Epi-Partial Normality

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Abstract. The main aim of this work is to study a new version of normality called epi-partial normality, which lies between epi-almost normality and epi-mild normality. A space $(X, \mathcal{T})$ is called an epi-partially normal space if there exists a topology $\mathcal{T}'$, which is coarser than $\mathcal{T}$, such that $(X, \mathcal{T}')$ is Hausdorff partially normal. In this work, we investigate this property and present some examples that illustrate the relationships between epi-partial normality and other weaker kinds of both normality and regularity. We show that this property is a topological, a semi regularization and an additive property. Some properties and relationships of epi-partial normality are presented and proved.

1. Introduction

Throughout this article, a space $X$ means a topological space on which no separation axioms are assumed, unless explicitly stated. The set of real numbers is denoted by $\mathbb{R}$. For a subset $A$ of $X$, $\overline{A}$ and int$(A)$ denote the closure and the interior of $A$ in $X$, respectively. A subset $A$ of $X$ is open-domain if it is the interior of its own closure or, equivalently, if it is the interior of some closed subsets [1]. A complement of an open domain subset is called closed-domain. A subset $A$ of $X$ is a $\pi$-closed subset if it is a finite intersection of closed domain subsets [2]. A complement of a $\pi$-closed subset is called $\pi$-open. Two sets $A$ and $B$ of $X$ are said to be separated if there exist two disjoint-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$ [3,4,5]. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a non-empty set $X$ such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called a topology coarser than $\mathcal{T}$, and $\mathcal{T}$ is called finer [4]. A space $X$ is normal [4], if any pair of disjoint-closed subsets $A$ and $B$ of $X$ can be separated. A space $X$ is almost-normal [6], if any two disjoint-closed subsets $A$ and $B$ of $X$, one of which is closed-domain, can be separated. A space $X$ is mildly normal [7], if any pair of disjoint closed-domain subsets $A$ and $B$ of $X$ can be separated. A space $X$ is partially-normal [8], if any two disjoint-closed subsets $A$ and $B$ of $X$, one of which is closed-domain, and the other is $\pi$-closed, can be separated. A space $X$ is Hausdorff or a $T_2$-space, if for each two distinct points $x, y \in X$, there exist two open-subsets $U$ and $V$ of $X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$ [4]. A space $X$ is completely Hausdorff or a Urysohn space [4,9], if for each two distinct-points $a, b \in X$ there exist two open subsets $U$ and $V$ of $X$ such that $a \in U, b \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A space $(X, \mathcal{T})$ is an almost-regular space if for each $x \in X$ and for each closed-domain set $F$ in $X$ such that $x \not\in F$, there exist two disjoint-open subsets $U$ and $V$ such that $x \in U$ and $F \subseteq V$ [10]. A space $(X, \mathcal{T})$ is an $\varepsilon$-normal [11], (resp. $\varepsilon$-mildly normal [12], $\varepsilon$-almost normal [13], $\varepsilon$-regular [14], $\varepsilon$-almost regular [15]) space if there exists a topology $\mathcal{T}'$ on $X$ that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is a $T_4$ (resp. Hausdorff mildly-normal, Hausdorff almost-normal, $T_1$-regular, $T_3$-almost regular) space. A space $(X, \mathcal{T})$ is sub-metrizable [16], if there exists a metric $d$ on $X$ such that the topology $\mathcal{T}_d$ on $X$ generated by $d$ is coarser than $\mathcal{T}$. A topology on $X$ generated by the family of all open-domain subsets, denoted by $\mathcal{T}_s$, is called the semi-regularization of $X$. A space $(X, \mathcal{T})$ is called semi-regular if $\mathcal{T} = \mathcal{T}_s$ [17].
In this paper, we introduce and study a weaker version of epi-normality called epi-partial normality. We show that this property is different from the other weaker kinds of epi-normality. Some properties, relationships, counterexamples and virous results are presented.

2. Definition and Examples
First, we introduce the definition of epi-partial normality:

2.1 Definition A space \((X, T')\) is said to be an \textit{epi-partially normal} space if there exists a topology \(T''\) on \(X\), which is coarser than \(T'\), such that \((X, T'')\) is Hausdorff partially-normal.

