QUASILINEAR AND HESSIAN EQUATIONS OF LANE–EMDEN TYPE

NGUYEN CONG PHUC AND IGOR E. VERBITSKY∗

Abstract. The existence problem is solved, and global pointwise estimates of solutions are obtained for quasilinear and Hessian equations of Lane–Emden type, including the following two model problems:

\[-\Delta_p u = u^q + \mu, \quad F_k[-u] = u^q + \mu, \quad u \geq 0,\]
on R^n, or on a bounded domain Ω ⊂ R^n. Here ∆_p is the p-Laplacian defined by ∆_p u = div(∇u|∇u|^{p-2}), and F_k[u] is the k-Hessian defined as the sum of k × k principal minors of the Hessian matrix D^2 u (k = 1, 2, ..., n); μ is a nonnegative measurable function (or measure) on Ω.

The solvability of these classes of equations in the renormalized (entropy) or viscosity sense has been an open problem even for good data μ ∈ L^s(Ω), s > 1. Such results are deduced from our existence criteria with the sharp exponents \( s = \frac{n(q-p+1)}{pq} \) for the first equation, and \( s = \frac{n(q-k)}{2kq} \) for the second one. Furthermore, a complete characterization of removable singularities is given.

Our methods are based on systematic use of Wolff’s potentials, dyadic models, and nonlinear trace inequalities. We make use of recent advances in potential theory and PDE due to Kilpeläinen and Malý, Trudinger and Wang, and Labutin. This enables us to treat singular solutions, nonlocal operators, and distributed singularities, and develop the theory simultaneously for quasilinear equations and equations of Monge-Ampère type.

1. Introduction

We study a class of quasilinear and fully nonlinear equations and inequalities with nonlinear source terms, which appear in such diverse areas as quasi-regular mappings, non-Newtonian fluids, reaction-diffusion problems, and stochastic control. In particular, the following two model equations are of substantial interest:

\[ -\Delta_p u = f(x, u), \quad F_k[-u] = f(x, u), \]

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on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$, where $f(x,u)$ is a non-negative function, convex and nondecreasing in $u$ for $u \geq 0$. Here $\Delta_p u = \text{div} (\nabla u |\nabla u|^{p-2})$ is the $p$-Laplacian ($p > 1$), and $F_k[u]$ is the $k$-Hessian ($k = 1, 2, \ldots, n$) defined by

$$F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$, which coincides with the Laplacian $F_1[u] = \Delta u$ if $k = 1$, and the Monge–Ampère operator $F_n[u] = \det (D^2 u)$ if $k = n$.

The form in which we write the second equation in (1.1) is chosen only for the sake of convenience, in order to emphasize the profound analogy between the quasilinear and Hessian equations. Obviously, it may be stated as $(-1)^k F_k[u] = f(x,u)$, $u \geq 0$, or $F_k[u] = f(x,-u)$, $u \leq 0$.

The existence and regularity theory, local and global estimates of sub- and super-solutions, the Wiener criterion, and Harnack inequalities associated with the $p$-Laplacian, as well as more general quasilinear operators, can be found in [HKM], [IM], [KM2], [M1], [MZ], [S1], [S2], [SZ], [TW4] where many fundamental results, and relations to other areas of analysis and geometry are presented.

The theory of fully nonlinear equations of Monge–Ampère type which involve the $k$-Hessian operator $F_k[u]$ was originally developed by Caffarelli, Nirenberg and Spruck, Ivochkina, and Krylov in the classical setting. We refer to [CNS], [GT], [Gu], [Iv], [Kr], [Ur], [Tru2], [TW1] for these and further results. Recent developments concerning the notion of the $k$-Hessian measure, weak convergence, and pointwise potential estimates due to Trudinger and Wang [TW2]–[TW4], and Labutin [L] are used extensively in this paper.

We are specifically interested in quasilinear and fully nonlinear equations of Lane–Emden type:

1. $-\Delta_p u = u^q$, and $F_k[-u] = u^q$, $u \geq 0$ in $\Omega$,

where $p > 1$, $q > 0$, $k = 1, 2, \ldots, n$, and the corresponding nonlinear inequalities:

1. $-\Delta_p u \geq u^q$, and $F_k[-u] \geq u^q$, $u \geq 0$ in $\Omega$.

The latter can be stated in the form of the inhomogeneous equations with measure data,

1. $-\Delta_p u = u^q + \mu$, $F_k[-u] = u^q + \mu$, $u \geq 0$ in $\Omega$.

where $\mu$ is a nonnegative Borel measure on $\Omega$. 
The difficulties arising in studies of such equations and inequalities with competing nonlinearities are well known. In particular, (1.3) may have singular solutions [SZ]. The existence problem for (1.5) has been open ([BV2], Problems 1 and 2; see also [BV1], [BV3], [Gr]) even for the quasilinear equation \(-\Delta_p u = u^q + f\) with good data \(f \in L^s(\Omega), s > 1\). Here solutions are generally understood in the renormalized (entropy) sense for quasilinear equations, and viscosity, or \(k\)-convexity sense, for fully nonlinear equations of Hessian type (see [BMMP], [DMOP], [JLM], [TW1]–[TW3], [Ur]). Precise definitions of these classes of admissible solutions are given in Sec. 3, Sec. 6, and Sec. 7 below.

In this paper, we present a unified approach to (1.3)–(1.5) which makes it possible to attack a number of open problems. It is based on global pointwise estimates, nonlinear integral inequalities in Sobolev spaces of fractional order, and analysis of dyadic models, along with the weak convergence and Hessian measure results [TW2]–[TW4]. The latter are used to bridge the gap between the dyadic models and partial differential equations. Some of these techniques were developed in the linear case, in the framework of Schrödinger operators and harmonic analysis [ChWW], [Fe], [KS], [NTV], [V1], [V2], and applications to semilinear equations [KV], [VV], [V3].

Our goal is to establish necessary and sufficient conditions for the existence of solutions to (1.5), sharp pointwise and integral estimates for solutions to (1.4), and a complete characterization of removable singularities for (1.3). We are mostly concerned with admissible solutions to the corresponding equations and inequalities. However, even for locally bounded solutions, as in [SZ], our results yield new pointwise and integral estimates, and Liouville-type theorems.

In the “linear case” \(p = 2\) and \(k = 1\), problems (1.3)–(1.5) with nonlinear sources are associated with the names of Lane and Emden, as well as Fowler. Authoritative historical and bibliographical comments can be found in [SZ]. An up-to-date survey of the vast literature on nonlinear elliptic equations with measure data is given in [Ver], including a thorough discussion of related work due to D. Adams and Pierre [AP], Baras and Pierre [BP], Berestycki, Capuzzo-Dolcetta, and Nirenberg [BCDN], Brezis and Cabré [BC], Kalton and Verbitsky [KV].

It is worth mentioning that related equations with absorption,

\[
-\Delta u + u^q = \mu, \quad u \geq 0 \quad \text{in} \, \Omega,
\]

were studied in detail by Bénilan and Brezis, Baras and Pierre, and Marcus and Véron analytically for \(1 < q < \infty\), and by Le Gall, and Dynkin and Kuznetsov using probabilistic methods when \(1 < q \leq 2\).
For a general class of semilinear equations

(1.7) \[-\Delta u + g(u) = \mu, \quad u \geq 0 \text{ in } \Omega,\]

where \(g\) belongs to the class of continuous nondecreasing functions such that \(g(0) = 0\), sharp existence results have been obtained quite recently by Brezis, Marcus, and Ponce [BMP]. It is well known that equations with absorption generally require “softer” methods of analysis, and the conditions on \(\mu\) which ensure the existence of solutions are less stringent than in the case of equations with source terms.

Quasilinear problems of Lane–Emden type (1.3)–(1.5) have been studied extensively over the past 15 years. Universal estimates for solutions, Liouville-type theorems, and analysis of removable singularities are due to Bidaut-Véron, Mitidieri and Pohozaev [BV1], [BV2], [BV3], [BVP], and Serrin and Zou [SZ]. (See also [BiD], [Gre], [Ver], and the literature cited there.) The profound difficulties in this theory are highlighted by the presence of the two critical exponents,

(1.8) \[q_* = \frac{n(p-1)}{n-p}, \quad q^* = \frac{n(p-1)+p}{n-p},\]

where \(1 < p < n\). As was shown in [BVP], [MP], and [SZ], the quasilinear inequality (1.3) does not have nontrivial weak solutions on \(\mathbb{R}^n\), or exterior domains, if \(q \leq q_*\). For \(q > q_*\), there exist \(u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}\) which obey (1.4), as well as singular solutions to (1.3) on \(\mathbb{R}^n\). However, for the existence of nontrivial solutions \(u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}\) to (1.3) on \(\mathbb{R}^n\), it is necessary and sufficient that \(q \geq q^*\) [SZ]. In the “linear case” \(p = 2\), this is classical [GS], [BP], [BCDN].

The following local estimates of solutions to quasilinear inequalities are used extensively in the studies mentioned above (see, e.g., [SZ], Lemma 2.4). Let \(B_R\) denote a ball of radius \(R\) such that \(B_{2R} \subset \Omega\). Then, for every solution \(u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}\) to the inequality \(-\Delta_p u \geq u^q\) in \(\Omega\),

(1.9) \[\int_{B_R} u^\gamma \, dx \leq C R^n \frac{\gamma p}{q-p+1}, \quad 0 < \gamma < q,\]

(1.10) \[\int_{B_R} |\nabla u|^{q+1} \, dx \leq C R^n \frac{\gamma p}{q-p+1}, \quad 0 < \gamma < q,\]

where the constants \(C\) in (1.9) and (1.10) depend only on \(p, q, n, \gamma\). Note that (1.9) holds even for \(\gamma = q\) (cf. [MP]), while (1.10) generally fails in this case. In what follows, we will substantially strengthen (1.9) in the end-point case \(\gamma = q\), and obtain global pointwise estimates of solutions.
In [PV], we proved that all compact sets $E \subset \Omega$ of zero Hausdorff measure, $H^{n-p/q-1}(E) = 0$, are removable singularities for the equation $-\Delta_p u = u^q$, $q > q^*$, and a more general class of nonlinear equations. Earlier results of this kind, under a stronger restriction $\text{cap}_{1,p}^q q-p+1 + \epsilon (E) = 0$ for some $\epsilon > 0$, are due to Bidaut-Véron [BV3]. Here $\text{cap}_{1,s}^q \cdot$ is the capacity associated with the Sobolev space $W^{1,s}$.

In fact, much more is true. We will show below that a compact set $E \subset \Omega$ is a removable singularity for $-\Delta_p u = u^q$ if and only if it has zero fractional capacity: $\text{cap}_{p,q}^{q-p+1} (E) = 0$. Here $\text{cap}_{\alpha,s} \cdot$ stands for the Bessel capacity associated with the Sobolev space $W^{\alpha,s}$ which is defined in Sec. 2. We observe that the usual $p$-capacity $\text{cap}_{1,p}^q \cdot$ used in the studies of the $p$-Laplacian [HKM], [KM2] plays a secondary role in the theory of equations of Lane–Emden type. Relations between these and other capacities used in nonlinear PDE are discussed in [AH], [M2], and [V4].

Our characterization of removable singularities is based on the solution of the existence problem for the equation

$$-\Delta_p u = u^q + \mu, \quad u \geq 0,$$

with nonnegative measure $\mu$ obtained in Sec. 3. Main existence theorems for quasilinear equations are stated below (Theorems 2.3 and 2.10). Here we only mention the following corollary in the case $\Omega = \mathbb{R}^n$:

If (1.11) has an admissible solution $u$, then

$$\int_{B_R} d\mu \leq C R^{n-p/q-p+1},$$

for every ball $B_R$ in $\mathbb{R}^n$, where $C = C(p,q,n)$, provided $1 < p < n$ and $q > q^*$; if $p \geq n$ or $q \leq q^*$, then $\mu = 0$.

Conversely, suppose that $1 < p < n$, $q > q^*$, and $d\mu = f \, dx$, $f \geq 0$, where

$$\int_{B_R} f^{1+\epsilon} \, dx \leq C R^{n-(1+\epsilon)p/q-p+1},$$

for some $\epsilon > 0$. Then there exists a constant $C_0(p,q,n)$ such that (1.11) has an admissible solution on $\mathbb{R}^n$ if $C \leq C_0(p,q,n)$.

The preceding inequality is an analogue of the classical Fefferman–Phong condition [FP], which appeared in applications to Schrödinger operators. In particular, (1.13) holds if $f \in L^{n(p+1)/pq, \infty}(\mathbb{R}^n)$. Here $L^{n,\infty}$ stands for the weak $L^s$ space. This sufficiency result, which to the best of our knowledge is new even in the $L^s$ scale, provides a comprehensive solution to Problem 1 in [BV2]. Notice that the exponent
$s = \frac{n(q-p+1)}{pq}$ is sharp. Broader classes of measures $\mu$ (possibly singular with respect to Lebesgue measure) which guarantee the existence of admissible solutions to (1.11) will be discussed in the sequel.

A substantial part of our work is concerned with integral inequalities for nonlinear potential operators, which are at the heart of our approach. We employ the notion of Wolff’s potential introduced originally in [HW] in relation to the spectral synthesis problem for Sobolev spaces. For a nonnegative Borel measure $\mu$ on $\mathbb{R}^n$, $s \in (1, +\infty)$, and $\alpha > 0$, the Wolff potential $W_{\alpha, s} \mu$ is defined by

$$W_{\alpha, s} \mu(x) = \int_0^\infty \left[ \frac{\mu(B_t(x))}{t^{n-\alpha s}} \right]^{\frac{1}{s-1}} dt, \quad x \in \mathbb{R}^n.$$  

We write $W_{\alpha, s} f$ in place of $W_{\alpha, s} \mu$ if $d\mu = f dx$, where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$. When dealing with equations in a bounded domain $\Omega \subset \mathbb{R}^n$, a truncated version is useful:

$$W_{\alpha, s}^r \mu(x) = \int_r^\infty \left[ \frac{\mu(B_t(x))}{t^{n-\alpha s}} \right]^{\frac{1}{s-1}} dt, \quad x \in \Omega,$$

where $0 < r \leq 2 \text{diam}(\Omega)$. In many instances, it is more convenient to work with the dyadic version, also introduced in [HW]:

$$W_{\alpha, s}^D \mu(x) = \sum_{Q \in D} \left[ \frac{\mu(Q)}{\ell(Q)^{n-\alpha s}} \right]^{\frac{1}{s-1}} \chi_Q(x), \quad x \in \mathbb{R}^n,$$

where $D = \{Q\}$ is the collection of the dyadic cubes $Q = 2^i(k+[0, 1]^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\ell(Q)$ is the side length of $Q$.

An indispensable source on nonlinear potential theory is provided by [AH], where the fundamental Wolff inequality and its applications are discussed. Very recently, an analogue of Wolff’s inequality for general dyadic and radially decreasing kernels was obtained in [COV]; some of the tools developed there are employed below.

The dyadic Wolff potentials appear in the following discrete model of (1.5) studied in Sec. 3:

$$u = W_{\alpha, s} u^q + f, \quad u \geq 0.$$  

As it turns out, this nonlinear integral equation with $f = W_{\alpha, s} \mu$ is intimately connected to the quasilinear differential equation (1.11) in the case $\alpha = 1$, $s = p$, and to its $k$-Hessian counterpart in the case $\alpha = \frac{2k}{k+1}$, $s = k+1$. Similar discrete models are used extensively in harmonic analysis and function spaces (see, e.g., [NTV], [St2], [V1]).

The profound role of Wolff’s potentials in the theory of quasilinear equations was discovered by Kilpeläinen and Malý [KM1]. They established local pointwise estimates for nonnegative $p$-superharmonic
functions in terms of Wolff’s potentials of the associated $p$-Laplacian measure $\mu$. More precisely, if $u \geq 0$ is a $p$-superharmonic function in $B(x,3r)$ such that $-\Delta_p u = \mu$, then
\begin{equation}
C_1 W^r_{1,p}(x) \mu(x) \leq u(x) \leq C_2 \inf_{B(x,r)} u + C_3 W^{2r}_{1,p}(x),
\end{equation}
where $C_1, C_2$ and $C_3$ are positive constants which depend only on $n$ and $p$.

In [TW1], [TW2], Trudinger and Wang introduced the notion of the Hessian measure $\mu[u]$ associated with $F_k[u]$ for a $k$-convex function $u$. Very recently, Labutin [L] proved local pointwise estimates for Hessian equations analogous to (1.18), where the Wolff potential $W^{2r}_{k+1, k+1}\mu$ is used in place of $W^r_{1,p}\mu$.

In what follows, we will need global pointwise estimates of this type. In the case of a $k$-convex solution to the equation $F_k[u] = \mu$ on $\mathbb{R}^n$ such that $\inf_{x \in \mathbb{R}^n} (-u(x)) = 0$, one has
\begin{equation}
C_1 W^{2k}_{\frac{k}{k+1}, k+1}\mu(x) \leq -u(x) \leq C_2 W^{2k}_{\frac{k}{k+1}, k+1}\mu(x),
\end{equation}
where $C_1$ and $C_2$ are positive constants which depend only on $n$ and $k$. Analogous global estimates are obtained below for admissible solutions of the Dirichlet problem for $-\Delta_p u = \mu$ and $F_k[-u] = \mu$ in a bounded domain $\Omega \subset \mathbb{R}^n$.

In the special case $\Omega = \mathbb{R}^n$, our criterion for the solvability of (1.11) can be stated in the form of the pointwise condition involving Wolff’s potentials:
\begin{equation}
W^{(W_1^p + \mu)}_1 \mu(x) \leq C W \mu(x) < +\infty \quad \text{a.e.,}
\end{equation}
which is necessary with $C = C_1(p,q,n)$, and sufficient with another constant $C = C_2(p,q,n)$. Moreover, in the latter case there exists an admissible solution $u$ to (1.11) such that
\begin{equation}
c_1 W_1^p \mu(x) \leq u(x) \leq c_2 W_1^p \mu(x), \quad x \in \mathbb{R}^n,
\end{equation}
where $c_1$ and $c_2$ are positive constants which depend only on $p, q, n$, provided $1 < p < n$ and $q > q_*$; if $p \geq n$ or $q \leq q_*$ then $u = 0$ and $\mu = 0$.

The iterated Wolff potential condition (1.20) plays a crucial role in our approach. As we will demonstrate in Sec. 5 it turns out to be equivalent to the fractional Riesz capacity condition
\begin{equation}
\mu(E) \leq C \text{Cap}_{\frac{q}{q-p}}(E),
\end{equation}
where $C$ does not depend on a compact set $E \subset \mathbb{R}^n$. Such classes of measures $\mu$ were introduced by V. Maz’ya in the early 60-s in the framework of linear problems.
It follows that every admissible solution $u$ to (1.11) on $\mathbb{R}^n$ obeys the inequality

$$\int_E u^q \, dx \leq C \text{Cap}_p \frac{q}{q-p+1} (E),$$

for all compact sets $E \subset \mathbb{R}^n$. We also prove an analogous estimate in a bounded domain $\Omega$ (Sec. 6). Obviously, this yields (1.9) in the end-point case $\gamma = q$:

$$\int_{B_R} u^q \, dx \leq C R^{n-\frac{qp}{q-p+1}},$$

where $B_{2R} \subset \Omega$. In the critical case $q = q^*$, we obtain an improved estimate:

$$\int_{B_r} u^{q^*} \, dx \leq C \left( \log \left( \frac{2R}{r} \right) \right)^{\frac{1-p}{q-p+1}},$$

for every ball $B_r$ of radius $r$ such that $B_r \subset B_{2R}$, and $B_{2R} \subset \Omega$. Cer-

In particular, (1.26) holds if $d\mu = f \, dx$, where $f \geq 0$ and $f \in L^n(q-k)/2kq, \infty (\mathbb{R}^n)$; the exponent $\frac{n(q-k)}{2kq}$ is sharp.

In Sec. 7 we will obtain precise existence theorems for equation (1.27) in a bounded domain $\Omega$ with the Dirichlet boundary condition $u = \phi, \phi \geq 0$, on $\partial \Omega$, for $1 \leq k \leq n$. Furthermore, removable singularities $E \subset \Omega$ for the homogeneous equation $F_k[-u] = u^q, u \geq 0$, will be
characterized as the sets of zero Bessel capacity $\text{cap}_{2k, \frac{q}{q-k}}(E) = 0$, in the most interesting case $q > k$.

The notion of the $k$-Hessian capacity introduced by Trudinger and Wang proved to be very useful in studies of the uniqueness problem for $k$-Hessian equations [TW3], as well as associated $k$-polar sets [L]. Comparison theorems for this capacity and the corresponding Hausdorff measures were obtained by Labutin in [L] where it is proved that the $(n - 2k)$-Hausdorff dimension is critical in this respect. We will enhance this result (see Theorem 2.20 below) by showing that the $k$-Hessian capacity is in fact locally equivalent to the fractional Bessel capacity $\text{cap}_{\frac{n}{2}, k+1}$.

