Relativistic stars in vector-tensor theories

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We study relativistic star solutions in second-order generalized Proca theories characterized by a $U(1)$-breaking vector field with derivative couplings. In the models with cubic and quartic derivative coupling, the mass and radius of stars become larger than those in general relativity for negative derivative coupling constants. This phenomenon is mostly attributed to the increase of star radius induced by a slower decrease of the matter pressure compared to general relativity. There is a tendency that the relativistic star with a smaller mass is not gravitationally bound for a low central density and hence dynamically unstable, but that with a larger mass is gravitationally bound. On the other hand, we show that the intrinsic vector-mode couplings give rise to general relativistic solutions with a trivial field profile, so the mass and radius are not modified from those in general relativity.

I. INTRODUCTION

The increasing evidence of dark sectors in the Universe\textsuperscript{1,2} implies that there may be some extra propagating degrees of freedom (DOFs) beyond the realm of General Relativity (GR). The new DOFs arising in modified gravitational theories can be potentially harmful as they generally mediate fifth forces with ordinary matter. In the local Universe with a weak gravitational field, however, there are several screening mechanisms of fifth forces known in the literature—such as Vainshtein\textsuperscript{3} and chameleon\textsuperscript{4} mechanisms. This screening property does not necessarily persist in the regime of strong gravity, reflecting the fact that the behavior of new DOFs can be modified by large nonlinearities in the field equations of motion. The direct detections of gravitational waves by Advanced LIGO and Virgo\textsuperscript{5,6} have opened up a new window for testing GR in strong gravity regimes.

Besides black holes (BHs), relativistic stars are also important compact objects which allow one to search possible deviation from GR in strong gravity regimes\textsuperscript{7,8}. Especially, neutron stars (NSs) are the representative relativistic stars. Inside a NS, the gravitational force balances the degeneracy pressure of fermions\textsuperscript{9}. The properties of NSs, including the mass and radius, depend on the equation of state (EOS) of strong interacting matter, i.e., the relation between the matter pressure and density\textsuperscript{10,11}. The microscopic determination of the EOS of NSs from underlying nuclear interactions in an extremely high-density regime remains a challenging theoretical problem.

In modified gravitational theories, the existence of extra propagating DOFs can also influence the properties of relativistic stars. In scalar-tensor theories where a scalar field $\phi$ has a direct coupling with the Ricci scalar $R$, the Einstein-frame metric $g_{\mu\nu}$ felt by the matter sector is different from the Jordan-frame metric $\tilde{g}_{\mu\nu}$. The relation of these two metrics can be parametrized by the form $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$, where $A(\phi)$ is a function of $\phi$\textsuperscript{12}. Inside a star, the conformal coupling to matter can trigger a tachyonic instability of the scalar field, and spontaneously scalarizes the relativistic star. Damour and Esposito-Farèse\textsuperscript{13} showed that such a scalarization, which occurs for the coupling $e.g.$, $A(\phi) = \exp(\beta\phi^2/2)$ (where $\beta$ is a constant), significantly modifies the properties of relativistic stars with respect to GR. Such a nontrivial excitation of the scalar field is a consequence of the absence of a no-hair theorem for stars. The scalarization can occur only for $\beta \lesssim -4.35$\textsuperscript{14,15}, whereas binary-pulsar observations\textsuperscript{17} have set stringent bounds on $\beta$, as $\beta \gtrsim -4.5$. For NSs, the existence of EOS-independent relations\textsuperscript{18} will be important to resolve the degeneracies between the effects associated modified gravity and uncertainties in EOSs, and test modified gravitational theories with future observations of NSs.

In shift-symmetric Horndeski (and beyond Horndeski) theories\textsuperscript{19,20} with a minimally coupled matter component, the no-hair theorem for relativistic stars was argued in Ref.\textsuperscript{21}. The theorem holds under the assumptions same as those used for proving the no-hair theorem of BHs in shift-symmetric Horndeski theories\textsuperscript{22}, with the regularity of metric functions and the scalar field at the center of stars. Thus, as in the case of hairy BH solutions\textsuperscript{23,31}, nontrivial NS configurations have been studied by violating at least one of those assumptions. For example, there exist relativistic star solutions for a linearly time-dependent scalar field $\phi = qt + \psi(r)$\textsuperscript{32,33}. Relativistic stars for other modified gravitational theories have been extensively studied in Refs.\textsuperscript{34,35}. In this paper, we will study relativistic star solutions in generalized Proca theories described by a $U(1)$-breaking vector field with derivative couplings. We show that the star configuration with nontrivial influence of the extra DOFs can be constructed more easily compared to scalar-tensor theories.
The action of generalized Proca theories with second-order equations of motion was first constructed in Refs. [39, 40] from the demand of keeping three propagating DOFs besides two tensor polarizations. The theories were further extended [41] to include intrinsic vector-mode couplings with the double dual Riemann tensor \( L^{\mu \nu \alpha \beta} \), such that the \( U(1) \)-invariant interactions derived by Horndeski [43] can be accommodated as a specific case. It is also possible to go beyond the second-order domain by keeping the five propagating DOFs [44, 45]. In such (beyond) generalized Proca theories, the derivative interactions can drive the late-time cosmic acceleration [46] with some distinct observational signatures [47, 48], while satisfying local gravity constraints in Solar System [49, 50].

In the Einstein-Maxwell theory with a massless vector field, the unique static and spherically symmetric BH solution corresponds to the Reissner-Nordström (RN) metric with mass and electric charge. In the Einstein-Proca theory with a massive vector field described by the Lagrangian \(-m^2 A^\mu A_\mu/2\), Bekenstein showed that only the static and spherically symmetric BH solution is given by the Schwarzschild metric without the vector hair [51]. This no-hair theorem cannot be applied to vector-tensor theories with derivative self-interactions and nonminimal couplings to the spacetime curvature. Indeed, it is known that there are a bunch of hairy BH solutions in generalized Proca theories [52–63]. In theories with a nonminimal coupling to the Einstein-tensor, \( \beta_4 G^{\mu \nu} A_\mu A_\nu \), Chagoya et al. [55] derived an exact static and spherically symmetric BH solution for the specific coupling \( \beta_4 = 1/4 \). This exact BH solution was further extended to asymptotically non-flat solutions [57, 58], non-exact solutions for \( \beta_4 \neq 1/4 \) [54, 58], and slowly-rotating solutions [59]. There are also exact BH solutions in a subclass of generalized Proca theories with new internal symmetries [61, 62].

In Refs. [59, 60], analytic and numerical BH solutions have been systematically constructed for a wide class of generalized Proca theories. The power-law coupling models, which include the case of vector Galileons, can give rise to a variety of hairy BH solutions. The cubic and quartic couplings provide BH solutions with a primary Proca hair, whereas the sixth-order and intrinsic vector-mode couplings lead to BH solutions with a secondary Proca hair. On the other hand, there are no regular BHs for quintic power-law couplings due to the divergence of the longitudinal mode at a finite radius.

While both BHs and stars are compact objects with strong gravitational forces, the internal structures of them are different. For static and spherically symmetric BHs the metric and curvature generally exhibit the divergence at the center of spherical symmetry, but this is not the case for stars. Moreover, the configuration of stars is affected by different choices of the EOS through the change of the matter pressure. In this paper, we will study how the presence of derivative couplings in generalized Proca theories affects the mass and radius of relativistic stars. In Ref. [57], the authors studied NS solutions in a subclass of generalized Proca theories with the Lagrangian \( \beta_4 G^{\mu \nu} A_\mu A_\nu \). We extend the analysis to more general cubic and quartic power-law derivative couplings and elucidate general properties of their effects on the mass and radius of relativistic stars.

For our purpose of investigating the effects of cubic and quartic derivative couplings on the mass and radius of relativistic stars in comparison with GR, we will restrict our numerical analysis to the case of the simplest polytropic EOS with two constant parameters [64]. We derive analytic solutions deep inside the star by imposing regular boundary conditions at the origin. The validity of analytic solutions will be confirmed by numerical integrations across the surface of star for the polytropic EOS. We will also study the effects of sixth-order and intrinsic vector-mode couplings on the configuration of relativistic stars. However, we will not consider quintic derivative couplings because of the absence of regular BHs [57, 60] as well as pathological behavior in the regime of weak gravity [50]. The essential qualitative features of relativistic stars in generalized Proca theories are not sensitive to the choice of EOSs.

We organize our paper as follows. In Sec. [II] we derive a set of equations in generalized Proca theories with matter on the static and spherically symmetric background, and briefly review relativistic stars in GR and the polytrope EOS. In Secs. [III] and [IV] we study how the mass and radius are modified by the presence of cubic and quartic power-law couplings, respectively. In Sec. [V] we show that sixth-order and intrinsic vector-mode couplings lead to the relativistic star solutions identical to those in GR with a trivial vector field. We conclude in Sec. [VI].

We work in the CGS units, where the speed of light, the reduced Planck constant, the gravitational constant, and the neutron mass are given by \( c = 2.9989 \times 10^{10} \text{ cm} \text{s}^{-1} \), \( \hbar = 1.0546 \times 10^{-27} \text{ erg} \text{s} \), \( G = 6.6741 \times 10^{-8} \text{ g}^{-1} \text{cm}^3 \text{s}^{-2} \), and \( m_n = 1.6749 \times 10^{-24} \text{ g} \), respectively.

