Computing Minimal Sets on Propositional Formulae I: Problems & Reductions

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Abstract

Boolean Satisfiability (SAT) is arguably the archetypical NP-complete decision problem. Progress in SAT solving algorithms has motivated an ever increasing number of practical applications in recent years. However, many practical uses of SAT involve solving function as opposed to decision problems. Concrete examples include computing minimal unsatisfiable subsets, minimal correction subsets, prime implicates and implicants, minimal models, backbone literals, and autarkies, among several others. In most cases, solving a function problem requires a number of adaptive or non-adaptive calls to a SAT solver. Given the computational complexity of SAT, it is therefore important to develop algorithms that either require the smallest possible number of calls to the SAT solver, or that involve simpler instances. This paper addresses a number of representative function problems defined on Boolean formulas, and shows that all these function problems can be reduced to a generic problem of computing a minimal set subject to a monotone predicate. This problem is referred to as the Minimal Set over Monotone Predicate (MSMP) problem. This exercise provides new ways for solving well-known function problems, including prime implicates, minimal correction subsets, backbone literals, independent variables and autarkies, among several others. Moreover, this exercise motivates the development of more efficient algorithms for the MSMP problem. Finally the paper outlines a number of areas of future research related with extensions of the MSMP problem.
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1 Introduction

The practical success of Propositional Satisfiability (SAT) solvers has motivated an ever increasing range of applications. Many applications are naturally formulated as decision procedures. In contrast, a growing number of applications involve solving function and opposed to decision problems. Representative examples include computing maximum satisfiability, minimum satisfiability, minimal unsatisfiable subsets of clauses, minimal correction subsets of clauses, minimal and maximal models, the backbone of a formula, the maximum autark assignment, prime implicants and implicates, among many others.

In recent years, the most widely used approach for solving a comprehensive range of function problems defined on Boolean formulas consists of using the SAT solver as an oracle, which is called a number of times polynomial in the size of the problem representation. It is interesting to observe that this approach matches in some sense well-known query complexity characterizations of functions problems [31, 79].

Moreover, as shown in [13, 14, 69], representative function problems can be viewed as computing a minimal set given some monotone predicate, and develop a number of algorithms for computing a minimal set subject to a monotone predicate. This work motivates the question of whether more function problems can be represented as computing a minimal set subject to a monotone predicate. The implications could be significant, both in terms of new algorithms for different problems, as well as possible new insights into some of these problems. This paper addresses this question, and shows that a large number of function problems defined on Boolean formulae can be cast as computing a minimal set subject to a monotone predicate. As a result, all the algorithms developed in [69] for some function problems can also be used for this much larger set of function problems.

The paper is organized as follows. Section 2 introduces the notation and definitions used throughout the paper. Section 3 defines and overviews the function problems studied in later sections. Section 4 shows how all the functions described in Section 3 can be reduced to solving an instance of the more general MSMP problem. Section 5 concludes the paper, by summarizing the main contributions and outlining a number of research directions.

2 Preliminaries

This section introduces the notation used in the remainder of the paper, covering propositional formulae, monotone predicates, complexity classes, and problem reductions.

2.1 Propositional Formulae

Standard propositional logic definitions are used throughout the paper (e.g. [48, 10]), some of which are reviewed in this section.

Sets are represented in caligraphic font, e.g. \( \mathcal{R}, \mathcal{T}, \mathcal{I}, \ldots \) Propositional formulas are also represented in caligraphic font, e.g. \( \mathcal{F}, \mathcal{H}, \mathcal{S}, \mathcal{M}, \ldots \) Propositional variables are represented with letters from the end of the alphabet, e.g. \( x, w, y, z \), and indeces can be used, e.g. \( x_1, w_1, \ldots \) An atom is a
propositional variable. A literal is a variable \( x_i \) or its complement \( \neg x_i \). A propositional formula (or simply a formula) \( F \) is defined inductively over a set of propositional variables, with the standard logical connectives, \( \neg, \land, \lor \), as follows:

1. An atom is a formula.
2. If \( F \) is a formula, then \( \neg F \) is a formula.
3. If \( F \) and \( G \) are formulas, then \( F \lor G \) is a formula.
4. If \( F \) and \( G \) are formulas, then \( F \land G \) is a formula.

The inductive step could be extended to include other propositional connectives, e.g. \( \to \) and \( \leftrightarrow \). (The use of parenthesis is not enforced, and standard binding rules apply (e.g. [48]), with parenthesis being used only to clarify the presentation of formulas.) The inductive definition of a propositional formula allows associating a parse tree with each formula, which can be used for evaluating truth assignments. The variables of a propositional formula \( F \) are represented by \( \text{var}(F) \). For simplicity, the set of variables of a formula will be denoted by \( X \equiv \text{var}(F) \). A clause \( c \) is a non-tautologous disjunction of literals. A term \( t \) is a non-contradictory conjunction of literals. Commonly used representations of propositional formulas include conjunctive and disjunctive normal forms (resp. CNF and DNF). A CNF formula \( F \) is a conjunction of clauses. A DNF formula \( F \) is a disjunction of terms. CNF and DNF formulas can also be viewed as sets of sets of literals. Both representations will be used interchangeably throughout the paper. In the remainder of the paper, propositional formulas are referred to as formulas, and this can either represent an arbitrary propositional formula, a CNF formula, or a DNF formula. The necessary qualification will be used when necessary. The following sets are used throughout. \( \mathbb{F} \) denotes the set of propositional formulas, \( \mathbb{C} \subset \mathbb{F} \) denotes the set of CNF formulas, and \( \mathbb{D} \subset \mathbb{F} \) denotes the set of DNF formulas.

When the set of variables of a formula \( F \) is relevant, the notation \( F[X] \) is used, meaning that \( F \) is defined in terms of variables from \( X \). Replacements of variables will be used. The notation \( F[x_i/y_i] \) represents formula \( F \) with variable \( x_i \) replaced by \( y_i \). This definition can be extended to more than one variable. For the general case, if \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_n\} \), then the notation \( F[X/Y] \) represents formula \( F \) with \( x_1 \) replaced by \( y_1 \), \( x_2 \) replaced by \( y_2 \), ..., and \( x_n \) replaced by \( y_n \). Alternatively, one could write \( F[x_1/y_1,x_2/y_2, \ldots, x_n/y_n] \). Given \( v_i \in \{0,1\} \), \( F[x_i=v_i] \) represents formula \( F \) with variable \( x_i \) replaced with \( v_i \). Alternatively, the notation \( F_{x_i=v_i} \) can be used.

The paper addresses mainly plain CNF formulas, i.e., formulas where any clause can be dropped (or relaxed) from the formula, being referred to as soft clauses [59]. Nevertheless, in some settings, CNF formulas can have hard clauses, i.e. clauses that cannot be dropped (or relaxed). In these cases, \( F \) can be viewed as a 2-tuple \( (\mathcal{H}, \mathcal{B}) \), where \( \mathcal{H} \) denotes the hard clauses, and \( \mathcal{B} \) denotes the soft (or relaxable, or breakable) clauses. (Observe that any satisfiability test involving \( F \) requires the hard clauses to be satisfied, whereas some of the clauses in \( \mathcal{B} \) may discarded.) Moreover, weights can be associated with the soft clauses as follows. A weight function \( \omega : \mathcal{H} \cup \mathcal{B} \to \mathbb{R} \cup \{\top\} \) associates a weight with each clause. For any soft clause \( c \in \mathcal{B} \), \( \omega(c) \neq \top \), whereas for any hard clause \( c \in \mathcal{H} \), \( \omega(c) = \top \), where \( \top \) is such that it exceeds \( \sum_{c \in \mathcal{B}} \omega(c) \), meaning that hard clauses are too costly to falsify. Throughout the paper, and unless otherwise stated, it is assumed that either \( \mathcal{H} = \emptyset \) or \( \mathcal{B} = \emptyset \), i.e. all clauses are soft or all clauses are hard, and so \( F \) corresponds to \( \mathcal{B} \) or to \( \mathcal{H} \), respectively. Moreover, the (implicit) weight function assigns cost 1 to each (soft) clause. Finally, the above
definitions can be extended to handle groups of clauses, e.g. [64, 36].

Given a formula $F$, a truth assignment $\nu$ is a map from the variables of $F$ to $\{0, 1\}$, $\nu : \text{var}(F) \rightarrow \{0, 1\}$. Given a truth assignment $\nu$, the value taken by a formula, denoted $F^\nu$ is defined inductively as follows:

1. If $x$ is a variable, $x^\nu = \nu(x)$.
2. If $F = (\neg G)$, then
   
   $F^\nu = \begin{cases} 0 & \text{if } G^\nu = 1 \\ 1 & \text{if } G^\nu = 0 \end{cases}$

3. If $F = (E \lor G)$, then
   
   $F^\nu = \begin{cases} 1 & \text{if } E^\nu = 1 \text{ or } G^\nu = 1 \\ 0 & \text{otherwise} \end{cases}$

4. If $F = (E \land G)$, then
   
   $F^\nu = \begin{cases} 1 & \text{if } E^\nu = 1 \text{ and } G^\nu = 1 \\ 0 & \text{otherwise} \end{cases}$

The inductive step could be extended to include additional propositional connectives, e.g. $\rightarrow$ and $\leftrightarrow$.

A truth assignment $\nu$ such that $F^\nu = 1$ is referred to as a \textit{satisfying truth assignment}. A formula $F$ is \textit{satisfiable} if it has a satisfying truth assignment; otherwise it is \textit{unsatisfiable}. As a result, the problem of propositional satisfiability is defined as follows:

\textbf{Definition 1 (Propositional Satisfiability (SAT))} Given a formula $F$, the decision problem SAT consists of deciding whether $F$ is satisfiable.

