AN ANALYTIC CHEVALLEY-SHEPHARD-TODD THEOREM

SHIBANANDA BISWAS, SWARNENDU DATTA, GARGI GHOSH, AND SUBRATA SHYAM ROY

ABSTRACT. In this note, we prove two theorems extending the well known Chevalley-Shephard-Todd Theorem about the action of pseudo-reflection groups on the ring of polynomials to the setting of the ring of holomorphic functions. In the process, we obtain a purely algebraic determinantal formula that may also be of independent interest.

1. Introduction and Statement of the results

Recall that a pseudo-reflection on \( \mathbb{C}^n \) is a linear automorphism \( \rho : \mathbb{C}^n \to \mathbb{C}^n \) such that the rank of \( 1 - \rho \) is 1, that is, \( \rho \) is not the identity map and fixes a hyperplane pointwise. Let \( G \) be a finite group generated by pseudo-reflections. Then \( G \) also acts on the set of functions on \( \mathbb{C}^n \) by \( \rho(f)(z) = f(\rho^{-1} \cdot z) \). Let \( A = \mathbb{C}[z_1, \ldots, z_n] \) be the ring of polynomial functions on \( \mathbb{C}^n \) and let \( B = A^G \) be the ring of \( G \)-invariant elements of \( A \). It is a well known theorem due to Chevalley-Shephard-Todd that:

**Theorem A.** [3, Theorem 3, p.112] \( B = \mathbb{C}[\theta_1, \ldots, \theta_n] \), where \( \theta_i \) are algebraically independent homogeneous polynomials. In particular, \( B \) is itself a polynomial algebra in \( n \) variables.

**Theorem B.** [3, Theorem 1, p.110] \( A \) is a free \( B \) module of rank \( d \), where \( d \) is the cardinality of \( G \). Further, one can choose a basis of \( A \) consisting of homogeneous polynomials.

**Example 1.1.** Let \( G = S_n \), the symmetric group on \( n \) elements, acting on \( \mathbb{C}^n \) by permuting the coordinates. It is generated by transpositions, which are actually reflections (that is, pseudo-reflections of order 2; equivalently, it fixes a hyperplane and acts on a complementary line by \(-1\)). Then \( B \) is the ring of symmetric polynomials and one has \( B = \mathbb{C}[s_1, \ldots, s_n] \), where \( s_i \) are the elementary symmetric polynomials. It can be shown that the \( n! \) elements

\[ z_1^{k_1} \cdots z_{n-1}^{k_{n-1}}, \quad 0 \leq k_i \leq i \]

form a basis of \( A \) as a \( B \) module [12, p. 94].

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Theorems 1.2 and 1.3 stated below correspond to generalizations of Theorems A and B above to the setting of holomorphic functions on $\mathbb{C}^n$. Let $\theta : \mathbb{C}^n \to \mathbb{C}^n$ be the function defined by

$$\theta(z) = (\theta_1(z), \ldots, \theta_n(z)), \quad z \in \mathbb{C}^n.$$

**Theorem 1.2.** For a $G$-invariant holomorphic function $f$ on $\mathbb{C}^n$, there exists a unique holomorphic function $g$ on $\mathbb{C}^n$ such that $f = g \circ \theta$.

Let $f_1, \ldots, f_d$ (recall that $d = \text{Card}(G)$) be a basis of $A$ as a $B$ module.

**Theorem 1.3.** Let $g$ be a holomorphic function on $\mathbb{C}^n$. Then there exist unique $G$-invariant holomorphic functions $g_1, \ldots, g_d$ such that $g = g_1 f_1 + \cdots + g_d f_d$.

**Remark 1.4.** A choice of different basis of $A$ is related to the chosen basis by a matrix (with entries in $B$) whose determinant is invertible in $B$ and consequently is a non-zero complex number. Thus the result is independent of the basis chosen.

