Solutions of Conformal Turbulence on a Half Plane

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ABSTRACT

Exact solutions of conformal turbulence restricted on a upper half plane are obtained. We show that the inertial range of homogeneous and isotropic turbulence with constant enstrophy flux develops in a distant region from the boundary. Thus in the presence of an anisotropic boundary, these exact solutions of turbulence generalize Kolmogorov’s solution consistently and differ from the Polyakov’s bulk case which requires a fine tuning of coefficients. The simplest solution in our case is given by the minimal model of $p = 2, q = 33$ and moreover we find a fixed point of solutions when $p, q$ become large.
In the theory of turbulence, Kolmogorov’s theory of the inertial range provides a simple but powerful tool in obtaining the scaling laws of locally isotropic turbulence.\(^1\) Nevertheless, the microscopic, field theoretic foundation of the scaling law so far has been lacking. Recently, Polyakov has proposed a completely new approach to this problem based on the assumption of conformal invariance of velocity correlators in the case of two dimensional fully developed turbulence.\(^2,3\) He took the non-unitary minimal conformal field theory as a field theoretic framework for turbulence and obtained exact solutions of turbulence with specific scaling behaviors, different from the Kolmogorov’s one, which seem to fit better with experimental results.\(^4\) However, due to the nature of non-unitarity, one point functions of operators in the theory do not necessarily vanish unlike the unitary case. The arbitrariness of one point functions in the Polyakov’s approach makes the theory largely undetermined and remains as an open problem.

In this letter, we show that one point functions are determined by boundary conditions, as suggested by Polyakov, in particular when they are confined on a upper half plane. This allows us to give a field theoretic account for the Kolmogorov’s hypothesis of inertial range.\(^5\) We show that the inertial range of homogeneous and isotropic turbulence with constant enstrophy flux develops in a distant region from the boundary. Moreover we argue that exact solutions obtained by Polyakov and others\(^6-9\) for the bulk case require an unnatural fine tuning of coefficients. Without such a fine tuning, we obtained series of solutions of conformal turbulence which are different from the bulk case. The simplest solution in our case is given by the minimal model of \(p = 2, q = 33\). More interestingly, we find a fixed point of conformal solutions when \(p, q\) become very large whose energy spectrum exponent is \(-67/15 \sim -4.4667\).

The basic assumption of Polyakov’s conformal turbulence is to identify the stream function \(\psi\), arising from the velocity vector \(v_i\) of incompressible fluid such that \(v_i = \epsilon_{ij}\partial_j \psi\), with a certain primary operator of a non-unitary minimal model which, in the inviscid and static case, satisfies the Navier-Stokes equation as well as the constant enstrophy condition. The use of two dimensional conformal field
theory (2-d CFT) allows us to compute exactly the velocity correlation functions, thereby the kinetic energy of fluid motion. In particular, the velocity two point function can be obtained directly from the stream two point function by differentiation, for example, 
\[ \langle v_x(z_1)v_x(z_2) \rangle = \partial_y \partial_{y_2} \langle \psi(z_1)\psi(z_2) \rangle; z = x + iy. \]

In principle, the two point function of stream function can be obtained exactly. However, in practice, for the brevity of our discussion and also in order to avoid the difficulty in solving for exact solutions, we consider only the short distance behavior of the velocity two point function which can be obtained easily from the operator product expansion (OPE) of two \( \psi \)'s;

\[ \langle \psi(z_1)\psi(z_2) \rangle = \sum_i C_{\psi\psi}^{\phi_i} \frac{\langle \phi_i \rangle}{|z_1 - z_2|^{4\Delta_\psi - 2\Delta_{\phi_i}}} + \text{descendents}, \] (1)

where \( C_{\psi\psi}^{\phi_i} \) are structure constants and \( \phi_i \) are primary operators of conformal dimension \( \Delta_{\phi_i} \). The one point functions \( \langle \phi_i \rangle \) except for that of the identity operator vanish identically when the vacuum is invariant under \( SL(2,C) \). However, in the presence of large scale structures of scale \( L \), so that the vacuum is no longer invariant under \( SL(2,C) \), the one point function behaves as

\[ \langle \phi_i \rangle \sim C_i L^{-2\Delta_{\phi_i}}. \] (2)

For the unitary case, \( \Delta_{\phi_i} \geq 0 \) and \( \langle \phi_i \rangle \) approaches zero when \( L \) becomes large. However this is no longer true for the non-unitary model where \( \Delta_{\phi_i} \) could be negative. Moreover non zero values of one point functions modify significantly other correlation functions so that they become essential ingredients in defining a non-unitary field theory.

