We verify the Rota–Heron–Welsh conjecture for matroids realizable as $c$-arrangements: the coefficients of the characteristic polynomial of the associated matroid are log-concave. This family of matroids strictly contains that of complex hyperplane arrangements. Our proof combines the study of intrinsic volumes of certain extensions of arrangements and the Lévy–Milman measure concentration phenomenon on realization spaces of arrangements.

In generalization of Birkhoff’s chromatic polynomial of a graph [Bir13], one defines for a matroid $M$ of rank $r$ the characteristic polynomial

$$\chi(M; \lambda) := \sum_{x \in L_M} \mu(x) \lambda^{r(M) - \text{rk}(x)} = \gamma_0(M) \lambda^r - \gamma_1(M) \lambda^{r-1} + \cdots + (-1)^r \gamma_r(M),$$

where $L_M$ is the intersection poset or lattice of flats of $M$ with Mőbius function $\mu(x) = \mu_{L_M}(\hat{0}, x)$ and rank function $\text{rk}(\cdot)$. The coefficients $\gamma_i(M)$ — the (unsigned) Whitney numbers of the first kind — carry a variety of combinatorial information of $M$ and have been subject to extensive study; see, for example, Chapters 7 and 8 of [Whi87]. The coefficients $\gamma_i$ coincide with the Betti numbers of the Orlik-Solomon algebra associated to $M$, and they are closely related to Milnor numbers and Chern–Schwartz–MacPherson classes of complements of complex hyperplane arrangements. This paper is devoted to the following property of characteristic polynomials of matroids:

**Rota–Heron–Welsh conjecture.** For any matroid $M$, the coefficients of the characteristic polynomial $\chi(M; \lambda)$ are log-concave, that is,

$$\gamma_{i-1}(M) \cdot \gamma_{i+1}(M) \leq \gamma_i(M)^2$$

for all $1 \leq i \leq n - 1$.

By Rota’s sign theorem [Rot64], $\gamma_i(M) > 0$ for all $i$ and hence the conjecture implies that the sequence of Whitney numbers is unimodal, i.e.,

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{i-1} \leq \gamma_i \geq \gamma_{i+1} \geq \cdots \geq \gamma_{n-1} \geq \gamma_n$$

for some $0 \leq i \leq n$. Following Aigner [Aig87], we define the absolute characteristic polynomial of $M$

$$\psi(M; \lambda) = \gamma_0(M) \lambda^r + \gamma_1(M) \lambda^{r-1} + \cdots + \gamma_r(M).$$

Spectacular progress towards a resolution of the conjecture has been achieved by Huh [Huh12] for matroids that can be realized over a field of characteristic 0 and in full generality by Adiprasito–Huh–Katz [AHK15]. The proof in [AHK15] is set in algebraic geometry and the aim of this note is to prove the following weaker result by appealing to methods from convex geometry.

**Theorem 1.** If $M$ is a matroid realizable by a $c$-arrangement, then the sequence of Whitney numbers $\gamma_0(M), \gamma_1(M), \ldots, \gamma_n(M)$ is log-concave.
Here, a \( c \)-arrangement is a collection of codimension-\( c \) linear subspaces of \( \mathbb{R}^d \) all whose non-empty intersections have codimension divisible by \( c \) [GM88, Part III]. It is easy to check that (rank functions of) \( c \)-arrangements give matroids; see Section 1. For \( c = 2 \), \( c \)-arrangements include complex hyperplane arrangements but are strictly more general (see [Zie93]). In particular, there are matroids not realizable over any field that can be realized as \( c \)-arrangements [GM88, Sec. III.5.2]. In this sense, Theorem 1 is not a complete resolution of HRW-conjecture. For example, it is known that the Vámos matroid [Oxl92, Example 2.1.22] does not satisfy Ingleton’s inequality [Ing71] and is therefore not realizable as a \( c \)-arrangement [Bjö94]. For some related development, compare also [Adi14b], where the Lefschetz hyperplane theorem is extended from the complex-algebraic case to \( c \)-arrangements.

