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On the functional limits for sums of a function of partial sums

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Abstract
We derive a functional central limit theorem (fclt) for normalised sums of a function of the partial sums of independent and identically distributed random variables. In particular, we show, using a technique presented in Huang and Zhang (Electron. Comm. Probab. 12, 51–56), that the result from Qi (Statist. Probab. Lett. 62, 93–100), for normalised products of partial sums, can be generalised in this fashion to a fclt.

Key words: Limit distributions, Product of sums, Stable laws

1. Introduction
While considering limiting properties of sums of records, Arnold and Villaseñor (1998) obtained the following version of the central limit theorem (clt) for a sequence \((X_n)\) of independent, identically distributed (iid) exponential random variables (rv’s) with the mean equal one:

\[
\left( \prod_{k=1}^{n} \frac{S_k}{k} \right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2} N} \quad \text{as } n \to \infty,
\]

where \(S_n = \sum_{k=1}^{n} X_k\) and \(N\) is a standard normal random variable.

Later Rempała and Wesołowski (2002) extended such a clt to general iid positive rv’s \((X_n)\). Namely, provided that

\[
\mathbb{E}X_1^2 < \infty,
\]

\[
\left( \prod_{k=1}^{n} \frac{S_k}{k\mu} \right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2} N} \quad \text{as } n \to \infty,
\]

(1)

where \(\mu = \mathbb{E}X_1\) and \(\gamma = \mu/\sigma\) with \(\sigma^2 = \text{Var}X_1 > 0\).

This result was generalised by Qi (2003) by assuming that the underlying distribution of \(X_1\) is in the domain of attraction of a stable law with index \(\alpha \in (1,2]\). In this case

\[
\left( \prod_{k=1}^{n} \frac{S_k}{k\mu} \right)^{1/\sqrt{n}} \xrightarrow{d} e^{(\Gamma(\alpha+1))^{1/\alpha} \mathcal{L}} \quad \text{as } n \to \infty,
\]

(2)

where \(\Gamma(\alpha+1) = \int_{0}^{\infty} x^{\alpha} e^{-x} dx\) and the sequence \(a_n\) is taken such that

\[
\frac{S_n - n\mu}{a_n} \xrightarrow{d} \mathcal{L},
\]

where \(\mathcal{L}\) is one of the stable distributions with index \(\alpha \in (1,2]\). Lu and Qi (2004) obtained a similar result in the case \(\alpha = 1\) with \(\mathbb{E}|X_1| < \infty\). In a paper by Huang and Zhang (2007) it is shown that (1) follows from the weak invariance principle and the whole result can be reformulated to a functional theorem.

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discussion, the only fact that plays the crucial role is the fact that if (6) holds, then
\[ a \mu/\sigma \]
Clearly by scaling it suffices to let \( \sigma = 1 \) and \( \mu = 0 \), hence only the parameters \( \alpha \) and \( \beta \) are unaltered by scaling. In case \( \alpha = 2 \) as mentioned above, we understand \( S_2(1, \beta, 0) \) as \( N \). Moreover, in case \( \alpha \in (1, 2] \), the sequence \( b_n \) can be taken equal to \( \mu \). On how to choose \( a_n \) we refer to (Whitt, 2002, Theorem 4.5.1). The choice is irrelevant for our discussion, the only fact that plays the crucial role is the fact that if (6) holds, then \( a_n = n^{1/\alpha}L(n) \), where \( L \) is slowly varying.

2. Preliminaries

The following theorem is well known and can be easily found in the literature (see e.g. Bingham et al., 1987).

**Theorem 1** (Stability Theorem). The general stable law with index \( \alpha \in (0, 2] \) is given by a characteristic function of one of the following forms:

1. \( \phi(t) = \exp(-|t|^\alpha(1-i\beta \text{sgn } t) \tan(\pi\alpha/2) + i\alpha t) \), \( \alpha \neq 1 \),
2. \( \phi(t) = \exp(-|t|^\alpha(1+i\beta \text{sgn } t) \log |t| + i\alpha t) \), \( \alpha = 1 \),
3. \( \phi(t) = \exp(-|t|^2/2 + i\alpha t) \), \( \alpha = 2 \)

with \( \beta \in [-1, 1] \), \( \mu \in \mathbb{R} \) and \( \sigma > 0 \).

