Quantitative quasiperiodicity

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Abstract

The Birkhoff ergodic theorem concludes that time averages, i.e. Birkhoff averages, $B_N(f) := \frac{1}{N} \sum_{n=0}^{N-1} f(x_n)$ of a function $f$ along a length $N$ ergodic trajectory $(x_n)$ of a function $T$ converge to the space average $\int f \, d\mu$, where $\mu$ is the unique invariant probability measure. Convergence of the time average to the space average is slow. We use a modified average of $f(x_n)$ by giving very small weights to the ‘end’ terms when $n$ is near 0 or $N − 1$. When $(x_n)$ is a trajectory on a quasiperiodic torus and $f$ and $T$ are $C^\infty$, our weighted Birkhoff average (denoted $WB_N(f)$) converges ‘super’ fast to $\int f \, d\mu$ with respect to the number of iterates $N$, i.e. with error decaying faster than $N^{-m}$ for every integer $m$. Our goal is to show that our weighted Birkhoff average is a powerful computational tool, and this paper illustrates its use for several examples where the quasiperiodic set is one or two dimensional. In particular, we compute rotation numbers and conjugacies (i.e. changes of variables) and their Fourier series, often with 30-digit accuracy.

Keywords: quasiperiodicity, Birkhoff ergodic theorem, Hamiltonian systems, rotation number, KAM Tori

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(Some figures may appear in colour only in the online journal)

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1. Introduction

Quasiperiodicity is a key type of observed dynamical behavior in a diverse set of applications. We say a map $T$ is $(d$-dimensionally) quasiperiodic (for $d \geq 1$) if (i) $T : \mathbb{T}^d \to \mathbb{T}^d$ and (ii) each trajectory is dense in $\mathbb{T}^d$ and (iii) there is a continuous choice of coordinates $\theta = (\theta_1, \cdots, \theta_d)$ and some $\rho = (\rho_1, \cdots, \rho_d) \in \mathbb{T}^d$ for which the $T$ has the form

$$T(\theta) = \theta + \rho \mod 1.$$  

(1)

Condition (ii) can be replaced by saying in dimension $d = 1$ that $\rho$ is irrational and in dimension $d > 1$ that all of the coordinates $\rho_j$ of $\rho$ are irrational and they are ‘independent’ over the reals; that is if $a = (a_1, \cdots, a_d)$ is a vector of integers and $\sum_{j=1}^d a_j \rho_j = 0$, then every $a_j = 0$. We then say such a $\rho$ is irrational.

Let $T$ be a $C^\infty$ quasiperiodic map. The quasiperiodicity persists for most small perturbations by the Kolmogorov–Arnold–Moser theory. We believe that quasiperiodicity is one of only three types of invariant sets with a dense trajectory that can occur in typical smooth maps. The other two types are periodic sets and chaotic sets. See [1] for the statement of our formal conjecture of this triumvirate. For example, quasiperiodicity occurs in a system of weakly coupled oscillators, in which there is an invariant smooth attracting torus in phase space with behavior that can be described exclusively by the phase angles of rotation of the system. Indeed, it is the property of the motion being described using only a set of phase angles that always characterizes quasiperiodic behavior. In a now classical set of papers, Newhouse, Ruelle, and Takens demonstrated a route to chaos through a region with quasiperiodic behavior, causing a surge in the study of the motion [2]. There is active current interest in development of a systematic numerical and theoretical approach to bifurcation theory for quasiperiodic systems. Our goal in this paper is to present a numerical method for the fast calculation of the limit of Birkhoff averages in quasiperiodic systems, allowing us to compute various key quantities.

If $f$ is integrable and the dynamical system is ergodic on the set in which the trajectory lives, then the Birkhoff ergodic theorem asserts that the Birkhoff average $B_N$ defined as

$$B_N(f) := \frac{1}{N} \sum_{n=0}^{N-1} f(x_n)$$  

(2)

defined on a function $f$ along an ergodic trajectory $(x_n)$ converges to the space average $\int f \, d\mu$ as $N \to \infty$ for $\mu$-almost every $x_0$, where $\mu$ is the unique invariant probability measure. In particular for quasiperiodic systems all trajectories with initial point in the ergodic set have the same limit of their Birkhoff averages. We develop a numerical technique for calculating the limit of such averages, where instead of weighting the terms $f(x_n)$ in the average equally, we weight the early and late terms of the set $\{0, \ldots, N-1\}$ much less than the terms with $n \sim N/2$ in the middle. That is, rather than using the equal weighting $(1/N)$ in the Birkhoff average, we use a weighting function $w(n/N)$.

1.1. Weighted Birkhoff averaging method

A function $w : \mathbb{R} \to [0, \infty)$ will be called a $C^\infty$ weighting function if $w$ is infinitely differentiable and $w > 0$ on $(0, 1)$ and $= 0$ elsewhere. The example of such a function that we will use in this paper is in equation (3) (figure 1) defined as

$$w(t) := \begin{cases} \exp\left(\frac{-1}{\eta(1-t)}\right), & \text{for } t \in (0, 1) \\ 0, & \text{for } t \notin (0, 1). \end{cases}$$  

(3)

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See equation (16) for a family of weighting functions $w^p(t)$ that converge even faster when many digits of precision are required. This is an example of what is often referred to as ‘window’ functions in spectral analysis or a ‘bump’ function in the theory of partitions of unity.

For $d = 1$, $\mathbb{T}$ is a simple closed curve. For a continuous function $f$ and a $C^\infty$ quasiperiodic map $T$ on $\mathbb{T}$, let $x_n \in \mathbb{T}$ be such that $x_n = T(x_{n-1})$ for all $n \geq 1$. We define a weighted Birkhoff ($WB_N$) average of $f$ as

$$WB_N(f) (x) := \sum_{n=0}^{N-1} \hat{w}_{n,N} f(x_n),$$

where $\hat{w}_{n,N} = \frac{w(n/N)}{\sum_{j=0}^{N-1} w(j/N)}$.

Note that $WB_N(f)$ is indeed an average of the values $f(x_n)$ since $\sum_{n=0}^{N-1} \hat{w}_{n,N} = 1$.

1.2. Main convergence result

The main convergence result we are using is theorem 3.1. It is proved in [3] and an outline of the proof is given here in section 4. We now state a special case of the theorem that avoids unnecessary terminology and states only the $C^\infty$ case.

Assume $f$ and $T$ are $C^\infty$ and $w$ is a $C^\infty$ weighting function, then for almost every rotation number $\rho$ and for every positive integer $m$ there is a constant $C_m > 0$ such that

$$\left| WB_N(f)(\theta) - \int f(\theta) d\theta \right| \leq C_m N^{-m}.$$

We refer to the above as super-fast (super polynomial) convergence or exponential convergence. The above constant $C_m$ depends on (i) $w(t)$ and its first $m$ derivatives; (ii) the function $f(t)$; and (iii) the rotation number(s) of the quasiperiodic trajectory or more precisely, the small divisors arising out of the rotation vector. Our method of averaging does not give improved convergence results for chaotic systems.
In [4–7], Laskar employs a Hanning data weighting function and the analogue of our \( \sin^2 \) weighting function for his computations, which lead to the convergence of order \( 1/N^2 \) or \( 1/N^3 \) respectively, (p 136 in [4]) where \( N \) is the length of a orbit. Specifically, the weighting function \( \cos^2(\pi x) \) for \( x \in (-1, 1) \) and averages over iterates from \(-N\) to \(N\). He mentions the \( C^\infty \) filter we use in Remark 2 of the appendix of [4], p. 146, though he does not use it. There seems to be no advantage to using his lower order methods than \( C^\infty \) filter. In particular the programming of both is quite simple. We will compare the two methods in figures 7(b), 9, 12(b) and 14.

Other authors have considered related numerical methods (see section 3.9), in particular [8–10], which we will compare to our approach when we introduce our averaging method in section 3. See also [11–20]. We announced some of the results presented here in [21].

