HOMOTOPY DECOMPOSITION OF A SUSPENDED REAL TORIC SPACE

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Abstract. We give $p$-local homotopy decompositions of the suspensions of real toric spaces for odd primes $p$. Our decomposition is compatible with the one given by Bahri, Bendersky, Cohen, and Gitler for the suspension of the corresponding real moment-angle complex, or more generally, the polyhedral product. As an application, we obtain a stable rigidity property for real toric spaces.

1. Introduction

For a simplicial complex $K$ on $m$-vertices $[m] = \{1, \ldots, m\}$ the real moment-angle complex $\mathbb{R}Z_K$ (or the polyhedral product $(D^1, S^0)^K$) of $K$ is defined as follows:

$$\mathbb{R}Z_K = (D^1, S^0)^K := \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma\},$$

where $D^1 = [0, 1]$ is the unit interval and $S^0 = \{0, 1\}$ is its boundary. It should be noted that $\mathbb{R}Z_K$ is a topological manifold if $K$ is a simplicial sphere [2, Lemma 6.13], and that there is a canonical $\mathbb{F}_2^n$-action on $\mathbb{R}Z_K$ which comes from the $\mathbb{F}_2$-action on the pair $(D^1, S^0)$.

Let $n \leq m$. A map $\lambda: V = [m] \to \mathbb{F}_2^n$ is called a (mod 2) characteristic function of $K$ if it has the property that

$$\lambda(i_1), \ldots, \lambda(i_t) \text{ are linearly independent in } \mathbb{F}_2^n \text{ if } \{i_1, \ldots, i_t\} \in K.$$

For convenience, a characteristic function $\lambda$ is frequently represented by an $(n \times m)$ $\mathbb{F}_2$-matrix $\Lambda = (\lambda(1) \cdots \lambda(m))$, called a characteristic matrix. Define a map $\theta: [m] \to \mathbb{F}_2^n$ so that $\theta(i)$ is the $i$-th coordinate vector of $\mathbb{F}_2^n$. Then the homomorphism $\Lambda$ (viewed as a matrix multiplication) satisfies $\Lambda \circ \theta = \lambda$. We will see in Lemma 3.1 that Condition (1) ensures that the group $\ker \Lambda \cong \mathbb{F}_2^{m-n}$ acts freely on $\mathbb{R}Z_K$. We denote by $M_\lambda$ the associated real toric space, which is defined to be $\mathbb{R}Z_K/\ker \Lambda$. If $K$ is a polytopal $(n - 1)$-sphere then $M_\lambda$ is known as a...
small cover and if $K$ is a star-shaped $(n - 1)$-sphere then $M_\lambda$ is known as a real topological toric manifold.

In [1, Theorem 2.21] it is shown that there is a homotopy equivalence

$$\Sigma R^*_K \simeq \bigvee_{I \in K} \Sigma |K_I|,$$

where $K_I$ is the full subcomplex of $K$ on the vertex set $I$ and $|K_I|$ is its geometric realization. In this short note, we give an analogous odd primary decomposition of the suspension of $M_\lambda$.

**Theorem 1.1.** Let $M_\lambda$ be a real toric space. Localized at an odd prime $p$ or the rationals (denoted by $p = 0$) there is a homotopy equivalence

$$\Sigma(M_\lambda) \simeq_p \bigvee_{I \in \text{Row}(\lambda)} \Sigma|K_I|,$$

where $\text{Row}(\lambda)$ is the space of $m$-dimensional $\mathbb{F}_2$-vectors spanned by the rows of $\Lambda$ associated to $\lambda$.

The restriction to odd primes arises because the free action of $\ker \Lambda$ on $R^*_K$ implies that when $|\ker \Lambda|$ is inverted in a coefficient ring $R$ then the quotient map $R^*_K \to M_\lambda$ induces an injection in cohomology with image the invariant subring $H^*(R^*_K; R)^{\ker \Lambda}$. This will be used to help analyze the topology of $R^*_K$. As $|\ker \Lambda|$ has order a power of 2 we can take $R$ to be $\mathbb{Z}(p)$ or $\mathbb{Q}$. In fact, Theorem 1.1 fails when $p = 2$ in simple cases. For example, if $K$ is the boundary of a triangle and $\lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, then $M_\lambda = \mathbb{R}P^2$ but each $\Sigma|K_I|$ is contractible.

