THE FIRST DIFFERENTIAL OF THE FUNCTOR
“ALGEBRAIC K-THEORY OF SPACES”

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ABSTRACT. In his “Algebraic K-theory of topological spaces II” Waldhausen proved that his functor $A(X)$ splits: There is a canonical map from the stable homotopy of $X$ which has a retraction up to weak equivalence. We adapt Waldhausen’s proof to obtain a calculation of the Differential (in the sense of Goodwillie’s “Calculus I”) of $A(X)$ at any path-connected base space.

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1. Introduction

Waldhausen proves in [Wal79] that his functor $A(X)$, the “Algebraic K-theory of space”, splits: The canonical “Barratt-Priddy-Quillen-Segal” map

$$\Omega^\infty \Sigma^\infty X_+ \to A(X)$$

from the stable homotopy of $X$ has a retraction up to weak homotopy equivalence. We adapt Waldhausen’s proof to obtain a calculation of the Differential of $A(X)$, in the sense of Goodwillie [Goo90], at any path-connected base space. Our main result says that there is a weak equivalence

$$D_B A(X) \simeq C^S(F, L(B))$$

where the left-hand side is the Goodwillie Differential of $A(-)$ at the space $B$, evaluated at $X$ over $B$. The right-hand side is the stabilization of a plus-construction of a cyclic bar construction. It takes as input the homotopy fiber $F$ of $X \to B$ and the Kan Loop Group $L(B)$ of $B$. We refer to Definition 4.2 for the quite technical definition of $C(-, -)$. The quoted result is Theorem 7.1.

Perhaps of as much interest as the calculation is the improvement of the methods of [Wal79] we do here. In Theorem 3.1 of [Wal79], which is the main step in the proof of the splitting of $A(X)$, Waldhausen had to assume that the space $X$ he started with is a suspension. Later, he observed that a reordering of the arguments makes this assumption unnecessary. This work is an implementation of that observation. The generalized version of [Wal79, Theorem 3.1] is our Theorem 4.4.

For completeness we also reprove the splitting result in our updated language. The proof proceeds roughly as follows: There is a map from $A(X)$ into its Goodwillie Differential at a point. Then Theorem 4.3 identifies this Differential with the functor $C^S(X, *)$ from above. This functor is built using the cyclic bar construction and using this structure we can construct a “trace map” to $\Omega^\infty \Sigma^\infty X_+$. Careful examination shows that this a retraction for $\Omega^\infty \Sigma^\infty X_+ \to A(X)$.

As we are describing the Differential at any space in a very similar way to the Differential at a point, it is an interesting question if the Differential at other base spaces would admit similar trace maps which are useful to get information on $A(X)$. In general, trace methods have been quite successful to obtain information about $A(X)$, see e.g. [Goo91].

For completeness we mention that later work (see [Wal87]) showed that second factor of the splitting of $A(X)$ can be identified as $\text{Wh}^\text{Diff}(X)$, the DIFF Whitehead space. It is deeply connected to automorphisms of manifolds, see e.g. the recent book [WJR13].

Outline of the paper. The structure of this article follows closely [Wal79]. We need all the definitions from there, so we introduce them here again. We correct a minor mistake and give more detailed proofs of most of the lemmas. Further, we use the language of Goodwillie’s Calculus of Functors [Goo90], which did not exist at the time when [Wal79] was written. We hope this streamlines the exposition a bit. Here is the outline of the chapters.

Section 2 contains general prerequisites. In Subsection 2.1 we introduce the model of algebraic K-theory of spaces we work with and collect a few related properties about the plus-construction. Subsection 2.2 contains a quick review of Goodwillie Calculus. As we will use bisimplicial sets a lot, Subsection 2.3 discusses
a realization lemma for bisimplicial sets and $k$-connected maps. Finally, Subsection 2.4 contains the statement of the Freudenthal Suspension Theorem and several corollaries which we will need.

Section 3 contains the definition of a generalized wedge and the cyclic bar construction. Both are crucial constructions for our comparison. The generalized wedge construction replaces the notion of a partial monoid from [Wal79 2.2], as it does not capture the correct notion, see Remark 3.12. We examine how both constructions behave under $k$-connected maps and relate them via a “semi-direct” product.

Section 4 contains the technical heart of this article. This section introduces a functor $C(-,-)$ and relates its Goodwillie Differential at the point to the Goodwillie Differential of $A(X)$ at the point via Theorem 4.4. As we already mentioned, this is the main improvement compared to [Wal79], as we do not need to assume the space $X$ is a suspension.

In Sections 5 and 6 we construct the trace map and prove the splitting of $A(X)$. We give more details than [Wal79], use Goodwillie’s language throughout, and are able to avoid products in the construction of the splitting. Otherwise this is not new but included for completeness.

Finally, Section 7 contains the calculation of the differential of $A(-)$ at any path-connected base space.

**Notation and conventions.** Let us explain some notions we use throughout. We will deal with topological spaces and simplicial sets and freely replace them with each other when necessary. As most spaces arise naturally as simplicial sets this will usually mean taking the geometric realization. We will use the term “spaces” when we do not want to specify which notion we use, as we already did so in the last sentence.

We will also assume that all topological spaces are compactly generated and thus geometric realization commutes with products. We denote the category of compactly generated spaces by $\textbf{Top}$ and the category of simplicial sets by $\textbf{sSet}$. The corresponding pointed categories are denoted by $\textbf{Top}_*$ and $\textbf{sSet}_*$ respectively.

We further use the notions of $k$-connected and $k$-equivalent. A map $f : A \to B$ between spaces is called $k$-connected if $\pi_l(f)$ is bijective for $l \leq k - 1$ and surjective for $l = k$. The map $f$ is called $k$-equivalent if $\pi_l(f)$ is bijective for $l \leq k$.

Whereas the first notion can be expressed by the vanishing of homotopy groups of the homotopy fiber for $l \leq k - 1$ and is for example used in the Freudenthal Suspension Theorem (cf. Thm. 2.11), the second notion is more convenient when comparing spaces. For example it has a two-out-of-three property and it is easier to deal with highly connected maps “in the wrong direction”. (That is what we do to prove Theorem 4.4.) For us these notions are interchangeable as we are interested in the case $k \to \infty$ anyway. A pointed space $(A, a_0)$ is called $k$-connected if the canonical map $a_0 \to A$ is so.

A map is $\infty$-connected if it is $k$-connected for all $k \in \mathbb{N}$. An $\infty$-connected map is also called a weak equivalence.

**Acknowledgements.** This work is a revised version of the author’s diploma thesis written under the direction of Friedhelm Waldhausen some years ago. Waldhausen showed the author a handwritten page with the maps of Theorem 3.1 of [Wal79] in
a new order such that one does not need to assume that $X$ is a suspension. The work at hand is the worked-out version of that page.

Therefore I would like to thank Friedhelm Waldhausen for suggesting this project and the support and advise during it. I also would like to thank Ross Staffeldt for helpful conversations for the revised version and the help in producing a final, publishable version.

2. PREREQUISITES

2.1. Algebraic K-Theory of (Topological) Spaces.

2.1.1. The Plus-Construction and Algebraic K-Theory of Topological Spaces. The plus-construction of Quillen is a way to topologically factor out a perfect subgroup of the fundamental group of a topological spaces without changing the cohomology type of the space. The main thing we will need of the construction is the following theorem.

Theorem 2.1 (Quillen’s plus-construction). Let $X$ be a connected pointed topological space with fundamental group $\pi_1$. Let $P$ be the maximal perfect normal subgroup of $\pi_1$. Then there exists a space $X^+$ and map $q_X : X \to X^+$ with the property that $q_X$ induces the quotient map $\pi_1 \to \pi_1/P$ on the fundamental groups and an isomorphism on cohomology of the universal coverings. $q_X$ is universal up to homotopy under maps killing $P$ out of the fundamental group of $X$.

For the category of spaces under $BA_5$ where the image of $A_5$ normally generates the maximal perfect normal subgroup $P$ this construction can be done functorially and $q_X$ is a natural transformation.

Proof. The first part is stated and proven in the language above in [Ros94, Thm. 5.2.2]. The more recent [DGM13] also contains a detailed discussion: The plus-construction is introduced in [DGM13, 1.1.6.2 ff.] and discussed in some detail in [DGM13, 3.1.1]. The last part is [DGM13, Proposition 3.1.2.3].

The only properties we will need are that for the spaces considered there is a natural map $X \to X^+$, and that the latter spaces fulfill the following connectivity condition.

Lemma 2.2 ([DGM13 Lemma 3.1.1.8]). Let $X \to Y$ be a $k$-connected map of connected spaces. Then $X^+ \to Y^+$ is also $k$-connected.

Proof. As cited or one uses the Gluing Lemma (Lemma 2.9) together with the attaching cells description of the plus-construction.

Now we can use the plus-construction to give a description of Algebraic K-Theory of Topological Spaces in a matrix-like style ([Wal85 Section 2.2]).

Let $X$ be a pointed connected simplicial set. Then the Kan Loop Group $L(X)$ is defined which is a simplicial group representing the loop space of $X$ up to weak equivalence (see [Wal96] for a construction). We call the realization of that group $L$ for short, it is a topological group with $L \simeq \Omega|X|$. The algebraic K-Theory for a pointed connected simplicial set $X$ can now be described (up to weak equivalence) as the topological space

\[
A(X) = \mathbb{Z} \times \left( \colim_{n,k} B\text{Aut}_L(\vee^n S^m \wedge L_+) \right)^+
\]
Here $B(-)$ denotes the classifying space (i.e. the nerve) of the simplicial monoid $\text{Aut}_L(\vee^k S^n \wedge L_+)$, which consists of the $L$-equivariant maps $\vee^k S^n \wedge L_+ \to \vee^k S^n \wedge L_+$ which are weak equivalences; $(-)^+$ is the Quillen plus-construction. See Section 3.2 for a review of the nerve construction. The factor $\mathbb{Z}$ has to be produced artificially to make this definition agree with the definition from [Wal85, 2.1].

2.1.2. The Barratt-Priddy-Quillen-Segal Map. There is a map from stable homotopy to algebraic K-Theory. It originates from the description of the stable homotopy by symmetric groups

$$\Omega^\infty \Sigma^\infty (X_+) \simeq \mathbb{Z} \times \left( \text{colim}_{k} B\text{Aut}_L(\vee^k S^0 \wedge L_+) \right)^+,$$

(the Barratt-Priddy-Quillen-Segal Theorem, see e.g. [Seg74]). It is the map

$$\Omega^\infty \Sigma^\infty (X_+) \simeq \mathbb{Z} \times \left( \text{colim}_{k} B\text{Aut}_L(\vee^k S^n \wedge L_+) \right)^+ \to \mathbb{Z} \times \left( \text{colim}_{k,n} B\text{Aut}_L(\vee^k S^n \wedge L_+) \right)^+ \simeq A(X) \tag{2}$$

induced by the inclusion (for $n = 0$)

$$\text{colim}_{k} B\text{Aut}_L(\vee^k S^0 \wedge L_+) \to \text{colim}_{k,n} B\text{Aut}_L(\vee^k S^n \wedge L_+).$$

We will provide a retraction up to weak equivalence for the map (2).

In view of the weak equivalence $X \xrightarrow{\sim} BL(X)$ (cf. [GJ99] Propositions V.6.3, V.6.4) we have a canonical map

$$X \xrightarrow{\sim} B\text{Aut}_L(\vee^1 S^0 \wedge L_+) \to \mathbb{Z} \times \left( \text{colim}_{k} B\text{Aut}_L(\vee^k S^0 \wedge L_+) \right)^+ \simeq \Omega^\infty \Sigma^\infty (X_+).$$

**Lemma 2.3.** This is up to homotopy the same map as the canonical stabilization map

$$X \to X_+ \to \Omega^\infty \Sigma^\infty (X_+).$$

**Proof.** This is Lemma 1.1 of [Wal79]. \hfill \Box

2.2. Goodwillie Calculus. Goodwillie defined the **differential** of a homotopy functor $F$ at any space $B$ in [Goo90, Section 1]. Goodwillie’s definition used homotopy functors from (compactly generated) topological spaces to (compactly generated) pointed topological spaces. We want to apply the theory to homotopy functors from pointed simplicial sets to (compactly generated) topological spaces, which works in the same way.

We first need to discuss reduced functors, then we briefly recall Goodwillie’s original definition.

2.2.1. **Reduced functors.** Let $\text{Top}$ denote the category of unpointed compactly generated topological spaces, $\text{Top}_+$ the pointed ones. Let $s\text{Set}$ denoted the category of unpointed simplicial sets, $s\text{Set}_+$ the pointed ones. Let $F: s\text{Set} \to \text{Top}_+$ be a homotopy functor, that is, it maps weak equivalences to weak equivalences. $F$ is called **reduced** if $F(*) \simeq *$. 
Often functors which arise “in nature” are not reduced. There is a canonical way to assign a reduced functor \( \tilde{F} \) to an arbitrary homotopy functor \( F \). We define \( \tilde{F} \) as the homotopy fiber of \( F(X) \to F(\ast) \), that is the homotopy pullback of

\[
\begin{array}{ccc}
F(X) & \to & F(\ast) \\
\downarrow & & \downarrow \\
\ast & \to & F(\ast)
\end{array}
\]

As this can be done functorially (e.g. by replacing the inclusion \( \ast \to F(\ast) \) by the path fibration over \( F(\ast) \)), \( \tilde{F} \) is a reduced homotopy functor.

Sometimes there is an inverse procedure. For certain functors \( F \) the functor \( X \mapsto \tilde{F}(X^+) \) is weakly equivalent to \( F \). This is especially true for \( \Omega^\infty \Sigma^\infty X^+ \) and \( A(X) \). We will prove this for \( \Omega^\infty \Sigma^\infty X^+ \) in Lemma 2.8 below and for \( A(X) \) in Lemma 6.1.

We need a further property about the reduction of a functor.

**Lemma 2.4.** Assume we have a natural transformation \( f : F(X) \to G(X) \) of homotopy functors which is, say, \((3m-1)\)-connected whenever \( X \) is \((m-1)\)-connected. Then the induced natural transformation \( f^* : \tilde{F}(X) \to \tilde{G}(X) \) also has this property.

**Proof.** The assumption implies that \( F(\ast) \simeq G(\ast) \). Further it is equivalent to say that the homotopy fiber of \( F(X) \to G(X) \) is \((3m-1)\)-connected. This homotopy fiber is the same as the homotopy fiber of \( f^* : \tilde{F}(X) \to \tilde{G}(X) \), hence \( f^* \) is \((3m-1)\)-connected, too. \( \square \)

### 2.2.2. The Goodwillie Differential

Let \( sSet/B \) be the category of spaces over a base space \( B \). In \( sSet/B \) we can define the suspension over \( B \), or fiberwise suspension of a space \( f : X \to B \) as the pushout

\[
\begin{array}{ccc}
X & \to & C_B X \\
\downarrow & & \downarrow \\
C_B X & \to & S_B X
\end{array}
\]

where \( C_B X \) is the cone on \( f \) which is the usual mapping cylinder of \( f \). As the name suggests, this yields the usual (unreduced) suspension on the fibers of \( f \).

**Definition 2.5** ([Goo90, 1.10]). If \( \Phi : sSet/B \to Top_* \) is a homotopy functor, then the functor \( \mathcal{T}\Phi : sSet/B \to Top_* \) is given by

\[
(\mathcal{T}\Phi)(X) = \operatorname{holim}(\Phi(C_B X) \to \Phi(S_B X) \leftarrow \Phi(C_B X)).
\]

We have an induced map \( t(\Phi) : \Phi \to \mathcal{T}\Phi \). Now \( P\Phi : Top/B \to Top_* \) is defined by

\[
(P\Phi)(X) = \operatorname{hocolim}(\Phi(X) \to (\mathcal{T}\Phi)(X) \to (\mathcal{T}^2\Phi)(X) \to \cdots).
\]

We define \( D\Phi \), the linearization of \( \Phi \), as the reduction of \( P\Phi \).

For a functor \( F : sSet \to Top_* \) one obtains a functor \( sSet/B \to Top_* \) by means of the forgetful functor \( sSet/B \to Top \).