**Remark 1.5.** An approach to prove the Theorem above is to prove similar theorem for formal power series and then one deals with the convergence issues. A similar approach was made in [1, Theorem 1.2] where it was proved that analytic solution to an arbitrary system of analytic equation exists provided a solution in formal power series without constant term exists! In our case, it is indeed true that the ring $\mathbb{C}[[z]]$ of formal power series is a free module over $\mathbb{C}[[z]]^G$, the ring of invariants under the action of $G$, of rank $d$. Using $\mathbb{C}[[z]]^G = \mathbb{C}[[\theta_1, \theta_2, \ldots, \theta_n]]$ (following [4, Theorem 2.a), A.IV.68]) and the fact that $\mathbb{C}[[z]]$ (respectively $\mathbb{C}[[z]]^G$) is the completion of the polynomial ring $A$ (respectively $B$) in $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal generated by the indeterminates [13, Example 1, page 260] and a commutative algebra technique [2, Proposition 10.13], the result for formal power series follows. However, it is not clear, a priori, whether the components of a holomorphic functions in terms of formal power series that we obtain via the decomposition above are necessarily holomorphic, that is, whether they converge on $\mathbb{C}^n$! We rather approach Theorem 1.3 independently by obtaining the formulae for the components $g_1, \ldots, g_d$.

We note that both theorems 1.2 and 1.3 can be extended to the case of $G$-invariant domain $\Omega \subseteq \mathbb{C}^n$. The necessary modifications in the proofs are indicated in section 4.

2. PROOF OF THEOREM 1.2

Note that $\theta$ is $G$-invariant, by definition, and thus factors through the quotient topological space $\mathbb{C}^n/G$. Let $\theta' : \mathbb{C}^n/G \to \mathbb{C}^n$ be the resulting map. If $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ is the quotient map, then one has $\theta = \theta' \circ \pi$. The following result is well known.

**Proposition 2.1.** $\theta'$ is bijective.

**Proof.** Equivalently, one needs to show that:

(i) $\theta$ is surjective;

(ii) The fibers of $\theta$ are precisely the $G$-orbits in $\mathbb{C}^n$. 


For (i), let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ and let $m$ be the maximal ideal of $B$ generated by $\theta_1 - \mu_1, \ldots, \theta_n - \mu_n$. Note that $B/m \cong \mathbb{C}$ and thus $A/mA$ is a complex vector space of dimension $d$, in view of Theorem B above. Thus $mA \neq A$ and the ideal $mA$ of $A$ is contained in a maximal ideal $n$ of $A$. By nullstellensatz, $n$ is generated by $z_1 - \lambda_1, \ldots, z_n - \lambda_n$ for suitable $\lambda_i \in \mathbb{C}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$. If $p \in n$, then $p(\lambda) = 0$. By taking $p = \theta_i - \mu_i$, we have $\theta_i(\lambda) = \mu_i$ for each $i$, whence $\theta(\lambda) = \mu$.

For (ii), let $O$ and $O'$ be two distinct orbits of $G$ and $p$ be a polynomial taking value 0 in $O$ and 1 in $O'$. Then the polynomial

$$q = \prod_{p \in G} \rho(p)$$

has the same property and is moreover $G$-invariant. Thus, by Theorem A, it is a polynomial in the $\theta_i$'s. It then follows that there exists a $j$ such that $\theta_j(\lambda) \neq \theta_j(\lambda')$. Thus $\theta$ takes $O$ and $O'$ to different orbits, whence (ii).

It follows from a Theorem of H. Cartan (cf. [5, Theorem 7.2]) that the quotient space $\mathbb{C}^n/G$ can be given the structure of a complex analytic space and that the map $\theta'$ is holomorphic. Further note that dim($\mathbb{C}^n/G$) = $n$ (cf. [9, E. 49p.]). Now we complete the proof of the theorem.

(Proof of Theorem 1.2). It is enough to show that $\theta'$ is biholomorphic. It is well known (cf. [9, Proposition 46.A.1]) that a bijective holomorphic mapping from an equidimensional reduced complex analytic space to a complex manifold of the same dimension is biholomorphic. Since $\theta'$ is bijective and $\mathbb{C}^n/G$ is reduced (cf. [6, p. 246]) of pure dimension $n$, the result follows.

3. PROOF OF THEOREM 1.3

3.1. Proof of uniqueness. Let $G = \{\rho_1, \ldots, \rho_d\}$. Applying $\rho_i$ to the equation $g = g_1f_1 + \cdots + g_df_d$ gives

$$\rho_i(g) = g_1\rho_i(f_1) + \cdots + g_d\rho_i(f_d), 1 \leq i \leq d.$$  

Let $y$ be the column vector $(g_1, \ldots, g_d)^t$ and let $x = (\rho_1(g), \ldots, \rho_d(g))^t$. Note that $My = x$, where

$$M = \left(\rho_i(f_j)\right)_{i,j=1}^d. \quad (3.1)$$

Lemma 3.1. The matrix $M$ is non-singular.

