The Triple-Pair Construction for Weighted $\omega$-Pushdown Automata

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1 Introduction and preliminaries

Weighted pushdown automata were introduced by Kuich, Salomaa [14]. Many results on classical pushdown automata and context-free grammars can be generalized to weighted pushdown automata and algebraic systems. Classic pushdown automata can also be used to accept infinite words (see Cohen, Gold [3]) and it is this aspect we generalize in our paper. We consider weighted $\omega$-pushdown automata and their relation to algebraic systems over a complete semiring-semimodule pair $((S \ll \Sigma^*)^n \times (S \ll \Sigma^\omega)^n)$. It turns out that the well known triple construction for pushdown automata can be generalized to a triple-pair construction for weighted $\omega$-pushdown automata. Our paper generalizes results of Droste, Kuich [5].

The paper consists of this and three more sections. In Section 2 pushdown transition matrices are introduced and their properties are studied. The main result of this section is that, for such a matrix $M$, the $p$-blocks, $p$ a pushdown symbol, of the infinite column vector $M^{\omega, l}$ satisfy a special equality. In Section 3 weighted $\omega$-pushdown automata are introduced. We show that for a weighted $\omega$-pushdown automaton $\mathcal{P}$ there exists a mixed algebraic system such that the behavior $\|\mathcal{P}\|$ of $\mathcal{P}$ is a component of a solution of this system. In Section 4 we consider the case that the complete star-omega semiring $S$ is equal to $\mathbb{B}$ or $\mathbb{N}^\omega$. Then for a given weighted $\omega$-pushdown automaton $\mathcal{P}$ a mixed context-free grammar is constructed that generates $\|\mathcal{P}\|$. The construction is a generalization of the well known triple construction and is called triple-pair construction for $\omega$-pushdown automata.

For the convenience of the reader, we quote definitions and results of Ésik, Kuich [8, 9, 10, 11] from Ésik, Kuich [7]. The reader should be familiar with Sections 5.1-5.6 of Ésik, Kuich [7].

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A semiring $S$ is called complete starsemiring if sums for all families $(s_i \mid i \in I)$ of elements of $S$ are defined, where $I$ is an arbitrary index set, and if $S$ is equipped with an additional unary star operation $^* : S \to S$ defined by $s^* = \sum_{j \geq 0} s^j$ for all $s \in S$. Moreover, certain conditions have to be satisfied making sure that computations with “infinite” sums can be performed analogous to those with finite sums.

A pair $(S, V)$, where $S$ is a complete starsemiring and $V$ is a complete $S$-semimodule is called a complete semiring-semimodule pair if products for all sequences $(s_i \mid i \in \mathbb{N})$ of elements of $S$ are defined and if $S$ and $V$ are equipped with an omega operation $^\omega : S \to V$ defined by $s^\omega = \prod_{j \geq 1} s$ for all $s \in S$. Moreover, certain conditions (e.g., “infinite” distributive laws) have to be satisfied making sure that computations with “infinite” sums and “infinite” products can be performed analogous to those with finite sums and finite products. (For details see Conway [4], Eilenberg [6], Bloom, Ésik [1], Ésik, Kuich [7], pages 30 and 105-107.)

A semiring $S$ is called complete star-omega semiring if $(S, S)$ is a complete semiring-semimodule pair.

For the theory of infinite words and finite automata accepting infinite words by the Büchi condition consult Perrin, Pin [15].

## 2 Pushdown transition matrices

In this section we introduce pushdown transition matrices and study their properties. Our first theorem generalizes Theorem 10.5 of Kuich, Salomaa [14]. Then we show in Theorems 3 and 6 that, for a pushdown transition matrix $M$, $(M^\omega)_p$ and $(M^\omega)^l$, $0 \leq l \leq n$, $p \in \Gamma$, introduced below satisfy the same specific equality. In Theorem 1 $S$ denotes a complete starsemiring; afterwards in this section, $(S, V)$ denotes a complete semiring-semimodule pair.

Following Kuich, Salomaa [14] and Kuich [13], we introduce pushdown transitions matrices. Let $\Gamma$ be an alphabet, called pushdown alphabet and let $n \geq 1$. A matrix $M \in (S^{n \times n})^\Gamma \times \Gamma^*$ is termed a pushdown transition matrix (with pushdown alphabet $\Gamma$ and stateset $\{1, \ldots, n\}$) if

(i) for each $p \in \Gamma$ there exist only finitely many blocks $M_{p, \pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$M_{\pi_1, \pi_2} = \begin{cases} M_{p, \pi} & \text{if there exist } p \in \Gamma, \pi, \pi' \in \Gamma^* \text{ with } \pi_1 = p\pi' \text{ and } \pi_2 = \pi\pi', \\ 0 & \text{otherwise}. \end{cases}$$

For the remaining of this paper, $M \in (S^{n \times n})^\Gamma \times \Gamma^*$ will denote a pushdown transition matrix with pushdown alphabet $\Gamma$ and stateset $\{1, \ldots, n\}$.

Our first theorem generalizes Theorem 10.5 of Kuich, Salomaa [14] and Theorem 6.2 of Kuich [13] to complete starsemirings. First observe that for all $p_1 \in \Gamma^+$, $\rho_2, \pi \in \Gamma^*$, we have $M_{p_1, \pi, \rho_2, \pi} = M_{p_1, \pi}$. Intuitively, our next theorem states that, emptying the pushdown tape with contents $p\pi$ by finite computations has the same effect (i.e., $(M^+)_{p, \pi, \epsilon}$) as emptying first the pushdown tape with contents $p$ (i.e., $(M^+)_{p, \epsilon}$) by finite computations and afterwards (i.e., multiplying) emptying the pushdown tape with contents $\pi$ (i.e., $(M^+)_{\pi, \epsilon}$) by finite computations.

