Some Notes on Relative Commutators

Masoumeh Ganjali and Ahmad Efranian*
Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures, Fedowsi University of Mashhad, P.O.Box 1159-91775 Mashhad, Iran
Email: m.ganjali20@yahoo.com, *erfanian@math.um.ac.ir

Abstract
Let $G$ be a group and $\alpha \in Aut(G)$. An $\alpha$-commutator of elements $x, y \in G$ is defined as $[x,y]_\alpha = x^{-1}y^{-1}xy^\alpha$. In 2015, Barzegar et al. introduced an $\alpha$-commutator of elements of $G$ and defined a new generalization of nilpotent groups by using the definition of $\alpha$-commutators which is called an $\alpha$-nilpotent group. They also introduced an $\alpha$-commutator subgroup of $G$, denoted by $D_\alpha(G)$ which is a subgroup generated by all $\alpha$-commutators. In 2016, an $\alpha$-perfect group, a group that is equal to its $\alpha$-commutator subgroup, was introduced by authors of this paper and the properties of such group was investigated. They proved some results on $\alpha$-perfect abelian groups and showed that a cyclic group $G$ of even order is not $\alpha$-perfect for any $\alpha \in Aut(G)$. In this paper, we may continue our investigation on $\alpha$-perfect groups and in addition to studying the relative perfectness of some classes of finite $p$-groups, we provide an example of a non-abelian $\alpha$-perfect 2-group.

Keywords: Auto-commutator subgroup; finite $p$-group; normal subgroup; perfect group.

Abstrak
Misalkan $G$ grup dan $\alpha \in Aut(G)$. Suatu $\alpha$-komutator dari unsur-unsur $x, y \in G$ didefinisikan sebagai $[x,y]_\alpha = x^{-1}y^{-1}xy^\alpha$. Pada tahun 2015, Barzegar et al. memperkenalkan $\alpha$-komutator dari unsur-unsur di $G$ dan mendefinisikan sebuah perumuman baru dari grup-grup nilpoten dengan menggunakan definsi dari $\alpha$-komutator yang dinamakan grup $\alpha$-nilpoten. Mereka juga memperkenalkan suatu subgroup $\alpha$-komutator dari $G$ yang dilambangkan dengan $D_\alpha(G)$ yang merupakan subgroup yang dibangun dari semua $\alpha$-komutator. Pada tahun 2016, grup $\alpha$- sempurna, yaitu grup yang subgroup $\alpha$-komutatornya sama dengan grup itu sendiri, diperkenalkan oleh penulis paper ini dan sifat-sifat grup tersebut juga diselidiki. Mereka membuktikan beberapa sifat dari grup abel $\alpha$-sempurna dan memperlihatkan bahwa suatu grup siklis $G$ dengan order genap bukan grup $\alpha$-sempurna untuk setiap $\alpha \in Aut(G)$. Di paper ini kita akan melanjutkan investigasi kita pada grup-grup $\alpha$-sempurna dan sebagai tambahan dalam mempelajari kesempurnaan relatif dari kelas-kelas dari $p$-grup berhingga, kita akan melihat contoh dari 2-grup $\alpha$-sempurna yang non abel.

Kata kunci: subgroup auto-komutator; $p$-grup berhingga; subgroup normal; grup sempurna.

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1. INTRODUCTION

In 1994, an auto-commutator $[x, \alpha] = x^{-1}x^\alpha$ of elements $x \in G$ and $\alpha \in \text{Aut}(G)$ was introduced by Hegarty, [1]. If $\alpha_g$ is an inner automorphism such that $x^{\alpha_g} = g^{-1}xg$ then auto-commutator $[x, \alpha_g] = x^{-1}g^{-1}xg$ is the ordinary commutator of two elements $x, g \in G$. Hegarty generalized the definition of the center of $G$, $Z(G) = \{x \in G : x^\alpha = x, \forall y \in G\}$ to the absolute center $L(G) = \{x \in G : x^\alpha = x, \forall \alpha \in \text{Aut}(G)\}$ of $G$. One can check that $L(G)$ is a characteristic subgroup of $G$ which is contained in $Z(G)$. He also introduced the auto-commutator subgroup of $G$, denoted by $K(G)$, which is a characteristic subgroup generated by all auto-commutators. Clearly, the commutator subgroup $G'$ is contained in $K(G)$.

