CONTROLLED SINGULAR EVOLUTION EQUATIONS AND PONTRYAGIN TYPE MAXIMUM PRINCIPLE WITH APPLICATIONS

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Abstract. Due to the propagation of new coronavirus (COVID-19) on the community, global researchers are concerned with how to minimize the impact of COVID-19 on the world. Mathematical models are effective tools that help to prevent and control this disease. This paper mainly focuses on the optimal control problems of an epidemic system governed by a class of singular evolution equations. The mild solutions of such equations of Riemann-Liouville or Caputo types are special cases of the proposed equations. We firstly discuss well-posedness in an appropriate functional space for such equations. In order to reduce the cost caused by control process and vaccines, and minimize the total number of susceptible people and infected people as much as possible, an optimal control problem of an epidemic system is presented.

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And then for associated control problem, we use a generalized Liapunov type theorem and the spike perturbation technique to obtain a Pontryagin type maximum principle for its optimal controls. In order to derive the maximum principle for an optimal control problems, some techniques from analytical semigroups are employed to overcome the difficulties. Finally, we discuss the potential applications.

1. Introduction. Many epidemic diseases, if not controlled, may be really dangerous all over the world due to the increasing number of traveling people, such as influenza, tuberculosis and now COVID-19. In recent decades, many researchers have realized that the optimal control theory of mathematical epidemic models could help to quantify possible disease control strategies with further improvements and recommendations. So the researchers have began to apply the obtained optimal control theories to the prevention and control of infectious diseases.

Optimal control theory of infinite dimensional systems is one of most important part of control theory. Roughly speaking, an optimal control of a system is a control which minimizes a given cost functional. Hence, an optimal control problem can be regarded as an optimization problem. In optimal control theory, a set of necessary conditions for optimal controls is referred to as a maximum principle which is a basic and crucial mathematical tool. In this research area, many mathematicians and control theorists have made great contributions in both deterministic \([27, 5, 6]\) and stochastic settings \([14, 15, 42]\). An extensive survey on control theory of infinite dimensional systems can be found in the book by Li and Yong \([28]\).

On the other hand, fractional calculus has attracted a lot of researchers’ attention due to its interesting applications in physical \([21]\), biology \([43]\), anomalous diffusion \([32, 17]\), finance \([26, 16]\), electrical circuits \([4]\), control \([11]\) and other science fields. An extensive study on fractional differential equations can referred to the books by Samko \([37]\) and Kilbas \([25]\). In recent years, optimal control problems for singular control systems and fractional differential systems have been studied by many authors. Relevant works can be traced back to those by Carlson \([10]\), Burnap and Kazemi \([9]\), De La Vega \([12]\). Later some authors began to consider the optimal control problems for fractional differential equations. We mention the works \([2, 3, 22, 30, 8, 23]\). In 2020, Lin and Yong \([29]\) investigated the well-posedness of a class of controlled singular Volterra integral equations and established a Pontryagin type maximum principle for optimal controls for such equations. The well-known fractional differential equations are special case of the equations. As far as we know, there are no results on necessary optimality conditions for controlled singular evolution equations.

In this work, motivated by the above consideration, we are concerned with the following controlled singular evolution equation with sectorial operator

\[
x(t) = \eta(t) + \int_0^t (t-s)^{\alpha-1} \mathcal{E}_{\alpha, \alpha}(-At^\alpha) f(s, x(s), u(s)) \, ds, \quad t \in [0, T],
\]

where \(0 < \alpha \leq 1\), \(T > 0\) is a constant, \(\eta(\cdot)\) and \(f(\cdot, \cdot, \cdot)\) are two given maps, \(A\) is a sectorial operator defined on a Banach space \(X\), \(\mathcal{E}_{\alpha, \alpha}(-At^\alpha)\) is defined as

\[
\mathcal{E}_{\alpha, \alpha}(-At^\alpha) = \frac{1}{2\pi i} \int_{\Gamma_\theta} \mathcal{E}_{\alpha, \alpha}(-\lambda t^\alpha)(\lambda I + A)^{-1} d\lambda,
\]

\(x(\cdot)\) is called the state trajectory, and \(u(\cdot)\) is called the control taking values in \(U\), where \(U\) is a separable metric space. (In Section 2, we will explain the operator.
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$E_{\alpha,\alpha}(-At^\alpha)$ on the space $X$ is well-defined.) To measure the performance of the control, we define the cost functional as

$$J(u(\cdot)) = \int_0^T g(t, x(t), u(t)) dt,$$

where $g$ is a given map.

Equation (1) can be used to describe many phenomena involving memories in the real world. Particularly, when using fractional parabolic evolution equations to stimulate some physical problems, it is often assumed that the partial differential operator in the linear part is a sectorial operator. For example, one can find from [30, 19, 36] that many elliptic operators equipped with homogeneous boundary conditions are sectorial when they are considered in the Lebesgue spaces (e.g. $L^p$-spaces) or in the space of continuous functions.

The rest of the paper is organized as follows. In Section 2, we give some necessary preliminaries. In Section 3, we discuss the well-posedness of the state equation in the space $L^p$ and the continuity of the solutions. In Section 4, we look at some special cases. In Section 5, we present a Pontryagin type maximum principle for our optimal control problem of the singular evolution equations. In Section 6, we demonstrate the applications in COVID-19 outbreaks. Some concluding remarks are collected in Section 7.

2. Preliminaries. In this section, we will introduce some basic definitions and useful results, which will be used in the later discuss.

As usual, for a linear operator $A$, we denote by $\sigma(A)$ its spectrum, while $\rho(A) := \mathbb{C} - \sigma(A)$ is the resolvent set of $A$, and denote by $R(z; A) = (zI - A)^{-1}$ the resolvent of $A$, where $z \in \rho(A)$.

We introduce the following space $L^p(0, T)$

$$\bigcup_{r>p} L^r(0, T), \ 1 \leq p < \infty,$$

where $L^r(0, T)$ (1 $\leq r < \infty$) denotes the space of all measurable functions $\varphi$ which map the interval $(0, T)$ into $\mathbb{R}$ with the norm $\|\varphi(\cdot)\|_r = \left( \int_0^T |\varphi(t)|^r dt \right)^{\frac{1}{r}} < \infty$. In particular,

$$L^\infty(0, T) = \{ \varphi : [0, T] \to \mathbb{R} | \varphi(\cdot) \text{ is measurable, and } \|\varphi(\cdot)\|_\infty \equiv \sup_{t \in [0, T]} |\varphi(t)| < \infty \}.$$

For convenience, sometimes we simply write $L^p(0, T)$ as $L^p(0, T)$ omitting $\mathbb{R}$.

The Euler gamma function $\Gamma(z)$ is defined by the Euler integral of the second kind:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ \Re(z) > 0.$$

This integral is convergent for all complex $z \in \mathbb{C}$ with $\Re(z) > 0$. The quotient expansion of two gamma functions at infinity is

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} \left(1 + O\left(\frac{1}{z}\right)\right), \ |\arg(z + a)| < \pi, \ |z| < \infty. \ \ (4)$$

The beta function is defined by the Euler integral of the first kind:

$$B(z, w) = \int_0^1 t^{z-1}(1 - t)^{w-1} dt, \ \Re(z) > 0, \ \Re(w) > 0. \ \ (5)$$
There exists a well-known relation between the Euler gamma function and the beta function:

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}, \quad z, w \notin \mathbb{Z}_0^- := \{0, -1, -2, -3, \ldots\}. \quad (6)$$

The generalized Mittag-Leffler function is defined as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0. \quad (7)$$

When $\beta = 1$, $E_{\alpha, 1}(z)$ coincides with the classical Mittag-Leffler function, i.e., $E_{\alpha, 1}(z) = E_{\alpha}(z)$. The integral representation of the generalized Mittag-Leffler function is

$$E_{\alpha, \beta}(z) = \frac{1}{2\pi i} \int_{H_\alpha} \frac{\lambda^{-\beta}e^{\lambda z}}{\lambda^{\alpha} - z} d\lambda, \quad \alpha, \beta > 0, \quad (8)$$

where the path of integration $H_\alpha$ (the Hankel path) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\lambda| < \frac{1}{\alpha}$ in the positive sense: $-\pi \leq \arg(\lambda - a) \leq \pi$ on $H_\alpha$.

