COMPACT GROUPS AND THEIR REPRESENTATIONS

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Abstract. This is an overview article on compact Lie groups and their representations, written for the Encyclopedia of Mathematical Physics to be published by Elsevier.

In this article we describe the structure and representation theory of compact Lie groups. Throughout the article, $G$ is a compact real Lie group with Lie algebra $\mathfrak{g}$. Unless otherwise stated, $G$ is assumed to be connected. The word “group” will always mean a “Lie group” and the word “subgroup” will mean a closed Lie subgroup. The notation $\text{Lie}(H)$ stands for the Lie algebra of a Lie group $H$. We assume that the reader is familiar with the basic facts of the theory of Lie groups and Lie algebras, which can be found in the article Lie groups: general theory in this Encyclopedia, or in the books listed in the bibliography.

1. EXAMPLES OF COMPACT LIE GROUPS

Examples of compact groups include

- Finite groups
- Quotient groups $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or more generally, $V/L$, where $V$ is a finite-dimensional real vector space and $L$ is a lattice in $V$, i.e. a discrete subgroup generated by some basis in $V$. Groups of this type are called tori; it is known that every commutative connected compact group is a torus.
- Unitary groups $U(n)$ and special unitary groups $SU(n)$, $n \geq 2$.
- Orthogonal groups $O(n)$ and $SO(n)$, $n \geq 3$.
- The groups $U(n, \mathbb{H})$, $n \geq 1$, of unitary quaternionic transformations, which are isomorphic to $\text{Sp}(n) := \text{Sp}(n, \mathbb{C}) \cap SU(2n)$.

The groups $O(n)$ have two connected components, one of which is $SO(n)$. The groups $SU(n)$ and $Sp(n)$ are connected and simply-connected.

The groups $SO(n)$ are connected but not simply-connected: for $n \geq 3$, the fundamental group of $SO(n)$ is $\mathbb{Z}_2$. The universal cover of $SO(n)$ is a simply-connected compact Lie group denoted by $\text{Spin}(n)$. For small $n$, we have isomorphisms: $\text{Spin}(3) \simeq SU(2)$, $\text{Spin}(4) \simeq SU(2) \times SU(2)$, $\text{Spin}(5) \simeq Sp(4)$, $\text{Spin}(6) \simeq SU(4)$.

2. RELATION TO SEMISIMPLE LIE ALGEBRAS AND LIE GROUPS

2.1. Reductive groups. A Lie algebra $\mathfrak{g}$ is called

- simple if it is non-abelian and has no ideals different from $\{0\}$ and $\mathfrak{g}$ itself.
- semisimple if it is a direct sum of simple ideals.
- reductive if it is a direct sum of semisimple and commutative ideals.

We call a connected Lie group $G$ simple or semisimple if $\text{Lie}(G)$ has this property.

Theorem 1. Let $G$ be a connected compact Lie group and $\mathfrak{g} = \text{Lie}(G)$. Then

1. The Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is reductive: $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g'}$, where $\mathfrak{a}$ is abelian and $\mathfrak{g'} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple.
2. The group $G$ can be written in the form $G = (A \times K)/Z$, where $A$ is a torus, $K$ is a connected, simply-connected compact semisimple Lie group, and $Z$ is a finite central subgroup in $A \times K$.
3. If $G$ is simply-connected, it is a product of simple compact Lie groups.

The proof of this result is based on the fact that the Killing form of $\mathfrak{g}$ is negative semidefinite.
Example 1. The group \( U(n) \) contains as the center the subgroup \( C \) of scalar matrices. The quotient group \( U(n)/C \) is simple and isomorphic to \( SU(n)/\mathbb{Z}_n \). The presentation of Theorem 1 in this case is

\[
U(n) = (T^1 \times SU(n))/\mathbb{Z}_n = (C \times SU(n))/(C \cap SU(n)).
\]

For the group \( SO(4) \) the presentation is \((SU(2) \times SU(2))/\{\pm(1 \times 1)\}\).

This theorem effectively reduces the study of the structure of connected compact groups to the study of simply-connected compact simple Lie groups.

2.2. Complexification of a compact Lie group. Recall that for a real Lie algebra \( g \), its complexification is \( g_\mathbb{C} = g \otimes \mathbb{C} \) with obvious commutator. It is also well-known that \( g_\mathbb{C} \) is semisimple or reductive iff \( g \) is semisimple or reductive respectively. There is a subtlety in the case of simple algebras: it is possible that a real Lie algebra is simple, but its complexification \( g_\mathbb{C} \) is only semisimple. However, this problem never arises for Lie algebras of compact groups: if \( g \) is a Lie algebra of a real compact Lie group, then \( g \) is simple if and only if \( g_\mathbb{C} \) is simple.

The notion of complexification for Lie groups is more delicate.

Definition 1. Let \( G \) be a connected real Lie group with Lie algebra \( g \). A complexification of \( G \) is a connected complex Lie group \( G_\mathbb{C} \) (i.e. a complex manifold with a structure of a Lie group such that group multiplication is given by a complex analytic map \( G_\mathbb{C} \times G_\mathbb{C} \to G_\mathbb{C} \)) which contains \( G \) as a closed subgroup, and such that \( \text{Lie}(G_\mathbb{C}) = g_\mathbb{C} \). In this case, we will also say that \( G \) is a real form of \( G_\mathbb{C} \).

It is not obvious why such a complexification exists at all; in fact, for arbitrary real group it may not exist. However, for compact groups we do have the following theorem.

Theorem 2. Let \( G \) be a connected compact Lie group. Then it has a unique complexification \( G_\mathbb{C} \supset G \). Moreover, the following properties hold:

1. The inclusion \( G \subset G_\mathbb{C} \) is a homotopy equivalence. In particular, \( \pi_1(G) = \pi_1(G_\mathbb{C}) \) and the quotient space \( G_\mathbb{C}/G \) is contractible.
2. Every finite-dimensional representation of \( G \) can be uniquely extended to a complex analytic representation of \( G_\mathbb{C} \).

