Reconstruction of the joint state of a two-mode Bose-Einstein condensate

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We propose a scheme to reconstruct the state of a two-mode Bose-Einstein condensate, with a given total number of atoms, using an atom interferometer that requires beam splitter, phase shift and non-ideal atom counting operations. The density matrix in the number-state basis can be computed directly from the probabilities of different counts for various phase shifts between the original modes, unless the beamsplitter is exactly balanced. Simulated noisy data from a two-mode coherent state is produced and the state is reconstructed, for 49 atoms. The error can be estimated from the singular values of the transformation matrix between state and probability data.

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In quantum physics, a system can be completely described by its quantum state. On the other hand, in a given experiment one can only measure the probability of a quantity, and the process of doing so usually destroys any knowledge of the complementary aspect of that quantity. Thus to reveal the whole state a variety of measurements must be made on an ensemble of identically prepared systems. Even when the systems are not identical one can recover a mixture of quantum states in the form of the density matrix. Optical homodyne tomography (OHT) is one such method of revealing the density matrix, tailored to determine the state of an electromagnetic field mode \( |\psi\rangle \). In that case the availability of coherent states from lasers allows the measurement of the probability of a quadrature value, for a given phase difference between the coherent state and the signal state. It also lends itself to a simple geometric interpretation: the measured distributions are the “shadows” of rotations of the Wigner function. Recent work has shown that it may be easier numerically to reconstruct the density matrix in the number-state basis directly from the distributions, without first going through the Wigner function \( \rho \).

In the field of atom optics, breakthroughs in the evaporative cooling of atomic vapors have resulted in observation of Bose-Einstein condensates (BEC) \( |\psi\rangle \). We may view these as the analog of a single-mode optical field, except that the field now has a finite mass and a quadratic interaction term. The “semiclassical” aspects of BEC such as collective excitations and condensate shape are now a subject of thorough experimental investigation \( \rho \). However, perhaps the most interesting aspects are due to the quantum states themselves. We might expect unusual quantum states for several reasons. One is, the collisions are analogous to nonlinear optical susceptibilities which are known to produce squeezing \( \rho \). Second, an entangled state with a random relative phase can emerge between two condensates, when they produce spatial interference fringes in the atom counts \( |\psi\rangle \). Also, in a closed system, quantum superpositions of different total numbers are not permitted (although mixtures of different numbers are). For this reason, the OHT method cannot be applied to these BEC since coherent states are such superpositions of number states. A recent experiment \( \rho \) has demonstrated coherence within a single output pulse of atoms; however, the phase varies from one pulse to the next, and the exact nature of the output state is not yet known. Although theoretical work on atom lasers suggests that coherent atom fields will eventually be available \( |\psi\rangle \), we instead take a different tack and reconstruct the density matrix from a two-mode system with fixed atom number.

We describe an atom interferometer, and a set of measurements sufficient to reconstruct the state. Using an angular momentum formalism suitable for the two-mode states, the reconstruction process amounts to the solution of a linear algebra problem. This is accomplished with standard Fourier transform and least-squares methods. Finally we reconstruct a two-mode coherent state from simulated noisy data, and compare the actual error with an analytical estimate.

We consider a fixed number of atoms which have been cooled to below the critical temperature for BEC. The atoms are prepared in the two-mode quantum state

\[
\rho = \sum_a p_a |\psi_a\rangle \langle \psi_a| \tag{1}
\]

\[
|\psi_a\rangle = \sum_{n=0}^{N} c_{n,a} |N - n\rangle_1 \otimes |n\rangle_2 \tag{2}
\]

that is to be measured. The modes may correspond to e. g. different hyperfine states; such a BEC has just been produced in an experiment \( \rho \). One of the modes has its phase shifted relative to the other by \( \phi \), that can be varied. Then an operation corresponding to that of a beam splitter is applied to the two modes. The number of atoms in one of the output modes is counted with the detector (see Fig. \( \rho \)). If the initial state preparation is considered to be a generalization of a beam splitter (which entangles a product state), the interferometer resembles a Mach-Zehnder arrangement. To collect data for tomography, for each of \( K \) phase shifts, we repeat the
experiment τ times, resulting in probability distributions of counts at each value of ϕ.

At this stage we are making several idealisations. We neglect the collisions between the atoms, especially at the beam splitter where this would result in mixing the two modes. The beamsplitter is assumed to be lossless, with transmission cos^2 θ. The action of the phase shifter and beamsplitter together on the density matrix ρ for the two modes is given by

\[ \rho_{\text{out}} = U(\theta, \phi)\dagger \rho U(\theta, \phi) \]  

with

\[ U(\theta, \phi) = \exp[iθ(a_1^†a_2e^{iφ} + a_2^†a_1e^{iφ})]. \]  

Finally we assume the detector has unit efficiency. This assumption is not crucial, as we can accumulate the same set of data when the efficiency is less than unity by counting atoms in both modes and keeping only the data for which the sum of counts was N.

