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Partial Eigenvalue Assignment for Gyroscopic Second-Order Systems with Time Delay

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Abstract: In this paper, the partial eigenvalue assignment problem of gyroscopic second-order systems with time delay is considered. We propose a multi-step method for solving this problem in which the undesired eigenvalues are moved to desired values and the remaining eigenvalues are required to remain unchanged. Using the matrix vectorization and Hadamard product, we transform this problem into a linear systems of lower order, and analysis the computational costs of our method. Numerical results exhibit the efficiency of our method.

Keywords: inverse problem; gyroscopic second-order systems; partial eigenvalue assignment; time delay

1. Introduction

Vibration is often the direct cause of malignant damage to most engineering structures such as aviation, aerospace, vessel, mechanical, electron, large bridges and super high-rise buildings [1–3]. Therefore, vibration design and control are critical in large structural designs. In order to combat undesirable effects of vibrations, caused by a few undesired eigenvalues of the systems, we need to reassign those undesired eigenvalues, leaving the rest unchanged, by using a suitable feedback control. This is a possible way to construct feedback control, especially leasing the rest eigenpairs unchanged. However, in the process of structural vibration control, each link, such as the measurement of the sensor, the calculation of the processor and the drive of the actuator, would consume time, resulting in a time delay of the control force.

In this paper, we consider the following gyroscopic second-order systems in control

\[ M \ddot{z}(t) + G \dot{z}(t) + Kz(t) = Bu(t - \tau), \]

where \( M, G, K \in \mathbb{R}^{n \times n} \) are, respectively, mass, gyroscopic and stiffness matrices with \( M \) symmetric positive definite, \( G \) skew-symmetric and \( K \) symmetric nonsingular; \( z(t) \in \mathbb{R}^n \) and its time derivatives are vectors of displacement, velocity and acceleration respectively. \( B \in \mathbb{R}^{n \times m} \) is the full column rank control matrix, \( u(t - \tau) \in \mathbb{R}^m \) is the control vector and \( \tau > 0 \) is time delay. The associated open loop pencil is given by \( P(\lambda) = \lambda^2 M + \lambda G + K \).

In practice, helicopter rotor blades or spin stabilized satellites with elastic appendages such as solar panels or antennas can be seen as the gyroscopic systems. In general, gyroscopic systems are second-order systems, and the problems of gyroscopic second-order systems have aroused much public attention [4–6].

One of feedback control is the following state feedback control

\[ u(t - \tau) = F_1^T \dot{z}(t - \tau) + F_2^T \ddot{z}(t - \tau) + F_3^T z(t - \tau), \]
where $F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$ are, respectively, acceleration, velocity, displacement state feedback matrices. In order to design the controller conveniently, the feedback matrices $F_1, F_2, F_3$ are constant matrices \([7,8]\). Note that Equation (1) can be replaced by the following equation

$$M \ddot{z}(t) + G \dot{z}(t) + Kz(t) = B \left[ F_1^T \ddot{z}(t) + F_2^T \dot{z}(t - \tau) + F_3^T z(t - \tau) \right].$$  

(3)

Separation of variables

$$z(t) = xe^{\lambda t},$$  

(4)

where $\lambda \in \mathbb{C}, x \in \mathbb{C}^n$, then Equation (3) yields the associated quadratic eigenvalue problem with time delay

$$P_t(\lambda) z(t) = 0,$$  

(5)

where $P_t(\lambda) = \lambda^2 (M - BF_1^T e^{-\lambda \tau}) + \lambda (G - BF_2^T e^{-\lambda \tau}) + (K - BF_3^T e^{-\lambda \tau})$.

For $\tau = 0$, Datta, Ram and Sarkissian \([9]\) solved the multi-input partial pole placement problem for undamped gyroscopic systems and gave the explicit solution of this problem. However, the gyroscopic second-order systems with time delay may have infinite eigenvalues so that the solution space of feedback systems with time delay may have infinite basis. This is the essential difference from feedback systems without time delay. The time delay not only weakens the dynamic characteristics of the control systems, furthermore, it causes a series of problems such as system stability and bifurcation. The partial eigenvalue assignment problem for gyroscopic second-order systems with time delay is to find the matrices $F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$ such that a few undesired eigenvalues of the open loop pencil

$$P(\lambda) = \lambda^2 M + \lambda G + K,$$  

(6)

are altered as required and the resting eigenpairs remain unchanged. These lead to the following problem.

**Problem GPEAP-TD.** Given the system matrices $M, G, K \in \mathbb{R}^{n \times n}$ with $M$ symmetric positive definite, $G$ skew-symmetric and $K$ symmetric nonsingular, the full column rank control matrix $B \in \mathbb{R}^{n \times m}$, $\tau > 0$ is the time delay, and given the self-conjugate subset $\{\lambda_i\}_{i=1}^p (p \ll n)$ of the open-loop eigenvalues $\{\lambda_i\}_{i=1}^n$ and the corresponding eigenvector set $\{x_i\}_{i=1}^p$ and a self-conjugate set $\{\mu_i\}_{i=1}^p$, find the state feedback matrices $F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$ such that the closed-loop pencil

$$P_t(\lambda) = \lambda^2 (M - BF_1^T e^{-\lambda \tau}) + \lambda (G - BF_2^T e^{-\lambda \tau}) + (K - BF_3^T e^{-\lambda \tau})$$

has the desired eigenvalues $\{\mu_i\}_{i=1}^p$ and the eigenpairs $\{\lambda_i, x_i\}_{i=p+1}^{2n}$.

Note that the number of eigenvalues of the open loop pencil $P(\lambda)$ is finite, but the number of eigenvalues of $P_t(\lambda)$ is infinite. Problem GPEAP-TD is to find state feedback matrices $F_1, F_2, F_3$, such that the $2n - p$ eigenpairs of $P(\lambda)$ remain unchanged. One method for solving the Problem GPEAP-TD is to transform the quadratic control problem to a standard first-order control problem and then solve the partial eigenvalue assignment problem for the first-order systems. However, there are several computational problems with this approach. For instance, it would compute the inverse of mass matrix, which may be ill-conditioned. Moreover, this transformation would, in most cases, destroy all the matrix structures inherent in most practical problems, such as symmetry, definiteness, sparsity, etc. \([10]\). Ram and Mottershead first proposed the receptance method in active vibration control \([11]\) and solved the multi-input partial pole placement with active vibration control by using the receptance method \([12]\). The research of receptance method can be seen in \([13–16]\). For gyroscopic systems, Datta \([17]\) researched the spectrum modification and further extended to the distributed parameter systems \([18]\). Liu \([19]\) proposed a multi-step method for solving the partial quadratic eigenvalue problem with time delay.

