ON THE CONSISTENCY OF THE DEFINABLE TREE PROPERTY ON $\aleph_1$

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Abstract. In this paper we prove the equiconsistency of “Every $\omega_1$—tree which is first order definable over $(H_{\omega_1}, \varepsilon)$ has a cofinal branch” with the existence of a $\Pi^1_1$ reflecting cardinal. We also prove that the addition of $MA$ to the definable tree property increases the consistency strength to that of a weakly compact cardinal. Finally we comment on the generalization to higher cardinals.

1. Introduction

A well known result of Aronszajn is the existence of an Aronszajn tree on $\aleph_1$, i.e. there exist an $\omega_1$—tree with no uncountable branch. The construction uses the axiom of choice and therefore does not give a definable such tree.

Sierpinski [12] and Kurepa [8] proved that if Ramsey theorem holds for $\kappa$ then $\kappa$ is a strong limit cardinal. Erdős [4] proved that such a $\kappa$ is inaccessible. They also provided counterexamples to Ramsey theorem on small cardinals. These counterexamples explicitly used a well-ordering of $P(\kappa)$. The proof raises the question whether we can find a definable counterexample.

In a straightforward generalization of Aronszajn’s proof, Specker proved the existence of Aronszajn trees for every successor $\kappa^+$ s.t $\kappa^{<\kappa} = \kappa$. This raised the question whether the GCH was needed for this result. Mitchell and Silver [11] have proved that the tree property on $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. Magidor and Shelah [11] have proved the consistency of the tree property on $\aleph_{\omega+1}$ from the existence of very large cardinals.

Mitchell’s forcing for the tree property works well if one tries to obtain the tree property on two non-consecutive cardinals (e.g. $\aleph_2$ and $\aleph_3$). However, his methods fail to prove the consistency of the tree property on $\aleph_2$ and $\aleph_3$ together. By a result of Magidor this is indeed a substantial difficulty since the consistency strength of the tree property on $\aleph_2$ and $\aleph_3$ together is much higher then a weakly compact (e.g. it implies the existence of $\theta^\#$). Abraham [1] has proved the consistency of the tree property on both $\aleph_2$ and $\aleph_3$ from a supercompact cardinal and a weakly compact cardinal above it. This result was extended by Cummings and Foremann [2] which proved the consistency of the tree property on all $\aleph_n$’s from many supercompacts.

Kunen, and Shelah and Harrington [3] considered the consistency strength of obtaining Lebesgue measurability of projective sets together with Martin’s axiom. They proved the equiconsistency of this theory with the existence of weakly compact cardinals. This shows that adding Martin’s axiom to projective measurability increases the consistency strength.

In this paper we consider the consistency strength of the definable tree property on $\aleph_1$, i.e. the existence of a model in which every $\omega_1$—tree which is first order definable (with parameters) over $(H_{\omega_1}, \varepsilon)$, has a cofinal branch. Our proof method answers also a related question regarding definable counterexamples to Ramsey theorem on $\aleph_1$.

In section (2) we define the exact meaning of a definable $\kappa$—tree, and study some variations used in this paper. In section (3) we define the $\Pi^1_1$ reflecting cardinals and derive an extension property which is similar to the extension property of weakly compact cardinal. We also bound the consistency strength of the existence of a $\Pi^1_1$ reflecting cardinal by the existence of a Mahlo cardinal. In section (4) we show that by forcing with the well known Levy collapse of a $\Pi^1_1$ reflecting cardinal $\kappa$ to $\aleph_1$, we obtain a model of the definable tree property. We also prove that in this model a definable Ramsey theorem on $\aleph_1$ holds. In section (5) we show

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that our assumptions on $\kappa$ are necessary by proving that if the definable tree property on $\aleph_1$ holds, then $L \models \aleph_1$ is a $\Pi^1_1$ reflecting cardinal.

In section (5) we prove the equiconsistency of the definable tree property on $\aleph_1 + \text{MA}$ with the existence of a weakly compact cardinal. This is done by exploiting the methods of Kunen and Shelah and Harrington (6). We also comment that adding any reasonable failure of GCH does not add to the consistency strength. Finally in section (7) we comment on the consistency of the definable tree property on higher cardinals with GCH. We describe a forcing to get the definable tree property on all $\aleph_n$’s together with GCH by using $\omega$ many $\Pi^1_1$ reflecting cardinals.

2. Definable $\kappa$-trees

In this section we define various notions of definable $\kappa$-trees and study the relationship between these notions. First define the usual notion of a $\kappa$-tree.

Definition 2.1. • A tree is a partially ordered set $(T, <_T)$ such that for any $t \in T$ the set $\{s \in T | s <_T t\}$ of predecessors of $t$ is well-ordered under $<_T$, and there is a root $r \in T$ such that for any $t \in T$, such that $t \neq r$, $r <_T t$.

