Bounds on quantum non-locality via partial transposition

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We explore the link between two concepts: the level of violation of a Bell inequality by a quantum state and discrimination between two states by means of local operations and classical communication (LOCC). For any bipartite Bell inequality, we show that its value on a given quantum state can not exceed the classical bound by more than the maximal quantum violation shrunk by a factor reporting distinguishability of this state from the separable set by means of LOCC. We then consider the general scenarios where the parties are allowed to perform a local pre-processing of many copies of the state before the Bell test (asymptotic and hidden-non-locality scenarios). We define the rate of non-locality and, for PPT states, we bound this quantity by the relative entropy of entanglement of the partially transposed state. The bounds are strong enough to limit the use of certain states containing private key in the device-independent scenario.

Non-locality is one of the most interesting phenomena emerging from quantum multiparticle states, extensively studied in recent years \cite{1}. The quantitative study of non-locality has two different approaches, one is to ask, for a fixed Bell scenario, what is the best one can obtain optimizing over all possible quantum resources (states and measurements) \cite{2,3}. A converse approach is to ask for a fixed quantum state, or a class of states, what is the best one can obtain using this state as a resource, i.e. optimizing over all Bell scenarios. Some references in this second approach include the seminal work of Werner \cite{4} exhibiting a local model for projective measurements for $U \otimes U$-invariant states (see also \cite{5}). Another result showing that typically the violation of correlation Bell inequalities by multipartite qudit states is very small \cite{6}. And an hierarchy of semidefinite programs that allows to bound the violation achievable by PPT states \cite{7}.

Here we follow the second approach, with the aim to show that certain states, despite being entangled, exhibit very limited gain of non-locality. To achieve this, we base on the concept of state discrimination by means of local operations and classical communication (LOCC) \cite{12}, a subject extensively studied in the last decade (see e.g. \cite{14} for recent results, and references therein). It has been shown, that there exist pairs of states which are hardly distinguishable from each other by means of LOCC, although being almost orthogonal, i.e. almost perfectly distinguishable by means of global operations \cite{12,14,15}. In \cite{15} it is shown that there exist even entangled states containing bit of privacy, which are almost indistinguishable from some separable (insecure) states. This fact has been shown recently to rule them out as a potential resource for swapping of a private key, in the so called quantum key repeaters \cite{16}.

We base also on the idea stated in \cite{19,20}, where the Bell inequality is considered as a particular witness of entanglement. The link between quantum non-locality and state discrimination that we start from, amounts to a simple observation that if a given state is hardly distinguishable from some separable one, it can not exhibit large violation in any Bell scenario, or else, one could use the procedure of checking the violation of a Bell inequality to discriminate between these two states (a quantitative version of this fact has been derived in \cite{21}). Here we refine these ideas, using partial transposition to explore the fact that Bell inequalities are implemented by a much smaller class of operations, the local ones. We obtain non-trivial upper bounds on the amount of quantum non-locality both in the single copy case for arbitrary bipartite states, as well as in the asymptotic and hidden-non-locality scenarios for states with positive partial transpose. To the best of our knowledge, this is the first quantitative approach for the latter two scenarios.

We start by deriving bounds on the violation of a Bell inequality on a single copy of a quantum state $\rho$. It exceeds the classical value by the maximal quantum value shrunk by a factor related to distinguishability between the state and separable states. Since it is hard to compute distinguishability by means of LOCC, following \cite{16,17}, we bound it using partial transposition. We then generalize this result to the asymptotic case of a large number of copies of the state, showing that the regularized (asymptotic) rate of non-locality \cite{22} is upper bounded by the relative entropy of entanglement of the partially transposed state, $\rho^T$. We also consider a hidden-non-locality scenario, and we prove the same upper bound for the rate of hidden non-locality, which we define here. The bounds are strong enough to guarantee very limited non-locality for some entangled states which contain secure quantum key. For this reason the use of these states for device independent security appears to be limited. Interestingly, to achieve the results, we apply techniques of \cite{18}, which were developed for the quantum key repeaters scenario. Details of the proofs of the above results can be found in the Supplemental Material.

\textit{Notation.} — By bipartite box we refer to the conditional probability distribution of outputs $a$ and $b$ of Alice and Bob given inputs $x$ and $y$ respectively, $P(ab|xy)$. In what follows, by the set $S \equiv \{s_{a,b}\}$ we denote the coefficients of a particular Bell inequality, so that
Theorem 1. Given two bipartite states $\rho, \sigma \in B(C^d \otimes C^d)$, a Bell inequality $s_{x,y}^a$ and a set of quantum POVMs $\{A_{a|x} \otimes B_{b|y}\}$, it holds that:

$$S(\rho) - S(\sigma) \leq ||S^T||_{\infty}||\rho^T - \sigma^T||,$$

where $||.||$ denotes the trace norm, $||X||_{\infty}$ is the largest eigenvalue in modulus of operator $X$, and $\Gamma$ denotes partial transposition.