II. GENERALIZED PROCA THEORIES AND RELATIVISTIC STARS

A. Equations of motion on the static and spherically symmetric background

The action of generalized Proca theories with a vector field \( A_\mu \) is given by [39, 42]

\[
S = \int d^4 x \sqrt{-g} \left[ F + \sum_{i=2}^{6} \mathcal{L}_i + \mathcal{L}_m \right],
\]  

(2.1)
where $g$ is a determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{L}_m$ is a matter Lagrangian, and

\begin{align*}
\mathcal{L}_2 &= G_2(X,F,Y), \\
\mathcal{L}_3 &= G_3(X)\nabla_\mu A^\mu, \\
\mathcal{L}_4 &= G_4(X)R + G_{4,X}(X) \left[(\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu\right], \\
\mathcal{L}_5 &= G_5(X)[G_{\mu\nu} \nabla_\mu A^\nu - \frac{1}{6}G_5,X(X) \left[(\nabla_\mu A^\mu)^3 - 3\nabla_\mu A^\nu \nabla_\rho A_\sigma \nabla^\sigma A^\rho + 2\nabla_\rho A_\sigma \nabla^\rho A^\nu \nabla^\sigma A_\nu\right] \\
&\quad - g_5(X)\tilde{F}^{\alpha\beta} \tilde{F}_\alpha A_\beta], \\
\mathcal{L}_6 &= G_6(X)L^{\alpha\beta\gamma\delta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2}G_{6,X}(X)\tilde{F}^{\alpha\beta} \tilde{F}_\mu A_\nu \nabla_\alpha A_\beta \nabla_\gamma A_\nu, \\
\end{align*}

with

\begin{align*}
F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu, \\
F &= -\frac{1}{4}F_{\mu\nu} F^{\mu\nu}, \\
X &= -\frac{1}{2}A_\mu A^\mu, \\
Y &= A^\mu A^\nu F_{\mu\alpha} F_{\nu\alpha}. \\
\end{align*}

Here, $\nabla_\mu$, $R$, and $G_{\mu\nu}$ represent the covariant derivative, the Ricci scalar, and the Einstein tensor associated with the four-dimensional metric $g_{\mu\nu}$, respectively. While the function $G_2$ is generally dependent on $X,F,Y$, the functions $G_{3,4,5,6}$ and $g_5$ depend on $X$ alone with the notation of partial derivatives $G_{i,X} \equiv \partial G_i/\partial X$. The dual strength tensor $\tilde{F}^{\mu\nu}$ and the double dual Riemann tensor $L^{\mu\alpha\beta\gamma\delta}$ are defined, respectively, by

\begin{align*}
\tilde{F}^{\mu\nu} &= \frac{1}{2}E^{\mu\nu\alpha\beta} F_{\alpha\beta}, \\
L^{\mu\alpha\beta\gamma\delta} &= \frac{1}{4}E^{\mu\nu\rho\sigma} E^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}, \\
\end{align*}

where $E^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor satisfying the normalization $E^{\mu\nu\alpha\beta} E_{\mu\nu\alpha\beta} = -4!$, and $R_{\rho\sigma\gamma\delta}$ is the Riemann tensor. The Lagrangians containing the functions $g_5(X)$ and $G_{6}(X)$ correspond to intrinsic vector-modes.

We consider a static and spherically symmetric background characterized by the line element

\begin{equation}
ds^2 = -f(r)c^2dt^2 + h^{-1}(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\end{equation}

where $f$ and $h$ are functions of the distance $r$ from the center of symmetry. On this background, the vector field can be expressed in the form

\begin{equation}
A_\mu = (cA_0(r), A_1(r), 0, 0),
\end{equation}

where $A_1(r)$ is the $r$-derivative of a longitudinal scalar $\chi$, such that $A_1(r) = d\chi/dr = \chi'(r)$. The transverse mode $A^{(T)}_1$ in the spatial components $A_i$ needs to vanish due to the regularity at the origin. On the static and spherically symmetric background (2.9) with the vector components (2.10) there is the relation $Y = 4FX$, so the additional dependence of $Y$ in Eq. (2.2) can be removed.

We assume that the matter sector is described by a perfect fluid minimally coupled to gravity. Defining the matter energy-momentum tensor

\begin{equation}
T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}},
\end{equation}

the mixed tensor $T^{\mu}_{\nu}$ is expressed in the form

\begin{equation}
T^{\mu}_{\nu} = \text{diag} \left(-\rho c^2, P, P, P \right),
\end{equation}

where $\rho$ is the total mass density and $P$ is the pressure.

Varying the action (2.1) with respect to $f, h, A_0, A_1$, respectively, we obtain

\begin{align*}
\left(c_1 + \frac{c_2}{r} + \frac{c_3}{r^2}\right) h' + c_4 + \frac{c_5}{r} + \frac{c_6}{r^2} &= \frac{\rho}{c^2}, \\
-\frac{h}{f} \left(c_1 + \frac{c_2}{r} + \frac{c_3}{r^2}\right) f' + c_7 + \frac{c_8}{r} + \frac{c_9}{r^2} &= \frac{P}{c^2},
\end{align*}

where $c_i$ are constants.
where \( c_1, c_2, \ldots, c_9 \) are given in Appendix A and
\[
rf [2fhA_0'' + 2A_0' + r(fh' - f'h)A_0'] (1 + 2G/r) + 2rfA_0'' [2fhA_0'' - (f'h - f'h')A_0'] G_{2,FF} - 2rf^2 A_0 G_{2,XX} \\
- 2rfA_0' (fh^2 A_1' - hA_0 A_0' + f'hX_0 - f'hX_1) G_{2,XX} - rf A_0 [2rf hA_1' + (r'f' + rfh') + 4fhA_1] G_{3,XX} \\
+ 4fhA_0(rh' + h - 1) G_{4,XX} - 8fA_0 [rfh^2 A_1' - (r'fh + rf h') + 4fhA_1] \frac{G}{2} G_{2,XX} \\
- fA_0 [f(3h - 1)h' A_1 + h(h - 1)] (f A_1' + 2f A_1') G_{5,XX} - 2rfhA_0 X [2fh A_1' + (f'h + h') A_1] G_{5,XX} \\
- 2f [(f(3h - 1)h') A_1' + h(h - 1)] (2fA_0'' - f' A_0') G_{6} - 4fA_0 X [hA_0 A_0' - 2fh^2 A_1' - 2f'h X_0 + 2f'h X_1] G_{6,XX} \\
- 2f [4fh^2 X_1 A_0'' - 2h(h - X_0) f A_0'' + 2f(6h - 1)h' X_1 A_0' + h(1 - 1) A_0 A_0'' - 2fh^2 (3h - 1) A_0 A_1' ] G_{6,XX} \\
- 4fh [2rf h A_1 A_0'' - (r'fh - 3rfh - 2f'h) A_1 - 2rfh A_1' A_0'] g_5 \\
- 4rf A_0' [2fh A_0' A_1' + 4fh X_1 A_1'' - 2A_1 (f'h X_0 - f'h X_1)] g_{5,XX} = 0 ,
\]
\[
A_1 \left[ r^2 f G_{2,XX} - 2r f'h + f h - f \right] G_{4,XX} + 4h(r A_0 A_0' - r fX - f X_1) G_{4,XX} - h A_0'' G_{6,XX} - 2h^2 X_1 A_0'' G_{6,XX} \\
= r [r fX - A_0 A_0' + 4f X_1] G_{3,XX} + 2f'h X_1 G_{5,XX} + (A_0 A_0' - f'X) \left[ (1 - h) G_{5,XX} - 2hX_1 G_{5,XX} \right] \\
- 2rh A_0'' (g_5 + 2X_1 g_5) .
\]

The quantity \( X \) is given by \( X = X_0 + X_1 \), where
\[
X_0 = \frac{A_0^2}{2f} , \\
X_1 = - \frac{h A_0^2}{2} .
\]

From the matter continuity equation, it follows that
\[
P' + \frac{P'}{2f} \left( \rho c^2 + P \right) = 0 .
\]

For a given EOS
\[
P = P(\rho) ,
\]
Eqs. (2.13)–(2.16) with Eq. (2.18) form a closed set of equations to determine \( f, h, A_0, A_1, \rho, \) and \( P \) as functions of \( r \).

### B. Relativistic stars in GR

Here, we briefly review relativistic stars in GR without the vector field \( A_\mu \). This corresponds to the functions
\[
G_4 = \frac{1}{16\pi G} , \\
G_2 = G_3 = G_5 = G_6 = 0 , \\
g_5 = 0 .
\]

In this case, Eqs. (2.13) and (2.14) reduce, respectively, to
\[
\frac{h'}{r} + \frac{h - 1}{r^2} = - \frac{8\pi G \rho}{c^2} ,
\]
\[
\frac{h f'}{f} + \frac{h - 1}{r^2} = \frac{8\pi G P}{c^4} .
\]

Introducing the mass function \( M(r) \), as
\[
h(r) = 1 - \frac{2GM(r)}{c^2 r} ,
\]
we can express Eq. (2.21) in the simple form
\[
M'(r) = 4\pi \rho r^2 .
\]

