M-theory and automorphic scattering

Gordon Chalmers

Argonne National Laboratory
High Energy Physics Division
9700 South Cass Avenue
Argonne, IL 60439-4815
e-mail: chalmers@pcl9.hep.anl.gov

Abstract

The strongly coupled limit of string scattering and the automorphic construction of the graviton S-matrix is compared with the eleven dimensional formulation of M-theory. In a particular scaling limit at strong string coupling, M-theory is described by eleven-dimensional supergravity which does not possess a dilaton, but rather a perturbative expansion in the gravitational coupling and derivatives. The latter theory provides an off-shell description of the string, upon dimensional reduction.
1 Introduction

M-theory has emerged as the unifying framework of the five fundamental consistent string theories [1]. At low energies, M-theory limits to $\mathcal{N} = 1$ eleven dimensional supergravity [2]; more precisely graviton scattering elements have been computed to an order of twelve derivatives where it has been shown to be in agreement with the superstring theory to this order upon dimensional reduction. $\mathcal{N} = 1$ $d = 11$ supergravity contains the massless gravitational multiplet, and massive multiplets have not been consistently formulated in $d = 11$ which is a strong constraint on the matter structure (there is one exception for a massless matter multiplet [3].) In addition, string theory graviton scattering elements must obey consistency requirements with respect to the U-duality groups [4]; the strongly coupled limit of IIA on a circle (T-dualized to IIB superstring theory on a circle) defines M-theory at all scales. Given the consistency with supersymmetry and automorphic functions, a question is to what extent M-theory is described solely by eleven dimensional supergravity (an examination of some of the superspace constraints is found in [5, 6, 7], and there, further corrections beyond the massless gravitational multiplet are not permitted). A consistent quantum field theory description of scattering amplitudes also generates an off-shell description of string theory scattering elements.

M-theory is built from the U-duality structures, and the S-matrix which is compliant with S- and U-duality may be expanded in the strongly coupled regime to obtain a perturbative description of M-theory. M-theory is known to agree with maximal supergravity in $d = 11$ up to twelve derivatives [8, 11, 14], as explicit computations in supergravity and superstring theory indicate [11, 3, 12, 13, 14, 15]. We examine the perturbative form of the automorphic functions in this limit and show compatibility with $d = 11$ supergravity. A question we address is how the description differs from maximal $\mathcal{N} = 1$ ($\mathcal{N} = 32$) supergravity in eleven dimensions within a specific first-quantized string-inspired regulator (the form of which is very controlled by supersymmetry constraints). The generating functional discussed previously is a means to deriving dynamics at all values of the coupling constants in addition to providing an off-shell description.

The automorphic string scattering reformulates the expansion at multi-genera and packages coefficients in the derivative expansion (including instanton results) in terms of automorphic functions. Taking these results and demanding self-consistency allows for a determination of these coefficients without resort to the path integral.

$\textsuperscript{1}$These calculations prohibit massive modes from contributing at tree-level in eleven dimensions, as would be expected based on general covariance.
The outline of this work is as follows. In section 2 we review the automorphic basis upon which a string S-matrix may be expanded, with duality manifest. In section 3 we examine the perturbative structure of M-theory arising from the automorphic functions and demonstrate compatibility with supergravity. In section 4 we take the strongly coupled limit of the T-dualized formulation of this theory, IIA to M-theory. We then briefly comment on the reformulated string expansion and sewing to obtain these coefficients.

2 Automorphic scattering of graviton scattering

We briefly review the construction of the S-duality compliant graviton scattering at the four-point order in type IIB superstring theory (reviewed in [16, 17] to eight derivatives) at all genera [18]. T-dualizing along a compact direction generates the scattering in type IIA. The automorphic construction relies upon a basis of automorphic functions that is compatible with the perturbative structure of the superstring (see [19] for a review of these functions). A basis is determined from the ring of functions,

\[ g_k \in \prod_{j=1}^{n} E_{s_j, -s_j}(\tau, \bar{\tau}) , \]

satisfying the properties,

\[ \sum_{j=1}^{n} q_j = 0 \quad \sum_{j=1}^{n} s_j = s , \]

and invariance under

\[ \tau \rightarrow \frac{a + b\tau}{c + d\tau} , \]

with \( bc - ad = 1 \). The first property enforces modular invariance, and \( q_i \) is related to the \( U(1)_R \) charge. The second parameterizes the asymptotic structure. The ring of functions is specified by partitioning an integer \( s \) into half-integers with \( |q| \leq s \), and the functions themselves are algebraic,

\[ E_{s}(q, -q)(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{(m + n\tau)^{s+q}(m + n\bar{\tau})^{s-q}} . \]