Proof. We show the sequence of (in)equalities and comment it below:

$$S(\rho) - S(\sigma) = \sum_{a,b,x,y} Tr s_{x,y}^a A_{a|x} \otimes B_{b|y}(\rho - \sigma) = (2)$$

$$\sum_{a,b,x,y} Tr s_{x,y}^a A_{a|x} \otimes B_{b|y}^T(\rho - \sigma)^T = (3)$$

$$Tr S^T(\rho^T - \sigma^T) = ||S^T||_{\infty}Tr \frac{S^T}{||S^T||_{\infty}}(\rho^T - \sigma^T) \leq (4)$$

$$\sup_{M \geq 0, M \leq 1} ||S^T||_{\infty}Tr M(\rho^T - \sigma^T) = ||S^T||_{\infty}||\rho^T - \sigma^T||. (5)$$

In the first equality we use the linearity of trace function, then the well known identity $Tr XY = Tr X^T Y^T$. In eq (2), we use the fact that Bell inequality is non-negative, i.e. $s_{x,y}^a \geq 0$, so the operator $S$ is positive as a sum of positive operators $A_{a|x} \otimes B_{b|y}^T$ (note that transposition does not change positivity). Further, by definition of infinity norm, we have $\frac{S^T}{||S^T||_{\infty}} \leq 1$. Finally we use the fact that trace norm is the supremum over positive operators less than identity (see [23]), and the assertion follows. □

Remark 1. Let us note here, that in eq. (4) we can have much tighter bound: $S(\rho) - S(\sigma) \leq ||S^T||_{\infty}||\rho^T - \sigma^T||_{sep}$, where $||X||_{sep} = \sup_{M \geq 0, M \leq 1} Tr MX$, where $M = \sum_\alpha A_\alpha \otimes B_\alpha$ for $\alpha \geq 0$ and $A_i$, $B_i$ are positive operators. It is however hard to evaluate the latter quantity, hence we focus here on the upper bound on it.

From Theorem 1 we have an immediate corollary related to the fact that separable states, i.e. states of the form $\sigma_{AB} = \sum_i \gamma_i A_i \otimes \sigma_i B_i$, yield boxes that have local hidden variable model. In consequence, $S(\sigma_{AB}) \leq C(S) \forall \sigma_{AB} \in SEP$, with SEP denoting the set of separable states.

Also, let us denote

$$Q_S(\rho) := \sup_{\{A_{a|x} \otimes B_{b|y}\}} \sum_{a,b,x,y} s_{x,y}^a Tr A_{a|x} \otimes B_{b|y} \rho, (6)$$

with supremum taken over all POVM elements $\{A_{a|x} \otimes B_{b|y}\}$. Note that $Q(S) = \sup_{\rho} Q_S(\rho)$, and it is straightforward that $||S^T||_{\infty}$ is upper bounded by $Q(S)$. Hence we have:

Corollary 1. For any bipartite Bell expression $S$, and state $\rho$, it holds that:

$$Q_S(\rho) \leq C(S) + Q(S) \inf_{\sigma \in SEP} ||\rho^T - \sigma^T||. (7)$$

One can however expect that the same bound should hold for all states which are to the same extent indistinguishable from separable ones. To this end we introduce the hierarchy of sets $D(\epsilon)$:

$$D(\epsilon) := \{\rho : \exists \sigma \in SEP \ \ ||\rho^T - \sigma^T|| \leq \epsilon\}. (8)$$

Observe that $D(\epsilon)$ is a convex set, which includes SEP for any $\epsilon > 0$. In consequence, due to Corollary 1 we have the following dependence (see Fig. 1):

Corollary 2. For any bipartite Bell expression $S$ and $\epsilon > 0$, it holds that:

$$\sup_{\rho \in D(\epsilon)} Q_S(\rho) \leq C(S) + \epsilon Q(S). (9)$$

Note here, that according to Remark 1 a stronger version of the above result holds for the set $D(\epsilon)$ defined with respect to $||.||_{sep}$ norm instead of the norm based on partial transposition.

Examples.— Basing on [17] and [18] we exhibit some examples of entangled states that have negligible violation. In our construction we base on private states [23][25]. A private state can be described as follows:

$$\gamma_X = \frac{1}{2} [\langle 00|00 \rangle \otimes \sqrt{XX^T} + |00\rangle \langle 11| \otimes X +$$

$$|11\rangle \langle 00| \otimes X^T + |11\rangle \langle 11| \otimes \sqrt{XX^T}]. (10)$$

where $X$ is an arbitrary operator with trace norm 1. We then see that a private bit resembles a singlet state with “operator amplitudes” which are functions of $X$. 

$\sum_{a,b,x,y} s_{x,y}^a P(ab|xy)$ is the value of the Bell inequality $S$ on a particular box $P(ab|xy)$. From now on, when we refer to a Bell inequality $S$ we assume that all coefficients are positive, $s_{x,y}^a \geq 0$ (note that using the normalization condition we can rewrite any Bell inequality with positive coefficients only). Denoting the maximal value of the Bell expression $S$ over all boxes in classical case $C(S)$, quantum $Q(S)$ and super-quantum $NS(S)$, we have the following relation: $C(S) \leq Q(S) \leq NS(S)$.

For a bipartite state $\rho_{AB}$ and the set of POVMs $\{M_{x,y}\}$, $M_{x,y} = \{A_{a|x} \otimes B_{b|y}\}$, we represent the corresponding box by $\{Tr M_{x,y} \rho_{AB}\}$, and denote the value of the Bell expression $S$ for this particular POVMs by $S(\rho_{AB})$, i.e. $S_{\rho_{AB}} = Tr S_{\rho_{AB}}$ where $S = \sum_{a,b,x,y} s_{x,y}^a A_{a|x} \otimes B_{b|y}$.
Proposition 1. The proposition below are defined in [24] (see Supplemental (containing private key). The states considered in under partial transposition, and at the same time entanglement, which are (up to local unitary transformation) invariant by equality is limited by the maximum quantum value, $D$.

