Bayesian Comparison of the Cosmic Duality Scenarios

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The cosmic distance duality relation (CDDR), \( D_L(1+z)^{-2}/D_A = \eta = 1 \), with \( D_L \) and \( D_A \), being the luminosity and angular diameter distances, respectively, is a crucial premise in cosmological scenarios. Many investigations try to test CDDR through observational approaches, even some of these also consider a deformed CDDR, i.e., \( \eta = \eta(z) \). In this paper, we use type Ia supernovae luminosity distances and galaxy cluster measurements (their angular diameter distances and gas mass fractions) in order to perform a Bayesian model comparison between \( \eta(z) \) functions. We show that the data here used is unable to pinpoint, with a high degree of Bayesian evidence, which \( \eta(z) \) function best captures the evolution of CDDR.

I. INTRODUCTION

Measuring distances in cosmology is of crucial importance when one wants to relate observational data with theoretical models. The two types of distance most used in cosmology are the luminosity distance, \( D_L \), and, the angular diameter distance, \( D_A \). The former is a distance measurement associated with an object based on the decrease of its brightness and, the latter one is associated with the measurement of the angular size of the object projected on the celestial sphere. These cosmological distances are functions of the redshift \( z \) of the astronomical object considered and are connected by a relation known as cosmic distance duality relation (CDDR), \( \frac{D_L(z)}{D_A(z)} = 1 \), or as Etherington’s reciprocity law in the background of the astronomical observation [1, 2].

The CDDR is obtained in the context of Friedmann-Lemaître-Robertson-Walker metric but holds for general metric theories of gravity in any background, in which photons travel in null geodesics and, the number of photons is conserved during cosmic evolution [3]. In fact, the generality of this relationship is of crucial importance in the context of observational cosmology and, a little deviation from it may indicate the possibility of a new physics or the presence of systematic errors in observations [4]. Simultaneously with the increase in the number and the quality of astronomical data, different methods have been proposed to test the validity of the CDDR. We can divide them in two classes: cosmological model-dependent tests based on ΛCDM framework [5–10], and cosmological model-independent ones. The last ones have been performed by using combinations of several astronomical data: angular diameter distance of galaxy clusters, galaxy cluster gas mass fraction1, type Ia supernovae (SNe Ia), strong gravitational lensing, cosmic microwave background, gamma ray bursts, radio compact sources, baryon acoustic oscillations, gravitational waves, etc [11–36].

In order to test the CDDR, the basic approach has been considered a deformed expression, given by \( \frac{D_L(z)}{D_A(z)} = \eta(z) \) and to obtain constraints on some \( \eta(z) \) functions2. In this context, the authors of the Ref. [12] assumed two \( \eta(z) \) functions, such as: \( \eta(z) = 1 + \eta_0 z \) and \( \eta(z) = \eta_0 + \eta_1 z/(1 + z) \). Actually, these functions are clearly inspired by similar expressions for the equation of state parameter of dark energy models. By using angular diameter distance samples of galaxy clusters jointly with luminosity distances of SNe Ia, they obtained that CDDR is valid within 2\( \sigma \) (\( \eta_0 \approx 0 \)). However, other \( \eta(z) \) functions were also proposed, e.g. \( \eta(z) = \eta_0 + \eta_1 z \), \( \eta(z) = \eta_0 + \eta_1 z/(1 + z) \), \( \eta(z) = \eta_0 + \eta_1 \ln(1 + z) \) and \( \eta(z) = (1 + z)^{\epsilon} \) (see, for instance, Refs. [7, 14, 19]). As basic result from literature, the CDDR validity has been verified, at least, within 2\( \sigma \) c.L.. However, it is very worth to stress that the current analyses can not distinguish which \( \eta(z) \) function describes better the data.

Recently, the Bayesian inference has been widely used as an useful tool in order to search several problems in Physics [37], Cosmology and Astronomy [38–48]. Specifically, issues associated with cosmological scenarios, the Bayesian model comparison has been a powerful technique to study many issues, e.g. by proposing alternative models of a Cold Dark Matter (ΛCDM), it is possible to provide the effect of bulk viscosity in the context of dark matter and dark energy for different models [45, 46], by calculating the Bayesian evidence for models through the Population Monte Carlo [49], by comparing the dark energy models [42, 43] and, studying several models, assuming the interaction in the dark sector [47], among others models which have used this technique (See [50] and references therein). An interesting question here is trying to analyse which \( \eta(z) \) should be viable from the Bayesian inference standpoint.

