Notes on Jordan-Hölder property for exact categories

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Abstract

We prove a generalised version of Jordan-Hölder theorem for some exact categories; the nice exact categories, which allow us to study the relative length function.

1 INTRODUCTION

In 1869 Camille Jordan announced, and demonstrated in 1870, that for two composition sequences of the same finite group, the order sequence (the number of elements) of quotients is the same, up to permutation. In 1889 Otto Hölder reinforced this result by proving the theorem known as the Jordan-Hölder-Schreier theorem, which states that any two composition series of a given group are equivalent, that is, they have the same composition length and the same composition factors, up to permutation and isomorphism. It is proved using the Schreier refinement theorem (which uses the Zassenhaus lemma), see [Rot95] for the standard proof. In [Ba06], Baumslag gives a short proof of the Jordan–Hölder theorem, for groups, by intersecting the terms in one subnormal series with those in the other series.

In this article we introduce the nice exact categories and follow Baumslag’s method to prove a generalised version of Jordan-Hölder theorem for these categories, which are a generalisation of the abelian categories, namely pre-abelian additive categories with a choice of a Quillen exact structure [Qu73] which is given by a class of short exact sequences, called admissible pairs of morphisms, satisfying Quillen’s axioms and moreover satisfying an additional axiom, that we will call the nice axiom. This new axiom says that the pull-back of two admissible monics exists and yields two admissible monics and the push-out along this pull-back implies the admissibility of the unique monic giving by the universal property of this push-out, making the sum of two admissible subobjects also an admissible subobject.

To do so we generalise the Schur lemma to pre-abelian exact categories. Then we study the notion of admissible intersection and sum for the nice categories and prove some of their properties and the isomorphism theorems relatively to a quasi-nice exact structure.

The $\mathcal{E}$–Jordan-Hölder theorem, in this general context, allows us to define relative Jordan-Hölder length function for nice exact categories and then to prove that the Hopkins–Levitski theorem still holds.

We then study this length on different exact categories and prove that it also get reduced under reduction of exact structures on a fixed additive category.

Note that Jordan-Hölder property does not hold in general for any exact category, see [BHLR] Example 6.9] for a counter-example. In this work we fix $\mathcal{A}$, a pre-abelian additive category and we prove that the relative $\mathcal{E}$–Jordan-Hölder property holds for all nice exact categories $(\mathcal{A}, \mathcal{E})$ for any choice of a quasi-nice exact structure $\mathcal{E}$ in the lattice of exact
structures \((\text{Ex}(\mathcal{A}), \subseteq)\) studied in our last work \cite{BHLR} Section 5.\)

And since the minimal split exact structure is quasi-nice, then the Krull-Shmidt property also holds for pre-abelian additive categories.

We mention that while preparing this note, we learned of the parallel but independent results of Enomoto \cite{E19}.

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\section{Background}

In this section we recall from \cite{GR92, B10} the definition of Quillen exact structures and the definition of a pre-abelian additive category.

\begin{definition}
Let \(\mathcal{A}\) be an additive category. A kernel-cokernel pair \((i, d)\) in \(\mathcal{A}\) is a pair of composable morphisms such that \(i\) is kernel of \(d\) and \(d\) is cokernel of \(i\). If a class \(\mathcal{E}\) of kernel-cokernel pairs on \(\mathcal{A}\) is fixed, an \textit{admissible monic} is a morphism \(i\) for which there exist a morphism \(d\) such that \((i, d)\in\mathcal{E}\). An \textit{admissible epic} is defined dually. Note that admissible monics and admissible epics are referred to as inflation and deflation in \cite{GR92}, respectively. We depict an admissible monic by \(\rightarrow \rightarrow \rightarrow \rightarrow\) and an admissible epic by \(\rightarrow \rightarrow \rightarrow \rightarrow\). An \textit{exact structure} \(\mathcal{E}\) on \(\mathcal{A}\) is a class of kernel-cokernel pairs \((i, d)\) in \(\mathcal{A}\) which is closed under isomorphisms and satisfies the following axioms:

(A0) For all objects \(A \in \text{Obj}\,\mathcal{A}\) the identity \(1_A\) is an admissible monic

(A0)\textsuperscript{op} For all objects \(A \in \text{Obj}\,\mathcal{A}\) the identity \(1_A\) is an admissible epic

(A1) the class of admissible monics is closed under composition

(A1)\textsuperscript{op} the class of admissible epics is closed under composition

(A2) The push-out of an admissible monic \(i : A \rightarrow B\) along an arbitrary morphism \(f : A \rightarrow C\) exists and yields an admissible monic \(j:\)

\[\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{j} & D
\end{array}\]

(A2)\textsuperscript{op} The pull-back of an admissible epic \(h\) along an arbitrary morphism \(g\) exists and yields an admissible epic \(k:\)

\[\begin{array}{ccc}
A & \xleftarrow{h} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xleftarrow{k} & D
\end{array}\]

