DC current in a circularly shaking optical lattice

Sergey S. Pershoguba\textsuperscript{a}, Victor M. Yakovenko\textsuperscript{b}

\textsuperscript{a}Department of Physics and Astronomy, University of New Hampshire, Durham, New Hampshire 03824, USA
\textsuperscript{b}JQI, Department of Physics, University of Maryland, College Park, Maryland 20742, USA

Abstract

We study an effect of circular shaking of a two-dimensional optical lattice loaded with bosonic atoms. We evaluate the renormalization of the energy dispersion perturbatively to second order in the strength of shaking. If the lattice breaks mirror symmetry, the renormalized energy dispersion has a finite slope at the center of the Brillouin zone. So, the bosonic atoms move with a finite group velocity \( \tilde{v} \) under the influence of shaking. This effect could be used to transport neutral bosonic atoms in optical lattices in arbitrary directions.

1. Introduction

An applied ac electric field \( E(t) \) causes a shift of quantum energy levels. For localized systems with discrete levels, such as atoms or molecules \([1, 3]\), this effect is known as the Stark-Bloch-Siegert (SBS) energy shift. But in extended systems, such as electrons in solids \([4, 5]\), delocalized wavefunctions are characterized by quasimomentum \( \hbar k \), which is not conserved in the presence of \( E(t) \). That leads to additional contributions to the energy shift besides the conventional SBS shift. A systematic derivation of the energy band renormalization in solids to the second order in \( E(t) \) was recently done in Ref. \([6]\). The paper identified a term that couples the Berry curvature \( \Omega(k) \) of the electrons in a crystal and the helicity \( h \) of circularly polarized light. Using this coupling, Ref. \([6]\) proposed optical control of the sign of spontaneous orbital magnetization in the recently discovered Chern insulators in Moiré materials \([7]\).

Here we discuss further physical implications of the second-order energy shift \( \varepsilon^{(2)}(k) \) derived in Ref. \([6]\) and evaluate the corresponding correction to the group velocity of quasiparticles. Under suitable conditions, it may result in a dc current induced by circularly polarized ac electric field, which is known as the circular photogalvanic effect \([8]\). This effect was discussed in many papers, e.g., in Ref. \([9, 11]\), for resonant transitions between energy bands in the case where the driving frequency \( \omega \) is lower than the gap, so there are no resonant transitions, but quantum states in different bands form coherent superpositions due to the presence of \( E(t) \).

It should be emphasized that the total dc current due to the group velocity modification vanishes for a fully occupied band, so it is not of interest for electronic insulators. Instead, here we study an analog of the circular photogalvanic effect for neutral bosonic atoms in an optical lattice, where they occupy the lowest energy state at \( k = 0 \) at low temperature. An analogue of the electric force \( eE(t) \) for neutral atoms is obtained by shaking of the optical lattice, which produces the force of inertia \( F(t) = -m \frac{d^2 R(t)}{dt^2} \), as experimentally realized in Ref. \([12]\). Here \( m \) is the mass of an atom, and the time-dependent vector \( R(t) \) represents displacement of the optical lattice.

We show that, in the presence of circular shaking, the energy minimum may shift away from \( k = 0 \), so the atoms with \( k = 0 \) would acquire a nonzero group velocity \( \tilde{v} \) and thus carry current. This current is not persistent but transient and eventually vanishes when the atoms relax to a new energy minimum at \( k \neq 0 \) in the presence of dissipation. Nevertheless, the induced current may be of interest for transporting bosonic atoms in optical lattices over a fixed distance by turning the circular shaking on and off for a limited time.

We study the geometry where bosonic atoms are loaded into a two-dimensional (2D) optical lattice in the \( xy \) plane, as shown in Fig. \([1]\). Circular shaking...
of the lattice is described by the in-plane displacement vector $R(t)$. The direction of circular shaking is characterized by the out-of-plane helicity vector $h \parallel \hat{z}$. A nonzero current can be obtained when the optical lattice breaks mirror symmetry and admits a polar vector $P$. For decoupled one-dimensional (1D) zigzag chains with alternating on-site energies, shown in red and blue in Fig. 1, the vector $P$ is the in-plane dipole moment perpendicular to the chains. Then the induced velocity $\hat{v}$ is along the chains and perpendicular to both $h$ and $P$:

$$\hat{v} \propto [h \times P]. \quad (1.1)$$

Since $h$ is an axial time-reversal-odd vector, while $P$ is polar and time-reversal even, their vector product has the symmetry consistent with the velocity $\hat{v}$. In Eq. (1.1), $\hat{v}$ can be interpreted as an analog of the Hall current, whereas $h$ and $P$ are the analogs of magnetic and electric fields. The 1D geometry shown in Fig. 1 is a limiting case of the 2D honeycomb lattice with anisotropic couplings between the sites studied experimentally in Ref. 12.

The outline of the paper is as follows. In Sec. 2, we derive the effect of shaking in the non-inertial reference frame of the lattice. In Sec. 3, we discuss renormalization of the energy dispersion to the second order in $F$ following Ref. 6. After introducing the anisotropic honeycomb lattice in Sec. 4, we evaluate the velocity $\hat{v}$ in the 1D and 2D limits of the model in Secs. 5 and 6 respectively. Conclusions are given in Sec. 7.

