GALOIS TREES IN THE GRAPH OF $p$-GROUPS OF MAXIMAL CLASS

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Abstract. The investigation of the graph $\mathcal{G}_p$ associated with the finite $p$-groups of maximal class was initiated by Blackburn (1958) and became a deep and interesting research topic since then. Leedham-Green & McKay (1976–1984) introduced skeletons of $\mathcal{G}_p$, described their importance for the structural investigation of $\mathcal{G}_p$ and exhibited their relation to algebraic number theory. Here we go one step further: we partition the skeletons into so-called Galois trees and study their general shape. In the special case $p \geq 7$ and $p \equiv 5 \pmod{6}$, we show that they have a significant impact on the periodic patterns of $\mathcal{G}_p$ conjectured by Eick, Leedham-Green, Newman & O’Brien (2013). In particular, we use Galois trees to prove a conjecture by Dietrich (2010) on these periodic patterns.

1. Introduction

The classification of $p$-groups of maximal class is a long-standing research problem. It was first investigated by Blackburn [1] and later continued in various publications. In a broader context, this research problem is part of coclass theory. The monograph by Leedham-Green & McKay [16] provides an introduction to this research area, whereas their work [12–15] puts particular emphasis on the maximal class gated by Blackburn [1] and later continued in various publications. In a broader context, this research.

The graph $\mathcal{G}_p$. The $p$-groups of maximal class can be visualised by a graph $\mathcal{G}_p$: its vertices correspond one-to-one to the isomorphism type representatives of finite $p$-groups of maximal class, and there is an edge $G \to H$ if and only if $H/\gamma(H) \cong G$, where $\gamma(H)$ denotes the last non-trivial term in the lower central series of $H$. The graph $\mathcal{G}_p$ is an infinite graph and the broad structure of $\mathcal{G}_p$ is as follows: It consists of an isolated vertex corresponding to the cyclic group of order $p$, and an infinite tree $\mathcal{T}_p$ with root corresponding to the elementary abelian group of order $p^2$. The tree $\mathcal{T}_p$ has a unique infinite path $S_p(2) \to S_p(3) \to \cdots$ starting at its root. Each group $S_p(n)$ has order $p^n$ and is isomorphic to a lower central series quotient of the infinite pro-$p$-group of maximal class. The graph $\mathcal{G}_p$ was fully determined for $p \leq 3$ by Blackburn [1], and it is experimentally well investigated for $p = 5$, see [3, 5, 18]. For $p \geq 7$ its detailed structure is still widely unknown and our results will be focused on that case.

Branches. A group $H$ is a descendant of $G$ in $\mathcal{T}_p$ if there is a path from $G$ to $H$ in $\mathcal{T}_p$. If $H$ has distance $m$ to $G$, then $H$ is an $m$-step descendant of $G$; if $m = 1$ then $H$ is an immediate descendant of $G$; we define $m$-step ancestors analogously. A group is capable if it has immediate descendants. The branch $B_p(n)$ of $\mathcal{T}_p$ is the full subgraph of $\mathcal{T}_p$ consisting of all descendants of $S_p(n)$ that are not descendants of $S_p(n + 1)$. Thus, $\mathcal{T}_p$ consists of the branches $B_p(2), B_p(3), \ldots$ which are connected by its infinite path.

The depth of a group $G$ in $B_p(n)$ is its distance to the root $S_p(n)$. The depth $\text{dep}(B_p(n))$ is the maximum of the depths of groups in $B_p(n)$. Blackburn’s classification [1] implies that the depth of $B_p(n)$ is bounded for $p \leq 3$. For $p \geq 5$, the depth of $B_p(n)$ grows linearly in $n$. More precisely, for $p \geq 5$ Dietrich [2] proved that with $c = 2p - 8$ for $p \geq 7$ and $c = 4$ for $p = 5$ the depth satisfies $n - c \leq \text{dep}(B_p(n)) \leq n + c - 3$.
Periodic patterns. Throughout we write \( d = p - 1 \). A pruned branch \( B_p(n, k) \) is the subtree of \( B_p(n) \) containing all groups of depth at most \( k \) in \( B_p(n) \). Extending the results of du Sautoy \([7]\) and Eick & Leedham-Green \([8]\), Dietrich \([2]\) proved that there exists \( n_0 = n_0(p) \) such that for all \( n \geq n_0 \) and \( c \) as defined above there exists a graph isomorphism
\[
i : B_p(n + d, n - c) \to B_p(n, n - c).
\]

This is called the first periodicity of the graphs of \( p \)-groups of maximal class. It remains to study \( B_p(n + d) \) at ‘large’ depth at least \( n - c \). For \( G \) in \( B_p(n) \) we denote by \( D(G) \) the tree of all descendants of \( G \) in \( B_p(n) \) and we write \( D(G, m) \) for its pruned version. Two groups \( G \) and \( H \) in \( T_p \) are twins if \( D(G) \cong D(H) \). They are \( m \)-step twins if \( D(G, m) \cong D(H, m) \). The following conjecture is a main open problem in the theory of \( p \)-groups at current and a driving force of current research in this area, see \([3, 9]\).

**Conjecture 1.1.** Let \( p \geq 5 \). There exists \( n_0 = n_0(p) \) such that for all \( n \geq n_0 \) and each group \( G \) at depth \( n - c \) in \( B_p(n + d) \) there exists a twin \( H \) at depth \( n - c - d \) in \( B_p(n) \).

If this conjecture is true, then \( G_p \) is determined by a finite subgraph and two periodicities. The available computational evidence for small primes supports Conjecture \([14]\) but unfortunately this evidence is rather thin. If one wants to prove Conjecture \([14]\) then a construction for twins is needed. A first idea was to use \( H = i(K) \) as potential twin, where \( K \) is a \( d \)-step ancestor of \( G \). This is the reason why twins have been called periodic parents in earlier work. This idea turned out to be wrong, see Section \([7]\) for details.

Skeletons. The skeleton \( S_p(n) \) is the full subtree of \( B_p(n) \) consisting of all capable groups of depth at most \( n - c \) in \( B_p(n) \). Leedham-Green & McKay introduced and studied these groups in a more general setting under the name constructible groups: they described a construction of these groups based on algebraic number theory and they also translated the isomorphism problem of these groups to a problem in number theory; we recall some of this theory in Section \([3]\).

It is the central aim of this paper to exhibit new structural results for the graph \( G_p \) and its skeletons, with a view towards a construction of twins. From now on, \( p \geq 7 \) is a prime and write
\[
d = p - 1, \quad c = 2p - 8, \quad \text{and} \quad \ell = (p - 3)/2.
\]

2. Main results

We define the Galois order \( \mathrm{galois}(G) \) of a \( p \)-group \( G \) as the \( p' \)-factor of its automorphism group order. If \( n \geq 3 \), then \( \mathrm{galois}(S_p(n)) = d^2 \), and \( \mathrm{galois}(G) \) divides \( d \) for every other group \( G \) in \( S_p(n) \). Further, if \( H \) is a descendant of \( G \), then \( \mathrm{galois}(H) \) divides \( \mathrm{galois}(G) \); see Theorem \([3, 4]\). We call a subtree \( \mathcal{R} \) of \( S_p(n) \) a Galois tree with Galois order \( h \) if \( \mathrm{galois}(G) = h \) for all \( G \) in \( \mathcal{R} \) and \( \mathcal{R} \) is maximal with this property; that is, neither an ancestor of the root of \( \mathcal{R} \) nor a descendant of a leaf of \( \mathcal{R} \) has Galois order \( h \). By definition the Galois trees (with all possible Galois orders) form a partition of \( S_p(n) \).

The work by Dietrich \([3\text{ Section 6}]\) suggests that the investigation of Galois trees may provide a way to determine twins. Motivated by this, the Galois trees with Galois order \( d \) have been fully determined by Dietrich & Eick \([4]\): each skeleton \( S_p(n) \) consists of \( \ell \) such Galois trees, with all roots at depth \( 1 \) and all leaves at depth \( \text{dep}(S_p(n)) \). Furthermore, all of these trees are distance regular, that is, the number of immediate descendants of a group in such a Galois tree depends on the distance to its root only. The work of this paper is guided by an investigation of Galois trees for Galois orders \( h \mid d \).

We start by revisiting and refining the construction of skeleton groups by Leedham-Green & McKay. Our new construction simplifies the isomorphism problem and also allows us to read off information about the Galois order of the groups. Stating these new results in detail requires more notation, which is why we postpone this to Theorem \([4.1]\) and Proposition \([5.2]\). Both of these results are key ingredients in the proofs of our three main results stated below. The first of these, Theorem \([2.1]\) shows that the Galois trees for arbitrary Galois order also extend to the full depth of the skeleton.
Theorem 2.1. Let $p \geq 7$ and $n \geq \max\{c, 8\}$. Every leaf of a Galois tree in $S_p(n)$ has depth $\text{dep}(S_p(n))$.