Since a Lie algebra of a compact Lie group \( G \) is reductive, we see that \( G_\mathbb{C} \) must be reductive; if \( G \) is semisimple or simple, then so is \( G_\mathbb{C} \). The natural question is whether every complex reductive group can be obtained in this way. The following theorem gives a partial answer.

Theorem 3. Every connected complex semisimple Lie group \( H \) has a compact real form: there is a compact real subgroup \( K \subset H \) such that \( H = K_\mathbb{C} \). Moreover, such a compact real form is unique up to conjugation.

Example 2. The unitary group \( U(n) \) is a compact real form of the group \( GL(n, \mathbb{C}) \).

The orthogonal group \( SO(n) \) is a compact real form of the group \( SO(n, \mathbb{C}) \).

The group \( Sp(n) \) is a compact real form of the group \( Sp(n, \mathbb{C}) \).

The universal cover of \( GL(n, \mathbb{C}) \) has no compact real form.

These results have a number of important applications. For example, they show that study of representations of a semisimple complex group \( H \) can be replaced by the study of representations of its compact form; in particular, every representation is completely reducible (this argument is known as Weyl's unitary trick).

2.3. Classification of simple compact Lie groups. Theorem 1 essentially reduces such classification to classification of simply-connected simple compact groups, and Theorem 2. Theorem 3 reduce it to the classification of simple complex Lie algebras. Since the latter is well-known, we get the following result.

Theorem 4. Let \( G \) be a connected, simply-connected simple compact Lie group. Then \( g_\mathbb{C} \) must be a simple complex Lie algebra and thus can be described by a Dynkin diagram of one of the following types: \( A_n \), \( B_n \), \( C_n \), \( D_n \), \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \), \( G_2 \).

Conversely, for each Dynkin diagram in the above list, there exists a unique up to isomorphism simply-connected simple compact Lie group whose Lie algebra is described by this Dynkin diagram.
For types \(A_n, \ldots, D_n\), the corresponding compact Lie groups are well-known classical groups shown in the table below:

| \(A_n, n \geq 1\) | \(B_n, n \geq 2\) | \(C_n, n \geq 3\) | \(D_n, n \geq 4\) |
|-----------------|-----------------|-----------------|-----------------|
| SU\((n + 1)\)   | Spin\((2n + 1)\) | Sp\((n)\)       | Spin\((2n)\)    |

The restrictions on \(n\) in this table are made to avoid repetitions which appear for small values of \(n\). Namely, \(A_1 = B_1 = C_1\), which gives SU\((2) = \text{Spin}\((3) = \text{Sp}(1)\); \(D_2 = A_1 \cup A_1\), which gives \text{Spin}\((4) = \text{SU}(2) \times \text{SU}(2)\); \(B_2 = C_2\), which gives \text{SO}(5) = \text{Sp}(4)\) and \(A_3 = D_3\), which gives SU\((4) = \text{Spin}(6)\). Other than that, all entries are distinct.

Exceptional groups \(E_6, \ldots, G_2\) also admit explicit geometric and algebraic descriptions which are related to the exceptional non-associative algebra \(\mathbb{O}\) of so-called octonions (or Cayley numbers). For example, the compact group of type \(G_2\) can be defined as a subgroup of SO\((7)\) which preserves an almost complex structure on \(S^6\). It can also be described as the subgroup of GL\((7, \mathbb{R})\) which preserves one quadratic and one cubic form, or, finally, as a group of all automorphisms of \(\mathbb{O}\).

3. Maximal tori

3.1. Main properties. In this section, \(G\) is a compact connected Lie group.

**Definition 2.** A maximal torus in \(G\) is a maximal connected commutative subgroup \(T \subset G\).

The following theorem lists the main properties of maximal tori.

**Theorem 5.**

1. For every element \(g \in G\), there exists a maximal torus \(T \ni g\).
2. Any two maximal tori in \(G\) are conjugate.
3. If \(g \in G\) commutes with all elements of a maximal torus \(T\), then \(g \in T\).
4. A connected subgroup \(H \subset G\) is a maximal torus iff the Lie algebra \(\text{Lie}(H)\) is a maximal abelian subalgebra in \(\text{Lie}(G)\).

**Example 3.** Let \(G = U(n)\). Then the set \(T\) of diagonal unitary matrices is a maximal torus in \(G\); moreover, every maximal torus is of this form after a suitable unitary change of basis. In particular, this implies that every element in \(G\) is conjugate to a diagonal matrix.

**Example 4.** Let \(G = \text{SO}(3)\). Then the set \(D\) of diagonal matrices is a maximal commutative subgroup in \(G\), but not a torus. Here \(D\) consists of 4 elements and is not connected.

3.2. Maximal tori and Cartan subalgebras. The study of maximal tori in compact Lie groups is closely related to the study of Cartan subalgebras in reductive complex Lie algebras (see \[\text{Se}\] for a definition of a Cartan subalgebra).

**Theorem 6.** Let \(G\) be a connected compact Lie group with Lie algebra \(\mathfrak{g}\), and let \(T \subset G\) be a maximal torus in \(G\), \(t = \text{Lie}(T) \subset \mathfrak{g}\). Let \(\mathfrak{g}_C, G_C\) be the complexification of \(\mathfrak{g}, G\) as in Theorem 2.

Let \(\mathfrak{h} = t_C \subset \mathfrak{g}_C\). Then \(\mathfrak{h}\) is a Cartan subalgebra in \(\mathfrak{g}_C\). Conversely, every Cartan subalgebra in \(G_C\) can be obtained as \(t_C\) for some maximal torus \(T \subset G\).