In order to understand the role of the one point function and its physical implications more clearly, we first note that they can be determined exactly by a specific boundary condition.\[11\] Consider, for example, the two dimensional turbulence restricted on a upper half plane; \( \text{Im } z = y > 0 \). A typical boundary condition
is such that the normal component of the velocity vector vanishes at the boundary. In order to maintain the spirit of 2-d CFT approach to turbulence on a upper half plane, we assume that the conformal symmetry of correlation functions persists in the presence of boundary, but now in a reduced form which preserves the boundary condition. This is tantamount to reducing $SL(2, C)$ to $SL(2, R)$ in the case of the global conformal transformation so that only real translations and dilatations are involved. The domain under consideration, the upper half plane, is invariant under global real conformal transformations of $SL(2, R)/Z_2$ with an added point at infinity. This boundary condition also requires correlation functions to be invariant under $SL(2, R)$. In the case of two-point function, one could obtain this by forming an $SL(2, R)$ invariant $u$

$$u = \frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{(z_1 - \bar{z}_2)(z_2 - \bar{z}_1)}, \quad (3)$$

where two points $z_1$ and $z_2$ are given in the upper half plane and $\bar{z}_1$ and $\bar{z}_2$ are their images in the lower half plane. In general, it is possible to show that for the degenerate conformal field theories, the $n$-point function $\langle \phi(z_1, \bar{z}_1) \cdots \phi(z_n, \bar{z}_n) \rangle_b$ can be obtained systematically from the bulk $2n$-point function, where $\langle \cdots \rangle_b$ denotes correlation functions in the upper half plane.\textsuperscript{[10]} This is so because the $n$-point function $\langle \phi(z_1, \bar{z}_1) \cdots \phi(z_n, \bar{z}_n) \rangle_b$ satisfies the same differential equation as does the bulk $2n$ point function consisting of charges in the upper half plane as well as their images in the lower half plane. In particular, one point functions on a upper half plane can be obtained exactly from the known bulk two point functions such that

$$\langle \phi_i(z) \rangle_b = d_{\phi_i} y^{-2\Delta_{\phi_i}}; \quad y = \text{Im} z, \quad (4)$$

where $d_{\phi_i}$ are arbitrary constants. These constants $d_{\phi_i}$, as well as the negative energy states ($\Delta_{\phi_i} < 0$) in non-unitary minimal models, seem to be unphysical at first sight. However, they acquire physical meaning when the non-unitary model is applied for an effective description of an open physical system. In our case, $d_{\phi_i}$ obtain physical meaning through the velocity correlators and become physical
parameters of turbulence. From Eq.(4), for example, we have an average velocity profile;

\[ \langle v_y \rangle_b = -\partial_x \langle \psi \rangle_b = 0; \quad \langle v_x \rangle_b = \partial_y \langle \psi \rangle_b = -2d_\psi \Delta_\psi y^{-2\Delta_\psi - 1}, \quad (5) \]

where \( d_\psi \) is a parameter which controls the magnitude of average horizontal velocity. Another example where \( d_{\phi_i} \) manifest physical meaning is the kinetic energy of turbulent fluid. It is commonly expressed in terms of the energy density at point \((x, y)\) given by

\[ \frac{1}{2} \langle v_\alpha^2 (x, y) \rangle_b = \int dk E_{(x,y)}(k). \quad (6) \]

In the momentum space, the energy spectrum \( E_{(x,y)}(k) \) is given in terms of velocity correlators:

\[ E_{(x,y)}(k) = \frac{1}{8\pi^2} \int d^2 x' e^{ik\alpha(x-x')_\alpha} \langle v_{\beta}(x', y') v_{\beta}(x, y) \rangle_b. \quad (7) \]

For a large \( k \), \( E_{(x,y)}(k) \) can be obtained directly from Eqs.(4) & (1) such that

\[ E_{(x,y)}(k) \sim \sum_i C_{\psi \psi}^{\phi_i} k^{4\Delta_\psi - 2\Delta_{\phi_i} + 1} \langle \phi_i \rangle_b + \text{descendents} \]

\[ = \sum_i C_{\psi \psi}^{\phi_i} k^{4\Delta_\psi - 2\Delta_{\phi_i} + 1} d_{\phi_i} y^{-2\Delta_{\phi_i}} + \text{descendents}. \quad (8) \]

