An octonionic formulation of the M-theory algebra

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We give an octonionic formulation of the $\mathcal{N}=1$ supersymmetry algebra in $D=11$, including all brane charges. We write this in terms of a novel outer product, which takes a pair of elements of the division algebra $A$ and returns a real linear operator on $A$. More generally, with this product comes the power to rewrite any linear operation on $\mathbb{R}^n$ ($n=1, 2, 4, 8$) in terms of multiplication in the $n$-dimensional division algebra $A$. Finally, we consider the reinterpretation of the $D=11$ supersymmetry algebra as an octonionic algebra in $D=4$ and the truncation to division subalgebras.

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INTRODUCTION

A recurring theme in the study of supersymmetry and string theory is the connection to the four division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. See, for example, [1–8]. The octonions are of particular interest in this context since they may be used to describe representations of the Lorentz group in spacetime dimensions $D=10, 11$, where string and M-theory live. Furthermore, the octonions provide a natural explanation [9, 10] for the appearance of exceptional groups as the U-dualities of supergravities and M-theory live. Furthermore, the octonions are of particular interest in this context since they may be used to describe representations of the Lorentz group in spacetime dimensions $D=8$. The octonions are of particular interest in this context.

In the present paper we tackle this problem by introducing a novel outer product, which takes a pair of elements belonging to a division algebra $A$ and returns a real linear operator on $A$, expressed using multiplication in $A$. This product enables one to rewrite any expression involving $n \times n$ matrices and $n$-dimensional vectors in terms of multiplication in the $n$-dimensional division algebra $A$. We solve the problem of the octonionic M-algebra using this product, which allows a derivation of the correct $\{Q, Q\}$ bracket. In the final section we consider “Cayley-Dickson halving” the octonionic M-algebra, which corresponds to its reinterpretation as the maximal supergravity algebra in $D=7, 5, 4$. For example, the M-algebra may be considered to be an octonionic rewriting of the $D=4$, $\mathcal{N}=8$ supersymmetry algebra; from this perspective the $D=4$, $\mathcal{N}=1$ algebra comes from a truncation $O \rightarrow R$.

THE DIVISION ALGEBRAS

A normed division algebra is an algebra $A$ equipped with a positive-definite norm satisfying the condition

$$||xy|| = ||x|| ||y||.$$  \hspace{1cm} (1)

Remarkably, there are only four such algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, with dimensions $n=1, 2, 4$ and $8$, respectively.

A division algebra element $x \in A$ is written as the linear combination of $n$ basis elements with real coefficients: $x = x_a e_a$, with $x_a \in \mathbb{R}$ and $a = 0, \cdots, (n-1)$. One basis element $e_0 = 1$ is real; the other $(n-1)$ $e_i$ are imaginary:

$$e_i^2 = 1, \quad e_i \neq e_j.$$ \hspace{1cm} (2)

where $i = 1, \cdots, (n-1)$. In analogy with the complex case, we define a conjugation operation indicated by $\ast$, which changes the sign of the imaginary basis elements:

$$e_0^\ast = e_0, \quad e_i^\ast = -e_i.$$ \hspace{1cm} (3)

The multiplication rule for the basis elements of a division algebra is given by:

$$e_a e_b = (\delta_{ab} \delta_{bc} + \delta_{ab} \delta_{ac} - \delta_{ab} \delta_{bc} - C_{abc}) e_c \equiv \Gamma_{bc}^{a} e_c, \quad e_a^\ast e_b = (\delta_{ab} \delta_{bc} - \delta_{ab} \delta_{ac} + \delta_{ab} \delta_{bc} - C_{abc}) e_c \equiv \Gamma_{bc}^{a \ast} e_c.$$ \hspace{1cm} (4)

where we define the structure constants$^1$

$$\Gamma_{bc}^{a} = \delta_{ab} \delta_{bc} + \delta_{ab} \delta_{ac} - \delta_{ab} \delta_{bc} - C_{abc}, \quad \Gamma_{bc}^{a \ast} = \delta_{ab} \delta_{bc} - \delta_{ab} \delta_{ac} + \delta_{ab} \delta_{bc} - C_{abc} \Rightarrow \Gamma_{bc}^{a} = \Gamma_{bc}^{a \ast}.$$ \hspace{1cm} (5)

