ON THE BOUNDED APPROXIMATION PROPERTY IN BANACH SPACES

JESÚS M. F. CASTILLO AND YOLANDA MORENO

Abstract. We prove that the kernel of a quotient operator from an $L_1$-space onto a Banach space $X$ with the Bounded Approximation Property (BAP) has the BAP. This completes earlier results of Lusky –case $\ell_1$– and Figiel, Johnson and Peczynski –case $X^*$ separable. Given a Banach space $X$, we show that if the kernel of a quotient map from some $L_1$-space onto $X$ has the BAP then every kernel of every quotient map from any $L_1$-space onto $X$ has the BAP. The dual result for $L_\infty$-spaces also hold: if for some $L_\infty$-space $E$ some quotient $E/X$ has the BAP then for every $L_\infty$-space $E$ every quotient $E/X$ has the BAP.

1. Preliminaries

An exact sequence $0 \to Y \to X \to Z \to 0$ in the category of Banach spaces and bounded linear operators is a diagram in which the kernel of each arrow coincides with the image of the preceding; the middle space $X$ is also called a twisted sum of $Y$ and $Z$. By the open mapping theorem this means that $Y$ is isomorphic to a subspace of $X$ and $Z$ is isomorphic to the corresponding quotient. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there exists an operator $T : X \to X_1$ making commutative the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
\| & \downarrow T & \| & & & & & & \\
0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0.
\end{array}
$$

The classical 3-lemma (see [4, p. 3]) shows that $T$ must be an isomorphism. An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. There is a correspondence (see [4, Thm. 1.5.c, Section 1.6]) between exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces and the so-called $z$-linear maps which are homogeneous maps $\omega : Z \curvearrowright Y$ (we use this notation to stress the fact that these are not linear maps) with the property that there exists some constant $C > 0$ such that for all finite sets $x_1, \ldots, x_n \in Z$ one has $\|\omega(\sum_{n=1}^N x_n) - \sum_{n=1}^N \omega(x_n)\| \leq C \sum_{n=1}^N \|x_n\|$. The infimum of those constants $C$ is called the $z$-linearity constant of $F$ and denoted $Z(\omega)$.

The process to obtain a $z$-linear map out from an exact sequence $0 \to Y \to X \to Z \to 0$ is the following: get a homogeneous bounded selection $b : Z \to X$ for the quotient map $q$, and then a linear $\ell : Z \to X$ selection for the quotient map. Then $\omega = b - \ell$ is a $z$-linear map. A $z$-linear map $\omega : Z \curvearrowright Y$ induces the exact sequence of Banach spaces $0 \to Y \to Y \oplus Z \to Z \to 0$ in which $Y \oplus Z$ means the completion of the vector space $Y \times X$ endowed with the quasi-norm $\|(y, x)\|_\omega = \|y - \omega(x)\| + \|x\|$. The $z$-linearity of $\omega$ makes this quasi-norm equivalent to a norm (see [3]). The embedding is $j(y) = (y, z)$.
while the quotient map is \( p(y, z) = z \). The exact sequences

\[
\begin{align*}
0 & \longrightarrow Y \overset{i}{\longrightarrow} X \overset{q}{\longrightarrow} Z \longrightarrow 0 \\
0 & \longrightarrow Y \overset{j}{\longrightarrow} Y \oplus_{\omega} Z \overset{p}{\longrightarrow} Z \longrightarrow 0
\end{align*}
\]

are equivalent setting as \( T : X \rightarrow Y \oplus_{\omega} Z \) the operator \( T(x) = (x - \ell qx, qx) \). We will use the notation \( 0 \rightarrow Y \overset{j}{\rightarrow} X \overset{i}{\rightarrow} Z \rightarrow 0 \equiv \omega \) to mean that \( \omega \) is a \( z \)-linear map associated to that exact sequence. Two \( z \)-linear maps \( \omega, \omega' : Z \rightarrow Y \) are said to be equivalent, and we write \( \omega \equiv \omega' \), if the induced exact sequences are equivalent. Two maps \( \omega, \omega' : Z \rightarrow Y \) are equivalent if and only if the difference \( \omega - \omega' \) can be written as \( B + L \), where \( B : Z \rightarrow Y \) is a homogeneous bounded map and \( L : Z \rightarrow Y \) is a linear map.

