The Capacity of a Class of Linear Deterministic Networks

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Abstract—In this paper, we investigate optimal coding strategies for a class of linear deterministic relay networks. The network under study is a relay network, with one source, one destination, and two relay nodes. Additionally, there is a disturbing source of signals that causes interference with the information signals received by the relay nodes. Our model captures the effect of the interference of message signals and disturbing signals on a single relay network, or the interference of signals from multiple relay networks with each other in the linear deterministic framework. For several ranges of the network parameters we find upper bounds on the maximum achievable source–destination rate in the presence of the disturbing node and in each case we find an optimal coding scheme that achieves the upper bound.

I. INTRODUCTION

Finding the capacity of relay networks is a basic open problem in network information theory [1]. One way to approximate the function of relay networks is through deterministic models that was first introduced by Aref [2]. Avestimehr, Diggavi and Tse have recently [3], [4] introduced a linear deterministic model for wireless relay networks that treats noise as a deterministic thresholding function and the interference of signals as a linear transformation over a finite field. They have successfully applied this model to several relay networks and have shown that the capacity of the deterministic model is within a constant gap from the corresponding wireless network. Furthermore, they have shown that a max–flow min–cut result holds for the capacity of the relay deterministic network in the case of a single multicast session. While the capacity achieving scheme in [3], [4] is a random coding over long blocks of signals, recent works [5], [6], [7], [8] have devised low complexity and deterministic schemes that achieve the maximum capacity in the case of a unicast session.

In the case of multiple messages, Mohajer et al., [9] have considered two unicast sessions on a relay network with two relays, two sources, and two destinations. They study two special cases of this setting, namely ZS and ZZ channels and give a full characterization of the capacity region along with capacity achieving schemes in each case.

In this paper, we are also interested in the deterministic relay network with interference at each node. The network here consists of a source node, a destination node, and two relay nodes. Additionally, there is a disturbing node that sends signals to the two relays and causes interference with message signals. Our goal is to characterize the capacity region from source to destination in the presence of the disturbing node. Our model differs from the model in [9] in two ways. First they consider a two dimensional capacity region in which each source node tries to send messages to its designated destination. But in our model, we only try to find the one dimensional capacity from one source to the corresponding destination. On the other hand, while in both the ZS and ZZ channels the interference of the two messages only occur in one relay node, in our model we assume that the disturbing signals have interference with both relays which makes it more difficult to handle. We find the capacity and offer capacity achieving schemes that have low complexities for this relaying problem.

The organization of the material in this paper is as follows. In Section II we describe the deterministic model of relay networks and the model of the network that we study in this paper. Furthermore, we discuss the general linear coding and decoding strategies and the achievable rate of the linear schemes. In Section III we find optimal linear coding schemes and their corresponding achievable rates.

II. PROBLEM SETTING

First we briefly state the linear deterministic model of relay networks from [3], [4], then we introduce our considered network model.

A. Deterministic model

Consider a directed graph \(N(V,E)\) where \(V\) denotes the set of nodes in the network including source, relays and destination and \(E\) is the set of edges. Communication from node \(i\) to \(j\) has a nonnegative gain \(n_{i,j}\) associated with it. This number models the channel gain in the corresponding Gaussian setting. Each node \(i\) transmits a vector \(x_i \in \mathbb{F}_2^q\) and receives a vector \(y_i \in \mathbb{F}_2^q\) where \(q = \max_i (n_{i,j})\).

The received signal at each node is a deterministic function of transmitted signals at the other nodes with the following input–output relation:

\[
y_j = \sum_{k:(k,j) \in E} Q^{y_j-n_{(k,j)}} x_k \tag{1}
\]
where \( Q \) is the \( q \times q \) shift matrix given by
\[
Q = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

In this paper, we are interested in linear coding schemes where at every node \( j \) the transmitted signal is a linear function of the received signal

\[
x_j = G_j y_j. \tag{2}
\]

The goal is to design coding functions \( G_j \) such that the destination node receives enough information for decoding the message sent by the source.