From Definition 2.1 above, clearly that every epi-partially normal space is Hausdorff and every Hausdorff partially-normal space is an epi-partially normal space. Note that, every sub-metrizable space is an epi-normal space [11]. Also, we conclude the following implications:

\[ \text{epi-normality} \Rightarrow \text{epi-almost normality} \Rightarrow \text{epi-partial normality} \Rightarrow \text{epi-mild normality} \]

Note that, none of the above implications is reversible, as shown by the following examples:

2.2 Example The space \(X = \mathbb{R}^{ω_1}\) [18, Example 0.3], is Tychonoff, separable, not normal and every closed domain (resp. \(π\)-closed) in \(\mathbb{R}^{ω_1}\) depends on a countable set. Kalantan in [18] has shown that \(\mathbb{R}^{ω_1}\) is a quasi normal space. Thus, \(\mathbb{R}^{ω_1}\) is partially-normal. Since \(\mathbb{R}^{ω_1}\) is a Tychonoff partially normal space, it is epi-partially normal. So, \(\mathbb{R}^{ω_1}\), which is not normal.

It can be observed that: every epi-almost normal space is Urysohn and every Hausdorff almost normal space is epi-almost normal [13].

2.3 Example The simplified arens square topology [9, Example 81], is Hausdorff, semi regular, Lindelöf, \(σ\)-compact and with a \(σ\)-locally finite base, and it is neither completely Hausdorff, Urysohn, regular, normal nor locally compact. Since \(X\) is semi regular and not a regular space, we get \(X\) is not almost regular. Since \(X\) is \(T_1\) and not almost regular, we obtain \(X\) is not an almost normal space. Since \(X\) is a quasi normal space [19], we get \(X\) is partially normal. Thus, \(X\) is a Hausdorff partially normal space. Hence, it is an epi-partially normal space. Therefore, the simplified arens square topology is an example of an epi-partially normal space that is neither epi-almost normal nor epi-regular. It is also a Hausdorff semi regular partially normal space. Thus, the simplified arens square topology is an epi-partially normal \(T_1\)-partially normal space, which is neither epi-almost regular, almost regular nor almost normal.

2.4 Example The telophase topology [9, Example 73], is a \(T_1\), compact, Lindelöf, paracompact space and with a \(σ\)-locally finite base, and it is neither Hausdorff, normal, regular, semi regular nor semi normal. It is easy to show that \(X\) is an almost-regular space. Since \(X\) is a paracompact almost-regular space, it is almost-normal, partially-normal and mildly-normal. Since \(X\) is not Hausdorff, it is neither epi-mildly normal, epi-partially normal nor epi-regular. Since it is \(T_1\) almost regular, we get \(X\) is an epi-almost regular space. Thus, the telophase topology is an example of an epi-almost regular partially normal compact (Lindelöf, paracompact) space with a \(σ\)-locally finite base, which is neither epi-partially normal nor epi-regular.

2.5 Example The relatively prime integer topology and the prime integer topology [9, Example 60, 61], are Hausdorff, semi regular and Lindelöf spaces, which are not Urysohn. Since the only disjoint closed domain subsets in both spaces are \(X\) and \(∅\), both spaces are mildly normal. Thus, the relative prime integer and the prime integer topologies are epi-mildly normal being Hausdorff mildly normal spaces. Since the closures of any two proper-open subsets in both spaces have a non-empty intersection, they are not partially normal. Hence, they are neither quasi normal nor almost normal spaces. Since they are semi regular and not regular spaces, they are neither almost regular nor almost
completely regular. Since the two spaces are not Urysohn, they are neither epi-normal, epi-almost normal nor epi-regular spaces. Now, we show that the given spaces are not epi-partially normal.

**Claim 1**: In the given two spaces, any Hausdorff topology that is coarser than $\mathcal{T}$ cannot be partially normal.

**Proof of the Claim 1**: Let $\mathcal{T}'$ be any Hausdorff topology that is coarser than $\mathcal{T}$ in the given-spaces. Then, $\mathcal{T}' \subseteq \mathcal{T}$ and $(X, \mathcal{T}')$ is a Hausdorff semi-regular space because a topology that is coarser than a semi-regular space is semi-regular. Let $U$ and $V$ be any two disjoint proper open subsets of $(X, \mathcal{T}')$. Then, $U$ and $V$ are disjoint open subsets of $(X, \mathcal{T})$ as $\mathcal{T}' \subseteq \mathcal{T}$. Since the closures of any two proper open subsets of $(X, \mathcal{T})$ have a non-empty intersection [9], we get $\overline{U} \cap \overline{V} \neq \emptyset$. Since $\overline{U} \subseteq \overline{U}'$ and $\overline{V} \subseteq \overline{V}'$, we have $\overline{U}' \cap \overline{V}' \neq \emptyset$. Thus, the closures of any two proper open subsets in $(X, \mathcal{T}')$ have a non-empty intersection. Therefore, $(X, \mathcal{T}')$ is not a partially normal space. Thus, any Hausdorff topology coarser that $\mathcal{T}$ cannot be partially normal. Hence, $(X, \mathcal{T})$ is not an epi-partially normal space.

