A measure of non-Gaussianity for quantum states

J. Solomon Ivan · M. Sanjay Kumar · R. Simon

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Abstract We propose a measure of non-Gaussianity for quantum states of a system of $n$ oscillator modes. Our measure is based on the quasi-probability $Q(\alpha), \alpha \in \mathbb{C}^n$. Since any measure of non-Gaussianity is necessarily an attempt at making a quantitative statement on the departure of the shape of the $Q$ function from Gaussian, any good measure of non-Gaussianity should be invariant under transformations which do not alter the shape of the $Q$ functions, namely displacements, passage through passive linear systems, and uniform scaling of all the phase space variables: $Q(\alpha) \rightarrow \lambda^{2n} Q(\lambda \alpha)$. Our measure which meets this ‘shape criterion’ is computed for a few families of states, and the results are contrasted with existing measures of non-Gaussianity. The shape criterion implies, in particular, that the non-Gaussianity of the photon-added thermal states should be independent of temperature.

Keywords Non-Gaussianity measure · Quasiprobability · $Q$-function · Wehrl entropy · Photon-added thermal states

J. S. Ivan
Raman Research Institute, C. V. Raman Avenue, Sadashivanagar, Bangalore 560 080, India
e-mail: solomon@rri.res.in

M. S. Kumar
S.N. Bose National Centre for Basic Sciences, Sector-III, Block-JD, Salt Lake, Kolkata 700 098, India
e-mail: sanjay@bose.res.in

R. Simon (✉)
Optics and Quantum Information Group, The Institute of Mathematical Sciences,
C.I.T Campus, Tharamani, Chennai, 600 113, India
e-mail: simon@imsc.res.in
1 Introduction

Quantum information theory of continuous variable systems has been actively pursued in recent years, especially in the context of Gaussian states [1–3]. Such states are the ones which occur naturally in most experimental situations, particularly in quantum optics. While these states live in an infinite-dimensional Hilbert space, they are remarkably easy to handle since they are fully described by their covariance matrix (and first moments). Further, their evolution under quadratic Hamiltonians is easily cast in the language of symplectic groups and (classical) phase space [4–6]. The fundamental protocol of quantum teleportation has been achieved using these states [7,8]. However, there are situations wherein one deals with (nonclassical) non-Gaussian states to generate entanglement [9–16]. They arise naturally in nonlinear evolutions like passage through a Kerr medium [17,18]. The potential use of non-Gaussian entangled states towards quantum information processing has been explored in [19,20], and more recently, the robustness of non-Gaussian entanglement against Gaussian noise has been demonstrated in [21], suggesting that non-Gaussian states are valuable from the quantum information processing perspective. It is thus imperative that one is able to qualify and quantify the non-Gaussianity of a state in a well defined manner.

Effort in this direction has already been initiated in some recent work [22–24], where various measures of non-Gaussianity have been explored. In [22,23], Genoni et al. introduced non-Gaussianity measures based on the Hilbert–Schmidt distance and the quantum relative entropy. The main idea of their approach was to use these two quantities to evaluate how different a non-Gaussian state is from its Gaussian counterpart. And this difference is a measure of the state’s non-Gaussianity. Given the fact that both these quantities are defined on the density matrix, the corresponding measures of non-Gaussianity are evaluated directly on the density matrix. In this work we explore an alternative measure of non-Gaussianity for quantum states that has its roots in classical probability theory.

From the perspective of classical probability theory, Gaussian distributions are those probability distributions that are completely specified by their first and second moments; all their higher-order moments are determined by these lower-order moments. Non-Gaussian probabilities do not enjoy this special property. An easier, and possibly more effective, way to distinguish the two is through cumulants: every non-vanishing cumulant of order greater than two serves as an indicator of non-Gaussianity of the probability distribution under consideration [25,26]. The purpose of any good measure of non-Gaussianity in the context of classical probability theory is thus to capture the essence of the non-vanishing higher-order cumulants. A non-Gaussianity measure should thus manifestly depend on the higher-order cumulants.

The notion of non-Gaussianity can be extended to a quantum mechanical state through its definition on the associated $Q$ function, a member of the one-parameter family of $s$-ordered quasi-probabilities [27]. That this is an appropriate route is endorsed by the fact that the Marcinkiewicz theorem [see below] holds for the $s$-ordered quasi-probabilities as well. It turns out that the cumulants of order greater than 2 for the various $s$-ordered quasi-probabilities corresponding to a fixed state
\(\hat{\rho}\) are independent of \(s\), indicating that the higher order cumulants are intrinsic to the state. Moreover, all higher-order cumulants of order greater than 2 vanish identically for Gaussian states. Thus any non-vanishing higher-order cumulant of the quasi-probability indicates non-Gaussianity of the state, and this conclusion is independent of the ordering parameter \(s\).

The above considerations will suggest that any good measure of non-Gaussianity relevant in the context of classical probability theory can, with suitable modification, lead to a good measure of non-Gaussianity of quantum mechanical states, provided a state is identified through its \(Q\) function (For a brief review on such measures in classical probability theory, see [26]). The purpose of such a quantum measure would be to capture the essence of the non-vanishing higher-order cumulants of the \(Q\) function associated with the state.

A feature that is naturally demanded on any good measure of non-Gaussianity in the classical probability theory context is the invariance of the measure under overall scaling of the probability distribution. Ultimately, non-Gaussianity measure is a quantitative statement of the departure of the shape of a probability distribution from Gaussian. But uniform scaling of all the variables of a probability distribution does not alter the ‘shape’ of the distribution, and hence it should not affect its non-Gaussianity. This feature is worth demanding in the quantum mechanical context too, through the \(Q\) function for, as has been proved recently, scaling of the \(Q\) function is indeed a physical process, namely a quantum-limited amplifier channel [28,29].

The purpose of this paper is to motivate and present a measure of non-Gaussianity of quantum states. Our measure is based on the Wehrl entropy [30], the quantum analogue of differential entropy [31] well-known from the context of classical information theory of continuous variables [Differential entropy itself is a generalisation of Shannon entropy from discrete to continuous variables].

