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DISTRIBUTION OF RANDOM VARIABLES ON THE SYMMETRIC GROUP

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Introduction

Actuality

Distribution of many characteristics of different combinatorial structures can be very well approximated by the distribution of some variables defined on the symmetric group $S_n$. One example of such approximation is the distribution of the degree of the splitting field of a random polynomial with integer coefficients. The degree of the splitting field of a polynomial is an important characteristic which allows us to estimate how many steps it would take to decompose the polynomial into the product of prime polynomials. Thus the information about the distribution of the degree of the splitting field of a random polynomial would allow us to estimate how much time would it take to decompose a randomly chosen polynomial into the product of prime polynomials.

Aims and problems

The aim of this work is to obtain the estimates for the remainder term in the Erdős Turán law and also to prove analogous result for distribution of the logarithm of the order of a random permutation on some subsets $S_n^{(k)}$ of the symmetric group. To obtain these aims we also prove some estimates for the mean values of multiplicative functions on $S_n$ and $S_n^{(k)}$, which are of independent interest, and which allow us to estimate the convergence rate to normal law of additive functions on $S_n$.

We also study the distribution of the degree of the splitting field of a random polynomial and obtain sharp estimates for the convergence rate of it to normal law.
Methods

In research we apply both probabilistic and analytic methods. Some analytic methods used here have their origins in the probabilistic number theory, and some have their roots in the theory of summation of divergent series. We also prove some tauberian theorem for Voronoi summability of divergent series.

Novelty

All obtained results are new. They generalize and improve the earlier results of Erdős and Turán, Manstavičius, Nicolas, Pavlov, Barbour and Tavaré.

Results

Let $S_n$ be the symmetric group. Each $\sigma \in S_n$ can be represented as a product of independent cycles

$$\sigma = \kappa_1 \kappa_2 \ldots \kappa_{\omega(\sigma)}. \quad (1)$$

This representation is unique up to the order of cycles.

On $S_n$ we can define a uniform probability measure $P_n$ by assigning to each subset $A \subset S_n$ probability

$$P_n(A) = \frac{|A|}{|S_n|} = \frac{|A|}{n!},$$

here $|A|$ is the number of elements of the set $A$. Let us denote by $\alpha_j(\sigma)$ the number of cycles in the decomposition of $\sigma$ into product of independent cycles (1) whose length is equal to $j$. Then, obviously

$$\alpha_1(\sigma) + 2\alpha_2(\sigma) + \cdots + n\alpha_n(\sigma) = n. \quad (2)$$

And $\omega(\sigma)$ – the number of independent cycles in the decomposition of $\sigma$ in (1) can be expressed as

$$\omega(\sigma) = \alpha_1(\sigma) + \alpha_2(\sigma) + \cdots + \alpha_n(\sigma).$$

In 1942 V. L. Goncharov [7] proved that

$$P_n \left( \frac{\omega(\sigma) - \log n}{\sqrt{\log n}} < x \right) \to \Phi(x), \quad \text{as} \quad n \to \infty$$

here $\Phi(x)$ is the standard normal distribution. He also proved that the distribution of $\alpha_j(\sigma)$ when $j$ is fixed is asymptotically distributed as poissonian random variable with parameter $1/j$. 
The uniform probability measure is not the only probability measure with respect to which one can study the distribution of random variables on \( S_n \). Let \( \theta > 0 \) be a fixed parameter. By putting
\[
P_{n,\theta}(\sigma) = \frac{\theta^{\omega(\sigma)}}{\theta(\theta + 1) \cdots (\theta + n - 1)}
\]
we define the Ewens probability measure \( P_{n,\theta} \).

As in [10] we will call a function \( f : S_n \to \mathbb{C} \) multiplicative if for \( \sigma \in S_n \) having representation (1),
\[
f(\sigma) = f(\kappa_1)f(\kappa_2) \cdots f(\kappa_{\omega})
\]
and \( f(\kappa) \) depends on cycle length \( |\kappa| \) only. For any multiplicative function \( f \) on \( S_n \) we will denote by \( \hat{f}(j) \) the value of \( f \) on the cycles whose length is equal to \( j \), \( 1 \leq j \leq n \). That means that \( f(\kappa) = \hat{f}(|\kappa|) \), where \( |\kappa| \) is the length of cycle \( \kappa \).

Suppose \( d(\sigma) \) is a non-negative multiplicative function, \( d(\sigma) \not\equiv 0 \). Then we can define a probabilistic measure \( \nu_{n,d} \) on \( S_n \) by the formula
\[
\nu_{n,d}(\{\sigma\}) = \frac{d(\sigma)}{\sum_{w \in S_n} d(w)}.
\]

If \( \hat{d}(j) = 1 \), we obtain the uniform probability measure, and for \( \hat{d}(j) \equiv \theta > 0 \) we obtain Ewens probability measure.

Let us denote by \( M_n^d(f) \) the weighted mean of a multiplicative function \( f : S_n \to \mathbb{C} \) with respect to the measure \( \nu_{n,d}(\sigma) \):
\[
M_n^d(f) = \sum_{\sigma \in S_n} f(\sigma) \nu_{n,d}(\sigma) = \frac{\sum_{\sigma \in S_n} f(\sigma) d(\sigma)}{\sum_{\sigma \in S_n} d(\sigma)}.
\]

In 2002 E. Manstavičius proved the following result.

**Theorem A** ([12]). Let \( f : S_n \to \mathbb{C} \) be a multiplicative function, such that \( |f(\sigma)| \leq 1 \), satisfying the conditions:
\[
\sum_{j \leq n} \frac{1 - \Re \hat{f}(j)}{j} \leq D
\]

and
\[
\frac{1}{n} \sum_{j=1}^n |\hat{f}(j) - 1| \leq \mu_n = o(1),
\]
for some positive constant $D$ and some sequence $\mu_n$.

Suppose that the measure defining multiplicative function $d(\sigma)$ satisfies the condition $0 < d^- \leq d(\kappa) \leq d^+$ on cycles $\kappa \in S_n$ for some fixed positive constants $d^-$ and $d^+$, then there exist positive constants $c_1 = c_1(d^-, d^+)$ and $c_2 = c_2(d^-, d^+)$ such that

$$M_n^d(f) = \exp \left\{ \sum_{j \leq n} d_j \hat{f}(j) - \frac{1}{j} \right\} + O \left( \mu_n^c + \frac{1}{n^{c_2}} \right),$$

where $d_j = \hat{d}(j)$ is the value of $d$ on the cycles of length $j$.

We prove the following result.

**Theorem 1.** Let $f : S_n \to \mathbb{C}$ be a multiplicative function satisfying the condition $|f(\sigma)| \leq 1$ for all $\sigma \in S_n$. Suppose that the measure defining multiplicative function $d(\sigma)$ is such that $0 < d^- \leq d_j \leq d^+$, where $d_j$ is the value of $d(\sigma)$ on cycles of length $j$. Then we have

$$\Delta_n := \left| M_n^d(f) - \exp \left\{ \sum_{k=1}^{n} d_k \frac{\hat{f}(k) - 1}{k} \right\} \right|$$

$$\leq c \left( \left( \sum_{j=0}^{n} p_j \right)^{-1} \sum_{k=1}^{n} |\hat{f}(k) - 1| p_{n-k} + \frac{1}{n^{d^-}} \sum_{k=1}^{n} |\hat{f}(k) - 1| k^{d^--1} \right)$$

$$+ \frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - 1| \left( 1 + \log \frac{n}{k} \right)$$

for $d^- < 1$ and

$$\Delta_n \leq c \left( \left( \sum_{j=0}^{n} p_j \right)^{-1} \sum_{k=1}^{n} |\hat{f}(k) - 1| p_{n-k} + \frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - 1| \left( 1 + \log \frac{n}{k} \right) \right)$$

for $d^- \geq 1$, where $c = c(d^-, d^+)$ is a positive constant which depends on $d^-$ and $d^+$ only, and

$$p_n = \frac{1}{n!} \sum_{\sigma \in S_n} d(\sigma)$$

Thus Theorem 1 shows that condition (5) in Theorem A is superfluous. Our result yields more accurate estimates of the remainder term than Theorem A. The proof of theorem 1 is different from that of Theorem A of Manstavičius. It is based on some properties of Voronoi means of divergent series. In chapter 1 we establish a tauberian type theorem for Voronoi summability which is of independent interest.
As in [10], we will call a function \( h : S_n \rightarrow \mathbb{R} \) additive if for \( \sigma \in S_n \) having decomposition (1) we have
\[
h(\sigma) = h(\kappa_1) + h(\kappa_2) + \cdots + h(\kappa_\omega)
\]
and we assume that the value of \( h \) on cycles \( \kappa \) depend on the length of cycles only. This means that there exist \( n \) real numbers \( \hat{h}(1), \hat{h}(2), \ldots, \hat{h}(n) \) such that \( h(\kappa) = \hat{h}(|\kappa|) \) for all cycles \( \kappa \in S_n \).

In 1998 E. Manstavičius proved the following result.

**Theorem B ([10])**. Let \( d_j \equiv 1 \). Suppose \( \hat{h}_n(k) \) satisfy the condition
\[
\sum_{k=1}^{n} \frac{\hat{h}_n^2(k)}{k} = 1.
\]
Then
\[
\sup_{x \in \mathbb{R}} \left| \nu_n(x) - \Phi(x) - \frac{D_n x e^{-x^2/2}}{2\sqrt{2\pi}} \right| \ll L_n,
\]
where \( \nu_n(x) := \nu_{n,1}(h(\sigma) - a(n) < x) \),
\[
D_n = \sum_{1 \leq k, l \leq n, k + l > n} \frac{\hat{h}_n(k)\hat{h}_n(l)}{kl}, \quad L_n = \sum_{k=1}^{n} \frac{(|\hat{h}_n(k)|^3)}{k}, \quad a(n) = \sum_{k=1}^{n} \frac{\hat{h}_n(k)}{k}.
\]

**Corollary A ([10])**. We have
\[
R_n := \sup_{x \in \mathbb{R}} |\nu_{n,1}(x) - \Phi(x)| \ll \frac{L_n^2}{3}.
\]

There exists a sequence of constants \( \hat{h}_n(k) \), satisfying the normalizing condition (6), such that \( L_n = o(1) \) and
\[
R_n \gg \frac{L_n^2}{3}.
\]

This result has been generalized for \( d(j) \equiv \theta > 0 \) in our paper [21].

We now generalize this result for the case when the probabilistic measure on the symmetric group is \( \nu_{n,d} \).

Let us denote
\[
A(n) = \sum_{k=1}^{n} d_k \frac{\hat{h}_n(k)}{k}, \quad C_n = \sum_{j=1}^{n} d_j \frac{\hat{h}_n(j)}{j} \left( \frac{p_{n-j}}{p_n} - 1 \right),
\]
and
\[
L_{n,p} = \sum_{k=1}^{n} \frac{|\hat{h}_n(k)|^p}{k}, \quad L_{n,2}' = \sum_{j=1}^{n} \frac{\hat{h}_n^2(j)}{j} \left( \frac{p_{n-j}}{p_n} - 1 \right).
\]
Henceforth assume that \( \tilde{h}_n(k) \) satisfies the normalizing condition
\[
\sum_{k=1}^{n} d_k \frac{\tilde{h}_n^2(k)}{k} = 1. \tag{7}
\]

**Theorem 2.** Suppose \( 0 < d^- \leq d_j \leq d^+ \), and \( p \) is a fixed number such that \( \infty \geq p > \max \{2, 1/d^-\} \). Suppose
\[
F_n(x) = \nu_{n,d}(h(\sigma) - A(n) < x),
\]
where \( h(\sigma) \) is an additive function satisfying condition (7). Then we have
\[
\sup_{x \in \mathbb{R}} \left| F_n(x) - \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} C_n \right| \ll L_{n,3} + L_{n,p}^{2/p} + L_{n,2}',
\]
here we assume that
\[
L_{n,\infty}^{1/\infty} = \lim_{p \to \infty} L_{n,p}^{1/p} = \max_{1 \leq j \leq n} |\hat{h}(j)|
\]
for \( p = \infty \).

The condition \( L_{n,3} = o(1) \) is not necessary to ensure the convergence of an additive function to the normal law. In [2] it was presented the example of an additive function for which \( L_{n,3} \gg 1 \) and whose distribution nevertheless converges to the normal law. In section 1.3 we investigate the convergence rate for this special additive function. Surprisingly the rate is essentially faster than could be obtain from Theorem 2.

Let \( a = (a_1, a_2, \ldots, a_n) \) be a vector with nonnegative integer components. Following [3], we denote
\[
O_n(a) = \text{l. c. m. } \{i: 1 \leq i \leq n, \ a_i > 0\} \quad \text{and} \quad P_n(a) = \prod_{i=1}^{n} i^{a_i}.
\]
One can easily see that \( O_n(a) \leq P_n(a) \).

Let us denote \( \alpha = \alpha(\sigma) = (\alpha_1(\sigma), \alpha_2(\sigma), \ldots, \alpha_n(\sigma)) \). Then \( O_n(\alpha(\sigma)) \) is equal to the order of the permutation \( \sigma \), i. e.
\[
O_n(\alpha(\sigma)) = \min \{m \geq 0 | \sigma^m = e\},
\]
where \( e - \text{unit of the group } S_n \).

Erdős and Turán [3] proved that if \( P_n \) is the uniform probability measure on \( S_n \), then
\[
P_n \left( \Phi(x) = \frac{\log O(\alpha(\sigma)) - \frac{1}{2} \log^2 n}{\sqrt{\frac{1}{3} \log^{3/2} n}} \right) \to \Phi(x) \quad \text{as } n \to \infty
\]
the distribution of log $O_n(\alpha)$ converges to the standard normal law. Their
proof consisted of two steps. First they proved that the distributions of
log $O(\alpha(\sigma))$ and log $P(\alpha(\sigma))$ are close enough to allow to replace investigation
of log $O(\alpha(\sigma))$ by that of log $P(\alpha(\sigma))$. And then they proved that log $P(\alpha(\sigma))$
in asymptotically normally distributed. log $P_n(\alpha(\sigma)) = \sum_{j=1}^{n} \alpha_j(\sigma) \log j$ being an additive function, its distribution is much easier to investigate.

Nicolas [16] obtained the estimate $O((\log^{-1/2} n \log \log \log n))$ for the convergence rate in their theorem. He has also conjectured that the iterated logarithm in his estimate is superfluous. Barbour and Tavaré [3] proved that

$$\sup_{x \in \mathbb{R}} |\nu_{n,\theta} \left\{ \frac{\log O_n(\alpha(\sigma)) - (\theta/2) \log^2 n + \theta \log n \log \log n}{(\theta/\sqrt{3}) \log^{3/2} n} < x \right\} - \Phi(x)| \leq \frac{1}{\sqrt{\log n}}, \quad (8)$$

where $\nu_{n,\theta}$ is the Ewens probability measure.

In chapter 3 we prove the following theorems.

**Theorem 3.**

$$\sup_{x \in \mathbb{R}} |\nu_{n,1} \left\{ \frac{\log O_n(\alpha) - M \log O_n(\alpha)}{(1/\sqrt{3}) \log^{3/2} n} < x \right\} - \Phi(x) - \frac{3^{3/2}}{24 \sqrt{2 \pi}} \frac{(1-x^2)e^{-x^2/2}}{\sqrt{\log n}}| \leq \left( \frac{\log \log n}{\log n} \right)^{2/3}. \quad (9)$$

**Theorem 4.**

$$M \log O_n(\alpha) = \frac{1}{2} \log^2 n - \log n (\log \log n - 1)$$

$$+ \sum_{\rho} (\log n)^\rho \Gamma(-\rho) + O((\log \log n)^2),$$

where $\sum_{\rho}$ is the sum over all nontrivial zeros of the Riemann zeta-function.

Let $F_q$ be a finite field with $q$ elements. We denote by $E_n$ the set of
normed polynomials of degree $n$ with coefficients in $F_q$. Then each element
$P \in E_n$ can be decomposed into a product of prime (over $F_q$) and normed polynomials. This decomposition is unique up to the order of multiplicands. We denote by $\xi_k = \xi_k(P)$ the number of prime polynomials of degree $k$ in the canonical decomposition of a polynomial $P$. On $E_n$, we define the uniform probability measure

$$\nu_n(\ldots) = \frac{1}{q^n} |\{P \in E_n: \ldots\}|.$$
Then $O_n(\xi)$, where $\xi := \xi(P) = (\xi_1(P), \xi_2(P), \ldots, \xi_n(P))$, is equal to the degree of the splitting field of $P$. Nicolas [15] has proved that

$$\sup_{x \in \mathbb{R}} \nu_{\xi} \left\{ \log O_n(\xi) - \frac{1}{2} \log^2 n < x \right\} - \Phi(x) \ll \frac{(\log \log n)^4}{\sqrt{\log n}}. \quad (10)$$

The investigation of $\xi(P)$ on $E_n$ has much in common with that of the random vector $\alpha(\sigma) = (\alpha_1(\sigma), \alpha_2(\sigma), \ldots, \alpha_n(\sigma))$ defined on the symmetric group $S_n$ with the uniform probability measure. In fact theorems 3 and 4 will be just simple consequences of the following theorems.

**Theorem 5.**

$$\sup_{x \in \mathbb{R}} \nu_{\xi} \left\{ \log O_n(\xi) - M \log O_n(\xi) \right\} (1/\sqrt{3}) \log^{3/2} n < x \right\} - \Phi(x) - \frac{3^{3/2}}{24 \sqrt{2 \pi}} \frac{(1 - x^2)e^{-x^2/2}}{\log n} \right\} \ll \left( \frac{\log \log n}{\log n} \right)^{2/3}. \quad (11)$$

**Theorem 6.**

$$M \log O_n(\xi) = \frac{1}{2} \log^2 n - \log n (\log \log n - 1) + \sum_\rho (\log n)^\rho \Gamma(-\rho) + O((\log \log n)^2),$$

where $\sum_\rho$ denotes the sum over all nontrivial zeros of the Riemann zeta-function.

Assuming that the Riemann hypothesis is true, the sum over the nontrivial zeros of the Riemann zeta-function in Theorems 6 and 4 can be estimated as $O(\sqrt{\log n})$. Using the well-known fact that $\zeta(\sigma + it) \neq 0$ for $\sigma \geq 1 - \frac{c}{\log(2 + |t|)}$, one can estimate those sums over the nontrivial zeros of $\zeta(s)$ as $O(\log n \exp\{-c \frac{\log \log n}{\log \log \log n}\})$.

From theorems 5 and 3 it follows that, for $\theta = 1$, the convergence rate in 8 cannot be improved in the sense of order.

The proof of 8 of Barbour and Tavare is based on approximating the distribution of the random vector $\alpha = (\alpha_1(\sigma), \alpha_2(\sigma), \ldots, \alpha_n(\sigma))$ by the distribution of a random vector $Z = (Z_1, Z_2, \ldots, Z_n)$, where $Z_j$ are independent Poisson random variables with parameters $1/j$. The distributions of $O_n(\alpha)$ and $O_n(Z)$ are approximated by the distributions of $O_n(Z)$ and $P_n(Z)$, respectively. In the proof of Theorem 3 below, we directly approximate the
variable $O_n(\xi(P))$ by $P_n(\xi(P))$, without using any auxiliary independent random variables.

Let us denote by $S_n^{(k)} = \{ \sigma = \sigma^k | x \in S_n \}$ the subset of $S_n$, which consists of all permutations $\sigma \in S_n$, from which one can extract root of degree $k$.

On the subset $S_n^{(k)}$ we can define the uniform probability measure $\nu_n^{(k)}$ by means of formula

$$\nu_n^{(k)}(A) = \frac{|A|}{|S_n^{(k)}|}$$

for any $A \subset S_n^{(k)}$.

In Chapter 3 we investigate the distribution additive and multiplicative functions with respect to measure $\nu_n^{(k)}(\sigma)$, establishing the analog of Theorem 1 for the mean values of multiplicative functions on $S_n^{(k)}$, thus generalizing earlier results of Pavlov [17].

Pavlov [17] proved that

$$\log O_n(\alpha(\sigma)) - M_n \log O_n(\alpha(\sigma))$$

$$\approx \frac{3}{3k} \log^{3/2} n$$

is asymptotically normally distributed on $S_n^{(k)}$ when $n \to \infty$. Here $\phi(k)$ – Euler’s function. In chapter 3 we estimate the convergence rate.

**Theorem 7.** For fixed $k$ we have

$$\sup_{x \in \mathbb{R}} \left| \nu_n^{(k)} \left\{ \frac{\log O_n(\alpha(\sigma)) - M_n \log O_n(\alpha(\sigma))}{\sqrt{3k} \log^{3/2} n} < x \right\} \right| - \Phi(x) - r_k(x) \frac{e^{-x^2/2}}{\sqrt{\log n}}$$

$$\ll \left( \frac{\log \log n}{\log n} \right)^{2/3},$$

where $r_k(x) = \sqrt{\frac{3}{8\sqrt{2\pi}}} \frac{(1-8C_0-x^2)}{8\sqrt{2\pi}}$ and

$$C_0 = \gamma_0 \int_0^1 \frac{(1-y)^{-1} - 1}{y} \, dy,$$

here $\gamma_0 = \frac{\phi(k)}{k}$.

**Theorem 8.**

$$M_n \log O(\alpha) = \frac{\gamma_0}{2} \log^2 n - \gamma_0 \log n (\log \log n + C(k))$$

$$+ \sum_{\rho} \Gamma(-\rho) (\gamma_0 \log n)^{\rho} + O((\log \log n)^2),$$

where

$$C(k) = \log \gamma_0 - 1 - \int_0^1 \frac{(1-y)^{-1} - 1}{y} \, dy - \sum_{p|k} \frac{\log p}{p-1},$$

and $\sum_{\rho}$ is the sum over the non-trivial zeroes of the Riemann Zeta function.
Chapter 1

Additive and multiplicative functions on $S_n$

1.1 Voronoi sums

Let $\alpha_k(\sigma)$ be the number of cycles in $\sigma$ whose length is equal to $k$. Then $\alpha_1(\sigma) + 2\alpha_2(\sigma) + \cdots + n\alpha_n(\sigma) = n$ and we have the following representation for multiplicative function

$$f(\sigma) = \hat{f}(1)^{\alpha_1(\sigma)} \hat{f}(2)^{\alpha_2(\sigma)} \cdots \hat{f}(n)^{\alpha_n(\sigma)},$$

where as before $\hat{f}(j)$ are the values of multiplicative function $f(\sigma)$ on cycles of length $j$. We assume that $0^0 = 1$ in the above relationship.

Since the quantity of $\sigma \in S_n$ such that $\alpha_k(\sigma) = s_k$ for $1 \leq k \leq n$ is equal to

$$n! \prod_{j=1}^{n} \frac{1}{s_j! s_j!},$$

when $s_1 + 2s_2 + \cdots + ns_n = n$, we have for any multiplicative function $f(\sigma)$

$$\sum_{\sigma \in S_n} f(\sigma) = n! \sum_{k_1+2k_2+\cdots+nk_n=n} \prod_{j=1}^{n} \left( \frac{\hat{f}(j)}{j} \right)^{k_j} \frac{1}{k_j!}.$$

One can easily see that if $N_m = \frac{1}{m!} \sum_{\sigma \in S_m} f(\sigma)$ then

$$\sum_{j=0}^{\infty} N_j z^j = \exp \left\{ \sum_{j=1}^{\infty} \frac{\hat{f}(j)}{j} z^j \right\}$$

(1.1)

Here we assume that $N_0 = 1$. Therefore

$$M_n^d(f) = \frac{\sum_{\sigma \in S_n} f(\sigma) d(\sigma)}{\sum_{\sigma \in S_n} d(\sigma)} = \frac{M_n}{p_n},$$

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where $M_j$ and $p_j$ are defined by relationships
\[
\sum_{j=0}^{\infty} M_j z^j = \exp \left\{ \sum_{j=1}^{\infty} \frac{d_j f(j)}{j} z^j \right\} \quad \text{and} \quad p(z) = \sum_{j=0}^{\infty} p_j z^j = \exp \left\{ \sum_{j=1}^{\infty} \frac{d_j}{j} z^j \right\}.
\]

We have
\[
F(z) = \exp \left\{ \sum_{j=1}^{\infty} \frac{d_j f(j)}{j} z^j \right\} = \sum_{j=0}^{\infty} M_j z^j = \exp \{L(z)\} p(z), \tag{1.2}
\]

here $L(z) = \sum_{j=1}^{\infty} \frac{d_j f(j-1)}{j} z^j$. Let us denote $\exp \{L(z)\} = \sum_{j=0}^{\infty} m_j z^j$, then
\[
M_n^d(f) = \frac{M_n}{p_n} = \frac{1}{p_n} \sum_{j=0}^{n} m_j p_{n-j}.
\]

The proof of the theorem \[ will be based on the following theorem.

**Theorem 9.** Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$, when $|z| < 1$, and $0 < d^- < d_j < d^+$. Then for $n \geq 1$ we have
\[
\left| \frac{1}{p_n} \sum_{k=0}^{n} a_k p_{n-k} - f(e^{-1/n}) - \frac{S(f; n)}{np_n} \right| \leq c \left( \frac{1}{n^\theta} \sum_{j=1}^{n} |S(f; j)| j^{\theta-1} + \frac{1}{p(e^{-1/j})} \sum_{j>n} |S(f; j)| j e^{-j/n} \right),
\]

where $S(f; m) = \sum_{k=1}^{m} a_k k p_{m-k}$, $\theta = \min \{d^-, 1\}$ and $c = c(d^+, d^-)$ is a constant which depends on $d^+$ and $d^-$ only.

Hence we obtain

**Theorem 10.** Let $f(z)$ and $p(z)$ be the same as in theorem \[. Then the relation
\[
\lim_{n \to \infty} \frac{1}{p_n} \sum_{k=0}^{n} a_k p_{n-k} = A \in \mathbb{C},
\]

holds if and only if the following two conditions are satisfied:

1) $\lim_{x \to 1^+} f(x) = A$

2) $S(f; n) = \sum_{k=1}^{n} a_k k p_{n-k} = o(np_n)$ as $n \to \infty$. 

1.1. VORONOI SUMS

Theorem 10 can be reformulated in terms of Voronoi summation theory (see [8]). Suppose \( \sum_{j=0}^{\infty} a_k \) is a formal series and \( r_j \) is a sequence of non-negative real numbers. If

\[
\lim_{n \to \infty} \frac{r_0 s_n + r_1 s_{n-1} + \cdots + r_n s_0}{r_0 + r_1 + \cdots + r_n} = s \in \mathbb{R},
\]

where \( s_k = a_0 + a_1 + \cdots + a_k \), then we say that series \( \sum_{j=0}^{\infty} a_k \) can be summed in the sense of Voronoi and its Voronoi sum is equal to \( s \). In such case we write

\[
(W, r_n) \sum_{j=0}^{\infty} a_k = s.
\]

Theorem 10 yields the necessary and sufficient conditions for a series to have a Voronoi sum when \( r_n \) are defined by formula

\[
\sum_{m=0}^{\infty} r_m z^m = \exp \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j}{j} z^j \right\},
\]

where \(-1 < \lambda^- \leq \lambda_j \leq \lambda^+ < \infty\). For such \( r_j \) we have that

\[
(W, r_n) \sum_{j=0}^{\infty} a_k = A
\]

if and only if

\[
\lim_{x \uparrow 1} \sum_{j=0}^{\infty} a_j x^j = A \in \mathbb{C}
\]

and

\[
\frac{r_0 D_n + r_1 D_{n-1} + \cdots + r_n D_0}{r_0 + r_1 + \cdots + r_n} = o(n) \quad \text{as} \quad n \to \infty,
\]

where \( D_n = 1a_1 + 2a_2 + \cdots + na_n \). When \( \lambda_j \equiv 0 \), this condition takes form \( D_n = o(n) \) and we obtain the classical theorem of Tauber.

Note that in such case \( r_n \) can be negative, though condition \(-1 < \lambda^- \leq \lambda_j \) ensures that \( r_0 + r_1 + \cdots + r_m \geq \left( \frac{m+\lambda^-}{m} \right) > 0 \) for \( m \geq 1 \).

Let us now find the generating function of the characteristic function of the distribution of \( h_n(\sigma) \). Since for additive function we have \( h(\sigma) = \hat{h}(1)\alpha_1(\sigma) + \hat{h}(2)\alpha_2(\sigma) + \cdots + \hat{h}(n)\alpha_n(\sigma) \) and

\[
\nu_{n,d}(\alpha_1(\sigma) = s_1, \ldots, \alpha_n(\sigma) = s_n) = \frac{1}{p_n} \prod_{j=1}^{n} \left( \frac{d_j}{j} \right)^{s_j} \frac{1}{s_j!},
\]
therefore

\[ g_n(t) = \sum_{\sigma \in S_n} \exp \{ it \tilde{h}_n(\sigma) \} \nu_{n,d}(\sigma) = \frac{1}{p_n} \sum_{m_1+2m_2+\ldots+n_m=n} \prod_{j=1}^{n} \left( d_j \frac{\hat{f}(j)}{j} \right)^{m_j} \]

where \( \hat{f}(k) = \exp \{ it \tilde{h}_n(k) \} \). Thus the characteristic function of the random variable \( \tilde{h}_n(k) \) is the weighted mean of a multiplicative function.

Let \( p(z) \) be denoted as before. In what follows we assume that \( 0 < d^- \leq d_k \leq d^+ \), and \( \theta = \min \{1, d^-\} \), where \( d^-, d^+ \) are fixed positive numbers. We also denote \( \tilde{d}_k = d_k - \theta \) and

\[ \tilde{p}(z) = \exp \left\{ \sum_{k=1}^{\infty} \frac{\tilde{d}_k z^k}{k} \right\} = \sum_{n=0}^{\infty} \tilde{p}_n z^n. \]

One can easily see that

\[ \tilde{p}(z) = p(z)(1 - z)\theta. \]

**Lemma 1.** If \( m \geq n \geq 1 \), then

\[
\left( \frac{m}{n} \right)^{d^-} e^{-d^-/n} \leq \frac{p(e^{-1/m})}{p(e^{-1/n})} \leq \left( \frac{m}{n} \right)^{d^+} e^{d^+/m},
\]

and

\[
\left( \frac{m}{n} \right)^{\tilde{d}^-} e^{-\tilde{d}^-/n} \leq \frac{\tilde{p}(e^{-1/m})}{\tilde{p}(e^{-1/n})} \leq \left( \frac{m}{n} \right)^{\tilde{d}^+} e^{\tilde{d}^+/m},
\]

where \( \tilde{d}_k^+ = d_k^+ - \theta \), \( \tilde{d}_k^- = d_k^- - \theta \) and \( \tilde{d}_k^+ = d_k^+ - \theta \).

**Proof.** We have

\[
\frac{p(e^{-1/m})}{p(e^{-1/n})} = \exp \left\{ \sum_{k=1}^{\infty} \frac{d_k}{k} (e^{-k/m} - e^{-k/n}) \right\} \leq \exp \left\{ \tilde{d}_k^+ \sum_{k=1}^{\infty} \frac{e^{-k/m} - e^{-k/n}}{k} \right\}
\]

\[
= \exp \left\{ \tilde{d}_k^+ \log \frac{1 - e^{-1/n}}{1 - e^{-1/m}} \right\} = \left( \frac{1 - e^{-1/n}}{1 - e^{-1/m}} \right)^{\tilde{d}_k^+} \leq \left( \frac{m}{n} \right)^{\tilde{d}_k^+} e^{\tilde{d}_k^+/m}.
\]

here we have used the inequalities \( e^{-x} x \leq 1 - e^{-x} \leq x \) for \( x \geq 0 \).

In the same way we obtain the lower bound estimate.

The proof of the second inequality is analogous.

The lemma is proved.

Further we will often use the inequality

\[ a_0 + a_1 + \cdots + a_n \leq e g(e^{-1/n}), \]
where \( g(x) = \sum_{j=0}^{\infty} a_k x^k \) and \( a_k \geq 0, k \geq 0 \). Differentiating \( p(z) \) and \( \tilde{p}(z) \) we obtain
\[
z p'(z) = p(z) \sum_{k=1}^{\infty} d_k z^k \quad \text{and} \quad \tilde{z} p'(z) = \tilde{p}(z) \sum_{k=1}^{\infty} \tilde{d}_k z^k
\]
hence we have for \( n \geq 1 \)
\[
p_n = \frac{1}{n} \sum_{k=1}^{n} d_k p_{n-k} \quad \text{and} \quad \tilde{p}_n = \frac{1}{n} \sum_{k=1}^{n} \tilde{d}_k \tilde{p}_{n-k}.
\]

Hence we have
\[
p_n \leq \frac{d^+ e}{n} \frac{p(e^{-1/n})}{f(x)} \quad \text{and} \quad \tilde{p}_n \leq \frac{d^+ e}{n} \frac{\tilde{p}(e^{-1/n})}{f(x)}.
\]

It has been proved in [12] that there is a positive constant \( c(d^+) \) such that
\[
p_n \geq d^- c(d^+) \frac{p(e^{-1/n})}{n}.
\]

For the sake of completeness we will give here another proof of this estimate based on the following theorem which is of interest in itself.