It is well-known that for $x > 0$, $E_{\alpha, \beta}(-x)$ is completely monotone if and only if $0 < \alpha \leq 1$, and $\alpha \leq \beta$. (See [13, 31])

The Wright function is defined as

$$W_{\lambda, \mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n\lambda + \mu)}, \quad \lambda > -1, \quad \mu > 0, \quad z \in \mathbb{C}. \quad (9)$$

Note that the case $\lambda = 0$ is trivial since $W_{0, \mu}(z) = e^z / \Gamma(\mu)$. In particular, for the case $0 < \nu < 1$, $W_{-\nu, 1-\nu}(-z) = M_\nu(z)$, where $M_\nu(z)$ is the Mainardi’s function defined as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-n\nu + 1 - \nu)}. \quad (10)$$

The Laplace transform of the Mainardi’s function is

$$\int_0^{\infty} e^{-st}M_\nu(t)dt = E_{\nu, \nu}(-s). \quad (11)$$

On the other hand, $M_\nu(z)$ satisfies the following three relationships

$$\int_0^{\infty} \frac{t^\nu}{t^\nu+1} M_\nu(1/t^\nu)e^{-st}dt = e^{-s^\nu}, \quad \int_0^{\infty} t^\delta M_\nu(t)dt = \frac{\Gamma(\delta + 1)}{\Gamma(\nu\delta + 1)}, \quad \delta > -1, \quad (12)$$

$$\int_0^{\infty} \nu t M_\nu(t)e^{-st}dt = E_{\nu, \nu}(-s), \quad 0 < \nu < 1. \quad (13)$$

In the following we will show that the operator family $\{E_{\alpha, \alpha}(-At^\alpha)\}_{t \geq 0}$ on the space $X$ is well-defined. To prove this, we need the definition of sectorial operator and a key lemma (Theorem 1.3.4 in [19]).

**Definition 2.1.** A linear operator $A$ in a Banach space $X$ is called a sectorial operator if it is a closed densely defined operator such that, for some $\phi$ in $(0, \frac{\pi}{2})$ and some $M \geq 1$ and a real number $a$, the sector

$$S_{a, \phi} = \{\lambda \phi \leq \arg(\lambda - a) \leq \pi, \quad \lambda \neq a\} \quad (14)$$
is in the resolvent set of $A$ and for all $\lambda \in S_{\alpha,\phi}$, it has
\[
\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - a|}, \quad (15)
\]

**Lemma 2.2.** If $A$ is a sectorial operator on $X$, then $-A$ is the infinitesimal generator of an analytic semigroup \(\{e^{-At}\}_{t \geq 0}\), where
\[
e^{-At} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda t}(\lambda I + A)^{-1} d\lambda,
\]
where $\Gamma$ is a contour in $\rho(-A)$ with $\arg \lambda \to \pm \theta$ as $|\lambda| \to \infty$ for some $\theta$ in $(\pi/2, \pi)$. Further, $e^{-At}$ can be continued analytically into a sector \(\{t \neq 0 : |\arg t| < \epsilon\}\) containing the positive real axis, and if $\Re(\lambda) > \omega > 0$ whenever $\lambda \in \sigma(A)$, then for $t > 0$, it has
\[
\|e^{-At}\| \leq C e^{-\omega t}, \quad (16)
\]
for some constant $C$.

Following the ideas of Theorem 3.1 in [40] and combining Lemma 2.2, we have
\[
\int_0^\infty M_\alpha(s)e^{-\lambda t s} ds = \frac{1}{2\pi i} \int_0^\infty M_\alpha(s) \int_{\Gamma} e^{-\lambda s t s}(\lambda I + A)^{-1} d\lambda ds
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} E_\alpha(-\lambda t s)(\lambda I + A)^{-1} d\lambda =: E_\alpha(-At s), \quad (17)
\]
which implies that the operator family \(\{E_\alpha(-At s)\}_{t \geq 0}\) on the space $X$ is well-defined.

A similar argument shows that
\[
\int_0^\infty \alpha s M_\alpha(s)e^{-\lambda t s} ds = \frac{1}{2\pi i} \int_0^\infty \alpha s M_\alpha(s) \int_{\Gamma} e^{-\lambda s t s}(\lambda I + A)^{-1} d\lambda ds
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} E_{\alpha,\alpha}(-\lambda t s)(\lambda I + A)^{-1} d\lambda =: E_{\alpha,\alpha}(-At s),
\]
which implies that the operator family \(\{E_{\alpha,\alpha}(-At s)\}_{t \geq 0}\) on the space $X$ is also well-defined.

Moreover, by (13) and (16), we can get
\[
\|E_{\alpha,\alpha}(-At s)\| \leq C \int_0^\infty \nu s M_\nu(s)e^{-\omega s t s} ds = CE_{\alpha,\alpha}(-\omega t s), \quad t > 0. \quad (18)
\]

Note that in the case where $A$ is densely defined, the adjoint operator $A^*$ is well defined. And according to Theorem 2(P.225, [41]), it has $R(\lambda; A)^* = R(\lambda; A^*)$. It follows that the adjoint operators $(e^{-At})^*$ and $(E_{\alpha,\alpha}(-At s))^*$ of $e^{-At}$ and $E_{\alpha,\alpha}(-At s)$ are well-defined, respectively, and simply denote them by $e^{-A^* t}$ and $E_{\alpha,\alpha}(-A^* t)$, respectively.

Next, let us recall the well-known Young’s inequality for convolution. (For details, one can refer to Theorem 3.9.4 in [7])

**Lemma 2.3.** Let $p, q, r \geq 1$ satisfy $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then, for any $f \in L^p(0, T)$ and $g \in L^r(0, T)$, it has
\[
\| f \ast g \|_p \leq \| f \|_q \| g \|_r,
\]
where $\ast$ denotes the convolution operator.

Using Lemma 2.3, we can get the following corollary.
Corollary 1. Let \(0 < \alpha < 1\), \(\omega > 0\), \(p, q \geq 1\), \(1 \leq r < \frac{1}{\alpha}\) and they satisfy \(\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}\). Then, for any \(a < b\), \(0 < \delta \leq b - a\), and \(\varphi \in L^q(a, b)\), it has the following estimate

\[
\left( \int_a^{a+\delta} \left| \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \varphi(s) ds \right|^p dt \right)^{\frac{1}{p}} \leq \frac{1}{\Gamma(\alpha)} \left( \frac{\delta^{1+r(\alpha-1)}}{1 + r(\alpha - 1)} \right)^{\frac{1}{r}} \|\varphi(\cdot)\|_{L^r(a, b)}.
\]

Proof. For convenience, denote \(\theta_\delta(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\omega t^\alpha) \mathbf{1}_{[0, \delta]}(t)\) for all \(t \in \mathbb{R}\). Let \(\varphi \in L^q(a, b)\). Then, according to the definitions of convolution and \(\theta_\delta(t)\), we have

\[
(\varphi(\cdot) \mathbf{1}_{[a, b]}(\cdot) * \theta_\delta(\cdot))(t) = \int_a^b (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \mathbf{1}_{[0, \delta]}(t-s) \varphi(s) ds
\]

\[
= \left\{ \begin{array}{ll}
\int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \varphi(s) ds, & \text{a.e. } t \in [a, b+\delta], \\
0, & \text{if } t \notin [a, b + \delta].
\end{array} \right.
\]

Therefore, for almost everywhere \(t \in [a, a+\delta]\), it holds

\[
(\varphi(\cdot) \mathbf{1}_{[a, b]}(\cdot) * \theta_\delta(\cdot))(t) = \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \varphi(s) ds.
\]

Furthermore, by Lemma 2.3, we can get

\[
\left( \int_a^{a+\delta} \left| \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \varphi(s) ds \right|^p dt \right)^{\frac{1}{p}} = \|\varphi(\cdot) \mathbf{1}_{[a, b]}(\cdot) * \theta_\delta(\cdot)\|_{L^p(a, a+\delta)}
\]

\[
\leq \|\varphi(\cdot) \mathbf{1}_{[a, b]}(\cdot) * \theta_\delta(\cdot)\|_{L^p(\mathbb{R})}
\]

\[
\leq \|\varphi(\cdot)\|_{L^r(a, b)} \|\theta_\delta(\cdot)\|_{L^r(0, \delta)}.
\]

Note that \(E_{\alpha,\alpha}(-\omega t^\alpha)\) is a decreasing continuous function on \(t \in [0, \delta]\), and \(t^{\alpha-1}\) is integral and has constant sign, then, according the mean value theorem, there exists a \(\xi \in (0, \delta)\) such that

\[
\|\theta_\delta(\cdot)\|_{L^r(0, \delta)} = \left( \int_0^\delta s^{\alpha-1}(E_{\alpha,\alpha}(-\omega s^\alpha)) r ds \right)^{\frac{1}{r}} = E_{\alpha,\alpha}(-\omega \xi^\alpha) \left( \int_0^\delta s^{\alpha-1} ds \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma(\alpha)} \left( \frac{\delta^{1+r(\alpha-1)}}{1 + r(\alpha - 1)} \right)^{\frac{1}{r}}.
\]

Combining inequalities (20) and (21) proves inequality (19). \(\square\)
Lemma 2.4. Let \( \alpha \in (0, 1) \), \( \varphi(\cdot) \in L^p(0, T; \mathbb{R}) \) with \( p \geq 1 \). Define \( \psi : [0, T] \rightarrow \mathbb{R} \)
as
\[
\psi(t) = \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha) \varphi(s) \, ds, \quad \text{a.e. } t \in [0, T].
\] (22)
Then \( \psi(\cdot) \in L^p(0, T; \mathbb{R}) \) and
\[
\|\psi(\cdot)\|_p \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|\varphi(\cdot)\|_p.
\] (23)
In addition, if \( p > \frac{1}{\alpha} \), then \( \psi(t) \) is a continuous function defined on \([0, T]\).

Proof. By Corollary 1, it is obvious that inequality (23) holds.