Proof. Let $K$ (resp. $L$) be the fraction field of $A$ (resp. $B$). From [3, Chapter. V, $\S$5.2, p. 110], we note that $K$ is a Galois extension of $L$ with Galois group $G$ and that $f_1, \ldots, f_d$ is a basis of $K$ over $L$. The $(i,j)$-th entry of the matrix $M^tM$ is $\rho_i(f_j) + \cdots + \rho_d(f_i f_j) = \operatorname{tr}(f_i f_j)$. (cf. [10, Corollary 8.13]), where $\operatorname{tr} : K \to L$ denotes the trace function. Thus, it is the matrix of the bilinear form $B : K \times K \to L, (f, f') \mapsto \operatorname{tr}(f f')$. This is non-degenerate since for nonzero $f$, one has $B(f, 1/f) = \operatorname{tr}(1) = [K : L] \neq 0$. Thus $\det(M)^2 = \det(M^tM)$ (it is the discriminant of the extension $K/L$) is nonzero.

To prove uniqueness, let $y'$ be another vector such that $My' = x$. Then $M(y - y') = 0$. Left multiplication by the adjoint of $M$, it follows that $\det(M)(y - y') = 0$. By the lemma above, it follows that $y - y' = 0$. 

AN ANALYTIC CHEVALLEY-SHEPHARD-TODD THEOREM
since the ring of holomorphic functions on $\mathbb{C}^n$ is an integral domain. This completes the proof.

### 3.2. Proof of existence

We shall first establish an explicit formula (up to a constant) for $\det(M)$, stated in Proposition 3.6 below. We begin by recalling the definition of Poincaré series (cf. [3, p. 108]), followed by a couple of lemmas.

Let $V$ be a (complex) vector space. A grading of $V$ is a direct sum decomposition

$$V = \oplus_{k \geq 0} V_k$$

such that $\dim(V_i)$ is finite for each $i$. The Poincaré series of the graded vector space $V$, denoted by $P_V$, is defined by

$$P_V(t) = \sum_{k \geq 0} \dim(V_k) t^k.$$

If $W = \oplus_{k \geq 0} W_k$ is another graded vector space, define a grading of $V \oplus W$ and $V \otimes W$ by

$$(V \oplus W)_k = V_k \oplus W_k,$$

$$(V \otimes W)_k = \sum_{p+q=k} V_p \otimes W_q,$$

and thus

$$P_{V \oplus W} = P_V + P_W,$$

$$P_{V \otimes W} = P_V \cdot P_W.$$

**Example 3.2.**

(a) Define a grading of $A$ by letting $A_k$ be the space of homogeneous polynomials of degree $k$. One has $A \cong \mathbb{C}[z_1] \otimes \cdots \otimes \mathbb{C}[z_n]$ and since

$$P_{\mathbb{C}[z]}(t) = \sum_{k \geq 0} t^k = \frac{1}{1-t},$$

a repeated application of the second formula shows that

$$P_A(t) = \frac{1}{(1-t)^n}.$$  

(b) Note that $B$ is a homogeneous subspace of $A$, that is, if $B_k = B \cap A_k$, then $B = \oplus B_k$. Let $d_1, \ldots, d_n$ be the degrees of $\theta_1, \ldots, \theta_n$. Then one has

$$P_B(t) = \frac{1}{1 - t^{d_1}} \cdots \frac{1}{1 - t^{d_n}}.$$

This follows from the equality $P_{\mathbb{C}[\theta]}(t) = \sum_{k \geq 0} t^{kd_i} = (1 - t^{d_i})^{-1}$ together with the isomorphism $B \cong \mathbb{C}[\theta_1] \otimes \cdots \otimes \mathbb{C}[\theta_n]$.

**Lemma 3.3.** Let $m$ denote the number of pseudo-reflections in $G$. Then $\det(M)$ is a homogeneous polynomial of degree $dm/2$.