From the above theorem one can see that every stable law with index \( \alpha \in (0, 2] \) can be parametrized by four parameters and written as \( S_\alpha(\sigma, \beta, \mu) \). We distinguish the case \( \alpha = 2 \) because otherwise \( S_2(\sigma, \beta, \mu) \overset{d}{=} \mathcal{N}(\mu, 2\sigma^2) \) and \( \beta \) plays no role, moreover one would like to think of \( S_2(1, \beta, 0) \) as \( N \) not \( N(0, 2) \).

Let \( (X_n) \) be a sequence of iid rv’s, set \( S_n = \sum_{k=1}^n X_k \) and assume \( X_1 \) is in the domain of attraction of a stable law with index \( \alpha \in (1, 2] \). Note that for such \( X_1 \) we have \( \mathbb{E}|X_1| < \infty \). Recall that a sequence of iid rv’s \( (X_n) \) is said to be in the domain of attraction of a stable law \( S_\alpha(\sigma, \beta, \mu) \), if there exists constants \( a_n > 0 \) and \( b_n \in \mathbb{R} \) such that

\[ S_n - b_n \overset{d}{\rightarrow} a_n S_\alpha(\sigma, \beta, \mu). \]  

Clearly by scaling it suffices to let \( \sigma = 1 \) and \( \mu = 0 \), hence only the parameters \( \alpha \) and \( \beta \) are unaltered by scaling. In case \( \alpha = 2 \) as mentioned above, we understand \( S_2(1, \beta, 0) \) as \( N \). Moreover, in case \( \alpha \in (1, 2] \), the sequence \( b_n \) can be taken equal to \( \mu \). On how to choose \( a_n \) we refer to (Whitt, 2002, Theorem 4.5.1). The choice is irrelevant for our discussion, the only fact that plays the crucial role is the fact that if (6) holds, then \( a_n = n^{1/\alpha}L(n) \), where \( L \) is slowly varying.
Furthermore, in addition to stable clt (6), there is convergence in distribution

\[ S_n(t) := \frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \mu}{a_n} \xrightarrow{d} \mathcal{L}(t) \quad \text{in} \quad D[0, 1], \tag{7} \]

where \( \mathcal{L} \) is a standard \((\alpha, \beta)\)-stable Lévy motion, with

\[ \mathcal{L}(t) \overset{d}{=} t^{1/\alpha} S_\alpha(1, \beta, 0) \overset{d}{=} S_\alpha(t, \beta, 0) \]

and as before, for \( \alpha = 2 \), \( \mathcal{L} \) is a standard Wiener process (c.f. Whitt, 2002, Theorem 4.5.3).

### 3. Main Results

**Theorem 2.** Let \((X_n)\) be a sequence of iid rv’s in the domain of attraction of the stable law \(S_\alpha(1, \beta, 0)\) with \(\alpha \in (1, 2)\), so that (6) holds for some sequence \(a_n\) and \(b_n = n\mu\), where \(\mu = \mathbb{E}X_1\). Let \(f\) be a real function defined on an interval \(I\) such that \(\mathbb{P}(X_1 \in I) = 1\) and \(f'(\mu)\) exists. Then, as \(n \to \infty\)

\[ \frac{1}{a_n} \sum_{k=1}^{\lfloor nt \rfloor} (f(S_k/k) - f(\mu)) \xrightarrow{d} f'(\mu) \int_0^t \frac{\mathcal{L}(x)}{x} \, dx \quad \text{in} \quad D[0, 1], \tag{8} \]

where \(S_k\) denotes the \(k\)-th partial sum.

