ON RESOLUTIONS OF HIGHEST WEIGHT MODULES OVER THE \( N = 2 \) SUPERCONFORMAL ALGEBRA

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Abstract. In this paper we construct Bernstein–Gelfand–Gelfand type resolution of simple highest weight modules over the simple \( N = 2 \) vertex operator superalgebra of central charge \( c_{p,p'} = 3 \left( 1 - \frac{2p'}{p} \right) \) by means of the Kazama–Suzuki coset construction. As an application, we compute the twisted Zhu algebras of the simple \( N = 2 \) vertex operator superalgebra. We also compute the Frenkel–Zhu bimodule structure associated with a certain simple highest weight module of central charge \( c_{3,2} = -1 \).

1. Introduction

1.1. Background. One of the most fundamental problems in the study of 2-dimensional conformal field theory is to compute fusion rules between primary fields (see e.g. \([\text{Ver}88]\)). In \([\text{FM}94]\), B. Feigin and F. Malikov computed the fusion rules of the Wess–Zumino–Witten model associated with the affine Lie algebra \( \widehat{sl}_2 \) at Kac–Wakimoto admissible levels. Though this model has the remarkable modular invariance property (see \([\text{KW}88]\) for detail), a naive generalization of the Verlinde formula, which is originally proposed in \([\text{Ver}88]\), fails in general (see \([\text{KS}88]\, [\text{MFW}90]\, [\text{AY}92]\)). After more than two decades of studies (see \([\text{CR}12]\) for historical detail), in \([\text{CR}13]\), T. Creutzig and D. Ridout conjectured a consistent generalization of the Verlinde formula (partially proved in \([\text{CHY}17, \text{Corollary 7.7}]\)) which includes some non-standard simple \( \widehat{sl}_2 \)-modules, called relaxed highest weight modules. This type of \( \widehat{sl}_2 \)-module was initially studied by B. Feigin, A. Semikhatov, and I. Tipunin \([\text{FST}98]\) in connection with the \( N = 2 \) superconformal algebra by the Kazama–Suzuki coset construction. In fact, as was observed in \([\text{FST}98]\), relaxed highest weight \( \widehat{sl}_2 \)-modules turn out to correspond to standard \( N = 2 \) highest weight modules. Recently, in \([\text{Sat}18]\), the second author proposed a conjectural Verlinde formula in the \( N = 2 \) side, which is a counterpart to Creutzig–Ridout’s formula in the \( \widehat{sl}_2 \) side. It is worth noting that, by some technical reasons, the computation of fusion rules in the \( N = 2 \) side seems to be easier than that in the \( \widehat{sl}_2 \) side. However, the structure of the fusion rules has not yet been determined in the both sides, except for few cases. For example, see \([\text{Rid}11]\) for a detailed analysis in the case of \( \widehat{sl}_2 \) at level \( k = -\frac{1}{2} \).

In terms of vertex operator superalgebras, fusion rules can be estimated by the theory of Frenkel–Zhu’s bimodule developed in \([\text{FZ}92]\) (see also \([\text{KW}94]\, [\text{Li}99a]\)). In the theory, twisted Zhu’s algebra, which is originally introduced by Y. Zhu
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in [Zhu96] and generalized by several authors (e.g. [KW94], [DLM98a], [Xu98], [DLM98b], [DSK06]), plays a prominent role. Unlike in the case of the Virasoro algebra, unfortunately, highest weight modules over the \( N = 2 \) superconformal algebra (even in the Neveu–Schwarz sector) may contain subsingular vectors discovered in [GRR97]. See e.g. [Dör98, §1] for further information. Due to such a circumstance, the twisted Zhu algebra of the simple \( N = 2 \) vertex operator superalgebra (see Appendix A.1 for the definition) has not yet been completely determined to the best of our knowledge. See [EG97, §3] for partial results.

1.2. Main Results. In this paper we study the representation theory of the \( N = 2 \) vertex operator superalgebra and determine its twisted Zhu algebra(s) as a first step towards the full understanding of the corresponding Verlinde formula.

Our main tool is an exact functor induced by the Kazama–Suzuki coset construction (see [Sat16, §4.1] for the definition). More precisely, this functor is defined as one from a certain full subcategory of \( \hat{\mathfrak{sl}}_2 \)-modules of level \( k \in \mathbb{C} \setminus \{-2\} \) to a similar full subcategory of modules over the \( N = 2 \) superconformal algebra of central charge \( c = \frac{3k}{k + 2} \). From this viewpoint, the structure of \( N = 2 \) Verma modules turns out to reflect that of relaxed Verma modules over \( \hat{\mathfrak{sl}}_2 \) with spectral flow twists (see [FST98], [Sat16] for more detail).

In the present paper, the level \( k \) is supposed to be a Kac–Wakimoto admissible level and the corresponding central charge \( c \) is given by

\[
c = c_{p,p'} := 3 \left( 1 - \frac{2p'}{p} \right),
\]

where \((p, p')\) is a pair of coprime positive integers such that \( p \geq 2 \). Then our main results in §3 are summarized as follows (see Remark 3.6).

Theorem 1.1. Let \( L_c \) be the simple \( N = 2 \) vertex operator superalgebra of central charge \( c = c_{p,p'} \). Then every simple \( L_c \)-module admits Bernstein–Gelfand–Gelfand (BGG) type resolution in terms of spectral flow twisted modules (see Appendix A.2 for the definition) of certain Verma type modules.

As a corollary, in §4 we prove the following (see Theorem 4.4 for detail).

Theorem 1.2. Every highest weight module over the \( N = 2 \) superconformal algebra of central charge \( c = c_{p,p'} \) corresponding to a generalized Verma module for \( \hat{\mathfrak{sl}}_2 \) (see Lemma 3.2 for the precise definition) contains a unique singular vector which generates its maximum proper submodule.

In §5 we give two applications of the above singular vectors. First, for general \( c = c_{p,p'} \), we determine the twisted Zhu algebras \( A_\sigma(L_c) \) and \( A_{id}(L_c) \) (see [B.1] for the definition), where \( \sigma \) and \( id \) are the parity involution and the identity automorphism of \( L_c \), respectively (see Theorem 5.2 and Corollary 5.9 for detail).

Theorem 1.3. For \( c = c_{p,p'} \), we have the following.

1. The \( \sigma \)-twisted Zhu algebra \( A_\sigma(L_c) \) is isomorphic to \( U(\mathfrak{gl}_{1|1})/\langle \phi_c \rangle \) ideal for some \( \phi_c \in U(\mathfrak{gl}_{1|1}) \) corresponding to \( N_{p,p'} \).

2. The \( id \)-twisted Zhu algebra \( A_{id}(L_c) \) is isomorphic to \( \mathbb{C}[h,q]/\langle f_c, g_c \rangle \) ideal for some \( f_c, g_c \in \mathbb{C}[h,q] \) corresponding to \( N_{p,p'} \) and \( G^+ \frac{1}{2} G^- \frac{1}{2} N_{p,p'} \).

1Let \( I \) be the proper submodule of a highest weight module \( M \) generated by singular vectors. Then a non-singular vector in \( M \) is called subsingular if its image in \( M/I \) is singular.
Note that the latter has been conjectured by W. Eholzer and M. Gaberdiel in [EG97, §3]. See Remark 1.16 and Remark 5.10 for more detail.

Second, in the case of $c = c_{3,2} = -1$, we compute the Frenkel–Zhu bimodule structure associated with a certain simple highest weight module. We should mention that the simple vertex operator superalgebra $L_c$ is not $C_2$-cofinite if $p' \neq 1$ by [Sat18, Corollary 2.3] (see also Corollary 5.10 in this paper). As a consequence of the computation, we obtain a non-trivial upper bound for conjectural fusion rules in [Sat18, Example 5.2]. See Remark 5.14 for detail.

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2. BGG type resolution for simple $\widehat{sl}_2$-modules

In this section we give a brief review on the existence of several BGG type resolutions for certain simple weight $\widehat{sl}_2$-modules. At the end of this section, we give a new resolution for certain relaxed highest weight $\widehat{sl}_2$-modules in terms of relaxed Verma modules (see 2.3 for the definition).