Consider a private state defined by $\rho = \frac{1}{d^2} \sum_{i,j=0}^{d-1} |ij\rangle \langle ji|$ being the (normalized) swap operator. Then for the CHSH inequality we have the following bound:

$$Q_{CHSH}(\gamma_X) \leq 2 + \frac{\sqrt{2} + 1}{2\sqrt{2d}}.$$  

(11)

Now we consider the following PPT state:

$$\rho_p = (1-p)\gamma_X + \frac{p}{2} |01\rangle \langle 01| \otimes \sqrt{YY^\dagger} + |10\rangle \langle 10| \otimes \sqrt{Y^\dagger Y}$$  

(12)

with $X = \frac{1}{d^2} \sum_{i,j=0}^{d-1} u_{ij} |ij\rangle \langle ji|$ and $Y = \sqrt{\frac{1}{d}} X^\dagger$, and $u_{ij}$ are elements of an unitary matrix with all $|u_{ij}| = \frac{1}{\sqrt{d}}$ (an example is the quantum Fourier transform), and $p = \frac{1}{\sqrt{d+1}}$. By Corollary 1 we have

$$Q_S(\rho_p) \leq C(S) + Q(S) \frac{1}{\sqrt{d_s}}.$$

(13)

Following [18], we can also obtain a bound for states which are (up to local unitary transformation) invariant under partial transposition, and at the same time entangled (containing private key). The states considered in the proposition below are defined in [24] (see Supplemental Material).

Proposition 1. There exist bipartite states $\rho \in B(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes m})$ with $d = m^2$, $k = m$ satisfying $K_D(\rho^F \otimes \rho) \to 1$ with increasing $m$, such that:

$$Q_S(\rho \otimes \rho^F) \leq C(S) + \frac{Q(S)}{2^{m-1}}.$$  

(14)

Proposition 1 shows that for some class of states invariant under partial transposition, although the rate of distillable key can be made arbitrarily close to 1 by increasing the dimension of the systems, the possibility of violating any Bell inequality is bounded by an amount vanishing with the dimension of the system.

In [7] it is shown that a bipartite Bell inequality with $n$ inputs and $k$ outputs satisfy $Q(S) \leq C(S) \min \{n, k\}$, up to some universal constant independent of the parameters of the scenario. With this we see that the bound [7] ensures that for any fixed Bell scenario, as we wish to increase the key rate obtained from the exhibited families of states, the possibility of observing a violation of a Bell inequality vanishes.

Bound on asymptotic non-locality.— In considerations above, we have provided bounds which hold for single copy of a quantum state. However, in case of the first example, the state $\rho$ is distillable [20], hence, as it was noted by Peres [27], a pre-processing of many copies of a state by local operations, before the Bell test, could lead to the violation of a Bell inequality, even for states that have local model for the single copy level. Here we quantify the asymptotic non-locality by defining a rate of non-locality (see [22]) and applying methods of [18] to bound it. In the first step, we will bound this quantity by a function of the relative entropy distance under restrictive measurements introduced in [25].

In [22] a measure of non-locality, based on the relative entropy, was introduced (analogous measure was also used to quantify contextuality [29]). It captures quantitatively how “similar” is a given probability distribution to a local one. Given a box $P = (ab|xy)$, where for fixed $x, y$ we have distribution $P_{xy}(ab|xy)$, its non-locality is quantified by:

$$N(P) = \sup_{\{p(x,y)\}} \inf_{P_L \in L} \sum_{x,y} p(x,y)D(P_{xy}(ab|xy)||P_L(ab|x,y))$$

(15)

where infimum in the above is taken over all boxes admitting a local model (belonging to set $L$) and $D(P||Q)$ is the relative entropy between distributions $P$ and $Q$, $D(P||Q) = \sum_i P(i) \log \frac{P(i)}{Q(i)}$.

We are interested in quantifying how much non-locality $N$ one can obtain from $n$ copies of a given state $\rho_{AB}$, per number of copies, in the asymptotic limit, after processing it by LOCC.

Definition 1. For a bipartite state $\rho_{AB}$ its Rate of Non-locality, $R(\rho_{AB})$, is given by:

$$R(\rho_{AB}) \equiv \lim_{n \to \infty} \frac{1}{n} \sup_{\{M_{xy}\}_{x\neq y}} N_0(\{TrM_{xy}A(\rho_{AB}^\otimes n)\}),$$

(16)

where $\lim$ denotes the supremum limit.

Now we want to set bounds for the non-locality attainable in the asymptotic scenario. To state the bound we will need a well known entanglement measure, called relative entropy of entanglement [30]: $E_r(\rho) = \inf_{\sigma \in SEP} S(\rho||\sigma)$, where $S(\rho||\sigma) = Tr\rho \log \rho - Tr\rho \log \sigma$ is the quantum relative entropy, and infimum is taken over separable states.
Our main results state an upper bound on the rate of (hidden) non-locality of a PPT bipartite quantum state by the relative entropy of the partially transposed state. To achieve these results, we first introduce another non-locality measure, which is at the same time entanglement monotone, denoted as $T^\infty$.

**Definition 2.** For a bipartite state $\rho_{AB}$, its restricted regularized relative entropy of non-locality is given by:

$$T^\infty(\rho_{AB}) \equiv \lim_{n \to \infty} \frac{1}{n} \sup_{\Lambda \in \text{LOCC}} \sup_{\{M_{xy}\}} \sup_{\sigma \in \text{SEP}} \inf_{p(x,y)} \sum_{x,y} p(x,y) D(\{TrM_{xy}\Lambda(\rho^{\otimes n})\}\{TrM_{xy}\Lambda(\sigma^{\otimes n})\}).$$

Note that the definition of $T^\infty$ originates from $R$ by relaxing the optimization over local boxes to an optimization over separable states and same local measurements.

Now we are ready to state our main result.