2. Shaking of the lattice

We start with analyzing the effect of periodic shaking of a lattice. It may be described by the time-dependent Schrödinger equation

$$i\hbar \partial_t \psi(r, t) = \left\{ \frac{\hat{p}^2}{2m} + U[r - R(t)] \right\} \psi(r, t), \quad (2.1)$$

where $U[r - R(t)]$ represents the potential of the lattice moving along the trajectory $R(t)$; $\hat{p} = -i\hbar \partial / \partial r$ is the momentum operator. Let us go to the reference frame, where the lattice is at rest [13]. For that purpose, we substitute

$$\psi(r, t) = e^{-i\hat{p} \cdot \overline{R}/\hbar} e^{i\hat{\varphi}(r, t)} \psi_{ni}(r, t), \quad (2.2)$$

$$h \varphi(r, t) = m\dot{R} \cdot r + \frac{1}{2} \int^t dt' m\dot{R}^2, \quad (2.3)$$

where $\psi_{ni}(r, t)$ is a wavefunction in the noninertial (ni) co-moving frame of reference. Then Eq. (2.1) becomes

$$i\hbar \partial_t \psi_{ni}(r, t) = \left\{ \frac{\hat{p}^2}{2m} + U[r] - F(t) \cdot r \right\} \psi_{ni}(r, t), \quad (2.4)$$

As expected, the effect of shaking enters via a time-dependent force of inertia $F(t)$. In the rest of the paper, we work with Eq. (2.4) and drop the subscript “ni” in $\psi_{ni}(r, t)$ to simplify notations.

Throughout this work, we assume a monochromatic time dependence of the driving force

$$F(t) = \frac{F(\omega) e^{-i\omega t} + F(-\omega) e^{i\omega t}}{2} \quad (2.5)$$

at nonzero frequency $\omega$. Here the amplitudes are related $F(-\omega) = F^*(\omega)$ to ensure that $F(t)$ is real. We focus on the case of circular shaking in $xy$ plane, which may be described by

$$F(\omega) = (1, \pm i, 0) \overline{F}_\pm. \quad (2.6)$$

Here, the signs + and − correspond to the clockwise or counterclockwise direction of shaking. We define helicity

$$h = \text{Im} \left[ F^*(\omega) \times F(\omega) \right] = \pm 2|F_{\pm}|^2 \langle 0, 0, 1 \rangle, \quad (2.7)$$

where the first equality sign represents the definition, and the second is the result of evaluation for Eq. (2.6). Helicity (2.7) is an axial time-reversal odd vector, which, therefore, has the symmetry properties of magnetic field.
The problem given by Eq. (2.4) is equivalent to that of a particle with charge $e$ in a time-dependent electric field $E(t) = F(t)/\epsilon$, which was considered in Ref. [6]. We summarize the results of that work in the following section.

3. Renormalization of the energy spectrum due to shaking

3.1. General case

For the Schrödinger equation with periodic potential $U(r)$, the eigenstates are Bloch function labeled by the discrete band index $n$ and continuous quasimomentum $k$. The wavefunctions for the energies $\varepsilon_n(k)$ have the Bloch form $\psi_n,m(r) = e^{ik_r r} u_{n,m}(r)$, where $u_{n,m}(r)$ is periodic in $r$.

In the presence of the time-dependent force $F$ acting on a particle, its momentum satisfies Newton’s equation of motion

$$\frac{d\mathbf{k}(t)}{dt} = \frac{F(t)}{\hbar}. \quad (3.1)$$

For clarity, we distinguish the time-dependent momentum $\mathbf{k}$ and its time-averaged value $\langle \mathbf{k}(t) \rangle_t$.

Using expansion (2.5) one may relate the two momenta

$$\mathbf{k}(t) = \mathbf{k} + \delta \mathbf{k}(t), \quad (3.2)$$

$$\delta \mathbf{k}(t) = \frac{\mathbf{F}(-\omega) e^{i\omega t} - \mathbf{F}(\omega) e^{-i\omega t}}{2i\hbar \omega}. \quad (3.3)$$

It is then practical [14] to expand the wavefunction as

$$\psi(t) = e^{i\mathbf{r} \cdot \mathbf{k}(t)} \sum_m c_{m,k}(t) e^{-\frac{i}{\hbar} \int dt' \varepsilon_m(\mathbf{k}(t')) u_{m,k}(r)}. \quad (3.3)$$

We substitute Eq. (3.3) in the Schrödinger equation Eq. (2.4) and obtain

$$i\hbar \dot{c}_{m,k} = -\sum_{m'} F_{\alpha}(t) r_{mn}^{\alpha}(\mathbf{k}) e^{\frac{i}{\hbar} \int dt' \varepsilon_{m,m'}(\mathbf{k})} c_{m',k}. \quad (3.4)$$

Equation (3.4) reflects the notion that the matrix elements of the coordinate operator $\mathbf{r}$ in a crystal are expressed in terms of the Berry connections [15, 16]

$$r_{mn}^{\alpha}(\mathbf{k}) = i (u_{n,k} | \frac{\partial}{\partial k_{\alpha}} | u_{m,k}). \quad (3.5)$$

