ERGODIC EXTENSIONS AND HILBERT MODULES ASSOCIATED TO ENDMORPHISMS OF MASAS

EVGENIOS T.A. KAKARIADIS AND JUSTIN R. PETERS

ABSTRACT. We show that a class of ergodic transformations on a probability measure space \((X, \mu)\) extends to a representation of \(\mathcal{B}(L^2(X, \mu))\) that is both implemented by a Cuntz family and ergodic. This class contains several known examples, which are unified in this work.

During the analysis of the existence and uniqueness of such a Cuntz family we give several results of individual interest. Most notably we prove a decomposition of \(X\) for \(N\)-to-one local homeomorphisms that is connected to the orthonormal basis of Hilbert modules. We remark that the trivial Hilbert module of the Cuntz algebra \(\mathcal{O}_N\) does not have a well-defined Hilbert module basis (moreover that it is unitarily equivalent to the module sum \(\sum_{n=1}^{\infty} \mathcal{O}_N\) for infinitely many \(n \in \mathbb{N}\)).

1. Introduction

In this paper we continue with the examination of representations of dynamical systems implemented by Cuntz families [6]. We are strongly motivated by the recent work and aspect of Courtney, Muhly and Schmidt [2] in which the general theory of Hilbert modules is used as an alternate route to examine specific examples, in association with earlier work of Laca [7] where ergodic transformations of \(\mathcal{B}(H)\) in general are examined. In contrast to our previous work [6], which is directed to the abstract operator algebraic point of view, here we analyze a class of particular transformations \(\varphi: X \to X\) of a probability measure space \((X, \mu)\).

There are several well known examples (including the backward shift on infinite words on \(N\) symbols, and finite Blaschke products with \(N\) factors) where such transformations yield a representation \(\alpha: L^\infty(X, \mu) \to L^\infty(X, \mu)\) that is implemented by a Cuntz family. That is, there is a Cuntz family \(\{S_i\}_{i=1}^{N}\) in \(\mathcal{B}(L^2(X, \mu))\) such that

\[
\alpha(M_f) = \sum_{i=1}^{N} S_i M_f S_i^*,
\]

2010 Mathematics Subject Classification. 47C15, 37A55, 46L55, 46C10.

Key words and phrases. Maximal abelian selfadjoint algebras, ergodic transformations, Hilbert modules.

The second author acknowledges partial support from the National Science Foundation, DMS-0750986.
where \( M_f \in \mathcal{B}(L^2(X, \mu)) \) is the multiplication operator associated to \( f \in L^\infty(X, \mu) \), and therefore \( \alpha \) extends to a representation \( \alpha_S \) of \( \mathcal{B}(L^2(X, \mu)) \).

Our goal here is two-fold. First we give conditions under which the transformation \( \varphi : X \to X \) defines such a representation \( \alpha : L^\infty(X, \mu) \to L^\infty(X, \mu) \) (Proposition 2.2). Secondly we show that ergodicity of \( \varphi : X \to X \) (as a transformation of a probability measure space) implies ergodicity of the induced \( \alpha_S : \mathcal{B}(L^2(X, \mu)) \to \mathcal{B}(L^2(X, \mu)) \) (as a representation of a von Neumann algebra) (Theorem 3.2).

The existence of a Cuntz family implementing \( \alpha : L^\infty(X, \mu) \to L^\infty(X, \mu) \) is connected to a decomposition of the space \( X \) based on a maximal family of sets (Lemma 3.1). There is a question whether different decompositions yield the same extension. We show that the answer to this question is connected to the existence of an orthonormal basis of a suitable W*-module (Proposition 4.5). As a consequence we obtain a complete invariant on Stacey’s multiplicity \( n \) crossed products \([11]\) (Corollary 4.6).

A useful tool for the study of the endomorphism \( \alpha : L^\infty(X, \mu) \to L^\infty(X, \mu) \) is the intertwining Hilbert module \( \mathcal{E}(X, \mu) \) introduced in section 4. Under certain conditions on the transformation \( \varphi : X \to X \) there is a transfer operator; our setting encompasses several cases, including that described in \([2, \text{Theorem 5.2}]\). We conclude by showing that the existence of a basis for the Hilbert module \( \mathcal{E}(X, \mu) \) is equivalent to the existence of a Cuntz family implementing \( \alpha : L^\infty(X, \mu) \to L^\infty(X, \mu) \), and in turn is equivalent to the existence of a basis for \( L^\infty(X, \mu) \) viewed as a Hilbert module, where the inner product is defined by the transfer operator (Theorem 5.2).

Hilbert modules may not have a well defined (up to unitary equivalence) basis, in contrast to Hilbert spaces. Therefore it is central for (and non-trivial in) our analysis to achieve a well defined basis. For example \( \mathcal{O}_2 \) is unitarily equivalent to \( \sum_{k=1}^n \mathcal{O}_2 \) for all \( n \in \mathbb{N} \), as Hilbert modules over \( \mathcal{O}_2 \) (Remark 4.2). This phenomenon is also connected to the multiplicity of multivariable C*-dynamics \([5]\) and produces a fascinating obstacle (so far) for the classification of these objects. For encountering this problem, Gipson \([4]\) develops the notion of the invariant basis number for C*-algebras, along with an in-depth analysis of C*-algebras that do (or do not) attain such a number.

2. Preliminaries

Let us begin with a general comment on *-endomorphisms \( \alpha_S \) of \( \mathcal{B}(H) \) that are implemented by a Cuntz family \( \{S_1, \ldots, S_N\} \), i.e.,

\[
\alpha_S(T) = \sum_{i=1}^N S_i TS_i^*, \quad \text{for all } T \in \mathcal{B}(H).
\]

We write \( \mathcal{O}_N = C^*(S_1, \ldots, S_N) \) for the Cuntz algebra \([1]\) inside \( \mathcal{B}(H) \). Both \( \alpha_S \) and the restriction \( \alpha_S|_{\mathcal{O}_N} \) of \( \alpha_S \) to \( \mathcal{O}_N \) are injective, but they are not
onto for \( N > 1 \). Indeed, if there is a \( T \in \mathcal{B}(H) \) such that \( \alpha_S(T) = 0 \), then
\[
T = S_1^*S_1TS_1^*S_1 = S_1^*\alpha_S(T)S_1 = 0.
\]
Furthermore if there is a \( T \in \mathcal{O}_N \) such that \( \alpha_S(T) = S_1 \), then
\[
I = S_1^*S_1 = S_1^*\alpha_S(T) = TS_1^*,
\]
hence \( S_1 \) is a unitary, which holds if and only if \( N = 1 \).

Let \( (X, \mu) \) and \( (Y, \nu) \) be compact, Hausdorff measure spaces, endowed with their Borel structure. Then a continuous map \( \varphi: (X, \mu) \to (Y, \nu) \) is a Borel homomorphism. However the mapping
\[
\alpha: (L^\infty(Y, \nu), \| \cdot \|_\infty) \to (L^\infty(X, \mu), \| \cdot \|_\infty) : f \mapsto f \circ \varphi
\]
where \( \| \cdot \|_\infty \) is the essential sup-norm, may not even be well defined. In particular one can show that \( \alpha \) is well-defined if and only if \( \mu \circ \varphi^{-1} \ll \nu \) (i.e., \( \varphi^{-1} \) preserves the \( \nu \)-null sets). When \( \varphi(Y) \) is in addition a Borel set, then \( \alpha \) is well-defined and injective if and only if \( \nu(\varphi(Y)^c) = 0 \) (i.e., \( \varphi \) is almost onto \( X \)) and \( \mu \circ \varphi^{-1} \sim \nu \).

