Annihilator Essential Submodules

Yousef A. Qasim, Sahira M. Yaseen

Department of Mathematics, College of science, Baghdad University, Baghdad, Iraq.

Email: alqazzazyousef@gmail.com

Abstract. Through this paper R represent a commutative ring with identity and all R-modules are unitary left R-modules. In this work we consider a generalization of the class of essential submodules namely annihilator essential submodules. We study the relation between the submodule and his annihilator and we give some basic properties. Also we introduce the concept of annihilator uniform modules and annihilator maximal submodules.

Keywords: Essential submodules, Annihilator essential submodules, Annihilator maximal submodules, Uniform modules, Annihilator uniform modules.

1. Introduction
Through this paper R will be a commutative ring with identity and all R-modules will be unitary left R-modules. It well known that A proper submodule N of an R-module M is called essential in M if for every non zero submodule K of M we have N∩K≠0 [1]. And M is called uniform if every non zero submodule N of an R-module M is essential [2]. The singular of an R-module M is the set Z(M)={x∈M: ann(x)≤eR} where ann(x)= {r∈R: rx=0}, if Z(M)=M then M is called singular module, if Z(M)=0 then M is called non singular module see[2]. Many authors have been interested in studying different definitions generalization of essential submodules see[3], [4], [5]. In section one we introduce the notion of annihilator essential submodule as a generalization of essential submodules where a submodule N of an R-module M is called annihilator essential (shortly ann-essential) denoted by N≤a.eM if N∩L=0 then ann(L)≤eR for every submodule L of M, in this section we give many properties of annihilator essential submodules . In section two we present an annihilator uniform modules concept as a generalization of uniform modules where an R-module M is called annihilator uniform (shortly ann-uniform) if every submodule of M is annihilator essential submodule. Authors in [6],[7],[8] gave a generalization for maximal submodules where a submodule N of an R-module M is called maximal submodule if whenever N<W≤M, where W is a submodule of M, then W=M. In the last section we introduce the concept of annihilator maximal submodules as generalization of maximal submodules ,where a submodule N of an R-module M is called annihilator maximal submodule (shortly ann-maximal) if whenever N<W≤M, W is ann-essential submodule of M, implies that W=M .And we investigate some properties of this type of submodules.
1.1 Annihilator essential submodules

In this section we introduce a class of annihilator essential submodules as a generalization of essential submodules. We give some basic properties for this concept. Firstly we begin by the following definition.

**Definition (1.1):** Let \( N \) be an non zero submodule of an \( R \)-module \( M \). \( N \) is said to be annihilator essential submodule of \( M \) (shortly ann-essential submodule) denoted by \( N \leq a.e \) \( M \), if \( N \cap L = 0 \) then \( \text{ann}(L) \leq e \) \( R \) for every submodule \( L \) of \( M \). That is a non zero submodule \( N \) is annihilator essential if \( N \cap L \neq 0 \) for every submodule \( L \) of \( M \) such that \( \text{ann}(L) \) is not essential in \( R \). Anon zero ideal \( I \) of a ring \( R \) is ann-essential ideal if \( I \cap E = 0 \) then \( \text{ann}(E) \leq e \) \( R \) for every ideal \( E \) of \( R \) as an \( R \)-module.

**Remarks and Examples (1.2):**

(1). Every essential submodule of an \( R \)-module \( M \) is ann-essential submodule.

Proof: let \( K \) be an essential submodule of \( M \) such that \( K \cap L = 0 \), Since \( K \) is essential then \( L = 0 \) and \( \text{ann}(0) = \{ r \in R: r.0 = 0 \} = R \leq e \) \( R \). Thus, \( K \) is ann-essential submodule. But the converse in general is not true as the following example shows: Let \( M = \mathbb{Z}_6 \) as \( \mathbb{Z} \)-module, Let \( K = \{ 0, 2, 4 \} \), the submodules of \( \mathbb{Z}_6 \) such that \( K \cap L = 0 \) are \( L_1 = \{ 0, 3 \} \) and \( L_2 = \{ 0 \} \), \( K \cap L_1 = 0 \) and \( \text{ann}(L_1) = 2\mathbb{Z} \leq e \mathbb{Z} \) then and \( K \cap L_2 = 0 \) and \( \text{ann}(L_2) = Z \leq eZ \), thus \( K \) is ann-essential submodule. But \( K \) is not essential submodule in \( \mathbb{Z}_6 \) since \( K \cap L = 0 \) and \( L \neq 0 \).

