FINITE-TIME EXPONENTIAL SYNCHRONIZATION OF REACTION-DIFFUSION DELAYED COMPLEX-DYNAMICAL NETWORKS

M. SYED ALI* AND L. PALANISAMY
Department of Mathematics, Thiruvalluvar University
Vellore-632115, Tamil Nadu, India

NALLAPPAN GUNASEKARAN
Department of Mathematical Sciences, Shibaura Institute of Technology
Saitama 337-8570, Japan

AHMED ALSAEDI AND BASHIR AHMAD
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group
Department of Mathematics Faculty of Science
King Abdulaziz University, P.O. Box 80257, Jeddah 21589, Saudi Arabia

ABSTRACT. This investigation looks at the issue of finite time exponential synchronization of complex dynamical systems with reaction diffusion term. This report studies complex networks consisting of $N$ straightly and diffusively coupled networks. By building a new Lyapunov Krasovskii functional (LKF), using Jensen’s inequality and convex algorithms approach, stability conditions frameworks are determined. At last, a numerical precedent is given to demonstrate the practicality of the theoretical results.

Complex dynamical systems (CDNs) have gotten impressive consideration because of their broad applications in useful society, for example, guideline of intensity framework, parallel picture preparing, the activity of no-man air vehicle, the acknowledgment of chain explosion, and so on., (see [12, 35, 27, 2, 3, 5] and references in that). As of late, complex systems can be seen all over the place and are turning into an essential piece of our day by day life. The absolute most surely understood models incorporate nourishment networks, correspondence systems, interpersonal organizations, control matrices, cell systems, WWW, metabolic networks, illness transmission systems, and so on. Along these lines, the topology and dynamical conduct of complex dynamical systems have been broadly considered by the scientists ([8, 17, 21, 9]). Specifically, the synchronization issue of complex dynamical systems has gotten much concentration as of late ([34, 25]).

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* Corresponding author: M. Syed Ali.
The synchronization marvels are normal and essential in certifiable systems, for example, synchronization on the Internet, synchronization exchange of computerized or simple flags in correspondence systems and synchronization identified with natural neural models ([32, 18]). In this way, synchronization investigation of complex systems is critical both in principle and practice ([10]). In flow years, so as to make a profound comprehension of the synchronization wonder and to utilize the synchronization conduct, specialists have paid expanding considerations to the synchronization issue of deferred complex dynamical systems (For instance [30, 14], and references in that). In the earlier years, the synchronization issues in coupled systems have been generally examined because of its applications in secure correspondence and flag generators plan [24, 7].

Numerous critical plants, for example, concoction reactors and warmth exchange forms, are represented by fractional differential conditions. In reality, there are bunches of response dispersion marvels in nature and building fields. Lamentably, the response dispersion impact isn’t mulled over in the current works [26, 20]. Essentially, the reaction diffusion impact is basic in neural systems [15]. Thus, it is important to study reaction diffusion results with coupled networks. Clearly, it is fascinating and huge to explore dynamical practices of CDNs with reaction diffusion terms. Along these lines, synchronization of CDNs with reaction diffusion terms have been dissected by the scientists [6, 13].

In addition, the finite time control procedures have shown better power and unsettling influence dismissal properties. As of late, finite time synchronization of complex systems researched in [29, 9, 4, 1]. In flag transmission, the flag will end up frail because of dispersion in flag transmission, so it is critical to think about that the enactment fluctuates in space just as in time and the response dissemination impacts can’t be ignored in both organic and man-made models [31, 16]. As electrons transport in a nonuniform electromagnetic field, the dispersion marvels couldn’t be overlooked. So it is critical to examine the intricate systems with diffusion terms, which can be depicted by incomplete differential conditions. To the best of creators information exponential result for finite time synchronization of complex dynamical systems with reaction diffusion have not been concentrated still at this point.

Persuaded by these perceptions, In this report we explore the exponential stability result for finite time synchronization of complex dynamical systems with internal coupling. A novel Lyapunov-Krasovskii funcational is built and after that the finite time based synchronization criteria are inferred for the proposed CDNs. The finite time stability criteria is given as far as direct network imbalances, which can be proficiently unraveled through standard numerical programming. New limited strategies are abused for the basic terms and the connection between the terms are completely taken inside the course of action of Linear matrix inequalities (LMIs). At last, a numerical precedent is given to demonstrate the adequacy of the hypothetical results.