In general a Borel map \( \varphi: X \to Y \) is said to preserve the \( \nu \)-null sets if \( \nu \circ \varphi \ll \mu \). In this case \( \nu \ll \mu \circ \varphi^{-1} \) and \( \varphi(E) \) is Borel for every Borel subset \( E \) of \( X \). Indeed for the latter, observe that a Borel subset \( E \) of \( X \) is the union of an \( F_\sigma \) set \( A \) and a \( \mu \)-null set \( N \). Then \( \varphi(N) \) is a \( \nu \)-null set and compactness of \( X \) implies that \( A \) is \( \sigma \)-compact hence \( \varphi(A) \) is Borel; thus \( \varphi(E) \) is measurable.

Recall that if \( \varphi: X \to Y \) is a Borel map, then a mapping \( \psi: \varphi(X) \to X \) is called a Borel (cross) section of \( \varphi \) if \( \psi \) is a Borel map and \( \varphi \circ \psi = \text{id}_{\varphi(X)} \).

**Proposition 2.1.** Let \( \varphi: X \to Y \) be an onto map, such that \( \varphi \) and \( \varphi^{-1} \) preserve the null sets, and let \( \psi: Y \to X \) be a Borel section of \( \varphi \). Then \( X_0 := \psi(Y) \) is Borel and there is an isometry \( S: L^2(Y, \nu) \to L^2(X, \mu) \) such that
\[
M_{f \circ \varphi}|_{L^2(X_0, \nu|_{X_0})} = SM_fS^* \text{ for all } f \in L^\infty(Y, \nu).
\]

**Proof.** Observe that \( \psi \) preserves the null sets (which implies that \( X_0 \) is Borel). Since \( \mu \circ \varphi^{-1} \ll \nu \) then \( \mu \ll \nu \circ \varphi \). For a null set \( E \subseteq Y \) we get that \( \nu \circ \varphi(\psi(E)) = \nu(E) = 0 \), thus \( \mu \circ \psi(E) = 0 \). Note that since \( X_0 \) is Borel then \( \varphi|_{X_0} \) is a Borel isomorphism with \( \psi: Y \to X_0 \) as an inverse.

On the other hand if \( \mu \circ \psi(E) = 0 \) then \( \nu(E) = \nu \circ \varphi(\psi(E)) = 0 \) since \( \varphi \) preserves the null sets. Therefore \( \nu \) is equivalent to \( \mu \circ \psi = \mu|_{X_0} \circ \psi \), and the Radon-Nykodim derivative \( u = d(\mu|_{X_0} \circ \psi)/d\nu \) is defined. It is a standard fact that the operator \( S_0^* : L^2(X_0, \mu|_{X_0}) \to L^2(Y, \nu) \) defined by
\[
S_0^*(g) = g \circ \psi \cdot u^{1/2}, \text{ for all } g \in L^2(X_0, \mu|_{X_0}),
\]
is a unitary such that
\[
M_{f \circ \varphi}|_{L^2(X_0, \nu|_{X_0})} = S_0 M_f S_0^* \text{ for all } f \in L^\infty(Y, \nu).
\]
Extend \( S_0^* \) trivially to \( S^* \) on \( L^2(X, \mu) = L^2(X_0, \mu|_{X_0}) \oplus L^2(X^c, \mu|_{X_0^c}) \). Then the adjoint \( S^* \) of \( S^* \) is an isometry and gives the required equation. 

\[ \blacksquare \]
**Proposition 2.2.** Let \( \varphi : X \to Y \) be an onto map, such that \( \varphi \) and \( \varphi^{-1} \) preserve the null sets. Suppose that there is a family \( \{\psi_1, \ldots, \psi_N\} \) of \( N \) Borel sections of \( \varphi \) such that \( \psi_i(Y) \cap \psi_j(Y) = \emptyset \) for \( i \neq j \), and \( \cup_i \psi(Y) \) is almost equal to \( X \). Then there is a Cuntz family that implements \( \alpha \).

**Proof.** For every \( i = 1, \ldots, N \) let \( X_i = \psi_i(Y) \) and let \( S_i \) be constructed as above on \( X_i = L^2(X_i, \mu|\mathcal{X}_i) \). Note that \( X_i \perp X_j \) for \( i \neq j \), thus if \( X_0 = \cup_i X_i \) and \( x_0 = \oplus_i X_i \) we get \( M_1|x_i = \alpha(1)|x_i = S_i S_i^* = S_i^* S_i \), therefore

\[
M_{\alpha(f)}|x_0 = \sum_{i=1}^{N} M_{\alpha(f)}|x_i = \sum_{i=1}^{N} S_i M_i S_i^*.
\]

Since \( X = \cup_i X_i \) a.e. we obtain that \( I_{x_0} = M_1|x_0 = \sum_{i=1}^{N} M_1|x_i = \sum_{i=1}^{N} S_i S_i^* \). Finally \( x_0 \) is almost equal to \( X \), hence \( L^2(X, \nu) = x_0 \) and the proof is complete.

**3. Ergodic Extensions**

Let \((X, \mu)\) be a probability measure space such that \( X \) is a compact, Hausdorff space and \( \mu \) is a regular Borel measure on \( X \). Then a measure preserving map \( \varphi : X \to X \) induces an injective \(*\)-homomorphism \( \alpha : L^\infty(X, \mu) \to L^\infty(X, \mu) \). We are interested in the case where \( \alpha \) is implemented by a Cuntz family \( \{S_i\}_{i=1}^{N} \) in \( B(L^2(X, \mu)) \). In this case \( \alpha \) extends to an injective \(*\)-endomorphism \( \alpha_S \) of \( B(L^2(X, \mu)) \). A natural question is whether ergodicity (of the mapping) \( \varphi \) implies ergodicity (of the \(*\)-endomorphism) \( \alpha_S \). Recall that \( \alpha_S \) is **ergodic** if the von Neumann algebra \( \mathcal{N}_{\alpha_S} := \{ T \in B(H) \mid \alpha_S(T) = T \} \) is trivial. We aim to give a positive answer for a class of ergodic mappings that includes central examples.

Recall that a map \( \varphi : X \to X \) is called a **local homeomorphism** if for every point \( x \in X \) there is a neighborhood \( U \) such that \( \varphi|_U \) is a homeomorphism onto its image. Clearly, local homeomorphisms are continuous and open. We begin with a decomposition lemma that fits in our study.

**Lemma 3.1.** Let \( \varphi \) be a local homeomorphism of a compact Hausdorff space \( X \) such that \( |\varphi^{-1}(x)| = N > 1 \) for all \( x \in X \). Then there exist pairwise disjoint open subsets \( U_1, \ldots, U_N \) such that

(1) \( \varphi|_{U_i} \) is one-to-one for all \( i = 1, \ldots, n \);
(2) \( \varphi(U_i) = \varphi(U_j) \) for all \( i, j = 1, \ldots, N \);
(3) \( X = \cup_{i=1}^{N} (U_i \cup \partial U_i) \);
(4) \( X = \varphi(U_i) \cup \partial \varphi(U_i) \) for all \( i = 1, \ldots, N \).

