QUASISTRICT SYMMETRIC MONOIDAL 2-CATEGORIES VIA WIRE DIAGRAMS

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Abstract. In this paper we give an expository account of quasistrict symmetric monoidal 2-categories as introduced by Schommer-Pries. We reformulate the definition using a graphical calculus called wire diagrams, which facilitates computations and emphasizes the central role played by the interchanger coherence isomorphisms.

1. Introduction

Establishing the definition of a symmetric monoidal bicategory, and proving associated coherence and strictification results, has been a considerable effort by a number of authors [18, 19, 2, 7, 6, 1, 22, 11, 12, 14, 13, 21, 25]; see also the references in [23]. Recently, Schommer-Pries has defined a stricter version of a symmetric monoidal bicategory, called a quasistrict symmetric monoidal 2-category, and proved the following strictification result:

\textbf{Theorem 1 ([23])}. Every symmetric monoidal bicategory is equivalent to a quasistrict symmetric monoidal 2-category.

In this paper we give an expository account of this result, by introducing a graphical notation which we call wire diagrams. The utility of this notation is twofold. Firstly, wire diagrams offer a simple visual explanation for what is going on. Secondly, wire diagrams facilitate working with these structures and making actual computations. In fact, this was the motivation for the coherence result above. As part of a project related to three-dimensional topological quantum field theory, we found ourselves working in a symmetric monoidal bicategory presented by generators and relations [5, 4, 3]. The calculations involved were all expressed in this graphical calculus, and it would have been intractable to perform them without it.

Ordinary algebra is about manipulating a string of symbols on a line. One can think of algebraic manipulations in a symmetric monoidal bicategory as being a form of stable 3-dimensional algebra. Wire diagrams are one possible notation for this. The basic idea is that the tensor product direction runs out of the page, composition of 1morphisms and (horizontal) composition of 2morphisms runs up the page, and (vertical) composition of 2morphisms runs from left-to-right\textsuperscript{1}:

\textsuperscript{1}Unfortunately, what is usually called vertical composition of 2morphisms runs horizontally in wire diagrams, and what is usually called horizontal composition runs vertically!
To make such a diagram clearer, it will usually just be drawn flat in the page (but the three-dimensional picture should be kept in mind), like this:

\[
\begin{array}{c}
\begin{array}{c}
\alpha
\end{array}
\end{array}
\]
Slimming down the definition of quasistrict symmetric monoidal 2-category in this way makes it more suitable for a diagrammatic calculus, as well as, we hope, psychologically more pleasant. However, we view this distinction between the ‘stringent’ and ‘quasistrict’ forms of the definition as only a technical one, which is explicitly made in this paper for the purpose of precision; other authors may choose not to make this distinction, leaving it implicitly understood.

This paper is structured as follows. In Section 2 we introduce wire diagrams in the familiar setting of 2-categories. In Section 3 we review semistrict monoidal 2-categories. In Section 4 we introduce stringent monoidal 2-categories, extend the wire diagram notation to this setting, and prove that a stringent monoidal 2-category is the same thing as a semistrict monoidal 2-category. In Section 5 we introduce stringent symmetric monoidal 2-categories, extend the wire diagram notation to this setting, and prove that a stringent symmetric monoidal 2-category is the same thing as a quasistrict symmetric monoidal 2-category.

**Notation.** We will use the convention that ‘2-category’ refers to a *strict* bicategory. Bicategories and 2-categories $\mathbf{M}$ will be written in bold font and categories $C$ in plain font.

**Remark.** The wire diagram notation can be extended in a straightforward way to give a natural graphical calculus for semistrict braided monoidal 2-categories (in the sense of [18, 6, 2, 12] too, though we do not do this here.

## 2. Wire diagrams for 2-categories

In this section we introduce wire diagrams in the setting of 2-categories. Let $\mathbf{M}$ be a 2-category. The objects $A, B, \ldots$ of $\mathbf{M}$ are drawn as:

\[
\begin{array}{c}
A \\
\downarrow \\
A
\end{array}
\]

A 1-morphism $f: A \to B$ is drawn as:

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

Note that composition of 1-morphisms runs from *bottom to top*! If $f, g: A \Rightarrow B$ are 1-morphisms, then a 2-morphism $\alpha: f \Rightarrow g$ is drawn as:

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array}
\quad
\begin{array}{c}
\bigcirc \\
\downarrow
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
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\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\bigcirc
\end{array}
\]
If \( f : A \to B \) and \( g : B \to C \) are 1-morphisms, then their composite \( g \circ f : A \to C \) is drawn by stacking them on top of each other:

\[
\begin{array}{c}
A \\
\downarrow \quad g \circ f \quad \equiv \\
B \\
\downarrow \quad f \quad \equiv \\
C \\
\end{array}
\]

