Global controllability with a single local actuator

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We show that we can achieve global density-operator controllability for most $N$-dimensional bilinear Hamiltonian control systems with general fixed couplings using a single, locally-acting actuator that modulates one energy-level transition. Controllability depends upon the position of the actuator and relies on the absence of either decompositions into non-interacting subgroups or symmetries restricting the dynamics to a subgroup of $\mathfrak{U}(N)$. These results are applied to spin-chain systems and used to explicitly construct control sequences for a single binary-valued switch actuator.

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I. INTRODUCTION

The ability to control the dynamics of quantum systems is a long established objective in areas as diverse as molecular chemistry and quantum computing among others. Control in practice comprises various related tasks such as transforming a system from a given initial state to a desired target state, implementing a desired unitary operator, or optimising the expectation value of a selected observable. The manner in which control is effected depends on the system but a common approach for quantum systems is the application of external electromagnetic fields. In the diabatic control regime these drive transitions between different states of the system, and control can be achieved by adjusting the amplitude and phase of the field(s) as a function of time in a way that maximizes constructive interference of various excitation pathways that lead to a desired outcome, while maximizing destructive interference for all others.

Although the ultimate goal of control is usually to find a control field that steers the system in the manner required to achieve the objective, the question of what tasks can be accomplished for a given system with a given set of actuators, is of fundamental interest. A key concept in this regard is that of controllability. A substantial number of papers have been devoted to studying this issue for both classical and quantum systems, establishing various notions of controllability and general algebraic criteria for them, and showing that particular types of systems are controllable [1, 2, 3, 4, 5, 6, 7, 8]. On the latter front, it has been shown, for instance, that any system with distinct transition frequencies and a connected transition graph is controllable [9, 10]. For an $n$-level system this requires at least $n-1$ transitions with non-zero probabilities. It has also been shown that these requirements can be relaxed in many cases [11], and more recently indirect controllability has been studied [12].

One remaining area of interest is global controllability with a small number of local actuators. A motivation for this type of scenario could be a chain or array of quantum dots with control electrodes to locally manipulate the dynamics of one or a few quantum dots. In the ideal case, one might consider separate control electrodes for each quantum dot, as well as separate electrodes to modulate all the interactions between pairs of adjacent quantum dots, as proposed by Kane in [13] and many other quantum computing architectures since. Leaving aside the often considerable challenge of finding optimal control schemes and fighting decoherence, with sufficiently many local actuators almost any (Hamiltonian) quantum system is controllable, at least in principle. However, in many cases it is impractical or even impossible to have a large number of individual local actuators such as control electrodes. Rather, one would like to make do with as few local actuators as possible to simplify the engineering design and reduce deleterious effects such as decoherence and crosstalk, for example.

Motivated by this problem we investigate the question of controllability of a finite-dimensional model system with the smallest number of simple actuators whose effect is strictly confined a local perturbation of the Hamiltonian. We also note here that by local we mean localized in space, affecting a single transition, for instance, not simultaneous local operations on many individual elements such as qubits as is common in global control schemes. We effectively show that in most cases a single local actuator is sufficient to ensure controllability of the system as a whole, provided the latter is not decomposable into non-interacting parts, and does not exhibit dynamical symmetries that its evolution to a subgroup of the unitary group $(\mathfrak{U}(n)\text{ or }\mathfrak{U}(N))$. Many systems with fixed interactions connecting its parts such as chains of quantum dots etc with fixed non-zero couplings between adjacent dots satisfy this connectedness requirement, and the disorder present in most realistic systems is likely to ensure that there are no special dynamical symmetries to worry about in most cases. For these systems our controllability analysis suggests that the entire system can be controlled by modulating a single transition with a local actuator. Although the explicit controllability proofs given apply to specific model systems, the
same arguments are applicable to many other model systems, suggesting that a large class of systems with fixed couplings may be controllable using a very small number of fixed local actuators. We conclude with an explicit example of constructive control with a single binary switch actuator.

II. DEFINITIONS AND BASIC RESULTS ON CONTROLLABILITY

We restrict ourselves here to control problems that can be classified as open-loop Hamiltonian engineering problems and systems subject to Hamiltonian dynamics. Open-loop control engineering means that we aim to design control fields relying only on (presumed) knowledge of the initial state of the system and the dynamic laws governing its evolution in the presence of the control fields, without any feedback from measurements. We furthermore assume the state space of the system is a finite-dimensional Hilbert space, \( \mathcal{H} \cong \mathbb{C}^N \). The state of the system in this case can be represented by a density operator \( \hat{\rho} \), i.e., a positive unit-trace operator acting on \( \mathcal{H} \), and its evolution is governed by the quantum Liouville equation

\[
\frac{d}{dt} \hat{\rho}(t) = \left[ \hat{H}[f(t)], \hat{\rho}(t) \right] + i\hbar \mathcal{L}_D[\hat{\rho}(t)],
\]

where \( [A,B] = AB - BA \) is the usual matrix commutator and \( \mathcal{L}_D = 0 \) for a Hamiltonian control system. The operator \( \hat{H}[f(t)] \) is the total Hamiltonian of the system subject to the control fields \( f(t) \). For control-linear systems we have the perturbative expansion

\[
\hat{H}[f(t)] = \hat{H}_0 + \sum_{m=1}^{M} f_m(t) \hat{H}_m,
\]

where \( \hat{H}_0 \) is the internal Hamiltonian of the system and \( \hat{H}_m, m > 0 \), are the interaction terms.

