1].

Let \( P(S) \) be the collection of all subsets of \( S \), partially ordered by inclusion. Recall that any matroid \( M \) on \( S \) is determined by a nonempty set \( \mathcal{I}(M) \subset P(S) \) (or \( \mathcal{I} \) if clear from the context) whose elements are called the independent sets of \( M \). The set \( \mathcal{I} \) has the following three properties:
- \( \emptyset \in \mathcal{I} \)
- if \( I \subseteq J \) and \( J \in \mathcal{I} \) then \( I \in \mathcal{I} \)
- if \( J \in P(S) \) and \( I_1, I_2 \) are two subsets of \( J \) maximal with respect to membership in \( \mathcal{I} \) then \( |I_1| = |I_2| \). (This common size is called the rank of \( J \) in \( M \) and is denoted by \( r(J) \)).

When a map of vector spaces \( \phi : U_S \to W \) is given, one obtains a matroid \( M(\phi) \) on \( S \) by letting the set \( \mathcal{I} \) consist of all subsets \( I \) of \( S \) for which \( |I| = \dim_k V_I \). In this case for any subset \( J \) of \( S \) we let \( r(J) = \dim_k V_J \). One says that \( M(\phi) \) is \emph{represented} by \( \phi \) over \( k \) and that \( \phi \) is a representation over \( k \) of \( M(\phi) \).

Recall that a \emph{circuit} of a matroid \( M \) is a minimal dependent set, and a \emph{loop} of \( M \) is an element \( a \in S \) so that \( \{a\} \) is a circuit. A \emph{T-flat} is a subset of \( S \) that is a union of circuits. A \emph{flat} of \( M \) is a subset \( B \subset S \) such that \( r(B) = r(B) + 1 \) for each \( c \notin B \). A \emph{hyperplane} of \( M \) is a maximal proper flat, i.e., a flat \( H \) such that \( r(H) = r(S) - 1 \). The collection of T-flats forms a lattice with respect to inclusion. The intersection of flats is a flat. The \emph{matroid closure} \( \overline{B} \) of a subset \( B \subseteq S \) is the
smithallest flat containing $B$ it equals the intersection of all flats containing $B$. Two elements $a, b \in S$ are called \textit{parallel} if they are not loops and $\{a\} = \{b\}$. Let $J \subset S$. The \textit{restriction} of $M$ to $J$ is the matroid $M|J$ whose independent sets form the set $\mathcal{I}(M|J) = \mathcal{I}(M) \cap P(J)$. The \textit{contraction} of $M$ to $J$ is the matroid $M, J$ whose independent sets form the set $\mathcal{I}(M, J) = \{I \subset J \mid I \cup I' \in \mathcal{I}(M), \forall I' \in \mathcal{I}(M|S \setminus J)\}$. The $\beta$-invariant of $M$ is

$$\beta(M) = (-1)^{r(S)} \sum_{J \subset S} (-1)^{|J|} r(J).$$

One of the important properties of the $\beta$-invariant is that $\beta(M) = 0$ is zero whenever $M$ has a loop, see [8, Theorem II].

1.4. Multigraded resolutions. For the rest of the paper we denote by $T_\bullet(\Phi, S)$ the free multigraded resolution of $L$ from [12, Theorem 4.5]. In homological degrees 0 and 1 the resolution $T_\bullet(\Phi, S)$ is simply the minimal presentation $\Phi$. For $n \geq 2$, the $R$-components of $T_\bullet(\Phi, S)$ are:

$$T_n(\Phi, S) = \bigoplus_I (T_I \otimes R)[-\deg I]$$

where the index $I \subset S$ runs through all T-flats of $M$ such that $r(I) = |I| - n + 1$, $\deg I$ is the componentwise maximum of the multidegrees of the elements of $I$, $T_I$ is a certain $k$-vector space associated to $I$, see [12, Definition 2.2.3], and $(T_I \otimes R)[-\deg I]$ is the free module $T_I \otimes R$ shifted by multidegree $\deg I$: $(T_I \otimes R)[-\deg I],_\alpha = (T_I \otimes R),_\alpha + \deg I$ for any $a \in \mathbb{Z}^m$. While in general the resolution $T_\bullet(\Phi, S)$ is not minimal, we will use it to obtain information about the minimal multigraded resolution of $L$. We denote by $\beta_{i, \alpha}(L)$ the $i$th multigraded Betti number of $L$:

$$\beta_{i, \alpha}(L) = \dim_k H_i(T(\Phi, S) \otimes k),_\alpha = \dim_k \text{Tor}_i^R(L, k),_\alpha.$$

2. Multigraded modules of generic type

First we recall the definition of $LCM$-lattice of the multigraded module $L$.