We then have

**Definition 2.6** ([Goo90, 1.11]).

1. The functor \( P_B F = P\Phi : sSet/B \to Top_* \) is called the 1-jet of \( F \) at \( B \).
(2) The functor \( D_B F = D\Phi : sSet/B \to Top_* \) is called the \textit{differential} of \( F \) at \( B \).

The functors \( P_B F \) and \( D_B F \) have certain special properties. So \( P_B F \) is \textit{excisive} and \( D_B F \) is \textit{linear}, which implies \( \pi_* D_B F \) is a generalized homology theory. After \textbf{[Goo90]} one can reduce \( F \) first to \( \tilde{F} \) and then \( P_B \tilde{F} \) is already reduced.

In the case of homotopy functors from \textit{pointed} simplicial sets to pointed topological spaces everything works analogously. We can replace the unreduced suspension \( S_B \) by the reduced suspension \( \Sigma_B \) when convenient, since they are weakly equivalent for simplicial sets. We will especially need the case \( B \) equals the point \( pt \). Therefore we elaborate what happens when we specialize above definition to that case.

\[ 2.2.3. \text{The Goodwillie Differential at a point.} \]

Let \( F : sSet_* \to Top_* \) be a reduced homotopy functor. The fiberwise suspension over a point is the usual suspension, i.e., \( SX \) can be described as the pushout of two unreduced cones:

\[
\begin{array}{ccc}
X & \longrightarrow & CX \\
\downarrow & & \downarrow \\
CX & \longrightarrow & SX
\end{array}
\]

Applying \( F \) to it gives the map \( t(F) \) from \( F(X) \) to the homotopy limit \( TF(X) \simeq \Omega F(SX) \) of the lower right part of that diagram. One can iterate this infinitely often by taking the homotopy colimit and one obtains

\[ PF(X) = \operatorname{hocolim}_n \Omega^n F(S^n X). \]

Clearly for every \( X \) the space \( PF(X) \) is an infinite loop space.

We will need a fact from \textbf{[Goo90], Def. 1.13, Proposition 1.17(ii)}:

**Lemma 2.7.** Suppose we have a natural transformation \( F(X) \xrightarrow{\tau_X} G(X) \) which is \((2k - i)\)-connected when \( X \) is \( k \)-connected, for a fixed natural number \( i \). Then \( \tau_X \) induces a weak equivalence \( PF \xrightarrow{\tau_P} PG \).

In the situation of the lemma one says \( F \) and \( G \) \textit{agree to first order}.

The following lemma collects a few properties about the case \( F = \text{Id} \) which we will need later. It also provides the promised inverse procedure to the reduction of the functor \( X \mapsto \Omega^\infty \Sigma^\infty X_+ \).

**Lemma 2.8.** Let \( X \) be pointed. Define \( Q(X) = \Omega^\infty \Sigma^\infty X_+ \). Then

\[ \bar{Q}(X) \simeq \Omega^\infty \Sigma^\infty \]

(hence \( \bar{Q}(X_+) \simeq Q(X) \)).

**Furthermore** \( P\Omega^\infty \Sigma^\infty \) is weakly equivalent to \( \Omega^\infty \Sigma^\infty \).

**Proof.** We only need to prove \( Q(X) = \Omega^\infty \Sigma^\infty X_+ \simeq \Omega^\infty \Sigma^\infty X \times \Omega^\infty \Sigma^\infty S^0 \). Then \( \bar{Q}(X) \) is the homotopy fiber of \( \Omega^\infty \Sigma^\infty X \times \Omega^\infty \Sigma^\infty S^0 \xrightarrow{pr} \Omega^\infty \Sigma^\infty S^0 \) and hence equal to \( \Omega^\infty \Sigma^\infty X \).

But the first is clear since stable homotopy is a homology theory. (Or one can use Lemma \textbf{4.15} and stabilize).

For the second part consider the map \( Y \to \Omega^\infty \Sigma^\infty Y \) which is \((2m + 1)\)-connected for \( Y \) \( m \)-connected. Hence \( P\text{Id} \to P\Omega^\infty \Sigma^\infty \) is a weak equivalence but \( P\text{Id}(Y) \) is weakly equivalent to \( \Omega^\infty \Sigma^\infty Y \). \( \square \)
For a (possibly unreduced) functor $F$, we call the differential $DF$ of $F$ at the point also the stabilization of $F$ and denote it by $F^S$. Our notation is somewhat for historical reasons. I hope it might be less confusing in larger diagrams. Note that there is a canonical transformation $F \to F^S$.

2.3. Some General Facts on Simplicial and Bisimplicial Sets. A nice introduction to bisimplicial sets is given in [GJ99, IV.1]. We need the following two lemmas repeatedly.

**Lemma 2.9 (Gluing Lemma).** Given the diagram

$$
\begin{array}{ccc}
B & & C \\
\downarrow & & \downarrow \\
A & & C \\
\downarrow & & \downarrow \\
B' & & C'
\end{array}
$$

of simplicial sets where the left horizontal maps are cofibrations. If the vertical maps are all $k$-connected (resp. $k$-equivalences) the induced map on the pushout

$$B \cup_A C \to B' \cup_{A'} C'$$

is $k$-connected (resp. a $k$-equivalence).

**Proof.** This follows from the gluing lemma ([GJ99, Lemma II.8.8]) for weak (i.e. $\infty$-) equivalences. After geometric realization we can assume $T$ is a diagram in CW-complexes. Now if $A \to A'$ is $k$-connected, it can be factored as $A \to T_A \to A'$ where $T_A \to A'$ is a weak equivalence and $A \to T_A$ is the identity on the $k$-skeleton. We can factor the maps in diagram $T$ compatibly, with $T_A \to T_B$ being a cofibration, and get a factorization

$$B \cup_A C \to T_B \cup_{T_A} T_C \to B' \cup_{A'} C'$$

The first map is the identity on the $k$-skeleton and hence $k$-connected. The second is a weak equivalence by the gluing lemma for weak equivalences. Hence the composition is $k$-connected.

If $A \to A'$ is a $k$-equivalence, it factors similarly as $A \to T_A' \to A'$, where now $A \to T_A'$ is an inclusion of the $k$-skeleton and all $k + 1$-cells are attached with constant maps. Then $B \cup_A C \to T_B \cup_{T_A} T_C$ has the same property. The second case follows. □

**Lemma 2.10 (Realization Lemma).** Let $f_n: X_n \to Y_n$ be a map of bisimplicial sets. If $f_n$ is $k$-connected (or $k$-equivalent) for every $n$ then so is $f$.

**Proof.** Proposition IV.1.7 in [GJ99] shows the Realization Lemma for weak equivalences. The proof indicated there uses the Gluing Lemma in an induction. If we plug in the $k$-connected or $k$-equivalent Gluing Lemma 2.9 instead, it gives a proof of our statement. □

2.4. About the Freudenthal Suspension Theorem. At a rough guess we need the Freudenthal Theorem and some easy corollaries of it about a dozen times. For the convenience of the reader (and of the author) we collect what we need.

**Theorem 2.11 (Freudenthal).** Let $X$ be a $m$-connected pointed topological space. Suspension induces a map

$$\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X).$$
This map is an isomorphism for \( n \leq 2m \) and surjective for \( n = 2m + 1 \).

**Corollary 2.12.** For \( X \) an \( m \)-connected pointed topological space suspension gives a map

\[
\text{Map}(S^n, S^n \wedge X) \xrightarrow{\Sigma} \text{Map}(S^{n+1}, S^{n+1} \wedge X)
\]

which is \((2m + 1)\)-connected.

**Corollary 2.13.** For \( X \) an \( m \)-connected pointed topological space the canonical map

\[
X \to \Omega^\infty \Sigma^\infty X = \colim \Omega^n \Sigma^n X
\]

is \(2m + 1\)-connected.

**Corollary 2.14.** For \( X \) an \( m \)-connected pointed topological space the unit of the adjunction \( \Sigma \rightleftharpoons \Omega \), the map \( X \to \Omega \Sigma X \), is \((2m + 1)\)-connected. Similarly the counit \( \Sigma \Omega X \to X \) is \((2m + 1)\)-connected.

### 3. The Generalized Wedge and the Cyclic Bar Construction

#### 3.1. A Review of Monoids and Operations.

**Definition 3.1.** A pointed set \((H, h)\) is a monoid if there is a multiplication map \( \mu : H \times H \to H \) such that

\[
\begin{align*}
H \times H \times H & \xrightarrow{\mu \times \text{id}} H \times H \\
\text{id} \times \mu & \downarrow \quad \mu \downarrow \\
H \times H & \xrightarrow{\mu} H
\end{align*}
\]

and

\[
\begin{align*}
H & \xrightarrow{(h, \text{id})} H \times H \\
\mu & \downarrow \quad \text{id} \downarrow \\
H & \xrightarrow{\mu} H
\end{align*}
\]

commute. Property (4) is called associativity and (5) says that \( h : * \to H \) is a two-sided identity.

\((H, h)\) operates from the left on a set \( M \) if there is a map \( a : H \times M \to M \) (action of \( H \) on \( M \)) such that

\[
\begin{align*}
H \times H \times M & \xrightarrow{\mu \times \text{id} \times a} H \times M \\
\mu \times \text{id} & \downarrow \quad a \downarrow \\
H \times M & \xrightarrow{a} M \\
M & \xrightarrow{\text{id}} M
\end{align*}
\]

and

\[
\begin{align*}
M & \xrightarrow{(h, \text{id})} H \times M \\
\text{id} & \downarrow \quad a \downarrow \\
M & \xrightarrow{a} M
\end{align*}
\]

commute. These properties mean that the operation is associative and unitary. There is the dual notion of operation from the right.

The definition applies verbatim if we replace \( \text{Set} \) by any category with twofold products.
Remark 3.2. We are mainly interested in the cases of pointed simplicial sets, pointed simplicial topological spaces or pointed bisimplicial sets. It is clear that in a category of simplicial objects these definitions apply degree-wise. So, for example, a simplicial monoid is a simplicial set which is a monoid in each degree and the multiplication is compatible with the simplicial structure maps.

We have – of course – a notion of a maps of monoids or operations.

Definition 3.3. Let \( H, G \) be monoids with multiplications \( \mu_H, \mu_G \).

(1) A map \( f : H \to G \) is called a morphism of monoids if it commutes with the multiplication and respects the basepoint. Stated as a diagram this is

\[
\begin{array}{ccc}
H \times H & \xrightarrow{\mu_H} & H \\
\downarrow f \times f & & \downarrow f \\
G \times G & \xrightarrow{\mu_G} & G
\end{array}
\]

commutes.

(2) Let \( H \) operate on \( M \) via \( \alpha_M \) and \( G \) operate on \( N \) via \( \alpha_N \). A pair of maps \( f : H \to G \) and \( g : M \to N \) is called a morphism of operations if \( f \) is a morphism of monoids and \( g \) commutes with the operations, i.e.,

\[
\begin{array}{ccc}
H \times M & \xrightarrow{\alpha_M} & M \\
\downarrow f \times g & & \downarrow g \\
G \times N & \xrightarrow{\alpha_N} & N
\end{array}
\]

commutes. We have a dual notion for an operation from the right.

Definition 3.4. Let \( G, H \) be monoids and \( M \) an object such that \( G \) operates from the left on \( M \) and \( H \) operates from the right. Call the operations \( l \) and \( r \) respectively. We say that these operations are compatible if they commute, meaning the following diagram commutes.

\[
\begin{array}{ccc}
G \times M \times H & \xrightarrow{l \times \text{id}} & M \times H \\
\downarrow \text{id} \times r & & \downarrow r \\
G \times M & \xrightarrow{r} & M
\end{array}
\]

Remark 3.5 (Categorical Viewpoint). Note that talking about right and left operations is just a convenient way to distinguish them. As any operation from the right is the same as an operation of the opposite monoid from the left we could just express compatibility in a diagram corresponding to the one above where \( G \) and \( H \) actually commute and which is easily generalized to compatible operations of \( n \) monoids on \( M \).

From a more categorical viewpoint we can view compatibility as follows. In the situation of Definition 3.3 the diagram (9) can be interpreted as a diagram in the category of \( G \)-operations: Let \( G \) operate trivially on \( H \), i.e., \( a_H(g, h) = h \) for all \( g \in G, h \in H \). Then \( G \) operates on \( M \times H \) like shown in the first line of (9). The commutativity of the diagram then says that \( r : M \times H \to M \) is a map of \( G \)-operations. This viewpoint helps to identify how an operation from both sides in a general situation will look like.
3.2. **Review: The Nerve Construction.** Let $H$ be monoid (of sets). We can consider $H$ as a category with one object and then apply the usual nerve construction to get a simplicial set. This is sometimes called the *bar construction of $H$* or the *nerve of $H$*. We will give an explicit definition.

**Definition 3.6.** Let $H$ be a monoid. The *nerve of $H$* is the following simplicial object

$$[k] \rightarrow H \times \cdots \times H$$

with face maps

$$d_0(h_1, \ldots, h_k) = (h_2, \ldots, h_k)$$
$$d_i(h_1, \ldots, h_k) = (h_1, \ldots, h_i h_{i+1}, \ldots, h_k)$$
$$d_k(h_1, \ldots, h_k) = (h_1, \ldots, h_{k-1})$$

and degeneration maps

$$s_i(h_1, \ldots, h_k) = (h_1, \ldots, h_{i-1}, 1, h_i, \ldots, h_k).$$

3.3. **The Definition of a Generalized Wedge.** We now construct a simplicial object which serves as an intermediate step in a following comparison. It can be viewed as a generalization of the nerve construction of a monoid or of the iterated wedge, this will be discussed below. We first introduce a notation. In the following we always assume we have a (bi-)simplicial monoid operating on a (bi-)simplicial set.

**Definition 3.7.** Let $H$ be a (bi-)simplicial monoid operating on the (bi-)simplicial set $M$ from both sides and compatibly. Further assume there is an inclusion $H \hookrightarrow M$ which is compatible with the operation. We then write $H \vee M$ and call this an *operation situation*.

**Remark 3.8.** This inclusion hypothesis is needed as the generalized wedge construction below uses set-theoretic union of simplicial sets. It may be possible to remove that, but we do not need the generality here and hence restrict to our case. The situation is for example given if we have the inclusion of a submonoid into a monoid, which will be our main example.

The next definition provides a simplicial object which is given in each degree by what one may call a “fat wedge”. See the remark below for an explicit description in low degrees. See the definition after that for a description of maps of operation situations. This definition essentially goes back to Waldhausen.

**Definition 3.9 (Generalized Wedge).** Let $H \vee M$ an operation situation. We define the generalized wedge as the following simplicial object. Fix a degree $p$. For each integer $i$ with $1 \leq i \leq p$ we can form the product of $p$ factors

$$\bigvee_{i}^{p}(H \vee M) = H \times \cdots \times H_{i-1} \times M \times H_{i+1} \times \cdots \times H_p.$$  

We have the usual “bar-construction style” boundary maps given by forgetting, multiplication in $H$ or left/right operation on $M$ (cf. [3,2]) which gives us maps

$$\bigvee_{i}^{p}(H \vee M) \rightarrow \bigvee_{i}^{p-1}(H \vee M) \cup \bigvee_{i-1}^{p-1}(H \vee M).$$

By setting $\bigvee^{p}(H \vee M) := \bigcup_{i}^{p}(H \vee M)$ the structure maps give a simplicial object which we call generalized wedge. We denote it by $\bigvee(H \vee M)$. 

Definition 3.13. partial monoids. generalized wedge definition. It is not clear if a corresponding lemma holds for the proof of [Wal79, 2.2.1], which corresponds to our Lemma 3.14 below, needs the technically results to handle generalized wedges. So a map of operations gives a map of generalized wedges in a natural way. We then claims that the degree $n$ on the nerve of the partial monoid, which by definition is the set of composable $n$-tuples, is the degree $n$ of a generalized wedge.

This is not true in general. For example, assume that $M$ has a zero “0”, i.e., $0 \cdot x = 0 = x \cdot 0$, and assume $0 \in A$. Then $(x, 0, z) \in M \times M \times M$ is always composable, but for $x, z \in M \setminus A$ the tuple $(x, 0, y)$ is neither in $M \times A \times A$ nor in $A \times A \times M$, hence not in the generalized wedge. Waldhausen actually uses the generalized wedge definition in the course of his paper. Note that the proof of [Wal79 2.2.1], which corresponds to our Lemma 3.14 below, needs the generalized wedge definition. It is not clear if a corresponding lemma holds for partial monoids.

