Linking numbers in local quantum field theory

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Abstract. Linking numbers appear in local quantum field theory in the presence of tensor fields, which are closed two-forms on Minkowski space. Given any pair of such fields, it is shown that the commutator of the corresponding intrinsic (gauge invariant) vector potentials, integrated about spacelike separated, spatial loops, are elements of the center of the algebra of all local fields. Moreover, these commutators are proportional to the linking numbers of the underlying loops. If the commutators are different from zero, the underlying two-forms are not exact (there do not exist local vector potentials for them). The theory then necessarily contains massless particles. A prominent example of this kind, due to J.E. Roberts, is given by the free electromagnetic field and its Hodge dual. Further examples with more complex mass spectrum are presented in this article.

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1 Introduction

Skew symmetric tensor fields, which are closed two-forms on Minkowski space, are familiar from local quantum field theory, the most prominent example being the electromagnetic field satisfying the homogeneous Maxwell equation. One frequently argues that such two-forms are exact and introduces corresponding local vector potentials, which are in general defined on indefinite metric spaces. It was pointed out by J.E. Roberts [?] that this step might not always be possible, however, if one is dealing with several closed two-forms, such as the free electromagnetic field and its Hodge dual.

Obstructions to the existence of local vector potentials manifest themselves in commutators of the given tensor fields, which are integrated over compact two-
surfaces with spacelike separated boundaries. These integrals can be rewritten in terms of intrinsically defined (gauge invariant) vector potentials, cf. [2], which are integrated along the boundaries. If these commutators are different from zero, the intrinsic vector potentials, being defined on loop functions, cannot be extended to vector potentials which are point-like and local with respect to each other. Phrased differently, the underlying closed two-forms can not be exact in the setting of local quantum field theory. Let us explain this point in somewhat more detail.

Let \( F_{\mu \nu}(f) \) be a local, hermitean, skew symmetric rank-two tensor field in the framework of quantum field theory. It is assumed to be linear with regard to the tensor-valued test functions \( f \) and closed as a two-form on Minkowski space, i.e.

\[
F_{\mu \nu}(\partial_{\rho} f_{\mu \nu \rho}) = 0, \tag{1.1}
\]

where \( f_{\mu \nu \rho} \) are test functions with values in skew symmetric rank-three tensors and \( \partial_{\rho} \) are the spacetime derivatives. There then exists a corresponding intrinsic (gauge invariant) vector potential \( A_{\mu}(h) \). It is defined for all co-closed vector-valued test functions \( h \), satisfying \( \partial_{\nu} h_{\mu} = 0 \), which form a vector space denoted by \( \mathcal{C}_1(\mathbb{R}^4) \). Given \( h \in \mathcal{C}_1(\mathbb{R}^4) \), the relation between the tensor field and the intrinsic vector potential is established by the formula \( F_{\mu \nu}(f) = A_{\mu}(h) \), where \( f \) is any test function satisfying \( \partial_{\nu} f = h \). Such functions exist by the Poincaré lemma and \( F_{\mu \nu}(f) \) depends only on \( h \) in view of the fact that \( F_{\mu \nu} \) is a closed two-form, cf. [2].

Einstein causality is expressed by the condition of locality according to which the commutator of local fields vanishes at spacelike distances. The intrinsic vector potentials inherit from the underlying local tensor fields certain locality properties: if \( h \in \mathcal{C}_1(\mathbb{R}^4) \) is any test function which has support in some bounded, contractible region \( \mathcal{O} \), then \( A_{\mu}(h) \) commutes with all local operators which are localized in similar regions in the spacelike complement of \( \mathcal{O} \). Yet if the support properties of \( h \) are topologically non-trivial, this commutativity may fail. Consider for example two local, hermitean, skew symmetric and closed rank-two tensor fields \( F_{\mu \nu}, G_{\rho \sigma} \) with corresponding intrinsic vector potentials \( A_{\mu} \) and \( B_{\rho} \), respectively. As was shown in [3, 7] by examples, the commutator

\[
[A_{\mu}(h), B_{\rho}(k)]
\]

need not vanish for test functions \( h, k \in \mathcal{C}_1(\mathbb{R}^4) \) having support in linked, spacelike separated loop-shaped regions. The intrinsic vector potentials can then not be extended to local point fields which are defined on the space of all vector-valued test functions \( \mathbb{D}(\mathbb{R}^4) \supset \mathcal{C}_1(\mathbb{R}^4) \).

In the present article we study the properties of the commutators \( [A_{\mu}(h), B_{\rho}(k)] \) of intrinsic vector potentials in detail and refer to them as causal commutators if the supports of the underlying test functions are spacelike separated. Our analysis is organized as follows. Denoting by \( \mathfrak{P} \) the polynomial \(*\)-algebra generated by all local fields in the theory and making use of results in [2], we will show in a first step that
the commutators are central (superselected) elements of $\mathfrak{g}$ whenever the supports of $g$ and $h$ are spacelike separated. For certain specific test functions, having supports in linked, spacelike separated loop-shaped regions (loop functions), the causal commutators are then shown to be stable under deformations of the loops. Thus these commutators encode topological information and are proportional to linking numbers associated with the underlying loops. Similar results were stated in [7] without proofs; these are supplied here. Our main result consists of the demonstration that the causal commutators can be different from zero only in theories involving massless particles. This resembles Swieca’s theorem on the existence of massless particles in theories with an electric charge [12], but it is not directly related to it. Examples of theories with non-trivial causal commutators are then exhibited and the article closes with some expository remarks.