B. **Diamond network with a disturbing node**

We consider a diamond relay network with a disturbing node which is depicted in Figure 1. Here \( S \) and \( D \) are respectively the source and the destination nodes and \( M \) is the source of disturbing signals. Also nodes \( A \) and \( B \) are the relay nodes. For simplicity of our analysis we assume that the links from \( M \) to the two relays have the same gain \( m \) that is also realistic in the situation where the distance between the nodes in the diamond wireless network is relatively small compared to the distance from node \( M \) which might be operating in a different network. As before, we denote the transmitted signal from node \( i \) by \( x_i \) and the received signal by \( y_i \).

Let \( G_A \) and \( G_B \) be the coding matrices at nodes \( A \) and \( B \) respectively. The signal \( y_D \) received by destination is a linear combination of the transmitted signals \( x_S \) and \( x_M \). Let

\[
y_D = G_S x_S + G_M x_M.
\]

where by application of (1) we have
\[
G_S = Q^{q-n_3} G_A Q^{q-n_1} + Q^{q-n_4} G_B Q^{q-n_2}, \tag{3}
\]
\[
G_M = Q^{q-n_3} G_A Q^{q-m} + Q^{q-n_4} G_B Q^{q-m}. \tag{4}
\]

For every choice of \( G_A \) and \( G_B \) we let \( R = R(G_A, G_B) \) denote the rate of transmission of information from \( S \) to \( D \). Also we let \( \max_{G_A, G_B} R(G_A, G_B) \) denote the capacity of transmission from \( S \) to \( D \). In the following, we explicitly find \( R(G_A, G_B) \) and in the next section we find the capacity \( C \) in terms of the network parameters.

In our analysis, we usually work with the range and dimension of matrices. For matrix \( H \) we let \( \text{range}(H) \) denote the linear span of the columns of \( H \). Also for a subspace \( S \subseteq \mathbb{F}_2^q \) we let \( \dim(S) \) denote its dimension. Obviously \( \dim(\text{range}(H)) = \text{rank}(H) \). For two matrices \( H_1, H_2 \) we also use the shorthand \( \text{rank}(H_1 \cap H_2) \) to denote \( \dim(\text{range}(H_1) \cap \text{range}(H_2)) \).

**Theorem 1.** For any choice of \( G_A \) and \( G_B \) we have
\[
R(G_A, G_B) = \text{rank}(G_S) - \text{rank}(G_S \cap G_M)
\]
where \( G_S \) and \( G_M \) are defined in (3) and (4).

For two subspaces \( S_1 \) and \( S_2 \) of \( \mathbb{F}_2^q \), let \( S_1 + S_2 = \{ s_1 + s_2 : s_1 \in S_1, s_2 \in S_2 \} \). To prove the Theorem 1 we first prove the following lemma.

**Lemma 1.** If \( S_1 \) and \( S_2 \) are two subspaces of \( \mathbb{F}_2^q \) then for any \( s \in S = S_1 + S_2 \) there exists a unique pair \( s_1 \in S_1 \) and \( s_2 \in S_2 \) with \( s_1 + s_2 = s \) if and only if \( S_1 \cap S_2 = \{ 0 \} \).

**Proof:** First suppose that \( S_1 \cap S_2 \neq \{ 0 \} \); hence, there exists at last \( t \neq 0 \) such that \( t \in S_1, t \in S_2 \). Then if \( s = s_1 + s_2 \) with \( s_1 \in S_1 \) and \( s_2 \in S_2 \), we also have \( s_1 + t \in S_1 \) and \( s_2 - t \in S_2 \) and \( (s_1 + t) + (s_2 - t) = s \). Therefore the condition is necessary. To prove the sufficiency, we notice that \( S \) is a subspace of \( \mathbb{F}_2^q \) and if \( S_1 \cap S_2 = \{ 0 \} \), we can form a basis for \( S \) by union of a basis of \( S_1 \) and a basis of \( S_2 \). Now if \( s = s_1 + s_2 = t_1 + t_2 \) and \( s_1 \neq t_1, s_2 \neq t_2 \) we can form two different expansions for \( s \) in the basis of \( S \) by using either the expansion of \( s_1 \) and \( s_2 \) in the bases of \( S_1 \) and \( S_2 \) or the expansions of \( t_1 \) and \( t_2 \) in the bases of \( S_1 \) and \( S_2 \). This contradicts the fact that each \( s \in S \) has a unique expansion in the basis of \( S \).