A Galois tree $\mathcal{R}$ has ramification level $e$ if there is a group in $\mathcal{R}$ at depth $e$ in $S_p(n)$ that has more than one immediate descendant in $\mathcal{R}$. Our second main result Theorem 2.2 shows that for $p \equiv 5 \mod 6$ the ramification levels and depths of roots of Galois trees occur periodically with periodicity $d$. We prove Theorem 2.2 in Sections 5 and 6. The distinction between $p \equiv 1 \mod 6$ and $p \equiv 5 \mod 6$ dates back to work of Leedham-Green & McKay and, in particular, the assumption $p \equiv 5 \mod 6$ implies a key property of certain stabilisers, see [3 Theorem 5.1] and Section 7.

Theorem 2.2. Let $p \geq 7$ with $p \equiv 5 \mod 6$ and $n \geq \max\{c, 8\}$. Let $\mathcal{R}$ be a Galois tree in $S_p(n)$ with Galois order $h$. There exists $e_0 = e_0(p)$ such that for all $e_0 \leq e < \text{dep}(S_p(n))$ the following hold:

a) If $G$ is a group at depth $e$ in $\mathcal{R}$, then the number of immediate descendants of $G$ in $\mathcal{R}$ is a power of $p$.
b) Let $e + d < \text{dep}(S_p(n))$. Then $\mathcal{R}$ has a ramification level $e$ if and only if it has a ramification level $e + d$.
c) Let $e + d < \text{dep}(S_p(n))$. If the root of $\mathcal{R}$ has depth $e$ in $S_p(n)$, then there is a Galois tree with Galois order $h$ and root at depth $e + d$ in $S_p(n)$.

Our new description of skeleton groups can also be used to prove a first existence result for twins. As before, we assume $p \equiv 5 \mod 6$. We prove Theorem 2.3 in Section 7.

Theorem 2.3. Let $p \geq 7$ with $p \equiv 5 \mod 6$. There exists $n_0 = n_0(p)$ so that for all $n \geq n_0$ every skeleton group $G$ at depth $\text{dep}(S_p(n))$ in $B_p(n + d)$ has a $d$-step twin at depth $\text{dep}(S_p(n)) - d$ in $B_p(n)$.

This result provides a first serious step towards a construction of twins. It is also the first periodicity result that yields a full description of the growth of the (pruned) branches of a coclass tree with growing depths and widths, see [4 Appendix A.2] for an overview of known periodicity results.

We have used computational methods to explore further properties of Galois trees. The examples exhibited in Appendix A allow us to make the following observations.

Remark 2.4. a) Galois trees with Galois order $d$ are distance regular; see [1]. This result does not extend to Galois trees with arbitrary Galois order; see Figure 2.
b) If $p \equiv 5 \mod 6$, then the number of immediate descendants in a Galois tree is a power of $p$. It seems that this does not extend to $p \equiv 1 \mod 6$; see Figure 2.
c) All Galois trees with Galois order $d$ have roots of depth 1. For arbitrary Galois order, there seems to be no bound to the depths of the roots of Galois trees; see Figure 3.

3. Preliminaries

We recall some results from algebraic number theory and then the description of skeleton groups and their isomorphism problem. Along the way, we introduce the notation that is fixed throughout this paper. To assist the reader with the notation, Appendix B provides a short glossary.

Algebraic number theory. Most of the following results can be found in standard books such as Neukirch [17 Chapter II], but we also refer to [4] for a recent treatment and more details. Let $\mathbb{Q}_p$ denote the $p$-adic rational numbers with $p$-adic integers $\mathbb{Z}_p$, and let $\theta$ be a primitive $p$-th root of unity over $\mathbb{Q}_p$. The field $K = \mathbb{Q}_p(\theta)$ is an extension of degree $d$ over $\mathbb{Q}_p$ with $\mathbb{Q}_p$-basis $\{1, \theta, \ldots, \theta^{d-1}\}$. The maximal order of $K$ is $\mathcal{O} = \mathbb{Z}_p[\theta]$. It contains a unique maximal ideal $p = (\theta - 1)$, which yields the unique series of ideals given by the ideals $p^m$, where $p^0 = \mathcal{O}$. One can extend this definition to fractional ideals $p^m$ for all $m \in \mathbb{Z}$. An important property is that $pp^m = p^{m+d}$ for all $m$. Below we write $(p^m)^\ell$ for the direct product of $\ell$ copies of $p^m$, and not for the ideal $p^{m\ell}$. The group of units $\mathcal{U}$ of $\mathcal{O}$ has the form $\mathcal{U} = (\omega) \times (\theta) \times \mathcal{U}_2$, where $\omega$ is a primitive $d$-th root of unity in $\mathbb{Z}_p$ and $\mathcal{U}_2 = 1 + p^\mathcal{U}$.

For $i \in \mathbb{Z}$ coprime to $p$ we define the automorphism $\sigma_i : K \rightarrow K$, $\theta \mapsto \theta^i$, so that $G = \{\sigma_i : 1 \leq i \leq d\}$ is the Galois group of $K$; it is cyclic of order $d$. Throughout, we fix a generator $\sigma$ of $G$ and set $\tau = \sigma^k$ for a fixed divisor $k$ of $d$; note that $\tau$ has order $h = d/k$ in $G$ and $d = hk$ holds.
The automorphism $\tau$ induces a diagonalisable $\mathbb{Z}_p$-linear map on $p^2$; there are $h$ distinct eigenspaces of $\mathbb{Z}_p$-dimension $k$ each, with eigenvalues $\{\omega^i : 0 \leq i \leq d-1\}$. The map $\chi : U \to U$, $u \mapsto ur(u)^{-1}$ is a group homomorphism with $U = \ker(\chi) \times \im(\chi)$. If $h < d$, then $\ker(\chi) = \langle \omega \rangle \times (1 + F)$ where $F = \{a \in p^2 : \tau(a) = a\}$ is a $\mathbb{Z}_p$-subspace of dimension $k$ in $O$; if $h = d$, then $\ker(\chi) = \mathbb{Z}_p^*$. The results above are proved analogously to [4 Lemmas 2.1 & 2.2], which cover the case $h = d$.

### Skeleton groups

Skeleton groups play a crucial role in almost all recent treatments of coclass graphs. These groups were first described by Leedham-Green & McKay using the name constructible groups and we refer to [16] for details. For a recent discussion of skeleton groups we refer to [6]. Below we follow a slight modification that has already been used in [4]; proofs can be found in said references.

Let $\Hom = \text{Hom}_O(O \wedge O, O)$ be the group of homomorphisms $O \wedge O \to O$ that are compatible with multiplication by $\theta$. The image of $f \in \Hom$ is an ideal in $O$, so $\im(f) = p^{n-1}$ for some $n \in \mathbb{N}$. Given $e \in \{0, \ldots, \text{dep}(S_p(n))\}$ and $f \in \Hom$ with image $p^{n-1}$, one defines a multiplicative group structure on $O/p^{n+e-1}$ via $(a + p^{n+e-1}) \circ (b + p^{n+e-1}) = (a + b + \frac{1}{2}f(a \wedge b)) + p^{n+e-1}$.

The resulting group $M_{n,e}(f)$ is a group of order $p^{n+e-1}$ and class 2. The group $\langle \theta \rangle$ is cyclic of order $p$; since $f$ is compatible with multiplication by $\theta$, it follows that $\langle \theta \rangle$ acts naturally on $M_{n,e}(f)$. For $n \geq \max\{c, 8\}$ and $0 \leq e \leq \text{dep}(S_p(n))$ a homomorphism $f \in \Hom$ with image $p^{n-1}$ defines a group of depth $e$ in $S_p(n)$ via $M_{n,e}(f) \times \langle \theta \rangle$ and, conversely, every group of depth $e$ in $S_p(n)$ can be obtained as $M_{n,e}(f) \times \langle \theta \rangle$ for some $f \in \Hom$ with image $p^{n-1}$; see e.g. [6 Corollary 3.5].

The following is taken from [16 Sections 8.2 & 8.3], see also [3 Section 4.1]. The group $\Hom$ is a free $O$-module and has a natural basis $\{T_1, \ldots, T_\ell\}$ such that the homomorphism $x_1T_1 + \ldots + x_\ell T_\ell$ has image $p^n$ if and only if each $x_i \in p^n$ and at least one $x_j \notin p^{n+1}$. Leedham-Green & McKay noted that this basis is not well adapted to the isomorphism problem, so they introduced a different generating set $\{S_2, \ldots, S_{\ell+1}\}$. (The precise definitions can be found on [4 p. 235]; we do not repeat them here because we do not use them explicitly and want to avoid unnecessary notation.) For every $f \in \Hom$ there is a unique tuple $x = (x_1, \ldots, x_\ell) \in \mathbb{K}^\ell$ such that $f = S(x) = x_1S_2 + \ldots + x_\ell S_{\ell+1}$ and we define the corresponding skeleton group as

$$C_{n,e}(x) = M_{n,e}(S(x)) \times \langle \theta \rangle.$$  

However, $x_i \notin O$ is possible, and the base change matrix $B$ from $\{S_2, \ldots, S_{\ell+1}\}$ to $\{T_1, \ldots, T_\ell\}$ is not invertible over $O$. We define

$$\Gamma_n = (p^{n-1})^\ell B^{-1} \quad \text{and} \quad \Delta_n = \Gamma_n \setminus \Gamma_{n+1},$$

so that tuples in $\Delta_n$ parametrise the homomorphisms that have image equal to $p^{n-1}$. To discuss the isomorphism problem for the groups $C_{n,e}(x)$, the following is required. For $i \in \{2, \ldots, \ell+1\}$ let $\rho_i : U \to U$ be defined by $\rho_i(u) = u^{-1}\sigma_i(u)\sigma_{i-1}(u)$. It is shown in [4 Lemma 4.3] that $(u, \varphi) \in U \times \mathcal{G}$ acts on $x = (x_1, \ldots, x_\ell) \in \mathbb{K}^\ell$ via

$$(u, \varphi)(x) = (\rho_2(u)^{-1}\varphi(x_1), \ldots, \rho_{\ell+1}(u)^{-1}\varphi(x_\ell)).$$

The next theorem summarises some key results for skeleton groups and their isomorphism problem.

