Analytic results in 2+1-dimensional Finite Temperature LGT

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Abstract

In a 2+1-dimensional pure LGT at finite temperature the critical coupling for the deconfinement transition scales as \( \beta_c(n_t) = J_c n_t + a_1 \), where \( n_t \) is the number of links in the “time-like” direction of the symmetric lattice. We study the effective action for the Polyakov loop obtained by neglecting the space-like plaquettes, and we are able to compute analytically in this context the coefficient \( a_1 \) for any SU(\( N \)) gauge group; the value of \( J_c \) is instead obtained from the effective action by means of (improved) mean field techniques. Both coefficients have already been calculated in the large \( N \) limit in a previous paper. The results are in very good agreement with the existing Monte Carlo simulations. This fact supports the conjecture that, in the 2+1-dimensional theory, space-like plaquettes have little influence on the dynamics of the Polyakov loops in the deconfined phase.

1 Introduction and setting

Let us consider a finite temperature lattice gauge theory (LGT) with gauge group SU(\( N \)), defined on a 2 + 1 dimensional cubic lattice. The “time” dimension is compactified with periodic boundary conditions and compactification length \( \frac{1}{T} = N_t a_t \), where \( N_t \) is the number of lattice spacings in the time direction, and \( a_t \) the corresponding lattice spacing; \( T \) is the temperature. We consider in general asymmetric lattices; denoting by \( a_s \) the lattice spacing in the space directions, the

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asymmetry parameter \( \rho \) is defined as:
\[
\rho = \frac{a_s}{a_t} = TN_t a_s .
\] (1)

Moreover we introduce different bare couplings \( \beta_t \) and \( \beta_s \) in the time and space directions. The Wilson action is then:
\[
S_W = N \sum_{\vec{x},t} \left( \sum_{i=1}^2 \beta_t \text{Re} \text{Tr} G_{0i}(\vec{x},t) + \beta_s \text{Re} \text{Tr} G_{12}(\vec{x},t) \right) ,
\] (2)

where \( G_{0i} \) and \( G_{12} \) are the time-like and space-like plaquette variables:
\[
\begin{align*}
G_{0i}(\vec{x},t) &= V(\vec{x},t)U_i(\vec{x},t+1)V^\dagger(\vec{x}+\hat{i},t)U_i^\dagger(\vec{x}+\hat{i},t) , \\
G_{12}(\vec{x},t) &= U_1(\vec{x},t)U_2(\vec{x}+\hat{i},t)U_1^\dagger(\vec{x}+\hat{j},t)U_2^\dagger(\vec{x},t) .
\end{align*}
\] (3)

We have denoted by \( U_i(\vec{x},t) \) the link variables in the space directions and by \( V(\vec{x},t) \) the link variables in the time direction.

For a given \( \rho \) the relation between the couplings \( \beta_t \) and \( \beta_s \) can be obtained by requiring that the Wilson action (2) reproduces in the continuum limit the usual Yang–Mills action. At the classical level, this leads to the relations
\[
\frac{2}{Ng^2} = a_s \sqrt{\beta_s \beta_t} , \quad \rho = \sqrt{\frac{\beta_t}{\beta_s}} ,
\] (4)

which show that in \( d = 2 \) the coupling constant \( g^2 \) has the dimensions of a mass. This is the main difference with the \( d = 3 \) case and it is the reason for which the \( d = 2 \) models are much easier to study. In fact in this case the coupling constant itself sets the overall mass scale. Near the continuum limit a physical observable, like the critical temperature \( T_c \), with the dimensions of a mass can be written, according to the renormalization group equations, as a series in powers of \( g^2 \). Hence in terms of the coupling \( \beta_t \) we have
\[
a_t T_c = \frac{c_1}{\beta_t} + \frac{c_2}{\beta_t^2} + \cdots ,
\] (5)

with \( c_1 \) and \( c_2 \) suitable constants. The critical temperature is obtained by looking at the critical coupling \( \beta_c \) at which the deconfinement transition occurs. Since the lattice size in the time direction is \( N_t \equiv \frac{1}{T_{at}} \), we can rephrase eq. (3) as a scaling law for the critical coupling \( \beta_c \) as a function of \( N_t \):
\[
\beta_c(N_t) = J_c N_t + a_1 \cdots ,
\] (6)

\footnote{The notations throughout this paper are the same as the ones of Ref. [1], to which we refer for a more detailed exposition of the general setting of the theory. In particular we choose the normalization of the couplings \( \beta_s \) and \( \beta_t \) so as to have a smooth large \( N \) limit, since we will compare the results at finite \( N \) with the ones obtained in the large \( N \) limit.}

\footnote{It is not mandatory to measure \( T_c \) in units of \( a_t \) and to choose \( \beta_t \) instead of \( \beta_s \) in eq. (3), but these choices simplify the comparison with the results of Sec. 3.}
where we have introduced the two constants \( J_c \equiv c_1 \) and \( a_1 \equiv \frac{c_2}{c_1} \) for future convenience. Finding the precise values of these two constants is one of the aims of the present paper.

