Local linear estimator for stochastic differential equations
driven by $\alpha$-stable Lévy motions

Song Yu-Ping, Lin Zheng-Yan *
Department of Mathematics, Zhejiang University, Hangzhou, China, 310027

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Abstract: We study the local linear estimator for the drift coefficient of stochastic differential equations
driven by $\alpha$-stable Lévy motions observed at discrete instants letting $T \to \infty$. Under regular conditions, we
derive the weak consistency and central limit theorem of the estimator. Compare with Nadaraya-Watson
estimator, the local linear estimator has a bias reduction whether kernel function is symmetric or not under
different schemes.

Keyword: local linear estimator; stable Lévy motions; bias reduction; consistency; central limit theorem.

Mathematics Subject Classification: 60J52; 62G20; 62M05; 65C30.

1 Introduction

Continuous-time models play an important role in the study of financial time series. Especially, many
models in economics and finance, like those for an interest rate or an asset price involve continuous-time
diffusion processes. Particularly, their theoretical and empirical applications to finance are quite extensive
(see Jacod and Shiryaev [18]). However, growing evidence shows that stochastic processes with jumps are
becoming more and more important (see Andersen et al. [2]; Baskshi et al. [7]; Duffie et al. [11]). Recently,
stochastic processes with jumps as an extension of continuous-path ones have been studied by more and
more statisticians since the financial phenomena can be better characterized (see Aït-Sahalia and Jacod [1];
Bandi and Nguyen [3]).

A diffusion model with continuous paths is represented by the following stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

(1.1)

where $W_t$ is a standard Brownian motion, $\mu : \mathbb{R} \to \mathbb{R}$ is an unknown measurable function and $\sigma : \mathbb{R} \to \mathbb{R}_+$
is an unknown positive function. Many authors have investigated nonparametric estimations for the drift
function $\mu(x)$ and the diffusion function $\sigma(x)$, which to some extend prevent the misspecification of the model
(1.1) compare with the parametric estimations. Prakasa Rao [27] constructed a non-parametric estimator
similar as the Nadaraya-Watson estimator for $\mu(x)$. Bandi and Phillips [4] discuss the Nadaraya-Watson
estimator for these functions of non-stationary recurrent diffusion processes. Fan and Zhang [15] proposed
local linear estimators for them and obtained bias reduction properties. In a finite sample, Xu [34] extended
re-weighted idea proposed by Hall and Presnell [16] to estimate $\sigma(x)$ under recurrence. Xu [33] discussed the

*Corresponding author, zlin@zju.edu.cn
empirical likelihood-based inference for nonparametric recurrent diffusions to construct confidence intervals. Furthermore, Bandi and Phillips [5] proposed a simple and robust approach to specify a parameter class of diffusions and estimate the parameters of interest by minimizing criteria based on the integrated squared difference between kernel estimates of the drift and diffusion functions and their parametric counterparts.

Recently, stochastic processes with jumps have been paid more attention in various applications, for instance, financial time series to reflect discontinuity of asset return (see Baskshi et al. [7]; Duffie et al. [11]; Johannes [20]; Bandi and Nguyen [5]). In this paper, we consider the stochastic process with jumps through the stochastic differential equation driven by an α-stable Lévy motion (1 < α < 2):

\[ dX_t = \mu(X_{t-})dt + \sigma(X_{t-})dZ_t, \quad X_0 = \eta, \]

(1.2)

where \( \{Z_t, t \geq 0\} \) is a standard α-stable Lévy motion defined on a probability space \( (\Omega, \mathcal{F}, P) \) equipped with a right continuous and increasing family of \( \sigma \)-algebras \( \{\mathcal{F}_t, t \geq 0\} \) and \( \eta \) is a random variable independent of \( \{Z_t\} \). \( Z_1 \) has an \( \alpha \)-stable distribution \( S_\alpha(1, \beta, 0) \) with the characteristic function:

\[ E\exp\{iuZ_1\} = \exp\left\{ -|u|^\alpha \left( 1 - i\beta \text{sgn}(u) \tan \frac{\alpha\pi}{2} \right) \right\}, \quad u \in \mathbb{R}, \]

(1.3)

where \( \beta \in [-1, 1] \) is the skewness parameter. One can refer to Sato [30], Barndorff-Nielsen et al. [6] for more detailed properties on stable distributions. Usually, we get observations \( \{X_{t_i}, t_i = i\Delta_n, i = 0, 1, \ldots, n\} \) for model (1.2), where \( \Delta_n \) is the time frequency for observation and \( n \) is the sample size. This paper is devoted to the nonparametric estimation of the unknown drift function. Our estimation procedure for model (1.2) should be based on \( \{X_{t_i}, t_i = i\Delta_n, i = 0, 1, \ldots, n\} \).

The stochastic differential equation driven by Lévy motion has received growing interest from both theoreticians and practitioners recently, such as applications to finance, climate dynamics et al.. Masuda ([23], [24]) proved some probabilistic properties of a multidimensional diffusion processes with jumps and provided mild regularity conditions for a multidimensional Ornstein-Uhlenbeck process driven by a general Lévy process for any initial distribution to be exponential \( \beta \)-mixing. When model (1.2) is specially a mean-reverted Ornstein-Uhlenbeck process driven by a Lévy process, i.e. \( \mu(x) \) is known to be linear with the form \( \mu(x) = \gamma - \lambda x \) and \( \sigma = 1 \), where \( (\gamma, \lambda) \) is unknown parameters to be estimated. Based on \( \{X_{t_i}, t_i = i\Delta_n, i = 0, 1, \ldots, n\} \), Hu and Long [17] studied the least-squares estimator for \( \lambda > 0 \) when \( Z \) is symmetric \( \alpha \)-stable and \( \gamma = 0 \). Masuda [25] considered an approximate self-weighted least absolute deviation type estimator for \( (\gamma, \lambda) \). Zhou and Yu [37] proved the asymptotic distributions of the least squares estimator of the mean reversion parameter \( \lambda \) allowing for nonlinearity in the diffusion function under three sampling schemes. However, in model (1.2), the drift function \( \mu(x) \) is seldom known and the diffusion function may be nonlinear in reality. With no prior specified form of the drift function, Long and Qian [22] discussed the Nadaraya-Watson estimator for it and obtained the weak consistency and central limit theorem.