2.1. Notations. We fix a pair of coprime integers $(p, p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ and set $a := \frac{p'}{p}$. Unless otherwise specified, the letter $k$ stands for the Kac–Wakimoto admissible level $-2 + a^{-1}$ of the affine Lie algebra $\widehat{sl}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t^\pm] \oplus \mathbb{C}K$. We denote the standard $\mathfrak{sl}_2$-triple by $\{E, F, H\}$. We put the following two finite sets:

- $\mathcal{I}_{KW} := \{(m, n) \in \mathbb{Z}^2 \mid 1 \leq m \leq p - 1, 0 \leq n \leq p' - 1\}$,
- $\mathcal{I}_{BPZ} := \{(m, n) \in \mathcal{I}_{KW} \mid n \neq 0, p'm + pn \leq pp'\}$.

For $j \in \mathbb{C}$, the Verma module $M_{j,k}$ of $\widehat{sl}_2$ is the $\widehat{sl}_2$-module which is freely generated by a vector $|j, k\rangle$ subject to the relations

$$X_n |j, k\rangle = E_0 |j, k\rangle = 0, \quad H_0 |j, k\rangle = 2j |j, k\rangle, \quad K |j, k\rangle = k |j, k\rangle$$

for any $X \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}_{>0}$, where $X_n := X \otimes t^n$. It is clear that we have

$$L_0 |j, k\rangle = \Delta(j) |j, k\rangle \quad (\Delta(j) := \frac{j(j + 1)}{k + 2}),$$

where the operator $L_0$ is given by the Sugawara construction.

Finally we set

$$M(m, n) := M_{jm, n, k} \quad \left(j_{m,n} := \frac{m - 1}{2} - \frac{n}{2a}\right)$$

for $m, n \in \mathbb{Z}$. Denote by $L(m, n)$ the simple quotient of $M(m, n)$.

2.2. Parabolic BGG Resolution. We first recall the following BGG resolution of the Kac–Wakimoto admissible highest weight $\widehat{sl}_2$-module $L(r, s)$ for $(r, s) \in \mathcal{I}_{KW}$ constructed by F. Malikov.

Theorem 2.1. [Mal91, Theorem A] We set $M_n := M(n) \oplus M(-n)$ for $n \in \mathbb{Z}_{>0}$, where $M(2m) := M(2pm + r, s)$ and $M(2m - 1) := M(2pm - r, s)$ for $m \in \mathbb{Z}$. Then there exists an exact sequence

$$\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M(r, s) \rightarrow L(r, s) \rightarrow 0.$$
Next, for $m \in \mathbb{Z}_{>0}$, we define the generalized Verma module $V(m)$ to be the \( \hat{\mathfrak{sl}}_2 \)-module freely generated by the vector $|m\rangle$ subject to the relations

$$X_n |m\rangle = E_0 |m\rangle = E_0^n |m\rangle = 0, \quad H_0 |m\rangle = (m-1)|m\rangle, \quad K |m\rangle = k |m\rangle$$

for any $X \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}_{>0}$. Then the following parabolic BGG resolution of $L(r,0)$ for $(r,0) \in \mathfrak{h}_{KW}$ is obtained as a slight generalization of Theorem 2.1.

**Theorem 2.2.** We set $V_{2m} := V(2pm+r)$ and $V_{2m-1} := V(2pm-r)$ for $m \in \mathbb{Z}_{>0}$. Then there exists an exact sequence

$$\cdots \rightarrow V_n \rightarrow \cdots \rightarrow V_2 \rightarrow V_1 \rightarrow V(r) \rightarrow L(r,0) \rightarrow 0.$$

**Remark 2.3.** When $p' = 1$, the resolutions (2.1) and (2.2) for $L(r,0)$ are obtained by H. Garland and J. Lepowsky in [GL76, Theorem 8.6] (see also [RCW82]).

### 2.3. Relaxed Verma modules.

In this subsection we consider $k \in \mathbb{C}$. Relaxed Verma modules are firstly introduced by [FST98] in the case of $\mathfrak{sl}_2$ (cf. [Fut96]). Their quotient modules, which are called relaxed highest weight modules, have been studied in [RW15, §2.1], [ADM17], [Ada18], [KR18]. For the reader’s convenience, we specify the precise definition of relaxed Verma modules used in this paper (cf. [RW15, §2.1], [KR18, §2]).

Let $\mathfrak{g}$ be a simple complex Lie algebra (or a basic classical Lie superalgebra, in general), $\mathfrak{h}$ be its Cartan subalgebra, and $U_0$ be the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. For a simple finite-dimensional $U_0$-module $M$, we define the action of $\hat{\mathfrak{g}}_{\geq 0} := \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ on the induced $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U_0} M$ by $X_n \mapsto \delta_{n,0}X$ and $K \mapsto k$ for $X \in \mathfrak{g}$ and $n \in \mathbb{Z}_{\geq 0}$. We denote this $\hat{\mathfrak{g}}_{\geq 0}$-module by $\hat{M}_k$. Then we call the parabolic induction $\hat{M}_k := \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{U(\mathfrak{g})} \hat{M}_k$ the relaxed Verma module induced from $M$. When the level $k$ is not critical, it is clear that the Virasoro algebra acts on $\hat{M}_k$ via the Sugawara construction and the operator $L_0$ acts diagonally on $\hat{M}_k$.

When $\mathfrak{g} = \mathfrak{sl}_2$ and $k \in \mathbb{C} \setminus \{ -2 \}$, the algebra $U_0$ is freely generated by $H$ and the quadratic Casimir element and it is not hard to verify that the definition of relaxed Verma modules can be rephrased as follows (see [Sat16, §3.3.2] for detail):

**Definition 2.4.** For $(h,j) \in \mathbb{C}^2$, the relaxed Verma module $R_{h,j,k}$ is the $\hat{\mathfrak{sl}}_2$-module freely generated by a relaxed highest weight vector $|h,j,k\rangle$ subject to the relations

$$X_n |h,j,k\rangle = 0, \quad L_0 |h,j,k\rangle = h |h,j,k\rangle,$$

$$H_0 |h,j,k\rangle = 2j |h,j,k\rangle, \quad K |h,j,k\rangle = k |h,j,k\rangle$$

for any $X \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}_{>0}$. We write $L_{h,j,k}$ for its unique simple quotient module.

### 2.4. Twisting functor.

In this subsection we recall some known methods (see e.g. [KT98, §2.4]) to construct relaxed Verma modules from ordinary Verma modules.

Let $U_F$ be the (right) localization of the non-commutative algebra $U := U(\hat{\mathfrak{sl}}_2)$ with respect to the multiplicative set $\{(F_0)^n \mid n \in \mathbb{Z}_{\geq 0}\}$ in $U$. For $j \in \mathbb{C}$, we define the functor $F^j$ by

$$F^j := \text{Res}^{U_F}_U \circ \text{Ad}(F_0)^j \circ \text{Ind}^{U_F}_U : U\text{-mod} \rightarrow U\text{-mod},$$
where $\text{Ad}(F_0^j)$ is the algebra automorphism of $U_F$ defined by

$$\text{Ad}(F_0^j)(u) := \sum_{\ell \geq 0} \binom{j}{\ell} \text{ad}(F_0)^{\ell} F_0^{-\ell} \in U_F$$

for $u \in U_F$. Since the functor $F^j$ is the composition of three exact functors, it is exact. For an $\hat{\mathfrak{sl}}_2$-module $M$, we denote each element $v$ of the underlying space $U_F \otimes_U M$ of the twisted module $F^jM := F^j(M)$ by $F^jv$.

**Lemma 2.5.** Let $j_0, j \in \mathbb{C}$. Then there exists a unique $\hat{\mathfrak{sl}}_2$-module homomorphism

$$(2.3) \quad R_{\Delta(j_0), j, k} \to F^{-j-j_0} M_{j_0, k}: \Delta(j_0), j, k) \mapsto F^{-j-j_0}(j_0, k).$$

Moreover it is an isomorphism if and only if $j \notin \{j_0, -j_0 - 1\} + \mathbb{Z}_{\leq 0}$.