On using Eqs. (2.22) and (2.23), the continuity equation (2.18) reduces to the Tolman-Oppenheimer-Volkoff (TOV) equation
\[
P'(r) = - \frac{G(\rho + P/c^2)(M + 4\pi r^3 P/c^2)}{r^2 [1 - 2GM/(c^2 r)]} .
\]
Around the center of star, we expand \( f, h, \rho, \) and \( P \) in the following forms
\[
\begin{align*}
    f(r) &= 1 + \sum_{i=2}^{\infty} f_i r^i, \\
    h(r) &= 1 + \sum_{i=2}^{\infty} h_i r^i, \\
    \rho(r) &= \rho_c + \sum_{i=2}^{\infty} \rho_i r^i, \\
    P(r) &= p_c + \sum_{i=2}^{\infty} p_i r^i, 
\end{align*}
\]  
where \( f_i, h_i, \rho_i, p_i \) are constants. Then, the regularity conditions \( f'(0) = h'(0) = \rho'(0) = P'(0) = 0 \) are satisfied with \( \rho(r) \) and \( P(r) \) converging to constant values \( \rho_c \) and \( p_c \), respectively, as \( r \to 0 \). By solving Eqs. (2.21), (2.22), and (2.28) iteratively, the boundary conditions around \( r = 0 \) can be found as
\[
\begin{align*}
    f(r) &= 1 + \frac{4\pi G(c^2 \rho_c + 3p_c)}{3c^3} r^2 + O(r^4), \\
    h(r) &= 1 - \frac{8\pi Gp_c}{3c^2} r^2 + O(r^4), \\
    P(r) &= p_c - \frac{2\pi G(c^2 \rho_c + 3p_c)(c^2 \rho_c + p_c)}{3c^3} r^2 + O(r^4). 
\end{align*}
\]  
The numerical integration is performed until reaching the surface of star \( r = R_* \), where \( P(R_*) = 0 \). By requiring the continuity of metric functions and their first-order derivatives across the surface \( r = R_* \), the internal solution is smoothly joined to the exterior Schwarzschild solution given by the metric (2.9) with
\[
f = h = 1 - \frac{2GM_*}{c^2 r},
\]  
where the Arnowitt-Deser-Misner (ADM) mass is given by \( M_* = M(R_*) \). Provided that the EOS (2.19) inside the star is known, it is practically more convenient to integrate Eqs. (2.22), (2.24), and (2.25) to determine \( \rho(r), P(r) \) and \( M(r) \). In Secs. III-V, the mass and radius of relativistic stars in generalized Proca theories will be compared to those in GR.

C. The polytrope equation of state

As we will see later, the qualitative results of relativistic stars in generalized Proca theories do not depend on the choice of EOSs. Thus, in this paper, we focus on one of the simplest EOSs known as the polytropic EOS which is given by
\[
P = K\rho^\Gamma,
\]  
where \( \rho_0 \) is the rest-mass density, and \( K, \Gamma \) are constants. In general, the total energy density \( \rho c^2 \) is expressed in the form \( \rho c^2 = \rho_0 c^2 (1 + \epsilon) \), where \( \epsilon \) is the dimensionless internal energy density per unit mass. For baryons with number density \( n_b \) and the mean rest mass \( m_b \), the rest-mass density is given by \( \rho_0 = n_b m_b \). On using the first law of thermodynamics for the adiabatic process, the baryon pressure is expressed as \( P = n_b^2 m_b c^2 \partial \epsilon / \partial n_b \) [64]. For the polytropic EOS (2.31), i.e., \( P = K(n_b m_b)^\Gamma \), we obtain the integrated solution \( \epsilon = K\rho_0^{\Gamma-1}/[c^2(\Gamma - 1)] \), so the total mass density yields \( \rho = \rho_0 + K\rho_0^{\Gamma}/[c^2(\Gamma - 1)] \). We define the dimensionless rest-mass density \( \chi \) and the rescaled polytropic gas constant \( K \), as
\[
\chi \equiv \frac{\rho_0}{\rho} = \frac{n_b}{n_0}, \quad K \equiv \frac{K}{\rho_0^{\Gamma-1} c^2},
\]  
with
\[
\tilde{\rho}_0 = n_0 m_b,
\]  
where \( n_0 = 0.1 \) (fm)\(^{-3} \) is the typical nuclear number density of relativistic stars. As a result, the polytropic EOS can be expressed in the form [13]
\[
\rho = \tilde{\rho}_0 \left( \chi + \frac{K}{\Gamma - 1} \chi^\Gamma \right), \quad P = K\rho_0 c^2 \chi^\Gamma,
\]  
with
\[
w \equiv \frac{P}{\rho c^2} = \frac{K\chi^{\Gamma-1}}{1 + K \chi^{\Gamma-1}/(\Gamma - 1)}.
\]
In the nonrelativistic regime characterized by \( K \chi^{\Gamma - \frac{1}{2}} \ll 1 \), we have \( w \simeq K \chi^{\Gamma - \frac{1}{2}} \), so \( w \) grows with the increase of mass density \( \rho \). In the relativistic regime, \( w \) approaches a constant value \( \Gamma - 1 \) for increasing \( \rho \).

For the numerical propose, it is convenient to introduce the dimensionless quantities:

\[
x = \frac{r}{r_0}, \quad y = \frac{\rho}{\rho_0}, \quad w_0 = \frac{P}{\rho_0 c^2}, \quad m(r) = \frac{3M(r)}{4\pi\rho_0^2 r_0^3},
\]

(2.36)

where

\[
r_0 = \sqrt{\frac{\frac{c^2}{G \rho_0}}{\Gamma}}.
\]

(2.37)

In the following, we identify \( m_b \) with the neutron mass \( m_n = 1.6749 \times 10^{-24} \text{ g} \). Then, the distance (2.37) corresponds to \( r_0 = 89.696 \text{ km} \) with \( \rho_0 = 1.6749 \times 10^{14} \text{ g cm}^{-3} \). The polytropic EOS (2.34) can be expressed in the form

\[
w_0(x) = K \chi(x)^\Gamma, \quad y(x) = \left( \frac{w_0(x)}{K} \right)^{1/\Gamma} + \frac{w_0(x)}{\Gamma - 1}.
\]

(2.38)

Specifying the value of \( w_0(0) \), the associated dimensionless density \( y_c = \rho_c/\rho_0 \) is also fixed at the center of star. The star radius \( R_* \) is defined by

\[
w_0(x_*) = 0,
\]

(2.39)

where \( x_* = R_*/r_0 \). By choosing different boundary conditions of \( w_0 \) at \( x = 0 \), we obtain the configuration of relativistic stars with different mass \( M_* \) and radius \( R_* \). In terms of the solar mass \( M_\odot = 1.9884 \times 10^{33} \text{ g} \), we can express the ADM mass \( M_* \) in the form

\[
M_* = 2.5462 \times 10^2 m(x_*) M_\odot.
\]

(2.40)

For the comparison with observational data of NSs, however, we would need phenomenologically parametrized EOSs specifying the stiffness of the star in several density intervals [65]. In this paper, we will not perform the comparison with observational data of NSs, but we focus on how vector-field derivative couplings modify the mass-radius relation of relativistic stars from GR by considering the polytropic EOS (2.34) with two constant parameters \( \Gamma \) and \( K \). As we will see below, the qualitative behavior of vector-field derivative couplings on the mass and radius of relativistic stars, which can be analytically understood in some degree, is generally insensitive to the choice of EOSs. For numerics, we choose the index \( \Gamma = 2.34 \) in Secs. [III] and [IV].

III. CUBIC COUPLINGS

Let us begin with the cubic derivative interaction \( G_3(X) \). For concreteness, we study the power-law coupling given by

\[
G_3 = \beta_3 X^n,
\]

(3.1)

where \( \beta_3 \) is a constant and \( n \) is a positive integer. We also take into account the Einstein-Hilbert term \( G_4 = 1/(16\pi G) \) in the action (2.1) with \( G_2 = G_5 = G_6 = 0 \) and \( g_5 = 0 \). We consider the models with positive integer \( n \), which includes the vector Galileon as a special case \( (n = 1) \). From Eq. (2.10), the longitudinal component is related to \( A_0, f, h \) and their derivatives as

\[
A_1 = \epsilon \sqrt{\frac{r A_0 (f' A_0 - 2 f A_0')}{f h (r f' + 4 f)}},
\]

(3.2)

where \( \epsilon = \pm 1 \).

A. Analytic solutions around the center of star

We first derive analytic solutions to the metrics, the vector field, and the pressure around \( r = 0 \). We take the positive branch of Eq. (3.2) and differentiate it with respect to \( r \). Then, \( A_1 \) and \( A'_1 \) are substituted into Eqs. (2.13) - (2.15) to eliminate the dependence of the longitudinal mode.
Around the center of star, we expand $f, h, \rho, P$ in the forms (2.26). The temporal vector component is also expanded as

$$A_0 = a_0 + \sum_{i=2}^{\infty} a_i r^i, \quad \text{(3.3)}$$

where $a_0$ and $a_i$ are constants. These solutions satisfy the regular boundary conditions $f'(0) = h'(0) = \rho'(0) = P'(0) = 0$ and $A_0'(0) = 0$. Without loss of generality, we will assume that $a_0 > 0$. We also require the condition $P''(0) < 0$ for the pressure [66]. Expanding the continuity equation (2.18) around the origin, we obtain

$$p_2 = -\frac{\rho c^2}{2} + \frac{p_c}{2} f_2. \quad \text{(3.4)}$$

The condition $P''(0) < 0$, which corresponds to $p_2 < 0$, is satisfied for

$$f_2 > 0. \quad \text{(3.5)}$$

From Eq. (3.2), the leading-order solution of the longitudinal mode around the center of star is given by

$$A_1 = \sqrt{\frac{a_0(a_0 f_2 - 2a_2)}{2}} r, \quad \text{(3.6)}$$

which ensures the regularity of $A_1$ at $r = 0$. For the existence of this solution, we require that

$$a_0(a_0 f_2 - 2a_2) > 0. \quad \text{(3.7)}$$

Substituting Eq. (3.3) into Eqs. (2.18) and (2.19) and solving them iteratively, we obtain the following solutions around the origin:

$$f(r) = 1 + \frac{4\pi}{3} (1 + 3w_c + \mathcal{F}) \frac{r^2}{r_c^2} + \mathcal{O}(r^4), \quad \text{(3.8)}$$

$$h(r) = 1 - \frac{8\pi}{3} (1 + \mathcal{F}) \frac{r^2}{r_c^2} + \mathcal{O}(r^4), \quad \text{(3.9)}$$