Whereas Huh’s proof is set in algebraic and tropical geometry (see also [HK12]), our proof is in the realm of classical convex geometry. The key idea follows a recent geometric approach to the MacPherson conjecture [Adi14a]: The main result gives a geometric representation of the Whitney numbers of a \( c \)-arrangement \( \mathcal{A} \) as the intrinsic volumes of a high-dimensional convex body. The log-concavity then simply follows from the Alexandrov–Fenchel inequalities. To establish this, we first describe what we call an extension of an arrangement (Section 2). This yields a sequence of probability spaces of arrangements. We prove that the associated convex bodies (zonotopes for hyperplane arrangements, discotopes for \( c \)-arrangements) have a Wills polynomial resembling the characteristic polynomial of \( \mathcal{A} \) asymptotically almost surely (a.a.s.) using the most basic form of Lévy–Milman measure concentration. The curiosity of this proof is underscored by the fact that arrangements in general have complicated realization spaces (see Remark 8) but a geometry that nevertheless allows for a probabilistic treatment.

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1. Convex geometry of \( c \)-arrangements

In this paper, we focus on matroids that can be realized by some central \( c \)-arrangement in \( \mathbb{R}^d \). For a general arrangement \( \mathcal{A} \) of subspaces in \( \mathbb{R}^d \), we write \( L_{\mathcal{A}} \) for the intersection poset, that is, the nonempty intersections of elements in \( \mathcal{A} \) ordered by reverse inclusion. The minimum is thus \( \mathbb{R}^d \) and if \( \mathcal{A} \) is central, then the maximum is \( \hat{1} = \bigcap_{H \in \mathcal{A}} H \). A central arrangement is essential if \( \hat{1} = \{0\} \). In analogy to hyperplane arrangements, we define the absolute characteristic polynomial of a subspace arrangement \( \mathcal{A} \) as

\[
\psi(\mathcal{A}; \lambda) := \sum_{x \in L_{\mathcal{A}}} \mu(x)(-1)^{d-\dim(x)} \lambda^{\dim(x)}
\]

where \( \mu(x) = \mu(0, x) \) is the Möbius function of \( L_{\mathcal{A}} \); see [Bjö94, Sect. 4.4]. The arrangement \( \mathcal{A} \) is a \( c \)-arrangement if all subspaces are of codimension \( c \) and \( \codim(x) \) is divisible by \( c \) for all \( x \in L_{\mathcal{A}} \). It was first noted in [GM88] that for a central \( c \)-arrangement, \( x \mapsto \frac{1}{c} \codim(x) \) is the rank function of a matroid \( M(\mathcal{A}) \).

For an element \( H \in \mathcal{A} \), we define the deletion and the contraction

\[
\mathcal{A} \setminus H := \{ H' \in \mathcal{A} : H' \not\subset H \} \quad \text{and} \quad \mathcal{A} / H := \{ H' \cap H : H' \in \mathcal{A} \setminus H \}.
\]

The absolute characteristic polynomial satisfies the deletion-contraction identity

\[
\psi(\mathcal{A}; \lambda) = \psi(\mathcal{A} \setminus H; \lambda) + \psi(\mathcal{A} / H; \lambda).
\]
If $\mathcal{A}$ is a central c-arrangement in $\mathbb{R}^d$ realizing a matroid $M$ of rank $r$, then $\psi(\mathcal{A}; \lambda) = \lambda^{d-r} \psi(M_d; \lambda^c)$. We refer the reader to Stanley’s lecture notes on hyperplane arrangements [Sta07] (see also [Sta12]) and Björner’s excellent treatment of subspace arrangements [Bjo94].

1.1. Zonotopes and discotopes. We denote by $\kappa_d = \mathrm{vol}_d(B^d) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}$ the volume of the unit $d$-ball. For an affine subspace $H \subset \mathbb{R}^d$ of dimension $k$, let us write $H^\perp$ for $(d - k)$-dimensional linear subspace orthogonal to $-p + H$ for $p \in H$. We write $n_H := \kappa_{d-k} B^d \cap H^\perp$ to denote the ball in $H^\perp$ of volume 1. To a subspace arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ in $\mathbb{R}^d$ we associate the convex body

$$Z(\mathcal{A}) := \mathbf{n}_1 + \mathbf{n}_2 + \cdots + \mathbf{n}_n.$$ If $\mathcal{A}$ is a hyperplane arrangement, then $Z(\mathcal{A})$ is the zonotope corresponding to the unit normals to the hyperplanes in $\mathcal{A}$. For subspace arrangements this is more general and $Z(\mathcal{A})$ is called the discotope of $\mathcal{A}$. In analogy, we call $n_H$ the generalized (unit) normal of $H$ or the $k$-normal if we want to emphasize the dimension $k = \dim H$.