Moreover, \( \varphi(\partial U_i) \subseteq \partial \varphi(U_i) \) and \( \varphi^{-1}(\partial \varphi(U_i)) = \cup_{j=1}^{N} \partial U_j \), for all \( i = 1, \ldots, N \).

**Proof.** First let us construct a family that satisfies (1) and (2). Let \( \mathcal{F} \) be the collection that consists of \( \{U_1, \ldots, U_N\} \) such that \( U_i \) are open, disjoint, \( \varphi|_{U_i} \) is one-to-one for all \( i = 1, \ldots, N \), and \( \varphi(U_i) = \varphi(U_j) \).

**Claim.** The collection \( \mathcal{F} \) is non-empty.
Proof of Claim. Let a \( y \in X \) and suppose that \( x_1, \ldots, x_N \) are the \( N \) pre-images of \( y \). Let \( V_i \) be a neighborhood of \( x_i \) such that \( \varphi|_{V_i} \) is one-to-one. Since \( X \) is a Hausdorff space we can choose \( V_i \) be disjoint. Moreover \( \varphi(U_i) \) are open sets, since \( \varphi \) is an open map. Let \( V = \cap_{i=1}^N \varphi(V_i) \) which is open and let \( U_i = \varphi^{-1}(V) \cap V_i \). Then the \( U_i \) are disjoint and \( \varphi(U_i) \) is one-to-one, since the \( U_i \) are subsets of the \( V_i \). In addition

\[
\varphi(U_i) = \varphi \circ \varphi^{-1}(V) \cap \varphi(V_i) = V \cap \varphi(V_i) = V,
\]

and the proof of the claim is complete.

The collection \( \mathcal{F} \) is endowed with the partial order "\( \leq \)" such that

\[
\{U_1, \ldots, U_N\} \leq \{V_1, \ldots, V_N\} \text{ if } U_i \subseteq V_i, \text{ for all } i = 1, \ldots, N,
\]

after perhaps a re-ordering. Let \( \mathcal{C} = \{\{U^k_1, \ldots, U^k_N\} \mid k \in I\} \), be a chain in \( \mathcal{F} \), with the understanding that when \( \{U^k_1, \ldots, U^k_N\} \leq \{U^l_1, \ldots, U^l_N\} \) then \( U^k_i \subseteq U^l_i \) for all \( i = 1, \ldots, N \). Then the element \( \{\cup_k U^k_1, \ldots, \cup_k U^k_N\} \) is an upper bound for \( \mathcal{C} \) inside \( \mathcal{F} \). Indeed, what suffices to prove is that the \( \cup_k U^k \) are disjoint (with respect to the indices \( i \)). If there were an \( x \) in two such unions then there would be some \( k, l \in I \) such that \( x \in U^k_i \cap U^l_i \). Without loss of generality assume that \( \{U^k_1, \ldots, U^k_N\} \leq \{U^l_1, \ldots, U^l_N\} \) therefore \( x \in U^k_i \cap U^l_i \subseteq U^l_i \cap U^l_j = \emptyset \) which is absurd. Then the collection \( \mathcal{F} \) has a maximal element by Zorn’s Lemma. From now on fix this maximal element to be \( \{U_1, \ldots, U_N\} \). By definition the sets \( U_1, \ldots, U_N \) satisfy the properties (1) and (2) of the statement.

Secondly we prove that \( X = \varphi(U_i) \cup \partial \varphi(U_i) \), for all \( i = 1, \ldots, N \), where \( \{U_1, \ldots, U_N\} \) is the maximal family constructed above. Since \( X \setminus \varphi(U_i) \) is closed it suffices to show that it has empty interior. To this end let \( V \) be an open neighborhood of some \( y \in \text{int}(X \setminus \varphi(U_i)) \) with \( N \) pre-images \( x_1, \ldots, x_N \). Then \( \varphi^{-1}(V) \) is open, contains the \( x_i \) and \( \varphi^{-1}(V) \cap (\cup_{i=1}^N U_i) = \emptyset \). Indeed, if there was a \( z \in \varphi^{-1}(V) \cap (\cup_{i=1}^N U_i) \), then \( \varphi(z) = V \cap \varphi(U_i) = \emptyset \), which is absurd. As in the proof of the claim above, we can find neighborhoods \( V_i \) of \( x_i \) inside \( \varphi^{-1}(V) \) such that they are disjoint, \( \varphi|_{V_i} \) is one-to-one and \( \varphi(V_i) = V \), perhaps by passing to a sub-neighborhood of \( y \). Therefore the family \( \{U_1 \cup V_1, \ldots, U_N \cup V_N\} \) is in \( \mathcal{F} \), which contradicts to the maximality of \( \{U_1, \ldots, U_N\} \).

Thirdly, we show that \( X = \cup_{i=1}^N (U_i \cup \partial U_i) \). It suffices to show that the closed set \( X \setminus \cup_{i=1}^N (U_i \cup \partial U_i) \) has empty interior. Indeed, in this case it will coincide with its boundary, hence with \( \partial (\cup_{i=1}^N U_i) \). Since the \( U_i \) are open and disjoint we get that this boundary will be \( \cup_{i=1}^N \partial(U_i) \). To this end, let \( U \) be an open neighborhood of an element \( x \) in the interior of \( X \setminus (\cup_{i=1}^N (U_i \cup \partial U_i)) \). If there were an \( x' \in U \) such that \( \varphi(x') \in \varphi(U_i) \), then \( \varphi(x') \) would have \( N+1 \) pre-images which is a contradiction. Indeed, recall that \( \varphi(U_i) = \varphi(U_j) \) and \( U_i \cap U_j = \emptyset \). Therefore \( \varphi(U) \) is contained in the interior of \( X \setminus \varphi(U_i) \). But \( X \setminus \varphi(U_i) \) has empty interior, which gives the contradiction.
Finally, let $x \in \partial U_i$. If $\varphi(x) \in \varphi(U_i)$ then the element $\varphi(x)$ would have $N + 1$ pre-images, which is a contradiction. Therefore $\varphi(\partial U_i) \subseteq X \setminus \varphi(U_i) = \partial \varphi(U_i)$. Note also that by construction we obtain

$$\varphi^{-1}(\partial \varphi(U_i)) = \varphi^{-1}(X \setminus \varphi(U_i)) = X \setminus \bigcup_{j=1}^{N} U_j = \bigcup_{j=1}^{N} U_j,$$

for all $i = 1, \ldots, N$, and the proof of the lemma is complete.