There is a large class of complexes $K$ (see Section 4) where the homotopy equivalence (2) desuspends. For such a $K$ there is a homotopy equivalence $M_\lambda \simeq_p \bigvee_{I \in \text{Row}(\lambda)} \Sigma|K_I|$.

Recent work of Yu [11] gave a different decomposition of the suspension of certain quotient spaces of $R^*_K$. He considers a homomorphism $\Lambda : \mathbb{F}_2^m \to \mathbb{F}_2^n$ which is associated to a partition on the vertices of $K$, and proves that $\Sigma R^*_K/\ker \Lambda$ decomposes analogously to the Bahri, Bendersky, Cohen and Gitler decomposition. Yu’s decomposition has the advantage of working integrally and also for some non-free actions, but it has the disadvantage of working only for particular homomorphisms $\Lambda$. Our decomposition, by contrast, works only after localizing at an odd prime but holds for all characteristic maps derived from free actions.

## 2. Polyhedral product and its stable decomposition

Let us first recall Bahri, Bendersky, Cohen and Gitler’s argument in [1]. To make it more clear, we present it in its full polyhedral product form. Let $K$ be a simplicial complex on the vertex set $[m]$ and for $1 \leq i \leq m$ let $(X_i, A_i)$ be pairs of pointed $CW$-complexes. If $\sigma$ is a face of $K$ let

$$(X, A)^\sigma = \prod_{i=1}^m Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in K \\ A_i & \text{if } i \notin K. \end{cases}$$
The polyhedral product is
\[(X, A)^K = \bigcup_{\sigma \in K} (X, A)^{\sigma}.\]

Notice that \((X, A)^K\) is a subspace of the product \(\prod_{i=1}^m X_i\). There is a canonical quotient map from the product to the smash product, \(\prod_{i=1}^m X_i \to \wedge_{i=1}^m X_i\). The smash polyhedral product \(\overbrace{(X, A)}^K\) is the image of the composite \((X, A)^K \to \prod_{i=1}^m X_i \to \wedge_{i=1}^m X_i\). In particular, mapping onto the image gives a map \((X, A)^K \to \overbrace{(X, A)}^K\).

Let \(I \subset [m]\). There is an induced inclusion \(K_I \to K\), which in turn induces a map of polyhedral products \((X, A)^K \to (X, A)^{K_I}\). We then obtain a composition into a smash polyhedral product:
\[p_I : (X, A)^K \to (X, A)^{K_I} \to \overbrace{(X, A)}^{K_I}.\]

Suspending, we can add every such composition over all full subcomplexes of \(K\), giving a composition
\[\overline{H} : \Sigma(X, A)^K \xrightarrow{\text{compul}} \bigvee_{I \subset [m]} \Sigma(X, A)^{K_I} \xrightarrow{\vee \Sigma p_I} \bigvee_{I \subset [m]} \Sigma(\overbrace{(X, A)}^{K_I}.\]

Bahri, Bendersky, Cohen and Gitler \([\text{I}]\) Theorem 2.10] show that \(\overline{H}\) is a homotopy equivalence.

Further, in the special case when each \(X_i\) is contractible, they show that there is a homotopy equivalence \(\overbrace{(X, A)}^{K_I} \simeq \Sigma(K_I \wedge \widehat{A}^I)\) \([\text{I}]\) Theorem 2.19], where \(\widehat{A}^I = \wedge_{j=1}^k A_{i_j}\) for \(I = (i_1, \ldots, i_k)\). Consequently, when each \(X_i\) is contractible the map \(\overline{H}\) specializes to a homotopy equivalence
\[H : \Sigma(X, A)^K \to \bigvee_{I \subset [m]} \Sigma(\overbrace{(X, A)}^{K_I} \xrightarrow{\vee \Sigma p_I} \bigvee_{I \subset [m]} \Sigma(\overbrace{(X, A)}^{K_I}.\]

