Local unambiguous discrimination of symmetric ternary states

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We investigate unambiguous discrimination between given quantum states with a sequential measurement, which is restricted to local measurements and one-way classical communication. If the given states are binary or those each of whose individual systems is two-dimensional, then it is in some cases known whether a sequential measurement achieves a globally optimal unambiguous measurement. In contrast, for more than two states each of whose individual systems is more than two-dimensional, the problem becomes extremely complicated. This paper focuses on symmetric ternary pure states each of whose individual systems is three-dimensional, which include phase shift keyed (PSK) optical coherent states and a lifted version of “double trine” states. We provide a necessary and sufficient condition for an optimal sequential measurement to be globally optimal for the bipartite case. A sufficient condition of global optimality for multipartite states is also presented. One can easily judge whether these conditions hold for given states. Some examples are given, which demonstrate that, despite the restriction to local measurements and one-way classical communication, a sequential measurement can be globally optimal in quite a few cases.

I. INTRODUCTION

Discrimination between quantum states as accurately as possible is a fundamental issue in quantum information theory. It is a well-known property of quantum theory that perfect discrimination among nonorthogonal quantum states is impossible. Then, given a finite set of nonorthogonal quantum states, we need to find an optimal measurement with respect to a reasonable criterion. Unambiguous discrimination is one of the most common strategies to distinguish between quantum states [1–3]. An unambiguous measurement achieves error-free (i.e., unambiguous) discrimination at the expense of allowing for a certain rate of inconclusive results. Finding an unambiguous measurement that maximizes the average success probability for various quantum states has been widely investigated (e.g., [4–12]).

When given quantum states are shared between two or more systems, measurement strategies can be classified into two types: global and local. A local measurement is performed by a series of individual measurements on the subsystems combined with classical communication. In particular, sequential measurements, in which the classical communication is one-way only, have been widely investigated under several optimality criteria (e.g., [13–22]). Although the performance of an optimal sequential measurement is often strictly less than that of an optimal global measurement even if given states are not entangled, a sequential measurement has the advantage of being relatively easy to implement with current technology. As an example of a realizable sequential measurement for optical coherent states, a receiver based on a combination of a photon detector and a feedback circuit, which we call a Dolinar-like receiver, has been proposed [23] and experimentally demonstrated [24]. Also, unambiguous discrimination using Dolinar-like receivers has been studied [25–27].

Several studies on optimal unambiguous sequential measurements have also been carried out [28–31]. For binary pure states with any prior probabilities, it has been shown that an optimal unambiguous sequential measurement can achieve the performance of an optimal global measurement [28, 29]. For short, we say that a sequential measurement can be globally optimal. As for more than two states, in the case in which each of the individual systems is two-dimensional, whether a sequential measurement can be globally optimal has been clarified for several cases [30, 31]. However, in the case in which individual systems are more than two-dimensional, the problem becomes extremely complicated. Due to the restriction of local measurements and one-way classical communication, it would not be surprising if a sequential measurement cannot be globally optimal except for some special cases. It is worth mentioning that, according to Ref. [32], in the case of a minimum-error measurement, which maximizes the average success probability but sometimes returns an incorrect answer, an optimal sequential measurement does not seem to be globally optimal for any ternary phase shift keyed (PSK) optical coherent states.

In this paper, we focus on symmetric ternary pure states each of whose individual systems is three-dimensional. These states include PSK optical coherent states and a lifted version of “double trine” states [33]. We provide a necessary and sufficient condition that a sequential measurement can be globally optimal for the bipartite case, using which one can easily judge whether global optimality is achieved by a sequential measurement for given states. We use the convex optimization approach reported in Ref. [34] to derive the condition. We also give a sufficient condition of global optimality for the multipartite case. Some examples of symmetric ternary pure states are presented, which show that a sequential measurement can be globally optimal in quite a few cases. One of the examples shows that the problem of whether a sequential measurement for bipartite ternary PSK optical coherent states can be globally optimal is completely solved analytically, while its minimum-error measurement version has been solved only numerically [32]. Moreover, we show that a Dolinar-like receiver for any ternary PSK optical coherent states cannot be globally optimal unambiguous measurement.

The paper is organized as follows. In Sec. II, we formu-
late the problem of finding an optimal unambiguous measurement and its sequential-measurement version as convex programming problems. In Sec. III, we present our main theorem. Using this theorem, we derive a necessary and sufficient condition for an optimal sequential measurement for bipartite symmetric ternary pure states to be globally optimal. A sufficient condition of global optimality for multipartite symmetric ternary pure states is also derived. In Sec. IV, we prove the main theorem. Finally, we provide some examples to demonstrate the usefulness of our results in Sec. V.

II. OPTIMAL UNAMBIGUOUS SEQUENTIAL MEASUREMENTS

In this section, we first provide an optimization problem of finding optimal unambiguous measurements. Then, we discuss a sequential-measurement version of the optimization problem. We also provide a necessary and sufficient condition for an optimal sequential measurement to be globally optimal. Note that this condition is quite general but requires extra effort to decide whether global optimality is achieved by a sequential measurement for given quantum states. In Sec. III, we will use this condition to derive a formula that is directly applicable to symmetric ternary pure states.

A. Problem of finding optimal unambiguous measurements

We here consider unambiguous measurements without restriction to sequential measurements. Consider a quantum system prepared in one of $R$ quantum states represented by density operators $\{\hat{\rho}_r\}_{r \in R}$ on a complex Hilbert space $\mathcal{H}$, where $R := \{0, 1, \ldots, R - 1\}$. The density operator $\hat{\rho}_r$ satisfies $\hat{\rho}_r \geq 0$ and $\text{Tr} \hat{\rho}_r = 1$, where $\hat{A} \geq 0$ denotes that $\hat{A}$ is positive semidefinite (similarly, $\hat{A} \geq \hat{B}$ denotes $\hat{A} - \hat{B} \geq 0$). To unambiguously discriminate the $R$ states, we can consider a measurement represented by a positive-operator-valued measure (POVM), $\hat{M} := \{\hat{M}_\lambda\}_{\lambda \in \Omega}$, consisting of $R + 1$ detection operators, on $\mathcal{H}$, where $\hat{M}_\lambda$, satisfies $\hat{M}_\lambda \geq 0$ and $\sum_{\lambda \in \Omega} \hat{M}_\lambda = \hat{I}$ ($\hat{I}$ is the identity operator on $\mathcal{H}$). The detection operator $\hat{M}_\lambda$ with $r < R$ corresponds to the identification of the state $\hat{\rho}_r$, while $\hat{M}_R$ corresponds to the inconclusive answer. Any unambiguous measurement $\hat{M}$ satisfies $\text{Tr}(\hat{\rho}_r \hat{M}_k) = 0$ for any $k \in R \setminus \{r\}$, where \$ denotes set difference. Given possible states $\{\hat{\rho}_r\}$ and their prior probabilities $\{c_r\}$, we want to find an unambiguous measurement maximizing the average success probability, which we call an optimal unambiguous measurement or just an optimal sequential measurement for short. Reference [6] shows that the problem of finding an optimal measurement can be formulated as a semidefinite programming problem, which is a special case of a convex programming problem. For analytical convenience, instead of the formulation of Ref. [6], we consider the following semidefinite programming problem:

$$P_G : \text{maximize} \quad P(\hat{\Pi}) := \lim_{J \to 0} \sum_{r=0}^{R-1} \text{Tr}[\hat{\rho}_r - \lambda \hat{\nu}_r] \hat{\Pi}_r$$

subject to $\hat{\Pi} : \text{POVM}$

where $\hat{\rho}_r := \xi_r \hat{\rho}_r$ and $\hat{\nu}_r := \sum_{k \in I_R \setminus \{r\}} \hat{\rho}_k$. Since $P(\hat{\Pi}) = -\infty$ holds if there exists $r \in I_R$ such that $\text{Tr}(\hat{\nu}_r \hat{M}_r) \neq 0$ (i.e., $\hat{\Pi}$ is not an unambiguous measurement), any optimal solution to Problem $P_G$ is guaranteed to be an unambiguous measurement. The optimal value, which is the average success probability of an optimal measurement, is larger than zero if and only if at least one of the operators $\hat{\rho}_r$ has a nonzero overlap with the kernels of $\hat{\nu}_r$ [35].

The dual problem to Problem $P_G$ can be written as $^1$

$$\text{DP}_G : \text{minimize} \quad \text{Tr} \hat{Z}_G$$

subject to $\hat{Z}_G \geq \lim_{J \to 0} \hat{\rho}_r - \hat{\lambda} \hat{\nu}_r$ ($\forall r \in I_R$),

where $\hat{Z}_G$ is a positive semidefinite operator on $\mathcal{H}$. The optimal values of Problems $P_G$ and $\text{DP}_G$ are the same.

B. Problem of finding optimal unambiguous sequential measurements

Now, let us assume that $\mathcal{H}$ is a bipartite Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let us restrict our attention to a sequential measurement from Alice to Bob. In a sequential measurement, Alice performs a measurement on $\mathcal{H}_A$ and communicates her result to Bob. Then, he performs a measurement on $\mathcal{H}_B$, which can depend on Alice’s outcomes, and obtains the final measurement result. This sequential measurement can be considered from a different point of view [32]. Let $\omega$ be an index associated with Bob’s measurement $\hat{B}(\omega) := \{\hat{B}_r(\omega)\}_{r=0}^R$, and $\omega$ be the entire set of indices $\omega$. Alice performs a measurement, $\hat{A} := \{\hat{A}(\omega)\}_{\omega \in \Omega}$, with continuous outcomes, and sends the result $\omega \in \Omega$ to Bob. Then, he performs the corresponding measurement $\hat{B}(\omega)$, which is uniquely determined by the result $\omega$. This sequential measurement is denoted as $\hat{\Pi}^{(A)} := \{\hat{\Pi}^{(A)}_r\}_{r=0}^R$ with

$$\hat{\Pi}^{(A)}_r := \int_{\hat{A}(\omega)} \hat{B}_r(\omega) \otimes \hat{\nu}_r$$

which is uniquely determined by Alice’s POVM $\hat{A}$.

