INTERIOR $C^{1,\alpha}$ REGULARITY ON THE LINEARIZED MONGE-AMPRÈRE EQUATION WITH VMO TYPE COEFFICIENTS

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Abstract. In this paper, we establish interior $C^{1,\alpha}$ estimates for solutions of the linearized Monge-Ampère equation

$$L_{\phi}u := \text{tr}[\Phi D^2u] = f,$$

where the density of the Monge-Ampère measure $g := \det D^2\phi$ satisfies a VMO-type condition and $\Phi := (\det D^2\phi)(D^2\phi)^{-1}$ is the cofactor matrix of $D^2\phi$.

1. Introduction

This paper is concerned with interior regularity of solutions of the linearized Monge-Ampère equation

$$(1.1) \quad L_{\phi}u := \text{tr}[\Phi D^2u] = f,$$

where $\phi$ is a solution of the Monge-Ampère equation

$$(1.2) \quad \det D^2\phi = g, \quad \lambda \leq g \leq \Lambda \quad \text{in } \Omega,$$

for some constants $0 < \lambda \leq \Lambda < \infty$.

The operator $L_{\phi}$ appears in several contexts including affine geometry, complex geometry and fluid mechanics, see for example [1, 8, 9, 10, 11]. In particular, the authors in [11] resolved Chern’s conjecture in affine geometry concerning affine maximal hypersurfaces in $\mathbb{R}^3$.

Concerning regularity of (1.1) a fundamental result is the Harnack inequality for nonnegative solutions of $L_{\phi}u = 0$ established in [2], which yields interior Hölder continuity of solutions of (1.1). By using this result and perturbation arguments, Gutiérrez and Nguyen established in [4] interior $C^{1,\alpha}$ estimates for solutions of (1.1) when $g \in C(\Omega)$. Their main result has the form

$$(1.3) \quad \|u\|_{C^{1,\alpha}(\Omega')} \leq C\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p_{\text{loc}}(\Omega)},$$

for any $0 < \alpha' < \alpha$. Here $\Omega' \Subset \Omega$ and $\|f\|_{L^p_{\text{loc}}(\Omega)}$ is defined in Theorem 1 below.

In [5], interior $W^{2,p}$ estimates for solutions of (1.1) were established for general $1 < p < q, f \in L^q(\Omega)$ and continuous density $g = \det D^2\phi$. The main result in this paper has the form

$$(1.4) \quad \|D^2u\|_{L^p(\Omega')} \leq C\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)}.$$

By the imbedding theorem, when $1 < n < p < q$ the above estimate holds if we replace the left hand side by $\|u\|_{C^{1,\gamma}(\Omega')}$ with $\gamma < 1 - n/p$. Since $\|f\|_{L^p(\Omega)} \leq C\|f\|_{L^q(\Omega)}$ for $1 - \alpha - \frac{n}{2q} \geq 0$, hence for $f \in L^q(\Omega)$ the inequality (1.3) gives a better $C^{1,\gamma}$ estimate for $\gamma < 1 - n/p$ than (1.4).

On the other hand, in the case that $g$ is discontinuous, Huang [7] proved interior $W^{2,p}$ estimates for solutions $\phi$ of (1.2) where $g = \det D^2\phi$ belongs to a VMO-type space $\text{VMO}_{\text{loc}}(\Omega, \phi)$ (see Section 2 for the definition). Using this result we recently established in [12] global $W^{2,p}$ estimates for solutions of (1.1) when $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$ defined in [7]. Our result has a similar form to (1.4) where the $L^p$ norm of $D^2u$ is estimated in terms of $\|f\|_{L^q(\Omega)}$. By the imbedding theorem again, interior $C^{1,\alpha}$ estimate when $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$ follows. But this $C^{1,\alpha}$ estimate is in terms of $\|f\|_{L^q(\Omega)}$ rather than $\|f\|_{L^q_{\text{loc}}(\Omega)}$ in (1.3).

Therefore, we are interested in establishing the interior $C^{1,\alpha}$ estimates for solutions of (1.1) in terms of $\|f\|_{L^p_{\text{loc}}(\Omega)}$ under the assumption that $g$ belongs to $\text{VMO}_{\text{loc}}(\Omega, \phi)$. Namely, we extend the result in [4] from the case that $g \in C(\Omega)$ to the case that $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$. Our main result can be stated as follows.

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Theorem 1. Let \( B_{a_1} \subset \Omega \subset B_1 \) be a normalized convex domain and \( \phi \in C(\overline{\Omega}) \) be a convex solution of (1.2) with \( \phi = 0 \) on \( \partial \Omega \), where \( g \in \text{VMO}_{\text{loc}}(\Omega, \phi) \). Assume that \( u \in W^{2,n}_{\text{loc}}(\Omega) \) is a solution of \( L_\phi u = f \) in \( \Omega \) with
\[
[f]_{s,\Omega}^n := \sup_{S_\phi(x,r) \subset \Omega} \frac{1}{r^{n-1}} \left( \int_{S_\phi(x,r)} |f|^n \, dx \right)^{\frac{1}{n}} < \infty
\]
for some \( 0 < \alpha < 1 \). Then for any \( \alpha' \in (0, \alpha) \) and any \( \Omega' \subset \Omega \) we have
\[
(1.5) \quad ||d||_{C^{1,\alpha'}(\Omega')} \leq C(||d||_{L^\infty(\Omega)} + [f]_{s,\Omega}^n),
\]
where \( C \) depends only on \( n, \alpha, \alpha', \lambda, \Lambda, \text{dist}(\Omega', \partial \Omega) \) and the VMO-type property of \( g \).