**Theorem 3.1.** Let $n \geq \max\{c, 8\}$ and $0 \leq e \leq \text{dep}(S_p(n))$. Let $x, y \in \Delta_n$.

a) The map $x \mapsto C_{n,e}(x)$ yields an onto map from $\Delta_n$ to the groups of depth $e$ in $S_p(n)$.

b) We have $C_{n,e}(x) \cong C_{n,e}(y)$ if and only if $(u, \varphi)(x) \equiv y \mod \Gamma_{n+e}$ for some $(u, \varphi) \in U \times \mathcal{G}$.

c) We have $h \mid \text{gal}(C_{n,e}(x))$ if and only if $(u, \varphi)(x) \equiv x \mod \Gamma_{n+e}$ for some $u \in U$.

d) If $G$ has depth $e \geq 1$ in $S_p(n)$, then $\text{gal}(G)$ divides $d$ and the Galois order of every descendant of $G$ in $S_p(n)$ divides $\text{gal}(G)$.

**Proof.** Part a) is proved in [16 Sections 8.2 & 8.3] and [4 Section 4.1]. A proof for part b) can be found in [4 Section 4.2], based on ideas of Leedham-Green & McKay. By [3 Lemma 5.4], the $p^i$-part of $|\text{Aut}(C_{n,e}(x))|$ is the same as the order of the image of the projection $\text{Stab}_{\ell+1}O(x + \Gamma_{n+e}) \to \mathcal{G}$; thus, $h$ divides the Galois
order of $C_{n,e}(x)$ if and only if there is some $u \in U$ with $(u, \tau)(x) \equiv x \mod \Gamma_{n+e}$; see [4] Theorem 4.4] for additional information on Aut$(C_{n,e}(x))$. This implies c) and d).

4. A local to global principle

Theorem 3.1 describes the groups of depth $e$ in $S_p(n)$ whose Galois order is divisible by $h$ via fixed points under some action modulo $\Gamma_{n+e}$; that is, they are fixed points in a local setting. The aim of this section is to prove the following theorem, which shows that they can also be described via global fixed points. This result is our first main result and a central ingredient for the proofs of our main theorems.

**Theorem 4.1.** Let $p \geq 7$ and $n \geq \max\{c, 8\}$. Every group in $S_p(n)$ is defined by a 'global Galois fixed point': if $G$ has depth $e$ in $S_p(n)$ and Galois order $h$, then $G \cong C_{n,e}(x)$ for some $x \in \Delta_n$ that is fixed by some $(v, \varphi) \in U \times G$, where $\varphi$ has order $h$ and $v \in \mathbb{Z}_p$ is a $d$-th root of unity.

Throughout this section, let $n \geq \max\{c, 8\}$ and $0 \leq e \leq \text{dep}(S_p(n))$. Recall that $\tau = \sigma^k$ where $d = hk$, so $\tau$ generates the subgroup of order $h$ in the Galois group $G$. In some results below we use that the homomorphism $\chi : U \to U$, $u \mapsto u\tau(u)^{-1}$ induces the decomposition $U = \ker(\chi) \times \text{im}(\chi)$.

**Lemma 4.2.** If $G$ has depth $e$ in $S_p(n)$, then $h | \text{gal}(G)$ if and only if $G \cong C_{n,e}(x)$ for some $x \in \Delta_n$ such that $(v, \tau)(x) \equiv x \mod \Gamma_{n+e}$ for some $v \in \ker(\chi)$.

**Proof.** Theorem 3.1(b) shows that $G \in S_p(n)$ and $h | \text{gal}(G)$ if and only if $G \cong C_{n,e}(y)$ for some $y \in \Delta_n$ such that $(u, \tau)(y) \equiv y \mod \Gamma_{n+e}$ for some $u \in U$. Write $u = vw$ with $v \in \ker(\chi)$ and $w \in \text{im}(\chi)$, that is, $w^{-1} = s\tau(s)^{-1}$ for some $s \in U$. Now define $x = (s, 1)(y)$; Theorem 3.1(b) shows that $G \cong C_{n,e}(x)$ and, by construction, we have

$$x \equiv (s, 1)(u, \tau)(y) \equiv (s, 1)(u, \tau)(s, 1)^{-1}(x) \equiv (su\tau(s)^{-1}, \tau)(x) \equiv (v, \tau)(x) \mod \Gamma_{n+e}.$$ 

As a next step, we investigate the action of $\tau$ on $\Gamma_n$. Let $L = \{a \in \mathbb{K} : \tau(a) = a\}$ be the fixed field in $\mathbb{K}$ under the action of $\tau$. Let $\mathcal{O}_L = \mathcal{O} \cap L$ the maximal order in $L$ and let $U_L = U \cap L$ be its unit group; note that $U_L = \ker(\chi)$. For $i \in \{0, \ldots, d - 1\}$ let

$$\Sigma_{n,i} = \{x \in \Gamma_n : (1, \tau)(x) = \omega^i x\}$$

and

$$\Omega_{n,i} = \{x \in \Delta_n : (1, \tau)(x) = \omega^i x\}.$$

Note that $\Omega_{n,i} = \Sigma_{n,i} \setminus \Sigma_{n+1,i}$ for all $i \in \{0, \ldots, d - 1\}$.

**Lemma 4.3.** Let $\mathcal{I} = \{(kj) \mod d : 0 \leq j \leq d - 1\}$. For each $i \in \mathcal{I}$ the set $\Sigma_{n,i}$ is an $\mathcal{O}_L$-sublattice of $\Gamma_n$ such that $(u, 1)(x) \in \Sigma_{n,i}$ for each $x \in \Sigma_{n,i}$ and $u \in U_L$. One can decompose

$$\Gamma_n = \bigoplus_{i \in \mathcal{I}} \Sigma_{n,i}.$$

**Proof.** For $h = d$ this is [4] Lemma 5.5]. It follows readily from that lemma that $\Sigma_{n,i}$ is an $\mathbb{Z}_p$-sublattice of $\Gamma_n$; note that $(1, \sigma)(x) = \omega^i x$ implies that $(1, \tau)(x) = \omega^k x$, and this also asserts that $\Gamma_n = \bigoplus_{i \in \mathcal{I}} \Sigma_{n,i}$. It remains to show that each $\Sigma_{n,i}$ is invariant under the scalar action of $\mathcal{O}_L$ and the twisted action of $U_L$. Let $x \in \Sigma_{n,i}$. If $a \in \mathcal{O}_L$, then $(1, \tau)(ax) = \tau(a)(1, \tau)(x) = a\omega^i x = \omega^i(ax)$, so $ax \in \Sigma_{n,i}$. For $u \in U_L$ we have $(u, 1)(x) = (\rho_2(u)^{-1}x_1, \ldots, \rho_{d-1}(u)^{-1}x_d)$. Since $u \in U_L$, each $\rho_j(u) \in U_L$, so $(1, \tau)((u, 1)(x)) = (u, 1)((1, \tau)(x)) = (u, 1)(\omega^i x) = \omega^i((u, 1)(x))$, as claimed.

Next, we give a proof for a number-theoretic condition for the existence of global fixed points.
Lemma 4.4. We have $\Omega_{n,i} \neq \emptyset$ if and only if $(1, \tau)$ has an eigenvector in $\Delta_1$ with eigenvalue $\omega^{i+k(-n+1)}$; in particular, the eigenvalues of $(1, \tau)$ on $\Delta_1$ are $\{\omega^{i-k(j+1)} : j = 1, \ldots, \ell\}$, so $\Omega_{n,i} \neq \emptyset$ if and only if $i \equiv k(n-2j) \mod d$ for some $j = 1, \ldots, \ell$.

