Quasiperiodicity in time evolution of Bloch vector under thermal Jaynes-Cummings model

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Abstract
We study a quasiperiodic structure in the time evolution of the Bloch vector, whose dynamics is governed by the thermal Jaynes-Cummings model (JCM). Putting the two-level atom into a certain pure state and the cavity field into a mixed state in thermal equilibrium at initial time, we let the whole system evolve according to the JCM Hamiltonian. During this time evolution, motion of the Bloch vector seems to be in disorder. Because of the thermal photon distribution, both a norm and a direction of the Bloch vector change hard at random. In this paper, taking a different viewpoint compared with ones that we have been used to, we investigate quasiperiodicity of the Bloch vector’s trajectories. Introducing the concept of the quasiperiodic motion, we can explain the confused behaviour of the system as an intermediate state between periodic and chaotic motions. More specifically, we discuss the following three facts: (1) We show that the dynamics of the Bloch vector is equivalent to that of a compressible fluid. (2) If we adjust the time interval $\Delta t$ properly, figures consisting of plotted dots at constant time intervals acquire scale invariance under replacement of $\Delta t$ by $s\Delta t$, where $s(>1)$ is an arbitrary real but not transcendental number. (3) We can compute values of the time variable $t$, which let $|S_z(t)|$ (the absolute value of the $z$-component of the Bloch vector) be very small, with the Diophantine approximation (a rational approximation of an irrational number).

1 Introduction

The Jaynes-Cummings model (JCM) describes a two-level atom interacting with a single-mode radiation field. Because it is a soluble fully quantum mechanical model, the JCM has
been attracting many researchers’ attention since it was proposed in 1963 [1, 2, 3, 4, 5]. Considering the Hamiltonian for a magnetic dipole in a magnetic field, assuming near resonance and applying the rotating-wave approximation to it, we obtain the Hamiltonian of the JCM. Thus, the JCM is suitable for describing interaction of the radiation with the matter, for example, the spontaneous emission given rise to by a single two-level atom in the cavity field.

If we put the single-mode field into the state with sharply defined photon number at initial time, the JCM causes the Rabi oscillations in the atomic population inversion. Moreover, if we put the field into the coherent state initially, the JCM shows the collapse and the revival of the Rabi oscillations. Because the classical theory cannot explain the revival of the Rabi oscillations, we can regard it as an evidence of the quantum nature of the radiation, that is, the discreteness of photons [6, 7, 8].

However, in the case where we prepare the single-mode field in a thermal state initially, we hardly predict how the system evolves. Because of the thermal fluctuation of the field, the Bloch vector develops in time in a confusing manner. Both its norm and direction change hard at random, so that it seems to be in disorder. A number of researchers analyze this problem in details and try to find distinct properties, which characterize its unpredictable behaviour.

In Ref. [9], von Foerster computes time evolution of the probability for finding the atom in the ground state with preparing the radiation field into various states at initial time. Knight and Radmore investigate both the thermal JCM and its corresponding semiclassical model [10]. They compare time evolution of the atomic population inversion of the quantum model with that of the semiclassical model by plotting their graphs numerically. In Ref. [11], Knight analyzes the time-dependence of the atomic inversion of the JCM with an initial field state in thermal equilibrium. In the work, he states the following observation: The thermal fluctuation causes the collapse of the Rabi oscillations and their collapse time is very short. In addition, the revival overlaps it, so that interference occurs and it generates a very irregular time evolution. Liu and Tombesi discuss thermodynamics of the JCM [12]. In their analysis, the grand partition function of both the atom and the boson field is considered.

When we discuss the thermal JCM, we have to handle an intractable infinite series. If the single-atom field is resonant with the atom, the \( n \)th term of this intractable infinite series is given by a trigonometric function of \( \sqrt{n}t \) for \( n = 0, 1, 2, \ldots \), where \( t \) represents the variable of the time. Buck and Sukumar emphasize this fact in Ref. [13]. Because the infinite series is a superposition of the trigonometric function of \( \sqrt{n}t \) for \( n = 0, 1, 2, \ldots \) and it cannot be a Fourier series, the value of the sum of the series varies in an unpredictable manner as the time \( t \) progresses. Arroyo-Correa and Sanchez-Mondragon try to discuss the thermal JCM and evaluate the atomic population inversion, which is described by this intractable infinite series, using a technique of the complex analysis [14]. Chumakov et al. examine a new analytical approach to obtain an approximate sum of this series, which is reliable for a small initial mean photon number [15]. In Ref. [16], Klimov and Chumakov rewrite this intractable infinite series as a sum of two integrals by using the Abel-Plana formula.

Not only the thermal fluctuation but also a continuous measurement lets the JCM exhibit chaotic behaviour. Fukuo et al. study time evolution of non rotating-wave ap-
proximation JCM under a continuous quantum-nondemolition (QND) measurement [17]. Counting the number of photons inside the cavity with the QND photodetector, the system evolves in an irreversible way and yields chaos.

In this paper, we study the time evolution of the Bloch vector governed by the thermal JCM. Preparing the atom and the single-mode field initially in a certain pure state and a mixed state with a Bose-Einstein photon number distribution respectively, we let the whole system develop according to the JCM Hamiltonian. Thus, the system never suffers from dissipation. As mentioned before, under these circumstances, the behaviour of the Bloch vector is thrown into disorder. In this paper, taking a different viewpoint compared with ones that we have been used to, we investigate quasiperiodicity of the Bloch vector’s trajectories. Using the concept of the quasiperiodicity, we can explain the confused behaviour of the system as an intermediate state between periodic and chaotic motions. The revealed quasiperiodic structures hidden in the thermal fluctuation of the Bloch vector have close relations with topics of the number theory, which is one of the oldest branches of the pure mathematics. This appearance of unexpected natures of the thermal JCM itself is our motivation for this work.

In the current paper, we discuss the following three facts:

- We draw an analogy between the dynamics of the Bloch vector and that of a compressible fluid. We show that the time evolution of the Bloch vector is equivalent to a compressible inviscid flow that has no vorticity.

- If we adjust the time displacement $\Delta t$ properly, figures consisting of plotted dots at the constant interval $\Delta t$ acquire scale invariance under replacement of $\Delta t$ by $s\Delta t$, where $s(> 1)$ is an arbitrary real but not transcendental number. This fact is derived by making use of the Weyl criterion, which gives a necessary and sufficient condition for a sequence of real numbers to be uniformly distributed modulo unity.

- We can compute values of the time variable $t$, which let $|S_z(t)|$ (the absolute value of the $z$-component of the Bloch vector) be very small, with the Diophantine approximation (a rational approximation of an irrational number).

In Ref. [18], Azuma examines a histogram of $\{S_z(n\Delta t) : n = 0, 1, ..., N\}$, where $S(t)$ is the Bloch vector of the thermal JCM. Plotting the variance of the histogram of samples $\{S_z(n\Delta t)\}$ against the inverse of the temperature $\beta = 1/(k_B T)$, Azuma finds a scaling property. In the current paper, we also take a constant interval $\Delta t$ for the time variable $t$, and this prescription takes an important role. Thus, the present paper is a sequel of Ref. [18].

This paper is organized as follows: In Sec. 2, we give a brief review of the thermal JCM. In Sec. 3, we examine trajectories of the Bloch vector numerically. In Sec. 4, we explain the quasiperiodicity, which we can observe in the trajectories of the Bloch vector. In Sec. 5, we draw an analogy between the Bloch vector and the compressible fluid. In Sec. 6, we discuss the scale invariance of the figures generated as discrete plots of the Bloch vector. In Secs. 7 and 8, we investigate a graph of $S_x(t)$ versus the inverse of the temperature $\beta$ for the time $t$ such that $|S_z(t)| \ll 1$ by the numerical experiments and the perturbative evaluation, respectively. [$S(t)$ stands for the Bloch vector.] In Sec. 9, we give
brief discussions. In Appendix A, we give physical meanings of quantities of a fictitious fluid, for instance, the density $\rho$, the pressure $p$ and the external force per unit mass $K$, which we introduce for the analogy between the dynamics of the Bloch vector and that of the compressible fluid. In Appendix B, we consider how to build the Hamiltonian, which yields dynamics of the compressible fluid with zero vorticity. Moreover, we consider the reason why the trajectory of the Bloch vector appears to intersect itself as shown in Sec. 3. In Appendix C, we give a proof of the Weyl criterion. In Appendix D, we consider physical transient spectra of the atom in the cavity. In Appendix E, we prove two theorems, which are related to the Diophantine approximation.

2 A brief review of the thermal JCM

In this section, we give a brief review of the thermal JCM. To describe its dynamics, we use the notation of Ref. [18].

The Hamiltonian of the JCM is expressed in the form,

$$H = \frac{\hbar}{2} \omega_0 \sigma_z + \hbar \omega a^\dagger a + \hbar g (\sigma_+ a + \sigma_- a^\dagger),$$

(1)

$$\sigma_\pm = \frac{1}{2} (\sigma_x \pm i \sigma_y),$$

(2)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(3)

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0,$$

(4)

where $\sigma_x$, $\sigma_y$ and $\sigma_z$ given by Eqs. (2) and (3) are the Pauli matrices acting on atomic state vectors, and $a^\dagger$ and $a$ given by Eq. (4) are the photon creation and annihilation operators, respectively. We write the state of the two-level atom as a two-component column vector. We describe the state of the cavity field as a superposition of number states of photons.

Let us divide the JCM Hamiltonian defined in Eq. (1) into two parts as follows:

$$H = \hbar (C_1 + C_2),$$

$$C_1 = \omega \left( \frac{1}{2} \sigma_z + a^\dagger a \right),$$

$$C_2 = g (\sigma_+ a + \sigma_- a^\dagger) - \frac{\Delta \omega}{2} \sigma_z,$$

(5)

where $\Delta \omega = \omega - \omega_0$. Then, we obtain the following relation:

$$[C_1, C_2] = 0.$$ (6)

Because we can diagonalize the Hermitian operator $C_1$ at ease, we take the following interaction picture for describing the state of both the atom and the field. First, we write a state vector of the whole system in the Schrödinger and interaction pictures as $|\psi_S(t)\rangle$ and $|\psi_I(t)\rangle$, respectively. Second, assuming $|\psi_I(0)\rangle = |\psi_S(0)\rangle$, we define $|\psi_I(t)\rangle$ as follows:

$$|\psi_I(t)\rangle = \exp (i C_1 t) |\psi_S(t)\rangle.$$ (7)
Thus, because of Eq. (6), the time evolution of $|\psi(t)\rangle$ is given by

$$ |\psi(t)\rangle = U(t)|\psi(0)\rangle, \quad (8) $$

where

$$ U(t) = \exp(-iC_2t). \quad (9) $$

Here, we give an explicit form of $U(t)$ as follows:

$$ U(t) = \exp\left[-it \left( \begin{array}{cc} -\Delta \omega /2 & ga \\ ga^\dagger & \Delta \omega /2 \end{array} \right) \right] $$

$$ = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \left( \begin{array}{cc} (D + g^2)^n & 0 \\ 0 & D^n \end{array} \right) $$

$$ + \sum_{n=0}^{\infty} \frac{(-i)(-1)^n t^{2n+1}}{(2n+1)!} \left( \begin{array}{cc} -(\Delta \omega /2)(D + g^2)^n & gaD^n \\ ga^\dagger(D + g^2)^n & (\Delta \omega /2)D^n \end{array} \right) $$

$$ = \left( \begin{array}{cc} u_{00} & u_{01} \\ u_{10} & u_{11} \end{array} \right), \quad (10) $$

$$ D = (\frac{\Delta \omega}{2})^2 + g^2a^\dagger a, \quad (11) $$

$$ u_{00} = \cos(t\sqrt{D + g^2}) + \frac{i}{2} \Delta \omega \frac{\sin(t\sqrt{D + g^2})}{\sqrt{D + g^2}}, $$

$$ u_{01} = -iga \frac{\sin(t\sqrt{D})}{\sqrt{D}}, $$

$$ u_{10} = -iga^\dagger \frac{\sin(t\sqrt{D + g^2})}{\sqrt{D + g^2}}, $$

$$ u_{11} = \cos(t\sqrt{D}) - \frac{i}{2} \Delta \omega \frac{\sin(t\sqrt{D})}{\sqrt{D}}. \quad (12) $$

We put the whole system into the following initial states:

$$ \rho_{AP}(0) = \rho_A(0) \otimes \rho_P, \quad (13) $$

$$ \rho_A(0) = \sum_{i,j\in\{0,1\}} \rho_{A,ij}(0)|i\rangle_A\langle j|, \quad (14) $$

$$ \rho_P = \frac{\exp(-\beta \hbar \omega a^\dagger a)}{\text{Tr} \exp(-\beta \hbar \omega a^\dagger a)} $$

$$ = (1 - e^{-\beta \hbar \omega}) \exp(-\beta \hbar \omega a^\dagger a), \quad (15) $$

$$ |0\rangle_A = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad |1\rangle_A = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad (16) $$

where $\{\rho_{A,ij}\}$ represent an arbitrary $2 \times 2$ Hermitian matrix with nonnegative eigenvalues and trace unity. The indices A and P stand for the atom and the photon, respectively. The
density operator $\rho_P$ given by Eq. (15) represents the thermal equilibrium state weighted by the Bose-Einstein distribution for the inverse of the temperature $\beta = 1/(k_B T)$.

After these preparations, the time evolution of the atomic density operator is given by

$$\rho_A(t) = \sum_{i,j \in \{0, 1\}} \rho_{A,ij}(t) |i\rangle_A \langle j|,$$

(17)

$$\rho_{A,ij}(t) = \sum_{k,l \in \{0, 1\}} \rho_{A,kl}(0) A_{kl,ij}(t) \quad \text{for} \ i, j \in \{0, 1\},$$

(18)

$$A_{kl,ij}(t) = \langle i | \text{Tr}_P [U(t) (|k\rangle_A \langle l| \otimes \rho_P) U^\dagger(t)] | j \rangle_A.$$

(19)

Here, we pay attention to the following facts. Clearly, relations $\rho_{A,10}(t) = \rho_{A,01}(t)^*$ and $\rho_{A,11}(t) = 1 - \rho_{A,00}(t)$ hold. Thus, we need to estimate only two components, $\rho_{A,00}(t)$ and $\rho_{A,01}(t)$, so that

$$A_{00,00}(t) = (1 - e^{-\beta \hbar \omega}) \sum_{n=0}^{\infty} \frac{(\Delta \omega/2)^2 + g^2(n + 1) \cos^2(t \sqrt{\tilde{D}(n + 1)})}{\tilde{D}(n + 1)} e^{-n \beta \hbar \omega},$$

$$A_{11,00}(t) = (1 - e^{-\beta \hbar \omega}) \sum_{n=1}^{\infty} g^2 n \frac{\sin^2(t \sqrt{\tilde{D}(n)})}{\tilde{D}(n)} e^{-n \beta \hbar \omega},$$

$$A_{01,01}(t) = (1 - e^{-\beta \hbar \omega}) \sum_{n=0}^{\infty} \left[ \cos(t \sqrt{\tilde{D}(n + 1)}) + i \frac{\Delta \omega}{2} \frac{\sin(t \sqrt{\tilde{D}(n + 1)})}{\sqrt{\tilde{D}(n + 1)}} \right]$$

$$\times \left[ \cos(t \sqrt{\tilde{D}(n)}) + i \frac{\Delta \omega}{2} \frac{\sin(t \sqrt{\tilde{D}(n)})}{\sqrt{\tilde{D}(n)}} \right] e^{-n \beta \hbar \omega},$$

(20)

$$A_{01,00}(t) = A_{10,00}(t) = A_{00,01}(t) = A_{10,01}(t) = A_{11,01}(t) = 0,$$

(21)

$$\tilde{D}(n) = (\frac{\Delta \omega}{2})^2 + g^2 n.$$

(22)

Then, we introduce the Bloch vector $S(t) = (S_x(t), S_y(t), S_z(t))$, which provides us a visual description of the dynamics of the atomic state in a convenient way,

$$\rho_A(t) = \frac{1}{2} [I + S(t) \cdot \sigma],$$

(23)

$$S(t) = \begin{pmatrix} L^{(1)}_{\Delta \omega}(t) & L^{(2)}_{\Delta \omega}(t) & 0 \\ -L^{(2)}_{\Delta \omega}(t) & L^{(1)}_{\Delta \omega}(t) & 0 \\ 0 & 0 & L^{(3)}_{\Delta \omega}(t) \end{pmatrix} S(0) + \begin{pmatrix} 0 \\ 0 \\ L^{(4)}_{\Delta \omega}(t) \end{pmatrix},$$

(24)

$$L^{(1)}_{\Delta \omega}(t) = \text{Re}[A_{01,01}(t)],$$

$$L^{(2)}_{\Delta \omega}(t) = \text{Im}[A_{01,01}(t)],$$

$$L^{(3)}_{\Delta \omega}(t) = A_{00,00}(t) - A_{11,00}(t),$$

$$L^{(4)}_{\Delta \omega}(t) = A_{00,00}(t) + A_{11,00}(t) - 1.$$  

(25)
3 Trajectories of the Bloch vector

In this section, we examine the time evolution of the Bloch vector given by Eqs. (24) and (25) numerically. From now on, for simplicity, we assume $\Delta \omega = 0$. Then, we can rewrite the equations that represent the time evolution of the Bloch vector as follows:

\[
S(t) = \begin{pmatrix}
L_1(t) & 0 & 0 \\
0 & L_1(t) & 0 \\
0 & 0 & L_3(t)
\end{pmatrix} S(0) + \begin{pmatrix}
0 \\
0 \\
L_4(t)
\end{pmatrix}, \quad (26)
\]

\[
L_1(t) = (1 - e^{-\beta}) \sum_{n=0}^{\infty} \cos(\sqrt{n+1}t) \cos(\sqrt{n}t)e^{-n\beta},
\]

\[
L_3(t) = \frac{1}{2}(1 - e^{-\beta}) + \frac{e^{2\beta} - 1}{2e^{\beta}} \sum_{n=1}^{\infty} \cos(2\sqrt{n}t)e^{-n\beta},
\]

\[
L_4(t) = -\frac{1}{2}(1 - e^{-\beta}) + \frac{(e^{\beta} - 1)^2}{2e^{\beta}} \sum_{n=1}^{\infty} \cos(2\sqrt{n}t)e^{-n\beta}, \quad (27)
\]

where we assume $\omega \neq 0$ and $g \neq 0$, and we replace parameters $\beta \hbar \omega$ and $t|g|$ with $\beta$ and $t$, respectively. They imply that the time $t$ is in units of $|g|^{-1}$ and the inverse of the temperature $\beta$ is in units of $(\hbar \omega)^{-1}$. As a result of these replacements, the Bloch vector $S(t)$ depends only on two dimensionless variables, $t$ and $\beta$.

Here, putting the initial state of the atom into $(1/\sqrt{2})(|0\rangle_A + |1\rangle_A)$, we obtain the initial Bloch vector $S(0) = (1, 0, 0)$ and its time evolution

\[
S(t) = \begin{pmatrix}
L_1(t) \\
0 \\
L_4(t)
\end{pmatrix}. \quad (28)
\]

Equation (28) tells us that the Bloch vector $S(t)$ always lies on the $xz$-plane $\forall t(\geq 0)$, so that it is convenient for tracing the trajectory of $S(t)$ as time passes. Thus, from now on, we only examine the case where the Bloch vector is given by Eqs. (27) and (28).

Figure 1 shows a trajectory of the Bloch vector $S(t)$ given by Eqs. (27) and (28) for $\beta = 1.0$ and $0 \leq t \leq 250$. To obtain Fig. 1, we replace an infinite summation $\sum_{n=0}^{\infty}$ with a finite summation $\sum_{n=0}^{150}$ for an actual numerical calculation of $L_1(t)$ defined in Eq. (27). For calculating $L_4(t)$ numerically, we give a similar treatment. Throughout this paper, whenever we carry out numerical calculations of $L_1(t)$ and $L_4(t)$, we always apply this approximation to them. At the end of this section, we argue numerical errors given rise to by this approximation.

Looking at Fig. 1, we feel that both a norm and a direction of $S(t)$ change hard at random and it is in a state of disorder. With a careful observation of the trajectory, we notice that the figure drawn by $S(t)$ is nearly symmetrical with respect to the $z$-axis and the trajectory lies within a particular limited area on the $xz$-plane. However, it is difficult for us to find a regular form and order any more from Fig. 1.

As shown in Fig. 1, introducing the non-zero temperature and its thermal fluctuation, we can observe that the behaviour of $S(t)$ becomes in disorder. However, if we look at
the time evolution of \( S(t) \) from a new viewpoint, which is different from ones that we have been used to, we can find novel structures hidden in the trajectory of \( S(t) \). We show these secret structures in the following paragraphs.

We plot trajectories of the Bloch vector given by Eqs. (27) and (28) on the \( xz \)-plane as \( S(t) = (S_x(t), S_z(t)) \) in Figs. 2, 3 and 4. In these figures, we plot \( S(t) \) at constant time intervals, so that the time variable takes discrete values as \( t_n = n\Delta t \) for \( n = 0, 1, 2, ..., N \). Moreover, we have the number of dots plotted \( N \) as large as possible. Turning our eyes on Figs. 2, 3 and 4, we understand that the trajectories of the Bloch vector reveal regular forms, distinct orders and novel structures, that we have not been able to discover before.