As in Ref. [6], we use perturbation theory to second-order in $\mathbf{F}$ to evaluate the correction to eigenstate energy

$$\varepsilon_{n}^{(2)}(\mathbf{k}) = \frac{1}{4} \frac{\partial^2 \varepsilon_n(\mathbf{k})}{\partial k_{\alpha} \partial k_{\beta}} \text{Re} [F_{\alpha}^* \omega F_{\beta}(\omega)]$$

$$- \frac{\Omega_{n,\gamma}(\mathbf{k})}{4 \hbar \omega} \text{Im} [F_{\alpha}^* \omega F_{\beta}(\omega)]$$

$$- \frac{1}{4} \text{Re} \sum_{m \neq n} r_{mn}^{\alpha}(\mathbf{k}) r_{mn}^{\beta}(\mathbf{k}) F_{\alpha}^* (\omega) F_{\beta}(\omega)$$

$$- \frac{1}{4} \text{Re} \sum_{m \neq n} \left[ r_{mn}^{\alpha}(\mathbf{k}) r_{mn}^{\beta}(\mathbf{k}) \right] \times F_{\alpha}^* (\omega) F_{\beta}(\omega). \quad (3.6)$$

Here we use a concise notation $\varepsilon_{m,m}(\mathbf{k}) = \varepsilon_m(\mathbf{k}) - \varepsilon_n(\mathbf{k})$ and the antisymmetric Levi-Civita tensor $\epsilon^{\alpha\beta\gamma}$. We also introduce the Berry curvature

$$\Omega_{n,\gamma}(\mathbf{k}) = \frac{\partial}{\partial k_{\gamma}} \times \mathbf{r}_{n,m}(\mathbf{k}). \quad (3.7)$$

The subscripts of $\Omega_{n,\gamma}(\mathbf{k})$ appearing in Eq. (3.6) label the band $n$ and the spatial component $\gamma$. In contrast, $\Omega_{n,\gamma}(\mathbf{k})$ in Eq. (3.7) does not have the index $\gamma$, because it is written in a vector form. Let us briefly comment on the origin of the terms appearing in expansion (3.6).

(i) We start from the conventional last two terms in Eq. (3.6) with $[\varepsilon_{mm}(\mathbf{k}) - \omega]$ and $[\varepsilon_{mn}(\mathbf{k}) + \omega]$ known as the Stark-Bloch-Siegert (SBS) shifts [6, 15]. They were originally discussed in the systems with discrete energy spectrum such as atoms and molecules [13].

(ii) In contrast, the first two terms in Eq. (3.6) occur only in extended systems because $\delta \mathbf{k}(t)$ is driven by force $\mathbf{F}(t)$ according to Eq. (3.2). The first term in Eq. (3.6) appears because the instantaneous energy $\varepsilon_m(\mathbf{k}(t))$ in Eq. (3.3) depends on time $t$. The effective renormalized energy is obtained by averaging $\varepsilon_m = \langle \varepsilon_m(\mathbf{k}(t)) \rangle_t$ over time. In particular, in the case when momentum variation $\delta \mathbf{k}$ is small, we may expand the instantaneous energy

$$\varepsilon_m(\mathbf{k}(t)) \approx \varepsilon_m(\mathbf{k}) + \delta k_{\alpha}(t) \frac{\partial \varepsilon_m(\mathbf{k})}{\partial k_{\alpha}}$$

$$+ \delta k_{\alpha}(t) \delta k_{\beta}(t) \frac{\partial^2 \varepsilon_m(\mathbf{k})}{\partial k_{\alpha} \partial k_{\beta}}, \quad (3.8)$$

average it over time and get the first term in Eq. (3.6). An interesting corollary of this term is that, being proportional to the curvature tensor $\partial^2 \varepsilon_n(\mathbf{k})/\partial k_{\alpha} \partial k_{\beta}$, it leads to flattening of the energy dispersion. Therefore, shaking could be used.
as an experimental tool to tune flatness of the energy dispersion. The degree of the dispersion flatness is known as a crucial parameter controlling the novel strongly-interacting phases in the recently-discovered Moiré materials [7].

(iii) The second term in Eq. (3.6) occurs due to the accumulation of the Berry phase with time. To illustrate it, consider the accumulation of the Berry phase with time. To make it clear, consider $F(\omega)$ given by Eq. (2.6), which causes the momentum displacement $\delta k_R$ to move along a closed circular trajectory of radius $\delta k_R = F_+/\hbar \omega$. Upon making a full cycle along the closed loop, the wavefunction accumulates the Berry phase

$$
\phi \approx \Omega_{n_z}(k) \pi (\delta k_R)^2
$$

(3.9)

proportional to the flux of the Berry curvature through the circular area enclosed by the orbit. Since that phase is accumulated over one oscillation period $T = 2\pi/\omega$, it corresponds to energy shift $\Delta \varepsilon_n = \hbar \phi/T$, i.e. the second term in Eq. (3.6).

To the best of our knowledge, this term was first identified in Ref. [6].