In order to face this issue, in this paper, we use SNe Ia luminosity distances and galaxy cluster measurements (angular diameter distances and gas mass fractions) to perform a Bayesian model comparison between \( \eta(z) \) functions used in...

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1 In the Ref. [11] only massive galaxy clusters observations were considered in its method to test the CDDR.

2 The Ref. [22], by applying a non-parametric method, namely, Gaussian process, proposed a test based on galaxy clusters observations and \( H(z) \) measurements (see also the Ref. [24]) without using \( \eta(z) \) functions.
Figure 1. The panels show the data used in this work. The Figure (a) shows $f_{X\text{-ray}}$, the Figure (b) shows the SNe Ia data measure at the same redshift of the X-ray mass fraction (black) and, the angular diameter distances of the galaxy clusters (blue). The Figure (c) shows the angular distance of the galaxy clusters.

literature, such as: $\eta = \eta_0$, $\eta(z) = 1 + \eta_1 z$, $\eta(z) = 1 + \eta_1 z/(1 + z)$, $\eta(z) = \eta_0 + \eta_1 z$ and $\eta(z) = \eta_0 + \eta_1 z/(1 + z)$. The basic idea is to estimate the Bayesian evidence and compute the Bayes factor of each $\eta(z)$ function with respect to $\eta = \eta_0$. The $\eta$ constant is chosen as standard model because if $\eta_0 = 1$ the standard CDDR is recovered and we obtain this value within 2$\sigma$ c.l. with the data set used in our analyses.

This paper is divided in the following way: in Section II, we describe the data used in this work: SNe Ia and galaxy clusters observations. In Section IV, we show the $\eta(z)$ parameterizations assumed in this work. Next, in Section V, we achieve the Bayesian analysis by considering the data and parameterizations previously presented. Finally, Section VI presents the main results of the statistical analysis, and in Section VII we show the conclusions of the work.

II. DATA
A. Type Ia Supernovae

The luminosity distances are obtained from the SNe Ia sample called Pantheon [51]. The full compilation consists of 1049 spectroscopically confirmed SNe Ia and covers a redshift range of $0.01 \leq z \leq 2.3$, being the most recent wide refined sample of SNe Ia. However, to perform the appropriate tests on the CDDR, it must be used SNe Ia at the same (or approximately) redshift of the galaxy clusters (see below). Then, for each galaxy cluster, we make a selection of SNe Ia according to the criterion: $|z_{GC} - z_{SNe \text{ Ia}}| \leq 0.005$. Then, we perform the weighted average by each galaxy cluster by:

$$\tilde{\mu} = \frac{\sum_i \mu_i / \sigma_{\mu_i}^2}{\sum_i 1 / \sigma_{\mu_i}^2},$$

(1)

$$\sigma_{\mu}^2 = \frac{1}{\sum_i 1 / \sigma_{\mu_i}^2},$$

(2)

where $\mu_i(z)$ is the distance module of SNe Ia. Hence, the luminosity distance follows from $D_L(z) = 10^{\tilde{\mu} - 25}/[Mpc]$, and its error is given by error propagation, $\sigma_{D_L}^2 = (\partial D_L / \partial \tilde{\mu})^2 \sigma_{\mu}^2$ (see Fig. 1(b)).

B. Galaxy clusters

In order to perform the analyses, we also use two different observations of galaxy clusters, namely: angular diameter and gas mass fraction. The data set are:

- The $D_A(z)$ data of 25 galaxy clusters obtained via their Sunyaev-Zeldovich effect plus X-ray observations and presented by the Ref. [52]. The X-ray surface brightness of these clusters were described by the elliptical isothermal $\beta$-model. The galaxy clusters are distributed over the redshift interval $0.023 \leq z \leq 0.784$ (see Fig. 1(c)).

