The Mathematician’s Bias – and the Return to Embodied Computation

S. Barry Cooper
School of Mathematics, University of Leeds,
Leeds LS2 9JT, United Kingdom

Abstract
There are growing uncertainties surrounding the classical model of computation established by Gödel, Church, Kleene, Turing and others in the 1930s onwards. The mismatch between the Turing machine conception, and the experiences of those more practically engaged in computing, has parallels with the wider one between science and those working creatively or intuitively out in the ‘real’ world. The scientific outlook is more flexible and basic than some understand or want to admit. The science is subject to limitations which threaten careers. We look at embodiment and disembodiment of computation as the key to the mismatch, and find Turing had the right idea all along – amongst a productive confusion of ideas about computation in the real and the abstract worlds.

When we get out of bed in the morning, we approach a complicated world of information with a determination not just to survive the day – though that may be hard enough: we mean to “compute” our way towards various vaguely defined objectives. The process will likely be messy, but we certainly experience it as a computational one. Our conception of the computation is very flexible, but we host no in principle rejection of Turing’s notion of mechanical intelligence.

But our sense of ownership of a computational process deserts us somewhat when we think about what it is that makes our daily computing adventure so complicated. The world outside has both predictability and a lack of it bordering on randomness. Here is Nassim Taleb, in his best-selling book “The Black Swan” [23]:

I have spent my entire life studying randomness, practicing randomness, hating randomness. The more that time passes, the worse things seem to me, the more scared I get, the more disgusted I am with Mother Nature. The more I think about my subject, the more I see evidence that the world we have in our minds is different from the one playing outside. Every morning the world appears to me more random than it did the day before, and humans seem to be even more fooled by it than they were the previous day. It is becoming unbearable. I find writing these lines painful; I find the world revolting.

Taleb distrusts mathematicians and their models, from personal experience of their failures, and of their perceived unwillingness to face up to the realities. But we need to give the professionals a chance. Let us look more closely at the
modelling process, and how we deal with computing in a material world. Can we absorb Taleb’s computational context into a classical model based on logical structure. Or does embodied information need to be separately modelled? And does this take us beyond the mathematician’s focus on computable functions.

1 Computation Disembodied

What was clearly new about the Turing model of computation was its successful disembodiment of the machine. For practical purposes, this was not as complete as some post-Turing theoreticians like to pretend: the re-embodied computer which is now a familiar feature of the modern world was hard won by pioneering engineers. But, for the purposes of the stored program computer, and for the proof of incomputability of the Halting Problem, the essential disembodiment was that delivering program-data convergence. It was this universality that John von Neumann recognised as a theoretical anticipation of the stored program computer. The apparent omnipotence even led Turing to talk of his post-war ACE project as aimed at building ‘a brain’.

This paradigm has achieved a strong grip on subsequent thinking. Within the philosophy of mind there is a strong tendency towards physicalism and functionalism, both of which open the door to some version of the Turing model. The functionalist (see Hilary Putnam [18]) stresses what a computer does as something realisable in different hardware. An important expression of the functionalist view in computer science is provided by the notion of a virtual machine, whereby one expects to achieve software implementation of a given programmable machine. Aaron Sloman [20] and others have usefully applied the concept to AI.

This playing down of distinction between information and process has been taken further, and become a familiar feature of programming and theory. As Samson Abramsky describes (private communication):

Turing took traditional mathematical objects, real numbers, functions etc. as the things to be computed. In subsequent work in Computer Science, the view of computation has broadened enormously. In the work on concurrent processes, the behaviour is the object of interest. There is indeed a lack of a clear-cut Church-Turing thesis in this wider sphere of computation – computation as interaction, as Robin Milner put it.

In the quantum world there is a parallel convergence between matter and law-like energy. All this has given rise to a standard computational paradigm vulnerable to surprises from the natural world. Physical processes not subject to data-process convergence will not be recognisably different. But beneath the ‘normal science’, theoretical inadequacies may be brewing – or not, according to the viewpoint. The challenges to the standard model are varied, but most seem to have an impact on universality.