Notations. In this report, $R^n$ signifies Euclidean space of n dimension, and $R^{m \times n}$ is the arrangement of all $m \times n$ matrices. The documentation $X \geq Y$ gives that $X - Y$ means semi definite for symmetric matrices X and Y. The transpose of matrix R is $R^T$. I is the Identity matrix with suitable measurements. In symmetric square networks or long framework articulations, we use ‘$*$’ to speak to a term that is prompted by symmetry. ‘$\otimes$’ is a network’s Kronecker item.
1. System description and preliminaries. The Complex network with reaction diffusion is described as follows

\[
\frac{\partial w_i(x,t)}{\partial t} = A\nabla^2 w_i(x,t) - Cw_i(x,t) + W_0f(w_i(x,t)) + W_1f(w_i(x,t - \tau(t)))
\]

\[
+\sigma_1 \sum_{j=1}^{M} G_{ij}^{(1)} \Gamma_1 w_j(x,t) + \sigma_2 \sum_{j=1}^{M} G_{ij}^{(2)} \Gamma_2 w_j(x,t - \tau(t)), \quad i = 1, 2, \ldots, M,
\]

\[
w(x,0) = \phi(x), \quad x \in \Gamma,
\]

\[
w(x,0) = 0, \quad (x,t) \in \partial\Gamma \times [0, +\infty).
\]

The state of neuron at time \( t \) is defined by \( x \in \Gamma \subset \mathbb{R}^q; w_i(x,t) \in \mathbb{R} \). \( \nabla^2 = \sum_{k=1}^{q} \frac{\partial^2}{\partial x_k^2} \) is Laplace diffusion operator on \( \Gamma \). \( f(w_i(x,t)) \in \mathbb{R}^n \) represent the neuron activation function. Initial and boundary value \( \phi_i(x)(i = 1, 2, \ldots, n) \) is defined on \( \Omega \). \( C = \text{diag}(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \) with \( c_i > 0 (i = 1, 2, \ldots, n) \) is the rate with which the \( i \)-th neuron will reset its capability to the resting state when separated from the system; \( A = \text{diag}(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) with \( a_i > 0 (i = 1, 2, \ldots, n) \) speaks to the of \( i \)-th neuron dissemination coefficient in the transmission; The weight matrix is \( W_0 \in \mathbb{R}^{n \times n} \) and the delayed weight matrix is \( W_1 \in \mathbb{R}^{n \times n} \); \( L_q \in \mathbb{R}^{n \times n} \) represent the inner connection matrix. \( G_q = (G_{ij}^q)_{M \times M}, \quad (q = 1, 2) \) represents matrix of outer-coupling. For association of node \( i \) with node \( j (i \neq j) \), \( G_{ij}^q = G_{ji}^q = 1 \); if there is no association, then \( G_{ij}^q = G_{ji}^q = 0, (i \neq j) \). The matrix \( G_q \) are defined as

\[
\begin{align*}
G_{ii}^{(1)} &= \sum_{j=1, j \neq i}^{M} G_{ij}^{(1)} = -\sum_{j=1, j \neq i}^{M} G_{ij}^{(1)}, \\
G_{ii}^{(2)} &= \sum_{j=1, j \neq i}^{M} G_{ij}^{(2)} = -\sum_{j=1, j \neq i}^{M} G_{ij}^{(2)}, \quad i = 1, 2, \ldots, M.
\end{align*}
\]

The time-varying delay \( \tau(t) \) is differentiable and it holds:

\[
0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \mu_1 \leq \dot{\tau}(t) \leq \mu_2,
\]

where \( \tau_1, \tau_2, \mu_1 \) and \( \mu_2 \) are known constants.

**Assumption (A).** The activation functions of neuron \( f_i(\cdot) \) are continuous, bounded and satisfies,

\[
l_i^- \leq \frac{f_i(x) - f_i(\bar{x})}{x - \bar{x}} \leq l_i^+,
\]

where \( l_i^- \) and \( l_i^+ \) are constants. Denote \( L_1 = \text{diag}\{l_1^- l_1^+, \ldots, l_n^- l_n^+\}, L_2 = \text{diag}\{\frac{l_1^- l_1^+}{2}, \ldots, \frac{l_n^- l_n^+}{2}\} \).

