The IC-indices of Some Complete Multipartite Graphs

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Abstract

A coloring of a connected graph $G$ is a function $f$ mapping the vertex set of $G$ into the set of all integers. For any subgraph $H$ of $G$, we denote the sum of the values of $f$ on the vertices of $H$ as $f(H)$. If for any integer $k \in \{1, 2, 3, \ldots, f(G)\}$, there exists an induced connected subgraph $H$ of $G$ such that $f(H) = k$, then the coloring $f$ is called an IC-coloring of $G$. The IC-index of $G$, denoted as $M(G)$, is the maximum value of $f(G)$ over all possible IC-colorings $f$ of $G$. In this paper, we present a useful method from which a lower bound on the IC-index of any complete multipartite graph can be derived. Subsequently, we show that, for $m \geq 2$ and $n \geq 2$, our lower bound on $M(K_{m,n})$ is the exact value of it.

Keywords: IC-coloring; IC-index; complete multipartite graph

1 Introduction

Given a connected simple graph $G$, a coloring of $G$ is a function $f$ mapping $V(G)$ into $\mathbb{N}$. For any subgraph $H$ of $G$, we denote the sum $\sum_{v \in V(H)} f(v)$ as $f(H)$. If for any integer $k \in \{1, 2, 3, \ldots, f(G)\}$, there exists an induced connected subgraph $H$ of $G$ such that $f(H) = k$, then the coloring $f$ is called an IC-coloring of $G$. Every connected graph $G$ admits a trivial IC-coloring which assigns the value 1 to every vertex of $G$. The highest possible value of $f(G)$ is referred to as the IC-index of a graph $G$, denoted as $M(G)$, that is,

$$M(G) = \max\{f(G) \mid f \text{ is an IC-coloring of } G\}.$$ 

An IC-coloring $f$ satisfying $f(G) = M(G)$ is called a maximal IC-coloring of $G$. In this paper, we only consider simple graphs. For the terminologies and notations in graph theory, please refer to [13].

The problem of IC-coloring of finite graphs originated from the postage stamp problem in number theory, which has been studied in some literature [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12]. In 1992, Glenn Chappel formulated the IC-coloring problem as a “subgraph sums problem” and showed that $M(C_n) \leq n^2 - n + 1$. Later, in 1995, Penrice [7] introduced the IC-coloring as the stamp covering and showed that $M(K_n) = 2^n - 1$ for $n \geq 1$ and $M(K_{1,n}) = 2^n + 2$ for $n \geq 2$. In 2005, Salehi et. al. [8] proved that $M(K_{2,n}) = 3 \cdot 2^n + 1$ for $n \geq 2$. Along with the result by Shiue and Fu [11] who showed that $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$,
for $2 \leq m \leq n$, in 2008, the problem regarding complete bipartite graphs was completely settled. In this present paper, we deal with complete multipartite graphs. A complete multipartite graph $K_{m_1,m_2,\cdots,m_k}$ is a graph whose vertex set can be partitioned into $k$ partite sets $V_1, V_2, \cdots, V_k$, where $|V_i| = m_i$ for all $i \in \{1, 2, \cdots, k\}$, such that there are no edges within each $V_i$ and any two vertices from different partite sets are adjacent. A complete multipartite graph with $k$ partite sets is called a complete $k$-partite graph as well. We also denote as $K_1(n), m_{n+1}, m_{n+2}, \cdots, m_k$, $n \leq k$, the complete $k$-partite graphs in which there are $n$ partite sets which are of size one and the rest $(k - n)$ partite sets have sizes $m_{n+1}, m_{n+2}, \cdots$, and $m_k$. Therefore $K_1(n)$ represents the complete graph $K_n$.

In this paper, we introduce some useful lemmas in Section 2. In Section 3, our main results are presented. We start with a useful proposition which gives a lower bound on the IC-index of the join of an independent set and a given connected graph. Consequently, a lower bound on the IC-index of any complete multipartite graph can be deduced. We shall show that our lower bound on $M(K_1(n), m)$ is indeed the exact value of it for $m \geq 2$ and $n \geq 2$. A concluding remark will be given in Section 4.

2 Preliminaries

Some basic know results from [11] are introduced in this section. They are very useful in the discussion of our main results. For brevity, we let $[1, \ell]$, $\ell \in \mathbb{N}$, denote the set $\{1, 2, \cdots, \ell\}$. A sequence consisting of 0’s and 1’s is called a binary sequence.

Lemma 2.1. [11] If $a_1, a_2, \cdots, a_n$ are $n$ positive integers which satisfy that $a_1 = 1$ and $a_i \leq a_{i+1} \leq \sum_{j=1}^{i-1} a_j + 1$ for all $i \in [1, n - 1]$. Then, for each $\ell \in [1, \sum_{i=1}^{n} a_i]$, there exists a binary sequence $c_1, c_2, \cdots, c_n$ such that $\ell = \sum_{i=1}^{n} c_i a_j$.

Lemma 2.2. [11] If $s_0, s_1, \cdots, s_n$ is a sequence of integers, then for each $i \in [1, n]$ there exists an integer $r_i \in \mathbb{Z}$ such that $s_i = \sum_{j=0}^{i-1} s_j + r_i$ and $\sum_{j=0}^{n} s_j = 2^n s_0 + \sum_{j=1}^{n} 2^{n-j} r_j$.

Lemma 2.3. [11] Let $V(G) = \{u_1, u_2, \cdots, u_n\}$. If $f$ is an IC-coloring of $G$ such that $f(u_i) \leq f(u_{i+1})$ for all $i \in [1, n - 1]$, then $f(u_1) = 1$ and $f(u_{i+1}) \leq \sum_{j=1}^{i} f(u_j) + 1$ for all $i \in [1, n - 1]$.

Lemma 2.4. [11] Let $f$ be an IC-coloring of a graph $G$ such that $f(u_i) < f(u_{i+1})$ for $i \in [1, n - 1]$, where $V(G) = \{u_1, u_2, \cdots, u_n\}$. For each pair $(i_1, i_2)$ where $1 \leq i_1 < i_2 \leq n$, if $f(u_{i_1}) = \sum_{j=1}^{i_1-1} f(u_j) + 1$ and $u_{i_1}u_{i_2} \notin E(G)$, then either $f(u_{i_2}) \leq \sum_{j=1}^{i_1-1} f(u_j) - f(u_{i_1})$ or $f(u_{i_2}) \leq f(u_{i_1}) + f(u_{i_2})$.