Magnifying the upper right part of Fig. 3, we obtain Fig. 5. And magnifying the upper right part of Fig. 5, we obtain Fig. 6. Looking at Figs. 2, 3, 4, 5 and 6 carefully, we notice that points of \( |S_z| \ll 1 \) increases in number as \( \beta \) becomes smaller, that is, the temperature becomes larger. Moreover, we observe the following facts: On the one hand, points of \( |S_z| \ll 1 \) for \( \beta \gg 1 \) satisfy \( |S_x| \approx 1 \). On the other hand, points of \( |S_z| \ll 1 \) for \( 0 < \beta \ll 1 \) are distributed widely in the range of \(-1 \leq S_x \leq 1 \). In Fig. 7, we examine this tendency of the Bloch vector \( S(t) \) numerically.

How to draw a graph of Fig. 7 is as follows: First, we consider a set of samples, 
\[
\{ S_z(t_n) : n \in \{0, 1, 2, ..., N\} \},
\]
where \( t_0, t_1, ..., t_N \) are discrete values of the time variable generated with the constant interval \( \Delta t \) as mentioned before. Second, we take a certain positive number being small enough, \( 0 < \epsilon \ll 1 \). Third, we select values of \( t_n \), each of which satisfies \( |S_z(t_n)| \leq \epsilon \). Because of Eqs. (27) and (28), we can regard \( S_x(= L_1) \) as a function depending only on two variables, \( t_n \) and \( \beta \). Fourth, we plot \( (\beta, S_x(t_n)) \) such that \( |S_z(t_n)| \leq \epsilon \) and obtain the graph of Fig. 7. (In Sec. 7, we explain details for drawing Fig. 7, for example, how to choose \( \epsilon \) and \( N \) properly for given \( \beta \) such that \( 0.5 \leq \beta \leq 5.0 \), with giving a concrete example.) Figure 7 shows dependence of \( S_x \) on \( \beta \) on condition that \( |S_z| \ll 1 \) holds. In the graph of Fig. 7, we can observe successive splits leading to different branches. It gives us

![Figure 1: A trajectory of \( S(t) \) given by Eqs. (27) and (28) for \( 0 \leq t \leq 250 \) and \( \beta = 1.0 \).](image-url)
Figure 2: A figure consisting of dots of $S(t)$ given by Eqs. (27) and (28). We plot them at the constant time interval $\Delta t = 3.5$ for $\beta = 2.0$. The number of dots are equal to $N = 128\,000$. We give all points a diameter being $1/1000$ of the width of the whole graph.

Figure 3: A figure consisting of dots of $S(t)$ given by Eqs. (27) and (28). We plot them at the constant time interval $\Delta t = 3.5$ for $\beta = 1.0$. The number of dots are equal to $N = 384\,000$. We give all points a diameter being $1/1000$ of the width of the whole graph.
Figure 4: A figure consisting of dots of $S(t)$ given by Eqs. (27) and (28). We plot them at the constant time interval $\Delta t = 3.5$ for $\beta = 0.5$. The number of dots are equal to $N = 1 \, 024 \, 000$. We give all points a diameter being $1/1000$ of the width of the whole graph.

Figure 5: A figure consisting of dots of $S(t)$ given by Eqs. (27) and (28). We plot them at the constant time interval $\Delta t = 3.5$ for $\beta = 1.0$ and $N = 12 \, 800 \, 000$. The horizontal and vertical ranges are given by $0.7 \leq S_x \leq 1.0$ and $-0.1 \leq S_z \leq 0.0$, respectively. This figure corresponds to an enlargement of the upper right part of Fig. 3. The number of dots plotted actually for this figure is equal to 274 692. We give all points a diameter being $1/1000$ of the width of the whole graph.
Figure 6: A figure consisting of dots of $S(t)$ given by Eqs. (27) and (28). We plot them at the constant time interval $\Delta t = 3.5$ for $\beta = 1.0$ and $N = 1 \times 10^4$. The horizontal and vertical ranges are given by $0.95 \leq S_x \leq 1.0$ and $-0.02 \leq S_z \leq 0.0$, respectively. This figure corresponds to an enlargement of the upper right part of Fig. 5. The number of dots plotted actually for this figure is equal to 507 596. We give all points a diameter being $1/1000$ of the width of the whole graph.

Figure 7: Taking a certain positive number being small enough as $0 < \epsilon \ll 1$, we plot $(\beta, S_z(t_n))$ for discrete time variable $t_n = n \Delta t$ for $\Delta t = 3.5$ on condition that $|S_z(t_n)| \leq \epsilon$ holds. The horizontal axis represents $\beta$ for the range of $0.5 \leq \beta \leq 5.0$ with the logarithmic scale. We plot all points at a constant interval 0.025 for $\beta$. We give them a diameter being $4/1000$ of the width of the whole graph. (We explain how to draw this graph precisely in Sec. 7.)
an impression that the Bloch vector $S(t)$ hides further secret orders and structures inside itself.

Closing this section, let us argue the numerical precisions of $L_1(t)$ and $L_4(t)$. As mentioned before, whenever we carry out the numerical calculations of $L_1(t)$ and $L_4(t)$ defined in Eq. (27), we replace the infinite summation $\sum_{n=0}^{\infty}$ with the finite summation $\sum_{n=0}^{150}$, which contains the first 151 terms. Here, we estimate numerical errors caused by this treatment. For example, we can evaluate the upper bound of numerical errors for $L_1(t)$ as

$$E[L_1] \leq (1 - e^{-\beta}) \left| \sum_{n=151}^{\infty} \cos(\sqrt{n+1}t) \cos(\sqrt{n}t) e^{-n\beta} \right|$$

$$\leq (1 - e^{-\beta}) \sum_{n=151}^{\infty} e^{-n\beta}$$

$$= (1 - e^{-\beta}) e^{-151\beta} \sum_{n=0}^{\infty} e^{-n\beta}$$

$$= e^{-151\beta}. \quad (30)$$

Thus, if we assume $0.5 \leq \beta \leq 5.0$, we obtain

$$E[L_1] \leq 1.63 \times 10^{-33}. \quad (31)$$

Therefore, our calculation of $L_1(t)$ serves a precision of 33 significant decimal digits to us. Similar things hold for $L_4(t)$.

To obtain numerical data for drawing graphs of Figs. 1, 2, 3, 4, 5, 6 and 7, we utilize the Fortran compiler, which can manipulate the data type of the real quadruple precision. This implies that the compiler sorts out numerical calculations with a precision of about 33 significant decimal digits, so that it is fit to suppress numerical errors as Eqs. (30) and (31).

To obtain numerical data for drawing graphs of Figs. 13, 14, 15, 16, 17, 22, 23, 24, 25, 26, 27 and 28, and to carry out calculations required in Sec. 8, we utilize a high-performance computer algebra system, the Mathematica. The Mathematica has an option allowing us to specify any precision we want, so that we let it execute numerical calculations with 50 significant decimal digits for keeping extra enough amount of digits. In Sec. 8, we often have to manipulate integers, whose number of decimal digits is equal to 15, in computations. Under these circumstances, we need 15 digits for integers and 33 digits for the decimal representation of the floating point, so that we have to prepare 48 digits in total for computations over real numbers. Thus, utilizing the Mathematica with the 50 significant decimal digits, we can keep away from danger caused by the roundoff and truncation errors. Therefore, we can obtain numerical values of $L_1(t)$ and $L_4(t)$ with 33 significant decimal digits at least, and we can obtain stable results of the numerical calculations.
4 Quasiperiodicity in the dynamics of the Bloch vector

In this section, we discuss the quasiperiodicity of the Bloch vector, which is given by Eqs. (26) and (27). Here, we examine the behaviour of the Bloch vector under the low temperature limit. Assuming that the temperature is low enough as $\beta = 1/(k_B T) \gg 1$ and neglecting the terms of order $O((e^{-\beta})^2)$, we obtain approximations of $L_i(t)$ for $i = 1, 3, 4$ as follows:

\begin{align*}
L_1(t) &\simeq (1 - e^{-\beta}) \cos t + e^{-\beta} \cos(\sqrt{2}t) \cos t, \\
L_3(t) &\simeq \frac{1}{2}(1 - e^{-\beta}) + \frac{1}{2} \cos(2t), \\
L_4(t) &\simeq -\frac{1}{2}(1 - e^{-\beta}) + \frac{1}{2}(1 - 2e^{-\beta}) \cos(2t).
\end{align*}

The above approximations of $L_3(t)$ and $L_4(t)$ are periodic functions of $t$ with period $\pi$. In contrast, the function $L_1(t)$ is made of $\cos t$ and $\cos(\sqrt{2}t)$, that is, a complex structure of trigonometric functions with periods $2\pi$ and $\sqrt{2}\pi$, so that its behaviour is unpredictable. Thus, from now on, we concentrate on analyzing the function $L_1(t)$. Equation (32), the approximation of $L_1(t)$, has two angular frequencies, $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$, so that the ratio of $\omega_2$ to $\omega_1$ is irrational as $\omega_2/\omega_1 = \sqrt{2}$. We call $m$ angular frequencies $\omega_1, \omega_2, \ldots, \omega_m$ are incommensurate if no one of the angular frequencies $\omega_i$ can be expressed as a linear combination of the others using coefficients that are rational numbers. And if the system has incommensurate angular frequencies, we call it quasiperiodic.

The notion of the quasiperiodicity means an intermediate state between periodic and chaotic ones [19, 20, 21, 22, 23, 24]. To understand the quasiperiodicity of Eq. (32) giving the approximation of $L_1(t)$, we examine a dynamical system that has two incommensurate angular frequencies, $\omega_1$ and $\omega_2$. In the case of two-angular-frequency quasiperiodic motion, we can describe a dynamical variable $f(t)$ as a function of two independent variables, $G(t_1, t_2)$, for example,

\begin{align*}
f(t) &= L_1(t) = (1 - e^{-\beta}) \cos t + e^{-\beta} \cos(\sqrt{2}t) \cos t, \\
G(t_1, t_2) &= (1 - e^{-\beta}) \cos t_1 + e^{-\beta} \cos(\sqrt{2}t_2) \cos t_1, \\
G(t_1 + 2\pi, t_2) &= G(t_1, t_2 + \sqrt{2}\pi) = G(t_1, t_2), \\
f(t) &= G(t_1, t_2)|_{t_1 = t_2 = t}.
\end{align*}

Letting $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$, the equation of the form,

$$m_1\omega_1 + m_2\omega_2 = 0,$$

does not hold for arbitrary integers, $m_1$ and $m_2$, except for $m_1 = m_2 = 0$, so that $\omega_1$ and $\omega_2$ are incommensurate. Thus, the dynamics of the system is specified by two independent angles $\theta_1$ and $\theta_2$ as

$$G\left(\frac{\theta_1}{\omega_1}, \frac{\theta_2}{\omega_2}\right),$$

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where \( 0 \leq \theta_i < 2\pi \) for \( i = 1, 2 \). This implies that the orbit of the motion lies on the torus embedded in the phase space. From Eq. (38), we understand that \( f(t) \) is the quantity obtained along the orbit of \( \theta_1 = \omega_1 t \) and \( \theta_2 = \omega_2 t \) being realized on the torus in the phase space. In the following, we consider these matters in a general and precise manner.

Let us consider an arbitrary Hamiltonian \( H(q_1, q_2, p_1, p_2, t) \), whose number of degrees of freedom is equal to two. Because the system evolves according to this Hamiltonian, the trajectory that the system follows is described with Hamilton’s equations,

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H(q_j, p_j, t)}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H(q_j, p_j, t)}{\partial q_i}
\end{align*}
\]  

for \( i = 1, 2 \). (41)

The phase space of the system is four-dimensional and its co-ordinate system is given by \((q_1, q_2, p_1, p_2)\). Then, we remember the Poincaré-Cartan integral invariant and Liouville’s theorem, which are fundamental results in the field of the analytical mechanics [21, 24, 25]. They guarantee that the density of the system points in an infinitesimal volume element is preserved while they are traveling with time according to the canonical equations of motion. In other words, the volume of a region of the phase space is invariant under evolution with Hamilton’s equations of the motion in time. Thus, for example, \( dq_1 \wedge dp_1, dq_2 \wedge dp_2 \) and \( (dq_1 \wedge dp_1) \wedge (dq_2 \wedge dp_2) \) are preserved along the trajectories in the four-dimensional phase space.

Now, we assume that the motion of the system is bounded in the four-dimensional phase space. Furthermore, we suppose the Hamiltonian system to be integrable. Thus, the system has two functions \( I_i(q_j, p_j) \) for \( i = 1, 2 \), which are in involution and independent. Moreover, these functions are time-independent and preserved. Hence, they satisfy the following commutation relations:

\[
[I_1, I_2] = 0, \quad [I_i, H] = 0 \quad \text{for} \quad i = 1, 2.
\]  

(42)

In the above commutation relations, the bilinear operation \([u, v]\) denotes the Poisson bracket, so that \( I_i \) for \( i = 1, 2 \) are constants of the motion.

Here, let us define \( I_i \) for \( i = 1, 2 \) at initial time \((t = 0)\) as follows:

\[
I_i(0) = I_i(q_j(0), p_j(0)) = \text{constant} \quad \text{for} \quad i = 1, 2.
\]  

(43)

This implies that the motion of the system is confined to a two-dimensional surface embedded in the four-dimensional phase space \((q_1, q_2, p_1, p_2)\). In addition, the surface is determined by the two integral of the motion, \( I_1(0) \) and \( I_2(0) \), and all possible states of the system have to lie on it.

In this section, we often use two technical words, a constant of motion and an integral of motion. The constant of motion is a quantity that is conserved through the motion. On the other hand, the integral of motion is a function which is constant along a trajectory in the phase space. Clearly, an integral of the motion is a constant of the motion. However, a constant of the motion is not necessarily an integral of the motion.

From now on, we let \( \mathcal{M}_{I(0)} \) represent the above surface specified with \( I_1(0) \) and \( I_2(0) \). Then, we suppose \( \mu_0 \in \mathcal{M}_{I(0)} \) to be an initial point of the system at \( t = 0 \) in the phase
space. Moreover, we define an open neighbourhood with centre $\mu_0$ and radius $0 < \epsilon \ll 1$ on the surface $\mathcal{M}_{I(0)}$, such that

$$\sigma_{\mu_0}(\epsilon) = \{ \mu \in \mathcal{M}_{I(0)} : d(\mu, \mu_0) < \epsilon \},$$

where $d(\mu, \mu_0)$ represents a distance between $\mu$ and $\mu_0$ on $\mathcal{M}_{I(0)}$.

A point in the phase space takes a set of values for the canonical co-ordinates and momenta. It evolves through the phase space in time. We let the point travel from $\mu_0$ at $t = 0$ to $\mu(\Delta t)$ for a period of time $\Delta t$, where $0 < \Delta t \ll 1$. Furthermore, we let $\sigma_{\mu(\Delta t)}$ be a set of points which come from $\sigma_{\mu_0}(\epsilon)$, the open neighbourhood of $\mu_0$, along the trajectories of the system evolving for a period of time $\Delta t$. Clearly, $\mu(\Delta t) \in \sigma_{\mu(\Delta t)}$ and $\sigma_{\mu(\Delta t)} \in \mathcal{M}_{I(0)}$ hold.

Because of the Poincaré-Cartan integral invariant and Liouville’s theorem, an area of $\sigma_{\mu(\Delta t)}$ has to be equal to that of $\sigma_{\mu_0}(\epsilon)$ on the surface $\mathcal{M}_{I(0)}$. Then, we repeat this time evolution many times with a constant interval $\Delta t$. At the same time, we adjust some parameters as follows. For an arbitrary but fixed time interval $\Delta t$, we can always take a sufficiently small radius $\epsilon(> 0)$ of the open neighbourhood $\sigma_{\mu_0}(\epsilon)$, such as any two successive open sets never intersect in the sequence of $\sigma_{\mu_0}(\epsilon), \sigma_{\mu(\Delta t)}, \sigma_{\mu(2\Delta t)}, \ldots$. Namely, making $\epsilon(> 0)$ small enough, we can let the following relations hold,

$$\sigma_{\mu(n\Delta t)} \cap \sigma_{\mu((n+1)\Delta t)} = \emptyset \quad \text{for } n = 0, 1, 2, \ldots$$

From the above considerations, the sequence of the open neighbourhoods, $\sigma_{\mu(n\Delta t)}$ for $n = 0, 1, 2, \ldots$, lies on the surface $\mathcal{M}_{I(0)}$ as shown in Fig. 8.

Here, we remember the following fact. We assume that the motion of the system is bounded in the phase space. Thus, an area, where all points of actually possible states of the system belong to, is finite. This implies that the sequence of the open neighbourhoods, $\sigma_{\mu_0}(\epsilon), \sigma_{\mu(\Delta t)}, \sigma_{\mu(2\Delta t)}, \ldots$, lies on the region of finite size. Hence, there has to exist a finite
positive integer $N(>0)$ such that

$$\sigma_{\mu_0}(\epsilon) \cap \sigma_{\mu(N\Delta t)} \neq \emptyset. \quad (46)$$

This is because every open neighbourhood of $\sigma_{\mu(n\Delta t)}$ for $n = 0, 1, 2, \ldots$ has a finite equal area, which is determined by radius $\epsilon$. If the sequence of open neighbourhoods covers the entire finite region of actually possible states on the surface $M_{I(0)}$, there must exist a certain finite positive integer $N$ that satisfies Eq. (46).

The results obtained above hold even if we take arbitrary small $\Delta t$ and $\epsilon$. Hence, we understand that the motion of the integrable system bounded on the surface $M_{I(0)}$ has to be periodic. This conclusion implies that the motion of the system is periodic along the trajectory of the time evolution being determined by Hamilton’s equations.

Moreover, we can extend this statement as follows. At first, we assume that $I_1(0)$ and $I_2(0)$ are the integrals of the motion of the system. Next, we regard these integrals as general angular momenta $I_i(\{q_j\}, \{p_j\})$ for $i = 1, 2$. Let $\theta_1$ and $\theta_2$ be canonical angular variables of these angular momenta, so that $(\theta_1, I_1)$ and $(\theta_2, I_2)$ form canonical co-ordinates. Then, putting parameters $\theta_i$ along the loop as $\theta_i \in [0, 2\pi)$ for $i = 1, 2$, we can consider both variables of $\theta_1$ and $\theta_2$ to be periodic,

$$\theta_i = \omega_i t \pmod{2\pi} \quad \text{for} \quad i = 1, 2. \quad (47)$$

We can regard Eq. (47) as a definition of $\omega_i$ for $i = 1, 2$. In the following paragraphs, we explain the reason why Eq. (47) holds more precisely.

On the surface $M_{I(0)}$ being determined by the two integrals of the motion, $I_1(0)$ and $I_2(0)$, we can put a two-dimensional co-ordinate system $(\theta_1, \theta_2)$, where $\theta_1$ and $\theta_2$ are independent of each other, as shown in Fig 9. At this moment, we suppose $\theta_1$ and $\theta_2$ do not need to be periodic. Here, we pay attention to the following fact. Using the Poisson bracket, we can describe a differential operator with respect to a variable $\theta_i$ on the surface $M_{I(0)}$ as

$$D_{I_i} = -[I_i, \cdot] = -\sum_{j \in \{1, 2\}} \left( \frac{\partial I_i}{\partial \theta_j} \frac{\partial}{\partial I_j} - \frac{\partial I_i}{\partial I_j} \frac{\partial}{\partial \theta_j} \right) = \frac{\partial}{\partial \theta_i}. \quad (48)$$
This implies that a derivative of a function \( f(\theta_1, \theta_2, I_1(0), I_2(0)) \) in \( \theta_i \) direction at an arbitrary point on the surface \( M_{I(0)} \) is given by

\[
D_{I_i} f = -[I_i, f] \quad \text{for} \ i = 1, 2. \quad (49)
\]

Because we suppose \([I_1, I_2] = 0\) in Eq. (42), the above two differential operators commute obviously as

\[
D_{I_1}D_{I_2} - D_{I_2}D_{I_1} = -[I_1, [I_2, \cdot]] + [I_2, [I_1, \cdot]] = -D_{[I_1, I_2]} = 0. \quad (50)
\]

Moreover, \( D_{I_1} \) and \( D_{I_2} \) are linearly independent because \( \theta_1 \) and \( \theta_2 \) are independent variables. We explain this fact in detail as follows. Let us suppose there exist non-zero coefficients \( \alpha_1 \) and \( \alpha_2 \) such as \( \alpha_1 D_{I_1} + \alpha_2 D_{I_2} = 0 \). Furthermore, we consider an arbitrary function \( f(\theta_1, \theta_2) \) defined on \( M_{I(0)} \), and we obtain an identity, \((\alpha_1 D_{I_1} + \alpha_2 D_{I_2})f(\theta_1, \theta_2) = 0\). This implies

\[
[\alpha_1 I_1 + \alpha_2 I_2, f] = (\alpha_1 \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_2})f(\theta_1, \theta_2) = 0 \quad (51)
\]

The above identity has to hold for any function \( f(\theta_1, \theta_2) \) defined on \( M_{I(0)} \), so that we obtain \( \alpha_1 = \alpha_2 = 0 \). Hence, we can conclude that \( D_{I_1} \) and \( D_{I_2} \) are linearly independent.