In conclusion, we remark that our methods provide a promising approach for a wide class of nonlinear problems, including curvature and subelliptic equations, and more general nonlinearities.

2. Main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We study the existence problem for the quasilinear equation

$$\begin{cases}
-\text{div} \mathcal{A}(x, \nabla u) = u^q + \omega, \\
u \geq 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}$$

(2.1)

where $p > 1$, $q > p - 1$ and

$$\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}.$$ 

(2.2)

for some $\alpha, \beta > 0$. The precise structural conditions imposed on $\mathcal{A}(x, \xi)$ are stated in Sec. 4, formulae (4.1)–(4.5). This includes the principal model problem

$$\begin{cases}
-\Delta_p u = u^q + \omega, \\
u \geq 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases}$$

(2.3)

Here $\Delta_p$ is the $p$-Laplacian defined by $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$. We observe that in the well-studied case $q \leq p - 1$ hard analysis techniques are not needed, and many of our results simplify. We refer to [Gre], [SZ] for further comments and references, especially in the classical case $q = p - 1$. 
Our approach also applies to the following class of fully nonlinear equations

\[
\begin{align*}
F_k[-u] &= u^q + \omega, \\
u &\geq 0 \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( k = 1, 2, \ldots, n \), and \( F_k \) is the \( k \)-Hessian operator,

\[
F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.
\]

Here \((\lambda_1, \ldots, \lambda_n)\) are the eigenvalues of the Hessian matrix \( D^2 u \), and \(-u\) belongs to the class of \( k \)-subharmonic (or \( k \)-convex) functions on \( \Omega \) introduced by Trudinger and Wang in [TW1]–[TW2]. Analogues of equations (2.1) and (2.4) on the entire space \( \mathbb{R}^n \) are studied as well.

To state our results, let us introduce some necessary definitions and notations. Let \( \mathcal{M}_B^+(\Omega) \) (resp. \( \mathcal{M}^+(\Omega) \)) denote the class of all non-negative finite (respectively locally finite) Borel measures on \( \Omega \). For \( \mu \in \mathcal{M}_B^+(\mathbb{R}^n) \) and a Borel set \( E \subset \mathbb{R}^n \), we denote by \( \mu_E \) the restriction of \( \mu \) to \( E \): \( d\mu_E = \chi_E d\mu \) where \( \chi_E \) is the characteristic function of \( E \).

We define the Riesz potential \( I_\alpha \) of order \( \alpha \), \( 0 < \alpha < n \), on \( \mathbb{R}^n \) by

\[
I_\alpha \mu(x) = c(n, \alpha) \int_{\mathbb{R}^n} |x - y|^{\alpha - n} d\mu(y), \quad x \in \mathbb{R}^n,
\]

where \( \mu \in \mathcal{M}_B^+(\mathbb{R}^n) \) and \( c(n, \alpha) \) is a normalized constant. For \( \alpha > 0 \), \( p > 1 \), such that \( \alpha p < n \), the Wolff potential \( W_{\alpha, p} \) is defined by

\[
W_{\alpha, p} \mu(x) = \int_0^\infty \left[ \frac{\mu(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} dt, \quad x \in \mathbb{R}^n.
\]

When dealing with equations in a bounded domain \( \Omega \subset \mathbb{R}^n \), it is convenient to use the truncated versions of Riesz and Wolff potentials. For \( 0 < r \leq \infty, \alpha > 0 \) and \( p > 1 \), we set

\[
I_\alpha^r \mu(x) = \int_0^r \frac{\mu(B_t(x))}{t^{n-\alpha}} dt, \quad W_{\alpha, p}^r \mu(x) = \int_0^r \left[ \frac{\mu(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} dt.
\]

Here \( I_\alpha^\infty \) and \( W_{\alpha, p}^\infty \) are understood as \( I_\alpha \) and \( W_{\alpha, p} \) respectively. For \( \alpha > 0 \), we denote by \( G_\alpha \) the Bessel kernel of order \( \alpha \) (see [AH], Sec. 1.2.4). The Bessel potential of a measure \( \mu \in \mathcal{M}_B^+(\mathbb{R}^n) \) is defined by

\[
G_\alpha \mu(x) = \int_{\mathbb{R}^n} G_\alpha(x - y) d\mu(y), \quad x \in \mathbb{R}^n.
\]

Various capacities will be used throughout the paper. Among them are the Riesz and Bessel capacities defined respectively by

\[
\text{Cap}_{I_\alpha, s}(E) = \inf \{ \| f \|_{L^s(\mathbb{R}^n)} : I_\alpha f \geq \chi_E, \ 0 \leq f \in L^s(\mathbb{R}^n) \},
\]
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\[ \text{Cap}_{G_{\alpha}, s}(E) = \inf \{ \| f \|_{L^s(\mathbb{R}^n)} : G_{\alpha} f \geq \chi_E, \ 0 \leq f \in L^s(\mathbb{R}^n) \}, \]

for any \( E \subset \mathbb{R}^n \).

Our first two theorems are concerned with global pointwise potential estimates for quasilinear and Hessian equations on a bounded domain \( \Omega \) in \( \mathbb{R}^n \).

**Theorem 2.1.** Suppose that \( u \) is a renormalized solution to the equation

\[
\begin{cases}
-\text{div} A(x, \nabla u) = \omega & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with data \( \omega \in \mathcal{M}^+_B(\Omega) \). Then there is a positive constant \( K \) which does not depend on \( u \) and \( \Omega \) such that

\[
\frac{1}{K} \mathcal{W}^{\frac{\text{dist}(x, \partial \Omega)}{1, p}} \omega(x) \leq u(x) \leq K \mathcal{W}^{2\text{diam}(\Omega)} \omega(x),
\]

for all \( x \) in \( \Omega \).

**Theorem 2.2.** Let \( \omega \) be a nonnegative finite measure on \( \Omega \) such that \( \omega \in L^s(\Omega \setminus E) \) for a compact set \( E \subset \Omega \). Here \( s > \frac{n}{2k} \) if \( 1 \leq k \leq \frac{n}{2} \), and \( s = 1 \) if \( \frac{n}{2} < k \leq n \). Suppose that \( -u \) is a nonpositive \( k \)-subharmonic function in \( \Omega \) such that \( u \) is continuous near \( \partial \Omega \), and solves the equation

\[
\begin{cases}
F_k[-u] = \omega & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Then, for all \( x \in \Omega \),

\[
\frac{1}{K} \mathcal{W}^{\frac{\text{dist}(x, \partial \Omega)}{2k, k+1}} \omega(x) \leq u(x) \leq K \mathcal{W}^{2\text{diam}(\Omega)} \omega(x),
\]

where \( K \) is a constant which does not depend on \( x, u, \) and \( \Omega \).

We remark that the upper estimate in (2.6) does not hold in general if \( u \) is merely a weak solution of (2.5) in the sense of [KM1]. For a counter example, see [Kil, Sec. 2]. Upper estimates similar to the one in (2.7) hold also for \( k \)-subharmonic functions with non-homogeneous boundary condition as well (see Sec. 7). Equivalent definitions of renormalized solutions to the problem (2.5) are given in Sec. 6. For definitions of \( k \)-subharmonic functions, see Sec. 7.

Note also that in the case of the entire space \( \Omega = \mathbb{R}^n \), if \( -u \) is a non-positive \( k \)-subharmonic function such that \( F_k[-u] = \mu \) and \( \inf_{x \in \mathbb{R}^n} u(x) = 0 \), then

\[
\frac{1}{K} \mathcal{W}^{\frac{2k}{k+1}, k+1} \mu(x) \leq u(x) \leq K \mathcal{W}^{\frac{2k}{k+1}, k+1} \mu(x).
\]
An analogous two-sided estimate holds for $\mathcal{A}$-superharmonic functions as well, with $W_{1,p}^{k}$ in place of $W_{\frac{2k}{k+1},k+1}^{k+1}$. These global estimates are deduced from the local ones given in [L], [KM2].

In the next two theorems we give criteria for the solvability of quasilinear and Hessian equations on the entire space $\mathbb{R}^n$.

**Theorem 2.3.** Let $\omega$ be a measure in $\mathcal{M}^+(\mathbb{R}^n)$, $1 < p < n$ and $q > p-1$. Then the following statements are equivalent.

(i) There exists a nonnegative $\mathcal{A}$-superharmonic solution $u \in L^q_{\text{loc}}(\mathbb{R}^n)$ to the equation

$$\begin{cases}
\inf_{x \in \mathbb{R}^n} u(x) = 0 \\
-\text{div} \mathcal{A}(x, \nabla u) = u^q + \epsilon \omega \quad \text{in} \quad \mathbb{R}^n,
\end{cases}$$

for some $\epsilon > 0$.

(ii) For all compact sets $E \subset \mathbb{R}^n$,

$$\omega(E) \leq C \text{Cap}_{p, \frac{q}{q-p+1}}(E).$$

(iii) The testing inequality

$$\int_B \left[ W_{1,p}^q(\omega^q_B(x)) \right]^q dx \leq C \omega(B)$$

holds for all balls $B$ in $\mathbb{R}^n$.

(iv) There exists a constant $C$ such that

$$W_{1,p}(W_{1,p}^q(x))^q(x) \leq C W_{1,p}^q(x) < \infty \quad \text{a.e.}$$

Moreover, there is a constant $C_0 = C_0(n,p,q,\alpha,\beta)$ such that if any one of the conditions (2.9)–(2.11) holds with $C \leq C_0$, then equation (2.8) has a solution $u$ with $\epsilon = 1$ which satisfies the two-sided estimate

$$\frac{1}{K} W_{1,p}(x) \leq u(x) \leq K W_{1,p}(x), \quad x \in \mathbb{R}^n,$$

where $K$ depends only on $n, p, q, \alpha, \beta$. Conversely, if (2.8) has a solution $u$ as in statement (i) with $\epsilon = 1$, then conditions (2.9)–(2.11) hold with $C = C_1(n,p,q,\alpha,\beta)$. Here $\alpha$ and $\beta$ are the structural constants of $\mathcal{A}$ defined in (2.2).

Using condition (2.9) in the above theorem, we can now deduce from the isoperimetric inequality:

$$|E|^{1-\frac{pq}{q-p+1}} \leq C \text{Cap}_{p, \frac{q}{q-p+1}}(E),$$

(see [AH] or [M2]), a simple sufficient condition for the solvability of (2.8).
Corollary 2.4. Suppose that $f \in L^{n(q-p+1)/pq, \infty}(\mathbb{R}^n)$ and $d\omega = f \, dx$. If $q > p-1$ and $\frac{pq}{q-p+1} < n$, then equation (2.8) has a nonnegative solution for some $\epsilon > 0$.

Remark 2.5. The condition $f \in L^{n(q-p+1)/pq, \infty}(\mathbb{R}^n)$ in Corollary 2.4 can be relaxed by using the Fefferman–Phong condition [Fef]:

$$\int_{B_R} f^{1+\delta} \, dx \leq CR^{n-(1+\delta)\frac{pq}{q-p+1}},$$

for some $\delta > 0$, which is known to be sufficient for the validity of (2.9); see, e.g., [KS], [V2].

Theorem 2.6. Let $\omega$ be a measure in $\mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$, and $q > k$. Then the following statements are equivalent.

(i) There exists a solution $u \geq 0$, $-u \in \Phi^k(\Omega) \cap L^q_{\text{loc}}(\mathbb{R}^n)$, to the equation

$$\inf_{x \in \mathbb{R}^n} u(x) = 0 \quad F_k[-u] = u^q + \epsilon \omega \quad \text{in} \quad \mathbb{R}^n,$$

for some $\epsilon > 0$.

(ii) For all compact sets $E \subset \mathbb{R}^n$,

$$\omega(E) \leq C \text{Cap}_{2k, \frac{n}{q-k}}(E).$$

(iii) The testing inequality

$$\int_{B} \left[W^{2k, k+1}_k \omega(x)\right]^q \, dx \leq C \omega(B)$$

holds for all balls $B$ in $\mathbb{R}^n$.

(iv) There exists a constant $C$ such that

$$W^{2k, k+1}_k(W^{\frac{k}{2q-1}}_k \omega)^q(x) \leq C W^{\frac{k}{2q-1}}_k \omega(x) < \infty \quad \text{a.e.}$$

Moreover, there is a constant $C_0 = C_0(n, k, q)$ such that if any one of the conditions (2.13)–(2.15) holds with $C \leq C_0$, then equation (2.12) has a solution $u$ with $\epsilon = 1$ which satisfies the two-sided estimate

$$\frac{1}{K} W^{\frac{k}{2q-1}}_k \omega(x) \leq u(x) \leq K W^{\frac{k}{2q-1}}_k \omega(x), \quad x \in \mathbb{R}^n,$$

where $K$ depends only on $n, k, q$. Conversely, if there is a solution $u$ to (2.12) as in statement (i) with $\epsilon = 1$, then conditions (2.13)–(2.15) hold with $C = C_1(n, k, q)$.

Corollary 2.7. Suppose that $f \in L^{n(q-k)/2q, \infty}(\mathbb{R}^n)$ and $d\omega = f \, dx$. If $q > k$ and $\frac{2k}{q-k} < n$ then the equation (2.12) has a nonnegative solution for some $\epsilon > 0$. 
Since $\text{Cap}_{I_{\alpha,s}}(E) = 0$ in the case $\alpha s \geq n$ for all Borel sets $E \subset \mathbb{R}^n$ (see [AH]), we obtain the following Liouville-type theorems for quasi-linear and Hessian differential inequalities.

**Corollary 2.8.** If $q \leq \frac{n(p-1)}{n-p}$, then the inequality $-\text{div}A(x,\nabla u) \geq u^q$ admits no nontrivial nonnegative $A$-superharmonic solutions in $\mathbb{R}^n$. Analogously, if $q \leq \frac{nk}{n-2k}$, then the inequality $F_k[-u] \geq u^q$ admits no nontrivial nonnegative solutions in $\mathbb{R}^n$.

**Remark 2.9.** When $1 < p < n$ and $q > \frac{n(p-1)}{n-p}$, the function $u(x) = c|x|^{-\frac{p}{q-p+1}}$ with
\[
c = \left[ \frac{p^{p-1}}{(q-p+1)p} \right]^{\frac{1}{q-p+1}} \left[ q(n-p) - n(p-1) \right]^{\frac{1}{q-p+1}},
\]
is a nontrivial admissible (but singular) global solution of $-\Delta_p u = u^q$ (see [SZ]). Similarly, the function $u(x) = c'|x|^{-\frac{2k}{q-k}}$ with
\[
c' = \left[ \frac{(n-1)!}{k!(n-k)!} \right]^{\frac{1}{q-k}} \left[ \frac{(2k)^k}{(q-k)^{k+1}} \right]^{\frac{1}{q-k}} \left[ q(n-2k) - nk \right]^{\frac{1}{q-k}},
\]
where $1 \leq k < n/2$ and $q > \frac{nk}{n-2k}$, is a singular admissible global solution of $F_k[-u] = u^q$ (see [Tso] or [Tru1], formula (3.2)). Thus, we see that the exponent $\frac{n(p-1)}{n-p}$ (respectively $\frac{nk}{n-2k}$) is also critical for the homogeneous equation $-\text{div}A(x,\nabla u) = u^q$ (respectively $F_k[-u] = u^q$) in $\mathbb{R}^n$. The situation is different when we restrict ourselves only to locally bounded solutions in $\mathbb{R}^n$ (see [GS], [SZ]).

Existence results on a bounded domain $\Omega$ analogous to Theorems 2.3 and 2.6 are contained in the following two theorems, where Bessel potentials and the corresponding capacities are used in place of respectively Riesz potentials and Riesz capacities.

**Theorem 2.10.** Let $\omega$ be a measure in $\mathcal{M}_B^+(\Omega)$ which is compactly supported in $\Omega$. Let $p > 1$, $q > p-1$, and let $R = \text{diam}(\Omega)$. Then the following statements are equivalent.

(i) There exists a nonnegative renormalized solution $u \in L^q(\Omega)$ to the equation
\[
-\text{div}A(x,\nabla u) = u^q + \epsilon \omega \quad \text{in} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]
for some $\epsilon > 0$.

(ii) For all compact sets $E \subset \text{supp}\omega$,
\[
\omega(E) \leq C \text{Cap}_{G_p,\frac{q}{q-p+1}}(E).
\]
The testing inequality

\[ \int_B \left[ W_{1,p}^{2R} \omega_B(x) \right]^q \, dx \leq C \omega(B) \]

holds for all balls \( B \) such that \( B \cap \text{supp} \omega \neq \emptyset \).

There exists a constant \( C \) such that

\[ W_{1,p}^{2R}(W_{1,p}^{2R} \omega)^q(x) \leq C W_{1,p}^{2R} \omega(x) < \infty \quad \text{a.e. on } \Omega. \]

Remark 2.11. In the case where \( \omega \) is not compactly supported in \( \Omega \), it can be easily seen from the proof of this theorem that any one of the conditions (ii), (iii), and (iv) above is still sufficient for the solvability of (2.16) for some \( \epsilon > 0 \). Moreover, in the subcritical case \( \frac{pq}{q-p+1} > n \), these conditions are redundant since the Bessel capacity \( \text{Cap}_{G_p, q} \) of a single point is positive (see [AH]). This ensures that statement (ii) of Theorem 2.10 holds for some constant \( C > 0 \) provided \( \omega \) is a finite measure.

Corollary 2.12. Suppose that \( f \in L^{\frac{n(q-p+1)}{pq}}(\Omega) \) and \( d\omega = f \, dx \). If \( q > p - 1 \) and \( \frac{pq}{q-p+1} < n \) then the equation (2.16) has a nonnegative renormalized (or equivalently, entropy) solution for some \( \epsilon > 0 \).

Theorem 2.13. Let \( \Omega \) be a uniformly \((k-1)\)-convex domain in \( \mathbb{R}^n \), and let \( \omega \in \mathcal{M}_B^+(\Omega) \) be compactly supported in \( \Omega \). Suppose that \( 1 \leq k \leq n \), \( q > k \), \( R = \text{diam}(\Omega) \), and \( \varphi \in C^0(\partial \Omega) \), \( \varphi \geq 0 \). Then the following statements are equivalent.

(i) There exists a solution \( u \geq 0 \), \( -u \in \Phi^k(\Omega) \cap L^q(\Omega) \), continuous near \( \partial \Omega \), to the equation

\[ \begin{aligned}
F_k[-u] &= u^q + \epsilon \omega \quad \text{in } \Omega, \\
u &= \epsilon \varphi \quad \text{on } \partial \Omega
\end{aligned} \]

for some \( \epsilon > 0 \).

(ii) For all compact sets \( E \subset \text{supp} \omega \),

\[ \omega(E) \leq C \text{Cap}_{G_p, \frac{q}{q-p+1}}(E). \]

(iii) The testing inequality

\[ \int_B \left[ W_{2k, q-k}^{2R} \omega_B(x) \right]^q \, dx \leq C \omega(B) \]

holds for all balls \( B \) such that \( B \cap \text{supp} \omega \neq \emptyset \).

(iv) There exists a constant \( C \) such that

\[ W_{2k, q-k}^{2R}(W_{2k, q-k}^{2R} \omega)^q(x) \leq C W_{2k, q-k}^{2R} \omega(x) < \infty \quad \text{a.e. on } \Omega. \]
Remark 2.14. As in Remark 2.11 suppose that $\omega \in \mathcal{M}_B^+(\Omega) \cap L^s(\Omega \setminus E)$, for a compact set $E \subset \Omega$, where $s > \frac{n}{2k}$ if $k \leq \frac{n}{2}$, and $s = 1$ if $k > \frac{n}{2}$. Then any one of the conditions (ii), (iii), and (iv) in Theorem 2.13 is still sufficient for the solvability of (2.18) for some $\epsilon > 0$. Moreover, in the subcritical case $\frac{2kq}{q-k} > n$ these conditions are redundant.

Corollary 2.15. Suppose that $f \in L^{\frac{n(q-k)}{2kq}}(\Omega) \cap L^s(\Omega \setminus E)$ for some $s > \frac{n}{2k}$ and for some compact set $E \subset \Omega$. Let $d\omega = fd\mathbf{x}$. If $q > k$ and $\frac{2kq}{q-k} < n$ then the equation (2.18) has a nonnegative solution for some $\epsilon > 0$.