1.3. The Babylonian problem of quasiperiodic rotation numbers

What constitutes a ‘big-data’ problem depends on the speed of computation available. With this understanding, the first big-data problem was 2500+ years ago when the Babylonians computed the three periods of the Moon from data on the position of the Moon collected almost daily for many years. The Moon’s position through the fixed stars can be viewed as a quasiperiodic trajectory with \( d = 3 \) and their problem was to compute the rotation numbers from such a trajectory, which they did with high accuracy. See [22].

1.4. Applications

We demonstrate our weighted Birkhoff averaging method and its convergence rate by computing rotation numbers, conjugacies (i.e. changes of variables), and their Fourier series in dimensions one and two. We will refer to a one-dimensional quasiperiodic curve as a curve.

We start by describing our results for a key example of quasiperiodicity: the (circular, planar) restricted three-body problem (R3BP). This is an idealized model of the motion of a planet, a large moon, and an asteroid governed by Newtonian mechanics, in a model studied by Poincaré [23, 24]. In particular, we consider a planar three-body problem consisting of two massive bodies (‘planet’ and ‘moon’) moving in circles about their center of mass and a third body (‘asteroid’) whose mass is infinitesimal, having no effect on the dynamics of the other two.

We assume that the moon has mass \( \mu \) and the planet mass is \( 1 - \mu \) where \( \mu = 0.1 \), and writing equations in rotating coordinates around the center of mass. Thus the planet remains fixed at \((q_1, p_1) = (-0.1, 0)\), and the moon is fixed at \((q_2, p_2) = (0.9, 0)\). In these coordinates, the satellite’s location and velocity are given by the \emph{generalized position vector} \((q_1, q_2)\) and \emph{generalized velocity vector} \((p_1, p_2)\). The equations of motion are as follows.

\[
\begin{align*}
\frac{dq_1}{dt} &= p_1 + q_2, \\
\frac{dp_1}{dt} &= p_2 - q_1, \\
\frac{dq_2}{dt} &= p_2 - \mu \frac{d^2 q_1}{d \text{moon}^2} - (1 - \mu) \frac{q_1}{d \text{planet}} \\
\frac{dp_2}{dt} &= -p_1 - \mu \frac{dq_2}{d \text{moon}} - (1 - \mu) \frac{q_2}{d \text{planet}},
\end{align*}
\]

where

\[
d^2_{\text{moon}} = (q_1 - 1 + \mu)^2 + q_2^2
\]
The following function $H$ is a Hamiltonian for this system

$$H = \frac{p_1^2 + p_2^2}{2} + (p_1q_2 - p_2q_1) + \left(\frac{1 - \mu}{d_{\text{planet}}} - \frac{\mu}{d_{\text{moon}}}\right),$$

(6)

where $p_1 = \dot{q}_1 - q_2$ and $p_2 = \dot{q}_2 + q_1$ (see [25] p 59 equations (63)–(66)).

The three terms in equation (6) are respectively the kinetic energy, angular moment, and the potential. For fixed $H$, Poincaré reduced this problem to the study of the Poincaré return map for a fixed value of $H$, only considering a discrete trajectory of the values of $(q_1, p_1)$ on the section $q_2 = 0$ and $dq_2/dt > 0$. Thus we consider a map in two dimensions rather than a flow in four dimensions. Figure 2 shows one possible motion of the asteroid for the full flow. The orbit is spiraling on a torus. The black curve shows the corresponding trajectory on the Poincaré return map. Figure 3 shows the Poincaré return map for the asteroid for a variety of starting points. A variety of orbits are shown, most of which are quasiperiodic invariant curves.
An exception is trajectory A in figure 3(a), which is an invariant recurrent set consisting of 42 curves. Each curve is an invariant quasiperiodic curve under the 42nd iterate of the map.

The paper proceeds as follows: in section 2, we give the formal definition of quasiperiodicity, rotation number, and the conjugacy map to the rigid rotation. In section 3, we describe
our numerical technique in detail. We illustrate our weighted Birkhoff averaging method for a series of four examples, including an example of a two-dimensionally quasiperiodic map. In all cases, we get fast convergence and are in most cases able to give results with about 30-digit accuracy. Section 4 describes what happens when a rotation number is unusually well approximated by a fraction with small denominator. In such cases we elliptically say the rotation number is ‘nearly rational’. Finally, section 5 contains our concluding remarks.

2. Quasiperiodicity

In the introduction, we described quasiperiodic motion as motion that could be fully understood through a set of angles of rotation. We now formalize that idea in the following definition.

2.1. Quasiperiodicty

For a dimension $d \geq 1$, let $\rho = (\rho_1, \rho_2, \ldots, \rho_d)$ be a vector whose coordinates are irrational and are independent over the integers (see equation (1)). The following map $T_\rho : \mathbb{T}^d \to \mathbb{T}^d$ is called a rigid rotation:

$$T_\rho(\theta) = \theta + \rho \mod 1, \text{ where } \text{‘mod’ is applied to each coordinate.}$$

(7)

A rigid rotation is the simplest, albeit least interesting example of a map with quasiperiodicity. Since $T_\rho$ gives the same values on opposite sides of the unit cube, we identify the sides and refer to the domain of the rigid rotation as a curve in one dimension and a $d$-torus in dimension $d > 1$. In this paper, we will sometimes refer to the curve as a $1$-torus. We define a map $T$ on an ambient space $\mathbb{R}^D$ to be quasiperiodic if either $T$ or some iterate $T^k$ is topologically conjugate to a rigid rotation. We will assume $k = 1$ in the rest of this description. That is, a map $T$ is quasiperiodic if there is a rigid rotation map $T_\rho$ and an invertible conjugacy map (i.e. change of coordinates) $h : \mathbb{T}^d \to \mathbb{R}^D$ such that

$$T(h(\theta)) = h(T_\rho(\theta)).$$

(8)

A flow has quasiperiodic behavior if its associated Poincaré return map has quasiperiodic behavior.

For an invertible map $T$ to be quasiperiodic on a curve $C$, it is necessary and sufficient that $T$ has a dense trajectory, as shown in [26]. In general, a one-time differentiable invertible map on a curve without periodic points may not be quasiperiodic. However, if we assume that the map $T$ and the curve $C$ are twice continuously differentiable, then Denjoy [27] showed that these conditions are both necessary and sufficient. Furthermore, clearly any rigid irrational rotation map is a real analytic map, but even if we assume that a quasiperiodic function is analytic, Arnold showed that the conjugacy map $h$ may only be continuous for some atypical rotation number. However, Herman (see [28]) proved that for homeomorphisms on a circle, most conjugacy maps $h$ are analytic. Yamaguchi and Tanikawa [29] and Hunt, Khanin, Sinai and Yorke [30] show that the critical KAM curve may not be $C^2$.

2.2. Diophantine rotations

An irrational vector $\rho \in \mathbb{R}^d$ is said to be Diophantine if for some $\beta > 0$ it is Diophantine of class $\beta$ (see [28], definition 3.1), which means there exists $C_\beta > 0$ such that for every $k \in \mathbb{Z}^d$, $k \neq 0$ and every $n \in \mathbb{Z}$,

\[ 1/n^\beta \leq C_\beta. \]
\[ |k \cdot \rho - n| \geq \frac{C \cdot 2^{d+\beta}}{|k|^{d+\beta}}. \]  

We conjecture that if the map is analytic, then almost every quasiperiodic torus having a rotation number that is Diophantine (i.e. far from rational) is real analytic.

### 2.2.1. Assigning angular coordinates.

Let \((x_n)\) be the forward orbit under \(T\), and \((\theta_n)\) the forward orbit under \(T_\rho\). That is, \(x_{n+1} = T(x_n)\) and \(\theta_{n+1} = \theta_n + \rho \mod 1\).

