SCATTERING THEORY FOR THE LAPLACIAN ON MANIFOLDS WITH BOUNDED CURVATURE

WERNER MÜLLER AND GORM SALOMONSEN.

Abstract. In this paper we study the behaviour of the continuous spectrum of the Laplacian on a complete Riemannian manifold of bounded curvature under perturbations of the metric. The perturbations that we consider are such that its covariant derivatives up to some order decay with some rate in the geodesic distance from a fixed point. Especially we impose no conditions on the injectivity radius. One of the main results are conditions on the rate of decay, depending on geometric properties of the underlying manifold, that guarantee the existence and completeness of the wave operators.

0. Introduction.

Let $(M, g)$ be a complete Riemannian manifold and let $\Delta_g$ be the Laplacian on functions attached to $g$. Then $\Delta_g$ is an essentially self-adjoint operator in $L^2(M)$ [Cn]. If $M$ is non-compact, then $\Delta_g$ may have a nonempty continuous spectrum. The purpose of this paper is to study the behavior of the continuous spectrum of $\Delta_g$ with respect to perturbations of the metric $g$. If $h$ is a compactly supported perturbation of $g$, it is well known that the wave operators

$$W_\pm(\Delta_g, \Delta_h) := \text{s-lim}_{t \to \pm \infty} e^{it\Delta_g} J e^{-it\Delta_h} P_{ac}(\Delta_h)$$

exist. Therefore, the absolute continuous parts $\Delta_{g, ac}$ and $\Delta_{h, ac}$ of $\Delta_g$ and $\Delta_h$, respectively, are unitarily equivalent. Our goal is to study non-compactly supported perturbations of $g$, it is well known that the wave operators

$$W_\pm(\Delta_g, \Delta_h)$$

exist. Therefore, the absolute continuous parts $\Delta_{g, ac}$ and $\Delta_{h, ac}$ of $\Delta_g$ and $\Delta_h$, respectively, are unitarily equivalent. Our goal is to study non-compactly supported perturbations of the metric. To this end we introduce a certain class of functions, called functions of moderate decay, which describe the rate of decay of the perturbation of a given metric. Let $\beta: [1, \infty) \to \mathbb{R}^+$ be a function of moderate decay (see Definition 1.4). Then two complete metrics $g$ and $h$ are said to be equivalent up to order $k \in \mathbb{N}$, if there exist $C > 0$ and $p \in M$ such that

$$|g - h|_g(x) + \sum_{j=0}^{k-1} |(\nabla^g)^j(\nabla^g - \nabla^h)|_g(x) \leq C\beta(1 + d_g(x, p)),$$

where $d_g(x, p)$ is the geodesic distance of $x$ and $p$ with respect to $g$, and $\nabla^g$ (resp. $\nabla^h$) the Levi-Civita connection with respect to $g$ (resp. $h$). This condition turns out to be

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an equivalence relation in the set of complete metrics on $M$. We denote this equivalence relation by $g \sim^h_\beta h$. It implies, in particular, that the two metrics are quasi-isometric.

To develop scattering theory for the Laplacian we need to impose additional assumptions on the metrics. In this paper we restrict attention to the class of complete metrics with bounded sectional curvature. In some cases we will also demand that higher derivatives of the curvature tensor are bounded. The assumption that the metric has bounded sectional curvature allows us to control the behavior of the injectivity radius $\tilde{\imath}(x)$ sufficiently well.

One of the main results is the following theorem.

**Theorem 0.1.** Assume $g$ and $h$ be complete metrics on $M$ with bounded curvature up to order 2. Let $\beta$ be a function of moderate decay. Suppose that $g \sim^h_\beta h$. Assume that there exist real numbers $a, b$ satisfying

i) $b \geq 1$ and $a + b = 2$,

ii) $\frac{\beta}{\tilde{\imath}} \in L^1(M)$,

iii) $\frac{\beta}{\tilde{\imath}}(x) \frac{a(x)}{2} \in L^\infty(M)$.

Then $e^{-t\Delta_g} - e^{-t\Delta_h}$ is a trace class operator.

Here $\tilde{\imath}(x)$ is the modified injectivity radius, defined by (2.1), which is bounded from above by a constant which depends on the bound of the sectional curvature. Moreover, $e^{-t\Delta_h}$ is regarded as bounded operator in $L^2(M, g)$. This is possible, because $g$ and $h$ are quasi-isometric.

By the invariance principle for wave operators [Ka], Theorem 0.1 implies that the wave operators $W_\pm(\Delta_g, \Delta_h)$ exist and are complete (see Theorem 7.1). Under additional assumptions on $(M, g)$, the conditions on $\beta$ can be relaxed. This is, for example, the case for manifolds with cusps and manifolds with cylindrical ends. In either case, the method of Enss can be used to prove the existence and completeness of the wave operators.

We also study the analytic continuation of the resolvent. This result can be used, for example, to construct generalized eigenfunctions as in [Mu2].

The structure of the paper is as follows. In section 1 we introduce our class of functions of moderate decay and study some of its elementary properties. Then we set up the equivalence relation mentioned above and prove some facts about equivalent metrics. In section 2 we study the behavior of the injectivity radius on manifolds with bounded sectional curvature.

Then we introduce and study weighted Sobolev spaces in section 3. In section 4 we show that certain functions of the Laplacian including the heat kernel and the resolvent extend to bounded operators in weighted $L^2$-spaces. Section 5 deals with the comparison of weighted Sobolev spaces with respect to equivalent metrics. Then we prove Theorem 0.1 in section 6. In section 7 we deal with the existence and completeness of wave operators. First we prove a general result which is based on Theorem 0.1. Then we consider the case of a manifold with cusps and use the method of Enss to establish the existence and
completeness of the wave operators under weaker assumptions on $\beta$. The final section 8
deals with the analytic continuation of the resolvent, regarded as operator in weighted $L^2$-spaces.

1. Equivalent of Riemannian Metrics.

Let $M$ be an open, connected $C^\infty$–manifold of dimension $n$ and let $\mathcal{M} = \mathcal{M}(M)$ be the
space of all complete Riemannian metrics on $M$. Eichhorn [Ei1] has shown that $\mathcal{M}$ can
be endowed with a canonical topology given by a metrizable uniform structure. We briefly
recall its definition.

For a given Riemannian metric $g$ on $M$, denote by $\nabla^g$ the Levi–Civita connection of $g$
and by $|\cdot|_g$ the norm induced by $g$ in the fibers of $\oplus_{p,q \geq 0} (TM^{\otimes p} \otimes T^*M^{\otimes q})$. Let $h$
be any other Riemannian metric on $M$. For $k \geq 0$ set

\begin{equation}
|g - h|_g(x) = |g - h|_g(x) + \sum_{j=0}^{k-1} |(\nabla^g)^j(\nabla^g - \nabla^h)|_g(x), \quad x \in M.
\end{equation}

and

\begin{equation}
k\|g - h\|_g = \sup_{x \in M} k|g - h|_g(x).
\end{equation}

Recall that two metrics $g, h$ are said to be quasi-isometric if there exist $C_1, C_2 > 0$
such that

\begin{equation}
C_1g(x) \leq h(x) \leq C_2g(x), \quad \text{for all } x \in M,
\end{equation}
in the sense of positive definite forms. We shall write $g \sim h$ for quasi–isometric metrics $g$
and $h$. If $g$ and $h$ are quasi–isometric, then (1.3) implies that for all $p, q \geq 0$, there exist
$A_{p,q}, B_{p,q} > 0$ such that for every tensor field $T$ on $M$ of bidegree $(p, q)$, we have

\begin{equation}
A_{p,q}|T|_g(x) \leq |T|_h(x) \leq B_{p,q}|T|_g(x), \quad x \in M.
\end{equation}

Put $\nabla := \nabla^g$ and $\nabla' := \nabla^h$. Let $\nabla^{p,q}$ and $\nabla'^{p,q}$ be the canonical extension of $\nabla$ and $\nabla'$,
respectively, to the tensor bundle $T^{p,q}(M)$. Then for all $p, q \in \mathbb{N}$ there exists $C_{p,q} > 0$
such that

\begin{equation}
|\nabla^{p,q} - \nabla'^{p,q}|_g(x) \leq C_{p,q}|\nabla - \nabla'|_g(x), \quad x \in M.
\end{equation}

For $k \geq 1$ and $\delta > 0$, set

\[ V_\delta = \{(g, g') \in \mathcal{M} \times \mathcal{M} \mid g \sim g' \text{ and } k\|g - g'\|_g < \delta \}. \]

It is proved in [Ei1], Proposition 2.1, that $\{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform
structure on $\mathcal{M}$. 
Lemma 1.1. Let \( g, h \in \mathcal{M} \). Assume that there exists a compact subset \( K \subset M \) and \( 0 < \delta < 1 \) such that \( |g - h|_g(x) \leq \delta \) for all \( x \in M \setminus K \). Then \( g \) and \( h \) are quasi-isometric.

Proof: Let \( x \in M \setminus K \). Choose geodesic coordinates w.r.t. \( g \), centered at \( x \). Then \( g_{ij}(x) = \delta_{ij} \). Let \( H = (h_{ij}(x)) \) be the matrix representing \( h(x) \) in these coordinates. Denote by \( \| \cdot \| \) the supremum norm of linear maps in \( \mathbb{R}^n \). Then by assumption, we have \( \| H - \text{Id} \| \leq \delta < 1 \). Hence the Neumann series for \( H^{-1} = (\text{Id} - (\text{Id} - H))^{-1} \) converges in norm which implies that \( \| H^{-1} \| \leq 1/(1 - \delta) \). Thus for all \( \xi \in \mathbb{R}^n \), we get
\[
(1 - \delta) \| \xi \|^2 \leq \left( \| H^{-1} \| \right)^{-1} \| \xi \|^2 \leq \langle H\xi, \xi \rangle \leq \| H \| \| \xi \|^2 \leq (1 + \delta) \| \xi \|^2.
\]
This implies that
\[
(1 - \delta)g(x) \leq h(x) \leq (1 + \delta)g(x), \quad \text{for all } x \in M \setminus K.
\]
Since \( K \) is compact, it follows that \( g \) and \( h \) are quasi-isometric. \( \square \)

We need two results from the proof of Proposition 2.1 in [Ei1] which we state as lemmas. For the convenience of the reader we repeat the proofs.

Lemma 1.2. Let \( g, h \in \mathcal{M} \) be quasi-isometric. For every \( k \geq 0 \), there exists a polynomial \( P_k(X_1, \ldots, X_k) \), depending on the quasi-isometry constants, with nonnegative coefficients and vanishing constant term, such that
\[
k|g - h|_h(x) \leq P_k(|g - h|_g(x), |\nabla^g - \nabla^h|_g(x), \ldots, |(\nabla^g)^{k-1}(\nabla^g - \nabla^h)|_g(x)), \quad x \in M.
\]

Proof: From (1.4) follows that
\[
|g - h|_h(x) \leq C_3|g - h|_g(x)
\]
and
\[
|\nabla^g - \nabla^h|_h(x) \leq C_4|\nabla^g - \nabla^h|_g(x), \quad x \in M.
\]
This takes care of the first two terms in (1.1) and settles the question for \( k = 0, 1 \). Now we shall proceed by induction. Let \( k \geq 2 \) and suppose that the lemma holds for \( l \leq k - 1 \). For each \( p \geq 0 \), we have
\[
(\nabla^h)^p(\nabla^h - \nabla^g) = (\nabla^g)(\nabla^h)^{p-1}(\nabla^h - \nabla^g) + (\nabla^h - \nabla^g)(\nabla^h)^{p-1}(\nabla^h - \nabla^g).
\]
Let \( p \leq k \). Using (1.7), (1.5) and the induction hypothesis, we can estimate the pointwise \( h \)-norm of the second term on the right hand side of (1.8) in the desired way. To deal with the first term, we use the formula
\[
(\nabla^g)^p(\nabla^h)(\nabla^h - \nabla^g) = (\nabla^g)^{p+1}(\nabla^h)^{l-1}(\nabla^h - \nabla^g)
\]
\[
+ (\nabla^g)^p(\nabla^h - \nabla^g)(\nabla^h)^{l-1}(\nabla^h - \nabla^g).
\]
Applying the Leibniz rule, we get
\[(\nabla^g)^p(\nabla^h - \nabla^g)(\nabla^h)^{l-1}(\nabla^h - \nabla^g) = \sum_{i=0}^{p} \binom{p}{i} ((\nabla^g)^i(\nabla^h - \nabla^g))(\nabla^g)^{p-i}(\nabla^h)^{l-1}(\nabla^h - \nabla^g)).\]

Inserting (1.8) and iterating these formulas reduces everything to the induction hypothesis. \(\square\)

**Lemma 1.3.** Let \(g_i \in \mathcal{M}, \ i = 1, 2, 3\), and suppose that \(g_1 \sim g_2 \sim g_3\). For every \(k \geq 0\), there exists a polynomial \(Q_k\), depending on the quasi-isometry constants, in the variables \(|g_1 - g_2|_{g_1}(x)\) and \(|g_2 - g_3|_{g_2}(x)\), \(i, j = 0, \ldots, k\), with nonnegative coefficients and vanishing constant term, such that

\[k|g_1 - g_3|_{g_1}(x) \leq Q_k(|g_1 - g_2|_{g_1}(x), |g_2 - g_3|_{g_2}(x)), \quad x \in M.\]

If there exists \(\delta < 1\) such that \(\|g_1 - g_2\|_{g_1} \leq \delta\) and \(\|g_2 - g_3\|_{g_2} \leq \delta\), the dependence on the quasi-isometry constants can be removed.

**Proof:** Since \(g_1 \sim g_2\), it follows from (1.4) that

\[|g_1 - g_3|_{g_1}(x) \leq |g_1 - g_2|_{g_1}(x) + C_1|g_2 - g_3|_{g_2}(x).\]

Set \(\nabla_i = \nabla^{g_i}, \ i = 1, 2, 3\). By the same argument, we get

\[|\nabla_1 - \nabla_3|_{g_1}(x) \leq |\nabla_1 - \nabla_2|_{g_1}(x) + C_2|\nabla_2 - \nabla_3|_{g_2}(x).\]

Thus, the lemma holds for \(k = 0, 1\), and we can use induction to prove the lemma. First observe that for \(p \geq 0\),

\[\nabla_1^p(\nabla_1 - \nabla_3) = \nabla_1^p(\nabla_1 - \nabla_2) + \nabla_1^p(\nabla_2 - \nabla_3).\]

The pointwise \(g_1\)-norm of the first term on the right hand side gives already what we want. The second term can be written as

\[\nabla_1^p(\nabla_2 - \nabla_3) = (\nabla_1 - \nabla_2)\nabla_1^{p-1}(\nabla_2 - \nabla_3) + \nabla_2\nabla_1^{p-1}(\nabla_2 - \nabla_3).\]

Iteration of this formula and application of the Leibniz rule reduces again everything to the induction hypothesis. The last statement again follows from Lemma 1.1. \(\square\)

To set up our equivalence relation in \(\mathcal{M}\), we introduce an appropriate class of functions.

**Definition 1.4.** Let \(\beta : [1, \infty) \to \mathbb{R}\) be a positive, continuous, non-increasing function. Then \(\beta\) is called a function of *moderate decay*, if it satisfies the following conditions

\[1) \quad \sup_{x \in [1, \infty)} x\beta(x) < \infty; \]

\[2) \quad \exists C_\beta > 0 : \beta(x + y) \geq C_\beta \beta(x)\beta(y), \quad x, y \geq 1.\]

Furthermore, \(\beta\) is called of *sub-exponential decay* if for any \(c > 0\), \(e^{cx}\beta(x) \to \infty\) as \(x \to \infty\).
Remark 1. The class of functions which are of moderate or sub-exponential decay are closed under multiplication, and also under raising to positive powers. The function $e^{-tx}$, $t \geq 0$, is of moderate decay and the functions $x^{-1}$ and $\exp(-x^\alpha)$, $0 < \alpha < 1$, are of sub-exponential decay. Thus the class of functions introduced in Definition 1.4 is not empty.

Next we establish some elementary properties of $\beta$.

Lemma 1.5. Let $\beta$ be of moderate decay. Then there exist constants $C > 0$ and $c \geq 0$ such that
\begin{equation}
\beta(x) \geq Ce^{-cx}, \quad x \in [1, \infty).
\end{equation}

Proof: Given $x \in [1, \infty)$, write $x$ as $x = y + n$, where $y \in [1, 2)$ and $n \in \mathbb{N}$. Applying condition 2) of (1.10) repeatedly, we get
\begin{equation}
\beta(x) \geq \beta(y)(C_\beta \beta(1))^n.
\end{equation}
By assumption, $\beta$ is continuous. Hence there exists $C > 0$ such that $\beta(y) \geq C$ for $y \in [1, 2]$. Since $\beta$ is non-increasing, it follows that $C_\beta \beta(1) \leq 1$. Thus there exists $c \geq 0$ such that $C_\beta \beta(1) = e^{-c}$. Together with (1.12) the claim follows. \hfill \Box

Thus for a function $\beta$ of moderate decay there exist constants $c, C_1, C_2 > 0$ such that
\begin{equation}
C_1 e^{-cx} \leq \beta(x) \leq C_2 x^{-1}, \quad x \geq 1.
\end{equation}

Lemma 1.6. Let $\beta$ be a function of moderate decay. Then for all $x, y, q \in M$, we have
\begin{equation}
C_\beta \beta(1 + d(x, y)) \leq \frac{\beta(1 + d(x, q))}{\beta(1 + d(y, q))} \leq \frac{1}{C_\beta \beta(1 + d(x, y))}.
\end{equation}
Moreover, for every $q' \in M$ there exists a constant $C > 0$, depending only on $q$ and $q'$, such that
\begin{equation}
C^{-1} \beta(1 + d(x, q')) \leq \beta(1 + d(x, q)) \leq C \beta(1 + d(x, q')).
\end{equation}

Proof: Since $\beta$ is non-increasing, it follows from (1.10) that
\begin{align*}
\frac{\beta(1 + d(x, q))}{\beta(1 + d(y, q))} & \leq \frac{\beta(1 + d(x, q))}{\beta(1 + d(x, q) + 1 + d(x, y))} \\
& \leq \frac{\beta(1 + d(x, q))}{C_\beta \beta(1 + d(x, q))\beta(1 + d(x, y))} \\
& = \frac{1}{C_\beta \beta(1 + d(x, y))}.
\end{align*}
Switching the roles of $x$ and $y$, we obtain the other inequality in (1.13). Furthermore, switching the roles of $x$ and $q$ and putting $y = q'$ in (1.13) gives (1.14). \hfill \Box
Lemma 1.7. Let $\beta$ be a function of moderate decay. Let $g, h \in \mathcal{M}$, $q \in M$, and suppose that
\begin{equation}
|g - h|_g(x) \leq \beta(1 + d_g(x, q)), \quad x \in M.
\end{equation}
Then $g$ and $h$ are quasi-isometric and there exist constants $C_1, C_2 > 0$ such that
\begin{equation}
C_1 d_g(x, y) \leq d_h(x, y) \leq C_2 d_g(x, y), \quad x, y \in M,
\end{equation}
and
\begin{equation}
C_1 \beta(1 + d_g(x, q)) \leq \beta(1 + d_h(x, q)) \leq C_2 \beta(1 + d_g(x, q)), \quad x \in M.
\end{equation}

Proof: Let $0 < \delta < 1$. From condition 1) of (1.10) follows that there exists $r_0$ such that $\beta(1 + r) \leq \delta$ for $r \geq r_0$. Thus by Lemma 1.1, $g$ and $h$ are quasi-isometric and this implies (1.16). To prove the second part, we first note that it follows from the proof of Lemma 1.1 that
\begin{equation*}
d_h(x, q) \leq (1 + \beta(1 + d_g(x, q))d_g(x, q), \quad d_g(x, q) \geq r_0.
\end{equation*}
Moreover, by condition 1) of (1.10) there exists $C > 0$ such that
\begin{equation*}
\beta(1 + d_g(x, q))d_g(x, q) \leq C, \quad x \in M.
\end{equation*}
Then using (1.10), (1.16) and the assumption that $\beta$ is non-increasing, we get
\begin{equation*}
\beta(1 + d_h(x, q)) \geq \beta(1 + (1 + \beta(1 + d_g(x, q))d_g(x, q)) \geq C_\beta \beta(C)\beta(1 + d_g(x, q)).
\end{equation*}
Switching the roles of $g$ and $h$, we obtain the other inequality. \hfill \Box

Let $k \geq 0$, and consider the following relation for metrics $g, h \in \mathcal{M}$:
\begin{equation}
 There exist $q \in M$ and $C > 0$ such that for all $x \in M$ we have
\begin{equation}
k |g - h|_g(x) \leq C \beta(1 + d_g(x, q)).
\end{equation}

Proposition 1.8. The relation (1.18) defines an equivalence relation in $\mathcal{M}$.

