Parameter free predictions within the proxy-SU(3) model

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Abstract.

Using a new approximate analytic parameter-free proxy-SU(3) scheme, we make predictions of shape observables for deformed nuclei, namely $\beta$ and $\gamma$ deformation variables, and compare them with empirical data and with predictions by relativistic and non-relativistic mean-field theories. Furthermore, analytic expressions are derived for $B(E2)$ ratios within the proxy-SU(3) model, free of any free parameters, and/or scaling factors. The predicted $B(E2)$ ratios are in good agreement with the experimental data for deformed rare earth nuclides.

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1 Introduction

The proxy-SU(3) model has been recently introduced in Refs. [1][2]. The approximations used in this scheme have been discussed and justified through a Nilsson calculation in Ref. [1], while in Ref. [2] the way to predict the $\beta$ and $\gamma$ deformation parameters for any nucleus, using as input only the proton number $Z$ and the neutron number $N$ of the nucleus, as well as the quantum numbers $\lambda$ and $\mu$ appearing in the SU(3) irreducible representation (irreps) characterizing this nucleus within the proxy-SU(3) scheme, has been described in detail.
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In Section 2 of the present paper we carry out in the rare earth region a detailed comparison of the proxy-SU(3) predictions to detailed results obtained with the D1S Gogny interaction, tabulated in Ref. [3], while in Section 3 we calculate \( B(E2) \) ratios within ground state bands and \( \gamma_1 \) bands of some deformed rare earth nuclei and compare them to the existing data [4]. In both sections, no free parameters are used.

2 Predictions for the deformation parameters

2.1 Connection between deformation variables and SU(3) quantum numbers

A connection between the collective variables \( \beta \) and \( \gamma \) of the collective model [5] and the quantum numbers \( \lambda \) and \( \mu \) characterizing the irreducible representation \( (\lambda, \mu) \) of SU(3) [6, 7] has long been established [8, 9], based on the fact that the invariant quantities of the two theories should possess the same values.

The relevant equation for \( \beta \) reads [8, 9]

\[
\beta^2 = \frac{4\pi}{5} \frac{1}{(Ar^2)^2} (\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu + 3),
\]

where \( A \) is the mass number of the nucleus and \( r^2 \) is related to the dimensionless mean square radius [10], \( \sqrt{r^2} = r_0 A^{1/6} \). The constant \( r_0 \) is determined from a fit over a wide range of nuclei [11, 12]. We use the value in Ref. [8], \( r_0 = 0.87 \), in agreement to Ref. [12]. The quantity in Eq. (1) is proportional to the second order Casimir operator of SU(3) [13],

\[
C_2(\lambda, \mu) = \frac{2}{3} (\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu).
\]

The relevant equation for \( \gamma \) reads [8, 9]

\[
\gamma = \arctan \left( \frac{\sqrt{3}(\mu + 1)}{2\lambda + \mu + 3} \right).
\]

2.2 Numerical results

In Fig. 1 (Fig. 2) results for the collective variable \( \beta \) (\( \gamma \)) are shown, calculated from Eq. (1) [Eq. (3)] and rescaled in the case of \( \beta \) as described in detail in Ref. [2]. Experimental results obtained from Ref. [15] are also shown for comparison, as in Ref. [2]. Furthermore, comparison to the detailed results provided by the D1S Gogny force, tabulated in Ref. [3], is made. By “Gogny D1S mean” we label the mean ground state \( \beta \) (\( \gamma \)) deformation [entry 11 (12) in the tables of [3]], while the error bars correspond to the variance of the ground state \( \beta \) (\( \gamma \)) deformation.
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deformation [entry 13 (14) in [3]]. By “Gogny D1S min.” we label the $\beta$ ($\gamma$) deformation at the HFB energy minimum [entry 4 (5) in [3]]. In the case of $\beta$, predictions obtained with relativistic mean field theory (RMF) [14] are also shown.

In the case of $\beta$ we note that the HFB minimum lies always within the error bars of the D1S Gogny mean g.s. deformation (except for $N = 84$), while in the case of $\gamma$ we see that the HFB minimum lies well below the error bars of the D1S Gogny mean g.s. deformation for most of the $N$ values, but jumps suddenly to very high values, close to 60 degrees, at or near $N = 116$.

