Backward Stochastic Differential Equations and Feynman-Kac Formula for Multidimensional Lévy Processes, with Applications in Finance

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Abstract

In this paper we show the existence and form uniqueness of a solution for multidimensional backward stochastic differential equations driven by a multidimensional Lévy process with moments of all orders. The results are important from a pure mathematical point of view as well as in the world of finance: an application to Clark-Ocone and Feynman-Kac formulas for multidimensional Lévy processes is presented. Moreover, the Feynman-Kac formula and the related partial differential integral equations provide an analogue of the famous Black-Scholes partial differential equation and thus can be used for the purpose of option pricing in a multidimensional Lévy market.

Keywords: backward stochastic differential equations, multidimensional Lévy processes, orthogonal polynomials, option Pricing.

AMS Subject Classification: 60J30, 60H05.

1. Introduction

A linear version of Backward Stochastic Differential Equations (BSDEs in short) driven by Brownian motion was initially considered by Bismut(1973) in the context of optimal control. Nonlinear BSDEs were later introduced by Pardoux and Peng (1990) and independently by Duffie and Epstein(1992). The BSDE theory has found wide applications in partial differential equation theory, stochastic controls and, particularly, mathematical finance (see El Karoui and Quenez(1997); Ma and Yong(1999)).

Situ (1997) studied BSDEs driven by a Brownian motion and a Poisson point process. Ouknine (1998) considered BSDEs driven by a Poisson random measure. Nualart and Schoutens (2000) proved a martingale representation theorem for Lévy processes satisfying some exponential moment condition. By using this martingale representation result, Nualart and Schoutens (2001) established the existence and uniqueness of solutions for BSDEs driven by a Lévy process of the kind considered in Nualart and Schoutens (2000), furthermore Nualart and Schoutens (2001) presented the Clark-Ocone and the Feynman-Kac formulas, the related Partial Differential Integral Equation (PDIE) and their applications in finance.

In the past twenty years, there is already a growing interest for multidimensional Lévy Processes. Some concepts and basic properties about multidimensional Lévy Processes were summarized in Sato (1999). Applications of multidimensional Lévy Processes to analyzing biomolecular (DNA and protein) data and one-server light traffic queues were explored by Dembo, Karlin and Zeitouni (1994). A small deviations property of multidimensional Lévy Processes were discussed by Simon (2003). In finance research, practically all financial applications require a multidimensional model with dependence between components: examples are basket option pricing, portfolio optimization, simulation of risk scenarios for portfolios. In most of these applications, jumps in the price process must be taken into account.

This research was supported by the National Basic Research Program of China (973 Program) (Program No.2007 CB814903) and the National Natural Science Foundation of China (Program No.70671069).

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account. Cont and Tankov (2004) systematically investigated these problems in multidimensional Lévy market. In addition, the optimal portfolios in multidimensional Lévy market is discussed by Emmer and Klüppelberg (2004), and option pricing is investigated by Reich, N., Schwab, C. and Winter, C. (2009). Some simulation approaches for multidimensional Lévy processes are also investigated in Cohen and Rosiński (2007). Lévy copulas was also suggested by Kallsen and Tankov (2006) in order to characterize the dependence among components of multidimensional Lévy Processes.

Recently, a martingale representation theorem for multidimensional Lévy processes was also proved in Lin (2011), and the obtained representation formula was similar as that in Nualart and Schoutens (2000). The purpose of this paper is to use this martingale representation result obtained in Lin (2011) to establish the existence and form uniqueness of solutions for BSDE’s driven by a multivariate Lévy process considered in Lin (2011). Although the proof techniques are similar to those in Nualart and Schoutens (2001), the results are important from a pure mathematical point of view as well as in the world of finance. This is illustrated in the applications. The resulting Clark-Ocone and Feynman-Kac formulas, and option pricing in a multivariate Lévy market. Finally, in the appendix one can find detailed proofs of the main results.

The paper is organized as follows. Section 2 contains some preliminaries on multidimensional Lévy processes. Section 3 contains the main result on BSDEs driven by multidimensional Lévy processes. In Section 4 we have included some applications of BSDE’s driven by multidimensional Lévy processes to the Clark-Ocone, the Feynman-Kac formulas, and option pricing in a multivariate Lévy market. Moreover, the Feynman-Kac formula and the related Partial Di

2. Preliminary

A $\mathbb{R}^n$-valued stochastic process $X = \{X(t) = (X_1(t), X_2(t), \ldots, X_n(t))^t, t \geq 0\}$ defined in complete probability space $(\Omega, \mathcal{F}, P)$ is called Lévy process if $X$ has stationary and independent increments and $X(0) = \mathbf{0}$. A Lévy process possesses a càdlàg modification and we will always assume that we are using this càdlàg version. If we let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}_t$, where $\mathcal{G}_t = \sigma\{X(s), 0 \leq s \leq t\}$ is the natural filtration of $X$, and $\mathcal{N}_t$ are the $\mathcal{F}$-null sets of $\mathcal{F}$, then $\{\mathcal{F}_t, t \geq 0\}$ is a right continuous family of $\sigma$-fields. We assume that $\mathcal{F}$ is generated by $X$. For an up-to-date and comprehensive account of Lévy processes we refer the reader to Bertoin (1996) and Sato (1999).

Let $X$ be a Lévy process and denote by $X(t-) = \lim_{s \to t, s < t} X(s)$, $t > 0$, the left limit process and by $\Delta X(t) = X(t) - X(t-)$ the jump size at time $t$. It is known that the law of $X(t)$ is infinitely divisible with characteristic function of the form

$$E[\exp(i\theta \cdot X(t))] = (\phi(\theta))^t,$$ 

where $\phi(\theta)$ is the characteristic function of $X(1)$. The function $\psi(\theta) = \log(\phi(\theta))$ is called the characteristic exponent and it satisfies the following famous Lévy-Khintchine formula (Bertoin, 1996):

$$\psi(\theta) = -\frac{1}{2} \theta \cdot \Sigma \theta + i a \cdot \theta + \int_{\mathbb{R}_0^n} (\exp(i\theta \cdot x) - 1 - i\theta \cdot x \mathbb{1}_{|x| \leq 1}) \nu(dx).$$

where $a, x \in \mathbb{R}_0^n$, $\Sigma$ is a symmetric nonnegative-definite $n \times n$ matrix, and $\nu$ is a measure on $\mathbb{R}_0^n \setminus \{0\}$ with $\int(\|x\|^2 \wedge 1)\nu(dx) < \infty$. The measure $\nu$ is called the Lévy measure of $X$.