Our results on local integral estimates for quasilinear and Hessian inequalities are given in the next two theorems. We will need the capacity associated with the space $W^{\alpha,s}$ relative to the domain $\Omega$ defined by

$$\text{cap}_{\alpha,s}(E, \Omega) = \inf \{ \| f \|^s_{W^{\alpha,s}(\mathbb{R}^n)} : f \in C_0^\infty(\Omega), f \geq 1 \text{ on } E \}.$$  

Theorem 2.16. Let $u$ be a nonnegative $A$-superharmonic function in $\Omega$ such that $-\text{div}A(x, \nabla u) \geq u^q$, where $p > 1$ and $q > p-1$. Then there exists a constant $C$ which depends only on $p,q,n$, and the structural constants of $A$ such that

$$\int_{B_R} u^q \, dx \leq C R^{n-\frac{pq}{q-p+1}}$$

if $\frac{pq}{q-p+1} < n$, and

$$\int_{B_r} u^q \, dx \leq C (\log \frac{2R}{r})^{\frac{1-p}{q-p+1}}$$

if $\frac{pq}{q-p+1} = n$. Here $0 < r \leq R$, $B_r \subset B_R$, and $B_R$ is a ball such that $B_{2R} \subset \Omega$.

Moreover, if $\frac{pq}{q-p+1} < n$, and $\Omega$ is a bounded $C^\infty$-domain then

$$\int_E u^q \leq C \text{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$$

for any compact set $E \subset \Omega$.

Theorem 2.17. Let $u \geq 0$ be such that $-u$ is $k$-subharmonic and that $F_k[-u] \geq u^q$ in $\Omega$ with $q > k$. Then there exists a constant $C$ which depends only on $k, q, n$ such that

$$\int_{B_R} u^q \, dx \leq C R^{n-\frac{2kq}{q-k}}$$
if \( \frac{2kq}{q-k} < n \), and
\[
\int_{B_r} u^q \, dx \leq C \left( \log \frac{2R}{r} \right)^{-k}
\]
if \( \frac{2kq}{q-k} = n \). Here \( 0 < r \leq R \), \( B_r \subset B_R \), and \( B_R \) is a ball such that \( B_{2R} \subset \Omega \).

Moreover, if \( \frac{2kq}{q-k} < n \) and \( \Omega \) is a bounded \( C^\infty \)-domain then
\[
\int_E u^q \leq C \text{cap}_{2k, \frac{q}{q-k}}(E, \Omega)
\]
for any compact set \( E \subset \Omega \).

As a consequence of Theorems 2.10 and 2.13, we deduce the following results concerning removable singularities of quasilinear and fully nonlinear equations.

**Theorem 2.18.** Let \( E \) be a compact subset of \( \Omega \). Then any solution \( u \) to the problem
\[
\begin{cases}
  u \text{ is } \mathcal{A}\text{-superharmonic in } \Omega \setminus E, \\
  u \in L^q_{\text{loc}}(\Omega \setminus E), \quad u \geq 0, \\
  -\text{div} \mathcal{A}(x, \nabla u) = u^q \text{ in } \mathcal{D}'(\Omega \setminus E),
\end{cases}
\]
is also a solution to
\[
\begin{cases}
  u \text{ is } \mathcal{A}\text{-superharmonic in } \Omega, \\
  u \in L^q_{\text{loc}}(\Omega), \quad u \geq 0, \\
  -\text{div} \mathcal{A}(x, \nabla u) = u^q \text{ in } \mathcal{D}'(\Omega),
\end{cases}
\]
if and only if \( \text{Cap}_{G, \frac{q}{q-k}}(E) = 0 \).

**Theorem 2.19.** Let \( E \) be a compact subset of \( \Omega \). Then any solution \( u \) to the problem
\[
\begin{cases}
  -u \text{ is } k\text{-subharmonic in } \Omega \setminus E, \\
  u \in L^q_{\text{loc}}(\Omega \setminus E), \quad u \geq 0, \\
  F_k[-u] = u^q \text{ in } \mathcal{D}'(\Omega \setminus E),
\end{cases}
\]
is also a solution to
\[
\begin{cases}
  -u \text{ is } k\text{-subharmonic in } \Omega, \\
  u \in L^q_{\text{loc}}(\Omega), \quad u \geq 0, \\
  F_k[-u] = u^q \text{ in } \mathcal{D}'(\Omega),
\end{cases}
\]
if and only if \( \text{Cap}_{G, 2k, \frac{q}{q-k}}(E) = 0 \).
In [TW3], Trudinger and Wang introduced the so called $k$-Hessian capacity $\text{cap}_k(\cdot, \Omega)$ defined by

$$\text{cap}_k(E, \Omega) = \sup \left\{ \int_E F_k[u] : u \text{ is } k\text{-subharmonic in } \Omega, -1 < u < 0 \right\}.$$  

Our next theorem asserts that locally the $k$-Hessian capacity is equivalent to the Bessel capacity $\text{Cap}_{G^{2k,k+1}}$.

In what follows, $\mathcal{Q} = \{Q\}$ will stand for a Whitney decomposition of $\Omega$ into a union of disjoint dyadic cubes (see Sec. 6).

**Theorem 2.20.** Let $1 \leq k < \frac{n}{2}$ be an integer. Then there are constants $M_1, M_2$ such that

$$M_1 \text{ Cap}_{G^{2k,k+1}}(E) \leq \text{cap}_k(E, \Omega) \leq M_2 \text{ Cap}_{G^{2k,k+1}}(E),$$

for any compact set $E \subset \Omega$ with $Q \in \mathcal{Q}$. Furthermore, if $\Omega$ is a bounded $C^\infty$-domain then

$$\text{cap}_k(E, \Omega) \leq C \text{ cap}_{G^{2k,k+1}}(E, \Omega),$$

for any compact set $E \subset \Omega$, where $\text{cap}_{G^{2k,k+1}}(E, \Omega)$ is defined by (2.19) with $\alpha = 2k$ and $s = \frac{2k}{k+1}$.

3. Discrete models of nonlinear equations

Let $\mathcal{D}$ be the family of all dyadic cubes in $\mathbb{R}^n$. For $\omega \in \mathcal{M}^+(\mathbb{R}^n)$ we define its dyadic Riesz and Wolff potentials respectively by

$$I_\alpha \omega(x) = \sum_{Q \in \mathcal{D}} \frac{\omega(Q)}{|Q|^{1-\alpha/n}} \chi_Q(x), \quad \alpha > 0,$$

$$W_{\alpha,p} \omega(x) = \sum_{Q \in \mathcal{D}} \left[ \frac{\omega(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \chi_Q(x).$$

In this section we are concerned with nonlinear inhomogeneous integral equations of the type

$$u = W_{\alpha,p}(u^q) + f, \quad u \in L^q_{\text{loc}}(\mathbb{R}^n),$$

where $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, $q > p - 1$, and $W_{\alpha,p}$ is defined as in (3.2) with $\alpha > 0$ and $p > 1$ such that $0 < \alpha p < n$.

It is convenient to introduce a nonlinear operator $\mathcal{N}$ associated with the equation (3.3) defined by

$$\mathcal{N} f = W_{\alpha,p}(f^q), \quad f \in L^q_{\text{loc}}(\mathbb{R}^n),$$
so that (5.3) can be rewritten as
\[ u = \mathcal{N}u + f, \quad u \in L^q_{\text{loc}}(\mathbb{R}^n). \]
It is obvious that \( \mathcal{N} \) is monotonic, i.e., \( \mathcal{N}f \geq \mathcal{N}g \) whenever \( f \geq g \geq 0 \) a.e., and \( \mathcal{N}(\lambda f) = \lambda^{\frac{q}{p-1}} \mathcal{N}f \) for all \( \lambda \geq 0 \). Since
\[ (a + b)^{p'-1} \leq \max\{1, 2^{p'-2}\}(a^{p'-1} + b^{p'-1}) \]
for all \( a, b \geq 0 \), it follows that
\[ \left( \mathcal{N}(f + g) \right)^{1/q} \leq \max\{1, 2^{p'-2}\} \left[ (\mathcal{N}f)^{1/q} + (\mathcal{N}g)^{1/q} \right]. \]

**Proposition 3.1.** Let \( \mu \) be a measure in \( M^+(\mathbb{R}^n) \), \( \alpha > 0 \), \( p > 1 \) and \( q > p - 1 \). Then the following quantities are equivalent:
\[ (a) \quad A_1(P, \mu) = \sum_{Q \subset P} \left[ \frac{\mu(Q)}{|Q|^{1-2\alpha/n}} \right]^{\frac{1}{p-1}} |Q|; \]
\[ (b) \quad A_2(P, \mu) = \int_P \left[ \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{(1-2\alpha/n)\frac{1}{p-1}}} \chi_Q(x) \right]^q dx, \]
\[ (c) \quad A_3(P, \mu) = \int_P \left[ \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{1-2\alpha/n}} \chi_Q(x) \right]^{\alpha p/n} dx, \]
where \( P \) is a dyadic cube in \( \mathbb{R}^n \) or \( P = \mathbb{R}^n \) and the constants of equivalence do not depend on \( P \) and \( \mu \).

**Proof.** The equivalence of \( A_1 \) and \( A_3 \) follows from Wolff’s inequality (see [HW], [COV]). Moreover, it has been proven in [COV] that
\[ A_3(P, \mu) \asymp \int_P \left[ \sup_{x \in Q \subset P} \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} dx. \]
Since
\[ \left[ \sup_{x \in Q \subset P} \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \leq \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{(1-\alpha p/n)\frac{1}{p-1}}} \chi_Q(x), \]
from (3.7) we obtain \( A_3 \leq CA_2 \). In addition, for \( p \leq 2 \) we clearly have \( A_2 \leq A_3 \leq CA_1 \). Therefore, it remains to check that, in the case \( p > 2 \), \( A_2 \leq CA_1 \) for some \( C > 0 \) independent of \( P \) and \( \mu \). By Proposition 2.2 in [COV] we have (note that \( q > p - 1 > 1 \))
\[ A_2(P, \mu) = \int_P \left[ \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{(1-\alpha p/n)\frac{1}{p-1}}} \chi_Q(x) \right]^q dx \]
\[ \leq C \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{(1-\alpha p/n)\frac{1}{p-1}+q-2}} \left[ \sum_{Q' \subset Q} \frac{\mu(Q')}{|Q'|^{(1-\alpha p/n)\frac{1}{p-1}-1}} \right]^{q-1}. \]
On the other hand, by Hölder’s inequality,

\[
\sum_{Q' \subset Q} \frac{\mu(Q')}{|Q'|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - 1}} \leq \sum_{Q' \subset Q} \left( \mu(Q')^{\frac{1}{p-1}} |Q'|^{\epsilon} \right)^{\frac{1}{\epsilon}} \left( \sum_{Q' \subset Q} |Q'|^{-r(1 - \frac{\alpha p}{n})\frac{1}{p-1} + r - r\epsilon} \right)^{\frac{1}{r}}.
\]

where \( r' = p - 1 > 1 \), \( r = \frac{p-1}{p-2} \) and \( \epsilon > 0 \) is chosen so that \( -r(1 - \frac{\alpha p}{n})\frac{1}{p-1} + r - r\epsilon > 1 \), i.e., \( 0 < \epsilon < \frac{\alpha p}{(p-1)n} \). Therefore,

\[
\sum_{Q' \subset Q} \frac{\mu(Q')}{|Q'|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - 1}} \leq C \frac{\mu(Q)^{\frac{1}{p-1}} |Q'|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} + \epsilon}}{|Q'|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - 1}}.
\]

Hence, combining this with (3.8) we obtain

\[
A_2(P, \mu) \leq C \sum_{Q \subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - q - 2}} \left[ \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - 1}} \right]^{q-1} = C \sum_{Q \subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1} - 1}} = CA_1(P, \mu).
\]

This completes the proof of the proposition. \( \square \)

**Theorem 3.2.** Let \( \alpha > 0 \), \( p > 1 \) be such that \( 0 < \alpha p < n \), and let \( q > p - 1 \). Suppose \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \) and \( d\omega = f^q dx \). Then the following statements are equivalent.

(i) The equation

\[
(3.9) \quad u = W_{\alpha, p}(u^q) + \epsilon f
\]

has a solution \( u \in L^q_{\text{loc}}(\mathbb{R}^n) \) for some \( \epsilon > 0 \).

(ii) The testing inequality

\[
(3.10) \quad \int_P \left[ \sum_{Q \subset P} \frac{\omega(Q)}{|Q|^{\left(1 - \frac{\alpha p}{n}\right)\frac{1}{p-1}}} \chi_Q(x) \right]^{\frac{q}{p-1}} dx \leq C |P|_\omega
\]
holds for all dyadic cubes $P$.

(iii) The testing inequality

$$\int_P \left[ \sum_{Q \subset P} \frac{\omega(Q)^{1 \over \alpha p}}{|Q|^{(1-\alpha p)\over p}} \chi_Q(x) \right]^q dx \leq C |P|_\omega$$

holds for all dyadic cubes $P$.

(iv) There exists a constant $C$ such that

$$W_{\alpha, p}[W_{\alpha, p}(f^q)]^q(x) \leq CW_{\alpha, p}(f^q)(x) < \infty \quad \text{a.e.}$$

Proof. We show that $(iv) \implies (i) \implies (ii) \implies (iii) \implies (iv)$. Note that by Proposition 3.1 we have $(ii) \iff (iii)$. Therefore, it is enough to prove that $(iv) \implies (i) \implies (iii) \implies (iv)$.

Proof of $(iv) \implies (i)$. Note that the pointwise condition (3.12) can be rewritten as

$$N^2 f \leq CN f < \infty \quad \text{a.e.},$$

where $N$ is the operator defined by (3.4). The sufficiency of this condition for the solvability of (3.9) can be proved using simple iterations:

$$u_{n+1} = Nu_n + \epsilon f, \quad n = 0, 1, 2, \ldots$$

starting from $u_0 = 0$. Since $N$ is monotonic it is easy to see that $u_n$ is increasing and that $\epsilon^{p-1} Nf + \epsilon f \leq u_n$ for all $n \geq 2$. Let $c(p) = \max\{1, 2^{p-1}\}$, $c_1 = 0$, $c_2 = [\epsilon^{p-1} c(p)]^q$ and

$$c_n = \left[ \epsilon^{p-1} c(p)(1 + C^{1/q}) c_{n-1}^{p-1} \right]^q, \quad n = 3, 4, \ldots$$

where $C$ is the constant in (3.12). Here we choose $\epsilon$ so that

$$\epsilon^{p-1} c(p) = \left( \frac{q - p + 1}{q} \right)^{q-p+1\over q} \left( \frac{p - 1}{q} \right)^{p-1\over q} C^{1-p^q}.$$

By induction and using (3.6) we have

$$u_n \leq c_n N f + \epsilon f, \quad n = 1, 2, 3, \ldots$$

Note that

$$x_0 = \left[ \frac{q}{p - 1} \epsilon^{p-1} c(p) C^{1/q} \right]^{q-p+1\over q-1}$$

is the only root of the equation

$$x = \left[ \epsilon^{p-1} c(p)(1 + C^{1/q}x) \right]^q$$

and thus $\lim_{n \to \infty} c_n = x_0$. Hence there exists a solution

$$u(x) = \lim_{n \to \infty} u_n(x)$$
to equation (3.9) (with that choice of \( \varepsilon \)) such that
\[
\varepsilon f + \varepsilon^{\frac{q}{p'}} W_{\alpha, p}(f^q) \leq u(x) \leq \varepsilon f + x_0 W_{\alpha, p}(f^q).
\]

**Proof of (i) \( \implies \) (iii).** Suppose that \( u \in L^q_{\text{loc}}(\mathbb{R}^n) \) is a solution of (3.9). Let \( P \) be a cube in \( D \) and \( d\mu = u^q dx \). Since
\[
[u(x)]^q \geq [W_{\alpha, p}(u^q)(x)]^q \quad \text{a.e.,}
\]
we have
\[
\int_P [W_{\alpha, p}(u^q)(x)]^q dx \leq \int_P [u(x)]^q dx.
\]
Thus,
\[
(3.13) \quad \int_P \left[ \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{(1 - \frac{\alpha p}{n})}^{p-1}} \chi_Q(x) \right]^q dx \leq C |P|_{\mu},
\]
for all \( P \in D \). By Proposition 3.1, inequality (3.13) is equivalent to
\[
\int_P \left[ \sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{1 - \frac{\alpha p}{n}}} \chi_Q(x) \right]^{\frac{q}{p-1}} dx \leq C |P|_{\mu}
\]
for all \( P \in D \), which in its turn is equivalent to the weak-type inequality
\[
(3.14) \quad \| \mathcal{I}_{\alpha p}(g) \|_{L^{q, (dx)} \to L^{q, (d\omega)}} \leq C \| g \|_{L^{q, (dx)}},
\]
for all \( g \in L^{\frac{q}{p-1}}(\mathbb{R}^n), \ g \geq 0 \) (see [NTV], [VW]). Note that by (3.9),
\[
d\mu = u^q dx \geq \varepsilon^q f^q dx = \varepsilon^q d\omega.
\]
We now deduce from (3.14),
\[
(3.15) \quad \| \mathcal{I}_{\alpha p}(g) \|_{L^{q, (d\omega)} \to L^{q, (dx)}} \leq \frac{C}{\varepsilon^{q-\frac{q}{p+1}}} \| g \|_{L^{q, (dx)}}
\]
Similarly, by duality and Proposition 3.1 we see that (3.15) is equivalent to the testing inequality (3.11). The implication (i) \( \implies \) (iii) is proved.

**Proof of (iii) \( \implies \) (iv).** We first deduce from the testing inequality (3.11) that
\[
|P|_{\omega} \leq C \left| P \right|^{1 - \frac{\alpha p}{q(q+1)}},
\]
for all dyadic cubes \( P \). In fact, this can be verified by using (3.11) and the obvious estimate
\[
\int_P \left[ \frac{\omega(P)}{|P|^{1-\frac{\alpha p}{n}}} \right]^{\frac{q}{p-1}} dx \leq \int_P \left[ \sum_{Q \subset P} \frac{\omega(Q)}{|Q|^{(1 - \frac{\alpha p}{n})}^{p-1}} \chi_Q(x) \right]^q dx.
\]
Following [KV], [V3], we next introduce a certain decomposition of the dyadic Wolff potential $W_{\alpha, \mu}$. To each dyadic cube $P \in \mathcal{D}$, we associate the “upper” and “lower” parts of $W_{\alpha, \mu}$ defined respectively by

$$U_P \mu(x) = \sum_{Q \subset P} \left[ \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \chi_Q(x),$$

$$V_P \mu(x) = \sum_{Q \supset P} \left[ \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \chi_Q(x).$$

Obviously,

$$U_P \mu(x) \leq W_{\alpha, \mu}(x), \quad V_P \mu(x) \leq W_{\alpha, \mu}(x),$$

and for $x \in P$,

$$W_{\alpha, \mu}(x) = U_P \mu(x) + V_P \mu(x) - \left[ \frac{\mu(P)}{|P|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}}.$$

Using the notation just introduced, we can rewrite the testing inequality (3.11) in the form:

$$\int_P [U_P \omega(x)]^q \, dx \leq C \, |P|_\omega,$$

for all dyadic cubes $P$. Recall that $d\omega = f^q \, dx$. The desired pointwise inequality (3.12) can be restated as

$$\sum_{P \in \mathcal{D}} \left[ \int_P [W_{\alpha, \mu} \omega(y)]^q \, dy \right] \frac{1}{|P|^{1-\alpha p/n}} \chi_P(x) \leq C \, W_{\alpha, \mu} \omega(x).$$

From the discussion above we have, for $y \in P$,

$$W_{\alpha, \mu} \omega(y) \leq U_P \omega(y) + V_P \omega(y)$$

while from the testing inequality (3.17),

$$\sum_{P \in \mathcal{D}} \left[ \int_P [U_P \omega(y)]^q \, dy \right] \frac{1}{|P|^{1-\alpha p/n}} \chi_P(x) \leq C \, W_{\alpha, \mu} \omega(x).$$

Therefore, to prove (3.18) it enough to prove

$$\sum_{P \in \mathcal{D}} \left[ \int_P [V_P \omega(y)]^q \, dy \right] \frac{1}{|P|^{1-\alpha p/n}} \chi_P(x) \leq C \, W_{\alpha, \mu} \omega(x).$$