Take any \( t_0 \in (0, T) \) and let \( \delta > 0 \) small so that \([t_0 - \delta, t_0 + \delta] \subseteq [0, T]\). Then for any \( t, t' \) with \( t_0 - \delta < t < t' < t_0 + \delta \), we have
\[
|\psi(t) - \psi(t')| = \left| \int_0^{t-\delta} (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha) \varphi(s) \, ds + \int_{t-\delta}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha) \varphi(s) \, ds - \int_0^{t'-\delta} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \varphi(s) \, ds - \int_{t'-\delta}^{t'} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \varphi(s) \, ds - \int_0^{t-\delta} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \varphi(s) \, ds + \int_{t-\delta}^{t'-\delta} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \varphi(s) \, ds \right|
\leq I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = \int_0^{t-\delta} (t - s)^{\alpha - 1} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha) \varphi(s) \, ds,
\]
\[
I_2 = \int_0^{t-\delta} (t' - s)^{\alpha - 1} \left(E_{\alpha, \alpha}(-\omega(t - s)^\alpha) - E_{\alpha, \alpha}(-\omega(t' - s)^\alpha)\right) \varphi(s) \, ds,
\]
\[
I_3 = \int_{t-\delta}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha) \varphi(s) \, ds,
\]
\[
I_4 = \int_{t'-\delta}^{t'} (t' - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \varphi(s) \, ds.
\]
For \( I_1 \), we get
\[
I_1 \leq \sup_{s \in [0, t-\delta]} E_{\alpha, \alpha}(-\omega(t' - s)^\alpha) \frac{(t' - t)^{1-\alpha}}{\delta^{2(1-\alpha)}} \|\varphi(\cdot)\|_1.
\]
For \( I_2 \), we use Hölder inequality to obtain
\[
I_2 \leq \sup_{s \in [0, t-\delta]} \left(E_{\alpha, \alpha}(-\omega(t - s)^\alpha) - E_{\alpha, \alpha}(-\omega(t' - s)^\alpha)\right) \frac{(t')^\nu - (t' - t + \delta)^\nu}{\nu} \|\varphi(\cdot)\|_p,
\]
where \( \nu = \frac{p\alpha - 1}{p - 1} \). Similarly, for \( I_3 \) and \( I_4 \), it has
\[
I_3 \leq \frac{\delta \nu}{\nu \| \varphi(\cdot) \|_p}, \quad I_4 \leq \frac{(t' - t + \delta) \nu}{\nu \| \varphi(\cdot) \|_p}.
\]
In the case \( p > \frac{1}{\alpha} \), it has \( \nu > 0 \). Hence, for any \( \varepsilon > 0 \), we take \( \delta > 0 \) sufficiently small and \( \delta_1 > 0 \) even smaller with \( \delta_1 < \delta \) so that \( I_1 + I_3 + I_4 < \frac{\varepsilon}{2} \). Also, Since \( E_{\alpha,\alpha}(\cdot) \) is continuous, we can take \( \delta_2 > 0 \) sufficiently small with \( \delta_2 < \delta \) so that \( I_2 < \frac{\varepsilon}{2} \). Then we take \( \delta = \min\{\delta_1, \delta_2\} \) so that \( \| \psi(t) - \psi(t') \| < \varepsilon \). This shows that \( \psi(t) \) is continuous for any \( t \in [0, T] \). The proof is completed.

In the following, we present a generalized Gronwall inequality with a singular kernel.

**Lemma 2.5.** Let \( \alpha \in (0, 1) \), \( p > \frac{1}{\alpha} \), and let \( L(\cdot), a(\cdot), x(\cdot) \) be nonnegative functions with \( L(\cdot) \in L^p(0, T) \) and \( a(\cdot), x(\cdot) \in L^{\frac{p}{p-\alpha}}(0, T) \). And suppose that they satisfy the following inequality
\[
x(t) \leq a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) x(s) ds, \quad a.e. t \in [0, T]. \tag{24}
\]
Then there exists a constant \( K > 0 \) such that for a.e. \( t \in [0, T] \),
\[
x(t) \leq a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) a(s) ds + K \int_0^t (t - s)^{\alpha - 1} L(s) a(s) ds. \tag{25}
\]

**Proof.** Since \( L(\cdot) \in L^p(0, T) \) and \( x(\cdot) \in L^{\frac{p}{p-\alpha}}(0, T) \), we get \( L(\cdot) x(\cdot) \in L^1(0, T) \), which implies that the right-hand side of equation (24) is well-defined. Note that \( L(\cdot) \) is a nonnegative function, then for a.e. \( t \in [0, T] \) we can get
\[
x(t) \leq a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) x(s) ds
\]
\[
\leq a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) a(s) ds
\]
\[
+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) \cdot \int_0^s (s - \tau)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(s - \tau)^{\alpha}) L(\tau) x(\tau) d\tau ds
\]
\[
= a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha}) L(s) a(s) ds
\]
\[
+ \int_0^t L(\tau) \left( \int_0^\tau (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^{\alpha})(s - \tau)^{\alpha - 1} \cdot E_{\alpha,\alpha}(-\omega(s - \tau)^{\alpha}) L(s) ds \right) x(\tau) d\tau.
\]
With the help of Hölder inequality and mean value theorem, we obtain that
\[
\int_{\tau}^{t} (t-s)^{\alpha-1} E_{\alpha,a}(-\omega(t-s)^{\alpha})(s-\tau)^{\alpha-1} E_{\alpha,a}(-\omega(s-\tau)^{\alpha}) L(s) ds \\
\leq \|L(\cdot)\|_{p} \left( \int_{\tau}^{t} (t-s)^{\frac{\alpha-1}{\alpha}} \left( E_{\alpha,a}(-\omega(t-s)^{\alpha}) \right)^{\frac{p}{\alpha}} (s-\tau)^{\frac{p(\alpha-1)}{p-1}} \right)^{\frac{p-1}{p}} \\
\cdot \left( E_{\alpha,a}(-\omega(s-\tau)^{\alpha}) \right)^{\frac{p}{\alpha}} ds \\
\leq \frac{\|L(\cdot)\|_{p}}{(\Gamma(\alpha))^{2}} \left( \int_{\tau}^{t} (t-s)^{\frac{\alpha-1}{\alpha}} (s-\tau)^{\frac{\alpha-1}{\alpha}} ds \right)^{\frac{p-1}{p}} \\
= \frac{\|L(\cdot)\|_{p}}{(\Gamma(\alpha))^{2}} (t-\tau)^{2(\alpha-1)+\frac{p+1}{p}} B\left( \frac{p\alpha-1}{p-1}, \frac{p\alpha-1}{p-1} \right)^{\frac{p-1}{p}},
\]
where $B(\cdot, \cdot)$ is the beta function. Denote
\[
c_{1} = \frac{\|L(\cdot)\|_{p}}{(\Gamma(\alpha))^{2}} B\left( \frac{p\alpha-1}{p-1}, \frac{p\alpha-1}{p-1} \right)^{\frac{p-1}{p}},
\]
\[
\alpha_{1} = 2\alpha - \frac{1}{p}.
\]
This implies that
\[
x(t) \leq a(t) + \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,a}(-\omega(t-s)^{\alpha}) L(s) a(s) ds \\
+ c_{1} \int_{0}^{t} (t-s)^{\alpha_{1}-1} L(s) x(s) ds. \tag{26}
\]
Next, we will use the mathematical induction on $k$ to prove that
\[
x(t) \leq a(t) + \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,a}(-\omega(t-s)^{\alpha}) L(s) a(s) ds \\
+ \sum_{i=1}^{k-1} c_{i} \int_{0}^{t} (t-s)^{\alpha_{i}-1} L(s) a(s) ds + c_{k} \int_{0}^{t} (t-s)^{\alpha_{k}-1} L(s) x(s) ds, \tag{27}
\]
where $k = 2, 3, \ldots$, and
\[
\alpha_{i} = \alpha_{i-1} + \frac{p\alpha-1}{p} = \alpha + i \left( \alpha - \frac{1}{p} \right), \quad \alpha_{0} = \alpha, \\
c_{i+1} = c_{i} \frac{\|L(\cdot)\|_{p}}{(\Gamma(\alpha))^{2}} B\left( \frac{p\alpha-1}{p-1}, \frac{p\alpha_{i}-1}{p-1} \right)^{\frac{p-1}{p}}, \quad i = 1, 2, \ldots, k.
\]
Obviously, inequality (27) is true for $k = 1$. Now let us suppose that inequality (27) is true for any fixed $k \in \mathbb{N}^{+}$. In the following we will show that inequality (27) is also true for $k + 1$. Using the induction hypothesis and changing the order of
integrals, we can obtain that

\[
x(t) \leq a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)L(s) \left[ a(s) + \int_0^s (s-\tau)^{-1} E_{\alpha,\alpha}(-\omega(s-\tau)^\alpha)L(\tau)a(\tau)d\tau \right. \\
+ \sum_{i=1}^{k-1} c_i \int_0^s (s-\tau)^{\alpha_i-1} L(\tau)a(\tau)d\tau + c_k \int_0^s (s-\tau)^{\alpha_k-1} L(\tau)x(\tau)d\tau \left. \right] ds \\
= a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)L(s) \\
+ \sum_{i=1}^{k-1} c_i \int_0^s (s-\tau)^{\alpha_i-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)L(s) \left[ \int_0^s (s-\tau)^{\alpha_i-1} L(\tau)a(\tau)d\tau \right] ds \\
+ c_k \int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)L(s) \left[ \int_0^s (s-\tau)^{\alpha_k-1} L(\tau)x(\tau)d\tau \right] ds \\
\leq a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)L(s) a(s) ds \\
+ c_1 \int_0^t (t-s)^{\alpha_1-1} L(s)a(s) ds \\
+ \sum_{i=1}^{k-1} c_i \int_0^t L(\tau) \left[ \int_0^t (t-s)^{\alpha_i-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)(s-\tau)^{\alpha_i-1} L(s) ds \right] a(\tau)d\tau \\
+ c_k \int_0^t L(\tau) \left[ \int_0^t (t-s)^{\alpha_i-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)(s-\tau)^{\alpha_k-1} L(s) ds \right] x(\tau)d\tau.
\]

