Characterization of continuous endomorphisms of the space of entire functions of a given order

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\textbf{ABSTRACT}

In this paper we show that continuous endomorphisms on the space of entire functions with a given order can be expressed as differential operators of infinite order satisfying suitable growth conditions.

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1. Introduction

The aim of this paper is to characterize continuous endomorphisms of the space of entire functions of exponential type of order $p > 0$. Let $A_p$ denote the space of entire functions of $n$ complex variables of order $p$ and normal type with its DFS-topology (see [1], for example). We consider an endomorphism $F$ of this space, and we show that there is a unique linear differential operator $P$ of infinite order with coefficients in $A_p$, and satisfying suitable growth conditions, which realizes $F$, that is such that $Ff = Pf$ holds for any $f \in A_p$. Conversely, if a formal differential operator $P$ of infinite order has coefficients in $A_p$ that satisfy these conditions, then the operator induces a continuous endomorphism on $A_p$.

The problem of characterizing linear continuous operators acting on a given function space has a long history. Peetre [2,3] proved that any morphism of sheaves of smooth functions can be represented by a differential operator locally of finite order. Note that Peetre did not make any continuity assumption. In the analytic category, or in the ultradifferentiable category, operators of infinite order naturally appear as continuous endomorphisms of function spaces. For example, any continuous sheaf endomorphism (local operator) of the
sheaf of holomorphic functions is expressed by a differential operator of infinite order satisfying certain estimates for coefficients. See [4,5] for the analytic category, and [6] for the ultradifferentiable category. For spaces of entire functions of several variables, the problem of characterizing continuous endomorphisms was first considered by the second author [7], who introduced a special class of differential operators of infinite order and then discussed the characterization of continuous endomorphisms. A similar class was introduced in [8,9] with an eye towards to superoscillations (see the references cited in these articles). We also note that several similar classes of infinite order differential operators with constant coefficients or with polynomial coefficients were introduced and studied in [10] and our work is inspired by this thesis. In this article, we give a complete answer to the characterization problem of the continuous endomorphisms in the space of entire functions of a given order. A partial answer to this problem, for the case where \( p > 1 \) and of normal type, was announced in [11].

The plan of this paper is as follows. In the second section, we recall some definitions of special spaces of entire functions and we give a refined estimate for the derivatives of functions in these spaces. In the third section, we introduce a class of differential operators of infinite order and show that it can be identified with the space of continuous endomorphisms of the space of entire functions of a given order of normal type. In the fourth section, we give a similar result for the case of minimal type. The last section contains some examples and points out an immediate generalization of our results.

### 2. The spaces of entire functions of a given order

Let \( p \) and \( \tau \) be positive numbers. Let \( A_{p,\tau} \) denote the set of all entire functions \( f \) of \( n \) complex variables \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \) such that

\[
\|f\|_{p,\tau} := \sup_{z \in \mathbb{C}^n} |f(z)| \exp(-\tau |z|^p) < \infty, (1)
\]

where \( |z| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots |z_n|^2} \). We call \( \|f\|_{p,\tau} \) the \((p,\tau)\)-norm of \( f \). The set \( A_{p,\tau} \) becomes a Banach space with this norm. If \( \tau < \tau' \), there is a natural inclusion mapping \( A_{p,\tau} \hookrightarrow A_{p,\tau'} \). This mapping is a compact operator. Let \( A_p \) (resp. \( A_{p,0} \)) denote the inductive limit (resp. the projective limit) of the family \( \{A_{p,\tau}\}_{\tau > 0} \) of the Banach spaces:

\[
A_p := \lim_{\tau \to 0} A_{p,\tau} \quad (\text{resp. } A_{p,0} := \lim_{\tau \to 0} A_{p,\tau}). (2)
\]

This becomes a DFS (resp. FS) space (cf. [1]). An entire function \( f \in A_p \) (resp. \( f \in A_{p,0} \)) is said to be of order at most \( p \) of normal type (resp. of order at most \( p \) of minimal type). A linear operator \( F : A_p \longrightarrow A_p \) is continuous if and only if for any \( \tau > 0 \) there exist \( C > 0 \) and \( \tau' > 0 \) for which \( F(A_{p,\tau}) \subset A_{p,\tau'} \) and

\[
\|Ff\|_{p,\tau'} \leq C\|f\|_{p,\tau}
\]

holds for any \( f \in A_{p,\tau} \) ([12, Chap 4, Part 1, 5, Corollary 1]).