**Theorem 1.** Let $S$ be a complete starsemiring and $M \in (S^{n \times n})^\Gamma \times \Gamma^*$ be a pushdown transition matrix. Then, for all $p \in \Gamma$ and $\pi \in \Gamma^*$,

$$(M^+)_{p, \pi, \epsilon} = (M^+)_{p, \epsilon}(M^+)_{\pi, \epsilon}.$$
Proof. Since the case $\pi = \varepsilon$ is trivial, we assume $\pi \in \Gamma^*$. We obtain

$$(M^*)_{p,\pi,\varepsilon} = \sum_{m \geq 0} (M^{m+1})_{p,\pi,\varepsilon}$$

$$= \sum_{m \geq 0} \sum_{\pi_1, \ldots, \pi_m \in \Gamma^+} M_{p,\pi_1,\pi_2} \cdots M_{\pi_{m-1},\pi_m,\varepsilon}$$

$$= \left( \sum_{m \geq 0} \sum_{\pi_1, \ldots, \pi_m \in \Gamma^+} M_{p,\pi_1,\pi_2} \cdots M_{\pi_{m-1},\pi_m,\varepsilon} \right) \cdot \left( \sum_{m \geq 0} \sum_{\pi_1, \ldots, \pi_m \in \Gamma^+} M_{\pi_1,\pi_2} \cdots M_{\pi_{m-1},\pi_m,\varepsilon} \right)$$

$$= \left( \sum_{m \geq 0} \sum_{\pi_1, \ldots, \pi_m \in \Gamma^+} M_{p,\pi_1,\pi_2} \cdots M_{\pi_{m-1},\pi_m,\varepsilon} \right) (M^*)_{p,\pi} = (M^*)_{p,\pi,\varepsilon}.$$
and

\[ S(j)(t) = \sum_{(j)(t)} M_{p_1 \ldots p_k, p_1} M_{p_1, p_2} M_{p_2, p_3} \ldots, 2 \leq j, t \geq 1, \]

where \( \sum_{(1)} \) and \( \sum_{(j)(t)} \) means summation over all runs in the classes (1) and (j)(t), respectively. We obtain

\[ S(1) = \sum_{p_1, p_1' \in \Gamma^+} M_{p_1, p_1'} M_{p_1', p_1'} \cdots = (M^\omega)_{p_1}. \]

For \( 2 \leq j \leq k - 1, t \geq 1, \) we obtain

\[ S(j)(t) = \left( \sum_{p_1, p_1' \in \Gamma^+} M_{p_1, p_1'} \cdots M_{p_{j-1}, p_j} \right) \cdot \left( \sum_{p_j, p_j' \in \Gamma^+} M_{p_j, p_j'} \cdots \right) = (M^j)_{p_1 \ldots p_{j-1}, e}(M^\omega)_{p_j}. \]

For \( t \geq 1, \)

\[ S(k)(t) = \left( \sum_{p_1, p_1' \in \Gamma^+} M_{p_1, p_1'} \cdots M_{p_{k-1}, p_k} \right) \cdot \left( \sum_{p_k, p_k' \in \Gamma^+} M_{p_k, p_k'} \cdots \right) = (M^k)_{p_1 \ldots p_k, e}(M^\omega)_{p_k}. \]

Hence, we obtain

\[ (M^\omega)_{p_1 \ldots p_k} = S(1) + \sum_{2 \leq j \leq k - 1} \sum_{t \geq 1} S(j)(t) = (M^\omega)_{p_1} + \sum_{2 \leq j \leq k} (M^j)_{p_1 \ldots p_{j-1}, e}(M^\omega)_{p_j} \]

\[ = \sum_{1 \leq j \leq k} (M^j)_{p_1 \ldots p_{j-1}, e}(M^\omega)_{p_j}. \]

Intuitively, our next theorem states that the infinite computations starting with \( p \) on the pushdown tape yield the same matrix \( (M^\omega)_p \) as summing up, for all \( \pi = p_1 \ldots p_k \) and all \( 1 \leq j \leq k \) the product of \( M_{p, \pi} \) (i.e., changing the contents of the pushdown tape from \( p \) to \( \pi \)) with the matrix \( (M^j)_{p_1 \ldots p_{j-1}, e} \) (i.e., emptying the pushdown tape with contents \( p_1 \ldots p_{j-1} \) by finite computations) and eventually with the matrix \( (M^\omega)_{p_j} \) (i.e., the infinite computations starting with \( p_j \) on the pushdown tape).

This means that in \( \pi \) the pushdown symbols \( p_1, \ldots, p_{j-1} \) are emptied by finite computations and \( p_j \) is chosen for starting the infinite computations. Clearly, \( p_{j+1}, \ldots, p_k \) are not read.

**Theorem 3.** Let \( (S, V) \) be a complete semiring-semimodule pair and let \( M \in \left(S^{\alpha \times \beta}\right)^{\Gamma^+ \times \Gamma^+} \) be a pushdown transition matrix. Then, for all \( p \in \Gamma, \)

\[ (M^\omega)_p = \sum_{p_1 \ldots p_k \in \Gamma^+} M_{p, p_1 \ldots p_k} \sum_{1 \leq j \leq k} (M^j)_{p_1 \ldots p_{j-1}, e}(M^\omega)_{p_j}. \]

**Proof.** We obtain, by Lemma 2

\[ \sum_{p_1 \ldots p_k \in \Gamma^+} M_{p, p_1 \ldots p_k} \sum_{1 \leq j \leq k} (M^j)_{p_1 \ldots p_{j-1}, e}(M^\omega)_{p_j} = \sum_{p_1 \ldots p_k \in \Gamma^+} M_{p, p_1 \ldots p_k} (M^\omega)_{p_1 \ldots p_k} \]

\[ = \sum_{\pi \in \Gamma^+} M_{p, \pi} (M^\omega)_{\pi} = (MM^\omega)_{\pi} = M^\omega. \]

\[ \square \]
We define the matrices \((A_M)_{p,p'} \in S^{n \times n}, M \in (S^{n \times n})^{G \times \Gamma^*}\) a pushdown transition matrix, by

\[
(A_M)_{p,p'} = \sum_{\pi=p_1 \cdots p_k \in \Gamma^+} M_{p,p}(M^\pi)_{p_1,\cdots,p_{k-1},x},
\]

and \(A_M \in (S^{n \times n})^{G \times \Gamma}\) by \(A_M = ((A_M)_{p,p'})_{p,p' \in \Gamma}\). Whenever we use the notation \(A_M\) we mean the matrix just defined.