**Theorem 2.** For any bipartite state it holds that

$$R(\rho_{AB}) \leq T^\infty(\rho_{AB}) \leq E_r(\rho_{AB}).$$

For $\rho_{AB}$ a PPT state, it holds that

$$T^\infty(\rho_{AB}) \leq E_r(\rho^r_{AB}).$$

Which leads to the corollary:

**Corollary 3.** For a bipartite PPT state $\rho_{AB}$ it holds that:

$$R(\rho_{AB}) \leq \min \{E_r(\rho_{AB}), E_r(\rho^r_{AB})\}.$$  

Since $E_r$ is asymptotically continuous [31], the bound $R(\rho_{AB}) \leq E_r(\rho_{AB})$ is meaningful only when the state is close to separable states under global operations. More important is the second bound which, as we show here, leads to non-trivial examples.

**Bound on post-selected non-locality.** — We can also extend the results of the previous section to non-trace-preserving maps, i.e. when the parties can perform a ‘filtering’ operation before the Bell test, the so called hidden non-locality scenario [32]. Popescu [32] showed that by performing a ‘filtering’ operation, and given that this operation succeeds, it is possible to obtain much larger violation of the CHSH game on the resulting state, not bounded as we claimed. However, we note that it is also important to take into account the probability of obtaining the ‘filtered’ result. For this reason, in order to quantify the effect of postselection, we propose to consider a rate of hidden non-locality, $R_H(\rho_{AB})$, defined as follows:

$$R_H(\rho_{AB}) \equiv \lim_{n \to \infty} \frac{1}{n} \sup_{\Lambda \in \text{LOCC}} \sup_{\{M_{xy}\}} \sup_{\rho_0} F_0 N(\{TrM_{xy}F_0(\Lambda(\rho^r_{AB}))\}).$$

Where a filtering process, $F_0$, takes state $\Lambda(\rho^r_{AB})$ to flag form, $F_0 = \sum_i |i\rangle \langle i| \otimes F_i r F_i^\dagger$ and later ensures all other results except the “good” one that leads to the highest violation of the Bell inequality. $\rho_0 = TrF_0 \Lambda(\rho^r_{AB}) F_0^\dagger$ is the probability that the filter results in the desired outcome. We can have the same bound for $R_H$, as for $R$.

**Theorem 3.** For any bipartite state $\rho_{AB}$ it holds that

$$R_H(\rho_{AB}) \leq T^\infty(\rho_{AB}) \leq E_r(\rho_{AB}).$$

For a bipartite PPT state $\rho_{AB}$ it holds that

$$R_H(\rho_{AB}) \leq E_r(\rho^r_{AB}).$$

**Application.** — An application of the Corollary [3] follows from the fact, that $E_r$ is asymptotically continuous [31], hence generally, if $\rho_0 \in \text{D}(\epsilon)$, for $\epsilon < \frac{1}{2}$ we have:

$$R(\rho^r_{AB}) \leq 4\epsilon \log d + 2h(\epsilon)$$

where $h(p) = -p \log p - (1-p) \log (1-p)$ is the binary Shannon entropy, and $d$ is the dimension of the system (due to Theorem [3] the same bound holds for $R_H$). Hence, if $\epsilon$ decreases with $d$ faster than $\frac{1}{\log d}$, the rate of non-locality vanishes with increasing dimension (note that for example, the family of states shown in eq. (12) have this property).

**Implications for cryptography.** — In [33, 34] it is shown that one can launch quantum key distribution (QKD) protocols based on shared private bits. Here we ask if such a protocol can be made device independent (DI). To this end, the private bit should violate some Bell inequality. Indeed any known DI QKD protocol [55–67] bases on some Bell inequality $S$, and admits some level of violation, say $\epsilon_v$, below which it aborts. Now, due to eqs. (11, 14) there are (approximate) private bits, which exhibit violation of inequality $S$ only up to $\epsilon' < \epsilon_v$, and hence will be aborted. This rules out such states from usage in this DI QKD protocol. Moreover, every realization of DI QKD has inevitable errors due to decoherence. In such a case, the level of violation $\epsilon'$ can be even below the precision of the experiment.

**Discussion.** — We have presented bounds on quantum non-locality, both, in the single copy case for arbitrary bipartite states, as well as in the asymptotic and hidden non-locality scenarios for states with positive partial transpose. To achieve this, we have explored the link between state discrimination by restricted class of operations and the subject of non-locality.

As future directions, for the single copy scenario, instead of discrimination from separable states, a refinement would be to consider the distance from states admitting local hidden-variable model, e.g. the class of Werner states with local model [9], which could possibly lead to tighter bounds. For the asymptotic and hidden non-locality scenarios, it would be interesting to extend
the bound for the rate of (hidden) non-locality to the case of NPT states. Also for these scenarios it would be worth finding new bounds for states invariant under partial transposition (especially for the ones containing private key [38, 39]).

It is worth noting, that our results are strongly related to the so called Peres conjecture [40], recently disproved in [41]. Namely, we have asked a quantitative rephrasing of the original question posed by Asher Peres: how much one can violate a Bell inequality with PPT states? We have shown that, for certain PPT states, the level of violation both for single copy as well as in terms of the relative entropy of (hidden) non-locality in the asymptotic cases, can be negligible. Notably, as we showed in the examples, even some states containing privacy admit such limited non-locality content, which limits their usage in DI QKD protocols. This is a counterintuitive result showing a fundamental gap, on the level of resources, between quantum and device independent cryptography due to presence of noise.