**Claim 2**: Any Hausdorff topology that is coarser than $\mathcal{T}$ cannot be almost regular.

**Proof of the Claim 2**: Suppose that there exists a Hausdorff topology $\mathcal{T}'$, which is coarser than $\mathcal{T}$, such that $(X, \mathcal{T}')$ is a Hausdorff almost regular space. Since any topology coarser than a semi-regular space is semi-regular and $(X, \mathcal{T})$ is semi-regular, we have $(X, \mathcal{T}')$ is semi-regular. Thus, $(X, \mathcal{T}')$ is a Hausdorff semi regular almost regular space. Therefore, $(X, \mathcal{T}')$ is a Hausdorff regular space. Hence, $(X, \mathcal{T}^*)$ is a $T_3$-space and thus Urysohn. Since $\mathcal{T}^* \subseteq \mathcal{T}'$, we get $(X, \mathcal{T})$ is a Urysohn space, which is a contradiction as $(X, \mathcal{T})$ is not Urysohn. Thus, any Hausdorff topology that is coarser than $\mathcal{T}$ cannot be almost regular. Hence, $(X, \mathcal{T})$ is not an epi-regular space. Therefore, the relatively prime integer and the prime integer topologies are examples of epi-mildly normal spaces, which are neither epi-partially normal, epi-almost normal nor epi-regular.

2.6 **Example** The ordinal space $\omega_1 + 1$ is an epi-normal space (hence epi-partially normal), which is not sub-metrizable [11]. Any uncountable indiscrete space is an example of a partially-normal space that is not epi-partially normal being not Hausdorff. The Niemytzki plane topology [9], is an epi-partially normal space, which is not normal.

2.7 **Example** The particular point topology $(\mathbb{R}, \mathcal{T}_p)$, the finite complement topology $(\mathbb{R}, \mathcal{C}F)$ and the countable-complement topology $(\mathbb{R}, \mathcal{C}C)$ are almost-regular and partially normal spaces because the only closed domain sets in which are $\mathbb{R}$ and $\emptyset$ [9]. Thus, $(\mathbb{R}, \mathcal{T}_p)$, $(\mathbb{R}, \mathcal{C}F)$ and $(\mathbb{R}, \mathcal{C}C)$ are partially normal spaces, which are not epi-partially normal because they are not Hausdorff.

2.8 **Example** The integer broom topology [9, Example 121], is a $T_0$, normal, semi normal, compact, Lindelöf and paracompact space, which is neither $T_1$, regular nor semi regular. Thus, $X$ is an almost normal, partially normal and mildly normal space. Since $X$ is not a $T_1$-space, it is neither an epi-regular, epi-mildly normal, epi-partially normal nor epi-normal space. Therefore, the integer broom topology is an example of a partially normal compact (Lindelöf, paracompact) space, which is neither epi-partially normal nor epi-regular.

2.9 **Example** Consider the space of the Example 6 in [12], we have $X$ is an epi-normal sub-metrizable Tychonoff zero-dimensional space, which is not mildly-normal. Thus, it is an epi-partially normal Tychonoff space, which is not partially normal.

Next, we present the following results:

2.10 **Theorem** Epi-partial normality is a topological property.

**Proof.** Let $(X, \mathcal{T})$ be an epi-partially normal space. Assume that $(X, \mathcal{T}) \equiv (Y, \mathcal{S})$. Then, there exists a homeomorphism $f: X \to Y$ and there exists a topology $\mathcal{T}'$ on $X$ that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is Hausdorff partially-normal. Define $\mathcal{S}'$ on $Y$ by: $\mathcal{S}' = \{ f(U): U \in \mathcal{T}' \}$. Then, $\mathcal{S}'$ is a topology on $Y$, which is coarser than $\mathcal{S}$, and $(Y, \mathcal{S}')$ is a Hausdorff partially-normal space. Therefore, $(Y, \mathcal{S})$ is an epi-partially normal space. □
2.11 Theorem Epi-partial normality is a hereditary property with respect to closed domain subspaces.