Usually in a 2-category, we think of there being two composition laws for 2-morphisms: horizontal and vertical composition. In wire diagrams we will single out ‘vertical composition’ as the primary operation (which we will just call composition of 2-morphisms for simplicity), and describe horizontal composition in terms of whiskering. So, if \( \alpha : f \Rightarrow g \) and \( \beta : g \Rightarrow h \) are 2-morphisms, then their composite \( \beta \circ \alpha : f \Rightarrow h \) is drawn as:

\[
\begin{array}{c}
A \\
\downarrow \quad f \quad \equiv \\
B \\
\downarrow \quad \alpha \quad \equiv \\
C \\
\downarrow \quad g \quad \equiv \\
B \\
\downarrow \quad \beta \quad \equiv \\
D \\
\end{array}
\]

Suppose \( A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D \) are a composable triple of 1-morphisms, and that \( \alpha : f_2 \Rightarrow g \) is a 2-morphism. Then as usual we can whisker \( \alpha \) with the identity 2-morphisms on \( f_1 \) and \( f_3 \) respectively to obtain

\[
\text{id}_{f_3} * \alpha * \text{id}_{f_1} : f_3 \circ f_2 \circ f_1 \Rightarrow f_3 \circ g \circ f_1.
\]

Whiskering is drawn by enclosing the source of the 2-morphism with a box. So, the diagram

\[
\begin{array}{c}
D \\
\downarrow \quad f_3 \\
C \\
\downarrow \quad \alpha \quad \equiv \\
B \\
\downarrow \quad g \quad \equiv \\
B \\
\downarrow \quad f_2 \\
A \\
\end{array}
\]

stands for the 2-morphism \( \text{id}_{f_3} * \alpha * \text{id}_{f_1} \). As is well known, the usual ‘horizontal composition’ of 2-morphisms in a 2-category can be described solely in terms of ‘vertical composition’, and whiskering. So, if \( f_1, g_1 : A \to B \) and \( f_2, g_2 : B \to C \) are 1-morphisms, and \( \alpha : f_1 \Rightarrow g_1 \) and \( \beta : f_2 \Rightarrow g_2 \) are 2-morphisms, then we have:

\[
(2) \quad \beta * \alpha = (\beta * \text{id}_{f_1}) \circ (\text{id}_{f_2} * \alpha) = (\text{id}_{f_2} * \alpha) \circ (\beta * \text{id}_{f_1})
\]
We can view this equation as defining horizontal composition. In wire diagrams, we will draw $\beta \ast \alpha$ as if it is being applied simultaneously. So, the equations (2) look as follows in wire diagrams:

\[
\begin{array}{ccc}
\begin{array}{c}
C \\
\downarrow f_2 \\
B
\end{array} & \xrightarrow{\alpha} & \\
\begin{array}{c}
C \\
\downarrow f_1 \\
A
\end{array} & := & \\
\begin{array}{c}
C \\
\downarrow g_2 \\
B
\end{array} & \xleftarrow{\beta} & \\
\begin{array}{c}
C \\
\downarrow g_1 \\
A
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
C \\
\downarrow f_2 \\
B
\end{array} & \xrightarrow{\beta} & \\
\begin{array}{c}
C \\
\downarrow f_1 \\
A
\end{array} = & \\
\begin{array}{c}
C \\
\downarrow g_2 \\
B
\end{array} & \xleftarrow{\alpha} & \\
\begin{array}{c}
C \\
\downarrow g_1 \\
A
\end{array}
\end{array}
\]

3. Semistrict monoidal 2-categories

In this section we recall the notion of a semistrict monoidal 2-category \cite{2, 7, 19, 18} in the formulation of Crans \cite{6}.

The category $2\text{Cat}$ of strict 2-categories and strict 2-functors can be equipped with the Gray tensor product $\otimes^G$ making it into a monoidal category \cite{22} (see \cite{11} for an exposition). The most important feature of the Gray tensor product $C \otimes^G D$ of two strict 2-categories is that the objects of $C \otimes^G D$ are the same as the objects of $C \times D$, and that for every pair of 1-morphisms $f : A \to A'$ in $C$ and $g : B \to B'$ in $D$ there is a 2-isomorphism\footnote{Note that our convention runs counter to that of Baez and Neuchl \cite{2}, but when thought of as a cubical functor fits the standard definition of pseudofunctor \cite{20} correctly.}

\[
\begin{array}{ccc}
(A, B) & \xrightarrow{\gamma_{f, g}} & (A', B') \\
\downarrow \text{id}_A \otimes^G g & & \downarrow \text{id}_{A'} \otimes^G g \\
(A, B') & \xleftarrow{\gamma_{f, g}^{-1}} & (A', B')
\end{array}
\]

in $C \otimes^G D$. A **semistrict monoidal 2-category** is then usually defined as a monoid in the monoidal category $(2\text{Cat}, \otimes^G)$. In this way, Gray categories are used as a technical construct to avoid leaving the world of strict 2-categories and strict 2-functors. However, the explicit algebraic definition of the Gray tensor product $C \otimes^G D$ is rather awkward, given by a long
list of generators and relations [11, Section 5.1]. In practice, the notion of a cubical functor is used instead.