The asymptotics of these functions are,

\[ E_{s}(q, -q)(\tau, \bar{\tau}) = g_0\tau_2^s + g_1\tau_2^{1-s} + O(e^{-2\pi\tau_2}) , \]
where the coefficients in (2.4) are,

\[ g_0 = 2\zeta(2s) \quad g_1 = 2\sqrt{\pi}\zeta(2s - 1) \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.6} \]

and describe the perturbative terms together with exponentially decaying instanton (and wrapped solitons on compact spaces) corrections to graviton scattering. We restrict to the cases of half-integral or integral \( s \) and do not include the cusp forms; cusp functions have only instantonic or exponential terms in the asymptotic expansion. The functions in the ring in (2.1) have the asymptotics,

\[ g_k = g_k^{(0)} \tau_2^{\frac{3}{2} + \frac{h}{2}} + g_k^{(1)} \tau_2^{-\frac{1}{2} + \frac{h}{2}} + \ldots + g_k^{(h)} \tau_2^{\frac{3}{2} + \frac{h}{2} - 2h} + O(e^{-2\pi\tau_2}) , \tag{2.7} \]

where \( h \) is correlated with a maximum genus on the string S-matrix expansion. S-duality and a direct convergence of the dual planar limit of \( \mathcal{N} = 4 \) super Yang-Mills theory defined through covariantized holographic string scattering \cite{20} in the AdS/CFT correspondence \cite{21, 22, 23} support that there are no further coefficients beyond the maximum genus \( h \), \( h = \frac{1}{2}(k + 2) \) for \( k \) even and \( \frac{1}{2}(k + 1) \) for \( k \) odd; further modular forms in the complete basis, (2.4), composed from \( s \) on the complex plane do not agree in general with the perturbative structure of the string.

These modular invariant functions may be generalized from the fundamental domain \( U(1) \setminus SL(2, R)/SL(2, Z) \) describing the vacuum of uncompactified type IIB superstring theory to toroidally compactified spaces, and pertain to other examples, including type IIB propagating on K3\footnote{The metric on K3 is not known, but the moduli space of manifolds is known, being a simple coset, and the latter is sufficient to defining automorphic functions on a K3. This potentially allows for a reconstruction of the K3 metric from the scattering.} The graviton scattering S-matrix must be expanded on these functions. At the four-point order, on-shell supersymmetry restricts the tensor structure to be composed of the Bel-Robinson form \cite{24}, with the tensor \( R^4 \) built out of contracting eight Weyl tensors as in \cite{25},

\[ S = \int d^{10}x \sqrt{g} \left[ \frac{1}{3!} R^4 + \sum_{n=0}^{\infty} g_k(\tau, \bar{\tau}) (\alpha' \Box)^n R^4 \right] , \tag{2.8} \]

where the tensor structure of the boxes is left undetermined. At tree-level the derivative structure has the form,

\[ s^k + t^k + u^k , \tag{2.9} \]

but in general there are mixed products of the invariants,

\[ s^{k_1} t^{k_2} u^{k_3} \quad k = k_1 + k_2 + k_3 . \tag{2.10} \]
The $R^4$ tensor has the form,

$$t^\mu_{\nu_1...\nu_8} = \frac{1}{2} \delta^\mu_{\nu_1...\nu_8}$$

(2.11)

with

$$t^\mu_{\nu_1...\nu_8} = -\frac{1}{2} \epsilon^\mu_{\nu_1...\nu_8}$$

(2.12)

and

$$-\frac{1}{2} \left( (\eta^\mu_{\nu_1...\nu_4} \eta^{\nu_5...\nu_8} - \eta^\mu_{\nu_1...\nu_8} \eta^{\nu_5...\nu_4}) (\eta^\mu_{\nu_1...\nu_5} \eta^{\nu_7...\nu_8} - \eta^\mu_{\nu_1...\nu_8} \eta^{\nu_7...\nu_5}) - (\eta^\mu_{\nu_1...\nu_5} \eta^{\nu_4...\nu_8} - \eta^\mu_{\nu_1...\nu_8} \eta^{\nu_4...\nu_5}) (\eta^\mu_{\nu_1...\nu_4} \eta^{\nu_6...\nu_8} - \eta^\mu_{\nu_1...\nu_8} \eta^{\nu_6...\nu_4}) + (\eta^\mu_{\nu_1...\nu_4} \eta^{\nu_5...\nu_8} - \eta^\mu_{\nu_1...\nu_8} \eta^{\nu_5...\nu_4}) + \text{perms} \right)$$