The space \( \text{VMO}_{\text{loc}}(\Omega, \phi) \) above is defined in Section 2.

We follow the perturbation arguments as in [4]. The main lemma in our case is the stability of the cofactor matrix of \( D^2 \phi \) under a VMO-type condition of \( g = \det D^2 \phi \). For this we use the interior \( W^{2,p} \) estimates for solutions of (1.2) in [7]. We also need a result from [7] which concerns the eccentricity of sections of (1.2) under the VMO-type condition of \( g \).

The paper is organised as follows. In Section 2, we establish the stability of the cofactor matrix of \( D^2 \phi \) under a VMO-type condition of \( g = \det D^2 \phi \). In Section 3, we give an approximation lemma and investigate the eccentricity of sections of solutions of (1.2) when \( g \) is in VMO-type spaces. In Section 4, we prove the \( C^{1,\alpha} \) estimate of solutions of (1.1) at the minimum point of \( \phi \), and finally, we give the complete proof of Theorem 1.

2. Preliminary results and stability of cofactor matrices

We first introduce some notation. Let \( \phi \in C(\overline{\Omega}) \) be a solution of (1.2). A section of \( \phi \) centered at \( x_0 \in \Omega \) with height \( h \) is defined by
\[
S_\phi(x_0, h) := \{ x \in \Omega : \phi(x) < \phi(x_0) + \nabla \phi(x_0) \cdot (x - x_0) + h \}.
\]
If \( \phi = 0 \) on \( \partial \Omega \), then for \( 0 < \alpha < 1 \), we define
\[
\Omega_\alpha := \{ x \in \Omega : \phi(x) < (1 - \alpha) \min_{\Omega} \phi \}.
\]
Let \( B_{a_1}(x_0) \) be the ball centered at \( x_0 \in \mathbb{R}^n \) with radius \( r \) and denote for simplicity \( B_r = B_r(0) \).

We always use the following assumption:
\( \text{(H)} \) \( B_{a_2} \subset \Omega \subset B_{a_1} \) is a convex domain and \( \phi \in C(\overline{\Omega}) \) is a solution of (1.2) with \( \phi = 0 \) on \( \partial \Omega \), where \( 0 < a_1 \leq a_2 < \infty \).

Under the assumption (H) we often take \( w \) to be the convex solution of
\[
\begin{cases}
\det D^2 w = 1 & \text{in} \ \Omega, \\
w = 0 & \text{on} \ \partial \Omega.
\end{cases}
\]

The following Hölder estimate for (1.1) is from [2].

Lemma 2.1. (See [4, Lemma 2.5 and (2.2)].) Assume that condition (H) holds. Let \( u \in W^{2,n}_{\text{loc}}(\Omega) \) be a solution of \( L_\phi u = f \) in \( \Omega \). Let \( \Omega' \subset \Omega \). Then for any \( x_0 \in \Omega' \) and \( h \leq c \), we have
\[
|u(x) - u(y)| \leq C^* h^{-\beta} |x - y|^{\beta} \left( ||d||_{L^\infty(S_\phi(x_0, 2h))} + (2h)^{\frac{1}{2}} ||f||_{L^\infty(S_\phi(x_0, 2h))} \right) \quad \forall x, y \in S_\phi(x_0, h),
\]
where \( C^*, c > 0, 0 < \beta < 1 \) are constants depending only on \( n, \lambda, \Lambda, a_1, a_2 \) and \( \text{dist}(\Omega', \partial \Omega) \).

The lemma below concerns classical \( C^{1,1} \) interior estimate for uniformly elliptic equations.

Lemma 2.2. (See [4, Theorem 2.7].) Assume that \( B_{a_1} \subset \Omega \subset B_{a_2} \) is a convex domain and \( w \) is the solution of (2.2). Then for any \( \varphi \in C(\partial B_{a_2}) \) there exists a solution \( h \in C^2(B_{a_2}) \cap C(\overline{B_{a_2}}) \) of \( L_\varphi h = 0 \) in \( B_{a_2} \) such that
\[
||h||_{C^{1,1}(\overline{B_{a_2}})} \leq c_r ||\varphi||_{L^\infty(\partial B_{a_2})},
\]
where \( c_r > 0 \) is a constant depending only on \( n, a_1, a_2 \).

The following Lemmas 2.3, 2.4 and Theorem 2.1 were proved in [7].
Lemma 2.3. (See [7, Lemma 2.1].) Assume that condition (H) holds. Then for any $\Omega' \Subset \Omega$, there exist positive constants $h_0, C$ and $q$ such that for $x_0 \in \Omega'$, and $0 < h \leq h_0$,
\[ B_{C^{-1}h}(x_0) \subset S_{g}(x_0, h) \subset B_{Ch}(x_0), \]
where $q = q(n, \lambda, A, a_1, a_2)$ and $h_0, C$ depends only on $n, \lambda, A, a_1, a_2 \text{ and } \text{dist}(\Omega', \partial \Omega)$.

Assume that condition (H) holds. The space $\text{VMO}_{\text{loc}}(\Omega, \phi)$ is defined in [7] as follows. Given a function $g \in L^1(\Omega)$, we say that $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$ if for any $\Omega' \Subset \Omega$,
\[ Q_g(r, \Omega') := \sup_{x_0 \in \Omega'} \text{mosc}_{S_{g}(x_0, h)} g \rightarrow 0 \quad \text{as } r \rightarrow 0. \]
Here the mean oscillation of $g$ over a measurable subset $A \subset \Omega$ is defined by
\[ \text{mosc}_A g := \int_A |g(x) - g_A| dx, \]
where $g_A = \int_A g dx$ denotes the average of $g$ over $A$.