Proof. Let \((X, \mathcal{T})\) be an epi-partially normal space and \((M, \mathcal{T}_M)\) be a closed domain subspace of \(X\). Then, there exists a topology \(\mathcal{T}'\) on \(X\) coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is Hausdorff partially normal. To show \((M, \mathcal{T}_M)\) is an epi-partially normal subspace, define \(\mathcal{T}_M' = \{U \cap M : U \in \mathcal{T}'\}\). Then, \(\mathcal{T}_M' \subseteq \mathcal{T}_M\). Hence, \(\mathcal{T}_M'\) is a topology on \(M\), which is coarser than \(\mathcal{T}_M\). Since \((X, \mathcal{T}')\) is a Hausdorff partially normal space and \((M, \mathcal{T}_M')\) is a closed domain subspace of \(X\), we get \((M, \mathcal{T}_M')\) is a Hausdorff partially normal subspace of \((X, \mathcal{T}')\). Therefore, \((M, \mathcal{T}_M)\) is epi-partially normal. □

Since every closed-and-open (clopen) subset is closed domain, we get:

2.12 Corollary Epi-partial normality is a hereditary with respect to clopen subspaces.

2.13 Theorem Epi-partial normality is an additive-property.

Proof. Let \(X = \bigoplus_{s \in S} X_s\) be an epi-partially normal space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) coarser than \(\bigoplus_{s \in S} \mathcal{T}_s\) such that \((X, \mathcal{T}')\) is Hausdorff partially normal. Since \(X_s\) is a clopen subset of \(X\), we have \(X_s\) is an epi-partially normal subspace of \(X\) for each \(s \in S\). Conversely, suppose that \(X_s\) is an epi-partially normal space for each \(s \in S\). Then, there exists a topology \(\mathcal{T}_s'\) on \(X_s\) that is coarser than \(\mathcal{T}_s\) such that \((X_s, \mathcal{T}_s')\) is Hausdorff partially-normal. Since both Hausdorffness and partial normality are additive properties, we get \((X, \bigoplus_{s \in S} \mathcal{T}_s')\) is a Hausdorff partially normal space. Since \(\bigoplus_{s \in S} \mathcal{T}_s'\) is a topology that is coarser than \(\bigoplus_{s \in S} \mathcal{T}_s\), we obtain \((X, \bigoplus_{s \in S} \mathcal{T}_s)\) is epi-partially normal. □

The following result can be proved easily by using the same arguments of the corresponding result of the almost normality in [20].

2.14 Theorem If \(X\) is a partially-normal countably-compact space and \(M\) is a Hausdorff paracompact first-countable space, then the product \(X \times M\) is partially-normal.

2.15 Theorem If \(X\) is an epi-partially normal countably-compact space and \(M\) is a Hausdorff paracompact first-countable space, then the product \(X \times M\) is an epi-partially normal space.

Proof. Let \((X, \mathcal{T})\) be an epi-partially normal countably-compact space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is Hausdorff partially normal. Since \((X, \mathcal{T}')\) is countably-compact, we get \((X, \mathcal{T}')\) is countably-compact. Since \(M\) is a Hausdorff paracompact space, we obtain \((X, \mathcal{T}') \times M\) is Hausdorff partially-normal. Thus, \(X \times M\) is an epi-partially normal space. □

Since every compact space is paracompact and every metrizable space is a paracompact first-countable space [4], we get:

2.16 Corollary If \(X\) is an epi-partially normal countably-compact (resp. compact) space and \(M\) is a metrizable space, then the product \(X \times M\) is an epi-partially normal space.

3. Relationships of epi-partial normality

Next, we study the relationships among epi-partial normality, paracompactness and almost regularity. First, we need to recall the following definitions:

3.1 Definition Let \((X, \mathcal{T})\) be a topological space. Then: \(X\) is said to be:

1. almost-compact [21], if each open cover of \(X\) has a finite subfamily such that the closures of whose members cover \(X\).

2. nearly-compact if every open domain cover \(\mathcal{U}\) of \(X\) has a finite subcover [21,22,23].
3. **H-closed**, if $X$ is a Hausdorff almost-compact space [24,25].

4. **nearly-paracompact** if every open domain cover of $X$ has a locally-finite open refinement [10].

Note that: every compact (countably-compact, nearly compact) space is a paracompact (countably-paracompact, nearly-paracompact) space, respectively. Every compact space is a countably-compact space and every nearly-compact space is almost-compact. Recall that: a space $X$ is called a weakly-regular space if every point $x \in X$ and every open-domain subset $U$ of $X$ such that $x \in U$, there exists an open set $V$ in $X$ such that $x \in V \subseteq \overline{V} \subseteq U$ [10]. Some characterizations of weakly-regular spaces by using $\pi$-open and $\pi$-closed sets, have been given in [26]. Since every weakly-regular nearly paracompact space is mildly normal [27], we can improve this fact by the following theorem that can be proved by using the same arguments of the corresponding result in [27].