One possible realisation of the interferometer is an extension of the recent output coupler built by the MIT group et al [8]. After the condensate is prepared in a superposition of \(|gmF\rangle\) and \(|gmF + 2\rangle\) hyperfine states, the atoms are dropped out of the trap. The operation \(U(\theta, \phi)\) is accomplished via two off-resonance optical pulses, connecting the states in a Raman transition, which also introduces a relative phase shift due to the phase between the pulses. Finally an on-resonance rf pulse transfers atoms from the untrapped \(|gmF\rangle\) state to the trapped \(|g'F\rangle\) state; the separate groups of atoms can then be counted.

To simplify the notation and the calculation of matrix elements, we use the formal equivalence between the algebra for two harmonic oscillators and that for angular momentum [11]. We write the state

\[ |n\rangle \otimes |N - n\rangle = |j + m\rangle_1 \otimes |j - m\rangle_2 \]  

as \(|m\rangle\) where \(j = N/2\) and \(m = n - j\). The 2\(j + 1\) states \(|m\rangle\) have all the properties of the eigenstates of \(J^2\) and \(J_z\) and important operators are

\[ J_+ = a_1^†a_2 \]  
\[ J_- = a_2^†a_1 \]  
\[ J_z = \frac{1}{2} \left( a_1^†a_1 - a_2^†a_2 \right). \]

The combined effect of the beamsplitter and phase shift is seen to be a rotation by \(-2θ\) about an axis \(\hat{\mathbf{n}}_ϕ = \hat{\mathbf{x}} \cos ϕ - \hat{\mathbf{y}} \sin ϕ\)

\[ U(\theta, \phi) = \exp \left[ iθ (J_+e^{iφ} + J_-e^{-iφ}) \right] = \exp \left[ i2θ \mathbf{J} \cdot \hat{\mathbf{n}}_ϕ \right]. \]

An explicit expression for the number-basis matrix elements of the rotation is

\[ D_{lm}^{(j)} = \langle l|U(\theta, \phi)|m\rangle \]  

\[ = \sum_k \frac{\sqrt{(j + m)!/(j - m)!}}{k!(j + m - k)!(j - k - l)!(k + l - m)!} \times (\cos θ)^{2j - 2k + m - l} (i\sin θ)^{2k + l - m} e^{i(l - m)ϕ}. \]

where the sum is over all values for which the arguments of the factorials are nonnegative.

The probability of \(m\) counts at the detector, for a phase shift setting of \(ϕ\) is

\[ P_m(ϕ) = \langle m|U(θ, ϕ)\dagger ρ_{\text{in}}U(θ, ϕ)|m\rangle \]

or in the number-state basis

\[ P_m(ϕ) = \sum_{l,l'} T_{m,ϕl,l}ρ_{ll'}, \]

where

\[ T_{m,ϕl,l} = D_{lm}^{(j)*}(θ, ϕ)D_{lm}^{(j)}(θ, ϕ). \]

This last equation shows that \(T\) is simply a linear operator taking the density matrix (an \((N + 1)^2\) element complex vector) to the probability data (an \((N + 1)K\) element real vector); we wish to invert this operator to reconstruct the density matrix. This is formally done by defining the positive semidefinite, hermitian \((N + 1)^2 \times (N + 1)^2\) matrix

\[ M = T^\dagger T, \]

where \(T^\dagger\) means the conjugate transpose of the matrix \(T\), and computing

\[ ρ = M^{-1}T^\dagger P, \]

provided \(M\) is nonsingular. This is actually the best solution in the least squares sense [12].

For the value \(θ = \frac{π}{4}\), corresponding to a balanced beamsplitter, the matrix \(M\) is singular. Since this might be somewhat surprising we prove this result, and give an example of two states which will produce identical probability data. First we note that if there exists a vector \(v\) such that \(Mv = 0\), this implies \(Tv = 0\). Making the Ansatz

\[ v = f(J_z) \]
\[ f(-q) = -f(q), \]

and using the formula for \(T\) in the number state basis Eq. (13) we find

\[ Tv = \sum_{q=-j}^i |D_{j,q}^{(j)}(\frac{π}{4}, ϕ)|^2 f(q). \]

Then, using the fact from Eq. (13) that

\[ |D_{j,q}^{(j)}(\frac{π}{4}, ϕ)| = |D_{j,q}^{(j)}(\frac{π}{4}, ϕ)|, \]
yields zero for the sum of an odd function in Eq. (18), for any values of \( l \) and \( \phi \). This proves that the matrix \( M \) is singular for \( \theta = \frac{\pi}{4} \). To illustrate this physically, we choose the pure states

\[
|\psi_1\rangle = \frac{N}{2} + m\rangle_1 \otimes \frac{N}{2} - m\rangle_2
\]

\[
|\psi_2\rangle = \frac{N}{2} - m\rangle_1 \otimes \frac{N}{2} + m\rangle_2
\]

having the respective density matrices

\[
\rho_1 = |m\rangle\langle m|
\]

\[
\rho_2 = |-m\rangle\langle -m|.
\]

Then the difference vector of the density matrices is in the nullspace of \( T \):

\[
\rho_2 - \rho_1 = f(J_z) \text{ where } f(x) = \begin{cases} 1 & \text{if } x = m \\ -1 & \text{if } x = -m \\ 0 & \text{otherwise} \end{cases}
\]

so that \( T\rho_1 = T\rho_2 \). Clearly the balanced scheme cannot distinguish these two states because there is an ambiguity about which number of atoms belongs to which mode. This difficulty does not arise in OHT because in that case one of the inputs is already known to be a coherent state. Of course, varying \( \theta \) will enable these two situations to be distinguished, but we would like to keep the experimental setup the same for all of the data collection, so we require \( \theta \neq \frac{\pi}{4} \).

