A fully nonlinear free transmission problem

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Abstract
We examine a free transmission problem driven by fully nonlinear elliptic operators. Since the transmission interface is determined endogenously, our analysis is two-fold: we study the regularity of the solutions and geometric properties of the free boundary. We prove that strong solutions are locally of class $C^{1,1}$, locally. As regards the free boundary we start by establishing weak results, such as its non-degeneracy, and proceed with the characterization of global solutions. Then, we turn our attention to the set of non-degenerate points. We find this set inherits the regularity of the solutions. That is, it is locally the graph of a $C^{1,1}$-regular function, with universal estimates.

Keywords: Free transmission problems; fully nonlinear operators; regularity of the solutions; regularity of the free boundary.

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1 Introduction
We consider a fully nonlinear transmission problem of the form

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = 1 \quad \text{in} \quad \Omega^+(u) \cup \Omega^-(u),$$

where $F_1, F_2 : S(d) \to \mathbb{R}$ are $(\lambda, \Lambda)$-elliptic operators, $\Omega^+(u) := \{u > 0\}$ and $\Omega^-(u) := \{u < 0\}$. We prove optimal regularity results for the strong solutions to (1) and examine the associated free boundary. In particular, we prove that solutions are locally of class $C^{1,1}$ and establish non-degeneracy of the free interface. The latter result unlocks the analysis of global solutions. Finally, we
focus on the set of non-degenerate points of the free boundary and show that it is, locally, the graph of a $C^{1,1}$ function.

Transmission problems comprise a class of models aimed at examining a variety of phenomena in heterogeneous media. The problems under the scope of this formulation include thermal and electromagnetic conductivity, composite materials and, more generally, diffusion processes driven by discontinuous laws.

Given a domain $\Omega \subset \mathbb{R}^d$, it gets split into mutually disjoint subregions $\Omega_i \Subset \Omega$ for $i = 1, \ldots, k$, for some $k \in \mathbb{N}$. The mechanism governing the problem is smooth within $\Omega_i$, though possibly discontinuous across $\partial \Omega_i$. A paramount, subtle, aspect of the theory concerns the nature of those subregions.

In fact, $(\Omega_i)_{i=1}^k$ and the geometry of $\partial \Omega_i$ can be prescribed a priori. The alternative is $(\Omega_i)_{i=1}^k$ to be determined endogenously. The latter setting frames the theory in the context of free boundary problems. Both cases differ substantially; as a consequence, their analysis also requires distinct techniques. The vast majority of former studies on transmission problems presupposes a priori knowledge of the subregions $\Omega_i$ and their geometric properties. A work-horse of the theory is the divergence-form equation

$$\text{div} \left( a(x) Du \right) = 0 \quad \text{in} \quad \Omega,$$

where the matrix-valued function $a(\cdot)$ is defined as

$$a(x) := a_i \quad \text{for} \quad x \in \Omega_i,$$

for constant matrices $a_i$ and $i = 1, \ldots, k$. Though smooth within every $\Omega_i$, the coefficients of (2) can be discontinuous across $\partial \Omega_i$. This feature introduces genuine difficulties in the analysis.

The first formulation of a transmission problem appeared in [30] and addressed a topic in the realm of material sciences. More precisely, in elasticity theory. In that paper, the author proves the uniqueness of solutions for a model consisting of two subregions, which are known a priori. The existence of solutions is discussed in [30], although not examined in detail. See also [31].

The formulation in [30] motivated a number of subsequent studies [6, 13, 14, 15, 25, 19, 28, 34, 35, 38]. Those papers present a wide range of developments, including the existence of solutions for the transmission problem in [30] and the analysis of several variants. We refer the reader to [7] for an account of those results and methods.
Estimates and regularity results for the solutions to transmission problems have also been treated in the literature. In [23] the authors consider a bounded subdomain $\Omega \subset \mathbb{R}^d$, which is split into a finite number of subregions $\Omega_1, \Omega_2, \ldots, \Omega_k$, known a priori. The motivation is in the study of composite materials with closely spaced inclusions. A two-dimensional example is the cross-section of a fiber-reinforced material; see Figure 1. The mathematical analysis amounts to the study of

$$\frac{\partial}{\partial x_i} \left( a(x) \frac{\partial}{\partial x_j} u \right) = f \quad \text{in} \quad \Omega, \quad (3)$$

where

$$a(x) := \begin{cases} a_i(x) & \text{for} \quad x \in \Omega_i, \quad i = 1, \ldots, k \\ a_{k+1}(x) & \text{for} \quad x \in \Omega \setminus \bigcup_{i=1}^k \Omega_i. \end{cases}$$

Under natural assumptions on the data, the authors establish local Hölder continuity for the gradient of the solutions. From the applied perspective, the gradient encodes information on the stresses of the material. Their findings imply bounds on the gradient independent of the location of the fibers. C.f. [4].

Fig. 1: The cross-section of a fiber-reinforced material provides an example in $\mathbb{R}^2$ of a bounded domain with a finite number of inclusions. The grey subregions in the cross-section represent the fibers, whereas the remainder of the material is the matrix.

The vectorial setting is the subject of [24]. In that paper the authors extend the developments reported in [23] to systems. Moreover, they produce bounds for higher derivatives of the solutions.

In [2] the authors consider a domain with two subregions, which are supposed
to be $\varepsilon$-apart, for some $\varepsilon > 0$. Within each subregion, the divergence-form equation is governed by a constant coefficient $k$. Conversely, outside those subregions the diffusivity coefficient is equal to 1. By setting $k = +\infty$, the authors frame the problem in the context of perfect conductivity.

In this setting, it is known that bounds on the gradient deteriorate as the two subregions approach each other. The analysis in [2] yields blow up rates for the gradient bounds as $\varepsilon \to 0$. The case of multiple inclusions, covering perfect conductivity and insulation ($k = 0$), is discussed in [3]. See also [8].

Recently, new developments have been obtained under minimal regularity requirements for the transmission interfaces. In [10] the authors consider a smooth and bounded domain $\Omega$ and fix $\Omega_1 \subseteq \Omega$, defining $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. They suppose the boundary of the transmission interface $\partial \Omega_1$ to be of class $C^{1,\alpha}$ and prove existence, uniqueness and $C^{1,\alpha}(\overline{\Omega}_i)$-regularity of the solutions to the problem, for $i = 1, 2$. Their argument imports regularity from flat problems, through a new stability result; see [10, Theorem 4.2].

Another class of transmission problems concerns models where the subregions of interest are determined endogenously. For example, given $\Omega \subset \mathbb{R}^d$, one would consider

$$\Omega_1 : \{x \in \Omega \mid u(x) < 0\} \quad \text{and} \quad \Omega_2 : \{x \in \Omega \mid u(x) > 0\},$$

where $u : \Omega \to \mathbb{R}$ solves a prescribed equation. Roughly speaking, knowledge of the solution is required to determine the subregions of the domain where distinct diffusion phenomena take place. In this context, a further structure arises, namely, the free interface, or free boundary. Here, in addition to the analysis of the solutions, properties of the free boundary are also of central interest.