**Theorem 4.** Let \((S,V)\) be a complete semiring-semimodule pair and let \(M \in (S^{n \times n})^{G \times \Gamma^*}\) be a pushdown transition matrix. Then, for all \(p \in \Gamma\),

\[
(M^\omega)_{p} = \sum_{p' \in \Gamma} (A_M)_{p,p'}(M^{\omega})_{p'}.
\]

**Proof.** We obtain by Theorem 3

\[
\sum_{p' \in \Gamma} (A_M)_{p,p'}(M^{\omega})_{p'} = \sum_{p' \in \Gamma} \sum_{\pi=p_1 \cdots p_k \in \Gamma^+} \sum_{1 \leq j \leq k} \delta_{p_j,p'}M_{p,p}(M^\pi)_{p_1,\cdots,p_{j-1},x}(M^{\omega})_{p'}
\]

\[
= \sum_{p_1 \cdots p_k \in \Gamma^+} M_{p,p_1 \cdots p_k} \sum_{1 \leq j \leq k} \sum_{p' \in \Gamma} \delta_{p_j,p'}(M^\pi)_{p_1,\cdots,p_{j-1},x}(M^{\omega})_{p'}
\]

\[
= \sum_{p_1 \cdots p_k \in \Gamma^+} M_{p,p_1 \cdots p_k} \sum_{1 \leq j \leq k} (M^\pi)_{p_1,\cdots,p_{j-1},x}(M^{\omega})_{p_j} = (M^\omega)_{p}.
\]

\[\square\]

When we say “\(G\) is the graph with adjacency matrix \(M \in (S^{n \times n})^{G \times \Gamma^*}\), then it means that \(G\) is the graph with adjacency matrix \(M' \in S^{(\Gamma^* \times n) \times (G \times \Gamma^*)}\), where \(M\) corresponds to \(M'\) with respect to the canonical isomorphism between \((S^{n \times n})^{G \times \Gamma^*}\) and \(S^{(\Gamma^* \times n) \times (G \times \Gamma^*)}\).

Let now \(M\) be a pushdown transition matrix and \(0 \leq l \leq n\). Then \(M^{\omega,l}\) is the column vector in \((V^n)^\Gamma\) defined as follows: For \(p \in \Gamma^*\) and \(1 \leq i \leq n\), let \(((M^{\omega,l})_j)_{p}^{i}\) be the sum of all weights of paths in the graph with adjacency matrix \(M\) that have initial vertex \((\pi,i)\) and visit vertices \((\pi',i')\), \(\pi' \in \Gamma^*, 1 \leq i' \leq l\), infinitely often. Observe that \(M^{\omega,0} = 0\) and \(M^{\omega,n} = M^{\omega}\).

Let \(P_l = \{(j_1,j_2,\cdots) \in \{1,\ldots,n\}^l | j_i \leq l\}\) for infinitely many \(i \geq 1\).

Then for \(p \in \Gamma^+, 1 \leq j \leq n\), we obtain

\[
((M^{\omega,l})_j)_{p} = \sum_{(\pi_1,\pi_2,\cdots) \in \Gamma^+(j_1,j_2,\cdots)} (M_{\pi_1,\pi_2})_{j_1,j_2} (M_{\pi_2,\pi_3})_{j_3,j_4} \cdots.
\]

**Lemma 5.** Let \((S,V)\) be a complete semiring-semimodule pair and let \(M \in (S^{n \times n})^{G \times \Gamma^*}\) be a pushdown transition matrix. Then, for all \(p_1,\ldots,p_k \in \Gamma, 0 \leq l \leq n\),

\[
(M^{\omega,l})_{p_1 \cdots p_k} = \sum_{1 \leq j \leq k} (M^*)_{p_1 \cdots p_{j-1},x}(M^{\omega,l})_{p_j}.
\]

**Proof.** By the proof of Lemma 2 and the following summation identity: Assume that \(A_1,A_2,\ldots\) are matrices in \(S^{n \times n}\). Then, for \(0 \leq l \leq n, 1 \leq j \leq n,\) and \(m \geq 1\),

\[
\sum_{(j_1,j_2,\cdots) \in P_l} (A_1)_{j_1} (A_2)_{j_2} \cdots \sum_{1 \leq j_1,\ldots,j_m \leq n} (A_1)_{j_1} \cdots (A_m)_{j_m} \sum_{(j_{m+1},j_{m+2},\cdots) \in P_l} (A_{m+1})_{j_{m+1}j_{m+2} \cdots}.
\]

\[\square\]
Theorem 6 generalizes Theorem 4 from $M^{\omega,n}$ to $M^{\omega,l}$, $0 \leq l \leq n$.

**Theorem 6.** Let $(S, V)$ be a complete semiring-semimodule pair and let $M \in (S^{n \times n})^{\Gamma \times \Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$, $0 \leq l \leq n$,

$$(M^{\omega,l})_p = \sum_{p' \in \Gamma} (A_M)_{p,p'} (M^{\omega,l})_{p'}.$$

### 3 Algebraic systems and $\omega$-pushdown automata

In this section, we define $\omega$-pushdown automata and show that for an $\omega$-pushdown automaton $\mathcal{P}$ there exists an algebraic system over a quemiring such that the behavior $\|\mathcal{P}\|$ of $\mathcal{P}$ is a component of a solution of this system.

For the definition of an $S'$-algebraic system over a quemiring $S \times V$ we refer the reader to [7], page 136, and for the definition of quemirings to [7], page 110. Here we note that a quemiring $T$ is isomorphic to a quemiring $S \times V$ determined by the semiring-semimodule pair $(S, V)$, cf. [7], page 110.

