Analysis of Gegenbauer kernel filtration on the hypersphere

Louis Omenyi 1,*, McSylvester Omaba 2, Emmanuel Nwaeze 3, Michael Uchenna 1

1Department of Mathematics and Statistics, Alex Ekwueme Federal University, Ndifu-Alike, Nigeria
2Department of Mathematics, College of Science, University of Hafr Al Batin, Hafr Al Batin, Saudi Arabia

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ABSTRACT

In this study, we aim to construct explicit forms of convolution formulae for Gegenbauer kernel filtration on the surface of the unit hypersphere. Using the properties of Gegenbauer polynomials, we reformulated Gegenbauer filtration as the limit of a sequence of finite linear combinations of hyperspherical Legendre harmonics and gave proof for the completeness of the associated series. We also proved the existence of a fundamental solution of the spherical Laplace-Beltrami operator on the hypersphere using the filtration kernel. An application of the filtration on a one-dimensional Cauchy wave problem was also demonstrated.

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1. Introduction

Spherical harmonic analysis is basically the spectral theory of a differential operator such as the spherical Laplacian, which we denote by \( \Delta_{\mathbb{S}} \), on the hypersphere. In this analysis, spherical harmonics play salient roles. Spherical harmonic analysis is a process of decomposing a function on a sphere into components of various wavelengths using surface spherical harmonics as base functions (Bogdanova et al., 2005; Bulow, 2004).

The role of classical orthogonal polynomials such as the Gegenbauer polynomials as reproducing kernels for the spaces of spherical harmonics of a given degree, or more generally, as providing an explicit construction of symmetry adapted basis functions for those spaces has been studied extensively (Camporei, 1990; Bezubik and Strasburger, 2006; Omenyi and Uchenna, 2019). Also, Strasburger (1993) studied the connection of the Fourier transform on the Euclidean space to the Hankel transform obtained via a restriction to \( SO(n) \)-finite functions and various integral identities of the Hecke-Bochner type resulting there. The generalized concept of convolution on groups is intimately related to the concept of filtering on homogeneous spaces. Some insight into spherical filtering with particular emphasis on wavelet transform can be found in Driscoll and Healy (1994), Antoine and Vandergheynst (1999), Bogdanova et al. (2005), and more recently in Dai and Xu (2013) and Claessens (2016).

The goal of the present paper is to present a novel form of the Gegenbauer kernel filtration of harmonic functions on the hypersphere. It is known, in general, that there is no explicit expression for the fundamental solution of a Laplace-type operator on a Riemannian manifold (Aubin, 1998; Cohl and Palmer, 2015). In this work, we aim to demonstrate that with the Gegenbauer filtration kernel, a closed-form of fundamental solution can be constructed. This puts in limelight signal processing methods on non-Euclidean spaces and in particular on the hypersphere. The most basic is the notion of Fourier transform that on the sphere corresponds to the expansion of functions into series of familiar spherical harmonics. A vast amount of literature is available on such expansions, mostly from quantum mechanics and mathematical physics (Szegöes, 2004; Assche et al., 2000; Cohl and Palmer, 2015; Drake et al., 2008; Healy et al., 2003). We also derive some general formulae for the Gegenbauer filtration of functions on \( \mathbb{S}^n \), including recent generalizations of Fourier spherical harmonic expansions and discuss their function theoretic consequences.

In the next subsections, we fix basic notations and concepts before proceeding to present our results in subsequent sections.

1.1. The hypersphere

The hypersphere, \( \mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : x^Tx = 1 \} \), \( n > 3 \) in \( \mathbb{R}^n \) is a set of points whose Euclidean distance from the origin is equal to unity. The hypersphere \( \mathbb{S}^{n-1} \) may be parameterized by a set of
hyperspherical polar coordinates. If \((x_1, x_2, \cdots, x_n)\) are Cartesian coordinates in \(\mathbb{R}^n\), then we define the angles \(\theta_1, \theta_2, \cdots, \theta_{n-1}\) with \(\theta_1, \theta_2, \cdots, \theta_{n-2} \in [0, \pi]\) and \(\theta_{n-1} \in [0, 2\pi]\) such that:

\[
\begin{align*}
x_1 &= \cos\theta_1 \\
x_2 &= \sin\theta_1 \cos\theta_2 \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
x_4 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \\
&\vdots \\
x_{n-1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\
x_n &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1}
\end{align*}
\]  

(1)

This is a natural generalization of spherical polar coordinates in \(\mathbb{R}^3\). In the familiar case of \(S^2 \subset \mathbb{R}^3\), \(\theta_1\) corresponds to the elevation and \(\theta_2\) corresponds to the azimuth.

