Magnetic Neutron Stars in $f(R)$ gravity

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Neutron stars with strong magnetic fields are considered in the framework of $f(R)$ gravity. In order to describe dense matter in magnetic field, the model with baryon octet interacting through $\sigma\rho\omega$-fields is used. The hyperonization process results in softening the equation of state (EoS) and in decreasing the maximal mass. We investigate the effect of strong magnetic field in models involving quadratic and cubic corrections in the Ricci scalar $R$ to the Hilbert-Einstein action. For large fields, the Mass-Radius relation differs considerably from that of General Relativity only for stars with masses close to the maximal one. Another interesting feature is the possible existence of more compact stable stars with extremely large magnetic fields ($\sim 6 \times 10^{18} \text{G}$ instead of $\sim 4 \times 10^{18} \text{G}$ as in General Relativity) in the central regions of the stars. Due to cubic terms, a significant increasing of the maximal mass is possible.

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I. INTRODUCTION

Neutron stars are observed as several classes of self-gravitating systems: as radio and X-ray pulsars, as X-ray bursters, as compact thermal X-ray sources in supernova remnants, as rotating radio transients. In general, the structure of neutron stars and the relation between the mass and the radius are determined by equations of state (EoS) of dense matter.

The maximal mass of neutron star is still an open question. Recent observations allows to estimate this limit at least as $2M_\odot$: the well-measured limit of the pulsar PSR J1614-2230 is $1.97M_\odot$ [1], while, for the pulsar J0348+0432, it is $2.01 M_\odot$ [2]. Other examples of massive neutron stars are Vela X-1 ($\sim 1.8M_\odot$ [3]) and 4U 1822-371 ($\sim 2M_\odot$, [4]). There are some indications in favor of the existence of more massive neutron stars with masses $\sim 2.4M_\odot$ (the possible masses of B1957+20 [5] and 4U 1700-377 [6] or even $\sim 2.7M_\odot$ (J1748-2021B [7]).

It is interesting to note that for various EoS including hyperons, the maximal mass limit for non-magnetic neutron stars is considerably below than of the two-solar masses limit. The hyperonization process softens EoS and then the maximal allowable mass results reduced $\sim 1.5M_\odot$.

There are several ways to approach the solution of this problem (the so called “hyperon puzzle”).

Firstly, the extensions of the simple model of hyperonic matter (with three exchange meson fields - the so called $\rho\omega\sigma$-model) allow to achieve the increasing of maximal mass. Various approaches are proposed following this track. For instance, larger hyperon-vector couplings (in comparison with quark counting rule) require stiffness of the EoS [8–10]. Similar effect occurs in model with chiral quark-meson coupling [11]. The quartic vector-meson terms in the Lagrangian [12] or the inclusion of an additional vector-meson mediating repulsive interaction amongst hyperons [13] also lead to the increasing of the maximal mass limit. Authors of Ref. [14] proposed an EoS with maximum mass $\sim 2.1M_\odot$ using the quark-meson coupling model, which naturally incorporates hyperons without additional parameters. A model with in-medium hyperon interactions is considered in [15].

Another source for increasing the maximal mass limit is the existence of strong magnetic fields inside the star. The existence of soft gamma-ray repeaters and anomalous X-ray pulsars can be linked to neutron stars with very strong magnetic fields of the order $10^{15}$ G on the surface. In these cases, the maximum magnetic field in the central regions of neutron star can exceed $10^{18}$ G, according to the scalar virial theorem. Such magnetic fields affect considerably the EoS for dense matter and result in increasing the maximal mass of neutron stars.

Various models of dense nuclear matter in presence of strong magnetic fields have been considered in literature. The simplest model with interacting $npe\mu$ gas is investigated in [16]. Models with hyperons and quarks are considered in [17–21]. It has been demonstrated that the Landau quantization leads to the softening of the EoS for matter
but account for contributions of magnetic field into pressure and density. This fact leads, on the other hand, to the stiffening of EoS.