**Proof.** In view of Remark 1.4, choosing a different basis of $A$ (as a $B$ module) changes $\det(M)$ only by a constant and thus we may assume that $f_1, \ldots, f_d$
are homogeneous (cf. Theorem B). Let $e_j$ be degree of $f_j$. It is clear that
$\det(M)$ is homogeneous and is of degree $\sum_j e_j$. Now one has
\[ A = f_1 B \oplus \cdots \oplus f_d B \]
and by considering Poincaré series, we get
\[ \frac{1}{(1-t)^n} = \frac{1}{(1-t)^{d_1} \cdots (1-t)^{d_n}} \cdot (t^{e_1} + \cdots + t^{e_d}) \]
and thus
\[ \sum t^{e_i} = \prod_i \frac{1-t^{d_i}}{1-t} = \prod_i (1 + t + \cdots + t^{d_i-1}). \tag{3.2} \]
We now take derivative with respect to $t$ and evaluate at $t = 1$. This gives
$\sum e_j$ on the left side. Further, derivative of $1 + t + \cdots + t^{d_i-1}$ is $1 + 2t + \cdots + (d_i - 1)t^{d_i - 2}$, which when evaluated at $t = 1$ gives $d_i(d_i - 1)/2$. Thus
the right side becomes
\[ \sum_i d_1 \cdots \hat{d_i} \cdots d_n (d_i(d_i - 1)/2), \]
where $\hat{\cdot}$ denotes omission. This is equal to
\[ d_1 \cdots d_n \left( \sum_i (d_i - 1) \right)/2. \]
But it is well known that $d_1 \cdots d_n = d$ (cf. [3, Chapter V, p.115]) and
$\sum(d_i - 1) = m$ (cf. [3, Chapter V, p.116]), whence the result follows.

**Remark 3.4.** It follows from the Equation (3.2) that the degrees $e_j$ are
independent of the chosen homogeneous basis. Moreover, if $G$ is a reflection
group, then Equation (3.2) shows that $e_1, \ldots, e_d$ are the lengths of the
elements of $G$ (cf. [8, Theorem in Section 3.15]).

We call a hyperplane $H$ of $\mathbb{C}^n$ reflecting if there exists a pseudo-reflection
in $G$ acting trivially on $H$. The set of elements of $G$ acting trivially on $H$
is a subgroup; it is cyclic (for instance, choose a positive Hermitian form
invariant under the action of $G$. Then the line $\ell$ orthogonal to $H$ is invariant
under the subgroup, thus $H$ embeds in $\text{Aut}(\ell) = \mathbb{C} \setminus \{0\}$, hence being a
subgroup of the group of $d$-th roots of unity, it is cyclic). If its order is $k$, then
the subgroup is of the form $\{1, \rho, \ldots, \rho^{k-1}\}$ where each $\rho^i, 1 \leq i < k$
is a pseudo-reflection. Let $L$ denote a nonzero homogeneous linear function
on $\mathbb{C}^n$ vanishing on $H$ so that $H = \{ z \in \mathbb{C}^n : L(z) = 0 \}$. It is well defined
up to a non-zero constant multiplier.

**Lemma 3.5.** Let $N$ be a square matrix with entries in $A$ and let $v$ be a row
of $N$. If $v, \rho(v), \ldots, \rho^j(v)$ are distinct rows of $N$, then $\det(N)$ is divisible
by $L^{j(j+1)/2}$.

**Proof.** We argue by induction on $j$, the case $j = 0$ being trivial. Using row
reduction, we may replace the rows by
\[ v - \rho(v), \rho(v) - \rho^2(v), \ldots, \rho^{j-1}(v) - \rho^j(v), \rho^j(v). \]
Note that if $f$ is a polynomial, then $f - \rho(f)$ is identically zero on $H$ and thus
is divisible by $L$ (cf. [8, Lemma in Section 3.3]). Let $u = (v - \rho(v))/L$. The
rows are \( L\mathbf{u}, L\rho(\mathbf{u}), \ldots, L\rho^{j-1}(\mathbf{u}), \rho^j(\mathbf{v}) \). By induction, the determinant of a matrix with rows \( \mathbf{u}, \rho(\mathbf{u}), \ldots, \rho^{j-1}(\mathbf{u}) \) is divisible by \( L^{(j-1)/2} \) and thus \( \det(N) \) is divisible by \( L^j \cdot L^{(j-1)/2} \).

Let \( H_1, \ldots, H_t \) be the reflecting hyperplanes of \( G \) and let \( m_1, \ldots, m_t \) be the orders of the corresponding subgroups \( K_1, \ldots, K_t \). One has

\[
m = \sum_{i=1}^t (m_i - 1).
\]

Let \( L_i \) be a nonzero linear function on \( \mathbb{C}^n \) vanishing on \( H_i \).

**Proposition 3.6.**

\[
\det(M) = c \prod_{i=1}^t L_i^{d(m_i - 1)/2},
\]

where \( c \) is a nonzero constant.

