SU(2|1) Supersymmetric Mechanics

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Abstract. We review the salient features of the new kind of $\mathcal{N}=4$ supersymmetric mechanics associated with the supergroup SU(2|1), proceeding from the worldline SU(2|1) superspace formalism.

1. Motivations

Supersymmetric Quantum Mechanics (SQM) [1] is the simplest $(d=1)$ supersymmetric theory. It probes the structure of higher-dimensional supersymmetric theories via the dimensional reduction and, in addition, displays a number of distinguishing features on its own. For instance, it provides superextensions of integrable models like Calogero-Moser systems and Landau-type models. An extended $d=1$ supersymmetry exhibits interesting specific properties, such as dualities between various supermultiplets [2], the existence of nonlinear “cousins” of off-shell linear multiplets [3], etc.

Symmetry algebra of the standard “flat” $\mathcal{N}$ extended SQM is

$$\{Q^A, Q^B\} = 2\delta^{AB}H, \quad [H, Q^A] = 0, \quad A, B = 1 \ldots \mathcal{N}. \tag{1.1}$$

Recently, a considerable attention was paid to rigid supersymmetric theories based on curved analogs of the Poincaré supergroup in diverse dimensions (see, e.g., [4]). With this in mind, it is of obvious interest to consider “curved” analogs of SQM.

A way to define such generalized SQM models is suggested by $\mathcal{N}=2, d=1$ “Poincaré” superalgebra in the complex notation

$$Q = \frac{1}{\sqrt{2}}(Q^1 + iQ^2), \quad \bar{Q} = \frac{1}{\sqrt{2}}(Q^1 - iQ^2),$$

$$\{Q, \bar{Q}\} = 2H, \quad Q^2 = \bar{Q}^2 = 0, \quad [H, Q] = [H, \bar{Q}] = 0. \tag{1.2}$$

The relations (1.2) form the superalgebra su(1|1), with $H$ as the central charge generator.

This two-fold interpretation of $\mathcal{N}=2, d=1$ “Poincaré” superalgebra suggests two ways of extending it to higher-rank $d=1$ supersymmetries.

A. The standard extension:

$$\mathcal{N}=2, d=1 \quad \Rightarrow \quad \mathcal{N}>2, d=1 \text{ “Poincaré”},$$

B. A non-standard extension:

$$\mathcal{N}=2, d=1 \equiv u(1|1) \subset su(2|1) \subset su(2|2) \subset \ldots .$$
The superspace coordinates $SU(2\,|\,1)$ primary aim was to construct the worldline superfield realizations of $SU(2\,|\,1)$ and to show that all off-shell multiplets of $N = 4, d = 1$ supersymmetry have the well-defined $SU(2\,|\,1)$ analogs [5]. It turned out, e.g., that the ”weak supersymmetry” models [9] are associated with the $SU(2\,|\,1)$ multiplet (1, 4, 3).

The supergroup $SU(2\,|\,1)$ has also invariant chiral subspaces which are carriers of the chiral multiplets (2, 4, 2) [5, 6]. The relevant component actions naturally provide the bosonic $d = 1$ Wess-Zumino-type terms. Besides this, $SU(2\,|\,1)$ also admits a supercoset [8] which is an analog of the harmonic analytic superspace of the $N = 4, d = 1$ supersymmetry [10]. There, $SU(2\,|\,1)$ counterparts of the multiplets (4, 4, 1) can be defined.

In [7] we demonstrated that the superconformal $D(2, 1; \alpha)$ invariant models in the $SU(2\,|\,1)$ superspace formulation naturally yield trigonometric realization of the $d = 1$ conformal group $SO(2, 1)$ [11, 12].

2. $SU(2\,|\,1)$ superspace

The (central-extended) superalgebra $su(2\,|\,1)$ is written as:

$$\{Q^i, \bar{Q}_j\} = 2m (I^i_j - \delta^i_j F) + 2\delta^i_j H, \quad [I^i_j, I^k_l] = \delta^k_l I^i_j - \delta^i_l I^k_j,$$

$$[I^i_j, \bar{Q}_l] = \frac{1}{2} \delta^i_l Q_j - \delta^i_j Q_l, \quad [I^i_j, Q^k_l] = \delta^i_j Q^k - \frac{1}{2} \delta^i_j Q^k,$$

$$[F, \bar{Q}_l] = -\frac{1}{2} \bar{Q}_l, \quad [F, Q^k_l] = \frac{1}{2} Q^k.$$

The basic $SU(2\,|\,1)$ worldline superspace is identified with the supercoset:

$$\frac{SU(2\,|\,1)}{SU(2) \times U(1)} \sim \{Q^i, \bar{Q}_j, H, I^i_j, F\} / \{I^i_j, F\}.$$ 

The superspace coordinates $\{t, \theta_i, \bar{\theta}^i\}$ are the parameters associated with the coset generators. The supercoset element can be conveniently parametrized as

$$g = \exp \left( itH + i\bar{\theta}^i Q^i - i\bar{\theta}^i \bar{Q}_j \right), \quad \bar{\theta}_i = \left[ 1 - \frac{2m}{3} \left( \bar{\theta} \cdot \bar{\theta} \right) \right] \theta_i.$$

The $SU(2\,|\,1)$ superspace coordinates are transformed under the $Q, \bar{Q}$ left shifts as

$$\delta \theta_i = \epsilon_i + 2m (\epsilon \cdot \theta) \theta_i, \quad \delta \bar{\theta}^i = \bar{\epsilon}^i - 2m (\epsilon \cdot \bar{\theta}) \bar{\theta}^i,$$

$$\delta t = i \left[ (\epsilon \cdot \theta) + (\bar{\epsilon} \cdot \bar{\theta}) \right].$$