Recall from [7] the definition and basic properties of the \( z \)-dual of a Banach space \( X \). We define the \( z \)-dual of \( X \) as the Banach space \( X^z = [Z_L(X, \mathbb{R}), Z()] \) of \( z \)-linear maps \( \omega : X \rightarrow \mathbb{R} \) such that \( \omega(e_\gamma) = 0 \) for a prefixed Hamel basis \( (e_\gamma) \) of \( X \). The space \( co_z(X) \) is defined as the closed linear span in \((X^z)^*\) of the evaluation functionals \( \delta_x : X^z \rightarrow \mathbb{R} \) given by \( \delta_x(\omega) = \omega(x) \). Given a \( z \)-linear map \( \omega : X \rightarrow Y \) and a Hamel basis \( (e_\gamma) \) we denote \( \nabla \omega \) the so-called canonical form of \( \omega \), namely, the map

\[
\nabla \omega(p) = \omega(p) - \sum \lambda_j \omega(e_j)
\]

where \( p = \sum \lambda_j e_j \). The two basic properties of \( co_z(X) \) are displayed in the following proposition; the proof can be found in [7, Prop. 3.2]:

**Proposition 1.1.** There is a \( z \)-linear map \( \Omega_X : X \rightarrow co_z(X) \) with the property that given a \( z \)-linear map \( \omega : X \rightarrow Y \) then there exists an operator \( \phi_\omega : co_z(X) \rightarrow Y \) such that \( \phi_\omega \Omega_X = \nabla \omega \) and \( \|\phi_\omega\| = Z(\omega) \).

It is easy to give examples of \( z \)-linear maps on finite dimensional spaces with infinite-dimensional range: set \( B : \mathbb{R}^2 \rightarrow C[0, 1] \) defined by \( B(e^{i\theta}) = x^\theta \), \( 0 \leq \theta < \pi \), and by homogeneity on the rest. It is however possible to modify a \( z \)-linear map defined on a finite-dimensional Banach space so that the image is finite dimensional [7, Lemma 2.3]. We need an improvement of that result:

**Lemma 1.1.** Let \( \Omega : X \rightarrow Y \) be a \( z \)-linear map and let \( F \) be a finite dimensional subspace of \( X \). Given a finite set of points \( x_1, \ldots, x_n \) in the unit sphere of \( F \) so that \( x_i \neq \pm x_j \) for \( i \neq j \) and \( \varepsilon > 0 \), there is a \( z \)-linear map \( \Omega_F : X \rightarrow Y \) verifying:

1. \( \Omega_F x_i = \Omega x_i \) for all \( 1 \leq i \leq n \).
2. \( \|\Omega_F - \Omega\| \leq (1 + \varepsilon)Z(\Omega) \).
3. \( Z(\Omega_F) \leq (3 + \varepsilon)Z(\Omega) \).
4. The image of \( F \) by \( \Omega_F \) spans a finite dimensional space.

**Proof.** Let \( (e_\gamma) \) be a Hamel basis for \( X \) formed by norm one vectors. Assume that \( F \subset [e_{\gamma_1}, \ldots, e_{\gamma_n}] \). Fix a finite set \( A = \{x_1, \ldots, x_n\} \) of elements of norm at most \( 1 + \varepsilon \) such that the unit ball of \( F \) is contained in the convex hull of \( A \). Be sure that the set \( A \) contains all \( e_{\gamma_1}, \ldots, e_{\gamma_n} \) and that \( x_i \neq \pm x_j \) for \( i \neq j \). Modify now \( \Omega \) as follows: \( \Omega_F a = \Omega a \) for all \( a \in A \); if \( p \) is a norm one element of \( F \) and \( p = \sum \theta_i x_i \) then \( \Omega_F(\sum \theta_i x_i) = \sum \theta_i \Omega x_i \). It is necessary to establish some order between the convex combinations of the \( x_i \) to have the value of \( \Omega_F \) univocally defined and equal to \( \Omega a \) at the points of \( A \), but this can be easily done in many ways. Condition (1) is therefore fulfilled. We show (2): if
ON THE BOUNDED APPROXIMATION PROPERTY IN BANACH SPACES

$p = \sum \theta_i x_i$ is a norm one element of $F$ then

$$
\left\| \Omega \left( \sum \theta_i x_i \right) - \Omega \left( \sum \theta_i x_i \right) \right\| = \left\| \sum \theta_i \Omega x_i - \Omega \left( \sum \theta_i x_i \right) \right\|
\leq Z(\Omega) \sum \theta_i \|x_i\|
\leq (1 + \varepsilon) Z(\Omega).
$$

If $p$ is a point not in $F$ then $\Omega_F(p) = \Omega(p)$, and thus (2) is proved. Observe that if $\Lambda$ is a bounded homogeneous map then $Z(\Lambda) \leq 2\|\Lambda\|$. This and the previous estimate yield (3):

$$
Z(\Omega_F) = Z(\Omega_F - \Omega + \Omega)
\leq Z(\Omega_F - \Omega) + Z(\Omega)
\leq 2\|\Omega_F - \Omega\| + Z(\Omega)
\leq 2(1 + \varepsilon)Z(\Omega) + Z(\Omega)
= 3(1 + \varepsilon)Z(\Omega).
$$

Condition (4) is clear.

A few facts about the connections between $z$-linear maps and the associated exact sequences will be needed in this paper. Given an exact sequence $0 \to Y \to X \to Z \to 0 \equiv \omega$ and an operator $\alpha : Y \to Y'$, there is a commutative diagram

$$
0 \longrightarrow Y \xrightarrow{j} Y \oplus \omega Z \xrightarrow{p} Z \longrightarrow 0
$$

in which $\tau(y, z) = (\alpha y, z)$. Moreover,

**Lemma 1.2.** If one has a commutative diagram

$$
0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \longrightarrow 0 \equiv \omega
$$

(1)

then

$$
0 \longrightarrow Y' \xrightarrow{i'} X' \xrightarrow{q'} Z \longrightarrow 0 \equiv \alpha \omega
$$

i.e., $\alpha \omega$ is a $z$-linear map associated with the lower sequence in (1).