The tensor $C_{abc}$ is totally antisymmetric with $C_{0ab} = 0$, so it is identically zero for $A = \mathbb{R}, \mathbb{C}$. For the quaternions

$^1$ The unusual choice of index structure is for later convenience - see equations (6) and (7).
$C_{ijk}$ is simply the permutation symbol $\varepsilon_{ijk}$, while for the octonions the non-zero $C_{ijk}$ are specified by the set of oriented lines of the Fano plane, see [15].

One of the most important properties of the division algebras is that they provide a representation of the SO($n$) Clifford algebra. This is reflected in the structure constants, which satisfy

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}\mathbb{1},$$
$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}\mathbb{1}.\quad(6)$$

In other words, we have the interpretation that multiplying a division algebra element $\psi$ by the basis element $e_a$ has the effect of multiplying $\psi$'s components by the gamma matrix $\Gamma^a$:

$$e_a\psi = e_a e_b \psi_b = \Gamma^a_{bc} e_c \psi_b = e_c \Gamma^a_{cb} \psi_b.\quad(7)$$

This property is essential for many of the applications of division algebras to physics, including that of this paper.

A natural inner product [15] on $\mathbb{A}$ is given by:

$$\langle x | y \rangle = \frac{1}{2} (x^* y + y^* x) = x_a y_a \quad\text{i.e.} \quad \langle e_a | e_b \rangle = \delta_{ab}.\quad(8)$$

This is just the canonical inner product on $\mathbb{R}^n$.

**A NEW OUTER PRODUCT**

It is interesting to see what other linear operations on $\mathbb{R}^n$ look like when written in terms of the division-algebraic multiplication rule. This was explored in [16], but we take a different approach here. Consider the following general problem. Given some linear operator on $\mathbb{R}^n$ expressed as an $n \times n$ matrix $M_{ab}$, we would like to find an operator $\hat{M}$ on the division algebra $\mathbb{A}$ such that $\hat{M}$ has the effect of multiplying the components of $x = x_a e_a \in \mathbb{A}$ by $M_{ab}$:

$$\hat{M} x = e_a M_{ab} x_b. \quad(9)$$

An explicit form for this operator can be found using the inner product above. First we rewrite

$$M_{ab} = M_{cd} \langle e_a | e_c \rangle \langle e_b | e_d \rangle = \frac{1}{2} M_{cd} \langle e_a | e_c e_d e_b \rangle + e_c (e_d^* e_b).\quad(10)$$

Now it is clear that the operator

$$\hat{M} = \frac{1}{2} M_{cd} \langle e_c | e_d^* \cdot + e_c (\cdot)^* e_d \rangle,\quad(11)$$

where a dot represents a slot for an octonion, has matrix elements

$$\langle e_a | \hat{M} e_b \rangle = M_{ab}.\quad(12)$$

This suggests that we write the outer product for division algebra elements using their multiplication rule, defining:

$$\times : \mathbb{A} \otimes \mathbb{A} \to \text{End}(\mathbb{A})$$

$$e_a \otimes e_b \mapsto e_a \times e_b \equiv \frac{1}{2} \left( e_a (e_b^* \cdot) + e_a (\cdot)^* e_b \right).\quad(13)$$

With the new product comes the power to rewrite any expression involving $n \times n$ matrices and $n$-dimensional vectors in terms of multiplication in the $n$-dimensional division algebra $\mathbb{A}$.