The Nadaraya-Watson estimator given for \( \mu(x) \) is locally approximating \( \mu(x) \) by a constant (a zero-degree polynomial). However, in the context of nonparametric estimator with finite-dimensional auxiliary variables, local polynomial smoothing has become the golden standard (see Fan [12], Wand and Jones [36]). The local polynomial estimator is known to share the simplicity and consistency of the kernel estimators as Nadaraya-Watson or Gasser-Müller estimators but avoids boundary effects, at least when convergence rates are concerned. Local polynomial smoothing at a point \( x \) fits a polynomial to the pairs \((X_i; Y_i)\) for those \( X_i \) falling in a neighborhood of \( x \) determined by a smoothing parameter \( h \). The local polynomial estimator has received increasing attention and it has gained acceptance as an attractive method of nonparametric estimation function and its derivatives. This smoothing method has become a powerful and useful diagnostic
tool for data analysis. In particular, the local linear estimator locally fits a polynomial of degree one. In this paper, we propose the local linear estimators for drift function in model (1.2). As a nonparametric methodology, local polynomial estimator makes use of the observation information to estimate corresponding functions not assuming the function form. The estimator is obtained by locally fitting a polynomial of degree one to the data via weighted least squares and it shows advantages compared with Nadaraya-Watson approach (see Fan and Gijbels [13]). For further motivation and study of the local linear estimator, see Fan and Gijbels [14], Ruppert and Wand [29], Stone [32], Cleveland [10].

The remainder of this paper is organized as follows. In Section 2, local linear estimator and appropriate assumptions for model (1.2) are introduced. In Section 3, we present some technical lemmas and asymptotic results. The proofs will be collected in Section 4.

2 Local Linear Estimator and Assumptions

We lay out some notations. For simplify, $X_i$ denotes $X_{t_i}$ and we shall omit the subscript $n$ in the notation if no confusion will be caused. We will use notation $\overset{L_p}{\rightarrow}$ to denote “convergence in probability”, notation $\overset{a.s.}{\rightarrow}$ to denote “convergence almost surely” and notation $\Rightarrow$ to denote “convergence in distribution”.

Local polynomial estimator firstly introduced in Fan [12] has been widely used in regression analysis and time series analysis. It has gained acceptance as an attractive method of nonparametric estimation of regression function and its derivatives. The estimator is obtained by locally fitting $p$-th polynomial to the data via weighted least squares and it shows advantages compared with other kernel nonparametric regression estimators. The idea of weighted local polynomial regression is the following: under some smoothness conditions of the curve $m(x)$, we can expand $m(x)$ in a neighborhood of the point $x_0$ as follows:

$$m(x) \approx m(x_0) + m'(x_0)(x - x_0) + \frac{m''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{m(p)(x_0)}{p!}(x - x_0)^p$$

$$\equiv \sum_{j=0}^{p} \beta_j (x - x_0)^j,$$

where $\beta_j = \frac{m^{(j)}(x_0)}{j!}$.

Thus, the problem of estimating infinite dimensional $m(x)$ is equivalent to estimating the $p$-dimensional parameter $\beta_0, \beta_1, \ldots, \beta_p$. Consider a weighted local polynomial regression:

$$\arg \min_{\beta_0, \beta_1, \ldots, \beta_p} \sum_{i=0}^{n-1} \left\{ Y_i - \sum_{j=0}^{p} \beta_j (X_i - x)^j \right\}^2 K_{h_n} (X_i - x),$$

where $Y_i = \frac{X_{i+1} - X_i}{\Delta}$ and $K_{h_n} (\cdot) = \frac{1}{h_n}K(\frac{\cdot}{h_n})$, is kernel function with $h_n$ the bandwidth.

What we are interested in is to estimate $\hat{\mu}(x) = \hat{\beta}_0$, hence as Fan and Gijbels [14] remarked, it is reasonable for us to discuss $p = 1$: the local linear estimator for the drift function $\mu(\cdot)$ in this paper. The local linear estimator for $\mu(x)$ is the solution $\beta_0$ of the optimal problem:

$$\arg \min_{\beta_0, \beta_1} \sum_{i=0}^{n-1} \left\{ Y_i - \sum_{j=0}^{1} \beta_j (X_i - x)^j \right\}^2 K_{h} (X_i - x).$$
The solution of $\beta_0$ is

$$\hat{\mu}(x) = \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{n^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (X_{i+1} - X_i),$$

where $S_{n,k} = \sum_{i=0}^{n-1} K_h(X_i - x)(X_i - x)^k$, $k = 1, 2$.

We can also write $\hat{\mu}(x) = \frac{\hat{g}_n(x)}{h_n(x)}$,

$$\hat{h}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{n^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\},$$

$$\hat{g}_n(x) = \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{n^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (X_{i+1} - X_i).$$

We now present some assumptions used in this paper.