Our main contribution in this paper is to give the solvable conditions and explicit solutions to Problem GPEAP-TD, and construct a new multi-step method for solving this problem. Our method
only need solve the small scale linear systems so that the computational costs are much lower than
that of the traditional multi-step method.

The following notations will be used in this paper. The 2n eigenvalues of the open-loop pencil

\[ P(\lambda) = \lambda_1, \ldots, \lambda_{2n}, \]  

and the corresponding eigenvectors are \( x_1, \ldots, x_{2n} \). We let

\[ \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2n}) \]

whose diagonal elements are eigenvalues of the open-loop systems.

\[ \Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p) \]

whose diagonal elements are the eigenvalues to be altered.

\[ \Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n}) \]

whose diagonal elements are the eigenvalues kept unchanged.

\[ \Lambda_c = \text{diag}(\mu_1, \ldots, \mu_p) \]

whose diagonal elements are the eigenvalues to be assigned.

\( Y_c = [y_1, \ldots, y_p] \)

whose columns are corresponding eigenvectors of the eigenvalues in \( \Lambda_c \).

\( X = [x_1, \ldots, x_{2n}] \)

whose columns are corresponding right eigenvectors of the open-loop systems.

\[ X_1 = [x_1, \ldots, x_p] \]

\[ X_2 = [x_{p+1}, \ldots, x_{2n}] \]

\( I_n \) represents the unit matrix of order \( n \).

\( A \times B \) represents the Hadamard product of the matrix \( A \) and the matrix \( B \).

\( A \otimes B \) represents the Kronecker product of the matrix \( A \) and the matrix \( B \).

Throughout this paper, we use the following assumptions.

1. The system (1) is partially controllable with respect to \( \{\lambda_i\}_{i=1}^p \);
2. \( \{\lambda_i \neq 0\}_{i=1}^p \cap \{-\lambda_i\}_{i=1}^p = \emptyset \; \{\lambda_i\}_{i=1}^p \cap \{\lambda_i\}_{i=p+1}^{2n} = \emptyset \; \{\mu_i \neq -1\}_{i=1}^p \);
3. \( \lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_p \) are all distinct;
4. \( \mu_1, \ldots, \mu_p \) are finite eigenvalues.

2. Materials and Methods

2.1. Single-Input Control

We first solve Problem GPEAP-TD by single-input state feedback control, that means

\[ F_1 = f_1, F_2 = f_2, F_3 = f_3, B = b \in \mathbb{R}^n. \]

**Lemma 1.** [20] (The conception of partially controllability) The system (1) is partially controllable with respect to the subset \( \{\lambda_i\}_{i=1}^p \) of the spectrum of \( P(\lambda) \) if it is controllable with respect to each of the eigenvalues \( \lambda_i, i = 1, \ldots, p \).

**Lemma 2.** [21] Suppose that the eigenvalues of the open-loop pencil \( P(\lambda) \) are partitioned into the disjoint sets \( \{\lambda_i\}_{i=1}^p \) and \( \{\lambda_i\}_{i=p+1}^{2n} \), that is \( \{\lambda_i\}_{i=1}^p \cap \{\lambda_i\}_{i=p+1}^{2n} = \emptyset \), then

\[ \Lambda_1 X_1^T M X_2 \Lambda_2 + X_1^T K X_2 = 0. \]  \( \text{(7)} \)

From lemma 1, we have the following theorem.

**Theorem 1.** Given \( M, G, K \in \mathbb{R}^{n \times n} \) with \( M \) symmetric positive definite, \( G \) skew-symmetric and \( K \) symmetric nonsingular, \( b \in \mathbb{R}^n \), the self-conjugate subset \( \{\lambda_i\}_{i=1}^p \) of the open-loop spectrum \( \{\lambda_i\}_{i=1}^{2n} \) and the corresponding eigenvector set \( \{x_i\}_{i=1}^p \). Define

\[ f_1 = M X_1 \Lambda_1 \phi, f_2 = (M X_1 \Lambda_1 + K X_1) \phi, f_3 = K X_1 \phi, \]

where \( \phi \in \mathbb{R}^p \), then

\[ M X_2 \Lambda_2^2 - b f_1^T X_2 \Lambda_2 e^{-\tau \Lambda_2} + G X_2 \Lambda_2 - b f_2^T X_2 \Lambda_2 e^{-\tau \Lambda_2} + K X_2 - b f_3^T X_2 e^{-\tau \Lambda_2} = 0. \]  \( \text{(9)} \)
**Proof.** Use the orthogonality relation (7), and consider the \((\Lambda_2, X_2)\) is eigenpairs of systems \((M, G, K)\),

\[
\Lambda_1 X_1^T MX_2 \Lambda_2 + X_1^T KX_2 = 0, \\
MX_2 \Lambda_2^2 + GX_2 \Lambda_2 + KX_2 = 0,
\]

we have

\[
MX_2 \Lambda_2^2 - b_{f_1}^T X_2 \Lambda_2^2 e^{-\tau \Lambda_2} + GX_2 \Lambda_2 - b_{f_2}^T X_2 \Lambda_2 e^{-\tau \Lambda_2} + KX_2 - b_{f_3}^T X_2 e^{-\tau \Lambda_2} = 0,
\]

the Theorem 1 is proved. □

In order to solve Problem GPEAP-TD completely, we need to choose \(\phi\), such that

\[
MY_2 \Lambda_2^2 + GY_c \Lambda_c + KY_c - b_{f_1}^T Y_c \Lambda_c^2 e^{-\tau \Lambda_c} - b_{f_2}^T Y_c \Lambda_c e^{-\tau \Lambda_c} - b_{f_3}^T Y_c e^{-\tau \Lambda_c} = 0,
\]

Substituting for \(f_1, f_2, f_3\), we have

\[
MY_2 \Lambda_2^2 + GY_c \Lambda_c + KY_c = b_{f_1}^T [\Lambda_1 X_1^T MY_c \Lambda_c^2 + (\Lambda_1 X_1^T M + X_1^T K) Y_c \Lambda_c + X_1^T KY_c] e^{-\tau \Lambda_c}
\]

\[
\Lambda = b_{f_1},
\]

where

\[
\gamma = \phi^T [\Lambda_1 X_1^T MY_c \Lambda_c^2 + (\Lambda_1 X_1^T M + X_1^T K) Y_c \Lambda_c + X_1^T KY_c] e^{-\tau \Lambda_c}.
\]