In the sequel, $(S, V)$ is a complete semiring-semimodule pair and $S'$ is a subset of $S$ containing 0 and 1. Let $M \in (S^{n \times n})^{\Gamma \times \Gamma^*}$ be a pushdown matrix. Consider the $S'^{n \times n}$-algebraic system over the complete semiring-semimodule pair $(S'^{n \times n}, V^n)$, i.e., over the quemiring $S'^{n \times n}$.

**Theorem 6.** Let $(S, V)$ be a complete semiring-semimodule pair and let $M \in (S^{n \times n})^{\Gamma \times \Gamma^*}$ be a pushdown transition matrix. Then, for all $p \in \Gamma$, $0 \leq l \leq n$,

$$(M^{\omega,l})_p = \sum_{p' \in \Gamma} (A_M)_{p,p'} (M^{\omega,l})_{p'}.$$

(See Section 5.6 of Ésik, Kuich [7].) The variables of this system (1) are $y_p, p \in \Gamma$, and $x_p, \pi \in \Gamma^*$, is defined by $y_p = y_p \pi = y_p \pi_1$ for $p \in \Gamma$, $\pi \in \Gamma^*$ and $y_\pi = \varepsilon$. Hence, for $\pi = p_1 \ldots p_k$, $y_\pi = y_{p_1} \ldots y_{p_k}$. The variables $y_p$ are variables for $(S'^{n \times n}, V^n)$.

Let $x = (x_p)_{p \in \Gamma}$, where $x_p, p \in \Gamma$, are variables for $S'^{n \times n}$. Then, for $p \in \Gamma$, $\pi = p_1 p_2 \ldots p_k$, $(M_{p, \pi} y_\pi)_x$ is defined to be

$$(M_{p, \pi} y_\pi)_x = (M_{p, \pi} y_{p_1} \ldots y_{p_k})_x = M_{p, \pi} z_{p_1} + M_{p, \pi} x_{p_1} z_{p_2} + \cdots + M_{p, \pi} x_{p_1} \ldots x_{p_{k-1}} z_{p_k}.$$

Here $z_p, p \in \Gamma$, are variables for $V^n$.

We obtain, for $p \in \Gamma$, $\pi = p_1 \ldots p_k$,

$$(M_{p, \pi} y_\pi)_x = \sum_{p' \in \Gamma} M_{p, \pi} x_{p_1} \ldots x_{p_{k-1}} z_{p'},$$

and

$$(M_{p, \pi} y_\pi)_x = \sum_{p' \in \Gamma} M_{p, \pi} x_{p_1} \ldots x_{p_{k-1}} z_{p'}.$$

The system (1) induces the following mixed $\omega$-algebraic system:

$$(M_{p, \pi} y_\pi)_x = \sum_{p' \in \Gamma} M_{p, \pi} x_{p_1} \ldots x_{p_{k-1}} z_{p'}.$$

Here (2) is an $S'^{n \times n}$-algebraic system over the semiring $S'^{n \times n}$ (see Section 2.3 of Ésik, Kuich [7]) and (3) is an $S'^{n \times n}$-linear system over the semimodule $V^n$ (see Section 5.5 of Ésik, Kuich [7]).
In the classical theory of automata and formal languages, equation (2) plays a crucial role in the transition from pushdown automata to context-free grammars. It is, in the form of matrix notation, the well-known triple construction. (See Harrison [12], Theorem 5.4.3; Bucher, Maurer [2], Sätze 2.3.10, 2.3.30; Kuich, Salomaa [14], pages 178, 306; Kuich [13], page 642; Ésik, Kuich [7], pages 77, 78.) By Theorem 5.6.1 of Ésik, Kuich [7], \((A, U) \in ((S^{n \times n})^\Gamma, (V^n)^\Gamma)\) is a solution of (1) iff \(A\) is a solution of (2) and \((A, U)\) is a solution of (3). We now compute such solutions \((A, U)\).

**Theorem 7.** Let \(S\) be a complete starsemiring and \(M \in (S^{n \times n})^\Gamma \times \Gamma^n\) be a pushdown transition matrix. Then \(((M^*)_{p,e})_{p \in \Gamma}\) is a solution of (2).

**Proof.** By Theorem 1.

We now substitute in (3) for \((x_p)_{p \in \Gamma}\) the solution \(((M^*)_{p,e})_{p \in \Gamma}\) of (1) and obtain the \(S^{n \times n}\)-linear system (4) over the semimodule \(V^n\)

\[ z_p = \sum_{p' \in \Gamma} (A_M)_{p,p'} z_{p'}, \quad p \in \Gamma. \]  

(4)

**Theorem 8.** Let \((S, V)\) be a complete semiring-semimodule pair and \(M \in (S^{n \times n})^\Gamma \times \Gamma^n\) be a pushdown transition matrix. Then, for all \(0 \leq l \leq n\), \(((M^{\omega,l})_{p})_{p \in \Gamma}\) is a solution of (4).

**Proof.** By Theorem 6.

**Corollary 9.** Let \((S, V)\) be a complete semiring-semimodule pair and \(M \in (S^{n \times n})^\Gamma \times \Gamma^n\) be a pushdown transition matrix. Then, for all \(0 \leq l \leq n\),

\[ (((M^*)_{p,e})_{p \in \Gamma}, (M^{\omega,l})_{p \in \Gamma}) \]

is a solution of (1).

We can write the system (4) in matrix notation in the form

\[ z = A_M z \]

(5)

with column vector \(z = (z_p)_{p \in \Gamma}\).

**Corollary 10.** Let \((S, V)\) be a complete semiring-semimodule pair and \(M \in (S^{n \times n})^\Gamma \times \Gamma^n\) be a pushdown transition matrix. Then for all \(0 \leq l \leq n\), \(((M^{\omega,l})_{p})_{p \in \Gamma}\) is a solution of (5).

We now introduce pushdown automata and \(\omega\)-pushdown automata (see Kuich, Salomaa [14], Kuich [13], Cohen, Gold [3]).