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[1] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[2] B. S. Tsirelson, J. Soviet. Math. 36, 557 (1987).
[3] M. Navascues, S. Pironio, and A. Acín, New J. Phys. 10, 073013 (2008).
[4] M. Junge, C. Palazuelos, D. Prez-Garcia, I. Villanueva, and M. Wolf, Communications in Mathematical Physics 300, 715 (2010).
[5] M. Junge and C. Palazuelos, Communications in Mathematical Physics 306, 695 (2011).
[6] C. Palazuelos (2012), arXiv:1206.3695.
[7] D. Prez-Garcia, M. Wolf, C. Palazuelos, I. Villanueva, and M. Junge, Communications in Mathematical Physics 279, 455 (2008).
[8] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[9] J. Barrett, Phys. Rev. A 65, 042302 (2002).
[10] R. C. Drumond and R. I. Oliveira, Phys. Rev. A 86, 012117 (2012).
[11] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Phys. Rev. Lett. 111, 030501 (2013).
[12] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 59, 1070 (1999).
[13] N. Yu, R. Duan, and M. Ying, IEEE Trans. Inf. Theory 60, 2069 (2014).
[14] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, IEEE Trans. Inf. Theory 48, 580 (2002).
[15] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, Phys. Rev. Lett. 86, 5807 (2001).
[16] T. Eggeling and R. F. Werner, Phys. Rev. Lett. 76, 097905 (2002).
[17] K. Horodecki, Ph.D. thesis, University of Warsaw (2008).
[18] S. Bäuml, M. Christandl, K. Horodecki, and A. Winter (2014), arXiv:1402.5927.
[19] P. Hyllus, O. Gühne, D. Bruß, and M. Lewenstein, Phys. Rev. A 72, 012321 (2005).
[20] B. M. Terhal, Physics Letters A 271, 319 (2000).
[21] N. Brunner and T. Vértesi, Phys. Rev. A 86, 042113 (2012).
[22] W. van Dam, R. Gill, and P. Grunwald, IEEE Trans. Inf. Theory 51, 2812 (2005).
[23] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2011), 10th ed., ISBN 1107002176, 9781107002173.
[24] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. Lett. 94, 160502 (2005).
[25] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, IEEE Trans. Inf. Theory 55, 1898 (2009).
[26] P. Horodecki and R. Augusiak, Phys. Rev. A 74, 010302 (2006).
[27] A. Peres, Phys. Rev. A 54, 2685 (1996).
[28] M. Piani, Phys. Rev. Lett. 103, 160504 (2009).
[29] A. Grudka, K. Horodecki, M. Horodecki, P. Horodecki, R. Horodecki, P. Joshi, W. Kłobus, and A. Wójcik, Phys. Rev. Lett. 112, 120401 (2014).
[30] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[31] B. Synak-Radtke and M. Horodecki, J. Phys. A: Math. Gen. 39, L423 (2006).
[32] S. Popescu, Phys. Rev. Lett. 74, 2619 (1995).
[33] K. Horodecki, M. Horodecki, P. Horodecki, D. Leung, and J. Oppenheim, IEEE Trans. Inf. Theory 54, 2604 (2008).
[34] K. Horodecki, D. Leung, H.-K. Lo, and J. Oppenheim, Phys. Rev. Lett. 96, 070501 (2006).
[35] J. Barrett, L. Hardy, and A. Kent, Phys. Rev. Lett. 95, 100503 (2005).
[36] E. Hanggi, R. Renner, and S. Wolf, EUROCRYPT pp. 216–234 (2010).
[37] J. Barrett, R. Colbeck, and A. Kent, Phys. Rev. A 86, 062326 (2012).
[38] K. Horodecki, L. Pankowski, M. Horodecki, and P. Horodecki, IEEE Trans. Inf. Theory 54, 2621 (2008).
[39] M. Ozols, G. Smith, and J. A. Smolin, Phys. Rev. Lett. 112, 110502 (2014).
[40] A. Peres, Found Phys. 29, 589 (1999).
[41] T. Vértesi and N. Brunner (2014), arXiv:1405.4502.
Supplemental Material:
Bounds on quantum non-locality via partial transposition

Here we present the detailed proofs of the results stated in the main text. We follow the same notation and all the numberings of equations and statements refer to the main text.

By bipartite box we refer to the conditional probability distribution of outputs $a$ and $b$ of Alice and Bob given inputs $x$ and $y$ respectively, $P(ab|xy)$. By the set $\mathcal{S} = \{s_{a,b}^{x,y}\}$ we denote the coefficients of a particular Bell inequality, so that $\sum_{a,b,x,y} s_{a,b}^{x,y} P(ab|xy)$ is the value of the Bell inequality $S$ on a particular box $P(ab|xy)$. When we refer to a Bell inequality $S$, we assume that all coefficients are positive, $s_{a,b}^{x,y} \geq 0$. We denote the maximal value of the Bell expression $S$ over all boxes in the classical case $C(S)$, quantum $Q(S)$ and super-quantum $NS(S)$. For a bipartite state $\rho_{AB}$ and the set of POVMs $\{M_{xy}\}$, $M_{xy} = \{A_{a|x} \otimes B_{b|y}\}$, we represent the corresponding box by $\{Tr_{AB} M_{xy} \rho_{AB}\}$, and denote the value of the Bell expression $S$ for this particular POVMs by $S(\rho_{AB})$, i.e. $S(\rho_{AB}) = Tr S_{AB}$ where $S = \sum_{a,b,x,y} s_{a,b}^{x,y} A_{a|x} \otimes B_{b|y}$.

Appendix A: Bounds for Bell inequalities.

We start by giving upper bounds on the maximum violation of a Bell inequality achieved by a quantum state in the single copy scenario.