Next we prove Theorem 1.

**Proof:** For a linear scheme we choose a subspace \( \mathcal{X} \subseteq \mathbb{F}_2^q \) as the set of our codewords \( x_S \). Let \( \mathcal{S} = \{ G_S x_S : x_S \in \mathcal{X} \} \). For a successful decoding \( \mathcal{X} \) has to satisfy two properties; first, each \( x_S \in \mathcal{X} \) should be mapped into a unique vector in \( \mathcal{S} \). This implies that \( \dim(\mathcal{S}) = \dim(\mathcal{X}) \). Second, for every \( x_S \in \mathcal{X} \), and every \( x_M \in \mathbb{F}_2^q \), \( y_D = G_S x_S + G_M x_M \) corresponds to the unique pair of \( G_S x_S \) and \( G_M x_M \). These two conditions guarantee that each \( y_D \) corresponds to a unique codeword \( x_S \). Then, the maximum dimension of \( \mathcal{X} \) that satisfies the two conditions is the rate \( R \) of the code. Since \( G_M x_M \) can be any vector in \( \text{range}(G_M) \), to satisfy the second condition, by Lemma 1, \( \text{range}(G_M) \cap \mathcal{S} = \{ 0 \} \). \( \mathcal{S} \) has the maximum dimension when it is the largest subspace of the set \( (\text{range}(G_S) - \text{range}(G_M)) \cup \{ 0 \} \) which has a dimension of \( \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \). Next we choose \( \mathcal{X} \) to be the subspace of \( \mathbb{F}_2^q \) with dimension of \( \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \) such that \( G_S \) maps it to the set \( \mathcal{S} \). Notice that since \( \mathcal{S} \) is a subspace of \( \text{range}(G_S) \) we can always find such \( \mathcal{X} \). Therefore \( R = \dim(\mathcal{X}) = \dim(\mathcal{S}) = \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \).

**III. CAPACITY OF THE NETWORK**

In this section, we find the linear capacity of the network, that is the maximum achievable rate by a linear coding scheme.
In several steps of our capacity calculation we will use the following useful lemma:

**Lemma 2.** Let $F_{m \times n}$ and $G_{m \times n}$ be two matrices that are the same in at least their first $m - \alpha$ rows. Then

$$\text{rank}(F) - \text{rank}(F \cap G) \leq \min(n, \alpha).$$

**Proof:** Let $F = \begin{bmatrix} A \\ B \end{bmatrix}$ and $G = \begin{bmatrix} A \\ D \end{bmatrix}$ where $A$ is a $(m - \alpha) \times n$ matrix. It is easy to verify that

$$\text{rank}(F) - \text{rank}(F \cap G) = \text{rank}(\begin{bmatrix} F \\ G \end{bmatrix}) - \text{rank}(G).$$

We have

$$\text{rank}(\begin{bmatrix} F \\ G \end{bmatrix}) = \text{rank}(\begin{bmatrix} A \\ A \end{bmatrix}) \leq \text{rank}(\begin{bmatrix} A \\ B \end{bmatrix}) \leq \text{rank}(A) + \alpha.$$ 

Also, $\text{rank}(G) = \text{rank}(\begin{bmatrix} A \\ D \end{bmatrix}) \geq \text{rank}(A)$. Therefore

$$\text{rank}(\begin{bmatrix} F \\ G \end{bmatrix}) - \text{rank}(G) \leq \alpha. \text{ Also, } \text{rank}(F) \leq n \text{ and thus } \text{rank}(F) - \text{rank}(F \cap G) \leq n.$$ 

Next we derive the linear capacity of the network in Figure 1. By symmetry, we only derive the capacity for $n_1 \geq n_2$.