In [1] the authors examine a transmission problem with free interface. They consider the functional

$$I(v) := \int_{\Omega} \frac{1}{2} \langle A(x, v)Dv, Dv \rangle + \Lambda(v) + fv \, dx,$$

where

$$A(x, u) := A_+(x)\chi_{\{u > 0\}} + A_-(x)\chi_{\{u \leq 0\}},$$

$$\Lambda(u) := \lambda_+(x)\chi_{\{u > 0\}} + \lambda_-(x)\chi_{\{u \leq 0\}},$$

and

$$\Omega_1 : \{x \in \Omega \mid u(x) < 0\} \quad \text{and} \quad \Omega_2 : \{x \in \Omega \mid u(x) > 0\},$$
and

\[ f := f_+ (x) \chi_{\{ u > 0 \}} + f_- (x) \chi_{\{ u \leq 0 \}} \]

with \( A_\pm \) matrix-valued mappings and \( \lambda_\pm \) and \( f_\pm \) given functions. Local minimizers for (4) satisfy

\[
\text{div} \left( A_+ (x) D u \right) = f_+ \quad \text{in} \quad \Omega_+ := \{ u > 0 \},
\]

\[
\text{div} \left( A_- (x) D u \right) = f_- \quad \text{in} \quad \Omega_- := \{ u < 0 \}^o,
\]

while Hadamard’s-type of arguments yield a flux condition across the free interface \( F(u) := \partial \Omega_+ \cap \Omega \), depending on \( \lambda_+ \) and \( \lambda_- \). The authors prove the existence of minimizers, with \( L^\infty \)-bounds. In fact the proof of existence bypasses the lack of convexity of the functional and yields estimates in \( L^\infty \) as a by-product. Those local minima are proved to have a local modulus of continuity. Under the assumption that \( A_+ \) and \( A_- \) are close, in a sense made precise in that paper, the authors prove that solutions are indeed asymptotically Lipschitz. We emphasize that improved regularity under such small-jump condition follows through a set of methods known as geometric tangential analysis. We refer the reader to the following survey papers on this class of techniques [40, 32, 41].

The problem examined in [4] profits from the existence of an associated functional and the properties derived for its minima. We remark those structures are not available in the context of (1).

In the present paper we study \( W^{2,d} \)-strong solutions to (1). Inspired by ideas firstly put forward in [17], we notice that a \( W^{2,d} \)-solution to (1) solves

\[ G(D^2 u) = g \quad \text{in} \quad B_1 \]

in the \( L^d \)-sense, where \( g \in L^\infty (B_1) \). If we suppose either \( F_1 \) or \( F_2 \) to be convex, we obtain Hessian regularity in BMO-spaces, i.e., \( u \in W^{2,\text{BMO}}_{\text{loc}} (B_1) \).

In addition, by requiring both operators to be convex and supposing they are positively homogeneous of degree one, we produce quadratic growth for the solutions. It follows from a dyadic analysis combined with the maximum principle. The argument relies on a scaling strategy, using the \( L^\infty \)-norms of the solutions as a normalization factor. This machinery was introduced in [12] in the context of an obstacle problem driven by the Laplacian. In [22] the authors took this perspective to the fully nonlinear setting and developed a fairly complete analysis of the obstacle problem governed by fully nonlinear equations. We also refer the reader to [21].
The quadratic growth results developed in [12] and [22] rely on a smallness condition on the density of the region where solutions are negative. Our argument resorts to a similar assumption. In fact, we consider the quantity
\[ V_r(x^*, u) := \frac{\text{vol}(B_r(x^*) \cap \Omega^- (u))}{r^d}; \]
by supposing \( V_r(x, u) \) is controlled for every \( x \in \partial(\Omega^+(u) \cup \Omega^-(u)) \cap B_{1/2} \), we are capable of proving quadratic growth for the solutions, away from the free boundary. A further scaling argument – depending on the square of the distance to the free boundary – is capable of relating \( B_1 \) with each connected component associated with the transmission problem. This fact extrapolates regularity information for \( u \); namely, we prove that strong solutions to (1) are of class \( C^{1,1} \) in \( B_{1/2} \). This is the content of our first main result.

**Theorem 1** (Regularity of the solutions). Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1). Suppose A1-A4, to be detailed further, hold true. Then, \( u \in C^{1,1}_{\text{loc}}(B_1) \) and there exists a universal constant \( C > 0 \) such that
\[ \|D^2u\|_{L^{\infty}(B_{1/2})} \leq C. \]

After examining the regularity of the solutions, we turn our attention to the free transmission interface. Set \( \Omega := \Omega^+(u) \cup \Omega^-(u) \). At this point, an alternative arises. In fact, we can either suppose \( \{Du \neq 0\} \subset \Omega \) or \( \{Du \neq 0\} \not\subset \Omega \). In the former case, we are capable of producing a characterization of the global solutions to (1). In the latter, we produce a regularity result for the non-degenerate portion of the free boundary.

We start our analysis by supposing \( \{Du \neq 0\} \subset \Omega \) and establishing a non-degeneracy result. Very much based on the maximum principle, it follows along the same lines put forward in [22] and [17]. The non-degeneracy property combines with Theorem 1 to control quadratically the growth of the solutions from above and from below.

A further consequence of non-degeneracy concerns global solutions to (1); it relies on a condition concerning the thickness of the free boundary. For \( A \subset \mathbb{R}^d \), denote with MD\((A)\) the smallest distance between two hyperplanes enclosing \( A \). Our second main result reads as follows.

**Theorem 2** (Characterization of global solutions). Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1) in \( \mathbb{R}^d \). Suppose A1-A4, to be detailed below, are in force.
Suppose further there exists $\varepsilon_0 > 0$ such that
\[
\frac{\text{MD} \left( (B_1 \setminus \Omega) \cap B_r(x) \right)}{r} > \varepsilon_0,
\]
for $0 < r \ll 1$ and $x \in \partial \Omega$. Then $u$ is a half-space solution. That is, up to a rotation,
\[
u(x) = \frac{\gamma[(x_1)_+]^2}{2} + C,
\]
where $C \in \mathbb{R}$ and $\gamma \in (1/\Lambda, 1/\lambda)$ is such that either $F_1(\gamma e_1 \otimes e_1) = 1$ or $F_2(\gamma e_1 \otimes e_1) = 1$.

Finally we drop the condition $\{Du \neq 0\} \subset \Omega$ and examine the regularity of the free boundary. Our analysis focuses on the non-degenerate points, see [20].