**Proof.** What one has to check is that the two sequences

$$
0 \longrightarrow Y' \xrightarrow{j'} Y' \oplus \alpha \omega Z \xrightarrow{p'} Z \longrightarrow 0
$$

are equivalent; and this happens via the map $\tau'(y, z) = y' + T(0, z)$. Indeed, that the map $\tau'$ makes the diagram commutative is clear; its continuity follows from the estimate:
\[ \|y' + T(0, z)\| \leq \|y' - \alpha_\omega z\| + \|\alpha_\omega z + T(0, z)\| \]
\[ = \|y' - \alpha_\omega z\| + \|\alpha_\omega z + T(\omega z, z) + T(-\omega z, 0)\| \]
\[ \leq \|y' - \alpha_\omega z\| + \|T(\omega z, z)\| \]
\[ \leq \|y' - \alpha_\omega z\| + \|(\omega z, z)\|_\omega \]
\[ = \|y' - \alpha_\omega z\| + \|z\| \]
\[ = \|(y', z)\|_{\alpha\omega}. \]
\[ \square \]

Analogously, given an exact sequence \(0 \to Y \to X \to Z \to 0 \equiv \omega\) and an operator \(\gamma : Z' \to Z\), there is a commutative diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \overset{j}{\longrightarrow} & Y \oplus \omega Z & \overset{p}{\longrightarrow} & Z & \longrightarrow & 0 \\
\| & & \| & & \tau' & & \| & & \| & & \gamma \|
\end{array}
\]
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y' & \overset{i'}{\longrightarrow} & Y' \oplus \omega Z' & \overset{q'}{\longrightarrow} & Z' & \longrightarrow & 0 \\
\| & & \| & & \tau' & & \| & & \| & & \gamma \|
\end{array}
\]
in which \(\tau(y, z') = (y, \gamma z')\). Moreover,

**Lemma 1.3.** If one has a commutative diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \overset{i}{\longrightarrow} & X & \overset{q}{\longrightarrow} & Z & \longrightarrow & 0 \equiv \omega \\
\| & & \| & & T & & \| & & \| & & \gamma \|
\end{array}
\]
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y' & \overset{i'}{\longrightarrow} & X' & \overset{q'}{\longrightarrow} & Z' & \longrightarrow & 0 \\
\| & & \| & & \tau' & & \| & & \| & & \gamma \|
\end{array}
\]
then
\[
0 \longrightarrow Y \overset{\ell}{\longrightarrow} X' \overset{q'}{\longrightarrow} Z' \longrightarrow 0 \equiv \omega \gamma
\]
i.e., \(\omega \gamma\) is a \(z\)-linear map associated with the lower sequence in (2).

**Proof.** One has to check that the two sequences
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \overset{j'}{\longrightarrow} & Y \oplus \omega Z' & \overset{q'}{\longrightarrow} & Z' & \longrightarrow & 0 \\
\| & & \| & & \tau' & & \| & & \| & & \gamma \|
\end{array}
\]
\[
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \overset{i'}{\longrightarrow} & X' & \overset{q'}{\longrightarrow} & Z' & \longrightarrow & 0 \\
\| & & \| & & \tau' & & \| & & \| & & \gamma \|
\end{array}
\]
are equivalent; and this happens via the map \(\tau'(x') = (Tx' - \ell \gamma q' x', q' x')\), where \(\ell\) is a linear selection for \(q\) such that for some bounded selection \(b\) for \(q\) one has \(\omega = b - \ell\). Which means that \(\omega \gamma\) is a \(z\)-linear map associated with the lower sequence. The commutativity of the diagram is clear and the continuity of \(\tau'\) follows from the estimate:
\[
\|(Tx' - \ell \gamma q' x', q' x')\|_{\omega \gamma} = \|Tx' - \ell \gamma q' x' - \omega \gamma q' x'| + \|q' x'| \]
\[= \|Tx' - b \gamma q' x'| + \|q' x'| \]
\[\leq (\|T\| + \|b\| \|\gamma\| + 1) \|x'|. \]
\[ \square \]

As an immediate consequence we have.
Lemma 1.4.

(1) Given a commutative diagram like (1) the exact sequence \( \omega q' \) is equivalent to the exact sequence

\[
0 \to Y \xrightarrow{d} Y' \oplus X \xrightarrow{m} X' \to 0
\]

where \( d(y) = (-\alpha y, i'y) \) and \( m(y', x) = y' + Tx \).