1.2. Wills polynomials and normal cones. Let $K$ denote any closed $r$-dimensional convex body in $\mathbb{R}^d$ and let $B^d$ be the unit ball. Steiner’s formula asserts that the volume of the Minkowski sum of $K$ and the dilated ball $\lambda B^d$ is given by

$$\mathrm{vol}_d(K + \lambda B^d) = \nu_d(K) \kappa_0 + \nu_{d-1}(K) \kappa_1 \lambda + \cdots + \nu_0(K) \kappa_d \lambda^d. \quad (4)$$

This is called the Steiner polynomial of $K$. The coefficients $\nu_i(K)$, called the intrinsic volumes of $K$, will be of great importance to us. For a polytope $P \subset \mathbb{R}^d$, they have a simple interpretation: For a face $F \subseteq P$ of dimension $k$, let

$$N_F(P) := \{\omega \in \mathbb{R}^d : \omega^t x \leq \omega^t y, x \in P, y \in F\}$$

be the normal cone of $F$ at $P$ and define the external angle of $F$ at $P$ as

$$\alpha_F(P) := \frac{\mathrm{vol}_{d-k}(N_F(P) \cap B^d)}{\kappa_{d-k}}.$$ The intrinsic volumes of $P$ can now be expressed as

$$\nu_i(P) = \sum_{F \text{ i-face of } P} \alpha_F(P) \cdot \mathrm{vol}_i(F). \quad (5)$$

A central result concerning the coefficients of Steiner polynomials is the following consequence of the Alexandrov–Fenchel inequalities (cf. [Sch93]).

**Theorem 2.** The coefficients $(\nu_i(K) \kappa_{d-i})_{i=0,\ldots,d}$ of the $d$-dimensional Steiner polynomial of a $r$-dimensional convex body, $r \leq d$, form a log-concave sequence.

It is clear that the Steiner polynomial makes reference to the ambient space whereas the intrinsic volumes do not. This leads to the so-called Wills polynomial [Wil73, Had75]: For a $d$-dimensional convex body $K$ we define

$$W(K; \lambda) := \nu_d(K) + \nu_{d-1}(K) \lambda + \cdots + \nu_0(K) \lambda^d. \quad (6)$$

For a zonotope, the Wills polynomial carries quite some combinatorial information: Let $Z = \sum_{i=1}^n [-z_i, z_i]$ be a zonotope. The $k$-faces of a zonotope $Z$ can be grouped in belts. Any two $k$-faces $F_1, F_2 \subset Z$ in the same belt are translates and hence $\mathrm{vol}_k(F_1) = \mathrm{vol}_k(F_2).$ The belts of $Z$ are in bijection with the flats of the corresponding hyperplane arrangement $\mathcal{A}$. Moreover, the sum of the external angles of all $k$-faces in a belt sums to 1 and therefore

$$W(Z; \lambda) = \sum_{L \in \mathcal{L}(\mathcal{A})} \mathrm{vol}_{\dim F_L}(F) \lambda^{d - \dim F_L}. \quad (7)$$
where $F_L$ is a representative of a face of the belt corresponding to $L$. Let $[-z_i, z_i]$ be a generating segment of $Z$ and denote by $Z\setminus i$ the deletion and by $Z/i$ the contraction (i.e., projection onto $z_i^\perp$), then

$$W(Z; \lambda) = W(Z\setminus i; \lambda) + \|z_i\|W(Z/i; \lambda).$$

As an example, let $\mathcal{A}_\delta$ be the arrangement of the $d$ coordinate hyperplanes in $\mathbb{R}^d$. The corresponding zonotope $Z_d$ is a translate of the unit cube $[0, 1]^d$. Hence

$$W(Z_d; \lambda) = (1 + \lambda)^d = \sum_{i=0}^{d} \binom{d}{i} \lambda^{d-i}.$$

Observe that $W(Z_d; \lambda) = \psi(\mathcal{A}_\delta; \lambda)$. It is natural to attempt to find a convex body $K$ for every matroid $M$ such that $W(K; \lambda) = \psi(M; \lambda)$. On second thought, this is likely to fail, since the Wills polynomial encodes geometric information rather than combinatorial and for general zonotopes do not satisfy the appropriate deletion-contraction recurrence. Repairing these defects will be the purpose of this note.

It was shown by McMullen [McM91] that the intrinsic volumes are also log-concave.

**Corollary 3.** For a $d$-dimensional convex body $K$, the coefficients of the Wills polynomial $\nu_0(K), \ldots, \nu_d(K)$ form a log-concave sequence.