The problem of finding an unambiguous sequential measurement maximizing the average success probability, which we call an optimal unambiguous sequential measurement or just an optimal sequential measurement, can be formulated as the following optimization problem:

$$P : \text{maximize} \quad P(\hat{\Pi}^{(A)})$$

subject to $\hat{A} \in \mathcal{M}_A$

with Alice’s POVM $\hat{A}$, where $\mathcal{M}_A$ is the entire set of Alice’s continuous measurements $\{\hat{A}(\omega)\}_{\omega \in \Omega}$. Compared to Problem $P_G$, this problem restricts $\hat{\Pi}$ to the form $\hat{\Pi} = \hat{\Pi}^{(A)}$. We can

\[1\] One can obtain this problem from Eq. (12) in Ref. [36] with $M = R + 1$, $J = 0$, $\xi_m = \rho_m - \lambda \xi_m$ ($m < R$), $\xi_R = 0$, and $A \to \infty$. 
easily see that this problem is a convex programming problem and obtain the following dual problem [34]:

\[
\text{DP : minimize } \text{Tr} \hat{X} \text{ subject to } \hat{\Gamma}(\omega, \hat{X}) \geq 0 \quad (\forall \omega \in \Omega)
\]

with a Hermitian operator \( \hat{X} \), where

\[
\hat{\Gamma}(\omega, \hat{X}) := \hat{X} - \lim_{\lambda \to \infty} \sum_{r=0}^{R-1} \text{Tr}_B \left[ (\hat{\rho}_r - \lambda \hat{\nu}_r) \hat{B}_r^{(\omega)} \right].
\]

\( \text{Tr}_B \) is the partial trace over \( \mathcal{H}_B \). The optimal values of Problems P and DP are also the same.

C. Condition for sequential measurement to be globally optimal

Let \( \hat{Z}^*_G \) be an optimal solution to Problem P_G and \( \hat{X}^*_G := \text{Tr}_B \hat{Z}^*_G \). Also, let \( \hat{\Gamma}^*(\omega) := \hat{\Gamma}(\omega, \hat{X}^*_G) \). We now want to know whether a sequential measurement can be globally optimal, i.e., whether an optimal solution to Problem P is also optimal to Problem P_G. To this end, we utilize the following remark:

Remark 1 A sequential measurement \( \hat{\Gamma}^*(\hat{A}(\hat{\omega})) \) is an optimal unambiguous measurement if and only if it satisfies

\[
\hat{\Gamma}^*(\omega) \hat{A}(\omega) = 0, \quad \forall \omega \in \Omega.
\]  

Proof Assume that \( \hat{X}^*_G \) is a feasible solution to Problem DP, i.e., \( \hat{\Gamma}^*(\omega) \geq 0 \) holds for any \( \omega \in \Omega \). It is known that \( \hat{\Gamma}^*(\omega) \) and \( \hat{X}^*_G \) are respectively optimal solutions to Problems P and DP if and only if \( \hat{\Gamma}(\omega, \hat{X}) \geq 0 \) and \( \hat{\Gamma}(\omega, \hat{X}) \hat{A}(\omega) = 0 \) hold for any \( \omega \in \Omega \) (see Theorem 2 of Ref. [34] \(^2\)). Thus, \( \hat{\Gamma}^*(\omega) \) and \( \hat{X}^*_G \) are respectively optimal solutions to Problems P and DP if and only if Eq. (7) holds. If Eq. (7) holds, then, since \( \hat{P}[\hat{\Gamma}^*(\hat{A}(\hat{\omega}))] = \text{Tr} \hat{X}^*_G = \text{Tr} \hat{Z}^*_G \) is equal to the optimal value of Problem P_G, \( \hat{\Gamma}^*(\omega) \) is globally optimal. Therefore, to prove this remark, it suffices to show that \( \hat{X}^*_G \) is a feasible solution to Problem DP.

Multiplying \( [\hat{B}_r^{(\omega)}]^{1/2} \) on both sides of the constraint of Problem DP_G and taking the partial trace over \( \mathcal{H}_B \) gives

\[
\text{Tr}_B \left[ \hat{Z}^*_G \hat{B}_r^{(\omega)} \right] \geq \lim_{\lambda \to \infty} \text{Tr}_B \left[ (\hat{\rho}_r - \lambda \hat{\nu}_r) \hat{B}_r^{(\omega)} \right].
\]

Therefore, we have

\[
\sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{Z}^*_G \hat{B}_r^{(\omega)} \right] \geq \lim_{\lambda \to \infty} \sum_{r=0}^{R-1} \text{Tr}_B \left[ (\hat{\rho}_r - \lambda \hat{\nu}_r) \hat{B}_r^{(\omega)} \right].
\]

Also, from \( \hat{Z}^*_G = \text{Tr}_B \hat{Z}^*_G \), we have

\[
\hat{X}^*_G = \sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{Z}^*_G \hat{B}_r^{(\omega)} \right] \geq \sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{Z}^*_G \hat{B}_r^{(\omega)} \right].
\]

From these equations and Eq. (6), \( \hat{\Gamma}^*(\omega) \geq 0 \) holds for any \( \omega \in \Omega \), and thus \( \hat{X}^*_G \) is a feasible solution to Problem DP.

We will further investigate Alice’s POVM \( \hat{A} \) satisfying Eq. (7). Let

\[
K_\omega := \text{Ker} \left( \sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{\nu}_r \hat{B}_r^{(\omega)} \right] \right).
\]

Let us consider \( |\gamma\rangle \in \text{supp} \hat{A}(\omega) \). Suppose that Eq. (7) holds; then, from Eqs. (6) and (11), \( |\gamma\rangle \in K_\omega \) and \( \hat{P}_\omega \left[ \hat{X}^*_G - \sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{\rho}_r \hat{B}_r^{(\omega)} \right] \right] |\gamma\rangle = 0 \) hold, where \( \hat{P}_\omega \) is the projection operator onto \( K_\omega \). Conversely, if these two equations hold for any \( |\gamma\rangle \in \text{supp} \hat{A}(\omega) \), then Eq. (7) holds. Therefore, Eq. (7) is equivalent to the following equations:

\[
K_\omega := \left\{ \hat{P}_\omega \left[ \hat{X}^*_G - \sum_{r=0}^{R-1} \text{Tr}_B \left[ \hat{\rho}_r \hat{B}_r^{(\omega)} \right] \right] \hat{A}(\omega) = 0 \right\}.
\]

Let us consider the case in which each state \( \hat{\rho}_r \) is separable, i.e., it is in the form of

\[
\hat{\rho}_r = \xi_r \hat{a}_r \otimes \hat{b}_r,
\]

where \( \hat{a}_r \) and \( \hat{b}_r \) are respectively density operators on \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Then, Eq. (6) reduces to

\[
\hat{\Gamma}(\omega; \hat{X}) = \hat{X} - \sum_{r=0}^{R-1} p_r^{(\omega)} \xi_r \hat{b}_r + \lambda \sum_{r=0}^{R-1} e_r^{(\omega)} \xi_r \hat{a}_r,
\]

where \( p_r^{(\omega)} := \text{Tr}[\hat{b}_r \hat{B}_r^{(\omega)}] \) is the probability of Bob correctly identifying the state \( \hat{b}_r \), and \( e_r^{(\omega)} := \sum_{k \in \mathcal{F}_r(\omega)} \text{Tr}[\hat{b}_k \hat{B}_k^{(\omega)}] \) is the probability of Bob misidentifying the state \( \hat{b}_r \). Also, it follows from \( K_\omega = \text{Ker} \sum_{r=0}^{R-1} e_r^{(\omega)} \xi_r \hat{a}_r \), that the first line of Eq. (12) can be expressed as

\[
\text{Tr}[\hat{a}_r \hat{A}(\omega)] = 0, \quad \forall r \in T^{(\omega)},
\]

where \( T^{(\omega)} \) is the entire set of indices \( r \in I_R \) such that Bob’s measurement never gives incorrect results, i.e.,

\[
T^{(\omega)} := \left\{ r \in I_R : e_r^{(\omega)} = 0 \right\}.
\]

Equation (15) implies that, for any \( r \in I_R \) and \( \omega \in \Omega \) such that the state \( \hat{b}_r \) will be incorrectly identified by Bob’s measurement \( \hat{B}_r^{(\omega)} \) (i.e., \( e_r^{(\omega)} \neq 0 \)), Alice’s outcome must not be \( \omega \) for the state \( \hat{a}_r \) (i.e., \( \text{Tr}[\hat{a}_r \hat{A}(\omega)] = 0 \)). Thus, Eq. (15) ensures that the measurement \( \hat{\Gamma}^*(\hat{A}(\hat{\omega})) \) never gives erroneous results.

III. SEQUENTIAL MEASUREMENTS FOR SYMMETRIC TERNARY PURE STATES

Remark 1 is useful in determining whether a sequential measurement can be globally optimal. Concretely, it is possible to decide whether a sequential measurement can be globally optimal by examining whether there exists \( \hat{A} \in M_3 \) satisfying Eq. (7). However, in general, it is quite difficult to examine this for all continuous values \( \omega \in \Omega \). In this section, we consider sequential measurements for bipartite symmetric ternary pure states and derive a formula that can directly
determine whether a sequential measurement can be globally optimal. Extending our results to the multipartite case enables us to obtain a sufficient condition that a sequential measurement can be globally optimal.

A. Main results

Let us consider bipartite ternary separable pure states, \( |Ψ_r⟩ := |a_r⟩ ⊗ |b_r⟩ \) for \( r = 0, 1 \), which are the special case of Eq. (13) with \( a_r = |a_r⟩ \) and \( b_r = |b_r⟩ \). Assume that \(|a_r⟩\) and \(|b_r⟩\) respectively span three-dimensional Hilbert spaces, \( H_A \) and \( H_B \). Also, assume that \(|Ψ_r⟩\) is symmetric in the following sense: the prior probabilities are equal (i.e., \( ξ_r = 1/3 \)) and there exist unitary operators \( V_A \) on \( H_A \) and \( V_B \) on \( H_B \) satisfying

\[
|a_{r0}⟩ = V_A |a_r⟩, \quad |b_{r0}⟩ = V_B |b_r⟩.
\]

where \( ⊕ \) denotes addition modulo 3. These states are characterized by the inner products \( K_A := ⟨a_0|a_r⟩ \) and \( K_B := ⟨b_0|b_r⟩ \), which are generally complex values. For any \( r ∈ I_3 \), we have

\[
⟨a_r|a_{r0}⟩ = K_A, \quad ⟨b_r|b_{r0}⟩ = K_B.
\]

\(|a_r⟩\) and/or \(|b_r⟩\) can be PSK optical coherent states, pulse position modulated (PPM) optical coherent states, and lifted trine states [33]. If \(|a_r⟩\) or \(|b_r⟩\) is mutually orthogonal (i.e., \( K_A = 0 \) or \( K_B = 0 \)), then an optimal sequential measurement perfectly discriminates \(|Ψ_r⟩\), and thus is globally optimal. So, assume that \(|a_r⟩\) and \(|b_r⟩\) are not mutually orthogonal.

We shall present a theorem that can be used to determine whether a sequential measurement can be globally optimal for given bipartite symmetric ternary pure states. Let us consider the following set with seven elements

\[
Ω^* := \{ω_{1,j}, ω_{2,j}, ω_3 : j ∈ I_3\},
\]

where \( I_3 \) is the set of integers from 0 to 2.