Hamiltonian dynamics constrains the evolution of density operators \( \rho(t) \) to isospectral flows

\[
\dot{\rho}(t) = \hat{U}(t,t_0) \hat{\rho}(t_0) \hat{U}(t,t_0)^\dagger,
\]

since the evolution operator \( \hat{U}(t,t_0) \) must satisfy the related Schrödinger equation

\[
ih \frac{d}{dt} \hat{U}(t,t_0) = \hat{H}[f(t)] \hat{U}(t,t_0)
\]

and is hence restricted to the unitary group \( \mathfrak{U}(N) \). Due to this fundamental restriction it is clear that the maximum degree of state control we can achieve for this system is the ability to interconvert density operators with the same spectrum, which is achieved if we can implement any unitary operator in the special unitary group \( \mathfrak{SU}(N) \) of unitary operators with determinant 1 as abelian factors do not affect the isospectral flow. It is also not difficult to show that any proper subgroup of \( \mathfrak{SU}(N) \) is not sufficient to interconvert any two generic density operators with the same spectrum.

To properly define the notion of controllability we need some concepts from Lie group / algebra theory. A Lie algebra is a vector space over a field endowed with a bilinear composition \([x,y] \) that satisfies the Jacobi identity

\[
[[x,y],z] + [[[y,z],z] + [[z,x],y] = 0.
\]

It is easy to see that the anti-Hermitian matrices \( i\hat{H}_0 \) and \( i\hat{H}_1 \) generate a Lie algebra \( \mathfrak{L} \) which must be a subalgebra of the Lie algebra of skew-hermitian matrices \( \mathfrak{u}(N) \), and if \( i\hat{H}_0 \) and \( i\hat{H}_1 \) have zero trace, \( \mathfrak{L} \) will be a subalgebra of the trace-zero, anti-Hermitian matrices \( \mathfrak{su}(N) \), which can be regarded as the tangent space to the Lie group \( \mathcal{G} \) at the identity via the exponential map \( x \in \mathfrak{u}(N) \mapsto \exp(x) \in \mathcal{G} \). Therefore, we can argue that if the \( i\hat{H}_0 \) and \( i\hat{H}_1 \)—or their trace-zero counterparts \( \hat{H}_m = \hat{H}_m - N^{-1} Tr(\hat{H}_m) \mathbb{I}_N \)—generate the entire Lie algebra \( \mathfrak{su}(N) \) then we can in principle dynamically generate any matrix \( \hat{U} \in \mathcal{G} \). Hence, a system is said to be density matrix controllable or simply controllable if the Lie algebra generated by \( i\hat{H}_0 \) and \( i\hat{H}_1 \) is \( \mathfrak{su}(N) \). These Lie algebraic criteria are useful as they are easy to check by quite straightforward calculations. In principle, these can be done numerically for a given set of Hamiltonians but for higher dimensional systems the calculations can be time-consuming and the accuracy very limited. It is therefore desirable to have more explicit criteria that guarantee controllability for certain classes of systems, and several such results exist.

For example, consider a simple finite-dimensional system (dim \( \mathcal{H} = N \)) with a control-linear Hamiltonian of the form \( \hat{H}_0 + f(t) \hat{H}_1 \). Choose a basis such that \( \hat{H}_0 \) is diagonal with energy levels \( E_n, n = 1, \ldots, N \), and transition frequencies \( \omega_{mn} = E_n - E_m \). If \( \hat{H}_0 \) is regular, i.e., has non-degenerate eigenvalues then we can associate each 1-dimensional eigenspace with the vertex of a graph and interpret the non-zero elements in the matrix representation of the interaction Hamiltonian \( \hat{H}_1 \) (with respect to the eigenbasis of \( \hat{H}_0 \)) as edges of a transition graph. In this case a sufficient condition for controllability is that \( \hat{H}_0 \) be strongly regular, i.e., have distinct transition frequencies \( \omega_{mn} \neq \omega_{m'n'} \) unless \( (m,n) = (m',n') \), and the transition graph as defined above be connected [9]. The conditions of uniqueness of the transition frequencies can be slightly relaxed in that we only need to consider the transition frequencies of those transitions that occur with non-zero probability. This is a useful result as it is very easy to check, although it is important to remember that it provides only a sufficient, not a necessary condition, and indeed many systems that do not satisfy these conditions are controllable. For example, given a system with a Hamiltonian of the form \( \hat{H}[f(t)] = \hat{H}_0 + f(t)\hat{H}_1 \),
where