Investigation of the relative commutators are interesting for some authors, for instance Barzegar et al. [2] also introduced a new generalization of commutators with respect to a fixed automorphism of group $G$. Let $\alpha \in \text{Aut}(G)$, then an $\alpha$-commutator of two elements $x, g \in G$ is defined as $[x, y]_\alpha = x^{-1}y^\alpha y^{-1}$ which is equal to the ordinary commutator $[x, y] = x^{-1}y^{-1}xy$ whenever $\alpha$ is the identity automorphism. In [2], the subgroup which is generated by all $\alpha$-commutators was denoted by $D_\alpha(G)$ and called $\alpha$-commutator subgroup of $G$. It is not difficult to prove that $D_\alpha(G)$ is a normal subgroup of $G$ that is contained in $K(G)$. Authors of [2] also introduced a new generalization of a nilpotent group $G$, which is called an $\alpha$-nilpotent group for a fixed automorphism $\alpha$ of $G$. Here, we may present the definition of an $\alpha$-nilpotent group $G$. We start by the definition of a lower central $\alpha$-series. Put $\Gamma_0^\alpha(G) = G$ and $\Gamma_1^\alpha(G) = D_\alpha(G)$ and define inductively $\Gamma_{n+1}^\alpha(G) = [G, \Gamma_n^\alpha(G)]_\alpha = \{[x, y]_\alpha : x \in G, y \in \Gamma_n^\alpha(G)\}$, $n \geq 1$. We can see that $\Gamma_n^\alpha(G)$ is a normal subgroup of $G$ which is invariant under $\alpha$ and $\Gamma_{n+1}^\alpha(G) \leq \Gamma_n^\alpha(G)$, for all $n \geq 1$. Following normal series is called a lower central $\alpha$-series $G \geq \Gamma_2^\alpha(G) \geq \cdots \geq \Gamma_n^\alpha(G) \geq \cdots$.

A group $G$ is called an $\alpha$-nilpotent group of nilpotency class $n$ if $\Gamma_n^\alpha(G) = \{1\}$ and $\Gamma_{n+1}^\alpha(G) \neq \{1\}$. Clearly, if $\alpha$ is considered as the identity automorphism, then an $\alpha$-nilpotent group is the ordinary one. In [2], it was proved that an $\alpha$-nilpotent group is nilpotent, but the converse is not valid in general. For instance, authors proved that the cyclic group of order $n = p_1p_2\cdots p_t$ is $\alpha$-nilpotent if and only if $\alpha$ is the identity automorphism, for distinct primes $p_1, p_2, \ldots, p_t$. Authors of [3] continued investigation on $\alpha$-nilpotent groups and proved some new results on this new concept. For example, they proved that an extra special $p$-group, $p$ is an odd prime number, is nilpotent with respect to a non-identity automorphism $\alpha$ but is not nilpotent relative to all its automorphisms. For an inner automorphism $\alpha_g \in \text{Inn}(G)$, we can see that nilpotency and $\alpha_g$-nilpotency are equivalent. Therefore, we may ask the following question.

**Question.** Is there a non-inner automorphism $\alpha$ of nilpotent group $G$ such that $G$ is $\alpha$-nilpotent?
This question was answered for finitely generated abelian groups, for more details see [3]. Actually, authors classified all finitely generated abelian groups which are nilpotent with respect to a non-inner automorphism. Furthermore, they proved some results on relative normal and absolute normal subgroups of some classes of finite groups. In [4], they introduced an $\alpha$-perfect group $G$, a group which is equal to its $\alpha$-commutator subgroup, for a fixed automorphism $\alpha$ of $G$. If $G'$ is the ordinary commutator subgroup of $G$, then $G' \leq D_\alpha(G)$ for all $\alpha \in Aut(G)$. It follows that if $G$ is a perfect group, then it is perfect with respect to all its automorphisms. One can check that an $\alpha$-nilpotent group cannot be $\alpha$-perfect, but the symmetric group of order $n!$, $S_n$ is an example of a non-nilpotent group where is not $\alpha$-perfect, because $D_\alpha(S_n) = (S_n) = A_n$ for all $\alpha \in Aut(G)$. The relative perfectness of abelian groups was studied by authors of [4]. In this paper, we may continue our investigation on relative perfect groups and prove some new results on some classes of finite non-abelian $p$-groups.