• The $\alpha$’th level of $T$ denoted by $T_\alpha$ is the set of elements of $T$ whose set of $T-$predecessors has order type $\alpha$.

• A tree $(T, <_T)$ is a $\kappa$-tree if $|T| = \kappa$ and for every $\alpha |T_\alpha| < \kappa$.

• A branch is a $<_T$ linearly ordered subset of $T$.

• A cofinal branch is a branch which intersects every level of $T$.

Next we would like to give a definition of a definable $\kappa$-tree. Three different notions naturally arise

Definition 2.2. • A $\kappa$-tree is definable in the strict sense if its underlying set is $\kappa$, and $<_T$ is $\Sigma^\omega_\omega((H_\kappa, \in))$.

• A $\kappa$-tree is definable in the wide sense if its underlying set $T$ and $<_T$ are both $\Sigma^\omega_\omega((H_\kappa, \in))$ and $T$ has definability cardinal $\kappa$, i.e. there is a bijection $f : \kappa \leftrightarrow T$ which is $\Sigma^\omega_\omega((H_\kappa, \in))$.

• A $\kappa$-tree is definable in the very wide sense if its underlying set $T$ and $<_T$ are both $\Sigma^\omega_\omega((H_\kappa, \in))$.

Obviously being definable in the wide sense implies being definable in the very wide sense, and being definable in the strict sense implies being definable in the wide sense. The following proposition states that being definable in the wide sense is almost equivalent to being definable in the strict sense.

Proposition 2.1. If $(T, <_T)$ is a $\kappa$-tree definable in the wide sense then there is a tree $(\kappa, <_{T'})$ isomorphic to $(T, <_T)$ which is definable in the strict sense.

Proof:

Let $f$ be the definable bijection $f : \kappa \rightarrow T$, and let $\psi(x, y, z)$ be the definition of $<_T$ from the parameter $z$. Now define $<_{T'}$ by $\alpha <_{T'} \beta$ iff $\psi(f(\alpha), f(\beta), z)$. The rest trivially follows. □

Using this proposition we will not distinguish between the strict and the wide notions of definability. Also notice that if there is a well-ordering of $H_\kappa$ which is $\Sigma^\omega_\omega((H_\kappa, \in))$ then the last two notions coincide. However in general this is not the case, and we will distinguish between definability in the very wide sense and definability in the strict sense which we will adopt as the definition of definability. So from now on, a definable $\kappa$-tree is a tree definable in the strict sense.

3. $\Pi^1_1$ Reflecting Cardinals

Let $\kappa$ be a cardinal. We say that $\kappa$ is $\Pi^m_1$ reflecting, if $\kappa$ is inaccessible and for every $A \subseteq V_\kappa$ definable over $V_\kappa$ (with parameters) and for every $\Pi^m_1$ sentence $\Phi$, such that

$$(V_\kappa, \in, A) \models \Phi$$

there is an $\alpha < \kappa$ such that

$$(V_\alpha, \in, A \cap V_\alpha) \models \Phi.$$ The $\Pi^m_1$ reflecting cardinals are a lightface analog of the $\Pi^m_1$ indescribable cardinals. However they have a much weaker consistency strength. If $\kappa$ is $\Pi^1_1$ reflecting, then an easy consequence is the fact that $\kappa$ is the $\alpha$’th inaccessible cardinal, since inaccessibility is a $\Pi^1_1$ property, and for each $\alpha < \kappa$ being the $\alpha$’th inaccessible is also a $\Pi^1_1$ property. Next we prove the following lemma:
Lemma 3.1. Suppose \( \kappa \) is a Mahlo cardinal then for every \( m, n \), 
\( \{ \alpha < \kappa \mid \alpha \) is a \( \Pi^m_n \) reflecting cardinal \} is stationary.

Proof:
Let \( \kappa \) be Mahlo, and let \( C \) be a club. We shall find a \( \mu \in C \) such that \( \mu \) is a \( \Pi^m_n \) reflecting cardinal. Let \( e : (\Phi, \psi, a) \to \kappa \) be an enumeration of triples \((\Phi, \psi, a)\) s.t. \( \Phi \) is a \( \Pi^m_n \) sentence, \( \psi(x, a) \) is a first order formula in the free variable \( x, a \in V_\kappa \). Also assume without loss of generality that \( a \in V_\kappa(\Phi, \psi, a) \) and \( e(\Phi, \psi, a) < \text{rank}(a)^\kappa \), where \( a^\alpha \) is the least inaccessible above \( \alpha \), and that \( e \) is \( 1-1 \) and onto \( \kappa \). For each triple \((\Phi, \psi, a)\) define \( g(e(\Phi, \psi, a)) \) by:

\[
g(e(\Phi, \psi, a)) = \begin{cases} 
\text{The least } \rho \in C \text{ such that } e(\Phi, \psi, a) < \rho \text{ and } & \\
(V_\rho, \epsilon, S) \models (x \in S \iff \psi(x, a)) \land \Phi, & \text{if there is such a } \rho.
\end{cases}
\]