**Corollary 1.** For any bipartite Bell expression $S$, and state $\rho$, it holds that:

$$Q_S(\rho) \leq C(S) + Q(S) \inf_{\sigma \in SEP} \|\rho^\Gamma - \sigma^\Gamma\|.$$

**Proof of Corollary 1.** First note that by substituting any separable state $\sigma$ in (1), and using the fact that $S(\sigma_{AB}) \leq C(S) \forall \sigma_{AB} \in SEP$ we have:

$$S(\rho) \leq C(S) + ||S^\Gamma|| \inf_{\sigma \in SEP} ||\rho^\Gamma - \sigma^\Gamma||. \quad (A1)$$

Now taking supremum over POVMs $\{A_{a|x} \otimes B_{b|y}\}$ on both sides we have the desired result.

Based on [17][18], we have by Corollary 1 an immediate observation that certain private states have a limited possibility of violating a Bell inequality.

**Observation 1.** For any bipartite Bell inequality $S$, if the states $\sqrt{XX^\dagger}$ and $\sqrt{X^\dagger X}$ are separable, then a private state $\gamma_X$, described by $X$ according to eq. (11), satisfies:

$$Q(\gamma_X) \leq C(S) + Q(S)||X^\Gamma||. \quad (A2)$$

While, as shown before, $||X^\Gamma||$ can be vanishing exponentially fast in number of qubits that composes $\gamma_X$.

**Proposition 1.** There exist bipartite states (see eq. (A7) below) $\rho \in B(C^2 \otimes C^2 \otimes (C^d \otimes C^d)^\otimes m)$ with $d = m^2$, $k = m$ satisfying $K_D(\rho^\Gamma \otimes \rho) \rightarrow 1$ with increasing $m$, such that:

$$Q_S(\rho \otimes \rho^\Gamma) \leq C(S) + \frac{Q(S)}{2^{m-1}}. \quad (A3)$$

**Proof of Proposition 1.** Consider $\rho$ defined in eq. (149) of [18]. It has a property, that its distillable key is almost 1, and there is a separable state $\sigma_\rho$ such that $||\rho^\Gamma - \sigma_\rho^\Gamma|| \leq p$, with $p = \frac{(q - q^m)}{2^{m} + 2(\frac{q^m}{2 - q})}$. Note that $p \leq \frac{1}{2m}$ for natural $m$. Knowing this, we bound the violation achieved by the above states in two steps. We first apply Theorem 1 to state $\rho \otimes \rho^\Gamma$, with $\sigma = \sigma_\rho \otimes \rho^\Gamma$, in order to obtain:

$$S(\rho \otimes \rho^\Gamma) \leq S(\sigma_\rho \otimes \rho^\Gamma) + Q(S)||\rho \otimes \rho^\Gamma|| - (\sigma_\rho \otimes \rho^\Gamma)||,$$

which in turn is bounded by

$$S(\sigma_\rho \otimes \rho^\Gamma) + Q(S)||\rho^\Gamma - \sigma^\Gamma||. \quad (A4)$$
We now use the fact that \( \sigma_r \) is separable, and that we can write \( \rho^\Gamma = (1 - r)\rho_{\text{sep}} + r\rho_{n\text{sep}} \), with \( \rho_{\text{sep}} \in \text{SEP} \) and \( r \leq p \). By linearity of trace we obtain:

\[
S(\sigma_r \otimes \rho^\Gamma) = (1 - r)S(\sigma_r \otimes \rho_{\text{sep}}) + rS(\sigma_r \otimes \rho_{n\text{sep}}).
\]  
(A5)

First term of RHS is bounded by \( C(S) \), as the state \( \sigma_r \otimes \rho_{\text{sep}} \) is separable. The second term is bounded by \( Q(\sigma_r \otimes \rho_{n\text{sep}}) \), which is in fact equal to \( Q(\rho_{n\text{sep}}) \).

This lead us to the following bound:

\[
S(\rho \otimes \rho^\Gamma) \leq C(S) + pQ(\rho_{n\text{sep}}) + pQ(S) \leq C(S) + 2pQ(S) \leq C(S) + \frac{Q(S)}{2^{m-1}},
\]  
(A6)

which proves the result.

An example of states satisfying Proposition 1 is the family \( \hat{\rho}_{p,d,k,m} \) on \( B \left( C^2 \otimes C^2 \otimes (C^d \otimes C^d)^{\otimes m} \right) \) [24] (see [18]).

Their matrix form is given below, up to the normalization factor \( N_m = 2(p^m) + 2(\frac{1}{2} - p)^m \):

\[
\begin{bmatrix}
[p(\frac{1}{2} - \tau_2)]^{\otimes m} & 0 & 0 & [p(\frac{1}{2} - \tau_2)]^{\otimes m} \\
0 & [(\frac{1}{2} - p)\tau_2]^{\otimes m} & 0 & 0 \\
0 & 0 & [(\frac{1}{2} - p)\tau_2]^{\otimes m} & 0 \\
[p(\frac{1}{2} + \tau_2)]^{\otimes m} & 0 & 0 & [p(\frac{1}{2} + \tau_2)]^{\otimes m}
\end{bmatrix}.
\]  
(A7)

\( \tau_1 = (\frac{\rho_a + \rho_s}{2})^{\otimes k} \) and \( \tau_2 = (\rho_s)^{\otimes k} \), while \( \rho_s \) and \( \rho_a \) are the \( d \)-dimensional symmetric and antisymmetric Werner state, respectively.