The photon-added thermal states [32] play a key role in our considerations. These nonclassical states have been generated experimentally [33–36]. Their special importance to the present work arises from the fact that the \(Q\) functions of these states are scaled versions of those of the Fock states, and therefore one will expect any good measure of non-Gaussianity to return the same values for both classes of states.

The plan of the paper is as follows. In Sect. 2 we briefly introduce the definition of moments and cumulants, and recall two well-known theorems in the context of these notions. The one-parameter family of \(s\)-ordered quasi-probabilities corresponding to quantum density operators is briefly considered in Sect. 3, with particular emphasis on the \(Q\) function, and in Sect. 4 we review the relationship between differential entropy and the Kullback–Leibler distance of classical probability theory. As a final item of preparation, we review briefly in Sect. 5 the Wehrl entropy [30] and some of its properties. With these preparations, we introduce in Sect. 6 our non-Gaussianity measure and explore some of its more important properties, including its invariance under uniform scaling of the underlying phase space. In Sect. 7 we evaluate this measure for three families of quantum states, and in Sect. 8 we compare our measure with the two other measures of non-Gaussianity which are already available [22,23]. The paper concludes in Sect. 9 with some additional remarks.
2 Moments and cumulants

For a multivariate probability distribution \( \mathcal{P}(x) \), where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), the characteristic function \( \chi(\xi) \), \( \xi \in \mathbb{R}^n \), is given by the Fourier transform of \( \mathcal{P}(x) \) [25]:

\[
\chi(\xi) = \int d^n x \mathcal{P}(x) \exp[i \cdot x] = \sum_{m_1m_2\ldots m_n} \left( \prod_{k=1}^{n} \frac{(i \xi_k)^{m_k}}{m_k!} \right) \langle x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \rangle,
\]

\[
\langle x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \rangle = \int d^n x x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \mathcal{P}(x). \tag{2.1}
\]

It follows from the invertibility of Fourier transformation that the characteristic function retains all the information contained in the probability distribution. The characteristic function is often called the moment generating function, since one obtains from it all the moments of the underlying probability distribution through this compact expression:

\[
\langle x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \rangle = \left( \prod_{k=1}^{n} \frac{d^{m_k}}{d(i \xi_k)^{m_k}} \right) \chi(\xi) \mid_{\xi=0}. \tag{2.2}
\]

Another equivalent description of a probability distribution is through the cumulant generating function. This is defined through the logarithm of the characteristic function

\[
\Gamma(\xi) = \log \chi(\xi) = \sum_{m_1m_2\ldots m_n} \left( \prod_{k=1}^{n} \frac{(i \xi_k)^{m_k}}{m_k!} \right) \gamma_{m_1,m_2\ldots,m_n}, \tag{2.3}
\]

or, equivalently, through

\[
\chi(\xi) = \exp(\Gamma(\xi)). \tag{2.4}
\]

From Eq. (2.3), it is easy to see that the cumulants \( \gamma_{m_1,m_2\ldots,m_n} \) can be expressed as

\[
\gamma_{m_1,m_2\ldots,m_k} = \left( \prod_{k=1}^{n} \frac{d^{m_k}}{d(i \xi_k)^{m_k}} \right) \Gamma(\xi) \mid_{\xi=0}. \tag{2.5}
\]

Thus, the cumulants are related to \( \Gamma(\cdot) \) in precisely the same way as the moments are related to \( \chi(\cdot) \). The set of all moments \( \langle x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \rangle \) gives a complete characterisation of a probability distribution \( \mathcal{P}(x) \), and the same is true of the set of all cumulants \( \gamma_{m_1,m_2\ldots,m_n} \) as well. Indeed, one can describe one set in terms of the other.
and given a set of real numbers $\mu_{m_1,m_2,\ldots,m_n}$ (or $\gamma_{m_1,m_2,\ldots,m_n}$), one may ask if there exists a probability distribution for which this could be the set moments (or cumulants). This is the ‘problem of moments’. See, for instance [38]).

With these notations and definitions on hand, we now recall two important results from classical probability theory.

**Theorem 1** The cumulant generating function of a Gaussian probability distribution in $n$ variables is a multinomial of degree equal to 2 [25].

**Theorem 2** (Marcinkiewicz Theorem). If the cumulant generating function of a (normalised) function in $n$ variables is a multinomial of finite degree greater than 2, then the function will not be point wise non-negative, and hence will fail to be a probability distribution [39,40].

Theorem 1 is a statement of the fact that a Gaussian probability is fully determined by its moments of order $\leq 2$; all the higher-order cumulants are identically zero for a Gaussian probability. Theorem 2 is a much stronger statement. It implies that any true probability distribution other than the Gaussian distribution has a cumulant generating function which cannot truncate at any (finite) order. That is, a non-Gaussian probability distribution has non-vanishing cumulants of arbitrarily high order. We note in passing that non-vanishing cumulants of order greater than 2 serve as indicators of the non-Gaussianity of the underlying probability.

### 3 Quasi-probabilities and the $Q$ function

A state of a quantum mechanical system specified by density operator $\hat{\rho}$ can be faithfully described by any member of the one-parameter family of $s$-ordered quasi-probability distributions $-1 \leq s < 1$ [27]. In other words, an $s$-ordered quasi-probability captures all the information present in the density operator $\hat{\rho}$. However, it is not a genuine probability distribution in general; in particular, it is not point wise non-negative. The prefix quasi underscores precisely this aspect. Nevertheless, the $s$-ordered family of quasi-probability distributions gives us a framework wherein one could give a phase space description of quantum mechanical systems in the language of classical probability theory.