In our case, each pair \((X_i, A_i)\) equals \((D^1, S^0)\) and \(D^1\) is contractible. As there is a homotopy equivalence \(S^0 \wedge S^0 \simeq S^0\), each \(\widehat{A}^I\) is homotopy equivalent to \(S^0\). Therefore there are homotopy equivalences
\[(3) \quad \overline{\mathbb{R}Z}_{K_I} := (\overbrace{D^1, S^0}^{K_I}) \xrightarrow{\simeq} \Sigma(K_I \wedge \widehat{A}^I) \simeq \Sigma(K_I) \wedge S^0 \simeq |K_I| \wedge S^0 \simeq |K_I| \wedge S^1 \simeq |K_I|]\n
Thus the map \(H\) becomes a homotopy equivalence
\[H : \Sigma \mathbb{R}Z_K \to \bigvee_{I \subset [m]} \Sigma \mathbb{R}Z_{K_I} \xrightarrow{\simeq} \bigvee_{I \subset [m]} \Sigma(\overbrace{\mathbb{R}Z}_{K_I}.\]

It is in this form that we will use the Bahri, Bendersky, Cohen and Gitler decomposition because, as we will see shortly, it corresponds to a module decomposition of a differential graded algebra \(R_K\) whose cohomology equals \(H^*(\mathbb{R}Z_K)\). But it is worth pointing out that in \([\text{I}]\) Theorem 2.21] it was shown that when each \(X_i\) is contractible then \(\overbrace{(X, A)}^{K_I}\) is contractible.
if $I \in K$. So the usual Bahri, Bendersky, Cohen and Gitler decomposition is of the form

$$\Sigma(\mathcal{X}, \mathcal{A})^K \simeq \bigvee_{I \in K} \Sigma^2(|K_I| \wedge \tilde{A}^I),$$

giving the special case

$$\Sigma RZ_K \simeq \bigvee_{I \notin K} \Sigma|K_I|,$$

which is the statement in [2].

3. Proof of the Main Theorem

First, recall that $M_\lambda$ is the quotient of $RZ_K$ by $\ker \Lambda$.

**Lemma 3.1.** Under Condition (1), $\ker \Lambda$ acts on $RZ_K$ freely.

**Proof.** Let $\bar{g} = (x_1, x_2, \ldots, x_m) \in RZ_K = (D^1, S^0)^K$ be the fixed point of an element $g = (g_1, g_2, \ldots, g_m) \in \ker \Lambda \subset \mathbb{F}_2^m$. This means either $g_i = 0$ or $x_i \in (D^1)^{\mathbb{F}_2} = \{1/2\}$ for all $i \in [m]$. Let $\sigma \in K$ be the maximal simplex such that $x \in (D^1, S^0)^{\sigma}$ and $\Lambda_\sigma$ be the sub-matrix of $\Lambda$ consisting of columns corresponding to $\sigma$. Let $g_\sigma$ be the sub-vector of $g$ corresponding to $\sigma$. Since $g \in \ker \Lambda$, we have

$$\Lambda g = \Lambda_\sigma g_\sigma + \Lambda_{[m] \backslash \sigma} g_{[m] \backslash \sigma} = 0.$$

Since $\mathbb{F}_2$ acts on $S^0$ freely, we have $g_i = 0$ for $i \notin \sigma$. Then, by the previous equation we have $\Lambda_\sigma g_\sigma = 0$. Therefore Condition (1) implies $g_\sigma = 0$ and we have $\bar{g} = 0$. \hfill \Box

Next, consider the following diagram

$$
\begin{array}{c}
\Sigma RZ_K \xrightarrow{H} \Sigma \bigvee_{I \subseteq [m]} RZ_{K_I} \xrightarrow{\sigma} \Sigma \bigvee_{I \subseteq [m]} \Sigma|K_I| \\
\Sigma M_\lambda \xrightarrow{\phi} \Sigma \bigvee_{I \in \text{Row}(\lambda)} RZ_{K_I} \xrightarrow{\sigma} \Sigma \bigvee_{I \in \text{Row}(\lambda)} \Sigma|K_I|
\end{array}
$$

where, by definition, $\phi = \Sigma q \circ H^{-1} \circ \Sigma \text{incl}$.