Namely, \( \{D_{I_1}, D_{I_2}\} \) is a set of linearly independent and commuting differential operators. Moreover, we can regard these differential operators as generators of the Abelian group of displacements on \( M_{I(0)} \). Here, we put a reference point (an origin) of the co-ordinate system \((\theta_1, \theta_2)\) at an initial point \( \mu_0 \) for \( t = 0 \) on \( M_{I(0)} \). Moreover, we consider \((t_1 I_1 + t_2 I_2)\) to be a generator of a canonical transformation. This canonical transformation causes the following displacement of the reference point. We let \( \mu_0 \) denote the two-dimensional generalized co-ordinates of \( \mu_0 \in M_{I(0)} \), and \( \mu(\theta) = \mu(\theta_1, \theta_2) \) denote the two-dimensional generalized co-ordinates specified with \((\theta_1, \theta_2)\) on \( M_{I(0)} \). Then, we obtain

\[
\mu_0 = \mu|_{\theta=(0,0)} \quad \rightarrow \quad g^t(\mu_0) \equiv \mu_0 - \sum_{i \in \{1,2\}} t_i [I_i, \mu(\theta)]
\]

\[
= \mu(0,0) + \sum_{i \in \{1,2\}} t_i \left[ \frac{\partial}{\partial \theta_i} \mu(\theta) \right]_{\theta=(0,0)}. \quad (52)
\]

From the above equation, we notice that derivatives concerning the general angular momenta \( I_i \) for \( i = 1, 2 \) are equal to zero during the displacement of \( g^t(\mu_0) \) from \( \mu_0 \). Thus, \( g^t(\mu_0) \) is always confined on the surface \( M_{I(0)} \).

Here, we pay attention to the following fact. A change of an arbitrary function \( F \) with time \( t \) is given by the following Hamilton’s equation:

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]. \quad (53)
\]

If the function \( F \) is independent of time \( t \) in an explicit manner, we obtain

\[
\frac{dF}{dt} = [F, H]. \quad (54)
\]

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Figure 10: We can consider the surface $M_{I(0)}$ to be topologically equivalent to a two-dimensional torus, on which every point is specified with the co-ordinate system $(\theta_1, \theta_2)$.

The above equation implies

$$F(dt) = F(0) + dt [F, H]|_{t=0}.$$  \hfill (55)

We regard the above equation as a canonical transformation whose generator is given by $dtH$. Then, a canonical transformation of $\mu_0$ caused by a generator $\sum_{i \in \{1,2\}} t_i I_i$ can be written as

$$g^t(\mu_0) = \mu_0 + \sum_{i \in \{1,2\}} t_i [\mu, I_i] \bigg|_{\mu=\mu_0}.$$  \hfill (56)

The above equation coincides with Eq. (52). Hence, let us consider $\sum_{i \in \{1,2\}} t_i I_i$ to be another new Hamiltonian. Then, regarding Eq. (56) as its time evolution, we can apply the Poincarè-Cartan integral invariant and Liouville’s theorem to the canonical transformation caused by $\sum_{i \in \{1,2\}} t_i I_i$.

Thus, when the open neighbourhood with centre $\mu_0$ and radius $\epsilon$ is mapped to an open neighbourhood with centre $g^t(\mu_0)$, its area is preserved along a path of the canonical transformation. This means that there exists an certain positive integer $N$, such as an open neighbourhood with centre $(g^t)^\otimes N(\mu_0)$ intersects the initial open neighbourhood with centre $\mu_0$ in the sequence of $\mu_0 \to g^t(\mu_0) \to g^t(g^t(\mu_0)) \to ... \to (g^t)^\otimes N(\mu_0)$.

In this discussion, we can suppose $t = (t_1, t_2)$ to be an arbitrary two-dimensional real vector. Hence, we understand that the two angles $\theta_1$ and $\theta_2$ are independent and periodic in times $t_1$ and $t_2$, respectively. Thus, we obtain $\theta_i = \omega_i t_i$ for $i = 1, 2$. Then, reminding Eqs. (35), (36), (37) and (38), we set $t_1 = t_2 = t$ and obtain Eq. (47). From the above discussion, we can consider that the surface $M_{I(0)}$ embedded in the four-dimensional phase space to be topologically equivalent to a two-dimensional torus, $T^2 = S^1 \times S^1$, as shown in Fig. 10.

Here, let us think about the special case $\omega_2/\omega_1 = 1/2$. Then, an orbit winds around the torus twice in the short direction $\theta_1$ and simultaneously once in the long direction $\theta_2$ as time progresses. Thus, the orbit closes on itself after finite period of time as shown in Fig. 11. Hence, actually possible states of the system are confined on the one-dimensional closed path lying on the torus.

However, if we think about the case where $\omega_2/\omega_1$ is irrational, for example, $\omega_2/\omega_1 = \sqrt{2}$, the situation with respect to the orbit of the system becomes different drastically. If $\omega_2/\omega_1$ is irrational, the orbit never closes on itself as shown in Fig. 12. As time goes
Figure 11: In the case of $\omega_2/\omega_1 = 1/2$, an orbit moves around the torus twice in the short direction $\theta_1$ and once in the long direction $\theta_2$ as time proceeds, so that the orbit forms a finite closed path.

Figure 12: In the case of $\omega_2/\omega_1 = \sqrt{2}$, an orbit never closes on itself in a finite time period. The trajectory becomes dense on the torus as time proceeds to infinity.
to infinity, the orbit on the torus will eventually come arbitrary close to every point on the toroidal surface. Thus, the orbit fills up the torus in the phase space finally, and the trajectory of the system becomes dense and uniformly distributed on the torus.

Furthermore, if $\omega_2/\omega_1$ is irrational, the trajectory of the system has both fractal and ergodic properties that periodic systems never show. We explain these characteristics of quasiperiodic systems in the following. At first, let us consider an arbitrary continuous, smooth and bounded function defined on the surface $M_I(0)$, such as $f(\mu) \forall \mu \in M_I(0)$. Next, we define the following two averages of $f(\mu)$.

- The time average of $f(\mu)$. Getting a reference point as a starting-point (an initial point of $t = 0$), we take an average of $f(\mu)$ along the path, which is parameterized with time $t \in [0, +\infty)$ as

$$\langle f \rangle_t(\mu_0) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\omega t) dt,$$

where $\omega = (\omega_1, \omega_2)$. In the above calculation, we put the reference point $\mu_0$ at the origin of the two-dimensional co-ordinate system $\theta = (\theta_1, \theta_2) = (0, 0)$.

- The average of $f(\mu)$ over the phase space. We take the average of $f(\mu)$ over all points of $\mu$ belonging to the surface $M_I(0)$ as

$$\langle f \rangle_{M_I(0)} = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 f(\theta).$$

Then, we can show the following fact. If $\omega_2/\omega_1$ is irrational, the above two averages become equivalent to each other [24],

$$\langle f \rangle_t = \langle f \rangle_{M_I(0)}.$$  

The reason why is as follows.

We introduce a Fourier-series expansion of $f(\theta)$ on $M_I(0)$ as

$$f(\theta) = \sum_{k = (k_1, k_2) \in \mathbb{Z}^2} f_ke^{ik\cdot\theta},$$

where $\mathbb{Z}$ represents the set of all integers. Taking care of

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{ik\theta} = \delta_{k,0},$$

we obtain

$$\langle f \rangle_{M_I(0)} = f_0,$$

at ease. On the other hand, we can rewrite the time average as

$$\langle f \rangle_t(\mu_0) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T dt \sum_{k_1 \in \{0, \pm1, \pm2, \ldots\}} \sum_{k_2 \in \{0, \pm1, \pm2, \ldots\}} f_ke^{ik\cdot\omega t}$$

$$= f_0 + \sum_{k \neq 0} f_k \lim_{T \to +\infty} \frac{1}{T} \int_0^T dt e^{ik\cdot\omega t}$$

$$= f_0 + \sum_{k \neq 0} f_k \lim_{T \to +\infty} \frac{1}{T} e^{ik\cdot\omega T} - 1 \frac{1}{ik \cdot \omega}.$$ (63)
In the derivation of the above equation, we use the fact that \( \mathbf{k} \cdot \mathbf{\omega} \neq 0 \) \( \forall \mathbf{k} \neq \mathbf{0} \) if \( \omega_2/\omega_1 \) is irrational. Moreover, using the following formula,

\[
\lim_{T \to +\infty} \frac{1}{T} \frac{e^{i \mathbf{k} \cdot \mathbf{\omega} T} - 1}{i \mathbf{k} \cdot \mathbf{\omega}} = 0 \quad \text{for} \quad \mathbf{k} \cdot \mathbf{\omega} \neq 0,
\]

we arrive at

\[
\langle f \rangle_t(\mu_0) = f_0.
\]

From Eqs. (62) and (65), we can conclude Eq. (59).

These results tell us the following. If \( \omega_2/\omega_1 \) is irrational, the trajectory of the system on \( \mathcal{M}_{I(0)} \) has the Hausdorff dimension \( D_H = 2 \), which is related with the integral of a function defined on the surface \( \mathcal{M}_{I(0)} \). However, the topological dimension of the trajectory on \( \mathcal{M}_{I(0)} \) is obviously given by \( D_T = 1 \). Thus, we obtain \( D_T < D_H \) and we have to conclude that the trajectory on \( \mathcal{M}_{I(0)} \) is fractal [26, 27]. Moreover, the relation \( \langle f \rangle_t(\mu_0) = \langle f \rangle_{\mathcal{M}_{I(0)}} \) implies that the time average and the average over phase space are equivalent to each other, so that the trajectory has ergodicity. From these discussions, we can expect that the trajectory drawn by the Bloch vector has fractal properties if \( \omega_2/\omega_1 \) is irrational. We examine this point in Sec. 6.

Here, we pay attention to the following fact. As mentioned before, if \( \omega_2/\omega_1 \) is irrational, the trajectory never closes on itself in the phase space. However, taking the long-term limit, the trajectory on the torus in the phase space will eventually come arbitrary close to every point on the toroidal surface. Hence, if we take an arbitrary small positive number \( \epsilon(>0) \), there actually exists time \( t \) such as

\[
d(\mu_0, \mu(t)) < \epsilon.
\]

The question is how to estimate time \( t \), which satisfies Eq. (66). Let us think about obtaining a rational approximate number of irrational \( \omega_2/\omega_1 \) by its continued fraction expression. For example, in case of \( \omega_2/\omega_1 = \sqrt{2} \), we can describe its continued fraction as

\[
\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}},
\]

By truncating the above expression with the first some terms, we can obtain the rational approximate number \( p/q \) with high precision such as

\[
|\sqrt{2} - \frac{p}{q}| < \frac{1}{q^2}.
\]

(This fact is assured by the theorem of Roth [28].)

Thus, because of \( \omega_2/\omega_1 = \sqrt{2} \simeq p/q \), we obtain \( p\omega_1 \simeq q\omega_2 \). This implies the following. If the orbit winds around the torus \( p \)th times in the short direction \( \theta_1 \) and simultaneously \( q \)th times in the long direction \( \theta_2 \), it arrives at a point which is very close to the initial point \( \mu_0 \). We investigate this question in Sec. 8.

In this section, so far, we suppose the low-temperature limit and evaluate the contribution to \( L_1(t) \) up to \( O(e^{-\beta}) \). Here, we let the temperature become higher slightly
and estimate the contribution to \( L_1(t) \) up to \( O((e^{-\beta})^2) \). Then, we can regard \( L_1(t) \) as a superposition of \( \cos t, \cos(\sqrt{2}t) \) and \( \cos(\sqrt{3}t) \), and it causes quasiperiodicity of three incommensurate angular frequencies. Moreover, if we make the temperature become much higher and estimate the contribution to \( L_1(t) \) up to \( O((e^{-\beta})^3) \), \( L_1(t) \) has four angular frequencies, \( \omega_1 = 1 \), \( \omega_2 = \sqrt{2} \), \( \omega_3 = \sqrt{3} \) and \( \omega_4 = 2 \). Then, \( \omega_4/\omega_1 \) is equal to two, so that it is rational. Let us think about the four-dimensional torus \( T^4 = S^1 \times S^1 \times S^1 \times S^1 \), where the orbit of the motion lies, and its two-dimensional submanifold \((\theta_1, \theta_4)\). Then, the orbit is periodic on the submanifold \((\theta_1, \theta_4)\). So that, as the temperature becomes higher, the motion of the Bloch vector causes a complicated trajectory, which is a mixture of periodic and quasiperiodic orbits.

Finally, we note the fact that \( L_1(t) \) has countably infinite incommensurate angular frequencies, \( \omega_n = \sqrt{n} \) for \( n = 1, 2, 3, 5, ... \). This means that the system has countably infinite angular momenta as integrals of the motion, \( I_n \) for \( n = 1, 2, 3, 5, ... \). Thus, we have to conclude that we cannot describe the motion of the Bloch vector as the \( n \)-body problem in classical mechanics, which has a finite number of degrees of freedom. Hence, we can expect the motion of the Bloch vector to be equivalent to a classical field theory, which has a countably infinite number of degrees of freedom. In fact, we confirm that the motion of the Bloch vector is equivalent to the dynamics of a compressible fluid in Sec. 5.

## 5 Equivalence between the Bloch vector and compressible fluid

In this section, we discuss equivalence between the dynamics of the Bloch vector and a compressible fluid flow. In Sec. 4, we emphasize that the quasiperiodic motion of the Bloch vector has countably infinite angular momenta, so that we cannot describe it as the \( n \)-body problem of the classical mechanics.

We derive the Bloch vector \( \mathbf{S}(t) \) given by Eqs. (26) and (27) from the Schrödinger equation, whose Hamiltonian is defined in Eqs. (1), (2), (3) and (4), originally. As explained in Sec. 2, for obtaining \( \mathbf{S}(t) \), what we apply to the quantum mechanical state in fact are only providing the initial state of the cavity field at \( t = 0 \) as a density matrix of a mixed state and tracing out its Hilbert space \( \forall t > 0 \).

Because we trace out the Hilbert space of the cavity field, the entanglement between the single atom and the cavity field is removed. Thus, the dynamics of the Bloch vector has to be described with classical theory. Hence, we can expect that the trajectory of the Bloch vector to be described with a classical field theory, which has a countably infinite number of degrees of freedom. In this section, we demonstrate that it is exactly equivalent to hydrodynamics of a compressible fluid, which is inviscid and has zero vorticity [29, 30].

From now on, we describe \( \mathbf{S}(t) \) as \( \mathbf{x}(t) = (x(t), y(t), z(t)) \). Thus, we rewrite Eq. (26) as

\[
\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} L_1(t)x(0) \\ L_1(t)y(0) \\ L_3(t)z(0) + L_4(t) \end{pmatrix}, \tag{69}
\]

where \( |\mathbf{x}(0)| \leq 1 \). We consider Eq. (69) to be the Lagrangian forms of the dynamical equations of a fluid. In Eq. (69), we let \( \mathbf{x}(0) \) be the initial co-ordinates of any particle.
of the fluid and \( \mathbf{x}(t) \) be its co-ordinates at time \( t \). Hence, Eq. (69) expresses the time evolution of the fluid particles being put initially inside a unit circle, that is, \( |\mathbf{x}(0)| \leq 1 \).

From Eq. (69), the velocity of the fluid particle at point \( \mathbf{x}(t) \) at time \( t \) is given by

\[
\dot{\mathbf{x}}(t) = \begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{z}(t)
\end{pmatrix} = \begin{pmatrix}
\hat{L}_1(t)x(0) \\
\hat{L}_1(t)y(0) \\
\hat{L}_3(t)z(0) + \hat{L}_4(t)
\end{pmatrix}.
\]

(70)

Thus, from Eqs. (27) and (70), calculating an initial velocity at every point inside the unit circle explicitly, we obtain

\[
v(t = 0; x(0), y(0), z(0)) = \dot{\mathbf{x}}(t = 0; x(0), y(0), z(0)) = 0 \quad \forall \mathbf{x}(0) \text{ such that } |\mathbf{x}(0)| \leq 1.
\]

(71)

To obtain the dynamical equations of the fluid, we rewrite Eq. (69) as the following relations,

\[
x(0) = \frac{x(t)}{L_1(t)}, \quad y(0) = \frac{y(t)}{L_1(t)}; \quad z(0) = \frac{z(t) - L_4(t)}{L_3(t)}.
\]

(72)

Substitution of Eq. (72) into Eq. (70) yields

\[
\dot{\mathbf{x}}(t) = \begin{pmatrix}
[\frac{\hat{L}_1(t)}{L_1(t)}]x(t) \\
[\frac{\hat{L}_1(t)}{L_1(t)}]y(t) \\
[\frac{\hat{L}_3(t)}{L_3(t)}]z(t) - \frac{L_4(t)}{L_3(t)} + \hat{L}_4(t)
\end{pmatrix}.
\]

(73)

Therefore, the velocity of the fluid at point \( \mathbf{x} = (x, y, z) \) at time \( t \) is described as

\[
v(t, x, y, z) = \begin{pmatrix}
[\frac{\hat{L}_1(t)}{L_1(t)}]x \\
[\frac{\hat{L}_1(t)}{L_1(t)}]y \\
[\frac{\hat{L}_3(t)}{L_3(t)}]z - \frac{L_4(t)}{L_3(t)} + \hat{L}_4(t)
\end{pmatrix}.
\]

(74)

Here, let us point out some characteristic properties of the fluid obtained by the analogy of the Bloch vector. From Eq. (74), we obtain

\[
\nabla \cdot \mathbf{v} = 2\frac{\hat{L}_1(t)}{L_1(t)} + \frac{\hat{L}_3(t)}{L_3(t)}.
\]

(75)

Thus, we understand that \( \nabla \cdot \mathbf{v} \) depends only on \( \beta \) and \( t \). By numerical calculations, we can confirm \( \nabla \cdot \mathbf{v} \neq 0 \) at almost all times \( t \) for \( 0 < \beta < +\infty \). Thus, we consider that the following relation holds in general,

\[
\nabla \cdot \mathbf{v} \neq 0.
\]

(76)

In addition, from Eq. (74), we can obtain the following equation, at ease:

\[
\nabla \times \mathbf{v} = 0 \quad \forall (t, \mathbf{x}).
\]

(77)

Then, let us think about the Eulerian equation of continuity,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.
\]

(78)
If we assume that the fluid is incompressible, that is, $\rho = \text{constant}$, we obtain $\nabla \cdot \mathbf{v} = 0 \quad \forall (t, \mathbf{x})$. However, it contradicts Eq. (76), so that we conclude that the fluid has to be compressible. Moreover, Eq. (77) implies that there is no vorticity in any portion of the fluid at any time. In general, a concept of turbulence closely relates to randomness and complexity of vorticity. Thus, the fluid never shows turbulence motion.

Let us examine dynamics of the fluid further. From Eqs. (74) and (77), we obtain the following velocity-potential $\Phi$:

$$\mathbf{v} = \nabla \Phi,$$

$$\Phi = \frac{\dot{L}_1(t)}{L_1(t)} \left( \frac{x^2}{2} + \frac{y^2}{2} \right) + \frac{\dot{L}_3(t)}{L_3(t)} \left( \frac{z^2}{2} - z L_4(t) \right) + \dot{L}_4(t) z.$$  (80)

Then, from Eq. (80), we obtain the following relation:

$$\triangle \Phi = 2 \frac{\dot{L}_1(t)}{L_1(t)} + \frac{\dot{L}_3(t)}{L_3(t)}.$$  (81)

Thus, using Eq. (79) and (81), we arrive at

$$\triangle \mathbf{v} = \triangle \nabla \Phi = \nabla \triangle \Phi = 0.$$  (82)

Equation (82) implies that the viscosity of the fluid has no effects on evolution of an actual motion of a fluid element. The reason why is given in the following.

In general, the equations of motion of a compressible fluid are given by

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{K} - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \triangle \mathbf{v},$$  (83)

which are called the Navier-Stokes equations. In Eq. (83), $\rho$, $p$, $\mathbf{K}$ and $\mu$ represent the density of the fluid, the pressure, an external force per unit mass and the viscosity of the fluid, respectively. However, because of Eq. (82), we can drop the term including $\mu$ from the Navier-Stokes equations. Thus, we understand that the viscosity $\mu$ has no effects on the motion of the fluid, in fact.

Putting the above considerations together, we obtain the following observations. If we regard the Bloch vector $\mathbf{S}(t)$ as a fluid, it is compressible and inviscid. Moreover, it has no vorticity. By reason of the fact that the fluid is inviscid and has zero vorticity, its motion seems to be simple for us. However, this impression is not true for the compressible fluid. Indeed, sometimes we can hardly predict the motion of the compressible fluid, even if it does not have the viscosity.

Figures 13, 14 and 15 show the velocity $\mathbf{v}(x, y, z, t)$ given by Eq. (74) on the $xz$-plane for $|x| \leq 1$ and $z \leq 0$ with $\beta = 1.0$ at $t = 0.5$, 1.0 and 1.5, respectively. Thus, we can think that Figs. 13, 14 and 15 represent the tangent vectors of $\mathbf{S}(t)$ at $t = 0.5$, 1.0 and 1.5, respectively. Looking at these figures, we understand that the velocity of the fluid at any point at any time changes hard, so that the particle of the fluid draws a very complicated trajectory.

In Appendix A, we give physical meanings of $\rho$, $p$ and $\mathbf{K}$ defined in Eqs. (78) and (83). In Appendix B, we consider how to build the Hamiltonian, which generates the Navier-Stokes equations (83). Moreover, we consider the reason why the trajectory in Fig. 1 appears to intersect itself. We explain that intersections never violate existence and uniqueness of solutions for the Navier-Stokes equations.
Figure 13: This figure represents the velocity $\mathbf{v}$ given by Eq. (74) at points of $|x| \leq 1$ and $z \leq 0$ on the $xz$-plane at time $t = 0.5$ with $\beta = 1.0$. The lengths of arrows (vectors) represent only just the ratios of their norms compared with each other.

Figure 14: This figure represents the velocity $\mathbf{v}$ given by Eq. (74) at points of $|x| \leq 1$ and $z \leq 0$ on the $xz$-plane at time $t = 1.0$ with $\beta = 1.0$. The lengths of arrows (vectors) represent only just the ratios of their norms compared with each other.
Figure 15: This figure represents the velocity $v$ given by Eq. (74) at points of $|x| \leq 1$ and $z \leq 0$ on the $xz$-plane at time $t = 1.5$ with $\beta = 1.0$. The lengths of arrows (vectors) represent only just the ratios of their norms compared with each other.