**3.2 Theorem** Every weakly-regular, nearly-paracompact space is partially normal.

Since every almost-regular space is weakly-regular, every nearly-paracompact Hausdorff space is almost-regular (hence weakly-regular) and every nearly-compact space is nearly-paracompact, we obtain the following corollaries:

**3.3 Corollary** Every almost-regular, nearly-paracompact (resp. nearly-compact) space is a partially norma spacial.

**3.4 Corollary** Every weakly-regular, nearly-compact space is partially normal.

**3.5 Corollary** Every nearly-paracompact Hausdorff space is partially normal.

Since every almost-regular $\theta$-compact (resp. almost compact) space is nearly-compact [24,28], we conclude:

**3.6 Corollary** Every almost-compact almost-regular (resp. weakly-regular) space is partially normal.

A partially normal nearly-paracompact space is not necessarily almost-regular, as shown by the following counterexample:

**3.7 Example** Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ be a topology on $X$. It can be observed that: $(X, \mathcal{T})$ is a partially normal nearly-paracompact space, which is not almost regular because $\{b, c\}$ is closed domain, $a \not\in \{b, c\}$ and there do not exist two disjoint-open subsets containing $a$ and $\{b, c\}$, respectively.

Note that: every epi-almost regular space is $T_1$, every epi-regular space is epi-almost regular and every epi-regular space is Urysohn [15].

**3.8 Theorem** If $(X, \mathcal{T})$ is an epi-almost regular almost-compact space and its semi regularization $(X, \mathcal{T}_e)$ is $T_1$, then $(X, \mathcal{T})$ is an epi-partially normal space.

**Proof.** Let $(X, \mathcal{T})$ be an epi-almost regular almost-compact space and its semi regularization $(X, \mathcal{T}_e)$ be $T_1$. Then, $(X, \mathcal{T}_e)$ is an almost-regular $T_1$ almost-compact space. Hence, it is a $T_3$-almost compact space [17]. Since every almost-regular almost-compact space is partially-normal, Corollary 3.6, we get $(X, \mathcal{T}_e)$ is Hausdorff partially-normal. Hence, $(X, \mathcal{T})$ is an epi-partially normal space. □

Since every nearly compact space is almost compact, we get:

**3.9 Corollary** If $(X, \mathcal{T})$ is an epi-almost regular nearly compact space and its semi regularization $(X, \mathcal{T}_e)$ is $T_1$, then $(X, \mathcal{T})$ is an epi-partially normal space.

**3.10 Theorem** Every epi-almost regular Hausdorff Lindelöf space is an epi-partially normal space.
Proof. Let \((X, \mathcal{T})\) be an epi-almost regular Hausdorff Lindelöf space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) that is coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is a Hausdorff almost-regular Lindelöf space. Since every almost-regular Lindelöf space is partially-normal being quasi-normal [29], we get \((X, \mathcal{T}')\) is a partially-normal space. Thus, \((X, \mathcal{T}')\) is a Hausdorff partially-normal space. Therefore, \((X, \mathcal{T})\) is epi-partially normal. □

3.11 Theorem If \((X, \mathcal{T})\) is an epi-almost regular Hausdorff space and the witness of epi-almost regularity \((X, \mathcal{T}')\) is with a \(\sigma\)-locally finite base, then \((X, \mathcal{T})\) is an epi-partially normal space.

Proof. Let \((X, \mathcal{T})\) be an epi-almost regular Hausdorff space and \((X, \mathcal{T}')\) be the witness of epi-almost regularity. Then, \((X, \mathcal{T}')\) is Hausdorff almost-regular with a \(\sigma\)-locally finite base. Since every almost-regular space with a \(\sigma\)-locally finite base is quasi-normal [29] (hence partially-normal), we get \((X, \mathcal{T}')\) is a Hausdorff partially-normal space. Therefore, \((X, \mathcal{T})\) is epi-partially normal. □

3.12 Theorem If \((X, \mathcal{T})\) is an epi-almost regular Hausdorff space and the witness of epi-almost regularity \((X, \mathcal{T}')\) is a paracompact space., then \((X, \mathcal{T})\) is epi-partially normal.