\textbf{Definition 2.1.} Let $\Lambda = \Lambda(L)$ be the lattice in $\mathbb{Z}^m$ (with the join operation being componentwise maximum) join-generated by the multidegrees of the elements of $S$. We call $\Lambda$ the $LCM$-lattice of $L$. Since the collection of multidegrees $\{\deg a \mid a \in S\}$ of the elements of $S$ is independent of the choice of the basis $S$, the $LCM$-lattice is an invariant of $L$.

\textbf{Remark 2.2.} The multidegrees of the free modules in $T_\bullet(\Phi, S)$ are elements of the $LCM$-lattice $\Lambda(L)$. It follows that the minimal syzygies of $L$ can occur only in multidegrees $\alpha$ that belong to $\Lambda(L)$. Consequently, for $i \geq 1$ the Betti numbers $\beta_{i, \alpha}(L)$ can be nonzero only if $\alpha \in \Lambda(L)$.

Consider the \textit{degree map} of posets

$$\deg : P(S) \longrightarrow \mathbb{Z}^m$$

given by $\deg A = \bigvee \{\deg a \mid a \in A\}$ for each subset $A \subset S$, and note that $\Lambda(L)$ is precisely the image of the map $\deg$. For each $\alpha \in \Lambda(L)$ there always is a unique maximal set $I^\alpha$ in $P(S)$ of degree $\alpha$: $I^\alpha$ equals the union of all sets of degree $\leq \alpha$. 

Definition 2.3. We say that \( \alpha \in \Lambda(L) \) is a *generic element of \( \Lambda(L) \) if the fiber \( \deg^{-1}(\alpha) \) is a closed interval in \( P(S) \), i.e. if there is a unique minimal subset of \( S \) of degree \( \alpha \) denoted by \( I_\alpha \) and \( \deg^{-1}(\alpha) = [I_\alpha, I^\alpha] \).

We say that \( \alpha \in \mathbb{Z}^n \) is *generic relative to \( L \) if either \( \alpha \notin \Lambda(L) \) or if \( \alpha \) is a generic element of \( \Lambda(L) \).

We say that \( \Lambda(L) \) is of *generic type* if each \( \alpha \in \mathbb{Z}^n \) is generic relative to \( L \) and in this case we say that \( L \) is also of *generic type*.

It is immediate that if \( L \) is of generic type then no two elements of \( S \) have the same multidegree. The notion of generic type only depends on the multidegrees of the basis elements of \( S \) and is independent of the choice of the particular basis \( S \). Below we give some examples to differentiate between the different notions of generic.

Examples 2.4.

- The Scarf simplicial complex of \( \Phi \), \( \Delta(\Phi) \) is the subcomplex of \( P(S) \) consisting of all subsets \( I \) so that \( \deg^{-1}(\deg I) = \{1\} \).
- Let \( J = (x^2, xy, xz) \). Then \( R/J \) is of generic type as can be readily checked, and \( J \) is not generic in the sense of [1] or [10].
- If \( J \) is a monomial ideal generic in the sense of [1], then \( R/J \) is of generic type. Indeed, for \( \alpha \in \Lambda(R/J) \) take \( s_i \in S \) to be the unique monomial generator of \( I \) that agrees with \( \alpha \) in the \( i \)-th coordinate. The unique minimal set \( I_\alpha \) of degree \( \alpha \) is the collection of the distinct \( s_i \) obtained this way.
- Let \( J = (x^3z^2, x^2y^3, xy^2z, y^3z^2) \). Then \( (3, 3, 2) \) is not a generic element of \( \Lambda(R/J) \) as \( \{1, 2\}, \{1, 4\} \) are minimal in \( \deg^{-1}(3, 3, 2) \). We note that \( J \) is generic in the sense of [10].
- Let \( I \) be a monomial ideal, \( L = R/I \), and \( \Delta_L \) be the Scarf complex of \( L \), see [1] or [10]. If \( \sigma \in \Delta_L \) then the multidegree \( \alpha \) of \( \sigma \) is a generic element of \( \Lambda(L) \) and \( \deg^{-1}(\alpha) = I^\alpha = I_\alpha \) is just a point.
- Let \( I \) be a monomial ideal in \( R \) and \( J \) the polarization of \( I \) in a polynomial ring \( S \). It is clear that \( R/I \) is of generic type if and only if \( S/J \) is of generic type. This is not the case when \( I \) is generic in the sense of [10] as the simple example \( (x^2, xy) \) demonstrates.
- If \( I \) is generic in the sense of [10] then \( I^* = I + (x^D, \ldots, x^D_m) \) (where \( D \) is sufficiently large) is also generic in the sense of [10]. Let \( I = \langle xy, xz \rangle \). \( R/I \) is of generic type and as we will see below the algebraic Scarf complex is a minimal free resolution of \( R/I \). However for \( D > 1 \) the ideal \( I^* = I + (x^D, y^D, z^D) \) is not of generic type since \( \deg^{-1}((1, D, D)) \) is not an interval.

