**c-Axis Penetration Depth in the Cuprates: Additional Evidence for Incoherent Hopping**

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Measurements of the c-axis penetration depth $\lambda_c$ in the cuprates reveal a low-temperature $T$ dependence which is inconsistent with simple models of coupling between the CuO$_2$ layers. In this paper, we examine whether a model based on *incoherent* hopping between the layers can account for this low-$T$ behavior. We compute $\lambda_c$ directly from linear response theory and compare our results with recent experimental measurements on YBa$_2$Cu$_3$O$_{7-\delta}$ as a function of temperature and doping. We find that the data can be reproduced within this model providing the inter-layer scattering is anisotropic and the pairing is $d$-wave. In addition, our calculations demonstrate that $1/\lambda_c^2$ is proportional to the $c$-axis critical current, which seems to be a generic feature in weakly coupled layered superconductors.

**Introduction.** Studies of the $c$-axis properties of the cuprate superconductors have revealed many unusual features in both the normal and superconducting states. 

In the normal state, the $c$-axis resistivity $\rho_c(T)$ can show either a metallic or a semiconducting temperature dependence as a function of the doping, while the in-plane resistivity is always metallic. 

In the superconducting state, strong evidence exists in a variety of cuprates that the CuO$_2$ planes are coupled by the Josephson effect, in contrast to the behavior of conventional superconductors. 

Additionally, $c$-axis penetration depth measurements on YBa$_2$Cu$_3$O$_{7-\delta}$ (YBCO) crystals have become available which agree with neither the conventional theory nor with each other.

Recent work has been directed toward accounting for these anomalous features within a theoretical framework based on incoherent quasiparticle hopping between the CuO$_2$ planes. 

In this picture, the coupling between nearest-neighbor CuO$_2$ planes is so weak that quasiparticle transport occurs through uncorrelated tunneling (or hopping) of the quasiparticles. The origin of this incoherence is unclear, but it may be related to strong intra-layer scattering, 

or strong electromagnetic fluctuations. 

Based on these ideas, several phenomenological models have been developed to study the implications of this incoherence and have had some success in describing both $\rho_c(T)$ and the $c$-axis critical current $j_c$. 

In this paper, we extend the calculations of Refs. 

by deriving expressions for the electromagnetic penetration depth along the $c$-axis directly from linear response theory and compare our results with recent experiments. 

Following Ref. 

we postulate the existence of three mechanisms for quasiparticle transport along the $c$-direction: a direct hopping induced by overlap of the quasiparticle wave functions, impurity-assisted hopping due to disorder in the intercalating layers separating the CuO$_2$ planes, and boson-assisted hopping due to, for example, certain optical phonons in the cuprates.

We find that the inverse $c$-axis penetration depth $1/\lambda_c$ is the sum in quadrature of $1/\lambda_c$ due to each mechanism in isolation, and we derive expressions for these terms. Our calculations demonstrate that $1/\lambda_c^2 \propto j_c$ as hypothesized earlier based on calculations within the Lawrence-Doniach model, 

indicating that this Josephson-like relation between $j_c$ and $\lambda_c$ is a general property of weakly coupled layered superconductors. Finally, we examine the temperature dependence of the contributions of each interlayer hopping process to the penetration depth for both $s$- and $d$-wave superconductors. Comparing with the data of Bonn *et al.*, 

we find that the doping and temperature dependence of the experimental $\lambda_c^2(0)/\lambda_c^2(T)$ is qualitatively consistent with $d$-wave pairing and anisotropic interlayer scattering.

**Penetration depth in the incoherent hopping model.** To begin, consider electrons on a three-dimensional tetragonal lattice governed by the Hamiltonian $H_{el} = \sum_m H_m + H_{\perp}$. The lattice is defined by the primitive vectors $a\hat{x}$, $a\hat{y}$, and $d\hat{z}$ with $d \gg a$, so that it resembles a stack of layers, which we index by $m$. $H_m$ is the Hamiltonian for the motion in the $m$th layer and $H_{\perp}$ describes the interlayer coupling. For the purposes of this paper, it is not necessary to specify $H_m$ completely; we need only assume that it becomes a BCS-type superconductor below a critical temperature $T_c$. The interlayer coupling Hamiltonian may then be written in terms of the annihilation and creation operators for the quasiparticles obtained from $H_m$ at site $i \equiv (i_x, i_y)$ in layer $m$ with spin projection $\sigma$, $c^\dagger_{im\sigma}$ and $c_{im\sigma}$:

