Exact Canonically Conjugate Momenta Approach to a One-Dimensional Neutron-Proton System, I

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October 14, 2014

Abstract

Applying Tomonaga’s idea to nuclei with the aid of Sunakawa’s method, we developed a collective description of nuclear surface oscillations. It gives a microscopic foundation of nuclear collective motions in relation to Bohr-Mottelson model. Introducing collective variables, collective description is formulated with the first quantized language, contrary to the second quantized manner in Sunakawa’s method for a Bose system. It overcomes difficulties remaining in traditional theoretical treatments of nuclear collective motions: Collective momenta in Tomonaga’s manner are not exact canonically conjugate to collective coordinates and they are not independent. In contrast to the above collective description, Tomonaga first gave the another basic idea to approach elementary excitations in a Fermi system. The similar idea was also proposed by Luttinger. The Sunakawa’s method for a Fermi system is anticipated to work well for such a problem. In this paper, on the isospin space, we define a density operator and following Tomonaga, introduce collective momentum. Then we develop an exact canonically momenta approach to a one-dimensional neutron-proton system.

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1 Introduction

To study quarks and $SU(N)$ colored non-Abelian gluons (QCD) in the large-$N$ limit, Jevicki and Sakita developed a collective field formalism involving only gauge-invariant variables[1]. The QCD describes successfully short-distance phenomena due to the property of asymptotic freedom, one still has the difficult problem of color confinement occurring as a large-distance phenomenon. Their basic idea consists in reformulating the quantum collective field theory in terms of gauge-invariant variables. It leads to an effective Hamiltonian which determines the behaviour in the large-$N$ limit by classical stationary point solutions.

On the other hand, in studies of collective motions in nuclei, very difficult problems of large-amplitude collective motion, which are strongly non-linear phenomena in quantum nuclear dynamics, still remain unsolved. How to go beyond the usual mean field theories towards a construction of a theory for large-amplitude collective motions in nuclei [2]?

Applying Tomonaga’s idea in his collective motion theory [3] to nuclei with the aid of the Sunakawa’s discrete integral equation method [4], we developed a collective description of surface oscillations of nuclei [5]. It gives a possible microscopic foundation of nuclear collective motions in relation to Bohr-Mottelson model [6]. Introducing collective variables, a collective description is formulated by using the first quantized language, contrary to the second quantized manner in the Sunakawa’s method for a Bose system. It overcomes the difficulties still remaining in traditional theoretical treatments of nuclear collective motions: Collective momenta in Tomonaga’s manner are not exact canonically conjugate to collective coordinates and they are not independent. Our exact canonically conjugate momenta to collective coordinates are found from a viewpoint different from the canonical transformation and the group theory [7]. Recently we got exact canonical variables revisiting the Tomonaga’s method and described a collective motion also in two-dimensional nuclei [8].

In constructing a collective field theory for $SU(N)$ quantum system [9], we are standing on a situation similar to the above one. It is appreciable as a common feature of strongly nonlinear physics. Applying Tomonaga’s idea, a collective description of an $SU(N)$ system is plausible in terms of collective variables invariant under $SU(N)$ transformation. One of the present authors (S.N.) gave exact canonical variables using the discrete integral equation method [10]. They are regarded as a natural extension of Sunakawa’s variables to the ones of $SU(N)$ quantum system but are derived in the first quantized manner.

In contrast to the above collective description, to approach elementary excitations in a Fermi system, Tomonaga first gave another basic idea [11]. A similar idea was also given by Luttinger [12]. The Sunakawa’s method for a Fermi system [4] may be anticipated to work well for such a problem. In this paper, on the isospin space $(T, T_z)$, we define the density operator $\rho_k^{T,T_z}$ and following Tomonaga, introduce collective momenta. Then we develop an exact canonical momenta approach to a one-dimensional neutron-proton system.

In §2 we introduce collective variables $\rho_k^{0,0}$ and associated variables $\pi_k^{0,0}$ and give commutation relations between them. In §3 exact canonically conjugate momenta $\Pi_k^{0,0}$ is defined by a discrete integral equation. §3 is also devoted to a proof of the exact canonical commutation relation between collective variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$. In §4 dependence of the original Hamiltonian on $\Pi_k^{0,0}$ and $\rho_k^{0,0}$ is determined. §5 is devoted to a calculation of a constant term in the collective Hamiltonian. Finally in §6 some discussions and further perspectives are given. In the Appendices the calculation of some commutators are presented.
2 Collective variables and the associated relations

Let $H$ be the Hamiltonian of a one dimensional proton-neutron system:

$$H = T + V = \int dx \psi^\dagger(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) \psi(x) + \frac{1}{2} \int dx dx' \psi^\dagger(x) \psi^\dagger(x') V(x-x') \psi(x') \psi(x). \quad (2.1)$$

Let us introduce the $z$-component of isospin to distinguish the Neutron and the Proton:

$$\tau_z = \begin{cases} 1/2, & \text{for Neutron,} \\ -1/2, & \text{for Proton.} \end{cases} \quad (2.2)$$

In (2.1), the operators $\psi(x)$ and $\psi^\dagger(x)$ are expanded respectively, as

$$\begin{align*}
\psi(x) &= \frac{1}{\sqrt{L}} \sum_{kz} a_{kz} e^{ikx} \phi_{kz} = \frac{1}{\sqrt{L}} \sum_{kz} \frac{1+2\tau_z}{2} a_{kz} e^{ikx} \phi^+_{kz} + \frac{1}{\sqrt{L}} \sum_{kz} \frac{1-2\tau_z}{2} a_{kz} e^{ikx} \phi^-_{kz}, \\
\psi^\dagger(x) &= \frac{1}{\sqrt{L}} \sum_{kz} a^\dagger_{kz} e^{-ikx} \phi^*_{kz} = \frac{1}{\sqrt{L}} \sum_{kz} \frac{1+2\tau_z}{2} a^\dagger_{kz} e^{-ikx} \phi^*_+_{kz} + \frac{1}{\sqrt{L}} \sum_{kz} \frac{1-2\tau_z}{2} a^\dagger_{kz} e^{-ikx} \phi^*_-_z,
\end{align*} \quad (2.3)$$

and the interaction potential $V(x)$ is also expanded as

$$V(x) = \frac{1}{L} \sum_k \nu(k) e^{ikx}, \quad (2.4)$$

together with the following orthogonal relations:

$$\int (e^{ikx})^* e^{ik'x} dx = \int e^{i(k-k')x} dx = L \delta_{k,k'}, \quad \int \phi^*_z \phi_{z'} d\tau = \delta_{z,z'}. \quad (2.5)$$

where $L$ is the length of a one-dimensional periodic box. The anti-commutation relations among $a_{kz}$ and $a^\dagger_{kz}$ are given as

$$\begin{align*}
\{a_{kz}, a^\dagger_{k'z}\} &= \delta_{k,k'} \delta_{z,z'}, \\
\{a_{kz}, a_{k'z}\} &= \{a^\dagger_{kz}, a^\dagger_{k'z}\} = 0.
\end{align*} \quad (2.6)$$

