Proof of Sarkar–Kumar’s conjectures on average entanglement entropies over the Bures–Hall ensemble

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Abstract
Sarkar and Kumar recently conjectured (2019 J. Phys. A: Math. Theor. 52 295203) that for a bipartite system of Hilbert dimension $mn$, the mean values of quantum purity and von Neumann entropy of a subsystem of dimension $m \leq n$ over the Bures–Hall measure are given by
$$\frac{2n(2n + m) - m^2 + 1}{2n(2mn - m^2 + 1)}$$ and
$$\psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right)$$, respectively, where $\psi_0(\cdot)$ is the digamma function. We prove the above conjectured formulas in this work. A key ingredient of the proofs is Forrester and Kieburg’s discovery on the connection between the Bures–Hall ensemble and the Cauchy–Laguerre biorthogonal ensemble studied by Bertola et al.

Keywords: quantum entanglement, von Neumann entropy, quantum purity, Bures–Hall measure, random matrix theory, biorthogonal ensemble

1. Introduction and the conjectures

Quantum information theory aims at understanding the theoretical underpinnings of quantum science and technology. One of the most fundamental features of quantum mechanics is the phenomenon of entanglement, which is the essential enabler for quantum technologies. In this work, we wish to understand the degree of entanglement as measured by average entanglement entropies over the distance metric of Bures–Hall measure, firstly discussed in [1, 2]. The Bures–Hall measure enjoys the property that, without any prior knowledge on a density matrix, the optimal way to estimate the density matrix is to generate a state at random with respect to this measure [2–4]. It is the only monotone metric that is simultaneously Fisher adjusted and Fubini–Study adjusted [2, 3]. In particular, we study the averages of quantum purity and von Neumann entropy over such a measure, the exact formulas of which were conjectured by Sarkar
and Kumar recently [5]. The focus of this paper is to prove the two conjectured average entropy formulas.

The density matrix formalism that leads to the Bures–Hall ensemble is the following. Consider a composite quantum system that consists of two subsystems A and B of Hilbert space dimensions \(m\) and \(n\), respectively. The Hilbert space \(\mathcal{H}_{A+B}\) of the composite system is given by the tensor product of the Hilbert spaces of the subsystems, \(\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B\). Define a state of the composite system as a linear combination of the random coefficients \(z_{i,j}\) and the complete basis \(\{|i_A\rangle\}\) and \(\{|j_B\rangle\}\) of \(\mathcal{H}_A\) and \(\mathcal{H}_B\) [6],

\[
|\psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} z_{i,j} |i_A\rangle \otimes |j_B\rangle, \tag{1}
\]

we then consider a superposition of the state (1) as

\[
|\varphi\rangle = |\psi\rangle + (U \otimes I_m)|\psi\rangle, \tag{2}
\]

where \(U\) is an \(m \times m\) unitary random matrix with the measure proportional to \(\det(I_m + U)^{2\alpha + 1}\) [5]. The corresponding density matrix is

\[
\rho = |\varphi\rangle \langle \varphi|, \tag{3}
\]

which has the natural probability constraint

\[
\text{tr}(\rho) = 1. \tag{4}
\]

We assume without loss of generality that \(m \leq n\). The reduced density matrix \(\rho_A\) of the smaller subsystem A is computed by partial tracing of the full density matrix (3) over the other subsystem B (interpreted as the environment) as

\[
\rho_A = \text{tr}_B \rho. \tag{5}
\]

The resulting density of eigenvalues of \(\rho_A\) (\(\lambda_i \in [0,1]\), \(i = 1, \ldots, m\)) is the (generalized) Bures–Hall measure [1, 2, 5]

\[
f(\lambda) = \frac{1}{c} \delta \left( 1 - \sum_{i=1}^{m} \lambda_i \right) \prod_{1 \leq i < j \leq m} \left( \lambda_i - \lambda_j \right)^2 \prod_{i=1}^{m} \lambda_i^{\alpha}, \tag{6}
\]

where the parameter \(\alpha\) takes half-integer values

\[
\alpha = n - m - \frac{1}{2}, \tag{7}
\]

and the constant \(c\) is

\[
c = \frac{2^{-m(m+2\alpha)} \pi^{m/2}}{\Gamma \left(m(m+2\alpha+1)/2\right)} \prod_{i=1}^{m} \Gamma(i + 1) \Gamma(i + 2\alpha + 1) \Gamma(i + \alpha + 1/2). \tag{8}
\]