The standard semantic entailment notation is used throughout. Let $A, C \in \mathcal{F}$. $A \models C$ denotes that for every truth assignment $\nu$ to the variables in $\text{var}(A) \cup \text{var}(C)$, $(A^\nu = 1) \Rightarrow (C^\nu = 1)$. The equivalence notation $A \equiv C$ is used to denote that $A \models C \land C \models A$, indicating that $A$ and $C$ have the same satisfying truth assignments, when $\text{var}(A) = \text{var}(C)$. If a formula $F \in \mathcal{F}$ is satisfiable, we write $F \not\models \bot$. If a formula $F$ is unsatisfiable, we write $F \models \bot$. Moreover, if $\nu$ is a satisfying truth assignment of $F$, the notation $\nu \models F$ is also used. Finally, if $F$ is a tautology, then the notation $\top \models F$ is used, whereas $\top \not\models F$ denotes that $F$ is not a tautology.

The following results are well-known, e.g. [48], and will be used throughout.

\textbf{Proposition 1} Let $F \in \mathcal{C}$, with $F \not\models \bot$. Then, $\forall E \in \mathcal{C}, (E \subseteq F) \Rightarrow (E \not\models \bot)$.

\textbf{Proposition 2} Let $U \in \mathcal{C}$, with $U \not\models \bot$. Then, $\forall T \in \mathcal{C}, (T \supseteq U) \Rightarrow (T \models \bot)$.

\textbf{Proposition 3} Let $F \in \mathcal{D}$, with $\top \not\models F$. Then, $\forall E \in \mathcal{D}, (E \subseteq F) \Rightarrow (\top \not\models E)$.

\textbf{Proposition 4} Let $U \in \mathcal{D}$, with $\top \models U$. Then, $\forall T \in \mathcal{D}, (T \supseteq U) \Rightarrow (\top \models T)$.

Given a formula $F$, with set of variables $X = \text{var}(F)$, the following additional definitions apply. $L(X) \triangleq \{x, \neg x \mid x \in X\}$ represents the set of literals given the variables in $X$. A truth assignment $\nu$
can be represented as a set of literals $V \subseteq \mathbb{L}$, interpreted as a conjunction of literals or a term, where each literal in $V$ encodes the value assigned to a given variable $x \in X$, i.e. literal $x$ if $\nu(x) = 1$ and literal $\neg x$ if $\nu(x) = 0$. Clearly, $|V| = |X|$. In the remainder of the paper, an assignment can either be represented as a map or as a set of literals as defined above. This will be clear from the context.

Sets of literals are used to represent partial truth assignments. The set of all partial truth assignments $\mathbb{A}$, given $X \triangleq \text{var}(F)$, is defined by $\mathbb{A}(X) \triangleq \{ V \subseteq L(X) \mid (\forall x \in X), (x \notin V) \lor (\neg x \notin V)\}$. (For simplicity, and when clear from the context, the dependency of $\mathbb{A}$ and $X$ will be omitted.) Sets of literals can also be used to satisfy or falsify CNF or DNF formulas. For a set of literals $V \in \mathbb{A}$ and $F \in \mathbb{F}$, the notation $V \models F$ denotes whether assigning 1 to the literals in $V$ satisfies $F$. Similarly, the notation $V \not\models F$ denotes whether assigning 1 to the literals in $V$ falsifies $F$.

As noted above, sets of literals are in most cases interpreted as the conjunction of the literals, e.g. as a term. However, in some situations, it is convenient to interpret a set of literals as a disjunction of the literals, e.g. as a clause. Throughout the paper, the following convention is used. A set of literals qualified as a term, as a truth assignment, or as an implicant (defined below) is interpreted as a conjunction of literals. A set of literals qualified as a clause, or as an implicat (defined below) is interpreted as a disjunction of literals. Given $F \in \mathbb{F}$, a term $t \in \mathbb{A}$ is an implicant of $F$ iff $t \models F$. (Alternatively, we could write $t \in \mathbb{A}$ is an implicant of $F$ iff $t \models F$.) Similarly, a clause $c \in \mathbb{A}$ is an implicate of $F$ iff $F \vdash c$. (Alternatively, we could write $c \in \mathbb{A}$ is an implicate of $F$ iff $F \vdash (\lor_{l \in c} l)$.)

In some settings it is necessary to reason with the variables assigned value 1. This is the case for example when reasoning with minimal and maximal models. Given a truth assignment, $\nu$, with $V$ the associated set of literals, the variables assigned value 1 are given by $M = V \cap X$. The function $\nu(M, X)$ allows recovering the truth assignment associated with a set $M$ of variables assigned value 1. If the assignment $\nu$, associated with a set $M$ of variables assigned value 1, is satisfying, then $M$ is referred to as a model. Moreover, the notation $\nu(M, X) \models F$ is used to denote that, given a set $M$ of variables assigned value 1, the associated truth assignment is satisfying. In contrast, $\nu(M, X) \not\models F$ is used to denote that the truth assignment falsifies the formula.

Most of the algorithms described in this paper use sequences of calls to a SAT solver. A SAT solver accepts a (CNF-encoded) propositional formula $F$ as a single argument and returns a 2-tuple $(\text{st}, \alpha)$, i.e. $(\text{st}, \nu) = \text{SAT}(F)$, where $\text{st} \in \{0, 1\}$ denotes whether the formula is unsatisfiable ($\text{st} = 0$, or simply unsat) or satisfiable ($\text{st} = 1$, or simply sat), and $\nu$ is a (satisfying) truth assignment in case the formula is satisfiable. $\nu$ will also be referred to as a witness of satisfiability. For simplicity, in this paper it is assumed the SAT solver does not return unsatisfiable subformulas when the outcome is unsat, although this feature is available in many modern SAT solvers, e.g. [27, 9]. Throughout the paper, it will be implicitly assumed that a SAT solver call yields not only the 0/1 outcome, but the actual 2-tuple $(\text{st}, \nu)$.

Modern SAT solvers typically accept CNF formulas [71]. Procedures for CNF-encoding (or clausifying) arbitrary propositional formulas are well-known (e.g. [97, 80]). Throughout the paper the propositional formulas passed to SAT solvers are often not in CNF. For simplicity, it is left implicit that a clausification procedure would be invoked if necessary. Moreover, additional non-clausal constraints will be used. These include pseudo-Boolean constraints and cardinality constraints,
e.g. [87, 81, 28]. Examples of clausification approaches are described in [81, 28].

2.2 Monotone Predicates & MSMP

A predicate \( P : 2^\mathbb{R} \rightarrow \{0, 1\} \), defined on \( \mathbb{R} \), is said to be monotone (e.g. [13]) if whenever \( P(\mathcal{R}_0) \) holds, with \( \mathcal{R}_0 \subseteq \mathcal{R} \), then \( P(\mathcal{R}_1) \) also holds, with \( \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R} \). Observe that \( P(\mathcal{R}) \) can be assumed, but this is not required. Also, \( P(\mathcal{R}) \) can be tested with a single predicate test. Moreover, observe that, if there exists a set \( \mathcal{R}_0 \subseteq \mathcal{R} \) such that \( P(\mathcal{R}_0) \) holds, and \( P \) is monotone, then \( P(\mathcal{R}) \) also holds.

**Definition 2** Let \( P \) be a monotone predicate, and let \( \mathcal{M} \subseteq \mathcal{R} \) such that \( P(\mathcal{M}) \) holds. \( \mathcal{M} \) is minimal iff \( \forall \mathcal{M}' \subseteq \mathcal{M}, \neg P(\mathcal{M}') \).

**Example 1** It is simple to conclude that, given a finite set \( \mathcal{W} \), predicate \( P(\mathcal{W}) \triangleq [|\mathcal{W}| \geq K] \), for some \( K \geq 0 \), is monotone. In contrast, predicate \( P(\mathcal{W}) \triangleq [|\mathcal{W}| \mod 2 = 1] \) is not monotone.

**Definition 3 (MSMP Problem)** Given a monotone predicate \( P \), the Minimal Set over a Monotone Predicate (MSMP) problem consists in finding a minimal subset \( \mathcal{M} \) of \( \mathcal{R} \) such that \( P(\mathcal{M}) \) holds.

Monotone predicates were used in [13, 14] for describing algorithms for computing a prime implicate of CNF formula given a clause. The MSMP problem was introduced in [69]. [69] also showed that a number of additional function problems defined on propositional formulas could be reduced to the MSMP problem, including the function problems of computing minimal unsatisfiable subsets, minimal correction subsets and minimal models.

2.3 Function Problems

For many computational problems, the main goal is not to solve a decision problem, but to solve instead a function problem [79, Section 10.3]. The goal of a function problem is to compute some solution, e.g. a satisfying truth assignment in the case of SAT, the size of a prime implicate, the actual prime implicate, the largest number of clauses that any truth assignment can satisfy in a CNF formula, etc. This paper addresses function problems defined on propositional formulas, which are defined in Section 3. The main focus of the paper are function problems that can be related with computing a minimal set over a monotone predicate, i.e. function problems that can be reduced to the MSMP problem. Optimization problems will also be studied, with the purpose of illustrating the modeling flexibility of monotone predicates.

2.4 Problem Reductions

The notation \( A \leq_p B \) is used to denote that any instance of a function problem \( A \) can be solved by solving instead a polynomially-related instance of another function problem \( B \). In this paper, \( B \)

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1Function problems are also referred to as search problems [31, Section 5.1].
denotes the MSMP problem, and any reduction $A \leq_p B$ indicates that the solution to the function problem $A$ for a concrete instance $a$ can be obtained by computing a minimal set (subject to a monotone predicate) for some resulting concrete instance $b$ of the MSMP problem. Moreover, all reductions described in the paper are provided with enough detail to make it straightforward to conclude that the problem instances are polynomially related.

2.5 Related Work

The work in this paper is motivated by the use of monotone predicates for computing a prime implicate of a CNF formula given an implicate (e.g. a clause) [13, 14]. This work was recently extended to show that monotone predicates can be applied to other problems, namely MUSes, MCSes, minimal models and (also) prime implicants given a clause [69]. [69] also proposed the Progression algorithm for the MSMP problem.

A tightly related concept in computational complexity is hereditarity [17, 18], which was proposed in the 90s to develop lower and upper bounds on the query complexity of computing maximal solutions. A concrete example is the computation of minimal unsatisfiability and maximal satisfiability [17, 18]. In contrast, other function problems, e.g. minimal and maximal models, involve different definitions from those standard in AI (e.g. [7]). Moreover, one concern of this paper is to develop a rigorous characterization of a SAT solver as an oracle. As shown in Section 2, if the time spent on the SAT solver is ignored, then a SAT solver can viewed as a witness oracle [16].