- The most recent X-ray mass fraction measurements of 40 galaxy clusters in redshift range $0.078 \leq z \leq 1.063$ from the Ref. [53], Fig. 1(c). These authors measured the gas mass fraction in spherical shells at radii near $r_{2500}$, rather than integrated at all radii ($< r_{2500}$) as in previous works. As consequence, the theoretical uncertainty in the gas depletion obtained from hydrodynamic simulations is reduced [53, 54] (see Fig. 1(a)).

III. METHODOLOGY

In this section, we present the equations used in our analyses. It is important to stress that previous works discussed how the expression $D_L(1 + z)^2 / D_A = \eta = 1$ has to be modified if one wishes to test it by using X-ray and Sunyaev-Zeldovich effect (SZE) observations of galaxy clusters[11]. These observations are affected if there are deviations from the CDDR validity and variation in the fine structure constant. In the follows, we discussed briefly this point (see details in Refs. [11, 28, 55, 56]).
A. $D_A$ from galaxy clusters × SNe Ia

In Ref. [55] was discussed as modifications of gravity via the presence of a scalar field with a multiplicative coupling to the electromagnetic Lagrangian affect the CDDR validity and induce a variation in the fine structure constant. In this context, the Ref. [11] discussed how the angular diameter distance of a galaxy cluster obtained from its SZE and X-ray observations ($D_A^{\text{data}}$) is affected by a such multiplicative coupling. The authors showed that the observations do not give the true distance, but $D_A^{\text{data}} = \eta(z)D_A$. Then, if one wants to test the CDDR by using $D_A(1+z)^2D_A^{-1} = \eta$ and galaxy clusters data, the angular diameter distance $D_A(z)$ must be replaced by $D_A(z) = \eta^{-4}D_A^{\text{data}}$ (see the Refs. [6, 11] for more details). Then, the equation basic to test the CDDR using $D_L$ from SNe Ia and $D_A^{\text{data}}$ from galaxy clusters is:

$$\frac{D_L(z)}{(1+z)^2D_A^{\text{data}}(z)} = \eta^{-3}(z),$$

or, equivalently,

$$\eta_{\text{obs}}(z) = \left(\frac{D_L(z)}{(1+z)^2D_A^{\text{data}}(z)}\right)^{-1/3}.$$  

(3)

B. Gas mass fractions × SNe Ia

The test by using gas mass fraction performed here is completely based on the equations obtained in the Refs. [28, 56]. Likewise, the authors of those references showed that the usual expression used in gas mass fraction measurements (where $\eta = 1$, see Ref. [57]) have to be replaced by

$$f_{\text{X-ray}}^{\text{obs}}(z) = N \left[ \frac{\eta(z)^{3/2}D_L^{3/2}}{D_L^{3/2}} \right],$$

if one wishes taking into account possible deviation of the CDDR validity and a variation of the fine structure constant. One may defines yet:

$$\eta_{\text{obs}}(z) = \left[ \frac{f_{\text{gas}}^2 D_L(z)}{N^{2/3}D_L^*} \right]^{3/7},$$

the symbol $\ast$ denotes quantities from a fiducial cosmological model used in the observations (usually a flat $\Lambda$CDM model where $\eta = 1$). The $N$ factor corresponds to the parameters: $K(z)$, which quantifies inaccuracies in instrument calibration, as well as any bias in the masses measured due to substructure, bulk motions and/or non-thermal pressure in the cluster gas; $\gamma$, the depletion factor, which corresponds to the ratio by which $f_{\text{gas}}$ is depleted with respect to the universal baryonic mean and the $\Omega_b/\Omega_M$ ratio. The $K(z)$ parameter for this sample was estimated to be $K = 0.96 \pm 0.12$ (statistical + systematic errors) and no significant trends with mass, redshift or the morphological indicators were verified [58]. The $\gamma$ factor was taken to be $\gamma = 0.848 \pm 0.085$ in agreement with the most recent estimates via observational data (SNe Ia, gas mass fraction and, Hubble parameter) [29] and in agreement with simulations [54]. We also use priors on the $\Omega_b$ and $\Omega_M$ parameters, i.e., $\Omega_b = 0.049 \pm 0.0001$ and $\Omega_M = 0.315 \pm 0.007$, as given by current CMB experiments [59]. These priors are from analyses by using exclusively CMB observations on the flat $\Lambda$CDM model.