Let $\varphi$ be as in Lemma 3.1 such that $\varphi$ and $\varphi^{-1}$ preserve the null sets. If $\{U_i\}_{i=1}^{N}$ is the family satisfying the properties of Lemma 3.1 and $\partial U_i$ (equivalently $\partial \varphi(U_i)$) are null sets then the $*$-endomorphism $\alpha: L^\infty(X, \mu) \to L^\infty(X, \mu): f \mapsto f \circ \varphi$ is implemented by a Cuntz family. Indeed, let $X_0 = \bigcup_{i=1}^{N} U_i$ and $Y_0 = \varphi(U_i)$. Then $\varphi_0 := \varphi|_{X_0}$ has $N$ Borel sections $\psi_i$, for $i = 1, \ldots, N$, with

$$\psi_i = [\varphi|_{U_i}]^{-1}: Y_0 \to X_0.$$

Moreover $\varphi_0$ and $\varphi_0^{-1}$ preserve the null sets. By Proposition 2.2 there is a Cuntz family $\{S_i\}_{i=1}^{N}$ with

$$S_i: L^2(Y_0, \mu|_{Y_0}) \to L^2(X_0, \mu|_{X_0})$$

that implements the representation $L^\infty(X_0, \mu|_{X_0}) \ni f \mapsto f \circ \varphi_0 \in L^\infty(Y_0, \mu|_{Y_0})$.

Since $X_0$ and $Y_0$ are almost equal to $X$ then the family $\{S_i\}_{i=1}^{N}$ implements $\alpha$.

Given a decomposition of $X$ as above and a finite word $i = i_1 \ldots i_k$ in $\{1, \ldots, N\}$ we can define the Borel sets

$$U_{i_1 i_2 \ldots i_k} = \{x \in X \mid x \in U_{i_1}, \ldots, \varphi^{k-1} \in U_{i_k}\}.$$

This is extended to infinite words $i = i_1 i_2 \ldots i_k \ldots$ with the understanding that $U_i = \bigotimes_{k=1}^{\infty} U_{i_1 \ldots i_k}$.

**Theorem 3.2.** Let $(X, \mu, \varphi)$ be a dynamical system such that:

1. $\varphi$ is a local homeomorphism of $X$ such that each point of $X$ has $N > 1$ pre-images;
2. $\{U_i\}_{i=1}^{N}$ is a decomposition of $X$ as in Lemma 3.1 such that the $\partial U_i$ are null sets;
3. $\varphi$ is ergodic and preserves the null sets;
4. the sets $U_i$, for $i \in \{i_1 i_2 \ldots i_k \}$ generate the $\sigma$-algebra up to sets of measure zero.

Then $\alpha: M_f \to M_{f \circ \varphi}$ admits an extension $\alpha_S$ to $B(L^2(X, m))$ which is ergodic. Furthermore $\alpha_S$ defines (by restriction) an irreducible representation of $O_N$. 
Proof. Under these assumptions there is a Cuntz family \( \{ S_i \}_{i=1}^{N} \) that implements \( \alpha \). Let \( \alpha_S(T) = \sum_{i=1}^{N} S_i T S_i^* \) be the extension of \( \alpha \) to \( \mathcal{B}(L^2(X, \mu)) \). Since \( \alpha_S \) is a weak*-continuous endomorphism of \( \mathcal{B}(L^2(X, \mu)) \), then \( \mathcal{N}_{\alpha_S} = \{ T \in \mathcal{B}(\mathcal{H}) \mid \alpha_S(T) = T \} \) is a von Neumann algebra. Fix a projection \( P \in \mathcal{N}_{\alpha_S} \). Then \( \alpha_S(P) = P \) implies that \( S_i P = PS_i \), and \( S_i^* P = PS_i^* \), for all \( i = 1, \ldots, N \). In particular \( P \) commutes with the range projections of the \( S_i \) and the products of the \( S_i \). But these projections are the characteristic functions of the sets \( U_i \), for the words \( i \) on the symbols \( \{1, \ldots, N\} \). Since the sets \( U_i \) generate the \( \sigma \)-algebra up to null sets, the linear span of these projections is weak*-dense in \( L^\infty(X, \mu) \). It follows that \( P \) is in the MASA \( L^\infty(X, \mu) \), hence \( P = \chi_E \) for a measurable set \( E \). However
\[
M_{X, E} = P = \alpha_S(P) = \alpha(P) = M_{X, E} \circ \varphi
\]
and ergodicity of \( \varphi \) implies that \( E \) is either \( X \) or \( \emptyset \). Thus \( \mathcal{N}_{\alpha_S} = CI \). The second part of the theorem follows by the comments after [7, Definition 3.2].

We give examples of dynamical systems which satisfy the conditions of Theorem 3.2.

Examples 3.3. The first example is the canonical Cuntz-Krieger example of a dynamical system associated with Cuntz isometries. Let \( N \in \mathbb{N} \) and
\[
X = \prod_{k=1}^{N} \{1, \ldots, N\}_k
\]
with measure \( \mu = \prod_{k=1}^{N} \mu_k \)
where each \( \mu_k = \mu_j \) for all \( j, k \) such that \( \mu_k(A) = |A|/N \) for all \( A \subset \{1, \ldots, N\} \). If we consider \( X \) as a compact abelian group, with “odometer” addition, then \( \mu \) is the Haar measure on \( X \). Let \( \varphi \) be the shift map \( \varphi(i_1, i_2, \ldots) = (i_2, i_3, \ldots) \) which is a \( N \)-to-one local homeomorphism. Then \( \varphi \) is ergodic and the conditions of the theorem are satisfied for the cylinder sets \( U_i := \{(i_1, i_2, \ldots) \mid i_1 = i\} \) (which are clopen so that \( \partial U_i = \emptyset \)).

A second example arises when \( X \) is the circle \( \mathbb{T} \), \( \mu \) is Lebesgue measure, and \( \varphi \) is a finite Blaschke product with \( N > 1 \) factors and zero Denjoy-Wolf fixed point (i.e., at least one of the Blaschke factors is \( z \)). Then \( \varphi \) is ergodic and the sets \( U_i \) are arcs on the circle, so the condition \( \mu(\partial U_i) = 0 \) is satisfied. This example is considered in [2].

In view of Theorem 3.2 one can ask whether the \( \sigma \)-algebra generated by the sets \( U_i \) with \( i \in \mathbb{F}_N^+ \) always generates the full \( \sigma \)-algebra of measurable sets, up to measure zero. This is not true, as the following example shows.

Example 3.4. Let \( (X, \mu, \varphi) \) be the canonical Cuntz-Krieger example as above. Also let \( \tau \) be an irrational rotation on the circle \( \mathbb{T} \) with Lebesgue measure. Set \( Y = X \times \mathbb{T} \), \( \sigma(x, z) = (\varphi(x), \tau(z)) \) and \( \nu = \mu \times \lambda \). Then \( (Y, \nu, \sigma) \) is ergodic as the product of the mixing shift map with the ergodic irrational rotation. If \( U_i = \{(i_1, i_2, \ldots) \in X \mid i_1 = i\} \) let \( V_i = U_i \times [0, 1] \). Then the \( V_i \) are as in Lemma 3.1, but the \( V_i \) with \( i \in \mathbb{F}_N^+ \) do not suffice to generate the \( \sigma \)-algebra of measurable sets up to measure zero.
4. Uniqueness of the Extension

The reader is referred to the work of Paschke [10] for an introduction to W*-modules and to [8, 9] for the general theory of C*-modules.

