Polymer quantization versus the Snyder noncommutative space

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Abstract
We study a noncanonical Hilbert space representation of the polymer quantum mechanics. It is shown that the Heisenberg algebra gets some modifications in the constructed setup, from which a generalized uncertainty principle will naturally come out. Although the extracted physical results are the same as those obtained from the standard canonical representation, the noncanonical representation may be notable in view of its possible connection with the generalized uncertainty theories suggested by string theory. In this regard, by considering a Snyder-deformed Heisenberg algebra, we show that since the translation group is not deformed, it can be identified with a polymer-modified Heisenberg algebra. In the classical level, it is shown that the noncanonical Poisson brackets are related to their canonical counterparts by means of a Darboux transformation on the corresponding phase space.

Keywords: polymer quantization, noncommutative geometry, Snyder space

(Some figures may appear in colour only in the online journal)

1. Introduction

When one takes into account the gravity and quantum theory issues in special relativity, the general theory of relativity and the quantum field theory can be achieved, respectively. However, the need for a quantum theory of gravity is inevitable either in general relativity to resolve the space–time singularities or in quantum field theories to overcome the ultraviolet divergencies. Evidently, both of these problems have the same origin and may be resolved by taking an effective cutoff on ultraviolet, i.e., a minimal observable length scale (for singularity resolution in general relativity, see [1], and for the ultraviolet regularization of the quantum field theory, see [2]). This issue is also proposed by the quantum gravity candidates such as...
loop quantum gravity [3] and string theory [4], both of which suggest some modifications to the structure of the standard quantum mechanics. From the string theory point of view, a minimum length scale $\alpha$, which is responsible for the strings’ size [5], will appear due to the corrections to the standard uncertainty principle. Such generalized uncertainty relations can also be realized from a noncanonical (deformed) Heisenberg algebra on the corresponding Hilbert space [6, 7]. Noncommutativity between space–time coordinates was first introduced by Snyder in 1947 [8], which was a Lorentz invariant theory with a minimal-length scale in its formalism. The associated noncanonical Heisenberg algebra is also very similar to the one obtained in the context of string theory [9].

In recent years, another formalism of quantum mechanics, called polymer quantum mechanics, is proposed, which supports the idea of the existence of a minimal-length scale $\mu$, known as the polymer length, without any attribution to the noncanonical Heisenberg algebra [10]. In this approach for quantization of a given dynamical system, one uses methods very similar to the effective models of loop quantum gravity. Here, the main role is played by the mentioned polymer length scale in such a way that, unlike the deformed algebraic structure usually coming from the noncommutative phase-space variables, it enters into the Hamiltonian of the system to deform its functional form into a new one called the polymeric Hamiltonian. However, a question may be raised: can a polymer-deformed quantum mechanical theory be compatible with a theory with noncanonical (deformed) Heisenberg algebra in its formalism? In this paper we are going to answer this question by concentrating on the noncanonical Hilbert space representation of the polymer quantum mechanics. In particular, we show that the noncanonical representation of the polymer quantum mechanics coincides with a Snyder-deformed (noncanonical) Heisenberg algebra just by reasonable identification of the deformation parameters as $\mu = \alpha \approx l_n$, where $l_n \sim 10^{-33}$ cm is the Planck length.

The structure of the paper is as follows: in section 2, the general formulation of the polymer quantization is briefly reviewed. The standard canonical representation of the polymer quantum mechanics is presented in section 3. In section 4, the noncanonical representation of the polymer quantum mechanics is investigated, and the corresponding Hilbert space representation is also studied. The generalized uncertainty principle is obtained in this setup, and the simple harmonic oscillator is considered as a relevant example. In section 5, inspired by the theory of Snyder noncommutative spaces, we first construct a noncanonical Heisenberg algebra and then show that it can be identified with the polymer-modified Heisenberg algebra obtained from the noncanonical polymeric representation. The classical limits of the canonical and noncanonical polymeric representation are the subject of section 6, in which it is shown that these two setups are related to each other just by a Darboux transformation on the associated phase space. Section 7 is devoted to the summary and conclusions.

2. Polymer quantization

Polymer quantum mechanics is an alternative framework of the commutation relation that is investigated in the context of loop quantum gravity [10]. In contrast to the standard Schrödinger picture, this representation gives an ultraviolet cutoff due to the existence of the polymer length as a possible minimum length scale for the system under consideration. In this section, we briefly review the kinematics and dynamics of the polymer quantum mechanics. (see references [10–13] for more details.)
2.1. Kinematics

Consider a classical mechanical system consisting of a single particle that is characterized by the position $\tilde{q}$ and its conjugate momentum $\tilde{p}$ (we will use the notation $(\tilde{q}, \tilde{p})$ for the classical variables through this paper). In the two-dimensional phase space $(\tilde{q}, \tilde{p})$, these quantities satisfy the ordinary canonical Poisson algebra

$$\{\tilde{q}, \tilde{p}\} = 1. \quad (1)$$

Canonical quantization is straightforward. Classical variables $(\tilde{q}, \tilde{p})$, are replaced by quantum operators $(q, p)$ and the role of the Poisson brackets will be played by the Dirac commutators. Then one is led to the quantum Heisenberg algebra

$$[q, p] = i, \quad (2)$$

in which we have set $\hbar = 1$. The operators $(q, p)$ are unbounded, and the associated Hilbert space should necessarily be infinite-dimensional. However, in the quantum gravity regime, in order to take the natural cutoffs such as minimal length and maximal momentum into account, one deals with bounded variables. One way out of this difficulty is implementing the Weyl algebra [10, 12]. Thus, in an alternative way, but more admissible for the polymer representation, one can adopt the complex exponential versions of the operators $q$ and $p$ (Weyl operators) as

$$U(\alpha) = e^{i\alpha q}, \quad V(\beta) = e^{i\beta p}, \quad (3)$$

with

$$U(\alpha_1). U(\alpha_2) = U(\alpha_1 + \alpha_2), \quad V(\beta_1). V(\beta_2) = V(\beta_1 + \beta_2). \quad (4)$$