1. \( \hat{B}^{(ω_{1,j})} \) is the measurement that always returns \( j \), i.e., \( \hat{B}^{(ω_{1,j})} = δ_{r,j}I_b \), where \( δ_{r,j} \) is the Kronecker delta and \( I_b \) is the identity operator on \( H_B \).
2. \( \hat{B}^{(ω_{2,j})} \) is an optimal unambiguous measurement for binary states \(|b_{r0})⟩, |b_{r2})⟩\) with equal prior probabilities of 1/2.
3. \( \hat{B}^{(ω_3)} \) is an optimal unambiguous measurement for ternary states \(|b_{r0})⟩, |b_{r1})⟩, |b_{r2})⟩\) with equal prior probabilities of 1/3.

For simpler notation, we write \( ω_k \) for \( ω_{k,0} \) for each \( k ∈ \{1, 2\} \).

When a sequential measurement can be globally optimal, there can exist a large (or even infinite) number of optimal sequential measurements. However, as we shall show in the following theorem, if a sequential measurement can be globally optimal, then there always exists an optimal sequential measurement in which Alice never returns an index \( ω \) with \( ω ∈ Ω^* \) (proof in Sec. IV):

**Theorem 2** Suppose that, for bipartite symmetric ternary pure states \(|Ψ_r⟩ := |a_r⟩ ⊗ |b_r⟩\) for \( r = 0, 1 \), a sequential measurement can be globally optimal. Then, there exists an optimal sequential measurement \( \hat{A}^* \) with \( \hat{A}^* ∈ M_Ω^* \) such that

\[
\hat{A}^*(ω) = 0, \quad ∀ω ∈ Ω^*.
\]

The measurement \( \hat{A}^* \) is schematically illustrated in Fig. 1. Due to the definition of \( ω_{1,j}, ω_{2,j}, ω_3 ∈ Ω^*, T(ω) \), defined by Eq. (16), satisfies \( T(ω_{1,j}) = |j⟩, T(ω_{2,j}) = |j ⊕ 1, j ⊕ 2⟩, \) and \( T(ω_3) = |0, 1, 2⟩ \). From Eq. (15), \( Tr[\hat{A}^* T(ω_{1,j})] = 0 \) must hold for any distinct \( r,j ∈ I_3 \). Thus, if Alice returns the index \( ω_{1,j} \), then the given state must be \(|Ψ_r⟩\). (In this case, the given state is uniquely determined before Bob performs the measurement.) Also, from Eq. (15), \( Tr[\hat{A}^* T(ω_{2,j})] = 0 \) holds for any \( j ∈ I_3 \), which indicates that if Alice returns the index \( ω_{2,j} \), then the state \(|Ψ_r⟩\) is unambiguously filtered out. In this case, Alice’s measurement result does not indicate which of the two states \(|Ψ_{r0})⟩\) and \(|Ψ_{r2})⟩\) is given. If Alice returns the index \( ω_3 \), then Alice’s result provides no information about the given state.

Using Theorem 2, we can derive a simple formula for determining whether a sequential measurement can be globally optimal. Before we state this formula, we shall give some preliminaries. Let \( τ := \exp(i2π/3) \), where \( i := √−1 \). Also, let \(|ϕ_n⟩\) and \(|ϕ'_n⟩\), respectively, denote the normalized eigenvectors corresponding to the eigenvalues \( τ^n \) \( (n ∈ I_3) \) of \( V_A \) and \( V_B \). Moreover, let

\[
x_n := |⟨ϕ_n|a_0⟩|, \quad y_n := |⟨ϕ'_n|b_0⟩|.
\]

(21)

Note that \( x_n, y_n > 0 \) holds for any \( n ∈ I_3 \). By selecting appropriate global phases of \(|a_r⟩\) and \(|b_r⟩\) and permuting \(|Ψ_1⟩\) and \(|Ψ_2⟩\) if necessary, we may assume

\[
x_0 > x_1, \quad x_1 ≥ x_2, \quad y_0 ≥ y_1 ≥ y_2, \quad y_0 ≠ y_2.
\]

(22)

Also, by selecting global phases of \(|ϕ_n⟩\) and \(|ϕ'_n⟩\) such that \( ⟨ϕ_n|a_0⟩ \) and \( ⟨ϕ'_n|b_0⟩ \) are positive real numbers, \(|a_r⟩\) and \(|b_r⟩\) are written as

\[
|a_r⟩ = ∑_{n=0}^{2} x_n τ^n |ϕ_n⟩, \quad |b_r⟩ = ∑_{n=0}^{2} y_n τ^n |ϕ'_n⟩.
\]

(23)

For simpleness, let \( x_n \) and \( y_n \) be uniquely determined by \( K_A \) and \( K_B \). Let \( K'_A := ⟨a_0|a_1⟩ \) and \( K'_B := ⟨b_0|b_1⟩ \), where \(|a_r⟩\) and \(|b_r⟩\) are expressed by Eq. (22) and (23); then, we have

\[
x_n = \sqrt{1 + τ^2 K'_A + τ^4(K'_A)^3}, \quad y_n = \sqrt{1 + τ^2 K'_B + τ^4(K'_B)^3},
\]

(24)

FIG. 1. Schematic diagram of an optimal sequential measurement \( \hat{A}^* \).
where * designates complex conjugate. Let \( \eta := (1 - |K_B|)/3 \). Note that \( 3\eta = 1 - |K_B| \) equals the average success probability of the optimal unambiguous measurement \( B^{(\omega)} \) for binary states \((|b_1\rangle, |b_2\rangle)\) with equal prior probabilities of \(1/2\).

We get the following corollary (proof in Appendix A):

**Corollary 3** For bipartite symmetric ternary pure states \((|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle)\) expressed by Eqs. (22) and (23), the following two statements are equivalent.

1. A sequential measurement can be globally optimal.
2. Either \( y_1 = y_2 \) or
   \[
   x_{20} - x_1 z_1 \geq 0,
   \]
   \[
   \sum_{k=0}^2 x_k^2 (z_{10k} - z_{20k})^2 \geq 0
   \]
   \[
   (25)
   \]
   holds, where \( z_k := \sqrt{\eta^2 - \eta} \) and \( \oplus \) denotes subtraction modulo 3.

Using this corollary, we can easily judge whether a sequential measurement can be globally optimal for bipartite symmetric ternary pure states.

**B. Extension to multipartite states**

We can extend the above results to multipartite states. As a simple example, we consider tripartite symmetric ternary pure states \((|\Psi_r\rangle = |a_r\rangle \otimes |b_r\rangle \otimes |c_r\rangle)\) with equal prior probabilities. There exist unitary operators \( \hat{V}_A, \hat{V}_B, \) and \( \hat{V}_C \) on \( H_A, H_B, \) and \( H_C \) satisfying Eq. (17) and \( |c_r\rangle = \hat{V}_C |c_r\rangle \).

Here, we consider the composite system of \( H_A \otimes H_B \otimes H_C \), \( H_B \otimes H_C \), and interpret these states as bipartite states \((|\Psi_r\rangle = |a_r\rangle \otimes |b_r\rangle)\) in \( H_B \otimes H_C \). It is obvious that a sequential measurement can be globally optimal for the tripartite states, then it is also true for the bipartite states. Assume that it is true for the bipartite states; then, from Theorem 2, there exists a sequential measurement \( \hat{V}^{(\omega)} \) satisfying Eq. (20), which is globally optimal. It follows that \( \hat{V}^{(\omega)} \) can be realized by a sequential measurement on the tripartite system \( H_A \otimes H_B \otimes H_C \) if and only if, for any \( \omega \in \Omega^* \), the measurement \( B^{(\omega)} \) on \( H_B \otimes H_C \) can be realized by a sequential measurement on \( H_B \otimes H_C \).

By using Corollary 3, one can easily judge whether a sequential measurement for the bipartite states \((|\psi_r^{(0)}\rangle \otimes |\psi_r^{(0)}\rangle)\) can be globally optimal. Note that the above sufficient condition may not be necessary. For example, let us again consider the tripartite states \((|a_r\rangle \otimes |b_r\rangle \otimes |c_r\rangle)\). For an optimal sequential measurement for these states to be globally optimal, it is sufficient that there exists a globally optimal sequential measurement \( \hat{V}^{(\omega)} \) for the bipartite states \((|a_r\rangle \otimes |b_r\rangle)\), such that the measurement \( B^{(\omega)} \) can be realized by a sequential measurement on the bipartite system \( H_B \otimes H_C \) for any \( \omega \) with \( A(\omega) \neq 0 \), where \( A \) can be different from \( A^* \).

**IV. PROOF OF THEOREM 2**

We now prove Theorem 2 using Remark 1. We provide an overview of our proof in this section, leaving some technical details to the appendix. As a starting point, we first obtain \( \bar{X}_G^* \) in Sec. IV A. Next, in Sec. IV B, we consider the case \( y_1 = y_2 \). After that, we consider the case \( y_1 \neq y_2 \). A sufficient condition for this theorem to hold and its reformulation are given in Secs. IV C and IV D, respectively. In Sec. IV E, we prove that this sufficient condition holds.