**Proof.** Since the vector $F^{-j-j_0}(j_0, k)$ satisfies the same annihilation relation as that of $[\Delta(j_0), j, k]$, the $\hat{\mathfrak{sl}}_2$-module homomorphism $(2.3)$ uniquely exists. By a direct computation, we have

$$E_0 F^{-j-j_0} F_0^{-m}(j_0, k) = -(m + j - j_0)(m + j + j_0 + 1) F^{-j-j_0} F_0^{-m-1}(j_0, k)$$

for $m \in \mathbb{Z}_{>0}$. Since the image of $(2.3)$ is given by $U(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]) F^{-j-j_0}(j_0, k)$ and the set $\{F^{-j-j_0} F_0^m | n \in \mathbb{Z}\}$ forms a free $U(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}])$-basis of $F^{-j-j_0} M_{j_0, k}$, the mapping $(2.3)$ is bijective if and only if $j \notin \{j_0, -j_0 - 1\} + \mathbb{Z}_{\leq 0}$.

The following lemma is used in the next subsection.

**Lemma 2.6.** Assume that $j \notin \{j_0, -j_0 - 1\} + \mathbb{Z}$. Then, for any $n \in \mathbb{Z}$, the module $F^{-j-n} j_0 M_{j_0, k}$ is isomorphic to $R_{\Delta(j_0), j, k}$.

**Proof.** By the previous lemma, it suffices to prove that $R_{\Delta(j_0), j+n, k}$ is isomorphic to $R_{\Delta(j_0), j, k}$. This follows from direct calculation.

### 2.5. Relaxed BGG Resolution.

Since $-j_{m,n} - 1 = j_{p,m',p'} - n$, we have by Lemma 2.6

$$R_{m,n;j} := R_{\Delta(j_m,n), j, k} \simeq F^{-j+j_{m,n}} M(m, n)$$

if $j \notin \{j_{m,n}, j_{p,m',p'} - n\} + \mathbb{Z}_{\leq 0}$. Then, by applying the exact functor $F^{-j+j_{r,s}}$ to the BGG resolution $(2.1)$, we obtain the following BGG type resolution for the simple quotient $L_{r,s;j}$ of $R_{r,s;j}$:

**Proposition 2.7.** Assume that $(r, s) \in \mathcal{BPZ}$ and $j \notin \{j_{r,s}, j_{r-p,p'-s}\} + \mathbb{Z}$. We set $R_n := R(n; j) \oplus R(-n; j)$ for $n \in \mathbb{Z}_{>0}$, where $R(2m; j) := R_{2pm-r,s;j}$ and $R(2m-1; j) := R_{2pm-r,s;j}$ for $m \in \mathbb{Z}$. Then there exists an exact sequence

$$(2.4) \quad \cdots \to R_n \to \cdots \to R_2 \to R_1 \to R_{r,s;j} \to L_{r,s;j} \to 0.$$  

**Proof.** Since $F^{-j+j_{r,s}} M_n$ is isomorphic to $R_n$ by Lemma 2.6 there exists an exact sequence

$$\cdots \to R_n \to \cdots \to R_2 \to R_1 \to R_{r,s;j} \to F^{-j+j_{r,s}} L(r, s) \to 0.$$  

Therefore the formal character of $F^{-j+j_{r,s}} L(r, s)$ is given by

$$(2.5) \quad \text{ch} F^{-j+j_{r,s}} L(r, s) = \sum_{n \in \mathbb{Z}} (-1)^n \text{ch} R_n,$$

where $R_0 := R_{r,s;j}$. It is clear that the formal series (2.5) coincides with the character of $L_{r,s;j}$ given by T. Creutzig and D. Ridout in [CR13, Corollary 5]. Since Creutzig–Ridout’s character formula is proved in [Ada18, Proposition 5.4] for
some special cases and in [KR18] Theorem 5.2 for general cases\footnote{The character formula in [CR13] Corollary 5 also follows from [Sat18] Theorem 3.1] together with [Sat16] Proposition 7.1 (4) and Theorem 7.8.}, this completes the proof.

3. BGG TYPE RESOLUTION FOR SIMPLE $\mathfrak{ns}_2$-MODULES

In this section we prove the existence of BGG type resolution for certain simple highest weight modules over the $N = 2$ superconformal algebra. As we mentioned in [12] we use the functor $\Omega^+_c : \mathcal{C}_sf \to \mathcal{C}_sc$ for $j \in \mathbb{C}$ defined in [Sat16] §4.1, where $\mathcal{C}_sf$ and $\mathcal{C}_sc$ are certain full subcategories of $\mathfrak{sl}_2$-modules of level $k$ and $\mathfrak{ns}_2$-modules of central charge $c$, respectively. Our discussion below is essentially based on the exactness of the functor $\Omega^+_c$ (see [Sat16] Theorem 4.4) and the spectral flow equivariance property (see [Sat16] Corollary 6.3)).

3.1. Notations. The Neveu-Schwarz sector of the $N = 2$ superconformal algebra is the Lie superalgebra

$$\mathfrak{ns}_2 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}J_n \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G^+_r \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G^-_r \oplus \mathbb{C}C$$

whose $\mathbb{Z}_2$-grading is given by

$$(\mathfrak{ns}_2)^0 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}J_n \oplus \mathbb{C}C, \quad (\mathfrak{ns}_2)^1 = \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G^+_r \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}G^-_r$$

with the following (anti-)commutation relations:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)G\delta_{n+m,0},$$

$$[L_n, J_m] = -mJ_{n+m}, \quad [L_n, G^+_r] = \left(\frac{n}{2} - r\right)G^+_r,$$

$$[J_n, J_m] = \frac{n}{3}C\delta_{n+m,0}, \quad [J_n, G^+_r] = \pm G^+_r,$$

$$[G^+_r, G^-_s] = 2L_{r+s} + (r - s)J_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)C\delta_{r+s,0},$$

$$[G^+_r, G^+_s] = [G^-_r, G^-_s] = 0, \quad (\mathfrak{ns}_2, C) = \{0\},$$

for $n, m \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. Define a triangular decomposition $\mathfrak{ns}_2 = (\mathfrak{ns}_2)_+ \oplus (\mathfrak{ns}_2)_0 \oplus (\mathfrak{ns}_2)_-$ by

$$(\mathfrak{ns}_2)_+ := \bigoplus_{n > 0} \mathbb{C}L_n \oplus \bigoplus_{n > 0} \mathbb{C}J_n \oplus \bigoplus_{r > 0} \mathbb{C}G^+_r \oplus \bigoplus_{r > 0} \mathbb{C}G^-_r,$$

$$(\mathfrak{ns}_2)_- := \bigoplus_{n < 0} \mathbb{C}L_n \oplus \bigoplus_{n < 0} \mathbb{C}J_n \oplus \bigoplus_{r < 0} \mathbb{C}G^+_r \oplus \bigoplus_{r < 0} \mathbb{C}G^-_r,$$

$$(\mathfrak{ns}_2)_0 := \mathbb{C}L_0 \oplus \mathbb{C}J_0 \oplus \mathbb{C}C.$$

For $(h, j, c) \in \mathbb{C}^3$, the Verma module $\mathcal{M}_{h,j,c}$ of $\mathfrak{ns}_2$ is the $\mathbb{Z}_2$-graded $\mathfrak{ns}_2$-module freely generated by an even vector $[h, j, c]^{sc}$ subject to the relations

$$(\mathfrak{ns}_2)_+ [h, j, c]^{sc} := \{0\}, \quad L_0 [h, j, c]^{sc} := h [h, j, c]^{sc},$$

$$J_0 [h, j, c]^{sc} := j [h, j, c]^{sc}, \quad C [h, j, c]^{sc} := c [h, j, c]^{sc}.$$
Similarly, the chiral Verma module $\mathcal{M}^{+}_{j,c}$ is the $\mathbb{Z}_2$-graded $\mathfrak{ns}_2$-module freely generated by an even vector $|j,c\rangle^{\text{sc}}$ subject to the relations
\[
(n_{\mathfrak{ns}_2}^+)_{|j,c\rangle^{\text{sc}}} = \{0\}, \quad G_{\frac{1}{2}}^+|j,c\rangle^{\text{sc}} := 0,
\]
\[
L_0|j,c\rangle^{\text{sc}} := \frac{j}{2}|j,c\rangle^{\text{sc}}, \quad J_0|j,c\rangle^{\text{sc}} := j|j,c\rangle^{\text{sc}}, \quad C|j,c\rangle^{\text{sc}} := c|j,c\rangle^{\text{sc}}.
\]

3.2. BGG type resolution for atypical modules. For $m, n \in \mathbb{Z}$, we set
\[
\mathcal{M}^+(m,n) := \mathcal{M}^{+}_{2aj_{m,n},3(1-2a)}
\]
and denote its simple quotient by $\mathcal{L}(m,n)$. For the sake of completeness, we recall the following BGG type resolution of $\mathcal{L}(r,s)$ for $(r,s) \in J_{KW}$, which is firstly given by B. Feigin, A. Semikhatov, V. Sirota, and I. Tipunin in [FSST99, Theorem 3.1] (see [Sat16, Theorem 7.11] for the proof).