Note that, for $y \in P$,

$$V_P \omega(y) = \sum_{Q \supset P} \left[ \frac{\omega(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} = \text{const.}$$
An application of the elementary inequality
\[
\left( \sum_{k=1}^{\infty} a_k \right)^s \leq s \sum_{k=1}^{\infty} a_k \left( \sum_{j=k}^{\infty} a_j \right)^{s-1}
\]
where \(1 \leq s < \infty\) and \(0 \leq a_k < \infty\), then gives
\[
|\mathcal{V}_P \omega(y)|^{\frac{q}{p-1}} \leq C \sum_{Q \supset P} \left\{ \frac{\omega(Q)}{|Q|^{1-\alpha p/n}} \right\}^{\frac{1}{p-1}} \sum_{R \supset Q} \left\{ \frac{\omega(R)}{|R|^{1-\alpha p/n}} \right\}^{\frac{1}{p-1}} \chi_P(x).
\]
Using this inequality we see that the left-hand side of (3.19) is bounded from above by a constant multiple of
\[
\sum_{P \in \mathcal{D}} |P|^{\frac{\alpha p}{n(p-1)}} \sum_{Q \supset P} \left\{ \frac{\omega(Q)}{|Q|^{1-\alpha p/n}} \right\}^{\frac{1}{p-1}} \sum_{R \supset Q} \left\{ \frac{\omega(R)}{|R|^{1-\alpha p/n}} \right\}^{\frac{1}{p-1}} \chi_P(x).
\]
Changing the order of summation, we see that it is equal to
\[
\sum_{Q \in \mathcal{D}} \left\{ \frac{\omega(Q)}{|Q|^{1-\alpha p/n}} \right\}^{\frac{1}{p-1}} \chi_Q(x) \left\{ \sum_{P \subset Q} |P|^{\frac{\alpha p}{n(p-1)}} \chi_P(x) |\mathcal{V}_Q \omega(x)|^{\frac{q}{p-1}} \right\}.
\]
By (3.16), the expression in the curly brackets above is uniformly bounded. Therefore, the proof of estimate (3.19), and hence of (iii) \(\Rightarrow\) (iv), is complete. \(\square\)

4. \(A\)-superharmonic functions

In this section, we recall for later use some facts on \(A\)-superharmonic functions, most of which can be found in [HKM], [KM1], [KM2], and [TW3]. Let \(\Omega\) be an open set in \(\mathbb{R}^n\), and \(p > 1\). We will mainly be interested in the case where \(\Omega\) is bounded and \(1 < p \leq n\), or \(\Omega = \mathbb{R}^n\) and \(1 < p < n\). We assume that \(\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is a vector valued mapping which satisfies the following structural properties:

(4.1) the mapping \(x \to \mathcal{A}(x, \xi)\) is measurable for all \(\xi \in \mathbb{R}^n\),
(4.2) the mapping \(\xi \to \mathcal{A}(x, \xi)\) is continuous for a.e. \(x \in \mathbb{R}^n\),

and there are constants \(0 < \alpha \leq \beta < \infty\) such that for a.e. \(x \in \mathbb{R}^n\),

(4.3) \(\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p\), \(\mathcal{A}(x, \xi) \leq \beta |\xi|^{p-1}\),
(4.4) \((\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0\), if \(\xi_1 \neq \xi_2\),
(4.5) \(\mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi)\), if \(\lambda \in \mathbb{R}\ \setminus \{0\}\).
For $u \in W^{1,p}_{\text{loc}}(\Omega)$, we define the divergence of $A(x, \nabla u)$ in the sense of distributions, i.e., if $\varphi \in C_0^\infty(\Omega)$, then
\[
\text{div}A(x, \nabla u)(\varphi) = -\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx.
\]

It is well known that every solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to the equation
\[
-\text{div}A(x, \nabla u) = 0 \quad (4.6)
\]
has a continuous representative. Such continuous solutions are said to be $A$-harmonic in $\Omega$. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ and
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0,
\]
for all nonnegative $\varphi \in C_0^\infty(\Omega)$, i.e., $-\text{div}A(x, \nabla u) \geq 0$ in the distributional sense, then $u$ is called a supersolution of the equation (4.6) in $\Omega$.

A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ is called $A$-superharmonic if $u$ is not identically infinite in each component of $\Omega$, and if for all open sets $D$ such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $A$-harmonic in $D$, it follows that $h \leq u$ on $\partial D$ implies $h \leq u$ in $D$.

It is worth mentioning that $p$-superharmonicity can also be defined equivalently using the language of viscosity solutions (see [JLM]).

We recall here the fundamental connection between supersolutions of (4.6) and $A$-superharmonic functions [HKM].

**Proposition 4.1** ([HKM]). (i) If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is such that
\[
-\text{div}A(x, \nabla u) \geq 0,
\]
then there is an $A$-superharmonic function $v$ such that $u = v$ a.e.. Moreover,
\[
\text{(4.7)} \quad v(x) = \text{ess lim inf}_{y \to x} v(y), \quad x \in \Omega.
\]

(ii) If $v$ is $A$-superharmonic, then (4.7) holds. Moreover, if $v \in W^{1,p}_{\text{loc}}(\Omega)$, then
\[
-\text{div}A(x, \nabla v) \geq 0.
\]

(iii) If $v$ is $A$-superharmonic and locally bounded, then $v \in W^{1,p}_{\text{loc}}(\Omega)$, and
\[
-\text{div}A(x, \nabla v) \geq 0.
\]
Note that an $\mathcal{A}$-superharmonic function $u$ does not necessarily belong to $W^{1,p}_{\text{loc}}(\Omega)$, but its truncation $\min\{u, k\}$ does, for every integer $k$, by Proposition 4.1(iii). Using this we set
\[ Du = \lim_{k \to \infty} \nabla \min\{u, k\}, \]
defined a.e. If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\text{loc}}(\Omega)$, then $Du$ coincides with the regular distributional gradient of $u$. In general we have the following gradient estimates [KM1] (see also [HKM], [TW4]).

**Proposition 4.2** ([KM1]). Suppose $u$ is $\mathcal{A}$-superharmonic in $\Omega$ and $1 \leq q < \frac{n}{n-1}$. Then both $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $L^q_{\text{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then $Du$ is the distributional gradient of $u$.

We can now extend the definition of the divergence of $\mathcal{A}(x, \nabla u)$ if $u$ is merely an $\mathcal{A}$-superharmonic function in $\Omega$. For such $u$ we set
\[ -\text{div}\mathcal{A}(x, \nabla u)(\varphi) = \int_{\Omega} \mathcal{A}(x, Du) \cdot \nabla \varphi \, dx, \]
for all $\varphi \in C_0^\infty(\Omega)$. Note that by Proposition 4.2 and the dominated convergence theorem,
\[ -\text{div}\mathcal{A}(x, \nabla u)(\varphi) = \lim_{k \to \infty} \int_{\Omega} \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla \varphi \, dx \geq 0 \]
whenever $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geq 0$.

Since $-\text{div}\mathcal{A}(x, \nabla u)$ is a nonnegative distribution in $\Omega$ for an $\mathcal{A}$-superharmonic $u$, it follows that there is a positive (not necessarily finite) Radon measure denoted by $\mu[u]$ such that
\[ -\text{div}\mathcal{A}(x, \nabla u) = \mu[u] \text{ in } \Omega. \]
Conversely, given a positive finite measure $\mu$ in a bounded $\Omega$, there is an $\mathcal{A}$-superharmonic function $u$ such that $-\text{div}\mathcal{A}(x, \nabla u) = \mu$ in $\Omega$ and $\min\{u, k\} \in W^{1,p}_{\text{loc}}(\Omega)$ for all integers $k$. Moreover, if $\mu$ is a positive finite measure in $\mathbb{R}^n$ we can also find a positive $\mathcal{A}$-superharmonic function $u$ such that $-\text{div}\mathcal{A}(x, \nabla u) = \mu$ in $\mathbb{R}^n$. We refer to [KM1] and [KM2] for details.

The following weak continuity result in [TW4] will be used later in Sec. 5 to prove the existence of $\mathcal{A}$-superharmonic solutions to quasilinear equations.

**Theorem 4.3** ([TW4]). Suppose that $\{u_n\}$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in $\Omega$ that converges a.e. to an $\mathcal{A}$-superharmonic function $u$. Then the sequence of measures $\{\mu[u_n]\}$
converges to \( \mu[u] \) weakly, i.e.,
\[
\lim_{n \to \infty} \int_\Omega \varphi \, d\mu[u_n] = \int_\Omega \varphi \, d\mu[u],
\]
for all \( \varphi \in C_0^\infty(\Omega) \).

In [KM2] (see also [Mi, Theorem 3.1] and [MZ]) the following pointwise potential estimate for \( \mathcal{A} \)-superharmonic functions was established, which serves as a major tool in our study of quasilinear equations of Lane–Emden type.

**Theorem 4.4 ([KM2]).** Suppose \( u \geq 0 \) is an \( \mathcal{A} \)-superharmonic function in \( B(x, 3r) \). If \( \mu = -\text{div}\mathcal{A}(x, \nabla u) \), then
\[
C_1 \mathcal{W}_{1,p}^r(x) \leq u(x) \leq C_2 \inf_{B(x,r)} u + C_3 \mathcal{W}_{1,p}^{2r}(x),
\]
where \( C_1, C_2 \) and \( C_3 \) are positive constants which depend only on \( n, p \) and the structural constants \( \alpha \) and \( \beta \).

A consequence of Theorem 4.4 is the following global version of the above potential pointwise estimate.

**Corollary 4.5 ([KM2]).** Let \( u \) be an \( \mathcal{A} \)-superharmonic function in \( \mathbb{R}^n \) with \( \inf_{\mathbb{R}^n} u = 0 \). If \( \mu = -\text{div}\mathcal{A}(x, \nabla u) \), then
\[
\frac{1}{K} \mathcal{W}_{1,p}(x) \leq u(x) \leq K \mathcal{W}_{1,p}(x),
\]
for all \( x \in \mathbb{R}^n \), where \( K \) is a positive constant depending only on \( n, p \) and the structural constants \( \alpha \) and \( \beta \).

5. **Quasilinear equations on \( \mathbb{R}^n \)**

In this section, we study the solvability problem for the quasilinear equation
\[
-\text{div}\mathcal{A}(x, \nabla u) = u^q + \omega
\]
in the class of nonnegative \( \mathcal{A} \)-superharmonic functions on the entire space \( \mathbb{R}^n \), where \( \mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p \) is defined precisely as in Sec. 4. Here we assume \( 1 < p < n, q > p - 1 \), and \( \omega \in \mathcal{M}^+(\mathbb{R}^n) \). In this setting, all solutions are understood in the “potential-theoretic” sense, i.e., \( 0 \leq u \in L^q_{\text{loc}}(\mathbb{R}^n) \) is a solution to (5.1) if \( u \) is an \( \mathcal{A} \)-superharmonic function and
\[
\int \lim_{k \to \infty} \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla \varphi \, dx = \int u^q \varphi \, dx + \int \varphi \, d\omega,
\]
for all test functions \( \varphi \in C_0^\infty(\mathbb{R}^n) \).

First we prove the continuous counterpart of Proposition 3.1. Here
we use the well-known argument due to Fefferman and Stein [FS] which is based on the averaging over shifts of the dyadic lattice $D$.

**Proposition 5.1.** Let $0 < r \leq \infty$. Let $\mu \in M^+(\mathbb{R}^n)$, $\alpha > 0$, $p > 1$, and $q > p - 1$. Then the following quantities are equivalent.

\begin{align*}
(a) \quad & \left\| W_{\alpha,p}^r \right\|_{L^q(dx)}^{q} = \left\| P_{\alpha,p}^r \right\|_{L^q(dx)}^{q} = \int_{\mathbb{R}^n} \left[ \frac{\mu(B_t(x))}{t^{n-\alpha}} \right]^\frac{q}{p-1} \, dt, \\
(b) \quad & \left\| W_{\alpha,p}^r \right\|_{L^q(dx)}^{q} = \int_{\mathbb{R}^n} \{ \int_{0}^{r} \left[ \frac{\mu(B_t(x))}{t^{n-\alpha}} \right]^\frac{q}{p} \, dt \}^q \, dx, \\
(c) \quad & \left\| P_{\alpha,p}^r \right\|_{L^q(dx)}^{q} = \int_{\mathbb{R}^n} \{ \int_{0}^{r} \mu(B_t(x)) \, dt \}^\frac{q}{p} \, dx,
\end{align*}

where the constants of equivalence do not depend on $\mu$ and $r$.

**Proof.** We will prove only the equivalence of (b) and (c), i.e., there are constants $C_1, C_2 > 0$ such that

\begin{equation}
C_1 \left\| W_{\alpha,p}^r \right\|_{L^q(dx)}^{q} \leq \left\| P_{\alpha,p}^r \right\|_{L^q(dx)}^{q} \leq C_2 \left\| W_{\alpha,p}^r \right\|_{L^q(dx)}^{q}.
\end{equation}

The equivalence of (a) and (c) which follows from Wolff’s inequality (see [AH], [HW]), can also be deduced by a similar argument. We first restrict ourselves to the case $r < \infty$. Observe that there is a constant $C > 0$ such that

\begin{equation}
\left\| P_{\alpha,p}^r \right\|_{L^q(dx)}^{q} \leq C \left\| \mu(B_t(x)) \right\|_{L^q(dx)}^{q}.
\end{equation}

In fact, since

\begin{equation}
\int_{0}^{r} \frac{\mu(B_t(x))}{t^{n-\alpha}} \, dt \leq C \int_{0}^{r} \frac{\mu(B_t(x))}{t^{n-\alpha}} \, dt + C \frac{\mu(B_{2r}(x))}{r^{n-\alpha}},
\end{equation}

(5.4) will follow from the estimate

\begin{equation}
\int_{\mathbb{R}^n} \left[ \frac{\mu(B_{2r}(x))}{r^{n-\alpha}} \right]^\frac{q}{p} \, dx \leq C \int_{\mathbb{R}^n} \left[ \int_{0}^{r} \frac{\mu(B_t(x))}{t^{n-\alpha}} \, dt \right]^\frac{q}{p} \, dx.
\end{equation}

Note that for a partition of $\mathbb{R}^n$ into a union of disjoint cubes $\{Q_j\}$ such that $\text{diam}(Q_j) = r/4$ we have

\begin{align*}
\int_{\mathbb{R}^n} \mu(B_{2r}(x)) \, dx &= \sum_j \int_{Q_j} \mu(B_{2r}(x)) \, dx \\
&\leq C \sum_j \int_{Q_j} \mu(Q_j) \, dx.
\end{align*}
where we have used the fact that the ball \( B_{2r}(x) \) is contained in the union of at most \( N \) cubes in \( \{ Q_j \} \) for some constant \( N \) depending only on \( n \). Thus

\[
\int_{\mathbb{R}^n} \left[ \frac{\mu(B_{2r}(x))}{r^{n-\alpha p}} \right]^\frac{q}{p-1} \, dx \leq C \sum_j \int_{Q_j} \left[ \frac{\mu(B_{r/2}(x))}{r^{n-\alpha p}} \right]^\frac{q}{p-1} \, dx
\]

\[
\leq C \sum_j \int_{Q_j} \left[ \int_0^r \frac{\mu(B_t(x)) \, dt}{t^{n-\alpha p}} \right]^\frac{q}{p-1} \, dx,
\]

which gives (5.5).

By arguing as in [COV], we can find constants \( a, C \) and \( c \) depending only on \( n \) such that

\[
W_{r, \alpha, p} \mu(x) \leq C r^{-n} \int_{|t| \leq cr} \sum_{Q \in D_t \atop \ell(Q) \leq 4r/a} \left[ \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^\frac{q}{p-1} \chi_Q(x) \, dt
\]

where \( D_t, t \in \mathbb{R}^n \), denotes the lattice \( \mathcal{D} + t = \{ Q = Q' + t : Q' \in \mathcal{D} \} \) and \( \ell(Q) \) is the side length of \( Q \). Using Proposition 2.2 in [COV] and arguing as in the proof of Theorem 3.1 we obtain

\[
\int_{\mathbb{R}^n} \sum_{Q \in D_t \atop \ell(Q) \leq 4r/a} \left[ \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \chi_Q(x) \right]^q \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \left[ \sum_{Q \in D_t \atop \ell(Q) \leq 4r/a} \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \chi_Q(x) \right]^\frac{q}{p-1} \, dx
\]

where the constant of equivalence is independent of \( \mu, r \) and \( t \). The last two estimates together with the integral Minkowski inequality then give

\[
\| W_{r, \alpha, p} \mu \|_{L^q(dx)} \leq C r^{-n} \int_{|t| \leq cr} \left\{ \int_{\mathbb{R}^n} \left( \sum_{Q \in D_t \atop \ell(Q) \leq 4r/a} \left[ \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \right]^\frac{q}{p-1} \chi_Q(x) \right)^q \, dx \right\}^\frac{1}{q} \, dt
\]

\[
\leq C r^{-n} \int_{|t| \leq cr} \left[ \int_{\mathbb{R}^n} \left( \sum_{Q \in D_t \atop \ell(Q) \leq 4r/a} \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \chi_Q(x) \right)^q \, dx \right]^\frac{1}{q} \, dt.
\]
Note that
\[
\sum_{Q \in \mathcal{D}_t \ell(Q) \leq 4r/a} \frac{\mu(Q)}{|Q|^{1-\alpha p/n}} \chi_Q(x) \leq C \sum_{2^k \leq 4r/a} \frac{\mu(B(x, \sqrt{n2^k}))}{2^{k(n-\alpha p)}} \leq CT_{\alpha p \sqrt{n/a}}^{\alpha p } \mu(x),
\]
where \(C\) is independent of \(t\). Thus, in view of (5.4), we obtain the lower estimate in (5.3).

Now by letting \(R \to \infty\) in the inequality
\[
||W_{\alpha, p, \mu}^R||_{L^q(dx)} \leq C||I_{\alpha p, \mu}^R||_{L^{q-1}(dx)}, \quad 0 < R < \infty,
\]
we get the lower estimate in (5.3) with \(r = \infty\). The upper estimate in (5.3) can be deduced in a similar way. This completes the proof of Proposition 5.1. □

The next theorem gives a characterization of the existence of non-negative solutions to the equation \(-\text{div} A(x, \nabla u) = \mu\) in terms of Wolff’s potentials.

**Theorem 5.2.** Let \(\mu\) be a measure in \(\mathcal{M}^+(\mathbb{R}^n)\). Suppose that \(W_{1, p, \mu} < \infty\) a.e. Then there is a nonnegative \(A\)-superharmonic function \(u\) in \(\mathbb{R}^n\) such that
\[
(5.6) \quad -\text{div} A(x, \nabla u) = \mu \quad \text{in} \quad \mathbb{R}^n,
\]
and
\[
(5.7) \quad \frac{1}{K} W_{1, p, \mu}(x) \leq u(x) \leq K W_{1, p, \mu}(x),
\]
for all \(x\) in \(\mathbb{R}^n\), where \(K\) is the constant in (4.8). Conversely, if \(u\) is a nonnegative \(A\)-superharmonic function in \(\mathbb{R}^n\) which solves (5.7) then \(W_{1, p, \mu} < \infty\) a.e. on \(\mathbb{R}^n\).

**Proof.** The second statement of the theorem follows immediately from the lower Wolff potential estimate. To prove the first statement, we let \(\mu_k = \mu_{B_k}\), the restriction of \(\mu\) on the ball \(B_k\) of radius \(k\) and centered at the origin, so that \(\mu_k \to \mu\) weakly as measures and \(\mu_k\) are finite positive Borel measures on \(\mathbb{R}^n\). Thus arguing as in the proof of Theorem 2.4 in [KML] we can find nonnegative \(A\)-superharmonic functions \(u_k\) in \(\mathbb{R}^n\) such that
\[
-\text{div} A(x, \nabla u_k) = \mu_k
\]
in \(\mathbb{R}^n\). By replacing \(u_k\) with \(u_k - \inf_{\mathbb{R}^n} u_k\) we can assume that \(\inf_{\mathbb{R}^n} u_k = 0\) so that \(u_k(x) \leq K W_{1, p, \mu_k}(x) \leq K W_{1, p, \mu}(x) < \infty\) for a.e. \(x\) in \(\mathbb{R}^n\) by Corollary 4.5. Let \(\{u_{k_j}\}\) be a subsequence of \(\{u_k\}\) such that \(u_{k_j} \to u\)
a.e. on \( \mathbb{R}^n \) for some nonnegative \( A \)-superharmonic function \( u \) in \( \mathbb{R}^n \) (see [KMT, Theorem 1.17]). Then by Theorem 4.3 we see that

\[-\text{div} A(x, \nabla u) = \mu\]

in \( \mathbb{R}^n \). The estimate (5.7) then follows from the Wolff potential estimate, which completes the proof of the theorem. □

**Theorem 5.3.** Let \( \omega \in \mathcal{M}^+(\mathbb{R}^n) \), \( 1 < p < n \), and \( q > p - 1 \). Assume that

\[(5.8) \quad W_{1,p}(W_{1,p}\omega)^q \leq C W_{1,p}\omega < \infty \quad \text{a.e.,}\]

where

\[(5.9) \quad C \leq \left( \frac{q - p + 1}{qK \max\{1, 2p'-2\}} \right)^{q(p' - 1)} \left( \frac{p - 1}{q - p + 1} \right),\]

and \( K \) is the constant in (4.8). Then there is an \( A \)-superharmonic function \( u \in L^q_{\text{loc}}(\mathbb{R}^n) \) such that

\[(5.10) \quad \begin{cases}
\inf_{x \in \mathbb{R}^n} u(x) = 0 \\
-\text{div} A(x, \nabla u) = u^q + \omega,
\end{cases}\]

and

\[1 \leq \frac{1}{M} W_{1,p}(x) \leq u(x) \leq M W_{1,p}(x),\]

for all \( x \) in \( \mathbb{R}^n \), where the constant \( M \) depends only on \( n, p, q \) and the structural constants \( \alpha \) and \( \beta \).