The parameters $\alpha$ and $\beta$ have dimensions of momentum and length, respectively. Using the Baker–Hausdorff lemma, the Heisenberg algebra (2) for the operators $(q, p)$ takes the following Weyl algebra form

$$U(\alpha). V(\beta) = e^{-i\alpha\beta} V(\beta). U(\alpha), \quad (5)$$

generated by the exponentiated position and momentum operators (3). Then, the ordinary Schrödinger representation can be obtained via the standard Gel’fand–Naimark–Segal construction [10]. In position polarization of the Schrödinger representation, the corresponding Hilbert space is $H_{\text{Sch}} = L^2(\mathbb{R}, \, dq)$, which is the space of the square integrable functions with respect to the Lebesgue measure $dq$ on the real line $\mathbb{R}$. Therefore, the Hilbert space in the momentum polarization will be

$$H_{\text{Sch}} = L^2(\mathbb{R}, \, dp), \quad (6)$$

in which the operators are represented as

$$p. \, \psi(p) = p\psi(p), \quad q. \, \psi(p) = i\frac{\partial \psi(p)}{\partial p}, \quad (7)$$

where $\psi(p) = \langle p | \psi \rangle$ is a typical wave function. According to the Stone–Von Neumann uniqueness theorem, every irreducible representation of the Weyl algebra (5) that is weakly continuous in the parameters $\alpha$ and $\beta$ is unitarily equivalent to the standard Schrödinger representation. But the so-called polymer representation is unitarily inequivalent to the Schrödinger one [10]. Evidently, the construction should violate at least one assumption of the Stone–Von Neumann uniqueness theorem. Indeed, the exponentiated momentum operator $V(\beta)$ is no longer weakly continuous in $\beta$ in the polymer representation. Taking this
hypothesis into account, the polymer representation can be obtained from the Weyl algebra \((5)\) via the Gel’fand–Naimark–Segal construction \([10, 12]\).

In this regard, we define an abstract ket \(\langle \lambda \rangle\), labeled by a real number \(\lambda\), which shall belong to the non-separable Hilbert space \(\mathcal{H}_{\text{poly}}\). An appropriate state can be obtained by taking a linear combination of a finite collection of vectors \(\langle \lambda_i \rangle\), where \(\lambda_i \in \mathbb{R}\) and \(i = 1, 2, \ldots, N\) as

\[
|\psi\rangle = \sum_{i=1}^{N} a_i \langle \lambda_i |.
\]

Then, the polymer inner product between the fundamental kets will be

\[
\langle \lambda | \nu \rangle = \delta_{\lambda, \nu}.
\]

One can work in position or momentum polarization. In momentum polarization, the states are denoted by \(\psi(p) = \langle p | \psi \rangle\), where

\[
\psi_x(p) = \langle p | i \rangle = e^{i \lambda p}.
\]

Since here the operator \(V(\beta)\) is not weakly continuous in \(\beta\), the associated momentum operator \(p\) does not exist. However, \(V(\beta)\) itself is a well-defined operator that acts on the states \((10)\) as

\[
V(\nu). \psi_x(p) = e^{i \lambda p} e^{i \nu p} = e^{i (\lambda + \nu) p} = \psi_{\lambda + \nu}(p).
\]

On the other hand, the operator \(q\) is well-defined in the sense that the corresponding exponential operator \(U(\alpha)\) is continuous in \(\alpha\). Therefore, the operator \(q\) can be identified as

\[
q. \psi_x(p) = i \frac{\partial}{\partial p} \psi_x(p) = -\lambda e^{i \lambda p} = -\lambda \psi_x(p),
\]

which can be interpreted as a discrete operator in the sense that its eigenvalues are labeled by \(\lambda\) which, in turn, may take their values from a continuum such that the states \((10)\) are orthonormal. Now, what remains is the definition of an inner product with respect to an appropriate measure on the abstract Hilbert space \(\mathcal{H}_{\text{poly}}\). The above definitions correspond to the Bohr compactification of the real line \(\mathbb{R}\), which is a compact group, and the Haar measure \(d\mu_H\) is a natural measure defined on it. Therefore, the polymer Hilbert space \(\mathcal{H}_{\text{poly}}\) in momentum polarization will be \([12]\)

\[
\mathcal{H}_{\text{poly}} = L^2(\mathbb{R}, d\mu_H).
\]

Therefore, one may define the following inner product for the periodic functions \(\psi_{\beta}(p)\) on this Hilbert space,

\[
\langle \psi_{\lambda} | \psi_{\nu} \rangle_{\text{poly}} = \int_{\mathbb{R}} d\mu_H \psi^*_\lambda(p) \psi_\nu(p) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dp \ psi^*_\lambda(p) \psi_\nu(p) = \delta_{\lambda, \nu}.
\]

In contrast to the standard Schrödinger representation, the Dirac delta function is replaced by the Kronecker delta in this setup.

2.2. Dynamics

Our starting point in this section is to consider a classical Hamiltonian function and see how one can pass from this point to the polymeric Hamiltonian. To do this, let us begin with the classical nonrelativistic Hamiltonian function
where \( m \) is a mass parameter and \( W \) denotes a potential function. At the classical level, the pair \((\tilde{q}, \tilde{p})\) satisfies the Poisson algebra (1), and at quantum level, it should be replaced with its operator counterpart \((q, p)\) and the standard Heisenberg algebra (2). Both of the position \( q \) and momentum \( p \) are well-defined operators in the standard Schrödinger representation, since their exponentiated operators \( U(\alpha) \) and \( V(\beta) \) are weakly continuous in \( \alpha \) and \( \beta \), respectively. In momentum polarization, the position and momentum operators are given by the relations (7) on the Hilbert space (6). In polymer representation, however, this argument fails to be applicable, since the exponentiated momentum operator \( V(\beta) \) is no longer weakly continuous in \( \beta \), and, consequently, the corresponding momentum operator does not exist. The main task now is to define a suitable momentum operator and its square in a polymer framework to achieve a quantum Hamiltonian operator from the classical version (15). The situation is very similar to that in loop quantum gravity, where, while the connection is not a well-defined operator, its holonomy is and plays the role of the conjugate momentum for the volume operator with discrete eigenvalues [14]. The standard prescription is to define the lattice \( \gamma_\mu \) on the configuration space as

\[
\gamma_\mu = \{ q \in \mathbb{R} | q = n \mu, \forall n \in \mathbb{Z} \},
\]

which ensures the self-adjointness of the position operator and also the discreteness of its eigenvalues. In order to regulate a momentum operator in this setup, we consider the action of its exponentiated operator \( V(\beta) = e^{i\beta p} \) on the basic kets as

\[
V(\mu) | \mu \rangle = e^{i\mu p} e^{i\beta p} = e^{i(\mu + \beta)p} = | \mu + \mu \rangle = | \mu_{n+1} \rangle.
\]

The physical states in the separable Hilbert space \( \mathcal{H}_\mu \) are then of the form

\[
| \psi \rangle = \sum_n b_n | \mu \rangle,
\]

with coefficients \( b_n \) satisfying \( \sum_n |b_n|^2 < \infty \). In this respect, one can regulate the momentum operator in the polymer framework by means of the exponential shift operator as [10]

\[
p_\mu | \mu \rangle = \frac{1}{2i\mu} (V(\mu) - V(-\mu)) | \mu \rangle = \frac{1}{2i\mu} (| \mu_{n+1} \rangle - | \mu_{n-1} \rangle).
\]