The $SU(2\,|\,1)$ generators in the superspace realization read

$$Q^i = -i \frac{\partial}{\partial \theta_i} + 2im\bar{\theta} \bar{\theta}^i \frac{\partial}{\partial \bar{\theta}^i} + \bar{\theta}^i \frac{\partial}{\partial t}, \quad \bar{Q}_j = i \frac{\partial}{\partial \bar{\theta}^j} + 2im\theta_j \theta_k \frac{\partial}{\partial \theta_k} - \theta_j \frac{\partial}{\partial t},$$

$$I^i_j = \left( \bar{\theta}^i \frac{\partial}{\partial \bar{\theta}^j} - \theta_j \frac{\partial}{\partial \theta_i} \right) - \frac{\delta^i_j}{2} \left( \bar{\theta}^k \frac{\partial}{\partial \bar{\theta}^k} - \theta_k \frac{\partial}{\partial \theta_k} \right),$$

$$F = \frac{1}{2} \left( \bar{\theta}^k \frac{\partial}{\partial \bar{\theta}^k} - \theta_k \frac{\partial}{\partial \theta_k} \right), \quad H = i\bar{\theta}^i.$$
The corresponding covariant derivatives are defined by
\[
D_i = \left[ 1 + m (\bar{\theta} \cdot \theta) - \frac{3m^2}{4} (\bar{\theta} \cdot \theta)^2 \right] \frac{\partial}{\partial \theta_i} - m\bar{\theta}_j \frac{\partial}{\partial \theta_j} - i\bar{\theta} \frac{\partial}{\partial t} + \ldots ,
\]
\[
D_j = - \left[ 1 + m (\bar{\theta} \cdot \theta) - \frac{3m^2}{4} (\bar{\theta} \cdot \theta)^2 \right] \frac{\partial}{\partial \theta^j} + m\bar{\theta}^k \frac{\partial}{\partial \theta^k} + i\theta_j \frac{\partial}{\partial t} + \ldots .
\]
Here, “dots” stand for matrix $U(2)$ connection parts. The invariant integration measure is defined by the expression $d\zeta = dt d^2 \theta d^2 \bar{\theta} (1 + 2m \theta \cdot \bar{\theta})$, \quad $\delta d\zeta = 0$.

3. (1, 4, 3) multiplet
The (1, 4, 3) multiplet is described by the real neutral superfield $G(t, \theta, \bar{\theta})$ satisfying
\[
\varepsilon^{ij} D_i D_j G = \varepsilon_{ij} D^i D^j G = 0 \Rightarrow
\]
\[
G = x - mx (\bar{\theta} \cdot \theta) [1 - 2m (\bar{\theta} \cdot \theta)] + \frac{x}{2} (\bar{\theta} \cdot \theta)^2 - i (\theta \cdot \bar{\theta}) \left( \theta_i \bar{\psi}^i + \bar{\theta}^i \tilde{\bar{\psi}}_j \right)
\]
\[
+ \left[ 1 - 2m (\bar{\theta} \cdot \theta) \right] \left( \theta_i \bar{\psi}^i - \bar{\theta}^i \tilde{\bar{\psi}}_j \right) + \bar{\theta}^j \theta_i B_{ij}^1, \quad B_{ik}^1 = 0 .
\]
The irreducible set of off-shell fields is \(x(t), \psi^i(t), \bar{\psi}_i(t), B_{ij}^1(t)(B_{ik}^1 = 0)\).

The $\epsilon$ transformation properties of the component fields are as follows
\[
\delta x = (\bar{\epsilon} \cdot \psi) - (\epsilon \cdot \bar{\psi}), \quad \delta \psi^i = i\bar{\epsilon} \bar{\psi}^i - m\bar{\epsilon}^i \bar{x} + \bar{\epsilon}^k B_{ik}^1 ,
\]
\[
\delta B_{(i)j} = -2i \left[ \epsilon_{(i)j} \psi_j + \bar{\epsilon}_{(i)j} \bar{\psi}_j \right] + 2m \left[ \epsilon_{(i)j} \bar{\psi}_j - \bar{\epsilon}_{(i)j} \psi_j \right] .
\]
The general off-shell Lagrangian is written as
\[
\mathcal{L}_{\text{off}} = - \int dt^2 \theta^2 \bar{\theta} (1 + 2m \theta \cdot \bar{\theta}) f(G), \quad S = \int dt \mathcal{L}_{\text{off}} .
\]
After doing $\theta$ integral and eliminating the auxiliary field, we obtain $\mathcal{L}_{\text{on}}$ as
\[
\mathcal{L}_{\text{on}} = \dot{x}^2 g(x) + i \left( \bar{\psi}_i \bar{\psi}^i - \bar{\psi}_i \psi^i \right) g(x) - \frac{1}{2} (\bar{\psi}_i \psi^i)^2 \left[ g''(x) - \frac{3(g'(x))^2}{2g(x)} \right]
\]
\[
- m^2 \bar{x}^2 g(x) + 2m \bar{\psi}_i \psi^i g(x) + mx \bar{\psi}_i \psi^i g'(x), \quad g(x) := f''(x) .
\]
It can be simplified by passing to the new variables $y(x), \zeta^i$,
\[
\dot{x}^2 g(x) = \frac{1}{2} y^2 , \quad \Rightarrow y'(x) = \sqrt{2g(x)} , \quad \zeta^i = \psi^i y'(x),
\]
\[
\mathcal{L}_{\text{on}} = \frac{\dot{y}^2}{2} + i \left( \bar{\zeta}_i \zeta^i - \bar{\zeta}_i \zeta^i \right) - \frac{m^2}{2} V^2 (y) + m \bar{\zeta}_i V' (y)
\]
\[
- \frac{1}{2} (\bar{\zeta}_i \zeta^i)^2 \partial_y \left( \frac{V'(y) - 1}{V(y)} \right), \quad V(y) := \frac{x(y)}{x'(y)} .
\]
This Lagrangian defines the SQM model with “weak” $\mathcal{N} = 4$ supersymmetry [9].

3.1 Quantization
The simplest choice in (3.1) is $f(x) = \frac{\dot{x}^2}{4}$:
\[
\mathcal{L}_{\text{on}}^{(\text{free})} = \frac{\dot{x}^2}{2} - \frac{m^2 \bar{x}^2}{2} + i \left( \bar{\psi}_i \bar{\psi}^i - \bar{\psi}_i \psi^i \right) + m \bar{\psi}_i \psi^i ,
\]
\[
\delta x = (\bar{\epsilon} \cdot \psi) - (\epsilon \cdot \bar{\psi}) , \quad \delta \psi^i = i\bar{\epsilon} \bar{\psi}^i - m\bar{\epsilon}^i \bar{x} .
\]
The corresponding conserved Noether charges and Hamiltonian are computed to be
\[ Q^i = \psi^i (p - imx), \quad \hat{Q}_i = \bar{\psi}_i (p + imx), \]
\[ H = \frac{\hat{p}^2}{2} + \frac{m^2 x^2}{2} + m \hat{\psi}^i \bar{\psi}_i, \quad F = \frac{1}{2} \hat{\psi}^k \bar{\psi}_k, \quad I_j^i = \hat{\psi}^i \bar{\psi}_j - \frac{1}{2} \delta^i_j \hat{\psi}^k \bar{\psi}_k. \]