**Definition 2.** Suppose $C$, $C'$ and $D$ are strict 2-categories. A cubical functor $F: C \times C' \to D$ is a pseudofunctor whose coherence isomorphisms

$\Phi_{(f',g'),(f,g)}: F(f', g') \circ F(f, g) \Rightarrow F(f'f, g'g)$

are the identity 2-morphism if $f' = \text{id}$ or if $g = \text{id}$.

Note that we have not listed unit coherence 2-isomorphisms $u_{(A,B)}: \text{id}_F(F(A, B)) \Rightarrow F(\text{id}_A, \text{id}_B)$ as part of the data of a cubical functor, since it follows from (a) the fact that all the 2-categories are strict, (b) the cubical condition, and (c) the unit equation on $u_{(A,B)}$ in a pseudofunctor, that each $u_{(A,B)}$ must be the identity.

**Proposition 3.** ([10], [9], [11, Thm 5.2.5]) There is a canonical isomorphism

$$\text{Cub}(C \times C', D) \cong \text{2Cat}(C \otimes_G C', D)$$

between the set of cubical functors from $C \times C'$ to $D$ and the set of strict 2-functors from $C \otimes_G C'$ to $D$.

Let us write $2\text{Cat}^\text{ps}$ for the category whose objects are strict 2-categories and whose morphisms are pseudofunctors. It forms a monoidal category $(2\text{Cat}^\text{ps}, \times)$ under Cartesian product of 2-categories. With the above discussion in mind, the following definition is normally used in practice (if not explicitly so then implicitly so!).

**Definition 4.** A semistrict monoidal 2-category is a monoid $(M, 1, \otimes, \{\Phi_{(f',g'),(f,g)}\})$ in the monoidal category $(2\text{Cat}^\text{ps}, \times)$ whose tensor product pseudofunctor $(\otimes, \Phi): M \times M \to M$

is cubical.

An alternative, possibly more natural, way to define a semistrict monoidal 2-category is to start with the definition of a fully weak monoidal bicategory [25, 23] and then impose strictness conditions on the coherence data.

**Definition 5.** A semistrict monoidal 2-category is a monoidal bicategory $M$ such that:

- $M$ is a strict 2-category;
- The transformations $\alpha, r, \pi, \mu$ and $\rho$ are identities. Moreover the inverse adjoint equivalences $\alpha^*, l^*$ and $r^*$ are also identities with trivial adjunction data.
- The functor $\otimes = (\otimes, \Phi_{(f',g'),(f,g)}, \Phi_{A,B})$ is cubical, and $\Phi_{A,B}$ is the identity for all objects $A, B$.

If we unravel the many diagrams defining a monoidal bicategory from [25, 23], and impose the above equations, we conclude that these two definitions are identical.

In fact, this definition contains redundant information.

**Definition 6.** Let $M$ be a semistrict monoidal 2-category. The underlying interchanger is the collection of 2-isomorphisms

$\phi_{f,g} := \Phi_{(f,\text{id}), (\text{id}, g)}: (f \otimes \text{id}) \circ (\text{id} \otimes g) \Rightarrow (\text{id} \otimes g) \circ (f \otimes \text{id})$

where $f, g$ are 1-morphisms in $M$. 

(4)
Note that the target of $\Phi_{(f, id), (id, g)}$ in (4) makes sense, since if we unravel the definitions, we obtain:

\[
\begin{array}{c}
(f \otimes id) \circ (id \otimes g) \\
\Phi_{(f, id), (id, g)} \\
\end{array}
\Rightarrow
\begin{array}{c}
(f \circ id) \otimes (id \circ g) \\
\end{array}
\]
\[
\begin{array}{c}
(id \otimes g) \circ (f \otimes id) \\
\Phi_{(id, g), (f, id)}^{-1} = id \\
\end{array}
\Rightarrow
\begin{array}{c}
(id \circ f) \otimes (g \circ id) \\
\end{array}
\]

The following lemma is standard. We will give a graphical proof in terms of wire diagrams in part 3 of Proposition 9 below.

**Lemma 7.** ([23], [16, Lemma 2.15]) The coherence 2-isomorphisms $\Phi_{(f', g'), (f, g)}$ are uniquely determined by the underlying interchanger 2-isomorphisms $\phi_{f, g}$.

### 4. Stringent Monoidal 2-Categories

In this section we introduce stringent monoidal 2-categories, extend the wire diagram notation to them, and prove that they are equivalent to semistrict monoidal 2-categories.

#### 4.1. The Definition

In the light of Lemma 7, it is convenient to formulate the notion of a semistrict monoidal 2-category purely in terms of the interchanger 2-isomorphisms. This has been the approach in [2, 16]. We make the following definition, apologizing to the reader for the burden of excessive terminology, in the hope that it is compensated for by the boon of greater precision.