The squared momentum operator can be defined in the same way just by twice acting the polymer momentum operator (19) on the states as

\[
p_\mu^2 | \mu \rangle = \frac{1}{4\mu^2} (2 - V(2\mu) - V(-2\mu)) | \mu \rangle = \frac{1}{4\mu^2} (2 | \mu \rangle - | \mu_{n+2} \rangle - | \mu_{n-2} \rangle).
\]

Then, the quantum polymeric Hamiltonian operator can be deduced by means of theopolymeric squared momentum operator (20) from the classical Hamiltonian function (15) as

\[
H_\mu = \frac{1}{8 m \mu^2} (2 - V(2\mu) - V(-2\mu)) + W(q).
\]

Evidently, the momentum is periodic in the sense that the functions of \( p \) that preserve the lattice \( \gamma_\mu \) are of the form \( e^{im\mu} \) for integer \( m \). There is a maximal value for the momentum,

\[\text{1 The operator } p_\mu, p_\mu^2 \text{ shifts the states two steps in the lattice to both sides. However, one can rescale this operator as } p_\mu \rightarrow \mu/2 \text{ to give an approximation for the squared momentum operator that shifts the states one step in the lattice [10, 12].}\]
which is defined by the lattice spacing $\mu$. Thus the momentum operator is a bounded self-adjoint operator on the lattice. In the classical regime, it means that the momentum part of the phase space is compactified to a circle $S^1$ (see the section for the classical phase space). Thus, the separable Hilbert space $H_{\gamma \mu}$ in momentum polarization will be [12]

$$H_{\gamma \mu} = L^2 \left( S^1, d\mu_{S^1} \right).$$

(22)

where $d\mu_{S^1}$ is now the Haar measure for $S^1$. This Hilbert space is a subspace of the nonseparable polymeric Hilbert space (13). One can also define the momentum operator and its square on the nonseparable Hilbert space $H_{\text{Poly}}$, but their physical interpretation is problematic [12]. For instance, as we will see, the position eigenvectors are orthonormal only when one considers the lattice (16) on the separable Hilbert space (22).

Like the standard Schrödinger representation, one can solve the Hamiltonian eigenvalue problem in position or momentum polarization in the polymer setup. But, we prefer to work in momentum polarization in this paper. Indeed, the Hamiltonian eigenvalue problem

$$H_{\mu} \left| \psi \right\rangle = E \left| \psi \right\rangle$$

(23)

takes the form of a difference equation for the Fourier coefficient of a general state $\psi(q)$ in position polarization, and it becomes a differential equation when it acts on the state (18) in momentum polarization of the Hilbert space $H_{\gamma \mu}$. (see reference [10, 15, 16] for more details.)

### 3. Canonical polymeric representation

Using definition (3) for the shift operator $V(\mu) = e^{i\mu p}$ in relation (19) gives an approximation for the polymeric momentum as

$$p_\mu \approx \frac{1}{\mu} \sin(\mu p),$$

(24)

with $p \in (-\pi/\mu, +\pi/\mu)$. Note also that this relation implies a bounded range for the polymeric momentum as $p_\mu \in (-1/\mu, +1/\mu)$. In the limit of $\mu p \ll 1$, the polymeric momentum operator $p_\mu$ coincides with its standard unbounded canonical one $p \in (-\infty, +\infty)$. This result shows that the compactification of the momentum by the circle $S^1$ induces an ultraviolet cutoff defined by the lattice spacing. In the same way, the polymeric Hamiltonian operator (21) can be approximated as

$$H_{\mu} = \frac{1}{4\mu^2} (1 - \cos(2\mu p)) + W(q).$$

(25)

Interestingly, this Hamiltonian gives both the maximal momentum and also maximal energy for the case of a free particle [12].

Note that one can represent the separable polymeric Hilbert space (22) with either of the pairs $(q, p)$ and $(\mu q, p_\mu)$ on the lattice (16). The pair $(q, p)$ satisfies the canonical Heisenberg algebra (2), and we refer to its components as canonical operators. In the next section, we will show that the pair $(\mu q, p_\mu)$ satisfies a noncanonical Heisenberg algebra, and, consequently, we call them noncanonical operators.

In the momentum polarization of the canonical polymeric representation of the quantum mechanics, the position and momentum act as a derivative and multiplication operators, respectively.
\[ q \psi(p) = i \frac{\partial}{\partial p} \psi(p), \quad \mu \psi(p) = \frac{1}{\mu} \sin(\mu p) \psi(p). \] (26)

Also, the corresponding Hilbert space is nothing but (22) with the uniform measure

\[ H_{\mu, p} = L^2(S^1, dp). \] (27)

Therefore, the inner product for the periodic functions \( \psi_\mu(p) \) on the Hilbert space (27) can be obtained via the definition (14) as

\[ \langle \psi_1 \mid \psi_2 \rangle_{L^2} = \frac{\mu}{2\pi} \int_{-\pi / \mu}^{+\pi / \mu} dp \psi_1^\ast(p) \psi_2(p), \] (28)

with \( p \in (-\pi / \mu, \pi / \mu) \). The completeness relation then will be

\[ 1 = \frac{\mu}{2\pi} \int_{-\pi / \mu}^{+\pi / \mu} dp \mid p \rangle \langle p \mid, \] (29)

and the inner product of momentum eigenvectors remains unchanged as \( \langle p \mid p' \rangle = \delta(p - p') \).

The position eigenvalue problem \( q \psi(p) = \lambda \psi(p) \) takes the form of a differential equation

\[ i \frac{\partial \psi}{\partial p} = \mu \psi(p), \] (30)

because of this, we call the components of the pair \( (q, p_\mu) \) the noncanonical operators. It is important to note that the relation (30) would be only considered on the lattice to ensure the self-adjointness of the position and momentum operators. Nevertheless, we consider this issue in more detail to compare the results with the case of the Snyder noncommutative space in the next section. In the noncanonical representation, the Hamiltonian operator takes the standard functional form

\[ H_p = \frac{p^2}{2m} + W(q). \] (31)

Thus, one may work within two equivalent pictures on the separable Hilbert space (22): (i) working with the standard canonical momentum \( p \), which satisfies the standard Heisenberg algebra (2), and (ii) working with the polymeric momentum \( p_\mu \), which satisfies the polymer-modified Heisenberg algebra (30) and the corresponding Hamiltonian operator (31) with the standard functional form. The former is the standard canonical polymeric representation of the quantum mechanics \([10, 12]\). The latter, however, is the noncanonical polymeric representation of the quantum mechanics; we are going to consider its Hilbert space representation in this paper.