3.2. Two-band model

Let us now apply the expression (3.6) to a model with two bands referred to as top and bottom labeled as $t$ and $b$ for brevity. We note that the product of fields $\text{Re} [F_+^{\sigma}(\omega)F_\beta(\omega)]$ and $\text{Im} [F_+^{\sigma}(\omega)F_\beta(\omega)] = \epsilon_{n\beta} \hbar^2/2$ are symmetric and antisymmetric under the exchange of indices $\alpha \leftrightarrow \beta$. In the latter term, we invoked the definition of helicity (2.7). It is practical to split the second-order energy shift (3.6) in two terms

$$
\varepsilon_b^{(2)}(k) = \varepsilon_b^{(s)}(k) + \varepsilon_b^{(a)}(k),
$$

(3.10)

which absorb the symmetric ($s$) and antisymmetric ($a$) contributions, respectively. Explicitly, we obtain from Eq. (3.6)

$$
\varepsilon_b^{(s)}(k) = \text{Re} [F_+^{\sigma}(\omega)F_\beta(\omega)]
$$

(3.11)

$$
\times \left\{ \frac{1}{4(\hbar \omega)^2} \partial^2 \varepsilon_b(k) \partial k_\alpha \partial k_\beta \right\}
$$

$$
\times \left\{ \frac{\varepsilon_{tb}(k) \text{Re} \left[ r_{tb}(k) r_{tb}^\ast(k) \right]}{2 [\epsilon_{tb}(k) - (\hbar \omega)^2]} \right\},
$$

$$
\varepsilon_b^{(a)}(k) = -\frac{\varepsilon_{tb}^2(k)}{4\hbar \omega [\epsilon_{tb}(k) - (\hbar \omega)^2]} \Omega_{tb}(k) \cdot \hbar,
$$

(3.12)

where in the second equation we used the sum-rule applicable for a two-band model

$$
\text{Im} \left[ r_{tb}^n r_{tb}^{n\beta} \right] = -\frac{\epsilon_{n\beta}^2 \Omega_{tb}}{2}.
$$

(3.13)

If the bare Hamiltonian (in the absence of driving force) has the time-reversal symmetry, the antisymmetric term is odd $\varepsilon_b^{(a)}(-k) = -\varepsilon_b^{(a)}(k)$, whereas the symmetric is even $\varepsilon_b^{(s)}(-k) = \varepsilon_b^{(s)}(k)$ in momentum $k$. One interesting consequence of this parity property is that the renormalized energy dispersion

$$
\varepsilon_b(k) = \varepsilon_b(k) + \varepsilon_b^{(2)}(k)
$$

(3.14)

acquires a finite group velocity at $k = 0$:

$$
\tilde{v}_b(k)|_{k=0} = \frac{\partial \varepsilon_b(k)}{\hbar \partial k} |_{k=0}
$$

(3.15)

$$
= -\frac{\varepsilon_{tb}^2(k)}{4\hbar \omega^2 [\epsilon_{tb}(k) - (\hbar \omega)^2]} \partial \left[ \Omega_{tb}(k) \cdot \hbar \right] |_{k=0}.
$$

Notice that a non-zero gradient of the Berry curvature leads to a non-zero group velocity.

These results can be straightforwardly extended to the case of a finite occupation of states in the momentum space described by the function $f_n(k)$, where $n$ labels bands. In this case, the current density is given by the integral

$$
j = \sum_n \int \frac{d^2 k}{(2\pi)^2} f_n(k) \tilde{v}_n(k).
$$

(3.16)

However, for a fully-occupied bottom and empty top band, i.e. for $f_b(k) = 1$ and $f_t(k) = 0$, the integrand becomes a full derivative, so $j$ vanishes.

4. The honeycomb lattice model

We are motivated by the experimental realization of a shaking optical honeycomb lattice in Ref. [12]. In this section, we introduce a 2D tight-binding lattice shown in Fig. 2. The distinct tunneling amplitudes $w_1$, $w_2$, and $w_3$ along the nearest-neighbour vectors $a_1$, $a_2$, and $a_3$ render the model anisotropic. The two inequivalent sublattices A and B have different onsite energies $W$ and $-W$. This model breaks inversion symmetry but preserves time-reversal symmetry.

In the momentum $k = (k_x,k_y)$ space, the Hamiltonian is a $2 \times 2$ matrix

$$
H(k) = \sigma \cdot w(k),
$$

(4.1)

$$
w(k) = \left[ \sum_{j=1}^3 w_j \cos(k \cdot a_j), \sum_{j=1}^3 w_j \sin(k \cdot a_j), W \right],
$$
which we expanded in the Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. The Hamiltonian acts on the wavefunction $u(k) = [u_x(k), u_y(k)]^T$ with two components corresponding to the two sublattices. The spectrum of Hamiltonian (4.1) has two eigenvalues

$$\varepsilon_{t,b}(k) = \pm |w(k)|$$

with positive and negative values referred to as top and bottom bands.

Anticipating the application of Eqs. (3.10)-(3.15), we express the Berry curvature of the bottom band

$$\Omega_{b,\gamma}(k) = \frac{\epsilon_{\alpha\beta\gamma}}{4} \hat{w}(k) \cdot \left[ \frac{\partial \hat{w}(k)}{\partial k_\alpha} \times \frac{\partial \hat{w}(k)}{\partial k_\beta} \right],$$

as well as the product of the off-diagonal matrix elements

$$\text{Re} \left[ r_{\alpha\beta}^R(k) r_{\alpha\beta}^I(k) \right] = \frac{1}{4} \left[ \frac{\partial \hat{w}(k)}{\partial k_\alpha} \cdot \frac{\partial \hat{w}(k)}{\partial k_\beta} \right],$$

where $\hat{w}(k) = w(k)/|w(k)|$ is a unit vector. The value of these relations is that they do not rely on the explicit eigenstates as shown in Appendix A.