2 Causal commutators are elements of the center

In this section we analyze algebraic properties of the causal commutators (1.1). To this end, we recall some standard notation for skew symmetric tensor functions in order to express properties of their quantum counterparts in precise mathematical terms.

In the sequel, we shall adopt the short hand notation $t \in D_n(\mathbb{R}^4)$ for skew symmetric $n$-tensor valued test functions $t = (t^{\mu_1 \ldots \mu_n})$, $n = 0, \ldots, 4$. Considering their supports, we write $\text{supp}(t) \perp \text{supp}(t')$ to denote the spacelike separation of the supports of pairs $t, t' \in D_n(\mathbb{R}^4)$ in Minkowski spacetime. As already mentioned in the introduction, a stronger condition of spacelike separation is obtained by requiring that the supports of $t, t'$ are contained in spacelike separated, bounded, contractible regions, which then are separated by two characteristic planes; in this case we write $\text{supp}(t) \times \text{supp}(t')$. The difference between $\perp$ and $\times$ becomes manifest when the supports of $t, t'$ have non-trivial topological properties.

On the tensor valued test functions acts the co-derivative $\delta$, whose action on $f \in D_2(\mathbb{R}^4)$ is given by $(\delta f)^\mu = -2 \partial_\nu f^{\nu \mu} \in \mathcal{C}_1(\mathbb{R}^4)$, where $\mathcal{C}_1(\mathbb{R}^4)$ consists of all co-closed functions in $\mathcal{D}_1(\mathbb{R}^4)$, satisfying $\delta h = -\partial_\nu h^\nu = 0$. According to the Poincaré Lemma one has $\mathcal{C}_1(\mathbb{R}^4) = \delta(\mathcal{D}_2(\mathbb{R}^4))$, cf. [2].

Passing to the quantum level, we consider hermitean operators $F(f), G(g)$, linear in $f, g \in D_2(\mathbb{R}^4)$, which fulfill the homogeneous Maxwell equation

$$dF(t) \equiv F_{\mu \nu}(\partial_\rho t^{\rho \mu \nu}) = 0, \quad dG(t) = 0, \quad t \in D_3(\mathbb{R}^4),$$

(2.1)

and the causality relations

$$[F(f), G(g)] = 0, \quad \text{supp}(f) \perp \text{supp}(g), \quad f, g \in D_2(\mathbb{R}^4).$$

(2.2)
The homogeneous Maxwell equation implies that for any \( h,k \in \mathcal{C}_1(\mathbb{R}^4) \) the operators
\[
A(h) \doteq F(f), \quad B(k) \doteq G(g)
\]
are well-defined for any choice of test functions \( f,g \in \mathcal{D}_2(\mathbb{R}^4) \), satisfying \( \delta f = h \) and \( \delta g = k \), c.f. \cite{2}. We call \( A \) and \( B \) the intrinsic vector potentials defined by \( F \) and \( G \), respectively.

We assume that the tensor fields \( F,G \) are elements of some polynomial *-algebra \( \mathfrak{B} \), which is generated by local field operators and on which the Poincaré group acts covariantly by automorphisms. Based on this input, we want to clarify the properties of commutators of the corresponding intrinsic vector potentials, cf (1.1). We begin with a technical lemma which slightly generalizes a result in \cite{2}.

**Lemma 2.1.** Let \( h \in \mathcal{C}_1(\mathbb{R}^4) \) and \( g \in \mathcal{D}_2(\mathbb{R}^4) \) such that \( \text{supp}(g) \perp \text{supp}(h) \). Then
\[
[A(h), G(g)] = [B(h), F(g)] = 0.
\]

**Proof.** Using a partition of unity we can decompose \( g \) into a finite sum \( \sum_i g_i \) where \( g_i \) is supported in a double cone \( \mathcal{O}_i \) such that \( \mathcal{O}_i \perp \text{supp}(h) \) for any \( i \). By the Causal Poincaré Lemma (see \cite{2}, Appendix) we can find for any \( i \) a co-primitive \( f_i \) of \( h \), \( \delta f_i = h \), such that \( \text{supp}(f_i) \perp \mathcal{O}_i \). Hence, by (2.2), one has
\[
[A(h), G(g)] = \sum_i [A(h), G(g_i)] = \sum_i [F(f_i), G(g_i)] = 0.
\]
The same argument applies to the commutator between \( B \) and \( F \). \hfill \( \square \)

Using this result it is easily seen that the causal commutator of the intrinsic vector potentials, defined on test functions with spacelike separated supports, is a central element, hence a superselected quantity. The subsequent proposition is a slight generalization of results in \cite{2}, Appendix.

**Proposition 2.2.** Let \( h,k \in \mathcal{C}_1(\mathbb{R}^4) \) with \( \text{supp}(h) \perp \text{supp}(k) \). Then the commutator \( [A(h), B(k)] \) is translation invariant and hence an element of the center of \( \mathfrak{B} \).