This allows us to use results about Cartan subalgebras (such as root decomposition, properties of root systems, etc.) when studying compact Lie groups. From now on, we will denote by \(R \subset \mathfrak{h}^*\) the root system of \(\mathfrak{g}_C\); it can be shown that in fact, \(R \subset i\mathfrak{t}^*\). We will also use notation \(\alpha^\vee \subset i\mathfrak{t}\) for the coroot corresponding to a root \(\alpha \in R\). Definitions of these and related notions can be found in \[\text{Se}\] or in the article *Lie groups: general theory* in this encyclopedia.

3.3. Weights and roots. Let \(G\) be semisimple. Recall that the root lattice \(Q \subset i\mathfrak{t}^*\) is the abelian group generated by roots \(\alpha \in R\), and let the coroot lattice \(Q^\vee \subset i\mathfrak{t}\) be the abelian group generated by coroots \(\alpha^\vee\), \(\alpha \in R\). Define also the weight and coweight lattices by

\[
P = \{ \lambda \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \text{ } \forall \alpha \in R \} \subset i\mathfrak{t}^*
\]

\[
P^\vee = \{ t \mid \langle t, \alpha \rangle \in \mathbb{Z} \text{ } \forall \alpha \in R \} \subset i\mathfrak{t}
\]
where \( \langle \cdot, \cdot \rangle \) is the pairing between \( t \) and the dual vector space \( t^* \).

It follows from the definition of root system that we have inclusions

\[
Q \subset P \subset i t^*
\]

\[
Q^\vee \subset P^\vee \subset i t
\]

Both \( P, Q \) are lattices in \( it^* \); thus, the index \( (P : Q) \) is finite. It can be computed explicitly: if \( \alpha_i \) is a basis of the root system, then the fundamental weights \( \omega_i \) defined by

\[
\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}
\]

form a basis of \( P \). The simple roots \( \alpha_i \) are related to fundamental weights \( \omega_j \) by the Cartan matrix \( A \): \( \alpha_i = \sum A_{ij} \omega_j \). Therefore, \( (P : Q) = (P^\vee : Q^\vee) = |\det A| \).

Definitions of \( P, Q, P^\vee, Q^\vee \) if \( g \) also make sense when \( g \) reductive but not semisimple. However, in this case they are no longer lattices: \( \text{rk} Q < \dim t^* \), and \( P \) is not discrete.

We can now give more precise information about the structure of the maximal torus.

**Lemma 1.** Let \( T \) be a compact connected commutative Lie group, and \( t = \text{Lie}(T) \) its Lie algebra. Then the exponential map is surjective and preimage of unit is a lattice \( L \subset t \). There is an isomorphism of Lie groups

\[
\exp : t/L \to T.
\]

In particular, \( T \cong \mathbb{R}^r / \mathbb{Z}^r = T^r, \quad r = \dim T \).

Let \( X(T) \subset it^* \) the lattice dual to \( \frac{1}{2\pi i} L \):

\[
X(T) = \{ \lambda \in it^* \mid \langle \lambda, l \rangle \in 2\pi i \mathbb{Z} \quad \forall l \in L \}.
\]

It is called the **character lattice** for \( T \) (see Section 4.3).

**Theorem 7.** Let \( G \) be a compact connected Lie group, and let \( T \subset G \) be a maximal torus in \( G \).

Then \( Q \subset X(T) \subset P \). Moreover, the group \( G \) is uniquely determined by the Lie algebra \( g \) and the lattice \( X(T) \in t^* \) which can be any lattice between \( Q \) and \( P \).

**Corollary.** For a given complex semisimple Lie algebra \( a \), there are only finitely many (up to isomorphism) compact connected Lie groups \( G \) with \( g_C = a \).

The largest of them is the simply-connected group, for which \( T = t/2\pi i Q^\vee \), \( X(T) = P \); the smallest is the so-called **adjoint group**, for which \( T = t/2\pi i P^\vee \), \( X(T) = Q \).

**Example 5.** Let \( G = U(n) \). Then \( it = \{ \text{real diagonal matrices} \} \). Choosing the standard basis of matrix units in \( it \), we identify \( it \cong \mathbb{R}^n \), which also allows us to identify \( it^* \cong \mathbb{R}^n \). Under this identification,

\[
Q = \{ (\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}, \sum \lambda_i = 0 \},
\]

\[
P = \{ (\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{R}, \lambda_i - \lambda_j \in \mathbb{Z} \},
\]

\[
X(T) = \mathbb{Z}^n.
\]

Note that \( Q, P \) are not lattices: \( Q \cong \mathbb{Z}^{n-1} \), \( P \cong \mathbb{R} \times \mathbb{Z}^{n-1} \).

Now let \( G = SU(n) \). Then \( it^* = \mathbb{R}^n / \mathbb{R} \cdot (1, \ldots, 1) \), and \( Q, P \) are the images of \( Q, P \) for \( G = U(n) \) in this quotient. In this quotient they are lattices, and \( (P : Q) = n \). The character lattice in this case is \( X(T) = P \), since \( SU(n) \) is simply-connected. The adjoint group is \( PSU(n) = SU(n) / C \), where \( C = \{ \lambda \cdot \text{id} \mid \lambda^n = 1 \} \) is the center of \( SU(n) \).

3.4. **Weyl group.** Let us fix a maximal torus \( T \subset G \). Let \( N(T) \subset G \) be the normalizer of \( T \) in \( G \):

\[
N(T) = \{ g \in G \mid gTg^{-1} = T \}.
\]

For any \( g \in N(T) \) the transformation \( A(g) : t \mapsto gtg^{-1} \) is an automorphism of \( T \). According to Theorem 5, this automorphism is trivial iff \( g \in T \). So in fact, it is the quotient group \( N(T) / T \) which acts on \( T \).

**Definition 3.** The group \( W = N(T) / T \) is called the **Weyl group** of \( G \).
Since the Weyl group acts faithfully on $t$ and $t^*$, it is common to consider $W$ as a subgroup in $GL(t^*)$. It is known that $W$ is finite.