(2) Given a commutative diagram like (2) the exact sequence \( i'\omega \) is equivalent to the exact sequence

\[
0 \to X' \xrightarrow{J} X \oplus Z' \xrightarrow{Q} Z \to 0
\]

where \( J(x') = (Tx', q'x') \) and \( Q(x, z') = qx - \gamma z' \).

Proof. To prove (1) observe that there is a commutative diagram

\[
\begin{array}{c}
0 \to Y \xrightarrow{i} X \\
\| \downarrow \tau \downarrow \| \\
0 \to Y \xrightarrow{d} Y' \oplus X \xrightarrow{m} X' \to 0
\end{array}
\]

where \( \tau(y', x) = x \) and apply Lemma 1.3. To prove (2) observe that there is a commutative diagram

\[
\begin{array}{c}
0 \to Y \xrightarrow{i} X \\
\| \downarrow \tau \downarrow \| \\
0 \to X' \xrightarrow{J} X \oplus Z' \xrightarrow{Q} Z \to 0
\end{array}
\]

where \( \tau(x) = (x, 0) \) and apply Lemma 1.2. \( \square \)

Following [12], an exact sequence \( 0 \to Y \to X \to Z \to 0 \) is said to locally split if its dual sequence \( 0 \to Z^* \to X^* \to Y^* \to 0 \) splits.

Lemma 1.5. An exact sequence \( 0 \to Y \to X \to Z \to 0 \equiv \omega \) (locally) splits if and only if for every operator \( \alpha : Y \to Y \) and \( \gamma : Z' \to Z \) the sequence \( \alpha \omega \gamma \) (locally) splits.

Proof. The sufficiency is obvious in both cases. To prove the necessity is simple for the splitting: if \( \omega = B + L \) with \( B \) homogeneous bounded and \( L \) linear then \( \alpha \omega \gamma = \alpha B \gamma + \alpha L \gamma \) with \( \alpha B \gamma \) homogeneous bounded and \( \alpha L \gamma \) linear. The case of local spliting follows from this and the observation that if one has the commutative diagram

\[
\begin{array}{c}
0 \to Y \xrightarrow{\alpha} X \\
\| \downarrow \| \\
0 \to Y' \xrightarrow{i} E \xrightarrow{\gamma} Z \to 0 \equiv \alpha \omega
\end{array}
\]

Following [12], an exact sequence \( 0 \to Y \to X \to Z \to 0 \equiv \omega \) (locally) splits if and only if for every operator \( \alpha : Y \to Y \) and \( \gamma : Z' \to Z \) the sequence \( \alpha \omega \gamma \) (locally) splits.
then the biduals form also a commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & Y^{**} & \longrightarrow & X^{**} & \longrightarrow & Z^{**} & \longrightarrow & 0 = \Omega \\
\alpha^{**} & \downarrow & & \downarrow & & & & \downarrow & \\
0 & \longrightarrow & Y^{**} & \longrightarrow & E^{**} & \longrightarrow & Z^{**} & \longrightarrow & 0 = \alpha^{**} \Omega \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \longrightarrow & Y^{**} & \longrightarrow & X^{**} & \longrightarrow & Z^{**} & \longrightarrow & 0 = \alpha^{**} \Omega \gamma^{**}.
\end{array}
\]
Kalton shows in [12] Thm. 3.5 that an exact sequence locally splits if and only if its bidual sequence splits. Thus, since \( \Omega \) splits, so does \( \alpha^{**} \Omega \gamma^{**} \).

\[ \square \]

**Definition 1.1.** A Banach space \( X \) has the \( \lambda \)-BAP if for each finite dimensional subspace \( F \subset X \) there is a finite rank operator \( T : X \rightarrow X \) such that \( \| T \| \leq \lambda \) and \( T(f) = f \) for each \( f \in F \).

It is well known that in a locally splitting sequence \( 0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0 \) one has: i) if \( Y,Z \) have the BAP then also \( X \) has the BAP [10]; ii) if \( X \) has the BAP then \( Y \) has the BAP [12] Thm. 5.1.

2. Results for \( L_1 \)-spaces

We assume in what follows that \( L_1 \) denotes an arbitrary \( L_1 \)-space. In [16, 17] Lusky shows that when \( X \) is separable and has the BAP then the kernel of every quotient map \( \ell_1 \rightarrow X \) has the BAP. Theorem 2.1 (b) of [8] asserts that if \( q : L_1 \rightarrow X \) is a quotient map and \( X^{\ast} \) has the BAP then both \( \ker q \) and \((\ker q)^{\ast}\) have the BAP. It is well-known that when \( X^{\ast} \) has the BAP then also \( X \) has the BAP, but the converse fails since there exist spaces with basis whose dual do not have AP [15, 1.e.7(b)]. Therefore, the missing case is to show that the kernel of an arbitrary quotient map \( L_1 \rightarrow X \) has the BAP when \( X \) has the BAP.

**Lemma 2.1.** Let \( X \) be a Banach space with the \( \lambda \)-BAP. Then \( \text{co}_2(X) \) has the \((3\lambda + \varepsilon)\)-BAP.

