Abstract. We present a homotopy theory for a weak version of modular operads whose compositions and contractions are only defined up to homotopy. This homotopy theory takes the form of a Quillen model structure on the collection of simplicial presheaves for a certain category of undirected graphs. This new category of undirected graphs, denoted $\mathbf{U}$, plays a similar role for modular operads that the dendroidal category $\Omega$ plays for operads. We carefully study properties of $\mathbf{U}$, including the existence of certain factorization systems. Related structures, such as cyclic operads and stable modular operads, can be similarly treated using categories derived from $\mathbf{U}$.

A modular operad, as introduced by Getzler and Kapranov [GK98], is a kind of cyclic operad equipped with self-compositions of operations. That is, it is an algebraic structure consisting of a sequence of sets $P(n)$, indexed on nonnegative integers $n$, together with families of ‘composition operations’ $P(n) \times P(m) \to P(n + m - 2)$ and ‘contraction operations’ $P(n) \to P(n - 2)$. The canonical example is when $P(n)$ is the moduli space of Riemann surfaces with $n$ marked points. This paper, along with its companion [HRY], center around a new category of graphs that permit a Segalic approach to the study of modular operads.

The main goal of the present paper is to propose a precise definition for up-to-homotopy modular operads and provide a homotopy theory for such objects. Why might one pursue such a program? One motivation comes from the work of Mann and Robalo who show, by passing to correspondences in derived stacks, that the operad of stable curves of genus zero acts on any smooth projective complex variety (see [MR18, Theorem 1.1.2]). This allows them to construct (genus zero) Gromov–Witten invariants on the derived category of the variety in question. We anticipate this work will be used in the study of higher genus version of that result; see Remark 1.2.1 of [MR18], which is prefigured in the ordinary case in [Bar07, 11.2].

An additional motivation comes from work of the second author with Horel and Boavida de Brito on subgroups of the profinite Grothendieck–Teichmüller group. They conjecture that the homotopy automorphisms of a profinite completion of Getzler–Kapronov’s modular operad $\mathcal{M}_{*,*}$ [GK98] will be isomorphic to the group $\Lambda$ which is a subgroup of the profinite Grothendieck–Teichmüller group which still contains the absolute Galois group [HLS00, Theorem A]. The profinite completion of $\mathcal{M}_{*,*}$ cannot be a strict modular operad as the profinite completion is often not product preserving [BdBHR19, Proposition 3.9]. The material in this paper and
its companion \cite{HRY} provides the homotopy-theoretic framework required for this project.

Around half of this paper is devoted to the introduction of a modular graphical category \(U\) and a study of its properties. The objects of this category are undirected, connected graphs with loose ends, while morphisms are given by ‘blowing up’ vertices of the source into subgraphs of the target in a way that reflects iterated operations in a modular operad. The category \(U\) is actually a proper subcategory of the category of Feynman graphs studied by Joyal and Kock in \cite{JK11}. The restriction we make is partly to disallow ‘duplication of variables’ from appearing in morphisms.\footnote{This follows the theory of algebras over operads, which generally can model types of algebras where each variable term appears exactly once in the defining equations.} As in our earlier work on graph categories \cite{HRY15, HRY18, HRY19}, which made similar restrictions in other contexts, this bears fruit. Namely, the weak factorization system that exists on the category of Joyal and Kock becomes an orthogonal factorization system on \(U\), and, moreover, there is a generalized Reedy structure on \(U\). These two facts constitute the main theorems of the first half.

The heart of this paper is Section 3, where we investigate the homotopy theory of simplicial \(U\)-presheaves. Roughly, such a presheaf \(X\) will be said to satisfy the Segal condition if the value of \(X\) at a graph \(G\) is determined up to homotopy by the value of \(X\) at each of the vertices of \(G\). If, additionally, the value of \(X\) at an edge is contractible, then we say that \(X\) is a Segal modular operad.\footnote{This terminology is chosen to be consonant with ‘Segal operad’ from \cite{BH14}.}

**Theorem A.** The category of simplicial \(U\)-presheaves admits a model structure whose fibrant objects are the Segal modular operads.

If \(X\) is a Segal modular operad, then after passing to the homotopy category of spaces one has an honest (unital, symmetric) modular operad. Indeed, the two types of operations from the first paragraph can be found by working with those connected graphs that have precisely one internal edge. On the other hand, there is a strict analogue of the Segal condition, and in the companion paper \cite{HRY} we prove the following nerve theorem. It shows that the strict Segal condition gives a characterization of (colored) modular operads.

**Theorem B.** Each graph freely generates a (colored) modular operad. This process gives a functor \(U \to \text{ModOp}\) which induces a fully-faithful functor \(N : \text{ModOp} \to \text{Set}^{U^{op}}\). The essential image of \(N\) consists precisely of those presheaves which satisfy a strict Segal condition.

An attentive reader may have noticed that there is no notion of genus for operations in the modular operads discussed above. In the original definition \cite{GK98} of modular operad, the graded objects \(P = \{P(n)\}_{n \geq 0}\) had an additional genus grading as \(P(n) = \{P(n, g)\}_{g > 1 - \frac{n}{2}}\). The composition operations are to be interpreted as additive on genus, while the contraction operations increase genus by one. Of course we also have applications in mind where it is beneficial to keep track of this geometric information, so we provide a variant of \(U\) whose objects are stable graphs. There are analogues of Theorem \[A\] and Theorem \[B\] for the category of stable graphs.