Furthermore, we can use Hölder inequality to get the following estimate

\[
\int_\tau^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)(s-\tau)^{\alpha_i-1} L(s) ds \\
\leq \|L(\cdot)\|_p \left[ \int_\tau^t (t-s)^{\frac{p(\alpha_i-1)}{p+1}} (s-\tau)^{\frac{p(\alpha_i-1)}{p+1}} ds \right]^{\frac{p-1}{p}} \\
= \frac{\|L(\cdot)\|_p}{\Gamma(\alpha)} (t-\tau)^{\frac{p\alpha_i-\alpha_i-1}{p}} B \left( \frac{p\alpha_i-1}{p-1}, \frac{p\alpha_i-1}{p-1} \right) \frac{p-1}{p} = c_{i+1} (t-\tau)^{\alpha_{i+1}-1},
\]

where

\[
\alpha_{i+1} = \alpha_i + \frac{p\alpha_i - 1}{p}, \quad (28) \\
c_{i+1} = c_i \frac{\|L(\cdot)\|_p}{\Gamma(\alpha)} B \left( \frac{p\alpha_i-1}{p-1}, \frac{p\alpha_i-1}{p-1} \right) \frac{p-1}{p}, \quad i = 1, 2, \ldots, k. \quad (29)
\]
Therefore, we have the following inequality
\[
x(t) \leq a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s))L(s)a(s)ds \\
+ c_1 \int_0^t (t-s)^{\alpha_1-1} L(s)a(s)ds \\
+ \sum_{i=1}^{k-1} c_i \int_0^t (t-\tau)^{\alpha_i-1} L(\tau)a(\tau)d\tau + c_{k+1} \int_0^t (t-\tau)^{\alpha_{k+1}-1} L(\tau)x(\tau)d\tau
\]
\[
= a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s))L(s)a(s)ds \\
+ \sum_{j=1}^k c_j \int_0^t (t-\tau)^{\alpha_j-1} L(\tau)a(\tau)d\tau + c_{k+1} \int_0^t (t-\tau)^{\alpha_{k+1}-1} L(\tau)x(\tau)d\tau,
\]
which shows that inequality (27) is valid for \( k+1 \). Therefore, inequality (27) holds for any \( k \in \mathbb{N}^+ \).

Finally, using the fact: there exists a positive constant \( K \) such that
\[
(t-s)^{\alpha_k-1} \leq K(t-s)^{-\alpha}, \quad 1 \leq i \leq k, \quad 0 \leq s < t \leq T,
\]
we have
\[
x(t) \leq a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s))L(s)a(s)ds \\
+ K \sum_{i=1}^{k-1} c_i \int_0^t (t-\tau)^{\alpha_i-1} L(\tau)a(\tau)d\tau \\
+ c_k \int_0^t (t-\tau)^{\alpha_k-1} L(\tau)x(\tau)d\tau.
\]
On the other hand, we can prove that the series \( \sum_{i=1}^{k-1} c_i \) converges as \( k \to \infty \). Since \( p > \frac{1}{\alpha} \), and \( 0 < \alpha < 1 \), it has \( \alpha_i \to \infty \) as \( i \to \infty \). On the other hand, by the relationships (28) and (29), we have
\[
\lim_{i \to \infty} \frac{c_{i+1}}{c_i} = \lim_{i \to \infty} \frac{\|L()\|_p B(\frac{p\alpha-1}{p-1}, \frac{p\alpha_i-1}{p-1}) \frac{p-1}{n}}{\Gamma(\alpha)} \\
= \lim_{i \to \infty} \frac{\|L()\|_p}{\Gamma(\alpha)} \left( \frac{\Gamma(\frac{p\alpha-1}{p-1}) \Gamma(\frac{p\alpha_i-1}{p-1})}{\Gamma(\frac{p\alpha_i+1}{p-1})} \right)^{\frac{p-1}{p}} \\
\leq M \frac{\|L()\|_p}{\Gamma(\alpha)} \left( \frac{p\alpha-1}{p-1} \right)^{\frac{p-1}{p}} \lim_{i \to \infty} \left( \frac{p\alpha_i-1}{p-1} \right)^{-\frac{p-1}{p}} = 0.
\]
Thus, the series \( \sum_{i=1}^{k-1} c_i \) is convergent and \( \lim_{k \to \infty} c_k = 0 \). This implies that inequality (25) holds for a.e. \( t \in [0, T] \). The proof is completed.

3. Well-posedness in \( L^p \) for the state equation and continuity of the state trajectory. In this section, we discuss the well-posedness for the state equation (1) in the case \( X = L^p(0, T; \mathbb{R}) \). Let \( U \) be a separable metric space with the metric
d. With the Borel σ-field, $U$ is regard as a measurable space. Let $u_0 \in U$ be given. For any $p \geq 1$, we introduce

$$U^p[0, T] = \{ u : [0, T] \to U | u(\cdot) \text{ is measurable}, \, d(u(\cdot), u_0) \in L^p(0, T; \mathbb{R}) \}.$$  

To establish the well-posedness for the state equation (1) in $L^p(0, T; \mathbb{R})$ and continuity of the state trajectory, we need the following assumption for the nonlinear term $f$ of the state equation.

(H1) let $f : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ be a map with $t \mapsto f(t, y, u)$ being strongly measurable, $y \mapsto f(t, y, u)$ being continuously Fréchet differential, and $(y, u) \mapsto f(t, y, u)$ being continuous. Moreover, there exist nonnegative functions $L_0(\cdot)$ and $L(\cdot)$ with

$$L_0(\cdot) \in L^1((\frac{1}{p-1})^{\frac{1}{p}}(0, T; \mathbb{R}), \, L(\cdot) \in L^1((\frac{1}{p-1} + 1)(0, T; \mathbb{R})$$

such that for some $p \geq 1$, $\alpha \in (0, 1)$ and $u_0 \in U$, it has

$$|f(t, 0, u_0)| \leq L_0(t), \quad t \in [0, T],$$

and

$$|f(t, y, u) - f(t, y', u')| \leq L(t)(|y - y'| + d(u, u')), \quad t \in [0, T], \quad y, y' \in \mathbb{R}, \quad u, u' \in U.$$  

Now we give a theorem about the well-posedness of the state equation in $L^p(0, T)$.

**Theorem 3.1.** Let (H1) hold with some $p \geq 1$ and $\alpha \in (0, 1)$. Then for any $\eta(\cdot) \in L^p(0, T)$ and $u(\cdot) \in \mathcal{U}^p(0, T)$, equation (1) has a unique solution $x(\cdot) \equiv x(\cdot; \eta(\cdot), u(\cdot)) \in L^p(0, T)$.

If $(\eta_1(\cdot), u_1(\cdot))$, $(\eta_2(\cdot), u_2(\cdot)) \in L^p(0, T) \times \mathcal{U}^p(0, T)$ and $x_1(\cdot)$, $x_2(\cdot)$ are the corresponding solutions, then there exists a constant $K > 0$ such that for a.e. $t \in [0, T]$,

$$|x_1(t) - x_2(t)| \leq a(t) + \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha)L(s)a(s)ds$$

$$+ K \int_0^t (t - s)^{\alpha - 1} L(s)a(s)ds,$$

where

$$a(t) = |\eta_1(t) - \eta_2(t)| + \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha)L(s)d(u_1(s), u_2(s))ds.$$

**Proof.** We firstly discuss the case $\alpha \in (0, 1)$. We will use Picard iteration method to prove the existence and uniqueness of the solution of equation (1) in the space $L^p(0, T)$.

**Uniqueness.** Let $x(\cdot)$ and $y(\cdot)$ be two solutions of equation (1), respectively. Then we have

$$|x(t) - y(t)| \leq \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-\omega(t - s)^\alpha)L(s)|x(s) - y(s)|ds, \quad \text{a.e. } t \in [0, T].$$

Hence, applying Lemma 2.5 to inequality (34) to obtain that

$$|x(t) - y(t)| = 0, \quad \text{a.e. } t \in [0, T].$$

(35)

It follows that $|x(\cdot) - y(\cdot)|_p \equiv 0$. The uniqueness of the solution has been proved.