The paper addresses function problems defined on Boolean formulas, which intersect a wide range of areas of research. References to related work, both on the actual function problems, and associated areas of research are included throughout the paper.

3 Target Function Problems

This section provides a brief overview of the function problems studied in the rest of the paper. All function problems studied in this section are defined on Boolean formulas. Nevertheless, the work can be extended to function problems defined on expressive domains, e.g. ILP, SMT, CSP, etc.

3.1 Problem Definitions

The function problems described below can be organized as follows: (i) minimal unsatisfiability (and maximal satisfiability); (ii) irredundant subformulas; (iii) maximal falsifiability (and minimal satisfiability); (iv) minimal and maximal models; (v) prime implicates and implicants; (vi) backbone literals; (vii) formula entailment; (viii) variable independence; and (ix) maximum autarkies. Afterwards, Section 4 shows that all these function problems can be reduced to the MSMP problem. In what follows, and unless otherwise stated, $\mathcal{F}$ denotes an arbitrary Boolean formula. In some specific cases, $\mathcal{F}$ is restricted to be in CNF (or in DNF), and this will be noted.
3.1.1 Minimal Unsatisfiability & Maximal Satisfiability

Minimal unsatisfiability and maximal satisfiability have been extensively studied in a number of contexts (e.g. [47, 59, 5, 76] and references therein). In this paper, the following definitions are used.

**Definition 4 (MUS; FMUS)** Let $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \models \bot$. $M \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $M \models \bot$ and $\forall M' \subseteq M$, $M' \not\models \bot$. FMUS is the function problem of computing an MUS of $\mathcal{F}$.

**Definition 5 (MCS; FMCS)** Let $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \models \bot$. $C \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus C \not\models \bot$ and $\forall C' \subseteq C$, $\mathcal{F} \setminus C' \models \bot$. FMCS is the function problem of computing an MCS of $\mathcal{F}$.

**Definition 6 (MSS; FMSS)** Let $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \models \bot$. $S \subseteq \mathcal{F}$ is a Maximal Satisfiable Subset (MSS) iff $S \not\models \bot$ and $\forall F \supseteq S$, $F \models \bot$. FMSS is the function problem of computing an MSS of $\mathcal{F}$.

The relationship between MCSes and MSSes is well-known, e.g. [11, 64]:

**Remark 1** $C$ is an MCS of $\mathcal{F}$ iff $S = \mathcal{F} \setminus C$ is an MSS of $\mathcal{F}$.

The MaxSAT problem is defined assuming unweighted clauses.

**Definition 7 (LMSS/MaxSAT)** Let $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \models \bot$. An MSS $S$ of $\mathcal{F}$ is a largest MSS (LMSS) iff for any MSS $S'$ of $\mathcal{F}$, $|S'| \leq |S|$. FLMSS (or MaxSAT) is the function problem of computing an LMSS of $\mathcal{F}$.

**Observation 1** Observe that MaxSAT is defined as the function problem of computing one largest MSS. In some contexts, other definitions are used, namely as the problem of computing the largest number of simultaneously satisfied clauses [32]. This distinction is quite significant for the unweighted case of MaxSAT. The definition used in this paper aims to model the function problem actually solved by modern MaxSAT solvers. Since practical MaxSAT solvers always compute a witness of the reported solution, this avoids the issue with reproducing a witness in the case an NP oracle were considered.

**Definition 8 (SMCS)** Let $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \models \bot$. An MCS $C$ of $\mathcal{F}$ is a smallest MCS (SMCS) iff for any MCS $C'$ of $\mathcal{F}$, $|C'| \geq |C|$. FSMCS is the function problem of computing an SMCS of $\mathcal{F}$.

**Remark 2** Clearly, $S \subseteq \mathcal{F}$ is an LMSS of $\mathcal{F}$ iff $C = \mathcal{F} \setminus S$ is an SMCS of $\mathcal{F}$.

Moreover, it is straightforward to extend the above definitions to cases where there are hard clauses (i.e. clauses that cannot be dropped from the formula) and where soft clauses have weights [59]. There exists a large body of theoretical and practical work on computing MUSes, MCSes and on solving MaxSAT, including a well-known minimal hitting set duality relationship between MUSes and MCSes [84, 11, 64]. Recent references describing new algorithms and surveys for these problems include [32, 59, 67, 5, 1, 68, 76]. There is a growing number of practical applications of both minimal unsatisfiability and maximal satisfiability (e.g. see [32, 67, 5, 76] and references therein).
Recent examples of applications include [88, 42, 43, 50, 40, 34]. Extraction of MUSes for non-clausal formulas and for SMT formulas is addressed in [47, 6, 102, 50, 40, 34].

Finally, we should note that similar definitions can be developed for DNF formulas, by considering minimal validity instead of minimal unsatisfiability. This would also allow developing several tightly related function problems, as above.

3.1.2 Irredundant Subformulas

The definitions for minimal unsatisfiability (see previous section) can be reformulated for the case where the goal is to remove redundancy from a CNF formula, as follows.

Definition 9 (MES; FMES) Let $F \in \mathbb{C}$. $E \subseteq F$ is a Minimal Equivalent Subset (MES) iff $E \equiv F$ and $\forall E' \subseteq E$, $E' \not\equiv F$. FMES is the function problem of computing an MES of $F$.

Definition 10 (MDS; FMDS) Let $F \in \mathbb{C}$. $D \subseteq F$ is a Minimal Distinguishing Subset (MDS) iff $F \setminus D \not\equiv F$ and $\forall D' \subseteq D$, $F \setminus D' \equiv F$. FMDS is the function problem of computing an MDS of $F$.

Definition 11 (MNS; FMNS) Let $F \in \mathbb{C}$. $N \subseteq F$ is a Maximal Non-equivalent Subset (MNS) iff $N \not\equiv F$ and $\forall F \supseteq N'$ $N' \supseteq N$, $N' \equiv F$. FMNS is the function problem of computing an MNS of $F$.

Remark 3 $D$ is an MDS of $F$ iff $N = F \setminus D$ is an MNS of $F$.

Definition 12 (LMNS) Let $F \in \mathbb{C}$. An MNS $N$ of $F$ is a largest MNS (LMNS) iff for any MNS $N'$ of $F$, $|N'| \leq |N|$. FLMNS is the function problem of computing an LMNS of $F$.

Definition 13 (SMDS) Let $F \in \mathbb{C}$. An MDS $D$ of $F$ is a smallest MDS (SMDS) iff for any MDS $D'$ of $F$, $|D'| \geq |D|$. FSMDS is the function problem of computing an SMDS of $F$.

Remark 4 Clearly, $N \subseteq F$ is an LMNS of $F$ iff $D = F \setminus N$ is an SMDS of $F$.

Moreover, it is straightforward to extend the above definitions to cases where there are hard clauses (i.e. clauses that cannot be dropped from the formula) and where soft clauses have weights.

Complexity characterizations of computing irredundant subformulas were studied in [61, 62, 63, 56]. Recent practical algorithms include [12, 19, 4].

Finally, we should note that definitions of irredundancy can be developed for DNF formulas in terms of a subset-minimal set of terms equivalent to the original formula. As before, this would allow defining additional function problems, as above.

3.1.3 Minimal Satisfiability & Maximal Falsifiability

In some settings, the goal is to minimize the number of satisfied clauses. This is generally referred to as the Minimum Satisfiability (MinSAT) problem [49]. By analogy with the MaxSAT case, one can also consider extremal sets [38].
**Definition 14 (All-Falsifiable)** Let $F \in \mathbb{C}$. $U \subseteq F$ is All-Falsifiable if there exists a truth assignment $\nu$, to $\text{var}(U)$, such that $\nu$ falsifies all clauses in $U$.

**Definition 15 (MFS; FMFS)** Given $F \in \mathbb{C}$, a Maximal Falsifiable Subset (MFS) of $M \subseteq F$ is all-falsifiable and, $\forall F \supseteq N \supseteq M$, $N$ is not all-falsifiable. FMFS is the function problem of computing an MFS of $F$.

**Definition 16 (MCFS; FMCFS)** Given $F \in \mathbb{C}$, a Minimal Correction (for Falsifiability) Subset (MCFS) is a set $C \subseteq F$ such that $F \setminus C$ is all-falsifiable and $\forall C' \subseteq C$, $F \setminus C'$ is not all-falsifiable. FMCFS is the function problem of computing an MCFS of $F$.

**Remark 5** $M$ is an MFS of $F$ iff $N = F \setminus D$ is an MCFS of $F$.

**Definition 17 (FLMFS/MaxFalse)** Let $F \in \mathbb{C}$. An MFS $M$ of $F$ is a largest MFS (LMFS) iff for any MFS $M'$ of $F$, $|M'| \leq |M|$. FLMFS (or Maximum Falsifiability, MaxFalse) is the function problem of computing an LMFS of $F$.

**Definition 18 (FSMCFS/MinSAT)** Let $F \in \mathbb{C}$. An MCFS $C$ of $F$ is a smallest MFS (SMCFS) iff for any MCFS $C'$ of $F$, $|C'| \geq |C|$. FSMCFS (or Minimum Satisfiability, MinSAT) is the function problem of computing an SMCFS of $F$.

**Remark 6** Clearly, $M \subseteq F$ is an LMFS of $F$ iff $C = F \setminus M$ is an SMCFS of $F$.

**Remark 7** Given Remark 5, one can conclude that the MinSAT problem consists of computing the smallest MCFS, and this represents the complement of the MaxFalse solution. Thus, for $F \in \mathbb{C}$, where $n_t$ and $n_f$ denote respectively the MinSAT and the MaxFalse solutions, then $|F| = n_t + n_f$.

As with minimal unsatisfiability and irredundancy, maximal falsifiability can be generalized to the case when some clauses are hard and when soft clauses have weights. Similarly, one could consider the DNF formulas, and function problems related with all-true terms.

The MinSAT problem has been studied in [49, 66, 2, 60]. The problem of maximal falsifiability is studied in [38].