**Definition 8.** A stringent monoidal 2-category $\mathbf{M} = (M, 1, \otimes, \{\phi_{f, g}\})$ consists of:

- a strict 2-category $M$,
- an object $1 \in M$, drawn as the invisible wire:

(i) An object $1 \in M$, drawn as the invisible wire:

(ii) For any two objects $A, B \in M$, an object $A \otimes B \in M$, drawn as:

\[
\begin{array}{c}
A \\
\downarrow \\
A \otimes B \\
\end{array}
\]

(iii) The tensor product of objects is strictly associative and unital. Moreover, for each object $A \in M$, it extends to a strict 2-functor $L_A := A \otimes -$ and $R_A := - \otimes A$. These 2-functors satisfy $L_AL_B = L_{A\otimes B}$, $R_BR_A = R_{A\otimes B}$ and $L_AR_B = R_BL_A$ for all $A, B \in M$. 

Let us pause here to explain the wire diagram notation more precisely. Each wire diagram representing a 1-morphism is to be evaluated into a 1-morphism in \( \mathbf{M} \) according to the prescription \textit{tensor first, then compose}. For instance, the diagram

\[
\begin{array}{c}
A'' \\
\downarrow^f \\
A' \\
\downarrow^g \\
A \\
\end{array} \quad \begin{array}{c}
B'' \\
\downarrow^{f'} \\
B' \\
\end{array} 
\]

is to be evaluated as follows. First, draw horizontal lines to separate the diagram into its indecomposable pieces. The regions between the horizontal lines evaluate to tensor products of objects, and the horizontal lines evaluate to tensor products of 1-morphisms. Then, compose the 1-morphisms together:

\[
\begin{array}{c}
A'' \\
\downarrow^f \\
A' \\
\downarrow^g \\
A \\
\end{array} \otimes \begin{array}{c}
B'' \\
\downarrow^{f'} \\
B' \\
\end{array} = \begin{array}{c}
(id_{A''} \otimes g') \circ (f' \otimes id_B) \circ (f \otimes g) \\
\end{array}.
\]

So, the wire diagram (5) evaluates to the 1-morphism

\[
(id_{A''} \otimes g') \circ (f' \otimes id_B) \circ (f \otimes g).
\]

in \( \mathbf{M} \).

With this prescription, we can interpret (iii) as follows. Functoriality at the level of 1-morphisms means that equations between composites of 1-morphisms are \textit{local}, that is they remain true after arbitrarily tensoring on the left and right and pre- and post-composition:
Functoriality at the level of 2-morphisms is similar: if an equation between composites of 2-morphisms holds, then it continues to hold after arbitrarily tensoring the left and right hand sides and pre- and post-composing with 1-morphisms.

If a 2-morphism $\alpha : f \Rightarrow g$ is surrounded by tensor products and composites of 1-morphisms, then we use a box to indicate where $\alpha$ is acting. So for instance,

\[
\begin{array}{c}
\begin{array}{ccc}
A' & B''' & C' \\
\downarrow h_1 & \downarrow h_2 & \downarrow h_3 \\
A & B & C
\end{array}
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
\begin{array}{ccc}
A' & B'' & C' \\
\downarrow h_1 & \downarrow h_2 & \downarrow h_3 \\
A & B' & C
\end{array}
\end{array}
\]

evaluates in gory detail as

\[
\text{id}_{\text{id}_{A'} \otimes h_2 \otimes \text{id}_{C'}} \circ (\text{id}_{h_3} \otimes \alpha \otimes \text{id}_{h_1}) \circ \text{id}_{\text{id}_{A} \otimes h_4 \otimes \text{id}_{C'}}.
\]

At this point the utility of the wire diagrams notation starts to become clear!

(iv) For every pair of 1-morphisms $f : A \to A'$ and $g : B \to B'$, an interchanger 2-isomorphism

\[
\phi_{f,g} : (f \otimes \text{id}_{B'}) \circ (\text{id}_{A} \otimes g) \Rightarrow (\text{id}_{A'} \otimes g) \circ (f \otimes \text{id}_{B})
\]

drawn as:

\[
\begin{array}{c}
\begin{array}{ccc}
A' & B' & A' & B' \\
\Downarrow f & \Downarrow \phi_{f,g} & \Downarrow i & \Downarrow f \\
A & B & A & B
\end{array}
\end{array}
\]

We pause here to unpack a crucial identity. If $\phi$ is the underlying interchanger of the coherence isomorphisms $\Phi$ in a semistrict monoidal 2-category, then the cubical equation on $\Phi$ implies the following in wire diagrams:

\[
\begin{array}{c}
\begin{array}{ccc}
A' & B' & A' & B' \\
\Downarrow \phi & \Downarrow \text{id} & \Downarrow \text{id} & \Downarrow \phi \\
A & B & A & B
\end{array}
\end{array}
\]

The first equation follows from left-tensoring and right-tensoring being strict 2-functors. The third equation follows since $\mathbf{M}$ is a strict 2-category. The second equation follows from $\Phi_{(\text{id}_{g}, (f, \text{id}))} = \text{id}$. To emphasize: although the interchanger (6) is nontrivial, we at least have the following identity, which we take as an axiom in a stringent monoidal 2-category.
(v) For all 1-morphisms $f: A \to A'$ and $g: B \to B'$, we have:

\[
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow f & \downarrow g \\
\text{A} & \text{B}
\end{array}
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow f & \downarrow g \\
\text{A} & \text{B}
\end{array} =
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow f & \downarrow g \\
\text{A} & \text{B}
\end{array}
\]

This is called nudging in [22]. Note that if we had adopted the ‘opcubical’ convention on cubical functors as in [16], nudging would have worked in the opposite direction.

