INFLUENCE OF KALUZA SCALAR ON THE RAYCHAUDHURI EQUATION

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\textit{Abstract}

It is shown that the influence of Kaluza scalar is to induce expansion in the Raychaudhuri equation for two representative solutions of the Kaluza theory.

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The 4-dimensional geodesic equation from the 5-dimensional Kaluza theory, in the absence of electromagnetism, is [1]

\[
d\frac{2}{ds}x^\mu + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = \frac{1}{2} \frac{a^2}{G_{55}^2} g^{\mu\lambda}(\partial_{\lambda} G_{55}),
\]

where \(a\) is a constant along the geodesic. The metric used is

\[
G_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & G_{55} \end{pmatrix},
\]

where \(g_{\mu\nu}\) is the metric in 4-dimensional spacetime. We shall consider \(G_{55}\) to be a function of \(r\) only. The right side of (1) shows that the particle is not free; it has an acceleration due to Kaluza scalar \(G_{55}\). This consequence of Kaluza scalar in the 4-dimensional geodesic equation has been studied by the author in [1] to restore causality in the 'brane universe' scenario.

The Raychaudhuri equation [2,3] describing the evolution of a collection of particles following their geodesics, is

\[
\dot{\Theta} = -2\sigma^2 + 2\omega^2 - \frac{\Theta^2}{3} - \frac{4\pi G}{c^4} (\rho c^2 + 3p) + (\dot{u}^\mu)_{;\mu},
\]

where \(u^\mu = \frac{dx^\mu}{ds}\), \(\Theta = u^\mu_{;\mu}\) characterizes the volume of the collection of particles having 4-velocity \(u^\mu\) as they fall under their own gravity, \(2\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}\) with \(\sigma_{\mu\nu}\) as the symmetric shear tensor (distortion without change in volume), \(2\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}\) with \(\omega_{\mu\nu}\) as antisymmetric vortex tensor (rotation without change in shape), \(\rho\) and \(p\) are the density and pressure in the collection of particles and the subscript ; stands for covariant derivative. For simplicity we have set the cosmological constant zero.

In (3), \(\dot{u}^\mu = u^\mu_{;\mu}\) possible acceleration (orthogonal to \(u^\mu\)) of the collection of particles due to non-gravitational forces and the last term in (3) involves \(\dot{u}^\mu\). Now we observe that

\[
\dot{u}^\mu = u^\mu_{;\nu} u^\nu = \left(\frac{du^\mu}{dx^\nu} + \Gamma^\mu_{\nu\lambda} u^\lambda\right) u^\nu,
\]
\[
\frac{du}{ds} + \Gamma^{\mu}_{\lambda \nu} u^\lambda u^\nu,
\]

where (1) is used for \( \frac{du}{ds} + \Gamma^{\mu}_{\lambda \nu} u^\lambda u^\nu \) with \( u^\mu = \frac{dx^\mu}{ds} \). In the usual 4-dimensional geodesic equation, the right side of (1) is zero and hence \( \dot{u}^\mu = 0 \), as these geodesics correspond to free particles under gravity. However, the 4-dimensional geodesic equation emerging from Kaluza’s 5-dimensional theory, which is (1), allows the possibility \( \ddot{u}^\mu \neq 0 \) as in (4); the geodesics now experience an acceleration due to Kaluza scalar \( G_{55} \). So, in a 5-dimensional spacetime with \( G_{55} = G_{55}(x^\mu) \), the Raychaudhuri equation (3) receives contribution from its last term. We write the last term in (3) as

\[
(\dot{u}^\mu)_{;\mu} = \frac{a^2}{2} \left( g^{\mu \rho} (\partial_{\rho} \frac{1}{G_{55}}) \right)_{;\mu},
\]

where \( D_{\mu} \) stands for covariant derivative as \( D_{\mu}(\partial_{\rho} \frac{1}{G_{55}}) = \partial_{\mu} \partial_{\rho} \frac{1}{G_{55}} - \Gamma^{\sigma}_{\mu \rho} (\partial_{\sigma} \frac{1}{G_{55}}) \). Thus, (3) becomes

\[
\dot{\Theta} = -2\sigma^2 + 2\omega^2 - \frac{\Theta^2}{3} - \frac{4\pi G}{c^4} (\rho c^2 + 3p) - \frac{a^2}{2} g^{\mu \rho} (D_{\mu}(\partial_{\rho} \frac{1}{G_{55}})).
\]