For a quantum state describing the radiation field of $n$ modes ($n$ oscillators) the characteristic function of the $s$-ordered quasi-probability, for any $-1 \leq s \leq 1$, is defined through [27]

$$\chi_\rho(\xi; s) = \exp\left[\frac{s}{2} |\xi|^2 \right] \text{Tr}(\hat{\rho} D(\xi)), \quad (3.1)$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$, and $D(\xi)$ is the $n$-mode (phase space) displacement operator:

$$D(\xi) = \exp\left[ \sum_j (\xi_j \hat{a}_j^\dagger - \xi_j^* \hat{a}_j) \right]. \quad (3.2)$$
The $s$-ordered quasi-probability itself is just the Fourier transform of this characteristic function $\chi_\rho(\xi; s)$:

$$W_\rho(\alpha; s) = \int \exp \left[ \sum_j (\alpha_j^* \xi_j - \alpha_j \xi_j^*) \right] \chi_\rho(\xi; s) \prod_j d^2 \xi_j.$$  \hspace{1cm} (3.3)

Here $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators of the $j$th mode, $\alpha_j$ represents the (c-number) phase space variables $q_j$, $p_j$ corresponding to the $j$th mode through $\alpha_j = (q_j + ip_j)/\sqrt{2}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$. The particular cases $s = -1, 0, 1$ correspond, respectively, to the better known $Q$ function, the Wigner function, and the $P$ function.

The $Q$ function corresponding to a density operator $\hat{\rho}$ has a particularly simple expression in terms of coherent state projections:

$$Q(\alpha) = \langle \alpha | \hat{\rho} | \alpha \rangle, \quad \alpha \in \mathbb{C}^n. \hspace{1cm} (3.4)$$

It may be noted that the $Q$ function is manifestly nonnegative for all $\alpha \in \mathbb{C}^n$.

Reality of $W_\rho(\alpha; s)$ is equivalent to hermiticity of the density operator $\hat{\rho}$, and the fact that $\hat{\rho}$ is of unit trace faithfully transcribes to

$$\frac{1}{\pi^n} \int W_\rho(\alpha; s) d^2 \alpha = 1. \hspace{1cm} (3.5)$$

While these two properties hold for every $s$-ordered quasi-probability, point wise non negativity for all states is a distinction which applies to the $Q$ function alone. In other words, the $Q$ function is a genuine probability distribution; every other $W_\rho(\alpha; s)$ is only a quasi-probability. Gaussian pure states are the only pure states for which the Wigner function is a classical probability [41]; in the case of $P$ function, the coherent states are the only pure states with this property.

However, not every probability distribution is a $Q$ function. This is evident, for instance, from the obvious fact that $Q(\alpha) \leq 1, \forall \alpha \in \mathbb{C}^n$.

The next result captures, in a concise form, the manner in which members of the one-parameter family of $s$-ordered quasi-probabilities $W_\rho(\alpha; s)$ differ from one another for a given state $\hat{\rho}$.

**Theorem 3** Only the second order cumulants of the quasi-probability of a given state depend on the order parameter $s$; all the other cumulants are independent of the quasi-probability under consideration.

This result is already familiar in the case of a single-mode radiation field [17]. But the proof is, as outlined below, immediate in the multi-mode case as well. The characteristic functions of a state $\hat{\rho}$ for two different values of the order parameter $s_1$ and $s_2$ are obviously related in the following manner [27]:

$$\chi_\rho(\xi; s_1) = \exp \left( (s_1 - s_2) |\xi|^2 \right) \chi_\rho(\xi; s_2). \hspace{1cm} (3.6)$$
On taking logarithm of both sides to obtain the corresponding cumulant generating functions we have

$$\log \chi_{\rho}(\xi ; s_1) = (s_1 - s_2)|\xi|^2 + \log \chi_{\rho}(\xi ; s_2).$$

That is,

$$\Gamma_{\rho}(\xi ; s_1) = (s_1 - s_2)|\xi|^2 + \Gamma_{\rho}(\xi ; s_2). \quad (3.7)$$

Thus the cumulant generating function for different $s$-ordered quasi-probabilities differ only in second order, completing proof of the theorem.

In these equations $|\xi|^2$ stands, as usual, for $\sum_{j=1}^{n} |\xi_j|^2$. As an immediate consequence of this theorem we have

**Theorem 4** For no quasi-probability can the cumulant generating function be a multinomial of finite order $>2$.

**Proof** Since the $Q$ function, for every state $\rho$, is a genuine probability distribution, it follows from the Marcinkiewicz theorem that the cumulant generating function of $Q$ cannot be a multinomial of finite order $>2$. Since the different $s$-ordered quasi-probabilities differ only in second-order cumulants, this conclusion holds for all $s$-ordered quasi-probabilities, thus proving the theorem.

We conclude this Section with the following remarks. The above considerations show that quasi-probabilities fail to be true probabilities only in this limited sense: they differ from genuine probabilities only in cumulants of order two. The distributions, however, can be quite different from classical probabilities, particularly for $s > 0$, and they can become as singular as the Fourier transform of $\exp[\sigma y^2]$, $\sigma > 0$, a Gaussian with the wrong signature for the variance.

Since the higher-order cumulants, which should play an essential role in any reasonable definition of non-Gaussianity measure, do not depend on the value of the parameter $s$, they may be viewed as attributes intrinsic to the state under consideration; we may therefore use any convenient quasi-probability to capture their essence.

## 4 Differential entropy and the Kullback–Leibler distance

The role of Shannon entropy of probability distributions over discrete random variables is taken over by differential entropy in the case of continuous variables. Given a multivariate probability distribution $P(x)$ in $n$ variables $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the associated differential entropy $H(P(x))$ is defined by [31]

$$H(P(x)) = -\int d^n x P(x) \log P(x). \quad (4.1)$$

But unlike the Shannon entropy, the differential entropy can be negative. This is manifest, for instance, for a uniform distribution over a region of less than unit volume in $\mathbb{R}^n$. 

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Among all the probability distributions with a fixed set of first and second moments, the Gaussian probability distribution has the maximum differential entropy [31]. This fact may be used to modify differential entropy to construct a non-negative quantity

\[ J(\mathcal{P}(x)) = H(\mathcal{P}_G(x)) - H(\mathcal{P}(x)). \]  

(4.2)

Here \( \mathcal{P}_G(x) \) is the Gaussian probability distribution with the same first and second moments as those of the given probability distribution \( \mathcal{P}(x) \).