(vi) For all 1-morphisms $f: A \to A'$, $g: B \to B'$, $h: C \to C'$ we have

\[\phi_{\text{id}_A \otimes g, h} = \text{id}_A \otimes \phi_{g, h}, \quad \phi_{f \otimes \text{id}_B, h} = \phi_{f, \text{id}_B} \otimes h, \quad \text{and} \quad \phi_{f, g} \otimes \text{id}_C = \phi_{f, g} \otimes \text{id}_C.\]

The first equation says that

\[
\begin{array}{c}
\text{A} & \text{B'} & \text{C'} \\
\downarrow f & \downarrow \text{id} & \downarrow \phi_{\text{id}_A \otimes g, h} \\
\text{A} & \text{B} & \text{C}
\end{array}
\begin{array}{c}
\text{A} & \text{B'} & \text{C'} \\
\downarrow f & \downarrow \text{id} & \downarrow \phi_{g, h} \\
\text{A} & \text{B} & \text{C}
\end{array} =
\begin{array}{c}
\text{A} & \text{B'} & \text{C'} \\
\downarrow f & \downarrow \text{id} & \downarrow \phi_{g, h} \\
\text{A} & \text{B} & \text{C}
\end{array}
\]

and similarly for the other two equations.

(vii) For all 1-morphisms $f: A \to A'$ and $g: B \to B'$ we have $\phi_{f, \text{id}} = \text{id}$ and $\phi_{\text{id}, g} = \text{id}$. In diagrams:

\[
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow \phi_{f, \text{id}_B} & \downarrow \text{id} \\
\text{A} & \text{B}
\end{array}
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow \phi_{f, \text{id}_B} & \downarrow \text{id} \\
\text{A} & \text{B}
\end{array} =
\begin{array}{c}
\text{A'} & \text{B'} \\
\downarrow \phi_{f, \text{id}_B} & \downarrow \text{id} \\
\text{A} & \text{B}
\end{array}
\]

plus the other version of this equation (where $f$ occurs on the right).
(viii) For all 1-morphisms \( f: A \rightarrow A' \), \( g: B \rightarrow B' \) and 2-morphisms \( \alpha: f \Rightarrow f' \), the following equation holds:

\[
\begin{array}{c}
A' & \overset{f}{\rightarrow} & B' \\
\downarrow & & \downarrow \\
A & \overset{g}{\rightarrow} & B
\end{array}
\begin{array}{c}
A' & \overset{f'}{\rightarrow} & B' \\
\downarrow & & \downarrow \\
A & \overset{g'}{\rightarrow} & B
\end{array}
\]

\[
\alpha
\]

Similarly for a 2-morphism \( \beta: g \Rightarrow g' \).

(ix) For all 1-morphisms \( f: A \rightarrow A' \), \( g: B \rightarrow B' \), \( h: B' \rightarrow B'' \), the following diagram commutes:

\[
\begin{array}{c}
A' & \overset{f}{\rightarrow} & B' & \overset{g}{\rightarrow} & A' & \overset{f'}{\rightarrow} & B' \\
\downarrow & \phi_{f,g} & \downarrow & \phi_{f,g'} & \downarrow & \phi_{f,g} & \downarrow \\
A & \overset{g}{\rightarrow} & B & \overset{g'}{\rightarrow} & A & \overset{g'}{\rightarrow} & B
\end{array}
\]

Note that we have used colours to differentiate the source of the 2-morphisms. There is also a corresponding rotated version of this diagram.

4.2. Example. A stringent monoidal 2-category with one object \( M \) is the same thing as a symmetric monoidal category \( M \), after reindexing the 1-morphisms and
2-morphisms in $\mathcal{M}$ as objects and morphisms in $\mathcal{M}$ respectively. Wire diagrams make this very clear. Let $A$ and $B$ be 1-endomorphisms of the unit object $1 \in \mathcal{M}$. The tensor product $\odot$ in $\mathcal{M}$ is defined as $A \odot B := A \otimes B$ in $\mathcal{M}$. The braiding $\sigma_{A,B} : A \odot B \to B \odot A$ in $\mathcal{M}$ is defined by using the interchanger $\phi_{B,A}$ in $\mathcal{M}$, as follows:

$$\phi_{B,A} = \begin{aligned} A \otimes B &= (\text{id}_1 \otimes B) \circ (A \otimes \text{id}_1) \\ &= (B \otimes \text{id}_1) \circ (\text{id}_1 \otimes A) \\ \phi_{B,A} &= (\text{id}_1 \otimes A) \circ (B \otimes \text{id}_1) \\ &= B \otimes A \end{aligned}$$