There are two simple facts about the definition above.

Proposition 2.1. For any function $g^1$ such that $g := (g^1)^n \in L^1(\Omega)$, the following hold:

(i)
\[ \left( \int_{\Omega} |g - (g^1)|_1^n dx \right)^{\frac{1}{n}} \leq 2 \left( \int_{\Omega} |g - g_1| dx \right)^{\frac{1}{n}}. \]

(ii) For any measurable subset $A \subset \Omega$, we have
\[ \left( \int_{\Omega} |g - (g^1)|_1^n dx \right)^{\frac{1}{n}} \leq \left\{ 1 + \left( \frac{|A|}{|A|} \right)^{\frac{1}{n}} \right\} \left( \int_{\Omega} |g - (g^1)|_{\Omega} dx \right)^{\frac{1}{n}}. \]

The maximum principle below is used to compare solutions $\phi$ of (1.2) and $w$ of (2.2), where $g = \det D^2 \phi \in \text{VMO}_{\text{loc}}(\Omega, \phi)$ defined above.

Lemma 2.4. (See [7, Lemma 3.1].) Assume $\Omega$ is a bounded convex domain in $\mathbb{R}^n$. Let $\phi$ and $w$ be the weak solutions to $\det D^2 \phi = g_1^0 \geq 0$ and $\det D^2 w = g_2^0 \geq 0$ in $\Omega$, respectively. Assume that $g_1, g_2 \in L^q(\Omega)$. Then
\[ \max_{\Omega} (\phi - w) \leq \max_{\partial \Omega} (\phi - w) + C_n \text{diam}(\Omega) \left( \int_{\Omega} (g_2 - g_1)^+ dx \right)^{1/n}. \]

The theorem below gives $W^{2,p}$ estimates of solutions $\phi$ of (1.2) under a VMO-type condition of $g$.

Theorem 2.1. (See [7, Theorem A(i)].) Assume the condition (H) holds. Let $0 < \alpha < 1, 1 \leq p < \infty$. There exist constants $0 < \epsilon < 1$ and $C > 0$ depending only on $n, \lambda, \Lambda, a_1, a_2, \alpha$ and $\epsilon$ such that if $\text{mosc}_S g \leq \epsilon$ for any $S = S_{\phi}(x_0, h) \Subset \Omega$, then
\[ ||D^2 \phi||_{L^p(\Omega)} \leq C. \]

Next we establish stability of cofactor matrices of $D^2 \phi$ under a VMO-type condition of $g$.

Lemma 2.5. Assume that $0 < \lambda \leq \Lambda < \infty$ and $a_1, a_2 > 0$. Let $B_{a_1} \subset \Omega^k \subset B_{a_2}$ be a sequence of convex domain converging in the Hausdorff metric to a convex domain $B_{a_1} \subset \Omega \subset B_{a_2}$. For each $k \in \mathbb{N}$, let $\phi_k \in C(\overline{\Omega^k})$ be a convex function satisfying
\[ \left\{ \begin{array}{ll}
\det D^2 \phi_k = g_k & \text{in } \Omega^k, \\
\phi_k = 0 & \text{on } \partial \Omega^k.
\end{array} \right. \]

where $0 < \lambda \leq g_k = (g_k^1)^n \leq \Lambda$ in $\Omega^k$,
\[ \text{mosc}_{\Omega^k} g_k \leq \frac{1}{k} \quad \text{and} \quad \sup_{S_{\phi_k}(x, h) \subset \Omega^k} \text{mosc}_{S_{\phi_k}(x, h)} g_k \leq \frac{1}{k}. \]

Suppose that $\phi_k$ converges uniformly on compact subsets of $\Omega$ to a convex function $\phi \in C(\overline{\Omega})$ which is a solution of
\[ \left\{ \begin{array}{ll}
\det D^2 \phi = 1 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{array} \right. \]
Then there exists a subsequence which we still denote by $\phi_k$ such that for any $1 \leq p < \infty$,
\[
\lim_{k \to \infty} \|D^2\phi_k - (g^1_k)_{B_{a_k}} D^2\phi\|_{L^p(B_{a_k})} = 0,
\]
and
\[
\lim_{k \to \infty} \|\Phi_k - (g^1_k)^{v-1}_B\Phi\|_{L^p(B_{a_k})} = 0,
\]
where $\Phi_k$ and $\Phi$ are the cofactor matrices of $D^2\phi_k$ and $D^2\phi$ respectively.

**Proof.** First we note that since $\text{dist}(B_{a_k}, \partial \Omega^k) \geq c(n, a_1, a_2)$, then $B_{a_k} \subset \Omega^k$, where $c$ is a constant depending only on $n, \lambda, \Lambda, a_1, a_2$. For any $1 \leq p < \infty$, let $\epsilon(p) = \epsilon(n, \lambda, \Lambda, a_1, a_2, p, \alpha) = \epsilon(n, \lambda, \Lambda, a_1, a_2, p)$ be the constant in Theorem 2.1, then for any $k \geq k_{\epsilon(p)} := \left\lceil \frac{1}{\epsilon(p)} \right\rceil + 1$ we have
\[
\sup_{S_{a_k}(x, h) \in \Omega^k} \text{mosc}_{S_{a_k}(x, h)} g_k \leq \epsilon(p).
\]
Thus Theorem 2.1 implies that
\[
\|D^2\phi_k\|_{L^p(B_{a_k})} \leq \|D^2\phi_k\|_{L^p(\Omega^k)} \leq C(n, \lambda, \Lambda, a_1, a_2, p) \quad \forall k \geq k_{\epsilon(p)}.
\]