IV. PARAMETERIZATIONS

The $\eta(z)$ functions considered here are [7, 12, 14, 19]:

$$\eta(z) = \eta_0,$$  

(7)

$$\eta(z) = 1 + \eta_1 z,$$  

(8)

$$\eta(z) = 1 + \eta_1 \frac{z}{1+z},$$  

(9)

$$\eta(z) = \eta_0 + \eta_1 z,$$  

(10)

$$\eta(z) = \eta_0 + \eta_1 \frac{z}{1+z}.$$  

(11)

These are the main $\eta(z)$ functions widely used in the literature. Actually, they effectively parametrize our ignorance of the underlying process responsible for a possible CDDR violation. As commented earlier, the current analyses can not distinguish which $\eta(z)$ function describes better the data. Then, The basic idea here is to estimate the Bayesian evidence and compute the Bayes factor of the $\eta(z)$ functions with respect to $\eta = \eta_0$, which we verify to be $\approx 1$ within 2$\sigma$ c.l. with the current SNe Ia and galaxy cluster data discussed in Sec. (II). We will briefly describe what procedure follows to determine the Bayesian evidence and compare the $\eta(z)$ functions in the next section.

V. BAYESIAN ANALYSIS

Now, let us briefly introduce a summary of the Bayesian Inference (BI). From the probability standpoint, BI is based on a measure of the degree of belief about a proposition. This method describes the connection between the competing models, the data, and the prior information concerning model parameters. The core of BI is the Bayes’ theorem, which updates our prior knowledge about the model in light of newly available data, being a consequence of the axioms of theory
of probability. This hypothesis relates the posterior distribution $P(\Phi|D, M)$, likelihood $\mathcal{L}(D|\Phi, M)$, the prior distribution $\pi(\Phi|M)$, and the Bayesian evidence $\mathcal{E}(D|M)$ [41]

$$P(\Phi|D, M) = \frac{\mathcal{L}(D|\Phi, M)\pi(\Phi|M)}{\mathcal{E}(D|M)}, \quad (12)$$

where $\Phi$ is the set of parameters, $D$ represents the data and $M$ is the model.

In the context of parameter constraint, the Bayesian evidence $\mathcal{E}(D|M)$ is just a normalization constant, and it does not affect the profile of posterior distribution since it does not depend upon the model parameters. However, it becomes an essential ingredient in the Bayesian model comparison viewpoint. So, the Bayesian evidence of a model in the continuous parameter space $\Omega$ can be written as

$$\mathcal{E}(D|M) = \int_{\Omega} \mathcal{L}(D|\Phi, M)\pi(\Phi|M)d\Phi. \quad (13)$$

Therefore, the evidence is the average probability value across the allowed model parameter space before considering the data.

Table I. The table shows the prior distribution of each parameter used in this work.

| $\ln B_{ij}$ | Interpretation                     |
|---------------|------------------------------------|
| > 5           | Strong evidence for model $i$      |
| [2.5, 5]      | Moderate evidence for model $i$    |
| [1, 2.5]      | Weak evidence for model $i$        |
| [-1, 1]       | Inconclusive                       |
| [-2.5, -1]    | Weak evidence for standard model   |
| [-5, -2.5]    | Moderate evidence for standard model |
| < -5          | Strong evidence for standard model |

The most significant feature in the Bayesian model comparison is associated with the comparison of two models that describe the same data. The models fit the data well and are also predictive, shifting the average of the likelihood in Eq. (13) in the direction of higher values. Instead, if a model which fits poorly or is not very predictive, moving the average down [40]. The application of Bayesian analysis has been widely applied in Cosmology [38, 39, 42–47]. When comparing two models, $M_i$ versus $M_j$, given a set of data, we use the Bayes’ factor defined in terms of the ratio of the evidence of models $M_i$ and $M_j$

$$B_{ij} = \frac{\mathcal{E}_i}{\mathcal{E}_j}, \quad (14)$$

where $\mathcal{E}_j$ is the standard model, and $\mathcal{E}_i$ are the competing models in which we want to compare. Here, we will compare the $\eta(z)$ functions defined in the Section IV by assuming that parameterization (7) is the standard model (Model 1) and, $\mathcal{E}_i$ are the others parameterizations Eqs. (8), (9), (10) and, (11). Models 2, 3, 4 and, 5 respectively, in which we want to compare. If each model is assigned an equal prior probability, the Bayes factor gives the posterior odds of the two models.

