Self-Accelerating Matter Waves

C. Yuce

Department of Physics, Anadolu University, Turkey

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The free particle Schrödinger equation admits a non-trivial self-accelerating Airy wave packet solution. Recently, the Airy beams that freely accelerate in space was experimentally realized in the field of atomic matter waves. Here we present self-accelerating waves for the Bose-Einstein condensate in a time dependent harmonic oscillator potential. We show that parity and time reversal symmetries for self accelerating waves are spontaneously broken.

I. INTRODUCTION

In 1979, Berry and Balazs theoretically showed that the Schrödinger equation describing a free particle admits a non-trivial Airy wave packet solution. This free particle wave packet is unique in the sense that it accelerates. Furthermore, the Airy wave packet doesn’t spread out as it accelerates. The Airy wave packet is also called self-accelerating wave packet since it accelerates in the absence of an external potential. The accelerating behavior is not consistent with the Ehrenfest theorem, which describes the motion of the center of mass of the wave packet. The reason of this inconsistency is the non-integrability of the Airy function. The Berry and Balazs’ paper initiated extensive theoretical investigation on such waves. The self-accelerating Airy wave packet was also experimentally realized within the context of optics three decades after its theoretical prediction. Since then, self-accelerating wave packets have stimulated growing research interest. Soon after the first experimental realization of the self-accelerating Airy beams in optics, the same types of waves using free electrons instead of light was generated. Using a nanoscale hologram technique, the Airy wave packet for electrons is obtained by diffraction of electrons. The self-accelerating and non-spreading solution is not restricted to the linear Schrödinger equation and nonlinear generalization was considered by some authors. The ultracold systems are used to study the effects in quantum and statistical physics both theoretically and experimentally. Experimental achievements and theoretical studies of Bose Einstein condensates (BEC) of weakly interacting atoms have stimulated intensive interest in the field of atomic matter waves. Recently, self accelerating matter waves are considered in the absence of external potential. In experiments of BEC, harmonic potential is mostly used to trap atoms. Therefore self accelerating matter waves in the presence of external harmonic potential is worth studying. In this paper, we find and examine self accelerating wave packets for ultracold atoms in a time dependent harmonic trap. We derive a formula for self-acceleration of such waves and show that the self-acceleration depends on the initial form of the wave packet.

II. SELF ACCELERATING AIRY WAVES

In the present study, we consider ultracold atoms in a time dependent trap, which can be realizable experimentally. The physics of ultracold gases in a time dependent trap was theoretically investigated more than a decade ago. Another physical freedom on the study of ultracold gases is the temporal tunability of nonlinear interaction. Some experiments on ultracold experiments have demonstrated that tuning of the nonlinear interaction strength can be achieved by applying an external magnetic field, known as the Feshbach resonance. With these experimental degrees of freedom, it is possible to study self-accelerating waves for ultracold atoms. The physics of ultracold atoms is well described by the Gross Pitaevskii (GP) equation. The 1-D GP equation for a BEC with time dependent nonlinear interaction strength in a time dependent harmonic trap reads

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2(t)}{2} x^2 + g(t)|\psi|^2 \right) \psi \]  

where \( \omega(t) \) and \( g(t) \) are the time dependent angular frequency and nonlinear interaction strength, respectively. We assume that the nonlinear interaction strength is positive for all time. In the non-interacting limit, the system obeys the Ehrenfest’s theorem provided that the wave function is square integrable. According to the Ehrenfest’s theorem, the acceleration of wave packet can be found using the equation

\[ \frac{d^2 <x>} {dt^2} + \omega^2(t) <x> = 0. \]

If either the wave packet is initially displaced from the origin, i.e., \( <x(0)> \neq 0 \) or the initial velocity of the wave packet is different from zero, i.e., \( <\dot{x}(0)> = 0 \), then the wave packet accelerates. Although the additional presence of the nonlinear interaction changes the density profile, it has nothing to do with the acceleration of the wave packet. The Ehrenfest’s theorem shows that the expectation value obeys the classical dynamical laws. Therefore the acceleration of the wave packet matches the acceleration of the classical particle moving in one dimension under the influence of harmonic potential. This statement is true only when the wave packet is square integrable. Below we will find non-integrable wave packet solution of the GP equation. In other words, we are looking for a wave packet that doesn’t obey the Ehrenfest’s theorem. To get self-accelerating wave packet solution, let us first
rewrite the GP equation in an accelerating frame,
\[ x' = \frac{x - x_c(t)}{L(t)} \]  
(2)

where the time dependent function \( x_c(t) \) describes translation and \( L(t) \) is a time dependent dimensionless scale factor to be determined later. More precisely, we will see that the width of the wave packet changes according to \( L(t) \). Initially, we take \( L(0) = 1 \). Under this coordinate transformation, the time derivative operator transforms as \( \partial_t \rightarrow \partial_t - (\dot{L} x' + \dot{x}_c)/L \partial_{x'} \), where dot denotes time derivation.