It may be recalled that Kullback–Leibler distance between two probabilities \( \mathcal{P}_1(x) \) and \( \mathcal{P}_2(x) \) is defined as the difference of their differential entropies [31]:

\[ S(\mathcal{P}_1(x)||\mathcal{P}_2(x)) = H(\mathcal{P}_2(x)) - H(\mathcal{P}_1(x)) = -\int \mathcal{P}_1(x) \log(\mathcal{P}_1(x)) \, dx + \int \mathcal{P}_2(x) \log(\mathcal{P}_2(x)) \, dx. \]  

(4.3)

Thus \( J(\mathcal{P}(x)) \) can be regarded as the Kullback–Leibler distance between the given probability \( \mathcal{P}(x) \) and the associated Gaussian distribution \( \mathcal{P}_G(x) \):

\[ J(\mathcal{P}(x)) = S(\mathcal{P}_G(x)||\mathcal{P}(x)). \]  

(4.4)

\( J(\mathcal{P}(x)) \) is sometimes known by the name negentropy.

5 Wehrl entropy

Wehrl entropy [30,42] may be viewed as the extension of differential entropy to the quantum mechanical context, but the Wehrl entropy has interesting properties which distinguish it from differential entropy. The distinction arises from the fact that while every \( Q \) function certainly qualifies to be a classical probability distribution, every classical probability is not a \( Q \) function. The uncertainty principle has a fundamental role to play in this aspect [30]. The potential use of Wehrl entropy as a measure of the ‘coherent’ component of a state has been discussed in Ref. [43]. And its possible role in defining an entanglement measure has also been explored [44,45].

For a state \( \hat{\rho} \) describing \( n \) modes of radiation field, the Wehrl entropy is defined as

\[ H_W(\hat{\rho}) = -\frac{1}{\pi^n} \int \prod d^2 \alpha_j Q_\rho(\alpha) \log Q_\rho(\alpha), \]  

(5.1)

where \( Q_\rho(\alpha) \) is the \( Q \) function corresponding to \( \hat{\rho} \). This definition may be compared with that of differential entropy; the role of \( \mathcal{P}(x) \) in differential entropy is played by \( Q_\rho(\alpha) \) in Wehrl entropy.

However, in contradistinction to differential entropy, the Wehrl entropy is always positive. This is an immediate consequence of the fact that the \( Q \) function is bounded from above by unity. It turns out that the Wehrl entropy is always greater than or
equal to unity \([46]\); indeed, it attains its least value of unity for the coherent states and only for these states. This property can be thought of as a manifestation of the uncertainty principle, which the coherent states saturate. Further, the Wehrl entropy is always greater than the von Neumann entropy \([30]\):

\[
H_W(\hat{\rho}) \geq S(\hat{\rho}) = -\text{Tr}(\hat{\rho} \log \hat{\rho}).
\] (5.2)

While the von Neumann entropy is zero for pure states, we have just noted that the Wehrl entropy \(H_W(\hat{\rho})\) is greater than or equal to unity for all states. Several aspects of the Wehrl entropy have been explored in Ref. \([43]\).

6 A non-Gaussianity measure for quantum states

As is well-known, a quantum state \(\hat{\rho}\) is said to be Gaussian iff the associated Wigner distribution is Gaussian. This will suggest that the non-Gaussianity of a state is coded into the non-vanishing cumulants of order \(> 2\) of the Wigner function. Since the Wigner and \(Q\) functions are related by convolution by a Gaussian, the \(Q\) function of a state is Gaussian iff the Wigner function is, and the non-Gaussianity should thus be found coded in the higher-order cumulants of the \(Q\) function as well. The consistency of these statements is ensured by the fact that the higher-order cumulants are the same for the Wigner and the \(Q\) functions \(\text{[Indeed, as we have shown earlier, the higher-order cumulants are intrinsic to the state, and hence are the same for all } s\)-ordered quasi-probabilities].

Non-Gaussianity can thus be described using either the Wigner function or the \(Q\) function. The fact that the \(Q\) function is everywhere non-negative, rendering it a genuine probability in the classical sense, makes it our preferred choice. We employ therefore the Wehrl entropy to capture the essence of the higher-order cumulants.

Given a state \(\hat{\rho}\), our measure of non-Gaussianity \(\mathcal{N}(\hat{\rho})\) is defined as the difference of two Wehrl entropies:

\[
\mathcal{N}(\hat{\rho}) = H_W(\hat{\rho}_G) - H_W(\hat{\rho}).
\] (6.1)

Here \(H_W(\hat{\rho})\) is the Wehrl entropy of the given state \(\hat{\rho}\) and \(H_W(\hat{\rho}_G)\) is the Wehrl entropy of the Gaussian state \(\hat{\rho}_G\) that has the same first and second moments as \(\hat{\rho}\). Since \(\mathcal{N}(\hat{\rho})\) measures the departure of the Wehrl entropy of \(\hat{\rho}\) from that of its Gaussian partner \(\hat{\rho}_G\), it can be viewed as a quantum Kullback–Leibler distance. \(\mathcal{N}(\hat{\rho})\) could also be viewed as a relative Wehrl entropy. But we prefer to call it simply a non-Gaussianity measure.

This measure of non-Gaussianity enjoys several interesting properties. We will now list some of them:

(i) \(\mathcal{N}(\hat{\rho}) \geq 0\), equality holding iff \(\hat{\rho}\) is Gaussian.

Proof This is a restatement of the fact that the Wehrl entropy of a Gaussian state is greater than that of all states with the same first and second moments as the Gaussian. In fact the Wehrl entropy shares this property with the differential entropy, in view of the fact that the \(Q\) function is essentially a classical probability \(\text{[See the statement before Eq. (4.2)]}\).
ii) $\mathcal{N}(\hat{\rho})$ is invariant under phase space displacements:

\[ \mathcal{N}(\hat{\rho}) = \mathcal{N} \left( D(\xi) \hat{\rho} D(\xi)^\dag \right). \]  

(6.2)

*Proof* Let $D(\xi) \hat{\rho} D(\xi)^\dag$ be denoted, for brevity, by $\hat{\rho}'$. The $Q$ function of $\hat{\rho}'$ is related to that of $\hat{\rho}$ in this simple manner:

\[ Q_{\hat{\rho}'}(\alpha) = Q_{\hat{\rho}}(\alpha - \xi). \]  

(6.3)

That is, displacement $D(\xi)$ acts as a rigid translation in phase space [30,46,47]. Thus it has no effect on the Wehrl entropy of any state, and hence leaves $\mathcal{N}(\hat{\rho})$ invariant for every state.