Throughout this paper, we will use the standard multi-index notation. We denote by $\mathbb{N}_0$ the set of nonnegative integers. A multi-index is usually denoted by $\mathbf{p}$, $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n$. Whenever $\mathbf{p}$ appears with subscript or superscript, it means a multi-index. In this spirit, for example, for $x = (x_1, \ldots, x_n)$, a monomial in variables $x_1, \ldots, x_n$ is denoted by $x^p = x_1^{p_1} \cdots x_n^{p_n}$. In addition, we also define $\mathbf{p}! = p_1! \cdots p_n!$ and $|\mathbf{p}| = p_1 + \cdots + p_n$; and if $\mathbf{p}, \mathbf{q} \in \mathbb{N}_0^n$, then we define $\delta_{\mathbf{p}, \mathbf{q}} = \delta_{p_1, q_1} \cdots \delta_{p_n, q_n}$. 

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Hypothesis 1. We will suppose in the remaining of the paper that the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$,

$$\int_{|x| \geq \varepsilon} \exp(\lambda |x|) v(dx) < \infty.$$ 

This implies that

$$\int x^p v(dx) < \infty, \quad |p| \geq 2$$

and that the characteristic function $E[\exp(i \theta \cdot X(t))]$ is analytic in a neighborhood of origin $o$. As a consequence, $X(t)$ has moments of all orders and the polynomials are dense in $L^2(\mathbb{R}^n, \mathbb{P} \circ X(t)^{-1})$ for all $t > 0$.

Fix a time interval $[0, T]$ and set $L^2_t = L^2(\mathbb{F}_T, \mathbb{P})$. We will denote by $\mathcal{P}$ the predictable sub-$\sigma$-field of $\mathbb{F}_T \otimes \mathcal{B}_{[0,T]}$. First we introduce some notation:

- Let $H^2_T$ denote the space of square integrable and $\mathbb{F}_t$--progressively one-dimensional measurable processes $\phi = \{\phi(t), t \in [0, T]\}$ such that

$$\|\phi\|^2 = E \left[ \int_0^T \|\phi(t)\|^2 dt \right] < \infty.$$ 

- Set $M^2_T$ will denote the subspace of $H^2_T$ formed by predictable processes.

- $(H^2_T(\mathbb{F}))^m$ and $(M^2_T(\mathbb{F}))^m$ are the corresponding spaces of $m$--dimensional $\mathbb{F}^m$--valued processes equipped with the norm

$$\|\phi\|^2_{L^2(\mathbb{F}), m} = \sum_{k=1}^m \|\phi_k(t)\|^2 \quad \text{where } \phi = (\phi_1, \phi_2, \ldots, \phi_m)'$$

- Set $\mathcal{H}_T^2 = H^2_T \times (M^2_T(\mathbb{F}))^m$.

Following Lin (2011) we introduce power jump monomial processes of the form

$$X(t)^{(p_1, \ldots, p_m)} \overset{def}{=} \sum_{0 < s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_m(s))^{p_m}.$$ 

The number $|p|$ is called the total degree of $X(t)^p$. Furthermore define

$$Y(t)^{(p_1, \ldots, p_m)} \overset{def}{=} X(t)^{(p_1, \ldots, p_m)} - E[X(t)^{(p_1, \ldots, p_m)}] = X(t)^{(p_1, \ldots, p_m)} - m_p t ,$$

the compensated power jump process of multi-index $p = (p_1, p_2, \ldots, p_m)$ where $m_p = \int \cdots \int x^p v(dx)$. Under hypothesis 1, $Y(t)^{(p_1, \ldots, p_m)}$ is a normal martingale, since for an integrable Lévy process $Z$, the process $[Z_t - E[Z_t], t \geq 0]$ is a martingale. We call $Y(t)^{(p_1, \ldots, p_m)}$ the Teugels martingale monomial of multi-index $(p_1, \ldots, p_m)$.

We can apply the standard Gram-Schmidt process with the graded lexicographical order to generate a biorthogonal basis $\{H^p, p \in \mathbb{N}^m\}$, such that each $H^p(|p| = d)$ is a linear combination of the $Y^q$, with $|q| \leq |p|$ and the leading coefficient equal to 1. We set

$$H^p = Y^p + \sum_{q < p, |q| = |p|} c_q Y^q + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q Y^q,$$
where \( p = \{p_1, \ldots, p_n\}, q = \{q_1, \ldots, q_n\} \) and \( \prec \) represent the relation of graded lexicographical order between two multi-indexes. Some details about the technique and theory of orthogonal polynomials of several variables refer to Dunkl and Xu (2001).

Set

\[
p(x)^p = x^p + \sum_{q \prec p, |q| = |p|} c_q x^q + \sum_{k=1}^{[p]-1} \sum_{q \in \mathbb{N}_0^k} c_q x^q,
\]

\[
\tilde{p}(x)^p = x^p + \sum_{q \prec p, |q| = |p|} c_q x^q + \sum_{k=2}^{[p]-1} \sum_{q \in \mathbb{N}_0^k} c_q x^q,
\]

Set

\[
H^p(t) = \sum_{0 < s \leq t} \left( (\Delta X_1)^{p_1} \cdots (\Delta X_n)^{p_n} + \sum_{q \prec p, |q| = |p|} c_q (\Delta X_1)^{q_1} \cdots (\Delta X_n)^{q_n} + \sum_{k=1}^{[p]-1} \sum_{q \in \mathbb{N}_0^k} c_q (\Delta X_1)^{q_1} \cdots (\Delta X_n)^{q_n} \right),
\]

\[
-\text{rE}\left[ X^p(1) + \sum_{q \prec p, |q| = |p|} c_q X^q(1) + \sum_{k=1}^{[p]-1} \sum_{q \in \mathbb{N}_0^k} c_q X^q(1) \right]
\]

\[
= (c_{e_1} X_1(1) + \cdots + c_{e_n} X_n(1)) + \sum_{0 < s \leq t} \tilde{p}(\Delta X(s))
\]

\[
-\text{rE}\left[ \sum_{0 < s \leq t} \tilde{p}(\Delta X(s)) \right] - \text{rE} \left[ c_{e_1} X_1(1) + \cdots + c_{e_n} X_n(1) \right].
\]

Specially we have

\[
H^{e_1}(t) = c_{e_1}(1)(X_1(t) - \text{rE}(X_1(1))),
\]

\[
H^{e_2}(t) = c_{e_2}(2)(X_2(t) - \text{rE}(X_2(1))) + c_{e_1}(2)(X_1(t) - \text{rE}(X_1(1))),
\]

\[
\vdots
\]

\[
H^{e_n}(t) = c_{e_n}(n)(X_n(t) - \text{rE}(X_n(1))) + c_{e_{n-1}}(n)(X_{n-1}(t) - \text{rE}(X_{n-1}(1))) + \cdots + c_{e_1}(n)(X_1(t) - \text{rE}(X_1(1))).
\]

(1)

The main results in Lin (2011) is the Predictable Representation Property (PRP): Every random variable \( F \) in \( L^2(\Omega, \mathcal{F}) \) has a representation of the form

\[
F = \mathbb{E}(F) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} \int_0^T \Phi^p(s)dH^p(s)
\]

where \( \Phi^p(s) \) is predictable. It is worthwhile to emphasize that \( \Phi^p(s) \) is not uniqueness, that is to say that \( \Phi^p(s) \) is different for different Gram-Schmidt process, but the form of \( \Phi^p(s) \) is uniqueness. This type of uniqueness is called “form uniqueness”. This result is an extended version for the corresponding Theorem in Nualart and Schouten (2000).