In Fig. 1 the proxy-SU(3) predictions for $\beta$ lie within the error bars of the D1S Gogny mean g.s. deformation, with the following few exceptions: a) The first ($N = 84$) point in Gd-Hf, b) the last two ($N = 120, 122$) points in Gd and Dy, as well as the last point ($N = 122$) in Er, c) a few isolated cases, like the $N = 110$ point in Er, the $N = 102$ point in W and Os, and the $N = 100, 102$ points in Pt. We stress, however, that proxy-SU(3) is only valid for deformed nuclei and therefore some of these differences [items a) and b)] may not be meaningful.

Similar observations can be made for $\gamma$ in Fig. 2, where the exceptions occur in: a) The first three points ($N = 84, 86, 88$) in Gd-Yb, b) the last three ($N = 118, 120, 122$) points in W-Pt, which agree with the HFB minimum rather than with the mean g.s. deformation, c) a few isolated cases, like some of the $N = 106$ in Gd, Dy, Er, Hf, W, and several points in Yb.

2.3 Discussion

The above observations can be summarized as follows:

1) While the $\beta$ deformation at the HFB energy minimum remains always close to the mean ground state $\beta$ deformation, the behavior of the $\gamma$ deformation is strikingly different. In most of the region the $\gamma$ deformation at the HFB energy minimum remains close to zero, but it suddenly jumps to values close to 60 degrees near the end of the shell ($N = 116-122$). This jump is sudden in Gd-Hf, while it becomes more gradual in W, Os, Pt.

2) In the beginning of the region we see some failures of proxy-SU(3) at $N = 84, 86, 88$ in Gd-Hf. These failures are expected, since these nuclei are not well deformed, as known from their $R_{4/2}$ ratios.

3) In most of the region, the proxy-SU(3) predictions for both $\beta$ and $\gamma$ are in good agreement with the D1S Gogny mean g.s. deformations.

4) The agreement of the proxy-SU(3) predictions with the D1S Gogny mean g.s. deformations remains good up to the end of the shell for $\beta$, while for $\gamma$ in W, Os, Pt it is observed that the proxy-SU(3) predictions for $\gamma$ jump at the end of the shell from close agreement to the D1S Gogny mean g.s. deformations to close agreement with $\gamma$ at the HFB energy minimum, i.e., close to 60 degrees.
3 B(E2) ratios

As discussed in Appendix A, $B(E2)$s within the proxy-SU(3) model are proportional to the square of the relevant reduced matrix element of the quadrupole operator $Q$. If ratios of $B(E2)$s within the same nucleus and within the same irreducible representation are considered, only the relevant SU(3)$\rightarrow$SO(3) coupling coefficients remain, while all other factors cancel out, leading to

$$
\frac{B(E2; L_i \rightarrow L_f)}{B(E2; 2g \rightarrow 0g)} = 5 \frac{2L_f + 1}{2L_i + 1} \frac{((\lambda, \mu)K_iL_i; (1, 1)2||((\lambda, \mu)K_fL_f))}{((\lambda, \mu)02; (1, 1)2||((\lambda, \mu)00))}^2, \quad (4)
$$

where normalization to the $B(E2)$ connecting the first excited $2^+$ state to the $0^+$ ground state of even-even nuclei is made. The needed SU(3)$\rightarrow$SO(3) coupling coefficients are readily obtained from the SU3CGVCS code [16], as described in Appendix A.

It should be noticed that the ratios given by Eq. (4) are completely free of any free parameters and/or scaling factors.

3.1 Numerical results

Calculations have been performed for the proxy-SU(3) irreps (54,12), (52,14), and (50,10). The irrep (54,12) accommodates $^{168}$Er, for which complete spectroscopy has been performed [17], and $^{160}$Gd, for which little data on $B(E2)$s exist [4]. The irrep (52,14) accommodates $^{162}$Dy, for which complete spectroscopy has been performed [18], and $^{166}$Er, for which rich data exist [4]. It also accommodates $^{172}$Er, for which little data on $B(E2)$s exist [4]. The irrep (50,10) accommodates $^{156}$Gd, which has been cited as the textbook example of the bosonic SU(3) in the IBM-1 framework [13]. The Alaga values [19], derived from the relevant Clebsch-Gordan coefficients alone, are also given for comparison.

3.2 Comparisons to experimental data for specific nuclei

$B(E2)$s within the ground state band are shown in Fig. 3. Agreement between the proxy-SU(3) predictions and the data is excellent in the cases of $^{156}$Gd, $^{162}$Dy, and $^{166}$Er, while in $^{168}$Er three points are missed. It appears that nuclear stretching [20] has been properly taken into account.

In Fig. 4 three pairs of nuclei, each pair accommodated within a single proxy-SU(3) irrep, are shown. These are the only pairs for which adequate data exist [4] in the region of 50-82 protons and 82-126 neutrons. Agreement within the experimental errors is seen in almost all cases.