(3.1)

(3.2)

For the other direction let \( \kappa \) be a \( \Pi^1_1 \) reflecting cardinal, and fix some \( n < \omega \). Let \( \sigma \) be a \( \Pi^1_1 \) formula expressing the failure of the extension property relative to \( n \), that is there is no transitive \( \Sigma_n \)-elementary
extension $X$ of $V_κ$ containing $κ$ (see the above remark). Let $τ$ be a $Π^1_1$ formula expressing the inaccessibility of $κ$. Let 

$$C = \{ \alpha < κ | (V_α, ε) \prec_α (V_κ, ε) \}. $$

$C$ is a club subset of $κ$ which is $Σ_{n+1}$ definable in $V_κ$. Let $ψ$ be the formula expressing the fact that $C$ is a club. Therefore,

$$(V_κ, ε, C) \models σ ∧ τ ∧ ψ.$$ 

By $Π^1_1$ reflection there exists an $α$ such that

$$(V_α, ε, C ∩ α) \models σ ∧ τ ∧ ψ.$$ 

Hence $α$ is an inaccessible cardinal below $κ$ which is a limit point of $C$. Hence $α ∈ C$, so $(V_α, ε) \prec_α (V_κ, ε)$. Let $X_0 = V_α ∪ \{ V_α \}$ and construct an elementary submodel $(X’, E) \prec (V_κ, ε)$, containing $X_0$ of cardinality $α$. Let $(X’, E)$ be the Skolem hull of $X_0$ inside $(V_κ, ε)$. Let $(X, ε)$ be the transitive collapse of $(X’, E)$. It follows that $(V_α, ε) \prec (X, ε)$, and $V_α ∈ X$ hence

$$(V_α, ε) \models ¬σ.$$ 

and this is a contradiction. □ 

We finish this section with the following observation on $Π^1_1$ reflecting cardinals:

**Lemma 3.3.** Let $κ$ be a $Π^1_1$ reflecting cardinal. Let $T$ be a $κ$—tree definable in the very wide sense, then $T$ has a cofinal branch.

**Proof :**

Note that since $κ$ is inaccessible $V_κ = H_κ$. Let $T$ be a definable $κ$—tree definable in the very wide sense. Let $n$ be large enough such that the assertion “$∀α T$ has a cofinal branch of length $α$.” is $Σ_n$ over $(V_κ, ε, T)$. By theorem 3.2 there is a transitive structure $(X, ε, T)$ such that

$$(V_κ, ε, T) \prec_κ (X, ε, T^X).$$

(3.3)

Since (for $n$ large enough) $V_κ^X = V_κ ∈ X$, it follows that $T^X ∩ V_κ^X = T$. Now, since $T$ is a $κ$—tree it follows that

$$(V_κ, ε, T) \models ∀α T \text{ has a cofinal branch of length } α.$$ 

Therefore,

$$(X, ε, T^X) \models ∀α T^X \text{ has a branch of length } α.$$ 

Since $κ ∈ X$ we see that $(X, ε, T^X) \models T^X \text{ has a branch } b \text{ of length } κ$. Because $T^X ∩ V_κ = T$, this branch $b$ is really a cofinal branch through $T$. □

4. THE FORCING CONSTRUCTION

In this section we describe the forcing construction and prove that the extended model satisfies the tree property for $ω_1$—trees first order definable over $(H_{ω_1}, ε)$. Let $κ$ be a $Π^1_1$ reflecting cardinal in $V$. Let

$$P = \text{Coll } (ω, < κ)$$

be the Levy collapse of $κ$ to $ω_1$ and for every $α < κ$ let

$$P_α = \text{Coll } (ω, < α)$$

be an initial segments of the forcing. Let $G$ be a $P$ generic filter. Our main theorem is

**Theorem 4.1.** $V[G] \models \text{ “every definable } ω_1$—tree $T \text{ has a cofinal branch”}. Moreover if $V = L$ then $V[G] \models \text{ “every } ω_1$—tree $T \text{ definable in the wide sense over } (H_{ω_1}, ε) \text{ has a cofinal branch”}.$

Define the following definable partition relation :

**Definition 4.1.** $N_1 \overset{def}{=} (N_1)_1^{m}$ iff every partition of $[N_1]^m$ into $α$ sets which is first order definable over $(H_{ω_1}, ε)$ (with parameters in $H_{ω_1}$), has a homogeneous set of size $N_1$.