**A. Derivation of \( \bar{X}_G^* \)**

Let us consider \( |a_r\rangle \otimes |b_r\rangle \) in the form of Eqs. (22) and (23). A simple calculation gives

\[
|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle = \sum_{n=0}^2 \bar{x}_n r^n |\bar{\phi}_n\rangle \, ,
\]

where

\[
|\bar{\phi}_n\rangle := \frac{1}{\bar{x}_n} \sum_{k=0}^2 x_k y_{nk} |\phi_k\rangle \otimes |\phi'_{nk}\rangle \, ,
\]

\[
\bar{x}_n := \left| \sum_{k=0}^2 x_k^2 y_{nk} \right|^{1/2} \, .
\]

Obviously, \( |\bar{\phi}_n\rangle \) is an orthonormal basis and \( \bar{x}_n \) is positive real. We have

\[
\bar{x}_{1}^2 - \bar{x}_{2}^2 = \bar{x}_{1}^2 \left( x_0 + y_0 \right) \left( x_0 + y_0 \right) - \bar{x}_{1}^2 \left( x_0 - y_0 \right) \left( x_0 - y_0 \right) \geq 0 \, ,
\]

(28)
where the inequality follows from Eq. (22). Thus, \( \hat{x}_1 \geq \hat{x}_2 \) holds. Note that whether \( \hat{x}_0 \geq \hat{x}_2 \) or not depends on given states. Let

\[
[\nu_0, \nu_1, \nu_2] := \begin{cases} 
[2, 1, 0], & \hat{x}_0 \geq \hat{x}_2, \\
[0, 2, 1], & \text{otherwise};
\end{cases}
\tag{29}
\]

then, \( \hat{x}_1 \geq \hat{x}_\nu \) and \( \hat{x}_\nu \geq \hat{x}_0 \) hold. We can easily see that an optimal solution to Problem DP \( G \) is given by (see Appendix B)

\[
\hat{Z}^*_{c} = 3\hat{x}_\nu^2 (|\phi_{\nu}\rangle \langle \phi_{\nu}|). 
\tag{30}
\]

From Eq. (30), we have

\[
\hat{X}_G = \text{Tr}_B \hat{Z}^*_{c} = 3 \sum_{n=0}^{2} \hat{x}_n^2 \langle \phi_{n}| \langle \phi_{n}|. 
\tag{31}
\]

### B. Case of \( y_1 = y_2 \)

We here show that, in the case of \( y_1 = y_2 \) (i.e., \( K_B \) is positive real), there exists a globally optimal sequential measurement \( \hat{\Pi}(\hat{A}^*) \) satisfying Eq. (20). Let

\[
\hat{A}^*(\omega) := \begin{cases} 
\hat{A}_r, & \omega = \omega_{1,r} (r \in I_3), \\
\hat{A}_3, & \omega = \omega_3, \\
0, & \text{otherwise};
\end{cases}
\tag{32}
\]

where \( \{\hat{A}_r\}_{r=1}^3 \) is an optimal unambiguous measurement for \( \{|a_r\}\) with equal prior probabilities. Obviously, \( \hat{A}^* \) is in \( M_A \) and satisfies Eq. (20). It follows that the sequential measurement \( \hat{\Pi}(\hat{A}^*) \) can be interpreted as follows: Alice and Bob respectively perform optimal measurements for \( \{|a_r\}\) and \( \{|b_j\}\) with equal prior probabilities and get the results, \( r_A \) and \( r_B \). \( \hat{\Pi}(\hat{A}^*) \) returns \( r_A \) if \( r_A \in I_3 \), \( r_B \) if \( r_B \in I_3 \), and \( r = 3 \) otherwise. Note that \( r_A = r_B \) holds whenever \( r_A \) and \( r_B \) are in \( I_3 \).

The average success probabilities of optimal measurements for \( \{|a_r\}\) and \( \{|b_j\}\) with equal prior probabilities are respectively \( P_A := 3\hat{x}_2^2 \) and \( P_B := 3\hat{y}_2^2 \) (see Theorem 4 of Ref. [37]); thus, the average success probability of \( \hat{\Pi}(\hat{A}^*) \) is

\[
1 - (1 - P_A)(1 - P_B) = 3(\hat{x}_2^2 + \hat{y}_2^2 - 3\hat{x}_2\hat{y}_2^2). 
\tag{33}
\]

On the other hand, that of an optimal measurement for \( \{|\Psi_r\rangle\} \) with equal prior probabilities is given by

\[
3\hat{x}_\nu^2 = 3\hat{x}_2^2 = 3 \left[ (1 - \hat{x}_2^2)\hat{y}_2^2 + \hat{x}_2^2(1 - 2\hat{y}_2^2) \right] 
= 3(\hat{x}_2^2 + \hat{y}_2^2 - 3\hat{x}_2\hat{y}_2^2),
\tag{34}
\]

where the first equality follows from the fact that \( \hat{x}_0 \geq \hat{x}_2 \) (i.e., \( \nu_0 = 2 \)) holds when \( y_1 = y_2 \), and the second equality follows from the definition of \( \hat{x}_2 \). Thus, \( \hat{\Pi}(\hat{A}^*) \) is globally optimal.

We should note that the same discussion is applicable to the case of \( x_1 = x_2 \) (i.e., \( K_A \) is positive real); in this case, there also exists a globally optimal sequential measurement \( \hat{\Pi}(\hat{A}^*) \) satisfying Eq. (20).

### C. Sufficient condition for Theorem 2

Since we have already proved the theorem in the case of \( y_1 = y_2 \), in what follows, we only consider the case \( y_1 \neq y_2 \). (We do not have to assume \( x_1 \neq x_2 \); the following proof is also valid for \( x_1 = x_2 \).) After some algebra using Eq. (31), we get from Eq. (11) \(^3\), we set \( \Gamma^*(\omega) = 2 \) for any \( \omega \in \Omega^* \) and \( \hat{\Pi}(\omega) |\pi^*_{\omega} \rangle = 0 \), where \( |\pi^*_{\omega} \rangle \in \mathcal{H}_A (\omega \in \Omega^*) \) is the normal vector defined as

\[
|\pi^*_{\omega} \rangle := \begin{cases} 
C_1 \hat{V}_A^j \sum_{n=0}^{2} \hat{x}_n^{-1} |\phi_n\rangle, & \omega = \omega_{1,j}, \\
C_2 \hat{V}_A^j \sum_{n=0}^{2} \hat{x}_n^{-1} \hat{y}_n^{-1} |\phi_n\rangle, & \omega = \omega_{2,j}, \\
0, & \omega = \omega_3,
\end{cases}
\tag{35}
\]

and \( C_1 \) and \( C_2 \) are normalization constants. Thus, it follows that \( \hat{A}^* \) satisfies Eq. (20) and Eq. (7) with \( \hat{A} = \hat{A}^* \) if only if \( \hat{A}^* \) is expressed as

\[
\hat{A}^*(\omega) := \begin{cases} 
\kappa_{\omega}^* |\pi^*_{\omega} \rangle \langle \pi^*_{\omega}|, & \omega \in \Omega^*, \\
0, & \text{otherwise},
\end{cases}
\tag{36}
\]

where, for each \( \omega \in \Omega^* \), \( \kappa_{\omega}^* \) is a nonnegative real number. Therefore, from Remark 1, to prove that \( \hat{\Pi}(\hat{A}^*) \) is an optimal measurement, it suffices to show that there exists Alice’s POVM \( \hat{A}^* \) (i.e., \( \hat{A}^* \in M_A \) in the form of Eq. (36).

Let \( \hat{A} \) be an optimal solution to Problem P. Due to the symmetry of the states, we assume without loss of generality that \( \hat{A} \) is symmetric in the following sense: for any \( \omega \in \Omega \), \( \hat{A}(\omega') = \hat{V}_A \hat{A}(\omega) \hat{V}_A^\dagger \) and \( \hat{A}(\omega'') = \hat{V}_A \hat{A}(\omega) \hat{V}_A^\dagger \), where \( \omega', \omega'' \in \Omega \) are the indices such that \( \tilde{B}_{r_0}^{(\omega')} = \hat{V}_B r_{r_0}^{(\omega')} \tilde{B}_B \) and \( \tilde{B}_{r_0}^{(\omega'')} = \hat{V}_B r_{r_0}^{(\omega'')} \tilde{B}_B \) for any \( r \in I_3 \) (see Theorem 4 of Ref. [34] in detail).

Let

\[
\hat{S} (\hat{T}) := \frac{1}{3} \sum_{k=0}^{2} \hat{V}_B^k \hat{T} (\hat{V}_A^k)^\dagger, 
\tag{37}
\]

where \( \hat{T} \) is a positive semidefinite operator on \( \mathcal{H}_A \). \( \hat{S} (\hat{T}) \) is also a positive semidefinite operator on \( \mathcal{H}_A \) satisfying \( \text{Tr}[\hat{S}(\hat{T})] = \text{Tr} \hat{T} \) and commuting with \( \hat{V}_A \). For notational simplicity, we denote \( \hat{S}(\hat{A}(\omega)) \) by \( \hat{S}(\omega) \). Due to the symmetry of \( \hat{A} \), \( \hat{S}(\omega) = \hat{S}(\omega') = \hat{S}(\omega'') \) holds. Let

\[
\hat{E}_k := \hat{S}(\kappa_{\omega}^* |\pi^*_{\omega} \rangle \langle \pi^*_{\omega}|), \ k \in \{1, 2, 3\};
\tag{38}
\]

then, Eqs. (35) and (37) give

\[
\sum_{j=0}^{2} |\pi^*_{\omega_j} \rangle \langle \pi^*_{\omega_j}| = 3\hat{E}_3^*, \ k \in \{1, 2\},
\tag{39}
\]

\[
|\pi^*_{\omega_j} \rangle \langle \pi^*_{\omega_j}| = \hat{E}_3^*.
\tag{39}
\]

\(^3\) We also use the fact that \( \rho_{j_0}^{(\omega_j)} = 1, \rho_{j_0}^{(\omega_2)} = \rho_{j_2}^{(\omega_2)} = 3\pi, \) and \( \rho_{\omega_3} = 3\pi^2 \).
Here, assume that \( \hat{S}(\omega) \) can be expressed as
\[
\hat{S}(\omega) = \sum_{k=1}^{3} w_{\omega,k} \hat{E}_k, \quad \forall \omega \in \Omega_+,
\]
\[
\text{where } w_{\omega,k} \geq 0, \quad \forall \omega \in \Omega_+, k \in \{1, 2, 3\},
\]
which is the point in the \( s - \)plane that corresponds to \( \hat{E}_k \). The point in the \( s - \)plane is given by Eq. (40). For any positive semidefinite operator \( T \neq 0 \), \( s_n(T) \) is defined as follows:
\[
s_n(T) := \left( \phi_n, \frac{\hat{S}(T)}{\text{Tr}[\hat{S}(T)]} \right) \theta_n.
\]
(44)

From \( \sum_{n=1}^{\infty} \left( \phi_n, \hat{S}(T) \right) \theta_n = \text{Tr}[\hat{S}(T)] \), \( \sum_{n=1}^{\infty} s_n(T) = 1 \) holds. Let us consider the following point:
\[
s(T) := [s_n(T), s_0(T)],
\]
(45)
which is in a two-dimensional space (we call it the \( s - \)plane). Let us show that \( s(T) \) is in the first quadrant of the \( s - \)plane. We can easily verify that the point \( s(T) \) has a one-to-one correspondence with \( \hat{S}(T)/\text{Tr}[\hat{S}(T)] \). Let
\[
\epsilon^*_k := s \left( \left\{ \pi^*_n \right\} \right), \quad k \in \{1, 2, 3\},
\]
(46)
which is the point in the \( s - \)plane that corresponds to \( \hat{E}_k \). Defined by Eq. (38), \( \epsilon^*_k = [0, 0] \) holds from Eq. (35). Also, let \( T^\ast \) be the triangle formed by \( \epsilon^*_1 \), \( \epsilon^*_2 \), and \( \epsilon^*_3 \). Note that \( T^\ast \) may degenerate to a straight line segment in special cases. For simplicity, we denote \( s_n(\omega) := s_n(\hat{A}(\omega)) \) and \( s(\omega) := s(\hat{A}(\omega))/\text{Tr}[\hat{S}(\omega)] \). From the first line of Eq. (40), we have
\[
s(\omega) = \frac{1}{\text{Tr}[\hat{S}(\omega)]} \sum_{k=1}^{3} w_{\omega,k} \epsilon^*_k.
\]
(47)