**Proof.** Let \( \{u_k\}_{k \geq 0} \) be a sequence of \( A \)-superharmonic functions such that \( \inf_{\mathbb{R}^n} u_k = 0 \), \( u_k \in L^q_{\text{loc}}(\mathbb{R}^n) \),

\[\int A(x, \nabla u_0) \cdot \nabla \varphi dx = \int \varphi d\omega,\]

and

\[(5.11) \quad \int A(x, \nabla u_{k+1}) \cdot \nabla \varphi dx = \int u_k^q \varphi dx + \int \varphi d\omega,\]

for all integer \( k \geq 0 \) and \( \varphi \in C_0^\infty(\mathbb{R}^n) \). The existence of such a sequence is guaranteed by Theorem 5.2 and condition (5.8). Put \( c_0 = K \), where \( K \) is the constant in (4.8). By the potential estimate we see that \( u_0 \leq c_0 W_{1,p}\omega \) and \( u_{k+1} \leq K W_{1,p}(u_k^q + \omega) \) for all \( k \geq 0 \). From these estimates and (5.8) we get

\[u_1 \leq K \max\{1, 2p'-2\}[W_{1,p}(u_0^q) + W_{1,p}\mu] \leq K \max\{1, 2p'-2\}(c_0^q(p' - 1)C + 1)W_{1,p}\mu = c_1 W_{1,p}\mu,\]
where \( c_1 = K \max\{1, 2^{p'-2}\} (c_0^{q'(p'-1)} C + 1) \). By induction we can find a sequence \( \{c_n\}_{n \geq 0} \) of positive numbers such that \( u_n \leq c_n W_{1,p} \) for all \( n \geq 0 \) with \( c_0 = K \) and \( c_{n+1} = K \max\{1, 2^{p'-2}\} (c_n^{q'(p'-1)} C + 1) \) for all \( n \geq 0 \). It is then easy to see that \( c_n \leq c_0^{(q'(p'-1))^{n+1}} \) for all \( n \geq 0 \) as long as \( 5.9 \) is satisfied. Thus

\[
\begin{align*}
  u_k &\leq K \max\{1, 2^{p'-2}\} q W_{1,p} \\
  &\text{for all } k \geq 0.
\end{align*}
\]

By Theorem 1.17 in [KM1], we can find a subsequence which is also denoted by \( \{u_k\}_{k \geq 0} \) and an \( A \)-superharmonic function \( u \) such that \( u_k \to u \) almost everywhere. As \( k \to \infty \) in \( 5.9 \), the left-hand side tends to \( \int A(x, \nabla u) \cdot \nabla \varphi dx \) by weak continuity results in [TW1] while the right hand side tends to \( \int u^q \varphi dx + \int \varphi d\omega \) by the dominated convergence theorem. Therefore \( u \) is a solution to \( 5.10 \), which completes the proof of the theorem. □

**Theorem 5.4.** Let \( \omega \) be a locally finite positive measure on \( R^n \), \( 1 < p < n \), and \( q > p - 1 \). Then the following statements are equivalent.

(i) There exists a nonnegative solution \( u \in L^q_{\text{loc}}(R^n) \) to the equation

\[
\begin{align*}
  \inf_{x \in R^n} u(x) &= 0 \\
  -\text{div} A(x, \nabla u) &= u^q + \epsilon \omega
\end{align*}
\]

for some \( \epsilon > 0 \).

(ii) The testing inequality

\[
\int_B [I_{p} \omega_B(x)]^{\frac{q}{p-1}} dx \leq C \omega(B)
\]

holds for all balls \( B \) in \( R^n \).

(iii) The testing inequality

\[
\int_B \left[ W_{1,p} \omega_B(x) \right]^q dx \leq C \omega(B)
\]

holds for all balls \( B \) in \( R^n \).

(iv) There exists a constant \( C \) such that

\[
W_{1,p}(W_{1,p} \omega)^q \leq C W_{1,p} \omega < \infty \quad \text{a.e.}
\]

Moreover, if the constant \( C \) in \( 5.14 \) satisfies

\[
C \leq \left( \frac{q - p + 1}{q K \max\{1, 2^{p'-2}\}} \right)^{q'(p'-1)} \left( \frac{p - 1}{q - p + 1} \right),
\]

where \( K \) is the constant in \( 4.8 \), then the equation \( 5.12 \) has a solution \( u \) with \( \epsilon = 1 \) which obeys the two-sided estimate

\[
C_1 W_{1,p} \omega(x) \leq u(x) \leq C_2 W_{1,p} \omega(x)
\]
for all \( x \in \mathbb{R}^n \).

**Remark 5.5.** It is of interest to note that statement (ii) in Theorem 5.4 is also equivalent to the following capacitary condition (see e.g. [V2]):

(v) There exists a constant \( C > 0 \) such that

\[
\omega(E) \leq C \text{Cap}_{1, \frac{2}{q-p+1}}(E)
\]

for all compact sets \( E \subset \mathbb{R}^n \).

**Proof of Theorem 5.4.** We show that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i). Note that (5.13) is also equivalent to the testing inequality (see e.g. [VW]):

\[
\int_{\mathbb{R}^n} \left[ I_p \omega_B(x) \right]^{\frac{q}{p-1}} dx \leq C \omega(B).
\]

By applying Proposition (5.1) we deduce (ii) \( \Rightarrow \) (iii). The implication (iv) \( \Rightarrow \) (i) clearly follows from Theorem 5.3. Therefore, it remains to check (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (iv).

**Proof of (i) \( \Rightarrow \) (ii).** Let \( u \) be a nonnegative solution of (5.12) and let \( \mu = u^q + \epsilon \omega \). Then \( \mu \) is a positive measure such that \( \mu \geq u^q \), \( \mu \geq \epsilon \omega \) and \( u(x) \geq \frac{1}{K} W_{1,p} \mu(x) \) where \( K \) is the constant in (4.8). Therefore,

\[
\int_P d\mu \geq \int_P u^q dx \geq C \int_P (W_{1,p} \mu)^q dx
\]

\[
\geq C \int_P \left[ \sum_{Q \subset P} \left( \frac{\mu(Q)}{|Q|^{1-n}} \right)^{\frac{1}{p-1}} \chi_Q(x) \right]^q dx,
\]

for all dyadic cubes \( P \) in \( \mathbb{R}^n \). Using this and Proposition 3.1 we get

\[
\sum_{Q \subset P} \left[ \frac{\mu(Q)}{|Q|^{1-n}} \right]^{\frac{q}{p-1}} |Q| \leq C \mu(P), \quad P \in \mathcal{D}.
\]

It is known that the preceding condition is equivalent to the inequality (see [V1, Sec. 3])

\[
\|I_p(f)\|_{L_{\frac{q}{p-1}}(d\mu)} \leq C \|f\|_{L_{\frac{q}{p-1}}(dx)},
\]

where \( C \) does not depend on \( f \in L_{\frac{q}{p-1}}(dx) \). Since \( \mu \geq \epsilon \omega \), from this we have

\[
\|I_p(f)\|_{L_{\frac{q}{p-1}}(d\omega)} \leq \epsilon \frac{q+1}{q} C \|f\|_{L_{\frac{q}{p-1}}(dx)}.
\]

Therefore, by duality we obtain the testing inequality (5.13). This completes the proof of (i) \( \Rightarrow \) (ii).
Proof of (iii)⇒(iv). We first claim that (5.14) yields
\[
\int_0^\infty \left[ \omega(B_t(x)) \right] \frac{1}{t^{n-p}} \frac{dt}{t} \leq C \frac{r^{n-p}}{r^{n-p+1}},
\]
where $C$ is independent of $x$ and $r$. Note that for $y \in B_t(x)$ and $\tau \geq 2t$, we have $B_t(x) \subset B_\tau(y)$. Thus,
\[
\omega(B_t(x)) \leq C \frac{\tau^{n-p}}{\tau^{n-p+1}}.
\]
Combining this with (5.14), we obtain
\[
(5.17) \quad \omega(B_t(x)) \leq C t^{n-p}.
\]
which clearly implies (5.16).

Next, we introduce a decomposition of the Wolff potential $W_{1,p}$ into its lower and upper parts defined respectively by
\[
L_r \mu(x) = \int_r^\infty \left[ \mu(B_t(x)) \right] \frac{1}{t^{n-p}} \frac{dt}{t}, \quad r > 0, \ x \in \mathbb{R}^n,
\]
and
\[
U_r \mu(x) = \int_0^r \left[ \mu(B_t(x)) \right] \frac{1}{t^{n-p}} \frac{dt}{t}, \quad r > 0, \ x \in \mathbb{R}^n.
\]
Let $d\nu = (W_{1,p} \omega)^q dx$. For each $r > 0$ let $d\mu_r = (U_r \omega)^q dx$ and $d\lambda_r = (L_r \omega)^q dx$. Then
\[
(5.18) \quad \nu \leq C(q)(\mu_r + \lambda_r)
\]
Let $x \in \mathbb{R}^n$ and $B_r = B_r(x)$. Since $W_{1,p}(W_{1,p} \omega)^q = W_{1,p} \omega$, we have to prove that
\[
W_{1,p} \omega(x) = \int_0^\infty \left[ \nu(B_r) \right] \frac{1}{r^{n-p}} \frac{dr}{r} \leq C W_{1,p} \omega(x).
\]
For $r > 0$, $t \leq r$ and $y \in B_r$ we have $B_t(y) \subset B_{2r}$. Therefore it is easy to see that $U_r \omega = U_r \omega_{B_{2r}}$ on $B_r$. Using this together with (5.14), we have
\[
\mu_r(B_r) = \int_{B_r} (U_r \omega)^q dx = \int_{B_r} (U_r \omega_{B_{2r}})^q dx \leq C \omega(B_{2r}).
\]
Hence,
\[
\int_0^\infty \left[ \frac{\mu_r(B_r)}{r^{n-p}} \right] \frac{1}{r^{n-p}} \frac{dr}{r} \leq C \int_0^\infty \left[ \frac{\omega(B_{2r})}{r^{n-p}} \right] \frac{1}{r^{n-p}} \frac{dr}{r} \leq C' W_{1,p} \omega(x).
\]
On the other hand, for \( y \in B_r \) and \( t \geq r \), we have \( B_t(y) \subset B_{2t} \), and consequently

\[
(5.20) \quad L_t \omega(y) \leq \int_r^\infty \left[ \frac{\omega(B_{2t})}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \leq C \int_2^\infty \left[ \frac{\omega(B_t)}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \leq C L_t \omega(x).
\]

Using (5.20), we obtain

\[
\lambda_r(B_r) = \int_{B_r} (L_r \omega(y))^q dy \leq C (L_r \omega(x))^q |B_r|.
\]

Thus,

\[
\int_0^\infty \left[ \frac{\lambda_r(B_r)}{r^{n-p}} \right]^{\frac{1}{p-1}} \frac{dr}{r} \leq C' \int_0^\infty (L_r \omega(x))^{\frac{q}{p-1}} \left( \frac{|B_r|}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\]

Using now (5.16), we get

\[
\int_0^\infty \left[ \frac{\nu(B_r)}{r^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \leq C W_{1,p} \omega(x),
\]

where we have used integration by parts in the last equality. Combining (5.18), (5.19) and (5.21) gives

\[
W_{1,p}(x) = \int_0^\infty \left[ \frac{\nu(B_r)}{r^{n-p}} \right]^{\frac{1}{p-1}} \frac{dr}{r} \leq C W_{1,p} \omega(x),
\]

for a suitable constant \( C \) independent of \( \omega \). Thus, (iii) implies (iv) as claimed. This completes the proof of the theorem. \( \square \)
6. Renormalized solutions of quasilinear Dirichlet problems

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$. We denote by $\mathcal{M}_B(\Omega)$ (respectively $\mathcal{M}_B^+(\Omega)$) the set of all Radon measures (respectively nonnegative Radon measures) with bounded variation in $\Omega$. Let $\mathcal{A}$ be as in Sec. 4 and let $1 < p < \infty$. In this section we consider the Dirichlet problem

\[
\begin{cases}
-\text{div} \mathcal{A}(x, \nabla u) = u^q + \omega, \\
u \geq 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where $\omega \in \mathcal{M}_B^+(\Omega)$ and $q > p - 1$.

It is well known that when the data is not regular enough, a solution of nonlinear Leray-Lions type equations does not necessarily belong to the Sobolev space $W^{1,p}_0(\Omega)$. Therefore, we use the framework of renormalized solutions which seems proper for such problems with measure data (see, e.g., [DMOP]).

For a measure $\mu$ in $\mathcal{M}_B(\Omega)$, its positive and negative parts are denoted by $\mu^+$ and $\mu^-$, respectively. We say that a sequence of measures $\{\mu_n\}$ in $\mathcal{M}_B(\Omega)$ converges in the narrow topology to $\mu \in \mathcal{M}_B(\Omega)$ if

\[
\lim_{n \to \infty} \int_{\Omega} \varphi \, d\mu_n = \int_{\Omega} \varphi \, d\mu
\]

for every bounded and continuous function $\varphi$ on $\Omega$.

Denote by $\mathcal{M}_0(\Omega)$ (respectively $\mathcal{M}_s(\Omega)$) the set of all measures in $\mathcal{M}_B(\Omega)$ which are continuous (respectively singular) with respect to the capacity $\text{cap}_{1,p}(\cdot, \Omega)$. Here $\text{cap}_{1,p}(\cdot, \Omega)$ is the capacity relative to the domain $\Omega$ defined by

\[
\text{cap}_{1,p}(E, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \phi|^p \, dx : \phi \in C^\infty_0(\Omega), \phi \geq 1 \text{ on } E \right\}
\]

for any compact set $E \subset \Omega$. Recall that, for every measure $\mu$ in $\mathcal{M}_B(\Omega)$, there exists a unique pair of measures $(\mu_0, \mu_s)$ with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. If $\mu$ is nonnegative, then so are $\mu_0$ and $\mu_s$ (see [FST, Lemma 2.1]).

For $k > 0$ and for $s \in \mathbb{R}$ we denote by $T_k(s)$ the truncation $T_k(s) = \max\{-k, \min\{k, s\}\}$. Recall also from [BBC] that if $u$ is a measurable function on $\Omega$ which is finite almost everywhere and satisfies $T_k(u) \in \mathcal{W}_0^{1,p}(\Omega)$ for every $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that

\[
\nabla T_k(u) = v \chi_{\{|u| < k\}} \text{ almost everywhere in } \Omega, \text{ for all } k > 0.
\]
Moreover, \( v \) is unique up to almost everywhere equivalence. We define the gradient \( Du \) of \( u \) as this function \( v \), and set \( Du = v \).

In [DMOP], several equivalent definitions of renormalized solutions are given. In what follows, we will need the following ones.

**Definition 6.1.** Let \( \mu \) be in \( \mathcal{M}_B(\Omega) \). Then \( u \) is said to be a renormalized solution of

\[
\begin{aligned}
-\text{div} A(x, \nabla u) &= \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

if the following conditions hold:

(a) The function \( u \) is measurable and finite almost everywhere, and \( T_k(u) \) belongs to \( W^{1,p}_0(\Omega) \) for every \( k > 0 \).

(b) The gradient \( Du \) of \( u \) satisfies \( |Du|^{p-1} \in L^q(\Omega) \) for all \( q < \frac{n}{n-1} \).

(c) If \( w \) belongs to \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) and if there exist \( k > 0, w^+ \) and \( w^- \) in \( W^{1,q}(\Omega) \cap L^\infty(\Omega) \), with \( r > N \), such that

\[
\begin{aligned}
w &= w^+ \quad \text{a.e. on the set } \{u > k\}, \\
w &= w^- \quad \text{a.e. on the set } \{u < -k\}
\end{aligned}
\]

then

\[
\int_\Omega A(x, Du) \cdot \nabla \varphi dx = \int_\Omega \varphi d\mu_0 + \int_\Omega \varphi^+ d\mu_k^+ - \int_\Omega \varphi^- d\mu_k^-.
\]

**Definition 6.2.** Let \( \mu \) be in \( \mathcal{M}_B(\Omega) \). Then \( u \) is a renormalized solution of (6.3) if \( u \) satisfies (a) and (b) in Definition 6.1 and if the following conditions hold:

(c) For every \( k > 0 \) there exist two nonnegative measures in \( \mathcal{M}_0(\Omega) \), \( \lambda_k^+ \) and \( \lambda_k^- \), concentrated on the sets \( \{u = k\} \) and \( \{u = -k\} \), respectively, such that \( \lambda_k^+ \to \mu_k^+ \) and \( \lambda_k^- \to \mu_k^- \) in the narrow topology of measures.

(d) For every \( k > 0 \)

\[
\int_{\{u < k\}} A(x, Du) \cdot \nabla \varphi dx = \int_{\{u < k\}} \varphi d\mu_0 + \int_\Omega \varphi^+ d\lambda_k^+ - \int_\Omega \varphi^- d\lambda_k^-.
\]

for every \( \varphi \) in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

**Remark 6.3.** By [DMOP] Remark 2.18, if \( u \) is a renormalized solution of (6.3) then (the \( \text{cap}_{1,p} \)-quasi continuous representative of) \( u \) is finite \( \text{cap}_{1,p} \)-quasieverywhere. Therefore, \( u \) is finite \( \mu_0 \)-almost everywhere.

**Remark 6.4.** By (6.5), if \( u \) is a renormalized solution of (6.3) then

\[
-\text{div} A(x, \nabla T_k(u)) = \mu_k \quad \text{in } \Omega,
\]

where

\[
\mu_k = \chi_{\{|u| < k\}} \mu_0 + \lambda_k^+ - \lambda_k^-.
\]
Since $T_k(u) \in W^{1,p}(\Omega)$, by (4.3) we see that $-\text{div} A(x, \nabla T_k(u))$ and hence $\mu_k$ belongs to the dual space $W^{-1,p'}(\Omega)$ of $W^{1,p}(\Omega)$. Moreover, by Remark 6.3 $|u| < \infty \mu_0$-almost everywhere and hence $\chi_{\{|u|<k\}} \to \chi_{\Omega}$ $\mu_0$-almost everywhere as $k \to \infty$. Therefore, by the monotone convergence theorem, $\mu_k$ converges to $\mu$ in the narrow topology of measures.

**Remark 6.5.** If $\mu \geq 0$, i.e., $\mu \in \mathcal{M}_B^+(\Omega)$, and $u$ is a renormalized solution of (6.3) then $u$ is nonnegative. To see this, for each $k > 0$ we "test" (6.4) with $w = -T_k(u^-)$ where $u^- = -\min\{u, 0\}$, $w^+ = 0$ and $w^- = -k$:

$$- \int_{\Omega} A(x, Du) \cdot \nabla T_k(u^-) \, dx = - \int_{\Omega} T_k(u^-) \, d\mu_0 + \int_{\Omega} k \, d\mu_s^- = - \int_{\Omega} T_k(u^-) \, d\mu_0 \leq 0,$$

since $\mu_s^- = 0$. Thus using (4.3) we get

$$\int_{\Omega} |\nabla T_k(u^-)|^p \, dx \leq 0$$

for every $k > 0$. Therefore $u^- = 0$ a.e., i.e., $u$ is nonnegative.

**Remark 6.6.** Let $\mu \in \mathcal{M}_B^+(\Omega)$ and let $u$ be a renormalized solution of (6.3). Since $u^- = 0$ a.e. (by Remark 6.5) and hence $u^- = 0$ cap$_{1,p}$-quasi everywhere (see [HKM, Theorem 4.12]), in Remark 6.4 we may take $\lambda_k^- = 0$, and thus $\mu_k$ is nonnegative. Hence by (6.6) and Proposition 4.1, the functions $v_k$ defined by $v_k(x) = \text{ess lim inf}_{y \to x} T_k(u)(y)$ are $\mathcal{A}$-superharmonic and increasing. Using Lemma 7.3 in [HKM], it is then easy to see that $v_k \to v$ as $k \to \infty$ everywhere in $\Omega$ for some $\mathcal{A}$-superharmonic function $v$ on $\Omega$ such that $v = u$ a.e.. In other words, $v$ is an $\mathcal{A}$-superharmonic representative of $u$.