**Proof.** Let \( h \in \mathcal{C}_1(\mathbb{R}^4) \) and let \( h_y \) be its translate for any \( y \in \mathbb{R}^4 \). We proceed to the test function \( f^y \in \mathcal{D}_2(\mathbb{R}^4) \) given by
\[
f^y(x) \doteq (1/2) \int_0^1 du \left( \epsilon^\mu h^\nu (x-u y) - y^\mu h^\nu (x-u y) \right), \quad x \in \mathbb{R}^4.
\]
It is a co-primitive of \( (h-h_y) \in \mathcal{C}_1(\mathbb{R}^4) \), that is, \( \delta f^y = (h-h_y) \). Moreover, \( f^y \) has support in the cylindrical region \( \{ \text{supp}(h)+uy : 0 \leq u \leq 1 \} \). So, for sufficiently small translations \( y \in \mathbb{R}^4 \), we have \( \text{supp}(f^y) \perp \text{supp}(k) \). This implies by Lemma 2.1
\[
[A(h), B(k)] = [A(\delta f^y + h_y), B(k)] = [F(f^y), B(k)] + [A(h_y), B(k)] = [A(h_y), B(k)].
\]
Applying the same argument to the translates of $k$ one finds that

$$[A(h), B(k)] = [A(h_y), B(k_y)]$$

for sufficiently small $y$. This equality extends to arbitrary translations $y \in \mathbb{R}^4$ by iteration. It then follows from translation covariance, locality, and the Jacobi identity that, for any given $h, k \in \mathcal{C}_1(\mathbb{R}^4)$ with $\text{supp}(h) \perp \text{supp}(k)$, the commutator $[A(h), B(k)]$ commutes with any other element of $\mathfrak{p}$. Hence, being itself an element of $\mathfrak{p}$, it lies in the center of $\mathfrak{p}$, completing the proof.

3 Homology invariance of causal commutators

Turning to the geometrical analysis of the causal commutators, we recall from [3, Sect. 3] some definitions and facts about loops, surfaces and test functions. Given a test function $s \in R_0(\mathbb{R}^4)$ and a loop $\gamma : [0, 1] \rightarrow \mathbb{R}^4$, one can define a test function in $l_s, \gamma \in \mathcal{C}_1(\mathbb{R}^4)$, called loop function, putting

$$l_s^\mu(x) = \int_0^1 du s(x + \gamma(u)) \dot{\gamma}^\mu(u),$$

where $\dot{\gamma}^\mu$ is the tangent vector. It is apparent that $\text{supp}(l_s, \gamma) \subset \text{supp}(s) + \gamma$. One can perform a similar construction for surfaces $\sigma : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^4$ and define corresponding test functions $f_s, \sigma \in \mathcal{D}_2(\mathbb{R}^4)$, putting

$$f_s^{\mu\nu}(x) = -1/2 \int_0^1 \int_0^1 d^2 u s(x + \sigma(u)) \sigma^{\mu\nu}(u),$$

where $\sigma^{\mu\nu}$ is the Jacobian fixed by $\sigma$. Clearly, $\text{supp}(f_s, \sigma) \subset \text{supp}(s) + \sigma$. One then has the relation

$$\delta f_s, \sigma = l_s, \partial \sigma,$$

where $\partial \sigma$ is the boundary of $\sigma$. Since this is a straightforward consequence of Stokes’ theorem, its proof is omitted.

In the next step we prove two invariance properties of the commutators of interest here. The first one concerns the dependence of the commutator on the cohomology class of the loop functions entering in the smearing of the operators; to be precise: the 0-co-cohomology class of the underlying scalar functions $s \in \mathcal{D}_0(\mathbb{R}^4)$, which is fixed by its integral $\int dx s(x)$.

**Lemma 3.1.** Let $\gamma_1, \gamma_2$ be two spacelike separated loops, let $\partial_1, \partial_2$ be two open balls centered about the origin such that $\partial_1 + \gamma_1 \perp \partial_2 + \gamma_2$, and let $s_1, s_2 \in \mathcal{D}_0(\mathbb{R}^4)$ be test functions with support in $\partial_1$, respectively $\partial_2$. Then, for any test function $\hat{s}_1$ with $\text{supp}(\hat{s}_1) \subset \partial_1$ and $\int dx \hat{s}_1(x) = 1$, one has

$$[A(l_{s_1, \gamma_1}), B(l_{s_2, \gamma_2})] = \kappa [A(l_{s_1, \gamma_1}), B(l_{s_2, \gamma_2})],$$

where $\kappa = \int dx s_1(x)$. 

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Proof. Since the integral of the function $(s_1 - \kappa \hat{s}_1)$ is zero, there is a test function $h \in \mathcal{D}_1(\mathbb{R}^4)$ such that $\text{supp}(h) \subset \mathcal{O}_1$ and $\hat{\delta} h = (s_1 - \kappa \hat{s}_1)$. Thus the associated test function $f \in \mathcal{D}_2(\mathbb{R}^4)$, given by
\[
f^{\mu\nu}(x) \equiv (1/2) \int_0^1 du \left( h^\mu (x - \gamma_1(u)) \gamma'_1(u) - h^\nu (x - \gamma_1(u)) \gamma'_0(u) \right),
\]
has support in $\mathcal{O}_1 + \gamma_1$ and $\delta f = (l_{s_1, \gamma} - \kappa l_{\hat{s}_1, \hat{\gamma}})$. According to Lemma 2.1, this implies
\[
[A(l_{s_1, \gamma}), B(l_{s_2, \gamma})] = \kappa [A(l_{\hat{s}_1, \hat{\gamma}}), B(l_{s_2, \gamma})] + [A(\delta f), B(l_{s_2, \gamma})]
\]
\[
= \kappa [A(l_{\hat{s}_1, \hat{\gamma}}), B(l_{s_2, \gamma})] + [F(f), B(l_{s_2, \gamma})]
\]
\[
= \kappa [A(l_{\hat{s}_1, \hat{\gamma}}), B(l_{s_2, \gamma})],
\]
completing the proof. □

Since the commutator in the lemma vanishes if either $\int dx s_1(x) = 0$ or $\int dx s_2(x) = 0$, we adopt below the following convention.

Standing assumption: The scalar functions $s \in \mathcal{D}_0(\mathbb{R}^4)$ entering in the loop functions $l_{s, \gamma}$ are normalized, i.e. $\int dx s(x) = 1$.