(iii) $\mathcal{N}(\hat{\rho})$ is invariant under passage through any passive linear system.

*Proof* A passive linear system is represented by a $n \times n$ unitary matrix $U$. It maps a coherent state $|\alpha\rangle$ into a new coherent state $|\alpha'\rangle = |U\alpha\rangle$ [30,46,47], where $\alpha \in \mathbb{C}^n$ is to be viewed as a column vector. Let $\hat{U}_U$ be the unitary operator in the $n$-mode Hilbert space which represents the passive linear system labelled by the matrix $U$. Let us denote by $\hat{\rho}'$ the transformed state $\hat{U}_U \hat{\rho} \hat{U}_U^\dag$ at the output of this passive system. Then the output $Q$ function is related to the input $Q$ function in this manner:

\[ Q_{\hat{\rho}'}(\alpha) = Q_{\hat{\rho}}(U^{-1}\alpha) = Q_{\hat{\rho}}(U^\dag \alpha). \]  

(6.4)

That is, the action of a passive linear system is a rigid $SO(2n)$ rotation in the $2n$-dimensional phase space. It follows immediately that this transformation does not change the Wehrl entropy of any state, and hence does not affect $\mathcal{N}(\hat{\rho})$.

Remark While in the single-mode case of two-dimensional phase space all proper rotations are canonical transformations, this is not true in the multi-mode case. That is, $\text{Sp}(2n, \mathbb{R}) \cap SO(2n)$ is a proper subgroup of $SO(2n)$ isomorphic to $U(n)$, the $n^2$-parameter group of $n \times n$ unitary matrices, whereas $SO(2n)$ is a much larger $(2n^2 - n)$-parameter group [47]. Only those phase space rotations which are elements of this intersection act as unitary transformations in the Hilbert space of $n$ oscillators.

(iv) $\mathcal{N}(\hat{\rho})$ is invariant under a uniform phase space scaling $\lambda$ defined at the level of the $Q$ function in the following manner:

\[ \lambda : \quad Q(\alpha) \rightarrow Q'(\alpha) = \lambda^{2n} Q(\lambda \alpha). \]  

(6.5)

*Proof* Under this uniform phase space scaling of the $Q$ function, the Wehrl entropy changes by a simple additive part that is independent of the state:
\[ H_W(\hat{\rho}) = -\frac{1}{\pi n} \int Q(\alpha) \log Q(\alpha) \prod_{j=1}^{n} d^2\alpha_j \]
\[ \rightarrow -\frac{1}{\pi n} \int \lambda^{2n} Q(\lambda \alpha) \log (\lambda^{2n} Q(\lambda \alpha)) \prod_{j=1}^{n} d^2\alpha_j \]
\[ = H_W(\hat{\rho}) - 2n \log \lambda. \] (6.6)

Note that in arriving at the last equation we have made a change of variables in the integral and made use of the normalisation of the \( Q \) function. Now it trivially follows from this result that \( N(\hat{\rho}) \), being a difference of two Wehrl entropies, remains invariant.

**Remark** While the above conclusion holds mathematically for all \( \lambda > 0 \), the scaled \( Q \) function fails to be a physical \( Q \) function if \( \lambda > 1 \) [28]. Therefore we restrict this scale parameter to the physically relevant range \( 0 < \lambda \leq 1 \). In this range, scaling of the \( Q \) function corresponds to the action of an amplifier channel on the state [28].

(v) \( N(\hat{\rho}) \) is additive on tensor product states:
\[ N(\hat{\rho}_1 \otimes \hat{\rho}_2) = N(\hat{\rho}_1) + N(\hat{\rho}_2). \] (6.7)

**Proof** Under tensor product the \( Q \) functions go as product probabilities by definition. This is true of their associated Gaussian probabilities as well.

**Corollary** For a bipartite state of the form \( \hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_G \), where \( \hat{\rho}_G \) is a Gaussian state
\[ N(\hat{\rho}) = N(\hat{\rho}_a \otimes \hat{\rho}_G) = N(\hat{\rho}_a). \] (6.8)

**Proof** From (v) we have
\[ N(\hat{\rho}) = N(\hat{\rho}_a \otimes \hat{\rho}_G) = N(\hat{\rho}_a) + N(\hat{\rho}_G). \]

and from (i)
\[ N(\hat{\rho}_a) + N(\hat{\rho}_G) = N(\hat{\rho}_a). \] (6.9)

**Proposition** For a bipartite state of the form \( \hat{\rho}_{out} = \hat{U}_U(\hat{\rho}_a \otimes |\alpha\rangle\langle \alpha|)\hat{U}_U^\dagger \), where \( U \) represents a passive linear system and \( |\alpha\rangle \) is a coherent state, we have
\[ N(\hat{\rho}_{out}) = N(\hat{\rho}_a). \] (6.10)

**Proof** From (iii) we have
\[ N(\hat{\rho}_{out}) = N(\hat{U}_U(\hat{\rho}_a \otimes |\alpha\rangle\langle \alpha|)\hat{U}_U^\dagger) = N(\hat{\rho}_a \otimes |\alpha\rangle\langle \alpha|). \]
We have from (v)
\[ N(\hat{\rho}_a \otimes |\alpha\rangle \langle \alpha|) = N(\hat{\rho}_a) + N(|\alpha\rangle \langle \alpha|). \]

Since the coherent state $|\alpha\rangle$ is Gaussian, we have from (i)
\[ N(\hat{\rho}_a) + N(|\alpha\rangle \langle \alpha|) = N(\hat{\rho}_a). \]  

(6.11)

This result is useful in evaluating the non-Gaussianity of bipartite states produced by the action of beam splitters, as we shall illustrate in the next Section.