If \(F : \mathbf{R} \to \mathbf{S}\) is a functor between small categories and \(M\) is a bicomplete category, then there is a restriction functor \(F^* : M^S \to M^R\) that has both adjoints (given
by left and right Kan extension). Suppose further that these diagram categories have model category structures. As $F^*$ is both a left adjoint and a right adjoint, one would like to know whether or not $F^*$ is left Quillen, right Quillen, or neither. For example, if $M$ is itself a model category and the two diagram categories both have the injective model structure (with cofibrations and weak equivalences defined to be levelwise), then it is immediate that $F^*$ is left Quillen. In [Bar10, HV19], this question was considered when $F$ is a Reedy functor between (strict) Reedy categories. Barwick classified those Reedy functors $F$ so that $F^*$ is left Quillen (resp. right Quillen) for every model category $M$. A natural question is whether this classification can be adapted to the setting of generalized Reedy categories [BM11].

In Section 5 we show that Barwick’s characterization does not extend in the obvious way to the setting of generalized Reedy categories, by means of an explicit counterexample. Our observation arose out of a careful comparison of the present paper to [HRY19]. In that paper, we introduced a category $Ξ$ of undirected trees for the purposes of studying higher cyclic operads. On the other hand, one could consider the subcategory $U_{cyc}$ of $U$ on the simply connected graphs. There is a functor $U_{cyc} \rightarrow Ξ$, but it is not an equivalence and constitutes our counterexample. The key difference between the two categories is in what type of colored cyclic operads they can be used to model. The color sets of cyclic operads in [HRY19] do not have any additional structure, whereas in other settings [CGR14, DCH18] the color sets will come with an involution. Indeed, in [HRY], our modular operads have involutive color sets, so one should expect $U_{cyc}$ to model cyclic operads with involution.

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1. Graphs and the category $U$

All graphs in this paper are undirected and are allowed to have ‘loose ends,’ that is, it is not necessary for both ends (or either end) of an edge to touch a vertex. One possible concise definition for such a graph (compare [JS91, §2]) is a pair $(X, V)$ where $X$ is a space, $V$ is a finite set of points of $X$, and $X \setminus V$ is a one-manifold (without boundary) having only a finite set of connected components. Components of $X \setminus V$ are the edges of the graph, and elements of $V$ are the vertices. Thus we may have loops divorced from any vertex (those components of $X \setminus V$ homeomorphic to $S^1$), edges loose at one end (those with one missing limit point in $X$), and free floating edges (components of $X$ homeomorphic to $(0, 1)$ which contain no vertices).

\footnote{For sanity, take $X$ to be a locally finite one-dimensional CW complex, though we will not use the CW structure.}
We now give some basic definitions. An arc of a graph is an edge together with a chosen orientation. Thus for any graph there a set $A$ of arcs, which comes equipped with a free involution $i$ given by reversing orientation. There is a partially-defined function $t : A \to V$ which takes an arc to the vertex it points towards; we write $D \subseteq A$ for the domain of $t$. For each vertex $v \in V$, there is a corresponding neighborhood $nb(v) = t^{-1}(v) \subseteq D \subseteq A$ consisting of arcs which point towards $v$.

**Remark 1.1.** It is important to note that knowledge of $A$, $V$, $i$, and $t$ does not allow us to reconstruct the original graph. The only point of ambiguity is when $a$ and $ia$ are both in the complement of the domain of $t$; we cannot tell if the associated edge is $S^1$ or $(0, 1)$. One way to account for this difference is by considering the boundary $\partial(G) \subseteq A \setminus D$ of the graph. Concretely, $\partial(G)$ may be identified with the set of ends (as in Definition 1 of [HR96]) of $X$; in the one-dimensional setting one can consider a free compactification $\overline{X}$ of $X$ and then we have $\partial(G)$ is in bijection with the discrete space $\overline{X} \setminus X$. Abstractly, the boundary $\partial(G)$ is a subset of the complement of $D$ so that $i \partial(G) \subseteq D \cap \partial(G)$ and $iD \setminus D \subseteq \partial(G)$.

**Example 1.2.** Let us give geometric descriptions of several important graphs.

- The **exceptional edge**, denoted $\uparrow$, is the graph where $X$ is the open interval $(0, 1)$ and $V = \emptyset$.
- The **nodeless loop** is the graph where $X$ is the circle $S^1$ and $V = \emptyset$.
- Let $n \geq 0$ be an integer. The **$n$-star** $\star_n$ is the graph $(X, V)$ where $V = \{0\} \subseteq \mathbb{C}$ and $X = V \cup \{re^{2\pi i} \mid 0 \leq r < 1 \text{ and } pn \text{ is an integer with } 1 \leq pn \leq n \} \subseteq \mathbb{C}$.

Equipping $X$ with the usual topology, we then have that $X \setminus V$ is homeomorphic to $n$ copies of $(0, 1)$.

- Let $n \geq 0$ be an integer. The linear graph $L_n$ has $X = (0, 1)$ and
  
  $$V = \left\{ \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1} \right\}.$$

In particular, $L_0$ is the exceptional edge.

- Let $n \geq 0$ be an integer. There is a graph with $X = S^1$ and
  
  $$V = \{e^{2\pi i} \mid pn \in \mathbb{Z} \text{ and } 1 \leq pn \leq n \} \subseteq S^1;$$

the case $n = 0$ recovers the nodeless loop.

Henceforth, we will use combinatorial definitions of graphs, of which there are several competing definitions [BB17, JK11, JS91, YJ15]. In this paper we primarily use Definition 1.3 due to Joyal and Kock, as it is both extremely simple and also allows us to express the notion of étale map (Definition 1.11). In light of Remark 1.1, we see that this definition does not capture those graphs where $X \setminus V$ has some $S^1$ components (we will return to this issue in §4.1). With that proviso, all of these combinatorial graph definitions are equivalent (Proposition 15.2, Proposition 15.6, and Proposition 15.8 of [BB17]), so we may use them interchangeably.