The Euclidean norm \( s(x,t) \in \mathbb{R}^n, \quad (x,t) \in \Omega \times \mathbb{R} \) is a synchronization complex, which can be either a equilibrium point, a periodic orbit of a chaotic attractor and fulfills

\[
\frac{\partial s(x,t)}{\partial t} = A\nabla^2 s(x,t) - Cs(x,t) + W_0f(s(x,t)) + W_1f(s(x,t - \tau(t))),
\]

with error vectors as follows:

\[
e_i(x,t) = w_i(x,t) - s(x,t), \quad i = 1, 2, \ldots, M.
\]

The error dynamics would be

\[
\frac{\partial e_i(x,t)}{\partial t} = A\nabla^2 e_i(x,t) - Ce_i(x,t) + W_0f(e_i(x,t)) + W_1f(e_i(x,t - \tau(t)))
\]
\[ + \sigma_1 \sum_{j=1}^{M} G_{ij}^{(1)} e_j(x, t) + \sigma_2 \sum_{j=1}^{M} G_{ij}^{(2)} e_j(x, t - \tau(t)), \]

where \( f(e_i(x, t)) = f(w_i(x, t)) - f(s(x, t)). \)

By Kronecker product

\[
\frac{\partial e(x, t)}{\partial t} = (I_M \otimes A) \nabla^2 e(x, t) - (I_M \otimes C)e(x, t)
\]

\[
+ (I_M \otimes W_0)f(e(x, t)) + (I_M \otimes W_1)f(e(x, t - \tau(t)))
\]

\[ + \sigma_1(G^{(1)} \otimes \Gamma_1)e(x, t) + \sigma_2(G^{(2)} \otimes \Gamma_2)e(x, t - \tau(t)), \]

in which \( I_M \) denotes the \( M \times M \) identity matrix.

The lemmas and definitions useful in deriving the main results are given below:

**Definition 1.1.** [28] The system (6) is called synchronized in finite-time (FTS) subject to to \( (c_1, c_2, L, T) \), if for scalar \( T > 0 \) given, scalars \( c_1 > 0 \) with \( c_1 > c_2 \) there exists matrix \( L > 0 \), and satisfies

\[ e^T(0)Le(0) \leq c_1 \Rightarrow e^T(t)Le(t) \leq c_2, \forall t \in [0, T]. \]

**Lemma 1.2.** [19] (Wirtinger’s inequality) Let \( f(x) \) be a real-valued function defined on \( [a, b] \subset \mathbb{R} \) with \( f(a) = f(b) = 0 \). If \( f(x) \in C[a, b] \), then

\[
\int_a^b f(x)^2dx \leq \frac{b-a)^2}{\pi^2} \int_a^b f'(x)^2dx.
\]

**Lemma 1.3.** [22] For the Kronecker product \( \otimes \). Then the following satisfies:

(i). \((\alpha X) \otimes Y = X \otimes (\alpha Y), \forall \alpha \in \mathbb{R}; \)

(ii). \( X \otimes Y + X \otimes Z = X \otimes (Y + Z); \)

(iii). \((X \otimes Y)(Z \otimes D) = (XZ) \otimes (YD); \)

(iv). \((X \otimes Y)^T = X^T \otimes Y^T. \)

**Lemma 1.4.** [11] Define \( \rho = [1, 1, \ldots, 1]^T, F_M = \rho \rho^T, U = MI_N - F_M, P \in \mathbb{R}^{n \times n}, x = [x_1^T, x_2^T, \ldots, x_M]^T \) and \( y = [y_1^T, y_2^T, \ldots, y_M]^T \) with \( x_k, y_k \in \mathbb{R}^n (k = 1, 2, \ldots, M) \), then

\[
x^T(U \otimes P)y = \sum_{1 \leq i \leq j \leq N} (x_i - x_j)^T P(y_i - y_j).
\]

**Lemma 1.5.** [33] Let \( \rho = [1, 1, \ldots, 1]^T \in \mathbb{R}^M, U = MI_M - \rho \rho^T, P \in \mathbb{R}^{n \times n}, x = [x_1^T, x_2^T, \ldots, x_M]^T \) and \( y = [y_1^T, y_2^T, \ldots, y_M]^T \) with \( x_k, y_k \in \mathbb{R}^n (k = 1, 2, \ldots, M) \), \( H = (h_{ij}) \in \mathbb{R}^{N \times N} \) satisfies the condition of (3), then

\[
x^T(UH \otimes P)y = \sum_{1 \leq i \leq j \leq N} (x_i - x_j)^T (-Nh_{ij} P)(y_i - y_j).
\]