**Definition 4.1.** Let \( \mathcal{M} \) be a Hilbert module over a unital C*-algebra \( A \). A subset \( \{\xi_1, \ldots, \xi_N\} \) of \( \mathcal{M} \) is said to be an **orthonormal basis** for \( \mathcal{M} \) if \( \xi_i \in \mathcal{M} \), \( \langle \xi_i, \xi_j \rangle = \delta_{ij}1_A \) and

\[
\xi = \sum_{i=1}^{N} \xi_i \cdot \langle \xi_i, \xi \rangle, \text{ for all } \xi \in \mathcal{M}.
\]

In the case where \( N = \infty \) the sum is understood as norm-convergent.

As a consequence \( \sum_{i=1}^{N} \theta_{\xi_i, \xi} = \text{id}_{\mathcal{M}} \) with the understanding that the sum is convergent in the strong topology when \( N = \infty \). When \( A \) is non-unital, we define the basis of \( \mathcal{M} \) by using the unitization \( A^1 = A + \mathbb{C} \). Indeed we can extend the right action to \( A^1 \) by

\[
\xi \cdot (a + \lambda) = \xi \cdot a + \lambda \xi,
\]

for all \( a \in A \) and \( \lambda \in \mathbb{C} \). Then the basis of \( \mathcal{M} \) over \( A \) is defined as the basis of \( \mathcal{M} \) over \( A^1 \). This is just to ensure that the formula \( \langle \xi_i, \xi_j \rangle = \delta_{ij}1_{A^1} \) makes sense.

**Remark 4.2.** In general, a Hilbert module may not have an orthonormal basis. However, W*-modules have a basis \( \{\xi_i\} \) such that \( \langle \xi_i, \xi_j \rangle \) is a projection [10, Theorem 3.12]. Moreover, the size of an orthonormal basis is not well defined, meaning that there may be bases \( \{s_i\}_{i \in I} \) and \( \{t_j\}_{j \in J} \) with \( |I| \neq |J| \). The reason is that the uniqueness of the linear combinations is not guaranteed. For a counterexample let \( \mathcal{M} = \mathcal{O}_2 \) be the trivial Hilbert module over itself, where \( \mathcal{O}_2 \) is the Cuntz algebra on two generators, say \( s_1 \) and \( s_2 \). Then the sets \( \{1_{\mathcal{O}_2}\} \) and \( \{s_1, s_2\} \) are both bases for the Hilbert module. Indeed for \( \xi \in \mathcal{O}_2 \) we trivially have that \( \xi = 1_{\mathcal{O}_2} \cdot \langle 1_{\mathcal{O}_2}, \xi \rangle \), and that

\[
\xi = (s_1 s_1^* + s_2 s_2^*) \xi = s_1 \cdot (s_1, \xi) + s_2 \cdot (s_2, \xi),
\]

since \( s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{O}_2} \).

Similarly, one can show that the trivial Hilbert module \( \mathcal{M} = \mathcal{O}_2 \) over \( \mathcal{O}_2 \) is unitarily equivalent to the (interior) direct sum \( \mathcal{M} + \mathcal{M} \) over \( \mathcal{O}_2 \) by the unitary \( U = [s_1 \quad s_2] \). Inductively we get that \( \mathcal{M} \) is unitarily equivalent to \( \sum_{k=1}^{n} \mathcal{M} \) for all \( n \in \mathbb{N} \).

**Remark 4.3.** Nevertheless, when the Hilbert module is over a stably finite C*-algebra \( A \), then the size is unique. Indeed, let \( \{\xi_i\}_{i \in I} \) and \( \{\eta_j\}_{j \in J} \) be two orthonormal bases of such a Hilbert module \( \mathcal{M} \) and form the rectangular matrix \( U = [(\xi_i, \eta_j)] \). Then, the \( (i, j) \)-entry of the \( |I| \times |J| \) matrix \( UU^* \) is

\[
\sum_{k=1}^{|J|} \langle \xi_i, \eta_k \rangle \langle \eta_k, \xi_j \rangle = \sum_{k} \langle \xi_i, \eta_k \langle \eta_k, \xi_j \rangle \rangle = \langle \xi_i, \xi_j \rangle = \delta_{ij}1_A.
\]
Analogous computations for $U^*U$ show that $U$ is a unitary in $M_{[I_1,J]}(A)$. Since $A$ is stably finite we get that $|I| = |J|$. In fact we get the following formula

$$[\eta_1, \ldots, \eta_N] = [\xi_1, \ldots, \xi_N][U_{ij}],$$

and the unitary $U$ is in $M_N(A)$. In contrast to [7] the unitary $U$ may not be in $M_N(\mathbb{C})$.

Let $\alpha: L^\infty(X,\mu) \rightarrow L^\infty(X,\mu)$ be a *-homomorphism and let the linear space

$$\mathcal{E}(X,\mu) = \{T \in B(L^2(X,\mu)) \mid Ta = \alpha(a)T, \text{ for all } a \in L^\infty(X,\mu)\}. $$

Then $\mathcal{E}(X,\mu)$ becomes a Hilbert module over $L^\infty(X,\mu)$ by defining

$$S \cdot a := Sa \quad \text{and} \quad \langle S, T \rangle := S^*T$$

for all $a \in L^\infty(X,\mu)$ and $S, T \in \mathcal{E}(X,\mu)$. Indeed, for $b \in L^\infty(X,\mu)$ we obtain

$$\langle (Sa)b = S(ab) = (Sb)a = (\alpha(b)S)a = \alpha(b)(Sa),$$

thus $Sa \in \mathcal{E}(X,\mu)$. Also,

$$\langle S, T \rangle \cdot b = (S^*T)b = S^*Tb = S^*\alpha(b)T = bS^*T = b \cdot \langle S, T \rangle,$$

for all $b \in L^\infty(X,\mu)$, which implies that $\langle S, T \rangle \in L^\infty(X,\mu)' = L^\infty(X,\mu)$. Thus the inner product and the right action are well defined and routine calculations show that $\mathcal{E}(X,\mu)$ is a Hilbert module over $L^\infty(X,\mu)$. In particular the Hilbert module $\mathcal{E}(X,\mu)$ becomes a W*-correspondence over $L^\infty(X,\mu)$ by defining

$$a \cdot S = \alpha(a)S, \text{ for all } a \in L^\infty(X,\mu) \text{ and } S \in \mathcal{E}(X,\mu).$$

Indeed, for $b \in L^\infty(X,\mu)$ we obtain that

$$(a \cdot S)b = \alpha(a)Sb = \alpha(a)\alpha(b)S = \alpha(b)\alpha(a)S = \alpha(b)(a \cdot S),$$

hence $a \cdot S \in \mathcal{E}(X,\mu)$.

It is evident that $\mathcal{E}(X,\mu)$ is a weak*-closed subspace of $B(L^2(X,\mu))$. Hence as a self-dual W*-correspondence it receives a basis $\{S_i\}_{i \in I}$ such that $\langle S_i, S_j \rangle = S_i^*S_j = 0$ when $i \neq j$, $\langle S_i, S_i \rangle = S_i^*S_i$ is a projection in $L^\infty(X,\mu)$, and $T = \sum_i S_i S_i^*T$, for all $T \in \mathcal{E}(X,\mu)$ [10, Theorem 3.12].

**Lemma 4.4.** Let $\{S_i\}_{i=1}^n$ be a basis for $\mathcal{E}(X,\mu)$ with $N < \infty$. Then the following are equivalent:

1. $\{S_i\}_{i=1}^N$ is an orthonormal basis for $\mathcal{E}(X,\mu)$;
2. $\{S_i\}_{i=1}^N$ is a Cuntz family that implements $\alpha$ of $L^\infty(X,\mu)$.