It is useful to note various equivalent ways of writing the outer product above:

$$e_a \times e_b = \frac{1}{2} \left( e_a (e_b^* \cdot) + e_a (\cdot)^* e_b \right) = \frac{1}{2} \left( (\cdot) e_b + (e_b^* \cdot) e_a \right) = \frac{1}{2} \left( (\cdot) e_b + e_b \cdot (\cdot)^* e_a \right).\quad(14)$$

Due to the alternativity of the division algebras we also have

$$e_a (e_b^* \cdot) + e_a (\cdot)^* e_b = (e_a e_b^* \cdot) + (e_a (\cdot)^* e_b),\quad(15)$$

and similarly for the other four possibilities above.

**OCTONIONIC SPINORS IN $D = 11$**

In $D = 11$ the Majorana spinor may be written as a 32-component real column vector. However, if we consider $\mathbb{R}^{32}$ as the tensor product $\mathbb{R}^4 \otimes \mathbb{R}^8 \cong \mathbb{R}^4 \otimes \mathbb{O}$ then we can write this as a 4-component octonionic column vector

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \quad \lambda_a \in \mathbb{O}, \quad \alpha = 1, 2, 3, 4. \quad(16)$$

A natural set of generators $\{\gamma^M\} = \{\gamma^0, \gamma^{a+1}, \gamma^9, \gamma^{10}\}$, $M = 0, 1, \ldots, 10$ for the $4 \times 4$ octonionic Clifford algebra is then given by

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma^{a+1} = \begin{pmatrix} 0 & 0 & 0 & e_a^* \\ 0 & 0 & e_a & 0 \\ 0 & e_a & 0 & 0 \\ e_a & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^9 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma^{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with $a = 0, 1, \ldots, 7$. These matrices satisfy

$$\gamma^M \gamma^N + \gamma^N \gamma^M = 2\eta^{MN} \mathbb{1},$$

(18)
and the infinitesimal Lorentz transformation of the spinor $\lambda$ is

$$\delta \lambda = \frac{1}{4} \omega_{MN} \gamma^M (\gamma^N \lambda),$$

(19)

where $\omega_{MN} = -\omega_{NM}$. In general, the action of the rank $r$ Clifford algebra element on $\lambda$ can be written

$$\gamma^{[M_1} \hat{\gamma}^{M_2} \cdots (\gamma^{M_{r-1}} (\gamma^{M_r]} \lambda)) \cdots)).$$

(20)

The positioning of the brackets in the above expression follows from repeated application of (7); non-associativity matters only for the imaginary gamma matrices $\gamma^{i+1}$, which provide a representation of the SO(7) Clifford algebra. If we define an operator $\hat{\gamma}^M$, whose action is left-multiplication by $\gamma^M$, then we can think of the rank $r$ Clifford algebra element as the operator

$$\hat{\gamma}^{[M_1} \hat{\gamma}^{M_2} \cdots \hat{\gamma}^{M_r]},$$

(21)

where the operators $\hat{\gamma}^M$ must be composed as

$$\hat{\gamma}^M \hat{\gamma}^N \lambda = \gamma^M (\gamma^N \lambda) \neq (\gamma^M \gamma^N) \lambda.$$

(22)

This ensures that the action of $\hat{\gamma}^{[M_1} \hat{\gamma}^{M_2} \cdots \hat{\gamma}^{M_r]}$ on a spinor is given by (20), as required.

THE OCTONIONIC M-ALGEBRA

The anti-commutator of two supercharges in the $D = 11$ supergravity theory is conventionally written as the ‘M-algebra’ [17, 18]

$$\{Q_\alpha, Q_\beta\} = (\gamma^M C)_{\alpha\beta} P_M + (\gamma^{MN} C)_{\alpha\beta} Z_{MN} + (\gamma^{MNPQR} C)_{\alpha\beta} Z_{MNPQR},$$

(23)

where $\alpha, \beta = 1, \ldots, 32$, $P_M$ is the generator of translations and $Z_{MN}$ and $Z_{MNPQR}$ are the brane charges. The charge conjugation matrix $C_{\alpha\beta}$ serves to lower an index on each of the gamma matrices.