The model (4.1) allows to interpolate between the cases of 1D decoupled zigzag chains ($w_3 = 0$) and the 2D anisotropic honeycomb model, where all hopping amplitudes are of the same order. We study these two limiting cases in the following sections.

### 5. One-dimensional limit: Decoupled chains

As a warm up, we start from a simpler case, where one of the hopping amplitudes vanishes, e.g. $w_3 = 0$. Then, the 2D lattice illustrated in Fig. 2 reduces into a set of 1D zigzag chains aligned along the $x$ direction. In this limit, it is known as the Rice-Mele model [17] in the literature. The energy spectrum (4.2) becomes

$$\varepsilon_{t,b}(k) = \pm |w(k)|$$

$$= \pm \sqrt{W^2 + w_1^2 + w_2^2 + 2 w_1 w_2 \cos k \cdot (a_2 - a_1)}.$$  

By considering the coordinate representation of the argument

$$k \cdot (a_2 - a_1) = k_x \sqrt{3} a_1,$$  

note that the energy dispersion (5.1) depends only on momentum $k_x$ but not on $k_y$ because the chains are decoupled in $y$ direction. Nevertheless the Hamiltonian (4.1) retains the dependence on both components $k_x$ and $k_y$, which is crucial for the evaluation of the Berry curvature (4.3)

$$\Omega_{b}(k) = \frac{W w_1 w_2}{2|w(k)|} \left[ a_1 \times a_2 \right] \sin k \cdot (a_2 - a_1).$$

Similar to Eq. (5.1), the Berry curvature (5.3) depends only on momentum $k_x$. We plot both the energy spectrum (5.1) and the Berry curvature (5.3)
in Fig. 3. Although the model consists of decoupled 1D chains, the Berry curvature is nonzero. To satisfy the time-reversal symmetry of the tight-binding model, the energy spectrum (5.1) is even, whereas the Berry curvature (5.3) is odd. The Berry curvature (5.3) is a product of the two momentum-dependent functions $1/|w(k)|^3$ and $\sin k \cdot (a_2 - a_1) = \sin k_x \sqrt{3}a$. The former function is maximal at the Brillouin zone edge $k_{z,a} = \pi/\sqrt{3}$, whereas the latter is shifted due to the linear-in-$k$ curvature (5.3) is a product of the two momentum-space. It is instructive to examine the behavior of the bare and renormalized dispersion in the vicinity of $k_x = 0$, where the bottom ($b$) band reaches its minimum. Expansion of the bare spectrum (5.1) yields

$$\varepsilon_b(k) \approx \varepsilon_b(0) + \frac{\hbar^2 k_x^2}{2m_x}, \quad (5.4)$$

where $\varepsilon_b(0) = -|w(0)| = -\sqrt{W^2 + (w_1^2 + w_2^2)}$ and $m_x = \frac{\hbar^2 |w(0)|}{3w_1 w_2 a^2}$. The renormalized dispersion (3.14) acquires a linear-in-$k$ term

$$\tilde{\varepsilon}_b(k) \approx \tilde{\varepsilon}_b(0) + \frac{\hbar^2 k_x^2}{2m_x} + h \cdot k \cdot \nu, \quad (5.5)$$

where the velocity is evaluated using Eqs. (3.15) and (5.3)

$$\nu_b = C_{12} w_1 w_2 (a_1 - a_2), \quad (5.6)$$

$$C_{12} = \frac{W [h \cdot (a_1 \times a_2)]}{2 \hbar^2 |w(0)| |4| [w(0)]^2 - (h\omega)^2|}$$.  

Both bare (5.4) and renormalized (5.5) energy dispersions are plotted in Fig. 4. The former dispersion is given by a parabola centered at $k_x = 0$, whereas the latter is shifted due to the linear-in-$k$ term. In Appendix B, we present an alternative derivation of Eq. (5.6) using the conventional perturbation theory without resorting to the concept of the Berry curvature.

The $1/\omega$ divergence at $\omega \to 0$ in Eq. (5.6) is superficial. It was derived in the opposite limit of small momentum displacement $\delta k \sim F/\omega \ll 1/a$, which cannot be used to extract the $\omega \to 0$ behavior. Further, note that helicity $h$ scales as the forth power of frequency $h \propto F^2 \propto \omega^4$ according to Eqs. (2.4) and (2.7). So, Eq. (5.6) formally scales as $\tilde{\varepsilon}_b \propto \omega^3$ at $\omega \to 0$.

A couple of observations about velocity (5.6) are in order. (i) In order to move along the zigzag chains, the atoms traverse the nearest-neighbor bonds along the vectors $a_1$ and $a_2$ which furnish the zigzag chains. That explains the appearance of the product of the corresponding hopping amplitudes $w_1 w_2$ in Eq. (5.6). (ii) It is crucial that the vectors $a_1$ and $a_2$, which form the zigzag chain, are non-collinear. Otherwise, the vector product appearing in the definition of $C_{12}$ renders the velocity (5.6) zero. In other words, a 1D chain aligned along a straight line, i.e. if $a_1 \parallel a_2$, would not couple to the two components of the force $F$, and the velocity would vanish. (iii) The velocity (5.6) changes sign if the direction of shaking is reversed, i.e. if $h \to -h$.