Moreover, it should be noted that, although this paper addresses MUSes/MCSes/MSSes/MESe/-MDSe/MSes/MFSes/MCFSes defined solely on sets of clauses, other variants could be considered, namely groups of clauses or variables. The formalizations of these variants as instances of the MSMP problem mimic the formalizations developed for the case of sets of clauses. (See [64, 25, 77, 35, 3, 36] for related work on groups of clauses and variables.)

### 3.1.4 Minimal & Maximal Models

Minimal and maximal models are additional examples of function problems associated with propositional formulas.
Definition 19 (Minimal Model; FMnM) Given $F \in \mathbb{F}$, a model $M \subseteq X$ of $F$ is minimal iff $\nu(M, X) \models F$ and $\forall M' \supseteq M$, $\nu(M', X) \not\models F$. FMnM is the function problem of computing a minimal model of $F$.

Definition 20 (Maximal Model; FMxM) Given $F \in \mathbb{F}$, a model $M \subseteq X$ of $F$ is maximal iff $\nu(M, X) \models F$ and $\forall F \supseteq M'$, $\nu(M', X) \not\models F$. FMxM is the function problem of computing a maximal model of $F$.

Observation 2 Both minimal and maximal models can be computed subject to a set $Z$ of variables other than $X \triangleq \text{var}(F)$ [7], i.e. the so-called $Z$-minimal and $Z$-maximal models. The above definitions can easily be modified for this more general definition.

Minimal models find a wide range of applications, including non-monotonic reasoning and bioinformatics (e.g. [29, 95]). Algorithms for computing minimal and maximal models have been studied in the past (e.g. [7, 78, 8, 45, 51]). In some contexts, minimal and maximal models have been referred to as MIN-ONE$_\subseteq$ and MAX-ONE$_\subseteq$ solutions (e.g. [86]).

3.1.5 Implicants & Implicates

Two relevant function problems associated with propositional formulas consist of computing prime implicants, starting from an implicant represented as a term, and prime implicants, starting from an implicate represented as a clause.

Definition 21 (Prime Implicant (given term); FPIt) Given $F \in \mathbb{F}$ and an implicant $t \in \mathbb{A}$ of $F$, term $u \subseteq t$ is a prime implicant of $F$ iff $u \models F$ and $\forall v \subseteq u$, $v \not\models F$. FPIt is the function problem of computing a prime implicant of $F$ given an implicant $t$ of $F$.

Definition 22 (Prime Implicate (given clause); FPIc) Given $F \in \mathbb{F}$ and an implicate $c \in \mathbb{A}$ of $F$, clause $p \subseteq c$ is a prime implicate of $F$ iff $F \models p$ and $\forall q \subseteq p$, $F \not\models q$. FPIc is the function problem of computing a prime implicate of $F$ given an implicate $c$ of $F$.

Observe that computing a prime implicant from a given implicant for a formula in CNF can be done in linear time (e.g. [83, 24]). Similarly, computing a prime implicate from a given implicate for a formula in DNF can also be done in linear time.

For the general case of arbitrary Boolean formulas these function problems become significantly harder, and the algorithms described in this paper require a number of SAT solver calls that is linear in the number of variables in the worst case (see [18, 98] for similar approaches and conclusions).

Prime implicants and implicates find many practical applications, including truth maintenance systems [85, 23], knowledge compilation [22], conformant planning [96], the simplification of Boolean functions [82, 73], abstraction in model checking [14, 13], minimization of counterexamples in model checking [83, 91, 90], among many others. A wealth of algorithms exist for computing prime implicants and implicants (e.g. [72] and references therein).
Another problem of interest in practice is the Longest Extension of Implicant problem \([98]\) for DNF formulas. By analogy, we also consider the Longest Extension of Implicate problem for CNF formulas.

**Definition 23 (Longest Extension of Implicant; FLEIt)** Let \( t \) be an implicant of \( F \in \mathbb{D} \). Term \( A \supseteq u \supseteq t \) is a longest extension of implicant \( t \) iff \( (F \setminus \{t\}) \cup \{u\} \equiv F \) and \( \forall_{u' \supseteq u}, (F \setminus \{t\}) \cup \{u'\} \neq F \). The FLEIt function problem consists of computing a longest extension of implicant \( t \) of \( F \).

**Definition 24 (Longest Extension of Implicate; FLEIc)** Let \( c \) be an implicate of \( F \in \mathbb{C} \). Clause \( A \supseteq u \supseteq c \) is a longest extension of implicate \( c \) iff \( (F \setminus \{c\}) \cup \{u\} \equiv F \) and \( \forall_{u' \supseteq u}, (F \setminus \{c\}) \cup \{u'\} \neq F \). The FLEIc function problem consists of computing a longest extension of implicate \( c \) of \( F \).

### 3.1.6 Formula Entailment

A number of problems related with formula entailment can also be considered. We address two function problems: computing a minimal set entailing a formula and computing a maximal set entailed by a formula.

**Definition 25 (Minimal Entailing Subset; FMnES)** Let \( J \in \mathbb{C} \) and \( I \in \mathbb{F} \) be such that \( J \models I \). \( M \subseteq J \) is a Minimal Entailing Subset (MnES) of \( I \) iff \( M \models I \) and \( \forall_{M' \subsetneq M} M' \not\models I \). FMnES is the function problem of computing a minimal entailing subset of \( J \) given \( I \).

**Observation 3** The function problem FMnES from Definition 25 can be generalized to capture the case of logic-based abduction \([30]\) when the universe of hypotheses is consistent with the theory, and the minimality criterion is subset minimality.

**Definition 26 (Maximal Entailed Subset; FMxES)** Let \( J \in \mathbb{F} \) and \( N \in \mathbb{C} \) be such that \( J \not\models N \). \( I \subseteq N \) is a Maximal Entailed Subset (MxES) iff \( J \models I \) and \( \forall_{N' \supseteq I} N' \not\models I \). FMxES is the function problem of computing a maximal entailed subset of \( N \) given \( J \).

### 3.1.7 Backbone Literals

The problem of computing the backbone of a Boolean formula, i.e. the literals that are common to all satisfying assignments of the formula, finds a wide range of applications. Two definitions are considered.

**Definition 27 (Backbone; FBB)** Given \( F \in \mathbb{F} \), the backbone of \( F \) is a maximal set of literals \( B \in \mathbb{A} \) which are true in all models of \( F \), i.e. \( F \models \land_{l \in B}(l) \). FBB is the function problem of computing the backbone of \( F \in \mathbb{F} \).

In practice, algorithms for computing the backbone of a formula often start from a reference model \([70]\). This allows a slightly different formulation of the backbone function problem.
Definition 28 (FBBr) Let \( \nu \) be a model of \( F \in \mathbb{F} \) and \( V \) the associated set of literals. The backbone of \( F \) is a maximal set of literals \( B \subseteq V \) such that \( F \models \bigwedge_{l \in B}(l) \). FBBr is the function problem of computing the backbone of \( F \in \mathbb{F} \) given \( V \).

Backbones of Boolean formulas were first studied in the context of work on phase transitions and problem hardness [74, 94, 46]. In addition, backbones find several practical applications, that include configuration [92, 93], abstract argumentation [101], cyclic causal models [37], debugging of fabricated circuits [103], among others. A number of recent algorithms have been proposed for computing the backbone of Boolean formulas [44, 39, 70, 104]. Recently, generalized backbones were studied in [20].

3.1.8 Variable Independence

A formula \( F \) is (semantically) independent of a variable \( x \) if the set of models of \( F \) does not change by fixing \( x \) to any truth value [58, Def. 4, Prop. 7].

Definition 29 (Variable Independence; FVInd) A \( F \in \mathbb{F} \) is independent from \( x \in \text{var}(F) \) iff \( F \equiv F_{x=0} \) (or, \( F \equiv F_{x=1} \)). FVInd is the function problem of computing a maximal set of variables of which \( F \) is independent from.

Observe that, as indicated earlier, the definition of MESes can be extended to variables. However, FVInd is defined in a more general setting, since \( F \) need not be in CNF. Variable independence (and also literal independence) are important in a number of settings [58]. Variables declared independent are also known as redundant or inessential, e.g. [26, 15, 21]. In later sections, and for simplicity, if \( F \) is independent from \( x \), then we write that \( x \) is redundant for \( F \).

3.1.9 Maximum Autarkies

This section provides a brief overview of the problem of identifying the (maximum) autarkies of unsatisfiable CNF formulas.

Definition 30 (Autarky; FAut) Given \( F \in \mathbb{C} \), with \( F \models \bot \), a set \( A \in \text{var}(F) \) is an autarky iff there exists a truth assignment to the variables in \( A \) that satisfies all clauses containing literals in the variables of \( A \). FAut is the function problem of computing the maximal autarky of \( F \).

Autarkies were first proposed in the context of improving exponential upper bounds of SAT algorithms [75, 100], and have later been studied in the context of minimal unsatisfiable subformulas [53, 47, 55]. More recently, autarkies were used in model elimination [99] and for speeding up MCS/MUS enumeration algorithms [65]. Algorithms for computing autarkies were studied in [54, 57, 65, 47].
3.2 Properties

This section summarizes properties of some of the function problems presented in the previous section, which are essential for some of the results presented later.

For some function problems, the number of maximal/minimal sets is very restricted. In fact, for FLEIt, FLEIc, FMxES, FBBr, FBB and FAut, the following holds.

**Proposition 5** For the function problems FLEIt, FLEIc, FMxES, FBB, FBBr, and FAut there is a unique maximal set, which is maximum.

**Proof.** For each case, the proof is by contradiction.

1. For FLEIt, let $u$ be a subset-maximal extension of $t$ such that $u \models F$, and let $l_x$ be a literal not included in $u$ such that $t \land l_x \not\models F$. Then, $u \land l_x \not\models F$; a contradiction.

2. For FLEIc the proof is similar to the previous case.

3. For FMxES, let $I$ be a maximal set such that $J \models I$, and assume there exists clause $c \in \mathcal{N} \setminus I$ such that $J \models c$. Then, any model of $J$ satisfies both $I$ and $c$, and so $J \models I \cup \{c\}$; a contradiction.

4. For FBB/FBBr, the proof is similar. Let $B$ denote a maximal set of backbone literals, and let $l \not\in B$ be a backbone literal. Then, for any model of the formula, all literals in $B \cup \{l\}$ are true; a contradiction.