Thus, it follows that Eq. (40) is equivalent to the following:
\[
s(\omega) \in T^\ast, \quad \forall \omega \in \Omega_+.
\]
(48)

![Figure 2](image.png)

**Figure 2.** \( \hat{S}(\omega) \) in the case of \( K_A = K_B = 0.2 \exp(\pi i/10) \). The entire sets of points \( s(\omega) \) \( (\omega \in \Omega_+) \) with \( |T(\omega)| = 2 \) and 3, denoted by \( D_2 \) and \( D_3 \), are depicted by the green and blue regions in this figure, respectively. Also, \( s(\omega) = e^*_1 \) returns when \( \omega \) satisfies \( |T(\omega)| = 1 \). Indeed, in this case, since \( T(\omega) = \{j\} \) holds for certain \( j \in I_1 \), we can easily see that \( \hat{A}(\omega) \propto \{\pi^*_n, \langle \pi^*_n \rangle \} \) must hold from Eq. (15), which gives \( s(\omega) = e^*_1 \). The triangle \( T^\ast \) is also shown in the dashed line in Fig. 2. One case to see that \( e^*_1, D_2, \) and \( D_3 \) are all included in \( T^\ast \). Note that we can show, under the assumption that Theorem 2 holds, that a sequential measurement can be globally optimal if and only if \( s(\hat{1}_A) = (1/3, 1/3) \) \( \in T^\ast \) holds (see Appendix C). In the case shown in Fig. 2, \( s(\hat{1}_A) \in T^\ast \), and thus a sequential measurement can be globally optimal.

**E. Proof of Eq. (48)**

We shall prove that Eq. (48), which is a sufficient condition of Theorem 2, holds. \( \hat{S}^\ast \) can be rewritten as the following form:
\[
\hat{S}^\ast = S^\ast G - \sum_{r=0}^{\infty} \frac{\mu_r^\omega}{\mu_r^\omega} |a_r \rangle \langle a_r|.
\]
(49)

From Eq. (14), \( \mu_r^\omega = -\infty \) if \( e_r^\omega \neq 0 \); otherwise, \( \mu_r^\omega = \mu_r^\omega/3 \) holds. It is very hard to show in a naive way that each \( s(\omega) \) with \( \omega \in \Omega_+ \) is included in \( T^\ast \). However, we can rather easily show that Eq. (48) holds by considering the following...
two cases: (1) the case in which at least two of \(\mu_{(o)}^i\), are the same and (2) the other case in which \(\mu_{(o)}^i\), are all different.

Case (1): at least two of \(\mu_{(o)}^i\), are the same

Due to the symmetry of the states, we assume \(\mu_{(o)}^1 = \mu_{(o)}^2 \); where \(q\) without loss of generality; then, \(\hat{\Gamma}^\ast(\omega)\) can be expressed as

\[
\hat{\Gamma}^\ast(\omega) = \mathcal{X}_G - q\hat{\Psi} - p|a_0\rangle\langle a_0|,
\]

where \(\hat{\Psi} := \sum_{k=0}^{\eta} |a_k\rangle\langle a_k|\) and \(p\) is a real number. If \(|T^{(o)}| = 3\), then \(q = p_{(o)}^1/3 = p_{(o)}^2/3\) holds to satisfy Eq. (50). Also, in this case, we can easily see that \(p_{(o)}^1 = p_{(o)}^2 \leq p_{(o)}^1 = p_{(o)}^2 = 3\eta\) holds, which gives \(0 \leq q \leq \eta\). Moreover, \(q = -\infty\) holds if \(|T^{(o)}| = 1\), and \(q = \eta\) holds if \(|T^{(o)}| = 2\). Thus, \(q \leq \eta\) always holds.

For each \(q \leq \eta\), let \(|\gamma_q\rangle\) be a normal vector satisfying

\[
|\gamma_q\rangle \in \text{Ker} \hat{\Gamma}^\ast(q),
\]

\[
\hat{\Gamma}^\ast(q) := \mathcal{X}_G - q\hat{\Psi} - p_q|a_0\rangle\langle a_0|,
\]

\[
\hat{\gamma}^\ast(q) = \frac{1}{\sqrt{3}} \sum_{n=0}^{2} \frac{1}{\sqrt{\nu_n}} (|\psi_n\rangle - q|\phi_n\rangle), \quad q \neq \gamma_0^2,
\]

\[
|\gamma_q\rangle = \left\{ \begin{array}{ll}
C_q \sum_{n=0}^{2} \frac{1}{\sqrt{\nu_n}} (|\psi_n\rangle - q|\phi_n\rangle), & q \neq \gamma_0^2, \\
|\phi_1\rangle, & \text{otherwise},
\end{array} \right.
\]

where \(C_q\) is a normalization constant. Let \(C\) be the set defined as

\[
C := \{ s|\gamma_q\rangle \langle \gamma_q| : q \leq \eta\}.
\]

In Fig. 2, C is shown in the blue dotted line. Since \(\hat{\Gamma}^\ast(\omega)\) is in the form of Eq. (50) satisfying rank \(\hat{\Gamma}^\ast(\omega) < 3\), \(\hat{\Gamma}^\ast(\omega)\) is equivalent to \(\hat{\Gamma}^\ast(\mu_{(o)}^1)\). Thus, \(\hat{\Lambda}(\omega) \propto |\gamma_{\mu_{(o)}^1}\rangle \langle \gamma_{\mu_{(o)}^1}|\) holds, which yields \(s(\omega) \in C\). Therefore, to prove \(s(\omega) \in T^\ast\), it suffices to show \(C \subseteq T^\ast\). We can prove this using Eq. (52) (see Appendix D 3).

Case (2): \(\mu_{(o)}^i\), are all different

In this case, we can show that each \(s(\omega)\) is on a straight line segment whose endpoints are in \(C\) (see Appendix D 5). Since \(C \subseteq T^\ast\) holds, such a line segment is in the triangle \(T^\ast\). Therefore, \(s(\omega)\) is in \(T^\ast\) holds.

The two cases (1) and (2) exhaust all possibilities; thus, from the above arguments, Eq. (48) holds, and thus we complete the proof.

V. EXAMPLES

In this section, we present some examples of symmetric ternary pure states in which a sequential measurement can be globally optimal. In Secs. V A and V B, we consider the bipartite case. In Sec. V C, we consider the multipartite case.

A. Case of \(K_A = K_B\)

We first give some examples of bipartite states \(|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\), with \(K_A = K_B = K\). Note that when \(|a_r\rangle\) and \(|b_r\rangle\) are in the form of Eqs. (22) and (23), \(x_n = y_n\) holds for each \(n \in I_3\), and thus \(x_0 \geq x_1\) holds from \(y_0 \geq y_1\).

The region of the complex plane where a sequential measurement for the states \(|\Psi_r\rangle\) with \(K_A = K_B = K\) can be globally optimal is shown in red in Fig. 3. This region is easily obtained from Corollary 3. The horizontal and vertical directions are the real and imaginary axes, respectively. The region of all possible \(K\) is represented as the dotted equilateral triangle. This figure implies that, at least in the case of \(K_A = K_B\), a sequential measurement can be globally optimal in quite a few cases.

As a concrete example, let us consider the symmetric ternary pure states in which \(|a_r\rangle\) and \(|b_r\rangle\) are the lifted trine states, \(|L_r\rangle\) which are expressed by \[|L_r\rangle = \sqrt{1 - g} \left( \cos \frac{2\pi}{3} |u_0\rangle + \sin \frac{2\pi}{3} |u_1\rangle \right) + \sqrt{g} |u_2\rangle, \]

where \(|u_\eta\rangle\) is an orthonormal basis. The real parameter \(g\) is in the range \(0 < g < 1\). Equation (54) gives \(K = (3g - 1)/2\), and thus \(K\) is real in the range \(-1/2 < K < 1\). It follows that the states \(|\Psi_r\rangle = |L_r\rangle \otimes |L_r\rangle\), are also regarded as lifted trine states. The region of possible values of \(K\) is shown in dashed green line in Fig. 3. From this figure, a sequential measurement for \(|\Psi_r\rangle\) can be globally optimal if and only if \(K \geq 0\) (i.e., \(g \geq 1/3\)).

Another example is the states in which \(|a_r\rangle\) and \(|b_r\rangle\) are the ternary pure states \(|\alpha_r\rangle\) and \(|\alpha_r\rangle\), where \(|\alpha_r\rangle\) is a normalized eigenvector of the photon annihilation operator with the eigenvalue \(\alpha_r := \sqrt{\gamma^2}\), and \(S = |\alpha_r|^2\) is the average photon number of \(|\alpha_r\rangle\). In this case, the states \(|\Psi_r\rangle = |\alpha_r\rangle \otimes |\alpha_r\rangle\), are also regarded as the ternary PSK optical coherent states with the average photon number \(2S\). We have

\[
K = |\alpha_0|\alpha_1 = e^{-\frac{2\pi}{3} S} e^{-\frac{2\pi}{3} \frac{2\pi}{3}}
\]

for some global phase. The solid blue line in Fig. 3 displays the region of possible values of \(K\). It follows from Eq. (55) that \(\arg K = \frac{\sqrt{3}}{\sqrt{3}} S\) is proportional to \(S\). From Fig. 3, a necessary and sufficient condition that a sequential measurement for \(|\Psi_r\rangle\) can be globally optimal is \(2\pi k/3 \leq \arg K + \pi/6 \leq 2\pi k/3 + \pi/3\), i.e.,

\[
\frac{(4k - 1)\pi}{3\sqrt{3}} \leq S \leq \frac{(4k + 1)\pi}{3\sqrt{3}}, \quad k \in \{0, 1, 2, \cdots\}.
\]

The average success probability of an optimal sequential measurement for \(|\Psi_r\rangle = |\alpha_r\rangle \otimes |\alpha_r\rangle\), is plotted in solid blue line in Fig. 4. Also, that of an optimal measurement is shown in dashed black line. These probabilities can be numerically computed using a modified version of the method given in Ref. [32]. The region of \(S\) satisfying Eq. (56), in which a sequential measurement can be globally optimal, is shown in red. It is worth mentioning that, as shown in Fig. 2 of
Ref. [32], in the strategy for minimum-error discrimination, an optimal sequential measurement for the ternary PSK optical coherent states is unlikely to be globally optimal, at least when $S$ is small. In the strategy for unambiguous discrimination, a sequential measurement can be globally optimal if (and only if) $S$ satisfies Eq. (56).