4. Noncanonical polymeric representation

As we have mentioned above, one may also work with the pair \( (q, p_\mu) \) on the separable Hilbert space (22). We note that while the canonical momentum \( p \) satisfies the canonical Heisenberg algebra (2), it is easy to show that the polymeric momentum \( p_\mu \) satisfies the noncanonical (deformed) Heisenberg algebra

\[ \left[ q, p_\mu \right] = i \sqrt{1 - \left( \mu p_\mu \right)^2}. \] (30)

Because of this, we call the components of the pair \( (q, p_\mu) \) the noncanonical operators. It is important to note that the relation (30) would be only considered on the lattice to ensure the self-adjointness of the position and momentum operators. Nevertheless, we consider this issue in more detail to compare the results with the case of the Snyder noncommutative space in the next section. In the noncanonical representation, the Hamiltonian operator takes the standard functional form

\[ H_p = \frac{p^2}{2m} + W(q). \] (31)

Thus, one may work within two equivalent pictures on the separable Hilbert space (22): (i) working with the standard canonical momentum \( p \), which satisfies the standard Heisenberg algebra (2) together with the polymer-modified Hamiltonian operator (25), and (ii) working with the polymeric momentum \( p_\mu \), which satisfies the polymer-modified Heisenberg algebra (30) and the corresponding Hamiltonian operator (31) with the standard functional form. The former is the standard canonical polymeric representation of the quantum mechanics \([10, 12]\). The latter, however, is the noncanonical polymeric representation of the quantum mechanics; we are going to consider its Hilbert space representation in this paper.
4.1. Hilbert space representation

Clearly, this Hilbert space is unitarily equivalent to the Hilbert space (27). This is because they may be considered as the different faces of the separable Hilbert space (22), and, consequently, all the physical results should be the same. Nevertheless, as we will show, the noncanonical representation would be interesting due to its possible connection with the theories based on the noncommutative Heisenberg algebra such as the generalized uncertainty relations suggested by string theory [5–7]. Also, the noncanonical representation is more admissible from the statistical point of view when the polymer quantization is applied to the thermodynamical systems [17, 18].

In momentum polarization, the position and the polymeric momentum operators act on the physical states as

\[ p_\mu \psi(p_\mu) = p_\mu \psi(p_\mu), \quad q \cdot \psi(p_\mu) = i\sqrt{1 - (\mu p_\mu)^2} \frac{\partial \psi(p_\mu)}{\partial p_\mu}, \]

where \( \psi(p_\mu) = \langle p_\mu | \psi \rangle \) and \( | \psi \rangle \) is given by (18). The requirement that the operators should be self-adjoint in the noncanonical chart means that we have to define an appropriate Haar measure on the Hilbert space (22), which will be

\[ H_{r,r_\mu} = L^2 \{ S^1, dp_\mu / \sqrt{1 - (\mu p_\mu)^2} \}. \]

The operators \( q \) and \( p_\mu \) are self-adjoint and also symmetric with respect to the inner product

\[ \langle \psi_1 | \psi_2 \rangle_{r,r_\mu} = \frac{\mu}{2} \int_{-1/\mu}^{1/\mu} dp_\mu \psi_1^*(p_\mu) \psi_2(p_\mu) / \sqrt{1 - (\mu p_\mu)^2} \]

in which we have used the definition (14). The completeness relation in this setup will be

\[ 1 = \frac{\mu}{2} \int_{-1/\mu}^{1/\mu} \frac{dp_\mu}{\sqrt{1 - (\mu p_\mu)^2}} | p_\mu \rangle \langle p_\mu |, \]

and the inner product of the momentum eigenvectors becomes

\[ \langle p_\mu | p'_\mu \rangle = \sqrt{1 - (\mu p_\mu)^2} \delta(p_\mu - p'_\mu). \]

The eigenvalue problem for the position operator in the noncanonical representation (32) takes the form of the following differential equation

\[ i\sqrt{1 - (\mu p_\mu)^2} \frac{\partial \psi(p_\mu)}{\partial p_\mu} = \lambda \psi(p_\mu), \]

with solution

\[ \psi_\lambda(p_\mu) = c \exp \left[ -i \sin^{-1}(\mu p_\mu) \right]. \]
The constant \( c \) should be fixed by the normalization condition

\[
1 = cc^\mu \int_{-1/\mu}^{1/\mu} \frac{dp}{\sqrt{1 - (\mu p)^2}} = cc^\pi, \tag{39}
\]

in which definition (34) is used. Thus, the normalized eigenvectors for the position operator in the noncanonical polymeric representation will be

\[
\psi_\lambda(p) = \sqrt{\frac{\mu}{\pi}} \exp \left[ -i\frac{\sin^{-1}(\mu p)}{\mu} \right]. \tag{40}
\]

One would be tempted to conclude that the plane waves of the standard Schrödinger representation \( \psi_\lambda(p) \sim e^{-ipl} \), with \( \lambda \in \mathbb{R} \) and \( p \in (-\infty, +\infty) \), are recovered in the limit \( \mu \to 0 \), since the polymeric momentum \( \mu p \) reduces to the standard unbounded canonical momentum \( p \) in this limit. However, since the approximation (24) is reliable for \( \mu p \ll 1 \), one should be careful when taking the continuum limit in the polymer framework [12]. But it can be shown that the continuum limit will be correctly recovered for a given small value of the lattice spacing \( \mu \) (e.g., \( \mu \sim l_\mu \)) [11]. Also, the position eigenstates (40) should be orthogonal in momentum polarization. To see this, we consider their inner product as

\[
\langle \psi_\lambda | \psi_{\lambda'} \rangle = \frac{\mu}{\pi} \int_{-1/\mu}^{1/\mu} \frac{\exp \left[ -(\lambda - \lambda') \frac{\sin^{-1}(\mu p)}{\mu} \right]}{\sqrt{1 - (\mu p)^2}} dp = \frac{\sin \left( \frac{\pi (\lambda - \lambda')}{\mu} \right)}{\pi (\lambda - \lambda')/2\mu} = \delta_{\lambda,\lambda'}, \tag{41}
\]

in agreement with our previous general definition (14). In the last equality we have used the fact that \( \frac{\lambda - \lambda'}{\mu} = n \in \mathbb{Z} \), since \( \lambda \) and \( \lambda' \) are the eigenvalues of the position operator, which are restricted to belong to the lattice \( \gamma\mu \) (16). Thus, the position eigenvectors leave the lattice (16) invariant and are naturally orthogonal in the polymer framework. As we will show, this is not the case for the theories based on the deformed Heisenberg algebra, and an extra assumption is needed to recover the orthogonality of the position eigenvectors.

Unlike the standard Schrödinger representation, the expectation value of the kinetic term of the Hamiltonian operator in the polymer framework converges. Indeed, with the help of (34) we get

\[
\langle \psi_\lambda | \frac{p^2 w}{2m} | \psi_\lambda \rangle = \frac{1}{4m\mu^2}. \tag{42}
\]

This result indicates the existence of an upper bound for the energy of the free particle in the polymer framework [12]. In contrast, the expectation value of the kinetic term of the Hamiltonian diverges in the minimal length uncertainty relation framework [6].

4.2. The modified uncertainty relation

The position operator \( q \) is a self-adjoint and symmetric operator in the polymer framework. Also, it is diagonal in the basis of the eigenstates (40), which means that there is no uncertainty in measurement of position. To be more precise, let us take a look at the uncertainty principle in this setup.