It is clear from eqs. (1) and (4) that equivalent regularizations with different values of \( \rho \) require different values of \( N_t \). Hence, to maintain the equivalence, \( N_t \) must be a function of \( \rho \) according to eq. (1).

Among all these equivalent regularizations a particular role is played by the symmetric one, which is defined by

\[
\beta \equiv \frac{2}{aNg^2} \tag{7}
\]

(from now on we shall distinguish the symmetric regularization from the asymmetric ones by eliminating the subscripts \( t \) and \( s \) in \( \beta \) and \( a \)). By comparing eq.s (1,4,7) we see that all regularizations are equivalent to the symmetric one provided

\[
\beta = \rho \beta_s = \frac{\beta_t}{\rho} , \tag{8}
\]

\[
N_t(\rho) = \rho n_t , \tag{9}
\]

where \( n_t \) is the number of links in the time direction with a symmetric regularization: \( n_t = N_t(\rho = 1) \).

Notice however that these equivalence relations have been derived in the naive or “classical” continuum limit, and quantum corrections are in general present. They lead in the large-\( \rho \) limit to the following expressions:

\[
\beta_t = \rho(\beta + \alpha^0_\tau) + \alpha^1_\tau + \ldots , \quad \beta_s = \frac{\beta + \alpha^0_\sigma}{\rho} + \frac{\alpha^1_\sigma}{\rho^2} + \ldots . \tag{10}
\]

The constants \( \alpha^0,1_\tau,\sigma \) encode the quantum corrections to eq. (8). These corrections were studied and calculated in the (3+1) dimensional case by F. Karsch in [2].

It is well known that, as a consequence of the periodic boundary conditions in the time direction, the Wilson action enjoys a new \( \mathbb{Z}_N \) global symmetry. Moreover, it becomes possible to define gauge invariant observables which are topologically non-trivial, such as the Polyakov loop:

\[
\hat{P}(\vec{x}) = \text{Tr} P(\vec{x}) = \text{Tr} \prod_{t=1}^{N_t} V(\vec{x},t) . \tag{11}
\]

The Polyakov loop is a natural order parameter for the \( \mathbb{Z}_N \) symmetry. In the high temperature, deconfined phase (\( T > T_c \)) we have \( \langle \hat{P}(\vec{x}) \rangle \neq 0 \), while in the low temperature, confining domain (\( T < T_c \)), the symmetry is unbroken, namely \( \langle \hat{P}(\vec{x}) \rangle = 0 \).

\footnote{We will refer to the untraced quantity \( P(\vec{x}) \) as the Polyakov line.}
The way to approach the description of the deconfinement transition is to construct an effective action for the relevant degrees of freedom, namely the Polyakov loops. This would require to perform the exact integration over all the gauge degrees of freedom on the space-like links, a clearly impossible task. In \([1, 3]\), we discussed a perturbative approach, in which the effective action for the Polyakov loop is calculated exactly order by order in an expansion in powers of \(\beta_s\). Except for the theory with \(N = 2\) (see \([3]\)) only the zeroth order of this expansion, which corresponds to neglecting the space-like plaquettes altogether, has been studied and used to determine the critical temperature. The point is that the approximation of throwing away the space-like plaquettes appears to be a rather crude one if one wants to go beyond purely qualitative results. It is true however that far away from the continuum limit, that is for very small \(N_t\), the deconfinement transition occurs for small values of \(\beta\) (we refer here to a symmetric lattice) and that the couplings among Polyakov loops induced by the space-like plaquettes, being of higher order in \(\beta\), can be safely neglected. As we proceed towards the continuum limit (large \(\beta\)) there is no obvious reason why the space-like plaquettes should be uninfluential. In fact we know that in \(3 + 1\) dimensions the effect of the space-like plaquettes is crucial, as the rescaling of the critical coupling with \(N_t\), which is linear in their absence, becomes logarithmic, according to asymptotic freedom, if they are taken into account. The situation is quite different in \(2 + 1\) dimensions, where the rescaling of the critical coupling is linear both with and without space-like plaquettes. This is clearly related to the fact that the dimensional coupling constant sets the mass scale of the theory. So in \(2 + 1\) dimensions we are in a situation where the zeroth order approximation is on one hand a good approximation of the full theory for very small \(N_t\), and on the other hand it has the same scaling properties of the full theory near the continuum limit. It is natural to conjecture at this point that the effective action for the Polyakov loop obtained by neglecting the space-like plaquettes is a good approximation even in the continuum limit and up to the temperature where the deconfinement transition occurs. Checking this conjecture by comparing the analytic predictions based on the zeroth order approximation with the Monte Carlo simulations based on the full theory is the purpose of this paper.