The Weyl group can also be defined in terms of Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}_\mathbb{C}$.

**Theorem 8.** The Weyl group coincides with the subgroup in $GL(t^*)$ generated by reflections $s_\alpha : x \mapsto x - \frac{2(\alpha, x)}{\langle \alpha, \alpha \rangle}$, $\alpha \in R$. Here $(, )$ is the bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$.

**Theorem 9.**

1. Two elements $t_1, t_2 \in T$ are conjugate in $G$ iff $t_2 = w(t_1)$ for some $w \in W$.
2. There exists a natural homeomorphism of quotient spaces $G/AdG \simeq T/W$, where $AdG$ stands for action of $G$ on itself by conjugation. (Note, however, that these quotient spaces are not manifolds: they have singularities.)
3. Let us call a function $f$ on $G$ central if $f(gh^{-1}) = f(g)$ for any $g, h \in G$. Then the restriction map gives an isomorphism
   \[
   \{\text{continuous central functions on } G\} \simeq \{W\text{-invariant continuous functions on } T\}
   \]

**Example 6.** Let $G = U(n)$. The set of diagonal unitary matrices is a maximal torus, and the Weyl group is the symmetric group $S_n$ acting on diagonal matrices by permutations of entries. In this case, Theorem 9 shows that if $f(U)$ is a central function of a unitary matrix, then $f(U) = \hat{f} (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$ are eigenvalues of $U$ and $\hat{f}$ is a symmetric function in $n$ variables.

### 4. Representations of Compact Groups

**4.1. Basic notions.** By a representation of $G$ we understand a pair $(\pi, V)$ where $V$ is a complex vector space and $\pi$ is a continuous homomorphism $G \to \text{Aut}(V)$. This notation is often shortened to $\pi$ or $V$. In this article we only consider finite-dimensional representations; in this case, the homomorphism $\pi$ is automatically smooth and even real-analytic.

We associate to any f.d. representation $(\pi, V)$ of $G$ the representation $(\pi_*, V)$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which is just the derivative of the map $\pi : G \to \text{Aut} V$ at the unit point $e \in G$. In terms of the exponential map we have the following commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & \text{Aut} V \\
\exp & \uparrow & \uparrow \exp \\
\mathfrak{g} & \xrightarrow{\pi_*} & \text{End} V.
\end{array}
\]

Choosing a basis in $V$, we can write the operators $\pi(g)$ and $\pi_*(X)$ in matrix form and consider $\pi$ and $\pi_*$ as matrix valued functions on $G$ and $\mathfrak{g}$. The diagram above means that

\[
\pi(\exp X) = e^{\pi_*(X)}.
\]

Recall that if $G$ is connected, simply-connected, then every representation of $\mathfrak{g}$ can be uniquely lifted to a representation of $G$. Thus, classification of representations of connected simply-connected Lie groups is equivalent to the classification of representations of Lie algebras.

Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be two representation of the same group $G$. An operator $A \in \text{Hom}(V_1, V_2)$ is called an intertwining operator, or simply an intertwiner, if $A \circ \pi_1(g) = \pi_2(g) \circ A$ for all $g \in G$. Two representations are called equivalent if they admit an invertible intertwiner. In this case, using an appropriate choice of bases, we can write $\pi_1$ and $\pi_2$ by the same matrix-valued function.

Let $(\pi, V)$ be a representation of $G$. If all operators $\pi(g)$, $g \in G$, preserve a subspace $V_1 \subset V$, then the restrictions $\pi_1(g) = \pi(g)|_{V_1}$ define a subrepresentation $(\pi_1, V_1)$ of $(\pi, V)$. In this case, the quotient space $V_2 = V/V_1$ also has a canonical structure of a representation, called the quotient representation.

A representation $(\pi, V)$ is called reducible if it has a non-trivial (different from $V$ and $\{0\}$) subrepresentation. Otherwise it is called irreducible.
We call representation \((\pi, V)\) **unitary** if \(V\) is a Hilbert space and all operators \(\pi(g), g \in G\), are unitary, i.e. given by unitary matrices in any orthonormal basis. We use a short term “unirrep” for a “unitary irreducible representation”.

### 4.2. Main theorems

The following simple but important result was one of the first discoveries in representation theory. It holds for representations of any group, not necessarily compact.

**Theorem 10** (Schur Lemma). Let \((\pi_i, V_i), i = 1, 2\), be any two irreducible finite-dimensional representations of the same group \(G\). Then any intertwiner \(A: V_1 \to V_2\) is either invertible or zero.

**Corollary 1.** If \(V\) is an irreducible f.d. representation, then any intertwiner \(A: V \to V\) is scalar: \(A = c \cdot \text{id}, c \in \mathbb{C}\).

**Corollary 2.** Every irreducible representation of a commutative group is one-dimensional.

The following theorem is the fundamental results of the representation theory of compact groups. Its proof is based on the technique of invariant integrals on a compact group, which will be discussed in the next section.

**Theorem 11.**

1. Any f.d. representation of a compact group is equivalent to a unitary representation.
2. Any f.d. representation is completely reducible: it can be decomposed into direct sum

\[ V = \bigoplus n_i V_i \]

where \(V_i\) are pairwise non-equivalent unirreps. Numbers \(n_i \in \mathbb{Z}_+\) are called **multiplicities**.

### 4.3. Examples of representations

The representation theory looks rather different for abelian (i.e. commutative) and non-abelian groups. Here we consider two simplest examples of both kinds.

Our first example is a 1-dimensional compact connected Lie group. Topologically it is a circle which we realize as a set \(T \cong U(1)\) of all complex numbers \(t\) with absolute value 1.