In the accelerating frame, we will seek the solution of the form
\[ \psi(x', t) = \frac{1}{\sqrt{L}} e^{i\Lambda(x', t)} \psi(x') \],
(3)

where the position dependent phase reads \( \Lambda(x', t) = \frac{m}{\hbar} \left( ax' + \frac{\beta}{2} x'^2 + S \right) \) and the time dependent functions are given by \( \alpha(t) = L \dot{x}_c, \beta(t) = L \ddot{L} \)

and \( \dot{S}(t) = \frac{1}{2} x'^2 - \frac{\omega^2}{2} x'^2 \). Substitute these transformations into the GP equation and assume that the following equations are satisfied
\[ \ddot{L} + \omega^2(t) L = 0, \]
\[ \ddot{x}_c + \omega^2(t) x_c = \frac{a_0}{L^3}, \]
(5)

where \( a_0 > 0 \) is an arbitrary constant in units of acceleration that can be experimentally manipulated as we shall see below.

Suppose the nonlinear interaction changes according to \( g(t) = g_0 / L(t) \), where \( g_0 > 0 \) is a constant. Then the time dependent GP equation is transformed to the time independent second Painlevé equation
\[ \left( -\frac{d^2}{dx'^2} + \frac{2m^2a_0}{\hbar^2} x' + G_0 |\psi|^2 \right) \psi = 0. \]

where \( G_0 = 2m/\hbar^2 g_0 \).

One can find a non-integrable stationary solution of this equation. By transforming backwards, the wave packet in the lab. frame can be obtained. However, such a non-integrable solution can’t be used to find the acceleration of the wave packet, \( \langle x' \rangle \). Fortunately, the equations describe time dependent width and acceleration of the original wave packet. Before getting stationary solutions of the second Painlevé equation, let us discuss these equations in more detail.

The equation for \( L(t) \) implies that spreading of the wave-packet is uniquely determined by the external time dependent harmonic potential. The wave packet remains non-dispersive in the absence of external potential, \( \omega = 0 \). Depending on the angular frequency \( \omega(t) \), one can also obtain breathing or expanding/contracting wave packet solution. The equation implies that the motion of the wave packet depends not only on the external harmonic potential but also on the scale factor \( L(t) \) and the constant \( a_0 \). The constant \( a_0 \) plays a vital role for self accelerating waves since wave packets follow the trajectory of the classical motion in the limit \( a_0 = 0 \). The self-acceleration depends also on \( L(t) \), which is determined by the external potential. The presence of external potential expands/contracts the self accelerating wave and the expansion/contraction contributes acceleration. It is interesting to note that an initial phase of the wave function plays a role on the self-acceleration. As can be seen from the equation, different initial conditions \( L(0) \) lead to different scale functions \( L(t) \) and this contributes self acceleration according to \( 1 \). In other words, imprinting the phase on the initial wave function, i.e., \( \beta(t = 0) \neq 0 \), changes the self-acceleration of the wave packet.

Let us now discuss the solutions of the equations for some certain cases. The self accelerating solution obtained by Berry and Balasz is recovered when \( \omega(t) = 0 \). In this case, the scale function reads \( L(t) = 1 + ut \), where the constant \( u \) measures how fast the width of the wave packet changes (Berry and Balasz considered only the special case with \( u = 0 \)). If \( u = 0 \), which is possible by preparing the initial wave function such that \( \beta(0) \) vanishes, the intensity profile of the wave packet remains invariant while it experiences constant acceleration. In this case, \( x_c(t) = v_0 t + 1/2a_0 t^2 \), where \( v_0 \) is the initial velocity and acceleration matches the constant \( a_0 \). Observe that acceleration is not due to the external force. If \( u \neq 0 \), then the acceleration changes in time and accelerating wave packet expands/shrinks ballistically depending on the sign of \( u \).

Consider next the constant harmonic potential, \( \omega^2(t) = \omega_0^2 \). In this case, no physically acceptable self accelerating solution exists since \( x' \) becomes singular when the scale function \( L(t) = \cos(\omega_0 t) \) equals to zero. If the harmonic potential is inverted, \( \omega^2(t) = -\omega_0^2 \), the self accelerating wave packet expands or contracts exponentially \( L(t) = e^{\mp \omega_0 t} \) depending on the initial condition \( \beta(0) = \mp \omega_0 t \).

Consider now time dependent harmonic potential. An interesting case is the non-accelerating wave packet solution in the presence of time dependent potential. Let us set \( \dot{x}_c = 0 \) in the coupled equations. We find that the acceleration of the wave packet is zero if \( \omega^2(t) = \frac{2v_0^2}{9(x_0 + v_0 t)^2} \), where \( v_0^2 = 9/2 \ a_0 \) and \( x_0 \) is an arbitrary constant. This result contradicts with the Ehrenfest’s theorem, which states that the acceleration of the moving wavepacket is zero only for the free particle case.