Similarly, a linear operator \( F : A_{p,0} \longrightarrow A_{p,0} \) is continuous if and only if for any \( \tau > 0 \), there exist \( C > 0 \) and \( \tau' > 0 \) for which

\[
\|Ff\|_{p,\tau} \leq C\|f\|_{p,\tau'}
\]

holds for any \( f \in A_{p,0} \).
To unify descriptions, we set \( s_p = \max\{2^{p-1}, 1\} \). The following estimates for derivatives will be effectively used:

**Lemma 2.1:** Let \( f \) be an element of \( A_{p, \tau} \), and write

\[
\partial^\alpha_z = \left( \frac{\partial}{\partial z_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial z_2} \right)^{\alpha_2} \cdots \left( \frac{\partial}{\partial z_n} \right)^{\alpha_n}
\]

for a multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0} \), \( \alpha! := \alpha_1! \alpha_2! \cdots \alpha_n! \), and \( |\alpha| := \alpha_1 + \alpha_2 + \cdots + |\alpha_n| \). Then \( \partial^\alpha_z f \in A_{p, s_p \tau} \) and we have

\[
\|\partial^\alpha_z f\|_{p, s_p \tau} \leq \frac{\alpha!}{|\alpha|^{1/p}} (e \tau p)^{|\alpha|/p} (2n^{1/2})^{|\alpha|} \|f\|_{p, \tau}.
\]

**Proof:** The Cauchy formula shows that

\[
|\partial^\alpha_z f(z)| \leq \frac{\alpha!}{r|\alpha|} \sup_{|\zeta| = r} |f(z + \zeta)|
\]

holds for any \( r > 0 \), where we set \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \). Since \( f \in A_{p, \tau} \), we have

\[
|f(z)| \leq \|f\|_{p, \tau} \exp(\tau |z|^p)
\]

for all \( z \in \mathbb{C}^n \). Hence we have

\[
|f(z + \zeta)| \leq \begin{cases} 
\|f\|_{p, \tau} \exp(\tau (|z|^p + |\zeta|^p)) & \text{if } 0 < p \leq 1, \\
\|f\|_{p, \tau} \exp(2^{p-1} \tau (|z|^p + |\zeta|^p)) & \text{if } 1 < p.
\end{cases}
\]

Here we have used the Hölder inequality for the case \( p > 1 \). Combining (4) and (6), we have

\[
|\partial^\alpha_z f(z)| \leq \begin{cases} 
\frac{\alpha!}{r|\alpha|} \|f\|_{p, \tau} \exp(\tau |z|^p) \exp(\tau n^{p/2} r^p) & \text{if } 0 < p \leq 1, \\
\frac{\alpha!}{r|\alpha|} \|f\|_{p, \tau} \exp(2^{p-1} \tau |z|^p) \exp(2^{p-1} \tau n^{p/2} r^p) & \text{if } 1 < p.
\end{cases}
\]

for all \( r > 0 \). Taking the minimum of the right-hand sides of (7) with respect to \( r \), we have (3). \( \blacksquare \)

### 3. The case of normal type

Consider a multisequence \( \{a_\alpha(z)\} \) of entire functions where \( \alpha \) runs over \( \mathbb{Z}^n_{\geq 0} \). Let \( p \) be a positive number and \( q \) a real number satisfying \( (1/p) + (1/q) = 1 \) if \( p \neq 1 \), while if \( p = 1 \), we set \( (1/q) = 0 \). We will use the standard convention for multi-indices, that is, for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0} \), we set \( \alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \ldots, \alpha_n \pm \beta_n) \) and \( \alpha^\beta = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \) for \( z = (z_1, z_2, \ldots, z_n) \), etc. We write \( \beta \leq \alpha \) if and only if \( \beta_k \leq \alpha_k \) for all \( k = 1, 2, \ldots, n \).
**Definition 3.1:** We will denote by $D_p$ the set of all formal differential operator $P$ of the form