The surface area of the hypersphere satisfies the recursive relation:

\[
|S^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}, \quad n \geq 3,
\]

(2)

see e.g. Jost and Jost (2008) and Lee (2003) for details.

Let \(\mathcal{H}_{l,n}\) denote the space homogeneous Legendre polynomials of degree \(l\) in dimension \(n\). We call function \(f \in \mathcal{H}_{l,n}\) such that \(\Delta_S f = 0\) a hyperspherical harmonic, where \(\Delta_S\) is the spherical Laplacian defined as:

\[
\Delta_S := \frac{1}{\sin^{n-2} \theta} \frac{\partial}{\partial \theta} (\sin^{n-1} \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \Delta_{S^{n-1}}.
\]

(3)

The space of hyperspherical harmonic polynomials restricted to the unit hypersphere, \(S^n\), is denoted by \(Y_{l,n}\). So, any \(Y_l \in Y_{l,n}\) is related to a homogeneous harmonic \(h_l \in \mathcal{H}_{l,n}\) by \(h_l(\mathbf{r}) = r^l Y_l(\mathbf{r})\) where \(r = |\mathbf{r}|\). So, they have the same dimension.

1.2. Hyperspherical harmonics

To construct a Gegenbauer kernel for filtration on the hypersphere, one needs to clarify the concept of Gegenbauer defined as:

\[
P_{l,n}(t) = (-1)^l R_{l,n}(1 - t^2)^{\frac{1}{2} - \frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{l+1}{2}\right) \Gamma\left(\frac{l+n+1}{2}\right)} (1 - t^2)^{\frac{n-1}{2}} \quad \text{with } n \geq 2,
\]

(5)

where the Rodrigues constant \(R_{l,n}\) is given by:

\[
R_{l,n} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^l \Gamma\left(l + \frac{n+1}{2}\right)}
\]

We remark that for \(n = 3\), one recovers from 5 the standard Rodrigues representation formula for the standard Legendre polynomials as:

\[
P_{l,3}(t) = \frac{1}{2^l l!} (t^2 - 1)^l, \quad l \in \mathbb{N}_0.
\]

(6)

Moreover, Morimoto (1998) proved an integral representation of \(P_{l,n}\) to be:

\[
P_{l,n}(t) = \int_{-1}^{1} [t(i + (1 - t^2)\bar{s})]'(1 - s^2)^{n-\frac{1}{2}} ds, \quad l \in \mathbb{N}_0, n \geq 3, t \in [-1,1].
\]

(7)

We recall that \(P_{l,n}(t) \in \mathcal{H}_l(S^n)\) and note that the dimension, \(d_l(n)\), of \(\mathcal{H}_l(S^n)\) is given by the formula:

\[
d_l(n) = \left(\begin{array}{c} l+n \\end{array}\right) - \left(\begin{array}{c} l+n-2 \\end{array}\right) = \frac{(2l + n - 1)(l + n - 2)!}{l(n-1)!},
\]

for \(l \in \mathbb{N}\).

(8)

2. Technical lemmas and basic assumptions

In what follows, we briefly review basic technical lemmas and assumptions on hyperspherical harmonic polynomials that will lead us to the main result of this work.

**Lemma 2.1:** (Addition lemma). Let \(\{\psi_j; 1 \leq j \leq d_l(n)\}\) be an orthonormal basis of \(\mathcal{H}_l(S^n)\), i.e.

\[
\int_{S^n} \psi_{j}(x) \psi_{j'}(x) d\mu(x) = \delta_{jj'}, \quad 1 \leq j, m \leq d_l(n).
\]

(9)

Then,

\[
\sum_{j=1}^{d_l(n)} \psi_{j}(x) \psi_{j}(y) = \frac{d_l(n)}{|S^n|} P_{l,n}(x \cdot y).
\]

(10)