Proof. Let \((X, \mathcal{T})\) be an epi-almost regular Hausdorff space and \((X, \mathcal{T}')\) be the witness of epi-almost regularity. Then, \((X, \mathcal{T}')\) is a Hausdorff almost-regular paracompact space. Since every almost-regular paracompact space is quasi-normal [29], we get \((X, \mathcal{T}')\) is a Hausdorff partially normal space. Therefore, \((X, \mathcal{T})\) is an epi-partially normal space. □

Clearly that, if \(X\) is a completely Hausdorff (Urysohn) \(H\)-closed space, then \(X\) is an epi-mildly normal space. We can improve this fact by the following:

3.13 Theorem Every Urysohn \(H\)-closed space is an epi-partially normal space.

Proof. Let \((X, \mathcal{T})\) be a Urysohn \(H\)-closed space. Then, \(X\) is a Hausdorff almost compact space. Since every almost compact Urysohn space is an almost regular space, thus \(X\) is almost regular. Since every almost-compact almost-regular space is partially normal, Corollary 3.6, we get \(X\) is a Hausdorff partially normal space. Therefore, \(X\) is an epi-partially normal space. □

Since every nearly paracompact Hausdorff space is a quasi-normal space [30], and hence it is a partially normal space, we get:

3.14 Corollary Every nearly paracompact (resp. nearly compact, compact) Hausdorff space is an epi-partially normal space.

Since every epi-regular space is a Urysohn space [14], we obtain:

3.15 Corollary Every epi-regular almost-compact (resp. Lindelöf ) space is an epi-partially normal space.

3.16 Corollary If \((X, \mathcal{T})\) is an epi-regular space and the witness of epi-regularity \((X, \mathcal{T}')\) is paracompact (resp. with a \(\sigma\)-locally finite base), then \((X, \mathcal{T})\) is an epi-partially normal space.

3.17 Corollary Every epi-regular nearly-compact space is an epi-partially normal space.

Since every almost-regular nearly-compact space is partially-normal and every almost-compact Urysohn space is almost-regular, we obtain:

3.18 Corollary Every almost-regular Hausdorff nearly compact space is epi-partially normal.

3.19 Corollary Every nearly-compact Urysohn space is epi-partially normal.

It can be observed that, any epi-almost regular with a \(\sigma\)-locally finite base is not necessary to be an epi-partially normal space and any epi-almost regular paracompact space is not necessary to be epi-partially normal. For example, the telophase topology, Example 2.4, is an epi-almost regular paracompact space with a \(\sigma\)-locally finite base, which is not epi-partially normal.
3.20 **Theorem** Every epi-partially normal almost-regular space is a Urysohn space.

**Proof.** It is obvious because every Hausdorff almost-regular space is Urysohn. □

3.21 **Corollary** Every almost regular non Urysonh space cannot be an epi-partially normal space.

Since every epi-partially normal space is Hausdorff and every almost normal $T_1$-space is almost regular, we get:

3.22 **Corollary** Every epi-partially normal almost-normal space is an almost-regular space.

Since every nearly compact $H$-closed space is an epi-normal space, we obtain:

3.23 **Corollary** Every nearly-compact $H$-closed space is an epi-partially normal space.

Since every epi-partially normal space is Hausdorff, any topology coarser than a nearly-paracompact space is paracompact [17], and every Hausdorff paracompact space is normal, we conclude:

3.24 **Corollary** Every nearly paracompact epi-partially normal space is an epi-normal space.

3.25 **Theorem** If $(X, 𝒯)$ is an epi-almost regular nearly paracompact space and its semi regularization $(X, 𝒯_2)$ is $T_1$, then $(X, 𝒯)$ is an epi-partially normal space.

**Proof.** Let $(X, 𝒯)$ be any epi-almost regular nearly paracompact space and its semi regularization $(X, 𝒯_2)$ be $T_1$. Then, $(X, 𝒯_2)$ is an almost-regular $T_1$ paracompact space [17]. Thus, $(X, 𝒯_2)$ is a Hausdorff partially normal space. Therefore, $(X, 𝒯)$ is an epi-partially normal space. □

3.26 **Corollary** If $(X, 𝒯)$ is an epi-almost regular nearly compact space and its semi regularization $(X, 𝒯_2)$ is $T_1$, then $(X, 𝒯)$ is an epi-partially normal space.

4. **Properties of epi-partial normality**

Note that, every extremally-disconnected space is an almost normal space [6], and hence it is a partially normal space. Then, we conclude that:

4.1 **Corollary** Every Hausdorff extremally-disconnected space is an epi-partially normal space.