Let us elaborate on the significance of the renormalized dispersion (5.5) for experiments. Initially, the bosonic atoms are loaded in the optical lattice.
At low temperatures, the atoms occupy the lowest energy state of the bare dispersion at $k_x = 0$ marked as $A$ in Fig. 4, where their group velocity is zero. Upon turning on the circular shaking, the energy dispersion becomes renormalized as shown in Fig. 4. The bosonic atoms now occupy a state at $k_x = 0$ of the renormalized dispersion marked as $B$ in Fig. 4. Notice that the state $B$ has a nonvanishing group velocity, so the atoms move along the chains, i.e., parallel to the vector $(a_2 - a_1) = \sqrt{3} a \hat{x}$. Since any optical lattice realized in experiments is spatially confined, the atoms will hit the edge of the system and scatter backward. In that process, they occupy a new state $C$, which is characterized by the same energy as $B$ and opposite velocity $-\tilde{v}_b$. The atoms travel back and forth within the system by switching between the states $B$ and $C$. It should be noted, however, that the discussed current is transient. Eventually, due to dissipation processes, the atoms relax to the minimum of the renormalized dispersion $D$, where the group velocity vanishes. Nevertheless, the discussed effect could be used to transport neutral bosonic atoms in optical lattices over fixed distances by turning the circular shaking on and off for a limited time.

6. Anisotropic 2D case

In this section, we discuss the general situation where all three hopping amplitudes $w_1$, $w_2$, and $w_3$ are nonzero. To relate with the previous section, we start with the case $w_1$, $w_2 \gg w_3$. It corresponds to zigzag chains aligned along $x$ direction coupled by a weak interchain coupling $w_3$ in $y$ direction. The corresponding energy spectrum and Berry curvature have a weak modulation in $k_y$ momentum as shown in Fig. 5. The case of anisotropic honeycomb lattice with all hopping amplitudes of the same order is shown in Fig. 6. The Berry curvature has pronounced peaks near the corners of the Brillouin zone.

Notice that the Berry curvature in both cases, shown in both Figs. 5(b) and 6(b), have a nonvanishing gradient at $k = 0$. Therefore, according to Eq. (3.15), the group velocity of the renormalized state at $k = 0$ becomes nonzero. We follow the steps of the previous section and obtain a general
ization of group velocity \( \text{Eq. (3.15)} \) at \( k = 0 \)

\[
\ddot{\mathbf{v}}_j = C_{ij} w_1 w_2 (\mathbf{a}_1 - \mathbf{a}_2) + C_{ij} w_2 w_3 (\mathbf{a}_2 - \mathbf{a}_3)
+ C_{ij} w_3 w_1 (\mathbf{a}_3 - \mathbf{a}_1), \quad (6.1)
\]

\[
C_{ij} = \frac{W}{2 \hbar^2 |\mathbf{w}(0)|} \frac{|\mathbf{w}(0)|^2 - (\hbar \omega)^2}{|\mathbf{w}(0)|^2 - (\hbar \omega)^2}.
\]

Equation (6.1) has the following geometric interpretation. The honeycomb lattice, shown in Fig. 2, may be viewed as a collection of three distinct families of zigzag chains: (i) chains formed by vectors \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \), (ii) chains formed by vectors \( \mathbf{a}_2 \) and \( \mathbf{a}_3 \) and (iii) chains formed by vectors \( \mathbf{a}_3 \) and \( \mathbf{a}_1 \). Equation (6.1) implies that each chain family sustains a 1D current (5.6) flowing in the corresponding direction, and the total current is a superposition of all the currents.

Equation (6.1) simplifies further when all nearest-neighbor vectors have equal magnitude \( |\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = a \). In this case, all prefactors \( C_{ij} = C \) are equal, and it is instructive to write the velocity (6.1) in components parallel [i.e. along the vector \( \{\mathbf{a}_1 - \mathbf{a}_2\} \)] and perpendicular [i.e. along the vector \( \mathbf{a}_3 \)] to zigzag chains

\[
\ddot{\mathbf{v}}_i = \frac{C}{2} \left[ 2 w_1 w_2 - w_3 (w_1 + w_2) \right] (\mathbf{a}_1 - \mathbf{a}_2) + \frac{C}{2} w_3 (w_1 - w_2) \mathbf{a}_3. \quad (6.2)
\]

Some comments about Eq. (6.2) are in order.

(i) For the isotropic honeycomb lattice, where all hopping amplitudes are equal \( w_1 = w_2 = w_3 \), the current vanishes, because the lattice has \( C_3 \) rotational symmetry.

(ii) If \( w_1 = w_2 \), the velocity is purely along the “zigzag” direction \( \mathbf{v}_0 \propto (\mathbf{a}_1 - \mathbf{a}_2) \parallel \hat{x} \). In this case, the lattice in Fig. 2 is uniaxial and has a ferroelectric order parameter \( P \parallel y \) perpendicular to the zigzag chains, so the current direction is given by Eq. (5.1).

(iii) If \( w_3 = 2 w_1 w_2/(w_1 + w_2) \), the current is along the “armchair” direction \( \mathbf{a}_3 \). In general, the velocity \( \mathbf{v}_0 \) can be pointed in arbitrary direction by choosing the tunneling amplitudes appropriately.