5. For FAut, the proof is again similar. Let $A_1$ be a maximal autarky, let $A_2$ be another autarky, and let $A_2 \setminus A_1 \neq \emptyset$. Then, $K = A_1 \cup A_2$ is an autarky and $A_1 \subseteq K$; a contradiction.

Thus, for FLEIt, FLEIc, FMxES, FBB, FBBr and FAut there is a unique maximal set which is maximum.

**Proposition 6** A formula $F$ is independent from $x_i \in X$ iff $F \equiv F[x_i/y_i]$, where $y_i$ is a new variable with $y_i \not\in X$.

**Proof.** By definition, $F$ is independent from $x_i$ iff $F \equiv F_{x=0}$ or $F \equiv F_{x=1}$. Hence, $\forall y_i \in \{0,1\} F \equiv F[x_i/y_i]$; with $y_i \not\in X$. Thus, $F \equiv F[x_i/y_i]$ with $y_i \not\in X$.

4 Reductions to MSMP

This section shows how each of the function problems defined in Section 3 can be represented as an instance of the MSMP problem. Section 4.1 introduces general predicate forms, which are shown to be monotone, and which simplify the presentation of the reductions in the Section 4.2. Section 4.2 is structured similarly to Section 3.1: (i) minimal unsatisfiability (and maximal satisfiability); (ii) irredundant subformulas; (iii) maximal falsifiability (and minimal satisfiability); (iv) minimal and maximal models; (v) prime implicates and implicants; (vi) backbone literals; (vii) formula entailment; (viii) variable independence; and (ix) maximum autarkies.
4.1 Predicate Forms

In the next section, several function problems are reduced to the MSMP. The reduction involves specifying a reference set $R$ and a monotone predicate $P$ defined in terms of a working set $W \subseteq R$. To simplify the description of the different monotone predicates, this section develops general predicate forms, which capture all of the monotone predicates developed in the next sections, and proves that all predicates of any of these forms are monotone. As a result, for any concrete predicate, monotonicity is an immediate consequence of the monotonicity of the general predicate forms.

Let element $u_i \in R$ represent either a literal or a clause. Moreover, $\sigma(u_i)$ represents a Boolean formula built from $u_i$, where new variables may be used, but such that $u_i$ is the only element from $R$ used in $\sigma(u_i)$. For example, $\sigma(u_i)$ can represent the negation of a literal or a clause, etc. Let $\mathcal{G}$ be a propositional formula that is independent from the elements in $W$, i.e. $\mathcal{G}$ does not change with $W$. Then, the following general predicate forms are defined.

**Definition 31 (Predicates of Form $L$)** A predicate is of form $L$ iff its general form is given by,

$$P(W) \triangleq \text{SAT}(\mathcal{G} \land \land_{u_i \in R \setminus W} (\sigma(u_i)))$$

(1)

**Definition 32 (Predicates of Form $P$)** A predicate is of form $P$ iff its general form is given by,

$$P(W) \triangleq \lnot\text{SAT}(\mathcal{G} \land \land_{u_i \in W} (\sigma(u_i)))$$

(2)

**Definition 33 (Predicates of Form $B$)** A predicate is of form $B$ iff its general form is given by,

$$P(W) \triangleq \lnot\text{SAT}(\mathcal{G} \land (\lor_{u_i \in R \setminus W} (\sigma(u_i))))$$

(3)

**Proposition 7** Predicates of the forms $L$, $B$ and $P$ are monotone.

*Proof.*

1. Let $P$ be a predicate of form $L$. Let $P(R_0)$ hold, with $R_0 \subseteq R$, i.e. the argument to the SAT oracle is satisfiable. Then, by Proposition 1, for any $R_1$, with $R_1 \supseteq R_0$ (and so $R \setminus R_1 \subseteq R \setminus R_0$), $P(R_1)$ also holds, since the argument to the SAT oracle call is the conjunction of a subset of the constraints used for the case of $R_0$, and so also satisfiable. Thus, $P$ is monotone.

2. Let $P$ be a predicate of form $P$. Let $P(R_0)$ hold, with $R_0 \subseteq R$, i.e. the argument to the SAT oracle is unsatisfiable. Then, by Proposition 2, for any $R_1$, with $R_1 \supseteq R_0$ (and so $R \setminus R_1 \subseteq R \setminus R_0$), $P(R_1)$ also holds, since the argument to the SAT oracle call is the conjunction of a superset of the constraints used for the case of $R_0$, and so also unsatisfiable. Thus, $P$ is monotone.

3. Let $P$ be a predicate of form $B$. Let $P(R_0)$ hold, with $R_0 \subseteq R$, i.e. the argument to the SAT oracle is unsatisfiable. Then, for any $R_1$, with $R_1 \supseteq R_0$ (and so $R \setminus R_1 \subseteq R \setminus R_0$), $P(R_1)$ also holds, since the clause created from $R \setminus R_1$ has fewer elements than for the case of $R \setminus R_0$, and so it is also unsatisfiable (i.e. all literals in the clause will also be resolved away). Thus, $P$ is monotone.
4.2 Minimal Sets

In the remainder of this section the reference set is denoted $\mathcal{R}$ and the monotone predicate $P$ is defined in terms of a working set $\mathcal{W} \subseteq \mathcal{R}$. All function problems considered are defined in Section 3.

4.2.1 Minimal Unsatisfiability & Maximal Satisfiability

**Proposition 8** $\text{FMUS} \leq_p \text{MSMP}$.

**Proof.**

*Reduction.* The reduction is defined as follows. $\mathcal{R} \triangleq \mathcal{F}$ and,

$$P(\mathcal{W}) \triangleq \neg \text{SAT}(\land_{c \in \mathcal{W}}(c))$$

with $\mathcal{W} \subseteq \mathcal{R}$.

*Monotonicity.* The predicate (see (4)) is of form $\mathcal{P}$, with $G \triangleq \emptyset$, $u_i \triangleq c$, and $\sigma(c) \triangleq c$. Thus, by Proposition 7 the predicate is monotone.

*Correctness.* Let $\mathcal{M}$ be a minimal set for which $P(\mathcal{M})$ holds, i.e. $\mathcal{M}$ is unsatisfiable. By Definition 2, since $\mathcal{M}$ is minimal for $P$, then for any $\mathcal{M}' \subset \mathcal{M}$ the predicate does not hold, i.e. $\mathcal{M}'$ is satisfiable. Thus, by Definition 4, $\mathcal{M}$ is an MUS of $\mathcal{F}$.

**Proposition 9** $\text{FMCS} \leq_p \text{MSMP}$.

**Proof.**

*Reduction.* The reduction is defined as follows. $\mathcal{R} \triangleq \mathcal{F}$ and,

$$P(\mathcal{W}) \triangleq \text{SAT}(\land_{c \in \mathcal{R} \setminus \mathcal{W}}(c))$$

with $\mathcal{W} \subseteq \mathcal{R}$.

*Monotonicity.* The predicate (see (5)) is of form $\mathcal{L}$, with $G \triangleq \emptyset$, $u_i \triangleq c$, and $\sigma(c) \triangleq c$. Thus, by Proposition 7 the predicate is monotone.

*Correctness.* Let $\mathcal{M}$ be a minimal set such that $P(\mathcal{M})$ holds. By Definition 2, since $\mathcal{M}$ is minimal for $P$, then for any $\mathcal{M}' \subset \mathcal{M}$ the predicate does not hold, i.e. $\mathcal{R} \setminus \mathcal{M}'$ is unsatisfiable. Hence, $\mathcal{F} \setminus \mathcal{M}$ is satisfiable, and for any $\mathcal{M}' \subset \mathcal{M}$, $\mathcal{F} \setminus \mathcal{M}'$ is unsatisfiable. Thus, by Definition 5, $\mathcal{M}$ is an MCS of $\mathcal{F}$.

**Remark 8** By Remark 1, an MSS of $\mathcal{F} \in \mathcal{C}$ can be computed as follows. Compute an MCS $\mathcal{M}$ of $\mathcal{F}$ and return $\mathcal{F} \setminus \mathcal{M}$.

4.2.2 Irredundant Subformulas

**Proposition 10** $\text{FMES} \leq_p \text{MSMP}$.
Reduction. The reduction is defined as follows. $R \triangleq \mathcal{F}$ and,

$$P(W) \triangleq \neg \text{SAT} (\neg \mathcal{F} \land \land_{c \in W} (c))$$

with $W \subseteq R$.

Monotonicity. The predicate (see (6)) is of form $\mathcal{P}$, with $G \triangleq \neg \mathcal{F}$, $u_i = c$, and $\sigma(c) \triangleq c$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Let $M$ be a minimal set such that $P(M)$ holds. By Definition 2, since $M$ is minimal for $P$, then $P$ holds for $M$, i.e. $M \models \mathcal{F}$, and for any $M' \subseteq M$ the predicate does not hold, i.e. $M' \not\models \mathcal{F}$. Thus, by Definition 9, $M$ is an MES of $\mathcal{F}$.

Observation 4 Observe that the problem of computing an irredundant subformula can be reduced to the problem of computing a (group) MUS $[4]$. Thus, an MUS for the resulting problem is an MES for the original problem.

Also, note that the argument of the predicate in (6) can be simplified:

$$
\neg \mathcal{F} \land \land_{c \in W} (c) \Leftrightarrow (\lor_{c \in \mathcal{F}} (\neg c)) \land \land_{c \in W} (c) \\
\Leftrightarrow (\lor_{c \in \mathcal{F} \setminus W} \neg c) \land \land_{c \in W} (c)
$$

Thus, the predicate in (6) can be formulated as follows:

$$P(W) \triangleq \neg \text{SAT} (\neg (\mathcal{F} \setminus W) \land \land_{c \in W} (c))$$

(7)

Throughout the paper, (6) is used, since it facilitates relating this predicate with others. However, for practical purposes (7) would be preferred.

Proposition 11 $\text{FMDS} \leq_p \text{MSMP}$.

Proof.

Reduction. The reduction is defined as follows. $R \triangleq \mathcal{F}$ and,

$$P(W) \triangleq \text{SAT} (\neg \mathcal{F} \land \land_{c \in \mathcal{F} \setminus W} (c))$$

(8)

with $W \subseteq R$.