It can be observed that, any epi-partially normal space may not be an extremally-disconnected space, for example: the rational-sequence topology $(\mathbb{R}, \mathcal{R}_S)$ [9, Example 65], is a Tychonoff epi-partially normal space, being an epi-normal space, which is not extremally-disconnected [9]. Also, a minimal-Hausdorff space may not be a compact or an epi-partially normal space, for example: the space in [12, Example 6] is a minimal Hausdorff space that is neither compact nor epi-partially normal because it is not partially normal. Note that, if $(X, 𝒯)$ is an almost-regular space, then its semi regularization $(X, 𝒯_2)$ is a regular space [17], and every almost-compact almost regular space is mildly-normal [27].

4.2 **Theorem** If $(X, 𝒯)$ is an almost-regular nearly-compact space and its semi regularization $(X, 𝒯_2)$ is a $T_1$-space, then $(X, 𝒯)$ is an epi-normal space and hence it is epi-partially normal.

**Proof.** Let $(X, 𝒯)$ be an almost-regular nearly-compact space. Then, its semi regularization $(X, 𝒯_2)$ is a regular space [12]. Since $(X, 𝒯_2)$ is a $T_1$-space, we get $(X, 𝒯_3)$ is a $T_3$-space. In fact, a topology that is coarser than a nearly-compact space is compact. Thus, $(X, 𝒯_3)$ is a regular compact space. Since every regular-compact space is normal, we obtain $(X, 𝒯_2)$ is a Hausdorff normal space. Therefore, $(X, 𝒯)$ is an epi-normal space and hence it is an epi-partially normal space. □

4.3 **Theorem** If $(X, 𝒯)$ is an almost-regular almost-compact space and its semi regularization $(X, 𝒯_2)$ is a $T_1$-space, then $(X, 𝒯)$ is an epi-partially normal space.
Proof. Let \((X, \mathcal{T})\) be an almost-regular almost-compact space. Then, \((X, \mathcal{T}_s)\) is a regular space [12]. Since \((X, \mathcal{T}_s)\) is a \(T_1\)-space, we have \((X, \mathcal{T}_s)\) is a \(T_3\)-space. It is well known that a topology that is coarser than an almost-compact space is almost-compact. Thus, \((X, \mathcal{T}_s)\) is an almost-regular almost-compact space. Since every almost-regular almost-compact space is partially normal, Corollary 3.6, we obtain \((X, \mathcal{T}_s)\) is a Hausdorff partially-normal space. Therefore, \((X, \mathcal{T})\) is epi-partially normal. \(\Box\)

4.4 Theorem \((X, \mathcal{T})\) is a partially normal space if and only if its semi regularization \((X, \mathcal{T}_s)\) is a partially normal space.

Proof. Let \(A\) and \(B\) be any two disjoint-closed sets in the semi-regularization \((X, \mathcal{T}_s)\) of \((X, \mathcal{T})\) such that \(A\) is \(\pi\)-closed and \(B\) is closed domain. Then, \(A\) is \(\pi\)-closed and \(B\) is closed domain in \((X, \mathcal{T})\). By partial-normality of \((X, \mathcal{T})\), there exist two open-sets \(U\) and \(V\) in \((X, \mathcal{T})\) such that \(A \subseteq U\), \(B \subseteq V\) and \(U \cap V = \emptyset\). Thus, there exist two disjoint-open sets \(U_s\) and \(V_s\) in \((X, \mathcal{T}_s)\) such that \(U \subseteq U_s\) and \(V \subseteq V_s\) [17]. So, we have \(A \subseteq U_s\), \(B \subseteq V_s\) and \(U_s \cap V_s = \emptyset\). Hence, \((X, \mathcal{T}_s)\) is a partially-normal space. Conversely, let \((X, \mathcal{T}_s)\) be a partially-normal space. Let \(A\) and \(B\) be any two disjoint-closed sets in \((X, \mathcal{T})\) such that \(A\) is \(\pi\)-closed and \(B\) is closed domain. Then, \(A\) and \(B\) are disjoint closed-sets in \((X, \mathcal{T}_s)\) such that \(A\) is \(\pi\)-closed and \(B\) is closed domain. By partial-normality of \((X, \mathcal{T}_s)\), there exist two disjoint open-sets \(U_s\) and \(V_s\) in \((X, \mathcal{T}_s)\) such that \(A \subseteq U_s\) and \(B \subseteq V_s\). Since \(T_s \subseteq T\) and \(U_s, V_s \in T_s\), we get \(U_s\) and \(V_s\) are disjoint open-sets in \((X, \mathcal{T})\). Hence, \((X, \mathcal{T})\) is a partially normal space. \(\Box\)

4.5 Corollary Partial normality is a semi-regularization property.

Note that, \((X, \mathcal{T})\) is a Hausdorff space if and only if its semi-regularization \((X, \mathcal{T}_s)\) is Hausdorff [17]. Thus, we conclude:

4.6 Corollary \((X, \mathcal{T})\) is an epi-partially normal space if and only if \((X, \mathcal{T}_s)\) is an epi-partially normal space.