The left-hand side is a symmetric 32 $\times$ 32 matrix with 528 components, while the terms on the right-hand side consist of the rank 1, 2 and 5 Clifford algebra elements, which form a basis for such symmetric Clifford matrices. In terms of SO(1, 10) representations:

$$\begin{pmatrix} 32 & 32 \end{pmatrix}_{\text{Sym}} = 11 + 55 + 462.$$

(24)

We would like to write this algebra in terms of $4 \times 4$ octonionic matrices. However, the space of octonionic $4 \times 4$ matrices is of dimension $16 \times 8 = 128$, and hence naively does not carry nearly enough degrees of freedom to write.

The solution to this problem is to use the octonionic Clifford algebra operators $\hat{\gamma}^{[M_1} \hat{\gamma}^{M_2} \cdots \hat{\gamma}^{M_r]}$ defined in the previous section. These operators (including all ranks $r$) span a space of dimension $32 \times 32 = 1024$. In other words, their octonionic matrix elements are

$$\langle e_a | \gamma^M \gamma^b e_b \rangle = \gamma^M \delta_{ab}, \quad \alpha, \beta = 1, 2, 3, 4,$$

(25)

and if we think of $aa$ as a composite spinor index $\alpha = 1, \ldots, 32$, then the set of $\{\gamma^M \gamma^a\}$ generates the usual real Clifford algebra as in [23].

For the charge conjugation matrix, we define the $4 \times 4$ real matrix (which is numerically equal to $\gamma^0$ but with a different index structure)

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

(26)

The octonionic matrix elements of this are then trivially

$$C_{\alpha a \beta b} = \langle e_a | C_{\alpha\beta} e_b \rangle = C_{\alpha\beta} \delta_{ab},$$

(27)

which can be identified with the $32 \times 32$ matrix:

$$C_{\alpha\beta} = C_{\alpha a \beta b} = C_{\alpha\beta} \delta_{ab}.$$

(28)

Armed with these tools, the right-hand side can then be written over $O$ simply by replacing $\alpha \rightarrow a$ and putting hats on the gammas:

$$(\hat{\gamma}^M C)_{\alpha\beta} P_M + (\hat{\gamma}^{MN} C)_{\alpha\beta} Z_{MN} + (\hat{\gamma}^{MNPQR} C)_{\alpha\beta} Z_{MNPQR},$$

(29)

With the identification $\tilde{\alpha} = \alpha a$, we can also write the left-hand side of (23) in terms of the composite indices:

$$\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\} \equiv \{Q_{\alpha a}, Q_{\beta b}\}.$$

(30)

Now, the expression (29) is an octonionic operator with matrix elements as on the right-hand side of (23), so on the left we require an octonionic operator

$$\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\}$$

(31)

with matrix elements given by (30). The required operator is obtained simply by contracting (30) with the outer product $e_a \times e_b$ defined in (13):

$$\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\} \equiv \{Q_{\alpha a}, Q_{\beta b}\} e_a \times e_b.$$

(32)

The octonionic formulation of the $M$-algebra is then

$$\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\} = (\hat{\gamma}^M C)_{\alpha\beta} P_M + (\hat{\gamma}^{MN} C)_{\alpha\beta} Z_{MN} + (\hat{\gamma}^{MNPQR} C)_{\alpha\beta} Z_{MNPQR}.$$

(33)

Using the first two versions of the outer product given in (14), we could write the left-hand side as

$$\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\} = \frac{1}{2} \left( \lbrace Q_{\alpha a}, Q_{\beta b} \rbrace \cdot \cdot + \{ Q_{\alpha} \cdot | Q_{\beta} \cdot \cdot + Q_{\alpha} \cdot | Q_{\beta} \cdot \} \right).$$

(34)