This paper is organized as follows. In the next section we shall construct the effective action for the Polyakov loop mentioned above. This section contains material which has been already presented elsewhere \([1]\), and has been inserted so as to make this paper self-contained. In Sec. 3 we shall obtain in the framework of the \(\beta_s = 0\) approximation the scaling law \((\ref{scaling})\) and the explicit value of \(a_1\) coefficient. The exact calculation of \(a_1\) is the main new analytic result of the paper. In the last section we shall estimate by means of suitably improved mean field techniques the \(J_c\) coefficient and compare our results with those obtained with Monte Carlo simulations. The agreement of the analytic predictions with the Monte Carlo simulations, which seems to confirm the above conjecture, is the main physical result of our analysis.
2 The effective action

In order to derive the effective action for the Polyakov loops at $\beta_s = 0$, one starts from the time-like part of the Wilson action (2) and performs the character expansion

$$e^{S_W(\beta_s=0)} = \prod_{\bar{x},t,s} \left\{ \sum_r d_r D_r(\beta_t) \chi_r(G_{0i}(\bar{x}, t)) \right\}. \quad (12)$$

In the above equation $d_r$ is the dimension of the representation $r$ and $\chi_r(U)$ the character of a matrix $U$ in this representation; the coefficients $D_r(\beta_t)$ are given by

$$D_r(\beta) = \frac{F_r(\beta)}{d_r F_0(\beta)}, \quad (13)$$

with

$$F_r(\beta) \equiv \int DU e^{N\beta \text{ReTr}U} \chi_r^*(U) = \sum_{n=-\infty}^{\infty} \text{det} I_{r_j+i+n}(N\beta). \quad (14)$$

The $r_j$’s are a set of integers labelling the representation $r$ and they are constrained by $r_1 > \cdots > r_N = 0$. The indices $1 \leq i, j \leq N$ label the entries of the $N \times N$ matrix the determinant of which is taken and $I_n(\beta)$ denotes the modified Bessel function of order $n$.

As a result of neglecting the space-like plaquettes, each space link variable appears only in two time-like plaquettes, and the integration over the variables $U_i(\bar{x}, t)$ can be easily performed by using the rules for the integration of characters (see [1] for the details). One gets the following effective action for the Polyakov loops:

$$e^{S_{\text{Pol}}(\beta_s=0)} = \prod_{\bar{x},i} \sum_r [D_r(\beta_t)]^{N_i} \chi_r(P(\bar{x} + \hat{i})) \chi_r(P^\dagger(\bar{x})). \quad (15)$$

From the effective action we obtain some general informations on the scaling behaviour that we may expect for the critical coupling. They are obtained by requiring that the physics is unaffected by changes of $N_t$ provided $\beta_t$ is suitably rescaled. In particular it is easy to derive that the rescaling of $\beta_t$ is linear.

In fact if we just assume that $\beta_t$ becomes large for large $N_t$, we can replace in (15) the coefficients $D_r(\beta_t)$ with their large $\beta_t$ behaviour which is well known and is given by

$$\lim_{\beta_t \to \infty} D_r(\beta_t) = e^{-\frac{C_r}{2N^{\beta_t}}} \ldots, \quad (16)$$

where $C_r$ denotes the quadratic Casimir in the $r^{th}$ representation:

$$C_r = \sum_{i=1}^{N} r_i^2 - \sum_{i=1}^{N} r_i - \frac{N(N^2 - 1)}{12}. \quad (17)$$

\footnote{For the derivation of this and of the following formulas, see for instance Ref. [4].}
So we have at the leading order

\[ [D_r(\beta_t)]^{N_t} \beta_t \rightarrow \infty \exp \left( -\frac{C_r N_t}{2N \beta_t (N_t)} \right). \]  

(18)

Clearly the above discussion is independent from the asymmetry of the lattice, and in particular the case of the symmetric lattice can be recovered by simply replacing \( \beta_t \) and \( N_t \) with \( \beta \) and \( n_t \) respectively. The r.h.s. of (18), and hence the effective action (15), are independent from \( N_t \) in the large \( N_t \) limit provided \( \beta_t(N_t) \) depends linearly from \( N_t \). If we define, consistently with eq. (6),

\[ J = \lim_{N_t \rightarrow \infty} \frac{\beta_t(N_t)}{N_t} \]  