Let

\[
H = \Lambda_1 X_1^T MY_c \Lambda_c^2 + (\Lambda_1 X_1^T M + X_1^T K) Y_c \Lambda_c + X_1^T KY_c,
\]

and choose \(\gamma = (1, \ldots, 1)\), then

\[
\phi^T H = e^{\tau \Lambda_c \gamma} = e^{\tau \Lambda_c} (1, \ldots, 1).
\]

First, we can prove that the vectors \(f_1, f_2, f_3\) obtained by this way are real vectors. Since the sets \(\{\lambda_i\}_{i=1}^p\) and \(\{x_i\}_{i=1}^p\) are self-conjugate, considering the matrices \(M, G, K\) are real, we have

\[
\bar{X}_1 = X_1 T_1, \bar{\Lambda}_1 = \bar{T}_1^T \Lambda_1 T_1,
\]

where \(T_1\) is a nonsingular permutation matrix. Similarly, there exists a nonsingular permutation matrix \(T_2\) such that

\[
\bar{Y}_c = Y_c T_2, \bar{\Lambda}_c = \bar{T}_2^T \Lambda_c T_2.
\]

Conjugating (13) and (14), we get

\[
\bar{H} = \bar{\Lambda}_1 \bar{X}_1^T MY_c \bar{\Lambda}_c^2 + (\bar{\Lambda}_1 \bar{X}_1^T M + \bar{X}_1^T K) \bar{Y}_c \bar{\Lambda}_c + \bar{X}_1^T K \bar{Y}_c = \bar{T}_1^T HT_2,
\]

\[
\bar{H}^T \bar{\phi} = \bar{HT} \bar{\phi} = \bar{T}_1^T H^T T_1 \bar{\phi} = e^{\tau \Lambda_c \gamma^H} = \bar{T}_2^T e^{\tau \Lambda_c \gamma^T} = \bar{T}_2^T H^T \phi,
\]

which implies \(\bar{\phi} = \bar{T}_1^T \phi\). Therefore

\[
\bar{f}_1 = MX_1 T_1 \bar{T}_1^T \Lambda_1 T_1 \bar{T}_1^T \bar{\phi} = f_1,
\]

\[
\bar{f}_2 = \left( MX_1 T_1 \bar{T}_1^T \Lambda_1 T_1 + KX_1 T_1 \right) \bar{T}_1^T \bar{\phi} = f_2,
\]

\[
\bar{f}_3 = \left( MX_1 T_1 \bar{T}_1^T \Lambda_1 T_1 + KX_1 T_1 \right) \bar{T}_1^T \bar{\phi} = f_3
\]
\[ \dot{f}_3 = K X_1 T_1 \phi = f_3, \]

which implies \( f_1, f_2, f_3 \) are real.

Therefore we give the following theorem.

**Theorem 2.** If the system \((M, G, K)\) is partially controllable with respect to the subset \( \{ \lambda_i \}_{i=1}^p, \{ \lambda_i \neq 0 \}_{i=1}^p \), \( \{ \mu_i \}_{i=1}^p \cap \{ -\lambda_i \}_{i=1}^p = \emptyset \), \( \{ \lambda_i \}_{i=1}^{2n} \cap \{ \lambda_i \}_{i=p+1}^p = \emptyset \), and \( \lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_p \) are all distinct, assume that \( \{ \mu_i \neq -1 \}_{i=1}^p \) are finite eigenvalues, then

1. For any arbitrary vector \( \phi \in \mathbb{R}^p \), the closed-loop pencil \( P_t(\lambda) = \lambda^2 (M - B f_1^T e^{-\lambda T}) + \lambda (G - B f_2^T e^{-\lambda T}) + (K - B f_3^T e^{-\lambda T}) \) has eigenpairs \( \{(\lambda_i, x_i)\}_{i=1}^{2n} \), where the feedback vectors \( f_1, f_2, f_3 \) are defined by (8).
2. Let \( y_i (i = 1, \ldots, p) \) satisfy

\[
(\mu_i^2 M + \mu_i G + K) y_i = b, \tag{15}
\]

Define \( H = \lambda_1 X_1^T MY_c \Lambda_2^2 + (\lambda_1 X_1^T M + X_1^T K) Y_c \Lambda_c + X_1^T KY_c \), then \( H \) is nonsingular and Problem GPEAP-TD has a solution in the form (8), where \( \phi \) is a solution of the linear systems (14).

**Proof.** The first part can be proved by Theorem 1. From (11) and \( \gamma = (1, \ldots, 1) \), we have

\[
(\mu_i^2 M + \mu_i G + K) y_i = b,
\]

for \( i = 1, \ldots, p \). Since \( \text{rank} (\mu_i^2 M + \mu_i G + K, b) = n \) and \( \{ \mu_i \}_{i=1}^p \cap \{ -\lambda_i \}_{i=1}^p = \emptyset \), then the linear systems (15) has a solution \( y_i \).

From (13), we have

\[
h_{rs} = (\mu_s + 1) x_r^T (\mu_s \lambda_r M + K) y_s, r, s = 1, \ldots, p, \tag{16}
\]

where \( h_{rs} \) is the entry in the \( r \)-th row and \( s \)-th column of matrix \( H \).

Since \( (\lambda_r^2 M + \lambda_r G + K) x_r = 0, r = 1, \ldots, p \). Then

\[
-\lambda_r G x_r = (\lambda_r^2 M + K) x_r.
\]

Multiplying both sides by \( \frac{\mu_s}{\lambda_r} \), we get

\[
-\mu_s G x_r = \left( \mu_s \lambda_r M + \frac{\mu_s}{\lambda_r} K \right) x_r.
\]

Adding \( (\mu_s^2 M + K) x_r \) to both sides, we obtain

\[
\left( \mu_s^2 M - \mu_s G + K \right) x_r = \left[ \mu_s (\mu_s + \lambda_r) M + \frac{\mu_s}{\lambda_r} K \right] x_r.
\]

Therefore

\[
\frac{\lambda_r}{\mu_s + \lambda_r} \left( \mu_s^2 M - \mu_s G + K \right) x_r = (\mu_s \lambda_r M + K) x_r.
\]