6.1 Shape criterion for good measure of non-Gaussianity

Properties (ii), (iii), and (iv) deal with transformations which do not change the shape of the $Q$ functions. Since non-Gaussianity is a quantitative statement regarding the departure of the shape of the $Q$ function from Gaussian, it will appear that any good measure of non-Gaussianity should return the same value for all states connected by these transformations. In particular, two quantum states whose $Q$ functions are related by a uniform scaling of all the phase space coordinates should be assigned the same amount of non-Gaussianity. We will call this the shape criterion, and we have seen that our measure $N(\hat{\rho})$ meets this requirement.

7 Examples

In this Section we evaluate our non-Gaussianity measure $N(\hat{\rho})$ for three families of states, namely the Fock states, the photon-added thermal states, and the phase-averaged coherent states of a single-mode of radiation. While the first two families consist of nonclassical states, the third one is a family of classical states.

7.1 Photon number states

The $Q$ function of the Fock state (energy eigenstate) $\hat{\rho} = |m\rangle \langle m|$ of the oscillator is given by the phase space distribution
\[ Q_{|m\rangle}(\alpha) = \frac{|\alpha|^{2m}}{m!} \exp(-|\alpha|^2), \]  

(7.1)
whose only non-vanishing moment of order $\leq 2$ is $\langle |\alpha|^2 \rangle = \text{Tr}(\hat{\rho}\hat{a}\hat{a}^\dagger) = m + 1$. The phase space average $\langle |\alpha|^2 \rangle$ is with respect to the probability distribution $Q_{|m\rangle}(\alpha)$ and, by definition, it equals the (quantum) expectation value of the associated anti-normally ordered operator $\hat{a}\hat{a}^\dagger$. The Gaussian state which has the same moments of order $\leq 2$ as $\hat{\rho}_{\langle m\rangle} = |m\rangle \langle m|$ is clearly the thermal state with mean photon number $\langle \hat{n} \rangle \equiv \langle \hat{a}^\dagger \hat{a} \rangle = m$. The $Q$ function of such a thermal state $\hat{\rho}_G$ is given by
\[ Q_G(\alpha) = \frac{1}{\langle \hat{n} \rangle + 1} \exp \left( -\frac{|\alpha|^2}{\langle \hat{n} \rangle + 1} \right), \quad \langle \hat{n} \rangle = m. \]  

(7.2)
The Wehrl entropy corresponding to $\hat{\rho}_G$ is easily computed:

$$H_W(\hat{\rho}_G) = 1 + \log(1 + \langle \hat{n} \rangle)$$
$$= 1 + \log(1 + m). \tag{7.3}$$

The Wehrl entropy of the photon number state $\hat{\rho} = |m\rangle\langle m|$ is

$$H_W(\hat{\rho}_{|m\rangle}) = -\frac{1}{\pi} \int d^2\alpha Q_{|m\rangle}(\alpha) \log Q_{|m\rangle}(\alpha). \tag{7.4}$$

This can be computed explicitly by going to the polar coordinates, and one obtains [43]

$$H_W(\hat{\rho}_{|m\rangle}) = 1 + m + \log m! - m\psi(m + 1),$$
$$\psi(m + 1) = \sum_{k=1}^m \frac{1}{k} - \gamma, \tag{7.5}$$

where $\psi(m)$ is the digamma function, and $\gamma = 0.5772 \cdots$ is the Euler constant. Hence the non-Gaussianity of the photon number state $\hat{\rho} = |m\rangle\langle m|$ is

$$\mathcal{N}(\hat{\rho}_{|m\rangle}) = H_W(\hat{\rho}_G) - H_W(\hat{\rho}_{|m\rangle}),$$
$$= \ln(m + 1) - m - \log m! + m\psi(m + 1). \tag{7.6}$$

In Fig. 1 we have plotted this non-Gaussianity as a function of the photon number $m$. It is clear that the non-Gaussianity of $|m\rangle$ increases monotonically with the photon number $m$, and goes to $\infty$ as $m$ tends to $\infty$. That this was to be expected can be seen as follows. For large $m$ values $\psi(m + 1) \sim \ln(m + 1)$, and $\log m! \sim m\log m - m$, and hence $\mathcal{N}(\hat{\rho}_{|m\rangle}) \sim \log(m + 1)$. We shall be returning to this result in the next Section.

Now consider a bipartite state of two modes with one mode in the Fock state and the other in the vacuum. Non-Gaussianity of this product state is the same as that of the Fock state, and this follows from Eq. (6.8). Let this bipartite state be passed through

![Fig. 1 Variation of $\mathcal{N}(\rho)$ with number of photons $m$ for the Fock state $\rho = |m\rangle\langle m|$](image-url)
a beam splitter. The state at the output will be entangled due to the nonclassicality of the Fock state [9, 10], but in view of Eq. (6.10), this two-mode state will have the same non-Gaussianity as the original single-mode Fock state.

7.2 Photon-added thermal states

In this subsection we evaluate the non-Gaussianity of the photon-added thermal state (PATS) [32]. The PATS is defined through

\[ \hat{\rho} = C \hat{\alpha}^m \hat{\rho}_\text{th} \hat{\alpha}^m, \tag{7.7} \]

where \( C \) is the normalisation constant which ensures Tr (\( \hat{\rho} \)) = 1, and \( \hat{\rho}_\text{th} \) is the thermal state given by

\[ \hat{\rho}_\text{th} = (1 - x) \sum_{n=0}^{\infty} x^n |k\rangle \langle k|; \quad x = \exp \left[ -\frac{\hbar \omega}{kT} \right]. \tag{7.8} \]

One can alternatively define the PATS through parametric differentiation:

\[ \hat{\rho} = \frac{(1 - x)^{m+1}}{m!} \frac{d^m}{dx^m} \sum_{k=0}^{\infty} x^k |k\rangle \langle k|. \tag{7.9} \]

PATS are thus parametrised by two parameters: \( 0 \leq x < 1 \), and \( m = 0, 1, 2, \ldots \). The limit \( x \to 0 \) corresponds to Fock states, and the limit \( m \to 0 \) corresponds to thermal states.