$$P = \sum_{\alpha} a_{\alpha}(z) \partial^\alpha_z,$$

where the multisequence \( \{a_{\alpha}(z)\} \) is such that for any $\varepsilon > 0$, there exist $B > 0$ and $C > 0$ such that

$$\|a_{\alpha}\|_{p,B} \leq C \frac{|\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!}$$

holds for any $\alpha \in \mathbb{Z}_n^n_{\geq 0}$.

For the sake of completeness, we offer the following simple result of independent interest:

**Proposition 3.2:** Let \( \{a_{\alpha}(z)\} \) be a multisequence of entire functions. The condition given in Definition 3.1 is equivalent to any of the following alternative conditions:

1. For any $\varepsilon > 0$, there exist $B > 0$ and $C > 0$ for which

$$|a_{\alpha}(z)| \leq C \frac{\varepsilon |\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!}$$

holds for any $z \in \mathbb{C}^n$, $\alpha \in \mathbb{Z}_n^n_{\geq 0}$.

2. For any $\varepsilon > 0$, there exist $B > 0$ and $C > 0$ for which

$$|a_{\alpha}(z)| \leq C \frac{\varepsilon |\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!}$$

holds for any $z \in \mathbb{C}^n$, $\alpha \in \mathbb{Z}_n^n_{\geq 0}$.

3. For any $\varepsilon > 0$, there exist $B > 0$ and $C > 0$ such that

$$\|a_{\alpha}\|_{p,B} \leq C \frac{\varepsilon |\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!}$$

holds for any $\alpha \in \mathbb{Z}_n^n_{\geq 0}$.

**Proof:** The equivalence of the first two conditions follows from the following estimates, which are an immediate consequence of the Stirling formula and hold for any $\alpha \in \mathbb{Z}_n^n_{\geq 0}$:

$$n^{-|\alpha|} e^{-|\alpha|/p} \frac{|\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!} \leq \frac{1}{\alpha!^{1/\varepsilon}} \leq \frac{|\alpha|^{\varepsilon/|\alpha|} \exp(B|z|^p)}{\alpha!}.$$

The equivalence of the first and the third (resp. the second and the condition given in Definition 3.1) conditions comes directly from the definition of the $(p, \tau)$-norm $\| \cdot \|_{p,\tau}$.

**Remark:** In what follows we will use the terminology ‘linear continuous operator’ instead of ‘continuous endomorphism’ which is included in the title of this paper. These two terms
have the same meaning: the latter is shorter but the former seems to be more familiar in functional analysis.

**Theorem 3.3:**

(i) Let $P \in D_p$ be of the form (8). For any entire function $f \in A_p$, the series

$$Pf := \sum_{\alpha} a_{\alpha}(z) \partial^{\alpha} f$$

converges and $Pf \in A_p$. Moreover, $f \mapsto Pf$ defines a linear continuous operator $P : A_p \to A_p$.

(ii) Let $F : A_p \to A_p$ be a linear continuous operator. Then there is a unique $P \in D_p$ such that $Ff = Pf$ holds for any $f \in A_p$.

**Proof:** Using Lemma 2.1 and estimate (9) for $a_{\alpha}$, we have for any $f \in A_{p,\tau}$,

$$\|Pf\|_{p,B+s_p\tau} \leq \sum_{\alpha} \|a_{\alpha}\|_{p,B} \|\partial^{\alpha} f\|_{p,s_p\tau}$$

$$\leq C \sum_{\alpha} \frac{|\alpha|^{\alpha/p}}{\alpha!} e^{\alpha} \frac{\alpha!}{|\alpha|^{\alpha/p}} (e \tau p)^{\frac{\alpha}{p}} (2n^{\frac{1}{2}})^{\alpha} \|f\|_{p,\tau}$$