Proof. For $k = d$ this is [4, Lemma 5.7]. Let $x \in \Delta_1$ be an eigenvector of $(1, \tau)$ with eigenvalue $\omega^{i+k(-n+1)}$. By [4, Lemma 2.1] there exists an eigenvector $a \in \mathbb{F}_p^{n-1} \setminus \mathbb{F}_p^n$ of $\sigma$ with eigenvalue $\omega^{n-1}$. Since $\tau = \sigma^k$, it follows that $a$ is an eigenvector of $\tau$ with eigenvalue $\omega^{k(-n+1)}$. We have $ax \in \Delta_n$ and $$(1, \tau)(ax) = \omega^{k(-n+1)} ax = \omega^i(ax),$$ so $ax \in \Omega_{n,i}$. Conversely, if $x \in \Omega_{n,g}$ is an eigenvector of $(1, \tau)$ with eigenvalue $\omega^i$, then choose an eigenvector $a \in \mathbb{F}_p^{n-1} \setminus \mathbb{F}_p^{n+2}$ of $\tau$ with eigenvalue $\omega^{k(-n+1)}$, and observe that $ax \in \Delta_1$ satisfies $$(1, \tau)(ax) = \omega^{k(-n+1)} \omega^i x = \omega^{i+k(-n+1)} ax.$$ Lastly, [4, Lemma 5.8] shows that the eigenvalues of $(1, \tau)$ on $\Delta_1$ are $\{\omega^{2j+1} : j = 1, \ldots, \ell\}$, hence the final claim follows from $\tau = \sigma^k$.

The following proposition summarises our ‘local-to-global’ principle for skeleton groups: every skeleton group at depth $e$ in $S_p(n)$ can be defined as $C_{n,e}(z)$ where $z \in \Omega_{n,j}$ is some ‘global fixed point’ of $(\omega^j, \tau)$ for some $j$.

Proposition 4.5. If $G$ has depth $e$ in $S_p(n)$, then the following are equivalent:

a) $G \in S_p(n)$ with $h \mid \text{gal}(G)$;

b) $G \cong C_{n,e}(x)$ for some $x \in \Delta_n$ and $u \in \mathcal{U}$ with $(u, \tau)(x) \equiv x \mod \Gamma_{n+e}$;

c) $G \cong C_{n,e}(y)$ for some $y \in \Delta_n$ and $v \in \ker(\chi)$ with $(v, \tau)(y) \equiv y \mod \Gamma_{n+e}$;

d) $G \cong C_{n,e}(z)$ for some $j \equiv k(n-2i) \mod d$ : $i = 1, \ldots, \ell$ and $z \in \Omega_{n,j}$.

Proof. Parts a,b,c are Theorem 3.1 and Lemma 4.2 and listed for completeness only. It is trivially true that d) implies b), and it remains to show that c) implies d); we proceed with an argument similar to the proof of [4, Lemma 5.6]. In the following suppose that $G \cong C_{n,e}(y)$ for some $y \in \Delta_n$ with $(v, \tau)(y) \equiv y \mod \Gamma_{n+e}$ for a suitable $v \in \ker(\chi)$. Lemma 4.3 implies that we can decompose $y = y_1 + \ldots + y_k$, with each $y_i \in \Sigma_{n,i}$. Write $v = \omega^j \theta^q u$ for some $j, q \in \mathbb{Z}$ and $u \in \mathcal{U}_2$, so that each $\rho_i(v) = \omega^j \rho_i(u)$ with $\rho_i(u) \in \mathcal{U}_2$. In the following let $i \in \ker(\chi)$, that is, $\tau(v) = v$, we have $$(1, \tau)(v, 1)(y_i) = (v, \tau)(y_i) = (v, 1)(1, \tau)(y_i) = \omega^i(v, 1)(y_i) = \omega^i \omega^j(u, 1)(y_i).$$ Note that $(v, 1)(y_i) \in \Sigma_{n,i}$, hence $(v, \tau)(y_i) \in \Sigma_{n,i}$, so the assumption $y - (v, \tau)(y) \in \Gamma_{n+e}$ together with Lemma 4.3 implies that for every $i$ we have $y_i - (v, \tau)(y_i) = y_i - \omega^j - 1(u, 1)(y_i) \in \Gamma_{n+e}$, that is, $$(u, 1)(y_i) \equiv \omega^j - 1(y_i) \mod \Gamma_{n+e}.$$ This shows that $(u^d, 1)(y_i) \equiv y_i \mod \Gamma_{n+e}$ and, since the action of $(u, 1)$ on $\Gamma_1 / \Gamma_{n+e}$ has $p$-power order, it follows that there is some $m$ such that $(u^d, 1)^m$ and $(u, 1)$ induce the same action. The former acts trivially on $y_i$ while the latter acts by multiplication by $\omega^j - 1$. Thus, if $\omega^j - 1 \neq 1$, then this forces $y_i \in \Gamma_{n+e}$. In conclusion, $y \equiv y_j \mod \Gamma_{n+e}$, and since $y \in \Delta_n$ we also have $y_j \in \Delta_n$. Thus, $G \cong C_{n,e}(y) = C_{n,e}(y_j)$, and we can take $z = y_j$; the claim follows with Lemma 4.4.

Theorem 4.1 is an immediate consequence of Proposition 4.5 and allows us to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathcal{R}$ be a Galois tree in $S_p(n)$ with Galois order $h$. Recall that $\text{dep}(S_p(n)) = n - e$. Let $G$ be a group in $\mathcal{R}$ at depth $e$ in $S_p(n)$. By Theorem 4.1 we have $G \cong C_{n,e}(x)$ for some $x \in \Delta_n$ with $(1, \tau)(x) = \omega^i x$ for some $i$. The group $H = C_{n,n-e}(x)$ is a descendant of $G$ in $S_p(n)$, hence $\text{gal}(H)$ divides $\text{gal}(G)$. On the other hand, Proposition 4.5 shows that $\text{gal}(G)$ divides $\text{gal}(H)$. Thus, $H$ is a leaf in $\mathcal{R}$, and every leaf of $\mathcal{R}$ has depth $\text{dep}(S_p(n))$. 

5. Ramification of Galois trees

To describe the structure of Galois trees in more detail, it is required to solve the isomorphism problem of skeleton groups defined by global Galois fixed points; see Theorem 4.1. The aim of this section is to give a generalisation of Lemma 5.11, and to indicate why it appears to be a lot more difficult to prove an explicit structure result for Galois trees that generalises the results for $h = d$ in Theorem 6.2. We start with a preliminary lemma; as before, $\tau = \sigma^k$ with $d = p - 1 = hk$.

Lemma 5.1. Let $x,y \in \Sigma_{n,i}$ such that $(u,1)(x) \equiv y \mod \Gamma_{n,e}$ for some $u \in U$. If $u = vw$ with $v \in \ker(\chi)$ and $w \in \im(\chi)$, then $(v,1)(x) \equiv y \mod \Gamma_{n,e}$.

Proof. Since $y$ and $(u,1)(x)$ lie in $\Sigma_{n,i}$, it follows from $y \equiv (u,1)(x) \mod \Gamma_{n,e}$ that

$$\omega^iy \equiv \omega^i(\tau(u),1)(x) \equiv \omega^i((\chi(u))^{-1},1)(u,1)(x) \equiv \omega^i(\chi(u)^{-1},1)(y) \mod \Gamma_{n,e},$$

hence $(\chi(u),1)$ fixes $y$ modulo $\Gamma_{n,e}$ and, by symmetry, also $x$ modulo $\Gamma_{n,e}$. Note that $\chi(u) = \chi(w)$, so also $(\chi(u),1)$ fixes both $x$ and $y$ modulo $\Gamma_{n,e}$. Now iterate the argument: since $(\chi(u),1)$ fixes $x$ modulo $\Gamma_{n,e}$, it follows that $(\chi^2(u),1)$ fixes $x$ modulo $\Gamma_{n,e}$ and, by induction, $(\chi^m(u),1)$ does the same for all $m \in \mathbb{N}$. Let $j$ be large enough such that $U_j = 1 + \mathfrak{p}^j$ acts trivially on $\Gamma_n/\Gamma_{n,e}$. Similar to the decomposition of $U$, we have $U_j = \ker(\chi| U_j) \times \im(\chi| U_j)$. Since $U = \ker(\chi) \times \im(\chi)$, it follows that $\chi$ induces an automorphism of $J = \im(\chi)/\im(\chi| U_j)$, denoted $\psi$. Note that $J$ is finite since it is a quotient of $\im(\chi)/\im(\chi)^p$ for some $r \geq 1$ and the latter group is finite because it is finitely generated abelian with finite exponent. It follows that $\psi$ has finite order $m \in \mathbb{N}$. This shows that $\psi^m(\hat{w}) = \hat{w}$ for the coset $\hat{w} = \psi^m(\chi| U_j) \in J$, that is, $\chi^m(w) = ws$ for some $s \in U$. Since $(s,1)$ and $(\chi^m(w),1) = (ws,1)$ fix $x$ modulo $\Gamma_{n,e}$, it follows that $(w,1)$ fixes $x$ modulo $\Gamma_{n,e}$. The claim follows.

Proposition 5.2. Let $x \in \Omega_{n,i}$ and $y \in \Omega_{n,j}$. Then $C_{n,e}(x) \cong C_{n,e}(y)$ if and only if $i = j$ and $(v,\sigma^m)(x) \equiv y \mod \Gamma_{n,e}$ for some $v \in \ker(\chi)$ and $m \in \{0,\ldots,k - 1\}$.