**Remark 1.** If \( \nu = 0 \), we are in the classical Brownian case and \( H^p(t) = 0, |p| \geq 2 \). If \( \mu \) has only mass in 1, we are in the Poisson case; and also here \( H^p(t) = 0, |p| \geq 2 \). Both case are degenerate cases in this Lévy framework.

From these observations, it is not so hard to see that the PRP property shows that financial markets based on a non-Brownian or non-Poissonian Lévy process, i.e. with a stock price behaviour \( S_i(t) = S_i(0) \exp(rt + X_i(t)) \), \( i = 1, 2, \cdots, n \), are so called incomplete, meaning that perfectly replicating or hedging strategies do not exists for all relevant contingent claims.
3. Multidimensional BSDEs Driven by Multidimensional Lévy Processes

Taking into account the results and notation presented in the previous section, it seems natural to consider the

\[-dY(t) = f(t, Y(t^-), Z(t))dt - \sum_{d=1}^{\infty} \sum_{p \in \mathcal{P}_d} z^p(s)dH^p(s), \quad Y(T) = \xi,\]  

(2)

where

- \(Y(t) = (Y_1(t), Y_2(t), \ldots, Y_m(t))'\).
- \(Z(t) = (z^p(t))_{p \in \mathcal{P}_d}\) each component \(z^p(t) = (z^p_1, \ldots, z^p_m)'\) is a \(m\)-variables \(\mathcal{F}_t\) predictable function;
- \(f = (f_1, f_2, \ldots, f_m) : \Omega \times [0, T] \times \mathbb{R}^m \times \left(M^{2}_d(P)\right)^m \to \mathbb{R}^m\) is a measurable \(m\)-dimensional vector function such that \(f(\cdot, 0, 0) \in \left(H^2_s\right)^m\).
- \(f\) is uniformly Lipschitz in the first two components, i.e., there exists \(C_k > 0, k = 1, 2, \ldots, m\) such that \(dt \otimes d\mathbb{P}\) a.s., for all \((y_1, z_1)\) and \((y_2, z_2)\) in \(\mathbb{R}^m \times (F)^m\)

\[|f_k(t, y_1, z_1) - f_k(t, y_2, z_2)| \leq C_k \left(||y_1 - y_2||_2 + ||z_1 - z_2||_{\mathcal{F}^p}\right), \quad k = 1, 2, \ldots, m.\]
- \(\xi \in L^2_t(\Omega, \mathbb{P}).\)

If \((f, \xi)\) satisfies the above assumptions, the pair \((f, \xi)\) is said to be standard data for BSDE. A solution of the BSDE is a pair of processes, \((Y(t), Z(t)), 0 \leq t \leq T\) \(\in H^2 \times \left(M^{2}_d(P)\right)^m\) such that the following relation holds for all \(t \in [0, T]\):

\[Y(t) = \xi + \int_{t}^{T} f(s, Y(s^-), Z(s))ds - \sum_{d=1}^{\infty} \sum_{p \in \mathcal{P}_d} \int_{t}^{T} z^p(s)dH^p(s).\]  

(3)

Note that the progressivity measurability of \((Y(t), Z(t)), 0 \leq t \leq T\) implies that \((Y(0), Z(0))\) is deterministic.

When the orthogonal polynomials are fixed, a first key result concerns the existence uniqueness of solution of BSDE:

**Theorem 1.** Given standard data \((f, \xi)\), there exists a unique form solution \((Y, U, Z)\) which solves the BSDE (3)

The proof can be found in the Appendix, as the proof of the continuous dependency of the solution on the final data \(\xi\) and the function \(f\).

**Theorem 2.** Given standard data \((f, \xi)\) and \((f', \xi')\), let \((Y, Z)\) and \((Y', Z')\) be the unique adapted form solutions of the BSDE(3) corresponding to \((f, \xi)\) and \((f', \xi')\). Then

\[E \left[\int_{0}^{T} \left(||Y(s^-) - \hat{Y}(s^-)||^2 + \sum_{d=1}^{\infty} \sum_{p \in \mathcal{P}_d} ||z^p(s) - \hat{z}^p(s)||^2\right)ds\right] \leq C \left(E||\xi - \hat{\xi}||^2\right) + E \left[\int_{0}^{T} ||f(s, Y(s^-), Z(s)) - \hat{f}(s, Y(s^-), Z(s)||^2ds\right].\]

The definition of “form solution” in Theorem 1 and Theorem 2 is deferred until later in the proofs of Theorem 1 and Theorem 2 in Appendix.

4. Applications

Suppose our n-dimensional Lévy process \(X(t)\) has no Brownian part, i.e. \(X(t) = at + L(t)\), where \(a = (a_1, \ldots, a_n)'\)

and \(L(t)\) is n-dimensional pure jump process with Lévy measure \(\nu(dx)\).
4.1. Clark-Ocone Formula and Feynman-Kac Formula

Let us consider the simple case of a BSDE where \( f = 0 \), and the terminal random vector \( \xi \) is a function of \( X(T) \), that is,

\[
-dY(t) = -\sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} z^p(t) dH^p(t), \quad Y(T) = g(X(T))
\]

or equivalently

\[
Y(t) = g(X(T)) - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} \int_t^T z^p(s) dH^p(s), \quad (4)
\]

where \( g = (g_1, g_2, \ldots, g_m)' \) and \( \mathbb{E}([|g(X(T))|^2]) < \infty \). Let \( \theta_k = \theta_k(t, x), k = 1, 2, \ldots, m \), be the solution of the following PDIE(Partial Differential Integral Equation) with terminal value \( g_k \):

\[
\frac{\partial \theta_k}{\partial t}(t, x) + \int_{\mathbb{R}^n} \left( \theta_k(t, x + y) - \theta_k(t, x) - \frac{\partial \theta_k}{\partial x}(t, x) \cdot y \right) \nu(dy) + \mathbf{a} \cdot \frac{\partial \theta_k}{\partial x}(t, x) = 0, \quad (5)
\]

where \( \mathbf{a} = \int_{[|y|] \geq 1} y \nu(dy) \). Set

\[
\theta^{(1)}_k(t, x, y) = \theta_k(t, x + y) - \theta_k(t, x) - \frac{\partial \theta_k}{\partial x}(t, x) \cdot y. \quad (6)
\]

The following result is a version of the Clark-Ocone formula for functions of a Lévy process. Again the proof can be found in the Appendix.