2. RELATIVE PERFECT GROUPS

In this section, we recall the definition of an $\alpha$-perfect group for a fixed automorphism $\alpha$. At first, we present some results on relative perfect groups that were proved in [4]. Finally, we may add some new results on non-abelian relative perfect groups.

**Definition 2.1.** Let $G$ be a group and $\alpha \in Aut(G)$. A group $G$ is called an $\alpha$-perfect group, whenever $G = D_\alpha(G)$.

**Definition 2.2.** If $G$ is a finite group and $\alpha \in Aut(G)$, then a subgroup $H$ of $G$ is called an $\alpha$-normal subgroup of $G$, denoted by $H \trianglelefteq G$, if $g^{-1}h\alpha g \in H$ for all $g \in G$ and $h \in H$. If $H$ is $\alpha$-normal with respect to all automorphisms $\alpha \in Aut(G)$, then $G$ is called an absolute normal subgroup of $G$.

**Lemma 2.3.** ([4]) Let $H$ be a subgroup of finite group $G$, then (i) if there exists an $\alpha \in Aut(G)$ such that $H \trianglelefteq G$, then $H$ is a normal subgroup of $G$, (ii) $H$ is an absolute normal subgroup of $G$ if and only if $K(G) \leq H$.

It might be important to find all proper absolute normal subgroups of given finite group $G$. In [3] and [4], the structure of absolute normal subgroups of some classes of finite groups were given. For instance, we have the following results.

**Lemma 2.4.** ([4]) If $G \cong \mathbb{Z}_{2^n}$, such than $(2, m) = 1$, then the proper subgroup $H$ of $G$ is absolute normal if and only if $H = 2G$.

**Theorem 2.5.** ([3]) (i) If $D_{2^n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} \rangle$, then $\langle x \rangle$ is the only proper absolute normal subgroup of $D_{2^n}$. (ii) Semi-dihedral 2-group $SD_{2^n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} = x^{2^n-1} \rangle, n \geq 3$ has the only proper absolute normal subgroups given by $\langle x \rangle, \langle x^2, y \rangle, \langle x^2, yx \rangle$. (iii) Generalized
quaternion 2-group $Q_{2^n} = \langle x, y : x^{2n} = 1, x^{2n} = y^2, yxy^{-1} = x^{-1} \rangle, n \geq 3$ has the only proper absolute normal subgroup $\langle x \rangle$. (iv) Twisted dihedral 2-group $S_D_{2^n} = \langle x, y : x^{2n} = y^2 = 1, yxy^{-1} = x^{2n+1} \rangle, n \geq 3$ has the only proper absolute normal subgroup $\langle x^2, y \rangle$.

**Theorem 2.6.** ([3]) If $p$ is an odd prime number and $M_n(p) = \langle x, y : x^p = y = 1, xy = yx \rangle$, $n \geq 3$, then $M_n(p)$ does not possess a proper absolute normal subgroup.

Next lemma, talks about the existence of an $\alpha$-normal subgroup in abelian $\alpha$-perfect group $G$.

**Lemma 2.7.** ([4]) Let $G$ be a finite abelian group. Then $G$ is $\alpha$-perfect if and only if $G$ does not possess a proper $\alpha$-normal subgroup.

If $G$ is a finite cyclic group of order $n$, then $\alpha \in Aut(G)$ if and only if $x^\alpha = ux$ such that $(u, n) = 1$, for all $x \in G$. We denote such $\alpha$ by $\alpha_u$.

**Lemma 2.8.** ([4]) Let $G$ be a cyclic group of order $n$ and $\alpha_u \in Aut(G)$ be a non-identity automorphism. Then $G$ is $\alpha_u$-perfect if only if $(u - 1, n) = 1$.

By Lemma 2.8, we can conclude that there is no $\alpha$-perfect cyclic group of even order, for all automorphisms $\alpha$ of such group. If $p$ is an odd prime number, then $Z_{p^r}, r > 1$, is $\alpha_u$-perfect for each $1 < u < p$, but it is not $\alpha_{p+1}$-perfect.

Now, we are ready to prove some new results on relative perfect groups.

**Lemma 2.9.** If $\alpha_g$ is an inner automorphism and $\beta = \alpha \circ \alpha_g$, then $G$ is $\alpha$-perfect if and only if $\beta$ is $\alpha$-perfect.