**Theorem 3.1** ([FSST99], [Sat16]). We set $\mathcal{M}^+_n := \mathcal{M}^+(n) \oplus \mathcal{M}^+(-n)$ for $n > 0$, where $\mathcal{M}^+ (2m) := \mathcal{M}^{+} (2pm + r,s)^{pm}$ and $\mathcal{M}^+ (2m - 1) := \mathcal{M}^{+} (2pm - r,s)^{pm-r}$ for $m \in \mathbb{Z}$. Then there exists an exact sequence
\[
\cdots \to \mathcal{M}^+_n \to \cdots \to \mathcal{M}^+_2 \to \mathcal{M}^+_1 \to \mathcal{M}^+(r,s) \to \mathcal{L}(r,s) \to 0
\]
of $\mathfrak{ns}_2$-modules.

In what follows, we construct a new BGG type resolution of $\mathcal{L}(r,0)$, which is the counterpart of (2.2) in the $N = 2$ side. For convinience of later use, we give an explicit description of $\Omega^+_{j_{m,0}}(V(m))$ as follows.

**Lemma 3.2.** For $m \in \mathbb{Z}_{>0}$, the $\mathfrak{ns}_2$-module $\Omega^+_{j_{m,0}}(V(m))$ is isomorphic to the $\mathfrak{ns}_2$-module $V(m)$ freely generated by $|m\rangle^{\text{sc}}$ subject to the relations
\[
(n_{\mathfrak{ns}_2}^+)_{|m\rangle^{\text{sc}}} = \{0\}, \quad G_{\frac{1}{2}}^+|m\rangle^{\text{sc}} := G_{-\frac{1}{2}}|m\rangle^{\text{sc}}, \quad G_{-\frac{1}{2}}^+|m\rangle^{\text{sc}} := 0,
\]
\[
L_0|m\rangle^{\text{sc}} := a_{j_{m,0}}|m\rangle^{\text{sc}}, \quad J_0|m\rangle^{\text{sc}} := 2a_{j_{m,0}}|m\rangle^{\text{sc}}, \quad C|m\rangle^{\text{sc}} := 3(1-2a)|m\rangle^{\text{sc}}.
\]

Since the proof is same as that of [Sat16, Proposition 7.1]), we omit it.

**Remark 3.3.** The $\mathfrak{ns}_2$-module $V(1)$ is by definition isomorphic to the vacuum $\mathfrak{ns}_2$-module $V_c$ of central charge $c = c_{p,p'}$ (see Appendix A.1 for the definition).

**Theorem 3.4.** We set $\mathcal{V}_{2m} := \mathcal{V}(2pm + r)^{pm}$ and $\mathcal{V}_{2m-1} := \mathcal{V}(2pm - r)^{pm-r}$ for $m \in \mathbb{Z}_{>0}$. Then there exists an exact sequence
\[
\cdots \to \mathcal{V}_n \to \cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{V}(r) \to \mathcal{L}(r,0) \to 0.
\]

**Proof.** By [Sat16, Example 6.4], we have
\[
\Omega^{+}_{j_{r,0}}(\mathcal{V}_{2m}) = \Omega^{+}_{j_{2pm+r,0}-pm}\left(\mathcal{V}(2pm + r)\right) \simeq \Omega^{+}_{j_{2pm+r,0}}\left(\mathcal{V}(2pm + r)\right)^{pm}
\]
for any $m \in \mathbb{Z}_{>0}$. Then, by Lemma 3.2 we obtain $\mathcal{V}_{2m} \simeq \Omega^{+}_{j_{r,0}}(\mathcal{V}_{2m})$. We also obtain $\mathcal{V}_{2m-1} \simeq \Omega^{+}_{j_{r,0}}(\mathcal{V}_{2m-1})$ in the same way. This completes the proof. \hfill \Box

\footnote{In the rest of this section, for simplicity of notation, we do not consider the $\mathbb{Z}_2$-graded structure given in the previous subsection.}
3.3. BGG type resolution for typical modules. For \((m, n) \in \mathbb{Z}^2\) and \(j \in \mathbb{C}\), we set
\[
M_{m,n;j} := M_{\Delta(jm, -aj^2, 2aj, 3(1-2a)}
\]
and denote its simple quotient by \(L_{m,n;j}\). Note that by [Sat16, Proposition 7.1 (1) and (4)] we have \(M_{m,n;j} \simeq \Omega_j^+(R_{m,n;j})\) and \(L_{m,n;j} \simeq \Omega_j^+(L_{m,n;j})\).

By applying the exact functor \(\Omega_j^+\) to the BGG type resolution (3.1), we obtain the following resolution of \(L_{r,s;j}\):

**Theorem 3.5.** Assume that \((r, s) \in \mathcal{J}_{BPZ}\) and \(j \notin \{j_{r,s}, j_{p-r,p'-s}\} + \mathbb{Z}\). We set \(\mathcal{M}_n := \mathcal{M}(n; j) \oplus \mathcal{M}(-n; j)\) for \(n \in \mathbb{Z}_{>0}\), where \(\mathcal{M}(2m; j) := (\mathcal{M}_{2pm+r,s;j})^{pm}\) and \(\mathcal{M}(2m-1; j) := (\mathcal{M}_{2pm-r,s;j})^{pm-\kappa}\) for \(m \in \mathbb{Z}\). Then there exists an exact sequence
\[
\cdots \rightarrow \mathcal{M}_n \rightarrow \cdots \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_{r,s;j} \rightarrow L_{r,s;j} \rightarrow 0.
\]

The proof is similar to that of Theorem 3.4 and we omit it.

**Remark 3.6.** As a consequence of [Ada99, Theorem 7.1 and 7.2], simple \(L_v\)-modules are exhausted by the following (see [Sat18, Theorem 2.1 and Lemma 4.1] for detail):

1. \(L(r, s)^\theta\) for \((r, s) \in \mathcal{J}_{KW}\) and \(\theta \in \mathbb{Z}\),
2. \(L_{r,s;j}\) for \((r, s) \in \mathcal{J}_{BPZ}\) and \(j \notin \{j_{r,s}, j_{p-r,p'-s}\} + \mathbb{Z}\).

Since we obtain BGG type resolution for \(L(r, s)^\theta\) by applying the exact functor (3.1) to the above exact sequence (3.1), we conclude Theorem 1.1.

4. Uniqueness of singular generator

4.1. Affine VOA side. We recall the following singular vector formula for the generalized Verma module \(V(r)\) at the Kac–Wakimoto admissible level \(k = -2 + a^{-1}\), which is a direct corollary of [MFF86, Proposition 2.1] and Theorem 2.2.

**Proposition 4.1.** The maximum proper \(\widehat{\mathfrak{sl}_2}\)-submodule of \(V(r)\) is generated by the Malikov–Feigin–Fuchs (MFF) singular vector
\[
v_{p,p'}(r) := E_{-1}^{(2p'-1)\kappa-r-p'}F_0^{(2p'-2)\kappa-r}F_0^{2\kappa-r}E_{-1}^{r}\mid r\rangle \in V(r)
\]
in the \((L_0^{\mathcal{Sog}}, H_0)\)-eigenspace with eigenvalue \((p-r)p', 2p-r-1\), where \(\kappa := a^{-1} = \frac{2}{p}\) and \(\mid r\rangle\) is the highest weight vector in \(V(r)\).