We quantize as
\[ [\hat{x}, \hat{p}] = i, \quad \{\hat{\psi}^i, \bar{\psi}_j\} = \delta^i_j, \quad \hat{p} = -i \partial_x, \quad \bar{\psi}_j = \partial / \partial \hat{\psi}^j. \]
The quantum Hamiltonian and (super)charges which form the superalgebra \( su(2|1) \) are
\[ \hat{H} = \frac{1}{2} (\hat{p} + im \hat{x}) (\hat{p} - im \hat{x}) + m \hat{\psi}^i \bar{\psi}_i, \quad \hat{F} = \frac{1}{2} \hat{\psi}^k \bar{\psi}_k, \quad \hat{I}_j^i = \hat{\psi}^i \bar{\psi}_j - \frac{1}{2} \delta^i_j \hat{\psi}^k \bar{\psi}_k, \]
\[ \hat{Q}^i = \hat{\psi}^i (\hat{p} - im \hat{x}), \quad \hat{Q}_i = \bar{\psi}_i (\hat{p} + im \hat{x}). \]

3.2 Spectrum

The super wave-function \( \Omega^{(\ell)} \) at the energy level \( \ell \geq 2 \), reveals the four-fold degeneracy
\[ \Omega^{(\ell)} = a^{(\ell)} |\ell\rangle + b^{(\ell)} |\ell - 1\rangle + \frac{1}{2} c^{(\ell)} \varepsilon_{ij} \psi^i \bar{\psi}^j |\ell - 2\rangle, \quad \ell \geq 2, \]
where \( |\ell\rangle, |\ell - 1\rangle, |\ell - 2\rangle \) are the harmonic oscillator functions. We treat the operators \( \hat{p} \pm im \hat{x} \) in \( \hat{H} \) as the creation and annihilation operators and impose the standard conditions
\[ \hat{\psi}_i |\ell\rangle = 0, \quad (\hat{p} - im \hat{x}) |0\rangle = 0, \quad (\hat{p} + im \hat{x}) |\ell\rangle = |\ell + 1\rangle. \]

The spectrum of the Hamiltonian is then
\[ \hat{H} \Omega^{(\ell)} = m \ell \Omega^{(\ell)}, \quad m > 0. \]

The ground state \( (\ell = 0) \) and the first excited states \( (\ell = 1) \) are special, they encompass non-equal numbers of bosonic and fermionic states:
\[ \Omega^{(0)} = a^{(0)} |0\rangle, \quad \Omega^{(1)} = a^{(1)} |1\rangle + b^{(1)} |\psi^i \bar{\psi}_i\rangle. \]
The ground state is annihilated by all \( SU(2|1) \) generators including \( Q^i \) and \( \hat{Q}_i \), so it is \( SU(2|1) \) singlet. The states with \( \ell = 1 \) form the fundamental \( (2|1) \) representation of \( SU(2|1) \). The supercharges act on them as
\[ Q^i \psi^k |0\rangle = 0, \quad \hat{Q}_i \psi^k |0\rangle = \delta^k_i |1\rangle, \]
\[ Q^i |1\rangle = 2m \psi^i |0\rangle, \quad \hat{Q}_i |1\rangle = 0. \]
The states with \( \ell > 1 \) form the representations \( (2|2) \), with equal numbers of bosonic and fermionic states.

For the considered model, the \( SU(2|1) \) Casimirs are nicely expressed as
\[ m^2 C_2 = \hat{H} (\hat{H} - m), \quad m^3 C_3 = \hat{H} (\hat{H} - m) (\hat{H} - \frac{m}{2}) \]
\[ C_2(\ell) = (\ell - 1) \ell, \quad C_3(\ell) = (\ell - 1/2) (\ell - 1) \ell. \]
The ground state with \( \ell = 0 \) is atypical [13], because Casimir operators are zero on it. On the states with \( \ell = 1 \) both Casimirs as well vanish, so these states also form an atypical (fundamental) \( SU(2|1) \) representation. On the \( \ell > 1 \) states both Casimirs are non-zero, so these states form typical [13] \( SU(2|1) \) representations.
4. Chiral multiplet

The $SU(2|1)$ counterpart of the $\mathcal{N}=4$, $d=1$ chiral multiplet $(2,4,2)$ can also be defined. This is due to the existence of the chiral coset

$$\{Q^i, \bar{Q}^i, H, R^i, F\} \sim (t_L, \theta_i), \quad t_L = t + \frac{i}{2m} \ln(1 + 2m \bar{\theta} \cdot \theta),$$

$$\delta \theta_i = \epsilon_i + 2m (\bar{\epsilon} \cdot \theta) \theta_i, \quad \delta t_L = 2i (\bar{\epsilon} \cdot \theta).$$

The multiplet $(2,4,2)$ is described by the chiral superfield $\Phi$

$$\mathcal{D}_j \Phi = 0, \quad \bar{\mathcal{D}}^j \Phi = 0, \quad \bar{F} \Phi = 2\kappa \Phi,$$

where, in general, $\kappa \neq 0$. The solution of this constraint is

$$\Phi = \left[1 + m (\bar{\theta} \cdot \theta)\right]^{-\kappa} \varphi_L(t_L, \theta), \quad \varphi_L(t_L, \theta) = z + \sqrt{2} \theta_1 \xi^i + \bar{\epsilon}^i \theta_j \xi_j F.$$  

The superfield $\varphi_L$ and its components transform as

$$\delta^* \varphi_L = 4\kappa m (\bar{\epsilon}^i \theta_j) \varphi_L \Rightarrow$$

$$\delta z = -\sqrt{2} \epsilon_i \xi^i, \quad \delta \xi^i = \sqrt{2} i \epsilon^i \nabla_t z - \sqrt{2} \bar{\epsilon}^i \epsilon^j F,$$

$$\delta F = -\sqrt{2} \bar{\epsilon}^i \epsilon^j \left[ m \xi^i + i \nabla_t \xi^i \right], \quad \nabla_t := \partial_t + 2i \kappa m.$$  

The general superfield Lagrangian reads

$$\mathcal{L}^{(\kappa)}_{\text{off}} = \frac{1}{4} \int d^2 \theta d^2 \bar{\theta} (1 + 2m \bar{\theta} \cdot \theta) f(\Phi, \Phi^\dagger).$$