**Proposition 3.** Suppose that \( \theta_k \) is a \( C^{1,2} \) function for \( k = 1, 2, \ldots, m \) such that \( \frac{\partial \theta_k}{\partial x} \) and \( \frac{\partial^2 \theta_k}{\partial x^2} \) are bounded by a polynomial function of \( x \), uniformly in \( t \), then the unique adapted form solution of (4) is given by

\[
Y_1(t) = \theta_0(t, X(t))
\]

\[
z_k^p = \int_{\mathbb{R}^n} \theta^{(1)}_k(t, X(t), y) p^p(y) \nu(dy), \quad |p| \geq 2,
\]

\[
z_k^i = \int_{\mathbb{R}^n} \theta^{(1)}_k(t, X(t), y) p^i(y) \nu(dy) + \sum_{j=1}^n \frac{\partial \theta_k}{\partial x_j}(t, x) \tilde{e}_j, \quad i = 1, 2, \ldots, n
\]

where \( \theta_k = \theta_k(t, x) \) for \( k = 1, 2, \ldots, m \) are the solutions of the system of PDIE(5), \( \theta^{(1)}_k(t, x, y) \) for \( k = 1, 2, \ldots, m \) are given by (6) and \( [\tilde{e}_i] \) is the inverse matrix of the coefficient matrix Gram-Schmidt transformation (1).

Now by taking expectations we derive that the solution \( \theta(t, x) = (\theta_1(t, x), \ldots, \theta_m(t, x))' \) to our PDIE(5) equation has the stochastic representation

\[
\theta(t, x) = \mathbb{E} [g(X(T)) | X(t) = x].
\]

This is an extension of the classical Feynman-Kac Formula.

If \( \int_{\mathbb{R}^n} |y| \nu(dy) < \infty \), and we take \( \mathbf{a} = \int_{[|y|] \geq 1} y \nu(dy) \), then the equation (5) reduces to

\[
\frac{\partial \theta}{\partial t}(t, x) + \int_{\mathbb{R}^n} (\theta(t, x + y) - \theta(t, x)) \nu(dy) = 0,
\]

\[
\theta(T, x) = g(x),
\]

and we have for \( k = 1, 2, \ldots, m \)

\[
z_k^p = \int_{\mathbb{R}^n} (\theta_k(t, X(t) + y) - \theta_k(t, X(t))) p^p(y) \nu(dy), \quad |p| \geq 2,
\]

\[
z_k^i = \int_{\mathbb{R}^n} (\theta_k(t, X(t) + y) - \theta_k(t, X(t))) p^i(y) \nu(dy) + \sum_{j=1}^n \frac{\partial \theta_k}{\partial x_j}(t, x) \tilde{e}_j, \quad i = 1, 2, \ldots, n.
\]
**Example:** In this example, we define a two-dimensional Poisson process by using Lévy copulas. All the concepts and notations are adopted from the Kallsen and Tankov (2006). Here the two marginal processes are given respectively by two Poisson process $N_i(t)$ where $i = 1, 2$. In particular, the Lévy copula $F(u_1, u_2) : \mathbb{R}^n \to \mathbb{R}$ is taken as

$$F(u_1, u_2) = (|u_1|^\mu + |u_2|^\mu)^{-1/\mu}(\eta I_{[u_1, u_2 \geq 0]} - (1 - \eta)I_{[u_1, u_2 < 0]}) \quad (7)$$

where $\mu > 0$ and any $\eta \in [0, 1]$. It defines a two parameter family of Lévy copulas which resembles the Clayton family of ordinary copulas. Thus its Lévy measure can be calculated as:

$$\nu(dx_1 dx_2) = \frac{\eta(1 + \mu)(\lambda_1 \lambda_2)^\mu}{(\lambda_1^\mu + \lambda_2^\mu)^{1/2}} \delta_1(x_1)\delta_1(x_2). \quad (8)$$

Set two compensated Poisson process $X_i(t) = N_i(t) - \lambda_i t$ for $i = 1, 2$, then we can construct a set of martingales $H^p, p \in \mathbb{N}_0, i = 1, 2$ by orthogonalizing procedure proposed by Lin (2011) such that $H^P$ is strongly orthogonal to $H^q$ with respect to Lévy measure $\nu(dx_1 dx_2)$, for $p \neq q$. Moreover the PDIE (5) reduces to

$$\frac{\partial \theta}{\partial t}(t, x) + \theta(t, x + 1) - \theta(t, x) - \frac{\partial \theta}{\partial x}(t, x) \cdot 1 \eta(1 + \mu)(\lambda_1 \lambda_2)^\mu = 0,$$

$$\theta(T, x) = g(x).$$

The Clark-Ocone Formula is now given by

$$g_\lambda(X(T)) = \mathbb{E}[g_\lambda(X(T))] + \sum_{i=1}^n \int_0^T (\theta_i(s, X(s) + 1) - \theta_i(s, X(s))) \, dX_i(s)$$

### 4.2 Nonlinear Clark-Haussman-Ocone Formula and Feynman-Kac Formula

Let us consider the BSDE

$$-dY(t) = f(t, Y(t), Z(t)) + \sum_{p=1}^\infty \sum_{\nu \in \mathbb{N}_0^d} z^\nu(t) dH^\nu(t), \quad Y(T) = g(X(T)) \quad (9)$$

or equivalently

$$Y(t) = g(X(T)) + \int_0^T f(s, Y(s-), Z(s))ds - \sum_{p=1}^\infty \sum_{\nu \in \mathbb{N}_0^d} \int_0^T z^\nu(s) dH^\nu(s),$$