In text form, this is expressed as follows:

$$A \otimes B = (\text{id}_1 \otimes B) \circ (A \otimes \text{id}_1) = (B \otimes \text{id}_1) \circ (\text{id}_1 \otimes A)$$

The first equation is nudging (Axiom (v)), the second equation is the fact that $1$ is a strict unit (Axiom (iii)), the third is the interchanger, and the fourth is nudging again. It is then a pleasant exercise in the graphical calculus that $\sigma_{A,B}$ is natural and bilinear, so that $(\mathcal{M}, \text{id}_1, \odot, \{\sigma_{A,B}\})$ is a symmetric monoidal category. The reverse procedure works in the same way.

### 4.3. Equivalence with semistrict monoidal 2-categories.

We can now prove the following.

**Proposition 9.**

1. If $(\mathcal{M}, 1, \otimes, \{\Phi(f',g'),(f,g)\})$ is a semistrict monoidal 2-category, then restricting to the underlying interchanger 2-isomorphisms $\phi_{f,g}$ gives a stringent monoidal 2-category.

2. If $(\mathcal{M}, 1, \otimes, \{\phi_{f,g}\})$ is a stringent monoidal 2-category, then the interchanger 2-isomorphisms $\phi_{f,g}$ can be extended to coherence isomorphisms $\Phi(f',g'),(f,g)$ making $\mathcal{M}$ into a semistrict monoidal 2-category.

3. The processes in (1) and (2) are inverse to each other, on-the-nose.

**Proof.**

1. Axiom (iii) follows since $\otimes$ is a cubical functor, from which it follows that $L_A := A \otimes -$ and $L_B := - \otimes B$ are strict 2-functors. The equation $L_A L_B = L_{A \otimes B}$ follows from the associativity equation coming from $\mathcal{M}$ being a monoid in $\mathcal{2Cat}^{ps}$. Similarly $R_B R_A = R_{A \otimes B}$, as well as $L_A R_B = R_B L_A$.

Axiom (v) follows from the cubical identity, as explained above. Axiom (vi) follows from $\mathcal{M}$ being a monoid in $\mathcal{2Cat}^{ps}$. Axiom (vii) follows from the cubical equation. Axiom (viii) follows from the naturality of $\Phi(f',g'),(f,g)$. Axiom (ix) follows from the coherence equation on $\Phi(f',g'),(f,g)$.

2. We define

$$\Phi(f',g'),(f,g) := \begin{aligned} & \phi_{f',g'} & \phi_{f',g'} & \phi_{f',g'} & \phi_{f',g'} \\ & f' \circ f & g' \circ g & f' \circ f & g' \circ g \end{aligned}$$
The coherence equation that $\Phi$ must satisfy in order for $\otimes : M \times M \to M$ to be a pseudofunctor looks as follows in wire diagrams:

$$ \Phi(f'' \cdot g'', (f' \cdot g')) $$

It is straightforward to verify graphically that the definition (7) of $\Phi$ satisfies the above, using axiom (ix). Also, $\Phi$ is natural because of axiom (viii). This establishes that $(\otimes, \Phi) : M \times M \to M$ is a pseudofunctor. It is cubical because of axiom (vii). Associativity follows from axiom (ix).

3. We need to check that $\Phi$ is uniquely determined by its underlying interchanger 2-isomorphisms $\phi$. This follows from the following commutative diagram:

Each square is an instance of (8) and hence commutes. Due to the cubical condition, all the $\Phi$ terms are the identity except for the one on the far right, hence we have $\Phi(f'' \cdot g'', (f \cdot g)) = \phi f'' \cdot g''$. In other words, the formula (7) holds in every semistrict monoidal 2-category.

5. **Stringent symmetric monoidal 2-categories**

In this section we define *stringent symmetric monoidal 2-categories*, and extend the wire diagram calculus to them.
5.1. The definition.

Definition 10. A stringent symmetric monoidal 2-category \((\mathcal{M}, 1, \otimes, \{\phi_{f,g}\}, \{\beta_{A,B}\})\) is a stringent monoidal 2-category \((\mathcal{M}, 1, \otimes, \phi_{f,g})\) equipped with, for every pair of objects \(A, B \in \mathcal{M}\), a 1-morphism

\[\beta_{A,B} : A \otimes B \to B \otimes A,\]

drawn as

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (1,1) {C};
  \draw (A) edge (B);
  \end{tikzpicture}
\]

satisfying the following equations between 1-morphisms on-the-nose:

(i) \(\beta_{A,B} \circ \beta_{B,A} = \text{id}_{A \otimes B}\). In wire diagrams:

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (1,1) {C};
  \draw (A) edge (B);
\end{tikzpicture}
= \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (1,1) {C};
  \draw (A) edge (B);
\end{tikzpicture}
\]

(ii) \(\beta_{A \otimes B, C} = (\beta_{(A,C)} \otimes \text{id}) \circ (\text{id} \otimes \beta_{B,C})\). In wire diagrams:

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
= \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
\]

There is a similar equation for \(\beta_{A,B \otimes C}\).