Therefore neutron stars are very peculiar objects for testing theories of matter at high density regimes and in strong magnetic fields. It is interesting to note that data about neutron stars (mainly mass-radius \((M - R)\) relation) can be used for investigating possible deviations from General Relativity (GR).

The initial motivation for studying modified gravity came from the discovered accelerated expansion of the universe confirmed by numerous independent observations. These observations include Hubble diagram for Ia type supernovae [28–30], cosmic microwave background radiation (CMBR) data [31], surveys of gravitational weak lensing [32] and data on Lyman alpha forest absorption lines [33].

This acceleration takes place at relatively small distances ("Hubble flow") and requires (in GR) non-standard cosmic fluid (dark energy) filling the universe with negative pressure but not clustered in large scale structure. The nature of dark energy is unclear. Although from an observational viewpoint, the so called ΛCDM model (where dark energy is considered as Einstein Cosmological Constant) is in agreement with data coming from observations there are various problems and shortcomings at theoretical level. One of this issues is the "smallness" of cosmological constant i.e. the difference of 120 orders of magnitude between its observed value and the one predicted by quantum field theory [34].

An alternative approach to dark energy problem consists of extending of GR. In this case, the accelerated expansion can be obtained without using "dark energy" but enlarging the gravitational sector [35–43]. Therefore theories of modified gravity can be considered as real alternative to GR.

The study of relativistic stars in modified gravity is interesting from several reasons and could constitute a formidable probe for such theories. Firstly one can reject some models that do not allow the existence of stable star configurations [44–49] (however one has to note that stability can be achieved due to "chameleon mechanism" [50, 51] and may depend on the choice of the EoS). Secondly there is the possibility for the existence of new stellar structures, in the framework of modified gravity, escaping the standard stellar models. The observation of such self-gravitating anomalous structures could provide strong evidence for the Extended Gravity (see e.g. [52–54]).

The present paper is devoted to neutron stars with strong magnetic fields in framework of analytic \(f(R)\) gravity. Assuming a simple model for strong interactions, one can obtain the EoS for dense matter in magnetic field. Landau quantization, due to magnetic fields, results to have significant effects. We consider the cases of slowly and fast varying fields.

The paper is organized as follows. In Sec. II, we briefly consider the the field equations for \(f(R)\) gravity and the modified Tolman–Oppenheimer–Volkoff (TOV) equations. Then relativistic mean field theory for dense matter in strong magnetic fields is presented (Sec.III).

In Sec. IV, the neutron star models for strong magnetic fields in quadratic \((f(R) = R + \alpha R^2)\) and cubic \((f(R) = R + \alpha R^3)\) gravity are presented. The \(M - R\) relation is derived and compared with the one in GR. Conclusions and outlooks are reported in Sec.V.

## II. MODIFIED TOV EQUATIONS IN \(f(R)\) GRAVITY

The action of \(f(R)\)-gravity is

\[
S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_{\text{matter}} .
\]  

It can be expressed as \(f(R) = R + \alpha h(R)\). The field equations are

\[
(1 + \alpha h_R)G_{\mu\nu} - \frac{1}{2} \alpha (h - h_R R)g_{\mu\nu} - \alpha (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) h_R = 8\pi G T_{\mu\nu}/c^4 .
\]

Here \(g\) is the determinant of the metric \(g_{\mu\nu}\) and \(S_{\text{matter}}\) is the action of the standard perfect fluid matter. The Einstein tensor is \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\) and \(h_R = \frac{dh}{dR}\).