Suppose that $\theta_k = \theta_k(t, x)$ for $k = 1, 2, \cdots, m$ satisfy the following system of PDIE:

$$\frac{\partial \theta_k}{\partial t}(t, x) + \int_{\mathbb{R}^d} \theta_k^{(1)}(t, x, y)v(dy) + a \cdot \frac{\partial \theta_k}{\partial x}(t, x) + f_k(t, \theta_k(t, x), \Theta_k(t, x)) = 0, \quad (10)$$

$$\theta_k(T, x) = g_k(x).$$

where as in the previous section, we define $\theta_k^{(1)}(t, x, y)$ by (6), and $\Theta_k(t) = [\theta_k(t)]_{p \in \mathbb{N}_0^d}$ where

$$\theta_k^{(1)}(t, x, y) = \int_{\mathbb{R}^d} \theta_k^{(1)}(t, x, y)p^k(y) v(dy), \quad (11)$$

**Proposition 4.** Suppose that $\theta_k$ is a $C^{1,2}$ function such that $\frac{\partial \theta_k}{\partial t}$ and $\frac{\partial \theta_k}{\partial x}$ are bounded by a polynomial function of $x$, uniformly in $t$, then the unique adapted form solution of (9) is given by

$$\begin{align*}
Y_k(t) &= \theta_k(t, X(t)) \\
\zeta^p_k &= \int_{\mathbb{R}^d} \theta_k^{(1)}(t, X(t), y)p^k(y)v(dy), \quad |p| \geq 2, \\
\zeta^i_k &= \int_{\mathbb{R}^d} \theta_k^{(1)}(t, X(t), y)p^k(y)v(dy) + \sum_{j=1}^n \frac{\partial \theta_k}{\partial x^i} \tilde{c}_{ij}, \quad i = 1, 2, \cdots, n.
\end{align*}$$

where $\theta_k = \theta_k(t, x)$ for $k = 1, 2, \cdots, m$ are the solution of the system of PDIE(10), $\theta_k^{(1)}(t, x, y)$ is given by (6) and $[\tilde{c}_{ij}]$ is the inverse matrix of the coefficient matrix Gram-Schmidt transformation (1).
Notice that taking expectations we get

\[ \theta_k(t, x) = \mathbb{E} \left[ g_k(X(T)) | X(t) = x \right] + \mathbb{E} \left[ \int_t^T f_k(s, \theta_k(s, X(s-)), \Theta_k(s, X(s-))) \, ds | X(t) = x \right]. \]

**Example:** Consider again the very special case where have a two-dimensional Poisson process \((N_1(t), N_2(t))\) by using Lévy copulas (7). Set \(X_i(t) = N_i(t) - \lambda_i t\) where \(i = 1, 2\). Then the PDE (10) reduces to

\[ \frac{\partial \theta_k}{\partial t}(t, x) + \left( \theta_k(t, x + 1) - \theta_k(t, x) - \frac{\partial \theta_k}{\partial x}(t, x) \cdot 1 \right) \frac{\eta(1 + \mu)(\lambda_1 \lambda_2)^\mu}{(\lambda_1^\mu + \lambda_2^\mu)^\mu + 2} \\
+ f_k(t, \theta_k(t, x), \theta_k(t, x + 1) - \theta_k(t, x)) = 0 \\
\theta_k(t, x) = g_k(x), \\
k = 1, 2, \ldots, m \]

And we derive the nonlinear Feynman-Kac Formula:

\[ \theta_k(t, x) = \mathbb{E} \left[ g_k(X(T)) | X(t) = x \right] + \mathbb{E} \left[ \int_t^T f_k(s, \theta_k(s, X(s-)), \Theta_k(s, X(s-))) \, ds | X(t) = x \right]. \]

where \(k = 1, 2, \ldots, m\).

### 4.3. Option Pricing

In the last two decades several particular choices for one-dimensional non-Brownian Lévy processes were proposed. Madan and Seneta (1990) have proposed a Lévy process with variance gamma distributed increments. We mention also the Hyperbolic Model proposed by Eberlein and Keller (1995). In the same year Barndorff-Nielsen (1995) proposed the normal inverse Gaussian Lévy process. The CMGY model was also introduced in Carr et al. (2000). Finally, we mention the Meixner model (see Grigelionis 1999) and Schoutens (2001). All models give a much better fit to the data and lead to an improvement with respect to the Black-Scholes model.

Multidimensional models with jumps are more difficult to construct than one-dimensional ones. A simple method to introduce jumps into a multidimensional model is to take a multivariate Brownian motion and time change it with a univariate subordinator (refer to Cont and Tankov (2004)). The multidimensional versions of the models include variance gamma, normal inverse Gaussian and generalized hyperbolic processes. The principal advantage of this method is its simplicity and analytic tractability; in particular, processes of this type are easy to simulate. Another method to introduce jumps into a multidimensional model is so-called method of Lévy copulas proposed by Kallsen and Tankov (2006). The principle advantage in this way lies in that the dependence among components of the multidimensional Lévy processes can be completely characterized with a Lévy copula. This allows us to give a systematic method to construct multidimensional Lévy processes with specified dependence.

Here we define a multivariate Meixner process by using Lévy copulas, and all the concepts and notations are adopted from the Kallsen and Tankov (2006). In particular, for \(n \geq 2\), the Lévy copula \(F(u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}\) is taken as

\[ F(u_1, \ldots, u_n) = 2^{\frac{1}{2-n}} \left( \sum_{j=1}^n \left| u_j \right|^{\mu} \right)^{-\frac{1}{\mu}} \left( \eta I_{[u_1, \ldots, u_n \geq 0]} - (1 - \eta) I_{[u_1, \ldots, u_n < 0]} \right). \]

It defines a two parameter family of Lévy copulas which resembles the Clayton family of ordinary copulas. It is in fact a Lévy copula homogeneous of order 1, for any \(\mu > 0\) and any \(\eta \in [0, 1]\).

In addition, we know that if the tail integrals \(U_i(x_i)\) related to the marginal Lévy densities \(\nu_i(d x_i), i = 1, \ldots, n\), are absolutely continuous, we can compute the Lévy density of the Lévy copula process by differentiation as follows:

\[ \nu(dx_1, \ldots, dx_n) = \partial_1 \cdots \partial_n F[\xi = U_1(x_1), \ldots, \xi = U_n(x_n)] \nu_1(dx_1) \cdots \nu_n(dx_n) \]
where $v_i(dx_i), \ldots, v_n(dx_n)$ are marginal Lévy densities.