The second property of the causal commutators we want to establish is their homology invariance with regard to deformations of the underlying loops.

Lemma 3.2. Let $\gamma_1, \gamma_2$ be two spacelike separated loops, let $\mathcal{O}_1, \mathcal{O}_2$ be open balls centered about the origin such that $\mathcal{O}_1 + \gamma_1 \perp \mathcal{O}_2 + \gamma_2$, and let $s_1, s_2 \in \mathcal{D}_0(\mathbb{R}^4)$ be normalized functions with support in $\mathcal{O}_1$, respectively $\mathcal{O}_2$. Then, for any loop $\gamma_1$ which is homologous to $\gamma_2$ in the causal complement of $\gamma_2$ and any normalized function $l_{\hat{s}_1}$ having support in an open ball $\mathcal{O}_1$ about the origin such that $\mathcal{O}_1 + \hat{\gamma}_1 \perp \mathcal{O}_2 + \gamma_2$, one has
\[
[A(l_{s_1, \gamma}), B(l_{s_2, \gamma})] = [A(l_{\hat{s}_1, \hat{\gamma}}), B(l_{s_2, \gamma})].
\]
An analogous relation holds if $(s_2, \gamma_2)$ is replaced by a homologous pair $(\hat{s}_2, \hat{\gamma}_2)$.

Proof. According to the hypothesis on $\hat{\gamma}_1$, there is a surface $\sigma \subset \mathbb{R}^4$ such that $\sigma \perp \gamma_2$ and $\partial \sigma = \gamma_1 - \hat{\gamma}_1$. Let $\mathcal{O}_1$ be any open ball about the origin such that $\mathcal{O}_1 \subset \mathcal{O}_1$ and $\mathcal{O}_1 + \sigma \perp \mathcal{O}_2 + \gamma_2$, and pick a function $\hat{s}_1$ with support in $\mathcal{O}_1$. Then, by Lemma 3.1, we have
\[
[A(l_{s_1, \gamma}), B(l_{s_2, \gamma})] = [A(l_{\hat{s}_1, \hat{\gamma}}), B(l_{s_2, \gamma})].
\]
As in the preceding lemma, we consider the test function $f_{\hat{s}_1, \sigma} \in \mathcal{D}_2(\mathbb{R}^4)$. It has support in $\mathcal{O}_1 + \sigma$, and $\delta f_{\hat{s}_1, \sigma} = l_{\hat{s}_1, \hat{\gamma}} - l_{\hat{s}_1, \gamma}$ by (3.1). Since $O_1 + \sigma \perp O_2 + \gamma_2$, it
follows from Lemma 2.1 that

\[ [A(l_1, \gamma_1), B(l_2, \gamma_2)] = [A(l_1, \gamma_1), B(l_2, \gamma_2)] = [F(f_1, \sigma), B(l_2, \gamma_2)] + [A(l_1, \gamma_1), B(l_2, \gamma_2)] = [A(l_1, \gamma_1), B(l_2, \gamma_2)], \]

proving the Lemma.

It follows from this lemma that the commutators are stable under deformations of the underlying loop functions within the given limitations. With this information we can turn now to the analysis of their relation to the linking numbers, where we begin by recalling some topological definitions. For any simple loop \( \gamma \) in \( \mathbb{R}^3 \), the first homology group \( H_1(\mathbb{R}^3 \setminus \gamma) \) is isomorphic to \( \mathbb{Z} \) (Alexander duality, [5, Chpt. 3]). Given any pair of disjoint linked loops \( \gamma_1, \gamma_2 \subset \mathbb{R}^3 \), where \( \gamma_1 \) is simple, their linking number \( L(\gamma_1, \gamma_2) \) is equal to the homology class of \( \gamma_2 \) in \( H_1(\mathbb{R}^3 \setminus \gamma_1) \cong \mathbb{Z} \), respectively of \( \gamma_1 \) in \( H_1(\mathbb{R}^3 \setminus \gamma_2) \), cf. [5, Chpt. 1], [8, Chpt. 5]. It coincides with the analytical definition given in terms of the Gauss integral,

\[
L(\gamma_1, \gamma_2) \triangleq (1/4\pi) \int_0^1 du \int_0^1 dv \frac{\det(\gamma_1(u) - \gamma_2(v), \gamma_1(u) - \gamma_2(v))}{|\gamma_1(u) - \gamma_2(v)|^3}. \tag{3.2}
\]

Although Lemma 3.2 implies that the commutators of the intrinsic vector potentials \( A, B \) are stable under deformations of the loop functions, there is some subtle point to be observed. It is known that loops in \( \mathbb{R}^4 \setminus \gamma \) can be disentangled (trivialized), namely \( H_1(\mathbb{R}^4 \setminus \gamma) = 0 \). Yet the homology invariance of Lemma 3.2 refers to the causal complement and not to the complement of the curves in \( \mathbb{R}^4 \). A direct connection between this notion and the notion of linking numbers is obtained by considering a particular class of loops in Minkowski space. A loop \( \gamma : [0, 1] \to \mathbb{R}^4 \) is said to be spatial whenever the points \( \gamma(u), \gamma(v) \) are spacelike with respect to each other for any \( u \neq v \) and \((u,v) \neq (0,1),(1,0)\). Such loops are simple and so are their projections onto the time zero plane, given by \( \mathbb{R}^4 \supset \gamma = (\gamma_0, \gamma) \to \gamma \subset \mathbb{R}^3 \).