**Proof.** Fix a Hamel basis \( (e_\gamma)_{\gamma \in \Gamma} \) for \( X \) and let \( \Omega_X : X \cap \text{co}_2(X) \) the universal map appearing in Proposition 1.1 verifying \( \Omega_X e_\gamma = 0 \) for all \( \gamma \in \Gamma \). Let \( \mathfrak{F} \) be a finite dimensional subspace of \( \text{co}_2(X) \). We can assume without loss of generality that \( \mathfrak{F} \subset [\Omega_X x_1, \ldots, \Omega_X x_m] \). Take \( [x_1, \ldots, x_m] \subset X \) and let \( F \) be a finite set of \( \gamma \) for which \( [x_1, \ldots, x_m] \subset [e_\gamma : \gamma \in F] \). Let \( B_F : X \rightarrow X \) be a finite rank operator fixing \( [e_\gamma : \gamma \in F] \). Let \( \Omega_{B_F(X)} \) be the version of \( \Omega_X \) verifying that the image of \( B_F(X) \) is finite-dimensional, which has moreover been done so that \( \Omega_{B_F(X)} x_k = \Omega_X x_k \) and \( \Omega_{B_F(X)} e_\gamma = \Omega_X e_\gamma \) for all \( 1 \leq k \leq m \) and all \( \gamma \in F \). By the properties of \( \text{co}_2(X) \) there is an operator \( \phi_F : \text{co}_2(X) \rightarrow \text{co}_2(X) \) such that

\[
\phi_F \Omega_X = \nabla(\Omega_{B_F(X)} B_F).
\]

Given any \( p \in X \), if \( p = \sum_j \lambda_j e_j \) then

\[
\nabla(\Omega_{B_F(X)} B_F)(p) = \Omega_{B_F(X)} B_F p - \sum_j \lambda_j \Omega_{B_F(X)} B_F e_j
\]

and therefore the image of \( \nabla(\Omega_{B_F(X)} B_F) \) spans a finite dimensional space. This means that also the range of \( \phi_F \) is finite dimensional. Moreover, \( \phi_F \) fixes \( \mathfrak{F} \) since for \( \Omega_X x_k, 1 \leq k \leq m \) one has that if \( x_k = \sum \lambda_i e_\gamma_i \) with \( \gamma_i \in F \) one has
ON THE BOUNDED APPROXIMATION PROPERTY IN BANACH SPACES

\[ \phi_F(\Omega X x_k) = \nabla(\Omega_{B_F(X)}B_F)(x_k) \]
\[ = \Omega_{B_F(X)}B_F(x_k) - \sum \lambda_i \Omega_{B_F(X)}B_F e_{\gamma_i} \]
\[ = \Omega_{B_F(X)}x_k - \sum \lambda_i \Omega_{B_F(X)}e_{\gamma_i} \]
\[ = \Omega X x_k \]

Finally,
\[ \|\phi_F\| = Z(\nabla(\Omega_{B_F(X)}B_F)) = Z(\Omega_{B_F(X)}B_F) \leq Z(\Omega_{B_F(X)})\|B_F\| \leq (3 + \varepsilon)Z(\Omega_X)\lambda. \]

\[ \square \]

**Theorem 2.1.** Let \( 0 \to Y \to \mathcal{L}_1 \to X \to 0 \) be an exact sequence in which \( X \) has the BAP. Then \( Y \) has the BAP.

**Proof.** The universal property of the \( \text{co}_z(\cdot) \) construction mentioned in Proposition 1.1 yields a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & \text{co}_z(X) & \to & \Sigma & \to & X & \to & 0 \\
& & \downarrow \phi & & \downarrow & & \equiv & \Omega_X \\
0 & \to & Y & \to & \mathcal{L}_1 & \to & X & \to & 0.
\end{array}
\]

Therefore, by virtue of Lemma 1.4 (1) there is an exact sequence

\[ 0 \longrightarrow \text{co}_z(X) \longrightarrow \Sigma \oplus Y \longrightarrow \mathcal{L}_1 \longrightarrow 0 \equiv \Omega_X p. \]

This sequence locally splits: the dual sequence splits since the dual of an \( \mathcal{L}_1 \)-space is injective. Thus, \( \Sigma \oplus Y \) has the BAP, as well as both \( \Sigma \) and \( Y \). \[ \square \]

**Question.** Does a similar result hold for other well-known variations of the BAP such as the commuting bounded approximation property (CBAP), the uniform approximation property (UAP) or the existence of finite dimensional decomposition (FDD)? The previous proof cannot be translated to cover the case of the CBAP or FDD since these properties do not pass to complemented subspaces \[ \square \]; also, the result cannot be translated to the case of the UAP since Lemma 1.1 spoils the estimate on the dimension of the required finite dimensional operator.