To prove Theorem 1.1 we will show that $\phi^*$ induces an isomorphism on cohomology with $\mathbb{Z}_p$-coefficients. From now on, assume that coefficients in cohomology are $\mathbb{Q}$ or $\mathbb{Z}_p$, where $p$ is an odd prime.

First, by [4] the cohomology ring of $RZ_K$ is given as follows. Let $\mathbb{Z}_p\langle u_1, \ldots, u_m, t_1, \ldots, t_m \rangle$ be the free associative algebra over the indeterminants of degree $u_i = 1, \deg t_i = 0$ ($i = 1, \ldots, m$). Define a differential graded algebra $R_K$ by

$$R_K = \mathbb{Z}_p\langle u_1, \ldots, u_m, t_1, \ldots, t_m \rangle / \langle u_\sigma \mid \sigma \notin K, u_i^2, u_iu_j + u_ju_i, u_it_i - t_i, t_iu_i, t_iu_j - u_jt_i, t_i^2 - t_i, t_it_j - t_jt_i \rangle$$

where $d(t_i) = u_i$ for each $i = 1, \ldots, m$. Then $H^*(RZ_K) = H^*(R_K)$. We shall use the notation $u_\sigma$ (respectively, $t_\sigma$) for the monomial $u_{i_1} \cdots u_{i_k}$ (respectively, $t_{i_1} \cdots t_{i_k}$) where $\sigma = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, is a subset of $[m]$. For $I \subseteq [m]$, denote by $R_{K_I}$ the differential graded sub-module
Lemma 3.2. There is an additive isomorphism
\[ H^*(R_{K_I}) \simeq \tilde{H}^*(\mathbb{R}Z_{K_I}) \]
and the projection \( p_I : \mathbb{R}Z_K \to \mathbb{R}Z_{K_I} \) induces the inclusion \( p_I^* : H^*(R_{K_I}) \hookrightarrow H^*(R_K) \).

Proof. The first assertion follows from \( \mathbb{R}Z_K \simeq \Sigma |K_I| \) (see (3)) and the isomorphism \( H^*(R_{K_I}) \simeq \tilde{H}^{*-1}(|K_I|) \) given by
\[ R_{K_I} \to C^*(K_I) \]
\[ u_{\sigma t_I \sigma} \mapsto \sigma^* \]
where \( C^*(K_I) \) is the simplicial cochain complex of \( K_I \) (see (9) in [3]).

To show the second assertion, we look more closely at the isomorphism \( H^*(R_K \simeq H^*(\mathbb{R}Z_K) \).
From the proof of [4, Theorem 1.4], the monomials \( u_{\sigma t_I \sigma} \) are mapped into the image of \( p_I^* : C^*_e(\mathbb{R}Z_{K_I}) \to C^*_e(\mathbb{R}Z_{K_I}) \), where \( C^*_e \) denotes the cellular cochain complex. By combining this with the first assertion, we deduce the second assertion. \( \square \)

Now we investigate the maps appearing in [4]. Since the action of \( \ker \Lambda \) on \( \mathbb{R}Z_K \) is free and \( |\ker \Lambda| \) is a unit in the coefficient ring \( \mathbb{Z}_p \), the map \( q^* \) is injective with image \( H^*(\mathbb{R}Z_K)^{\ker \Lambda} \). Notice that in cohomology \( incl \) induces the projection \( incl^* : \bigoplus_{I \subseteq [m]} H^*(R_{K_I}) \to \bigoplus_{I \in \text{Row}(\lambda)} H^*(R_{K_I}) \).

Recall that \( H = \Sigma \bigvee_{I \subseteq [m]} p_I \circ \comul \) and \( \phi = \Sigma q \circ \tilde{H}^{-1} \circ \Sigma incl \). So \( \phi^* \) is the composite
\[ \phi^* : H^*(\Sigma M_\Lambda) \simeq H^*(\Sigma \mathbb{R}Z_K)^{\ker \Lambda} \hookrightarrow H^*(\Sigma \mathbb{R}Z_K) \simeq \bigoplus_{I \subseteq [m]} H^*(R_{K_I}) \to \bigoplus_{I \in \text{Row}(\lambda)} H^*(R_{K_I}), \]
where \( \Sigma \) for graded modules means the degree shift in the positive degree parts.