**Theorem 11.** Suppose \( f(x) = \sum_{k=0}^{\infty} a_k x^k \), where \( a_k \geq 0 \) and
\[
\frac{f'(x)}{f(x)} \leq \frac{c}{1 - x}
\]
when \( 0 \leq x < 1 \). Then there exists such a positive constant \( K = K(c) \) that
\[
\sum_{j=0}^{N} a_k \geq K(c) f(e^{-1/N})
\]
when \( N \geq 2c \).

**Proof.** If \( 0 \leq x < 1 \), we have
\[
f(x) \leq \sum_{k=0}^{N} a_k x^k + \frac{1}{N} \sum_{k=0}^{\infty} k a_k x^k \leq \sum_{k=0}^{N} a_k + \frac{xf'(x)}{N}
\]
\[
\leq \sum_{k=0}^{N} a_k + \frac{f(x)}{N} \frac{xf'(x)}{f(x)} \leq \sum_{k=0}^{N} a_k + f(x) \frac{cx}{N(1 - x)}.
\]
Inserting here \( x = e^{-1/n} \) with \( n = \lceil \frac{N}{2c} \rceil \), we obtain

\[
    f(e^{-1/n}) \leq \sum_{k=0}^{N} a_k + \frac{cn}{N} f(e^{-1/n}) \leq \sum_{k=0}^{N} a_k + \frac{1}{2} f(e^{-1/n}),
\]

therefore

\[
    \frac{1}{2} f(e^{-1/n}) \leq \sum_{k=0}^{N} a_k.
\]

If \( c \leq 1/2 \), then \( N \leq \lceil \frac{N}{2c} \rceil = n \) and

\[
    \frac{1}{2} f(e^{-1/N}) \leq \frac{1}{2} f(e^{-1/n}) \leq \sum_{k=0}^{N} a_k,
\]

therefore in such a case the theorem will be true with \( K(c) = \frac{1}{2} \).

Suppose now that \( c > \frac{1}{2} \), then \( N \geq \lceil \frac{N}{2c} \rceil = n \) and we have

\[
    \frac{f(e^{-1/n})}{f(e^{-1/N})} = \exp \{ \log f(e^{-1/n}) - \log f(e^{-1/N}) \} = \exp \left\{ - \int_{e^{-1/n}}^{e^{-1/N}} \frac{f'(x)}{f(x)} \, dx \right\}
\]

\[
    \geq \exp \left\{ -c \int_{e^{-1/n}}^{e^{-1/N}} dx \frac{dx}{1-x} \right\} = \left( \frac{1 - e^{-1/N}}{1 - e^{-1/n}} \right)^c \geq e^{-c/N} \left( \frac{n}{N} \right)^c
\]

\[
    = e^{-c/N} \left( \frac{1}{N} \left\lfloor \frac{N}{2c} \right\rfloor \right)^c \geq K(c) > 0,
\]

where \( K(c) = \inf_{m \geq 2c} e^{-c/m} \left( \frac{1}{m} \left\lceil \frac{m}{2c} \right\rceil \right)^c \).

The theorem is proved.

The application of this theorem for \( f(z) = p(z) \) together with (1.3) yields the proof of estimate (1.5).

**Lemma 2.** If \( 0 \leq s \leq n/2 \), then

\[
    |p_{n+s} - p_n| \ll \frac{p(e^{-1/n})}{n} \left( \frac{s}{n} \right) \theta \ll p_n \left( \frac{s}{n} \right) \theta,
\]

where \( \theta = \min \{d^-, 1\} \).

**Proof.** Since \( p(z) = \frac{\tilde{p}(z)}{1-z} \), then we have

\[
    p_n = \sum_{k=0}^{n} \tilde{p}_k \left( \frac{n-k+\theta-1}{n-k} \right),
\]
therefore
\[ p_{n+s} - p_n = \sum_{k=0}^{n} \tilde{p}_k \left( \begin{array}{c} n+s-k+\theta-1 \\ n+k \end{array} \right) - \left( \begin{array}{c} n-k+\theta-1 \\ n-k \end{array} \right) \]
\[ + \sum_{n+s+k>n} \tilde{p}_k \left( \begin{array}{c} n+s-k+\theta-1 \\ n+s-k \end{array} \right) =: S_1 + S_2. \]

If \( s = 0 \), then the estimate of the theorem is trivial, therefore we assume that \( s > 0 \). Applying here the estimate (1.4) together with Lemma \( \text{I} \) we have
\[ S_2 \leq \sum_{l=0}^{s} \left( \frac{l+\theta-1}{l} \right) \max_{n+s+k>n} \tilde{p}_k \leq \left( \frac{s+\theta}{s} \right) \tilde{p}(e^{-1/n})e^{d^+} \max_{n+s+k>n} \tilde{p}(e^{-1/k}) \frac{1}{k} \]
\[ \ll s^\theta e^{\tilde{p}(e^{-1/n})} = s^\theta \frac{p(e^{-1/n})(1-e^{-1/n})^\theta}{n} \ll \left( \frac{s}{n} \right)^\theta \frac{p(e^{-1/n})}{n}. \]

If \( \theta = 1 \), then \( S_1 = 0 \), therefore estimating \( S_1 \) we may assume that \( \theta < 1 \).

It is well-known that \( \left( \frac{n-k+\theta-1}{n-k} \right) = \frac{n^\theta}{\Gamma(\theta)} \left( 1 + O\left( \frac{1}{n} \right) \right) \) (see e. g. [6]). Once again applying Lemma \( \text{I} \) and the estimate (1.4) we have
\[ S_1 \ll \sum_{k=0}^{n-s} \tilde{p}_k \left( \begin{array}{c} n-k+\theta-1 \\ n-k \end{array} \right) \left( \frac{s}{n-k} \right)^\theta \frac{p(e^{-1/n})}{n} \frac{s}{n-k} \]
\[ \ll \sum_{k=0}^{n-s} \tilde{p}_k \left( \begin{array}{c} n-k+\theta-1 \\ n-k \end{array} \right) \frac{s}{n} + \tilde{p}(e^{-1/n}) \frac{s}{n} \]
\[ \ll \sum_{k \leq n/2} \tilde{p}_k \frac{s}{n} k^\theta \leq n/2 \sum_{n/2 < k \leq n-s} \tilde{p}_k s(n-k)^\theta - 2 + \frac{p(e^{-1/n})}{n} \left( \frac{s}{n} \right)^\theta \]
\[ \ll \frac{p(e^{-1/n})}{n} s + \frac{p(e^{-1/n})}{n} \left( \frac{s}{n} \right)^\theta \ll \frac{p(e^{-1/n})}{n} \left( \frac{s}{n} \right)^\theta. \]

The lemma is proved.

For \( 0 \leq x \leq 1 \) we denote
\[ G_x(z) = \frac{p(z)}{p(zx)} = \sum_{k=0}^{\infty} g_{k,x} z^k \quad \text{and} \quad \tilde{G}_x(z) = \frac{\tilde{p}(z)}{\tilde{p}(zx)} = \sum_{k=0}^{\infty} \tilde{g}_{k,x} z^k, \]
and

\[ C_x(z) = \left( \frac{1 - zx}{1 - z} \right)^\theta = \sum_{k=0}^\infty c_{k,x} z^k. \]

Since \( \tilde{p}(z) = p(z)(1 - z)^\theta \), we have

\[ G_x(z) = \tilde{G}_x(z) \left( \frac{1 - zx}{1 - z} \right)^\theta. \]

Differentiating \( C_x(z) \) and \( G_x(z) \) with respect to \( z \) we have

\[ zC'_x(z) = C_x(z) \theta \sum_{k=1}^\infty z^k (1 - x^k) \quad \text{and} \quad zG'_x(z) = G_x(z) \sum_{k=1}^\infty d_k z^k (1 - x^k). \]

Whence we obtain that

\[ c_{n,x} = \theta \frac{n}{n} \sum_{k=1}^n c_{n-k,x} (1 - x^k) \quad \text{and} \quad g_{n,x} = \frac{1}{n} \sum_{k=1}^n g_{n-k,x} d_k (1 - x^k), \]

for \( n \geq 1 \) and \( c_{0,x} = g_{0,x} = 1 \). Hence we deduce that \( c_{n,x}, g_{n,x} \geq 0 \) and therefore

\[ \sum_{m=0}^n c_{m,x} \leq eC_x(e^{-1/n}) \quad \text{and} \quad \sum_{m=0}^n g_{m,x} \leq eG_x(e^{-1/n}). \]

It follows hence

\[ c_{n,x} \leq e\theta C_x(e^{-1/n}) \quad \text{and} \quad g_{n,x} \leq ed^+ G_x(e^{-1/n}). \]

**Lemma 3.** Suppose \( 0 < x < 1 \) and \( s \leq m/2 \), then we have

\[ |c_{m,x} - c_{m-s,x}| \leq sm^{\alpha-2} (1 - x)^\theta + \frac{s}{m^2}, \]

for \( m \geq 1 \).

**Proof.** Suppose \( L_\epsilon \) is a contour \( L_\epsilon = L_1 \cup L_2 \cup L_3 \cup L_4 \), where

\[ L_1 = \{ z | z = 2e^{it}, \pi \geq |t| \geq \epsilon \}, \quad L_2 = \{ z | z = 1 + \frac{e^{it}}{m}, \pi \geq |t| \geq \epsilon \}, \]

\[ L_3 = \{ z | z = t \left( 1 + \frac{e^{it}}{m} \right) + (1 - t)2e^{it}, \quad 0 \leq t \leq 1 \}. \]
Applying Cauchy formula we have
\[
L_4 = \left\{ z | z = t \left( 1 + \frac{e^{-ix}}{m} \right) + (1-t)2e^{-ix}, \ 0 \leq t \leq 1 \right\}.
\]

By the lemma proved
\[
|c_{m,x} - c_{m-1,x}| = \left| \frac{1}{2\pi i} \int_{L_4} C_x(z) \left( \frac{1-z}{z^{m+1}} \right) dz \right| 
\leq \frac{1}{2\pi} \int_{L_4} \left| \frac{1-x|z|^\theta |1-z|^{-\theta}}{|z|^{m+1}} \right| dz
\leq \int_{L_4} \frac{|1-z| + |1-z|^{1-\theta}|1-x|^\theta}{|z|^{m+1}} |dz|
\]

Allowing now \( \epsilon \to 0 \), we have
\[
|c_{m,x} - c_{m-1,x}| \ll \frac{1}{2m} + \int_{1+\frac{1}{m}}^{2m} \frac{(y-1) + (y-1)^{1-\theta}|1-x|^\theta}{y^{m+1}} dy
\]

Now we have for \( s \leq m/2 \)
\[
|c_{m,x} - c_{m-s,x}| \ll |c_{m,x} - c_{m-1,x}| + |c_{m-1,x} - c_{m-2,x}| + \cdots + |c_{m-s+1,x} - c_{m-s,x}|
\ll 2^s s^{\theta-2}(1-x)^\theta + \frac{s}{m^2}.
\]

The lemma is proved.

**Lemma 4.** For \( 0 \leq x \leq e^{-1/n} \) and \( k \leq n/8 \), we have
\[
g_{n,x} - g_{n-k,x} \ll p(e^{-1/n}) \left( \left( \frac{k}{n} \right)^\theta + \frac{1}{(n(1-x))^\theta} \right).
\]

**Proof.** Since \( G_x(z) = \tilde{C}_x(z) C_x(z) \), we obtain
\[
g_{n,x} - g_{n-k,x} = \sum_{s=0}^{n} \tilde{g}_{s,x} c_{n-s,x} - \sum_{s=0}^{n-k} \tilde{g}_{s,x} c_{n-k-s,x} = \sum_{s=0}^{n-2k} \tilde{g}_{s,x} (c_{n-s,x} - c_{n-k-s,x})
+ \sum_{n-2k < s \leq n} \tilde{g}_{s,x} c_{n-s,x} - \sum_{n-2k < s \leq n-k} \tilde{g}_{s,x} c_{n-k-s,x} =: S_1 + S_2 + S_3.
\]
If \( \theta = 1 \) then \( c_j = 1 - x \), for \( j \geq 1 \), it follows hence that in this case \( S_1 = 0 \). Therefore, while estimating \( S_1 \), we may assume that \( \theta < 1 \). Applying Lemma 3 we have

\[
S_1 \ll \sum_{0 \leq s \leq n-2k} \tilde{g}_{s,x} \left( k(n-s)^{\theta-2}(1-x)^{\theta} + \frac{k}{(n-s)^2} \right) 
\]

\[
\ll kn^{\theta-2}(1-x)^{\theta} \sum_{0 \leq s \leq n/2} \tilde{g}_{s,x} 
\]

\[
+ \sum_{n/2 \leq s \leq n-2k} \tilde{g}_{s} \left( k(n-s)^{\theta-2}(1-x)^{\theta} + \frac{k}{(n-s)^2} \right) 
\]

\[
\ll kn^{\theta-2}(1-x)^{\theta} \tilde{G}_x(e^{-1/n}) 
\]

\[
+ k \frac{\tilde{G}_x(e^{-1/n})}{n} \sum_{n/2 \leq s \leq n-2k} \left( (n-s)^{\theta-2}(1-x)^{\theta} + \frac{1}{(n-s)^2} \right) 
\]

\[
\ll kn^{\theta-2}(1-x)^{\theta} \tilde{G}_x(e^{-1/n}) \left( \frac{1-e^{-1/n}}{1-xe^{-1/n}} \right)^{\theta} 
\]

\[
+ k \frac{G_x(e^{-1/n})}{n} \left( \frac{1-e^{-1/n}}{1-xe^{-1/n}} \right)^{\theta} \left( k^{\theta-1}(1-x)^{\theta} + \frac{1}{k} \right). 
\]

Since \( 1 - xe^{-1/n} \geq 1 - x \), we have

\[
S_1 \ll \frac{k}{n^2} \frac{p(e^{-1/n})}{p(xe^{-1/n})} + \frac{1}{n} \frac{p(e^{-1/n})}{p(xe^{-1/n})} \left( \frac{k^n}{n} + \frac{1}{(n(1-x))^\theta} \right). 
\]

In a similar way we obtain

\[
S_2 + S_3 \ll \frac{1}{n} \tilde{G}_x(e^{-1/n}) \sum_{0 \leq l \leq 2k} c_{x,l} 
\]

\[
\ll \frac{1}{n} \frac{p(e^{-1/n})}{p(xe^{-1/n})} \left( \frac{1-e^{-1/n}}{1-xe^{-1/n}} \right)^{\theta} \left( \frac{1-xe^{-1/2k}}{1-e^{-1/2k}} \right)^{\theta} 
\]

\[
\ll \frac{1}{n} \frac{p(e^{-1/n})}{p(xe^{-1/n})} \left( \frac{1-x+x(1-e^{-1/2k})}{1-x} \right)^{\theta} \left( \frac{k^n}{n} \right)^{\theta} 
\]

\[
\ll \frac{1}{n} \frac{p(e^{-1/n})}{p(xe^{-1/n})} \left( 1 + \frac{1}{k(1-x)} \right)^{\theta} \left( \frac{k^n}{n} \right)^{\theta} 
\]

\[
\ll \frac{1}{n} \frac{p(e^{-1/n})}{p(xe^{-1/n})} \left( \frac{k^n}{n} + \frac{1}{(n(1-x))^\theta} \right). 
\]

Since \( p(xe^{-1/n}) \gg p(x) \) if \( 0 \leq x \leq e^{-1/n} \), the proof of the lemma follows. \( \square \)
Lemma 5. Suppose $u(x) = \exp \left\{ \sum_{k=1}^{\infty} \frac{u_k}{k} x^k \right\}$ and $0 \leq u_k \leq A$, then the following estimates hold:

1) $\int_0^{x/j} \frac{x^{j-1}}{u(x)} \, dx \ll \frac{1}{ju(e^{-x/j})}$ if $j \geq 1$;

2) $\int_0^{e^{-1/n}} \frac{x^{j-1}}{u(x)} \, dx \ll \frac{e^{-j/n}}{j u(e^{-1/n})}$ if $j \geq n$.

Proof. 1) For $j \geq 1$ we have

$$
\int_0^{x/j} \frac{x^{j-1}}{u(x)} \, dx = \int_0^{x/j} \frac{x^{j-1}}{u(x)} \, dx + \int_{x/j}^{1} \frac{x^{j-1}}{u(x)} \, dx \\
\leq \frac{1}{u(e^{-1/j})} \int_0^{x/j} \frac{u(e^{-1/j})}{u(x)} x^{j-1} \, dx + \frac{1}{u(e^{-1/j})} \\
\leq \frac{1}{u(e^{-1/j})} \int_0^{x/j} x^{j-1} \exp \left\{ \sum_{k=1}^{\infty} \frac{u_k}{k} (e^{-k/j} - x^k) \right\} \, dx + \frac{1}{j u(e^{-1/j})} \\
\leq \frac{1}{u(e^{-1/j})} \int_0^{x/j} x^{j-1} \left( \frac{1 - x}{1 - e^{-1/j}} \right)^A \, dx + \frac{1}{j u(e^{-1/j})} \\
\leq \frac{e^{A/j} \Gamma(A + 1) + 1}{j u(e^{-1/j})},
$$

here we have used the inequalities $e^{-y} y \leq 1 - e^{-y} \leq y$, for $y \geq 0$.

2) Suppose now that $j \geq n$, then

$$
\int_0^{e^{-1/n}} \frac{x^{j-1}}{u(x)} \, dx \ll \frac{1}{u(e^{-1/n})} \int_0^{e^{-1/n}} x^{j-1} \left( \frac{1 - x}{1 - e^{-1/n}} \right)^A \, dx \\
\leq \frac{e^{A/n} \Gamma(A + 1) + 1}{j u(e^{-1/n})},
$$

where we have used the inequalities $e^{-y} y \leq 1 - e^{-y} \leq y$, for $y \geq 0$.
since \( \int_w^\infty y^A e^{-y} \, dy \ll w^A e^{-w} \), as \( w \to \infty \).

The lemma is proved.

\[ \square \]

**Lemma 6.**

\[ \int_0^{e^{-1/n}} \left| g_{x,n} - \frac{p_n}{p(x)} \right| x^{j-1} \, dx \ll \frac{p(e^{-1/n})}{n} \left( \frac{j^\theta}{n^\theta} \right) \frac{1}{p(e^{-1/j})}, \]

when \( 1 \leq j \leq n \).

**Proof.** Since \( p(z) = p(xz) \frac{p(z)}{p(xz)} = p(xz)G_x(z) \), therefore \( p_n = \sum_{k=0}^n p_k x^k g_{n-k,x} \) and

\[ \left| g_{x,n} - \frac{p_n}{p(x)} \right| = \frac{1}{p(x)} \left| p(x)g_{n,x} - \sum_{k=0}^n p_k x^k g_{n-k,x} \right| \]

\[ \leq \frac{1}{p(x)} \left| \sum_{k=0}^n p_k x^k (g_{n,x} - g_{n-k,x}) \right| + \frac{g_{n,x}}{p(x)} \sum_{k>n} p_k x^k \]

\[ \leq \frac{1}{p(x)} \sum_{k \leq n/8} p_k x^k |g_{n,x} - g_{n-k,x}| + \frac{g_{n,x}}{p(x)} \sum_{k>n/8} p_k x^k \]

\[ + \frac{1}{p(x)} \sum_{n/8 < k \leq n} p_k x^k g_{n-k,x}, \]

Suppose \( 0 \leq x \leq e^{-1/n} \). Applying here Lemma 4 we have

\[ \left| g_{x,n} - \frac{p_n}{p(x)} \right| \ll \frac{p(e^{-1/n})}{np(x)^2} \sum_{k \leq n/8} p_k x^k \left( \left( \frac{k}{n} \right)^\theta + \frac{1}{(n(1-x))^\theta} \right) \]

\[ + \frac{G_x(e^{-1/n})}{np(x)} \sum_{k > n/8} p_k x^k + \frac{p(e^{-1/n}) x^{n/8}}{np(x)} G_x(e^{-1/n}) \]

\[ =: S_1(x) + S_2(x) + S_3(x). \]
1.1. VORONOI SUMS

Applying Lemma 5 with \( u(x) = p(x)^2 \) and \( u(x) = p(x)^2(1 - x)^\theta \), we have

\[
\int_0^{e^{-1/n}} S_1(x)x^{j-1} \, dx \\
= \frac{p(e^{-1/n})}{n} \sum_{k \leq n/8} p_k \left( \frac{k}{n} \right)^\theta \int_0^{e^{-1/n}} \frac{x^{k+j}}{p(x)^2} \, dx + \frac{1}{n^\theta} \int_0^{e^{-1/n}} \frac{x^{k+j} \, dx}{p(x)^2(1 - x)^\theta}
\
\ll \frac{p(e^{-1/n})}{n} \sum_{k \leq n/8} p_k \left( \frac{k}{n} \right)^\theta \left( \frac{1}{(k + j)p(e^{-1/(k+j)})^2} \right)
\ll \frac{p(e^{-1/n})}{n} \left( \frac{1}{j} \frac{(j/n)^\theta}{p(e^{-1/j})} \right) 1 \frac{1}{p(e^{-1/j})} \sum_{k \leq j} \frac{p_k}{k} \left( \frac{k}{n} \right)^\theta \frac{1}{p(e^{-1/k})^2}.
\]

Applying here inequality \( \sum_{k \leq j} p_k \ll e p(e^{-1/j}) \) in the first sum and the estimate (1.4) in the second one we have

\[
\int_0^{e^{-1/n}} S_1(x)x^{j-1} \, dx \\
\ll \frac{p(e^{-1/n})}{n} \left( \frac{1}{j} \frac{(j/n)^\theta}{p(e^{-1/j})} \right) \frac{1}{p(e^{-1/j})} + \frac{1}{p(e^{-1/j})} \sum_{k \geq j} \frac{1}{k^2} \left( \frac{k}{n} \right)^\theta \frac{p(e^{-1/j})}{p(e^{-1/k})}.
\]

Lemma 4 yields the estimate \( \frac{p(e^{-1/j})}{p(e^{-1/k})} \leq \left( \frac{k}{j} \right)^{d^+} e^{d^-/j} \), therefore

\[
\int_0^{e^{-1/n}} S_1(x)x^{j-1} \, dx \\
\ll \frac{p(e^{-1/n})}{n} \left( \frac{1}{j} \frac{(j/n)^\theta}{p(e^{-1/j})} \right) \frac{1}{p(e^{-1/j})} + \frac{1}{p(e^{-1/j})} \sum_{k \geq j} \frac{1}{k^2} \left( \frac{k}{n} \right)^\theta \left( \frac{j}{k} \right)^{d^-} \leq \frac{p(e^{-1/n})}{n} \left( \frac{j^{\theta-1}}{n^\theta} \right) \frac{1}{p(e^{-1/j})}.
\]

Let us now estimate \( S_2(x) \)

\[
S_2(x) \ll \frac{p(e^{-1/n})}{np(x)^2} \sum_{k > n/8} p_k x^k \leq \frac{p(e^{-1/n})x^{n/16}}{np(x)^2} p(\sqrt{x}) \ll \frac{p(e^{-1/n})x^{n/16}}{np(x)}.
\]
since \( p(y) \ll p(y^2) \) uniformly for \( 0 \leq y < 1 \). Hence, applying Lemma \( \mathcal{L} \) we have
\[
\int_0^{e^{-1/n}} S_2(x)x^{j-1} \, dx \ll \frac{p(e^{-1/n})}{n} \int_0^{e^{-1/n}} \frac{x^{n/16}}{p(x)} \, dx \ll \frac{1}{n^2}.
\]
In a similar way we obtain the estimate
\[
\int_0^{e^{-1/n}} S_3(x)x^{j-1} \, dx \ll \frac{p(e^{-1/n})^2}{n} \int_0^{e^{-1/n}} \frac{x^{n/8}}{p(x)^2} \, dx \ll \frac{1}{n^2}.
\]
Collecting the obtained estimates and noticing that
\[
\frac{p(e^{-1/n})}{n} \left( \frac{j^\theta - 1}{n^\theta} \right) \frac{1}{p(e^{-1/j})} \gg \frac{1}{nj} \geq \frac{1}{n^2},
\]
for \( 1 \leq j \leq n \), we obtain the proof of the lemma.

Let us define
\[
F_j(z) = p(z) \int_0^1 x^{j-1} \, dx = \sum_{m=0}^{\infty} f_{m,j} z^m.
\]
Denoting
\[
q(z) = \frac{1}{p(z)} = \sum_{m=0}^{\infty} q_m z^m
\]
we can easily see that
\[
F_j(z) = p(z) \int_0^1 x^{j-1} q(x) \, dx = \sum_{m=0}^{\infty} p_m z^m \sum_{s=0}^{\infty} \frac{q_s}{s+j} z^s = \sum_{m=0}^{\infty} f_{m,j} z^m,
\]
thus
\[
f_{m,j} = \sum_{s=0}^{m} \frac{p_m s^m}{s+j}.
\]
On the other hand,
\[
F_j(z) = \int_0^1 x^{j-1} G_x(z) \, dx = \sum_{m=0}^{\infty} z^m \int_0^1 g_{m,x} x^{j-1} \, dx = \sum_{m=0}^{\infty} f_{m,j} z^m,
\]
therefore
\[
f_{m,j} = \int_0^1 g_{m,x} x^{j-1} \, dx \geq 0.
\]
Then the following lemma holds
Lemma 7. We have
\[ f_{m,j} \ll \frac{1}{j^2} \text{ when } j \geq m \geq 1 \]
and \( f_{0,j} = \frac{1}{j} \).

Proof. Differentiating \( F_j(z) \) we can easily obtain, that
\[ zF'_j(z) = F_j(z) \sum_{k=1}^{\infty} d_k z^k + 1 - jF_j(z). \]
Putting here \( z = 0 \), we have \( f_{0,j} = 1/j \).

For \( m \geq 1 \), we have for \( j \geq m \)
\[ f_{m,j} = \frac{1}{m+j} \sum_{k=1}^{m} d_k f_{m-k,j} \ll \frac{d^+ e F_j(e^{-1/m})}{m+j} = \frac{d^+ e}{m+j} p(e^{-1/m}) \int_0^1 \frac{x^{j-1} dx}{p(xe^{-1/m})} \]
\[ = \frac{d^+ e}{m+j} p(e^{-1/m}) e^j/m \int_0^{e^{-1/m}} x^{j-1} dx = \frac{1}{j^2}, \]
here we have applied Lemma 5.

The lemma is proved.

Proof of Theorem 9. One can easily see, that
\[ \sum_{m=1}^{\infty} S(f; m) z^m = zf'(z)p(z), \]
therefore
\[ ma_m = \sum_{k=1}^{m} S(f; k) q_{m-k}, \]
where
\[ q(z) = \frac{1}{p(z)} = \sum_{m=0}^{\infty} q_m z^m. \]
Therefore we have
\[ \sum_{k=0}^{n} a_k p_{n-k} - p_n f(e^{-1/n}) \]
\[ = \sum_{k=1}^{n} p_{n-k} \frac{1}{k} \sum_{j=1}^{k} S(f; j) q_{k-j} - p_n \sum_{k=1}^{\infty} \frac{e^{-k/n}}{k} \sum_{j=1}^{k} S(f; j) q_{k-j} \]
\[ = \sum_{j=1}^{n} S(f; j) \sum_{k=j}^{n} \frac{p_{n-k} q_{k-j}}{k} - p_n \sum_{j=1}^{\infty} S(f; j) \sum_{k=j}^{\infty} \frac{e^{-k/j}}{k} q_{k-j} \]
\[ = \frac{S(f; n)}{n} + \sum_{j=1}^{n-1} S(f; j) \sum_{s=0}^{n-j} \frac{p_{n-j-s} q_s}{s + j} - p_n \sum_{j=1}^{\infty} S(f; j) \sum_{k=j}^{\infty} \frac{e^{-k/j}}{k} q_{k-j}, \]
recalling that
\[ f_{m,j} = \sum_{s=0}^{m} \frac{p_{m-s} q_s}{s + j} = \int_0^1 g_{m,x} x^{j-1} \, dx \]
we have
\[ \sum_{k=0}^{n} a_k p_{n-k} - p_n f(e^{-1/n}) - \frac{S(f; n)}{n} \]
\[ = \sum_{j=1}^{n-1} S(f; j) \left( f_{n-j,j} - p_n \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \right) \]
\[ + p_n \sum_{j=n}^{\infty} S(f; j) \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx. \]
Let us denote
\[ R_n = \sum_{k=0}^{n} a_k p_{n-k} - p_n f(e^{-1/n}) - \frac{S(f; n)}{n}, \]
then we have
\[ |R_n| \leq \sum_{1 \leq j \leq n/2} |S(f; j)| \left| \int_0^1 x^{j-1} g_{n-j,x} \, dx - p_{n-j} \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \right| \]
\[ + \sum_{1 \leq j \leq n/2} |S(f; j)||p_n - p_{n-j}| \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \]
\[ + \sum_{n/2 \leq j \leq n-1} |S(f; j)| f_{n-j,j} + p_n \sum_{j \geq n} |S(f; j)| \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx. \]
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Since \( g_{m,x} \leq ed^t \frac{G_{j(e^{-1/m})}}{m} = ed^t \frac{p(e^{-1/m})}{m} \), therefore we have

\[
\int_{e^{-1/n}}^1 x^{j-1} g_{n-j,x} \, dx \leq \frac{ep(e^{-1/(n-j)})}{n-j} \int_{e^{-1/n}}^1 \frac{dx}{p(xe^{-1/(n-j)})} \ll \frac{p(e^{-1/n})}{n^2} \frac{1}{p(e^{-1/n})} = \frac{1}{n^2},
\]

when \( j \leq n/2 \).

Applying here Lemma 5, Lemma 6 and Lemma 7, we have

\[
|R_n| \ll \sum_{1 \leq j \leq n/2} |S(f;j)| \frac{p(e^{-1/(n-j)})}{n-j} \left( \frac{j^{\theta-1}}{(n-j)^\theta} \right) \frac{1}{p(e^{-1/j})} + \sum_{1 \leq j \leq n/2} \frac{|S(f;j)|}{jp(e^{-1/j})} + \frac{1}{n^2} \sum_{n/2 \leq j \leq n-1} |S(f;j)|
\]

\[
+ p_n \sum_{j > n/2} |S(f;j)| e^{-j/n} \frac{e^{-j/n}}{jp(e^{-1/n})} \ll p_n \left( \frac{1}{n^\theta} \sum_{j=1}^n |S(f;j)| j^{\theta-1} + \frac{1}{p(e^{-1/n})} \sum_{j > n} \frac{|S(f;j)|}{j} e^{-j/n} \right).
\]

The theorem is proved.

**Proof of theorem [10]** 1) **Sufficiency.** Applying Theorem 9 one can easily see that condition 2 of Theorem 10 implies that

\[
\frac{1}{p_n} \sum_{k=0}^n a_k p_{n-k} = f(e^{-1/n}) + o(1).
\]

Condition 1 finally proves the sufficiency of conditions 1 and 2 of Theorem 10.

2) **Necessity.** Suppose now that

\[
\lim_{n \to \infty} \frac{1}{p_n} \sum_{k=0}^n a_k p_{n-k} = A.
\]

Let us denote

\[ w_n = \sum_{k=0}^n a_k p_{n-k}. \]

Under our assumption we have \( w_n = A p_n (1 + \epsilon_n) \), where \( \epsilon_n \to \infty \) as \( n \to \infty \).
We have

\[
\sum_{m=1}^{\infty} S(f; m)z^m = zf'(z)p(z) = z(p(z)f(z))' - zp'(z)f(z)
\]

\[
= z(p(z)f(z))' - p(z)f(z) \sum_{k=1}^{\infty} d_k z^k
\]

\[
= \sum_{n=1}^{\infty} nw_n z^n - \sum_{n=1}^{\infty} w_n z^n \sum_{k=1}^{\infty} d_k z^k.
\]

Hence we obtain

\[
S(f; n) = nw_n - \sum_{k=1}^{n} d_k w_{n-k}
\]

\[
= Anp_n - A \sum_{k=1}^{n} d_k p_{n-k} + Anp_n \epsilon_n - A \sum_{k=1}^{n} d_k p_{n-k} \epsilon_{n-k}
\]

\[
= Anp_n \epsilon_n - A \sum_{k=1}^{n} d_k p_{n-k} \epsilon_{n-k} = o(np_n),
\]

since \( np_n = \sum_{k=1}^{n} d_k p_{n-k} \) and \( np_n \gg p(e^{-1/n}) \to \infty \) as \( n \to \infty \).

The necessity of condition 2) is proved.

The necessity of condition 1) is well known, see e. g. [8]. It is obtained by noticing that

\[
f(x) = \frac{f(x)p(x)}{p(x)} = \frac{w_0 + w_2 x + \cdots + w_n x^n + \cdots}{p_0 + p_2 x + \cdots + p_n x^n + \cdots}.
\]

Putting here estimate \( w_n = Ap_n(1 + o(1)) \) and using the fact that \( p(x) \to \infty \) as \( x \not\to 1 \) we finally obtain

\[
f(x) \to A \quad \text{for} \quad x \not\to 1.
\]

The theorem is proved.

\[\square\]

1.2 Mean value theorems

Proof of Theorem 7. Let \( M_j \) be defined by

\[
F(z) = \exp \left\{ \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k} z^k \right\} = \sum_{n=0}^{\infty} M_n z^n,
\]
where \( \hat{f}(k) \in C, |\hat{f}(k)| \leq 1 \).