We define the Fourier component of the density operator ($\rho(x) = \psi^\dagger(x) \psi(x)$) on the isospin space $(T, T_z)$ as

$$\rho^T_{k,T_z} = \frac{\sqrt{2}}{\sqrt{A}} \sum_{p,z} \frac{1}{2} \tau_z (TT_z) a^\dagger_{p+\frac{1}{2},z} (-1)^{\frac{1}{2}+\tau_z} a^\dagger_{p-\frac{1}{2},-z}, \quad \rho^T_{k,T_z} = (-1)^{T_z} \rho^T_{-k,-T_z},$$

$$\rho^0_{0,0} = \frac{2}{\sqrt{2} A} \sum_{p,z} a^\dagger_{p,z} a_{p,z} = \frac{\sqrt{2}}{2 \sqrt{A}} (N+Z) = \sqrt{A},$$

$$\rho^0_{0,1} = \frac{1}{\sqrt{A}} \sum_p \left( a^\dagger_{p,N} a_{p,N} - a^\dagger_{p,p} a_{p,p} \right) = \frac{1}{\sqrt{A}} (N-Z),$$

$$\rho^1_{0,1} = -\frac{\sqrt{2}}{\sqrt{A}} \sum_p a^\dagger_{p,N} a_{p,P}, \quad \rho^0_{0,-1} = \frac{\sqrt{2}}{\sqrt{A}} \sum_p a^\dagger_{p,P} a_{p,N}. \quad (2.7)$$
where \( N, Z \) and \( A \) are the total numbers of the Neutron, the Proton and the N-P System under consideration, respectively [13]. Substituting (2.3) and (2.4) into the expression for \( V \) in (2.1), the \( V \) is expressed as

\[
V = \frac{1}{2L} \sum_{\{k\}, \{\tau_z\}, T, T_z} \nu_T(k) \left\{ \frac{1}{2} \tau'_z \frac{1}{2} \tau''_z | TT_z \rangle \langle \frac{1}{2} \tau''_z \frac{1}{2} \tau'_z | TT_z \rangle a_{k' \tau'_z}^+ a_{k'' \tau''_z} a_{k'' \tau''_z} a_{k' \tau'_z} \right\}
\]

Using (2.6), the commutation relation between \( \rho_{1T, 1T} \) and \( \rho_{2T, 2T} \) is calculated as

\[
\left[ \rho_{1T, 1T}, \rho_{2T, 2T} \right] = \frac{\sqrt{\bar{A}}}{\sqrt{A}} \sum_{T_{1T}, T_{2T}, k} \left\{ \left(-1\right)^{T_{1}+T_{2}+T_{3} - 1} \right\}
\]

\[
\times \sqrt{(2T_{1}+1)(2T_{2}+1)} W \left( \frac{1}{2} \right) (T_{1} T_{2} T_{3}) \rho_{k+1k} \]  

\[
\text{(2.10)}
\]

detailed calculations of which are given in Appendix A. With the aid of (2.9) and (2.10), the commutation relation between \( V \) and \( \rho_{T, T} \) is computed as

\[
\left[ V, \rho_{T_{1}, T_{1}} \right] = \frac{\sqrt{\bar{A}}}{2 \sqrt{2L}} \sum_{T_{1}, T_{2}, k} \nu_{T_{1}}(k) \left\{ \left(-1\right)^{T_{1}+T_{2}+T_{3} - 1} \right\} \sqrt{(2T_{1}+1)(2T_{2}+1)}
\]

\[
\times W \left( \frac{1}{2} \right) (T_{1} T_{2} T_{3}) \rho_{k+1k} \]  

\[
\text{(2.11)}
\]

In the definition of the Fourier component of the density operator (2.7), we concentrate on the two cases of

(I): \( T=0, T_z=0 \),

(II): \( T=1, T_z=0 \).

Then we have

(I): \( \rho_{0,0} = \frac{\sqrt{\bar{A}}}{\sqrt{A}} \sum_{p, \tau_z} \left\{ \left( -1 \right)^{\tau_z - p + \frac{1}{2} \tau_z} a_{p+\frac{1}{2} \tau_z}^+ a_{p+\frac{1}{2} \tau_z} + \frac{1}{\sqrt{A}} \sum_{p, \tau_z} a_{p+\frac{1}{2} \tau_z}^+ a_{p+\frac{1}{2} \tau_z} \right\} \frac{1}{\sqrt{2}} \frac{1}{2} \frac{1}{2} - \tau_z |00\rangle \) \]