What is happening in both situations is a side-stepping of the mathematically familiar type structure, whereby numbers, functions and relations, and relations over functions etc. give rise to a hierarchy of fundamentally different objects, increasingly hard to handle as one closes up hierarchically the universe of definable entities. In the real world we require a level of constructibility of
computability even, which forces approximations and a rejection of paths to the unknown.

1.1 The Mathematician’s Bias

A symptom of the inadequacy of a type-constrained world-view is the October 2010 ACM Ubiquity Symposium on What is Computation? Part of the Editor’s Introduction by Peter J. Denning [10] reads:

By the late 1940s, the answer was that computation was steps carried out by automated computers to produce definite outputs. That definition did very well: it remained the standard for nearly fifty years. But it is now being challenged. People in many fields have accepted that computational thinking is a way of approaching science and engineering. The Internet is full of servers that provide nonstop computation endlessly. Researchers in biology and physics have claimed the discovery of natural computational processes that have nothing to do with computers. How must our definition evolve to answer the challenges of brains computing, algorithms never terminating by design, computation as a natural occurrence, and computation without computers?

In another contribution to the Symposium, Lance Fortnow [12] asks: “So why are we having a series now asking a question that was settled in the 1930s?” And continues:

A few computer scientists nevertheless try to argue that the [Church-Turing] thesis fails to capture some aspects of computation. Some of these have been published in prestigious venues such as Science, the Communications of the ACM and now as a whole series of papers in ACM Ubiquity. Some people outside of computer science might think that there is a serious debate about the nature of computation. There isn’t.

Undeterred, Dennis J. Frailey thinks it’s the mathematicians have got it wrong:

The concept of computation is arguably the most dramatic advance in mathematical thinking of the past century . . . Church, Gödel, and Turing defined it in terms of mathematical functions . . . They were inclined to the view that only the algorithmic functions constituted computation. I’ll call this the “mathematician’s bias” because I believe it limits our thinking and prevent us from fully appreciating the power of computation.

Clearly, we do not have much of a grip on the issue. It is the old story of the Blind Men and the Elephant again. On the one hand computation is seen as an essentially open, contextual, activity with the nature of data in question. Others bring out a formidable armoury of mathematical weapons in service of the reductive project – for them there is little in concurrency or interaction or continuous data or mental-recursions to stretch the mathematical capabilities of the Turing machine.
Of course, there has always been creative play on the paradoxical misfit between the Turing machine model and the realities of the real world. One of the best-known is Jin Wicked’s wonderful image of Alan Turing himself embodying his universal machine. How well this fits with David Leavitt’s perception, in his book *The Man Who Knew Too Much*, of Turing actually identifying with his computing machines:

![Figure 1: Alan Turing as a Universal Turing Machine.](image)

Essentially, what is happening is that some observers are reviving the mathematical type-structure in a real-world context, others are denying that the mathematics is capable of inhabiting the material world. Some are struck by the sheer globality of how the world computes, the computational fruits of complex-
ity, connectivity and interaction. And by the loss of a simple inductive structure implicit in non-linearity and the failure of computable approximations implicit in too inclusive a view of computationally based environments.

In the past though, it has been the mathematics that has clarified difficult problems and vague intuitions. Apparently, there is a lot of mathematics we have not found out how to use in the material context, though the ownership of the mathematics of reality is beginning to slip from the grasp of those unwilling to adapt.

At the cutting edge there is new mathematics being developed, and questioning of previously sacrosanct conceptual frameworks. Here are some uncomfortable observations (private communication) from Samson Abramsky:

Formally, giving a program + data logically implies the output (leaving aside non-determinism or randomness), so why actually bother computing the result! …

… Can information increase in computation? Information theory and thermodynamics seem to tell us that it can’t, yet intuitively, this is surely exactly why we compute – to get information we didn’t have before.

And in mathematics, our operations and definitions certainly do give us new information. Is our wonderful universe so constrained it cannot go where even high-school arithmetics leads us (as the Davis-Matiyasevich-Putnam-Robinson negative solution to Hilbert’s Tenth Problem tells us)?