In order to prove theorem 4.1 we shall first prove a definable Ramsey theorem in $V[G]$.

**Lemma 4.2.** $V[G] \models N_1 \overset{def}{=} (N_1)_2^{2}$
Proof:
Let $F : [\aleph_1]^2 \to 2$ be a definable function in $V[G]$, defined by

$$
F(\langle \alpha, \beta \rangle) = i \iff \Phi(x, \alpha, \beta, i)
$$

(4.2)

where $\Phi$ is a $\Sigma_2$ formula relativized to $(H_{\omega_1})^{V[G]}$, and the parameter $x$ can be taken as a real, that is, a function $x : \omega \to \omega$. By the $\kappa$-cc of the Levy collapse for every real $x \in V[G]$ there is an $\varepsilon < \kappa$ such that $x \in V[G_{\varepsilon}]$, where $G_{\varepsilon}$ is an initial segment of the generic, which is generic for $P_\varepsilon = \text{Coll}(\omega, < \varepsilon)$. Moreover by a result of Solovay (see [6] proposition 10.21) $V[G] = V[G_{\varepsilon}][H]$, and $H$ is generic for the Levy collapse $\text{Coll}(\omega, < \kappa)$. By the homogeneity of the Levy collapse every set of ordinals definable in $V[G]$ with parameters in $V[G_{\varepsilon}]$ is definable in $V[G_{\varepsilon}]$, and for every formula $\Psi$ we can compute another formula $\Phi$ such that:

$$
V[G] \models \Psi(x, \alpha, \beta, i) \iff V[G_{\varepsilon}] \models \Phi(x, \alpha, \beta, i)
$$

(4.3)

for all $\alpha, \beta < \kappa$ and $i \in \{0, 1\}$. Fix such an $\varepsilon$. Let $\tilde{\sigma}$ be a $P_\varepsilon$ name for $x$. Now we define a $\kappa$–tree $T$ which is definable in $V_\varepsilon$. For each $\alpha < \kappa$ let

$$
\tilde{h} \in T_\alpha \iff
\tilde{h} \text{ is a } P_\varepsilon \text{ name for a function from } \alpha \to \{0, 1\} \text{ and }
\exists \mu \exists \mu^0 \exists \beta > \alpha \forall \nu \forall \beta < \alpha \forall p \in P_\varepsilon
$$

\[ p \Vdash_{P_\varepsilon} \Phi(\tilde{\sigma}, \beta, \mu_0, \tilde{h}(\beta)) \iff p \Vdash_{P_\varepsilon} \Phi(\tilde{\sigma}, \beta, \delta, \tilde{h}(\beta)) \]

(4.4)

and define $T = \cup_{\alpha < \kappa} T_\alpha$. We shall write $\tilde{h}_\alpha$ to denote that $\tilde{h}_\alpha \in T_\alpha$.

The ordering of $T$ is

$$
\tilde{h}_\alpha \leq \tilde{h}_\beta \iff ||P_\varepsilon \tilde{h}_\alpha \subseteq \tilde{h}_\beta
$$

(4.5)

The tree $T$ is a $\kappa$–tree. To prove this first observe that for every $\alpha < \kappa$, $T_\alpha \neq \emptyset$ since for every $\alpha$ there are at most $2^{\alpha + P_\varepsilon}$ many $P_\varepsilon$ names for such functions, and therefore we are partitioning $\kappa$ into less than $\kappa$ many subsets according to the possible values of $(F(\langle \beta, \beta \rangle) : \beta < \alpha)$. Secondly $|T_\alpha| \leq 2^{\alpha} < \kappa$. $T$ is definable in the very wide sense in $H_\varepsilon$ by (4.4). Therefore by lemma (3.3) $T$ has a cofinal branch $\langle \tilde{h}_\alpha : \alpha < \omega_1 \rangle$. Work now in $V[G]$. Let $\tilde{h}_\alpha(G_{\varepsilon})$ denote the realization of the name $\tilde{h}_\alpha$ in $V[G_{\varepsilon}]$. Let $h = \cup_{\alpha < \kappa} \tilde{h}_\alpha(G_{\varepsilon})$, $h$ is a function from $\kappa = \aleph_1^{V[G]}$ to $\{0, 1\}$. Define

$$
A_\alpha = \{ \alpha < \gamma < \aleph_1 \mid \forall \beta < \alpha F(\langle \beta, \gamma \rangle) = h(\beta) \}
$$

