STABILITY FOR A MAGNETIC SCHRÖDINGER OPERATOR ON A RIEMANN SURFACE WITH BOUNDARY

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Abstract. We consider a magnetic Schrödinger operator \((\nabla^X)^*\nabla^X + q\) on a compact Riemann surface with boundary and prove a log log-type stability estimate in terms of Cauchy data for the electric potential and magnetic field under the assumption that they satisfy appropriate a priori bounds. We also give a similar stability result for the holonomy of the connection 1-form \(X\).

1. Introduction. Let \((M, g)\) be a compact Riemann surface with boundary \(\partial M\). We will consider a connection \(\nabla^X\) on a complex line bundle over \(M\), with connection 1-form \(X\). By the associated connection Laplacian we mean the differential operator

\[
\Delta^X := (\nabla^X)^* \nabla^X = (d + i \ast X)^* (d + i \ast X) = - \ast (d \ast + i X \wedge \ast)(d + i X),
\]

where \(d\) denotes the exterior derivative, \(i = \sqrt{-1}\) and \(\ast\) is the Hodge star with respect to the metric \(g\). In particular, when \(X\) is real valued, \(\Delta^X\) is often called the magnetic Laplacian associated with the magnetic field \(dX\). We will restrict our attention to the case of real-valued \(X\) in this work.

By adding a complex-valued potential \(q\) we get the magnetic Schrödinger operator associated with the couple \((X, q)\)

\[
L = L_{X,q} := \Delta^X + q.
\]

We denote by \(H^s(M)\) the Sobolev space containing functions on \(M\) with \(s\) derivatives in \(L^2(M)\). The Cauchy data space \(C_L\) of \(L\) is defined by

\[
C_L := \{(u, \nabla^X u)|_{\partial M} \in H^{1/2}(\partial M) \times H^{-1/2}(\partial M); u \in H^1(M), Lu = 0\},
\]

where \(\nu\) denotes the outward pointing unit normal vector field to \(\partial M\) and \(\nabla^X u := (\nabla^X u)(\nu)\) is the normal derivative associated with \(X\).

In this work we assume that we are working with two different magnetic Schrödinger operators \(L_j := L_{X_j, q_j}\) and their corresponding Cauchy data spaces \(C_j := C_{L_j}, j = 1, 2\). Assuming certain a priori bounds for the norms of \(X_j\) and \(q_j\), we illustrate that if the Cauchy data spaces are sufficiently similar, then so are the \(q_j\)s and \(X_j\)s respectively. The main results of this paper are:

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Theorem 1.1. Suppose that \( q_1, q_2 \) are complex-valued functions and \( X_1, X_2 \) are real-valued 1-forms such that for some \( p > 2 \),
\[
\|q_j\|_{W^{1,p}(M)} \leq K, \quad \|X_j\|_{W^{2,p}(T^*M)} \leq K, \quad j = 1, 2.
\]
Denote by \( C_j := C_{L_j} \) the Cauchy data spaces as defined in 3 for the corresponding magnetic Schrödinger operators \( L_j := L_{X_j, q_j} \), as defined in 1-2. Then if the distance \( d(C_1, C_2) \) is small enough, there is an \( \alpha > 0 \) such that
\[
\|q_1 - q_2\|_{L^2(M)} + \|d(X_1 - X_2)\|_{L^2(\Lambda^2(M))} \leq \frac{C}{\log^a \log \frac{1}{d(C_1, C_2)}},
\]
where \( C = C(K, M, \alpha) \).

By interpolation it is then also quite immediate to deduce:

Corollary 1. If in addition to the assumptions in Theorem 1.1 we have for some \( k \geq 3 \)
\[
\|q_j\|_{H^k(M)} \leq K, \quad \|X_j\|_{H^k(T^*M)} \leq K, \quad j = 1, 2.
\]
Then there is an \( \alpha > 0 \) such that
\[
\|q_1 - q_2\|_{H^s(M)} + \|d(X_1 - X_2)\|_{H^s(\Lambda^2(M))} \leq \frac{C}{\log^a \log \frac{1}{d(C_1, C_2)}},
\]
where \( C = C(K, M, \beta), 0 \leq s < k \).

These results further quantify the uniqueness results by Guillarmou and Tzou from 2011, [7]. They showed that the Cauchy data of the magnetic Schrödinger operator \( L_{X,q} \) uniquely determines the potential \( q \) and uniquely determines the connection \( X \) up to so-called gauge isomorphism. This result was extended, by the same authors together with Albin and Uhlmann [1], to identification of coefficients (up to gauge) in elliptic systems in 2013. We have borrowed plenty of notations and conventions from these works.

The main idea in [7] is to rewrite \( L_{\bar{\partial}u} = 0 \) to \( \bar{\partial} \)-systems with matrix-valued potentials, and then apply the idea by Bukhgeim [2]. In the identifiability case, one is able to do this due to a certain orthogonality condition on the boundary which allows one to judiciously choose conjugation factors so that they agree on the boundary. In this work the orthogonality condition will only be an approximate one and we quantify the boundary conditions of the conjugation factors by this approximate orthogonality condition (see Section 5). This process unfortunately causes the extra logarithm in the end result.

Another additional feature which we must consider is the uniformity of the estimates in the stationary phase expansion. As such we must construct slightly different phase functions than in [7] to be used in these solutions and refine several estimates from mentioned works. This work is done in Section 2.

To see that the Cauchy data space cannot determine the couple \((X, q)\) completely, consider introducing a real-valued function \( f \) (say in \( H^2(M) \)) whose restriction to the boundary \( \partial M \) is zero. Then the Cauchy data spaces associated with the magnetic Schrödinger operators \( L_{X,q} \) and \( L_{X+d\bar{\partial}f,q} \) can be seen to coincide.

In [7] it is shown that the Cauchy data space determines the relative cohomology class of \( X \). This is done through a number of steps. First by showing that \( C_1 = C_2 \) implies that \( d(X_1 - X_2) = 0 \) and \( q_1 = q_2 \). If \( M \) is simply connected this would imply that \( X_1 \) and \( X_2 \) differ by an exact 1-form. Furthermore, by boundary determination, \( \iota_{\partial M}^* (X_1 - X_2) = 0 \), where \( \iota_{\partial M}^* \) denotes pullback by inclusion to the boundary.
case of a more general Riemann surface, not necessarily simply connected, there are further obstructions so that \( X_1 \) and \( X_2 \) may no longer only differ by an exact 1-form. Namely, if \( E \) is a complex line bundle over \( M \) and there is a so-called unitary bundle isomorphism \( F : E \to E \) that preserves the Hermitian inner product and satisfies \( \nabla X_1 = F^* \nabla X_2 F \) with \( F = \text{Id} \) on \( E|_{\partial M} \) it can again be seen that \( C_1 = C_2 \). Having such a unitary bundle isomorphism corresponds to multiplication by a function \( F \) on \( M \) with the properties \(|F| = 1\) and \( F|_{\partial M} = 1 \). This is again equivalent with \( \iota_{\partial M} (X_1 - X_2) = 0 \) on \( \partial M \), \( d(X_1 - X_2) = 0 \) and that all periods of \( X_1 - X_2 \) is an integer multiple of \( 2\pi \). I.e. for every closed loop \( \gamma \) in \( M \) it holds that

\[
\int_\gamma (X_1 - X_2) \in 2\pi \mathbb{Z}.
\]

In this case one can deduce that

\[
X_1 - X_2 = df + 2\pi \sum_{k=1}^{N} n_k \omega_k, \int_\gamma \omega_k = \delta_{jk}
\]

where \( n_k \in \mathbb{Z}, \{\omega_k\}_{k=1}^{N} \) which is a basis for the first relative cohomology \( H^1(M, \partial M) \), dual to \( \{\gamma_j\}_{j=1}^{N} \) which are some non-homotopically equivalent loops on \( M \), \( f \) is a function whose restriction to the boundary \( \partial M \) is zero and \( \delta_{jk} \) is the usual Kronecker delta.

In our framework we quantify the above statement by showing:

**Theorem 1.2.** There is an \( \alpha > 0 \) such that if \( d(C_1, C_2) \) is small enough, then for every closed loop \( \gamma \) in \( M \) it holds that

\[
(5) \inf_{k \in \mathbb{Z}} \left| \int_\gamma (X_1 - X_2) - 2\pi k \right| \leq C|\gamma| \frac{1}{\log \alpha \log \frac{1}{d(C_1, C_2)}},
\]

where \( C = C(K, M, p, \alpha) \).

The above result is a consequence of Theorem 1.1 and the Picard-Lindelöf Theorem. Some precaution is needed when defining relevant functions on \( M \) since in our case \( X_1 - X_2 \) is not necessarily closed, compare with [7]. This introduces extra complications and a suitable expression for \( X_1 - X_2 \) is derived to quantify its distance from an exact gauge (see Section 6).

We also refer to [8] for more background on the Calderón inverse problem for Schrödinger operators in dimension 2. In particular, [5] proves uniqueness for the usual Schrödinger operator \( \Delta_g + q \) on a Riemann surface and [6] handles the corresponding partial data case. For similar results in Euclidean domains, see [15, 12, 11]. Related stability estimates can also be found in [16, 17]. A constructive method for the reconstructing an isotropic conductivity on a Riemann surface was first obtained in [9].
where \( \Lambda^{1,0}(M) = T^{*}_{1,0}M := \operatorname{ker}(\star + i) \) and \( \Lambda^{0,1}(M) = T^{*}_{0,1}M := \operatorname{ker}(\star - i) \) are eigenspaces to \( \star \) corresponding to it’s eigenvalues \( \pm i \). We say that a 1-form belonging to \( \Lambda^{p,q}(M) \) is of type \((p,q) \in \{(1,0),(0,1)\}\). In holomorphic coordinates, \( z = x + iy \) in a chart \((U,\phi)\), the Hodge star \( \star : \Lambda^{k}(M) \to \Lambda^{2-k}(M) \), acts according to:

\[
\star(u \, dz + v \, d\bar{z}) = -i \, u \, dz + i \, v \, d\bar{z},
\]

where \( u, v \) are functions. So locally \( \Lambda^{1,0}(M) \) is spanned by \( dz \) while \( \Lambda^{0,1}(M) \) is spanned by \( d\bar{z} \). We have the natural projections

\[
\pi_{1,0} : \Lambda^{1}(M) \to \Lambda^{1,0}(M), \quad \omega \mapsto \omega_{1,0} = \pi_{1,0}\omega,
\]

\[
\pi_{0,1} : \Lambda^{1}(M) \to \Lambda^{0,1}(M), \quad \omega \mapsto \omega_{0,1} = \pi_{0,1}\omega,
\]

and by definition \( \star\omega_{1,0} = -i\omega_{1,0} \) and \( \star\omega_{0,1} = i\omega_{0,1} \).

A Hermitian inner product can be defined on the vector space \( \Lambda^{k}(M) \) according to

\[
(\lambda,\eta)_{L^{2}(\Lambda^{k}(M))} := \int_{M} \lambda \wedge \bar{\eta}, \quad \lambda,\eta \in \Lambda^{k}(M).
\]

We will often assume that our functions and forms belong to certain Sobolev spaces. Recall that for non-negative integers \( k \) and \( 1 \leq p \leq \infty \), a function \( f \) is said to belong to the Sobolev space \( W^{k,p}(M) \) if it is \( k \) times weakly differentiable and all partial derivatives up to order \( k \) belong to \( L^{p}(M) \). For \( k = 2 \) we use the common notation \( H^{k}(M) := W^{k,2}(M) \), or \( H^{s}(M) \) when considering non-integers \( s = k \), and for \( k = 0 \) we have \( L^{p}(M) = W^{0,p}(M) \). For a more thorough discussion on Sobolev spaces we refer to e.g. Section 1.3 of [18], that also covers the spaces \( W^{k,p}(\Lambda^{l}(M)) \) that we also consider in the cases \( l = 1,2 \).

### 2.1. Cauchy-Riemann operators on \( M \)

Next we will introduce the Cauchy Riemann operators \( \partial, \bar{\partial} \) as mappings of \( k \)-forms to \((k+1)\)-forms, \( k = 0,1 \).

On functions their action is defined by

\[
\partial f := \pi_{1,0} df, \quad \bar{\partial} f := \pi_{0,1} df.
\]

In holomorphic coordinates this is simply

\[
\partial f := \partial_{z} f \, dz, \quad \bar{\partial} f := \partial_{\bar{z}} f \, d\bar{z},
\]

where

\[
\partial_{z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

are the usual Wirtinger derivatives. The space of holomorphic functions over \( M \) is denoted by \( \mathcal{H}(M) \) and consists of functions \( f \) that satisfy \( \bar{\partial} f = 0 \).