(19)

we obtain the large \( N_t \) limit\(^5\) of the effective action (14):

\[ e^{S_{\text{hk}}^{\text{Pol}}} = \prod_{\bar{x},\bar{i}} \sum_r \chi_r(P(\bar{x} + \bar{i})) \chi_r(P^\dagger(\bar{x})) \exp \left( -\frac{C_r}{2NJ} \right). \]  

(20)

The above effective action is the same one that we would have obtained, for any value of \( N_t \) and not just in the large \( N_t \) limit, if we had started, always within the \( \beta_s = 0 \) approximation, from the heat kernel action instead of the Wilson action and with a coupling \( \beta_{\text{hk}}(N_t) \) given by

\[ \beta_{\text{hk}}(N_t) = JN_t. \]  

(21)

The critical value of \( J \) at the deconfinement transition will be determined in Sec. 4 by using an improved mean field method and a Poisson resummation of eq. (20). In the next section instead we shall calculate the next to leading term in the rescaling of \( \beta_t \), namely the coefficient \( a_1 \) of eq. (6). Unlike \( J \), this depends on the particular form of the action on the lattice (it is zero if we start from the heat kernel action as shown in (21)) and it is completely determined by the form of the coefficients \( D_r \).

3 The scaling law: exact calculation of the subleading term

We have shown in the last section that the effective action (15) has a smooth large \( N_t \) limit, given in eq. (20), provided a linear rescaling of \( \beta_t \) is assumed. In this section we want to go further, and assuming a linear rescaling of the form

\[ \beta_t(N_t) = JN_t + a_1 \]  

(22)

\(^5\)The large \( N_t \) limit can correspond either to the continuum limit, for instance in the symmetric lattice where \( \rho = 1 \) and \( N_t = n_t \), or to the limit \( \rho \rightarrow \infty \), which is a continuum limit only in the time dimension. These two limits, although conceptually completely different, produce here the same effect as a consequence of having neglected the space-like plaquettes.
we shall determine the next to leading coefficient \( a_1 \) by requiring that the effective action is independent from \( N_t \) also in the next to leading order of its \( \frac{1}{N_t} \) expansion. More precisely we shall show that it exists, for each \( N_t \), a value of \( a_1 \), independent from the representation \( r \), for which \((D_r(\beta_t))^{N_t}\) is independent from \( N_t \) in the large \( N_t \) limit up to order \( \frac{1}{N_t^2} \).

In order to do that we need to find the large \( \beta_t \) asymptotic expansion of \( D_r(\beta_t) \) up to the order \( \frac{1}{\beta_t^2} \), namely one order higher than the leading one, already given in eq. (16) and used in the derivation of the heat kernel action (20).

To begin with let us write \( F_r(\beta) \), defined in (14), as

\[
F_r(\beta) = \sum_{n=\infty}^\infty \sum_P (-1)^{\sigma(P)} \int_{-\pi}^{\pi} \prod_i d\theta_i e^{i \sum_j (r_j + P(j) + n) \theta_j + N \beta \sum_j \cos \theta_j} .
\]  

(23)

The infinite sum over \( n \) implies the costraint \( \sum_i \theta_i = 0 \); as a consequence a shift of all the integers \( r_j \) by an arbitrary, not necessarily integer, quantity does not affect the r.h.s. of (23). We can therefore replace the \( r_j \)'s in (23) with

\[
\hat{r}_j = r_j - \frac{1}{N} \sum_{k=1}^N r_k ,
\]

(24)

which satisfy the relation

\[
\sum_j \hat{r}_j = 0 .
\]

(25)

We then proceed to the following redefinitions:

\[
\xi_j = \sqrt{N \beta} \theta_j, \quad \nu_j = \frac{\hat{r}_j}{\sqrt{N \beta}}, \quad \nu = \frac{n}{\sqrt{N \beta}}.
\]

(26)

With these replacements eq. (23) can be rewritten as

\[
F_r(\beta) = \sum_{\nu} \int_{-\pi}^{\pi} d\xi \mathcal{J}(\frac{\xi}{\sqrt{N \beta}}) e^{\sum_j (\nu_j + \nu) \xi_j + N \beta \sum_j \cos \xi_j} .
\]

(27)

where \( \mathcal{J}(\frac{\xi}{\sqrt{N \beta}}) \) is the unitary Vandermonde determinant defined by

\[
\mathcal{J}(\theta) = \prod_{i<j} 2 \sin \frac{\theta_i - \theta_j}{2} .
\]

(28)

We shall now derive the asymptotic expansion in \( \frac{1}{\beta} \) of (27), keeping \( \nu_j \) fixed, in spite of the \( \beta \) dependence hidden in the definition of \( \nu_j \). Only at the end of the calculation we shall replace \( \nu_j \) with its expression.