(4.6)

then for every $\alpha$, $|A_\alpha| = \aleph_1$ by the definition of $\tilde{h}_\alpha$. Moreover $(A_\alpha : \alpha < \aleph_1)$ is a decreasing sequence of sets. We construct $H_0$ by induction on $\alpha < \aleph_1$. Let $\beta_0 = 0$. For each $\alpha$ let $\gamma_\alpha = \sup \{ \beta_\beta i < \alpha \}$. Let $\beta_\alpha = \min A_\gamma$. Let $H_0 = \{ \beta_\alpha | \alpha < \aleph_1 \}$. By the definition of $H_0$, for every $\alpha < \beta \in H_0$ we have $F(\langle \alpha, \beta \rangle) = h(\alpha)$. Let $l$ be minimal such that $|h^{-1}(l) \cap H_0| = \aleph_1$. Now $H = \{ \alpha \in H_0 | h(\alpha) = l \}$ is the homogeneous set. $\square$

Note that the proof really gives the following consequence

$$
V[G] \models \aleph_1 \overset{\text{def}}{=} \langle \aleph_1 \rangle^2_{\aleph_0}.
$$

If $V = L$, or there is a definable well-ordering of $H_\omega$ in $V$, then the same proof yields the following:

Lemma 4.3. Assume $V = L$. Let $h, A \in V[G]$ be such that $|A| = \aleph_1$, and $h : [A]^2 \to 2$ is a partition of $[A]^2$ into two parts, where both $A$ and $h$ are first order definable (with parameters) over $(H_{\omega_1}, \varepsilon)^{V[G]}$. Then there is a $B \subseteq A$ homogeneous for $h$ and $|B| = \aleph_1$.

To derive theorem 4.1 we follow the proof that a weakly compact cardinal has the tree property (see lemma 29.6 of [1]), replacing the partition property, of a weakly compact cardinal with the definable partition lemma 4.3.

Proof of theorem 4.1
Let $T = (\aleph_1, <_T)$ be a definable tree on $\aleph_1$, i.e. there is a formula $\Psi(\alpha, \beta, z)$ such that

$$
\alpha <_T \beta \iff (H_{\omega_1}, \varepsilon) \models \Psi(\alpha, \beta, z).
$$

(4.7)

We extend the partial tree ordering $<_T$ into a linear ordering as follows:

1. $\alpha < \beta$ if
2. $\alpha < T \beta$ or

(i) $\alpha < T \beta$ or
(ii) $\alpha, \beta$ are $<_T$ incomparable and if $\zeta$ is the first level where the predecessors of $\alpha, \beta, \alpha_\zeta, \beta_\zeta$ are distinct then $\alpha_\zeta < \beta_\zeta$.

$\prec$ is first order definable in $(\mathcal{H}_{\omega_1}, \varepsilon)$ using the definition of $<_T$. Now define a partition of $[\aleph_1]^2$ by

$$F(\{\alpha, \beta\}) = 1 \iff \alpha < \beta \text{ agrees with } \alpha \prec \beta.$$  \hspace{1cm} (4.8)

Since both $\prec$ and $<_T$ are definable $F$ is definable as well. Hence there is $H \subset \aleph_1$ which is homogenous for $F$, with $|H| = \aleph_1$. Let

$$B = \{ x \in \aleph_1 \mid \{ \alpha \in H \mid x <_T \alpha \} = \aleph_1 \}. \hspace{1cm} (4.9)$$

Since every level is countable there are members of $B$ of every level. If we’ll prove that any two members of $B$ are $<_T$ comparable, then $B$ will be the $\aleph_1$-branch. Let $x, y \in B$ be $<_T$ incomparable elements. Assume, without loss of generality, that $x \prec y$. Since both $x, y$ have $\aleph_1$ many $<_T$ successors in $H$ we can find $\alpha, \beta, \mu \in H$ such that $\alpha <_T \beta <_T \mu, x <_T \alpha, \mu$ and $y <_T \beta$. By the definition of $\prec$ we get $\alpha \prec \beta$ and $\mu \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\beta, \mu\}) = 0$, contradicting the fact that $H$ is homogeneous for $F$. Finally note that if we force over $L$ theorem $\ref{thm:reflecting}$ can be strengthened to trees definable in the very wide sense. The proof is identical using lemma $\ref{lem:tree}$ instead of $\ref{lem:tree}$. \hfill \Box

5. The lower bound

In this section we prove that the definable tree property implies the consistency of a $\Pi^1_1$ reflecting cardinal. In this section let $\aleph_1$ denote $\aleph_1^V$.

**Theorem 5.1.** If $\aleph_1$ has the definable tree property then

$L \models \aleph_1$ is a $\Pi^1_1$ reflecting cardinal.