Lemma 2.5. [11] Let $r_1, r_2, \cdots, r_n$ be $n$ numbers. If there are two integers $i$ and $k$ such that $1 \leq i < k \leq n$ and $r_i < r_k$, then

$$\sum_{j=1}^{n} 2^{n-j} r_j < \sum_{j=1}^{n} 2^{n-j} r_j - (2^{n-i} r_i + 2^{n-k} r_k) + (2^{n-i} r_k + 2^{n-k} r_i).$$

Lemma 2.6. [11] Let $f$ be an IC-coloring of a graph $G$. If $G$ has $\ell$ induced connected subgraphs and there are $2k$ distinct induced connected subgraphs $H_1, G_1, H_2, G_2, \cdots, H_k, G_k$ of $G$ such that $f(H_i) = f(G_i)$ for all $i \in [1, k]$, then $f(G) \leq \ell - k$. 
3 Main Result

We start this section with a useful method for deriving a meaningful lower bound on the IC-index of the join of an independent set and a given graph. Subsequently, we show that the lower bound on $M(K_{1(n,m)})$ derived from our method, for $m \geq 2$ and $n \geq 2$, also serves as an upper bound on it. This determines the exact value of $M(K_{1(n,m)})$.

3.1 Lower Bounds on the IC-indices of Complete Multipartite Graphs

For the derivation of lower bounds, we view complete multipartite graphs as being generated by a graph operation starting with graphs with some vertices and no edges. The join of two disjoint graphs $H_0$ and $H_1$, written $H_0 \lor H_1$, is the graph with vertex set $V = V(H_0) \cup V(H_1)$ and edge set $E = E(H_0) \cup E(H_1) \cup \{(u,v) \mid u \in V(H_0), v \in V(H_1)\}$.

Let $O_m$ be the graph with $m$ vertices and no edges, then the join of $O_1$ and $K_n$, or $O_1 \lor K_n$, is the complete graph $K_{n+1}$ and the graph $O_m \lor K_n$ is exactly the complete multipartite graph $K_{1(n,m)}$. Observe that the join of $O_m$ and $O_n$ is the complete bipartite graph $K_{m,n}$. The join of $O_m$ and $K_{n_1,n_2}$ forms the complete tripartite graph $K_{n_1,n_2,m}$. Since joining $O_m$ with a complete $(k-1)$-partite graph generates a complete $k$-partite graph, we are concerned about how the value of the IC-index of a graph changes as we joining $O_m$ to that graph.

Proposition 3.1. If $g$ is an IC-coloring of a connected graph $G$, then there exists an IC-coloring $f$ of $O_m \lor G$ such that $f(O_m \lor G) = 2^m g(G) + 1$ for $m \geq 1$.

Proof. Let $V_0 = V(O_m) = \{w_1, w_2, \ldots, w_m\}$ and $V_1 = V(G) = \{v_1, v_2, \ldots, v_n\}$. We define $f$ on $V(O_m \lor G)$ as $f(v_i) = g(v_i)$ for $i \in [1,n]$, $f(w_j) = 2^{j-1} g(G)$ for $j \in [1,m-1]$ and $f(w_m) = 2^{m-1} g(G) + 1$. The value of $f(O_m \lor G)$ can be calculated as follows.

\[
\begin{align*}
    f(G) &= g(G) + \sum_{j=1}^{m} 2^{j-1} g(G) + 1 \\
      &= g(G) + (2^{m-1} - 1) g(G) + 1 \\
      &= 2^m g(G) + 1.
\end{align*}
\]

Given any integer $k \in [1,2^m g(G) + 1]$, we need to identify an induced connected subgraph $H$ of $O_m \lor G$ such that $f(H) = k$. Since $g$ is an IC-coloring of $G$ and $f(v) = g(v)$ for each vertex in $G$, the desired induced connected subgraph exists for each $k \in [1,g(G)]$. For $k \in [g(G) + 1,2^{m-1} g(G)]$, we rewrite $k$ into the form $k = kg(G) + r$ where $1 \leq r \leq g(G)$. Hence, $1 \leq q \leq 2^{m-1} - 1$ and then there exists a binary sequence $c_1, c_2, \ldots, c_{m-1}$, which are not all zero, such that $q = \sum_{i=1}^{m-1} c_i 2^{i-1}$. It follows that \(qg(G) = \sum_{i=1}^{m-1} c_i 2^{i-1} g(G) = \sum_{i=1}^{m-1} c_i f(w_i)\). Now, let $H'$ be a connected subgraph of $G$ such that $f(H') = r$ and let $W = V(H') \cup \{w_i \mid c_i = 1, \ i \in [1,m-1]\}$. Since $H'$ is connected, the subgraph $H$ induced by $W$ is also connected and satisfies $f(H) = k$. Next, if $k = 2^{m-1} g(G) + 1$, then the subgraph $H$ induced by the single vertex $w_m$ fits our need. Finally, for $k \in [2^m g(G) + 2, 2^m g(G) + 1]$, we write $k$ as $k = (2^{m-1} g(G) + 1) + k'$, where $1 \leq k' \leq 2^{m-1} g(G)$. By the above argument, there is a connected subgraph $H'$ such that $f(H') = k'$ and $V(H') \cap V_1 \neq \emptyset$. Let $W = V(H') \cup \{w_m\}$, then the subgraph $H$ induced by $W$ is connected and we have $f(H) = k$. The result follows. \(\Box\)
Using the known result $M(K_n) = 2^n - 1$ for $n \geq 1$ [7], lower bounds on the IC-indices of $K_{1(n),m}$ and $K_{1(n),m_1,m_2,\ldots,m_t}$ can be easily obtained by Proposition 3.1.