Let \( r = \text{rank } \Phi \), \( g = \text{rank}_R(G) \). We recall from [7] that \( \Phi \) is of *uniform rank* if all \( g \times r \) submatrices of the coefficient matrix of \( \Phi \) have rank equal to \( r \). In [7] the Scarf complex of \( \Phi \), \( S_\bullet(\Phi) \) was introduced. When \( g = 1 \), \( S_\bullet(\Phi) \) is the algebraic Scarf complex, \( F_{\Delta(\Phi)} \), of [1]. In [7, Theorem 5.6] it is shown that when \( \Phi \) is of uniform rank and \( L \) is generic in the sense of [1] then \( S_\bullet(\Phi) \) is a minimal free multigraded resolution of \( L \). The condition needed for the proof of [7, Theorem 5.6] to work is that there is a unique minimal face of degree \( \alpha \), so that \( i \in I^\alpha \setminus I_\alpha \) if and only if \( \deg(I^\alpha \setminus i) = \deg(I^\alpha) = \alpha \) ([7, pg 547]). This condition is is equivalent to \( \alpha \) being a generic element of \( \Lambda(L) \). Thus the next theorem holds:
Theorem 2.5. Let $\Phi : E \rightarrow G$ be a minimal free multigraded presentation of the multigraded module $L$ so that $\Phi$ is of uniform rank and $L$ is of generic type. Then $S_\ast(\Phi)$ is a minimal free multigraded resolution of $L$.

We apply the above when $R/I$ is of generic type:

Corollary 2.6. Let $I$ be a monomial ideal so that $R/I$ is of generic type and $\Phi : R^n \rightarrow R$ a minimal multigraded presentation of $R/I$. The algebraic Scarf complex $F_{\Delta(\Phi)}$ is a minimal free multigraded resolution of $R/I$.

Proof. Minimality of the presentation $\Phi$ implies that $\Phi$ is of uniform rank. □

3. AFFINE SIMPLICIAL COMPLEXES AND THE $\beta$-INVARIANT

Let $M$ be an arbitrary matroid on the set $S$, $\phi : U_S \rightarrow W$ a representation of $M$ and $\{e_a \mid a \in S\}$ a basis of $U_S$. For any $b \in S$ we introduce a simplicial complex and compute its homology.

Definition 3.1. Let $b \in S$. We let $\Delta_b$ be the simplicial complex

$$\Delta_b = \{ J \subset P(S) \mid b \notin J \}$$

We call $\Delta_b$ the affine simplicial complex of $M$ away from $b$.

We note that the facets of $\Delta_b$ are the hyperplanes of the matroid $M$ that do not contain $b$ and that $J \in \Delta_b$ if and only if $V_J \neq V_J \cup b$.

Theorem 3.2. Let $b \in S$. Then for $i \geq 0$

$$\dim_k \tilde{H}_i(\Delta_b, k) = \begin{cases} \beta(M) & \text{if } i = r(M) - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $b$ is not a loop then the above equality is true for all $i$.

Proof. If $M$ has a loop $c$ then $\beta(M) = 0$. Furthermore, in that case $\Delta_b$ is either the empty simplicial complex (when $b$ is a loop) or a cone with apex $c$. Thus in the sequel we assume that $M$ has no loops.

Next we define a new complex $\overline{\Delta}_b$ as follows: the set of vertices of $\overline{\Delta}_b$ is

$$\text{Vert}(\overline{\Delta}_b) = \{ J \mid J \text{ is a flat in } M \text{ of rank 1 such that } b \notin J \}.$$  

The faces of $\overline{\Delta}_b$ are the sets

$$\{ J_1, \ldots, J_l \} : J_1 \cup \cdots \cup J_l \subset H, \text{ for some facet } H \text{ of } \Delta_b.$$  