We may note that PATS (with \( m \geq 1 \)) is nonclassical for all values of \( x \) [9]. Indeed, it violates a three-term classicality condition [14].

The \( Q \) function of PATS can be easily calculated and is given by

\[ Q_{\text{PATS}}^{(m,x)}(\alpha) = \frac{(1 - x)^{m+1}}{m!} |\alpha|^2 \exp[-(1 - x)|\alpha|^2]. \tag{7.10} \]

It is evident that the \( Q \) function of the PATS is a scaled version of the \( Q \) function of the Fock state:

\[ Q_{\text{PATS}}^{(m,x)}(\alpha) = \lambda^2 Q_{|m\rangle}(\lambda \alpha), \quad \lambda = \sqrt{1 - x}. \tag{7.11} \]

Since our measure of non-Gaussianity respects the shape criterion put forward in the previous Section, it is immediate that the non-Gaussianity of the PATS is the same as that of the photon number state:

\[ N(\hat{\rho}_{\text{PATS}}^{(m,x)}) = \ln(m + 1) - m - \log m! + m \psi(m + 1) = N(\hat{\rho}_{|m\rangle}). \tag{7.12} \]
It is worth emphasising here that the PATS is a special state with regard to the question of verifying whether a given measure of non-Gaussianity is a good measure, i.e., whether it satisfies the shape criterion. The test is as simple as checking whether the measure in question evaluated for the PATS is independent of the temperature parameter $x$ or not.

Finally we consider, as in the previous Subsection, a bipartite state of two modes, with one mode in the PATS $\hat{\rho}_{\text{PATS}}^{(m,x)}$ and the other in the vacuum state. Let us pass this two-mode state through a beam splitter. That the state at the output of the beam splitter is entangled follows from the nonclassicality of the PATS [9,10]. It follows from Eq. (6.10) that non-Gaussianity of this entangled state is the same as that of the PATS, and hence is fully determined by $m$.

We have already noted that PATS violates a three-term classicality condition. This implies that the output state is entangled and, moreover, that it is single-copy distillable [9].

### 7.3 Phase-averaged coherent states

As our final example, we evaluate the non-Gaussianity for the phase-averaged coherent states. Given a coherent state $|\beta\rangle$ its phase-averaged version is

$$
\hat{\rho}_{|\beta\rangle} = \int \frac{d\theta}{2\pi} \exp[-i \theta \hat{\alpha}^\dagger \hat{\alpha}] |\beta\rangle\langle \beta| \exp[i \theta \hat{\alpha}^\dagger \hat{\alpha}]
$$

$$
= \exp(-|\beta|^2) \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{n!} |n\rangle\langle n|.
$$

(7.13)

Since $\hat{\rho}_{|\beta\rangle}$ is a convex sum of Fock states, its $Q$ function is a corresponding convex sum:

$$
Q_{|\beta\rangle}^{\beta\alpha}(\alpha) = \exp[-(|\alpha|^2 + |\beta|^2)] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} |\beta|^{2n}}{n! n!}
$$

$$
= \exp[-(|\alpha|^2 + |\beta|^2)] I_0(2|\alpha||\beta|),
$$

(7.14)

where $I_0(.)$ is the modified Bessel function of integral order zero. The only non-zero moment of order $\leq 2$ is $\langle |z|^2 \rangle = \text{Tr}(\hat{\rho}_{|\beta\rangle} \hat{\alpha}^\dagger \hat{\alpha}) = 1 + |\beta|^2$. The associated Gaussian probability $Q_{G}^{\beta\alpha}(\alpha)$ that has the same first and second moments is thus the thermal state with average photon number $\langle \hat{n} \rangle = |\beta|^2$. As we have shown earlier in Eq. (7.3), the Wehrl entropy of this Gaussian state is $H_W(\hat{\rho}_{G}^{\beta\alpha}) = 1 + \log(1 + |\beta|^2)$. To compute the Wehrl entropy corresponding to the original phase-averaged coherent state, however, we resort to numerical evaluation. In Fig. 2 we present the non-Gaussianity of $\hat{\rho}_{|\beta\rangle}$ as a function of $|\beta|^2$, the energy of the state. It is seen to be a monotone increasing function of $|\beta|^2$.

Note that the phase-averaged coherent states are classical since they are, by definition, convex sums of coherent states. Thus if a bipartite state consisting of a
phase-averaged coherent state in one mode and vacuum in the other is passed through a beam splitter, the two-mode mixed state at the output will remain separable (since the phase-averaged coherent state is classical [9]), with the same non-Gaussianity as the original phase-averaged coherent state.

8 Comparison with other measures

In this Section we compare our non-Gaussianity measure $\mathcal{N}(\hat{\rho})$ with two non-Gaussianity measures which have been proposed recently.

8.1 Measure based on Hilbert–Schmidt distance

Genoni et al. [22], have proposed a non-Gaussianity measure based on the Hilbert–Schmidt distance. They define non-Gaussianity of a state $\hat{\rho}$ as

$$\delta_1(\hat{\rho}) = \frac{\text{Tr}[\hat{\rho} - \hat{\tau}]^2}{2\text{Tr}(\hat{\rho}^2)},$$

(8.1)
where $\hat{\tau}$ is the Gaussian state with the same first and second moments as $\hat{\rho}$. Let us compare this measure with ours in the specific case of the PATS $\hat{\rho}_{\text{PATS}}^{(m,x)}$. In Fig. 3 we plot $\delta_1(\hat{\rho}_{\text{PATS}}^{(m,x)})$ as a function of the Boltzmann parameter $x$, for fixed value of $m = 1$. It is seen that $\delta_1(\hat{\rho}_{\text{PATS}}^{(m,x)})$, for $m = 1$, is not a constant but varies with the temperature parameter $x$. This shows that this measure of Genoni et al. does not satisfy our shape criterion.