7. Conclusions

In this paper, we study a 2D honeycomb optical lattice shaken along an in-plane circular trajectory \( \mathbf{R}(t) \) as illustrated in Figs. 1 and 2. We focus on the case where the bosonic atoms are loaded in the lattice. In the frame of reference of the lattice, the atoms experience force of inertia \( \mathbf{F}(t) = -m \mathbf{v}^2 \mathbf{R}/dt^2 \), where \( m \) is the mass of an atom. We consider the case of low shaking frequencies \( \omega \), so interband transitions do not occur. Following our previous work [3], we use perturbation theory to second order in \( \mathbf{F} \) to derive the renormalization \( \text{Eq. (3.6)} \) of the energy dispersion. We identify two main effects produced by shaking. (i) The renormalized energy dispersion is flattened with respect to the bare one as shown in Fig. 3. (ii) The minimum of the renormalized dispersion shifts from \( k = 0 \) as shown in Fig. 4. The latter effect is proportional to the helicity \( \hbar \) of shaking and produces a nonzero group velocity \( \mathbf{v} \) at \( k = 0 \). It could be used to transport neutral bosonic atoms in optical lattices.

The results of the paper can be easily extended to electrons in semiconductors driven by ac electric field \( \mathbf{E}(t) \), where the electric force \( \mathbf{F}(t) = e \mathbf{E}(t) \) acts on electrons. In the presence of the finite occupation function \( f_n(k) \) in momentum space, the current density is given by the integral \( \text{Eq. (6.16)} \). It may be interpreted as a dc current driven by the ac electric field \( j \propto E(\omega)E(-\omega) \). However, the current density \( j \) vanishes for a fully-occupied band.

Acknowledgments. We thank L. Glazman for useful discussions. SP was supported by the U.S. Department of Energy (DOE), Office of Science, Basic Energy Sciences (BES) under Award No. DE-SC0020221.

References

[1] D. H. Kobe, Gauge invariant derivation of the AC Stark shift, J. Phys. B: At. Mol. Phys. 16 (1983) 1159.
[2] N. B. Delone, V. P. Krainov, AC Stark shift of atomic energy levels, Phys.-Usp. 42 (1999) 669.
[3] M. Haas, U. D. Jentschura, C. H. Keitel, Comparison of classical and second quantized description of the dynamic Stark shift, Am. J. Phys. 74 (2006) 77.
[4] E. J. Sie, J. W. McIver, Y.-H. Lee, L. Fu, J. Kong, N. Gedik, Valley-selective optical Stark shift in monolayer WS₂, Nat. Mat. 14 (2015) 290.
[5] E. J. Sie, C. H. Lui, Y.-H. Lee, L. Fu, J. Kong, N. Gedik, Large, valley-exclusive Bloch-Siegert shift in monolayer WS₂, Science 355 (6329) (2017) 1086.
[6] S. S. Peshoguba, V. M. Yakovenko, Optical control of topological memory based on orbital magnetization, Phys. Rev. B 105 (2022) 064423.
[7] E. Y. Andrei, D. K. Efetov, P. Jarillo-Herrero, A. H. MacDonald, K. F. Mak, T. Senthil, E. Tutuc, A. Yazdani, A. F. Young, The marvels of moiré materials, Nat. Rev. Mat. 6 (2021) 201.
[8] B. I. Sturman, V. M. Fridkin, The Photovoltaic and Photorefractive Effects in Noncentrosymmetric Materials, 1st Edition, Gordon and Breach Science Publishers, Philadelphia, Reading, Paris, 1992.
Figure B.7: Perturbation theory in the nearest-neighbor hopping amplitudes $w_1$ and $w_2$ generates an effective coupling $\Omega$ between sites $B_1$ and $B_2$ with imaginary part.

\[ w/|w| \] representing the Hamiltonian

\[
\begin{align*}
|t\rangle\langle t| &= \frac{1}{2} \left[ I + \sigma \cdot \hat{w} \right] , \\
|b\rangle\langle b| &= \frac{1}{2} \left[ I - \sigma \cdot \hat{w} \right].
\end{align*}
\] (A.2)

Substituting the projection operators \([A,2]\) in Eq. (A.1) and using the well-known identity for the Pauli matrices $\text{Tr}[\sigma_\alpha \sigma_\beta \sigma_\gamma] = 2i \epsilon_{\alpha\beta\gamma}$, we get

\[ \Omega_{b,\gamma}(k) = \frac{\epsilon_{\alpha\beta\gamma}}{4} \hat{w}(k) \cdot \left[ \partial^\alpha \hat{w}(k) \times \partial^\beta \hat{w}(k) \right] , \]

i.e. the Eq. (4.3) of the main text.

Now, let us derive Eq. (4.4) for the product of Berry connections. We rewrite it using trace

\[
\text{Re} \left[ r^{\alpha}_{\beta}(k) r^{\beta}_{\alpha}(k) \right] = \langle b | \langle t | i \partial^\alpha t \langle t | i \partial^\beta b \rangle \\
= -\frac{1}{2} \text{Tr} \left[ \partial^\alpha (| t \rangle\langle t |) \partial^\beta (| b \rangle\langle b |) \right]
\]

substitute the projection operators \([A,2]\), use the identity for the Pauli matrices $\text{Tr}[\sigma_\alpha \sigma_\beta \sigma_\gamma] = 2 \delta_{\alpha\beta}$ and obtain

\[ \text{Re} \left[ r^{\alpha}_{\beta}(k) r^{\beta}_{\alpha}(k) \right] = \frac{1}{4} \left[ \partial^\alpha \hat{w}(k) \cdot \partial^\beta \hat{w}(k) \right] , \]

i.e. the Eq. (4.4) of the main text.