On 1-forms we define

\[
\partial \omega := d\pi_{1,0}\omega = d\omega_{1,0}, \quad \bar{\partial} \omega := d\pi_{0,1}\omega = d\omega_{0,1}.
\]

In coordinates this is

\[
\partial(u \, dz + v \, d\bar{z}) = \partial u \wedge dz, \quad \bar{\partial}(u \, dz + v \, d\bar{z}) = \bar{\partial} u \wedge dz.
\]

It is clear that \( d = \partial + \bar{\partial} \) holds for both functions and 1-forms. The adjoints of \( \partial \) and \( \bar{\partial} \) are simply given by \( \partial^{*} = i \star \bar{\partial} \) and \( \bar{\partial}^{*} = -i \star \partial \) respectively. We can now define the Laplacian of a function \( f \) on \( M \) by

\[
\Delta f := -2i \star \bar{\partial} \partial f = \delta df
\]

where \( \delta = d^{*} \) is the codifferential, i.e. the adjoint of \( d \) with respect to the inner product, and the \( \star \) is the induced Hodge star that maps 2-forms to 0-forms.
2.2. Existence of suitable phase functions on $M$. We will now construct functions that are holomorphic and Morse, with uniformly bounded from below Hessian outside a neighborhood of their stationary points. These functions will be used as phase functions in latter arguments that allow us to estimate the degeneracy near stationary points. This must be taken into account when deriving correct remainder estimates in e.g. stationary phase expansions.

Start out with any $\Phi \in \mathcal{H}(M)$ which is Morse, meaning that if $\hat{p} \in M$ is a stationary point, $\partial \Phi(\hat{p}) = 0$, then the Hessian $\partial^2 \Phi(\hat{p}) \neq 0$. For the construction of such functions see [5]. Suppose that $\{p_1, \ldots, p_n\}$ is the set of stationary points of $\Phi$ and for $k = 1, \ldots, n$ denote by $p_{j,k}, j = 1, \ldots, m_k$ the set of points for which the values $\Phi(p_{j,k})$ coincide with the critical values $\Phi(p_k)$. Then we have the following lemma:

Lemma 2.1. Let $\Phi$ be a holomorphic Morse function on a compact Riemann surface $M$ with boundary, having critical points $\{p_k\}_{k=1}^n \subset M$, and let

$$P_{cv} = \{p_{j,k} \in M; \Phi(p_{j,k}) = \Phi(p_k), j = 1, \ldots, m_k, k = 1, \ldots, n\}$$

be the set of points where the values of $\Phi$ coincides with a critical value. For any fixed $\delta > 0$ sufficiently small we let $N_\delta$ be a neighborhood of $P_{cv}$ defined by:

$$N_\delta = \bigcup_{j,k} B_{j,k,\delta}, \quad B_{j,k,\delta} = \phi^{-1}(B(\phi(p_{j,k}), \delta)), \quad \text{where} \ (U, \phi) \text{ are charts such that } \phi : U \to C$$

and

$$B(\phi(p_{j,k}), \delta) = \{z \in C; |z - z_{j,k}| < \delta, z_{j,k} = \phi(p_{j,k})\} \subset \phi(U).$$

Then for any $\hat{p} \in M \setminus N_\delta$ there exists a holomorphic Morse function $\Phi_{\hat{p}}$ with a critical point at $\hat{p}$ such that at all critical points $p$ of $\Phi_{\hat{p}}$ there is a $c > 0$ independent of the choice of $\hat{p} \in M \setminus N_\delta$ such that

$$|\partial^2 \Phi_{\hat{p}}(p)| \geq c\delta^4.$$

Proof. The sought after function will be defined by

$$\Phi_{\hat{p}}(p) = (\Phi(p) - \Phi(\hat{p}))^2, \quad \hat{p} \in M \setminus N_\delta.$$

Then

$$\partial \Phi_{\hat{p}}(p) = 2(\Phi(p) - \Phi(\hat{p}))\partial \Phi(p),$$

so clearly $\partial \Phi_{\hat{p}}(\hat{p}) = 0$, and we will show that $\Phi_{\hat{p}}$ is Morse. Observe that, in a coordinate system $(U, \phi)$ containing $p = \phi(z)$

$$\partial^2 \Phi_{\hat{p}}(\phi^{-1}(z)) = 2(\partial_\phi \phi^{-1}(z))^2 + 2(\Phi(\phi^{-1}(z)) - \Phi(\hat{p}))\partial^2 \Phi(\phi^{-1}(z)).$$

Suppose that $p \in M$ is a point such that

$$0 = \partial \Phi_{\hat{p}}(p) = 2(\Phi(p) - \Phi(\hat{p}))\partial \Phi(p),$$

Then, since $\hat{p}$ is bounded away from critical values of $\Phi$, we have the following two mutually exclusive cases

1. $\Phi(p) = 0$ in which case $p = p_k$ for some $k \in \{1, \ldots, n\}$ or,
2. $\Phi(p) = \Phi(\hat{p}).$

In case 1, we have with $z_k = \phi(p_k),

$$\partial^2 \Phi_{\hat{p}}(\phi^{-1}(z_k)) = 2(\Phi(\phi^{-1}(z_k)) - \Phi(\phi^{-1}(z)))\partial^2 \Phi(\phi^{-1}(z_k)).$$
and since $\Phi$ is Morse there is a holomorphic chart in which $\partial^2_ϕ(ϕ^{-1}(z_k)) = a_k \neq 0$. Also since $\hat{p} \notin N_δ$, $Φ(p_k) - Φ(\hat{p}) \neq 0$ and furthermore (for small enough $δ > 0$ and $\hat{p}$ close enough to $p_k$) we have in local coordinates,

$$|Φ(ϕ^{-1}(z_k)) - Φ(ϕ^{-1}(\hat{z}))| = |a_k||z_k - \hat{z}|^2 + O(|z_k - \hat{z}|^3).$$

(Observe that $|Φ(p_k) - Φ(\hat{p})| → 0$ only if $\hat{p} → p_{j,k}$ for some $p_{j,k} ∈ P_{ce}$, and thus $|Φ(p_k) - Φ(\hat{p})| = |Φ(p_k) - Φ(p_{j,k}) + Φ(p_{j,k}) - Φ(\hat{p})| = |Φ(p_{j,k}) - Φ(\hat{p})|.$) Since we only have finitely many critical points we can thus choose an $a > 0$, such that in a coordinate where $p = ϕ(z)$

$$|\partial^2_ϕ(ϕ^{-1}(z))| > a δ^2,$$

for all $\hat{p} ∈ M \setminus N_δ$, for $p ∈ M : ∂Φ(p) = 0$.

In case 2, when $Φ(p) = Φ(\hat{p})$ it follows that $p ≠ p_k$ for any $k$, since $\hat{p} ∉ N_δ$. Furthermore,

$$\partial^2_ϕ(ϕ^{-1}(z)) = 2(∂_ϕ(ϕ^{-1}(z)))^2.$$ Clearly the right hand side approaches 0 only when $p → p_k$ for some $k ∈ \{1, \ldots, n\}$ and we have by a similar argument as in case 1 that

$$|z - z_k| ≥ \frac{1}{c} |Φ(ϕ^{-1}(z)) - Φ(ϕ^{-1}(z_k))| = \frac{1}{c} |Φ(ϕ^{-1}(\hat{z})) - Φ(ϕ^{-1}(z_k))|$$

$$|a_k||\hat{z} - z_k|^2 + O(|\hat{z} - z_k|^3),$$

where $c > 0$ is the Lipschitz constant for $Φ ∘ ϕ^{-1}$. So the lower bound $|\hat{z} - z_k| ≥ δ$ will yield a lower bound on $|z - z_k| ≥ δ^2$, hence near $p_k$ we have (in local coordinates)

$$|∂_ϕ(ϕ^{-1}(z))| = 2a_k|z - z_k| + O(|z - z_k|^2) ≥ \sqrt{b}|\hat{z} - z_k|^2 = \sqrt{b}δ^2.$$ Thus for $p$ near $p_k$ we can again choose $b > 0$ such that

$$|\partial^2_ϕ(ϕ^{-1}(z_k))| > b δ^4,$$ for all $\hat{p} ∈ M \setminus N_δ$, for $p ∈ M : Φ(p) = Φ(\hat{p})$, and this holds uniformly in $\hat{p} ∈ M \setminus N_δ$.

**Remark 1.** When the Riemann surface $M$ is of a particularly simple type, e.g. if it is equipped with a global holomorphic coordinate $z$ (such as when $M$ is just a domain in $C$) we do not need the construction in the above lemma. Since in the latter case it holds that for every $\hat{z} = ϕ(\hat{p})$, $Φ(z) = (z - \hat{z})^2$ is a Morse holomorphic function with a critical point at $\hat{z}$.

We are going to use the method of stationary phase with the above constructed phase function in order to later derive estimates. For convenience we replace $δ$ by $\sqrt{δ}$ in the above lemma, i.e. we consider the phase function $Φ = Φ_ϕ$ on $M \setminus N_\sqrt{δ}$, where $|∂^2_ϕΦ_ϕ| ≥ c δ^2$. Recall that $C_0^∞$ as usual denotes smooth and compactly supported functions/form. The dependence on the parameters $h$ and $δ$ will be important.

**Lemma 2.2** (Stationary phase). Suppose $ψ$ is a smooth function and $K ⊂ C$ is a set containing only one critical point $\hat{z}$ of $ψ$ and that $|∂^2_ϕψ(\hat{z})| ≥ δ^2 > 0$. Then for every $u ∈ C_0^∞(K)$,

1. $$\left| \int u(z)e^{2ıψ(z)/h} \, dz \, d\hat{z} \right| ≤ \frac{Ch^2}{δ^2} \|u\|_{W^2,∞(K)},$$

2. $$\left| \int u(z)e^{2ıψ(z)/h} \, dz \, d\hat{z} - \frac{ch}{δ} u(\hat{z}) \right| ≤ \frac{Ch^2}{δ^2} \|u\|_{W^4,∞(K)}.$$ The constants $C > 0, c > 0$ are uniformly bounded with respect to $\hat{z}$. 

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Proof. This follows from [10], Theorem 7.7.5. \( \square \)

2.3. The inverses of \( \partial \) and \( \bar{\partial} \). Section 2 in [7] contain lemmas regarding the construction and boundedness of right inverses of the Cauchy-Riemann operators. These results ensure that the constructed solutions to the Dirac-systems that are considered in the paper are well-behaved. As we will use a very similar approach in Section 3, we recite some of these essential lemmas below.

**Lemma 2.3** (Right inverse to \( \bar{\partial} \), [7]). There exists an operator

\[
\bar{\partial}^{-1} : C_0^\infty(\Lambda^{1,0}(M)) \to C^\infty(M)
\]

such that

(i) \( \bar{\partial}\bar{\partial}^{-1} \omega = \omega \) for all \( \omega \in C_0^\infty(\Lambda^{1,0}(M)) \).

(ii) If \( \chi_j \in C_c^\infty(M) \) are supported in complex charts \( U_j \), bi-holomorphic to a bounded open set \( \Omega \subset \mathbb{C} \) with complex coordinate \( z \), and such that \( \chi = \sum \chi_j \) is equal to 1 on \( M \), then as operators

\[
\bar{\partial}^{-1} \chi = \sum \hat{\chi}_j \hat{T} \chi_j + K,
\]

where \( \hat{\chi}_j \in C_c^\infty(U_j) \) are such that \( \hat{\chi}_j \chi_j = \chi_j \), \( K \) has smooth kernel on \( M \times M \) and \( \hat{T} \) is given in the complex coordinate \( z \in U_i \) by

\[
\hat{T}(f \, dz) = \frac{1}{\pi} \int_\mathbb{C} \frac{f(\zeta)}{z - \zeta} \, d\zeta \wedge d\bar{\zeta}.
\]

(iii) \( \bar{\partial}^{-1} \) is bounded from \( L^p(\Lambda^{1,0}(M)) \) to \( W^{1,p}(M) \) for all \( p > 1 \).

So by (ii), the inverse can be expressed by the usual Cauchy operator, which is called \( \hat{T} \) here, plus a smoothing term, in local coordinates. A similar result for \( \bar{\partial}^* \) is given by

**Lemma 2.4** (Right inverse to \( \bar{\partial}^* \), [7]). Let \( \bar{\partial}^* : W^{1,p}(\Lambda^{0,1}(M)) \to L^p(M) \), then there exists an operator

\[
\bar{\partial}^{*-1} : C_0^\infty(M) \to C^\infty(\Lambda^{0,1}(M))
\]

such that

(i) \( \bar{\partial}^* \bar{\partial}^{*-1} \varphi = \varphi \) for all \( \varphi \in C_0^\infty(M) \).

(ii) If \( \chi_j \in C_c^\infty(M) \) are supported in complex charts \( U_j \), bi-holomorphic to a bounded open set \( \Omega \subset \mathbb{C} \) with complex coordinate \( z \), and such that \( \chi = \sum \chi_j \) is equal to 1 on \( M \), then as operators

\[
\bar{\partial}^{*-1} \chi = \sum \hat{\chi}_j \hat{T} \chi_j + K,
\]

where \( \hat{\chi}_j \in C_c^\infty(U_j) \) are such that \( \hat{\chi}_j \chi_j = \chi_j \), \( K \) has smooth kernel on \( M \times M \) and \( \hat{T} \) is given in the complex coordinate \( z \in U_i \) by

\[
T f(z) = \left( \frac{1}{\pi} \int_\mathbb{C} \frac{f(\zeta)}{z - \zeta} \, d\zeta \wedge d\bar{\zeta} \right) \, dz.
\]

(iii) \( \bar{\partial}^{*-1} \) is bounded from \( L^p(M) \) to \( W^{1,p}(\Lambda^{0,1}(M)) \) for all \( p > 1 \).