2 The Mathematics of Embodiment?

So it is not that mathematicians are only interested in simple mathematical objects. Or that computability only deals with functions over the natural numbers. What we do have is a computational paradigm which dominates our view of the landscape. We even have higher-type computation in various forms, including that mapped out by Stephen Kleene in his three late papers on the topic. What we did not get was any suggestion that higher-type computability might play a role in modelling the processes which it is now suggested might stretch the old Turing model. And the suspicion is that this simple connection can only be explained in the context of a powerful counter-paradigm.

The other by-product of the counter-paradigm is a tendency to desert basic mathematical theory in favour of less focused descriptive arguments for new computational phenomena, accompanied by attempts at models derived from these descriptions unrooted in any classical analysis of their power. Here is another of the Ubiquity symposium contributors, Peter Wegner writing with Dina Goldin [28] on The Church-Turing Thesis: Breaking the Myth:

One example of a problem that is not algorithmic is the following instruction from a recipe [quote Knuth, 1968]:
‘toss lightly until the mixture is crumbly.’

This problem is not algorithmic because it is impossible for a computer to know how long to mix: this may depend on conditions such as humidity that cannot be predicted with certainty ahead of time. In the function-based mathematical worldview, all inputs must be
specified at the start of the computation, preventing the kind of feedback that would be necessary to determine when it’s time to stop mixing.

At the level of a human trying to carry out the cooking instructions, we have a strong impression that something non-algorithmic is going on. But a determined reductionist would not be at all convinced that this is more than a superficial testing of classical modelling based on a particular mode of observation. The recipe is certainly implementable. But will be executed differently by different cooks under different conditions. What is actually happening in a global sense can be made precise and potentially reducible to the Turing model by modelling the whole context, cook and kitchen and ingredients. Who cares that the language used in the description of the recipe is a bit imprecise?

Even quantum physics has received effective attention from the reductionists, via David Deutsch’s [11] placing of the standard model of quantum computing firmly within the scope of the Turing model. Of course, the model does not make full use of wave function collapse, and Calude and Svozil [5] have shown, under reasonable assumptions, that quantum randomness entails incomputability. Meanwhile, the mind is very hard to pin down theoretically, but Deutsch is following a well-established reductive tradition when he argues (in Question and Answers with David Deutsch, on the New.Scientist.com News Service, December, 2006):

I am sure we will have [conscious computers], I expect they will be purely classical, and I expect that it will be a long time in the future. Significant advances in our philosophical understanding of what consciousness is, will be needed.

Maybe starting at the other end, and attempting to embody the richness of trans-computational mathematics, can bring more convincing results? Here we have a very different problem. Our examples of incomputable objects are very simple, even “natural” from some perspectives. But there is a huge gap between the universal Turing machine, which we can see embodied in a very persuasive sense by a modern computer, and the halting problem, which is incomputable but very abstract to the point of having no visible embodiment at all. And the lazy perception is that such examples of incomputability are nothing to do with the material world, even if we are moved to suspect trans-Turing computation by our impressionistic observation of natural phenomena.

The key to bringing some clarity to the situation is to examine the mathematics of the incomputable, in all its basic simplicity. And to look for qualitatively similar mathematics bringing with it a level of embodiment, and an apparent avoidance of computability, to give us a better handle on the natural world and its candidate trans-Turing phenomena.

A meaningful first example, residing at the border between mathematics and the embodied world is the Mandelbrot set. In fact, there are many other such fractal-like objects. But the Mandelbrot set has attracted special attention for good reasons: It has a simple definition over the complex plane involving basic arithmetical operations and a couple of quantifiers – in fact, with a little fine-tuning, just one universal quantifier; it is graphically beautiful and complex, being both approximable via a computer screen, and containing endlessly explorable inner structure; and, its computability as a set of complex numbers
provides us with a challenging open problem. As Roger Penrose says in his best-selling 1994 book [16] *The Emperor’s New Mind*:

Now we witnessed . . . a certain extraordinarily complicated looking set, namely the Mandelbrot set. Although the rules which provide its definition are surprisingly simple, the set itself exhibits an endless variety of highly elaborate structures.