We consider three cases to prove the existence of the solution in $L^p(0, T)$.

**Case 1.** $p > \frac{1}{1-\alpha}$. In this case, we divide the proof into three steps.
Step 1. Construct a Picard iteration sequence \( \{x_k(t)\}_{k=0}^{\infty} \subseteq L^p(0, T) \)

\[
x^{k+1}(t) = \eta(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)\alpha) f(s, x^k(s), u(s))ds,
\]

where the initial iterative function \( x^0(t) = \eta(t), t \in [0, T] \).

Now we will use the mathematical induction on \( k \) to prove \( \{x_k(\cdot)\} \subseteq L^p(0, T) \). Obviously, it has \( x^0(t) \in L^p(0, T) \). Let us suppose that for any fixed \( k \in \mathbb{N} \), it has \( x^k(\cdot) \in L^p(0, T) \). Then we verify that \( x^{k+1}(\cdot) \in L^p(0, T) \). Then by Corollary 1, for any \( q \geq 1 \), and \( 0 \leq \varepsilon < \frac{\alpha}{1-\alpha} \) with \( \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{1+\varepsilon} \), it has

\[
\|x^{k+1}(\cdot)\|_p \leq \|\eta(\cdot)\|_p + \frac{1}{\Gamma(\alpha)} \left( \frac{T^{1-(1+\varepsilon)(1-\alpha)}}{1 - (1 + \varepsilon)(1 - \alpha)} \right)^{\frac{1}{1+\varepsilon}} \|L_0(\cdot)\|
\]

\[+ L(\cdot)\|x^k(\cdot)\| + d(u(\cdot), u_0)\|_p.
\]

(37)

In this case, \( p > \frac{1}{1-\alpha} \) implies that \( \frac{1}{\alpha} > \frac{p}{p-1} \) and \( \frac{p}{1+\alpha p} > 1 \). Then it has

\[
L_0(\cdot) \in L^{\frac{p}{1+\alpha p}}(0, T), \quad L(\cdot) \in L^{\frac{1}{1+\alpha p}}(0, T).
\]

(38)

On the other hand, for any \( 0 \leq \varepsilon < \frac{\alpha}{1-\alpha} \), it has

\[
\frac{1}{q} < \frac{1}{p} + \alpha < 1, \quad \frac{1}{q} - \frac{1}{p} < \alpha.
\]

(39)

Then, as \( \varepsilon \to \frac{\alpha}{1-\alpha}^- \), it has

\[
q \to \frac{p}{1+\alpha p} + \frac{pq}{p-q} \to \frac{1}{\alpha}.
\]

(40)

Combining (38) and (40), we could find \( \varepsilon \) close enough to \( \frac{\alpha}{1-\alpha} \) so that \( L_0(\cdot) \in L^q(0, T) \) and \( L(\cdot) \in L^{\frac{p}{p-q}}(0, T) \). Then, by Hölder inequality, we can obtain

\[
\|L_0(\cdot) + L(\cdot)\|\|x^k(\cdot)\| + \|d(u(\cdot), u_0)\|_p \leq \|L_0(\cdot)\|_q + \|L(\cdot)\|^{\frac{p}{p-q}} \|x^k(\cdot)\| + d(u(\cdot), u_0)\|_p.
\]

This inequality and inequality (37) imply that \( x^{k+1}(\cdot) \in L^p(0, T) \). Hence, the sequence \( \{x^k(\cdot)\} \subseteq L^p(0, T) \).

Step 2. In this step, we will show that the sequence \( \{x^k(\cdot)\} \) is convergent uniformly in \( L^p(0, T) \). To do this, let \( \delta \) be sufficiently small such that

\[
Q := \frac{1}{\Gamma(\alpha)} \cdot \left( \frac{\delta^{1-(1+\varepsilon)(1-\alpha)}}{1 - (1 + \varepsilon)(1 - \alpha)} \right)^{\frac{1}{1+\varepsilon}} < 1.
\]

(41)

Note that

\[
\|x^{1}(\cdot) - x^{0}(\cdot)\|_{L^p(0, \delta)} \leq \frac{1}{\Gamma(\alpha)} \cdot \left( \frac{\delta^{1-(1+\varepsilon)(1-\alpha)}}{1 - (1 + \varepsilon)(1 - \alpha)} \right)^{\frac{1}{1+\varepsilon}} \|L_0(\cdot)\|_q
\]

\[+ \|L(\cdot)\|^{\frac{p}{p-q}} \|\eta(\cdot)\| + d(u(\cdot), u_0)\|_p.
\]

Then

\[
\|x^{2}(\cdot) - x^{1}(\cdot)\|_{L^p(0, \delta)} \leq \frac{1}{\Gamma(\alpha)} \cdot \left( \frac{\delta^{1-(1+\varepsilon)(1-\alpha)}}{1 - (1 + \varepsilon)(1 - \alpha)} \right)^{\frac{1}{1+\varepsilon}} \|L(\cdot)\|^{\frac{p}{p-q}} \|x^1(\cdot) - x^0(\cdot)\|_p
\]

\[\leq bC,
\]
where
\[ b = Q(L(\cdot)) \| \frac{\eta(\cdot)}{p} \|_q, \quad C = Q(\| L_0(\cdot) \|_q + \| L(\cdot) \|_{\frac{pq}{p+q}} \| \eta(\cdot) \| + \| \omega(u(\cdot), u_0) \|_p). \] (42)
Using the mathematical induction on \( k \), we can obtain that
\[ \| x^{k+1}(\cdot) - x^k(\cdot) \|_{L^p(0, \delta)} \leq b^k C. \] (43)
Under the condition (41), the series \[ \sum_{k=0}^{\infty} \| x^{k+1}(\cdot) - x^k(\cdot) \|_{L^p(0, \delta)} \] is convergent uniformly in \( L^p(0, \delta) \). It follows that the sequence \( \{x^k(\cdot)\} \) is convergent uniformly in \( L^p(0, \delta) \). Denote the limit by \( x(\cdot) \). Clearly, \( x(\cdot) \in L^p(0, \delta) \).

Finally, it remains to show that \( x(\cdot) \) satisfies equation (1). Note that
\[
\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(A(t-s)^\alpha) \left( f(s, x^{k-1}(s), u(s)) - f(s, x(s), u(s)) \right) ds \right\|
\leq \frac{1}{\Gamma(\alpha)} \left( \frac{\delta^{1-(1+\varepsilon)(1-\alpha)}}{1 - (1+\varepsilon)(1-\alpha)} \right)^{\frac{1}{\alpha p}} \| L(\cdot) \|_{\frac{pq}{p+q}} \| x^{k-1}(\cdot) - x(\cdot) \|_p \to 0,
\]
as \( k \to \infty \) in \( L^p(0, \delta) \).
This implies that \( x(\cdot) \) is the solution to equation (1).

Step 3. By Step 2 again, there is a solution to equation (1) on \( [\delta, 2\delta] \). Repeating this procedure we see that there is a solution to equation (1) on the entire interval \( [0,T] \).

Case 2. \( 1 < p \leq \frac{1}{1-\alpha} \). In this case, it has
\[ 1 - \alpha \leq \frac{1}{p} < 1, \quad \frac{1}{\alpha} \leq \frac{p}{p-1}, \quad \frac{p}{1+\alpha p} \leq 1. \]
Thus, we can take \( \varepsilon \in (0, p-1) \) so that
\[ 1 - \alpha \leq \frac{1}{p} < \frac{1}{1+\varepsilon}. \]
It follows that
\[ \frac{1}{p} < \frac{1}{q} = \frac{1}{p} + 1 - \frac{1}{1+\varepsilon} \to 1^-, \quad \frac{p-q}{pq} = \frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{1+\varepsilon} \to \frac{p-1^-}{p}, \quad \text{as} \ \varepsilon \to p-1^-.
\]
This relationship and condition (30) imply that we could find \( \varepsilon \) close enough to \( p-1 \) such that \( L_0(\cdot) \in L^q(0, T; X) \) and \( L(\cdot) \in L^{\frac{p}{1+\varepsilon}}(0, T; X) \). The rest of the proof is similar to that of Case 1. Here we omit it.

Case 3. \( p = 1 \). In this case, condition (30) becomes \( L_0(\cdot) \in L^{1+}(0, T) \) and \( L(\cdot) \in L^\infty(0, T) \). Then we can pick \( \varepsilon = 0 \), and inequality (37) can be reduced
\[ \| x^{k+1}(\cdot) \|_1 \leq \| \eta(\cdot) \|_1 + \frac{T^\alpha}{\Gamma(1+\alpha)} \left( \| L_0(\cdot) \|_1 + \| L(\cdot) \|_{L^\infty(0,T)} \left( \| x^k(\cdot) \| + \| \omega(u(\cdot), u_0) \|_1 \right) \right). \] (44)

The rest of the proof is similar to that of Case 1. Here we omit it.