The phenomenon described in the proposition –all the kernels of all the quotient maps from an \( \mathcal{L}_1 \)-space onto a space with the BAP have the BAP– is part of a general stability result

**Proposition 2.1.** Given two exact sequences

\[ \begin{align*}
0 & \longrightarrow Y \longrightarrow \mathcal{L}_1 \longrightarrow Z \longrightarrow 0 \\
0 & \longrightarrow Y' \longrightarrow \mathcal{L}'_1 \longrightarrow Z \longrightarrow 0
\end{align*} \]

Then \( Y \) has the BAP if and only if \( Y' \) has the BAP.
Proof. Observe that if one has a commutative diagram formed by short exact sequences

\[
\begin{array}{c}
\omega' q \\
\parallel \parallel \\
0 \\
\uparrow \\
0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0 \equiv \omega \\
\parallel \\
\uparrow \\
0 \longrightarrow Y \longrightarrow E \longrightarrow X' \longrightarrow 0 \equiv \omega q' \\
\parallel \\
\uparrow \\
Y' = Y' \\
\parallel \\
\uparrow \\
0 \\
\end{array}
\]

in which we assume that:

1. \(\omega q'\) and \(\omega' q\) locally split.
2. \(X\) has the BAP
3. \(Y'\) has the BAP.

then also \(Y\) has the BAP: Since \(\omega' q\) locally split and both \(X, Y'\) have the BAP, then \(E\) has the BAP \([10]\), see also \([4, \text{Thm. 7.3.e}]\). Since \(\omega q'\) locally splits, then \(Y\) has the BAP. Apply now this schema to diagram \((3)\) depicted in the form

\[
\begin{array}{c}
\omega' q \\
\parallel \parallel \\
0 \\
\uparrow \\
0 \longrightarrow Y \longrightarrow L_1 \longrightarrow Z \longrightarrow 0 \equiv \omega \\
\parallel \\
\uparrow \\
0 \longrightarrow Y \longrightarrow E \longrightarrow L_1' \longrightarrow 0 \equiv \omega q' \\
\parallel \\
\uparrow \\
Y' = Y' \\
\parallel \\
\uparrow \\
0 \\
\end{array}
\]

and recall that both \(\omega' q\) and \(\omega q'\) locally split since the quotient is an \(L_1\)-space. □

The following lemma, rather its dual version, will be required to work with \(L_\infty\)-spaces; we include its proof for the sake of completeness.
**Lemma 2.1.** Given an exact sequence $0 \to Y \to X \to \mathcal{L}_1 \to 0$, in which $X$ has the BAP then the space $Y$ has the BAP.

**Proof.** General properties of $\ell_1(\Gamma)$ spaces yield a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & K & \longrightarrow & \ell_1(\Gamma) & \longrightarrow & \mathcal{L}_1 & \longrightarrow & 0 \equiv \omega \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & \mathcal{L}_1 & \longrightarrow & 0.
\end{array}
$$

By Lemma 1.4 (1) the exact sequence associated to $\omega q$ has the form

$$
0 \longrightarrow K \longrightarrow Y \oplus \ell_1(\Gamma) \longrightarrow X \longrightarrow 0.
$$

This sequence locally splits since $\omega$ locally splits. The space $K$ must have the BAP and, thus, if $X$ has the BAP then $Y \oplus \ell_1(\Gamma)$ has the BAP, which implies that also $Y$ has the BAP. \hfill \Box

### 3. Dual results for $\mathcal{L}_\infty$-spaces and $z$-duals

We assume in what follows that $\mathcal{L}_\infty$ denotes an arbitrary $\mathcal{L}_\infty$-space. For the case of $\mathcal{L}_\infty$-spaces the result (and proof) of Figiel, Johnson and Pełczyński [8, Thm. 2.1.(a)] is already optimal:

**Proposition 3.1.** Let $0 \to Y \to \mathcal{L}_\infty \to Z \to 0$ be an exact sequence in which $Y$ has the BAP. Then $Z$ has the BAP.

Let us show that the dual result of Proposition 2.1 for $\mathcal{L}_\infty$-spaces also holds. We need a lemma, dual of Lemma 2.1.

**Lemma 3.1.** Given an exact sequence $0 \to \mathcal{L}_\infty \to X \to Z \to 0$ in which $X$ has the BAP then the space $Z$ has the BAP.

**Proof.** By the injectivity properties of $\ell_\infty(\Gamma)$ spaces, there is a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{L}_\infty & \longrightarrow & \ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/\mathcal{L}_\infty & \longrightarrow & 0 \equiv \omega \\
0 & \longrightarrow & \mathcal{L}_\infty & \longrightarrow & \ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/\mathcal{L}_\infty & \longrightarrow & 0.
\end{array}
$$

By Lemma 1.4 (2) the exact sequence associated to $j\omega$ has the form

$$
0 \longrightarrow \mathcal{L}_\infty \longrightarrow \ell_\infty(\Gamma) \oplus Z \longrightarrow \ell_\infty(\Gamma)/\mathcal{L}_\infty \longrightarrow 0.
$$

This sequence locally splits since $\omega$ locally splits. The space $\ell_\infty(\Gamma)/\mathcal{L}_\infty$, as a quotient of two $\mathcal{L}_\infty$-spaces, is an $\mathcal{L}_\infty$-space, hence it has the BAP. Thus, if $X$ has the BAP then $\ell_\infty(\Gamma) \oplus Z$ has the BAP, which implies that also $Z$ has the BAP. \hfill \Box

Recall that the fact that both $Y, X$ have the BAP in a locally splitting sequence $0 \to Y \to X \to Z \to 0$ does not imply that $Z$ has the BAP: indeed, every separable Banach space $Z$ is a quotient $X^{**}/X$ in which both $X, X^{**}$ have a basis [14]. Taking as $Z$ a space without BAP provides the example since every sequence $0 \to X \to X^{**} \to Z \to 0$ locally splits.