The Raychaudhuri equation (3) has been derived [3] from ‘general considerations’ and no specific metric is used. It is known that the vorticity \( \omega \) induces expansion while the shear \( \sigma \) induces contraction. The acceleration term \(-\frac{a^2}{2} g^{\mu \rho} (D_{\mu}(\partial_{\rho} \frac{1}{G_{55}}))\) can possibly induce expansion if \(-\frac{a^2}{2} g^{\mu \rho} (D_{\mu}(\partial_{\rho} \frac{1}{G_{55}}))\) turns out to be positive. We examine this for a static spherically symmetric ansatz for the 5-dimensional metric (2) as

\[
(ds)^2 = e^\rho e^\nu(dt)^2 - e^\nu(dr)^2 - r^2((d\theta)^2 + \sin^2\theta(d\phi)^2) - \psi(r)(dx^5)^2,
\]
where \( \mu, \nu \) are functions of \( r \) only and \( G_{55} = \psi(r) \). \( \mu(r), \nu(r), \psi(r) \) are unknown functions of \( r \) to be determined by the 5 - dimensional Einstein equations.

For the metric in (7), the non - vanishing five dimensional Christoffel connections are:

\[
\begin{align*}
\Delta^t_{rt} &= \frac{\mu'}{2}; & \Delta^r_{rr} &= \frac{\nu'}{2}; & \Delta^r_{tt} &= \frac{1}{2} e^{(\mu-\nu)} c^2 \mu'; \\
\Delta^r_{\theta\theta} &= -e^{-\nu} r; & \Delta^r_{\phi\phi} &= -e^{-\nu} r \sin^2 \theta; & \Delta^r_{55} &= -\frac{1}{2} e^{-\nu} \psi'; \\
\Delta^r_{r\theta} &= \frac{1}{r}; & \Delta^r_{\phi\phi} &= -\sin \theta \cos \theta; & \Delta^{\phi}_{r\phi} &= \frac{1}{r}; \\
\Delta^5_{r5} &= \psi' \psi; & \Delta^5_{55} &= \frac{\psi'}{2\psi},
\end{align*}
\]

where the prime \( ' \) stands for differentiation with respect to \( r \). The Ricci tensor

\[
\tilde{R}_{AB} = \partial_C \Delta^C_{AB} - \partial_B \Delta^C_{AC} + \Delta^C_{DC} \Delta^D_{AB} - \Delta^C_{DB} \Delta^D_{CA},
\]

gives the components as

\[
\begin{align*}
\tilde{R}_{tt} &= \frac{1}{2} e^{(\mu-\nu)} c^2 \{ \mu'' - \frac{\mu' \nu'}{2} + \frac{\mu'^2}{2} + \frac{2 \mu' \psi'}{r} + \frac{\mu' \psi'}{2\psi} \}, \\
\tilde{R}_{rr} &= -\frac{\mu''}{2} + \frac{\mu' \nu'}{4} + \frac{\nu'}{r} + \frac{\mu'^2}{4} - \frac{\psi''}{2\psi} + \frac{\nu' \psi'}{4\psi} + \frac{\psi'^2}{4\psi^2}, \\
\tilde{R}_{\theta\theta} &= 1 - e^{-\nu} - r e^{-\nu} \{ \frac{\mu'}{2} - \frac{\nu'}{2} + \frac{\psi'}{2\psi} \}, \\
\tilde{R}_{\phi\phi} &= \sin^2 \theta \tilde{R}_{\theta\theta}, \\
\tilde{R}_{55} &= e^{-\nu} \{ -\frac{\psi''}{2} - \frac{\mu' \psi'}{4} + \frac{\nu' \psi'}{4} - \frac{\psi'}{r} + \frac{\psi'^2}{4\psi} \}.
\end{align*}
\]