In (6), the presence of the Dirac delta function \(\delta(\cdot)\) reflects the constraint (4). Note that each random coefficient \(z_{i,j}\) in (1) follows the standard complex Gaussian distribution [6] with zero mean and unit variance with the probability constraint \(\sum_{i,j} |z_{i,j}|^2 = 1\). This leads
to the interpretation of the Bures–Hall ensemble (6) as eigenvalue density of the Hermitian matrix [5]

\[
\frac{XX^\dagger}{\text{tr} (XX^\dagger)},
\]

where \(X = (I_m + U)G\) is given by a product of a unitary random matrix \(U\), cf (2), and an \(m \times n\) complex Ginibre random matrix \(G\).

The degree of entanglement of subsystems can be measured by the entanglement entropies, which are functions of eigenvalues of the reduced density matrix (5). An entanglement entropy should monotonically change from the separable state (\(\lambda_1 = 1, \lambda_2 = \cdots = \lambda_m = 0\)) to the maximally-entangled state (\(\lambda_1 = \lambda_2 = \cdots = \lambda_m = 1/m\)). A standard one is the quantum purity [7]

\[
S_P = \text{tr} (\rho_A^2) = \sum_{i=1}^{m} \lambda_i^2,
\]

supported in \(S_P \in [1/m, 1]\), which measures how far a state is from a pure state \(\rho_A^2 = \rho_A\) that corresponds to \(S_P = 1\). Quantum purity (10) is an example of polynomial entropies, whereas a well-known non-polynomial entropy is the von Neumann entropy [7]

\[
S_{\text{vN}} = -\text{tr} (\rho_A \ln \rho_A) = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i,
\]

supported in \(S_{\text{vN}} \in [0, \ln m]\), which achieves the separable state and maximally-entangled state when \(S_{\text{vN}} = 0\) and when \(S_{\text{vN}} = \ln m\), respectively.

Statistical information of entropies is encoded through their moments: the first moment (average value) implies the typical behavior of entanglement and the higher moments specify fluctuation around the typical values. For the Hilbert–Schmidt measure [7], that corresponds to the density without the interaction term

\[
\prod_{1 \leq i < j \leq m} (\lambda_i + \lambda_j)
\]

in (6), the moments of quantum purity [8, 9] and von Neumann entropy [10–15] have been well-investigated. However, knowledge on the behavior of entanglement entropies over the Bures–Hall measure is quite limited. In the special case of equal subsystem dimensions \(m = n\), i.e., \(\alpha = 1/2\) in (7), the resulting moments of purity were derived in [3, 4]. For arbitrary subsystem dimensions \(m < n\), Sarkar and Kumar recently conjectured [5, equations (61) and (59)] that the average quantum purity and the average von Neumann entropy are given by (notice the notational difference here and in [5])

\[
\mathbb{E}_f [S_P] = \frac{2n(2n + m) - m^2 + 1}{2n(2mn - m^2 + 2)}
\]

and

\[
\mathbb{E}_f [S_{\text{vN}}] = \psi_0 \left( mn - \frac{m^2}{2} + 1 \right) - \psi_0 \left( n + \frac{1}{2} \right),
\]

\(^{1}\)For a comprehensive treatment of the density matrix formalism including the discussed measures and entropies, we refer readers to [6, 7] and references therein.
respectively, where the expectations $\mathbb{E}_f [\cdot]$ are taken over the Bures–Hall ensemble (6). Here, 
\[
\psi_0(x) = \frac{d \ln \Gamma(x)}{dx} \text{ is the digamma function (psi function) [16] and for a positive integer } l,
\]
\[
\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \tag{15a}
\]
\[
\psi_0\left(l + \frac{1}{2}\right) = -\gamma - 2 \ln 2 + 2 \sum_{k=0}^{l-1} \frac{1}{2k+1}, \tag{15b}
\]
where $\gamma \approx 0.5772$ is the Euler’s constant. In the next section, we show that the conjectured formulas (13) and (14) are indeed correct.