We repeat the proof since it fits perfectly into our setting.

**Proof.** Let $K \subset \mathbb{R}^d$ be a $d$-dimensional convex body. For every $n \geq d$, the have an isometric embedding $K \subset \mathbb{R}^n$. Thus, we can consider the coefficients of the Steiner polynomials $S_n(K; \lambda) = \text{vol}_n(K + \lambda B_n)$ for $n \to \infty$. Since the sequence $\nu_i(K) \kappa_{n-i}$ is log-concave, so is the sequence

$$\tilde{\nu}_{i,n}(K) = \nu_i(K) \cdot \kappa_{n-i} \cdot \pi^{-\frac{n^2}{2}} \sqrt{n! (\frac{n}{2\pi})^{\frac{n}{2}}}$$

By the first Stirling formula, we infer that $\tilde{\nu}_{i,n}(K) \xrightarrow{n \to \infty} \nu_i(K)$. \qed

1.3. Measure concentration. The philosophy of measure concentration makes our use of this principle quite clear: If $X$ is a random variable in a metric probability space depending on sufficiently many, sufficiently independent variables then $X$ is virtually constant. We argue here that if the normals generating an arrangement are sufficiently independent, then (the Wills polynomial of) a random arrangement is essentially independent of the realization, and hence “combinatorial”. We refer to [GM00] and [MS86] for the necessary background.

The underlying principle of measure concentration is geometric, and in this context goes back to Lévy and later Milman (cf. [GM00]), who revealed the connection to isoperimetric properties. Ultimately, we shall only need a very special case of this technology: $S^d$ with the natural angular distance $\delta$ and uniform distribution $\mu$ for hyperplane arrangements, and, more generally, the Grassmannians $G_{r,d}$ with the uniform measure $\mu$ and metric $\delta$ defined as the Hausdorff distance between unit balls. For a subset $A \subset G_{r,d}$, we denote by $A_\varepsilon = \{ x \in G_{r,d} : \delta(x, A) < \varepsilon \}$ the $\varepsilon$-neighborhood of $A$.

**Proposition 4** (cf. [MS86, Sec. 6.6]). The space of $(G_{r,d}, \delta, \mu)$ is a normal Lévy family (w.r.t. $d$), i.e. for every Borel subset $A \subset G_{r,d}$ with $\mu(A) = 1/2$, we have

$$\mu(A_\varepsilon) = 1 - \sqrt{\frac{\pi}{8}} \cdot e^{-\frac{1}{2} \delta^2}$$

for all $\varepsilon > 0$.

For the use of this proposition, note that for a metric probability space $\mathcal{X} = (X, \delta, \mu)$ with

$$\alpha_{\mathcal{X}}(\varepsilon) := 1 - \inf \{ \mu(A) : A \subset X \text{ Borel}, \mu(A) \geq \frac{1}{2} \},$$
a $r$-Lipschitz function $f$ on $X$ satisfies
\[ \mu(|f(x) - M_f| > \varepsilon) \leq 2\varepsilon \chi(\frac{\varepsilon}{r}), \]
for $M_f$ the Lévy mean of $f$. Recall that the Lévy mean $M_f$ satisfies
\[ \mu(f(x) \leq M_f) \geq \frac{1}{2} \quad \text{and} \quad \mu(f(x) \geq M_f) \geq \frac{1}{2}. \]

Let us mention another feature of measure concentration on $S^d$ (and the Grassmannian): Consider $A_{d,k}$ any $(d-k)$-dimensional totally geodesic subspace of $S^d$, endowed with its natural intrinsic metric and uniform measure $\mu$. Then there are uniform constants $\tilde{c}_k, \tilde{c}_k > 0$ such that
\[ \mu(A_{d,k}) = 1 - \tilde{c}_k e^{-\tilde{c}_k d \varepsilon^2}. \]
In addition to measure concentration, this inequality makes clear that if $A$ is a totally geodesic subspace of small dimension in $S^d$, then most of the measure lies in the orthogonal complement to $A$.

2. Extensions of arrangements and Wills polynomials of the Lévy mean

In this section we construct for every $c$-arrangement $\mathcal{A}$ a parametrized family of arrangements. Viewed as a probability space, we can use measure concentration to verify that the Wills polynomials corresponding to the associated discotopes satisfy the deletion-contraction property of characteristic polynomials and asymptotically almost surely coincide with them. Ultimately, the proof of Theorem 1 is probabilistic but the intuition of measure concentration allows for a simple enough explanation.