(A.1). The drift function $\mu(\cdot)$ is twice continuously differentiable with bounded first and second order derivatives; the diffusion function $\sigma(\cdot)$ satisfy a global Lipschitz condition, there exists a positive constant $C > 0$ such that

$$|\sigma(y) - \sigma(x)| \leq C|y - x|, \quad y, \ x \in \mathbb{R}.$$

(A.2). There exist positive constants $\sigma_0$ and $\sigma_1$ such that $0 < \sigma_0 \leq \sigma(x) \leq \sigma_1$ for each $x \in \mathbb{R}$.

(A.3). The solution $X_t$ admits a unique invariant distribution $F(x)$ and is geometrically strong mixing, i.e. there exist $c_0 > 0$ and $\rho \in (0, 1)$ such that $\alpha_X(t) \leq c_0\rho^t$, $t \geq 0$.

(A.4). The density function $f(x)$ of the stationary distribution $F(x)$ is continuously differentiable and $f(x) > 0$.

(A.5). The kernel function $K(\cdot)$ is nonnegative probability density function with compact support satisfying: $K_2 := \int_{-\infty}^{+\infty} u^2 K(u)du < \infty$, $\int_{-\infty}^{+\infty} K^2(u)du < \infty$.

(A.6). As $n \to \infty$, $h \to 0$, $\Delta \to 0$, $nh\Delta \to \infty$.

Remark 2.1. The condition (A.1) ensures that (1.2) admits a unique non-plosive càdlàg adapted solution, see Jacod and Shiryaev [18]. (A.3) implies $X_t$ is ergodic and stationary. The mixing property of a stochastic process describes the temporal dependence in data. One can refer to Bradley [9] for different kinds of mixing properties. For some sufficient conditions which guarantee (A.3), one can refer to Masuda [24]. The kernel function is not necessarily to be symmetric. Sometimes, unilateral kernel function may make predictor easier (see Fan and Zhang [15]).

3 Some Technical Lemmas and Asymptotic Results

We say that a continuous function $G : [0, \infty) \to [0, \infty)$ grows more slowly than $u^\alpha$ ($\alpha > 0$) if there exist positive constants $c, \lambda_0 \alpha_0 < \alpha$ such that $G(u) \leq c\lambda_0^{\alpha_0}G(u)$ for all $u > 0$ and all $\lambda \geq \lambda_0$.

Lemma 3.1. Let $\phi(t)$ be a predictable process satisfying $\int_0^T |\phi(t)|^\alpha dt < \infty$ almost surely for $T < \infty$. We assume that either $\phi$ is nonnegative or $Z$ is symmetric. If $G(u)$ grows more slowly than $u^\alpha$, then there exist positive constants $c_1$ and $c_2$ depending only on $\alpha, \alpha_0, c$ and $\lambda_0$ such that for each $T > 0$

$$c_1 E[G(\int_0^T |\phi(t)|^\alpha dt)^{1/\alpha}] \leq E[G(\sup_{t \leq T} |\int_0^t \phi(s) dZ_s|)] \leq c_2 E[G(\int_0^T |\phi(t)|^\alpha dt)^{1/\alpha}].$$
Remark 3.1. This lemma can be viewed as a generalization of Theorem 3.2 in Rosonski and Wozyckynski [28], where they only dealt with the case that $Z$ is symmetric.

Lemma 3.2. Suppose that there is a deterministic and nonnegative function $\Phi$ such that

$$\Phi^\alpha(T) \int_0^T |\phi(t)|^\alpha dt \overset{P}{\to} 1 \quad as \quad T \to \infty.$$ 

Then, we have

$$\Phi(T) \int_0^T |\phi(t)|dZ_t \Longrightarrow S_\alpha(1, \beta, 0).$$

Remark 3.2. This lemma can be regarded as an extending to $\alpha$-stable case of Theorem 1.19 in Kutoyants [21].

Lemma 3.3. Assumptions (A.1) - (A.6) lead to the following result:

$$\frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \overset{a.s.}{\longrightarrow} f(x) \int_{-\infty}^{+\infty} u^k K(u)du$$

Remark 3.3. In Long and Qian [22], they proved a weaker case: $\frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \overset{P}{\longrightarrow} f(x)$. One can easily obtain $\hat{h}_n(x) \overset{a.s.}{\longrightarrow} K_2 f^2(x) - (K_1 f(x))^2$ based on this lemma.

Theorem 3.1. Assume that (A.1)-(A.6) hold and $\alpha \in (1, 2)$, then $\hat{\mu}(x) \overset{P}{\longrightarrow} \mu(x)$ as $n \to \infty$.

Theorem 3.2. Let $\alpha \in [1, 2)$ and assume that (A.1) - (A.6) are satisfied.

(i) If $(n\Delta h)^{-\alpha} h = O(1)$ and $(n\Delta h)^{-\alpha} \Delta^\frac{1}{2} = O(1)$ for some $\kappa > \alpha$, then

$$(n\Delta h)^{-\alpha} \Lambda(x)(\hat{\mu}(x) - \mu(x)) \Rightarrow S_\alpha(1, \beta, 0)$$