**Example 4.2.** When \(r = 1\), we obtain the following (see [AM95, Theorem 3.3]): the maximum proper ideal of the universal affine vertex algebra \(V_k(\widehat{\mathfrak{sl}_2}) \simeq V(1)\) is generated by the MFF singular vector \(v_{p,p'} := v_{p,p'}(1) \in V_k(\mathfrak{sl}_2)\).

The next lemma plays a key role in this section.

**Lemma 4.3.** There exists a unique \(\widehat{\mathfrak{sl}_2}\)-module homomorphism
\[
f_{p,p'}(r) : R(r) \rightarrow V(r); \quad v(r) \mapsto F_0^{p-r}v_{p,p'}(r),
\]
where \(R(r) := R_{(p-r)p'+\Delta(jr, k), jr, a, k}\) and \(v(r)\) is the relaxed highest weight vector of \(R(r)\). Moreover we have
\[
\text{Hom}_{\widehat{\mathfrak{sl}_2}}(R(r), V(r)) = \mathbb{C}f_{p,p'}(r).
\]
Proof. Since \( v_{p,p'}(r) \) is a singular vector, the vector \( F_{0}^{p-r}v_{p,p'}(r) \) satisfies the same annihilation relations as those for \( v(r) \). Thus the former half follows.

Let \( f: R(r) \to V(r) \) be an \( \mathfrak{sl}_{2} \)-module homomorphism. Then the vector \( f(v(r)) \) lies in the \( (L_{0}^{\text{sing}}, H_{0}) \)-eigenspace with eigenvalue \( ((p-r)p', r-1) \) in the maximum proper \( \mathfrak{sl}_{2} \)-submodule of \( V(r) \). By the Poincaré–Birkoff–Witt (PBW) theorem, the eigenspace is \( \mathbb{C}F_{0}^{p-r}v_{p,p'}(r) \). This proves the latter half. \( \square \)

4.2. \( \mathbb{N} = 2 \) VOSA side. In this subsection, the letter \( c \) always stands for the central charge \( c_{p,p'} = 3(1 - 2a) \). The next proposition gives a generator of the maximum proper submodule of \( V(r) \).

**Theorem 4.4.** Set \( M(r) := M_{(p-r)p'+aj_{r,0},2aj_{r,0},c} \). Then we have

\[
\dim \text{Hom}_{\mathbb{N}_{2}}(M(r), V(r)) = 1,
\]

that is, there exists a unique singular vector \( N_{p,p'}(r) \in V(r)_{\mathbb{N}} \) with \((L_{0}, H_{0})\)-eigenvector \( ((p-r)p'+aj_{r,0},2aj_{r,0}) \) up to non-zero scalar multiple. In addition, the maximum proper submodule of \( V(r) \) is generated by \( N_{p,p'}(r) \).

**Proof.** Since the functor \( \Omega_{jr}^{+} \) is fully faithful by [Sat16, Theorem 4.4], we have

\[
\text{Hom}_{\mathbb{N}_{2}}(M(r), V(r)) = \mathbb{C}g_{p,p'}(r),
\]

where \( g_{p,p'}(r) := \Omega_{jr,0}^{+}(f_{p,p'}(r)) \). The exact functor \( \Omega_{jr,0}^{+} \) sends the exact sequence

\[
R(r) \xrightarrow{f_{p,p'}(r)} V(r) \xrightarrow{\pi} L(r,0)
\]

to the following exact sequence

\[
M(r) \xrightarrow{g_{p,p'}(r)} V(r) \xrightarrow{\pi'} L(r,0),
\]

where \( \pi \) and \( \pi' \) are the natural projections. This completes the proof. \( \square \)

**Example 4.5.** When \( r = 1 \), we have

\[
\dim \text{Hom}_{\mathbb{N}_{2}}(M_{(p-1)p',0,c}, V_{c}) = 1,
\]

that is, there exists a unique singular vector \( N_{p,p'}(1) := N_{p,p'}(1) \in V_{c}^{0} \) with \((L_{0}, H_{0})\)-eigenvector \( ((p-1)p', 0) \) up to non-zero scalar multiple. In addition, the maximum proper ideal \( I_{c} \) of the vertex operator superalgebra \( V_{c} \) is generated by \( N_{p,p'} \).

**Remark 4.6.** When \( p' = 1 \), the simple highest weight module \( L(r,0) \) lies in the \( \mathbb{N} = 2 \) unitary minimal series [BFK86] and \( V(r) \) contains no subsingular vectors by [Dor98, Theorem 4]. Therefore Theorem 4.4 for \( p' = 1 \) is a direct corollary of [Dor98, Theorem 1]. On the other hand, when \( p' > 1 \), it has not been known that \( V(r) \) contains no subsingular vectors (cf. [EG97, Remark 8]).

5. Applications

5.1. Application 1: Structure of Zhu’s algebra. In this subsection, we compute the twisted Zhu algebras of \( I_{c} \) by using the singular vector \( N_{p,p'} \) in \( V_{c} \). See [B.1] for the definition of the twisted Zhu algebras.
5.1.1. The \( \sigma \)-twisted (Ramond) case. Take a \( \mathbb{Z}_2 \)-homogeneous \( \mathbb{C} \)-basis of the general linear Lie superalgebra \( \mathfrak{gl}_{1|1} = \text{End}_\mathbb{C}(\mathbb{C}^{1|1}) \) as
\[
Z := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Psi^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Psi^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
By direct computations, one can verify the following:

**Proposition 5.1.** For \( c \in \mathbb{C} \), there exists a unique superalgebra isomorphism
\[
i_\sigma : U(\mathfrak{gl}_{1|1}) \xrightarrow{\cong} A_\sigma(V_c)
\]
such that \( i_\sigma(Z) = [L] - \frac{1}{2c}[1^e], \quad i_\sigma(J) = [J], \quad \text{and} \quad i_\sigma(\Psi^\pm) = [G^\pm]. \)

Then we obtain the following description of \( A_\sigma(L_c) \).

**Theorem 5.2.** When \( c = p,p' \), the kernel of the natural projection
\[
\pi_\sigma : A_\sigma(V_c) \to A_\sigma(L_c)
\]
is generated by the coset \( [N_{p,p}] \) of the singular vector \( N_{p,p'} \in V_c^0 \).

**Proof.** Let \( J_{c,\sigma} \) be the two-sided ideal of \( A_\sigma(V_c) \) which is generated by \( [N_{p,p}] \). Since \( J_{c,\sigma} \subseteq \ker \pi_\sigma = I_c + O_\sigma(V_c) \) is clear, it suffices to prove that \( J_{c,\sigma} \supseteq I_c + O_\sigma(V_c) \). Since \( I_c = U((\mathfrak{n}_{2,2})_-)N_{p,p'} \) by Example 4.3, it is proved by a super analog of Zhu96 Lemma 2.1.2 (see e.g. DZ06 Lemma 3.1 (i)) and by induction with respect to the PBW filtration of \( U((\mathfrak{n}_{2,2})_-) \). \( \square \)

Then Theorem 1.3 (1) follows from Proposition 5.1 and Theorem 5.2.

**Example 5.3.** By the PBW theorem, there exist \( P_1, P_2 \in \mathbb{C}[x,y] \) such that
\[
\phi_c := i_\sigma^{-1}([N_{p,p'}]) = P_1(Z,J) + P_2(Z,J)\Psi^- \Psi^+ \in U(\mathfrak{gl}_{1|1})^0.
\]
We give some examples of \( \phi_c \) explicitly. For brevity, we write \( v \propto w \) if \( v \) and \( w \) are proportional.