Let $\delta > 0$ be an arbitrary small constant, and let $\Omega(\delta) := \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}$. Then there exists $k_\delta \in \mathbb{N}$ such that for all $k \geq k_\delta$,
\[
\text{dist}(x, \partial \Omega^k) \leq 2\delta, \quad \forall x \in \partial(\Omega(\delta)).
\]
Then Aleksandrov’s estimate ([3, Theorem 1.4.2]) implies that
\[
|\phi_k(x) - (g^1_k)_{B_{a_k}} \phi(x)| \leq C(n, \lambda, \Lambda, a_1, a_2)\delta^{1/n} \quad \forall x \in \partial(\Omega(\delta)).
\]

By choosing $k_\delta$ even larger, we have $\Omega(\delta) \subset \Omega^k$ for $k \geq k_\delta$. It follows from Proposition 2.1 that,
\[
\left(\int_{\Omega(\delta)} |g^1_k - (g^1_k)_{B_{a_k}}|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega_k} |g^1_k - (g^1_k)_{B_{a_k}}|^p dx \right)^{\frac{1}{p}} \leq C(n, a_1, a_2) \left(\int_{\Omega_k} |g^1_k - (g^1_k)_{\Omega_k}|^p dx \right)^{\frac{1}{p}} \leq \frac{C(n, a_1, a_2)}{k^{\frac{1}{n}}},
\]
\[
(2.5)
\]

Using the above two estimates and applying Lemma 2.4 with $\phi \rightsquigarrow \phi_k, w \rightsquigarrow (g^1_k)_{B_{a_k}} \phi$, we get
\[
\max_{\Omega(\delta)} |\phi_k - (g^1_k)_{B_{a_k}} \phi| \leq \max_{\partial(\Omega(\delta))} |\phi_k - (g^1_k)_{B_{a_k}} \phi| + C_\Omega \text{diam}(\Omega(\delta)) \left(\int_{\Omega(\delta)} |g^1_k - (g^1_k)_{B_{a_k}}|^p dx \right)^{\frac{1}{p}} \leq C(n, \lambda, \Lambda, a_1, a_2) \left[\delta^{\frac{1}{n}} + \frac{1}{k^{\frac{1}{n}}} \right],
\]
\[
(2.6)
\]
for all $k \geq k_\delta$.

Using (2.4), (2.5), (2.6) and similar arguments to the proof of [4, Lemma 3.4], we can obtain the first conclusion of the lemma. For the second conclusion, we write
\[
\Phi_k - (g^1_k)^{v-1}_B \Phi = \left[ 1 - \frac{(g^1_k)^v_{B_{a_k}}}{\det D^2 \phi_k} \right] \Phi_k - \frac{(g^1_k)^{v-1}_{B_{a_k}}}{\det D^2 \phi_k} \Phi_k \left( D^2 \phi_k - (g^1_k)_{B_{a_k}} D^2 \phi \right) \Phi.
\]

For any $1 \leq q, r < \infty$, if $qr \leq n$ then by (2.5) and Hölder inequality,
\[
\left(\int_{B_{a_k}} |g^1_k - (g^1_k)_{B_{a_k}}|^p dx \right)^{\frac{1}{p}} \leq C(n, \lambda, \Lambda, a_1, a_2, q, r) \left(\int_{B_{a_k}} |g^1_k - (g^1_k)_{B_{a_k}}|^q dx \right)^{\frac{1}{q}} \leq \frac{C(n, \lambda, \Lambda, a_1, a_2, q, r)}{k^{\frac{1}{n}}}.
\]
On the other hand, if \( qr > n \) then
\[
\left( \int_{B_{\frac{1}{2}}} |g_k^1 - (g_k^1)_{B_{\frac{1}{2}}}|^{qr} \, dx \right)^{\frac{1}{qr}} \leq C(n, q, r, \lambda, \Lambda) \left( \int_{B_{\frac{1}{2}}} |g_k^1 - (g_k^1)_{B_{\frac{1}{2}}}|^{n} \, dx \right)^{\frac{1}{n}} \leq \frac{C(n, q, r, \lambda, \Lambda, a_1, a_2)}{K^{\frac{1}{n}}}.
\]

Note that
\[
|(g_k^1)^{\rho} - (g_k^1)_{B_{\frac{1}{2}}}| \leq C(n, \lambda, \Lambda)|g_k^1 - (g_k^1)_{B_{\frac{1}{2}}}|.
\]

Therefore,
\[
\left\| 1 - \frac{(g_k^1)_{B_{\frac{1}{2}}}}{\det D^2 \phi_k} \right\|_{L^{n/(n-2)}(B_{\frac{1}{2}})} \to 0, \quad \text{as } k \to \infty.
\]

The rest of the proof is similar to that of [4, Lemma 3.5], using (2.4) and the first conclusion of the lemma.

\[\square\]

3. Main lemmas

3.1. A approximation lemma. Next we compare solutions \( v \) of (1.1) and \( h \) of \( \mathcal{L}_w h = 0 \). The lemma below can be proved using similar arguments as in [4, Lemma 4.1]. The difference is that we estimate \( \|v - h\|_{L^\infty} \) in terms of \( \|\Phi - (g^{\frac{1}{2}})^{n-1} W\|_{L^\infty} \) rather than \( \|\Phi - W\|_{L^\infty} \).