**Lemma 1.6.** [19] Let \( \Omega \) be a cube \(|y_k| < l_k (k = 1, 2, \ldots, q) \) and let \( f(y) \) be a real-valued function belonging to \( C^1(\Omega) \) which vanishes on the boundary \( \partial \Omega f \Omega \), i.e., \( f(x)|_{\partial \Omega} = 0 \). Then

\[
\int_{\Omega} f^2(y)dy \leq l_k^2 \int_{\Omega} \left( \frac{\partial f}{\partial y_k} \right)^2 dy,
\]

where \( y = (y_1, y_2, \ldots, y_q)^T. \)
Lemma 1.7. [23] Let $g_1 > 0, g_2 > 0, \ldots, g_N > 0 : \mathbb{R}^m \rightarrow \mathbb{R}$ takes inputs in a set $\mathcal{F} \in \mathbb{R}^m$. The combination of reciprocally convex for $g_i$ over the set $\mathcal{F}$ then holds the following

$$\min_{\{\beta_i | \beta_i > 0, \sum_i \beta_i = 1\}} \sum_i \frac{1}{\beta_i} g_i(t) = \sum_i g_i(t) + \max_{h_{i,j}(t)} h_{i,j}(t),$$

subject to

$$\left\{ h_{i,j} : \mathbb{R}^m \rightarrow \mathbb{R}, h_{i,j}(t) \geq h_{i,j}(t), \begin{bmatrix} g_i(t) & h_{i,j}(t) \\ h_{i,j}(t) & g_j(t) \end{bmatrix} \geq 0 \right\}.$$  

2. Main results. In this section, we derive our stability results as follows.

**Theorem 2.1.** For the given scalar $0 < \alpha$, the system (6) is finite time stable with $0 < c_1 < c_2$ if there exist matrices $P > 0, Q_1 > 0, Q_2 > 0, Q_3 > 0, Z > 0$, and diagonal matrices $M_1 > 0, M_2 > 0$ such that the following inequalities hold

$$\Phi < 0,$$  
$$e^{\alpha t} c_1 \nu < \lambda_1 c_2,$$

where

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & 0 & 0 & \phi_{15} & \phi_{16} & 0 & 0 \\ * & \phi_{22} & 0 & 0 & 0 & \phi_{26} & 0 & 0 \\ * & * & \phi_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \phi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \phi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \phi_{66} & 0 & 0 \\ * & * & * & * & * & * & \phi_{77} & \phi_{78} \\ * & * & * & * & * & * & * & \phi_{88} \end{bmatrix},$$

$$\phi_{11} = \frac{-\pi^2}{(\epsilon_2 - \epsilon_1)} (PA + AT^TP) - 2PC - 2Mg_{ij}^{(1)} \sigma_1 P_{T1} - \alpha P + Q_1 + Q_2 + Q_3 - L_1 M_1 + (\tau_2 - \tau_1)^2 Z_1,$$

$$\phi_{12} = -2Mg_{ij}^{(2)} \sigma_1 P_{T2}, \ \phi_{15} = PW_0 + L_2 M_1, \ \phi_{16} = PW_1, \ \phi_{26} = L_2 M_2,$$

$$\phi_{55} = -M_1, \ \phi_{66} = -M_2,$$

$$\phi_{22} = \begin{cases} -e^{\alpha \tau_1}(1 - \tau(t))Q_1 - L_1 M_2, \ \alpha \geq 0 \\ -e^{\alpha \tau_2}(1 - \tau(t))Q_1 - L_1 M_2, \ \alpha < 0 \end{cases}, \ \phi_{33} = \begin{cases} -e^{\alpha \tau_1} Q_2, \ \alpha \geq 0 \\ -e^{\alpha \tau_2} Q_2, \ \alpha < 0 \end{cases},$$

$$\phi_{44} = \begin{cases} -e^{\alpha \tau_1} Z_1, \ \alpha \geq 0 \\ -e^{\alpha \tau_2} Z_1, \ \alpha < 0 \end{cases}, \ \phi_{77} = \begin{cases} -e^{\alpha \tau_1} Z_1, \ \alpha \geq 0 \\ -e^{\alpha \tau_2} Z_1, \ \alpha < 0 \end{cases},$$

$$\gamma_1 = \gamma_{\min}(P), \ \gamma_2 = \gamma_{\max}(P), \ \gamma_3 = \gamma_{\max}(Q_1), \ \gamma_4 = \gamma_{\max}(Q_2), \ \gamma_5 = \gamma_{\max}(Q_3), \ \gamma_6 = \gamma_{\max}(Z_1).$$
\[ \nu = \gamma_2 + \tau \gamma_3 + \tau_1 \gamma_4 + \tau_2 \gamma_5 + \frac{(\tau_2 - \tau_1)^3}{2} \gamma_6. \]