(2.13)

(2.14)

(2.15)

(2.16)

There are non-analytic terms that may be constructed by unitarity from the local terms in (2.8), and the form in (2.8) has manifest duality invariance under the fractional linear transformations when the energy scales are below the regime $s_{ij} \leq 4/\alpha'$. The massive modes appear upon resumming the higher derivative terms, as in the expansion of $\ln(1 - \alpha's)$.

The generalization to the $SL(d, Z)$ modular forms is parameterized by the moduli of a $d$-dimensional torus and in the limit of one of the radii becoming large we have the expansion \[13, 14,\]

$$E_{R=d,s}^{SL(d,Z)}(\phi_j) = E_{R=d-1,s}^{SL(d-1,Z)}(\phi_j) + \frac{2\pi^s \Gamma(s - \frac{d-1}{2}) \zeta(2s - d + 1)}{\pi^{s-\frac{d-1}{2}} \Gamma(s) R^{2s-d+1} V_{d-1}}$$

(2.17)

$$+ \sum_{m^n, n^b} \frac{n^{a} g_{ab} n^{b}}{m^2} |K_{s-\frac{d-1}{2}} (2\pi |m| R \sqrt{n^{a} g_{ab} n^{b}})|$$

(2.18)

In the case of $SL(2,Z)$ the expansion relevant for the M-theory limit from IIB superstring theory compactified on a circle of radius $R$ is,

$$E_{R=2,s}^{SL(2,Z)}(\phi_j) = \frac{2\pi^s \Gamma(s - \frac{d-1}{2}) \zeta(2s - d + 1)}{\pi^{s-\frac{d-1}{2}} \Gamma(s) R^{2s-d+1} V_{d-1}}$$

(2.19)

$$+ \sum_{m^n, n^b} \frac{n^{a} g_{ab} n^{b}}{m^2} |K_{s-\frac{d-1}{2}} (2\pi |m| R \sqrt{n^{a} g_{ab} n^{b}})|$$

(2.20)

with $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}$. The first term is relevant for the eleven-dimensional limit and the latter terms decouple.
3 Eleven dimensional quantum limit

The IIB superstring compactified on a circle and T-dualized describes the M-theory limit upon taking the compact volume to infinity. The coupling constants are identified as

\[ V = R_{10} R_{11}, \quad \tau = C^{(0)} + i \frac{R_{10}}{R_{11}} = \frac{\theta}{2\pi} + i \frac{4\pi}{g_s^2}, \quad (3.1) \]

\[ R_{11} = e^{2\phi A/3}, \quad R_{10} = r_A e^{\phi A/3}, \quad (3.2) \]

and

\[ l_{11} = (g^A)^{\frac{4}{3}} l_{10}, \quad (3.3) \]

where \( r_A \) is a dimensionless radii of the tenth dimension as measured in Planck units.

Furthermore, the kinematic invariants in the ten-dimensional Einstein frame of the string and the eleven dimensional supergravity ones are,

\[ s_{ij} = S_{ij} \frac{l_{11}^2}{l_{10}^2 R_{11}}, \quad (3.4) \]

in the string frame with the inverse power of the eleven dimensional radius (at finite values of the compact eleven dimensional radius) arising from the inverse metric in \( S_{ij} = -G^{\mu\nu}(k_i + k_j)_{\mu}(k_i + k_j)_{\nu} \). It will be useful for us to S-dualize the torus so that the coupling has the inverse \( \tau_2 \rightarrow 1/\tau_2 \). Then a term generated at large volume from the automorphic functions has the coupling dependence from,