It results in the following component on-shell Lagrangian, with $g := f_{zz}$,

$$\mathcal{L}^{(\kappa)}_{\text{on}} = g \dot{z} \dot{z} + 2i \kappa m (\dot{\bar{z}} \bar{z} - \dot{z} \bar{z}) g - \frac{i m}{2} (\dot{\bar{z}} f_z - \dot{z} f_{\bar{z}}) - \frac{i}{2} (\bar{\xi} \cdot \bar{\xi}) (\dot{\bar{z}} g_{\bar{z}} - \dot{z} g_z) + \frac{i}{2} (\bar{\xi} \cdot i \xi^i - \bar{\xi} \bar{\xi}^i) g - m^2 V - m (\bar{\xi} \cdot \bar{\xi}) U + \frac{1}{2} (\bar{\xi} \cdot \bar{\xi}) R, \quad R = g_{zz} - g_{\bar{z} \bar{z}} g, \quad V = \kappa (\dot{\bar{z}} \partial_{\bar{z}} + z \partial_z) f - \kappa^2 (\dot{z} \partial_{\bar{z}} + \bar{z} \partial_z)^2 f, \quad U = \kappa (\dot{z} \partial_{\bar{z}} + z \partial_z) g - (1 - 2\kappa) g.$$  

It is invariant, up to a total derivative, under the transformations

$$\delta z = -\sqrt{2} \epsilon_i \xi^i, \quad \delta \xi^i = \sqrt{2} i \epsilon^i \nabla_t z + \sqrt{2} \epsilon_i \epsilon^j \xi^j g_z g,$$

The bosonic “core” of this Lagrangian is

$$\mathcal{L}^{(\kappa)}_{\text{bos}} = g \dot{z} \dot{z} + 2i \kappa m (\dot{\bar{z}} \bar{z} - \dot{z} \bar{z}) g - \frac{i m}{2} (\dot{\bar{z}} f_z - \dot{z} f_{\bar{z}}) - m^2 V,$$

$$V = \kappa (\dot{\bar{z}} \partial_{\bar{z}} + z \partial_z) f - \kappa^2 (\dot{z} \partial_{\bar{z}} + \bar{z} \partial_z)^2 f.$$  

We observe that the standard $\mathcal{N}=4$, $d=1$ kinetic term is deformed to a non-trivial Lagrangian with WZ-term, as well as a potential term. The basic novel feature compared to the standard $\mathcal{N}=4$ Kähler sigma model Lagrangian for the multiplet $(2,4,2)$ is just the necessary presence of this WZ term with the strength $\sim m$.

4.1 Model on a complex plane
The relevant quantum Hamiltonian and (super)charges are:

\[ L_{\text{free}}^{(s)} = \dot{\bar{z}} \dot{z} + i m \left( 2\kappa - \frac{1}{2} \right) (\dot{\bar{z}} \dot{z} - \bar{z} \dot{z}) + \frac{i}{2} \left( \bar{\xi}_i \dot{\xi}^i - \xi^i \dot{\xi}_i \right) + 2\kappa (2\kappa - 1) m^2 \bar{z} z + (1 - 2\kappa) m (\dot{\xi} \cdot \dot{\xi}) . \]

This Lagrangian is invariant (modulo a total derivative) under the transformations

\[ \delta z = -\sqrt{2} \epsilon_i \xi^i , \quad \delta \xi^i = \sqrt{2} i \epsilon_i \dot{z} - 2\sqrt{2} \kappa m \epsilon^3 z . \]

The relevant quantum Hamiltonian and (super)charges are:

\[ \hat{H} = \nabla \bar{z} \nabla z - 2\kappa m (\dot{\bar{z}} \partial_z - \dot{z} \partial_{\bar{z}}) + m (1 - 2\kappa) \hat{\eta}^k \hat{\eta}_k , \]

\[ \hat{Q}^i = \sqrt{2} \hat{\eta}^j \nabla \bar{z} , \quad \hat{Q}_j = \sqrt{2} \hat{\eta}_j \nabla z , \]

\[ \hat{F} = -2\kappa (\dot{\bar{z}} \partial_z - \dot{z} \partial_{\bar{z}}) - \left( 2\kappa - \frac{1}{2} \right) \hat{\eta}^k \hat{\eta}_k , \quad \hat{F}_j = \hat{\eta}^i \hat{\eta}_j - \frac{1}{2} \delta^i_j \hat{\eta}^k \hat{\eta}_k , \]

with

\[ \nabla z = -i \partial_z - \frac{i}{2} m \bar{z} , \quad \nabla \bar{z} = -i \partial_{\bar{z}} + \frac{i}{2} m z , \quad [\nabla z, \nabla \bar{z}] = m . \]

These generators form \( su(2|1) \) superalgebra.

### 4.2 Wave functions and spectrum

We take advantage of the fact that there exists an extra \( U(1) \) charge generator commuting with all \( SU(2|1) \) generators,

\[ \hat{E} = - (\dot{\bar{z}} \partial_z - \dot{z} \partial_{\bar{z}}) - \hat{\eta}^k \hat{\eta}_k . \]

Then the relevant wave functions can be constructed in terms of bosonic eigenfunctions of this external generator

\[ \Omega^{(\alpha)} = \bar{z}^\alpha A(z \bar{z}) , \quad \hat{E} \Omega^{(\alpha)} = \alpha \Omega^{(\alpha)} , \quad (4.2) \]

\( \alpha \) being some positive real number. Requiring this set to simultaneously form the full set of the eigenfunctions of the bosonic part of the Hamiltonian yields

\[ \Omega^{(\alpha)} \rightarrow \Omega^{(\ell; \alpha)} , \quad \hat{H} \Omega^{(\ell; \alpha)} = m(\ell + 2\kappa \alpha) \Omega^{(\ell; \alpha)} , \]

\[ \Omega^{(\ell; \alpha)} = \bar{z}^\alpha e^{-\frac{m z}{2}} L^{(\alpha)}_\ell (m z \bar{z}) , \]

where \( L^{(\alpha)}_\ell \) are Laguerre polynomials and \( \ell \) is Landau level.