(ii) If $(n\Delta h)^{-\alpha} h^2 = O(1)$ and $(n\Delta h)^{-\alpha} \Delta^\frac{1}{2} = O(1)$ for some $\kappa > \alpha$, then

$$(n\Delta h)^{-\alpha} \Lambda(x)(\hat{\mu}(x) - \mu(x) - h^2 \Gamma_\mu(x)) \Rightarrow S_\alpha(1, \beta, 0)$$

where $\Lambda(x) = [K_2 - (K_1)^2]f(x)^{1 - \alpha} \left\{ \frac{\int_{-\infty}^{+\infty} K(u)(K_2 - uK_1)\sigma(u)du}{\int_{-\infty}^{+\infty} K(u)\sigma(u)du} \right\}^{\frac{1}{2}}$, and $\Gamma_\mu(x) = \mu''(x) [K_2 - (K_1)^2]$. We can easily observe that the bias in the local linear case is smaller than the one in the Nadaraya-Watson case in comparison to the results between this paper and Long & Qian [22] whether $K(\cdot)$ is symmetric or not. Furthermore, when $\alpha = 1$, $\hat{\mu}(x)$ is inconsistent with $\mu(x)$ easily obtained from Theorem 3.2.
4 Proofs

Proof of Lemma 3.1. See Long and Qian (Lemma 2.7).

Proof of Lemma 3.2. See Long and Qian (Lemma 2.6).

Proof of Lemma 3.3.

We first note that

$$
\frac{1}{n}\sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k - f(x) \int_{-\infty}^{+\infty} u^k K(u) du
$$

$$
= \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k - \frac{1}{n} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right]
$$

$$
+ \frac{1}{n} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right] - f(x) \int_{-\infty}^{+\infty} u^k K(u) du.
$$

From the stationarity of $X_t$, we have:

$$
\frac{1}{n} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right] = E \left[ K_h(X_1 - x) \left( \frac{X_1 - x}{h} \right)^k \right]
$$

$$
= \int_{-\infty}^{+\infty} K_h(y - x) \left( \frac{y - x}{h} \right)^k f(y) dy
$$

$$
= \int_{-\infty}^{+\infty} K(u) u^k f(x + uh) du
$$

$$
\rightarrow f(x) \int_{-\infty}^{+\infty} u^k K(u) du.
$$

Thus, from (4.1) and (4.2) it suffices to prove that

$$
\frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k - \frac{1}{n} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right]
$$

$$
= \frac{1}{n} \sum_{i=0}^{n-1} \left\{ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k - E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right] \right\}
$$

$$
= \frac{1}{n} \sum_{i=0}^{n-1} \delta_{n,i}(x) \overset{a.s.}{\longrightarrow} 0.
$$

Note that $\sup_{0 \leq i \leq n-1} \left| \delta_{n,i}(x) \right| \leq C_0 h^{-1}$ a.s. for some positive constant $C_0 < \infty$ by the compact support of $K(\cdot)$. Applying Theorem 1.3 (2) in Bosq [8], we have for each integer $q \in \left[ 1, \frac{n}{2} \right]$ and each $\varepsilon > 0$

$$
P \left( \frac{1}{n} \left| \sum_{i=0}^{n-1} \delta_{n,i}(x) \right| > \varepsilon \right) \leq 4 \exp \left( -\frac{\varepsilon^2 q}{8 \nu^2(q)} \right) + 22 \left( 1 + \frac{4C_0 h^{-1}}{\varepsilon} \right)^{1/2} q^{1/2} C_X([p] \Delta),
$$

where

$$
\nu^2(q) = \frac{2}{p^2 s(q)} + \frac{C_0 h^{-1} \varepsilon}{2}
$$

with $p = \frac{n}{2q}$ and

$$
s(q) = \max_{0 \leq j \leq 2q-1} E [(j\nu + 1 - j\nu) \delta_{n,j\nu+1}(x) + \delta_{n,j\nu+2}(x) + \cdots]
$$
Using the Hölder inequality and stationarity of $X_i$, one can easily obtain that $s(q) = O(p^2 h^{-1})$.

By choosing $q = [\sqrt{n\Delta}/\sqrt{h}]$ and $p = \frac{n}{2q} = O(\sqrt{nh}/\sqrt{\Delta})$, we get

$$
\frac{\varepsilon^2 q}{8\nu^2(q)} = \varepsilon^2 \cdot O(q h) = O(\varepsilon^2 \sqrt{n\Delta h}).
$$

Moreover, we can obtain

$$
22 \left(1 + \frac{4C_0h^{-1}}{\varepsilon}\right)^{1/2} q \alpha_X([p]\Delta) \leq C(\varepsilon) \exp(-O(\varepsilon^2 \sqrt{n\Delta h})).
$$

under the mixing properties of $X_t$ in (A.3) and (A.6).

(4.4), (4.5) and (4.6) imply

$$
P \left( \frac{1}{n} \left| \sum_{i=0}^{n-1} \delta_{n,i}(x) \right| > \varepsilon \right) \leq C(\varepsilon) \exp(-O(\varepsilon^2 \sqrt{n\Delta h})).
$$

Therefore, $\frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k - \frac{1}{n} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^k \right] \xrightarrow{n \to 0} 0$ based on Borel-Cantelli lemma and (A.6).