The deformed Heisenberg algebra (30) in the noncanonical polymeric representation naturally provides a modification to the standard uncertainty relation through the standard
(quantization scheme-independent) definition

\[(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4} \langle [A, B] \rangle^2, \quad (43)\]

for two arbitrary operators \(A\) and \(B\), where \(\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2}\) and similarly for \(B\). For the noncanonical pair \((q, p_p)\) with commutation relation (30), the definition (43) gives

\[\Delta q \Delta p_p \geq \frac{1}{2} \left\{ \sqrt{1 - \left( \mu p_p \right)^2} \right\}, \quad (44)\]

which is in agreement with the result obtained in [10, 19]. In the limit \(\mu \to 0\) we get

\[\Delta q \Delta p_p \geq \frac{1}{2} \left( 1 - \frac{\mu^2}{2} (\Delta p_p)^2 + o(\mu^4) \right), \quad (45)\]

in which we have set \(\langle p_p \rangle = 0\). Although this relation, in some senses, is very similar to the results obtained from the generalized uncertainty relations which are investigated in the context of the string theory [5–7], there is a crucial difference. The generalized uncertainty principle theories predict a minimal (nonzero) uncertainty \(\Delta q_0 > 0\) in the position measurement such that (see [6] for details)

\[(\Delta q)|_\psi = \langle \psi | q - \langle q | \psi \rangle | \psi \rangle^2 |\psi \rangle \geq \Delta q_0, \quad (46)\]

which clearly implies that there cannot be any physical state that is a position eigenvector, since such an eigenstate should, of course, have zero uncertainty in position. On the other hand, the polymer-modified uncertainty relation (44) does not predict any minimal uncertainty in position measurement. Here, the existence of a minimal length (the polymer length scale \(\mu\)) is encoded in the discrete position eigenvalues rather than the minimal uncertainty in position measurement. The advantage of this viewpoint to the minimal length becomes more clear when we note that one can work either in position or in momentum polarization in the polymer framework. But the position polarization fails to be applicable in the generalized uncertainty theories due to the existence of the nonzero minimal uncertainty \(\Delta q_0\) in position measurement [6].

### 4.3. Simple harmonic oscillator

In this section we consider the well-known example of the harmonic oscillator in the noncanonical polymeric representation of the quantum mechanics, whose energy eigenvalues are given by equation (23). This eigenvalue problem may be solved in position or momentum polarization, and, as we have mentioned before, it becomes a difference equation in position polarization while taking the form of a differential equation in momentum polarization. In momentum polarization, one can consider the eigenvalue problem (23) in canonical or noncanonical polymeric representations on the Hilbert spaces (27) and (33), respectively. Let us first consider this problem in the noncanonical representation, and then we show the equivalence of the results with those obtained from the standard canonical representation.

In the noncanonical representation, the action of the position and momentum operators on a typical wave function are defined by relations (32), and it is straightforward to show that the eigenvalue problem (23) for the harmonic oscillator with potential \(W = \frac{1}{2} m o^2 q^2\) takes the form of the following differential equation
\[
\frac{d^2\psi(p_\mu)}{dp_\mu^2} - \frac{\mu^2 p_\mu}{1 - (\mu p_\mu)^2} \frac{d\psi(p_\mu)}{dp_\mu} + \frac{d^4(\epsilon - p_\mu^2)}{1 - (\mu p_\mu)^2} \psi(p_\mu) = 0,
\]  
(47)  

where we have defined \(d := 1/\sqrt{m_0\epsilon}\) and \(\epsilon := 2mE\). Setting  
\[
\phi := \sin^{-1}(\mu p_\mu) + \frac{\pi}{2}
\]  
(48)  
simplifies the above equation as  
\[
\frac{d^2\psi(\phi)}{d\phi^2} + \left(a - 2h^2 \cos(2\phi)\right)\psi(\phi) = 0,
\]  
(49)  

with  
\[
a = 2h\left(\epsilon d^2 - h\right) \quad \text{and} \quad h = \frac{d^2}{2\mu^2}.
\]  
(50)  

The above differential equation is a Mathieu equation whose solutions can be written in terms of the Mathieu cosine and sine functions as \([20]\)  
\[
\psi(\phi) = c_1 C(a, h^2, \phi) + c_2 S(a, h^2, \phi),
\]  
(51)  

where \(c_1\) and \(c_2\) are the constants of integration. Note that the Fourier transform of the wave function (51) gives the corresponding wave function in the position polarization, which clearly is periodic, since it is restricted on the lattice (16) by definition. In this respect, the solution (51) should be periodic to ensure appropriate fall-off for the associated Fourier coefficient \([10]\). (See also \([20]\) for the properties of the periodic solution of the Mathieu equation.) In the limit \(\mu/d \ll 1\), one can use the asymptotic formula  
\[
a_n = -2h^2 + 2h(2n + 1) - \frac{2n^2 + 2n + 1}{4} + O(h^{-1})
\]  
(52)  

to arrive at the following energy eigenvalues  
\[
E_n = (2n + 1) \frac{\omega}{2} - \frac{2n^2 + 2n + 1}{4} \left(\frac{\mu}{d}\right)^2 + O\left(\frac{\mu^4}{d^4}\right).
\]  

The first term in the right-hand side of the above equation is nothing but the energy eigenvalues of the harmonic oscillator in the standard Schrödinger representation. The second term is the first polymeric correction, although it is smaller than can be detected \([10]\). The energy eigenvalues (52) are in agreement with the result obtained in \([10]\) from the standard canonical polymeric representation of the quantum mechanics\(^2\). This coincidence refers to the fact that two Hilbert spaces (27) and (33) are unitarily equivalent. This is because they are just the canonical and noncanonical representations of the unique Hilbert space (22). Furthermore, the result (52) is the same as the harmonic oscillator eigenvalues that were obtained in the Snyder-deformed Heisenberg algebra setup. This coincidence signals a connection between these two apparently different setups, and we will explore this connection in the next section.

\(^2\) The relation (52) exactly coincides with the relation (5.10) of \([10]\) just by substituting \(\mu = \mu_1/2\). This is because of the different definition of the momentum square (20). More precisely, the squared of the momentum (20) shifts the states two steps in both sides in the lattice (16). But, the momentum squared defined in \([10]\) shifts the states one step in both sides.
5. Connection with noncommutative geometry

5.1. Snyder-deformed Heisenberg algebra

In 1947, Snyder formulated a Lorentz invariant space–time which admits a minimal length scale (ultraviolet cutoff) [8]. Snyder’s reasonable assumptions can be also translated into a noncanonical Heisenberg algebra in order to achieve a minimal length scale in standard quantum mechanics [9]. In this regard, we first introduce a Snyder-deformed Heisenberg algebra idea, and then we will show how it can be identified with the noncanonical polymeric representation of the quantum mechanics. The strategy is based on the following arguments [21]:

- Fix the noncommutativity between coordinates as
  \[ [Q_i, Q_j] = \alpha^2 J_{ij}, \]
  where \( i, j = 1, \ldots, n \) and \( \alpha \) is a deformation parameter with dimension of length. This parameter is usually assumed to be of the order of the Planck length \( l_p \) to ensure that these effects becomes only important in the high-energy regimes and also being negligible in the limit of low energies (correspondence principle). \( J_{ij} = -J_{ji} = i(q_ip_j - q_jp_i) \) are the generators of rotation in the \( n \) dimension. Such a form of the noncommutativity between position coordinates was first proposed by Snyder himself [8].