Remark 3.10. We thus have $\mathcal{V}^0(H \times M) = \ast$, $\mathcal{V}^1(H \times M) = M$, $\mathcal{V}^2(H \times M) = M \times H \cup H \times M$ and so on. Note that this construction takes a operation situation of simplicial sets to a bisimplicial set.

Example 3.11. (1) If we take $H = M$ to be a monoid or a group we get the usual bar construction or the nerve of $H$, see 3.2

(2) If we take $H = \ast$ and $M$ any pointed simplicial set we get $\mathcal{V}^p(*) \times M = \ast \times A \times \ast \times M$, which may explain the name. This bisimplicial set is isomorphic to the “external smash product” $(S^1 \wedge M)_{k,l} = S^1 \wedge M_{k,l}$. Therefore the diagonalization of that bisimplicial set is equivalent to the suspension of $M$. This uses that our model for the simplicial circle $S^1$ is $\Delta^1/\partial \Delta^1$.

Remark 3.12. In [Wal79 2.2] Waldhausen recalls Segal’s notion of a partial monoid and its nerve. He then considers the case of a monoid $M$ and a submonoid $A$ and makes it into a partial monoid by declaring $x, y \in M$ to be composable if and only if one of them is in $A$. He then claims that the degree $n$ on the nerve of the partial monoid, which by definition is the set of composable $n$-tuples, is the degree $n$ of a generalized wedge.

This is not true in general. For example, assume that $M$ has a zero “0”, i.e., $0 \cdot x = 0 = x \cdot 0$, and assume $0 \in A$. Then $(x, 0, z) \in M \times M \times M$ is always composable, but for $x, z \in M \setminus A$ the tuple $(x, 0, y)$ is neither in $M \times A \times A$, $A \times M \times A$ nor in $A \times A \times M$, hence not in the generalized wedge. Waldhausen actually uses the generalized wedge definition in the course of his paper. Note that the proof of [Wal79 2.2.1], which corresponds to our Lemma 3.14 below, needs the generalized wedge definition. It is not clear if a corresponding lemma holds for partial monoids.

Definition 3.13 (Map of operations). Given two operation situations $H \times M$ and $G \times N$. A map $(f_h, f_m): H \times M \to G \times N$ from the first to the second is a pair of maps $f_h: H \to G$, $f_m: M \to N$ such that $f_h$ is a map of monoids, $(f_h, f_m)$ is a map of left and right operations, and $f_h$, $f_m$ are compatible with the inclusions.

So a map of operations gives a map of generalized wedges in a natural way. We will prove a connectedness result on this map and create a new operation situation out of old ones in the next section.

3.4. Some Results on Generalized Wedges. In this section we discuss some technical results to handle generalized wedges.

Lemma 3.14. Let $(f_h, f_m): H \times M \to H' \times M'$ be a map of operations. Assume that $f_h$ is an inclusion. If $f_h$ is k-connected and $f_m$ is l-connected, then the resulting map of bisimplicial sets $\mathcal{V}(H \times M) \to \mathcal{V}(H' \times M')$ is $n$-connected with $n = \min(2k + 1, l)$.

Proof. We can factor the map as

$$\mathcal{V}(H \times M) \xrightarrow{\mathcal{V}(id_H \times f_m)} \mathcal{V}(H \times M') \xrightarrow{\mathcal{V}(f_h \times id_M)} \mathcal{V}(H' \times M')$$

since $H$ operates naturally on $M'$ via $H \to H'$. The assumption for $f_h$ being an inclusion implies that this operation $H \times M'$ is still an operation situation (in the sense of Definition 3.7). We analyse the maps $\mathcal{V}(id_H \times f_m)$ and $\mathcal{V}(f_h \times id_M)$, which are maps of simplicial sets. By the Realization Lemma 2.10 it suffices to
show \( n \)-connectedness in for every \( p \) separately, so we can forget the operation and multiplication maps in the following. We will use that an \( n \)-connected map can be written as an inclusion which is the identity on the \( n \)-skeleton, cf. Lemma \([\text{S2}]\) in the appendix.

The map \( \bigvee^p(H \uplus M) \to \bigvee^p(H, M') \) is an isomorphism for \( p = 0 \) and it is the map \( f_m: M \to M' \) for \( p = 1 \). Hence it is \( l \)-connected there. For larger \( p \) (and after realization) we can assume that \( M' \) is built from \( M \) by attaching only cells of dimension \( l + 1 \) or higher. As \( \bigvee^p(H \uplus M) \to \bigvee^p(H \uplus M') \) is just a product of \( f_m \) with identity maps, the newly attached cells are still of high dimension and this remains true after the union over \( i \). Hence the map

\[
\bigvee^p(\text{id}_H \uplus f_m): \bigvee^p(H \uplus M) \to \bigvee^p(H \uplus M')
\]

is \( l \)-connected.

The map \( \bigvee^p(H \uplus M') \to \bigvee^p(H' \uplus M') \) is the identity for \( p \) equals 0 and 1. We can assume that \( H' \) is built from \( H \) by attaching cells of dimension greater than or equal to \( k + 1 \) if we allow modification of \( M \), cf. Lemma \([\text{S2}]\). We then look at the inclusion for \( p = 2 \) which is

\[
M' \times H \cup H \times M' \to M' \times H' \cup H' \times M'.
\]

Thus the first additional cells are in dimension \((k + 1) + (k + 1)\), thus this map is \( 2k + 1 \)-connected. An analogous argument shows that the map for \( p = 3 \) is even \( 3k + 2 \)-connected and so on.

3.4.1. The Semi-Direct Product. We will now define what we will call the semi-direct product after Waldhausen \([\text{Wal79}]\). It is a way to construct a new operation situation out of an old one. It takes an operation situation \( H \uplus M \) on which a second monoid \( G \) operates to a new operation situation \( G \times (H \uplus M) = (G \times H) \uplus (G \times M) \). For that we need to define what the semi-direct product of monoids should be first.

**Definition 3.15** (Waldhausen). Let \( H \) and \( G \) be monoids, let \( G \) operate on \( H \) from both sides and compatibly, and let the operation be compatible with the monoid structure (“an operation in the category of monoids”). Then the semi-direct product of \( G \) and \( H \) is the monoid \( G \rtimes H \) with underlying set \( G \times H \) and multiplication

\[
(g, h) \cdot (g', h') = (gg', (h \cdot g') \cdot (g \cdot h')).
\]

**Remark 3.16.** See the remark after Lemma 2.2.1 of \([\text{Wal79}]\) why this is called the semi-direct product and why it is the common notion in the case that \( G \) is a group.

The compatibility with multiplication of \( H \) is in fact needed for the associativity of the new monoid, which is left to the reader.

**Remark 3.17.** What we really want is to construct a new operation situation out of an old one. So suppose we have an operation situation \( H \uplus M \) and let \( G \) operate on that from both sides and compatibly. Let us enumerate what that means:

1. \( G \) operates on the monoid \( H \) from both sides and compatibly. This especially means the operation is distributive with respect to the multiplication of \( H \), i.e. \( g \cdot (hh') = (g \cdot h)(g \cdot h') \). Also \( g \cdot 1_H = 1_H \).
2. \( G \) operates on \( M \) from both sides and compatibly.
3. The operation of \( G \) on \( M \) is compatible with the operation of \( H \) on \( M \), that is \( g \cdot (h \cdot m) = (g \cdot h) \cdot (g \cdot m) \) and so on. (“\( G \) operates distributively.”)

We can now define what we mean by the semi-direct product of \( G \) with \( H \uplus M \).
Definition 3.18. The semi-direct product of $G$ operating on $H \triangleright M$ is the operation situation $(G \times H) \triangleright (G \times M)$. The operations are given by
\[
(g, h) \cdot (\hat{g}, m) = (g\hat{g}, (h \cdot \hat{g}) \cdot (g \cdot m)) \quad \text{and} \\
(\hat{g}, m) \cdot (g, h) = (\hat{g}g, (m \cdot g) \cdot (\hat{g} \cdot h))
\]
respectively. It is denoted by $G \ltimes (H \triangleright M)$.

We leave the straightforward but lengthy proof that these are indeed compatible operations to the reader.

Remark 3.19. In our main application the situation will be much simpler, as we will have $H = \ast$ to be the trivial group. Then $G$ operates on $\ast \triangleright M = M$ and the semi-direct product is the operation situation $G \triangleright (G \times M)$. The formulas above reduce to
\[
g \cdot (\hat{g}, m) = (g\hat{g}, (g \cdot m)) \\
(\hat{g}, m) \cdot g = (\hat{g}g, (m \cdot g)).
\]

3.5. The Cyclic Bar Construction. We need another method to construct a simplicial object out of a monoid operating on an object from both sides and compatibly. This construction is the cyclic bar construction. As usual we are mainly interested in the case when a simplicial monoid acts on a simplicial set or sometimes even a bisimplicial monoid acts on a bisimplicial set. So here is the definition.

Definition 3.20 (Cyclic bar construction, [Wal79, 2.3]). Let $G$ be a monoid which acts on a set $X$ from both sides and compatibly. The cyclic bar construction $N^\text{cy}(G, X)$ is the simplicial set

\[
[k] \mapsto G \times \ldots \times G \times X
\]

with face maps
\[
d_0(g_1, \ldots, g_k, x) = (g_2, \ldots, g_k, g_1x) \\
d_i(g_1, \ldots, g_k, x) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_k, x) \\
d_k(g_1, \ldots, g_k, x) = (g_1, \ldots, g_{k-1}, g_kx)
\]

and degeneracy maps the insertion of the unit
\[
s_1(g_1, \ldots, g_k, x) = (g_1, \ldots, g_{i-1}, 1_F, g_i, \ldots, g_k, x).
\]

Remark 3.21. We have written the definition in element notation, although the definition is sensible and intended for any category. Call the left and right operations $l$ and $r$ respectively and the multiplication $\mu$. Then in a more precise map-notation the face maps are
\[
d_0 = (\text{pr}_2, \ldots, \text{pr}_k, r \circ (\text{pr}_X, \text{pr}_1)) \\
d_i = (\text{pr}_1, \ldots, \mu \circ (\text{pr}_1, \text{pr}_{i+1}), \ldots, \text{pr}_k, \text{pr}_X) \\
d_k = (\text{pr}_1, \ldots, \text{pr}_{k-1}, l \circ (\text{pr}_k, \text{pr}_X))
\]

and similarly for the degeneracy maps. We will continue to use elements.

Remark 3.22. The cyclic bar construction contains other bar constructions as a special case. For example one obtains the two-sided bar construction by taking the product of the two spaces on which the monoid now operates from both sides.
Remark 3.23. We will use that definition in the case of (bi-)simplicial sets in which the formulas of the definition – of course – evaluate to exactly the same formulas in each (bi-)degree.

We have a lemma similar to that for the generalized wedge construction (Lemma 3.14). Let $X, Y, G$ be simplicial sets or spaces and let additionally $G$ be a monoid. Let $G$ operate on $X$ and $Y$ from both sides and compatibly and let $f : X \to Y$ be a map which is compatible with the operations. Then the cyclic bar constructions $N^\text{cy}(G, X)$ and $N^\text{cy}(G, Y)$ are defined and $f$ gives a map of these. The connectivity of $f$ then transfers to the cyclic bar constructions:

**Lemma 3.24.** Let $f : X \to Y$ a $k$-connected map. Then the induced map of the cyclic bar constructions $N^\text{cy}(G, X) \to N^\text{cy}(G, Y)$ is also $k$-connected.

**Proof.** By the Realization Lemma 2.10 it is enough to show that in each degree. But this is clear since these are the maps

\[ X \xrightarrow{f} Y, \quad G \times X \xrightarrow{id \times f} G \times Y, \quad \ldots. \]

\[ 3.6. \text{ A Comparison Lemma.} \] We now compare the cyclic bar construction to the generalized wedge construction. Before stating the lemma we remark that an operation on a operation situation carries over to an operation on the generalized wedge of that operation situation:

**Remark 3.25.** Let $G$ operate on the operation situation $H \vee M$ from both sides and compatibly. What that means is unwrapped in 3.17. Then there is a canonical operation of $G$ on $\vee(H \vee M)$: The operation in degree $p$ is given by operation on each factor. Inspecting 3.17 shows that this operation is compatible with the structure maps of the generalized wedge. So the cyclic bar construction of $G$ and $\vee(H \vee M)$ is defined. It is a trisimplicial set. We diagonalize the two directions which come from the generalized wedge and from the cyclic bar construction. That is, we consider

\[ (\text{diag} N^\text{cy}(G, \vee(H \vee M)))_{pq} := N^\text{cy}_p(G_q, \vee_p(H_q \vee M_q)). \]

That leaves the original simplicial direction unaffected.

So the following lemma makes sense.

**Lemma 3.26** (Adaption of [Wal79, 2.3.1]). Let $G$ be a simplicial monoid acting on the operation situation $H \vee M$ from both sides and compatibly. Then there is a map of bisimplicial sets

\[ u : \text{diag} N^\text{cy}(G, \vee(H \vee M)) \to \vee(G \ltimes (H \vee M)). \]

It is an isomorphism if $G$ acts invertibly. If $\pi_0G$ is a group then it is a weak equivalence.

Let $H \vee M$ and $H' \vee M'$ be operation situations on which the monoids $G$, resp. $G'$, act. Given

\[ a : G \to G' \quad f : H \vee M \to H' \vee M' \]

where $a$ is a map of monoids and $f$ is a map of operation situations which is compatible with the $G$- and $G'$-action via $a$. Then $u$ is natural with respect to $(a, f)$. 

Remark 3.27. We will suppress the internal simplicial direction in the following discussion as it is not interesting for the arguments.

Let us start with some remarks on the objects involved before we turn to the proof. We will see that the underlying sets of the two bisimplicial objects are the same. Therefore only the structure maps can cause trouble. So let us write out both simplicial objects. Remember that \( \sqrt[k]{H \uplus M} \) equals \( \bigcup_i \sqrt[k]{i}(H \uplus M) \) where \( \sqrt[k]{i}(H \uplus M) = H \times \cdots \times M \cdots \times H \). So in degree \( k \) of \( \sqrt[k]{H \uplus M} \) we have

\[
G \times \overset{k}{\cdots} \times G \times \sqrt[k]{H \uplus M}
\]

and face maps

\[
d_0(g_1, \ldots, g_k; h_1, \ldots, h_k) = (g_2, \ldots, g_k; h_2g_1, \ldots, h_i g_1, \ldots, h_k g_1)
\]

\[
d_j(g_1, \ldots, g_k; h_1, \ldots, h_i, \ldots, h_k) = (g_1, \ldots, g_j g_{j+1}, \ldots, g_k; h_1, \ldots, h_j \cdot h_{j+1}, \ldots, h_k)
\]

\[
d_k(g_1, \ldots, g_k; h_1, \ldots, h_i, \ldots, h_k) = (g_1, \ldots, g_{k-1}; g_k h_1, \ldots, g_k h_i, \ldots, g_k h_{k-1}).
\]

There is an abuse of notation. As the generalized wedge is a union, the map is defined by maps on each part \( i \); these are defined by assuming the element \( h_i \) actually lies in \( M \) instead of \( H \). As the inclusion of \( H \) into \( M \) is compatible with the operations when doing the generalized wedge, the formulas above work fine. So for example \( h_i \cdot h_{i+1} \) unambiguously means either multiplication in \( H \) or operation of \( H \) on \( M \) from the left or the right, respectively.

On the other hand, degree \( k \) of \( \sqrt[k]{H \uplus M} \) resolves to

\[
\bigcup_i \sqrt[k]{i}(G \times (H \uplus M))
\]

where

\[
\sqrt[k]{i}(G \times (H \uplus M)) = G \times H \times \cdots \times G \times M \times \cdots M \times G \times H
\]

with face maps

\[
d_0(g_1, h_1; \ldots; g_i, h_i; \ldots; g_k, h_k) = (g_2, h_2; \ldots; g_i, h_i; \ldots; g_k, h_k)
\]

\[
d_i(g_1, h_1; \ldots; g_i, h_i; \ldots; g_k, h_k) =
\]

\[
d_k(g_1, h_1; \ldots; g_i, h_i; \ldots; g_k, h_k) = (g_1, h_1; \ldots; g_i, h_i; \ldots; g_{k-1}, h_{k-1}).
\]

As before we abuse notation, as “\( \cdots \)” may mean multiplication in \( H \) or operation on \( M \) from the left or right. One can read that as explained before. We can now produce the map and tell which conditions it must fulfill. This involves quite a lot of lengthy formulas, the reader is advised to consult the last section of the proof and especially equation (10) to see how they are constructed out of simpler maps.