$$A_0(r) = \frac{a_0}{\sqrt{8\pi G}} \left(1 + \frac{4\pi}{3} \mathcal{F} \frac{r^2}{r_c^2} \right) + \mathcal{O}(r^4), \quad \text{(3.10)}$$

$$P(r) = p_c - \frac{2\pi}{3} (\rho c^2 + p_c) (1 + 3w_c + \mathcal{F}) \frac{r^2}{r_c^2} + \mathcal{O}(r^4), \quad \text{(3.11)}$$

where

$$\mathcal{F} \equiv \frac{3\pi^2 a_0^{2n+1} \beta_3}{2^{2n+3} \pi} \left[-\beta_3 a_0^{2n-1} \left(1 - \frac{\tilde{a}_0^2}{2} \right) \pm \sqrt{\beta_3^{2} \tilde{a}_0^{4n-2} \left(1 - \frac{\tilde{a}_0^2}{2} \right)^2 + \frac{2^{2n+3} \pi}{3n^2} (1 + 3w_c) \right], \quad \text{(3.12)}$$

with the dimensionless constants defined by

$$\beta_3 \equiv \frac{\beta_3 c_r}{(\sqrt{8\pi G})^{2n-1}}, \quad \tilde{a}_0 \equiv \sqrt{8\pi G} a_0, \quad c_r \equiv \sqrt{\frac{c^2}{G\rho_c}}, \quad w_c \equiv \frac{p_c}{\rho_c c^2}. \quad \text{(3.13)}$$

In the limit that $\beta_3 \to 0$, the iterative solutions (3.8), (3.9), and (3.11) recover the general relativistic solutions (2.27), (2.28), and (2.29), respectively. The density $\rho(r)$ is known for a given EOS. On using Eq. (3.8) with Eq. (3.12), the condition (3.7) translates to

$$|\beta_3 a_0^{2n} | < \frac{2^{n+1}}{n} \sqrt{\frac{\pi(1 + 3w_c)}{3}}, \quad \text{(3.14)}$$

Under this bound, the condition (3.7) is automatically satisfied. The EOS $w_c$ is bounded from above with the maximum value of order 1. For the polytropic EOS (2.35), we have that $w_c < 1$. Then, $|\beta_3 a_0^{2n} | \lesssim 2^{n+1}/n$ from Eq. (3.14). For $n = O(1)$, the product $|\beta_3 a_0^{2n} |$ is constrained to be smaller than the order of 1.
For the branch of the positive sign in Eq. (3.12) the upper bound (3.14) corresponds to the negative value of $\bar{\beta}_3$, whereas, for the negative sign, the upper limit of $\beta_3$ is positive. In the following, we will focus on the case of positive sign in Eq. (3.12) without loss of generality. Then, for $\beta_3 < 0$, the term $\mathcal{F}$ in Eq. (3.11) is negative, so the negative coupling $\beta_3$ effectively increases the pressure. In other words, the positive term $1 + 3w_c$ in Eq. (3.11) is partially compensated by the negative term $\mathcal{F}$. This means that, with increasing $r$, the pressure $P(r)$ decreases more slowly relative to the case $\beta_3 = 0$ at least around the center of body. Then, we expect that the negative coupling $\beta_3$ may lead to a larger radius of star than that for $\beta_3 = 0$.

Indeed, the negative value of $\beta_3$ close to the upper bound of Eq. (3.14) gives rise to the pressure (3.11) which is nearly constant around the center of star. Then, we may anticipate that the radius of star can be infinitely large. However, we will show that this is not the case. From Eqs. (2.23) and (3.9) the mass function around $r = 0$ is given by

$$M(r) = \frac{4}{3} \pi \rho_c r^3 (1 + \mathcal{F}) + \mathcal{O}(r^5).$$

(3.15)

The negative coupling $\beta_3$ leads to the decrease of $M(r)$ relative to the case of GR. For the theoretical consistency, we require that $M(r) > 0$ around the center of body. This amounts to the condition $\mathcal{F} > -1$, which translates to

$$|\beta_3| a_0^{2n} < \frac{2^{n+1}}{n} \sqrt{\frac{2\pi}{3(2 + 3w_c a_0^2)}},$$

(3.16)

which is tighter than the bound (3.14). Substituting $\mathcal{F} = -1$ into Eq. (3.11), the pressure corresponding to the maximum value of $|\beta_3|$ in Eq. (3.16) is given by

$$P_{\text{max}}(r) = \rho_c \left[ 1 - 2\pi(1 + w_c) \frac{r^2}{r_c^2} \right],$$

(3.17)

which decreases for increasing $r$. This expression is valid around $r = 0$, but we extrapolate it to the surface of star to provide a crude criterion for the upper limit of the radius $R_\star$. Then, we obtain the bound

$$R_\star \lesssim \frac{r_c}{\sqrt{2\pi(1 + w_c)}},$$

(3.18)

which means that $R_\star$ is constrained to be smaller than the order of $r_c$. Since the r.h.s. of Eq. (3.18) does not depend on the power $n$, the maximum radius is insensitive to the form of cubic couplings $G_3(X)$.

To discuss the dynamical stability of relativistic stars, we define the proper mass

$$M_p \equiv \int_{R \leq R_\star} d^3x \rho \sqrt{(3)g} = \int_0^{R_\star} 4\pi \rho r^2 \sqrt{\frac{1}{r}} dr,$$

(3.19)

where $(3)g$ is the determinant of three-dimensional spatial metric. The gravitational binding energy is defined by the difference between $M_p$ and the ADM mass $M_\star$, i.e.,

$$\Delta \equiv (M_p - M_\star) c^2.$$

(3.20)

The star with $\Delta > 0$ is gravitationally bound and the condition $\Delta > 0$ can be regarded as a necessary condition for its dynamical stability, whereas the star with $\Delta < 0$ is not bound and hence dynamically unstable. For $\beta_3 < 0$ the r-derivative of the leading-order term on the r.h.s. of Eq. (3.19) is smaller than $4\pi \rho_c r^2$, whereas the term inside the integral of Eq. (3.19) is larger than $4\pi \rho r^2$. This implies that the condition $\Delta > 0$ may hold for $\beta_3 < 0$, but we need to caution that Eq. (3.19) is valid only around the central region of star.

### B. Numerical solutions

The above analytic solutions have been derived under the expansion around $r = 0$. In order to study the effect of the coupling $\beta_3$ on the mass $M_\star$ and the radius $R_\star$ of relativistic stars more precisely, we numerically solve Eqs. (2.13)-(2.16) with Eq. (2.18) for the polytropic EOS (2.34) by using the boundary conditions (3.8)-(3.11) around the origin. For numerical computations, we will focus on the case of vector Galileons, i.e., $n = 1$ in Eq. (3.1). The numerical integration is performed until reaching the surface $r = R_\star$ characterized by the condition $w_0(R_\star) = 0$, where $w_0$ is
FIG. 1. Variation of the pressure in cubic Galileons ($G_3 = \beta_3 X$) for the polytropic EOS (2.31) with $K = 0.0130$ and $\Gamma = 2.34$. The two cases (i) and (ii) correspond to (i) $\beta_3 = -1$, $a_0 = 2.2$, $\chi(r) = 10.471$ at $r/r_0 = 10^{-3}$ and (ii) $\beta_3 = 1$, $a_0 = 2.0$, $\chi(r) = 17.783$ at $r/r_0 = 10^{-3}$, respectively. We also show the case of GR with $\beta_3 = 0$, $a_0 = 0$, $\chi(r) = 15.136$ at $r/r_0 = 10^{-3}$. The boundary conditions of $f, h, A_0, P$ are chosen to be consistent with Eqs. (5.10)-(5.13) of Ref. [60].

In Fig. 1 we plot the normalized pressure $P/(\rho_0 c^2)$ versus the distance $r$ from the center of star with $K = 0.013$ and $\Gamma = 2.34$ for three different values of $\beta_3 \equiv \beta_3 r_0 / \sqrt{8\pi G} = \beta_3 r_0 / r_c$. In GR, the pressure varies according to Eq. (2.29) at small distances. As we observe in Fig. 1, $P(r)$ starts to decrease rapidly around the surface of star. In the numerical simulation of Fig. 1 the star radius is $R_* \simeq 9.3$ km for $\beta_3 = 0$. In the presence of negative coupling $\beta_3$, the pressure decreases more slowly with increasing $r$, see case (i) of Fig. 1. In case (i), we have chosen a smaller value of the central pressure relative to that in GR, but the smaller decreasing rate of $P(r)$ in the former leads to the larger radius, $R_* \simeq 12.3$ km. The case (ii) in Fig. 1 corresponds to a positive value of $\beta_3$ with a larger central pressure compared to the GR case. The decreasing rate of $P(r)$ in case (ii) is faster than that in GR, so the resulting radius is smaller, $R_* \simeq 8.3$ km.

Similarly, the density $\rho(r)$ also decreases as a function of $r$. The central density $\rho_c$ in case (i) is smaller than that in GR, while the radius $R_*$ is larger. Since the density $\rho(r)$ in case (i) decreases more slowly relative to the case of GR, the former catches up with the latter at an intermediate distance ($r \simeq 6$ km). The $r$-derivative of the mass function $M(r)$ can be generally written in the form

$$M'(r) = 4\pi \rho(r)r^2[1 + \tilde{F}(r)],$$

(3.21)

where $\tilde{F}(r)$ is a function of $r$ containing the dependence of $\beta_3$. As we estimated in Eq. (3.15), the functions $\rho(r)$ and $\tilde{F}(r)$ around $r = 0$ reduce to the constants $\rho_c$ and $F$, respectively. When we integrate Eq. (3.21) with respect to $r$, the first term on the r.h.s. gives rise to a contribution to $M_*$ which is roughly proportional to $4\pi/3\rho_c R^2_\circ$. The increase of $R_*$ induced by the negative coupling $\beta_3$ leads to a larger contribution to $M_*$ relative to the decrease of $\rho_c$. In case (i) the mass contribution arising from the integration of the term $4\pi \rho(r)r^2$ in Eq. (3.21) is $M_{*1} = 2.87 M_\odot$, which is larger than the value $M_{*1} = 1.67 M_\odot$ of GR in Fig. 1. The ratio of $R^2_\circ$ between the case (i) and GR is given by $(12.3/9.3)^3 = 2.31$. This increase is slightly compensated by the smaller density in the central region with the decrease about 25%, so the resulting ratio of $M_{*1}$ between the two cases becomes $2.87/1.67 = 1.72 < 2.31$.