Now, let \( (\eta_1(\cdot), u_1(\cdot)), (\eta_2(\cdot), u_2(\cdot)) \in L^p(0, T; X) \times U^p(0, T) \) and \( x_1(\cdot), x_2(\cdot) \) are the corresponding solutions. Then, by (H1), we have
\[
|x_1(t) - x_2(t)| \leq |\eta_1(t) - \eta_2(t)| + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\omega(t-s)^\alpha)L(s)d(u_1(s), u_2(s))ds
+ \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\omega(t-s)^\alpha)L(s)|x_1(s) - x_2(s)|ds.
\]
Then, by Lemma 2.5, we can obtain the estimate (33).
In final, one sees that the conclusion of this theorem is true for the case \( \alpha = 1 \). For this case, \( E_{\alpha,\alpha}(z) = e^z \), \((t - s)\alpha - 1 = 1 \) and the integral is non-singular. The proof is complete.

Finally, we discuss the continuity of the state trajectory. To do this, we need the following assumption.

\((H1)^{'}\) Let \((H1)\) hold with \( p > \frac{1}{\alpha} \) such that

\[
L_0(\cdot) \in L^{\frac{p}{p\alpha - 1}}(0,T), \quad L(\cdot) \in L^{\frac{p}{p\alpha - 1}}(0,T).
\]

Note that in the case \( p > \frac{1}{\alpha} \),

\[
\frac{p}{pa - 1} > \frac{1}{\alpha}, \quad \frac{p}{p\alpha - 1} > \frac{p}{p - 1}, \quad \frac{1}{\alpha} > \frac{p}{1 + \alpha p},
\]

which implies that

\[
L^{\frac{p}{p\alpha - 1}}(0,T) \subset L^{\left(\frac{p}{p\alpha - 1}\right) + 1}(0,T), \quad L^{\frac{p}{p\alpha - 1}}(0,T) \subset L^{\left(\frac{1}{\alpha}\right) + 1}(0,T).
\]

Hence, \((H1)^{'}\) is stronger than \((H1)\). Under the condition \((H1)^{'}\), we have the following result.

**Proposition 1.** Let \((H1)^{'}\) hold. Then \( x(\cdot) - \eta(\cdot) \in C(0,T;\mathbb{R}) \).

**Proof.** Define

\[
\psi(t) = \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^{\alpha}) f(s, x(s), u(s)) ds, \quad t \in [0,T].
\]

By Lemma 2.4, we only need to show that \( \psi \in L^q(0,T) \) for some \( q > \frac{1}{\alpha} \).

Note that

\[
|f(s, x(s), u(s))| \leq L_0(s) + L(s)(|x(s)| + d(u(s), u_0)) =: \tilde{\varphi}(s), \quad t \in [0,T].
\]

Then, by \((H1)^{'}\), we have \( L(\cdot) \in L^r(0,T) \) for some \( r > \frac{p}{p\alpha - 1} \). Let \( q = \frac{rp}{r + p} \). Then it has

\[
\frac{1}{\alpha} < q < p, \quad \frac{pq}{p - q} = r.
\]

Therefore, using Hölder inequality, we get

\[
\|	ilde{\varphi}(\cdot)\|_q \leq \|L_0(\cdot)\|_q + \left( \int_0^T L(s)^{\frac{pq}{p\alpha - q}} ds \right)^{\frac{p\alpha - q}{p}} \left( \int_0^T (|x(s)| + d(u(s), u_0))^p ds \right)^{\frac{q}{p}} < \infty.
\]

This shows that \( \tilde{\varphi}(\cdot) \in L^q(0,T) \) for some \( q > \frac{1}{\alpha} \). It follows from Lemma 2.4 that our conclusion is true. The proof is complete.

4. **Special case.** In this section, we consider some special cases.

**Case 1.** \( \alpha = 1 \). In this case, the state equation (1) becomes the abstract evolution equations, which has been discussed in [27] and [29].

**Case 2.** \( A = 0 \). In this case, the state equation (1) becomes the controlled singular Volterra integral equations, which has been discussed in [28].

**Case 3.** Fractional evolution equations. Let us recall some definitions about factional calculus.
**Definition 4.1.** Let $\alpha > 0$, $f \in L^1(0,T)$. Then the left-side Riemann-Liouville (R-L) fractional integral and the right-side R-L fractional integral of order $\alpha$ with respect to $t$ are defined as

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \geq 0, \quad (45)$$

$$I_{T-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds, \quad t \leq T, \quad (46)$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

**Property 1.** Let $\alpha > 0$, $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{\alpha}$ (If the equality holds, then $p \neq 1$ and $q \neq 1$.) If $f \in L^p(0,T)$ and $g \in L^q(0,T)$, then

$$\int_0^T f(t)(I_{0+}^\alpha g)(t) dt = \int_0^T g(t)(I_{T-}^\alpha f)(t) dt.$$

For the fractional derivatives, there are two types that are commonly used: the left-side (right-side) R-L derivative and the left-side (right-side) Caputo derivative.

**Definition 4.2.** Let $m-1 < \alpha \leq m$, $m \in \mathbb{N}^+$. The left-side R-L derivative and left-side Caputo derivative of order $\alpha$ with respect to $t$ are defined, respectively, as

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} f(s) ds,$$

$$(^{C}D_{0+}^\alpha f)(t) = D_{0+}^\alpha \left( f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k \right), \quad t \geq 0. \quad (47)$$

Further, if $f(t) \in C^m([0,T])$, then Caputo derivative can also be defined as

$$^{C}D_{0+}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}^+, \quad (48)$$

which is known as a smooth fractional derivative.

Note that if $f^{(k)}(0) = 0$, $k = 0, 1, \ldots, m-1$, then $(^{C}D_{0+}^\alpha f)(t)$ coincides with $(D_{0+}^\alpha f)(t)$.

Similarly, the right-side R-L derivative operator $D_{T-}^\alpha$ and the right-side Caputo derivative operator $^{C}D_{T-}^\alpha$ have been defined. (For details, refer to the books [25, 37])

**Property 2.** Let $0 < \alpha \leq 1$, and $f \in L^1([0,T])$. Then the following equality

$$(I_{0+}^\alpha D_{0+}^\alpha f)(t) = f(t) - \frac{(I_{0+}^{1-\alpha} f)(0+)}{\Gamma(\alpha)} t^{\alpha-1},$$

holds almost everywhere on $[0,T]$.

**Property 3.** Let $0 < \alpha \leq 1$, and $f \in C([0,T])$. Then the following equality

$$(I_{0+}^\alpha (^{C}D_{0+}^\alpha f))(t) = f(t) - f(0), \quad (49)$$

holds on $[0,T]$.

**Property 4.** Let $\alpha > 0$, $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{\alpha}$ (If the equality holds, then $p \neq 1$ and $q \neq 1$.) If $f \in L^p(0,T)$ and $g \in L^q(0,T)$, then

$$\int_0^T f(t)(D_{0+}^\alpha g)(t) dt = \int_0^T g(t)(D_{T-}^\alpha f)(t) dt.$$
Let $\alpha \in (0, 1]$, $A \in \mathbb{R}^{n \times n}$. The solution to the initial value problem with the R-L derivative
\[(D_0^\alpha x)(t) +Ax(t) = f(t), \quad (I_{0^+}^{1-\alpha} x)(0+) = \tilde{x}_0 \in \mathbb{R}^n,\]
has the form
\[x(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha)\tilde{x}_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)f(s)ds. \quad (49)\]

Let $\alpha \in (0, 1]$, $A \in \mathbb{R}^{n \times n}$. The solution to the initial value problem with the Caputo derivative
\[\left(\frac{d^\alpha}{dt^\alpha}x\right)(t) +Ax(t) = f(t), \quad x(0) = \tilde{x}_0 \in \mathbb{R}^n,\]
has the form
\[x(t) = E_{\alpha}(-At^\alpha)\tilde{x}_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)f(s)ds. \quad (50)\]

Let $\alpha \in (0, 1]$. Now consider the following abstract Cauchy problem with the Caputo derivative
\[\left(\frac{d^\alpha}{dt^\alpha}x\right)(t) +Ax(t) = f(t, x(t), u(t)), \quad t \in [0, T], \quad (51)\]
with the initial condition $x(0) = \pi_0$, where $A$ is a sectorial operator on the space $X$.

Using Property 3 and the similar arguments to Lemma 4.1 in [40], the mild solution of the Cauchy problem (51) satisfies the following integral equation
\[x(t) = E_{\alpha,\alpha}(-At^\alpha)\pi_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)f(s, x(s), u(s))ds, \quad (52)\]
where $E_{\alpha,\alpha}(-At^\alpha)$ is defined as (2), and
\[E_{\alpha}(-At^\alpha) = E_{\alpha,1}(-At^\alpha), \quad t \in [0, T]. \quad (53)\]

Likewise, consider the following abstract Cauchy problem with the R-L derivative
\[(D_0^\alpha x)(t) +Ax(t) = f(t, x(t), u(t)), \quad t \in [0, T], \quad (54)\]
with the initial condition $(I_{0^+}^{1-\alpha} x)(0+) = \pi_0$, where $A$ is a sectorial operator on the space $X$.