### Appendix B: Bound on asymptotic non-locality

To treat the asymptotic scenario we introduce the restricted regularized relative entropy of non-locality. This quantity is an entanglement measure, and is related to the relative entropy of non-locality:

\[
T^\infty(\rho_{AB}) \equiv \lim_{n \to \infty} \frac{1}{n} \sup_{\{P(x,y)\} \in \text{LOCC}} \sup_{\Lambda \in \text{SEP}} \sup_{\sigma \in \text{SEP}} \sum_{x,y} p(x,y)D(\{\text{Tr}M_{xy}\Lambda(\rho)\}||\{\text{Tr}M_{xy}\Lambda(\sigma)\}).
\]  
(B1)

Note that in the expression \( D(\{\text{Tr}M_{xy}\Lambda(\rho)\}||\{\text{Tr}M_{xy}\Lambda(\sigma)\}) \) we can treat \( \{\text{Tr}M_{xy}\Lambda(\rho)\} \) as a diagonal matrix with elements given by the probability distribution \( P(ab|xy) \), and then \( D(\cdot||\cdot) \) is the quantum relative entropy. A similar quantity has been introduced by Piani in [23]. This function is easier to deal with than \( R(\rho) \).

Recalling the definitions introduced in the main text, we use the relative entropy as a measure of non-locality

\[
\mathcal{N}(P) = \sup_{\{p(x,y)\}} \inf_{P_L(\cdot)} \sum_{x,y} p(x,y)D(P_{xy}(ab|xy)||P_L(ab|x,y)),
\]  
(B2)

and for the asymptotic scenario we define the rate of non-locality:

\[
R(\rho_{AB}) \equiv \lim_{n \to \infty} \frac{1}{n} \sup_{\Lambda \in \text{LOCC}} \sup_{\{M_{xy}\}} \mathcal{N}(\{\text{Tr}M_{xy}\Lambda(\rho_{AB}^{\otimes n})\}).
\]  
(B3)

In all the following proofs we consider optimization over the probability distribution of the inputs \( \{p(x,y)\} \), but one can also restrict to the uniform case [22] and all the results follow in the same way.

We are now ready to prove Theorems 2.

**Theorem 2.** For any bipartite state it holds that

\[
R(\rho_{AB}) \leq T^\infty(\rho_{AB}) \leq E_r(\rho_{AB}).
\]  
(B4)

For \( \rho_{AB} \) a PPT state, it holds that

\[
T^\infty(\rho_{AB}) \leq E_r(\rho_{AB}^\Gamma).
\]  
(B5)
Proof of Theorem 2. We first prove that \( R \leq T^\infty \). In the first step let us note that \( \Lambda (\sigma^{\otimes n}) \) is a separable state, since \( \Lambda \) is an LOCC operation. Hence, if we place infimum over all separable states \( \sigma \) instead of that of the form \( \Lambda (\sigma^{\otimes n}) \) in definition of \( T^\infty \), we may only decrease the quantity. Second, we observe that instead of obtaining the local quantum box via the same POVMs, \( \{ M_{xy} \} \), as for \( \Lambda (\rho^{\otimes n}) \), we can place also infimum over all \( M_{xy} \) acting on \( \sigma \), which also can only lower the quantity. In the last step we observe that the set of such obtained quantum boxes is included in the set of the local ones, hence we can place infimum over the latter instead, which proves the desired result.

Now we prove the relation \( T^\infty (\rho) \leq E_r (\rho^f) \) for PPT states. The proof of \( T^\infty (\rho) \leq E_r (\rho^f) \) follows in the same way, without use of partial transposition. To prove that \( T^\infty (\rho) \leq E_r (\rho^f) \) note that, since \( \Lambda \in LOCC \), it has a separable representation: \( \Lambda (\rho) = \sum_{i,j} C_i \otimes D_j (\rho^{\otimes n}) \). Using properties of trace and the separable representation we focus now on the term:

\[
D(\{ Tr \sum_{ij} C_i^{\dagger} A_{a|x} C_i \otimes D_j^{\dagger} B_{b|y} D_j \rho^{\otimes n} \}) \leq \{ Tr \sum_{ij} C_i^{\dagger} A_{a|x} C_i \otimes D_j^{\dagger} B_{b|y} D_j \sigma^{\otimes n} \}).
\]  

(B6)

Applying to both its components the identity \( Tr XY = Tr X^T Y^T \), we have

\[
D(\{ Tr \sum_{ij} C_i^{\dagger} A_{a|x} C_i \otimes (D_j^{*})^{\dagger} B_{b|y}^{\dagger} D_j^{*} (\rho^T)^{\otimes n} \}) \leq \{ Tr \sum_{ij} C_i^{\dagger} A_{a|x} C_i \otimes (D_j^{*})^{\dagger} B_{b|y}^{\dagger} D_j^{*} (\sigma^T)^{\otimes n} \}),
\]  

(B7)

which implies that \( T^\infty \) can be written as:

\[
\lim_{n \to \infty} \frac{1}{n} \sup_{\Lambda ' \in LOCC} \sup_{\{ M_{xy} \}} \sup_{\rho (x,y)} \inf_{\sigma \in SEP} \sum_{x,y} p(x,y) D(\{ Tr M_{xy}^{\dagger} \Lambda ' (\rho^T)^{\otimes n} \}) || \{ Tr M_{xy}^{\dagger} \Lambda ' (\sigma^T)^{\otimes n} \}) \]  

with \( \Lambda ' \) being a new separable operation with \( D_j \) operators complex conjugated, and \( M' \) being a new set of POVMs with \( B_{b|y} \) transposed. Now, since \( \sigma^T \) is also a separable state, and by the fact that the relative entropy is non-increasing under completely positive trace-preserving maps, we have:

\[
\inf_{\sigma ' \in SEP} \sum_{x,y} p(x,y) D(\{ Tr M_{xy}^{\dagger} \Lambda ' (\rho^T)^{\otimes n} \}) || \{ Tr M_{xy}^{\dagger} \Lambda ' (\sigma^T)^{\otimes n} \}) \leq \inf_{\sigma ' \in SEP} \sum_{x,y} p(x,y) D(\{ (\rho^T)^{\otimes n} || \sigma^{\otimes n} \}).
\]