1. $n_1 > n_2$, $n_3 \geq n_4$ and $m \leq n_1$: Let $A$ be a $q \times q$ matrix. For two integers $n_l$ and $n_r$ consider a partition of the elements of $A$ as follows

$$A = \begin{bmatrix} q - n_r & n_r \\ q - n_l & * \\ * & * \\ * & * \end{bmatrix}$$

$$Q^{q-m}A^{q-n_r} = \begin{bmatrix} 0 & 0 \\ n_r & q - n_r \\ * & A_1 \\ * & 0 \end{bmatrix}.$$ 

Now let $G_A$ be partitioned as follows

$$G_A = \begin{bmatrix} q - m & m \\ q - n_3 + n_4 & * \end{bmatrix}$$

Then

$$Q^{q-n_3}G_AQ^{q-n_4} = \begin{bmatrix} n_1 - m & m & q - n_1 \\ 0 & 0 & 0 \\ * & A & 0 \\ * & * & 0 \end{bmatrix},$$

$$Q^{q-n_4}G_BQ^{q-n_2} = \begin{bmatrix} n_1 - m & m & q - n_1 \\ 0 & 0 & 0 \\ * & * & 0 \end{bmatrix}.$$ 

Therefore,

$$G_S = \begin{bmatrix} n_1 - m & m & q - n_1 \\ 0 & 0 & 0 \\ B_1 & A & 0 \\ B_2 & D & 0 \end{bmatrix}.$$ 

Also

$$Q^{q-n_3}G_AQ^{q-m} = \begin{bmatrix} q - n_3 \\ n_3 - n_4 \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix},$$

$$Q^{q-n_4}G_BQ^{q-m} = \begin{bmatrix} q - n_3 \\ n_3 - n_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

Therefore,

$$G_M = \begin{bmatrix} q - n_3 \\ n_3 - n_4 \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix},$$

where (a) follows by Lemma 2. On the other hand, $R$ can not exceed the value of any cut set in the absence of the noisy source, hence

$$R \leq \text{max}(n_3, n_4) = n_3$$

if we let $k = \min(m, n_4)$ then we have

$$R \leq \text{min}(n_1 - m + k, n_3). \quad (5)$$

We let matrices $G_A$ and $G_B$ be of the following forms and show that they achieve the above upper bound. Let $j = \text{min}(n_1 - m, n_3 - k)$ and for any $t$ let $I_t$ be the $t \times t$ identity matrix.

$$G_A = \begin{bmatrix} n_1 - m & m & q - n_1 \\ 0 & I_j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_B = \begin{bmatrix} n_4 - k & q - k \\ k & I_k \\ 0 & 0 \end{bmatrix}.$$ 

Then it is easy to verify that $G_S$ is of the following form

$$G_S = \begin{bmatrix} q - n_3 \\ n_3 - j - k \\ k \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

$$G_M = \begin{bmatrix} q - n_3 \\ n_3 - j - k \\ k \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. $$

Therefore,

$$Q^{q-n_3}G_AQ^{q-m} = \begin{bmatrix} q - n_3 \\ n_3 - n_4 \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix},$$

$$Q^{q-n_4}G_BQ^{q-m} = \begin{bmatrix} q - n_3 \\ n_3 - n_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

Next we find an upper bound on any achievable rate $R = \text{rank}(G_S) - \text{rank}(G_S \cap G_M)$ in terms of the network parameters and then find $G_A$ and $G_B$ that achieve the bound.
where $L$ is a lower triangular matrix with ones on its diagonal. This implies that $\text{rank}(G_S) = j + k$. Also $G_M = 0_{q \times q}$ and hence a rate of $R = j + k$ is achievable by this coding scheme. This matches our bound \((5)\).

2) $n_1 > n_2$, $n_3 \geq n_4$ and $m \geq n_1$: In this case $G_S$ and $G_M$ are as the following formats:

$$G_S = \begin{pmatrix} n_1 & q - n_1 \\ n_3 - n_4 & A \\ n_4 & D \\ \end{pmatrix},$$

$$G_M = \begin{pmatrix} q - n_3 \\ n_3 - n_4 & C_1 \\ n_4 & C_2 E \end{pmatrix}.$$

We have the following upper bound on any achievable rate $R$:

$$R = \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \leq \text{rank}(A) - \text{rank}(A \cap E) \leq \min(n_1, n_4),$$
\((6)\)

where \((a)\) follows by Lemma 2. Let $k = \min(n_1, n_4)$. We design $G_A$ and $G_B$ as follows:

$$G_A = \begin{pmatrix} q - n_1 & n_1 - k & k \\ n_3 - k & 0 & 0 \\ q - n_3 & 0 & 0 \end{pmatrix},$$

$$G_B = \begin{pmatrix} n_4 - k \\ q - n_4 \\ n_4 - k \\ q - n_4 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & 0 & I_k \end{pmatrix}.$$