Since $M$ has no loops we can define the simplicial map $\pi : \Delta_b \rightarrow \overline{\Delta}_b$ that sends each vertex $y$ of $\Delta_b$ to its matroid closure $[y]$, which is a flat of rank 1. Since a facet of $\Delta_b$ contains $y$ if and only if it contains $[y]$, it is straightforward that $\pi$ is a quotient map arising from partitioning the vertices of $\Delta_b$ into classes of parallel elements. Therefore the Contractible Subcomplex Lemma [3, (2.2)] yields that $\pi$ is a homotopy equivalence. Let $L_b$ be the poset obtained by removing from the lattice of flats of $M$ those flats that contain $b$. Then by the Crosscut Theorem [2, Theorem 2.3], $\overline{\Delta}_b$ is homotopy equivalent with the order complex of the poset.
\( L_b^o \) obtained by removing from \( L_b \) its minimal element. By the results of Wachs and Walker [13, Theorem 3.2 and Corollary 7.2] this order complex is a shellable simplicial complex, and its reduced homology has already been computed, see e.g. [3] and [15, Theorem 2.6] or [5, Theorem 3.12]. In particular, it is possibly nonzero only in dimension \( r(\mathbf{M}) - 2 \), and its rank there is precisely the \( \beta \)-invariant of \( \mathbf{M} \). □

Next we compare the homology of \( \Delta_b \) with the homology of a complex of vector spaces determined by the subspaces of \( V_S \). Let \( \omega \) be an ordering on \( S \) and we use this ordering to identify each subset of \( S \) with the increasing sequence of its elements. For each \( J \subset S \) and \( c \notin J \), we have that \( V_J \) is a subset of \( V_{J,\{c\}} \). We let

\[
V(\phi, \omega)_i = \bigoplus_{B \subset S, |B| = |S| - i} V_B
\]

and \( V(\phi, \omega)_\bullet \) be the complex

\[
0 \to \bigoplus_{B \subset S, |B| = 1} V_B \to \ldots \to \bigoplus_{B \subset S, |B| = |S| - 1} V_B \to V_S \to 0
\]

where at the \( i \)th stage the maps componentwise are the inclusions \( V_B \to V_{B,\{c\}} \) times the sign of the permutation that arranges the sequence \((c, B)\) in increasing order then followed by composition with the natural inclusion \( V_{B,\{c\}} \hookrightarrow V(\phi, \omega)_{i+1} \).

**Lemma 3.4.** If \( b \in S \) is not a loop of \( \mathbf{M} \) then

\[
\text{H}_i(V(\phi, \omega)_\bullet) \cong \text{H}|S|_{i-1}(\Delta_b, k)
\]

for all \( i \).

**Proof.** Let \( \bar{Y} \) be the reduced chain complex of \( \Delta_b \) over \( k \). Since the complexes \( V(\phi, \omega)_\bullet \) are canonically isomorphic for different choices of \( \omega \), we may assume that \( b \) is the greatest element of \( S \) and that \( \omega \) induces the orientation on the faces of \( \Delta_b \) used to construct \( \bar{Y} \). We write \( V_* \) for \( V(\phi, \omega)_\bullet \). We consider a certain subcomplex \( D_* \) of \( V_* \). We let \( D_i = \bigoplus_j V_j \) where \( b \in J \) and \( |J| = |S| - i \). In particular \( D_0 = V_S \) and

\[
D_i : 0 \to V_b \to \ldots \to V_S \to 0
\]

where \( D_i \to D_i-1 \) are the restrictions of the maps from (3.3). Let

\[
E_* = (V_*/D_*)[1].
\]

If \( b \notin J \), we let \( \psi_J \) be the inclusion \( V_J \subset V_{J,\{b\}} \) taken with the sign \((-1)^{|J|}\). Then \( \psi : E_* \to D_* \) defined componentwise by the maps \( \psi_J \) is an injective map of complexes and the complex \( V_* \) is precisely the mapping cone of \( \psi \). We let

\[
\overline{D}_* \equiv D_* / \psi(E_*).
\]

It follows from the injectivity of \( \psi \) and the standard properties of mapping cones that \( \text{H}_i(V_*) \cong \text{H}_i(\overline{D}_*) \) for each \( i \). Next we note that the nonzero summands of \( \overline{D}_i \) are of the form \( V_{J,\{b\}}/V_J \) where \( V_J \neq V_{J,\{b\}} \) and \( |J| = |S| - 1 - i \). Thus the sets \( J \) involved are precisely the faces of \( \Delta_b \). We consider \( Y' = \bar{Y}'(\{S\} + 2) \), (so that that \( \text{Hom}_k(\overline{Y}_{i-1}, k) \) is in homological degree \(|S| - 1 \). For \( J \in \Delta_b \) we let \( \sigma_J^* \) be the standard generator of \( Y' \) associated to \( J \). Identifying \( \sigma_J^* \) with \( \phi(e_b) + V_J \) in \( V_{J,\{b\}}/V_J \) we see that \( Y'^{\{\{S\} + 2\}} \) can be identified with \( \overline{D}_* \) and the lemma follows. □