We aim to show that \( \phi^* \) is an isomorphism. To see this, first observe that \( H^*(\mathbb{R}Z_K)^{\ker \Lambda} \simeq H^*(R_{K_{\ker \Lambda}}) \). We need two lemmas.

Lemma 3.3 ([7, Section 4]). The Reynolds operator
\[ N(x) := \frac{1}{|\ker \Lambda|} \sum_{g \in \ker \Lambda} gx \]
induces an additive isomorphism \( \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I} \simeq R_{K_{\ker \Lambda}} \), where \( R_{K_{\ker \Lambda}} \) is the \( \ker \Lambda \)-invariant ring of \( R_K \). Furthermore, for a monomial \( x = u_{\sigma t_I \sigma} \), \( N(x) \) has the unique maximal term \( x \), where the order is given by the containment of the index set. \( \square \)

Lemma 3.4. The composite
\[ \Phi : R_{K_{\ker \Lambda}} \hookrightarrow R_K \simeq \bigoplus_{I \subseteq [m]} R_{K_I} \to \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I} \]
is an isomorphism, where \( \pi \) is the projection.
Proof. We first show this is surjective. Take an element $x \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$. We induct on the size of the index set of $x$. By Lemma 3.3, the terms in $\pi(N(x) - x) \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$ has an index set strictly smaller than that for $x$. By induction hypothesis, there is an element $y \in R^\text{ker}\lambda$ such that $\Phi(y) = \pi(N(x) - x)$. Put $z = N(x) - y \in R^\text{ker}\lambda$ and we have $\Phi(z) = \pi(N(x)) - \Phi(y) = \pi(x) = x$.

On the other hand, suppose $\Phi(y) = 0$ for some $y \in R^\text{ker}\lambda$. By Lemma 3.3, there is $x \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$ such that $y = N(x)$ and $y$ must contain the maximal terms in $x$. Thus, $\Phi(y) = 0$ implies $x = 0$ and $y = N(x) = 0$.

Proof of Theorem 1.1. Since $H^*(\mathbb{R}Z_K)^\text{ker}\lambda \cong H^*(R^\text{ker}\lambda)$, the definitions of $\phi$ and $\Phi$ imply that $\Phi^* = \phi^*$. Therefore, by Lemma 3.4 $\phi^*$ is an isomorphism.

Suppose that (2) desuspends to give $\mathbb{R}Z_K \cong \bigvee_{I \in [m]} \Sigma|K_I|$. Then $\mathbb{R}Z_K$ is equipped with a co-multiplication $c : \mathbb{R}Z_K \rightarrow \bigvee_{I \in [m]} \mathbb{R}Z_K$ coming from the suspension on the right hand side. Thus, we can replace $\tilde{H}$ with its desuspension $\Sigma^{-1} \tilde{H} := \bigvee_{I \in [m]} p_I \circ c$ in the above argument to obtain

$$M_\lambda \cong_p \bigvee_{I \in \text{Row}(\lambda)} \Sigma|K_I|.$$

4. Stable rigidity of real toric spaces

In this section, we give an application of Theorem 1.1 to a stable rigidity property of real toric spaces.

Corollary 4.1. Let $M_\lambda$ be a real toric space over $K$. When $K_I$ for any $I \in \text{Row}(\lambda)$ suspends to a wedge of spheres after localization at an odd prime $p$, $\Sigma M_\lambda$ is homotopy equivalent to a wedge of spheres after localization at $p$. Let $N_\mu$ be another real toric spaces over $K'$, where $K'_I$ for any $I \in \text{Row}(\mu)$ suspends to a wedge of spheres after localization at $p$. Then, if $H^*(M_\lambda; \mathbb{F}_p) \cong H^*(N_\mu; \mathbb{F}_p)$ as modules, we have $\Sigma M_\lambda \cong_p \Sigma N_\mu$. In particular, if $M_\lambda$ is a $\mathbb{F}_p$-homology sphere (or acyclic) over such $K$, then $M_\lambda$ is homotopy equivalent to a sphere (or a point) after localization at $p$.