\[
\text{(2.12)}
\]
It is possible to prove the last relation in the case (II) of (2.12). Using (2.11), the commutator 
\[ [V, \rho_k^{1,0}] \] is calculated as \[ [V, \rho_k^{1,0}] = -\frac{\sqrt{A}}{\sqrt{2L}} \sum_{k,l} \nu_{T_1}^E(k_1) \left\{ \rho_{k_1}^{1,1} \rho_{k+k_1}^{1,1} + \rho_{k+k_1}^{1,1} \rho_{k_1}^{1,1} - \rho_{k+k_1}^{1,1} \rho_{k_1}^{1,1} - \rho_{k+k_1}^{1,1} \rho_{k_1}^{1,1} \right\} \]
which vanishes if the condition \( \nu_{T_1}^E(k_1) = \nu_{T_1}^E(k+k_1) \) is satisfied. This means \( \nu_{T_1}^E(k_1) \) is constant. Then the variables \( \rho_k^{0,0} \) and \( \rho_k^{1,0} \) become good collective variables. From now on we will find exact canonically conjugate momenta to these collective variables. Following Tomonaga, first we introduce collective momenta associated with the \( \rho_k^{0,0} \) through
\[
\pi_k^{0,0} = \frac{m}{k^2} \rho_{-k}^{0,0} = \frac{m}{k^2} \hbar \left[ H, \rho_{-k}^{0,0} \right] = \frac{m}{k^2} \hbar [T, \rho_{-k}^{0,0}] = \pi_{-k}^{0,0}, \quad (k \neq 0). \tag{2.13}
\]
Calculating the commutator (2.13), we obtain explicit expressions for the associated collective variables \( \pi_k^{0,0} \) as
\[
\pi_k^{0,0} = -i \sqrt{2 \hbar} \sum_{p,\tau_z} \frac{1}{2} \left( \tau_z \right) \sum_{\tau_z} \left[ \phi_{-k,\tau_z}^{0,0} (0) a_p^{\dagger} (-1)^{\frac{1}{2} - \tau_z} a_{p+\frac{1}{2},\tau_z} \right] = -i \sqrt{2 \hbar} \sum_{p,\tau_z} \frac{1}{2} \left( \tau_z \right) \left[ \sum_{\tau_z} \left[ \phi_{-k,\tau_z}^{0,0} (-1)^{\frac{1}{2} - \tau_z} a_{p+\frac{1}{2},\tau_z} \right] \right], \tag{2.14}
\]
where we have used the explicit expression for the kinetic operator \( T \) in (2.9) and
\[
[T, \rho_{-k}^{0,0}] = \sqrt{2 \hbar} \sum_{p,\tau_z} \left\{ \left( \frac{p-k}{2} \right)^2 - \left( \frac{p+k}{2} \right)^2 \right\} \left[ \sum_{\tau_z} \left[ \phi_{-k,\tau_z}^{0,0} (0) a_p^{\dagger} (-1)^{\frac{1}{2} - \tau_z} a_{p+\frac{1}{2},\tau_z} \right] \right], \tag{2.15}
\]
At first \( \pi_k^{0,0} \) is regarded as the collective momenta conjugate to \( \rho_k^{0,0} \) in the sense of Tomonaga [3]. The commutation relations among the variables \( \rho_k^{0,0} \) and \( \pi_k^{0,0} \), however, become as follows:
\[
\begin{align*}
[\rho_k^{0,0}, \rho_{k'}^{0,0}] &= 0, \\
[\pi_k^{0,0}, \rho_{k'}^{0,0}] &= -i \hbar \frac{k'}{\sqrt{A} k} \rho_{k'-k}^{0,0}, \\
[\pi_k^{0,0}, \pi_{k'}^{0,0}] &= -i \hbar \frac{k'}{\sqrt{A} k} (k^2 - k'^2) \pi_{k+k'}^{0,0}.
\end{align*}
\tag{2.16}
\]
These commutation relations have quite the same structures as those of the commutation relations which Sumakawa et al. obtained at the first stage in their earlier work [4]. As is shown from the above, the R.H.S. of the second line does not take the value \(-i \hbar \delta_{kk'}\) and the third one does not vanish. Detailed calculations for them are given in the Appendix B. Then from these facts, it is self-evident that the variables \( \rho_k^{0,0} \) and \( \pi_k^{0,0} \) are not canonically conjugate to each other if we take into account contributions of the order of \( \frac{1}{\sqrt{A}} \). For \( T = 1, T' = 0 \), we have the following commutation relations:
\[
\begin{align*}
[\rho_k^{1,0}, \rho_{k'}^{1,0}] &= 0, \quad [\rho_k^{1,0}, \rho_{k'}^{1,0}] = 0, \\
[\pi_k^{1,0}, \rho_{k'}^{0,0}] &= -i \hbar \frac{k'}{\sqrt{A} k} \rho_{k'-k}^{0,0}, \quad [\pi_k^{1,0}, \rho_{k'}^{0,0}] = -i \hbar \frac{k'}{\sqrt{A} k} \rho_{k'-k}^{0,0}, \\
[\pi_k^{1,0}, \pi_{k'}^{1,0}] &= -i \hbar \frac{(k^2 - k'^2)}{\sqrt{A} k} \pi_{k+k'}^{1,0}, \quad [\pi_k^{1,0}, \pi_{k'}^{1,0}] = -i \hbar \frac{(k^2 - k'^2)}{\sqrt{A} k} \pi_{k+k'}^{1,0}.
\end{align*}
\tag{2.17}
\]
The structures of the commutators (2.17) are seen to have twisted properties comparing with those of the commutators (2.16).
3 Exact canonically conjugate momenta

In order to overcome the difficulties mentioned in the preceding section, we define the exact canonically conjugate momenta $\Pi_{k,0}^{0,0}$ by

$$\Pi_{k,0}^{0,0} = \pi_{k,0}^{0,0} - \frac{1}{\sqrt{A}} \sum_{p \neq k} p \rho_{p-k}^{0,0} \Pi_{p,0}^{0,0} (k \neq 0), \quad \Pi_{k,0}^{0,0} = \Pi_{k,0}^{0,0}.$$  \hspace{1cm} (3.1)

This type of discrete integral equation was first proposed by Sunakawa et al. in their second quantized collective formalism for an interacting boson system [4]. It was also proposed by the present author in his first quantized description of a quadrupole type nuclear collective motion [5]. As is clear from their structures, the new variables $\Pi_{k,0}^{0,0}$ are no longer one-body operators but essentially many-body operators. From (3.1), we get the exact canonical commutation relations and the commutators between the new $\Pi_{k,0}^{0,0}$ and the old $\pi_{k,0}^{0,0}$ as

$$[\rho_{k}^{0,0}, \rho_{k'}^{0,0}] = 0, \quad [\Pi_{k}^{0,0}, \rho_{k'}^{0,0}] = -i\hbar \delta_{kk'}, \quad [\Pi_{k}^{0,0}, \Pi_{k'}^{0,0}] = 0, \quad (3.2)$$

and the commutator between $\pi_{k,0}^{0,0}$ and $\Pi_{k,0}^{0,0}$ as

$$[\pi_{k,0}^{0,0}, \Pi_{k,0}^{0,0}] = \frac{i\hbar}{\sqrt{A}} (k + k') \Pi_{k,k'}^{0,0}, \quad (3.3)$$

derivation of (3.3) is given in Appendix B. Following Sunakawa’s method, the above exact canonical commutation relations are easily proved as follows. First, iterating the discrete integral equation (3.1) and using (2.16), we get

$$\begin{align*}
[\Pi_{k,0}^{0,0}, \rho_{k'}^{0,0}] &= [\pi_{k,0}^{0,0}, \rho_{k'}^{0,0}] - \frac{1}{\sqrt{A}} \sum_{p \neq k} \rho_{p-k}^{0,0} [\pi_{p,0}^{0,0}, \rho_{k'}^{0,0}] \\
&+ \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p} q \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{q-k'}^{0,0} \rho_{k'}^{0,0} \\
&- \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p, r \neq q} r \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{r-q}^{0,0} \rho_{k'}^{0,0} \\
&= -\frac{i\hbar \rho_{k'}^{0,0}}{\sqrt{A}} + \frac{1}{\sqrt{A}} \sum_{p \neq k} p \rho_{p-k}^{0,0} \rho_{k'}^{0,0} \\
&- \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p} q \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{k'}^{0,0} \\
&+ \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p, r \neq q} r \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{r-q}^{0,0} \rho_{k'}^{0,0} \\
&= -\frac{i\hbar \rho_{k'}^{0,0}}{\sqrt{A}} + \frac{i\hbar \rho_{k'}^{0,0}}{\sqrt{A}} (1 - \delta_{kk'}) \\
&+ \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, k \neq p} \rho_{p-k}^{0,0} \rho_{k'}^{0,0} - \frac{i\hbar \rho_{k'}^{0,0}}{\sqrt{A}} \sum_{p \neq k, k \neq p} \rho_{p-k}^{0,0} \rho_{k'}^{0,0} - \cdots \\
&= -\frac{i\hbar \rho_{k'}^{0,0} \rho_{k'}^{0,0}}{\sqrt{A}} \delta_{kk'} = -i\hbar \delta_{kk'}, \quad (\rho_{0,0}^{0,0} = \sqrt{A}).
\end{align*}$$