Hence we are able to construct a $n$-variate Lévy process $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))'$ with state space $\mathbb{R}^n$ and characteristic triple $(\mathbb{E}, \nu, \alpha)$ by using the above complete dependent Lévy copula or Clayton Lévy copulas proposed by Kallsen and Tankov (2006). Here, in particular, we will consider Lévy marginal measure $v_i$ of Meixner type for Lévy copula. We recall the marginal density $f_i$, the cumulative generating function $K_i$, the drift $a_i$, and the marginal Lévy measure $\nu_i$, for the $i$th component of the Meixner Process $X(t), t \geq 0$, $i = 1, 2, \cdots, n$, are illustrated as follows,

$$\frac{\mathbb{P}_{Mec}}{dx_i} = f_i(x_i; \alpha_i, \beta_i, \delta_i, \mu_i) = \left(2\cos \frac{\pi}{2}\right)^{\frac{\beta_i}{2}} e^{-\left(\frac{1}{\alpha_i} x_i^2 + \mu_i x_i\right)} \left(1 + \frac{\alpha_i x_i^2}{2}\right)^{-\frac{\beta_i}{2}},$$

$$K_i(\theta; \alpha_i, \beta_i, \delta_i, \mu_i) = \mu_i \theta_i + 2\delta_i \left(\log \cos \frac{\beta_i}{2} - \log \cos \frac{\alpha_i \theta_i + \beta_i}{2}\right),$$

$$a_i(\alpha_i, \beta_i, \delta_i, \mu_i) = \mu_i + \alpha_i \delta_i \tan \frac{\beta_i}{2} = 2\delta_i \int_1^\infty \frac{\sinh \frac{\beta_i}{2}}{\sinh \frac{\alpha_i}{2}} dx_i,$$

$$v_i(dx_i; \alpha_i, \beta_i, \delta_i, \mu_i) = \frac{\delta_i e^{-\frac{\beta_i}{2}}}{x_i \sinh \frac{\alpha_i}{2}} dx_i,$$

where $\alpha_i > 0, -\pi < \beta_i < \pi, \mu_i \in \mathbb{R}$, and $\delta_i > 0$.

From the form of the cumulative generating function one easily deduces that the density at any time $t$ can be calculated by multiplying the parameters $\delta_i$ and $\mu_i$ by $t$ for both cases.

In addition, Reich, etc.(2009) had proved that $e^{X_t}$ is a martingale with respect to the canonical filtration $\mathcal{F}$ of $X$ if only if

$$\frac{\sigma_{ij}}{2} + a_j + \int_{\mathbb{R}^n} (e^{z_j} - 1 - z_j I_{|z|>1}) \nu(dz) = 0.$$

Following Reich, etc.(2009), we assume a risk-neutral market which consists of one riskless asset (the bond) with price process given by $B(t) = e^r$, where $r$ is compund interest rate, and $n(\geq 1)$ risky assets (the stocks), with price process:

$$S_i(t) = S_i(0) \exp(rt + X_i(t)), \quad i = 1, 2, \cdots, n,$$

where $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))'$ is a $n$-variate Lévy process and characteristic triple $(\mathbb{E}, \nu, \alpha)$ under a risk-neutral measure $\mathbb{P}$ such that $e^{X_t}$ is a martingale with respect to the canonical filtration $\mathcal{F}_t^{\mathbb{P}}$ of $X(t)$. Denote by $\mathbb{P}(dz)$ the probability measure of $X(1)$.

We consider a European option with maturity $T < \infty$ and payoff $G(S)$ where $S = (S_1, \cdots, S_n)'$, and we assume that $G(S)$ is Lipschitz. According to the fundamental theorem of asset pricing (see Delbaen and Schachermayer (1994)) the value $V(t, S)$ of this option is given by

$$V(t, S) = \mathbb{E}[e^{-r(T-t)}G(S(T))|S(t) = S].$$

If $V(t, S)$ satisfies

$$V(t, S) \in C^{1,2}((0, T) \times \mathbb{R}^n_0) \cap C^0([0, T] \times \mathbb{R}^n_0)$$

Then Reich,etc.(2009) had proved that $V(t, S)$ is a classical solution of the backward Kolmogorov equation:

$$\frac{\partial V}{\partial t}(t, S) + \frac{1}{2} \sum_{i,j=1}^n S_i S_j \sigma_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j}(t, S) + r \sum_{i=1}^n S_i \frac{\partial V}{\partial S_i}(t, S) - rV(t, S)$$

$$+ \int_{\mathbb{R}^n} \left(V(t, S \cdot e^z) - V(t, S) - \sum_{i=1}^n S_i (e^z_i - 1) \frac{\partial V}{\partial S_i}(t, S) \right) \nu(dz) = 0,$$

on $(0, T) \times \mathbb{R}^n_0$ where $V(t, S \cdot e^z) \overset{def}{=} V(t, S_1 e^{z_1}, \cdots, S_n e^{z_n})$, and the terminal condition is given by

$$V(T, S) = g(S), \quad \forall S \in \mathbb{R}^n_0.$
5. Appendix: Proofs of the Results

Proof of Theorem 1:

We define a mapping $\Phi$ from $\mathcal{H}^2_T$ into itself such that $(Y, Z) \in \mathcal{H}^2_T$ is a solution of the BSDE if only if it is a fixed point of $\Phi$. Given $(U, V) \in \mathcal{H}^2_T$, we define $(Y, Z) = \Phi(U, V)$ as follows:

$$Y(t) = E \left[ \xi + \int_t^T f(s, U(s-), V(s))ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

and $(Z(t), 0 \leq t \leq T)$ is given by the martingale representation of Lin (2011) applied to the square integrable random variable

$$\xi + \int_0^T f(s, U(s-), V(s))ds,$$

i.e.,

$$\xi + \int_0^T f(s, U(s-), V(s))ds = E \left[ \xi + \int_0^T f(s, U(s-), V(s))ds \right] + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}} \int_0^T \phi^p(s)dH^p(s),$$

In according to the result in Lin (2011), a different Gram-Schmidt process will generate a different $\Phi^p(s)$. Here and hereafter, for a given $(U, V) \in \mathcal{H}^2_T$, we always mean that the $\Phi^p(s)$ is generated by arbitrarily selecting a fixed Gram-Schmidt process. It is to say that the Gram-Schmidt process is fixed once the Gram-Schmidt process is first arbitrarily selected. In the following, the existence and uniqueness of the solution for BSDE(3) will related to this selected $\Phi^p(s)$. Although a $\Phi^p(s)$ is different for a different Gram-Schmidt process, the expression forms of $\Phi^p(s)$ are the same, the corresponding solutions are called “form solution”.

Taking the conditional expectation with respect to $\mathcal{F}_t$ in the last identity yields

$$Y_t + \int_0^t f(s, U(s-), V(s))ds = Y_0 + \sum_{d=1}^\infty \sum_{p \in \mathbb{N}} \int_0^T \phi^p(s)dH^p(s),$$

from which we deduce that

$$Y_t = \xi + \int_t^T f(s, U(s-), V(s))ds - \sum_{d=1}^\infty \sum_{p \in \mathbb{N}} \int_t^T \phi^p(s)dH^p(s),$$

and we have shown that $(Y, Z) \in \mathcal{H}^2_T$ solves our BSDE if only if it is a fixed point of $\Phi$.