Let \(S\) be a complete semiring and \(S' \subseteq S\) with \(0, 1 \in S'\). An \(S'\)-pushdown automaton over \(S\)

\[ \mathcal{P} = (n, \Gamma, I, M, P, p_0) \]

is given by

(i) a finite set of states \(\{1, \ldots, n\}, \ n \geq 1\),

(ii) an alphabet \(\Gamma\) of pushdown symbols,

(iii) a pushdown transition matrix \(M \in (S^{n \times n})^\Gamma \times \Gamma^n\),

(iv) an initial state vector \(I \in S^{1 \times n}\),
(v) a final state vector \( P \in S^{n \times 1} \),

(vi) an initial pushdown symbol \( p_0 \in \Gamma \).

The behavior \( \|P\| \) of \( P \) is an element of \( S \) and is defined by \( \|P\| = I(M^*)_{p_0,e}P \).

For a complete semiring-semimodule pair \((S, V)\), an \( S' - \omega \)-pushdown automaton (over \( (S, V) \))

\[
\mathcal{P} = (n, \Gamma, I, M, P, p_0, l)
\]
is given by an \( S' \)-pushdown automaton \((n, \Gamma, I, M, P, p_0)\) and an \( l \in \{0, \ldots, n\} \) indicating that the states \( 1, \ldots, l \) are repeated states.

The behavior \( \|\mathcal{P}\| \) of the \( S' - \omega \)-pushdown automaton \( \mathcal{P} \) is defined by

\[
\|\mathcal{P}\| = I(M^*)_{p_0,e}P + I(M^{\omega,l})_{p_0}.
\]

Here \( I(M^*)_{p_0,e}P \) is the behavior of the \( S' - \omega \)-pushdown automaton \( \mathcal{P}_1 = (n, \Gamma, I, M, P, p_0, 0) \) and \( I(M^{\omega,l})_{p_0} \) is the behavior of the \( S' - \omega \)-pushdown automaton \( \mathcal{P}_2 = (n, \Gamma, I, M, 0, p_0, l) \). Observe that \( \mathcal{P}_2 \) is an automaton with the Büchi acceptance condition: if \( G \) is the graph with adjacency matrix \( M \), then only paths that visit the repeated states \( 1, \ldots, l \) infinitely often contribute to \( \|\mathcal{P}_2\| \). Furthermore, \( \mathcal{P}_1 \) contains no repeated states and behaves like an ordinary \( S' \)-pushdown automaton.

**Theorem 11.** Let \((S, V)\) be a complete semiring-semimodule pair and let \( \mathcal{P} = (n, \Gamma, I, M, P, p_0, l) \) be an \( S' - \omega \)-pushdown automaton over \((S, V)\). Then \( \|(\|\mathcal{P}\|, (I(M^*)_{p,e})_{p \in \Gamma}, (I(M^{\omega,l})_{p})_{p \in \Gamma})\| \) is a solution of the \( S''^{n \times n} \)-algebraic system

\[
y_0 = Iy_{p_0}P, y_p = \sum_{\pi \in \Gamma^*} M_{p, \pi} y_{\pi}, p \in \Gamma
\]

over the complete semiring-semimodule pair \((S''^{n \times n}, V^n)\).

**Proof.** By Corollary [9] \((I(M^*)_{p,e})_{p \in \Gamma}, (I(M^{\omega,l})_{p})_{p \in \Gamma}\) is a solution of the second equation. Since

\[
I((I(M^*)_{p_0,e}), (I(M^{\omega,l})_{p_0}))P = (I(M^*)_{p_0,e}P, I(M^{\omega,l})_{p_0}) = \|\mathcal{P}\|,
\]

\( \|(\|\mathcal{P}\|, (I(M^*)_{p,e})_{p \in \Gamma}, (I(M^{\omega,l})_{p})_{p \in \Gamma})\| \) is a solution of the given \( S''^{n \times n} \)-algebraic system. \( \square \)

Let \( S \) be a complete star-omega semiring and \( \Sigma \) be an alphabet. Then by Theorem 5.5.5 of Ésik, Kuich [7], \((S \ll \Sigma^*, S \ll \Sigma^\omega)\) is a complete semiring-semimodule pair. Let \( \mathcal{P} = (n, \Gamma, M, I, P, p_0, l) \) be an \( S(\Sigma \cup \{\varepsilon\}) - \omega \)-pushdown automaton over \((S \ll \Sigma^*, S \ll \Sigma^\omega)\). Consider the algebraic system over the complete semiring-semimodule pair \( ((S \ll \Sigma^*)^{n \times n}, (S \ll \Sigma^\omega)^n) \)

\[
y_0 = Iy_{p_0}P, y_p = \sum_{\pi \in \Gamma^*} M_{p, \pi} y_{\pi}, p \in \Gamma
\]

and the mixed algebraic system [7] over \((S \ll \Sigma^*)^{n \times n}, (S \ll \Sigma^\omega)^n) \) induced by (6)

\[
x_0 = Ix_{p_0}P, x_p = \sum_{\pi = p_1 \ldots p_k \in \Gamma^*} M_{p, \pi} x_{p_1} \ldots x_{p_k}, p \in \Gamma,
\]

\[
z_0 = Iz_{p_0}P, z_p = \sum_{\pi = p_1 \ldots p_k \in \Gamma^*} M_{p, \pi} \sum_{1 \leq j \leq k} x_{p_1} \ldots x_{p_{j-1}} z_{p_j}, p \in \Gamma.
\]