\begin{align}
H_0 &= \sum_{n=1}^{N} E_n |n\rangle\langle n|, \\
H_1 &= \sum_{n=1}^{N-1} d_n |n\rangle\langle n+1| + |n+1\rangle\langle n|,
\end{align}

the graph connectivity result allows us to conclude that the system is controllable if the energies of the states are such that the frequencies of all transitions between adjacent states are distinct and \(d_n \neq 0\) for \(n = 1, 2, \ldots, N-1\). In principle this controllability result can be explained in terms of frequency-selective control. If all the possible transitions have different frequencies then we can imagine a field resonant with a particular transition frequency as selectively driving only the resonant transition. Thus, we can implement any selective control of the graph connectivity result allows us to conclude that the system is controllable provided the energy levels of the system are such that the frequencies of all transitions (see Fig. 1b), we cannot draw any conclusions even if the transition graph of the system appears connected and the transition frequencies are distinct. Of course, if we have many local control fields, each selectively driving a single transition, as shown in Fig. 1c then it is again obvious that the system is controllable. It is not obvious, however, under which conditions the ability to control a single transition of a large connected system as in Fig. 1c is sufficient for controllability.

### III. CONTROLLABILITY FOR SINGLE LOCAL ACTUATOR

In this section we consider a model system with \(N\) distinct states that are permanently coupled in some form, such as an array of quantum dots. For our model we first assume coupling of nearest neighbour type, which leads to a drift Hamiltonian of the form

\[ A_0 = H_0 + H_1 \]

with \(H_0\) and \(H_1\) as in Eq. (5), and a single local actuator modulating the coupling between states \(|r\rangle\) and \(|r+1\rangle\),

\[ A_r = |r\rangle\langle r+1| + |r+1\rangle\langle r|. \]

Thus, for a single local actuator positioned between \(r\) and \(r+1\) we have the total Hamiltonian

\[ H[f(t)] = A_0 + f(t)A_r. \]

**Example 1.** The Hamiltonian of the first excitation subspace of a spin chain of length \(N\) with nearest neighbour coupling of isotropic Heisenberg form given by the coupling constants \(d_n > 0\) for \(n = 1, \ldots, N-1\) is

\[ A_0 = \sum_{n=1}^{N} E_n |n\rangle\langle n| + \sum_{n=1}^{N-1} d_n |n\rangle\langle n+1| + |n+1\rangle\langle n|, \]

where the energy levels are explicitly

\[ E_n = \frac{1}{2} \sum_{\ell \neq n-1,n} d_\ell - \frac{1}{2} (d_{n-1} + d_n) \]

and we set \(d_0 = d_N = 0\). Assuming we have a local actuator that allows us to modulate the coupling between spins \(r\) and \(r+1\), the total Hamiltonian is of the form Eq. (6) with \(A_r\) of the form Eq. (7). The first excitation subspace Hamiltonian for spin chains with dipole-dipole interactions is also of form (7) but with different energy levels. Similar results hold for any spin chain decomposable into excitation subspaces.

Again, it is quite obvious that \(N-1\) independent local actuators of this type, controlling the coupling between...
spins $n$ and $n+1$ in the chain, will suffice for the system to be controllable, but in fact, a single such actuator suffices in most cases.

**Theorem 1.** A quantum system with Hamiltonian $H[f(t)] = A_0 + f(t)A_r$ with $A_0$ and $A_r$ as above is controllable if $\omega_r \neq 0$, $d_n \neq 0$ and $d^2_{r+1} \neq d^2_{r-1}$.

Proof. We show that the trace-zero anti-Hermitian matrices $iV_0$ and $iV_r$ defined by

$$V_0 = A_0 - \frac{\text{Tr}(A_0)}{N} I_N, \quad V_1 = A_r - \frac{\text{Tr}(A_r)}{N} I_N$$

generate the Lie algebra $\mathfrak{su}(N)$. To this end it suffices to show that the Lie algebra $\mathfrak{su}(N)$ contains the $2(n-1)$ generators $x_n \equiv x_{n,n+1}$ and $y_n \equiv y_{n,n+1}$ of $\mathfrak{su}(N)$, where the basis elements of $\mathfrak{su}(N)$ are defined as usual,

$$x_{mn} = |m\rangle \langle n| - |n\rangle \langle m|, \quad y_{mn} = i(|m\rangle \langle n| + |n\rangle \langle m|),$$