For star configurations, one can assumes a spherically symmetric metric with two independent functions of radial coordinate, that is:

\[
ds^2 = -e^{2\lambda} c^2 dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .
\]

For the exterior solution, we assume a Schwarzschild metric. Therefore it is convenient to define the variable [53, 54]

\[
e^{-2\lambda} = 1 - \frac{2GM}{c^2 r} .
\]
The value of variable $M$ on the star surface is the gravitational mass. For a perfect fluid, the energy-momentum tensor is $T_{\mu\nu} = \text{diag}(e^{2\phi}p, c^2, e^{2\lambda}P, r^2 P, r^2 \sin^2 \theta P)$, where $\rho$ is the matter density and $P$ is the pressure. The field equations of interest are

$$-8\pi G \rho/c^2 = -r^{-2} + e^{-2\lambda}(1 - 2r\lambda)'r^{-2} + a h_R(-r^{-2} + e^{-2\lambda}(1 - 2r\lambda)'r^{-2})$$

$$-\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha [h'_R r^{-1}(2 - 2r\lambda') + h''_R r^{-1}],$$

$$(9)$$

$$8\pi G P/c^4 = -r^{-2} + e^{-2\lambda}(1 + 2r\phi)'r^{-2} + a h_R(-r^{-2} + e^{-2\lambda}(1 + 2r\phi)'r^{-2})$$

$$-\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha h'_R r^{-1}(2 + 2r\phi'),$$

where $' \equiv d/dr$. The second TOV equation follows from the conservation law $T^\mu_{\nu,\mu} = 0$ and Eq. (9). As result, the modified TOV equations can be written as [59]

$$\left(1 + a h_R + \frac{1}{2}\alpha h'_R r\right) \frac{dm}{dr} = 4\pi pr^2 - \frac{1}{4} \alpha^2 \left[h - h_R R - 2 \left(1 - \frac{2m}{r}\right) \left(\frac{2h'_R}{r} + h''_R r\right)\right],$$

$$(7)$$

$$8\pi p = -2(1 + a h_R) \frac{m}{r^3} - \left(1 - \frac{2m}{r}\right) \left(\frac{2}{r}(1 + a h_R) + \alpha h'_Rh'_R\right)(\rho + p)^{-1} \frac{dp}{dr} -$$

$$-\frac{1}{2}\alpha \left[h - h_R R - 4 \left(1 - \frac{2m}{r}\right) \frac{h'_R}{r}\right],$$

$$(8)$$

Here we have introduced the dimensionless variables $M = m M_\odot$, $r \to r g$, $\rho \to \rho M_\odot/r_g^3$, $P \to \rho M_\odot c^2/r_g^3$, $R \to R/r_g$, $\alpha h'_R R \to h(R)$, where $r_g = GM_\odot/c^2 = 1.47473$ km. The third independent equation for Ricci curvature scalar is

$$3\alpha h'_R \left[\frac{2}{r} \frac{3m}{r^3} - \frac{dm}{r dr} - \left(1 - \frac{2m}{r}\right) \frac{dp}{(\rho + p) dr}\right] \frac{d}{dr} + \left(1 - \frac{2m}{r}\right) \frac{d^2}{dr^2} hR + \alpha h'_R hR - 2\alpha h - R = -8\pi(\rho - 3p).$$

Eqs. (7), (8), and (9) can be solved numerically for given EOS. In order to get solution, one can use perturbative approach (see for details [57–60]). In the framework of perturbative approach, terms containing $h(R)$ are assumed to be of first order in the small parameter $\alpha$, so all such terms should be evaluated at $O(\alpha)$ order. The Ricci curvature scalar at zero order is $R^{(0)} = 8\pi(\rho^{(0)} - 3p^{(0)})$. Therefore the deviation from GR strongly depends from the assumed form of EoS.

III. RELATIVISTIC MEAN FIELD THEORY FOR DENSE MATTER IN PRESENCE OF STRONG MAGNETIC FIELD

Let us assume a simple model for describing nuclear matter in magnetic field. The magnetic field $B$ is assumed along z-axis i.e. the 4-potential is $A^{\mu} = (0, 0, Bx, 0)$. For nuclear matter consisting of baryon octet $(b = p, n, \Lambda, \Sigma^{0, \pm}, \Xi^{0, -})$ interacting with magnetic field and scalar $\sigma$, isoscalar-vector $\omega_\mu$, and isovector-vector $\rho_\mu$ meson fields and leptons $(l = e^-, \mu^-)$, it is [61]