\[ E_s = 2\zeta(2s) \left( \frac{R_{11}}{R_{10}} \right)^s + 2\sqrt{\pi} \zeta(2s - 1) \left( \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right) \left( \frac{R_{11}}{R_{10}} \right)^{s-1} + O(e^{-2\pi \tau_2}), \quad (3.5) \]

\[ g_k(\tau, \bar{\tau}) l_{10}^{2k-1} \Box^k R^4 \rightarrow l_{10}^{2k-1} \Box^k R^4 = l_{10}^{2k-1} \left( \frac{R_{11}}{R_{10}} \right)^{-3-k+2g_{\text{max}}} \Box_{11}^k R_{11}^4, \quad (3.6) \]

where a transformation to string frame from Einstein frame is taken and \( g_{\text{max}} = \frac{1}{2}(k + 2) \) for g even and \( \frac{1}{2}(k + 1) \) for g odd. There are subleading terms in (3.6) and a factor of \( \sqrt{\tau_2} \) is also absorbed into the latter limit from the determinant of the metric.

We next examine the coupling structure in the loop expansion of supergravity.

Translating to loop counting via the Einstein-Hilbert term, \( \frac{1}{l_{11}^2} \int d^{11} x \sqrt{g} R \), generates the power series from the above ten-dimensional derivative terms,

\[ l_{11}^{2k-1+5L} g_A^{\frac{2k-1}{3}} \left( \frac{R_{11}}{R_{10}} \right)^{-3-k+2g_{\text{max}}} \Box_{11}^k R_{11}^4 l_{11}^{-5L}, \quad (3.7) \]
where a factor of $l^{-5L}$ is taken out of the prefactor. This latter factor nullifies the primitive divergence in a $\phi^3$ graph in eleven dimensions; the divergence structure of maximal supergravity is conjectured to be that of $\phi^3$ theory with $2(L-1)$ kinematic invariant factors in front when regulated in a string inspired regulator (e.g. $R^4$ at one-loop, $s^2R^4$ at two loops, etc...) \cite{24,27}. The string-inspired regulator is elucidated in \cite{14} at two-loops. The divergence structure in eleven dimensions of maximal supergravity then agrees with the couplings in (3.7) if $k = 2L$, as the coupling constant in maximal supergravity in eleven dimensions is $\kappa_{11}^2 = l_{11}^9$. Thus the maximal loop graph contributing according to the M-theory limit is in agreement with the conjectured form of the supergravity expansion, after including the additional eleventh dimensional volume element. (The additional factor of $l_{11}^{-1}$ is a remnant of taking the limit from nine dimensions and two additional powers appear in formulating in eleven dimensions.)

Example maximum couplings up to order $k = 4$ are,
\begin{align}
\frac{1}{\Box^3}R^4 : & \quad L = 0, \\
E_2 R^4 : & \quad L = 1, \\
E_2 \Box^2 R^4 : & \quad L = 2, \\
E_3 \Box^3 R^4, |E_{3/2}^{(1,-1)}|\Box^3 R^4 : & \quad L = 2, \\
\ldots : & \quad \ldots ,
\end{align}
and at higher powers the terms are descending power series in multiples of four. The genus one and two results for the $E_{5/2}$ function have been examined in \cite{12,13,14,15}. The $1/\Box^3 R^4$ is two derivatives and arises at tree-level in the graviton scattering. The string perturbative expansion is in powers of $\tau_2^2$. This table is in agreement with the known supergravity structure in $d = 11$ dimensions. The coefficients at arbitrary genera are not known but the cancellation properties of the expansion are in agreement with this power series and the instanton corrections.

Matrix theory \cite{28} is a partonic description of the D-brane in M-theory, defined by taking the dimensional reduction of ten-dimensional supergravity, in the same vein as Nahm’s equations are the dimensional reduction of gauge theory and describe solitons. The automorphic construction, and the quantum regulated supergravity, agrees with the eleven dimensional origin of Matrix theory in the large $N$ limit. Here the analogy is between four-dimensional gauge theory and eleven dimensional supergravity and the states of the dyons or solitons in the gauge theory or string theory described by the dimensional reduced theory to one dimension.
4 Recursive construction of the amplitudes

The automorphic properties of the scattering and a recursive formulation without a direct resort to the functional integral also obey this correspondence with the $N = 1 d = 11$ supergravity. In this section we reformulate the expansion and derive a recursive algorithm for the coefficients. We expand on this in future work [29].