Transposing the above formula, we can get

\[
x_r^T \frac{\lambda_r}{\mu_s + \lambda_r} \left( \mu_s^2 M + \mu_s G + K \right) = x_r^T (\mu_s \lambda_r M + K). \tag{17}
\]
Substituting (15) and (17) into (16), we have

\[ h_r = \frac{(\mu_i + 1)\lambda_i}{\mu_i + \lambda_i} x_i^T \left( \mu_i^2 M + \mu_i G + K \right) y_s \]

\[ = \frac{(\mu_i + 1)\lambda_i}{\mu_i + \lambda_i} x_i^T b, \]

it follows that

\[ H = H_1 H_2 H_3, \]

where

\[ H_1 = \begin{bmatrix} \mu_1 + 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_p + 1 \end{bmatrix}, \]

\[ H_2 = \begin{bmatrix} \frac{1}{\mu_1 + \lambda_1} & \cdots & \frac{1}{\mu_1 + \lambda_p} \\ \vdots & \ddots & \vdots \\ \frac{1}{\mu_p + \lambda_1} & \cdots & \frac{1}{\mu_p + \lambda_p} \end{bmatrix}, \]

\[ H_3 = \begin{bmatrix} \lambda_1 x_1^T b & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \lambda_p x_p^T b \end{bmatrix}. \]

Since $H_2$ is a Cauchy matrix and $\{\mu_i\}_{i=1}^p \cap \{-\lambda_i\}_{i=1}^p = \emptyset$, then $H_2$ is nonsingular. Note that the system (1) is partially controllable with respect to the subset $\{\lambda_i\}_{i=1}^p$, then we can get $x_t^T b \neq 0 \ (r = 1, \ldots, p)$ [18]. According to the assumption, $\{\lambda_i \neq 0\}_{i=1}^p, \{\mu_i \neq -1\}_{i=1}^p$, then $H_1, H_3$ are also nonsingular, therefore $H$ is nonsingular and the linear systems (14) has a unique solution $\phi$. \[ \square \]

Therefore, $f_1, f_2, f_3$ are the solutions to Problem GPEAP-TD by single-input state feedback control. Based on Theorem 1, we can get the following Algorithm 1.

**Algorithm 1** An algorithm for Problem GPEAP-TD by single-input control.

**Input:**
- The $n \times n$ real matrices $M, G, K$ with $M$ symmetric positive definite, $G$ skew-symmetric and $K$ symmetric nonsingular;
- The control vector $b$ and the time delay $\tau > 0$;
- The self-conjugate subset $\{\lambda_i\}_{i=1}^p$ of the open-loop spectrum $\{\lambda_i\}_{i=1}^{2n}$ and a self-conjugate set $\{\mu_i \neq -1\}_{i=1}^p$. Assume that $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_p$ are all distinct.

**Output:**
- Find the state feedback vectors $f_1, f_2, f_3$ such that the closed-loop pencil $P_c(\lambda) = \lambda^2 (M - b f_1^T e^{-\lambda \tau} + \lambda (G - b f_2^T e^{-\lambda \tau}) + (K - b f_3^T e^{-\lambda \tau})$ has the desired eigenvalues $\{\mu_i\}_{i=1}^p$ and the eigenpairs $\{(\lambda_i, x_i)\}_{i=1}^{2n+p+1}$.

**1.** Form $\Lambda_1 = diag(\lambda_1, \ldots, \lambda_p);$
**2.** Form $X_1 = \{x_1, \ldots, x_p\}$;
**3.** Form $\Lambda_c = diag(\mu_1, \ldots, \mu_p)$;
**4.** for $i = 1, 2, \ldots, p$ do
**5.** solve for $y_i: (\mu_i^2 M + \mu_i G + K) y_i = b$;
**6. end for**
**7.** Compute $Y_c = \{y_1, \ldots, y_p\}$;
**8.** Form $H = \Lambda_1 X_1^T M Y_c \Lambda_2^2 + (\Lambda_1 X_1^T M + X_1^T K) Y_c \Lambda_c + X_1^T K Y_c$;
**9.** Compute $\phi: \phi = e^{t \Lambda_c} (1, \ldots, 1)$;
**10.** Compute $f_1, f_2, f_3: f_1 = M X_1 \Lambda_1 \phi, f_2 = (M X_1 \Lambda_1 + K X_1) \phi, f_3 = K X_1 \phi$.

### 2.2. Multi-Input Control

In this section, we consider Problem GPEAP-TD by multi-input state feedback control. Note that Equation (3) is equivalent to

\[ M \ddot{z}(t) + G \dot{z}(t) + K z(t) = \sum_{k=1}^{m} b_k \left( f_{ik}^T \dot{z}(t - \tau) + f_{ik}^T \dot{z}(t - 2\tau) + f_{ik}^T \dot{z}(t - \tau) \right), \]

(20)
where $b_k$ and $f_{1k}, f_{2k}, f_{3k}$ are the $k$-th columns of $B$ and $F_1, F_2, F_3$ respectively. The associated closed-loop pencil is given by

$$P_{rk}(\lambda) = \lambda^2 \left( M - \sum_{k=1}^{m} b_k f_{1k}^T e^{-\lambda \tau} \right) + \lambda \left( G - \sum_{k=1}^{m} b_k f_{2k}^T e^{-\lambda \tau} \right) + \left( K - \sum_{k=1}^{m} b_k f_{3k}^T e^{-\lambda \tau} \right). \quad (21)$$

Define

$$\eta_{ik} = \alpha_{ik} + \frac{k}{m} (\mu_i - \alpha_{ik}), \quad i = 1, \ldots, p, \quad k = 1, \ldots, m, \quad (22)$$

where $\alpha_{ik} \in \mathbb{C}$. Note that $\eta_{im} = \mu_i, i = 1, \ldots, p$. Let

$$M_k = M - \sum_{l=1}^{k-1} b_l f_{1l}^T e^{-\lambda \tau}, \quad G_k = G - \sum_{l=1}^{k-1} b_l f_{2l}^T e^{-\lambda \tau}, \quad K_k = K - \sum_{l=1}^{k-1} b_l f_{3l}^T e^{-\lambda \tau}, \quad k = 1, \ldots, m \quad (23)$$

and $M_1 = M, G_1 = G, K_1 = K$. Therefore, the Problem GPEAP-TD is equivalent to the following problem.

**Problem MGPEAP-TD.** Given $M, G, K, B, X_1, \Lambda_1, \Lambda_2$. Let $\eta_{ik}$ and $\{M_k, G_k, K_k\}_{k=1}^{m}$ be defined by (22) and (23) respectively. For $k = 1, \ldots, m$, find the feedback vectors $f_{1k}, f_{2k}, f_{3k}$ such that the single-input closed-loop pencil

$$P_{rk}(\lambda) = \lambda^2 \left( M_k - b_k f_{1k}^T e^{-\lambda \tau} \right) + \lambda \left( G_k - b_k f_{2k}^T e^{-\lambda \tau} \right) + \left( K_k - b_k f_{3k}^T e^{-\lambda \tau} \right) \quad (24)$$

has the desired eigenvalues $\{\eta_{ik}\}_{i=1}^{p}$ and the eigenpairs $\{(\lambda_i, x_i)\}_{i=1}^{2n}$.