Proxy-SU(3) predictions for $B(E2)$s within the $\gamma_1$ band, with $\Delta L = -2$ (increasing with $L$) and $\Delta L = -1$ (decreasing with $L$), are shown in Fig. 5, and
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are compared to the data for nuclei for which sufficient data exist \cite{4}. The distinction between increasing $B(E2)$s with $\Delta L = -2$ and decreasing $B(E2)$s with $\Delta L = -1$ is seen clearly in the data.

3.3 Discussion

The main findings of the present section can be summarized as follows.

Analytic expressions for $B(E2)$ ratios for heavy deformed nuclei providing numerical results in good agreement with experiment are derived within the proxy-SU(3) scheme without using any free parameters and/or scaling factors. The derivation, described in Appendix A, is exact. The only quantities appearing in the final formula are the relevant SU(3)$\rightarrow$SO(3) coupling coefficients, for which computer codes are readily available \cite{16,21}.

Concerning further work, spectra of heavy deformed nuclei will be considered within the proxy-SU(3) scheme, involving three- and/or four-body terms in order to break the degeneracy between the ground state and $\gamma_1$ bands \cite{22–24}. Furthermore, $B(M1)$ transition rates can be considered along the proxy-SU(3) path, using the techniques already developed \cite{25} in the framework of the pseudo-SU(3) scheme.

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Appendix A. Formulae used for $B(E2)$s

In most of the earlier work, effective charges

$$e_\pi = e + e_{eff}, \quad e_\nu = e_{eff}, \quad (5)$$

have been used, where the effective charge $e_{eff}$ is usually fixed so that the calculated $B(E2)$ transition rate for the $2^+_1 \rightarrow 0^+_1$ transition reproduces the experimental value \cite{25}. In the present approach we make the choice $e_{eff} = 0$, which leads to $e_\pi = e$ and $e_\nu = 0$.

The needed matrix elements of the relevant quadrupole operators, $Q^\pi$ and $Q^\nu$ for protons and neutrons respectively, are given in detail in Appendix D of Ref. \cite{23}, with SU(3)$\rightarrow$SO(3) coupling coefficients \cite{16,21,26,27}, as well as 9-$(\lambda, \mu)$ coefficients \cite{21,26,28} appearing in the relevant expressions. Codes for calculating
these coefficients are readily available, given in the references just cited. With $e_{eff} = 0$ one sees that only the matrix elements of $Q^\pi$ are needed. Furthermore, if we use ratios of $B(E2)$ transition rates within a given nucleus, the $9\,\left(\lambda, \mu\right)$ coefficients will cancel out and the only nontrivial term remaining in the $B(E2)$ ratios will be the ratio of the relevant SU(3)$\rightarrow$SO(3) coupling coefficients, which remarkably involve only the highest weight $(\lambda, \mu)$ irrep characterizing the whole nucleus, while they are independent of the $(\lambda_\pi, \mu_\pi)$ and $(\lambda_\nu, \mu_\nu)$ irreps characterizing the protons and the neutrons separately.

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Figure 1. Proxy SU(3) predictions for Gd-Pt isotopes for $\beta$, obtained from Eq. (1), as described in detail in Ref. [2], compared with results by the D1S-Gogny interaction (D1S-Gogny) [3] and by relativistic mean field theory (RMF) [14], as well as with empirical values (exp.) [15]. See subsection 2.2 for further discussion.
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Figure 2. Same as Fig. 1, but for $\gamma$, derived from Eq. (3). See subsection 2.2 for further discussion.
Figure 3. B(E2) values within the ground state band are shown for the indicated proxy-SU(3) irreps and for four nuclei, with data taken from [4]. All values are normalized to $B(E2; 2_1^+ \rightarrow 0_1^+)$. Results for (54,12) are not shown in the upper left panel, because for this band they are very similar to those of (52,14). See subsection 3.2 for further discussion.
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Figure 4. Experimental values of $B(E2)$s within the ground state bands of three pairs of nuclei, each pair accommodated within the same proxy-SU(3) irrep. Data are taken from [4]. All values are normalized to $B(E2; 2^+_1 \rightarrow 0^+_1)$. See subsection 3.2 for further discussion.
Figure 5. B(E2)s within the $\gamma_1$ band are shown for the indicated proxy-SU(3) irreps and for two nuclei, for which sufficient data exist \[4\]. All values are normalized to $B(E2; 2^+ \rightarrow 0^+)$.

See subsection 3.2 for further discussion.