Proof of Lemma 3.26. We define the map \( u \) from the first object to the second as the map

\[
u_n(g_1, \ldots, g_n; h_1, \ldots, h_n) = \]

\[
(g_1, g_1 \cdot g_1 h_1 g_1; g_2, g_2 \cdot g_2 h_2 g_2; \ldots; g_n, g_n h_n g_1 \cdot \cdots \cdot g_n).
\]

It is a map of simplicial sets. We factor \( u \) into two maps which are themselves maps of simplicial sets. In each degree we factor each of these maps further into maps which are easily seen to be isomorphisms (respectively weak equivalences). To
simplify the description of the maps we reorder the factors of the target to match
the ordering of the source (as the underlying sets of the two spaces are canonically
isomorphic). In this description \( u \) is written as:

\[
u_n(g_1, \ldots, g_n; h_1, \ldots, h_n) = (g_1, g_2, \ldots, g_n; g_1 \cdots g_nh_1g_1, g_2 \cdots g_nh_2g_1g_2, \ldots, g_nh_ng_1 \cdots g_n)
\]

Note that \( u \) is natural with respect to maps of operations.

**Factoring \( u \) into \( v \) and \( w \).** We construct an intermediate simplicial set \( T \). It has
the same underlying sets as before and has face maps

\[
d_0(g_1, \ldots, g_i, \ldots, g_k; h_1, \ldots, h_i, \ldots, h_k) = (g_2, \ldots, g_i, \ldots, g_k; h_2, \ldots, h_i, \ldots, h_k)
\]

\[
d_i(g_1, \ldots, g_i, \ldots, g_k; h_1, \ldots, h_i, \ldots, h_k) = (g_1, \ldots, g_{i+1}, \ldots, g_k; h_1, \ldots, h_{i+1}, \ldots, h_k)
\]

\[
d_k(g_1, \ldots, g_i, \ldots, g_k; h_1, \ldots, h_i, \ldots, h_k) = (g_1, \ldots, g_i, \ldots, g_{k-1}; gkh_1, \ldots, gh_i, \ldots, gh_{k-1}).
\]

One checks that the simplicial identities are satisfied. (We did not need to do
that for the other two objects since the description of the face maps were merely
calculations.) Then we can define a map \( v \) from \( \text{diag} \mathcal{N}^\text{cy}(G, \mathcal{V}(H \setminus M)) \) to \( T \) by

\[
v_n(g_1, \ldots, g_n; h_1, \ldots, h_n) = (g_1, \ldots, g_n; h_1g_1, h_2g_1g_2, \ldots, h_ng_1 \cdots g_n)
\]

and a corresponding map \( w \) from \( T \) to \( \mathcal{V}(G \ltimes (H \setminus M)) \) by

\[
w_n(g_1, \ldots, g_n; h_1, \ldots, h_n) = (g_1, \ldots, g_n; g_1 \cdots g_nh_1, g_2 \cdots g_nh_2, \ldots, g_nh_n).
\]

Here we reordered the factors of the last term as mentioned before.

One checks that \( v \) and \( w \) are maps of simplicial sets; this part is left to the
reader. It is obvious that the composition \( w \circ v \) is equal to \( u \).

Note that we could have defined the simplicial object \( T \) in another way for which
the maps \( v \) and \( w \) were interchanged and the multiplication by the face maps of \( T \)
were from the other side, so there is a kind of symmetry.

To show that \( v \) and \( w \) are isomorphisms (respectively weak equivalences) we
factor them further in each degree. To show that \( v \) and \( w \) are weak equivalences it
suffices to show that in each degree \( n \) the maps \( v_n \) and \( w_n \) are weak equivalences
because of the Realization Lemma.\(^{10} \)

We factor the maps \( v_n \) and \( w_n \) into \( v_n = r_1r_2 \cdots r_n \) and \( w_n = l_nl_{n-1} \cdots l_1 \) where
\( r_i \) is

\[
r_i(g_1, \ldots, g_n; h_1, \ldots, h_n) = (g_1, \ldots, g_n; h_1, \ldots, h_{i-1}, h_ig_i, \ldots, h_ng_i)
\]

and similarly for \( l_i \):

\[
l_i(g_1, \ldots, g_n; h_1, \ldots, h_n) = (g_1, \ldots, g_n; g_1h_1, \ldots, gh_i, h_{i+1}, \ldots, h_n).
\]

One can restrict \( r_i \) to the factors \( G_i \times \mathcal{V}^k(H \setminus M) \) and then \( r_i \) is the shear map
of a suitable operation of \( G = G_i \) on \( \mathcal{V}^k(H \setminus M) \). Namely it is the operation on the
last \( i \) factors on \( \mathcal{V}^k(H \setminus M) \). (We consider this in each degree so we do not need
any compatibility with face maps here which we actually do not have: in general
the $r_i$ are not maps of simplicial objects. Note that $\sqrt[k]{H}$ is just concise description of the underlying sets, it it not meant as part of a simplicial object.)

Therefore each $r_i$ and also each $l_i$ is a shear map after restriction to some factors. By Lemma 8.3 if $G$ acts invertibly they are isomorphisms; if $\pi_0 G$ is a group they are weak equivalences. As products commute with homotopy groups the unrestricted maps $r_i, l_i$ are also isomorphisms, resp. weak equivalences. As mentioned,

$$w_n = l_n \cdots l_1, \quad v_n = r_1 \cdots r_n \quad \text{and} \quad u_n = w_n v_n.$$  

So $u$ is an isomorphism if each $r_i, l_i$ is one. If each $r_i, l_i$ is a weak equivalence the Realization Lemma 2.10 implies that $u$ is a weak equivalence. \hfill \Box

4. The Comparison Theorem

We want to compare the Goodwillie differentials of the functor $A(-)$ and a functor $C(-)$ which we will define using the cyclic bar construction. As we will see in Section 7 the Goodwillie differential of $A(-)$ at a connected space $B$ can be calculated by looking at spaces of the form $Y \times L(B)$ where $L(B)$ is the Kan Loop Group of the space $B$ over which the differential is taken. Therefore we introduce a functor $A(Y, G)$, which is essentially $A(Y \times BG)$, and a functor $C(Y, G)$ to incorporate the base space $G$ as $G = L(B)$ into the comparison.

We give the definitions of $A(Y, G)$ and $C(Y, G)$ next. For the following let $G$ be a simplicial group. In the applications it is the realization of the Kan Loop Group, so we can and will assume it is a CW complex. Let $Y$ be a pointed simplicial set. In our applications we vary $Y$ such that its connectedness increases.

**Definition 4.1.** Let $Y, G$ be as above and $n, k \geq 0$. Define

$$A^k(Y, G) := \mathcal{N} H_{[G]}([L(Y)])(\sqrt[k]{S^n} \wedge |G| \prod |L(Y)|)$$

and

$$A(Y, G) := \mathbb{Z} \times \left( \colim_{n,k} A^k_n(Y, G) \right)^+,$$

where $-^+$ is the plus-construction. Note that $A^k_n(Y, G)$ is natural in $Y, G$ and with respect to $k, n$.

If $G = *$ the definition of $A(Y, G)$ agrees with our definition for $A(Y)$ from [1]. We use a slightly different notation here, which is more convenient for the following and follows [Wal79]. We let $H_G(-)$ denote the simplicial monoid of $G$-equivariant pointed weak self-equivalences, and let $\mathcal{N}$ the nerve of a monoid. Detailed explanations will be given at the beginning of Section 4.1. Similarly let $\text{Map}_G(-, -)$ denote the $G$-equivariant mapping space and $\mathcal{N}^{cy}$ the cyclic bar construction of Section 3.5. Define $C(Y, G)$ in analogy to (11) as follows.

**Definition 4.2.** Let $Y, G$ be as above and $n, k \geq 0$. Define

$$C^k_n(Y, G) := \mathcal{N}^{cy}(H_{[G]}(\sqrt[k]{S^n} \wedge |G|), \text{Map}_{[G]}(\sqrt[k]{S^n} \wedge |G|, \sqrt[k]{S^n} \wedge |G| \prod |Y|))$$

and

$$C(Y, G) := \left( \colim_{n,k} C^k_n(Y, G) \right)^+,$$

where $-^+$, as usual, denotes the plus-construction. Note that $C^k_n(Y, G)$ is natural in $Y, G$ and with respect to $k, n$. 


The main ingredient in the comparison of $A(Y,G)$ and $C(Y,G)$ is Theorem 4.3 below which will be used to provide a long chain of maps between $A(Y,G)$ and $C(Y,G)$ which is natural in $Y$ and $G$ and roughly twice as highly connected as $Y$. Therefore it will induce an equivalence of the Goodwillie Differentials of $A(−, G)$ and $C(−, G)$ at a point. We will prove:

**Theorem 4.3.** The Goodwillie Differentials at a point of $Y \mapsto A(Y,G)$ and $Y \mapsto C(Y,G)$ are weakly equivalent. In other words

$$A^S(Y,G) \simeq C^S(Y,G)$$

where we stabilize in $Y$.

In Section 4.1 we introduce some notation and state Theorem 4.4. The rest of this section will be concerned with technical preparations which are needed in the proof of this theorem. The actual chain of maps and its connectedness is described in Section 4.2 followed by a section on Theorem 4.5, the addendum to Theorem 4.4. In Section 4.4 we compare $A(−)$ and $C(−)$ and complete the proof that their differentials are weakly equivalent.

4.1. **Prerequisites and the Statement of the Theorem.** We collect notation and definitions first. For $X, Y$ pointed topological spaces we denote the simplicial mapping space of pointed maps by $\text{Map}(X,Y)$. This is the singular complex of the topological mapping space or more explicitly the simplicial set $[n] \mapsto \text{Hom}(X \wedge |\Delta^n|_+, Y)$.

Analogously, when $X$ and $Y$ are pointed $G$-spaces, for $G$ a topological group, we denote by $\text{Map}_G(X,Y)$ the simplicial mapping space of $G$-equivariant pointed maps. Note that for $G$ the trivial group we get the non-equivariant notion back.

By $H(Y)$ (respectively $H_G(Y)$) we denote the simplicial monoid of pointed weak self-equivalences (respectively $G$-equivariant pointed weak self-equivalences) of a topological space (respectively $G$-space) $Y$. These are maps which induce isomorphisms on all homotopy groups for each basepoint. We need that $H_G(Y)$ consists of connected components of $\text{Map}_G(Y,Y)$ (Lemma 8.5).

By $L$ we denote the realization of the Kan Loop group of the $(m+1)$-connected simplicial set $W$, which is an $m$-connected topological group. If we consider $L$ as a pointed space, we choose as basepoint the neutral element $1_L \in L$. As usual, $−_+$ denotes an added disjoint basepoint and $S^n \wedge −$ is the topological smash product with an $n$-sphere which represents $n$-fold suspension.

By $K$ we denote a finite free $G$-space. So $K = \Pi^k G = G \amalg \ldots \amalg G$. For the pointed version it is $K_+ = \vee^k G_+ = G_+ \vee \ldots \vee G_+$, the $k$-fold wedge of $G_+$.

By $N$ we denote the nerve-construction of a monoid which we discussed in section 3.2. By $N^{cy}$ we denote the cyclic bar construction we introduced in section 3.5. By $S^1_s$ we denote the simplicial 1-sphere $\Delta^1/\partial\Delta^1$. If it is clear from the context the subscript $s$, which distinguishes it from the topological 1-sphere, will be dropped.

The following Table 1 shows an overview of all the notation used in this section.

Now we can state the theorem.

**Theorem 4.4.** Let the notation be as defined above and let $L$, the realized Kan Loop Group of the simplicial set $W$, be $m$-connected, $m \geq -1$. There is a natural chain of maps between

$$N^\ast H_G \times L(S^n \wedge K_+ \wedge L_+)$$
pointed simplicial set, \((m+1)\)-connected, \(m \geq 0\).

realization of Kan Loop Group of \(W\), \(m\)-connected

topological group (which can be given a CW-structure)

\(H^k G\), free \(G\)-space

\(v^k G_+\), free pointed \(G\)-space

pointed weak \(G\)-equivalences \(Y \to Y\)

pointed \(G\)-equivariant maps \(Y \to Z\)

nerve of a monoid

cyclic bar construction of Section 3.5

topological \(n\)-sphere

simplicial 1-sphere, \(\Delta^1/\partial\Delta^1\)

directions in which to stabilize

| \(W\) | pointed simplicial set, \((m+1)\)-connected, \(m \geq 0\). |
| \(L\) | realization of Kan Loop Group of \(W\), \(m\)-connected |
| \(G\) | topological group (which can be given a CW-structure) |
| \(K\) | \(H^k G\), free \(G\)-space |
| \(K_+\) | \(v^k G_+\), free pointed \(G\)-space |
| \(H_G(Y)\) | pointed weak \(G\)-equivalences \(Y \to Y\) |
| \(Map_G(Y, Z)\) | pointed \(G\)-equivariant maps \(Y \to Z\) |
| \(N\) | nerve of a monoid |
| \(N^{cy}\) | cyclic bar construction of Section 3.5 |
| \(S^n\) | topological \(n\)-sphere |
| \(S^1_+\) | simplicial 1-sphere, \(\Delta^1/\partial\Delta^1\) |
| \(n, k\) | directions in which to stabilize |

**Table 1.** Notation used in this section

\[ N^{cy}(H_G(S^n \wedge K_+), Map_G(S^n \wedge K_+, S^n \wedge |W| \wedge K_+)) \]

and each map in the chain is a \(q\)-equivalence with

\[ q = \min(n - 3, 2m). \]

This chain of maps is natural in \(W\), \(G\) and with respect to \(k\) and \(n\).

We have some further properties of this chain, which eventually describes the Barratt-Priddy-Quillen-Segal map.

**Theorem 4.5** (Addendum to Theorem 4.4). Furthermore, for \(k = 1\), \(n = 0\), the chain of Theorem 4.4 specializes to a chain between

\[ N(G \times L) \quad \text{and} \quad N^{cy}(G, G_+ \wedge |W|) \]

which induces a weak map in the \(2m\)-connected range (see below). Increasing \(k\) and \(n\) gives a natural transformation to the chain of Theorem 4.4.

In case \(G\) is the trivial group the induced weak map is indeed a \(2m\)-equivalence and is essentially the map

\[ N(L(W)) \leftarrow \Sigma \Omega W \to W. \]

**Remark 4.6.** By “induces a weak map in the \(2m\)-connected range” we mean the following: There is a chain of maps which are not necessary all high (i.e., \(2m\))-equivalences but the maps in the wrong direction are. Thus after stabilizing with respect to \(m\) this chain gives a genuine map (at least in the homotopy category).

We will prove the theorem in Section 4.2 and its addendum in Section 4.3 for the addendum part we can just apply the arguments of the first part in a special case. The stabilization maps are described in Remark 4.24.

We need as intermediate construction the generalized wedge construction of Section 3.3. We will first provide some preparatory lemmas which we need in the course of the proof. Unfortunately the proof of Theorem 4.4 is quite technical so some patience of the reader is required; we therefore split it into several parts which are as independent as possible. As a guide we provide a large diagram in Figure 1 which depicts the maps we will use. Each of the maps will be described in its own subsection.
\[ |N_{\mathcal{G} \times L}(S^n \wedge (K \times L)_+)| \]

\[ |\mathcal{V}(H_G(S^n \wedge K_+) \mathcal{V} H_{\mathcal{G} \times L}(S^n \wedge (K \times L)_+))| \]

\[ |\mathcal{V}(H_G(S^n \wedge K_+) \mathcal{V} \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L \wedge S^n \wedge K_+))| \]

\[ \cong \]

\[ |\mathcal{V}(H_G(S^n \wedge K_+) \mathcal{V} \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L) \times H_G(S^n \wedge K_+))| \]

\[ \cong \]

\[ |\text{diag} \mathcal{N}^{\text{cy}}(H_G(S^n \wedge K_+), \mathcal{V}(\ast \mathcal{V} \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)))| \]

\[ \cong \]

\[ |\mathcal{N}^{\text{cy}}(H_G(S^n \wedge K_+), \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L))| \]

\[ \cong \]

\[ |\mathcal{N}^{\text{cy}}(H_G(S^n \wedge K_+), \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge |W|))| \]

**Figure 1.** The (realized) maps providing the equivalence of Theorem 4.4. The decorations of the arrows indicate the connectedness.