We are now ready to compute the homology of the complex \( V(\phi, \omega)_\bullet \):
Theorem 3.5.

\[ \dim_k H_i(V(\phi, \omega)_*) = \begin{cases} 
\beta(M) & \text{if } i = |S| - r(M) \\
0 & \text{otherwise.} 
\end{cases} \]

Proof. If \( M \) has an element \( b \) that is not a loop, then the theorem is immediate by combining Theorem 3.2 and Lemma 3.4. If all elements of \( S \) are loops in \( M \) then \( r(M) = 0 \), the complex \( V(\phi, \omega)_* \) is zero, and the desired conclusion is immediate from the fact that \( \beta(M) = 0 \).

\[ \square \]

4. Minors of \( M(\Phi, S) \) Associated with a Generic Multidegree

Let \( \alpha \in \Lambda(L) \) be a generic element. For the rest of this paper we fix \( M = M(\Phi, S) \), we set \( W = W(\Phi) \), and \( \phi = \phi(\Phi) : U_S \rightarrow W \) (see Section 1). We introduce some new matroids associated with \( \alpha \). Recall that \( \deg^{-1}(\alpha) = [I_\alpha, I_\alpha] \) is a closed interval in the boolean poset \( P(S) \).

Definition 4.1.

- We set \( I(\alpha) = I^\alpha \setminus I_\alpha \).
- We denote by \( M^\alpha \) the matroid that is the restriction of \( M \) to \( I^\alpha \). In standard matroid notation we have \( M^\alpha = M|I^\alpha \).
- We denote by \( M_\alpha \) the matroid that is the contraction of \( M^\alpha \) to \( I_\alpha \). In standard matroid notation we have \( M_\alpha = M^\alpha/I_\alpha \).

We discuss the above matroids in terms of some linear transformations associated with \( \alpha \).

Remarks 4.2.

- \( M^\alpha \) is represented by \( \phi_{I^\alpha} : U_{I^\alpha} \rightarrow W \) over \( k \).
- Let \( \pi_\alpha : W \rightarrow W/V_{I(\alpha)} \) be the canonical projection map, and let \( \tilde{\phi}_{I_\alpha} := \pi_\alpha \circ \phi_{I_\alpha} : U_{I_\alpha} \rightarrow W/V_{I(\alpha)} \).

The matroid \( M_\alpha \) is represented by \( \tilde{\phi}_{I_\alpha} \) over \( k \).

- We set \( \nabla_{I_\alpha} := V_{I_\alpha}/V_{I(\alpha)} \). For each \( B \subset I_\alpha \), \( V_{I(\alpha), I_\alpha} \) is a subspace of \( V_{I(\alpha), I_\alpha, B} \) and we set \( \nabla_B := V_{I(\alpha), I_\alpha, B}/V_{I(\alpha)} = \tilde{\phi}_{I_\alpha}(U_B) \).
- Let \( \omega \) be an ordering on \( I_\alpha \). According to the definition

\[ V(\phi_{I_\alpha}, \omega)_* : 0 \rightarrow \bigoplus_{B \subset I_\alpha, |B|=1} \nabla_B \rightarrow \cdots \rightarrow \nabla_{I_\alpha} \rightarrow 0. \]

Next we define a new complex with the same homology as \( V(\phi_{I_\alpha}, \omega)_* \).

Definition 4.4. Let \( \omega \) be an ordering on \( I_\alpha \). We let

\[ V(\alpha, \phi, \omega)_* = \bigoplus_{A \subset I_\alpha, |A|=1} V_{I_\alpha \setminus A} \]

and define a complex of vector spaces \( V(\alpha, \phi, \omega)_* \) as the sequence

\[ 0 \rightarrow V_{I(\alpha)} \rightarrow \cdots \rightarrow \bigoplus_{A \subset I_\alpha, |A|=1} V_{I_\alpha \setminus A} \rightarrow V_{I_\alpha} \rightarrow 0 \]
with maps that are componentwise the inclusions $V_{I^a \setminus (A \cup \{c\})} \subseteq V_{I^a \setminus A}$ times the
sign of the permutation that arranges the sequence $(c, I_\alpha \setminus A)$ in increasing order
then followed by composition with the natural inclusion.