Here we again observe that (ii) says that the inverse can be expressed as a Cauchy-type operator plus a smoothing term, in local coordinates. For the proof we again refer to [7]. The main use of Lemma 2.3 and 2.4 in [7] is to prove Lemma 2.5 and 2.7. We could also make use of Lemma 2.3 and 2.4 in order to prove estimates for the solutions of the \( \bar{\partial} \)-systems we will consider in later sections. In
order to prove Theorem 1.1 we will require more refined and explicit estimates than what is needed in order to prove identifiability of the pair \((X, q)\), so this is another reason for restating also the above lemmas.

Let us now assume that \(M\) is strictly contained in some larger surface \(N\). Suppose \(p, q \in [1, \infty]\) and define the continuous extension operator from \(M\) to \(N\) by
\[
E : W^{k,p}(\Lambda^{0,1}(M)) \to W^{k,p}_c(\Lambda^{0,1}(N)),
\]
where \(W^{k,p}_c(\Lambda^{0,1}(N))\) denotes the subspace of compactly supported type \((0,1)\)-forms in \(W^{k,p}(\Lambda^{0,1}(N))\), \(k = 1, 2\), with a range made of type \((0,1)\)-forms with support in \(N_\delta = \{n \in N; d(n, M) \leq \delta\}\) for some \(\delta > 0\). We also denote by
\[
R : L^q(N) \to L^q(M)
\]
the restriction map from \(N\) to \(M\).

**Lemma 2.5** (Lemma 2.2 from [7]). Let \(\psi\) be a real-valued smooth Morse function on \(N\) and let \(\bar{\partial}^{-1}_\psi := R\bar{\partial}^{-1}_\psi e^{-2i\psi/h} E\). For \(q > 1, p > 2\) there exists \(C > 0\) that is independent of \(h\) such that for all \(\omega \in W^{1,p}(\Lambda^{0,1}(M))\),
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^q(M)} \leq C h^{2/3} \|\omega\|_{W^{1,p}(\Lambda^{0,1}(M))}, \quad 1 \leq q < 2,
\]
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^q(M)} \leq C h^{1/3} \|\omega\|_{W^{1,p}(\Lambda^{0,1}(M))}, \quad 2 \leq q \leq p.
\]

By interpolation, there is thus an \(\varepsilon > 0\) and \(C > 0\) such that for all \(\omega \in W^{1,p}(\Lambda^{0,1}(M))\)
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^2(M)} \leq C h^{1/2+\varepsilon} \|\omega\|_{W^{1,p}(\Lambda^{0,1}(M))}.
\]

As we will be required to use the special phase functions constructed in Lemma 2.1 we state below a more explicit version of the above lemma in order to see the \(\delta\)-dependence.

**Lemma 2.6** (Refinement of Lemma 2.2 from [7]). Let \(\Phi_\beta\) be a holomorphic Morse function as in Lemma 2.1 and let \(\psi = \text{Im} \Phi_\beta\). Define \(\bar{\partial}^{-1}_\psi := R\bar{\partial}^{-1}_\psi e^{-2i\psi/h} E\) where \(\bar{\partial}^{-1}_\psi\) is the right inverse of \(\bar{\partial} : W^{1,p}(M) \to L^p(\Lambda^{0,1}(M))\). Let \(p > 2\), then there are constants \(\varepsilon > 0, C > 0\) independent of \(h\), \(\delta\), and \(\bar{\partial}\) such that for all \(\omega \in W^{1,p}(\Lambda^{0,1}(M))\),
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^q(M_0)} \leq C \frac{h^{2/3}}{\delta^4} \|\omega\|_{W^{1,p}(M_0, T_{0,1}^* M_0)} \quad \text{if } 1 \leq q < 2
\]
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^q(M_0)} \leq C \frac{h^{1/3}}{\delta^4} \|\omega\|_{W^{1,p}(M_0, T_{0,1}^* M_0)} \quad \text{if } 2 \leq q \leq p.
\]
\[
\|\bar{\partial}^{-1}_\psi \omega\|_{L^2(M)} \leq \frac{C h^{1/2+\varepsilon}}{\delta^4} \|\omega\|_{W^{1,p}(\Lambda^{0,1}(M))}.
\]

**Proof.** The last inequality comes from interpolating the estimates 8 and 9 so we will prove 8 and 9. We recall the Sobolev embedding \(W^{1,p}(M) \subset C^\alpha(M)\) for \(\alpha \leq 1 - 2/p\) if \(p > 2\), and we shall denote by \(T\) the Cauchy-Riemann inverse of \(\partial\) in \(\mathbf{C}\):
\[
Tf(z) := \frac{1}{\pi} \int_{\mathbf{C}} \frac{f(\xi)}{z - \xi} \, d\xi_1 d\xi_2
\]
where \(\xi = \xi_1 + i\xi_2\). If \(\Omega, \Omega' \subset \mathbf{C}\) are bounded open sets, then the operator \(1_{\Omega'} T\) maps \(L^p(\Omega)\) to \(L^p(\Omega')\). Clearly, since \(E\) and \(R\) are continuous operators, it suffices to prove the estimates for compactly supported forms \(\omega \in W^{1,p}(T_{0,1}^* M)\) on \(M\). Thus by partition of unity, it suffices to assume that \(\omega\) is compactly supported in a chart biholomorphic to a bounded domain \(\Omega \subset \mathbf{C}\), and since the estimates will be
localized, we can assume with no loss of generality that $\psi$ has only one critical point, say $z_0 \in \Omega$ (in the chart). The expression of $\tilde{\partial}_\psi^{-1}(f dz)$ in complex local coordinates in the chart $\Omega$ satisfies

$$
\tilde{\partial}_\psi^{-1}(f(z)dz) = \chi(z)T(e^{-2i\psi/h}f) + K(e^{-2i\psi/h}f dz)
$$

where $K$ is an operator with smooth kernel and $\chi \in C^\infty_0(\mathbb{C})$.

Let us first prove 8. Let $\varphi \in C^\infty_0(\mathbb{C})$ be a function which is equal to 1 for $|z - z_0| > 2\epsilon_0$ and to 0 in $|z - z_0| \leq \epsilon_0$, where $\epsilon_0 > 0$ is a parameter that will be chosen later (it will depend on $h$). Using Minkowski inequality, one can write when $q < 2$

$$
\|\chi T((1 - \varphi)e^{-2i\psi/h}f)\|_{L^q(\mathbb{C})} \leq \int_\Omega \left|\frac{\chi(1)}{1 - \varphi}\right|_{L^q(\mathbb{C})}|(1 - \varphi(\xi))f(\xi)|d\xi_1d\xi_2
$$

(10)

$$
\leq C\|f\|_{L^\infty(\Omega)}\int_\Omega |(1 - \varphi(\xi))|d\xi_1d\xi_2 \leq C\epsilon_0^2\|f\|_{L^\infty(\Omega)}.
$$

On the support of $\varphi$, we observe that since $\varphi = 0$ near $z_0$, we can use

$$
T(e^{-2i\psi/h}\varphi f) = \frac{1}{2}ih\left[e^{-2i\psi/h}\frac{\partial f}{\partial \psi} - T(e^{-2i\psi/h}\bar{\psi}\frac{\partial f}{\partial \psi})\right]
$$

and the boundedness of $T$ on $L^q$ to deduce that for any $q < 2$

$$
\|\chi T(\varphi e^{-2i\psi/h}f)\|_{L^q(\mathbb{C})} \leq C\hbar\left(\left\|\frac{\varphi f}{\partial \psi}\right\|_{L^q} + \left\|\frac{\partial f}{\partial \psi}\right\|_{L^q} + \left\|\frac{\bar{\psi} \frac{\partial f}{\partial \psi}}{\partial \psi}\right\|_{L^q} + \left\|\frac{\varphi f}{\partial \psi}\right\|^2_{L^q}\right).
$$

The first term is clearly bounded by $\frac{\epsilon_0^{-1}}{2}\|f\|_{L^\infty}$ due to the lower bound on the Hessian of the phase function in Lemma 2.1. For the last term, observe that for the same reason, $\frac{1}{|\partial \psi|} \leq \frac{\epsilon_0}{\sqrt{|z - z_0|}}$ near $z_0$, therefore

$$
\left\|\frac{\varphi f}{\partial \psi}\right\|_{L^q} \leq \frac{C}{\delta^4}\|f\|_{L^\infty}\left(\int_0^1 r^{1-2q}dr\right)^{1/q} \leq \frac{C}{\delta^4}\epsilon_0^{-2}\|f\|_{L^\infty}.
$$

The second term can be bounded by $\|\frac{\partial f}{\partial \psi}\|_{L^q} \leq \|f\|_{L^\infty}\|\frac{\partial \psi}{\partial \psi}\|_{L^q}$. Observe that while $\|\frac{\partial \psi}{\partial \psi}\|_{L^\infty}$ grows like $\frac{\epsilon_0^2}{\delta^2}$, $\bar{\partial}\varphi$ is only supported in a neighbourhood of radius $2\epsilon_0$. Therefore we obtain

$$
\left\|\frac{\bar{\partial}\varphi}{\partial \psi}\right\|_{L^q} \leq \frac{\epsilon_0^{2/q-2}}{\delta^4}\|f\|_{L^\infty}.
$$

The third term can be estimated by

$$
\left\|\frac{\varphi \partial f}{\partial \psi}\right\|_{L^q} \leq \frac{C}{\delta^4}\|\partial f\|_{L^p}\left\|\frac{\varphi}{\partial \psi}\right\|_{L^\infty} \leq \frac{C}{\delta^4}\epsilon_0^{-1}\|\partial f\|_{L^p}.
$$

Combining these four estimates with 11 we obtain

$$
\|\chi T(\varphi e^{-2i\psi/h}f)\|_{L^q(\mathbb{C})} \leq \frac{h}{\delta^4}\|f\|_{W^{1,q}(\epsilon_0^{-1} + \epsilon_0^2/q-2)}.
$$

Combining this and 10 and optimizing by taking $\epsilon_0 = h^{1/3}$, we deduce that

$$
\|\chi T(e^{-2i\psi/h}f)\|_{L^q(\mathbb{C})} \leq \frac{h^{2/3}}{\delta^4}\|f\|_{W^{1,p}}.
$$

(12)
if $q < 2$. We now move on to the smoothing part given by $K(e^{-2i\psi/h} f)$. Take $\chi$ to be a compactly supported function in $\Omega$ such that it is equal to 1 on the support of $f$, we see that $K(e^{2i\psi/h} f) = K(e^{-2i\psi/h}(f - \chi f(z_0)) + f(z_0)K(e^{-2i\psi/h}\chi)$. By applying stationary phase and Part 1 of Lemma 2.2, we easily see that $\|f(z_0)K(e^{-2i\psi/h}\chi)\|_{L^q} \leq \frac{C}{h}||f||_{L^q}$ for any $q \in [1, \infty]$. For the first term, we write $\tilde{f} := f - \chi f(z_0)$ and we integrate by parts to get, for some smoothing operator $K'$

$$K(e^{-2i\psi/h}\tilde{f}) = hK'(e^{-2i\psi/h}\bar{f} \frac{\partial}{\partial z^j}) + \frac{h}{2i}K(e^{-2i\psi/h}\partial_z(\bar{f} \frac{\partial}{\partial z^j})),$$

By the fact that $K$ and $K'$ are smoothing, we see that for all $k \in \mathbf{N}$

$$\|K(e^{2i\psi/h}\tilde{f})\|_{C^k} \leq hC\left(\|\tilde{f}\|_{L^1} + \|\bar{f}\|_{L^1}\right)$$

Using the Hessian estimate of the phase in Lemma 2.1, the Sobolev embedding $W^{1,p} \subset C^{\alpha}$ for $\alpha = 1 - 2/p$ and $\tilde{f}(z_0) = 0$, we can estimate both terms by $\frac{C}{h}||f||_{W^{1,p}}$ if $p > 2$. Therefore,

$$\|K(e^{2i\psi/h} f)\|_{L^q} \leq \frac{C}{h}||f||_{W^{1,p}}$$

for any $q \in [1, \infty]$ and $p > 2$. Combining 13 and 12 we see that 8 is established.

Let us now turn our attention to the case when $\infty > q \geq 2$, one can use the boundedness of $T$ on $L^q$ and thus

$$\|\chi T((1 - \varphi)e^{-2i\psi/h} f)\|_{L^q(\mathbf{C})} \leq \|(1 - \varphi)e^{-2i\psi/h} f\|_{L^q(\Omega)} \leq C\epsilon_0^q \|f\|_{L^\infty}.$$  

Now since $\varphi = 0$ near $z_0$, we can use

$$T(e^{-2i\psi/h}\varphi f) = \frac{1}{2i}[e^{-2i\psi/h}(\varphi f) - T(e^{-2i\psi/h}\varphi f)]$$

and the boundedness of $T$ on $L^q$ to deduce that for any $q \leq p$, 11 holds again with all the terms satisfying the same estimates as before so that

$$\|T(e^{-2i\psi/h}\varphi f)\|_{L^q} \leq \frac{C}{h}||f||_{W^{1,p}}(\epsilon_0^q - \epsilon_0^{-1}) \leq \frac{C}{h}||f||_{W^{1,p}}$$

since now $q \geq 2$. Now combine the above estimate with 14 and take $\epsilon_0 = h^{-\frac{1}{2}}$ we get

$$\|T(e^{-2i\psi/h} f)\|_{L^q} \leq \frac{C}{h}||f||_{W^{1,p}}$$

for $2 \leq q \leq p$. The smoothing operator $K$ is controlled by 13 for all $q \in [1, \infty]$ and therefore we obtain 9.