$$\mathcal{L} = \sum_b \bar{\psi}_b \left(\gamma_\mu (i\partial^\mu - q_b A^\mu - g_{\omega b}\omega^\mu - \frac{1}{2}g_{\rho b} \rho^\mu) - (m_b - g_{\sigma b}\sigma)\right) \psi_b + \sum_l \bar{\psi}_l (\gamma_\mu (i\partial^\mu - q_l A^\mu) - m_l) \psi_l +$$

$$+ \frac{1}{2} \left((\partial_\mu \sigma)^2 - m_\sigma^2 \sigma^2\right) - V(\sigma) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega^\mu - \frac{1}{4} \omega^{\mu\nu} \omega^{\mu\nu} - \frac{1}{4} \rho^{\mu\nu} \rho^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho_\mu^2.$$ 

Here the mesonic and electromagnetic field strength tensors are defined by the usual relations $\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$, $\rho_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For the sake of simplicity, we consider frozen-field configurations of electromagnetic field. Also we neglect the anomalous magnetic moments (AMM) of baryons and leptons because their effect is very small. The strong interaction couplings $g_{\omega b}, g_{\rho b}$ and $g_{\rho b}$ depend from density. We use the parameterization adopted in [61]:

...
\[ g_i(\rho) = g_{i0} f_i(x), x = \rho/\rho_0, \] (11)

where

\[ f_i(x) = a_i \frac{1 + b_i(x + d_i)^2}{1 + c_i(x + d_i)^2}. \]

For the isovector field, it is

\[ g_b = g_{b0} \exp[-a_b(x - 1)]. \]

The values of constants \(a_i, b_i, c_i, d_i\) are given in [61]. Using the mean-field approximation, one can obtain the following equations for meson fields:

\[ m_\sigma^2 \frac{d}{d\sigma} + \frac{dV}{d\sigma} = \sum_b g_{sb} n_b^2, \quad m_\omega^2 \omega_0 = \sum_b g_{\omega b} n_b, \quad m_{\omega 0}^2 \rho_0 = \sum_b g_{\rho b} n_b. \] (12)

Here \(\sigma, \omega, \rho\) are the expectation values of the meson fields in uniform matter. The quantities \(n_{sb}^2, n_b\) are the scalar and vector baryon number densities, correspondingly. The simplest scalar field potential is defined as

\[ V(\sigma) = \frac{1}{3} b m_N (g_{\sigma N} \sigma)^3 + \frac{1}{4} c (g_{\sigma N} \sigma)^4, \] (13)

where \(b, c\) are dimensionless constants. The values of nucleon-meson couplings and parameters \(b, c\) are given in Table I. From the Dirac equations for charged and neutral baryons and leptons, we have the energy spectra:

\[ E^{b}_{\nu} = (k^2 + m_b^2 + 2\nu|q_b|) \sqrt{1 + \gamma_{3b} \omega^0 + \gamma_{3b} \omega^0 + \gamma_{3b} \omega^0 + \Sigma^R_0}, \] (14)

\[ E^{b} = (k^2 + m_b^2)^{1/2} + g_{\omega b} \omega^0 + \gamma_{3b} \gamma_{3b} \omega^0 + \Sigma^R_0, \] (15)

\[ E^{b}_{\nu} = (k^2 + m_b^2)^{1/2} + 2\nu|q_b| \sqrt{1}. \] (16)

The number \(\nu = n + 1/2 - \text{sgn}(q) s/2\) denotes the Landau levels of the fermions with electric charge \(q\), spin number \(s = \pm 1\) for spin up and spin down cases correspondingly. The spin degeneracy is \(g_{\nu} = 1\) for lowest Landau level (\(\nu = 0\)) and 2 for all other levels. The effective mass for baryons is \(m_b^* = m_b - g_{sb} \sigma\).

The rearrangement self-energy term is defined by

\[ \Sigma^R_{\sigma} = -\frac{\partial \ln g_{\sigma N}}{\partial n} m_\sigma^2 \sigma^2 + \frac{\partial \ln g_{\omega N}}{\partial n} m_\omega^2 \omega_0^2 + \frac{\partial \ln g_{\rho N}}{\partial n} m_{\rho 0}^2 \rho_0^2. \] (17)

Here \(n = \sum_b n_b\).