The asymptotic expansion of the integral at the r.h.s. of (27) is obtained in two steps. First, by noticing that under the integral \( \xi_j = -i \frac{\partial}{\partial \nu_j} \), we extract the Vandermonde determinant and write

\[
F_r(\beta) = \mathcal{J}(\frac{-i}{\sqrt{N \beta}} \frac{\partial}{\partial \nu}) \sum_{\nu} \int_{-\pi}^{\pi} \frac{d\xi}{\sqrt{N \beta}} \prod_i e^{i \sum_j (\nu_j + \nu) \xi_j + N \beta \sum \cos \frac{\xi_j}{\sqrt{N \beta}}} .
\]

(29)
The asymptotic behaviour of the above integral can now be easily calculated by expanding the cosine and using the saddle point method. Also, the integrals over the \( \xi_i \)'s can be taken from \(-\infty \) to \(+\infty \) up to terms which are exponentially depressed. The result is:

\[
F_r(\beta) = J\left(\frac{-i}{\sqrt{N\beta}} \frac{\partial}{\partial \nu}\right) \sum_{\nu} e^{-\frac{i}{2} \sum_j (\nu_j + \nu)^2} \\
\times \left[ 1 + \frac{1}{8N\beta}(1 - 2 \sum_j (\nu_j + \nu)^2 + \frac{1}{3} \sum_j (\nu_j + \nu)^4) + C + O\left(\frac{1}{N^2\beta^2}\right) \right],
\]

where \( C \) collects all terms that do not depend on the representation, that is that are independent from \( \nu_j \). These are in fact irrelevant for our calculation as only the ratio \( \frac{F_r(\beta)}{F_0(\beta)} \) appears in (13). Similarly the terms of order \( \frac{1}{N^2\beta} \) have not been evaluated because they are either representation independent or of order higher than \( \frac{1}{\beta^2} \) once the \( \beta \) dependence of \( \nu_j \) is taken into account.

The next step is the summation over \( \nu \). It is easy to see that up to non perturbative terms we can perform the replacement

\[
\sum_{\nu} \rightarrow \sqrt{N\beta} \int_{-\infty}^{\infty} d\nu
\]

and that the resulting gaussian integrals can be calculated. With the condition \( \sum_j \nu_j = 0 \), the result is:

\[
F_r(\beta) = J\left(\frac{-i}{\sqrt{N\beta}} \frac{\partial}{\partial \nu}\right) e^{-\frac{i}{2} \sum_j \nu_j^2} \left[ 1 - \frac{N - 1}{4\beta N^2} \sum_j \nu_j^2 + \frac{1}{24\beta} \sum_j \nu_j^4 + C + O\left(\frac{1}{\beta^2}\right) \right].
\]

We have now to perform the derivatives included in the unitary Vandermonde determinant \( J \). By expanding the sines (see eq. (28)) we can first write it, up to the relevant order in \( \frac{1}{\beta} \), as

\[
J\left(\frac{-i}{\sqrt{N\beta}} \frac{\partial}{\partial \nu}\right) = \left(1 + \frac{1}{48\beta} \sum_j \frac{\partial^2}{\partial \nu_j^2} + O\left(\frac{1}{\beta^2}\right)\right) \Delta\left(\frac{-i}{\sqrt{N\beta}} \frac{\partial}{\partial \nu}\right),
\]

where \( \Delta \) is the ordinary Vandermonde determinant. By operating with the r.h.s. of (32) we get

\[
F_r(\beta) = \Delta\left(\frac{-i}{\sqrt{N\beta}} \frac{\partial}{\partial \nu}\right) e^{-\frac{i}{2} \sum_j \nu_j^2} \left[ 1 + \frac{N^2 - 6N + 6}{24\beta N^2} \sum_j \nu_j^2 + \frac{1}{24\beta} \sum_j \nu_j^4 + C \right].
\]

\[\text{In deriving eq. (32) the relation } \sum_j \frac{\partial}{\partial \nu_j} = 0, \text{ which follows from } \sum_j \nu_j = 0, \text{ has been taken into account.}\]
Finally we have to operate with $\Delta(-\frac{\partial}{\partial \nu})$ and keep in the result only the terms quadratic in $\nu_j^2$. In fact terms independent from $\nu_j$ are irrelevant for the reason given above, and terms of the type $\beta^{-1} \sum_j \nu_j^2$ are of order $\beta^{-3}$ once the dependence of $\nu_j$ from $\beta$ is taken into account. We shall make use of the following relations:

$$
\Delta(-\frac{\partial}{\partial \nu}) e^{-\frac{1}{2} \sum_j \nu_j^2} = \Delta(\nu) e^{-\frac{1}{2} \sum_j \nu_j^2},
$$

(34)

$$
\Delta(-\frac{\partial}{\partial \nu}) \sum_j \nu_j^2 e^{-\frac{1}{2} \sum_j \nu_j^2} = \Delta(\nu)(\sum_j \nu_j^4 - \frac{N(N-1)}{4}) e^{-\frac{1}{2} \sum_j \nu_j^2},
$$