**Proof.** We construct the following Lyapunov-Krasovskii functional as:

\[ V(t) = \sum_{k=1}^{5} V_k(t), \quad (10) \]

where

\[
\begin{align*}
V_1(t) &= \int_{\Omega} e^T(x,t)(U \otimes P)e(x,t)dx, \\
V_2(t) &= \int_{t-\tau(t)}^{t} \int_{\Omega} e^{\alpha(t-s)}e^T(x,s)(U \otimes Q_1)e(x,s)dxds, \\
V_3(t) &= \int_{t-\tau_1}^{t} \int_{\Omega} e^{\alpha(t-s)}e^T(x,s)(U \otimes Q_2)e(x,s)dxds, \\
V_4(t) &= \int_{t-\tau_2}^{t} \int_{\Omega} e^{\alpha(t-s)}e^T(x,s)(U \otimes Q_3)e(x,s)dxds, \\
V_5(t) &= (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \int_{\Omega} e^{\alpha(t-s)}e^T(x,s)(U \otimes Z_1)e(x,s)dxdsd\theta.
\end{align*}
\]

We can calculate the derivative of \( V(t) \) on the system (6) as

\[
V_1(t) = 2 \int_{\Omega} \left[ e^T(x,t)(U \otimes P) \left( \frac{\partial e(x,t)}{\partial t} \right) \right] dx
\]

\[
= 2 \int_{\Omega} e^T(x,t)(U \otimes P) \left( [(I_M \otimes A) \nabla^2 e(x,t) - (I_M \otimes C)e(x,t) \right.
\]

\[
+ (I_M \otimes W_0)f(e(x,t))(I_M \otimes W_1)f(e(x,t - \tau(t)) + \sigma_1(G^{(1)} \otimes \Gamma_1)e(x,t)
\]

\[
+ \sigma_2(G^{(2)} \otimes \Gamma_2)e(x,t - \tau(t))] \right) + \alpha V_1(t) - \alpha V_1(t).
\]

By using Lemma (1.4), we get

\[
\begin{align*}
&= \int_{\Omega} \left\{ \sum_{1 \leq i < j \leq M} [2e_{ij}^T(x,t)(PA)\nabla^2 e_{ij}(x,t) + 2e_{ij}^T(x,t)(-PC)e_{ij}(x,t) \\
+ 2e_{ij}^T(x,t)(PW_0)f(e_{ij}(x,t)) + 2e_{ij}^T(x,t)(PW_1)f(e_{ij}(x,t - \tau(t)) \\
+ 2e_{ij}^T(x,t)[-MG_{ij}^{(1)}\sigma_1 \Gamma_1]e_{ij}(x,t) + 2e_{ij}^T(x,t)[-MG_{ij}^{(2)}\sigma_2 \Gamma_2]e_{ij}(x,t - \tau(t)) \\
e_{ij}^T(x,t)(-\alpha P)e_{ij}(x,t)] \right\} dx.
\end{align*}
\]

From Green’s formula and the boundary condition, we have

\[
\int_{\Omega} e_{ij}(x,t)\nabla^2 e_{ij}(x,t)dx = -\sum_{k=1}^{q} \int_{\Omega} \frac{\partial e_{il}(x,t)}{\partial x_k} \frac{\partial e_{lj}(x,t)}{\partial x_k} dx,
\]

where \( l, j \in \{1, 2, \cdots, n\}, i = 1, 2, \cdots, M \). Letting \( P^k = (p^k_{ij})_{n \times n} \), then we can obtain

\[
\int_{\Omega} e_{ij}^T(x,t)(PA)\nabla^2 e_{ij}(x,t)dx = \sum_{i=1}^{M} \int_{\Omega} e_{ij}^T(x,t)(P^k A)\nabla^2 e_{ij}(x,t)dx
\]
\[
= \sum_{i=1}^{M} \sum_{j=1}^{n} \sum_{l=1}^{n} (p_{ij}^{T} a_l) \int_{\Omega} e_{ij}^{T}(x, t) \nabla^2 e_{il}(x, t) \, dx
\]