**Corollary 12.** Let \((S, V)\) be a complete semiring-semimodule pair, \( \Sigma \) be an alphabet and \( \mathcal{P} = (n, \Gamma, M, I, P, p_0, l) \) be an \( S(\Sigma \cup \{\varepsilon\}) - \omega \)-pushdown automaton over \((S \ll \Sigma^*, S \ll \Sigma^\omega)\). Then \( (I(M^*)_{p_0,e}P, (I(M^*)_{p,e})_{p \in \Gamma}, I(M^{\omega,l})_{p_0}, ((I(M^{\omega,l})_{p})_{p \in \Gamma}) \) is a solution of (7). It is called solution of order \( l \).
Let now in (7)
\[ x = ([i, p, j])_{1 \leq i, j \leq n}, p \in \Gamma, \]
be \( n \times n \)-matrices of variables and
\[ z = ([i, p])_{1 \leq i \leq n}, p \in \Gamma \]
be \( n \)-dimensional column vectors of variables. If we write the mixed algebraic system (7) component-wise, we obtain a mixed algebraic system over \((S \ll \Sigma^* \gg, (S \ll \Sigma^\omega \gg))\) with variables \([i, p, j]\) over \( S \ll \Sigma^* \gg \), where \( p \in \Gamma, 1 \leq i, j \leq n \), and variables \([i, p]\) over \( S \ll \Sigma^\omega \gg \), where \( p \in \Gamma, 1 \leq i \leq n \).

Writing the mixed algebraic system (7) component-wise, we obtain the system (8):
\[
\begin{align*}
x_0 &= \sum_{1 \leq m_1, m_2 \leq n} I_{m_1} [m_1, p_0, m_2] P_{m_2}, \\
[i, p, j] &= \sum_{k \geq 0, p_1, \ldots, p_k \in \Gamma} \sum_{1 \leq m_1, \ldots, m_2 \leq n} (M p, p_1 \ldots p_k)_{i, m_1} [m_1, p_1, m_2] [m_2, p_2, m_3] \ldots [m_k, p_k, j], \\
z_0 &= \sum_{1 \leq m \leq n} I_{m} [m, p_0] \\
[i, p] &= \sum_{k \geq 1, p_1, \ldots, p_k \in \Gamma} \sum_{1 \leq m_1, \ldots, m_k \leq n} (M p, p_1 \ldots p_k)_{i, m_1} [m_1, p_1, m_2] \ldots [m_{j-1}, p_{j-1}, m_j] [m_j, p_j], \\
\end{align*}
\]
\( p \in \Gamma, 1 \leq i, j \leq n \) \quad (8)

Theorem 13. Let \((S, V)\) be a complete semiring-semimodule pair and \( \mathcal{P} = (n, \Gamma, M, I, p_0, P, l) \) be a \( S'\)-\( \omega \)-pushdown automaton. Then
\[
(I (M^*)_{p_0} e P_1 (\ldots (I (M^*)_{p, \epsilon} i, j)_{p \in \Gamma} i, j \leq n)) (I (M^\omega)_{p_0}) \]
is a solution of the system (8) called solution of order \( l \) with \( \| \mathcal{P} \| = (I (M^*)_{p_0} e P_1 (I (M^\omega)_{p_0})) \).

4 Mixed algebraic systems and mixed context-free grammars

In this section we associate a mixed context-free grammar with finite and infinite derivations to the algebraic system (8). The language generated by this mixed context-free grammar is then the behavior \( \| \mathcal{P} \| \) of the \( \omega \)-pushdown automaton \( \mathcal{P} \). The construction of the mixed context-free grammar from the \( \omega \)-pushdown automaton \( \mathcal{P} \) is a generalization of the well known triple construction and is called now triple-pair construction for \( \omega \)-pushdown automata. We will consider the commutative complete star-omega semirings \( \mathbb{B} = (\{0, 1\}, \lor, \land, *, 0, 1) \) with \( 0^* = 1^* = 1 \) and \( \mathbb{N}^\omega = (\mathbb{N} \cup \{\infty\}, +, *, 0, 1) \) with \( 0^* = 1 \) and \( a^* = \infty \) for \( a \neq 0 \).

If \( S = \mathbb{B} \) or \( S = \mathbb{N}^\omega \) and \( 0 \leq l \leq n \), then we associate to the mixed algebraic system (8) over \((S \ll \Sigma^* \gg, (S \ll \Sigma^\omega \gg))\), and hence to the \( \omega \)-pushdown automaton \( \mathcal{P} = (n, \Gamma, I, M, P, p_0, l) \), the mixed context-free grammar
\[ G_l = (X, Z, \Sigma, P_X, P_Z, x_0, z_0, l). \]

(See also Ésik, Kuich [7, page 139].) Here
(i) \( X = \{x_0\} \cup \{[i, p, j] \mid 1 \leq i, j \leq n, p \in \Gamma\} \) is a set of variables for finite derivations;
(ii) \( Z = \{z_0\} \cup \{[i, p] \mid 1 \leq i \leq n, p \in \Gamma\} \) is a set of variables for infinite derivations;
(iii) \( \Sigma \) is an alphabet of terminal symbols;
(iv) $P_X$ is a finite set of productions for finite derivations given below;
(v) $P_Z$ is a finite set of productions for infinite derivations given below;
(vi) $x_0$ is the start variable for finite derivations;
(vii) $z_0$ is the start variable for infinite derivations;
(viii) $\{[i, p] | 1 \leq i \leq l, p \in \Gamma\}$ is the set of repeated variables for infinite derivations.

In the definition of $G_l$ the sets $P_X$ and $P_Z$ are as follows:

$$P_X = \{x_0 \rightarrow a_1[m_1, p_0, m_2]a_2 |$$
$$1 \leq m_1, m_2 \leq n, (l_m, a_1) \neq 0, (P_{m_2}, a_2) \neq 0, a_1, a_2 \in \Sigma \cup \{\varepsilon\} \} \cup$$
$$\{[i, p, j] \rightarrow a[m_1, p_1, m_2][m_2, p_2, m_3] \ldots [m_k, p_k, j] | p \in \Gamma, 1 \leq i, j \leq n, k \geq 0,$n
$$p_1, \ldots, p_k \in \Gamma, 1 \leq m_1, \ldots, m_k \leq n, ((M_{p, p_1, \ldots, p_k})_{l_{m_l}}, a) \neq 0, a \in \Sigma \cup \{\varepsilon\} \},$$

$$P_Z = \{z_0 \rightarrow a[m, p_0] | 1 \leq m \leq n, (l_m, a) \neq 0, a \in \Sigma \cup \{\varepsilon\} \} \cup$$
$$\{[i, p] \rightarrow a[m_1, p_1, m_2] \ldots [m_{j-1}, p_{j-1}, m_j][m_j, p_j] | p, p_1, \ldots, p_k \in \Gamma, 1 \leq i \leq n,$n
$$k \geq 1, 1 \leq j \leq k, 1 \leq m_1, \ldots, m_j \leq n, ((M_{p, p_1, \ldots, p_k})_{l_{m_l}}, a) \neq 0, a \in \Sigma \cup \{\varepsilon\} \}. $$