The final lemma of this section follows Lemma 2.5, and is proved in exactly the same way, but for a corresponding $\tilde{\partial}_{\psi}^{-1}$. Here the restriction and extension operators must be interpreted in a different way, namely that $R'$ restricts sections of $A^{0,1}(N)$ to $M$ and $E'$ is a continuous extension from $W^{k,p}(M)$ to $W^{k,p}(N)$, $k = 0, 1$ where the functions in its range are supported in some $N_\delta$.

**Lemma 2.7** (Refinement of Lemma 2.3 in [7]). Let $\Phi_\beta$ be a holomorphic Morse function as in Lemma 2.1 and let $\psi = \text{Im} \Phi_\beta$. Define $\tilde{\partial}_{\psi}^{-1} := R'\tilde{\partial}_{\psi}^{-1}e^{2i\psi/h} E'$. For
q > 1, p > 2 there exists constants ε > 0, C > 0 independent of h, δ, and ̂p such that for all v ∈ W^{1,p}(M),
\begin{align*}
\|\tilde{\partial}^{-1}v\|_{L^q(\Lambda^{0,1}(M))} &\leq \frac{C}{\delta^4} h^{2/3}\|v\|_{W^{1,p}(M)}, \quad 1 \leq q < 2, \\
\|\tilde{\partial}^{-1}v\|_{L^q(\Lambda^{0,1}(M))} &\leq \frac{C}{\delta^4} h^{1/q}\|v\|_{W^{1,p}(M)}, \quad 2 \leq q \leq p.
\end{align*}
By interpolation, there is thus an ε > 0 and C > 0 such that for all v ∈ W^{1,p}(M)
\|\tilde{\partial}^{-1}v\|_{L^2(\Lambda^{0,1}(M))} \leq \frac{C}{\delta^4} h^{1/2+\varepsilon}\|v\|_{W^{1,p}(M)}.

3. Inverse boundary problems for systems. Our approach mimic the idea of Bukhgeim [2] that makes it possible to study a first order differential operator represented by a matrix instead of 2.

The idea is to consider the bundle Σ(M) := Λ^0(M) ⊕ Λ^{0,1}(M) and the ̂∂-system
\[(D + V)U = 0,\]
where
\[D = \begin{bmatrix} 0 & ̂\partial \\ ̂\partial & 0 \end{bmatrix}, \quad V = \begin{bmatrix} Q^+ & A' \\ A & Q^- \end{bmatrix}, \quad U = \left( \begin{array}{c} u \\ ω_{0,1} \end{array} \right) ∈ Σ(M).\]
The operator D is often called a Dirac operator and is formally self-adjoint. The potential V will be built up by functions Q± on the diagonal and 1-forms A, A' on the antidiagonal. The action of V on Σ(M) must be interpreted in the correct way, e.g. in our case it will be of the form
\[V = \begin{bmatrix} Q & ̂\ast(A ∧ ·) \\ A & -1 \end{bmatrix}, \quad A ∈ Λ^{0,1}(M).\]
Hence
\[VU = \left( \begin{array}{c} Qu + ̂\ast(A ∧ ω_{0,1}) \\ uA - ω_{0,1} \end{array} \right).\]

In [7] it is assumed that V is a diagonal endomorphism of Σ(M). This condition was relaxed in [1] and we will mainly follow the methodology of this work in this section.

The inner products on Λ^0(M) and Λ^1(M) induce a natural inner product on Σ(M) as
\[\langle ·, · \rangle_{L^2(Σ(M))} := \langle ·, · \rangle_{L^2(Σ(M))} + \langle ·, · \rangle_{L^2(Λ^1(M))}\]
where the two inner products in the right hand side are defined by 6.

We will make use of the following boundary integral identity, which is easily proved by integration by parts.

**Lemma 3.1.** Suppose that U' is a solution to the system (D + V*)U' = 0 in M. Then
\[\langle (D + V)U, U' \rangle_{L^2(Σ(M))} = \langle U, U' \rangle_{∂M},\]
where
\[\langle U, U' \rangle_{∂M} := \int_{∂M} \iota^*_M \left( u * \overline{ω_{0,1}} - ̂\ast(ω_{0,1} u') \right).\]
In the above, \(\iota^*_M\) denotes pullback by inclusion and
\[U = \left( \begin{array}{c} u \\ ω_{0,1} \end{array} \right), \quad U' = \left( \begin{array}{c} u' \\ ω_{0,1} \end{array} \right).\]
Our goal will be to reduce to the case when \( V \) is in fact diagonal. This is the situation first studied by Bukhgeim in the planar case in [2], laying the foundation to the manifold version in Proposition 2.5 in [1]. In our case, due to the modified phase function \( \Phi \) that we must resort to, our version of that proposition reads:

**Proposition 1.** Let \( \Phi_\ell \) be a holomorphic Morse function as in Lemma 2.1.

1. If

\[
V = \begin{bmatrix}
\tilde{Q} & 0 \\
0 & \tilde{F}
\end{bmatrix}, \quad \tilde{Q}, \tilde{F} \in W^{1,p}(M),
\]

for some \( p > 2 \), then there exist solutions to \((D + V)F_h = 0 \) on \( M \) of the form

\[
F_h = \begin{pmatrix}
\exp^{\Phi_\ell r_h} \\
\exp^{\Phi_\ell (b + s_h)}
\end{pmatrix},
\]

for any anti-holomorphic one form \( b \) and so that for some \( \varepsilon > 0 \)

\[
(15) \quad \|r_h\|_{L^2(M)} + \|s_h\|_{L^2(M)} \leq C h^{1/2 + \varepsilon} \delta^4.
\]

2. There exists solutions to \((D + V)G_h = 0 \) on \( M \) of the form

\[
G_h = \begin{pmatrix}
\exp^{\Phi_\ell (a + r_h)} \\
\exp^{\Phi_\ell s_h}
\end{pmatrix},
\]

for any holomorphic function \( a \) and so that for some \( \varepsilon > 0 \), 15 still hold.

**Proof.** We prove item 1 only as item 2 is analogous. The proof closely resembles the one of Proposition 3.1 in [7] but makes use of the refined Lemma 2.6 (or similar) instead of Lemma 2.2 in [7] (and its variants respectively).

We make use of the fact that the mentioned lemmas contain very explicit expressions for the remainder terms \( r_h \in \Lambda^0(M), s_h \in \Lambda^{0,1}(M) \). It can be seen, following the same computation as in [7], that the remainder terms must solve the system

\[
(16) \quad \begin{cases}
r_h + \bar{\partial}^{-1}_\psi (\tilde{F} s_h) = - \bar{\partial}^{-1}_\psi (\tilde{F} b) \\
\bar{s}_h + \bar{\partial}^{-1}_{\psi} (\tilde{Q} r_h) = - \bar{\partial}^{-1}_{\psi} (\tilde{Q} a).
\end{cases}
\]

In the case \( \tilde{Q} \in L^\infty(M), \tilde{F} \in W^{1,p}(M) \) we can choose \( a = 0 \) and it follows that \( r_h \)

must satisfy

\[
(17) \quad (I - S_h) r_h = - \bar{\partial}^{-1}_\psi (\tilde{F} b), \quad S_h := \bar{\partial}^{-1}_\psi \tilde{F} \bar{\partial}^{-1}_{\psi} \tilde{Q}.
\]

The idea is now to solve through a Neumann series. From Lemma 2.6 it follows that \( \|S_h\|_{L^2 \to L^2} \leq \frac{C}{\delta^4} (h^{1/2 - \varepsilon}), 0 < \varepsilon < 1/2 \), c.f. Lemma 2.4 in [1]. We remark that also this result (or the corresponding Lemma 3.1 in [7]) must be modified slightly. However, as we will later require that \( h^\varepsilon \delta^{-7/2} \) to be small for some (preferably as large as possible) \( 0 < \varepsilon < 1/2 \). This will ensure that the bound:

\[
\|S_h\|_{L^2 \to L^2} \leq \frac{C h^{1/2 - \varepsilon}}{\delta^4}
\]

still makes the Neumann series argument valid. Furthermore we may establish the estimate 15 in an analogous way. So we can solve equation 17 for small \( h \),

\[
r_h = - \sum_{j=0}^{\infty} S^j_h \bar{\partial}^{-1}_\psi \tilde{F} b.
\]
The above $r_{h} \in L^{q}(M)$ for any $q \geq 2$ and substituting this solution into the equation for $s_{h}$ in 16 we get
\[ s_{h} = -\tilde{\partial}_{\psi}^{-1}(\tilde{Q}r_{h}). \]
Now another application of Lemma 2.7 gives the $L^{2}$-estimates in Proposition 1. Solution of the other from can be constructed by a similar argument after choosing $b = 0$ and $a \neq 0$. □

3.1. **The distance between Cauchy data for systems.** We would like to compare the Cauchy data for different potentials $V_{1}, V_{2}$ in a meaningful and quantitative manner. One standard method is to use a pseudo-distance inspired by the so-called Hausdorff distance. Recalling the definition of the Cauchy data spaces $C_{L_{j}}$, as in 3. Assuming that $u_{j} \in H^{k}(M)$ solves
\[ \begin{cases} L_{j}u_{j} = 0, \\ u_{j}|_{\partial M} = f_{j}, \end{cases} \tag{18} \]
for $f_{j} \in H^{k-1/2}(\partial M)$, for some $k \geq 1$. Then $\nabla_{v}^{X_{j}}u_{j} = g_{j} \in H^{k-3/2}(\partial M)$ and we may consider a norm on $C_{L_{j}}$ defined by,
\[ \| (f_{j}, g_{j}) \|_{H^{s}(\partial M) \oplus H^{s-1}(\partial M)} := \| f_{j} \|_{H^{s}(\partial M)} + \| g_{j} \|_{H^{s-1}(\partial M)}, \quad s \geq 1/2. \]
Then we set, for $(f_{j}, g_{j}) \in C_{L_{j}}$,
\[ d((f_{1}, g_{1}), (f_{2}, g_{2})) := \frac{\| (f_{1}, g_{1}) - (f_{2}, g_{2}) \|_{H^{s}(\partial M) \oplus H^{s-1}(\partial M)}}{\| f_{1} \|_{H^{s}(\partial M)}} \]
and define
\[ d(C_{L_{1}}, C_{L_{2}}) := \max \left\{ \sup_{C_{L_{1}}} \inf_{C_{L_{2}}} d((f_{1}, g_{1}), (f_{2}, g_{2})), \sup_{C_{L_{2}}} \inf_{C_{L_{1}}} d((f_{2}, g_{2}), (f_{1}, g_{1})) \right\}. \]
Correspondingly for the system formulation, we think of the Cauchy data $C_{V}$ is made up of boundary values $\tilde{\partial}_{M}(u, u_{\omega})^{T}$ for $H^{k}$-solutions $U = (u, u_{\omega})^{T}$ to $(D + V)U = 0$. We may consider $C_{V}$ being a subset of $H^{s}(\partial M) \oplus H^{s-1}(\partial M)$, whose norm we can use to introduce a distance when considering two potentials. Suppose we have two traces, $(f_{j}, \lambda_{j})^{T} \in C_{V_{j}}, j = 1, 2$, then we can consider the quantity
\[ d_{\partial M}((f_{1}, \lambda_{1}), (f_{2}, \lambda_{2})) := \frac{\| (f_{1}, \lambda_{1}) - (f_{2}, \lambda_{2}) \|_{H^{s}(\Sigma(\partial M))}}{\| f_{1} \|_{H^{s}(\partial M)}}. \]
The $H^{s}(\Sigma(\partial M))$-norm is defined, for $s \geq 1/2$, by
\[ \| (f, \lambda) \|_{H^{s}(\Sigma(\partial M))} := \| f \|_{H^{s}(\partial M)} + \| \lambda \|_{H^{s-1}(\partial M)}, \quad U = (f, \lambda)^{T} \in \Sigma(\partial M). \]
Then we can define a distance between Cauchy data as
\[ d'(C_{V_{1}}, C_{V_{2}}) = \max \left\{ \sup_{C_{V_{1}}} \inf_{C_{V_{2}}} d_{\partial M}((f_{1}, \lambda_{1}), (f_{2}, \lambda_{2})), \sup_{C_{V_{2}}} \inf_{C_{V_{1}}} d_{\partial M}((f_{2}, \lambda_{2}), (f_{1}, \lambda_{1})) \right\}. \]
**Proposition 2.** If $(D + V_{j})U_{j} = 0$ are the system formulations of the problems $L_{j}u_{j} = 0$, then
\[ d'(C_{V_{1}}, C_{V_{2}}) \leq d(C_{L_{1}}, C_{L_{2}}). \]
4. System reduction and estimates. Suppose now that we are given two magnetic Schrödinger operators $L_j := L_{X_j,q_j}, j = 1, 2$. Introduce $A_j := \pi_{0,1}X_j, B_j := \pi_{1,0}X_j,$ where $\pi_{0,1}$ and $\pi_{1,0}$ are the projections discussed in Section 2. We first observe that we can rewrite $L_j$ from the form 2 into

$$L_j = -2i(\partial + iA_j \wedge \partial + iA_j) + Q_j,$$

If $\alpha_j$ are primitive functions in the sense that $\delta \alpha_j = A_j$ (the existence of such $\alpha_j$ is guaranteed by Lemma 2.3), then we can furthermore rewrite 19 (using integrating factor) to

$$L_j = 2e^{-i\alpha_j} \delta^* e^{i\alpha_j} e^{-i\alpha_j} \delta + Q_j.$$  

In order to abbreviate, let us denote by $F_j = e^{i\alpha_j}$, then we can once again rewrite 20 as

$$L_j = 2\tilde{F}_j \delta^* \tilde{F}_j^{-1} \tilde{F}_j^{-1} \delta \tilde{F}_j + Q_j, \quad Q_j = -\star dX_j + q_j.$$  