**Corollary 3.2.** $M(K_{1(n),m}) = M(O_m \lor K_n) \geq 2^{m+n} - 2^m + 1$ for $m \geq 1$ and $n \geq 1$.

For $m = 1$, this lower bound matches the known value of the IC-index of $K_{n+1}$. We will show that this inequality is in fact an equality when $m \geq 2$ and $n \geq 2$.

**Corollary 3.3.** For $m_1 \geq m_2 \geq \cdots \geq m_\ell$,

$$M(K_{1(n),m_\ell,m_{\ell-1},\ldots,m_2,m_1}) = M(O_{m_1} \lor O_{m_2} \lor \cdots \lor O_{m_\ell} \lor K_n) \geq (2^{m_1} (2^{m_2} (\cdots (2^{m_\ell} (2^n - 1) + 1) \cdots) + 1) + 1).$$

Furthermore, since the IC-index of a bipartite graph is known, a lower bound on $M(K_{m_1,m_2,m_3})$ can be derived as well. By successively applying Proposition 3.1, a lower bound on the IC-index of any complete multipartite graph can be easily found.

### 3.2 The Exact Value of the IC-index of $K_{1(n),m}$

Now, we consider the graph $K_{1(n),m}$, or $O_m \lor K_n$. We shall show that the IC-index of this graph is $2^{m+n} - 2^m + 1$. In the remainder of the this paper, we let $G = K_{1(n),m} = O_m \lor K_n$, $V_0 = V(O_m)$ and $V_1 = V(K_n)$. We introduce some properties of the graph $K_{1(n),m}$ and any maximal IC-coloring of it first.

**Lemma 3.4.** $K_{1(n),m}$ has $(2^{m+n} - 2^m + m)$ induced connected subgraphs.

**Proof.** Any induced connected subgraph $H$ of $G$ must satisfy exactly one of the following three conditions: (i) $V(H) \subseteq V_1$ and $V(H) \neq \emptyset$; (ii) $V(H) \subseteq V_0$ and $|V(H)| = 1$; (iii) $V(H) \cap V_1 \neq \emptyset$ and $V(H) \cap V_0 \neq \emptyset$. Therefore, the number of induced connected graphs of $G$ is $(2^n - 1) + m + (2^m - 1)(2^n - 1) = 2^{m+n} - 2^m + m$. 

**Proposition 3.5.** If $f$ is a maximal IC-coloring of $K_{1(n),m}$, then $f(u) \neq f(v)$ for each pair of distinct vertices $u$ and $v$ in $V(K_{1(n),m})$.

**Proof.** Suppose that there are two distinct vertices $u$ and $v$ in $V(G)$ such that $f(u) = f(v)$. For subsets of vertices $V'_0 \subseteq V_0 \setminus \{u,v\}$ and $V'_1 \subseteq V_1 \setminus \{u,v\}$, we denote as $H_u$ and $H_v$ the subgraphs of $G$ induced by $V'_0 \cup V'_1 \cup \{u\}$ and $V'_0 \cup V'_1 \cup \{v\}$ respectively. Then, $H_u \neq H_v$ and $f(H_u) = f(H_v)$. Let $S$ be the set of all possible pairs $(V'_0, V'_1)$ such that $H_u$ and $H_v$ are both connected, and let $p = |S|$. One can see from Lemma 2.6 and Lemma 3.4 that $f(G) \leq (2^{m+n} - 2^m + m) - p$.

Observe that any connected subgraph $H$ of $G$ must satisfy exactly one of the two conditions: (i) $V(H) \cap V_1 \neq \emptyset$, or (ii) $V(H) \cap V_1 = \emptyset$ and $|V(H) \cap V_0| = 1$. Base on this fact, we now evaluate the number $p$ in the following three possible cases.

Case (a): if $u, v \in V_1$, then $p = 2^{|V_1 \setminus \{u,v\}|} \cdot 2^{|V_0|} = 2^{m+n-2}$.

Case (b): if $u, v \in V_0$, then either $V'_1 \neq \emptyset$ or $V'_0 = V'_1 = \emptyset$. Thus we have $p = 2^{|V_0 \setminus \{u,v\}|} \cdot (2^{|V_1|} - 1) + 1 = 2^{m-2} (2^n - 1) + 1 = 2^{m+n-2} - 2^{m-2} + 1$.

Case (c): if exactly one of $u$ and $v$ is in $V_0$, then either $V'_1 \neq \emptyset$ or $V'_0 = V'_1 = \emptyset$. It follows that $p = 2^{m-1} (2^{n-1} - 1) + 1 = 2^{m+n-2} - 2^{m-1} + 1$. 


The value of $p$ in Case (c) is the minimum among these three cases. This leads to an upper bound on $f(G)$ as follows.

$$f(G) \leq (2^{m+n} - 2^m + m) - p$$

$$\leq (2^{m+n} - 2^m + m) - (2^{m+n-2} - 2^{m-1} + 1)$$

$$= 2^{m+n} - 2^{m+n-2} - 2^{m-1} + m - 1.$$}

Since $m, n \geq 2$, we have $2^{m+n-2} \geq 2^m$ and $-2^{m-1} + m - 1 < 0$. This implies that $f(G) < 2^{m+n} - 2^m + 1$ which is a contradiction to Corollary 3.2 and we have the result. □

Now we are in a position to prove that the lower bound given in Corollary 3.2 also serves as an upper bound on $M(K_{1(n),m})$. However, the proof is too involved that we need some more notations to facilitate the whole discussion process. In what follows, for the given maximal IC-coloring $f$ of $G$, we always assume that $\{u_1, u_2, \ldots, u_{m+n}\}$ is the vertex set of $G$ such that $f(u_i) < f(u_{i+1})$ for all $i \in [1, m+n-1]$. Thus, $f(u_1) = 1$ and $f(u_2) = 2$ are always true. Besides, we also define $f_0 = 0$ and denote the sum $\sum_{j=1}^{i} f(u_j)$ as $f_i$ for $i \in [1, m+n]$. Let us introduce some useful facts first.

**Lemma 3.6.** If $f$ is a maximal IC-coloring of $K_{1(n),m}$, then

(1) $f_j \leq 2^{i-1}(f_i + 1) - 1$ for every pair $(i, j)$ such that $1 \leq i < j \leq m+n$.