\[\Phi_{\Lambda} = \{a_t\} \otimes \{b_r\}\]

is also the PSK optical coherent state, whose average photon number is $S_A$. Note that $n$ identical copies of $|\alpha_t\rangle$ are regarded as $|\alpha_t\rangle$ whose average photon number is $S$ (i.e., $|\alpha_t\rangle = |\alpha_t\rangle^{\otimes N}$). We here want to know...
whether a Dolinar-like receiver can be globally optimal. We consider the bipartite ternary states \(|\alpha_2\rangle = |\alpha_2\rangle \otimes |b_i\rangle\rangle\), where

\[ |\alpha_2\rangle = |\sqrt{t} \alpha_2\rangle = |\alpha_2\rangle^{\otimes N} \] 

and \(|b_i\rangle = |\sqrt{1-t} \alpha_i\rangle = |\alpha_i\rangle^{\otimes (1-t)N} \) with \(0 < t < 1\) are optical coherent states with average photon numbers \(tS\) and \((1-t)S\), respectively. The average success probability of an optimal sequential measurement for the bipartite states with any \(t\) is an upper bound on that of an optimal sequential measurement for \(N\)-partite states \(|\alpha_2\rangle^{\otimes N}\) with \(N \to \infty\), and thus is an upper bound on that of a Dolinar-like receiver. We here show that there exists \(t\) such that an optimal sequential measurement for the corresponding bipartite states \(|\alpha_2\rangle \otimes |b_i\rangle\rangle\) is not globally optimal, which means that a Dolinar-like receiver cannot be globally optimal. In the case in which \(\langle \alpha_0|\alpha_1\rangle\) is nonnegative real (i.e., \(S = 4\pi k/\sqrt{3}\) with \(k = 1, 2, \cdots\)), we choose \(t = 1/2\); then, from Eq. \((55)\), \(K_A = K_B\) and arg \(K_A = \pi\) holds, and thus a sequential measurement cannot be globally optimal, as already shown in Fig. 3. In the other case, we choose \(t \to 0\); formulating \(|\alpha_2\rangle\rangle\) and \(|b_i\rangle\rangle\) in the form of Eqs. \((22)\) and \((23)\), we have that for each \(k \in \{1, 2\}\)

\[
x^2_k = \frac{1}{3} + \frac{2}{3} e^{-\frac{2\pi}{3} \cos \left[\frac{2\pi}{3} + \sqrt{3}tS\right].} \tag{58}
\]

Taking the limit of \(t \to 0\), we obtain \(x_2/x_1 \to 0\). From Corollary 3, it is necessary to satisfy \(x_2z_0 - x_1z_1 \geq 0\) for a sequential measurement to be able to be globally optimal. When \(t \to 0\), from \(x_2/x_1 \to 0\), \(z_1 \to 0\) must hold. However, \(z_1\) converges to a positive number. \(z_1 \to 0\) holds only if \(\langle b_0|b_1\rangle\) converges to a nonnegative real number, i.e., \(y_1 - y_2 \to 0\); however, \(\langle b_0|b_j\rangle\) converges to \(\langle \alpha_0|\alpha_1\rangle\) (which is not a nonnegative real number.) Therefore, a Dolinar-like receiver cannot be globally optimal for any ternary PSK optical coherent states.

C. Case of multipartite states

As an example of multipartite states, let us address the problem of multiple-copy state discrimination \([13, 16, 39-41]\). We again consider \(N\)-partite ternary PSK optical coherent states \(|\alpha_2\rangle^{\otimes N}\) (\(|\alpha_2\rangle := |\alpha_2\rangle/\sqrt{N}\)). As described in Sec. V B, in the limit of \(N \to \infty\), a sequential measurement cannot be globally optimal. In this section, we consider \(N\) to be finite.

By using Corollaries 3 and 4, we can judge whether a sequential measurement can be globally optimal. The region of the average photon number \(S\) of \(|\alpha_2\rangle\) for which the sufficient condition holds is shown in red in Fig. 6. We here consider the range \(S \leq 1.3\). We can see that this figure that a sequential measurement can be globally optimal even for large \(N\) (such as \(N = 20\)) if \(S\) is sufficiently small (such as \(S \leq 0.1\)).

VI. CONCLUSION

An unambiguous sequential measurement for bipartite symmetric ternary pure states has been investigated. We have shown that a certain type of sequential measurement can always be globally optimal whenever there exists a globally optimal sequential measurement. From this result, we have derived a formula that can easily determine whether an optimal sequential measurement is globally optimal. Moreover, our results have been extended to multipartite states and have given a sufficient condition that a sequential measurement can be globally optimal.

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Appendix A: Proof of Corollary 3

Since, as already described in Sec. IV B, a sequential measurement can be globally optimal when \(y_1 = y_2\), we only have to consider the case \(y_1 \neq y_2\).

(1) ⇒ (2): From the discussion in Sec. IV C, there exists an optimal solution, \(\hat{A}^*\), to Problem P that is expressed by Eq. \((36)\) with \(k_{\omega_j}^*\) \((k \in \{1, 2, 3\})\) independent of \(j \in I_3\). Since \(\hat{A}^*\) is a POVM, we have

\[
\sum_{j=0}^{2} [\hat{A}^*(\omega_{1,j}) + \hat{A}^*(\omega_{2,j})] + \hat{A}^*(\omega_{3}) = \hat{I}_A. \tag{A1}
\]

Substituting Eqs. \((35)\) and \((36)\) into Eq. \((A1)\) gives

\[
\begin{bmatrix}
\begin{array}{ccc}
\nu_{0}^* & \nu_{2}^* & 0 \\
\nu_{0}^* & \nu_{2}^* & 0 \\
\nu_{0}^* & \nu_{2}^* & 1
\end{array}
\end{bmatrix}
\begin{bmatrix}
3k_{\omega_j}^* |C_{1}\rangle^2 \\
3k_{\omega_j}^* |C_{2}\rangle^2 \\
k_{\omega_j}^*
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \tag{A2}
\]

where we use \(\nu_{\omega} = k\) \((k \in I_3)\), which follows from Eq. \((29)\). After some algebra, we can see that Eq. \((25)\) must hold if and only if there exists \(k_{\omega_j}^* \geq 0\) satisfying Eq. \((A2)\).

(2) ⇒ (1): Let \(k_{\omega_j}^*\) be the solution to Eq. \((A2)\); then, \(\hat{A}^*\) defined by Eq. \((36)\) is a POVM. Since, as already described in Sec. IV C, \(\hat{G}^*(\omega) |\pi^*\rangle = 0\) holds for any \(\omega \in \Omega^*\), Eq. \((7)\)
with $\hat{A} = \hat{A}^*$ obviously holds. Therefore, from Remark 1, the sequential measurement $\Pi(\hat{A}^*)$ is globally optimal.

Note that one can obtain an analytical expression of $\hat{A}^*$ by substituting the solution $\kappa_{*n}^\gamma$ to Eq. (A2) into Eq. (35).

**Appendix B: Deriving of Eq. (30)**

From Theorem 4 of Ref. [37], since the states $|\Psi_r\rangle$ with equal prior probabilities are geometrically uniform states, the equal-probability measurement $\Pi^* := \Pi|\Psi_r\rangle\langle\Psi_r|$, given by

$$\Pi^* = |\pi_0^*\rangle\langle\pi_0^*|, \quad r \in I_3,$$

$$|\pi_0^*\rangle := \frac{x_{0n}}{\sqrt{3}} \sum_{n=0}^{2} \tilde{x}_n^{-1}|\phi_n\rangle,$$  \hspace{1cm} \text{(B1)}

is an optimal unambiguous measurement for $\{|\Psi_r\rangle\}$. Also, its average success probability is $P(\Pi^*) = 3\tilde{x}_{00}^{-1}$. Therefore, $\hat{Z}_G^*$ of Eq. (30), which is a feasible solution to Problem DP$_G$, satisfies $\text{Tr} \hat{Z}_G^* = P(\Pi^*)$, and thus is an optimal solution to Problem DP$_G$. Note that $\hat{Z}_G^*$ of Eq. (30) is always an optimal solution to Problem DP$_G$, while there could be other optimal solutions.

**Appendix C: Supplement of the $S$-plane**

Under the assumption that Theorem 2 holds, we shall show that $s(\hat{I}_A) \in T^*$ is a necessary and sufficient condition that a sequential measurement can be globally optimal.

First, we show the necessity. Assume that a sequential measurement can be globally optimal. From Theorem 2, there exists $\hat{A}^* \in M_A$ satisfying Eq. (20) such that $\Pi(\hat{A}^*)$ is globally optimal. As described in Sec. IV C, $\hat{A}^*$ is expressed by Eq. (36). Thus, let $\kappa_k^* := \kappa_{*k}^\gamma$ and $\kappa_k^* := 3\kappa_{*k}^\gamma$ for $k \in \{1, 2\}$; then, since $\hat{A}^*$ is a POVM, we have

$$\sum_{k=1}^{3} \kappa_k^* \hat{E}_k^* = \int_{\Omega} \hat{A}^*(d\omega) = \hat{I}_A.$$  \hspace{1cm} \text{(C1)}

Premultiplying and postmultiplying this equation by $\langle \phi_n |$ and $| \phi_n \rangle$, respectively, gives

$$\sum_{k=1}^{3} \kappa_k^* e_k^* = s(\hat{I}_A).$$  \hspace{1cm} \text{(C2)}

This indicates that $s(\hat{I}_A)$ is the weighted sum of $e_k^*$ with the weights $\kappa_k^*/3 \geq 0$, and thus $s(\hat{I}_A) \in T^*$ holds.

Next, we show the sufficiency. The above argument can be applied in the reverse direction. Assume $s(\hat{I}_A) \in T^*$; then, there exists $\kappa_k^*/3 \geq 0$ satisfying Eq. (C2). Consider $\hat{A}^*$ expressed by Eq. (36) with $\kappa_{*k}^\gamma = \kappa_k^*$ and $\kappa_{*k}^\gamma = \kappa_k^*/3$ ($k \in \{1, 2\}$). It follows that $\hat{A}^*$ is a POVM satisfying Eq. (20) and $\hat{A}^*(\omega)\hat{A}^*(\omega) = 0$. Thus, from Remark 1, $\Pi(\hat{A}^*)$ is globally optimal, and thus a sequential measurement can be globally optimal.