To quantify whether the model has favorable evidence or not, we adopted Jeffrey’s scale showed in Table I to interpret the values of the Bayes’ factor in terms of the strength of the evidence in comparing two competing models. This scale was suggested by Ref. [41] as a revised and conservative version of the original Jeffrey scale [60]. Note that this scale is empirically calibrated, i.e., it depends on the problem being investigated. Therefore, for an experiment for which $|\ln B_{ij}| < 1$, the evidence in favor of the model $M_i$ relative to model $M_j$ is interpreted as inconclusive. On the other hand, in the case of $\ln B_{ij} < -1$, we have support in favor of the model $M_j$. In this work, we consider the model 1 as the reference model $M_j$. For a complete discussion about this scale, see Ref. [41].

Furthermore, we assume that both type Ia supernovae and galaxy clusters, and gas mass fraction datasets follow a Gaussian likelihood, such as

$$\mathcal{L}(D|\Phi, M) \propto \exp \left[ -\frac{\chi^2(D|\Phi, M)}{2} \right], \quad (15)$$

whose $\chi^2$ reads

$$\chi^2(D|\Phi, M) = \sum_i \left( \frac{\eta_{\text{obs}}(z_i) - \eta_{\text{mod}}(z_i)}{\eta_{\text{err}}} \right)^2, \quad (16)$$

where $\eta_{\text{obs}}$ is a vector of the observed $\eta_{\text{obs}}$ function defined by Eqs. (6), and (4). $\eta_{\text{mod}}$ are the theoretical values obtained from the parameterizations, Eqs. (7), (8), (9), (10) and, (11) that we will test, and $\eta_{\text{err}}$ is the error given by propagation of uncertainty.

In order to perform out the Bayesian analysis, we use PyMultiNest [61], a Python module for MultiNest [62–64], a generic tool that uses Importance Nested Sampling [64, 65] to calculate the evidence, but which still allows for posterior inference as a consequence. We plot and analyze the results using GetDist [66]. Additionally, to increase the efficiency in the estimate of the evidence, we have chosen to perform all analysis by working with a set of 2000 live points, so that the number of samples for posterior distributions was of the order $O(10^5)$.

It should be pointed out that BI depends on the priors distributions $\pi(\Phi|M)$ adopted for the free parameters. This feature accounts for the predictive power of each model (parameterization), transforming this dependence in a property instead of a defect of the Bayesian inference framework. Albeit in the Bayesian analysis, the use of uniform (flat) priors can be acceptable in some cases, this type of prior can leads to issues of the point of view of model comparison. Uniform priors with distinct domain intervals change the evidence and can affect the Bayes’s factor between two competing models if it has not shared parameters. To use well-grounded priors, we considered values that reflect our actual state of knowledge about
the parameters of the models investigated. Moreover, we assume the following flat priors on the set of parameters: \( \eta_0 \sim \text{Uniform}(0, 2) \) and \( \eta_1 \sim \text{Uniform}(-1, 1) \).