(2). If \( N = M \) then \( M \leq a.e M \)

(3). Consider \( Z_6 \) as \( Z_6 \)-module, Let \( K = \{ 0, 2, 4 \} \) and let \( L = \{ 0, 3 \} \). \( K \) is not ann-essential submodule since \( K \cap L = 0 \) and \( \text{ann}(L) = K \) which is not essential ideal of \( Z_6 \).

(4). If \( M \) is a singular \( R \)-module then every submodule is ann-essential submodule.

Proof: Let \( N \leq M \) such that \( N \cap L = 0 \). Since \( M \) is singular then \( \text{Ann}(L) \leq e \) \( R \). Hence \( N \) is ann-essential submodule of \( M \). For example let \( M = \mathbb{Z} \) as \( \mathbb{Z} \)-module, \( \mathbb{Z} \) is singular module, Every submodule of \( \mathbb{Z} \) is ann-essential submodule.

The converse in general is not true as the following example shows: Let \( M = \mathbb{Z}_6 \) as \( \mathbb{Z}_6 \)-module, Every submodule of \( \mathbb{Z}_6 \) is ann-essential submodule but \( M \) is non singular module.

**Proposition (1.3):** Let \( A \), \( B \) be a submodules of an \( R \)-module \( M \) such that \( A \leq B \leq M \), If \( A \leq a.e M \) then \( B \leq a.e M \).

Proof: Let \( L \leq M \) such that \( B \cap L = 0 \) since \( A \leq B \) then we have \( A \cap L = 0 \) and since \( A \leq a.e M \) then we get \( \text{ann}(L) \leq e \) \( R \). Thus \( B \leq a.e M \). The converse in general is not true as the following example shows: Let \( M = \mathbb{Z}_{12} \) as \( \mathbb{Z}_{12} \)-module, Let \( K = \{ 0, 6 \} \leq a.e \mathbb{Z}_{12} \) since \( 2 \mathbb{Z} \leq e \mathbb{Z}_{12} \), But \( K \) is not ann-essential in \( \mathbb{Z}_{12} \), Since \( K \cap (4) = 0 \) but \( \text{ann}(4) = (3) \) which is not essential in \( \mathbb{Z}_{12} \).

**Corollary (1.4):** If \( N_1 \cap N_2 \leq a.e M \) then both of \( N_1 \) and \( N_2 \) are ann-essential submodules of \( M \).

Proof: Since \( N_1 \cap N_2 \leq N_1 \leq M \) and \( N_1 \cap N_2 \leq N_2 \leq M \), Then by proposition (1.3) we get \( N_1 \leq a.e M \) and \( N_2 \leq a.e M \). The converse in general is not true for example: Let \( M = \mathbb{Z}_6 \) as \( \mathbb{Z}_6 \)-module and let \( K = \{ 0, 2, 4 \} \leq a.e M \) and \( L = \{ 0, 3 \} \leq a.e M \) but \( K \cap L = 0 \leq a.e M \).

**Proposition (1.6):** Let \( M \) and \( N \) be \( R \)-modules and let \( f: M \to N \) be an isomorphism. If \( B \) is ann-essential submodule of \( M \) then \( f(B) \) is ann-essential submodule of \( N \).

Proof: Let \( L \leq N \) such that \( f(B) \cap L = 0 \), since \( f \) is monomorphism we have \( B \cap f^{-1}(L) = 0 \), since \( B \leq a.e M \) then \( \text{ann}(f^{-1}(L)) \leq e \) \( R \). Since \( f \) is epimorphism then we have \( \text{ann}(f^{-1}(L)) \leq \text{ann}(L) \leq e \) \( R \). Therefore, \( f(N) \) is ann-essential submodule of \( M \).

**Proposition (1.7):** If \( A \leq B \leq M \) and \( A \leq a.e B \leq M \) where \( M \) is an \( R \)-module then \( A \cap A_1 \leq a.e B \cap B_1 \).

Proof: Let \( L \leq (B \cap B_1) \) such that \( (A \cap A_1) \cap L = 0 \). Then \( A \cap (A_1 \cap L) = 0 \) since \( (A_1 \cap L) \leq B \) and \( A \leq B \) then \( A_1 \cap L = 0 \) since \( L \leq B_1 \) and \( A \leq a.e B_1 \) then we get \( \text{ann}(L) \leq e \) \( R \).
Recall that: M is called multiplication module if for every submodule N of an R-module M there exist an ideal I of R such that N=IM see[12] Equivalently , M is multiplication module if and only if for every sub module N of M , N=(N:RM)M .