First we prove that

$$L \models \aleph_1 \text{ is inaccessible.} \hspace{1cm} (5.1)$$

Assume that $\aleph_1$ is not inaccessible in $L$ then there is an $x \in \omega^\omega$ such that $\aleph_1 = \aleph_1^{L[x]}$ (see $\ref{prop:inaccessible}$ proposition 11.5). However inside $L[x]$ there is a special Aronszajn tree $T$ which is definable from the well ordering of $(\mathcal{H}_{\omega_1}, \varepsilon)^{L[x]}$, which is itself $\Sigma_1$ definable over $(\mathcal{H}_{\omega_1}, \varepsilon)^{L[x]} \ref{thm:tree}$. However $T$ cannot have a cofinal branch in $V$ since this implies $\aleph_1^{L[x]} < \aleph_1$. Similarly, by relativization, for every real $x$ $\aleph_1$ is inaccessible in $L[x]$.

Next we prove that $\aleph_1$ is a $\Pi^1_1$-reflecting cardinal in $L$. The proof is based on an idea from $\ref{thm:reflecting}$. We define a tree using the $\Sigma_n$ definable power set of $\aleph_1$. Note that since $\aleph_1$ is inaccessible in $L$, by $(5.1)$, we have

$$(\mathcal{H}_{\aleph_1}^L)^L = L_{\aleph_1}^L = (V_{\aleph_1}^L)^L.$$  

From a cofinal branch in the tree we define an ultrafilter on the $\Sigma_n$ definable subsets of $\aleph_1$, and construct an “ultrapower” of $L_{\aleph_1}$ using only functions $\Sigma_n$ definable over $(\mathcal{H}_{\omega_1}, \varepsilon)$. Note that this is a definable tree in the strong sense since there is a definable well-ordering of the underlying set. Let $(A_\alpha \mid \alpha < \aleph_1)$ be a definable enumeration of $P^L(\aleph_1) \cap \Sigma_n$ of order type $\aleph_1$. Define a tree $T$ of functions by

$$f \in T \iff f : \tau \to \{0, 1\}, \tau < \aleph_1 \text{ and } | \cap_{\alpha < \tau} A^{f(\alpha)}_\alpha| = \aleph_1$$

where $A^0 = A$, and $A^1 = \aleph_1 \setminus A$. The ordering on $T$ is $f <_T g$ iff $f \subseteq g$. Since $\aleph_1$ is inaccessible in $L$ the tree is an $\omega_1$-tree. Since the truth of $\Sigma_n$ formulas is a $\Sigma_{n+1}$ definable, the tree $T$ is $\Sigma_k$ definable over $(\mathcal{H}_{\omega_1}, \varepsilon)$, for some $k$. Hence by the definable tree property it has a cofinal branch producing a function $b : \aleph_1 \to \{0, 1\}$. $b$ defines an ultrafilter $U$ on $P^L(\aleph_1) \cap \Sigma_n$ by

$$A_\alpha \in U \iff b(\alpha) = 0$$

and this ultrafilter is countably complete on $P^L(\aleph_1) \cap \Sigma_n$. The “ultrapower” is now defined by

$$f \in \text{ult}(L_{\aleph_1}, U) \iff f : \aleph_1 \to L_{\aleph_1}, f \in L \text{ and } f \text{ is } \Sigma_n \text{ definable over } (\mathcal{H}_{\omega_1}, \varepsilon) \hspace{1cm} (5.2)$$

and

$$f \equiv g \iff \{ \alpha \mid f(\alpha) = g(\alpha) \} \in U \hspace{1cm} (5.3)$$

$$f E g \iff \{ \alpha \mid f(\alpha) \in g(\alpha) \} \in U. \hspace{1cm} (5.4)$$

The “ultrapower” is wellfounded by the completeness of the ultrafilter and there is a $\Sigma_n$ embedding $j : L_{\aleph_1} \prec_n \text{ ult}(L_{\aleph_1}, U)$, by the proof of Los theorem. However since $L_{\aleph_1} \models V = L$ if $n$ is large enough
ult(L_{R_1}, U) \models V = L$, and hence its transitive collapse is really $L_\alpha$ for some $\alpha > \aleph_1$. Therefore $L_{\aleph_1} \prec L_\alpha$. This gives the desired extension property, and by the equivalence of the extension property and $\Pi^1_1$ reflection of $\aleph_1$ is $\Pi^1_1$ reflecting in $L$. Finally by relativizing we obtain that for every real $x$, $\aleph_1$ is $\Pi^1_1$ reflecting in $L[x]$. 