In all of the \(d = 1\) examples discussed here, the quasiperiodic curves are simple, closed, convex curves, and hence, angular coordinates can be obtained using polar coordinates \((\phi, r)\) with respect to a fixed and pre-defined center, where \(r\) is uniquely determined by \(\phi\). More generally we may have an image of a quasiperiodic curve or torus that is not an embedding as a closed, convex curve. Such is not needed, since we have a method based on the Takens delay coordinate maps that solves the problem. This general method of obtaining rotation numbers from the image of a quasiperiodic curve is announced in [21] with a complete description in [22].

Because of our nice embeddings the orbits \((\theta_n = n\rho)\), \((x_n)\) and \((\phi_n)\) are all conjugate and there is a continuous map \(V : S^1 \rightarrow S^1\) such that

\[ x_n = h(\theta_n), \quad \phi_n = V(\theta_n), \quad \theta_n = n\rho \mod 1 \text{ for all } n = 0, 1, 2, 3, \ldots \]  

Since \(V\) is invertible, the following map is periodic.

\[ g(\theta) := V(\theta) - \theta \]  

For example, in figure 4(a), the angular coordinate \(\phi\) for the quasiperiodic curve \(B_1\) of the restricted three-body problem is measured from a point \((q_1, p_1) = (-0.2, 0)\).

### 3. Weighted Birkhoff averaging \(WB_X\) and its applications

As mentioned in section 1, our approach is to modify the regular Birkhoff average in equation (2) using the weighting function \(w\) in equation (3) (see figure 1) so that for quasiperiodic dynamical systems, the weighted Birkhoff average in equation (4) converges much faster to the same limit \(\int f(\theta) d\theta\). We will formalize this claim using the main result from the companion paper [3].

**Definition.** A function \(w : \mathbb{R} \rightarrow [0, \infty)\) is said to be a \(C^\infty\) bump function if \(w\) is \(C^\infty\) and the support of \(w\) is \([0, 1]\) and \(\int_0^1 w(x) dx \neq 0\) and \(w\) and all of its derivatives vanish at 0 and 1.

**Theorem 3.1 (Theorems 1.1, 3.1 in [3]).** For \(r \in [1, \infty]\), let \(X\) be a \(C^r\) manifold and \(T : X \rightarrow X\) be a \(C^r\) map which is \(d\)-dimensionally quasiperiodic on an invariant set \(X_0 \subseteq X\), with invariant probability measure \(\mu\) and a rotation vector of Diophantine class \(\beta(> 0)\). Let \(f : X \rightarrow \mathbb{R}\) be a \(C^r\) map. Let \(m > 1\) be an integer such that \(r \geq d + m(d + \beta)\), and let \(w : \mathbb{R} \rightarrow \mathbb{R}\) be a \(C^m\) bump function. Then there is a constant \(C_m\) depending upon \(w, f, m, M\), and \(\beta\) but independent of \(x_0 \in X_0\) such that

\[ \left| (WB_X f)(x_0) - \int_{X_0} f d\mu \right| \leq C_m N^{-m}. \]  

In particular, if \(r = \infty\) and \(w\) is a \(C^\infty\) bump function, then equation (12) holds for every \(m \in \mathbb{N}\).

Theorem 3.1 is proved in section 4 in a way that lets us determine what happens when an irrational rotation number is near a rational number.
3.1. Diophantine rotation numbers

The assumption on the rotation numbers being Diophantine means that the rotation numbers cannot be closely approximated by rational numbers with small denominators. For numbers which are not Diophantine, the trajectories have ergodic properties ‘close’ to periodic orbits and therefore, do not converge slower than the rate $N^{-k}$ for some $k$. 

Figure 4. Quasiperiodicity for the R3BP. For the quasiperiodic curve $B_1$ from figure 3, panel (a) shows the curve $B_1$ along with the projection mapping from $x_n$, the $n$th iterate in the $(q_1, p_1)$ coordinates, to angular coordinate $\phi_n$. It is parameterized by coordinates $\phi \in S^1 \equiv [0, 1)$. Panel (b) depicts the periodic part $g(\theta)$ of the one-dimensional conjugacy map (equation (11)) between the quasi-periodic behavior and rigid rotation by $\rho$. The curve represents the true Fourier series up to 30 decimal digits. Panel (c) shows the convergence rate of the error in the rotation number $\rho_N$ as a function of the number of iterates $N$. The ‘error’ is the difference $|\rho_N - \rho_N^*|$, where $N^* = 400\,000$ is large enough so that $\rho_N$ appears to have converged. The straight line is an upper bound for the curve and its exponent indicates the rate of convergence. Panel (d) shows the norm of the Fourier coefficients of the conjugacy as a function of index. This exponential decay indicates that the conjugacy function is analytic, up to numerical precision. The step size used for the eighth order Runge–Kutta scheme is $2 \times 10^{-5}$. 

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3.2. Robustness of assumptions

It is well known that for every $\beta > 0$ the set of Diophantine vectors of class $\beta$ have full Lebesgue measure in $\mathbb{R}^d$ (see for example, [28], 4.1). Thus, the assumption of the rotation number being Diophantine is robust in a measure theoretic sense, i.e. in physical experiments, the rotation number will be Diophantine with probability 1.

3.3. Computing a rotation number or rotation vector

We now show how to apply this averaging method in computation. We observe that $N$ must generally be larger for $T^2$ than for $T$ to get a 30-digit accuracy.

According to the definition of ‘quasiperiodicity’, a quasiperiodic map is conjugate to a rigid rotation of the form equation (7) with rotation vector $\rho$. However, this vector $\rho$ is not unique. When the dimension $d = 1$, there are two choices of $\rho$ depending on whether you move clockwise or counterclockwise around the circle, and for $d > 1$, it is shown in [22] that there is a set of choices for $\rho$ which are related to each other by unimodular transformations of $\mathbb{R}^d$ and are dense in $T^d$. Our goal will be to find any one of these equivalent rotation vectors. The rotation vector of a quasiperiodic trajectory $(y_n)_{n=0}^{N-1}$ can be viewed as the average rotation traversed by the sequence of $d$-vectors $(T(\theta_n) - \theta_n)$, which by equation (10) is equivalent to the average of the angular increments $(\phi_{n+1} - \phi_n)$. To make the notion of the angular distance between $\phi_n$ and $\phi_{n+1}$ consistent, we must make this choice continuously. By conjugating with $(hV^{-1})$ from equation (10) if necessary, we may assume that the map $T$ is in angular coordinates on $T^d$.

We then associate with $T$ a continuous map $\tilde{T}$ on the full Euclidean space $\mathbb{R}^d$ such that

$$\tilde{T}(z) \pmod{1} = T(z) \pmod{1).$$

The map $\tilde{T}$ is called a lift of $T$, and $z \in \mathbb{R}^d$ is a lift of $\phi \in T^d$.

Since $T$ is invertible, the map $\tilde{T}(z) - z$ has period one in every coordinate direction. For example, the rigid rotation $T(\phi) = \phi + \rho \pmod{1}$, for $\theta$ in $T^d$ has a corresponding lift map $\tilde{T}(z) = z + \rho$. Of course if $\rho$ was $\sqrt{2}$, we would have the same map $T$ as for $\rho = \sqrt{2} - 1$ so we define $\tilde{T}$ using rotation numbers are in $(0, 1)$. Using the lift, we now give a formula for the rotation vector for the trajectory $(y_n)$ starting at $y_0$:

$$\rho(T) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} (\tilde{T}(z_n) - z_n).$$

(13)

This average converges slowly as $N \to \infty$, with order of at most $1/N$. However, since equation (13) can be written as a Birkhoff average by writing $f(z_n) = \tilde{T}(z_n) - z_n$, we can apply our method to this function. That is, let $(z_n)_{n=0}^{N-1}$ be an orbit for $T$. Our approximation of $\rho$ is given by the weighted Birkhoff average of $f$,

$$WB_N(z_{n+1} - z_n) := \sum_{n=0}^{N-1} \tilde{w}_{n,N}(z_{n+1} - z_n) \to \rho \text{ as } N \to \infty.$$  