1. When \( (p,p') = (4,1) \), we have \( c_{4,1} = \frac{3}{2} \) and
\[
N_{4,1} \propto \left( 10J_{-3} - 3L_{-3} + 3G^+_{-\frac{3}{2}} G^-_{-\frac{3}{2}} - 12L_{-2}J_{-1} + 8J_{-1}^3 \right) 1^c
\]
(see [EG97] p.72). By some computations, we obtain
\[
\phi_{\frac{3}{2}} \propto (4J - 1)(J(4J + 1) - 6Z) - 6\Psi^- \Psi^+.
\]

2. When \( (p,p') = (2,3) \), we have \( c_{2,3} = -6 \) and
\[
N_{2,3} \propto \left( -10J_{-3} - 6L_{-3} + 6G^+_{-\frac{3}{2}} G^-_{-\frac{3}{2}} + 6L_{-2}J_{-1} + J_{-1}^3 \right) 1^c
\]
(see [EG97] p.72). By some computations, we obtain
\[
\phi_{-6} \propto (J + 1)(J(J - 1) + 6Z) - 6\Psi^- \Psi^+
\]
(3) When \((p, p') = (3, 2)\), we have \(c_{3,2} = -1\) and

\[
N_{3,2} \propto \left( 42J_{-4} + 24L_{-4} + 27J_{-2}J_{-2} - 84J_{-3}J_{-1} - 6G^+_{-\frac{3}{2}}G^-_{-\frac{3}{2}} + 6G^+_{-\frac{3}{2}}G^-_{-\frac{3}{2}} - 32L_{-2}L_{-2} - 36L_{-3}J_{-1} + 36J_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{3}{2}} + 12L_{-2}J_{-1}^2 + 9J_{-1}^4 \right) 1^c
\]

(see [EG97, p.72]). By some computations, we obtain

\[
\phi_{-1} \propto \left((6J + 1)(6J + 5) - 48Z\right)\left((6J - 1)(6J - 5) + 96Z\right) - 72^2 J\Psi - \Psi^+.
\]

In the same way as [Ada99, Theorem 7.1 and 7.2], we obtain the classification of simple \(\sigma\)-twisted \(L_c\)-modules by the Kazama–Suzuki coset construction. Then, by a generalization of [Zhu96, Theorem 2.2.2] (see e.g. [Xu98, Corollary 5.1.8], [DZ06, Theorem 6.5]), we also obtain the classification of finite-dimensional simple \(A_\sigma(L_c)\)-modules as follows.

**Proposition 5.4.** Let \(c = c_{p,p'}\). For \((z, j) \in \mathbb{C}^2\), let \(L_{z,j}\) be the simple highest weight \(\mathfrak{gl}_{1|1}\)-module (with respect to the standard Borel subalgebra \(\mathfrak{b} = \mathbb{C}Z \oplus \mathbb{C}j \oplus \mathbb{C}\Psi^+)\) of highest weight \(z\mathbf{Z}^* + jJ^*\), where \((\mathbf{Z}^*, J^*)\) is the dual basis of \((\mathbf{Z}, J)\). Then the action of \(U(\mathfrak{gl}_{1|1})\) on \(L_{z,j}\) factors through that of \(A_\sigma(L_c)\) if and only if the pair \((z, j)\) lies in the disjoint union of the following sets:

\[
(5.1) \quad \left\{ \left( z_{r,\theta}, \mu_{r,\theta} + \frac{1}{2} \right) \left| (r, 0) \in J_{KW}, 0 \leq \theta \leq r - 1, \theta \in \mathbb{Z} \right\},
\]

\[
(5.2) \quad \left\{ \left( \frac{(ar-s)^2 - \mu^2}{4a}, \mu + \frac{1}{2} \right) \left| (r, s) \in J_{BPZ}, \mu \in \mathbb{C} \right\},
\]

where \(z_{r,\theta} := a(\theta + 1)(r - 1 - \theta)\) and \(\mu_{r,\theta} := a((r - 1 - \theta) - (\theta + 1))\).

**Remark 5.5.** When \(p' = 1\), the set \(\{5.1\}\) corresponds to the class \(P_0^{-}\) unitary minimal series of central charge \(\hat{c} = \frac{4}{3}\) in [BFK86] and the set \(\{5.2\}\) is empty.

As a consequence, we obtain the following.

**Corollary 5.6.** For \(c = c_{p,p'}\), the following conditions are all equivalent:

\[(1) \quad p' = 1,\]
\[(2) \quad A_\sigma(L_c) \text{ is finite-dimensional},\]
\[(3) \quad \text{every finite-dimensional } A_\sigma(L_c)-\text{module is completely reducible}.\]

**Proof.** First the implication \((1) \Rightarrow (2)\) follows from the \(\sigma\)-twisted regularity of \(L_c\) (cf. [Ada01, Theorem 8.1]) and a natural generalization of [L99b, Theorem 3.8]. Next the implication \((2) \Rightarrow (3)\) follows from Proposition 5.3. At last we prove that \((3)\) implies \((1)\). We suppose that \(p' \neq 1\). Let \(M_{z,j}\) be the Verma module of \(\mathfrak{gl}_{1|1}\) of highest weight \((z, j) \in \mathbb{C}^2\). By Proposition 5.3 and the existence of a non-split exact sequence \(0 \rightarrow L_{0,j-1} \rightarrow M_{0,j} \rightarrow L_{0,j} \rightarrow 0\) of \(\mathfrak{gl}_{1|1}\)-modules, the element \(\phi_c\) acts trivially on the indecomposable non-simple \(\mathfrak{gl}_{1|1}\)-module \(M_{0,\mu,\tau, -1 + \frac{1}{2}}\). Thus \((3)\) implies \((1)\).
5.1.2. The id-twisted (= Neveu–Schwarz) case. The following isomorphism seems to be well known (e.g. [Ada99 Remark 1.1]) and is proved in the same way as [KW94 Lemma 3.1].

**Proposition 5.7.** For \(c \in \mathbb{C}\), there exists a uniquely purely even \(\mathbb{Z}_2\)-graded algebra isomorphism

\[
i_{id}: \mathbb{C}[h, q] \xrightarrow{\cong} A_{id}(V_c)
\]

such that \(i_{id}(h) = [L]\) and \(i_{id}(q) = [J]\).

Then we obtain a similar result on the structure of \(A_{id}(L_c)\).

**Theorem 5.8.** When \(c = c_{p,p'}\), the kernel of the natural projection

\[
\pi_{id}: A_{id}(V_c) \rightarrow A_{id}(L_c)
\]

is generated by the cosets \([N_{p,p'}]\) and \([G_+^{-\frac{p}{2}}, G_-^{\frac{p}{2}} N_{p,p'}]\) in \(A_{id}(V_c)\).

**Proof.** Let \(J_{c, id}\) be the ideal of \(A_{id}(V_c)\) generated by \([N_{p,p'}]\) and \([G_+^{-\frac{p}{2}}, G_-^{\frac{p}{2}} N_{p,p'}]\). We only need to verify \(L_c + O_{id} \subseteq J_{c, id}\). Since the even singular vector \(N_{p,p'}\) generates the maximum proper submodule of \(V_c\) by Example 4.5 it suffices to prove that \(U_+ N_{p,p'} + O_{id} \subseteq J_{c, id}\) and \(G_+^{-\frac{p}{2}}, G_-^{\frac{p}{2}} U_+ N_{p,p'} + O_{id} \subseteq J_{c, id}\), where \(U_- := U((\mathrm{ns}_2)^0)\). The former follows from a direct computation and the latter is proved by induction with respect to the PBW filtration of \(U_-\). \(\square\)

**Corollary 5.9 (EG97).** The kernel of the mapping

\[
\pi_{id} \circ i_{id}: \mathbb{C}[h, q] \xrightarrow{\cong} A_{id}(V_c) \rightarrow A_{id}(L_c)
\]

generated by the two polynomials

\[
f_c := i_{id}^{-1}([N_{p,p'}]), \quad g_c := i_{id}^{-1}([G_+^{-\frac{p}{2}}, G_-^{\frac{p}{2}} N_{p,p'}]) \in \mathbb{C}[h, q].
\]

**Remark 5.10.** In [EG97 §3], W. Eholzer and M.R. Gaberdiel discussed the relation between the structure of the Zhu algebra \(A_{id}(L_c)\) and the existence of an “additional independent bosonic singular vector” \(N\) in \(V_c\). By the uniqueness, their singular vector \(N\) coincides with \(N_{p,p'}\) (up to non-zero scalar multiple).