An \textit{exact category} is a pair \((\mathcal{A}, \mathcal{E})\) consisting of an additive category \(\mathcal{A}\) and an exact structure \(\mathcal{E}\) on \(\mathcal{A}\). The pairs \((i, d)\) forming the class \(\mathcal{E}\) are called \textit{admissible short exact sequences}, or just \textit{admissible sequences}.

\[\text{2}\]
Definition 2.2. [BHLR] Definition 8.1] A morphism $f : A \to B$ in an exact category is called admissible if it factors as a composition of an admissible monic with an admissible epic. Admissible morphisms will sometimes be displayed as

$$A \longrightarrow f \longrightarrow B$$

in diagrams, and the classes of admissible arrows of $A$ will be denoted as $\text{Hom}_{\mathcal{E}}^A(-, -)$.

Proposition 2.3. [BHLR Proposition 2.16] Suppose that $i : A \to B$ is a morphism in $\mathcal{A}$ admitting a cokernel. If there exists a morphism $j : B \to C$ such that the composite $j \circ i: A \longrightarrow C$ is an admissible monic, then $i$ is an admissible monic.

Definition 3.2. [BHLR Definition 3.3] A non-zero object $S$ in $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$–simple if $S$ admits no $\mathcal{E}$–subobjects except 0 and $S$, that is, whenever $A \subset \mathcal{E} S$, then $A$ is the zero object or isomorphic to $S$.

Remark 3.3. Let $A$ be an $\mathcal{E}$–subobject of $B$ given by the monic $A \longrightarrow i \longrightarrow B$. We denote by $B/A$ (or simply $B/A$ when $i$ is clear from the context) the Cokernel of $i$, thus we denote the corresponding admissible sequence as

$$A \longrightarrow i \longrightarrow B \longrightarrow B/A.$$

Remark 3.4. An admissible monic $A \longrightarrow i \longrightarrow B$ is relatively proper precisely when its cokernel is non-zero. In fact, by uniqueness of kernels and cokernels, the exact sequence

$$B \longrightarrow i \longrightarrow B \longrightarrow 0$$

is, up to isomorphism, the only one with zero cokernel. Thus an admissible monic $i$ has $\text{Coker}(i) = 0$ precisely when $i$ is an isomorphism. Dually, an admissible epic $B \longrightarrow d \longrightarrow C$ is an isomorphism precisely when $\text{Ker}(d) = 0$. In particular a morphism which is at the same time an admissible monic and epic is an isomorphism.

Note that a subobject is proper means all admissible monics are proper.

Definition 3.5. [BHLR Definition 6.4] An object $X$ of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$–Noetherian if any increasing sequence of $\mathcal{E}$–subobjects of $X$

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X \cdots$$

becomes stationary. Dually, an object $X$ of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$–Artinian if any descending sequence of $\mathcal{E}$–subobjects of $X$

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

becomes stationary. The exact category $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{E}$–Artinian (respectively $\mathcal{E}$–Noetherian) if every object is $\mathcal{E}$–Artinian (respectively $\mathcal{E}$–Noetherian).

3 The $\mathcal{E}$-Schur lemma
Lemma 3.6. (The $E$-Schur lemma) Let $X \rightarrowtail Y$ be an admissible non-zero morphism.

- If $X$ is $E$–simple, then $f$ is an admissible monic,
- If $Y$ is $E$–simple, then $f$ is an admissible epic.

Proof. Let $X \xrightarrow{e} S \xrightarrow{m} Y$ be the factorisation of $f$ as a composition of an admissible epic $e$ with an admissible monic $m$.

- If $X$ is $E$–simple then either $\text{Ker}(e) = X$ or $\text{Ker}(e) = 0$, but in the first case $e = 0$ and so $f = 0$, contradicting the assumption $f \neq 0$. Hence $\text{Ker}(e) = 0$, and by Remark 3.4 $e \cong 1_X$ and $f \cong m$ and therefore $f$ is an admissible monic.
- If $Y$ is $E$–simple, then the $E$–subobject $S$ is either zero or equal to $Y$, but in case $S = 0$, $e = 0$, we get $f = m \circ e = 0$ which contradicts $f \neq 0$. Therefore $S = Y$ and $m : Y \twoheadrightarrow Y$ is an admissible monic with $\text{Im}(m) = Y$. By Remark 3.4 $m \cong 1_Y$, and $f \cong e$, which means that $f$ is an admissible epic.

Corollary 3.7. Let $S$ be an $E$–simple object, then the non-zero admissible endomorphisms $S \rightarrowtail S$ form the group $\text{Aut}(S)$ of automorphisms of $S$.

Proof. It follows from Lemma 3.6 that any non-zero admissible morphism $S \rightarrowtail S$ is an admissible monic and an admissible epic, thus $f$ is an isomorphism. Conversely, every isomorphism is admissible, so we get the group of automorphisms of $S$ which is closed under composition by $(A2)$ or $(A2)^{op}$.