(2) $f_i \geq 3 \cdot 2^{i-2}$ for each $i \in [2, m+n]$.

**Proof.** (1) Given $i \in [1, m+n-1]$, we let $s_0 = f_i$ and $s_k = f(u_{i+k})$ for each $k \in [1, m+n-i]$. By Lemma 2.3, we have $s_k \leq f_{i+k-1} + 1 = \sum_{\ell=0}^{k-1} s_\ell + 1$. For each $j \in [i+1, m+n]$, one can see from Lemma 2.2 that

$$f_j = \sum_{\ell=0}^{j-i} s_\ell \leq 2^{j-i} f_i + \sum_{\ell=1}^{j-i} 2^{j-i-\ell} \cdot 1 = 2^{j-i}(f_i + 1) - 1.$$ 

(2) Suppose that there exists some $i \in [2, m+n]$ such that $f_i \leq 3 \cdot 2^{i-2} - 1$. According to part(1), we have

$$f(G) = f_{m+n} \leq 2^{m+n-i}(f_i + 1) - 1 \leq 2^{m+n-i}(3 \cdot 2^{i-2}) - 1 = 2^{m+n} - 2^{m+n-2} - 1 < 2^{m+n} - 2^m + 1.$$ 

This contradicts to Corollary 3.2 and we have the result. □

**Lemma 3.7.** Suppose that $f$ is a maximal IC-coloring of $K_{1(n),m}$. Let $s_0 = f_0 = 0$ and $s_i = f(u_i)$ for $i \in [1, m+n]$. If each $r_i$ is the integer such that $s_i = \sum_{j=0}^{i-1} s_j + r_i$, $i \in [1, m+n]$, then $r_i \leq 1$. Furthermore, if $r_i \leq 0$ for all $i \in \{j \mid u_j \in V_0 \setminus \{u_k_0\}\}$, where $k_0 = \max \{j \mid u_j \in V_0\}$, then $f(K_{1(n),m}) \leq 2^{m+n} - 2^m + 1$.

**Proof.** The first result is trivial from Lemma 2.3. To prove that second result, we describe the IC-coloring $\bar{f}$ defined in the proof of Proposition 3.1 for $O_m \vee K_n$ explicitly. Let $V_0 = V(O_m) = \{w_1, w_2, \ldots, w_m\}$ and $V_1 = V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Since we are given a maximal IC-coloring $g$ of $K_n$ defined as $g(v_i) = 2^{i-1}$ for each $v_i \in V_1$ [7], we have an IC-coloring $\bar{f}$ of $O_m \vee K_n$ defined as $\bar{f}(v_i) = g(v_i) = 2^{i-1}$ for $i \in [1, n]$, $\bar{f}(w_j) = 2^{j-1}(2^n-1)$ for $j \in [1, m-1]$ and $\bar{f}(w_m) = 2^{m-1}(2^n-1)+1$. Then $\bar{f}(O_m \vee K_n) = 2^{m+n} - 2^m + 1$. Let us
rearrange the vertices of $G$ into a new order $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{m+n}\}$ such that $\bar{f}(\bar{u}_i) < \bar{f}(\bar{u}_{i+1})$ for all $i \in [1, m+n-1]$. Then $\bar{f}(\bar{u}_i) = \sum_{j=1}^{i-1} \bar{f}(u_j) + \bar{r}_i$, where $\bar{r}_i = 1$ for $i \in [1, n] \cup \{n+m\}$, and $\bar{r}_i = 0$ for $i \in [n+1, n+m-1]$. According to the assumption in this Lemma, $r_i \leq 0$ for all $i \in \{j \mid u_j \in V_0 \setminus \{u_{k_0}\}\}$. If $r_{m+n} = 1$, then among $r_1, r_2, \ldots, r_{m+n}$ there are at most $(n+1)$ of them taking the value one and at least $(m-1)$ of them having the values no more than zero. By comparing the distributions of the 1’s in $\bar{r}_i$’s and $r_i$’s, we conclude from Lemma 2.5 that $f(G)$ does not exceed the value $\bar{f}(G) = 2^{m+n} - 2^m + 1$. If $r_{m+n} = 0$, then there are at most $n$ $r_i$’s being one and at least $m$ $r_i$’s not exceeding zero. Lemma 2.5 again guarantees the truth of the inequality $f(G) \leq \bar{f}(G)$ in this case.

Next, we consider the case where the assumption in Lemma 3.7 is violated, namely, there is a $r_i$ having the value one for some $i \in \{j \mid u_j \in V_0 \setminus \{u_{k_0}\}\}$.

**Lemma 3.8.** Let $f$ be a maximal IC-coloring of $K_{1(n),m}$. Let $s_0 = f_0 = 0$, $s_i = f(u_i)$ and $r_i = s_i - \sum_{j=0}^{i-1} s_j$ for $i \in [1, m+n]$. Suppose that $S_1 = \{i < k_0 \mid r_i = 1 \text{ and } u_i \in V_0\} \neq \emptyset$, where $k_0 = \max \{j \mid u_j \in V_0\}$. Let $i_1 = \min S_1$, $t = |\{u_i \in V_0 \mid i < i_1\}|$ and $S_2 = \{i \geq i_1 + 1 \mid u_i \in V_0 \text{ and } f(u_i) > f_{i-1} - f(u_{i_1})\}$.

(i) If $S_2 = \emptyset$, then $f(K_{1(n),m}) \leq 2^{m+n} - 2^m + 1$.

(ii) If $S_2 \neq \emptyset$, then $f(K_{1(n),m}) \leq 2^{m+n} - 2^{m+n-2}(1 - 2^{-(i_2-i_1)}) - (2^t - 1)(3 \cdot 2^{m+n-i_1-1} + 2^{m+n-i_2-1})$, where $i_2 = \min S_2$.

(iii) If $S_2 \neq \emptyset$ and $n \geq 3$, then $f(K_{1(n),m}) \leq 2^{m+n} - 2^m + 1$.