**Lemma 3.1.** Let \( \rho^* : [0, \infty) \to [0, \infty) \) be a nondecreasing continuous function with \( \lim_{r \to 0^+} \rho^*(\varepsilon) \to 0 \). Assume the condition (H) holds and \( \omega \in C(\Omega) \) is the solution of (2.2). Suppose \( \nu \in W^{2, n}_{\text{loc}}(B_{\frac{1}{2}}) \cap C(\overline{B}_{\frac{1}{2}}) \) is a solution of \( \mathcal{L}_w \nu = f \) in \( B_{\frac{1}{2}} \) with \( |\nu| \leq 1 \) in \( B_{\frac{1}{2}} \), and \( h \in W^{2, n}_{\text{loc}}(B_{\frac{1}{2}}) \cap C(\overline{B}_{\frac{1}{2}}) \) is a solution of
\[
\begin{cases}
\mathcal{L}_w h = 0 & \text{in } B_{\frac{1}{2}}, \\
h = \nu & \text{on } \partial B_{\frac{1}{2}},
\end{cases}
\]

Assume that \( \nu \) and \( h \) have \( \rho^* \) as a modulus of continuity in \( \partial B_{\frac{1}{2}} \). Then for any \( 0 < \tau < \frac{1}{4} \), we have
\[
\|v - h\|_{L^\infty(B_{\frac{1}{2}})} \leq C(n, \lambda, \Lambda, a_1, a_2) \rho^* \left( \|\Phi - (g^{\frac{1}{2}})^{n-1} W\|_{L^\infty(B_{\frac{1}{2}})} \right) + \|f\|_{L^\infty(B_{\frac{1}{2}})}
\]
provided that \( \|\Phi - (g^{\frac{1}{2}})^{n-1} W\|_{L^\infty(B_{\frac{1}{2}})} \leq \tau^2 \). Here \( \Phi \) and \( W \) are the cofactor matrices of \( D^2 \nu \) and \( D^2 w \) respectively.

Using Lemma 3.1, the stability of the cofactor matrix Lemma 2.5 and arguing as in [4, Lemma 4.2], we obtain the following approximation lemma when \( g = \det D^2 \phi \) satisfies a VMO-type small oscillation.

**Lemma 3.2.** Let \( \rho : [0, \infty) \to [0, \infty) \) be a nondecreasing continuous function with \( \lim_{r \to 0^+} \rho(\varepsilon) \to 0 \). Given \( K > 0, \varepsilon > 0 \). Assume the condition (H) holds and \( \omega \in C(\Omega) \) is the solution of (2.2). Let \( \varphi \in C(\partial B_{\frac{1}{2}}) \) have \( \rho \) as a modulus of continuity on \( \partial B_{\frac{1}{2}} \) and satisfy \( \|\varphi\|_{L^\infty(\partial B_{\frac{1}{2}})} \leq K \). Then there exists \( \delta = \delta(\varepsilon, n, \rho, \lambda, \Lambda, a_1, a_2, K) > 0 \) such that if
\[
\text{mosc}_\Omega g \leq \delta \quad \text{and} \quad \sup_{S_{\varepsilon}(x, h) \in \Omega} \text{mosc}_{S_{\varepsilon}(x, h)} g \leq \delta,
\]
and \( f \in L^n(B_{\frac{1}{2}}) \) with \( \|f\|_{L^n(B_{\frac{1}{2}})} \leq \delta \), then any classical solutions \( v, h \) of
\[
\begin{cases}
\mathcal{L}_w \varphi = f & \text{in } B_{\frac{1}{2}}, \\
v = \varphi & \text{on } \partial B_{\frac{1}{2}},
\end{cases}
\quad \text{and} \quad \begin{cases}
\mathcal{L}_w h = 0 & \text{in } B_{\frac{1}{2}}, \\
h = \nu & \text{on } \partial B_{\frac{1}{2}},
\end{cases}
\]
satisfy
\[
\|v - h\|_{L^\infty(B_{\frac{1}{2}})} \leq \varepsilon.
\]
3.2. Eccentricity of cross sections. The two lemmas below concern the eccentricity of sections of (1.2) if $g$ is in VMO-type spaces. They are slight modifications of [7, Lemmas 4.1, 4.2]. Note that [7, Lemma 4.1] gives an affine transformation $Tx = A(x - z_0)$. But $z_0$ is not necessarily the minimum point of $\phi$. This is not convenient when we prove the $C^{1,\alpha}$ estimate in Theorem 4.1 in the next section. In the following two lemmas we replace $z_0$ in [7, Lemma 4.1] by the minimum point of $\phi$.

**Lemma 3.3.** Assume the condition (H) holds, where $\text{mosc}_{\partial \Omega} g \leq \varepsilon$. Then there exist $c_0, C_0 > 0$ depending only on $n, \lambda, \Lambda, a_1, a_2$ and a positive definite matrix $M = A^tA$ satisfying

$$\det M = 1, \quad 0 < c_0 I \leq M \leq C_0 I,$$

such that for $0 < \mu \leq c_0$ and $\varepsilon^{\frac{1}{2}} \leq c_0 \mu^2$, we have

$$B_{(1 - \delta) \sqrt{\frac{1}{\mu}}} \subset \mu^{\frac{1}{2}} T S_{\phi}(x_0, \mu) \subset B_{(1 + \delta) \sqrt{\frac{1}{\mu}}},$$

where $\delta = C_0(\mu^\frac{1}{2} + \mu^{-1} \varepsilon^{\frac{1}{2}})$, $x_0 \in \Omega$ is the minimum point of $\phi$ and $Tx = A(x - x_0)$.