4 NICE EXACT CATEGORIES

In this section we introduce the notion of nice exact categories and give some examples of such categories.

4.1 Definition of a quasi-nice exact structure and properties

Let us first recall the definitions of intersection and sum for abelian categories in general as mentioned in [G62, section 5] or as defined in [P63] Definition 2.6):

Definition 4.1. Let $(X_1, i_1)$, $(X_2, i_2)$ be two subobjects of an object $X$ in an abelian category, that is, we consider monics $i_1 : X_1 \rightarrow X$ and $i_2 : X_2 \rightarrow X$. We denote by $X_1 +_X X_2$ (or simply $X_1 + X_2$ when there is no possibility of confusion) the sum of $X_1$ and $X_2$, which is defined as the image $\text{Im}(s)$ of the morphism

$$s = [i_1 \ i_2] : X_1 \oplus X_2 \rightarrow X.$$
**Definition 4.2.** Let \((X_1, i_1), (X_2, i_2)\) be two subobjects of an object \(X\) in an abelian category. We denote by \(X_1 \cap X_2\) (or simply \(X_1 \cap X_2\)) the intersection of \(X_1\) and \(X_2\), defined as the kernel \(\ker(t)\) of the morphism
\[
t = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} : X \to Y_1 \oplus Y_2
\]
where \(d_1 : X \to Y_1\) and \(d_2 : X \to Y_2\) are the cokernels of the monics \(i_1\) and \(i_2\), respectively.

Notice that this intersection, which exists and is well defined in a pre-abelian exact category, is not necessarily an admissible subobject. So let us introduce the following new exact categories which are N.I.C.E in the sense that they are Necessarily Intersection Closed Exact categories:

**Definition 4.3.** An exact category \((A, E)\) is **quasi-nice** if it satisfies the following additional axiom:

\((A_{\text{nice}})\) The pull-back of two admissible monics \(j : C \to D\) and \(g : B \to D\) exists and yields two admissible monics \(i\) and \(f\):

\[
\begin{array}{c}
A \xrightarrow{i} B \\
\downarrow f \\
C \xrightarrow{PB} D
\end{array}
\]

and moreover, the push-out along these pull-backs yields an admissible monic \(u\)

\[
\begin{array}{c}
PB \xrightarrow{i} B \\
\downarrow f \quad \downarrow j \quad \downarrow g \\
PQ \\
C \xrightarrow{k} PQ \xrightarrow{l} D
\end{array}
\]

And in this case we say that \(E\) is a **quasi-nice** exact structure on \(A\).

**Definition 4.4.** Let \((A, E)\) be an exact category, it is said to be **nice** when \(A\) is pre-abelian additive category and \(E\) is a quasi-nice exact structure.

Now let us define relative notions of intersection and sum, so we consider the following constructions as definitions of admissible intersection and sum of admissible subobjects in these nice categories:

**Definition 4.5.** Let \((X_1, i_1), (X_2, i_2)\) be two \(E\)-subobjects of an object \(X\). We define their **intersection**, \(X_1 \cap X_2\), to be the pullback

\[
\begin{array}{c}
X_1 \cap X_2 \xrightarrow{\pi_2} X_1 \\
\downarrow \pi_1 \\
X_2 \xrightarrow{i_2} X.
\end{array}
\]

We then define their **sum**, \(X_1 + X_2\), to be the pushout

\[
\begin{array}{c}
X_1 \cap X_2 \xrightarrow{\pi_2} X_1 \\
\downarrow \pi_1 \\
X_2 \xrightarrow{i_2} X_1 + X_2.
\end{array}
\]

\(^1\)The existence of \(u\) is given by the universal property of the push-out.
Remark 4.6. Equivalently, for two \( \mathcal{E} \)-subobjects, \((X_1, i_1), (X_2, i_2)\) of an object \(X\) we have

\[
X_1 \cap_X X_2 = \text{Ker} \left( X_1 \oplus X_2 \xrightarrow{[i_1 - i_2]} X \right)
\]

and

\[
X_1 +_X X_2 = \text{Coker} \left( X_1 \cap_X X_2 \xrightarrow{[i_1 - i_2]^t} X_1 \oplus X_2 \right)
\]

when they exist. Thus, as the direct sum is an associative operation, so are the sum and intersection operations.

Now let us show how this definition generalises the abelian versions \(^{[1,2]}\) and \(^{[1,1]}\).

**Proposition 4.7.** Let \((\mathcal{A}, \mathcal{E}_{\text{all}})\) be an abelian exact category and let \((X_1, i_1)\) and \((X_2, i_2)\) be two \(\mathcal{E}\)-subobjects of an object \(X\). Then \(\text{Ker} t\) forms the pull-back of \((X, i_1, i_2)\), where

\[
t = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} : X \rightarrow X/X_1 \oplus X/X_2
\]

is given by the cokernels \(d_1, d_2\) of \(i_1, i_2\) as in \(^{[1,2]}\).