2. Average entropies over Bures–Hall ensemble

2.1. Moment relations

The first step is a rather standard calculation, briefly outlined below (see also, e.g., [4, 5, 10, 12, 14, 15]), that relates the moment computation over an ensemble with the constraint $\delta (1 - \sum_{i=1}^{m} \lambda_i)$ to a one without. As will be seen, the corresponding unconstrained ensemble of the Bures–Hall ensemble (6) is [5]
\[
h(x) = \frac{1}{c'} \prod_{1 \leq i < j \leq m} \frac{(x_i - x_j)^2}{x_i + x_j} \prod_{i=1}^{m} x_i^{\alpha} e^{-x_i}, \tag{16}
\]
where $x_i \in [0, \infty), i = 1, \ldots, m$, and the constant $c'$ is related to the constant (8) by
\[
c' = c \Gamma \left(m(m + 2\alpha + 1)/2\right). \tag{17}
\]
Despite being only interested in the half-integer values of $\alpha$ in (7), the following results, in particular the expressions (52) and (55), are valid for $\alpha > -1$ that the density (16) is defined. We start by finding the first moment relation for the von Neumann entropy, where, by multiplying an auxiliary gamma density integral, one has
\[
\mathbb{E}_f [S_{vN}] = \int_{0}^{\infty} e^{-\theta \delta - 1} \Gamma(d) \int_{\lambda} S_{vN f}(\lambda) \prod_{i=1}^{m} d\lambda_i. \tag{18}
\]
Inserting the change of variables
\[
\lambda_i = \frac{x_i}{\theta}, \quad i = 1, \ldots, m, \tag{19}
\]
into (18), some simplification leads to
\[
\mathbb{E}_f [S_{vN}] = \psi_0(d) - \frac{c^{-1}}{\Gamma(d)} \int_{\alpha} \prod_{1 \leq i < j \leq m} \frac{(x_i - x_j)^2}{x_i + x_j} \prod_{i=1}^{m} x_i^{\alpha} \times \int_{0}^{\infty} e^{-\theta \delta - m(m+1)/2 - \alpha m - 1} \delta \left( \theta - \sum_{i=1}^{m} x_i \right) d\theta \prod_{i=1}^{m} dx_i, \tag{20}
\]
where we also used
\[ \int_0^\infty e^{-\theta d - 1} \ln \theta d\theta = \Gamma(d)\psi_0(d), \quad \Re(d) > 0, \] (21)
and the random variable
\[ T_{\text{vN}} = \sum_{i=1}^m x_i \ln x_i \] (22)
is understood as the induced von Neumann entropy over the unconstrained ensemble (16). By setting \( d = m(m+1)/2 + \alpha m + 1 \), the integral over \( \theta \) in (20) can be conveniently evaluated that leads to the first moment relation as
\[ \mathbb{E}_f [S_{\text{vN}}] = \psi_0 \left( \frac{m(m+1)}{2} + \alpha m + 1 \right) - \frac{2}{m(m+2\alpha+1)} \mathbb{E}_h [T_{\text{vN}}], \] (23)
where we used the identity (17). In a similar but more straightforward manner, the first moment relation for quantum purity is obtained as (see also [5])
\[ \mathbb{E}_f [S_{\text{P}}] = \frac{4}{m(m+2\alpha+1)(m^2+2\alpha m+m+2)} \mathbb{E}_h [T_{\text{P}}], \] (24)
where \( T_{\text{P}} \) is the induced purity
\[ T_{\text{P}} = \sum_{i=1}^m x_i^2 \] (25)
over the unconstrained ensemble (16).

2.2. Computation of some related integrals

Proving (13) and (14) now boils down to computing the induced first moments \( \mathbb{E}_h [T_{\text{P}}] \) in (24) and \( \mathbb{E}_h [T_{\text{vN}}] \) in (23), respectively. Computing these average values requires the one-point correlation function [17, 18], i.e., the density of an arbitrary eigenvalue, of the unconstrained Bures–Hall ensemble (16). In fact, its \( k \)-point correlation function was recently derived in [19], which is written in terms of the correlation functions of the Cauchy–Laguerre biorthogonal ensemble [20]. In particular, the needed an arbitrary eigenvalue density of the unconstrained ensemble (16) is [19]
\[ h_1(x) = \frac{1}{2m} \left( G_\alpha(x) + G_{\alpha+1}(x) \right), \] (26)
where we denote
\[ G_\alpha(q) = \int_0^1 G_{2,3}^{1,1}(q|x)G_{2,3}^{2,1}(q|tx)dt \] (27)
with
\[ G_{2,3}^{1,1}(q|x) = G_{2,3}^{1,1}\left(\begin{array}{c} -m; m + 2\alpha + 1 \\ 2\alpha + 1; 0, q \end{array}\right|x), \] (28)
\[ G_{2,3}^{2,1}(q|x) = G_{2,3}^{2,1}\left(\begin{array}{c} -m - 2\alpha - 1; m \\ 0, -q; -2\alpha - 1 \end{array}\right|x) \] (29)