- The rotation generators should satisfy the ordinary \( SO(n) \) algebra
  \[ [J_{ij}, J_{kl}] = \delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk}. \]

- The translation group is not deformed, i.e.,
  \[ [P_i, P_j] = 0. \]
  This assumption is reasonable, since we are looking just for the ultraviolet effects such as minimal length and maximal momentum (high-energy regime modification). The noncommutativity between the momentum operators is evidence of the existence of the infrared cutoff, which is important only for the low-energy regimes [9]. Therefore, since the ordinary momentum operators also satisfy the commutation relations \([p_i, p_j] = 0\), the momenta are the same either in noncommutative or in commutative proposals.

- In addition, it is natural to assume [8]
  \[ [J_{ij}, Q_k] = Q_i \delta_{jk} - Q_j \delta_{ik} \]
  \[ [J_{ij}, P_k] = P_i \delta_{jk} - P_j \delta_{ik}. \]

The above statements cannot fix the commutation relation between variables \( Q \) and \( P \), and there are many noncommutative Heisenberg algebras which satisfy all the above conditions and also are closed in virtue of the Jacobi identity. Nevertheless, relations (53) and (56) will be satisfied, which are the only requirement of the setup. It is always possible to find noncommutative variables \((Q, P)\) in terms of the commutative pair \((q, p)\) such that equation (53) is satisfied (Darboux theorem). The most general \( SO(n) \) covariant form of such
noncommutative variables compatible with all of the above-mentioned conditions is
\[ Q_i = q_i \varphi_1 + \alpha^2 \left( q_i p_j \right) p_j \varphi_2, \]
\[ P_i = p_i, \quad \text{(58)} \]

where \( \varphi_1 = q_1(\alpha^2 p^2) \) and \( \varphi_2 = q_2(\alpha^2 p^2) \) are two unknown functions of the momenta. The function \( \varphi_1 \) should satisfy the condition \( \varphi_1(0) = 1 \) for \( \alpha \to 0 \) to recover the standard commutation relations in the low-energy regimes (correspondence principle). Relations (58) give the commutation relation between coordinates and momenta as
\[ \left[ Q_i, P_j \right] = i \left( \delta_{ij} \varphi_1 + \alpha^2 p_j \varphi_2 \right). \quad \text{(59)} \]

Our next challenge is how to determine two functions \( \varphi_1 \) and \( \varphi_2 \). In fact, they should be obtained in such a way that the relations (53) and (56) are satisfied. Substituting (58) into relation (56), one can see that this relation is automatically satisfied without any restriction on the functions \( \varphi_1 \) and \( \varphi_2 \). Substituting into relation (53), however, gives
\[ \frac{d\varphi_1}{d\left( \alpha^2 p^2 \right)} + \frac{1}{2} \frac{1 - \varphi_1 \varphi_2}{\varphi_1 + \alpha^2 p^2 \varphi_2} = 0. \quad \text{(60)} \]
It is easy to see that the commutation relations (59), (53), and (55) satisfy the Jacobi identity and constitute the noncommutative Heisenberg algebra in this setup. This noncommutative algebra can support a number of deformed Heisenberg algebras just by choosing an appropriate form for the functions \( \varphi_1 \) and \( \varphi_2 \) [22, 21]. Here, we would like to construct a connection between the noncommutative Heisenberg algebra (59) and the polymer quantum mechanics, and to do this, it is plausible to consider a one-dimensional system for which the functions \( \varphi_1 \) and \( \varphi_2 \) are uniquely fixed. Indeed, in this case we may find these functions as \( \varphi_1 = \sqrt{1 - (\alpha p)^2} \) and \( \varphi_2 = 0 \), leading to the noncanonical Heisenberg algebra
\[ \left[ Q, P \right] = i \sqrt{1 - (\alpha P)^2}, \quad \text{(61)} \]
in which the fact that \( P = p \) is used. It is also important to note that the sign of the deformation parameter \( \alpha^2 \) is not fixed, which may lead to different physical results. Evidently the momentum is bounded, as \( P \in \left( -\frac{1}{\alpha}, +\frac{1}{\alpha} \right) \) if \( \alpha^2 > 0 \), while a minimum nonzero uncertainty in position as \( \Delta Q_0 = \left( \sqrt{-\alpha^2/2} \right) \) can be realized in the case of \( \alpha^2 < 0 \) [22]. The former one is very similar to the noncanonical polymeric representation with deformed Heisenberg algebra (30), and we only consider this case in this paper.

The Snyder-deformed Heisenberg algebra (61), together with the associated Hamiltonian operator
\[ H = \frac{p^2}{2m} + W(Q), \quad \text{(62)} \]
complete our construction to study the kinematics and dynamics of the physical system in this setup. However, one should be careful that while the Hamiltonian operator (62) has the standard functional form, the associated momentum is bounded as \( P \in \left( -1/\alpha, +1/\alpha \right) \).

3 More precisely, the variables \((q, p)\) are Darboux (canonical) variables, and transformation (58) is a Darboux transformation [23].
5.2. Hilbert space representation

Comparing the commutation relation (30) with the Snyder-deformed Heisenberg algebra (61), one immediately recognizes a correspondence between momenta $p_\mu$ and $P$; they satisfy the same modified Heisenberg algebra, and also they are bounded in the same way by the parameters $\mu$ and $\alpha$, respectively. Thus, we identify the polymeric momentum $\mu p$ with $P$ as

$$Q \equiv q, \quad P \equiv p_\mu = \frac{1}{\mu} \sin(\mu p),$$

in which we have also set the reasonable identification $\alpha = \mu$ for the deformation parameters.