**Appendix B. Perturbative calculation in real space**

In this section, we evaluate the velocity \([5.6]\) without resorting to the concept of the Berry curvature. It is instructive to derive the effect in real space.
space. For that purpose, we pick the three nearest sites labeled as $B_1$, $B_2$ and $A_1$ from the whole lattice and illustrate them in Fig. B.7. Although the sites $B_1$ and $B_2$ are not coupled by the direct hopping amplitudes, we derive an effective hopping amplitude $w_{\text{eff}}$ between them using perturbation theory in small $w_1$ and $w_2$. We find that $w_{\text{eff}}$ has imaginary part due to the circular shaking, which produces nonzero velocity \footnote{[5,6]}. We start with the unperturbed Hamiltonian in the “bra-ket” notation

$$H_0 = W \ | A_1 \rangle \langle A_1 | - W \ | B_1 \rangle \langle B_1 | - W \ | B_2 \rangle \langle B_2 | .$$

The coupling between the sites is treated as perturbation

$$V = w_1 e^{iA(t) a_1} | A_1 \rangle \langle B_1 | + w_2 e^{-iA(t) a_2} | B_2 \rangle \langle A_1 | + \text{h.c.}, \quad \text{(B.1)}$$

where we use a gauge with the driving force entering as a “vector potential” satisfying the equation $P(t) = -\dot{A}(t)$. It produces nonzero phases of the hopping amplitudes. Assuming the harmonic driving force \footnote{[2,5]}, we may find the vector potential explicitly

$$A(t) = A(\omega)e^{-i\omega t} + A(-\omega)e^{i\omega t},$$

$$A(\omega) = \frac{F(\omega)}{i \omega}.$$ \quad \text{(B.2)}

Let us evaluate the evolution operator $U_t = \mathcal{T} \exp \left[ -i \int dt' V_1(t') \right]$ perturbatively in $V_1$ using the interaction (I) representation. We use units $\hbar = 1$ to simplify notations in this section. We start with evaluating the first-order contribution to the evolution operator, which moves the particle from site $B_1$ to $A_1$:

$$U^{(1)}_t = \frac{1}{i} \int dt' V_1(t') = | A_1 \rangle \langle B_1 | \times$$

$$w_1 \left[ \frac{A_\beta(\omega) e^{i(2W - \omega)t}}{2W - \omega} + \frac{A_\beta(-\omega) e^{i(2W + \omega)t}}{2W + \omega} \right] a_\beta^\dagger + \ldots ,$$

where we retain only $\alpha A$ term. Next, we evaluate the second-order correction to the evolution operator and choose the process that moves the particle from the site $A_1$ to $B_2$:

$$U^{(2)}_t = \frac{1}{i} \int dt' V_1(t') \frac{1}{i} \int dt'' V_1(t'')$$

$$= | B_2 \rangle \langle A_1 | A_1 \rangle \langle B_1 | \frac{1}{i} \int dt' \ w_{\text{eff}} + \ldots \quad \text{(B.3)}$$

where the effective coupling is

$$w_{\text{eff}} = -w_1 w_2 \frac{W \ \text{Re} \ [A_\alpha(-\omega) A_\beta(\omega)]}{4W^2 - \omega^2} a_\alpha^\dagger a_\beta^\dagger - \frac{i}{2} \ w_1 w_2 \ \text{Im} \ [A_\alpha(-\omega) A_\beta(\omega)] a_\alpha^\dagger a_\beta^\dagger. \quad \text{(B.4)}$$

According to Eq. \footnote{[B.3]}, the net result is that the particle from the site $B_1$ hops to site $B_2$ through the intermediate site $A_1$. The strength of that effective coupling is described by the hopping amplitude $w_{\text{eff}}$. Notice that $w_{\text{eff}}$ has an imaginary part that couples with $\text{Im} [A_\alpha(-\omega) A_\beta(\omega)] = h^2 \epsilon_{\alpha \beta \gamma z}/2\omega^2$, which is related to helicity $h = h \hat{z}$, so we get

$$\text{Im} [w_{\text{eff}}] = \frac{w_1 w_2 [h \cdot (a_1 \times a_2)]}{4\omega [4W^2 - \omega^2]}.$$ \quad \text{(B.5)}

Now, let us consider the whole lattice, which has infinitely many sites. Then, we go to the momentum $k$ space, and find that the imaginary part of $w_{\text{eff}}$ produces an odd-in-$k$ contribution

$$\tilde{\epsilon}_{k, \text{odd}}(k) = 2 \text{Im} [w_{\text{eff}}] \sin k \cdot (a_1 - a_2). \quad \text{(B.6)}$$

It has a non-zero group velocity at $k = 0$

$$\tilde{v}_k(k) \big|_{k=0} = \frac{\partial \tilde{\epsilon}_{k, \text{odd}}(k)}{\partial k} \big|_{k=0} = \frac{w_1 w_2 [h \cdot (a_1 \times a_2)]}{2\omega [4W^2 - \omega^2]} (a_1 - a_2). \quad \text{(B.7)}$$

This equation is consistent with Eq. \footnote{[5,6]} to the lowest order in $w_1 w_2$. 