One can easily see that

\[
F(z) = p(z)m(z),
\]

where

\[
m(z) = \exp \left\{ \sum_{k=1}^{\infty} \frac{d_k(\hat{f}(k) - 1)}{k} z^k \right\} = \sum_{j=1}^{\infty} m_j z^j,
\]

then

\[
M_n = \sum_{k=0}^{n} p_k m_{n-k}.
\]

In the notations of Theorem 9 we have

\[
\sum_{j=1}^{\infty} S(m; j) z^j = zm'(z)p(z) = m(z)p(z) \sum_{k=1}^{\infty} z^k d_k(\hat{f}(k) - 1) = F(z) \sum_{k=1}^{\infty} z^k d_k(\hat{f}(k) - 1)
\]

hence we have

\[
S(m; j) = \sum_{k=1}^{j} d_k(\hat{f}(k) - 1) M_{j-k}.
\]

Since \( |\hat{f}(k)| \leq 1 \) implies \( |M_j| \leq p_j \), therefore

\[
|S(m; j)| \leq d^j \sum_{k=1}^{j} |\hat{f}(k) - 1| p_{j-k}.
\]

Since \( \hat{f}(k) \) for \( k > n \) do not influence \( M_n \) we assume that \( \hat{f}(k) = 1 \) for \( k > n \).

Applying here Theorem 9 with \( f(z) = m(z) \) we have

\[
\left| \frac{M_n}{p_n} - m(e^{-1/n}) \right| = \left| \frac{M_n}{p_n} - \exp \left\{ \sum_{k=1}^{n} \frac{d_k(\hat{f}(k) - 1)}{k} e^{-k/n} \right\} \right|
\]

\[
\leq c \left( \frac{1}{np_n} \sum_{k=1}^{n} |\hat{f}(k) - 1| p_{n-k} + \frac{1}{n^\theta} \sum_{j=1}^{n} \frac{j^{j-1}}{p(e^{-1/j})} \sum_{k=1}^{j} |\hat{f}(k) - 1| p_{j-k} \right.
\]

\[
+ \frac{1}{p(e^{-1/n})} \sum_{j>n} \frac{e^{-j/n}}{j} \sum_{k=1}^{j} |\hat{f}(k) - 1| p_{j-k} \right).
\]
here \( \theta = \min\{1, d^-\} \) and \( c = c(d^+, d^-) \). We have

\[
\frac{1}{n^\theta} \sum_{j=1}^{n} \frac{j^{\theta-1}}{p(e^{-1/j})} \sum_{k=1}^{j} |\hat{f}(k) - 1| p_{j-k} = \frac{1}{n^\theta} \sum_{k=1}^{n} j^{\theta-1} |\hat{f}(k) - 1| \frac{1}{p(e^{-1/j})} p_{j-k}
\]

\[
\ll \frac{1}{n^\theta} \sum_{k=1}^{n} |\hat{f}(k) - 1| \sum_{n \gg j \gg 2k} j^{\theta-1} \frac{p_{j-k}}{p(e^{-1/j})}
\]

\[
\ll \frac{1}{n^\theta} \sum_{k=1}^{n} |\hat{f}(k) - 1| \left( j^{\theta-1} + \int_{k}^{n} x^{\theta-2} dx \right)
\]

and

\[
\frac{1}{p(e^{-1/n})} \sum_{j>n} e^{-j/n} \sum_{k=1}^{n} |\hat{f}(k) - 1| p_{j-k} = \frac{1}{p(e^{-1/n})} \sum_{k=1}^{n} |\hat{f}(k) - 1| \sum_{j>n} \frac{e^{-j/n}}{j} p_{j-k}
\]

\[
\leq \frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - 1| e^{-k/n} \sum_{j>n} e^{-(j-k)/n} p_{j-k} \leq \frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - 1|.
\]

The theorem is proved.

Let us define

\[ L_n(z) = \sum_{j=1}^{n} d_j \frac{\hat{f}(j) - 1}{j} z^j \text{ and } \rho(p) = \left( \sum_{k=1}^{n} \frac{|\hat{f}(j) - 1|^p}{j} \right)^{1/p}, \]

moreover we assume that

\[ \rho(\infty) = \lim_{p \to \infty} \rho(p) = \max_{1 \leq k \leq n} |\hat{f}(k) - 1|. \]

**Lemma 8.** For \( m, n \geq 1 \) and \( 1/p + 1/q = 1 \) with \( \infty \geq p > 1 \), we have

\[ |L_n(1) - L_m(1)| \leq d^+ \rho(p) \left| \log \frac{n}{m} \right|^{1/q}. \]
Proof. Suppose \( n \geq m \) and \( p < \infty \) then applying Cauchy’s inequality with parameters \( p, q \) we have

\[
|L_n(1) - L_m(1)| \leq d^+ \sum_{m < j \leq n} \left| \frac{\hat{f}(j) - 1}{j} \right| \leq d^+ \left( \sum_{m < j \leq n} \left| \frac{\hat{f}(j) - 1}{j} \right|^p \right)^{\frac{1}{p}} \left( \sum_{m < j \leq n} \frac{1}{j} \right)^{\frac{1}{q}}
\]

\[
\leq d^+ \left( \sum_{1 \leq j \leq N} \left| \frac{\hat{f}(j) - 1}{j} \right|^p \right)^{\frac{1}{p}} \left( \int_{m}^{n} \frac{dx}{x} \right)^{\frac{1}{q}} \leq d^+ \rho(p) |\log \frac{n}{m}|^{-\frac{1}{q}}.
\]

Allowing \( p \to \infty \) we see that this inequality is true for \( p = \infty \) also.

The lemma is proved.

Lemma 9. For \( n \geq 1 \) and \( q(d^--1) > -1 \), we have

\[
\sum_{j=1}^{n} \frac{1}{j} \left| \frac{p_{n-j}}{p_n} - 1 \right|^q \ll 1.
\]

Proof. We have

\[
\sum_{j=1}^{n} \frac{1}{j} \left| \frac{p_{n-j}}{p_n} - 1 \right|^q \ll \sum_{1 \leq j \leq n/4} \frac{1}{j} \left| \frac{p_{n-j}}{p_n} - 1 \right|^q + \frac{1}{n} \sum_{n/4 \leq j \leq n} \left| \frac{p_{n-j}}{p_n} \right|^q + 1,
\]

applying Lemma 2 and Lemma 1 together with (1.4) and (1.5) we have

\[
\sum_{j=1}^{n} \frac{1}{j} \left| \frac{p_{n-j}}{p_n} - 1 \right|^q \ll \sum_{1 \leq j \leq n/4} \frac{1}{j} \left| \frac{p_{n-j}}{p_n} - 1 \right|^q + \frac{1}{n} \sum_{1 \leq j \leq 3n/4} \left| \frac{p(e^{-1/s})}{p(e^{-1/n})} n \right|^q + 1,
\]

\[
\ll 1 + \frac{1}{n} \sum_{s \leq n/4} \left( \frac{s}{n} \right)^{d^-} \frac{n}{s} \ll 1 + \frac{1}{n} \sum_{1 \leq s \leq 3n/4} \left| \frac{s}{n} \right|^{q(d^-)-1} \ll 1.
\]

The lemma is proved.

The following theorem has been proved for \( d_j \equiv 1 \) by E. Manstavičius [13], later generalized for \( d_j \equiv \theta > 0 \) in [21].

Theorem 12. For any fixed \( \infty \geq p > \max \{1, 1/d^-\} \), there exists such a positive \( \delta = \delta(d^-, d^+, p) \) that, if \( \rho \leq \delta \), then

\[
\frac{M_N}{p_N} = \exp\{L_N(1)\} \left( 1 + \sum_{j=1}^{N} d_j \frac{\hat{f}(j) - 1}{j} \left( \frac{p_{N-j}}{p_N} - 1 \right) + O(p^2) \right). \tag{1.6}
\]
CHAPTER 1. ADDITIVE AND MULTIPLICATIVE FUNCTIONS ON $S_N$

Proof. We will assume during this proof that $\hat{f}(j) = 1$, if $j > N$ and as before $M_j$ will be defined by (1.1). We will suppose that $R_m$ are complex numbers satisfying the relation

$$\frac{M_m}{p_m} = \exp\{L_m(1)\} \left(1 + \sum_{j=1}^{m} d_j \frac{\hat{f}(j) - 1}{j} \left(\frac{p_{m-j}}{p_m} - 1\right) + R_m\right). \quad (1.7)$$

We set

$$R := \sup_{m \geq 0} |R_m|.$$

One can easily see that $R$ is finite since $L_k(z) = L_N(z)$ if $k \geq N$ and $|M_m| \leq p_m$.

Suppose $h_n(z) = \exp\{L_N(z)\} - \exp\{L_n(1)\} L_N(z)$, where $n \geq 1$. It is easy to see that the generating function of $S(h_n; j)$ is

$$\sum_{j=1}^{\infty} S(h_n; j) z^j = z h_n'(z) p(z) = F_N(z) z L'_N(z) - \exp\{L_n(1)\} z L'_N(z) p(z),$$

therefore

$$S(h_n; m) = \sum_{k=1}^{m} d_k (\hat{f}(k) - 1) (M_{m-k} - p_{m-k} \exp\{L_n(1)\}).$$

Inserting here expression (1.7) of $M_n$, we get

$$|S(h_n; m)| \leq \sum_{k=1}^{m} d_k |\hat{f}(k) - 1| |\exp\{L_{m-k}(1)\} - \exp\{L_n(1)\}| p_{m-k}$$

$$+ d^+ \rho \sum_{k=1}^{m-1} d_k |\hat{f}(k) - 1| |\exp\{L_{m-k}(1)\}| \left(\sum_{j=1}^{m-k} \frac{1}{j} \left|\frac{p_{m-k-j}}{p_{m-k}} - 1\right|^q\right)^{1/q} p_{m-k}$$

$$+ R \sum_{k=1}^{m-1} d_k |\hat{f}(k) - 1| |\exp\{L_{m-k}(1)\}| p_{m-k}.$$

Here we have applied Cauchy’s inequality with parameters $1/p + 1/q = 1$ and used the fact that $R_0 = 0$.

Applying once more Cauchy inequality, we have

$$|S(h_n; m)| \ll \rho m^{1/p} \left(\sum_{k=0}^{m-1} |\exp\{L_k(1)\} - \exp\{L_n(1)\}|^q p_k^q\right)^{1/q}$$

$$+ (\rho + R) \rho m^{1/p} \left(\sum_{k=1}^{m-1} |\exp\{L_k(1)\}|^q p_k^q\right)^{1/q}.$$
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Applying here lemma 8 we have

\[ |S(h_n; m)| \ll (\rho^2 + \rho R) m^{1/p} \exp\{L_n(1)\} \left( \exp\{q \delta d^+ \log n^{1/q}\} \right. \]

\[ \left. + \sum_{k=1}^{m-1} \left( 1 + \left| \log \frac{n}{k} \right| \right) \exp \left\{ q \delta d^+ \left| \log \frac{n}{k} \right|^{1/q} \right\} p_k^q \right)^{1/q}. \]

Here and further the constant in symbol \( \ll \) will depend on \( \theta \) and \( p \) only. As \( q(d^- - 1) > -1 \) so we can chose such a positive \( \epsilon = \epsilon(p, d^-, d^+) < \theta \), that \( q(d^- - 1) - \epsilon > -1 \). When \( \theta = 1 \) we might take, for example, \( \epsilon = \frac{1}{2} \) and for \( d^- < 1 \) we might take \( \epsilon = \min\left\{ \frac{1 + q(d^- - 1)}{2}, \frac{1}{2} \right\} \). It is easy to see that there exists such a positive constant \( C_\epsilon \), that inequality

\[ (1 + |\log x|) \exp \left\{ \epsilon/2 \right\} \exp |\log x|^{1/q} \leq C_\epsilon \left( x^\epsilon \right. \]

\[ \left. + \frac{1}{x^\epsilon} \right), \]

holds for \( x > 0 \). Supposing that \( \rho < \delta \leq \frac{\epsilon}{2q\theta} \) and applying this inequality we get

\[ |S(h_n; m)| \ll (\rho^2 + \rho R) m^{1/p} \exp\{L_n(1)\} \left( 1 + \sum_{k=1}^{m-1} \left( \left( \frac{n}{k} \right)^\epsilon + \left( \frac{k}{n} \right)^{\epsilon/k} \right) p_k^q \right)^{1/q} \]

\[ \ll (\rho^2 + \rho R) m^{1/p} \exp\{L_n(1)\} \left( 1 + p(e^{-1/m}) \sum_{k=1}^{m-1} \left( \left( \frac{n}{k} \right)^\epsilon + \left( \frac{k}{n} \right)^{\epsilon/k} \right) \left( \frac{p(e^{-1/k})}{kp(e^{-1/m})} \right)^q \right)^{1/q} \]

\[ \ll (\rho^2 + \rho R) m^{1/p} \exp\{L_n(1)\} \left( 1 + p(e^{-1/m}) \sum_{k=1}^{m-1} \left( \left( \frac{n}{k} \right)^\epsilon + \left( \frac{k}{n} \right)^{\epsilon/k} \right) \right)^{1/q} \]

here we have estimated \( p_k \) by means of (1.4) and have applied Lemma 1. Estimating the sums occurring in this estimate we have

\[ |S(h_n; m)| \ll |\exp\{L_n(1)\}| \rho(\rho + R) p e^{-1/m} \left( \left( \frac{n}{m} \right)^{\epsilon/q} + \left( \frac{m}{n} \right)^{\epsilon/q} \right), \quad (1.8) \]

Applying Theorem 9 with \( f(z) = h_n(z) \) and using the estimate (1.8) we have

\[ \frac{M_n}{p_n} - \exp\{L_n(1)\} \sum_{j=1}^{n} d_j \frac{\hat{f}(j) - 1}{j} p_{n-j} - \]

\[ - (\exp\{L_N(e^{-1/n})\} - \exp\{L_n(1)\} L_N(e^{-1/n})) \ll \rho(\rho + R) |\exp\{L_n(1)\}|. \]
Inserting here estimates
\[
\exp\{L_N(e^{-1/n})\} = \exp\{L_n(e^{-1/n})\} \left(1 + (L_N(e^{-1/n}) - L_n(e^{-1/n})) + O(\rho^2)\right)
\]
and
\[
\exp\{L_n(e^{-1/n})\} = \exp\{L_n(1)\} \left(1 + (L_n(e^{-1/n}) - L_n(1)) + O(\rho^2)\right),
\]
we obtain
\[
\frac{M_n - \exp\{L_n(1)\}}{p_n} \left(1 + \sum_{j=1}^{n} d_j \frac{\hat{f}(j) - 1}{j} \frac{p_{n-j}}{p_n} - 1\right)
\ll \exp\{L_n(1)\} |\rho(\rho + R)|.
\]
Recalling the definition of \( R_n \), we see that there exists such a constant \( A = A(d^-, d^+, p) \), that
\[
|\exp\{L_n(1)\}| |R_n| \leq A(d^-, d^+, p) |\exp\{L_n(1)\}| \rho(\rho + R).
\]
Dividing each part of this inequality by \( \exp\{L_n(1)\} \) and taking the supremum by \( n \) we obtain
\[
R \leq A(d^-, d^+, p)(\rho^2 + \rho R).
\]
Supposing now that \( \delta = \min \left\{ \frac{1}{2A(d^-, d^+, p)}, \frac{\epsilon}{2 \rho d^-} \right\} \), we have
\[
R \leq 2A(d^-, d^+, p) \rho^2.
\]
The theorem is proved. \( \square \)

For \( u > 0 \) we define
\[
E(u) := \exp \left\{ 2 \sum_{k=1}^{n} \frac{|\hat{f}(k) - 1|}{k} \text{ for } |\hat{f}(k) - 1| > u \right\}
\]

**Theorem 13.** There exists such a constant \( \eta = \eta(d^-, d^+) \) that for any \( u \leq \eta \) we have
\[
|M_n p_n^{-1}| \ll |\exp \{L_n(1)\}| (E(u))^{d^+},
\]
**Proof.** Let us denote for \( u > 0 \)
\[
\hat{f}_u(j) = \begin{cases} 
1, & \text{if } |\hat{f}(j) - 1| > u; \\
\hat{f}(j), & \text{if } |\hat{f}(j) - 1| \leq u,
\end{cases}
\]
and

\[ F_u(z) := \exp \left\{ \sum_{j=1}^{\infty} d_j \frac{\hat{f}_u(j)}{j} z^j \right\} = \exp \{ L_n^{(u)}(z) \} p(z) = \sum_{k=0}^{\infty} M_k^{(u)} z^k \]

here

\[ L_n^{(u)}(z) = \sum_{j=1}^{n} d_j \frac{\hat{f}_u(j) - 1}{j} z^j = \sum_{j \leq n} d_j \frac{\hat{f}(j) - 1}{j} z^j. \]

Then

\[ F(z) = F_u(z) \exp \left\{ \sum_{|\hat{f}(j)-1| > u} d_j \frac{\hat{f}(j) - 1}{j} z^j \right\}. \]

Therefore

\[ M_n = \sum_{k=0}^{n} M_k^{(u)} m_{n-k}^{(u)}, \]

where \( m_k^{(u)} \) are defined by

\[ m^{(u)}(z) = \sum_{j=0}^{\infty} m_j^{(u)} z^j = \exp \left\{ \sum_{|\hat{f}(j)-1| > u} d_j \frac{\hat{f}(j) - 1}{j} z^j \right\}. \]

Differentiating \( m^{(u)}(z) \) one can easily see that \( m_k \) satisfy the recurrent relationship

\[ m_j^{(u)} = \frac{1}{j} \sum_{1 \leq k \leq j} d_j (\hat{f}(j) - 1) m_{j-k}^{(u)}, \]

for \( j \geq 1 \). From which we have

\[ |m_j^{(u)}| \leq \frac{2d^+}{j} \sum_{|\hat{f}(j)-1| > u} \left| d_j (\hat{f}(j) - 1) m_{j-k}^{(u)} \right| \leq \frac{2d^+}{j} \sum_{k=0}^{\infty} |m_k^{(u)}| = \frac{2d^+}{j} \sum_{j=0}^{\infty} |m_j^{(u)}| \leq \frac{2d^+}{j} \exp \left\{ d^+ \sum_{|\hat{f}(j)-1| > u} \left| \frac{\hat{f}(j) - 1}{j} \right| \right\} = \frac{2d^+}{j} E^{d^+/2}(u). \]

Suppose that \( \eta \leq \delta(d^-, d^+, \infty) \), then Theorem 12 implies that \( M_k^{(u)} = p_k \exp \{ L_k^{(u)}(1) \} (1 + O(\eta)) \) and we have
\[ |M_n| = \sum_{k=0}^{n} |M_k^{(u)}m_{n-k}^{(u)}| \]
\[ \ll E^{d+/2}(u) \sum_{k \leq n/2} \frac{p_k}{n-k} |\exp\{L_k^{(u)}(1)\}| + |\exp\{L_n^{(u)}(1)\}|p_n\left(\sum_{k=0}^{\infty} |m_k^{(u)}| \right) \]
\[ \ll E^{d+/2}(u)|\exp\{L_n^{(u)}(1)\}||E^{d+/2}(u)| \left(\frac{1}{n} \sum_{k \leq n/2} \frac{p}{k} \left(\frac{e^{-1/k}}{n/k}\right)^{ud^+} + 1 \right) \]
\[ \ll \frac{p(e^{-1/n})}{n} |\exp\{L_n^{(u)}(1)\}||E^{d+/2}(u)|E^{d+/2}(u) \left(\sum_{k \leq n/2} \frac{1}{k} \left(\frac{e^{-1/n}}{k}\right)^{ud^+-d^-} + 1 \right) , \]

here we have used the estimate \(|\exp\{\{L_k^{(u)}(1) - L_n^{(u)}(1)\}\}| \leq \left(\frac{u}{k}\right)^{ud^+} \) for \( k \leq n \) and Lemma 1. Assuming that \( \eta \) fixed such that \( u \leq \eta < \frac{d^-}{d^+} \), we obtain
\[ |M_n| \ll p_n|\exp\{L_n^{(u)}(1)\}|E^{d+/2}(u) \leq p_n|\exp\{L_n^{(1)}\}|E^{d+/2}(u) . \]
Thus we have that the theorem holds with \( \eta = \min\{\delta(d^-, d^+, \infty), d^+/2d^-(\}\). \( \square \)

**Proof of theorem**. Putting \( \hat{f}(k) = e^{ith_n(k)} \) in Theorem 12, for \( |t| \leq \delta L_n^{1/p} \)

we have
\[ \phi_n(t) := \int_{-\infty}^{\infty} e^{itx} dF_n(x) \]
\[ = e^{-itA(n)} \exp\{L_n(1)\} \left(1 + \sum_{j=1}^{n} d_j \frac{\hat{f}(j) - 1}{j} \left(\frac{p_{n-j}}{p_n} - 1 \right) + O(p^2) \right) \]
\[ = e^{-t^2/2 + O(|t|^4L_{3,n})} \left(1 + C_n it + O\left(|t|^2 \frac{n}{p_n} \left(\frac{p_{n-j}}{p_n} - 1 \right) \right) + O\left(|t|^2 L_n^{2/p} \right) \right) \]
\[ = e^{-t^2/2} \left(1 + C_n it + O\left(|t|^2(1 + |t|^3)(L_n^{2/p} + L_{n,3} + L_{n,2}) \right) \right) , \]
\[ (1.9) \]

As in [11], from Theorem 13 we deduce the existence of some sufficiently small \( c = c(d^-, d^+) \), that if \( |t| \leq cL_n^{-1} =: T \) then
\[ |\phi(t)| \ll e^{-ct^2} , \]
\[ (1.10) \]
here $c_1 = c_1(d^-, d^+)$ is some fixed positive constant. Applying the generalized Eseen inequality (see for example [18]), we obtain

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} C_n \right| \ll \int_{-T}^{T} \frac{\phi_n(t) - e^{-t^2/2} (1 + C_n it)}{|t|} dt + \frac{1}{T}.$$  

Representing the integral on the right hand side of this inequality as a sum of integrals over the intervals $|t| \leq \delta L_{n,p}^{-1/p}$ and $\delta L_{n,p}^{-1/p} < |t| \leq T$ and applying estimates (1.9) and (1.10) in those intervals we obtain the proof of the theorem.

The theorem is proved.

### 1.3 Example of a special function

Let us consider now the uniform probability measure $\nu_{n,1}$ on $S_n$. As noted in [10], the Cauchy inequality yields that $L_n \gg \frac{1}{\sqrt{\log n}}$ for any sequence $\hat{h}_n(k)$ satisfying the normalizing condition (6). Thus, the convergence rate is at most of the logarithmic order.

In the work [2] G. J. Babu and E. Manstavičius investigated the additive function $h_n(\sigma)$ with $\hat{h}_n(j) = d(j) (j/n)^{1/2}$, where

$$d(j) = \begin{cases} 
\Phi^{-1} (\{j\sqrt{2}\}), & \text{if } |\Phi^{-1}(\{j\sqrt{2}\})| \leq \log j, \\
0, & \text{in other cases.}
\end{cases}$$

Here $\Phi(x)$ is the standard normal distribution and $\{x\}$ is the fractional part of a real number $x$. They showed that, although for this function $L_n \gg 1$, distribution converges to the normal law. Now the summands corresponding to the long cycles are not negligible for the asymptotic distribution. The in this section we will estimate the convergence rate of the distribution $F_n(x) = \nu_n(h_n(\sigma) < x)$ of this special additive function.

**Theorem 14.** There exists a positive constant $c > 0$ such that

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \ll \frac{1}{n^c}.$$  

The proof allows to obtain some numerical estimate of $c$.

**Lemma 10.** Suppose $F(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic for $|z| < 1$. Then we have for $T \geq 2$

$$\sum_{k=0}^{n} a_k = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} e^s F(e^{-s/n}) \frac{ds}{s} + O \left( \sum_{k=0}^{\infty} \frac{|a_k| e^{-k/n}}{1 + T |\frac{k}{n} - 1|} \right).$$
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Proof. The proof of this result is the same as that of Peron’s formula for Dirichlet series (see e. g. \[19\]).

Applying the well known estimates

$$\frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{xs}}{s} \, ds = \begin{cases} 1 + O \left( \frac{e^{|x|}}{T} \right) & \text{if } x > 0, \\ O \left( \frac{e^{|x|}}{T} \right) & \text{if } x < 0 \end{cases} \quad (1.11)$$

and noticing that

$$\frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{xs} F(e^{-s/n})}{s} \, ds = \sum_{k=0}^{\infty} a_k \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{s(1-k/n)}}{s} \, ds$$

we have

$$\frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{xs} F(e^{-s/n})}{s} \, ds = \sum_{k=0}^{\infty} a_k \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{s(1-k/n)}}{s} \, ds$$

Estimating $I_1$ and $I_3$ we use (1.11). Estimating $I_2$ by means of (1.12), we finally obtain the desired result.

The lemma is proved. \qed

Suppose $k_j(\sigma)$ is the number of cycles in the $\sigma$ whose length is equal to $j$. Then obviously $\sum_{j=1}^{n} j k_j(\sigma) = n$, and any additive function can be represented as $h_n(\sigma) = \sum_{j=1}^{n} \hat{h}_n(j) k_j(\sigma)$. Therefore

$$\phi_n(t) = M_n e^{it\sigma} = \frac{1}{n!} \sum_{\sigma \in S_n} e^{it\sigma} = \sum_{s_1+2s_2+\cdots+ns_n = n} \prod_{j=1}^{n} \left( \frac{e^{ith_j(j)}}{j^{s_j}} \right) \frac{1}{s_j!},$$

here we have used the well known fact that

$$\nu(k_1(\sigma) = s_1, k_2(\sigma) = s_2, \ldots, k_n(\sigma) = s_n) = \prod_{j=1}^{n} \frac{1}{j^{s_j}s_j!},$$
for \( s_1 + 2s_2 + \cdots + ns_n = n \). Hence
\[
1 + \sum_{n=1}^{\infty} \phi_n(t) z^n = \exp \left\{ \sum_{j=1}^{\infty} \frac{e^{it \hat{h}_n(j)}}{j} z^j \right\}.
\]
Suppose
\[
F(z) = \exp \left\{ \sum_{k=1}^{\infty} \frac{\hat{f}(k)}{k} z^k \right\} = \frac{1}{1 - z} \exp \left\{ \sum_{k=1}^{\infty} \frac{\hat{f}(k) - 1}{k} z^k \right\} = \sum_{k=0}^{\infty} M_k z^k.
\]
Let us define \( m_j \) by relationship
\[
m(z) = \exp \left\{ \sum_{k=1}^{\infty} \frac{\hat{f}(k) - 1}{k} z^k \right\} = \sum_{k=0}^{\infty} m_k z^k.
\]
Hence we have \( M_n = m_0 + m_1 + \cdots + m_n \).

**Lemma 11.** Suppose \( 2 \leq T \leq n \) and \( |\hat{f}(j)| \leq 1 \), then
\[
M_n = \frac{1}{2\pi i} \int_{1 iT}^{1 + iT} \frac{e^s}{s} \exp \left\{ \sum_{k=1}^{n} \frac{\hat{f}(k) - 1}{k} e^{-sk/n} \right\} \frac{ds}{T} \exp \left\{ \log \frac{T}{\log \frac{T}{T}} \right\}.
\]

**Proof.** Since \( \hat{f}(j) \) for \( j > n \) do not influence the value of \( M_n \) we may assume that \( \hat{f}(j) = 1 \) for \( j > n \). One can easily see that
\[
\sum_{k=0}^{\infty} |m_k| e^{-k/n} \leq \exp \left\{ \sum_{k=1}^{n} \frac{|\hat{f}(k) - 1|}{k} e^{-k/n} \right\}. \tag{1.13}
\]
Differentiating \( m(z) \) one can easily verify that \( m_j \) satisfy the recurrence relationship
\[
m_N = \frac{1}{N} \sum_{k=1}^{N} (\hat{f}(j) - 1)m_{N-k},
\]
for \( N \geq 1 \). Hence, recalling that \( |\hat{f}(j)| \leq 1 \), we have
\[
|m_N| \leq 2 \frac{N}{N} \sum_{k=1}^{N} |m_{N-k}| \leq \frac{2}{N} \exp \left\{ \sum_{k=1}^{n} \frac{|\hat{f}(k) - 1|}{k} \right\}. \tag{1.14}
\]
These estimates yield
\[
\sum_{k=0}^{\infty} \frac{|m_k| e^{-k/n}}{1 + T |\frac{k}{n} - 1|} \ll \sum_{n/2 < k < 3n/2} \frac{|m_k| e^{-k/n}}{1 + T |\frac{k}{n} - 1|} + \frac{1}{T} \sum_{k=0}^{\infty} |m_k| e^{-k/n}
\ll \exp \left\{ \sum_{k=1}^{n} \frac{|\hat{f}(k) - 1|}{k} \right\} \left( \frac{\log T}{T} + \frac{1}{T} \right)
\]
here we have applied (1.13) to estimate the sum over \( k \) such that \(|n-k| \geq n/2\) and used (1.14) to estimate the sum over \( k \) such that \( n/2 < k < 3n/2\).

Applying now lemma 10 with \( a_n = m_n \) together with the last estimate we complete the proof of the lemma.

The lemma is proved.

Proof of Theorem 14. In the work [2] it has been shown that

\[
\sum_{1 \leq j \leq n} \frac{1}{d(j)} 1 
\leq \Phi(x) + O(n^{-1/2}) \quad \text{and} \quad \sum_{j=1}^{n} |d(j)| = \int_{-\infty}^{\infty} |x| d\Phi(x) + O\left(\log^2 \frac{n}{\sqrt{n}}\right).
\]

Let us denote \( \hat{f}(j) = \exp\left\{itd\left(\frac{j}{n}\right)\right\} \) for \( 1 \leq j \leq n \). Then we have for \( 1 < |t| \leq \sqrt{n} \)

\[
\sum_{j=1}^{n} \frac{|\hat{f}(j) - 1|}{j} \leq |t| \sum_{1 \leq j < \frac{n}{|t|}} \frac{|d(j)|}{j} \left(\frac{j}{n}\right)^{1/2} + 2 \sum_{\frac{n}{|t|} \leq j \leq n} \frac{1}{j} \leq 4 \log |t| + O(1).
\]

For \( |t| \leq 1 \), we have \( \sum_{j=1}^{n} \frac{|\hat{f}(j) - 1|}{j} = O(|t|) \).