**Proof.** Let $w$ be the solution of (2.2). Then from Lemma 2.4 and Proposition 2.1, we obtain

$$\max_{\Omega} |\phi - g^1_{\Omega}w| \leq C(n, \lambda, \Lambda, a_1, a_2)\varepsilon^{\frac{1}{2}}.$$

Using this and arguing as in [4, Lemma 3.2] we can obtain that there exist constants $C, c_0 > 0$ depending only on $n, \lambda, \Lambda, a_1, a_2$ such that if $0 < \mu \leq c_0, 0 < \gamma \leq \frac{\mu}{4}$ and $0 < \varepsilon^{\frac{1}{2}} \leq c_0 \mu^2$, then

(3.1) \quad $S_w(x_0, \mu - C\varepsilon^{\frac{1}{2}}) \subset S_w(x_0, \mu + C\varepsilon^{\frac{1}{2}}),$

(3.2) \quad $\partial S_w(x_0, \mu + \gamma) \subset C\frac{\partial S_w(x_0, \mu)}{\partial \mu}, \quad \partial S_w(x_0, \mu - \gamma) \subset C\frac{\partial S_w(x_0, \mu)}{\partial \mu},$

(3.3) \quad $B_{C \sqrt{\gamma}}(x_0) \subset S_w(x_0, \mu) \subset B_{C \sqrt{\gamma}}(x_0),$

(3.4) \quad $\partial S_w(x_0, \mu) \subset C\frac{\partial S_w(x_0, \mu)}{\partial \mu}(\partial \sqrt{E}).$

where $E := \{ x : \frac{1}{4}(D^2w(x_0))(x - x_0), x - x_0) \leq 1 \}.$

From (3.1), (3.2) and (3.4), we obtain

$$\partial S_{\phi}(x_0, \mu) \subset C\frac{\partial S_w(x_0, \mu)}{\partial \mu}(\partial \sqrt{E}).$$

Since $S_{\phi}(x_0, \mu) = S_{\phi}(x_0, \varepsilon^\frac{1}{2})$, we obtain

(3.5) \quad $\partial S_{\phi}(x_0, \mu) \subset C\frac{\partial S_w(x_0, \mu)}{\partial \mu}(\partial \sqrt{\varepsilon^\frac{1}{2}}) E.$

for any $0 < \mu \leq c_0$ and $\varepsilon^{\frac{1}{2}} \leq c_0 \mu^2$.

Write $D^2w(x_0) = A^tA$ for some positive definite matrix $A > 0$ and $M = D^2w(x_0)$, and then the conclusion follows.

**Lemma 3.4.** Assume that $\lambda \leq b^2 \leq \Lambda$. Let $B_{(1 - \delta) \sqrt{\frac{1}{\mu}}} \subset \Omega \subset B_{(1 + \delta) \sqrt{\frac{1}{\mu}}}$ be a convex domain with $\delta > 0$ small and $\phi \in C(\overline{\Omega})$ be a solution of (1.2) with $\phi = 0$ on $\partial \Omega$, where $\text{mosc}_{\partial \Omega} g \leq \varepsilon$. Then there exist $c_0, C_0 > 0$ depending only on $n, \lambda, \Lambda$ and a positive definite matrix $M = A^tA$ satisfying

$$\det M = 1, \quad (1 - C_0\delta)I \leq M \leq (1 + C_0\delta)I,$$

such that for $0 < \mu \leq c_0$ and $\varepsilon^{\frac{1}{2}} \leq c_0 \mu^2$, we have

$$B_{(1 - \delta_1) \sqrt{\frac{1}{\mu}}} \subset \mu^{\frac{1}{2}} T S_{\phi}(x_0, \mu) \subset B_{(1 + \delta_1) \sqrt{\frac{1}{\mu}}} \subset \mu^{\frac{1}{2}} T S_{\phi}(x_0, \mu) \subset B_{(1 + \delta_1) \sqrt{\frac{1}{\mu}}}(0),$$

where $\delta_1 = C_0(\delta^\frac{1}{2} + \mu^{-1} \varepsilon^{\frac{1}{2}})$, $x_0 \in \Omega$ is the minimum point of $\phi$ and $Tx = A(x - x_0)$. 

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Theorem 4.1. Assume that normalized convex domain and \( \phi \) for any \( (3.6) \), we have

\[
\delta S_w(x_0, \mu) \subset N_{C_\delta \sqrt{\mu}E}.
\]

Indeed, we argue as in the proof of [4, (3.16)] and find that in order to prove (3.6), we only need to prove that for any \( \xi \in (1 + C_\delta \sqrt{\mu}) \sqrt{\mu}B \), we have

\[
|D^3w(\xi)| \leq C(n, \lambda, \Lambda)\delta.
\]

For this, we note that \( (1 + C_\delta \sqrt{\mu}) \sqrt{\mu}B \subset \Omega' \subset \Omega \) for some \( \Omega' \). Then from the proof of [7, Lemma 4.2], we obtain

\[
||w - P||_{C^1(\Omega')} \leq C(n, \lambda, \Lambda)\delta,
\]

where \( P(x) = \frac{1}{2}x^2 - \frac{1}{2}g \). Thus, the estimate (3.7) holds and therefore (3.6) is true. Moreover, the last estimate implies that

\[
|D^2w(x_0) - I| \leq C(n, \lambda, \Lambda)\delta.
\]

Similar to Lemma 3.3, (3.1), (3.2) and (3.6) imply that

\[
\delta S_{\phi + \frac{1}{\gamma}}(x_0, \mu) \subset N_{C_\delta \sqrt{\mu}E}(\delta S_w(x_0, \mu)) \subset N_{C_\delta \sqrt{\mu}E}(\delta S_{\phi}(x_0, \mu))
\]

or

\[
\delta S_{\phi}(x_0, \mu) \subset N_{C_\delta \sqrt{\mu}E}(\delta S_{\phi}(x_0, \mu))
\]

for any \( 0 < \mu \leq c_0 \) and \( \epsilon \leq c_0\mu^2 \).