We consider the set
\[
\mathcal{N}(u) := \left\{ x \in B_1 \mid u(x) = 0 \text{ and } \limsup_{z \to x} \frac{|u(z)|}{|x - z|} > 0 \right\},
\]
and examine its geometric properties. Our second main result reads as follows.

**Theorem 3 (Regularity of the free boundary).** Let $u \in W^{2,d}(B_1)$ be a strong solution to (1). Suppose A1-A4, to be detailed further, hold true. Then,

1. The set of non-degenerate points $\mathcal{N}(u)$ is locally the graph of a $C^{1,1}$-regular function.
2. For every $z \in \mathcal{N}(u)$ there exists a positive radius $r_z \in (0, 1/2)$ such that
\[
|\nu(x) - \nu(y)| \leq C|x - y|,
\]
for every $x, y \in B_{r_z}(z) \cap \{u = 0\}$, where $C > 0$ is the universal constant from Theorem 1 and $\nu$ denotes the inward unit vector normal to $\partial \Omega^\pm(u)$.

The findings reported in Theorem 3 are related to recent developments concerning nodal sets for broken quasilinear equations; see [20]. We mention that a complete result on the regularity of the free boundary as well as the analysis of singular sets are not included in this paper; see, for instance, [22, 33].

The remainder of this paper is organized as follows: Section 2 gathers elementary results and details the main assumptions under which we work. In Section 3 we study the regularity of the strong solutions to (1) and present the proof of Theorem 1. A fourth section reports a preliminary analysis of the free
boundary, establishes the non-degeneracy property and proves Theorem 2. In Section 5 we set forth the proof of Theorem 3.

2 Elementary material and main assumptions

This section presents some preliminary material, as well as the main hypotheses we use in the paper. With \( \mathcal{S}(d) \) we denote the space of symmetric matrices of order \( d \); when convenient, we identify \( \mathcal{S}(d) \sim \mathbb{R}^{d(d+1)/2} \). We start with the uniform ellipticity of the operators \( F_i \).

\[ \text{A 1 (Uniform ellipticity and convexity). For } i = 1, 2, \text{ we suppose the operator } F_i : \mathcal{S}(d) \to \mathbb{R} \text{ to be } (\lambda, \Lambda)-\text{uniformly elliptic. That is, for } 0 < \lambda \leq \Lambda, \text{ it holds} \]

\[ \lambda \| N \| \leq F_i(M + N) - F_i(M) \leq \Lambda \| N \|, \]

\[ \text{for every } M, N \in \mathcal{S}(d), N \geq 0, \text{ and } i = 1, 2. \text{ We also suppose } F_i \text{ are convex operators satisfying } F_i(0) = 0. \]

When deriving an elliptic equation satisfied in the entire \( B_1 \) by the strong solutions to (1), we suppose the operators \( F_1 \) and \( F_2 \) are comparable. This is the content of the next assumption.

\[ \text{A 2 (Comparable diffusions). We suppose the operators } F_1 \text{ and } F_2 \text{ are comparable in the } L^\infty \text{-topology. I.e., there exists } C > 0 \text{ such that} \]

\[ \sup_{M \in \mathcal{S}(d)} |F_1(M) - F_2(M)| \leq C. \]

The former assumption is instrumental in proving that \( u \) solves an elliptic equation with right-hand side in \( L^\infty(B_1) \). We stress that A2 does not require \( F_1 \) and \( F_2 \) to be close to each other; i.e., the constant \( C > 0 \) in the assumption does not satisfy any smallness regime. C.f. [1].

The next assumption concerns homogeneity of degree 1. It plays a major role in the quadratic growth of the solutions. The argument towards quadratic growth in [12] uses the linearity of the Laplacian operator. In [22] the authors notice that in the fully nonlinear case the condition that parallels linearity is the homogeneity of degree 1.

\[ \text{A 3 (Homogeneity of degree one). We suppose } F_i \text{ to be homogeneous of degree} \]

1.
one for \( i = 1, 2 \); that is, for every \( \tau \in \mathbb{R} \) and \( M \in \mathcal{S}(d) \), we have

\[
F_i(\tau M) = \tau F_i(M),
\]

for \( i = 1, 2 \).

Before proceeding with further assumptions, we introduce some notation used throughout the paper. We denote by \( \Omega^+(u) \) the subset of the unit ball where \( u > 0 \), whereas \( \Omega^-(u) \) stands for the set where \( u < 0 \). That is,

\[
\Omega^+(u) := \{ x \in B_1 \mid u(x) > 0 \} \quad \text{and} \quad \Omega^-(u) := \{ x \in B_1 \mid u(x) < 0 \}.
\]

When referring to the set where \( u \neq 0 \) it is convenient to use the notation \( \Omega(u) := \Omega^+(u) \cup \Omega^-(u) \). With \( \partial \Omega(u) \) we denote the union of the topological boundaries of \( \Omega^+ \) and \( \Omega^- \). I.e.,

\[
\partial \Omega(u) := (\partial \Omega^+(u) \cup \partial \Omega^-(u)) \cap B_1.
\]

Also, we denote with \( \Sigma(u) \) the set where \( u \) vanishes:

\[
\Sigma(u) = \{ x \in B_1 \mid u(x) = 0 \}.
\]

Finally, we introduce the set of non-degenerate points, denoted by \( \mathcal{N}(u) \) and defined as

\[
\mathcal{N}(u) := \left\{ z \in \Sigma(u) : \limsup_{x \to z} \frac{|u(x)|}{|x - z|} > 0 \right\}.
\]

A further condition imposed on the problem regards the subregion \( \Omega^+(u) \). It is critical in proving quadratic growth of the solutions through the set of methods used in the paper. For \( x^* \in \partial \Omega \) and \( 0 < r \ll 1 \), we consider the quantity

\[
V_r(x^*, u) := \frac{\text{vol}(B_r(x^*) \cap \Omega^-(u))}{r^d}.
\]

(6)

For ease of notation, we set \( V_r(0, u) =: V_r(u) \).

A 4 (Normalized volume of \( \Omega^-(u) \)). We suppose there exists \( C_0 > 0 \), to be determined later, such that

\[
V_r(x^*, u) \leq C_0
\]

for every \( x^* \in \partial \Omega(u) \) and every \( r \in (0, 1/2) \).

The former assumption imposes a control on the size of the subregion where
$u$ is negative. It resonates on the geometry of the free boundary; see Figure 2.

Fig. 2: The geometry depicted on the left is within the scope of (11). In fact, as the radii of the balls centered at $x^*$ decrease from $r_1$ to $r_2$, $V(x^*, r)$ decreases even faster. The case on the right behaves differently. Here, the normalized volume is constant, independent of the radii of the ball; hence, it might fail to satisfy a prescribed smallness regime as in (11).