\[
= - \sum_{k=1}^{q} \sum_{l=1}^{M} \sum_{j=1}^{n} \sum_{l=1}^{n} (p_{ij}^{T} a_l) \int_{\Omega} \frac{\partial e_{il}(x, t)}{\partial x_k} \frac{\partial e_{il}(x, t)}{\partial x_k} \, dx
\]

\[
= - \sum_{k=1}^{q} \int_{\Omega} \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)^{T} (PA) \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right) \, dx.
\]

Similarly

\[
\int_{\Omega} (\nabla^2 e_{ij}(x, t))^{T} (A^{T} P^{T}) e_{ij}(x, t) \, dx = - \sum_{k=1}^{q} \int_{\Omega} \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)^{T} (A^{T} P^{T}) \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right) \, dx.
\]

Then we can get

\[
\int_{\Omega} 2e_{ij}^{T}(x, t)(PA)\nabla^2 e_{ij}(x, t) \, dx = - \sum_{k=1}^{q} \int_{\Omega} \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)^{T} (PA + A^{T} P^{T}) \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right) \, dx.
\]

Again, there exists a real matrix \( B \in \mathbb{R}^{nM \times nM} \) such that \( PA + A^{T} P^{T} = B^{T} B \), then we have,

\[
\left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)^{T} (PA + A^{T} P^{T}) \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right) = \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)^{T} (B^{T} B) \left( \frac{\partial e_{ij}(x, t)}{\partial x_k} \right)
\]

\[
= \left( \frac{\partial (Be_{ij}(x, t))}{\partial x_k} \right)^{T} \left( \frac{\partial (Be_{ij}(x, t))}{\partial x_k} \right) \cdot
\]

Let \( \lambda(x, t) = Be_{ij}(x, t) \), for \((x, t) \in \partial \Omega \times [-\tau, +\infty) \) follow the boundary condition (7), we get \( \lambda(x, t) = Be_{ij}(x, t) = 0 \). Lemma 1.2 gives,

\[
\sum_{k=1}^{q} \int_{\Omega} \left( \frac{\partial \lambda(x, t)}{\partial x_k} \right)^{T} \left( \frac{\partial \lambda(x, t)}{\partial x_k} \right) \, dx \geq \pi^2 \int_{\Omega} e_{ij}^{T}(x, t)(PA + A^{T} P^{T})e_{ij}(x, t) \, dx
\]

\[
= \pi^2 \int_{\Omega} e_{ij}^{T}(x, t)(PA + A^{T} P^{T})e_{ij}(x, t) \, dx.
\]

Thus,

\[
\dot{V}_{1}(t) = \int_{\Omega} \left\{ \sum_{1 \leq i < j \leq M} \left[ \frac{\pi^2}{(e_{ij} - e_{il})^2} e_{ij}^{T}(x, t)(PA + A^{T} P^{T})e_{ij}(x, t) + 2e_{ij}^{T}(x, t)(-PC)e_{ij}(x, t) \right. \right.
\]

\[
+ 2e_{ij}^{T}(x, t)(PW_0)f(e_{ij}(x, t)) + 2e_{ij}^{T}(x, t)(PW_1)f(e_{ij}(x, t) - \tau(t)) \left. \right] e_{ij}(x, t) \, dx + \alpha V_{1}(t),
\]

\[
\dot{V}_{2}(t) = \alpha V_{2}(t) + \int_{\Omega} \left[ e_{ij}^{T}(x, t)(U \otimes Q_1) e_{ij}(x, t) \right.
\]

\[
- e^{\alpha \tau(t)}(1 - \tau(t)) e_{ij}^{T}(x, t - \tau(t))(U \otimes Q_1) e_{ij}(x, t - \tau(t)) \left. \right] \, dx
\]

\[
\leq \alpha V_{2}(t) + \int_{\Omega} \sum_{1 \leq i < j \leq M} \left[ e_{ij}^{T}(x, t) Q_1 e_{ij}(x, t) \right. \]
Moreover, from Assumption (A), for any positive diagonal matrices $M_1$ and $M_2$

\begin{align}
0 \leq \begin{bmatrix}
    e_{ij}(x,t) \\
    f(e_{ij}(x,t))
\end{bmatrix}^T 
    \begin{bmatrix}
    -L_1M_1 & L_2M_1 \\
    * & -M_1
\end{bmatrix} 
    \begin{bmatrix}
    e_{ij}(x,t) \\
    f(e_{ij}(x,t))
\end{bmatrix},
\end{align}