$$H_{\perp} = \sum_{im\sigma} t_{im} \left[ c^\dagger_{im+1,\sigma} c_{im\sigma} + \text{h.c.} \right],$$

where
The terms in this Hamiltonian represent interlayer hopping due to overlap of the quasiparticle wave functions (parameterized by \( t_{im} \)), impurity scattering (modeled by the random variable \( V_{im} \)), and bosonic scattering (written in terms of the bosonic field operator \( \phi_{im} \) and its coupling strength to the quasiparticles \( g_{im} \)).

As discussed in the Introduction, a significant number of experiments suggest that the interlayer coupling in the cuprates is in general weak and in particular incoherent. A fully microscopic theory of this incoherence is beyond the scope of this article, but its effects can be simulated by performing calculations to second order in \( j \), and has been used successfully to study the normal-state resistivity[12,13] and the diamagnetic current[16].

The terms in this Hamiltonian represent interlayer hopping amplitude proportional to the Fourier transform of a retarded current-current correlation function and

\[
D_{Q,Q'}^{zz} = -\frac{e^2d}{\hbar^2a^2} \sum_{i < j, \sigma} e^{-i(Q-Q') \cdot R_{im}} 
\times \left\langle \left[ j^{zz}_{im} \right]_{\sigma} \right\rangle 
\]

the diamagnetic term. In these expressions, \( N \) is the total number of sites in the lattice, \( Q = (q_x,q_z) \), and \( R_{im} \) is the position vector of the lattice site indexed by \( im \). (Throughout this paper, we use capital letters to denote 3D vectors and small letters to denote 2D vectors.) Note that the conductivity is not diagonal in wave vector due to the impurity term; after impurity averaging, translational invariance will be restored and we will find \( \sigma_{Q,Q'} \propto \delta_{Q,Q'} \).

To proceed further we must account for the incoherent nature of charge transport along the c-axis by expanding both \( \Pi_{Q,Q'}^{zz} \) and \( D_{Q,Q'}^{zz} \) to second order in the interlayer hopping amplitude \( t_{im} \). Thus, the paramagnetic current-current correlation function, evaluated using the standard diagrammatic techniques in Matsubara space, is represented by a bare particle-hole bubble (vertex corrections being of higher order in \( t_{im} \)) with purely intra-layer Green’s functions renormalized only by the intra-layer self energy. Although the diamagnetic term does not play any substantial role in the optical conductivity because it is independent of frequency, it is important for the penetration depth calculation and must be treated carefully. In fact, evaluating this term perturbatively yields a form nearly identical to \( \Pi_{Q,Q'}^{zz} \) except for factors at the vertices, as we shall see below. Working in Matsubara space with Nambu Green’s functions for the intra-layer propagators[21] and assuming that the layers \( m \) are identical, we obtain

\[
\Pi_{Q,Q'}^{zz}(i\nu_n) = \frac{2e^2d}{\hbar^2a^2} \frac{\delta_{q_x,q'_x}}{N^3} \sum_{k,k'} \int_0^\beta d\tau e^{i\nu_n\tau} 
\times \left\langle T_{[\tau]} T_{[-\tau]} \right\rangle \tau_{-k+q-k'+q'} 
\times \text{Tr} \left[ \hat{G}_{k'}(\tau) \hat{G}_{k}(\tau) \right] 
\]

and

\[
D_{Q,Q'}^{zz} = -\frac{e^2d}{\hbar^2a^2} \frac{\delta_{q_x,q'_x}}{N^3} \sum_{k,k'} \int_0^\beta d\tau 
\times \left\langle T_{[\tau]} T_{[-\tau]} \right\rangle \tau_{-k+q-k'+q'} 
\times \text{Tr} \left[ \hat{G}_{k'}(\tau) \hat{G}_{k}(\tau) \right] \tau_{-3} 
\]

where \( N^3 \) is the number of lattice sites in a single layer, \( \tau_3 \) is the third Pauli matrix, and \( t_{q} = \sum_{m} e^{-i q \cdot r_{im}} \) is assumed to be independent of \( m \). \( (k_B = 1 \text{ throughout this paper}, \text{and the rest of the notation is standard.}) \)