The first two terms look similar to the more intuitive anti-commutator $\{Q_{\tilde{\alpha}}, Q_{\tilde{\beta}}\}$, explored in [14], but to reproduce the full $M$-algebra we require all four terms above.
RELATION TO LOWER DIMENSIONS

It is interesting to consider the octonionic version of the supersymmetry algebra after an $11 = 4 + 7$ split:

$$\text{SO}(1,10) \supset \text{SO}(1,3) \times \text{SO}(7).$$  \hfil (35)

Seven of the Clifford algebra generators $\gamma^{i+1}$ are imaginary, while the other four are real. This suggests that we split the dimensions as follows:

$$M = 0,1,\ldots,10 \rightarrow i = 1,\ldots,8, \quad \mu = 0,1,9,10. \hfil (36)$$

In $D = 4$ we regard the $D = 11$ octonionic spinor $Q_{\alpha a}e_a$ as eight 4-component Majorana spinors $Q_{\alpha a}$, which we may leave packaged as an ‘internal’ octonion. This transforms as the spinor $8$ of SO(7). The $D = 4$ interpretation of the octonionic gamma matrices is as follows:

$$\hat{\gamma}_{i+1} = \gamma_i \hat{e}_i,$$  \hfil (37)

where $\hat{e}_i$ denotes the operator whose action is left-multiplication by $e_i$ and $\gamma_i$ (otherwise known as $\gamma_5$) is the highest rank Clifford element:

$$\gamma_i = -\gamma^0 \gamma^i \gamma^9 \gamma^{10} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \hfil (38)$$

The matrix $C_{\alpha \beta}$ is just the charge conjugation matrix in $D = 4$.

We do not split the $M, N$ indices of equation (33) into $\mu$ and $i$ parts here, as the expression of the right-hand side itself is not particularly illuminating. The result is a copy of the $N = 8$ supersymmetry algebra written over the octonions. The interesting point is that the $D = 11$ supersymmetry algebra can be reinterpreted as an octonionic $D = 4$ algebra.

More generally, the spinor and associated gamma matrices defined in (16) and (17) correspond to those of $D = 4,5,7$ if we replace $O$ with $R, C, H$, respectively - see Table I. This means that in this framework the minimal supersymmetry algebra in these dimensions is written over $R, C, H$, while doubling the amount of supersymmetry corresponds to Cayley-Dickson doubling the division algebra. This process terminates when we reach maximal supersymmetry, i.e. when the Cayley-Dickson process takes us to $O$, the largest normed division algebra.

$$D \setminus N \setminus 1 R^{32} C^{16} H^{16} C^{8} H^{8} C^{4} H^{4} O^{4}$$

| $D \setminus N$ | 1 | 2 | 4 | 8 |
|---------------|---|---|---|---|
| 11            | $O^4$ | $H^4$ | $O^4$ | $H^4$ | $O^4$ | $H^4$ | $O^4$ |
| 7             | $H^4$ | $O^4$ | $H^4$ | $O^4$ | $H^4$ | $O^4$ | $H^4$ | $O^4$ |
| 5             | $C^4$ | $H^4$ | $C^4$ | $H^4$ | $C^4$ | $H^4$ | $C^4$ | $H^4$ |
| 4             | $R^4$ | $C^4$ | $H^4$ | $C^4$ | $H^4$ | $C^4$ | $H^4$ | $C^4$ |

TABLE I. A summary of the division algebraic parameterisation of spinors used in $D = n + 3$ supersymmetry algebras. Note that supersymmetry algebras sharing the same $\Lambda$ are equivalent and that Cayley-Dickson doubling $\Lambda$ corresponds to doubling $N$, or equivalently climbing upwards in dimension $D$.

The above discussion serves to emphasise the correspondence between the octonions and maximal supersymmetry in various dimensions. Rather than thinking of the M-theory algebra as an eleven-dimensional real algebra, it may be fruitful to think of it as a four-dimensional octonionic one, as in Table II.

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