Real toric spaces associated to graphs. Given a connected simple graph $G$ with $n + 1$ nodes $[n + 1]$, the graph associahedron $P_G$ (6) of dimension $n$ is a convex polytope whose facets correspond to the connected subgraphs of $G$. Let $K$ be the boundary complex of $P_G$. We can describe $K$ directly from $G$: the vertex set of $K$ consists proper subsets $T \subset [n + 1]$ such that $G|_T$ are connected and the simplices are the tubings of $G$. We define a mod 2 characteristic map $\lambda_G$ on $K$ as follows:

$$\lambda_G(T) = \begin{cases} \sum_{t \in T} e_t, & \text{if } n + 1 \notin T; \\ \sum_{t \notin T} e_t, & \text{if } n + 1 \in T, \end{cases}$$

where $e_t$ is the $t$-th coordinate vector of $\mathbb{F}_2^n$. Then we have a real toric manifold $M(G) := M_{\lambda_G}$ associated to $G$. 
The signed \( a \)-number \( sa(G) \) of \( G \) is defined recursively by

\[
sa(G) = \begin{cases} 
1, & \text{if } G = \emptyset; \\
0, & \text{if } G \text{ has a connected component with odd number of nodes} \\
- \sum_{T \subseteq [n+1]} sa(G|_T), & \text{otherwise,}
\end{cases}
\]

and the \( a \)-number \( a(G) \) of \( G \) is the absolute value of \( sa(G) \). As shown in [6], there is a bijection \( \varphi \) from Row(\( \lambda G \)) to the set of subgraphs of \( G \) having an even number nodes and \( |K_I| \) for \( I \in \text{Row}(\lambda G) \) is homotopy equivalent to \( \bigvee^{a(\varphi(I))} S^{|\varphi(I)|/2-1} \) where \( |\varphi(I)| \) is the number of nodes of \( \varphi(I) \). By Theorem 1.1 we obtain the following.

**Corollary 4.2.** We have a homotopy equivalence

\[
\Sigma M(G)(p) \simeq_p \bigvee_{I \in \text{Row}(\lambda G)} a(\varphi(I)) S^{|\varphi(I)|/2+1} \quad \text{for any odd prime } p.
\]

Now, we define the \( a_i \)-number \( a_i(G) \) of \( G \) by

\[
a_i(G) = \sum_{T \subseteq [n+1]} a(G|_T).
\]

Then, \( a_i(G) \) coincides the \( i \)th Betti number \( \beta_i(M(G); \mathbb{F}_p) \) of \( M(G) \). It should be noted that, by Corollary 4.2, if two graphs \( G_1 \) and \( G_2 \) have the same \( a_i \)-numbers for all \( i \)'s, then \( \Sigma M(G_1) \simeq_p \Sigma M(G_2) \) for any odd prime \( p \).

**Example 4.3.** Let \( P_4 \) be a path graph of length 3, and \( K_{1,3} \) a tree with one internal node and 3 leaves (known as a claw). One can compute \( a_i(G) := \sum_{T \subseteq [n+1]} a(G|_T) \) as follows:

\[
a_0(P_4) = a_0(K_{1,3}) = 1 \\
a_1(P_4) = a_1(K_{1,3}) = 3 \\
a_2(P_4) = a_2(K_{1,3}) = 2 \\
a_i(P_4) = a_i(K_{1,3}) = 0 \quad \text{for } i > 2.
\]

Hence, by Corollary 4.2, \( \Sigma M(P_4) \simeq_p \Sigma M(K_{1,3}) \) for any odd prime \( p \) although \( \Sigma M(P_4) \) and \( \Sigma M(K_{1,3}) \) are not homotopy equivalent since they have different mod-2 cohomology.

**Real toric spaces over fillable complexes.** There is a wide class of simplicial complexes on which every real toric space satisfies the assumption in Corollary 4.1.