Using these estimates together with lemma 11 we have

\[
M_n = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^z}{z} \exp\left\{S_n(t, z)\right\} \, dz + O\left(\frac{\log T}{T} (1 + |t|^4)\right),
\]

(1.15)

here, as in [2], we denote \( S_n(t, z) = \sum_{j=1}^{n} \exp\left\{itd\left(\frac{j/n}{j}\right)^{1/2}\right\} - 1 e^{-zj/n} \) where \( z = 1 + iu \), and \( u \in \mathbb{R} \). Further we assume that \( p \) is a natural number such that \( p < n \) and we put \( H = n/p \). We have

\[
S_n(t, z) = \sum_{k=1}^{H-1} \sum_{Hk < j < H(k+1)} \frac{\exp\left\{itd\left(\frac{j/n}{j}\right)^{1/2}\right\} - 1}{j} e^{-zj/n} + O\left(\frac{|t|}{p^{1/2}}\right),
\]

\[
= \sum_{k=1}^{H-1} \sum_{Hk < j < H(k+1)} \frac{\exp\left\{itd\left(\frac{Hk/n}{Hk}\right)^{1/2}\right\} - 1}{Hk} e^{-zHk/n}
\]

+ \( R_n + O\left(\frac{|t|}{p^{1/2}}\right)\),
where

\[
R_n \ll \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left| \frac{\exp\left\{ \text{itd}(j)(j/n)\right\}^{1/2} - 1}{j} \right| |e^{-zj/n} - e^{-zHk/n}| \\
+ \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left| \frac{\exp\left\{ \text{itd}(j)(Hk/n)\right\}^{1/2} - 1}{Hk} \right| \exp\left\{ \text{itd}(j)(Hk/n)\right\}^{1/2} - 1 |j-Hk| \\
=: R_{n1} + R_{n2}.
\]

Now

\[
R_{n1} \ll \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left| \frac{z}{j} \right| \frac{j-Hk}{n} \ll \left| \frac{z}{n} \right| \sum_{k=1}^{p-1} \frac{1}{Hk} \sum_{Hk < j \leq H(k+1)} |j-Hk| \\
\ll \left| \frac{z}{n} \right| H \log p = \left| z \right| \frac{\log p}{p}.
\]

In a similar way we obtain

\[
R_{n2} \ll \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left| \exp\left\{ \text{itd}(j)(j/n)\right\}^{1/2} - \exp\left\{ \text{itd}(j)(Hk/n)\right\}^{1/2} \right| \frac{1}{j} \\
+ \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left| \exp\left\{ \text{itd}(j)(Hk/n)\right\}^{1/2} - 1 \right| \left| \frac{1}{j} - \frac{1}{Hk} \right| =: R'_{n2} + R''_{n2}
\]

We have

\[
R'_{n2} \ll |t| \left| \sum_{k=1}^{p-1} \left( \frac{Hk}{n} \right)^{1/2} \sum_{Hk < j \leq H(k+1)} \frac{|d(j)|}{j} \left( \frac{j}{Hk} \right)^{1/2} - 1 \right| \\
\ll |t| \left| \sum_{k=1}^{p-1} \left( \frac{Hk}{n} \right)^{1/2} \frac{1}{Hk} \sum_{Hk < j \leq H(k+1)} |d(j)| \right| \\
\ll |t| \log n \left( \frac{H}{n} \right)^{1/2} = |t| \frac{\log n}{p^{1/2}}.
\]
since $|d(j)| \leq \log j$. Using similar considerations we obtain

$$R''_{n2} \ll |t| \sum_{k=1}^{p-1} \sum_{Hk < j \leq H(k+1)} \left( \frac{Hk}{n} \right)^{1/2} |d(j)| \left| \frac{1}{j} - \frac{1}{Hk} \right|$$

$$\ll |t| \log n \sum_{k=1}^{p-1} \left( \frac{Hk}{n} \right)^{1/2} \frac{1}{(Hk)^2} \sum_{Hk < j \leq H(k+1)} |j - Hk|$$

$$\ll |t| \log n \sum_{k=1}^{p-1} \left( \frac{H}{n} \right)^{1/2} \frac{1}{k^{3/2}} \ll |t| \frac{\log n}{p^{1/2}}.$$

Therefore

$$R_n \ll |z| \frac{\log p}{p} + |t| \frac{\log n}{p^{1/2}}.$$

One can see that

$$V_k(x) := \frac{1}{H} \sum_{Hk < j \leq H(k+1)} \frac{1}{d(j)^{1/2}} = \Phi(x) + \alpha_k(x),$$

where $|\alpha_k(x)| = O(H^{-1/2})$, because

$$\frac{1}{H} \sum_{Hk < j \leq H(k+1)} \frac{1}{d(j)^{1/2}} = \frac{1}{H} \sum_{0 < s \leq H} 1 + O \left( \frac{1}{H} \right) = x + O \left( \frac{1}{H^{1/2}} \right),$$

for $0 < x < 1$. Now we can estimate

$$\frac{1}{H} \sum_{Hk < j \leq H(k+1)} \left( \exp \left\{ \frac{itd(j)(Hk/n)^{1/2}}{2} \right\} - 1 \right)$$

$$= \int_{-\log n}^{\log n} \left( \exp \left\{ \frac{it(Hk/n)^{1/2}x}{2} \right\} - 1 \right) dV_k(x)$$

$$= \int_{-\log n}^{\log n} \left( \exp \left\{ \frac{it(Hk/n)^{1/2}x}{2} \right\} - 1 \right) d\Phi(x)$$

$$+ \int_{-\log n}^{\log n} \left( \exp \left\{ \frac{it(Hk/n)^{1/2}x}{2} \right\} - 1 \right) d\alpha_k(x)$$

$$= \int_{-\infty}^{\infty} \left( \exp \left\{ \frac{it(Hk/n)^{1/2}x}{2} \right\} - 1 \right) d\Phi(x)$$

$$+ O \left( \frac{1}{n} \right) + O(H^{-1/2}) + O \left( |t| \left( \frac{Hk}{n} \right)^{1/2} \frac{\log n}{H^{1/2}} \right).$$
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Hence recalling that $H = n/p$ and $|z| > 1$, we have

$$ S(t, z) = \sum_{k=1}^{p} \frac{e^{-zk/p}}{k} \int_{-\infty}^{\infty} \left( \exp\{it(k/p)^{1/2}x\} - 1 \right) d\Phi(x) $$

$$ + O \left( |z| \frac{\log p}{p} + \frac{|t| \log n}{p^{1/2}} + (1 + |t|) \frac{\log n}{H^{1/2}} \right). $$

Now we will apply the well-known formula

$$ \sum_{n=a}^{b} f(n) = f(a) + \int_{a}^{b} f(x) dx + \int_{a}^{b} \{x\} f'(x) dx, $$

where $a, b \in \mathbb{Z}$. Putting here

$$ f(y) = \frac{e^{-zy/p}}{y} \int_{-\infty}^{\infty} \left( \exp\{it(y/p)^{1/2}x\} - 1 \right) d\Phi(x), $$

we have

$$ \sum_{k=1}^{p} \frac{e^{-zk/p}}{k} \int_{-\infty}^{\infty} \left( \exp\{it(k/p)^{1/2}x\} - 1 \right) d\Phi(x) $$

$$ = \int_{1}^{p} \frac{e^{-zy/p}}{y} \int_{-\infty}^{\infty} \left( \exp\{it(y/p)^{1/2}x\} - 1 \right) d\Phi(x) dy $$

$$ + O \left( \int_{1}^{p} |f'(y)| dy \right) + O \left( \frac{|t|}{p^{1/2}} \right) $$

$$ = \int_{0}^{1} \frac{e^{-zy}}{y} \int_{-\infty}^{\infty} \left( \exp\{ity^{1/2}x\} - 1 \right) d\Phi(x) dy + O \left( \frac{|t| |z|}{p} + \frac{|t|}{p^{1/2}} \right), $$

since $|f'(y)| \ll \frac{|t| |z|}{y^{1/2} p^{1/2}}$.

As noted in [2]

$$ \int_{0}^{1} \frac{e^{-zy}}{y} \int_{-\infty}^{\infty} \left( \exp\{ity^{1/2}x\} - 1 \right) d\Phi(x) dy = \log \frac{z}{z + t^{2}/2} - \int_{z + t^{2}/2}^{\infty} \frac{e^{-w}}{w} dw, $$

therefore, inserting the obtained estimates into (1.15) and taking $p = n^{1/2}$ and $T = n^{1/4}$, we have for $|t| < \frac{n^{1/4}}{\log n}$

$$ M_n = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} \frac{e^{z}}{z + \frac{iT}{2}} \exp \left\{ - \int_{z + \frac{iT}{2}}^{\infty} \frac{e^{-w}}{w} dw \right\} dz $$

$$ + O \left( (1 + |t|) \frac{\log^{2} n}{n^{1/4}} + (1 + |t|^{4}) \frac{\log n}{n^{1/4}} \right). $$
Since as shown in [2]

\[ \frac{1}{2\pi i} \int_{1-\infty^{1/4}}^{1+\infty^{1/4}} \frac{e^z}{z + \frac{t^2}{2}} \exp \left\{ - \int_{t^2}^{z} \frac{e^{-w}}{w} dw \right\} dz = e^{-t^2/2} + \left( \frac{\log n}{n^{1/4}} \right), \]

we finally have the following estimate for \( M_n \)

\[ M_n = e^{-t^2/2} + O \left( (1 + |t|) \frac{\log^2 n}{n^{1/4}} + (1 + |t|^4) \frac{\log n}{n^{1/4}} \right). \quad (1.16) \]

We will also need some crude estimate of \( g_n(t) = M_n \) for small \( |t| \leq 1 \).

\[ |M_n - 1| \leq \sum_{k=1}^{\infty} |m_k| \leq \exp \left\{ \sum_{k=1}^{n} \frac{|\hat{f}(k) - 1|}{k} \right\} - 1 \ll |t|. \quad (1.17) \]

The Berry-Eseen inequality (see e. g. [18]) gives

\[ \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \ll \int_{-U}^{U} \frac{|g_n(t) - e^{-t^2/2}|}{|t|} dt + \frac{1}{U}. \]

Putting here \( U = n^{1/17} \), we split the integration contour into two parts \( |t| \leq \frac{1}{n} \) and \( \frac{1}{n} \leq |t| \leq U \) and applying in each of these interval correspondingly the estimates (1.17) and (1.16) we complete the proof of the theorem.

The theorem is proved.
Chapter 2

Erdős Turán law

2.1 Proofs

Let \( f: (\mathbb{Z}^+)^n \to \mathbb{C} \) be a function of the form

\[
f((a_1, a_2, \ldots, a_n)) = \prod_{j=1}^{n} \hat{f}(j)^{a_j},
\]

where \( \hat{f}(j) \in \mathbb{C} \), and we set \( 0^0 = 1 \). Then we have the identity

\[
\sum_{n=0}^{\infty} M_n f(\xi) z^n = \prod_{j=0}^{\infty} \left(1 + \sum_{n=1}^{\infty} \left(\frac{z}{q}\right)^{nj} \hat{f}(j)^n\right)^{I_j} = \prod_{j \geq 1} \left(1 - \hat{f}(j) \left(\frac{z}{q}\right)^j\right)^{-I_j},
\]

(2.1)

where

\[
M_n f(\xi) = \frac{1}{q^n} \sum_{P \in E_n} f(\xi(P)), \quad n \geq 1,
\]

and \( M_0 f(\xi) = 1 \); here \( \xi = \xi(P) = (\xi_1(P), \xi_2(P), \ldots, \xi_n(P)) \), \( \xi_k(P) \) denotes the number of normed prime polynomials of degree \( k \) in the canonical decomposition of \( P \), and \( I_n \) is the number of prime polynomials in \( E_n \). Relation (2.1) can be obtained by calculating the coefficient at \( z^n \) in the Taylor expansion of the infinite product on the right-hand side of (2.1). Putting here \( \hat{f}(j) \equiv 1 \), we obtain the well-known relation (see, e.g., [15])

\[
\prod_{n \geq 1} \left(1 - \left(\frac{z}{q}\right)^n\right)^{-I_n} = \frac{1}{1 - z},
\]

(2.2)

from which it follows, in particular, that
\[ I_n = \frac{q^n}{n} + A_n \frac{q^{n/2}}{n}, \]  

(2.3)

where \(-2 \leq A_n \leq 0\). Putting, in (2.1),

\[ \hat{f}(j) = \begin{cases} 
  e^{it}, & \text{if } j = k, \\
  1, & \text{if } j \neq k,
\end{cases} \]

and using (2.2), we obtain

\[
\sum_{n=0}^{\infty} z^n M_n e^{it \xi_k} = \left( 1 - \left( \frac{z}{q} \right) e^{it} \right)^{-I_k} \prod_{n \atop n > 0, n \neq k} \left( 1 - \left( \frac{z}{q} \right)^n \right)^{-I_n} = \frac{1}{1 - \frac{z^k}{q^k}} \frac{I_k}{1 - z} \left( 1 - \left( \frac{z}{q} \right)^k \right) e^{it I_k}.
\]

Differentiating the obtained formula with respect to \(t\) and putting \(t = 0\), we obtain

\[
\sum_{n=1}^{\infty} z^n M_n \xi_k = \frac{z^k}{(1 - z)} \left( 1 - \left( \frac{z}{q} \right)^k \right) \frac{I_k}{q^k}.
\]

(2.4)

Hence, it follows that, for \(k \leq n\), we have

\[
M_n \xi_k = \frac{I_k}{q^k} \sum_{j: 1 \leq kj \leq n} \frac{1}{q^{kJ-1}} = \frac{1}{k} + O \left( \frac{1}{q^{k/2}} \right).
\]

(2.5)

Similarly, we obtain

\[
\sum_{m \geq 1} z^m M_m \xi_k^2 = \frac{1}{(1 - z)} \left[ \frac{z^k}{1 - \left( \frac{z}{q} \right)^k} \frac{I_k}{q^k} + \frac{z^{2k}}{1 - \left( \frac{z}{q} \right)^k} \frac{I_k}{q^k} q^k + \frac{I_k I_k}{q^k} q^k + 1 \right]
\]

and, for \(k \leq n\),

\[
M_n \xi_k^2 = \frac{1}{k} + O \left( \frac{1}{k^2} \right).
\]

(2.6)
Estimating the closeness of \( \log P_n(\xi) \) to \( \log O_n(\xi) \) we will use, as in [3], the formula

\[
\log P_n(a) - \log O_n(a) = \sum_p \sum_{s \geq 1} (D_{np^s} - 1)^+ \log p,
\]

(2.7)

where the sum is taken over all prime numbers, \( a \in (\mathbb{Z}^+)^n \), \( (d - 1)^+ = d - 1 + I[d = 0] \), and

\[
D_{nk} = D_{nk}(a) = \sum_{j \leq n: k | j} a_j,
\]

Let us find the generating function of \( \nu_n(D_{nk} = 0) \). Taking, in (2.1),

\[
\hat{f}(j) = \begin{cases} 
0, & \text{if } k | j, \\
1, & \text{if } k \nmid j,
\end{cases}
\]

and using (2.2), we get

\[
\sum_{n=1}^{\infty} \nu_n(D_{nk} = 0) z^n = \prod_{n: k | n} \left(1 - (\frac{z}{q})^n\right)^{-I_n} = \frac{1}{(1 - z)} \prod_{n: k | n} \left(1 - \left(\frac{z}{q}\right)^n\right)^{I_n}
= \frac{1}{(1 - z)} \exp \left\{ - \sum_{n: k | n} \frac{z^n}{n} + \sum_{n: k | n} I_n \left[ \log \left(1 - \left(\frac{z}{q}\right)^n\right) \right] + \left(\frac{z}{q}\right)^n - \sum_{n: k | n} \frac{A_n}{nq^{n/2}z^n} \right\} = \frac{(1 - z^k)^{\frac{1}{q}}}{1 - z} \exp \{ F_k(z) \},
\]

(2.9)

where

\[
F_k(z) = \sum_{n: k | n} I_n \left[ \log \left(1 - \left(\frac{z}{q}\right)^n\right) + \left(\frac{z}{q}\right)^n \right] - \sum_{n: k | n} \frac{A_n}{nq^{n/2}z^n}.
\]
We further use the following notation: if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then we denote \([f(z)]_n = a_n\). We shall often use the following elementary property of this notation. Let \( f_1(z), f_2(z), g_1(z), \) and \( g_2(z) \) be analytic functions in a neighborhood of zero. If \([|f_i(z)|_n] \leq |g_i(z)|_n\) for \( s \geq 0 \) and \( i = 1, 2 \), then

\[
[|f_1(z)f_2(z)|_n] \leq |g_1(z)g_2(z)|_n \quad \text{for} \quad s \geq 0.
\]

The notation of the form \( u(...; v(... \ll v(n, q, t...) = O(v(n, q, t...) \). The constants in the symbols \( O(...) \) and \( \ll \) are always assumed to be absolute and independent of \( q \).

**Lemma 12.** If \( k \geq 2 \), then

\[
[\exp\{F_k(z)\}]_n \leq \left[ 1 - \left( \frac{z}{\sqrt{2}} \right)^k \right]^{-1}, \quad s \geq 0. \tag{i} \]

\[
[(1 - z^k)_{1/k} \exp\{F_k(z)\}]_n \ll \left[ 1 + \sum_{n=1}^{\infty} \frac{z^{kn}}{kn} \right], \quad m \geq 0. \tag{ii}
\]

**Proof.** Using estimate (2.3), we obtain

\[
F_k(z) = - \sum_{n: k|n} I_n \sum_{j=2}^{\infty} \frac{1}{j} \left( \frac{z}{q} \right)^{nj} - \sum_{n: k|n} \frac{A_n}{nq^{n/2}} z^n
\]

\[
= - \sum_{m=1}^{\infty} \left( \frac{z}{\sqrt{q}} \right)^{km} \frac{1}{km} \left( \frac{1}{q^{km/2}} \sum_{l \geq 1, j \geq 2, l_j = m} \frac{kmI_{kl}}{j} + A_{km} \right).
\]

Since \( 0 < I_n \leq \frac{q_n}{n} \) and \(-2 \leq A_j \leq 0\), we have

\[
\left| \frac{1}{q^{km/2}} \sum_{l \geq 1, j \geq 2, l_j = m} \frac{kmI_{kl}}{j} + A_{km} \right| \leq 2,
\]

whence, for \( s \geq 0 \), we have

\[
[F_k(z)]_n \leq \left[ \frac{2}{k} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{z}{\sqrt{q}} \right)^{km} \right]_n \leq \left[ \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{z}{\sqrt{2}} \right)^{km} \right]_n
\]

\[
= \left[ \log \left( 1 - \left( \frac{z}{\sqrt{2}} \right)^k \right)^{-1} \right]_n.
\]
Hence, for $s \geq 0$, we obtain

$$||\exp\{F_k(z)\||(s) = \left| \sum_{j=0}^{\infty} \frac{[F_k(z)]^j(s)}{j!} \right| \leq \sum_{j=0}^{\infty} \frac{1}{j!} \left[ \log \left( 1 - \left( \frac{z}{\sqrt{2}} \right)^k \right) \right]^j = \left[ \left( 1 - \left( \frac{z}{\sqrt{2}} \right)^k \right)^{-1} \right] \right|_{(s)} .$$

In the proof of estimate (ii), we shall use estimate (i). Since

$$||[\left( 1 - z \right)^{1/k}]||(j) = \frac{1}{k} \prod_{l=2}^{j} \left( 1 - \frac{1 + 1/k}{l} \right) \leq \frac{1}{k} \prod_{l=2}^{j} \left( 1 - \frac{1}{l} \right) = \frac{1}{kj}$$

for $j \geq 1$, we have that

$$||[\left( 1 - z^k \right)^{1/k}]||(j) \leq \left[ 1 + \sum_{s=1}^{\infty} \frac{z^{ks}}{ks} \right]_{(j)}$$

for $j \geq 0$. Using this estimate and (i), we obtain

$$||[\left( 1 - z^k \right)^{1/k} \exp\{F_k(z)\}]||(m) \leq \left[ \left( 1 + \sum_{s=1}^{\infty} \frac{z^{ks}}{ks} \right) \left( \sum_{s=0}^{\infty} \frac{z^{ks}}{2ks/2} \right) \right]_{(m)} \leq \left[ 1 + \sum_{n=1}^{\infty} \frac{z^{kn}}{kn} \right]_{(m)},$$

and the lemma is proved.

**Lemma 13.** If $k \geq 2$, then

$$\nu_n(D_{nk}(\xi) = 0) = \exp \left\{ -\frac{1}{k} \sum_{j=1}^{\lfloor n/k \rfloor} \frac{1}{j} \right\} + O \left( \frac{1}{k^2} \right).$$

**Proof.** In [4], an exact expression has been obtained for

$$\left[ \frac{(1 - z^k)^{1/k}}{1 - z} \right]_{(n)} = \sum_{j \leq n} \left[ (1 - z^k)^{1/k} \right]_{(j)} = \sum_{j: \text{jk} \leq n} \left[ (1 - z)^{1/k} \right]_{(j)}$$

$$= \left[ \frac{(1 - z)^{1/k}}{1 - z} \right]_{(\lfloor n/k \rfloor)} = \prod_{j=1}^{\lfloor n/k \rfloor} \left( 1 - \frac{1}{jk} \right) = \exp \left\{ -\frac{1}{k} \sum_{j=1}^{\lfloor n/k \rfloor} \frac{1}{j} \right\} \left( 1 + O \left( \frac{1}{k^2} \right) \right).$$
CHAPTER 2. ERDŐS TURÁN LAW

Hence, using (2.9) and Lemma 12, we obtain

\[ \nu_n(D_{nk}(\xi) = 0) = \left[ \frac{(1 - z^k)^{1/k}}{1 - z} \exp \{ F_k(z) \} \right]_{(n)} = \left[ \frac{(1 - z^k)^{1/k}}{1 - z} \right]_{(n)} \]

\[ + \sum_{m=1}^{n} \left[ \frac{(1 - z^k)^{1/k}}{1 - z} \right]_{(n-m)} \exp \{ F_k(z) \} \right]_{(m)} \]

\[ = \left[ \frac{(1 - z^k)^{1/k}}{1 - z} \right]_{(n)} + O \left( \frac{1}{2k^2} \right) = \exp \left\{ - \frac{1}{k} \sum_{j=1}^{[n/k]} \frac{1}{j} \right\} + O \left( \frac{1}{k^2} \right). \]

The lemma is proved.

Proof of theorem 6. From (2.6) and Lemma 13 it follows that

\[ \mu_n := M_n \left( \log P_n(\xi) - \log O_n(\xi) \right) = \sum_{p} \sum_{s \geq 0} \left( M_n D_{np^s} - 1 + \nu_n(D_{np^s} = 0) \right) \log p, \]

\[ = \sum_{m \leq n} \Lambda(m) \left[ \sum_{j=1}^{[n/m]} \frac{1}{jm} - \left( 1 - e^{-\sum_{j=1}^{[n/m]} \frac{1}{jm}} \right) \right] + O(1) \]

\[ = \sum_{m \leq n} \Lambda(m) \left[ \log \frac{n}{m} + \gamma + O \left( \frac{m}{n} \right) \right] \]

\[ - \left( 1 - e^{-\frac{\log n}{m}} \left( 1 - \frac{\gamma}{m} + O \left( \frac{1}{m^2 + \frac{1}{n}} \right) \right) \right) + O(1) \]

\[ = \sum_{m \leq n} \Lambda(m) \left[ \log \frac{n}{m} - \left( 1 - \exp \left\{ -\frac{\log \frac{n}{m}}{m} \right\} \right) \right] \]

\[ + \gamma \sum_{m \leq n} \Lambda(m) \frac{1 - \exp \left\{ -\frac{\log \frac{n}{m}}{m} \right\}}{m} + O(1), \]

where \( \gamma \) is the Euler constant, and \( \Lambda(m) \) is the Mangoldt function. We further denote by \( B \) bounded constants (not necessarily the same in different places). Since

\[ \sum_{m \leq n} \Lambda(m) \frac{1 - \exp \left\{ -\frac{\log \frac{n}{m}}{m} \right\}}{m} = \sum_{m \leq \log n} \Lambda(m) \frac{1 - \exp \left\{ -\frac{\log \frac{n}{m}}{m} \right\}}{m} \]

\[ + B \sum_{m > \log n} \frac{\Lambda(m)}{m^2} \log n + O(1) = O(\log \log n), \]
we have
\[
\mu_n = \sum_{m \leq n} \Lambda(m) \left[ \frac{\log \frac{n}{m}}{m} - \left( 1 - \exp \left\{ -\frac{\log \frac{n}{m}}{m} \right\} \right) \right] + O(\log \log n)
\]
\[
= \sum_{m \leq n} \Lambda(m) \left[ \frac{\log n}{m} - \frac{\log m}{m} - \left( 1 - \exp \left\{ -\frac{\log n}{m} \right\} \right) \right]
\times \left( 1 + \frac{\log m}{m} + O \left( \frac{\log m}{m} \right)^2 \right) + O(\log \log n)
\]
\[
= \sum_{m \leq n} \Lambda(m) \left[ \frac{\log n}{m} - \left( 1 - \exp \left\{ -\frac{\log n}{m} \right\} \right) \right]
+ B \sum_{m \leq n} \Lambda(m) \frac{\log m}{m} \left( 1 - \exp \left\{ -\frac{\log n}{m} \right\} \right) + O(\log \log n)
\]
\[
= S(x) + O((\log x)^2),
\]
where
\[
S(x) = \sum_{m=1}^{\infty} \Lambda(m) \phi \left( \frac{m}{x} \right),
\]
\[
\phi(y) = e^{-\frac{1}{y}} - 1 + \frac{1}{y}, \text{ and } x = \log n. \text{ As in the proof of one formula of Linnik (see, e.g., [9], p. 83, Problem 6), we represent } S(x) \text{ by a sum over nontrivial zeros of } \zeta(s). \text{ Calculating the Mellin transform of the function } \phi(y), \text{ we obtain }
\]
\[
\int_{0}^{\infty} \phi(x)x^{s-1}dx = \Gamma(-s)
\]
for \(1 < \Re s < 2\); here \(\Gamma(s)\) is the Euler Gamma function. Then we have
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \Gamma(-s)x^{s}ds = \sum_{m=1}^{\infty} \Lambda(m) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(-s) \left( \frac{m}{x} \right)^{-s} ds = S(x)
\]
for \(1 < \sigma < 2\). Let us change the integration line \(\Re s = \sigma\) in the above integral by the line \(\Re s = -\frac{1}{2}\) in the following way. Let us consider the integral over the rectangle with vertices \(-\frac{1}{2} + iT, -\frac{1}{2} - iT, \sigma + iT, \text{ and } \sigma - iT\) with \(T\) chosen so that the distance from the closest zeros of \(\zeta(s)\) is \(\gg \frac{1}{\log T}\). Letting now \(T\) to infinity and applying the well-known estimates of \(\frac{\zeta'(s)}{\zeta(s)}\) in the critical strip, we see that the integrals over \(\left[ -\frac{1}{2} + iT, \sigma + iT \right] \) and \(\left[ -\frac{1}{2} - iT, \sigma - iT \right] \) tend to zero as \(T \to \infty\). Applying now the residue theorem, we obtain
\[ S(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \Gamma(-s)x^s ds = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \Gamma(-s)x^s ds \]

\[ + \sum_{\rho} \text{res}_{s=\rho} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \Gamma(-s)x^s + \left( \text{res}_{s=0} + \text{res}_{s=1} \right) \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \Gamma(-s)x^s \]

\[ = x(\log x - 1) + \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \Gamma(-\rho)x^\rho + O\left( \frac{1}{\sqrt{x}} \right), \]

where \( \sum_{\rho} \) is the sum over nontrivial zeros of the Riemann zeta-function.

Now using estimate (2.4), we calculate

\[ M_n \log P_n(\xi(P)) = \sum_{j=1}^{n} \frac{M_n \xi_j \log j}{j} + O(1) = \frac{1}{2} \log^2 n + O(1). \]

The theorem is proved.

We now estimate the sum over the nontrivial zeros of the Riemann zeta-function. Since \( \Re \rho \leq 1 - \frac{\epsilon}{\log T} \) for \( |\Im \rho| \leq T \) (see [9] or [19]) and \( |\Gamma(-(\sigma + it))| \ll e^{-\frac{\epsilon}{2}|t|} |t|^{-1/2} \) uniformly in \( 0 \leq \sigma \leq 1 \), putting \( T = \log \log n \), we get

\[ R_n := \left| \sum_{\rho} \Gamma(-\rho)(\log n)^\rho \right| \ll (\log n)^{1 - \frac{\epsilon}{\log T}} \left| \sum_{|\Im \rho| \leq T} \Gamma(-\rho) \right| + \log n \sum_{|\Im \rho| > T} |\Gamma(-\rho)| \]

\[ \ll \log n \left( (\log n)^{1 - \frac{\epsilon}{\log T}} + e^{-T} \right) \ll (\log n)e^{-r \frac{\log n \log \log \log n}{\log \log n}}; \]

here we used the well-known fact that \( \sum_{n<|\Im \rho|\leq n+1} 1 \ll \log n \).

To estimate the covariances of \( (D_{nk} - 1)^+ \) the following elementary lemma will be useful.

**Lemma 14.** Let \( f(z) \) and \( g(z) \) be analytic functions. Then

\[ \left[ \frac{f(z)g(z)}{1 - z} \right]_{(n)} = \left[ \frac{f(z)}{1 - z} \right]_{(n)} \left[ \frac{g(z)}{1 - z} \right]_{(n)} = -\sum_{1 \leq i,j \leq n \atop i+j > n} [f(z)]_{(i)} [g(z)]_{(j)}. \]

In the following two lemmas, we prove the same estimates for the covariances of \( (D_{nk}(\xi(P)) - 1)^+ \) as those obtained in [11] for \( (D_{nk}(Z) - 1)^+ \), where \( Z \) is a vector with components that are independent random Poisson variables with parameters \( \theta/j \).
Lemma 15. For \((k, l) = 1, k, l \geq 2\), we have the estimate
\[
\text{cov}((D_{nk}(\xi) - 1)^+, (D_{nl}(\xi) - 1)^+) \ll \frac{\log n}{kl}.
\]

Proof. Since \((d - 1)^+ = d - 1 + I(d = 0)\), we have
\[
\text{cov}((D_{nk} - 1)^+, (D_{nl} - 1)^+) = \text{cov}(D_{nk}, D_{nl}) + \text{cov}(D_{nk}, I[D_{nl} = 0])
+ \text{cov}(I[D_{nk} = 0], D_{nl} = 0) + \text{cov}(I[D_{nk} = 0], I[D_{nl} = 0]).
\]
Putting, in (2.1),
\[
\hat{f}(m) = \begin{cases} 0, & \text{if } k|m \text{ or } l|m, \\ 1, & \text{otherwise}, \end{cases}
\]
and using (2.1), we get
\[
U_{k,l}(z) := \sum_{n=0}^{\infty} \nu_n(D_{nk}(\xi) = 0, D_{nl}(\xi) = 0) z^n = \frac{1}{1 - z} \prod_{n \geq 1: k|n \land l|n} \left(1 - \left(\frac{z}{q}\right)^n\right)^{I_n}
= \frac{1}{1 - z} \prod_{n \geq 1: k|n} \left(1 - \left(\frac{z}{q}\right)^n\right)^{I_n} \prod_{n \geq 1: l|n} \left(1 - \left(\frac{z}{q}\right)^n\right)^{I_n}
= \frac{1}{1 - z} \frac{(1 - z^k)^{1/k}(1 - z^l)^{1/l}}{(1 - z^{kl})^{1/kl}} \exp\{F_k(z) + F_l(z) - F_{kl}(z)\}.
\]

Hence, we have
\[
\nu_n(D_{nk}(\xi) = 0, D_{nl}(\xi) = 0) = \left[\frac{(1 - z^k)^{1/k}(1 - z^l)^{1/l}}{(1 - z)} \exp\{F_k(z) + F_l(z)\}\right]_{(n)}
+ [U_{k,l}(z)\left(1 - \exp\{F_{kl}(z)\}\right)]_{(n)} + [U_{k,l}(z) \exp\{F_{kl}(z)\} (1 - (1 - z^{kl})^{1/kl})]_{(n)}
= [S_1(z) + S_2(z) + S_3(z)]_{(n)}.
\]

Since \(0 \leq [U_{k,l}(z)]_{(j)} \leq \left[\frac{1}{1 + z}\right]_{(j)}\) and, by Lemma 12, \([1 - \exp\{F_{kl}(z)\}]_{(j)} \leq [(1 - (z^j/\sqrt{2})^{kl})^{-1} - 1]_{(j)}\) for \(j \geq 0\), we have
\[
|[S_2(z)]_{(n)}| \leq \left[\frac{1}{1 - z} \left(\left(1 - \left(\frac{z^j}{\sqrt{2}}\right)^k\right)^{1/2} - 1\right)\right]_{(n)}
= \left[\frac{1}{1 - z} \left(\sum_{j \geq 1} \left(\frac{z^j}{\sqrt{2}}\right)^{kl}\right)\right]_{(n)} \ll \frac{1}{2^{kl/2}}.
\]
Similarly, since \( |[(1 - z^{kl})^{1/2} - 1]_j| \leq \left( \sum_{s=1}^{\infty} \frac{z^{kl}}{kl} \right)_j \) for \( j \geq 0 \), we have
\[
|S_3(z)|_n \leq \left[ \frac{1}{1 - z} \left( 1 - \left( \frac{z}{\sqrt{2}} \right)^k \right)^{-1} \left( \sum_{j \geq 1} \frac{z^{kJ}}{klj} \right) \right]_n \ll \log n \frac{1}{kl}.
\]

Applying the estimates obtained and using (2.8), we get
\[
\text{cov}(I[D_{nk}(\xi) = 0], I[D_{nl}(\xi) = 0]) = \left[ (1 - z^k)^{1/k} \exp \{ F_k(z) \} (1 - z^l)^{1/l} \exp \{ F_l(z) \} \right]_{(n)} (1 - z) \ll \log n \frac{1}{kl}.
\]

Applying estimate (ii) of Lemma 12 and Lemma 14, we obtain
\[
\text{cov}(I[D_{nk}(\xi) = 0], I[D_{nl}(\xi) = 0]) = - \sum \left[ (1 - z^k)^{1/k} \exp \{ F_k(z) \} \right]_{(i)} \left[ (1 - z^l)^{1/l} \exp \{ F_l(z) \} \right]_{(j)} + O \left( \frac{\log n}{kl} \right).
\]

Therefore, we have
\[
\text{cov}(I[D_{nk}(\xi) = 0], I[D_{nl}(\xi) = 0]) \ll \frac{\log n}{kl} + \sum_{i,j: ik + jl > n \atop 1 \leq i, j \leq n} \frac{1}{kijl}.
\]

Since
\[
\frac{1}{kl} \sum_{i,j: ik + jl > n \atop 1 \leq i, j \leq n} \frac{1}{ij} = \frac{[n/k]}{ikl} \sum_{i=1}^{[n/k]} \frac{1}{i} \sum_{n\leq k < j \leq n} \frac{1}{j}
\]
\[
= \frac{1}{kl} \sum_{1 \leq i \leq \frac{n}{k}} \frac{1}{i} \sum_{\frac{n}{k} < j \leq n} \frac{1}{j} + \frac{1}{kl} \sum_{\frac{n}{k} < i \leq \frac{n}{k}} \frac{1}{i} \sum_{\frac{n}{k} < j \leq n} \frac{1}{j}
\]
\[
\ll \frac{1}{kl} \sum_{1 \leq i \leq \frac{n}{k}} \frac{1}{i} + \frac{1}{kl} \sum_{\frac{n}{k} < i \leq \frac{n}{k}} \frac{\log n}{i} \ll \frac{\log n}{kl},
\]

Therefore, we have
\[
\text{cov}(I[D_{nk}(\xi) = 0], I[D_{nl}(\xi) = 0]) \ll \frac{\log n}{kl} + \sum_{i,j: ik + jl > n \atop 1 \leq i, j \leq n} \frac{1}{kijl}.
\]
we finally have
\[
\text{cov} \left( I[D_{nk}(\xi) = 0], I[D_{nl}(\xi) = 0] \right) \ll \frac{\log n}{kl}.
\] (2.10)

To estimate \( \text{cov}(I[D_{nk}(\xi) = 0], \xi_j) \) we consider two cases: \( k \nmid j \) and \( k | j \).