Another interesting difference appears when one compares our measure $N(\hat{\rho})$ with $\delta_1(\hat{\rho})$ in the case of the photon number states $\hat{\rho} = |m\rangle\langle m|$. As we have shown earlier [see Fig. 1], our measure monotonically increases with the photon number $m$ and tends to infinity as $m$ tends to infinity. In contrast, as Genoni et al. have shown and emphasised [22], their measure $\delta_1(\hat{\rho})$ saturates at the value $\frac{1}{2}$.

### 8.2 Measure based on quantum relative entropy

Genoni et al. [23] have proposed, in a subsequent paper, a second measure of non-Gaussianity, this one based on quantum relative entropy. They define non-Gaussianity of a state $\hat{\rho}$ as

$$\delta_2(\hat{\rho}) = S(\hat{\tau}) - S(\hat{\rho}),$$

(8.2)

where $S(\cdot)$ is the von Neumann entropy of the state in question and $\hat{\tau}$ is the Gaussian state with the same first and second moments as the given state $\hat{\rho}$.

At first sight it would seem that $\delta_2(\hat{\rho})$ and our measure $N(\hat{\rho})$ are very similar, the only difference being that $\hat{\rho}$ is replaced by $Q(z)$ and that the trace operation in the formula for the von Neumann entropy is replaced in our measure by a phase space integral. A closer look reveals that this is not the case; $\delta_2(\hat{\rho})$ does not reduce to $N(\hat{\rho})$ under this kind of ‘quantum-classical correspondence’. And $\delta_2(\hat{\rho})$ and $N(\hat{\rho})$ turn out to be quite different entities.

A qualitative difference between $\delta_2(\hat{\rho})$ and $N(\hat{\rho})$ becomes manifest when one compares these two measures in the context of a pure state. As the von Neumann entropy of a pure state is zero, $\delta_2(\hat{\rho})$ reduces to $S(\hat{\tau})$, the von Neumann entropy of the Gaussian state with the same first and second moments as $\hat{\rho}$. In other words $\delta_2(\hat{\rho})$ does not consult, in the case of pure states, moments or cumulants of $\hat{\rho}$ of order higher than 2. Consequently, all pure states which have the same set of first and second moments but differ in higher moments will get assigned the same non-Gaussianity $\delta_2(\rho)$. This is not the case with our measure $N(\hat{\rho})$.

To bring out a second qualitative difference we check if $\delta_2(\hat{\rho})$ satisfies the shape criterion. To this end we ask if $\delta_2(\hat{\rho})$ will ascribe the same amount of non-Gaussianity to the PATS and the photon number state, i.e., whether $\delta_2(\rho)$ evaluated for the PATS $\hat{\rho}_{\text{PATS}}^{(m,x)}$ is independent of the temperature parameter $x$. We find that this is not the case. This is shown in Fig. 4 wherein we present $\delta_2(\hat{\rho}_{\text{PATS}}^{(m,x)})$, for fixed value $m = 1$, as a function of $x$.

We conclude this Section with a further remark. With reference to Figs. 3 and 4, while the non-Gaussianity measures $\delta_1(\hat{\rho}_{\text{PATS}}^{(m,x)})$ and $\delta_2(\hat{\rho}_{\text{PATS}}^{(m,x)})$, for fixed $m$, vary with
Fig. 4 Variation of $\delta_2(\hat{\rho})$ as a function of the Boltzmann parameter $x$ for the photon-added thermal state

the temperature (or scale) parameter $x$, thus failing the shape criterion, the variation is not monotone. The significance of the temperatures at which these measures assume their respective minimum values is not clear.

9 Concluding remarks

We have presented a measure of non-Gaussianity of quantum states based on the $Q$ function. In doing so we have been guided by the fundamental principle that any measure of non-Gaussianity is an attempt to make a quantitative statement on the departure of the shape of the $Q$ function from Gaussian, and the measure must therefore remain invariant under all transformations which do not change the shape of the $Q$ function.

Uniform scaling of all the phase space coordinates at the level of the $Q$ function has proved to be an important shape preserving transformation, and our shape criterion demands that non-Gaussianity of the photon-added thermal states should be independent of temperature.

We have explored various properties our measure which meets the shape criterion. We have presented analytical and numerical results on the non-Gaussianity of a few families of quantum states. We have also compared our measure with other measures of non-Gaussianity available in the literature.

Our measure $\mathcal{N}(\hat{\rho})$ meets the shape criterion which, in our opinion, should be respected by every good measure of non-Gaussianity. We hasten to add, however, that this is not the only measure that meets this criterion. For instance, if $\gamma^{(2n)}$ is an appropriate linear combination of the cumulants of order $2n$, and $\gamma^{(2)}$ an appropriate linear combination of the cumulants of order 2, it is clear that the ratio between $\gamma^{(2n)}$ and the $n$th power of $\gamma^{(2)}$ will meet this criterion, for every $n \geq 2$. Our choice $\mathcal{N}(\hat{\rho})$ has the attraction of being immediately related to well-known entities like the Wehrl entropy and Kullback–Leibler distance.

In the case of classical probability defined on a $2n$ dimensional space $\mathbb{C}^n$, one would have required the non-Gaussianity measure to be invariant under the full Euclidean group consisting of translations and all SO(2n) rotations. In the case of phase space,
SO(2n) rotations which fall outside the subgroup $\text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n)$ are unphysical, and hence the restriction to this subgroup of passive linear systems.

As noted in [48], our measure rests on the invariance semi-group of $Q$ functions which is different from the invariance semi-group of the Gaussian family of states—operations which map Gaussian states into Gaussians. The latter semigroup includes the full $\text{Sp}(2n, \mathbb{R})$, and not just the intersection subgroup $\text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n)$. It further includes a whole family of completely positive maps known as Gaussian channels.

Finally, it may be noted that a scaling transformation similar to the one we have implemented on $Q$ functions cannot be implemented on Wigner functions [49]. This follows from the following fact: that $W(\alpha)$ is a Wigner function does not imply that $\lambda^{2n} W(\lambda \alpha)$ is necessarily a Wigner function.

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