Let $\tilde{C}$ be the reduced chain complex over $k$ for the full simplex on the set $I_\alpha$ as
oriented by $\omega$. Let $C' = (\tilde{C})^* (-|I_\alpha| + 1)$, (so that $\text{Hom}_k(\tilde{C}_{-1}, k)$ is in homological
degree $|I_\alpha|$). We will consider the complex $V_{I(\alpha)} \otimes C'$. Let $\sigma_A^*$ be the standard
generator of $C'$ associated to $A \subset I_\alpha$. We identify $V_{I(\alpha)} \otimes \sigma_A^*$ with the subspace
$V_{I(\alpha)}$ of $V_{I(\alpha) \cup B}$ via the map $v \otimes \sigma_A^* \mapsto v$, where $B = I_\alpha \setminus A$. With this identification
it is easy to see that the following holds:

**Proposition 4.6.** The complex $V_{I(\alpha)} \otimes C'$ is a subcomplex of $V(\alpha, \phi, \omega)_\bullet$ and we
have the short exact sequence of complexes

\[ 0 \to V_{I(\alpha)} \otimes C' \to V(\alpha, \phi, \omega)_\bullet \to V(\bar{\phi}_{I_\alpha}, \omega)_\bullet \to 0. \]

We note that $V_{I(\alpha)} \otimes C'$ is an exact complex. Thus taking the long exact sequence
on the homology of (4.7) yields

**Lemma 4.8.** $H_i(V(\alpha, \phi, \omega)_\bullet) = H_i(V(\bar{\phi}_{I_\alpha}, \omega)_\bullet)$.

Combining Theorem 3.5 and Lemma 4.8 we obtain the following

**Corollary 4.9.** Let $\omega$ be an ordering on $I_\alpha$. Then for all $i$ we have

\[ \dim_k H_i(V(\alpha, \phi, \omega)_\bullet) = \begin{cases} 
\beta(M_\alpha) & \text{if } i = |I_\alpha| - r(M_\alpha) \\
0 & \text{otherwise.}
\end{cases} \]

We finish this section with an example to demonstrate the above.

**Example 4.10.** Let $R = \mathbb{Q}[x, y, z]$, $E \cong R^4$ a multigraded free module with basis
$S = \{a, b, c, d\}$ where $\deg a = (3, 1, 1)$, $\deg b = (1, 3, 1)$, $\deg c = (1, 1, 3)$, $\deg d = (1, 2, 2)$, $G \cong R^2$ and $L$ multigraded with minimal multigraded free presentation

\[ E \xrightarrow{\Phi} G \xrightarrow{\pi} L \xrightarrow{} 0, \]

and such that the matrix of $\phi(\Phi)$ according to the bases $\{e \otimes 1 \mid e \in S\}$ and the
canonical basis of $G$ is given by the matrix

\[ \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 3 \end{bmatrix}. \]

First we examine the case where $\alpha = (3, 3, 3)$. Here $I^a = \{a, b, c, d\}$, $I_\alpha = \{a, b, c\}$
and $I(\alpha) = \{d\}$. Thus $\beta(M_\alpha) = 1$, $r(M_\alpha) = 1$ and for any ordering $\omega$ on $I_\alpha$ the
homology of the complex

\[ V(\alpha, \phi, \omega) : 0 \to \mathbb{Q} \to \mathbb{Q}^6 \to \mathbb{Q}^6 \to \mathbb{Q}^2 \to 0 \]

is nonzero precisely for $i = 2$.

When $\alpha = (3, 2, 3)$, $\deg^{-1}(\alpha)$ equals the point $\{a, c, d\}$, while $r(M_\alpha) = 2$ and
$\beta(M_\alpha) = 1$. For any ordering $\omega$ on $I_\alpha$ the homology of the complex

\[ V(\alpha, \phi, \omega) : 0 \to \mathbb{Q}^3 \to \mathbb{Q}^6 \to \mathbb{Q}^2 \to 0 \]

is nonzero precisely for $i = 1$. 
5. The Betti numbers of \( L \)

Let \( T_*(\Phi, S) \) be the multigraded free resolution of \( L \), see Section 1.4 and let \( \alpha \in \Lambda(L) \). We examine the \( \alpha \)-graded piece of \( T_n(\Phi, S) \) for \( n \geq 2 \) in order to compute \( \beta_{i,\alpha}(L) \). We have that

\[
T_n(\Phi, S) = \bigoplus_{J, \deg J + \beta = \alpha} (T_J \otimes R)[-\deg J]_\beta
\]

where the index \( J \subset S \) runs through all \( T \)-flats of \( M \) such that \( r(J) = |J| - n + 1 \). It follows that \( \deg J \leq \alpha \) and that \( J \subset I_{\alpha} \). Let \( m \) be the maximal multigraded ideal of \( R \). It is clear that

\[
mT_i(\Phi, S) \cap T_i(\Phi, S) = \bigoplus_{J, \deg J + \beta = \alpha} (T_J \otimes R)[-\deg J]_\beta.
\]

When \( \alpha \) is a generic element the condition \( \deg J < \alpha \) is equivalent to the existence of an element \( b \in I_{\alpha} \) such that \( J \subset I_{\alpha} \setminus \{b\} \).