6 The scale invariance of the figures generated by the Bloch vector

In this section, we show that the figures drawn in Figs. 2, 3, 4, 5 and 6 are invariant under a scale change in $\Delta t$. We draw these figures by plotting the trajectories of the Bloch vector at the constant interval $\Delta t$. This scale invariance is manifestation of the fractal structure of the Bloch vector’s dynamics, as mentioned in Sec. 4.

To show the scale invariance of these discrete plots, we use some properties of random sequences of real numbers, which are uniformly distributed modulo $2\pi$. In the former half of this section, we give two considerations in preparation for explaining the scale invariance of the figures. First, we discuss the physical meaning of plotting the trajectories of the Bloch vector at the constant time interval $\Delta t$. We examine differences between continuous and discrete plots. Second, we discuss the random sequences that are uniformly distributed modulo $2\pi$. After these preparations, in the latter half of this section, we discuss the scale invariance of figures generated by plotting the Bloch vector at the interval $\Delta t$.

First, we consider physical meaning of dividing time into a lattice. With a discrete time slice $\Delta t$, we have a chain of time co-ordinates $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots, N$. Then, as shown in Figs. 2, 3, 4, 5 and 6, we plot trajectories of the Bloch vector $S(t)$ as $\{S(t_n) : n \in \{0, 1, 2, \ldots, N\}\}$. Through this procedure, we introduce a unit of time $\Delta t$. From now on, we regard it as a scale of the time variable. Indeed, assuming $\Delta t$ to be a real but not transcendental number and $\Delta t > \pi$, the graphs of Figs. 2, 3, 4, 5 and 6 are invariant under the following change of the scale,

$$\Delta t \to s\Delta t,$$

where $s$ is a real but not transcendental number and $s > 1$. We show this fact in the latter half of this section.

As an alternative plan to Eq. (84), we can consider a change of a scale of the continuous
time variable,

\[ t \rightarrow st \quad (s > 0). \tag{85} \]

In this case, we handle the time variable \( t \) as a continuous quantity. Throughout the current paper, we adopt Eq. (84) rather than Eq. (85). Here, we think about a difference between the change of the scale of Eq. (84) and that of Eq. (85). On the one hand, if we choose Eq. (84), we have to neglect all events which relate to the quantity of the time being less than \( \Delta t \). On the other hand, if we select Eq. (85), we have to admit an infinitesimal time displacement for the system. Thus, taking Eq. (84), we get rid of notions of infinitesimal time evolution. In the latter half of this section, we consider \( \Delta t \) to be in the range between \( \pi \) and \( 2\pi \), that is, \( \pi < \Delta t < 2\pi \). This treatment generates a random sequence

\[ t_n = n\Delta t \quad (\text{mod} \ 2\pi) \]

for \( n = 0, 1, \ldots, N \) and lets the graphs of discrete dots in Figs. 2, 3, 4, 5 and 6 be invariant under the change of the scale \( \Delta t \).

Moreover, the above treatment produces an unexpected effect that removes a history of the sequence \( \{S(t_n) : t_n = n\Delta t, n \in \{0, 1, 2, \ldots, N\}\} \) actually. This is because \( \Delta t \) is rather larger, such as \( \pi < \Delta t < 2\pi \). If we let the time displacement be small enough, the sequence of \( (S(t_0), S(t_1), S(t_2), \ldots) \) approximates to the continuous trajectory of \( S(t) \) well. However, if we take \( \pi < \Delta t < 2\pi \), we cannot find neither trace nor history of continuous trajectory of \( S(t) \) except for its discrete dots.

Next, we think about random sequences uniformly distributed in \([0, 2\pi)\). At first, we study some properties of sequences of real numbers, which are uniformly distributed modulo \( 2\pi \) (but need not be random). We begin by giving the following set \( S_N \), which consists of \((N + 1)\) elements,

\[ S_N = \{x_j : 0 \leq x_j < 2\pi, j \in \{0, 1, \ldots, N\}\}. \tag{86} \]

Rewriting the elements of \( S_N \) as an ordered list \((x_0, x_1, \ldots, x_N)\), we can consider it to be a real sequence. Let us examine conditions for the set \( S_N \) to be uniformly distributed in the range of \([0, 2\pi)\) under the limit of \( N \to +\infty \). If the set \( S_N \) is uniformly distributed, the following relation holds,

\[ \lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} \phi(x_j) = \frac{1}{2\pi} \int_{0}^{2\pi} dx \phi(x), \tag{87} \]

where \( \phi(x) \) is an arbitrary real valued function being bounded, differentiable and continuous for \( 0 \leq x < 2\pi \).

Here, let us give a Fourier-series expansion of \( \phi(x) \) as follows:

\[ \phi(x) = \sum_{m=-\infty}^{+\infty} c_m e^{imx}, \]

\[ c_m = \frac{1}{2\pi} \int_{0}^{2\pi} dx \phi(x)e^{-imx}. \tag{88} \]

Then, substitution of Eq. (88) into the left-hand side of Eq. (87) yields

\[ \lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} \phi(x_j) = \lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} \sum_{m=-\infty}^{+\infty} c_m \exp(imx_j). \tag{89} \]
In a similar way, substitution of Eq. (88) into the right-hand side of Eq. (87) yields

\[
\frac{1}{2\pi} \int_0^{2\pi} dx \phi(x) = \frac{1}{2\pi} \int_0^{2\pi} dx \sum_{m=-\infty}^{+\infty} c_m e^{imx}.
\]

(90)

Thus, putting Eqs. (87), (89) and (90) together, we obtain

\[
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} \sum_{m=-\infty}^{+\infty} c_m \exp(imx_j) = \frac{1}{2\pi} \int_0^{2\pi} dx \sum_{m=-\infty}^{+\infty} c_m e^{imx}.
\]

(91)

Then, we remember that \(\phi(x)\) is an arbitrary function in Eq. (87). Hence, we can regard \(\{c_m : m \in \{\pm 1, \pm 2, \ldots\}\}\) as arbitrary coefficients in Eq. (91), so that we obtain the following relations:

\[
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} \exp(imx_j) = \frac{1}{2\pi} \int_0^{2\pi} dxe^{imx}
\]

\[
= \frac{1}{2\pi} \left[ \frac{e^{imx}}{im} \right]_{x=0}^{x=2\pi} = 0 \quad \forall m \in \{\pm 1, \pm 2, \ldots\}.
\]

(92)

These relations are called the Weyl criterion [28, 31, 32, 33, 34].

From the above discussion, we obtain the necessary condition for the sequences uniformly distributed. Moreover, if the set \(\lim_{N \to +\infty} S_N\) given by Eq. (86) is uniformly distributed, Eq. (92) holds. Indeed, the Weyl criterion given by Eq. (92) is a necessary and sufficient condition for \(\lim_{N \to +\infty} S_N\) to be uniformly distributed. We give a rigorous proof of this fact in Appendix C.

Let us consider a concrete example of \(S_N\) as follows:

\[
S_N = \{n\Delta t \pmod{2\pi} : n \in \{0, 1, 2, \ldots, N\}\},
\]

(93)

where \(\Delta t\) is a positive constant. Substitution of Eq. (93) into the Weyl criterion given by Eq. (92) yields

\[
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{n=0}^{N} e^{imn\Delta t} = \lim_{N \to +\infty} \frac{1}{N+1} \frac{1 - e^{im(N+1)\Delta t}}{1 - e^{im\Delta t}}.
\]

(94)

If \(m\Delta t \pmod{2\pi} \neq 0\ \forall m \in \{\pm 1, \pm 2, \ldots\}\), the right-hand side of Eq. (94) is equal to zero.

Hence, at least, letting \(\Delta t\) be written as

\[
\Delta t = \sigma \frac{p}{q},
\]

(95)

where arbitrary positive integers \(p\) and \(q\) are coprime, and \(\sigma\) is unity or an arbitrary positive irrational but not transcendental number, we consider \(\lim_{N \to +\infty} S_N\) to be uniformly distributed for \([0, 2\pi]\). To derive this result, we make use of facts that \(\pi\) is a transcendental number and a multiple of \(2\pi\) is never equal to

\[
n\Delta t = n\sigma \frac{p}{q},
\]

(96)
where \( n \) is an arbitrary positive integer. In fact, not only \( \Delta t \) given by Eq. (95) but also a large variety of other time displacements lets \( \lim_{N \to +\infty} S_N \) given by Eq. (93) be uniformly distributed for \([0, 2\pi)\). However, throughout this paper, we concentrate on \( \Delta t \) given by Eq. (95).

From the above considerations, we can conclude that the sequence \( \lim_{N \to \infty} S_N \) such that Eq. (93) is uniformly distributed modulo \( 2\pi \), where \( \Delta t \) is given by Eq. (95). Moreover, we pay attention to the following fact. If we assume \( \pi < \Delta t < 2\pi \), (97)

we can consider \( \lim_{N \to +\infty} S_N \) to be a pseudorandom sequence uniformly distributed for \([0, 2\pi)\). This is because a relation \( 2\Delta t > 2\pi \) generates an effect of the linear congruence method on the sequence \( n\Delta t \) modulo \( 2\pi \) for \( n = 0, 1, 2, ... \).

We can confirm this effect by giving a concrete counter-example for \( 0 < \Delta t \ll \pi \). For example, taking \( \Delta t = 0.05 \), we obtain

\[
125\Delta t = 6.25, \quad 126\Delta t = 6.3, \quad 125\Delta t < 2\pi < 126\Delta t,
\]

so that \((0, \Delta t, 2\Delta t, ..., 125\Delta t)\) form a finite arithmetic progression. Elements of the progression do not suffer from an effect of a constant modulus \( 2\pi \). When we reach at \( 126\Delta t \), we observe an effect of the linear congruence method first. In this case, evidently, we cannot think \( \lim_{N \to +\infty} S_N \) to be a random sequence.

Using the results obtained above, we discuss an invariance of \( S_z(t) = L_4(t) \) under a change of the scale \( \Delta t \). Here, we introduce a variable \( b = e^{-\beta} \). Because of \( 0 < \beta < +\infty \), we obtain \( 0 < b < 1 \). Thus, we can rewrite \( L_4(t) \) given by Eq. (27) as

\[
L_4(t) = -\frac{1}{2}(1 - b) + \frac{b}{2}(1 - \frac{1}{b})^2 f(t),
\]

\[
f(t) = \sum_{n=1}^{\infty} b^n \cos(2\sqrt{n}t).
\]

In the following paragraphs, we show that the function \( f(t) \) given by Eq. (100) has a special scaling property.

First, we consider a set \( S \) of pseudorandom numbers (or a sequence) distributed uniformly for \([0, 2\pi)\) as follows:

\[
S = \{ t_n \pmod{2\pi} : n \in \{0, 1, 2, \ldots\} \}.
\]

Substitution of elements of \( S \) into \( f(t) \) defined in Eq. (100) yields a set of \( F \) as

\[
F = \{ f(t_0), f(t_1), f(t_2), \ldots \}.
\]

Here, let us consider an arbitrary real but not transcendental number \( s(> 1) \) and generate the following set,

\[
S' = \{ st_n \pmod{2\pi} : n \in \{0, 1, 2, \ldots\} \}.
\]

Moreover, we construct a new set \( F' \) from \( S' \) as follows:

\[
F' = \{ f(st_0), f(st_1), f(st_2), \ldots \}.
\]
Then, we cannot distinguish \( S \) from \( S' \) and \( F \) from \( F' \) actually on condition that both sets \( S \) and \( S' \) are countably infinite and their cardinalities are equal to each other, that is, \( |S| = |S'| \). We argue this statement in the following paragraphs.

First, we think about \( \Delta t \) given by

\[
\Delta t = \frac{p}{q}, \quad \frac{\pi}{2} < \Delta t < \pi,
\]

where arbitrary positive integers \( p \) and \( q \) are coprime. For example, we can choose \( \Delta t = 7/4 = 1.75 \). From the rational number \( \Delta t \), we can generate a set as follows:

\[
S_{M,\Delta t} = \{m\Delta t \mod\pi : m \in \{0, 1, 2, \ldots, M\}\}.
\]

As explained before, if we take sufficient large \( M \), we can consider \( S_{M,\Delta t} \) to be uniformly distributed for \([0, \pi)\). In addition, because of the effect of the constant modulus \( \pi \), we can regard \( S_{M,\Delta t} \) as a sequence of the pseudorandom numbers. Moreover, replacing \( \sigma \) with \( \sqrt{n} \) in Eq. (95) that gives the definition of \( \Delta t \), we obtain new pseudorandom sequences uniformly distributed for \([0, \pi)\),

\[
S_{M,\sqrt{n}\Delta t} = \{m\sqrt{n}\Delta t \mod\pi : m \in \{0, 1, 2, \ldots, M\}\} \quad \text{for } n = 1, 2, 3, \ldots
\]

Here, using the function \( f(t) \) given by Eq. (100), let us construct a set,

\[
F_{M,\Delta t} = \{f(m\Delta t \mod\pi) : m \in \{0, 1, 2, \ldots, M\}\}.
\]

In Eq. (100), the \( n \)th term of the series \( f(t) \) is given by \( b^n \cos(2\sqrt{n}t) \). Thus, a set of arguments \( \sqrt{n}t \) of this cosine function is equivalent to \( S_{M,\sqrt{n}\Delta t} \) defined in Eq. (107).

Furthermore, taking an arbitrary real but not transcendental number \( s(>1) \), we think about a change of the scale \( \Delta t \to s\Delta t \). This change of the scale yields the following sets:

\[
S_{M, s\sqrt{n}\Delta t} = \{sm\sqrt{n}\Delta t \mod\pi : m \in \{0, 1, 2, \ldots, M\}\} \quad \text{for } n = 1, 2, 3, \ldots
\]

\[
F_{M, s\Delta t} = \{f(sm\Delta t \mod\pi) : m \in \{0, 1, 2, \ldots, M\}\}.
\]

In the limit of \( M \to +\infty \), both \( S_{M, s\sqrt{n}\Delta t} \) and \( S_{M, s\sqrt{n}\Delta t} \) become pseudorandom sequences uniformly distributed \([0, \pi)\), because \( s\Delta t \) satisfies the definition of \( \Delta t \) given by Eq. (95). This implies that we cannot distinguish \( S_{M, \sqrt{n}\Delta t} \) and \( S_{M, s\sqrt{n}\Delta t} \) actually in the limit of \( M \to +\infty \). Thus, we cannot distinguish \( F_{M, \Delta t} \) and \( F_{M, s\Delta t} \) actually under the limit of \( M \to +\infty \), too. Let us describe these results as \( S_{M, \sqrt{n}\Delta t} \simeq S_{M, s\sqrt{n}\Delta t} \) and \( F_{M, \Delta t} \simeq F_{M, s\Delta t} \).

From these discussions, we can conclude as follows: taking the change of the scale \( \Delta t \to s\Delta t \) for \( s > 1 \), we observe the scale invariance, \( S \simeq S' \) and \( F \simeq F' \), for Eqs. (101), (102), (103) and (104).

We can find similar scale invariance for \( S_x(t) = L_1(t) \). Substituting \( b = e^{-\beta} \) into Eq. (27), we can rewrite \( L_1(t) \) as

\[
L_1(t) = (1 - b)g(t),
\]

\[
g(t) = \sum_{n=0}^{\infty} b^n \cos(\sqrt{n} + 1)t) \cos(\sqrt{n}t).
\]
Substitution of elements of $S$ given by Eq. (101) into $g(t)$ yields a set of
$G = \{g(t_0), g(t_1), g(t_2), \ldots \}$. Then, the set $G$ also acquires scaling properties, which are similar to $\mathcal{F} \simeq \mathcal{F}'$ for Eqs. (102) and (104).

To provide pseudorandom sequences to functions $L_4(t)$ and $f(t)$ defined in Eqs. (99) and (100), we choose $(\pi/2) < \Delta t < \pi$. However, if we think about $L_1(t)$ and $g(t)$ defined in Eqs. (111) and (112), we have to choose $\pi < \Delta t < 2\pi$ for generating pseudorandom sequences safely. The reason why we have to be careful in adjusting $\Delta t$ is the difference of a numerical factor two between arguments of the $n$th terms of series $f(t)$ and $g(t)$, that is, $\cos(2\sqrt{n}t)$ and $\cos(\sqrt{n + 1}t)\cos(\sqrt{n}t)$. Hence, we set $\Delta t = 7/2 = 3.5$ for Figs. 2, 3, 4, 5 and 6.

In Appendix D, we examine physical transient spectra of the atom in the cavity. Moreover, to understand the physical meanings and the scale invariance of the discrete plots of the trajectories of the Bloch vector, we investigate difference between continuous and discrete Fourier transforms for the atomic fluorescence. To confirm the scale invariance of the discrete plots, we examine histograms of the samples of the atomic fluorescence at constant time intervals.

7 The graph of $S_x(t)$ versus the inverse of the temperature $\beta$ for the time $t$ such that $|S_z(t)| \ll 1$: the numerical experiments

Figures 2, 3 and 4 suggest to us that distribution of points with $|S_x| \ll 1$ depends on $\beta$ strongly. More precisely, on the one hand, for $\beta \gg 1$, points of $|S_x| \ll 1$ are localized around $|S_x| \simeq 1$. On the other hand, for $\beta \ll 1$, points of $|S_x| \ll 1$ are spread over a range of $S_x \in [-1, 1]$. Figure 7 illustrates this characteristic feature of the function $L_4(t) = S_z(t)$ given by Eqs. (99) and (100) well. How to draw Fig. 7 is as follows. First, taking $0 < \epsilon \ll 1$, we look for $t_n(= n\Delta t)$, each of which satisfies $|L_4(t_n)| \leq \epsilon$. Second, we plot points $(\beta, S_x(t_n))$ for these times $t_n$.

First of all, we show that the following relation holds,

$$L_4(t) = 0 \Leftrightarrow \cos(2\sqrt{n}t) = 1 \quad \forall n \in \{1, 2, 3, \ldots\}. \quad (113)$$

We can derive Eq. (113) as follows. On the one hand, if we assume the right-hand statement of Eq. (113) holds, we can rewrite Eq. (100) as

$$f(t) = \sum_{n=1}^{\infty} b^n = \frac{b}{1 - b}. \quad (114)$$

Substitution of Eq. (114) into Eq. (99) yields $L_4(t) = 0$. On the other hand, taking care of $0 < b < 1$ and $-1 \leq \cos(2\sqrt{n}t) \leq 1 \quad \forall n \in \{1, 2, 3, \ldots\}$ in Eq. (100), we can conclude $f(t) = b/(1 - b)$ if and only if the right-hand statement in Eq. (113) holds. Thus, from these discussions, we arrive at $L_4(t) = 0$ if and only if $\cos(2\sqrt{n}t) = 1 \quad \forall n \in \{1, 2, 3, \ldots\}$.

Therefore, we can consider the right-hand statement of Eq. (113) to be a necessary and sufficient condition for $L_4(t) = 0$. Then, we can rewrite this necessary and sufficient
Equation (116) insists that

\[ 2t = 2n_1\pi, \]
\[ 2\sqrt{2}t = 2n_2\pi, \]
\[ 2\sqrt{3}t = 2n_3\pi, \]
\[ \ldots, \]  

(115)

where \( n_1, n_2, n_3, \ldots \) are integers. Thus, if we assume \( n_1 \neq 0 \), we obtain

\[ \sqrt{2} = \frac{n_2}{n_1}, \quad \sqrt{3} = \frac{n_3}{n_1}, \quad \ldots \]  

(116)

Equation (116) insists that \( \sqrt{2}, \sqrt{3}, \ldots \) are rational numbers, so that this result causes a contradiction. Hence, we achieve a conclusion that \( L_4(t) = 0 \) if and only if \( t = 0 \).

However, as mentioned in Sec. 4, it is possible that \( |L_4(t)| \ll 1 \) holds for some \( t(> 0) \). Thus, let us think about a problem whether or not we can find \( t \) such that

\[ \exists \epsilon \ll 1, \quad |L_4(t)| \leq \epsilon \quad \text{for} \quad 0 < \epsilon \ll 1. \]

Before we try a treatment of an algebraic analysis, we examine this problem with numerical calculations. First, we choose an arbitrary rational number as a time displacement \( \pi < \Delta t < 2\pi \). Second, taking a sequence of the time variable, \( t_n = n\Delta t \) for \( n \in \{0, 1, 2, \ldots, N\} \), we construct a set,

\[ \{L_4(t_n) : n \in \{0, 1, 2, \ldots, N\}\}. \]  

(117)

Third, choosing \( 0 < \epsilon \ll 1 \), we gather \( t_n \), each of which satisfies \( |L_4(t_n)| \leq \epsilon \). Because of Eqs. (27) and (28), we can consider \( L_1(=S_x) \) to be a function of \( t_n \) and \( \beta \). Fourth, we plot \( (\beta, L_4(t_n)) \) for these times \( t_n \) and obtain a graph of Fig. 7. The graph of Fig. 7 represents a dependence of \( S_x \) on \( \beta \) under the condition \( |S_x| \leq \epsilon \).