(B9)

We finally use the identity \( D(\{ (\rho^T)^{\otimes n} || \sigma^{\otimes n} \}) = n D(\rho^T || \sigma) \). Since the latter term does not depend on \( \Lambda \) and \( M \) as well as \( p(x,y) \), and the number of copies \( n \) cancels with the regularization term \( \frac{1}{n} \), we obtain via (B8) and (B9) the bound:

\[
T^\infty (\rho) \leq \inf_{\sigma ' \in SEP} D(\rho^T || \sigma) \equiv E_r (\rho^f).
\]

(B10)

□

Appendix C: Bound on post-selected non-locality

We can also treat the case where a filtering (non-trace-preserving) operation is performed before the Bell test. To quantity the non-locality achieved in this case we define the rate of hidden non-locality:

\[
R_H (\rho_{AB}) = \lim_{n \to \infty} \frac{1}{n} \sup_{\Lambda \in LOCC} \sup_{\{ M_{xy} \}} \sup_{F_0} p_{F_0} (\{ Tr M_{xy} \Lambda (F_{0}(\rho_{AB}^{\otimes n})) \}),
\]

(C1)

where a filtering process, \( F_0 \), takes state \( \Lambda (\rho_{AB}^{\otimes n}) \) to flag form, \( F (\rho) = \sum_i | i \rangle \langle i | \otimes F_i \rho F_i^{\dagger} \), and later erases all other results except the “good” one that leads to the highest violation of the Bell inequality. \( p_{F_0} = Tr F_0 \Lambda (\rho_{AB}^{\otimes n}) F_0^{\dagger} \) is the probability that the filter results in the desired outcome.

Analogously to the Theorem 2, for the hidden non-locality scenario, we have the following result:

Theorem 3. For any bipartite state \( \rho_{AB} \) it holds that

\[
R_H (\rho_{AB}) \leq T^\infty (\rho_{AB}) \leq E_r (\rho_{AB}).
\]

(C2)

For a bipartite PPT state \( \rho_{AB} \) it holds that

\[
R_H (\rho_{AB}) \leq E_r (\rho_{AB}^f).
\]

(C3)
Proof of Theorem 3. We just have to show that \( R_H(\rho_{AB}) \leq T^\infty(\rho_{AB}) \) and (C3) follows from (B5).

Let us consider:

\[
T_n(\rho_f) \equiv \frac{1}{n} \sup_{\Lambda \in \text{LOCC}} \sup_{M_{xy}} \sup_{\sigma \in \text{SEP}} \inf_{p(x,y)} \sum_{x,y} p(x,y) D(\{ \text{Tr} M_{xy} \Lambda(\rho^{\otimes n}) \} || \{ \text{Tr} M_{xy} \Lambda(\sigma^{\otimes n}) \}). \tag{C4}
\]

Now let us restrict to a map \( \Lambda \) of the form \( \Lambda = F \circ \Lambda_0 \), where \( \Lambda_0 \) is an arbitrary LOCC operation that acts on \( n \) copies of the system, and is followed by measurement \( F \), i.e. \( \Lambda(\rho^{\otimes n}) = \sum_i p^F_i |i\rangle\langle i| \otimes F_i(\Lambda_0(\rho^{\otimes n}))F_i^\dagger \).

This restriction just decrease the RHS, so we have

\[
T_n(\rho^{\otimes n}) \geq \frac{1}{n} \sup_{\Lambda_0 \in \text{LOCC}} \sup_{F} \sup_{\{ M_{xy} \}} \sup_{p(x,y)} \sup_{\sigma \in \text{SEP}} \inf_{q(x,y)} \sum_{x,y} p(x,y) D(\{ \text{Tr} M_{xy} \Lambda_0(\rho^{\otimes n}) \} || \{ \text{Tr} M_{xy} \Lambda_0(\sigma^{\otimes n}) \} || \{ \text{Tr} M_{xy} F_i(\Lambda_0(\rho^{\otimes n}))F_i^\dagger \} || \{ \text{Tr} M_{xy} F_i(\Lambda_0(\sigma^{\otimes n}))F_i^\dagger \}). \tag{C5}
\]

where \( p^{F_i} = \text{Tr} F_i(\Lambda_0(\rho^{\otimes n}))F_i^\dagger \) and \( q^{F_i} = \text{Tr} F_i(\Lambda_0(\sigma^{\otimes n}))F_i^\dagger \).

Using the following property of relative entropy [28]:

\[
D \left( \sum_i p_i \rho_i \otimes |i\rangle\langle i| \bigg| \sum_i q_i \sigma_i \otimes |i\rangle\langle i| \right) = \sum_i p_i D(\rho_i || \sigma_i) + D(p||q), \tag{C6}
\]

we obtain

\[
T_n(\rho^{\otimes n}) \geq \frac{1}{n} \sup_{\Lambda_0 \in \text{LOCC}} \sup_{F_0} \sup_{\{ M_{xy} \}} \sup_{p(x,y)} \sup_{\sigma \in \text{SEP}} \inf_{q(x,y)} \sum_{x,y} p(x,y) D(\{ \text{Tr} M_{xy} F_0(\Lambda(\rho^{\otimes n}))F_0^\dagger \} || \{ \text{Tr} M_{xy} F_0(\Lambda(\sigma^{\otimes n}))F_0^\dagger \})). \tag{C7}
\]

where we have dropped the terms \( D(p||q) \geq 0, \sum_{F_i \neq F_0} p^{F_i} D(\rho_i || \sigma_i) \geq 0 \).

Now note that the RHS is an upper bound for \( R_H \) and then we have the desired result. \( \square \)