$$h_n = |n\rangle \langle n| - |n+1\rangle \langle n+1|,$$

for $1 \leq m < n \leq N$. Let $V_0(0) = i(V_0 - d_r V_1)$. We have $iV_1 = y_r \in \mathfrak{su}(2)$ and

$$X_0 \equiv [y_r, V_0(0)] = d_{r-1} x_{r-1,r+1} - d_{r+1} x_{r,r+2} - \omega_r x_r$$

$$Y_0 \equiv [x_r, y_r] = d_{r-1} y_{r-1} + d_{r+1} y_{r+1} - 2\omega_r h_r$$

are Hermitian with $X_0^T = -X_0$, $Y_0^T = -Y_0$, and $[X_0, Y_0] = X_0 Y_0 - Y_0 X_0 = -2\omega_r h_r X_r$. By induction we have

$$X_k \equiv [Y_{k-1}, X_0] = \sum_{m=1}^{k-1} d_{k-m}^2 x_{m-1,m+1} + 2\omega_r h_{r+k} x_{r+k},$$

$$Y_k \equiv [X_{k-1}, Y_0] = \sum_{m=1}^{k-1} d_{k-m}^2 y_{m-1,m+1} + 2\omega_r h_{r+k} y_{r+k},$$

for $2 \leq k \leq N-r-2$. To show that the elements $x_{r-k}, y_{r-k}$ for $2 \leq k \leq r-1$ are in $\mathfrak{su}(2)$, we note that

$$x_{r-k} = d_{r-k+1} y_{r-k+1}, \quad y_{r-k} = [x_{r-k}, h_{r-k}]$$

and setting $W_j = V_0^{(j+1)} - V_0^{(j)}$ for $0 \leq j \leq N-1$ and $W_0(k) = W_r(k)$ shows

$$x_{r-k} = d_{r-k+1} y_{r-k+1}, \quad y_{r-k} = [x_{r-k}, h_{r-k}]$$

for $2 \leq k \leq r-2$, as desired. \!

For a Heisenberg spin chain $d_n > 0$ for $n = 1, \ldots, N-1$ and $d_0 = d_N = 0$. Thus $\omega_r \neq 0$ is equivalent to $d_{r+1} \neq d_r$ and we have the following.

**Corollary 1.** The first excitation subspace of a Heisenberg spin chain of length $N$ with coupling constants $d_n$ is controllable with single local actuator between spins $r$ and $r+1$ if $d_{r+1} \neq d_r$.

A Heisenberg spin chain with non-uniform couplings almost certainly satisfies $d^2_{r+1} \neq d^2_{r-1}$ for any $r$ between $1$ and $N-1$. A chain with uniform coupling $d_n = d$, $n = 1, \ldots, N-1$, $d_0 = d_N = 0$, satisfies this condition only if the actuator is placed near the end of the chain, i.e., $r = 1$ or $r = N-1$. However, we can generalize the previous theorem.

**Theorem 2.** A quantum system with Hamiltonian $H[f(t)] = A_0 + f(t)A_r$ with $A_0$ and $A_r$ as above is controllable if $\omega_r \neq 0$, $d_n \neq 0$ and $d^2_{r-k-1} \neq d^2_{r+k+1}$ for some $k \in \mathbb{N}_0$. 

Proof. For $k = 0$, i.e., if $d^2_{r-1} \neq d^2_{r+1}$, the result follows from Thm 3 and $d^2_{r-1} = d^2_{r+1}$ we begin as in the proof of Thm 1 to conclude that $y_r \in \mathcal{L}$, $x_r = (3\omega)^{-1}(X_0 + X'_0) \in \mathcal{L}$ and $h_r = 2^{-1}[x_r, y_r] \in \mathcal{L}$, and set

$$V^{(0)}_0 = iV_0 - d_r y_r$$

$$V^{(0)}_1 = 3^{-1}(4Y_0 + Y'_0) = d_r y_r - d_{r+1} y_{r+1}$$

$$X^{(0)}_1 = [x_r, Y^{(0)}_1, y_r] = d_r y_{r-1} + d_{r+1} y_{r+1}$$

$$Z^{(0)}_1 = 2^{-1}[X^{(0)}_1, Y^{(0)}_1] = d^2_{r-1} h_{r-1} + d^2_{r+1} h_{r+1}$$

$$V^{(0)}_0 = V^{(0)}_0 - Y^{(0)}_1 = iH_0 - \sum_{n \in I^{(1)}} d_n y_n$$

where $I^{(1)}$ is the index set $\{1, \ldots, N-1\}$ minus the subset \{r-1, r, r+1\}.