**Proof.** We can see that $[x, y]_\beta = [x, y]_\alpha [y, g]_\alpha$ and since $[y, g] \in G' \leq D_\alpha(G)$ and $D_\alpha(G)$ is $\alpha$-invariant, then $[x, y]_\beta \in D_\alpha(G)$ and so $D_\beta(G) \leq D_\alpha(G)$. We can write $\alpha = \beta \circ \alpha_g^{-1}$ and prove $D_\alpha(G) \leq D_\beta(G)$. Now, we are done. $\blacksquare$

**Example 2.10.** (i) If $G$ is isomorphic to one of the groups where are defined in Theorem 2.5, then $G$ possesses a proper absolute normal subgroup. Furthermore, we know that $D_\alpha(G) \leq K(G)$, for all $\alpha \in Aut(G)$. So by Lemma 2.3, $D_\alpha(G)$ is a proper subgroup of $G$ and $G$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$. (ii) Assume that that $Q_s = \langle xy : x^{4} = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle, n \geq 3$ and $\alpha \in Aut(Q_s)$ is an automorphism by argument $x^\alpha = y$ and $y^\alpha = xy$. Then $D_\alpha(Q_s) = Q_s$ and $Q_s$ is an $\alpha$-perfect group.
By Theorem 2.6, if $G \cong M_n(p)$, then $G$ does not possess any proper absolute normal subgroup. Here, we may prove that $M_n(p)$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Theorem 2.11.** If $G \cong M_n(p) = \langle x, y : x^{p^{n+1}} = y^p = 1, xy = yx^{p^{-1}} \rangle$ for an odd prime number $p$ and $n \geq 3$, then $G$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Proof.** We can see that $|G| = p^n, Z(G) = \langle x^p \rangle, G' = \langle x^{p^{n-1}} \rangle$. The automorphism group of $G$,

$Aut(G)$ is equal to $\{ \alpha_{ij} : x^{a_{ij}} = x^i, y^{a_{ij}} = x^{b_{ij}}, 0 \leq i \leq p^{n-1}, 0 \leq j \leq 0 \}$. It is not difficult to see that $[x, x]_{a_{ij}} = x^{-i}y^i, [x, y]_{a_{ij}} = x^{(k+1)p^{n-2}}, [y, x]_{a_{ij}} = x^{p^{n-2}}x^{-j}y^j, [y, y]_{a_{ij}} = x^{b_{ij}}$.

Therefore, $D_{a_{ij}}(G) = \langle x^{-i}y^j, x^{p^{n-2}} \rangle$. If $i \neq 1$, then $|x^{-i}y^j| = |x|$, also $(i-1)p^{n-2} = p^{n-2}$ and for a positive integer $m$ we have $\left( x^{-i}y^j \right)^m = x^{m(i-1)(i-1)/2}y^{p^{n-2}}$.

Now, if we put $m = p^{n-2}$, then since $n \geq 3, p^{n-2} \geq p$ and $(x^{-i}y^j)^{p^{n-2}} = x^{(i-1)p^{n-2}} = x^{p^{n-2}}$. It means that $x^{p^{n-2}} \in \langle x^{-i}, y^j \rangle$ and $D_{a_{ij}}(G) \subseteq \langle x^{-i}, y^j \rangle$ is a proper subgroup of $G$, because $|x^{-i}y^j| \neq p^{n-1}$. Now if, $i = 1$ and $j \neq 0$, then $D_{a_{ij}}(G) \subseteq \langle y^i, x^{p^{n-2}} \rangle$ and since $|y| = |x^{p^{n-1}}| = p$, then $D_{a_{ij}}(G) \neq G$. In case $j = 0$, we have $D_{a_{ij}}(G) = G \neq G$. If $i \neq 1$ and $i \neq 1$ then $i = 1 + lp^s$, where $(l, p) = 1, s = 1, 2, \ldots, n - 2$. In this case, $G$ is $\alpha_{ij}$-nilpotent by Theorem 4.10 of [2] and so $D_{a_{ij}}(G) \neq G$. Hence, we are finished.