We proceed by introducing the notion of thickness. For any set $A$, we denote by $\text{MD}(A)$ the smallest possible distance between two parallel hyperplanes containing $A$. For a function $u \in W^{2,d}(\mathbb{R}^d)$, solving (11) in $\mathbb{R}^d$, we define the thickness of $\Sigma(u)$ in $B_r(x)$ as

$$
\delta_r(u, x) := \frac{\text{MD}(\Sigma(u) \cap B_r(x))}{r}. \quad (7)
$$

The thickness $\delta_r$ satisfies some properties which we list below. We refer to [29, Chapter 5] for more details. See also [17].

**Proposition 1.** Let $u \in W^{2,d}(\mathbb{R}^d)$ be a solution to (11) in $\mathbb{R}^d$. The measure of thickness $\delta_r$, introduced in (7), satisfies the following properties:

1. Let $u_r : B_1 \to \mathbb{R}$ be defined as

$$
u_r(x) := \frac{u(rx)}{r^2},$$

for $r > 0$. Then,

$$\delta_1(u_r, 0) = \delta_r(u, x);$$

2. Let $P_2 : B_1 \to \mathbb{R}$ be a polynomial global solution of the form

$$P_2(x) := \Sigma a_j x_j^2,$$
with $a_j$ such that either $F_1(D^2 P_2) = 1$ or $F_2(D^2 P_2) = 1$. Then, we have $\delta_r(P_2, 0) = 0$;

3. If $u_r$ converges to some function $u_0$ then

$$\lim_{r \to 0} \sup \delta_r(u, x_0) \leq \delta_1(u_0, 0).$$

We proceed by recalling the definitions of solution used in the paper. Let $G : B_1 \times \mathcal{S}(d) \to \mathbb{R}$ be a given operator.

**Definition 1** ($L^d$-viscosity solution). We say that $u \in \text{USC}(B_1)$ is an $L^d$-viscosity sub-solution to

$$G(D^2 u, x) = 0 \quad \text{in} \quad B_1$$

if, for every $\varphi \in W^{2,d}_\text{loc}(B_1)$ and $x_0 \in B_1$, such that $u - \varphi$ attains a local maximum at $x_0$, we have

$$\text{ess lim inf}_{x \to x_0} G(x, D^2 \varphi(x)) \leq 0.$$

Similarly, we say that $u \in \text{LSC}(B_1)$ is an $L^d$-viscosity super-solution to if, for every $\varphi \in W^{2,d}_\text{loc}(B_1)$ and $x_0 \in B_1$, such that $u - \varphi$ attains a local minimum at $x_0$, we have

$$\text{ess lim sup}_{x \to x_0} G(x, D^2 \varphi(x)) \geq 0.$$

For a comprehensive account of the theory of $L^d$-viscosity solutions, we refer the reader to [9]. We proceed with the definition of strong solution.

**Definition 2** ($W^{2,d}_\text{loc}$-strong solution). We say that $u \in W^{2,d}_\text{loc}(B_1)$ is a strong solution to

$$G(x, D^2 u) = 0 \quad \text{in} \quad B_1$$

if $u$ satisfies the equation at almost every $x \in B_1$.

We refer the reader to [18, Chapter 9] for further details on this class of solutions and their properties. We close this section with an example.

**Example 1** (Radial solutions). Consider the function $v$ given by

$$v(x) := \frac{|x|^2}{2d} - \frac{1}{8d}$$
restricted to \( B_1 \). We have \( \Omega^+(v) = \{ x \in B_1, |x| > 1/2 \} \) and \( \Omega^-(v) = \{ x \in B_1, |x| < 1/2 \} \). If we set \( F_1 = F_2 := \Delta \), then \( v \) solves \((1)\). In addition, the free transmission is given by \( \{|x| = 1/2\} \).

In the next section we examine the regularity of strong solutions to \((1)\). In particular, we present the proof of Theorem 1.

## 3 Regularity of the solutions

In this section we detail the proof of Theorem 1. We start by noticing that a \( W^{2,d} \)-strong solution to \((1)\) solves a uniformly elliptic PDE in \( B_1 \). Moreover, the source term for such equation is bounded. Then we establish quadratic growth for the solutions away from the free boundary.

**Proposition 2.** Let \( u \in W^{2,d}(B_1) \) be a strong solution to \((1)\). Suppose \( F_i \) are \((\lambda, \Lambda)\)-elliptic operators, with \( F_i(0) = 0 \). Suppose further \( A2 \) holds true. There exists a \((\lambda, \Lambda)\)-elliptic operator \( G : S(d) \to \mathbb{R} \) and a function \( g \in L^\infty(B_1) \) such that \( u \) is a strong solution to

\[
G(D^2u(x)) = g(x) \quad \text{in} \quad B_1. \tag{9}
\]

**Proof.** Because \( u \in W^{2,d}(B_1) \), we have \( D^2u(x) = 0 \) for almost every \( x \in \{ u = 0 \} \). Without loss of generality, consider \( G := F_1 \). For a.e. \( x \in \{ u > 0 \} \) we have \( G(D^2u(x)) = 1 \). In addition, the last condition in \( A1 \) yields \( G(D^2u(x)) = 0 \) for almost every \( x \in \{ u = 0 \} \). Finally, we consider \( x \in \{ u < 0 \} \); because of \( A1 \), we know that \( F_2(D^2u(x)) = 1 \) for a.e. \( -x \in \{ u < 0 \} \). It follows from \( A2 \) that

\[
|F_1(D^2u(x))| \leq C + 1 \quad \text{a.e.} -x \in \{ u < 0 \}.
\]

By defining \( g : B_1 \to \mathbb{R} \) as

\[
g(x) := \begin{cases} 
1 & \text{if } x \in \{ u > 0 \} \\
0 & \text{if } x \in \{ u = 0 \} \\
F_1(D^2u(x)) & \text{if } x \in \{ u < 0 \},
\end{cases}
\]

we have \( g \in L^\infty(B_1) \) and \( G(D^2u(x)) = g(x) \) almost everywhere in \( B_1 \).

The next result states that \( u \) is an \( L^d \)-viscosity solution to \( G = g \) in \( B_1 \); it follows from \([9, Lemma 2.5]\). For the sake of completeness, we include it here as a Proposition.
Proposition 3. Let \( u \in W^{2,d}(B_1) \) be a strong solution to (11). Suppose \( F_i \) are \((\lambda, \Lambda)\)-elliptic operators, with \( F_i(0) = 0 \) and \( F_1 \) convex. Suppose further \( A^2 \) holds true. Then, \( u \) is an \( L^d \)-viscosity solution to (9).