Write \( D^2w(x_0) = A'\Lambda A \) for some positive definite matrix \( A > 0 \) and \( M = D^2w(x_0) \), then (3.8) gives

\[
(1 - C_\delta I) \leq M \leq (1 + C_\delta I),
\]

and the conclusion follows. \( \square \)

Remark 3.1. Under the assumptions in Lemmas 3.3 and 3.4, it follows from (3.1) and (3.3) that for any \( 0 < \mu \leq c_0 \) and \( \epsilon \leq c_0\mu^2 \), we have

\[
\delta S_{\phi}(x_0, \mu) \subset B_{\sqrt{\mu(c_0 + \epsilon)}}(x_0),
\]

where \( c_0 \) depend only on \( n, \lambda, \Lambda \) (Under the assumptions in Lemma 3.3 these constants also depend on \( a_1, a_2 \) )

4. Interior \( C^{1,\alpha} \) estimate for linearized equation

4.1. Estimate at the minimum point of the convex function. In this subsection, we prove \( C^{1,\alpha} \) estimate of (1.1) at the minimum point of \( \phi \) under a VMO-type condition of \( \det D^2\phi \).

Theorem 4.1. Assume that \( 0 < \alpha' < \alpha < 1, r_0, C_1 > 0 \) and \( 0 < \lambda \leq \Lambda < \infty \). Assume \( B_{\alpha'} \subset \Omega \subset B_1 \) is a normalized convex domain and \( \phi \in C(\Omega) \) is a convex solution of (1.2) with \( \phi = 0 \) on \( \partial\Omega \), where

\[
\text{mosc}_{\Omega} g \leq \theta \quad \text{and} \quad \sup_{S_{\phi}(x, \lambda) \subset \Omega} \text{mosc}_{S_{\phi}(x, \lambda)} g \leq \theta.
\]

Let \( u \in W^{2,\alpha}_{\text{loc}}(\Omega) \) be a solution of \( L_{\phi}u = f \) in \( \Omega \) with

\[
\left( \frac{1}{|S_{\phi}(\phi)|} \int_{S_{\phi}(\phi)} |f|^\alpha d\lambda \right)^{\frac{2}{\alpha}} \leq C_1 \frac{\lambda^{\alpha'}}{\lambda^{\alpha'}}
\]

for all \( S_{\phi}(\phi) = S_{\phi}(x_0, r) \subset \Omega, r \leq r_0 \),

where \( x_0 \) is the minimum point of \( \phi \), then \( u \) is \( C^{1,\alpha'} \) at \( x_0 \), more precisely, there is an affine function \( l(x) \) such that

\[
r^{-(1+\alpha')}|u - l|_{L^\infty(B_{\alpha'}(x_0))} + |Dl| \leq C|u|_{L^\infty(\Omega)} + C_1
\]

for all \( r \leq \mu^* \), where \( \theta \in (0, 1), C > 0, \mu^* > 0 \) depend only on \( n, \lambda, \Lambda, \alpha, \alpha', r_0 \).
**Proof.** We can assume that \( |u| \in L^\infty(\Omega) \leq 1 \) and

\[
\left( \frac{1}{|S_\epsilon(\phi)|} \int_{S_\epsilon(\phi)} |f|^p \, dx \right)^{\frac{1}{p}} \leq 2 \theta^{\frac{\alpha-1}{2}} \quad \text{for all } S_\epsilon(\phi) = S_\phi(x_0, r) \Subset \Omega, r \leq r_0.
\]

And we only need to prove that

\[
r^{-1(1+a')} |u - l|_{L^\infty(B(x_0))} + |Dl| \leq C \quad \forall r \leq \mu^*,
\]

where \( C > 0, \mu^* > 0 \) depend only on \( n, \lambda, \Lambda, \alpha, \alpha', r_0 \).

Define \( a_1 := \frac{1}{r_0} \sqrt{c_0} \) and \( a_2 := 2 \sqrt{\frac{c_0}{r_0}} \). Then Lemma 2.1 gives constants \( C^*, \beta > 0 \), Lemma 2.2 gives \( c_\epsilon > 0 \), Lemma 3.3 and 3.4 give \( c_0, C_0 > 0 \). All these constants depend only on \( n, \lambda, \Lambda \). Applying Lemma 3.3, 3.4 and 3.2 and similar arguments to the proof of [4, Theorem 4.5], we can prove that there exist \( 0 < \mu \leq 1 \) depending only on \( n, \lambda, \Lambda, r_0 \), a sequence of positive matrices \( A_k \) with \( \det A_k = 1 \), a sequence \( b_k > 0 \) and a sequence of affine functions \( l_k(x) = a_k + b_k \cdot (x - x_0) \) satisfying for \( k \geq 1 \),

\[
\|A_{k-1}^{-1}A_k^{-1}\| \leq \frac{1}{\sqrt{c_0}}, \quad \|A_k\| \leq \sqrt{C_0(1 + C_0\delta_0)(1 + C_0\delta_1) \cdots (1 + C_0\delta_{k-1})};
\]

\[
B_{a_1} \subset B_{(1-\delta_0)\sqrt{\frac{c_0}{r_0}}} \subset \mu^{\frac{1}{2}} A_0(S_{l_0}(\phi) - x_0) \subset B_{(1+\delta_1)\sqrt{\frac{c_0}{r_0}}} \subset B_{a_2}, \quad \lambda \leq b_k \leq \Lambda;
\]