**Proof.** Observe that, among $r_1, r_2, \ldots, r_{i_1-1}$, there are at most $(i_1 - t - 1)$ of them having the value one. By Lemma 2.2 and Lemma 2.5 we have that

\[
\begin{align*}
  f_{i_1-1} &= 2^{i_1-1} \cdot s_0 + \sum_{j=1}^{i_1-1} 2^{j_1-1-j} \cdot r_j \\
  &\leq 2^{i_1-1} \cdot 0 + \sum_{j=1}^{i_1-1-t} 2^{i_1-1-j} \cdot 1 \\
  &= 2^{i_1-1} - 2^t 
\end{align*}
\]

and

\[
\begin{align*}
  f_{i_1} &= f_{i_1-1} + f(u_{i_1}) \\
  &= 2 f_{i_1-1} + 1 \\
  &\leq 2^{i_1} - 2^{t+1} + 1 .
\end{align*}
\]

(i) Consider the case $s_0' = f_i$ and $s_j' = f(u_{i_1+j})$ for $j \in [1, m+n-i_1]$. Let $r_j' = s_j' - \sum_{i=0}^{j-1} s_i'$, then $r_j' \leq 1$. Furthermore, if $S_2 = \emptyset$, that is, $f(u_{i+1+j}) \leq f_{i+1+j-1} - f(u_i)$ for all $u_{i_1+j} \in V_0$, then $f(u_{i_1+j}) - \sum_{i=1}^{i_1-j} f(u_i) \leq -f(u_{i_1}) \leq -1$. Since $r_j' \leq -1$ for all $j \in \{j \mid u_{i_1+j} \in V_0\}$, there are at most $(n-(i_1-t-1))$ 1’s in the values of $r_1', r_2', \ldots, r_{m+n-i_1}'$. By Lemma 2.2, 2.5 and Inequality (2), we have

\[
\begin{align*}
  f(G) &= \sum_{j=0}^{m+n-i_1} s_j' \\
  &= 2^{m+n-i_1} f_{i_1} + \sum_{j=1}^{m+n-i_1} 2^{m+n-i_1} - j \cdot r_j' \\
  &\leq 2^{m+n-i_1} (2^{i_1} - 2^{t+1} + 1) + \sum_{j=1}^{n-(i_1-t-1)} 2^{m+n-i_1-1} \cdot 1 + \sum_{j=(i_1-t-1)+1}^{m+n-i_1-1} 2^{m+n-i_1-j}(-1) \\
  &\leq 2^{m+n-i_1} (2^{i_1} - 2^{t+1} + 1) + 2^{m+n-i_1} - 2^{m-t-1} - (2^{m-t-1} - 1) \leq 2^{m+n} + 2^{m+n-i_1-1} - 2^{m+n-i_1+1} - 2^{m-t} + 1.
\end{align*}
\]

Note that $n \geq i_1-t-1 \geq 0$ and $t \geq 0$. If $n = i_1-t-1$ or $t = 0$, then $f(G) \leq 2^{m+n} - 2^m + 1$. 


Otherwise, \( n \geq i_1 - t \) and \( t \geq 1 \), then \( 2^{m+n-i_1+1} - 2^{m+n-i_1+t+1} \leq 2^{m+n-i_1+t} - 2^{m+n-i_1+t+1} = -2^{m+n-i_1+t} \leq -2^m \). This implies that \( f(G) \leq 2^{m+n} - 2^m - 2^{m-t} + 1 < 2^{m+n} - 2^m + 1 \) and the result in (i) is asserted.

(ii) If \( S_2 \neq \emptyset \), then \( f(u_{i_2}) > f(u_{i_2-1}) \). By Lemma 2.4 we have that \( f(u_{i_2+1}) \leq f(u_{i_1}) + f(u_{i_2}) \) and then

\[
\begin{align*}
  f_{i_2+1} &= f_{i_2-1} + f(u_{i_2}) + f(u_{i_2+1}) \\
  &\leq f_{i_2-1} + 2f(u_{i_2}) + f(u_{i_1}) \\
  &\leq f_{i_2-1} + 2(f_{i_2-1} + 1) + (f_{i_1} + 1) \\
  &= 3f_{i_2-1} + f_{i_1} + 3. \hspace{1cm} \text{(3)}
\end{align*}
\]

Now, the desired upper bound on \( f(G) \) can be derived as follows.

\[
\begin{align*}
f(G) &= f_{m+n} \leq 2^{m+n-(i_2+1)}(f_{i_2+1} + 1) \hspace{1cm} \text{(by Lemma 3.6(1))} \\
&\leq 2^{m+n-(i_2+1)}[(3f_{i_2-1} + f_{i_1} + 3) + 1] \hspace{1cm} \text{(by Inequality (3))} \\
&\leq 2^{m+n-(i_2+1)}[3(2^{i_2-i_1}(f_{i_1} - 1) + 1) + f_{i_1} + 4] \hspace{1cm} \text{(by Lemma 3.6(1))} \\
&= 2^{m+n-(i_2+1)}[(3 \cdot 2^{i_2-i_1} + 1)(f_{i_1} - 1) + 1] \\
&= 2^{m+n-(i_2+1)}(3 \cdot 2^{i_2-i_1} + 1)(2^{i_1} - 1 - (2^t - 1)) \hspace{1cm} \text{(by Inequality (1))} \\
&= 3 \cdot 2^{m+n-2} + 2^{m+n-(i_2-i_1)-2} - (2^t - 1)(3 \cdot 2^{m+n-i_1-1} + 2^{m+n-i_2-1}) \\
&= 2^{m+n} - 2^{m+n-2}(1 - 2^{-(i_2-i_1)}) - (2^t - 1)(3 \cdot 2^{m+n-i_1-1} + 2^{m+n-i_2-1}) \hspace{1cm} \text{(4)}
\end{align*}
\]

(iii) Since \( i_2 - i_1 \geq 1, 2^{m+n-2}(1 - 2^{-(i_2-i_1)}) \geq 2^{m+n-3}. \) It follows that \( f(G) \leq 2^{m+n} - 2^{m+n-3} < 2^{m+n} - 2^m + 1 \) when \( n \geq 3 \). The proof is completed.