**Proof.** Let us consider the following diagram

\[
\begin{array}{ccc}
\text{Ker} t & \xrightarrow{k_1} & X_1 \\
\downarrow k_2 & & \downarrow i \\
X_2 & \xrightarrow{i_2} & X \\
\downarrow d_1 & & \downarrow d_2 \\
X/X_2 & \xrightarrow{t} & X/X_1 \oplus X/X_2
\end{array}
\]

where \(i_1 \circ k_1 = i\) and \(i_2 \circ k_2 = i\).

And then consider an object \(V\) and two morphisms \(v_1, v_2\) such that \(i_1 \circ v_1 = i_2 \circ v_2\):

\[
\begin{array}{ccc}
V & \xrightarrow{v_1} & X_1 \\
\downarrow v_2 & & \downarrow k_2 \\
\text{Ker} t & \xrightarrow{k_1} & X_2 \\
\downarrow i_2 & & \downarrow i \\
X & \xrightarrow{d_2} & X/X_2 \\
\downarrow d_1 & & \downarrow t \\
X/X_2 & \xrightarrow{v_1} & X/X_1 \oplus X/X_2
\end{array}
\]

Since \(t \circ i_1 \circ v_1 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \circ i_1 \circ v_1 = \begin{bmatrix} d_1 \circ i_1 \circ v_1 \\ d_2 \circ i_1 \circ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ d_2 \circ i_2 \circ v_2 \end{bmatrix} = 0\)

then by the universal property of the kernel there exists a unique morphism \(v\) such that \(i_1 \circ v_1 = i \circ v = i_2 \circ k_2 \circ v\). Since \(i_1\) is mono, we conclude \(v_1 = k_2 \circ v\).

By symmetry we also have that there exists a unique morphism \(v\) such that \(v_2 = k_1 \circ v\).

We conclude that \(\text{Ker} t, k_1, k_2\) is the pull-back of \((X, i_1, i_2)\).

\(\Box\)

**Proposition 4.8.** Let \((\mathcal{A}, \mathcal{E}_{\text{all}})\) be an abelian exact category and let \((X_1, i_1)\) and \((X_2, i_2)\) be two \(\mathcal{E}\)-subobjects of an object \(X\). Then \(\text{Im} s\) forms the push-out of \((X_1 \cap_X X_2, s_1, s_2)\) where \(s\) is as in \(^{[1,2]}\) and \(s_1, s_2\) are given by the pull-back as in \(^{[1,1]}\)
Proof.

\[
\begin{array}{c}
\text{Ker}[i_1, i_2] \xrightarrow{\left[\begin{array}{c}s_1 \\ -s_2\end{array}\right]} X_1 \oplus X_2 \xrightarrow{\pi = [i_1, i_2]} X \\
\text{Coker}[s_1, -s_2]^t \\
\end{array}
\]

In the abelian case, the pull-back along \((X, i_1, i_2)\) is defined to be \(\text{Ker}[i_1, i_2] = \left[\begin{array}{c}s_1 \\ -s_2\end{array}\right]\) such that \([i_1, i_2] \circ \left[\begin{array}{c}s_1 \\ -s_2\end{array}\right] = 0\). \(X_1 \cap X_2 = \text{Ker}[i_1, i_2] = \left[\begin{array}{c}s_1 \\ -s_2\end{array}\right]\)

And the push-out along this pull-back \((X_1 \cap X_2, s_1, s_2)\) is defined to be \((\text{Coker} \left[\begin{array}{c}s_1 \\ -s_2\end{array}\right], j_1, j_2)\) such that \([j_1, j_2] \circ \left[\begin{array}{c}s_1 \\ -s_2\end{array}\right] = 0\).

So the push-out \(X_1 + X_2\) is the \(\text{Coker}(\text{ker}(s_1)) = \text{Coim}(s)\). And since \(\text{Coim}(s) \cong \text{Im}(s)\) in an abelian category, then \(\text{Im}(s)\) coincide with the general admissible sum on a nice category.

**Remark 4.9.** For two \(E\)-subobjects \((X_1, i_1), (X_2, i_2)\) of an object \(X\):

i) \(X_1 \cap X = X_1 = X_1 + X_1\).

ii) If \(X_1 + X_2 = 0_A\) then \(X_1 = X_2 = 0_A\).

Now let us prove some properties of the intersection and the sum of \(E\)-subobjects of an object in a nice exact category:

**Lemma 4.10.** Let \(X, Y\) and \(Y'\) be \(E\)-subobjects of an object \(Z\) in a nice category \((A, E)\). If there exists an admissible monic \(i : Y \twoheadrightarrow Y'\) then there exists an admissible monic \(X \cap_2 Y \twoheadrightarrow X \cap_2 Y'\).