Proof: Let $g, h \in \mathcal{M}$ and suppose that (1.18) holds. Then by Lemma 1.7, $g, h$ are quasi-isometric. Then Lemma 1.2 combined with (1.17) implies that
\begin{equation*}
k |g - h|_h(x) \leq C \beta(1 + d_g(x, y)) \leq C \beta(1 + d_h(x, q)).
\end{equation*}
Thus the relation (1.18) is symmetric. The transitivity follows from Lemma 1.3 and (1.17). By Lemma 1.6, the relation is independent of $q$. \hfill \Box

This justifies the following definition.

Definition 1.9. Let $\beta$ be a function of moderate decay. Two metrics $g, h \in \mathcal{M}$ are said to be $\beta$-equivalemt up to order $k$ if (1.18) holds. In this case we write $g \sim^k \beta h$. 
Example 1.10. Let \((M,g)\) be a complete Riemannian manifold which is Euclidean at infinity, that is, there exists a compact subset \(K \subset M\) such that \((M \setminus K, g)\) is isometric to \(\mathbb{R}^n \setminus B_r(0)\) for some \(r > 0\), where \(\mathbb{R}^n\) is equipped with its standard metric. Let \(\beta(r) = r^{-a}, a > 1\), and let \(h\) be a complete Riemannian metric on \(M\) such that \(h \sim_\beta^k g\) for some \(k \in \mathbb{N}\). Then \(h|_{M \setminus K}\) may be regarded as metric on \(\mathbb{R}^n \setminus B_r(0)\) and if \(h_{ij}\) are the components of \(h|_{M \setminus K}\) with respect to the standard coordinates \(x_1, \ldots, x_n \in \mathbb{R}^n\), then the condition \(h \sim_\beta^k g\) is equivalent to

\[
\left| \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} (h_{ij}(x) - \delta_{ij}) \right| \leq C(1 + \|x\|)^{-a}
\]

for all multideces \(\alpha\) with \(|\alpha| \leq k\) and all \(x \in \mathbb{R}^n \setminus B_r(0)\). Such metrics are called asymptotically Euclidean.

To simplify notation, we will write \(\beta(x)\) in place of \(\beta(1 + d_g(x,q))\). If \(g \sim_\beta^k h\), it follows from Lemma 1.8, that we may use both \(d_g\) and \(d_h\) in (1.18).

Next we show that the \(\beta\)-equivalence can also be defined in a different manner. Namely we have the following proposition.

Proposition 1.10. Let \(k \geq 0\) and let \(g, h \in \mathcal{M}\). Then \(g \sim_\beta^k h\) holds if and only if there exists \(C_1 > 0\) such that

\[
\sum_{i=0}^{k} |(\nabla^g)^i(g - h)|_g(x) \leq C_1 \beta(x), \quad x \in M.
\]

Proof: Let \(g, h \in \mathcal{M}\). The lemma holds obviously for \(k = 0\). Let \(k \geq 1\). Recall that \(\nabla^g g = 0\) and \(\nabla^h h = 0\). Using this fact, we get

\[
(\nabla^g)^k(g - h) = -(\nabla^g)^k h = -(\nabla^g)^{-1}(\nabla^g - \nabla^h)h
\]

\[
= -\sum_{i=0}^{k-1} \binom{k-1}{i} ((\nabla^g)^i(\nabla^g - \nabla^h)) ((\nabla^g)^{k-1-i}(h)).
\]

Suppose that \(k|g - h|_g(x) \leq C \beta(x), x \in M\), for some constant \(C > 0\). Then \(|h|_g(x) \leq C'\) for some constant \(C' > 0\). By induction it follows from (1.5) and (1.20) that

\[
\sum_{i=0}^{k} |(\nabla^g)^i(g - h)|_g(x) \leq C_1 \beta(x), \quad x \in M,
\]

for some constant \(C_1 > 0\), depending on \(C\) and \(k\).

Now assume that (1.21) holds. We observe that for any smooth vector fields \(X, Y, Z\), the following formula holds
\[ h((\nabla^g_X - \nabla^h_X)Y, Z) = \frac{1}{2} \left\{ \nabla^g_X (g - h)(Y, Z) + \nabla^g_X (g - h)(X, Z) \right. \]
\[ \left. - \nabla^g_Z (g - h)(X, Y) \right\}. \]  

From this formula we get
\[ |\nabla^h - \nabla^g|_h \leq C |\nabla^g (g - h)|_h. \]

Taking covariant derivatives of (1.22) and using induction, we obtain
\[ k|h - g|_h(x) \leq C \sum_{i=0}^{k} |(\nabla^g)^i (g - h)|_h(x). \]

By (1.4) and (1.21), we get
\[ k|h - g|_h(x) \leq C \beta(x), \]
and Lemma 1.2 implies that
\[ k|g - h|_g(x) \leq C_1 \beta(x), \quad x \in M, \]
for some constant \( C_1 > 0. \)

Thus, we may define \( \beta \)-equivalence also by requiring that (1.21) holds for some constant \( C_1 \). It follows from the previous proposition that this gives rise to an equivalence relation.

Finally, we study the behavior of the curvature tensor and its covariant derivatives under \( \beta \)-equivalence. Given \( g \in \mathcal{M} \), denote by \( R^g \) the curvature tensor of \( g \).

**Lemma 1.11.** Let \( k \geq 2 \) and let \( g, h \in \mathcal{M} \). Suppose that \( g \sim^k \beta h \). Then there exists \( C_k > 0 \) such that
\[ |(\nabla^g)^i (R^g - R^h)|_g(x) \leq C_k \beta(x), \quad x \in M, \quad i = 0, \ldots, k - 2. \]

**Proof:** Set \( \nabla = \nabla^g, \nabla' = \nabla^h \). We define the exterior differential
\[ d^\nabla : C^\infty(\Lambda^p(T^*M) \otimes TM) \to C^\infty(\Lambda^{p+1}(T^*M) \otimes TM) \]
associated with \( \nabla \) by the following formula
\[ (d^\nabla \alpha)(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i \nabla_{X_i} \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) \]
\[ - \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \]

Then, regarded as operators \( C^\infty(TM) \to C^\infty(\Lambda^2(T^*M) \otimes TM) \), we have
\[ R^\nabla = d^\nabla \circ d^\nabla, \]
and a corresponding formula holds for $\nabla'$. Set $A = \nabla' - \nabla$ and let $X, Y$ be smooth vector fields on $M$. Then we have [Be, p. 25]

$$R^{\nabla'}(X,Y) - R^{\nabla}(X,Y) = \nabla_X (A(Y)) - \nabla_Y (A(X)) - A([X,Y]) - A(X) \circ A(Y) + A(Y) \circ A(X) = (\nabla A)(X,Y) - (\nabla A)(Y,X) - A(X) \circ A(Y) + A(Y) \circ A(X).$$

Differentiating this equality and using induction gives the desired result. \hfill $\square$

Recall that a Riemannian manifold $(M, g)$ is said to have bounded curvature of order $k$, if the covariant derivatives $\nabla^i R$, $0 \leq i \leq k$, of the curvature tensor $R$ are uniformly bounded on $M$, i.e., there exists $C > 0$ such that $|\nabla^i R| (x) \leq C$, $x \in M$, $0 \leq i \leq k$.

**Corollary 1.12.** Let $k \geq 2$ and let $g, h \in \mathcal{M}$. Suppose that $g \sim^k_h h$. Then

1) $(M, g)$ has bounded curvature of order $k - 2$ if and only if $(M, h)$ has bounded curvature of order $k - 2$.
2) The sectional curvature of $(M, g)$ is bounded from below (resp. from above) if and only if the sectional curvature of $(M, h)$ is bounded from below (resp. above).
3) The Ricci curvature of $(M, g)$ is bounded from below (resp. from above) if and only if the Ricci curvature of $(M, h)$ is bounded from below (resp. above).

### 2. Injectivity radius and bounded curvature.

In this section we establish some properties of the injectivity radius on a manifold with bounded sectional curvature. Let $(M, g)$ be a complete, $n$-dimensional Riemannian manifold with bounded sectional curvature, say $|K_M| \leq K$. Let $p \in M$. Recall that the injectivity radius $i(p)$ at $p$ equals the minimal distance from $p$ to its cut locus $C(p)$ (see [CE], [Kl]). Also note that $i(p)$ is a continuous function of $p \in M$ [Kl, Proposition 2.1.10].

**Proposition 2.1.** Let $h$ be another complete Riemannian metric on $M$ with bounded sectional curvature $|K^h_M| \leq K$ and assume that $g$ and $h$ are equivalent. Given $p \in M$, let $i_g(p)$ and $i_h(p)$ denote the injectivity radii at $p$ with respect to $g$ and $h$, respectively. Then there exist constants $c, c' > 0$ such that

$$i_h(p) \geq \min\{ci_g(p), c'\}, \quad p \in M.$$

**Proof:** Since $g$ and $h$ are assumed to be equivalent, there exists $\varepsilon > 0$ such that $e^{-\varepsilon} g \leq h \leq e^{\varepsilon} g$.

Let $x \in M$ and suppose that $i_h(x) < \min \left\{ e^{-2\varepsilon}/(2\sqrt{K}), e^{-\varepsilon}i_g(x)/2 \right\}$. It follows from [CE, Corollary 1.30] that distinct conjugate points along a geodesic (with respect to $h$) have
distance $\geq \pi/\sqrt{K}$. Therefore, by [CE, Lemma 5.6], there exists a closed geodesic loop $\gamma^h$ at $x$ with respect to the metric $h$, with

$$h\text{-length}(\gamma^h) < \min \left\{ e^{-2\pi/\sqrt{K}}, e^{-\epsilon_i(x)} \right\}.$$ 

Hence, we have

$$g\text{-length}(\gamma^h) < \min \left\{ e^{-\epsilon\pi/\sqrt{K}}, i_g(x) \right\}.$$ 

In particular, $g\text{-length}(\gamma^h) < \pi/\sqrt{K}$. Let $r_{\text{max}}^h$ be the maximal rank radius of $\exp_x$ with respect to $g$. Then we obtain $g\text{-length}(\gamma^h) < \pi/\sqrt{K} \leq r_{\text{max}}(x)$. By [BK], Proposition 2.2.2, there exists a unique $g$-geodesic loop $\tilde{\gamma}: [0, 1] \to M$ at $x$ with $g\text{-length}(\tilde{\gamma}) < r_{\text{max}}(x)$, which is obtained from $\gamma^h$ by a length decreasing homotopy $H: [0, 1] \times [0, 1] \to M$ (cf. [BK], 2.1.2). Hence, we have

$$g\text{-length}(\tilde{\gamma}) \leq g\text{-length}(\gamma^h) < \min \left\{ e^{-\epsilon\pi/\sqrt{K}}, i_g(x) \right\}.$$ 

Since $h\text{-length}(H(\cdot, s)) \leq e^s g\text{-length}(\gamma^h) < 2\pi/\sqrt{K}$ for $s \in [0, 1]$, it follows from [Kl, Lemma 2.6.4], that $g\text{-length}(\tilde{\gamma}) > 0$. Parameterize $\tilde{\gamma}$ by $g$-arc length. Then either $\tilde{\gamma}(t)$ or $\tilde{\gamma}(\text{length}(\tilde{\gamma}) - t)$ belongs to the cut locus of $x$ for some $t \leq \frac{1}{2} g\text{-length}(\tilde{\gamma})$. Therefore $i_g(x) < i_g(x)$, a contradiction. 

Let $\beta$ be a function of moderate decay. Suppose that $g \sim_0 h$. Then by Lemma 1.7, $g$ and $h$ are quasi-isometric. Therefore, if $h$ has bounded sectional curvature, then Proposition 2.1 can be applied to $g, h$. For $x \in M$ set

$$(2.1) \quad \bar{i}(x) := \min \left\{ \frac{\pi}{12\sqrt{K}}, i(x) \right\}.$$ 

Then it follows, that under the assumptions of Proposition 2.1, there exists $c_2 > 0$ such that

$$i_h(p) \geq c_2 \bar{i}_g(p), \quad p \in M.$$ 

Next recall the Bishop–Günther inequalities [Gra, Theorem 3.17], [Gro, Lemma 5.3], which give estimates of the volume of small balls from above and below. 

**Lemma 2.2.** For $r \leq \bar{i}(x_0)$,

$$\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r \left( \frac{\sin t\sqrt{K}}{\sqrt{K}} \right)^{n-1} dt \leq \text{Vol}(B_r(x_0)) \leq \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r \left( \frac{\sinh t\sqrt{K}}{\sqrt{K}} \right)^{n-1} dt.$$ 

We note that the inequality on the right hand side holds for all $r \in \mathbb{R}_+$. In particular

$$(2.2) \quad \text{Vol}(B_r(x_0)) = O\left(e^{(n-1)\sqrt{K}r}\right)$$

as $r \to \infty$. 

It is also important to know the maximal possible decay of the injectivity radius.

**Lemma 2.3.** There exists a constant \( C > 0 \), depending only on \( K \), such that
\[
\tilde{\iota}(x) \geq C \tilde{\iota}(p)^n e^{-(n-1)\sqrt{K} d(x,p)}
\]
for all \( x, p \in M \).

**Proof:** Let \( p \in M \) and fix \( r, r_0, s \), with \( r_0 + 2s < \pi/\sqrt{K} \), \( r_0 \leq \pi/4\sqrt{K} \). By [CGT, Theorem 4.7] we get
\[
\tilde{\iota}(x) \geq \frac{r_0}{2} \cdot \frac{1}{1 + \left( V_K^R/\text{Vol}(B_r(p)) \right) \left( V_K^R d(x,p) + \sqrt{K} \tilde{\iota}(p) \right)},
\]
where \( V_s^K \) denotes the volume of a ball of radius \( s \) in the \( n \)-dimensional hyperbolic space of curvature \( -K \). Set \( r_0 = s = \frac{\pi}{5\sqrt{K}} \), \( r = \tilde{\iota}(p) \) and apply Lemma 2.2 to estimate \( \text{Vol}(B_{\tilde{\iota}(p)}(p)) \) from below. Then (2.4) implies
\[
\tilde{\iota}(x) \geq C_1 \tilde{\iota}(p)^n e^{-(n-1)\sqrt{K} (d(x,p) + \tilde{\iota}(p))} \geq C \tilde{\iota}(p)^n e^{-(n-1)\sqrt{K} d(x,p)}.
\]
\( \Box \)

**Corollary 2.4.** Given \( p \in M \), there exists a constant \( C = C(p) > 0 \) such that
\[
\tilde{\iota}(x) \geq C e^{-(n-1)\sqrt{K} d(x,p)}, \quad x \in M.
\]
\( \Box \)

**Lemma 2.5.** There exists a constant \( C \), depending only on \( K \), such that for each \( x, y \in M \) we have the inequality
\[
\tilde{\iota}(y) \geq C \tilde{\iota}(x) e^{-(n-1)\sqrt{K} d(x,y)}/\tilde{\iota}(x).
\]

**Proof:** Let \( \lambda = \max \{ 1, \frac{\pi^2}{144 \tilde{\iota}(x)^2} \} \). Then the injectivity radius \( \iota_\lambda \) at \( x \) with respect to \( \lambda g \) is given by
\[
\iota_\lambda(x) = \lambda \iota(x) = \begin{cases} \iota(x) & \text{if } \iota(x) > \frac{\pi}{12\sqrt{K}}; \\ \frac{\pi}{12\sqrt{K}} & \text{if } \iota(x) \leq \frac{\pi}{12\sqrt{K}}. \end{cases}
\]
Since \( \lambda^{-1} \leq 1 \), the sectional curvature \( K_M^{\lambda g} \) with respect to \( \lambda g \) also satisfies \( |K_M^{\lambda g}| \leq K \).

Let \( r = \frac{\pi}{\sqrt{K}} \), \( r_0 = s = \frac{r}{12} = \frac{\pi}{12\sqrt{K}} \) and set \( d = \lambda^{\frac{1}{2}} d_g(x,y) \). Then \( d \) is the distance between \( x \) and \( y \) with respect to \( \lambda g \).
Let $V_s(y)$ be the volume of the geodesic ball of radius $s$ and center $y$ with respect to $\lambda g$ and let $V^K_s$ denote the volume of a ball of radius $s$ in the $n$-dimensional simply connected space of constant curvature $-K$. Then by [CGT, Theorem 4.3] we get

\[ i_\lambda(y) \geq \frac{r_0}{2} \frac{1}{V^K_{r_0+s}} \geq \frac{r_0 V_s(y)}{4 V^K_{r_0+s}}. \]

Now, [CGT, Proposition 4.1, i)] states that

\[ \frac{V_s(y)}{V^K_s} \geq \frac{V_{d+s}(y)}{V^K_{d+s}}. \]

Together with (2.6) this gives

\[ i_\lambda(y) \geq \frac{r_0 V_{d+s}(y)V^K_s}{4 V^K_{d+s} V^K_{r_0+s}}. \]

From the definition of $d$ it follows that, with respect to the metric $\lambda g$, the ball of radius $d + s$ around $y$ contains the ball of radius $s$ around $x$. Hence $V_{d+s}(y) \geq V_s(x)$. Since $s = \frac{\pi}{12\sqrt{K}} = i_\lambda(x)$, it follows from Lemma 2.2 that there exists $c > 0$ such that $V_s(x) \geq c$ for all $x \in M$. Hence, we get

\[ i_\lambda(y) \geq \frac{r_0 V_s(x) V^K_s}{4 V^K_{d+s} V^K_{r_0+s}} \geq C e^{-((n-1)\sqrt{K}d)} \]

\[ \geq C e^{-((n-1)\max\{\sqrt{K}, \frac{\pi}{12\sqrt{K}}\} d(x,y)}} = C e^{-\frac{(n-1)x d(x,y)}{\tilde{i}(x)}}, \]

for some constant $C > 0$. Now the lemma follows by dividing both sides of this inequality by $\lambda^\frac{1}{2}$.

\[ \Box \]

We can now establish the following basic result about the existence of uniformly locally finite coverings on manifolds with bounded curvature.