We have shown that \( f(G) \leq 2^{m+n} - 2^m + 1 \) is valid in many cases. With some further discussion, the final conclusion can be achieved.

**Theorem 3.9.** \( M(K_{1(n),m}) = 2^{m+n} - 2^m + 1. \)

**Proof.** Let \( f \) be a maximal IC-coloring of \( G \). We adopt the notation used in Lemma 3.8. By Lemma 3.7 and Lemma 3.8, it suffices to show that \( f(G) \leq 2^{m+n} - 2^m + 1 \) holds when \( S_2 \neq \emptyset \) and \( n = 2 \). First note that when \( n = 2 \), we have \( 0 \leq i_1 - t - 1 \leq 2 \) and the upper bound on \( f(G) \) in (4) can be rewritten as

\[
\begin{align*}
f(G) &\leq 2^{m+2} - 2^m(1 - 2^{-(i_2-i_1)}) - (2^t - 1)(3 \cdot 2^{m-i_1+1} + 2^{m-i_2+1}) \\
&= (2^{m+2} - 2^m) + 2^{m-(i_2-i_1)} - 3 \cdot 2^{m-i_1+t+1}(1 - 2^{-t}) + 2^{m-i_2+1}(2^t - 1) \\
&= 2^{m+2} - 2^m - 3 \cdot 2^{m-(i_2-i_1)}(1 - 2^{-t}) + 2^{m-(i_2-i_1)}(1 - 2^{-t})(2^t - 1)) \hspace{1cm} \text{(5)}
\end{align*}
\]

(1) If \( t \geq 2 \), then

\[
\begin{align*}
f(G) &\leq (2^{m+2} - 2^m) - 3 \cdot 2^{m-2}(1 - \frac{1}{4}) + 2^{m-1} \\
&< 2^{m+2} - 2^m + 1.
\end{align*}
\]

(2) If \( t = 1 \) and \( i_1 \leq 3 \), then

\[
\begin{align*}
f(G) &\leq (2^{m+2} - 2^m) - 3 \cdot 2^{m-1}(1 - 2^{-1}) + 2^{m-1}(1 - 2^{-2}(2 - 1)) \\
&= (2^{m+2} - 2^m) + (3/4 - 3/2) \cdot 2^{m-1} \\
&< 2^{m+2} - 2^m + 1.
\end{align*}
\]

(3) We have so far consider all but the following four cases: \((i_1,t) = (1,0),(2,0),(3,0),(4,1)\).
Let us have a closer investigation of the value \( f(G) \) again before literally starting the discussion of these cases. Now, consider the the situation where \( s_0 = f_0 \), \( s_i = f(u_i) \) and \( s_i = \sum_{j=0}^{i-1} s_j + r_i \) for \( i \in [1, i_2 - 1] \). From the definition of \( i_1 \) and \( i_2 \), we see that \( r_i \leq 0 \) for all \( i \in \{ i \leq i_2 - 1 \mid u_i \in V_0 \} \setminus \{ i_1 \} \). Since \( n = 2 \), among \( r_1, r_2, \ldots, r_{i_2-1} \), there are at most three of them taking the value one. We therefore have

\[
\begin{align*}
f_{i_2-1} & \leq 2^{i_2-1} s_0 + \sum_{j=1}^{i_2-1} 2^{i_2-1-j} \cdot r_j \\
& \leq 2^{i_2-2} + 2^{i_2-3} + 2^{i_2-4} \\
& = 7 \cdot 2^{i_2-4}.
\end{align*}
\]

Making use of this fact and Inequality (1), an upper bound on \( f_{i_2+1} \) can be derived from Inequality (3) as follows.

\[
\begin{align*}
f_{i_2+1} & \leq 3f_{i_2-1} + f_{i_1-1} + 3 \\
& \leq 21 \cdot 2^{i_2-4} + 2^{i_1-1} - 2^t + 3.
\end{align*}
\]

This leads to an upper bound on \( f(G) \).

\[
f(G) < 2^{m+2-(i_2+1)}(f_{i_2+1} + 1) \quad \text{(by Lemma 3.6(1))} \\
= 2^{m+2-(i_2+1)}(21 \cdot 2^{i_2-4} + 2^{i_1-1} - 2^t + 4) \\
= (21 \cdot 2^{m-3} + \left[2^{m-(i_2-i_1)} - 2^{m+1-i_2+t} + 2^{m+i_2-i_1+3} \right] \\
= (2^{m+2} - 2^m - 3 \cdot 2^{m-3}) + \left[2^{m-(i_2-i_1)}(1 - 2^{-i_2+i+1} + 2^{-i_1+3}) \right] \ldots \ldots \ldots (6)
\]

Now, let us have the discussion for the remaining four cases.

**Case 1.** \((i_1, t) = (1, 0)\).

In this case, \( f(u_i) = f(u_1) = 1 \). If \( i_2 \geq 5 \), then Inequality (6) gives

\[
f(G) < (2^{m+2} - 2^m) - 3 \cdot 2^{m-3} + 2^{m-4}(1 - 2^0 + 2^t) \\
< 2^{m+2} - 2^m + 1.
\]

When \( i_2 = 4 \), since \( f(u_2) = 2 > f_1 - f(u_1) \) and \( f(u_3) > f(u_2) = f_2 - f(u_1) \), we have \( \{u_2, u_3\} \subseteq V_1 \) by the definition of \( i_2 \). So, \( V_1 = \{u_2, u_3\} \) and \( V_0 = \{u_i \mid i \in \{1\} \cup [4, m+n]\} \). Now, \( f(u_3) \leq f_2 + 1 = 4 \) and \( f_3 \leq 7 \). Since \( i_2 = 4 \), we know that \( f(u_4) > f_3 - f(u_1) \) holds. Lemma 2.4 guarantees that \( m + n \geq 5 \) and \( f(u_5) \leq f(u_1) + f(u_4) \leq 9 \), giving \( f_5 = f_3 + f(u_4) + f(u_5) \leq f_3 + (f_2 + 1) + f(u_5) \leq 7 + 8 + 9 \leq 24 \). Now, suppose that \( m + n \geq 6 \), then the fact \( f_6 \geq 3 \cdot 2^{n-2} = 48 \) from Lemma 3.6(2) implies that \( f(u_6) \geq 48 - f_5 \geq 24 \), which means \( f(u_6) > f_5 - f(u_1) \). One can then obtain from Lemma 2.4 that \( m + n \geq 7 \) and \( f(u_7) \leq f(u_1) + f(u_6) \leq 1 + (f_5 + 1) = 26 \). However, this leads to \( f_7 = f_5 + f(u_6) + f(u_7) \leq 24 + 25 + 26 = 75 < 3 \cdot 2^{7-2} \), contradicting to Lemma 3.6(2). We therefore conclude that the only possible situation is \( m + n = 5 \) and \( f(G) = f_5 \leq 24 < 2^{m+2} - 2^m + 1 \).