The correlation function \( \left\langle T_{[\tau]} T_{[\tau]} \right\rangle \) represents a combination of an impurity average over \( V_{im} \) and a thermodynamic average over the bosonic field \( \phi_{im} \). Since we go
to second order in the hopping amplitudes and take the mean of $V_{nm}$ to be zero, this correlation function decomposes into a sum of three components: a direct hopping term, an impurity-assisted hopping term, and a boson-assisted hopping term. The conductivity is therefore a sum of three terms, as obtained previously.\[14\] Since the penetration depth is given by

$$\frac{e^2}{4\pi\lambda_c^2} = \lim_{Q \to 0} \lim_{\omega \to 0} \left[ \omega \text{Im} \sigma_{Q}^{zz}(\omega) \right]$$

$$= \lim_{\omega \to 0} \lim_{Q \to 0} \left[ \omega \text{Im} \sigma_{Q}^{zz}(\omega) \right]$$

(10)

after impurity averaging, this decomposition allows us to write

$$\left( \frac{1}{\lambda_c^2} \right)_{\text{Total}} = \left( \frac{1}{\lambda_c^2} \right)_{\text{direct}} + \left( \frac{1}{\lambda_c^2} \right)_{\text{imp}} + \left( \frac{1}{\lambda_c^2} \right)_{\text{inel}}$$

(11)

and consider each term separately.

Direct contribution. For the direct term, $t_{im} \to t_\perp$ [cf. Eq. (2)], from which

$$\langle T_r \sigma_{\mathbf{q}}(\tau) \sigma_{\mathbf{q}'}(\tau') \rangle_{\text{direct}} = N^2 \delta_{q,0} \delta_{q',0} t_\perp^2$$

(12)

Inserting this relation into Eqs. (8)-(9) gives

$$\Pi_{Q, Q'}^{zz}(i\nu_n)_{\text{direct}} = \delta_{Q, Q'} \frac{2e^2d}{h^2\alpha^2} t_\perp^2 \frac{T}{N_{\parallel}} \times \sum_{kl} \text{Tr} \left[ \hat{G}_{k+q}(i\omega_{l+n})\hat{G}_k(i\omega_l) \right]$$

(13)

and

$$D_{Q, Q'}^{zz}_{\text{direct}} = -\delta_{Q, Q'} \frac{2e^2d}{h^2\alpha^2} t_\perp^2 \frac{T}{N_{\parallel}} \times \sum_{kl} \text{Tr} \left[ \hat{G}_k(i\omega_l)\hat{\phi}_q\hat{G}_k(i\omega_l)\hat{\phi}_q \right]$$

(14)

The resulting conductivity is, as expected, diagonal in wave vector. Inserting Eqs. (13)-(14) into Eq. (10), we have

$$\frac{e^2}{4\pi\lambda_c^2} = \frac{8e^2d}{h^2\alpha^2} t_\perp^2 \frac{T}{N_{\parallel}} \sum_{kl} F_k(i\omega_l)F_k(i\omega_l),$$

(15)

where $F_k(i\omega_l)$ is the Gor'kov propagator.\[21\]

Several features of this result are noteworthy. First and foremost, Eq. (5) has the same form as the expression for the c-axis critical current; specifically, $(1/\lambda_c^2)_{\text{direct}} \propto j_c^{\text{direct}}$.\[14\] As we shall see, this conclusion holds for the other two processes as well. This behavior arises from the cancellation of the normal $(\hat{\phi}_n$ and $\hat{\phi}_n)$ components of the quasiparticle propagators between the paramagnetic and diamagnetic terms. This cancellation is a special property of the c-axis charge transport in weakly coupled layered superconductors and reproduces the results obtained by Bulaevskii and Clem based on the Lawrence-Daniach model.\[15\]

Second, if we take the BCS form for the Gor'kov propagators $F_k(i\omega_l) = \Delta_k/(i\omega_l^2 - E_k^2)$, in the limit where the superconducting energy scales are smaller than the intra-layer electronic energy scales, we obtain

$$\left( \frac{1}{\lambda_c^2} \right)_{\text{direct}} = \frac{16\pi^2e^2d}{h^2c^2} N(0)t_\perp^2 T \sum_{l} \left( \frac{\Delta_k^2}{|\omega_l^2 + \Delta_k^2|^{3/2}} \right)_k,$$