Then $G_M = 0_{q \times q}$ and $G_S$ is of the following form

$$G_S = \begin{pmatrix} n_1 - k \\ k & q - n_1 \end{pmatrix},$$

where $L$ is a lower triangular matrix with ones on its diagonal. Therefore $\text{rank}(G_S) = k$ and $R = k$ is achievable. This matches bound \((5)\).

3) $n_1 > n_2$, $n_4 \geq n_3$ and $m \leq n_2$: In this case for any choices of $G_A$ and $G_B$, $G_S$ and $G_M$ are of the following forms:

$$G_S = \begin{pmatrix} n_2 - m & m & n_1 - n_2 & q - n_1 \\ n_3 & B_1 & A & 0 \\ n_3 & B_2 & D_1 & D_2 \\ \end{pmatrix},$$

$$G_M = \begin{pmatrix} q - n_4 \\ n_4 - n_3 & A \\ n_3 & E \end{pmatrix}.$$

Let us define

$$\hat{G}_M = \begin{pmatrix} n_4 - n_3 & A \\ n_3 & E \end{pmatrix}.$$

The following bound holds for any rate $R$:

$$R = \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \leq \text{rank}(B_1) + \text{rank}(A, D_1, D_2) - \text{rank}(A, D_1, D_2 \cap \hat{G}_M) \leq n_2 - m + n_3,$$
\((a)\)

where \((a)\) follows by Lemma 2. This bound, together with the cutset bound $R \leq \min(n_1, n_4, n_2 + n_3)$ in the absence of the node $M$, results in the following bound

$$R \leq \min(n_1, n_4, n_2 + n_3 - m).$$
\((7)\)

Let $k = \min(n_3, m)$. We design $G_A$ and $G_B$ as follows:

$$G_A = \begin{pmatrix} n_3 - k \\ q - n_3 & 0 & 0 & 0 & F_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_B = \begin{pmatrix} n_4 - n_3 \\ n_4 - n_3 & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 \\ q - n_4 & 0 & 0 & 0 \end{pmatrix}.$$

Then $F_1, F_2$ and $F_3$ are chosen so that $\text{rank}(F)$ is maximum

$$F = \begin{pmatrix} n_4 - n_3 \\ n_3 - k \end{pmatrix},$$

where $L$ is a lower triangular matrix with ones on its diagonal. Therefore $\text{rank}(F) = \min(n_4 - k, n_1 - m, n_2 + n_3 - m - k)$.

By this choice, $G_M = 0_{q \times q}$ and $G_S$ is of the following form

$$G_S = \begin{pmatrix} n_4 - n_3 \\ n_3 - k \end{pmatrix},$$

where $L$ is a lower triangular matrix with ones on its diagonal. Therefore

$$\text{rank}(G_S) = k + \text{rank}(F) = \min(n_4 - k, n_1 - m, n_2 + n_3 - m - k) + k = \min(n_4, n_1 - m + k, n_2 + n_3 - m) = \min(n_4, n_1 - m + n_3, n_1 - m + m, n_2 + n_3 - m) = \min(n_1, n_4, n_2 + n_3 - m)$$

and $R = \min(n_1, n_4, n_2 + n_3 - m)$ is achievable. This matches our bound \((7)\).
4) $n_1 > n_2$, $n_4 \geq n_3$ and $m > n_2$: In this case $G_S$ and $G_M$ are of the following forms

\[ G_S = n_4 - n_3 \begin{pmatrix} q - n_4 & n_2 & n_1 - n_2 & q - n_1 \\ n_3 & A & 0 & 0 \\ 0 & D_1 & D_2 & 0 \end{pmatrix}, \]

\[ G_M = n_4 - n_3 \begin{pmatrix} q - n_4 & m - n_2 & n_2 & q - m \\ n_3 & C_1 & A & 0 \\ 0 & C_2 & E & 0 \end{pmatrix}. \]