The basic identity for recursively deriving the coefficients follows from sewing the S-matrix onto itself: $\sum SS = S$ with a summation over the intermediate lines both in number and particle type. This sewing relation involves only one-loop integrals in order to obtain multi-genus string theory results; the string loop integrals have been performed and encoded in the automorphic function basis, extracting the manifest non-perturbative duality and symmetries of the string as well as the instanton coefficients related to the genus expansion via modular invariance. The primary complication is the tensor contractions associated with the massive string modes, however, string field theory techniques may be adapted to generate this sewing.

The generating functional for $\Phi^3$ theory is an example that does not have the complications with the tensors. We integrate out the loops and perform a derivative expansion, obtaining the on-shell amplitudes,

$$A_4(x_j) = \sum_{k=2}^{\infty} \int A_{2+k}(x_1, x_2; y_j) \prod_{j=1}^{k} \Delta(y_j - z_j) \left| A_{2+k}(z_j; x_3, x_4) \right|_{y_j = y, z_j = z}$$ (4.1)

power series expanded in derivatives from,

$$A_n = \sum_{a,b} c^b_{a,n} t^b_{a,n}$$ (4.2)

with the invariants,

$$t_{i,m} = (k_i + \ldots + k_{i+m-1})^2 \quad (t_{i,2} = s_{i,i+1})$$ (4.3)

(the series in (4.2) is a truncated one and not the most general). The graphs in (4.1) is the s-channel recursion. Inserting the derivative expansion of the amplitude into the sewing relation (4.1) gives recursive relations relating loops at different orders ($l_1 + l_2 = l_3$ with $l_1$ and $l_2$ the loop orders of the two amplitudes),

$$\sum_{a,b} t^{b}_{a;4} t^{b}_{a;4} = \sum_{i,l,l,m} \int c_{i,m} \left( \sum_{n=1}^{m-1} \partial^{i}_{i+n} \right)^{2l} \prod_{j=1}^{k} \Delta(y_j - z_j) \sum_{l', i} \left( \sum_{n=1}^{m-1} \partial^{x}_{i+n} \right)^{2l'} \left| y_j = y, z_j = z \right.$$(4.4)

integrated with $d^4(y - z)$ and the directions of the derivatives implied on the propagators and the uninserted external lines. The derivatives in (4.4) and their tensor
form are written in compact form because of the many propagators. The $k$ massive propagators runs up to $b - 2l - d$, with $d$ the dimension of the integrated $d^l(y - z)$ spacetime. The expression is more familiar in k-space, however, in x-space the integrals are more explicit; An interesting feature is that in k-space the integrals are multi-loops and complicated to evaluate but in x-space all of the integrals are one-loop and may be performed explicitly. The coefficients may be iteratively constructed from one loop order to the next. This is the sewing relation we generalize now to the context of the superstring, which lifts directly into eleven dimensions, together with manifest duality properties encoded in the automorphic functions. In the superstring this formalism also avoids complications with superconformal ghost systems; however, superconformal insertions are potentially useful in mechanizing cancellations in supergravity from the superstring [27].

The graviton scattering from the covariantized S-matrix in (2.8) is generated by the derivatives,

$$A_4(k_i, \epsilon_i) = \prod_{j=1}^{4} \epsilon_{\mu_j \nu_j} \frac{\partial}{\partial g_{\mu_j \nu_j}} S[\hat{g} + \eta]|_{\hat{g}=0}, \quad (4.5)$$

after including the non-analytic terms derived from iterating unitarity cuts in (2.8) [18]. The four-point amplitude in (4.5) holds at arbitrary genera and in order to formulate the sewing for the generating functional we require the similarly formulated amplitudes with massive external string states and higher-point amplitudes.