**Theorem 2.6.** Assume that $M$ is non-compact. Let $h$ be a continuous real valued function on $M$ such that

- \( \forall x \colon 0 < h(x) \leq \tilde{i}(x) \).
- \( \text{ii)} \) There exists constants $C_1, C_2 > 0$ such that
  \[ h(x) \geq C_1 h(x_0)e^{-C_2 \frac{d(x,x_0)}{\tilde{i}(x_0)}} \]
  for all $x, x_0 \in M$.

Then for each $a \geq 1$, there exists a sequence $\{x_i\}_{i=0}^\infty \subset M$ and a constant $C_3 < \infty$, depending only on $K$, $a$, $C_1$ and $C_2$ such that

1) \[ \bigcup_{i=0}^\infty B_{h(x_i)}(x_i) = M. \]
2) ∀i ∈ N : |{ j | B_{ah(x_i)}(x_i) ∩ B_{ah(x_j)}(x_j) ≠ ∅} | ≤ C_3.

**Proof:** Let x_0 ∈ M. For k ∈ N define recursively

\[ m(k) = \min \{ m ∈ N | B_m(x_0) \setminus \bigcup_{i<k} B_{h(x_i)}(x_i) ≠ ∅ \} \]

and pick x_k ∈ B_{m(k)} \setminus \bigcup_{i<k} B_{h(x_i)}(x_i). In this way we get a sequence \{x_i\}_{i=0}^\infty of points of M. From the construction it follows that this sequence satisfies the following condition:

\[ ∀i, j ∈ N : d(x_i, x_j) ≥ \min\{h(x_i), h(x_j)\}. \tag{2.7} \]

Let m ∈ N. Then by ii), there exists c > 0 such that h(x) ≥ c for all x ∈ B_m(x_0). Hence it follows from (2.7) that d(x_i, x_j) ≥ c if x_i, x_j ∈ B_m(x_0). Since B_m(x_0) is compact, this implies that only finitely many of the x_i’s, say x_1, ..., x_m, are contained in B_m(x_0). Hence

\[ B_m(x_0) ⊂ \bigcup_{i=0}^{m-1} B_{h(x_i)}(x_i) \]

which implies that

\[ M = \bigcup_{i=0}^{∞} B_{h(x_i)}(x_i). \]

It remains to prove 2). Let a ≥ 1. For j ∈ N put B_j = B_{ah(x_j)}(x_j). Let i ∈ N be given and put

\[ Ω_i = \{ x_j | B_i ∩ B_j ≠ ∅ \}. \]

Since h is bounded from above, Ω_i is contained in a compact subset Y of M. By ii) there exists c > 0 such that h(x) ≥ c for all x ∈ Y. Using (2.7), it follows that Ω_i is a discrete subset of Y and hence, Ω_i is a finite set. Let x_{j_1} ∈ Ω_i be such that

\[ h(x_{j_1}) = \max\{h(x_j) | x_j ∈ Ω_i\}. \]

Since B_i ∩ B_{j_1} ≠ ∅, it follows that B_i ⊂ B_{3ah(x_{j_1})}(x_{j_1}) which in turn implies that

\[ B_{h(x_{j_1})}(x_j) ⊂ B_{(4a+1)h(x_{j_1})}(x_{j_1}) \]

for all x_j ∈ Ω_i. Therefore by ii) we get

\[ h(x_j) ≥ C_1 h(x_{j_1}) e^{-\frac{C_2}{h(x_{j_1})}} ≥ C_1 h(x_{j_1}) e^{-4aC_2}. \]

Thus there exists C_4 > 0 such that

\[ h(x_j) ≥ C_4 h(x_{j_1}) \tag{2.8} \]

for all x_j ∈ Ω_i. Obviously C_4 ≤ 1. Hence by i), we obtain

\[ \frac{C_4 h(x_{j_1})}{2} ≤ \frac{i(x_{j_1})}{2} ≤ \frac{π}{24\sqrt{K}}. \tag{2.9} \]
Moreover, by (2.7) and (2.8) we have $d(x_i,x_j) \geq C_4 h(x_j)$ for all $x_k, x_l \in \Omega_i$. Therefore, the balls $B_{c_4 h(x_j)}(x_j), x_j \in \Omega_i$, are pairwise disjoint. Using Lemma 2.2, we get

$$\# \{x_j \mid B_i \cap B_j \neq \emptyset\} \leq \frac{\int_0^{(4a+1)h(x_j)} \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}{\int_0^{c_4 h(x_j)} \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}.$$  

(2.10)

There exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\sinh t\sqrt{K} \leq c_1 t, \quad 0 \leq t \leq \frac{(4a+1)\pi}{12C_4\sqrt{K}};$$

$$\sinh t\sqrt{K} \geq c_2 t, \quad 0 \leq t \leq \frac{\pi}{24\sqrt{K}}.$$  

Hence by (2.9), it follows that the right hand side of (2.10) is bounded by $\frac{c_2}{c_1} \left(\frac{(8a+2)C_4}{C_4 c_2}\right)^n$. This proves the lemma.

Finally we will define and estimate some global invariants of $(M,g)$.

**Definition 2.7.** Let $s > 0$. For $s > \varepsilon > 0$, let $\kappa_\varepsilon(M,g; s) \in \mathbb{N} \cup \{\infty\}$ be the smallest number such that there exists a sequence $\{x_i\}_{i=1}^\infty$ such that $\{B_{s-\varepsilon}(x_i)\}_{i=1}^\infty$ is an open covering of $M$ and

$$\sup_{x \in M} \# \{i \in \mathbb{N} \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \kappa_\varepsilon(M,g; s).$$

(2.11)

Further, let $\kappa(M,g; s) = \kappa_0(M,g, s)$. Put $\kappa(M,g,0) = 1$.

**Lemma 2.8.** $\kappa_\varepsilon(M,g; s)$ is finite for all $s > \varepsilon$. Moreover, there exist constants $C, c > 0$, which depend only on $K$, such that for $s > \frac{2\varepsilon}{\sqrt{K}} + \varepsilon$, we have

$$\kappa_\varepsilon(M,g; s) \leq Ce^{cs}.$$

**Proof:** We may proceed as in the proof of Theorem 2.6 and construct a sequence $\{x_i\}_{i=1}^\infty \subset M$ such that $d(x_i, x_j) \geq s-\varepsilon$ for all $i,j \in \mathbb{N}$ and $\{B_{s-\varepsilon}(x_i)\}_{i=1}^\infty$ is a covering of $M$. Let $x \in M$. If $x \in B_{3s+\varepsilon}(x_i)$, it follows that $B_{s-\varepsilon}(x_i) \subset B_{5s}(x)$. Moreover, $B_{s-\varepsilon}(x_i) \cap B_{s-\varepsilon}(x_j) = \emptyset$ if $i \neq j$. Hence, we get

$$\# \{i \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \frac{\Vol(B_{5s}(x))}{\min_i \Vol(B_{s-\varepsilon}(x_i))}.$$  

(2.12)

Next observe that for any $x_i$ with $d(x,x_i) \leq 5s$ we have $B_{5s}(x) \subset B_{10s}(x_i)$. Moreover, by Lemma 5.3 of [Gro], we have

$$\frac{\Vol(B_{10s}(x_i))}{\Vol(B_{s-\varepsilon}(x_i))} \leq \frac{\int_0^{10s} \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}{\int_0^{s-\varepsilon} \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}.$$  

(8a+2)C_4 \left(\frac{1}{C_4 c_2}\right)^n.
Then combined with (2.12) we obtain
\[
\#\{i \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \frac{\int_0^{10s} \left( \sinh t\sqrt{K} \right)^{n-1} dt}{\int_0^{\frac{\pi}{2}} \left( \sinh t\sqrt{K} \right)^{n-1} dt}.
\]
If \((s - \varepsilon)/2 \geq \pi/\sqrt{K}\), the right hand side can be estimated by \(Ce^{cs}\) for certain constants \(C, c > 0\) depending on \(K\).

\[\Box\]

3. Weighted Sobolev Spaces

In this section we introduce weighted Sobolev spaces on manifolds with bounded curvature.

Let \((M, g)\) be a Riemannian manifold. Let \(\nabla\) be the Levi-Civita connection of \(g\) and let \(\Delta = d^*d\) be the Laplacian on functions with respect to \(g\). Let \(\xi\) be a positive, measurable function on \(M\), which is finite a.e. Given \(m \in \mathbb{N}_0\), and \(p \in \mathbb{N}\), we define the weighted \(L^p\)-space \(L^p_\xi(M, TM^{\otimes m})\) by

\[L^p_\xi(M, TM^{\otimes m}) = \{ \varphi \in L^p_{\text{loc}}(M, TM^{\otimes m}) \mid \xi^{1/p}\varphi \in L^p(M, TM^{\otimes m}) \}.\]

Then for \(k \in \mathbb{N}\) we define the weighted Sobolev space \(W^{p,k}_\xi(M)\) by

\[W^{p,k}_\xi(M) = \left\{ f \in L^p_\xi(M) \mid \nabla^m f \in L^p_\xi(M, TM^{\otimes m}) \text{ for all } m = 1, \ldots, k \right\},\]

where \(\nabla\) is applied iteratively in the distributional sense and the norm of \(f \in W^{p,k}_\xi(M)\) is given by

\[\| f \|_{W^{p,k}_\xi} = \left( \sum_{i=0}^{k} \int_M |\nabla^i f(x)|^p \xi(x) \, dv_g(x) \right)^{1/p}.
\]

Then \(W^{p,k}_\xi(M)\) is a Banach space. In this paper we will only consider the case \(p = 2\). To simplify notation we shall write \(W^{k}_\xi(M)\) in place of \(W^{2,k}_\xi(M)\). The closure of \(C^\infty_0(M)\) in \(W^{k}_\xi(M)\) will be denoted by \(W^{k}_\xi_0(M)\). We shall write \(W^k(M)\) for \(W^{1}_1(M)\) and \(W^0_0(M)\) for \(W^{0,1}_0(M)\). Since 0 is not a weight, this cannot lead to any confusion. Note that \(W^{k}_\xi(M)\) and \(W^{k}_0(M)\) are Hilbert spaces. The weighted Sobolev space \(H^l_\xi(M)\) is defined for even integers \(l\). Let \(k \in \mathbb{N}\). Then

\[H^{2k}_\xi(M) = \left\{ f \in L^2_\xi(M) \mid \Delta^lf \in L^2_\xi(M) \text{ for all } l = 1, \ldots, k \right\}.
\]

The norm is given by

\[\| f \|^2_{H^{2k}_\xi} = \sum_{j=0}^{k} \int_M |\Delta^j f(x)|^2 \xi(x) \, dv_g(x).
\]
As an equivalent norm one can use the norm defined by
\begin{equation}
\| f \|_{H^2_k(\xi)} = \| (\Delta + \text{Id})^k f \|_{L^2_\xi}.
\end{equation}

The closure of $C^\infty_0(M)$ in $H^2_k(\xi)$ will be denoted by $H^2_k(\xi)$. If $\xi \equiv 1$, the Sobolev space $H^2_k(M)$ will be denoted by $H^2_k(M)$ and $H^2_k(\xi)$ by $H^2_k(\xi)$. Note that the Laplacian $\Delta$ induces a bounded operator
\begin{equation}
\Delta_\xi : H^2_k(\xi) \to L^2_\xi(M)
\end{equation}
which is defined in the obvious way.

Next we establish some elementary properties of weighted Sobolev spaces.

**Lemma 3.1.** Assume that $\xi$ is continuous. Let $p, k \in \mathbb{N}$. Then $C^\infty(M) \cap W^{p,k}(\xi)$ is dense in $W^{p,k}(\xi)$ and $C^\infty(M) \cap H^2_k(\xi)$ is dense in $H^2_k(\xi)$. 

**Proof:** We proceed as in the proof of Theorem 1 in [Ma, 1.1.5]. Let $\{U_i : i \in I\}$ be a locally finite covering of $M$ such that for each $i \in I$ there exists an open subset $V_i$ with $\overline{U}_i \subset V_i$ and $V_i$ is diffeomorphic to the unit ball in $\mathbb{R}^n$. Let $\{\varphi_i : i \in I\}$ be an associated partition of unity. Let $u \in W^{p,k}(\xi)$ and let $\varepsilon \in (0, 1/2)$. For each $i \in I$ let $u_i = \varphi_i u$. Then $u_i$ belongs to $W^{p,k}(\xi)$ with $\text{supp } u_i \subset U_i$. Since $\xi$ is continuous, it follows that $u_i \in W^{p,k}(U_i)$ and $\text{supp } u_i$ is contained in the interior of $U_i$. Hence there exists a mollification $g_i \in C^\infty_c(U_i)$ of $u_i$ such that
\[\| g_i - u_i \|_{W^{p,k}} \leq \frac{\varepsilon^i}{\max_{x \in \overline{U}_i} \xi(x)}.
\]
[Ev, Section 5.3]. Then
\[\| g_i - u_i \|_{W^{p,k}} \leq \varepsilon^i.
\]
Clearly $g = \sum_i g_i$ belongs to $C^\infty(M)$. Let $\omega \subset M$ be a relatively compact open subset. Then we have
\[u|_\omega = \sum_i u_i|_\omega,
\]
and the sum is finite. Hence
\[\| g - u \|_{W^{p,k}(\omega)} \leq \sum_i \| g_i - u_i \|_{W^{p,k}} \leq \varepsilon (1 - \varepsilon)^{-1} \leq 2\varepsilon.
\]
This implies that $\| u \|_{W^{p,k}(\omega)} \leq \| u \|_{W^{p,k}} + 2\varepsilon$ for all relatively compact open subsets $\omega \subset M$. Hence by the theorem of Beppo-Levi, we have $g \in C^\infty \cap W^{p,k}(\xi)$ and
\[\| g - u \|_{W^{p,k}} \leq 2\varepsilon.
\]
The proof that $C^\infty(M) \cap H^2_k(\xi)$ is dense in $H^2_k(\xi)$ is similar. \qed
Therefore we can use the following alternative definition of the Sobolev spaces. Let \( C^\infty_k(M) \) denote the space of all \( f \in C^\infty(M) \) such that \( |\nabla^j f| \in L^p_\xi(M) \) for \( j = 0, \ldots, k \). Then \( W^{p,k}_\xi(M) \) is the completion of \( C^\infty_k(M) \) with respect to the norm (3.2). Similarly let \( \tilde{C}^\infty_k(M) \) the space of all \( f \in C^\infty(M) \) such that \( (\Delta + \text{Id})^k f \in L^2_\xi(M) \). Then \( H^{2k}_\xi(M) \) is the completion of \( \tilde{C}^\infty_k(M) \) with respect to the norm (3.3). Similarly let \( \tilde{C}^\infty_{s,k}(M) \) be the space of all \( f \in C^\infty(M) \) such that \( (\Delta + \text{Id})^{s/2} f \in L^2_\xi(M) \). Let \( H^{2s}_\xi(M) \) be the completion of \( \tilde{C}^\infty_{s,k}(M) \) with respect to the norm
\[
\| f \|_{H^{2s}_\xi(M)} = \| (\Delta + \text{Id})^{s/2} f \|_{L^2_\xi}. 
\]

In general the Sobolev spaces \( W^{k}_\xi(M) \) and \( W^{k}_{0,\xi}(M) \) (resp. \( H^{2k}_\xi(M) \) and \( H^{2k}_{0,\xi}(M) \)) will not coincide. If \( (M,g) \) is complete and \( \xi \equiv 1 \), the following is known [Sa].

**Lemma 3.2.** Assume that \( (M,g) \) is complete. Then for all \( k \in \mathbb{N} \) we have
\[
W^k(M) = W^k_0(M), \quad H^{2k}(M) = H^{2k}_0(M), \quad \text{and} \quad W^{2k}(M) = H^{2k}(M).
\]

**Proof:** For the proof we refer to [Sa]. The fact that \( C^\infty_0(M) \) is dense in \( H^{2k}_0(M) \) is an immediate consequence of [Cn]. Indeed by [Cn], \( (\Delta + \text{Id})^k \) is essentially self-adjoint on \( C^\infty_0(M) \) for all \( k \in \mathbb{N} \). Thus
\[
(\Delta + \text{Id})^k(C^\infty_0(M)) = L^2(M).
\]
Let \( f \in H^{2k}(M) \). Then \( (\Delta + \text{Id})^k f \in L^2(M) \) and hence, by (3.6) there exists a sequence \( \{\varphi_j\} \subset C^\infty_0(M) \) such that
\[
\| f - \varphi_j \|_{H^{2k}} = \| (\Delta + \text{Id})^k(f - \varphi_j) \|_{L^2} \to 0
\]
as \( j \to \infty \). \( \square \)

Under additional assumptions on \( \xi \), similar results hold for weighted Sobolev spaces [Sa]. In general the following weaker results hold.

**Lemma 3.3.** For all \( k \in \mathbb{N} \), the natural inclusion \( W^{2k}_\xi(M) \hookrightarrow H^{2k}_\xi(M) \) is bounded.

**Proof:** Let \( k \in \mathbb{N} \). Let \( f \in W^{2k}_\xi(M) \). Then we have \( \nabla^j f \in L^2_\xi(M) \) for \( j = 0, \ldots, 2k \). Recall that
\[
\Delta = -\text{Tr}(\nabla^2 f)
\]
and \( \nabla \text{Tr} = 0 \). Hence it follows that there exists \( C > 0 \) such that
\[
|\Delta^j f|(x) \leq C|\nabla^{2j} f|(x)
\]
for all \( j = 0, \ldots, k \) and \( x \in M \). This implies \( \Delta^j f \in L^2_\xi(M) \) for \( j = 0, \ldots, k \), and
\[
\| f \|_{H^{2k}_\xi} \leq C \| f \|_{W^{2k}_\xi}.
\]
\( \square \)
In order to deal with the inclusion in the other direction, we need some preparation. Let $B_s \subset \mathbb{R}^n$ denote the ball of radius $s > 0$ around the origin in $\mathbb{R}^n$. Given $m \in \mathbb{N}$ and $r, K, \lambda > 0$, denote by $\mathcal{Ell}^m(r, K, \lambda)$ the set of elliptic differential operators

$$P = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha$$

of order $m$ in $B_r$ such that the coefficients of $P$ satisfy:

1. $a_\alpha \in C^m(B_r)$.
2. $\sum_{|\alpha| < m} \| a_\alpha \|_{C^0(B_r)} \leq K$, $\sum_{|\alpha| = m} \| a_\alpha \|_{C^1(B_r)} \leq K$.
3. $\lambda^{-1} \| \xi \|^m \leq \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha \leq \lambda \| \xi \|^m$ for all $\xi \in \mathbb{R}^n$ and $x \in B_r$.