Acting by supercharges on \( \Omega^{(\ell; \alpha)} \) and imposing the obvious vacuum condition,

\[ \hat{\eta}_j \Omega^{(\ell; \alpha)} = 0 \Rightarrow \hat{Q}_j \Omega^{(\ell; \alpha)} = 0 , \]

we obtain other eigenstates of \( \hat{H} \) and \( \hat{E} \):

\[ \Psi^{(\ell; \alpha)} = \left[ a^{(\ell; \alpha)} + b^{(\ell; \alpha)} \eta^j \nabla \bar{z} + \frac{1}{2} c^{(\ell; \alpha)} \varepsilon_{ij} \eta^i \eta^j \nabla^2 \right] \Omega^{(\ell; \alpha)} , \quad \ell \geq 2 , \]

\[ \Psi^{(1; \alpha)} = a^{(1; \alpha)} \Omega^{(1; \alpha)} + b^{(1; \alpha)} \eta^j \nabla \bar{z} \Omega^{(1; \alpha)} , \quad \Psi^{(0; \alpha)} = a^{(0; \alpha)} \Omega^{(0; \alpha)} . \]
The ground state ($\ell = 0$) is $SU(2|1)$ singlet. The wave functions with $\ell = 1$ form the atypical fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states), while those with $\ell \geq 2$ form typical $(2, 2)$ representations.

The Casimir operators in the case at hand are represented as

$$m^2 C_2 = \left( \hat{H} - 2\kappa m \hat{E} \right) \left( \hat{H} - 2\kappa m \hat{E} - m \right),$$
$$m^3 C_3 = \left( \hat{H} - 2\kappa m \hat{E} \right) \left( \hat{H} - 2\kappa m \hat{E} - m \right) \left( \hat{H} - 2\kappa m \hat{E} - \frac{m}{2} \right),$$

whence

$$C_2(\ell) = (\ell - 1) \ell, \quad C_3(\ell) = (\ell - 1/2) (\ell - 1) \ell.$$  

They are vanishing for the wave functions with $\ell = 0, 1$, confirming the interpretation of the corresponding representations as atypical, and are non-vanishing on the wave functions with $\ell \geq 2$, implying them to form typical representations of $SU(2|1)$.

5. Generalized $SU(2|1)$ chirality

One can choose another $SU(2|1)$ coset as the basic superspace

$$\frac{SU(2|1) \times U(1)_{\text{ext}}}{SU(2) \times U(1)_{\text{ext}}} = \frac{SU(2|1)}{SU(2)} \sim \{Q^i, \bar{Q}_j, \hat{H}, I_j^i \} \sim \{I_j^i \}.$$  

The Hamiltonian is now the full internal $U(1)$ generator $\hat{H} = H - m F$. The covariant derivatives $D_i, \bar{D}^i$ are $U(1)$ inert and a generalized chirality condition can be imposed

$$(\cos \lambda \bar{D}_i - \sin \lambda D_i) \Phi = 0,$$  

(5.1)

$\lambda$ being a new real parameter. Eq. (5.1) is solved by $\Phi = \varphi_L(\hat{t}_L, \hat{\theta}_L)$, where $(\hat{t}_L, \hat{\theta}_L)$ is the proper set of the chiral coordinates. The components of $\varphi_L$ are transformed as

$$\delta z = -\sqrt{2} \cos \lambda (\epsilon \cdot \xi) e^{\frac{m t}{2}} + \sqrt{2} \sin \lambda (\epsilon \cdot \xi) e^{-\frac{m t}{2}},$$
$$\delta \xi^a = \sqrt{2} e^{\epsilon} [i \cos \lambda \zeta - \sin \lambda B] e^{-\frac{m t}{2}} - \sqrt{2} e^{-\epsilon} [i \sin \lambda \zeta + \cos \lambda B] e^{\frac{m t}{2}},$$
$$\delta B = \sqrt{2} \cos \lambda [i (\bar{\epsilon} \cdot \bar{\xi}) + \frac{m}{2} (\bar{\epsilon} \cdot \xi)] e^{-\frac{m t}{2}} + \sqrt{2} \sin \lambda [i (\epsilon \cdot \bar{\xi}) - \frac{m}{2} (\epsilon \cdot \xi)] e^{\frac{m t}{2}}.$$  

The most general $SU(2|1)$ invariant action of $\varphi^a(\hat{t}_L, \hat{\theta}_L), a = 1, \ldots, N$, is

$$S_{\text{kin}} = \int dt \mathcal{L}_{\text{kin}} = \frac{1}{4} \int d\zeta f(\varphi^a, \bar{\varphi}^\alpha).$$  

Its on-shell bosonic core

$$\mathcal{L}_{\text{kin}}^{\text{on}} = g_{ab} \bar{z}^a z^b - \frac{i}{2} m \cos 2\lambda (\bar{z}^a f_a - z^a f_a) - \frac{m^2}{4} g^{ab} \sin^2 2\lambda f_a f_b$$

is recognized as the Lagrangian of the Kähler oscillator [14] extended by a coupling to an external magnetic field.

The supercharges do not commute with the Hamiltonian $\hat{H}$, but are still conserved due to their explicit $t$-dependence

$$\frac{d}{dt} Q^i = \partial_t Q^i + \{Q^i, \hat{H}\}_{\text{P.B.}} = 0, \quad \frac{d}{dt} \bar{Q}_j = \partial_t \bar{Q}_j + \{\bar{Q}_j, \hat{H}\}_{\text{P.B.}} = 0.$$  

7
6. Superconformal models

The most general \( \mathcal{N} = 4, d = 1 \) superconformal algebra is \( D(2, 1; \alpha) \) [15]:

\[
\{ Q_{\alpha i'}, Q_{\beta j'} \} = 2 \left( \epsilon_{ij} \epsilon_{i'j'} T_{\alpha\beta} + \alpha \epsilon_{\alpha\beta} \epsilon_{i'j'} J_{ij} - (1 + \alpha) \epsilon_{\alpha\beta} \epsilon_{ij} L_{i'j'} \right),
\]

\[
[T_{\alpha\beta}, Q_{\gamma i'}] = -i \epsilon_{\gamma(\alpha} Q_{\beta)i'}, \quad [T_{\alpha\beta}, T_{\gamma\delta}] = i (\epsilon_{\alpha\gamma} T_{\beta\delta} + \epsilon_{\beta\delta} T_{\alpha\gamma}),
\]

\[
[J_{ij}, Q_{\alpha k'}] = -i \epsilon_{k'i} Q_{\alpha ij'}, \quad [J_{ij}, J_{kl}] = i (\epsilon_{ik} J_{jl} + \epsilon_{il} J_{jk}),
\]

\[
[L_{i'j'}, Q_{\alpha k'}] = -i \epsilon_{k' i} Q_{\alpha ij'}, \quad [L_{i'j'}, L_{k'l'}] = i (\epsilon_{i'k'} L_{j'l'} + \epsilon_{j'l'} L_{i'k'}).
\]

Here, \( Q_{\alpha i'} \) are eight supercharges and the bosonic subalgebra is

\[
\text{su}(2) \oplus \text{su}(2') \oplus \text{so}(2, 1) \equiv \{ J_{ik} \} \oplus \{ L_{i'k'} \} \oplus \{ T_{\alpha\beta} \}.
\]