(16)

where $N(0)$ is two-dimensional density of states at the Fermi surface (for a parabolic spectrum $\epsilon_k = h^2k^2/2m - E_F$, $N(0) = m/2\pi h^2$), $\Delta_k$ is the superconducting gap function, $E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}$, and the angle brackets denote a normalized Fermi surface average over the indicated variable. Comparing with the standard BCS result,\[23\] we see that this term has the same temperature dependence but a different magnitude.\[14\] This behavior arises from a combination of the dimensionality of the layers and the conservation of the transmitted wave vector.\[4\]

Impurity-assisted contribution. The impurity-assisted term is computed in an analogous way. In this case, $t_{im} \to V_{im}$ [cf. Eq. (4)] and the impurity average yields $\bar{V}_{im}V_{jm} = \bar{V}^2_{i-j}$. The correlation function in Eqs. (8)-(9) is then

$$\langle T_r \sigma_{\mathbf{q}}(\tau) \sigma_{\mathbf{q}'}(\tau') \rangle_{\text{imp}} = N\delta_{q,-q'} \bar{V}^2_{q}$$

(17)

from which we obtain

$$\Pi_{Q, Q'}^{zz}(i\nu_n)_{\text{imp}} = \delta_{Q, Q'} \frac{2e^2d}{h^2\alpha^2} T \sum_{k'k} \bar{V}_{k-k'} \times \text{Tr} \left[ \hat{G}_{k+q}(i\omega_{l+n})\hat{G}_k(i\omega_l) \right]$$

(18)

and

$$D_{Q, Q'}^{zz}_{\text{imp}} = -\delta_{Q, Q'} \frac{2e^2d}{h^2\alpha^2} T \sum_{kk'\ell} \bar{V}_{k-k'} \times \text{Tr} \left[ \hat{G}_k(i\omega_l)\hat{\phi}_{q}\hat{G}_k(i\omega_l)\hat{\phi}_{q} \right].$$

(19)

The penetration depth follows from these relations as

$$\frac{e^2}{4\pi\lambda_c^2} = \frac{8e^2d}{h^2\alpha^2} T \sum_{kk'\ell} \bar{V}_{k-k'} F_k(i\omega_l)F_k(i\omega_{l+n}),$$

(20)

Again, the penetration depth has the same form as the corresponding critical current calculation: $(1/\lambda_c^2)_{\text{imp}} \propto j_c^{\text{imp}}$.\[14\] In the BCS limit taken above, Eq. (20) may be written

$$\left( \frac{1}{\lambda_c^2} \right)_{\text{imp}} = \frac{32\pi^3e^2v}{h^2c^2} N^2(0) T \sum_{l} \left\langle \left| \bar{V}_{k-k'} \right| \Delta_k \Delta_{k'} \right\rangle_k,$$

(21)
where \( v = a^2 d \) is the unit cell volume.

The contribution of the impurity-assisted component to the total penetration depth depends on both the pairing symmetry and the anisotropy of the matrix element. If the pairing is isotropic (\( \Delta_k = \Delta \)), this expression reduces to the Ambegaokar-Baratoff result for SIS tunnel junctions. [23] However, if the pairing is d-wave (or any pairing state with \( \langle \Delta_k \rangle_0 = 0 \)) and the matrix element is isotropic \( (V^2_{k-k'} = V^2) \), \( 1/\lambda^2_{\text{imp}} \) vanishes by symmetry. [11,14] This result is a special consequence of an isotropic scattering matrix element and does not hold when anisotropy, which should be present in any real material, is included.