**Proof.** By definition we have the two following pull-back diagrams

\[
\begin{array}{c}
X \cap_2 Y \xrightarrow{f} Y \\
\xrightarrow{g} \\
X \xrightarrow{h} Z
\end{array}
\]

and

\[
\begin{array}{c}
X \cap_2 Y \xrightarrow{f} Y \\
\xrightarrow{g} \\
X \xrightarrow{h} Z
\end{array}
\]
where \( h' \circ i = h \).
So we have a monic \( i \circ f = l \) that commutes the following diagram

\[
\begin{array}{c}
X \cap_Z Y \ar{r}{f} \ar{d}{g'} \ar{d}{h'} & Y' \ar{d}{g} \\
X \ar{r}{k} & Z
\end{array}
\]

By the universal property of the pull-back, there exist a morphism

\[ r : X \cap_Z Y \to X \cap_Z Y'. \]

such that \( f' \circ r = l \) and \( g' \circ r = g \).
Since \( l \) is an admissible monic, and the cokernel of \( r \) exists, then the obscure axiom 2.3 implies that the morphism \( r \) is also an admissible monic.

**Lemma 4.11.** Let \( X, Y \) and \( Y' \) be \( \mathcal{E} \)-subobjects of an object \( Z \) in a nice category \((A, \mathcal{E})\).
If there exists an admissible monic

\[ i : Y \rightarrowrightarrow Y' \]

then there exists an admissible monic

\[ Y +_Z X \rightarrowrightarrow Y' +_Z X. \]

**Proof.** By definition we have the two following push-out diagram

\[
\begin{array}{ccc}
X \cap_Z Y \ar{r}{f} \ar{d}{g} & Y' \ar{d}{d} \\
X \ar{r}{u} \ar{d}{e} & Y +_Z X \ar{r}{r'} & Y' +_Z X
\end{array}
\]

where \( d' \circ i = l' \), and by the universal property of the push-out, there exists a unique morphism

\[ r' : Y +_Z X \to Y' +_Z X. \]

such that \( r' \circ e = e' \) and \( r' \circ d = d' \). The unique two admissible monics

\[ u : Y +_Z X \rightarrowrightarrow Z \]
\[ u' : Y' +_Z X \rightarrowrightarrow Z \]

such that \( u' \circ r' = u \) are admissibles by \((A_{nice})\), and since \( u \) is an admissible monic and the cokernel of \( r' \) exists, then the obscure axiom 2.3 implies that the morphism \( r' \) is also an admissible monic.

\[ \square \]
Hence we will show the existence of some special admissible short exact sequences, which will play an important role in the proof of the \( \mathcal{E} \)-Jordan-Hölder property next. 

**Lemma 5.1.** Let \( X \) and \( Y' \xrightarrow{S} Y'' \xrightarrow{T} Z \) be three \( \mathcal{E} \)-subobjects of an object \( Z \). Then there exists an admissible short exact sequence

\[
(X' + zX)/X \xrightarrow{S'} (Y'' + zX)/X \xrightarrow{T'} (Y'' + zX)/(Y' + zX)
\]
Proof. The admissible monic that exists by \[4.11\] fit into the commutative diagram below, where the arrow on the right exists by the universal property of a Cokernel, then by the dual of [Bu10 Proposition 2.12] the right square is bicartesian, and by (A2) (or by [Bu10 Proposition 2.15]) the morphism

\[
Y' + z X / X \xrightarrow{c} Y'' + z X / X
\]

is also an admissible monic.

Since the first two horizontal rows and the middle column are short exact, then by the Noether Isomorphism for exact categories [Bu10 Lemma 3.5] the third column is a well defined admissible short exact sequence, and is uniquely determined by the requirement that it makes the diagram commutative. Moreover, the upper right hand square is bicartesian;

\[
\begin{array}{ccccccccc}
0 & \to & X & \to & Y' + z X & \to & (Y' + z X) / X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & X & \to & Y'' + z X & \to & (Y'' + z X) / X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(Y'' + z X) / (Y' + z X) & \to & (Y'' + z X) / (Y' + z X) & \to & 0 & \to & 0
\end{array}
\]

In particular \((Y'' + z X) / (Y' + z X)\) is the admissible Cokernel of the admissible monic \(c\). □

Lemma 5.2. (The $\mathcal{E}$–second isomorphism theorem) Let \(X\), and \(Y' \longrightarrow Y''\) be three \(\mathcal{E}\)–subobjects of an object \(Z\). The following is an admissible short exact sequence

\[
Y' \cap z X \longrightarrow Y' \longrightarrow (Y' + z X) / X
\]