Given an open subset $\Omega \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, $W^k(\Omega)$ is the usual Sobolev space.

**Lemma 3.4.** Let $K, \lambda > 0$ be given. There exists $r_0 = r_0(K, \lambda) > 0$ and $C = C(\lambda) > 0$ such that for all $r \leq r_0$, $P \in \mathcal{Ell}^m(r, K, \lambda)$ and $x_0 \in B_r$:

$$\| u \|_{W^m(B_r)} \leq C \left( \| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right)$$

for all $u \in C_0^\infty(B_r)$.

**Proof:** Let $1 \geq r > 0$ and let $P \in \mathcal{Ell}^m(r, K, \lambda)$. Put

$$P_0 = \sum_{|\alpha| = m} a_\alpha(0)D^\alpha.$$

By Lemma 17.1.2 of [H] there exists $C_1 > 0$ which depends only on $\lambda$ such that for all $u \in C_0^\infty(B_r)$:

$$\| u \|_{W^m(B_r)} \leq C \left( \| P_0 u \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right).$$

Now $Pu = P_0u + (P - P_0)u$. Thus

$$\| u \|_{W^m(B_r)} \leq C \left( \| Pu \|_{L^2(B_r)} + \| (P - P_0)u \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right).$$

Next observe that

$$(P - P_0)u = \sum_{|\alpha| = m} (a_\alpha(x) - a_\alpha(0))D^\alpha u + \sum_{|\alpha| < m} a_\alpha(x)D^\alpha u.$$

Hence by 2):

$$\| (P - P_0)u \|_{L^2(B_r)} \leq r \sum_{|\alpha| = m} \| a_\alpha \|_{C^1(B_r)} \| u \|_{W^m(B_r)}$$

$$+ \sum_{|\alpha| < m} \| a_\alpha \|_{C^0(B_r)} \| u \|_{W^{m-1}(B_r)} \leq K \left( r \| u \|_{W^m(B_r)} + \| u \|_{W^{m-1}(B_r)} \right).$$

By the Poincaré inequality there exists $C_2 > 0$ which is independent of $r \leq 1$ such that for all $u \in C_0^\infty(B_r)$:

$$\| u \|_{W^{m-1}(B_r)} \leq r C_2 \| u \|_{W^m(B_r)}.$$
Using this inequality, it follows from (3.9) that
\[ \| (P - P_0)u \|_{L^2(B_r)} \leq r C(K) \| u \|_{W^m(B_r)}. \]
Together with (3.8) we get
\[ (1 - rC(C(K))) \| u \|_{W^m(B_r)} \leq C \left( \| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right). \]
Set
\[ r_0 = \min\{1, \frac{1}{2C(K)}\}. \]
Then it follows that for all \( r \leq r_0 \) and \( u \in C_0^\infty(B_r) \):
\[ \| u \|_{W^m(B_r)} \leq 2C \left( \| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right). \]

Lemma 3.5. Let \( k \geq 1 \) be even. Assume that \( M \) has bounded curvature of order \( k \). Let \( K > 0 \) be such that \( \sup_{x \in M} |\nabla_l R(x)| \leq K, \ l = 0, \ldots, 2k \). There exist constants \( r_0 = r_0(K) > 0 \) and \( C = C(K) > 0 \) such that for all \( x \in M \) and \( r \leq \min\{r_0, \bar{i}(x_0)\} \) one has
\[ \| u \|_{W^{2k}(B_r(x_0))} \leq C \| u \|_{H^{2k}(B_r(x_0))} \]
for all \( u \in C_0^\infty(B_r(x_0)) \).

Proof: By [Ei2, Corollary 2.6 and 2.7] there exists a constant \( C_1 > 0 \), which depends only on \( K \), such that for every \( x_0 \in M \), every \( r \leq \bar{i}(x_0) \), and all \( i, j, k = 1, \ldots, n \), one has
\[ \sup_{x \in B_r(x_0)} |D^\alpha g_{ij}(x)| \leq C_1, \ |\alpha| \leq 2k, \sup_{x \in B_r(x_0)} |D^\beta \Gamma^i_{jk}(x)| \leq C_1, \ |eta| \leq 2k - 1, \]
where the \( g_{ij} \) and \( \Gamma^i_{jk} \) denote the coefficients of \( g \) and \( \nabla \), respectively, with respect to normal coordinates on the geodesic ball \( B_r(x_0) \) of radius \( r \) with center \( x_0 \).

Let \( x_0 \in M \) and \( r \leq \bar{i}(x_0) \). Let \( B_r \subset T_{x_0}M \) denote the ball of radius \( r \) around the origin. Let \( W^{2k}(B_r) \) be the Sobolev space with respect to the flat connection. Then it follows from (3.10) that there exists \( C_2 = C_2(K) > 0 \) such that
\[ C_2^{-1} \| u \circ \exp_{x_0} \|_{W^{2k}(B_r(x_0))} \leq \| u \|_{W^{2k}(B_r(x_0))} \leq C_2 \| u \circ \exp_{x_0} \|_{W^{2k}(B_r)} \]
for all \( x_0 \in M, r \leq \bar{i}(x_0) \), and \( u \in C_c^\infty(B_r(x_0)) \). Let \( \tilde{g} \) be the metric on \( B_r \) which is the pull-back of \( g \mid B_r(x_0) \) with respect to \( \exp_{x_0} : B_r(x_0) \to B_r(x_0) \). Let \( \tilde{\Delta} \) be the Laplacian on \( B_r \) with respect to \( \tilde{g} \). Then by (3.11) it is sufficient to show that there exists \( C_3 = C_3(K) > 0 \) such that
\[ \| f \|_{W^{2k}(B_r)} \leq C_3 \| (\tilde{\Delta} + \text{Id})^j f \|_{L^2(B_r)} \]
for all \( x_0 \in M, r \leq \bar{i}(x_0) \), and \( f \in C_0^\infty(B_r) \). Set \( P = (\tilde{\Delta} + \text{Id})^k \). By (3.10) there exists \( C_4 > 0 \), which depends only on \( K \), such that \( P \in \mathcal{E} H^{2k}(r, 1, C_4) \). Then by Lemma 3.4, there exist \( r_0 > 0 \) and \( C_5 > 0 \) such that (3.12) holds for all \( x_0 \in M \) and \( r \leq \min\{r_0, \bar{i}(x_0)\} \). This completes the proof of the lemma.
Lemma 3.6. Let $k \in \mathbb{N}$ be even. Suppose that $(M,g)$ has bounded curvature of order $2k$. Let $\beta : M \to \mathbb{R}^+$ be a function of controlled decay. Then there exists a canonical bounded inclusions

\begin{equation}
H^k_{\beta,2kn}(M) \hookrightarrow W^k_\beta(M) \quad \text{and} \quad H^k_\beta(M) \hookrightarrow W^k_{\beta,2kn}(M).
\end{equation}

**Proof:** By Theorem 2.6, there exists a covering $\{B_{\frac{1}{2k^2}}(x_i)| x_i \in \tilde{M}\}_{i=1}^\infty$ of $M$ by balls and a constant $C > 0$ such that

\begin{equation}
\forall x \in M : \#\{x_i \mid x \in B_{\frac{1}{2k^2}}(x_i)\} \leq C.
\end{equation}

Let $\varphi \in C^\infty(\mathbb{R})$ be such that $\varphi = 1$ on $[0,1]$ and $\varphi = 0$ on $[2,\infty)$. For $x \in M$ and $1 \leq j \leq k$, we define

\begin{equation}
\varphi_{j,x}(y) = \begin{cases} 
\varphi(2^j \frac{d(x,y)}{d(x)}) & y \in B_{\frac{1}{2k^2}}(x); \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Then $\varphi_{j,x} \in C^\infty_0(M)$. Let $f \in H^k(M)$. Using Lemma 3.1, it follows that $\varphi_{j,x}f \in H^k(B_{\frac{1}{2k^2}}(x))$. Then by Lemma 3.5 we get $\varphi_{j,x}f \in W^k(B_{\frac{1}{2k^2}}(x))$ and

\begin{equation}
\nabla^j(\varphi_{j,x}f) = \sum_{p=0}^{j} \begin{pmatrix} j \\ p \end{pmatrix} (\nabla^p \varphi_{j,x})(\nabla^{j-p}f), \quad j = 0, \ldots, k.
\end{equation}

By estimating the supremum-norm of the derivatives of $\varphi_{k,x}$ and using Lemma 3.5, we get

\begin{equation}
\|\varphi_{k,x}f\|_{W^k} \leq C\|f\|_{W^k(B_{\frac{1}{2k^2}}(x))} + C' \sum_{p=1}^{k} \begin{pmatrix} k \\ p \end{pmatrix} i^{-p}(x)\|\varphi_{k-1,x}f\|_{W^{k-p}}
\end{equation}

\begin{equation}
\leq C\|f\|_{H^k(B_{\frac{1}{2k^2}}(x))} + C'' \sum_{p=1}^{k} \begin{pmatrix} k \\ p \end{pmatrix} i^{-p}(x)\|\varphi_{k-1,x}f\|_{H^{k-p}}.
\end{equation}

By induction, this yields

\begin{equation}
\|\varphi_{k,x}f\|_{W^k} \leq C'i^{-k}(x)\|f\|_{H^k(B_{\frac{1}{2k^2}}(x))}.
\end{equation}

Let $f \in H^k_{\beta}$. By Lemma 1.6, Lemma 3.5, (3.14) and (3.17) we get

\begin{equation}
\|f\|_{W^k_{\beta}} \leq C \sum_{i=1}^{\infty} \beta_i^{\frac{1}{2}}(x_i)\|\varphi_{k,x_i}f\|_{W^k_{\beta}(x_i)} \leq C \sum_{i=1}^{\infty} \beta_i^{\frac{1}{2}}(x_i)\|\varphi_{k,x_i}f\|_{W^k}
\end{equation}

\begin{equation}
\leq C \sum_{i=1}^{\infty} \beta_i^{\frac{1}{2}}(x_i)i^{-k}(x_i)\|f\|_{H^k(B_{\frac{1}{2k^2}}(x_i))}.
\end{equation}

By (2.3) there exists a constant $C_1 > 0$ such that

$i(x_i)^{-k}(x_i)^k \leq C_1$
for all $i \in \mathbb{N}$ and $x \in B_i(x_i)$. This implies
\[ \sum_{i=1}^{\infty} \beta_{\frac{1}{2}}^i(x_i) \| x \|_{H^k(B_i(x_i))} \leq C_2 \| f \|_{H^k_{\gamma-2k\beta}}, \]
which together with (3.18) gives the first inclusion. The proof of the second inclusion is analogous.

\[ \square \]

Remark 2. Lemma 3.6 is not optimal. Under additional assumptions on $\beta$ one can show that $W_{2k}^2(M) = H_{2k}^2(M)$ [Sa].

4. Functions of the Laplacian.

Assume that $(M, g)$ is complete. Then $\Delta: C_0^\infty(M) \to L^2(M)$ is essentially self-adjoint and functions $f(\sqrt{\Delta})$ can be defined by the spectral theorem for unbounded self-adjoint operators by
\[ f(\sqrt{\Delta}) = \int_0^\infty f(\lambda) dE_\lambda, \]
where $dE_\lambda$ is the projection spectral measure associated with $\sqrt{\Delta}$. Let $f \in L^1(\mathbb{R})$ be even and let
\[ \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) \cos(\lambda x) \, dx. \]
Then $f(\sqrt{\Delta})$ can also be defined by
\[ (4.1) \]
\[ f(\sqrt{\Delta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) \cos(\lambda \sqrt{\Delta}) \, d\lambda. \]
This representation has been used in [CGT] to study the kernel of $f(\sqrt{\Delta})$. We will use (4.1) to study $f(\sqrt{\Delta})$ as operator in weighted $L^2$-spaces. To this end we need to study $\cos(\lambda \sqrt{\Delta})$ as operator in $L^2_{\beta}(M)$. Given $s > 0$, let $\kappa(M, g, s)$ be the constant introduced in Definition 2.7.

Theorem 4.1. Assume that $(M, g)$ has bounded curvature. Let $\beta$ be a function of moderate decay. Then $\cos(s \sqrt{\Delta})$ extends to a bounded operator in $L^2_{\beta}(M)$ for all $s \in \mathbb{R}$ and there exist $C, c > 0$ such that
\[ (4.2) \]
\[ \| \cos(s \sqrt{\Delta}) \|_{L^2_{\beta}, L^2_{\beta}} \leq C e^{c|s|}, \quad s \in \mathbb{R}. \]
Moreover $\cos(s \sqrt{\Delta}): L^2_{\beta}(M) \to L^2_{\beta}(M)$ is strongly continuous in $s$.

\textbf{Proof:} Let $s > 0$. Choose a sequence $\{x_k\}_{k=1}^\infty \subset M$ which minimizes $\kappa(M, g; s)$. For $k \in \mathbb{N}$ let $P_k$ denote the multiplication by the characteristic function of $B_s(x_k) \setminus \bigcup_{j=0}^{k-1} B_s(x_j)$. Then each $P_k$ is an orthogonal projection in $L^2(M)$ and $L^2_{\beta}(M)$, respectively. Moreover the projections satisfy $P_k P_{k'} = 0$ for $k \neq k'$ and $\sum_{k=1}^{\infty} P_k = 1$, where the series is strongly
convergent. Obviously the image of $P_k$ consists of functions with support in $B_s(x_k)$. Now recall that $\cos(t\sqrt{\Delta})$ has unit propagation speed [CGT, p.19], i.e.,

$$\text{supp } \cos(s\sqrt{\Delta})\delta_x \subset B_t(x)$$

for all $x \in M$ and $t \in \mathbb{R}$. Let $f \in L^2(M)$. Then it follows that

$$\text{supp } \cos(s\sqrt{\Delta})P_k f \subset B_{2s}(x_k)$$

and

$$\text{supp } \cos(s\sqrt{\Delta})\left((1 - \chi_{B_{3s}(x_k)})f\right) \subset M - B_{2s}(x_k).$$

Hence

$$\| \cos(s\sqrt{\Delta}) f \|_\beta^2 = \sum_{k=1}^{\infty} \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta}) f \rangle_\beta$$

(4.3)

$$= \sum_{k=1}^{\infty} \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f) \rangle_\beta.$$ 

Now observe that the norm of $\cos(s\sqrt{\Delta})$ as an operator in $L^2(M)$ is bounded by 1. This implies

$$\left| \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f) \rangle_\beta \right| \leq \sup_{y \in B_{3s}(x_k)} \beta(y) \| P_k f \|_{L^2} \cdot \| \chi_{B_{3s}(x_k)} f \|_{L^2}.$$ 

To estimate the right hand side, we write

$$\sup_{y \in B_{3s}(x_k)} \beta(y) \| P_k f \|_{L^2}^2 = \int_M |P_k f(x)|^2 \sup_{y \in B_{3s}(x_k)} \frac{\beta(y)}{\beta(x)} \beta(x) \, dx.$$ 

Since the support of $P_k f$ is contained in $B_s(x_k)$, we can use (1.13) to estimate the right hand side. This gives

$$\sup_{y \in B_{3s}(x_k)} \beta(y) \| P_k f \|_{L^2}^2 \leq C_{\beta}^{-1} \frac{1}{1 + 4s} \| P_k f \|_{L^2}^2.$$ 

A similar inequality holds with respect to $\| \chi_{B_{3s}(x_k)} f \|_{L^2}$. Putting the estimations together, we get

$$\left| \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f) \rangle_\beta \right| \leq C_{\beta}^{-1} \frac{1}{1 + 6s} \| P_k f \|_{L^2}^2 \cdot \| \chi_{B_{3s}(x_k)} f \|_{L^2}^2.$$ 

(4.4)

Now recall that by Lemma 2.8 we have $\kappa(M, g; s) < \infty$. Hence we get

$$\sum_{k=1}^{\infty} \| \chi_{B_{3s}(x_k)} f \|_{L^2}^2 \leq \kappa(M, g; s) \| f \|_{L^2}^2 < \infty.$$
Together with 4.3 and (4.4) we obtain
\[
\| \cos(s \sqrt{\Delta}) f \|_{L_\beta^2}^2 \leq C^{-1}_\beta \frac{1}{\beta(1 + 6s)} \| f \|_{L_\beta^2} \sum_{k=1}^{\infty} \| \chi_{B_{3s}(x_k)} f \|_{L_\beta^2} \\
\leq C^{-1}_\beta \frac{1}{\beta(1 + 6s)} \kappa(M, g, s)^{1/2} \| f \|_{L_\beta^2}^2.
\]
(4.5)

Recall that by (1.10) we have \( \beta(x) \leq C(1 + d(x, p))^{-1} \), \( x \in M \). Therefore \( L^2(M) \subset L^2_\beta(M) \), and \( L^2(M) \) is a dense subspace of \( L^2_\beta(M) \). This implies that \( \cos(s \sqrt{\Delta}) \) extends to a bounded operator in \( L^2_\beta(M) \). Moreover by (1.11) and Lemma 2.8 it follows that there exist constants \( C, c > 0 \) such that
\[
\| \cos(s \sqrt{\Delta}) \|_{L^2_\beta, L^2_\beta} \leq Ce^c, \quad s \in [0, \infty).
\]
Since \( \cos(-s \sqrt{\Delta}) = \cos(s \sqrt{\Delta}) \), this extends to all \( s \in \mathbb{R} \) such that (4.2) holds. The strong continuity is a consequence of the local bound of the norm and the strong continuity on the dense subspace \( L^2(M) \subseteq L^2_\beta(M) \). \( \square \)

Using Theorem 4.1, we can study \( f(\sqrt{\Delta}) \) as an operator in \( L^2_\beta(M) \). Given \( c \geq 0 \), let
\[
\mathcal{F}^1(c) = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} |\hat{f}(\lambda)| e^{c|\lambda|} d\lambda < \infty \right\}.
\]

**Lemma 4.2.** Assume \((M, g)\) has bounded curvature and let \( \beta \) be a function of moderate decay. Then there exists a constant \( c = c(M, g, \beta) \), such that for all even functions \( f \in \mathcal{F}^1(c) \), the operator \( f(\sqrt{\Delta}) \) extends to a bounded operator in \( L^2_\beta(M) \). Moreover, there exists a constant \( C_1 = C_1(M, g, \beta) > 0 \) such that
\[
\| f(\sqrt{\Delta}) \|_{L^2_\beta, L^2_\beta} \leq C_1 \| \hat{f} \|_{L^1_{\| \cdot \|}}
\]
for all \( f \) as above. If \( \kappa(M, g; s) \) is at most sub-exponentially increasing, then \( c(M, g, \beta) > 0 \) can be chosen arbitrarily.