For the remainder of this section, $\mathcal{P}$ always denotes the $\omega$-pushdown automaton $\mathcal{P} = (n, \Gamma, I, M, P, p_0, l)$. Especially this means that $I$ is a fixed parameter. Observe that $((M_{p, p_1, \ldots, p_k})_{l_{m_l}}, a) \neq 0$ iff $(m_1, \ldots, p_k) \in \delta(i, a, p)$ in the usual $\delta$-notation for the transition function of a classical pushdown automaton. (See Harrison [12] and Kuich [13] pages 638/639.) Here we have to reverse $p_1, \ldots, p_k$ since the pushdown tape of classical pushdown automata has its rightmost element as top element.

A finite leftmost derivation $\alpha_1 \Rightarrow^L \alpha_2$, where $\alpha_1, \alpha_2 \in (X \cup \Sigma)^*$, by productions in $P_X$ is defined as usual. An infinite (leftmost) derivation $\pi : z_0 \Rightarrow^\omega_l w$, for $z_0 \in Z, w \in \Sigma^\omega$, is defined as follows:

$$\pi : z_0 \Rightarrow_\alpha \alpha_0[i_0, p_0] \Rightarrow^L w_0[i_0, p_0] \Rightarrow_\alpha w_0 \alpha_1[i_1, p_1] \Rightarrow^L w_0 w_1[i_1, p_1] \Rightarrow^L \ldots$$
$$\Rightarrow^L w_0 w_1 \ldots w_m[i_m, p_m] \Rightarrow^L w_0 w_1 \ldots w_m \alpha_{m+1}[i_{m+1}, p_{m+1}] \Rightarrow^L \ldots,$$

where $z_0 \rightarrow \alpha_0[i_0, p_0], [i_0, p_0] \rightarrow \alpha_1[i_1, p_1], \ldots, [i_m, p_m] \rightarrow \alpha_{m+1}[i_{m+1}, p_{m+1}], \ldots$ are productions in $P_Z$ and $w = w_0 w_1 \ldots w_m$.

We now define an infinite derivation $\pi_l : z_0 \Rightarrow^\omega_l w$ for $0 \leq l \leq n, z_0 \in Z, w \in \Sigma^\omega$: We take the above definition for $\pi : z_0 \Rightarrow^L w$ and consider the sequence of the first elements $i$ of the triple variables $[i, p, j]$ of $X$ that are rewritten in the finite leftmost derivation $\alpha_m \Rightarrow^L w_m, m \geq 0$. Assume this sequence is $i_1, i_2, \ldots, i_m$ for some $t_m, m \geq 1$. Then, to obtain $\pi_l$ from $\pi$, the condition $i_0, i_1, i_2, \ldots, i_l, i_1, i_2, \ldots, i_l, i_1, i_2, \ldots, i_l, i_1, i_2, \ldots, i_l$ and $[i, p, j] \in \Gamma$ is to be satisfied.

Then we define

$$L(G_l) = \{w \in \Sigma^* | x_0 \Rightarrow^L w \} \cup \{w \in \Sigma^\omega | \pi : z_0 \Rightarrow^\omega_l w \}.$$ 

Observe that the construction of $G_l$ from $\mathcal{P}$ is $\omega$-nothing else than a generalization of the triple construction for $\omega$-pushdown automata, since the construction of the context-free grammar $G = (X, \Sigma, P_X, x_0)$ is the triple construction. (See Harrison [12], Theorem 5.4.3; Bucher, Maurer [2], Sätze 2.3.10, 2.3.30; Kuich, Salomaa [14], pages 178, 306; Kuich [13], page 642; Esik, Kuich [7], pages 77, 78.)

We call the construction of the mixed context-free grammar $G_l$ from $\mathcal{P}$ the triple-pair construction for $\omega$-pushdown automata. This is justified by the definition of the sets of variables $\{[i, p, j] | 1 \leq i, j \leq n, p \in \Gamma\}$ and $\{[i, p] | 1 \leq i \leq n, p \in \Gamma\}$ of $G_l$ and by the forthcoming Corollary 15.

In the next theorem we use the isomorphism between $B \ll \Sigma^* \gg B \ll \Sigma^\omega \gg$ and $2^{\Sigma^*} \times 2^{\Sigma^\omega}$. 
Theorem 14. Assume that $(\sigma, \tau)$ is the solution of order $l$ of the mixed algebraic system (8) over $(B \ll \Sigma^*, B \ll \Sigma^0)$ for $k \in \{0, \ldots, n\}$. Then