For proof, one may see Omenyi and Uchenna (2019), Omenyi (2014), and Morimoto (1998). This means in particular that \(P_{l,(l-1)/2}(x \cdot y)\) is a harmonic function on \(S^n\) with eigenvalue \(\lambda_l = l(l+1)/2\) for the eigenvalue problem:

\[
\Delta_{S^n} P_{l,(n-1)/2}(\theta) = \lambda_l P_{l,(n-1)/2}(\theta).
\]
Lemma 2.2 (Morimoto, 1998): The hyperspherical harmonic polynomials are orthogonal:

\[
\int_{\mathbb{S}^n} P_{ln}(\xi \cdot \eta)P_{jm}(\xi \cdot \eta) dV_n(\xi) = \begin{cases} 
\frac{|S^n|}{a(n)} & \text{if } l = j, \\
0 & \text{if } l \neq j. 
\end{cases}
\]  

(11)

An interesting assumption comes from the projection of integrable function onto the space of spherical harmonics on the hypersphere. We make the following definition.

Definition 2.3: A projection \( P_{ln} \) of \( f \in L^1(\mathbb{S}^{n-1}) \) into \( \mathcal{Y}_{ln} \) is defined to be:

\[
(P_{ln}f)(\xi) = \frac{d(n)}{|S^{n-1}|} \int_{\mathbb{S}^{n-1}} P_{ln}(\xi \cdot \eta)f(\eta)dS_{n-1}(\eta), \ \eta \in \mathbb{S}^{n-1}.
\]

(12)

Assumption 2.4: If \( f \in L^2(\mathbb{S}^{n-1}) \) then for any \( \xi \in \mathbb{S}^{n-1} \) we have:

\[
\| (P_{ln}f) \|_{L^2(\mathbb{S}^{n-1})} \leq \| f \|_{L^2(\mathbb{S}^{n-1})}.
\]

Proof: Let \( f \in L^2(\mathbb{S}^{n-1}) \) and \( \xi \in \mathbb{S}^{n-1} \) given, we have:

\[
\| (P_{ln}f)(\xi) \|_{L^2(\mathbb{S}^{n-1})}^2 \leq \frac{d(n)}{|S^{n-1}|} \int_{\mathbb{S}^{n-1}} |P_{ln}(\xi \cdot \eta)|^2 dS_{n-1}(\eta).
\]

Similarly,

\[
\| (P_{ln}f) \|_{L^2(\mathbb{S}^{n-1})} \leq \frac{d(n)}{|S^{n-1}|} \| f \|_{L^2(\mathbb{S}^{n-1})}.
\]

These imply that \( \| (P_{ln}f) \|_{L^2(\mathbb{S}^{n-1})} \leq \| f \|_{L^2(\mathbb{S}^{n-1})} \). The orthogonal decomposition of the hyperspherical harmonics 9 and the addition lemma 2.1 imply that any \( f \in L^2(\mathbb{S}^{n-1}) \) can be uniquely represented as:

\[
f(\xi) = \sum_{l=0}^{\infty} f_l(\xi); \ \text{with } f_l \in \mathcal{Y}_{ln}; \ l \geq 0.
\]

(13)

We call \( f_l \in \mathcal{Y}_{ln} \) hyperspherical component of \( f \) given by:

\[
f_l(\xi) = \frac{d(n)}{|S^{n-1}|} \int_{\mathbb{S}^{n-1}} f(\eta)P_{ln}(\xi \cdot \eta)dS_{n-1}(\eta); \ \eta \geq 0.
\]

(14)

Lemma 2.5 (Morimoto, 1998): The Gegenbauer function \( G_l^n \) is indeed a polynomial and has a representation in terms of the hyperspherical harmonics as:

\[
C_{l}^{\frac{n-2}{2}}(t) = \left( t + \frac{n-3}{2} \right) P_{ln}(t); \ \text{for } n \geq 3.
\]

(15)

3. Results and discussion

We now present the main results of this study. Let \( f \in C(\mathbb{S}^{n-1}) \). We define:

\[
f(\xi) = \int_{\mathbb{S}^{n-1}} \delta(1 - \xi \cdot \eta)f(\eta)dS_{n-1}(\eta), \ \xi \in \mathbb{S}^{n-1}
\]

using a Dirac delta function \( \delta(t) \) whose value is defined as:

\[
\delta(t) = \begin{cases} 
0 & \text{if } t \neq 0, \\
\frac{1}{\infty} & \text{if } t = 0.
\end{cases}
\]

and satisfies

\[
\int_{\mathbb{S}^{n-1}} \delta(1 - \xi \cdot \eta)dS_{n-1}(\eta) = 1 \ \forall \xi \in \mathbb{S}^{n-1}.
\]