**Proof:** By Theorem 4.1 there exist constants \( C, c > 0 \), depending on \((M, g, \beta)\), such that
\[
\| \cos(\sqrt{\Delta}) \|_{L^2_\beta, L^2_\beta} \leq C e^{c|s|},
\]
for all \( s \in \mathbb{R} \). Let \( \varphi \in L^2(M) \). Using (4.1), it follows that
\[
\| f(\sqrt{\Delta}) \varphi \|_{L^2_\beta} \leq \frac{C}{\sqrt{2\pi}} \| \hat{f} \|_{L^1_{\| \cdot \|}} \| \varphi \|_{L^2_\beta}.
\]
(4.7)

Since \( L^2(M) \) is dense in \( L^2_\beta(M) \), it follows from (4.7) that \( f(\sqrt{\Delta}) \) extends to a bounded operator in \( L^2_\beta(M) \). The last statement is obvious. \( \square \)

**Remark 3.** It is not difficult to see, that (4.1) is in fact strongly convergent in \( L^2_\beta \).
Corollary 4.3. Suppose that \((M, g)\) has bounded curvature and let \(\beta\) be a function of moderate decay. Then the following holds:

a) For every \(t > 0\), the heat operator \(e^{-t\Delta}\) extends to bounded operator in \(L^2_\beta(M)\). Its norm is uniformly bounded in \(t\) on compact intervals of \(\mathbb{R}^+\).

b) In the region \(\{\lambda \in \mathbb{C}: \text{Re}(\sqrt{-\lambda}) > c(M, g, \beta)\}\) the resolvent \((\Delta - \lambda)^{-1}\) extends to a bounded operator in \(L^2_\beta(M)\). The function of \(\lambda \mapsto (\Delta - \lambda)^{-1}\) is locally bounded and holomorphic on this domain.

c) If \(\beta\) is of sub-exponential decay and \(\kappa(M, g; s)\) is at most sub-exponentially increasing for \(s > s_0\), then \((\Delta - \lambda)^{-1} : L^2_\beta(M) \mapsto L^2_\beta(M)\) is defined and bounded for all \(\lambda \in \mathbb{C} \setminus [0, \infty)\).

Proof: This follows from Lemma 4.2 and

\[
(4.8) \quad \int_{-\infty}^{\infty} e^{-tx^2} \cos(xy) dx = \sqrt{\frac{\pi}{t}} e^{-\frac{\pi^2}{4t}}; \quad \int_{-\infty}^{\infty} \frac{1}{\lambda + x^2} \cos(xy) dx = \frac{\pi}{\sqrt{\lambda}} e^{-\sqrt{\lambda} |y|}.
\]

Let \(\beta\) be of moderate decay. There is a canonical pairing \((\cdot, \cdot)\) between \(L^2_\beta(M)\) and \(L^2_{\beta-1}(M)\) given by

\[
(f, g) = \int_M f(x)g(x) \, dx, \quad f \in L^2_\beta(M), \ g \in L^2_{\beta-1}(M).
\]

This pairing is non-degenerate so that \(L^2_{\beta-1}(M)\) is canonically isomorphic to the dual of \(L^2_\beta(M)\). Moreover, we have the following inclusions

\[
L^2_{\beta-1}(M) \subset L^2(M) \subset L^2_\beta(M).
\]

By duality it follows that Theorem 4.1, Lemma 4.2 and Corollary 4.3 also hold w.r.t. \(\beta^{-1}\). Especially, it follows that \(f(\sqrt{\Delta})\) defined on \(L^2_{\beta^{-1}}(M)\) is the restriction of \(f(\sqrt{\Delta})|_{L^2}\). Moreover, we have the identity

\[
(4.9) \quad f(\sqrt{\Delta})|_{L^2_{\beta^{-1}}} = \left(\tilde{f}(\sqrt{\Delta})|_{L^2_\beta}\right)^*.
\]

Lemma 4.4. Let \(\beta\) be a function of moderate decay. If \(\lambda\) and \(\bar{\lambda}\) satisfy condition b) of Corollary 4.3, then

\[
H^2_\beta(M) = (\Delta - \lambda)^{-1}(L^2_\beta(M)).
\]

Proof: First note that \(C^\infty_0(M)\) is dense in \(L^2_\beta(M)\). Indeed \(C^\infty_0(M)\) is dense in \(L^2(M)\), and \(L^2(M)\) is dense in \(L^2_\beta(M)\). Let \(f = (\Delta - \lambda)^{-1}g, \ g \in L^2_\beta(M)\). Then there exists a sequence \(\{\varphi_i\}_{i \in \mathbb{N}} \subset C^\infty_0(M)\) which converges to \(g\) in \(L^2_\beta(M)\) and \((\Delta - \lambda)^{-1}\varphi_i\) converges to \(f\) in \(L^2(M)\). Let \(\varphi \in C^\infty_0(M)\). Then

\[
\langle f, \Delta \varphi \rangle = \lim_{i \to \infty} \langle (\Delta - \lambda)^{-1}\varphi_i, \Delta \varphi \rangle = \lim_{i \to \infty} \langle \varphi_i + \lambda(\Delta - \lambda)^{-1}\varphi_i, \varphi \rangle = \langle g + \lambda f, \varphi \rangle.
\]
Thus $\Delta f = g + \lambda f \in L^2_\beta(M)$ and hence $f \in H^2_\beta(M)$. Now suppose that $f \in H^2_\beta(M)$ and set $g = (\Delta - \lambda)f$. Then $g \in L^2_\beta(M)$ and we need to show that $f = (\Delta - \lambda)^{-1}g$. Let $\varphi \in C_0^\infty(M)$. By definition of $(\Delta - \lambda)^{-1}g$, there exists a sequence $\{g_i\}_{i \in \mathbb{N}} \subset L^2(M)$ such that $(\Delta - \lambda)^{-1}g_i$ converges to $(\Delta - \lambda)^{-1}g$ in $L^2_\beta(M)$ as $i \to \infty$. Using this fact, we get

$$\langle (\Delta - \lambda)^{-1}g, \varphi \rangle = \langle g, (\Delta - \lambda)^{-1}\varphi \rangle = \langle (\Delta - \lambda)f, (\Delta - \lambda)^{-1}\varphi \rangle.$$  \hspace{1cm} (4.10)

Now observe that $(\Delta - \lambda)^{-1}\varphi$ belongs to $H^2(M)$. By Lemma 3.1, there exists a sequence $\{\varphi_i\}_{i \in \mathbb{N}} \subset C_0^\infty(M)$ which converges to $(\Delta - \lambda)^{-1}\varphi$ in $H^2(M)$. Thus

$$\langle (\Delta - \lambda)f, (\Delta - \lambda)^{-1}\varphi \rangle = \lim_{i \to \infty} \langle (\Delta - \lambda)f, \varphi_i \rangle = \langle f, (\Delta - \lambda)\varphi_i \rangle = \langle f, \varphi \rangle.$$

Together with (4.10) this implies that $f = (\Delta - \lambda)^{-1}g$. \hfill $\Box$

**Lemma 4.5.** Let $\beta$ be a function of moderate decay. Then $(\Delta + \lambda)(C_0^\infty(M))$ is dense in $H^2_\beta(M)$ for every $\lambda \in \mathbb{R}^+$. \hfill $\Box$

**Proof:** As in (3.6) it follows from the essential self-adjointness of $\Delta + \lambda$ Id that $(\Delta + \lambda)(C_0^\infty(M))$ is dense in $L^2(M)$. Moreover since $\beta$ is monotonically decreasing, we have that $L^2(M) \subset L^2_\beta(M)$ is dense and $\| f \|_\beta \leq C \| f \|$ for $f \in L^2(M)$. This implies that $(\Delta + \lambda)(C_0^\infty(M))$ is also dense in $L^2_\beta(M)$. \hfill $\Box$

**Corollary 4.6.** Let $\beta$ be of moderate decay. Then $C_0^\infty(M)$ is dense in $H^2_\beta(M)$.

**Proof:** Let $f \in H^2_\beta(M)$. Let $\lambda \gg 0$. By Lemma 4.4 there exists $g \in L^2_\beta(M)$ such that $f = (\Delta + \lambda)^{-1}g$. By Lemma 4.5 there exists a sequence $\{\varphi_i\}_{i \in \mathbb{N}} \subset C_0^\infty(M)$ such that $(\Delta + \lambda)\varphi_i$ converges to $g$ in $L^2_\beta(M)$ as $i \to \infty$. Thus $\varphi_i \to f$ in $L^2_\beta(M)$ and $(\Delta + \lambda)\varphi_i$ converges to $g = (\Delta + \lambda)f$ as $i \to \infty$. This implies that $\varphi_i$ converges to $f$ in $H^2_\beta(M)$. \hfill $\Box$

### 5. Equivalent Metrics and Sobolev Spaces.

In this section we study the dependence of the Sobolev spaces on the metric. We will prove, that if $g \sim^k_\beta h$ for an appropriate $\beta$, then the Sobolev spaces defined with respect to $g$ and $h$ are equivalent up to order $k$. We assume that all metrics have bounded sectional curvature. To indicate the dependence of the corresponding Sobolev space on the Riemannian metric $g$, we will write $W^k_\xi(M; g)$ and $H^k_\xi(M; g)$, respectively.

**Lemma 5.1.** Let $\beta$ be of moderate decay. Assume that $g \sim^k_\beta h$. Then the Sobolev spaces $W^k_\xi(M; g)$ and $W^k_\xi(M; h)$ are equivalent.

**Proof:** First note that by Lemma 1.7 the metrics $g$ and $h$ are quasi-isometric. This implies that $L^2_\xi(M, g)$ and $L^2_\xi(M; h)$ are equivalent. So the statement of the lemma holds for $k = 0$. 
Let \( f \in C^\infty(M) \). Let \( k \geq 1 \). By induction we will prove that for \( l \leq k \) there exists \( C_l > 0 \) such that for \( a, b \in \mathbb{N}_0 \), \( a + b = l \),

\[
(5.1) \quad \left| (\nabla^g)^a (\nabla^h)^b f|_h(x) \right| \leq C_l \sum_{i=0}^{a+b} \left| (\nabla^g)^i f|_g(x) \right|, \quad x \in M.
\]

Let \( l = 1 \). Since on functions the connections equal \( d \), \((5.1)\) follows from quasi-isometry of \( g \) and \( h \).

Next suppose that \((5.1)\) holds for \( 1 \leq l < k \). To establish \((5.1)\) for \( l + 1 \), we proceed by induction with respect to \( a \). Let \( a, b \in \mathbb{N} \) with \( a + b = l + 1 \). If \( a = l + 1 \) there is nothing to prove. Let \( a < l + 1 \). Then

\[
(5.2) \quad (\nabla^g)^a (\nabla^h)^b f = (\nabla^g)^a (\nabla^h - \nabla^g)(\nabla^h)^{b-1} f + (\nabla^g)^a (\nabla^h)^{b-1} f.
\]

and therefore, we get

\[
\left| (\nabla^g)^a (\nabla^h)^b f|_h(x) \right| \leq \left| (\nabla^g)^a (\nabla^h - \nabla^g)(\nabla^h)^{b-1} f|_h(x) \right| + \left| (\nabla^g)^{a+1} (\nabla^h)^{b-1} f|_h(x) \right|, \quad x \in M.
\]

Using \( g \sim_h h \) together with the binomial formula and the induction hypothesis, it follows that \((5.1)\) holds for \( l + 1 \). Especially, putting \( a = 0 \) we get

\[
(5.3) \quad \left| (\nabla^h)^l f|_h(x) \right| \leq C_l \sum_{i=0}^{l} \left| (\nabla^g)^i f|_g(x) \right|, \quad x \in M, \; l \leq k.
\]

Suppose that \( f \in C^\infty(M) \cap W^k(M; g) \). Then \((5.3)\) implies that \( f \in C^\infty(M) \cap W^k(M; h) \) and

\[
\| f \|_{W^k(M; h)} \leq C \| f \|_{W^k(M; g)}.
\]

By Lemma 3.1, \( C^\infty(M) \cap W^k(M; g) \) is dense in \( W^k(M; g) \). Therefore this inequality holds for all \( f \in W^k(M, g) \). By symmetry, a similar inequality holds with the roles of \( g \) and \( h \) interchanged. This concludes the proof.

Next we compare the Sobolev spaces \( H^2_k(M; g) \) and \( H^2_k(M; h) \). Let \( \Delta_g \) denote the Laplace operator with respect to the metric \( g \). Recall, that

\[
\Delta_g = (\nabla^g)^* \nabla^g;
\]

and that the formal adjoint \((\nabla^g)^* \) of \( \nabla^g \) is given by

\[
(5.4) \quad (\nabla^g)^* = -\text{Tr}(g^{-1} \nabla^g),
\]

where \( g^{-1} : T^*M \to TM \) is the isomorphism induced by the metric and \( \text{Tr} : T^*M \otimes TM \to \mathbb{R} \) denotes the contraction. Since \( \nabla^g \text{Tr} = 0 \) and \( \nabla^g g^{-1} = 0 \), we get

\[
(5.5) \quad \Delta^k_g = (-1)^k (\text{Tr} \; g^{-1})^k (\nabla^g)^{2k}.
\]
Lemma 5.2. Assume that $g \sim_{\beta}^{2k} h$. Then for each $l$, $0 \leq l \leq 2k$ and $j$, $0 \leq j \leq 2l$, there exist sections $\xi_{jl}^g, \xi_{jl}^h \in C^\infty(\text{Hom}((T^*M)^{\otimes j}, \mathbb{R}))$ such that

\begin{equation}
(5.6) \quad \Delta^l_g - \Delta^l_h = \sum_{j=0}^{2l} \xi_{jl}^g \circ (\nabla^g)^j = \sum_{j=0}^{2l} \xi_{jl}^h \circ (\nabla^h)^j
\end{equation}

and there exists $C > 0$ such that for $0 \leq p \leq l$

\begin{equation}
(5.7) \quad |(\nabla^g)^p \xi_{jl}^g|_g(x) \leq C \beta(x), \quad |(\nabla^h)^p \xi_{jl}^h|_h(x) \leq C \beta(x), \quad x \in M.
\end{equation}

Proof: Using (5.5) we get

\begin{equation}
(5.8) \quad (\nabla^g)^j - (\nabla^h)^j = (\nabla^g)^{j-1} (\nabla^g - \nabla^h) + (\nabla^h)^{j-1} (\nabla^h - \nabla^g).
\end{equation}

First consider the second term. Note that there exists $C > 0$ such that

\begin{equation}
(5.9) \quad |(\nabla^g)^p ((\nabla^g)^l - (\nabla^h)^l) |_g(x) \leq C |(\nabla^g)^p (g - h)|_g(x).
\end{equation}

Since $g \sim_{\beta}^{2k} h$, the right hand side is bounded by $C \beta(x)$. By symmetry, the same estimation holds with respect to $h$.

To deal with the first term on the right hand side of (5.8), we use

\begin{equation}
(\nabla^g)^j - (\nabla^h)^j = (\nabla^g)^{j-1} (\nabla^g - \nabla^h) + (\nabla^h)^{j-1} (\nabla^h - \nabla^g)
\end{equation}

and proceed by induction with respect to $j$. \hfill \Box

Corollary 5.3. Let $\beta$ be of controlled decay. Assume that $\beta \sim_{\beta}^{-2kn}$ is bounded, $g \sim_{\beta}^{2k} h$ and $(M,g)$ and $(M,h)$ have both bounded curvature of order $2k$. Then $H^{2k}(M,g)$ and $H^{2k}(M,h)$ are equivalent for all functions $\rho$ of controlled decay.

Proof: Let $f \in C^\infty(M) \cap H^{2k}(M;g)$. Using Lemma 3.6 and Lemma 5.1 we get

\begin{equation}
\| f \|_{H^{2k}(M;g)} \geq C_1 \| f \|_{W^{2k}_{\text{loc}}(M;g)} \geq C_2 \| f \|_{W^{2k}_{\text{loc}}(M;h)} \geq C_3 \| f \|_{W^{2k}_{\text{loc}}(M;h)}.
\end{equation}

By Lemma 5.2 it follows that $f \in C^\infty(M) \cap H^{2k}(M,h)$ and there exists a constant $C > 0$, which is independent of $f$, such that

\begin{equation}
\| f \|_{H^{2k}(M;h)} \leq C \| f \|_{H^{2k}(M;g)}.
\end{equation}

By symmetry, a similar inequality holds with $g$ and $h$ interchanged. \hfill \Box
Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with bounded sectional curvature, \(|K_M| < K\). Let \(e^{-t\Delta_g}(x, y)\) denote the heat kernel of \(\Delta_g\). Let \(0 < a_1 < a_2 < \infty\). Let \(\bar{i}\) be the modified injectivity radius defined by (2.1). It follows from [CGT, Proposition 1.3], that there exist \(C, c_1 > 0\) such that
\[
e^{-t\Delta_g}(x, y) \leq C \bar{i}(x)^{-\frac{3}{2}} \bar{i}(y)^{-\frac{3}{2}} e^{-c_1 d^2(x,y)}, \quad t \in [a_1, a_2].
\]

Let \(c < c_1\). Then by (6.1) and (2.3) there exists \(C > 0\) such that
\[
e^{-t\Delta_g}(x, y) \leq C \bar{i}(x)^{-\frac{n(n+1)}{2}} e^{-c_1 d^2(x,y)}, \quad t \in [a_1, a_2].
\]

**Lemma 6.1.** Let \(\beta\) be a function of moderate decay. Assume that there exist real numbers \(a, b\) such that
\[
\begin{align*}
&\text{i)} \quad a + b = 2, \\
&\text{ii)} \quad \beta^b \in L^1(M), \\
&\text{iii)} \quad \beta^{\alpha - \frac{n(n+1)}{2}} \in L^\infty(M).
\end{align*}
\]
Let \(M_\beta\) the operator of multiplication by \(\beta\). Then for every \(p \in \mathbb{N}_0\), the operator \(M_\beta \Delta_g^p e^{-t\Delta_g}\) is Hilbert-Schmidt. For \(t\) in a compact interval in \(\mathbb{R}^+\), the Hilbert-Schmidt norm is bounded.