By standard computations (see e.g. [KW94 Lemma 1.1]), one can verify that the pair \((f_c, g_c)\) coincides with \((p_1, p_2)\) appeared in [EG97 Table 3.1].

5.2. Application 2: Structure of Frenkel–Zhu’s bimodule. In this subsection, we compute the Frenkel–Zhu bimodule structure on a certain simple module over the simple \(N = 2\) vertex operator superalgebra of central charge \(c = c_{3,2} = -1\).

We first compute the \(\mathbb{Z}_2\)-graded \(A_{id}(V_c)\)-bimodule structure of \(A_{id}(\mathcal{M}_{j,c}^+)\). We always identify \(A_{id}(V_c)\) with \(\mathbb{C}[h, q]\) by the isomorphism \(i_{id}\) in Proposition 5.7.

**Lemma 5.11.** Set a \(\mathbb{Z}_2\)-graded structure on \(\mathbb{C}[x_\ell, x_r, y] \oplus \mathbb{C}[x_\ell, x_r, y]\) by \(\deg f = 0\) and \(\deg f \psi = 1\) for \(f \in \mathbb{C}[x_\ell, x_r, y]\). For \(j \in \mathbb{C}\), we define a \(\mathbb{Z}_2\)-graded \(\mathbb{C}[h, q]\)-bimodule structure on \(\mathbb{C}[x_\ell, x_r, y] \oplus \mathbb{C}[x_\ell, x_r, y]\) by

\[
h.(f + g \psi) := x_\ell (f + g \psi), \quad q.(f + g \psi) := (y + j) f + (y + j - 1) g \psi,
\]

\[
(f + g \psi) h := x_\ell (f + g \psi), \quad (f + g \psi) q := y (f + g \psi)
\]

for \(f, g \in \mathbb{C}[x_\ell, x_r, y]\). Then there exists a unique \(\mathbb{Z}_2\)-graded \(\mathbb{C}[h, q]\)-bimodule isomorphism

\[
\mathbb{C}[x_\ell, x_r, y] \oplus \mathbb{C}[x_\ell, x_r, y] \psi \rightarrow A_{id}(\mathcal{M}_{j,c}^+)
\]

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such that $1 \mapsto [j, c]^{\infty}$ and $\psi \mapsto [G_{\frac{3}{2}}^{-1}, j, c]^{\infty}$.

**Proof.** The existence and uniqueness of such a bimodule homomorphism are easily verified by calculation. By some computations (cf. [Wan93 §5], [KW94] Lemma 3.1), one can prove that $\mathcal{O}_d(M^+_{j,c})$ is linearly spanned by vectors of the form $(L_{(-\ell-2)} + 2L_{(-\ell-1)} + L_{(-\ell)})v$, $(J_{(-\ell-2)} + J_{(-\ell-1)})v$, or $(G^\pm_{(-\ell-1)} + G^\pm_{(-\ell)})v$ for $\ell \in \mathbb{Z}_{\geq 0}$ and $v \in M^+_{j,c}$. Then, by the PBW theorem, we can explicitly construct the inverse mapping of the induced linear mapping between the associated graded vector spaces (note that we consider the left-hand side is canonically identified with its associated graded space). This proves the bijectivity of the bimodule homomorphism.

Next, as a corollary of the classification in [Ada99 Theorem 7.2], the set of isomorphism classes of $\mathbb{Z}_{2}$-graded simple $L_{-1}$-modules is given by

$$\left\{ \mathcal{L}_{\frac{3}{2}, \frac{1}{2}, -1}^\psi, \mathcal{L}_{\frac{3}{2}, -1}^\psi \mid \psi \in \{-1, 0, 1\} \right\} \cup \left\{ \mathcal{L}_{-\frac{1}{2}, \frac{3}{2}, -1}^\psi, \mathcal{L}_{-\frac{1}{2}, -1}^\psi \mid \psi \in \mathbb{C} \right\},$$

where $\Pi$ is the parity changing functor on the category of $\mathbb{Z}_{2}$-graded vector spaces. In what follows, we denote the corresponding $\mathbb{Z}_{2}$-graded simple left $A_{\mathbb{C}}(L_{-1})$-modules (cf. [KW94 Theorem 1.3]) by

$$\{ C(\epsilon), \Pi C(\epsilon) \mid \epsilon \in \{-1, 0, 1\} \} \cup \{ C_j, \Pi C_j \mid j \in \mathbb{C} \}.$$ 

Note that we have $\mathcal{L}(1, 0) \simeq \mathcal{L}_{0,0,-1}$, $\mathcal{L}(2, 0) \simeq \mathcal{L}_{\frac{3}{2}, \frac{1}{2}, -1}$, and $\mathcal{L}(2, 0)^{\dagger} \simeq \mathcal{L}_{\frac{3}{2}, -\frac{1}{2}, -1}$. Then, by using the isomorphism in Lemma 5.11, we can describe the structure of $A_{\mathbb{C}}(\mathcal{L}_{\frac{3}{2}, -\frac{1}{2}, -1})$ as follows.

**Lemma 5.11.** The kernel of the natural surjective $\mathbb{C}[\hbar, q]$-bimodule homomorphism

$$\mathbb{C}[x_{\ell}, x_{r}, y] \oplus \mathbb{C}[x_{\ell}, x_{r}, y] \psi \xrightarrow{\zeta} A_{\mathbb{C}}(M^+_{\frac{3}{2}, -1}) \rightarrow A_{\mathbb{C}}(\mathcal{L}_{\frac{3}{2}, -\frac{1}{2}, -1})$$

is generated by

$$f_1 := 3P(P + R) - Q,$$

$$f_2 := (2P + R)(P - R),$$

$$f_3 := (2P + R)((3P - 4)(P - R) - 3Q + 2$$

$$g_1 \psi := (2P + 2R - 1) \psi,$$

$$g_2 \psi := (Q - P - R) \psi,$$

$$g_3 \psi := (4P^2 + (1 - 2R)P - 2R^2 + 2R - 3Q) \psi,$$

where $P := x_{\ell} - x_{r}$, $Q := x_{\ell} + x_{r} + \frac{1}{3}$, and $R := y + \frac{1}{3}$.

**Proof.** Set $w_1 := G^{-\frac{1}{2}}_{-\frac{3}{2}} G^{-\frac{1}{2}}_{-\frac{1}{2}} [(2/3, -1)^{\infty}$ and

$$w_2 := (4J_{-2} - 3G^{-\frac{1}{2}}_{-\frac{3}{2}} G^{-\frac{1}{2}}_{-\frac{1}{2}} - 2L_{-1} J_{-1} - 2J_{-1}^2 + 4L_{-1}^2) [(2/3, -1)^{\infty}.$$ 

By direct calculation, one can verify $w_2$ is a singular vector in $M^+_{\frac{3}{2}, -1}$, which is a lift of $N_{3,2}(2)$ with respect to the natural projection from $M^+_{\frac{3}{2}, -1}$ to $V(2)$. By Lemma 5.2 and Theorem 4.3, $w_1$ and $w_2$ generate the maximum proper submodule of $M^+_{\frac{3}{2}, -1}$. Then, by [KW94 Proposition 1.2] and the PBW theorem, the kernel
of the natural projection from $A_{id}(M_{\frac{\lambda}{\phi}}^{-})$ to $A_{id}(L_{\frac{\lambda}{\phi}}^{-})$ is generated by $f'_{1} := [G_{-\frac{1}{2}}^{+} G_{-\frac{1}{2}} w_{1}], f'_{2} := [w_{2}], f'_{3} := [G_{-\frac{1}{2}}^{+} G_{-\frac{1}{2}} w_{2}], g_{L} \psi := [G_{-\frac{1}{2}}^{+} w_{1}], g_{L}^{2} \psi := [G_{-\frac{1}{2}}^{+} w_{1}], g_{L}^{3} \psi := [G_{-\frac{1}{2}}^{+} w_{2}]$. By some computations, one can verify that $f_{i} \propto f'_{i}$ and $g_{L} \psi \propto g_{L}^{i} \psi$ for $i \in \{1, 2, 3\}$. This completes the proof. □

At last, by Lemma 5.12 and direct calculation, we obtain the following.