We consider the vacuum solution \( \tilde{R}_{AB} = 0 \) and this gives a set of four equations

\[
\tilde{R}_{tt} = 0 \Rightarrow \frac{\mu''}{2} - \frac{\mu' \nu'}{4} + \frac{\mu'^2}{4} + \frac{\mu' \psi'}{r} + \frac{\mu' \psi'}{4\psi} = 0,
\]
Thus, we have four equations (11) with three unknown functions $\mu, \nu, \psi$. The system is over determined. We need to construct out of these four equations, a set of three equations. One way, which we find useful, is to introduce one more equation $\tilde{R} = G^{AB} \tilde{R}_{AB} = 0$ and use it for the reduction. Using (7) and (10), we find $\tilde{R}_{rr} = 0$ gives

$$\frac{\mu''}{2} + \frac{\mu' \nu'}{4} + \frac{\nu'}{r} - \frac{\mu'^2}{4} - \frac{\psi''}{2 \psi} + \frac{\nu' \psi'}{4 \psi} + \frac{\psi'^2}{4 \psi^2} = 0,$$

(11)

The left side of (12) is exactly the same as in the curly bracket of the $\tilde{R}_{tt}$ equation (10) and so

$$\tilde{R}_{tt} = \frac{1}{2} e^{(\mu-\nu)} c^2 \left\{ \frac{2 \nu'}{r} - \frac{\psi''}{\psi} - \frac{2}{r^2} + \frac{\nu' \psi'}{2 \psi} - \frac{2 \nu'}{r \psi} + \frac{2 \psi'}{2 \psi^2} + \frac{2}{r^2} e^\nu \right\}. \quad (13)$$

Now, $\tilde{R}_{55} = 0$ equation in (11) gives $\psi''$ as

$$\psi'' = -\frac{\mu' \psi'}{2} + \frac{\nu' \psi'}{2} - \frac{2 \psi'}{r} + \frac{\psi'^2}{2 \psi}, \quad (14)$$

and substituting this for $\psi''$ in (13), we obtain

$$\tilde{R}_{tt} = \frac{1}{2} e^{(\mu-\nu)} c^2 \left\{ \frac{2 \nu'}{r^2} e^\nu + \frac{2 \nu'}{r} - \frac{2}{r^2} + \frac{\mu' \psi'}{2 \psi} \right\}. \quad (15)$$

Again, the $\tilde{R} = 0$ equation (12), gives (rearranging)

$$-\frac{\mu''}{2} + \frac{\mu' \nu'}{4} - \frac{\mu'^2}{4} + \frac{\nu'}{r} - \frac{\psi''}{2 \psi} = \frac{\mu'}{r} + \frac{\mu' \psi'}{4 \psi} + \frac{\nu' \psi'}{4 \psi} + \frac{\psi'^2}{r \psi}$$

$$- \frac{\psi'^2}{4 \psi^2} + \frac{1}{r^2} - \frac{e^\nu}{r^2}. \quad (16)$$
Substituting the left side of (16) in \( \tilde{R}_{rr} \) equation in (10), we obtain
\[
\tilde{R}_{rr} = -\frac{e^\nu}{r^2} + \frac{1}{r^2} + \frac{\mu'}{r} + \frac{\mu' \psi'}{4\psi} + \frac{\psi'}{r\psi}.
\] (17)