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further denoting some Meijer G-functions [16]. In general, the Meijer G-function is defined by the following contour integral [16]

\[
G_{m,n}^{p,q} \left( \begin{array}{c}
a_1, \ldots, a_n; a_{n+1}, \ldots, a_p \\
b_1, \ldots, b_m; b_{m+1}, \ldots, b_q
\end{array} \right| x \right) = \frac{1}{2\pi i} \int_\mathcal{L} \prod_{j=1}^m \Gamma (b_j + s) \prod_{j=m+1}^q \Gamma (1 - b_j - s) x^{-s} ds,
\]

(30)

where the contour \( \mathcal{L} \) separates the poles of \( \Gamma (1 - a_j - s) \) from the poles of \( \Gamma (b_j + s) \).

It will become clear that as intermediate steps to obtain \( \mathbb{E}_q[T_p] \) and \( \mathbb{E}_q[T_{\infty}] \), we need to compute the integral below involving the Meijer G-functions (28) and (29)

\[
I_q^{(\beta)}(t) = \int_0^\infty x^\beta G_{2,3}^{1,1}(q|tx)G_{2,3}^{1,1}(q|tx)dx, \quad t > 0,
\]

(31)

for \( \beta = 0, 1, 2 \), as well as its derivative for \( \beta = 1 \),

\[
H_q(t) = \frac{d}{dt} I_q^{(\beta)}(t) \bigg|_{\beta=1},
\]

(32)

where \( q \) will take the values \( \alpha \) and \( \alpha + 1 \) in both (31) and (32). To compute (31), we use the fact that the Meijer G-function (28) can be written as a terminating hypergeometric function [16] (see also [19, 20])

\[
G_{2,3}^{1,1}(q|tx) = \frac{\Gamma (m + 2\alpha + 2)(tx)^{2\alpha+1}}{\Gamma (m)\Gamma (2\alpha + 2)\Gamma (2\alpha + 2 - q)} \, _2F_1 \left( \begin{array}{c}
1 - m, m + 2\alpha + 2 \\
2\alpha + 2, 2\alpha + 2 - q
\end{array} \right| tx \right),
\]

(33)

\[
= \frac{\Gamma (m + 2\alpha + 2)(tx)^{2\alpha+1}}{\Gamma (m)\Gamma (2\alpha + 2)\Gamma (2\alpha + 2 - q)} \sum_{k=0}^{m-1} \frac{(1 - m)(m + 2\alpha + 2)k(2\alpha + 2 - q)k!}{(2\alpha + 2)k(2\alpha + 2 - q)k!},
\]

(34)

where \( (a)_n = \Gamma (a + n)/\Gamma (a) \) is the Pochhammer symbol. Inserting (34) into (31), the integral can now be evaluated by using the Mellin transform of the Meijer G-function [16] (cf (30))

\[
\int_0^\infty x^{\eta - 1} G_{p,q}^{m,n} \left( \begin{array}{c}
a_1, \ldots, a_n; a_{n+1}, \ldots, a_p \\
b_1, \ldots, b_m; b_{m+1}, \ldots, b_q
\end{array} \right| \eta \right) dx = \frac{\eta^{-\eta} \prod_{j=m+1}^q \Gamma (b_j + s) \prod_{j=1}^n \Gamma (1 - a_j - s)}{\prod_{j=m+1}^q \Gamma (a_j + s) \prod_{j=1}^n \Gamma (1 - b_j - s)}
\]

(35)

valid for \( \Re(s) > -\min_{1 \leq j \leq m} \Re(b_j) \) and \( \eta > 0 \), as

\[
I_q^{(\beta)}(t) = t^{-\beta - 1} I_q^{(\beta)},
\]

(36)

where \( I_q^{(\beta)} \) denotes the \( \beta \) independent part

\[
I_q^{(\beta)} = \sum_{k=0}^{m-1} \frac{(-1)^k \eta^k \Gamma (k + 2\alpha + m + 2)\Gamma (k + \beta + 1) \Gamma (k + \beta + 2\alpha + 2 - q)}{\Gamma (k + 2\alpha + 2)\Gamma (k + 2\alpha + 2 - q)\Gamma (m - k)k! \Gamma (k + \beta + 2\alpha + m + 2)}.
\]