(14)

3.4. Convergence rate of the weighted Birkhoff average $WB_N$

In order to illustrate the speed of convergence of our weighted Birkhoff average $WB_N$ as $N \to \infty$, we introduce four different possible choices for the weighting function $w$, depicted
in figure 5, and compare the convergence results for computing the rotation number for each of these choices of $w$.

$$w_{\text{quad}}(t) := t(1-t)$$

$$w_{\text{sin}^2}(t) := \sin^2(\pi t)$$

$$w_{[1]}(t) := w_{\text{exp}}(t) := \exp\left(-\frac{1}{t(1-t)}\right).$$

(15)

The weighting functions described above are defined to be 0 outside $(0, 1)$ and equal to the specified function inside $(0, 1)$. Recall that the last function in the list, $w_{\text{exp}}$, is the function used in our calculations. It is the only one in the list that is $C^\infty$. The others fail to be $C^\infty$ at $t = 0$ and $t = 1$. A family of $C^\infty$ weighting functions can be defined for $p \geq 1$ as

$$w_{[p]}(t) := \exp\left(-\frac{1}{t^p(1-t)^p}\right) \text{ for } t \in (0, 1)$$

(16)

and = 0 elsewhere. This paper mainly uses $w = w_{[1]}$. The function $w_{[2]}$ results in an averaging method which converges noticeably faster than $w_{[1]}$ when using 30-digit precision, but not in 15-digit precision. We will write the weighted Birkhoff averages as $WB_{N}^{[1]}$ (or just $WB_{N}$) and $WB_{N}^{[2]}$ when using $w_{[1]}$ and $w_{[2]}$ respectively. See figure 7 where the two are compared.

When we compute with the first choice of $w$, we recover the truncated sum in the definition of the Birkhoff average. To estimate the error, we expect the difference $f(x_{N+1}) - f(x_N)$ to be of order one, implying that for $w_{\text{quad}}$, the error in the average is generally proportional to $N^{-1}$ in our figures. The choice of a particular starting point also creates a similar uncertainty of order $1/N$. Every function $w$ is always positive between 0 and 1. For all but the first choice, the function vanishes as $t$ approaches 0 and 1. In addition, going down the list, increasing
number of derivatives of $w$ vanish for $t \to 0$ and $t \to 1$, with all derivatives of $w_{\exp}$ vanishing at 0 and 1. We thus expect the effect of the starting and endpoints to decay at the same rate as this number of vanishing derivatives. Indeed, we find that $w_{\quad \text{quad}}$ corresponds approximately to order $1/N^2$ convergence, $w_{\sin^2}$ to $1/N^3$ convergence, and $w_{\exp}$ to convergence faster than any polynomial in $1/N$, i.e. for every integer $m$, there is a constant $C > 0$ such that for $N$ sufficiently large, $|\mathcal{H}_N - f| \leq CN^{-m}$. Figures 12(b) and 9 show this effect. We have not tried other $C^\infty$ weighting functions.

3.5. Estimating error when the true rotation number is known

In order to test the error in the calculation of rotation number, we present two examples below where we know the exact rotation number. This allows us to determine the actual error in the calculation for the $\mathcal{H}_N$ method as $N$ increases. In both cases the error decreases to less than $10^{-31}$ and then it grows as $N$ increases, apparently due to accumulated round-off error.

**Example 1.** Let $(\theta_n)$ be an orbit under the rigid rotation described in equation (7) for a rotation by $\rho = \sqrt{2} - 1$. Assume that what we observe is $\phi$, a perturbed version of $\theta$, namely,

$$
\phi_n = \theta_n + \alpha \cos(2\pi \theta_n) + \beta \sin(2\pi \theta_n), \quad \text{where } \theta_n = n\rho \pmod{1}.
$$

(17)

We use the weighted Birkhoff average as in equation ((14)) (changing $y$ to $\phi$) to obtain an estimate of the rotation number $\rho$ from this orbit. Figure 6 shows the results for $\alpha = 0.1$ and $\beta = 0.2$ in (a) and for the case $\alpha = 0.0$ and $\beta = 0.0$ in (b).

**Example 2.** Figure 7 shows a geometric version of the problem from the previous example, and again the error in the rotation number is small.
3.6. Fourier coefficients and change of coordinates reconstruction

For a quasiperiodic curve as shown in figure 4(a), there are two approaches to representing the curve. Firstly, we can write the coordinates \((X, Y)\) as a function of \(\theta \in S^1\), or secondly, we can reduce the dimension and represent the points on the curve by an angle \(\phi \in S^1\), that is, \(\phi(X(\theta), Y(\theta))\), which is also \(V(\theta) = \theta + g(\theta)\). We have shown \(g\) in figure 4(b) and the exponential decay of the norm of the Fourier coefficients in figure 4(d). We report the Fourier series for the periodic part \(g(\theta)\).

Given a continuous periodic map \(f : S^1 \to \mathbb{R}\), the Fourier series representation of \(f\) is the following:

\[
   f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{i2\pi k \theta},
\]

where the complex coefficient \(a_k\) is given by the formula

\[
   a_k = \int_{\theta \in S^1} f(\theta) e^{-i2\pi k \theta} d\theta.
\]

If we only have access to an ergodic orbit \((x_n)\) on a curve, then we cannot use the fast Fourier transform as we only have the function values \(f(x_n)\) along a quasiperiodic trajectory, and a rotation number \(\rho\). Using interpolation to get the grid needed to apply a fast Fourier
transform introduces significant interpolation errors. So instead, we obtain these coefficients
using a weighted Birkhoff average on a trajectory $(x_n)$ by applying the functional $W^N$. For
$k = 0$, we find $a_0$ by applying $W^N$ to the function 1. Note that for all $k$, $a_{-k} = a_k$. For $k > 0$,
we find $a_k$ as follows:

$$a_k = W^N(f(\theta)e^{-i2\pi k\theta}) = \sum_{n=0}^{N-1} f(x_n)e^{-i2\pi kn\rho} \tilde{w}_{nN}. \quad (20)$$

This is depicted for the R3BP in figure 4, for the standard map in figure 10, for the forced van
der Pol equation in figure 11. In all three one-dimensional cases, we depict $|a_k|$ as a function of
$k$ for $k \geq 0$ only, as for all $k$, $|a_{-k}| = |a_k|$. Our main observation is that the Fourier
coefficients decay exponentially; that is, for some positive numbers $\alpha$ and $\beta$, in dimension one, the Fourier
coefficients satisfy

$$|a_k| \leq \alpha e^{-\beta |k|} \quad \text{for all } k \in \mathbb{Z}. \quad (21)$$

This is characteristic of analytic functions. Therefore, the conjugacy functions of all our
examples are effectively, analytic, ‘effectively’ meaning within the precision of our quadruple
precision numerics. In two dimensions, the computation of Fourier coefficients is similar, but
instead of only having one exponential functions, for each $(j,k)$, we have two linearly inde-
dependent sets of exponentials.

$$e^{i(jx+ky)} \text{ and } e^{i(jx-ky)}.$$ 

We define $a_{j,k}$ and $a_{j,-k}$ to be the complex-valued coefficients corresponding to these two
functions.
3.7. Examples

3.7.1. The standard map. The standard map is an area preserving map on the two-dimensional torus, often studied as a typical example of analytic twist maps (see [29]). It is defined as follows

\[ S_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y + \alpha \sin(x + y) \end{pmatrix} \mod 2\pi. \quad (22) \]

In this paper, we only consider the case \( \alpha = 1.0 \). Figure 8(a) shows the trajectories starting at a variety of different initial conditions plotted in different colors. The shaded set is a large invariant chaotic set with chaotic behavior, but many other invariant sets consist of one or more topological circles, on which the system has quasiperiodic behavior. For example, initial condition \((\pi, 1.65)\) yields chaos while \((\pi, 1.5)\) yields a quasiperiodic trajectory. As is clearly the case here, one-dimensional quasiperiodic sets often occur in families for non-linear processes, structured like the rings of an onion. There are typically narrow bands of chaos between quasiperiodic onion rings. Usually these rings are differentiable images of the \( d \)-torus. We have computed the rotation number to be 0.120 552 721 973 755 133 002 981 643 698 39 for one such standard map orbit shown in the figure 8(b) using quadruple precision. Figure 9 shows a convergence rate of \( O(N^{-8.75}) \) in computing the rotation number using \( WB_N \). Figure 10(a) shows the periodic parts \( g \) of conjugacies of the quasiperiodic orbit, and figure 10(b) shows the absolute values of Fourier coefficients representing an exponential decay.