We are ready to show the dual of Proposition 2.1 if for some $\mathcal{L}_\infty$-space the quotient $\mathcal{L}_\infty/X$ has the BAP the same happens for all $\mathcal{L}_\infty$-spaces.
**Proposition 3.2.** Given two exact sequences
\[
\begin{align*}
0 & \longrightarrow Y \longrightarrow \mathcal{L}_\infty \longrightarrow Z \longrightarrow 0 \\
0 & \longrightarrow Y' \longrightarrow \mathcal{L}'_\infty \longrightarrow Z' \longrightarrow 0
\end{align*}
\]

Then \(Z\) has the BAP if and only if \(Z'\) has the BAP.

**Proof.** Draw the sequences forming a commutative diagram
\[
\begin{array}{c}
\omega' \\
\| \| \\
\| \|
\end{array}
\quad
\begin{array}{c}
\iota \omega' \\
\| \| \\
\| \|
\end{array}
\quad
\begin{array}{c}
0 \\
\| \|
\| \|
\downarrow \\
\downarrow \\
0 \\
\downarrow \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\iota' \\
\| \|
\| \|
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0 = Z'
\end{array}
\quad
\begin{array}{c}
\| \|
\| \|
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0
\end{array}
\]

and assume that \(Z'\) has the BAP. Since \(i\omega'\) locally splits, \(PO\) must have the BAP and thus Lemma 3.1 applies to conclude that also \(Z\) has the BAP.

The following result can be considered the nonlinear version of Proposition 3.1, and the dual of “\(X\) has the BAP implies \(co_z(X)\) has the BAP”. It is somehow surprising since \(X^*\) apparently disappears in the construction of \(X^z\).

**Proposition 3.3.** If \(X^*\) has the BAP then also \(X^z\) has the BAP.

**Proof.** Let \(Z(X, \mathbb{R})\) denote the space of \(z\)-linear maps, \(B(X, \mathbb{R})\) the space of bounded homogeneous maps, \(L(X, \mathbb{R})\) the space of linear maps and, as usual, \(\mathcal{L}(X, \mathbb{R}) = X^*\) is the space of linear continuous maps on \(X\). Since \(Z(X, \mathbb{R}) = B(X, \mathbb{R}) + L(X, \mathbb{R})\) and \(B(X, \mathbb{R}) \cap L(X, \mathbb{R}) = \mathcal{L}(X, \mathbb{R})\) the diamond lemma applied to
\[
\begin{array}{c}
Z(X, \mathbb{R}) \\
\uparrow \quad \downarrow \\
B(X, \mathbb{R}) \quad L(X, \mathbb{R}) \\
\downarrow \quad \uparrow \\
\mathcal{L}(X, \mathbb{R})
\end{array}
\]
yields \(Z(X, \mathbb{R})/L(X, \mathbb{R}) = B(X, \mathbb{R})/\mathcal{L}(X, \mathbb{R})\). But observe that, algebraically speaking, \(Z(X, \mathbb{R})/L(X, \mathbb{R})\) is the space \(Z_L(X, \mathbb{R})\); moreover, the uniform boundedness principle for exact sequences \([1]\) yields that
they are also isomorphic as Banach spaces. Hence
\[ X^z = Z_L(X, \mathbb{R}) \cong B(X, \mathbb{R})/X^*. \]

In the same form as the space of bounded functions on \( X \) is \( \ell_\infty(X) \), the space \( B(X, \mathbb{R}) \) is \( \ell_\infty(S^+) \), where \( S^+ \) is a half of the unit sphere (i.e., a subset of the unit sphere with the property that for every norm one \( x \), either \( x \) or \( -x \) is in \( S^+ \)). Therefore, Proposition 3.1 and the isomorphism
\[ X^z \cong \ell_\infty(S^+)/X^* \]
complete the proof.