(37)
In obtaining (37), we also used the result of gamma function of negative arguments

$$\Gamma(-l+\epsilon) = \frac{(-1)^l}{l!\epsilon} (1 + o(\epsilon))$$  \hspace{1cm} (38)

to resolve some indeterminacy by taking the limit $\epsilon \to 0$. Since the $q$ dependent term in (37) is $\left(k + 2\alpha + 2 - q\right)$, $I_q^{(1)}(t)$ becomes a $\beta$th degree polynomial in $q$ for a non-negative integer $\beta$. The needed cases when $\beta = 0, 1, 2$ can now be directly obtained as

$$I_q^{(0)}(t) = 0,$$  \hspace{1cm} (39a)

$$I_q^{(1)}(t) = -\frac{m(m + 2\alpha + 1)(m + 2\alpha + 1 - q)}{2m + 2\alpha + 1} t^{-2},$$  \hspace{1cm} (39b)

$$I_q^{(2)}(t) = -\frac{m(m + 2\alpha + 1)(m + 2\alpha + 1 - q)}{2(m + \alpha)(m + \alpha + 1)(2m + 2\alpha + 1)} \left((m + 2\alpha + 1)(5m^2 + 8\alpha m + 4m + 4\alpha^2 + 4\alpha)
- (3m^2 + 6\alpha m + 3m + 4\alpha^2 + 4\alpha q)\right) t^{-3},$$  \hspace{1cm} (39c)

where the non-zero contribution in (37) for $\beta = 1$ and $\beta = 2$ is from the terms $k = m - 1$ and $k = m - 2, m - 1$, respectively. As a consequence of (39a), the integral (27) can be also represented, by the symmetry of (31) in $t$ and $x$ when $\beta = 0$, as

$$G_q(x) = -\int_1^\infty G_{2,1}^1(q|tx)G_{2,1}^1(q|tx)dt.$$  \hspace{1cm} (40)

To evaluate (32), we first notice from (36) and (31) that

$$H_q(t) = t^{-2}H_q - I_q^{(1)}(t) \ln t,$$  \hspace{1cm} (41)

where $I_q^{(1)}(t)$ has been computed in (39b), and $H_q$ similarly denotes (cf (32))

$$H_q = \frac{d}{d\beta} I_q^{(1)} \bigg|_{\beta=1}.$$  \hspace{1cm} (42)

By invoking (38) and the limiting behavior of digamma function

$$\psi_0(-l+\epsilon) = -\frac{1}{\epsilon}(1 + o(\epsilon))$$  \hspace{1cm} (43)

to resolve some indeterminacy, $H_q$ is calculated as

$$H_q = -\frac{m(m + 2\alpha + 1)(m + 2\alpha + 1 - q)}{2m + 2\alpha + 1} (\psi_0(m + 1) + \psi_0(m + 2\alpha + 2)
+ \psi_0(m + 2\alpha + 2 - q) - \psi_0(2m + 2\alpha + 2) - \psi_0(1))
+ \sum_{k=0}^{m-2} \frac{(k + 1)(k + 2\alpha + 2)(k + 2\alpha + 2 - q)}{(m-k-1)(m + 2\alpha + 2)}.$$  \hspace{1cm} (44)

With the help of the identity

$$\psi_0(l+n) = \psi_0(l) + \sum_{k=0}^{n-1} \frac{1}{l+k},$$  \hspace{1cm} (45)
further simplification of (44) gives
\[
H_q = \frac{m(m + 2\alpha + 1)}{2m + 2\alpha + 1} \left( \frac{a_1 + 2aq}{2(m + 2\alpha + 1)(2m + 2\alpha + 1)} + (2\alpha + 1 - 2q)(\psi_0(2m + 2\alpha + 2) - \psi_0(m + 2\alpha + 2) - (m + 2\alpha + 1 - q)\psi_0(m + 2\alpha + 1 - q) \right),
\]
where we denote
\[
a_1 = -4m^3 - 24\alpha m^2 - 14m^2 - 36\alpha^2 m - 40\alpha m - 11m - 16\alpha^3 - 28\alpha^2 - 16\alpha - 3, \quad (47a)
\]
\[
a_2 = 4m^2 + 8\alpha m + 3m + 4\alpha^2 + 4\alpha + 1. \quad (47b)
\]

With the above preparations, we now derive expressions for \(E_h[T_P]\) in (24) and \(E_h[T_{vN}]\) in (23).