2.1. Uniform matroids – an illustration. The general philosophy of the proof is simple: Consider a uniformly distributed collection of $n$ random vectors in $S^{d-1} \subset \mathbb{R}^d$ for $n < d$. Let $Z_{n,d}$ be the corresponding probability space of zonotopes. What is the Wills polynomial of a typical zonotope in $Z_{n,d}$?

Clearly, for $d \gg 0$ large, measure concentration dictates that $W(Z; \lambda)$ for $Z \in Z_{n,d}$ almost surely equals the Wills polynomial of the Lévy mean, denoted by $W(n, d; \lambda)$. Moreover, for $d \to \infty$, the random vectors are essentially orthogonal to one another and the geometric deletion-contraction (8) of Wills polynomials yields
\[ W(n, d; \lambda) \propto W(n - 1, d; \lambda) + W(n - 1, d - 1; \lambda) \]
where $f \succ g :\iff |f - g| \xrightarrow{d \to \infty} 0$. For $n = d$ or $d = 1$ it is easy to verify that $W(n, d; \lambda) = (1 + \lambda)^d$.

The matroid corresponding to $n < d$ general vectors in $S^{d-1}$ is independent of the chosen vectors and is the uniform matroid $U_{n,n}$. Inspecting its characteristic polynomial now reveals that asymptotically almost surely, $W(n, d; \lambda) = \psi(U_{n,n}; \lambda)$. Log-concavity of $\psi(U_{n,d}; \lambda)$ then follows from Corollary 3.

This example illustrates the underlying idea of our proof but also pinpoints the obstacles that need to be overcome: The colinearities encoded by a typical matroid prevent an associated zonotope from being random. Consider the uniform matroid $U_{2,3}$ on three elements with rank 2. We realize it in $\mathbb{R}^{d+1}$ by choosing two unit vectors $x, y$ uniformly at random in $S^d$, and a third unit vector $z$ uniformly at random in their common span. Then $x$ and $y$ are almost orthogonal, but $z$ is not (since it is not sufficiently independent), so the Wills polynomial of a random zonotope does not concentrate.

To treat this problem, we rely on an extension operation, but one that changes the matroid to a more “flexible” matroid. Nevertheless, the original information shall not be lost completely.
2.2. Extensions of arrangements and characteristic polynomials. We consider three extension constructions for subspace arrangements.

The trivial extension. The trivial extension of an arrangement was already implicitly used in the proof of Corollary 3. For an arrangement $\mathcal{A} = \{H_i \subset \mathbb{R}^d : i = 1, \ldots, n\}$, the trivial extension is

$$T_\ell(\mathcal{A}) := \{H_i \times \mathbb{R}^\ell \subset \mathbb{R}^{d+\ell} : i = 1, \ldots, n\}.$$ 

The intersection poset of $\mathcal{A}$ is unchanged but the dimension of every element increases by $\ell$. In particular, for the characteristic polynomial we have

$$\psi(T_\ell(\mathcal{A}); \lambda) = \lambda^\ell \psi(\mathcal{A}; \lambda).$$

The next two extensions depend the choice of generic subspaces and hence produce a parametrized collection of arrangements.

The large product extension. Let $\mathcal{A}$ be a c-arrangement in $\mathbb{R}^d$ and let $k, h \geq 1$ be fixed parameters. Let $\mathcal{A}' = T_k(\mathcal{A})$ be the trivial extension to $\mathbb{R}^{d+k}$. Choose $k$ general directions $s_1, \ldots, s_k \in \mathbb{R}^{d+k-1}$, called the extension directions. For every $s_i$, let $(S_{i,j})_{j=1,\ldots,h} \subset \mathbb{R}^{d+k}$ be distinct affine hyperplanes parallel to $s_i$. The large product extension with respect to $k$ and $h$ is defined as

$$\text{Pr}_{k,h}(\mathcal{A}) := \mathcal{A}' \cup \{S_{ij} : i = 1, \ldots, k, j = 1, \ldots, h\}.$$ 

Note that $\text{Pr}_{k,h}(\mathcal{A})$ is not central in general and a c-arrangement only when $c = 1$. The generalized normal $n_i = \frac{1}{2}(s_i \cap B^d)$ corresponding to $s_i$ is called the extension direction and $(S_{i,j})_j$ are the extension hyperplanes. For fixed $\mathcal{A}$, this construction yields a collection of arrangements parametrized by $(\mathbb{P}^{d+k-1})^k$. The characteristic polynomial is readily available as follows.