**Definition 1.3** (Feynman graphs [JK11]). A graph $G$ is a diagram of finite sets $\xymatrix{ \& A \ar@/^/[rr]^i \ar[rr]_t & & V }$
Figure 1. The graph \( \star_5 \) with \( \text{nb}(v) = \{1, 2, 3, 4, 5\} \) and \( \partial(\star_5) = \{1^\dagger, 2^\dagger, 3^\dagger, 4^\dagger, 5^\dagger\} \).

where \( i \) is a fixedpoint-free involution and \( s \) is a monomorphism. We will nearly always consider \( D \) as a subset of \( A \), and suppress the natural inclusion function \( s : D \subseteq A \) from the notation.

1. The boundary of such a graph is the set \( \partial(G) = A \setminus sD \).
2. An edge is just an \( i \)-orbit \([a, ia] \), and we write \( E = A / i \) for the set of edges.
3. An internal edge is an edge of the form \([sd, sd']\) where \( d, d' \in D \).

Let us translate the geometric descriptions from Example 1.2 into this setting.

**Example 1.4.** If \( Z \) is a set, write \( 2Z \) for the set
\[ \{z, z^\dagger \mid z \in Z\} \cong Z \amalg Z \]
together with the evident involution. We consider \( Z \) as a subset of \( 2Z \), and write \( Z^\dagger \) for its complement.

- The exceptional edge, \( \ddot{\cdot} \), is the graph with \( A = 2\{\ast\} \) and \( V = D = \emptyset \). As this graph is so important in what follows, we will give special names to its arcs and write \( A = \{\ast, b\} \).
- The nodeless loop is not expressible in the Feynman graph formalism, as we would want to take \( A = 2\{\ast\} \), \( V = D = \emptyset \), but have an empty boundary.
- The \( n \)-star \( \star_n \) has \( V \) a one-point set, \( D = \{1, \ldots, n\} \), and \( A = 2D \). The function \( s : D \to A = 2D \) is just the subset inclusion. See Figure 4.
- The linear graph \( L_n \) has \( A = 2\{0, \ldots, n\}, D = A \setminus \{0, n^\dagger\}, V = \{1, \ldots, n\} \), \( t(k) = k \) for \( 1 \leq k \leq n \), and \( t(k^\dagger) = k + 1 \) for \( 0 \leq k \leq n - 1 \). Each vertex neighborhood is of the form \( \text{nb}(k+1) = \{k^\dagger, k+1\} \) and the boundary is \( \partial(L_n) = \{0, n^\dagger\} \).
- For the loop with \( n \) vertices, we must suppose that \( n > 0 \). Let \( V = \{1, \ldots, n\} \) and \( A = D = 2V \). The target function is given by \( t(k) = k \) for \( 1 \leq k \leq n \), \( t(k^\dagger) = k + 1 \) for \( 1 \leq k \leq n - 1 \), and \( t(n^\dagger) = 1 \). For vertex neighborhoods, we have \( \text{nb}(1) = \{n^\dagger, 1\} \) and otherwise \( \text{nb}(k+1) = \{k^\dagger, k+1\} \).

It is especially convenient to have, for each graph \( G \), a collection of stars. Each such star is isomorphic to a \( \star_n \) from the previous definition.

**Definition 1.5 (Stars associated to graphs).** Suppose that \( G \) is a graph. We tweak the definition of \( \star_n \) from Example 1.4 as follows:

- Let \( \star_G \) be the one-vertex graph with \( A = 2\partial(G) \) and \( D = \partial(G)^\dagger \). Notice that \( \partial(\star_G) = A \setminus D = \partial(G) \) and that the neighborhood of the unique vertex is \( D = \partial(G)^\dagger \).

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5To ensure that we have a set of graphs, insist that all of the sets \( A, D, V \) are taken to be subsets of some fixed infinite set.

6We will return to the nodeless loop in Definition 1.2.
• Suppose that $v$ is a vertex of $G$ and let nb$(v)$ be its neighborhood in $G$. We let $\star_v$ denote the graph with $V = \{v\}$, $D = \text{nb}(v)$, and $A = 2\text{nb}(v)$. The boundary of $\star_v$ is $\text{nb}(v)^\uparrow \subseteq 2\text{nb}(v)$.

1.1. Étale maps as natural transformations.

**Definition 1.6.** Let $\mathcal{J}$ denote the category with three objects and three generating arrows, of shape $\overset{\rightarrow}{\bullet} \leftrightarrow \bullet \rightarrow \bullet$ each graph $G$ from Definition 1.3 is a functor from $\mathcal{J}$ into finite sets so that the leftward arrow is sent to a monomorphism and the generating endomorphism is sent to a free involution. As we are considering graphs as functors, there is an obvious notion of graph map: a natural transformation of functors.

Feynman graphs thus span a full subcategory of $\text{FinSet}^{\mathcal{J}}$, and we will consider two subcategories (Definition 1.11 and Definition 1.13) in this subsection. Our ultimate graph morphisms, given in Definition 1.31, will not be morphisms in the functor category $\text{FinSet}^{\mathcal{J}}$. Indeed, in that definition vertices are not sent to vertices, but rather to ‘subgraphs’ of the codomain. Nevertheless, by viewing graphs as functors, we can make the following definition of connectedness (which coincides with usual topological connectedness of an associated geometric version of the graphs).

**Definition 1.7.** A graph $G$ is connected if it cannot be written as a nontrivial coproduct in the functor category $\text{FinSet}^{\mathcal{J}}$.

**Construction 1.8** (The graph $G \setminus X$). Suppose that $G$ is a graph and $X \subseteq E$ is a set of edges. Recall that in the current formalism, an edge $e$ is an orbit of the involution $i$ on $A$. We form a new graph $G'$ as follows, which in the future we will denote by $G \setminus X$. The set of vertices of $G'$ coincides with the set of vertices of $G$. The set of arcs of $G'$, denoted $A'$, is the $i$-closed subset of $A$ consisting of those arcs that do not appear in any edge $e$ in the set $X$. In symbols,

$$A' = \{a \in A \mid \forall e \in X, a \notin e\} \subseteq A.$$  

The involution on $A'$ is just the restriction of that on $A$. Finally $D' = D \cap A'$.