Proof. We consider the following push-out diagram

\[
\begin{array}{ccc}
Y' \cap z X & \xrightarrow{g} & Y' \\
\downarrow{f} & & \downarrow{f'} \\
X & \xrightarrow{g'} & Y' + z X
\end{array}
\]

and by [Bu10 Proposition 2.12] this square is part of the diagram

\[
\begin{array}{ccccccccc}
Y' \cap z X & \xrightarrow{g} & Y' & \xrightarrow{c} & Y' / (Y' \cap z X) \\
\downarrow{f} & & \downarrow{f'} & & \downarrow{f'} \\
X & \xrightarrow{g'} & Y' + z X & \xrightarrow{c'} & (Y' + z X) / X
\end{array}
\]

□

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**Proposition 5.3.** Let \(X, Y, Z\) be three \(E\)-subobjects of an object \(Z\). There exists an admissible short exact sequence

\[
\frac{(Y'' \cap z L)}{Y' \cap z L} \to \frac{Y''}{Y'} \to \frac{(Y'' + z X) / (Y' + z X)}{Y'' / Y'} = 0.
\]

**Proof.** Consider the commutative diagram below in which the three columns are admissibles short exact sequences by 5.1 and 3.3. In addition the first two rows are admissibles short exact sequences by 5.2, then the 3x3-lemma for exact categories [5,10] Corollary 3.6] implies the existence of the commutative diagram of admissible short exact sequences

\[
\begin{array}{c@{\quad}c@{\quad}c@{\quad}}
0 & \to & Y' \cap z X \to Y' \to \frac{(Y' + z X)}{X} \to 0 \\
0 & \to & Y'' \cap z X \to Y'' \to \frac{(Y'' + z X)}{X} \to 0 \\
0 & \to & \frac{(Y' \cap z X)}{Y' \cap z X} \to Y'' \to \frac{(Y' + z X)}{Y'' / Y'} \to 0
\end{array}
\]

and in particular the third row is an admissible short exact sequence. \(\square\)

# 6 THE \(E\)-JORDAN-HÖLDER PROPERTY

In this section we prove the equivalent of the Jordan-Hölder theorem on a nice exact category \((A, E)\), and show how \(A\) is a Krull-Schmidt category.

Inspired by the idea of [5,10] we use the same steps to prove it relatively to the admissible morphisms of an exact structure.

**Definition 6.1.** An \(E\)-composition series for an object \(X\) of \(A\) is a sequence

\[
0 = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} X_n = X
\]

where all \(i_i\) are proper admissible monics with \(E\)-simple cokernel.

**Theorem 6.2.** (\(E\)-Jordan-Hölder theorem) Let \((A, E)\) be a nice exact category. Any two \(E\)-composition series for a finite object \(X\) of \(A\)

\[
0 = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{m-1}} X_m = X
\]

and

\[
0 = X'_0 \xrightarrow{i'_0} X'_1 \xrightarrow{i'_1} \cdots \xrightarrow{i'_{n-1}} X'_n = X
\]

are equivalent, that is, they have the same length and the same composition factors, up to permutation and isomorphism.

**Proof.** By induction on \(m\). If \(m = 0\), then \(X = 0\) and \(n = 0\). If \(m = 1\), then \(M\) is \(E\)-simple: the only \(E\)-composition series is \(0 \to M\), and so \(n = 1\). If \(m \geq 1\), we consider the sequence on \(E\)-subobjects of \(X\):

\[
0 \to X'_0 \cap X \cap X_{m-1} \to \cdots \to X'_{n-1} \cap X \cap X_{m-1} \to X_{m-1} = X_m = X.
\]

Since the Cokernels \(X/X_{m-1} = X_m/X_{m-1}\) are \(E\)-simples, there exists a unique \(0 \leq k \leq n\) such that

\[
X_{m-1} = X'_0 + X_{m-1}.X'_{k+1} + X_{m-1} \cdots = X'_{n-1} + X X_{m-1} = X.
\]
By [5,3] there exists for each $0 \leq l \leq n$ an admissible short exact sequence

$$0 \to (X'_l/X_m)/(X'_l/X_{m-1}) \to (X'_{l+1}/X_k) \to (X'_{l+1}/X_{m-1}) \to 0.$$ 

In particular the middle term of this sequence is an $E$–simple object. By the $E$–Schur lemma [5,6] the admissible monic (respectively the admissible epic) of this sequence is either the zero morphism, or an isomorphism. For $l = k$, we have

$$X_m/X_{m-1} \cong (X'_{k+1}+X/X_{m-1})/(X'_{k+1}+X X_{m-1}) \cong (X'_{k+1}/X_k)$$

and then by [5,6] we have $X'_{k+1}\cap X X_{m-1} \cong X_k \cap X X_{m-1}$. While for $l \neq k$ we have

$$(X'_{l+1}\cap X X_{m-1})/(X'_{l+1}) \cong (X'_{l+1}/X'_{l+1})$$

which means that $X'_{l+1}\cap X X_{m-1} \neq X'_{l+1}\cap X X_{m-1}$ and $X'_{l+1}\cap X X_{m-1}/X'_{l+1}\cap X X_{m-1}$ is an $E$–simple object. This shows that the sequence