In each summation, we separate the part with $p = k'$ and that with $p \neq k'$ and use the relation $\rho_{0,0}^{0,0} = \sqrt{A}$. All terms which involve higher order powers of $\rho_k^{0,0}$ cancel out except for the term $\rho_{k'}^{0,0}$. The remaining terms lead to the desired canonical commutation relations.
the c-number term on the R.H.S. of the equation in the last line which arises due to the exclusion of the \( p = k' \) term if \( k = k' \) in the summation with respect to \( p \). Thus we could reach the \textit{exact} canonical commutation relation. In order to assert that the operators \( \Pi^{0,0}_k \)'s are \textit{exact} canonical conjugate to \( \rho^{0,0}_k \)'s, we must give a proof on the commutativity of the \( \Pi^{0,0}_k \)'s for different \( k \)'s. The commutation relation between the \( \Pi^{0,0}_k \)'s is calculated tediously but straightforwardly as

\[
[\Pi^{0,0}_k, \Pi^{0,0}_{k'}] = -\frac{i\hbar}{\sqrt{Ak''}}(k^2 - k'^2)p^{0,0}_{k+k'} + \frac{i\hbar}{\sqrt{Ak'}}(k^2 - k'^2)p^{0,0}_{k+k'}
\]

(3.5)

In the above we have introduced the operator-valued function \( F(k; k') \) defined below. It is shown that the \( F(k; k') \) vanishes exactly owing to the definition of the \textit{exact} canonical conjugate momenta \( \Pi^{0,0}_k \),

\[
F(k; k') \equiv -\frac{i\hbar}{\sqrt{Ak''}}(k^2 - k'^2)p^{0,0}_{k+k'} \left\{ \frac{\Pi^{0,0}_k}{\sqrt{A(k + k')}} - \frac{i\hbar}{\sqrt{A(k + k')}} \sum_{p \neq k} p\rho^{0,0}_{p-(k+k')} \right\}
\]

(3.6)

Thus we have proved the \textit{exact} canonical commutation relations for \( \rho^{0,0}_k \) and \( \Pi^{0,0}_k \). The hermiticity property in (3.1) can also be proved with the help of (2.12), (2.13) and (3.4). With the aid of the deductive method, it is shown easily to satisfy (3.3) by using (2.16).

For \( T = 1, \ T = 0 \), we have the following commutation relations:

\[
\Pi^{1,0}_k = \frac{1}{\sqrt{A}} \sum_{p \neq k} p\rho^{0,0}_{p-k} \Pi^{1,0}_p \quad (k \neq 0), \quad \Pi^{1,0}_p = \Pi^{1,0}_p \cdot (3.7)
\]

\[
[\rho^{1,0}_k, \rho^{1,0}_{k'}] = 0, \quad [\Pi^{1,0}_k, \Pi^{1,0}_{k'}] = -i\hbar \delta_{kk'}, \quad (3.8)
\]

Due to the twisted properties of (2.17), unfortunately, the commutators \( [\Pi^{1,0}_k, \Pi^{1,0}_{k'}] \) do not vanish. Then, strict speaking, the \( \rho^{1,0}_k \) and \( \Pi^{1,0}_k \) are not the \textit{exact} canonical variables.
4 The $\Pi_k$- and $\phi_k$-dependence of the Hamiltonian

From the original Hamiltonian (2.1) we derive here a collective Hamiltonian in terms of the exact canonical variables $\rho_{k}^{0,0}$ and $\Pi_{k}^{0,0}$. The potential part $V$ is already written in terms of $\rho_{k}^{0,0}$. Our task is therefore to express the kinetic part $T$ in terms of the exact canonical variables $\rho_{k}^{0,0}$ and $\Pi_{k}^{0,0}$. For this purpose, following Sunakawa’s method, first we expand it in a power series of the exact canonical conjugate momenta $\Pi_{k}^{0,0}$ as follows:

$$T = T_0(\rho) + \sum_{p \neq 0} T_1(\rho; p)\Pi_{p}^{0,0} + \sum_{p \neq 0, q \neq 0} T_2(\rho; p, q)\Pi_{p}^{0,0}\Pi_{q}^{0,0} + \cdots,$$

(4.1)

where $T_n(n \neq 0)$ are the unknown expansion coefficients. In order to get their explicit expressions, we take the commutators between $T$ and $\rho_{k}^{0,0}$. On the other hand, from Eqs. (2.12) and (2.13), we can calculate directly values of the commutators between $T$ and $\rho_{k}^{0,0}$ by using the definition (3.1) as follows:

$$[T, \rho_{-k}^{0,0}] = \frac{\hbar}{i} \rho_{-k}^{0,0} = -\frac{\hbar k^2}{m} \pi_{-k}^{0,0}$$

(4.2)

Using Eq. (3.4) successively, we can easily obtain the commutators

$$[[T, \rho_{k}^{0,0}], \rho_{k'}^{0,0}] = \begin{cases} -\frac{\hbar^2 k^2}{m}, & \text{for } k' = -k, \\
\frac{\hbar^2 k k'}{\sqrt{A m}} \rho_{k+k'}^{0,0}, & \text{for } k' \neq -k,
\end{cases}$$

(4.3)

and so on. Comparing the above results with the commutators between $T$ of (4.1) and $\phi_{k}^{0,0}$, we can determine the coefficients $T_n(n \neq 0)$. Then we can express the kinetic part $T$ in terms of the exact canonical variables $\rho_{k}^{0,0}$ and $\Pi_{k}^{0,0}$ as follows:

$$T = T_0(\rho) + \frac{1}{2m} \sum_{k} k^2 \Pi_{k}^{0,0}\Pi_{-k}^{0,0} - \frac{1}{\sqrt{A 2m}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_{p}^{0,0}\rho_{q}^{0,0}\Pi_{p}^{0,0}\Pi_{q}^{0,0}.$$ 

(4.5)

Here we should stress that up to the present stage, all the expressions are derived without any approximation.