One can construct a sequence of kernel functions \( G_l(t) \) such that \( G_l(\xi \cdot \eta) \) approaches \( \delta(1 - \xi \cdot \eta) \) and is such that for each \( l \in \mathbb{N} \), the function

\[
\int_{\mathbb{S}^{n-1}} G_l(\xi \cdot \eta)dS_{n-1}(\eta)
\]

is a linear combination of spherical harmonics of the order less than or equal to \( l \). One possibility is to choose \( G_l(t) \) proportional to \( \left(\frac{1+t}{2}\right)^l \). Thus, we let:

\[
G_l(t) = \alpha_{ln} \left(\frac{1+t}{2}\right)^l
\]

where \( \alpha_{ln} \) is a scaling constant so that:

\[
\int_{\mathbb{S}^{n-1}} G_l(\xi \cdot \eta)dS_{n-1}(\eta) = 1 \ \forall \xi \in \mathbb{S}^{n-1}.
\]

(18)

Proposition 3.1: The constant \( \alpha_{ln} \) is given by:

\[
\alpha_{ln} = \frac{(1+n-2)!}{(2n)!} \frac{2n!}{\Gamma(n+\frac{1}{2})}.
\]

(19)

Proof. From:

\[
\int_{\mathbb{S}^{n-1}} \left(\frac{1+t}{2}\right)^l dS_{n-1}(\eta) = |\mathbb{S}^{n-1}| \int_{-1}^{1} \left(\frac{1+t}{2}\right)^l (1-t^2)^{\frac{n-3}{2}} dt
\]

we deduce on change of variables: \( s = \frac{1+t}{2} \) that:

\[
\int_{\mathbb{S}^{n-1}} \left(\frac{1+t}{2}\right)^l dS_{n-1}(\eta) = 2^{n-2}|\mathbb{S}^{n-2}| \int_{0}^{1} s^{\frac{n-3}{2}} ds
\]

and from 2 we know that \( |\mathbb{S}^{n-2}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \). Moreover,

\[
\int_{0}^{1} s^{\frac{n-3}{2}} (1-s)^\frac{n-3}{2} ds = \beta(l+\frac{n-1}{2}, \frac{n-1}{2}) = \frac{\Gamma(l+\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(l+n-1)}.
\]

Thus:

\[
\int_{\mathbb{S}^{n-1}} \left(\frac{1+t}{2}\right)^l dS_{n-1}(\eta) = (4\pi)^{\frac{n-1}{2}} \frac{\Gamma(l+\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(l+n-1)}.
\]

Therefore, \( \alpha_{ln} \) has the proposed value.

Now we introduce a filtration operator \( G_{ln} \) defined as follows:

\[
(G_{ln}f)(\xi) = \alpha_{ln} \int_{\mathbb{S}^{n-1}} \left(\frac{1+t}{2}\right)^l f(\eta)dS_{n-1}(\eta), \ \forall f \in C(\mathbb{S}^{n-1}).
\]
We express $\left( G_{l,n}f \right)(\xi)$ as a linear combination of spherical harmonics of the order less than or equal to $l$. To do this, write:
\[ \alpha_{l,n}(\frac{1+t}{2})^l = \sum_{k=0}^{l} \mu_{l,k,n} \frac{d_{l}(n)}{\sqrt{n-l}} P_{l,n} (t) \] (20)
and thus from 15:
\[ \alpha_{l,n}(\frac{1+t}{2})^l = \sum_{k=0}^{l} \mu_{l,k,n} C_{l}^{n-2} (t). \] (21)

To determine the coefficients $\{\mu_{l,k,n}\}_{k=0}^{l}$, we multiply both sides by the function:
\[ p_{l,n}(t)(1-t^2)^{\frac{n-2}{2}}, \quad 0 \leq j \leq l, \]
itegrate from $t = -1$ to $t = 1$ and use the orthogonality condition of $P_{l,n}$ to obtain:
\[ \mu_{l,k,n} = \alpha_{l,n} \int_{-1}^{1} (1-t^2)^{\frac{n-2}{2}} P_{l,n}(t)(1-t^2)^{\frac{n-2}{2}} dt. \]

Computing this following the same procedure as in the derivation of $\alpha_{l,n}$ we get
\[ \mu_{l,k,n} = \frac{\left[ (l+n-2)! \right]}{(l-j)!(l+n-j)!}. \]

So we have the definition of Gegenbauer filtering given by
\[ \left( G_{l,n}f \right)(\xi) = \sum_{l=0}^{l} \mu_{l,k,n} F(l,n)(\xi) \] (22)
where $F$ is the projector defined in 12. In order words, $G_{l,n}$ is a linear combination of spherical harmonics of the order less or equal to $l$. We also observe that for $t \in [-1,1]$:
\[ \lim_{l \to \infty} \left[ \frac{1+t}{2} \right]^l = 0. \]

Now we state and prove one of the main results of this study.