Thus, the set of four equations in (10, 11) reduce to a set of three equations
\[
\tilde{R}_{tt} = 0 \Rightarrow \frac{1}{r^2} e^\nu + \frac{\nu'}{r} - \frac{1}{r^2} + \frac{\mu' \psi'}{4\psi} = 0,
\]
\[
\tilde{R}_{rr} = 0 \Rightarrow -\frac{1}{r^2} e^\nu + \frac{1}{r^2} + \frac{\mu'}{r} + \frac{\mu' \psi'}{4\psi} + \frac{\psi'}{r\psi} = 0,
\]
\[
\tilde{R}_{\theta\theta} = 0 \Rightarrow 1 - e^{-\nu} - r e^{-\nu} \left\{ \frac{\mu'}{2} - \frac{\nu'}{2} + \frac{\psi'}{2p} \right\} = 0.
\] (18)

In obtaining the above 'reduced set' of three equations, we have incorporated the \( \tilde{R}_{55} = 0 \) equation to eliminate \( \psi' \). This reduced set (18) is a set of three first order differential equations. This is one of the results of this paper. Further, in (18), if the 'mixing term' \( \frac{\mu' \psi'}{4\psi} \) is eliminated in the first two equations, then we obtain the third equation in (18).

If \( \psi \) were a constant, then (18) solves for \( \nu \) and \( \mu \) as
\[
e^\nu = \left( 1 + \frac{A}{r} \right)^{-1},
\]
\[
e^\mu = \left( 1 + \frac{A}{r} \right),
\] (19)
where \( A \) is an integration constant. These correspond to Schwarzschild solutions.

By adding the first two equations in (18), we obtain
\[
\mu' + \nu' + \frac{r \mu'}{2} \frac{\psi'}{\psi} + \frac{\psi'}{\psi} = 0,
\] (20)
and the third equation in (18) is rewritten as
\[
\nu' - \mu' - \frac{\psi'}{\psi} = 2 \frac{2}{r} - 2 \frac{e^\nu}{r}.
\] (21)
The strategy we adopt is to take an ansatz for $\psi$ and use (20) to express $\mu'$ in terms of $\nu'$, which is used in (21) to determine $\nu$. Then the relation between $\mu'$ and $\nu'$ is used to determine $\mu$.

**Solution.1:**

We choose

$$\psi(r) = \frac{b}{r^\alpha},$$

where $b$ is a real positive constant and $\alpha$ is a real constant. Then (20) gives

$$\mu' = \frac{2\alpha}{(2 - \alpha)r} - \frac{2\nu'}{(2 - \alpha)},$$

which is used in (21) to determine $\nu$ as

$$e^\nu = \left(\frac{2(2 - \alpha)}{(\alpha^2 - 2\alpha + 4)} + C \frac{r^{-\frac{(\alpha^2 - 2\alpha + 4)}{(4 - \alpha)}}}{(2 - \alpha)}\right)^{-1},$$

and using (23), we find

$$e^\mu = r^{\frac{2\alpha}{(2 - \alpha)}} \left(\frac{2(2 - \alpha)}{(\alpha^2 - 2\alpha + 4)} + C \frac{r^{-\frac{(\alpha^2 - 2\alpha + 4)}{(4 - \alpha)}}}{(2 - \alpha)}\right)^{-\frac{2}{(2 - \alpha)}},$$

where $C$ is a dimensionful (length) constant. For $\alpha = 0$, $\psi$ becomes a constant and $e^\nu = (1 + \frac{C}{r})^{-1}$; $e^\mu = (1 + \frac{C}{r})$, corresponding to Schwarzschild solution, as noted earlier. With $\alpha = 1$, we have

$$\psi = \frac{b}{r},$$

$$e^\nu = \left(\frac{2}{3} + \frac{C}{r}\right)^{-1},$$

$$e^\mu = \left(C + \frac{2r}{3}\right)^2$$

For this solution, the last term in the Raychaudhuri equation (6), the influence of Kaluza scalar, becomes ($G_{55} = \psi = \frac{b}{r}$)

$$-\frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{G_{55}}) = \frac{a^2}{2br} \left(2 + \frac{3C}{2r}\right).$$
For $C > 0$, the right side is positive. For $C < 0$, we have from (26), $e^\nu = (\frac{2}{3} - \frac{|C|}{r})^{-1}$ and this must be positive to preserve the signature of the metric (+ - - - ). Therefore, $r > \frac{3|C|}{2}$. Setting $r = \beta \frac{3|C|}{2}$ with $\beta > 1$, the right side of (27) becomes $\frac{a^2}{2br}(2 - \frac{1}{\beta})$. Since $\beta > 1$, this is clearly positive. Thus, the influence of Kaluza scalar, for the solution (26), on the Raychaudhuri equation (6) is to induce expansion.