Next we prove that $\Phi$ is a strict contraction on $\mathcal{H}^2_T$ equipped with the norm

$$\| (Y, Z) \| = \left( \int_0^T e^{\beta s} \left( \| Y(s-) \|^2 + \| Z(s) \|^2 \right) ds \right)^{1/2},$$

for a suitable $\beta > 0$. Let $(U, V)$ and $(U', V')$ be two elements of $\mathcal{H}^2_T$ and set $\Phi(U, V) = (Y, Z)$ and $\Phi(U', V') = (Y', Z')$. Denote $(\bar{U}, \bar{V}) = (U - U', V - V')$ and $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$.

Applying Itô’s formula from $s = t$ to $s = T$, to $e^{\beta s}\| Y(s) - Y(s') \|^2$, it follows that

$$e^{\beta s}\| Y(t) - Y(t') \|^2 = -\beta \int_t^T e^{\beta s}\| Y(s-') - Y'(s-') \|^2 ds - 2 \int_t^T e^{\beta s} (Y(s-') - Y'(s-')) \cdot d(Y(s) - Y(s')) - \sum_{j=1}^m \int_t^T e^{\beta s} d[Y_j - Y'_j, Y_j - Y'_j](s). \quad (12)$$
We have
\[
-d(Y(t) - Y(t)^') = (f(t, U(t-), V(t)) - f(t, U(t-'), V(t'))) dt + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \mathbf{v}_d^p (t) dH^p(t),
\]
\[
d(Y_j - Y'_j, Y_j - Y'_j)(t) = \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \mathbf{v}_d^p (t) d[H^p, H^q](t), \quad j = 1, 2, \ldots, m
\]
where the symbol “\(\odot\)” represents the Hadamard-Schur product for two vectors.

Hence, taking expectations in (12), we have
\[
< H^p, H^q > (t) = \delta_{p,q} t.
\]
Using the fact that \(f\) is Lipschitz with constant \(C\) yields
\[
\mathbb{E} \left[ e^{\beta\|Y(t) - Y(t)^\prime\|} \right] + \frac{m}{\beta} \mathbb{E} \left[ \int_t^T e^{\beta\|\phi(s)\|} ds \right] \leq -\beta \mathbb{E} \left[ \int_t^T e^{\beta\|Y(s) - Y'(s)\|} ds \right] + 2CE \left[ \int_t^T e^{\beta\|Y(s) - Y'(s)\| \cdot (\|U(s) - U'(s)\| + \|V(s)\|)} ds \right].
\]
If we now use the fact that for every \(c > 0\) and \(a, b \in \mathbb{R}\) we have that \(2ab \leq ca^2 + \frac{1}{c}b^2\) and \((a + b)^2 \leq 2a^2 + 2b^2\), we obtain
\[
\mathbb{E} \left[ e^{\beta\|Y(t) - Y(t)^\prime\|} \right] + \mathbb{E} \left[ \int_t^T e^{\beta\|\phi(s)\|} ds \right] \leq (4C^2 - \beta) \mathbb{E} \left[ \int_t^T e^{\beta\|Y(s) - Y'(s)\|} ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T e^{\beta\|U(s) - U'(s)\|^2 + \|V(s)\|^2} ds \right].
\]
Taking now \(\beta = 4C^2 + 1\), and noting that \(e^{\beta\|Y(t) - Y(t)^\prime\|} \geq 0\), we finally derive
\[
\mathbb{E} \left[ \int_t^T e^{\beta\|Y(s) - Y'(s)\|^2} ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta\|\phi(s)\|^2} ds \right] \leq \frac{1}{2} \mathbb{E} \left[ \int_t^T e^{\beta\|U(s) - U'(s)\|^2 + \|V(s)\|^2} ds \right],
\]
that is
\[
\|Y(Z)\|^2 \leq \frac{1}{2} \|U, V\|^2,
\]
from which it follows that \(\Phi\) is a strict contraction on \(\mathcal{H}_f^2\) equipped with the norm \(\| \cdot \|\) if \(\beta = 4C^2 + 1\). Then \(\Phi\) has a unique fixed point and the theorem is proved.

\textbf{Proof of Theorem 2:}
Applying Itô’s formula from \( s = t \) to \( s = T \), to \( \|Y(s) - Y(s')\|^2 \), it follows that

\[
\|Y(T) - Y(T')\|^2 - \|Y(t) - Y(t')\|^2 = 2 \int_t^T (Y(s) - Y(s')) \cdot dY(s) - Y(s') + \sum_{j=1}^m \int_t^T d[Y_j, Y_j](s).
\]

Taking expectations and using the relations

\[
-d(Y(t) - Y(t')) = (f(t, Y(t-), Z(t)) - f^*(t, Y(t-'), Z(t')))dt + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \partial f^*(t)dH^p(t),
\]

we have

\[
\begin{align*}
\mathbb{E}\left[\|Y(t) - Y(t')\|^2\right] &= \sum_{j=1}^m \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \mathbb{E}\left[\int_t^T \partial f^*(s) \xi^2 \, ds\right] \\
&= \mathbb{E}\left[\int_t^T \|\xi - \xi'\|^2 \, ds\right] + 2\mathbb{E}\left[\int_t^T (Y(s) - Y(s')) \cdot (f(s, Y(s-), Z(s)) - f^*(s, Y(s-'), Z(s')) \, ds\right].
\end{align*}
\]

Using the Lipschitz property of \( f' \), computations similar to those of the proof of Theorem 1 we obtain

\[
\begin{align*}
\mathbb{E}\left[\|Y(t) - Y(t')\|^2\right] &\leq \mathbb{E}\left[\|\xi - \xi'\|^2\right] + (1 + 2C' + 2C'^2)\mathbb{E}\left[\int_t^T \|Y(s) - Y(s')\|^2 \, ds\right] \\
&+ \mathbb{E}\left[\int_t^T \|f(s, Y(s-), Z(s)) - f^*(s, Y(s-'), Z(s'))\|^2 \, ds\right].
\end{align*}
\]

Then by Gronwall’s inequality the result follows.