Proposition(1.8): Let M be a faithful multiplication R- module and let N be a submodule of M such that N=EM for some ideal E of R , then if E is ann- essential ideal of R then N is ann- essential submodule of M .

Proof: Let L be a submodule of M such that N∩L=0 , since M is multiplication R- module, there exist an ideal I of R such that L=IM , N∩L=EM∩IM = (E∩I)M =0 .Since M is faithful module then (E∩I)=0 Since E is ann- essential ideal then we get Ann(I)≤eR ,then ann(IM)≤eR [12] , then ann(L)≤eR .Thus, N is ann- essential sub module of M.

2. Annihilator-uniform modules .

In this section we give an annihilator -uniform module concept as a generalization of uniform module . Also we generalize some properties of uniform modules to annihilator-uniform modules.

Definition(2.1): Anon zero R-module is called annihilator-uniform (shortly by ann-uniform) if every non zero submodule of M is ann-essential submodule.

A ring R is called annihilator-uniform if R is an annihilator-uniform R-module .

Remarks and Examples(2.2):

- Every uniform R-module is ann-uniform . but the converse in general is not true as the following example shows : Let M= Z6 as Z-module , M is ann-uniform since every submodule is ann-essential ,but M is not uniform since (2')∩(3')=0 .
- Let M = Z12 as Z-module , M is ann-uniform module but not uniform .
- Z6 as Z6-module is not ann-uniform module .Since (2') and (3') are submodules of Z6 and both them is not ann-essential submodule .
- Every simple R-module is ann-uniform module .
- Every submodule of ann-uniform module is ann-uniform module .
- The annihilator-uniformity preserve under isomorphism .

Proof: Let M1 be an ann-uniform R-module and let f:M1 →M2 be an isomorphism , let A be a non zero submodule of M2 then f-1(A) is a submodule of M1 , since M1 is ann-uniform then f-1(A)≤a.e M1 then by prop. (1.6 ) we get f(f-1(A))=A ≤a.e M2 , thus, M2 is ann-uniform module .
- The direct sum of ann-uniform modules need not be ann-uniform module as the following example shows : Let M= Z⨁2Z as Z-module , both of Z and 2Z are Ann-uniform modules but M is not ann-uniform module since 2Z⨁(0)≤M and 2Z⨁(0)∩ (0)⨁2Z=0 , ann((0)⨁2Z)=0 ≰e R thus, 2Z⨁(0)≰a.e M .

Proposition(2.3): If M and M/ are ann-uniform modules then M∩M/ is ann-uniform module .

Proof: since M∩M/ ≤M and M∩M/≤M/ , and both of M and M/ are ann-uniform modules then by Rmark and Examples(2.2,5) we get M∩M/ is ann-uniform module .

Proposition(2.4): Let M be a faithful multiplication R- module . If R is ann-uniform ring then M is an ann-uniform R-module .

Proof: Let N be a non zero submodule of M , since M is multiplication module then there exist an ideal E of R such that N=EM , since R is ann-uniform ring then E is ann-essential ideal of R , since M
is faithful multiplication module then by prop. (1.8) we get $N$ is ann-essential submodule. Thus, $M$ is ann-uniform module.

3. Annihilator- maximal submodules

In this section we introduce the concept of annihilator – maximal submodules which is generalization for than maximal submodules. Moreover we study some basic properties for this concept.

Definition(3.1): A proper submodule $N$ of an $R$-module $M$ is called Annihilator-maximal (shortly ann-maximal) if whenever ann-essential submodule $W$ of $M$ with $N < W \leq M$ then $W = M$.

Equivalently: There is no proper ann-essential submodule of $M$ containing $N$ properly.

Remarks and Examples(3.2):

(1). Every maximal submodule is ann-maximal submodule but the converse in general is not true for example: consider the $Z$-module $M = 2Z \oplus 2Z$, and let $N = 4Z \oplus (0) < 2Z \oplus (0) \leq 2Z \oplus 2Z$. $N$ is not maximal submodule in $M$ but $N$ is ann-maximal submodule since there is no proper ann-essential submodule containing $N$ properly. Since $(2Z \oplus (0)) \cap ((0) \oplus nZ) = 0$ but $\text{ann}((0) \oplus nZ) \neq \mathbb{R}$.

(2). $Z$ as $Z$-module is not ann-maximal submodule of $Q$, since $Z < \frac{1}{2}Z \leq Q$ and $\frac{1}{2}Z$ is ann-essential submodule of $Q$. Since $\frac{1}{2}Z \cap L = 0$ then $L = 0$ for every $L \leq Q$ so $\text{ann}(0) = \mathbb{R} \leq \mathbb{R}$.