For $\beta_3 < 0$ the function $\tilde{F}(r)$ in Eq. (3.21) is negative around $r = 0$, so the negative coupling works to reduce the mass term $M_{*1}$. In case (i) of Fig. 1 the mass $M_{*2}$ arising from the numerical integration of $4\pi \rho(r)r^2 \tilde{F}(r)$ is found
to be $M_{a2} \simeq -0.29 M_{a1}$, so the total mass $M_* = M_{a1} + M_{a2}$ can be estimated as $M_* \simeq 0.71M_{a1} \simeq 2.03 M_\odot$. The mass function $M(r)$ in case (i) is smaller than that in GR except for the distance $r$ around the surface of star. However, the increase of $M(r)$ in case (i) continuously occurs up to the radius $R_*$ larger than that in GR, so the resulting mass $M_*$ in the former is larger. Thus, the main reason for the increase of $M_*$ comes from the increase of $R_*$ induced by the negative coupling.

For $\beta_3 > 0$, the radius $R_*$ gets smaller compared to the value in GR, see case (ii) of Fig. 1. Since the function $\tilde{F}(r)$ in Eq. (3.21) is positive, the mass function $M(r)$ is larger than that in GR at small distances. However, the increase of $M(r)$ stops at a smaller radius $R_*$, which results in a smaller mass $M_*$. Hence the positive coupling $\beta_3$ generally leads to the decrease of mass $M_*$ relative to the GR case.

In Fig. 2 we plot the mass-radius relation for the polytropic EOS \cite{[2.3]} with $K = 0.0130$ and $\Gamma = 2.34$. The central density is chosen to be in the range $\rho_c = \rho_c/\rho_0 \leq 200$. In this case, the maximum ADM mass $M_*$ in GR is given by $M_{\text{max}} = 1.67 M_\odot$ with the radius $R_* = 9.3$ km and the central density $\rho_c = 3.5 \times 10^{15}$ g cm$^{-3}$ (plotted as the GR case in Fig. 1). For increasing negative coupling $|\beta_3|$, the maximum mass gets larger. This effect tends to be significant for $|\beta_3| \rho_c^2$ exceeding the order of 1. The maximum mass reached for $\beta_3 = -1$ and $\tilde{\alpha}_0 = 2.2$ (case (c) in Fig. 2) is $M_{\text{max}} = 2.03 M_\odot$ with the radius $R_* = 12.3$ km and the central density $\rho_c = 2.1 \times 10^{15}$ g cm$^{-3}$ (plotted as case (i) of Fig. 1). Even though $\rho_c$ is smaller than that in GR, the larger radius $R_*$ leads to the maximum mass $M_{\text{max}}$ which is about $2.03/1.67 = 1.22$ times as large as that in GR.

From Eq. (3.10), there is the constraint $|\beta_3| \tilde{\alpha}_0^2 < 4\sqrt{2}/[3(2 + 3w_c \tilde{\alpha}_0^2)]$ for $n = 1$. If $\beta_3 = -1$, $w_c = 0.247$, $\rho_c/\rho_0 = 12.8$, this bound translates to $\tilde{\alpha}_0 < 2.7$. For increasing $\tilde{\alpha}_0$, the resulting mass of star tends to be larger. In case (d) shown in Fig. 2 ($\beta_3 = -1$, $\tilde{\alpha}_0 = 2.4$), the maximum mass for the radius $R_* < 20$ km is given by $M_{\text{max}} = 2.22 M_\odot$. For $2.5 \leq \tilde{\alpha}_0 < 2.7$, $M_*$ changes to a continuously growing function with respect to $R_*$. This property may be understood by using Eq. (3.11) for $|\beta_3| \tilde{\alpha}_0^2$ close to the upper bound (3.10). In this case, the star radius can be crudely estimated as

$$R_* \approx r_0 \sqrt{\frac{\rho_0}{2\pi(1 + w_c)\rho_c}}. \quad (3.22)$$

In the regime $w_c \ll 1$, the radius has the dependence $R_* \propto \rho_c^{-1/2}$, so it increases for decreasing $\rho_c$. The quantity $\rho_c R_*^3$ also increases for smaller $\rho_c$, as $\rho_c R_*^3 \propto \rho_c^{-1/2} \propto R_*$. While the negative coupling $\beta_3$ suppresses the growth of

\[\text{FIG. 2. Mass-radius relations in cubic Galileons for the polytropic EOS (2.3) with $\Gamma = 2.34$, $K = 0.0130$. We choose the boundary conditions (3.3)-(3.11) at the distance $r = 10^{-3}r_0$. Each curve corresponds to (a) $\beta_3 = -1$, $\tilde{\alpha}_0 = 1.0$, (b) $\beta_3 = -1$, $\tilde{\alpha}_0 = 2.0$, (c) $\beta_3 = -1$, $\tilde{\alpha}_0 = 2.2$, (d) $\beta_3 = -1$, $\tilde{\alpha}_0 = 2.4$, (e) $\beta_3 = +1$, $\tilde{\alpha}_0 = 2.0$, and the GR case $\beta_3 = 0$, $\tilde{\alpha}_0 = 0$. With increasing $\rho_c$, the values of $M_*$ and $R_*$ shift to the direction shown as the arrow inside the figure.}\]
$M(r)$ around $r = 0$, this is compensated by the increase of $R_*$ in the region of small $\rho_c$. Hence, for $|\tilde{\beta}_3|\hat{a}_0^2$ close to the upper bound (3.10), the mass $M_*$ continuously grows with the increase of $R_*$. Unless $\hat{a}_0$ is very close to the upper limit 2.7, the maximum mass $M_{\text{max}}$ does not exceed $3M_\odot$ for $R_* < 20$ km with the model parameters used in Fig. 2.

If the quantity $|\tilde{\beta}_3|\hat{a}_0^2$ exceeds the upper limit set by Eq. (3.10), the mass function $M(r)$ is negative around the center of star. Indeed, we numerically confirmed that the mass function enters the region $M(r) < 0$ around $r = 0$ and then $M(r)$ becomes positive at the distance away from the center. We regard that this situation is unphysical.

As we see in case (e) of Fig. 2, the positive coupling $\beta_3$ leads to smaller $M_*$ and $R_*$ than those in GR. In Fig. 3 we plot the quantity $\Delta/(M_*c^2) = M_p/M_* - 1$ versus the radius $R_*$ for the same model parameters as those used in Fig. 2 where $M_p$ is the proper mass defined by Eq. (3.10). For $\beta_3 < 0$, the binding energy $\Delta$ is always positive, so the star is gravitationally bound. For $\beta_3 > 0$, the star tends to be dynamically unstable in the region of small $\rho_c$. The configuration of maximum mass $M_* = 1.52M_\odot$ in case (e) of Fig. 2 ($\beta_3 = 1$ and $a_0 = 2.0$), which corresponds to the central density $\rho_c = 4.3 \times 10^{15}$ g·cm$^{-3}$ and the radius $R_* = 8.3$ km, leads to a positive binding energy, but the sign of $\Delta$ changes to negative for $\rho_c < 2.6 \times 10^{15}$ g·cm$^{-3}$.

For the ranges of $\rho_c$ smaller than those plotted as the cases (b), (c), (d) of Fig. 3, we numerically find that there is a maximum value of $\Delta/(M_*c^2)$ and then the binding energy gets smaller for decreasing $\rho_c$ further. In the intermediate regime where $\Delta/(M_*c^2)$ decreases with the increase of $\rho_c$, there is the “repulsive” gravity effect induced by the negative coupling $\beta_3$. The pressure increased by the negative coupling $\beta_3$ can support the star with a stronger gravitational force. In other words, the increased binding energy in the intermediate regime of $\rho_c$ is compatible with the large effective pressure induced by $\beta_3$.

The above discussion shows that not only the sign and the strength of coupling $\beta_3$ but also the amplitude of $A_0$ plays an important role for increasing the mass and radius of star. Around $r = 0$, the temporal component is given by Eq. (3.10), so the derivative $|A_0'|$ grows in proportion to $r$. The longitudinal mode $A_1$ has the same $r$-dependence as $|A_0'|$ around $r = 0$, see Eq. (3.9). For increasing $|a_0|$, the amplitude of $A_1$ also tends to be larger. In Fig. 4 we plot $|A_0'|$ and $A_1$ versus $r$ for the cases (i) and (ii) shown in Fig. 1. In both cases, $|A_0'|$ and $A_1$ increase in proportion to $r$ up to the distance close to the surface of star. Outside the body ($r > R_*$), the behavior of vector field is similar to the vacuum solution around the static and spherically symmetric BHs derived in Refs. [51, 60]. Namely, both $|A_0'|$ and $A_1$ decrease as $\propto 1/r^2$ for $r \gg R_*$. As in Refs. [51, 60], the coupling $\beta_3$ induces some difference between the two metric components $f$ and $h$ around the surface of star, but the difference becomes negligible in the limit that $r \gg R_*$.