The referee has pointed out the question of whether “\( c_0*(X) \) has the BAP implies \( X \) has the BAP”, as it occurs in the case of the Lipschitz-free space of Godefroy and Kalton [9]. This is a tough question: By the results in this paper, if \( c_0*(X) \) has the BAP then the kernel of every quotient map \( q : L_1 \to X \) has the BAP. But there are Banach spaces \( Z \) with the BAP which are subspaces of some \( L_1 \)-space so that \( L_1/Z \) does not have the BAP. Indeed, Prof. Szankowski has kindly informed us the classical Enflo-Davie example provides a subspace \( S \) of \( c_0 \) without AP for which he proved that \( c_0/S \) has the BAP (and then, all quotients \( L_\infty/S \) have the BAP). Actually \( c_0/S \) is isomorphic to \( c_0(\ell_2^\infty) \) and thus its dual \( \ell_1(\ell_2^\infty) \) also has the BAP. Thus, there is an exact sequence
\[ 0 \to \ell_1(\ell_2^\infty) \to \ell_1 \to S^* \to 0 \]
in which \( S^* \) fails to have the BAP. It is likely that \( c_0(S^*) \) has the BAP while \( S^* \) does not, but we could not prove it. Since every infinite dimensional \( L_1 \)-space contains a complemented copy of \( \ell_1 \) one has

**Lemma 3.1.** Every infinite dimensional \( L_1 \)-space contains a subspace \( R_1 \) with the BAP so that \( L_1/R_1 \) does not have the BAP.

On the other hand, the space \( \ell_p \) admits for \( 1 < p < 2 \) a subspace without the BAP [13], so \( \ell_p \) admits a quotient without BAP for \( 2 < p < +\infty \). Since \( \ell_p \) is a quotient of \( C[0,1] \) for \( 2 < p < +\infty \), the existence of subspaces \( J \) of \( C[0,1] \) such that \( C[0,1]/J \) does not have the BAP is clear (this short line was mentioned to us by Bill Johnson). In particular, \( J \) does not have the BAP and no quotient \( L_\infty/J \) can have the BAP. The existence of \( J \) means that the claim [5, Thm. 7.1] “Every separable Banach space \( X \) can be embedded into some \( L_\infty \) space in such a way that they verify some additional properties and \( L_\infty/X \) has the BAP” is wrong. Thus, Zippin’s result—every separable Banach space \( X \) can be embedded into some \( Z(X) \) in such a way that they verify the same additional properties as before and \( Z(X)/X \) has FDD [20]—cannot be (easily) improved.

**References**

[1] F. Cabello Sánchez and J.M.F. Castillo, Uniform boundedness and twisted sums of Banach spaces, Houston J. Math. 30 (2004), 523–536.
[2] P.G. Casazza, Approximation Properties, in Handbook of the Geometry of Banach Spaces, vol. I, W.B. Johnson and J. Lindenstrauss (eds.), North Holland 2001, pp.271-316.
[3] J.M.F. Castillo, Banach spaces, a la recherche du temps perdu, Extracta Math. 15 (2000) 291-334.
[4] J.M.F. Castillo and M. González, Three-space problems in Banach space theory, Springer Lecture Notes in Math. 1667, 1997.
[5] J.M.F. Castillo and Y. Moreno, The category of exact sequences of Banach spaces, Proceedings of the V Conference on Banach spaces, Caceres 2004; J.M.F. Castillo and W.B. Johnson (eds.). London Mathematical Society Lecture Notes Series, Cambridge University Press.
[6] J.M.F. Castillo and Y. Moreno, On the Lindenstrauss-Rosenthal theorem, Israel J. Math. 140 (2004) 253-270.
[7] J.M.F. Castillo and Y. Moreno, Sobczyk’s theorem and the Bounded Approximation Property, Studia Math. 201 (2010), 1-19.
[8] T. Figiel, W.B. Johnson and A. Pelczyński, Some approximation properties of Banach spaces and Banach lattices, Israel J. Math 183 (2011) 199-292.
[9] G. Godefroy and N. J. Kalton, Lipschitz-free Banach spaces, Studia Math. 159 (2003), 121–141.
[10] G. Godefroy and P. Saphar, Three-space problems for the approximation properties, Proc. Amer. Math. Soc. 105 (1989) 70-75.
[11] E. Hilton and K. Stammbach, A course in homological algebra, Graduate Texts in Mathematics 4, Springer-Verlag 1970.
[12] N. J. Kalton, Locally complemented subspaces and $L_p$ spaces for $0 < p < 1$, Math. Nachr. 115 (1984) 71–97.
[13] N.J. Kalton and N.T. Peck, Twisted sums of sequence spaces and the three-space problem, Tran. Amer. Math. Soc. 255 (1979) 1-30.
[14] J. Lindenstrauss, On James’s paper “Separable conjugate spaces”, Israel J. Math. 9 (1971) 279-284.
[15] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, sequence spaces, Ergeb. Math. 92, Springer-Verlag 1977.
[16] W. Lusky, Three-space problems and basis extensions, Israel J. Math.107 (1988) 17-27.
[17] W. Lusky, Three-space problems and bounded approximation property, Studia Math. 159 (2003) 417-434.
[18] A. Szankowski, Subspaces without the approximation property, Israel J. Math. 30 (1978) 123-129.
[19] A. Szankowski, Three-space problems for the approximation property, J. Eur. Math. Soc. 11 (2009) 273-282.
[20] M. Zippin, The embedding of Banach spaces into spaces with structure, Illinois J. Math. 34 (1990) 586–606.