At \( \alpha = -1, 0 \), \( D(2, 1; \alpha) \) is reduced to the semi-direct product

\[
D(2, 1; \alpha) \cong \text{PSU}(1, 1|2) \times \text{SU}(2)_{\text{ext}}, \quad \alpha = 0, -1.
\]

How to implement \( D(2, 1; \alpha) \) in the \( SU(2|1) \) superspaces? The crucial property is the existence of two different \( su(2|1) \subset D(2, 1; \alpha) \), so that the latter is a closure of them:

I. \( \{ Q^i, \tilde{Q}_j \} = 2m(\mu) I^i_j + 2\delta^i_j [H(\mu) - m(\mu) F], \)

II. \( \{ S^i, \tilde{S}_j \} = 2m(-\mu) I^i_j + 2\delta^i_j [H(-\mu) - m(-\mu) F], \)

\[
m(\mu) := -\alpha \mu, \quad H(\mu) := H + \mu F, \quad H = \hat{H} + \frac{\mu^2}{4} \hat{K},
\]

\[
(\hat{H}, \hat{K}) \in \text{so}(2, 1), \quad F \in \text{su}(2'), \quad I^i_j \in \text{su}(2).
\]

The remaining \( D(2, 1; \alpha) \) generators appear in \( \{ Q, S \} \) and \( \{ Q, \tilde{S} \} \). The supergroup \( SU(2|1) \) (I) is identified with the manifest superisometry of the \( SU(2|1) \) superspace; then \( SU(2|1) \) (II) is realized on the superspace coordinates and superfields as a hidden symmetry. One faces the “trigonometric” realization of the \( d = 1 \) conformal generators [11, 12]:

\[
\hat{H} = \frac{i}{2} [1 + \cos \mu \tau] \partial_\tau, \quad \hat{K} = \frac{2i}{\mu^2} [1 - \cos \mu \tau] \partial_\tau, \quad \hat{D} = \frac{i}{\mu} \sin \mu \tau \partial_\tau.
\]

The basic superfield constraints are \( D(2, 1; \alpha) \) covariant, at least for some special values of \( \alpha \). The multiplet \( (1, 4, 3) \) is superconformal for any \( \alpha \), the chiral multiplet admits the superconformal symmetry only for \( \alpha = 0, -1 \). The superconformal subclasses of the general \( SU(2|1) \) actions are singled out by requiring them to be even functions of \( \mu \), in accord with the above structure of \( D(2, 1; \alpha) \) as a closure of its two \( SU(2|1) \) subgroups.

6.1 Some examples of superconformal actions

i. The multiplet \( (1, 4, 3) \):

\[
S^{(\alpha)}_{\text{conf}} = -\int d\zeta f(G), \quad f(G) = \begin{cases} \frac{1}{8(\alpha+1)} G^{-\frac{1}{\alpha}} & \text{for } \alpha \neq -1, 0, \\ \frac{1}{8} G \ln G & \text{for } \alpha = -1. \end{cases}
\]

The simplest choice is \( \alpha = -1/2 \) yielding the free Lagrangian

\[
L^{(\alpha=-1/2)}_{\text{conf}} = \frac{\dot{y}^2}{2} + i \left( \bar{\zeta} \dot{\zeta}^i - \dot{\bar{\zeta}} \dot{\zeta}^i \right) + \tilde{B}_j B^j - \frac{\mu^2}{8} y^2.
\]
The most general $G$ superfield superconformally covariant constraints read

$$D^2 \bar{G} = D^2 G = 0, \quad [D, \bar{D}] \bar{G} = 4m \bar{G} - 4c.$$ 

At $c \neq 0$ they are covariant only under the $\alpha = -1$ supergroup. The bosonic sector of $L^{(c,\alpha=-1)}_{conf}$ contains the standard conformal potential:

$$L^{(c,\alpha=-1)}_{conf} \Rightarrow \frac{\dot{y}^2}{2} - \frac{\mu^2 y^2}{8} - \frac{c^2}{8y^2}.$$ 

ii. The standard chiral multiplet $(2, 4, 2)$:

$$\hat{D}_i \Phi = 0, \quad \bar{D}^j \bar{\Phi} = 0, \quad \hat{F} \Phi = 2 \kappa \Phi.$$ 

The constraints are covariant only for $\alpha = -1$. At $\kappa \neq 0$:

$$S^{(\kappa)}_{conf} = \frac{1}{4} \int d\zeta \, f(\Phi, \bar{\Phi}), \quad f = (\Phi \bar{\Phi})^{\frac{1}{4n}}.$$ 

Neither WZ term, nor standard conformal potential appear, only the oscillator term is present. At $\kappa = 1/4$, we reproduce the free Lagrangian:

$$L^{(\kappa=1/4)}_{conf} = \frac{i}{\lambda} \dot{z} \bar{z} + \frac{i}{2} \left( \bar{\xi} \dot{\xi} - \dot{\bar{\xi}} \xi \right) - \frac{\mu^2}{4} \bar{z} \bar{z}.$$ 

(6.1)

iii. The generalized chiral multiplet $(2, 4, 2)$:

$$(\cos \lambda \hat{D}_i - \sin \lambda \bar{D}_i) \Phi = 0.$$ 

It is also covariant under $D(2, 1; \alpha = -1)$ only.

In both cases, the only superconformally invariant superpotential is the holomorphic integral $S^{bot}_{conf} \sim \nu \int d\zeta_L \ln \varphi_L + \text{c.c.}$ yielding the standard conformal potential $\sim -\nu^2 |z|^2$. The extra potential $\sim \mu^2 \bar{z} \bar{z}$ comes always from the sigma-model action. At any $\kappa$ the conformal action is reduced, by a field redefinition, to the free one (6.1) plus $S^{bot}_{conf}$.