Define

$$f_{1k} = M X_1 \Lambda_1 \phi_k, \quad f_{2k} = (M X_1 \Lambda_1 + K X_1) \phi_k, \quad f_{3k} = K X_1 \phi_k, \quad (25)$$

where $\phi_k \in \mathbb{R}^p$ is arbitrary.

**Theorem 3.** If $\{\eta_{ik}\}_{i=1}^{p}$ are not the eigenvalues of the system $(M_k, G_k, K_k)$, $\{\lambda_i \neq 0\}_{i=1}^{p}$, $\{\eta_{ik}\}_{i=1}^{m}$ $(k = 1, \ldots, m)$ and $\{-\lambda_i\}_{i=1}^{m}$ are all distinct, then

1. For $k = 1, \ldots, m$ and any arbitrary vector $\phi_k$, the closed-loop pencil

$$P_{rk}(\lambda) = \lambda^2 \left( M_k - b_k f_{1k}^T e^{-\lambda \tau} \right) + \lambda \left( G_k - b_k f_{2k}^T e^{-\lambda \tau} \right) + \left( K_k - b_k f_{3k}^T e^{-\lambda \tau} \right)$$

has eigenpairs $\{(\lambda_i, x_i)\}_{i=1}^{2n}$, where the feedback vectors $f_{1k}, f_{2k}, f_{3k}$ are defined by (25).
2. Let $\{(\eta_{ik}, y_{ik})\}_{i=1}^{p}$ are the eigenpairs of the closed-loop pencil $P_{rk}(\lambda)$. Define

$$\Lambda_{rk} = \text{diag}(\eta_{ik}, \ldots, \eta_{ik}), \quad Y_{rk} = [y_{1k}, \ldots, y_{pk}],$$

and

$$H_k = \Lambda_1 X_1^T M Y_{rk} \Lambda_{rk}^2 + \left( \Lambda_1 X_1^T M + X_1^T K \right) Y_{rk} \Lambda_{rk} + X_1^T K Y_{rk},$$

choose $\alpha_{ik}$ such that $\eta_{ik}^2 M_k + \eta_{ik} G_k + K_k (k = 1, \ldots, m)$ is nonsingular, then Problem MGPEAP-TD has a solution in the form (25), where $\phi_k$ is a solution of the linear systems $\phi^T H = e^{\tau \Lambda_k} (1, \ldots, 1)$.

**Proof.** By Theorem 1, we have

$$M_k X_2 \Lambda_{rk}^2 - b_k f_{1k}^T Y_{rk} \Lambda_{rk}^2 e^{-\tau \Lambda_k} + G_k X_2 \Lambda_{rk} - b_k f_{2k}^T Y_{rk} \Lambda_{rk} e^{-\tau \Lambda_k} + K_k X_2 - b_k f_{3k}^T Y_{rk} e^{-\tau \Lambda_k} = 0, \quad (26)$$

which proves the first part of Theorem 3. Since $\{(\eta_{ik}, y_{ik})\}_{i=1}^{p}$ are the eigenpairs of the closed-loop pencil $P_{rk}(\lambda)$, and in order to solve Problem MGPPAP-TD completely, we need to choose $\phi_k$, such that

$$M_k Y_{rk} \Lambda_{rk}^2 - b_k f_{1k}^T V_{rk} \Lambda_{rk}^2 e^{-\tau \Lambda_k} + G_k Y_{rk} \Lambda_{rk} - b_k f_{2k}^T V_{rk} \Lambda_{rk} e^{-\tau \Lambda_k} + K_k Y_{rk} - b_k f_{3k}^T V_{rk} e^{-\tau \Lambda_k} = 0. \quad (27)$$
Substituting for \( f_{1k}, f_{2k}, f_{3k} \), we have
\[
M_k Y_{rk} \Lambda_{rk}^2 + G_k Y_{rk} \Lambda_{rk} + K_k Y_{rk} = b_k \phi_k^T \left[ \Lambda_1 X_1^T M Y_{rk} \Lambda_{rk}^2 + \left( \Lambda_1 X_1^T M + X_1^T K \right) Y_{rk} \Lambda_{rk} + X_1^T K Y_{rk} \right] e^{-\tau \Lambda_{rk}}
\]  
(28)

where \( \gamma_k = \phi_k^T \left[ \Lambda_1 X_1^T M Y_{rk} \Lambda_{rk}^2 + \left( \Lambda_1 X_1^T M + X_1^T K \right) Y_{rk} \Lambda_{rk} + X_1^T K Y_{rk} \right] e^{-\tau \Lambda_{rk}} \).

Choose \( \gamma_k = (1, \ldots, 1) \), then
\[
\eta_k^2 M_k + \eta_{rk} G_k + K_k \right) y_{ik} = b_k.
\]  
(29)

Since the coefficient matrix of linear Equation (29) is nonsingular, We can get \( y_{ik} \). Let
\[
H_k = \Lambda_1 X_1^T M Y_{rk} \Lambda_{rk} + \left( \Lambda_1 X_1^T M + X_1^T K \right) Y_{rk} \Lambda_{rk} + X_1^T K Y_{rk},
\]  
(30)

then
\[
\phi_k^T H_k = e^{\tau \Lambda_{rk}} (1, \ldots, 1),
\]  
(31)

which means \( \phi_k \) is a solution of the linear systems (31). \( \square \)

**Note:** We can choose \( a_{ik} \) such that \( \eta_{rk} \) are not eigenvalues of the system \( (M_k, G_k, K_k) \), then \( \eta_{rk}^2 M_k + \eta_{rk} G_k + K_k \) is nonsingular. This is an easy way for choosing \( a_{ik} \). Another method is choosing \( a_{ik} = \lambda_i (i = 1, \ldots, p) \), and we need to verify whether \( \eta_{rk}^2 M_k + \eta_{rk} G_k + K_k \) is nonsingular.

The traditional multi-step method need to solve \( y_{ik} \) \((i = 1, \ldots, p, k = 1, \ldots, m) \) which requires \( \frac{2}{\tau} n^3 mp \) flops, \( H_k \) which requires \( n^2 mp + 2nmp^2 + 2mp^2 \) flops, \( \phi_k \) which requires \( \frac{2}{\tau} mp^3 \) flops. In general, \( m, p \ll n \), the total computational costs are \( O(n^2 mp + n^2 mp^2) \). In order to reduce the total computational costs, we propose a new multi-step method.