Using Property 2 and the similar arguments to Lemma 4.1 in [40], the mild solution of the Cauchy problem (54) satisfies the following integral equation
\[x(t) = t^{\alpha-1}E_{\alpha}(-At^\alpha)\pi_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)f(s, x(s), u(s))ds, \quad (55)\]
where $E_{\alpha,\alpha}(-At^\alpha)$ is defined as (2).

From the above analysis, one sees that integral equations (49), (50), (52) and (55) are special cases of equation (1).

5. Pontryagin’s maximum principle. In this section, our goal is to give the necessary conditions for the optimal control problem for (1) with the cost functional. The necessary conditions are usually referred to as Pontryagin’s maximum principle.

To do this, we introduce the following hypothesis.

(H2) Let (H1) hold with $(y, u) \mapsto f_y(t, y, u)$ being continuous.
(H3) Let $g : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}$ be a map with $t \mapsto g(t, y, u)$ being strongly measurable, $y \mapsto g(t, y, u)$ being continuously Fréchet differentiable, and $(y, u) \mapsto$
Lemma 5.1. be a Banach space. For any $\rho > 0$, let
\[ E_\rho = \{ E \subseteq [0,T] \mid |E| = \rho T \}, \]
where $|E|$ denotes the Lebesgue measure of $E$. Then, for any $h(\cdot) \in C([0, T; L^1(0, T; \mathbb{R})])$, it has
\[ \inf_{E_\rho} \left\| \int_0^T \left( \frac{1}{\rho} - 1 \right) h(\cdot, s) ds \right\|_{C(0, T; \mathbb{R})} = 0. \]  

We can follow the ideas in the proof of Lemma 4.2 in [29] to prove the following lemma.

Lemma 5.2. For any $\rho > 0$, let
\[ E_\rho = \{ E \subseteq [0,T] \mid |E| = \rho T \}, \]
where $|E|$ denotes the Lebesgue measure of $E$, and let $\varphi(\cdot) \in L^p(0, T)$ with $p > \frac{1}{\alpha}$. Then it has
\[ \inf_{E_\rho} \sup_{t \in [0,T]} \left\| \int_0^t \left( \frac{1}{\rho} - 1 \right) (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \varphi(s) ds \right\| = 0. \]

Theorem 5.3. Let (H2)$''$ and (H3) hold, and let $(x^\dagger(\cdot), u^\dagger(\cdot))$ be an optimal pair of Problem (P) detailed in (57). Then there exists a solution $\phi(\cdot) \in L^{\frac{p}{\alpha}}(0, T; \mathbb{R})$ of the following adjoint equation
\[ \phi(t) = \int_t^T (s-t)^{\alpha-1} E_{\alpha,\alpha}(-A^*(s-t)^\alpha) f_x(s, x(t), u(t))^\ast \phi(s) ds \\
- \int_t^T (s-t)^{\alpha-1} E_{\alpha,\alpha}(-A^*(s-t)^\alpha) g_x(s, x(t), u(t))^\ast \phi(s) ds \]
such that the following maximum condition holds
\[ H(t, x^\dagger(t), u^\dagger(t), \phi(t)) = \max_{u \in U} H(t, x^\dagger(t), u, \phi(t)), \]
where
\[ H(t, x^\dagger(t), u(t), \phi(t)) = \langle \phi, f(t, x^\dagger(t), u(t)) \rangle - g(t, x^\dagger(t), u(t)), \]
\[ \forall (t, x, u, \phi) \in [0, T] \times \mathbb{R} \times U \times \mathbb{R}. \]
Proof. In order to prove this theorem, we divide it into two steps.

Step 1. A variational inequality. We fix \( v(\cdot) \in \mathcal{U}[0,T] \), and define

\[
 u_\rho(t) = \begin{cases} 
 u^\dagger(t), & t \in [0,T] \setminus E_\rho, \\
 v(t), & t \in E_\rho.
\end{cases}
\] (63)

By the definition of \( u_\rho(\cdot) \), one can easily see that \( u_\rho(\cdot) \in \mathcal{U}[0,T] \), and

\[
 d(u_\rho(\cdot), u^\dagger(\cdot)) \leq |E_\rho| = \rho T.
\]

Let \( x_\rho(\cdot) = x(\cdot; \eta(\cdot), u_\rho(\cdot)) \) be the solution of the following state equation

\[
 x_\rho(t) = \eta(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)^\alpha) f(s, x_\rho(s), u_\rho(s)) ds.
\] (64)

Then, by the definitions of \( x_\rho(\cdot) \) and \( x^\dagger(\cdot) \), we have

\[
 |x_\rho(t) - x^\dagger(t)| \leq \rho T a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) L(s)|x_\rho(s) - x^\dagger(s)| ds,
\] (65)

where

\[
 a(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) L(s) ds.
\]

Applying Lemma 2.5 to (65), we get

\[
 |x_\rho(t) - x^\dagger(t)| \leq \rho T \left( a(t) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) L(s) a(s) ds \right.
\]

\[
 + K \int_0^t (t-s)^{\alpha-1} L(s) a(s) ds
\]

In the case \( p > \frac{1}{\alpha} \), it has \( L(\cdot) \in L^r[0,T] \) for some \( r > \frac{p}{\rho \alpha - 1} > \frac{1}{\alpha} \). Then, by Lemma 2.4, \( a(t) \) is continuous. Hence, there exists a constant \( C > 0 \) such that

\[
 |x_\rho(t) - x^\dagger(t)| \leq C \rho.
\] (66)

On the other hand, it has

\[
 z_\rho(t) := \frac{x_\rho(t) - x^\dagger(t)}{\rho}
\]

\[
 = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)^\alpha) \frac{f(s, x_\rho(s), u_\rho(s)) - f(s, x^\dagger(s), u^\dagger(s))}{\rho} ds
\]

\[
 = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)^\alpha) \frac{f(s, x_\rho(s), u_\rho(s)) - f(s, x^\dagger(s), u^\dagger(s))}{\rho} ds
\]

\[
 + \int_{[0,t]\cap E_\rho} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)^\alpha) \frac{f(s, x^\dagger(s), v(s)) - f(s, x^\dagger(s), u^\dagger(s))}{\rho} ds.
\]
It follows from Lemma 5.2 that

\[ z_\rho(t) = \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha) \left( \int_0^1 f_s(s, (1 - \sigma)x^k(s) + \sigma z_\rho(s), u_\rho(s))d\sigma \right) z_\rho(s)ds \]
\[
+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha) \left( \int_0^1 f_s(s, x^k(s), v(s)) - f(s, x^k(s), u^k(s)) \right) 1_{E_\rho}(s)ds \]
\[
= \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha) \left( \int_0^1 f_s(s, (1 - \sigma)x^k(s) + \sigma z_\rho(s), u_\rho(s))d\sigma \right) z_\rho(s)ds \]
\[
+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha) \left( f(s, x^k(s), v(s)) - f(s, x^k(s), u^k(s)) \right) ds + o(1). \quad (67)\]

Denote

\[ \int_0^1 f_s(s, (1 - \sigma)x^k(s) + \sigma z_\rho(s), u_\rho(s))d\sigma = \hat{f}(s). \]

By (H1)', we get

\[ |\hat{f}(s)| \leq L(s), \quad s \in [0, T], \quad u \in U. \quad (68) \]

Based on the facts (66) and (68), the following estimate can be obtained

\[ |z_\rho(t) - z(t)| \leq \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^\alpha)L(s)|z_\rho(s) - z(s)|ds \]
\[
+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^\alpha)|\hat{f}(s) - f_s(s, x^k(s), u^k(s))|z(s)ds \]
\[
+ o(1), \quad (69) \]

where

\[ z(t) = \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha)f_s(s, x^k(s), u^k(s))z(s)ds \]
\[
+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-A(t - s)^\alpha)\left( f(s, x^k(s), v(s)) - f(s, x^k(s), u^k(s)) \right) ds, \quad (70) \]

which is called variational equation. Note that

\[ |z(t)| \leq \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^\alpha)L(s)|z(s)|ds + \rho Ta(t). \]

Using a similar argument to (66), we obtain that there exists a constant \( C^* \) such that \( |z(t)| \leq C^* \rho \). On the other hand, since

\[ (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^\alpha)|\hat{f}(s) - f_s(s, x^k(s), u^k(s))|z(s)| \leq \frac{2L(s)(t - s)^{\alpha - 1}}{\Gamma(\alpha)}|z(s)|, \quad a.e., \quad s \in [0, t), \]

with

\[ \int_0^t \frac{2L(s)(t - s)^{\alpha - 1}}{\Gamma(\alpha)}|z(s)| < \infty, \]

by the dominated convergence theorem, we have

\[ \lim_{\rho \to 0} \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\omega(t - s)^\alpha)|\hat{f}(s) - f_s(s, x^k(s), u^k(s))|z(s)ds = 0, \quad t \in [0, T]. \]
Hence, by (67), (69) and Lemma 2.4, we have
\[
\lim_{\rho \to 0} |z_\rho(t) - z(t)| = 0, \quad t \in [0, T].
\] (71)

Also since \((x^\dagger(\cdot), u^\dagger(\cdot))\) is an optimal pair, we have
\[
0 \leq \frac{J(u_\rho(\cdot)) - J(u^\dagger(\cdot))}{\rho} = \int_0^T \left(g_x^\rho(t)z_\rho(t) + \frac{g(t, x^\dagger(t), v(t)) - g(t, x^\dagger(t), u^\dagger(t))}{\rho} 1_{E_\rho}(t)\right)dt,
\] (72)

where
\[
g_x^\rho(t) = \int_0^1 g_x(s, (1 - \sigma)x^\dagger(s) + \sigma x_\rho(s), u_\rho(s))d\sigma, \quad t \in [0, T].
\]

Thus, according to Lemma 5.1 and inequality (72), we have
\[
0 \leq \int_0^T \left(g_x(t, x^\dagger(t), u^\dagger(t))z(t) + g(t, x^\dagger(t), v(t)) - g(t, x^\dagger(t), u^\dagger(t))\right)dt, \quad t \in [0, T],
\]
where \(z(\cdot)\) is the solution of the variational equation.