When \( i_2 = 3 \), from the definition of \( i_2 \), we have \( u_2 \in V_1 \) and also \( f(u_4) = f(u_{i_2+1}) \leq f(u_i) + f(u_{i_2}) = f(u_1) + f(u_3) \) by Lemma 2.4. Now, \( 2 = f(u_2) < f(u_3) \leq f_2 + 1 = 4 \). From Lemma 3.6(2), we see that \( f_4 \geq 3 \cdot 2^{i_2-2} = 12 \) and thus \( f_4 \leq f_2 + f(u_3) + f(u_4) \leq 3 + f(u_3) + (f(u_1) + f(u_3)) = 2f(u_3) + 4 \) which implies that \( f(u_3) = 4 \). It follows that \( f_3 = 7 \) and \( f(u_4) \leq f(u_1) + f(u_3) = 5 \) by Lemma 2.4 because \( f(u_3) > f_2 - f(u_1) \) and \( u_1u_3 \notin E(G) \). Since \( f(u_4) > f(u_3) = 4 \), we see that \( f(u_4) = 5 \) and \( f_4 = 12 \). Suppose that \( m + n \geq 5 \), then Lemma 3.6(2) gives \( f_5 \geq 3 \cdot 2^{5-2} = 24 \). One can see that \( f(u_5) = f_5 - f_4 \geq 24 - 12 = 12 > 9 = f(u_4) + f(u_3) \) and \( f(u_4) > f_3 - f(u_3) \). Lemma 2.4 then guarantees that \( u_4 \) must be in \( V_1 \). We therefore conclude that \( V_1 = \{u_2, u_4\} \) and
Therefore, we conclude that the inequality $f(u_5) \geq 12 > f_4 - f(u_1)$, we have from Lemma 2.4 that $m + n \geq 6$ and $f(u_6) \leq f(u_1) + f(u_5) \leq 1 + (f_4 + 1) \leq 14$. However, this leads to $f_6 \leq f_4 + f(u_5) + f(u_6) \leq 12 + (12 + 1) + 14 = 39 < 3 \cdot 2^{6-2}$ which is a contradiction to Lemma 3.6(2). Therefore the only possible situation is $"m + n = 4"$ and then $f(G) = f_4 = 12 < 2^{m+2} - 2^m + 1$.

If $i_2 = 2$, then $f(u_{i_1}) = f(u_1) = 1$ and $f(u_{i_2}) = f(u_2) = 2 = f_1 + 1$. Since $f(u_3) = f(u_{i_2+1}) = f(u_1) + f(u_2) = 3$ and $f(u_3) > f(u_2) = 2$, we have $f(u_3) = 3$ and then $f(u_4) = f(u_1)$. In addition, the fact $f_4 \geq 3 \cdot 2^{q-2} = 12$ implies that $f(u_4) = f_4 = 6 > 4 = f(u_3) + f(u_1)$. From Lemma 2.4, we see that $u_3$ must be in $V_1$. When $m = 2$, $f(G) = f_4 \leq 1 + 2 + 3 + (f_3 + 1) = 13 = 2^{m+2} - 2^m + 1$.

Next, consider the situation when $m \geq 3$. Note that the inequality $f_3 \geq 3 \cdot 2^{q-2} = 24$ gives $f(u_5) = f_5 - f_4 \geq 24 - 13 = 11 > 8 = f(u_1) + f(u_4)$. Along with the fact that $f(u_4) \geq 6 > 5 = f_3 - f(u_1)$, one can see that $u_4$ must also be in $V_1$ from Lemma 2.4. Therefore, we have $V_1 = \{u_3, u_4\}$ and $V_0 = \{u_i \mid i \in \{2\} \cup \{5, m + n\}\}$. Suppose that $f(u_5) \geq 12$, then $f(u_5) > f_5 = f(u_1)$. Lemma 2.4 guarantees that $m + n \leq 6$ and $f(u_6) \leq f(u_2) + f(u_5) \leq 2 + (f_4 + 1) \leq 16$. However, these facts lead to $f_6 = f_4 + f(u_5) + f(u_6) \leq 13 + 14 + 16 = 43 < 3 \cdot 2^{6-2}$, contradicting to Lemma 3.6(2). Hence, $f(u_5) = 11$ is asserted. Now, suppose again that $m + n \geq 6$. The inequality $f_6 \geq 3 \cdot 2^{q-2} = 48$ from Lemma 3.6(2) implies $f(u_6) = f_6 - f_5 \geq 48 - 24 = 24 > f_5 - f(u_1)$. So we have from Lemma 2.4 that $m + n \geq 7$ and $f(u_7) \leq f(u_1) + f(u_6) \leq 1 + (f_3 + 1) \leq 26$. However, this gives $f_7 \leq f_5 + f(u_6) + f(u_7) \leq f_5 + (f_5 + 1) + 26 \leq 75 < 3 \cdot 2^{7-2}$, contradicting to Lemma 3.6(2) again. Therefore, "$m + n = 5"$ is the only possible situation and then $f(G) = f_5 = 24 < 2^{m+2} - 2^m + 1$.

Case 2. $(i_1, t) = (2, 0)$.