Monotonicity. The predicate (see (8)) is of form $\mathcal{P}$, with $G \triangleq \neg \mathcal{F}$, $u_i = c$, and $\sigma(c) \triangleq c$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Let $D$ be a minimal set such that $P(D)$ holds. By Definition 2, since $D$ is minimal for $P$, then $P$ holds for $D$, i.e. $\mathcal{F} \setminus D \not\models \mathcal{F}$, and for any $D' \subseteq D$ the predicate does not hold, i.e. $\mathcal{F} \setminus D' \not\models \mathcal{F}$. Thus, by Definition 10, $D$ is an MDS of $\mathcal{F}$. 


Observation 5  The problem of computing an irredundant subformula can be reduced to the problem of computing a (group) MUS (see Observation 4 and [4]), and so an MCS for the resulting problem is an MDS for the original problem.

Similarly to the FMES case, the argument to the predicate in (8) can be simplified:

\[ \neg \mathcal{F} \land \land_{c \in \mathcal{R} \setminus \mathcal{W}} (c) \iff (\lor_{c \in \mathcal{F}} (\neg c)) \land \land_{c \in \mathcal{F} \setminus \mathcal{W}} (c) \iff \neg \mathcal{W} \land \land_{c \in \mathcal{R} \setminus \mathcal{W}} (c) \]

Thus, the predicate in (8) can be formulated as follows:

\[ P(W) \triangleq \text{SAT}(\neg \mathcal{W} \land \land_{c \in \mathcal{R} \setminus \mathcal{W}} (c)) \quad (9) \]

Throughout the paper, (8) is used, since it facilitates relating this predicate with others. However, for practical purposes (9) would be preferred.

Remark 9  By Remark 3, an MNS of \( \mathcal{F} \in \mathcal{C} \) can be computed as follows. Compute an MDS \( \mathcal{D} \) of \( \mathcal{F} \) and return \( \mathcal{F} \setminus \mathcal{D} \).

4.2.3 Minimal Satisfiability & Maximal Falsifiability

Proposition 12  \( \text{FM} \leq_p \text{MSMP} \).

Proof.

Reduction. The reduction is defined as follows. \( \mathcal{R} \triangleq \mathcal{F} \) and,

\[ P(W) \triangleq \text{SAT}(\land_{c \in \mathcal{R} \setminus \mathcal{W}} (\neg c)) \quad (10) \]

with \( \mathcal{W} \subseteq \mathcal{R} \).

Monotonicity. The predicate (see (10)) is of form \( \mathcal{L} \), with \( \mathcal{G} \triangleq \emptyset \), \( u_i \triangleq c \), and \( \sigma(c) \triangleq \neg c \). Thus, by Proposition 7 the predicate is monotone.

Correctness. Let \( \mathcal{M} \) be a minimal set such that \( P(\mathcal{M}) \) holds. By Definition 2, since \( \mathcal{M} \) is minimal for \( P \), then for any \( \mathcal{M}' \subseteq \mathcal{M} \) the predicate does not hold, i.e. \( \mathcal{F} \setminus \mathcal{M} \) is all-falsifiable, and for any \( \mathcal{M}' \subseteq \mathcal{M} \), \( \mathcal{F} \setminus \mathcal{M}' \) is not all-falsifiable. Thus, by Definition 15, \( \mathcal{M} \) is an MCFS of \( \mathcal{F} \).

Remark 10  By Remark 5, an MFS of \( \mathcal{F} \in \mathcal{C} \) can be computed as follows. Compute an MCFS \( \mathcal{C} \) of \( \mathcal{F} \) and return \( \mathcal{F} \setminus \mathcal{C} \).

4.2.4 Minimal & Maximal Models

Proposition 13  \( \text{FM} \leq_p \text{MSMP} \).
Proof.

Reduction. The reduction is defined as follows. \( R \triangleq X \triangleq \text{var}(F) \) and,

\[
P(W) \triangleq \text{SAT}(F \land \land_{x \in R \setminus W} (\neg x))
\]  

with \( W \subseteq R \).

Monotonicity. The predicate (see (11)) is of form \( \varphi \), with \( G \triangleq F \), \( u_i \triangleq x \), and \( \sigma(x) \triangleq \neg x \). Thus, by Proposition 7 the predicate is monotone.

Correctness. Let \( M \) be a minimal set such that \( P(M) \) holds. By Definition 2, since \( M \) is minimal for \( P \), then for any \( M' \subsetneq M \) the predicate does not hold. By observing that \( M \) denotes the set of variables that can be assigned value 1 (since the other variables must be assigned value 0) and, because \( M \) is minimal, \( M \) cannot be further reduced. Thus, by Definition 19, \( M \) is a minimal model of \( F \).

**Proposition 14** \( \text{FMxM} \leq_p \text{MSMP} \).

**Proof.** Let \( F^C \) be constructed from \( F \) by flipping the polarity of all literals in \( F \), i.e. replace \( l \) with \( \neg l \) for \( l \in \{ x, \neg x \mid x \in \text{var}(F) \} \). Observe that, \( F \) and \( F^C \) have the same parse tree excluding the leaves. Now, compute a minimal model \( M^C \) for \( F^C \), e.g. using Proposition 13. Then, as shown next, a maximal model for \( F \) is given by \( \text{var}(F) \setminus M^C \).

Let \( M^C \) be any model of \( F^C \), and let \( \nu^C(M^C, X) \) denote the associated truth assignment. Consider the truth assignment \( \nu \) obtained by flipping the value of all variables, and let \( M \) denote the variables assigned value 1, i.e. \( M = X \setminus M^C \). Clearly, \( M^C \) is minimal iff \( M \) is maximal. Moreover, let \( F \) be obtained from \( F^C \) by complementing all of its literals. Thus, the leaves of the parse tree of \( F^C \) are complemented and the values assigned to the leaves are also complemented. Now, recall that, with the exception of the leaves, both \( F \) and \( F^C \) have the same parse tree, and the leaves are assigned the same values in both cases. Thus, by structural induction it follows that \( \nu^C \models F^C \) iff \( \nu \models F \).

**4.2.5 Implicants & Implicates**

**Proposition 15** \( \text{FPIt} \leq_p \text{MSMP} \).

**Proof.**

Reduction. The reduction is defined as follows. \( R \triangleq L(t) \triangleq \{ l \mid l \in t \} \) and,

\[
P(W) \triangleq \neg \text{SAT}(\neg F \land \land_{l \in W} (l))
\]  

with \( W \subseteq R \).

Monotonicity. The predicate (see (12)) is of form \( \varphi \), with \( G \triangleq \neg F \), \( u_i \triangleq l \), and \( \sigma(l) \triangleq l \). Thus, by Proposition 7 the predicate is monotone.

Correctness. Let \( M \) be a minimal set such that \( P(M) \) holds. Thus, the literals in \( M \) entail \( F \), and for any proper subset \( M' \) of \( M \), the literals in \( M' \) do not entail \( F \). Thus \( M \) is a prime implicant of
Similarly, we can reduce the computation of a prime implicate given a clause to MSMP. (This reduction of FPIc to MSMP was first described in [13, 14].)

**Proposition 16** FPIc ≤ₚ MSMP.

**Proof.**

*Reduction.* The reduction is defined as follows. \( R \triangleq \{ l \mid l \in c \} \) and

\[
P(W) \triangleq \neg\text{SAT}(F \land \land_{l \in W}(-l))
\]

with \( W \subseteq R \).

**Monotonicity.** The predicate (see (13)) is of form \( \mathcal{D} \), with \( G \triangleq F \), \( u_i \triangleq l \), and \( \sigma(l) \triangleq \neg l \). Thus, by **Proposition 7** the predicate is monotone.

**Correctness.** Let \( M \) be a minimal set such that \( P(M) \) holds. Thus, \( F \) entails the literals in \( M \), and for any proper superset \( M' \) of \( M \), \( F \) does not entail the literals in \( M' \). Thus \( M \) is a prime implicite of \( F \).

Regarding FLEIt, \( F \) is in DNF, \( F = \lor_{j=1}^{m} t_j \), and let the implicant to extend be \( t_k \), with \( 1 \leq k \leq m \). The literals that can be used to extend \( t_k \) are \( L_t \triangleq \{ l \mid l \in L \land \{ l \}, \land t_k = \emptyset \} \). Moreover, let \( D = \lor_{j=1, j \neq k}^{m} t_j \). Define \( F^{\mathcal{X}} \triangleq F \land (\neg D) \), which can be simplified to \( F^{\mathcal{X}} \triangleq t_k \land \land_{i=1, i \neq k}^{m} (-t_i) \).

**Proposition 17** FLEIt ≤ₚ MSMP.

**Proof.** (Sketch)

*Reduction.* The reduction is defined as follows. \( R \triangleq L_t \) and

\[
P(W) \triangleq \neg\text{SAT}(F^{\mathcal{X}} \land (\lor_{l \in R \setminus W} l))
\]

with \( W \subseteq R \).

**Monotonicity.** The predicate (see (14)) is of form \( \mathcal{D} \), with \( G \triangleq F^{\mathcal{X}} \), \( u_i \triangleq l \), and \( \sigma(l) \triangleq \neg l \). Thus, by **Proposition 7** the predicate is monotone.

**Correctness.** Let \( u = t_k \land q \), where \( q \) is a term. Clearly, since \( t_k \models F \), then \( u \models F \), and so \( D \lor u \models F \). The issue is whether \( F \models D \lor u \), or equivalently \( F \land (\neg D) \land (\neg u) \models \bot \). Expanding we get \( t_k \land \land_{i=1, i \neq k}^{m} (-t_i) \land (\neg t_k \lor \neg q) \models \bot \), which can also be simplified to \( t_k \land \land_{i=1, i \neq k}^{m} (-t_i) \land (\neg q) \models \bot \). Hence, the goal is to find a maximal set of literals \( q \) such that \( t_k \land \land_{i=1, i \neq k}^{m} (-t_i) \land (\neg q) \models \bot \). This can be converted to a minimization problem by removing all literals and then adding to \( q \) literals that can be included.