7. Harmonic $SU(2|1)$ superspace

The harmonic extension of the standard $SU(2|1)$ superspace is defined as follows:

$$\{t, \theta_i, \bar{\theta}^j\} \Rightarrow \{t, \theta_i, \bar{\theta}^j, w^+_i, w^-_i\}, \quad w^+_i w^-_i = 1,$$

$$\delta \theta_i = \epsilon_i + 2m \bar{\epsilon} \bar{\theta}_k \theta_k, \quad \delta \bar{\theta}^j = \bar{\epsilon}^j - 2m \epsilon_k \bar{\theta}^k \bar{\theta}^j, \quad \delta t = i \left( \epsilon_k \bar{\theta}^k + \bar{\epsilon} \theta_k \right),$$

$$\delta w^+_i = \lambda^+ w^-_i, \quad \delta w^-_i = 0, \quad \lambda^{+/-} = -m \left( 1 - m \bar{\theta}^k \theta_k \right) \left( \bar{\theta}^k \bar{\epsilon}^j + \theta^k \epsilon^j \right) w^+_k w^-_j.$$ 

Passing to the analytic basis is accomplished by

$$\{t, \theta_i, \bar{\theta}^j, w^+_i\} \Rightarrow \{t_{(A)}, \theta^\pm, \bar{\theta}^\pm, w^\pm_i\},$$

$$\delta \theta^+ = \epsilon^+ + m \bar{\theta}^+ \theta^+ \epsilon^-, \quad \delta \bar{\theta}^- = \bar{\epsilon}^+ - m \bar{\theta}^- \theta^+ \bar{\epsilon}^-, \quad \delta t_{(A)} = 2i \left( \epsilon^+ \bar{\theta}^+ + \bar{\epsilon}^- \theta^+ \right),$$

$$\delta \theta^- = \epsilon^- - m \bar{\epsilon}^- \theta^- \theta^+, \quad \delta \bar{\theta}^+ = \bar{\epsilon}^- - m \bar{\epsilon}^- \bar{\theta}^+ \bar{\theta}^-, \quad \delta w^+_i = -m \left( \bar{\theta}^+ \epsilon^+ + \theta^+ \bar{\epsilon}^+ \right) w^-_i, \quad \delta w^-_i = 0, \quad \epsilon^\pm = \epsilon^i w^\pm_i, \quad \bar{\epsilon}^\pm = \bar{\epsilon}^i w^\pm_i.$$ 

The analytic subspace closed under the $SU(2|1)$ transformation amounts to the set

$$\zeta_A := \{t_{(A)}, \bar{\theta}^+, \theta^+, w^\pm_i\}.$$
In the analytic basis, the algebra of covariant derivatives (both spinor and harmonic) contains the following closed subalgebra, the so called “CR-structure”:
\[
\{ \mathcal{D}^+, \mathcal{D}^+ \} = -2m \mathcal{D}^{++}, \quad [\mathcal{D}^{++}, \mathcal{D}^+] = [\mathcal{D}^{++}, \bar{\mathcal{D}}^+] = 0,
\]
where
\[
\mathcal{D}^+ = \frac{\partial}{\partial \theta^+} + m \bar{\theta}^- \mathcal{D}^{++}, \quad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial \bar{\theta}^-} - m \theta^- \mathcal{D}^{++}
\]
and \( \mathcal{D}^{++} = w^{++} \partial_{w^{--}} + \ldots \) is one of the three harmonic derivatives forming an \( su(2) \) algebra. Analytic superfields are defined as:
\[
\mathcal{D}^+ \varphi^{++} = \bar{\mathcal{D}}^+ \varphi^{++} = 0 \Rightarrow \mathcal{D}^{++} \varphi^{++} = 0.
\]
The simplest choice is \( q^{+a}, \bar{q}^{+a} = \varepsilon_{ab} q^{+b} \), describing the off-shell multiplet \( (4, 4, 0) \)
\[
q^{+a} (\zeta_A) = x^{ia} w^{+}_i + \theta^+ \psi^a + \bar{\theta}^- \bar{\psi}^a - 2i \theta^+ \bar{\theta}^- x^{ia} w^{-}_i.
\]
It transforms as
\[
\delta q^{+a} = -m \left( \bar{\theta}^- \varepsilon^+ + \theta^+ \varepsilon^- \right) q^{+a} \Rightarrow \delta x^{ia} = -e^i \psi^a - e^i \bar{\psi}^a, \quad \delta \bar{\psi}^a = 2i \epsilon_k x^{a}_k m \epsilon_k x^k, \quad \delta \psi^a = 2i \epsilon_k x^{a}_k m \epsilon^k x^k.
\]
The sigma-model type invariant action is written as:
\[
S (q^{\pm a}) = \int d^4 \zeta_{L} (q^{+a} q^{-a}), \quad q^{-a} = \mathcal{D}^{-} q^{+a}
\]
and, in components, yields:
\[
\mathcal{L} = G \left[ x^{ia} \dot{x}_{ia} + \frac{i}{2} \left( \bar{\psi}_a \dot{\psi}^a - \bar{\psi}^a \dot{\psi}_a \right) + \frac{m}{2} \psi^a \bar{\psi}_a \right] - \frac{i}{2} x^{ia} \partial_a G \left( \psi_a \bar{\psi}^c + \psi^c \bar{\psi}_a \right)
\]
\[
- \frac{\Delta_x G}{16} (\bar{\psi})^2 (\psi)^2 + \frac{m}{2} x^2 G \psi^a \bar{\psi}_a - \frac{m^2}{4} x^2 G, \quad G := \Delta_x L(x^2) = G(x^2),
\]
where \( \Delta_x = \varepsilon^{ik} \varepsilon_{ab} \partial_{ia} \partial_{kb} \). The simplest Lagrangian corresponds to \( L = q^{+a} q^{-a} \):
\[
\mathcal{L}_{\text{free}} = x^{ia} \dot{x}_{ia} + \frac{i}{2} \left( \bar{\psi}_a \dot{\psi}^a - \bar{\psi}^a \dot{\psi}_a \right) + \frac{m}{2} \psi^a \bar{\psi}_a - \frac{m^2}{4} x^2 x^{ia} x_{ia}.
\]
As distinct from the flat \( (4, 4, 0) \) case \([10]\), no explicit \( SU(2) \) invariant Wess-Zumino term can be constructed. An internal WZ term, coming from the kinetic sigma-model action, can be defined for \( F q^{+a} \neq 0 \), where \( F \) is the matrix part of the generator \( F \).