(iii) If \(f : A \to A'\) is a 1-morphism, then \(\beta_{A', B} \circ (f \otimes \text{id}_B) = (\text{id}_A \otimes f) \circ \beta_{A,B}\). In wire diagrams:

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
= \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
\]

There is a similar equation for \(g : B \to B'\).

Moreover, we require the following equation between 2-morphisms:

(iv) For every 1-morphism \(f : A \to A'\) and every pair of objects \(B, C, \phi_{f,\beta_{B,C}} = \text{id}\). In wire diagrams:

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
= \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};
  \node (D) at (2,1) {D};
  \draw (A) edge (B);
\end{tikzpicture}
\]

Similarly, \(\phi_{\beta_{A,B},g} = \text{id}\) for every \(g : C \to C'\).
Having given the definition, note that the naturality condition (iii) on $\beta_{A,B}$ does not hold for tensor products! In general, we need to insert the interchanger $\phi$ in order to commute $f \otimes g$ past the braiding:

\[
\begin{array}{c}
\begin{array}{c}
B' \quad A' \\
\downarrow \quad \downarrow \\
\quad A \quad B
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
B' \quad A' \\
\downarrow \quad \downarrow \\
\quad A \quad B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad f \\
\downarrow \\
\quad A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad A' \\
\downarrow \\
\quad B
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
B' \quad A' \\
\downarrow \quad \downarrow \\
\quad A \quad B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad f \\
\downarrow \\
\quad A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad \phi_{g,f} \\
\end{array}
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
B' \quad A' \\
\downarrow \quad \downarrow \\
\quad A \quad B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad g \\
\downarrow \\
\quad B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\quad A' \\
\downarrow \\
\quad B
\end{array}
\end{array}
\end{array}
\]

5.2. Example. This example is adapted from [23, Example 2.30]; see also [8, 15, 24, 17]. Let $S$ be the sphere spectrum, so that $\pi_i(S)$ are the stable homotopy groups of spheres:

$\pi_0^s = \mathbb{Z}$, $\pi_1^s = \mathbb{Z}/2$, $\pi_2^s = \mathbb{Z}/2$, $\pi_3^s = \mathbb{Z}/24$, ... 

We can conceive of the truncation $S_{[0,2]}$ for $0 \leq i \leq 2$ as a quasistrict symmetric monoidal 2-category $Q$, as follows. The objects of $Q$ are the integers $\mathbb{Z}$. The hom-categories are given by

$\text{Hom}_Q(m,n) = \begin{cases} P & \text{if } m = n \\ \text{empty} & \text{otherwise.} \end{cases}$

Here, $P$ is a skeletal version of the Picard category Pic$^{\mathbb{Z}/2}(\mathbb{Z})$ whose objects are $\mathbb{Z}/2$-graded free abelian groups of total rank 1 and whose morphisms are invertible graded homomorphisms, with the usual $\mathbb{Z}/2$-graded tensor product and the Koszul rule for the symmetry [8]. So, $P$ has two objects 0 and 1, and each object has two automorphisms, $I$ and $-I$. The tensor product in $P$ is given on objects by addition mod 2, and on morphisms by multiplication. The braiding $b$ on the symmetric monoidal category $P$ is given by the Koszul rule, with the only nonidentity braiding given by $b_{0,1} = -I$.

We reindex $P$ so as to form part of the 2-category $Q$. So, composition of 1-morphisms in $Q$ corresponds to tensor product inside $P$.

The tensor product on $Q$ is given on objects by addition in $\mathbb{Z}$, and on 1- and 2-morphisms by tensor product in $P$. Let $(m,i)$ be $(n,j)$ with $m,n \in \mathbb{Z}$ and $i,j \in \{0,1\}$ be automorphisms of $m$ and $n$ in $B$ respectively. The interchanger

$\phi_{(m,i),(n,j)}: i + j \to i + j$

is defined to be the braiding $b_{i,j}$ inside the symmetric monoidal category $P$. These constructions equip $Q$ as a stringent monoidal 2-category.

The braiding 1-morphisms

$\beta_{m,n}: m + n \to m + n$

in $Q$ are defined as $\beta_{m,n} = m + n \pmod{2}$. This completes the description of $Q$ as a stringent symmetric monoidal 2-category.

5.3. Equivalence with semistrict symmetric monoidal 2-categories. We now recall the definition of a quasistrict symmetric monoidal 2-category, and prove that they are equivalent to stringent ones. The following definition is taken from [23, 6] and builds on the definition of a symmetric monoidal bicategory from [23, 25]. The reader is referred to these references for the the definitions of the braiding ‘bilinearators’ $R_{A,B|C}$ and $S_{A|B,C}$ etc.