**Proof.** From Lemma 3.3 and Equation (3.3), it follows that the degree of both sides is \( dm/2 \). Thus, by Lemma 3.1, it suffices to check that \( \det(M) \) is divisible by \( L_i^{d(m_i - 1)/2} \). Let \( K_i \) be the subgroup of \( G \) fixing \( H_i \) and let \( \rho_i \) be a generator of \( K_i \). Partition \( G \) into right cosets of \( K_i \). Fix a right coset \( K_i \rho \) for some \( \rho \in G \). Let \( \mathbf{u} = (f_1, \ldots, f_d) \) and let \( \mathbf{v} = \rho(\mathbf{u}) \). The rows in \( M \) corresponding to \( K_i \rho \) are \( \mathbf{v}, \rho_1(\mathbf{v}), \ldots, \rho_{m_i}^{-1}(\mathbf{v}) \). Thus by the Lemma 3.5, \( \det(M) \) is divisible by \( L_i^{m_i - 1}m_i/2 \). But there are \( d/m_i \) right cosets, and repeating the argument in the lemma for each of these, one deduces that \( \det(M) \) is divisible by \( (d/m_i)\)-th power of \( L_i^{m_i - 1}m_i/2 \).

**Remark 3.7.**

(i) If \( G \) is a reflection group, then each \( m_i = 2 \) and \( t = m \). In this case, \( d \) is even and the product of the linear factors reduces to \( (L_1 \cdots L_m)^{d/2} \).

(ii) It is a result due to Steinberg (cf. [3, Ch. V, §5.4]) that the Jacobian \( J = \text{Jac}(\theta_1, \ldots, \theta_n) \) is a constant multiple of products of linear factors corresponding to each pseudo-reflection in \( G \). This is none other than \( \prod L_i^{m_i - 1} \). Thus \( \det(M) \) is a constant multiple of \( J^{d/2} \).

Note that if \( d \) is odd, then each \( m_i - 1 \) is even and \( J \) is already a square in \( A \).

To prove existence, we first obtain a formula for the \( g_i \), when \( g \) is a polynomial. Recall that \( \mathbf{x} = (\rho_1(\mathbf{g}), \ldots, \rho_d(\mathbf{g}))^t \). From Theorem B (in section 1), it follows that there exists \( \mathbf{y} = (g_1, \ldots, g_d)^t \), with each \( g_i \) \( G \)-invariant, such that \( M\mathbf{y} = \mathbf{x} \). If \( (\tilde{g}_1, \ldots, \tilde{g}_d)^t = \text{adj}(M)\mathbf{x} \), then \( \det(M)g_i = \tilde{g}_i \), that is, \( \det(M) \) divides \( \tilde{g}_i \) for each \( i \). Further, if \( g \) is homogeneous of degree \( e \), then \( g_i \) is homogeneous of degree \( e - e_i \) (by comparing the homogeneous part of degree \( e \) of the equation \( \mathbf{g} = \sum_{i=1}^d g_i f_i \)) and thus \( \tilde{g}_i \) is homogeneous polynomials (of degree \( dm/2 + e - e_i \)). Suppose now that \( g \) is a holomorphic function on \( \mathbb{C}^n \) and let \( g_{(k)} \) denote the homogeneous component of \( g \) of degree \( k \) in the power series expansion of \( g \). It follows from the above that:

1. the \( i \)-th component \( \tilde{g}_i \) of \( \text{adj}(M)\mathbf{x} \) is an entire function whose homogeneous components \( g_{(k,i)}, k \geq 0 \), and
(2) \( \det(M) \) divides \( \tilde{g}(k,i) \) and \( g(k,i) = \tilde{g}(k,i)/\det(M) \) is a \( G \)-invariant polynomial for each \( i \) and \( k \).

Since \( \tilde{g}_i = \sum_{k \geq 0} g(k,i) \), the proof of existence would be complete if we prove that the observations above would imply \( \det(M) \) divides each \( \tilde{g}_i \) and the resultants are \( G \)-invariant.

(Proof of Theorem 1.3). From the Proposition 3.6 it follows that to prove \( \det(M) \) divides each \( \tilde{g}_i \), it is enough to show that if \( L \) is a product of linear polynomials that divides the homogeneous components of the power series of a holomorphic function \( f \) on \( \mathbb{C}^n \), then \( L \) divides \( f \). Let \( \ell \) be a linear factor of \( L \). Using a linear change of coordinates, we may assume \( \ell = z_1 \), in which case \( f \) will have the power series expansion as \( \sum_{i_1 \geq 1, i_2, \ldots, i_n \geq 0} a_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \).