**Definition 1.9.** Suppose that $G$ is a graph.

• If $Y \subseteq A$ is a set of arcs, temporarily let $\bar{Y} \subseteq E$ be the collection of edges containing elements of $Y$. That is,

$$\bar{Y} = \{e \in E \mid \exists a \in Y \text{ with } a \in e\}.$$  

We define $G \setminus Y$ to be the graph $G \setminus \bar{Y}$ from Construction 1.8.

• If $G$ is any graph with boundary $\partial(G) = A \setminus D$, write $\text{core}(G)$ for the graph $G \setminus \partial(G)$.

**Example 1.10.** Suppose that $G$ is any graph and $X \subseteq E$. Then the inclusion $G \setminus X \rightarrow G$ is a natural transformation. Similarly, we have natural transformations $G \setminus Y \rightarrow G$ (when $Y \subseteq A$) and $\text{core}(G) \rightarrow G$ as well. Often the latter map fits into a square expressing $G$ as a pushout of graphs (in the functor category $\text{FinSet}^{\mathcal{J}}$); see Construction 1.27.

The following definition is also due to Joyal and Kock [JK11].

---

7Specifically, when $i\partial(G) \cap \partial(G) = \emptyset$.  

Definition 1.11 (Étale maps). A natural transformation $G \to G'$ is said to be \textit{étale} if the right-hand square of

$$
\begin{array}{ccc}
A & \xleftarrow{\alpha} & D \\
\downarrow & & \downarrow \\
A' & \xleftarrow{\alpha'} & D'
\end{array}
\quad
\begin{array}{c}
\downarrow \iota \\
V
\end{array}
\quad
\begin{array}{ccc}
V' & \xrightarrow{\iota'} & V \\
\downarrow & & \downarrow \\
\emptyset & \xleftarrow{\iota} & \{v\}
\end{array}
$$


is a pullback.

Example 1.12. Let us give some simple examples of maps which are not étale.

- Consider the map $\text{core}(\star_n) \to \star_n$ described in Example 1.10. This becomes the map $\emptyset \to \{v\}$

$$
\begin{array}{ccc}
\emptyset & \xleftarrow{\iota} & \{v\} \\
\downarrow & & \downarrow \\
\{1, \ldots, n, 1^\dagger, \ldots, n^\dagger\} & \xleftarrow{\iota'} & \{1, \ldots, n\}
\end{array}
$$

which is not étale unless $n = 0$.

- More generally, suppose that $G$ is a connected graph. Then $\text{core}(G) \to G$ is étale if and only if $G$ is the exceptional edge or $\partial(G)$ is empty. If $G$ is not connected, then $\text{core}(G) \to G$ splits as a sum over the set of connected components of $G$, and is thus étale if and only if each summand is étale.

- Let $L_n$ denote the linear graph with $n$ vertices. If $n > 0$, then there are no étale maps from $L_n$ to a graph with no vertices.

Definition 1.13 (Embeddings). Suppose that $G$ and $G'$ are connected graphs. An \textit{embedding} $G \to G'$ is an étale map (Definition 1.11) where $V \to V'$ is a monomorphism. If $G'$ is a connected graph, write $\tilde{\text{Emb}}(G')$ for the collection of embeddings $G \to G'$ (in particular, $G$ is also connected).

Since an embedding is, in particular, étale, the function $D \to D'$ is also a monomorphism. It may be the case, however, that $A \to A'$ is not. We will specify exactly when this can happen in Lemma 1.22.

Example 1.14. We can consider $V$ as a subset of $\tilde{\text{Emb}}(G)$. Indeed, for each vertex $v \in V$ we have the associated star $\star_v$ from Definition 1.5 which has a single vertex and $2\text{nb}(v) = \text{nb}(v) \amalg \text{nb}(v)^\dagger$ as its set of arcs. We write

$$
\iota_v : \star_v \to G
$$

for the étale map

$$
\begin{array}{ccc}
\text{nb}(v) \amalg \text{nb}(v)^\dagger & \xleftarrow{\iota} & \{v\} \\
\downarrow & & \downarrow \\
A & \xleftarrow{\iota'} & D
\end{array}
\quad
\begin{array}{c}
\downarrow \iota \\
V.
\end{array}
$$

The left-hand map in this diagram is just the inclusion $\text{nb}(v) \to D \to A$ on the first component, while the second component (which is forced by compatibility with the involutions) sends $a^\dagger$ to $ia$. As the right-hand map is a monomorphism, $\iota_v$ is an embedding.
Example 1.15 (Contracted star). Suppose that we take \(\star_5\) from Figure 1 and identify 4\(^\dagger\) with 5 (and likewise 5\(^\dagger\) with 4). That is, we consider the graph \(G\) with one vertex, set of arcs \(A = \{1, 2, 3, 4, 5, 1\dagger, 2\dagger, 3\dagger\}\) and \(D = \{1, 2, 3, 4, 5\}\). The involution is the same as that for \(\star_5\), except that \(i(4) = 5\) (and \(i(5) = 4\)). Thus there is one internal edge \(e = [4, 5]\). The natural embedding \(\iota_e : \star_5 \rightarrow G\) is not injective on arcs.

![Diagram](image)

In the previous example we showed how to connect up two boundary edges and still have an embedding. This can be done in the reverse direction, namely by starting with a graph and cutting at some internal edge. In Lemma 1.22 we give a general statement about embeddings that are not monomorphisms, and we see that they always come from such internal edge cuttings. The reader should contrast this example with Construction 1.8, where edges are deleted entirely.