Introducing $\omega = (\partial + iA_j)u$ we see that we can rewrite the second order partial differential equations $L_j u = 0$ as the first order $\partial$-system

$$\begin{bmatrix} Q_j/2 & -i \star (\partial + iA_j \wedge \cdot) \\ \tilde{\partial} + iA_j & -1 \end{bmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or equivalently

$$\begin{bmatrix} 0 & \tilde{\partial}^* \\ \partial & 0 \end{bmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} + \begin{bmatrix} Q_j/2 & \star (\tilde{A}_j \wedge \cdot) \\ iA_j & -1 \end{bmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

Using notations from Section 3 we can abbreviate 22 by $(D + V_j)U = 0$. By a similar argument leading to the form 21 of $L_j$ we can also observe that we can split the system 22 further into (c.f. [1])

$$\begin{bmatrix} \tilde{F}_j & 0 \\ 0 & F_j^{-1} \end{bmatrix} \begin{bmatrix} 0 & \tilde{\partial}^* \\ \partial & 0 \end{bmatrix} + \begin{bmatrix} F_j^{-1}Q_jF_j^{-1}/2 & 0 \\ 0 & -F_j\tilde{F}_j \end{bmatrix} \times \begin{bmatrix} F_j & 0 \\ 0 & F_j^{-1} \end{bmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
or equivalently, since the leftmost matrix is invertible
\begin{equation}
\left( \begin{bmatrix} 0 & \tilde{\omega} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} + \begin{bmatrix} |F_j|^{-2}Q_j/2 & 0 \\ 0 & -|F_j|^2 \end{bmatrix} \right) \begin{bmatrix} \tilde{u} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\end{equation}
where
\[
\tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} F_j \tilde{u} \\ F_j^{-1} \tilde{\omega} \end{bmatrix} \in \Sigma(M).
\]
Denoting the potential matrix in 23 by \( \tilde{V} \), the system can be abbreviated as \((D + \tilde{V}_j)\tilde{U} = 0.\) Now we are in the case discussed above, with diagonal potential matrices, that was treated by Bukhgeim.

Suppose now that we have a solution to \((D + \tilde{V}_1)\tilde{U}_h^1 = 0,\) then by Proposition 1 we can assume that the solution has the form
\[
\tilde{U}_h^1 = \begin{bmatrix} e^{\Phi/h}(a + r_h^1) \\ e^{\Phi/h}s_h^1 \end{bmatrix}.
\]
It follows that a solution to \((D + V_1)U_h^1 = 0\) can be found on the form
\begin{equation}
U_h^1 = \begin{bmatrix} F_1^{-1}e^{\Phi/h}(a + r_h^1) \\ F_1e^{\Phi/h}s_h^1 \end{bmatrix},
\end{equation}
where \(a\) is an arbitrary holomorphic function.

Similarly, a solution to \((D + V_2)U_h^2 = (D + V_2)U_h^2 = 0\) can be found on the form
\begin{equation}
U_h^2 = \begin{bmatrix} F_2^{-1}e^{-\Phi/h}r_h^2 \\ F_2e^{-\Phi/h}(b + s_h^2) \end{bmatrix}.
\end{equation}
Suppose now that \(U_h\) is a solution to \((D+V_2)U_h = 0,\) and \(U_h^1, U_h^2\) are the solutions described above. Then by Lemma 3.1
\begin{equation}
\langle (D + V_2)U_h^1, U_h^2 \rangle_{L^2(\Sigma(M))} = \langle (D + V_2)(U_h^1 - U_h), U_h^2 \rangle_{L^2(\Sigma(M))} = \langle U_h^1 - U_h, U_h^2 \rangle_{\partial M}.
\end{equation}
At the same time it is also true that
\[
\langle (D + V_2)U_h^1, U_h^2 \rangle_{L^2(\Sigma(M))} = \langle (V_2 - V_1)U_h^1, U_h^2 \rangle_{L^2(\Sigma(M))}.
\]
So we have derived the boundary integral identity
\begin{equation}
\langle (V_2 - V_1)U_h^1, U_h^2 \rangle_{L^2(\Sigma(M))} = \langle U_h^1 - U_h, U_h^2 \rangle_{\partial M},
\end{equation}
from which we will derive an estimate in order to compare Cauchy data for the potential matrices \(V_j\) and their diagonalized counterparts \(\tilde{V}_j\), in the sense of 22-23.

First we show the following auxiliary estimate.

**Lemma 4.1.** Assume that there is a constant \(K > 0\) such that for some \(p > 2,\)
\begin{equation}
\max\{\|q_j\|_{W^{1,p}(\Sigma(M))}, \|X_j\|_{W^{2,p}(\Sigma(M))}\} \leq K.
\end{equation}
and consider the systems corresponding to the problems \(L_ju = 0\) as described in 22. Then the boundary integral identity 27 implies the following inequality for small \(h > 0, \delta > 0:\)
\[
\left| \int_M F_1^{-1}F_2a(A_1 - A_2) \land \ast d\right| \leq C \left( \frac{h^{1/2+\epsilon}}{\delta^4} + d(C_1, C_2)\right)\|U_h^1\|_{H^1}\|U_h^2\|_{H^1},
\]
where \(C = C(K, M, p), c > 0, A_j = \pi_{0,1}X_j, d(C_1, C_2)\) is the distance between Cauchy data for the problems \(L_ju = 0\) and \(a, b, F_j\) are the quantities defined above appearing in the solutions \(U_h^j, j = 1, 2\) of the systems.
Remark 2. Let us first remark on the a priori estimate 28. Under the assumptions
\begin{align}
(29) \quad A_j \in C^{1+r} \cap W^{s,p}(\Lambda^{0,1}(M)), \quad p > 1, r > 1, r \notin \mathbb{N}, sp > 4, \\
(30) \quad Q_j \in W^{1,q}(M), \quad q > 2,
\end{align}
we will have that \( \hat{V}_j \in W^{1,q}(\text{End}(\Sigma(M))) \), c.f. [1].

Furthermore, the assumption
\[ \max\{\|q_j\|_{W^{1,p}(M)}, \|X_j\|_{W^{2,p}(\Lambda^{1}(M))}\} \leq K, \quad \text{for some} \quad p > 2. \]
implies in particular that
\[ \max\{\|Q_j\|_{W^{1,p}(M)}, \|A_j\|_{W^{1,p}(\Lambda^{0,1}(M))}\} \leq K \]
holds. Then by Sobolev embedding and elliptic regularity (for \( \partial, \) see e.g. Theorem 4.6.9 in [14]), it follows that
\begin{align}
(32) \quad \max\{\|Q_j\|_{L^\infty(M)}, \|A_j\|_{L^\infty(\Lambda^{0,1}(M))}\} \leq K, \\
(33) \quad \|\alpha_j\|_{L^\infty(M)} \leq C_0 \|\alpha_j\|_{W^{1,p}(\Lambda^{0,1}(M))} \leq C_0 K \leq C,
\end{align}

\[ \|F_j\|_{L^\infty(M)} \leq e^{C_0 K} \leq C, \quad C = C(K, M, p). \]

Proof of Lemma 4.1. Expanding the left hand side in 27, we find
\begin{align}
(35) \quad \langle (V_2 - V_1)U_h^1, U_h^2 \rangle_{L^2(\Sigma(M))} &= \frac{1}{2} \int_M e^{2i\psi/h} F_1^{-1} \bar{F}_2^{-1} Q(a + r_h^1)^\nu \bar{r}_h^1 + 2 \bar{F}_1 \bar{F}_2^{-1} \star (\bar{A} \wedge s_h^1) r_h^2 dV_g \\
&\quad + i \int_M F_1^{-1} F_2 (a + r_h^1) A \wedge \star(b + s_h^1).
\end{align}

where \( A = A_2 - A_1, Q = Q_2 - Q_1, a \) can be any holomorphic function and \( b \) any antiholomorphic 1-form. The next step will be to use the estimates from Proposition 1. Under the assumption in 28 and the following remark, we get (by applying Cauchy-Schwarz inequality) the estimates
\[ \left| \int_M e^{2i\psi/h} F_1^{-1} \bar{F}_2 Q(a + r_h^1)^\nu \bar{r}_h^1 dV_g \right| \leq C \left( \frac{h^{1/2+\epsilon}}{\delta^4} + \frac{h^{1+2\epsilon}}{\delta^8} \right) \leq \frac{Ch^{1+2\epsilon}}{\delta^4}, \]
\[ \left| \int_M \bar{F}_1 \bar{F}_2^{-1} \star (\bar{A} \wedge s_h^1) r_h^2 dV_g \right| \leq \frac{Ch^{1+2\epsilon}}{\delta^8} \leq \frac{Ch^{1+2\epsilon}}{\delta^4}, \]
\[ \left| \int_M F_1^{-1} F_2 (a + r_h^1) A \wedge \star(b + s_h^1)^2 \right| \leq \int_M a F_1^{-1} F_2 A \wedge \star b + \frac{Ch^{1+2\epsilon}}{\delta^4}. \]

By examining the boundary term in 27 we find
\[ \langle U_h^1 - U_h, U_h^2 \rangle_{\partial M} = \int_{\partial M} ((u_1 - u) \star \bar{\omega} - \star(\omega_1 - \omega) \bar{\omega}) \],
where we temporarily have abbreviated the solutions according to the earlier convention, \( U_h^j = (u_j, \omega_j)^T \). By Cauchy-Schwarz inequality,
\begin{align}
\langle (U_h^1 - U_h, U_h^2)_{\partial M} \rangle &\leq 2 \| (u_1 - u, \nabla_{\nu} u_1 - \nabla_{\nu} u) \|_{H^{1/2}(\partial M) \otimes H^{-1/2}(\partial M)} \| \nu_{\partial M} U_h^2 \|_{H^{1/2}(\partial M)} \\
&\leq C d((u_1, \nabla_{\nu} u_1), (u, \nabla_{\nu} u)) \| U_h^1 \|_{H^1(\Sigma(M))} \| U_h^2 \|_{H^1(\Sigma(M))}
\end{align}
by the boundedness of the trace operator. Since this holds for any solution \( U_h \) to 
\((D + V_2)U_h = 0\) it follows by taking infimum over the corresponding Cauchy data 
space that

\[
\|(V_2 - V_1)U_h, U_h^2\|_{L^2(\Sigma(M))} \leq Cd(C_1, C_2)\|U_h\|_{H^1(\Sigma(M))}\|U_h^2\|_{H^1(\Sigma(M))}.
\]

The proof is finished by rearranging the terms in 27 and applying the triangle 
inequality.

The next step is to estimate the Sobolev norms of the solutions appearing in the 
right hand side of 36. This requires a quite detailed discussion but will yield results 
that we will use more than once.

4.1. \( H^1 \)-estimates of solutions to \((D + V)U = 0\). We will need to estimate 
solutions of systems \((D + V)U = 0\) in \( H^1 \)-norm sense. For the general (non-
diagonal) potentials \( V \), that we must consider, we saw that solutions were given by 
24 and 25. Our goal will be to establish:

**Proposition 3.** Let \( U_1, U_2 \) be the solutions given in Proposition 1 
with \( \tilde{Q}, \tilde{F} \in W^{1, p}(M), p > 2 \). Then there are constants 
c > 0, \( D = D(K, M, p) > 0 \) such that

\[
\max\{\|U_1\|_{H^1(M, \Sigma(M))}, \|U_2\|_{H^1(M, \Sigma(M))}\} \leq De^{c/h},
\]

for small \( h > 0 \).