Lemma 5. For a c-arrangement $\mathcal{A}$ and parameters $k, h \geq 1$

$$\psi(\text{Pr}_{k,h}(\mathcal{A}); \lambda) = (\lambda + h) \cdot \psi(\text{Pr}_{k-1,h}(\mathcal{A}); \lambda).$$

Proof. Observe that

$$\text{Pr}_{k,h}(\mathcal{A})/S_{ij} \cong \text{Pr}_{k-1,h}(\mathcal{A}).$$

Iterating (3) yields the claim. \qed

Hence, we can recover the characteristic polynomial of $\mathcal{A}$ as

$$\psi(\mathcal{A}; \lambda) = \lim_{h \to \infty} \frac{\psi(\text{Pr}_{k,h}(\mathcal{A}); \lambda)}{h^k}.$$ 

The large product extension can be further augmented by a trivial extension, and we abbreviate $\text{Pr}_{k,h,\ell}(\mathcal{A}) := T_\ell(\text{Pr}_{k,h}(\mathcal{A}))$.

The semiflexible extension. Let $\mathcal{A}$ be an arrangement with a distinguished element $H_e \in \mathcal{A}$ and generalized normal $n_e$. For parameters $k, h \geq 1$, the semiflexible extension $\text{Sf}_{k,h}(\mathcal{A}, e)$ is obtained from the large product extension $\text{Pr}_{k,h}(\mathcal{A})$ as follows:

$$\text{Sf}_{k,h}(\mathcal{A}, e) := \left(\text{Pr}_{k,h}(\mathcal{A}) \setminus \{H_e\}\right) \cup \{H_e\},$$

where $H_e$ is a linear subspace of dimension $\dim H_e$ whose generalized normal is in general position in $n_e + \sum_{i=1}^k n_i$, where $n_1, \ldots, n_k$ are the extension normals. The element $e'$ is called the semiflexible element of the extension.

Lemma 6. Let $\mathcal{A}$ be an arrangement with distinguished element $e$. Then

$$\psi(\text{Sf}_{k,h}(\mathcal{A}, e); \lambda) = h \cdot \psi(\text{Sf}_{k-1,h}(\mathcal{A}, e); \lambda) + \psi(\text{Pr}_{k-1,h}(\mathcal{A}/e); \lambda) + \psi(\text{Pr}_{k-1,h}(\mathcal{A}\setminus e); \lambda).$$

Proof. For an extension hyperplane $S$ of $\text{Sf}_{k,h}(\mathcal{A}, e)$, we note that

$$\text{Sf}_{k,h}(\mathcal{A}, e)/S \cong \text{Sf}_{k-1,h}(\mathcal{A}, e).$$

Hence, iterating (3) for all $h$ extension hyperplanes of $S$ yields

$$\psi(\text{Sf}_{k,h}(\mathcal{A}, e); \lambda) = h \psi(\text{Sf}_{k-1,h}(\mathcal{A}, e); \lambda) + \psi(\mathcal{A}'; \lambda).$$
Now, $\mathcal{A}'$ is an arrangement in $\mathbb{R}^{d+k}$ with the distinguished subspace $H_{\mathcal{A}'}$. The deletion of $H_{\mathcal{A}'}$ results in an arrangement $\Pr_{k-1,h}(\mathcal{A} \setminus e)$ but embedded in $\mathbb{R}^{d+k}$. Since $H_{\mathcal{A}'}$ is in general position to the other subspaces, it follows that the restriction to $H_{\mathcal{A}'}$ yields $\Pr_{k-1,h}(\mathcal{A} / e)$ from which the claim follows. □

Combining Lemma 5 and Lemma 6 yields the following.

**Corollary 7.**

$$\lim_{h \to \infty} \frac{\psi(Sf_{k,h}(\mathcal{A}, e); \lambda)}{h^k} = \lim_{h \to \infty} \frac{\psi(Pr_{k,h}(\mathcal{A}); \lambda)}{h^k} = \psi(\mathcal{A}; \lambda).$$

The semiflexible extension can be further augmented by a trivial extension, and we abbreviate $Sf_{k,h,\ell}(\mathcal{A}, e) := T_{\ell}(Sf_{k,h}(\mathcal{A}, e))$.