3.7.2. The forced Van der Pol oscillator. Figure 11(a) shows attracting orbits for the time-2\( \pi/0.83 \) map of the following periodically forced Van der Pol oscillator with nonlinear damping [31]

\[ \frac{d^2x}{dt^2} - 0.2 \left(1 - x^2\right) \frac{dx}{dt} + 20x^3 = F \sin (0.83t), \quad (23) \]

for several values of \( F \). While the innermost orbit shown is chaotic, the outer orbits are topological circles with quasiperiodic behavior\(^7\). Each curve was assigned the angular coordinates of equation (13) by assigning the points on the curve the angle with respect to the origin \((0,0)\). Figure 11(b) shows the periodic parts \( g \) of conjugacies of quasiperiodic orbits, and figure 11(c) shows the absolute values of Fourier coefficients representing exponential decays.

3.7.3. A two-dimensional torus map. So far, the quasiperiodic sets studied here are closed curves. We now describe an example [32–34] of a two-dimensional quasiperiodic torus map on \( T^2 \). This is a two-dimensional version of Arnold’s family of one-dimensional maps (see [35]). The map is given by \((T_1, T_2)\) where

\[ T_1(x, y) = \left[ x + \omega_1 + \frac{\epsilon}{2\pi} P_1(x, y) \right] \mod 1, \]

\[ T_2(x, y) = \left[ y + \omega_2 + \frac{\epsilon}{2\pi} P_2(x, y) \right] \mod 1, \quad (24) \]

and \( P_i(x, y), i = 1, 2 \) are periodic functions with period one in both variables, defined by:

\(^7\) As with the standard map, we have specified all non-essential parameters rather than stating the most general form of the Van der Pol equation. Our computed rotation numbers for the three orbits \( F = 5.0, 15.0, \) and 25.0 are 0.292 061 263 291 995 892 855 775 787 189 59, 0.375 534 411 131 440 108 849 089 280 833 18 and 0.562 353 700 926 850 566 344 192 213 361 54 respectively.
The values of all coefficients are given in table 1. This choice of this function is based on [32, 33]. The papers use the same form of equation, though the constants are close to but not precisely the same as the ones used previously. This fits with the point of view advocated by these papers: that the constants should be randomly chosen. Since we are using higher precision, we have chosen constants that are irrational to the level of our precision. The forward orbit is dense on the torus, and the map is a nonlinear map which exhibits two-dimensional quasiperiodic behavior.

Figure 12(a) depicts iterates of the orbit, indicating that it is dense in the torus. We use our weighted Birkhoff average to compute the two Lyapunov exponents, which have super convergence to zero. Figure 12(b) shows one of them. In terms of method, this is just a matter of...
Figure 10. The standard map conjugacy. This figure shows the analysis of the quasiperiodic trajectory in Figure 8. Panel (a) depicts the periodic part $g(\theta)$ of the conjugacy between the quasiperiodic behavior and rigid rotation by $\rho$, measured with respect to the center point $(x,y) = (\pi,0)$. See equation (11) for a description of $g(\theta)$. Panel (b) shows the decay of the Fourier coefficients. Since the conjugacy is an odd function, the odd-numbered Fourier coefficients are zero and therefore have been omitted from the picture. The decay of the Fourier terms can be bounded from above by an exponential decay, which suggests that the conjugacy is analytic. An orbit of length $N = 10^7$ is used for these computations. A smaller orbit of length $N = 10^6$ does not lead to any significant changes.

Figure 11. Forced Van der Pol oscillator. Panel (a) shows attracting orbits for a number of different forcing values $F$ for the stroboscopic map of the van der Pol flow given in equation (23). The plot depicts points $(X,Y) = (x(t_k), x'(t_k))$, where $t_k = 2k\pi/0.83, k = 0, 1, 2, \ldots$. The chaotic orbit lying inside the cycles corresponds to $F = 45.0$. There are stable quasiperiodic orbits shown as curves, which from outermost to innermost correspond to $F = 5.0, 15.0, 25.0$ and $35.0$ respectively. Panel (b) is the periodic part $g(\theta)$ of the conjugacy (equation (10)) to a rigid rotation, measured with respect to the center point $(0,0)$ for $F = 5.0, 15.0, 25.0$. Panel (c) shows the absolute values of the Fourier coefficients of $g$. Their linear decay as $|k|$ increases in this log-linear plot means the coefficients decay exponentially fast, which is a signature of the analyticity $g$ and hence of the conjugacy to a rigid rotation. The decay is exponentially fast down to the resolution of the numerics.
changing the function $f$ used in $\text{WB}_N$ in equation (4). Likewise, finding rotation numbers in two dimensions uses the same technique as in the one-dimensional case (see figure 12(c)). In all of our calculations, the computation is significantly longer than in dimension one in order to get the same accuracy, perhaps because in dimension two sufficient coverage by a trajectory may vary like the square of the side length of the domain. Figure 12(b) shows a convergence rate of $O(N^{-10.0})$.

The reconstructed conjugacy function for the two-dimensional torus is depicted in figure 13(a). The decay of Fourier coefficients shown in figure 13(b) displays $\sqrt{j^2 + k^2}$ on the horizontal axis, and $|a_{j,k}|$ on the vertical axis, where both of these coefficients are complex.
meaning that $| \cdot |$ represents the modulus. Again here, the coefficients decay exponentially, though the decay of coefficients is considerably slower in two dimensions. The data set looks quite a lot more crowded in this case, since there are $10^4$ coefficients and many different values of $(j, k)$ have almost same values of $\sqrt{j^2 + k^2}$. In addition, the coefficients $a_{j,k}$ generally converge at different exponential rates. This is why there is a strange looking set consisting of an upper and a lower cloud of data in figure 13(b). While more information on the difference between these coefficients is gained by interactively viewing the data in three dimensions, we have not been able to find a satisfactory static flat projection of this data. We feel that in a still image, the data cloud shown conveys the maximum information.

3.7.4. The three-body problem revisited. We have computed trajectories for the Poincaré return map using an eighth order Runge–Kutta method with time step $2 \times 10^{-5}$, in quadruple precision and figure 4(c) is consistent with 30-digit accuracy of the rotation number (see item 1 below) for the quasiperiodic orbit of the three-body problem labeled $B_1$ in figure 3(b). The initial condition for this orbit is $(q_1, q_2, p_1, p_2) = (-0.15, 0, 0, p_2)$ where $p_2 \approx 5.41$ is chosen so that the Hamiltonian is about $-2.63$. While it is more straightforward to obtain a numerical trajectory when the quasiperiodic trajectory is stable, it is also possible when it is a saddle or a repellor. Our weighted Birkhoff average $W_{B_N}$ approach works equally well for both cases. The extent of convergence of $W_{B_N}$ is limited by the accuracy of the trajectory data. We now list results of our numerical methods for the restricted three-body problem.

1. The rotation number is $0.063961728757453097164077724400302$, computed to 30-digit accuracy, and Figure 4(c) shows the accuracy plateauing at about 30-digit accuracy.

2. We compute 200 terms of the Fourier series, the last 125 of which have magnitude near 0 (i.e. less than $10^{-30}$). There is a conjugacy map $h$ between the first return map and a rigid rotation on the circle. Evaluating the Fourier series allows us to reconstruct the conjugacy map (see figure 4(b)).
3. We find that the coefficients for the conjugacy map decrease exponentially fast (see figure 4(d)). That is a signature of a real analytic function as in figure 4(b).