Because \(\mathcal{L}(x)\) is càdlàg, it has at most countably many discontinuity points, so the integral on the right hand side of (8) exists and is finite almost surely if

\[ \int_0^t \frac{\mathcal{L}(x)}{x} \, dx < \infty \quad \text{a.s.} \]

To ensure this, note that for a positive nondecreasing function \(h\) we have

\[ \int_0^t \frac{\mathcal{L}(x)}{x} \, dx \leq \sup_{0 \leq s \leq t} \frac{|\mathcal{L}(s)|}{h(s)} \int_0^t h(x) \, dx. \]

Setting \(h(x) = x^\gamma\) with \(\gamma \in (0, 1/\alpha)\) we get

\[ \int_0^t \frac{\mathcal{L}(x)}{x} \, dx < \infty \quad \text{and} \quad \sup_{0 \leq s \leq t} \left\{ \frac{|\mathcal{L}(s)|}{h(s)} \right\} \to 0 \quad \text{a.s. as} \quad t \to 0, \]

by Khintchine’s Theorem (see e.g., Barndorff-Nielsen et al., 2001, Theorem 2.1). This guaranties the existence of the integral in (8) as well as implies that

\[ \sup_{0 \leq s \leq t} \left| \int_0^t \frac{\mathcal{L}(x)}{x} \, dx \right| = 0 \quad \text{a.s. as} \quad t \to 0, \]

a fact that is going to be used later in the proof.
Remark 1. Observe that
\[ \int_0^t \frac{L(x)}{x} \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{t}{n} L\left(\frac{tk}{n}\right) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} \left( \frac{L\left(\frac{t}{n}\right)}{n} - L\left(\frac{t(i-1)}{n}\right) \right) \]
\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \frac{1}{k} \right) \right) \frac{1}{\alpha} \]
\[ = \sum_{k=1}^{n} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \frac{1}{k} \right) \right) \frac{1}{\alpha} \]
\[ = \sum_{k=1}^{n} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \frac{1}{k} \right) \right) \frac{1}{\alpha} \]
\[ = S_\alpha (t, \beta, 0) \left( \int_0^1 \left( \log x \right)^\alpha \, dx \right) \]
\[ = S_\alpha (t, \beta, 0) \left( \Gamma (\alpha + 1) \right) \frac{1}{\alpha} . \]

If \( X_1 \) is a positive rv, then the limiting stable law has \( \beta = 1 \). Setting \( f(x) = \mu \log(x/\mu) \), Theorem 2 yields
\[ \left( \prod_{k=1}^{n} \frac{S_k}{k} \right)^{\mu/a_n} \to d \exp \left( \sum_{k=1}^{n} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \frac{1}{k} \right) \right) \right) \]
which is the result (2) obtained by Qi (2003).

Remark 2. If \( \mathbb{E} X_1^2 < \infty \), then \( X_1 \) is in the domain of attraction of normal distribution \( \mathcal{N} \) and \( \sigma_n \sim \sigma \sqrt{n} \), where \( \sigma^2 = \text{Var}(X_1) > 0 \). If furthermore \( X_1 \) is positive, then setting \( \gamma = \mu/\sigma \)
\[ \left( \prod_{k=1}^{n} \frac{S_k}{k} \right)^{\gamma} \sim \int_{0}^{1} \frac{W(x)}{x} \, dx \]
in \( D[0, 1] \) as \( n \to \infty \), which coincides with the result (5) by Huang and Zhang (2007).

Before proceeding to the proof of the main theorem, we need a technical lemma.

Lemma 3. Under the assumptions of Theorem 2
\[ \sum_{k=1}^{n} \frac{\mathbb{E}|S_k - k\mu|}{k} = O(a_n). \]

Proof. Note that
\[ \sum_{k=1}^{n} \frac{\mathbb{E}|S_k - k\mu|}{k} \leq \sup_{k \leq n} \left( \frac{\mathbb{E}|S_k - k\mu|}{a_k} \right) \sum_{k=1}^{n} \frac{a_k}{k} . \]

By Theorem 6.1 in DeAcosta and Giné (1979)
\[ \mathbb{E} \left( \frac{|S_n - n\mu|}{a_n} \right) = O(1). \] (9)