**Proof.** For convenience we write $I \in B(L^2(X,\mu))$ also for the unit of $L^\infty(X,\mu)$. Since $\{S_i\}_{i=1}^N$ is a basis we obtain $I = \sum_{i=1}^N \theta_i S_i = S_i S_i^*$. Moreover $S_i \in \mathcal{E}(X,\mu)$ thus $S_i a = \alpha(a) S_i$ for all $a \in L^\infty(X,\mu)$. Hence

$$\sum_{i=1}^N S_i a S_i^* = \alpha(a) \sum_{i=1}^N S_i S_i^* = \alpha(a).$$
On the other hand if \( \{S_i\}_{i=1}^N \) is a Cuntz family, then \( \langle S_i, S_j \rangle = \delta_{ij}I \) and 
\[
\sum_{i=1}^N S_i S_i^* = I, \text{ since } \alpha(I) = I \text{ for the unit } I \in L^\infty(X, \mu). \]
If \( \alpha(a) = \sum_{i=1}^N S_i a S_i^* \), then \( S_i a = \alpha(a) S_i \) for all \( a \in L^\infty(X, \mu) \), for \( i = 1, \ldots, N \).
Thus, \( S_i \in \mathcal{E}(X, \mu) \). For \( T \in \mathcal{E}(X, \mu) \) set \( a_i = \langle S_i, T \rangle = S_i^* T \). Then
\[
\sum_{i=1}^N S_i a_i = \sum_{i=1}^N S_i S_i^* T = T,
\]
and the proof is complete. \( \square \)

Let \( \{S_1, \ldots, S_N\} \) be an orthonormal basis of \( \mathcal{E}(X, \mu) \), and let the extension \( \alpha_S \) of \( \alpha \) be given by
\[
\alpha_S : \mathcal{B}(L^2(X, \mu)) \to \mathcal{B}(L^2(X, \mu)) : R \mapsto \sum_{i=1}^N S_i R S_i^*.
\]
We can then define the linear space
\[
\mathcal{H}_S := \{T \in \mathcal{B}(L^2(X, \mu)) \mid TR = \alpha_S(R)T \text{ for all } R \in \mathcal{B}(L^2(X, \mu)) \}.
\]
It becomes a Hilbert space endowed with the inner product
\[
\langle T_1, T_2 \rangle = T_1^* T_2, \text{ for all } T_1, T_2 \in \mathcal{H}_S.
\]
Indeed it is easy to check that \( \langle T_1, T_2 \rangle \in \mathcal{B}(L^2(X, \mu))' = \mathbb{C} \). Moreover it has dimension \( N \) and the Cuntz family \( \{S_i\}_{i=1}^N \) is in \( \mathcal{H}_S \). The proof is the same as in Remark 4.2 taking into account that \( \alpha_S(R)S_j = S_j R \), for all \( R \in \mathcal{B}(L^2(X, \mu)) \). These results were established by Laca [7].

**Proposition 4.5.** Let \( \{S_i\}_{i=1}^N \) and \( \{Q_i\}_{i=1}^N \) be two orthonormal bases for \( \mathcal{E}(X, \mu) \). Then the following are equivalent:
(1) The unitary \( U \) that induces a pairing of the bases is in \( M_N(\mathbb{C}) \);
(2) The extensions \( \alpha_S \) and \( \alpha_Q \) in \( \mathcal{B}(L^2(X, \mu)) \) coincide.

**Proof.** For convenience we write \( I \in \mathcal{B}(L^2(X, \mu)) \) also for the unit of \( L^\infty(X, \mu) \).

(1) \( \Rightarrow \) (2): We compute
\[
\alpha_Q(R) = \sum_{i=1}^N Q_i R Q_i^* = \sum_{i,j,k=1}^N S_j \langle S_j, Q_i \rangle R \langle Q_i, S_k \rangle S_k^* = \sum_{k,j=1}^N S_j R \sum_{i=1}^N \langle S_j, Q_i \rangle \langle Q_i, S_k \rangle S_k^* = \sum_{k,j=1}^N S_j R \delta_{j,k} S_k^* = \sum_{k=1}^N S_k R S_k^* = \alpha_S(R),
\]
since \( \sum_{i=1}^N \langle S_j, Q_i \rangle \langle Q_i, S_k \rangle \) is the \( (j, k) \)-entry of \( UU^* = I \), and we have used that the entry \( \langle S_j, Q_i \rangle \) of \( U \) is in \( \mathbb{C} \).
Proof. The fact that \(\pi\) implies that it is a \(L\text{-}

\text{envelope of immediate.}

\[(2) \Rightarrow (1)\] If \(\alpha_Q = \alpha_S\) then by definition \(H_S = H_Q\). Thus
\[
S_i^* Q_k R = S_i^* \alpha_Q(R)Q_k = S_i^* \alpha_S(R)Q_k = RS_i^* Q_k
\]
for all \(R \in \mathcal{B}(L^2(X,\mu))\), hence \(S_i^* Q_k \in \mathcal{B}(L^2(X,\mu))' = \mathbb{C}.

As a consequence we have a complete invariant for the multiplicity \(n\) crossed products on \(L^\infty(X,\mu)\) [11]. Recall that given a \(*\)-endomorphism \(\alpha: A \to A\) of a \(C^*\)-algebra \(A\) then the multiplicity \(n\) crossed product \(A \rtimes^\alpha_n \mathbb{N}\) is the enveloping \(C^*\)-algebra generated by \(\pi(A)\) and a Toeplitz-Cuntz family \(\{Q_i\}_{i=1}^n\) such that \(\pi\) is a non-degenerate representation of \(A\), and \(\pi(\alpha(a)) = \sum_{i=1}^n Q_i \pi(a) Q_i^*\), for all \(a \in A\). When \(\alpha\) is unital then non-degeneracy of \(\pi\) is redundant and \(\{Q_i\}_{i=1}^n\) can be considered to be a Cuntz family [6, Section 3 and Proposition 3.1]. In [6, Subsection 3.3] we introduced the semicrossed product \(A \rtimes^\alpha T_n^+\) as the non-involutive subalgebra of \(A \rtimes^\alpha \mathbb{N}\) generated by \(\pi(\alpha)\) and \(\{Q_i\}_{i=1}^n\).

**Corollary 4.6.** Let \(\alpha\) be a unital weak*-continuous isometric endomorphism of \(L^\infty(X,\mu)\) and suppose that there is a representation \((\text{id}, \{S_i\}_{i=1}^n)\) of Stacey's crossed product \(L^\infty(X,\mu) \rtimes^\alpha_n \mathbb{N}\) on \(L^2(X,\mu)\). Then the following are equivalent

1. \(L^\infty(X,\mu) \rtimes^\alpha_n \mathbb{N} \cong L^\infty(X,\mu) \rtimes^\alpha_m \mathbb{N}\) via a \(*\)-isomorphism that fixes \(L^\infty(X,\mu)\) elementwise;
2. There is a representation \((\text{id}, \{Q_i\}_{i=1}^m)\) of \(L^\infty(X,\mu) \rtimes^\alpha_m \mathbb{N}\) acting on \(L^2(X,\mu)\);
3. \(n = m\);
4. \(L^\infty(X,\mu) \rtimes^\alpha T_n^+ \cong L^\infty(X,\mu) \rtimes^\alpha T_m^+\) via a completely isometric isomorphism that fixes \(L^\infty(X,\mu)\) elementwise.