Setting $d^2_{r+j} = d^2_{r-j}$ for $j = 1 \ldots k-1$ and observing that we cannot separate the $r+1$ to $r+k$ and $r-1$ to $r-k$ terms, respectively, at this stage we continue along similar lines by iterating the following set of recurrence relations for $j = 1, \ldots, k-1$

$$Z^{(1)}_j = d^2_{r-2} Z^{(0)}_j + h_{r-j} + h_{r+j}$$

$$X^{(1)}_j = [Y^{(0)}_0, V^{(j)}_0] = d_{r-j} d_{r-j-1,x_{r-j-1},r-j+1} - d_{r+j} \omega_r h_{r-j} - d_{r-j+1} x_{r+j,r+j+1} - d_{r+j} \omega_r h_{r+j}$$

$$Y^{(1)}_j = [X^{(1)}_j, Y^{(0)}_j] = d^2_{r-2} d_{r-2,x_{r-2},r-2} - 2d^2_{r-2} \omega_r h_{r-2} + d^2_{r+2} d_{r+2,x_{r+2},r+2} - 2d^2_{r+2} \omega_r h_{r+2}$$

$$Y^{(1)}_j = d^2_{r-2} Y^{(1)}_j$$

$$X^{(0)}_j = d_{r-j} d_{r-j+1,x_{r-j+1},r-j+1} + d_{r+j} \omega_r h_{r-j} - 2d_{r+j} \omega_r h_{r+j}$$

$$Y^{(0)}_j = [X^{(j+1)}_j, Y^{(2)}_j] = d_{r-j} d_{r-j+1,x_{r-j+1},r-j+1} + d_{r+j} \omega_r h_{r-j} - 2d_{r+j} \omega_r h_{r+j}$$

$$Z^{(0)}_j = 2^{-1}[X^{(j+1)}_j, Y^{(2)}_j] = d^2_{r-2} d_{r-2,x_{r-2},r-2} - 2d^2_{r-2} \omega_r h_{r-2} + d^2_{r+2} d_{r+2,x_{r+2},r+2} - 2d^2_{r+2} \omega_r h_{r+2}$$

$$V^{(j+1)}_0 = V^{(j)}_0 - Y^{(j)}_j + iH_0 - \sum_{n \in I^{(j+1)}} d_n y_n$$

where $I^{(j+1)}$ is the index set $I^{(j)}$ with the subset \{r-j - 1, r + j + 1\} removed. Since $d^2_{r-k-1} \neq d^2_{r+k+1}$ and

$$X^{(0)}_k = d_{r-k} x_{r-k} + d_{r+k} x_{r+k}$$

$$Y^{(0)}_k = d_{r-k} y_{r-k} + d_{r+k} y_{r+k}$$

$$Z^{(0)}_k = d^2_{r-k} h_{r-k} + d^2_{r+k} h_{r+k}$$

$$V^{(k)}_0 = iH_0 - \sum_{n \in I^{(k+1)}} d_n y_n$$

where $I^{(k)}$ is the index set $\{1, \ldots, r - k, r + k, \ldots N\}$.

To complete the proof by showing that $y_{r \pm (k+1)}$ and $x_{r \pm (k+1)}$ are in $\mathcal{L}$, we calculate the commutators

$$X^{(1)}_k = [Y^{(0)}_k, V^{(k)}_0]$$

$$Y^{(1)}_k = [X^{(1)}_k, Y^{(0)}_k]$$

$$X^{(2)}_k = [Z^{(1)}_k, Y^{(1)}_k]$$

$$Y^{(2)}_k = [X^{(2)}_k, Z^{(1)}_k]$$

$$Z^{(2)}_k = 2^{-1}[X^{(2)}_k, Y^{(2)}_k]$$

$$Y^{(3)}_k = 2^{-1}[Z^{(2)}_k, X^{(3)}_k]$$

which gives

$$y_{r \pm (k+1)} = \frac{Y^{(3)}_k - d_{r \mp (k+1)} Y^{(2)}_k}{d_{r \mp (k+1)}(d_{r \pm (k+1)} - d_{r \mp (k+1)})}$$

$$x_{r \pm (k+1)} = \frac{[y_{r \pm (k+1)}, Z^{(2)}_k]/(2d_{r \pm (k+1)})}{[x_{r \pm (k+1)}, y_{r \pm (k+1)}]/2}$$

showing that these generators are in $\mathcal{L}$. To show that the generators $x_{r \pm j}, y_{r \pm j}$, and $h_{r \pm j}$ are in $\mathcal{L}$ for $j = 1, \ldots, k$ we set

$$V^{(k)}_0 = V^{(k)}_0 - d_{r-k-1} y_{r-k-1} - d_{r+k+1} y_{r+k+1}$$

$$= iH_0 - \sum_{n \in I^{(k+1)}} d_n y_n$$
where $I^{(k+1)}$ is the index set $I^{(k)}$ with the subset \{r − k − 1, r + k + 1\} removed, and note that
\[
    x_{r \pm j} = d_{r \pm j}^{-1}[y_{r \pm (j+1)}, x_{(0)}], \quad y_{r \pm j} = [x_{r \pm (j+1)}, x_{r \pm j}, y_{r \pm (j+1)}],
\]
\[
    h_{r \pm j} = 2^{-1}[x_{r \pm j}, y_{r \pm j}].
\]