Every unirrep of \(T\) is one-dimensional; thus, it is just a continuous multiplicative map \(\pi\) of \(T\) to itself. It is well-known that every such a map has the form

\[ \pi_k(t) = t^k \quad \text{for some} \quad k \in \mathbb{Z}. \]

The collection of all unirreps of \(T\) is itself a group, called **Pontrjagin dual** of \(T\) and denoted by \(\hat{T}\). This group is isomorphic to \(\mathbb{Z}\).

By Theorem 1 any f.d. representation \(\pi\) of \(T\) is equivalent to a direct sum of 1-dimensional unirreps. So, an equivalence class of \(\pi\) is defined by the multiplicity function \(\mu\) on \(\hat{T} = \mathbb{Z}\) taking non-negative values:

\[ \pi \simeq \bigoplus \mu(k) \cdot \pi_k. \]

The many-dimensional case of compact connected abelian Lie group can be treated in a similar way. Let \(T\) be a torus, i.e. an abelian compact group, \(t = \text{Lie}(T)\). Then every irreducible representation of \(T\) is one-dimensional and thus is defined by a group homomorphism \(\chi: T \to \mathbb{U}(1)\). Such homomorphisms are called **characters** of \(T\). One easily sees that such characters themselves form a group (Pontrjagin dual of \(T\)). If we denote by \(L\) the kernel of the exponential map \(t \to T\) (see Lemma 4), one easily sees that every character has a form

\[ \chi(\exp(t)) = e^{(t, \lambda)} \quad \lambda \in it^*, \quad \lambda \in X(T) \]

where \(X(T) \subset t^*\) is the lattice defined by \(4\). Thus, we can identify the group of characters \(\hat{T}\) with \(X(T)\). In particular, this shows that \(\hat{T} \cong \mathbb{Z}^{\dim T}\).

The second example is the group \(G = SU(2)\), the simplest connected, simply-connected non-abelian compact Lie group. Topologically \(G\) is a 3-dimensional sphere since the general element of \(G\) is a matrix of the form

\[ g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \]
Let $V$ be 2-dimensional complex vector space, realized by column vectors $\begin{pmatrix} u \\ v \end{pmatrix}$. The group $G$ acts naturally on $V$. This action induces the representation $\Pi$ of $G$ in the space $S(V)$ of all polynomials in $u, v$. It is infinite-dimensional, but has many f.d. subrepresentations. In particular, let $S^k(V)$, or simply $S^k$, be the space of all homogeneous polynomials of degree $k$. Clearly, $\dim S^k = k + 1$.

It turns out that the corresponding f.d. representations $(\Pi_k, S^k)$, $k \geq 0$, are irreducible, pairwise non-equivalent and exhaust the set $\hat{G}$ of all unirreps.

Some particular case are of special interest:

1. $k = 0$. The space $V_0$ consists of constant functions and $\Pi_0$ is the trivial 1-dimensional representation: $\Pi_0(g) \equiv 1$.
2. $k = 1$. The space $V_1$ is identical to $V$ and $\Pi_1$ is just the tautological representation $\pi(g) \equiv g$.
3. $k = 2$. The space $V_2$ is spanned by monomials $a^2, uv, v^2$. The remarkable fact is that this representation is equivalent to a real one. Namely, in the new basis $x = \frac{a^2 + b^2}{2}, y = \frac{a^2 - b^2}{2i}, z = iuv$ we have

$$\Pi_2 \begin{pmatrix} a \\ b \\ \sigma \end{pmatrix} = \begin{pmatrix} \Re(a^2 + b^2) & 2\Im(ab) & 2\Im(b^2 - a^2) \\ 2\Im(a\sigma) & |a|^2 - |b|^2 & 2\Re(a\sigma) \\ 3(a^2 + b^2) & 2\Re(ab) & \Re(a^2 - b^2) \end{pmatrix}.$$ 

This formula defines a homomorphism $\Pi_2 : SU(2) \to SO(3)$. It can be shown that this homomorphism is surjective, and its kernel is the subgroup $\{ \pm 1 \} \subset SU(2)$:

$$1 \to \{ \pm 1 \} \to SU(2) \to SO(3) \to 1.$$ 

The simplest way to see it is to establish the equivalence of $\Pi_1$ with the adjoint representation of $G$ in $\mathfrak{g}$. The corresponding intertwiner is

$$S^2 \ni (\alpha + i\gamma)u^2 + 2i\beta uv + (\alpha - i\gamma)v^2 \leftrightarrow \begin{pmatrix} i\beta & \alpha + i\gamma \\ -\alpha + i\gamma & -i\beta \end{pmatrix} \in \mathfrak{g}.$$ 

Note that $SU(2)$ and $SO(3)$ are the only compact groups associated with the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

The group $G$ contains the subgroup $H$ of diagonal matrices, isomorphic to $\mathbb{T}^1$. Consider the restriction of $\Pi_n$ to $\mathbb{T}^1$. It splits into the sum of unirreps $\pi_k$ as follows:

$$\text{Res}^\mathbb{T}^{\mathbb{T}^1}_n \Pi_n = \sum_{s=0}^{s=\lfloor n/2 \rfloor} \pi_{n-2s}.$$ 

The characters $\pi_k$ which enter this decomposition are called the weights of $\Pi_n$. The collection of all weights (together with multiplicities) forms a multiset in $\hat{\mathbb{T}}$ denoted by $P(\Pi_n)$, or $P(S^n)$.

Note the following features of this multiset:

1. $P(\Pi_n)$ is invariant under reflection $k \mapsto -k$.
2. All weights of $\Pi_n$ are congruent modulo 2.
3. The non-equivalent unirreps have different multisets of weights.

Below we show how these features are generalized to all compact connected Lie groups.

### 5. Fourier Transform

#### 5.1. Haar measure and invariant integral.

The important feature of compact groups is the existence of so-called “invariant integral”, or “average”.

**Theorem 12.** For every compact Lie group $G$, there exists a unique measure $dg$ on $G$, called **Haar measure**, which is invariant under left shifts $L_g$: $h \mapsto gh$ and satisfies $\int_G dg = 1$.