Example 1.16. We describe a family of embeddings obtained by “snipping” a single edge. Let \(G\) be a connected graph and let \(e = [a, ia]\) be a chosen internal edge in \(G\). We let \(G_e\) denote the graph obtained from \(G\) by snipping \(e\). Explicitly, we define \(G_e\) to have the same set of vertices as \(G\) and set of arcs \(A \cup \{b, c\}\). The involution \(i_e\) on \(G_e\) is given by \(i_e(a) = b\) and \(i_e(ia) = c\), while the rest of the structure remains the same. This results in one of two cases: either \(G_e\) is connected or not.

Case 1 If the graph \(G_e\) is no longer connected, this means that \(e\) was an edge between two distinct vertices \(u\) and \(v\), with \(a \in nb(u)\) and \(ia \in nb(v)\). Moreover, there are no other edges connecting \(u\) and \(v\). Snipping \(e\) thus results in two embeddings \(f_u : G_u \rightarrow G\) and \(f_v : G_v \rightarrow G\) which include the connected graph \(G_u\), the half containing \(u\), (respectively, \(G_v\)) into \(G\). Note that the embedding \(f_u : G_u \rightarrow G\) is injective on vertices and arcs; all additional graph structure on \(G_u\) is that of \(G\). The same is true for \(G_v\).

![Diagram](image)

Case 2 In the second case, we still have a connected graph after snipping \(e\). This means that the edge \(e\) was part of a cycle. This creates one embedding.
\( G_e \rightarrow G \) where \( G_e \) has two more arcs than \( G \). Specifically, if \( a \in \text{nb}(u) \) and \( ia \in \text{nb}(v) \) (allowing for the possibility that \( u = v \)), then we add a pair of arcs which disconnect the edge. In the picture below, \( G \) contains the new arcs \( b \) and \( c \) and the embedding \( f \) takes \( b \) to \( ia \) and \( c \) to \( a \).

1.2. **Graph substitution.** A key construction when dealing with graphs with loose ends and operadic structures is that of graph substitution. Suppose that we are given a graph \( G \), a collection of graphs \( H_v \) indexed by the vertices of \( G \), and specified bijections \( i_{\text{nb}(v)} \sim \varnothing(H_v) \). Then we can form a new graph \( G\{H_v\} \) by a process of graph substitution, where we replace each vertex \( v \) by the graph \( H_v \), identifying the edges at the boundary of \( H_v \) with the edges incident to the vertex \( v \) in \( G \). A detailed treatment may be found in the combinatorial settings in [YJ15, Ch. 5] and [BB17, §13].

Intuitively, one sees that there should be a canonical identification of the vertices of \( G\{H_v\} \) with \( \coprod_{v \in G} V(H_v) \), that all internal edges in \( H_v \) become internal edges in \( G\{H_v\} \), and that \( \partial(G\{H_v\}) \sim \partial(G) \). Theorem 5.32 and Lemma 5.31 in [YJ15] tell us that graph substitution is associative and unital. Unitality means that

\[ \star_G\{G\} \equiv G \equiv G\{\star_v\} \]

here \( \star_G \) and \( \star_v \) are as in Definition 1.5 and the bijections needed to define the graph substitutions are the identity on \( \partial(G) \) and in the the bijections \( i_{\text{nb}(v)} \sim \text{nb}(v) \), respectively. Associativity asserts that

\[ [G\{H_v\}]\{I_u^v\} \equiv G\{H_v\{I_u^v\} \}
\]

where the \( I_u^v \) are graphs indexed on \( u \in V(H_v) \) and with bijections left implicit.

**Remark 1.17.** The collection of graphs from Definition 1.3 are not closed under graph substitution operations. Indeed, if \( G \) is the loop with one vertex from Example 1.4 and \( \uparrow \) is the exceptional edge, then \( G\{\uparrow\} \) should be the nodeless loop. As we will only be dealing with connected graphs (see Definition 1.7 for the remainder of this paper, this example is the main one we need to worry about (since graph substitution can be done one vertex at a time). In working with disconnected graphs in generality, the result of a graph substitution may have many nodeless loops even when the original graphs involved have none.

For Proposition 1.38 and Lemma 1.41 it is helpful to have an explicit description of graph substitution for Feynman graphs. The remainder of this section is a little more difficult than what surrounds it, so it is recommended that most readers skip ahead to Section 1.3 for now, carrying with them the preceding intuitive discussion and referring back as needed. The following description is inspired by [Koc16] §1.5.
Construction 1.18 (Graph substitution). Suppose that $G$ is a graph where each component of $G$ contains at least one vertex and let $E_i$ be its set of internal edges. For each edge $e \in E_i$, choose an ordering $e = [x^1_e, x^2_e]$ for the two-element equivalence class of arcs comprising $e$. We can exhibit $G$ as a coequalizer (in the diagram category $\text{FinSet}^\mathcal{F}$)

$$
\prod_{e \in E_i} \downarrow \xrightarrow{\mathcal{R}} \prod_{v \in V} \star_v \xrightarrow{\pi} G,
$$

where the map on the right is $\prod_v i_v$.

- $\mathcal{R}$ is the coproduct of maps $\downarrow \rightarrow \star_{tx^1_e}$ with $\mathcal{R}_e(\mathfrak{z}(v)) = (x^1_e)^1 \in \mathcal{O}(\star_{tx^1_e})$ and $\mathcal{R}_e(b) = x^1_e \in D(\star_{tx^1_e})$;
- $\mathfrak{z}$ is the coproduct of maps $\downarrow \rightarrow \star_{tx^2_e}$ with $\mathfrak{z}_e(b) = (x^2_e)^1 \in \mathcal{O}(\star_{tx^2_e})$ and $\mathfrak{z}_e(z) = x^2_e \in D(\star_{tx^2_e})$.