Finally, we show that the generators $x_{r+k+j}, y_{r+k+j}$ and $h_{r+k+j}$ are in $\mathcal{L}$ for $j = 2, \ldots, N − r − k − 1$, by iterating the following set of recurrence relations for $j = 2, \ldots, N − r − k − 1$:
\[
    x_{r+k+j} = d_{r+k+j}^{-1}[h_{r+k+j-1}, V_0^{(k+j-1)}],
\]
\[
    y_{r+k+j} = [x_{r+k+j}, h_{r+k+j-1}],
\]
\[
    h_{r+k+j} = 2^{-1}[x_{r+k+j}, y_{r+k+j}],
\]
\[
    V_0^{(k+j)} = V_0^{(k+j-1)} - d_{r+k+j} y_{r+k+j}.
\]

Similarly, we show that the elements $x_{r-k-j}$, $y_{r-k-j}$ and $h_{r-k-j}$ are in $\mathcal{L}$ for $j = 2, \ldots, r − k − 1$, by setting $W_0^{(k-1)} = V_0^{(k+1)}$ and iterating the following recurrence relations for $j = 2, \ldots, r − k − 1$:
\[
    x_{r-k-j} = d_{r-k-j}^{-1}[h_{r-k-j+1}, W_0^{(k-j+1)}],
\]
\[
    y_{r-k-j} = [x_{r-k-j}, h_{r-k-j+1}],
\]
\[
    h_{r-k-j} = 2^{-1}[x_{r-k-j}, y_{r-k-j}],
\]
\[
    W_0^{(k-j)} = W_0^{(k-j+1)} - d_{r-k-j} y_{r-k-j}.
\]

We have now shown that $x_j, y_j \in \mathcal{L}$ for $j = 1, \ldots, N − 1$ as desired, completing the proof.

We note that $d_{r-k-j}^2 \neq d_{r+k+1}^2$ for some integer $k$ is always satisfied if the system dimension is odd, $N = 2\ell + 1$, no matter where we place the actuator. If $N = 2\ell$ then $d_{r-k-j}^2 = d_{r+k+1}^2$ for all $k$ is possible only if $r = \ell$, i.e., if the actuator is placed in the middle, and the coupling constants are symmetric around the centre, $d_{r-k}^2 = d_{r+k}^2$ for all $k$. For a spin chain with strictly isotropic Heisenberg interaction the requirement $\omega_r \neq 0$ is still a problem if $d_{r-1} = d_{r+1}$ but controllability could be restored by engineering a local perturbation of the energy levels in the vicinity of the actuator, which may indeed achieved by the actuator itself.

We can interpret these results in terms of transition graphs. Given a system with a tridiagonal drift Hamiltonian $H_0$ with respect to some Hilbert space basis \{|n\> : 1, \ldots, N\}, we can define a transition graph as before by taking the $N$ basis states as vertices and adding edges for each non-zero transition. Since the Hamiltonian is tridiagonal the resulting graph is either a linear chain or disconnected. Connectedness is a necessary condition, and the results above guarantee controllability in the following cases:

- If the chain is connected and has odd length then the actuator can be placed anywhere provided the vertices associated with the controlled edge have different energy levels.

It is worth pointing out here that the transition frequencies of the system need not be distinct. In fact, the most of the energy levels can be degenerate as is usually the case for uniform spin chains. We only require that the vertices of the controlled transition have different energy levels, which can generally be achieved by placing the actuator near either end of the chain.

**IV. CONSTRUCTIVE CONTROL WITH SINGLE BINARY SWITCH ACTUATOR**

The results of the previous section suggest that a single actuator is often sufficient to achieve the same degree of controllability that is achievable with many local actuators. This result is not too surprising on purely Lie algebraic grounds considering that two randomly choosen Hermitian $N \times N$ matrices, generically, will generate the entire Lie algebra $\mathfrak{u}(N)$. Of course, the Hamiltonian matrices in our model system are far from random, and our Lie algebra calculations indeed show that for certain systems such as a spin chain with uniform, isotropic Heisenberg coupling between adjacent spins, controllability depends on the type of interaction and the placement of

![Diagram](file:///C:/Users/.../image.png)

**FIG. 2**: Controllability of a chain with single actuator: model systems (a) and (d) are controllable, (b) is controllable if $E_3 \neq E_4$, while (c) is not controllable even if $E_3 \neq E_4$ if chain has reflection symmetry, i.e., $|d_{2r-k}| = |d_k|$ and $E_{2r-k+1} = E_k$ for all $k$ as the associated dynamical Lie group has symplectic symmetry.