\section{The definable tree property and Martin’s axiom}

In this section we investigate the consistency strength of the Definable Tree Property on $\aleph_1$ together with Martin’s Axiom and large continuum. The exposition is based on the Shelah-Harrington paper \cite{ShelahHarrington}. The main theorem is the following:

\begin{theorem}
Let $\aleph_0 < \lambda$ be a cardinal satisfying $\lambda^{<\lambda} = \lambda$. The following are equiconsistent:
\begin{enumerate}
  \item The definable tree property on $\aleph_1 + MA + 2^{\aleph_0} = \lambda$
  \item $\aleph_1$ is weakly compact in $L$.
\end{enumerate}
\end{theorem}

For the $2 \Rightarrow 1$ direction, we use lemma \cite{ShelahHarrington} which proves that the definable tree property implies that for every real $x$ $\aleph_1^{L[x]} < \aleph_1$. Now we finish by the following result from \cite{ShelahHarrington}:

\begin{theorem}
Assume MA then either there exists a real $x$ such that $\aleph_1^{L[x]} = \aleph_1$, or $\aleph_1$ is weakly compact in $L$.
\end{theorem}

The proof of the other direction follows closely Kunen’s forcing for the consistency of “MA+ Every set in $L(R)$ is Lebesgue measurable $+ 2^{\aleph_0} = \lambda$”. The model we use is Kunen’s model, and we only prove that the definable tree property holds in that model. For completeness we present the full proofs of Kunen’s basic observations, as presented in \cite{ShelahHarrington}. Let $\kappa$ be a weakly compact cardinal.

\begin{lemma}
If $B$ is a complete Boolean Algebra with the $\kappa$-c.c., and if $X \subseteq B$ satisfies $|X| < \kappa$, then there is a complete subalgebra $\bar{B}$, s.t. $X \subseteq \bar{B}$ and $|\bar{B}| < \kappa$.
\end{lemma}

\begin{proof}
Since $B$ satisfies the $\kappa$-c.c., and $\kappa^{<\kappa} = \kappa$, there is $B'$ a complete subalgebra of $B$ such that $X \subseteq B'$ and $|B'| \leq \kappa$. Without loss of generality assume that $B' \subseteq \kappa$. Let $D$ be the set of maximal antichains of $B'$. $D \subseteq [\kappa]^{<\kappa}$. By $\Pi^1_1$ reflection there is an $\alpha < \kappa$ s.t. $B' \cap \alpha$ is $< \alpha$ complete and $D \cap [\alpha]^{<\alpha}$ is the set of maximal antichains of $B' \cap \alpha$. Therefore $B' \cap \alpha$ is a complete subalgebra of $B$. If we choose $\alpha > \sup(X)$ then $X \subseteq B' \cap \alpha$.
\end{proof}

\begin{lemma}
If $P_0, P_1$ are two complete Boolean algebras with $\kappa$-c.c., then $P_0 \times P_1$ has the $\kappa$-c.c.
\end{lemma}

\begin{proof}
Let $\langle \mu_0, \mu_1 \rangle | \alpha < \kappa \rangle$ be a sequence of elements of $P_0 \times P_1$. Define $F : [\kappa]^2 \rightarrow 2 \times 2$. $F(\{\alpha, \beta\})(i) = 0$ iff $\mu_i$ are compatible. A size $\kappa$ homogeneous set for $F$ gives a size $\kappa$ set of pairwise compatible elements, since by the $< \kappa$-c.c. of $P_1$ the homogeneous color is $< 0, 0$.

Assume that $\nu < \lambda = \lambda^{<\lambda}$. To obtain the model we iterate $\lambda$ many times $\kappa$-c.c. forcings of size $< \lambda$ using finite support, the same way this is done for $MA$ (\cite{ShelahHarrington}). By lemma \cite{ShelahHarrington} we can assume, without loss of generality, that each forcing appears $\lambda$ many times in the iteration. Denote the iteration by $B$. To prove that every $\aleph_1$ tree which is ordinal definable from a real has a branch in $V^B$, we first prove that Ramsey theorem for $\aleph_1$ holds for $L(R)$ partitions. Then we finish the same way as in the proof of theorem \cite{ShelahHarrington}.

Let $F : [\aleph_1]^2 \rightarrow 2$ be definable in $V^B$ from a real $x$, and ordinal parameters. By lemma \cite{ShelahHarrington} $x$ is generic for a countable subalgebra $\bar{B}$. Moreover since each forcing in the iteration appears unboundedly many times we can assume , without loss of generality, that $B/\bar{B}$ is an homogeneous iteration over $V^B$ which is built the same way as $B$ is built over $V$. Therefore every value of $F$ is decided inside $V^B$. Since $|\bar{B}| < \kappa$ we can construct (in $V$) a $\kappa$ tree of $\bar{B}$ names of possible values of $F$, the same way as we have built it in lemma \cite{ShelahHarrington}. Now by the weak compactness of $\kappa$ we can find a branch through that tree. Finally we use the branch to obtain the homogeneous set the same as we have done in lemma \cite{ShelahHarrington}.