What we see is a complicated and undeniably physical object existing in front of us on our computer screen *because* it was described, and having complex form *because* the description had a slightly elevated level of complexity of language – namely the addition of a quantifier. Mathematically, the description is not much more complicated than that of the non-halting set of a universal Turing machine. The Mandelbrot set is of course made up of complex numbers rather than natural numbers, but the fact it appears on our computer screen gives it a digital kinship with familiar Turing-machine data. Leaving this abstract unembodied vista behind, and walking round to the other side of the Mandelbrot set, we see why such fractals have such significance for so many people – the scene is one of Nature in all its fascinating complexity, beautiful and embodied, like the Mandelbrot set, but this time produced by simple laws which are themselves embodied. This toy fractal provides a neat connection between natural complexity and the unrestrained type-structures of mathematics, with their power to take us definably far beyond what we can computably capture.

What indicates to the observer a phenomenon of computational complexity? It is the richness of *visible form* which impresses. It is the character of a higher order entity. On the other hand, despite the appearance pointing to the basic computational unpredictability, there is an overall identity to the appearance which we are capable of appreciating, and which is caught by the *definition* of the exotic shape. And it is natural to view this definition as *computed* by the totality of the underlying computational context. The mathematics both points to the atomic unpredictability of the universal Turing machine observed; and reassures us with a computation of a higher order, whereby we reaffirm a level of understanding of the underlying turbulence.

There is an obvious role here for some of the most abstract and little known mathematics logicians have developed. How many mathematicians have imagined any role at all in the real world for the conceptual development of higher order computation by Stephen Kleene, Gerald Sacks, Dag Normann and their successors? As a primary schoolboy, I was fascinated by the family folding mangle, and carried it off to possess and enjoy in my own little ‘camp’ in the undergrowth beyond the back gardens. The usefulness had no meaning, the loss I was made aware of later by the grown-ups was a shock. It is surely time for the generalised ‘recursion theorists’ to return this conceptually beautiful work to the real world which indirectly gave rise to it. And to make sense of the sort of physical mysteries that Alan Turing himself identified as important sixty years ago.

3 Emergent Natural Patterns

Back in the early 1950s, Alan Turing became interested in how certain patterns in Nature arise. His seminal importation [27] of mathematical techniques into
this area were to bring a new level of computability to natural phenomena, while improving our understanding of some of the mysteries pointed to by those who distrust the reductionism of those trapped by the Turing machine paradigm. Turing was able to relate such familiar features of everyday life as patterns on animals’ coats – stripes on zebras, patches on cow hides, moving patterns on tropical fish – to simple reaction-diffusion systems describing chemical reactions and diffusion. This brought mathematics to play in biology in a way that made his one published paper on the topic one of the most cited in the literature, and the most cited in recent years of all the papers he wrote – including the 1936 computable numbers paper, and the influential AI ‘Turing Test’ article \[26\] in Mind in 1950.

Turing’s mathematics links up with the relationship between emergence and descriptions pointed to by the fractal example. Of course, some halting problems are solvable, as some Julia sets are computable (see \[2\]). See \[7\] for a more detailed argument for the two-way correspondence between emergence and appropriate descriptions, opening the door to incomputability in Nature. And for the grounding of the observed robustness of emergent phenomena in mathematical framing of the descriptions as mathematical definability.

Turing’s approach is seminal, illuminating the connection between possible incomputability, mathematics, and natural phenomena. It has been carried forward by James D. Murray \[15\] and others, and though things get a lot more complex than the examples tackled by Turing, it is enough to make something more coherent from the confusion of intuitions and models. The embodiment does extend to the emergent halting set and possibly hierarchically beyond, taking us into a world beyond basic algorithms – see Chaitin’s recent take \[6\] on creativity and biology.

This knack Turing had for drawing fundamental computational aspects out of concrete contexts has become increasingly clear to us since he died. The relevance of this seminal work on morphogenesis becomes daily wider and deeper in import. And we can now see it as definability embodied, and hence identify definability as a very real form of computability.