Our remaining task in this section is to determine the term $T_0(\rho)$ in (4.1) which depends only on $\rho_{k}^{0,0}$. For this purpose, we also expand it in a power series of the collective coordinates $\rho_{k}^{0,0}$ in the form

$$T_0(\rho) = C_0 + \sum_{p \neq 0} C_1(p)\rho_{p}^{0,0} + \sum_{p \neq 0, q \neq 0} C_2(p, q)\rho_{p}^{0,0}\rho_{q}^{0,0} + \cdots,$$

(4.6)

where $C_2(p, q) = C_2(q, p)$. The expansion coefficients should be determined by a procedure similar to the one used in the previous section. From the definition (3.1), we get easily the discrete integral equation

$$[\Pi_{k}^{0,0}, T_0(\rho)] = f_k(\rho) - \frac{1}{\sqrt{A k}} \sum_{p \neq 0, k} p \rho_{p-k}^{0,0}[\Pi_{p}^{0,0}, T_0(\rho)],$$

(4.7)
and the inhomogeneous term $f_k(\rho)$ becomes

$$\begin{align*}
f_k(\rho) & \equiv [\pi^{0,0}_k, T_0(\rho)] \\
& = \left[ \pi^{0,0}_k, T - \frac{1}{2m} \sum_p p^2 \Pi^0_p \Pi^0_p + \frac{1}{\sqrt{A}} \frac{1}{2m} \sum_{p \neq 0, q \neq 0, p + q \neq 0} pq \rho^{0,0}_p \Pi^0_p \Pi^0_q \right] \\
& = [\pi^{0,0}_k, T] - \frac{i\hbar}{Am} \sum_{p \neq 0, q \neq 0} pq \rho^{0,0}_p \Pi^0_p \Pi^0_q, \quad (k \neq 0)
\end{align*}$$

(4.8)

with the aid of the result (4.5) and the commutation relation (3.3). At first glance the operator-valued function $f_k(\rho)$ seems to depend on $\Pi^0_k$. However, it turns out that $f_k(\rho)$ does not really depend on $\Pi^0_k$ because it commutes with $\rho^{0,0}_k$ as shown below

$$\begin{align*}
[\rho^{0,0}_k', f_k(\rho)] &= [\rho^{0,0}_k', [\pi^{0,0}_k, T]] + \frac{2\hbar^2 k'}{Am} \sum_{p \neq 0} p \rho^{0,0}_p \Pi^0_p \\
& = -2\hbar^3 \frac{k'(k - k')}{\sqrt{Am}} \pi^{0,0}_{k-k'} + \frac{2\hbar^2 k'}{Am} \sum_{p \neq 0} p \rho^{0,0}_p \Pi^0_p \\
& = -2\hbar \frac{k'(k - k')}{\sqrt{Am}} \left\{ \pi^{0,0}_{k-k'} - \frac{1}{\sqrt{A(k - k')}} \sum_{p \neq 0} p \rho^{0,0}_p \Pi^0_p \right\} = 0.
\end{align*}$$

(4.9)

In the above we have used the commutation relation

$$[\rho^{0,0}_k', [\pi^{0,0}_k, T]] = -2\hbar \frac{k'(k - k')}{\sqrt{Am}} \pi^{0,0}_{k-k'},$$

(4.10)

which is proved with the help of the commutation relation between $\pi^{0,0}_k$ and $T$,

$$[\pi^{0,0}_k, T] = -\frac{i\hbar}{\sqrt{Am}} \frac{h^2}{2m} \sum_{p, \tau_\zeta} p \left( p^2 \frac{\tau_\zeta}{2} - \frac{p}{2} \right) a^\dagger_{p-\frac{1}{2}, \tau_\zeta} a_{p+\frac{1}{2}, \tau_\zeta} - \frac{i\hbar}{\sqrt{Am}} \sum_{p, \tau_\zeta} p a^\dagger_{p-\frac{1}{2}, \tau_\zeta} a_{p+\frac{1}{2}, \tau_\zeta}.$$  

(4.11)

Up to the present stage, all the expressions are exact.

From now on, we make an approximation to calculate $T_0(\rho)$ up to the order of $\frac{1}{A}$. First we use $\rho^{0,0}_0 = \sqrt{A}$, $\Pi^0_k \approx \pi^{0,0}_k$ and give approximate expressions for $\pi^{0,0}_k$ and $\rho^{0,0}_k$ as $\pi^{0,0}_k \approx -\frac{i\hbar}{2} \sum_{\tau_\zeta} (\theta^* a_{k, \tau_\zeta} - a^\dagger_{k, \tau_\zeta} \theta)$ and $\rho^{0,0}_k \approx \sum_{\tau_\zeta} (\theta a_{k, \tau_\zeta} + a^\dagger_{k, \tau_\zeta} \theta)$, where $a_{0, \tau_\zeta} \approx \sqrt{A} \theta^*$ and $a_{0, \tau_\zeta} \approx \sqrt{A} \theta$. The $\theta$ and $\theta^*$ are Grassmann numbers and anti-commute with $a_{k, \tau_\zeta}$ and $a^\dagger_{k, \tau_\zeta}$. Then the second term in the last line of (4.8) is computed approximately as

$$\begin{align*}
-\frac{i\hbar}{Am} \sum_{p \neq 0, q \neq 0} pq \rho^{0,0}_p \Pi^0_p \Pi^0_q \approx -\frac{i\hbar}{\sqrt{Am}} \sum_{p \neq 0, p \neq k} p(k-p) \pi^{0,0}_p \pi^{0,0}_{k-p} \\
& = \frac{i\hbar^3}{4\sqrt{Am}} \sum_{p \neq 0, p \neq k} p(k-p) \sum_{\tau_\zeta} \left( \theta^* a_{p, \tau_\zeta} - a^\dagger_{p, \tau_\zeta} \theta \right) \sum_{\tau'_\zeta} \left( \theta^* a_{k-p, \tau'_\zeta} - a^\dagger_{k-p, \tau'_\zeta} \theta \right) \\
& = \frac{i\hbar^3}{4\sqrt{Am}} \sum_{p \neq 0, p \neq k} p(k-p) \sum_{\tau_\zeta} \left( \rho^{0,0}_0 - 2a^\dagger_{p, \tau_\zeta} \theta \right) \sum_{\tau'_\zeta} \left( \rho^{0,0}_0 - 2a^\dagger_{(k-p), \tau'_\zeta} \theta \right) \\
& \approx \theta^* \frac{i\hbar^3}{\sqrt{Am}} \sum_{p, \tau_\zeta} p^2 a^\dagger_{p-\frac{1}{2}, \tau_\zeta} a_{p+\frac{1}{2}, \tau_\zeta} - \theta^* \frac{i\hbar^3}{4\sqrt{Am}} \pi^{0,0}_0 \rho^{0,0}_{k-p} \\
& + \theta^* \frac{i\hbar^3}{\sqrt{Am}} \sum_{p, \tau_\zeta \neq \tau'_\zeta} \left( p^2 - \frac{k^2}{4A} \right) a^\dagger_{p-\frac{1}{2}, \tau_\zeta} a_{p+\frac{1}{2}, \tau'_\zeta} - \theta^* \frac{i\hbar^3}{\sqrt{Am}} \sum_{\tau_\zeta} p^2 + O(1.A).
\end{align*}$$