(3). Not every module has an ann-maximal submodule as the following example shows: Let $M = Z_{24}$ as $Z$-module, the submodules $(2 \overline{\cdot})$ and $(3 \overline{\cdot})$ are ann-maximal submodules of $Z_{24}$ since there is no proper ann-essential submodule containing $(2 \overline{\cdot})$ and $(3 \overline{\cdot})$ properly, but $(2 \overline{\cdot}) \cap (3 \overline{\cdot}) = 0$ which is not ann-maximal submodule of $M$.

Proposition(3.3): If $N$ and $W$ are proper submodules of an $R$-module $M$ such that $N \leq W \leq M$, if $N$ is an ann-maximal submodule then $W$ is ann-maximal submodule of $M$.

Proof: suppose that $W$ is not ann-maximal submodule of $M$ then there is an ann-essential submodule $U$ of $M$ such that $W \leq U \leq M$ then $N \leq U \leq M$ $N$, then $N$ is not ann-maximal submodule which is a contradiction.

Corollary(3.4): If $U$ and $V$ are proper submodules of an $R$-module $M$ such that $U \cap V$ is ann-maximal submodule then both of $U$ and $V$ are ann-maximal submodules of $M$.

Proof: Since $U \cap V \leq U \leq M$ and $U \cap V \leq V \leq M$ then by previous remark we get both of $U$ and $V$ are ann-maximal submodules.

The converse of corollary in general is not true as the following example shows: let $M = Z_{6}$ as $Z$-module, the submodules $(2 \overline{\cdot})$ and $(3 \overline{\cdot})$ are ann-maximal submodules of $Z_{6}$ since there is no proper ann-essential submodule containing $(2 \overline{\cdot})$ and $(3 \overline{\cdot})$ properly, but $(2 \overline{\cdot}) \cap (3 \overline{\cdot}) = 0$ which is not ann-maximal submodule of $M$.

Corollary(3.5): If $N$ and $K$ are proper submodules of an $R$-module $M$ such that $N + K$ is ann-maximal submodule of $M$ then $N + K$ is ann-maximal submodule of $M$.

Proof: It follows directly from proposition (3.3).

Remark(3.6): Let $M$ be an $R$-module and let $N$ and $K$ be a proper submodules of $M$ such that $N \leq K$.

If $N$ is ann-maximal submodule of $K$ and $K$ is ann-maximal submodule of $M$ then $N$ is not necessary ann-maximal submodule of $M$, for example:

Let $M = Z_{24}$ as $Z$-module, let $N = (4 \overline{\cdot})$ and $K = (2 \overline{\cdot})$, $N \leq K$ note that:

$N$ is ann-maximal submodule of $K$ and $K$ is ann-maximal submodule of $M$ but $N$ is not ann-maximal submodule of $M$. 

Remark (3.7): Let A be an ann- maximal submodule of an R- module M . If B is a submodule of M such that B \cong A then its not necessary that B is an ann- maximal submodule of M as the following example shows:
Example: consider the Z_ module Z . The sub module 2Z is an ann- maximal submodule of Z and 2Z\cong Z , but Z is not ann- maximal sub module of Z , since any ann- maximal submodule must be proper in any R- module.

Proposition(3.8): Let M1 and M2 be an R- modules and let N\leq M2 ,and f:M1 \rightarrow M2 be an isomorphism , if N is ann- maximal submodule of M2 then f-1(N) is ann- maximal submodule of M1 .
Proof: To prove f-1(N) \neq M1 , assume that f-1(N) = M1 ,Since N\neq M2 then there exist m\in M2 such that m\notin N . Then f-1(m) \in f-1(M2) =M1 = f-1(N) since f is epimorphism . There exist n \in N such that f-1(m)= f-1(n) , then f-1(m-n)=0 ,Then f(f-1(m-n))=f(0) then m-n =0 ,since f is isomorphism , then we get m=n so m\in N which is contradiction , thus , f-1(N) \neq M1 .
Now , to prove f-1(N) is ann- maximal submodule :
Assume that f-1(N) is not ann- maximal , then there exist an ann- essential submodule W of M1 such that f-1(N) \leq W \leq M1 
Then N \leq f(W) \leq f(M1) , f(W) is ann- essential submodule by proposition(1.6), since f is epimorphism Then N \leq f(W) \leq M2 , Then we get N is not ann- maximal submodule which is a contradiction .Thus, f-1(N) is an ann- maximal submodule of M1

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