Now suppose we are given graphs $H_v$ and isomorphisms $m_v$ from $i(\text{nb}(v)) \subseteq A(G)$ to $\hat{\mathcal{O}}(H_v)$. We then have induced maps $\mathcal{R}$ and $\mathfrak{z}$, where

- $\mathcal{R}$ is the coproduct of maps $\mathcal{R}_e : \downarrow \rightarrow H_{tx^1_e}$ with $\mathcal{R}_e(\mathfrak{z}(v)) = m_{tx^1_e}(ix^1_e) \in \mathcal{O}(H_{tx^1_e})$,
- $\mathfrak{z}$ is the coproduct of maps $\mathfrak{z}_e : \downarrow \rightarrow H_{tx^2_e}$ with $\mathfrak{z}_e(b) = m_{tx^2_e}(ix^2_e) \in \mathcal{O}(H_{tx^2_e})$.

We can then form the coequalizer

$$
\prod_{e \in E_i} \downarrow \xrightarrow{\mathcal{R}} \prod_{v \in V} H_v \xrightarrow{\pi} K.
$$

One can check that this object $K \in \text{FinSet}^\mathcal{F}$ is always graph in the sense of Definition 1.3. Since colimits in $\text{FinSet}^\mathcal{F}$ are computed levelwise, it is immediate that $V(K) = \prod_{v \in V(G)} V(H_v)$ and $D(K) = \prod_{v \in V(G)} D(H_v)$. Further, the graph substitution $G\{H_v\}$ is represented by $K$ as long as $G\{H_v\}$ can be represented by Feynman graphs. When all of the graphs $G$ and $H_v$ are connected, this is the case except when $G$ is a loop with $n$ vertices (Example 1.4), and all of the $H_v$ are edges.

To distinguish between the various involutions, we will write $i_v$ for the involution on the graph $H_v$. Let us analyze some of the structure of $A(K)$ by studying the preimages of certain elements. We have three situations whose behavior follows readily from the coequalizer description.

- (A) Suppose that $e$ is an internal edge of $G$ between vertices $v$ and $w$ (which may be equal). If $H_v$ and $H_w$ are not edges, then

  $$
  \pi^{-1} \pi(\mathcal{R}_e(\mathfrak{z})) = \{\mathcal{R}_e(\mathfrak{z}), \mathfrak{z}_e(\mathfrak{z})\} \text{ and } \pi^{-1} \pi(\mathcal{R}_e(b)) = \{\mathcal{R}_e(b), \mathfrak{z}_e(b)\}.
  $$

- (B) If $[d, i_v, d]$ is an internal edge of $H_v$, then $\pi^{-1} \pi(d) = \{d\}$.

- (C) If $x \in \mathcal{O}(G) \cap i \text{nb}(v)$ and $H_v$ is not an edge, then

  $$
  \pi^{-1} \pi(m_vx) = \{m_vx\} \text{ and } \pi^{-1} \pi(i_v(m_vx)) = \{i_v(m_vx)\}.
  $$

We now show how to recover a standard, intuitive fact about graph substitution in an elementary way from Construction 1.18. While reading the proof, note that $\mathcal{O}(G) \rightarrow \mathcal{O}(K)$ is always defined and injective, even when $K$ does not represent $G\{H_v\}$. Indeed, it is only in showing surjectivity that we must impose particular constraints on $G$ and $H_v$ (so that $G\{H_v\}$ is not a nodeless loop).

---

8Outside of this situation, we must modify the two coequalizers, adding in to the middle terms those components of $G$ that lack vertices.
Lemma 1.19. The function which sends $x \in \partial(G)$ to $\pi(m_{x^2}x) \in A(K)$ constitutes a bijection between $\partial(G)$ and $\partial(K)$.

Proof. This is easy to see in the case when there is a vertex $v_0$ such that $H_v = \star_v$ (and $m_v$ is the evident map) for $v \neq v_0$. We prove this only in that case. For the general case, one can either make an inductive argument from this case, introduce a variation on this proof involving paths, or appeal to something like Construction 2.8.

We now go deeper in our study of embeddings. We first show that if $\{m_u x, \tilde{\mathcal{S}}(\mathcal{S})(\mathcal{S})\} \subseteq \pi^{-1}(\pi(m_u x))$ and furthermore the sets $\pi^{-1}(\pi(m_u x))$ and $\pi^{-1}(\pi(i_u(m_u x)))$ are disjoint since the involution on $A(K)$ is free. Every other element of $\bigcup_{v \in G} A(H_v)$ is accounted for in \[ A \cup \mathcal{B} \cup \mathcal{C} \] (that is, each other $y$ in this set satisfies $|\pi^{-1}(y)| = 2$) so the inclusions in \[ A \cup \mathcal{B} \cup \mathcal{C} \] are actually equalities. Since neither of the two elements of $\pi^{-1}(\pi(m_u x))$ are in $\bigcup_{v \in G} D(H_v)$, it follows that $\pi(m_u x) \notin D(K)$. Hence $\pi(m_u x) \in \partial(K)$.

On the other hand, if $y \in \partial(K)$ then

\[
\pi^{-1}(y) \subseteq \bigcup_{v \in G} \partial(H_v) = \bigcup_{v \in G} m_v(v \mathrm{nb}(v)).
\]

Suppose that $m_u x \in \pi^{-1}(y)$ with $x \notin \partial(G)$. Then $e = [x, ix]$ is an internal edge of $G$. Without loss of generality about the ordering of the arcs of this internal edge, we have $\mathcal{R}_e(\mathcal{S}) = m_w x$ and $\tilde{\mathcal{S}}(\mathcal{S})(\mathcal{S}) = m_u x$, where $w = tx$. Since $\mathcal{R}_e(\mathcal{S})(\mathcal{S})$ is in $\pi^{-1}(y)$, we know by [2] that $\mathcal{R}_e(\mathcal{S})(\mathcal{S}) = i_w m_w(\mathcal{S})$ is an element of $\partial(H_w)$. Thus $H_w$ must be an edge, so $w = v_0$. We cannot pull off this same trick twice, so $m_w^{-1}(\mathcal{R}_e(\mathcal{S})(\mathcal{S}))$ is in $\partial(G)$ unless $w = u$. If $w = u$, then we are in the situation where $G$ is the loop with one node and $H_w$ is the exceptional edge, which is explicitly disallowed. Hence there is an $m_v(x) \in \pi^{-1}(y) \subseteq \partial(G)$.