**Theorem 3.2:** The Gegenbauer filtration operator $G_{l,n}$ is complete. That is, let $f \in C(S^{n-1})$ then
\[ \lim_{l \to \infty} ||G_{l,n}f - f||_{C(S^{n-1})} = 0. \]

**Proof:** Using modulus of continuity, we have:
\[ \omega(f; \delta) = \sup \{|f(\xi) - f(\eta)|: \xi, \eta \in S^{n-1}, |\xi - \eta| \leq \delta, \delta > 0\}, \]
and since $f \in C(S^{n-1})$, we have that $\omega(f; \delta) \to 0$ as $\delta \to 0$. Denote:
\[ M := \sup \{|f(\xi) - f(\eta)|: \xi, \eta \in S^{n-1}, |\xi - \eta| \leq \delta\} < \infty. \]

Let $\xi \in S^{n-1}$ be arbitrary but fixed. Using (18), we have:
\[ \left( G_{l,n}f \right)(\xi) - f(\xi) = \alpha_{l,n} \int_{S^{n-1}} (1+t\xi \eta)\frac{1}{2} |f(\xi) - f(\eta)| dS_{n-1}(\eta) = I_1(\xi) + I_2(\xi), \]
where,
\[ I_1(\xi) = \alpha_{l,n} \int_{S^{n-1}} (1+t\xi \eta)\frac{1}{2} |f(\eta)| - f(\xi) dS_{n-1}(\eta) \]
and,
\[ I_2(\xi) = \alpha_{l,n} \int_{S^{n-1}} (1+t\xi \eta)\frac{1}{2} |f(\eta)| - f(\xi) dS_{n-1}(\eta). \]

we bound each term as follows.
\[ I_1(\xi) \leq \omega(f; \delta) \alpha_{l,n} \int_{S^{n-1}} (1+t\xi \eta)\frac{1}{2} |dS_{n-1}(\eta)| = \omega(f; \delta) \]
and,
\[ I_2(\xi) \leq M \alpha_{l,n} |S^{n-1}(1-t^2)^{\frac{n-2}{2}}. \]

In bounding $I_2(\xi)$, we used the relation
\[ |\xi - \eta| > \delta \xi \cdot \eta < 1 - \frac{\delta^2}{2} \]
for $\xi, \eta \in S^{n-1}$. Thus, for any $\delta \in (0,1)$, applying Theorem 3.2, we have:
\[ \limsup_{l \to \infty} ||G_{l,n}f - f||_{C(S^{n-1})} \leq \omega(f; \delta). \]

note that $\omega(f; \delta) \to 0$ as $\delta \to 0$. So the statement holds.

Using the operator 22, we can recast Theorem 3.2 as 3.3 below.

**Theorem 3.3:** For any $f \in C(S^{n-1})$,
\[ f(\xi) = \limsup_{l \to \infty} \sum_{k=0}^{l} \mu_{l,k,n}(G_{l,n}f)(\xi) \]
uniformly in $\xi \in S^{n-1}$. If $G_{l,n}f = 0$ for all $l \in \mathbb{N}$, then $f$ must be the zero function.

Theorem 3.3 and the orthogonality of the hyperspherical polynomials imply that $\{Y_{l,n}^j| l \in \mathbb{N} \}$ is the only system of source spaces in $C(S^{n-1})$ since any primitive source not identical to one in $Y_{l,n}^j$, $l \in \mathbb{N}$ is orthogonal to all and is therefore trivial.