From (25) and (29), we have
\[
(h_k)_{rs} = \frac{\lambda_r (\eta_{rk}^2 + 1)}{\eta_{rk}^2 + A_r} x_{rk}^T (\eta_{rk}^2 M + \eta_{rk} G + K) y_{ik}
\]  
\[
= \frac{\lambda_r (\eta_{rk}^2 + 1)}{\eta_{rk}^2 + A_r} x_{rk}^T \left[ M_k + \sum_{l=1}^{k-1} b_l f_l e^{-\tau_{rl}} \right] + \eta_{rk} \left[ G_k + \sum_{l=1}^{k-1} b_l f_l e^{-\tau_{rl}} + K_k + \sum_{l=1}^{k-1} b_l f_l e^{-\tau_{rl}} \right] y_{ik}
\]  
(32)

From (15) and (25), we have
\[
(h_k)_{rs} = \frac{\lambda_r (\eta_{rk}^2 + 1)}{\eta_{rk}^2 + A_r} x_{rk}^T b_k + \frac{\lambda_r (\eta_{rk}^2 + 1)}{\eta_{rk}^2 + A_r} e^{-\tau_{rk}} x_{rk}^T \sum_{l=1}^{k-1} b_l \phi_l^T \left[ \Lambda_1 X_1^T M \eta_{rk}^2 + \left( \Lambda_1 X_1^T M + X_1^T K \right) \eta_{rk} + X_1^T K \right] y_{ik}.
\]  
(33)

Let
\[
A_k = \begin{bmatrix}
  x_1^T b_k \\
  \vdots \\
  x_p^T b_k
\end{bmatrix},
V_k = \begin{bmatrix}
  \lambda_1 \eta_{rk}^{p+1} / \eta_{rk} + A_1 \\
  \vdots \\
  \lambda_p \eta_{rk}^{p+1} / \eta_{rk} + A_p
\end{bmatrix},
T_k = \begin{bmatrix}
  e^{-\tau \eta_{rk}} \cdot \\
  \vdots \\
  e^{-\tau \eta_{rk}}
\end{bmatrix},
\]
and
\[
P_k = X_1^T \sum_{l=1}^{k-1} b_l \phi_l^T,
H_k = \Lambda_1 X_1^T M Y_{rk} \Lambda_{rk}^2 + \left( \Lambda_1 X_1^T M + X_1^T K \right) Y_{rk} \Lambda_{rk} + X_1^T K Y_{rk},
\]
where \( \phi_1 \) can be computed by (14), and \( \phi_{ik}, k = 2, \ldots, m \) can be computed by (31).
From (33), we can get
\[ H_k = U_k + R_k \star (P_k H_k), \]  
(34)
where \( U_k = A_k V_k, R_k = V_k T_k \) and \( \star \) represents the Hadamard product.

Let \( A = [a_1, a_2, ..., a_n] \) be a \( m \times n \) real matrix, \( a_i \) denotes the \( i \)-th column of the matrix \( A \), and the \( mn \) dimension vector \( \text{vec}(A) = (a_1^T, a_2^T, ..., a_n^T) \) be called the column straightening of matrix \( A \). The column straightening of the matrix has the following properties with the Hadamard product and Kronecker product of the matrix.

**Lemma 3.** [22] Let \( A \in \mathbb{R}^{m \times n} \).

1. If \( B \in \mathbb{R}^{m \times n} \), then \( \text{vec}(A \ast B) = \text{diag}(\text{vec}(A)) \text{vec}(B) \);
2. If \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{p \times q} \), then \( \text{vec}(ABC) = (C^T \otimes A) \text{vec}(B) \).

From lemma 3, matrix Equation (34) can be transformed into the following \( p^2 \) order linear systems
\[ \left[ I_{p^2} - \text{diag}(\text{vec}(R_k)) \left( I_p \otimes P_k \right) \right] \text{vec}(H_k) = \text{vec}(U_k). \]  
(35)

Because both \( P_k \) and \( U_k \) are known, the matrix \( H_k \) can be obtained by solving the linear systems (35), and then solving the linear systems (31) to get \( \phi_k \). Therefore, the calculation process of solving the partial eigenvalue assignment for multi-input second-order linear systems with time delay can be summarized as the following Algorithm 2.

**Algorithm 2** An algorithm for Problem MGPEAP-TD by multi-input control.

**Input:**
The \( n \times n \) real matrices \( M, G, K \) with \( M \) symmetric positive definite, \( G \) skew-symmetric and \( K \) symmetric nonsingular;
The full column rank control matrix \( B = [b_1, ..., b_m] \) and the time delay \( \tau > 0 \);
The self-conjugate subset \( \{\lambda_i\}_{i=1}^p \) of the open-loop spectrum \( \{\lambda_i\}_{i=1}^{2n} \) and a self-conjugate set \( \{\eta_{ik}\}_{i=1}^p \). Assume that \( \lambda_1, ..., \lambda_p, \eta_{ik}, ..., \eta_{pk} \) are all distinct.

**Output:**
Find the state feedback matrices \( F_1 = [f_{11}, ..., f_{1m}], F_2 = [f_{21}, ..., f_{2m}], F_3 = [f_{31}, ..., f_{3m}] \) such that the closed-loop pencil \( P_{ck}(\lambda) = \lambda^2 (M_k - b_k f_{ik}^T e^{-\lambda \tau}) + \lambda (G_k - b_k f_{ik}^T e^{-\lambda \tau}) + (K_k - b_k f_{ik}^T e^{-\lambda \tau}) \).

1. Form \( \Lambda_{yk} = [\eta_{yk}, ..., \eta_{pk}] \);
2. Compute \( Y_{yk} = [y_{yk}, ..., y_{yk}] \);
3. Chose \( a_{ik} \) and form \( \eta_{ik} = a_{ik} + k \mu \), \( i = 1, ..., p \), \( k = 1, ..., m \) such that \( \eta_{ik}^T M_k + \eta_{ik} G_k + K_k \) is nonsingular;
4. The vectors \( \phi_1, f_{11}, f_{21}, f_{31} \) are calculated by Algorithm 1;
5. for \( k = 2, ..., m \) do
6. Compute \( U_k, R_k, P_k \);
7. Compute \( H_k \) by solving the linear systems (35);
8. Compute \( \phi_k \) by solving the linear systems (31);
9. Compute \( f_{1k} = MX_k \Lambda_1 \phi_k, f_{2k} = (MX_k \Lambda_1 + KX_k) \phi_k, f_{3k} = KX_k \phi_k \);
end for
10. \( F_1 = [f_{11}, ..., f_{1m}], F_2 = [f_{21}, ..., f_{2m}], F_3 = [f_{31}, ..., f_{3m}] \).