The scalar densities for neutral baryons are

\[ n_{sb} = \frac{m_{sb}^2}{2\pi^2} \left( E^{b}_{\nu} k^{b}_{f} - m_{sb}^2 \ln \left| \frac{k^{b}_{f} + E^{b}_{\nu}}{m_{sb}^2} \right| \right). \] (18)

and for charged baryons, it is

\[ n_{sb} = \frac{|q_b| B m_{sb}^2}{2\pi^2} \sum_{\nu} g_{\nu} \ln \left| \frac{k^{b}_{f,\nu} + E^{b}_{f}}{\sqrt{m_{sb}^2 + 2\nu|q_b| B}} \right|. \] (19)

For the vector densities for neutral baryons, we have

\[ n_b = \frac{1}{3\pi^2} k^{b3}_{f}. \] (20)
and for charged baryons and leptons, it is

\[ n_{b,l} = \frac{|q_{b,l}| B}{2\pi^2} \sum_{\nu} g_{\nu} k_{f,\nu}^{b,l} \]  

(21)

Here \( E_f^{b,l} \) is the Fermi energy. For charged baryon, \( E_f^{b} \) is related to the Fermi momentum \( k_{f,\nu}^{b} \) as \( E_f^{b} = (k_{f,\nu}^{b})^2 + 2\nu |q_b| B \). For neutral baryon, it is \( E_f^{b} = (k_{f,\nu}^{b})^2 + m_b^2 \). The summation over \( \nu \) terminates at value \( \nu_{max} \) where the square of Fermi momenta is still positive. For large magnetic fields \( B \sim 10^{18} \text{ G} \), only few Landau levels are occupied.

For hyperon-meson couplings there are no well-defined rule. One can use for these constants quark counting rule \[62, 63]\.

\[ g_{\omega \Lambda} = g_{\omega \Sigma} = \frac{2}{3} g_{\omega N}, \quad g_{\omega \Xi} = \frac{1}{2} g_{\omega N}, \]

(22)

and

\[ g_{\rho \Sigma} = 2 g_{\rho N}, \quad g_{\rho \Xi} = g_{\rho N}. \]

(23)

Another choice is assuming that the fractions of nucleon-meson couplings, i.e. \( g_{iH} = x_i H g_{iN} \), is fixed. Here, it is \( x_{\pi H} = x_{\rho H} = 0.600, x_{\omega H} = 0.653, x_{\rho H} = 0.6 \) (see [23]). We use this definition for further calculations.

For chemical potential of baryons and leptons, one has

\[ \mu_b = E_f^b + g_{\omega b} \omega_0 + g_{\rho b} I_3 \rho_0 + \Sigma_0^R, \quad \mu_l = E_f^l. \]

The following conditions should be imposed on the matter in order to obtain the EoS with the following properties:

(i) baryon number conservation:

\[ \sum_b n_b = n, \]

(24)

(ii) charge neutrality:

\[ \sum_i q_i n_i = 0, \quad i = b, l, \]

(25)

(iii) beta-equilibrium conditions:

\[ \mu_n = \mu_\Lambda = \mu_\Sigma^0 = \mu_\Sigma^+, \quad \mu_p = \mu_\Sigma^- = \mu_n - \mu_e, \quad \mu_\Sigma^- = \mu_\Xi^- = \mu_n + \mu_e, \quad \mu_e = \mu_\mu. \]

(26)

At given \( n \), Eqs. (12)-(26) can be solved numerically and one can find the Fermi energy for particles and meson fields. Resulting matter energy density is

\[ \epsilon_m = \sum_b \epsilon_b + \sum_l \epsilon_l + \frac{1}{2} m_b^2 + \frac{1}{2} m_\omega^2 + \frac{1}{2} m_\rho^2 \rho_0^2 + U(\sigma). \]

(27)