As an example, the four-point classical Veneziano-Shapiro amplitude [25] in the limit $\alpha' \to 0$ is, in string frame,

$$A_4 = e^{-2\phi} R^4 \frac{\Gamma(1 + \alpha's)\Gamma(1 + \alpha't)\Gamma(1 + \alpha'u)}{\Gamma(1 - \alpha's)\Gamma(1 - \alpha't)\Gamma(1 - \alpha'u)} = e^{-2\phi} R^4 + \ldots, \quad (4.6)$$

with a derivative expansion from the covariantized scattering amplitude,

$$A_{4,g=0}^{IIB} = 64e^{-2\phi} \frac{R^4}{\alpha'^4stu} \exp\left(\sum_{p=1}^{\infty} \frac{2\zeta(2p + 1)}{2p + 1} \left(\frac{\alpha'}{4}\right)^{2p+1} \left(s^{2p+1} + t^{2p+1} + u^{2p+1}\right)\right). \quad (4.7)$$

The first term in (4.6) is deduced from sewing the three- and four-point variations of $\int d^{10}x \sqrt{g} R$. The expansion of the (4.6) gives contributions at all derivative orders in the expansion of (4.3).

An individual term in the expansion is, with $g = \eta + \hat{g}$,

$$V_{\{\mu_j \nu_j\}} = \prod_{j=1}^{4} \frac{\partial}{\partial \eta_{\mu_j \nu_j}} \sqrt{g} \eta^m R^4, \quad (4.8)$$
with a modular function prefactor. The higher point vertices we define in the following with string field techniques. Similar construction of the vertices can be found from the Koba-Nielsen formula. The propagators in de Donder gauge are

$$\Delta_{\mu_1 \nu_1, \mu_2 \nu_2}(x_1, x_2) = -\frac{1}{2} \frac{1}{(x_1 - x_2)^2} \left( \partial_\mu g_{\nu \sigma} + \partial_\nu g_{\mu \sigma} - \partial_\sigma g_{\mu \nu} \right)$$

(4.9)

The contribution to the four-point function is

$$A_4 = \tau_2^{\frac{3}{2} + \frac{1}{2} - 2g_1} \tau_2^{\frac{3}{2} + \frac{1}{2} - 2g_2} \varepsilon_{\mu_1 \nu_1} \varepsilon_{\mu_2 \nu_2} \varepsilon_{\alpha_3 \beta_3} \varepsilon_{\alpha_4 \beta_4} k_1^{\mu_1} k_1^{\nu_1} k_2^{\mu_2} k_2^{\nu_2} k_3^{\alpha_3} k_3^{\beta_3} k_4^{\alpha_4} k_4^{\beta_4}$$

$$\times \int d^{10} x V_{\mu_1 \nu_1, \mu_2 \nu_2, \mu_3 \nu_3, \mu_4 \nu_4} |_{x_1} \Delta_{\mu_3 \nu_3, \mu_3 \beta_3} \Delta_{\mu_4 \nu_4, \mu_3 \beta_4} V_{\alpha_1 \beta_1, \alpha_2 \beta_2; \alpha_3 \beta_3, \alpha_4 \beta_4} |_{x_2} ,$$

(4.10)

with \( x = x_1 - x_2 \). From the genus \( g_1 \) and \( g_2 \) results including the supersymmetric intermediate states, we may obtain a contribution to the genus \( g_1 + g_2 \) graviton scattering contribution.

$$A_4 = t_8, \mu_1, ... \mu_8 t_8, \nu_1, ... \nu_8 k_1^{\mu_5} k_1^{\nu_5} k_2^{\mu_6} k_2^{\nu_6} k_3^{\mu_7} k_3^{\nu_7} k_4^{\mu_8} k_4^{\nu_8} I(k_j) .$$

(4.11)

The inclusion of the massive multiplets allows for the derivation of arbitrary genus results. The sewing relations of the string scattering lift into the eleven-dimensional limit within the derivative expansion (where the massive string modes, or more generally branes on cycles, decouple).

### 5 Discussion

We have examined the eleven-dimensional quantum limit from the automorphic construction of the IIB superstring S-matrix elements. The limit generates a route to a description of M-theory in terms of perturbative eleven dimensional supergravity. Upon toroidal compactification the automorphic S-matrix elements map to string theory ones, as verified explicitly to twelve derivatives, and permits an off-shell description of the string scattering through a quantum field theory description. Further calculations at three-loops in supergravity or at genus two are required to test the duality equivalence. The coefficients in the expansion of the Eisenstein functions are generated from the massive and massless multiplets in the string, but the coefficients are in correspondence with the quantum eleven dimensional massless limit. The recursive construction of the amplitudes following from sewing the derivative expanded form of the S-matrix generates multi-genus results from field theory ones \(^{[29]}\) and is examined briefly in this work.
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