**Proof of Theorem 3.1.**

It suffices to prove that

$$
\hat{g}_n(x) \xrightarrow{p} [K_2f^2(x) - (K_1f(x))^2]\mu(x).
$$

By (1.2), we first note that

$$
\hat{g}_n(x) \equiv \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (X_{i+1} - X_i)
$$

$$
= \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \left( \int_{t_i}^{t_{i+1}} \mu(X_{s-}) ds + \int_{t_i}^{t_{i+1}} \sigma(X_{s-}) dZ_s \right)
$$

$$
= \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (\mu(X_i) \Delta + \int_{t_i}^{t_{i+1}} (\mu(X_{s-}) - \mu(X_i)) ds + \int_{t_i}^{t_{i+1}} \sigma(X_{s-}) dZ_s)
$$

$$
= g_{n,1}(x) + g_{n,2}(x) + g_{n,3}(x).
$$

To show the convergence of $\hat{g}_n(x)$, we should prove the following three results:

(i) $g_{n,1}(x) \xrightarrow{p} [K_2f^2(x) - (K_1f(x))^2]\mu(x)$, as $n \to \infty$;
(ii) $g_{n,2}(x) \xrightarrow{p} 0$, as $n \to \infty$;
(iii) $g_{n,3}(x) \xrightarrow{p} 0$, as $n \to \infty$;

**Proof of (i)**:

$$
g_{n,1}(x) = \mu(x) \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}
$$

$$
+ \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (\mu(X_i) - \mu(x))
$$
Proof of which one can refer to Long & Qian [22], Shimizu & Yoshida [31] and Jacod & Protter [19].

It follows that by Lemma 3.3 and the compact support of \( K \), by the Lipschitz property of \( \mu(x) \) and the stationarity of \( X_t \), we have

\[
|A_{n,2}(x)| \leq \frac{L}{n} \sum_{i=0}^{n-1} |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| + \frac{L}{n} \sum_{i=0}^{n-1} |X_i - x| K_h(X_i - x) \left\{ \frac{X_i - x}{h} \right\} \left| \frac{S_{n,1}}{nh} \right|, \tag{4.9}
\]

where \( L \) denotes the bound of the first derivative of \( \mu(x) \).

The two components of the right part are dealt with in the same way, so we only deal with the first one for convenience.

\[
\frac{1}{n} \sum_{i=0}^{n-1} |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| = \frac{1}{n} \sum_{i=0}^{n-1} \left( |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| - E \left[ |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| \right]\right)
\]

\[
+ \frac{1}{n} \sum_{i=0}^{n-1} E \left[ |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| \right]. \tag{4.10}
\]

We find that \( |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| - E \left[ |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| \right] \) is a.s. uniformly bounded for each \( i \) by Lemma 3.3 and the compact support of \( K(\cdot) \). Similar as the proof of (4.3), we can show that

\[
\frac{1}{n} \sum_{i=0}^{n-1} \left( |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| - E \left[ |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| \right]\right) \xrightarrow{p} 0. \tag{4.11}
\]

As for the second part,

\[
\lim_{h \to 0} \frac{1}{n} \sum_{i=0}^{n-1} E \left[ |X_i - x| K_h(X_i - x) \left| \frac{S_{n,2}}{nh^2} \right| \right] = \lim_{h \to 0} E \left[ |X_1 - x| K_h(X_1 - x) \left| \frac{S_{n,2}}{nh^2} \right| \right]
\]

\[
= \lim_{h \to 0} K_2 f(x) E \left[ |X_1 - x| K_h(X_1 - x) \right]
\]

\[
= \lim_{h \to 0} h K_2 f(x) \int_{-\infty}^{+\infty} |u| K(u) f(x + uh) du
\]

\[
= \lim_{h \to 0} h K_2 f^2(x) \int_{-\infty}^{+\infty} |u| K(u) du \to 0. \tag{4.12}
\]

It follows that \( A_{n,2} \xrightarrow{p} 0 \) as \( n \to \infty \). Hence \( g_{n,1}(x) \xrightarrow{p} [K_2 f^2(x) - (K_1 f(x))^2] \mu(x) \) by (4.7)-(4.12).

**Proof of (ii):**

We first introduce a basic inequality for (1.2):

\[
\sup_{t_i \leq s \leq t_{i+1}} |X_t - X_{t_i}| \leq e^L \Delta \left( |\mu(X_t)| \Delta + \sup_{t_i \leq s \leq t_{i+1}} \left| \int_{t_i}^t \sigma(X_s) dZ_s \right| \right), \tag{4.13}
\]

which one can refer to Long & Qian [22], Shimizu & Yoshida [31] and Jacod & Protter [19].
\[ |g_{n,2}(x)| \leq \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} \int_{t_i}^{t_{i+1}} |\mu(X_{s-}) - \mu(X_i)| ds \\
\leq \frac{L}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} \int_{t_i}^{t_{i+1}} |X_{s-} - X_i| ds \\
\leq \frac{L}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} \sup_{t_i \leq t \leq t_{i+1}} |X_t - X_i| \\
\leq \frac{Le^{t\Delta}}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} \left( |\mu(X_i)| \Delta + \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} \sigma(X_{s-})dZ_s \right| \right) \\
= A_{n,3}(x) + A_{n,4}(x) + A_{n,5}(x). \quad (4.14) \]

Similarly as the proof of (4.1), we know that

\[ L \Delta e^{t\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} |\mu(X_i)| \to 0, \]

where \( Q(x) = L \Delta e^{t\Delta} |\mu(x)| \left[ K_2 f^2(x) + \left( \int_{-\infty}^{t} |u| K(u)du \right)^2 f^2(x) \right] \to 0, \)

that is

\[ L \Delta e^{t\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n1}}{nh} \right| \right\} |\mu(X_i)| \to 0. \quad (4.15) \]

\( A_{n,4}(x), A_{n,5}(x) \) are dealt with the same approach, hence here we only verify \( A_{n,4}(x) \to 0. \)

As for \( A_{n,4}(x) \), according to Lemma 3.3, we only need to verify

\[ \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} \sigma(X_{s-})dZ_s \right| \overset{P}{\to} 0. \quad (4.16) \]