(13)

$$\leq 2^{n-1} C \sum_{k=0}^{\infty} (e \tau p)^{k/p} (4\varepsilon n^{\frac{1}{2}})^{k} \|f\|_{p,\tau}.$$

Here we have used the following two inequalities:

$$\|fg\|_{p,\tau+\sigma} \leq \|f\|_{p,\tau} \|g\|_{p,\sigma} \quad \text{for } f \in A_{p,\tau}, \ g \in A_{p,\sigma},$$

and

$$\sum_{|\alpha|=k} 1 = \binom{n+k-1}{k} \leq 2^{n+k-1}.$$

If $\varepsilon$ is sufficiently small, the last sum in (13) converges. For such an $\varepsilon > 0$, we set $\tau' = B + s_p \tau$ and

$$C' = \frac{2^{n-1} C}{1 - 4n^{\frac{1}{2}} (e \tau p)^{1/p} \varepsilon}.$$

Then we have

$$\|Pf\|_{p,\tau'} \leq C' \|f\|_{p,\tau}.$$

Hence $Pf \in A_{p,\tau'}$ and we have obtained the continuity of $P : A_p \to A_p$. 

(ii) Let \( F : A_p \to A_p \) be a linear continuous operator. This means that for any \( \tau > 0 \), there exist \( C > 0 \) and \( \tau' > 0 \) such that for any \( f \in A_{p,\tau} \), we have \( Ff \in A_{p,\tau'} \) and
\[
\|Ff\|_{p,\tau'} \leq C\|f\|_{p,\tau}.
\] (14)
Let us define a family of entire functions \( \{a_\alpha(z)\} (\alpha \in \mathbb{Z}_n\geq 0) \) by
\[
a_\alpha(z) = \sum_{\beta \leq \alpha} \frac{(-1)^{|\alpha - \beta|}z^{\alpha - \beta}}{(\alpha - \beta)!\beta!}Fz^\beta,
\] (15)
whose convergence in \( A_p \) will be proved together with their estimates. We define a formal differential operator \( P \) of infinite order by
\[
P = \sum_\alpha a_\alpha(z)\partial^\alpha_z.
\] (16)
First we show that \( P \in D_p \). For any \( \beta \in \mathbb{Z}_n\geq 0 \), we have
\[
\|z^\beta\|_{p,\tau} \leq \left( \frac{|\beta|}{e\tau p} \right)^{|\beta|/p}.
\] (17)
In fact, \( |z^\beta| \exp(-\tau|z|^p) \) is dominated by
\[
|z^\beta| \exp(-\tau r^p)
\] if \( |z| = r \). Taking the maximum of this function with respect to \( r \), we obtain (17). For any \( \tau > 0 \), there exist \( C > 0 \) and \( \tau' > 0 \) such that
\[
\|Fz^\beta\|_{p,\tau'} \leq C\|z^\beta\|_{p,\tau}.
\] (18)
Hence we have
\[
\|a_\alpha\|_{p,\tau+\tau'} \leq \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!\beta!} \|z^{\alpha - \beta}\|_{p,\tau} \|Fz^\beta\|_{p,\tau'}
\leq C \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)!\beta!} \left( \frac{|\alpha - \beta|}{e\tau p} \right)^{|\alpha - \beta|/p} \left( \frac{|\beta|}{e\tau p} \right)^{|\beta|/p}
\leq C \sum_{\beta \leq \alpha} \frac{\alpha!}{(\alpha - \beta)!\beta!} |\alpha|^{1/p} \left( \frac{1}{e\tau p} \right)^{|\alpha|/p}
\leq C \frac{2^{|\alpha|}}{\alpha!} \left( \frac{|\alpha|}{e\tau p} \right)^{|\alpha|/p}.
\] (19)
For a given \( \varepsilon > 0 \), we can take \( \tau > 0, C > 0 \) and \( \tau' > 0 \) so that (14) and
\[
\left( \frac{2p}{e\tau p} \right)^{1/p} < \varepsilon
\]
hold. If we set \( B = \tau + \tau' \), we have
\[
\|a_\alpha\|_{p,B} \leq C \frac{2^{|\alpha|}}{\alpha!} \varepsilon^{|\alpha|}
\]
for any \( \alpha \). This implies \( P \in D_p \).
\[\blacksquare\]
To finish the proof, we compute $Pf$ for $f \in A_{p,\tau}$. We take the Taylor expansion of $f$:

$$f(z) = \sum_{\mu} \frac{f_\mu}{\mu!} z^\mu \quad (f_\mu = \partial_z^\mu f(0)). \quad (20)$$