7.1 Quantization in the free case

The quantum Hamiltonian and supercharges are given by:
\[
H = -\frac{1}{4} (\partial^a - m x^a) (\partial_a + m x_a) + \frac{m}{2} \bar{\xi}^a \xi_a,
\]
\[
Q_i = -i \xi^a (\partial_i - m x_{ia}), \quad \bar{Q}_i = -i \bar{\xi}^a (\partial_i + m x_{ia}),
\]
\[
F = -\frac{1}{2} \bar{\xi}^a \xi^a, \quad I_{ik} = x^{i}_{(a} \partial_{k)a}, \quad E_{ab} = x^{i}_{(a} \partial_{b)} - \bar{\xi}_{(a} \xi_{b)}, \quad [E_{ab}, E_{cd}] = \varepsilon_{cb} E_{ad} - \varepsilon_{ad} E_{cb}.
\]
Here $E_{ab}$ are the extra $SU(2)_{PG}$ generators commuting with $SU(2|1)$. The ground state conditions are imposed as:

$$\xi^a |0\rangle = 0, \quad (\partial_{ia} + mx_{ia}) |0\rangle = 0 \quad \Rightarrow \quad |0\rangle = e^{-\frac{m^2}{2} x^2}, \quad Q^f |0\rangle = \tilde{Q}_f |0\rangle = 0.$$ 

The general bosonic state $|\ell; s\rangle$ is constructed as:

$$|\ell; s\rangle = A_{(i_1 i_2 \ldots i_{s})} (a_1 a_2 \ldots a_s) \nabla^{i_1 a_1} \nabla^{i_2 a_2} \ldots \nabla^{i_s a_s} (\nabla^{i a} \nabla_{ia})^{\ell} |0\rangle,$$

(7.1)

where $s/2$ is the highest weight ("isospin") of the irreducible representation of the group $SU(2)_{PG}$ acting on indices $a$. The spin content with respect to $SU(2)$ is the same.

The general wave function $\Omega^{(\ell,s)}$ is a collection of $|\ell; s\rangle$ and their fermionic descendants obtained through action of $\tilde{Q}_f$ on $|\ell; s\rangle$. The spectrum is:

$$H \Omega^{(\ell,s)} = \frac{m}{2} (2\ell + s) \Omega^{(\ell,s)}, \quad m > 0.$$

The Casimirs are given by the following expressions

$$m^2 C_2 = H (H + m) - \frac{m^2}{2} E^a E_a, \quad m^2 C_3 = \left( H + \frac{m^2}{2} \right) C_2,$$

$$\frac{1}{2} E^a E_a \Omega^{(\ell,s)} = \frac{s(s+1)}{2} \Omega^{(\ell,s)}.$$

All cases with $\ell \geq 0$ correspond to the typical representations, with the degeneracy $4(s+1)^2$ and equal number of bosonic and fermionic states. The wave function $\Omega^{(0,s)}$ describes atypical representations, with $(s+1)^2$ bosonic and $s(s+1)$ fermionic states and the degeneracy $(2s+1)(s+1)$.

### 7.2 Mirror $(4, 4, 0)$ multiplet

The standard $(4, 4, 0)$ multiplet has the content $(x^a, \xi^a), i$ being the doublet index of $SU(2)_{\text{int}}$ and $a$ - of $SU(2)_{PG}$; the mirror $(4, 4, 0)$ multiplet has the content $(y^A, \psi^A), A$ being the doublet index of some other $SU'(2)_{PG}$. In the flat $N = 4, d = 1$ supersymmetry these multiplets and the relevant SQM models are equivalent, up to switching two $SU(2)$ automorphism algebras. In the $SU(2|1)$ case, the models based on these two multiplets cease to be equivalent.

The salient feature of the mirror multiplet is the existence of the explicit $SU(2|1)$ invariant Wess-Zumino term:

$$\tilde{L}_{WZ} = 2\gamma \left\{ i (y^A \partial_A f - \bar{y}^A \bar{\partial}_A f) - \frac{m}{2} (y^A \partial_A f + \bar{y}^A \bar{\partial}_A f) - \frac{1}{2} \psi^A \psi^B \partial_A \bar{\partial}_B f \right\},$$

where $f(y, \bar{y})$ obey the constraints

$$\Delta_y f = 0, \quad m (y^B \partial_B - \bar{y}^B \bar{\partial}_B) f (y, \bar{y}) = 0.$$

It is interesting to study the corresponding SQM models.

### 8. Summary and outlook

We reviewed a new type of $N = 4$ SQM based on the supergroup $SU(2|1)$. It is a deformation of the standard $N = 4$ SQM by a mass parameter $m$. We presented the superfield formalism on two different coset supermanifolds of $SU(2|1)$ treated as the real and chiral $SU(2|1), d = 1$ superspaces. They are carriers of the off-shell multiplets $(1, 4, 3)$ and $(2, 4, 2)$. There are two non-equivalent types of the $(2, 4, 2)$ multiplet.
The $SU(2|1)$ SQM models reveal surprising features. For the $(1,4,3)$ multiplet, the oscillator-type potential terms are present in the sigma-model action. For the $(2,4,2)$ models, the kinetic term is accompanied by the $d = 1$ WZ term and potential terms.

We also constructed harmonic $SU(2|1)$ superspace and described two sorts of the “root” $(4,4,0)$ multiplets in its framework. The mirror $(4,4,0)$ multiplet admits an explicit $SU(2|1)$ invariant $d = 1$ super WZ term.

In all cases the sets of the quantum states reveal deviations from the standard rule of equality of the bosonic and fermionic states, in accordance with the existence of atypical $SU(2|1)$ representations.

Superconformally invariant subclasses of the $SU(2|1)$ actions were constructed, based on the property that the superconformal algebra $D(2,1;\alpha)$ is a closure of its two $su(2|1)$ subalgebras. The $d = 1$ conformal generators are always in the trigonometric realization.

In conclusion, we outline some further possible lines of study.

(a). Multi-particle extensions: to take a few superfields of one or different types, to construct the relevant off- and on-shell actions, to quantize, to identify the relevant target bosonic geometries ($m$-deformed?), etc.

(b). To inquire whether the remaining $\mathcal{N} = 4, d = 1$ multiplets (e.g. the multiplet $(3,4,1)$) have their $SU(2|1)$ counterparts and to construct the corresponding SQM models. By analogy with the flat $\mathcal{N} = 4, d = 1$ harmonic superspace approach [10], one can expect that all such multiplets and the associate SQM models follow from the multiplets $(4,4,0)$ and their SQM models through the well defined gauging procedure [16].

(c). To generalize all this to the next in complexity case of the supergroup $SU(2|2)$. It involves 8 supercharges and so can presumably be treated as a deformation of $\mathcal{N} = 8, d = 1$ supersymmetry (and of $\mathcal{N} = (4,4), d = 2$ supersymmetry, in fact).

(d). To establish possible links with the higher-dimensional theories with “curved” supersymmetries in the localization approach [4].

Finally, it is worth to mention a recent application of the $SU(2|1)$ SQM considerations for calculation of the vacuum energy in some $d > 1$ supersymmetric models [17, 18].

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