**Proof.** The fact that \(\alpha\) is an isometric endomorphism of a \(C^*\)-algebra implies that it is a \(*\)-homomorphism of the \(C^*\)-algebra \(L^\infty(X,\mu)\) and the multiplicity \(n\) crossed products are well defined. The implication \([(3) \Rightarrow (4)]\) is immediate.

\([(4) \Rightarrow (1)]\): By [6, Theorem 3.13] the \(C^*\)-algebra \(L^\infty(X,\mu) \rtimes^\alpha_n \mathbb{N}\) is the \(C^*\)-envelope of \(L^\infty(X,\mu) \rtimes^\alpha \mathbb{T}_n^+\), thus the completely isometric isomorphism extends to a \(*\)-isomorphism of the corresponding \(C^*\)-algebras.

\([(1) \Rightarrow (2)]\): If \(\Phi\) is the \(*\)-isomorphism, let \(Q_i := \Phi(S_i)\).

\([(2) \Rightarrow (3)]\): Let \((\text{id}, \{S_i\}_{i=1}^n)\) and \((\text{id}, \{Q_i\}_{i=1}^m)\) be two such representations. Then \(\alpha\) is implemented by \(\{S_i\}_{i=1}^n\) and \(\{Q_i\}_{i=1}^m\), thus they define a basis for \(\mathcal{E}(X,\mu)\). Therefore \(n = m\) by Remark 4.3.

5. Existence of a transfer operator

In general the mapping \(C(X) \ni f \rightarrow C_\varphi \circ f \circ \varphi \in L^2(X,\mu)\) may not extend to an operator on the Hilbert space \(L^2(X,\mu)\). However if

\[
c_0 \|\xi\|_2 \leq \|C_\varphi \xi\|_2 \leq c_1 \|\xi\|_2, \quad \text{for all } \xi \in L^2(X,\mu),
\]
then $C_\phi$ is an injective operator in $\mathcal{B}(L^2(X, \mu))$, and $\varphi^{-1}$ preserves the null sets. The map $\mu$ is called $\varphi$-bounded if there is a constant $K > 0$ such that $\mu(\varphi(E)) \leq K\mu(E)$, for all measurable sets $E \subset X$. In this case $\varphi$ preserves also the $\mu$-null sets.

Under these assumptions let the polar decomposition $C_\phi = S_\phi a_\varphi$. Then $S_\phi$ is an isometry and $a_\varphi$ is invertible. We can check that by definition $C_\phi a = \alpha(a)C_\phi$ for all $a \in L^\infty(X, \mu)$, hence $C_\phi \in \mathcal{E}(X, \mu)$. Hence $a_\varphi^2 = C_\phi^* C_\phi \in L^\infty(X, \mu)'$ so $a_\varphi \in L^\infty(X, \mu)$. Consequently, the isometry $S_\phi = C_\phi a_\varphi^{-1}$ is also in $\mathcal{E}(X, \mu)$ and the mapping

$$L : L^\infty(X, \mu) \to L^\infty(X, \mu) : a \mapsto S_\phi^* a S_\phi,$$

defines a transfer operator of $\alpha$, i.e. $L$ is positive and $aL(b) = L(\alpha(a)b)$ for all $a, b \in L^\infty(X, \mu)$. Following Exel [3] let the semi-inner-product on the $L^\infty(X, \mu)$-module $L^\infty(X, \mu)_L$ given by

$$\langle \eta, \xi \rangle_L = L(\eta^* \xi), \text{ and } \xi \cdot a = \xi \alpha(a),$$

for all $\eta, \xi, a \in L^\infty(X, \mu)$.

**Proposition 5.1.** Assume that $\mu$ is $\varphi$-bounded and that $C_\phi$ is a bounded below operator of $\mathcal{B}(L^2(X, \mu))$. Then $L^\infty(X, \mu)_L$ is a Hilbert module over $L^\infty(X, \mu)$, and as a vector space it coincides with $L^\infty(X, \mu)$.

**Proof.** It suffices to show that the norm $||| \cdot |||_L$ on the module $L^\infty(X, \mu)_L$ is equivalent to the norm $|| \cdot ||$ of $L^\infty(X, \mu)$.

First we show that there is a constant $M$ such that $||a|| \leq M ||aS_\phi||$ for every $a \in L^\infty(X, \mu)$. Since $||aS_\phi||^2 = ||a|S_\phi|^2$ and $||a|| = |||a|||$, it is enough to show that the relation $||a|| \leq M ||aS_\phi||$ holds for all positive $a$ in the norm-dense subspace of simple functions. To this end let $a = \sum_{i=1}^n d_i \chi_{E_i}$ where the sets $E_i$ are disjoint, of positive measure, and $d_1 > d_2 > \cdots > d_n > 0$; hence $||a|| = d_1$. To compute the norm $||aS_\phi||$ we let $\alpha$ act on unit vectors in the range of $C_\phi$; equivalently with unit vectors in the range of $S_\phi$. Let $E = E_1$ and $\xi = \frac{1}{\sqrt{\mu(\varphi^{-1}(\varphi(E)))}} \chi_{\varphi^{-1}(\varphi(E))}$. Then $\xi$ is a unit vector in the range of $S_\phi$. Also, the assumptions on $S_\phi$ and $\mu$ imply that $\mu(\varphi^{-1}(\varphi(E))) \leq c_1\mu(\varphi(E)) \leq c_1 K\mu(E)$. Therefore

$$||aS_\phi||^2 \geq ||a\xi||^2_2 = \int_X a^2|\xi|^2 \, d\mu \geq \int_X d_1^2 \chi_E |\xi|^2 \, d\mu = \frac{1}{\mu(\varphi^{-1}(\varphi(E)))} \int_X d_1^2 \chi_E \, d\mu = \frac{\mu(E)}{m(\varphi^{-1}(\varphi(E)))} d_1^2 \geq \frac{1}{c_1 K} d_1^2.$$

Since $||a|| = d_1$, we have that $||aS_\phi|| \geq \frac{1}{c_1 \sqrt{K}} ||a||$ on a norm dense subspace.

By the above inequality we obtain the equivalence of the norms $||| \cdot |||_L$ and $|| \cdot ||$. Indeed we have that

$$\frac{1}{M^2} ||a||^2 \leq |||a|S_\phi||^2 = ||S_\phi^* a^2 S_\phi|| = ||L(a^* a)|| = ||a||_L^2 = ||S_\phi^* a^* aS_\phi|| \leq ||a^* a|| = ||a||^2.$$
Theorem 5.2. Assume that $\mu$ is $\varphi$-bounded and that $C_\varphi$ is a bounded below operator of $B(L^2(X,\mu))$. Then the following are equivalent:

1. \{$\xi_i\}_{i=1}^n$ is an orthonormal basis for the Hilbert module $L^\infty(X,\mu)_C$;
2. \{$\xi_i S_\varphi\}_{i=1}^n$ is an orthonormal basis for the Hilbert module $\mathcal{E}(X,\mu)$;
3. \{$\xi_i S_\varphi\}_{i=1}^n$ is a Cuntz family that implements $\alpha$.