1) If \( k \nmid j \), then
\[
\text{cov}(I[D_{nk}(\xi) = 0], \xi_j) = \mathbf{M}_n(\xi_j I[D_{nk}(\xi) = 0]) - \mathbf{M}_n\xi_j \nu_n(D_{nk}(\xi) = 0).
\]

Putting, in (2.1),
\[
\hat{f}(s) = \begin{cases} e^{it}, & \text{for } s = j, \\ 0, & \text{if } k | s, \\ 1, & \text{for remaining } s \end{cases}
\]

we obtain
\[
\sum_{n \geq 0} \mathbf{M}_n(e^{it\xi_j} I[D_{nk}(\xi) = 0]) z^n = \frac{(1 - z^k)^{\frac{1}{k}}}{1 - z} \exp\{F_k(z)\} \frac{z^j I_j}{1 - (\frac{z}{q})^j q^j}.
\] (2.11)

Differentiating the obtained equality with respect to \( t \) and putting \( t = 0 \), we have
\[
\sum_{n \geq 0} \mathbf{M}_n(\xi_j I[D_{nk}(\xi) = 0]) z^n = \frac{(1 - z^k)^{\frac{1}{k}}}{1 - z} \exp\{F_k(z)\} \frac{z^j I_j}{1 - (\frac{z}{q})^j q^j}.
\]

Hence, applying Lemma 14 and (2.4), (2.9), and (2.11), we have
\[
\text{cov}(I[D_{nk}(\xi) = 0], \xi_j) = \left[ \frac{(1 - z^k)^{\frac{1}{k}}}{1 - z} \exp\{F_k(z)\} \frac{z^j I_j}{1 - (\frac{z}{q})^j q^j} \right]_{(n)}
\]
\[
- \left[ \frac{1}{1 - z} \frac{z^j I_j}{1 - (\frac{z}{q})^j q^j} \right]_{(n)} \left[ \frac{(1 - z^k)^{\frac{1}{k}}}{1 - z} \exp\{F_k(z)\} \right]_{(n)}
\]
\[
= - \sum_{r+s>n \atop 1 \leq r, s \leq n} \left[ \frac{z^j I_j}{1 - (\frac{z}{q})^j q^j} \right]_{(r)} \left[ (1 - z^k)^{\frac{1}{k}} \exp\{F_k(z)\} \right]_{(s)}
\]
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\[ \sum_{s, r : \frac{rj + sk}{n} \leq r_j, s_k \leq n} \frac{1}{j} \left[ \frac{z}{1 - \frac{z}{q^r}} \right]_{(r)} \left[ \left( 1 + \frac{1}{k} \sum_{m=1}^{\infty} \frac{z^m}{m} \right)_{(s)} \right] \]

\[ = \frac{1}{jk} \sum_{s, r : \frac{rj + sk}{n} \leq r_j, s_k \leq n} \frac{1}{q^{j(r-1)}} \sum_{\frac{n-j}{k} < s \leq \frac{n}{k}} \frac{1}{s} + \frac{1}{jk} \sum_{\frac{rj + sk}{n} \leq r_j, s_k \leq n, r \geq 2} \frac{1}{q^{j(r-1)}} \]

\[ \ll \frac{1}{jk} \left( 1 + \log \frac{n}{n-j+1} \right) + \log n \frac{\log n}{j q^k j}. \] (2.12)

2) Now suppose that \( k \mid j \). Since \( I[D_{nk}(\xi) = 0|\xi_j \equiv 0, \), in this case, we have

\[ \text{cov}(I[D_{nk}(\xi) = 0|\xi_j ] = -\nu_n(D_{nk}(\xi) = 0)M_n \xi_j \ll \frac{1}{j}. \] (2.13)

Now we can estimate \( \text{cov}(I[D_{nk}(\xi) = 0], D_{nl}(\xi)) \):

\[ \text{cov}(I[D_{nk}(\xi) = 0], D_{nl}(\xi)) = \sum_{j \leq n : \frac{j}{l_j} = \frac{k}{k_j}} \text{cov}(I[D_{nk}(\xi) = 0], \xi_j) + \sum_{j \leq n : \frac{j}{l_j} \neq \frac{k}{k_j}} \text{cov}(I[D_{nk}(\xi) = 0], \xi_j) \]

\[ \ll \sum_{j \leq n : k \mid j} \frac{1}{j} + \sum_{j \leq n : l \mid j} \left( \frac{1}{j k} \left( 1 + \log \frac{n}{n-j+1} \right) + \frac{\log n}{j q^k j} \right) \]

\[ \ll \frac{\log n}{k l} + \sum_{j \leq n / 2 : l \mid j} \frac{1}{k j n} + \sum_{n / 2 < j \leq n : l \mid j} \frac{1}{k n} \log n \ll \frac{\log n}{k l}; \]

here we used estimates (2.12) and (2.13).

Let \( i \neq j \). Then, as before, putting, in (2.1), \( \hat{f}(i) = e^{it_1}, \hat{f}(j) = e^{it_2}, \) and \( \hat{f}(m) = 1 \) for the remaining \( m \), we obtain the generating function of \( M_n e^{it_1 \xi_i + it_2 \xi_j} \). Differentiating the obtained formula with respect to \( t_1 \) and \( t_2 \) and putting \( t_1 = t_2 = 0 \), we get

\[ \sum_{n=1}^{\infty} M_n \xi_i \xi_j z^n = \frac{I_i I_j}{q^i q^j} \frac{1}{1 - z} \frac{z^{i+j}}{1 - \left( \frac{z}{q} \right)^i} \frac{z^j}{1 - \left( \frac{z}{q} \right)^j}. \]

From this and from (2.4), applying Lemma 14 we have

\[ \text{cov}(\xi_i, \xi_j) = \left[ \frac{I_i I_j}{q^i q^j} \frac{1}{1 - z} \frac{z^i}{1 - \left( \frac{z}{q} \right)^i} \frac{z^j}{1 - \left( \frac{z}{q} \right)^j} \right]_{(n)} \]
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\[ - \begin{bmatrix} I_i & \frac{1}{q^i} & \frac{z^i}{1 - z} \cdot (\frac{z}{q})^i \\ \end{bmatrix} \begin{bmatrix} I_j & \frac{1}{q^j} & \frac{z^j}{1 - z} \cdot (\frac{z}{q})^j \\ \end{bmatrix} \]

\[ = - \frac{I_i I_j}{q^i q^j} \sum_{i_k + j_l > n} \frac{1}{q^{i_k - 1} q^{j_l - 1}}. \]

Hence, it follows that, for \( i \neq j \), we have

\[ \text{cov}(\xi_i, \xi_j) \leq \begin{cases} \frac{1}{ij} q^{\min(i,j) - 1}, & \text{if } i + j \leq n, \\ \frac{1}{ij}, & \text{if } i + j > n. \end{cases} \quad (2.14) \]

For \( i = j \), by (2.6) we have \( \text{cov}(\xi_i, \xi_i) = D\xi_i \ll 1/i. \)

Applying the estimates obtained, we have

\[ \text{cov}(D_{nk}(\xi), D_{nl}(\xi)) \]

\[ \leq \sum_{k+i+l+j \leq n} \frac{1}{k i l j} q^{\min(k,i,l,j)} + \sum_{k+i+l+j > n} \frac{1}{k i l j} + \sum_{j=1}^{\lfloor n/kl \rfloor} \frac{1}{j k l} \ll \frac{\log n}{kl}. \quad (2.15) \]

Using estimates (2.10), (2.12), (2.13), and (2.15), we obtain the required estimate

\[ \text{cov}((D_{nk}(\xi) - 1)^+, (D_{nl}(\xi) - 1)^+) \ll \frac{\log n}{kl}. \]

The lemma is proved.

**Lemma 16.** Let \( p \) be a prime number, and let \( 1 \leq s \leq t \). Then we have

\[ \text{cov}((D_{np^s}(\xi) - 1)^+, (D_{np^t}(\xi) - 1)^+) \ll \frac{\log n}{p^t}. \]

**Proof.** Applying Lemma 13, we have

\[
\text{cov}(I[D_{np^s}(\xi) = 0], I[D_{np^t}(\xi) = 0]) \\
= \nu_n(D_{np^s}(\xi) = 0) - \nu_n(D_{np^s}(\xi) = 0) \nu_n(D_{np^t}(\xi) = 0) \\
= \nu_n(D_{np^s}(\xi) = 0) (1 - \nu_n(D_{np^t}(\xi) = 0)) \\
\ll 1 - \exp \left\{- \frac{1}{p^t} \sum_{j=1}^{[n/p^t]} \frac{1}{j} \right\} + \frac{1}{p^{2t}} \ll \frac{\log n}{p^t};
\]
here we have used the inequality $1 - e^{-x} \leq x$ for $x \geq 0$.

Since, for $s \leq t$, from $p^t|j$ it follows that $I_{D_{np^s}^*}(\xi) = 0|\xi_j = 0$, we have

$$\text{cov}(I_{D_{np^s}^*}(\xi) = 0, D_{np^t}(\xi)) = - \sum_{j \leq n: p^t|j} \nu_n(D_{np^s}^* = 0) M_n \xi_j \ll \frac{\log n}{p^t}.$$ 

Applying estimates (2.12) and (2.13), we have

$$\text{cov}(I_{D_{np^s}^*}(\xi) = 0, D_{np^t}(\xi)) = \sum_{j \leq n: p^t|j} \frac{1}{j} + \frac{\log n}{p^t} + \sum_{j \leq n: p^t|j} \frac{1}{jp^t} \log \frac{n}{n + 1 - j} \ll \frac{\log n}{p^t}.$$ 

Applying estimates (2.14), we have

$$\text{cov}(D_{np^s}(\xi), D_{np^t}(\xi)) = \sum_{1 \leq i, j \leq n} \text{cov}(\xi_i, \xi_j) = \sum_{j \leq n: p^t|j} \text{cov}(\xi_j, \xi_j)$$

$$+ \sum_{1 \leq i, j \leq n} \text{cov}(\xi_i, \xi_j) \ll \frac{\log n}{p^t} + \sum_{p^t|j, p^t|j, i \neq j} \frac{1}{ij} \frac{\log n}{\log \log n} \ll \frac{\log n}{p^t}.$$ 

From the estimates obtained it follows that

$$\text{cov}((D_{np^s}^* - 1)^+, (D_{np^t}^* - 1)^+) \ll \frac{\log n}{p^t}.$$ 

The lemma is proved.

The following proposition, which is completely similar to Proposition 2.3 of [3], gives the desired estimate of closeness of the random variables $\log \frac{O_n(\xi(P))}{\log \log n}$ and $\log \frac{P_n(\xi(P))}{\log \log n}$.

**Proposition 1.** For every fixed $K > 0$, we have

$$\nu_n \left( \frac{\log P_n(\xi) - \log O_n(\xi) - \mu_n}{\log^{3/2} n} > K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \left( \frac{\log \log n}{\log n} \right)^{2/3},$$

where $\mu_n = M(\log P_n(\xi) - \log O_n(\xi))$. 
Proof. In the proof of this proposition, we repeat the arguments of Barbour and Tavare in the proof of the corresponding result. We have

\[
\log P_n(\xi) - \log O_n(\xi) = \left( \sum_{p \leq \log^2 n} + \sum_{p > \log^2 n} \right) (D_n p(\xi) - 1)^+ \log p \\
+ \sum_p \sum_{s \geq 2} (D_n p_s(\xi) - 1)^+ \log p = V_1 + V_2 + V_3.
\]

Applying Lemmas 15 and 16, we have

\[
DV_1 \ll \sum_{p \leq \log^2 n} \frac{\log n}{p} \log^2 p + \sum_{p \neq q, p, q \leq \log^2 n} \frac{\log n}{pq} \log p \log q \ll \log n (\log \log n)^2.
\]

From Lemma 13 it follows that

\[
M(D_{nk} - 1)^+ \ll \min \left\{ \left( \frac{\log n}{k} \right), \left( \frac{\log n}{k} \right)^2 \right\}.
\]

Therefore,

\[
MV_2 \ll \sum_{p > \log^2 n} \frac{\log^2 n}{p^2} \log p \ll 1.
\]

We estimate

\[
DV_3 \ll \sum_p \log^2 p \sum_{s,t \geq 2, s \neq t} \frac{\log n}{p^{\max\{s,t\}}} + \sum_p \log p \log q \sum_{s,t \geq 2, s \neq t} \frac{\log n}{pq^{s+t}} \ll \log n.
\]

Applying the Chebyshev inequality, we get

\[
\nu_n \left( \frac{|V_1 - MV_1|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \left( \frac{\log \log n}{\log n} \right)^{2/3},
\]

\[
\nu_n \left( \frac{|V_2 - MV_2|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \frac{1}{(\log \log n)^{2/3} \log^{5/6} n},
\]

\[
\nu_n \left( \frac{|V_3 - MV_3|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \frac{1}{(\log \log n)^{4/3} \log^{2/3} n}.
\]

Hence, we obtain the proof of the proposition.
The proved proposition enables us to replace the investigation of closeness of \( \log O_n(\xi(P)) \) to the normal law by the investigation of the simpler quantity

\[
\log P_n(\xi(P)) = \sum_{i=1}^{n} \xi_i(P) \log i,
\]

which is a linear combination of the variables \( \xi_i(P) \).

Further we use the same notation \( M_n \) to denote the mean values of \( f(\alpha) \) and \( f(\xi) \) on \( S_n \) and \( E_n \), respectively.

We denote by \( \phi_{n,\alpha}(t) \) and \( \phi_{n,\xi}(t) \) the characteristic functions of the random variables

\[
\frac{\log P_n(\alpha(\sigma)) - \frac{1}{2} \log^2 n}{(1/\sqrt{3}) \log^{3/2} n} \quad \text{and} \quad \frac{\log P_n(\xi(P)) - \frac{1}{2} \log^2 n}{(1/\sqrt{3}) \log^{3/2} n},
\]

respectively.

To estimate \( \phi_{n,\alpha}(t) \) we apply Theorem 12 with \( d_j \equiv 1 \) and \( p = \infty \). Putting in (1.6)

\[
\hat{f}(k) = \exp \left\{ \frac{it \log k}{(\frac{1}{3} \log^3 n)^{1/2}} \right\}, \quad p = \infty;
\]

we get

\[
\rho = \rho(\infty) \ll \frac{|t|}{\log^{1/2} n}
\]

For \( |t| \leq \varepsilon \log^{1/2} n \), where \( \varepsilon \) is a sufficiently small fixed number, we have \( \rho \leq \delta \) and

\[
M_n \exp \left\{ it \frac{\log P_n(\alpha(\sigma))}{(1/\sqrt{3}) \log^{3/2} n} \right\} = \exp \left\{ S_n \left( \frac{t}{(1/\sqrt{3}) \log^{3/2} n} \right) \right\} \left( 1 + O \left( \frac{|t|^2}{\log n} \right) \right);
\]

here

\[
S_n(t) = \sum_{m=1}^{n} \frac{e^{it \log m} - 1}{m}.
\]

Let us estimate the quantity \( \exp\{S_n(t)\} \). We have

\[
S_n(t) = \sum_{m=1}^{n} \frac{e^{it \log m} - 1}{m} = \sum_{m=1}^{n} \frac{1}{m^{1-it}} - \log n - \gamma + O \left( \frac{1}{n} \right)
\]
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\[ \zeta(1 - it) + \frac{n^it}{it} - \log n - \gamma + O\left(\frac{1}{n}\right) \]

for \(|t| \leq 1\). Here we have used the formula

\[ \zeta(s) = \sum_{m \leq n} \frac{1}{m^s} + \frac{n^{1-s}}{s-1} + O\left(\frac{|s|}{n^\sigma}\right) \]

for \(\Re s = \sigma > 0\) (see, e.g., [9]).

Since \(\zeta(1 - it) = -\frac{1}{it} + \omega(t)\), where \(\omega(t)\) is an analytic function and \(\omega(0) = \gamma\), for \(|t| \leq 1\), we have

\[ S_n(t) = \frac{n^it - 1}{it} - \log n + O\left(\frac{1}{n} + |t|\right). \quad (2.16) \]

Applying the Taylor expansion for \(|t| \leq 1\), we have

\[ S_n(t) = \frac{1}{it} \left(\sum_{k=1}^{4} \frac{(it \log n)^k}{k!} + O(|t|^5 \log^5 n))\right) - \log n + O\left(\frac{1}{n} + |t|\right). \quad (2.17) \]

Let

\[ D_n(t) = S_n\left(\frac{t}{(1/\sqrt{3}) \log^{3/2} n}\right) - it \frac{\sqrt{3}}{2} \log^{1/2} n. \]

Then, for \(|t| \leq \log^{3/2} n\), we have

\[ D_n(t) = -\frac{t^2}{2} + (it)^3 \frac{3^{3/2}}{4!} \frac{1}{\log^{1/2} n} + O\left(\frac{1}{n} + \frac{|t|^4}{\log n} + \frac{|t|}{\log^{3/2} n}\right). \]

If \(|t| \leq \log^{1/6} n\), then

\[ \phi_{\alpha,n}(t) = \exp\{D_n(t)\} \left(1 + O\left(\frac{|t|^2}{\log n}\right)\right) = e^{-\frac{t^2}{2}} \left(1 + (it)^3 \frac{3^{3/2}}{4!} \frac{1}{\log^{1/2} n} \right) \]

\[ \quad + O\left(\frac{1}{n^\varepsilon} + \frac{|t|^4}{\log n} + \frac{|t|^6}{\log^{3/2} n} + \frac{|t|^2}{\log n}\right). \quad (2.18) \]

From (2.10) we have, for \(\varepsilon_1 > 0\),

\[ \left| \exp \left\{ S_n\left(\frac{t}{(1/\sqrt{3}) \log^{3/2} n}\right) \right\} \right| \]

\[ = \exp \left\{ \left(\frac{\sin\left(\frac{t\sqrt{3}}{\log^{1/2} n}\right)}{\frac{t\sqrt{3}}{\log^{1/2} n}} - 1\right) \log n + O(1) \right\} \ll \frac{1}{n^\lambda} \quad (2.19) \]
for \( \varepsilon_1 \log^{1/2} n \leq t \leq \log^{3/2} n \); here \( \lambda = \lambda(\varepsilon_1) = 1 - \sup_{u > \varepsilon_1} \sqrt{3} \frac{\sin u}{u} \).

Theorem 5 gives an estimate of the characteristic function of \( \phi_{n,\alpha}(t) \) for \( |t| \leq \log^{1/6} n \). We further estimate \( \phi_{n,\alpha}(t) \) for \( |t| \leq \log^{3/2} n \). The general Theorem A of Manstavičius \[10\] gives the desired estimate for \( |t| \leq \log^{1/2} n \) but its application becomes difficult in the region \( \log^{1/2} n \leq |t| \leq \log^{3/2} n \), since, for \( |t| \geq \log^{1/2} n \), its remainder term has an increasing multiplier.

**Theorem 15.**

\[
\phi_{n,\alpha}(t) \ll e^{-\frac{t^2}{3}} \quad \text{for} \quad |t| \leq \delta_1 \log^{1/2} n, \quad (2.20)
\]

\[
\phi_{n,\alpha}(t) \ll \frac{1}{n^u} \quad \text{for} \quad \delta_1 \log^{1/2} n < |t| \leq \log^{3/2} n, \quad (2.21)
\]

where \( u \) and \( \delta_1 \) are some fixed positive constants.

**Proof.** From (2.16) and (2.22) for \( |t| \leq \delta \sqrt{\log n} \) we have

\[
|\phi_{\alpha,n}(t)| \ll \exp\{D_n(t)\} = \exp\left\{S_n\left(\frac{t}{(1/\sqrt{3}) \log^{3/2} n}\right)\right\}
\]

\[
= \exp\left\{\left(\frac{\sin\left(\frac{t\sqrt{3}}{\log^{1/2} n}\right)}{t\sqrt{3}/\log^{1/2} n} - 1\right) \log n + O(1)\right\} \quad (2.22)
\]

Applying here the inequality \( \frac{\sin u}{u} \leq 1 - \frac{u^2}{9} \), which is true for \( |u| \leq 2 \), we have

\[
|\phi_{\alpha,n}(t)| \ll e^{-t^2/3},
\]

for \( |t| \leq \delta \sqrt{\log n} \).

The estimate (2.20) is proven.

In the proof of estimate (2.21), we use some ideas of [10] and [13]. Let

\[
\sum_{n \geq 0} N_n(t) z^n = \exp\left\{\sum_{k=1}^{n} \frac{e^{it \log k}}{k} z^k\right\}.
\]

Differentiating this identity with respect to \( z \), we can easily note that \( N_n \) satisfy the recurrent relation

\[
N_n(t) = \frac{1}{n} \sum_{k=1}^{n} e^{it \log k} N_{n-k}(t).
\]

Applying the Cauchy inequality and the Parseval identity, we have
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\[ |N_n(t)| \leq \left( \frac{1}{n} \sum_{k=1}^{\infty} |N_k(t)|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi n} \int_{-\pi}^{\pi} \left| \exp \left\{ \sum_{k=1}^{n} \frac{e^{it\log k}}{k} e^{ixk} \right\} \right|^2 \, dx \right)^{1/2}. \] (2.23)

Integrating by parts for \( \pi \geq |x| \geq \frac{1}{n} \) and \( t \in \mathbb{R} \), we have

\[ \sum_{k=1}^{n} \frac{1}{k^{1-it}} e^{ixk} = \int_{1/\pi}^{n} \frac{1}{y^{1-it}} d \left( \sum_{k<y} e^{ixk} \right) = (1-it) \int_{1/\pi}^{n} e^{ix} \frac{1-e^{ix|y|}}{1-e^{ix}} \frac{1}{y^{2-it}} dy + O(1) \]

\[ = \frac{1-it}{1-e^{ix}} \int_{1/\pi}^{1/|x|} \frac{1-e^{ix|y|}}{y^{2-it}} dy + O \left( \frac{|1-it|}{|e^{ix}-1|} \int_{1/|x|}^{n} dy \right) \]

\[ = \frac{1-it}{1-e^{ix}} \int_{1/\pi}^{1/|x|} \frac{y^{it}}{y^{2-it}} dy + O \left( \frac{|1-it|}{|e^{ix}-1|} \int_{1/|x|}^{1/|x|} \frac{e^{ix|y|}}{y^{2-it}} dy \right) + O(|1-it|) \]

\[ = -\frac{1-it}{e^{ix}-1} \int_{1/\pi}^{1/|x|} \frac{y^{it}}{y^{2-it}} dy + O \left( \frac{|1-it|}{|x|} \int_{1/|x|}^{1/|x|} |y|^2 dy \right) + O(|1-it|) \]

\[ = \frac{1-it}{e^{ix}-1} \int_{1/\pi}^{1/|x|} \frac{dy}{y^{1-it}} + O(|1-it|) = \frac{1-it}{e^{ix}-1} \int_{1/|x|}^{1/|x|} \frac{|x|^{-it} - 1}{it} + O(|1-it|) \]

\[ = \frac{e^{it \log \frac{1}{|x|}} - 1}{it} + O(|1-it|). \] (2.24)

Hence, it follows that

\[ \left| \exp \left\{ \sum_{k=1}^{n} \frac{e^{it\log k}}{k} e^{ixk} \right\} \right| \ll \exp \left\{ \sin \left( t \log \frac{1}{|x|} \right) \right\} \] (2.25)

for \( |t| \leq 1 \) and \( |x| > \frac{1}{n} \).
Since

\[ |\phi_{n,\alpha}(t)| = \left| N_n \left( \frac{t}{(\frac{1}{3} \log^3 n)^{1/2}} \right) \right|, \]

putting

\[ t' = \frac{t}{(\frac{1}{3} \log^3 n)^{1/2}} \]

for \( \log^{3/2} n \geq |t| \geq \delta_1 \log^{1/2} n \), from (2.23) and (2.25) we get

\[ |\phi_{n,\alpha}(t)|^2 \ll \frac{1}{n} \left( \int_{|x| \geq \frac{1}{n}} + \frac{1}{n} \int_{|x| \geq \frac{1}{n}} + \frac{1}{n} \int_{|x| < \frac{1}{n}} \right) \]

\[ \times \left| \exp \left\{ \sum_{k=1}^{n} \frac{e^{it' \log k}}{k} e^{ix_k} \right\} \right|^2 dx = I_1 + I_2 + I_3. \]

Since \( \frac{\sin(t \log (1/|x|))}{t} \leq \log (1/|x|) \), we have

\[ I_1 \leq \frac{1}{n} \int_{|x| \geq \frac{1}{n}} \exp \left( 2 \log \frac{1}{|x|} \right) dx \ll \frac{1}{\sqrt{n}}. \]

\[ I_2 \ll \frac{1}{n} \int_{\frac{1}{\sqrt{n}} > |x| > \frac{1}{n}} \exp \left\{ 2 \log \frac{1}{|x|} \left( \frac{\sin \left( t' \log \frac{1}{|x|} \right)}{t' \log \frac{1}{|x|}} \right) \right\} dx \]

\[ \leq \frac{1}{n} \int_{\frac{1}{\sqrt{n}} > |x| > \frac{1}{n}} \exp \left( 2 \log \frac{1}{|x|} (1 - \varepsilon) \right) dx \ll \frac{1}{n} \int_{1 > |x| > \frac{1}{n}} \frac{dx}{|x|^{2(1-\varepsilon)}} \ll \frac{1}{n^{2\varepsilon}}, \]

where \( 1 - \varepsilon := \max_{u > (\delta_1/2) \sqrt{3}} \frac{\sin u}{u} \).

Finally, since \( \exp \{ \sum_{k=1}^{n} \frac{e^{it' \log k}}{k} e^{ix_k} \} = n \exp \{ S_n(t') + O(1) \} \) for \( |x| \leq \frac{1}{n} \), applying (2.19) with \( \varepsilon_1 = \delta_1 \), we have

\[ I_3 \ll \exp \{ 2\Re S_n(t') \} \ll \frac{1}{n^{\lambda(\delta_1)}}. \]

The theorem is proved. \( \square \)

**Theorem 16.** Let \( a_1, a_2, \ldots, a_n \) be real numbers. We denote

\[ f_{S_n}(t) = M_n e^{it \sum_{k=1}^{n} a_k \alpha_k(\sigma)} \quad \text{and} \quad f_{E_n}(t) = M_n e^{it \sum_{k=1}^{n} a_k \xi_k(P)}. \]
2.1. PROOFS

If \( \max_{1 \leq k \leq n} |a_k|/q^{k/4} \leq \beta_n \), then, for \( |t| \leq \frac{1}{\beta_n} \), we have \( |f_{S_n}(t) - f_{E_n}(t)| \ll \beta_n|t| \).

Proof. Let \( a_k = 0 \) for \( k > n \). Taking \( f(k) = e^{ita_k} \) in (2.1), we have

\[
F_{E_n}(z) = \sum_{m=0}^{\infty} f_{E_m}(t) z^m = \prod_{k=1}^{\infty} \left( 1 - \left( \frac{z}{q} \right)^k e^{ita_k} \right)^{-I_k},
\]

\[
F_{S_n}(z) = \sum_{m=0}^{\infty} f_{S_m}(t) z^m = \exp \left\{ \sum_{k=1}^{\infty} \frac{e^{ita_k}}{k} z^k \right\},
\]

\[
F_{E_n}(z) = \exp \left\{ \sum_{k=1}^{\infty} \frac{e^{ita_k}}{k} z^k + H(z, t) \right\} = F_{S_n}(z) \exp \{ H(z, t) \}, \tag{2.26}
\]

where

\[
H(z, t) = -\sum_{k=1}^{\infty} \frac{q^k}{k} \log \left( 1 - \left( \frac{z}{q} \right)^k e^{ita_k} \right) + \left( \frac{z}{q} \right)^k e^{ita_k} \]

\[
+ \sum_{k=1}^{\infty} A_k \frac{q^{k/2}}{k} \log \left( 1 - \left( \frac{z}{q} \right)^k e^{ita_k} \right),
\]

here \( A_k \) are the same numbers as in (2.3). For \( t = 0 \), we have

\[
\frac{1}{1 - z} = \prod_{k=1}^{\infty} \left( 1 - \left( \frac{z}{q} \right)^k \right)^{-I_k} = \exp \{ H(z, 0) \}
\]

\[
= \frac{\exp \{ H(z, 0) \}}{1 - z},
\]

and, therefore, \( H(z, 0) = 0 \). Using this identity, from (2.26) we have

\[
F_{E_n}(z) = F_{S_n}(z) \exp \{ H(z, t) - H(z, 0) \}
\]

\[
= F_{S_n}(z) \exp \left\{ \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{j=2}^{\infty} \left( \frac{z}{q} \right)^{jk} \frac{(e^{ita_k} - 1)}{j} \right\}
\]

\[
+ \sum_{k=1}^{n} \frac{q^{k/2}}{k} A_k \sum_{j=1}^{\infty} \left( \frac{z}{q} \right)^{jk} \frac{(e^{ita_k} - 1)}{j} \}
\]

\[
= F_{S_n}(z) \exp \left\{ |t| \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{j=2}^{\infty} \left( \frac{z}{q} \right)^{jk} a_k b_{kj}(t) \right\}
\]

\[
+ |t| \sum_{k=1}^{\infty} \frac{q^{k/2}}{k} A_k \sum_{j=1}^{\infty} \left( \frac{z}{q} \right)^{jk} a_k b_{kj}(t) \right\},
\]
where \(|b_{kj}| \leq 1\). Therefore, we have

\[ F_{E_n}(z) = F_{S_n}(z) \exp \left\{ \beta_n|t| \sum_{m=1}^{\infty} \gamma_m(t)z^m \right\}, \]

where \(|\gamma_m(t)| \leq \frac{A}{q^{m/4}}\) with an absolute constant \(A\). Recalling that \(\beta_n|t| \leq 1\) for \(s \geq 1\), we get

\[
\left[ \exp \left\{ \beta_n|t| \sum_{m=1}^{\infty} \gamma_m(t)z^m \right\} \right]_{(s)} \leq \left[ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \beta_n|t| \sum_{m=1}^{\infty} \gamma_m(t)z^m \right)^k \right]_{(s)} \leq \left[ 1 + \beta_n|t| \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{\sum_{m=1}^{\infty} A}{q^{m/4}} z^m \right)^k \right]_{(s)} = \left[ \beta_n|t| \exp \left\{ \sum_{m=1}^{\infty} \frac{A}{q^{m/4}} z^m \right\} \right]_{(s)} \ll \frac{\beta_n|t|}{q^{s/8}}.
\]

Since \([F_{S_n}(z)]_{(k)} \leq 1\), using the inequality obtained before, we have

\[
f_{E_n}(t) = [F_{E_n}(z)]_{(n)} = \left[ F_{S_n}(z) \exp \left\{ \beta_n|t| \sum_{m=1}^{\infty} \gamma_m(t)z^m \right\} \right]_{(n)} = f_{S_n}(t) + O\left( \beta_n|t| \sum_{k=1}^{n} \frac{1}{q^{k/8}} \right) = f_{S_n}(t) + O(\beta_n|t|).
\]

The theorem is proved. \(\Box\)

Applying Theorem 16 with \(a_k = \frac{\log k}{(1/\sqrt{3}) \log^{3/2} n}\), we have

\[
|\phi_{n,\alpha}(t) - \phi_{n,\xi}(t)| \ll \frac{|t|}{\log^{3/2} n} \quad (2.27)
\]

for \(|t| \leq \log^{3/2} n\).

Let

\[
G_n(x) = \Phi(x) + \frac{3^{3/2}}{24\sqrt{2\pi}} (1 - x^2)e^{-x^2/2}\frac{1}{\sqrt{\log n}}.
\]

Then we have

\[
\int_{-\infty}^{+\infty} e^{itx} dG_n(x) = e^{-\frac{t^2}{4}} \left( 1 + \frac{3}{4!} \frac{3^{3/2}}{\log^{1/2} n} \right) =: g_n(t).
\]
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Let us also denote

\[ F_n(x) = \nu_n \left( \frac{\log P_n(\xi) - \frac{1}{2} \log^2 n}{(\frac{1}{3} \log^3 n)^{1/2}} < x \right). \]

Applying the generalized Esseen inequality (see e.g. [18]), we have

\[
\sup_{x \in \mathbb{R}} |G_n(x) - F_n(x)| \ll \int_{-\log^{3/4} n}^{\log^{3/4} n} \frac{\left| \phi_{n,\alpha}(t) - g_n(t) \right|}{|t|} dt + \frac{1}{\log^{3/4} n}.
\]

Using (2.27), we can estimate the second integral in this equality by \( O(\log^{-3/4} n) \).