4. The high rate of convergence of \( O(N^{-15.0}) \) for the rotation number in figure 4(c) suggests that we have an effective computational method that yields an accuracy that is close to the limit of numeric precision, provided \( N \) is sufficiently large.

3.7.5. Speed of convergence of Fourier series for a conjugacy. In a separate report [36], we examine conjugacies of the quasiperiodic curves in the Siegel disk. The map is a simple one-dimensional complex dynamical system \( z_{n+1} = f(z_n) \), where

\[
f(z) = z^2 + e^{2\pi i \rho} z
\]

and \( \rho = (\sqrt{5} - 1)/2 \) [9, 37].

It is found (see figure 3 in [36]) that the invariant curves get more irregular near the boundary of the disk. While typical smooth quasiperiodic curves require about 70 Fourier coefficients for 30-digit precision, near the boundary of the Siegel disk the curves are much more irregular and can require 24,000 coefficients (or more) for the same precision. The curves thus become increasingly fractal looking near the boundary and the Fourier series converges much more slowly. We remark that we would expect a similar slower convergence when exploring a case like the well known last KAM circle of the Standard Map.

3.7.6. Sources of error. We end this section by noting a few sources of error in the computation of Fourier coefficients. If the number of iterates \( N \) is too small, then we will not have sufficient coverage to get a good approximation of the coefficients, and the problem becomes more acute as the coefficient number \( |k| \) grows. If the approximation of the rotation number is not accurate, then we cannot expect the approximations of our Fourier coefficients to be good either, and given an error in the rotation number, there will be a \( k_{\text{max}} \) such that the Fourier coefficients \( a_k \) with \( |k| > k_{\text{max}} \) cannot be approximated with any reasonable accuracy. A more subtle form of an error comes from the fact that if the rotation number we are trying to estimate is close to being commensurate with the rotation number, then we will get unexpectedly insufficient coverage of the space when performing iteration. See section 4.2.

3.8. Lyapunov exponents computed as a weighted average

Lyapunov exponents are an important characterization of the dynamics resulting from the map \( T \). They measure the rate at which nearby trajectories diverge or converge and can be used to distinguish between chaos and quasiperiodicity, for example, the existence of a positive Lyapunov exponent implies chaos, while for quasiperiodic systems, all \( d \) Lyapunov exponents for equation (1) are zero. In this section, we will show a way to obtain ‘super-convergence’ to the Lyapunov exponents of a quasiperiodic dynamics on a 2D torus. Since we are considering the dynamics restricted to the torus, we do not calculate the Lyapunov exponents for the normal sub-bundle.

Lyapunov exponents are usually calculated numerically as an average of logarithms with all terms weighted equally, and the computations are therefore limited by extremely slow convergence rates. For quasiperiodic systems we would expect the error in the exponents to be \( O(1/N) \). For 2-dimensional quasiperiodic systems, the Lyapunov exponent can be expressed as a weighted Birkhoff average (using \( \text{WB}_N \)) and we observe that we obtain the Lyapunov exponents much faster. Figures 14(a) and (b) show convergence rates of \( O(N^{-20.0}) \) and \( O(N^{-16.7}) \) for the computation of the Lyapunov exponents using \( \text{WB}_N \). (We make no claim
that we can prove convergence in (a) is super fast.) Using the weighted Birkhoff average does not change the limit.

Recall that given a $d$-manifold $M$ and a map $T: M \rightarrow M$ with an invariant probability measure $\mu$, Oseledets’ multiplicative ergodic theorem (see [38]) states that there exists numbers $\lambda_1 \leq \ldots \leq \lambda_d$ such that for $\mu$-almost every point $x$ and every vector $v$ in the tangent space at $x$, the limit

$$\lim_{N \rightarrow \infty} \frac{\ln \| DT_N(x)v \|}{N}$$

exists and equals one among $\lambda_1, \ldots, \lambda_d$. These numbers are the Lyapunov exponents of the map. For the rest of the section, we will assume that we have a quasiperiodic orbit $(x_n)$ filling out a two-dimensional torus $\mathbb{T}^2$ and therefore $d = 2$ and $M = \mathbb{T}^2$.

One of the properties of the largest Lyapunov exponent $\lambda_2$ is that for almost every vector $v$ in the tangent space at $x_0$, the quantity $N^{-1} \ln \| DT_N(x_0)v \|$ converges to $\lambda_2$. Note that this limit can be expressed as an average along the trajectory $(x_n)$ in the following manner:

$$\frac{\ln \| DT_N(x_0)v \|}{N} = \sum_{n=0}^{N-1} \frac{\ln \| v_n \|}{N}, \text{ where } v_n = DT(x_{n-1})\frac{v_{n-1}}{\| v_{n-1} \|}, v_0 = v.$$ 

We obtain the same limit by taking our weighted Birkhoff average instead of a uniformly weighted average as above. We observe (without proof) that for quasiperiodic orbits, we get super-convergence to the same limit by taking a weighted Birkhoff average in the following manner:

$$\lambda_2 = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \hat{w}_{n,N} \ln \| v_n \|, \text{ where } v_n = DT(x_{n-1})\frac{v_{n-1}}{\| v_{n-1} \|}, v_0 = v.$$ 

See chapter 3 from [39] for an explanation of this method.
Once we have calculated $\lambda_2$, $\lambda_1$ can be obtained from the sum $\lambda_1 + \lambda_2$ of the Lyapunov exponents. $\lambda_1 + \lambda_2$ can be expressed as the Birkhoff average of the function $f_1(x) := \ln |\det DT(x)|$. Note that $\det DT(x_n) > 0$ for an orientation preserving map $T$. Therefore, we have

$$\lambda_1 + \lambda_2 = \lim_{N \to \infty} \frac{\ln |\det DT^N(x)|}{N} = \lim_{N \to \infty} B_N(f_1).$$

As mentioned before, the quantity $B_N(f_1)$ converges to the limit $\int_X \ln |\det DT(x)| \, d\mu(x)$ and we can obtain super-convergence to the same limit using the quantity $WB_N(f_1)$ as written below:

$$\lambda_1 + \lambda_2 = \lim_{N \to \infty} N^{-1} \sum_{n=0}^{N-1} \hat{w}_{n,N} \ln |\det DT(x_n)|. \quad (27)$$

3.9. Related methods

See [4–8, 10] for references to earlier methods for computing Birkhoff averages along a quasi-periodic orbit.

For higher dimensional quasiperiodicity ($d > 1$) Laskar [4] has an interesting technique of finding the frequency $\sigma$ that maximizes a function $\phi(\sigma)$. That corresponds to our rotation rate $\rho$. Each evaluation of $\phi(\sigma)$ requires application of the window filter (whereas our method uses only one application of $WB$). But his has the advantage of subtracting off this frequency and repeating this method to find the next frequency. Hence he has an automatic method for finding multiple rotation rates. Presumably our method could be combined with his to improve on both methods—when dealing with higher dimensional quasiperiodicity.

Luque and Villanueva [8, 10] develop fast methods for obtaining rotation numbers for smooth or analytic functions on a quasiperiodic torus, sometimes with quasiperiodic forcing with several rotation numbers. The paper [10] develops a technique to compute rotation numbers (but not other function integrals) with error satisfying $|\text{error}| \leq C_p N^{-p}$ for $N \geq 2^p$ where $C_p$ is a constant. Their method of computation is defined recursively, with the $p + 1$st method being defined in terms of the $p$th. As $p$ increases the computational complexity increases for fixed $N$. If $\tau(p, N_p)$ is their computation time when using $N = N_p$, it appears that $\tau(p, N_p)/N_p \to \infty$ as $p \to \infty$ (and $N_p \to \infty$). In comparison, computation time for our Weighted Birkhoff average is simply proportional to $N$ since it requires a sum of $N$ numbers. Figure 11 from [10] shows the rate of convergence to the rotation number for a quasiperiodic orbit arising from the R3BP. There, they get 30-digit accuracy for the rotation number using approximately 2000000 trajectory points while we get the same accuracy with 20000. The rate of convergence of their method is $\approx N^{-7.8}$. Their methods were extended in [40] to compute the derivatives of the rotation numbers for one-parameter families of circle diffeomorphisms, and in [41] to compute the Fourier coefficients of the conjugacy function (i.e. the change of variables between the map and rigid rotation).