In this case, $f(u_1) = 1$, $f(u_2) = 2$ and $u_1 \in V_1$. If $i_2 \geq 5$, then the upper bound on $f(G)$ in (6) can be rewritten as

$$f(G) \leq (2^{m+2} - 2^m) - 3 \cdot 2^{m-3} + 2^{m-3}(1 - 2^{-1} + 2^1) < 2^{m+2} - 2^m + 1.$$  

If $i_2 = 4$, then since $f(u_3) > 1 = f_2 - f(u_2)$, one can see that $u_3$ must be in $V_1$ from the definition of $i_2$. Therefore we have $V_1 = \{u_1, u_3\}$ and $V_0 = \{u_i \mid i \in \{2\} \cup [4, m + n]\}$. Note that $f(u_1) \leq f_3 + 1 = 8$. From the definition of $i_2$ and Lemma 2.4, we have $f(u_3) \leq f(u_2) + f(u_4) = 2 + f(u_4)$. Now, $3 \cdot 2^{q-2} \leq f_3 = f_4 + f(u_4) + f(u_5) \leq (1 + 2 + 2) + f(u_4) + (2 + f(u_4))$. This implies that $f(u_4) = 8 = f_3 + 1$ and $f(u_5) \leq 10$. Hence, $f(u_5) > f(u_4) > 7 \geq f_3 = f_4 - f(u_4)$. It follows from Lemma 2.4 that $f(u_6) \leq f(u_4) + f(u_5) \leq 8 + 10 = 18$ and then $f_6 = f_4 + f(u_5) + f(u_6) = 15 + 10 + 18 = 41 < 3 \cdot 2^{6-2}$ which contradicts to Lemma 3.6(2). We therefore conclude that $i_2 = 4$ can never occur in this case.

If $i_2 = 3$, then $u_1 \in V_1$ and $f(u_3) \leq f_2 + 1 = 4$. From the definition of $i_2$ and by virtue of Lemma 2.4, we have $f(u_4) \leq f(u_2) + f(u_3) \leq 6$. Suppose that $m + n \geq 5$. Since Lemma 3.6(2) gives that $f_4 \geq 3 \cdot 2^{q-2} = 12$ and $f_4 \leq 1 + 2 + f(u_3) + f(u_2) + f(u_3) = 5 + 2f(u_3)$, we obtain that $f(u_3) = 3$. Now, the inequality $f_5 \geq 3 \cdot 2^{q-2} = 24$ from Lemma 3.6(2) leads to $f(u_5) = f_5 - f_4 \geq 24 - (1 + 2 + 4 + 6) = 11 > f(u_3) + f(u_4)$. One can see that $u_4$ must be in $V_1$ by Lemma 2.4 because $f(u_4) > f(u_5) = 4 > f_3 - f(u_3)$ and $f(u_3) = f_2 + 1$. Hence, we have $V_1 = \{u_1, u_4\}$ and $V_0 = \{u_i \mid i \in \{2, 3\} \cup [5, m + n]\}$. In addition, since $f(u_5) \geq 11 > f_4 - f(u_3)$, we know from Lemma 2.4 that $m + n \geq 6$ and $f(u_6) \leq f(u_3) + f(u_5) \leq 4 + (f_3 + 1) \leq 18$. These facts together lead to $f_6 = f_4 + f(u_5) + f(u_6) \leq 13 + 14 + 18 = 45 < 3 \cdot 2^{6-2}$ which contradicts to Lemma 3.6(2). We therefore conclude that the inequality $m + n \geq 5$ is impossible to be true in Case 3. So,
m + n = 4 and then \( f(G) = f_4 \leq 13 = 2^{m+2} - 2^m + 1 \).

**Case 3.** \((i_1, t) = (3, 0)\).

In this case, \( V_1 = \{u_1, u_2\} \) can be easily seen from the definition of \( i_1 \). So, \( V_0 = \{u_i | i \in [3, m + n]\} \). Now, we have \( f(u_1) = 1, f(u_2) = 2 \) and \( f(u_3) = f_2 + 1 = 4 \). Let \( k = f(G) - (f(u_1) + f(u_2)) \) and let \( H \) be an induced subgraph of \( G \) such that \( f(H) = k \). If \( V_0 \setminus V(H) \neq \emptyset \), we denote as \( i_0 \) the minimum element in \( \{i | u_i \in V_0 \setminus V(H)\} \), then \( i_0 \geq 3 \) and \( V(H) \subseteq V_1 \cup \{u_i | i \geq i_0 + 1\} \). Therefore, \( f(H) \leq f(u_1) + f(u_2) + \sum_{i=i_0+1}^{m+n} f(u_i) \leq f(u_1) + f(u_2) + \sum_{i=3}^{m+n} f(u_i) < \sum_{i=3}^{m+n} f(u_i) = f(H) \), giving a contradiction. We then have that \( V_0 \subseteq V(H) \). Besides, since \( \sum_{i=3}^{m+n} f(u_i) = k \), we conclude that \( V(H) = V_0 \) and \( H \) is disconnected. Therefore, Case 3 is impossible to occur.

**Case 4.** \((i_1, t) = (4, 1)\).

In this case, \( V_1 = \{u_1, u_2\} \) and \( V_0 = \{u_i | i \in [3, m + n]\} \) are also true. Now, \( f(u_1) = 1, f(u_2) = 2 \) and \( f(u_3) = f_2 = 3 \) and \( f(u_4) = f_3 = 7 \). Let \( k = f(G) - (f(u_1) + f(u_2) + f(u_3)) \). In exactly the same way as we used in Case 3, one can show that Case 4 is not possible either.

Since \( f(G) \leq 2^{m+2} - 2^m + 1 \) is valid in all possible situations. The proof is completed.

4 Conclusion

In this work, we have given lower bounds on the IC-index of all complete multipartite graphs and have shown that our lower bound on \( M(K_{1(n), m}) \) is the exact value of it. Our coloring constructed in Proposition 3.1 is indeed a qualified maximal IC-coloring.

For further study of this problem, one can try to show that the lower bound given in Proposition 3.3 will also be an upper bound on \( M(K_{1(n), m_\ell, m_{\ell-1}, \ldots, m_2, m_1}) \) for all \( \ell \geq 2 \). We conjecture that the inequality in Proposition 3.3 is in fact an equality in the case where \( \ell \geq 2, m_\ell \geq 2 \) and \( n \geq 2 \).

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