As mentioned in [9], Proposition 3 follows from the ellipticity of \( G \) and the maximum principle for \( W^{2,d} \)-functions, as stated in [26]; see also [5]. At this point, by requiring \( F_1 \) to be convex, our analysis produces a first regularity result concerning the solutions to (1). In fact, the convexity of \( F_1 \) turns \( u \) into a viscosity solution to a convex equation with bounded right-hand side. From [11] we infer that \( u \in W^{2,\text{BMO}}_{\text{loc}}(B_1) \), with the appropriate estimates. As yet a further consequence, we also have \( u \in C^{1,\text{Log-Lip}}_{\text{loc}}(B_1) \); for a direct proof of this fact, see [37]. An alternative argument would be to relate functions with derivatives in BMO-spaces with the Zygmund class [27] and then notice the Zygmund class is a subset of the space of Log-Lipschitz functions [42].

In the sequel we prove that solutions to (1) satisfy a quadratic growth away from the free boundary.

3.1 Quadratic growth away from the free boundary

Let \( x^* \in B_1 \) be fixed. Consider the maximal subset of \( \mathbb{N} \) whose elements \( j \) are such that
\[
\sup_{x \in B_{2^{-j-1}}(x^*)} |u(x)| \geq \frac{1}{16} \sup_{x \in B_{2^{-j}}(x^*)} |u(x)|;
\]
we denote such set by \( \mathcal{M}(x^*, u) \).

Proposition 4. Let \( u \in W^{2,d}(B_1) \) be a strong solution to (11). Suppose \( A^1-A^3 \) hold true. Let \( x^* \in \partial \Omega \). There exists \( C_0 > 0 \) such that, if
\[
V_{2^{-j}}(x^*, u) < C_0,
\]
for every \( j \in \mathcal{M}(x^*, u) \), then
\[
\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{1}{C_0} 2^{-2j}, \quad \forall j \in \mathcal{M}(x^*, u).
\]

Proof. For ease of notation, we set \( x^* = 0 \) and \( \mathcal{M}(u) := \mathcal{M}(0, u) \). We resort to a contradiction argument; suppose the statement of the proposition is false. Then, there exist sequences \( (u_n)_{n \in \mathbb{N}} \) and \( (j_n)_{n \in \mathbb{N}} \) such that \( u_n \) is a normalized strong solution to (11),
\[
V_{2^{-j_n}}(u_n) < \frac{1}{n}.
\]
with
\[ \sup_{x \in B_{2^{-j_n}}} |u_n(x)| > \frac{n}{2j_n}, \tag{13} \]
for every \( j_n \in \mathcal{M}(u_n) \), and \( n \in \mathbb{N} \). Because \( \|u_n\|_{L^\infty(B_1)} \) is uniformly bounded, it follows from (13) that \( j_n \to \infty \). In particular, we may re-write (12) as
\[ V_{\frac{1}{2j_n}}(u_n) < \frac{1}{j_n}. \tag{14} \]

Now, we introduce an auxiliary function \( v_n : B_1 \to \mathbb{R} \), given by
\[ v_n(x) := \frac{u_n(2^{-j_n}x)}{\|u_n\|_{L^\infty(B_{2^{-j_n}})}}. \]
Clearly, \( v_n(0) = 0 \). In addition, \( V_1(v_n) \to 0 \). Moreover, it follows from the definition of \( v_n \) that
\[ \sup_{B_{1/2}} |v_n(x)| = 1 \tag{15} \]
and
\[ \sup_{B_1} |v_n(x)| \leq 16. \]
We notice that \( A3 \) yields
\[ G(D^2 v_n) = \frac{2^{-2j_n}}{\|u_n\|_{L^\infty(B_{2^{-j_n}})}} G(D^2 u_n(2^{-j_n}x)). \]
Therefore,
\[ |G(D^2 v_n)| \leq \frac{1}{n} \frac{C \|u_n\|_{L^\infty(B_{2^{-j_n}})}}{\|u_n\|_{L^\infty(B_{2^{-j_n}})}} \leq \frac{C}{n} \to 0, \tag{16} \]
as \( n \to \infty \).

It follows from standard results on the Hölder continuity of the solutions to \( G = 0 \) that \( (v_n)_{n \in \mathbb{N}} \) is equibounded in \( C^{1,\alpha}_{\text{loc}}(B_1) \), for some \( \alpha \in (0, 1) \); see [39]. Therefore, there exists \( v_\infty \) such that \( v_n \to v_\infty \) in \( C^{1,\beta}_{\text{loc}}(B_1) \), for every \( 0 < \beta < \alpha \). Since \( v_n(0) = 0 \) for every \( n \in \mathbb{N} \) we infer that \( v_\infty(0) = 0 \), whereas (15) leads to \( \|v_\infty\|_{L^\infty(B_{1/2})} = 1 \). Because \( V_1(v_n) \to 0 \), we conclude that \( v_\infty \geq 0 \) in \( B_1 \).

By the same token, standard stability results for viscosity solutions build
upon (16) to ensure
\[ G(D^2 v_\infty) = 0 \quad \text{in} \quad B_1. \]

We conclude that \( v_\infty \) is a viscosity solution to a homogeneous equation which attains an interior local minimum at the origin. As a consequence of the strong maximum principle, we obtain a contradiction and complete the proof. \( \square \)

In Proposition 4 the constant \( C_0 > 0 \) informing \( A_4 \) is determined. This quantity remains unchanged throughout the paper. The next result extrapolates the former analysis from \( \mathcal{M}(x^*, u) \) to the entire set of natural numbers.

**Proposition 5.** Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1). Suppose \( A_1-A_3 \) hold true. Let \( x^* \in \partial \Omega \). Suppose further that for every \( j \in \mathbb{M}(x^*, u) \) we have
\[ V_{2^{-j}}(x^*, u) < C_0, \]
for \( C_0 > 0 \) fixed in (11). Then
\[ \sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{4}{\varepsilon} 2^{-2j}, \quad \forall j \in \mathbb{N}. \]

**Proof.** As before we set \( x^* = 0 \) and argue through a contradiction argument. Suppose the proposition is false. Let \( m \in \mathbb{N} \) be the smallest natural number such that
\[ \sup_{B_{2^{-m}}} |u(x)| > \frac{4}{\varepsilon} 2^{-2m}. \]  
(17)

We claim that \( m - 1 \in \mathcal{M}(u) \). Indeed,
\[ \sup_{B_{2^{-m}}} |u(x)| \leq \frac{4}{\varepsilon} 2^{-2(m-1)} = \frac{16}{\varepsilon} 2^{-2m} < 4 \sup_{B_{2^{-m}}} |u(x)|. \]

We conclude
\[ \sup_{B_{2^{-m}}} |u(x)| \leq \sup_{B_{2^{-m}}} |u(x)| \leq \frac{1}{\varepsilon} 2^{-2(m-1)} = \frac{4}{\varepsilon} 2^{-2m}, \]
which contradicts (17) and completes the proof. \( \square \)

Consequential to Proposition 5 is the quadratic growth of \( u \) away from the free boundary. This is the content of the next corollary.
Corollary 1 (Quadratic growth). Let $u \in W^{2,d}(B_1)$ be a strong solution to (1). Suppose $A1-A5$ hold true. Let $x^* \in \partial \Omega \cap B_{1/2}$. Suppose further that, for every $j \in \mathcal{M}(x^*, u)$, we have

$$V_{2-j}(x^*, u) < C_0,$$

for $C_0 > 0$ as in Proposition 4. Then, for $0 < r < 1/2$ there exists $C > 0$ such that

$$\sup_{x \in B_r(x^*)} |u(x)| \leq C r^2,$$

where $C = C \left( d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)} \right)$.