(4.12)
From Eq. (4.8), combining (4.11) with (4.12), we can obtain an approximate expression for the operator-valued function $f_k(\rho)$ up to the order of $\frac{1}{\sqrt{A}}$ as

$$f_k(\rho) = -\frac{i\hbar^3}{4m}k^2\rho_{-k}^0 + \frac{i\hbar^3}{4m\sqrt{A}}\sum_{p\neq 0, q\neq 0} p(k^2 - pk)\rho_{-p-k-0}^0 - \frac{i\hbar^3}{\sqrt{Am}}\sum_{p} p^2 + O\left(\frac{1}{A}\right), \quad \text{(4.13)}$$

if the Grassmann numbers $\theta$ and $\theta^*$ satisfy a condition $\theta\theta^* = 1$. This is just the same form as that obtained by Sunakawa [4] except that there exists the last constant term. The condition $\theta\theta^* = 1$ is not good for a realistic fermion system. At the moment, however, we use this condition. Then substituting (4.13) into (4.14), we can get the R.H.S. of the discrete integral equation (4.14) as

$$[\Pi_k^0, T_0(\rho)] = -\frac{i\hbar^3 k^2}{4m} \rho_{-k}^0 + \frac{i\hbar^3}{4m\sqrt{A}} \sum_{p\neq 0, q\neq 0} p(k^2 - pk + p^2)\rho_{-p-k}^0 + O\left(\frac{1}{A}\right). \quad \text{(4.14)}$$

From (4.14) and the commutation relations (2.16) and (3.2), we get

$$[\Pi_{k', 0}, [\Pi_k^0, T_0(\rho)]] = \frac{-i\hbar^5}{4m\sqrt{A}}(k^2 + kk' + k'^2)\delta_{k', -k-k'}, \quad \text{(4.15)}$$

$$[\Pi_{k'', 0}, [\Pi_{k', 0}, [\Pi_k^0, T_0(\rho)]]] = 0.$$

By a procedure similar to the one in section 4, we can determine the coefficients $C_n(n \neq 0)$ and then get an approximate form of $T_0(\rho)$ in terms of variables $\rho_k^0$ in the following form:

$$T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{p \neq 0} p^2 \rho_p^0 \rho_{-p}^0$$

$$- \frac{\hbar^2}{24m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p + q \neq 0} (p^2 + pq + q^2) \rho_p^0 \rho_q^0 \rho_{-p-q}^0 + O\left(\frac{1}{A}\right). \quad \text{(4.16)}$$

Using the following identities:

$$\sum_{p \neq 0, q \neq 0, p + q \neq 0} p^2 \rho_p^0 \rho_q^0 \rho_{-p-q}^0 = \sum_{p \neq 0, q \neq 0, p + q \neq 0} (p + q)^2 \rho_p^0 \rho_q^0 \rho_{-p-q}^0,$$

$$\sum_{p \neq 0, q \neq 0, p + q \neq 0} p^2 \rho_p^0 \rho_q^0 \rho_{-p-q}^0 = -2 \sum_{p \neq 0, q \neq 0, p + q \neq 0} pq \rho_p^0 \rho_q^0 \rho_{-p-q}^0,$$ \quad \text{(4.17)}

Eq. (4.16) is rewritten as

$$T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{p \neq 0} p^2 \rho_p^0 \rho_{-p}^0 + \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p + q \neq 0} pq \rho_p^0 \rho_q^0 \rho_{-p-q}^0. \quad \text{(4.18)}$$

The constant term $C_0$ remains undetermined. The detailed calculation of $C_0$ will be given in the next section.
5 Calculation of the constant term

In this section, finally we calculate the constant term \( C_0 \) appearing firstly in Eq. (4.18). Substituting (4.18) into (4.5), the constant term \( C_0 \) is computed up to the order of \( \frac{1}{A} \):

\[
C_0 = T - \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \rho_k^0 \rho_{-k} - \frac{1}{2m} \sum_{k \neq 0} k^2 \Pi_k^0 \Pi_{-k}^0 + \frac{1}{2mA} \sum_{k \neq 0, p \neq 0, k + p \neq 0} k \rho_{k + p}^0 \Pi_k^0 \Pi_p^0 \tag{5.1}
\]

Using \( \Pi_k^0 \approx \pi_k^0 \), \( \rho_k^0 \approx \sum_\tau (\theta^* a_{k,\tau} + a_{k,\tau}^\dagger \theta) \) and \( \pi_k^0 \approx \frac{-i \hbar}{2m} \sum_\tau (\theta^* a_{k,\tau} - a_{-k,\tau}^\dagger \theta) \), we can easily calculate each term in the above as Sunakawa et al. did [4] and get the following results:

\[
- \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \rho_k^0 \rho_{-k} = - \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \sum_\tau (\theta^* a_{k,\tau} + a_{k,\tau}^\dagger \theta) \sum_\tau (\theta^* a_{k,\tau}^\dagger + a_{-k,\tau} \theta) \tag{5.2}
\]

\[
= -\theta \theta^* \sum_\tau \frac{\hbar^2 k^2}{4m} a_{k,\tau} a_{k,\tau}^\dagger - \theta \theta^* \sum_\tau \frac{\hbar^2 k^2}{4m} a_{k,\tau}^\dagger a_{k,\tau} \tag{5.3}
\]

\[
= -\theta \theta^* \sum_\tau \frac{\hbar^2 k^2}{4m} a_{k,\tau} a_{k,\tau}^\dagger - \theta \theta^* \sum_\tau \frac{\hbar^2 k^2}{4m} a_{k,\tau}^\dagger a_{k,\tau} \tag{5.4}
\]

\[
= 0.
\]

\[
- \frac{\hbar^2}{8m} \sum_{k \neq 0, p \neq 0, k + p \neq 0} k \rho_{k + p}^0 \Pi_k^0 \Pi_p^0 \tag{5.5}
\]

\[
= 0.
\]

Substituting Eqs (5.2) \( \sim \) (5.5) into (5.1), we have a result

\[
C_0 \cong (1 - \theta \theta^*) T + \theta \theta^* \frac{\hbar^2}{4m} \sum_k k^2 - \theta \theta^* \sum_\tau \frac{\hbar^2 k^2}{2m} a_{k,\tau} a_{k,\tau}^\dagger \tag{5.6}
\]

where we have used the relation \( \theta \theta^* = 1 \) and neglected the term \( \sum_{k,\tau \neq \tau'} \frac{\hbar^2 k^2}{2m} a_{k,\tau} a_{k,\tau}^\dagger \). The final result is not identical with Sunakawa’s one [4] for a Bose system. On the contrary, in this work, we treat a Fermion system. Then we get a result which is natural for a Fermion system.
6 Discussions and further perspectives