It is, however, important to note that the polymer length $\mu$ is the lattice parameter and does not have to always be very small, although we consider it to be small compared to other length scales present in the system under consideration. We have done such assumption before in the study of a harmonic oscillator, where we considered $\mu$ to be very small in comparison with the natural length scale $d$. On the other hand, in the Snyder-deformed Heisenberg algebra (61), the length scale $\alpha$ is always assumed to be very small (for instance, of the order of the Planck length) to induce a cutoff in the ultraviolet regime [8, 9]. Thus, we suppose that the identification (63) is possible for sufficiently small values of the polymer length scale $\mu$. Here, we consider $\mu = \alpha = \mathcal{O}(1)/\mu$, through the paper, where the numerical coefficient $\mathcal{O}(1)$ should be fixed by experiment [17, 24, 25]. Then, both the polymer quantum mechanics and the Snyder-deformed Heisenberg algebra (61) can induce a maximal value for the momentum around the Planck momentum $P_n = \mu^{-1}$ up to the numerical factor $\mathcal{O}(1)$.

As we say before, the eigenvalue problem of the harmonic oscillator in the Snyder-deformed framework gives the same result that we have already obtained in the previous section in the noncanonical polymeric representation [22]. But, the inner product of the position eigenstates (40) are no longer generally orthogonal. Indeed, we have [22]

$$\langle \psi_{\lambda} | \psi_{\lambda'} \rangle = \frac{\sin \left( \frac{\pi (\lambda - \lambda')}{2\alpha} \right)}{\pi \frac{\lambda - \lambda'}{2\alpha}},$$

and there is no reason to suppose $\frac{\lambda - \lambda'}{2\alpha} \in \mathbb{Z}$ in this setup (see figure 1). However, one can define a family of orthogonal eigenvectors that make the position operator diagonal [6, 22].

This is equivalent to considering a lattice such as (16) for the position operator. Thus, the identification (63) is reliable when one considers a lattice for the Snyder-deformed Heisenberg algebra on a separable Hilbert space being isomorphic to (22).

6. Classical phase space

Now, let us study the classical limit of the polymer quantum mechanics, which leads to a one-parameter family of $\mu$-dependent classical theory rather than the ordinary classical one. The standard classical theory would be recovered from the discrete $\mu$-dependent one in the continuum limit $\mu \to 0$. However, one can also study such effective $\mu$-dependent theories
The method is based on implementing the Weyl operator on the classical phase space in order to define an appropriate definition for the discrete derivative of the phase-space functions [17, 26]. However, here, we should study the classical limit of the polymer quantum mechanics, which we have considered in the previous sections. More precisely, we analyze the classical phase space corresponding to the Hilbert space (22) in two different pictures, i.e., in canonical and noncanonical pictures correspond to the Hilbert spaces (27) and (33), respectively. In this regard, the momentum part of the phase space should be compactified as a circle $S^1$, and the configuration space is the real line $\mathbb{R}$. Thus, we consider a two-dimensional symplectic space $\Gamma_\mu$ with topology $\mathbb{R} \times S^1$ as the polymeric phase space corresponding to the Hilbert space (22) and study its classical implications for two canonical and noncanonical cases.

### 6.1. Noncanonical chart

Let us start with the noncanonical representation of the polymeric phase space $\Gamma_\mu$ and deal with the classical variables $(\tilde{q}, \tilde{p}_\mu)$ corresponding to the operators $(q, p_\mu)$, which are expected to satisfy the following Poisson bracket
\[
\{ \tilde{q}, \tilde{p}_\mu \} = \sqrt{1 - \left( \mu \tilde{p}_\mu \right)^2}.
\]

The associated classical Hamiltonian function can easily be deduced from the corresponding Hamiltonian operator (62) as
\[
H_\mu(q, p_\mu) = \frac{\tilde{p}_\mu^2}{2m} + W(q).
\]
The Poisson algebra (65) together with the Hamiltonian function (66) determine the kinematics and dynamics of the system on the polymeric phase space $\Gamma_\mu$. Nevertheless, we implement the symplectic geometry in order to define the associated Hamiltonian system, which, as we will see, gives a more clear connection between classical and quantum frameworks for the polymeric systems.

Consider a symplectic manifold, thought of as the phase space $\Gamma_\mu$, equipped with a symplectic structure $\omega$, which is a closed nondegenerate two-form on $\Gamma_\mu$. We propose the symplectic two-form being the noncanonical form

$$\omega = \frac{d\tilde{q} \wedge d\tilde{p}_\mu}{\sqrt{1 - (\mu\tilde{p}_\mu)^2}}, \hspace{1cm} (67)$$

in this chart [17]. The reason for this proposition will become clear when we show that the symplectic structure (67) correctly generates the noncanonical Poisson algebra (65). The Hamiltonian triplet is defined as $\omega^\mu H = (H_\mu, \omega, x)$, where $x$ is the Hamiltonian vector field, which determines the time evolution of the system through the equation

$$\omega = \mu_i H d \omega , \hspace{1cm} (68)$$

Substituting the symplectic structure (67) and the associated Hamiltonian function (66) into equation (68) gives the solution for the vector field as

$$x_\mu = \sqrt{1 - (\mu\tilde{p}_\mu)^2} \left( \tilde{p}_\mu \frac{\partial}{\partial \tilde{q}} - \frac{\partial W}{\partial \tilde{q}} \frac{\partial}{\partial \tilde{p}_\mu} \right). \hspace{1cm} (69)$$

The integral curves of the vector field $x$, in turn, give the Hamilton’s equation in this chart

$$\frac{dq}{dt} = \frac{\tilde{p}_\mu}{m} \sqrt{1 - (\mu\tilde{p}_\mu)^2}, \hspace{0.5cm} \frac{d\tilde{p}_\mu}{dt} = -\frac{\partial W}{\partial \tilde{q}} \sqrt{1 - (\mu\tilde{p}_\mu)^2}, \hspace{1cm} (70)$$

which will be reduced to the standard Hamiltonian equations in the limit $\mu \to 0$. The Poisson bracket between two real-valued functions $F$ and $G$ on the phase space $\Gamma_\mu$ is defined as

$$\{F, G\} = \omega(x_\mu, x_\nu), \hspace{1cm} (71)$$

and closure of the symplectic structure ensures that the Jacobi identity is satisfied by the resultant Poisson brackets. With the help of relations (67) and (69), the definition (71) gives the Poisson brackets in this chart as

$$\{F, G\} = \sqrt{1 - (\mu\tilde{p}_\mu)^2} \left( \frac{\partial F}{\partial \tilde{q}} \frac{\partial G}{\partial \tilde{p}_\mu} - \frac{\partial F}{\partial \tilde{p}_\mu} \frac{\partial G}{\partial \tilde{q}} \right). \hspace{1cm} (72)$$

It is straightforward to check that the noncanonical Poisson algebra can be retrieved from relation (72) by substituting $F = \tilde{q}$ and $G = \tilde{p}$, which shows that the symplectic structure (67) is chosen correctly.

6.2. Darboux chart

According to the Darboux theorem, it is always possible to find a local chart in which any symplectic structure takes the canonical form. Thus, there is a (Darboux) transformation from
the noncanonical pair of variables \((\tilde{q}, \tilde{p}_\mu)\) to a canonical one \((\tilde{q}, \tilde{p})\) in which the symplectic two-form \((67)\) takes the canonical form.