$$L(G_1) = \sigma_{x_0} \cup \tau_{x_0}.$$  

Proof. By Theorem IV.1.2 of Salomaa, Soittola [16] and by Theorem 13 we obtain $\sigma_{x_0} = \{w \in \Sigma^* | x_0 \Rightarrow_L w\}$. We now show that $\tau_{x_0}$ is generated by the infinite derivations $\Rightarrow_{\omega L}$ from $z_0$. First observe that the rewriting by the typical $[i, p, j]$- and $[i, p]$- production corresponds to the situation that in the graph of the $\omega$-pushdown automaton $P$ the edge from $(\rho p, i)$ to $(p_1 \ldots p_j \rho, j)$, $\rho \in \Gamma^*$, is passed after the state $i$ is visited. The first step of the infinite derivation $\pi_i$ is given by $z_0 \Rightarrow_L \alpha_0[i_0, p]$ and indicates that the path in the graph of $P$ corresponding to $\pi_i$ starts in state $i_0$. Furthermore, the sequence of the first elements of variables that are rewritten in $\pi_i$, i.e., $i_0, i_1^1, \ldots, i_1^i, i_2^1, \ldots, i_2^j, \ldots, i_m, \ldots, i_{m+1}, \ldots, i_{m+1}, i_{m+1}, \ldots$ indicates that the path in the graph of $P$ corresponding to $\pi_i$ visits these states. Since this sequence is in $P_i$ the corresponding path contributes to $||P||$. Hence, by Theorem IV.1.2 of Salomaa, Soittola [16] and Theorem 13 for the finite leftmost derivations $\alpha_m \Rightarrow_L w_m, m \geq 1$, and by Theorem 5.5.9 of Ésik, Kuich [7] and Theorem 13 for the infinite derivations $[i_0, p_0] \Rightarrow \alpha_1[i_1, p_1] \Rightarrow \alpha_1 \alpha_2[i_2, p_2] \Rightarrow \cdots \Rightarrow \alpha_1 \alpha_2 \cdots \alpha_m[i_m, p_m] \Rightarrow \cdots$ we obtain

$$\tau_{x_0} = \{w \in \Sigma^0 | \pi : z_0 \Rightarrow_{\omega L} w\}.$$  

Corollary 15. Assume that the mixed context free grammar $G_i$ associated to the mixed algebraic system (8) is constructed from the $B(\Sigma \cup \{\varepsilon\})$-$\omega$-pushdown automaton $P$. Then

$$L(G_i) = ||P||.$$  

Proof. By Theorems 13 and 14.

For the remainder of this section our basic semiring is $\mathbb{N}^s$, which allows us to draw some stronger conclusions.

Theorem 16. Assume that $(\sigma, \tau)$ is the solution of order $l$ of the mixed algebraic system (8) over $(\mathbb{N}^s \ll \Sigma^*, \mathbb{N}^s \ll \Sigma^0)$ where the entries of $I, M, P$ are in \{0, 1\}\{$\Sigma \cup \{\varepsilon\}$}. Denote by $d(w)$, for $w \in \Sigma^*$, the number (possibly $\infty$) of distinct finite leftmost derivations of $w$ from $x_0$ with respect to $G_i$; and by $c(w)$, for $w \in \Sigma^0$, the number (possibly $\infty$) of distinct infinite leftmost derivations $\pi$ of $w$ from $z_0$ with respect to $G_i$. Then

$$\sigma_{x_0} = \sum_{w \in \Sigma^s} d(w)w \quad \text{and} \quad \tau_{x_0} = \sum_{w \in \Sigma^0} c(w)w.$$  

Proof. The proof of Theorem 16 is identical to the proof of Theorem 14 with the exceptions that Theorem IV.1.2 of Salomaa, Soittola [16] is replaced by Theorem IV.1.5 and Theorem 5.5.9 of Ésik, Kuich [7] is replaced by Theorem 5.5.10.

In the forthcoming Corollary 17 we consider, for a given \{0, 1\}\{$\Sigma \cup \{\varepsilon\}$}-$\omega$-pushdown automaton $P = (n, \Gamma, I, M, P, p_0, l)$ the number of distinct computations from an initial instantaneous description $(i, w, p_0)$ for $w \in \Sigma^*$, $I_i \neq 0$, to an accepting instantaneous description $(j, \varepsilon, \varepsilon)$, with $P_j \neq 0$, $i, j \in \{0, \ldots, n\}$.

Here $(i, w, p_0)$ means that $P$ starts in the initial state $i$ with $w$ on its input tape and $p_0$ on its pushdown tape; and $(j, \varepsilon, \varepsilon)$ means that $P$ has entered the final state $j$ with empty input tape and empty pushdown tape.
Furthermore, we consider the number of distinct infinite computations starting in an initial instantaneous description \((i, w, p_0)\) for \(w \in \Sigma^\omega\), \(I_i \neq 0\).

**Corollary 17.** Assume that the mixed context-free grammar \(G_l\) associated to the mixed algebraic system (8) is constructed from the \(\{0, 1\} \langle \Sigma \cup \{\varepsilon\} \rangle\)-\(\omega\)-pushdown automaton \(\mathcal{P}\). Then the number (possibly \(\infty\)) of distinct finite leftmost derivations of \(w\), \(w \in \Sigma^*\), from \(x_0\) equals the number of distinct finite computations from an initial instantaneous description for \(w\) to an accepting instantaneous description; moreover, the number (possibly \(\infty\)) of distinct infinite (leftmost) derivations of \(w\), \(w \in \Sigma^\omega\), from \(z_0\) equals the number of distinct infinite computations starting in an initial instantaneous description for \(w\).

**Proof.** By Corollary 6.11 of Kuich [13] and the definition of infinite derivations with respect to \(G_l\).

The context-free grammar \(G_l\) associated to (8) is called **unambiguous** if each \(w \in L(G_l)\), \(w \in \Sigma^*\) has a unique finite leftmost derivation and each \(w \in L(G_l)\), \(w \in \Sigma^\omega\), has a unique infinite (leftmost) derivation.

An \(\mathbb{N}^\omega (\Sigma \cup \{\varepsilon\})\)-\(\omega\)-pushdown automaton \(\mathcal{P}\) is called **unambiguous** if \((\|P\|, w) \in \{0, 1\}\) for each \(w \in \Sigma^* \cup \Sigma^\omega\).

**Corollary 18.** Assume that the mixed context-free grammar \(G_l\) associated to the mixed algebraic system (8) is constructed from the \(\{0, 1\} \langle \Sigma \cup \{\varepsilon\} \rangle\)-\(\omega\)-pushdown automaton \(\mathcal{P}\). Then \(G_l\) is unambiguous iff \(\|P\|\) is unambiguous.

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