By Markov inequality and Lemma 3.1, we have

\[ P \left( \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} \sigma(X_{s-})dZ_s \right| > \varepsilon \right) \leq \frac{1}{n^{\varepsilon}} \sum_{i=0}^{n-1} E \left[ \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} K_h(X_i - x) \sigma(X_{s-})dZ_s \right| \right] \\
\leq \frac{c_1}{n^{\varepsilon}} \sum_{i=0}^{n-1} E \left[ \int_{t_i}^{t_{i+1}} K_h^\alpha(X_i - x) \sigma^\alpha(X_{s-}) ds \right]^{1/\alpha} \\
\leq \frac{c_1}{n^{\varepsilon}} \sum_{i=0}^{n-1} E \left[ K_h(X_i - x) \sigma_1 \Delta^{1/\alpha} \right] \\
\leq O(\Delta^{1/\alpha}) \to 0. \quad (4.17) \]

Now, \( g_{n,2}(x) \to 0 \) by (4.14), (4.15) and (4.16).

Proof of (iii):

\[ |g_{n,3}(x)| = \left| \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n1}}{nh} \right| \right\} \left( \int_{t_i}^{t_{i+1}} \sigma(X_{s-})dZ_s \right) \right| \]
\[ \frac{S_{n,2}}{nh^2} \leq \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s \]
\[ + \frac{S_{n,1}}{nh} \cdot \frac{1}{n} \sum_{i=0}^{n-1} K_h(x_i - x) \left( \frac{X_i - x}{h} \right) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s. \]

By Lemma 3.3, we only need to prove
\[ \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s \overset{p}{\to} 0 \]
and
\[ \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \left( \frac{X_i - x}{h} \right) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s \overset{p}{\to} 0. \]

We only prove (4.19) for simplicity. Denote
\[ \phi_{n,1}(t, x) = \sum_{i=0}^{n-1} \frac{1}{h^{1/\alpha}} K \left( \frac{X_i - x}{h} \right) \sigma(X_{t_i+1})1_{(t_i, t_{i+1}]}(t), \]
we have
\[ \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s = \frac{1}{nh^{1/\alpha}} \left| \int_0^n \phi_{n,1}(t, x) dZ_t \right|. \]

With the same argument as the proof of \( A_{n,4}(x) \), we have that:
\[ P \left( \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \int_{t_i}^{t_{i+1}} \sigma(X_s-)dZ_s^s > \varepsilon \right) = O \left( (n\Delta h)^{1-n/\alpha} \right). \]

We get \( g_{n,3}(x) \overset{p}{\to} 0 \) by (4.18)-(4.21) and Assumption (A.6).

**Proof of Theorem 3.2.**
Note that
\[ (n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x)(\hat{\mu}(x) - \mu(x)) = \frac{(n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x) \left[ g_n(x) - \mu(x) \hat{h}_n(x) \right]}{\hat{h}_n(x)} = \frac{B_n(x)}{\hat{h}_n(x)}. \]

We have obtained \( \hat{h}_n(x) \to [K_2 - K_1^2]f^2(x) \) applying Lemma 3.3, so it is enough to study the asymptotic behavior of \( B_n(x) \).

\[ B_n(x) = (n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x) \left[ g_{n,1}(x) - \mu(x) \hat{h}_n(x) \right] + (n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x) g_{n,2}(x) + (n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x) g_{n,3}(x) \]
\[ =: B_{n,1}(x) + B_{n,2}(x) + B_{n,3}(x). \]

**Proof of \( B_{n,1}(x) \):**
We can express \( B_{n,1}(x) \) as
\[ B_{n,1}(x) = (n\Delta h)^{1-\frac{1}{\alpha}} \Lambda(x) \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right) (\mu(X_i) - \mu(x)). \]

Using Taylor's expansion, we get
\[ \mu(X_i) - \mu(x) = \mu'(x)(X_i - x) + \frac{1}{2} \mu''(x + \theta(X_i - x))(X_i - x)^2, \]
where $\theta_i$ is some random variable satisfying $\theta_i \in [0, 1]$.

Under a simple calculus, we have

$$\sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \cdot (X_i - x) \equiv 0,$$

so

$$B_{n,1}(x) = \underbrace{\left( n \Delta h \right)^{\frac{1}{2}} \Lambda(x) \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \frac{1}{2} \mu''(x)(X_i - x)^2}_{=: B_{n,1}^{(1)}(x)} + \underbrace{(n \Delta h)^{\frac{1}{2}} \Lambda(x) \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \frac{1}{2} \left\{ \mu''(x + \theta_i(X_i - x)) - \mu''(x) \right\}}_{=: B_{n,1}^{(2)}(x)}$$

By the stationary of $X_t$, we get

$$B_{n,1}^{(1)}(x) = \frac{1}{2} \mu''(x)(n \Delta h)^{\frac{1}{2}} \Lambda(x) E \left[ K_h(X_1 - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_1 - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (X_1 - x)^2 \right]$$

$$+ \frac{1}{2} \mu''(x)(n \Delta h)^{\frac{1}{2}} \Lambda(x) \frac{1}{n} \sum_{i=0}^{n-1} \left\{ K_h(X_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \frac{1}{2} \mu''(x)(X_i - x)^2 \right\}$$

$$- E \left[ K_h(X_1 - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_1 - x}{h} \right) \frac{S_{n,1}}{nh} \right\} (X_1 - x)^2 \right]$$

$$= D_{n,1}(x) + D_{n,2}(x).$$

One can easily obtain

$$D_{n,1}(x) = \frac{1}{2} \mu''(x) \Lambda(x) \left( K_2 \right)^2 - K_1 K_3 \right| (f(x))^2 (n \Delta h)^{\frac{1}{2}} h^2 (1 + o(1)).$$

Denote

$$D_{n,2}(x) := \frac{1}{2} \mu''(x) \Lambda(x) \frac{1}{n} \sum_{i=0}^{n-1} \xi_{n,i}.$$

Note that $\sup_{0 \leq \xi \leq n-1} |\xi_{n,i}| \leq M_1 (n \Delta h)^{-\frac{1}{4}} h$ a.s. for some positive constant $M_1 < \infty$.