To summarise: It is often possible to define emergent properties in terms of the elementary actions underlying them. While in mathematics, relations and even objects arise from descriptions via the notion of definability. And, if the language used is complicated enough, this can be a source of Turing incomputability. The key observation is that all that fancy higher type mathematics which the logicians carted off into the bushes has a vitally important practical usefulness. Emergent phenomena not only generate descriptions – there is no serendipity here – but derive from them.

One should not be misled by the morphogenesis into thinking of definability as exclusively relating higher order relations to basic algorithmic structure. As is well-known from the mathematics, definability can bring other quite basic aspects of a structure into play via descriptions in terms of objects of similar complexity. At the most basic level, we may find individual objects attaining their identity – that is, a unique role as observable entities – via their context within a wider causal framework.

The picture is one of simple computable rules, with a degree of connectivity underpinning a higher order computation, and emergent forms defined at the edge of computability. We see morphogenesis inhabit the same world a the Mandelbrot set; the same world as the halting problem for a Turing machine;
occupy the same world as the large scale structure we see in the wider universe; and, one speculates, the same world as human mental creativity, linking the synaptic connectivity of the brain via neural net modelling to the most surprising artistic and scientific achievements.

4 The Mind as Mathematics?

There is a strong inclination amongst both computer scientists and philosophers towards some kind of physicalist basis for mentality. For the philosopher, this is still an area with little to agree upon – except the supervenience of the mental upon the physicality of the brain. As Jaegwon Kim puts it in his book Mind in a Physical World (pp.14–15), supervenience:

represents the idea that mentality is at bottom physically based, and that there is no free-floating mentality unanchored in the physical nature of objects and events in which it is manifested.

Or, more mathematically, from the Stanford Encyclopedia of Philosophy:

A set of properties $A$ supervenes upon another set $B$ just in case no two things can differ with respect to $A$-properties without also differing with respect to their $B$-properties.

In this context, one has familiar questions, such as: How can mentality have a causal role in a world that is fundamentally physical? And the puzzle of causal ‘overdetermination’ – the problem of phenomena having both mental and physical causes. As Kim sums up the problem (in Physicalism, or Something Near Enough, 2005):

...the problem of mental causation is solvable only if mentality is physically reducible; however, phenomenal consciousness resists physical reduction, putting its causal efficacy in peril.

For computer scientists – and mathematicians such as Alan Turing – a physicalist approach takes one in the direction of computational models based on what we know of brain functionality.

The computational content of most of these models is not so radically different from that of a Turing machine, though the level of connectivity and parallelism present may present some problems for the reductionists. While back in the real world, the typical observer of brain functionality will probably not be so impressed, and will see a lot of what one knows about brains, from having one, missing. There are various models relevant here, most of them with a lot in common, and discussions which are transferable from one to another. Rodney Brooks comments:

...neither AI nor Alife has produced artifacts that could be confused with a living organism for more than an instant. AI just does not seem as present or aware as even a simple animal and Alife cannot match the complexities of the simplest forms of life.

The mind as a computational instrument presents a level of challenge to the modellers only matched by that facing those wanting to put the standard model
of particle physics. on a more secure footing. And many of us are persuaded of important connections between these modelling challenges, explored in various ways by different researchers – with, for example, the contrasting proposals of Roger Penrose and Henry Stapp having attracted widespread attention.

It is still true that we have very interesting connectionist models for the brain (see [24] for Turing’s contribution), and we are not surprised to see a leading researcher like Paul Smolensky [21] saying:

There is a reasonable chance that connectionist models will lead to the development of new somewhat-general-purpose self-programming, massively parallel analog computers, and a new theory of analog parallel computation: they may possibly even challenge the strong construal of Church’s Thesis as the claim that the class of well-defined computations is exhausted by those of Turing machines.

But, as Steven Pinker puts it: “…neural networks alone cannot do the job” – and we too expect a bit more than the emergence of embodied incomputability, occupying a different level of data to that used for further computation. That is no real advance on the familiar Turing machine. It is not very promising for trans-Turing computing machines. Pinker does not talk about incomputability, but does describe [17, p.124] human thinking as exhibiting “a kind of mental fecundity called recursion”, giving us a good impression of real life emergent phenomena re-entering the system – or the computational process as we would see it.