4.7 Corollary Epi-partial normality is a semi-regularization property.

4.8 Theorem If \((X, \mathcal{T})\) is an almost-regular (resp. almost-completely regular) partially-normal space and its semi regularization \((X, \mathcal{T}_s)\) is a \(T_1\)-space, then \((X, \mathcal{T})\) is a Urysohn epi-partially norma space.

Proof. Let \((X, \mathcal{T})\) be an almost-regular (resp. almost-completely regular) partially-normal space and its semi regularization \((X, \mathcal{T}_s)\) be a \(T_1\)-space. Since the semi-regularization of an almost-regular (resp. almost-completely regular) space is regular (resp. completely-regular) [17], we obtain \((X, \mathcal{T}_s)\) is a \(T_1\)-regular (resp. completely-regular) space, and hence it is a \(T_3\) (resp. Tychonoff) space. Thus, \((X, \mathcal{T}_s)\) is a Urysohn space, and hence it is Hausdorff. Therefore, \((X, \mathcal{T})\) is a Urysohn partially normal space. Hence, \((X, \mathcal{T})\) is a Urysohn epi-partially normal space. \(\Box\)

4.9 Theorem A semi regularization of a semi-normal space is semi-normal.

Proof. Let \(X\) be a semi-normal space and \((X, \mathcal{T}_s)\) be the semi-regularization space of \((X, \mathcal{T})\). Let \(A\) be any closed set in \(\mathcal{T}_s\) and \(U\) be any open set in \(\mathcal{T}_s\) such that \(A \subseteq U\). Since \(\mathcal{T}_s \subseteq \mathcal{T}\), we obtain \(A\) is a closed-set in \(\mathcal{T}\) and \(U\) is an open-set in \(\mathcal{T}\). By semi-normality of \((X, \mathcal{T})\), there exists an open-set \(V\) in \((X, \mathcal{T})\) such that \(A \subseteq V \subseteq \text{int}(V) \subseteq U\). Since \(\text{int}(V)\) is an open-domain set in \((X, \mathcal{T})\), we get \(\text{int}(V) \subseteq \mathcal{T}_s\). Put \(W = \text{int}(V)\). Thus, \(W\) is an open-set in \((X, \mathcal{T}_s)\) such that \(A \subseteq W \subseteq \text{int}(W) \subseteq U\). Hence, \((X, \mathcal{T}_s)\) is a semi-normal space. \(\Box\)

4.10 Theorem Every epi-partially normal semi-normal space is an epi-normal space.

Proof. Let \(X\) be an epi-partially normal semi-normal space and \((X, \mathcal{T}_s)\) be the semi-regularization of \((X, \mathcal{T})\). Then, \((X, \mathcal{T}_s)\) is a Hausdorff partially-normal semi-normal space. Since every partially-normal semi-normal space is normal, we get \((X, \mathcal{T}_s)\) is a Hausdorff normal space. Therefore, \((X, \mathcal{T})\) is an epi-normal space. \(\Box\)
Next, we conclude the following corollary:

**4.11 Corollary** Let $X$ be a semi-normal space. Then, the following statements are equivalent:

1. $X$ is epi-normal.
2. $X$ is epi-almost normal.
3. $X$ is epi-partially normal.
4. $X$ is epi-mildly normal.

Since every paracompact (resp. compact) space is a $\theta$-normal space, and every Hausdorff $\theta$-normal space is normal [31,32], we conclude:

**4.12 Corollary** Every epi-partially normal paracompact (resp. compact) space is $T_4$.

**4.13 Corollary** Every epi-regular paracompact (resp. compact) space is an epi-normal space, and hence it is epi-partially normal.

5. **Conclusion**
A new version of normality called epi-partial normality has been studied in this work. We have shown that epi-partial normality is different from epi-normality, epi-almost normality and epi-mild normality. Some properties, counterexamples and relationships with some other weaker forms of normality and regularity, were presented.

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6. **References**

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