(20)

Moreover, from Assumption (A), for any positive diagonal matrices $M_1$ and $M_2$

\begin{align}
0 \leq \begin{bmatrix}
    e_{ij}(x,t - \tau(t)) \\
    f(e_{ij}(x,t - \tau(t)))
\end{bmatrix}^T 
    \begin{bmatrix}
    -L_1M_2 & L_2M_2 \\
    * & -M_2
\end{bmatrix} 
    \begin{bmatrix}
    e_{ij}(x,t - \tau(t)) \\
    f(e_{ij}(x,t - \tau(t)))
\end{bmatrix}.
\end{align}

(21)
Combining (12)-(21) yields
\[ \dot{V}(t) \leq \alpha V(t) + \int_{\Omega} \sum_{1 \leq i < j \leq M} \xi_{ij}(x,t) \Phi \xi_{ij}(x,t) dx, \] (22)

Here
\[ \xi_{ij}(x,t) = \left[ e_{ij}^T(x,t), e_{ij}^T(x,t-\tau(t)), e_{ij}^T(x,t-\tau_1), e_{ij}^T(x,t-\tau_2), f^T(e_{ij}(x,t)), f^T(e_{ij}(x,t-\tau(t))), \int_{t-\tau(t)}^{t-\tau_1} e_{ij}^T(x,s) ds, \int_{t-\tau_2}^{t-\tau(t)} e_{ij}^T(x,s) ds \right]^T. \]

From (22), one can obtain that
\[ \dot{V}(t) < \alpha V(t), \] (23)
multiplying both the sides by \( e^{-\alpha t} \)
\[ e^{-\alpha t} \frac{d}{dt} V(t) < e^{-\alpha t} \alpha V(t) \]
\[ \frac{d}{dt} (e^{-\alpha t} V(t)) < 0. \]

By integrating we get,
\[ e^{-\alpha t} V(t) < V(0) \]
\[ V(t) < e^{\alpha t} V(0). \]

Now
\[ V(0) = \int_{\Omega} e^T(x,t)(U \otimes P)e(x,t) dx + \int_0^0 e^{-\alpha s} e^T(x,s)(U \otimes Q_1)e(x,s) ds ds \]
\[ + \int_{-\tau_1}^0 e^{-\alpha s} e^T(x,s)(U \otimes Q_2)e(x,s) ds ds \]
\[ + \int_{-\tau_2}^0 e^{-\alpha s} e^T(x,s)(U \otimes Q_3)e(x,s) ds ds \]
\[ + (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{-\tau_2}^{0} e^{-\alpha s} e^T(x,s)(U \otimes Z_1)e(x,s) ds ds d\theta \]
\[ \leq \left[ \lambda_{\text{max}}(P) + \tau e^{-\alpha s} \lambda_{\text{max}}(Q_1) + \tau_1 \lambda_{\text{max}}(Q_2) + \tau_2 e^{-\alpha s} \lambda_{\text{max}}(Q_2) \right. \]
\[ + \frac{(\tau_2 - \tau_1)^3}{2} e^{-\alpha s} \lambda_{\text{max}}(Z_1) \] \[ \left. \sum_{1 \leq i < j \leq M} \| e_{ij}(x,t) \|^2 \right] \]
\[ \leq \left[ \lambda_{\text{max}}(P) + \tau \lambda_{\text{max}}(Q_1) + \tau_1 \lambda_{\text{max}}(Q_2) + \tau_2 \lambda_{\text{max}}(Q_2) \right. \]
\[ + \frac{(\tau_2 - \tau_1)^3}{2} \lambda_{\text{max}}(Z_1) \] \[ \left. \sum_{1 \leq i < j \leq M} \| e_{ij}(x,t) \|^2 \right]. \]