For any $m \in \mathbb{N}$, we can write

$$f(z) - \sum_{|\mu| \leq m-1} \frac{f_\mu}{\mu!} z^\mu = \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} \partial_t^m f(tz) \, dt. \quad (21)$$

We set $f_{m,t}(z) := \partial_t^m f(tz)$. We have

$$\|f_{m,t}\|_{p,s_{\tau}+\tau'} \leq \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|\partial_z^\alpha f\|_{p,s_{\tau}} \|z^\alpha\|_{p,\tau'}$$

$$\leq m! 2^{m-1} \left( \frac{4n^{\frac{1}{2}} \tau^{1/p}}{\tau'_{1/p}} \right)^m \|f\|_{p,\tau}.$$

If $4n^{\frac{1}{2}} \tau^{1/p} < \tau'_{1/p}$, the $(p, s_{\tau} + \tau')$-norm of the right-hand side of (21) converges to zero when $m \to \infty$. This means that the Taylor expansion (20) converges in $A_{p,\tau''}$ for some $\tau'' > 0$. By the continuity of $F$, we have

$$Ff = \sum_{\mu} \frac{f_\mu}{\mu!} Fz^\mu. \quad (22)$$

On the other hand, by the definition of $a_\alpha$, we have

$$Pf = \sum_{\alpha} a_\alpha(z) \sum_{\alpha \leq \mu} \frac{f_\mu}{\mu!} \partial_z^\alpha z^\mu$$

$$= \sum_{\beta \leq \alpha \leq \mu} \frac{(-z)^{\alpha-\beta}}{(\alpha - \beta)! \beta!} (Fz^\beta) \cdot \frac{f_\mu}{(\mu - \alpha)!} z^{\mu-\alpha}$$

$$= \sum_{\mu} \frac{f_\mu}{\mu!} Fz^\mu$$

$$= Ff.$$

Here we have used the identity

$$\sum_{\gamma \leq \mu-\beta} \frac{(-1)^\gamma (\mu - \beta)!}{\gamma!(\mu - \beta - \gamma)!} = 0$$

which holds for any $\beta \leq \mu$ satisfying $\beta \neq \mu$. This completes the proof of the existence of $P$. As to the unicity, this follows immediately from the fact that if two such operators $P$ and $Q$ existed, then $P - Q$ would annihilate every function in $A_p$, and thus $P - Q$ would be the zero operator, namely each coefficient $a_\alpha(z)$ in $P - Q = \sum_\alpha a_\alpha(z) \partial_z^\alpha$ would vanish. Indeed if $P - Q$ annihilates every function in $A_p$, then in particular $P - Q$ annihilates every
monomial. This implies that we would have \((P - Q)z^\beta = a_\beta(z) + \sum_{\alpha \leq \beta, |\alpha| < |\beta|} a_\alpha(z) \cdot (\beta!/(\beta - \alpha)!))z^{\beta - \alpha} = 0\) for any \(\beta \in \mathbb{N}^n\). Then \(a_\beta(z) \equiv 0\) follows inductively in \(|\beta|\).

**Corollary 3.4:** The set \(D_p\) becomes a ring under natural addition and multiplication as differential operators.

### 4. The case of minimal type

Using the same notations as in the previous section, we now study the case in which minimal type growth is considered.

**Definition 4.1:** We will denote by \(D_{p,0}\) the set of all formal differential operators \(P\) of the form

\[
P = \sum_\alpha a_\alpha(z) \partial^\alpha z,
\]

where the multisequence \(\{a_\alpha(z)\}\) is such that for any \(\epsilon > 0\), there exist \(B > 0\) such that

\[
\|a_\alpha\|_{p,\epsilon} \leq \frac{|\alpha|}{\alpha!} B^{|\alpha|+1}
\]

holds for any \(\alpha \in \mathbb{Z}_n^\geq 0\).