It follows from the proof of (4.3) that $\overline{\frac{1}{n} \sum_{i=0}^{n-1} \xi_{n,i}} \to 0$ in exponential rate. However, in addition, we should calculate $\tilde{s}(q)$, which may be a little different from $s(q)$.

$$\tilde{s}(q) = \max_{0 \leq j \leq 2q-1} E \left[ \left( \sum_{i=0}^{n-1} \xi_{n,i} \right) + q + 1 - j p \right] + \xi_{n,(j+1)p} + \xi_{n,(j+1)p} \right) \right]^2.$$

Using Billingsley’s inequality in Bosq [8], $\tilde{s}(q) = O(p(n \Delta h)^2 (1 - \frac{1}{4}) \Delta^{-1} h^2)$.

$$|B_{n,1}^{(2)}(x)| \leq \frac{1}{2} \Lambda(x) \sup_{|x-y| \leq Mh} |\mu''(x) - \mu''(y)| (n \Delta h)^{-\frac{1}{4}} \left( \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x) \left( \left( \frac{X_i - x}{h} \right) \left( \frac{S_{n,1}}{nh} \right) \right)^2 \right)$$

$$+ \frac{1}{n h} \sum_{i=0}^{n-1} K_h(X_i - x) |X_i - x|^3 \left( \frac{S_{n,1}}{nh} \right)$$

$$= o(1) (n \Delta h)^{-\frac{1}{4}} h^2 \cdot \frac{1}{n} \sum_{i=0}^{n-1} K_h(X_i - x)$$
\[
= o_p(1)(n\Delta h)^{1-{\frac{1}{n}}}h^2. \tag{4.27}
\]

In conclusion, it follows from (4.23)-(4.27) that
\[
B_{n,1}(x) = o_p(1) + o_p(1)(n\Delta h)^{1-{\frac{1}{n}}}h^2 + \frac{1}{2}h^{\prime\prime}(x)\Lambda(x)[(K_2)^2 - K_1K_3](f(x))^2(n\Delta h)^{1-{\frac{1}{n}}}h^2(1 + o(1)). \tag{4.28}
\]

**Proof of B_{n,2}(x):** By (4.13), we have
\[
B_{n,2}(x) = (n\Delta h)^{1-{\frac{1}{n}}}\Lambda(x)g_{n,2}(x)
\leq (n\Delta h)^{1-{\frac{1}{n}}}\Lambda(x)\left[L\Delta e^{L\Delta} \frac{1}{n} \sum_{i=0}^{n-1} K_h(x_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n,1}}{nh} \right| \right\} \mu(X_i) \right]
+ L\Delta^{\frac{1}{n}} - \frac{1}{n\Delta^{\frac{1}{n}}} \sum_{i=0}^{n-1} K_h(x_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n,1}}{nh} \right| \right\} \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} \sigma(X_{s-})dZ_s \right|. \tag{4.29}
\]

Similar to the proof of (4.1), under given conditions and Lemma 3.3, we can obtain
\[
\frac{1}{n} \sum_{i=0}^{n-1} K_h(x_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n,1}}{nh} \right| \right\} \mu(X_i) \to \left[ K_2f^2(x) + |K_1|f^2(x) \int_{-\infty}^{+\infty} |u|K(u)du \right] |\mu(x)|. \tag{4.30}
\]

As in the proof of (4.17) and Lemma 3.3, we can show that
\[
P\left( \frac{1}{n\Delta^{\frac{1}{n}}} \sum_{i=0}^{n-1} K_h(x_i - x) \left\{ \left| \frac{S_{n,2}}{nh^2} \right| + \left| \frac{X_i - x}{h} \right| \left| \frac{S_{n,1}}{nh} \right| \right\} \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^{t} \sigma(X_{s-})dZ_s \right| > \varepsilon \right) \leq O(\Delta^{\frac{1}{n}} - \frac{\varepsilon}{h}). \tag{4.31}
\]

It follows from (4.29)-(4.31) and \(\kappa > \alpha\) that
\[
B_{n,2}(x) = O_p(1) \cdot (n\Delta h)^{1-{\frac{1}{n}}}\Delta + o_p(1) \cdot (n\Delta h)^{1-{\frac{1}{n}}}\Delta^{\frac{1}{n}}. \tag{4.32}
\]

**Proof of B_{n,3}(x):**

According to (4.18),
\[
B_{n,3}(x) = (n\Delta h)^{1-{\frac{1}{n}}}\Lambda(x) \frac{1}{n\Delta} \sum_{i=0}^{n-1} K_h(x_i - x) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \int_{t_i}^{t_{i+1}} \sigma(X_{s-})dZ_s \tag{4.33}
\]

Denote
\[
\phi_{n,2}(t, x) = \sum_{i=0}^{n-1} \frac{1}{h^{1/2}} K_h x h \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \sigma(X_{t-})1_{(t, t_{i+1}]}(t),
\]
then
\[
B_{n,3}(x) = \int_{t_0}^{t_n} \phi_{n,2}(t, x)dZ_t.
\]