Solution.2:

We now choose for $\psi(r)$ as

$$\psi(r) = 1 + \frac{b}{r},$$  \hspace{1cm} (28)

where $b$ is a real constant. Then (20) gives

$$\mu' = -\frac{2(r + b)}{(2r + b)}\nu' + \frac{2}{r} - \frac{4}{(2r + b)},$$  \hspace{1cm} (29)

which when used in (21) solves for $\nu$ as

$$e^{-\nu} = \frac{4C}{b} - 1 - \frac{b}{2r} + \frac{3C}{r} + \frac{(4r + 3b)}{(r + b)}(\frac{1}{2} - \frac{C}{b}),$$  \hspace{1cm} (30)

where $C$ is integration constant. By choosing $C = \frac{b}{2}$, we find

$$e^\nu = \left(1 + \frac{b}{r}\right)^{-1}.$$  \hspace{1cm} (31)

Using this in (29), we find $\mu' = 0$ or $\mu = constant$. Thus, the solution.2 corresponds to

$$\psi(r) = 1 + \frac{b}{r},$$

$$e^\nu = \left(1 + \frac{b}{r}\right)^{-1},$$

$$e^\mu = constant.$$  \hspace{1cm} (32)
For this solution, the last term in (6) with $G_{55} = \psi = 1 + \frac{b}{r}$, becomes

$$ -\frac{a^2}{2}g^{\mu\rho}(D_\mu(\partial_\rho \frac{1}{G_{55}})) = \frac{3a^2b}{4r^2(r+b)^2}, \quad (33) $$

which is positive, thereby making the influence of Kaluza scalar on the Raychaudhuri equation to induce expansion.

We have examined another solution for 5-dimensional Kaluza theory, by Chatterjee [4] which is

$$ e^\mu = \frac{\sqrt{r^2 + b} - \sqrt{b}}{\sqrt{r^2 + b} + \sqrt{b}}, $$

$$ e^\nu = \left(1 + \frac{b}{r^2}\right)^{-1} = \frac{r^2}{r^2 + b}, $$

$$ \psi = \frac{\sqrt{r^2 + b} + \sqrt{b}}{\sqrt{r^2 + b} - \sqrt{b}}, \quad (34) $$

where $b$ is a real positive constant. We have verified that the above solution (34) is consistent with (20) and (21). For the above solution (34), the last term in (6) is evaluated to be (with $G_{55} = \psi$)

$$ -\frac{a^2}{2}g^{\mu\rho}(D_\mu(\partial_\rho \frac{1}{G_{55}})) = \frac{6a^2b}{r^6}\left\{b + \frac{r^2}{2} - \sqrt{b}\sqrt{r^2 + b}\right\}. \quad (35) $$

It can be checked that the curly bracket in the right side of (35) is positive since $(b + \frac{r^2}{2})^2 > b(r^2 + b)$.

The influence of Kaluza scalar on the quantum mechanical phase shift has been studied by Weber [5] and Gopakumar [6]. Its influence on the rotation of the polarization of the light (optical activity from extra dimensions) has been studied by Ganguly and Parthasarathy [7]. In here, we report yet another influence on the Raychaudhuri equation.

In summary, we have shown that the influence of Kaluza scalar on the Raychaudhuri equation is to induce expansion, for three different solutions. We conjecture that this result may be relevant to inflation based on 5-dimensional Kaluza theory.
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