The following two lemmas will be needed for the fifth map. As they are quite technical they are stated and proved here. They describe as a special case how taking the loop-space and the suspension commute.

The first lemma will provide us with a map and calculate its connectivity in the very special case of a sphere, the second lemma will allow us to extend the result a little bit more, namely to finite wedges of spheres.

**Lemma 4.7.** Let \( A, Z \) be pointed topological spaces. There is a natural map

\[ S^1 \wedge \text{Map}(A, Z) \to \text{Map}(A, S^1 \wedge Z). \]

In case \( A = S^n \) and \( Z \) is \( m + n \)-connected, \( m \geq 0 \), this map is \((2m + 3)\)-connected.

**Proof.** Note that \( S^1 = |S^1_*| \). We will construct and analyse a map

\[ S^1 \wedge \text{Map}^!(A, Z) \to \text{Map}^!(A, S^1 \wedge Z), \]

where \( \text{Map}^! \) is the topological mapping space. (Recall that \( \text{Map} \) is the singular complex of \( \text{Map}^! \).) We describe how we can get the map \ref{eq:12} from \ref{eq:13}. Recall that the unit and counit of the adjunction singular complex \( S(-) \) and geometric realization \( |-| \) are weak equivalences, and that geometric realization commutes with smash products. The unit induces a map

\[ S^1_s \wedge \text{Map}(A, Z) \to S(|S^1_*| \wedge |\text{Map}(A, Z)|). \]
Now the counit gives a map \(|\text{Map}(A, Z)| \to \text{Map}^t(A, Z)|\) and hence a map
\[ S(|S^1| \wedge |\text{Map}(A, Z)|) \to S(|S^1| \wedge \text{Map}^t(A, Z)). \]

Both induced maps are weak equivalences. Now \(S(\cdot)\) applied to (13) gives the desired map and all connectivity properties transfer, as \(S(\cdot)\) preserves them.

There are several equivalent ways to describe (13). One can view \(\Sigma \Omega^n Z \to \Omega^n \Sigma Z\). Similarly one can first smash the map \(S^0 \to S^1\) with (the identity on) \(Z\) to obtain a map \(Z \to S^1 \wedge Z\) and then precompose with \(A \to Z\). Another way is to use the adjunction \(\cdot - \wedge A \rightleftharpoons \text{Map}^t(A, \cdot)\) and construct an adjoint map
\[ S^1 \wedge \text{Map}^t(A, Z) \wedge A \to S^1 \wedge Z \]
by the map which evaluates \(A\). We will work with this description of the map.

In case \(A = S^n\) the map (13) is a map \(\Sigma \Omega^n Z \to \Omega^n \Sigma Z\). A calculation which uses the adjoint description from above shows that this is the map
\[ \Sigma \Omega^n Z \overset{\text{coev}_{\Sigma^n}}{\longrightarrow} \Omega^n \Sigma^n (\Sigma \Omega^n Z) \overset{\text{twist}_{\Sigma^n}}{\longrightarrow} \Omega^n (\Sigma \Sigma^n \Omega^n Z) \overset{\text{ev}_{\Sigma^n}}{\longrightarrow} \Omega^n (\Sigma Z). \]

As the evaluation map \(\text{ev}_{\Sigma^n} : \Sigma \Omega^n T \to T\) is \((2l + 1)\)-connected for \(T\) being \(l\)-connected it follows that \(\text{ev}_{\Sigma^n} : \Sigma^n \Omega^n T \to T\) is \((2l - n + 2)\)-connected, hence the last map in (14) has connectedness \(2m + 3\). Similarly the coevaluation map \(\text{coev}_{\Sigma^n} : T \to \Omega^n T\) is \((2l + 1)\)-connected for \(T\) being \(l\)-connected and it follows that \(\text{coev}_{\Sigma^n}\) is \((2l + 1)\)-connected, too. Hence the first map in (14) is \(2(m + 1) + 1 = (2m + 3)\)-connected. Thus the whole map is \((2m + 3)\)-connected. \(\square\)

The next Lemma generalizes the one above to a finite wedge of spheres.

**Lemma 4.8.** Assume we have a map \(S^1_k \wedge \text{Map}(Y, B) \to \text{Map}(Y, S^1_k \wedge B)\) which is natural in \(Y\). Assume that for \(Y\) equal to \(A\) or \(A'\) this map is \(l\)-connected, that \(\text{Map}(A, B)\) and \(\text{Map}(A', B)\) are \(k\)-connected, and \(l \leq 2k + 1\). Then the map for \(Y = A \vee A'\)
\[ S^1_k \wedge \text{Map}(A \vee A', B) \to \text{Map}(A \vee A', S^1_k \wedge B) \]
is \(l\)-connected, too.

**Proof.** We have a cofiber sequence \(A \to A \vee A' \to A'\) which yields a fiber sequence of \(\text{Map}(-, S^1_k \wedge B)\) and hence a long exact sequence of homotopy groups
\[ \cdots \to \pi_i(\text{Map}(A', S^1_k \wedge B)) \to \pi_i(\text{Map}(A \vee A', S^1_k \wedge B)) \to \pi_i(\text{Map}(A, S^1_k \wedge B)) \to \pi_{i-1}(\text{Map}(A', S^1_k \wedge B)) \to \cdots. \]

As we have a retraction \(A \vee A' \to A\) and a coretraction \(A' \to A \vee A'\) this sequence is split exact.

We have a similar splitting sequence for \(\text{Map}(-, B)\). When we smash that sequence with \(S^1_k\) this may no longer be exact in all degrees but by the Freudenthal Suspension Theorem the corresponding sequence of homotopy groups \(\pi_i(S^1_k \wedge \text{Map}(-, B))\) is still a split exact sequence up to degree \(i \leq 2k + 1\). We then have the natural map from the second to the first sequence which is an isomorphism up to degree \(l\) on the outer terms by assumption. By the short five Lemma it is an isomorphism on the inner terms
\[ \pi_i(S^1_k \wedge \text{Map}(A \vee A', B)) \to \pi_i(\text{Map}(A \vee A', S^1_k \wedge B)) \]
for \(i \leq l\) and hence the map of the lemma is \(l\)-connected. \(\square\)
Corollary 4.9. Let $Z$ be $(n+m)$-connected and pointed, $m \geq 0$. There is a natural map

$$S^1 \wedge \text{Map}(\vee^k S^n, Z) \to \text{Map}(\vee^k S^n, S^1 \wedge Z)$$

which is $(2m+1)$-connected. □

4.2. The Chain of Maps. We now give the maps for Theorem 4.4, each in its own subsection. For all the following lemmas we will leave out the checking that the maps we give are in fact maps of operation situations, i.e., that they are compatible with the operation of $H_G(\ldots)$.

4.2.1. The First Map.

Lemma 4.10 (First Map). There is a map to

$$\mathcal{N}H_{G \times L}(S^n \wedge (K \times L)_+)$$

from

$$\vee (H_G(S^n \wedge K_+) \wedge H_{G \times L}(S^n \wedge (K \times L)_+))$$

which is $2m+1$-connected, hence $2m$-equivalent.

Proof. The map is given by the inclusion of monoids

$$(15) \quad H_G(S^n \wedge K_+) \to H_{G \times L}(S^n \wedge (K \times L)_+)$$

in view of $\mathcal{N}H_{G \times L}(S^n \wedge (K \times L)_+)$ being the same as

$$\vee (H_{G \times L}(S^n \wedge (K \times L)_+) \wedge H_{G \times L}(S^n \wedge (K \times L)_+)).$$

We show the map (15) is $m$-connected so the lemma follows by Lemma 3.14.

Lemma 4.11. Let $L$, $G$ be a topological groups. Let $L$ be $m$-connected, $m \geq 0$, $k \geq 1$. Then the map

$$H_G(\vee^k S^n \wedge G_+) \to H_{G \times L}(\vee^k S^n \wedge G_+ \wedge L_+)$$

given by $f \mapsto f \wedge \text{id}_{L_+}$ has a retraction and is $m$-connected.

Proof. The retraction is induced by the $L$-equivariant maps $L_+ \to S^0$ and the inclusion $S^0 \to L_+$. These come from the maps $1_L \to L \to 1_L$ by adding a basepoint.

We rewrite the maps using a non-equivariant mapping space.

The vertical maps are inclusions of components induced by adjunction, see Section 8.2. The lower map is induced by $S^0 \to L_+$ which is $m$-connected. By extracting the wedge out of the first component we can write it as

$$\Pi^k \Omega^n(\Sigma^n \vee^k G_+) \to \Pi^k \Omega^n(\Sigma^n \vee^k (G_+ \wedge L_+)).$$

We read off the connectivity: Taking coproduct does not change connectivity, taking $n$-fold suspension increases it to $m+n$. Taking the $n$-fold loopspace reduces connectivity again to $m$. Last, taking the $k$-fold product does not change the connectivity. Hence the lower map is $m$-connected.
As both vertical maps are inclusions of components we are done if we show that the upper map gives a bijection on components. We have to show surjectivity. Namely we have to show that each $G \times L$-equivariant weak equivalence $\hat{f}: \vee^k S^n \wedge G_+ \wedge L_+ \to \vee^k S^n \wedge G_+ \wedge L_+$ is homotopic to a map of the form $f \wedge \text{id}_{L_+}$ with $f$ a weak equivalence. As the vertical map respects homotopies, going down and using the established $0$-connectivity of the lower map shows indeed that there is an $f$ in $H_G(\vee^k S^n \wedge G_+)$ such that $\hat{f}$ is homotopic to $f \wedge \text{id}_{L_+}$. We have to show that $f$ is a weak equivalence. But this is clear since it is a retraction of one. \hfill \Box

Remark 4.12. Note that usually $f \wedge \text{id}$ being a (weak) homotopy equivalence does not imply that $f$ is one, the most prominent example being iterated suspension.

This finishes the proof of Lemma 4.10. \hfill \Box

4.2.2. The Second Map. Similar to what we already used in the proof of subsec-
tion 4.2.1 we can rewrite $H_G \times L(S^n \wedge K_+ \wedge L_+)$ as a union of connected components of $\text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L_+)$. We denote these connected components by $\overline{\text{Map}}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L_+)$.  

Lemma 4.13 (Second Map). There is a map from
$$\bigvee (H_G(S^n \wedge K_+) \vee \overline{\text{Map}}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L_+))$$
to
$$\bigvee (H_G(S^n \wedge K_+) \vee \overline{\text{Map}}_G(S^n \wedge K_+, (S^n \wedge K_+ \wedge L) \times (S^n \wedge K_+)))$$
which is $n-2$-connected. (Here the product in the last mapping space is a $G$-space by the diagonal action.)

Remark 4.14. The $\overline{\text{Map}}_G(\ldots)$ of the target denotes a suitable set of connected components. We do not specify them explicitly here, as we will examine it when we deal with the next map. It all amounts to a $\pi_0$-phenomenon which we may temporarily ignore.

Proof of Lemma 4.13. The map is induced by the map
$$S^n \wedge K_+ \wedge L_+ \to (S^n \wedge K_+ \wedge L) \times (S^n \wedge K_+)$$
which is itself induced by the two maps $L_+ \to S^n$, $L_+ \to L$. We will show that the map
$$\Omega^n \Sigma^n (K_+ \wedge L_+) \to \Omega^n \Sigma^n (K_+ \wedge L) \times \Omega^n \Sigma^n (K_+)$$
is $(n-2)$-connected and that therefore the map
$$\overline{\text{Map}}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L_+) \to \overline{\text{Map}}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L \times S^n \wedge K_+)$$
is also $(n-2)$-connected. Now the result follows from Lemma 4.11.

The connectedness of the map of mapping spaces is shown by first rewriting them to non-equivariant mapping spaces and then looking at them as loop spaces of suspensions. Similar to what is done in the proof of Lemma 4.11 we get a map
$$\Pi^k(\Omega^n \Sigma^n (K_+ \wedge L_+)) \to \Pi^k(\Omega^n \Sigma^n (K_+ \wedge L) \times \Omega^n \Sigma^n (K_+))$$.

The map we are interested in arises as a restriction to components. The restriction on the first term is given, we choose the connected components of the second term such that the map becomes a $\pi_0$-equivalence. (This determines what the term $\overline{\text{Map}}_G(\ldots)$ of the target means.) As taking the product does not change the connectivity, the next lemma finishes the proof of 4.13.
Lemma 4.15. Let Z be a pointed CW-complex, L be a pointed topological space. Then the map
\[ \Omega^n \Sigma^n (Z \wedge L) \rightarrow \Omega^n \Sigma^n (Z \wedge L) \times \Omega^n \Sigma^n (Z) \]
which is induced by \( L_+ \rightarrow L, L_+ \rightarrow S^0 \) is an \((n-2)\)-equivalence. (Here \( L_+ \) is pointed by the new point.)

Proof. The sequence
\[ S^0 \rightarrow L_+ \rightarrow L \]
is a cofiber sequence since the mapping cone of the first map is \( L \cup_+ I \) which is homotopy equivalent to \( L \). Similarly, the sequence
\[ Z \wedge S^0 \rightarrow Z \wedge L_+ \rightarrow Z \wedge L \]
is a cofiber sequence. (A fancy argument for this is that compactly generated pointed topological spaces with smash product form a monoidal model category. As \( Z \) is a CW-complex, hence cofibrant, the functor \( Z \wedge - \) is a Left Quillen Functor, hence it respects cofibrations.) Iterated suspension of the sequence gives again a cofiber sequence.

As stable homotopy is a homology theory we get a long exact sequence of stable homotopy groups which splits into split short exact sequences induced by the maps
\[ L_+ \rightarrow S^0, L \rightarrow L_+ \]. Hence
\[ \pi_i^s (\Sigma^n Z \wedge L_+) \cong \pi_i^s (\Sigma^n Z) \oplus \pi_i^s (\Sigma^n Z \wedge L). \]
The Freudenthal Suspension Theorem states that \( \pi_i (\Sigma^n Y) \rightarrow \pi_i^s (\Sigma^n Y) \) is an isomorphism for \( i \leq 2n - 2 \). Therefore we have for these \( i \) isomorphisms of unstable homotopy groups
\[ \pi_i (\Sigma^n (Z \wedge L_+)) \cong \pi_i (\Sigma^n (Z)) \oplus \pi_i (\Sigma^n (Z \wedge L)) = \pi_i (\Sigma^n (Z) \times \Sigma^n (Z \wedge L)). \]

As taking \( n \)-fold loop spaces commutes with products we get the desired isomorphism of homotopy groups up to degree \( i = n - 2 \). \( \square \)

This finishes the proof of Lemma 4.13. \( \square \)

4.2.3. The Third Map.

Lemma 4.16 (Third Map). There is a map from
\[ \bigvee (H_G(S^n \wedge K_+) \map T \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L \times S^n \wedge K_+)) \]
to
\[ \bigvee (H_G(S^n \wedge K_+) \map T \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L) \times H_G(S^n \wedge K_+)) \]
which is an isomorphism if \( L \) is connected.

Remark 4.17. The main point in the Lemma is to check the \( \pi_0 \)-condition we ignored so far. That is, we have to check that both mapping spaces \( \text{Map}_G(\ldots) \) and \( \text{Map}_G(\ldots) \times H_G(\ldots) \) give us the same components. Note that \( H_G(\ldots) \) acts from the left and the right on the product by the diagonal action.