3.9.1. Newton methods in the literature. An alternative approach to our approach for finding a conjugacy is considered in [42–45]. In the current paper, we are assuming that we are starting with only a set of iterates for a single finite length forward trajectory, rather than having access to the functional form of the defining equation. In contrast, the approach in the papers above assumes access to the full form of the original defining equations. The Fourier series for the conjugacy is obtained and validated by using automatic differentiation. In addition,
in this current paper, we assume that we start with a point in an invariant torus with quasiperiodic dynamics, whereas the methods referenced above include a fast Newton’s method for finding invariant tori, Lyapunov multipliers, and invariant stable and unstable manifolds. See also Jorba [46].

Several variants of the Newton’s method have been employed to determine quasiperiodic trajectories in different settings. In [47] a variant of Newton’s method was applied to locate periodic or quasi-periodic relative satellite motion in a non-linear, non-conservative setting.

A partial differential equation-based approach was taken in [48], where the authors defined an invariance equation which involves partial derivatives. The invariant tori are then computed using finite element methods. See also section 2 in [48] for more references on the numerical computation of invariant tori. In [49], the authors used an application of the Newton’s method to compute elliptical, low dimensional invariant tori in Hamiltonian systems.

Computing the conjugacy to the rigid rotation is key to the methods of de la Llave et al [42–45]. They use this to obtain a numerical ‘proof’ of the existence of tori, and, what they call ‘a posteriori KAM theory’.

4. Why our method works and when $N$ must be large

We will assume that we have a $C^\infty$ quasiperiodic map $F : \mathbb{T}^d \to \mathbb{T}^d$ and a $C^M$ map $f : \mathbb{T}^d \to \mathbb{C}$. To better understand convergence on an averaging method applied to an $f$, write

$$f(\theta) = \sum_k a_k f_k(\theta),$$

where $f_k(\theta) := e^{2\pi i k \theta}$. (28)

In particular $a_0 = \int_{\mathbb{T}^d} f$. $\text{WB}_N$ is of course a weighted average of $(f(n\rho))_{n=0}^{N-1}$ and by theorem 3.1 $\text{WB}_N(f) \to a_0$ as $N \to \infty$. Since an averaging process is linear, we can define $\psi_{N,k,\rho} = \text{WB}_N(f_k)$, a collection of numbers that is independent of the choice of $f$. (More generally one could define $\psi_{N,k,\rho}^{w}$ for any weighting function $w$ and the advantages of different choices of $w$ are reflected in the magnitudes $|\psi_{N,k,\rho}^{w}|$). They depend only on the averaging method (e.g. $\text{WB}_N$), $k$, $N$, and the rotation number $\rho$. This set encapsulates all errors that arise in the use of $\text{WB}_N$. In particular, $f(\theta) = \sum_k a_k f_k(\theta)$, we have

$$\text{WB}_N(f) = a_0 + \sum_{k \neq 0} a_k \psi_{N,k,\rho}.$$ (29)

In particular $\psi_{N,0,\rho} = 1$ and for each $k \neq 0$, $\lim_{N \to \infty} \psi_{N,k,\rho} = 0$. The rate of convergence to 0 depends on $\rho$ and $k$ and convergence can be slow when $e^{2\pi i k \rho} \approx 0$, as we show in the next section.

To investigate the effects of the rotation number $\rho$ on the numerical errors in calculating a Fourier coefficient, we show $\psi_{N,k,\rho} = \text{WB}_N(f_k)$ for each $k$ ($0 \leq k \leq 250$) for two cases of $\rho = \sqrt{2} - 1$ and $\pi - 3$ in figure 15. The figure suggests that $N = 10^5$ is sufficient to calculate Fourier coefficients for $\rho = \sqrt{2} - 1$, whereas for $\rho = \pi - 3$ even $N = 10^5$ is not sufficient.

4.1. Sketch of proof of theorem 3.1

Here we sketch a proof that enables us to determine what happens when a rotation number is near a rational number.

Note that for any constant $a_0$, $\text{WB}_N(a_0 + f) = a_0 + \text{WB}_N(f)$ so for simplicity we will assume $f$ has mean $a_0 = 0$. Let $I$ denote the identity operator and $U$ denote the Koopman operator on $L^2(X, \mu)$, defined as
\[ (Uf) : x \mapsto f(F(x)), \quad \text{for all } f \in L^2(X, \mu). \]

The idea of the proof is to provide two different estimates of the quantity

\[ \text{WB}_N(U - I)^m f = \frac{1}{N} \sum_n w\left(\frac{n}{N}\right) (U - I)^m f(n\rho). \quad (30) \]

First let \( m = 1 \) in equation (30). Then

\[
\frac{1}{N} \sum_n w\left(\frac{n}{N}\right) (U - I) f(n\rho) = \frac{1}{N} \sum_n w\left(\frac{n}{N}\right) f((n + 1)\rho) - \frac{1}{N} \sum_n w\left(\frac{n}{N}\right) f(n\rho) \\
= \frac{1}{N} \sum_n \left[w\left(\frac{n + 1}{N}\right) - w\left(\frac{n}{N}\right)\right] f(n\rho).
\]

Now taking absolute values on both sides give,

\[
\left| \frac{1}{N} \sum_n w\left(\frac{n}{N}\right) (U - I) f(n\rho) \right| \leq \frac{1}{N} \|f\|_{L^\infty} \sum_n \left| \frac{1}{N} w^{(1)}\left(\frac{n}{N}\right) \right| \leq \frac{1}{N} \|f\|_{L^\infty} \|w^{(1)}\|_{L^\infty}.
\]

**Figure 15.** Estimations of the numerical error in the calculations of Fourier coefficients. Panels (a) and (b) show the value of \( \psi_{k, N, \rho} = \text{WB}_N(f_k) \) for \( \rho = \sqrt{2} - 1 \), when the number of iterations is \( N = 10^4 \) and \( 10^5 \), respectively. Panels (c) and (d) show for \( \rho = \pi - 3 \), when \( N = 10^4 \) and \( 10^5 \), respectively.
Applying this procedure \( m \) times gives a constant \( c_m > 0 \) such that,
\[
|\text{WB}_N(U - I)^m f| \leq c_m N^{-m} \|f\|_{C^m},
\]
(31)
where \( c_m \) depends only on the first \( m \) derivatives of \( w \).

The second way to evaluate equation (30) is by using the Fourier series for \( f \). Note that \( f \) being \( C^M \) implies that there is a constant \( C_{f, m} > 0 \) such that \( |a_k| \leq C_{f, m} \|k\|^{-M} \). If \( k \neq 0 \), then
\[
\text{WB}_N(U - I)^m f_k = (e^{2\pi i k \cdot \rho} - 1)^m \text{WB}_N f_k.
\]
Hence, by equation (31),
\[
|\psi_N,k| = |\text{WB}_N f_k| \leq c_m \left( \frac{1}{N|e^{2\pi i k \cdot \rho} - 1|} \right)^m.
\]
(32)
This implies
\[
|\text{WB}_N(f)| = \left| \sum_{k \neq 0} a_k \text{WB}_N(f_k) \right| \leq \sum_{k \neq 0} |a_k| |\text{WB}_N(f_k)|
\leq c_m \left( \frac{1}{N} \right)^m \sum_{k \neq 0} |a_k| \left| \frac{1}{e^{2\pi i k \cdot \rho} - 1} \right|^m.
\]
(33)
The claim of the theorem now follows from equation (9) and the decay rate of \(|a_k|\).