The inner product of two complex-valued functions on the surface of $S^{n-1}$ is:
\[ (f,h)_{S^{n-1}} = \int_{S^{n-1}} (f(\xi) \eta^* h(\xi, \eta) dS_{n-1}(\eta) \]
where $\ast$ denotes complex conjugation. Using the fact that the Gegenbauer filtration is rotation invariant over $SO(n)$, one can move $h$ to any point $(\xi_0, \eta_0) \in S^{n-1}$. Then we define a generalized convolution as a function on the rotation group $SO(n)$ to be:
\[ (h \ast f)(R) = \int_{S^{n-1}} f(\xi, \eta) h(\xi, \eta) dS_{n-1}(\eta) \]
for $R \in SO(n)$ and where $h_\xi$ is $h$ rotated by $R$ defined as $h_\xi(A) = h(R^{-1}A)$. Thus, every well-behaved function $f \in L^2(S^{n-1})$ admits the expansion

$$f(\xi, \eta) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell m}(\xi) \mathcal{C}_\ell^m(\xi, \eta)$$

where the generalized Fourier transform on the surface of the hypersphere $\hat{f}_{\ell m}$ is defined to be:

$$\hat{f}_{\ell m}(\xi) = \int_{S^{n-1}} f(R) \rho_{\ell m}(R) dS_{n-1}(\eta)$$

Here, $\rho(R)$ is a function on $SO(n)$ containing fixed matrix-valued functions called the irreducible representations of $SO(n)$.

As a consequence of 12, we observe that for a suitable $\psi$, we have:

$$(F_{\ell n}\psi)(\xi) = \frac{d_\ell(n)}{\left[S^{n-1}\right]} \int_{S^{n-1}} P_{\ell n}(\xi \cdot \eta) \psi(\eta) dS_{n-1}(\eta) = \frac{d_\ell(n)}{\left[S^{n-1}\right]} \sum_{j=1}^{\max(\ell, n)} \langle \psi_\ell, \psi_j \rangle_{S^{n-1}} \psi_j(\xi)$$

$$= (F_{\ell n}\psi)(\xi) = \sum_{j=1}^{\max(\ell, n)} \lambda_j(\ell, n) \psi_j(\xi).$$

This leads to another interesting result of this work that Gegenbauer filtration on $S^{n-1}$ coincides

$$\mathcal{F} * h(l, a) = \int_{SO(n)} \left( \int_{S^{n-1}} h(R^{-1}\xi) \mathcal{G}_\ell^{(l)}(F_{\ell n}(F_{\ell n}(R^{-1} \xi) dS_{n-1}(\xi))) f(RN) dR \right)$$

where $N$ is the north pole. This implies that from the addition Lemma 2.1,

$$\mathcal{F} * h(l, a) = \int_{SO(n)} f(RN) \left( \int_{S^{n-1}} h(R^{-1}\xi) \mathcal{G}_\ell^{(l)}(F_{\ell n}(R^{-1} \xi) dS_{n-1}(\xi))) f(RN) dR \right)$$

which is 0 unless $a = 0$. This follows since the measure on $SO(n)$ is rotation-invariant.

Finally using the relationship$C_\ell^m$ and the hyperspherical harmonics $P_{\ell n}$ expressed in 15, we conclude that 24 holds.

In computing a fundamental solution, $u_1$ say, of Laplace’s equation on $S^n$ we know that:

$$\Delta_{S^{n-1}} u_1(\xi', \eta') = \delta(\xi, \eta'),$$

where $\delta(\xi, \eta')$ is the Dirac delta distribution on the manifold $S^n$. The Dirac delta distribution on the Riemannian manifold $S^n$ is defined for an open set $U \subseteq S^n$ with $\xi, \xi' \in S^n$ such that:

$$\int_{U} \delta(\xi, \eta') dV_n = \begin{cases} 1 & \text{if } \xi' \in U, \\ 0 & \text{if } \xi' \notin U. \end{cases}$$

Using the standard hyperspherical coordinates (1) on $S^n$, the Dirac delta distribution is given by

$$\delta(\xi, \xi') = \frac{\delta(\theta_n, -\theta_n, \cdots, \delta(\theta_0-\theta_0, \cdots, \delta(\theta_{-\ell}-\theta_{-\ell})}{\sin^{n-2} \theta_n \sin^{n-3} \theta_{n-1} \cdots \sin^{n-\ell} \theta_{-\ell}}.$$
Therefore, through 26, we can write,
\[
\delta(\xi, \xi') = \sum_{l=0}^{\infty} \sum_{\nu} C^l_{\nu}(\theta_1, \theta_2, \ldots, \theta_{n-1}) C^l_{\nu}(-\theta_1', \theta_2', \ldots, \theta_{n-1}').
\]
(27)