**Remark 6.7.** When we are dealing with pointwise values of a renormalized solution $u$ to the problem (6.3) with data $\mu \geq 0$, we always identify $u$ with its $\mathcal{A}$-superharmonic representative mentioned in Remark 6.6.

In Theorem 6.9 below, we give a global pointwise potential estimate for renormalized solutions on a bounded domain $\Omega$, whose proof is based on its local counterpart given in Theorem 4.4 and the following lemma.

**Lemma 6.8.** Suppose that $u$ is a renormalized solution of the problem (6.3) with data $\mu \in \mathcal{M}_B^+(\Omega)$. Let $B = B(x_0, 2\text{diam}(\Omega))$ be a ball...
centered at \(x_0 \in \Omega\). Then there exists a nonnegative \(A\)-superharmonic function \(w\) on \(B\) such that \(u \leq w\) on \(\Omega\), and

\[
\begin{aligned}
-\text{div} A(x, \nabla w) &= \mu \quad \text{in } B, \\
\|w\|_{L^{p-1}(B)} &\leq CR^{p-1} \mu(\Omega)^{-\frac{1}{p-1}}.
\end{aligned}
\]

Proof. Let \(u_k = \min\{u, k\}\), and let

\[
\mu_k = \chi_{u < k} \mu_0 + \lambda_k^+
\]

be as in Remark [6.4] (note that \(\lambda_k^- = 0\) by Remark [6.6]). We see that \(u_k \in W^{1,p}_0(\Omega)\) is the unique solution of problem (6.3) with data \(\mu_k\).

Since \(\mu_k\) is continuous with respect to the capacity \(\text{cap}_{1,p}(\cdot, B)\), we have a unique renormalized (or entropy) solution \(w_k\) to the problem

\[
\begin{aligned}
-\text{div} A(x, \nabla w_k) &= \mu_k \quad \text{in } B, \\
w_k &= 0 \quad \text{on } \partial B.
\end{aligned}
\]

We now extend \(u_k\) by zero outside \(\Omega\), and set

\[
\Psi = \min\{w_k - u_k, 0\} = \min\{\min\{w_k, k\} - u_k, 0\}.
\]

Note that \(\Psi \in W^{1,p}_0(\Omega) \cap W^{1,p}_0(B) \cap L^\infty(B)\) since \(|\Psi| \leq u_k\). Then using \(\Psi\) as a test function we have

\[
0 = \int_B A(x, \nabla w_k) \cdot \nabla \Psi dx - \int_\Omega A(x, \nabla u_k) \cdot \nabla \Psi dx
\]

\[
= \int_{B \cap \{w_k < u_k\}} A(x, \nabla w_k) \cdot \nabla \Psi dx - \int_{B \cap \{w_k < u_k\}} A(x, \nabla u_k) \cdot \nabla \Psi dx
\]

\[
= \int_{B \cap \{w_k < u_k\}} [A(x, \nabla w_k) - A(x, \nabla u_k)] \cdot (\nabla w_k - \nabla u_k) dx.
\]

Thus \(\nabla w_k = \nabla u_k\) a.e. on the set \(B \cap \{w_k < u_k\}\) by hypothesis (4.4) on \(A\). Hence \(\Psi = 0\) a.e., i.e.,

\[
(6.7) \quad u_k \leq w_k \quad \text{a.e.}
\]

Since \(\mu_k\) converges to \(\mu\) in the narrow topology of measures on \(\Omega\) (and hence also on \(B\)), arguing as in the proof of [KM1, Theorem 2.4], we can find a subsequence \(\{w_{k_j}\}\) of \(\{w_k\}\) such that \(w_{k_j} \to w\) a.e., where \(w\) is a nonnegative \(A\)-superharmonic function on \(B\) such that

\[
-\text{div} A(x, \nabla w) = \mu \quad \text{in } B.
\]

By (6.7) we have \(u \leq w\) a.e. on \(\Omega\), and hence \(u \leq w\) everywhere on \(\Omega\) due to Remark [6.7] and Proposition [4.1]. Note that for \(p < n\) we have

\[
\|w_k\|_{L^{\frac{n(p-1)}{n-p}, \infty}(B)} \leq C \mu_k(\Omega)^{-\frac{1}{p-1}},
\]
for some constant $C$ independent of $R$ and $k$ (see [DMOP, Theorem 4.1] or [BBC, Lemma 4.1]). Thus
\begin{equation}
\|w_k\|_{L^{p-1}(B)} \leq CR^{\frac{2}{p-1}} \mu_k(\Omega)^{\frac{1}{p-1}}.
\end{equation}
The inequality (6.8) also holds for $p \geq n$, see for example [Gre, Lemma 2.1]. Finally, using Fatou’s lemma and (6.8), we obtain
\begin{equation}
\|w\|_{L^{p-1}(B)} \leq CR^{\frac{2}{p-1}} \mu(\Omega)^{\frac{1}{p-1}}.
\end{equation}
This completes the proof of the lemma. □

**Theorem 6.9.** Suppose that $u$ is a renormalized solution of the problem (6.3) with data $\mu \in M_B^+(\Omega)$. Let $R = \text{diam}(\Omega)$. Then there is a constant $K$ independent of $\mu$ and $R$ such that
\begin{equation}
u(x) \leq K W_{1,p}^{2R}(x),
\end{equation}
for all $x$ in $\Omega$.

**Proof.** Let $w$ and $B$ be as in Lemma 6.8. Fix $x \in \Omega$. We denote by $d(x)$ the distance from $x$ to the boundary $\partial B$ of $B$. By Theorem 4.4, Lemma 6.8, and the fact that $d(x) \geq R$, we have
\begin{equation}
w(x) \leq C W_{1,p}^{2d(x)/3} \mu(x) + C \inf_{B(x,d(x)/3)} w
\end{equation}
\begin{align*}
&\leq C W_{1,p}^{2R} \mu(x) + C d(x)^{\frac{n}{p-1}} \|w\|_{L^{p-1}(B)}
\leq C W_{1,p}^{2R} \mu(x) + CR^{\frac{n}{p-1}} \mu(\Omega)^{\frac{1}{p-1}}
\leq C W_{1,p}^{2R} \mu(x).
\end{align*}
Therefore, from (6.10) and Lemma 6.8 we get the desired inequality (6.9). □

**Theorem 6.10.** Let $\omega \in M_B^+(\Omega)$. Let $p > 1$ and $q > p - 1$. Suppose that $R = \text{diam}(\Omega)$, and
\begin{equation}
W_{1,p}^{2R}(W_{1,p}^{2R})^q \leq C W_{1,p}^{2R} < \infty \quad \text{a.e.},
\end{equation}
where
\begin{equation}
C \leq \frac{q - p + 1}{qK \max\{1, 2^{p-2}\}}^{q(p'-1)} \left(\frac{p - 1}{q - p + 1}\right),
\end{equation}
and $K$ is the constant in Theorem 6.9. Then there is a renormalized solution $u \in L^q(\Omega)$ to the Dirichlet problem
\begin{equation}
\begin{cases}
-\text{div}A(x, \nabla u) = u^q + \omega \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\end{equation}
such that
\begin{equation}
u(x) \leq M W_{1,p}^{2R}(x),
\end{equation}
for all $x$ in $\Omega$, where the constant $M$ depends only on $n, p, q$ and the structural constants $\alpha$ and $\beta$.

**Proof.** Let $\{u_k\}_{k \geq 0}$ be a sequence of renormalized solutions defined inductively for the following Dirichlet problems:

\begin{equation}
-\text{div} A(x, \nabla u_0) = \omega \quad \text{in} \quad \Omega,
\end{equation}

\begin{equation}
\begin{aligned}
-\text{div} A(x, \nabla u_k) &= u_{k-1}^q + \omega \quad \text{in} \quad \Omega, \\
u_k &= 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}
\end{equation}

for $k \geq 1$. By Theorem 6.9 we have

\[ u_0 \leq K W_{1,p}^{2R} \omega, \quad u_k \leq K W_{1,p}^{2R} (u_{k-1}^q + \omega). \]

Thus by arguing as in the proof of Theorem 5.3 we obtain a constant $M > 0$ such that

\begin{equation}
\label{eq:6.15}
u_n \leq M W_{1,p}^{2R} \omega < \infty \quad \text{a.e.}
\end{equation}

for all $n \geq 0$. By passing to a subsequence (see [KM1, Theorem 1.17] or [DMOP, Sec. 5.1]), we can assume that $u_n \to u$ a.e. on $\Omega$ for some nonnegative function $u$. Note that by (6.15)

\[ u \leq M W_{1,p}^{2R} \omega < \infty \quad \text{a.e.} \]

and $u_n^q \to u^q$ in $L^1(\Omega)$. Finally, in view of (6.14), the stability result in [DMOP, Theorem 3.4] asserts that $u$ is a renormalized solution of (6.12), which proves the theorem.

Let $Q = \{Q\}$ be a Whitney decomposition of $\Omega$, i.e., $Q$ is a disjoint subfamily of the family of dyadic cubes in $\mathbb{R}^n$ such that $\Omega = \bigcup_{Q \in Q} Q$, where we can assume that $2^5 \text{diam}(Q) \leq \text{dist}(Q, \partial \Omega) \leq 2^7 \text{diam}(Q)$. Let $\{\phi_Q\}_{Q \in Q}$ be a partition of unity associated with the Whitney decomposition of $\Omega$ above: $0 \leq \phi_Q \in C_0^\infty(Q^*)$, $\phi_Q \geq 1/C(n)$ on $Q$, $\sum_Q \phi_Q = 1$ and $|D^\gamma \phi_Q| \leq A_\gamma (\text{diam}(Q))^{-|\gamma|}$ for all multi-indices $\gamma$. Here $Q^* = (1+\epsilon)Q$, $0 < \epsilon < \frac{1}{4}$ and $C(n)$ is a positive constant depending only on $n$ such that each point in $\Omega$ is contained in at most $C(n)$ of the cubes $Q^*$ (see [St1]).

**Theorem 6.11.** Let $\omega$ be a locally finite nonnegative measure on an open (not necessarily bounded) set $\Omega$. Let $p > 1$ and $q > p - 1$. Suppose that there exists a nonnegative $A$-superharmonic function $u$ in $\Omega$ such that

\begin{equation}
\label{eq:6.16}
-\text{div} A(x, \nabla u) = u^q + \omega \quad \text{in} \quad \Omega.
\end{equation}
Then, for each cube $P \in Q$ and compact set $E \subset \Omega$,

$$ \mu_P(E) \leq C \operatorname{Cap}_{I_p \frac{q}{q-p+1}}(E) $$

if $\frac{pq}{q-p+1} < n$, and

$$ \mu_P(E) \leq C(P) \operatorname{Cap}_{G_p \frac{q}{q-p+1}}(E) $$

if $\frac{pq}{q-p+1} \geq n$. Here $d\mu = u^q dx + d\omega$, and the constant $C$ in (6.17) is independent of $P$ and $E \subset \Omega$, but the constant $C(P)$ in (6.18) may depend on the side length of $P$.

Moreover, if $\frac{pq}{q-p+1} < n$ and $\Omega$ is a bounded $C^\infty$-domain, then

$$ \mu(E) \leq C \operatorname{cap}_{I_p \frac{q}{q-p+1}}(E, \Omega), $$

for all compact sets $E \subset \Omega$, where $\operatorname{cap}_{I_p \frac{q}{q-p+1}}(E, \Omega)$ is defined by (2.19).

**Proof.** Let $P$ be a fixed dyadic cube in $Q$. For a dyadic cube $P' \subset P$ we have

$$ \operatorname{dist}(P', \partial \Omega) \geq \operatorname{dist}(P, \partial \Omega) \geq 2^{5} \operatorname{diam}(P) \geq 2^{5} \operatorname{diam}(P'). $$

The lower estimate in Theorem 4.4 then yields

$$ u(x) \geq C \mathcal{W}^{2^{2} \operatorname{diam}(P')}_{-1, p} \mu(x) 
\geq C \sum_{k=0}^{\infty} \int_{2^{-k+3} \operatorname{diam}(P')} \left[ \frac{\mu(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \text{d}t
\geq C \sum_{Q \subset P'} \left[ \frac{\mu(Q)}{|Q|^{1-p/n}} \right]^{\frac{1}{p-1}} \chi_Q(x), $$

for all $x \in P'$. Thus it follows from Proposition 3.3 that

$$ \sum_{Q \subset P'} \left[ \frac{\mu(Q)}{|Q|^{1-p/n}} \right]^{\frac{q}{p-1}} |Q| \leq C \int_{P'} u^q dx \leq C \mu(P'), \quad P' \subset P. $$

Hence

$$ \mu(P') \leq C |P'|^{1-\frac{pq}{q-p+1}}, \quad P' \subset P. $$

To get a better estimate for $\mu(P')$ in the case $\frac{pq}{q-p+1} = n$, we observe that (6.19) is a dyadic Carleson condition. Thus by the dyadic Carleson imbedding theorem (see, e.g., [NTV], [V1]) we obtain, for $\frac{pq}{q-p+1} = n$,

$$ \sum_{Q \subset P} \mu(Q)^{\frac{q}{p-1}} \left[ \frac{1}{\mu(Q)} \int_Q f d\mu \right]^{\frac{q}{p-1}} \leq C \int_{P} f^p d\mu, $$

(6.21)
where \( f \in L^{\frac{q}{q-p+1}}(d\mu_P), f \geq 0 \). From (6.21) with \( f = \chi_{P'} \), one gets
\[
(6.22) \quad \mu(P') \leq C \left( \log \frac{2^n |P|}{|P'|} \right)^{\frac{1}{q-p+1}}, \quad P' \subset P,
\]
if \( \frac{pq}{q-p+1} = n \). Now let \( P' \) be a dyadic cube in \( \mathbb{R}^n \). From Wolff’s inequality for Riesz potentials (see [HW]) we have
\[
(6.23) \quad \int_{\mathbb{R}^n} (I_p\mu_{P' \cap P})^{\frac{q}{p-1}} \, dx \\
\leq \quad C \sum_{Q \in D} \left[ \frac{\mu_P(P' \cap Q)}{|Q|^{1-p/n}} \right]^{\frac{q}{p-1}} |Q| \\
= \quad C \sum_{Q \subset P'} \left[ \frac{\mu_P(Q)}{|Q|^{1-p/n}} \right]^{\frac{q}{p-1}} |Q| + C \sum_{P' \subset Q} \left[ \frac{\mu_P(P')}{|Q|^{1-p/n}} \right]^{\frac{q}{p-1}} |Q|.
\]
Thus, for \( \frac{pq}{q-p+1} < n \), by combining (6.19) and (6.23) we deduce
\[
(6.24) \quad \int_{\mathbb{R}^n} (I_p\mu_{P' \cap P})^{\frac{q}{p-1}} \, dx \leq C \mu_P(P').
\]
In the case \( \frac{pq}{q-p+1} \geq n \), a similar argument using (6.19), (6.20), (6.22) and Wolff’s inequality for Bessel potentials:
\[
(6.25) \quad \int_{\mathbb{R}^n} (G_p\mu_{P' \cap P})^{\frac{q}{p-1}} \, dx \leq C(P) \mu_P(P'),
\]
where the constant \( C(P) \) may depend on the side-length of \( P \). Note that (6.24), which holds for all dyadic cubes \( P' \subset \mathbb{R}^n \), is the well-known Kerman-Sawyer condition (see [KS]). Therefore,
\[
\|I_p(f)\|_{L^{\frac{q}{q-p+1}}(d\mu_P)} \leq C \|f\|_{L^{\frac{q}{q-p+1}}(dx)}
\]
for all \( f \in L^{\frac{q}{q-p+1}}(\mathbb{R}^n) \) which is equivalent to the capacitary condition:
\[
\mu_P(E) \leq C \text{Cap}_{p, \frac{q}{q-p+1}}(E)
\]
for all compact sets \( E \subset \mathbb{R}^n \). Thus we obtain (6.17). The inequality (6.18) is proved in the same way using (6.25). From (6.17) and the definition of \( \text{cap}_{p, \frac{q}{q-p+1}}(\cdot, \Omega) \), we see that, for each cube \( P \in \mathcal{Q} \),
\[
\mu_P(E) \leq C\text{cap}_{p, \frac{q}{q-p+1}}(E \cap P, \Omega)
\]
for all compact sets \( E \subset \Omega \). Thus
\[
\mu(E) \leq \sum_{P \in \mathcal{Q}} \mu_P(E)
\leq C \sum_{P \in \mathcal{Q}} \operatorname{cap}_{\frac{q}{q-p+1}}(E \cap P, \Omega)
\leq C \operatorname{cap}_{\frac{q}{q-p+1}}(E, \Omega),
\]
where the last inequality follows from the quasiadditivity of the capacity \( \operatorname{cap}_{\frac{q}{q-p+1}}(\cdot, \Omega) \) which is considered in the next theorem. \( \square \)

**Remark 6.12.** Let \( B_R \) be a ball such that \( B_{2R} \subset \Omega \). It is easy to see that there exists a constant \( c > 0 \) such that \( \ell(P) \geq cR \) for any Whitney cube \( P \) that intersects \( B_R \). On the other hand, if \( B_r \) is a ball in \( B_R \) then we can find at most \( N \) dyadic cubes \( P_i \) with \( cr/4 \leq \ell(P_i) \leq cr/2 \) that cover \( B_r \), where \( N \) depends only on \( n \). Thus if \( \frac{pq}{q-p+1} = n \) then from (6.22) we see that
\[
\mu(B_r) \leq C (\log \frac{2R}{r})^{1-p+q-p+1}
\]
for all balls \( B_r \subset B_R \). Here the constant \( C \) is independent of \( R \) and \( r \).

**Theorem 6.13.** Suppose that \( \Omega \) is a \( C^\infty \)-domain in \( \mathbb{R}^n \). Then there exists a constant \( C > 0 \) such that
\[
\sum_{Q \in \mathcal{Q}} \operatorname{cap}_{\frac{q}{q-p+1}}(E \cap Q, \Omega) \leq C \operatorname{cap}_{\frac{q}{q-p+1}}(E, \Omega)
\]
for all compact sets \( E \subset \Omega \).

**Proof.** Obviously, we may assume that \( \operatorname{cap}_{\frac{q}{q-p+1}}(E, \Omega) > 0 \). Then by definition there exists \( f \in C_0^\infty(\Omega) \), \( f \geq 1 \) on \( E \) such that
\[
2 \operatorname{cap}_{\frac{q}{q-p+1}}(E, \Omega) \geq \|f\|_{W^{\frac{q}{q-p+1}, q}(\mathbb{R}^n)}^q.
\]
By the refined localization principle on the smooth domain \( \Omega \) for the function space \( W^{\frac{q}{q-p+1}, q}(\mathbb{R}^n) \) we have
\[
\|f\|_{W^{\frac{q}{q-p+1}, q}(\mathbb{R}^n)} \geq C \sum_{Q \in \mathcal{Q}} \|f\|_{W^{\frac{q}{q-p+1}, q}(\mathbb{R}^n)},
\]
(see e.g. [Tri, Theorem 5.14]). Thus
\[
(6.26) \quad \sum_{Q \in \mathcal{Q}} \|f\|_{W^{\frac{q}{q-p+1}, q}(\mathbb{R}^n)} \leq C \operatorname{cap}_{\frac{q}{q-p+1}}(E, \Omega).
\]
Note that for \( x \in E \cap \overline{Q} \),
\[
f \phi_Q \geq \phi_Q \geq 1/C(n).
\]
Hence by definition we have
\[
\text{cap}_{p, q}(E \cap \overline{Q}, \Omega) \leq C \|f\|_{W^{p, 1}}^{q - p + 1} \|f\|_{W^{p, q, q - p + 1}}(R^n).
\]
From this and (6.26) we deduce the desired inequality. 

Theorem 6.14. Let \(\omega\) be a measure in \(\mathcal{M}_B(\Omega)\) with compact support in \(\Omega\). Let \(p > 1\), \(q > p - 1\) and \(R = \text{diam}(\Omega)\). Then the following statements are equivalent.

(i) There exists a nonnegative renormalized solution \(u \in L^q(\Omega)\) to the equation

\[
\begin{aligned}
-\text{div} A(x, \nabla u) &= u^q + \epsilon \omega & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]

for some \(\epsilon > 0\).

(ii) The testing inequality

\[
\int_B (G_{p, \omega B})^{\frac{q}{p - 1}} dx \leq C \omega(B)
\]

holds for all balls \(B\) such that \(B \cap \text{supp} \omega \neq \emptyset\).

(iii) For all compact sets \(E \subset \text{supp} \omega\),

\[
\omega(E) \leq C \text{Cap}_{G_{p, q}}(E).
\]

(iv) The testing inequality

\[
\int_B \left[ \mathcal{W}_{1, p}^2(\omega B(x)) \right]^q dx \leq C \omega(B)
\]

holds for all balls \(B\) such that \(B \cap \text{supp} \omega \neq \emptyset\).