Note that \( p^2 \) is always small, it is easy to solve the linear systems (35). The analysis of the computational costs of Algorithm 2 can be listed as follows. In our paper, we need to solve \( H_k \) which requires \( \frac{7}{2} mp^4 \) flops and \( \phi_k \) which requires \( \frac{7}{2} mp^3 \) flops. The total computational costs are \( O(mp^4) \). We already know that the total computational costs of traditional multi-step method are \( O(n^2 mp + n^2 mp) \). In general, \( m, p \ll n \), so the total computational costs of our new multi-step method are much lower than that of the traditional multi-step method.
3. Results

In this section, we give some numerical examples to illustrate the effectiveness of Algorithm 1 and Algorithm 2.

Example 1 ([17]). Consider the following gyroscopic second-order systems with time delay, the mass matrix, gyroscopic matrix and stiffness matrix are

\[
M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
G = \begin{bmatrix}
0 & -2 & 4 \\
2 & 0 & -2 \\
4 & 2 & 0
\end{bmatrix},
K = \begin{bmatrix}
13 & 2 & 1 \\
2 & 7 & 2 \\
1 & 2 & 4
\end{bmatrix}.
\]

The corresponding open loop pencil has 6 eigenvalues: \(\lambda_{12} = -0.0000 \pm 6.0860i, \lambda_{34} = -0.0000 \pm 3.1895i, \lambda_{56} = -0.0000 \pm 0.8878i\), the control vector \(b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\). The all eigenvalues should be reassigned to \(\mu_{12} = -1 \pm 6.0860i, \mu_{34} = -1 \pm 3.1895i, \mu_{56} = -1 \pm 0.8878i\). According to Algorithm 1, we can compute

\[
H = \begin{pmatrix}
0.1794 + 2.1831i & -26.7517 - 0.0006i & 0.6429 - 0.0693i & 1.8608 - 0.6424i & 0.2624 - 0.0276i & 0.3465 - 0.0665i \\
-26.7517 + 0.0006i & 0.1794 - 2.1831i & 1.8608 + 0.6424i & 0.6429 + 0.0693i & 0.3465 + 0.0665i & 0.2624 + 0.0276i \\
0.1611 + 1.4941i & -1.4931 - 4.3248i & 0.4837 + 0.0758i & 0.0001 - 3.1613i & 0.2264 - 0.0555i & 0.3577 - 0.1554i \\
-1.4931 + 4.3248i & 0.1611 - 1.4941i & 0.0001 + 3.1613i & 0.4837 - 0.0758i & 0.2264 + 0.0555i & 0.3577 + 0.1554i \\
0.0786 + 0.5483i & -0.1393 - 0.7239i & 0.2006 + 0.0499i & -0.3216 - 0.1397i & 0.1165 - 0.0656i & -0.0000 - 0.2725i \\
-0.1393 - 0.7239i & 0.0786 - 0.5483i & -0.3216 + 0.1397i & 0.2006 - 0.0499i & 0.1165 + 0.0656i & -0.0000 + 0.2725i
\end{pmatrix},
\]

\[
\phi = \begin{pmatrix}
-0.0806 - 0.0448i \\
-0.0806 + 0.0448i \\
0.1532 - 0.8458i \\
0.1532 + 0.8458i \\
-4.1935 - 6.4533i \\
-4.1935 + 6.4533i
\end{pmatrix},
\]

\[
f_1 = \begin{bmatrix}
-1.8760 \\
-1.8760 \\
-10.8801
\end{bmatrix},
f_2 = \begin{bmatrix}
-62.2999 \\
16.7801 \\
-4.5560
\end{bmatrix},
f_3 = \begin{bmatrix}
-60.4240 \\
18.6431 \\
-15.4361
\end{bmatrix},
\]

and

\[
\|MY\Lambda_2^2 - bf_1^T Y_c \Lambda_2 e^{-\tau\Lambda_c} + GY_c \Lambda_c - b f_2^T Y_c \Lambda_c e^{-\tau\Lambda_c} + KY_c - b f_3^T Y_c e^{-\tau\Lambda_c}\|_F = 4.3641 e - 15.
\]

Example 2 ([23]). Consider the following gyroscopic second-order systems (Figure 1) with time delay, the mass matrix, gyroscopic matrix and stiffness matrix are respectively

\[
M = \text{diag}(30.8113, 30.8113, 20.3712, 20.3712),
\]

\[
K = \text{diag}(2.4, 2.4, 0.024, 0.024),
\]

\[
G = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0.0385 \\
0 & 0 & -0.0385
\end{bmatrix},
\]

The corresponding open-loop pencil has 8 eigenvalues: \(\lambda_{12} = -0.0000 \pm 0.2791i, \lambda_{34} = -0.0000 \pm 0.2791i, \lambda_{56} = -0.0000 \pm 0.0353i, \lambda_{78} = -0.0000 \pm 0.0334i\), the control vector \(b = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 2 \end{bmatrix}\), the time delay \(\tau = 0.1\). The first two eigenvalues should be reassigned to \(\mu_{12} = -1 \pm 0.2791i\). Then

\[
\Lambda_1 = \text{diag} \begin{pmatrix}
-0.0000 + 0.2791i, -0.0000 - 0.2791i
\end{pmatrix},
\]
and the corresponding matrix of eigenvectors as
\[
X_1 = \begin{bmatrix}
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

According to Algorithm 1, we can compute
\[
f_1 = \begin{bmatrix}
-363.33 \\
0 \\
0 \\
0
\end{bmatrix},
\quad f_2 = \begin{bmatrix}
-779.62 \\
0 \\
0 \\
0
\end{bmatrix},
\quad f_3 = \begin{bmatrix}
-416.28 \\
0 \\
0 \\
0
\end{bmatrix},
\]

and
\[
\|MY_c^2 \Lambda_2^2 - b f_1^T Y_c \Lambda_2^2 e^{-\tau \Lambda_2} + GY_c^2 \Lambda_2^2 - b f_2^T Y_c \Lambda_2^2 e^{-\tau \Lambda_2} + KY_c^2 - b f_3^T Y_c \Lambda_2^2 e^{-\tau \Lambda_2}\|_F = 4.5471 \times 10^{-14},
\]

\[
\|MX_c^2 \Lambda_2^2 - b f_1^T X_c \Lambda_2^2 e^{-\tau \Lambda_2} + GX_c \Lambda_2^2 - b f_2^T X_c \Lambda_2^2 e^{-\tau \Lambda_2} + KX_c - b f_3^T X_c \Lambda_2^2 e^{-\tau \Lambda_2}\|_F = 7.0461 \times 10^{-16}.
\]

Figure 1. Cylindrical rotor with flexible bearings.