\[
|u - l_{k-1}|_{L^\infty(S_{l_k}(\phi))} \leq \mu^{\frac{1}{2}}(k-1);
\]

\[
|a_k - a_{k-1}| + \mu^{\frac{1}{2}} |(A_k^{-1})'(B_k - B_{k-1})| \leq 2c_\epsilon\mu^{\frac{1}{2}}(k-1);
\]

\[
\frac{|(u - l_{k-1})(\mu^{\frac{1}{2}} A_k^{-1}x + x_0) - (u - l_{k-1})(\mu^{\frac{1}{2}} A_k^{-1}y + x_0)|}{\mu^{\frac{1}{2}}(k-1)} \leq 2C^*(\sqrt{c_1\mu})^{-\beta} |x - y|^\beta,
\]

for any \( x, y \in \mu^{\frac{1}{2}} A_k(S_{l_k}(\phi) - x_0) \),

where \( A_0 := I, \quad l_0 := 0, \quad \delta_0 := 0, \quad \delta_1 := C_0(\mu^{\frac{1}{2}} + \mu^{-1}\theta^{\frac{1}{2}}) < \frac{1}{2}, \quad \delta_k := C_0(\delta_{k-1}\mu^{\frac{1}{2}} + \mu^{-1}\theta^{\frac{1}{2}}), \quad \delta_k < \delta_{k-1} \quad \text{for } k \geq 2.\]

The rest of the proof is the same as Part 4 (proof of (4.35)) in the proof of [4, Theorem 4.5]. \( \square \)

4.2. **Proof of Theorem 1.** By Lemma 2.3, for any \( \Omega' \Subset \Omega \), there exist positive constants \( h_0, C \) and \( q \) depending only on \( n, \lambda, \Lambda \) and \( \text{dist}(\Omega', \partial\Omega) \) such that for any \( x_0 \in \Omega' \), we have

\[
B_{C^{-1}h_0}(x_0) \subset S_{l_0}(x_0, h_0) \subset B_{C\Phi_0}(x_0).
\]

Choose \( h_0 \) smaller and we can assume \( S_{l_0}(x_0, h_0) \subset B_{C(h_0)^p}(x_0) \subset \Omega'' \Subset \Omega \). Since \( g \in \text{VMO}_{\text{loc}}(\Omega, \phi) \), we have

\[
\eta_{l_0}(r, \Omega'') := \sup_{S_{l_0}(x, h) \subset \Omega''} \text{mosc}_{S_{l_0}(x, h)} g \to 0, \quad r \to 0.
\]

Let \( \theta = \theta(n, \alpha, \alpha', \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)) > 0 \) be the constant in Theorem 4.1, then there exists \( 0 < r_1 < 1 \) such that \( \eta_{l_0}(r_1, \Omega'') < \theta \). Take \( h_0 \) smaller such that \( \text{diam}(B_{C\Phi_0}(x_0)) \leq r_1 \), then for any \( S_{l_0}(x, h) \subset S_{l_0}(x_0, h_0) \), we have \( S_{l_0}(x, h) \subset \Omega'' \) and \( \text{diam}(S_{l_0}(x, h)) \leq r_1 \), thus,

\[
\text{mosc}_{S_{l_0}(x, h)} g \leq \eta_{l_0}(r_1, \Omega'') \leq \theta.
\]

Fix such \( h_0 \) in the rest of the proof. Note that \( h_0 \) depends only on \( n, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega), r_1 \). Thus it depends only on \( n, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega), \alpha, \alpha' \) and the VMO-type property of \( g \).

Let \( T \) be an affine map such that

\[
B_{\alpha_0} \subset T(S_{l_0}(x_0, h_0)) \subset B_1.
\]

By (4.3) we have

\[
\|T\| \leq Ch_0^{-1}, \quad \|T^{-1}\| \leq Ch_0^2.
\]

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where \( C = C(n, \lambda, \Lambda, \text{dist}(\Omega', \partial \Omega)) > 0 \). Let \( \kappa_0 := |\det A|^{\frac{1}{n}} \), then we have \( \kappa_0 h_0 \geq r_0 \) for some constant \( r_0 \) depending only on \( n, \lambda, \Lambda \).

For \( y \in \Omega := T(S_d(x_0, h_0)) \), define
\[
\tilde{\varphi}(y) = \kappa_0[(\varphi - I_{a_0})(T^{-1}y) - h_0]
\]
and
\[
\nu(y) = \kappa_0^{-1} u(T^{-1}y),
\]
where \( I_{a_0}(x) \) is the supporting function of \( \varphi \) at \( x_0 \). Then,
\[
\det D^2\tilde{\varphi}(y) = \tilde{g}(y) = (\tilde{g}(y))^{n}, \quad \lambda \leq \tilde{g}(y) = g(T^{-1}y) \leq \Lambda \quad \text{in } \tilde{\Omega}
\]
and by (4.4), we have
\[
\text{mosc}_{\tilde{\Omega}} \tilde{g} = \text{mosc}_{S_d(x_0, h_0)}g \leq \theta \]
and
\[
\sup_{S_d(x_0, h_0)} \text{mosc}_{S_d^1(y, h)} \tilde{g} \leq \sup_{S_d(T^{-1}y, h h_0) \subset S_d(x_0, h_0)} \text{mosc}_{S_d^1(y, h)} g \leq \theta.
\]
Applying Theorem 4.1 to \( \nu \) and arguing as in the proof of [4, Theorem 4.7], we obtain the conclusion of Theorem 1.

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