Since the absolute sum of the terms of the power series is dominated by the absolute sum of the terms of those of \( \partial f/\partial z_1 \), it defines a holomorphic function, say \( h \) on \( \mathbb{C}^n \), and hence \( f(z) = z_1 h(z) \). The homogeneous components of \( f/z_1 \) are divisible by \( L/z_1 \) and one proceeds by induction on degree of \( L \) to see that \( f \) is divisible by \( L \). Finally, as \( g_i = \sum_{k \geq 0} g(k,i) \) where the sum converges (absolutely and uniformly on compact subsets of \( \mathbb{C}^n \)), in virtue of (2) above each \( g_i \) is \( G \)-invariant. \( \square \)

4. Generalisation to the case of \( G \)-invariant domains

Let \( \Omega \) be an arbitrary \( G \)-invariant domain in \( \mathbb{C}^n \). It follows from [11, Proposition 2.1], \( \theta(\Omega) \) is a open in \( \mathbb{C}^n \).

4.1. Generalisation of Theorem 1.2. Here, we want to show that for a \( G \)-invariant holomorphic functions \( f \) on \( \Omega \), there exists a unique holomorphic function on \( \theta(\Omega) \) such that \( f = g \circ \theta \). Defining \( \theta' \) as in section 2, the same argument shows that \( \theta' : \Omega/G \to \theta(\Omega) \) is biholomorphic, whence the result follows as before.

4.2. Generalisation of Theorem 1.3. This requires a little more work due to the fact that power series representation need not be possible on all of \( \Omega \) for every function holomorphic on an arbitrary \( G \)-invariant domain. We shall, thus, adopt a different approach here. We first prove the following lemma.

Lemma 4.1. Let \( g \) be a function on \( G \)-invariant domain \( \Omega \) and let \( (\tilde{g}_1, \ldots, \tilde{g}_d) \) be the column vector \( \text{adj}(M)x \), where \( x = (\rho_1(g), \ldots, \rho_d(g))^t \). Further, if there exists functions \( h_j : \Omega \to \mathbb{C}, 1 \leq j \leq d \), such that \( \tilde{g}_j = (\det(M))h_j \) on \( \Omega \), then \( h_j \) is \( G \)-invariant for each \( j, 1 \leq j \leq d \).

Proof. Note that \( \tilde{g}_j = \det M_j \), where \( M_j \) is a \( d \times d \) matrix obtained by replacing the \( j \)-th column of the matrix \( M \) by the column vector \( x \), that is,

\[
M_j = \begin{pmatrix}
\rho_1(f_1) & \ldots & \rho_1(f_{j-1}) & \rho_1(g) & \rho_1(f_{j+1}) & \ldots & \rho_1(f_d)
\rho_2(f_1) & \ldots & \rho_2(f_{j-1}) & \rho_2(g) & \rho_2(f_{j+1}) & \ldots & \rho_2(f_d)
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\rho_d(f_1) & \ldots & \rho_d(f_{j-1}) & \rho_d(g) & \rho_d(f_{j+1}) & \ldots & \rho_d(f_d)
\end{pmatrix}.
\]
Clearly if \( g = f_j \), then \( M_j = M \). Fix a \( k \) such that \( 1 \leq k \leq d \) and let \( \tilde{\rho}_i = \rho_k \rho_i \) for \( 1 \leq i \leq d \). Then \( \tilde{g}_j(\tilde{\rho}_k^{-1} \cdot z) = \det M_j(\rho_k^{-1} \cdot z) = \det \tilde{M}_j(z) \), where
\[
\tilde{M}_j = \begin{pmatrix}
\tilde{\rho}_1(f_1) & \ldots & \tilde{\rho}_1(f_{j-1}) & \tilde{\rho}_1(g) & \tilde{\rho}_1(f_{j+1}) & \ldots & \tilde{\rho}_1(f_d) \\
\tilde{\rho}_2(f_1) & \ldots & \tilde{\rho}_2(f_{j-1}) & \tilde{\rho}_2(g) & \tilde{\rho}_2(f_{j+1}) & \ldots & \tilde{\rho}_2(f_d) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{\rho}_d(f_1) & \ldots & \tilde{\rho}_d(f_{j-1}) & \tilde{\rho}_d(g) & \tilde{\rho}_d(f_{j+1}) & \ldots & \tilde{\rho}_d(f_d)
\end{pmatrix}
\]
is obtained by permuting the rows of the matrix \( M_j \) as \( \{\tilde{\rho}_1, \ldots, \tilde{\rho}_d\} \) is a permutation of the elements \( \rho_1, \ldots, \rho_d \) of the group \( G \). Therefore \( \det \tilde{M}_j \) and \( \det M_j \) are equal except a sign, that is, \( \tilde{g}_j(\tilde{\rho}_k^{-1} \cdot z) \) differs from \( \tilde{g}_j(z) \) by the same sign for all \( j, 1 \leq j \leq d \) and for all functions \( g \) on \( \Omega \). Thus \( \det (M \tilde{\rho}_k^{-1} \cdot z) \) also differs from \( \det \tilde{M}(z) \) by the same sign. Hence \( h_j(\tilde{\rho}_k^{-1} \cdot z) = h_j(z) \) for all \( z \in \Omega \) and for each \( j, 1 \leq j \leq d \). This completes the proof as \( k \) was chosen arbitrarily.