When we produce the graph of Fig. 7 with numerical calculations, we have to pay attention to the following facts. In general, if we fix \( \beta \) and \( N \) to certain values, the number of \( t_n \) such that \( |L_4(t_n)| \leq \epsilon \) decreases rapidly as \( \epsilon(> 0) \) becomes smaller. Thus, with fixing \( \beta \), we have to let \( N \) be larger as \( \epsilon(> 0) \) becomes smaller and approaches to zero. According to Eq. (117), we let \( L_4(t_0), L_4(t_1), L_4(t_2), \ldots, L_4(t_N) \) form a set of samples, whose number of elements is given by \((N + 1)\) for a certain fixed \( \beta \). Among the samples given by Eq. (117), the number of \( t_n \) with \( |L_4(t_n)| \leq \epsilon \) decreases rapidly as \( \beta \) becomes smaller. Thus, to let the number of \( t_n \) with \( |L_4(t_n)| \leq \epsilon \) remain constant, we have to cause \( N \) to be larger as \( \beta \) becomes smaller. Because of these circumstances, for actual numerical calculations, we introduce the following relation,

\[ N = N_0 e^{c_1/\beta}, \]  

(118)

where \( N_0 = 619.3 \) and \( c_1 = 13.37 \). We apply Eq. (118) to calculations with \( \epsilon = 7.5 \times 10^{-4} \) for \( 1.0 \leq \beta \leq 5.0 \). According to Eq. (118), we obtain \( N = 8978 \) for \( \beta = 5.0 \) and \( N = 396659904 \) for \( \beta = 1.0 \).

However, if we apply Eq. (118) to a case of \( 0.5 \leq \beta < 1.0 \), \( N \) becomes too large and we cannot carry out numerical calculations actually. Thus, for \( 0.5 \leq \beta < 1.0 \), we put \( N = 4 \times 10^8 \) as a constant and introduce an alternative relation,

\[ \epsilon = \epsilon_0 e^{c_2/\beta}, \]  

(119)
where \( \epsilon_0 = 1.875 \times 10^{-4} \) and \( c_2 = \ln 4 \). According to Eq. (119), we change \( \epsilon \) as a function of \( \beta \), so that we obtain \( \epsilon = 7.5 \times 10^{-4} \) for \( \beta = 1.0 \) and \( \epsilon = 3.0 \times 10^{-3} \) for \( \beta = 0.5 \).

Following the above prescriptions, we plot points of \((\beta, S_z(t_n))\) that satisfy \( |S_z(t_n)| \leq \epsilon \) for \( 0.5 \leq \beta \leq 5.0 \) in Fig. 7. For all points plotted in Fig. 7, we set \( \Delta t = 3.5 \). In Fig. 7, the horizontal and vertical axes are scaled logarithmically and linearly, respectively. Turning our eyes on Fig. 7, we observe that distinctive curves come into existence at \( S_z = \pm 1 \) in the limit of \( \beta \to +\infty \) and their branches grow out from their trunk and spread as \( \beta \) becomes smaller. In Sec. 8, we analyze properties of these curves using perturbative techniques.

### 8 The graph of \( S_z(t) \) versus the inverse of the temperature \( \beta \) for the time \( t \) such that \( |S_z(t)| \ll 1 \): the perturbative evaluation

In this section, we examine values of the time \( t \) that satisfy \( |L_4(t)| \ll 1 \) in perturbation theory. In Sec. 4, we suggest that Fig. 7 is an appearance of incommensurate angular frequencies under the quasiperiodic dynamics. Then, we explain how to obtain approximations of their ratio with continued fractions.

In Sec. 7, we obtain a result that \( L_4(t) = 0 \) holds if and only if \( t = 0 \). Let us regard \( L_4(t) \) as a power series in \( b \) from Eqs. (99) and (100). Then, we take \( 0 < \epsilon \ll 1 \) and investigate values of the time variable \( t \) which satisfy \( |L_4(t)| \leq \epsilon \). Thus, we begin the perturbation method with the following relation:

\[
\frac{b}{1-b} - \frac{2\epsilon b}{(1-b)^2} \leq \sum_{n=1}^{\infty} b^n \cos(2\sqrt{n}t) < \frac{b}{1-b}.
\]

(120)

Here, taking the low temperature limit \( \beta \gg 1 \), that is, \( 0 < b \ll 1 \), we investigate Eq. (120) in terms of the parameter \( b \), such that the zero, first etc., powers of \( b \) correspond to the zero, first, etc., orders of the perturbation calculation.

#### 8.1 The first and second-order perturbations

At first, we discuss the first-order perturbation. In first order, we can rewrite Eq. (120) as

\[
b - 2\epsilon b \leq b \cos(2t) < b.
\]

(121)

Taking the limit of \( \epsilon \to +0 \) for Eq. (121), we obtain \( t = 0, \pi, 2\pi, 3\pi, \ldots \). These results are not important for us indeed, because they do not give us any physical meanings.

Next, we discuss the second-order perturbation. In second order, we can rewrite Eq. (120) as

\[
0 < b[1 - \cos(2t)] + b^2[1 - \cos(2\sqrt{2}t)] \leq 2\epsilon b(1 + 2b).
\]

(122)

It is very difficult for us to find all values of \( t \), each of which satisfies Eq. (122). Thus, giving up our attempts to find all \( t \) with Eq. (122), we concentrate on specifying a subset of \( t \) such that Eq. (122).
Because of \( \cos(2t) \leq 1 \) and \( \cos(2\sqrt{2}t) \leq 1 \), both the first and second terms of Eq. (122) have to be larger than or equal to zero. Hence, let us consider a special case where the following two relations hold:

\[
\begin{align*}
b[1 - \cos(2t)] &= 2eb, \\
b^2[1 - \cos(2\sqrt{2}t)] &= 4eb^2.
\end{align*}
\]

We have to emphasize that some \( t \) with Eq. (122) may not satisfy Eqs. (123) and (124). However, we neglect this possibility throughout this section, and we give all attention on Eqs. (123) and (124).

Because Eq. (123) is essentially equivalent to Eq. (121), we obtain \( t = q\pi \) for \( q = 0, 1, 2, ... \). Here, we remember that we obtain \( q = t = 0 \) as a trivial root of \( L_4(t) = 0 \) in Sec. 7. Thus, we think about \( q = 1, 2, 3, ... \) only. Then, from Eq. (124), we obtain

\[
\cos(2\sqrt{2}q\pi) = 1 - 4\epsilon.
\]

This implies the following fact. There are a countably infinite number of inequalities,

\[
\begin{align*}
|2\sqrt{2}q\pi - 2p\pi| &< \delta(4\epsilon) \ll 1, \\
|2\sqrt{2}q\pi - 4p\pi| &< \delta(4\epsilon) \ll 1, \\
|2\sqrt{2}q\pi - 6p\pi| &< \delta(4\epsilon) \ll 1, \\
|2\sqrt{2}q\pi - 8p\pi| &< \delta(4\epsilon) \ll 1,
\end{align*}
\]

\[
\ldots,
\]

where an explicit form of \( \delta(4\epsilon) \) is given in a next paragraph. Among the above inequalities, only one inequality holds for a certain positive integer \( p \), where \( p \) and \( q \) are coprime.

An explicit form of a function \( \delta(4\epsilon) \) appearing in Eqs. (126), (127), (128) and (129) is given by

\[
\delta(4\epsilon) = |\arccos(1 - 4\epsilon)|.
\]

Equation (130) tells us that \( \delta(4\epsilon) \) is equal to a very small positive number. From Eq. (126), we can derive the following inequality,

\[
\frac{|\sqrt{2} - 1|}{p} < \frac{\delta(4\epsilon)}{2\pi} \ll 1.
\]

Thus, we can conclude \( p/q \simeq \sqrt{2} \), so that we arrive at the fact that \( p/q \) is a rational approximate number of \( \sqrt{2} \) finally. In other words, one of the time variables with Eqs. (123) and (124) is given by \( t = q\pi \), where \( p/q \) is a rational approximate number of \( \sqrt{2} \).

Then, the following question arises. How small the upper bound of \( |\sqrt{2} - (p/q)| \) is? How do we estimate the numerical precision of \( p/q \) as an approximate number of \( \sqrt{2} \)? We discuss the question of accuracy later.

From Eq. (127), we obtain

\[
\left| \frac{1}{\sqrt{2}} - \frac{q}{p} \right| < \frac{\delta(4\epsilon)}{4\pi p} \ll 1.
\]
This inequality yields a conclusion of \( p/q \simeq 1/\sqrt{2} \), so that we arrive at the fact that \( p/q \) is a rational approximate number of \( 1/\sqrt{2} \). This implies that one of the time variables with Eqs. (123) and (124) is given by \( t = q\pi \), where \( p/q \) is a rational approximate number of \( 1/\sqrt{2} \).

Applying a similar discussion to Eq. (128), we obtain \( p/q \simeq \sqrt{2}/3 \) and \( t = q\pi \). Moreover, applying a similar discussion to Eq. (129), we obtain \( p/q \simeq 1/(2\sqrt{2}) \) and \( t = q\pi \). Putting these results together, we can derive a general formula for \( k = 1, 2, 3, \ldots \),

\[
\frac{p}{q} \simeq \frac{\sqrt{2}}{k}, \quad t = q\pi \quad \text{for } k = 1, 2, 3, \ldots.
\] (133)

As a result of the above discussions, we obtain the following conclusion. In the second-order perturbation theory, the time variable \( t = q\pi \) satisfies \( |L_4(t)| \ll 1 \), where \( p/q \) is a rational approximate number of \( \sqrt{2}/k \) for \( k = 1, 2, 3, \ldots \), and \( p \) and \( q \) are coprime. Going into details, a set of the time variables \( t \) with \( 0 \leq |L_4(t)| \leq \epsilon (\ll 1) \) includes the following elements \( t = q\pi \). First, we consider rational approximations \( p/q \) of \( \sqrt{2}/k \) for \( k = 1, 2, 3, \ldots \). Second, using a very small positive value \( \delta (4\epsilon) \), we can give their accuracies as

\[
|\frac{\sqrt{2}}{k} - \frac{p}{q}| < \frac{\delta (4\epsilon)}{2k\pi q} \ll 1 \quad \text{for } k = 1, 2, 3, \ldots.
\] (134)

We can rewrite Eq. (134) as follows:

\[
|\frac{\sqrt{2}}{k} - \frac{p}{q}| < \frac{\delta (4\epsilon)}{2k\pi q} \ll 1 \quad \text{for } k = 1, 2, 3, \ldots.
\] (135)

In the above equation, \( p/q \) (where \( q > 0 \)) represents a rational approximate number of \( \sqrt{2}/k \). If we put \( t = q\pi \) with Eq. (135), \( 0 \leq |L_4(t)| \leq \epsilon \) holds. Therefore, Eq. (135) gives an upper bound of the error resulting from this approximation. Let us estimate the right-hand side of Eq. (135) at \( O(1/q) \). Then, Eq. (135) implies that the precision of the rational number approximation \( p/q \) to \( \sqrt{2}/k \) has to be less than \( O(1/q) \).

From the above considerations, we obtain the following result:

\[
|\frac{\sqrt{2}}{k} - \frac{p}{q}| = \frac{1}{cq^{1+\nu}} \quad \text{for } k = 1, 2, 3, \ldots,
\] (136)

where \( p \) and \( q(>0) \) are coprime and \( \nu > 1 \). In addition, we suppose that \( c \) is a constant and close to unity. Furthermore, we want to let \( \nu \) be as large as possible we can. This is because a rational approximate number \( p/q \) becomes closer to \( \sqrt{2}/k \) as \( \nu (>1) \) gets larger and larger.

Here, we make use of the following theorem [28, 35]. For an arbitrary irrational number \( \alpha \), there exist infinite sequences \( p_n \) and \( q_n(>0) \) for \( n \geq 0 \) such that \( p_n \) and \( q_n \) are coprime and

\[
|q_n \alpha - p_n| < \frac{1}{q_n}.
\] (137)

In other words, there exist infinitely many rational numbers \( p/q \) such that

\[
|\alpha - \frac{p}{q}| < \frac{1}{q^\nu},
\] (138)
where \( p \) and \( q (> 0) \) are coprime. (The rigorous proof of this theorem is given in Appendix E.) Moreover, the following fact is very useful for us in the remainder of this section. For an arbitrary irrational but not transcendental number \( \alpha \), if we suppose that there exist infinitely many rational numbers \( p/q \) such that

\[
|\alpha - p/q| < \frac{1}{q^\mu},
\]

the upper bound of \( \mu \) is equal to two. This fact is called the theorem of Roth, which is a fundamental result in the Diophantine approximation \([28]\). Because to prove this theorem is beyond our purpose of this paper, we do not touch it any more. Hence, from now on, we consider only a rational approximate number \( p/q \) of \( \sqrt{2}/k \) for \( k = 1, 2, 3, ... \) such that

\[
|\sqrt{2}/k - p/q| < \frac{1}{q^2},
\]

for \( k = 1, 2, 3, ... \),

and we choose \( t = q\pi \) for Eqs. (123) and (124).

In the following paragraphs, we confirm the above discussions with numerical calculations actually. By computing first some terms in a continued fraction representation of an irrational number, we can efficiently obtain its rational approximate number with high precision. Let us describe the continued fraction of an arbitrary irrational number \( \alpha \) as

\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}},
\]

where \( a_0 \) is an integer and \( a_1, a_2, ... \) are positive integers.

The continued fractions of irrational numbers have the following properties, whose proofs are given in Refs. [28, 35].

1. If a real number \( \alpha \) is irrational, its continued fraction expression \([a_0; a_1, a_2, ...]\) is infinite.

2. If and only if \( \alpha \) is an irrational solution of a quadratic equation with rational coefficients, its continued fraction expression is periodic.

3. For an arbitrary irrational number \( \alpha = [a_0; a_1, a_2, a_3, ...] \), let us consider infinite sequences of integers \((p_0, p_1, p_2, ...)\) and \((q_0, q_1, q_2, ...)\) such that

\[
[a_0; a_1, a_2, ..., a_n] = \frac{p_n}{q_n},
\]

Obviously, from Eq. (141), we can write down \( p_n \) and \( q_n \) as follows:

\[
p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2},
\]

\[
q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2}.
\]

Then, the following relation holds,

\[
\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_5}{q_5} < \frac{p_7}{q_7} < \frac{p_1}{q_1}.
\]
4. For a continued fraction expression of an arbitrary irrational number

\[ \alpha = [a_0; a_1, a_2, a_3, \ldots] \], let us consider a quantity obtained by including its first \((n+1)\) terms as shown in Eq. (142). Then the following inequalities hold,

\[ |\alpha - \frac{p_n}{q_n}| \leq \frac{1}{a_{n+1}q_n^2} \quad \text{for} \ n = 0, 1, 2, \ldots. \] (146)

Let us obtain a rational approximate number of \(\sqrt{2}/k\) for \(k = 1, 2, 3, \ldots\) by truncating its corresponding continued fraction. We describe the obtained rational approximate number as \(p/q\). Then, because of the fourth item written above, it satisfies Eq. (140).

[Here, we pay attention to the following fact. In general, we cannot find all rational approximate numbers of \(\sqrt{2}/k\) with Eq. (140) by using the continued fraction.]

The continued fraction of a quadratic irrational number \(\sqrt{2}\) is given by

\[ \sqrt{2} = [1; 2, 2, 2, \ldots] = [1; \dot{2}]. \] (147)

In Eq. (147), \(a_0 = 1\) and \(a_1 = a_2 = \ldots = 2\) hold, so that the figure "2" appears in succession with period unity. From now on, we indicate the repeating block by dots as shown in Eqs. (147) and (149). Moreover, we introduce the following notations,

\[ \chi_{\sqrt{2}}(0) = [1] = 1, \quad \chi_{\sqrt{2}}(1) = [1; 2] = 3/2, \quad \chi_{\sqrt{2}}(2) = [1; 2, 2] = 7/5, \]

\[ \chi_{\sqrt{2}}(n) = [1; 2, \ldots, 2]. \] (148)

Because 1/\(\sqrt{2}\), \(\sqrt{2}/3\) and 1/(2\(\sqrt{2}\)) are quadratic irrational numbers, we can write down them in the following continued fraction expressions, as well,

\[ 1/\sqrt{2} = [0; 1, 2, 2, 2, \ldots] = [0; 1, \dot{2}], \]
\[ \sqrt{2}/3 = [0; 2, 8, 4, 8, 4, \ldots] = [0; 2, \dot{8}, \dot{4}], \]
\[ 1/(2\sqrt{2}) = [0; 2, 1, 4, 1, 4, \ldots] = [0; 2, \dot{1}, \dot{4}]. \] (149)

Moreover, we obtain

\[ \chi_{1/\sqrt{2}}(0) = [0] = 0, \quad \chi_{1/\sqrt{2}}(1) = [0; 1] = 1, \quad \chi_{1/\sqrt{2}}(2) = [0; 1, 2] = 2/3, \]
\[ \chi_{1/\sqrt{2}}(n) = [0; 1, 2, \ldots, \dot{2}], \]
\[ \chi_{\sqrt{2}/3}(0) = [0] = 0, \quad \chi_{\sqrt{2}/3}(1) = [0; 2] = 1/2, \quad \chi_{\sqrt{2}/3}(2) = [0; 2, 8] = 8/17, \]
\[ \chi_{\sqrt{2}/3}(n) = [0; 2, 8, 4, \ldots, \dot{a}_n], \]
\[ \chi_{1/(2\sqrt{2})}(0) = [0] = 0, \quad \chi_{1/(2\sqrt{2})}(1) = [0; 2] = 1/2, \quad \chi_{1/(2\sqrt{2})}(2) = [0; 2, 1] = 1/3, \]
\[ \chi_{1/(2\sqrt{2})}(n) = [0; 2, 1, 4, \ldots, \dot{a}_n]. \] (150)
Let us consider Eq. (140) for \( k = 1, 2, 3, 4 \). (We neglect the cases where \( k = 5, 6, 7, \ldots \)). Then, we choose rational approximate numbers of \( \sqrt{2} \), \( 1/\sqrt{2} \), \( \sqrt{2}/3 \) and \( 1/(2\sqrt{2}) \) as follows. At first, thinking about rational approximate numbers of \( \sqrt{2} \), we dismiss \( \chi_{\sqrt{2}}(0), \ldots, \chi_{\sqrt{2}}(11) \) in order to obtain good accuracy. Thus, we select the following 28 rational numbers for the approximation of \( \sqrt{2} \),

\[
\chi_{\sqrt{2}}(12) = \frac{47321}{33461}, \quad \chi_{\sqrt{2}}(13) = \frac{114243}{80782}, \ldots, \\
\chi_{\sqrt{2}}(39) = \frac{1023286908188737}{723573111879672}.
\]

(151)

We do not select \( \chi_{\sqrt{2}}(40), \chi_{\sqrt{2}}(41), \chi_{\sqrt{2}}(42), \ldots \) because each number of digits in their denominators is larger than 15 and they are not tractable in the numerical calculations actually. In a similar way, we obtain rational approximate numbers of \( 1/\sqrt{2}, \sqrt{2}/3 \) and \( 1/(2\sqrt{2}) \) as follows:

\[
\chi_{1/\sqrt{2}}(12) = \frac{13860}{19601}, \quad \chi_{1/\sqrt{2}}(13) = \frac{33461}{47321}, \ldots, \\
\chi_{1/\sqrt{2}}(39) = \frac{299713796309065}{423859315570607}.
\]

(152)

\[
\chi_{\sqrt{2}/3}(12) = \frac{362226480}{768398401}, \quad \chi_{\sqrt{2}/3}(13) = \frac{1492851361}{3166815962}, \ldots, \\
\chi_{\sqrt{2}/3}(19) = \frac{58522759015841}{124145519261542}.
\]

(153)

\[
\chi_{1/(2\sqrt{2})}(12) = \frac{6930}{19601}, \quad \chi_{1/(2\sqrt{2})}(13) = \frac{33461}{94642}, \ldots, \\
\chi_{1/(2\sqrt{2})}(39) = \frac{299713796309065}{847718631141214}.
\]

(154)

In fact, we can verify that the following relations hold:

\[
|\sqrt{2} - \chi_{\sqrt{2}}(12)| \simeq 3.158 \times 10^{-10} < 1/(2 \times 33461)^2, \\
|\sqrt{2} - \chi_{\sqrt{2}}(13)| \simeq 5.418 \times 10^{-11} < 1/(2 \times 80782)^2,
\]

(155)

and

\[
1 - \cos(2\sqrt{2} \times 33461\pi) \simeq 2.204 \times 10^{-9}, \\
1 - \cos(2\sqrt{2} \times 80782\pi) \simeq 3.781 \times 10^{-10}.
\]

(156)

From Eqs. (151), (152), (153) and (154), we obtain 70 different positive integers as denominators. [Some integers appear as denominators twice or more in Eqs. (151), (152), (153) and (154), so that we have to avoid counting them over again.] Putting these 70 integers and zero together, we construct a set \( \mathcal{M} \), whose cardinality is equal to 71, as

\[
\mathcal{M} = \{0, 19601, 33461, 47321, 80782, 94642, \ldots, \\
723573111879672, \\
847718631141214\}.
\]

(157)
Figure 16: A linear-log plot of $(\beta, S_x(t))$ for $t = q\pi$ where $q \in \mathcal{M}$ given by Eq. (157) and $2.5 \leq \beta \leq 5.0$, and for $t = q\pi$ where $q \in \tilde{\mathcal{M}}$ given by Eq. (161) and $1.0 \leq \beta \leq 2.5$. All dots are drawn with changing $\beta$ at constant interval of 0.01. We give them a diameter being $4/1000$ of the width of the whole graph.

Then, we obtain 71 distinct times, $t = q\pi$ for $q \in \mathcal{M}$. According to Eqs. (27) and (28), we consider $S_x (= L_1)$ to be a function of $t$ and $\beta$ and plot the points of $(\beta, S_x(t))$ for $t = q\pi$ where $q \in \mathcal{M}$ and $2.5 \leq \beta \leq 5.0$ in Fig. 16. Figure 16 reproduces the graph of Fig. 7 for $2.5 \leq \beta \leq 5.0$ well.