(v) There exists a constant \(C\) such that

\[
\mathcal{W}_{1, p}^2(\omega B) \leq C \mathcal{W}_{1, p}^2 < \infty \quad \text{a.e. on } \Omega.
\]

Moreover, if the constant \(C\) in (6.30) satisfies

\[
C \leq \left( \frac{q - p + 1}{qK \max\{1, 2^{q' - 2}\}} \right)^{(q' - 1)} \left( \frac{p - 1}{q - p + 1} \right),
\]

where \(K\) is the constant in Theorem 6.9, then the equation (6.27) has a solution \(u\) with \(\epsilon = 1\) which obeys the estimate

\[
u(x) \leq M \mathcal{W}_{1, p}^2 \omega(x)
\]

for all \(x \in \Omega\).

Proof. It is well known that statements (ii) and (iii) above are equivalent (see e.g. [V2]). Thus it remains to prove that (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iv) \(\Rightarrow\) (v) \(\Rightarrow\) (i). Since \(\omega\) is compactly supported in \(\Omega\), using Theorem
we have (i) $\implies$ (iii) $\implies$ (ii). As before, the testing inequality \eqref{6.28} is also equivalent to the Kerman–Sawyer condition
\begin{equation}
\int_{\mathbb{R}^n} \left[ G_p \omega_B(x) \right]^{\frac{q}{p-1}} dx \leq C \omega(B),
\end{equation}
(see \cite{KS, V2}). Note that
\begin{equation}
\int_{\mathbb{R}^n} \left[ G_p \mu(x) \right]^{\frac{q}{p-1}} dx \simeq \int_{\mathbb{R}^n} \left[ \int_0^{2R} \frac{\mu(B_t(x))}{t^{n-p}} \right]^{\frac{q}{p-1}} dt,\quad 0 < r < 2R, x \in \mathbb{R}^n,
\end{equation}
where the constants of equivalence are independent of the measure \( \mu \), (see \cite{HW, AH}). From \eqref{6.31}, \eqref{6.32}, and Proposition \ref{prop:5.1} we deduce the implication (ii) $\implies$ (iv). Note that Theorem \ref{thm:6.10} gives (v) $\implies$ (i). Thus it remains to show that (iv) $\implies$ (v). In fact, the proof of this implication is similar to the proof of (iii) $\implies$ (iv) in Theorem \ref{thm:5.4}. We will only sketch some crucial steps here. We define the lower and upper parts of the truncated Wolff potential \( W_{1,p}^{2R} \) respectively by
\begin{align*}
L_r^{2R} \mu(x) &= \int_r^{2R} \left[ \frac{\mu(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad 0 < r < 2R, x \in \mathbb{R}^n \\
U_r^{2R} \mu(x) &= \int_0^r \left[ \frac{\mu(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad 0 < r < 2R, x \in \mathbb{R}^n.
\end{align*}
Since \( R = \text{diam}(\Omega) \) and \( \omega \in M_B^+(\Omega) \), to prove \eqref{6.30}, it is enough to verify that, for \( x \in \Omega \),
\begin{equation}
\int_0^{2R} \left[ \frac{\mu_r(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{r} \leq C W_{1,p}^{2R} \omega(x),
\end{equation}
and
\begin{equation}
\int_0^{2R} \left[ \frac{\lambda_r(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{r} \leq C W_{1,p}^{2R} \omega(x),
\end{equation}
where \( d\mu_r = (U_r^{2R} \omega)^q dx, \ d\lambda_r = (L_r^{2R} \omega)^q dx \) and \( 0 < r < 2R \). The proof of \eqref{6.33} is the same as before. For the proof of \eqref{6.34}, we need an estimate similar to \eqref{5.16}. Namely,
\begin{equation}
\int_r^{4R} \left[ \frac{\omega(B_t(x))}{t^{n-p}} \right]^{\frac{1}{q-1}} \frac{dt}{t} \leq C(R, \omega(\Omega)) \frac{r^{\frac{q-1}{q-p+1}}}{r^{q-p+1}}
\end{equation}
for all $0 < r \leq 4R$ and $x \in \Omega$. In fact, note that for $0 < t < R/2$ and $y \in B_t(x)$,

$$W_{1,p}^{2R} \omega_{B_t(x)}(y) \geq \int_{2t}^{2R} \left[ \frac{\omega(B_r(y) \cap B_t(x))}{\tau^{n-p}} \right]^{\frac{p}{n-p}} d\tau$$

$$\geq C(n,p) \frac{\omega(B_t(x))}{t^{n-p}}.$$  

As before, from this inequality and (6.29) one gets

$$(6.36) \quad \omega(B_t(x)) \leq Ct^{n-p+\frac{pq}{q-p+1}}, \quad 0 < t < R/2.$$  

To prove (6.35), we can assume that $0 < r < R/2$ and write the left-hand side of (6.35) as

$$(6.37) \quad \int_{R/2}^{R} \left[ \frac{\omega(B_t(x))}{t^{n-p}} \right]^{\frac{p}{n-p}} dt + \int_{R/2}^{4R} \left[ \frac{\omega(B_t(x))}{t^{n-p}} \right]^{\frac{p}{n-p}} dt.$$  

Applying (6.36) to the first term of (6.37) and using the fact that $\omega \in M^+_{p,q}(\Omega)$ in the second term of (6.37), we finally obtain (6.35). This completes the proof of (iv)$\implies$(v), and so Theorem 6.14 is proved. $\square$

**Remark 6.15.** From the proof of Theorem 6.14 we see that if $\omega$ is not assumed to be compactly supported in $\Omega$, then any one of the conditions (ii)–(v) is still sufficient for the solvability of (6.27) for some $\varepsilon > 0$.

**Theorem 6.16.** Let $E$ be a relatively closed subset of $\Omega$. Suppose that $\text{Cap}_{G_p, \frac{q}{q-p+1}}(E) = 0$. Then any solution $u$ of

$$\begin{cases}
\begin{aligned}
&u \text{ is } A\text{-superharmonic in } \Omega \setminus E, \\
&u \in L^q_{\text{loc}}(\Omega \setminus E), \\
&-\text{div} A(x, \nabla u) = u^q \quad \text{in } D'(\Omega \setminus E),
\end{aligned}
\end{cases}$$

is also a solution of

$$\begin{cases}
\begin{aligned}
&u \text{ is } A\text{-superharmonic in } \Omega, \\
&u \in L^q_{\text{loc}}(\Omega), \\
&-\text{div} A(x, \nabla u) = u^q \quad \text{in } D'(\Omega).
\end{aligned}
\end{cases}$$

Conversely, if $E$ is a compact set in $\Omega$ such that any solution of (6.38) is also a solution of (6.39) then $\text{Cap}_{G_p, \frac{q}{q-p+1}}(E) = 0$.

**Proof.** Let us prove the first part of the theorem. Since

$$\text{Cap}_{G_p, \frac{q}{q-p+1}}(E) = 0,$$

we have $\text{cap}_{1,p}(E; \Omega) = 0$ where the capacity $\text{cap}_{1,p}(\cdot; \Omega)$ is defined by (6.2) (see [HKM]). Thus $u$ can be extended so that it is a nonnegative
\(A\)-superharmonic function in \(\Omega\) (see [HKM]). Let \(\mu[u]\) be the Radon measure on \(\Omega\) associated with \(u\), and let \(\varphi\) be an arbitrary nonnegative function in \(C^\infty_0(\Omega)\). As in [BP, Lemme 2.2], we can find a sequence \(\{\varphi_n\}\) of nonnegative functions in \(C^\infty_0(\Omega - E)\) such that
\[
0 \leq \varphi_n \leq \varphi; \quad \varphi_n \to \varphi \quad \text{Cap}_{G_{p, q-1}}\text{-quasi everywhere.}
\]
By Fatou’s lemma we have
\[
\int_{\Omega} u^q \varphi \, dx \leq \liminf_{n \to \infty} \int_{\Omega} u^q \varphi_n \, dx = \liminf_{n \to \infty} \int_{\Omega} \varphi_n \, d\mu[u] \leq \int_{\Omega} \varphi \, d\mu[u] < \infty.
\]
Therefore \(u \in L^q_{\text{loc}}(\Omega)\), and \(\mu[u] \geq u^q\) in \(\mathcal{D}'(\Omega)\). It is then easy to see that
\[-\text{div} A(x, \nabla u) = u^q + \mu^E \quad \text{in} \quad \mathcal{D}'(\Omega)
\]
for some nonnegative measure \(\mu^E\) such that \(\mu^E(A) = 0\) for any Borel set \(A \subset \Omega - E\). Moreover, by Theorem 6.14 and Remark 6.15 we have, for any compact set \(K \subset E\),
\[
\mu^E(K) \leq C(K) \text{Cap}_{G_{p, q-1}}(K) = 0.
\]
Thus \(\mu^E = 0\) and \(u\) solves (6.39).

The second part of the theorem is proved in the same way as in the linear case \((p = 2)\) using the existence results in Theorem 6.14. We refer to [AP] for details. \(\square\)

7. Hessian equations

In this section, we study a fully nonlinear counterpart of the theory presented in the previous sections. Here the notion of \(k\)-subharmonic functions associated with the fully nonlinear \(k\)-Hessian operator \(F_k\), \(k = 1, \ldots, n\), introduced by Trudinger and Wang in [TW1–TW3] will play a role similar to that of \(A\)-superharmonic functions in the quasilinear theory.

Let \(\Omega\) be an open set in \(\mathbb{R}^n\), \(n \geq 2\). For \(k = 1, \ldots, n\) and \(u \in C^2(\Omega)\), the \(k\)-Hessian operator \(F_k\) is defined by
\[
F_k[u] = S_k(\lambda(D^2u)),
\]
where \(\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)\) denotes the eigenvalues of the Hessian matrix of second partial derivatives \(D^2u\), and \(S_k\) is the \(k^{th}\) symmetric
function on $\mathbb{R}^n$ given by

$$S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$ 

Thus $F_1[u] = \Delta u$ and $F_n[u] = \det D^2 u$. Alternatively, we may also write

$$F_k[u] = [D^2 u]_k,$$

where for an $n \times n$ matrix $A$, $[A]_k$ is the $k$-trace of $A$, i.e., the sum of its $k \times k$ principal minors. Several equivalent definitions of $k$-subharmonicity were given in [TW2], one of which involves the language of viscosity solutions: An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be $k$-subharmonic in $\Omega$, $1 \leq k \leq n$, if $F_k[q] \geq 0$ for any quadratic polynomial $q$ such that $u - q$ has a local finite maximum in $\Omega$. Equivalently, an upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is $k$-subharmonic in $\Omega$ if, for every open set $\Omega' \Subset \Omega$ and for every function $v \in C^2_{\text{loc}}(\Omega') \cap C^0(\overline{\Omega'})$ satisfying $F_k[v] \geq 0$ in $\Omega'$, the following implication holds:

$$u \leq v \text{ on } \partial \Omega' \implies u \leq v \text{ in } \Omega',$$

(see [TW2] Lemma 2.1). Note that a function $u \in C^2_{\text{loc}}(\Omega)$ is $k$-subharmonic if and only if

$$F_j[u] \geq 0 \text{ in } \Omega \text{ for all } j = 1, \ldots, k.$$ 

We denote by $\Phi^k(\Omega)$ the class of all $k$-subharmonic functions in $\Omega$ which are not identically equal to $-\infty$ in each component of $\Omega$. It was proven in [TW2] that $\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega)$ where $\Phi^1(\Omega)$ coincides with the set of all proper classical subharmonic functions in $\Omega$, and $\Phi^n(\Omega)$ is the set of functions convex on each component of $\Omega$.

The following weak convergence result proved in [TW2] is fundamental to potential theory associated with $k$-Hessian operators.

**Theorem 7.1 (TW2).** For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Borel measure $\mu_k[u]$ in $\Omega$ such that

(i) $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$, and

(ii) if $\{u_m\}$ is a sequence in $\Phi^k(\Omega)$ converging in $L^1_{\text{loc}}(\Omega)$ to a function $u \in \Phi^k(\Omega)$, then the sequence of the corresponding measures $\{\mu_k[u_m]\}$ converges weakly to $\mu_k[u]$.

The measure $\mu_k[u]$ in the theorem above is called the $k$-Hessian measure associated with $u$. Due to (i) in Theorem 7.1 we sometimes write $F_k[u]$ in place of $\mu_k[u]$ even in the case where $u \in \Phi^k(\Omega)$ does not belong to $C^2(\Omega)$. The $k$-Hessian measure is an important tool in potential theory for $\Phi^k(\Omega)$. It was used by D. A. Labutin to derive pointwise
estimates for functions in $\Phi^k(\Omega)$ in terms of the Wolff potential, which is an analogue of the Wolff potential estimates for $A$-superharmonic functions in Theorem 4.4.

**Theorem 7.2** ([4]). Let $u \geq 0$ be such that $-u \in \Phi^k(B(x, 3r))$, where $1 \leq k \leq n$. If $\mu = \mu_k[-u]$ then

$$C_1 W^{r/8}_{k+1, k+1}(x) \leq u(x) \leq C_2 \inf_{B(x, r)} u + C_3 W^{2r/8}_{k+1, k+1}(x),$$

where the constants $C_1, C_2$ and $C_3$ are independent of $x, u$, and $r$.

The following global estimate is deduced from the preceding theorem as in the quasilinear case.

**Corollary 7.3.** Let $u \geq 0$ be such that $-u \in \Phi^k(\mathbb{R}^n)$, where $1 \leq k < \frac{n}{2}$. If $\mu = \mu_k[-u]$ and $\inf_{\mathbb{R}^n} u = 0$ then for all $x \in \mathbb{R}^n$,

$$\frac{1}{K} W_{k+1, k+1}(x) \leq u(x) \leq K W_{k+1, k+1}(x),$$

for some constant $K$ independent of $x$ and $u$.

Let $\Omega$ be a bounded uniformly $(k - 1)$-convex domain in $\mathbb{R}^n$, that is, $\partial \Omega \in C^2$ and $H_j(\partial \Omega) > 0$, $j = 1, \ldots, k - 1$, where $H_j(\partial \Omega)$ denotes the $j$-mean curvature of the boundary $\partial \Omega$. We consider the following fully nonlinear problem:

$$\begin{cases} 
F_k[-u] = u^q + \omega \quad \text{in} \quad \Omega, \\
u \geq 0 \quad \text{in} \quad \Omega, \\
u = \varphi \quad \text{on} \quad \partial \Omega,
\end{cases}$$

in the class of functions $u$ such that $-u$ is $k$-subharmonic in $\Omega$. Here $\omega$ is a Borel measure compactly supported in $\Omega$, and the boundary condition in (7.1) is understood in the classical sense. Characterizations of the existence of $u \in \Phi^k(\Omega)$ continuous near $\partial \Omega$ which solves (7.1) can be obtained using the iteration scheme in the proof Theorem 6.13 together with the argument in the proof Theorem 6.14. To do so we need an analogue of the global upper potential estimates on a bounded domain given in Theorem 6.9 for quasilinear operators.

**Theorem 7.4.** Let $\mu$ be a nonnegative Borel measure compactly supported in a bounded domain $\Omega \subset \mathbb{R}^n$. Suppose that $u \geq 0, -u \in \Phi^k(\Omega)$ such that $u$ is continuous near $\partial \Omega$ and solves

$$\begin{cases} 
\mu_k[-u] = \mu + f \quad \text{in} \quad \Omega, \\
u = \varphi \quad \text{on} \quad \partial \Omega,
\end{cases}$$
where \(0 \leq \varphi \in C^0(\partial \Omega)\) and \(0 \leq f \in L^s(\Omega)\) with \(s > \frac{n}{2k}\) if \(1 \leq k \leq \frac{n}{2}\), and \(s = 1\) if \(\frac{n}{2} < k < n\). Then for all \(x \in \Omega\),
\[
u(x) \leq K \left[ W_{2R, \frac{2k}{k+1}}^{2k+1} (\mu + f)(x) + \max_{\partial \Omega} \varphi \right],
\]
where \(R = \text{diam}(\Omega)\) and \(K\) is a constant independent of \(x, \nu, \) and \(\Omega\).

**Proof.** Suppose that the support of \(\mu\) is contained in \(\Omega'\) for some open set \(\Omega' \subset \Omega\). Let \(M = \sup_{\Omega \setminus \Omega'} \nu\) and \(\nu_m = \min\{\nu, m\}\) for \(m > M\). Then 
\[
\begin{cases}
\mu_k[-\nu_m] = \mu_m & \text{in } \Omega, \\
\nu_m = \varphi & \text{on } \partial \Omega,
\end{cases}
\]
for some nonnegative Borel measure \(\mu_m\) in \(\Omega\). Since \(\nu_m \to \nu\) in \(L^1_{\text{loc}}(\Omega)\), by Theorem 7.1 we have
\[
\mu_m \to \mu + f \text{ weakly as measures in } \Omega.
\]
Note that \(\nu_m = \nu\) in \(\Omega \setminus \Omega'\) since \(m > M\). Thus \(\mu_m = \mu_k[\nu] = f\) in \(\Omega \setminus \Omega'\) for all \(m > M\). Using this and (7.2) it is easy to see that
\[
\int_{\Omega} \phi d\mu_m \to \int_{\Omega} \phi d\mu + \int_{\Omega} \phi f dx
\]
as \(m \to \infty\) for all \(\phi \in C^0(\Omega)\), i.e.,
\[
\mu_m \to \mu + f \text{ in the narrow topology of measures.}
\]
We now take a ball \(B \supset \Omega\) with \(B = B(x_0, 2R), x_0 \in \Omega\) and consider the solutions \(w_m \geq 0, -w_m \in \Phi^k(\Omega)\), continuous near \(\partial \Omega\), of
\[
\begin{cases}
\mu_k[-w_m] = \mu_m & \text{in } B, \\
w_m = \max_{\partial B} \varphi & \text{on } \partial B,
\end{cases}
\]
where \(m > M\). Since \(\nu_m\) is bounded in \(\Omega\) the measure \(\mu_m\) is continuous with respect to the capacity \(\text{cap}_k(\cdot, \Omega)\), and hence with respect to the capacity \(\text{cap}_k(\cdot, B)\) (see [TW3]). Here \(\text{cap}_k(\cdot, \Omega)\) is the \(k\)-Hessian capacity defined by
\[
\text{cap}_k(E, \Omega) = \sup \{ \mu_k[u](E) : u \in \Phi^k(\Omega), -1 < u < 0 \}
\]
for a compact set \(E \subset \Omega\). By a comparison principle (see [TW3, Theorem 4.1]), we have \(w_m \geq \max_{\partial \Omega} \varphi\) in \(B\), and hence \(w_m \geq \nu_m\) on \(\partial \Omega\). Thus, applying the comparison principle again, we have
\[
w_m \geq \nu_m \text{ in } \Omega.
\]
Since \(\mu_m \to \mu + f\) in the narrow topology of measures in \(\Omega\), we see that \(\mu_m \to \mu + f\) weakly as measures in \(B\). Therefore, arguing as in [TW2]
Sec. 6] we can find a subsequence \( \{w_{m_i}\} \) such that \( w_{m_i} \to w \) a.e. for some \( w \geq 0, -w \in \Phi^k(B) \) such that \( w \) is continuous near \( \partial B \) and

\[
\begin{aligned}
\mu_k[-w] &= \mu + f \quad \text{in } B, \\
w &= \max_{\partial B} \varphi \quad \text{on } \partial B.
\end{aligned}
\]

Note that from (7.4), \( w \geq u \) a.e. on \( \Omega \) and hence \( w \geq u \) everywhere on \( \Omega \). Using this and Theorem 7.2 applied to the function \( w \) on \( B(x, d(x)) \), where \( d(x) = \text{dist}(x, \partial B) \) we have, for \( x \in \Omega \) and \( d\nu = d\mu + fdx \),

(7.5) \[
\begin{aligned}
u(x) &\leq C W^2_{k+1, k+1} \left( \nu(B(x)) + \frac{\inf_{B(x, d(x)/3)} w}{d(x)/3} \right) + R^2_{n/k} \nu(\Omega)^{1/k},
\end{aligned}
\]

where the last inequality in (7.5) follows from the estimate (6.3) in [TW2]. The proof of the Theorem 7.4 is then complete by noting that

\[
\int_{R}^{2R} \left[ \frac{\nu(B_t(x))}{t^{n-2k}} \right] \frac{dt}{t} \geq CR^2_{n/k} \nu(\Omega)^{1/k}
\]

for all \( x \in \Omega \). □

The next theorem is a criterion for the existence of global solutions to fully nonlinear equations with general measure data, which is an analogue of Theorem 5.2

**Theorem 7.5.** Suppose that \( \mu \) is a measure in \( M^+(\mathbb{R}^n) \) such that \( W^2_{k+1, k+1} \mu < \infty \) a.e. on \( \mathbb{R}^n \). Then there exists \( u \geq 0, -u \in \Phi^k(\mathbb{R}^n) \) such that

(7.6) \[
F_k[-u] = \mu \quad \text{in } \mathbb{R}^n,
\]

and

(7.7) \[
\frac{1}{K} \frac{W^2_{k+1, k+1} \mu}{2k} \leq u \leq K \frac{W^2_{k+1, k+1} \mu}{2k}.
\]

Conversely, if \( u \geq 0, -u \in \Phi^k(\mathbb{R}^n) \) solves (7.6), then \( W^2_{k+1, k+1} \mu < \infty \) a.e. on \( \mathbb{R}^n \).