Instead of fixing the acceleration, one can also start with a pre-determined scale factor. As an example, one can study breathing wave packet by assuming \( L(t) = 1 + \epsilon \sin(\Omega t) \), where \( \epsilon \) is a small parameter and the constant \( \Omega \) is the angular frequency. We numerically find that acceleration oscillates between positive
and negative values with growing consecutive peaks. Finally, let us study the evolution of wave packets for a given trap frequency. As an example, suppose \( \omega^2 \) changes periodically as 
\[
\omega^2 = -2(1 + 0.2 \cos(10\pi t)).
\]
We solve the equations \( \text{(4,5)} \) with the initial conditions 
\[
x_c(0) = 0.5 \text{µm}, x_e(0) = L(0) = 0.
\]
We take \( a_0 = 0.1 \text{mm}/s^2 \), \( \epsilon = 0.2 \), \( \omega^2 = 2 \text{ Hz} \) and \( \Omega = 10\pi \text{ Hz} \).

In the noninteracting limit, \( G_0 = 0 \), an exact analytic solution is available. Let us find the stationary solution of the equation \( \text{(6)} \).

Let us first set the boundary conditions. The nonlinearity action changes the density profile of the self-accelerating Airy wave. The corresponding Hamiltonian for our system remains invariant under both the parity \( P \) operation and time reversal \( T \) operation provided that \( \omega(-t) = \mp \omega(t) \). However, the Airy function is not an even or odd function of position. Moreover, the time dependent function \( x_e(t) \) and the phase \( \Lambda(t) \) are not in general symmetric under time reversal. One can see from the solution \( \text{(3)} \) that \( P \) and \( T \) symmetries are spontaneously broken for self-accelerating waves.

So far we have considered the non-interacting limit where an exact analytical solution is available. Let us now study self-accelerating waves for the interacting case. It is worth saying that the relations \( \text{(5,6)} \) are not affected by the nonlinear interaction. Instead, the nonlinear interaction changes the density profile of the self-accelerating wave. We now aim at finding the solution of \( \text{(6)} \). Since no exact solution is available, we solve it numerically. Let us first set the boundary conditions. The nonlinear interaction is dominant around the main lobe and gets weaker away from the main lobe. Since the density and consequently the nonlinear interaction go to zero as \( x' \to \infty \), the solution of \( \text{(6)} \) asymptotically coincides with the Airy function at large positive \( x' \). Therefore we set the boundary condition 
\[
\psi^{'} \sim \lambda \text{Ai}((2m^2a_0/h^2)^{1/3}x') \quad \text{and} \quad \partial_{x'}\psi^{'} \sim \lambda \partial_{x'}\text{Ai}((2m^2a_0/h^2)^{1/3}x') \quad \text{at large} \quad x' \quad \text{to solve the second Painlevé equation numerically. Before going fur-}
\]
ther, we remark that the constant $\lambda$ is a nontrivial degree of freedom in the interacting case. Increasing $G_0$ at fixed $a_0$ and $\lambda$ increases the peak value of the main lobe slightly. If $G_0$ is very close to a critical value, $G_c$, then the peak value changes drastically. At exactly the critical value, $G_0 = G_c = \lambda^{-2} \frac{2m^2 a_0}{\hbar^2}^{3/2}$, the peak value goes to infinity. Therefore the solution of the equation (2) is not bounded if $G_0$ exceeds the critical value, $G_c$.

The Fig-2 plots the density of the self-accelerating solution at $\lambda = 5 \times 10^4$ for $G_0 = 500 m^{-1}$ (thin curve) and $G_0 = 0$ (thick curve). As expected, the effect of the nonlinearity can be mostly seen on the main lobe since the nonlinear interaction is weak for large $|x'|$. The peak value of the main lobe is increased by positive nonlinear interactions and the wave packet is extended with respect to noninteracting case. Furthermore, the main lobe is shifted to the left when compared to the linear one. Note that no normalization is made on these solutions since they are not integrable. To this end, we would like to emphasize that time evolution of the nonlinear solutions since they are not integrable. To this end, we would like to emphasize that time evolution of the nonlinear solution is exactly the same as the Airy wave packet (8).

In other words, the solution given in the Fig.2 translates while expanding/contracting in time according to (2), where $x_s(t)$ and $L(t)$ are the same for both noninteracting and interacting cases. We have shown that non-integrable wave packets lead to surprising physical results. However, realizing non-integrable wave function is impossible in practice. Introducing an exponential aperture function is one possible way for the physical realization of such waves $\psi \rightarrow \Psi = e^{\epsilon N} \psi$, where the decay factor $\epsilon > 0$ is a positive parameter to guarantee the convergence of the function on the negative branch. The Fourier transform of the truncated Airy wave packet is proportional to $-e^{-\epsilon k_0^2} e^{i(k/6a_0)} k^3$. In photonics community, the Airy wave was experimentally obtained by passing a Gaussian beam through a phase mask that adds a cubic phase modulation. Since the wave packet is truncated, acceleration maintains for a finite distance only. The decay factor $\epsilon$ determines how long the wave packet maintains its acceleration. For $\epsilon << 1$, accelerating behavior of the ideal Airy wave packet was shown to be observed.

To summarize, we have presented self-accelerating matter waves for the harmonically trapped condensate. We have derived formulas for self-acceleration and shown that the initial form of the wave function and an initial phase contribute the self-acceleration. We have discussed that nonlinear interaction has nothing to do with the self acceleration but changes the density profile of the self-accelerating wave. It is worth studying self-accelerating waves in an optical lattice potential.
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