Regarding FLEIc, \( F \) is in CNF, \( F = \land_{j=1}^{m} (c_j) \), and let the implicate to extend be \( c_k \), with \( 1 \leq k \leq m \). The literals that can be used to extend \( c_k \) are \( L_c \triangleq \{ l \mid l \in L \land \{ l \}, \land c_k = \emptyset \} \). Moreover, let \( D = \land_{j=1, j \neq k}^{m} (c_j) \). Define \( F^{\mathcal{X}} \triangleq (\neg F) \land (D) \), which can be simplified to \( F^{\mathcal{X}} \triangleq (\neg c_k) \land \land_{i=1, i \neq k}^{m} (c_i) \).

**Proposition 18** FLEIc ≤ₚ MSMP.
Proof. (Sketch)

Reduction. The reduction is defined as follows. $\mathcal{R} \triangleq \mathcal{L}$ and,

$$P(W) \triangleq \neg \text{SAT}(\mathcal{F}^{\text{in}} \land (\forall l \in \mathcal{W}) l) \quad (15)$$

with $\mathcal{W} \subseteq \mathcal{R}$.

Monotonicity. The predicate (see (15)) is of form $\mathscr{B}$, with $\mathcal{G} \triangleq \mathcal{F}^{\text{in}}$, $u_i \triangleq l$, and $\sigma(l) \triangleq l$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Let $u = c_k \lor q$, where $q$ is a clause. Clearly, since $\mathcal{F} \models c_k$, then $\mathcal{F} \models u$, and so $\mathcal{F} \models D \land (u)$. The issue is whether $D \land (u) \models \mathcal{F}$, or equivalently $(\neg \mathcal{F}) \land D \land (u) \models \bot$. Expanding we get $(\neg c_k) \land \land_{i=1,i \neq k} (c_i) \land (c_k \lor q) \models \bot$, which can also be simplified to $(\neg c_k) \land \land_{i=1,i \neq k} (c_i) \land (q) \models \bot$. Hence, the goal is to find a maximal set of literals $q$ such that $(\neg t_k) \land \land_{i=1,i \neq k} (c_i) \land (q) \models \bot$. This can be converted to a minimization problem by removing all literals and then adding to $q$ literals that can be included.

4.2.6 Formula Entailment

Proposition 19 $\text{FM}_{\text{ES}} \leq_p \text{MSMP}$.

Proof. (Sketch)

Reduction. The reduction is defined as follows. $\mathcal{R} \triangleq \mathcal{J}$ and,

$$P(W) \triangleq \neg \text{SAT}(\neg \mathcal{I} \land \land_{c \in \mathcal{W}} (c)) \quad (16)$$

with $\mathcal{W} \subseteq \mathcal{R}$.

Monotonicity. The predicate (see (16)) is of form $\mathscr{B}$, with $\mathcal{G} \triangleq \neg \mathcal{I}$, $u_i \triangleq c$, and $\sigma(c) \triangleq c$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Simple, based on previous proofs.

Proposition 20 $\text{FM}_{\text{xES}} \leq_p \text{MSMP}$.

Proof. (Sketch)

Reduction. The reduction is defined as follows. $\mathcal{R} \triangleq \mathcal{N}$ and,

$$P(W) \triangleq \neg \text{SAT}(\mathcal{J} \land (\forall c \in \mathcal{W} \neg c)) \quad (17)$$

with $\mathcal{W} \subseteq \mathcal{R}$.

Monotonicity. The predicate (see (17)) is of form $\mathscr{B}$, with $\mathcal{G} \triangleq \mathcal{J}$, $u_i \triangleq c$, and $\sigma(c) \triangleq \neg c$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Simple, based on previous proofs.
4.2.7 Backbone Literals

**Proposition 21** \( \text{FBBBr} \leq_p \text{MSMP} \).

**Proof.**

*Reduction.* Consider a set of literals \( V \), obtained from an initial satisfying assignment \( \nu \). The reduction is defined as follows. \( \mathcal{R} \triangleq V \) and,

\[
P(W) \triangleq \neg \text{SAT}(\mathcal{F} \land (\lor_{l \in \mathcal{R}\setminus W} \neg l))
\]

with \( W \subseteq \mathcal{R} \).

*Monotonicity.* The predicate (see (18)) is of form \( \mathcal{B} \), with \( \mathcal{G} \triangleq \mathcal{F} \), \( u_i \triangleq l \), and \( \sigma(l) \triangleq \neg l \). Thus, by Proposition 7 the predicate is monotone.

*Correctness.* Let \( \mathcal{T} \) be a minimal set such that \( P(\mathcal{T}) \) holds. By Definition 2, since \( \mathcal{T} \) is minimal for \( P \), then any proper superset \( \mathcal{T}' \) of \( \mathcal{T} \) the predicate does not hold, i.e. \( \mathcal{F} \not\models \mathcal{V} \setminus \mathcal{T}' \). Hence, \( \mathcal{F} \models \mathcal{V} \setminus \mathcal{T} \), and for any proper superset \( \mathcal{T}' \) of \( \mathcal{T} \), \( \mathcal{F} \not\models \mathcal{V} \setminus \mathcal{T}' \). Moreover, by Proposition 5, \( \mathcal{V} \setminus \mathcal{T} \) is maximal and unique. Thus, by Definition 28, \( \mathcal{V} \setminus \mathcal{T} \) is the set of backbone literals of \( \mathcal{F} \).

**Observation 6** Regarding the reduction in the proof Proposition 21, the computed minimal set \( \mathcal{T} \) is the complement of the set of backbone literals, which is given by \( \mathcal{V} \setminus \mathcal{T} \).

In practice, and for efficiency reasons, algorithms for computing the backbone of a Boolean formula start from a reference satisfying assignment (i.e. the FBBBr function problem) [41]. However, the reduction of the general backbone computation problem to MSMP yields interesting insights into the worst-case number of SAT oracle queries needed to compute the backbone of a Boolean formula.

**Proposition 22** \( \text{FBB} \leq_p \text{MSMP} \).

**Proof.** (Sketch)

*Reduction.* The reduction is defined as follows. \( \mathcal{R} \triangleq X \), with \( X = \text{var}(\mathcal{F}) \). Consider one auxiliary set of variables \( X' \), with \( |X'| = |X| \), and let:

\[
\mathcal{F}^{\text{BB}} \triangleq \mathcal{F}[X/X] \land \mathcal{F}[X/X']
\]

Finally, let:

\[
P(W) \triangleq \neg \text{SAT}(\mathcal{F}^{\text{BB}} \land (\lor_{x \in \mathcal{R}\setminus W} x \land \neg x'))
\]

*Monotonicity.* The predicate (see (19)) is of form \( \mathcal{B} \), with \( \mathcal{G} \triangleq \mathcal{F}^{\text{BB}} \), \( u_i \triangleq x \), and \( \sigma(x) \triangleq x \land x' \), where \( x' \) is a new variable not in \( \mathcal{R} \), but associated with \( x \). Thus, by Proposition 7 the predicate is monotone.

*Correctness.* The predicate holds for \( W = \mathcal{R} \). Similarly to the FBBBr case, \( (\lor_{x \in \mathcal{R}\setminus W} x \land \neg x') \) yields
an empty clause. As for the FBBp case, the literals to be removed from $\mathcal{R}$ are the ones that are backbone literals since adding literals ($\lor_{l \in \mathcal{R} \setminus \mathcal{W}} \neg l$) can only be done while keeping the formula unsatisfiable.

### 4.2.8 Variable Independence

**Proposition 23** $\text{FVInd} \leq_p \text{MSMP (form } \mathcal{P})$.

**Proof.** (Sketch)

*Reduction.* Consider the original set of variables $X$ and another set of variables $Y$, such that $|Y| = |X|$. $\mathcal{F}$ is to be checked for equivalence against $\mathcal{F}[X/Y]$, i.e. a copy of itself using new variables. An additional constraint is that some of these variables are equivalent. The non-equivalence between the two formulas is captured as follows:

$$\mathcal{F}^{\text{VInd}} \triangleq (\mathcal{F}[X/Y] \land \neg \mathcal{F}[X/X] \lor \neg \mathcal{F}[X/Y] \land \mathcal{F}[X/X])$$

(21)

Given $\mathcal{F}^{\text{VInd}}$, the reduction is defined as follows. $\mathcal{R} \triangleq X$ and,

$$P(W) \triangleq \neg \text{SAT}(\mathcal{F}^{\text{VInd}} \land \land_{x_i \in W} (x_i \leftrightarrow y_i))$$

(22)

*Monotonicity.* The predicate (see (22)) is of form $\mathcal{P}$, with $\mathcal{G} \triangleq \mathcal{F}^{\text{VInd}}, u_i \triangleq x_i$, and $\sigma(x_i) \triangleq x_i \leftrightarrow y_i$, where $y_i$ is a new variable not in $\mathcal{R}$, but associated with $x_i$. Thus, by Proposition 7 the predicate is monotone.

*Correctness.* For $W = \mathcal{R}$ each $x_i$ variable is equivalent to its corresponding $y_i$ variable. A variable $x_i$ is removed from $W$ if, by taking any possible value, it does not affect the equivalence of $\mathcal{F}$ with $\mathcal{F}$ defined on $Y$ variables. In such a case, $\mathcal{F}$ is independent from $x_i$.

### 4.2.9 Maximum Autarkies

Autarkies can be captured with two different monotone predicate forms. The first predicate is of form $\mathcal{L}$.