Proof. By Theorem 3.1b) the groups are isomorphic, if and only if there exists $(u,\varphi) \in U \times G$ with

$$(u,\varphi)(x) \equiv y \mod \Gamma_{n,e}.$$

Now observe that modulo $\Gamma_{n,e}$, we have

$$\omega^i y \equiv (1,\tau)(y) \equiv (1,\tau)(u,\varphi)(x) \equiv \omega^i((\chi(u))^{-1},1)(u,\varphi)(x) \equiv \omega^i(\chi(u)^{-1},1)(y),$$

that is, $(\chi(u)^{-1},1)(y) \equiv \omega^{-i}y \mod \Gamma_{n,e}$. Since the image of $\chi$ is contained in $\langle \theta \rangle \times U_2$, the action on $\Gamma_1/\Gamma_{n,e}$ induced by $(\chi(u)^{-1},1)$ has $p$-power order. This forces $\omega^{-i} = 1$, so $i = j$, as claimed.

Recall that $\tau = \sigma^k$ and write $\varphi = \sigma^r$ with $r = qk + m$ where $m \in \{0,\ldots,k - 1\}$. This shows that $(1,\varphi)(x) = (1,\sigma^m)(1,\sigma^q)(x) = \omega^q(1,\sigma^m)(x)$, which proves that

$$y \equiv (u,\varphi)(x) \equiv \omega^q(u,1)(1,\sigma^m)(x) \equiv (u\omega^{-q},\sigma^m)(x) \mod \Gamma_{n,e}.$$

Thus, without loss of generality, $\varphi = \sigma^m$ with $m \in \{0,\ldots,k - 1\}$. Note that $(1,\sigma^m)(x) \in \Sigma_{n,i}$, and $(u,1)$ maps $(1,\sigma^m)(x)$ to $y$ modulo $\Gamma_{n,e}$. Writing $u = vw$ with $v \in \ker(\chi)$ and $w \in \im(\chi)$, it follows from Lemma 5.1 that $(v,1)$ maps $(1,\sigma^m)(x)$ to $y$ modulo $\Gamma_{n,e}$. The claim follows.

Corollary 5.3. If $R$ is a Galois tree in $\mathcal{S}_p(n)$ with Galois order $h$, then all groups in $R$ can be defined by elements in $\Omega_{n,i}$ for a unique $i \in \{k(n - 2j) \mod d : j \in \{1,\ldots,\ell\}\}$.

Proof. Let $G$ be the root of $R$ and let $H$ be an $m$-step descendant of $G$ in $R$. By Theorem 4.1 we can define $G = C_{n,e}(x)$ and $H = C_{n,e+m}(y)$ for some $x \in \Omega_{n,i}$ and $y \in \Omega_{n,i_2}$. Since $H$ is a descendant of $G$, we have $C_{n,e}(x) \cong C_{n,e}(y)$, and now Proposition 5.2 implies that the claim is true for $i = i_1 = i_2$. It follows from Lemma 4.4 that $i \in \{k(n - 2j) \mod d : j \in \{1,\ldots,\ell\}\}$. \[\square\]
Lemma 5.4. Let \( \mathcal{R} \) be a Galois tree in \( \mathcal{S}_p(n) \) of Galois order \( h \) and type \( t \). If \( \mathcal{R} \) has ramification level \( e \), then
\[
eq \mod h \in \{2(j-i) \mod h : i \in I_t, j \in \{1, \ldots, \ell\}\},
\]
where \( I_t = \{i \in \{1, \ldots, \ell\} : t \equiv k(n-2i) \mod d\} \).

Proof. Let \( G_1 \) and \( G_2 \) be distinct groups in \( \mathcal{R} \) with the same immediate ancestor in \( \mathcal{R} \); suppose \( G_1 \) and \( G_2 \) have depth \( e + 1 \) in \( \mathcal{S}_p(n) \). Theorem 4.1 and Corollary 5.3 show that we can define each \( G_i = C_{n,e+1}(x_i) \) for some \( x_i \in \Omega_{n,t} \). Since \( C_{n,e}(x_1) \cong C_{n,e}(x_2) \) is the isomorphism type of the parent of \( G_1 \) and \( G_2 \), Proposition 5.2 shows that there is \( v \in \ker(\chi) \) and \( m \in \{0, \ldots, k-1\} \) with \( (v, \sigma^m)(x_1) \equiv x_2 \mod \Gamma_{n,e} \), that is, \((v, \sigma^m)(x_1) = x_2 + y \) for some \( y \in \Gamma_{n,e} \). It follows from the definition of \( \Sigma_{n,t} \) and the definition of \( \chi \) that \((v, \sigma^m)(x_1) \in \Sigma_{n,t} \), and so \( y = (v, \sigma^m)(x_1) - x_2 \in \Sigma_{n,e,t} \). Since \( x_1 \) and \((v, \sigma^m)(x_1) \) define isomorphic groups, we can assume, without loss of generality, that \( x_1 = x_2 + y \) with \( x_2 \in \Omega_{n,t} \) and \( y \in \Sigma_{n,e,t} \). Since \( x_1 \) and \( x_2 \) define non-isomorphic groups at depth \( e + 1 \), we have in fact \( y \in \Omega_{n+e,t} \). Since \( x_1 \in \Omega_{n,t} \), Lemma 4.4 shows that \( t \equiv k(n-2i) \mod d \) for some \( i \in \{1, \ldots, \ell\} \). Similarly, \( y \in \Omega_{n+e,t} \) implies that \( t \equiv k(n+e - 2j) \mod d \) for some \( j \in \{1, \ldots, \ell\} \). Together, \( h \) must divide \((j - i) \mod h \), as claimed.

Example 5.5. Let \( p = 7 \) and \( n = 24 \), and consider \( h = 6 \) and \( k = 1 \). By Lemma 4.4 the possible types of Galois trees are \( t \in \{2, 4\} \). If \( t = 2 \), then \( j = 2 \), and a ramification level \( e \) is only possible if \( e \mod 6 \in \{0, 4\} \). If \( t = 4 \), then \( j = 1 \), and a ramification level \( e \) can only occur for \( e \mod 6 \in \{0, 2\} \). However, Theorem 1 shows that the actual ramification levels are at depth \( e \) with \( e \mod 6 = 4 \) and \( e \mod 6 = 2 \), respectively, see also Figure 2.
(1) If \( z \in \mathcal{Z}(n, e, t) \), then \( u \cdot z \in \mathcal{Z}(n, e, t) \) if and only if \( \chi(u) \in \text{Stab}_d(x + y + \Gamma_{n+e+1}) \).

(2) We have \( u \cdot y \equiv v \cdot y \mod \Gamma_{n+e+1} \) if and only if \( uv^{-1} \in \text{Stab}_d(x + y + \Gamma_{n+e+1}) \).

We summarise these results in a lemma; this proves Theorem 2.2a).

**Lemma 5.6.** Let \( \mathcal{R} \) be a Galois tree in \( S_p(n) \) with Galois order \( h \) and type \( t \). Let \( G = C_{n,e}(x) \) be a group in \( \mathcal{R} \) with \( x \in \Omega_{n,t} \) and \( e < \text{dep}(S_p(n)) \). If \( M \subseteq \Sigma_{n+1} \) defines a set of representatives of the affine \( \text{Stab}_d(x + \Gamma_{n+e}) \)-orbits on \( \mathcal{Z}(n, e, t) \), then the immediate descendants of \( G \) in \( \mathcal{R} \), up to isomorphism, are

\[
\{ C_{n,e+1}(x + y) : y \in M \}.
\]

If \( p \equiv 5 \mod 6 \), then there exists \( e_0 = e_0(p) \) such that for all \( e_0 \leq e < n - c \) the number of immediate descendants of \( G \) in \( \mathcal{R} \) is a power of \( p \).

**Proof.** If \( z \in \Sigma_{n,t} \), then a direct calculation shows that \( (u, 1) \in \mathcal{U}_d \times G \) stabilises \( z + \Gamma_{n+e} \) if and only if \( \tau(u), 1 \) does. Also, note that if \( x \) and \( y \) are as in the lemma, then \( \text{Stab}_d(x + y + \Gamma_{n+e+1}) \) is a subgroup of \( \text{Stab}_d(x + \Gamma_{n+e+1}) \). A short calculation shows the following:

(3) If \( u \in \text{Stab}_d(x + \Gamma_{n+e}) \) and \( v \in \text{Stab}_d(x + y + \Gamma_{n+e+1}) \), then \( (\chi(u), 1)(x + y) \equiv x + y \mod \Gamma_{n+e+1} \).