Using (4.5), (4.18) and (5.6), we reach our final goal of expressing the original Hamiltonian $H$ in terms of the exact canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ in the following form:

$$H = \frac{A}{4L} (\alpha_T^F(0) - \nu_T(0)) - \frac{A}{4L} \sum_{k \neq 0} \nu_T(0) + \frac{\hbar^2 k^2}{4m}$$

$$+ \sum_{k \neq 0} \left\{ \frac{k^2}{2m} \Pi_k^{0,0} \Pi_{-k}^{0,0} + \frac{\hbar^2 k^2}{8m} + \frac{A}{4L} \Pi_T^F(0) \right\} \rho_k^{0,0} \rho_{-k}^{0,0}$$

$$- \frac{1}{2m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} p q \rho_p^{0,0} \rho_{q}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0} + \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} p q \rho_p^{0,0} \rho_{q}^{0,0} \rho_{p-q}^{0,0} + O\left(\frac{1}{A}\right),$$

(6.1)

which is just Sunakawa’s one up to the order of $\frac{1}{A}$ [4] except the sign of the term $\sum_k \frac{\hbar^2 k^2}{4m}$ in the R.H.S. of the first line of (6.1). This is because we treat a Fermion system. If we introduce Boson annihilation and creation operators $\alpha_k = \sqrt{\frac{mE_k}{2k^2 \hbar}} \rho_k^{0,0} + \frac{ik}{\sqrt{2mE_k}} \Pi_k^{0,0}$ and $\alpha_k^\dagger = \sqrt{\frac{mE_k}{2k^2 \hbar}} \rho_k^{0,0} - \frac{ik}{\sqrt{2mE_k}} \Pi_k^{0,0}$, the sum $\sum_{k \neq 0} \frac{\hbar^2 k^2}{4m} \alpha_T^F(0) + \frac{k^2}{2m} \Pi_k^{0,0} \Pi_{-k}^{0,0} + \frac{\hbar^2 k^2}{8m} + \frac{A}{4L} \Pi_T^F(0) \rho_k^{0,0} \rho_{-k}^{0,0}$ in (6.1) is diagonalized as $\sum_k E_k \alpha_k^\dagger \alpha_k$ $E_k = \sqrt{\frac{\hbar^2 k^2}{m} + \frac{\hbar^2 k^2}{4m} + \frac{A \nu_T(0)}{2L}}$, as shown in [14]. To convert Eq. (6.1) into a coordinate representation, we utilize collective field variables $\tilde{\rho}(x)$ and $\tilde{\Pi}(x)$ defined by the Fourier transformations of the exact canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ as

$$\tilde{\rho}(x) \equiv \frac{\sqrt{A}}{L} \sum_{k \neq 0} \rho_k e^{ikx}, \quad \tilde{\rho}'(x) \equiv \frac{\sqrt{A}}{L} \sum_{k \neq 0} \rho_k e^{ikx}, \quad \tilde{\rho}(x) = n + \tilde{\rho}'(x) \left(n = \frac{A}{L}\right),$$

(6.2)

$$\tilde{\Pi}(x) \equiv \frac{1}{\sqrt{A}} \sum_{k \neq 0} \Pi_k e^{ikx},$$

(6.3)

which reads the following coordinate representation of the collective field Hamiltonian:

$$H = \sum_k \frac{\hbar^2 k^2}{4m} + \int dx V(x) \tilde{\rho}(x) + \int dx \left[ \frac{1}{2} \partial_x \tilde{\Pi}(x) \tilde{\rho}(x) \partial_x \tilde{\Pi}(x) + \hbar^2 \frac{8}{m} \left( \frac{\tilde{\rho}'(x)}{n} + \frac{\tilde{\rho}'(x)}{n^2} \right) \partial_x \tilde{\rho}(x) \partial_x \tilde{\rho}(x) \right],$$

(6.4)

which is identical with the result given in [15] except the sign of the first term $\sum_k \frac{\hbar^2 k^2}{4m}$. As mentioned previously, Tomonaga first gave the basic idea to approach an elementary excitation in a Fermi system [11]. A similar idea was also proposed by Luttinger [12]. The Sunakawa’s method for a Fermi system [4] may be anticipated to work well for such a problem. So, following Tomonaga [11], we separate each density operator into two parts as

$$\rho_k^{0,0} = \rho_k^{0,0}_{+} + \rho_k^{0,0}_{-}, \quad \rho_k^{0,0}_{+} \equiv \frac{1}{\sqrt{A}} \sum_{\mu > 0, \tau z} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0}, \quad \rho_k^{0,0}_{-} \equiv \frac{1}{\sqrt{A}} \sum_{\mu < 0, \tau z} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0},$$

(6.5)

Further according to Tomonaga [11], we introduce collective momenta $\pi_k^{0,0(+)_{+}}$ defined as

$$\pi_k^{0,0(+)_{+}} \equiv \frac{m}{\hbar k^2} \rho_k^{0,0} - \frac{m i}{\hbar k^2} \left[ H, \rho_k^{0,0(+)_{+}} \right] = \frac{m i}{\hbar k^2} \left[ T, \rho_k^{0,0(+)_{+}} \right], \quad (k \neq 0).$$

(6.6)

Calculating the commutator (6.6), we obtain the explicit expressions for the associated collective variables $\pi_k^{0,0(+)_{+}}$ given as

$$\pi_k^{0,0(+)_{+}} = - \frac{\hbar}{\sqrt{Ak}} \sum_{\mu > 0, \tau z} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0}, \quad \pi_k^{0,0(+)_{+}} = - \frac{i \hbar}{\sqrt{Ak}} \sum_{\mu < 0, \tau z} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0} \rho_k^{0,0},$$

(6.7)

Under the above preliminaries, we can develop an exact canonically momenta approach to a one-dimensional neutron-proton system, for example, to a problem of describing a vibration of $T=0$ rod [16] in a one-dimensional nuclei.
Acknowledgements

One of the authors (S.N.) would like to express his sincere thanks to Professor Alex H. Blin for kind and warm hospitality extended to him at the Centro de Física Teórica, Universidade de Coimbra. This work was supported by the Portuguese Project POCTI/FIS/451/94. S.N. was supported by the Portuguese program POCTI/FIS/451/944.
Appendix

A Commutation relation between $\rho_{k_1}^{T_1, T_2}$ and $\rho_{k_2}^{T_2, T_2}$