4.2. Difficulties when \( \rho \) is approximately rational.

While theorem 3.1 requires \( \rho \) to be Diophantine to get fast convergence of \( \text{WB}_N(f) \) to \( \int f \, d\mu \) (the integral in the Birkhoff ergodic theorem), our computations have finite precision so we have to ask what this condition means in a finite precision world. With this in mind, we can restate the Diophantine condition, in equation (9), saying \( \rho \) is Diophantine (of class \( \beta \)) if there is some \( \beta > 0 \) for which
\[
\liminf_{|k| \to \infty} |k|^{d+\beta} |e^{2\pi i k \cdot \rho} - 1| > 0.
\]
In other words, the values of which \( k \) for which \(|k|^d |e^{2\pi i k \cdot \rho} - 1| \) should not go to 0 too fast. This idea suggests defining
\[
\Delta(k, \rho) := |k|^d |e^{2\pi i k \cdot \rho} - 1|.
\]
For fast convergence in computing Fourier series coefficients \( a_k \), the quantity \( \Delta(k, \rho) \) should not be too small for the relevant \( k \), those for which \(|a_k| \) is likely to be larger than our error threshold, which in this paper is about \( 10^{-30} \).

4.2.1. What values of \( \Delta(k, \rho) \) are we likely to encounter? For \( \rho_1 := (\sqrt{5} + 1)/2 \), the golden mean, we find
\[
\min_{k=2, \ldots, 10^6} \Delta(k, \rho_1) \approx 2.655.
\]
It appears that \( \lim_{|k| \to \infty} \Delta(k, \rho_1) = 2.809925 \ldots \). Note the last term in equation (33) is similar to \( \Delta \) except for the leading \(|k| \). Of course the power \(-m\) can greatly multiply the problem of \(|e^{2\pi i k \cdot \rho} - 1| \) being small. To offset \(|e^{2\pi i k \cdot \rho} - 1| \) being smaller by a factor of 100, we might expect that convergence would require \( N \) in equation (33) to be larger by a factor of 100.
Of course both are raised to the same power \( m \) in these equations. Indeed in figure 16 we must increase \( N \) by a factor of 100 to get 30-digit convergence. An illustration of the problem can be seen for \( \rho^2 = \pi - 3 \) since \( \pi \approx \frac{355}{113} \), and \( |k| = 113 \) yields the rather small value \( \Delta(\pm 113, \rho^2) \approx 0.021 \). Also \( \Delta(\pm 226, \rho^2) \approx 0.085 \). \( \Delta(\pm 339, \rho^2) \approx 0.193 \). These suggest slow convergence for those \( k \) values. When computing a Fourier series for \( f \) using \( \text{WB}_N \) with this \( \rho \), when \( N < 10^7 \), we obtain poor values for many Fourier coefficients as we illustrate in figure 16(a).

4.2.2. Comparing a function to its Fourier series. To estimate how accurate a computed Fourier series of a function \( f : T^d \to \mathbb{R}^1 \) is, define \( a_k = \text{WB}_N(ff_l) \) and \( f_l(\theta) := e^{2\pi i k \cdot \theta} \) (where \( k \cdot \theta \) denotes an inner product in \( \mathbb{R}^d \)), and

\[
 f^{K,N}(\theta) := \sum_{|k| \leq K} a_k f_k(\theta). 
\]

To test how similar \( f \) and \( f^{K,N} \) are in the \( L^2 \) and \( L^1 \) sense, we compute the errors

\[
 \delta_2^{K,N}(f) := \sqrt{\text{WB}_N((f - f^{K,N})^2)} \quad \text{and} \quad \delta_1^{K,N}(f) := \text{WB}_N(|f - f^{K,N}|). \tag{34}
\]

They are calculated in figure 16(b).

4.2.3. A saw-tooth pattern of errors. In the above example with \( N = 10^5 \), figure 16 shows a saw-tooth pattern with peaks at multiples of 113 where the slopes are all the same. Many coefficients have a big error, and all errors here are the result of \( \psi_{N,k} \) when \( k \) is a non-zero multiple of 113. To explain this saw-tooth effect, we note that we have found for this example...
that $\psi_{N,k} \approx 0$ for $N = 10^5$, when $k$ is not a multiple of 113 and in particular $k \neq 0$, and here we assume all those values are indeed 0 to simplify computation except for $k = 0$ and $|k| = k^*$ for some $k^* \gg 1$ such as $k^* = 113$.

Suppose we wish to compute the Fourier coefficients of $f$ and determine how accurate the result is. The computed $k$th coefficient, denoted $\hat{a}_k$ is

$$\hat{a}_k = \hat{a}_{\pm k} = WB_N(f(\theta)e^{\pm 2\pi ik \cdot \theta}) = \sum_{m} a_m \psi_{N,m \pm k}$$

which has significant contributions from $m \pm k = 0$ and $|m \pm k| = k^*$. Since $k^* \gg 1$, we can ignore $a_{\pm k}$, and we conclude that for small integers $n$,

$$\hat{a}_{|k^* \pm n|} \approx a_{|n|} \psi_{N,k^*}.$$ 

On the log-linear plot of the graph, the coefficients $a_k$ are almost linear, so it follows from this equation that the erroneous $\hat{a}_{|k^* \pm n|}$ has the same slope (the absolute value of the derivative), and that each non-zero multiple of $k^* = 113$ has the same error pattern. The heights of the peaks are $a_{0} \psi_{N,m k^*}$ at $k = m k^*$ where $m = 1, 2, \cdots$.

4.2.4. Remark on figure 10. The panel (b) of this figure shows a jump in the Fourier series terms at coefficient $|a_{506}|$ (i.e. $k = 253$ in that figure). This does not seem to be a numerical artifact. In fact $\Delta$ does have a local minimum at 506 but $\Delta(506, \rho) = 1.03$ is not particularly small for such minimum. Furthermore, changing the number of iterates (as mentioned in the caption) does not change the graph.

5. Concluding remarks

We have developed a straightforward but effective computational tool for quickly computing a large variety of quantities for quasiperiodic orbits. These quantities include rotation vectors, Fourier reconstruction of conjugacy maps, and in some cases Lyapunov exponents. The methods work well in one and higher dimensions. They are effective using both double and quadruple precision, though we have chosen to do most of our calculations in higher precision to show the full possibilities and quick convergence properties of our method.

The literature on quasiperiodicity is vast and windowing techniques analogous to ours are often used. But our goals in this paper are limited: to introduce the $C^\infty$ weighted Birkhoff averages $WB_N (= WB_N^{[1]}$) and $WB_N^{[2]}$ as numerically useful tools and to present some of its applications.

We note that the computational time for computing our weighting functions $w(t)$ is almost the same as for the weighting function $w_{\sin^2}$ that uses $\sin^2(\pi t)$. Both are equally easy to program. But convergence is far faster with the weighted Birkhoff averages, as seen in figures 7(b), 9, 12(b) and 14.

Quasiperiodic orbits can occur in many different situations, for example, subject to periodic forcing (see Luque and Villanueva [10]); as high-dimensional tori that are not simply embedded (see Medvedev et al [50]); in the presence of noise; etc. The question of whether our methods extend to these situations is worthy of further consideration.

5.1. When must $N$ be large to get convergence?

We have developed some diagnostics in section 4.2 to detect when $N$ must be chosen especially large to get high accuracy—at least for the $d = 1$ dimensional cases. Computation of
can be used to detect cases when $N$ must be large to get accurate values for Fourier coefficients. For example we found that because $\Delta(\pm 113, \rho_2)$ is so small, $N$ must be increased by a factor of 100 to get an accurate Fourier series for the conjugacy map. One might ask if there are other $k$ for which $\Delta(\pm 113, \rho_2)$ is quite small. We find $\Delta(\pm 113, \rho_2) < \Delta(k, \rho_2)$ for all $k \neq 113$ and $k < 10^7$.

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