### 2.3. Average quantum purity

By definition, the mean value \(E_h[T_P]\) is calculated by using the one-point density (26) as
\[
E_h[T_P] = m \int_0^\infty x^2 h_1(x) \, dx = -\frac{1}{2} \int_0^\infty x^2 \int_1^\infty G_{1,3}^{1,1}(\alpha |tx)G_{2,3}^{2,1}(\alpha |tx) \, dt \, dx
\]
\[
-\frac{1}{2} \int_0^\infty x^2 \int_1^\infty G_{1,3}^{1,1}(\alpha + 1|tx)G_{2,3}^{2,1}(\alpha + 1|tx) \, dt \, dx, \quad (48)
\]
where we used the representation (40) instead of (27). By changing the order of integration, we arrive at (cf (31))
\[
E_h[T_P] = -\frac{1}{2} \int_1^\infty \left( I_1^{(2)}(t) + I_{\alpha + 1}^{(2)}(t) \right) \, dt \quad (49)
\]
\[
= \frac{m(m + 2\alpha + 1)}{4(2m + 2\alpha + 1)} \left( 5m^2 + 10\alpha m + 5m + 4\alpha^2 + 4\alpha + 2 \right), \quad (50)
\]
where the last step was obtained by using (39c) and the fact that
\[
\int_1^\infty \frac{1}{t^3} \, dt = \frac{1}{2}. \quad (51)
\]

The change of the order of integration is justified since the integrals in (49) exist as a result of using the representation (40). Inserting (50) into (24), one obtains
\[
E_f[S_P] = \frac{5m^2 + 10\alpha m + 5m + 4\alpha^2 + 4\alpha + 2}{(2m + 2\alpha + 1)(m^2 + 2\alpha m + m + 2)}, \quad (52)
\]
Finally, evaluating the above expression with the value of $\alpha$ in (7) of the Bures–Hall ensemble, we prove the conjectured formula (13).

2.4. Average von Neumann entropy

Similarly to the steps that have led to (49), the mean value $E_{\alpha}[T_{vN}]$ is calculated via the relations (32) and (41) as

$$E_{\alpha}[T_{vN}] = -\frac{1}{2} \int_{1}^{\infty} (H_\alpha(t) + H_{\alpha+1}(t)) \, dt$$

$$= -\frac{1}{2} (H_\alpha + H_{\alpha+1}) \int_{1}^{\infty} \frac{1}{t^2} \, dt + \frac{1}{2} \int_{1}^{\infty} (I^{(1)}_\alpha(t) + I^{(1)}_{\alpha+1}(t)) \ln t \, dt.$$

The results (46) and (39b) give us

$$H_\alpha + H_{\alpha+1} = -m(m + 2\alpha + 1)(\psi_0(m + \alpha + 1) + 1),$$

$$I^{(1)}_\alpha(t) + I^{(1)}_{\alpha+1}(t) = -m(m + 2\alpha + 1)t^{-2},$$

and together with the fact that

$$\int_{1}^{\infty} \frac{1}{t^2} \, dt = 1, \quad \int_{1}^{\infty} \frac{\ln t}{t^2} \, dt = 1,$$  \hspace{1cm} (53)

one arrives at

$$E_{\alpha}[T_{vN}] = \frac{m(m + 2\alpha + 1)}{2} \psi_0(m + \alpha + 1).$$  \hspace{1cm} (54)

Inserting the above result into the moment relation (23), we finally obtain

$$E_f[S_{vN}] = \psi_0 \left( \frac{m(m + 1)}{2} + \alpha m + 1 \right) - \psi_0(m + \alpha + 1),$$  \hspace{1cm} (55)

which upon evaluated at the value of $\alpha$ in (7) proves the conjectured formula (14).

3. Conclusion and outlook

We proved the mean value formulas of quantum purity and von Neumann entropy over the Bures–Hall measure conjectured by Sarkar and Kumar. The starting point of our proofs is the recently discovered relation between the correlation functions of Bures–Hall ensemble and these of the Cauchy–Laguerre biorthogonal ensemble. Future work includes the study of higher order moments of the entanglement entropies in order to understand their statistical distributions.

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