The energy density for charged baryons is

\[ \epsilon_b^c = \frac{|q_b| B}{4\pi^2} \sum_{\nu} g_{\nu} \left( k_{f,\nu}^{b} E_f^b + (m_b^2 + 2\nu |q_b| B) \ln \left| \frac{k_{f,\nu}^{b} + E_f^b}{\sqrt{m_b^2 + 2\nu |q_b| B}} \right| \right) \]

(28)
TABLE II: Neutron star properties using TW model for quadratic gravity (slowly varying field). The energy density \(E_c\) and magnetic field \(B_c\) in the center for neutron star with maximal mass are given.

| \(B_0, 10^9\) G | \(\alpha, 10^9\) cm\(^2\) | \(M_{max, M_\odot}\) | \(R, \text{ km}\) | \(E_c, \text{ GeV/fm}^3\) | \(B_c, 10^{18}\) G |
|------------------|-----------------|-----------------|--------------|-----------------|------------------|
| 0                | 0               | 1.51            | 10.00        | 1.61            | 0                |
|                  | −5              | 1.55            | 10.00        | 1.61            | 0                |
|                  | 5               | 1.46            | 10.05        | 1.49            | 0                |
| 1                | 0               | 2.21            | 11.69        | 1.17            | 3.38             |
|                  | −5              | 2.30            | 11.58        | 1.27            | 3.56             |
|                  | 5               | 2.14            | 11.82        | 1.09            | 3.20             |
| 2                | 0               | 2.80            | 13.99        | 0.79            | 3.75             |
|                  | −5              | 2.91            | 13.44        | 0.97            | 3.93             |
|                  | 5               | 2.71            | 14.31        | 0.68            | 3.06             |
| 3                | 0               | 3.21            | 15.67        | 0.63            | 3.29             |
|                  | −5              | 3.33            | 15.24        | 0.68            | 3.75             |
|                  | 5               | 3.11            | 16.03        | 0.54            | 2.87             |

and for neutral baryons

\[
e_b^n = \frac{1}{4\pi^2} \left[ k_f^b(E_f^b)^3 - \frac{1}{2} m_b^* \left( m_b^* k_f^b E_f^b + m_b^* m_b^* \ln \left( \frac{k_f^b + E_f^b}{m_b^*} \right) \right) \right].
\]

The expression of energy density for leptons can be obtained from (28) by changing \(m_b^* \rightarrow m_l\). The pressure of dense matter is defined as

\[
p = \sum_b p_b + \sum_l p_l - \frac{1}{2} m_{\sigma}^2 + \frac{1}{2} m_{\omega}^2 + \frac{1}{2} m_{\rho}^2 - U(\sigma) + \Sigma_0^R.
\]

where pressure for charged baryons is

\[
p_c = \frac{|q_b| B}{12\pi^2} \sum_{\nu} g_{\nu} \left( k_f^{b_{\nu}} E_f^{b_{\nu}} - (m_b^*)^2 + 2\nu|q_b|B) \ln \left( \frac{k_f^{b_{\nu}} + E_f^{b_{\nu}}}{m_b^*} \right) \right)
\]

and for neutral baryons

\[
p_n = \frac{1}{12\pi^2} \left[ k_f^b(E_f^b)^3 - \frac{3}{2} m_b^* \left( m_b^* k_f^b E_f^b - m_b^* m_b^* \ln \left( \frac{k_f^b + E_f^b}{m_b^*} \right) \right) \right].
\]

In order to obtain the EoS, one needs to add the contribution of magnetic field, that is

\[
\epsilon = \epsilon_m + \frac{B^2}{8\pi}, \quad p = p_m + \frac{B^2}{8\pi}.
\]

We use a model where the magnetic field depends from baryon density only. The parameterization proposed in [23, 26] has the form

\[
B = B_s + B_0 \left[ 1 - \exp \left( -\beta(n/n_s)\gamma \right) \right],
\]

where \(B_s\) is the magnetic field on the star surface \((10^{15})\) G. For parameters \(\gamma\) and \(\beta\), one takes the values \(\gamma = 2, \beta = 0.05\) (slowly varying field) and \(\gamma = 3, \beta = 0.02\) (fast varying field). The value \(B_0\) is convenient to give in units of critical field for electron \(B_c = 4.414 \times 10^{13}\) G.