Step 2. In this step, we will prove the duality.

Note that \(z(\cdot)\) in (70) also satisfies the following differential equation
\[
(D^*_0; z)(t) + Az(t) = f_x(t, x^\dagger(t), u^\dagger(t))z(t) + (f(t, x^\dagger(t), v(t)) - f(t, x^\dagger(t), u^\dagger(t))).
\] (73)

On the other hand, by Property 4, we know that the adjoint equation can be written as
\[
(D^*_{x^\dagger}; \phi)(t) + A^*\phi(t) = f_x(t, x^\dagger(t), u^\dagger(t))^*\phi(t) - g_x(t, x^\dagger(t), u^\dagger(t)).
\] (74)

which is consistent with equation (62). Then, by equations (73) and (74), we have
\[
0 \leq \int_0^T g_x(t, x^\dagger(t), u^\dagger(t))z(t)dt + \int_0^T \left(g(t, x^\dagger(t), v(t)) - g(t, x^\dagger(t), u^\dagger(t))\right)dt
\]
\[
= \int_0^T \left(g_x(t, x^\dagger(t), u^\dagger(t)), z(t)\right)dt + \int_0^T \left(g(t, x^\dagger(t), v(t)) - g(t, x^\dagger(t), u^\dagger(t))\right)dt
\]
\[
= \int_0^T \langle \phi, f(s, x^\dagger(s), u^\dagger(s)) - f(s, x^\dagger(s), v(s))\rangle ds
\]
\[
+ \int_0^T \left(g(t, x^\dagger(t), v(t)) - g(t, x^\dagger(t), u^\dagger(t))\right)dt
\]
\[
= \int_0^T \left(H(t, x^\dagger(t), u^\dagger(t), \phi(t)) - H(t, x^\dagger(t), v(t), \phi(t))\right)ds,
\] (75)

where
\[
H(t, x^\dagger(t), u(t), \phi(t)) = \langle \phi, f(t, x^\dagger(t), u(t))\rangle - g(t, x^\dagger(t), u(t)).
\]

Since \(U\) is separable, there exists a countable dense set \(U_0 = \{u_i, i \geq 1\} \subset U\). For each \(u_i \in U_0\), we denote
\[
h_i(t) = H(t, x^\dagger(t), u_i, \phi^0, \phi(t)) - H(t, x^\dagger(t), u^\dagger(t), \phi^0, \phi(t)).
\]

Then \(h_i(\cdot) \in L^1[0, T; \mathbb{R})\). Then, according to Lebesgue point theorem (One can refer to Theorem 4.3 in [29]), there exists a measurable set \(F_i \subset [0, T]\) with \(|F_i| = T\)
such that for any $t \in F_i$, it has

$$
\lim_{\rho \to 0} \frac{1}{\rho} \int_{t-\rho}^{t+\rho} |h_i(t) - h_i(s)| ds = 0.
$$

Then for any $t \in F_i$, we define

$$
u_\rho(s) = \begin{cases} u_1(s), & t \in [0, T] \setminus (t - \rho, t + \rho), \\ u_2, & s \in (t - \rho, t + \rho). \end{cases}
$$

Then, by (75), we have $h_i(t) \leq 0$. This implies that

$$
H(t, x^1(t), u_i, \phi(t)) \leq H(t, x^1(t), u_1(t), \phi(t)).
$$

Consequently, by the Lebesgue point theorem, we can obtain the maximum condition. The proof is completed. \qed

6. Applications to COVID-19 model. The classical models of epidemic diseases generally include individuals and interaction changes among individuals in the population, such as SIR, COVID-19 and so on. Moreover, most of these models rely on integer-order differential equations [18, 39, 24, 44, 33]. However, in the past few years, many researchers found that fractional differential equations can be applied to model global phenomena with a greater grade of precision. For example, the authors have applied fractional-order ordinary differential equations to model the COVID-19 pandemic [20, 1, 34, 38, 35]. But one knows that individual’s diffusion behavior is inevitable in the actual propagation of epidemic disease. Motivated by the above analysis, we could attempt to use the following fractional reaction-diffusion equation instead of the classical reaction-diffusion equation to model the COVID-19 dynamical behaviour

$$
\begin{aligned}
(D_{0+}^\alpha S)(t, x) &= \theta_1 \triangle S(t, x) - \beta(x)S(t, x)I(t, x) - u_1(t, x)S(t, x), \\
(D_{0+}^\alpha I)(t, x) &= \theta_2 \triangle I(t, x) + \beta(x)S(t, x)I(t, x) - u_2(t, x)I(t, x),
\end{aligned}
$$

with the homogeneous Neumann boundary conditions and initial conditions are

$$
S(0, x) = S_0(x) > 0, \quad I(0, x) = I_0(x) > 0, \quad x \in \Omega,
$$

where $(t, x) \in (0, T) \times \Omega$, $(D_{0+}^\alpha S)(t, x)$ denotes the Caputo fractional derivative of order $\alpha$ with respect to $t$ with $0 < \alpha < 1$, the symbol $\triangle$ stands for the Laplacian operator in the spatial variable $x$. The functions $S(t, x)$ and $I(t, x)$ denote respectively the number of susceptible populations and the number of infected populations at location $x$ and time $t$; the positive constants $\theta_i$ ($i = 1, 2$) represent the diffusion rates for the susceptible and infected populations; $\beta(x)$ stands for the rates of disease transmission; $u_1(t, x)$ and $u_2(t, x)$ denote the vaccine rate and cure rate at location $x$ and time $t$. The homogeneous Neumann boundary condition means that there is no population flux across the boundary $\partial \Omega$.

For a given suitable admissible control set $U$, the main purpose is to minimize the number of susceptible and infected individuals and reduce the cost of vaccines and treatment. To do this, we define the following cost functional

$$
J(S, I, u)
$$

$$
= \int_{(0, T) \times \Omega} \left( l_1(t, x)S(t, x) + l_2(t, x)I(t, x) + \lambda_1(t, x)u_1(t, x) + \lambda_2(t, x)u_2(t, x) \right) dt dx
$$

$$
+ \int_{\Omega} \left( \delta_1(x)S(T, x) + \delta_2(x)I(T, x) + \rho_1(x)u_1(T, x) + \rho_2(x)u_2(T, x) \right) dx,
$$

$$
(77)
$$
where the positive functions $l_i$ and $\delta_i$ ($i = 1, 2$) denote weights, the positive functions $\lambda_i$ and $\rho_i$ ($i = 1, 2$) are the measure of the cost of interventions associated with the control for vaccination and treatment. Let $(S, I)$ be the solution of equation (76)-(77) with the control $u$. Then the optimal control problem is to minimize the cost functional $J(S, I, u)$ in (78) subject to the state equation (76)-(77).

Provided some assumptions on coefficients are satisfied, one can apply the abstract result directly in this case. Though there might not be apparent clinical interpretation, the optimal problem of singular evolution equations is of its own mathematical interest and we believe that it can be used for simulating the phase-based transmissibility of COVID-19 outbreaks.

7. Conclusions. In this paper, we considered the well-posedness of the singular evolution equations with sectorial operators in $L^p$, and presented a Pontryagin maximum principle for an optimal control problem of such equations. Here are some remarks as follows.

In this paper, the proposed equations and the related theory provide potential application values for the prevention and control of infectious diseases, such as COVID-19.

As we have analyzed, the Cauchy problems with the R-L derivative and the Caputo derivative are special cases of the proposed equations.

Note that the control domain $U$ is just a metric space and does not necessarily have any algebraic structure. Therefore, we only use the spike perturbation method to obtain the maximum principle.

Our results extend the existing results in references. In the case $\alpha = 1$, equation (1) turns to be the controlled abstract evolution equation discussed in [27]. It is easy to see that all the results that we presented will be validity for the abstract evolution equation.

Here we only discussed the case where the terminal state is free. Of course, we could consider the terminal state is constrained. In this case, one might need to use Ekeland’s variational principle to obtain the maximum principle. Some kind of transversality conditions will be obtained. Besides that, we could do some another extensions. For example, we could consider the maximum principle for an optimal control problem of stochastic singular equations.

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