**Proof.** Let us first recall what we mean with the \( H^1 \)-norm of some \( U = (u, \omega_{0,1})^T \in \Sigma(M) \).

\[
\|U\|_{H^1(\Sigma(M))}^{2} := \|u\|_{H^1(M)}^{2} + \|\omega_{0,1}\|_{H^1(\Lambda^1(M))}^{2},
\]

\[
\|u\|_{H^1(M)}^{2} = \|u\|_{L^2(M)}^{2} + \|\nabla u\|_{L^2(M)}^{2},
\]

\[
\|\omega_{0,1}\|_{H^1(\Lambda^1(M))}^{2} = \|\omega_{0,1}\|_{L^2(\Lambda^1(M))}^{2} + \|\nabla \omega_{0,1}\|_{L^2(\Lambda^1(M))}^{2}.
\]

For the solutions in 24 and 25, this means that we need to estimate

\[
\|F_1e^{\Phi/h}(a + r_h^1)\|_{H^1(M)}, \quad \|\tilde{F}_1e^{\tilde{\Phi}/h}s_h\|_{H^1(\Lambda^1(M))},
\]

\[
\|F_2e^{-\Phi/h}r_h^2\|_{H^1(M)}, \quad \|\tilde{F}_2e^{-\tilde{\Phi}/h}(b + s_h^2)\|_{H^1(\Lambda^1(M))},
\]

where we are free to choose the holomorphic function \( a \) and antiholomorphic 1-form \( b \). We start by observing that there is a constant \( c > 0 \) so that

\[
|e^{\pm\Phi/h}| \leq e^{c/h},
\]

since \( \varphi = \text{Re} \Phi \) is harmonic. By the a priori assumptions on the \( A_j \)'s in 29 and 
the discussion leading to 33, we will also have no trouble bounding the first order 
partial derivatives of the \( F_j \) by some constant depending on the a priori bounding 
constant \( K \).

We need to be a bit careful with the remainders \( r_h^1, s_h^1 \). By studying the system 
16 and invoking elliptic regularity we can however see that we are in no danger, 
assuming sufficient regularity on \( \tilde{Q}, \tilde{F} \). Rewriting the system we see that,

\[
\begin{cases}
    r_h = -\tilde{\partial}_\psi^{-1}(\tilde{F}(b + s_h)), \\
    s_h = -\tilde{\partial}_\psi^{-1}(\tilde{Q}(a + r_h)).
\end{cases}
\]

Now it is more clear that the right hand side should belong to \( H^1(M) \). To get 
an estimate, observe that by writing out the operators as in Section 2, \( \tilde{\partial}_\psi^{-1} = \)
Choosing for some \(1 - \varepsilon \leq \alpha < 1\) (by Lemma 2.3(i), 2.4(i) and Proposition 1),

\[
\|\tilde{\partial}r_h\|_{L^2} = \|\tilde{\partial}\tilde{\partial}^{-1}e^{2i\psi/h}E(\tilde{F}(b + s_h))\|_{L^2} \leq \tilde{C}(\|b\|_{L^2} + \|r_h\|_{L^2}) \leq C,
\]

\[
\|\partial s_h\|_{L^2} = \|\partial\tilde{\partial}^{-1}e^{2i\psi/h}E'Q(a + s_h)\|_{L^2} \leq \tilde{C}(\|b\|_{L^2} + \|r_h\|_{L^2}) \leq C.
\]

Furthermore,

\[
\|\partial r_h\|_{L^2} = \|\partial\tilde{\partial}^{-1}e^{2i\psi/h}E(\tilde{F}(b + s_h))\|_{L^2} \leq \tilde{C}(\|b\|_{L^2} + \|r_h\|_{L^2}) \leq C,
\]

\[
\|\partial s_h\|_{L^2} = \|\partial\tilde{\partial}^{-1} \ast e^{2i\psi/h}E'(Q(a + s_h))\|_{L^2} \leq \tilde{C}(\|b\|_{L^2} + \|r_h\|_{L^2}) \leq C.
\]

since \(\tilde{\partial}\tilde{\partial}^{-1}, \partial\tilde{\partial}^{-1}\) are bounded operators (related to the so-called Beurling transform) on \(L^p(M)\). In local coordinates the kernels are of Calderón-Zygmund type, modulo smoothing terms by Lemma 2.3 and 2.4, c.f. [19].

**Remark 3.** Continuing the argument in the above proof it is in fact also more or less almost immediate that if \(Q, F \in W^{k,p}(M)\), \(k \geq 1, p > 2\), then

\[
\|r_h\|_{H^k(M)} + \|s_h\|_{H^k(M)} = O(h^{1-k}).
\]

### 4.2. Conclusion of the system reduction step.

Adding up 26–37 we have managed to show that

\[
\left| \int_M a\Phi_1 F_2^{-1} A \wedge \tilde{\phi} \right| \leq C \frac{h^{1/2+\varepsilon}}{d^4} + Dc^{1/h} d(C_1, C_2).
\]

Choosing for some \(1 - \varepsilon < \alpha < 1\)

\[
h = \frac{c}{\alpha |\log d(C_1, C_2)|},
\]

we get the estimate (for some large enough \(C = C(K, M, p) > 0\))

\[
\left| \int_M aF_1^{-1} F_2 A \wedge \tilde{\phi} \right| \leq \frac{C}{\delta^4 |\log d(C_1, C_2)|^{1/2+\varepsilon}}.
\]

Looking at the left hand side integral we also observe that

\[
- i \int_M F_2 (A_2 - A_1) F_1^{-1} a \wedge \tilde{\phi} = \langle \partial (F_2 F_1^{-1} a), b \rangle_{L^2(A^1(M))}
\]

\[
= \int_{\partial M} \phi_0^a (F_2 F_1^{-1} a \ast \tilde{b}) + \langle F_2 F_1^{-1} a, \partial^\ast b \rangle_{L^2(M)} = \int_{\partial M} \theta_0^a (F_2 F_1^{-1} a \ast \tilde{b})
\]

The first equality follows from the fact that \(F_j^{-1} \tilde{\partial} F_j = -F_j \tilde{\partial} F_j^{-1} = A_j\) by construction, while the third equality is just Green’s integral identity. Finally, the last equality follows since \(b\) is antiholomorphic. Thus we have shown

**Lemma 4.2.** If for some \(p > 2\), an a priori assumption of type

\[
\max\{ \|\tilde{g}_j\|_{W^{1,p}(M)}, \|X_j\|_{W^{2,1}(M, \Lambda^0(M))} \} \leq K
\]

hold, then for some small \(\varepsilon > 0\),

\[
\left| \int_{\partial M} \theta_0^a (F_2 F_1^{-1} a \ast \tilde{b}) \right| \leq \frac{C(K, M, p)}{\delta^4 |\log d(C_1, C_2)|^{1/2+\varepsilon}}.
\]

where \(C_j\) is the Cauchy data associated with the operators \(L_j = L_{X_j, q_j}\), as defined in 3, and \(a \in H(M), b \in \Lambda^0(M)\).
Later we will choose \( \delta = (\log |\log d(C_1,C_2)|)^{-\epsilon'} \) for some \( \epsilon' < 1/2 + \epsilon \) so that 38 can be replaced by

\[
(39) \quad \left| \int_{\partial M} \iota_{\partial M}^* (F_2 F_1^{-1} a^* \bar{b}) \right| \leq \frac{C}{|\log d(C_1,C_2)|^2},
\]

with \( 0 < \beta < 1/2 + \epsilon - \epsilon' \gamma' \) where \( \gamma' > 0 \) can be chosen arbitrary small. We will from now on assume that this choice of \( \delta \) has been made and use 39 in place of 38 when referring to Lemma 4.2.

5. **Reduction to diagonal system.** One of the key observations in [1] is that if \( C_1 = C_2 \), then choosing \( a = 1 \)

\[
\int_{\partial M} \iota_{\partial M}^* (F_2 F_1^{-1} a^* \bar{b}) = 0.
\]

(This is also a consequence of Lemma 4.2 of course.) The following lemma is then used extensively:

**Lemma 5.1** (Lemma 2.8 in [1]). A complex-valued function \( f \in H^{1/2}(\partial M) \) is the restriction of a holomorphic function if and only if for all 1-forms \( \eta \in C^\infty(M, \Lambda^{1,0}(M)) \) satisfying \( \bar{\partial} \eta = 0 \),

\[
\int_{\partial M} f i_{\partial M}^* \eta = 0.
\]

The proof can be found in [7] where the above result is contained in Lemma 4.1. One must show that the harmonic extension of \( f \) is actually holomorphic, and this can be done by considering the so-called Hodge-Morrey decomposition of \((0,1)\)-forms.

Using the above lemma, the authors are thus able to conclude that \( (F_2^{-1} F_1)|_{\partial M} \) is indeed the restriction of a holomorphic function. Hence they may reduce the case of a general matrix potential \( V_j \) to the diagonalized case \( \bar{V}_j \).

Clearly, we are not in the same situation here so we need to motivate a similar reduction step in a slightly different way.

5.1. **Auxiliary estimate.** Let us introduce the subspace

\[
U = \{ \omega = \iota_{\partial M}^* b; \bar{\partial} b = 0, b \in \Lambda^1(M) \},
\]

of \( X = H^{1/2}(\Lambda^1(\partial M)) \) (or \( H^s(\Lambda^1(\partial M)) \)), for any \( s \geq 0 \) and argue abstractly from the viewpoint of Hilbert space theory. Consider the following subspace of the dual space \( X^* \),

\[
\ker U := \{ x^* \in X^*; x^*(x) = 0 \text{ for all } x \in U \}.
\]

Take a Schauder basis \( \{ x_n^* \}_{n=1}^\infty \) such that \( \ker U = \text{span} \{ x_n^* \}_{n=1}^\infty \) and a dual basis \( \{ x_n \}_{n=1}^\infty \) (defined by \( x_n^*(x_m) = \delta_{mn} \)). Now we claim that

\[
X = U \oplus \text{span} \{ x_n \}_{n=1}^\infty.
\]

Indeed, suppose there is \( 0 \neq x^* \in X^* \) such that \( x^*(x) = 0 \) for all \( x \in U \) \( \oplus \text{span} \{ x_n \}_{n=1}^\infty \). Then in particular \( x^*(x_n) = 0 \) for all \( n \in \mathbb{N} \). Thus \( x^* \notin \ker U \), but at the same time we must have \( x^*(x) = 0 \) for all \( x \in U \) and thus \( x^* \in \ker U \). An obvious contradiction unless \( x^* = 0 \). To see that it is indeed a direct sum, pick \( x \in U \cap \text{span} \{ x_n \}_{n=1}^\infty \), then \( x = \sum a_n x_n \in U \), but then \( 0 = x_n^*(x) = a_n \) for all \( n \in \mathbb{N} \).

Now take another Schauder basis \( \{ y_n \}_{n=1}^\infty \) such that \( \text{span} \{ y_n \}_{n=1}^\infty = U \) and a dual basis \( \{ y_n^* \}_{n=1}^\infty \subset X^* \), then we can in a very similar way show that
\[ X^* = \ker U \oplus \text{span} \{ y_n^* \}_{n=1}^\infty. \]

Introducing the projection \( \pi : X^* \to \ker U \)
we have by the above splitting of \( X^* = H^{-1/2}(\Lambda^1(\partial M)) \) that any linear functional on \( X \) may be written
\[ x^* = \pi x^* + (1 - \pi)x^*. \]

Now we claim that
\[ \|(1 - \pi)x^*\| = \sup_{\|x\| \leq 1} |(1 - \pi)x^*(x)| = \sup_{\|x\| \leq 1} |(1 - \pi)x^*(x)|. \]

This equality follows since \( (1 - \pi)x^*(x_n) = 0, n \in \mathbb{N} \) since \( (1 - \pi)x^* \in \text{span} \{ y_n^* \} \).

In particular, if we consider the linear functional
\[ A_f : H^{1/2}(\Lambda^1(\partial M)) \to \mathbb{C}, \quad A_f[\omega] := \int_{\partial M} f\omega, \quad f \in H^{-1/2}(\partial M), \]
we can interpret Lemma 4.2 as
\[ \|(1 - \pi)(F_2F_1^{-1})|_{\partial M}\|_{H^{-1/2}(\partial M)} \leq \frac{C}{|\log d(C_1, C_2)|^{\gamma}}, \]

since the operator norm of \( A_{(1 - \pi)(F_2F_1^{-1})|_{\partial M}} \) equals the norm in the left hand side by the argument above. Furthermore, by the a priori assumptions in 29-34, we have for some \( \delta' > 0 \), and interpolation
\[ \|(1 - \pi)(F_2F_1^{-1})|_{\partial M}\|_{H^r(\partial M)} \leq \frac{C}{|\log d(C_1, C_2)|^{\gamma(\alpha, \beta)}}, \]

where \( 0 < \alpha < 1, 0 < \gamma(\alpha, \beta) < (1 - \alpha)\beta \) and \( r > 3/2. \)

Now we have arrived at a stage where we have
\[ F_2F_1^{-1} = (1 - \pi)F_2F_1^{-1} + \pi F_2F_1^{-1} \]

and by Lemma 5.1 we know that
\[ \pi F_2F_1^{-1}|_{\partial M} = G|_{\partial M}, \quad \bar{\partial}G = 0. \]

Thus
\[ F_2|_{\partial M} = F_1(1 - \pi)F_2F_1^{-1}|_{\partial M} + F_1G|_{\partial M} \]

which is equivalent with
\[ F_2|_{\partial M} - \hat{F}_1|_{\partial M} = F_1(1 - \pi)F_2F_1^{-1}|_{\partial M}, \quad \hat{F}_1 = F_1G. \]

Clearly it also holds that
\[ \bar{\partial}\hat{F}_1 = iA_1\hat{F}_1, \quad \bar{\partial}F_2 = iA_2F_2 \]

and by our calculations above and a priori assumptions on the \( A_j \)'s
\[ \|(F_2 - \hat{F}_1)|_{\partial M}\|_{H^r(\partial M)} \leq \frac{C}{|\log d(C_1, C_2)|^{\gamma}}. \]

So by Sobolev embedding we can conclude the following lemma:

**Lemma 5.2.** Let \( \hat{F}_1, F_2 \) be as defined above, then there is a \( 0 < \gamma \leq 1/2 \) such that
\[ \|(F_2 - \hat{F}_1)|_{\partial M}\|_{C^1(\partial M)} \leq \frac{C}{|\log d(C_1, C_2)|^{\gamma}}, \]

where \( C = C(K, M, p) \) and \( d(C_1, C_2) \) measures the distance between the Cauchy data spaces as defined above.
5.2. An inequality between Cauchy data. From 24-25 we see that solutions to systems \((D + V_j)\tilde{U}_j = 0\) is related to the corresponding system \((D + \tilde{V}_j)\tilde{U}_j = 0\) with diagonalized potentials \(\tilde{V}_j\) (in the sense described above) via

\[
\tilde{U}_1 = \begin{bmatrix} \tilde{F}_1 & 0 \\ 0 & \tilde{F}_1^{-1} \end{bmatrix} U_1, \quad \tilde{U}_2 = \begin{bmatrix} F_2 & 0 \\ 0 & F_2^{-1} \end{bmatrix} U_2.
\]

The next lemma relates the distances for Cauchy data for the diagonalized potential \(\tilde{V}_j\) and the Cauchy data for the corresponding partial differential equations with the pairs \((X_j, q_j)\).