**FIG. 3.** The gravitational binding energy $\Delta$ normalized by $M_*c^2$ versus the radius $R_*$ in cubic Galileons for the polytropic EOS (2.34) with $K = 0.0130$ and $\Gamma = 2.34$ in the region of the central density $3 \leq y_c \leq 200$. Each curve corresponds to the cases plotted in Fig. 2. With increasing $\rho_c$, the values of $\Delta$ and $R_*$ shift to the direction shown as the arrow inside the figure.
IV. QUARTIC COUPLINGS

In this section, we study the effect of quartic derivative couplings $G_4(X)$ on the configuration of relativistic stars. We consider the power-law coupling model given by

$$G_4 = \frac{1}{16\pi G} + \beta_4 X^n,$$

with $G_2 = G_3 = G_5 = G_6 = 0$ and $g_5 = 0$, where $\beta_4$ is a constant and $n$ is a positive integer. In Ref. [40], the authors discussed the relativistic star solutions for the specific case $n = 1$. Now, we investigate the models of general power $n$ including the quartic vector Galileon ($n = 2$). From Eq. (2.16), the longitudinal mode obeys

$$\beta_4 A_1 \left(A_0^2 - f h A_1^2\right)^{n-2} [A_1^2 f h \{(1+h-2nh)f+(1-2n)rf'f\}+A_0^2 \{f(h-1)+(2n-1)rf'f\}-4r(n-1)A_0 A_0'fh] = 0. \quad (4.2)$$

This gives rise to the two branches characterized by $A_1 = 0$ or $A_1 \neq 0$. For the latter branch, our numerical analysis shows that the solutions are qualitatively similar to those of cubic derivative couplings discussed in Sec. III. Hence we will focus on the other branch

$$A_1 = 0,$$  

in the rest of this section.

A. Analytic solutions around the center of star

Let us first derive analytic solutions to $f, h, A_0, P$ by using the expansions (2.20) and (3.3) around $r = 0$. From the continuity equation (2.18), we obtain the relation same as Eq. (3.4) among the coefficients $p_2$ and $f_2$. Substituting

FIG. 4. Numerical solutions to the derivative $|\tilde{A}_0'| = \sqrt{8\pi G r_0 |\tilde{A}_0'|}$ (left) and the longitudinal mode $\tilde{A}_1 = \sqrt{8\pi G} A_1$ (right) in cubic Galileons for the polytropic (2.34) with $K = 0.0130$ and $\Gamma = 2.34$. The cases (i) and (ii) correspond to the same model parameters and boundary conditions as those used in Fig. [I].
From Eqs. (2.23) and (4.5), the mass function around surface of star, it is expected that the radius \( R \) decreases toward 0 (which corresponds to \( \bar{\beta} = 0 \) and \( f(r) < 0 \)). If \( \bar{\beta} > 0 \), then the first term in the square bracket of the numerator of Eq. (4.14) dominates the difference between \( \mathcal{F} \) and \( \mathcal{F}_+ \), giving the tighter bound than Eq. (4.13).

The coupling \( \bar{\beta} \) affects the decreasing rate of the pressure \( P(r) \) through the function \( f_2 \), whose value in GR is given by \( f_2^{\text{GR}} = 4\pi(1 + 3w_c)/(3r_c^2) \). The difference between \( f_2 \) and \( f_2^{\text{GR}} \) is

\[
f_2 - f_2^{\text{GR}} = \frac{8\pi\bar{\beta}_4 \bar{a}_0^2 \{2n\{4n - 1 + 3w_c(2n - 1)\} - 2\bar{\beta}_4 \bar{a}_0^{2n-2}\{8n^2 + \bar{a}_0^2(2n - 1)^2(1 + 3w_c)\}\}}{3\{2n - 2\bar{\beta}_4 \bar{a}_0^{2n-2}(2n - 1)^2r_c^2\}}.
\]

For \( \bar{\beta}_4 < 0 \), the r.h.s. of Eq. (4.14) is negative and hence \( f_2 < f_2^{\text{GR}} \). If the solution (4.47) is extrapolated up to the surface of star, it is expected that the radius \( R_* \) is larger than that in GR due to the slower decrease of \( P(r) \) toward 0. Since the amplitude of negative coupling is not constrained from the condition \( M(r) > 0 \), the radius \( R_* \) is not bounded from above. This property is different from that in cubic power-law couplings where \( R_* \) is constrained as Eq. (3.18) from the condition \( M(r) > 0 \).

If \( \bar{\beta}_4 > 0 \) and \( |\bar{\beta}_4| \bar{a}_0^{2n-2} \ll 1 \), then the first term in the square bracket of the numerator of Eq. (4.14) dominates over the second one, so that \( f_2 > f_2^{\text{GR}} \). In this regime, the radius \( R_* \) should be smaller than that in GR due to the faster decrease of \( P(r) \) toward 0. For increasing \( \bar{\beta}_4 \) and \( \bar{\rho}_0 \), the function \( f_2 \) reaches a maximum and then it starts to decrease toward 0 (which corresponds to \( \bar{\beta}_4 \bar{a}_0^{2n-2} = \mathcal{F}_+ \)). As a function of \( \bar{\beta}_4 \), \( f_2 \) has the maximum value

\[
f_2^{\text{max}} = \frac{\pi[32n^2(1 + 3w_c) + \bar{a}_0^2\{1 + 3(1 - 2n)w_c\}]}{6n[4n - \bar{a}_0^2(2n - 1)]r_c^2},
\]
at
\[
\bar{\beta}_4 = \frac{2^{n-1}[2(2 + 3w_c) n - 1 - 3w_c]}{16n^2 - \bar{a}_0^2(2n - 1)[1 + 3(1 - 2n)w_c]} \bar{a}_0^{2-2n}.
\] (4.16)

The coupling \(\bar{\beta}_4\) is smaller than the upper limit \(\bar{\beta}_4 = F_+ \bar{a}_0^{2-2n}\) determined by Eq. (4.10). This gives the following bound
\[
\bar{a}_0 < \bar{a}_{\text{max}} \equiv \sqrt[n]{\frac{4n}{2n-1}}.
\] (4.17)

The regime in which the condition \(f_2 < f_2^{\text{GR}}\) is satisfied is given by
\[
F_c \bar{a}_0^{2-2n} < \bar{\beta}_4 < F_+ \bar{a}_0^{2-2n},
\] (4.18)

where
\[
F_c \equiv \frac{2^{n-1}[2(2 + 3w_c) n - 1 - 3w_c]}{8n^2 + \bar{a}_0^2(2n - 1)^2(1 + 3w_c)}.
\] (4.19)

To realize the slower decrease of \(P(r)\) around \(r = 0\) relative to the GR case, we need to choose the large value of \(\bar{\beta}_4 \bar{a}_0^{2n-2}\) close to \(F_-\). For given \(\bar{\beta}_4\) and \(n > 1\), this amounts to choosing larger \(\bar{a}_0\) close to the upper bound (4.17).

For given \(\bar{\beta}_4\) and \(n > 1\), this amounts to choosing larger \(\bar{a}_0\) close to the upper bound (4.17). Taking the limit \(\bar{a}_0 \to \bar{a}_{\text{max}}\), however, both \(F_c \bar{a}_0^{2-2n}\) and \(F_+ \bar{a}_0^{2-2n}\) approach the same value \((2n - 1)^{-1}/(2^{n+1}n^n)\). In this limit, the parameter space consistent with Eq. (4.18) disappears with the divergence of \(f_2^{\text{max}}\). Even if we consider the value \(\bar{a}_0 = \bar{a}_{\text{max}} - \varepsilon\), where \(\varepsilon\) is a small positive parameter, the expansions of \(F_c \bar{a}_0^{2-2n}\) and \(F_+ \bar{a}_0^{2-2n}\) in terms of \(\varepsilon\) show that two terms are equivalent up to the order of \(\varepsilon\). Since the difference between \(F_c \bar{a}_0^{2-2n}\) and \(F_+ \bar{a}_0^{2-2n}\) appears only at the order of \(\varepsilon^2\), the parameter space consistent with Eq. (4.18) is restricted to be very narrow. This discussion shows that, for \(\bar{\beta}_4 > 0\), the function \(f_2\) is in the range \(f_2 > f_2^{\text{GR}}\) for most of the parameters under consideration, which should result in smaller \(R_\ast\) compared to the GR case.

In the following, we will confirm the above analytic estimation by numerically solving Eqs. (2.13)-(2.15) and (2.18) with \(A = 1\).

B. Numerical solutions

For the numerical computation, we focus on the case of quartic vector Galileons \((n = 2)\). The property of solutions in other power-law models \((n \neq 2)\) are qualitatively similar to those discussed below.

In Fig. 5 we plot the mass function \(M(r)\) versus \(r\) for several different values of \(\bar{\beta}_4\) and \(\bar{a}_0\) with the same central density \(\rho_c\). We employ the polytropic EOS (2.34) with \(\Gamma = 2.34\) and \(K = 0.01\). The mass \(M_\ast\) and the radius \(R_\ast\) of star can be identified by the point at which \(M(r)\) stops increasing, e.g., \(M_\ast \approx 1.4M_\odot\) and \(R_\ast \approx 9.5\) km in GR \((\bar{\beta}_4 = 0)\). As we analytically estimated above, the value of \(M(r)\) for \(\bar{\beta}_4 < 0\) is smaller than that in GR at small distances. However, as we see in case (i) of Fig. 5 the mass function in the former catches up with that in the latter at an intermediate distance inside the star, so the resulting mass \(M_\ast\) gets larger. Moreover, we have numerically confirmed that the negative coupling \(\bar{\beta}_4\) leads to a slower decrease of the pressure \(P(r)\) up to the star surface relative to the case \(\bar{\beta}_4 = 0\), which results in a greater radius \(R_\ast\). The case (i) in Fig. 5 shows that both \(R_\ast\) and \(M_\ast\) are larger than those in GR. When \(\bar{\beta}_4 > 0\), the mass function \(M(r)\) at small distances is larger than that for \(\bar{\beta}_4 = 0\). This property can be seen in case (ii) of Fig. 5 but the increase of \(M(r)\) stops at a smaller radius \(R_\ast\) because of a faster decrease of \(P(r)\). Hence the mass \(M_\ast\) in case (ii) is smaller than that in GR.