$$0 \subseteq E X'/X X_{m-1} \subseteq \cdots \subseteq E X'/X X_{m-1} = X'_{k+1}\cap X X_{m-1} \cdots \subseteq E X'/X X_{m-1} \subseteq E X X_{m-1}$$

is a composition series of $X_{m-1}$ of length $n-1$. By the recurrence hypothesis $m-1 = n-1$, and so $m = n$ and there exists a bijection

$$\sigma : \{0, 1, \ldots, k-1, k+1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\}$$

such that $X'_{l+1}/X'_{l} \cong X_{\sigma(l)+1}/X_{\sigma(l)}$ for $l \neq k$, and by taking $\sigma(i) = m-1$. 

\[\Box\]

**Corollary 6.3. (The Krull-Schmidt property)** Let $A$ be a pre-abelian additive category. If $X_1 \oplus X_2 \oplus \cdots \oplus X_r \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_s$, where the $X_i$ and $Y_i$ are all indecomposable objects in $A$, then $r = s$, and there exists a permutation $\sigma$ such that $X_{\sigma(i)} \cong Y_i$ for all $i$.

**Proof.** When $E = E_{\text{min}}$, the only admissible monics are embeddings of direct summands. And since $E_{\text{min}}$ is always quasi-nice, it gives rise to a nice exact category when the underlying additive category is pre-abelian, so the $E$-Jordan Hölder 6.2 theorem says that the number of factors is unique, and that the factors (the indecomposable summands) are also unique, up to isomorphism and permutation. 

\[\Box\]

## 7 The Relative Length Function

In this section, $(A, E)$ is a finite essentially small nice exact category and $l_E$ is the corresponding J.H-length function that the $E$–Jordan-Hölder theorem allows us to define over its set $\text{Obj}A$ of isomorphism classes of objects. Let $[X]$ be the isomorphism class of an object $X$ of $A$, we shall note it $X$ here to simplify.

### 7.1 The $E$–Jordan-Hölder length function

**Definition 7.1.** We define the function $l_E : \text{Obj}A \to \mathbb{N}$ as the length of an $E$–composition series of $X$. That is $l_E(X) = n$ if and only if there exists an $E$–composition series

$$0 = X_0 \Rightarrow X_1 \Rightarrow \cdots \Rightarrow X_{n-1} \Rightarrow X_n = X.$$ 

We say in this case that $X$ is $E$–finite. If no such bound exists, we say that $X$ is $E$–infinite. Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a function $l_E : \text{Obj}A \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Now we prove some corollaries of the $E$–Jordan-Hölder theorem:

**Corollary 7.2.** Every object is isomorphic to a finite direct sum of indecomposable objects.
Proof. Since \((\mathbb{A}, \mathcal{E}_{\text{fin}})\) is a finite essentially small nice exact category, then we apply \(6.2\) and the \(\mathcal{E}_{\text{fin}}\)-length of an object is the finite number of direct summands in this case.

**Corollary 7.3.** Let
\[
0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0
\]
be an admissible short exact sequence of finite length objects. Then,
\[
l_\mathcal{E}(Z) = l_\mathcal{E}(X) + l_\mathcal{E}(Y)
\]

**Proof.** We know that \(X\) is a subobject of \(Z\) and that \(Y \cong Z/X\). We consider the following composition sequences of \(X\) and \(Z\)
\[
0 = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = X
\]
\[
0 = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_{m-1} \rightarrow Z_m = Z
\]
then the following is a composition sequence of \(Y\) since by \([\text{Bi10}]\) Lemma 3.5 \((Z_{i+1}/X)/(Z_i/X) \cong (Z_{i+1}/Z_i)\)
\[
0 = Z_0/X \rightarrow Z_1/X \rightarrow \cdots \rightarrow Z_{l-1}/X \rightarrow Z_l/X \cong Y
\]
We have \(Z_0/X = 0\), then \(Z_0 = X\).

Now, we can construct the following composition sequence for \(Z\).
\[
0 = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = X_0
\]
\[
\cdots
\]
\[
Z_1 \rightarrow \cdots \rightarrow Z_{l-1} \rightarrow Z_l
\]
Then
\[
l_\mathcal{E}(Z) = m = n + l = l_\mathcal{E}(X) + l_\mathcal{E}(Y)
\]

**Corollary 7.4.** Let \(X, Y\) be two subobject of a finite length object \(Z\), then
\[
l_\mathcal{E}(X \cap Z Y) + l_\mathcal{E}(X + Z Y) = l_\mathcal{E}(X) + l_\mathcal{E}(Y)
\]