**Proof:** we have
\[
M_\beta \Delta_g^p e^{-t\Delta} = \left( M_\beta e^{-\frac{t}{2} \Delta} \right) \left( \Delta_g^p e^{-\frac{t}{2} \Delta} \right).
\]
Note that the operator norm of \(\Delta_g^p e^{-\frac{t}{2} \Delta}\) is bounded on compact subsets of \(\mathbb{R}^+\). Hence we may assume that \(p = 0\). By Corollary 4.3, 1), it follows that \(e^{-t\Delta}\) extends to a bounded operator in \(L^2(M)\) and its norm is uniformly bounded for \(0 < a < t < b\). The condition \(\beta^b \in L^1(M)\) implies that \(1 \in L^2(M)\). Hence \(e^{-t\Delta}1 \in L^2(M)\). Let \(e^{-t\Delta}(x, y)\) be the kernel of \(e^{-t\Delta}\). Then
\[
\langle 1, e^{-t\Delta}1 \rangle_{L^2(M)} = \int_M \int_M \beta^{b}(x) e^{-t\Delta_g}(x, y) dy dx.
\]
The integral converges since \(e^{-t\Delta}(x, y)\) is positive. Thus we get
\[
\int_M \int_M |\beta(x) e^{-t\Delta_g}(x, y)|^2 dy dx = \int_M \int_M \beta^2(x) e^{-2t\Delta_g}(x, y) dy dx
\leq \sup_{z,w \in M} |\beta^a(z) e^{-t\Delta_g}(z, w)| \int_M \int_M \beta^{b}(x) e^{-t\Delta_g}(x, y) dy dx
\leq C \sup_{z \in M} |\beta^a(z) \frac{n(n+1)}{2} | \int_M \beta^{b}(x) \left( e^{-t\Delta}(1) \right) (x) dx
\leq C_1 \| e^{-t\Delta}(1) \|_{L^2(M)}.
\]
This proves the lemma. \(\square\)
Lemma 6.2. Assume $\beta$ is a function of moderate decay and that there exist real numbers $a, b$ such that

i) $b \geq 1$ and $a + b = 2$,

ii) $\beta^\frac{3}{4} \in L^1(M)$,

iii) $\beta^\frac{3}{4} \cdot \frac{n(n+2)}{2} \in L^\infty(M)$.

Let $M_\beta$ be the operator of multiplication by $\beta$. Then the operator $M_{l-2n} M_\beta \Delta^p e^{-t\Delta}$ is a trace-class operator for $p \in \mathbb{N}$. For $t$ in a compact interval, the trace-class norm is bounded.

Proof: We decompose the operator as

$$M_{l-2n} M_\beta \Delta^p e^{-t\Delta} = \left\{M_{l-2n} M_\beta e^{-\frac{1}{2} \Delta} \mathbb{1}_{M_{\frac{1}{2}}} \right\} \cdot \left\{M_{\beta^\frac{3}{4}} \Delta^p e^{-\frac{1}{2} \Delta^2} \right\}.$$

Since $\beta$ is non-increasing and $\beta(x) \leq 1/2$ outside a compact set, it follows that $\beta^\frac{3}{4} \leq C \beta^\frac{1}{2}$ for $b \geq 1$. Hence by ii) we get $\beta^\frac{3}{4} \in L^1(M)$. Moreover by iii) it follows that $\beta^\frac{3}{4} \cdot \frac{n(n+2)}{2} \in L^\infty(M)$. Hence by Lemma 6.1, the second factor on the right hand side of (6.4) is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm is bounded for $t$ in a compact interval in $\mathbb{R}^+$. It remains to show that the first factor is Hilbert-Schmidt and that the Hilbert-Schmidt norm is bounded on compact intervals. By iii) we have

$$\beta^\frac{3}{4} \cdot \frac{n(n+2)}{2} - 2n \in L^\infty(M).$$

Using this observation together with (6.2), we get

$$\int_M \int_M |\tilde{r}^{-2n}(x) \beta(x) e^{-t\Delta}(x, y) \beta^{-\frac{1}{4}}(y)|^2 dxdy \leq C \sup_{z \in M} |\tilde{r}^{-\frac{n(n+1)}{2} - 2n} (z) \beta^a(z)| \int_M \int_M \beta^b(x) e^{-t\Delta}(x, y) \beta^{-\frac{3}{4}}(y) dxdy.$$

Now observe that by ii), $\beta^{-\frac{1}{4}}$ belongs to $L^2_{\beta^\frac{b+1}{4}}(M)$. Since $\beta^\frac{b+1}{4} \leq C \beta^\frac{b}{2}$, it follows from ii) that $\beta^\frac{b+1}{4}$ is integrable. Hence by Corollary 4.3, $e^{-t\Delta}$ extends to a bounded operator in $L^2_{\beta^\frac{b+1}{4}}(M)$. Therefore $\int_M e^{-t\Delta}(x, y) \beta^{-\frac{1}{4}}(y) dy \in L^2_{\beta^\frac{b+1}{4}}$, and the norm is uniformly bounded for $t$ in a compact interval of $\mathbb{R}^+$. Next note that $\beta^b \in L^2_{\beta^\frac{b+1}{4}}$. Hence

$$\int_M \int_M \beta^b(x) e^{-t\Delta}(x, y) \beta^{-\frac{3}{4}}(y) dxdy = \left\langle \beta^b, e^{-t\Delta} \beta^{-\frac{1}{4}} \right\rangle < \infty.$$

This implies the lemma. \qed

Lemma 6.3. Let $\beta$ be a function of moderate decay, satisfying the conditions of Lemma 6.2. Let $g, h$ be two complete metrics on $M$ such that $g \sim_{\beta^2} h$. Let $\Delta_g$ and $\Delta_h$ be the Laplacians of $g$ and $h$, respectively. Then

$$(\Delta_g - \Delta_h) e^{-t\Delta_g} \quad \text{and} \quad e^{-t\Delta_g} (\Delta_g - \Delta_h)$$
are trace class operators, and the trace norm is uniformly bounded for \( t \) in a compact subset of \((0, \infty)\).

**Proof:** We decompose \( e^{-t\Delta_g} \) as

\[
e^{-t\Delta_g} = \left( e^{-\frac{1}{2}\Delta_g} M_{\beta^{-\frac{1}{2}}} \right) \cdot \left( M_{\beta^{\frac{1}{2}}} e^{-\frac{1}{2}\Delta_g} \right).
\]

By Lemma 6.1, the second factor is a Hilbert-Schmidt operator and it suffices to show that \( (\Delta_g - \Delta_h) e^{-t\Delta_g} M_{\beta^{-\frac{1}{2}}} \) is Hilbert-Schmidt and that the Hilbert-Schmidt norm is bounded for \( t \) in a compact interval. Using Lemma 5.2 and Lemma 3.6, it follows that the Hilbert-Schmidt norm can be estimated by

\[
\| (\Delta_g - \Delta_h) e^{-t\Delta_g} M_{\beta^{-\frac{1}{2}}} \|_2^2 \leq C \sum_{i=0}^{2} \left( \int_M \int_M |(\nabla^g)^i e^{-t\Delta_g} (x, y) \beta^{-\frac{1}{2}}(y) |^2 \beta^2(x) \, dx \, dy \right)
\]

\[
= C \int_M \| e^{-t\Delta_g} (\cdot, y) \beta^{-\frac{1}{2}}(y) \|_{W^2_{\beta^2}}^2 \, dy
\]

\[
\leq C_1 \int_M \| e^{-t\Delta_g} (\cdot, y) \beta^{-\frac{1}{2}}(y) \|_{L^2_{\beta^{21-4n}}}^2 \, dy
\]

\[
\leq C_2 \sum_{q=0}^{1} \left( \int_M \| \beta(\cdot) i^{-2n}(\cdot) \Delta^q_g e^{-t\Delta_g} (\cdot, y) \beta^{-\frac{1}{2}}(y) \|_2^2 \, dy \right)
\]

\[
= C_2 \sum_{q=0}^{1} \| M_{\beta} M_{i^{-2n}} \Delta^q_g e^{-t\Delta_g} M_{\beta^{-\frac{1}{2}}} \|_2^2.
\]

By Lemma 6.2 the right hand side is finite and bounded for \( t \) in a compact interval of \( \mathbb{R}^+ \). To prove that \( e^{-t\Delta_g} (\Delta_g - \Delta_h) \) is a trace class operator, it suffices to establish it for its adjoint \( (\Delta_g - \Delta_h)^* e^{-t\Delta_g} \) with respect to \( g \). By (5.6) and (5.4) we have

\[
\Delta_g - \Delta_h)^* = (\zeta_0^g)^* + (\nabla^g)^* \circ (\zeta_1^g)^* + [(\nabla^g)^*]^2 \circ (\zeta_2^g)^*.
\]

Using (5.4) and (5.7), it follows that there exist \( \eta_j \in C^\infty(\text{Hom}((T^*M)^{\otimes j}, \mathbb{R})) \) such that

\[
\Delta_g - \Delta_h)^* = \eta_0 + \eta_1 \circ \nabla^g + \eta_2 \circ (\nabla^g)^2
\]

and these sections satisfy

\[
|\eta_j|_g(x) \leq C \beta(x), \quad 0 \leq j \leq 2, \ x \in M.
\]

Using (6.9) and (6.10) we can proceed as above and prove that \( (\Delta_g - \Delta_h)^* e^{-t\Delta_g} \) is a trace class operator. \( \square \)

We are now ready to prove Theorem 0.1. We note that for equivalent metrics, the Hilbert spaces \( L^2(M, g) \) and \( L^2(M, h) \) are equivalent. Hence we may regard \( e^{-t\Delta_h} \) as a bounded operator in \( L^2(M, g) \).
Proof of Theorem 0.1: By Duhamel’s principle we have
\[ e^{-t\Delta_g} - e^{-t\Delta_h} = \int_0^t e^{-s\Delta_g}(\Delta_h - \Delta_g)e^{-(t-s)\Delta_h} \, ds \]
(6.11)
\[ = \int_0^{t/2} e^{-s\Delta_g}(\Delta_h - \Delta_g)e^{-(t-s)\Delta_h} \, ds + \int_{t/2}^t e^{-s\Delta_g}(\Delta_h - \Delta_g)e^{-(t-s)\Delta_h} \, ds. \]
The integrals converge in the strong operator topology. By Lemma 6.3 the first integral is a trace class operator. In order to prove that the second integral is a trace class operator, it is sufficient to prove, that its adjoint with respect to \( h \) is of the trace class. This adjoint can be written as the strong integral
\[ \int_{t/2}^t (e^{-(t-s)\Delta_g})^* \, (\Delta_h - (\Delta_g)^*h) \, e^{-s\Delta_h} \, ds. \]
(6.12)
Since \((e^{-(t-s)\Delta_g})^*h\) is uniformly bounded in \( s \), it follows again from Lemma 6.3 that (6.12) is a trace class operator. \( \square \)

7. Existence and completeness of wave operators

In this section we study the wave operators associated to \((\Delta_g, \Delta_h)\) for equivalent metrics \( g \) and \( h \).

**Theorem 7.1.** Let \( g \) and \( h \) be two complete metrics of bounded curvature on \( M \) which satisfy the assumptions of Theorem 0.1. Let \( P_{ac}(\Delta_g) \) be the orthogonal projection onto the absolutely continuous subspace of \( \Delta_g \). Then the strong wave operators
\[ W_\pm(\Delta_h, \Delta_g) = s - \lim_{t \to \pm \infty} e^{it\Delta_h}e^{-it\Delta_g}P_{ac}(\Delta_g) \]
exist and are complete. In particular, the absolutely continuous parts of \( \Delta_g \) and \( \Delta_h \) are unitarily equivalent.

**Proof:** By Theorem 0.1, \( e^{-t\Delta_g} - e^{-t\Delta_h} \) is trace class. Then the existence and completeness of the wave operators follows from the invariance principle of Birman and Kato [Ka, Chapter X, Theorem 4.7]. \( \square \)

**Examples.** We give some examples to demonstrate Theorem 0.1:

1) Let \( M \) be a manifold with cylindrical ends. Then \( \tilde{t} \) is bounded from below, and we may take \( b = 2, a = 0 \). The condition \( \beta^2 \in L^1(M) \) is satisfied for \( \beta(t) = t^{\frac{3}{2} - \varepsilon} \) for any \( \varepsilon > 0 \).

2) More generally, let \( M \) be a manifold with bounded geometry of order 2. (i.e. there is a lower bound for the injectivity radius and the covariant derivatives of the curvature of order \( \leq 2 \) are bounded). Then we may choose \( x_0 \in M \) arbitrary and let \( \beta(t) \leq \frac{\text{vol}(B_t(x_0))}{t^{\frac{3}{2} - \varepsilon}} \) for any \( \varepsilon > 0 \). To see this we first notice that if \( M \) is non-compact, the
volume of such a manifold is infinite. This follows from Günthers inequality because we may find infinitely many disjoint balls of the same radius. Let \( a(r) := \frac{\partial}{\partial r} \text{vol}(B_r(x_0)) \). Then \( \int_1^\infty a(r)\beta(1 + r)^\frac{2}{3}dr < \infty \) and

\[
\int_1^\infty a(r)\beta(1 + r)^\frac{2}{3}dr \leq \int_1^\infty a(r)\beta(r)^\frac{2}{3}dr \leq \int_1^\infty a(r) \left( \int_0^r a(s)ds \right)^{-1/2}dr
\]

\[
= \int_0^\infty t^{-1 - \frac{2\pi}{2\pi}}dt < \infty.
\]

3) Let \( M \) be a Riemannian manifold with cusps in the sense of [Mu1]. Assume that \( M \) has bounded curvature. Then the injectivity radius is exponentially decreasing in the distance and the volume of \( M \) is finite. Thus we may take \( b = 1 \). It follows \( a = 1 \), and we may take

\[
\beta(t) = e^{-\left(\frac{n(n+1)}{2} + 4n\right)c\pi},
\]

where \( c \) is chosen such that \( \beta(x) \geq Ce^{-cd(x,\partial)} \).

\[ \square \]

The assumptions on \( \beta \) in Theorem 7.1 that guarantee the existence of the wave operators are not optimal. Under additional assumptions on \( (M, g) \), the conditions on \( \beta \) can be relaxed. For example, let \( (M, g) \) be a complete manifold which is Euclidean at infinity and let \( h \) be a metric on \( M \) which satisfies (1.19), that is \( (M, h) \) is an asymptotically Euclidean manifold. Then Cotta-Ramusino, Krüger, and Schrader [CKS] proved that the wave operators \( W_\pm(\Delta_g, \Delta_h) \) exist. The condition (1.19) is weaker than the assumption which is necessary in Theorem 7.1 in this case. The proof is based on Enss’s method [Si], which applies to this scattering system. An abstract version of Enss’s method has been developed by Amrein, Pearson and Wollenberg [APW], [BW, 16,IV,§15]. This method can be applied in cases where the structure of the continuous spectrum of the “free Hamiltonian” is sufficiently well known. To explain this in more detail we need to introduce some notation.

Let \( C_\infty(\mathbb{R}) \) be the space of all continuous functions on \( \mathbb{R} \) that vanish at infinity. For any closed countable subset \( I \subset \mathbb{R} \) let \( C_\infty(\mathbb{R} - I) \) of all functions \( f \in C_\infty(\mathbb{R}) \) satisfying \( f(x) = 0 \) for \( x \in I \). A subset \( \mathcal{A}_I \) of the space \( C(\mathbb{R}) \) of all bounded continuous functions on \( \mathbb{R} \) is called multiplicative generating for \( C_\infty(\mathbb{R} - I) \), if the linear span of the set

\[
\{ f \mid f = hg, h \in \mathcal{A}_I, g \in C_\infty(\mathbb{R} - I) \}
\]

is dense in \( C_\infty(\mathbb{R} - I) \) with respect to the norm \( || f || = \sup_{x \in \mathbb{R}} |f(x)| \). The main result of [APW] can be stated as follows.

**Theorem 7.2.** Let \( H \) and \( H_0 \) be two self-adjoint operators in a Hilbert space \( \mathcal{H} \). Let \( R_H(\lambda) \) and \( R_{H_0}(\lambda) \) denote the resolvents of \( H \) and \( H_0 \), respectively. Assume that there exist self-adjoint operators \( P_+ \) and \( P_- \) in \( \mathcal{H} \) and a set \( \mathcal{A}_I \) of multiplicative generating functions with respect to some closed countable subset \( I \subset \mathbb{R} \) satisfying the following properties

1. \( P_{ac}(H_0) = P_+ + P_- \) and \( s\text{-lim}_{t \to \pm \infty} e^{itH_0} P_- e^{-itH_0} P_{ac}(H_0) = 0 \).
2. \( (\text{Id} - P_{ac}(H_0))\alpha(H_0) \) is compact for all \( \alpha \in \mathcal{A}_I \).
3. \( R_H(i) - R_{H_0}(i) \) is compact.
(4) $\int_{0}^{\pm\infty} \| (R_H(i) - R_{H_0}(i)) e^{-itH_0} \alpha(H_0) P_\pm \| \ dt < \infty$ for all $\alpha \in \mathcal{A}_I$.

Then the wave operators $W_\pm(H, H_0)$ exist and are complete. Moreover $H$ and $H_0$ have no singularly continuous spectrum and each eigenvalue of $H$ and $H_0$ in $\mathbb{R} - I$ is of finite multiplicity. These eigenvalues accumulate at most at points of $I \cup \{\pm\infty\}$.

For the proof see Corollary 19 in [BW, 16, IV, §15].

As example, we consider a manifold $X$ with cusps as defined in [Mu1]. For simplicity we assume that $X$ has a single cusp. Then $X$ is a complete Riemannian manifold of dimension $n + 1$ that admits a decomposition

$$X = M \cup Y Z$$

in a compact Riemannian manifold $M$ with boundary $Y$ and a half-cylinder $Z = [1, \infty) \times Y$, and $M$ and $Z$ are glued along their common boundary $Y$. The metric $g$ on $X$ is such that its restriction to $Z$ is given by

$$g^Z = u^{-2}(du^2 + g^Y),$$

where $g^Y$ denotes the metric of $Y$. The metric $g$ is the fixed background metric and we consider perturbations $h$ of $g$. As free Hamiltonian $H_0$ we are taking a modification of the Laplacian $\Delta_g$ which is defined as follows. We regard $Y$ as a hypersurface in $X$ that separates $X$ into $M$ and $Z$. Let $C^\infty_c(X - Y)$ be the subspace of all $f \in C^\infty_c(X)$ that vanish in a neighborhood of $Y$. Let $\Delta_0$ denote Friedrichs’s extension of

$$\Delta_0: C^\infty_0(X - Y) \to L^2(X).$$

To begin with we need to study the spectrum of $\Delta_0$. With respect to the decomposition $L^2(X) = L^2(M) \oplus L^2(Z)$ we have

$$\Delta_0 = \Delta_{M,0} \oplus \Delta_{Z,0},$$

where $\Delta_{M,0}$ and $\Delta_{Z,0}$ are the Dirichlet Laplacians on $M$ and $Z$, respectively. Since $M$ is compact, $\Delta_{M,0}$ has pure point spectrum. Let

$$L^2_0(Z) := \{ f \in L^2(Z): \int_Y f(u, y) \ dy = 0 \text{ for almost all } u \in [1, \infty) \}.$$

The orthogonal complement $L^2_0(Z) \perp$ of $L^2_0(Z)$ in $L^2(Z)$ consists of functions which are independent of $y \in Y$ and therefore, can be identified with $L^2([1, \infty), u^{-(n+1)}du)$. The decomposition

$$L^2(Z) = L^2_0(Z) \oplus L^2_0(Z) \perp$$

is invariant under $\Delta_{Z,0}$.