Given any two spacelike separated, spatial loops \( \gamma_1 \perp \gamma_2 \), one can continuously deform both of them to disjoint loops in the time-zero plane \((0, \mathbb{R}^3)\), without affecting their spacelike separation. This is accomplished by the function \( H : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}^4 \), given by

\[
H(u, (x_0, x)) \triangleq ((1-u)x_0, x), \quad u \in [0, 1], (x_0, x) \in \mathbb{R}^4.
\]

Clearly, \( H \) is continuous, \( H(0, \cdot) \rhd \mathbb{R}^4 = id \) and \( H(1, \cdot) \rhd \mathbb{R}^4 = P_0 \), the projection onto \((0, \mathbb{R}^3) \subset \mathbb{R}^4 \). Putting \( \gamma_1, \gamma_2 \triangleq H(u, \gamma_1), \gamma_2, u \triangleq H(u, \gamma_2) \), one has \( \gamma_1, \gamma_2 \perp \gamma_2, u \) for any \( u \in [0, 1] \) and \( \gamma_1, 1 = (0, \gamma_1), \gamma_2, 1 = (0, \gamma_2) \) are disjoint loops lying in \((0, \mathbb{R}^3)\).
These observations allow us to introduce in $\mathbb{R}^4$ a causal linking number for spacelike separated, spatial loops $\gamma_1$, $\gamma_2$, given by

$$L_c(\gamma_1, \gamma_2) \doteq L(\gamma_1, \gamma_2) \in H_1(\mathbb{R}^3 \setminus \gamma_2) \simeq \mathbb{Z}.$$  (3.3)

We can establish now the main result of this section, stating the proportionality of the causal commutators of intrinsic vector potentials to the causal linking numbers of the underlying loops.

**Proposition 3.3.** Let $\gamma_1$, $\gamma_2$ be spacelike separated, spatial loops, let $\partial_1$, $\partial_2$ be open balls centered about the origin such that $\partial_1 + \gamma_1 \perp \partial_2 + \gamma_2$, and let $s_1, s_2 \in \mathcal{D}_0(\mathbb{R}^4)$ be normalized functions with support in $\partial_1$, respectively $\partial_2$. Then

$$[A(l_{s_1}, \gamma_1), B(l_{s_2}, \gamma_2)] = i L_c(\gamma_1, \gamma_2) Z_{AB},$$

where $L_c(\gamma_1, \gamma_2) \in \mathbb{Z}$ is the causal linking number of $\gamma_1$, $\gamma_2$ and $Z_{AB} = -Z_{BA}$ is a fixed hermitean element of the center of $\mathcal{D}^0_\gamma$ which does not depend on the choice of the loops and test functions within the above limitations.

**Proof.** According to Lemma 3.2 and the preceding remark concerning the deformation of spacelike separated, spatial loops, one has

$$[A(l_{s_1}, \gamma_1), B(l_{s_2}, \gamma_2)] = [A(l_{\hat{s}_1}, \gamma_1), B(l_{\hat{s}_2}, \gamma_2)],$$

where the normalized test functions $\hat{s}_1, \hat{s}_2$ have small enough supports, as required by Lemma 3.2. Let

$$\lambda \doteq L_c(\gamma_1, \gamma_2) = L(\gamma_1, \gamma_2) \in \mathbb{Z}$$

be the (without loss of generality positive) causal linking number for the given loops $\gamma_1, \gamma_2$. The projected loop $\gamma_1$ is homologous in $\mathbb{R}^3 \setminus \gamma_2$ to the $\lambda$-fold composition $\alpha_1 \doteq \alpha_1 \ast \cdots \ast \alpha_1$ of a generating circle $\alpha_1$ of the homology group $H_1(\mathbb{R}^3 \setminus \gamma_2)$, i.e. $L(\alpha_1, \gamma_2) = 1$. Thus, by another application of Lemma 3.2, one gets

$$[A(l_{\hat{s}_1}, \gamma_1), B(l_{\hat{s}_2}, \gamma_2)] = \lambda [A(l_{\hat{s}_1}', \alpha_1), B(l_{\hat{s}_2}', \gamma_2)],$$

where $\hat{s}_1', \hat{s}_2'$ are test functions complying with the support conditions in the lemma. The appearance of the factor $\lambda$ follows from the fact that loop functions are invariant under changes of the parametrization, so $l_{\hat{s}_1}, \alpha_1 = \lambda l_{s_1}, \alpha_1$. Now $\gamma_2$ can in turn be regarded as generator of the homology group $H_1(\mathbb{R}^3 \setminus \alpha_1)$, hence it is homologous in $\mathbb{R}^3 \setminus \alpha_1$ to a circle $\alpha_2$. By a final application of Lemma 3.2 we therefore arrive at

$$[A(l_{\hat{s}_1'}, \alpha_1), B(l_{\hat{s}_2'}, \gamma_2)] = [A(l_s, \alpha_1), B(l_s, \alpha_2)]$$

with a normalized test function $s$, satisfying the support conditions. Note that the circles $\alpha_1$, $\alpha_2$ and the function $s$ can be chosen independently of the initial data $\gamma_1$, $\gamma_2$ and $s_1, s_2$; in particular $[A(l_s, \alpha_1), B(l_s, \alpha_2)] = [A(l_s, \alpha_1), B(l_s, \alpha_2)]$. The operator $Z_{AB} \doteq -i [A(l_s, \alpha_1), B(l_s, \alpha_2)] = -Z_{BA}$ is contained in the center of $\mathcal{D}^0_\gamma$, cf. Lemma 2.1, and $Z_{AB}^* = Z_{AB}$ since $A, B$ are hermitean. The statement then follows from the preceding equalities. 

\[\square\]
This proposition shows that the causal commutators of intrinsic vector potentials, smeared with loop functions based on spacelike separated, spatial loops, can be interpreted as topological charges with values in \( \mathbb{Z} \). The fixed central element, multiplying the linking numbers, sets their scale, which in general depends on the underlying theory.