**Proposition 5.13.** For $\epsilon \in \{-1, 0, 1\}$ and $j \in \mathbb{C}$, we have

$$A_{id}(L_{\frac{\lambda}{\phi}, -1}) \otimes_{A_{id}(L_{-1})} \mathbb{C}(\epsilon) \simeq \begin{cases} \mathbb{C}(0) & \text{if } \epsilon = -1, \\
\mathbb{C}(1) & \text{if } \epsilon = 0, \\
\Pi \mathbb{C}_{\frac{\lambda}{\phi}} & \text{if } \epsilon = 1, \end{cases}$$

$$A_{id}(L_{\frac{\lambda}{\phi}, -1}) \otimes_{A_{id}(L_{-1})} \mathbb{C}_{j} \simeq \begin{cases} \mathbb{C}_{\frac{\lambda}{\phi}} \oplus \Pi \mathbb{C}(1) & \text{if } j = -\frac{1}{2}, \\
\mathbb{C}_{j + \frac{\lambda}{\phi}} & \text{if } j \neq -\frac{1}{2}. \end{cases}$$

**Remark 5.14.** In [Sat18 Example 5.2], the second author conjectures that the dimension of the space of $\mathbb{Z}_{2}$-graded intertwining operators (called the fusion rule) of type

$$\left( L_{-\frac{1}{2}(1+3j+\frac{\lambda}{\phi}), j, -1} \right)$$

is equal to 1 if $j \notin \left\{ \frac{1}{2} \right\} + \frac{\mathbb{Z}}{2}$. By a super analog of [Li99a Proposition 2.10] (e.g. [KW94 Theorem 1.5 (1)]) and Proposition 5.13, the conjectural fusion rule is actually bounded by

$$\dim_{\mathbb{C}} \text{Hom}_{A_{id}(L_{-1})} \left( A_{id}(L_{\frac{\lambda}{\phi}, -1}) \otimes_{A_{id}(L_{-1})} \mathbb{C}_{j}, \mathbb{C}_{j + \frac{\lambda}{\phi}} \oplus \Pi \mathbb{C}(1) \right) = 1.$$
where $L := L - 21^c$, $G^\pm := G^\pm 1^c$, and $J := J - 1^c$. In addition, the vertex superalgebra $(V_c, Y, 1^c)$ together with $L$ as a conformal vector forms a vertex operator superalgebra of central charge $c$.

When $c \neq 0$, we write $L_c$ for its simple quotient vertex operator superalgebra.

A.2. Spectral flow twists. In this subsection $V$ stands for $V_c$ or $L_c$. Let $\varepsilon, \varepsilon' \in \{0, \frac{1}{2}\}$ and let $(M, Y_M)$ be a $(1 - \varepsilon)\mathbb{Z}_{\geq 0}$-gradable $\sigma^{1-2\varepsilon}$-twisted $V$-module. The next lemma is proved in a similar way of [Li97, Proposition 3.2] (see [Xu98, Theorem 3.3.8] for detail).

**Lemma A.2.** For $\theta \in \mathbb{Z} + \varepsilon'$, we define

$$\Delta(\theta J; z) := z^{\theta J_0} \exp \left( \sum_{\ell=1}^{\infty} \frac{\theta J}{\ell} (-z)^{-\ell} \right) \in \text{End}(V)[[z^\pm (1-\varepsilon)]]$$

Then the mapping

$$Y_M(\theta J; z) := Y_M(\Delta(\theta J; z); z) : V \rightarrow \text{End}(M)[z^\pm (1-\varepsilon)]$$

gives rise to a $(1 - |\varepsilon - \varepsilon'|)\mathbb{Z}_{\geq 0}$-gradable $\mathbb{Z}^2$-graded $\sigma^{1-2|\varepsilon - \varepsilon'|}$-twisted $V$-module $M^\theta := (M, Y_M^\theta)$.

We call $M^\theta$ the spectral flow twisted module of $M$. Note that the spectral flow twisted module of a highest weight module is not a highest weight module in general.

Let $C^\varepsilon$ be the category of $(1 - \varepsilon)\mathbb{Z}_{\geq 0}$-gradable $\mathbb{Z}^2$-graded $\sigma^{1-2\varepsilon}$-twisted $V$-modules. Then it is clear that the assignments $M \mapsto M^\theta$ and $\text{Hom}(M_1, M_2) \rightarrow \text{Hom}(M_1^\theta, M_2^\theta) ; f \mapsto f$ give rise to an equivalence of categories $(\cdot)^\theta : C^\varepsilon \rightarrow C^{\varepsilon'}$ (cf. [Sat16, Lemma B.4]). Note that a similar equivalence holds also in the $\mathbb{Z}^2$-graded version.

**Appendix B. Zhu’s algebra and Frenkel–Zhu’s bimodule**

In this section, we first recall the definition of twisted Zhu’s algebra and Frenkel–Zhu’s bimodule originally introduced in [Zhu96] and [FZ92].

B.1. Definition. Let $V = V^0 \oplus V^1$ be a vertex operator superalgebra with the $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading

$$V^i = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} V_{\Delta} \ (i \in \{0, 1\})$$

with respect to the operator $L_0$. Let $\sigma$ denote the vertex operator superalgebra automorphism of $V$ defined by $\sigma|_{V^i} = (-1)^i \text{id}_{V^i}$ for $i \in \{0, 1\}$. Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}^2$-graded $g$-twisted $V$-module for $g \in \{\text{id}, \sigma\}$. For $A \in V_\Delta \cap V^i$ and $v \in M^j$,
we define

\[
A \ast v := \begin{cases} \\
\sum_{\ell=0}^{\infty} \left( \frac{\Delta}{\ell} \right) A(\ell-1) v & \text{if } g = \text{id}, \\
\sum_{\ell=0}^{\infty} \left( \frac{\Delta}{\ell} \right) A(\ell-1) v & \text{if } g = \sigma,
\end{cases}
\]

\[
v \ast A := \begin{cases} \\
\sum_{\ell=0}^{\infty} \left( - \frac{1}{\ell} \right) A(\ell-1) v & \text{if } g = \text{id}, \\
\sum_{\ell=0}^{\infty} \left( \frac{\Delta - 1}{\ell} \right) A(\ell-1) v & \text{if } g = \sigma,
\end{cases}
\]

\[
A \circ v := \begin{cases} \\
\sum_{\ell=0}^{\infty} \left( \frac{\Delta}{\ell} \right) A(\ell-2) v & \text{if } g = \text{id}, \\
\sum_{\ell=0}^{\infty} \left( \frac{\Delta}{\ell} \right) A(\ell-2) v & \text{if } g = \sigma
\end{cases}
\]

and extend them by linearity. We also define \(A_{V,g}(M) := M/O_{V,g}(M)\), where

\[
O_{V,g}(M) := \text{span}_{\mathbb{C}} \big\{ A \circ v \big| A \in V, v \in M \big\}.
\]

Then the next proposition is a natural generalization of \([Zhu96, \text{Theorem 2.1.1}]\) and \([FZ92, \text{Theorem 1.5.1}]\) (see e.g. \([Xu98]\)).

**Proposition B.1.** The bilinear mappings \(\ast : V \times M \to M\) and \(\ast : M \times V \to M\) give rise to the following structures:

1. a \(\mathbb{Z}_2\)-graded associative algebra structure on \(A_{V,g}(V)\),
2. a \(\mathbb{Z}_2\)-graded \(A_{V,g}(V)\)-bimodule structure on \(A_{V,g}(M)\).

Since \(I_c \circ M = \{0\}\) for any \(g\)-twisted \(L_c\)-module \(M\), we simply write \(A_{g}(M)\) and \(O_{g}(M)\) for \(A_{V,g}(M)\) and \(O_{V,g}(M)\), respectively. In this paper we use the notation \([A]\) for the image of \(A \in V\) under the natural projection from \(V\) to \(A_{g}(V)\).

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