Let \( \mathcal{O}(\Omega) \) denote the ring of all holomorphic functions from a domain \( \Omega \subset \mathbb{C}^n \) to \( \mathbb{C} \) and \( \mathcal{O}(\Omega)^G \) denote the ring of \( G \)-invariant holomorphic functions. We restate the generalisation of Theorem 1.3 to \( G \)-invariant domains as follows.

**Theorem 4.2.** If \( \Omega \) is a \( G \)-invariant domain in \( \mathbb{C}^n \), then \( \mathcal{O}(\Omega) \) is a finitely generated free module over \( \mathcal{O}(\Omega)^G \) of rank \( \text{Card}(G) \).

**Proof.** If \( \{f_1, \ldots, f_d\} \) is a basis of \( A \) as a \( B \) module, as in Theorem 1.3, we show that for each \( g \in \mathcal{O}(\Omega) \), there exist unique \( G \)-invariant holomorphic functions \( g_1, \ldots, g_d \) on \( \Omega \) such that \( g = g_1 f_1 + \cdots + g_d f_d \). The proof remains the same as the proof of Theorem 1.3 except that now we need to prove that the \( g_i \)'s exist as \( G \)-invariant holomorphic function on \( \Omega \). Lemma 4.1 shows that it is enough to find a global solution to the system of equations \( \det(M) y = \text{adj}(M) x \), where \( x = (\rho_1(g), \ldots, \rho_d(g))^T \). First we make the following elementary observation. Let \( H \) be the reflecting hyperplane fixed by a pseudoreflection \( \rho \in G \). Let \( f \in \mathcal{O}(\Omega) \) and let \( L \) be the linear factor of \( \det(M) \) corresponding to \( H \). Then we claim the \( L \) divides \( f - \rho(f) \) in \( \mathcal{O}(\Omega) \). This is because, if \( z \not\in H \cap \Omega \), then \( L(z) \neq 0 \) and this is true in a neighbourhood of \( z \). On the other hand, if \( H \cap \Omega \) is non-empty and \( z \in H \cap \Omega \), then it follows from Weierstrass Division Theorem [7, p.11] that \( L \) divides \( f - \rho(f) \) in a neighbourhood of \( z \) since \( L \) is irreducible. Therefore the claim follows as the holomorphic functions which are obtained locally patch to give a global holomorphic function, since both \( f - \rho(f) \) and \( L \) are defined on all of \( \Omega \). Using this result, one can apply the proof of Proposition 3.6 to \( \det(M_i) \) to deduce that \( \det(M) \) divides \( \det(M_i) \). As is observed in the proof of Lemma 4.1 that \( \text{adj}(M) x = \text{(det}(M_1), \ldots, \text{det}(M_d))^T \), it follows that \( \text{adj}(M) x \) is divisible by \( \det(M) \) which completes the proof. □

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(Biswas, Datta, Ghosh and Shyam Roy) Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur 741246, Nadia, West Bengal, India

E-mail address, Biswas: shibananda@gmail.com
E-mail address, Datta: swarnendu.datta@iiserkol.ac.in
E-mail address, Ghosh: gg13ip034@iiserkol.ac.in
E-mail address, Shyam Roy: ssroy@iiserkol.ac.in