**Theorem 2.12.** Let $p$ be an odd prime number. If $G = \langle a, b : a^{p^2} = b^{p^3} = 1, a^{-1}ba = b^{p+1} \rangle$ is a $p$-group of order $p^5$ and nilpotency class three, then $G$ is not $\alpha$-perfect for any $\alpha \in Aut(G)$.

**Proof.** The automorphism group of $G$ is $Aut(G) = \{ \alpha_{z, \omega, \mu} : a^{\alpha_{z, \omega, \mu}} = a^{z+\omega}, b^{\alpha_{z, \omega, \mu}} = b^{\omega + \mu z}, \mu \equiv 0, \omega \equiv 0, \mu p^3 \equiv 0 \}$. Let $\alpha = \alpha_{z, \omega, \mu} \in Aut(G)$, be an arbitrary automorphism such that $z = pt$ and $\mu = pk$ for some integers $t, k \in \mathbb{Z}$. Then

(i) $[a, a]_{a} = (a^{-1}b^\omega)^{p^k} = a^{p^k(b^{\omega}\omega^{(p+1)p^2-1})} = b^{\omega \n^{p^2+1-1}}, (ii) \quad [a, b]_{a} = a^{-1}b^{-1}ab^{\omega} = a^{-1}ab^{-1}a^{p^k}b^{\omega} = b^{p^k(b^{\omega})^{p^k-1}}, (iii) \quad [b, a]_{a} = b^{-1}a^{-1}ba(a^{-1}b^\omega)^{p^k} = b^{p^k(b^{\omega})^{p^k}}, (iv) \quad [b, b]_{a} = b^{p^k(b^{\omega})^{p^k}} = a^{p^k}b^{-(p+1)^n} = a^{p^k}b^{-(p+1)^{n+1}}.$


Assume that $\alpha \in D_a(G)$, then there exists an integer $j$ such that $a = (a^p b^{-p^{j+1}})^j$. Put $n = (p+1)^{p^{j+1}}$, then since $Z(G) = \{b^p\}$ we have $a = (a^p b^{-1})^j b^{-p^j}$ and $a^{p^j} b^{-p^j} = a$. We know that $ba = ab^{p+1}$, so we can conclude that $a^{p^j} b^{-p^j} = a^{p^j} b^{-p^j} b^{p+1}$, and so $a^{p^j} b^{(p+1)^{p^j}} b^{-m} = a^{p^j} b^{-p^j} b^{p+1}$ and $b^{(p+1)^{p^j}} = b^{p+1}$. But $b^{(p+1)^{p^j}} = b^{p^{j+1}}$, therefore we have $b^{p^{j+1}} = b^{p+1}$ and $b^{p^{j} - p} = 1$ which implies that $p^3 | p^2 t j - p$, a contradiction. Hence $D_a(G) < G$ and $G$ is not $\alpha$-perfect.

In [4], has been shown that for every finite abelian group $G$, there exists a finite abelian group $H$ and $\alpha \in Aut(H)$ such that $D_a(H) \cong G$. Here, we may improve this result to finitely generated abelian groups.

**Proposition 2.13.** If $\alpha \in Aut(G)$ and $\beta \in Aut(H)$, then $D_{\alpha \times \beta}(G \times H) = D_a(G) \times D_\beta(H)$.

**Proof.** It is straightforward. ■

**Theorem 2.14.** Assume that $G = \prod_{i=1}^{t} Z \times Z \times \cdots \times Z \times G_i$ such that $G_i$ is a finite abelian group. Then there exist an abelian group $H$ and $\alpha \in Aut(H)$ such that $D_a(H) \cong G$.

**Proof.** By Theorem 3.7 of [4], for finite group $G_i$, there exist abelian group $H$ and $\beta \in Aut(H)$ such that $D_\beta(H) \cong G_i$. Now, if $\alpha \in Aut(Z \times Z)$ by argument $(a, b)^{\alpha} = (a + b, b)$, then $D_a(Z \times Z) = \{(a, b) : a, b \in Z\}$ and $D_\beta(H) = D_\beta(H_1) \cong G_i$. Now, it is enough to put $H = \prod_{i=1}^{t} Z \times Z \times \cdots \times Z \times H_i$ and $\alpha = \alpha_i \times \cdots \times \alpha_i$, then $D_a(H) = \prod_{i=1}^{t} D_a(Z \times Z) \times D_\alpha(H_i) \cong G$, and the proof is completed. ■

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