Since \( \theta \in \mathcal{U}_d \) acts trivially on \( \Gamma_1 \) and \( \omega \in \mathcal{U}_d \) acts nontrivially on each \( \Gamma_i/\Gamma_{i+1} \), Properties (1)–(3) imply that the \( \text{Stab}_d(x + \Gamma_{n+e}) \)-orbit of \( y + \Gamma_{n+e+1} \in \mathcal{Z}(n, e, t) \) intersects \( \mathcal{Z}(n, e, t) \) in a set of \( p \)-power size

\[
|\{ \text{Stab}_d(x + y + \Gamma_{n+e+1}) : u \in \text{Stab}_d(x + \Gamma_{n+e})\}|.
\]

If \( p \equiv 5 \mod 6 \), then [3] Theorem 5.1 shows that (for sufficiently large \( e \)) we have

\[
\text{Stab}_d(x + \Gamma_{n+e+d}) = (\text{Stab}_d(x + \Gamma_{n+e+1}))^\lbrack p \rbrack \quad \text{and}
\]

\[
\text{Stab}_d(x + \Gamma_{n+e}) \leq \text{Stab}_d(\Gamma_1/\Gamma_{i+3d}) \quad \text{for all} \quad i;
\]

here \( (\text{Stab}_d(x + \Gamma_{n+e}))^\lbrack p \rbrack \) denotes the subgroup generated by all \( p \)-th powers. In this situation, \( \text{Stab}_d(x + \Gamma_{n+e}) \) stabilises \( \Gamma_{n+e}/\Gamma_{n+e+1} \), hence [5.2] is independent of \( y \) and each \( \text{Stab}_d(x + \Gamma_{n+e}) \)-orbit in \( \mathcal{Z}(n, e, t) \) has size

\[
\mathbf{e} = \left| \{ \text{Stab}_d(x + \Gamma_{n+e+1}) : u \in \text{Stab}_d(x + \Gamma_{n+e}), \chi(u) \in \text{Stab}_d(x + \Gamma_{n+e+1}) \} \right|.
\]

Thus, if \( p \equiv 5 \mod 6 \), then the number of affine orbits is \( |\mathcal{Z}(n, e, t)|/\mathbf{e} \). Since \( \mathcal{Z}(n, e, t) \) is a section of the \( p \)-group \( \Gamma_1/\Gamma_{n+e+1} \), it follows that \( |\mathcal{Z}(n, e, t)|/\mathbf{e} \) is a power of \( p \), as claimed.

This result implies that the ramification levels in a Galois tree occur with periodicity \( d \), at least when \( p \equiv 5 \mod 6 \). We prove this in the following proposition, which yields Theorem 2.2b). The proof is similar to the proof of Theorem 2.3, the difference is that in this proposition we only consider descendants in a Galois tree contained in \( S_p(n) \), whereas Theorem 2.3 considers descendants in the branch \( B_p(n) \), which leads to additional difficulties.

**Proposition 5.7.** If \( p \equiv 5 \mod 6 \), then there exists \( e_0 = e_0(p) \), such that for all \( e_0 \leq e < \text{dep}(S_p(n)) - d \) the following holds. If \( \mathcal{R} \) is a Galois tree in \( S_p(n) \) with Galois order \( h \) and type \( t \), then \( \mathcal{R} \) has a ramification level \( e \) if and only if \( \mathcal{R} \) has a ramification level \( e + d \). More precisely, if \( x \in \Omega_{n,t} \), then \( C_{n,e+d}(x) \) has exactly \( m \) immediate descendants in \( \mathcal{R} \) at depth \( e + d + 1 \) in \( S_p(n) \) if and only if \( C_{n,e}(x) \) has exactly \( m \) immediate descendants in \( \mathcal{R} \) at depth \( e + 1 \) in \( S_p(n) \).

**Proof.** We use Lemma 5.6 to describe the immediate descendants of \( C_{n,e}(x) \) and \( C_{n,e+d}(x) \) in \( \mathcal{R} \). Note that multiplication by \( p \) induces a bijection

\[
\phi: \mathcal{Z}(n, e, t) \to \mathcal{Z}(n, e + d, t).
\]
As in the proof of Lemma 5.6, the assumption \( p \equiv 5 \mod 6 \) implies that for sufficiently large \( e \) we have
\[
\text{Stab}_u(x + \Gamma_{n+e+d}) = (\text{Stab}_u(x + \Gamma_{n+e}))^{[p]}
\]
and every \( u \in \text{Stab}_u(x + \Gamma_{n+e+d}) \) acts trivially on \( \Gamma_i/\Gamma_{i+3d} \) for every \( i \), see Theorem 5.1. Now let \( u \in \text{Stab}_u(x + \Gamma_{n+e}) \). By assumption, \( (u, 1) (x) - x = y \) lies in \( \Gamma_{n+e} \), so \((u^p, 1) (x) - x \equiv py \mod \Gamma_{n+e+3d} \). This proves that for \( z + \Gamma_{n+e+1} \in \mathcal{Z}(n, e, t) \) we have
\[
\phi(u \cdot (z + \Gamma_{n+e+1})) = \phi((u, 1) (x) - x + (u, 1) (z) + \Gamma_{n+e+1}) = \phi(y + z + \Gamma_{n+e+1}) = py + pz + \Gamma_{n+e+d+1} = (u^p, 1) (x) - x + pz + \Gamma_{n+e+d+1} = u^p \cdot \phi(z + \Gamma_{n+e+1}).
\]
Note that \( \theta \) acts trivially on \( \Gamma_1 \); moreover, \( \omega^p = \omega \), and \( \text{Stab}_u(x + \Gamma_{n+e+d}) = (\text{Stab}_u(x + \Gamma_{n+e}))^{[p]} \) by assumption. Thus, \( \phi \) induces a bijection between the affine \( \text{Stab}_u(x + \Gamma_{n+e}) \)-orbits in \( \mathcal{Z}(n, e, t) \) and the affine \( \text{Stab}_u(x + \Gamma_{n+e+d}) \)-orbits in \( \mathcal{Z}(n, e + d, t) \). Now the claim follows with Lemma 5.6

\[\Box\]

6. The roots of Galois trees

In this section we investigate the structure of the roots of Galois trees. Throughout, we assume that \( G \) is a candidate for a root of a Galois tree in \( \mathcal{S}_p(n) \) with Galois order \( h \) and type \( t \). It will be convenient to describe \( G \) as a descendant of its immediate ancestor, which is why we assume that \( G \) has depth \( e + 1 \) in \( \mathcal{S}_p(n) \), so the parent has depth \( e \). Suppose this parent has Galois order \( b = ha \) with \( a \geq 1 \) a divisor of \( k \); thus, \( G \) is a root of a Galois tree if and only if \( a \geq 2 \). Throughout let \( \nu \in \mathcal{G} \) such that
\[
\nu^a = \tau;
\]
we can assume that \( \tau = \sigma^k \) and \( \nu = \sigma^{k/a} \) with \( d = hk = b(k/a) \). Recall that the sets \( \Sigma_{n,i} \) and \( \Omega_{n,i} \) have been defined with respect to our fixed generator \( \tau \in \mathcal{G} \) of order \( h \). We require the analogous definitions for \( \nu \), that is, for \( i \in \{1, \ldots, d - 1\} \) we set
\[
\Sigma'_{n,i} = \{x \in \Gamma_n : (1, \nu)(x) = \omega^i x\} \quad \text{and} \quad \Omega'_{n,i} = \{x \in \Delta_n : (1, \nu)(x) = \omega^i x\}
\]
and define \( \chi' : \mathcal{U} \rightarrow \mathcal{U} \) via \( \chi'(u) = u\nu(u)^{-1} \).

**Theorem 6.1.** With the previous notation, the following holds. The group \( G \) is the root of a Galois tree with immediate ancestor of Galois order \( b = ha \) for some \( a \geq 2 \) if and only if
\[
G \cong C_{n,e+1}(x + y)
\]
for some \( x \in \mathcal{O}^{\prime}_{n,s} \) and \( y \in \Omega_{n+e,t} \setminus \Omega'_{n+e,s} \) such that \( \omega^aa = \omega^t \) and \( \text{gal}(C_{n,e}(x)) = b \) and
\[
(\omega^{-s}, \nu)[(u, 1)(x + y)] \equiv (u, 1)(x + y) \mod \Gamma_{n+e+1} \quad \text{for all} \ u \in Q,
\]
where \( Q = \ker(\chi) \cap \text{Stab}_u(x + \Gamma_{n+e}) \).