To see the effects of anisotropy on the impurity-assisted contribution to the c-axis penetration depth, consider a scattering matrix element of the form

\[
\overline{V^2_{k-k'}} = V^2 \frac{k_F \delta k}{(k-k')^2 + \delta k^2}.
\] (22)

When \( \delta k \to \infty \), \( \overline{V^2_{k-k'}} \) becomes isotropic, and we recover the results discussed above. Physically, this limit corresponds to a total randomization of the wave vector of the scattered quasiparticle and is analogous to diffuse transmission through a tunnel junction. In the opposite limit where \( \delta k/k_F \to 0 \), the scattering matrix element preserves the direction of the scattered quasiparticle and is analogous to specular transmission through a tunnel junction. In this limit, Eq. (21) reduces to a generalization of the Ambegaokar-Baratoff [23] result:

\[
\left( \frac{\lambda^2_{c}(0)}{\lambda^2_{c}(T)} \right)_{\text{imp}} \to \left\langle \frac{\Delta_k}{\Delta_0} \tanh \left( \frac{\Delta_k}{2T} \right) \right\rangle_k,
\] (23)

where \( \Delta_0 = \langle \Delta_k(T = 0) \rangle_k \) is the average of the absolute value of the gap function over the Fermi surface at zero temperature. This form yields a finite contribution to the penetration depth for both s- and d-wave pairing due to the wave-vector-conserving nature of the scattering.

For intermediate \( \delta k/k_F \), we can evaluate \( 1/\lambda^2_{c} \) from Eqs. (21)-(22) for a d-wave superconductor with the results shown in Fig. 1. At zero temperature, we see from the inset to this figure that the magnitude of the penetration depth depends strongly on \( \delta k/k_F \), in keeping with the fact that \( (\lambda^2_{c}(0))_{\text{imp}} \) vanishes in the \( \delta k \to \infty \) limit. As seen from the main figure, however, the temperature dependence of the penetration depth ratio is not strongly affected by the amount of scattering anisotropy: the penetration depth ratios for \( \delta k/k_F = 0.01 \) and 2.0 lie very nearly on top of each other. Moreover, we see that the temperature dependence of the anisotropic impurity-assisted scattering is distinguishable from both the conventional BCS result and the s-wave Ambegaokar-Baratoff result.

Our calculations therefore suggest two experimental signatures of anisotropic impurity-assisted interlayer scattering: (1) the magnitude of the c-axis penetration depth should depend strongly on the type and quantity of disorder, but the temperature dependence should be largely insensitive to these factors, and (2) the temperature dependence of the penetration depth ratio should be between the BCS and Ambegaokar-Baratoff predictions.

**Boson-assisted contribution.** The boson-assisted hopping term is more involved due to the inelastic scattering (i.e., the retardation effects) induced by the boson. The hopping amplitude \( t_{im} \to \sum_{ij} g_{i-j,m} \phi_{jm} \) [cf. Eq. (3)], and so the thermodynamic average returns a factor of the boson propagator

\[
\mathcal{D}_{qm}(\tau) = -\sum_{ij} e^{-i\mathbf{r}_i \mathbf{r}_j} (T \tau |\phi_{im}(\tau)\phi_{jm}(0)|),
\]

allowing the correlation function in Eqs. (3)-(4) to be written

\[
\langle T\tau t_q(t\tau') \rangle_{\text{inel}} = -N_{||}\delta q_{\perp} - q_{\perp} |q_{qm}|^2 \mathcal{D}_{qm}(\tau).
\] (24)

If the layers are identical, we may drop the index \( m \) and compute the conductivity and the penetration depth as in the preceding two cases, giving

\[
\Pi_{Q',Q}(i\nu_n)_{\text{inel}} = -\delta q_{\perp} \frac{2e^2d}{h} \frac{T^2}{\hbar a^2 N_{||}}
\times \sum_{kk'\perp} |g_{k-k'}|^2 \mathcal{D}_{k-k'}(i\nu_{n+1})
\times Tr \left[ \hat{G}_{k+q}(i\omega_{n+1}) \hat{G}_{k}(i\omega_{l}) \right],
\] (25)
\begin{align}
D_{Q,Q}^{zz} (\omega)_{\text{inel}} &= \delta_{Q,Q'} \frac{2e^2 d}{\hbar^2 a^2 N_0^2} \sum_{kk'\ell\ell'} \frac{\vert g_{k-k'} \vert^2 D_{k-k'}(i\nu_{\ell-\ell'})}{T^2} \\
& \times \text{Tr} \left[ \hat{G}_{k'}(i\omega_{\ell'}) \hat{\tau}_3 \hat{G}_k(i\omega_{\ell}) \hat{\tau}_3 \right], \quad (26)
\end{align}

and
\begin{align}
\left( \frac{\epsilon^2}{4\pi \lambda_c^2} \right)_{\text{inel}} &= -\frac{8e^2 d}{\hbar^2 a^2 N_0^2} \sum_{kk'\ell\ell'} \frac{\vert g_{k-k'} \vert^2 D_{k-k'}(i\nu_{\ell-\ell'})}{T^2} \\
& \times F_{k}(i\omega_{\ell}) F_{k'}(i\omega_{\ell'}), \quad (27)
\end{align}