Using the definition $\rho_k^{T,T} = \frac{\sqrt{2}}{\sqrt{A}} \sum_{p, r_z, r_z'} \frac{1}{2} r_z \frac{1}{2} r_z' \langle TT | a_p^\dagger \rho_{p, k, \frac{1}{2}, r_z} (\frac{1}{2})^{zr_z} a_p - \frac{k}{2} - r_z' \rangle$ given in the first relation of (2.7), the commutation relation between $\rho_{k_1}^{T_1, T_1}$ and $\rho_{k_2}^{T_2, T_2}$ is calculated as

$$\left[ \rho_{k_1}^{T_1, T_1}, \rho_{k_2}^{T_2, T_2} \right]$$

$$= \frac{2}{\sqrt{A}} \sum_{p_1, r_z_1, r_z_1', r_2, r_2', r_2''} \frac{1}{2} r_{z_1} \frac{1}{2} r_{z_1}' \langle T_1 T_1 | \langle T_2 T_2 | (\frac{1}{2})^{\tau_1 + \tau_1'} - (\frac{1}{2})^{\tau_2 + \tau_2'} \rangle$$

$$\times \left( a_{p_1 + k_1, r_z_1} a_{p_1 + k_1, \frac{1}{2}, r_z_1'} a_{p_2 + k_2, r_2} a_{p_2 + k_2, \frac{1}{2}, r_2'} - a_{p_1 + k_1, r_z_1} a_{p_1 + k_1, \frac{1}{2}, r_z_1'} a_{p_2 + k_2, r_2} a_{p_2 + k_2, \frac{1}{2}, r_2'} \right)$$

$$= \frac{2}{\sqrt{A}} \sum_{p_z, r_z, r_z', r_2, r_2', r_2''} \frac{1}{2} r_{z_1} \frac{1}{2} r_{z_1}' \langle T_1 T_1 | \langle T_2 T_2 | (\frac{1}{2})^{\tau_1 + \tau_1'} - (\frac{1}{2})^{\tau_2 + \tau_2'} \rangle$$

$$\times \left( a_{p_z + k_1 + k_2, r_z} a_{p_z + k_1 + k_2, \frac{1}{2}, r_z'} a_{p_z + k_1 + k_2, r_2} a_{p_z + k_1 + k_2, \frac{1}{2}, r_2'} \right)$$

$$= \frac{2}{\sqrt{A}} \sum_{T_3, T_3} \sqrt{(2T_2 + 1)(2T_3 + 1)W (\frac{1}{2})^{T_2 T_3; T_1} (\frac{1}{2})^{T_3 T_3 T_2 - T_2 T_1} T_1 - T_2)}$$

$$\times \left( a_{p_z + k_1 + k_2, r_z} a_{p_z + k_1 + k_2, \frac{1}{2}, r_z'} a_{p_z + k_1 + k_2, r_2} a_{p_z + k_1 + k_2, \frac{1}{2}, r_2'} \right)$$

$$= \frac{2}{\sqrt{A}} \sum_{T_3, T_3} \left( (-1)^{T_1 T_2 + T_3} \sqrt{(2T_2 + 1)(2T_3 + 1)W (\frac{1}{2})^{T_2 T_3; T_1} (\frac{1}{2})^{T_3 T_3 T_2 - T_2 T_1} T_1 - T_2)} \right)$$

where we have used again the definition $\rho_k^{T, T}$. The equation (A.1) is just the commutation relation given by (2.10).
B Derivation of Eqs. (2.16) and (3.3)

In this Appendix, first we will give the proof of the second and last commutation relations in Eq. (2.16). From the explicit expressions for $\rho_{k'}^0$ (2.12) and $\pi_k^0$ (2.14), the commutators among them are computed as

$$
\left[\pi_k^0, \rho_{k'}^0\right] = -i\hbar \frac{1}{A k^2} \sum_{p, p', \tau_z} pk' \left[ a_{p - \frac{1}{2}, \tau_z} a_{p + \frac{1}{2}, \tau_z} - a_{p' - \frac{1}{2}, \tau_z} a_{p' + \frac{1}{2}, \tau_z} \right]
$$

$$
= -i\hbar k' k \sum_{p, \tau_z} a_{p - \frac{1}{2}, \tau_z} a_{p + \frac{1}{2}, \tau_z} + \pi_{p + \frac{1}{2}, \tau_z}
$$

$$
= -i\hbar \frac{\sqrt{A}}{k} \pi_{k', k'}^0
$$

and

$$
\left[\pi_k^0, \pi_{k'}^0\right] = -i\hbar^2 \frac{1}{A k^2} \sum_{p, p', \tau_z} pkp' k' \left[ a_{p - \frac{1}{2}, \tau_z} a_{p + \frac{1}{2}, \tau_z} - a_{p' - \frac{1}{2}, \tau_z} a_{p' + \frac{1}{2}, \tau_z} \right]
$$

$$
= -i\hbar^2 \frac{1}{A k^2} \sum_{p, \tau_z} \left\{ \left( p - \frac{k}{2} \right) \left( p - \frac{k'}{2} \right) - \left( p + \frac{k}{2} \right) \left( p + \frac{k'}{2} \right) \right\} k' a_{p - \frac{1}{2}, \tau_z} a_{p + \frac{1}{2}, \tau_z}
$$

$$
= -i\hbar^2 \frac{1}{\sqrt{A k}} \left( k^2 - k'^2 \right) \pi_{k+k'}^0
$$

The commutation relation between $\pi_k^0$ and $\Pi_{k'}^0$ is calculated tediously but straightforwardly as

$$
\left[\pi_k^0, \Pi_{k'}^0\right] = \left[\pi_k^0, \pi_{k'}^0\right] - \frac{1}{\sqrt{A k'}} \sum_{p \neq k'} p \left\{ \rho_{p-k}^0 [\pi_k^0, \pi_p^0] + \pi_{k, k'}^0 \right\}
$$

$$
+ \frac{1}{\sqrt{A k'}} \sum_{q \neq p} q \left\{ \rho_{p-k}^0 \rho_q^0 [\pi_k^0, \pi_q^0] + \pi_{k, q}^0 \right\} + \cdots
$$

$$
= i\hbar \sqrt{A (k + k')} \left\{ \pi_{k+k'}^0 - \frac{i\hbar}{\sqrt{A (k + k')}} \sum_{p \neq k+k'} \rho_{p-k}^0 \pi_p^0 \pi_{k-p}^0 \right\}
$$

$$
+ \frac{i\hbar}{\sqrt{A (k + k')}} \sum_{p \neq k+k', q \neq p} \rho_{p-k}^0 \rho_q^0 \pi_p^0 \pi_{k-p}^0 \pi_{p-q}^0
$$

$$
- \frac{i\hbar}{\sqrt{A (k + k')}} \sum_{p \neq k+k', q \neq p} \rho_{p-k}^0 \rho_q^0 \pi_p^0 \pi_{k-p}^0 \pi_{p-q}^0 \pi_{k-k'}^0
$$

$$
= \frac{i\hbar}{\sqrt{A k}} (k + k') \Pi_{k+k'}^0
$$

Thus we can get Eq. (3.3).
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