Note in particular that $A(K)$ is isomorphic, as a set, to $\partial(G) \amalg \bigcup_{v \in G} D(H_v)$. The involution on this set can be defined directly (in the case when $K$ represents $G(H_v)$) using the involutions on $G$ and $H_v$, the bijections $m_v$, and the function $x$ from Construction 2.8. We will never explicitly need this fact in this paper.

1.3. Embeddings and boundaries. We now go deeper in our study of embeddings. Our key result is Proposition 1.25 which tells us to which extent embeddings are determined by the images of their boundaries.

Lemma 1.20. Suppose that $f : G \to G'$ is in $\widetilde{\mathrm{Emb}}(G')$ and $\partial(G) = A \setminus D$. Then the composite $\partial(G) \hookrightarrow A \xrightarrow{f} A'$ is a monomorphism.
In particular, since \( f \) with \( fh \) when \( \left(\begin{array}{c}

Suppose that \( \text{Lemma 1.24.} \)

If \( A \) and \( A' \) expect to deduce that \( f \neq j \) Lemma 1.20, at least one of \( a \) one of \( G \) true. We thus suppose that \( A \) such that \( \left(\begin{array}{c}

Definition 1.42. \( \text{extend the following definition to the more general setting of ‘graphical maps’ in Definition 1.42.} \)

\begin{align*}
\text{Example 1.23.} \quad &\text{Consider the three graphs } G, G', G'' \text{ in Figure 2. Let } h, k : G \rightarrow G' \text{ and } f : G' \rightarrow G'' \text{ be the embeddings uniquely specified by } h(0) = 1, k(0) = 2, \text{ and } f(1) = f(2) = 3. \text{ Then } fh = fk, \text{ but } h \neq k. \\

\text{The main issue in the previous example was that } G \text{ was the exceptional edge. Indeed, we have the following.} \text{Lemma 1.24.} \quad &\text{Suppose that } \\

\end{align*}

Proof. If \( G \) is the exceptional edge with \( A = \partial(G) = \{g, h\} \) (see Example 1.4), then \( f(g) \neq i'(f(h)) = f(h) \) so \( \partial(G) = A \rightarrow A' \) is injective. For the remaining cases, simply notice that the indicated map is the following composite.

\[
\begin{array}{c}
\partial(G) \xrightarrow{i} D \xrightarrow{j} D' \xrightarrow{k} A' \\
\downarrow \quad \downarrow \quad \downarrow \\
A \xrightarrow{i} A' \xrightarrow{i'} A'
\end{array}
\]

In the preceding proof, we have relied on connectivity of \( G \) to ensure that \( i(\partial(G)) \subseteq D \); the only time this does not happen for connected graphs is when \( G \) is the exceptional edge.

It is useful to isolate a subset of \( A(G') \) that is isomorphic to \( \partial(G) \) via \( f \). We will extend the following definition to the more general setting of ‘graphical maps’ in Definition 1.42.

\begin{definition} \text{(Boundary of an embedding).} \quad &\text{If } f : G \rightarrow G' \text{ is an embedding, we write } \partial(f) \subset A' \text{ for the image of the (injective) function } f|_{\partial(G)} : \partial(G) \rightarrow A'. \end{definition}

\begin{lemma} \text{Suppose that } f : G \rightarrow G' \text{ is in } \overline{\text{Emb}}(G'). \text{ If } a_1 \neq a_2 \text{ are elements of } A \text{ such that } f(a_1) = f(a_2), \text{ then one of the following two situations holds:} \\
\text{(1) } a_1, ia_2 \in \partial(G) \text{ and } ia_1, a_2 \in D, \text{ or} \\
\text{(2) } a_1, ia_2 \in D \text{ and } ia_1, a_2 \in \partial(G).
\end{lemma}

\text{In particular, since } f|_{\partial(G)} \text{ is injective, } |f^{-1}(a)| \leq 2 \text{ for every } a \in A'. \text{ Proof.} \quad &\text{If } G = \uparrow, \text{ then } A \rightarrow A' \text{ is a monomorphism and the statement is vacuously true. We thus suppose that } G \neq \uparrow. \text{ Since } f : D \rightarrow D' \text{ is a monomorphism, at least one of } a_1, a_2 \text{ is not in } D. \text{ Similarly, since } f : \partial(G) \rightarrow A' \text{ is a monomorphism by Lemma 1.20, at least one of } a_1, a_2 \text{ is not in } \partial(G). \text{ Since } A = D \uplus \partial(G), \text{ there are } j \neq k \text{ in } \{1, 2\} \text{ so that } a_j, a_k \in D \text{ and } a_k \in \partial(G).
\text{Since } a_k \in \partial(G) \text{ and } G \neq \uparrow, \text{ we know that } ia_k \in D. \text{ Further, we have } f(ia_1) = f(ia_2) \text{ with } ia_1 \neq ia_2, \text{ so by the first paragraph we have } ia_j \in \partial(G). \text{ Case (1) occurs when } (j, k) = (2, 1), \text{ while case (2) occurs when } (j, k) = (1, 2). \quad \square
\end{align*}