Proof. The map is induced by the two projections
\[ S^n \wedge K_+ \wedge L \times S^n \wedge K_+ \rightarrow S^n \wedge K_+ \wedge L \]
\[ S^n \wedge K_+ \wedge L \times S^n \wedge K_+ \rightarrow S^n \wedge K_+ \]
which induce an isomorphism on \((G\text{-equivariant})\) mapping spaces. We claim we have an isomorphism
\[
\Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L \times S^n \wedge K_+) \cong \\
\Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L) \times H_G(S^n \wedge K_+).
\]
We have to show that restricting to connected components is possible. Therefore we have to track the choice of components we made for the previous maps. We have to check that a map in
\[
H_{G \times L}(S^n \wedge (K \times L)_+)
\]
is mapped to a map with second factor in
\[
H_G(S^n \wedge K_+)
\]
and such such map has a preimage.

We are treating a \(\pi_0\)-phenomenon so we can freely replace all maps by homotopic ones. Every map in \(H_{G \times L}(S^n \wedge (K \times L)_+)\) is homotopic to a map from \(H_G(S^n \wedge K_+)\), as we have seen in Lemma \(4.11\). That means, it is of the form \(f \wedge \id_{L_+}\) with \(f\) a weak equivalence in \(H_G(S^n \wedge K_+)\). Let us see what happens with this special kind of map through the course of the lemmas: First it is rewritten to \(f \wedge i_+\), where \(i_+: S^0 \to L_+\) is the map from the topological 0-sphere \([-1, 1]\) to \(L_+\) which maps the basepoint \(1 \in S^0\) to the new basepoint \(+ \in L_+\) and the non-basepoint \(-1 \in S^0\) to the old basepoint \(1_L\) of \(L \subseteq L_+\). Then \(f \wedge i_+\) is mapped to the pair of maps
\[
f \wedge 0_L, \quad f.
\]
Here \(0_L: S^0 \to L\) is the constant map to the basepoint \(1_L \in L\). (It follows that the first map is the constant map to the basepoint.)

As \(L\) is connected it follows that \(\Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)\) is connected, too, and so any two maps in it are homotopic. This shows we have our desired bijection of components and finishes the proof of Lemma \(4.16\).

4.2.4. The Fourth Map. The target of the third map is the generalized wedge of the operation situation
\[
H_G(S^n \wedge K_+) \vee (\Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L) \times H_G(S^n \wedge K_+)).
\]
This operation situation can be constructed as the semi-direct product (cf. Definition \(3.18\) of \(H_G(S^n \wedge K_+)\) and \(* \vee Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)\). Thus the target of the map of Lemma \(4.16\) can be identified with
\[
\vee(H_G(S^n \wedge K_+) \times (* \vee Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L))).
\]

Lemma 4.18 (Fourth Map). There is a map to
\[
\vee(H_G(S^n \wedge K_+) \times (* \vee Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)))
\]
from
\[
\text{diag} \, N^\vee(H_G(S^n \wedge K_+) \vee(* \vee Map_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)))
\]
which is a weak equivalence.

Recall from Remark \(3.26\) that the notation “\text{diag}” means that we diagonalizes the direction of the cyclic bar construction and the direction of the generalized wedge.

Proof. In view of \(\pi_0H_G(S^n \wedge K_+)\) being a group this is exactly Lemma \(3.26\) \(\Box\).
4.2.5. The Fifth Map.

Remark 4.19. As remarked before (Example 3.11)

\[ \bigvee(\ast \downarrow \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)) \]

is a bisimplicial set and its diagonalization is equivalent to the suspension of the mapping space.

**Lemma 4.20** (Fifth Map). There is a map from

\[ |N^c\gamma(H_G(S^n \wedge K_+), \bigvee(\ast \downarrow \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L)))| \]

to

\[ |N^c\gamma(H_G(S^n \wedge K_+), \text{Map}_G(S^n \wedge K_+, S^1 \wedge S^n \wedge K_+ \wedge L))| \]

which is \(2m + 1\)-connected. Here the bars indicate that we realize the trisimplicial sets there. This should mean diagonalization to a simplicial set.

The realization of the diagonalization is the same as the (iterated) topological realization of the trisimplicial set, so does not matter at which one we look (cf. [GJ99, Exercise IV.1.4]). Having a map of simplicial sets may be convenient, but to calculate connectedness we have to realize anyway.

As we remarked above (Remark 4.19) the generalized wedge is just the suspension of the mapping space. (As we made a diagonalization this is a genuine suspension, as we can realize everything to topological spaces and as we can realize the inner terms before performing the cyclic bar construction we can even view this as a suspension of the topological mapping space.)

We claim:

**Lemma 4.21.** There is a map

\[ S^1_s \wedge \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge L) \to \text{Map}_G(S^n \wedge K_+, S^n \wedge S^1 \wedge K_+ \wedge L) \]

which is \(2m + 1\)-connected.

**Proof of Lemma 4.20 assuming Lemma 4.21.** The map of Lemma 4.21 induces a map of cyclic bar constructions and the connectedness carries over to this map by Lemma 3.24. \(\square\)

**Proof of Lemma 4.21.** We need some effort which is contained in the Lemmas 4.7 and 4.8. As before we can pass over to non-equivariant mapping spaces. We thus have to produce and examine a map

\[ S^1_s \wedge \text{Map}(\vee^k S^n, S^n \wedge \vee^k G_+ \wedge L) \to \text{Map}(\vee^k S^n, S^1 \wedge S^n \wedge \vee^k G_+ \wedge L). \]

But this is provided by Corollary 4.20. The map is explained in Lemma 4.21. It basically comes from \(S^1_s \to S|S^1_s| \cong \text{Map}(S^0, S^1)\) and \(f, g \mapsto f \wedge g\). \(\square\)

This finishes the proof of Lemma 4.20.

4.2.6. The Sixth Map. Remember that \(L\) is the realization of the Kan Loop group of \(W\), hence it is weakly equivalent to \(\Omega W\). We assumed that \(W\) is \((m+1)\)-connected. Thus we have the evaluation map \(S^1 \wedge L \to X\) which is \((2(m+1)+1)\)-connected. Therefore we get:
Lemma 4.22 (Sixth Map). There is a map from
\[ N^\text{cy}(H_G(S^n \wedge K_+), \text{Map}_G(S^n \wedge K_+, S^1 \wedge S^n \wedge K_+ \wedge L)) \]

to
\[ N^\text{cy}(H_G(S^n \wedge K_+), \text{Map}_G(S^n \wedge K_+, S^n \wedge K_+ \wedge |W|)) \]
which is \(2m + 3\)-connected.

Proof. The map is induced by the map \(S^1 \wedge L \to W\). This is \((2m + 3)\)-connected. Thus the induced map on the mapping space is also at least \((2m + 3)\)-connected. Hence by Lemma 3.24 the map of the cyclic bar construction is \((2m + 3)\)-connected. \(\square\)

4.2.7. Naturality of the maps.

Lemma 4.23. The chain of maps is natural with respect to \(G\) and \(W\). Furthermore there are stabilization maps for \(k\) and \(n\) which also respect this chain of maps, i.e., the chain of maps is natural with respect to these stabilization maps, too.

Remark 4.24. The stabilization maps look as follows. For \(k\) they mimic the \(K\)-theoretic stabilization \(\text{GL}_k \to \text{GL}_{k+1}\). \(g \mapsto (\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix})\). Explicitly they are given as:

\[ N^\text{cy}(H_G \times L(\vee^k S^n \wedge (G \times L)_+)) \to N^\text{cy}(H_G \times L(\vee^{k+1} S^n \wedge (G \times L)_+)) \]

\[ f \mapsto f \vee \text{id}_{S^n \wedge (G \times L)_+} \]

Let \(0_{S^n \wedge G_+ \wedge |W|} : S^n \wedge G_+ \to S^n \wedge G_+ \wedge |W|\) be the map which maps everything to the basepoint. For the last term the stabilization map for \(k\) is given as above on the \(H_G(\ldots)\) part but by “adding the trivial map” \(0_{S^n \wedge G_+ \wedge |W|}\) on the \(\text{Map}_G(\ldots)\) part:

\[ \text{Map}_G(\vee^k S^n \wedge G_+, S^n \wedge \vee^k G_+ \wedge |W|) \to \]

\[ \text{Map}_G(\vee^{k+1} S^n \wedge G_+, S^n \wedge \vee^{k+1} G_+ \wedge |W|) \]

\[ g \mapsto g \vee 0_{S^n \wedge G_+ \wedge |W|} \]

We record for later that this map can be obtained by precomposing the projection \(\vee^{k+1} \to \vee^k\) and postcomposing the canonical inclusion \(\vee^k \to \vee^{k+1}\).

For \(n\) the stabilization maps are just the suspension maps on mapping spaces.

Proof of the lemma and the remark. We only prove the naturality in \(k\). The proof of the naturality in \(n\) is similar, but easier. The naturality with respect to \(G\) and \(W\) follows by inspection.

The first map in the chain of maps is induced by the inclusion of monoids
\[ H_G(\vee^k S^n \wedge G_+) \to H_G \times L(\vee^k S^n \wedge (G \times L)_+). \]

A map of the form \(f \vee \text{id}_{S^n \wedge G_+}\) is mapped to \((f \wedge \text{id}_{L_+}) \vee (\text{id}_{S^n \wedge G_+ \wedge L_+})\). So the first map is natural with respect to the stabilization maps for \(k\).

As the generalized wedge and the cyclic bar construction are natural constructions, and Lemma 3.20 is natural, too, we only have to look at the spaces in the second component. We look at the map

\[ f \vee \text{id}_{S^n \wedge (G \times L)_+} \]

(17)
and trace it through the rest of the chain. When we rewrite the mapping space $H_{G \times L}(S^n \wedge (G \times L)_+)$ as connected components of the mapping space $Map_G(S^n \wedge G_+, S^n \wedge G_+ \wedge L_+)$ the map $id_{S^n \wedge (G \times L)_+}$ becomes the inclusion

$$i: S^n \wedge G_+ \to S^n \wedge G_+ \wedge L_+$$

induced by $S^0 \to L_+$. The composition of the second and the third map maps it to the pair

$$0_{S^n \wedge G_+ \wedge L}, \quad id_{S^n \wedge G_+},$$

the zero map $S^n \wedge G_+ \to S^n \wedge G_+ \wedge L$ and the identity on $S^n \wedge G_+$, since it is induced by the two maps $L_+ \to L$ and $L_+ \to S^0$. Therefore the map of (17) is mapped to the pair

$$f \vee 0_{S^n \wedge G_+ \wedge L}, \quad f \vee id_{S^n \wedge G_+}.$$

This shows the composition of the second and the third map has the desired naturality property with respect to $k$. The naturality of the fourth map is provided by Lemma 3.26. The fifth map is induced by $Map(S^0, S^1) \wedge Map(A, B) \to Map(A, S^1 \wedge B)$, $f, g \mapsto f \wedge g$, so it is natural, too. As the sixth map is natural, too, the lemma follows.

This proves the first part of the Theorem.

4.3. The Addendum Part of the Theorem. We have to produce a chain of maps between

$$\mathcal{N}(G \times L)$$

and

$$\mathcal{N}^{cy}(G, G_+ \wedge |W|)$$

similar to that chain we have already produced. Fortunately we have already proved all the Lemmas we need for this chain during our work for the first chain. We obtain it by setting the $k$ which is implicitly contained everywhere to 1 and setting the $n$ which is (explicitly) contained everywhere to 0. We then discuss the case when $G$ is trivial.

Let us start with what we get if we set $k = 1$ and $n = 0$ in the Lemmas 4.10, 4.13, 4.16, 4.18, 4.20 and 4.22 (These are the Lemmas describing the six maps.) To get the chain of maps we diagonalize everything to simplicial sets (due to Lemma 4.20) and then have maps

$$\mathcal{N}(G \times L)$$

$$\xymatrix{ \mathcal{N}(G \times L) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}^{cy}(G, S^1 \wedge G_+ \wedge |W|) \ar@{<->}[r]^{(2m+3)} & \mathcal{N}^{cy}(G, G_+ \wedge |W|) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}^{cy}(G, S^1 \wedge G_+ \wedge L) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}(G, H_{G \times L}((G \times L)_+)) \\
\mathcal{N}^{cy}(G, S^1 \wedge G_+ \wedge L) \ar@{<->}[r]^{(n-2)} & \mathcal{N}^{cy}(G, (G_+ \wedge L) \times G) \ar@{<->}[r]^\sim & \mathcal{N}^{cy}(G, \mathcal{V}((G_+ \wedge L))) \ar@{<->}[r]^\sim & \mathcal{N}(G \times L) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}(G, H_{G \times L}((G \times L)_+)) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}^{cy}(G, S^1 \wedge G_+ \wedge |W|) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}^{cy}(G, G_+ \wedge |W|) \ar@{<->}[r]^{(2m+1)} & \mathcal{N}(G \times L) }

(18)

We used the simplifications $H_G(G_+) = G$ (since $G$ is a group) and $Map_G(G_+, Y) = Y$ to obtain a more concise notation. The labels on the arrows denote the connectedness which the corresponding Lemma provides. Note that in this case it might...
be higher, for example the fifth map is indeed an isomorphism here. Further note
that as we assume \( n = 0 \) the second map in general will not be invertible after
stabilization. The transformation to the chain of the theorem is now given by the
natural stabilization maps for \( n \) and \( k \).

In case \( G \) is the trivial group the situation simplifies a lot. Due to the fact that
\( \vee (\ast, -) \) is a suspension (cf. 4.19) the second line of (18) consists just of \( S^1 \land L \) and
all following maps are homeomorphisms except the last, which is just the evaluation
map.

This finishes the proof of the addendum.

4.4. Comparison of Differentials. Recall the definitions of \( A(Y, G) \) and \( C(Y, G) \)
from 4.1 and 4.2 respectively. We have the announced theorem:

**Theorem 4.3.** The Goodwillie Differentials at a point of \( Y \mapsto A(Y, G) \) and \( Y \mapsto C(Y, G) \) are weakly equivalent.

**Proof of Theorem 4.3.** This is mainly Theorem 4.4 with some adjustments. We
quickly recall the definitions from 4.1 and 4.2:

\[
\begin{align*}
A^n_k(Y, G) & := \mathcal{N}H_{|G| \times |L(Y)|}((\vee^k S^n \land |G|_+ \land |L(Y)|_+) \\
A(Y, G) & := \mathbb{Z} \times \left( \colim_{n,k} A^n_k(Y, G) \right)^+ \\
C^n_k(Y, G) & := \mathcal{N}^{cy}(H_{|G|}((\vee^k S^n \land |G|_+), Map_{|G|}((\vee^k S^n \land |G|_+, \vee^k S^n \land |G|_+ \land |Y|))) \\
C(Y, G) & := \left( \colim_{n,k} C^n_k(Y, G) \right)^+ 
\end{align*}
\]

For purposes of stabilization we can ignore the \( \mathbb{Z} \)-factor in \( A(Y, G) \) as we are taking
loop-spaces. Then Theorem 4.3 provides that there is a chain of \( q \)-equivalences
between \( A^n_k(Y, G) \) and \( C^n_k(Y, G) \), for \( q \) being the minimum on \( n \) and twice the
connectedness of \( Y \). Therefore after taking the colimit of \( n \) and \( k \) we get that the there
is a chain of \( 2m \)-equivalences between the resulting functors if \( Y \) is \( m \)-connected.

We can assume the plus-construction is functorial as the conditions of Theorem 2.1
are fulfilled in our case and so it preserves the connectedness by Lemma 2.2.
Hence we get that \( Y \mapsto C(Y, G) \) and \( Y \mapsto A(Y, G) \) agree to first order, if we ignore the
\( \mathbb{Z} \)-factor. The theorem follows by Lemma 2.7. \( \Box \)

**Remark 4.25.** Note that since \( L(B(G)) \simeq G \) we have \( A(Y \times BG) \simeq A(Y, G) \). Note
further, that \( G \) is fixed and is not involved in the stabilization.

We abbreviate the Goodwillie Derivatives \( A^S(Y, G) \) and \( C^S(Y, G) \), respectively.