Proof. It will be convenient to denote $S_i := \xi_i S_\varphi$. Note that by definition $S_i \in \mathcal{E}(X,\mu)$ and recall that the equivalence [(2) $\iff$ (3)] is Lemma 4.4. Moreover we write $I \in B(L^2(X,\mu))$ also for the unit of $L^\infty(X,\mu)$. The constant function of $L^2(X,\mu)$ will be denoted by $1$. 

[(1) $\implies$ (3)]: First we have that the $S_i$ have orthogonal ranges, since

$$S_i^* S_j = S_i^* \xi_i^* \xi_j S_\varphi = \mathcal{L}(\xi_i^* \xi_j) = \langle \xi_i, \xi_j \rangle = \delta_{ij}I.$$ 

Recall that the constant function $1: X \to \mathbb{C}$ is a separating vector and $C_\varphi(1) = 1 \circ \varphi = 1$. Therefore,

$$S_i a S_i^*(1) = S_i a S_i^* C_\varphi(1) = \xi_i S_\varphi a S_i^* \xi_i S_\varphi a_{\varphi}(1)$$

$$= \alpha(a) \xi_i S_\varphi \mathcal{L}(\xi_i^* \alpha(a))(1) = \alpha(a) \xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a))) S_\varphi(1).$$

for all $i = 1, \ldots, n$. Since \{$\xi_i\}_{i=1}^n$ defines a basis of $L^\infty(X,\mu)_C$ we have that

$$a = \sum_{i=1}^n \xi_i \cdot \langle \xi_i, a \rangle = \sum_{i=1}^n \xi_i \alpha(\mathcal{L}(\xi_i^* a))$$

for all $a \in L^\infty(X,\mu)$. Thus

$$\sum_{i=1}^n S_i a S_i^*(1) = \sum_{i=1}^n \alpha(a) \xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a))) S_\varphi(1)$$

$$= \alpha(a) \sum_{i=1}^n \xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a))) S_\varphi(1)$$

$$= \alpha(a) \alpha(a \varphi) S_\varphi(1) = \alpha(a) S_\varphi a_{\varphi}(1)$$

$$= \alpha(a) C_\varphi(1) = \alpha(a)(1).$$

Since $1$ is a separating vector we obtain that \{$S_i$\} implements $\alpha$.

[(3) $\implies$ (1)]: Note that the functions $\xi_i$ are orthonormal, since

$$\langle \xi_i, \xi_j \rangle = \mathcal{L}(\xi_i^* \xi_j) = C_\varphi^* \xi_i^* \xi_j C_\varphi = S_i^* S_j = \delta_{ij}I.$$ 

To see that the \{$\xi_i\}_{i=1}^n$ span $L^\infty(X,\mu)_C$, let an element $a \in L^\infty(X,\mu)$ with $\langle \xi_i, a \rangle = 0$ for all $i$. Then

$$(a S_\varphi)^* = S_\varphi^* a^* \cdot \sum_{i=1}^n S_i S_i^* = S_\varphi^* a^* \cdot \sum_{i=1}^n \xi_i S_\varphi S_i^*$$

$$= \sum_{i=1}^n (S_\varphi^* a^* \xi_i S_\varphi) S_i^* = \sum_{i=1}^n \langle a, \xi_i \rangle S_i^* = 0,$$

so that $a S_\varphi = 0$. Hence $a C_\varphi = 0$, thus $a(1) = a C_\varphi(1) = 0$. Since $1$ is a separating vector we obtain that $a = 0$. 

Remark 5.3. Assume that $\varphi: X \to X$ has $N$ Borel sections as in Proposition 2.2. Then the $N$ isometries $S_i$ of Proposition 2.2 can be written as

$$S_i = M_{X,Y_i} M_u \alpha(a_\varphi) C_\varphi = M_{X,Y_i} M_u M_{h_\varphi} C_\varphi,$$

where $u_i$ are as in Proposition 2.1 for $\psi_i$ and $a_\varphi = M_h \in L^\infty(X, \mu)$. Therefore, the elements $\xi_i = M_{X,Y_i} M_u M_{h_\varphi} C_\varphi$ define a basis for $L^\infty(X, \mu)$. 

There is also a converse of the above scheme that works at the level of $*$-homomorphisms. We would like to thank Philip Gipson for bringing this to our attention. If there is a Cuntz family $\{S_i\}_{i=1}^n$ in $B(L^2(X, \mu))$ that implements $\alpha$ then $S_i^* a S_i \in L^\infty(X, \mu)$ for all $i = 1, \ldots, n$. This follows because $L^\infty(X, \mu)$ is a MASA, $S_i b = \alpha(b) S_i$, and

$$S_i^* a S_i \cdot b = S_i^* \alpha(b) a S_i = b \cdot S_i^* a S_i,$$

for all $b \in L^\infty(X, \mu)$. Furthermore $S_i S_i^*$ commutes with every $a \in L^\infty(X, \mu)$, thus the $*$-homomorphisms $\beta_i: L^\infty(X, \mu) \to L^\infty(X, \mu)$ given by $\beta_i(a) = S_i^* a S_i$ are $n$ left inverses for $\alpha$.

References

[1] J. Cuntz, Simple C*-algebras generated by isometries, Comm. Math. Phys. 57:2 (1977), 173–185.
[2] D. Courtney, P.S. Muhly and W. Schmidt, Composition Operators and Endomorphisms, Complex Analysis and Operator Theory 6:1 (2012), 163–188.
[3] R. Exel, A new look at the crossed-product of a C*-algebra by an endomorphism, Ergodic Theory Dynam. Systems, 23:6 (2003), 1733–1750.
[4] P. Gipson, Invariant basis number for C*-algebras, preprint (2014), arXiv:1407.4713.
[5] E.T.A. Kakariadis and E.G. Katsoulis, Isomorphism invariants for multivariable C*-dynamics, J. Noncomm. Geom. 8:3 (2014), 771–787.
[6] E.T.A. Kakariadis and J.R. Peters, Representations of C*-dynamical systems implemented by Cuntz families, Münster J. Math. 6 (2013), 383–411.
[7] M. Laca, Endomorphisms of $B(H)$ and Cuntz algebras, J. Operator Th. 30 (1993), 85–108.
[8] C. Lance, Hilbert C*-modules. A toolkit for operator algebraists, London Mathematical Society Lecture Note Series, 210 Cambridge University Press, Cambridge, 1995.
[9] V.M. Manuilov and E.V. Troitsky, Hilbert C*-modules, Translated from the 2001 Russian original by the authors. Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.
[10] W.L. Paschke, Inner product modules over B*-algebras, Trans. Amer. Math. Soc. 182 (1973), 443–468.
[11] P.J. Stacey, Crossed products of C*-algebras by *-endomorphisms, J. Austral. Math. Soc. Ser. A, 54 (1993), 204–212.

School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne, NE1 7RU, UK
E-mail address: evgenios.kakariadis@ncl.ac.uk

Department of Mathematics, Iowa State University, Ames, Iowa, IA 50011, USA
E-mail address: peters@iastate.edu