**Definition 4.4** ([10, Definition 4.8]). Let \( K \) be a simplicial complex. Let \( K_1, \ldots, K_s \) be the connected components of \( K \), and let \( \hat{K}_i \) be a simplicial complex obtained from \( K_i \) by adding all of its minimal non-faces. Then \( K \) is said to be \( \mathbb{F}_p \)-\textit{homology fillable} if (1) for each \( i \) there are minimal non-faces \( M_1^i, \ldots, M_r^i \) of \( K \) such that \( K_i \cup M_1^i \cup \cdots \cup M_r^i \) is acyclic over \( \mathbb{F}_p \), and (2) \( \hat{K}_i \) is simply connected for each \( i \).
Moreover, we say that $K$ is \textit{totally $\mathbb{F}_p$-homology fillable} when $K_I$ is $\mathbb{F}_p$-homology fillable for any $\emptyset \neq I \subset [m]$.

**Proposition 4.5** ([10 Proposition 4.15]). If $K$ is $\mathbb{F}_p$-homology fillable, then $\Sigma |K|_{(p)}$ is a wedge of $p$-local spheres. \hfill $\square$

**Theorem 4.6** ([10 Corollary 4.7]). If $K$ is totally $\mathbb{F}_p$-homology fillable, the equivalence \eqref{equiv} desuspends after localization at $p$. \hfill $\square$

We immediately obtain the following two corollaries.

**Corollary 4.7.** If $K$ is totally $\mathbb{F}_p$-homology fillable, the equivalence in Theorem 1.1 desuspends and $M_\lambda$ is homotopy equivalent to a wedge of spheres after localization at an odd prime $p$. \hfill $\square$

**Corollary 4.8.** Let $M_\lambda$ and $N_\mu$ be real toric spaces over totally $\mathbb{F}_p$-homology fillable complexes $K$ and $K'$. If $H^*(M_\lambda; \mathbb{F}_p) \simeq H^*(N_\mu; \mathbb{F}_p)$ as modules, we have $M_\lambda \simeq_p N_\mu$. \hfill $\square$

There is a large class of simplicial complexes which are totally homology fillable.

**Proposition 4.9** ([10 Propositions 5.18 and 5.19]). If the Alexander dual of $K$ is sequentially Cohen-Macaulay over $\mathbb{F}_p$ \cite{3}, then $K$ is totally $\mathbb{F}_p$-homology fillable.

Note that the Alexander duals of shifted and shellable simplicial complexes are sequentially Cohen-Macaulay over $\mathbb{F}_p$.

**References**

[1] A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler, \textit{The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces}, Adv. Math. 225 (2010), no. 3, 1634–1668.

[2] V. M. Buchstaber and T. E. Panov, \textit{Torus actions and their applications in topology and combinatorics}, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002.

[3] A. Björner, M. Wachs, and V. Welker, \textit{On sequentially Cohen-Macaulay complexes and posets}, Israel J. Math. 169 (2009), 295-316.

[4] L. Cai, \textit{On the cohomology of polyhedral products with space pairs $(D^1, S^0)$}, arXiv:1301.1518.

[5] M. Carr and S. Devadoss, \textit{Coxeter complexes and graph-associahedra}, Topology Appl. 153 (2006) 2155–2168.

[6] S. Choi and H. Park, \textit{A new graph invariant arises in toric topology}, to appear in J. Math. Soc. Japan; arXiv:1210.3776.

[7] S. Choi and H. Park, \textit{On the cohomology and their torsion of real toric objects}, arXiv:1311.7056.

[8] M. W. Davis and T. Januszkiewicz, \textit{Convex polytopes, Coxeter orbifolds and torus actions}, Duke Math. J. 62 (1991), no. 2, 417–451.

[9] H. Ishida, Y. Fukukawa and M. Masuda, \textit{Topological toric manifolds}, Moscow Math. J. 13 (2013), no. 1, 57–98.

[10] K. Iriye and D. Kishimoto, \textit{The fat wedge filtration and a homotopy decomposition of a polyhedral product}, arXiv:1412.4866.

[11] L. Yu, \textit{On a class of quotient spaces of moment-angle complexes}, arXiv:1406.7392.
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