In Fig. 6 we show the mass-radius relation for the polytropic EOS with \(\Gamma = 2.34\) and \(K = 0.01\) in the presence of quartic Galileon couplings \(\bar{\beta}_4 = -0.1\) or \(\bar{\beta}_4 = 0.1\). Compared to GR, the negative coupling \(\bar{\beta}_4\) leads to larger values of \(M_\ast\) and \(R_\ast\). For this EOS, the maximum value of \(M_\ast\) in GR is given by \(M_{\text{max}} = 1.51M_\odot\) with the central density \(\rho_c = 4.1 \times 10^{15} \text{ g} \cdot \text{cm}^{-3}\) and the radius \(R_\ast = 8.48\) km. In the presence of negative \(\bar{\beta}_4\), the larger maximum mass can be realized with the smaller central density. In case (c) plotted in Fig. 6 which corresponds to \(\bar{\beta}_4 = -0.1\) and \(\bar{a}_0 = 1.3\), the maximum mass \(M_{\text{max}} = 2.06M_\odot\) with the radius \(R_\ast = 11.8\) km is reached at the density \(\rho_c = 1.6 \times 10^{15} \text{ g} \cdot \text{cm}^{-3}\). If we increase either \(\bar{\beta}_4\) or \(\bar{a}_0\) further, \(M_{\text{max}}\) becomes larger. Indeed, the condition \(M(r) > 0\) around \(r = 0\) does not restrict the amplitude of negative coupling \(\bar{\beta}_4\), so the mass \(M_\ast\) can be even larger than \(3M_\odot\) for \(\bar{\beta}_4 \sim 0\) close to the lower limit determined by the condition \(\bar{\beta}_4 \bar{a}_0^2 = F_-\).

In cases (e) and (f) depicted in Fig. 6 which correspond to \(\bar{\beta}_4 > 0\), the mass \(M_\ast\) and the radius \(R_\ast\) are smaller than those in GR, independent of the detail of EOSs. In these cases the condition \(f_2 > f_2^{\text{GR}}\) is satisfied, so the faster decrease of \(P(r)\) leads to the smaller radius \(R_\ast\) compared to that in GR. As shown in Fig. 5 the mass function \(M(r)\)
FIG. 5. The mass function $M(r)$ versus the distance $r$ in quartic Galileons ($n = 2$) for the polytropic EOS (2.34) with $\Gamma = 2.34$, $K = 0.010$, and the central density $\rho_c = 2.3 \times 10^{15} \text{ g cm}^{-3}$. The two curves at the top and bottom correspond to the model parameters (i) $\beta_4 = -0.06, a_0 = 1.5$, and (ii) $\beta_4 = 0.06, a_0 = 1.5$, while the solid curve corresponds to GR with $\beta_4 = 0, a_0 = 0$.

FIG. 6. Mass-radius relations in quartic Galileons for the polytropic EOS (2.34) with $\Gamma = 2.34$. We use Eqs. (4.4)-(4.7) as the boundary conditions at the distance $r = 10^{-3} r_0$. Each curve corresponds to (a) $\beta_4 = -0.1, a_0 = 1.0$, (b) $\beta_4 = -0.1, a_0 = 1.2$, (c) $\beta_4 = -0.1, a_0 = 1.3$, (d) $\beta_4 = -0.1, a_0 = 1.4$, (e) $\beta_4 = +0.1, a_0 = 1.0$, and (f) $\beta_4 = +0.1, a_0 = 1.5$. The GR case with $\beta_4 = 0$ and $a_0 = 1.0$ is plotted as the solid line. With increasing $\rho_c$, the values of $M_*$ and $R_*$ shift to the direction shown as the arrow inside the figure.

is larger than that for $\beta_4 = 0$ in the central region of star, but the decrease of $R_*$ induced by positive $\beta_4$ overwhelms this effect to end up with smaller $M_*$. We recall that there exists the restricted parameter range (4.18) in which the condition $f_2 < f_{2GR}^2$ can be satisfied for $\beta_4 > 0$. When $n = 2, \beta_4 = 0.1$, and $w_c = 0.4$, for example, the bound (4.18) translates to $1.600746 < a_0 < 1.600816$, whose parameter space is very narrow. Moreover, we find that the solutions in such a narrow parameter region are prone to numerical instabilities. Thus, the positive coupling $\beta_4$ generally leads to the suppression of $M_*$ and $R_*$ in most of the parameter space with stable solutions.
FIG. 7. The binding energy $\Delta$ normalized by $M_* c^2$ versus the radius $R_*$ in quartic Galileons for the polytropic EOS with $\Gamma = 2.34$ and $K = 0.01$ in the range $3 \leq y_c \leq 200$. Each curve corresponds to the cases plotted in Fig. 6.

The instability of star for large positive values of $\beta_4 \bar{a}_0^2$ close to $F_+$ can be also confirmed by computing the binding energy $\Delta$ defined by Eq. (3.20). In Fig. 7 we show $\Delta / (M_* c^2)$ versus the radius $R_*$ for $\Gamma = 2.34$ and $K = 0.01$ with several different values of $\beta_4$ and $\bar{a}_0$. When $\beta_4 < 0$, the binding energy is always positive, so the star is gravitationally bound. If $\beta_4 > 0$, $\Delta$ can be negative in the region of small central density $\rho_c$. In cases (e) and (f) shown in Fig. 7, $\Delta$ is negative for $\rho_c < 1.2 \times 10^{15}$ g · cm$^{-3}$ and $\rho_c < 8.2 \times 10^{15}$ g · cm$^{-3}$, respectively, so that the region of instability tends to be larger for $\beta_4 \bar{a}_0^2$ approaching the upper limit $F_+$. Thus, for $\beta_4 > 0$, it is difficult to realize the stable configuration of star with $M_*$ and $R_*$ larger than those in GR.

V. INTRINSIC VECTOR-MODE COUPLINGS

Finally, we investigate the relativistic star solutions in the presence of intrinsic vector-mode couplings given by

$$G_2 = -2g_4(X) F, \quad g_5 = g_5(X), \quad G_6 = G_6(X),$$

with $G_4 = 1/(16\pi G)$, where $g_4(X)$ is a function of $X$, and $F = hA_0'^2/(2f)$ on the background. From Eq. (2.16), it follows that

$$A_0'^2 \left[ \{ r^2 g_4, X + (3h - 1)G_6, X \} A_1 - 2 r g_5 + 2 h r g_5, X A_1^2 - G_6, XX h^2 A_1^3 \right] = 0.$$  (5.2)

We can write Eq. (2.15) in the following form

$$\alpha_1 A_0'' + \alpha_2 A_0' + \alpha_3 A_0^2 = 0,$$  (5.3)

where $\alpha_1, \alpha_2, \alpha_3$ are functions containing $A_0, A_1, A_1', f, h, f', h'$ and $g_4, g_5, G_6$ as well as their $X$-derivatives. The explicit expression of the coefficient $\alpha_1$ is given by

$$\alpha_1 = (2g_4 - 1)r^2 + 4 h r g_5 A_1 - 2 h^2 A_1^2 G_6, XX + 2(h - 1)G_6.$$  (5.4)

From Eq. (5.2), there is a branch characterized by $A_0'(r) = 0$, that is

$$A_0(r) = \text{constant},$$  (5.5)

which is consistent with Eq. (5.3).
There exist other branches where the terms in the square bracket of Eq. \((5.2)\) vanish, which can give rise to a nonvanishing longitudinal component \(A_1\). Even in such cases, the temporal vector component needs to obey the regular boundary condition \(A'_0(0) = 0\) at the center of star. Then, we obtain \(\alpha_1 A'_0(0) = 0\) from Eq. \((5.3)\), so that \(A'_0(0) = 0\) for \(\alpha_1 \neq 0\). This means that, when we integrate Eq. \((5.3)\) from \(r = 0\) with the boundary condition \(A'_0(0) = 0\), the derivative \(A'_0(r)\) remains to be 0 for arbitrary \(r\). Then, provided that \(\alpha_1 \neq 0\), we end up with the solution \((5.5)\) even for the branches other than \(A'_0(r) = 0\) in Eq. \((5.2)\). Substituting the solution \(A'_0(r) = 0\) into Eqs. \((2.13)\) and \((2.14)\), it follows that
\[
\frac{h'}{r} + \frac{h - 1}{r^2} = -\frac{8\pi G \rho}{c^2},
\]
\[
\frac{h f'}{f} + \frac{h - 1}{r^4} = \frac{8\pi G P}{c^4},
\]
which are exactly the same as Eqs. \((2.21)\) and \((2.22)\) in GR, respectively. This shows that the intrinsic vector-mode
nonvanishing longitudinal component
Eqs. \((2.13)\) and \((2.14)\), it follows that
\[
\frac{h'}{r} + \frac{h - 1}{r^2} = -\frac{8\pi G \rho}{c^2},
\]
\[
\frac{h f'}{f} + \frac{h - 1}{r^4} = \frac{8\pi G P}{c^4},
\]
where

\[
\frac{h'}{r} + \frac{h - 1}{r^2} = -\frac{8\pi G \rho}{c^2},
\]
\[
\frac{h f'}{f} + \frac{h - 1}{r^4} = \frac{8\pi G P}{c^4},
\]

In summary, the intrinsic vector-mode couplings do not give rise to any modifications to the metric components \(f\) and \(h\). The TOV equation also holds in the same form as Eq. \((2.23)\). Thus, for a given EOS, the solutions to \(f, h, P, \rho\) are the same as those in GR with \(A_0(r) = constant\). Requiring the smooth matching of the metric and vector field at the surface, Eq. \((5.5)\) remains the solution outside the star with the exterior metric given by the Schwarzschild solution \((2.30)\).