### 8.2 The third-order perturbation

In this subsection, we discuss the third-order perturbation. In third order, we can rewrite Eq. (120) as

$$0 < 1 - \tilde{f}(b, t) \leq \frac{2\epsilon b(1 + 2b + 3b^2)}{b + b^2 + b^3}, \quad (158)$$

$$\tilde{f}(b, t) = \frac{1}{b + b^2 + b^3} [b \cos(2t) + b^2 \cos(2\sqrt{2}t) + b^3 \cos(2\sqrt{3}t)]. \quad (159)$$

After discussions given in Sec. 8.1, we give up our attempts to find all $t$ with Eqs. (158) and (159). Thus, we concentrate on specifying a subset of $t$, each of which satisfies Eqs. (158) and (159). Obviously, the time $t$ with Eqs. (158) and (159) has to satisfy Eq. (122), as well. Thus, it is possible that some of $t$ with Eqs. (158) and (159) belong to the set of $t = q\pi$ for $q \in \mathcal{M}$ given by Eq. (157). Hence, we try finding the time $t$ with Eqs. (158) and (159) from $t = q\pi$ for $q \in \mathcal{M}$.

In fact, we select $q \in \mathcal{M}$ for $t = q\pi$ with Eqs. (158) and (159) on the following condition. Fixing the parameter for perturbation at $\epsilon \simeq 0.01478$ and $b = e^{-2.0} \simeq 0.1353$, that is, $\beta = 2.0$, and obtaining $2\epsilon b(1 + 2b + 3b^2)/(b + b^2 + 5b^3) \simeq 0.002$, we look for $q$ such that

$$1 - \tilde{f}(e^{-2.0}, q\pi) < 0.002 \quad \text{for } q \in \mathcal{M}. \quad (160)$$
As a result of numerical calculations, we find 14 integers for \( q \) satisfying Eq. (160),

\[
\tilde{M} = \{0, 19\,601, 470\,832, 1\,607\,521, 15\,994\,428, 18\,738\,638, 768\,398\,401, 10\,812\,186\,007, 21\,624\,372\,014, 627\,013\,566\,048, 8\,822\,750\,406\,821, 30\,122\,754\,096\,401, 299\,713\,796\,309\,065, 847\,718\,631\,141\,214\}.
\]

Thus, regarding \( S_x(= L_1) \) as a function of \( t \) and \( \beta \) because of Eqs. (27) and (28), we plot the points of \((\beta, S_x(t))\) for \( t = q\pi \) where \( q \in \tilde{M} \) of Eq. (161) and \( 1.0 \leq \beta \leq 2.5 \) in Fig. 16. We observe that Fig. 16 reproduces the graph of Fig. 7 for \( 1.0 \leq \beta \leq 2.5 \) well.

Before closing this subsection, we make two remarks concerning our treatments of the Diophantine approximation. First, we indicate a property of integers \( q \in \tilde{M} \) in Eq. (161). Remembering Eq. (140), we expect \( \forall q \in \tilde{M} \) to satisfy the following relation for certain positive integers \( k \) and \( q \),

\[
|\frac{\sqrt{3}}{k} - \frac{p}{q}| < \frac{1}{q^2}.
\]

In fact, for example, for \( q = 19\,601 \in \tilde{M} \) and \( q = 470\,832 \in \tilde{M} \), we have the following two relations, respectively,

\[
|\frac{\sqrt{3}}{2425} - \frac{14}{19\,601}| < \frac{1}{19\,601^2},
\]

\[
|\frac{\sqrt{3}}{163\,101} - \frac{5}{470\,832}| < \frac{1}{470\,832^2}.
\]

Second, we have to point out that values of the time co-ordinate \( t = q\pi \) for \( q \in \tilde{M} \) given by Eqs. (157) and (161) never fit in the distinct times \( t_n = n\Delta t, \Delta t = p/q \) and \( \pi < \Delta t < 2\pi \) discussed in Sec. 6. This implies that the graph of Fig. 7 is plotted at times of rational numbers and the graph of Fig. 16 is plotted at times of irrational (and transcendental) numbers. We cannot find an acceptable way of dealing this difference between Figs. 7 and 16.

9 Discussions

We can obtain the JCM by applying the rotating-wave approximation to a single two-level atom that interacts with a single mode of an optical cavity. In this paper, we investigate the quasiperiodicity, which the atom and the cavity field show during the time evolution according to the Jaynes-Cummings interaction, where the atom and the cavity field are initially put into a certain pure state and a mixed state in thermal equilibrium, respectively.

However, for actual experiments of the cavity quantum electrodynamics (QED) in the laboratory, we may not observe quasiperiodicity because of lack of the rotating-wave
approximation. For example, it is possible that we cannot reproduce robust experimental results of fractal properties and scale invariance, which we discuss in this paper.

In Ref. [36], Milonni et al. describe the interaction between a collection of two-level atoms and the single-mode classical electric field as the optical Bloch equations using the semiclassical approximation. They are given as follows:

\[
\begin{align*}
\dot{X}(t) &= -\omega_0 Y(t), \\
\dot{Y}(t) &= \omega_0 X(t) + (2/\hbar)pE(t)Z(t), \\
\dot{Z}(t) &= -(2/\hbar)pE(t)Y(t), \\
\dot{E}(t) + \omega_0^2 E(t) &= -4\pi Np\dot{X}(t),
\end{align*}
\]

where \((X(t), Y(t), Z(t))\) represents the Bloch vector, \(p\) is the transition dipole moment, \(E(t)\) represents the electric field, \(\omega_0\) and \(\omega\) are angular frequencies of the atoms and the electric field respectively, and \(N\) is a density of the atoms.

Milonni et al. investigate the optical Bloch equations numerically and obtain the following results. In the rotating-wave approximation, there is no predictions of chaos. On the other hand, if the initial conditions let the rotating-wave approximation fail, we can occasionally obtain the chaotic behaviour.

In Ref. [37], Prants et al. consider the recoil effect caused by the centre-of-mass motion of the atom in the cavity QED. They extend the Hamiltonian of the JCM as follows:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\hbar}{2}\omega_0 \hat{\sigma}_z + \hbar \omega_f \hat{a}^\dagger \hat{a} - \frac{h\Omega_0}{2}(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger) \cos k_f \hat{x},
\]

where \(\hat{x}\) and \(\hat{p}\) are the atomic position and momentum operators, respectively. In the above Hamiltonian, we consider the atom to be in a single-mode high-finesse standing-wave cavity, so that the dynamics of the system is sensitive to the centre-of-mass motion of the atom.

Prants et al. applies the semiclassical approximation to this extended JCM. They assume that an expectation value of \(\hat{x}\) varies in time slowly and put a certain special initial conditions. Then, they show that the expectation value of \(\hat{x}\) obeys the following equation:

\[
\ddot{x} + \omega^2(1 - \cos \Omega_N \tau) \sin x = 0.
\]

Moreover, they obtain the effective Hamiltonian, from which we can derive the above equation of motion as follows:

\[
H = \frac{1}{2} \dot{x}^2 - \omega^2 \cos x + \frac{\omega^2}{2} [\cos(x + \Omega_N \tau) + \cos(x - \Omega_N \tau)].
\]

In Eqs. (167) and (168), we set \(x = k_f \langle \hat{x} \rangle\) and \(\tau = \Omega_N t\). The new angular frequencies \(\omega\) and \(\Omega_N\) are quantities constructed from \(\omega_0\), \(\omega_f\) and \(\Omega_0\), and they satisfy the relation \(\omega \propto \Omega_N^{-1}\). The Hamiltonian given by Eq. (168) represents a particle moving in the field of three plane waves, \(\cos x\), \(\cos(x + \Omega_N \tau)\) and \(\cos(x - \Omega_N \tau)\). It is widely known that this Hamiltonian induce chaotic dynamics. We emphasize that Prants et al.’s results in Ref. [37] are derived under the rotating-wave approximation. In Ref. [38], Chotorlishvili and Toklikishvili generalize the Hamiltonian in Eq. (166) for a three level optical atom and discuss its chaotic dynamics.
In Refs. [36, 37, 38], the interactions between the atom and the cavity field are discussed under the semiclassical approximations. In general, it is very difficult to exactly solve the interaction between the two-level atom and the cavity field as a fully quantum mechanical model without the rotating-wave approximation. Hence, we cannot predict whether or not actual experiments of the cavity QED reproduce the stable results of quasiperiodicity, fractal properties and scale invariance. This remains to be solved in future.

In this paper, we investigate some remarkable properties of trajectories of the Bloch vector, which is governed by the thermal JCM. From careful observations of the quasiperiodic behaviour of the Bloch vector under the thermal JCM, we obtain novel and interesting facts, such as equivalence to the fluid dynamics, scale invariance and relation with the Diophantine approximation. Throughout the latter half of this paper, to examine quasiperiodic trajectories of the thermal Bloch vector precisely, we borrow some useful concepts from the number theory, which is a branch of the pure mathematics. In Sec. 6, we introduce a knowledge about uniform distribution of real sequences. In Sec. 8, we utilize methods for obtaining rational approximation of irrational numbers by means of their expressions of the continued fractions.

An appearance of Shor’s algorithm lets the quantum information theory attract many researchers’ attention. We remember that Shor’s algorithm is a quantum algorithm for solving integer factorization and discrete logarithm problems efficiently. These problems are regarded as important topics in the field of the number theory and the cryptography. Hence, progress of quantum information theory gives us new connections between ideas in the number theory and the quantum mechanics.

Although the JCM was proposed and studied by researchers of quantum optics, it has become familiar to those of quantum information theory. The JCM has given various information and understandings on quantum mechanics to physicists. The authors expect that we can bring out a great new variety of knowledge furthermore from the JCM, especially, in connection with both discrete mathematics and quantum theory.

### A Physical meanings of $\rho$, $p$ and $K$

In this section, we examine the physical meanings of $\rho(t, \mathbf{x})$, $p(t, \mathbf{x})$ and $K(t, \mathbf{x})$, which appear in Eqs. (78) and (83). We introduce the compressible fluid for explaining it to be equivalent to the evolution of the Bloch vector in Eq. (69). As a result of this procedure, we obtain these quantities, as the density of the fluid, the pressure and the external force per unit mass. Because the compressible fluid itself is fictitious, we need to clarify their physical meanings.

First of all, using Eq. (75), we rewrite Eq. (78) as follows:

$$\frac{\partial}{\partial t} \rho + \mathbf{v} \cdot \nabla \rho + \phi(t) \rho = 0, \quad (169)$$

$$\phi(t) = \nabla \cdot \mathbf{v} = 2 \frac{\dot{L}_1(t)}{L_1(t)} + \frac{\dot{L}_3(t)}{L_3(t)}. \quad (170)$$

To solve the partial differential equation (169), we separate the density of the fluid $\rho(t, \mathbf{x})$ into two parts,

$$\rho(t, \mathbf{x}) = R(t) W(\mathbf{x}). \quad (171)$$
Substitution of Eq. (171) into Eq. (169) yields
\[
\frac{1}{R(t)} \frac{\partial R(t)}{\partial t} + \phi(t) + \frac{1}{W(x)}(v \cdot \nabla)W(x) = 0. \tag{172}
\]

Looking at Eq. (172), we notice the following facts. On the one hand, the first and second terms depend on the variable \(t\) only. On the other hand, only the third term includes the variable \(x\). However, because of Eq. (74), the velocity is described as the function \(v = v(t, x)\), so that the third term of Eq. (172) depends on both \(x\) and \(t\). Putting these considerations together and letting \(\phi(t)\) be an arbitrary function, we can divide Eq. (172) into the following two differential equations:
\[
\frac{1}{R(t)} \frac{\partial R(t)}{\partial t} + \phi(t) = \phi(t), \tag{173}
\]
\[
\frac{1}{W(x)}(v \cdot \nabla)W(x) = -\phi(t). \tag{174}
\]

Then, let us solve Eq. (174) first. Dividing \(W(x)\) into three parts as
\[
W(x) = X(x)Y(y)Z(z), \tag{175}
\]
and taking care of Eq. (74), we rewrite Eq. (174) as follows:
\[
v_1(t, x) \frac{X'(x)}{X(x)} + v_2(t, y) \frac{Y'(y)}{Y(y)} + v_3(t, z) \frac{Z'(z)}{Z(z)} = -\phi(t), \tag{176}
\]
where
\[
v(t, x) = \begin{pmatrix} v_1(t, x) \\ v_2(t, y) \\ v_3(t, z) \end{pmatrix}, \tag{177}
\]
and
\[
v_1(t, x) = \frac{\dot{L}_1(t)}{L_1(t)} x, \\
v_2(t, y) = \frac{\dot{L}_1(t)}{L_1(t)} y, \\
v_3(t, z) = \frac{\dot{L}_3(t)}{L_3(t)} [z - L_4(t)] + \dot{L}_4(t). \tag{178}
\]

Moreover, we divide Eq. (176) into the following three differential equations,
\[
v_1(t, x) \frac{X'(x)}{X(x)} = -\varphi_1(t), \tag{179}
\]
\[
v_2(t, y) \frac{Y'(y)}{Y(y)} = -\varphi_2(t), \tag{180}
\]
\[
v_3(t, z) \frac{Z'(z)}{Z(z)} = -\varphi_3(t). \tag{181}
\]
where
\[ \varphi_1(t) + \varphi_2(t) + \varphi_3(t) = \varphi(t). \] (182)

Next, using Eq. (178), we rewrite Eq. (179) as
\[ \frac{x X'(x)}{X(x)} = - \frac{L_1(t)}{L_1(t)} \varphi_1(t). \] (183)

The left-hand side of Eq. (183) depends only on \( x \) and the right-hand side of Eq. (183) depends only on \( t \). Thus, they have to be equal to an arbitrary constant \((-C_1)\), and we obtain
\[ x X'(x) = -C_1 X(x), \] (184)
\[ C_1 \dot{L}_1(t) = L_1(t) \varphi_1(t). \] (185)

From Eq. (184), we obtain
\[ X(x) = D_1 x^{-C_1}, \] (186)
where \( D_1 \) is an arbitrary constant. In a similar way, from Eq. (180), we obtain
\[ Y(y) = D_2 y^{-C_2}, \] (187)
where \( C_2 \) and \( D_2 \) are arbitrary constants.

Next, we think about Eq. (181). Using Eq. (178), we rewrite Eq. (181) as follows:
\[ \frac{Z'(z)}{Z(z)} = - \frac{\varphi_3(t)}{[L_3(t)/L_3(t)] [z - L_4(t)] + L_4(t)}. \] (188)

The left-hand side of Eq. (188) depends only on \( z \). Thus, we have to let the right-hand side of Eq. (188) not rely on the variable \( t \), which is independent of \( z \). This implies \( \varphi_3(t) = 0 \). Thus, we obtain \( Z(z) = D_3 \), where \( D_3 \) is an arbitrary constant.

Putting these results together, we obtain
\[ W(x) = D_1 D_2 D_3 x^{-C_1} y^{-C_2}, \] (189)
\[ \varphi(t) = (C_1 + C_2) \frac{\dot{L}_1(t)}{L_1(t)}. \]

Because of Eq. (171), we obtain \( D_1 D_2 D_3 \neq 0 \), so that we can avoid letting the density of the fluid \( \rho(t, x) = R(t) W(x) \) be equal to zero at every point \( x \) at every time \( t \). Moreover, we have to put \( C_1 \leq 0 \) and \( C_2 \leq 0 \) because we do not want to let \( W(x) \) be singular at \( x = 0 \) and \( y = 0 \). Similarly, because we want to let \( W(x) \) converge to a finite value in the limits of \( |x| \to +\infty \) and \( |y| \to +\infty \), we put \( C_1 \geq 0 \) and \( C_2 \geq 0 \). Hence, from the above discussions, we set \( C_1 = C_2 = 0 \). Then, from Eq. (189), we obtain \( W(x) = W_0(= \text{constant}) \) and \( \varphi(t) = 0 \). Therefore, from Eq. (173), we arrive at
\[ \frac{\partial}{\partial t} R(t) + \phi(t) R(t) = 0. \] (190)
Taking care of \( L_1(0) = L_3(0) = 1 \) for Eq. (27) and Eq. (170), we obtain the solution of Eq. (190) as

\[
R(t) = R(0) \exp\left[- \int_0^t ds \phi(s) \right] = \frac{R(0)}{L_1(t)^2 L_3(t)}. \tag{191}
\]

Therefore, we can write down \( \rho(t, x) \) as follows:

\[
\rho(t, x) = \rho(t) = \frac{\rho(0)}{L_1(t)^2 L_3(t)}. \tag{192}
\]

Equation (192) tells us that the density of the fluid \( \rho \) depends only on \( t \), and it does not rely on \( x \).

Next, we think around the pressure \( p(t, x) \). At first, we have to carefully notice that we cannot determine \( p(t, x) \) only by the Navier-Stokes equations. To derive \( p(t, x) \) in the fluid precisely, we need an equation of state of the fluid given by classical thermodynamics. To let the discussion be simple, we assume the fluid to be barotropic, such as an ideal gas \[39\]. For a barotropic fluid, the pressure is a function of the density alone as

\[
p = f(\rho). \tag{193}
\]

From Eqs. (192) and (193), we obtain the pressure as

\[
p(t, x) = f(\rho(t)). \tag{194}
\]

Thus, we can conclude that the pressure \( p(t, x) [= p(t)] \) depends only on \( t \) and it never be a function of \( x \). Thus, we obtain

\[
\nabla p = 0. \tag{195}
\]

Putting these discussions and Eq. (82) together, we can rewrite Eq. (83) as follows:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{K}. \tag{196}
\]

From Eq. (196), we can compute the external force per unit mass. Substitution of Eq. (74) into Eq. (196) yields an explicit form of \( \mathbf{K} \) as

\[
\mathbf{K} = \left( \begin{array}{c} \frac{[\ddot{L}_1(t)/L_1(t)]_x}{[\ddot{L}_3(t)/L_3(t)]_y} \\ \frac{[\ddot{L}_1(t)/L_1(t)]_y}{[\ddot{L}_3(t)/L_3(t)]_z} \\ \frac{([\ddot{L}_3(t)/L_3(t)]_z - L_4(t)) + \ddot{L}_4(t)} \end{array} \right). \tag{197}
\]

Equations (196) and (197) mean that only the external force per unit mass \( \mathbf{K} \) causes and affects the dynamics of the velocity of the fluid \( \mathbf{v}(t, x) \).
B The Hamiltonian mechanics for the compressible fluid

In this section, we consider how to build the Hamiltonian, which yields dynamics of the compressible fluid being equivalent to the Bloch vector. In Sec. 5, we show that the dynamics of the Bloch vector can be described with the Navier-Stokes equations for the inviscid compressible fluid with zero vorticity. In the following, we construct the Hamiltonian that provides the Navier-Stokes equations. Moreover, we consider the reason why the trajectory in Fig. 1 appears to intersect itself. At the end of this section, we explain that the trajectory never intersects itself in the phase space. Thus, the intersections are the result of a projection operation and they never violate existence and uniqueness of solutions for the Navier-Stokes equations.

Now, let us consider how to construct the Hamiltonian for a fluid particle [40]. First, we prepare a three-dimensional real vector \( \mathbf{a} = (a_1, a_2, a_3) \) as a particle label at time \( \tau \). The vector \( \mathbf{a} \) is called a curvilinear labeling co-ordinates. They are co-ordinates fixed on the fluid particle. The time variable \( \tau \) is a proper time, which is shown by a clock attached to the fluid particle. Thus, the fluid particle is determined completely by \( (\mathbf{a}, \tau) \).

Let \( \mathbf{x}(\mathbf{a}, \tau) \) be the location of the fluid particle identified by \( \mathbf{a} \) and \( \tau \). Then, the vector \( \mathbf{a} \) specifies the mass element of the fluid particle, so that the Jacobian determinant of \( \mathbf{a} \) gives a mass-density of the fluid. The labeling co-ordinates \( \mathbf{a} \) remain constant following the motion of the fluid particle.

Here, we describe a mass element of the fluid as

\[
d(\text{mass}) = da_1 da_2 da_3. \tag{198}
\]

The partial differentiation of the proper time \( \tau \) is equal to the material derivative as

\[
\tau = t, \quad \frac{\partial}{\partial \tau} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \tag{199}
\]

where

\[
\mathbf{v} = (u, v, w) = \left( \frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}, \frac{\partial z}{\partial \tau} \right). \tag{200}
\]

Then, the mass-density of the fluid is given by

\[
\rho = \frac{\partial(a_1, a_2, a_3)}{\partial(x, y, z)} = \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}. \tag{201}
\]

Moreover, we define the specific volume of the fluid as

\[
\alpha = \frac{1}{\rho} = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}. \tag{202}
\]

From the above notations, we obtain

\[
\frac{\partial \alpha}{\partial \tau} = \frac{\partial(u, y, z)}{\partial(a_1, a_2, a_3)} + \frac{\partial(x, v, z)}{\partial(a_1, a_2, a_3)} + \frac{\partial(x, y, w)}{\partial(a_1, a_2, a_3)}
\]
\[
\frac{\partial}{\partial \tau} \frac{1}{\rho} = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial \tau} = \frac{1}{\rho} \nabla \cdot \mathbf{v},
\]  
(204)
so that we arrive at the continuity equation,

\[
\frac{\partial \rho}{\partial \tau} + \rho \nabla \cdot \mathbf{v} = 0.
\]  
(205)

Using Eq. (199), we can rewrite Eq. (205) as

\[
\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho + \rho \nabla \cdot \mathbf{v} = 0.
\]  
(206)

Next, we define the Lagrangian as follows:

\[
L = \int \int \int d^3 a \left[ 2 \left( \frac{\partial x}{\partial \tau} \right)^2 - \frac{\partial E(1/\rho)}{\partial x} - \frac{\partial K(x,t)}{\partial x} \right],
\]  
(207)
where \( E(1/\rho) \) and \( K(x,t) \) represent the specific internal energy and the potential for an external force, respectively. We describe Hamilton’s principle as

\[
\delta \int L d\tau = 0,
\]  
(208)
where \( \delta \) represents an arbitrary independent variation \( \delta x(a, \tau) \) for the fluid particle locations. Then, we can calculate \( \delta L \) as follows:

\[
\delta L = \delta \int d\tau \int \int d^3 a \left[ 2 \frac{\partial x}{\partial \tau} \cdot \frac{\partial x}{\partial \tau} - \frac{\partial E(1/\rho)}{\partial x} - \frac{\partial K(x,t)}{\partial x} \right]
\]

\[
= \int d\tau \int \int d^3 a \left[ -\frac{\partial^2 x}{\partial \tau^2} \cdot \delta x - \frac{\partial E(1/\rho)}{\partial \alpha} \frac{\partial (x)}{\partial \alpha} - \frac{\partial K(x,t)}{\partial x} \cdot \delta x \right].
\]  
(209)

To derive the above equation, we use Eq. (202).