Is there really something different happening here? If so, how do we model it? Does this finally sink the standard Turing machine model? Neuroscience gives us an impressively detailed picture of brain functionality. Antonio Damasio [9] vividly fills out the picture we get from Pinker:

As the brain forms images of an object - such as a face, a melody, a toothache, the memory of an event - and as the images of the object affect the state of the organism, yet another level of brain structure creates a swift nonverbal account of the events that are taking place in the varied brain regions activated as a consequence of the object-organism interaction. The mapping of the object-related consequences occurs in first-order neural maps representing the proto-self and object; the account of the causal relationship between object and organism can only be captured in second-order neural maps. . . . one might say that the swift, second-order nonverbal account narrates a story: that of the organism caught in the act of representing its own changing state as it goes about representing something else.

The book gives a modern picture of how the human body distributes its ‘second-order’ representations across the impressive connectivity of the human organism, and enables representations to play a role in further thought processes.

Remarkably, once again, we find clues to what is happening mathematically in the 1930s work of Kleene and Turing. In his 1939 Princeton paper, Turing [25] is in no doubt he is saying something mathematically about human mentality:

Mathematical reasoning may be regarded . . . as the exercise of a combination of . . . intuition and ingenuity. . . . In pre-Gödel times it
was thought by some that all the intuitive judgements of mathematics could be replaced by a finite number of ... rules. The necessity for intuition would then be entirely eliminated. In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been in much the same direction.

To cut things short, Turing’s analysis depends on Kleene’s trick of building the constructive ordinals by computably sampling infinitary data needed to compute higher ordinals. The sampling is not canonical, there are lots of ways of doing it computably. This means we have an iterative process analogous to that described by Pinker and Damasio, which exhibits the irreversibility we expect from Prigogine. It is a picture of definable data represented using constructive ordinals. Turing’s representation of data is scientifically standard, in terms of reals.

Buried away in this opaque but wonderful paper is a key idea, that of the oracle Turing machine, which remarkably is just what one needs to model computable ‘causality’ in science. One can represent any of the familiar physical transitions capturable on our computers – and this encompasses a comprehensive swathe of what we regard as scientific – using the appropriate kind of relative computation executable by the right oracle machine. The basic computational structure of the connectivity is captured by functionals modelled by oracle Turing machines. The mathematics delivers a complex structure – an extended Turing model – with a rich overlay of definable relations, corresponding to real world ubiquity of emergent form impacting non-trivially on the development of the organism.

Again, we may present definability as a form of computability via the strangely neglected models from generalised recursion theory. For those wanting to dig out the hidden treasures of this beautiful and suddenly relevant subject, probably the best guide is the 1990 Springer Perspectives in Logic book of Gerald E. Sacks [19] on Higher Recursion Theory.

The physical relevance of this extended model of computable causality fits nicely with the current return to basic notions of key figures concerned with quantum gravity and foundational questions in physics. As Lee Smolin affirms in his book [22] on The Trouble with Physics (p.241): “... causality itself is fundamental”.

Smolin is not, of course, thinking of computationally constrained models of causality in the sense of Turing. But there is a convergence of aims and value put on embodiment of higher order relations on very basic structures. Smolin alludes to the work of ‘early champions’ of the role of causality, such as Roger Penrose, Rafael Sorkin, Fay Dowker, and Fotini Markopoulou, and sets out a version of Penrose’s strong determinism (p.241 again):

It is not only the case that the spacetime geometry determines what the causal relations are. This can be turned around: Causal relations can determine the spacetime geometry ... Its easy to talk about space or spacetime emerging from something more fundamental, but those who have tried to develop the idea have found it difficult to realize in practice. ... We now believe they failed because they ignored the role that causality plays in spacetime. These days, many
of us working on quantum gravity believe that causality itself is fundamental - and is thus meaningful even at a level where the notion of space has disappeared.

5 Embodiment Restored

The difference between the extended Turing model of computation and what is commonly seen from the modellers of process is that there is a proper balance between process and information. The embodiment was a key problem for the early development of the computer, insufficiently recognised since the early days by the theorists, fixated on the universality paradigm.