Let
\[
\Phi_{t_n} = \left[ t_n \sigma^\alpha(x)f(x) \int_{-\infty}^{+\infty} K^\alpha(u) \{ K_2f(x) - uK_1f(x) \} \alpha du \right]^{-\frac{1}{\alpha}}.
\]

Then, we have
\[
\Phi_{t_n} \cdot \int_{0}^{t_n} \phi_{n,2}(t, x)dt = \Phi_{t_n} \cdot \sum_{i=0}^{n-1} \frac{1}{h^{1/2}} K_h x h \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\} \alpha \int_{t_i}^{t_{i+1}} \sigma(X_{s-})ds.
\]
Applying Lemma 3.3, and the similar proof of Lemma 3.3, we can obtain

\[
\Phi_{tn}^\alpha \cdot \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}^\alpha \sigma^\alpha(X_i) \Delta
\]

\[
+ \Phi_{tn}^\alpha \cdot \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}^\alpha \int_{t_i}^{t_{i+1}} (\sigma^\alpha(X_{s-}) - \sigma^\alpha(X_i)) ds
\]

\[
=: \ I + J. \quad (4.34)
\]

It follows from the proof of (4.1) that

\[
\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}^\alpha \sigma^\alpha(X_i) \xrightarrow{p} \sigma^\alpha(x)f(x) \int_{-\infty}^{+\infty} K^\alpha(u) \{K_2f(x) - uK_1f(x)\}^\alpha du.
\]

Therefore, we have

\[
I = \frac{1}{\sigma^\alpha(x)f(x) \int_{-\infty}^{+\infty} K^\alpha(u) \{K_2f(x) - uK_1f(x)\}^\alpha du} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}^\alpha \sigma^\alpha(X_i) \xrightarrow{p} 1. \quad (4.35)
\]

Next we deal with the second term J. By the mean-value theorem, the Lipschitz condition (A.1) and (4.13), we have

\[
|J| = \left| \Phi_{tn}^\alpha \cdot \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \left( \frac{X_i - x}{h} \right) \frac{S_{n,1}}{nh} \right\}^\alpha \int_{t_i}^{t_{i+1}} (\sigma^\alpha(X_{s-}) - \sigma^\alpha(X_i)) ds \right|
\]

\[
\leq \Phi_{tn}^\alpha \cdot \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha \int_{t_i}^{t_{i+1}} |\sigma^\alpha(X_{s-}) - \sigma^\alpha(X_i)| ds
\]

\[
\leq \frac{C}{n\Delta} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha \int_{t_i}^{t_{i+1}} |\sigma^\alpha(X_{s-}) - \sigma^\alpha(X_i)| ds
\]

\[
\leq \frac{C}{n\Delta} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha \sup_{t_i \leq s \leq t_{i+1}} |X_t - X_i| \Delta
\]

\[
\leq \frac{Ce^{\Delta}}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha |\mu(X_i)| \Delta
\]

\[
+ \frac{Ce^{\Delta}}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} - \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha \sup_{t_i \leq s \leq t_{i+1}} \int_{t_i}^{t} \sigma^\alpha(X_{s-}) dZ_s
\]

\[
=: \ J_1 + J_2. \quad (4.36)
\]

where \( \zeta \) is some random variable satisfying \( \zeta \in [\sigma(X_{s-}), \sigma(X_i)] \) or \( \zeta \in [\sigma(X_i), \sigma(X_{s-})] \).

Applying Lemma 3.3, and the similar proof of Lemma 3.3, we can obtain

\[
\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^\alpha \left( \frac{X_i - x}{h} \right) \left\{ \frac{S_{n,2}}{nh^2} + \frac{X_i - x}{h} \frac{S_{n,1}}{nh} \right\}^\alpha |\mu(X_i)| \xrightarrow{p} |\mu(x)| f(x) \int_{-\infty}^{+\infty} K^\alpha(u) \{K_2f(x) + |u||K_1f(x)|\}^\alpha du,
\]

hence \( J_1 \xrightarrow{p} 0 \). As for \( J_2 \xrightarrow{p} 0 \), one can refer to (4.31).

From (4.33)-(4.36) and Lemma 3.2, we have

\[
\Phi_{tn} B_{n,3}(x) \Rightarrow S_{\alpha}(1, \beta, 0). \quad (4.37)
\]
Therefore, it follows from (4.33), (4.37) and Lemma 3.3 that

\[
B_{n,3}(x) = [K_2 f^2(x) - (K_1 f(x))^2] \cdot \Phi_{t_n} \cdot B'_{n,3}(x)
\]

\[
\Rightarrow [K_2 f^2(x) - (K_1 f(x))^2] S_\alpha(1, \beta, 0).
\]

By Slutsky’s theorem and remark 3.3, we find

\[
(n\Delta h)^{1-\frac{1}{2}} \Lambda(x)(\hat{\mu}(x) - \mu(x) - h^2 \Gamma_{\mu}(x))
\]

\[
= \frac{B_n(x)}{h_n(x)} - (n\Delta h)^{1-\frac{1}{2}} h^2 \Lambda(x) \Gamma_{\mu}(x)
\]

\[
= \frac{B_n(x) - (n\Delta h)^{1-\frac{1}{2}} h^2 \Lambda(x) \Gamma_{\mu}(x) h(x)}{h_n(x)} + (n\Delta h)^{1-\frac{1}{2}} h^2 \Lambda(x) \Gamma_{\mu}(x) \left( \frac{h(x)}{h_n(x)} - 1 \right)
\]

\[
\Rightarrow S_\alpha(1, \beta, 0).
\]

This complete the proof of (ii) in Theorem 3.2. The proof of (i) is similar.

5 References

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