Proof. Find $j \in \mathbb{N}$ satisfying $2^{-j+1} \leq r < 2^{-j}$. It is straightforward to notice that

$$\sup_{B_r} |u(x)| \leq \sup_{B_{2^{-j}}} |u(x)| \leq C \left( \frac{1}{2} \right)^{j+1-1} \leq C r^2,$$

which ends the proof.

We close this section with the proof of Theorem 1; compare with [12, Theorem 1.1] and [22, Remark 1.3].

Proof of Theorem 1. Suppose $0 \in \partial \Omega$. Corollary 1 leads to

$$|u(x)| \leq C (\text{dist}(x, \partial \Omega))^2,$$

for every $x \in B_{1/2}$. Consider the auxiliary function $v : B_1 \to \mathbb{R}$ given by

$$v(y) := \frac{u(x + y \text{dist}(x, \partial \Omega))}{\text{dist}(x, \partial \Omega)^2};$$

clearly, $|D^2u(x)| = |D^2v(0)|$. Notice that

$$\{z \in B_1 \mid y \in B_1 \text{ and } z := x + y \text{dist}(x, \partial \Omega)\}$$

is contained in the same connected component to which $x$ belongs. Therefore, $F_i(D^2v) = 1$ or $F_i(D^2v) = 0$ in the unit ball. Hence, standard results in elliptic regularity theory produce

$$|D^2v(0)| \leq C,$$
for some universal constant $C > 0$, not depending on $x$, and the proof is complete.

In the next section, we turn our attention to the analysis of the free interface. We start working under the assumption $\{Du \neq 0\} \subset \Omega$ and produce a characterization of global solutions.

## 4 Classification of global solutions

In this section we examine the non-degeneracy of the free boundary. In addition we study properties of the global solutions. We start with the non-degeneracy property. This is the content of the next proposition.

**Proposition 6** (Non-degeneracy of the free boundary). Let $u \in W^{2,d}(B_1)$ be a strong solution to (1). Suppose that $A_1 - A_2$ are in force and $\{Du \neq 0\} \subset \Omega$. Let $x^* \in \partial \Omega \cap B_{1/2}$. There exists $C > 0$ such that

$$\sup_{x \in \partial B_r(x^*)} u(x) \geq Cr^2$$

for every $0 < r < 1/2$.

**Proof.** For ease of presentation, we split the proof in three steps.

**Step 1.** Without loss of generality, we take $x^* \in \Omega$. Furthermore, the set $\{Du \neq 0\}$ is dense in $\Omega$. It follows from the fact that $D^2u(x) = 0$ for almost every $x \in \{Du = 0\}$, the last condition in $A_1$ and (1). Therefore, we suppose $x \in \{Du \neq 0\} \cap \Omega$.

**Step 2.** We introduce an auxiliary function $w \in W^{2,d}(B_1)$, given by

$$w(x) := u(x) - \frac{|x - x^*|^2}{2d\lambda}.$$ 

We claim that

$$\max_{x \in \partial B_r(x^*)} w(x) = \sup_{x \in B_r(x^*)} w(x),$$

for every $0 < r < 1/2$. It follows from (18) that

$$\max_{x \in B_r(x^*)} u(x) \geq u(x^*) + Cr^2,$$
where \( C := 4d\Lambda \). By approximation, the former inequality yields the results. It remains to establish (18).

**Step 3.** Suppose (18) is false. There exists a maximum point \( y \in B_r(x^*) \) for \( w \). Hence,

\[
Dw(y) = Du(y) - \frac{|y - x^*|^2}{4d\Lambda} = 0.
\]

Were \( y = x^* \), it would be \( Du(x^*) = 0 \); we conclude that \( y \neq x^* \) and, in addition,

\[
Du(y) \neq 0.
\]

By assumption, we have \( y \in \Omega \).

On the other hand, \( w \) is a subsolution for \( G = 0 \) in \( \Omega \). In fact, for \( x \in \Omega \),

\[
G(D^2w(x)) \geq 1 - \mathcal{M}^+ \left( \frac{I_d}{2d\Lambda} \right) = \frac{1}{2}.
\]

Since \( y \in \Omega \), there exists a neighborhood of \( y \) where \( w \) is constant. We conclude \( w \) is constant in \( B_r(x^*) \) and (18) follows.

**Remark 1.** The proof of Proposition 6 follows closely the ideas put forward in [17, Lemma 3.1]. The former result has a number of standard consequences. Of particular interest is the negligibility of the free boundary in the sense of Lebesgue.

**Corollary 2** (Lebesgue negligibility of the free boundary). Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1). Suppose that A1-A2 are in force and \( \{ Du \neq 0 \} \subset \Omega \). Then \( \partial \Omega \) has Lebesgue measure zero.

Next, we establish the classification of global solutions. For completeness, we recall the definition of thickness \( \delta_r(u, x_0) \), stated in Section 2: let \( u \in W^{2,d}(B_1) \) be a strong solution to (1) and suppose \( x_0 \in \partial \Omega(u) \). Then,

\[
\delta_r(u, x_0) := \frac{\text{MD}(\Sigma(u) \cap B_r(x_0))}{r}.
\]

We proceed with a proposition on the geometry of \( u \).

**Proposition 7.** Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1) in \( \mathbb{R}^d \). Suppose A1-A4 are in force. Suppose further that there exists \( \varepsilon_0 > 0 \) such that

\[
\delta_r(u, x_0) \geq \varepsilon_0
\]

(19)
for all $r > 0$ and every $x_0 \in \partial \Omega$. Then $u$ is a convex function.

**Proof.** We argue by contradiction. Suppose that $u$ is not convex and define

$$
-m := \inf_{z \in \Omega, e \in S^{d-1}} \partial_{ee} u(z) < 0.
$$

(20)

We claim $m > 0$ is finite. In fact, because of Theorem 1, we have $u \in C^{1,1}(\mathbb{R}^d)$. Now, consider a minimizing sequence $(y_n, e_n) \subset \Omega \times S^{d-1}$ for (20); that is,

$$
\partial_{ee} u(y_n) \xrightarrow{n \to \infty} -m
$$

as $n \to \infty$. Define the rescaled function

$$
u_n(x) := \frac{u(d_n x + y_n) - u(y_n) - d_n Du(y_n) \cdot x}{d_n^2},
$$

where $d_n := \text{dist}(y_n, \partial \Omega)$. Define further

$$
\Omega_n := \frac{\Omega - y_n}{d_n} \quad \text{and} \quad \ell_n := -\frac{Du(y_n)}{d_n}.
$$

Since $Du = 0$ on $\partial \Omega$, we conclude $Du_n = \ell_n$ on $\partial \Omega_n$.