**Lemma 7.3.** The restriction of $\Delta_{Z,0}$ to $L^2_0(Z)$ has a compact resolvent. In particular, $\Delta_{Z,0}$ has pure point spectrum.
Lemma 7.4. Let $\Delta_Y$ be the Laplacian of $Y$. Let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal basis of eigenfunctions of $\Delta_Y$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $f \in C_c^\infty(Z) \cap L^2_0(Z)$. Then $f$ has an expansion of the form

$$f(u, y) = \sum_{k=1}^\infty a_k(u)\phi_k(y),$$

where the series converges in the $C^\infty$-topology. Let $b > 1$ and put $Z_b = [b, \infty) \times Y$. Let $C = \lambda_1^{-1}$. Then we have

$$\| f \|_{L^2(Z_b)}^2 = \sum_{k=1}^\infty \int_b^\infty |a_k(u)|^2 \frac{du}{u^{n+1}} \leq \frac{C'}{b^2} \sum_{k=1}^\infty \lambda_k \int_b^\infty |a_k(u)|^2 \frac{du}{u^n}. \tag{7.4}$$

Now observe that the Laplacian $\Delta_Z$ with respect to the metric (7.1) equals

$$-u^2 \frac{\partial^2}{\partial u^2} + nu \frac{\partial}{\partial u} + u^2 \Delta_Y. \tag{7.5}$$

Moreover, since $a_k \in C_c^\infty((1, \infty))$, we have

$$\int_1^\infty (-u^2 a_k''(u) + nu a_k'(u)) a_k(u) \frac{du}{u^{n+1}} = \int_1^\infty |a_k'(u)|^2 u^{1-n} du \geq 0. \tag{7.6}$$

This together with (7.4) implies

$$\| f \|_{L^2(Z_b)}^2 \leq \frac{C'}{b^2} \langle \Delta_Z f, f \rangle_{L^2(Z)} \leq \frac{C'}{b^2} \| \nabla f \|_{L^2(Z)} \leq \frac{C'}{b^2} \| f \|_{H^1(Z)}. \tag{7.6}$$

Let $H^1_0(Z) := H^1(Z) \cap L^2_0(Z)$. By continuity, (7.4) holds for all $f \in H^1_0(Z)$. By Rellich’s lemma, the embedding

$$i_b: H^1(Z - Z_b) \cap L^2_0(Z - Z_b) \to L^2(Z)$$

is compact. It follows from (7.6) that as $b \to \infty$, $i_b$ converges strongly to the embedding

$$i: H^1_0(Z) \to L^2(Z).$$

Hence $i$ is compact which implies the lemma. \qed

Let

$$D_0 := -u^2 \frac{d^2}{du^2} + nu \frac{d}{du}: C_c^\infty((1, \infty)) \to L^2([1, \infty), u^{-(n+1)}du)$$

and let $L_0$ be the self-adjoint extension of $D_0$ with respect to Dirichlet boundary conditions at 1. By (7.5), the restriction of $\Delta_{Z,0}$ to $L^2_0(Z) \leq L^2([1, \infty), u^{-(n+1)}du)$ is equivalent to $L_0$. The spectrum of $L_0$ is absolutely continuous and equals $[\frac{n^2}{4}, \infty)$. Thus we get the following lemma.

**Lemma 7.4.** The spectrum of $\Delta_0$ is the union of a pure point point spectrum and an absolutely continuous spectrum. The point spectrum consists of eigenvalues of finite multiplicity $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$. The absolutely continuous spectrum is equal to $[\frac{n^2}{4}, \infty)$ and the absolutely continuous part $\Delta_{0,ac}$ of $\Delta_0$ is equivalent to $L_0$. 
Let \( \varepsilon > 0 \) and let \( \beta(t) = e^{-\varepsilon t} \). Let \( h \) be a complete metric on \( X \). We put
\[
H := \Delta_h \quad \text{and} \quad H_0 := \Delta_0.
\]
Since \( H \) and \( H_0 \) are positive operators, we can replace \( i \) by \(-1\) in Theorem 7.2. So let
(7.7) \[
R_g := (\Delta_g + \text{Id})^{-1}, \quad R_h := (\Delta_h + \text{Id})^{-1} \quad \text{and} \quad R_0 := (\Delta_0 + \text{Id})^{-1}.
\]
First we have the following lemma.

**Lemma 7.5.** Suppose that \( h \sim_2 \beta g \). Then \( R_h - R_0 \) is a compact operator.

**Proof:** Since \( Y \subset X \) is a compact hypersurface, it follows that \( R_g - R_0 \) is a compact operator. So it suffices to show that \( R_h - R_g \) is compact. We have
(7.8) \[
R_h - R_g = -R_g (\Delta_h - \Delta_g) R_h.
\]
By Lemma 5.2 we have
(7.9) \[
\Delta_h - \Delta_g = \sum_{j=0}^2 \xi_j \circ (\nabla^h)^j
\]
and \( \xi_j \) satisfies
(7.10) \[
|\xi_j(x)| \leq C e^{-\varepsilon d(x,x_0)}, \quad x \in X.
\]
Now \( R_h : L^2(X) \to W^2(X) \) is continuous. Therefore by (7.9) and (7.10) it follows that
\[
(\Delta_h - \Delta_g) R_h : L^2(X) \to L^2(X)
\]
is a bounded operator. Using again that \( R_g - R_0 \) is compact, it follows from (7.8) that it suffices to show that \( R_0 (\Delta_h - \Delta_g) R_h \) is a compact operator.

For \( a > 1 \) let
\[
X_a = M \cup_Y ([1,a] \times Y).
\]
Denote by \( \chi_a \) the characteristic function of \( X_a \) in \( X \). We claim that \( R_0 \chi_a \) is a compact operator. By (7.2) we have
\[
R_0 = (\Delta_{M,0} + 1)^{-1} \oplus (\Delta_{Z,0} + 1)^{-1}.
\]
Since \( M \) is compact, \((\Delta_{M,0} + 1)^{-1}\) is compact. Let \( \chi_{[1,a]} \) be the characteristic function of the interval \([1,a]\) in \([1,\infty)\). By Lemma 7.4 it suffices to show that \((L_0 + \text{Id})^{-1} \chi_{[1,a]} \) is compact as operator in \( L^2([1,\infty), u^{-(n+1)} du) \). The kernel \( g(u,u') \) of \((L_0 + \text{Id})^{-1}\) is given by
(7.11) \[
g(u,u') = \frac{(uu')^{n/2}}{\sqrt{n^2/4 + 1}} \begin{cases} (u'/u)^{(n^2/4 + 1)/2}, & u > u'; \\ (u/u')^{(n^2/4 + 1)/2}, & u' > u. \end{cases}
\]
From this formula follows that \( g(u,u') \) is bounded on \([1,\infty) \times [1,a] \), and therefore square integrable with respect to the measure \( u^{-(n+1)} du \). This implies that \((L_0 + \text{Id})^{-1} \chi_{[1,a]} \) is a compact operator and hence, \( R_0 \chi_a \) is compact for all \( a > 1 \).
Let $M_{(1-\chi_a)\beta}$ denote the multiplication operator by $(1-\chi_a)\beta$. Using (7.9) and (7.10), we get
\begin{equation}
\| R_0(1 - \chi_a)(\Delta h - \Delta g) R_h \| \leq C \left( \sum_{j=0}^{2 \beta} \| (\nabla^j h) R_h \| \right) \cdot \| R_0 \| \cdot \| M_{(1-\chi_a)\beta} \|.
\end{equation}

(7.12)

Let $Z_a = [a, \infty) \times Y$. Then
\begin{equation}
\| M_{(1-\chi_a)\beta} \| \leq \sup_{x \in Z_a} \beta(x) = \sup_{x \in Z_a} e^{-\varepsilon d(x,x_0)}.
\end{equation}

Now observe that there exists $C_1 > 0$ such that for all $(u,y) \in Z_a$ we have
\begin{equation}
d((u,y),x_0) \geq d((u,y),(1,y)) - C = \log u - C_1.
\end{equation}
Hence together with (7.12) we get
\begin{equation}
\| R_0(1 - \chi_a)(\Delta h - \Delta g) R_h \| \leq C_2 a^{-\varepsilon}.
\end{equation}
Thus $R_0(\Delta h - \Delta g) R_h$ can be approximated in the operator norm by compact operators and hence, is a compact operator. \qed

Next we construct self-adjoint projections $P_\pm$ which satisfy the conditions of Theorem 7.2. Let
\begin{equation}
e(u,\lambda) := u^{n/2+i\lambda} - u^{n/2-i\lambda}, \quad u \in [1, \infty), \ \lambda \in \mathbb{R}.
\end{equation}
Then $e(u,\lambda)$ satisfies
\begin{equation}
D_0 e(u,\lambda) = (n^2/4 + \lambda^2) e(u,\lambda), \quad e(1,\lambda) = 0.
\end{equation}
Thus $e(u,\lambda)$ is the generalized eigenfunction for $L_0$. For $\varphi \in C_c^\infty(1,\infty))$ set
\begin{equation}
\hat{\varphi}(\lambda) := \frac{1}{2\pi} \int_{1}^{\infty} e(u,\lambda) \varphi(u) \frac{du}{u^{n+1}}.
\end{equation}
The map $\varphi \mapsto \hat{\varphi}$ extends to an isometry
\begin{equation}
F: L^2([1,\infty), u^{-(n+1)} du) \rightarrow L^2(\mathbb{R}^+)
\end{equation}
such that
\begin{equation}
F \circ L_0 \circ F^* = \tilde{L}_0,
\end{equation}
where $\tilde{L}_0$ is the multiplication operator by $(n^2/4 + \lambda^2)$. Let
\begin{equation}
U: L^2(\mathbb{R}^+) \rightarrow L^2([n^2/4, \infty))
\end{equation}
be defined by
\begin{equation}
(Uf)(\lambda) = \frac{f(\sqrt{\lambda - n^2/4})}{\sqrt{2(\lambda - n^2/4)}^{1/4}}.
\end{equation}
Then $U$ is an isometry such that $U \circ \tilde{L}_0 \circ U^* = \tilde{L}_0$, where $\tilde{L}_0$ is the multiplication operator by $\lambda$. Thus $U \circ F$ provides the spectral resolution of $L_0 = \Delta_{0,ac}$. Let
\begin{equation}
J: L^2([n^2/4, \infty)) \rightarrow L^2(\mathbb{R})
\end{equation}
denote the inclusion, let \( F : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the Fourier transform, and let \( \chi_{\pm} \) denote the characteristic function of \([0, \infty)\) and \((-\infty, 0]\), respectively. Set
\[
\widetilde{P}_{\pm} := J^* F \chi_{\pm} F^* J.
\]
Then \( \widetilde{P}_{+} + \widetilde{P}_{-} \) is the identity of \( L^2([n^2/4, \infty)) \). Let \( A = \frac{-id}{du} \), regarded as self-adjoint operator in \( L^2(\mathbb{R}) \). Then
\[
\widetilde{P}_{\pm} e^{-it\Delta_0} = J^* F \chi_{\pm} e^{-itA} F^* J.
\]
Let \( f \in L^2(\mathbb{R}) \). Using the Fourier transformation, it follows that \( (e^{-itA} f)(u) = f(u - t) \). Thus we get
\[
\| \chi_{\pm} e^{-itA} f \|^2 = \pm \int_{-t}^{\pm \infty} |f(u)|^2 \, du \to 0
\]
as \( t \to \mp \infty \). Hence we get
\[
(7.14) \quad \text{s-lim}_{t \to \pm \infty} e^{-itL_0} \widetilde{P}_{\pm} e^{-itL_0} = 0.
\]
Now put
\[
P_{\pm} := F^* U^* \widetilde{P}_{\pm} UF
\]
on \( L^2([1, \infty), u^{-(n+1)}du) \) and set \( P_{\pm} := 0 \) on the orthogonal complement of \( L^2_0(Z)^\perp = L^2([1, \infty), u^{-(n+1)}du) \) in \( L^2(X) \). Then \( P_{\pm} \) are self-adjoint projections that satisfy
\[
P_{+} + P_{-} = P_{ac}(\Delta_0).
\]
Furthermore we have
\[
e^{itH_0} P_{\pm} e^{-it\Delta_0} P_{ac}(\Delta_0) = F^* U \cdot e^{it\Delta_0} F^* e^{-it\Delta_0} P_{\pm} UF.
\]
So it follows from (7.14) that
\[
\text{s-lim}_{t \to \pm \infty} e^{itH_0} P_{\pm} e^{-itH_0} P_{ac}(H_0) = 0.
\]
Thus condition (1) of Theorem 7.2 is satisfied. Let \( I = \{n^2/4\} \). and put
\[
\mathcal{A}_I := C_c^\infty(\mathbb{R} - I).
\]
Then it is clear that \( \mathcal{A}_I \) is multiplicative generating for \( C_c^\infty(\mathbb{R} - I) \). By Lemma 7.4, \( \Delta_0 \) has pure point spectrum in the subspace \( (\text{Id} - P_{ac}(\Delta_0))L^2(X) \) consisting of eigenvalues of finite multiplicity with no finite points of accumulation. Let \( \alpha \in \mathcal{A}_I \). Then \( (\text{Id} - P_{ac}(\Delta_0))\alpha(\Delta_0) \) is a finite rank operator. This is condition (2) of Theorem 7.2. Condition (3) holds by Lemma 7.5. It remains to verify condition (4).

Given \( t > 0 \), let \( \chi_t \) be the characteristic function of \([e^t, \infty) \times Y \) in \( X \). Let \( \delta > 0 \). We have
\[
\| (R_h - R_0)e^{it\Delta_0} \alpha(\Delta_0) P_\pm \| \leq \| R_h - R_0 \| \cdot \| (1 - \chi_{\delta t}) e^{it\Delta_0} \alpha(\Delta_0) P_\pm \|
+ \| (R_h - R_0) \chi_{\delta t} \| \cdot \| \alpha(\Delta_0) \|. \tag{7.15}
\]
We will prove that for each \( \alpha \in C_c^\infty(\mathbb{R} - \{n^2/4\}) \) there exists \( \delta > 0 \) such that the right hand side is an integrable function of \( t \in \mathbb{R}^+ \). To estimate the first term on the right hand side we need the following auxiliary result.

**Lemma 7.6.** Let \( a \in \mathbb{R} \) and let \( f \in C_c^\infty(\mathbb{R} - \{a\}) \). Let \( \varepsilon > 0 \) such that \( f(\lambda^2 + a) = 0 \) for \( |\lambda| < \varepsilon \). Then for every \( m \in \mathbb{N} \) there exists \( C > 0 \) such that for \( t \in \mathbb{R} - \{0\} \) and \( |u| < \varepsilon |t|/2 \) one has

\[
\left| \int_0^\infty e^{2iu\lambda + i\lambda^2} f(\lambda^2 + a) \, d\lambda \right| \leq C|t|^{-m}.
\]

**Proof:** Let \( t \neq 0 \) and set \( x = u/t \). Then the left hand side of the inequality equals

\[
\left| \int_0^\infty e^{ix(\lambda + x)^2} f(\lambda^2 + a) \, d\lambda \right| = (2t)^{-m} \left| \int_0^\infty e^{ix(\lambda + x)^2} \left( \frac{1}{\lambda + x} \frac{d}{d\lambda} - \frac{1}{(\lambda + x)^2} \right)^m f(\lambda^2 + a) \, d\lambda \right|.
\]

Now assume that \( |u| < \varepsilon |t|/2 \). Then \( |x| < \varepsilon/2 \). On the other hand, we have \( f(\lambda^2 + a) = 0 \) for \( |\lambda| < \varepsilon \). Thus if \( f(\lambda^2 + a) \neq 0 \), then we have \( |\lambda + x| \geq |\lambda| - |x| > \varepsilon/2 \). Hence the right hand side can be estimated by \( C|t|^{-m} \).

\[ \Box \]

Let \( \varphi \in L^2((1, \infty), u^{-(n+1)} \, du) = P_{w_0}(\Delta_0)(L^2(X)) \). Then

\[
(e^{-it\Delta_0} \alpha(\Delta_0) \varphi)(u) = \frac{1}{2\pi} \int_0^\infty e(u, n/2 - i\lambda) e^{-it(\lambda^2 + n^2/4)} \alpha(\lambda^2 + n^2/4)(F\varphi)(\lambda) \, d\lambda.
\]

Let \( v \in C_c^\infty((1, \infty)) \). Put \( \varphi = P_+ v \) and \( w = F \ast JU \ast F_+ v \). Then \( w \in L^1(\mathbb{R}) \) and \( F\varphi = U^* J^* F(\chi_+ w) \). Using the definition of \( U, J \) and \( F \), we get

\[
(F\varphi)(\lambda) = \sqrt{2\lambda} \int_0^\infty e^{-is(\lambda^2 + n^2/4)} w(s) \, ds.
\]

Assume that \( t > 0 \). If we insert this expression into the right hand side of (7.16) and switch the order of integration, we obtain

\[
(e^{-it\Delta_0} \alpha(\Delta_0) P_+ v)(u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty w(s) \int_0^\infty e(u, n/2 - i\lambda) e^{-i(t+s)(\lambda^2 + n^2/4)} \alpha(\lambda^2 + n^2/4) \sqrt{\lambda} \, d\lambda \, ds.
\]

Now there exists \( \varepsilon > 0 \) such that \( \alpha(\lambda^2 + n^2/4) = 0 \) for \( |\lambda| < \varepsilon \). Assume that \( |\log(u)| < \varepsilon t/2 \). Using the definition (7.13) of \( e(u, \lambda) \) and Lemma 7.6, it follows that there exists \( C > 0 \) such that

\[
\left| e^{-it\Delta_0} \alpha(\Delta_0) P_+ v(u) \right|^2 \leq C \| w \|^2 u^n t^{-3} \leq C | v |^2 u^n t^{-3}.
\]
Thus for every \( \alpha \in C^c_c(\mathbb{R} - \{ n^2/4 \}) \) there exist \( C > 0 \) and \( \delta > 0 \) such that for \( t > \delta^{-1} \) one has
\[
\| (1 - \chi_{\delta t}) e^{-i t \Delta_0} \alpha(\Delta_0) P_+ \| \leq C t^{-3} \int_1^{e^{\delta t}} \frac{du}{u} = C \delta t^{-2}.
\]

Similarly one can show that
\[
\| (1 - \chi_{\delta t}) e^{-i t \Delta_0} \alpha(\Delta_0) P_- \| \leq C t^{-3} \int_1^{e^{\delta t}} \frac{du}{u} = C \delta t^{-2}, \quad t > \delta^{-1}.
\]

Hence for this choice of \( \delta \), the first term on the right hand side of (7.15) is an integrable function of \( t \in \mathbb{R}^+ \).