**Proposition 24** $\text{FAut} \leq_p \text{MSMP (form } \mathcal{L})$.

**Proof.** (Sketch)

*Reduction.* The reduction uses a simplified version of the model proposed in [65]. Consider a CNF formula $\mathcal{F}$ with a set of variables $X = \text{var}(\mathcal{F})$. Create new sets of variables $X^+, X^0, X^1$, such that for $x \in X$, $x^+$ indicates whether a variable is selected, and $x^1$ and $x^0$ replace, respectively, the literals $x$ and $\neg x$ in the clauses of $\mathcal{F}$. Thus, $\mathcal{F}$ is transformed into a new formula $\mathcal{F}^{0,1}$, where each literal in $x$ is translated either into $x^1$ or $x^0$. The resulting CNF formula, $\mathcal{F}^{\text{Aut}}$, consists of CNF-encoding the following sets of constraints. For each $x \in X$, add to $\mathcal{F}^{\text{Aut}}$ (the clauses resulting
from encoding) \( x^1 \leftrightarrow x^+ \land x \) and \( x^0 \leftrightarrow x^+ \land \neg x \). If a clause \( c^{0,1} \in F^{0,1} \) has a literal in \( x \), then add to \( F^{\text{Aut}} \) the clause \( (x^+ \rightarrow c^{0,1}) \). Now, \( R \triangleq X^+ \) and,

\[
P(W) \triangleq \text{SAT}(F^{\text{Aut}} \land \land x^+ \in R \setminus W(x^+))
\]

with \( W \subseteq R \).

**Monotonicity.** The predicate (see (23)) is of form \( \mathcal{L} \), with \( G \triangleq F^{\text{Aut}}, u_i \triangleq x^+, \) and \( \sigma(x^+) = x^+ \). Thus, by Proposition 7 the predicate is monotone.

**Correctness.** For \( W = R \), predicate holds, since the argument of the SAT oracle call consists of \( F^{\text{Aut}} \). The elements that are to be dropped from \( R \) are the variables for which there exists a truth assignment that identifies the maximum autarky, i.e. the autark variables. The minimal set is \( W = R \setminus K \), where \( K \) is the set of autark variables. The monotonicity properties in this case can be used to prove that the set of autark variables is maximum.

**Observation 7** Regarding the reduction in the proof Proposition 24, the computed minimal set \( T \) is the complement of the set of autark variables, which is given by \( \text{var}(F) \setminus T \).

An alternative predicate for \( F^{\text{Aut}} \) is of form \( \mathcal{B} \), as shown next.

**Proposition 25** \( F^{\text{Aut}} \leq_p \text{MSMP} \) (form \( \mathcal{B} \)).

**Proof.** (Sketch)

**Reduction.** As before, the reduction uses a simplified version of the model proposed in [65] (see proof of Proposition 24). Given these definitions, \( R \triangleq X^+ \) and,

\[
P(W) \triangleq \neg\text{SAT}(F^{\text{Aut}} \land (\lor x^+ \in R \setminus W(x^+)))
\]

with \( W \subseteq R \).

**Monotonicity.** The predicate (see (24)) is of form \( \mathcal{B} \), with \( G \triangleq F^{\text{Aut}}, u_i \triangleq x^+, \) and \( \sigma(x^+) = x^+ \). Thus, by Proposition 7 the predicate is monotone.

**Correctness.** Similar to previous proofs by showing that,

\[
F^{\text{Aut}} \models \land x^+ \in R \setminus W(\neg x^+)
\]

4.3 Optimization Problems

To illustrate the modeling flexibility of monotone predicates, this section investigates how to solve optimization problems, namely FSMCS, FSMDS, FLMFS and FSMnM. In contrast with the previous section, the objective here is not to develop efficient algorithms or insight, but to show other uses of monotone predicates. It should be noted that computing cardinality minimal sets for some of the
other function problems, e.g. FMUS, FMES, FPIt, FPIc, etc., is significantly harder, since these function problems are in the second level of the polynomial hierarchy, e.g. [33, 98].

In the remainder of this section, unweighted formulations are considered, i.e. each clause is soft with weight 1.

**Proposition 26** FSMCS ≤ₚ MSMP.

**Proof.**

*Reduction.* Let \( B = \{0, 1, 2, \ldots, |F|\} \) denote the possible numbers of clauses that are required to be satisfied. Furthermore, for each clause \( c_i \in F \), create a relaxed copy \( (\neg p_i \lor c_i) \) and let \( F^R \) denote the CNF formula where each clause \( c_i \) is replaced by its relaxed version. Let \( P \) denote the set of selection variables. For each \( b_j \in B \) create the constraint \( \sum_{p_i \in P} p_i \geq b_j \). Each of these constraints defines a lower bound on the number of satisfied clauses. Define \( R \triangleq B \). Moreover, given \( W \subseteq R \) let,

\[
Q(W) \triangleq \bigwedge_{b_j \in R \setminus W} \left( \sum_{p_i \in P} p_i \geq b_j \right)
\]

Finally, the predicate is defined as follows:

\[
P(W) \triangleq SAT \left( F^R \land Q(W) \right)
\]

**Monotonicity.** Given the definition of \( Q(W) \) in (26), the predicate (27) is of form \( \mathcal{L} \), with \( \mathcal{G} \triangleq F^R \), \( u_i = b_j \), and \( \sigma(b_j) \triangleq \left( \sum_{p_i \in P} p_i \geq b_j \right) \). Thus, by Proposition 7 the predicate is monotone.

**Correctness.** The set of true \( p_i \) variables picks a subset \( S \) of the clauses in \( F \), which is to be checked for satisfiability. Given the formulation, it holds that a subset-minimal set must correspond to a cardinality minimal set. If, by removing from \( W \) the element associated with some value \( b_j \) satisfies the predicate, then removing from \( W \) any element associated with the \( b_k \) of value no greater than \( b_j \) also satisfies the predicate, and this holds with the same truth assignment, namely the same set of selected clauses. Thus, any minimal set must exclude the element associated with the largest value \( b_j \) such that the predicate holds, and must also exclude any element associated with \( b_k \) of value no greater than \( b_j \). Thus, the elements not removed from \( W \) are associated with \( b_k \) representing sizes of sets of clauses such that not all can be simultaneously satisfied. Hence, the minimal set represents the smallest MCS of \( F \).

**Proposition 27** FSMDS ≤ₚ MSMP.

**Proof.** (Sketch)

*Reduction.* The definitions of the reduction in the proof of Proposition 26 apply. As a result, the predicate is defined as follows:

\[
P(W) \triangleq SAT \left( \neg F \land F^R \land Q(W) \right)
\]
Monotonicity. Given the definition of $Q(W)$ in (26), the predicate (28) is of form $L$, with $G \triangleq \neg F \land F^R$, $u_i = b_j$, and $\sigma(b_j) \triangleq \left(\sum_{p_i \in P} p_i \geq b_j\right)$. Thus, by Proposition 7 the predicate is monotone.

Correctness. The set of true $p_i$ variables picks a subset $S$ of the clauses in $F$. Thus, it must hold that $F \vDash S$. Thus one wants to pick the largest subset $N$ of $F$ such that $N \not\vDash F$. A minimal set of $W \subseteq R$ corresponds to a maximal set $R \setminus W$, containing all the constraints $\left(\sum_{p_i \in P} p_i \geq b_j\right)$ that can be satisfied. Hence, this gives the largest set $N$ of selected clauses such that $N \not\vDash F$.

**Proposition 28** $\text{FSMCS} \leq_p \text{MSMP}$.

**Proof.** (Sketch)

*Reduction.* The elements to relax in this case are the complements of the clauses, which will be satisfied (so that the clauses are falsified). As a result, define $N^R \triangleq \land_{c_i \in F}(\neg p_i \lor \neg c_i)$. The predicate can then be defined as follows:

$$P(W) \triangleq \text{SAT} \left(N^R \land Q(W)\right)$$

(29)

Monotonicity. Given the definition of $Q(W)$ in (26), the predicate (29) is of form $L$, with $G \triangleq N^R$, $u_i = b_j$, and $\sigma(b_j) \triangleq \left(\sum_{p_i \in P} p_i \geq b_j\right)$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Similar to previous proofs in this section.

**Proposition 29** $\text{FSMnM} \leq_p \text{MSMP}$.

**Proof.** (Sketch)

*Reduction.* The elements to relax in this case are the variables assigned value true. As a result, define $V^R \triangleq \land_{x_i \in V}(\neg p_i \lor \neg x_i)$. The predicate can then be defined as follows:

$$P(W) \triangleq \text{SAT} \left(F \land V^R \land Q(W)\right)$$

(30)

Monotonicity. Given the definition of $Q(W)$ in (26), the predicate (30) is of form $L$, with $G \triangleq F^R \land V^R$, $u_i \triangleq b_j$, and $\sigma(b_j) \triangleq \left(\sum_{p_i \in P} p_i \geq b_j\right)$. Thus, by Proposition 7 the predicate is monotone.

Correctness. Similar to previous proofs in this section.

### 4.4 Summary of Reductions

Table 1 summarizes the reductions described in Section 4.2 and in Section 4.3. For each problem, the associated reference provides the complete details of the reduction to the MSMP problem. The first part denotes function problems where the goal is to compute a minimal set. The second part denotes function problems that represent optimization problems. Besides the function problems
### Table 1: Overview of function problems and MSMP reductions

Additional problems include FMSS, FMNS, FMFS, FMxM, FLMSS, FLMNS, FLMFS and FLMxM. Other function problems, with reductions to MSMP similar to the ones above, are covered in Section 3.

summarized in Table 1, Section 3.1 and also Section 4.2 indicate that several other problems related with Boolean formulas can also be mapped into the MSMP problem. These include variants of the presented problems when considering groups of clauses, variables, hard clauses, etc.

It is important to observe that, for some problems, there are dedicated algorithms that require an asymptotically smaller number of queries to a SAT oracle than the most efficient of the MSMP algorithms described in the next section. Nevertheless, as shown above the reduction to MSMP yields new alternative algorithms and, for a number of cases, it allows developing relevant new insights.
5 Conclusions & Research Directions

This paper extends recent work [13, 14, 69] on monotone predicates and shows that a large number of function problems defined on Boolean formulas can be reduced to computing a minimal set over a monotone predicate. The paper also argues that monotone predicates find application in more expressive domains, including ILP, SMT and CSP.

A number of research directions can be envisioned. A natural question is to identify other function problems that can be reduced to the MSMP problem. Algorithms for MSMP are described elsewhere [69]. A natural research question is whether additional algorithms can be developed. In addition, most practical algorithms for solving minimal set problems exploit a number of pruning techniques [5, 69, 68]. Another natural research question is whether these techniques can be used in the more general setting of MSMP. Finally, another line of research is to develop precise query complexity characterizations of instantiations of the MSMP. Concrete examples include FMUS, FPlc, FPll, FBB, FVInd, among others.

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