Moreover since \(G_{l,n}\) is harmonic on its domain for fixed \(\theta_2, \theta_2, \ldots, \theta_{n-2} \in [0, \pi]\), its restriction is in \(C^2(S^{n-1})\) and therefore has a unique expansion in hyperspherical harmonics, namely
\[
(G_{l,n} \ast C^l_{\nu})(\xi, \xi') = \sum_{l=0}^{\infty} \sum_{\nu} u^l_{\nu}(\theta_1, \theta_2, \ldots, \theta_{n-1}) C^l_{\nu}(\theta_1, \theta_1', \ldots, \theta_{n-2}'),
\]
where \(u^l_{\nu} : [0, \pi]^n \rightarrow \mathbb{C}\). Furthermore, substitute 27, 28 into 25 and use the definition of \(\Delta_h^n\) satisfying 17 to obtain:
\[
G_{l,n} \ast \psi(\xi, \xi') = \sum_{l=0}^{\infty} \sum_{\nu} \psi_l(\theta, \theta') \psi^l_{\nu}(\theta_1, \theta_1', \ldots, \theta_{n-1}) C^l_{\nu}(-\theta_1', \theta_2', \ldots, \theta_{n-1}').
\]
(29)

Using the addition theorem for hyperspherical harmonics, Eq. 29 can now be simplified. Therefore,
\[
G_{l,n} \ast \psi(\xi, \xi') = \frac{1}{\sin^{n-1}\theta} \sum_{l=0}^{\infty} \sum_{\nu} \psi_l(\theta_1, \theta_1', \ldots, \theta_{n-1})(2l + n - 2)C^l_{\nu}(-\cos\gamma)
\]
where \(\gamma\) is the geodesic angle between \(\xi\) and \(\xi'\).

As an application, consider the Cauchy wave problem:
\[
\varphi_{tt} = \varphi_{xx}; \quad x \in \mathbb{R}, \quad t > 0
\]

This indicates the existence of \(\psi_l : [0, \pi]^2 \rightarrow \mathbb{R}\) such that
\[
\psi_l(\theta, \theta', \theta_{n-1}) = \psi_l(\theta, \theta') C^l_{\nu}(\theta_1', \theta_2', \ldots, \theta_{n-1}).
\]

From 28, the expression for a fundamental solution of the Laplace-Beltrami operator in standard hyperspherical coordinates on the hypersphere is therefore given by:
\[
\sum_{l=0}^{\infty} \sum_{\nu} \psi_l(\theta_1, \theta_2, \ldots, \theta_{n-1}) C^l_{\nu}(\theta_1', \theta_2', \ldots, \theta_{n-1}) = u_l.
\]

By the d’Alembert formula, (Pinchover and Rubinstein, 2005),
\[
\varphi(x, t) = \frac{1}{2} f(x + t) + \frac{1}{2} f(x - t).
\]

With the aid of MATHEMATICA, we compute and plot the Gegenbauer filtration of \(u\) at \(t = 1\) and \(t = 2\) using the synthesized product of the real parts of \(C^2_{\nu}(t)\) and \(Y_{2,1}(t, \phi)\) shown in Fig. 1.

**Fig. 1:** The synthesized smoothing kernel \(C^2_{\nu}(t) \ast Y_{2,1}(t, \phi)\)
Fig. 1 is the synthesized smoothing kernel $C^2_2(t) * Y_{2,1}(t, \phi)$.

For $t = 1$,

$$u(x, 1) = \begin{cases} \frac{1}{2}(x + 1) & \text{if } -1 \leq x < 0, \\ \frac{1}{2}(1 - x) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and so we have the profile sketch as Figs. 2A, 2B, and 2C.

Fig. 2 are the wave profile at $t = 1$, the synthesised profile at $t = 1$ and Gegenbauer convolved output respectively. For $t = 2$:

$$u(x, 2) = \begin{cases} \frac{1}{2}(x + 2) & \text{if } -2 \leq x < -1, \\ \frac{1}{2}(-x) & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and so we have the profile sketch as Figs. 3A, 3B and 3C.
4. Conclusion

We studied the Gegenbauer kernel filtration of harmonic functions on the hypersphere. We showed that under the filtration, an explicit expression for the fundamental solution of a Laplace-type operator on a Riemannian manifold can be constructed. A demonstration of the Gegenbauer filtration kernel with a closed-form fundamental solution was shown. This brings to the limelight an extension of signal processing methods on Euclidean spaces to non-Euclidean spaces of higher dimensions such as closed Riemannian manifolds, for example, the hypersphere.

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Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.
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