**Definition 11** ([23, 6]). A **Crans semistrict symmetric monoidal 2-category** is a symmetric monoidal bicategory $\mathbf{M}$ such that:
The underlying monoidal bicategory is a Gray monoid; the following additional normalization conditions apply:
(a) The 1-morphisms $\beta_{1,x}$ and $\beta_{x,1}$ are identity morphisms on $A$, for each object $A \in \mathcal{M}$.
(b) The isomorphisms $R_{1,A|B}$, $R_{A,1|B}$, $S_{A|1,B}$, and $S_{1|A,B}$ are the identity 2-isomorphism of $\beta_{A,B}$.
(c) The isomorphisms $R_{A,B|1}$ and $S_{1|A,B}$ are the identity 2-isomorphisms of $I_{A \otimes B}$.

A Crans semistrict symmetric monoidal 2-category comes equipped with 2-isomorphisms

$$\sigma_{A,B} : \text{id}_{A \otimes B} \Rightarrow \beta_{B,A} \circ \beta_{A,B}$$

for each pair of objects $A, B \in \mathcal{M}$, which witness the fact that the braiding is symmetric.

**Definition 12** ([23]). A quasistrict symmetric monoidal 2-category is a Crans semistrict symmetric monoidal 2-category $\mathcal{M}$ such that:

- **(QS.1)** The modifications $R$, $S$, and $\sigma$ are identities.
- **(QS.2)** For the transformation $\beta = (\beta_{A,B}, \beta_{f,g})$, the component $\beta_{f,g}$ is the identity if either $f$ or $g$ is an identity morphism.
- **(QS.3)** The 2-morphism witnessing naturality:

  $$\Phi(f',g',f,g) : (f' \otimes g') \circ (f \otimes g) \Rightarrow (f' \circ f) \otimes (g' \circ g)$$

  is an identity if either $f'$ or $g$ is a component of $\beta$, i.e. if $f' = \beta_{A,B}$ or $g = \beta_{A,B}$, for some pair of objects $A, B \in \mathcal{M}$.

Recall Theorem 1 of Schommer-Pries from the Introduction:

**Theorem 1** ([23]). Every symmetric monoidal bicategory is equivalent to a qua-sistrict symmetric monoidal 2-category.

We now show that quasistrict symmetric monoidal 2-categories are equivalent to stringent symmetric monoidal 2-categories.

**Proposition 13.**

1. If $(\mathcal{M}, 1, \otimes, \{\Phi(f',g',f,g)\}, \{\beta_{A,B}\}, \{\beta_{f,g}\})$ is a quasistrict monoidal 2-category, then restricting to the underlying interchangor 2-isomorphisms $\phi_{f,g}$ gives a stringent symmetric monoidal 2-category $(\mathcal{M}, 1, \otimes, \{\phi_{f,g}\}, \{\beta_{A,B}\})$.

2. If $(\mathcal{M}, 1, \otimes, \{\phi_{f,g}\}, \{\beta_{A,B}\})$ is a stringent symmetric monoidal 2-category, then coherence isomorphisms $\beta_{f,g}$ can be introduced, and the interchangor 2-isomorphisms $\phi_{f,g}$ can be extended to coherence isomorphisms $\Phi(f',g',f,g)$, so as to make $\mathcal{M}$ into a quasistrict symmetric monoidal 2-category.

3. The processes in (1) and (2) are inverse to each other, on-the-nose.

**Proof.**

1. The assertion that $\sigma$ is the identity gives Axiom (i) of a stringent symmetric monoidal 2-category. Similarly the assertion that $R$ and $S$ are identities gives Axiom (ii). Axiom (iii) follows from (QS.2), and Axiom (iv) follows from (QS.3).
2. We have already defined how to extend $\phi_{f,g}$ to $\Phi((f',g'),(f,g))$ in (7). We define $\beta_{f,g}$ by running (9) in reverse. That is, we define
\[ \beta_{f,g} := \phi_{f,g}^{-1} \]

as the following composite:

\[ \Phi((f',g'),(f,g)) \xrightarrow{\beta_{f,g}} \]

It is now routine to show that $\beta_{f,g}$ satisfies all the coherence equations listed in [23] for a quasistrict symmetric monoidal 2-category. Indeed, these equations can be translated into wire diagrams and the proof is entirely graphical. In particular, (QS.1) implies Axioms (i) and (ii), (QS.2) implies Axiom (iii), and (QS.3) implies Axiom (iv).

3. We need to show that $\beta_{f,g}$ is uniquely determined as the composite (11). Now, $\beta_{f,g}$ are the coherence 2-isomorphisms coming from the fact that $\beta$ is a transformation $\beta : \otimes \Rightarrow \otimes \circ \text{swap}$. Hence they satisfy the following coherence equation:

\[ \Phi((f',g'),(f,g)) \xrightarrow{\beta_{f,g}} \]

In (12), set $g' = \text{id}$ and $f = \text{id}$. Then, using $\beta_{\text{id},g} = \text{id}$ and $\beta_{f',\text{id}} = \text{id}$, we obtain precisely the formula (11) for $\beta_{f,g}$. \qed
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