Now, for a regularly varying function \( A > 0 \) with index \( \gamma > -1 \), its easy to see that
\[ \sum_{k \leq x} A(k) \sim \int_{1}^{x} \frac{1}{1 + \gamma} xA(x) \, dx \]
as \( x \to \infty \) if \( \gamma > -1 \),
where the last asymptotic equivalence follows from the Karamata’s Theorem (c.f. Bingham et al., 1987, Theorem 1.5.8). Recall that $a_n$ is slowly varying with index $1/\alpha > 0$, this implies

$$\sum_{k=1}^{n} \frac{a_k}{k} = O(a_n),$$

and proves the Lemma.

Now we may proceed to the proof of the main theorem. The proof follows the steps of the proof of (5) in Huang and Zhang (2007).

Proof of Theorem 2. Expand $f$ in the neighbourhood of $\mu$, then

$$\frac{1}{a_n} \sum_{k=1}^{[nt]} (f(S_k/k) - f(\mu)) = f'(\mu) \frac{a_n}{[nt]} \sum_{k=1}^{[nt]} (S_k/k - \mu) + \frac{1}{a_n} \sum_{k=1}^{[nt]} (S_k/k - \mu) r(S_k/k),$$

(10)

where $r(x) \to 0$ as $x \to \mu$. Note that $\mathbb{E}|X|$ < $\infty$ so by the SLLN $r(S_k/k) \to 0$ a.s.. It now follows from Lemma 3 that

$$\sup_{0 \leq t \leq 1} \left| \frac{1}{a_n} \sum_{k=1}^{[nt]} (S_k/k - \mu) r(S_k/k) \right| \leq \frac{1}{a_n} \sum_{k=1}^{n} \frac{|S_k - k\mu|}{k} |r(S_k/k)| = o_p(1).$$

So, according to (10) it suffices to show that, as $n \to \infty$

$$Y_\mu(t) := \frac{1}{a_n} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} \xrightarrow{d} \int_0^t \frac{L(x)}{x} \, dx, \quad \text{in } D[0,1].$$

Let

$$H_\epsilon(f)(t) = \left\{ \begin{array}{ll}
\int_t^{\min(t,\epsilon)} \frac{f(x)}{x} \, dx, & 0 \leq t \leq \epsilon \\
0, & \epsilon < t \leq 1
\end{array} \right.$$

and

$$Y_{\alpha,\epsilon}(t) = \left\{ \begin{array}{ll}
\frac{1}{a_n} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k}, & 0 \leq t \leq \epsilon \\
0, & \epsilon < t \leq 1
\end{array} \right.$$

It is obvious that

$$\sup_{0 \leq t \leq \epsilon} \left| \int_0^t \frac{L(x)}{x} \, dx - H_\epsilon(L)(t) \right| = \sup_{0 \leq t \leq \epsilon} \left| \int_0^t \frac{L(x)}{x} \, dx \right| \to 0 \text{ a.s. as } \epsilon \to 0$$

(12)

and

$$\mathbb{E} \max_{0 \leq t \leq \epsilon} |Y_\mu(t) - Y_{\alpha,\epsilon}(t)| \leq \frac{1}{a_n} \sum_{k=1}^{[nt]} \mathbb{E}\frac{|S_k - k\mu|}{k} \leq C\epsilon^{1/\alpha}$$

(13)

by the same argumentation as in the proof of Lemma 3.