All these considerations can be applied to models where curvature corrections appear in the TOV equations. Specifically, we adopt quadratic and cubic corrections.
IV. THE CASES OF QUADRATIC AND CUBIC CURVATURE CORRECTIONS

Let us firstly take into account models with quadratic gravity corrections, that is

$$f(R) = R + \alpha R^2.$$  \hfill (33)

Neutron stars with strong magnetic field in quadratic gravity was considered in [64] for relatively stiff EoS based on model with five meson fields. We consider the quadratic gravity case for EoS based on the above described model. For our calculations, the Typel-Wolter (TW) parametrization is used.

Let us note the following feature: for \( B = 0 \), the mass of neutron star increases with decreasing \( \alpha \) for the various

| \( B_0 \), \( 10^3 \) \( 10^9 \) cm\(^2\) | \( \alpha \), \( 10^{-5} \) | \( M_{\text{max}} \), \( M_\odot \) | \( R_\odot \), km | \( E_c \), GeV/fm\(^3\) | \( B_c \), \( 10^{18} \) G |
|---|---|---|---|---|---|
| 0 | 2.32 | 11.50 | 1.17 | 3.96 |
| -5 | 2.44 | 11.47 | 1.22 | 4.36 |
| 5 | 2.23 | 11.66 | 1.04 | 3.66 |
| 0 | 2.73 | 12.44 | 0.93 | 4.20 |
| -5 | 2.89 | 12.81 | 0.97 | 4.34 |
| 5 | 2.60 | 13.31 | 0.71 | 3.29 |
| 0 | 2.98 | 13.78 | 0.76 | 3.87 |
| -5 | 3.12 | 13.81 | 0.83 | 4.12 |
| 5 | 2.84 | 14.06 | 0.68 | 3.50 |

TABLE III: Neutron star properties using TW model for quadratic gravity (fast varying field).

| \( B_0 \), \( 10^3 \) \( 10^9 \) cm\(^2\) | \( \alpha \), \( 10^{-5} \) | \( M_{\text{max}} \), \( M_\odot \) | \( R_\odot \), km | \( E_c \), GeV/fm\(^3\) | \( B_c \), \( 10^{18} \) G |
|---|---|---|---|---|---|
| 2 | 10 | 2.30 | 11.82 | 1.49 | 5.97 |
| 3 | 10 | 2.52 | 12.86 | 1.40 | 6.08 |

TABLE IV: Compact star properties on the second “branch of stability” using TW model for quadratic gravity (fast varying field). The magnetic field at the center can exceed \( 6 \times 10^{18} \) G and the central energy density is approximately twice than in GR.

FIG. 1: The mass-radius diagram in model \( f(R) = R + \alpha R^2 \) for two values of \( \alpha \) without magnetic field in comparison with GR.
radii (see Fig. 1). For strong magnetic field, the $M - R$ relation for $M > 0.7M_\odot$ differs from such in GR only for masses close to the maximal one (see Figs. 2, 3). Another interesting feature appears for fast varying field. At high central densities, a second “branch” of stability can exists (Fig. 3, lower panel).

It is interesting to note that similar effects take place for non-magnetic neutron stars in the framework of a model like $f(R) = R + \alpha R^2 (1 + \gamma R)$ [59]. The stabilization of star configurations occurs thanks to the cubic term.

The maximal masses and corresponding radii are given in Tables II, III for some values of $\alpha$ and $B_0$. The maximal value of central density (and therefore magnetic field) decreases with increasing $\alpha$.

The parameters for compact (in comparison with GR) neutron stars on second “branch of stability” are given in Table IV.

For modified gravity with cubic term, $f(R) = R + \beta R^3$, the maximal value of neutron star mass for given EoS increases for $\beta < 0$ (Fig. 4). Some results are given in Tables V, VI. The maximal mass of neutron star can exceed $3M_\odot$. One can note that stars with magnetic field and cubic curvature corrections result stable for central energy density close to $\sim 1.8$ GeV/fm$^3$.