Now consider the second term on the right hand side of (7.15). We have
\[
(7.19) \quad \| (R_h - R_0) \chi_{\delta t} \| \leq \| (R_h - R_g) \chi_{\delta t} \| + \| (R_g - R_{\Delta_0}) \chi_{\delta t} \|.
\]

Let \( M_{\chi_{\delta t}} \) denote the multiplication operator by \( \chi_{\delta t} \). By (7.8) - (7.10) we get
\[
\| (R_h - R_g) \chi_{\delta t} \| \leq \| R_g \| \cdot \| \chi_{\delta t}(\Delta_h - \Delta_g) R_h \|
\]
\[
(7.20) \quad \leq C \| M_{\chi_{\delta t}} \left( \sum_{j=0}^2 \| (\nabla^h)^j R_h \| \right) \leq C_1 e^{-\varepsilon \delta t}.
\]

It remains to estimate the second term on the right of (7.19). Let \( \psi \in C^\infty(\mathbb{R}) \) such that \( f(u) = 0 \), if \( u \leq 2 \), and \( f(u) = 1 \), if \( u \geq 3 \). Define \( f \in C^\infty(Z) \) by \( f(u, y) = \psi(u) \) and extend \( f \) by zero to a smooth function on \( X \). Then we have
\[
R_g - R_0 = (f - 1) R_0 - R_g((\Delta_g + \text{Id})(f R_0) - \text{Id}).
\]

Observe that
\[
(\Delta_g + \text{Id})(f R_0) - \text{Id} = f - 1 + 2\nabla f \cdot \nabla R_0 + \Delta f \cdot R_0.
\]

Moreover note that \( (f - 1) \chi_{\delta t} = 0 \) if \( t \gg 0 \). Thus
\[
(7.21) \quad (R_g - R_0) \cdot \chi_{\delta t} = (f - 1) \cdot R_0 \cdot \chi_{\delta t} - R_g (2\nabla f \cdot \nabla R_0 \cdot \chi_{\delta t} + \Delta f \cdot R_0 \cdot \chi_{\delta t})
\]
for \( t \gg 0 \). It follows from (7.2) that \( R_0 \cdot \chi_{\delta t} \) acts in \( L^2(Z) \) and preserves the decomposition (7.3). Moreover \( \| R_0 \cdot \chi_{\delta t} \|_{L^2_0(Z)} \| = \| \chi_{\delta t} \cdot R_0 \|_{L^2_0(Z)} \|. \) Let \( \varphi \in L^2_0(Z) \). Then \( R_0 \varphi \in L^2_0(Z) \cap H^2(Z) \) and by (7.4) we obtain
\[
(7.22) \quad \| \chi_{\delta t} R_0 \varphi \| \leq C e^{-2\delta t} \| R_0 \varphi \| \leq C e^{-2\delta t} \| \varphi \|.
\]

On the orthogonal complement \( L^2_0(Z)^\perp \), the kernel of \( R_0 \) is given by (7.11). Let \( h \in C^\infty_c(Z) \). Then it follows from (7.11) that
\[
(7.23) \quad \| h \cdot R_0 \cdot \chi_{\delta t} \|_{L^2_0(Z)^\perp} \| \leq C e^{-\delta t \sqrt{n^2/4 + 1}} \leq C e^{-\delta t}.
\]

Combining (7.22) and (7.23) we obtain
\[
\| h \cdot R_0 \cdot \chi_{\delta t} \| \leq C e^{-\delta t}.
\]
Similar estimations hold for $\nabla R_0$. This proves that the second term on the right hand side of (7.15) is an integrable function of $t \in \mathbb{R}^+$. This is condition (4) of Theorem 7.2. Summarizing we have proved the following theorem.

**Theorem 7.7.** Let $(X, g)$ be a manifold with cusps and let $\Delta_0$ be defined by (7.2). Let $\varepsilon > 0$ and put $\beta(u) = e^{-\varepsilon u}$, $u \in \mathbb{R}$. Let $h$ be a complete metric on $X$ such that $h \sim_2 \beta^2 g$. Then we have

1. The wave operators $W_\pm(\Delta_h, \Delta_0)$ exist and are complete.
2. $\Delta_h$ has no singularly continuous spectrum.

**Corollary 7.8.** Let $g$ and $h$ be as above. Then the wave operators $W_\pm(\Delta_h, \Delta_g)$ exist and are complete.

This is a considerable improvement of the result that we get from Theorem 7.1 in this case.

**Remark.** Other cases of complete manifolds $(M, g)$ with a sufficiently explicit structure at infinity can be treated in the same way. This includes, for example, manifolds with cylindrical ends and asymptotically Euclidean manifolds.

### 8. $\beta$-Equivalence and Analytic Continuations of the Resolvent

In this section we study the existence of an analytic continuation of the resolvent in weighted $L^2$-spaces. Provided that such a continuation exists, we are able to study the behavior of the absolutely continuous spectrum under perturbation in more detail. The method is a modification of the method used in [Mu2].

**Definition 8.1.** Let $\mathcal{B}$ be a Banach space, $\Omega \subset \mathbb{C}$ a domain and $F : \Omega \mapsto \mathcal{B}$ a meromorphic function. Let $\Sigma$ be a Riemann surface and let $\pi : \Sigma \rightarrow \mathbb{C}$ be a ramified covering. A meromorphic continuation of $F$ to $\Sigma$ is a meromorphic function $\tilde{F} : \Sigma \rightarrow \mathcal{B}$ such that

a) There exists $\tilde{\Omega} \subseteq \Sigma$ such that $\pi : \tilde{\Omega} \rightarrow \Omega$ is biholomorphic.

b) $F \circ \pi = \tilde{F}$ on $\Omega$.

**Definition 8.2.** Let $\delta$ be a function of moderate decay and let $p \in \mathbb{N}$. By $H_{-\delta}^{-p}$ we denote the dual space of $H_\delta^p$, with respect to the extension of the $L^2$-pairing.

**Lemma 8.3.** Let $\zeta(u)$ be a non-increasing continuous function on $[1, \infty)$ with $\zeta(u) \rightarrow 0$ as $u \rightarrow \infty$ and let $\delta$ be a weight function. Then the canonical inclusion $j : L^2_{-\zeta^{-1}}(M) \rightarrow H_{-\delta}^{-2}(M)$ is compact.

**Proof:** It is enough to prove, that the adjoint $j^* : H_{\delta^{-1}}^2(M) \rightarrow L^2_{\delta^{-1}\zeta}(M)$ is compact. For $k \in \mathbb{N}$ let

$$\Omega_k = \{x \in M \mid \zeta(1 + d(x, x_0)) \geq 1/k\}.$$
Then each $\Omega_k$ is a compact subset of $M$. Let $P_k$ be the multiplication operator by the characteristic function of $\Omega_k$. By Rellich’s lemma, $j^*P_k$ is compact. For $f \in H^2_{\delta - 1}(M)$ we have

$$\int_{M - \Omega_k} |f(x)|^2 \delta^{-1}(x) \zeta(x) \, dx \leq \frac{1}{k} \|f\|_{H^2_{\delta - 1}}^2.$$ 

Thus $j^*P_k$ converges to $j^*$ in the operator topology. Hence $j^*$ is compact.

Let $\delta, \rho$ be functions of moderate decay. Then $L^2_{\delta - 1}(M) \subset L^2(M)$ and $H^2(M) \subset H^2_{\rho}(M)$. Thus for $\lambda \in \mathbb{C} - [0, \infty)$, the resolvent $(\Delta - \lambda)^{-1} : L^2(M) \to H^2(M)$ may be regarded as a bounded operator

$$(\Delta - \lambda)^{-1} : L^2_{\delta - 1}(M) \to H^2_{\rho}(M).$$

Denote by $\mathcal{L}(L^2_{\delta - 1}(M), H^2_{\rho}(M))$ the Banach space of all bounded operators from $L^2_{\delta - 1}(M)$ into $H^2_{\rho}(M)$, equipped with the strong operator norm.

**Theorem 8.4.** Let $g, h$ be complete Riemannian metrics on $M$ with bounded curvature of order 2. Let $\beta, \delta, \zeta$ and $\rho$ be functions of moderate decay on $M$ such that

$$(8.1) \beta^2(x) \leq C\gamma^4(x) \rho(x) \delta(x) \zeta(x), \quad x \in M,$$

and $g \sim^2 \beta h$. Let $\Omega \subset \mathbb{C} - [0, \infty)$ be open. Assume that there is a Riemann surface $\Sigma$ and a covering $\Sigma \to \Omega$ such that the operator valued function

$$\lambda \in \Omega \mapsto (\Delta_g - \lambda)^{-1} \in \mathcal{L}(L^2_{\delta - 1}(M, g), H^2_{\rho}(M, g))$$

admits an analytic continuation to a meromorphic function

$$\lambda \in \Sigma \mapsto R_g(\lambda) \in \mathcal{L}(L^2_{\delta - 1}(M, g), H^2_{\rho}(M, g))$$

with finite rank residues. Then

$$\lambda \in \Omega \mapsto (\Delta_h - \lambda)^{-1} \in \mathcal{L}(L^2_{\delta - 1}(M, h), H^2_{\rho}(M, h))$$

also admits a meromorphic continuation to $\Sigma$ with finite rank residues.

**Proof:** By assumption, $\beta\bar{\tau}^{-2n}$ is bounded. Hence by Corollary 5.3, $H^2(M, g)$ and $H^2(M, h)$ are equivalent and therefore, by duality, $H^{-2}(M, g)$ and $H^{-2}(M, h)$ are also equivalent. Let $\lambda \in \mathbb{C} - [0, \infty)$. Then

$$K(\lambda) := (\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g)$$

is a bounded operator in $L^2(M)$. Moreover $\text{Id} + K(\lambda) = (\Delta_g - \lambda)^{-1}(\Delta_h - \lambda)$ has a bounded inverse in $L^2(M)$ which is given by

$$(\text{Id} + K(\lambda))^{-1} = (\Delta_h - \lambda)^{-1}(\Delta_g - \lambda).$$

Thus for $\lambda \in \mathbb{C} - [0, \infty)$ we have

$$(8.2) (\Delta_h - \lambda)^{-1} = (\text{Id} + K(\lambda))^{-1}(\Delta_g - \lambda)^{-1}.$$
By Corollary 4.3 there exists $\lambda \in \mathbb{C} - [0, \infty)$ such that $(\Delta_h - \lambda)^{-1}$ extends to a bounded operator in $L^2_\rho(M)$. By Lemma 4.4 it follows that $(\Delta_h - \lambda)^{-1}$ maps $L^2_\rho(M)$ into $H^2_\rho(M)$. Moreover by definition $\Delta_g - \lambda$ is a bounded operator of $H^2_\rho(M)$ to $L^2_\rho(M)$. Hence $(\mathrm{Id} + K(\lambda))^{-1}$ extends to a bounded operator in $H^2_\rho(M)$. Let $\mu \in \Omega$. Then
\begin{equation}
\begin{aligned}
\mathrm{Id} + K(\mu) &= (\mathrm{Id} + K(\lambda)) - \{(\mathrm{Id} + K(\lambda)) - (\mathrm{Id} + K(\mu))\} \\
&= (\mathrm{Id} + K(\lambda)) - \{K(\lambda) - K(\mu)\} \\
&= (\mathrm{Id} + K(\lambda)) - (\lambda - \mu)(\Delta_g - \mu)^{-1}(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g).
\end{aligned}
\end{equation}

By Corollary 4.3 we may choose $\lambda$ such that $(\Delta_g - \lambda)^{-1}$ extends to a bounded operator in $L^2_{\delta_\zeta}(M)$. By duality, and Lemma 4.4, it defines a bounded operator
\begin{equation}
(\Delta_g - \lambda)^{-1} : L^2_{\delta^{-1}_1\zeta^{-1}}(M) \to H^2_{\delta^{-1}_1\zeta^{-1}}(M).
\end{equation}

Using Lemma 3.6, Lemma 5.2 and the assumption on $\beta$, it follows that the operator $(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g)$ is the composition of the following chain of bounded operators
\begin{equation}
\begin{aligned}
H^2_\beta(M) &\to W^2_{2i\rho}(M) \\
&\xrightarrow{\Delta_h - \Delta_g} L^2_{\beta-2i\rho}(M) \\
&\to L^2_{\delta^{-1}_1\zeta^{-1}}(M) \xrightarrow{(\Delta_g - \lambda)^{-1}} H^2_{\delta^{-1}_1\zeta^{-1}}(M) \xrightarrow{j} L^2_{\delta^{-1}}(M).
\end{aligned}
\end{equation}

By Lemma 8.3, the inclusion $j$ is a compact. Hence
\begin{equation}
(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g) : H^2_\rho(M) \to L^2_{\delta^{-1}}(M)
\end{equation}
is compact operator. Set
\begin{equation}
H_\lambda(\mu) = (\lambda - \mu)R_g(\mu) \circ (\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g), \quad \mu \in \Sigma.
\end{equation}

Then $H_\lambda(\mu), \mu \in \Sigma$, is a meromorphic family of compact operators and
\begin{equation}
\mathrm{Id} + K(\mu) = (\mathrm{Id} + K(\lambda)) \{\mathrm{Id} - (\mathrm{Id} + K(\lambda))^{-1} H_\lambda(\mu)\}.
\end{equation}

It then follows from [St], that $(\mathrm{Id} + K(\mu))^{-1}$ exists except for on a discrete set and is meromorphic in $\mu$. Thus, we may define
\begin{equation}
R_h(\mu) = (\mathrm{Id} + K(\mu))^{-1} \circ R_g(\mu).
\end{equation}

By (8.2) this is the desired meromorphic continuation of the resolvent $(\Delta_h - \lambda)^{-1}$.

\textbf{Examples.}

\textbf{1)} Let $M$ be a surface with cusps. Here by a cusp we mean a half-cylinder $[a, \infty) \times S^1$, $a > 0$, equipped with the Poincaré metric $g^{-2}(dx^2 + dy^2)$, and $M$ is a surface with a complete metric $g$ which in the complement of compact set is isometric to the disjoint union of finitely many cusps. Let $c > 0$ and let $x_0 \in M$. Set
\begin{equation}
\delta(x) := e^{-cd(x,x_0)}, \quad x \in M,
\end{equation}
and $\rho = \zeta = \delta$. Then $\delta$, $\rho$, and $\zeta$ are functions of moderate decay. Let
\begin{equation}
\Omega = \{s \in \mathbb{C} \mid \Re(s) > 1/2, \ s \notin (1/2, 1]\}.
\end{equation}
We consider the resolvent \( R_g(s) = (\Delta_g - s(1-s))^{-1} \) as a function of \( s \in \Omega \). Then it follows from [Mu2, Theorem 1] that \( R(s) \) admits an analytic continuation to a meromorphic function on \( \mathbb{C} \) with values in \( \mathcal{L}(L^2_{\delta^{-1}}(M), L^2_{\delta}(M)) \). Using the same method, one can show that the \( R_g(s) \) takes values in \( H^2_{\delta}(M) \). Now observe that the injectivity radius satisfies \( i(x) \sim e^{-d(x,x_0)} \). Let \( \beta > 0 \) and set \( \beta(x) = e^{-(4+\rho) d(x,x_0)} \). Choose the constant \( c > 0 \) in (8.8) such that \( c < \epsilon/4 \). Then \( \beta \) is a function of moderate decay which satisfies (8.1) with respect to our choice of the functions \( \delta, \rho, \) and \( \zeta \). Now note that the metric \( g \) has bounded curvature of all orders and the injectivity radius has a positive lower bound. Let \( \Delta_g \) admits a decomposition \( \Sigma \in \mathfrak{C} \) with values in \( \delta \). Define \( g \) of order 2 which satisfies (8.8). Then it follows from Theorem 8.4 that the resolvent \( R_{g}(s) = (\Delta_g - s(1-s))^{-1} \), \( s \in \Omega \), also admits a meromorphic extension to \( \mathbb{C} \) with values in \( \mathcal{L}(L^2_{\delta^{-1}}(M), H^2_{\delta}(M)) \). We think that the condition on \( \beta \) can be weakened.

2) Let \( M \) be a manifold with a cylindrical end. This means that \( M \) is a complete Riemannian manifold that admits a decomposition \( M = M_0 \cup_Y (\mathbb{R}^+ \times Y) \) into a compact manifold \( M_0 \) with boundary \( Y \) and a half-cylinder \( (\mathbb{R}^+ \times Y) \) which is glued to \( M_0 \) along the common boundary \( Y \). The restriction of the metric \( g \) of \( M \) to the half-cylinder is assumed to be the product metric. Then \( g \) is a metric with bounded geometry, that is, \( g \) has bounded curvature of all orders and the injectivity radius has a positive lower bound. Let \( \Delta_Y \) be the Laplacian of \( Y \) and let \( 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \) be the eigenvalues of \( \Delta_Y \). Let \( \Sigma \to \mathbb{C} \) be the Riemann surface to which the square roots \( \lambda \mapsto \sqrt{\lambda - \mu_j}, j \in \mathbb{N} \), extend holomorphically. Define \( \delta, \rho, \) and \( \zeta \) as in example 1. Then it follows as in [Mu2, Theorem 5] that the resolvent \( (\Delta_g - \lambda)^{-1} \) extends from \( \mathbb{C} - [0, \infty) \) to a meromorphic function \( \lambda \in \Sigma \mapsto R_g(\lambda) \) with values in \( \mathcal{L}(L^2_{\delta^{-1}}(M), H^2_{\delta}(M)) \). Now let \( \epsilon > 0, x_0 \in M, \) and set

\[
\beta(x) = e^{-\epsilon d(x,x_0)}, \quad x \in M.
\]

Choose \( c \) in the definition of \( \delta \) such that \( c < \epsilon/2 \). Then \( \beta \) satisfies (8.1) with respect to our choice of the functions \( \delta, \rho, \) and \( \zeta \). Let \( h \) be a complete metric on \( M \) with bounded curvature of order 2, and suppose that \( g \sim h^2 \). Then it follows from Theorem 8.4 that the resolvent \( (\Delta_h - \lambda)^{-1} \) also admits an extension from \( \mathbb{C} - [0, \infty) \) to a meromorphic function \( \lambda \in \Sigma \mapsto R_g(\lambda) \) with values in \( \mathcal{L}(L^2_{\delta^{-1}}(M), H^2_{\delta}(M)) \).

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Universität Bonn, Mathematisches Institut, Beringstrasse 1, D – 53115 Bonn, Germany

Cryptomathic A/S, Jaegergaardsgade 118, DK-8000 Aarhus C, Denmark

E-mail address: mueller@math.uni-bonn.de

E-mail address: gorm.salomonsen@cryptomathic.com