Rodney Brooks [4] tells how embodiment in the form of stored information has re-emerged in AI:

Modern researchers are now seriously investigating the embodied approach to intelligence and have rediscovered the importance of interaction with people as the basis for intelligence. My own work for the last twenty five years has been based on these two ideas.

The mathematics of the extended Turing model is notoriously for its technicality. And the mathematical character of the global structure based on it is disappointingly pathological from the point of view of mathematical expectations. But if we expect to model the emergence of global relations in terms of local structure – say that of large-scale structures in the real universe, or even of relations expressing globally observed natural laws – then the pathology provides the raw material for an embodied language within which to talk about the complexity of physical forms we discover about us. In retrospect, Hartley Rogers was remarkably prescient when he asked about the character of the Turing invariant relations in a talk entitled Some problems of definability in recursive function theory, at the Tenth Logic Colloquium, at the University of Leicester in 1965. Though at that time people had no appreciation of the physical significance of the question. It appearing increasingly likely that these relations are the key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure. One should be aware though of the schematic nature of the understanding they may provide. They are the equivalent of maps of the landscape provided by satellite scans, while the real work of exploring the substance of down below is at best guided and enlightened by an awareness of the overall structuring.

The character of the all-important automorphism group of the Turing universe is still to be pinned down. In the way of progress towards explanations of such scientific mysteries as the dichotomy between classical and quantum reality, and the removal of the need for speculative assumptions of ‘many-worlds’ and multiverses, is the so-called Bi-interpretability Conjecture, which has haunted us for nearly thirty years. Very roughly speaking, the conjecture says that the Turing definable relations are exactly those with information content describable in second-order arithmetic. One consequence of a positive solution to this problem would be the ruling out of non-trivial automorphisms of the Turing universe. And the breaking of the current match between the computability-theoretic structures and the physical reality we observe. As things are, we see and important partial validation of the conjecture, yielding a rigid substructure of the
Turing universe replete with uniquely defined entities – much like our classical reality we live in: and a wildly ill-defined context, with automorphic displacements corresponding to pre-measurement quantum ambiguity. Currently, the main clues to the outcome of the full Bi-interpretability Conjecture are a lack of more than incremental progress towards a positive answer for the last fifteen years, and an outline of a strategy for constructing a non-trivial automorphism in the public domain since the late 1990s.

We summarise some features of the mathematics, and refer the reader to sources such as [8] and [7] for further detail:

- Embodiment invalidating the ‘machine as data’ and universality paradigm.

- The organic linking of mechanics and emergent outcomes delivering a clearer model of supervenience of mentality on brain functionality, and a reconciliation of different levels of effectivity.

- A reaffirmation of the importance of experiment and evolving hardware, for both AI and extended computing generally.

- The validating of a route to creation of new information through interaction and emergence.

- The significance of definability and its breakdown for the physical universe giving a route to the determination of previously unexplained constants in the standard model of physics, and of quantum ambiguity in terms of a breakdown of definability, so making the multiverse redundant as an explanatory tool, and . . .

- . . . work by Slaman and Woodin and others on establishing partial rigidity of the Turing universe (see [1]) promising an explanation of the existence of our ‘quasi-classical’ universe.

As for building intelligent machines, we give the last word to Danny Hillis, quoted by Mark Williams in Red Herring magazine in April 03, 2001:

I used to think we’d do it by engineering. Now I believe we’ll evolve them. We’re likely to make thinking machines before we understand how the mind works, which is kind of backwards.

So, Turing’s computational modelling is showing good signs of durability – but within a turbulent natural environment, embodied, full of emergent wonders, and exhibiting a computational structure reaching far into unknown regions – via a revived type structure, the relevance of which is currently neglected by both mathematicians, computer scientists, and beyond. But, as I write this at the start of the centenary year of Alan Turing’s birth, there is every sign of a revival of the basic approach which underlay the great discoveries of the first half of the twentieth century.

Turing, as we know, anticipated much of what we are still clarifying about how the world computes. And, we hope, he is smiling down on us at the centenary of his birth, in a Little Venice nursing home (now the Colonnade Hotel).
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