**Remark 8.** The realization space of a subspace arrangement is defined as the space of coordinatizations, within respective Grassmannians, modulo affine transformations. The extension of an arrangement and the original arrangement have homotopy equivalent realization spaces almost surely. In fact, it is not hard to check that the realization space of a large product extension and the semiflexible extension is stably equivalent to the realization space of the original arrangement in the sense of Mnëv [Mnë88]. This allows us to make an interesting philosophical observation: Mnëv Universality and its refinement by Vakil [Vak06, LV12] and Kapovich–Millson [KM98] asserts that realization spaces of arrangements can be arbitrarily complicated. Hence, topologically realization spaces of extensions behave badly. On the other hand, measure concentration is unaffected by these topological pathologies as asymptotically this influence vanishes.

### 2.3. Pushforward measures on arrangement extensions.

We are now interested in the effect of large product and semiflexible extensions on the Wills polynomial. The trivial extension only increases the ambient dimension and hence leaves the Wills polynomial unaffected. Throughout this section let $\mathcal{A}$ be a fixed (linear) $c$-arrangement in $\mathbb{R}^d$ with elements labelled $e_1, \ldots, e_n$. For fixed $k, h, \ell$ define

$$Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n) := Sf_{k,h,\ell}(Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n-1), e_n)$$

with $Sf_{k,h,\ell}(\mathcal{A}, e_1)$ as defined in Section 2.2. This is an arrangement of $n + n \cdot k \cdot h$ subspaces in a Euclidean space of dimension $d + n \cdot (k + \ell)$. For every element $e_i$ there is a corresponding semiflexible element $e'_i$. More precisely, $Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n)$ is a collection of arrangements parametrized by

$$(\mathbb{R}^{d+n(k+\ell)-1})^k \times (\text{Gr}_{k+c,c})^n$$

corresponding to the choice of $kn$ (general) extension directions and $n$ semiflexible elements. Note that for chosen extension directions $s_1, \ldots, s_k$, a semiflexible element $H'_c$ for the codimension-$c$ subspace $H_c$ corresponds to the choice of a $c$-dimensional subspace in $H'_c + \text{span}\{s_1, \ldots, s_k\} \cong \mathbb{R}^{k+c}$. The particular choice of the extension hyperplanes is irrelevant for our purpose.

The uniform measure on (10) makes $Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n)$ into a probability space and the Wills polynomial of the discotope corresponding to $\mathcal{A}' \sim Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n)$ is a random variable. We note the following consequence of deleting, respectively contracting the semiflexible element $e'_n$:

$$Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n) \setminus e'_n \cong \Pr_{k,h,\ell}(Sf_{k-1,h,\ell}(\mathcal{A} \setminus e_n, e_1, \ldots, e_{n-1}))$$

and

$$Sf_{k,h,\ell}(\mathcal{A}, e_1, \ldots, e_n)/e'_n \cong \Pr_{k,h,\ell}(Sf_{k-1,h,\ell}(\mathcal{A} / e_n, e_1, \ldots, e_{n-1})).$$
The corresponding maps
\[
\left(\mathbb{R}P^{d+n(k+\ell)-1}\right)^{kn} \times (\text{Gr}_{k+c,c})^n \to \left(\mathbb{R}P^{d+n(k+\ell)-1}\right)^{kn} \times (\text{Gr}_{k+c,c})^{n-1}
\]
\[
\left(\mathbb{R}P^{d+n(k+\ell)-1}\right)^{kn} \times (\text{Gr}_{k+c,c})^n \to \left(\mathbb{R}P^{d+n(k+\ell)-2}\right)^{kn} \times (\text{Gr}_{k+c,c})^{n-1}
\]
yield a pushforward of the uniform measure that will be utilized in the proof of the following result.

For a polynomial \( p(\lambda) = \sum a_i \lambda^i \), let us denote by \([p(\lambda)]_i = a_i\) the coefficient of \( \lambda^i \). We also abbreviate \( W(\mathcal{A}'; \lambda) = W(\mathcal{Z}(\mathcal{A}'); \lambda) \).