Just as in the case of normal type, the condition that defines these operators can be rewritten in several equivalent forms.

**Proposition 4.2:** Let \(\{a_\alpha(z)\}\) be a multisequence of entire functions. The condition of Definition 4.1 is equivalent to any one of the following conditions:

(I)\(_0\) For any \(\epsilon > 0\), there exist \(B > 0\) for which

\[
|a_\alpha(z)| \leq \frac{B^{|\alpha|+1}}{|\alpha|^{1/q}} \exp(\epsilon |z|^p)
\]

holds for any \(z \in \mathbb{C}^n, \alpha \in \mathbb{Z}_n^\geq 0\).

(II)\(_0\) For any \(\epsilon > 0\), there exist \(B > 0\) for which

\[
|a_\alpha(z)| \leq \frac{|\alpha|}{\alpha!} B^{|\alpha|+1} \exp(\epsilon |z|^p)
\]

holds for any \(z \in \mathbb{C}^n, \alpha \in \mathbb{Z}_n^\geq 0\).

(III)\(_0\) For any \(\epsilon > 0\), there exist \(B > 0\) such that

\[
\|a_\alpha\|_{p,\epsilon} \leq \frac{B^{|\alpha|+1}}{|\alpha|^{1/q}}
\]

holds for any \(\alpha \in \mathbb{Z}_n^\geq 0\).
Theorem 4.3:  
(i) Let $P \in D_{p,0}$ be of the form (23). For an entire function $f \in A_{p,0}$, 
$$Pf := \sum_{\alpha} a_{\alpha}(z) \partial_{z}^{\alpha} f$$
converges and $Pf \in A_{p,0}$. Moreover, $f \mapsto Pf$ defines a linear continuous operator $P : A_{p,0} \rightarrow A_{p,0}$.
(ii) Let $F : A_{p,0} \rightarrow A_{p,0}$ be a linear continuous operator. Then there is a unique $P \in D_{p,0}$ such that $Ff = Pf$ holds for any $f \in A_{p,0}$.

Proof:  
(i) We assume condition (24) for $a_{\alpha}$. Similar computations in (13) yield
$$\|Pf\|_{p,\varepsilon + s_{p}\tau} \leq \sum_{\alpha} \|a_{\alpha}\|_{p,\varepsilon}\|\partial_{z}^{\alpha} f\|_{p,s_{p}\tau}$$
$$\leq \sum_{\alpha} \frac{\alpha!}{|\alpha|! |\alpha|/p} B^{\alpha+1} \frac{\alpha!}{|\alpha|/p} (e\tau p)^{|\alpha|/p} (2n^\frac{1}{2})^{|\alpha|/p} \|f\|_{p,\tau}$$
$$\leq 2^{n-1} B \sum_{k=0}^{\infty} (e\tau p)^{k/p} (4Bn^\frac{1}{2})^k \|f\|_{p,\tau}$$
for any $\varepsilon, \tau > 0, f \in A_{p,0}$. For any $\sigma > 0$, we can choose $\varepsilon > 0$ and $\tau > 0$ so that $\varepsilon + s_{p}\tau \leq \sigma$ and $4(e\tau p)^{1/p} Bn^\frac{1}{2} < 1$. Thus there exist $C > 0$ and $\tau > 0$ such that
$$\|Pf\|_{p,\sigma} \leq C\|f\|_{p,\tau}.$$ 

This means that $P : A_{p,0} \rightarrow A_{p,0}$ is continuous.

