Slice-by-slice numerical solution of 3D-vector tomography problem

Dr. Ivan Svetov
Researcher, Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia
E-mail: svetovie@math.nsc.ru

Abstract. We suggest a numerical solution of the problem by reconstruction of solenoidal part of the vector field defined in the unit ball. That is, approximation of solenoidal part of vector field is built by known ray transform, calculated along the lines parallel to one of the coordinate planes. Test calculations have shown good results of reconstruction solenoidal vector fields by using the proposed method.

1. Introduction
In this paper the 3D-vector tomography problem is considered in the following formulation. Let a bounded domain in $R^3$ be filled by a medium without refraction (probing rays permeate along straight lines). In the domain some vector field $v$ is given. It is require to find this field by its known ray transform.

We consider $R^3$ as the Euclidean vector space with the standard scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. The ray transform $I$ on $R^3$ maps a vector field $v = (v_j)$ to the function $[Iv]$ on the manifold of oriented lines $l = \{x + t\xi \mid x, \xi \in R^3, |\xi| = 1, \langle x, \xi \rangle = 0, t \in R\}$ by the formula

$$[Iv](x, \xi) = \sum_{j=1}^{3} \int_{-\infty}^{\infty} \xi_j v_j(x + t\xi) dt.$$ 

In calculating the ray transform along the ray only projection of the desired field on the ray makes contribution to the result. Therefore problem of reconstruction of vector fields $v$ by its ray transform $[Iv]$ has no unique solution. Namely, the operator $I$ possesses not trivial kernel consisting of potential vector fields with vanishing potentials on the boundary [1]. Therefore only the solenoidal part $s v$ of the field $v$ can be recovered from $[Iv]$.

The problem of recovering vector field $s v$ from $[Iv]$ is underdetermined in the number of unknown functions. Namely, three unknown functions $s v_j$ are connected by one differential equation $\delta^s v = \sum_{j=1}^{3} \frac{\partial (s v_j)}{\partial x_j} = 0$. This means that $s v$ depends on two functions. We need to recover these functions by one function $[Iv]$. This underdetermined does not complicate the vector field recover problem as compared with scalar field recover problem. On the other hand, the problem is overdetermined in dimension, because we try to recover functions $s v_j(x)$, where $x \in R^3$, from function $[Iv]$ on the four-dimensional manifold of oriented lines (variables $x, \xi$ have the dimension 6 in total, but there are two conditions: $|\xi| = 1, \langle x, \xi \rangle = 0$). Therefore it is
natural to pose the problem of recovering $^s v$ from incomplete data $[I v]|_{M^3}$, where $M^3$ is some three-dimensional submanifold of manifold of oriented lines.

Investigation of scalar incomplete data problems is a classical subject of mathematical tomography. As far as studying incomplete data problems of vector tomography is concerned, this direction is in its very beginning. To the author’s knowledge, there are very few mathematical papers in the direction. Denisjuk in his article [2] proved the possibility of recovery of a solenoidal part of $m$-tensor field, which is given in $R^3$, from incomplete data with usage of cone-beam scheme of observations. Namely, it is possible to recover a solenoidal part of $m$-tensor field by its known ray transforms computed along oriented lines from three-dimension manifold. Sharafutdinov in his article [3] proposes formulas for recovery of a solenoidal part of vector field by three-dimension data too, but with usage of parallel scheme of observations.

2. Definitions and statement of problem
In this paper we numerically solve the problem of recovery of a solenoidal part of vector field, which is given in unit ball $B = \{ x \mid x_1^2 + x_2^2 + x_3^2 < 1 \} \subset R^3$, from incomplete data. Namely, data have three dimensions. The algorithm of solving is based on the inversion formulas from paper [3], in which the problem of recovering a solenoidal part of a vector field on $R^3$ from ray transforms known over all lines parallel to one of the coordinate planes is considered.

We used coordinate planes $\pi_j = \{ x_j = 0 \} \subset R^3$ $j = 1, 2, 3$ only. We introduce coordinates $(\theta, s, z)$ on $M^3(\pi_j) = \{ (\xi, x) \in R^3 \times R^3 \mid |\xi| = 1, \xi_j = 0, \langle \xi, x \rangle = 0 \}$ such that

$$\xi = \cos \theta e_{j+1} + \sin \theta e_{j+2},$$

$$x = s(-\sin \theta e_{j+1} + \cos \theta e_{j+2}) + ze_j,$$

where $(e_1, e_2, e_3)$ is the standard basis.

For vector field $v$ a functions $[I_{(j)} v] = [I v]|_{M^3(\pi_j)}$ are defined by formula

$$[I_{(j)} v](\theta, s, z) = \int_{-\infty}^{\infty} \left( \cos \theta v_{j+1}(x + t\xi) + \sin \theta v_{j+2}(x + t\xi) \right) dt.$$
3. Schemes of algorithms of solution 2P- and 3P-problems

3P-problem. Let for vector field \( v \) three functions \( [I_{(j)}v], j = 1, 2, 3 \) are given. The following steps should take to restore the solenoidal part of \( v \) field \( v \) on these data.