Note that since $\kappa$ is weakly compact we are able to find homogeneous sets for every $L(R)$ coloring, and not just first order definable over $H_{\omega_1}$.

Finally we remark that like in Solovay’s proof of Lebesgue measurability \cite{Solovay}, just adding the negation of the Continuum Hypothesis does not add to the consistency strength. We have to notice that the product
of collapsing $\kappa$ to $\omega_1$ and then adding $\lambda$ Cohen reals satisfies the $\kappa$-c.c. The product is also homogeneous enough. Moreover every real belongs to a small generic extension, which is complemented by a homogeneous forcing. Hence the same situation as in theorem 4.1 still holds, and the argument there can be carried out.

7. The definable tree property on higher cardinals

This section contains several remarks regarding the definable tree property on cardinals above $\aleph_1$. By Mitchell’s and Silver’s results [11], the tree property on $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. However it is well known that assuming GCH, or even $\lambda^{<\lambda} = \lambda$, we have a $\lambda^+$ special Aronszajn tree. This is proved by the same construction as Aronszajn’s construction on $\aleph_1$, using the fact that under this hypothesis there is a universal linear order of cardinality $\lambda$.

We will prove that assuming a $\Pi^1_1$ reflecting cardinal the definable tree property on $\aleph_2$ is consistent with GCH. Then we consider the property of having the definable tree property on successive cardinals. We will generalize our forcing argument to prove that if it is consistent that there are $\omega$ $\Pi^1_1$ reflecting cardinals then it is consistent to have the definable tree property on $\langle \aleph_i : 1 \leq i < \omega \rangle$.

**Theorem 7.1.** Assume $\kappa$ is a $\Pi^1_1$ reflecting cardinal in $L$. Let $P = \text{Coll}(\aleph_1, < \kappa)$. Let $G$ be $P$ generic over $L$. Then

$L[G] \models \text{GCH and } \aleph_2 \text{ has the definable tree property.}$

**Proof :**

The proof is identical to the $\aleph_1$ case. Using the homogeneity of the Levy collapse, and the fact that $\kappa$ is $\Pi^1_1$ reflecting. We just build a tree of possible values for the definable function. Then using the definability of the function we obtain definability of the tree and thus we can use the branch to prove the definable version of Ramsey theorem for $\aleph_2$. The fact that GCH holds in $V[G]$, is proved in the usual way, by assuming $GCH$ in the ground model.

To obtain the definable tree property on all $\aleph_n$’s from a sequence of $\omega$ $\Pi^1_1$ reflecting cardinals, $\kappa_0 < \kappa_1 < \ldots$ just iterate the forcings $\text{Coll}(\aleph_n, < \kappa_n)$ with finite support. The proof that the definable tree property holds for every $n$, is a straightforward generalization and will not be given here.

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References

[1] U. Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Annals of Pure and Applied Logic*, 24, 1983.
[2] J. Cummings and M. Foreman. The tree property on all $\aleph_n$’s. preprint.
[3] K.J. Devlin. *Constructibility*. Perspectives in Mathematical Logic. Springer-Verlag, 1984.
[4] P. Erdős and A. Tarski. On families of mutually exclusive sets. *Annals of Mathematics*, 44:315–329, 1943.
[5] L. Harrington and S. Shelah. Some exact equiconsistency results in set theory. *Notre Dame Journal of Formal Logic*, 26:178–188, 1985. Proceedings of the 1980/1 Jerusalem Model Theory year.
[6] T. Jech. *Set Theory*. Academic Press, 1978.
[7] A. Kanamori. *The Higher Infinite*. Perspectives in Mathematical Logic. Springer-Verlag, 1994.
[8] D.R. Kurepa. Ensembles ordonnés et ramifiés, thèse. *Publications mathématique de l’université de Belgarda*, 4:1–138, 1935.
[9] M. Magidor, S. Shelah, and J. Stavi. On the standard part of nonstandard models of set theory. *Journal of Symbolic Logic*, 48, 1983.
[10] Menachem Magidor and Saharon Shelah. Successor of singular with no trees.
[11] W. Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5:473–478, 1973.
[12] W. Sierpinski. Sur un problème de la théorie des relations. *Annelli della Scuola Normale Superiore de Pisa*, 2:285–287, 1933.
[13] R. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92:1–56, 1970.
[14] R.M. Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin’s problem. *Annals of Mathematics*, 94, 1971.
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