VI. RESULTS

The results achieved considering the SNe Ia and \( D_A \) from galaxy clusters data are shown in the Fig. 2. As shown in the Fig. 2(a), the vertical traced line means \( \eta_0 = 1 \), i.e., CDDR validity. By considering the data, we obtain \( \eta_0 = 1.030 \pm 0.017 \) for model 1. This value obtained is compatible in 2\( \sigma \) c.l. with \( \eta_0 = 1 \) (light green region). In the Fig. 2(b), we show the results for model 2 and 3. Now, the vertical traced line means \( \eta_1 = 0 \) (CDDR validity). The values obtained for \( \eta_1 \) were \( \eta_1 = 0.091 \pm 0.059 \) for model 2, and \( \eta_1 = 0.134 \pm 0.082 \) for model 3. See that only model 2 is compatible with \( \eta_1 = 0 \) in 2\( \sigma \) of confidence (light green region), the model 3 is compatible in 3\( \sigma \) with \( \eta_1 = 0 \) (light blue region). Finally, in the Fig. 2(c), we present the triangle plot composed of the regions of confidence for \( \eta_0 \) and \( \eta_1 \), and, the posteriors distributions for models 4 and 5. The traced lines mean the values in which the CDDR is valid (\( \eta_0 = 1 \) and \( \eta_1 = 0 \)). The values obtained for the parameters of the model 4 were \( \eta_0 = 1.030 \pm 0.032 \) and \( \eta_1 = 0.00 \pm 0.11 \). These values are compatible in 2\( \sigma \) c.l. with the validity of CDDR. For model 5, we obtained \( \eta_0 = 1.030 \pm 0.037 \) and \( \eta_1 = 0.00 \pm 0.18 \), they are also compatible in 2\( \sigma \) c.l. with the validity of CDDR. We can see there is an anti-correlation between the parameters. Note that the data considered constrain the parameters of model 4 better than the model 5.

In the Fig. 3 we show the results obtained considering the X-ray gas mass fraction of galaxy clusters and SNe Ia. The model 1 is consistent in 1\( \sigma \) confidence with CDDR validity, \( \eta_0 = 0.977^{+0.025}_{-0.035} \). In the case of the models 2 and 3, Figs. 3(b), we obtained that ones are consistent in 1\( \sigma \) confidence with \( \eta_1 = 0 \). In the Figs. 3(c) we show the corner plot for models 4 and 5. The horizontal gray line means \( \eta_1 = 0 \) and the vertical line, \( \eta_0 = 1 \). The values obtained for the parameters of the model 4 were \( \eta_0 = 0.980^{+0.025}_{-0.035} \) and \( \eta_1 = -0.006 \pm 0.019 \) and, model 5 \( \eta_0 = 0.982^{+0.025}_{-0.031} \) and \( \eta_1 = -0.020 \pm 0.034 \). These values are compatible with CDDR validity in 1\( \sigma \) c.l. For the sake of Bayesian model comparison, we estimate the values of the logarithm of the Bayesian evidence (\( \text{Log} E \)) and, the Bayes’ factor (\( \text{Log} B \)), Table II and III. These results were obtained considering the priors defined in the last section and, we considered the model 1 as the reference one. In the case of \( D_A \) from galaxy clusters and, SNe Ia data, Table II, we first observe that model 2 has a positive value of Bayes’ factor (\( \text{Log} B = 0.858 \pm 0.052 \)). According to the Jefferys’ scale, Table I, we can conclude that this model has evidence inconclusive concerning to model 1. By considering the model 3, we obtain \( \text{Log} B = 1.296 \pm 0.052 \), so this one has weakly evidence favored by the data. Regarding the models 4 and 5, we obtain negative values for Bayes’ factor, which means that they have weakly evidence disfavored by the data. Thus we conclude that model 3 is weakly favored by \( D_A \) from galaxy clusters and SNe Ia data.

By considering the second data set, i.e., gas mass fraction and SNe Ia, we also implement Bayesian model comparison, Table III. The model 1 is the reference one. The model 2 and 3 have evidence inconclusive regarding the data. Concerning the other models, we note that they have moderate evidence disfavored by the data. From the Bayesian comparison model analysis point of view and the data considered, we conclude that all models have inconclusive and moderate evidence disfavored by X-ray gas mass fraction.

VII. CONCLUSIONS

In the last ten years, several works tested the cosmic distance duality relation (CDDR), \( \frac{D_A(z)}{D_L(z)} = 1 \) by considering a deformed CDDR, such as; \( \frac{D_A(z)}{D_L(z)} = \eta(z) \). Several \( \eta(z) \) functions were considered, however, the current analyses could not distinguish which \( \eta(z) \) function describes better the data.