4 Causal commutators and the mass spectrum

We turn now to the question under which circumstances the causal commutators can be different from zero. An answer is provided by relating it to the mass spectrum of the underlying theory. To this end we assume that the polynomial algebra \( \mathfrak{P} \) of local field operators is irreducibly represented on the vacuum Hilbert space (sector) \( \mathcal{H} \), cf. the Wightman framework [11]. On \( \mathcal{H} \) there acts a continuous, unitary representation \( U \) of the spacetime translations \( \mathbb{R}^4 \), implementing the action of the translations on the local field operators,

\[
U(x)F(f)U^{-1}(x) = F(f_x), \quad x \in \mathbb{R}^4,
\]

and analogously for the local field \( G \). The representation \( U \) satisfies the relativistic spectrum condition, i.e. the joint spectrum of its generators \( P \) has support in the forward lightcone \( \mathcal{V}^+ \), and there is a unique, one-dimensional, translational invariant subspace of \( \mathcal{H} \); it is generated by the vacuum vector \( \Omega \in \mathcal{H} \) and lies in the domain of the elements of \( \mathfrak{P} \).

These assumptions imply that the causal commutators, being central elements of \( \mathfrak{P} \) according to Proposition 2.2, are represented on \( \mathcal{H} \) by multiples of the identity, which coincide with their vacuum expectation values,

\[
[A(h),B(k)] = \langle \Omega, [A(h),B(k)] \Omega \rangle 1, \quad \text{supp } h \perp \text{supp } k.
\]

It enables us to determine their dependence on the mass spectrum by an application of the Jost-Lehmann-Dyson representation to the underlying local field operators \( F, G \), cf. for example [4, Lem. 6.2]. Namely, for any pair of test functions \( f, g \in \mathcal{D}_2(\mathbb{R}^4) \), having support in spacelike separated double cones, one has

\[
\langle \Omega, F(f)E(\Delta)G(g)\Omega \rangle = \langle \Omega, G(g)E(\Delta)F(f)\Omega \rangle,
\]

where \( E(\Delta) \) is the spectral projection of the mass operator \( M = (P^2)^{1/2} \) for any given Borel set \( \Delta \subset \mathbb{R} \). It follows by integration that this relation still holds if one replaces \( E(\Delta) \) by \( c(M)E(\Delta) \), where \( c(M) \) is any continuous, bounded function of the mass operator. Thus, the vacuum expectation values of the commutators of local fields

\[\text{[1]}\]

In [4] this relation was established in a framework of bounded local operators. Since the vector \( \Omega \) lies in the domain of the field operators, the argument given there extends to the present case.
the existence of massless particles, we can easily exhibit now such examples. As was

5 Examples of causal commutators

Having seen that the existence of non-trivial causal commutators is related to the existence of massless particles, we can easily exhibit now such examples. As was

With this information we can turn now to the analysis of the causal commutators. Given any test function \( h \in \mathcal{C}_1(\mathbb{R}^4) \), one has \( \Box h \in \mathcal{C}_1(\mathbb{R}^4) \), where \( \Box \) is the d’Alembertian. Since \( \partial_\nu h^\nu = 0 \), it follows that \( \partial_\nu (\partial^\nu h^\mu - \partial^\mu h^\nu) = \Box h^\mu \) and consequently \( A(\Box h) = A_\mu(\Box h^\mu) = F_{\mu\nu}(\partial^\nu h^\mu - \partial^\mu h^\nu) = F(dh) \). Similarly, one obtains for \( k \in \mathcal{C}_1(\mathbb{R}^4) \) the equality \( B(\Box k) = G(dk) \). Now given any Borel set \( \Delta_0 \subset \mathbb{R} \) which has a finite distance from \( 0 \in \mathbb{R} \), it follows from the preceding result, the covariance of the fields, and the invariance of \( \Omega \) under translations that

\[
M^2 E(\Delta_0) A(h) \Omega = - E(\Delta_0) A(\Box h) \Omega = - E(\Delta_0) F(dh) \Omega.
\]

In view of the choice of \( \Delta_0 \), this implies \( E(\Delta_0) A(h) \Omega = - M^{-2} E(\Delta_0) F(dh) \Omega \) and, in the same manner, one obtains \( E(\Delta_0) B(k) \Omega = - M^{-2} E(\Delta_0) G(dk) \Omega \). In view of the hermiticity of \( F, G \), and hence of \( A, B \), we therefore arrive at

\[
\langle \Omega, A(h) E(\Delta_0) B(k) \Omega \rangle = \langle \Omega, B(k) E(\Delta_0) A(h) \Omega \rangle = - M^{-2} E(\Delta_0) \langle \Omega, G(dk) \rangle - \langle \Omega, G(dk) M^{-2} E(\Delta_0) F(dh) \Omega \rangle.
\]

Now if \( h, k \in \mathcal{C}_1(\mathbb{R}^4) \) have spacelike separated supports, the same is true for their curls \( dh, dk \in \mathcal{D}(\mathbb{R}^4) \). Since \( F, G \) are local fields, it follows from the preceding equality and the above relation, based on the Jost-Lehmann-Dyson representation, that

\[
\langle \Omega, A(h) E(\Delta_0) B(k) \Omega \rangle - \langle \Omega, B(k) E(\Delta_0) A(h) \Omega \rangle = 0 \quad \text{if} \quad \text{supp} h \perp \text{supp} k.
\]

In view of the continuity properties of the spectral resolution \( \Delta_0 \mapsto E(\Delta_0) \) and the fact that the contributions of the intermediate vacuum state \( \Omega \) cancel in the commutator function, we have established the following result.