The problem of solving Eq. (12) is considerably simplified by taking the Fourier series of the probability data,

\[
F_m(r) = \frac{1}{2\pi} \int_0^{2\pi} P_m(\phi)e^{ir\phi} d\phi
\]

\[
= \sum_{l=r-j}^{l=r+j} T_{m0,l-r} \rho_{l,r}.
\]

(In practice, this is approximated by the discrete Fourier transform on a set of data for many \( \phi \) equally distributed about the circle.) By taking successive values of \( r \) from 0 to \( 2r \), each diagonal of the density matrix can be solved for independently. Since all the operations are linear the calculations are conveniently done in MATLAB.

We will see below that for elements far from the main diagonal, the data can be insufficient to accurately reconstruct them. Since the trace of \( \rho^2 \) is less than unity, each norm of a diagonal is also less than one. Taking this prior information into account, we can reduce the norm of each diagonal by using Tikhonov regularization [13].

We present a simulation of the probability data and reconstruct the density matrix from it, for the pure state

\[
|\psi\rangle = \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} \sin^{N-n} \theta \cos^n \theta e^{i\varphi} |n\rangle_1 \otimes |N-n\rangle_2.
\]

This state is the projection of coherent states for modes 1 and 2 onto the subspace with fixed total number \( N \) [14]; it can be produced by an rf pulse as in [8]. The parameters chosen were \( N = 49 \), \( \theta = 0.54 \), \( \varphi = 0.13 \). To simulate the noise associated with calculating probabilities from histograms, we add to each probability a term

\[
\delta P_m(\phi) = \sqrt{\frac{P_m(\phi)}{\tau}} g(P_m(\phi)),
\]

where \( \tau \) is the number of trials for that value of phase and \( g \) is a Gaussian distribution with zero mean and unit variance. In the example below we take \( \tau = 2000 \) and \( K = 180 \), i.e. 2000 trials every 2 degrees. In Fig. 2 we plot probability of \( m \) counts at each phase shift. The original and reconstructed density matrices are compared in Fig. 3. The reconstruction is most accurate (in both magnitude and phase) for the elements close to the main diagonal.

The error in the solution to the least squares problem can be found using the singular value decomposition of each matrix \( T \) [13]. Specifically, if the norm of the error in \( F(r) \) is \( ||\delta F(r)|| \) then the norm of the difference between the original and reconstructed \( r \)-th diagonal is

\[
||\delta \rho_r|| \equiv \sqrt{\sum_{i=-j}^{j-r} |\rho_{i,r+r}^0 - \rho_{i,r+r}|^2} \leq \frac{||\delta F(r)||}{\min \sigma},
\]

where \( \sigma \) is an eigenvalue of \( \sqrt{T^TT} \). Using the definition Eq. (25) to transform the variances \( \delta P_r \), and the fact that probabilities for a given phase must sum to one, we find

\[
||\delta F(r)|| = \frac{1}{\sqrt{K\tau}}.
\]

Thus our upper bound is independent of the actual data, depending only on \( N \), \( r \) and \( \theta \). In Fig. 4 we plot the actual and estimated norm of the error, divided by \( \sqrt{N - r + 1} \) so that it is per element, as a function of the distance \( r \) from the main diagonal. We have not been able to find a simple formula for the eigenvalues \( \sigma = \sigma(N, r, \theta) \), but the general trend is for smaller minimum values as \( r \) approaches \( N \). The most difficult states to reconstruct will be highly entangled ones, which depend on the high-frequency components of the probability. Provided one has some idea of which diagonals are needed this analysis is useful in choosing the optimal transmission.

A different type of error might occur if the number of atoms originally trapped is not known precisely, or (equivalently) if losses occur between the preparation of the original state and the detection of atoms. We are unable to quantify such errors, since states of fixed number are assumed. For mixed states with narrowly distributed total numbers, but no fine-scale oscillations in density matrix elements, we can expect the above algorithm will produce a qualitatively correct picture.
To conclude, we have proposed an atom interferometer for measuring the quantum state of a two-mode BEC. An algorithm for calculating the density matrix elements was applied to simulated probability data and successfully reconstructed a sample state. The errors can be estimated before data collection, and may be minimized by choosing an appropriate beamsplitter transmission if some qualitative features about the state are known.

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Fig. 1: Atom Interferometer

Fig. 2: Probability data
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Fig. 4: Errors vs. diagonal distance from main diagonal