On the other hand, it is easily seen that, for $n$ large enough such that $ne \geq 1$,

$$\sup_{0 \leq t \leq \epsilon} \left| \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} \right| \leq \sum_{k=1}^{[nt]} \frac{|S_k - k\mu|}{k} \leq \sup_{0 \leq t \leq \epsilon} \left| \int_{[nt]}^{[nt]+1} \frac{S_k - k\mu}{x} \, dx \right|

\leq \int_{[nt]}^{[nt]+1} \frac{|S_k - k\mu|}{x} \, dx + \sup_{0 \leq t \leq \epsilon} \left| \int_{[nt]}^{[nt]+1} \frac{|S_k - k\mu|}{x} \, dx \right|

\leq \max_{k \leq n} |S_k - k\mu| \sup_{0 \leq t \leq \epsilon} \left( \frac{2}{ne} + \frac{2}{nt} + \frac{1}{ne} \right)

\leq 5 \max_{k \leq n} |S_k - k\mu|/(ne) = O_p(a_n/n) = o_p(1),$$
by noticing that \( \max_{1 \leq i \leq n} |S_k - k\mu|/a_n \overset{d}{\rightarrow} \sup_{0 \leq t \leq 1} |\mathcal{L}(t)| \) according to (7). So
\[
\frac{1}{a_n} \sum_{k=\lceil n\epsilon \rceil + 1}^{nt} \frac{S_k - k\mu}{k} = \frac{1}{a_n} \int_{\lceil n\epsilon \rceil}^{nt} \frac{S_k - \lfloor x \rfloor \mu}{x} \, dx + o_P(1) = \int_{\epsilon}^{t} \frac{S_n(x)}{x} \, dx + o_P(1)
\]
uniformly in \( t \in [\epsilon, 1] \). Notice that \( H_L(\cdot) \) is a continuous mapping on the space \( D[0,1] \). Using the continuous mapping theorem (c.f. Theorem 2.7. of Billingsley (1999)) it follows that
\[
Y_{n,\epsilon}(t) = H_L(S_n(t)) + o_P(1) \overset{d}{\rightarrow} H_L(\mathcal{L}(t)) \quad \text{in} \quad D[0,1] \quad \text{as} \quad n \rightarrow \infty.
\]
Combining (12)-(14) yields (11) by Theorem 3.2 of Billingsley (1999). \( \square \)

4. Extensions

To prove Lemma 3, we have only used the property (9) (which is in fact the condition (4)) and the fact that \( a_n \) varies regularly with a positive index. The proof of Theorem 2 was based on the convergence (7) and the fact that \( S_k/k \rightarrow \mu \) a.s.. All those conditions are satisfied when \( S_k \) is defined to be the partial sum of a sequence of iid rv’s in the domain of attraction of a stable law with index greater than one. However, we do not need to assume anything about \( S_k \) and only require that it satisfies the aforementioned conditions. This leads to

**Theorem 4.** Let \( (S_k) \) be a sequence of random variables. Suppose there exists an \((\alpha, \beta)\)-stable Lévy process \( (\mathcal{L}(t))_{t \geq 0} \), a constant \( \mu \) and a sequence \( a_n \) such that as \( n \rightarrow \infty \)
\[
\frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \mu}{a_n} \overset{d}{\rightarrow} \mathcal{L}(t) \quad \text{in} \quad D[0,1],
\]
where \( a_n \) can be written as \( a_n = n^{1/\alpha} L(n) \) with \( \alpha \in (1,2] \) and \( L \) slowly varying. In addition, suppose that
\[
\sup_n \frac{\|S_n - n\mu\|}{a_n} = O(1),
\]
and \( S_n/n \rightarrow \mu \) a.s., then, as \( n \rightarrow \infty \)
\[
\frac{1}{a_n} \sum_{k=\lceil n\epsilon \rceil + 1}^{nt} (f(S_k/k) - f(\mu)) \overset{d}{\rightarrow} f'(\mu) \int_{0}^{t} \frac{\mathcal{L}(x)}{x} \, dx \quad \text{in} \quad D[0,1],
\]
for any real function \( f \) defined on an interval \( I \) such that \( \mathbb{P}(S_{k}/k \in I) = 1 \) for all \( k \), provided that \( f'(\mu) \) exists.

In their paper, Huang and Zhang showed that if \( (S_k) \) is a nondecreasing (in fact we only need monotonicity) sequence satisfying (15), then \( S_k/k \rightarrow \mu \) a.s.. Thus, Theorem 4 is an extension of the result (5) from Huang and Zhang (2007).

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