**Proposition 4.1.** In any theory having a mass spectrum with values \( m > 0 \) on the orthogonal complement of the vacuum \( \Omega \), one has \( [A(h), B(k)] = 0 \) for all test functions \( h, k \in \mathcal{C}_1(\mathbb{R}^4) \) with \( \text{supp} h \perp \text{supp} k \). The commutators can be different from 0 only in theories, where the spectral projection \( E(\{0\}) \) of the mass operator acts non-trivially on the orthogonal complement of \( \Omega \).

So we conclude that non-zero causal commutators, exhibiting the linking number of the underlying loop functions, only appear in the presence of massless particles; states with non-zero mass do not contribute to them.
shown in the preceding sections, it is sufficient to consider the vacuum expectation values of the commutators.

It follows from locality and Poincaré covariance of the fields \( F, G \) and the spectral properties of the translation operators that the (distributional) commutator functions on \( \mathbb{R}^4 \times \mathbb{R}^4 \) have a Källén-Lehman representation of the form

\[
\langle \Omega, [F_{\mu \nu}(x), G_{\rho \sigma}(y)] \Omega \rangle = \sum_Q \int d\sigma_Q(m) \int dp \epsilon(p_0) \delta(p^2 - m^2) Q_{\mu \nu \rho \sigma}(p) e^{-ip(x-y)},
\]

where the sum extends over tensors \( Q \) of rank four, built from polynomials in \( p \in \mathbb{R}^4 \) and the metric tensor \( g \), and \( d\sigma_Q(m) \) are (unbounded) measures which have support on the mass spectrum of the theory.

The assumption that \( F, G \) are hermitean, covariant and closed skew symmetric tensor fields of rank two imposes constraints on the tensors \( Q \). We do not need to discuss here these constraints in full generality since we know from the outset that contributions to the commutator functions with mass values \( m > 0 \) do not affect the linking numbers. So we can focus on those contributions to the above representation, which arise from an atomic value \( m = 0 \) in the mass spectrum. They can be combined into the expression

\[
K^{(0)}_{\mu \nu \rho \sigma}(x-y) = \int dp \epsilon(p_0) \delta(p^2) Q^{(0)}_{\mu \nu \rho \sigma}(p) e^{-ip(x-y)}, \tag{5.1}
\]

where the tensor \( Q^{(0)} \) is also built from polynomials in \( p \in \mathbb{R}^4 \) and the metric tensor \( g \). It follows after a straightforward computation that the above mentioned properties of the fields \( F, G \) imply that \( Q^{(0)} \) must have the form

\[
Q^{(0)}_{\mu \nu \rho \sigma}(p) = c_1 (p_\mu p_\rho g_{\nu \sigma} - p_\nu p_\rho g_{\mu \sigma} - p_\mu p_\nu g_{\rho \sigma} + p_\nu p_\sigma g_{\mu \rho}) + c_2 (p_\mu p_\tau g_{\nu \tau} - p_\nu p_\tau g_{\mu \tau} - p_\mu p_\mu g_{\nu \nu} - p_\nu p_\nu g_{\mu \mu} - p_\mu p_\rho g_{\nu \mu} + p_\nu p_\rho g_{\mu \nu}) \epsilon^{\tau \upsilon}_{\rho \sigma}, \tag{5.2}
\]

where \( c_1, c_2 \in \mathbb{R} \) and \( \epsilon \) is the totally skew symmetric Levi-Civita tensor.

The tensor in the first line is familiar from expectation values of the electromagnetic field and complies with all constraints, irrespective of the underlying mass spectrum. Its contribution to \( K^{(0)} \) in (5.1) is proportional to the commutator of the free electromagnetic field \( F^{(0)} \) with itself. The tensor in the second line requires some comment, however. It has the correct hermiticity properties, is skew symmetric in \( \mu, \nu \) as well as \( \rho, \sigma \) and manifestly encodes the fact that \( F \) is closed. That this condition is also satisfied for \( G \) follows from the fact that the tensor \( Q^{(0)} \) is restricted in \( K^{(0)} \) to the mass shell \( p^2 = 0 \). The contribution to \( K^{(0)} \), resulting from the second line in relation (5.2), is therefore proportional to the commutator of the free electromagnetic field \( F^{(0)} \) with its Hodge-dual \( \ast F^{(0)} \). But this dual tensor field is also closed since the electric current vanishes, \textit{i.e.} \( d \ast F^{(0)} = \ast \delta F^{(0)} = 0 \). Thus any tensor \( Q^{(0)} \)
of the form given above gives rise to an admissible contribution $K^{(0)}$ to commutator functions.

As has been shown by Roberts [7], cf. also [3], the commutator of the free electromagnetic field $F^{(0)}$ and its Hodge dual $\star F^{(0)}$ gives rise to non-trivial causal commutators for the corresponding intrinsic vector potentials; in contrast, the causal commutators, determined by the commutator of the free electromagnetic field with itself, vanish [3]. In view of the preceding results, we have thus arrived at the following proposition, characterizing all fields $F, G$ leading to intrinsic vector potentials with non-trivial causal commutators.