Now, observe that given $y_n \in \Omega$, we either have $y_n \in \Omega^+$ or $y_n \in \Omega^-$. It follows that $z = d_n x + y_n$ belongs either to $\Omega^+$ or $\Omega^-$. Therefore, we have

$$
F_1(D^2 u_n) = 1 \quad \text{or} \quad F_2(D^2 u_n) = 1
$$

in $\Omega_n$. In any case $u_n$ is $C^{2,\alpha}$-regular; hence, $\ell_n < C$ for every $n \in \mathbb{N}$ and some $C > 0$, universal. As a consequence, we have $\ell_n \to \ell_\infty$, through some subsequence if necessary.

Without loss of generality, suppose $e_n \to e_1$, as $n \to \infty$, through a subsequence, if required. Since $(u_j)_{n \in \mathbb{N}}$ is uniformly bounded in $C^{2,\alpha}(B_{1/2})$ there exists $u_\infty \in C^{2,\beta}(B_{1/2})$ such that $u_n \to u_\infty$ in the $C^{2,\beta}$-topology, for $0 < \beta < \alpha$. Moreover, $\partial_{11} u_\infty(0) = -m$.

Because $G$ is convex, $\partial_{11} u_\infty$ is a supersolution of the equation driven by the linear operator $G_{ij}(D^2 u_\infty) \partial_{ij}$. Let $\Omega_\infty$ be the connected component containing $B_1$. Since $\partial_{11} u_\infty(z) \geq -m$ in $B_1$, the strong maximum principle yields $\partial_{11} u_\infty \equiv -m$ in $\Omega_\infty$.

Without loss of generality we can assume $Du_\infty(x) = 0$ on $\partial \Omega_\infty$; indeed, it follows from an affine transformation of $u_\infty$. For any $e \in S^{d-1}$ we have
\( \partial_{x_1} u_\infty(z) \geq -m \) in \( B_1 \); in addition, the directional Hessian along \( e_1 \) attains \(-m\). We conclude \( e_1 \) is an eigenvector for \( D^2 u \) at every point, associated with the smallest eigenvalue. It follows that \( \partial_1 u_\infty = 0 \) along \( \partial \Omega_\infty \), for any \( j = 2, \ldots, d \).

Integrating \( u_\infty \) in the direction of \( e_1 \) we deduce

\[
u(x) = P(x) := -m \frac{x_1^2}{2} + ax_1 + b(x') \quad \text{in } \Omega_\infty,
\]

where \( x' = (x_2, \ldots, x_d) \), \( a \in \mathbb{R} \) is a fixed constant and \( b : \mathbb{R}^{d-1} \to \mathbb{R} \).

Observe that

\[
\frac{\partial}{\partial x_1} P(x) = -mx_1 + a;
\]

hence, \( \partial_1 P \) vanishes along the set \( \{x_1 = a/m\} \). On the other hand, the fact that \( Du_\infty = 0 \) on \( \partial \Omega_\infty \) yields \( \partial_1 u_\infty = \partial_1 P = 0 \) on \( \partial \Omega_\infty \). As a consequence, we infer \( \partial \Omega_\infty \subset \{x_1 = a/m\} \). At this point we distinguish two cases related to the former inclusion.

**Case 1**, \( \partial \Omega_\infty \neq \{x_1 = a/m\} \) - It follows that \( \mathbb{R}^d \setminus \{x_1 = a/m\} \subset \Omega_\infty \), and a further alternative is available, i.e.:

\[
F_1(D^2 u_\infty) = 1 \quad \text{or} \quad F_2(D^2 u_\infty) = 1
\]

almost everywhere in \( \mathbb{R}^d \). The Evans-Krylov Theorem applies to \( u_r(y) := u_\infty(ry)/r^2 \) inside \( B_1 \) to produce

\[
\sup_{x, z \in B_r} \frac{|D^2 u_\infty(x) - D^2 u_\infty(z)|}{|x - z|^\alpha} \leq \frac{C}{r^\alpha}.
\]

Letting \( r \to \infty \) we deduce that \( D^2 u_\infty \) is constant. Hence \( u_\infty(x) \) is a second order polynomial.

**Case 2**, \( \partial \Omega_\infty = \{x_1 = a/m\} \) - In this case, we have \( D_{x_1} P = 0 \) on \( \{x_1 = a/m\} \) (recall that \( Du_\infty = 0 \) on \( \partial \Omega_\infty \)). Thus, \( b \) is constant and we obtain

\[
u(x) = -m \frac{x_1^2}{2} + ax_1 + b \quad \text{in } \{x_1 > a/m\},
\]

which implies \( D^2 u_\infty \equiv -m \text{Id} \). Being negative-definite, \( D^2 u_\infty \) cannot satisfy either \( F_1(D^2 u_\infty) = 1 \) or \( F_2(D^2 u_\infty) = 1 \), which leads to a contradiction.

Therefore, if \( u \) is not convex, it has to be a second order polynomial. By combining [19] and Proposition 1 we conclude that \( u_\infty \) cannot be a second
degree polynomial, which leads us to a contradiction. 

**Corollary 3.** Let \( u \in W^{2,d}(B_1) \) be a strong solution to (1) in \( \mathbb{R}^d \). Suppose \( A_1, A_4 \) are in force and \( \{ Du \neq 0 \} \subset \Omega \). Suppose further (19) is in force. Then \( \Omega = \{ Du \neq 0 \} \).

**Proof.** Because \( u \) is convex, its set of critical points coincide with its set of minima; in addition, the set of minima of a convex function is trivially convex. Hence, \( \{ Du = 0 \} \) is convex. Since \( F_1(D^2u) = 1 \) in \( \Omega^+ \) and \( F_2(D^2u) = 1 \) in \( \Omega^- \) we have that \( |\Omega \setminus \{ Du \neq 0 \}| = 0 \). As a consequence, the convex set \( \{ Du = 0 \} \) has measure zero in \( \Omega \); if the former is nonempty, it must have co-dimension 1 and, therefore, violates the thickness condition (19). Hence, \( \Omega = \{ Du \neq 0 \} \).

In what follows, we produce the proof of Theorem 2.