Thus we have obtained an explicit expression of the energy spectrum as a function of the distance from the boundary. Unlike the bulk case, the spectrum is made of a sum of different powers controlled by \( \Delta_{\phi_i} \) and \( d_{\phi_i} \). The appearance of too many parameters, in general, is not desirable in the statistical description of turbulence. However, if we move far away from the boundary, i.e. when \( y \) becomes very large, only the first few terms in the series of Eq.(8) dominate and the theory is controlled by only few parameters. As we will discuss below, this is precisely in agreement with the traditional Kolmogorov’s idea of the inertial range.
The main idea of Kolmogorov is first to assume that with sufficiently large Reynolds number, turbulence becomes locally isotropic – i.e. small scale velocity field fluctuations are statistically homogeneous, isotropic, and stationary – regardless of the form of a finite space-time region bounding turbulence. Then Kolmogorov suggested that for locally isotropic turbulence, the energy spectrum in the intermediate scale (inertial range) depends only on the constant flux of energy (or enstrophy for the two dimensional turbulence\textsuperscript{[13]}). In our case, we note that for sufficiently large $y$, conformal turbulence becomes locally isotropic. This is so because even in the presence of the anisotropic half plane boundary, the small scale fluctuations of velocity field become isotropic when $y$ becomes large. In order to see this explicitly, one could compare for example a small scale velocity fluctuations $\langle v_x(x, y)v_x(x, y + \epsilon) \rangle_b$ with its $90^\circ$ rotation $\langle v_y(x, y)v_y(x + \epsilon, y) \rangle_b$; $\epsilon \ll 1$. In the leading order, they both agree to

$$\langle v_x(x, y)v_x(x, y + \epsilon) \rangle_b \approx \langle v_y(x, y)v_y(x + \epsilon, y) \rangle_b \approx \frac{2C_{\psi\psi}^\phi(2\Delta_\psi - \Delta_\phi)d_\phi}{y^{2\Delta_\phi}\epsilon^{4\Delta_\phi - 2\Delta_\psi + 2}}, \quad (9)$$

where $\phi$ is the minimal dimension operator of conformal dimension $\Delta_\phi$ coming from the operator product $[\psi][\psi]$. On the other hand, the constant enstrophy condition which follows from the $r$-independence of $\langle \dot{w}(r)w(0) \rangle$, can be met by a single condition when $y$ is large

$$\Delta_\phi + \Delta_\psi - \Delta_\chi = -3. \quad (10)$$

where $\chi$ is the minimal dimension operator coming from the OPE of $[\psi][\phi]$\textsuperscript{*}. In case $d_\chi = 0$, the constant enstrophy condition changes into

$$\Delta_\phi + \Delta_\psi = -3. \quad (11)$$

Therefore, we contend that the inertial range forms in a distant region from the

\textsuperscript{*} The minimal dimensional operator $\chi$ from the OPE of $[\psi][\phi]$ is not necessarily the same as the one coming from $[\psi][\psi][\psi]$, which were falsely identified in Ref.\textsuperscript{[3]}. In many cases, they are equal but as in the Table 1, $M_{5,73} M_{5,77}$ are exceptions. In the case of $M_{5,73}$, $\psi = \Phi_{(2,26)}$, $\phi = \Phi_{(1,15)}$, $\chi = \Phi_{(4,57)}$ and $\chi = [\phi][\psi]$ but $\chi \neq [\psi][\psi][\psi]$.\n
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boundary in the context of conformal turbulence while conformal turbulence generalizes turbulence to non-inertial regions. In general, the inertial range develops regardless the shape of a particular boundary. For example one point functions on a disk of radius $R$ can be obtained by finite conformal transformation of Eq.(4) such that

$$
\langle \phi_i(r) \rangle_{\text{circle}} = -\frac{(2R)^{2\Delta_{\phi}} d_{\phi_i}}{(R^2 - r^2)^{2\Delta_{\phi_i}}}.
$$

This again shows that the minimal dimension operator dominates when $R$ becomes large and $R \ll r$.

Having exemplified the role of one point functions, we note that there is no a priori reason to set parameters $d_{\phi_i}$ to zero. It occurs only in the specific case of $SL(2, C)$ invariant vacuum or by an unnatural fine tuning. Thus, it is more natural to assume that $d_{\chi} \neq 0$ in our case and use Eq.(10) for the constant enstrophy condition. This together with the Navier-Stokes condition $\Delta_{\phi} > 2\Delta_{\psi}$ leads us to the new set of solutions of conformal turbulence on a upper half plane which are listed in the Table 1. In the numerical computation, we have used the following notations and rules of 2-d CFT; $M_{p,q}$ denotes a minimal model with $p, q$ relatively prime and $p < q$, which is characterized by $1/2(p-1)(q-1)$ degenerate primary operators $\Phi_{m,n}(1 \leq n \leq q, \ 1 \leq m \leq p)$ of conformal dimension

$$
\Delta_{m,n} = \frac{(pn - qm)^2 - (p - q)^2}{4pq},
$$

and the fusion rules between two primaries;

$$
\Phi_{m_1,n_1} \times \Phi_{m_2,n_2} = \sum_{i=|m_1-m_2|+1}^{r} \sum_{j=|n_1-n_2|+1}^{s} C^{(i,j)}_{(m_1,n_1)(m_2,n_2)} \Phi_{i,j},
$$

with $r = \min(m_1 + m_2 - 1, 2p - m_1 - m_2 - 1)$, $s = \min(n_1 + n_2 - 1, 2q - n_1 - n_2 - 1)$, and $C^{(i,j)}_{(m_1,n_1)(m_2,n_2)}$ are structure coefficients and $i, j$ run over odd or even numbers if it is bounded by odd or even numbers respectively.
Finally, a couple of comments related to new solutions are in order.