We estimate the first integral using, in the interval \(|t| \leq \frac{1}{\log^2 n}\), the estimate

\[
\left| \phi_{n,\alpha}(t) - 1 \right| \leq \left| \phi_{n,\alpha}(t) - \phi_{n,\xi}(t) \right| + \left| 1 - \phi_{n,\xi}(t) \right| \ll |t| \left( \frac{1}{\log^{3/2} n} + M_n \left( \frac{\log P_n(\xi(\xi)) - (1/2) \log^2 n}{(1/\sqrt{3}) \log^{3/2} n} \right) \right) \ll |t| \log^{1/2} n.
\]

and, in the intervals \( \frac{1}{\log^2 n} < |t| \leq \log^{1/6} n \), \( \log^{1/6} n \leq |t| < \delta_1 \log^{1/2} n \), and \( \delta_1 \log^{1/2} n < |t| < \log^{2/3} n \), estimates (2.22), (2.20), and (2.21), respectively.

Finally, we have

\[
\sup_{x \in \mathbb{R}} |G_n(x) - F_n(x)| \ll \frac{1}{\log^{3/4} n}. \quad (2.28)
\]

To estimate the closeness between the distribution functions of \( \log O_n(\xi) \) and \( \log P_n(\xi) \), we use the following lemma, which is a generalization of Lemma 2.5 of [3].

**Lemma 17.** Let \( U \) and \( X \) be random variables. Suppose that \( \sup_{x \in \mathbb{R}} |P(U < x) - G(x)| \leq \eta \), where \( G(x) \) is a differentiable function satisfying the condition \( |G'(x)| \leq C \). Then, for each \( \varepsilon > 0 \), we have

\[
\sup_{x \in \mathbb{R}} |P(U + X < x) - G(x)| \ll \eta + \varepsilon + P(|X| > \varepsilon),
\]

where the constant in the symbol \( \ll \) depends on \( C \) only.
Proof of theorem \[4\] Putting, in Lemma \[17\]

\[X = -\frac{\log P_n(\xi) - \log O_n(\xi) - \mu_n}{(1/\sqrt{3}) \log^{3/2} n}\quad \text{and} \quad U = \frac{\log P_n(\xi) - M_n \log P_n(\xi)}{(1/\sqrt{3}) \log^{3/2} n},\]

\[\epsilon = ((\log \log n) / \log n)^{2/3}, \eta = 1/\log^{3/4} n\] and then applying (2.28) and Proposition 1, we obtain the proof of the theorem.

Proof of theorems \[3\] and \[4\]. In order to prove that the results of this paper hold in the case of the group \(S_n\), one can either repeat the proofs given above in a simplified way, or pass to the limit as \(q \to \infty\) for fixed \(n\) and using the facts that \((\xi_1(P), \ldots, \xi_n(P)) \to (\alpha_1(\sigma), \ldots, \alpha_n(\sigma))\) weakly as \(q \to \infty\) and that the constants in the symbols \(O\) and \(\ll\) do not depend on \(q\) in all the proofs given above.
Chapter 3

Functions on $S_n^{(k)}$

3.1 Means of multiplicative functions on $S_n^{(k)}$

As before we denote

$$S_n^{(k)} = \{ \sigma = x^k | x \in S_n \}.$$

In [14] Mineev and Pavlov proved the following criterion to determine whether $\sigma$ belongs to the set $S_n^{(k)}$.

**Theorem C.** Suppose $k$ has the following decomposition into the product of prime numbers $k = p_1^{l_1(p_1)} p_2^{l_2(p_2)} \ldots p_s^{l_s(p_s)}$. Let us define the function

$$q_k(j) = \prod_{p|k,j} p^{l_k(p)}.$$

Then $\sigma \in S_n^{(k)}$ if and only if

$$q_k(j)|\alpha_j(\sigma)$$

for all $1 \leq j \leq n$.

We have

$$M_n^{(k)} f = \frac{1}{|S_n^{(k)}|} \sum_{\sigma \in S_n^{(k)}} f(\sigma) = \frac{1}{|S_n^{(k)}|} \sum_{s_1 + 2s_2 + \ldots + ns_n = n} \prod_{j=1}^{n} \hat{f}(j)^{s_j} n! \prod_{j=1}^{n} \frac{1}{j^{s_j} s_j!}$$

$$= \frac{n!}{|S_n^{(k)}|} \sum_{s_1 + 2s_2 + \ldots + ns_n = n} \prod_{q_k(l)|s_l} \prod_{j=1}^{n} \left( \frac{\hat{f}(j)}{j} \right)^{s_j} \frac{1}{s_j!}.$$
CHAPTER 3. FUNCTIONS ON $S_N^{(K)}$

Here we have used the well known fact that the quantity of $\sigma \in S_n$ such that $\alpha_j(\sigma) = s_j$ for $1 \leq j \leq n$, equals

$$n! \prod_{j=1}^{n} \frac{1}{s_j! j^{s_j}},$$

when $s_1 + 2s_2 + \cdots + ns_n = n$. Hence one can easily see that the following identity holds

$$\sum_{n=0}^{\infty} \frac{|S_n^{(K)}|}{n!} M_n^{(k)} f z^n = \prod_{j=1}^{\infty} \left( 1 + \sum_{j \geq 1 \atop q_k(j)|s} \left( \frac{\hat{f}(j) z^j}{j} \right)^s \frac{1}{s!} \right) = \exp \left\{ \sum_{j \geq 1 \atop (j,k)=1} \frac{\hat{f}(j) z^j}{j} \right\} H_k(f; z).$$

Further we will assume that $|\hat{f}(j)| \leq 1$. We will denote by $M_n(f) = M_n^{(k)}(f)$ the mean value of $f(\sigma)$ on the subset $S_n^{(k)}$. Let us define

$$\mu_n(p) = \left( \frac{1}{n} \sum_{1 \leq j \leq n \atop (j,k)=1} |\hat{f}(j) - 1|^p \right)^{1/p}.$$

In the works [10], [13] and [23] there have been obtained the estimates of the mean values of the multiplicative functions on whole group $S_n$.

The following theorem establishes analogous estimate for $M_n^{(k)} f$.

**Theorem 17.** Suppose $|\hat{f}(j)| \leq 1$. Then

$$M_n^{(k)} f = \exp \left\{ \sum_{1 \leq j \leq n \atop (j,k)=1} \frac{\hat{f}(j) - 1}{j} \right\} \prod_{j \leq n \atop (j,k)>1} \left( 1 + \sum_{s \geq 1 \atop q_k(j)|s} \frac{\hat{f}(j)^s}{j^s s!} \right) + O \left( \mu_n(p) + \frac{1}{n^\beta} \right),$$

for $p > \frac{1}{\beta}$. Here $\beta = \frac{\phi(k)}{k}$, if $k$ is prime, and

$$\beta = \min_{d: d|k, d>1} \frac{\phi(k)}{k} \left( 1 - \mu(d) \prod_{p|d} \frac{1}{p-1} \right),$$

where

$$\mu(d) = \sum_{1 \leq j \leq \nu(d) \atop (j,k)=1} |\hat{f}(j) - 1|^p \right)^{1/p}.$$
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if $k$ is composite, where $\mu(d)$ – Moebius function, $\phi(k)$ – Euler function.

The next theorem is the analog of the well known Halsz - Wirsinig result for the multiplicative functions on natural numbers. Analogous result for multiplicative functions on permutations has been obtained in [13].

**Theorem 18.** Suppose we have a fixed sequence of complex numbers $\hat{f}(j)$ such that $|\hat{f}(j)| \leq 1$. Then there are two possible cases concerning the asymptotic behavior of the corresponding sequence of means $M_n^{(k)}f$.

1. If the series
   \[
   \sum_{j \geq 1 \atop (j,k) = 1} 1 - \Re(\hat{f}(j)e^{-ixj}) \frac{j}{j}
   \]  
   diverges for every $x \in [-\pi, \pi]$ then
   \[
   \lim_{n \to \infty} M_n^{(k)}f = 0
   \]

2. If
   \[
   \sum_{j \geq 1 \atop (j,k) = 1} 1 - \Re(\hat{f}(j)e^{-ix_0j}) \frac{j}{j} < \infty,
   \]
   for some $x_0 \in [-\pi, \pi]$, then
   \[
   M_n^{(k)}f = e^{ix_0n} \exp\left\{ \sum_{1 \leq j \leq n \atop (j,k) = 1} \frac{\hat{f}(j)e^{-ix_0j} - 1}{j} \right\} \prod_{j \leq n \atop (j,k) > 1} \left( 1 + \sum_{s \geq 1: q_k(j) | s} \frac{(\hat{f}(j)e^{-ix_0j})^s}{j^s s!} \right) + o(1).
   \]

In what follows we assume that $k$ is a fixed natural number, therefore we will often omit the index $k$ in the notation of the mean value $M_n f = M_n^{(k)} f$ and measure $\nu_n = \nu_n^{(k)}$.

Further we will denote $k_0 = \prod_{p | k} p$. Putting in (3.1) $\hat{f}(j) \equiv 1$, we obtain
\[
F(z) = \sum_{n=0}^{\infty} |S_n^{(k)}| \frac{z^n}{n!} = p(z) H_k(1; z)
\]

where
\[
p(z) = \sum_{j=0}^{\infty} p_j z^j = \exp\left\{ \sum_{j \geq 1 \atop (j,k) = 1} \frac{z^j}{j} \right\} = \prod_{m | k} \left( 1 - z^m \right)^{\mu(m)/m} = \prod_{j=0}^{k_0-1} \frac{1}{(1 - ze^{-2\pi i x_0})^{\gamma_j}},
\]
and $\gamma_j = \frac{\phi(k)}{k} \mu(l_j) \prod_{p\mid l_j} \frac{1}{p^{l_j}}$, where $l_j = \frac{k_0}{(j,k_0)}$ for $1 \leq j < k_0$, $\gamma_0 = \frac{\phi(k)}{k}$. In the work of Mineev and Pavlov it has been proved that $\gamma_j < \gamma_0 = \frac{\phi(k)}{k}$, if $j \neq 0$. One can easily see also that $|\gamma_j| \leq \gamma_0$, moreover, $|\gamma_j| = \gamma_0$ if and only if $l_j = 2$, that is when $\frac{j}{k_0} = \frac{1}{2}$.

Further we will denote by $\epsilon$ some positive, fixed number, not necessarily the same in different places.

Suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we will the following notation for the $n$-th coefficient in the Taylor expansion of $f$: $[f(z)](n) = a_n$. Further we will often use some simple properties of this notation, which we formulate in a form of the lemma.

**Lemma 18.** Suppose $u(z), v(z), U(z), V(z), \psi(z)$ are analytic in the vicinity of zero and such that $|[u(z)](n)| \leq [U(z)](n)$, $|[v(z)](n)| \leq [V(z)](n)$ and $[\psi(z)](n) \geq 0$ for $n \geq 0$. Then for $n \geq 0$ the following inequalities hold:

1. $|[u(z)v(z)](n)| \leq [U(z)V(z)](n),$

2. $|[e^{u(z)}](n)| \leq [e^{U(z)}](n),$

3. $|[e^{u(z)} - 1](n)| \leq \epsilon [e^{U(z)} - 1](n), \text{ for } 0 \leq \epsilon \leq 1,$

4. $[U(z)](n) \leq [U(z)(1 + V(z))] (n),$

5. $0 \leq \frac{1}{2} [\psi(z)(e^{\psi(z)} - 1)](n) \leq [1 + e^{\psi(z)}\psi(z) - e^{\psi(z)}](n) \leq [\psi(z)(e^{\psi(z)} - 1)](n).$

The estimates which are similar to those of the lemmas 19, 25 and 26 have been obtained in the work of Pavlov. For the sake of completeness we give here somewhat more elementary their proofs. We will estimate the $n$-th Taylor coefficient of the function $H_k(f; z)$.

**Lemma 19.** For $n \geq 1$ and $|\hat{f}(j)| \leq 1$ we have

$$[H_k(f; z)](n) = O \left( \frac{1}{n^2} \right).$$
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Proof. Applying lemma 18 one can easily see that for $n \geq 0$ and $|\hat{f}(j)| \leq 1$

$$[[H_k(f; z)](n)] \leq \left[\prod_{j \geq 1} \left(1 + \sum_{q_k(j) > 1} \left(\frac{z^j}{j} \frac{1}{s!}\right)^{q_k(j)}\right)\right]_{(n)}$$

$$\leq \left[\prod_{j \geq 1} \left(1 + \sum_{l=1}^{\infty} \left(\frac{z^j}{j}\right)^{lq_k(j)} \frac{1}{l!}\right)\right]_{(n)} = \left[\exp \left\{ \sum_{j \geq 1} \frac{z^{j}q_k(j)}{j^{q_k(j)}} \right\} \right]_{(n)}$$

$$ \leq \left[\exp \left\{ \sum_{j \geq 1} \frac{z^{j}q_k(j)}{j^{2}} \right\} \right]_{(n)} \leq \left[\exp \left\{ k^2 \sum_{j=1}^{\infty} \frac{z^{j}}{j^{2}} \right\} \right]_{(n)} ;$$

here we have used the fact that $2 \leq q_k(j) \leq k$ for $(j, k) > 1$. Therefore $[[H_k(f; z)](n)] \leq g_n$, where

$$g(z) = \sum_{n=0}^{\infty} g_n z^n = \exp \left\{ k^2 \sum_{j=1}^{\infty} \frac{z^j}{j^2} \right\}.$$ 

One can easily see that $g''(x) \ll \frac{1}{1-x}$ for $0 \leq x \leq 1$. Therefore

$$\sum_{j=1}^{\infty} j^2 g_j x^j \ll \frac{1}{1-x}.$$ 

Putting here $x = e^{-1/n}$, we obtain

$$\sum_{n \leq j \leq 2n} g_j \ll \frac{1}{n}.$$ 

Since $z g'(z) = k^2 g(z) \sum_{m=1}^{\infty} \frac{z^m}{m^2}$, then

$$ng_n = k^2 \sum_{j=0}^{n-1} \frac{g_j}{n-j} \leq 2k^2 \sum_{j \leq n/2} g_j + k^2 \sum_{n/2 < j \leq n} g_j,$$

therefore $g_n = O \left( \frac{1}{n^*} \right)$, whence it follows that $[[H_k(f; z)](n)] = O \left( \frac{1}{n^*} \right)$.

The lemma is proved.
CHAPTER 3. FUNCTIONS ON $S_{N_N}^{(k)}$

The Lemma 19 shows that the main contribution to the value of $M_n^{(k)}f$ is done by the coefficients of the function

$$F(z) = \sum_{m=0}^{\infty} M_m z^m = \exp \left\{ \sum_{(j,k)=1}^{\hat{f}(j)} z^j \right\}$$

This generating function may be regarded as a special case of more general generating function (1.2) of Chapter 1 if we put

$$d_j = \begin{cases} 1, & \text{if } (j, k) = 1 \\ 0, & \text{if } (j, k) > 1 \end{cases} \quad (3.2)$$

Unfortunately, Theorem 1 is not directly applicable here, as in our case the parameters $d_j$ are not bounded from below by a positive constant. The proof of Theorem 1 was based on estimate of Theorem 9, in the proof of which we used the condition $d_j \geq d^- > 0$. In a general case this condition can hardly be removed, because the behavior of corresponding $p_j$ might become irregular. Let us take for example in

$$p(z) = \exp \left\{ \sum_{j=1}^{\infty} d_j \frac{z^j}{j} \right\}$$

$$d_j = \begin{cases} 1, & \text{if } 2 | j \\ 0, & \text{if } 2 \not| j \end{cases}$$

we will have then $p(z) = \sum_{j=0}^{\infty} p_j = \frac{1}{(1-z)^2}$, which gives $p_{2j+1} = 0$ for $j \geq 0$.

However, in our case when $d_j$ are defined by (3.2) the function $p(z)$ has a good analytic continuation beyond the unit disc and good behavior of its special points which enables us to establish the analog of Theorem 9.

As in Chapter 1 we define $g_{j,x}$ by means of relation

$$G_x(z) = \frac{p(z)}{p(xz)} = \sum_{j=0}^{\infty} g_{j,x} z^j \quad (3.4)$$

Lemma 20. For $0 \leq x \leq e^{-1/n}$ we have

$$g_{n,x} = \frac{p_n}{p(x)} + O \left( n^{\gamma'-1}(1-x)^{\gamma'} + n^{\gamma-2}(1-x)^{\gamma-1} + \frac{1}{n} \right),$$

where $\gamma' = \max_{j \neq 0} \{ \max \{ \gamma_j, 0 \} \}$.
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Proof. Applying Cauchy formula we have

$$g_{n,x} = \frac{1}{2\pi i} \int_{C_{\phi,R}} \frac{p(z)}{p(zx)} \frac{dz}{z^{n+1}} = \sum_{j=0}^{k-1} \frac{1}{2\pi i} \int_{C_{\phi,R,j}} \frac{p(z)}{p(zx)} \frac{dz}{z^{n+1}} + O\left(\frac{1}{R^n}\right).$$

The integration contour in this formula is $C_{\phi,R} = C_R \cup \bigcup_{j=0}^{k-1} C_{\phi,R,j}$. Here for $\gamma_j > 0$, $C_{\phi,R,j}$ consists of the intervals on the complex plain, connecting the points $e^{2\pi ij/k}$ and $Re^{2\pi ij/k}$. For $\gamma_j < 0$, $C_{\phi,R,j}$ consists of the intervals \( \{z = e^{2\pi ij/k}(1 + re^{i\phi})\} \) and \( \{z = e^{2\pi ij/k}(1 + re^{-i\phi})\} \), where $1 \leq r \leq r_0$, and $r_0$ corresponds to the point of intersection with the circle $|z| = R$, $0 < \phi < \frac{\pi}{2}$ — fixed, sufficiently small angle.

Here $2 \leq R \leq 6$ is chosen in such a way that $|1 - Rx| \geq 1/2$.

Let $\gamma_j > 0$, then

$$\frac{1}{2\pi i} \int_{C_{\phi,R,j}} \frac{p(z)}{p(zx)} \frac{dz}{z^{n+1}} \ll \int_1^R \frac{1-xy}{1-y} \frac{\gamma_j}{y^{n+1}} \frac{dy}{n} = \frac{1}{n} \int_0^{Rn} \frac{1-xe^{u/n}}{1-e^{u/n}} \gamma_j e^{-u} du \ll \frac{1}{n} \int_0^{Rn} \left(\frac{n}{u}\right)^{\gamma_j} |1 - x + x(1 - e^{-u/n})|^\gamma_j e^{-u} du \ll n^{\gamma_j - 1}(1 - x)^{\gamma_j} + \frac{1}{n}$$

and for $\gamma_j < 0$

$$\frac{1}{2\pi i} \int_{C_{\phi,R,j}} \frac{p(z)}{p(zx)} \frac{dz}{z^{n+1}} \ll \int_{C_{\phi,R,j}} \frac{1-z}{1-xz} \frac{dz}{z^{n+1}} \ll \int_0^R \frac{r}{|1 - x(1 + re^{i\phi})|^{\gamma_j}} \frac{dr}{|1 + re^{i\phi}|^{n+1}} \ll \frac{1}{n},$$

because

$$|1 - x(1 + re^{i\phi})| = |(1 - x)e^{-i\phi/2} - rxe^{i\phi/2}| \geq \left|\frac{\sin \frac{\phi}{2}}{2}\right| |(1 - x) + rx|$$

and $|1 + re^{i\phi}| \geq 1 + r \cos \phi$.

Hence

$$g_{n,x} = \frac{1}{2\pi i} \int_{C_{\phi,R,0}} \frac{p(z)}{p(zx)} \frac{dz}{z^{n+1}} + O\left(\sum_{\gamma_j > 0} n^{\gamma_j - 1}(1 - x)^{\gamma_j} + \frac{1}{n}\right).$$
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Denoting $V_{x}(z) = \prod_{l=1}^{k-1} \left( \frac{1-xze^{-2\pi i/l}}{1-xe^{-2\pi i/l}} \right)^{\gamma_l}$, we obtain $\frac{p(z)}{p(x)} = V_{x}(z) \left( \frac{1-xz}{1-z} \right)^{\gamma_0}$.

From the above estimate we have

$$\frac{1}{2\pi i} \int_{C_{x,R,0}} \frac{p(z)}{p(x)} \frac{dz}{z^{n+1}} = \frac{V_x(1)}{2\pi i} \int_{C_{x,R,0}} \left( \frac{1-xz}{1-z} \right)^{\gamma_0} \frac{dz}{z^{n+1}} + \frac{1}{2\pi i} \int_{C_{x,R,0}} (V_x(z) - V_x(1)) \left( \frac{1-xz}{1-z} \right)^{\gamma_0} \frac{dz}{z^{n+1}} = I_1 + I_2.$$

Let us estimate $I_1$.

$$I_1 = \frac{V_x(1)(1-x)^{\gamma_0}}{2\pi i} \int_{C_{x,R,0}} \frac{dz}{(1-z)^{\gamma_0} z^{n+1}} + \frac{V_x(1)}{2\pi i} \int_{C_{x,R,0}} \left( \frac{1-x}{1-z} \right)^{\gamma_0} \left( \left( \frac{x(1-z)}{1-x} \right)^{\gamma_0} - 1 \right) \frac{dz}{z^{n+1}}$$

$$= V_x(1)(1-x)^{\gamma_0} \left( n + \gamma_0 - 1 \right) + O(n^{-1}) + O \left( \frac{|V_x(1)|}{2\pi i} \int_{C_{x,R,0}} \left( \frac{1-x}{1-z} \right)^{\gamma_0} \frac{|x(1-z)|}{1-x} \frac{dz}{z^{n+1}} \right)$$

$$= V_x(1)(1-x)^{\gamma_0} \left( n + \gamma_0 - 1 \right) + O(n^{-1}) + O((1-x)^{\gamma_0-1} n^{\gamma_0-2}).$$

Because $V_x'(z) \ll 1$ in the vicinity of the point $z = 1$, we have

$$I_2 \ll \int_{C_{x,R,0}} |1-z| \left| \frac{1-xz}{1-z} \right|^{\gamma_0} \frac{dz}{|z|^{n+1}} \ll \int_{C_{x,R,0}} |1-z|^{1-\gamma_0} \frac{dz}{|z|^{n+1}} \ll n^{\gamma_0-2}.$$

We have

$$V_x(1) = \lim_{x \to 1} \frac{p(z)}{p(x)} \left( \frac{1-z}{1-xz} \right)^{\gamma_0} = \frac{A_k}{p(x)(1-x)^{\gamma_0}},$$

where $A_k = \lim_{x \to 1} (1-x)^{\gamma_0} p(z)$.

Applying the earlier obtained estimates we have

$$g_{n,x} = \frac{A_k}{p(x)} \left( n + \gamma_0 - 1 \right) + O \left( \frac{1}{n} + (1-x)^{\gamma_0-1} n^{\gamma_0-2} + n^{\gamma'-1} (1-x)^{\gamma'} \right).$$

Putting here $x = 0$ and noting that $g_{n,0} = p_n$ we have

$$p_n = A_k \left( n + \gamma_0 - 1 \right) + O(n^{\gamma'-1}). \quad (3.3)$$

Inserting this estimate into the previous estimate we obtain the proof of the lemma.  \hfill \square
The following result has been proved in the work [17].

**Theorem D** ([17]). For $n \geq 1$ we have

$$c_n = \frac{|S^{(k)}_n|}{n!} = \frac{n^{\gamma_0 - 1}}{\Gamma(\gamma_0)} A_k h_k(1; 1) \left(1 + O(n^{-\beta})\right) = p_n h_k(1; 1) \left(1 + O(n^{-\beta})\right),$$

where $A_k = \lim_{x \to 1} p(x)(1-x)^{\gamma_0}$, $\beta = \gamma_0 - \gamma'$ and $\gamma' = \max_{j \neq 0} \{\gamma_j, 0\}$.

In formulation of this theorem in [17] the constant $\beta$ has not been written explicitly as $\beta = \gamma_0 - \gamma'$, although this formula could be easily obtained from the proof of the theorem there. Therefore, and also in order to make our exposition self-contained we present the proof of this theorem here.

**Proof of theorem D.** As $\sum_{n=0}^{\infty} \frac{|S^{(k)}_n|}{n!} = p(z) h_k(f, z)$ therefore applying Lemma [19] and estimate (3.3) we have

$$\frac{|S^{(k)}_n|}{n!} = \sum_{j=0}^{n} p_{n-j} [h_k(f, z)]_{(j)} = \sum_{j \leq n/2} A_k \left(\frac{n-j+\gamma_0-1}{n-j}\right) [h_k(f, z)]_{(j)}$$

$$+ O(n^{\gamma'-1}) + O\left(\frac{1}{n^2} \sum_{j \leq n/2} p_j\right) = A_k \left(\frac{n+\gamma_0-1}{n}\right) \sum_{j \leq n/2} [h_k(f, z)]_{(j)} + O\left(n^{\gamma_0-2} \log n\right) + O(n^{\gamma'-1})$$

$$+ O\left(\frac{p(e^{-1/n})}{n^2}\right) = \frac{A_k n^{\gamma_0-1}}{\Gamma(\gamma_0)} h_k(1, 1) + O\left(n^{\gamma_0-2} \log n\right) + O(n^{\gamma'-1}).$$

The theorem is proved.

**Theorem 19.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Then for $n \geq 1$ we have

$$\left| \frac{1}{p_n} \sum_{k=0}^{n} a_k p_{n-k} - f(e^{-1/n}) - \frac{S(f; n)}{np_n} \right| \leq C \left(\frac{1}{n^{\beta}} \sum_{j=1}^{n} \frac{|S(f; j)|}{p(e^{-1/n})} j^{\beta^{-1}} + \frac{1}{p(e^{-1/n})} \sum_{j>n} \frac{|S(f; j)|}{j} e^{-j/n}\right),$$

where $S(f; m) = \sum_{k=1}^{m} a_k k p_{m-k}$, $\beta = \gamma_0 - \gamma'$, and $C = C(k)$ - constant which depends on $k$ only, and $\gamma' = \max_{j \neq 0} \{\gamma_j, 0\}$.
**Proof.** The proof of this lemma is absolutely analogous to that of the theorem 9. Therefore we will repeat only the main steps.

From the proof of theorem 9 we have

\[ R_n = \sum_{k=0}^{n} a_k p_{n-k} - p_n f(e^{-1/n}) - \frac{S(f; n)}{n} \]

\[ = \sum_{j=1}^{n-1} S(f; j) \left( f_{n-j,j} - p_n \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \right) \]

\[ + p_n \sum_{j=n}^{\infty} S(f; j) \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx, \]

where

\[ f_{m,j} = \int_0^1 g_{m,x} x^{j-1} \, dx \geq 0. \]

Therefore

\[ |R_n| \leq \sum_{1 \leq j \leq n/2} |S(f; j)| \left| \int_0^1 x^{j-1} g_{n-j,x} \, dx - p_n \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \right| \]

\[ + \sum_{1 \leq j \leq n/2} |S(f; j)| |p_n - p_{n-j}| \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx \]

\[ + \sum_{n/2 < j < n-1} |S(f; j)| f_{n-j,j} \, dx + p_n \sum_{j \geq n} |S(f; j)| \int_0^{e^{-1/n}} \frac{x^{j-1}}{p(x)} \, dx. \]

In the proof of the Lemma 7 we did not use the condition \( d_j \geq d^- > 0 \) therefore its estimate \( f_{m,j} \ll \frac{1}{j} \) for \( j \geq m \geq 1 \) remains valid in the present case also.

The proof of Theorem 9 gives

\[ \int_0^{e^{-1/n}} x^{j-1} g_{n-j,x} \, dx \ll \frac{1}{n^2}, \]

for \( j \leq n/2. \)

Applying these estimates together with the Lemma 20 and Theorem 3.3 we obtain the proof of the theorem.

Let us denote

\[ L_n(z) = \sum_{1 \leq j \leq n} \hat{f}(j) \frac{1}{j} z^k \quad \text{and} \quad \rho(p) = \left( \sum_{1 \leq j \leq n} \frac{|\hat{f}(j) - 1|^p}{j} \right)^{1/p}; \]
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we will also assume that

$$\rho(\infty) = \lim_{p \to \infty} \max_{1 \leq j \leq n} |\hat{f}(j) - 1|.$$ 

Lemma 21. For any fixed $\infty \geq p > \frac{1}{\beta}$ we have

$$\frac{M_n}{p_n} = \exp\left\{ \sum_{1 \leq j \leq n} \frac{\hat{f}(j) - 1}{j} \right\} + O(\mu_n(p)).$$

Proof. We will suppose that $\hat{f}(j) = 1$ for $j > n$. We have

$$F(z) = \sum_{m=0}^{\infty} M_m z^m = \exp\left\{ \sum_{(j,k)=1}^{j \leq n} \frac{\hat{f}(j)}{j} z^j \right\} = p(z) \exp\{L_n(z)\} = p(z) h(z),$$

where $h(z) = \sum_{j=0}^{\infty} h_j z^j = \exp\{L_n(z)\}$. Therefore

$$\frac{M_n}{p_n} = \frac{1}{p_n} \sum_{j=0}^{n} h_j p_{n-j}.$$

Applying theorem 19 with $a_j = h_j$, we have

$$S(m; h) = [zp(z) h'(z)]_{(m)} = [zp(z) h(z)L'_n(z)]_{(m)} = [zF(z) L'_n(z)]_{(m)}$$

$$= \sum_{1 \leq j \leq m} (\hat{f}(j) - 1) M_{m-j}.$$

Taking into account that $|M_m| \leq p_m = \frac{A_k}{\Gamma(m)} m^{\gamma_0-1} (1 + o(1))$, and applying Cauchy inequality with parameters $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$|S(m; h)| \leq \left( \sum_{1 \leq j \leq m} |\hat{f}(j) - 1|^p \right)^{1/p} \left( \sum_{j=0}^{m} p_j^{q} \right)^{1/q} \ll \mu_n(p) n^{1/p} \left( 1 + \sum_{j=1}^{m} j^{(\gamma_0-1)q} \right)^{1/q}$$

$$\ll \mu_n(p) n^{1/p} m^{\frac{\gamma_0-1+1}{q}} \ll \mu_n(p) \left( \frac{n}{m} \right)^{1/p} m^{\gamma_0}.$$

Inserting this estimate into the inequality of theorem 19 we have

$$\left| \frac{M_n}{p_n} - h(e^{-1/n}) \right| \ll \mu_n(p).$$
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Since 
\[ h(e^{1/n}) = h(1)(1 + O(\mu_n(p))) \]
and 
\[ h(1) = \exp\{L_n(1)\} = \exp\left\{ \sum_{1 \leq j \leq n} \frac{\hat{f}(j) - 1}{j} \right\}, \]

hence we obtain the proof of the lemma. \( \square \)

**Proof of theorem 17.** Because
\[ \left| S_n^{(k)} \right| M_n(f) = \sum_{j=0}^{n} M_{n-j}[H_k(f; j)](j), \]
and 
\[ [H_k(f; z)](j) \leq [H_k(z)](j) = O(j^{-2}), \quad |M_m| \leq p_m = O(m^{\gamma_0-1}), \]
then
\[ \left| S_n^{(k)} \right| M_n(f) = \sum_{j \leq n/2} M_{n-j}[H_k(f; z)](j) + O(n^{\gamma_0-2}). \]

Applying here the asymptotic \( M_{n-j} = p_{n-j}(\exp\{L_{n-j}(1)\} + \mu_{n-j}(p)) \), we obtain the proof of the theorem. \( \square \)

The proof of the next theorem is obtained by applying theorem 19 to function \( h_n(z) = \exp\{L_N(z)\} - \exp\{L_n(1)\} L_N(z) \) and using the estimates of Lemma 19. The calculations are absolutely analogous to those of the proof of Theorem 12, therefore we will not repeat them here.

**Theorem 20.** For any fixed \( \infty \geq p > \frac{1}{\beta} \) there exists such positive \( \delta = \delta(k, p) \) that if \( \rho \leq \delta \) then

\[ M_n f = \frac{H(f; 1)}{H(1, 1)} \exp\{L_N(1)\} \left( 1 + \sum_{1 \leq j \leq N} \frac{\hat{f}(j) - 1}{j} \frac{p_{n-j}}{p_N} - 1 \right) + O\left( \rho^2 + \frac{1}{n^\epsilon} \right), \]

where \( \epsilon > 0 \) — fixed sufficiently small number.