Let \( A \subset S \). We let

\[
T_*(\phi_A) = T_*(\Phi_A, A) \cong 1 \otimes_R k.
\]

The complex \( T_*(\phi_A) \) was introduced in [12, Definition 2.4.1] where it was shown that

\[
T_*(\phi_A) \rightarrow V_A \rightarrow 0
\]

is exact. We note that \( T_0(\phi_A) = U_A \). We will need the following important property, see [12, Theorem 3.2(b) and Theorem 3.5]: if \( A \subset B \) then \( T_*(\phi_A) \) is canonically a subcomplex of \( T_*(\phi_B) \). The following Lemma is a straightforward consequence of the basic structure of these complexes.

**Lemma 5.1.** Let \( \omega \) be an ordering in \( Y \subset S \) and \( X_i \) a collection of subsets of \( Y \). There is a chain map \( p : \bigoplus_{b \in I_{\alpha}} T(\phi_{X_i}) \rightarrow T(\phi_Y) \) where \( p|_{T(\phi_X)} \) equals the canonical inclusion map times the sign determined by \( \omega \) to order the elements of \( (Y \setminus X_i, Y) \).

Let \( \alpha \in \Lambda(L) \) be a generic element. Fix an ordering \( \omega \) in \( I_{\alpha} \) and consider the chain map

\[
p : \bigoplus_{b \in I_{\alpha}} T(\phi_{I_{\alpha} \setminus \{b\}}) \rightarrow T(\phi_{I_{\alpha}})
\]

as in Lemma 5.1. We introduce a new complex:

**Definition 5.2.** Let \( C(\alpha)_* \) be the following complex of vector spaces:

\[
C(\alpha)_* = T(\phi_{I_{\alpha}}) / p(\bigoplus_{b \in I_{\alpha}} T(\phi_{I_{\alpha} \setminus \{b\}})).
\]

We note that for \( i \geq 1 \) we have

\[
mT_i(\Phi, S) \cap T_i(\Phi, S) = p(\bigoplus_{b \in I_{\alpha}} T(\phi_{I_{\alpha} \setminus \{b\}}))_{i-1}
\]

and thus the following lemma holds:

**Lemma 5.3.** Let \( \alpha \in \Lambda \) be a generic element. Then for \( i \geq 1 \)

\[
\beta_{i,\alpha}(L) = \dim_k H_{i-1}(C(\alpha)_*).
\]
Proof. Since $\beta_{i,\alpha}(L) = H_i(T(\Phi, S) \otimes_R k)_\alpha$, we combine the above remarks to get
\[(T_i(\Phi, S) \otimes_R k)_\alpha = T_i(\Phi, S)_\alpha/mT_i(\Phi, S) \cap T_i(\Phi, S)_\alpha = C_{i-1}(\alpha).\]
In view of the minimality of the presentation $\Phi$, the Lemma is now immediate. \[\square\]

We will reduce the study of the homology of $C(\alpha)$ to the study of a certain double complex. We will need the following lemma:

**Lemma 5.4.** Let $X \subset Y \subset S$, and $\omega$ be a linear ordering $Y$. Consider the sequence
\[
T(X, Y, \omega) : 0 \to T(\phi_X) \to \bigoplus_{b \in Y \setminus X} T(\phi_{X \cup \{b\}}) \to \bigoplus_{b,c \in Y \setminus X} T(\phi_{X \cup \{b,c\}}) \\
\to \ldots \to \bigoplus_{c \in Y \setminus X} T(\phi_{Y \setminus c}) \to T(\phi_Y) \to 0
\]
where the morphism component $T(\phi_{X \cup C}) \to T(\phi_{X \cup B})$ is 0 if $C \not\subset B$ and otherwise is the canonical inclusion times the sign determined by $\omega$. Then $T(X, Y, \omega)$ is an acyclic complex.