**Proof.** Consider the two following admissible short exact sequences
\[
0 \rightarrow X \cap Z Y \rightarrow X \rightarrow X/(X \cap Z Y) \rightarrow 0
\]
and
\[
0 \rightarrow Y \rightarrow X + Z Y \rightarrow (X + Z Y)/Y \rightarrow 0
\]
By \(5.2\) \(X/(X \cap Z Y) \cong (X + Z Y)/Y\), then by \(7.3\)
\[
l_\mathcal{E}(X + Z Y) - l_\mathcal{E}(Y) = l_\mathcal{E}((X + Z Y)/Y) = l_\mathcal{E}(X/(X \cap Z Y)) = l_\mathcal{E}(X) - l_\mathcal{E}(X \cap Z Y).
\]

Let us now prove that the function \(l_\mathcal{E}\) that we defined above is a **length function**:

**Definition 7.5.** \([\text{Kr07}]\) A **measure for a poset** \(S\) is a morphism of posets \(\mu : S \rightarrow P\) where \((P, \leq)\) is a totally ordered set.

And \(\mu\) is called a **length function** when \(P = \mathbb{N}\) with the natural order.

**Theorem 7.6.** The function \(l_\mathcal{E}\) of a finite essentially small nice exact category \((\mathbb{A}, \mathcal{E})\) is a **length function** for the poset \(\text{Obj}\mathbb{A}\).
Proof. The \( l_{\mathcal{E}} : \text{Obj} \mathcal{A} \to \mathbb{N} \) is defined on the set \( \text{Obj} \mathcal{A} \), which is a partially ordered set by \cite{BHLR} Proposition 6.11]. Moreover, consider \( X \) and \( Y \) in \( \text{Obj} \mathcal{A} \) such that \( X \subseteq_{\mathcal{E}} Y \). Then by \cite{7.3}
\[
l_{\mathcal{E}}(X) \leq l_{\mathcal{E}}(Y).
\]
so \( l_{\mathcal{E}} \) is a morphism of partially ordered sets, and so \( l_{\mathcal{E}} \) is a length function since \((\mathbb{N}, \leq)\) is totally ordered.

So an \( \mathcal{E} \)-finite object is an object with \( \mathcal{E} \)-finite length.

**Proposition 7.7.** \((\mathcal{E} \text{-Hopkins–Levitzki theorem})\) An object \( X \) of \((\mathcal{A}, \mathcal{E})\) is \(\mathcal{E}\)-Artinian and \(\mathcal{E}\)-Noetherian if and only if it has an \(\mathcal{E}\)-finite length.

**Proof.** For an \( \mathcal{E} \)-finite object \( X \) of length \( l_{\mathcal{E}}(X) = n \in \mathbb{N} \), the composition serie is of length \( n \). Thus any increasing or decreasing sequence of \( \mathcal{E} \)-subobjects of \( X \) must become stationary and \( X \) is \( \mathcal{E} \)-Artinian and \( \mathcal{E} \)-Noetherian.

Conversely, let \( X \) be an \( \mathcal{E} \)-Artinian and \( \mathcal{E} \)-Noetherian object. Then any composition serie ending with \( X \) has to be of finite length. So \( X \) is \( \mathcal{E} \)-finite.

**Corollary 7.8.** Let \( X \) be an \( \mathcal{E} \)-finite object in \((\mathcal{A}, \mathcal{E})\). Then any proper sequence of \( \mathcal{E} \)-subobjects of \( X \) can be refined to an \( \mathcal{E} \)-composition series.

**Proof.** We adapted the proof of \cite[Theorem 7.3]{P67} to our context.

**Remark 7.9.** Note that length functions for exact categories in general was studied in our last work \cite[Section 6]{BHLR}.

### 7.2 Reduction of exact structures

**Definition 7.10.** We denote by \((\text{Ex}(\mathcal{A}), \subseteq)\) the poset of exact structures \( \mathcal{E} \) on \( \mathcal{A} \), where the partial order is given by containment \( \mathcal{E}' \subseteq \mathcal{E} \).

This containment partial order is the reduction of exact structures discussed in \cite{BHLR} Section 4.

Now we study how the \( \mathcal{E} \)-Jordan Hölder length changes by reduction of exact structures:

**Lemma 7.11.** If \( \mathcal{E} \) and \( \mathcal{E}' \) are exact structures on \( \mathcal{A} \), such that \( \mathcal{E}' \subseteq \mathcal{E} \), then \( l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X) \) for all objects \( X \) in \( \mathcal{A} \).

**Proof.** Let us consider an \( \mathcal{E} \)-composition sequence of \( X \)
\[
0 = X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X = X_n
\]
when we reduce \( \mathcal{E}' \subseteq \mathcal{E} \), some of these pairs \((i_j, d_j)\), will not be in \( \mathcal{E}' \), while all the \( \mathcal{E} \)-simple Cokernels \( X_{j+1}/X_j \) are \( \mathcal{E}' \)-simples, so the sequence will be shorter but still an \( \mathcal{E}' \)-composition sequence, and therefore \( l_{\mathcal{E}}(X) = n \geq l_{\mathcal{E}'}(X) \).

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