**Theorem 9.** Let \( \mathcal{A} \) be a c-arrangement on \( n \) elements. For sufficiently fast growing sequences \((h_k)\) and \((\ell_k)\) the following holds asymptotically almost surely for \( k \to \infty \)
\[
h_k^{-kn} \left[ W(\mathcal{A}''; \lambda) \right]_{i-c} \asymp h_k^{-(n-1)} \left[ W(\mathcal{A}'''; \lambda) + W(\mathcal{A}'''; \lambda) \right]_{i-c}
\]
for all \( 0 \leq i \leq \frac{rk(\mathcal{A}')}{c} \) and where \( \mathcal{A}' \sim Sf_{k,h_k,\ell_k}(\mathcal{A}, e_1, \ldots, e_n) \), \( \mathcal{A}'' \sim Sf_{k,h_k,\ell_k}(\mathcal{A} \setminus e_n, e_1, \ldots, e_{n-1}) \), and \( \mathcal{A}''' \sim Sf_{k,h_k,\ell_k}(\mathcal{A}/e_n, e_1, \ldots, e_{n-1}) \).

**Proof.** Notice that if \( \ell_k \) is a sequence of positive integers large enough with respect to \( k \), then \( Sf_{k,h_k,\ell_k}(\mathcal{A}, e_1, \ldots, e_n) \) is a normal Lévy family following Proposition 4 (independent of the value of \( h \)).

The intrinsic volumes of the associated discotopes are Lipschitz continuous functions on the parameter space of \( Sf_{k,h_k,\ell_k}(\mathcal{A}, e_1, \ldots, e_n) \) with Lipschitz constants depending on \( h \) for the extension hyperplanes and on \( \mathcal{A}' \) for other elements. It follows that if \( \ell_k \) is large enough with respect to \( h_k \), then \( W(\cdot; \lambda) \) converges to the Lévy mean asymptotically almost surely. We may therefore treat the Wills polynomial of \( \mathcal{A}' \sim Sf_{k,h_k,\ell_k}(\mathcal{A}, e_1, \ldots, e_n) \) as virtually constant.

The specific geometry of the Grassmannian stronger dictates that the normal \( n_{\mathcal{A}'} \) is a.a.s. orthogonal to all other elements of the arrangement \( \mathcal{A}' \). Hence, for a random element \( \mathcal{A}' \)
\[
W(\mathcal{A}' \setminus e_n; \lambda) \asymp W(\mathcal{A}' \setminus e_n; \lambda) + W(\mathcal{A} / e_n; \lambda)
\]
for all \( 0 \leq i \leq \frac{rk(\mathcal{A}')}{c} \) by choice of normalization. By (11) and (12), we may further approximate
\[
h_k^{-kn} W(\mathcal{A}' \setminus e_n; \lambda) \asymp h_k^{-kn} W(\mathcal{A}' / e_n; \lambda)
\]
with \( \mathcal{B} \sim \text{Pr}_{k,h_k,\ell_k} Sf_{k,h_k,\ell_k}(\mathcal{A}' \setminus e_n, e_1, \ldots, e_{n-1}) \)
and
\[
h_k^{-kn} W(\mathcal{A} / e_n; \lambda) \asymp h_k^{-kn} W(\mathcal{C} / e_n; \lambda)
\]
with \( \mathcal{C} \sim \text{Pr}_{k,h_k,\ell_k} Sf_{k,h_k,\ell_k}(\mathcal{A}' / e_n, e_1, \ldots, e_{n-1}) \).

as \( k \to \infty \). Finally, observe that the asymptotic effect of a large product extension \( \text{Pr}_{k,h_k,\ell_k} \) on \( W \) is a multiplication of the Wills polynomial by \((h_k)^k\) (within a constant error term).

The relation between coefficients of Wills polynomials yields our main result.

**Theorem 10.** Let \( \mathcal{A} \) be a c-arrangement with elements \( e_1, \ldots, e_n \) and \((h_k)\), \((\ell_k)\) sufficiently fast growing sequences. For \( k \to \infty \) asymptotically almost surely
\[
(h_k)^{-kn} \cdot \nu_1(\mathcal{A}') \asymp \gamma_i(\mathcal{A}')
\]
where \( \mathcal{A}' \sim Sf_{k,h_k,\ell_k}(\mathcal{A}, e_1, \ldots, e_n) \), \( i = j c \), and \( 0 \leq j \leq d \). In particular, the sequence \((\gamma_0(\mathcal{A}'), \ldots, \gamma_r(\mathcal{A}'))\) of Whitney numbers is log-concave.

**Proof.** For \( n = 1 \) and \( \mathcal{A} = \{H\} \), the claim is immediate with the chosen normalization. For \( n > 1 \), it follows from Theorem 9 that for \( k \to \infty \), the Wills polynomial satisfies the same deletion-contraction relation (3) as the characteristic polynomial which completes the first claim. The log-concavity of the Whitney numbers \((\gamma_i)\) now follows from Corollary 3. □
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