**Proposition 5.1.** Let $F, G$ be local, hermitean, covariant and closed skew symmetric tensor fields. The causal commutators of the corresponding intrinsic vector potentials $A, B$ are different from zero, indicating the linking numbers of the underlying loop functions, if and only if the (distributional) commutator function $\langle \Omega, [F(\cdot), G(\cdot)] \Omega \rangle$ contains in its Källén-Lehmann representation a contribution $K^{(0)}(\cdot)$ as in (5.1), where $Q^{(0)}$ is of the form (5.2) with $c_2 \neq 0$.

We conclude this section by noting that there exists an abundance of quantum field theory models of fields $F, G$ with properties described in the preceding proposition. Examples can be easily exhibited in the class of generalized free field theories. There the fields have c-number commutation relations, so a generalized free field theory is completely fixed by specifying the two-point functions of the fields in the vacuum state. Thus one may put, for example,

$$\langle \Omega, F(f)G(g)\Omega \rangle = c \langle \Omega, F^{(0)}(f) \star F^{(0)}(g)\Omega \rangle, \quad f, g \in \mathcal{D}_2(\mathbb{R}^4),$$

where $c \in \mathbb{R} \setminus \{0\}$. One then chooses with the help of the Källén-Lehmann representation arbitrary two point functions for the field $F$, respectively $G$, satisfying

$$\langle \Omega, F(\overline{f})F(f)\Omega \rangle \geq |c| \langle \Omega, F^{(0)}(\overline{f})F^{(0)}(f)\Omega \rangle \geq 0, \quad f \in \mathcal{D}_2(\mathbb{R}^4),$$

$$\langle \Omega, G(\overline{g})G(g)\Omega \rangle \geq |c| \langle \Omega, F^{(0)}(\overline{g})F^{(0)}(g)\Omega \rangle \geq 0, \quad g \in \mathcal{D}_2(\mathbb{R}^4).$$

Since $\langle \Omega, \star F^{(0)}(\overline{g}) \star F^{(0)}(g)\Omega \rangle = \langle \Omega, F^{(0)}(\overline{g}) F^{(0)}(g)\Omega \rangle$ for $g \in \mathcal{D}_2(\mathbb{R}^4)$, it implies

$$|\langle \Omega, F(\overline{f})G(g)\Omega \rangle|^2 \leq \langle \Omega, F(\overline{f})F(f)\Omega \rangle \langle \Omega, G(\overline{g})G(g)\Omega \rangle, \quad f, g \in \mathcal{D}(\mathbb{R}^4),$$

in accordance with the condition of Wightman positivity [11]. The resulting generalized free field theories comply with all constraints on the tensor fields $F, G$, and the causal commutators of the corresponding intrinsic vector potentials are different from zero.
6 Conclusions

In the present investigation we have studied the appearance of linking numbers in quantum field theory, which arise from closed skew symmetric tensor fields \( F, G \). These linking numbers appear as “superselected charges” in commutators of the corresponding intrinsic vector potentials \( A, B \), which are smeared with loop functions, having supports in spacelike separated, spatial loops. They are different from zero only in the presence of massless particles in the theory. The intrinsic vector potentials \( A, B \), which are defined on the space of vector valued test functions \( \mathcal{C}_1(\mathbb{R}^4) \) with vanishing divergence, can then not both be extended to local pointlike vector fields, defined on the space of all vector valued test functions \( \mathcal{D}_1(\mathbb{R}^4) \); this is true even if one admits potentials in indefinite metric spaces. So such fields provide genuine examples of closed, but not exact tensor fields in the framework of local quantum field theory.

One can overcome this cohomological obstruction by relaxing the condition of pointlike locality for the vector potentials, admitting fields which are ray-localized. Such an approach has been proposed by J. Mund, B. Schroer and others, cf. [6] and references quoted there. There one proceeds from closed skew symmetric tensor fields \( F \) to vector potentials, defined as operator-valued distributions on \( \mathbb{R}^4 \times dS^3 \), where \( dS^3 = \{ e \in \mathbb{R}^4 : e^2 = -1 \} \) is de Sitter space. These potentials are given by

\[
A_{\epsilon \mu}(x) = \int_0^\infty du e^\rho F_{\mu \rho}(x + u e).
\]

They satisfy, as desired,

\[
\partial_\mu A_{\epsilon \nu}(x) - \partial_\nu A_{\epsilon \mu}(x) = \int_0^\infty du e^\rho (\partial_\mu F_{\nu \rho}(x + u e) - \partial_\nu F_{\mu \rho}(x + u e))

= - \int_0^\infty du e^\rho \partial_\rho F_{\mu \nu}(x + u e) = F_{\mu \nu}(x),
\]

where it is assumed that \( F \) vanishes at spacelike infinity in the states of interest. The locality properties of the underlying tensor fields then imply that the potentials \( A_{\epsilon}(x) \) commute with all local fields which are localized in the spacelike complement of the ray \( x + \mathbb{R}_+ e \). This approach yields gauge invariant vector potentials, which are defined on the test function space \( \mathcal{D}_1(\mathbb{R}^4) \times \mathcal{D}(dS^3) \). The price one has to pay is to give up the standard locality property of the vector potentials, which is, however, unavoidable in the cases considered in the present article. It is an intriguing question whether these ray localized vector potentials can serve as a substitute for the local vector potentials in gauge quantum field theory, as envisaged in [9].

Let us conclude with the remark that our analysis of linking numbers in commutator functions is based on purely topological arguments, avoiding computations of the
Gauss integral (3.2). We therefore hope that by refining our arguments one can establish results on knot invariants and Jones polynomials, as obtained by Witten in topological quantum field theory [13], also in case of relativistic quantum field theories with non-abelian gauge groups. Since the notion of homology, used in Lemma 3.2, is insensitive to knots, such an analysis has to be based on the finer concept of isotopy [1][10], however.

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