**Proof of Theorem 2** For simplicity we suppose that \( 0 \in \partial \Omega \). For \( r > 0 \) define

\[
    u_r(x) := \frac{u(rx)}{r^2}
\]

and let \( u_\infty \) be the limit, up to a sequence if necessary, of \( u_r \) as \( r \to \infty \). Notice that

\[
    \Sigma(u_\infty) = \{ \Sigma(u) : tx \in \Sigma(u) \quad \forall t > 0 \}.
\]

Next, we prove that \( \Sigma(u_\infty) \) is a half-space. As before, we resort to a contradiction argument. Suppose \( \Sigma(u_\infty) \) is not a half-space; then in some system of coordinates, we have

\[
    \Sigma(u_\infty) \subset C_{\theta_0} := \{ x \in \mathbb{R}^d ; x = (\rho \cos \theta, \rho \sin \theta, x_3, \ldots, x_d), \theta_0 \leq |\theta| \leq \pi \},
\]

for some \( \theta_0 > \pi/2 \).

Choose \( \theta_1 \in (\pi/2, \theta_0) \) and set \( \alpha := \pi/\theta_1 \). Then for \( \beta > 0 \) sufficiently large, the function

\[
    v := r^{\alpha}(e^{-\beta \sin(\alpha \theta)} - e^{-\beta})
\]

is a positive subsolution for the linear operator \( G_{ij}(D^2u)\partial_{ij} \) inside \( \mathbb{R}^d \setminus C_1 \), which vanishes on \( \partial C_{\theta_1} \). From Proposition 7 we have that \( u_\infty \) is convex; thus we deduce that \( \partial_1 u_\infty > 0 \) in \( \mathbb{R}^d \setminus C_{\theta_0} \). In addition \( \theta_0 > \theta_1 \). Hence, by the comparison principle we obtain that

\[
    v \leq \partial_1 u_\infty.
\]
Therefore, $\Sigma(u_{\infty})$ is a half-space that happens to be convex. As a consequence, it follows that $\Sigma(u)$ is also a half-space.

Finally, we apply global $C^{2,\alpha}$-estimates to $u$ inside the half-ball $B_1 \setminus \Sigma(u)$; see, for instance, [36]. We obtain

$$\sup_{x, z \in B_r \setminus \Sigma(u)} \frac{|D^2 u(x) - D^2 u(z)|}{|x - z|^\alpha} \leq C \frac{r^\alpha}{r^\alpha}.$$ 

Thus, letting $r \to \infty$ we conclude that $D^2 u$ is constant and hence $u$ is a second order polynomial inside the half-space $\mathbb{R}^d \setminus \Sigma(u)$. Recall that $Du = 0$ on the hyperplane $\partial \Sigma(u)$, because we have supposed $\{Du \neq 0\} \subset \Omega$. Hence, we conclude $u$ is a half-space solution and complete the proof.

In what follows we produce information on the regularity of the free boundary dropping the condition $\{Du \neq 0\} \subset \Omega$. Our analysis focuses on the non-degenerate points $x \in N(u)$.

## 5 Regularity of the free boundary at nondegenerate points

In this section we focus on non-degenerate points along $\partial \Omega$. It is clear that an underlying condition for those points to exist is that $\{Du \neq 0\} \not\subset \Omega$. In other words, it is necessary to admit points $x^* \in \partial \Omega$ at which $|Du(x^*)| > 0$. For a point $x^* \in N(u)$ we define the normal vector at $x^*$ as the direction $\nu : N(u) \to \mathbb{S}^{n-1}$ given by

$$\nu_{x^*} := \frac{Du(x^*)}{|Du(x^*)|}.$$ 

In the sequel our arguments build upon the local $C^{1,1}$-regularity of the solutions to (1).

**Proposition 8.** Let $u \in W^{2,d}(B_1)$ be a solution to (1). Suppose $A_1 - A_4$ hold true. Suppose further that $x^* \in N(u)$ and define $\delta > 0$ as

$$\limsup_{x \to x^*} \frac{|u(x)|}{|x - x^*|} = \delta.$$ 

There exists a universal constant $C > 0$, such that

$$B_r(x^*) \cap \Sigma(u) \subset \left\{ x \in B_1 \mid |(x - x^*) \cdot \nu_{x^*}| \leq \frac{C}{\delta} |x - x^*|^2 \right\}, \quad (21)$$
for every $0 < r \ll 1$.

Proof. Notice that for $x^* \in \Sigma(u)$ we have
\[
\limsup_{x \to x^*} \frac{|u(x) - u(x^*)|}{|x - x^*|} = \delta > 0.
\]
It implies that $|Du(x^*)| \geq \delta$. From the $C^{1,1}$-regularity of $u$ we infer that, for $x \in \Sigma(u)$, is holds
\[
|Du(z) \cdot (x - x^*)| \leq C|x - x^*|^2.
\]
Therefore
\[
|\nu_{x^*} \cdot (x - x^*)| \leq \frac{C}{\delta}|x - x^*|^2.
\]

In the light of Theorem\[11\] the former proposition also follows from a straightforward application of the Implicit Function Theorem for $C^{1,1}$-functions. See \[10\] Corollary A.4.

**Proposition 9.** Let $u \in W^{2,d}(B_1)$ be a solution to \[1\]. Suppose A1-A4 hold true and let $x^* \in \mathcal{N}(u)$. Set
\[
r_{x^*} := \frac{|Du(x^*)|}{2C},
\]
where $C$ is the same constant as in Theorem\[4\]. Then
\[
|Du(x)| = \limsup_{z \to x} \frac{|u(z)|}{|z - x|} \geq \frac{|Du(x^*)|}{2}.
\]
for every $x \in B_{r_{x^*}}(x^*) \cap \Sigma(u)$.

Proof. From Theorem\[4\] we infer that
\[
|Du(x^*)| - C|x^* - x| \leq |Du(x)|,
\]
where $C > 0$ is the upper bound for the Hessian of $u$ in $L^\infty(B_1)$, produced in Theorem\[11\] By imposing $x \in B_{r_{x^*}}(x^*)$, the result follows.

We notice Proposition\[9\] has spillovers on the definition of the mapping $\nu$. In fact, if $x^* \in \mathcal{N}(u)$, we have $|Du| > 0$ in $B_{r_{x^*}}(x^*)$; as a consequence, $\nu$ is well-defined inside this entire neighborhood.
**Proof of Theorem**

From Proposition we have that \( N(u) \) is locally the graph of a \( C^{1,1} \)-regular function. Suppose in the sequel that \( 0 \in N(u) \) and take \( r_0 > 0 \) as in Proposition. For \( x, y \in B_{r_0} \) we compute

\[
\frac{Du(x)}{|Du(x)|} - \frac{Du(y)}{|Du(y)|} = \frac{|Du(y)|(Du(x) - Du(y)) + (|Du(y)| - |Du(x)|)Du(y)}{|Du(x)||Du(y)|}
\]

Because \( x \) and \( y \) are in \( B_{r_0} \), the mapping \( \nu \) is well-defined at those points. As a consequence,

\[
|\nu(x) - \nu(y)| \leq C \frac{|x - y|}{|Du(0)|^2}
\]

where \( C > 0 \) is the constant in Theorem.

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