In this work, we relaxed the CDDR by assuming the \( \eta(z) \) functions as given in Sec. IV. In order to decide which \( \eta(z) \) function describes better the data, we implemented a Bayesian inference analysis in terms of the strength of the evidence according to Jefferys’s scale, Tab.I. We considered the priors defined in Sec. V and astronomical data such as SNe Ia, diameter distance angular of the galaxy clusters and, X-ray gas mass fraction. The results obtained are reported in Tabs. II and III, where we showed the mean, the error, the Bayesian evidence and, the Bayes’ factor for all models studied here. In the Figs. 2 and 3 we showed the 1\( \sigma \) and 2\( \sigma \) regions of confidence and the posteriors distributions for all models.

The statistical constraints on all the functions implied that the CDDR remains valid in 1\( \sigma \) in the analyses by using SNe Ia and galaxy cluster gas mass fractions and, in 2\( \sigma \) c.l. when \( D_A \) from galaxy clusters and SNe Ia data were considered. However, we concluded from the Bayesian comparison that \( \eta(z) = 1 + \eta_0 z / (1 + z) \) was weakly favored in the CDDR test considering the \( D_A \) from galaxy clusters and SNe Ia data with respect to our standard model \( \eta(z) = \eta_0 \). On the other hand, in the CDDR test considering the galaxy cluster gas mass fractions and SNe Ia, all the \( \eta(z) \) functions had or inconclusive evidence or moderate evidence(against) with respect to our standard model. In both methodologies \( \eta(z) = \eta_0 = 1 \) is in agreement within 2\( \sigma \) c.l. with the data.

Finally, we concluded that the present data used in our analyses failed to provide which function of \( \eta_0(z) \) better describes the evolution of the CDDR with redshift. Probably, this is a consequence of the galaxy cluster data set used in this paper. They still have significant statistical and systematic errors (\( \approx 20\% \)). We believe that when applied to upcoming galaxy cluster data, the analyses proposed here may be useful to probe a possible violation of the CDDR.
Table II. Confidence limits for the parameters using SNe Ia and Galaxy Clusters. The columns show the constraints on each model, whereas the rows show the parameter considering in this analysis.

| Parameter | Model 1 | Model 2 | Model 3 | Model 4 | Model 5 |
|-----------|---------|---------|---------|---------|---------|
| $\eta_0$  | 1.030 ± 0.017 | Fixed in 1 | Fixed in 1 | 1.030 ± 0.032 | 1.030 ± 0.037 |
| $\eta_1$  | - | 0.091 ± 0.059 | 0.134 ± 0.082 | 0.00 ± 0.11 | 0.00 ± 0.18 |
| $\ln E$   | -17.992 ± 0.041 | -17.134 ± 0.032 | -16.696 ± 0.029 | -19.910 ± 0.048 | -19.465 ± 0.047 |
| $\ln B$   | - | 0.858 ± 0.052 | 1.296 ± 0.052 | -1.918 ± 0.063 | -1.473 ± 0.062 |
| Interpretation | - | Inconclusive | Weak evidence (favored) | Weak evidence (against) | Weak evidence (against) |

Table III. Confidence limits for the parameters using gas mass fraction. The columns show the constraints on each model whereas the rows show the parameter considering in this analysis. Here, we marginalized $N$.

| Parameter | Model 1 | Model 2 | Model 3 | Model 4 | Model 5 |
|-----------|---------|---------|---------|---------|---------|
| $\eta_0$  | 0.977^{+0.025}_{-0.030} | Fixed in 1 | Fixed in 1 | 0.980^{+0.025}_{-0.031} | 0.982^{+0.025}_{-0.031} |
| $\eta_1$  | - | -0.008 ± 0.020 | -0.025 ± 0.033 | -0.006 ± 0.019 | -0.020 ± 0.034 |
| $\ln E$   | -38.540 ± 0.049 | -39.233 ± 0.050 | -38.521 ± 0.047 | -42.256 ± 0.063 | -41.560 ± 0.061 |
| $\ln B$   | - | -0.693 ± 0.070 | 0.019 ± 0.068 | -3.716 ± 0.080 | -3.020 ± 0.078 |
| Interpretation | - | Inconclusive | Inconclusive | Moderate evidence (against) | Moderate evidence (against) |

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