**Proof.** The second part of the theorem is trivial in view of Corollary 7.3. To prove the first part we denote by \( B_m \) the open ball in \( \mathbb{R}^n \) centered at the origin with radius \( m \), \( m = 1, 2, \ldots \). Let \( u_m \geq 0, -u_m \in \Phi^k(B_{m+1}) \), continuous near \( \partial B_{m+1} \) be a solution to the following Dirichlet problem

\[
\begin{aligned}
F_k[-u_m] &= \mu_{B_m} \quad \text{in } B_{m+1}, \\
u_m &= 0 \quad \text{on } \partial B_{m+1}.
\end{aligned}
\]
By Theorem 7.4 we have
\[ u_m \leq C W_{k+1, k+1} \mu < \infty \text{ a.e.,} \]
where $C$ is independent of $m$. Thus by passing to a subsequence we may assume that $u_m$ converges to $u$ a.e. for some $u \geq 0$ such that $-u \in \Phi^k(\mathbb{R}^n)$. Since $W_{k+1, k+1} \mu \in L^1_{\text{loc}}(\mathbb{R}^n)$, the weak continuity result (Theorem 7.1), (7.8) and Corollary 7.3 then imply that $u$ is a solution of (7.6) which satisfies (7.7). □

We are now in a position to establish the main results of this section.

**Theorem 7.6.** Let $\omega$ be a measure in $\mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < n/2$, and $q > k$. Then the following statements are equivalent.

(i) There exists a nonnegative solution $u$ to the equation
\[ \begin{align*}
\inf_{x \in \mathbb{R}^n} u(x) &= 0, \\
F_k[-u] &= u^q + \epsilon \omega \text{ in } \mathbb{R}^n,
\end{align*} \]

such that $-u \in \Phi^k(\Omega) \cap L^q_{\text{loc}}(\mathbb{R}^n)$, for some $\epsilon > 0$.

(ii) The testing inequality
\[ \int_B \left[ I_{2k} \omega_B(x) \right]^{q/k} dx \leq C \omega(B) \]
holds for all balls $B$ in $\mathbb{R}^n$.

(iii) For all compact sets $E \subset \mathbb{R}^n$,
\[ \omega(E) \leq C \text{Cap}_{I_{2k}, \frac{q}{q-k}}(E). \]

(iv) The testing inequality
\[ \int_B \left[ W_{2k+1, k+1} \omega_B(x) \right]^q dx \leq C \omega(B) \]
holds for all balls $B$ in $\mathbb{R}^n$.

(v) There exists a constant $C$ such that
\[ W_{2k+1, k+1}(W_{2k, k+1}^q, k+1)^q \leq C W_{2k, k+1}^{2k, k+1} \omega < \infty \text{ a.e.} \]

Moreover, if the constant $C$ in (7.13) satisfies
\[ C \leq \left( \frac{q-k}{qK} \right)^{q/k} \frac{k}{q-k}, \]

where $K$ is the constant in Corollary 7.3, then the equation (7.9) has a solution $u \geq 0$, $-u \in \Phi^k(\mathbb{R}^n)$ with $\epsilon = 1$ which obeys the two-sided estimate
\[ C_1 W_{2k, k+1} \omega(x) \leq u(x) \leq C_2 W_{2k, k+1} \omega(x) \]
for all $x \in \mathbb{R}^n$.

**Theorem 7.7.** Let $\Omega$ be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$. Suppose that $\omega \in \mathcal{M}^+_B(\Omega)$ such that $\omega = \mu + f$, where $\mu \in \mathcal{M}^+_B(\Omega)$ with $\text{supp} \mu \subseteq \Omega$ and $0 \leq f \in L^s(\Omega)$ with $s > n/2k$ if $1 \leq k \leq n/2$ and $s = 1$ if $n/2 < k \leq n$. Let $q > k$, $R = \text{diam}(\Omega)$ and $0 \leq \varphi \in C^0(\partial \Omega)$. Assume that

\begin{equation}
W_{2k^{k+1}}^{2R, k+1}(\mathcal{W}_{2k}^{2R, k+1} \omega)^q \leq A W_{2k^{k+1}}^{2R, k+1} \omega
\end{equation}

and

\begin{equation}
W_{2k^{k+1}}^{2R, k+1} \left[ W_{2k^{k+1}}^{2R, k+1} \left( \max_{\partial \Omega} \varphi \right)^q \right]^q \leq B W_{2k^{k+1}}^{2R, k+1} \left( \max_{\partial \Omega} \varphi \right)^q,
\end{equation}

where

\begin{equation}
A \leq \left( \frac{q - k}{3 + qK} \right)^{q/k} \left( \frac{k}{q - k} \right)
\end{equation}

and

\begin{equation}
B \leq \left( \frac{q - k}{3 + qK} \right)^{q/k} \left( \frac{k}{q - k} \right).
\end{equation}

Here $K$ is the constant in Theorem 7.4. Then there exists a function $u \geq 0$, $-u \in \Phi^k(\Omega) \cap L^q(\Omega)$, continuous near $\partial \Omega$ such that

\begin{equation}
\begin{cases}
F_k[-u] = u^q + \omega & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Moreover, there is a constant $C = C(n, k, q)$ such that

\begin{equation}
u \leq C \left\{ W_{2k^{k+1}}^{2k^{k+1}+1, k+1} \omega + W_{2k^{k+1}}^{2k^{k+1}+1, k+1} \left( \max_{\partial \Omega} \varphi \right)^q + \max_{\partial \Omega} \varphi \right\}.
\end{equation}

**Remark 7.8.** Condition (7.15) is redundant if $\varphi$ is small enough.

**Proof.** Let $\{u_m\}_{m \geq 0}$ be a sequence of nonnegative functions on $\Omega$ defined inductively by the following Dirichlet problems:

\begin{equation}
\begin{cases}
F_k[-u_0] = \omega & \text{in } \Omega, \\
u_0 = \varphi & \text{on } \partial \Omega,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
F_k[-u_m] = u_m^{q-1} + \omega & \text{in } \Omega, \\
u_m = \varphi & \text{on } \partial \Omega,
\end{cases}
\end{equation}
for \( m \geq 1 \). Here for each \( m \geq 0 \), \(-u_m\) is \( k\)-subharmonic and is continuous near \( \partial \Omega \). By Theorem 7.4 we have

\[
\begin{align*}
    u_0 & \leq K \frac{a}{2k+1} + K \max_{\partial \Omega} \varphi \\
        & = a_0 \frac{a}{2k+1} + b_0 \frac{a}{2k+1} \left( \max_{\partial \Omega} \varphi \right)^q + K \max_{\partial \Omega} \varphi,
\end{align*}
\]

where \( a_0 = K \) and \( b_0 = 0 \). Then by \((7.14)\) and \((7.15)\),

\[
\begin{align*}
    u_1 & \leq K \frac{a}{2k+1} \left( u_0^q + \varphi \right) + K \max_{\partial \Omega} \varphi \\
        & \leq K \left\{ \left( 3^{q-1} a_0^q \right)^\frac{1}{q} \frac{a}{2k+1} \left( \frac{a}{2k+1} \right)^q + (3^{q-1} b_0^q)^\frac{1}{q} \frac{a}{2k+1} \left( \frac{a}{2k+1} \right)^q \right\} \left( \max_{\partial \Omega} \varphi \right)^q + K \max_{\partial \Omega} \varphi \\
        & = a_1 \frac{a}{2k+1} + b_1 \frac{a}{2k+1} \left( \max_{\partial \Omega} \varphi \right)^q + K \max_{\partial \Omega} \varphi,
\end{align*}
\]

where

\[
\begin{align*}
    a_1 & = K \left\{ \left( 3^{q-1} a_0^q \right)^\frac{1}{q} A + 1 \right\}, & b_1 & = K \left[ (3^{q-1} b_0^q)^\frac{1}{q} B + K^\frac{k}{q} \right].
\end{align*}
\]

By induction we have

\[
\begin{align*}
    u_m & \leq a_m \frac{a}{2k+1} + b_m \frac{a}{2k+1} \left( \max_{\partial \Omega} \varphi \right)^q + K \max_{\partial \Omega} \varphi,
\end{align*}
\]

where

\[
\begin{align*}
    a_{m+1} & = K \left\{ \left( 3^{q-1} a_m^q \right)^\frac{1}{q} A + 1 \right\}, & b_{m+1} & = K \left[ (3^{q-1} b_m^q)^\frac{1}{q} B + K^\frac{k}{q} \right],
\end{align*}
\]

for all \( m \geq 0 \). It is then easy to see that

\[
\begin{align*}
    a_m & \leq \frac{K q}{q-k}, & b_m & \leq \frac{K^{\frac{k}{q}}}{q-k},
\end{align*}
\]

provided \((7.16)\) and \((7.17)\) are satisfied. Thus

\[
\begin{align*}
    (7.20) \quad u_m & \leq \frac{K q}{q-k} \frac{a}{2k+1} + \frac{K^{\frac{k}{q}}}{q-k} \frac{a}{2k+1} \left( \max_{\partial \Omega} \varphi \right)^q + K \max_{\partial \Omega} \varphi.
\end{align*}
\]

Using \((7.14)\), \((7.15)\), \((7.20)\), and passing to a subsequence, we can find a function \( u \geq 0 \) such that \(-u \) is \( k\)-subharmonic and \( u_m^q \rightarrow u^q \) in \( L^1(\Omega) \).
Thus in view of (7.19) and Theorem 7.1, we see that \( u \) is a desired solution of (7.18). \( \square \)

**Theorem 7.9.** Let \( \omega \) be a locally finite nonnegative measure on an open (not necessarily bounded) set \( \Omega \). Let \( q > k \), where \( 1 \leq k \leq n \). Suppose that \( u \geq 0, -u \in \Phi^k(\Omega) \) such that \( u \) is a solution of

\[
F_k[-u] = u^q + \omega \quad \text{in} \quad \Omega.
\]

Then for each cube \( P \in Q \), where \( Q = \{ Q \} \) is a Whitney decomposition of \( \Omega \) as before (see Sec. 6), we have

\[
(7.21) \quad \mu_P(E) \leq C \text{Cap}_{2k, \frac{q}{q-k}}(E),
\]

if \( \frac{2kq}{q-k} < n \), and

\[
(7.22) \quad \mu_P(E) \leq C(P) \text{Cap}_{G_{2k}, \frac{q}{q-k}}(E),
\]

if \( \frac{2kq}{q-k} \geq n \), for all compact sets \( E \subset \Omega \). Here \( d\mu = u^q dx + d\omega \), and the constant \( C \) in (7.21) does not depend on \( P \) and \( E \subset \Omega \); however, the constant \( C(P) \) in (7.22) may depend on the side length of \( P \).

Moreover, if \( \frac{2kq}{q-k} < n \), and \( \Omega \) is a bounded \( C^\infty \)-domain then

\[
\mu(E) \leq C \text{cap}_{2k, \frac{q}{q-k}}(E, \Omega)
\]

for all compact sets \( E \subset \Omega \), where \( \text{cap}_{2k, \frac{q}{q-k}}(E, \Omega) \) is defined by (2.19).

**Remark 7.10.** Let \( B_R \) be a ball such that \( B_{2R} \subset \Omega \). If \( \frac{2kq}{q-k} = n \) then as in Remark 6.12 we have

\[
\mu(B_r) \leq C(\log \frac{2R}{r})^{-\frac{k}{q-k}}\text{ for all balls } B_r \subset B_R.
\]

**Theorem 7.11.** Let \( \omega \) be a compactly supported measure in \( M^+_B(\Omega) \), where \( \Omega \) is a bounded uniformly \((k-1)\)-convex domain in \( \mathbb{R}^n \) \((1 \leq k \leq n)\). Let \( q > k \), \( R = \text{diam}(\Omega) \), and \( \varphi \in C^0(\partial \Omega) \), \( \varphi \geq 0 \). Then the following statements are equivalent.

(i) There exists a solution \( u \geq 0, -u \in \Phi^k(\Omega) \cap L^q(\Omega) \), continuous near \( \partial \Omega \), to the equation

\[
(7.23) \quad \left\{ \begin{array}{l}
F_k[-u] = u^q + \epsilon \omega \quad \text{in} \quad \Omega, \\
 u = \epsilon \varphi \quad \text{on} \quad \partial \Omega,
\end{array} \right.
\]

for some \( \epsilon > 0 \).

(ii) The testing inequality

\[
(7.24) \quad \int_B (G_{2k} \omega_B)^{\frac{q}{q-k}} dx \leq C \omega(B)
\]
holds for all balls $B$ such that $B \cap \text{supp} \omega \neq \emptyset$.

(iii) For all compact sets $E \subset \text{supp} \omega$,

$$\omega(E) \leq C \text{Cap}_{2k, \frac{q}{q-k}}(E).$$

(iv) The testing inequality

$$\int_B \left[ W_{2k+1}^{2k+1} \omega_B(x) \right]^q dx \leq C \omega(B)$$

holds for all balls $B$ such that $B \cap \text{supp} \omega \neq \emptyset$.

(v) There exists a constant $C$ such that

$$W_{2k+1}^{2k} \left( W_{2k+1}^{2k+1} \omega \right)^q \leq C W_{2k+1}^{2k+1} \omega < \infty \quad \text{a.e. on } \Omega.$$

**Remark 7.12.** As in Remark 6.15, if $\omega = \mu + f$, where $\text{supp} \mu \subset \Omega$, and $0 \leq f \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $k \leq n/2$, and $s = 1$ if $k > n/2$, then any one of the conditions (ii)–(v) is still sufficient for the solvability of the equation (7.23) for some $\epsilon > 0$.

**Theorem 7.13.** Let $E$ be a relatively closed subset of $\Omega$. Suppose that $\text{Cap}_{2k, \frac{q}{q-k}}(E) = 0$. Then any solution $u$ of

$$(7.25) \quad \begin{cases} -u \in \Phi^k(\Omega \setminus E) \cap L^q_{\text{loc}}(\Omega \setminus E), & u \geq 0, \\
F_k[-u] = u^q \quad \text{in } D'(\Omega \setminus E), 
\end{cases}$$

is also a solution of

$$(7.26) \quad \begin{cases} -u \in \Phi^k(\Omega) \cap L^q_{\text{loc}}(\Omega), & u \geq 0, \\
F_k[-u] = u^q \quad \text{in } D'(\Omega). 
\end{cases}$$

Conversely, if $E$ is a compact set in $\Omega$ such that any solution of (7.25) is also a solution of (7.26), then $\text{Cap}_{2k, \frac{q}{q-k}}(E) = 0$.

**Proof.** To prove this theorem, we proceed as in the proof of Theorem 2.18. For the first statement, note that if $\text{Cap}_{2k, \frac{q}{q-k}}(E) = 0$ then $\text{Cap}_{2k, \frac{q}{q-k}+1}(E) = 0$ and $k < \frac{n}{2}$ (see [AH]), which implies that

$$\text{cap}_k(E, B) = 0$$

for a ball $B \ni \Omega \supset E$ due to Theorem 7.14 below. Here $\text{cap}_k(\cdot, \Omega)$ is the (relative) $k$-Hessian capacity associated with the domain $\Omega$ (see (7.3)). Thus by [L] Theorem 4.2, $E$ is a $k$-polar set, i.e., $(-\infty)$-set of a $k$-subharmonic function in $\mathbb{R}^n$. It is then easy to see that the function $\bar{u}$ defined by

$$\bar{u}(x) = \begin{cases} u(x), & x \in \Omega \setminus E, \\
\limsup_{y \to x, \ y \notin E} u(y), & x \in E, 
\end{cases}$$

is a solution of (7.25). The second statement follows similarly.
belongs to $\Phi^k(\Omega)$, and $-\tilde{u}$ is an extension of $u$. The rest of the proof is then the same as before.

**Theorem 7.14.** Let $1 \leq k < \frac{n}{2}$ be an integer. Then
\begin{equation}
M_1 \operatorname{Cap}_{\frac{2k}{k+1}, k+1}(E) \leq \operatorname{cap}_k(E, \Omega) \leq M_2 \operatorname{Cap}_{\frac{2k}{k+1}, k+1}(E)
\end{equation}
for any compact set $E \subset \overline{Q}$ with $Q \in Q$, where the constants $M_1$, $M_2$ are independent of $E$ and $Q$.

**Proof.** Let $R$ be the diameter of $\Omega$. From Wolff’s inequality it follows that $\operatorname{Cap}_{\frac{2k}{k+1}, k+1}(E)$ is equivalent to
\[ \sup \{ \mu(E) : \mu \in M^+(E), \; W^{4R}_{\frac{2k}{k+1}, k+1} \mu \leq 1 \text{ on } \operatorname{supp} \mu \}, \]
for any compact set $E \subset \Omega$ (see [HW, Proposition 5]). To prove the left-hand inequality in (7.28), let $\mu \in M^+(E)$ such that $W^{4R}_{\frac{2k}{k+1}, k+1} \mu \leq 1$ on $\operatorname{supp} \mu$, and let $u \in \Phi^k(B)$ be a nonpositive solution of
\[ \begin{cases} F_k[u] = \mu & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \]
where $B$ is a ball of radius $R$ containing $\Omega$. By Theorem 7.1 and the boundedness principle for nonlinear potentials (see [AH]), we have
\[ |u| \leq C W^{2\operatorname{diam}(Q)}_{\frac{2k}{k+1}, k+1} \mu \leq C. \]
Thus
\[ \mu(E) = \mu_k[u](E) \leq C \operatorname{cap}_k(E, \Omega), \]
which shows that
\[ \operatorname{Cap}_{\frac{2k}{k+1}, k+1}(E) \leq C \operatorname{cap}_k(E, \Omega). \]
To prove the upper estimate in (7.28), we let $Q \in Q$, and fix a compact set $E \subset \overline{Q}$. Note that for $\mu \in M^+(E)$ and $x \in E$ we have
\[ W^{4R}_{\frac{2k}{k+1}, k+1} \mu(x) = W^{2\operatorname{diam}(Q)}_{\frac{2k}{k+1}, k+1} \mu(x) + \int_{2\operatorname{diam}(Q)}^{4R} \frac{[\mu(E)]}{t^{n-2k}} \frac{dt}{t}. \]
Thus, for $k < \frac{n}{2}$,
\begin{equation}
W^{4R}_{\frac{2k}{k+1}, k+1} \mu(x) \leq C W^{2\operatorname{diam}(Q)}_{\frac{2k}{k+1}, k+1} \mu(x), \quad \forall x \in E.
\end{equation}
Now for $u \in \Phi^k(\Omega)$ such that $-1 < u < 0$ by Theorem 7.2 we obtain
\[ W^{2\operatorname{diam}(Q)}_{\frac{2k}{k+1}, k+1} \mu_E(x) \leq W^{2\operatorname{diam}(Q)}_{\frac{2k}{k+1}, k+1} \mu(x) \leq C |u(x)| \leq C, \]
for all $x \in E$, where $\mu = \mu_k[u]$. Thus, we deduce from (7.29) that
\[ W^{AR}_{k^+,k+1} E(x) \leq C, \quad \forall x \in E, \]
which implies
\[ (7.30) \quad \mu(E) \leq C \text{Cap}_{G^{2k^+,k+1}}(E). \]
Finally, the definition of $\text{cap}_k(\cdot, \Omega)$ and (7.30) then give
\[ \text{cap}_k(E, \Omega) \leq C \text{Cap}_{G^{2k^+,k+1}}(E, \Omega), \]
which completes the proof of the theorem. \qed

Remark 7.15. If $\Omega$ is a $C^\infty$-domain in $\mathbb{R}^n$, and $1 \leq k < \frac{n}{2}$, then by the quasiadditivity of the capacity $\text{cap}_{G^{2k^+,k+1}}(\cdot, \Omega)$ (see Theorem 6.13) we have the following upper estimate for the $k$-Hessian capacity $\text{cap}_k(\cdot, \Omega)$: There exists a constant $C > 0$ such that for any compact set $E \subset \Omega$,
\[ \text{cap}_k(E, \Omega) \leq C \text{cap}_{G^{2k^+,k+1}}(E, \Omega). \]

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: nguyencp@math.missouri.edu

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: igor@math.missouri.edu