(ii) For a given endomorphism $F : A_{p,0} \rightarrow A_{p,0}$, we construct $P = \sum_{\alpha} a_{\alpha}(z) \partial_{z}^{\alpha}$ in the same way as in Theorem 3.3, (ii), namely,
$$a_{\alpha}(z) = \sum_{\beta \leq \alpha} (-1)^{|\alpha - \beta|} \frac{\alpha - \beta}{(\alpha - \beta)! \beta!} Fz_{\beta}.$$
The right-hand side converges in $A_{p,0}$ for any $\alpha$. For any $\sigma > 0$, there exist $C > 0$ and $\sigma' > 0$ so that $\|Fz_{\beta}\|_{p,\sigma} \leq C\|z_{\beta}\|_{p,\sigma'}$ holds for all $\beta$. We can prove estimates similar to (19) and obtain
$$\|a_{\alpha}\|_{p,\sigma + \tau} \leq \sum_{\beta \leq \alpha} \frac{\|z^{\alpha - \beta}\|_{p,\tau}}{(\alpha - \beta)! \beta!} \|Fz_{\beta}\|_{p,\sigma}$$
$$\leq C \sum_{\beta \leq \alpha} \frac{1}{(\alpha - \beta)! \beta!} \left(\frac{|\alpha - \beta|}{e\tau p}\right) |\alpha - \beta|/p \left(\frac{|\beta|}{e\sigma' p}\right)^{|\beta|/p}.$$ 

For any $\varepsilon > 0$, we choose $\sigma, \tau > 0$ so that $\sigma + \tau < \varepsilon$ holds. We can take $\tau$ as $\tau < \sigma'$. Hence $\|a_{\alpha}\|_{p,\varepsilon} \leq \|a_{\alpha}\|_{p,\sigma + \tau}$ is dominated by
$$C \frac{2^{\frac{|\alpha|}{p}}}{\alpha!} \left(\frac{|\alpha|}{e\tau p}\right)^{|\alpha|/p}.$$
If we set

\[ B = \max \left\{ C, 2 \left( \frac{1}{e \tau p} \right)^{1/p} \right\}, \]

we have

\[ \|a_\alpha\|_{p,\varepsilon} \leq \frac{|\alpha|^{1/p}}{\alpha!} B^{|\alpha| + 1}. \]

This implies \( P \in D_{p,0} \). Convergence of the Taylor expansion of \( f \) in \( A_{p,0} \) can be proved similarly to Theorem 3.3, (ii). Hence \( Ff = Pf \) follows. The uniqueness, again, is immediate.

\[ \blacksquare \]

**Corollary 4.4:** The set \( D_{p,0} \) is a ring under the natural addition and multiplication of differential operators.

### 5. Examples

We give some examples of operators in \( D_p \) or \( D_{p,0} \). Note that any differential operators of finite order with coefficients in \( A_p \) (resp. \( A_{p,0} \)) belong to \( D_p \) (resp. \( D_{p,0} \)) for any \( p > 0 \).

**Example 5.1:** For \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \), the translation operator \( P = \sum \frac{a_\alpha}{\alpha!} \partial^\alpha z \) belongs to \( D_p \) and to \( D_{p,0} \) for any \( p > 0 \). The action of \( P \) is given as \( Pf(z) = f(z + a) \) for \( f \in A_p \) or \( f \in A_{p,0} \).

We note that in [8,9], a class of differential operators of infinite order was introduced, which was denoted by \( D_{p,0} \). This is a subring of the ring \( D_p \) introduced in Section 3 and the subscript ‘0’ is not related to the ring \( D_{p,0} \) introduced in Section 4.

**Example 5.2:** Let \( \sigma \) be a positive number and set

\[ P = \sum_\alpha \frac{\sigma^{|\alpha|} z^\alpha}{\alpha!} \partial^\alpha z. \]

This is a dilation operator, namely,

\[ Pf(z) = f((1 + \sigma)z). \]

It follows from (17), and by the estimate in Definition 3.1, that this operator belongs to \( D_p \) for all \( p > 0 \). We can also see that \( P \notin D_{p,0} \).
Example 5.3: Let us consider the following initial value problem in $\mathbb{C}_{t,z}$:

$$
\begin{align*}
\frac{i}{t} \frac{\partial \psi}{\partial t} &= \left( -\frac{1}{2} \frac{\partial^2}{\partial z^2} + z \right) \psi, \\
\psi(0, z) &= \phi(z).
\end{align*}
$$