Here, we introduce a convenient formula. For an arbitrary function \( F = F(x,t) = F(a, \tau) \), the following relation holds:

\[
\int \int \int d^3 a F \frac{\partial (x)}{\partial (a)} = \int \int \int d^3 a F \left[ \frac{\partial (\delta x, y, z)}{\partial (a_1, a_2, a_3)} + \frac{\partial (x, \delta y, z)}{\partial (a_1, a_2, a_3)} + \frac{\partial (x, y, \delta z)}{\partial (a_1, a_2, a_3)} \right]
\]

\[
= \int \int \int d^3 a F \left[ \frac{\partial (\delta x, y, z)}{\partial (a_1, a_2, a_3)} + \frac{\partial (x, \delta y, z)}{\partial (a_1, a_2, a_3)} + \frac{\partial (x, y, \delta z)}{\partial (a_1, a_2, a_3)} \right]
\]  
(209)
\[
\begin{align*}
\delta L &= -\int d\tau \int d^3x \rho \left( \frac{\partial^2 x}{\partial \tau^2} + \frac{1}{\rho} \nabla p + \nabla K \right) \cdot \delta x,
\end{align*}
\] (211)

where we use the thermodynamical relation,
\[
p = -\frac{\partial E(\alpha)}{\partial \alpha}.
\] (212)

Hence, we obtain the following equations of motion:
\[
\frac{\partial^2 x}{\partial \tau^2} + \frac{1}{\rho} \nabla p + \nabla K = 0.
\] (213)

Then, using Eqs. (199) and (200), we can rewrite the above equations as
\[
\frac{Dv}{Dt} + \frac{1}{\rho} \nabla p + \nabla K = 0.
\] (214)

Consequently, we obtain the following equations of motion:
\[
\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \frac{1}{\rho} \nabla p + \nabla K &= 0, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0, \\
\frac{\partial E(1/\rho)}{\partial (1/\rho)} + p &= 0.
\end{align*}
\] (215-217)

Comparing Eq. (215) with Eq. (83), we obtain
\[
\nabla K = -K, \\
v = \nabla \Phi, \\
K = -\frac{L_1(t)}{L_1(t)} \left( \frac{x^2}{2} + \frac{y^2}{2} \right) - \frac{L_3(t)}{L_3(t)} \left[ \frac{z^2}{2} - zL_4(t) \right] - L_4(t)z,
\] (218)

and \( \rho, K, p \) and \( \Phi \) are given by Eqs. (192), (197), (194) and (80), respectively.
Because the Lagrangian density given by Eq. (207) is expressed as a function of \((x, t)\), some might consider the number of degrees of freedom to be equal to three. However, the integration over the labeling co-ordinates \(a\), that is, \(\iiint d^3a\) in Eq. (207), gives an infinite number of degrees of freedom to the system. The Lagrangian in Eq. (207) is a sum of Lagrangians, each of which determines dynamics of each single fluid particle. Here, let us rewrite the Lagrangian with a field of the flow \(\varphi(x, t)\) and a field of the density \(\rho(x, t)\) as

\[
L = \iiint d^3x \left[ \frac{1}{2} \rho (\nabla \varphi) \cdot (\nabla \varphi) - \tilde{E}(\rho) - K(x, t)\rho \right],
\]

(219)

where

\[
\frac{\partial^2 \tilde{E}}{\partial \rho^2} = \frac{1}{\rho} \frac{\partial p(\rho)}{\partial \rho}.
\]

(220)

In the derivation of the above Lagrangian, to replace labeling co-ordinates \(a(x, t)\) with the field of the density \(\rho(x, t)\) is essential.

Then, the Hamiltonian is given by

\[
H = \iiint d^3x \left[ \frac{1}{2} \rho (\nabla \varphi) \cdot (\nabla \varphi) + \tilde{E}(\rho) + K(x, t)\rho \right].
\]

(221)

Hence, we can derive equations of motion as

\[
\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \varphi} = -\nabla \cdot (\rho \nabla \varphi),
\]

\[
\frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta \rho} = -\frac{1}{2} (\nabla \varphi) \cdot (\nabla \varphi) - \tilde{E}'(\rho) - K(x, t),
\]

\[
v = \nabla \varphi.
\]

(222)

From the above results, we obtain

\[
\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \tilde{E}''(\rho)\nabla \rho - \nabla K.
\]

(223)

Because

\[
\tilde{E}''(\rho)\nabla \rho = \frac{1}{\rho} \frac{\partial p(\rho)}{\partial \rho} \nabla \rho = \frac{1}{\rho} \nabla p(\rho),
\]

(224)

we obtain the Navier-Stokes equations,

\[
\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \frac{1}{\rho} \nabla p + K.
\]

(225)

In Fig. 17, the trajectory of the Bloch vector intersects itself. As explained in Sec. 4, for the quasiperiodic system, the trajectory of the motion never intersects itself during a finite time interval in the phase space because of incommensurate angular frequencies. The quasiperiodic trajectory becomes dense on the surface (submanifold) specified by the integrals of motion in the phase space. This fact seems to contradict the intersection of the trajectory of the Bloch vector in Fig. 17. We examine this point with numerical calculations.
Figure 17: The trajectory of the Bloch vector $S(t)$ for $0 \leq t \leq 5$, where $S(0) = (1, 0, 0)$ and $\beta = 1.0$. The trajectory intersects itself at $t_1 = 1.644$ and $t_2 = 4.809$.

Figure 17 shows the trajectory of the Bloch vector $S(t)$ for $0 \leq t \leq 5$, where $S(0) = (1, 0, 0)$ and $\beta = 1.0$. Looking at Fig. 17, we notice that the trajectory intersects itself at $(S_x, S_z) = (0.06372, -0.4840)$ at $t_1 = 1.644$ and $t_2 = 4.809$. However, in the phase space, the orbit of the motion never closes on itself at time variables $t_1$ and $t_2$. We can confirm this fact as follows. Regarding the system as the inviscid compressible fluid with zero vorticity, we can understand dynamics of the system through the Hamiltonian given by Eq. (221). The velocity vectors $(v_x, v_z)$ at $t_1$ and $t_2$ are given numerically as

$$v_x(t_1) = -0.3632, \quad v_z(t_1) = 0.2681,$$
$$v_x(t_2) = -0.7494, \quad v_z(t_2) = -0.04348.$$  \hspace{1cm} (226)

Hence, in the phase space, states of the system at $t_1$ and $t_2$ are clearly different from each other.

C    A proof of the Weyl criterion

In this section, we give a rigorous proof that a real sequence is uniformly distributed modulo unity if and only if the Weyl criterion holds [28, 31, 32, 33, 34]. This result plays an important role in Sec. 6.

Let $(x_0, \ldots, x_N)$ be a real sequence, which consists of $(N+1)$ real numbers. Moreover, for the real number $x_j$, let $[x_j]$ denotes the integral part of $x_j$, that is, the greatest integer not larger than $x_j$. Then, let \{\{x_j\} = x_j - [x_j] be the fractal part of $x_j$, or the residue of $x_j$ modulo unity. Clearly, \{x_j\} \in [0, 1) holds.

Let the counting function $A(E; N+1)$ be defined as the number of terms $x_j$, $0 \leq j \leq N$, for which \{x_j\} $\in E$. Then, we define a sequence uniformly distributed modulo unity as follows. The real sequence $(x_0, \ldots, x_N)$ is uniformly distributed modulo unity if for every
pair $a$, $b$ of real numbers with $0 \leq a < b < 1$, we have

$$
\lim_{N \to +\infty} \frac{A([a, b]; N + 1)}{N + 1} = b - a. \quad (227)
$$

From now on, we prove that the following two statements (i) and (ii) are equivalent to each other. Especially, the statement (ii) is called the Weyl criterion, so that it is a necessary and sufficient condition of the statement (i).

(i) A real sequence of $(x_0, x_1, \ldots, x_N)$ is uniformly distributed modulo unity.

(ii) For all integers $l \neq 0$, the following relation holds:

$$
\lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} \exp(2\pi ix_j) = 0. \quad (228)
$$

Before we prove (i) $\Leftrightarrow$ (ii), we pay attention to the fact, $\exp(2\pi ix_j) = \exp(2\pi i\{x_j\})$. Thus, in the remainder of this section, we consider $\{x_j\}$ to be equivalent to $x_j$ without loss of generality.

First, we prove (i) $\Rightarrow$ (ii). At first, we suppose a real sequence $(x_0, x_1, \ldots, x_N)$ to be uniformly distributed modulo unity. Now, let $\chi_{[a,b]}$ be the characteristic function of the interval $[a, b] \subseteq [0, 1],

$$
\chi_{[a,b]}(x) = \begin{cases} 
1 & x \in [a, b], \\
0 & x \notin [a, b]. 
\end{cases} \quad (229)
$$

Using Eq. (229), we can rewrite Eq. (227), which defines uniform distribution of real sequences, as follows:

$$
\lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} \chi_{[a,b]}(x_j) = \int_{0}^{1} \chi_{[a,b]}(x)dx. \quad (230)
$$

Then, let $f(x)$ be a step function,

$$
f(x) = c_0\chi_{[0,a]}(x) + c_a\chi_{[a,b]}(x) + c_b\chi_{[b,c]}(x) + \ldots + c_z\chi_{[z,1]}(x), \quad (231)
$$

which corresponds to the subdivisions,

$$
[0, 1) = [0, a) + [a, b) + [b, c) + \ldots + [z, 1), \quad (232)
$$

where $c_0$, $c_a$, $c_b$, ..., $c_z$ are real numbers. [In Eq. (232), we divide $[0, 1)$ into 27 parts with points $a$, $b$, $c$, ..., $z$, for example. However, in fact, we can divide the interval $[0, 1)$ into any number of parts.] Thus, we can consider the step function to be a superposition of characteristic functions with real coefficients. From Eqs. (230), (231) and (232), we obtain

$$
\lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} f(x_j) = \int_{0}^{1} f(x)dx. \quad (233)
$$
Figure 18: These graphs show how to obtain the step function \( f(x) \), which approximates an arbitrary real function \( g(x) \) well.

Here, let \( g(x) \) be an arbitrary real valued function, which is continuous for every value of \( x \) such that \( x \in [0, 1) \) with \( g(0) = g(1) \). Then, \( \exists \epsilon > 0 \), we can always find a step function \( f(x) \) such that

\[
|g(x) - f(x)| \leq \epsilon, \quad 0 \leq \forall x < 1. \tag{234}
\]

We can understand this fact at ease by looking at Fig. 18. That is to say, \( f(x) \) is an approximate function of \( g(x) \), so that we can always let its accuracy be as fine as possible by adjusting widths of the steps and the real coefficients for the superposition.

Then, from Eq. (234), we obtain the following inequality,

\[
\left| \frac{1}{N + 1} \sum_{j=0}^{N} g(x_j) - \int_{0}^{1} g(x)dx \right|
\leq \left| \frac{1}{N + 1} \sum_{j=0}^{N} [g(x_j) - f(x_j)] \right| + \left| \frac{1}{N + 1} \sum_{j=0}^{N} f(x_j) - \int_{0}^{1} f(x)dx \right|
\leq 2\epsilon + \left| \frac{1}{N + 1} \sum_{j=0}^{N} f(x_j) - \int_{0}^{1} f(x)dx \right|. \tag{235}
\]

Thus, using Eq. (233), we can rewrite Eq. (235) as follows:

\[
\lim_{N \to +\infty} \sup_{N} \left| \frac{1}{N + 1} \sum_{j=0}^{N} g(x_j) - \int_{0}^{1} g(x)dx \right| \leq 2\epsilon. \tag{236}
\]

Then, remembering that \( \epsilon > 0 \) is an arbitrary constant, we obtain

\[
\lim_{N \to +\infty} \frac{1}{N + 1} \sum_{j=0}^{N} g(x_j) = \int_{0}^{1} g(x)dx. \tag{237}
\]

Let us substitute \( g(x) = e^{2\pi ilx} \) for all integers \( l \neq 0 \) into Eq. (237). Although we assume \( g(x) \) to be a real valued continuous function, we can put it into Eq. (237). The reason why
is that we can divide it into two trigonometric functions $e^{2\pi ilx} = \cos(2\pi lx) + i\sin(2\pi lx)$, where each of them $[\cos(2\pi lx) \text{ and } \sin(2\pi lx)]$ is real and continuous.

Obviously, the right-hand side of Eq. (237) is equal to zero for $l \neq 0$,

$$
\int_0^1 e^{2\pi ilx} \, dx = \left[ \frac{e^{2\pi ilx}}{2\pi il} \right]_{x=0}^{x=1} = 0.
$$

(238)

Thus, finally, we arrive at the following relations,

$$
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} \exp(2\pi ilx_j) = 0,
$$

(239)

where $l(\neq 0)$ is an arbitrary integer. Hence, we conclude that (i) $\Rightarrow$ (ii) is true.

Second, we prove (ii) $\Rightarrow$ (i). At first, we assume that the statement (ii) is true. Then, let us consider a function $g(x)$,

$$
g(x) = \sum_{k=1}^{m} \alpha_k \exp(2\pi il_k x),
$$

(240)

where $l_1, l_2, \ldots, l_m$ are arbitrary integers. Applying Eq. (228) to Eq. (240), we obtain

$$
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} g(x_j) = \int_0^1 g(x) \, dx.
$$

(241)

In the derivation of the above, we use

$$
\int_0^1 \exp(2\pi il_k x) \, dx = \begin{cases} 
0 & l_k \neq 0, \\
1 & l_k = 0.
\end{cases}
$$

(242)

Next, we consider an arbitrary function $f(x)$, which is continuous for every value of $x$ such that $x \in [0, 1]$ with $f(0) = f(1)$. Then, $\exists \varepsilon(>0)$, adjusting $\alpha_k$ and $l_k$ for $k = 1, 2, 3, \ldots, m$ properly in Eq. (240), we can always let $|f(x) - g(x)| \leq \varepsilon$ for $0 \leq x < 1$. This treatment means that we take a Fourier-series approximation for $f(x)$. As a result of these discussions, we obtain

$$
\left| \frac{1}{N+1} \sum_{j=0}^{N} f(x_j) - \int_0^1 f(x) \, dx \right|
\leq \left| \frac{1}{N+1} \sum_{j=0}^{N} [f(x_j) - g(x_j)] \right| + \left| \frac{1}{N+1} \sum_{j=0}^{N} g(x_j) - \int_0^1 g(x) \, dx \right|
+ \left| \int_0^1 g(x) \, dx - \int_0^1 f(x) \, dx \right|
\leq 2\varepsilon + \frac{1}{N+1} \sum_{j=0}^{N} g(x_j) - \int_0^1 g(x) \, dx.
$$

(243)

Moreover, substitution of Eq. (241) into Eq. (243) yields

$$
\lim_{N \to +\infty} \sup \left| \frac{1}{N+1} \sum_{j=0}^{N} f(x_j) - \int_0^1 f(x) \, dx \right| \leq 2\varepsilon.
$$

(244)
Figure 19: These graphs explain how to choose two continuous functions $f_1(x)$ and $f_2(x)$ with Eq. (246) for the characteristic function $\chi_{[a,b]}(x)$.

Here, remembering that $\epsilon$ is an arbitrary positive constant, we can rewrite Eq. (244) as

$$\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} f(x_j) = \int_{0}^{1} f(x) dx.$$  \hspace{1cm} (245)

Now, let us take an arbitrary interval $[a, b] \subset [0, 1)$. Then, $\exists \bar{\epsilon} (> 0)$, we can choose two continuous functions $f_1(x)$ and $f_2(x)$ such that

$$f_1(0) = f_1(1), \quad f_2(0) = f_2(1),$$

$$f_1 \leq \chi_{[a,b]} \leq f_2,$$

$$\int_{0}^{1} [f_2(x) - f_1(x)] dx \leq \bar{\epsilon}. \hspace{1cm} (246)$$

We can understand this fact at ease by looking at Fig. 19.

On the one hand, using Eqs. (245) and (246), we obtain

$$\lim_{N \to +\infty} \inf \frac{1}{N+1} \sum_{j=0}^{N} \chi_{[a,b]}(x_j) \geq \lim_{N \to +\infty} \inf \frac{1}{N+1} \sum_{j=0}^{N} f_1(x_j)$$

$$= \int_{0}^{1} f_1(x) dx$$

$$\geq \int_{0}^{1} f_2(x) dx - \bar{\epsilon}$$

$$\geq \int_{0}^{1} \chi_{[a,b]}(x) dx - \bar{\epsilon}. \hspace{1cm} (247)$$

On the other hand, from Eqs. (245) and (246), we obtain

$$\lim_{N \to +\infty} \sup \frac{1}{N+1} \sum_{j=0}^{N} \chi_{[a,b]}(x_j) \leq \lim_{N \to +\infty} \sup \frac{1}{N+1} \sum_{j=0}^{N} f_2(x_j)$$

$$= \int_{0}^{1} f_2(x) dx$$
\[
\begin{align*}
\int_0^1 f_1(x)dx + \tilde{\epsilon} \\
\leq \int_0^1 \chi_{[a,b]}(x)dx + \tilde{\epsilon}.
\end{align*}
\]

(248)

Here, remembering that \(\tilde{\epsilon}\) is an arbitrary positive constant and putting Eqs. (247) and (248) together, we arrive at

\[
\lim_{N \to +\infty} \frac{1}{N+1} \sum_{j=0}^{N} \chi_{[a,b]}(x_j) = \int_0^1 \chi_{[a,b]}(x)dx = b - a.
\]

(249)

This implies that the sequence \((x_0, x_1, ..., x_N)\) is uniformly distributed modulo unity. Hence, we conclude that \((\text{ii}) \Rightarrow (\text{i})\) holds.

**D Physical transient spectra of the atom in the cavity**

In this section, we consider spectra of the atom, which develops according to the Jaynes-Cummings interaction with the cavity field. We discuss whether or not we can detect quasiperiodicity in the spectral analyses. Moreover, to understand physical meanings of the discrete plots of the trajectories of the Bloch vector, we investigate difference between continuous and discrete Fourier transforms for the atomic fluorescence. At the end of this section, to confirm the scale invariance for the discrete plots of the trajectories of the Bloch vector, we examine histograms of the samples of the atomic fluorescence taken at constant time intervals.

First, we regard the time variable \(t\) as continuous one, and we define the physical transient spectrum of the radiation emitted by the atom as

\[
S(\tilde{\omega}) = 2\Gamma \int_{-\infty}^{T} dt_1 \int_{-\infty}^{T} dt_2 e^{-\left(\Gamma^{-1}\tilde{\omega}(T-t_1)-(\Gamma+i\tilde{\omega})(T-t_2)\right)} \langle \psi(t_1)|\sigma_+\sigma_-|\psi(t_2)\rangle,
\]

(250)

where \(\Gamma^{-1}\) represents the filter’s response time and \(T\) represents the time at which the measurement takes place [41, 42, 43]. The reason why the physical transient spectrum \(S(\tilde{\omega})\) is given by Eq. (250) is as follows.

Let us think about an experimental setup shown in Fig. 20 for measuring an atomic spectrum. We assume that we measure a state of the atom by detecting its fluorescence through the filter. Here, we neglect reduction of the wave function of the atom after each detection of its fluorescence.

At first, let \(f(t)\) represent the electric field of the fluorescence emitted from the atom. [Here, we assume that \(f(t)\) is not an operator but a function of the strength of the fluorescence.] Next, we suppose the filter is linear. Thus, we detect \(f(t)\) through the linear filter, which transforms the function \(f(t)\) into another function \(g(t)\) as

\[
g(t) = T\{f(t)\},
\]

(251)

where \(T\{\cdot\}\) denotes a linear transformation.
Figure 20: An experimental setup for measuring physical transient spectra of the atom in the cavity. The atomic fluorescence is detected through the filter.

Now, we think the input function \( f(t) \) to be an impulse concentrated at \( t = a \), so that we describe it as \( f(t) = \delta(t - a) \). Then, the impulse response of the filter is given by

\[
h(t; a) = T\{\delta(t - a)\}. \tag{252}\]

We can write down an arbitrary function \( f(t) \) as a superposition of the delta functions,

\[
f(t) = \int_{-\infty}^{+\infty} d\xi f(\xi) \delta(t - \xi). \tag{253}\]

Thus, the response of the filter for the above \( f(t) \) is given by

\[
g(t) = T\{f(t)\} = \int_{-\infty}^{+\infty} d\xi f(\xi) T\{\delta(t - \xi)\} = \int_{-\infty}^{+\infty} d\xi f(\xi) h(t; \xi). \tag{254}\]

Here, we assume that the linear transformation caused by the filter is stationary, so that it does not depend on the time variable \( t \). Now, let us describe the translation on the function of time \( f(t) \) by \( (-\tau) \) as \( f_{\tau}(t) = f(t - \tau) \). Then, the fact that the transformation of the linear filter is stationary implies

\[
(T\{f\})_{\tau} = T\{f_{\tau}\}. \tag{255}\]

Thus, if we set \( h(t) = T\{\delta(t)\} \), we obtain

\[
h(t; \xi) = T\{\delta_{\xi}(t)\} = h_{\xi}(t) = h(t - \xi). \tag{256}\]

Hence, the response of the filter, whose transformation is stationary, is given by

\[
g(t) = T\{f(t)\} = \int_{-\infty}^{+\infty} d\xi f(\xi) h(t - \xi). \tag{257}\]
Figure 21: The response of the filter, \( h(t) = \theta(t)e^{-\Gamma t} \). The passband width \( \Gamma^{-1} \) generates the major contribution of the response function to the output of the filter.