**Lemma 5.3.** Suppose that we have reduced the boundary value problems for \(L_ju_j = 0\) to boundary value problems with diagonal potential matrices \((D + \tilde{V}_j)\tilde{U}_j = 0\) as above. Then the distance of Cauchy data corresponding to the diagonalized problems is bounded in terms of the distance between Cauchy data for the initial problem according to

\[
d'(C_{\tilde{V}_1}, C_{\tilde{V}_2}) \leq C \left( d(C_1, C_2) + \frac{1}{\log d(C_1, C_2)^\gamma} \right).
\]

The quantities \(d'(C_{\tilde{V}_1}, C_{\tilde{V}_2})\) and \(d(C_1, C_2)\) are those defined in Section 3.1 and \(0 < \gamma \leq 1/2\).

**Proof.** We begin by writing

\[
\|(\tilde{u}_1|\partial_M, \tilde{v}_1^* \ast \hat{\omega}_1) - (\tilde{u}_2|\partial_M, \tilde{v}_2^* \ast \hat{\omega}_2)\|_{H^{1/2}(\Sigma(\partial_M))}
\]

\[
= \|(\tilde{F}_1u_1 - F_2u_2)|\partial_M\|_{H^{1/2}(\partial_M)} + \|\tilde{v}_1^* \ast (\tilde{F}_1^{-1} \omega_1 - F_2^{-1} \omega_2)\|_{H^{-1/2}(\Lambda^1(\partial_M))}
\]

Starting with the first term

\[
\|(\tilde{F}_1u_1 - F_2u_2)|\partial_M\|_{H^{1/2}(\partial_M)}
\]

\[
\leq \|\tilde{F}_1|\partial_M\|_{C^1(\partial_M)}\|u_1 - u_2\|_{\partial_M}\|_{H^{1/2}(\partial_M)}
\]

\[
+ \|(\tilde{F}_1 - F_2)|\partial_M\|_{C^1(\partial_M)}\|u_2\|_{\partial_M}\|_{H^{1/2}(\partial_M)}
\]

\[
\leq C \left( \|u_1 - u_2\|_{\partial_M}\|_{H^{1/2}(\partial_M)} + \|u_2\|_{\partial_M}\|_{H^{1/2}(\partial_M)} \right)
\]

where the last inequality follows from Lemma 5.2. Similarly, the second term satisfies

\[
\|\tilde{v}_1^* \ast (\tilde{F}_1^{-1} \omega_1 - F_2^{-1} \omega_2)\|_{H^{1/2}(\Lambda^1(\partial_M))}
\]

\[
= \left\| \tilde{v}_1^* \left( \tilde{F}_1 \tilde{F}_1^{-1} \ast \omega_1 - \tilde{F}_2 \tilde{F}_2^{-1} \ast \omega_2 \right) \right\|_{H^{1/2}(\Lambda^1(\partial_M))}
\]

\[
\leq C \left( \|\tilde{v}_1^* \ast (\tilde{F}_1 \ast \omega_1 - \tilde{F}_2 \ast \omega_2)\|_{H^{1/2}(\Lambda^1(\partial_M))} + \|\tilde{v}_1^* \ast \omega_2\|_{H^{1/2}(\Lambda^1(\partial_M))} \right)
\]

\[
\leq C \left( \|\tilde{v}_1^* \ast (\omega_1 - \omega_2)\|_{H^{1/2}(\Lambda^1(\partial_M))} + \|\tilde{v}_1^* \ast \omega_2\|_{H^{1/2}(\Lambda^1(\partial_M))} \right).
\]
Adding 40 and 41 we get
\[
\|((u_1 - \hat{u}_2)|_{\partial M}, t^*_\partial M \ast (\hat{\omega}_1 - \hat{\omega}_2))\|_{H^{1/2}(\Sigma(\partial M))} \\
\leq C \|((u_1 - u_2)|_{\partial M}, t^*_\partial M \ast (\omega_1 - \omega_2))\|_{H^{1/2}(\Sigma(\partial M))} \\
+ C \frac{\|(u_2|_{\partial M}, t^*_\partial M \ast \omega_2)\|_{H^{1/2}(\Sigma(\partial M))}}{\log d(C_1, C_2)^\gamma}.
\]
This implies that
\[
\|((u_1 - \hat{u}_2)|_{\partial M}, t^*_\partial M \ast (\hat{\omega}_1 - \hat{\omega}_2))\|_{H^{1/2}(\Sigma(\partial M))} \\
\leq C \left( \|((u_1 - u_2)|_{\partial M}, t^*_\partial M \ast (\omega_1 - \omega_2))\|_{H^{1/2}(\Sigma(\partial M))} + \frac{1}{\log d(C_1, C_2)^\gamma} \right),
\]
since $F_x^{-1}$ is bounded from below and $\|t^*_\partial M \ast \omega_2\|_{H^{-1/2}(\partial M)} \leq C \|u_2|_{\partial M}\|_{H^{1/2}(\partial M)}$. As we have a bijective correspondence between solutions to the $L_j, V_j$ and $\tilde{V}_j$ problems, it follows that
\[
d'(C_{V_1}, C_{V_2}) \leq C \left( d(C_1, C_2) + \frac{1}{\log d(C_1, C_2)^\gamma} \right)
\]
which is what we wanted to prove.

The important conclusion of Lemma 5.3 is that small differences in Cauchy data for the non-diagonalized $V_j$-case implies small differences in Cauchy data for the diagonal counterparts. This will allow us to deduce stability by considering the latter case.

5.3. Estimates for the potential $q$ and magnetic field $dX$. We are now ready to complete the proof of Theorem 1.1 (and 1). Recall our a priori assumptions 4:
\[
\|q_j\|_{W^{s,p}(M)} \leq K, \quad \|X_j\|_{W^{s,p}(T^* M)} \leq K, \quad j = 1, 2.
\]
If $C_j$ are the Cauchy data spaces as defined in 3 for the corresponding magnetic Schrödinger operators $L_j := L_{X_j,q_j}$, as defined in 1-2. Then if the distance $d(C_1, C_2)$ is small enough, there is an $\alpha \in (0, 1/2)$ such that
\[
\|q_1 - q_2\|_{L^2(M)} + \|d(X_1 - X_2)\|_{L^2(\Sigma(M))} \leq C \frac{1}{\log^\alpha \log \frac{1}{d(C_1, C_2)}},
\]
where $C = C(K, M, \alpha)$.

We will split up the proof into three parts, using our reduction to a diagonal system as described in the above sections.

Suppose that $\tilde{U}^1_h, \tilde{U}^2_h$ are the earlier constructed $H^1$-solutions to
\[
(D + \tilde{V}_1)\tilde{U}^1_h = 0, \quad (D + \tilde{V}_2^*)\tilde{U}^2_h = 0
\]
respectively and that $\tilde{U}_h$ solves $(D + \tilde{V}_2)\tilde{U}_h = 0$. By Lemma 3.1 we then have
\[
\langle \tilde{V}_2 - \tilde{V}_1, \tilde{U}^1_h, \tilde{U}^2_h, L^2(\Sigma(M)) \rangle = \langle \tilde{U}^1_h - \tilde{U}_h, \tilde{U}^2_h, \tilde{U}^1_h \rangle_{\partial M}.
\]
To abbreviate we introduce
\[
\tilde{\Psi} = \begin{bmatrix} 
\tilde{Q} & 0 \\
0 & \tilde{F} 
\end{bmatrix} = \begin{bmatrix} 
|F_2|^{-2}Q_2/2 - |F_1|^{-2}Q_1/2 & 0 \\
0 & -|F_2|^2 + |F_1|^2 
\end{bmatrix} = \tilde{V}_2 - \tilde{V}_1.
\]
Lemma 5.4. For $p_0 \in M \setminus N_{\sqrt{\delta}}$ there is an $\varepsilon > 0$ (that can be chosen uniformly with respect to $p_0$) such that

$$\max\{|\tilde{Q}(p_0)|, |\tilde{F}(p_0)|\} \leq \frac{C}{\delta^\varepsilon(\log(1/H))^{\gamma}},$$

where $H = |\log d(C_1, C_2)|^{-\gamma}$, and $C = C(K, M, p, \gamma, \varepsilon), \delta > 0, 0 < \gamma \leq 1/2$.

Proof. We will consider two cases, first when the solutions $\tilde{U}_h$ are of the forms

$$\tilde{U}_1 = \left(\frac{e^{\Phi/h}(a_1 + r_1)}{e^{\Phi/h}s_1}\right), \quad \tilde{U}_2 = \left(\frac{e^{-\Phi/h}(a_2 + r_2)}{e^{-\Phi/h}s_2}\right),$$

where $a_1, a_2$ are holomorphic functions. Then we have

$$\langle \tilde{V} \tilde{U}_1, \tilde{U}_2 \rangle_{L^2(\Sigma(\tilde{M}))} = \int_M \tilde{Q} e^{2\psi/h}(a_1 \tilde{a}_2 + a_1 \tilde{r}_2 + r_1 \tilde{a}_2 + r_1 \tilde{r}_2) dV_g + \tilde{F} e^{-2\psi/h}s_1 \wedge \tilde{s}_2.$$

In particular if $a_1 = a_2 = a$ we claim that we can estimate, for $h$ small,

$$\int_M \tilde{Q} e^{2\psi/h}(ar_1 \tilde{r}_2 + r_1 \tilde{r}_2) dV_g + \tilde{F} e^{-2\psi/h}s_1 \wedge \tilde{s}_2 \leq C h^{1+\varepsilon}. \tag{42}$$

Furthermore, if $\psi$ has a non-degenerate stationary point at $p_0 \in M$, we claim that

$$\left|\int_M \tilde{Q} e^{2\psi/h}|a|^2 dV_g - \frac{\delta}{\delta h} e^{2\psi(p_0)/h} \tilde{Q}(p_0)|a(p_0)|^2\right| \leq C h^2 \delta^{\varepsilon}, \tag{43}$$

where $C$ depends on the a priori bounds on the potentials.

Similarly, if we instead consider solutions

$$\tilde{U}_1 = \left(\frac{e^{\Phi/h}r_1}{e^{\Phi/h}(b_1 + s_1)}\right), \quad \tilde{U}_2 = \left(\frac{e^{-\Phi/h}r_2}{e^{-\Phi/h}(b_2 + s_2)}\right),$$

where $b_1, b_2$ are antiholomorphic 1-forms. Then

$$\langle \tilde{V} \tilde{U}_1, \tilde{U}_2 \rangle_{L^2(\Sigma(\tilde{M}))} = \int_M \tilde{Q} e^{2\psi/h}r_1 \tilde{r}_2 dV_g + \tilde{F} e^{-2\psi/h}(b_1 \wedge \tilde{s}_2 + b_1 \wedge \tilde{r}_2 + s_1 \wedge \tilde{b}_2 + s_1 \wedge \tilde{r}_2).$$

In particular if $b_1 = b_2 = b$ we claim that we can, similarly as for (42), estimate, for $h$ small,

$$\int_M \tilde{Q} e^{2\psi/h}r_1 \tilde{r}_2 dV_g + \tilde{F} e^{-2\psi/h}(b \wedge \tilde{s}_2 + b \wedge \tilde{r}_2 + s_1 \wedge \tilde{b}_2 + s_1 \wedge \tilde{r}_2) \leq C h^{1+\varepsilon}. \tag{44}$$

Again, if $\psi$ has a non-degenerate stationary point at $p_0 \in M$,

$$\left|\int_M \tilde{F} e^{-2\psi/h}b \wedge \tilde{s}_2 - \frac{\delta}{\delta h} e^{-2\psi(p_0)/h} \tilde{F}(p_0)|b(p_0)|^2\right| \leq C h^2 \delta^{\varepsilon}, \tag{45}$$

where $C$ depends on the a priori bounds on the potentials.