Let us define \( M_m \) by means of relation

\[ M(z) = \exp\left\{ \sum_{j \leq n} \frac{\hat{f}(j)}{j} z^j \right\} = \exp\{U_n(z)\} = \sum_{m=0}^{\infty} M_m z^m, \]

where \( |\hat{f}(j)| \leq 1; \) then \( |M_m| \leq p_m \) for \( m \geq 0. \)
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Lemma 22. For any multiplicative function $f$ such that $|\hat{f}(j)| \leq 1$, we have

$$M_n f \ll \left( \frac{1}{n^{2\gamma_0 + 1}} \int_{-\pi}^{\pi} |M(e^{ix})|^2 |U_n'(e^{ix})|^2 \, dx \right)^{\frac{\gamma_0}{2(\gamma_0 + 1)}} + \frac{1}{n^{\gamma_0}}.$$ 

Proof. Further we assume that $\hat{f}(j) = 0$ for $j > n$, since $\hat{f}(j)$ for $j > n$ do not influence the value of $M_n$. Differentiating $M(z)$ by $z$ one can easily check that $M_n$ satisfy the recurrence relation

$$M_m = \frac{1}{m} \sum_{(j,k) = 1 \atop 1 \leq j \leq m} \hat{f}(j) M_{m-j}$$

for $m \geq 1$. Therefore for $1 \leq T \leq n$ and $n/2 \leq m \leq n$ we have

$$|M_m| \leq \frac{2}{n} \sum_{j=0}^{n-1} |M_j| \leq \frac{2}{n} \sum_{0 \leq j \leq T} |M_j| + \frac{2}{nT} \sum_{T < j \leq n} j |M_j|$$

$$\leq 2 e^{p(e^{-1/T})} + 2 \sqrt{n} \left( \sum_{j=1}^{\infty} |jM_j|^2 \right)^{1/2}.$$ 

Putting here $T = [\epsilon n]$ with $1/n \leq \epsilon \leq 1$, for $n/2 \leq m \leq n$ we obtain

$$\frac{|M_m|}{n^{\gamma_0 - 1}} \ll \left( \frac{T}{n} \right)^{\gamma_0} + \frac{1}{n^{\gamma_0 - \frac{1}{2}T}} \left( \int_{-\pi}^{\pi} |M'(e^{ix})|^2 \, dx \right)^{1/2}$$

$$\ll \epsilon^{\gamma_0} \frac{1}{\epsilon} \left( \frac{1}{n^{2\gamma_0 + 1}} \int_{-\pi}^{\pi} |M(e^{ix})|^2 |U_n'(e^{ix})|^2 \, dx \right)^{1/2}.$$

$$\ll \left( \frac{1}{n^{2\gamma_0 + 1}} \int_{-\pi}^{\pi} |M(e^{ix})|^2 |U_n'(e^{ix})|^2 \, dx \right)^{\frac{\gamma_0}{2(\gamma_0 + 1)}} + \frac{1}{n^{\gamma_0}}.$$ 

Here we have taken the minimum by $1/n \leq \epsilon \leq 1$.

Lemma 19 gives $[H(f; z)](m) = O(m^{-2})$, and also $|M_m| \leq p_m = O(m^{\gamma_0 - 1})$, therefore

$$\frac{|S_n|}{n!} M_n f = \sum_{m=0}^{n} M_m [H(f; z)](n-m)$$

$$\ll n^{\gamma_0 - 1} \left( \frac{1}{n^{2\gamma_0 + 1}} \int_{-\pi}^{\pi} |M(e^{ix})|^2 |U_n'(e^{ix})|^2 \, dx \right)^{\frac{\gamma_0}{2(\gamma_0 + 1)}} + \frac{1}{n} + \frac{1}{n^2} \sum_{m \leq n/2} p_m$$

$$\ll n^{\gamma_0 - 1} \left( \frac{1}{n^{2\gamma_0 + 1}} \int_{-\pi}^{\pi} |M(e^{ix})|^2 |U_n'(e^{ix})|^2 \, dx \right)^{\frac{\gamma_0}{2(\gamma_0 + 1)}} + \frac{1}{n} + n^{\gamma_0 - 2}.$$ 

The lemma is proved. □
Corollary 1. Suppose \( \hat{f}(j) \in \mathbb{C} \) and \( |\hat{f}(j)| \leq 1 \) for \( 1 \leq j \leq n \). Let us denote

\[
J(n) := \min_{x \in [-\pi, \pi]} \sum_{1 \leq j \leq n} \frac{1 - \Re(\hat{f}(j)e^{-ix})}{j}
\]

Then for the mean value of the corresponding multiplicative function \( f \) we have

\[
M_n f \ll \exp \left\{ -\frac{\gamma_0}{\gamma_0 + 1} J(n) \right\} + \frac{1}{n^{\gamma_0}}
\]

Proof. Applying Lemma 22 and noticing that

\[
\max_{x \in [-\pi, \pi]} |M(e^{ix})| \ll n^{\gamma_0} \exp \{-J(n)\},
\]

we have

\[
M_n f \ll \exp \left\{ -\frac{\gamma_0}{\gamma_0 + 1} J(n) \right\} \left( \frac{1}{n} \int_{-\pi}^{\pi} |U'_n(e^{ix})|^2 \, dx \right)^{\frac{\gamma_0}{2(\gamma_0 + 1)}} + \frac{1}{n^{\gamma_0}}
\]

because by means Parseval identity have

\[
\int_{-\pi}^{\pi} |U'_n(e^{ix})|^2 \, dx = 2\pi \sum_{1 \leq j \leq n} |\hat{f}(j)|^2 \ll n.
\]

Hence we obtain the estimate of the corollary.

Proof of theorem 18. 1) Divergence of series (3.1) at every point \( x \in [-\pi, \pi] \) implies that \( J(n) \to \infty \), whence by Corollary 1 we have

\[
\lim_{n \to \infty} M^{(k)}_n f = 0.
\]

2) If series (3.1) converges at some point \( x_0 \in [-\pi, \pi] \) then

\[
\frac{1}{n} \sum_{1 \leq j \leq n} \left| \frac{\hat{f}(j)e^{-ix_0j} - 1}{j} \right|^2 \to \infty \quad \text{as} \quad n \to \infty.
\]

Application of Theorem 17 to \( \hat{f}(j) \to \hat{f}(j)e^{-itx_0} \) gives the desired estimate.

The theorem is proved. \( \square \)
3.2 Distribution of \( \log P_n(\alpha) \) on \( S_n^{(k)} \)

We now apply the results of the previous section to study the distribution of the additive function

\[
\log P_n(\alpha(\sigma)) = \sum_{j=1}^{n} \alpha_j(\sigma) \log j
\]

on \( S_n^{(k)} \).

Let us introduce the notation

\[
u_n(t) = M_n \exp \left\{ it \log \frac{P_n(\alpha)}{\sqrt{\phi(3k) \log^{3/2} n}}\right\}.
\]

Further we will denote

\[
t' = \frac{t}{\sqrt{\phi(3k) \log^{3/2} n}}.
\]

Lemma 23.

\[
u_n(t) \ll \frac{1}{{n^\epsilon}},
\]

for \( \delta \log^{1/2} n \leq t \leq \log^{3/2} n \). Here \( \delta \) – some fixed positive number.

Proof. Putting in lemma 22 \( \hat{f}(j) = e^{it' \log j} \), we obtain

\[
u_n(t) \ll \left( \frac{1}{n^{2\gamma_0+1}} \int_0^1 |\exp\{U_n(e^{2\pi ix})\}|^2 |U'_n(e^{2\pi ix})|^2 dx \right)^{\frac{m}{n^{2\gamma_0+1}}} + \frac{1}{n^{\gamma_0}}.
\]

We have

\[
U_n(e^{ix}) = \sum_{1 \leq j \leq n \atop (j,k)=1} e^{ixj} \sum_{d|j} \frac{1}{d^{1-\nu}} \mu(d) = \sum_{d|k} \mu(d) \sum_{1 \leq j \leq n \atop d|j} e^{ixj} d^{1-\nu} = \frac{\mu(d)}{d^{1-\nu}} \sum_{s=1}^{[n/d]} e^{ixds} d^{1-\nu} + O(1)
\]

\[
+ \sum_{d|k} \mu(d) S'_{\epsilon,n}(2\pi dx) + O(1),
\]

where

\[
S_{t,n}(x) = \sum_{s=1}^{n} e^{2\pi ixs}.
\]
It is clear that $S_{t,n}(x)$ is periodic with period 1: $S_{t,n}(x + 1) = S_{t,n}(x)$. It follows from (2.24) that for $\frac{1}{n} \leq |x| \leq \frac{1}{2}$ we have

$$S_{t,n}(x) = \sum_{s=1}^{n} \frac{e^{2\pi i xs}}{s^{1-it}} = \frac{2\pi x - it - 1}{it} + O(1 + |t|).$$

It follows hence

$$|S_{t,n}(x)| = O(1),$$

for $x \notin \bigcup_{m \in \mathbb{Z}} (m - \eta, m + \eta)$, if $\eta$—fixed such that $0 < \eta < \frac{1}{2}$ and $|t| \leq 1$.

Let us put $\eta := \frac{1}{4k_0}$ and

$$V_\eta = [0, 1] \cap \bigcup_{j=0}^{k_0} \left( j \frac{1}{k_0} - \eta, j \frac{1}{k_0} + \eta \right).$$

If $x \in [0, 1]$, then from $dx \in \bigcup_{m \in \mathbb{Z}} (m - \eta, m + \eta)$, for $d|k_0$ it follows that $x \in V_\eta$. Conversely, if $x \notin V_\eta$, then $dx \notin \bigcup_{m \in \mathbb{Z}} (m - \eta, m + \eta)$ for any $d|k_0$. Therefore for $x \notin V_\eta$ we have

$$U_n(e^{2\pi ix}) = \sum_{d|k} \frac{\mu(d)}{d|1-it} S_{t'}(dx) + O_\eta(1) = O_\eta(1).$$

It follows hence that

$$\frac{1}{n^{2\gamma_0 + 1}} \int_{x \in [1,0] \setminus V_\eta} |\exp\{U_n(e^{2\pi ix})\}|^2 |U_n'(e^{2\pi ix})|^2 dx \ll \frac{1}{n^{2\gamma_0}} \int_{x \in [1,0]} |U_n'(e^{2\pi ix})|^2 dx \ll \frac{1}{n^{2\gamma_0}}.$$

the last inequality follows from the Parseval identity.

Let us estimate the integral over $x \in V_\eta$. Let $x = \frac{s}{d_0} + u$, $|u| \leq \eta$, where $(s_0, d_0) = 1$ and $d_0|k_0$. If $dx \in \bigcup_{m \in \mathbb{Z}} (m - \eta, m + \eta)$, then there exists such $s$, $1 \leq s < k_0$ that $|x - \frac{s}{d}| < \frac{\eta}{d} \leq \eta$. Since the intervals $\left( \frac{s}{k_0} - \eta, \frac{s}{k_0} + \eta \right)$ do not intersect, then $\frac{s}{d} = \frac{s_0}{d_0}$, therefore in view of the fact that $s_0$ and $d_0$ are coprime we have $s_0|s$ and $d_0|d$. Therefore if $d_0 \nmid d|k_0$, then $dx \notin \bigcup_{m \in \mathbb{Z}} (m - \eta, m + \eta)$ and $S_t(dx) = O(1)$. 
Therefore for \( \frac{1}{n} < |u| < \eta \) we have

\[
U_n(e^{2\pi ix}) = \sum_{d : d_0 | d \mid k_0} \frac{\mu(d)}{d^{1-it'}} S_{t,n}(dx) + O(1)
\]

\[
= \sum_{d : d_0 | d \mid k_0} \frac{\mu(d)}{d^{1-it'}} S_{t,n} \left( d \left( x - \frac{s_0}{d_0} \right) + \frac{d}{d_0} s_0 \right) + O(1)
\]

\[
= \sum_{d : d_0 | d \mid k_0} \frac{\mu(s d_0)}{(s d_0)^{1-it'}} S_{t,n}(s d_0 u) + O(1)
\]

\[
= \sum_{s \mid s_0 \mid d_0} \frac{\mu(s d_0)}{sd_0} \left( \frac{|u|^{-it'} - |s d_0|^{-it'}}{it'} \right) + O(1)
\]

\[
= \frac{|u|^{-it'} - 1}{it'} \mu(d_0) \sum_{s \mid s_0 \mid d_0} \frac{\mu(s)}{s} + O(1)
\]

\[
= \frac{|u|^{-it'} - 1}{it'} \mu(d_0) \prod_{p \mid d_0} \left( 1 - \frac{1}{p} \right) + O(1)
\]

\[
= \gamma_0 \frac{|u|^{-it'} - 1}{it'} \mu(d_0) \prod_{p \mid d_0} \frac{1}{p-1} + O(1).
\]

If \( |u| \leq \frac{1}{n} \) similarly we have

\[
U_n(e^{2\pi ix}) = S_{t'}(0) \gamma_0 \mu(d_0) \prod_{p \mid d_0} \frac{1}{p-1} + O(1).
\]

Therefore if \( x \in \left[ \frac{s_0}{d_0} - \eta, \frac{s_0}{d_0} + \eta \right], \ (s_0, d_0) = 1 \) and \( d_0 \neq 2 \), then

\[
|U_n(e^{2\pi ix})| \leq \frac{\gamma_0}{2} \log n + O(1).
\]

Therefore

\[
|u_n(t)|^2 \ll \frac{1}{n^{2\gamma_0 + 1}} \int_{[-\eta, \eta]} |\exp\{U_n(e^{2\pi ix})\}|^2 |U_n(e^{2\pi ix})|^2 \, dx + O\left( \frac{1}{n^{\gamma_0}} \right),
\]

here

\[
I_{1/2} = \begin{cases} 
\emptyset, & \text{if } 2 \nmid k \\
\left[ \frac{1}{2} - \eta, \frac{1}{2} + \eta \right], & \text{if } 2 | k
\end{cases}
\]
CHAPTER 3. FUNCTIONS ON $S_N^{(K)}$

Since for $|t'| \leq n$ we have $\zeta(1 - it') = \sum_{s=1}^{n} \frac{1}{s^{1-it'}} - \frac{ni't'}{it'} + O(n^{-1})$ (see. [9]) and $\zeta(z) = \frac{1}{z-1} + O(1)$ for $|z-1| \leq 1$, then for $|t| \leq 1$ we have

$$S_t(0) = \sum_{s=1}^{n} \frac{1}{s^{1-it'}} = \frac{n^{it'} - 1}{it'} + O(1).$$

Now we have when $2 \nmid k$ we have $I_{1/2} = \emptyset$ and applying the same considerations as in Theorem 15 we have

$$|u_n(t)| \ll \frac{1}{n^{2\gamma_0+1}} \int_{-\eta}^{\eta} \left| \exp \left\{ U_n(e^{2\pi i x}) \right\} \right|^2 |U_n'(e^{2\pi i x})|^2 \, dx + \frac{1}{n^\gamma_0} \ll \frac{1}{n^\gamma_0}.$$
for $|t| \leq \delta \sqrt{\log n}$. Putting as before $t' = \frac{t}{\sqrt{\frac{\log(3/2)n}{t}}}$ we have

\[ L(1) = \sum_{1 \leq m \leq n} \frac{m^{it'} - 1}{m} = \sum_{d|k} \frac{\mu(d)}{d} \sum_{1 \leq t \leq n/d} \frac{(ld)^{it'} - 1}{l} \]

\[ = \sum_{d|k} \frac{\mu(d)}{d} \left( d^{it'} \sum_{1 \leq m \leq n/d} \frac{1}{m^{1-it'}} - \log \frac{n}{d} - c + O\left(\frac{1}{n}\right) \right) \]

\[ = \sum_{d|k} \frac{\mu(d)}{d} \left( d^{it'} \left( \zeta(1-it') + \frac{(n/d)^{it'}}{it'} \right) - \log \frac{n}{d} - c + O\left(\frac{1}{n}\right) \right), \]

where $c$ – Euler’s constant. Here we have used the well known estimate of the Riemann Zeta function $\zeta(s) = \sum_{m \leq x} \frac{1}{m^s} + x^{1-s} + O(x^{-\beta s})$ which is true for $0 < \sigma_0 < \Re s < 2$ and $x \geq \frac{|\Im s|}{\pi}$ (see [9]). As for $|s-1| \leq 1$ we have $\zeta(s) = \frac{1}{s-1} + c + O(|s-1|)$, therefore

\[ L(1) = \sum_{d|k} \frac{\mu(d)}{d} \left( \frac{n^{it'} - d^{it'}}{it'} - \log \frac{n}{d} \right) + O\left(\frac{|t'| + 1}{n}\right) \]

\[ = \gamma_0 \frac{n^{it'} - 1 - it' \log n}{it'} + O\left(\frac{|t'| + 1}{n}\right) \]

\[ = \gamma_0 \left( \frac{it'}{2!} \log^2 n + \frac{(it')^2}{3!} \log^3 n + \frac{(it')^3}{4!} \log^4 n \right) + O\left(\frac{|t'| + 1}{n} + |t'|^4 \log^5 n\right) \]

\[ = \frac{\frac{it_0}{2} \log^2 n}{\sqrt{\frac{\log(3/2)n}{t}}} - \frac{t^2}{2} + \frac{3^{3/2}(it')^3}{4! \sqrt{\gamma_0}} \frac{1}{\log^{1/2} n} + O\left(\frac{|t|}{\log^{3/2} n} + \frac{1}{n} + \frac{|t|^4}{\log n}\right). \]

Since $\frac{H(f;1)}{H(1;1)} = 1 + O \left( \frac{|t|}{\log^{3/2} n} \right)$ then finally we have

\[ u_n(t)e^{-it\sqrt{\frac{3}{4}\log n}} = e^{-t^2/2} \left( 1 + \frac{1}{\sqrt{\log n}} \sqrt{\frac{3}{\gamma_0}} \left( \frac{1}{8}(it)^3 + C_0it \right) \right) \]

\[ + O\left( \frac{1}{n^\epsilon} + \frac{|t| + |t|^8}{\log n} \right) \]

for $|t| \leq \log^{1/6} n$. If $|t| \leq \delta_0 \sqrt{\log n}$, where $\delta_0$ – fixed sufficiently small number then by means of similar calculations we have

\[ u_n(t) \ll e^{-t^2/2}. \]
For small \(t\) we will use crude estimate

\[
|u_n(t) - 1| \leq |t| \frac{M_n \log P_n(\alpha)}{\sqrt{2(\log n)^3/2}} \ll |t| \sqrt{\log n}.
\]

We have

\[
q_n(t) = e^{-t^2/2} \left(1 + \frac{1}{\sqrt{\log n}} \sqrt{\frac{3}{\gamma_0}} \left(\frac{1}{8}(it)^3 + C_0 i t\right)\right) = \int_{-\infty}^{\infty} e^{itx} dQ_n(x),
\]

where

\[
Q_n(x) = \Phi(x) + \frac{1}{\sqrt{\log n}} \sqrt{\frac{3}{\gamma_0}} \left(\frac{1 - 8C_0 - x^2}{8\sqrt{2\pi}} e^{-x^2/2}\right).
\]

Applying the generalized inequality of Eseen [18] we have

\[
\sup_{x \in \mathbb{C}} |F_n(x) - Q_n(x)| \ll \int_{-T}^{T} \frac{|u_n(t) e^{-it\sqrt{\log n}} - q_n(t)|}{|t|} + O\left(\frac{1}{T}\right)
\]

with \(T = \log n\) and using the earlier obtained estimates together with lemma 23, we obtain the proof of the theorem.

**3.3 Distribution of \(\log O_n(\alpha)\) on \(S_n^{(k)}\)**

While estimating the closeness of \(\log P_n(\alpha)\) and \(\log O_n(\alpha)\) we, as before, will use the formula

\[
\log P_n(a) - \log O_n(a) = \sum_{p} \sum_{s \geq 1} (D_{n,p^s} - 1) \log p,
\]

where the sum is taken over the all prime numbers, \(a \in (\mathbb{Z}^+)^n\), \((d - 1)^+ = d - 1 + I[d = 0]\) and

\[
D_{n,d} = D_{n,d}(a) = \sum_{j \leq n: d j} a_j.
\]

Since \((d - 1)^+ = d - 1 + I[d = 0] \geq 0\), therefore

\[
\Delta_{nd} := M_n(D_{n,d}(\alpha) - 1)^+ = \nu_n(D_{n,d} = 0) + M_n D_{n,d} - 1 \geq 0.
\]

Putting in formula \(\text{[3.1]}\) \(\hat{f}(j) = 1\) for \(d \nmid j\) and \(\hat{f}(j) = 0\) for \(d | j\), we obtain

\[
\sum_{n=0}^{\infty} \frac{|S_n^{(k)}|}{n!} \nu_n(D_{n,d} = 0) z^n = \prod_{d \mid j} \left(1 + \sum_{s \geq 1} \left(\frac{z_j}{j}\right)^s \frac{1}{s!}\right).
\]
3.3. DISTRIBUTION OF LOG $O_N(\alpha)$ ON $S^{(K)}_N$

In a similar way, putting $\hat{f}(j) = e^{it}$, for $d|j$ and $\hat{f}(j) = 1$ in other cases, we have

$$\sum_{n=0}^{\infty} \frac{|S^{(k)}_n|}{n!} M_n e^{itD_{n,d}z^n} = \prod_{j \geq 1: d|j} \left( 1 + \sum_{s \geq 1: q_k(j)||s} \left( \frac{z^j}{j} \right)^s \frac{1}{s!} \right) \prod_{j \geq 1: \overline{d}||j} \left( 1 + \sum_{s \geq 1: q_k(j)||s} \left( \frac{e^{it}z^j}{j} \right)^s \frac{1}{s!} \right).$$

Differentiating this inequality by $t$ and putting $t = 0$, we obtain

$$\sum_{n=0}^{\infty} \frac{|S^{(k)}_n|}{n!} M_n D_{n,d} z^n = \prod_{j=1}^{\infty} \left( 1 + \sum_{s \geq 1: q_k(j)||s} \left( \frac{z^j}{j} \right)^s \frac{1}{s!} \right) \sum_{j \leq n: d|j} \sum_{s \geq 1: q_k(j)||s} \left( \frac{z^j}{j} \right)^s \frac{1}{(s-1)!} \sum_{j \geq 1: \overline{d}||j} 1 + \sum_{s \geq 1: q_k(j)||s} \left( \frac{e^{it}z^j}{j} \right)^s \frac{1}{s!}.$$

We have

$$D_{n,d} = D'_{n,d} + D''_{n,d},$$

where

$$D'_{n,d} = \sum_{j \leq n: (j,k)=1} a_j \quad \text{and} \quad D''_{n,d} = \sum_{j \leq n: (j,k)>1} a_j.$$

Later we will often use the estimate of the following lemma

**Lemma 24.** Let as before $c_n = \frac{|S^{(k)}_n|}{n!}$ then we have

$$\sum_{j \leq n: d|j} c_{n-j} \ll c_{n-\lfloor n/d\rfloor} + \frac{n}{d} c_n,$$

for $d \leq n$.

**Proof.** Let $r = n - \lfloor n/d\rfloor$ then applying Theorem 4 we have

$$\sum_{j \leq n: d|j} c_{n-j} = \sum_{s=0}^{\lfloor n/d\rfloor} c_{r+sd} \ll c_r + \sum_{s=1}^{\lfloor n/d\rfloor} (r+sd)^{\gamma_0-1} \ll c_r + \sum_{s=1}^{\lfloor n/d\rfloor} (sd)^{\gamma_0-1} \ll c_r + d^{\gamma_0-1} \left( \frac{n}{d} \right)^{\gamma_0} \ll c_r + \frac{n^{\gamma_0}}{d} \ll c_{n-\lfloor n/d\rfloor} + \frac{n}{d} c_n.$$

The lemma is proved.
Lemma 25. For \(1 \leq j \leq n\) and \((d, k) = 1\) we have

\[
M_n D'_{n,d} = \frac{\gamma_0}{d} \log \frac{n}{d} + O \left( \frac{1}{d} + \frac{c_{n-d[n/d]}}{nc_n} \right);
\]

for any \(1 \leq d \leq n\) we have

\[
M_n D''_{n,d} \ll \frac{1}{d^2} + \frac{c_{n-d[n/d]}}{n^2c_n}.
\]

Proof. Suppose \((d, k) = 1\). Since

\[
\frac{|S_n^{(k)}|}{n!} \cdot M_n D'_{n,d} = \prod_{j=1}^{\infty} \left( 1 + \sum_{s \geq 1: q_k(j) | s} \left( \frac{z^j}{j} \right)^s \frac{1}{s!} \right) \sum_{j \geq 1: d | j} \frac{z^j}{j} \left( \begin{array}{c} \frac{n}{d} \\ (n) \end{array} \right) = F(z) \sum_{j \geq 1: d | j} \frac{z^j}{j} \left( \begin{array}{c} \frac{n}{d} \\ (n) \end{array} \right),
\]

where \(F(z) = \sum_{j=0}^{\infty} c_j z^j\) and \(c_m = \frac{|S_m^{(k)}|}{m!} = cm^{\gamma_0 - 1} (1 + O(m^{-\epsilon}))\), therefore

\[
c_n M_n D'_{n,d} = \sum_{j \leq n: d | j} \frac{c_{n-j}}{j} = \frac{1}{d} \sum_{s \leq \frac{n}{d}} \frac{c_{n-sd}}{s} + O \left( \frac{1}{n} \sum_{s \leq \frac{n}{d}} \frac{c_{n-sd}}{s} \right)
\]

\[
= \frac{c_n}{d} \sum_{s \leq \frac{n}{d}} \frac{1}{s} + O \left( \frac{1}{d} \sum_{s \leq \frac{n}{d}} \frac{|c_{n-sd} - c_n|}{s} \right) + O \left( \frac{c_{n-d[n/d]}}{n} + \frac{c_n}{d} \right).
\]

Since

\[
\sum_{s \leq x} \frac{1}{s} = \gamma_0 \log x + O(1),
\]

then

\[
c_n M_n D'_{n,d} = \gamma_0 \frac{c_n}{d} \log \frac{n}{d} + O \left( \frac{c_n}{d} \right) + O \left( \frac{c_n}{d} + \frac{1}{d} n^{\gamma_0 - 1 - \epsilon} \log n \right) + O \left( \frac{c_{n-d[n/d]}}{n} + \frac{c_n}{d} \right).
\]

Dividing each side of this equation by \(c_n\), we obtain the first assertion of the lemma.
3.3. DISTRIBUTION OF $LOG_{N}(\alpha)$ ON $S_{N}^{(K)}$

Let us prove the second estimate. We have

$$\frac{|S_{n}^{(k)}|}{n!} M_{n} D_{n,d}^{|n} \leq \left[ F(z) \sum_{j \geq 1; d|j} \sum_{s \geq 1; k\leq(j)|s} \frac{z^{sj}}{j^s} \frac{1}{(s-1)!} \right]_{(n)}.$$ 

As here $s \geq 2$ and $(s-1)! \gg s^4$, then

$$\frac{|S_{n}^{(k)}|}{n!} M_{n} D_{n,d}^{|n} \ll \left[ F(z) \sum_{m \geq 1; d|m} \sum_{j \geq 1; j \leq m} \frac{z^m}{j^2} \frac{1}{s^4} \right]_{(n)} = \left[ F(z) \sum_{m \geq 1; d|m} \frac{z^m}{m^2} \frac{1}{s^2} \right]_{(n)} \ll \frac{c_n}{d^2} + \frac{c_{n-d[n/d]}}{n^2}.$$

The lemma is proved. $\square$

**Lemma 26.** For $d \geq \log n$ and $(d,k) = 1$ we have

$$0 \leq M_{n}(D_{n,d}(\alpha) - 1)^{+} = \nu_{n}(D_{n,d} = 0) + M_{n} D_{n,d} - 1 \ll \left( \frac{\log n}{d} \right)^{2} + \frac{\log n c_{n-d[n/d]}}{nd c_n}.$$ 

If $(d,k) > 1$, then

$$0 \leq M_{n}(D_{n,d}(\alpha) - 1)^{+} \leq M_{n} D_{n,d} \ll \frac{1}{d^2} + \frac{c_{n-d[n/d]}}{n^2 c_n}.$$ 

**Proof.** Suppose $(d,k) = 1$. Let us denote $H_{k}(z) = H_{k}(1;z)$. Applying the
formulas which were obtained above together with lemma 18 we have
\[
\frac{|S_n^{(k)}|}{n!}\nu_n(D_{n,d} = 0) = \left[ \prod_{j \geq 1} \left( 1 + \sum_{s \geq 1} \frac{\left( \frac{z^j}{j} \right)^s}{s!} \right) \right]^{(n)}
\]
\[
= \frac{p(z)}{p(z^{d})^{1/d}} \prod_{j \geq 1: d \nmid j} \left( 1 + \sum_{s \geq 1} \frac{\left( \frac{z^j}{j} \right)^s}{s!} \right)^{(n)}
\]
\[
\leq \frac{p(z)}{p(z^{d})^{1/d}} \prod_{j \geq 1: d \nmid j} \left( 1 + \sum_{s \geq 1} \frac{\left( \frac{z^j}{j} \right)^s}{s!} \right)^{(n)}
\]
\[
= \frac{p(z)}{p(z^{d})^{1/d}} H_k(z)^{(n)}.
\]

In a similar way we have
\[
\frac{|S_n^{(k)}|}{n!}M_nD_{n,d} = \frac{|S_n^{(k)}|}{n!}M_nD'_{n,d} + \frac{|S_n^{(k)}|}{n!}M_nD''_{n,d}
\]
\[
= p(z)H_k(z) \sum_{j \geq 1: d \mid j} \frac{z^j}{j}^{(n)} + \frac{|S_n^{(k)}|}{n!}M_nD''_{n,d}
\]

Hence we have
\[
0 \leq \frac{|S_n^{(k)}|}{n!} \Delta_{n,d} \leq \left[ \frac{p(z)}{p(z^{d})^{1/d}} H_k(z) \left( 1 + \sum_{j \geq 1: d \mid j} \frac{z^j}{j}^{(n)} - p(z^{d})^{1/d} \right) \right]
\]
\[
+ \frac{|S_n^{(k)}|}{n!}M_nD''_{n,d} =: S_1 + S_2.
\]

Since the coefficients in the Taylor expansion of the functions \( \frac{p(z)}{p(z^{d})^{1/d}} \) and \( H_k(z) \) are positive, therefore \( \left[ \frac{p(z)}{p(z^{d})^{1/d}} H_k(z) \right]^{(m)} \geq 0 \) for \( m \geq 0 \). Putting \( \psi(z) = \sum_{j \geq 1: d \mid j} \frac{z^j}{j} \), we have \( e^{\psi(z)} = p(z^{d})^{1/d} \). Applying here the estimate 4)
3.3. DISTRIBUTION OF LOG $O_N(\alpha)$ ON $S_N^{(K)}$

of lemma [18] we obtain

$$S_1 \leq \left[ \frac{p(z)}{p(z^d)^{1/d}} H_k(z) \psi(z) \left( e^{\psi(z)} - 1 \right) \right]_{(n)}.$$ 

Since $[\psi(z)](s) \leq \left[ \frac{1}{d} \log \frac{1}{1-z^d} \right]_{(s)}$ and $\left[ \frac{p(z)}{p(z^d)^{1/d}} \right]_{(s)} \leq [p(z)](s)$ for $s \geq 0$, then applying here the estimates 1) and 3) of the lemma [18] we have

$$S_1 \leq \left[ p(z) H_k(z) \frac{1}{d} \log \frac{1}{1-z^d} \left( \frac{1}{(1-z^d)^{1/d}} - 1 \right) \right]_{(n)} \ll \left[ p(z) H_k(z) \frac{1}{d^2} \log^2 \frac{1}{1-z^d} \right]_{(n)}$$

for $d \geq \log n$. Here we have used the fact that for $0 < v < 1$

$$\left[ \frac{1}{(1-z)^v} \right]_{(m)} = \frac{v(v+1) \cdots (v+m-1)}{m!} = \frac{v}{m} \prod_{j=1}^{m-1} \left( 1 + \frac{v}{j} \right) \leq ve^2 \left[ \log \frac{1}{1-z} \right]_{(m)},$$

if $1 \leq m \leq n$ and $n$ is such that $v \log n \leq 1.$

Since $p(z) H_k(z) = F(z) = \sum_{j=0}^{\infty} c_j z^j$ with $c_m = \frac{|S_m|}{m!}$, then finally we obtain

$$S_1 \ll \frac{1}{d^2} \sum_{1 \leq j \leq \frac{n}{d^2}} \frac{\log j}{j} c_{n-jd} \ll \frac{c_n}{d^2} \sum_{1 \leq j \leq \frac{n}{d^2}} \frac{\log j}{j} + \frac{1}{d} \frac{\log j}{j} \sum_{\frac{n}{2d} \leq j \leq \frac{n}{d} - 1} \frac{\log j}{j} c_{n-jd}$$

$$+ \frac{\log n}{nd} c_{n-d[n/d]} \ll c_n \left( \log \frac{n}{d} \right)^2 + \frac{\log n}{nd} c_{n-d[n/d]}.$$

Lemma [25] yields the estimate of $S_2$: $S_2 \ll \frac{c_n}{d^2} + \frac{c_{n-d[n/d]}}{n^2}$.

Suppose now $(d,k) > 1$. Then $D_{n,d} = D_{n,d}^\prime$ and the desired estimate follows from the lemma [25].