The above property is in stark contrast with that in cubic and quartic couplings where the differential equation corresponding to Eq. \((5.3)\) contains the \(A_0\)-dependent terms which are not multiplied by the powers of \(A'_0\). As we discussed in Secs. \(\text{III}\) and \(\text{IV}\), the existence of such terms leads to the variation of \(A_0(r)\) for \(r > 0\). We also note that the presence of mass contribution \(m^2X\) to \(G_2\) gives rise to the terms \(m^2A_1\) and \(m^2A_0\) to Eqs. \((5.2)\) and \((5.3)\), respectively, so the general solution to \(A_0(r)\) is different from Eq. \((5.5)\).

In summary, the intrinsic vector-mode couplings \((5.1)\) only lead to the metric components in GR with the trivial temporal vector component \((1)\)-breaking vector field with derivative couplings, which leads to the propagation of fifth forces. On the weak gravitational background in Solar System, it is known that such fifth forces can be suppressed by derivative self-interactions under the operation of the Vainshtein mechanism \([49, 50]\). On the other hand, the deviation from GR can manifest itself in the strong gravitational regime like BHs \([50, 60]\). Indeed, there exist a bunch of hairy BH solutions in generalized Proca theories. Our interest in this paper was to show how the new “hair” induced by vector-field derivative couplings affects the configuration of relativistic stars.

In this paper, we studied how the mass-radius relation of relativistic stars is modified in generalized Proca theories. In these theories there exists a \(U(1)\)-breaking vector field with derivative couplings, which leads to the propagation of fifth forces. On the weak gravitational background in Solar System, it is known that such fifth forces can be suppressed by derivative self-interactions under the operation of the Vainshtein mechanism \([49, 50]\). On the other hand, the deviation from GR can manifest itself in the strong gravitational regime like BHs \([50, 60]\). Indeed, there exist a bunch of hairy BH solutions in generalized Proca theories. Our interest in this paper was to show how the new “hair” induced by vector-field derivative couplings affects the configuration of relativistic stars.

In Sec. \(\text{III}\) we considered the cubic power-law derivative coupling \((8.1)\) including the vector Galileon \((n = 1)\) as a specific case. In these models, the vector field has a nonvanishing longitudinal mode \(A_1\) related to the temporal component \(A_0\) according to Eq. \((5.2)\). Imposing the regularity of metrics, pressure, density, and vector field at the center of star \((r = 0)\), we derived the analytic solutions \((5.3)\)–\((5.11)\) around \(r = 0\). As we see in Eq. \((5.11)\), the negative coupling constant \(\beta_3\) leads to a slower decrease of the matter pressure \(P(r)\). This slower decrease continues up to the star surface, so the resulting radius \(R_*\) for \(\beta_3 < 0\) tends to be larger than that in GR. We also showed that the amplitude of negative coupling \(\beta_3\) is constrained as Eq. \((3.16)\) from the demand \(M(r) > 0\) around \(r = 0\). This limits the maximum radius reached by the cubic coupling, see Eq. \((5.18)\). These properties hold independently of the EOS of relativistic stars.

To compute the mass \(M_\ast\) and the radius \(R_*\) of relativistic stars precisely, we numerically solved Eqs. \((2.18)\)–\((2.10)\) for the cubic Galileon coupling \(G_3 = \beta_3 X\) by employing the polytropic EOS \((2.8)\) with \(\Gamma = 2.34\). We confirmed that the negative coupling \(\beta_3\) gives rise to \(R_*\) larger than in the case \(\beta_3 = 0\). Although the mass function \(M(r)\) is suppressed by negative \(\beta_3\) around \(r = 0\), the increase of \(R_*\) overwhelms this decrease to realize the mass \(M_\ast\) greater than that in GR. As we observe in Fig. \(\text{2}\) the maximum mass \(M_{\text{max}}\) increases for a larger temporal vector component \(a_0\) at \(r = 0\) and for an increasing amplitude of negative coupling \(\beta_3\). For \(\beta_3 > 0\), both \(M_\ast\) and \(R_*\) are smaller than those in GR. Moreover, the models with large positive values of \(\beta_3\) and \(a_0\) are prone to instabilities associated with a negative gravitational binding energy \(\Delta\) in the low-density regime.

In Sec. \(\text{IV}\) we studied the effect of quartic power-law couplings \((4.1)\) on the configuration of relativistic stars by considering the branch \(A_1 = 0\). Again, the negative coupling \(\beta_4\) leads to the larger mass \(M_\ast\) and the larger radius \(R_*\) relative to those in GR. The difference from cubic derivative interactions is that the amplitude of negative \(\beta_4\) is

VI. CONCLUSIONS
not constrained from the condition $M(r) > 0$. For $\beta_4 > 0$ we found that both $M_*$ and $R_*$ are smaller than those in GR for most of the parameter space. The solutions are also subject to instabilities in the low-density regime with increasing values of $\beta_4$ and $a_0$. This is not the case for negative $\beta_4$ where the necessary condition for the dynamical stability is satisfied.

In Sec. [V] we showed that the intrinsic vector-mode couplings (5.1) give rise to solutions same as those in GR with the constant value of $A_0$. This is attributed to the peculiar structure of the differential equation (5.3) as well as the regular boundary condition $A_0^* = 0$ at $r = 0$. Thus, the intrinsic vector modes do not modify the radius and mass of relativistic stars in GR.

There are several issues we did not address in this paper. We adopted the polytropic EOS (2.34) with $\Gamma = 2$ to compute the mass and radius of relativistic stars, but for the comparison of them with the observational data of NSs, we need to extend the analysis to more realistic EOSs by taking into account nuclear interactions and the composition of each layer of NSs. It is also possible to include the rotation of NSs in our analysis along the line of Ref. [67] and investigate the existence of EOS-independent relations [18] useful to test generalized Proca theories with NSs further. Although we have confirmed that most of the solutions obtained in this paper are gravitationally bound, the analysis of dynamical stabilities against odd- and even-parity perturbations may provide further constraints on couplings in generalized Proca theories. With this perturbative analysis on the spherically symmetric background, we should also be able to derive the local propagation speed $c_g$ of gravitational waves around NSs. If the vector-field derivative couplings studied in this paper are also responsible for today’s cosmic acceleration, the recent GW170817 bound of $c_g$ [68] on the cosmological background will provide tight constraints on quartic derivative couplings. These interesting issues will be left for future works.

Appendix A: Coefficients in the gravitational equations of motion

The coefficients $c_{1,2,\ldots,19}$ in Eqs. (2.13) and (2.14) are given by

\[
\begin{align*}
c_1 &= -A_1 X G_{3,X}, \\
c_2 &= -2G_4 + 4(X_0 + 2X_1)G_{4,X} + 8X_1 X G_{4,XX}, \\
c_3 &= -A_1 (3hX_0 + 5hX_1 - X)G_{5,X} - 2hA_1 X_1 X G_{5,XX}, \\
c_4 &= G_2 - 2X_0 G_{2,X} - \frac{h}{f}(A_0 A_1 A_0' + 2f X A_1')G_{3,X} - \frac{h A_0^2 (1 + 2G_{2,f})}{2f}, \\
c_5 &= -4hA_1 X_0 G_{3,X} - 4h^2 A_1 A_1' G_{4,X} + \frac{8h}{f} (A_0 X_1 A_0' - f h A_1 X A_1') G_{4,XX} + \frac{2h^2}{f} A_1 A_0^2 (g_5 + 2X_0 g_{5,X}), \\
c_6 &= 2(1 - h)G_4 + 4(hX - X_0)G_{4,X} + 8hX_0 X_1 G_{4,XX} - \frac{h}{f} [(h - 1)A_0 A_1 A_0' + 2f (3hX_1 + hX_0 - X) A_1'] G_{5,X} \\
&\quad - \frac{2h^2 X_1}{f} (A_0 A_1 A_0' + 2f X A_1') G_{5,XX} + \frac{h A_0^2}{f} [(h - 1) G_6 + 2(hX - X_0) G_{6,X} + 4hX_0 X_1 G_{6,XX}], \\
c_7 &= -G_2 + 2X_1 G_{2,X} - \frac{h}{f} A_0 A_1 A_0' G_{3,X} + \frac{h A_0^2 (1 + 2G_{2,f})}{2f}, \\
c_8 &= 4hA_1 X_1 G_{3,X} + \frac{4h}{f} A_0 A_0' (G_{4,X} + 2X_1 G_{4,XX}) - \frac{2h^2}{f} A_1 A_0^2 (3g_5 + 2X_1 g_{5,X}), \\
c_9 &= 2(h - 1)G_4 - 4(2h - 1) X_1 G_{4,X} - 8hX_1^2 G_{4,XX} - \frac{h}{f} A_0 A_1 A_0' [(3h - 1) G_{5,X} + 2hX_1 G_{5,XX}] \\
&\quad - \frac{h}{f} A_0^2 [(3h - 1) G_6 + 2(6h - 1) X_1 G_{6,X} + 4hX_1^2 G_{6,XX}].
\end{align*}
\]

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