In addition, this measure is also invariant under right shifts $h \mapsto hg$ and under involution $h \mapsto h^{-1}$.

Invariance of the Haar measure implies that for every integrable function $f(g)$, we have

$$\int_G f(g) dg = \int_G f(hg) dg = \int_G f(gh) dg = \int_G f(g^{-1}) dg$$
For a finite group $G$ the integral with respect to the Haar measure is just averaging over the group:

$$
\int_G f(g) \, dg = \frac{1}{|G|} \sum_{g \in G} f(g)
$$

For compact connected Lie groups the Haar measure is given by a differential form of top degree which is invariant under right and left translations.

For a torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with real coordinates $\theta_k \in \mathbb{R}/\mathbb{Z}$ or complex coordinates $t_k = e^{2\pi i \theta_k}$, the Haar measure is $d^n \theta := d\theta_1 d\theta_2 \cdots d\theta_n$ or $d^n t := \prod_{k=1}^n dt_k$.

In particular, consider a central function $f$ (see Theorem 4). Since every conjugacy class contains elements of the maximal torus $T$ (see Theorem 5), such a function is determined by its values on $T$, and integral of a central function can be reduced to integration over $T$. The resulting formula is called Weyl integration formula. For $G = U(n)$ it looks as follows:

$$
\int_{U(n)} f(g) \, dg = \frac{1}{n!} \int_T f(t) \prod_{i<j} |t_i - t_j|^2 d^n t
$$

where $T$ is the maximal torus consisting of diagonal matrices

$$
t = \text{diag}(t_1, \ldots, t_n), \quad t_k = e^{2\pi i \theta_k},
$$

and $d^n t$ is defined above.

Weyl integration formula for arbitrary compact group $G$ can be found in [Si] or [B, Section 18].

The main applications of the Haar measure is the proof of complete reducibility theorem (Theorem 11) and orthogonality relations (see below).

### 5.2. Orthogonality relations and Peter-Weyl theorem.

Let $V_1, V_2$ be unirreps of a compact group $G$. Taking any linear operator $A : V_1 \to V_2$ and averaging the expression $A(g) := \pi_2(g^{-1}) \circ A \circ \pi_1(g)$ over $G$, we get an intertwining operator $(A) = \int_G A(g) \, dg$. Comparing this fact with the Schur lemma, one obtains the following fundamental results.

Let $(\pi, V)$ be any unirrep of a compact group $G$. Choose any orthonormal basis $\{v_k, 1 \leq k \leq \dim V\}$ in $V$ and denote by $t_{kl}^\pi$, or $t_{kl}$, the function on $G$ defined by

$$
t_{kl}^\pi(g) = (\pi(g)v_l, v_k).
$$

The functions $t_{kl}^\pi$ are called **matrix elements** of the unirrep $(\pi, V)$.

**Theorem 13** (Orthogonality relations).

1. The matrix elements $t_{kl}^\pi$ are pairwise orthogonal and have norm $(\dim V)^{-1/2}$ in $L^2(G, dg)$.
2. The matrix elements corresponding to equivalent unirreps span the same subspace in $L^2(G, dg)$.
3. The matrix elements of two non-equivalent unirreps are orthogonal.
4. The linear span of all matrix elements of all unirreps is dense in $C(G), C^\infty(G)$ and in $L^2(G, dg)$. (Generalized Peter-Weyl theorem).

In particular, this theorem implies that the set $\hat{G}$ of equivalence classes of unirreps is countable.

For a f.-d. representation $(\pi, V)$ we introduce the **character** of $\pi$ as a function

$$
\chi_\pi(g) = \text{tr} \pi(g) = \sum_{k=1}^{\dim V} t_{kk}^\pi(g).
$$

It is obviously a central function on $G$.

**Remark.** Traditionally, in representation theory the word “character” has two different meanings: 1. A multiplicative map from a group to $U(1)$. 2. The trace of a representation operator $\pi(g)$. For 1-dimensional representations both notions coincide.

> From orthogonality relations we get the following result.

**Corollary.** The characters of unirreps of $G$ form an orthonormal basis in the subspace of central functions in $L^2(G, dg)$.
5.3. **Non-commutative Fourier transform.** The non-commutative Fourier transform on a compact group $G$ is defined as follows. Let $\hat{G}$ denote the set of equivalence classes of unirreps of $G$. Choose for any $\lambda \in \hat{G}$ a representation $(\pi_\lambda, V_\lambda)$ of class $\lambda$ and an orthonormal basis in $V_\lambda$. Denote by $d(\lambda)$ the dimension of $V_\lambda$.

We introduce the Hilbert space $L^2(\hat{G})$ as the space of matrix-valued functions on $\hat{G}$ whose value at a point $\lambda \in \hat{G}$ belongs to $\text{Mat}_{d(\lambda)}(\mathbb{C})$. The norm is defined as

$$\|F\|_{L^2(\hat{G})}^2 = \sum_{\lambda \in \hat{G}} d(\lambda) \cdot \text{tr} \left( F(\lambda)F(\lambda)^* \right).$$

For a function $f$ on $G$ define its Fourier transform $\tilde{f}$ as a matrix valued function on $\hat{G}$:

$$\tilde{f}(\lambda) = \int_G f(g^{-1})\pi_\lambda(g) dg.$$ 

Note that in the case $G = T^1$ this transform associates to a function $f$ the set of its Fourier coefficients. In general this transform keeps some important features of Fourier coefficients.

**Theorem 14.**

1. For a function $f \in L^1(G, dg)$ the Fourier transform $\tilde{f}$ is well defined and bounded (by matrix norm) function on $\hat{G}$.