4.4.1. Some remarks on the case \( G = \ast \). We now restrict to the case where \( G \) is the
trivial group. Waldhausen’s Theorem 3.1 from [Wal79, Thm. 3.1] states:

Let \( X \) be a pointed simplicial set which is \( m \)-connected, \( m \geq 0 \).
Let \( SX \) be its suspension. Then the two spaces

\[
\mathcal{N}H_{|G(SX)|}((\vee^k S^n \land |G(SX)|_+) \\
\mathcal{N}^{cy}(H(\vee^k S^n), Map(\vee^k S^n, \vee^k S^n \land |SX|))
\]

are naturally \( q \)-equivalent, where

\[
q = \min(n - 2, 2m + 1);
\]
that is, there is a chain of natural maps connecting these two spaces, and all the maps in the chain are \(q\)-connected.

Our Theorem 4.4 recovers this and is slightly more general: We do not need to work with the suspension of the space at which we are evaluating the functors. (This allowed us to introduce the groups \(G\) as an extra piece of data which we will need in Section 7.)

In view of the addendum 4.5 of Theorem 4.4 we have two natural maps \(N(L(Y)) \to A(Y,*)\) and \(Y \to C(Y,*)\). Taking the Goodwillie Differential then gives a commutative diagram

\[
\begin{array}{ccc}
\Omega^\infty \Sigma^\infty Y & \longrightarrow & A^S(Y,*) \\
\cong & \triangleright & \cong \\
\Omega^\infty \Sigma^\infty Y & \longrightarrow & C^S(Y,*)
\end{array}
\]

The first map is induced by the one of the Barratt-Priddy-Quillen-Segal Theorem due to Lemma 2.3. Hence we have the theorem which we fix for reference.

**Theorem 4.26.** Stabilization of the map \(Y \to C(Y,*)\) induces a map

\[\Omega^\infty \Sigma^\infty Y \to C^S(Y,*)\]

which in view of \(A^S(Y,*) \cong C^S(Y,*)\) (Thm. 4.25) is weakly equivalent to the Barratt-Priddy-Quillen-Segal map

\[\Omega^\infty \Sigma^\infty Y \to A(Y,*) \to A^S(Y,*)\].

5. **The Trace Map**

In this section we give a weak map \(C^S(Y,*) \to \Omega^\infty \Sigma^\infty Y\). By the theorem before (Theorem 4.3) we hence get a weak map \(A^S(Y,*) \to \Omega^\infty \Sigma^\infty Y\). We call this the **trace map** as its construction resembles the trace map of matrices; we will review that below. We show that it splits \(\Omega^\infty \Sigma^\infty Y\) off \(C^S(Y,*)\) by the inclusion of \(\Omega^\infty \Sigma^\infty Y\) (Theorem 4.29), up to homotopy. Hence we get a splitting of \(A^S(Y,*)\) which will produce the splitting of \(A(Y,*)\). In other words, we want produce the dashed arrow “trace” to complete the following diagram

\[
\begin{array}{ccc}
\Omega^\infty \Sigma^\infty Y & \longrightarrow & A(Y,*) \\
\cong & \triangleright & \cong \\
\Omega^\infty \Sigma^\infty Y & \longrightarrow & C^S(Y,*)
\end{array}
\]

5.1. **Review of the Algebraic Trace Map and a Topological Lemma.** Let \(F\) be a field and \(V\) be a finite-dimensional vector space. Let \(\text{End}(V)\) be the endomorphisms of \(V\), regarded as matrices. Then the trace can be defined as the sum of the diagonal entries. Recall that the algebraic trace map can also be defined (cf. [Wal79, p. 25]) as the composition

\[
\begin{align*}
\hom(V, V) & \xleftarrow{\cong} \hom(V, K) \otimes \hom(K, V) \\
& \xrightarrow{\text{twist}} \hom(K, V) \otimes \hom(V, K) \xrightarrow{\text{trace}} \hom(K, K).
\end{align*}
\]
where the last map is given by the composition of maps. The first part of the following lemma corresponds to the first isomorphism above, it will be applied with $Y$ a $k$-fold wedge. The second part will play the role of the last map in the case $V = K$.

**Lemma 5.1.** Suppose $Y$ is a $(m - 1)$-connected topological space. Then the map

$$\text{Map}(\vee^k S^n, S^{n+m}) \wedge \text{Map}(S^{n+m}, S^{n+m} \wedge Y) \to \text{Map}(\vee^k S^n, S^{n+m} \wedge Y)$$

given by composition is $(3m - 1)$-connected. Similarly the map

$$\text{Map}(S^{n+m}, S^{n+m} \wedge Y) \wedge \text{Map}(S^n, S^{n+m}) \to \text{Map}(S^{n+m}, S^{n+2m} \wedge Y)$$

given by first smashing the second factor with the identity on $S^m \wedge Y$ and then composing is also $(3m - 1)$-connected.

**Proof.** We treat the map (5.1.1) first. We give a commutative diagram

\[
\begin{array}{c}
\text{Map}(S^n, S^{n+m}) \wedge \text{Map}(S^{n+m}, S^{n+m} \wedge Y) \\
\vee^k S^0 \wedge S^m \wedge \text{Map}(S^{n+m}, S^{n+m} \wedge Y) \\
\vee^k S^0 \wedge S^m \wedge \text{Map}(S^m, S^m \wedge Y) \\
\end{array}
\]

in which the upper map is the map (5.1.1). The map (1) is the composition of the inclusion $\vee^k S^m \to \prod^k (S^m)$ with the iterated suspension on the first factor. On the second factor it is the identity on an $(m - 1)$-connected space. As the inclusion and the suspension map are $(2m - 1)$-connected it follows that the map (1) is $(3m - 1)$-connected.

Similarly, the map (2) is given by the iterated suspension on the second factor; on the first factor it is the identity on an $(m - 1)$-connected space. Therefore (2) is also $(3m - 1)$-connected. The horizontal map (3) is a wedge of an iteration of the evaluation (or counit) map, hence it is $3m$-connected (cf. the end of the proof of Lemma 4.7). The right map (4) (corresponding to map (1)) finally is given by the inclusion of a wedge into a product composed with iterated suspension. As such is $(4m - 1)$-connected. Inspection shows that the upper map (5.1.1) is $(3m - 1)$-connected.

For the map (5.1.2) we have a similar, but easier diagram.

\[
\begin{array}{c}
\text{Map}(S^{n+m}, S^{n+m} \wedge Y) \wedge \text{Map}(S^n, S^{n+m}) \\
\text{Map}(S^{n+m}, S^{n+m} \wedge Y) \wedge \text{Map}(S^0, S^m) \\
\text{Map}(S^0, Y) \wedge \text{Map}(S^0, S^m) \\
\end{array}
\]

All vertical maps are induced by (iterated) suspensions. So the map on the right is $(4m - 1)$-connected and the maps on the left are both $(3m - 1)$-connected. □
5.2. The Trace. The Addendum 4.5 provides a map $Y \to C(Y, \ast)$ which stabilizes to a map $\Omega^\infty \Sigma^\infty (Y) \to C^S(Y, \ast)$ (Theorem 4.26).

**Theorem 5.2.** This map has a retraction

$$C^S(Y, \ast) \to \Omega^\infty \Sigma^\infty (Y)$$

which exists up to weak equivalence.

For the proof of the theorem, which occupies the rest of this section, we need two ingredients. The first is an unstable description of the retraction, which will be called the “trace map”. The second is an adjustment of the stabilization procedure. Due to Lemma 5.1 we have to stabilize in two directions: these are the connectedness of $Y$ and of $S^m$, which, incidentally, are the same. To increase this distinction we denote the suspensions differently by $S^m_1 \wedge Y$ and $\Sigma^m_2 Y$. We could of course easily adapt the lemma to distinguish between them and get a connectedness of $m_1 + m_2 + \min(m_1, m_2) - 1$. But we do not need that as for the stabilization we can assume $m_1 = m_2$ and denote it by $m$ when we refer to the lemma.

5.2.1. The Unstable Trace. We will stabilize the diagram of Figure 2. We will explain it here and do the stabilization in the next section. Note that it represents only part of the data. We will have to reduce and take loopspaces.

![Diagram](image)

**Figure 2.** $\mathbb{D}\mathbb{A}\mathbb{G}(S^{m_1}, \Sigma^{m_2} Y)$: The diagram for the stabilization

The left horizontal maps are the stabilization maps of the Addendum 4.5 for $k$ and $n$ from 1 and 0 respectively. The right horizontal maps are the canonical maps $q(\_\_)$ into the plus-construction of the middle term in each row, see Theorem 2.1. Again by Theorem 2.1 the plus-construction can be made functorially here for $k \geq 5$, at least if $m_1 + m_2$ big enough. (In this special constellation we also could adapt the plus-construction on the outer spaces to fit to the middle space, which may not be functorial but still produces a commutative diagram, cf. [Ros94, 5.2.4] where this property is called “functorial”.) As for $m_1 + m_2$ big enough (2 or so) the fundamental group of the lower term is trivial, the plus-construction does not change its homotopy type. However, it keeps the diagram strictly commutative.
The vertical maps require more comment. The top vertical maps are induced by the first map of Lemma 5.1 Therefore the maps are \((3m - 1)\)-connected. The left bottom map is composition after switching of factors, which is also \((3m - 1)\)-connected by the second part of Lemma 5.1 The middle bottom map is not highly connected in general: this is the map where “the retraction happens”. It is the following map:

The degree \(p\) part of the cyclic bar construction consists of the space

\[
H(\vee^k S^n) \times \cdots \times H(\vee^k S^n) \times \\
\text{Map}(\vee^k S^n, S^{n+m_1}) \wedge \text{Map}(S^{n+m_1}, \vee^k S^{n+m_1} \wedge \Sigma^{m_2} Y).
\]

The smash product is formed by taking a quotient of the product, so it suffices to give a map from the product which is compatible with the corresponding equivalence relation. There is a map induced by cyclic permutation and composition after suitable smashing with the identity on additional terms. (This is very similar to what is done in the second part of Lemma 5.1.) The map is

\[
H(\vee^k S^n) \times \cdots \times H(\vee^k S^n) \times \\
\text{Map}(\vee^k S^n, S^{n+m_1}) \times \text{Map}(S^{n+m_1}, \vee^k S^{n+m_1} \wedge \Sigma^{m_2} Y) \\
\text{Map}(S^{n+m_1}, \vee^k S^{n+m_1} \wedge \Sigma^{m_2} Y) \times \\
H(\vee^k S^n) \times \cdots \times H(\vee^k S^n) \times \text{Map}(\vee^k S^n, S^{n+m_1}) \\
\text{Map}(S^{n+m_1}, S^{n+2m_1} \wedge \Sigma^{m_2} Y).
\]

The map is compatible with the equivalence relation for the smash product since composition with a trivial map is again a trivial map. It is compatible with the cyclic bar construction when we regard the last term as the trivial cyclic bar construction \(\mathcal{N}^{cy}(*, -)\) and so we get the middle map of Figure 2. The left bottom map of Figure 2 is exactly the middle map in the case \(k = 1\) and \(n = 0\) and Lemma 5.1 provides a connectedness of \((3m - 1)\) in that case.

The right vertical maps are the ones induced by the functorial plus-construction applied to the middle vertical maps. They retain the connectedness by Lemma 2.2

This describes the diagram for spaces of the form \(S^{m_1} \wedge \Sigma^{m_2} Y\). We will refer to the whole diagram as \(\text{DIAG}(S^{m_1}, \Sigma^{m_2} Y)\) for reasons which will become clear in the following.

5.2.2. The Stabilization. Note we have two different but related meanings of “stabilization”. First we have the “K-Theoretic”-like stabilization with respect to \(n\) and \(k\). Second we have the “Goodwillie”-style stabilization with respect to \(m_1\) and \(m_2\) for which we still have to produce the stabilization maps. We need to show all maps in \(\text{DIAG}(S^{m_1} \wedge \Sigma^{m_2} Y)\) are natural with respect to all these maps.

The directions \(k\) and \(n\) are less problematic. Stabilization with respect to \(n\) is the suspension map on mapping spaces, which is compatible with composition. Stabilization with respect to \(k\) is adding the identity on the \(H(-)\) term and adding the trivial map on the \(\text{Map}(-)\) terms (cf. Lemma 4.23). For the middle term this
means precomposing with the canonical projection $\wedge^{k+1} \to \wedge^k$ for the first $\text{Map}(-)$-term and postcomposing with the canonical inclusion $\wedge^k \to \wedge^{k+1}$ for the second one. This is compatible with composition and hence with both vertical maps.

The last two directions of the stabilization are the $m_1$-directions. The main point here is that in the middle term in $\text{Diag}(S^{m_1}, \Sigma^{m_2}Y)$ in Figure 2 the connectedness is split off into two terms. Hence we have to stabilize with respect to both. What requires a bit more care is taking the loop-spaces afterwards. The functors we look at are at not reduced at the moment, so we better take the reduced version (with respect to $Y$) everywhere. This does not affect connectivity considerations due to Lemma 2.4. So imagine a tilde over every term and a $\Omega^{m_1+m_2}$ before them (even the rightmost). We then have to construct maps for increasing $m_1$ and $m_2$.

Usually the Goodwillie-procedure uses unreduced suspensions. However we will use the reduced suspension because we will apply Lemma 5.1. As all spaces which are involved here are realizations of simplicial sets and hence well-pointed these two are weakly equivalent, so it does not matter.

To stabilize $m_2$ apply the Goodwillie procedure for $Y \mapsto \text{Diag}(S^{m_1}, \Sigma^{m_2}Y)$ on all terms. This provides a functorial map from each term for $m_2$ into the loopspace of the term for $m_2 + 1$.

For $m_1$ we have to take care of the middle row. However we can directly mimic Goodwillie’s method. (See Section 2.2.3 where we recall it for this purpose.) In $\text{Diag}(S^{m_1}, \Sigma Y)$ we replace all topological spheres of the form $S^{m_1}$ by cones $C^{m_1}$. Denote this resulting (3 × 3) diagram by $\text{Diag}(C^{m_1}, \Sigma Y)$. For the middle term for example this looks like

\[
[\mathcal{N}^{cy}(H(\wedge^k S^n), \text{Map}(\wedge^k S^n, S^n \wedge C^{m_1}) \wedge \text{Map}(S^n \wedge C^{m_1}, \wedge^k S^n \wedge C^{m_1} \wedge \Sigma Y))].
\]

This is contractible when we look at the reduced functor. We get two canonical maps to the original term, induced by the hemispheres. Similarly, we can replace the cones again by a lower-dimensional sphere $S^{m_1-1}$ and get an inclusion into the cone which is compatible with the maps to the original term. Namely we get a diagram (of diagrams)

\[
\begin{array}{ccc}
\text{Diag}(S^{m_1-1}, \Sigma Y) & \longrightarrow & \text{Diag}(C^{m_1}, \Sigma Y) \\
\downarrow & & \downarrow \\
\text{Diag}(C^{m_1}, \Sigma Y) & \longrightarrow & \text{Diag}(S^{m_1}, \Sigma Y)
\end{array}
\]

where the corners are diagrams of contractible spaces. Hence we get a map into the loopspace, in each entry in the diagram $\text{Diag}$. (For this we had to reduce beforehand.)

Now we just have a huge (four-plus-two)-dimensional diagram (which we will not give here) from which we take the homotopy colimit with respect to $k, n, m_1, m_2$.

We get the following diagram (in which the middle terms are not interesting for the
result and thus we do not identify them):

\[ \Omega^\infty \Sigma^\infty Y \xrightarrow{\cong} C^S(Y) \]

\[ \cong \]

\[ \Omega^\infty \Sigma^\infty Y \cong \Omega^\infty \Sigma^\infty Y \]

Let us explain how these spaces evolve. The spaces \( \Omega^\infty \Sigma^\infty Y \) are easy to identify. They are unaffected by the stabilization in \( k \) and \( n \) and reduction and hence given by \( \text{colim}_{m_1, m_2} \Omega^{m_1 + m_2} \Sigma^{m_1 + m_2} Y. \) (Actually do that for one term and you know by the diagram the other ones are homotopy equivalent.) When we stabilize first in the \( k, n \)-directions and then in the \( m_1, m_2 \)-direction the top right term of the stabilization is clearly \( C^S(Y) \). Taking partial colimits affects neither the passage to ordinary colimits nor does it affect the passage to the homotopy colimits that we need here. The reason is that the homotopy colimits can be formed by replacing the diagram by a "better one" and taking the colimit of that.