Before we prove the theorem, we note that Condition (6.1) can be replaced by
\[
(\omega^{-s}, \nu)[(u, 1)(x + y)] \equiv (u, 1)(x + y) \mod \Gamma_{n+e+1} \quad \text{for all} \ u \in Q,
\]
where \( Q \) is a set of coset representatives of \( \ker(\chi') \cap \text{Stab}_u(x + \Gamma_{n+e}) \) in \( Q \). This follows because if \( u \in \ker(\chi) \) satisfies \( (\omega^{-s}, \nu)[(u, 1)(x + y)] \equiv (u, 1)(x + y) \mod \Gamma_{n+e+1} \), then \( uv \) with \( v \in \ker(\chi') \) also satisfies
\[
(\omega^{-s}, \nu)(uv, 1)(x + y) \equiv (v, 1)(\omega^{-s}, \nu)(u, 1)(x + y) \equiv (v, 1)(u, 1)(x + y) \mod \Gamma_{n+e+1}.
\]
Moreover, Condition (6.1) with \( u = 1 \) implies that \( (1, \nu)(y) \not\equiv \omega^s y \mod \Gamma_{n+e+1} \); in particular, \( y \not\in \Omega'_{n+e,s} \).
Proof of Theorem 6.1. We split the proof in two parts.

\[ \Rightarrow: \] By Theorem 4.1 we can assume that \( G = C_{n,e+1}(z) \) for some \( z \in \Omega_{n,t} \). Since the immediate ancestor \( C_{n,e}(z) \) of \( G \) has Galois order \( b \) by assumption, Theorem 3.1b) shows that \( (u,v)(z) \equiv z \mod \Gamma_{n+e} \) for some \( u \in U \). Let \( \omega^1, \ldots, \omega^u \) be the \( d \)-th roots of unity with \( \omega^d = 1 \) and decompose

\[ \Sigma_{n,t} = \Sigma_{n,i,j} + \cdots + \Sigma_{n,t,u}. \]

Write \( z = z_{i_1} + \cdots + z_{i_u} \) with each \( z_{i_j} \in \Sigma_{n,i,j} \). Recall that \( (1,v)(z) = (u^{-1},1)(z) \mod \Gamma_{n+e} \), and we can write

\[ (u^{-1},1)(z_{i_1} + \cdots + z_{i_u}) \equiv (u^{-1},1)(z_{i_1} + \cdots + z_{i_u}) \mod \Gamma_{n+e}. \]

Write \( u = \omega^{s}\theta^q v \) with \( v \in U \) and note that \( z \equiv (u^{-d},1)(z) \equiv (\theta^q v^d,1)^{-1}(z) \mod \Gamma_{n+e} \). Since \( (\theta^q v,1) \) acts as an element of \( p \)-power order on \( \Gamma_1/\Gamma_{n+e} \), this implies that \( (\theta^q v,1)(z) \equiv z \mod \Gamma_{n+e} \), and therefore \( (u^{-1},1)(z) \equiv \omega^{u}z \mod \Gamma_{n+e} \). Together with (6.3), it follows that

\[ \omega^{1-s}z_{i_1} + \cdots + \omega^{u-s}z_{i_u} \equiv z_{i_1} + \cdots + z_{i_u} \mod \Gamma_{n+e}, \]

and with Lemma 4.3 we deduce that each \( (\omega^{j-s} - 1)z_{i_j} \in \Gamma_{n+e} \). Since \( (\omega^{j-s} - 1) \) is a unit if and only if \( s \neq i_j \), we have \( z_{i_j} \in \Gamma_{n+e} \) if and only if \( s \neq i_j \). This shows that

\[ z \equiv z_s \mod \Gamma_{n+e}, \]

and we can write \( z = z_s + y \) for some \( y \in \Gamma_{n+e} \). Recall that \( \tau = \nu^d \), so \( (1,\tau)(z) = \omega^\tau z \) shows that

\[ \omega^\tau z_s + \omega^\tau y = \omega^\tau z = (1,\nu^d)(z_s) + (1,\tau)(y) = \omega^{a_e}z_s + (1,\tau)(y). \]

From \( \omega^\tau = \omega^{a_e} \) we deduce that

\[ \omega^\tau y = (1,\tau)(y), \]

hence \( y \in \Sigma_{n,t,e} \). If we set \( x = z_s \), then \( G \cong C_{n,e+1}(x + y) \) as claimed. It remains to show that \( y \in \Omega_{n+e,t} \) and that (6.1) holds: if there is \( u \in \ker(\chi) \) such that \( (\omega^{-s},\nu)[(u,1)(x + y)] \equiv (u,1)(x + y) \mod \Gamma_{n+e+1} \), then Theorem 3.1b) shows that \( G \cong C_{n,e+1}((u,1)(x + y)) \) has Galois order divisible by \( b \). This is a contradiction to our assumption, hence (6.1) holds. Condition (6.1) with \( u = 1 \) implies that \( (1,\nu)(y) \neq \omega^s y \mod \Gamma_{n+e+1} \), hence \( y \notin \Omega_{n+e,s} \). Moreover, we must have \( y \in \Omega_{n+e,t} \) since \( C_{n,e+1}(x + y) \neq C_{n,e+1}(x) \); recall that the groups have different Galois orders.

\[ \Leftarrow: \] Let \( G \cong C_{n,e+1}(x + y) \) as in the theorem. The immediate ancestor of \( G \) is \( P = C_{n,e}(x) \) and has Galois order \( b \) by assumption. Since \( x + y \in \Sigma_{n,t} \), the Galois order of \( G \) is divisible by \( b \). Suppose, for a contradiction, that \( G \) is not the root of a Galois tree. Then \( G \) is a descendant of \( P \) that lies in the same Galois tree as \( P \), and Theorem 5.6 shows that \( G \cong C_{n,e+1}(x + z) \) for some \( z \in \Omega_{n,e,s} \). By Theorem 3.1b), there exists \( (u^{-1},\sigma^j) \in U \times G \) with

\[ (u^{-1},\sigma^j)(x + z) \equiv x + y \mod \Gamma_{n+e+1}. \]

Since \( x + z \in \Sigma_{n,s} \) is fixed by \( (\omega^{-s},\nu) \), an argument as in the proof of Proposition 5.1 shows that we can assume \( j \in \{0, \ldots, k/a - 1\} \); recall that \( \nu = \sigma^k/a \). If \( j \neq 0 \), then \( (u^{-1},\sigma^j)(x) \equiv x \mod \Gamma_{n+e} \), and Theorem 3.1b) implies that \( \text{gal}(P) \) is divisible by \( |\sigma^j| > |\nu| \), which contradicts our assumption \( \text{gal}(P) = b \). Thus, \( j = 0 \) and therefore

\[ (u,1)(x + y) \equiv x + z \mod \Gamma_{n+e+1}, \]

which already implies that \( u \in \text{Stab}_{U}(x + \Gamma_{n+e}) \). Since \( x + z \) and \( x + y \) both lie in \( \Sigma_{n,t} \), Lemma 5.1 shows that we can also assume that \( u \in \ker(\chi) \). Now (6.4) contradicts our assumption (6.1), because \( x + z \in \Sigma_{n,s} \) is fixed by \( (\omega^{-s},\nu) \). This final contradiction shows that \( G \) is the root of a Galois tree. \( \Box \)
The next corollary is analogous to Lemma 5.4 and gives a necessary (but in general not sufficient) criterion for the depths of Galois tree roots with Galois order $h$. Recall our assumptions that $\tau = \sigma^k$, $\nu = \sigma^{k/a}$ and $\omega^{sa} = \omega^i$.

**Corollary 6.2.** Using the previous notation, a Galois tree root at depth $e + 1$ with Galois order $h$ and immediate ancestor of Galois order $b = ha$ in a Galois tree of type $s$ can only exist if

$$e \equiv 2(j - i) \mod h$$

for some $i, j \in \{1, \ldots, \ell\}$ with $s \equiv (k/a)(n - 2i) \mod d$ and $t \equiv k(n + e - 2j) \mod d$.

**Proof.** From $d = dk$ and $b = ha$ it follows that $d = b(k/a)$. The groups in Theorem 6.1 can only exist if $\Omega'_{n,s} \neq \emptyset \neq \Omega_{n+e,t}$, and Lemma 5.4 shows that this holds if and only if $s \equiv (k/a)(n - 2i) \mod d$ and $t \equiv k(n + e - 2j) \mod d$ for some $i, j \in \{1, \ldots, \ell\}$. Since $sa \equiv t \mod d$, the claim follows from

$$t \equiv k(n - 2i) \equiv k(n + e - 2j) \mod d.$$ 

The following corollary concludes this section and proves Theorem 2.2c).

**Corollary 6.3.** If $p \equiv 5 \mod 6$, then for large enough $e$ the following hold. If $C_{n,e+1}(x + y)$ with $e + d < \text{dep}(S_p(n))$ is the root of a Galois tree as in Theorem 6.1, then $C_{n,e+d+1}(x + py)$ is the root of a Galois tree.

**Proof.** Write $G = C_{n,e+1}(x + y)$ and $\tilde{G} = C_{n,e+d+1}(x + py)$. The parent of $\tilde{G}$ is $\tilde{P} = C_{n,e+d}(x)$ with Galois order $b$. Suppose, for a contradiction, that $\tilde{G}$ has Galois order $b$. By Theorem 5.6, we have $\tilde{G} \equiv C_{n,e+d+1}(x + p^z)$ for some $z \equiv \Omega'_{n,s}$: recall that multiplication by $p$ induces an isomorphism $\Omega'_{i,s} \to \Omega'_{i+d,s}$ for all $i$. Now Theorem 2.1(b) shows that there is $(u, \sigma^j) \in U \times G$ with $(u, \sigma^j)(x + p^z) \equiv x + py \mod \Gamma_{n+e+d+1}$. Since $(u, \sigma^j)$ stabilises $x + \Gamma_{n+e+d}$, we can assume, as in earlier proofs, that $j = 0$ and so $u \in \text{Stab}_{\Omega_{n}}(x + \Gamma_{n+e+d})$. Thus, $py \equiv (u, 1)(x) - x + (u, 1)(p^z) \mod \Gamma_{n+e+d+1}$, that is, $pz$ and $py$ lie in the same orbit under the affine action defined in (5.1). The proof of Proposition 5.1 shows that there is $v \equiv \text{Stab}_{\Omega_{n}}(x + \Gamma_{n+e})$ with $v^p = u$ such that $y \equiv (v, 1)(x) - x + (v, 1)(z) \mod \Gamma_{n+e+1}$, which implies that $G = C_{n,e+1}(x + y) \cong C_{n,e+1}(x + z)$. Since $x + z \equiv \Omega'_{n,s}$, we have that $b$ divides $\text{gal}(G)$, a contradiction. Thus, $\tilde{G}$ is a root of a Galois tree.