(35)

$$
\Delta(-\frac{\partial}{\partial \nu}) \sum_j \nu_j^4 e^{-\frac{1}{2} \sum_j \nu_j^2} = \Delta(\nu)(\sum_j \nu_j^4 - 4(N - \frac{3}{2}) \sum_j \nu_j^2 + C(N)) e^{-\frac{1}{2} \sum_j \nu_j^2}.
$$

(36)

Eq. (34) simply follows from the remark that the polynomial in front of the exponential is globally of order $N(N-1)/2$ and that it is completely antisymmetric under exchange of the $\nu_j$’s. Eq. (35) is a simple consequence of (34): rescale the $\nu_j^2$’s in the exponent by a parameter $\gamma$ and take on both sides the derivative respect to $\gamma$. Eq. (36) can be checked directly for small values of $N$ and then proved by induction. Notice that in deriving eq. (36) the relation $\sum_j \nu_j = 0$ has been used.

By applying the above relations to eq. (33) we finally obtain

$$
F_r(\beta) = \tilde{C}(\beta, N) \Delta(\nu)e^{-\frac{1}{2} \sum_j \nu_j^2(1 + \frac{1}{2N}(\frac{N}{2} - \frac{1}{\sqrt{2}}) + O(\beta^{-2}))}.
$$

(37)

It is now sufficient to insert (37) into (13) and to keep into account the definition of $\nu_j$ given in (26) and the expression (17) of the quadratic Casimir in order to obtain

$$
D_r(\beta) = e^{\frac{\tilde{C}}{2N}\left[\frac{1}{N} + \frac{N^2 - 2}{4N^2} + \frac{1}{4N^2}\right] + O(\beta^{-3})}.
$$

(38)

This asymptotic behaviour can be inserted into the effective action (15), with the rescaling of the coupling given in eq. (22) to determine its large $N_t$ behaviour:

$$
e^{S_{pol}(\beta_{2\l})} = \prod_{x, i} \sum_r e^{-\frac{\tilde{C}}{2N_t}\left[1 - \frac{1}{2N_t}(a_1 - \frac{N^2 - 2}{4N_t})\right] + O(\frac{1}{N_t})} \chi_r(P(x_i + \hat{i})) \chi_r(P\dagger(x_i)).
$$

(39)

The leading order in $\frac{1}{N_t}$ gives, as we already knew, the heat-kernel action. As discussed earlier on, the rescaling of the coupling constant is determined by the requirement that the r.h.s. of (39) is $N_t$-independent up to the terms $O(\frac{1}{N_t})$. This fixes the value of $a_1$; in fact the $\frac{1}{N_t}$ term in the coefficients of the character expansion vanishes, for any representation, provided

$$
a_1 = \frac{N^2 - 2}{4N^2}.
$$

(40)
Let us remark that the above result in the large \( N \) limit \( (a_1 = 1/4) \) was already obtained in [5] by using an Eguchi–Kawai reduction scheme.

We have determined, in a purely analytic fashion, one of the two coefficients appearing in the scaling law (22). In the symmetric lattice \( (N_t = n_t) \) case eq. (23) describes the scaling towards the continuum limit \( (n_t \to \infty) \) of the critical temperature. This is the prediction that we are most interested in, since it can be tested against the Monte Carlo data, all of which are performed on symmetric lattices.

For asymmetric lattices \( (N_t = \rho n_t) \) eq. (22) gives the relation between the couplings at different values of the asymmetry parameter \( \rho \):

\[
\beta_t(\rho n_t) = J \rho n_t + a_1 = \rho (\beta_t(n_t) - a_1) + a_1 .
\]  

The last equation relates the value of \( a_1 \) which we just obtained with the parameters \( \alpha_{\rho}^{0,1} \) introduced in (10), and it gives

\[
\alpha_{\rho}^1 = a_1 , \quad \alpha_{\rho}^0 = -a_1 .
\]  

It is interesting to compare our prediction with the results obtained by Karsch [2] in 3 + 1 dimensions and including space-like plaquettes. Using the same normalization, they are given by

\[
\alpha_\tau^1 = a_1 = \frac{1}{4} - \frac{1}{2N^2} , \quad \alpha_\tau^0 = -(0.1305 - \frac{1}{2N^2}) = -a_1 + 0.1195 .
\]  

Our prediction for the subleading term \( \alpha_\tau^1 \) coincides with Karsch’s results, while a correction, presumably due to contribution of the space-like plaquettes appears in \( \alpha_\tau^0 \).