**Appendix D: Supplement of Theorem 2**

1. **Proof of $y_2^2 < \eta < y_1^2$**

Let

$$\chi := y_0^2 + y_1^2 y_2^2 + y_2^2 y_0^2;$$  \hspace{1cm} \text{(D1)}

then, we have

$$(1 - 3y_2^2)^2 - (1 - 3\chi)$$

$$= 3(3y_0^4 - 2y_0^2 + \chi)$$

$$= 3(y_0^4 - 2y_0^2(y_{k_{01}} + y_{k_{02}}) + y_0^2(y_{k_{01}} + y_{k_{02}}) + y_0^2 y_{k_{01}} y_{k_{02}})$$

$$= 3(y_0^4 - y_0^2)(y_{k_{01}} + y_{k_{02}}),$$  \hspace{1cm} \text{(D2)}

where the third line follows from $\sum_{n=0}^{2} y_n^2 = 1$. Substituting $k = 1$ into Eq. (D2) yields $(1 - 3y_1^2)^2 \leq 1 - 3\chi$. The equality holds when $y_0 = y_1$. In this case, from $y_2^2 = 1 - 2y_0^2$, we have $1 - 3y_1^2 = y_2^2 - y_0^2 < 0 \leq 1 - 3\chi$. Thus, $1 - 3y_1^2 < 1 - 3\chi$ always holds. Substituting the definition of $y_n$ in Eq. (24) into Eq. (D1) gives $|K_B|^2 = |K_B^\gamma|^2 = 1 - 3\chi$. Therefore, from the definition of $\eta$, we have

$$\eta = \frac{1}{3} (1 - \sqrt{1 - 3\chi}) < y_1^2.$$  \hspace{1cm} \text{(D3)}

In the same way, substituting $k = 2$ into Eq. (D2) yields $1 - 3y_2^2 > \sqrt{1 - 3\chi}$, which gives $\eta > y_2^2$.

2. **Derivation of $|y_q|$**

From Eqs. (23) and (31), we have

$$\hat{X}_G^* - q\hat{\Psi} = 3 \sum_{n=0}^{2} x_n^2 (y_{k_{0n}}^2 - q)|\phi_n\rangle\langle\phi_n|.$$  \hspace{1cm} \text{(D4)}

Also, since $y_2^2 < \eta < y_1^2$ holds (see Appendix D1), $\hat{X}_G^* - q\hat{\Psi}$ ($q \leq \eta$) is singular if and only if $q = y_2^2$ holds. One can easily see $|y_q\rangle = |\phi_{y_2^2}\rangle$ when $q = y_2^2$. Note that, in this case, one can define $p_q := 0$.

In what follows, assume $q \neq y_2^2$. From Eq. (51), we have

$$\langle \hat{X}_G^* - q\hat{\Psi} | \eta_q \rangle = p_q |a_0\rangle \langle a_0 | \eta_q \rangle \propto |a_0\rangle.$$  \hspace{1cm} \text{(D5)}

Thus, from Eqs. (23) and (D4), we have

$$|y_q\rangle \propto \langle \hat{X}_G^* - q\hat{\Psi} | a_0 \rangle \propto \sum_{n=0}^{2} \frac{1}{x_n(y_{k_{0n}}^2 - q)} |\phi_n\rangle.$$  \hspace{1cm} \text{(D6)}

Therefore, $|y_q\rangle$ is expressed by Eq. (52). One can verify rank $\hat{\Gamma}^{(1)}(q) < 3$ by letting $p_q := \langle a_0 | (\hat{X}_G^* - q\hat{\Psi})^{-1} | a_0 \rangle$ if $q < \eta$ and $p_q := -\infty$ if $q = \eta$. Note that since $|\eta_q\rangle$ is unique up to a global phase, rank $\hat{\Gamma}^{(1)}(q) = 2$ holds.
3. Proof of $C \subseteq T^*$

Since $s(\gamma_q) \langle \gamma_q \rangle = e_1^* \in T^*$ holds when $q = y_2^2$, we have only to consider the case of $q \neq y_2^2$. Let
\[
 u_n(q) := (\nu_n^2 - q)^{-1},
\]
Note that $q \neq y_2^2$ holds for any $n \in I_3$ since $y_1^2 < \eta < y_2^2$ holds (see Appendix D 1). From Eqs. (37), (44), and (52) and $v_n = n$, we have
\[
 s_n(v(q) \langle \gamma_q \rangle) \propto x_{0n}^2 u_n(q),
\]
and thus
\[
 s(\gamma_q) \langle \gamma_q \rangle \propto [x_{n0}^2 u_n^2(q), x_{n0}^2 u_n^2(q)].
\]

First, let us consider the case in which the three points $e_1^*$, $e_2^*$, and $e_3^*$ lie on a straight line. From Eq. (35), this case occurs only when $y_0 = y_1$. Since $s(\gamma_q) \langle \gamma_q \rangle \propto [x_{n0}^2, x_{n0}^2]$ holds from Eq. (D9), every point in $C$ is on the line joining the origin $e_2^*$ to the point $e_3^*$. From $u_n(q) = u_2(q) \leq u_2(q)$ (see Appendix D 4), we have
\[
 s_n(v(q) \langle \gamma_q \rangle) = \frac{x_{n0}^2 u_0^2(q)}{2} \leq \frac{x_{n0}^2 u_0^2(q)}{2} = s_n(\omega_1).
\]

Thus, $s(\gamma_q) \langle \gamma_q \rangle$ is an interior point between $e_1^*$ and $e_3^*$. Therefore, $C \subseteq T^*$ holds.

Next, let us consider the other case in which $e_1^*$, $e_2^*$, and $e_3^*$ do not lie on a straight line. Let $l_{jk}$ denote the straight line joining $e_2^*$ and $e_3^*$. It suffices to prove the following two statements: (a) $C$ is in the region between the two lines $l_{13}$ and $l_{23}$, and (b) $C$ is in the region between the two lines $l_{12}$ and $l_{13}$.

First, we prove the statement (a). The gradient of the line joining the origin to the point $s(\gamma_q) \langle \gamma_q \rangle$ is
\[
 \zeta(q) := \frac{s_n(v(q) \langle \gamma_q \rangle)}{s_n(\gamma_q) \langle \gamma_q \rangle)} = \frac{x_{0n}^2 (y_0^2 - q)^2}{x_{0n}^2 (y_0^2 - q)^2},
\]
where the last equality follows from Eq. (D8). Since $\eta < y_1^2 \leq y_0^2$ holds (see Appendix D 1), one can easily verify that $\zeta(q)$ monotonically decreases in the range $q \leq \eta$, which gives
\[
 \zeta(-\infty) \geq \zeta(q) \geq \zeta(\eta), \quad \forall q \leq \eta.
\]
Also, from $e_1^* = s(\gamma_{-\infty^2} \langle \gamma_{-\infty^2} \rangle)$ and $e_3^* = s(\gamma_q) \langle \gamma_q \rangle)$, the gradients of the lines $l_{13}$ and $l_{23}$ are, respectively, $\zeta(-\infty)$ and $\zeta(\eta)$. Therefore, from Eq. (D12), the statement (a) holds.

Next, we prove the statement (b). Let $c(q)$ denote the $s_{n0}$-coordinate of the intersection of the $s_{n0}$-axis and the line joining the two points $e_1^*$ and $s(\gamma_q) \langle \gamma_q \rangle$ in $C$. It follows that the statement (b) holds if and only if $c(q)$ satisfies
\[
 0 \leq c(q) \leq c(q), \quad \forall q \leq \eta.
\]
Since $s(\gamma_q) \langle \gamma_q \rangle$ is on the line joining $e_1^*$ and $[c(q), 0]$, we have that for some real number $w$
\[
 s_n(v(q) \langle \gamma_q \rangle) = ws_n(\omega_1) + (1 - w)c(q),
\]
Also, since $\sum_{i=0}^2 s_n(\hat{T}) = 1$ holds for any nonzero positive semidefinite operator $\hat{T}$, we have
\[
 s_n(v(q) \langle \gamma_q \rangle) = ws_n(\omega_1) + (1 - w)[1 - c(q)].
\]
After some algebra with Eqs. (D14), (D15), and (D8), we obtain
\[
 \bar{c}(q) := \frac{x_{0n}^2 c(q)}{x_{0n}^2 [1 - c(q)]} = \frac{u_2^2(q) - u_2^2(q)}{u_2^2(q) - u_2^2(q)}.
\]
It follows from the definition of $\bar{c}(q)$ that $\bar{c}(q)$ monotonically increases with $c(q)$. Thus, the statement (b), i.e. Eq. (D13), is equivalent to
\[
 0 \leq \bar{c}(q) \leq \bar{c}(\eta), \quad \forall q \leq \eta.
\]
Since $u_0^2(q) \leq u_2^2(q) \leq u_2^2(q)$ holds (see Appendix D 4), $\bar{c}(q) \geq 0$ obviously holds. Therefore, we need only show $\bar{c}(q) \leq \bar{c}(\eta)$.

Differentiating $\bar{c}(q)$ of Eq. (D16) with respect to $q$ gives
\[
 \frac{d\bar{c}(q)}{dq} = \frac{2[u_1^2(q) - u_2^2(q)]}{u_2^2(q) - u_2^2(q)} [f[u_1(q)] - f[u_2(q)]],
\]
which implies that $d\bar{c}(q)/dq \geq 0$ is equivalent to $f[u_1(q)] \geq f[u_2(q)]$. In the case of $q < y_2^2$, from $0 \leq u_1(q) \leq u_2(q)$, $f[u_1(q)] \leq f[u_2(q)]$ (i.e., $d\bar{c}(q)/dq \geq 0$) holds, which follows from the fact that $f(x)$ monotonically increases in the range $x \geq 0$. In the other case of $q > y_2^2$, from $u_2(q) \leq -u_1(q) \leq -u_0(q)$ (see Appendix D 4), $f[u_2(q)] < 0 \leq f[u_1(q)]$ (i.e., $d\bar{c}(q)/dq \geq 0$) holds, which follows from the fact that $f(x) < 0$ holds if and only if $x < -u_0(q)$. Therefore, $\bar{c}(q)$ attains its maximum at $q = -\infty$ and/or $q = \eta$, and thus, for the rest, it suffices to show $\bar{c}(\infty) \leq \bar{c}(\eta)$.