It is intriguing to observe that $\psi = \Phi_{(2,26)}$ occurs repeatedly with increasing frequency as we change $p$ and $q$; $2 \leq p \leq 25$, $p \leq q \leq 500$ all of whose energy spectrum exponents lie in between $-4$ and $-5$. Indeed, there exists a fixed point when $p \to \infty$ and $q/p \to 15$. In which case $\psi = \Phi_{(2,26)}$, $\Delta \psi = -3$; $\phi = \Phi_{(1,15)}$, $\Delta \phi = -49/15$; $\chi = \Phi_{(2,30)}$, $\Delta \chi = -49/15$, and the energy spectrum exponent is $-67/15$. The physical relevance of this fixed point is unknown.

Table 1 shows that the energy spectrum exponents can be greater than $-3$ whereas in the bulk case they were strictly less than $-3$. This could be understood as a boundary effect to the energy spectrum. Different type of boundaries; e.g. different shapes (strip or disk) or periodic boundary conditions can also modify the one point functions and the energy spectrum. Details will appear in Ref.[12].

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Table 1: Some Solutions for $\Delta_\psi + \Delta_\phi - \Delta_\chi = -3$

| $(p, q)$ | $\psi$ | $\phi$ | $\chi$ | exponent | $(p, q)$ | $\psi$ | $\phi$ | $\chi$ | exponent |
|---------|--------|--------|--------|----------|---------|--------|--------|--------|----------|
| (2, 33) | (1, 10)| (1, 17)| (1, 16)| $-3.727273$ | (3, 43) | (1, 12)| (1, 15)| (1, 14)| $-4.837209$ |
| (3, 44) | (2, 26)| (1, 15)| (2, 30)| $-4.613636$ | (3, 46) | (2, 26)| (1, 15)| (2, 30)| $-4.282609$ |
| (3, 52) | (2, 27)| (1, 17)| (2, 35)| $-3.307692$ | (3, 67) | (2, 31)| (1, 23)| (2, 45)| $-0.835821$ |
| (3, 121)| (1, 12)| (1, 23)| (1, 34)| $-2.000000$ | (3, 169)| (1, 14)| (1, 27)| (1, 40)| $-2.000000$ |
| (3, 196)| (1, 15)| (1, 29)| (1, 43)| $-2.000000$ | (3, 256)| (1, 17)| (1, 33)| (1, 49)| $-2.000000$ |
| (3, 289)| (1, 18)| (1, 35)| (1, 52)| $-2.000000$ | (3, 361)| (1, 20)| (1, 39)| (1, 58)| $-2.000000$ |
| (3, 400)| (1, 21)| (1, 41)| (1, 61)| $-2.000000$ | (3, 484)| (1, 23)| (1, 45)| (1, 67)| $-2.000000$ |
| (4, 59) | (2, 26)| (1, 15)| (2, 30)| $-4.580508$ | (4, 61) | (2, 26)| (1, 15)| (2, 30)| $-4.331967$ |
| (4, 75) | (2, 28)| (1, 19)| (2, 38)| $-2.590000$ | (4, 95) | (3, 56)| (1, 23)| (3, 72)| $-0.115789$ |
| (5, 71) | (4, 55)| (1, 15)| (4, 57)| $-4.929577$ | (5, 72) | (4, 55)| (1, 15)| (4, 57)| $-4.777778$ |
| (5, 73) | (2, 26)| (1, 15)| (2, 30)| $-4.638356$ | (5, 74) | (2, 26)| (1, 15)| (2, 30)| $-4.559459$ |
| (5, 76) | (2, 26)| (1, 15)| (2, 30)| $-4.360526$ | (5, 77) | (2, 26)| (1, 15)| (2, 30)| $-4.241558$ |
| (5, 91) | (4, 64)| (1, 19)| (4, 72)| $-2.890110$ | (5, 106)| (2, 30)| (1, 21)| (2, 42)| $-1.371698$ |
| (5, 114)| (4, 77)| (1, 23)| (4, 91)| $-0.578947$ | (5, 124)| (3, 58)| (1, 25)| (3, 74)| $0.424194$ |