In principle, calculations show that, for EoS based on GM2-GM3 parameterizations, we have similar results for models with $f(R) = R + \alpha R^2$ and $f(R) = R + \beta R^3$. For more stiff EoS the deviation from GR is larger.
FIG. 3: The Mass-Radius diagram in model $f(R) = R + \alpha R^2$ and in GR for fast varying field. On upper panel the cases $\alpha = -5 \times 10^9, 0, 5 \times 10^9$ cm$^2$ correspond to dotted, thick and thin lines correspondingly. On lower panel the mass radius relation for $\alpha = 10^{10}$ cm$^2$ is given (dotted lines). The second “branch of stability” with more compact (in comparison with GR) neutron stars exists.

| $B_0$, $10^5$ | $r_g$, $\beta$ | $M_{max}/M_\odot$ | $R$, km | $E_\epsilon$, GeV/fm$^3$ | $B_\epsilon$, $10^{18}$ G |
|---------------|----------------|--------------------|--------|----------------|----------------|
| 0             | 0              | 2.32               | 11.50  | 1.17           | 3.96           |
| -25           | -25            | 2.73               | 11.10  | 1.27           | 4.14           |
| 1             | -50            | 3.14               | 11.19  | 1.17           | 3.96           |
| -75           | -75            | 3.64               | 11.18  | 1.17           | 3.96           |
| 0             | 0              | 2.73               | 12.44  | 0.93           | 4.20           |
| -25           | -25            | 3.24               | 12.87  | 0.86           | 3.93           |
| 2             | -50            | 3.71               | 13.09  | 0.76           | 3.54           |

TABLE VI: Compact star properties using TW model for cubic gravity for several values of $\beta$ for fast varying magnetic field.
FIG. 4: The mass-radius diagram in model $f(R) = R + \beta R^3$ and in GR for slowly (upper panel) and fast (lower panel) varying field. One can see that for the deviation of M-R relation from GR is smaller for larger values of $B_0$.

V. CONCLUSIONS AND PERSPECTIVES

We presented neutron star models with strong magnetic fields in the framework of power-law $f(R)$ gravity models. For describing dense matter in magnetic field, a model with baryon octet interacting through $\sigma\rho\omega$-fields is used.

Although the softening of nucleon EoS, due to hyperonization, leads to the decrease of the upper limit mass of neutron star, the strong magnetic field can increase considerably the maximal mass of star.

In particular, we investigated the effect of strong magnetic field in models of quadratic, $f(R) = R + \alpha R^2$, and cubic, $f(R) = R + \alpha R^3$, gravity. For large fields, the $M - R$ relation differs considerably from such in GR only for stars with masses close to maximal. Another interesting feature is the possible existence of more compact stable stars with extremely large fields ($\sim 6 \times 10^{18}$ G instead $\sim 4 \times 10^{18}$ G in GR) in central regions of star. Due to the cubic term, the significant increasing of maximal mass ($M_{max} > 3M_\odot$) is possible. The central energy density can exceed $\sim 1.8$ GeV/fm$^3$.

However, it is worth stressing that the $f(R)$ models considered here can be related to the presence of strong gravitational fields where higher order curvature terms can emerge. Their origin is related to the effective actions
of quantum field theory formulated in curved spacetime. In the extreme field of neutron stars, it is realistic supposing the emergence of curvature corrections that improve the pressure effects and could explain supermassive self-gravitating systems.

As a next step, we will consider models of self-bounded quark stars and hybrid stars with quark cores. The EoS for quark matter (without magnetic field) is close to $p \sim \frac{1}{3} \rho c^2$ and therefore, in the framework of perturbative approach, the deviations from GR occur only for very large values of $\alpha$ in comparison with the above considered in quadratic gravity. However, for large magnetic fields, considerable effects can be induced on EoS and therefore the modified gravity effects can appear.

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