2. For a function $f \in L^1(G, dg) \cap L^2(G, dg)$ the following analog of the Plancherel formula holds:

$$\|f\|_{L^2(G, dg)}^2 := \int_G |f(g)|^2 dg = \sum_{\lambda \in \hat{G}} d(\lambda) \cdot \text{tr} \left( \tilde{f}(\lambda)\tilde{f}(\lambda)^* \right) =: \|\tilde{f}\|_{L^2(\hat{G})}^2.$$ 

3. The following inversion formula expresses $f$ in terms of $\tilde{f}$:

$$f(g) = \sum_{\lambda \in \hat{G}} d(\lambda) \cdot \text{tr} \left( \tilde{f}(\lambda)\pi_\lambda(g) \right).$$ 

4. The Fourier transform send the convolution to the matrix multiplication:

$$\widehat{f_1 \ast f_2} = \tilde{f}_1 \cdot \tilde{f}_2$$

where the convolution product $\ast$ is defined by

$$(f_1 \ast f_2)(h) = \int_G f_1(hg)f_2(g^{-1}) dg.$$ 

Note the special case of the inversion formula for $g = e$:

$$f(e) = \sum_{\lambda \in \hat{G}} d(\lambda) \cdot \text{tr} \left( \tilde{f}(\lambda) \right), \quad \text{or} \quad \delta(g) = \sum_{\lambda \in \hat{G}} d(\lambda) \cdot \chi_\lambda(g)$$

where $\delta(g)$ is Dirac’s delta function: $\int_G f(g)\delta(g) dg = f(e)$. Thus we get a presentation of Dirac’s delta function as a linear combination of characters.

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6. **Classification of finite-dimensional representations**

In this section, we give a classification of unirreps of a connected compact Lie group $G$.

6.1. **Weight decomposition.** Let $G$ be a connected compact group with maximal torus $T$, and let $(\pi, V)$ be a f.d. representation of $G$. Restricting it to $T$ and using complete reducibility, we get the following result.

**Theorem 15.** The vector space $V$ can be written in the form

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda, \quad V_\lambda = \{v \in V \mid \pi_\lambda(t)v = (\lambda, t)v \quad \forall t \in t\}.$$ 

where $X(T)$ is the character group of $T$ defined by $\chi$. 
The spaces $V_\lambda$ are called **weight subspaces**, vectors $v \in V_\lambda$ — **weight vectors** of weight $\lambda$. The set

$$P(V) = \{ \lambda \in X(T) \mid V_\lambda \neq \{0\} \}$$

is called the **set of weights** of $\pi$, or the **spectrum** of $\text{Res}_T^G \pi$, and

$$\text{mult}(\pi, V)(\lambda) := \dim V_\lambda$$

is called the **multiplicity** of $\lambda$ in $V$.

The next theorem easily follows from the definition of the Weyl group.

**Theorem 16.** For any f. d. representation $V$ of $G$, the set of weights with multiplicities is invariant under the action of the Weyl group:

$$w(P(V)) = P(V), \quad \text{mult}(\pi, V)(\lambda) = \text{mult}(\pi, V)(w(\lambda))$$

for any $w \in W$.

### 6.2. Classification of Unireps

Recall that $R$ is the root system of $\mathfrak{g}_C$. Assume that we have chosen a basis of simple roots $\alpha_1, \ldots, \alpha_r \subset R$. Then $R = R_+ \cup R_-$; roots $\alpha \in R_+$ can be written as a linear combination of simple roots with positive coefficients, and $R_- = -R_+$.

A (not necessarily finite-dimensional) representation of $\mathfrak{g}_C$ is called a **highest weight representation** if it is generated by a single vector $v \in V_\lambda$ (the highest weight vector) such that $\mathfrak{g}_\alpha v = 0$ for all positive roots $\alpha \in R_+$.

It can be shown that for every $\lambda \in X(T)$, there is a unique irreducible highest weight representation of $\mathfrak{g}_C$ with highest weight $\lambda$, which is denoted $L(\lambda)$. However, this representation can be infinite-dimensional; moreover, it may not be possible to lift it to a representation of $G$.

**Definition 4.** A weight $\lambda \in X(T)$ is called **dominant** if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+$ for any simple root $\alpha_i$. The set of all dominant weights is denoted by $X_+(T)$.

**Theorem 17.**

1. All weights of $L_\lambda$ are of the form $\mu = \lambda - \sum n_i \alpha_i$, $n_i \in \mathbb{Z}_+$.
2. Let $\lambda \in X_+$. Then the irreducible highest weight representation $L(\lambda)$ is finite-dimensional and lifts to a representation of $G$.
3. Every irreducible finite-dimensional representation of $G$ is of the form $L(\lambda)$ for some $\lambda \in X_+$.

Thus, we have a bijection $\{\text{unireps of } G\} \leftrightarrow X_+$.

**Example 7.** Let $G = SU(2)$. There is a unique simple root $\alpha$ and the unique fundamental weight $\omega$, related by $\alpha = 2\omega$. Therefore, $X_+ = \mathbb{Z}_+ \cdot \omega$ and unireps are indexed by non-negative integers. The representation with highest weight $k \cdot \omega$ is precisely the representation $\Pi_k$ constructed in Section [13].

**Example 8.** Let $G = U(n)$. Then $X = \mathbb{Z}_n$, and $X_+ = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_n \mid \lambda_1 \geq \cdots \geq \lambda_n \}$. Such objects are well known in combinatorics: if we additionally assume that $\lambda_n \geq 0$, then such dominant weights are in bijection with partitions with $n$ parts. They can also be described by **Young diagrams** with $n$ rows (see [PH]).