From the discussion in Section 4 and Lemma 5.3, we also have the estimate

$$|\langle \tilde{V}_2 - \tilde{V}_1, \tilde{U}_1 \rangle_{L^2(\Sigma(\tilde{M}))}| \leq C e^{c/h} H,$$

where we recall that

$$H = \frac{1}{|\log d(C_1, C_2)|^\gamma}, \quad \text{for some } 0 < \gamma \leq 1/2.$$
Choosing $\delta$ and use Lemma 2.6 and Proposition 1. Similarly one shows 42. There is an Lemma 5.5.

Combining either 42-43 or 44-45 with 46, we get
\[ |C \text{he}^{2iv(p_0)/h} \hat{Q}(p_0) |a(p_0)|^2 | \leq \frac{C}{\delta^7} (h^{1+\varepsilon} + e^{c/H}), \]
\[ |D \text{he}^{-2iv(p_0)/h} \hat{F}(p_0) |b(p_0)|^2 | \leq \frac{C}{\delta^7} (h^{1+\varepsilon} + e^{c/H}), \]
Equivalently,
\[ \max \{|\hat{Q}(p_0)|, |\hat{F}(p_0)|\} \leq \frac{C}{\delta^7} (h^{\varepsilon} + e^{c/H}). \]

Now, since we assume that $H > 0$ is very small, we can choose $h = c((1 - \varepsilon) \log(1/H))^{-1}$ and get from 47,
\[ \max \{|\hat{Q}(p_0)|, |\hat{F}(p_0)|\} \leq \frac{C}{\delta^7(\log(1/H))^\varepsilon} \]
for some large enough $C = C(K, M, p, \varepsilon)$.

To justify 44 for the terms that are not immediately obvious we use arguments similar to those in [7], e.g.
\[ \int_M \tilde{F} e^{-2iv/h} b \wedge \ast s^2 \tilde{F} = \int_M \tilde{F} e^{-2iv/h} b \wedge \ast -\tilde{\delta}^{-1} (Q_2 r_0^2) \]
\[ = -\int_M \tilde{\delta}^{-1} (\tilde{F} e^{-2iv/h} b) \wedge \ast Q_2 r_0^2 \]
and use Lemma 2.6 and Proposition 1. Similarly one shows 42. \qed

Next we will prove an $L^2$-estimate where we take care of those exceptional points that are not included in Lemma 5.4.

**Lemma 5.5.** There is an $\varepsilon > 0$ such that
\[ \| |F_1| - |F_2| \|_{L^2(M)} \leq \frac{C}{|\log H|^{2\varepsilon/15}}, \quad \text{where} \quad H = \frac{1}{|\log d(C_1, C_2)|^{\gamma}}, \]
$C = C(K, M, p, \gamma, \varepsilon)$ and $0 < \gamma \leq 1/2$.

**Proof.** By Lemma 5.4
\[ |\tilde{F}(p)| = |F_1(p)|^2 - |F_2(p)|^2 | \leq \frac{C}{\delta^7(\log H)^\varepsilon}, \quad p \in M \setminus N_{\sqrt{3}}. \]

It follows from this and the priori assumptions 4 that there is a $C = C(K, M, p, \gamma, \varepsilon)$ such that
\[ \| |F_1|^2 - |F_2|^2 \|_{L^2(M)} \leq \int_{M \setminus N_{\sqrt{3}}} |F_1|^2 - |F_2|^2^2 dV_g + \int_{N_{\sqrt{3}}} |F_1|^2 - |F_2|^2^2 dV_g \]
\[ \leq V_g(M) \left( \frac{C}{\delta^7(\log H)^\varepsilon} \right)^2 + C\delta^2 \leq C(\delta^{-14}|\log H|^{-2\varepsilon} + \delta). \]

Choosing $\delta = |\log H|^{-2\varepsilon/15}$, we get
\[ \| |F_1|^2 - |F_2|^2 |_{L^2(M)} \leq C|\log H|^{-2\varepsilon/15}. \]

Using that
\[ |F_1| - |F_2| = \frac{|F_1|^2 - |F_2|^2}{|F_1| + |F_2|} \]
together with the fact that \(|F_1| + |F_2|\) is uniformly bounded from below we can then also to conclude that
\[
\| |F_1| - |F_2| \|_{L^2(M)} \leq C |\log H|^{-2\varepsilon/15}.
\]

In the last steps of the proof we indicate how to also get higher Sobolev regularity estimates.

**Proof of Theorem 1.1/1.2.** Since we assume that \(X_j \in T^* M, j = 1, 2\) are real-valued, we can decompose \(X_j = A_j + \overline{A}_j\), and then
\[
dX_j = dA_j + d\overline{A}_j = d\pi_{0,1} A_j + d\pi_{1,0} \overline{A}_j = \partial A_j + \overline{\partial \overline{A}_j}.
\]
Let now \(\alpha_j\) be the primitive functions we introduced earlier, that is in the sense that \(\overline{\partial} = \partial A_j\) and \(F_j = e^{i\alpha_j}\). Then
\[
\partial F_j = i \partial \alpha_j F_j, \quad \overline{\partial} F_j = i A_j F_j,
\]
\[
\partial \overline{\partial} F_j = -i A_j F_j, \quad \overline{\partial} \partial F_j = -i \overline{\partial} \overline{\partial} F_j.
\]
Observe that \(-\overline{\partial} \overline{\partial} = \partial \overline{\partial} \) on functions so \(\partial \overline{\partial} \alpha_j = \partial A_j\) and \(-\overline{\partial} \partial \overline{\partial} = \overline{\partial} \overline{\partial} \overline{\partial} = \overline{\partial} \overline{\partial}\).

Next, under the assumptions that \(\varepsilon < \varepsilon / 15, 0 < s < 2 + \eta\), one can see that
\[
\|Q_j\|_{H^{2+n}(M)} = \|Q_j\|_{H^{s}(M)} \leq C|\log H|^{-\varepsilon'},
\]
\[
\|Q_j\|_{H^{s}(M)} \leq C|\log H|^{-\varepsilon'},
\]
for some \(0 < \varepsilon' < 2(1 - \alpha)\varepsilon / 15, \alpha > 0\) and \(0 < s < 2 + \eta\). Then from 48 we can conclude that there is an \(s' \geq 0\) so that
\[
\|d(X_1 - X_2)\|_{H^{s'}(M)} \leq C|\log H|^{-\varepsilon'}.
\]
Similarly we want to give an estimate for the potentials \(q_j\), starting from
\[
\|Q_j\|_{L^2(M)} = \|Q_j\|_{L^2(M)} \leq C|\log H|^{-2\varepsilon/15},
\]
where \(Q_j := -\star dX_j + q_j, j = 1, 2\). Some simple algebra and using that the \(|F_j|\)
are bounded away from zero one can see that
\[
\|Q_1 - Q_2\|_{L^2(M)} = \|q_1 - q_2 - \star d(X_1 - X_2)\|_{L^2(M)} \leq C|\log H|^{-2\varepsilon/15}.
\]
from which it simply follows, similarly as above, that
\[ \|q_1 - q_2\|_{H^{s'}(M)} \leq C|\log H|^{-\varepsilon}. \]
For \( s' = 0 \) we get Theorem 1.1, and \( \eta > 1 \) would allow for larger \( s' > 0 \).

6. Stability for the holonomy. We finally focus our attention towards the holonomy of the 1-form \( X := X_1 - X_2 \). We can for every closed loop \( \gamma \) based at any \( m \in M \), consider parallel transport for the connection \( \nabla^X = \partial + iX \) on the bundle \( M \times \mathbb{C} \). This defines an isomorphism \( P^X_\gamma : \mathbb{C} \to \mathbb{C} \), so we may view \( P^X_\gamma \) as a non-zero complex number and define the holonomy group of \( \nabla^X \) at \( m \) by
\[ \text{Hol}(\nabla^X, m) := \{ P^X_\gamma \in \mathbb{C} \setminus \{0\}; \gamma \text{ is a closed loop based at } m \}. \]
For real \( X \), \( P^X_\gamma \) is in fact unitary and it can be observed that \( P^X_\gamma = e^{\int \gamma \cdot J} \). When the curvature \( \text{d}X = 0 \) there is a natural group morphism
\[ \rho^X_m : \pi_1(M, m) \to \text{Hol}(\nabla^X, m), \]
where \( \pi_1(M, m) \) is the first fundamental group, consisting of equivalence classes of closed curves up to homotopy. The morphism \( \rho^X \) is called the holonomy representation into \( GL(\mathbb{C}) \) and it is trivial if and only if \( P^X_\gamma = 1 \) for all closed loops based at \( m \), independent of \( m \). We will show Theorem 1.2, or more precisely:
\[ \inf_{k \in \mathbb{Z}} \left| \int_\gamma X - 2\pi k \right| \leq |\gamma| |\omega(d(C_1, C_2))|, \]
where \( \omega = \omega(x) = C(\log |\log x|)^{-\varepsilon} \to 0 \) as \( x \to 0 \), \( C = C(K, Mp, \varepsilon) \). Thus the holonomy representation is in this sense close to being trivial.

\textbf{Proof of Theorem 1.2.} Let us again consider the functions \( F_j \) and define the new function
\[ \Theta := F_1 F_2^{-1}. \]
Then \( \partial \Theta = i(A_1 - A_2) \Theta \) and hence \( \partial \Theta^{-1} = i(\overline{A_1} - \overline{A_2}) \Theta^{-1} \). This implies that
\[ \frac{\partial \Theta}{\Theta} + \frac{\partial \Theta^{-1}}{\Theta^{-1}} = i(A_1 - A_2) + i(\overline{A_1} - \overline{A_2}) = i(X_1 - X_2) = iX. \]
By Sobolev embedding, \( \|F_1\|^2 - |F_2|^2 \leq \omega \), so
\[ |F_1 F_1 - F_2 F_2| < \omega. \]
Multiplying the above inequality with \( |F_1^{-1} F_2^{-1}| \) we equivalently have
\[ |F_1 F_2^{-1} - F_2 F_1^{-1}| = |\Theta - \Theta^{-1}| < \omega, \]
since the \( F_j^{-1} \) are bounded away from zero. So we may write
\[ \Theta^{-1} = \Theta + \theta, \quad |\theta| < \omega. \]
Thus we have,
\[ \frac{\partial \Theta}{\Theta} + \frac{\partial \Theta^{-1}}{\Theta^{-1}} = \frac{\partial \Theta}{\Theta} + \frac{\partial (\Theta + \theta)}{\Theta + \theta} = \frac{\partial \Theta}{\Theta} + \frac{\partial \Theta - \partial \theta}{\Theta(\Theta + \theta)} = \frac{d\Theta}{\Theta} + \bar{\Theta} = iX, \]
where \( |\bar{\theta}| \lesssim \omega \) by Sobolev embedding. (Here \( X_j \in W^{2,p}, p > 2 \) implies that \( F_j \in W^{3,p} \subset C^{1,\alpha} \) for some \( \alpha > 0 \).)
Let now $\gamma$ be any closed curve on $M$, it will be made up of a finite number of non-self-intersecting loops. So without loss of generality we can assume that $\gamma$ is made up of a single simple loop. We will show that for any such loop,

$$\int_{\gamma} \frac{d\Theta}{\Theta} \in 2\pi i \mathbb{Z},$$

and hence it will follow that

$$\inf_{k \in \mathbb{Z}} \left| \int_{\gamma} X - 2\pi k \right| \leq \left| \int_{\gamma} \frac{d\Theta}{\Theta} \right| = \left| \int_{\gamma} \dot{\theta} \right| \leq |\gamma| \omega,$$

which is what we want to show.

To show 49, suppose we remove a small subarc $\gamma_{q,p}$ between two points $q, p \in \gamma$, so that we are left with another arc $\gamma_{p,q}$ consisting of the remaining points on $\gamma$ between $p$ and $q$. Suppose then that we take a simply connected tubular neighborhood $N(\gamma_{p,q})$ around $\gamma_{p,q}$. In this neighborhood,

$$g(q) := \int_{p}^{q} \frac{d\Theta}{\Theta},$$

is well-defined (since the 1-form $\frac{d\Theta}{\Theta}$ is closed) and by the fundamental theorem of calculus,

$$dg(q) = \frac{d\Theta}{\Theta}(q).$$

Furthermore, $g(p) = 0$ so we can try to find a unique solution to this initial value problem. Take a parametrization of $\gamma = \gamma(t)$ so that $\gamma(0) = p, \gamma(1) = q$. Let $f(t) := \Theta(\gamma(t))$, then $f'(t) = \langle d\Theta(\gamma(t)), \gamma'(t) \rangle = f(t)\langle dg(\gamma(t)), \gamma'(t) \rangle$. Now $f(0) = \Theta(p)$ and solving the initial value problem for $f$ hence yields

$$f(t) = \Theta(p)e^{g(\gamma(t))}.$$

Then $\Theta(q) = \Theta(p)e^{g(q)}$. Taking the limit as $q \to p$ (strictly enlarging the neighborhood $N(\gamma_{p,q})$) we thus find

$$1 = \exp \left( \lim_{q \to p} \int_{p}^{q} \frac{d\Theta}{\Theta} \right).$$

We can then conclude that 49 must indeed be true and the theorem follows. \qed

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