Proof. We will do induction on $|Y \setminus X|$. If $|Y \setminus X| = 0$ then $Y = X$ and $T(X, Y, \omega) : 0 \to T(\phi_Y) \to 0$. Suppose now that $X \neq Y$ and let $b$ be the biggest element of $Y \setminus X$. We set $X' = X \cup \{b\}$, $Y' = Y \setminus \{b\}$ and let $\omega'$ be the induced ordering on $Y'$. Then we get the short exact sequence
\[
0 \to T(X', Y, \omega') \to T(X, Y, \omega) \to T(X, Y', \omega')(-1) \to 0.
\]
Using the induction hypothesis and the long exact sequence in homology we get that if $i > 1$ then $H_i(T(X, Y, \omega)) = 0$ while if $i = 1$ then
\[
0 \to H_1(T(X, Y, \omega)) \to H_0(T(X, Y', \omega')) \to H_0(T(X', Y, \omega)).
\]
Thus it suffices to show that the map of complexes
\[
H_0(T(X, Y', \omega')) \to H_0(T(X', Y, \omega))
\]
induced by the inclusion map $T(\phi_{Y'}) \subset T(\phi_Y)$ is injective. This however is immediate since
\[
T_i(\phi_{Y'}) \bigcap \bigoplus_{c \in Y \setminus X} T_i(\phi_{Y \setminus c}) = \bigoplus_{c \in Y' \setminus X} T_i(\phi_{Y' \setminus c}),
\]
as follows from the structure of these sets, see [12, Definition 2.4.1]. \[\square\]

We apply Lemma 5.4 to the special case where $X = I(\alpha)$ and $Y = I^\alpha$. We have that $Y \setminus X = I_\alpha$.

**Lemma 5.6.** Let $\alpha \in \Lambda$ generic, and let $\omega$ be an ordering on $I^\alpha$. Then the natural sequence of morphisms of complexes
\[
(5.7) \quad 0 \to T(\phi_{I(\alpha)}) \to \bigoplus_{b \in I_\alpha} T(\phi_{I(\alpha) \cup \{b\}}) \to \cdots \bigoplus_{b \in I_\alpha} T(\phi_{I^\alpha \setminus b}) \to T(\phi_{I^\alpha}) \to 0
\]
is acyclic.

We can now prove the main result of this paper:

**Theorem 5.8.** Let $\alpha \in \Lambda$ be generic. There is at most one $i \geq 1$ such that $\beta_{i,\alpha}(L) \neq 0$. More precisely, we have for each $i \geq 1$
\[
\beta_{i,\alpha}(L) = \begin{cases} 
\beta(M_\alpha) & \text{if } i = |I_\alpha| - \text{rank } M_\alpha + 1; \\
0 & \text{otherwise.}
\end{cases}
\]
Proof. Let $b \in I_a$ and let $\omega$ be an ordering on $I^a$ so that $b$ is the biggest element of $I_a$. The two standard spectral sequences associated with the double complex of Lemma 5.6 collapse. According to the first filtration we get the complex $V(\alpha, \phi, \omega)_*$, see the remarks preceding Lemma 5.1. According to the second filtration we get the complex $C(\alpha)_*$. Thus $H_i(C(\alpha)) = H_i(V(\alpha))$. Combining this with Lemma 5.3 and Corollary 4.9 we are done. □

We apply the theorem to monomial ideals. When $J$ is a monomial ideal and $\alpha$ is generic we prove that $\beta_{i,\alpha}(R/J) \neq 0$ if and only if $\alpha$ corresponds to a face of the Scarf complex of $R/J$.

Corollary 5.9. Let $J$ be a monomial ideal and let $\alpha \in \Lambda(R/J)$ be a generic element. If $I_{\alpha} \neq I^a$ then $\forall i \geq 1$, $\beta_{i,\alpha}(R/J) = 0$. Otherwise $\beta_{i,\alpha}(R/J) = \begin{cases} 1 & \text{if } i = |I_{\alpha}|; \\ 0 &\text{otherwise.} \end{cases}$

Proof. If $I_{\alpha} \neq I^a$ then $M_{\alpha}$ is the empty matroid and $\beta(M_{\alpha}) = 0$. Otherwise $\beta(M_{\alpha}) = \text{rank } M_{\alpha} = 1$. □

We finish this section with an example to show that in the general case it may be $I_{\alpha} \neq I^a$ and $\beta_{i,\alpha}(L) \neq 0$.

Example 5.10. Let $L$ be the module of Example 4.10. For $\alpha = (3,3,3)$ we have the following data: $\beta(M_{\alpha}) = \text{rank } M_{\alpha} = 1$, $|I_{\alpha}| = 3$. Thus $\beta_{3,\alpha}(L) = 1$.

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Department of Mathematics, Aristotle University of Thessaloniki, Greece
E-mail address: hara@math.auth.gr

Department of Mathematics, University at Albany, SUNY, Albany, NY 12222
E-mail address: tchernev@math.albany.edu