### 4 The leading term in the scaling law and comparison with the Monte Carlo data

Unlike the subleading term \( a_1 \) calculated in the previous section, the critical value \( J_c \) of the rescaled coupling \( J \) depends on the dynamics of the system and cannot be obtained exactly, since it would require solving the model defined by eq. (20). An estimate of \( J_c \) can be obtained by using a mean field approximation. With the form of the effective action given in eq. (20) this is not an easy task, due to the infinite sum over the group representations appearing in eq. (20). However the sum over the representations can be done explicitly by means of a Poisson resummation, and the action can be recast in the following form (see [6, 7]):

\[
e^{-S_{bh}(n_t=1)} = \prod_{\vec{x},\{i=1,2\}} \left( \frac{N}{4\pi} \right)^{1/2} \exp \left( \frac{-1}{24J} \right) \sum_P \frac{(NJ)^{(1-N)/2}}{J(\theta(\vec{x})) J(\theta(\vec{x}+\hat{i}))}
\]  

\( ^7 \)Eq. (43) has the same form as eq. (10); however, since we are neglecting the space-like plaquettes, the quantum corrections \( -a_1, a_1 \) in (11) can differ from \( \alpha_\tau^1, \alpha_\tau^0 \) in (10).
\[ x(-1)^{\sigma(P)} \sum_{\{l_i\}} \exp \left\{ -\frac{NJ}{2} \sum_{i=1}^{N} \left( \theta_i(\vec{x}) - \theta_{P(i)}(\vec{x} + \hat{i}) + 2\pi l_i \right)^2 \right\}, \]

where \( \theta_i(\vec{x}) \) are the invariant angles of the Polyakov lines: \( P(\vec{x}) = \text{diag}\{e^{i\theta(\vec{x})}\} \); \( J(\theta) \) is the Vandermonde determinant for a unitary matrix defined in (28) and \( P \) denotes a permutation of the indices. The set of integers \( \{l_i\} \) are winding numbers, related to the fact that the angles \( \theta_i(\vec{x}) \) are defined modulo \( 2\pi \). For special unitary groups the sum of the invariant angles must vanish modulo \( 2\pi \); if we choose them to satisfy the relation \( \sum_i \theta_i(\vec{x}) = \sum_i \theta_i(\vec{x} + \hat{i}) = 0 \), then the relation \( \sum_i l_i = 0 \) also follows.

An infinite sum over the integers \( \{l_i\} \) also appears in eq. (44), but the contribution of the winding modes is known to vanish in the deconfined phase in the large \( N \) limit and it is exponentially suppressed for finite \( N \). Hence an excellent approximation is obtained by truncating the sum keeping only very low values of \( \{l_i\} \). The mean field analysis is greatly simplified in comparison with the original form (20) of the effective action, to the extent that improvements of the mean field approximation, obtained by considering larger and larger clusters of “spins” (see Fig. 1), become viable. In this way more precise estimates of the critical coupling can be obtained (see [8] for further details). The results of our mean field analysis for the SU(2) and SU(3) models are reported in Tab. I together with the \( N = \infty \) result already obtained in [1]. Notice that the complexity of the analysis increases with \( N \); so we could reach a “step 3” cluster in the SU(2) case, while we could only go as far as a “step 2” cluster in the SU(3) case. Notice however that as the number of internal degrees of freedom increases the mean field approximation becomes more and more precise (in the large \( N \) limit it is exact), and smaller clusters are sufficient to extract rather good approximations of the exact result. In the SU(2) case, having at our disposal three steps we tried a (naive) extrapolation of our estimate toward the infinite cluster limit, assuming a simple geometric progression. This result is reported in the fourth line of Tab. I, while in the last line we have reported, for comparison, the Monte Carlo estimates of \( J_c \) (that we shall discuss below) for \( N = 2 \) and \( N = 3 \).

### 4.1 Comparison with the Monte Carlo results

It is interesting to compare our predictions for \( J_c \) and \( a_1 \) with the existing Monte Carlo data in (2+1) dimensions. Very precise estimates of the deconfinement temperature exist for the \( N = 2 \) and \( N = 3 \) models in the range \( 2 \leq n_t \leq 6 \). These can be found in [9, 10, 11] and are reported in Tab. II. All these simulations were made with the standard Wilson action and at \( \rho = 1 \).