**Example 3.** Consider the following 10th order gyroscopic system, the mass matrix, gyroscopic matrix, stiffness matrix and the control matrix are respectively
\[
M = 4I_n, \quad G = \begin{bmatrix}
0 & -2 & -4 \\
2 & 0 & -2 \\
4 & 2 & 0 \\
& & \ddots \\
& & & -2 \\
4 & 2 & 0
\end{bmatrix},
\quad K = \begin{bmatrix}
7 & 2 & 1 \\
2 & 7 & 2 \\
1 & 2 & 7 \\
& & \ddots \\
& & & 2 \\
1 & 2 & 7
\end{bmatrix},
\quad B = \begin{bmatrix}
1 & 2 \\
0 & 2 \\
0 & 0 \\
& & 2 & 3
\end{bmatrix},
\]

where \( n = 10, \ m = 2 \) and the time delay \( \tau = 0.1 \). The first two eigenvalues should be reassigned to \( \mu_{12} = -2 \pm 3.1699i \), and the other eigenpairs remain unchanged. According to Algorithm 2, we can compute
\[
H_1 = \begin{bmatrix}
-0.7052 + 0.0301i & 0.0737 + 0.0868i \\
0.0737 + 0.0868i & -0.7052 - 0.0301i
\end{bmatrix},
\quad \Phi_1 = \begin{bmatrix}
0.0737 + 0.0868i \\
-0.0737 - 0.0868i
\end{bmatrix},
\quad \Phi_2 = \begin{bmatrix}
0.0224 + 0.0389i \\
0.0224 - 0.0389i
\end{bmatrix},
\]

\[
U_2 = \begin{bmatrix}
-1.8549 - 0.0768i & 0.7889 + 1.5399i \\
2.3417 + 11.6630i & -1.8549 + 0.0768i
\end{bmatrix},
\quad R_2 = \begin{bmatrix}
0.7889 + 1.5399i & -10.5516 - 3.4614i \\
-10.5516 + 3.4614i & 0.7889 - 1.5399i
\end{bmatrix},
\]
According to Algorithm 2, we can compute

$$P_2 = \begin{pmatrix} -0.0438 + 0.0246i & 0.0314 + 0.0393i \\ 0.0314 - 0.0393i & -0.0438 - 0.0246i \end{pmatrix}, \quad H_2 = \begin{pmatrix} -3.4784 - 0.5223i & 1.9168 + 22.4796i \\ 1.9168 - 22.4796i & -3.4784 + 0.5223i \end{pmatrix},$$

$$F_1 = \begin{pmatrix} -0.4556 & -0.1653 \\ -0.2484 & -0.0519 \\ 0.3758 & 0.2078 \\ 1.0263 & 0.4238 \\ 1.0608 & 0.3741 \\ 0.3443 & 0.0530 \\ -0.5243 & -0.2712 \\ -0.9206 & -0.3734 \\ -0.5755 & -0.1985 \\ -0.0841 & 0.0012 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -0.7111 & -0.2829 \\ -0.7841 & -0.2645 \\ -0.2322 & -0.0011 \\ 0.8110 & 0.3973 \\ 1.4772 & 0.5919 \\ 1.3914 & 0.4016 \\ 0.2311 & -0.0027 \\ -0.6461 & -0.3140 \\ -0.7301 & -0.2953 \\ -0.3831 & -0.1296 \end{pmatrix}, \quad F_3 = \begin{pmatrix} -0.2554 & -0.1176 \\ -0.5357 & -0.2126 \\ -0.6081 & -0.2089 \\ 0.4164 & 0.2177 \\ 0.8471 & 0.3486 \\ 0.7554 & 0.2685 \\ 0.2745 & 0.0594 \\ -0.1546 & -0.0968 \\ -0.2990 & -0.1309 \end{pmatrix},$$

and

$$\|MY_i^2 - BF_1 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + G Y_i^1 - BF_2 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + K Y_i - BF_3 Y_i e^{-\tau \Lambda_{\tau_i}}\|_F = 7.2607e - 14,$$

$$\|MX_2^2 - BF_1^T X_2^2 e^{-\tau \Lambda_2} + G X_2^2 - BF_2^T X_2^2 e^{-\tau \Lambda_2} + K X_2 - BF_3^T X_2 e^{-\tau \Lambda_2}\|_F = 1.5784e - 13.$$

If we consider the gyroscopic system when $n = 1000$ in Example 2, and the time delay $\tau = 0.1$. The first two eigenvalues should be reassigned to $\mu_{12} = -1 \pm 3.4262i$, and the other eigenpairs remain unchanged. According to Algorithm 2, we can compute

$$\|MY_i^2 - BF_1 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + G Y_i^1 - BF_2 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + K Y_i - BF_3 Y_i e^{-\tau \Lambda_{\tau_i}}\|_F = 3.0787e - 12,$$

$$\|MX_2^2 - BF_1^T X_2^2 e^{-\tau \Lambda_2} + G X_2^2 - BF_2^T X_2^2 e^{-\tau \Lambda_2} + K X_2 - BF_3^T X_2 e^{-\tau \Lambda_2}\|_F = 2.4070e - 06.$$

**Example 4.** We consider the same gyroscopic system as in Example 2, and the time delay $\tau = 0.1$, $n = 500$, $m = 3$, the control matrix is $B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. The first two eigenvalues should be reassigned to $\mu_{12} = -1 \pm 3.4265i$, and the other eigenpairs remain unchanged. According to Algorithm 2, we can compute

$$\|MY_i^2 - BF_1 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + G Y_i^1 - BF_2 Y_i^2 e^{-\tau \Lambda_{\tau_i}} + K Y_i - BF_3 Y_i e^{-\tau \Lambda_{\tau_i}}\|_F = 6.6416e - 12,$$

$$\|MX_2^2 - BF_1^T X_2^2 e^{-\tau \Lambda_2} + G X_2^2 - BF_2^T X_2^2 e^{-\tau \Lambda_2} + K X_2 - BF_3^T X_2 e^{-\tau \Lambda_2}\|_F = 3.6188e - 07.$$

4. Conclusions

In this paper, we consider the partial eigenvalue assignment for undamped gyroscopic systems in control with time delay. We give the solvable condition and the explicit solutions to this problem and then we propose a multi-step method for solving this problem by which the undesired eigenvalues are altered as required and the resting eigenpairs are kept unchanged. Numerical examples show that our method is effective.
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