Let us define the following matrix

\[ \hat{G}_M = n_4 - n_3 \begin{pmatrix} q - n_4 & 0 & 0 \\ n_3 & 0 & 0 \\ 0 & A & 0 \\ 0 & E & 0 \end{pmatrix}. \]

Then we have the following bound on $R$:

\[
R = \text{rank}(G_S) - \text{rank}(G_S \cap G_M) \\
\leq \text{rank}(G_S) - \text{rank}(G_S \cap \hat{G}_M) \\
\leq \min(n_1, n_3), \tag{8}
\]

where (a) follows by Lemma 2. Let $k = \min(n_1, n_3)$. We design $G_A$ and $G_B$ as follows:

\[ G_A = k \begin{pmatrix} n_3 - k & 0 & 0 \\ k & 0 & I_k \\ q - n_3 & 0 & 0 \end{pmatrix}, \]

\[ G_B = k \begin{pmatrix} n_4 - k & q - k & 0 \\ k & 0 & I_k \\ q - n_4 & 0 & 0 \end{pmatrix}. \]

It is easy to verify that $G_M = 0_{k \times k}$ and $G_S$ is of the following form

\[ G_S = k \begin{pmatrix} n_1 - k & 0 & 0 \\ q - k & 0 & 0 \\ 0 & L & 0 \end{pmatrix}. \]

where $L$ is a lower triangular matrix with ones on its diagonal. Therefore $R = \text{rank}(G_S) = k$. This matches the bound (9).

5) $n_1 = n_2$, $m \geq n_1$ : In this case it is easy to verify that for any choice of $G_A$ and $G_B$, columns of $G_S$ are a subset of columns of $G_M$. Therefore $\text{rank}(G_S \cap G_M) = \text{rank}(G_S)$ and hence $R = \text{rank}(G_S) - \text{rank}(G_S \cap G_M) = 0$.

6) $n_1 = n_2$, $m < n_1$ : In this case for any choice of $G_A$ and $G_B$, columns of $G_M$ are a subset of columns of $G_S$. Therefore $\text{rank}(G_S \cap G_M) = \text{rank}(G_M)$, and $R = \text{rank}(G_S) - \text{rank}(G_M)$. Since there are at most $n_1 - m$ columns in $G_S$ that does not appear in $G_M$, $\text{rank}(G_S) - \text{rank}(G_M) \leq n_1 - m$ and therefore $R \leq n_1 - m$. Also from the cutset bound $R \leq \max(n_3, n_4)$ we have

\[ R \leq \min(n_1 - m, \max(n_3, n_4)). \tag{9} \]

Let $k = \min(n_1 - m, \max(n_3, n_4))$. Consider two cases:

- If $n_3 \geq n_4$ then we set $G_B = 0_{k \times q}$ and let

\[ G_A = k \begin{pmatrix} q - n_1 & k & n_1 - k \\ q - k & 0 & I_k \\ 0 & 0 & 0 \end{pmatrix}. \]

In this case $G_M = 0_{q \times k}$ and

\[ G_S = k \begin{pmatrix} q - n_3 & k & n_3 - k \\ q - k & 0 & I_k \\ 0 & 0 & 0 \end{pmatrix}. \]

Therefore $R = \text{rank}(G_S) - \text{rank}(G_M) = k$, that achieves bound (9).

- If $n_4 > n_3$ then we set $G_A = 0_{q \times k}$ and let

\[ G_B = k \begin{pmatrix} q - n_1 & k & n_1 - k \\ q - k & 0 & I_k \\ 0 & 0 & 0 \end{pmatrix}. \]

Similar to the previous case, in this case $G_M = 0_{q \times k}$ and

\[ G_S = k \begin{pmatrix} q - n_4 & k & n_4 - k \\ q - k & 0 & I_k \\ 0 & 0 & 0 \end{pmatrix}. \]

Therefore $R = \text{rank}(G_S) - \text{rank}(G_M) = k$, that achieves bound (9).

Remark 1. It is easy to verify that for cases 1, 3 and 6 where $m$ can be set to zero, the capacity of diamond network [3] is achievable by our coding scheme.

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