The data show that the expected linear dependence from \( n_t \) is very well fulfilled, and, what is more important, they are precise enough to allow to measure, besides the leading linear dependence, also the subleading correction. Both in the \( N = 2 \) and in the \( N = 3 \) case the Monte Carlo data fit the linear scaling law (eq.(11) for
Table I: Results for $J_c$ obtained by (improved) mean field techniques. For the $N = \infty$ case, see \cite{1}. The boldface entries represent our best estimates of the value of $J_c$. In the last line we have reported the Monte Carlo estimates of $J_c$ for $N = 2$ and $N = 3$.

|          | $N = 2$ | $N = 3$ | $N = \infty$ |
|----------|---------|---------|--------------|
| step 1   | 0.292   | 0.332   | 0.351        |
| step 2   | 0.332   | **0.359** |              |
| step 3   | 0.355   |          |              |
| extrapol. | **0.367** |          |              |
| Monte Carlo | 0.380(3) | 0.366(2) |              |

Table II: The critical coupling $\beta_c$ as a function of the lattice size in the t direction, $n_t$, in the (2+1) dimensional SU(2) and SU(3) LGT, taken from \cite{9}, \cite{11} and \cite{10}.

| $n_t$ | $N = 2$ | $N = 3$ |
|-------|---------|---------|
| 2     | \sim 0.866 | 0.906(2) |
| 3     | \sim 1.251 |        |
| 4     | 1.630(8)  | 1.638(6) |
| 6     | 2.388(10) | 2.371(17) |

Symmetric lattices

$$\beta_c(n_t) = J_c n_t + a_1$$ (45)

with a good confidence level. If we extract from the data the best fit values of $a_1$ and $J_c$ we obtain the results reported in Table III. The comparison with our theoretical prediction shows a quite remarkable agreement even in the subleading term.

Another way to compare the results of the Monte Carlo simulations with our analytic estimates is to factor out the leading linear dependence by plotting $\frac{\beta_c(n_t)}{n_t}$ for various values of $n_t$. The results are shown, for both Monte Carlo and analytic data, in Table IV and plotted in Fig. 2. The analytic data in Table IV are obtained by inserting in (45) our best estimate of $J_c$ and the exact result for $a_1$. The last row in Table IV is obtained from the best fit of the Monte Carlo data, using (45).

At the end of Sec. 1 we formulated the conjecture that in 2 + 1 dimensional LGT the dynamics of the Polyakov loop is dominated by the time-like plaquettes, and that the space-like plaquettes can therefore be neglected. This conjecture was based solely on the fact that the time-like plaquettes dominate for very low values of $n_t$ and that the scaling law for the critical coupling is the same for the theory with and without space-like plaquettes. The results of our analysis show that this conjecture al least predicts with very good approximation the location of the deconfinement
Table III: Best fit values of $J_c$ and $a_1$ compared to our analytic predictions.

|        | $N = 2$                               | $N = 3$                               | $N = 3$                               |
|--------|---------------------------------------|---------------------------------------|---------------------------------------|
|        | Fit of M.C. data Analytic              | Fit of M.C. data Analytic              | Analytic                              |
| $J_c$  | 0.380(3)                              | 0.367                                 | 0.366(2)                              |
| $a_1$  | 0.106(1)                              | 0.125                                 | 0.174(6)                              |

Table IV: MC data for $\beta_c(n_t)/n_t$ compared to our analytic predictions obtained inserting our best estimates for $J_c$ and our exact result $a_1$ in eq. (6). Entries in the last row are obtained by fitting the MC data (see Tab. III).

| $n_t$ | $N = 2$                               | $N = 3$                               | $N = \infty$                          |
|-------|---------------------------------------|---------------------------------------|---------------------------------------|
|       | M.C. data Analytic                    | M.C. data Analytic                    | Analytic                              |
| 2     | ~ 0.433                               | 0.430                                 | 0.4531(8)                             |
| 3     | ~ 0.417                               | 0.409                                 | 0.424                                 |
| 4     | 0.4075(19)                            | 0.398                                 | 0.4094(14)                            |
| 6     | 0.3979(16)                            | 0.388                                 | 0.3952(27)                            |
| $\infty$ | 0.380(3)                               | 0.367                                 | 0.366(2)                              |

transition. The problem of whether the order of the phase transition and its critical exponents can also be correctly predicted in the same approximation is an open one and it would deserve further investigation.

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Step 1 Step 2 Step 3

Fig. 1: Progressively more precise Bethe-like approximations. In Step 1 (standard mean field approximation) we integrate only on the invariant angles $\theta_i$ of the central site, while all the other are kept fixed to the mean value $\bar{\theta}$ and the consistency condition is $\langle P(\theta) \rangle = P(\bar{\theta})$, where $P$ is the Polyakov loop. In the successive steps also the invariant angles $\theta'_i$ of the nearest neighbours (Step 2) and $\theta''_i$ of the next to nearest neighbours (Step 3) are integrated over, and the consistency condition reads $\langle P(\theta) \rangle = \langle P(\theta') \rangle$. A link factor of the action $[14]$ is associated to each link drawn with a full line in the figure.

Fig. 2: The available M.C data for $\beta_c(n_t)/n(t)$ are compared to scaling laws obtained in the present paper.