As before, this expression is proportional to the corresponding one for the critical current. \[1\] In the BCS limit, we can introduce the spectral representation of the boson propagator familiar from the theory of superconductivity. \[24\]

\begin{align}
B_{k,k'}(\Omega) &= -\frac{1}{\pi} \vert g_{k-k'} \vert^2 \text{Im} D_{k-k'}(\Omega), \quad (28)
\end{align}

and write the result as
\begin{align}
\left( \frac{1}{\lambda_c^2} \right)_{\text{inel}} &= \frac{32\pi^3 e^2 \nu}{\hbar^2 c^2} N^2(0) \int_0^\infty d\Omega B_{k,k'}(\Omega) \\
& \times T^2 \sum_{\ell\ell'} \nu_{\ell-\ell'} + \Omega \sqrt{\omega_{\ell}^2 + \Delta_k^2} \\
& \times \frac{\Delta_{k'}^{\ell\ell'}}{\sqrt{\omega_{\ell'}^2 + \Delta_{k'}^{\ell\ell'}}} \langle k | k' \rangle. \quad (29)
\end{align}

For a simple Einstein phonon with a structureless coupling constant \( g_{q} \) or the dispersion of the boson itself, so it is difficult to obtain a simple form for \( B_{k,k'}(\Omega) \) similar to Eq. \[2\]. We therefore take an alternate approach and expand the wave vector dependence of all quantities in Eq. \[29\] in terms of Fermi surface harmonics \( F_L(k) \). \[23\] We further simplify our results by taking an Einstein spectrum for each component of \( B, B_{L,L'}(\Omega) = \lambda_{L,L'}(\Omega_{L,L'}/2N(0)) \delta(\Omega - \Omega_{L,L'}) \). This procedure gives \((1/\lambda_c^2)_{\text{inel}} = \sum_{L,L'} (1/\lambda_c^2)_{L,L'} \) and
\begin{align}
\left( \frac{1}{\lambda_c^2} \right)_{L,L'} &= \frac{32\pi^3 e^2 \nu}{\hbar^2 c^2} N^2(0) T^2 \sum_{\ell\ell'} \lambda_{L,L'} \Omega_{L,L'}^{\ell\ell'} \\
& \times A_{L}(i\omega_{\ell}) A_{L'}(i\omega_{\ell'}) \quad (30)
\end{align}

and \( A_{L}(i\omega) = \langle F_L(k) \Delta_{k}/\sqrt{\omega^2 + \Delta_{k}^2} \rangle \). For concreteness, we take a cylindrical Fermi surface within each layer, which gives \( F_L(k) = \sqrt{2cos(L\phi)} \), where \( k = kF(\cos \phi, \sin \phi) \).

We can make several general statements by looking at the form of Eq. \[30\]. First, although the total \((1/\lambda_c^2)_{\text{inel}} \) must be positive, the individual components \((1/\lambda_c^2)_{L,L'} \) may have either sign because of the difference in the signs of \( A_{L} \) and \( A_{L'} \). Second, for isotropic \( s \)-wave pairing \((\Delta_k = \Delta) \), only the \( L = L' = 0 \) component is non-zero, and we recover the results of Ref. \[14\] after analytic continuation. For \( d \)-wave pairing, on the other hand, only components with \( (L, L') = (4n + 2, 4n' + 2) \) \((n \text{ and } n' \text{ integers}) \) are non-zero.

These features are exhibited in Fig. \[2\] which shows \((1/\lambda_c^2)_{L,L'} \) for both \( s \) and \( d \)-wave pairing computed numerically from Eq. \[30\]. In these calculations, we take \( \Omega_{L,L'} = \Omega_{ph} = 40 \text{ meV} \) to reflect the expected importance of this phonon mode in \( c \)-axis transport. \[25\] For isotropic \( s \)-wave pairing, only the \((L, L') = (0,0) \) component contributes, and we obtain a result which is similar to, though slightly different from, the Ambegaokar-Baratoff result as long as the boson involved has an energy scale larger than the maximum of the gap function. \[13\] For \( d \)-wave pairing, many \((L, L') \) components are non-zero and some are negative, as suggested above.