7. A new periodicity result

We now consider the structure of the pruned branches in $T_p$, and proceed to prove our last main result. Recall from Section 1 that $\text{dep}(S_p(n)) = n - c$, and $B_p(n, n - c) \cong B_p(n + d, n - c)$ for all large enough $n$; in order to describe the sequence of all pruned branches $B_p(n, m - c)$ completely, it remains to specify the growth of the branches, that is, the additional $d$ layers of groups in $B_p(n + d, n + d - c)$ that are not covered by the isomorphism $\iota$. One way of doing that is to describe, for each skeleton group $G$ at depth $n - c$ in $B_p(n, n + d - c)$, the $d$-step descendant tree $D(G, d)$. It has been conjectured in [3] Conjecture 1 that for every such $G$ there exists a group $H$ at depth $n - d - c$ in $B_p(n + d, n + d + c)$ such that $D(G, d) \cong D(H, d)$. Recall that in [3] such a group $H$ is called a periodic parent of $G$, whereas we refer to such a group as a $d$-step twin.

A natural candidate for $H$ seems to be the $d$-step ancestor of $G$; indeed, it is shown in [3] Theorem 1.2] that if $p \equiv 5 \mod 6$ and if the $d$-step ancestor of $G$ has Galois order 1, then it is also a $d$-step twin. However, it is suggested in [3] Remark 4] and proved in [19] Section 6.2.7] that the $d$-step ancestor is infinitely many times not a $d$-step twin, so it is still an open problem to describe the growth of the pruned branches. A crucial ingredient in the proof of [3] Theorem 1.2] is that if a skeleton group has Galois order 1, then the same holds for all its skeleton group descendants. In fact, [3] Theorem 1.3] considers a slight generalisation: informally, it proves that if a group $G$ (as above) has distance at most $d$ from a certain path of groups with constant Galois order, then $G$ has a $d$-step twin. These results (together with [4]) highlight that the Galois order of skeleton groups seems to play a fundamental role for proving periodicity results concerning the growth of branches. The aim of this section is to exploit our new results on Galois trees.
to prove the following theorem, which yields a new periodicity result generalising [3, Theorem 1.3]; this proves [3, Conjecture 1] for $p \equiv 5 \bmod 6$ and implies Theorem 2.3.

**Theorem 7.1.** Let $p \geq 7$ with $p \equiv 5 \bmod 6$ and let $r$ be the total number of prime divisors of $d$. Then there exists $n_0 = n_0(p)$ such that for all $n \geq n_0$ the following holds: For every skeleton group $G$ at depth $n - c$ in $B_p(n + d)$ there is $m \leq pr - d$ such that the $m$-step ancestor $H$ of $G$ and the $d$-step ancestor $K$ of $H$ are $(m + d)$-step twins with the same Galois order; in particular, $G$ has a $d$-step twin at depth $n - d - c$ in $B_p(n + d)$.

**Proof.** First we show that $m$ (as in the statement of the theorem) exists, and demonstrate that the last claim of the theorem follows from the first. Let $G$ be a skeleton group at depth $n - c$ in $B_p(n + d, n + d - c)$. Since the Galois order of groups in $B_p(n)$ can change at most $r$ times, it follows that any path of length $pr$ in $B_p(n)$ has a subpath of length $d$ whose groups have the same Galois order. This proves that there exists $m \leq pr - d$ such that the $m$-step ancestor $H$ and $(m + d)$-step ancestor $K$ of $G$ have the same Galois order. We argue below that $D(H, m + d) \cong D(K, m + d)$; if $P$ is the group in $D(K, m + d)$ that corresponds to $G$ under this isomorphism of descendant trees, then $P$ is a $m$-step descendant of $K$ that is a $d$-step twin of $G$, that is, $D(G, d) \cong D(P, d)$. In particular, this proves that every skeleton group at depth $n - c$ in $B_p(n + d, n + d - c)$ has a $d$-step twin, as claimed.

It remains to show that $D(H, m + d) \cong D(K, m + d)$. By assumption, $H$ and $K$ have the same Galois order, say $h$, and by Theorem [4] we can assume that $H = C_{n + d, n - c - m}(f)$ for some $f \in \Omega_{n + d, n}$ such that $(1, \tau)(f) = \omega^y f$ for $\tau = \sigma^{d/h}$. To simplify the notation, in the following we write $\hat{n} = n + d$ and $e = n - c - m$, and we define $K_i = C_{\hat{n}, i}(f)$. Note that $S_h \rightarrow K_1 \rightarrow \ldots \rightarrow K_{\hat{n} - c}$ is a maximal path in $B_p(\hat{n}, \hat{n} - c)$ with $K = K_{e - d}$ and $H = K_e$. With this notation, it remains to show that

$$D(K_e, m + d) \cong D(K_{e - d}, m + d),$$

and our assumption is that $K_{e - d}, K_{e - d + 1}, \ldots, K_{e + m + d}$ all have the same Galois order $h$. This statement is proved in [3, Theorem 1.3] for $m + d \leq 2d$, but the same proof works for $m + d \leq pr$: the crucial ingredient is (5.3) which is formulated in [3] for $3d$, but the proof of (5.3) shows that the same result holds for $3pr$ if one increases the bound $e_0 = e_0(p)$. 

\[\square\]
Appendix A. Examples

We exhibit a few examples of skeletons which have been determined using the ANUPQ package [10] for GAP [11]. We use a compact form to draw trees: Each vertex is labeled with the Galois order of the corresponding group, and a number \( m \) on the right of a vertex indicates that this vertex and its full descendant tree appears \( m \) times in the graph.

**Figure 2.** A Galois tree in \( S_{11}(20) \) (left) and a Galois tree in \( S_7(11) \) (right)

**Figure 3.** Galois trees with Galois orders 3 and 6 in \( S_7(24) \)
### Appendix B. Notation

| Symbol | Definition |
|--------|------------|
| $p$    | a prime $\geq 7$ |
| $d$    | $p-1$ |
| $c$    | $2p-8$ |
| $\ell$ | $(p-3)/2$ |
| $h, k$ | fixed positive integers with $hk = d$ |
| $\text{gal}(G)$ | the Galois order of $G$ |
| $\mathcal{G}_p$ | the group associated with the finite $p$-groups of maximal class |
| $T_p$ | the maximal infinite tree in $\mathcal{G}_p$ |
| $S_p(n)$ | the group of order $p^n$ on the unique maximal infinite path in $T_p$ |
| $B_p(n)$ | the $n$-th branch of $T_p$ |
| $B_p(n, m)$ | the subtree of $B_p(n)$ consisting of all groups of depth at most $m$ |
| $\mathcal{D}(G)$ | the descendant tree of $G$ in $B_p(n)$ |
| $\mathcal{D}(G, m)$ | the $m$-step descendant tree of $G$ in $B_p(n)$ |
| $S_p(n)$ | the full subtree of $B_p(n)$ consisting of all capable groups of depth at most $n-c$ |
| $\omega$ | a primitive $d$-th root of unity in $\mathbb{Z}_p$ |
| $\theta$ | a primitive $p$-th root of unity in $\mathbb{Q}_p$ |
| $\mathbb{K}$ | $\mathbb{Q}_p(\theta)$ |
| $\mathcal{O}$ | the ideal in $\mathcal{O}$ generated by $\theta - 1$ |
| $\mathbb{U}$ | the unit group of $\mathcal{O}$ |
| $\mathcal{U}_j$ | the automorphism of $\mathbb{K}$ defined by $\theta \mapsto \theta^j$ |
| $\sigma_i$ | $1 + p^i$ for $j \geq 2$ |
| $\mathcal{G}$ | the Galois group of $\mathbb{K}/\mathbb{Q}_p$ |
| $\mathcal{G}^\ast$ | the Galois group of $\mathbb{K}/\mathbb{Q}_p$ |
| $\mathcal{U}$ | the Galois group of $\mathbb{K}/\mathbb{Q}_p$ |
| $\mathcal{U}_j$ | the unit group of $\mathcal{O}$ |
| $\sigma$ | a fixed generator of $\mathcal{G}$ |
| $\tau$ | $\sigma^k$ with $k$ fixed as above |
| $\chi$ | $U \mapsto U, u \mapsto u\tau(u)^{-1}$ |
| $\Gamma_n$ | $\left( p^{n-1} \right)^{\ell-1}$ |
| $\Delta_n$ | $\Gamma_n \backslash \Gamma_{n+1}$ |
| $\Sigma_{n,i}$ | $\{ x \in \Gamma_n : (1, \tau)(x) = \omega^i x \}$ |
| $\Omega_{n,i}$ | $\Sigma_{n,i} \backslash \Sigma_{n+1,i}$ |

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