To see how this statement may work, consider the following Darboux transformation on the phase space \(I_\mu\)

\[
\left( \tilde{q}, \tilde{p}_\mu \right) \rightarrow \left( \tilde{q}, \tilde{p}_\mu \right) = \left( \tilde{q}, \frac{1}{\mu} \sin^{-1}(\mu\tilde{p}_\mu) \right),
\]

which locally transforms the symplectic structure \((67)\) to the canonical form

\[
\omega_\mu \left( \tilde{q}, \tilde{p}_\mu \right) \rightarrow \omega \left( \tilde{q}, \tilde{p} \right) = dq \wedge d\tilde{p},
\]

and also transforms the Hamiltonian function \((66)\) to

\[
H_\mu \left( \tilde{q}, \tilde{p}_\mu \right) \rightarrow H_\mu \left( \tilde{q}, \tilde{p} \right) = \frac{1}{m\mu^2} (1 - \cos(\mu\tilde{p})) + W(\tilde{q}).
\]

This form for the Hamiltonian function is the same as the classical limit of the polymeric Hamiltonian operator \((25)\). Substituting the canonical symplectic structure \((74)\) and also the associated Hamiltonian function \((75)\) into equation \((68)\), we are led to the following solution for the Hamiltonian vector field

\[
x_H = \frac{\sin(\mu\tilde{p})}{m\mu} \frac{\partial}{\partial \tilde{q}} - \frac{\partial W}{\partial \tilde{q}} \frac{\partial}{\partial \tilde{p}}.
\]

The integral curves of the above Hamiltonian vector field are the polymer-modified Hamilton’s equations of motion in the canonical (Darboux) chart

\[
\frac{dq}{dt} = \frac{\sin(\mu\tilde{p})}{m\mu}, \quad \frac{d\tilde{p}}{dt} = -\frac{\partial W}{\partial \tilde{q}},
\]

which clearly reduce to the standard Hamilton’s equations in the continuum limit \(\mu \rightarrow 0\). It is straightforward to show that equations \((77)\) and \((70)\) are in agreement through the Darboux transformation \((73)\).

Substituting \(x_F\) and \(x_G\) from \((76)\) into \((71)\), one gets

\[
\{F, G\} = \frac{\partial F}{\partial \tilde{q}} \frac{\partial G}{\partial \tilde{p}} - \frac{\partial F}{\partial \tilde{p}} \frac{\partial G}{\partial \tilde{q}},
\]

which is nothing but the standard definition for the canonical Poisson bracket between two arbitrary functions \(F\) and \(G\). Choosing \(F(\tilde{q}, \tilde{p}) = \tilde{q}\) and \(G(\tilde{q}, \tilde{p}) = \tilde{p}\), we can easily deduce

\[
\{ \tilde{q}, \tilde{p} \} = 1,
\]

which coincides with the standard canonical Poisson algebra \((1)\).

Therefore, one deals with the unique Hamiltonian triplet \((H_\mu, \omega, x)\) on the polymeric phase space \(I_\mu\), which can be represented in two different local charts: (i) In the noncanonical chart, one can work with symplectic structure \((67)\), Hamiltonian function \((66)\), and the corresponding noncanonical Poisson algebra \((65)\). (ii) We can also work in the canonical chart and utilize the effective Hamiltonian function \((75)\), the symplectic structure \((74)\), and the ordinary canonical Poisson algebra \((79)\). The former is the classical limit of the noncanonical polymeric representation on the Hilbert space \((33)\), and the latter is the classical limit of the canonical representation on the Hilbert space \((27)\). The trajectories on the polymeric phase space \(I_\mu\) are the same in either noncanonical or canonical charts, since equation \((68)\) is held in a chart-independent manner for the Hamiltonian system \((H_\mu, \omega, x)\).
7. Summary and conclusions

The existence of a minimal measurable length is a common feature of quantum gravity candidates such as loop quantum gravity and string theory. This issue can also be realized from an algebraic deformation in the standard quantum mechanical structures. The so-called polymer quantum mechanics is investigated in the symmetric sector of the loop quantum gravity, which supports the existence of a kind of minimal length scale $\mu$, known as the polymer length. On the other hand, generalized uncertainty relations have been suggested in the context of string theory, which also predict a minimal length scale $\alpha$ (corresponding to the strings’ size) for the system under consideration. The minimal length scale is usually taken to be of the order of the Planck length, and, consequently, either the polymer quantum mechanics or the generalized uncertainty theories induce an ultraviolet cutoff on the Planck scale, where one expects the quantum gravitational effects would become important. Therefore, we have considered the noncanonical Hilbert space representation of the polymer quantum mechanics in order to explore a possible connection between the polymer quantum mechanics and the generalized uncertainty theories. All the physical results in the noncanonical representation, of course, are the same as those obtained in the canonical representation since these representations are unitarily equivalent. However, the relation with the generalized uncertainty theories immediately emerges, since the Heisenberg algebra gets modified in the noncanonical polymeric representation. The polymer-modified Heisenberg algebra (30) in the noncanonical representation leads to the polymer-modified uncertainty relation (44). We have shown that the polymer-modified Heisenberg algebra (30) coincides with the Snyder-deformed (noncommutative) Heisenberg algebra (61), in which the translation group is not deformed. Technically, however, there is a crucial difference between the polymer quantum mechanics and the generalized uncertainty theories. In polymer framework, the eigenvalues of the position operator are taken to be discrete with respect to the minimal (polymer) length scale $\mu$. In the generalized uncertainty theories, however, the minimal length scale is emerges through a nonzero minimal uncertainty in position measurement that forbids the position eigenvectors with zero uncertainty in position. The position eigenvectors are no longer orthogonal, and the position polarization fails to be applicable in this setup. In contrast, the position operator is restricted to belong to the lattice (16) in the polymer framework, and its eigenvectors are generally orthogonal and leave the lattice invariant. Thus, both the position and momentum polarization are relevant (self-adjoint) operators in the polymer framework. Therefore, the polymer viewpoint to the minimal length scale is more fundamental. At the classical level, the noncanonical representation of the polymer quantum mechanics is determined by the polymer-modified Poisson algebra (65) and the Hamiltonian function (66). In the canonical representation, however, the Poisson algebra takes its standard canonical form (79), but the corresponding Hamiltonian function gets modified as (75). We have shown that these two pictures are determined by the unique Hamiltonian system, and they are related by a Darboux transformation on the corresponding phase space. At the end, a natural question may arise: does it follow from the comparison to Snyder space–time that polymer quantization could be Lorentz-invariant? In this sense we would like to emphasize that since, like standard quantum mechanics, polymer quantum mechanics is not Lorentz-invariant, it is just equivalent to the quantum mechanics in the Snyder noncommutative setup. However, the Snyder noncommutative space–time may be equivalent to the polymer quantum field theory.
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