### 6.3. Explicit Construction of Representations

In addition to description of unireps as highest weight representations, they can also be constructed in other ways. In particular, they can be defined analytically as follows. Let $B = HN_+$ be the Borel subgroup in $G_C$; here $H = \exp \mathfrak{h}$, $N_+ = \exp \sum_{\alpha \in R_+} c_\alpha e_\alpha$. For $\lambda \in \mathfrak{h}_+$, let $\chi_\lambda : B \to \mathbb{C}^\times$ be a multiplicative map defined by

$$\chi_\lambda(hn) = e^{\langle h, \lambda \rangle}$$

**Theorem 18** (E.Cartan–Borel–Weil). Let $\lambda \in X(T)$. Denote by $V(\lambda)$ be the space of complex-analytic functions on $G_C$ which satisfy the following transformation property:

$$f(gb) = \chi_\lambda^{-1}(b)f(g), \quad g \in G_C, \quad b \in B.$$
The group $G_C$ acts on $V(\lambda)$ by left shifts:

\begin{equation}
(\pi(g)f)(h) = f(g^{-1}h).
\end{equation}

1. $V(\lambda) \neq \{0\}$ iff $-\lambda \in X_+$.
2. If $-\lambda \in X_+$, the representation of $G$ in $V(\lambda)$ is equivalent to $L(w_0(\lambda))$, where $w_0 \in W$ is the unique element of the Weyl group which sends $R_+$ to $R_-$.

This theorem can also be reformulated in more geometric terms: the spaces $V(\lambda)$ are naturally interpreted as spaces of global sections of appropriate line bundles on the flag variety $B = G_C/B = G/T$.

For classical groups irreducible representations can also be constructed explicitly as the subspaces in tensor powers $(C^n)^\otimes k$, transforming in a certain way under the action of the symmetric group $S_k$.

7. Characters and multiplicities

7.1. Characters. Let $(\pi, V)$ be a finite-dimensional representation of $G$ and let $\chi_\pi$ be its character as defined by (6). Since $\chi_\pi$ is central, and every element in $G$ is conjugate to an element of $T$, $\chi_\pi$ is completely determined by its restriction to $T$, which can be computed from the weight decomposition (6):

\begin{equation}
\chi_\pi|_T = \sum_{\lambda \in X(T)} \dim V_\lambda \cdot e_\lambda = \sum_{\lambda \in X(T)} \text{mult}_\pi \lambda \cdot e_\lambda
\end{equation}

where $e_\lambda$ is the function on $T$ defined by $e_\lambda(\exp(t)) = e^{(t,\lambda)}$, $t \in t$. Note that $e_{\lambda+\mu} = e_\lambda e_\mu$ and that $e_0 = 1$.

7.2. Weyl character formula.

Theorem 19 (Weyl character formula). Let $\lambda \in X_+$. Then

$$
\chi_{\lambda + \rho} = \frac{A_{\lambda + \rho}}{A_\rho}, \quad A_\mu = \sum_{w \in W} \varepsilon(w) e_{w(\mu)}
$$

where, for $w \in W$, we denote $\varepsilon(w) = \det w$ considered as a linear map $t^* \rightarrow t^*$, and $\rho = \frac{1}{2} \sum_{R_+} \alpha$.

In particular, computing the value of the character at point $t = 0$ by L'Hopital’s rule, it is possible to deduce the following formula for the dimension of irreducible representations:

\begin{equation}
\dim L(\lambda) = \prod_{\alpha \in R_+} \frac{\langle \alpha^\vee, \lambda + \rho \rangle}{\langle \alpha^\vee, \rho \rangle}.
\end{equation}

Example 9. Let $G = SU(2)$. Then Weyl character formula gives, for irreducible representation $\Pi_k$ with highest weight $k \cdot \omega$

$$
\chi_{\Pi_k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}} = x^k + x^{k-2} + \cdots + x^{-k}, \quad x = e_\omega
$$

which implies $\dim \Pi_k = k + 1$.

Weyl character formula is equivalent to the following formula for weight multiplicities, due to Kostant:

$$
\text{mult}_{L(\lambda)} \mu = \sum_{w \in W} \varepsilon(w) K(w(\lambda + \rho) - \rho - \mu),
$$

where $K$ is Kostant’s partition function: $K(\tau)$ is the number of ways of writing $\tau$ as a sum of positive roots (with repetitions).

For classical Lie groups such as $G = U(n)$, there are more explicit combinatorial formulas for weight multiplicities; for $U(n)$, the answer can be written in terms of the number of Young tableaux of a given shape. Details can be found in [FH].
7.3. Tensor product multiplicities. Let \((\pi, V)\) be a finite-dimensional representation of \(G\). By complete reducibility, one can write \(V = \sum n_{\lambda} L(\lambda)\). The coefficients \(n_{\lambda}\) are called multiplicities; finding them is an important problem in many applications. In particular, a special case of this is finding the multiplicities in tensor product of two unirreps:

\[
L(\lambda) \otimes L(\mu) = \sum N_{\lambda\mu}^\nu L(\nu).
\]

Characters provide a practical tool of computing multiplicities: since characters of unirreps are linearly independent, multiplicities can be found from the condition that \(\chi_V = \sum n_{\lambda}\chi_{L(\lambda)}\). In particular,

\[
\chi_{L(\lambda)}\chi_{L(\mu)} = \sum N_{\lambda\mu}^\nu \chi_{L(\nu)}.
\]

**Example 10.** For \(G = \text{SU}(2)\), tensor product multiplicities are given by

\[
\Pi_n \otimes \Pi_m = \bigoplus \Pi_l
\]

where the sum is taken over all \(l\) such that \(|m - n| \leq l \leq m + n, m + n + l\) is even.

For \(G = \text{U}(n)\), there is an algorithm for finding the tensor product multiplicities, formulated in the language of Young tableaux (Littlewood–Richardson rule). There are also tables and computer programs for computing these multiplicities; some of them are listed in the bibliography.

**See Also**

Classical groups and homogeneous spaces. Lie groups: general theory. Infinite-dimensional Lie algebras.

**Keywords**

Lie groups; compact groups; maximal tori; Lie algebras; characters; representations; roots; weights; highest weight; multiplicities;

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