In the following, we examine properties of \( h(t) \) with giving concrete examples.

First, we consider the case where the response of the filter is given by the Heaviside step function as \( h(t) = \theta(t) \). Then, we obtain

\[
g(t) = \int_{-\infty}^{+\infty} d\xi f(\xi) \theta(t - \xi) = \int_{-\infty}^{t} d\xi f(\xi). \tag{258}
\]

The above equation means that \( g(t) \) reflects the history of \( f(\xi) \) during \( \xi \in (-\infty, t] \). Because the output of the filter depends on past and current inputs but not future ones, we can say that the filter obtains causality.

Second, we consider a filter whose response is given by \( h(t) = \theta(t)e^{-\Gamma t} \). Then, we obtain

\[
g(t) = \int_{-\infty}^{+\infty} d\xi f(\xi) e^{-\Gamma(t-\xi)} \theta(t - \xi). \tag{259}
\]

As shown in Fig. 21, setting \( h(t) = \theta(t)e^{-\Gamma t} \), the output of the filter depends mainly on the inputs \( f(\xi) \) for \( \xi \in (t - \Gamma^{-1}, t) \). That is to say, \( \Gamma^{-1} \) corresponds to the passband width.

Third, we think about the case where the response of the filter is given by \( h(t) = e^{-i\tilde{\omega}t} \). Then, we obtain

\[
g(t) = \int_{-\infty}^{+\infty} d\xi f(\xi) e^{-i\tilde{\omega}(t-\xi)} = F(\tilde{\omega})e^{-i\tilde{\omega}t}, \tag{260}
\]

where \( F(\tilde{\omega}) \) represents the Fourier transform of \( f(t) \). The above equation implies that the filter extracts an angular frequency component \( F(\tilde{\omega}) \) from the input \( f(t) \) as the response.

Putting above discussions together, we obtain a real response of an actual filter as follows:

\[
h(t, \tilde{\omega}, \Gamma) \propto \theta(t)e^{-\Gamma t-i\tilde{\omega}t}. \tag{261}
\]

Thus, we can define the physical transient spectrum for the atom in the cavity as

\[
S(\tilde{\omega}) = 2\Gamma \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \theta(T - t_1)e^{-(\Gamma-i\tilde{\omega})(T-t_1)} \theta(T - t_2)e^{-(\Gamma+i\tilde{\omega})(T-t_2)} \times \langle \psi(t_1)|\sigma_+\sigma_-|\psi(t_2)\rangle, \tag{262}
\]

so that Eq. (250) is derived.
From now on, we calculate the physical transient spectrum of the atom evolving under the thermal JCM according to Eq. (250). At first, we rewrite Eq. (250) for cases where the state of the atom is described with a density operator,

\[
S(\tilde{\omega}) = 2\Gamma \int_{-\infty}^{T} dt_1 \int_{-\infty}^{T} dt_2 e^{-(\Gamma - i\tilde{\omega})(T-t_1) - (\Gamma + i\tilde{\omega})(T-t_2)} \times \text{Tr}_P[U(t_2)\rho_{AP}(0)U^\dagger(t_1)\sigma_+\sigma_-],
\]

where \(\rho_{AP}(0) = \rho_A(0) \otimes \rho_P\).

Here, as discussed in Sec. 3, we assume the initial state of the atom in the form

\[
\rho_A(0) = \frac{1}{2}(|0\rangle_A + |1\rangle_A)(\langle 0| + \langle 1|)
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Moreover, we assume that the initial state of the cavity field obeys the Bose-Einstein statistic as Eq. (15),

\[
\rho_P = (1 - e^{-\beta\omega}) \exp(-\beta\omega a^\dagger a).
\]

Then, we obtain

\[
\sigma_-U(t_2)\rho_A(0) \otimes \rho_P U^\dagger(t_1)\sigma_+ = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix},
\]

where

\[
\eta = [u_{00}(t_2) + u_{01}(t_2)]\rho_P[u_{00}(t_1) + u_{01}(t_1)],
\]

and \(u_{00}\) and \(u_{01}\) are given by Eq. (12).

Then calculating the partial traces of the operators, we obtain

\[
\text{Tr}_P[u_{00}(t_2)\rho_P u_{00}^\dagger(t_1)] = \sum_{n=0}^{\infty} [\cos(\sqrt{D(n)} + g^2 t_2) + \frac{i}{2} \Delta \omega \frac{\sin(\sqrt{D(n)} + g^2 t_2)}{\sqrt{D(n)} + g^2}] \\
\times (1 - e^{-\beta\omega})e^{-\beta\omega n} \\
\times [\cos(\sqrt{D(n)} + g^2 t_1) - \frac{i}{2} \Delta \omega \frac{\sin(\sqrt{D(n)} + g^2 t_1)}{\sqrt{D(n)} + g^2}],
\]

\[
\text{Tr}_P[u_{01}(t_2)\rho_P u_{01}^\dagger(t_1)] = \sum_{n=1}^{\infty} g^2 n(1 - e^{-\beta\omega})\frac{\sin(\sqrt{D(n)t_2})}{\sqrt{D(n)}} e^{-\beta\omega n} \frac{\sin(\sqrt{D(n)t_1})}{\sqrt{D(n)}},
\]

\[
\text{Tr}_P[u_{00}(t_2)\rho_P u_{01}^\dagger(t_1)] = 0,
\]

where \(D(n)\) is given by Eq. (22). Here, as discussed in Sec. 3, we set \(\Delta \omega = 0\), and we obtain

\[
\text{Tr}_P[u_{00}(t_2)\rho_P u_{00}^\dagger(t_1)] = (1 - e^{-\beta\omega}) \sum_{n=1}^{\infty} \cos(|g|\sqrt{n} t_2) \cos(|g|\sqrt{n} t_1)e^{-\beta\omega(n-1)},
\]

\[
\text{Tr}_P[u_{01}(t_2)\rho_P u_{01}^\dagger(t_1)] = (1 - e^{-\beta\omega}) \sum_{n=1}^{\infty} \sin(|g|\sqrt{n} t_2) \sin(|g|\sqrt{n} t_1)e^{-\beta\omega n}.
\]
Figure 22: The spectrum $S(\tilde{\omega})$ for $T = 100.0$, $\Gamma = 0.01$ and $\beta = 4.0$. We can observe two peaks at $\tilde{\omega} = 1$ and $\tilde{\omega} = \sqrt{2}$.

Thus, we arrive at

$$\text{Tr}[U(t_2)\rho_{AP}(0)U^\dagger(t_1)\sigma_+\sigma_-] = \frac{1}{2}(1 - e^{-\beta h\omega}) \sum_{n=1}^{\infty} [\cos(|g|\sqrt{nt_2}) \cos(|g|\sqrt{nt_1}) e^{-\beta h\omega(n-1)}$$

$$+ \sin(|g|\sqrt{nt_2}) \sin(|g|\sqrt{nt_1})] e^{-\beta h\omega n}].$$

(270)

Here, we set $\bar{h} = 1$. Then, replacing $\beta\omega$ and $|g|t$ with $\beta$ and $t$, we substitute Eq. (270) into Eq. (250). With tough calculations, we obtain

$$S(\tilde{\omega}) = \Gamma \int_{-\infty}^{T} dt_1 \int_{-\infty}^{T} dt_2 e^{-(\Gamma - i\tilde{\omega})(T-t_1)-(\Gamma + i\tilde{\omega})(T-t_2)} (1 - e^{-\beta})$$

$$\times \sum_{n=1}^{\infty} [\cos(\sqrt{nt_2}) \cos(\sqrt{nt_1}) e^{\beta} + \sin(\sqrt{nt_2}) \sin(\sqrt{nt_1})] e^{-\beta n}$$

$$= \Gamma \sum_{n=1}^{\infty} \frac{e^{-(n+1)\beta}}{2[n + (\Gamma - i\tilde{\omega})^2][n + (\Gamma + i\tilde{\omega})^2]}$$

$$\times \left[ (e^{2\beta} - 1)(n + \Gamma^2 + \tilde{\omega}^2) 
+ (e^\beta - 1)^2[(\Gamma^2 + \tilde{\omega}^2 - n) \cos(2\sqrt{n}T) + 2\sqrt{n} \sin(2\sqrt{n}T)] \right].$$

(271)

In the above equation, we assume that $\Gamma$ and $\tilde{\omega}$ are in units of $\omega$ and $T$ is in units of $|g|^{-1}$.

Looking at Eq. (271), we notice $S(-\tilde{\omega}) = S(\tilde{\omega})$ and $S^\dagger(\tilde{\omega}) = S(\tilde{\omega})$. We show the spectra $S(\tilde{\omega})$ of $\Gamma T = 1$, $T = 100.0$ and $\beta = 4.0$, $2.0$, $1.0$, $0.5$ in Figs. 22, 23, 24, 25, respectively. In Figs. 22, 23, 24 and 25, we can observe peaks of $\tilde{\omega} = \sqrt{n}$ for $n = 1, 2, 3, \ldots$. This is manifestation of the quasiperiodicity of the system. Moreover, if $\beta$ becomes smaller, that is, if the temperature becomes higher, the number of peaks of $S(\tilde{\omega})$ increases.

Here, let us think about the sampling theorem. The sampling theorem tells us that if a signal has an upper angular frequency limit of $\tilde{\omega}$, then we need sample points at time intervals being equal to or less than $\pi/\tilde{\omega}$ for enabling us to reconstruct the original signal.
Figure 23: The spectrum $S(\tilde{\omega})$ for $T = 100.0$, $\Gamma = 0.01$ and $\beta = 2.0$. We can observe four peaks at $\tilde{\omega} = 1,\sqrt{2},\sqrt{3},2$.

Figure 24: The spectrum $S(\tilde{\omega})$ for $T = 100.0$, $\Gamma = 0.01$ and $\beta = 1.0$. We can observe peaks at $\tilde{\omega} = 1,\sqrt{2},\sqrt{3},2,\sqrt{5},\sqrt{6},...$.

Figure 25: The spectrum $S(\tilde{\omega})$ for $T = 100.0$, $\Gamma = 0.01$ and $\beta = 0.5$. We can observe peaks at $\tilde{\omega} = 1,\sqrt{2},\sqrt{3},2,\sqrt{5},\sqrt{6},\sqrt{7},2\sqrt{2},3,...$. 
Thus, to obtain information about the component of $\tilde{\omega} = \sqrt{n}$, we have to take samples of the signal at time interval being equal to or less than $\Delta t = \pi / \sqrt{n}$. On the other hand, from the discussions about the scale invariance and the pseudorandom sequence uniformly distributed for $[0, 2\pi)$ in Sec. 6, we obtain $\pi < \Delta t < 2\pi$ in Eq. (97). Thus, for discretizing the dynamics of the Bloch vector, we have to set $\pi < \Delta t < 2\pi$. In fact, in Figs. 2, 3, 4, 5, 6 and 7 in Sec. 3, we take $\Delta t = 3.5$ to convert the continuous trajectory of the Bloch vector to a discrete-time sequence.

From these discussions, we understand that discretization of the dynamics of the Bloch vector with the time interval $\pi < \Delta t < 2\pi$ destroys information of all components of $\tilde{\omega} = \sqrt{n}$ for $n = 1, 2, 3, ...,$, so that we cannot recover the original trajectory. This implies that the discretization with $\pi < \Delta t < 2\pi$ erases the history of the trajectory of the Bloch vector completely. Because of this effect, figures consisting of the discrete-time sequence of the Bloch vector acquire the scale invariance and other fractal properties.

In the following, we discuss the Fourier transform of the discrete-time sequence obtained along the trajectory of the Bloch vector with the time interval $\pi < \Delta t < 2\pi$. Let us define the discrete Fourier transform of the fluorescence of the atom in the cavity as follows:

$$c_k = \frac{1}{N^2} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} e^{-i2\pi(l_2-l_1)k/N} \text{Tr}[U(l_2\Delta t)\rho_{AP}(0)U^\dagger(l_1\Delta t)\sigma_+\sigma_-].$$ (272)

Looking at the above equation, we notice $c_{k+N} = c_k$, so that we concentrate only on $\{c_k : k = 0, 1, ..., N-1\}$. With tough calculations, we obtain

$$c_k = \sum_{n=1}^{\infty} \frac{1}{N^2[\cos(2k\pi/N) - \cos(\sqrt{n}\Delta t)]^2} e^{-n\beta} \sinh(\frac{\beta}{2})$$

$$\times \left[ \cosh(\frac{\beta}{2})[(1 - \cos(\frac{2k\pi}{N}) \cos(\sqrt{n}\Delta t))(1 - \cos(2k\pi) \cos(\sqrt{n}\Delta t))$$

$$- \sin(2k\pi) \sin(\frac{2k\pi}{N}) \sin(\sqrt{n}\Delta t) \sin(\sqrt{n}\Delta t))]$$

$$+ \sinh(\frac{\beta}{2})[\cos(\frac{2k\pi}{N}) - \cos(\sqrt{n}\Delta t)] \cos(\sqrt{n}(N-1)\Delta t)$$

$$\times [\cos(2k\pi) - \cos(\sqrt{n}\Delta t)] \right].$$ (273)

In Figs. 26 and 27, we plot $\{|c_k| : k = 0, 1, 2, ..., N-1\}$ for $\beta = 1.0$, $\Delta t = 3.5$, $N = 4000$ and $\beta = 1.0$, $\Delta t = 4.5$, $N = 4000$, respectively. Comparing Figs. 26 and 27, the Fourier transforms of $\Delta t = 3.5$ and $\Delta t = 4.5$ are obviously different from each other. This observation seems to insist that the scale invariance does not hold. That is, under a rescaling from $\Delta t = 3.5$ to $\Delta t = 4.5$, the figure consisting of the discrete-time sequence of the Bloch vector changes. However, this discussion is not true.

In fact, it is not significant to apply the Fourier analysis to behaviour of the discrete-time sequence of the Bloch vector. This is because we take a time interval $\pi < \Delta t < 2\pi$ and we cannot recover entire information about components of $\tilde{\omega} = \sqrt{n}$ for $n = 1, 2, ..., N - 1$. That is to say, a set of samples taken at time interval $\Delta t$ lose the past history of the trajectory. When we apply the discrete Fourier transform to $N$ samples of
Figure 26: The discrete Fourier transform of the Bloch vector for $\beta = 1.0$, $\Delta t = 3.5$ and $N = 4000$.

Figure 27: The discrete Fourier transform of the Bloch vector for $\beta = 1.0$, $\Delta t = 4.5$ and $N = 4000$. 

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Figure 28: The histograms of sets given by Eq. (275) for $\Delta t = 3.5$, $\beta = 1.0$, $N = 500\,000$ and $\Delta t = 4.5$, $\beta = 1.0$, $N = 500\,000$ with a thin solid curve and thick dashed curve, respectively. In the graphs, we let the width of bins in the horizontal axis be equal to 0.0025.

A signal $f(t)$,

$$f(0), f(\Delta t), f(2\Delta t), \ldots, f((N-1)\Delta t),$$

we assume that they are samples arranged in chronological order. However, the samples of the trajectory of the Bloch vector lose their past history completely, so that $N$ samples cannot form a sequence of the chronological order shown in Eq. (274).

We can only regard the samples shown in Eq. (274) as a set

$$\{f(n\Delta t) : n = 0, 1, 2, \ldots, N - 1\},$$

which lose the information about their order. Here, thinking about the scale invariance $\Delta t \to s\Delta t$ for $s > 1$, we want to show that we cannot distinguish two sets of samples, $\{f(n\Delta t) : n = 0, 1, 2, \ldots, N - 1\}$ and $\{f(ns\Delta t) : n = 0, 1, 2, \ldots, N - 1\}$, under the limit of $N \to \infty$.

Thus, we analyze the above two sets of the samples in the following way. First, let us consider a histogram of the following set,

$$\{\text{Tr}[U(l\Delta t)\rho_{AP}(0)U^\dagger(l\Delta t)\sigma_+\sigma_-] : l = 0, 1, 2, \ldots, N - 1\}. \quad (275)$$

Second, we prepare two sets of Eq. (275) for two different time interval, $\Delta t$ and $s\Delta t$. If we cannot distinguish their histograms under the limit of $N \to \infty$, we can conclude that the scale invariance holds under $\Delta t \to s\Delta t$.

In Fig. 28, we plot histograms of sets given by Eq. (275) for $\Delta t = 3.5$, $\beta = 1.0$, $N = 500\,000$ and $\Delta t = 4.5$, $\beta = 1.0$, $N = 500\,000$ with a thin solid curve and a thick dashed curve, respectively. In Fig. 28, we let the width of bins in the horizontal axis be equal to 0.0025. Looking at Fig. 28, we can conclude that histograms of $\Delta t = 3.5$ and $\Delta t = 4.5$ are quite similar and we can hardly distinguish them. Thus, we can confirm the scale invariance from Fig. 28.
E Rational approximations of irrational numbers and their precisions

In this section, we prove two theorems, which are related to the Diophantine approximation [28, 35]. From these theorems, we can estimate precisions given by rational approximations of irrational numbers, as utilized in Sec. 8.

**Theorem 1.** (Dirichlet’s approximation theorem)

Let $\alpha$ be an arbitrary positive number, and let $Q$ be an arbitrary integer with $Q > 1$. Then, there is a rational number $p/q$ such that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{qQ},$$

where $p$ and $q$ are coprime and $0 < q < Q$.

**Proof.**

We prove the above theorem by the pigeon-hole principle. On the one hand, using $\alpha$, we put $(Q + 1)$ points (pigeons) in the interval $[0, 1]$ as follows:

$$0, \alpha - \lfloor \alpha \rfloor, 2\alpha - \lfloor 2\alpha \rfloor, ..., (Q - 1)\alpha - \lfloor (Q - 1)\alpha \rfloor, 1.$$  (277)

On the other hand, we split the interval $[0, 1]$ into $Q$ subintervals (pigeon-holes), each of length $1/Q$, as follows:

$$[0, 1/Q), [1/Q, 2/Q), ..., [(Q - 2)/Q, (Q - 1)/Q), [(Q - 1)/Q, 1].$$  (278)

Then, two or more pigeons have to belong to a particular pigeon-hole. Thus, one of the following two statements has to hold true. The first one is that there exist integers $j$ and $k$ such that

$$|(k\alpha - \lfloor k\alpha \rfloor) - (j\alpha - \lfloor j\alpha \rfloor)| \leq 1/Q,$$  (279)

where $0 \leq j < k < Q$. The second one is that there exists an integer $k$ such that

$$|(k\alpha - \lfloor k\alpha \rfloor) - 1| \leq 1/Q,$$  (280)

where $0 < k < Q$.

In the case where Eq. (279) holds, we define integers $p$ and $q$ as

$$q = k - j, \quad p = \lfloor k\alpha \rfloor - \lfloor j\alpha \rfloor.$$  (281)

Then, from Eq. (279), we obtain

$$|q\alpha - p| \leq 1/Q.$$  (282)

This implies that we achieve Eq. (276). In the case where Eq. (280) holds, we define integers $p$ and $q$ as follows:

$$q = k, \quad p = \lfloor k\alpha \rfloor + 1.$$  (283)
Then, from Eq. (280), we obtain

\[ |q\alpha - p| \leq 1/Q. \tag{284} \]

This implies that we arrive at Eq. (276).

Thus, we conclude that we have proved the theorem.

**Theorem 2.**

Let \(\alpha(> 0)\) be an arbitrary irrational number. Then, there are two infinite integer sequences, \(p_n\) and \(q_n\) (where \(n \geq 0\)), such that

\[ |q_n\alpha - p_n| < \frac{1}{q_n}, \tag{285} \]

where \(p_n\) and \(q_n\) are coprime and \(q_n > 0\). In other words, there are infinitely many rational numbers \(p/q\), where \(p\) and \(q\) are coprime and \(q > 0\), such that

\[ |\alpha - \frac{p}{q}| < \frac{1}{q^2}. \tag{286} \]

**Proof.**

We prove this theorem by assuming the opposite. At first, let us suppose that there are only a finite number of \(p_n/q_n\), say \(p_1/q_1, p_2/q_2, ..., p_N/q_N\) with Eq. (285). Then, we consider the following real number \(\rho\),

\[ \rho = \min_{1 \leq n \leq N} |q_n\alpha - p_n|. \tag{287} \]

Because of Theorem 1, there exists \(p/q\) such that

\[ |\alpha - \frac{p}{q}| \leq \frac{1}{qQ}, \quad 0 < q < Q, \tag{288} \]

where \(Q\) is an integer and larger than unity. Moreover, we can choose an arbitrary integer greater than unity as \(Q\). Thus, we set \(Q > 1/\rho\). Then, the following relation holds,

\[ |q\alpha - p| \leq \frac{1}{Q}, \quad \frac{1}{Q} < \rho. \tag{289} \]

This result contradicts Eq. (287), which defines \(\rho\) as the minimum number. Thus, using proof by contradiction, we can conclude Theorem 2.

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