**Step 1.** Calculate a value \( (B[I_{(j)}v]'_s)(x) \) of the 2D back-projection operator, \( j = 1, 2, 3 \) by formula

\[
(B[I_{(j)}v]'_s)(x) = \frac{1}{2\pi} \int_{0}^{2\pi} [I_{(j)}v]'_s(\theta, -x_{j+1}\sin\theta + x_{j+2}\cos\theta, x_j) \, d\theta.
\]

**Step 2.** Find a functions \( \lambda_{(j)}(y), \ j = 1, 2, 3 \) by formula

\[
\lambda_{(j)}(y) = \frac{i}{2}(y_{j+1}^2 + y_{j+2}^2)^{1/2}F_3[B[I_{(j)}v]'_s](y),
\]

where \( F_3[\cdot] \) is operator of the direct 3D-Fourier transform.

**Step 3.** Calculate \( u(y) = F_3[^b v](y) \) by formula

\[
u_i(y) = |y|^{-2}(\lambda_{(i+1)}(y)y_{i+1} - \lambda_{(i+2)}(y)y_{i+1}).
\]

**Step 4.** Find \( \lambda v(x) \) by applying the inverse 3D-Fourier transform.

2P-problem. Let for vector field \( v \) two functions \( [I_{(j)}v], j = 1, 2 \) are given. The following steps should take to restore the solenoidal part of \( v \) field \( v \) on these data.

**Step 1.** Calculate \( (B[I_{(j)}v]'_s)(x) \), \( j = 1, 2 \) by formula

\[
(B[I_{(j)}v]'_s)(x) = \frac{1}{2\pi} \int_{0}^{2\pi} [I_{(j)}v]'_s(\theta, -x_{j+1}\sin\theta + x_{j+2}\cos\theta, x_j) \, d\theta.
\]

**Step 2.** Find a functions \( \lambda_{(j)}(y), \ j = 1, 2 \) by formula

\[
\lambda_{(j)}(y) = \frac{i}{2}(y_{j+1}^2 + y_{j+2}^2)^{1/2}F_3[B[I_{(j)}v]'_s](y).
\]

A function \( \lambda_{(3)} \) are found by formula

\[
\lambda_{(3)} = -\frac{1}{y_3}(\lambda_{(1)}y_1 + \lambda_{(2)}y_2).
\]

**Step 3.** Calculate \( u(y) = F_3[^b v](y) \) by formula

\[
u_i(y) = |y|^{-2}(\lambda_{(i+1)}(y)y_{i+2} - \lambda_{(i+2)}(y)y_{i+1}).
\]

**Step 4.** Find \( \lambda v(x) \) by applying the inverse 3D-Fourier transform.

4. Numerical implementation of algorithms

**Input date.** We use discretization 100 for \( z \in [-1, 1] \) and 100, 200, 300, 400, 500 for \( s \in [-1, 1] \), and \( \theta \in [0, 2\pi] \).

**Regularization for solution of 2P-problem.** If we have \( \{y_3 = 0\} \) \( \{(y_3)_q\} \) is very small), then \( \lambda_{(3)}((y_1)_m, (y_2)_k, (y_3)_q) \) are calculated with usage regularization.

1) Cut: \( ((y_3)_q)^{-1} \) are replaced \( ((y_3)_q)^{-1} \) in formula for finding \( \lambda_{(3)}(y) \).
2) Linear interpolation:

\[
\lambda_{(3)}((y_1)_m, (y_2)_k, (y_3)_q) = \frac{\lambda_{(3)}((y_1)_m, (y_2)_k, (y_3)_{q-1}) + \lambda_{(3)}((y_1)_m, (y_2)_k, (y_3)_{q+1})}{2}.
\]
Without regularization for small $|(y_3)_q|$ components $v_1(y)$ and $v_2(y)$ can not restore.

**Numerical simulation.** We reconstruction continuous solenoidal vector field with components

$$v_i(x) = \begin{cases} 
48e^{-24|x|^2}(x_{i+2} - x_{i+1}), & \text{for } |x| < 1, \\
0, & \text{else.}
\end{cases}$$

Dependence relative error of reconstruction ($\%, L_2$-norm) on discretization of input data for $s, \theta$ is given in the Table 1.

| problem, regularization | 100 | 200 | 300 | 400 | 500 |
|------------------------|-----|-----|-----|-----|-----|
| 2P-problem, regularization 1 | 36.2 | 34.7 | 33.1 | 32.5 | 31.9 |
| 2P-problem, regularization 2 | 21.9 | 8.4 | 4.7 | 3.2 | 2.5 |
| 3P-problem | 4.3 | 2.4 | 1.3 | 0.9 | 0.7 |

5. Conclusion
Numerical simulations have shown good accuracy of the approximation by using the proposed algorithms for reconstruction of a solenoidal part of vector field, which is given in the unit ball, with incomplete data. It is sufficient to solve the 2P-problem for the uniqueness of a solenoidal part of the vector field reconstruction. On the other hand, the solution of 2P-problem is unstable. However, good choice of regularization allowed to recover a solenoidal part of the vector field with accuracy close to the accuracy in 3P-problem solving algorithm realization.

Acknowledgments
The work was supported partially by RFBR (grants 11-07-00447-a, 12-01-09335-mob z), SB RAS (grant 2012-32), DMS RAS (grant 2012-1.3.1)

References
[1] V. A. Sharafutdinov, *Integral Geometry of Tensor Fields*, Utrecht: VSP, 1994.
[2] A. Denisjuk, Inversion of the X-ray transform for 3D symmetric tensor fields with sources on a curve. *Inverse Problems*. No.22 (2006), pp. 399–411.
[3] V. A. Sharafutdinov, Slice-by-slice reconstruction algorithm for vector tomography with incomplete data. *Inverse Problems*. No.23 (2007), pp. 2603–2627.