From Eq. (D16), we have
\[
 \bar{c}(q) = \frac{(y_1^2 - \eta^2)(y_1^2 - q^2)^2 - (y_0^2 - q)^2}{(y_1^2 - \eta^2)(y_1^2 - q^2)^2 - (y_0^2 - q)^2}
\]
which gives
\[
 \bar{c}(\infty) = \frac{y_0^2 - y_2^2}{y_0^2 - y_2^2}.
\]
Also, we have
\[
 (y_1^2 - \eta^2)(y_0^2 + y_2^2 - 2q) - (y_0^2 - q^2)(y_0^2 + y_1^2 - 2q) = (y_0^2 + y_2^2)(3\eta^2 - 2\eta + \chi) = 0,
\]
where the second to fourth lines, respectively, follow from $\sum_{i=0}^2 q_i = 1$, Eq. (D1), and Eq. (D3). Thus, substituting $q = \eta$ into Eq. (D19) gives $\bar{c}(\eta) = (y_0^2 - y_1^2)/(y_0^2 - y_2^2)$. Therefore, $\bar{c}(\infty) = \bar{c}(\eta)$ holds.
4. Proof of $u_0(q) \leq u_1(q) \leq u_2(q) \quad (\forall q \leq \eta, q \neq y^2)$

Since $u_1(q) \geq u_0(q) > 0$ holds, it suffices to prove $u_2^2(q) \geq u_1^2(q)$. In the case of $q < y^2_2$, from $u_2(q) > u_1(q) > 0$, this is obvious. Let us consider the case of $q > y^2_2$. Since $u_2(q) < 0$ holds, it suffices to show $u_2(q) + u_1(q) \leq 0$. Let $\tilde{u}_k := u_k(\eta)$; then, we have

$$\tilde{u}_2 + \tilde{u}_1 \leq \tilde{u}_2 + \tilde{u}_0 + \tilde{u}_1 = \tilde{u}_2\tilde{u}_0\tilde{u}_0^{-1} + \tilde{u}_1\tilde{u}_2^{-1} + \tilde{u}_2\tilde{u}_0^{-1} = 0,$$

(D22)

where the third and fourth lines, respectively, follow from the definition of $\chi$ in Eq. (D1) and Eq. (D3). Since $u_2(q) \leq \tilde{u}_2$ and $u_1(q) \leq \tilde{u}_1$ hold, $u_2(q) + u_1(q) \leq \tilde{u}_2 + \tilde{u}_1 \leq 0$ holds.

5. Case (2)

Let us consider, without loss of generality, $\omega \in \Omega_+, \mu < \mu_0^{(1)}$ and $\mu < \mu_0^{(2)}$. In order to show that $s(\omega)$ is on a straight line segment whose endpoints are in $C$, we shall show the two statements: (a) $s(\omega)$ is on a certain straight line segment, and (b) the line segment is part of a straight line segment whose endpoints are in $C$.

Since we now consider the case (2), $|T^{(\omega)}|$ must be 2 or 3. Let

$$\hat{X} := \begin{cases} \hat{X}_G - \mu^{(2)} \psi, & |T^{(\omega)}| = 3, \\ \hat{X}_G + \infty |a_2| \langle a_2 \rangle, & |T^{(\omega)}| = 2. \end{cases}$$

(D33)

One can easily see that $\hat{X}$ is a positive definite operator. Let $|\sigma(q)|$ be a normal vector satisfying

$$|\sigma(q)| \in \text{Ker} \hat{\Gamma}^{(2)}(q),$$

$$\langle a_0 | \sigma(q) \rangle \geq 0,$$

$$\hat{\Gamma}^{(2)}(q) := \hat{X} - q |a_1 \rangle \langle a_1| - p_{\psi}^* |a_0 \rangle |a_0 \rangle,$$  

(D34)

where $p_{\psi}^*$ is the function of $q$ such that rank $\hat{\Gamma}^{(2)}(q) < 3$. (We can define such $p_{\psi}^*$ as $p_{\psi}^* := (a_0 | \hat{X} - q |a_1 \rangle \langle a_1| |a_0 \rangle)^{-1}$. Since $\hat{X} - q |a_1 \rangle \langle a_1|$ is positive definite, such $p_{\psi}^*$ always exists.) $p_{\psi}^*$ monotonically decreases with $q$. $\hat{\Gamma}^{(2)}(\omega) = \hat{\Gamma}^{(2)}[|\mu| - \mu^{(2)}]$ holds if $|T^{(\omega)}| = 3$; otherwise, $\hat{\Gamma}^{(2)}(\omega) = \hat{\Gamma}^{(2)}[|\mu^{(1)}|]$ holds.

First, we show the statement (a). Let $\hat{\Gamma}^{(2)}(0) := \hat{\Gamma}^{(2)}(\omega)$, $\hat{\Gamma}^{(2)}(\omega) := \hat{\Gamma}^{(2)}[p_{\psi}^* |a_0 \rangle |a_0 \rangle, |a_0 \rangle |a_0 \rangle], |a_1 \rangle := |a_1 \rangle |a_1 \rangle, |a_1 \rangle$ holds if $q = p_{\psi}^* (i.e., p_{\psi}^* = 0)$. We shall express $|\sigma(q)|$ in terms of $|\sigma_0|$ and $|\sigma_1|$. For each $k \in \{0, 1\}$, from $\hat{\Gamma}^{(2)}(\eta_k) = 0$, we have

$$|\eta_k \rangle = \frac{\hat{X} |\sigma_k \rangle}{p_{\psi}^* (a_k |\sigma_k \rangle)).$$

(D35)

Note that since $\hat{X}$ is positive definite, we have $\hat{X} |\sigma_k \rangle \neq 0$, which yields $p_{\psi}^* (a_k |\sigma_k \rangle) \neq 0$. Substituting Eq. (D24) into $\hat{\Gamma}^{(2)}(q) |\sigma(q)\rangle = 0$ and using Eq. (D25) yields

$$|\sigma(q)| = \hat{X}^{-1} \langle p_{\psi}^* |a_0 \rangle |a_0 \rangle + q_{\psi}^* |a_1 \rangle |a_1 \rangle,$$

$$q_{\psi}^* = \left( \frac{p_{\psi}^* (a_0 |a_0 \rangle)}{q_{\psi}^* (a_1 |a_1 \rangle)} \right)^2.$$(D36)

where $\eta_k := \langle a_1 |\sigma(q)\rangle$. Premultiplying this equation by $\langle a_0 \rangle$ and some algebra gives

$$\langle a_1 |\sigma(q)\rangle = \frac{p_{\psi}^* (a_0 |a_0 \rangle)}{q_{\psi}^* (a_1 |a_1 \rangle)}.$$(D37)

Substituting this equation into Eq. (D26) gives

$$|\sigma(q)| = \frac{p_{\psi}^* (a_0 |a_0 \rangle)}{q_{\psi}^* (a_1 |a_1 \rangle)}.$$(D38)

Since $r_0 \geq 0$ and $\langle a_0 |\sigma_0 \rangle \geq 0$ hold from Eq. (D24), it follows from Eq. (D28) that $|\sigma(q)|$ is expressed as

$$|\sigma(q)| = c_0 |\sigma_0 \rangle + c_1 |\sigma_1 \rangle,$$(D39)

with certain nonnegative real numbers $c_0$ and $c_1$. Let $q_2$ be the real number satisfying $p_{\psi}^* = q_2$. One can easily verify that, when $q = q_2$, Eq. (D29) with $c_0 = c_1 = c$ holds. Let $|\sigma_2 \rangle := |\sigma(q_2)\rangle$.

Due to the symmetry of the states, $\hat{S}(\langle a_0 \rangle |\sigma_0 \rangle) = \hat{S}(\langle a_1 \rangle |\sigma_1 \rangle) = \hat{S}_{\sigma_2}$. Thus, from Eq. (D29), we have

$$\hat{S}(\langle a_0 \rangle |\sigma_0 \rangle) = \langle c_0^2 + c_1^2 \rangle \hat{S}_{\sigma_0} + c_0 c_1 \hat{S}_{\sigma_2}.$$  

(D35)

where

$$\hat{S}_{\sigma_2} := \frac{1}{2} \sum_{j=0}^2 \hat{V}_j (|\sigma_0 \rangle \langle a_1 | + |\sigma_1 \rangle \langle a_0 |) (\hat{V}_j)^\dagger.$$(D36)

Substituting Eq. (D30) into $\hat{S}(\langle a_0 \rangle |\sigma_0 \rangle) = \hat{S}_{\sigma_2}$ yields

$$\hat{S}_{\sigma_2} = 2c^2 \hat{S}_{\sigma_0} + c^2 \hat{S}_{\sigma_2}.$$(D37)

Substituting this into Eq. (D30) gives

$$\hat{S}(\langle a_0 \rangle |\sigma_0 \rangle) = \langle c_0^2 \rangle \hat{S}_{\sigma_0} + \langle c_1^2 \rangle \hat{S}_{\sigma_2}.$$(D38)

where $c_0^2 \equiv (c_0 - 1)^2$ and $c_1^2 \equiv (c_0 - 1)^2$. Note that taking the trace of this gives $c_0^2 + c_1^2 = 1$ and that $c_0^2, c_1^2 \geq 0$ holds. Also, Eq. (D33) gives

$$\langle a_1 \rangle \sigma(q)\rangle = \langle c_0^2 \rangle \hat{S}_{\sigma_0} + \langle c_1^2 \rangle \hat{S}_{\sigma_2}.$$  

(D39)

where $s_{\sigma_0} := \langle a_0 |\sigma(q)\rangle$ and $s_{\sigma_2} := \langle a_1 |\sigma(q)\rangle$. Therefore, $s_{\sigma(q)} = s_{\sigma_0} + s_{\sigma_2}$ is the case (1), i.e., at least two of $|\mu_0^{(1)}|$, in Eq. (49) are the same; thus, $s_{\sigma_2}$ is in $C$. Also, if $s_0$ satisfies $|T^{(0)}| = 3$, then $q = 0$ is also the case (1), and thus $s_{\sigma_0}$ is in $C$. Therefore, in the case of $|T^{(\eta)}| = 3$, $\hat{L}$ is the line segment whose endpoints, $s_{\sigma_0}$ and $s_{\sigma_2}$, are in $C$. In what follows, assume $|T^{(\eta)}| = 2$. We shall show that $\hat{L}$ is part of the straight line segment whose endpoints are
\[ e^*_1 = s(\omega_1) \in C \text{ and } s_{\omega_2} \in C. \] 

Taking the limit as \( q \to -\infty \) in Eq. (D24) gives \( s(\sigma(-\infty))\langle \sigma(-\infty) | = e^*_1. \) Thus, repeating the above argument with \( q \to -\infty \) indicates that \( |\sigma(-\infty) \rangle \) is expressed as Eq. (D29) with \( c_0 > 0 \) and \( c_1 < 0, \) and that Eq. (D33) holds with \( c'_2 < 0. \) Thus, \( s_{\omega_2} \) is an interior point between \( e^*_1 \) and \( s_{\omega_2}. \) Therefore, \( \mathcal{L} \) is part of the line segment whose endpoints are \( e^*_1 \) and \( s_{\omega_2}. \)

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