On the other hand
\[ V(t) \geq \lambda_{\text{min}}(P) \sum_{1 \leq i < j \leq M} \| e_{ij}(x,t) \|^2. \]
Therefore
\[
\sum_{1 \leq i < j \leq M} \|e_{ij}(x, t)\|^2 \leq e^{at} c_1 \left[ \lambda_{\text{max}}(P) + \tau e^{-\alpha s} \lambda_{\text{max}}(Q_1) + \tau_1 \lambda_{\text{max}}(Q_2) + \frac{\tau_2 e^{-\alpha s} \lambda_{\text{max}}(Q_2) + \frac{(\tau_2 - \tau_1)^2}{2} e^{-\alpha s} \lambda_{\text{max}}(Z_1)}{\lambda_{\text{min}}(P)} \right] < c_2.
\]  
(24)

Then, it can be deduced that Eq. (24) together with Eq. (10) implies for any \(t \in [0, T]\),
\[
\sum_{1 \leq i < j \leq M} \|e_{ij}(x, t)\|^2 \leq c_2.
\]  
(25)

This completes the proof. □

**Remark 2.2.** Very few results have been published for the synchronization of reaction-diffusion networks [13, 15]. Finite-time synchronization of coupled reaction-diffusion networks have not been fully studied. We studied complex network consisting of \(M\) linearly and diffusively coupled identical reaction-diffusion systems. The considered network is different form the networks studied in [15]. Reaction-diffusion complex systems (6) studied in our paper is more general than the network studied in [15].

3. **Numerical examples.**

**Example 3.1.** The complex network system is considered as
\[
\frac{\partial e(x, t)}{\partial t} = (I_M \otimes A) \nabla^2 e(x, t) - (I_M \otimes C) e(x, t) + (I_M \otimes W_0) f(e(x, t))
\]
\[
+ (I_M \otimes W_1) f(e(x, t - \tau(t))) + \sigma_1 (G^{(1)} \otimes \Gamma_1) e(x, t) + \sigma_2 (G^{(2)} \otimes \Gamma_2) e(x, t - \tau(t)),
\]  
(26)

with parameters
\[
C = \begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1.5 & 2.3 \\ 1.22 & 1.35 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1.3 & 1.2 \\ 0.1 & 1.2 \end{bmatrix},
\]
\[
L_1 = 0, \quad \Gamma_1 = \Gamma_2 = L_2 = A = I.
\]

\(G_1\) and \(G_2\) is
\[
G_1 = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.5 & -1 & 0.5 \\ 0.5 & 0.5 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.5 & -1 & 0.5 \\ 0.5 & 0.5 & -1 \end{bmatrix}.
\]

Let \(\epsilon_1 = 0.1, \quad \epsilon_2 = 0.2, \quad \tau_1 = 0.1, \quad \tau_2 = 0.5, \quad \alpha = 2, \quad \sigma_1 = 0.1, \quad \sigma_2 = 0.2, \quad c_1 = 0.2, \quad \mu = 0.1\) and \(M = 3\). The feasible solutions obtained by solving the Theorem 3.1 using Matlab are
\[
P = \begin{bmatrix} 1.7571 & 0.0286 \\ 0.0286 & 1.8311 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 34.7638 & -1.6837 \\ -1.6837 & 33.9049 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 30.9366 & -1.4593 \\ -1.4593 & 29.9005 \end{bmatrix},
\]
\[
Q_3 = \begin{bmatrix} 20.6312 & -0.8448 \\ -0.8448 & 20.0193 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 25.5828 & -0.0528 \\ -0.0528 & 25.5473 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 8.9276 & -0.0136 \\ -0.0136 & 8.9182 \end{bmatrix},
\]
\[
M_1 = \begin{bmatrix} 15.0395 & 0 \\ 0 & 15.0395 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 30.2422 & 0 \\ 0 & 30.2422 \end{bmatrix}, \quad c_2 = 2.0551.
\]

The trajectories of the systems is given in Figure 1 for initial conditions of \(x(t) = [-0.6, -0.4, -0.3, 0, 0.1, 0.2]\).

4. Conclusion. The finite-time synchronization problem for complex networks with reaction-diffusion term is investigated. An oval Lyapunov functional is constructed. By applying the theory of Kronecker product of matrices, and using reciprocally convex combination technique stability criterion is derived. Sufficient conditions for the synchronization of coupled complex networks in terms of LMI is derived. By solving the LMIs we check the synchronization of complex system. A numerical example is provided to show the correctness of thermometrical results.

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E-mail address: syedgru@gmail.com
E-mail address: saamy143@gmail.com
E-mail address: gunasmaths@gmail.com
E-mail address: bashirahmad_qau@yahoo.com
E-mail address: aalsaedi@hotmail.com