The lemma is proved. \( \square \)

Lemma 27. There exists such a positive constant $\epsilon$, that

$$\left[ \frac{p(z)}{p(z^d)^{1/2}} \right]_{(n)} = \frac{d^\gamma_0/d}{\Gamma(\gamma_0 (1 - 1/2))} \int n^\gamma_0 (1 - 1/2)^{-1} (1 + O(n^{-\epsilon})), \quad C < d \leq \log^D n.$$ 

Here $D$ is an arbitrary fixed positive constant and $S$ sufficiently big positive number.
we have used the fact that for 1 \leq |z| \leq 2 and p(z) \ll d^m for \( z \in L_j, k_0 \nmid j \) then applying the formula of Cauchy, we have

\[
\begin{align*}
\left[ \frac{p(z)}{p(z^d)^\frac{1}{d}} \right]_{(n)} &= \frac{1}{2\pi i} \int_{K_d} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{L_j} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{L_j} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} = \frac{d_{k_0} - 1}{2\pi i} \int_{L_j} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} \\
&= \frac{1}{2\pi i} \int_{L_0} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} + O\left( \sum_{0 < j < d_{k_0}} \int_{L_j} \frac{dr}{(r - 1)^\frac{m}{d} r^{n+1}} \right) + O(2^{-n}) \\
&= I + O\left( d^m + \sum_{s=1}^{d_{k_0} - 1} n^{s+\frac{m}{d} - 1} \right) = I + O\left( d^m + n^{\gamma'/d} - 1 \right),
\end{align*}
\]

where \( \gamma' = \max_{j \neq 0} \gamma_j < \gamma_0 \) and

\[
\begin{align*}
I &= \frac{1}{2\pi i} \int_{L(1,1/2d)} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} + O\left( \int_{1+1/2d}^{2} \frac{1}{(r - 1)^\gamma_0 r^{n+1}} dr \right) = \frac{1}{2\pi i} \int_{L(1,1/2d)} \frac{p(z)}{p(z^d)^\frac{1}{d}} \frac{dz}{z^{n+1}} + O\left( d^m e^{-\frac{\gamma_0}{2d}} \right),
\end{align*}
\]

and \( L(1,1+1/2d) \) is the part of \( L_0 \), which lies inside of disc \( |z| < 1 < \frac{1}{2d} \). Here we have used the fact that \( |z^d| > c > 1 \) for \( |z| > 1 + \frac{1}{2d} \) and \( |p(w)| \rightarrow 1 \), if \( |w| \rightarrow \infty \).

Suppose \( S = \{ z \in \mathbb{C} : |\arg(z)| < 1/(2k_0) \} \). Then for \( z \in S \) we have

\[
p(z) = \prod_{j=0}^{d_{k_0} - 1} \frac{1}{(1 - ze^{-2\pi i j/k})^{\gamma_j}} = \frac{u(z)}{(1 - z)^{\gamma_0}}.
\]

here \( u(z) \) - analytic function in the sector \( S \). If \( z \) is such, that \( |\arg(z)| < 1/(4dk_0) \), then \( z \in S \) and \( z^d \in S \). The previous formula yields

\[
\frac{p(z)}{p(z^d)^\frac{1}{d}} = \frac{(1 - z^d)^{\frac{m}{d} u(z)}}{(1 - z)^{\gamma_0} u(z)^{1/d}} = \frac{1}{(1 - z)^{\gamma_0(1-1/d)}} \left( \frac{1 - z^d}{1 - z} \right)^{\frac{m}{d} u(z)} u(z)^{1/d}.
\]
With the same restrictions on \( z \) we have
\[
\left( \frac{1 - z^d}{1 - z} \right)^{\frac{20}{d}} - d^{\frac{20}{d}} = (1 + z + \cdots + z^{d-1})^{\frac{20}{d}} - d^{\frac{20}{d}}
\]
\[
= d^{\frac{20}{d}} \left( \left( \frac{1 + z + \cdots + z^{d-1}}{d} \right)^{\frac{20}{d}} - 1 \right).
\]
Since \( |z| \leq 1 + |z - 1| \leq e^{|z-1|} \), then
\[
|z^j - 1| = |z - 1||1 + z + \cdots + z^{j-1}| \leq j|z - 1|e^{|z-1|},
\]
therefore for \( |z - 1| \leq 1/(2d) \) we have
\[
\left| \frac{(z - 1) + (z^2 - 1) + \cdots + (z^{d-1} - 1)}{d} \right| 
\leq e \frac{|z - 1| + 2|z - 1| + \cdots + (d - 1)|z - 1|}{d} = \frac{ed}{2} |z - 1| \leq \frac{e}{4}.
\]
Applying this inequality together with the estimate \((1 + z)^\theta = 1 + O(\theta |z|)\) for \( z \) such that \( |\arg(z)| \leq 1/(4k_0d) \) and \( |1 - z| \leq \frac{1}{4d} \), we obtain
\[
\left| \left( \frac{1 - z^d}{1 - z} \right)^{\frac{20}{d}} - d^{\frac{20}{d}} \right| \ll |1 - z|.
\]
In a similar way, having the same restrictions on \( z \) we obtain
\[
\left| \frac{u(z)}{u(z^d)^{\frac{1}{d}}} - \frac{u(1)}{u(1)^{\frac{1}{d}}} \right| \ll |z - 1| + \left| \left( \frac{u(z^d)}{u(1)} \right)^{\frac{1}{d}} - 1 \right| \ll |z - 1|.
\]
Finally we obtain that for \( |\arg(z)| < 1/(4dk_0) \) and \( |1 - z| \leq \frac{1}{4d} \) holds the estimate
\[
\frac{p(z)}{p(z^d)^{\frac{1}{d}}} = \frac{d^{\gamma_0/d}u(1)^{1-1/d}}{(1 - z)^{\gamma_0(1-1/d)}} + O(|1 - z|).
\]
Putting this estimate in (3.4) and recalling that \( u(1) = A_k \), we obtain
\[
I = \frac{1}{2\pi i} \int_{L(1,1+\frac{1}{d})} \frac{d^{\gamma_0/d}A_k^{1-1/d} + O(|1 - z|)}{(1 - z)^{\gamma_0(1-1/d)}z^{\gamma_1+1}} dz + O \left( d^{\gamma_0} e^{-\frac{n}{2^{d+1}}} \right)
\]
\[
= \frac{1}{2\pi i} \int_{L(1,1+\frac{1}{d})} \frac{d^{\gamma_0/d}A_k^{1-1/d}}{(1 - z)^{\gamma_0(1-1/d)}z^{\gamma_1+1}} dz + O \left( n^{\gamma_0(1-1/d)-2} + d^{\gamma_0} e^{-\frac{n}{2^{d+1}}} \right)
\]
\[
= \frac{d^{\gamma_0/d}A_k^{1-1/d}}{\Gamma \left( \gamma_0 \left( 1 - \frac{1}{d} \right) \right)} n^{\gamma_0(1-\frac{1}{d})-1} \left( 1 + O \left( \frac{1}{n} \right) \right) + O \left( n^{\gamma_0(1-1/d)-2} + d^{\gamma_0} e^{-\frac{n}{2^{d+1}}} \right),
\]
Lemma 28. There exists such a positive constant \( \epsilon \), that for \( C < l, r \leq \log^D n \) and \( (l, k) = 1, (r, k) = 1 \), we have

\[
\left[ \frac{p(z)p(z^l)^{\frac{1}{\gamma}}} {p(z^r)^{\frac{1}{\gamma}}p(z^{l'})^{\frac{1}{\tau}}} \right] = \frac{1}{2\pi i} \int_{K_{rl}} \frac{p(z)p(z^l)^{\frac{1}{\gamma}}} {p(z^r)^{\frac{1}{\gamma}}p(z^{l'})^{\frac{1}{\tau}}} dz = O\left( (nl)\frac{1}{\gamma} - \frac{1}{\tau} + \frac{1}{\tau} \right) - \frac{(nl)^{\gamma_0(\frac{1}{\gamma} + \frac{1}{\tau} + \frac{1}{\tau'})}}{\gamma'_{n+1}}.
\]

The proof of the next lemma is analogous.
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where $\gamma' = \max_{j \neq 0} \gamma_j < \gamma_0$. Applying estimate (3.5) with $d = rl$ we obtain

$$
\frac{p(z)p(z')^{1/\tau}}{p(z')^{1/\tau} p(z'^{1/\tau})} = \frac{1}{(1 - z)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})}} \frac{(1 - z')^{\gamma_n(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})}}{(1 - z'^{1/\sigma})^{\gamma_n(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})}} u(z)u(z'^{1/\sigma})^{1/\tau} u(z')^{1/\sigma} u(z^{1/\sigma})^{1/\tau} = r^{\gamma_0/\tau} I^{\gamma_0/\tau}(1)^{\gamma_n(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})} + O(|z - 1|)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})},
$$

when $|z - 1| \leq \frac{1}{4r_0 rl}$. Denoting as before by $L(1, 1 + \frac{1}{4r_\tau})$ the part of $L_0$, which lies inside of disc $|z - 1| < \frac{1}{4r_1}$, we have

$$
\frac{1}{2\pi i} \int_{L_0} \frac{p(z)p(z')^{1/\tau}}{p(z')^{1/\tau} p(z'^{1/\tau})} \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{L(1, 1 + \frac{1}{4r_\tau})} \frac{p(z)p(z')^{1/\tau}}{p(z')^{1/\tau} p(z'^{1/\tau})} \frac{dz}{z^{n+1}} + O \left( \int_{1 + \frac{1}{4r_\tau}}^{2} \frac{dy}{(y - 1)^{\gamma_0(1 + \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})} y^{n+1}} \right)
$$

$$
= \frac{1}{2\pi i} \int_{L(1, 1 + \frac{1}{4r_\tau})} \frac{r^{\gamma_0/\tau} I^{\gamma_0/\tau} u(1)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})} + O(|z - 1|)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})}}{(rl)^{\gamma_0/\tau}(1 - z)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})} z^{n+1}} + O \left( (lr)^{\gamma_0} e^{-\frac{n}{4r_1+1}} \right)
$$

$$
= \frac{r^{\gamma_0/\tau} I^{\gamma_0/\tau} u(1)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})}}{(rl)^{\gamma_0/\tau} 2\pi i} \int_{L(1, 1 + \frac{1}{4r_\tau})} \frac{dz}{(1 - z)^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma})} z^{n+1}} + O \left( n^{\gamma_0(1 - \frac{4}{\tau} - \frac{1}{\tau} + \frac{1}{\sigma}) - 2} \right) + O \left( (lr)^{\gamma_0} e^{-\frac{n}{4r_1+1}} \right).
$$

Applying here (3.3) with $d = 2rl$ we complete the proof of the lemma.

Let us denote

$$
H_k^{(d)}(z) = \prod_{j \geq 1: d|j} \left( 1 + \sum_{s \geq 1: q_k(j)|s} \left( \frac{z^j}{j} \right) \frac{s}{s!} \right).
$$

**Lemma 29.** For $(k, d) = 1$ and $C < d \leq \log^D n$ we have

$$
\nu_n(D_{nd} = 0) = \frac{\Gamma(\gamma_0) d^{\gamma_0/d} A_k^{-1/d}}{\Gamma\left( \gamma_0 \left( 1 - \frac{1}{d} \right) \right)} H_k^{(d)}(1) n^{-\frac{4}{d}} \left( 1 + O(n^{-c}) \right).
$$
Proof. Since \( \left[ \frac{H_k(z)}{H_k^{(d)}(z)} \right]_{(m)} \) \( \leq \) \( [H_k(z)]_{(m)} = O(m^{-2}) \) for \( m \geq 1 \), then for \( (k, d) = 1 \)

\[
\frac{|S_n^{(k)}|}{n!} \nu_n(D_{n,d} = 0) = \left[ \frac{p(z)}{p(z^{d})^{\frac{1}{d}}} \frac{H_k(z)}{H_k^{(d)}(z)} \right]_{(n)} = \sum_{j=0}^{n} \left[ \frac{p(z)}{p(z^{d})^{\frac{1}{d}}} \right]_{(j)} \left[ \frac{H_k(z)}{H_k^{(d)}(z)} \right]_{(n-j)} = \sum_{n/2 < j < n} \left[ \frac{p(z)}{p(z^{d})^{\frac{1}{d}}} \right]_{(n-j)} \left[ \frac{H_k(z)}{H_k^{(d)}(z)} \right]_{(n-j)} + O \left( \frac{1}{n^2} \sum_{j=0}^{n/2} \left[ \frac{p(z)}{p(z^{d})^{\frac{1}{d}}} \right]_{(j)} \right)
\]

\[
= \sum_{s \leq n/2} \left( \frac{d^{n/d} A_k^{1-1/d}}{\Gamma(\gamma_0 (1 - \frac{1}{d}))} (n-s)^{\gamma_0 (1-\frac{1}{d}) - 1} (1 + O(n^{-\epsilon})) \right) \left[ \frac{H_k(z)}{H_k^{(d)}(z)} \right]_{(s)} + O \left( n^{\gamma_0 (1-\frac{1}{d}) - 2} \right)
\]

As \( \frac{|S_n^{(k)}|}{n!} = \frac{A_k n^{\gamma_0 - 1}}{\Gamma(n)} (1 + O(n^{-\epsilon})) \), we obtain the proof of the lemma.

Lemma 30. If \((k, l) = 1, (k, r) = 1\) and \(C < l, r \leq \log^D n\), then

\[
\nu_n(D_{n,l} = 0, D_{n,r} = 0) = \frac{H_k^{(1r)}(1)}{H_k^{(1)}(1) H_k^{(r)}(1)} \frac{\Gamma(\gamma_0 (1 - \frac{1}{d}) + \frac{1}{d} + \frac{1}{l})}{\Gamma(\gamma_0 (1 - \frac{1}{d}) + \frac{1}{d} + \frac{1}{l})} n^{\gamma_0 (1-\frac{1}{d} - \frac{1}{l} - \frac{1}{r})} (1 + O(n^{-\epsilon})).
\]

Proof. For \((k, l) = 1, (k, r) = 1\) we have

\[
\sum_{n=0}^{\infty} \frac{|S_n^{(k)}|}{n!} \nu_n(D_{n,r} = 0, D_{n,l} = 0) z^n = \frac{p(z)p(z^{lr})^{\frac{1}{lr}} H_k(z) H_k^{(lr)}(z)}{p(z^{lr})^{\frac{1}{lr}} H_k^{(l)}(z) H_k^{(r)}(z)}.
\]

Applying the same considerations as in lemma 29 we complete the proof of the lemma.

Lemma 31. Suppose \(f \in C^2[-\epsilon, \epsilon]\) and \(f(0) = 1\), then

\[
f(x + y) - f(x)f(y) \ll |xy|,
\]

for \(|x| \leq \frac{\epsilon}{2}\) and \(|y| \leq \frac{\epsilon}{2}\).
Proof. We have

\[ f(x + y) - f(x)f(y) = \int_0^1 \frac{d}{dt}(f(xt + yt) - f(x)f(y))dt \]

\[ = x\int_0^1 (f'(xt + yt) - f'(xt))dt \]

\[ + y\int_0^1 (f'(xt + yt) - f(x)f'(yt))dt. \]

Putting here \( f(yt) = 1 + O(t|y|) \) in the first integral, and \( f(xt) = 1 + O(t|x|) \) in the second, we have

\[ f(x + y) - f(x)f(y) = x\int_0^1 (f'(xt + yt) - f'(xt))dt \]

\[ + y\int_0^1 (f'(xt + yt) - f'(yt))dt + O(|xy|) \ll |xy|. \]

The lemma is proved.

The proof of theorem. Since for \((j, k) > 1\)

\[ \left| \frac{S_n^{(k)}}{n!} \right| M_n \alpha_j \ll \left[ F(z) \sum_{s \geq 1} \frac{z^{sj}}{js^4} \right]_{(n)}, \]

then

\[ M_n \alpha_j \ll \frac{1}{j^2} + \frac{1}{n^{\gamma_0 + 1}} \]

for \((j, k) > 1\). For \((k, j) = 1\) we have \( M_n \alpha_j = \frac{c_{n-j}}{jc_n} \). Hence we obtain

\[ M_n P_n(\alpha(\sigma)) = \sum_{j \leq n} \log j M_n \alpha_j = \sum_{j \leq n, (j,k)=1} \frac{c_{n-j} \log j}{jc_n} + O(1). \]
Applying here the estimate \( c_m = cm^{\gamma_0 - 1}(1 + O(m^{-\epsilon})) \) we obtain

\[
M_n \log P_n(\alpha(\sigma)) = \sum_{j \leq n-1 \atop (j,k)=1} \log \frac{j}{n} \left( 1 - \frac{j}{n} \right)^{\gamma_0 - 1} + O(1)
\]

\[
= \sum_{j \leq n-1 \atop (j,k)=1} \frac{\log j}{j} + \log n \sum_{j \leq n-1 \atop (j,k)=1} \frac{1}{j} \left( \left( 1 - \frac{j}{n} \right)^{\gamma_0 - 1} - 1 \right)
\]

\[
- \frac{1}{n} \sum_{j \leq n-1 \atop (j,k)=1} \log \frac{n}{j} \left( \left( 1 - \frac{j}{n} \right)^{\gamma_0 - 1} - 1 \right) + O(1)
\]

\[
= \frac{\gamma_0}{2} \log^2 n + C_0 \log n + O(1),
\]

where \( C_0 = \gamma_0 \int_0^1 \frac{(1-y)^{\gamma_0-1}-1}{y} \, dy. \)

Application of lemma 26 gives

\[
\mu_n = M_n(\log P_n(\alpha) - \log O(\alpha)) = \sum_{m \leq \log^2 n \atop (m,k)=1} \Lambda(m) M_n(D_{n,m} - 1)^+ + O(1).
\]

From lemmas 29 and 26 we have

\[
M_n(D_{n,m} - 1)^+ = \nu_n(D_{n,m} = 0) + M_n D_{n,m} - 1 = \lambda \left( \frac{m}{x} \right) + O \left( \frac{\log m}{m} \right),
\]

for \( m \leq \log^2 n \) and \( (m,k) = 1 \) where \( \lambda(y) = e^y - 1 + \frac{1}{y} \) and \( x = \gamma_0 \log n \).

Therefore

\[
\mu_n = \sum_{m=1 \atop (m,k)=1}^{\infty} \Lambda(m) \lambda \left( \frac{m}{x} \right) + O((\log \log n)^2) = S(x) - K(x) + O((\log \log n)^2),
\]

where

\[
S(x) = \sum_{m=1}^{\infty} \Lambda(m) \lambda \left( \frac{m}{x} \right) \quad \text{and} \quad K(x) = \sum_{m=1 \atop (m,k)>1}^{\infty} \Lambda(m) \lambda \left( \frac{m}{x} \right).
\]

Since

\[
\int_0^{\infty} \lambda(x)x^{s-1} \, dx = \Gamma(-s),
\]
for $1 < \Re s < 2$, then applying Mellin’s inversion formula we have

$$K(x) = \sum_{m=1}^{\infty} \Lambda(m) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(-s) \left( \frac{m}{x} \right)^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left( \sum_{p|k} \log p \right) \Gamma(-s) x^s ds,$$

for $1 < \sigma < 2$. Changing the integration contour in the above integral from the line $\Re s = \sigma$ to $\Re s = -\frac{1}{2}$ and calculating the residuals, we obtain

$$K(x) = x \left( \sum_{p|k} \frac{\log p}{p-1} \right) + O(\log x).$$

By means of similar considerations it has been proved in [10] that

$$S(x) = x(\log x - 1) + \frac{d'(0)}{\zeta(0)} - \sum_{\rho} \Gamma(-\rho) x^{\rho} + O \left( \frac{1}{\sqrt{x}} \right),$$

where $\sum_{\rho}$ denotes the sum over the non-trivial zeroes of the Riemann Zeta function. Therefore

$$M_n \log O(\alpha) = \frac{\gamma_0}{2} \log^2 n + C_0 \log n - S(\gamma_0 \log n) + K(\gamma_0 \log n) + O((\log \log n)^2)$$

$$= \frac{\gamma_0}{2} \log^2 n + C_0 \log n - \gamma_0 \log n(\log(\gamma_0 \log n) - 1)$$

$$+ \sum_{\rho} \Gamma(-\rho)(\gamma_0 \log n)^{\rho} + C_1 \gamma_0 \log n + O((\log \log n)^2),$$

here $C_1 = \sum_{p|k} \frac{\log p}{p-1}$.

Theorem 8 is proved. \hfill \Box

**Lemma 32.** For $1 \leq r, l, d \leq n^\epsilon$ and $(r, l) = 1$ we have

1) $\mathrm{cov}(D_{n,r}', D_{n,r}') \ll \frac{\log n}{rl},$

2) $M_n D_{n,r}' D_{n,l}' \ll \frac{\log n}{rl^2},$

3) $M_n D_{n,r}'' D_{n,l}'' \ll \frac{1}{r^2l^2}.$
4) \( D_n D_{n,d} \ll \frac{\log n}{d} \).

Here \( \epsilon \) is a sufficiently small fixed number.

**Proof.** During the proof of the assertion 1) we will assume that \( (rl, k) = 1 \), since otherwise whether \( D_{n,r} = 0 \), or \( D'_{n,l} = 0 \). Then for \( j_1 + j_2 \leq n \), \( j_1 \neq j_2 \) and \( (j_1 j_2, k) = 1 \) we have

\[
M_n \alpha_{j_1} \alpha_{j_2} = \frac{c_{n-j_1-j_2}}{c_{n j_1 j_2}}.
\]

Therefore

\[
\text{cov}(D'_{n,r}, D'_{n,r}) = M_n D'_{n,r} D'_{n,l} - M_n D'_{n,r} M_n D'_{n,l}
\]

\[
= \sum_{j_1 \neq j_2, j_1, j_2 \leq n/4} \left( \frac{c_n c_{n-j_1-j_2} - c_{n-j_1} c_{n-j_2}}{c_{n j_1 j_2}} \right)
\]

\[
+ \sum_{j_1 \neq j_2, j_1, j_2 \leq n/4} \left( \frac{c_{n-j_1} c_{n-j_2}}{c_{n j_1 j_2}} \right) + \sum_{j_1 \neq j_2, j_1, j_2 \leq n/4} \left( \frac{c_{n-j_1} c_{n-j_2}}{c_{n j_1 j_2}} \right)
\]

\[
+ \sum_{j \leq n: r l | j} D_n \alpha_j = S_1 + S_2 + S_3 + S_4.
\]

Since \( c_m = \frac{S^{(6)}_m}{m!} = c m^{\tau_0-1} (1 + O(m^{-\epsilon})) \) then one can easily obtain that

\[
S_1 \ll \sum_{j_1, j_2 \leq n/4} \left( \frac{1 - j_1 j_2}{n} \right)^{\tau_0-1} \frac{1}{j_1 j_2} + \frac{n^{-\epsilon} \log^2 n}{r l} \ll \frac{1}{r l}.
\]

As for any \( d \geq 1 \) and \( m \geq 1 \) we have

\[
\sum_{j \leq m: d | j} c_m - j \ll 1 + \frac{m^{\tau_0}}{d},
\]

therefore

\[
S_2 \ll \frac{1}{n} \sum_{j_1 + j_2 \leq n} \left( \frac{1}{j_1} + \frac{1}{j_2} \right) = \frac{1}{n c_n} \sum_{j_1 \leq n} \frac{1}{j_1} \sum_{j_2 \leq n-j_1} \frac{c_{n-j_1-j_2}}{j_2}
\]

\[
+ \frac{1}{n c_n} \sum_{j_2 \leq n} \frac{1}{j_2} \sum_{j_1 \leq n-j_2} \frac{c_{n-j_2-j_1}}{j_1} \ll \frac{\log n}{n^{\tau_0 r}} + \frac{\log n}{n^{\tau_0 l}} + \frac{\log n}{r l} \ll \frac{\log n}{r l}.
\]
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for $r, l \leq n^{\gamma_0}$.

In a similar way we obtain that for $r, l \leq n^{\gamma_0}$

$$S_3 \ll \frac{1}{n} \sum_{j_1, j_2 \leq n} \frac{c_{n-j_1} c_{n-j_2}}{c_n^2} \left( \frac{1}{j_1} + \frac{1}{j_2} \right) \ll \frac{\log n}{rl}.$$ 

Since for $(j, k) = 1$ we have $D_n \alpha_j \leq M_n \alpha_j^2 = \frac{c_{n-j}}{j} + \frac{c_{n-2j}}{j^2}$ (here we assume that $c_m = 0$ for $m < 0$), then

$$S_4 \ll \frac{\log n}{rl} + \frac{1}{n^{\gamma_0}} \ll \frac{\log n}{rl}$$

for $r, l \leq n^{\gamma_0/2}$.

Assertion 1) is proved.

Applying the same considerations we obtain

$$c_n M_n D'_{n,r} D''_{n,l} \ll \left[ \frac{F(z)}{\sum j \sum m} \right] = \sum_{j, m \leq n/4} \frac{c_{n-j-m}}{jm^2}$$

$$\ll \sum_{j, m \leq n/4} \frac{c_{n-j-m}}{jm^2} + \sum_{j, m \geq n/4} \frac{c_{n-j-m}}{jm^2}$$

$$+ \sum_{j, m \geq n/4} \frac{c_{n-j-m}}{jm^2} \ll c_n \frac{\log n}{rl^2}$$

for $r, l \leq n^{\gamma_0}$.

In a similar way we prove 3).

One can easily see that

$$D_n D_{n,d} \leq 2D_n D'_{n,d} + 2D_n D''_{n,d}.$$ 

Slightly changing the proof of assertion 1), one can check that $D_n D'_{n,d} \ll \frac{\log n}{d}$ for $d \leq n^{\gamma_0}$. In a similar way $D_n D''_{n,d} \leq M_n D^2_{n,d} \ll \frac{1}{d^2}$. Hence follows the proof of the estimate 4).

The lemma is proved. \(\square\)

**Lemma 33.** If $(r, l) = 1$, $C < l$ and $r \leq \log D n$, then

$$\text{cov}(I[D_{n,l} = 0], I[D_{n,r} = 0]) \ll \frac{\log n}{lr}.$$
Proof. Let us suppose at first that \((k, l) = 1, (k, r) = 1\), then

\[
\text{cov}(I[D_{n,l} = 0], I[D_{n,r} = 0]) = \nu_n(D_{n,l} = 0, D_{n,r} = 0) - \nu_n(D_{n,l} = 0)\nu_n(D_{n,r} = 0).
\]

Inserting here the estimates of the lemmas \([29, 30]\) and applying the lemma \([31]\) to function \(f(x) = \frac{r(\gamma_0)}{I(\gamma_0(1+x))}\) with \(x = 1/l\) and \(y = 1/r\), we obtain the desired estimate.

Suppose now \((k, r) > 1\). Then \(D_{n,r} = D''_{n,r}\).

Since \(1 - I[D_{n,d} = 0] = I[D_{n,d} > 0] \leq D_{n,d}\), therefore, applying the lemmas \([10, 3]\), we obtain

\[
|\text{cov}(I[D_{n,l} = 0], I[D_{n,r} = 0])| = |\text{cov}(I[D_{n,l} > 0], I[D_{n,r} > 0])| \leq M_n D_{n,l} D''_{n,r} + M_n D_{n,l} M_n D''_{n,r} + M_n D''_{n,l} D''_{n,r} + M_n D''_{n,l} M_n D''_{n,r} \ll \log n \frac{r^2}{l^2},
\]

for \(r, l \leq \log^D n\).

The lemma is proved.

Lemma 34. If \(1 \leq d \leq n\), then

\[
D_n(D_{n,d} - 1)^+ \leq D_n I[D_{n,d} = 0] + D_n D_{n,d} \ll \frac{\log n}{d}.
\]

Proof. Since \((D_{n,d} - 1)^+ = I[D_{n,d} = 0] + D_{n,d} - 1\), then

\[
D_n(D_{n,d} - 1)^+ = D_n I[D_{n,d} = 0] + D_n D_{n,d} + 2\text{cov}(I[D_{n,d} = 0], D_{n,d}) \leq D_n I[D_{n,d} = 0] + D_n D_{n,d};
\]

as \(\text{cov}(I[D_{n,d} = 0], D_{n,d}) = -M_n I[D_{n,d} = 0] M_n D_{n,d} \leq 0\).

Applying the inequality \(\nu_n(D_{n,d} > 0) \leq M_n D_{n,d}\) we obtain

\[
D_n(I[D_{n,d} = 0]) = \nu_n(D_{n,d} = 0) - \nu_n(D_{n,d} = 0)^2 \leq \nu_n(D_{n,d} > 0) \leq M_n D_{n,d} \ll \frac{\log n}{d};
\]

the last inequality follows from the lemma \([25]\). Hence and from the estimate 4) of lemma \([32]\) we obtain the proof of the lemma.
3.3. DISTRIBUTION OF $\log O_N(\alpha)$ ON $S_N^{(K)}$

**Proposition 2.** For any fixed $K > 0$ we have

$$
\nu_n \left( \frac{\left| \log P_n(\alpha) - \log O_n(\alpha) - \mu_n \right|}{\log^{3/2} n} > K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \left( \frac{\log \log n}{\log n} \right)^{2/3},
$$

where $\mu_n = M_n(\log P_n(\alpha) - \log O_n(\alpha))$.

**Proof.** The proof of this proposition uses the considerations of A. D. Barbour and S. Tavarè of the work [3].

$$
\log P_n(\alpha) - \log O_n(\alpha) = \sum_{m \leq C} \Lambda(m)(D_{n,m}(\alpha) - 1)^+ + \sum_{C < p, s \leq \log^2 n} (D_{n,p^s}(\alpha) - 1)^+ \log p + \sum_{m > \log^2 n} \Lambda(m)(D_{n,m}(\alpha) - 1)^+ = V_1 + V_2 + V_3.
$$

Applying lemma [26] we obtain

$$
M_n V_3 \ll 1.
$$

Applying the recently proved lemma we have

$$
D_n V_1 \leq \log^2 C D_n \left( \sum_{m \leq C} (D_{n,m}(\alpha) - 1)^+ \right) \leq C \log^2 C \sum_{m \leq C} D_n(D_{n,m}(\alpha) - 1)^+ \leq C \log^3 C \log n.
$$

Since $(D_{n,m}(\alpha) - 1)^+ = I[D_{n,m}(\alpha) = 0] + D_{n,m}(\alpha) - 1$ then denoting $V_2^{(1)} = \sum_{C < p, s \leq \log^2 n} I[D_{n,p^s} = 0] \log p$ and $V_2^{(2)} = \sum_{C < p, s \leq \log^2 n} D_{n,p^s} \log p$ we have

$$
D_n V_2^{(1)} = \sum_{C < p, q, l \leq \log^2 n} \text{cov}(I[D_{n,p^s} = 0], I[D_{n,q^l} = 0]) \log p \log q
$$

$$
\ll \sum_{C < p, q, l \leq \log^2 n} \frac{\log n}{p^q l} \log p \log q + \sum_{C < p, s \leq \log^2 n} \log^2 p D_n I[D_{n,p^s} = 0]
$$

$$
+ \sum_{C < p \leq \log^2 n} \sum_{s > l \geq 1} (D_n I[D_{n,p^s} = 0])^{1/2} (D_n I[D_{n,q^l} = 0])^{1/2} \log^2 p
$$

$$
\ll \log n(\log \log n)^2 + \log n \sum_{C < p \leq \log^2 n} \sum_{s > l \geq 1} \frac{\log^2 p}{p^{s+l}} \ll \log n(\log \log n)^2.
$$
Applying the same calculations we have

\[ D_n V_2^{(2)} \ll \log n (\log \log n)^2. \]

Therefore

\[ D_n V_2 \leq 2D_n V_2^{(1)} + 2D_2 V_2^{(2)} \ll \log n (\log \log n)^2. \]

Applying the Chebyshev inequality we have

\[
\nu_n \left( \frac{|V_1 - M_n V_1|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \frac{1}{(\log \log n)^{4/3} \log^{2/3} n},
\]

\[
\nu_n \left( \frac{|V_2 - M_n V_2|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \left( \frac{\log \log n}{\log n} \right)^{2/3},
\]

\[
\nu_n \left( \frac{|V_3 - M_n V_3|}{\log^{3/2} n} > \frac{1}{3} K \left( \frac{\log \log n}{\log n} \right)^{2/3} \right) \ll \frac{1}{(\log \log n)^{2/3} \log^{5/6} n}.
\]

Hence follows the proof of the proposition.

**Proof of theorem 7** Putting in lemma 17 \( U = \frac{\log P_n(\sigma) - M_n \log P_n(\sigma)}{\sqrt{\frac{\log \log n}{\log^{3/2} n}} \log^{3/2} n} \) and \( X = \frac{\log P_n(\sigma) - \log O_n(\sigma) - \mu_n}{\sqrt{\frac{\log \log n}{\log^{3/2} n}}} \) and applying the proposition 1, we obtain the proof of the theorem. \( \square \)
Conclusions

The asymptotic expansions proved in Chapters 2 and 3 show that the best possible estimate for the convergence rate of the distribution function of the logarithm of the order of a random permutation on $S_n$ and $S_n^{(k)}$ is $\frac{1}{\sqrt{\log n}}$.

The similar conclusion is true for the distribution of the logarithm of the degree of the splitting field of a random polynomial.
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