By these considerations the upper map is then the same as in Theorem 4.26, the Goodwillie differential at the point of the map \( Y \to C(Y) \). Thus the right column provides the retraction up to weak equivalence, which we promised. This proves Theorem 5.2.

\[ \square \]

6. THE SPLITTING OF \( A(X) \)

We now can combine our results from the previous sections to obtain the splitting. Assume that \( X \) is pointed for technical reasons. \( X_+ \) denotes \( X \) with disjoint added basepoint, as usual.

For the following we need a version of algebraic K-Theory which accepts non-connected spaces as inputs. Such a version is given by Waldhausen in [Wal85, Section 2.1]. It is given as the following loop space of the realization of a simplicial category

\[ A(X) = \Omega|wS, (R_f(X))|. \]

Here \( X \) is an unpointed simplicial set; \( R_f(X) \) is the category of spaces which have \( X \) as a retract, a specified section and arise by attaching only finitely many cells. \( S \) is Waldhausen’s construction from [Wal85, Section 1]. We will not explain it here. We only need three properties of this definition. They are:

1. \( A(X) \) respects weak equivalences and is an infinite loop space.
2. \( A(X) \) maps disjoint unions to products.
3. For \( X \) connected, \( A(X) \) can be described by Quillen’s plus-construction as in \([1]\).

Details can be found in [Wal85, Sections 1.3, 1.5, 2.2] for the first and third part. The second statement is more or less clear from the definition.

We have a natural map \( \Omega^\infty \Sigma^\infty (X_+) \to A(X) \) given by the Barratt-Priddy-Quillen-Segal Theorem \([2]\), which we will now prove to be coretraction up to weak equivalence. We will look at the reduced functor \( \tilde{A}(X) \).

**Lemma 6.1.** There is a natural weak equivalence

\[ A(X) \xrightarrow{\cong} \tilde{A}(X_+). \]
Proof. We use that $A(-)$ takes disjoint unions to products. Therefore $A(X_+) \simeq A(X) \times A(*)$. However, the induced map to the point

$$p_{X_+}: A(X) \times A(*) \to A(*)$$

is not the projection to the second factor. But we can repair this: Take the self-equivalence

$$s: A(X) \times A(*) \xrightarrow{\begin{pmatrix} \text{id} & -p_X \\ 0 & \text{id} \end{pmatrix}} A(X) \times A(*)$$

with $p_X: A(X) \to A(*)$ the induced map to the point. This uses that $A(*)$ is an invertible H-space so we can add maps. ($A(*)$ is an infinite loop space, so in particular an invertible H-space.)

We get the solid homotopy commutative diagram:

$$\begin{array}{ccc}
A(X) & \xrightarrow{\text{id}} & \tilde{A}(X_+) \\
\downarrow & \downarrow & \downarrow \\
A(X) \times A(*) & \xrightarrow{s} & A(X_+) \\
\downarrow_{\text{pr}_2} & & \downarrow_{p_{X_+}} \\
A(*) & \xrightarrow{\text{id}} & A(*)
\end{array}$$

Because $A(X)$ is the (homotopy) fiber of $\text{pr}_2: A(X) \times A(*) \to A(*)$, and $\tilde{A}(X_+)$ is the homotopy fiber of $p_{X_+}: A(X_+) \to A(*)$ the diagram induces a weak equivalence $A(X) \xrightarrow{\simeq} \tilde{A}(X_+)$ on the (dashed) homotopy fibers. \(\square\)

In the definition of the Barratt-Priddy-Quillen-Segal map (2)

$$\Omega^\infty \Sigma^\infty (X_+) \to A(X)$$

we used that $X$ is connected. However, as both functors map disjoint unions to products, we can extend the definition to arbitrary spaces. We can reduce both sides and obtain a map

$$\Omega^\infty \Sigma^\infty (Y) \to \tilde{A}(Y)$$

which we will also call the Barratt-Priddy-Quillen-Segal map. Note for the following that $A(X, *) = A(X)$, and hence $\tilde{A}(Y, *) = \tilde{A}(Y)$, by the notation of Definition 4.1.

By Theorem 4.26 the Barratt-Priddy-Quillen-Segal map fits into a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega^\infty \Sigma^\infty (Y) & \xrightarrow{\tilde{A}(Y, *)} & A^S(Y, *) \\
\downarrow & & \downarrow \simeq \\
\Omega^\infty \Sigma^\infty (Y) & \xrightarrow{C^S(Y, *)} &
\end{array}$$

where the lower map $\Omega^\infty \Sigma^\infty (Y) \to C^S(Y, *)$ is induced by the inclusion map $Y \to C(Y, *)$. The vertical map is only a weak map. It is a chain of weak equivalences and exists as genuine map only in the homotopy category where it is an isomorphism.

Further, Theorem 5.2 provides up to weak equivalence a retraction

$$C^S(Y, *) \to \Omega^\infty \Sigma^\infty (Y)$$

for the map. Therefore we have proved the Theorem:
Theorem 6.2. Let $Y$ be a pointed simplicial set. There is a diagram commutative up to weak equivalence

$$
\begin{array}{ccc}
\Omega^\infty \Sigma^\infty(Y) & \longrightarrow & A(Y) \\
& & \downarrow \\
& & \uparrow \\
& & C^S(Y) \\
& & \downarrow \\
& & \Omega^\infty \Sigma^\infty(Y)
\end{array}
$$

Therefore

$$\tilde{A}(Y) \simeq \Omega^\infty \Sigma^\infty(Y) \times R(Y)$$

for some space $R(Y)$.

Remark 6.3. Note that everything works also if $Y$ is not connected: For the Barratt-Priddy-Quillen-Segal map this is noted above and for the rest we only apply the functors involved to an iterated suspension of $Y$ which is always connected.

When we plug in $X_+$ for $Y$ we get a splitting of $A(X)$.

Corollary 6.4. The unreduced stable homotopy splits off $A(-)$:

$$A(X) \simeq \Omega^\infty \Sigma^\infty(X_+) \times R(X_+)$$

Proof. Set $Y = X_+$ in Theorem 6.2 and note $\tilde{A}(X_+) \simeq A(X)$ by Lemma 6.1. □

7. The Differential of $A(-)$ at a Connected Space

7.1. The Calculation of the Differential of the Functor $A(-)$. In Section 4 we calculated the Goodwillie differential at a point of $A(X)$. The essential ingredient was Theorem 4.4 which we used to describe $A^S(X)$ by $C^S(X)$, a stabilization of a cyclic bar construction, in Theorem 4.3. (Recall from Section 2.2.3 that the stabilization is the same as the Goodwillie differential at a point.)

This essentially recovers results of Waldhausen in [Wal79]. However, our Theorem 4.4 is more general than the corresponding Theorem 3.1 of [Wal79]: We do not need to assume that our space is a suspension. This allows us to calculate the differential of $A(X)$ at an arbitrary base space. We will see that the value of the differential at a base space $B$ evaluated at $f: X \to B$ only depends on the base space $B$ and the homotopy fiber of $f$. We will prove:

Theorem 7.1 (Goodwillie differential of algebraic K-theory of spaces). Let $X$ be a space over a connected space $B$, let $F$ be the homotopy fiber.

The differential of algebraic K-theory of spaces at a connected space $B$ may be expressed as the stabilization of a plus-construction of a cyclic bar construction.

More precisely, there is a weak equivalence:

$$D_B A(X) \simeq C^S(F, L(B))$$

where $D_B A(X)$ is the Goodwillie differential of $A(-)$ at $B$ evaluated at $X$ and $C^S(F, L(B))$ is the stabilization along $F$ of the construction $C(F, L(B))$ from Definition 4.2.
Because we use a definition of $A(X)$ which accepts only connected spaces as input we have to restrict to a connected base space. However, spaces over $B$ can be described as the disjoint union over the components. We briefly discussed the more general definition of $A(-)$ at the beginning of Section 6. It accepts arbitrary simplicial sets as input. Also, it takes disjoint unions to products. As fiberwise suspension respects disjoint unions, too, our theorem generalizes to arbitrary $B$.

For technical reasons we always assume all our spaces are pointed. We need a prerequisite for the proof of Theorem 7.1.

We recall that $S_B X$ denotes the suspension of $X$ over $B$ from Section 2.2, and that $L$ denotes the Kan Loop Group from Subsection 2.1.1.

**Lemma 7.2.** Let $f : X \to B$ an $n$-connected map, $n \geq 0$. Then $S_B X \to B$ is $(n+1)$-connected and has a section.

Let $g : Y \to B$ be a $2$-connected map. Assume that it has a section $s : B \to Y$, i.e. $g \circ s = \text{id}_B$. Then we get a splitting up to weak equivalence

$$L(Y) \simeq L(B) \times L(F)$$

where $F$ denotes the homotopy fiber of $g$.

**Proof.** The homotopy fiber of $S_B X \to B$ is homotopy equivalent to the suspension of the homotopy fiber of $X \to B$. Also, $C_B X$ and hence $S_B X$ receives a section from $B$. This shows the first part.

For the second part, the section $s$ together with the inclusion of the fiber gives a map

$$(s \times i) : B \times F \to Y$$

We have to show that this map is a weak equivalence, i.e., that it induces an isomorphism on homotopy groups. Hence we have to show

$$\pi_i(B) \oplus \pi_i(F) \to \pi_i(Y)$$

is an isomorphism for all $i$.

Looking at the long exact sequence of the fiber sequence $F \to Y \to B$

$$\cdots \to \pi_i(F) \to \pi_i(Y) \xrightarrow{\alpha} \pi_i(B) \xrightarrow{0} \pi_{i-1}(F) \to \pi_{i-1}(Y) \xrightarrow{\alpha} \pi_{i-1}(B) \xrightarrow{0} \cdots \to \pi_1(F) \to \pi_1(Y) \xrightarrow{\alpha} \pi_1(B) \xrightarrow{0} \pi_0(F) \to \pi_0(Y) \xrightarrow{\alpha} \pi_0(B) \to 0$$

shows that the boundary maps are zero due to the existence of the section $B \to Y$. Hence the long exact sequence breaks up into split short exact sequences

$$0 \to \pi_i(F) \to \pi_i(Y) \xrightarrow{\alpha} \pi_i(B) \to 0$$

For $i \leq 1$ this follows from the 2-connectedness of $f$ (and the resulting triviality of $\pi_i(F)$). For $i \geq 2$ this follows since we have a section $\pi_1(s) : \pi_1(B) \to \pi_1(Y)$ of abelian groups.

This gives that $s \times i$ is a weak equivalence. The loop group of $Y$ has the homotopy type of the loop space $\Omega$. But $\Omega$ respects products. Therefore $s \times i$ induces a weak equivalence on Kan Loop Groups.

We added an extra topological group $G$ in Theorem 4.4 and carried it through the chain of equivalences. While it was trivial in the applications of the last section, we will need it here.
Proof of Theorem 7.1. Remember taking the differential is done in the following way. We take a (non-empty) space $X \to B$ over a connected base space. Then we take the (unreduced) fiberwise suspension to get $S_B X \to B$ and iterate this. It follows by Lemma 7.2 that at least after the second suspension the map $S_B^2 X \to B$ is 2-connected and has a section. Hence there is a weak splitting $S_B^2 X \simeq B \times S^2 F$. The suspension from $S_B^i X$ to $S_B^{i+1} X$ corresponds to the suspension from $S^i F$ to $S^{i+1} F$. Therefore, via the splitting, suspension over $B$ corresponds to stabilization in $F$.

It follows that for $i \geq 2$ $A(S_B^i X)$ can be replaced by $A(B \times S^i F) = A(S^i F, L(B))$ and hence it follows (as reduction does not change anything) that the differential of $A(\cdot)$ at $B$ is equivalent to the differential of $A(\cdot, L(B))$ at the point:

$$D_B A(X) \simeq D_\bullet A(F, L(B)).$$

The last term is the same as the stabilization in $F$:

$$D_\bullet A(F, L(B)) = A^S(F, L(B)).$$

Then the theorem follows by Theorem 4.3 which says $A^S(F, L(B)) \simeq C^S(F, L(B))$. □

8. Appendix: Some simple results

8.1. CW-complexes. Here we collect some simple results which are used before but did not seem to fit there.

Lemma 8.1.

1. Let $X$ be a CW-complex which is $k$-connected. Then $X$ is homotopy equivalent to a CW-complex $X'$ whose $k$-skeleton $X_k$ is a point.

2. Let $X \to Y$ be a map of CW-complexes which is $k$-connected. Then it can be factored into $X \xrightarrow{\alpha} X' \xrightarrow{\beta} Y$ where $\beta$ is a homotopy equivalence and $\alpha$ is an inclusion which is the identity on the $k$-skeleton. (That is, the first additional cell is attached in dimension $k+1$).

Proof. The first case is a special case of the second for $* \to X$. The second case is proved by doing a CW-Approximation. This is well-known, see e.g. Swi02, 6.13]. □

Lemma 8.2. The last Lemma holds in a relative case: Assume we have a $k$-connected map $H \to G$ and a commutative diagram of CW-complexes

\[
\begin{array}{ccc}
H & \longrightarrow & G \\
\downarrow & & \downarrow \\
M & \longrightarrow & M
\end{array}
\]

where the vertical arrows are inclusions. Then $H \to G$ can be replaced by an inclusion of the $k$-skeleton $H \to G'$ such that we get a diagram

\[
\begin{array}{ccc}
H & \longrightarrow & G' \\
\downarrow & & \downarrow \\
M' & \longrightarrow & M'
\end{array}
\]
where we have a homotopy equivalence $M' \simeq M$ which is compatible with the inclusions.

**Proof.** We apply Lemma 8.1 from before. We can replace $H \to G$ by an inclusion

$$
\begin{array}{ccc}
H & \xrightarrow{\sim} & G' \\
\downarrow & & \downarrow \\
M & \xrightarrow{\sim} & M
\end{array}
$$

The composition $G' \to G \to M$ need not to be an inclusion but can be made into one as before by replacing $M$ by $M'$ by applying Lemma 8.1 again. We obtain our desired diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\sim} & G' \\
\downarrow & & \downarrow \\
M' & \xrightarrow{\sim} & M'
\end{array}
$$

\[\square\]

### 8.2. Topological Groups acting on Spaces.

**Lemma 8.3.** Let $G$ be a topological group. Then $- \wedge G_+$ from pointed spaces to pointed $G$-spaces is left adjoint to the forgetful functor. Furthermore the isomorphism of Hom-sets carries over to an homeomorphism of topological mapping spaces:

$$\text{Map}(A, B) \cong \text{Map}_G(A \wedge G_+, B).$$

We have a slight generalization of the previous lemma.

**Lemma 8.4.** Let $G \times L$ be a product of topological groups. Then $- \wedge L_+$ from pointed $G$-spaces to pointed $G \times L$-spaces is left adjoint to the forgetful functor from $G \times L$ to $G$-spaces. Furthermore the isomorphism of Hom-sets carries over to an homeomorphism of topological mapping spaces.

$$\text{Map}_G(A, B) \cong \text{Map}_{G \times L}(A \wedge L_+, B).$$

**Lemma 8.5.** Denote by $H(X)$ the submonoid of self-equivalences of $X$. Then $H(X) \to \text{Map}(X, X)$ is an inclusion of path-connected components. This also works $G$-equivariantly.

**Proof.** Every map homotopic to a weak equivalence is itself a weak equivalence and a homotopy is just a path in the mapping space. \[\square\]

A note on the shear map.

**Lemma 8.6.** Let the simplicial monoid $G$ operate on the simplicial set $X$. Then there is the shear map

$$G \times X \to G \times X$$

$$(g, x) \mapsto (g, gx).$$

If $G$ acts invertibly it is an isomorphism. If $\pi_0 G$ is a group then it is a weak equivalence.
Proof. $G$ acting invertibly on $X$ means there is an inverse $i$ to the operation $a$ which exactly means $G \times X \xrightarrow{(pr_G,a)} G \times X \xrightarrow{(pr_G,i)} G \times X$ is the identity.

If $\pi_0 G$ is a group every $\pi_i G$ is a group hence the induced operation on homotopy groups $\pi_i G \times \pi_i X \to \pi_i X$ is invertibly, so the same argument as before shows $\pi_i(pr_G,a) : \pi_i(G \times X) \to \pi_i(G \times X)$ is an isomorphism for every $i$ (and for every basepoint), hence a weak equivalence. □

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