- If the chain is connected and has even length then we must ensure in addition that the system does not admit symplectic symmetry, which is guaranteed as long as the actuator is not placed precisely in the middle of the chain.
- For a uniform chain for which the energy levels in the interior of the chain are always degenerate, controllability is ensured by placing the actuator near either end of the chain.
the actuator. Nonetheless, that a single local actuator in many cases results in the same degree of controllability than, say, $2N - 1$ local actuators to individually control all of the energy levels and transitions, is rather surprising when one considers the substantially reduced control that such an actuator affords, and it begs the question whether it is possible to control such a system constructively, i.e., whether we can find a local control field $f(t)$ that achieves the desired global system dynamics, and if there exists a solution, whether it can be practically implemented.

The type of control functions that are feasible generally depends on the specifics of the system and actuator. For laser-controlled quantum dots, for example, the availability of pulse shaping technology and the demonstrated superiority of shaped pulses over simple pulses in certain settings, suggests optimization routines designed to find an optimal time-dependent pulse shape $f(t)$, and many such algorithms based on gradients and variational techniques have been proposed (see for example [15] [16] [17]). For many other systems, especially voltage gate controlled systems, however, it is generally difficult to implement complicated time-varying potentials, and simple, piecewise constant controls that can be approximated by square pulses are preferable. In the following we consider the simplest type of such an actuator, a binary switch that switches the voltage between two possible values, corresponding to two fixed Hamiltonians

$$H^{(1)} = A_0 + f_0 A_1,$$  \hspace{1cm} (11a)

$$H^{(2)} = A_0 + f_1 A_1.$$  \hspace{1cm} (11b)

Given a sequence of switching times $t = (t_1, \ldots, t_K)$ the corresponding evolution of the system is given by

$$U(t) = U^{(1)}(t_1)U^{(2)}(t_2) \ldots U^{(1)}(t_{K-1})U^{(2)}(t_K)$$  \hspace{1cm} (12)

where $U^{(m)}(t_k) = \exp(-it_k H^{(m)})$ for $m = 1, 2$. The control task in this case is reduced to find the switching times $t$ to accomplish a desired task. Although analytical expressions for the optimal switching times are generally very difficult to obtain for all but very simple systems, numerical optimization techniques can be used to find suitable controls, and we have found them to be surprisingly effective in many cases.

As a specific example, we consider the first excitation subspace of a spin chain of length four with a single binary switch actuator placed between spins one and two, i.e., $r = 1$. This system is controllable with a single actuator at $r = 1$ according to Theorem 1. To show that we can constructively control this system with a single binary switch actuator, we find switching time sequences for a complete set of generators of $SU(4)$. Interpreting the first excitation subspace of the chain as a two-qubit system by setting

$$|0\rangle = |00\rangle, \quad |1\rangle = |01\rangle, \quad |2\rangle = |10\rangle, \quad |3\rangle = |11\rangle,$$

we show that it is possible to find vectors $t^{(s)}$ such that

$$\|U_T^{(s)} - U(t^{(s)})\| \leq 10^{-4}$$  \hspace{1cm} (13)

for the following set of six target operators

$$U_T^{(s)} \in \{ I \otimes I, \quad \text{Had} \otimes I, \quad T \otimes I, \quad I \otimes \text{Had}, \quad I \otimes T, \quad \text{CNOT} \},$$  \hspace{1cm} (14)

where $I$ is identity operator on a single two-level subspace (qubit), $T = \exp(-i\pi/8\sigma_z)$ is a $\pi/8$ phase gate, and Had and CNOT are the Hadamard and CNOT gate, respectively,

$$\text{Had} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{CNOT} = e^{-i\pi/4} \begin{pmatrix} I_2 & 0 \\ 0 & \sigma_z \end{pmatrix},$$

with $\sigma_x$ and $\sigma_z$ being the usual Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of target operators (14) was chosen because it is a universal set of elementary gates in that any other $SU(4)$ operator can be constructed from these elementary gates, and the ability to implement a universal set of gates for $SU(4)$ is equivalent to (density operator) controllability of the system.

Table I shows the time vectors $t^{(s)}$, as well as the gate operation times $T = \sum_{k=1}^{K} t_k$ and gate errors as defined above for a system with uniform isotropic Heisen-burg coupling. The optimization was performed using a Nelder-Mead downhill simplex algorithm [18] with multiple initial simplices. The table shows that it is possible to implement all of the six elementary gates with a fidelity $\geq 99.99\%$ with no more than 20 switches of a single on-off switch actuator in approximately 40 time units each, a surprisingly good result considering the minimal nature of the available control. Since solutions are obviously not unique, and the minimum control time or number of switches required to achieve the control objective are unknown, even better solutions probably exist. The non-uniqueness of the solutions can be exploited to satisfy additional constraints such as minimum pulse lengths (switching cannot be arbitrarily fast) etc.