This is formally solved as

$$
\psi(t, z) = \exp \left( \frac{t}{i} \left( -\frac{1}{2} \frac{\partial^2}{\partial z^2} + z \right) \right) \phi(z).
$$

The operator appeared in the right-hand side can be written in the form

$$
P := \exp \left( -i \left( tz + \frac{t^3}{6} \right) \right) \exp \left( \frac{it}{2} \frac{\partial^2}{\partial z^2} + \frac{t^2}{2} \frac{\partial}{\partial z} \right).
$$

If $1 \leq p < 2$, this is an element of $D_p$ on $\mathbb{C}_z$ for any $t$. Hence for every $\phi \in A_p$, (30) has a unique solution $\psi = P\phi$ which belongs to $A_p$ for any $t$.

Remark: In this article, we considered the spaces $A_p$ or $A_{p,0}$ for a constant order $p$. The proofs of the results obtained in Sections 3 and 4 are in fact valid for the case of the spaces of entire functions of exponential type with respect to a given proximate order $\rho(r)$ introduced by Valiron [13] in one variable case (for such spaces in several variable case, see for example [14]) and we can give the complete answer to the corresponding characterization problem of the continuous endomorphisms. We remark that in [7], one of the authors considered this problem but gave only a partial answer which is incomplete comparing with the results of present article.

Remark: The authors became interested in these problems as a consequence of recent work on the phenomenon of superoscillations. As it is shows, for example, in [8] and [9], the question of whether superoscillations persist when evolved according to physically significant Schrödinger equations can be reduced, in most cases, to the study of the continuity of specific infinite order differential operators that are naturally associated to the potential that appears in the Schrödinger equation. The results, and the examples, in this paper are therefore particularly significant in this context, and the reader interested in more details is referred to the recent literature on superoscillations.

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References

[1] Berenstein CA, Gay R. Complex analysis and special topics in harmonic analysis. New York: Springer-Verlag; 1995.

[2] Peetre J. Une caractérisation abstraite des opérateurs différentiels. Math Scand. 1959;7:211–218.
[3] Peetre J. Récification à l'article 'Une caractérisation abstraite des opérateurs différentiels'. Math Scand. 1960;8:116–120.

[4] Ishimura R. Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels. Mem Fac Sci Kyushu Univ. 1978;32:301–312.

[5] Sato M, Kawai T, Kashiwara M. Microfunctions and pseudo-differential equations. Berlin: Springer; 1973, p. 265–529. (Lecture Notes in Math.; 287).

[6] Komatsu H. Ultradistributions. II. The kernel theorem and ultradistributions with support in a submanifold. J Fac Sci Univ Tokyo Sect IA Math. 1977;24:607–628.

[7] Ishimura R. Endomorphisms of the space of higher-order entire functions and infinite-order differential operators. Kyushu J Math. 2007;61:83–94.

[8] Aoki T, Colombo F, Sabadini I, et al. Continuity theorems for a class of convolution operators and applications to superoscillations. Annali di Matematica Pura e Applicata. 2018;197:1533–1545.

[9] Aoki T, Colombo F, Sabadini I, et al. Continuity of some operators arising in the theory of superoscillations. Quantum Studies: Math Found. 2018;5(3):463–476.

[10] Van der Steen P. On differential operators of infinite order [Doctoral dissertation] Delft: Technical University of Delft Uitgeverij Waltman; 1968.

[11] Aoki T, Ishimura R, Struppa DC, et al. Linear continuous operators acting on the space of entire functions of a given order. RIMS Kōkyūroku. 2019;2101:1–6.

[12] Grothendieck A. Topological vector spaces. New York: Gordon and Breach Science Publishers; 1973. (Notes on mathematics and its applications).

[13] Valiron G. Lectures on the general theory of integral functions. Toulouse: Privat; 1923.

[14] Lelong P, Gruman L. Entire functions of several complex variables. Berlin: Springer-Verlag; 1980. (Grund. Mth. Wiss.; 282).