**Lemma 5.** Let \( h : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R} \) be a random function measurable with respect to \( \mathcal{P} \otimes \mathcal{B}_{\mathbb{R}^n} \) such that

\[
|h(s, y)| \leq a_s(y \cdot y + \|y\|) \quad a.s.,
\]

where \( a_s, 0 \leq s \leq T \) is a nonnegative predictable process such that \( \mathbb{E}\left[\int_0^T a_s^2 \, ds\right] < \infty \). Then for each \( t \in [0, T] \) we have

\[
\sum_{t < s \leq T} h(s, \triangle X(s)) = \int_t^T \int_{\mathbb{R}^n} h(s, y) \nu(dy) \, ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \int_0^T \left< h(s), \nu^p \right> \, dH^p(s).
\]

**Proof of Lemma 5:** Because (13) implies that \( \mathbb{E}\left[\int_0^T |h(s, y)|^2 \nu(dy) \, ds\right] < \infty \), we have that

\[
M(t) = \sum_{0 < s \leq t} h(s, \triangle X(s)) - \int_0^t \int_{\mathbb{R}^n} h(s, y) \nu(dy) \, ds.
\]

\[
\sum_{t < s \leq T} h(s, \triangle X(s)) = \int_t^T \int_{\mathbb{R}^n} h(s, y) \nu(dy) \, ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_d} \int_0^T \left< h(s), \nu^p \right> \, dH^p(s).
\]
is a square integrable martingale. By the Predictable Representation Theorem, there exists a process $\phi$ in the space $(M^2(I^2))^{\text{fin}}$ such that

$$M(t) = \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_0} \int_0^t \phi^p(s) dH^p(s)$$

Taking into account that $< H^p, H^q > = \delta_{pq}$, we have

$$< M, H^p > = \int_0^t \phi^p(s) ds. \quad (14)$$

On the other hand, using that $\Delta M(s) \Delta H^p(s) = h(s, \Delta X(s)) p^p(\Delta X(s))$ we obtain

$$< M, H^p > = \int_0^t \int_{\mathbb{R}^e} h(s, y) p^p(y) \nu(dy) ds. \quad (15)$$

Consequently, (14) and (15) imply

$$\phi^p(s) = \int_{\mathbb{R}^e} h(s, y) p^p(y) \nu(dy).$$

and the result follows. \(\square\)

**Proof of Proposition 3:**

Under the hypotheses of Proposition 3 the function $d_k(X, t, y) = h(s, X(s) + y) - h(s, X(s)) - \frac{\partial h}{\partial y}(s, X(s))y$, we obtain

$$g_k(T, X(T)) - g_k(t, X(t)) = \int_t^T \frac{\partial g_k}{\partial X}(s, X(s)) ds + \sum_{i=1}^n \int_t^T \frac{\partial g_k}{\partial x_i}(s, X(s)) dX_i(s)$$

$$+ \sum_{i=1}^n \int_t^T \left[ \theta_k(s, X(s)) - \theta_k(s, X(s)) - \frac{\partial g_k}{\partial x_i}(s, X(s)) \Delta X_i(s) \right] ds. \quad (16)$$

If we apply Lemma 5 to $h(s, X, y) = \theta_k(s, X(s) + y) - \theta_k(s, X(s)) - \frac{\partial h}{\partial y}(s, X(s))y$, we obtain

$$\sum_{t \leq s \leq t} \int_t^T \left( \int_{\mathbb{R}^e} \frac{d_k^{(1)}}{k}(s, X(s), y) p^p(y) \nu(dy) \right) dH^p(s)$$

$$= \int_t^T \int_{\mathbb{R}^e} \frac{d_k^{(1)}}{k}(s, X(s), y) \nu(dy) ds \quad (17)$$

Hence, substituting (17) into (16) yields

$$g_k(T, X(T)) - g_k(t, X(t)) = \int_t^T \frac{\partial g_k}{\partial X}(s, X(s)) ds + \sum_{i=1}^n \int_t^T \frac{\partial g_k}{\partial x_i}(s, X(s)) dX_i(s)$$

$$+ \sum_{d=1}^{\infty} \sum_{p \in \mathbb{P}_0} \left( \int_t^T \left( \int_{\mathbb{R}^e} \frac{d_k^{(1)}}{k}(s, X(s), y) p^p(y) \nu(dy) \right) dH^p(s) \right)$$

$$+ \int_t^T \int_{\mathbb{R}^e} \frac{d_k^{(1)}}{k}(s, X(s), y) \nu(dy) ds \quad (18)$$

Notice that

$$X_i(t) = Y_i^{(1)}(t) + tE(X_i(1)) = \sum_{j=1}^n \tilde{c}_{ij} H^j(t) + tE(X_i(1)),$$
and
\[ E(X(1)) = a_1 + \int_{|y| \leq 1} y \nu(dy). \]

By applying the condition (5) and \( Y(0) = E[Y(0)] = E[g(X(T))] \). We can rewrite (18) as
\[
\begin{align*}
g_{\nu}(X(T)) &= E[g_{\nu}(X(T))] + \sum_{j=1}^{n} \int_{t}^{T} \frac{\partial \theta_{j}}{\partial x_{j}}(s, X(s^{-})) \sum_{j=1}^{n} \tilde{c} v dH^{\nu}(t) \\
&+ \sum_{d=2}^{\infty} \sum_{p \in \mathbb{P}^{+}_{d}} \int_{t}^{T} \left( \int_{\mathbb{R}^{d}} \theta^{(1)}(s, X(s^{-}), y) p^{(d)}(y) v(dy) \right) dH^{\nu}(s)
\end{align*}
\]

which completes the proof of the Proposition. \( \diamond \)

**Proof of Proposition 4:**

Apply Itô’s lemma to \( \theta_{k}(s, X(s)) \) for \( k = 1, 2, \cdots, d \) from \( s = t \) to \( s = T \). By using Lemma 5, we obtain the equality (18). Now, using (10) we get
\[
\begin{align*}
g_{\nu}(X(T)) - \theta_{k}(t, X(t)) &= -\int_{t}^{T} f_{k}(s, Y(s^{-}), Z(s)) ds \\
&+ \sum_{j=1}^{n} \int_{t}^{T} \frac{\partial \theta_{j}}{\partial x_{j}}(s, X(s^{-})) \sum_{j=1}^{n} \tilde{c} v dH^{\nu}(s) \\
&+ \sum_{d=2}^{\infty} \sum_{p \in \mathbb{P}^{+}_{d}} \int_{t}^{T} \left( \int_{\mathbb{R}^{d}} \theta^{(1)}(s, X(s^{-}), y) p^{(d)}(y) v(dy) \right) dH^{\nu}(s)
\end{align*}
\]

which completes the proof of the Proposition.

**Acknowledgement:** The authors would like to thank a kind proposal given by Professor David Nualart for the initial version of this paper.

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