V. CONCLUSION

We have shown that a certain class of systems of Hilbert space dimension $N$ is controllable with a single local actuator. In particular, the results show that it is usually not necessary to be able to control all transitions, and a single local actuator in fact suffices in most cases to achieve controllability. That is to say, we do not require $N - 1$ or more independent local actuators, or global actuators acting on the entire system.

The results establish theoretical minimum requirements for controllability for a class of systems that includes many types of spin chains and other systems with
| Gate | Had | T | CNOT |
|------|-----|---|------|
| error | 6.04207e-05 | 9.93462e-06 | 9.41944e-10 | 8.86637e-07 | 1.10773e-06 | 1.31773e-06 |
| duration | 40.5351 | 37.9537 | 41.166 | 41.1328 | 42.5368 | 39.3569 |
| t_1 | 0.731996 | 3.94518 | 1.79446 | 3.08601 | 1.30764 | 3.34576 |
| t_2 | 2.03884 | 2.20021 | 1.79932 | 3.13305 | 1.10690 | 0.0179813 |
| t_3 | 3.52727 | 0.0384191 | 0.0730935 | 0.701478 | 0.518925 | 2.59171 |
| t_4 | 1.38628 | 1.07432e-07 | 1.71885 | 3.62498 | 5.85085 | 3.23448 |
| t_5 | 3.39919 | 0.680856 | 2.07051 | 2.45712 | 0.396729 | 1.46693 |
| t_6 | 0.951534 | 3.04816 | 0.747468 | 0.68558 | 7.37392 | 0.212212 |
| t_7 | 1.35113 | 1.292 | 1.84047 | 1.3746 | 1.23031 | 5.07851 |
| t_8 | 0.575672 | 1.86256 | 2.53341 | 1.12801 | 1.16712 | 3.07975 |
| t_9 | 3.38307 | 4.14879 | 4.73792 | 3.50997 | 0.802765 | 2.75667 |
| t_{10} | 0.0365974 | 0.356856 | 1.3432 | 1.92944 | 4.08279 | 0.439889 |
| t_{11} | 3.62131 | 1.02202 | 1.39084 | 5.57969 | 1.27132 | 3.25423 |
| t_{12} | 0.93505 | 0.0453206 | 0.320722 | 0.298252 | 2.70023 | 2.41685 |
| t_{13} | 1.75377 | 2.13701 | 4.15595 | 0.987279 | 4.67647 | 1.04768 |
| t_{14} | 5.19515 | 1.24291 | 0.533115 | 0.26934 | 0.705919 | 1.31426 |
| t_{15} | 4.50999 | 0.101593 | 1.03574 | 1.7998 | 1.01477 | 2.6859 |
| t_{16} | 1.01899 | 4.40131 | 7.58673 | 4.66334 | 1.78438 | 0.732592 |
| t_{17} | 4.01314 | 1.07241 | 4.77061 | 0.135612 | 1.02283 | 0.16703 |
| t_{18} | 0.991019 | 5.83516 | 0.857316 | 1.31499 | 1.02426 | 0.770284 |
| t_{19} | 0.705316 | 1.62299 | 1.7735 | 3.38444 | 2.16687 | 2.32287 |
| t_{20} | 0.409887 | 2.89999 | 0.082803 | 1.07044 | 2.43177 | 2.41471 |

TABLE I: Gate errors (1−gate fidelity), total time T required to implement respective gates, and vector of switching times t_k to implement a universal set of elementary gates with 20 switches for (the first excitation subspace of) a uniform isotropic Heisenberg spin chain of length four.

As systems with fixed interactions are generally much easier to engineer than systems with individually tunable transitions, this is a promising result. Although the controllability proof is an existence proof, we have further demonstrated that is is possible to constructively control a system with a local actuator, even if the actuator is limited to a binary switch, for a four-level system, where we have shown that it is possible to implement a complete set of generators of $SU(4)$ with fidelities ≥ 99.99% using a single binary switch actuator, with no more than 20 switches per gate required. Although it would be desirable to have analytic expressions for the switching times, it appears that numerical optimization techniques are quite effective in finding suitable controls.

Numerical simulations extending the technique to systems with non-tridiagonal Hamiltonians suggest that constructive control is generally still possible, and similar strong controllability results can almost certainly be obtained using very similar arguments for these systems. This class would include for example interesting systems such as spin-chains with non-nearest neighbour couplings. Beyond the extension of generic controllability results to other classes of Hamiltonians, interesting questions for future work—the answers to which could point the way to achieving effective control with much simpler control system designs—include what type of systems can be effectively controlled with a single local actuator, whether the placement of the actuator matters, and how different forms of coupling between the system and the actuator affect the control outcomes in practice.

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