Examining the \( d \)-wave curves more closely reveals several interesting features. At low temperatures, the non-zero components do not have the linear-in-\( T \) behavior typical of \( d \)-wave superconductors, but seem to follow a higher power law. This result is consistent with the fact that the nodes in a \( d \)-wave gap function must give rise to a power law in the penetration depth, but we see that the exact exponent for this power law is determined by the mechanism responsible for the hopping and the symmetry of its matrix elements. At higher temperatures, the contribution from higher harmonics \((L, L' > 2) \) vanishes. This feature results from the orthogonality of the Fermi surface harmonics in the following way. As \( T \to T_c \), the magnitude of \( \Delta_k \) decreases and \( A_{L}(i\omega_n) \sim \langle F_L(k) \Delta_k \rangle/k_{\omega_n} \). Because the gap function is pure \( d \)-wave \((L = 2) \), \( \Delta_k \propto F_2(k) \) and so \( A_{L}(i\omega_n) \), and hence \((1/\lambda_c^2)_{L,L'} \), goes to zero when \( L, L' > 2 \) much faster than for the \( L = L' = 2 \) component. We have also computed the magnitudes of the penetration depth components \((1/\lambda_c^2)_{L,L'} \) and find that they depend strongly on \((L, L') \), becoming considerably smaller with larger \( L \) and \( L' \). Thus, one expects that the \((L, L') = (2,2) \) component will dominate the boson-assisted contribution to the \( c \)-axis penetration depth in a \( d \)-wave superconductor.

The conclusion one can draw from this analysis is that boson-assisted hopping in a \( d \)-wave superconductor can contribute to the \( c \)-axis penetration depth if the coupling is anisotropic or if the boson has some dispersion. Moreover, the resulting contribution to the penetration depth
is not necessarily linear in $T$ at low $T$, contrary to what one expects and similar to what is observed experimentally.

Relation to experiment. In order to connect our results more firmly to experiment, we compare some representative curves in our model to the experimental $\lambda_c$ data of Bonn et al. [6] on YBCO in Fig. 3. The dashed line shows the BCS prediction for a $d$-wave superconductor: it is clearly inconsistent with these data. The solid and dotted curves show the penetration depth ratio for purely disorder-mediated hopping and a linear combination of disorder- and boson-assisted hopping. We see that the penetration depth ratio computed within the incoherent hopping model is qualitatively consistent with these data: despite having a $d$-wave order parameter, the low-temperature behavior shows a much smaller slope, and the penetration depth ratio falls much more slowly with temperature in the fully oxygenated samples while the disorder-assisted contribution due to the boson-assisted component in the de-oxygenated samples remains relatively constant. The net result is that the penetration depth ratio will fall more slowly with temperature in the de-oxygenated samples, and this is what is observed experimentally.

Summary. Viewing the CuO$_2$ layers in the cuprate superconductors as incoherently coupled explains the features of the anisotropic resistivity in these materials [12,13] and makes definite predictions for the $c$-axis critical current $j_c$. [14] We have extended this theory by providing a microscopic derivation of the $c$-axis penetration depth $\lambda_c$ and find that $j_c \propto 1/\lambda_c^2$, which seems to be a generic property of superconductors consisting of weakly coupled layers. [15] We have also examined the contributions to $\lambda_c$ arising from direct, impurity-assisted, and boson-assisted hopping. We find that anisotropy in the scattering matrix elements or boson dispersion are required in order to obtain a finite penetration depth in $d$-wave superconductors in the absence of direct hopping, although no such restriction arises for $s$-wave pairing. By computing the temperature dependence of the penetration depth ratio $\lambda_c^2(0)/\lambda_c^2(T)$ for $d$-wave pairing and anisotropic scattering, we find that the experimental $c$-axis penetration depth ratio as a function of both temperature and doping is qualitatively consistent with this.
model for at least some measurements of $\lambda_c$ in YBCO.

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