We next wish to consider a diagram of embeddings of the form

\[
G \xrightarrow{h} G' \xrightarrow{f} G''
\]

with \( fh = fk \). Since \( f \) need not be a monomorphism in \( \text{FinSet}^\mathbb{F} \), one wouldn’t expect to deduce that \( h = k \). Indeed, we have the following counterexample.

\begin{example} \text{Consider the three graphs } G, G', G'' \text{ in Figure 2. Let } h, k : G \rightarrow G' \text{ and } f : G' \rightarrow G'' \text{ be the embeddings uniquely specified by } h(0) = 1, k(0) = 2, \text{ and } f(1) = f(2) = 3. \text{ Then } fh = fk, \text{ but } h \neq k. \text{ The main issue in the previous example was that } G \text{ was the exceptional edge. Indeed, we have the following.} \text{Lemma 1.24.} \quad &\text{Suppose that } \\

\end{align*}

\begin{align*}
\text{Example 1.23.} \quad &\text{Consider the three graphs } G, G', G'' \text{ in Figure 2. Let } h, k : G \rightarrow G' \text{ and } f : G' \rightarrow G'' \text{ be the embeddings uniquely specified by } h(0) = 1, k(0) = 2, \text{ and } f(1) = f(2) = 3. \text{ Then } fh = fk, \text{ but } h \neq k. \\

\text{The main issue in the previous example was that } G \text{ was the exceptional edge. Indeed, we have the following.} \text{Lemma 1.24.} \quad &\text{Suppose that } \\

\end{align*}
is a diagram of embeddings, with $G, G', G''$ connected graphs and $G \neq \emptyset$. If $fh = fk$, then $h = k$.

**Proof.** We have a commutative diagram

$$D \xrightarrow{h} D' \xrightarrow{f} D''$$

with $f : D' \to D''$ a monomorphism, so $h$ and $k$ are identical on $D$. We must show that they agree for elements in $A \setminus D = \partial(G)$. Let $a \in \partial(G)$. Since $G$ is connected and not the exceptional edge, we know $ia \in D$. Thus we have the middle equality in $h(a) = h(ia) = ih(ia) = ik(ia) = k(ia) = k(a)$, so $h = k$ on $\partial(G)$. □

**Proposition 1.25.** Suppose that $f_1 : G_1 \to G$ and $f_2 : G_2 \to G$ are in $\overline{\text{Emb}}(G)$ with neither $G_1$ nor $G_2$ the exceptional edge. If $\partial(f_1) = \partial(f_2)$, then there is a unique isomorphism $z : G_1 \to G_2$ so that $f_1 = f_2 z$. The same statement is true if both $G_1$ and $G_2$ are the exceptional edge $\emptyset$.

It may be the case that $\partial(f_1) = \partial(f_2)$ but $G_1 \not\cong G_2$. For instance, in Example 1.23 we have $\partial(f) = \{i3, 3\} = \partial(fk)$ but $G \not\cong G'$. The following lemma addresses the empty boundary case of Proposition 1.25, which will be key in proving the general case.

**Lemma 1.26.** Suppose that $f : G' \to G$ is in $\overline{\text{Emb}}(G)$ and the boundary of $G'$ is empty. Then $f$ is an isomorphism.

**Proof.** By assumption, the inclusion $D' \subset A'$ is an equality. Since $G'$ is not empty, $V' \neq \emptyset$. Suppose that $V' \to V$ is not surjective; then there exists a pair $v', v$ with $f(v')$ connected to $v \notin f(V')$. Write $a \in \text{nb}(f(v'))$ and $i(a) \in \text{nb}(v)$ for the two orientations of the connecting edge. By the étale condition for $f$, there is a unique $d' \in D'$ with $f(d') = a$ and $t'(d') = v'$. Notice that $ia = if(d') = f(i'd')$. Since $A' = D'$, $\tilde{v} = t'(i'd') \in V'$ is defined, and, further, $f(\tilde{v}) = f(t'i'd') = tf(i'd') = t(ia) = v$. This is a contradiction, hence $V' \to V$ is surjective. Now $V' \to V$ is an bijection, so the étale condition implies that $D' \to D$ is a bijection as well. Connectedness of $G$ ensures $D = A$. □

The following construction will help reduce the proof of Proposition 1.25 to the special case from Lemma 1.26.

**Construction 1.27** (Determination by core and boundary). As in Definition 1.6, we write $\mathcal{F}$ for the category $\mathcal{C} \cdot \bullet \leftarrow \bullet \to \bullet$. Given a connected graph $H \neq \emptyset$, 

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{example.png}
\caption{Graphs $G, G', G''$ for Example 1.23}
\end{figure}
we have a pushout diagram in $\text{FinSet}^\ast$

\[
\begin{array}{ccc}
\coprod_{\partial(H)} \ast_0 & \to & \coprod_{\partial(H)} \ast_1 \\
\downarrow & & \downarrow \\
\text{core}(H) & \to & H,
\end{array}
\]

(3)

where $\text{core}(H) = H \setminus \partial(H)$ is from Definition 1.9, note that $\text{core}(H)$ and $H$ have an identical set of vertices. Here, the top map is induced from the unique morphism $\ast_0 \to \ast_1$ on each component. The left vertical map, at the component $a \in \partial(H)$, sends the unique vertex of $\ast_0$ to the vertex $\text{tia}$. For the diagram to commute, we know precisely what the right vertical map must do on vertices. At the component $a \in \partial(H)$, the right vertical map sends the unique boundary arc of $\ast_1$ to $a$.

Notice that the vertical maps need not be monomorphisms in the diagram category $\text{FinSet}^\ast$, and that the maps in the diagram are étale only when $\partial(H) = \emptyset$.

**Proof of Proposition 1.25.** It is very simple that the isomorphism $z$, if it exists, is unique: since $f_1 : V_1 \to V$ and $f_2 : V_2 \to V$ are monomorphisms, there is at most one map $z : V_1 \to V_2$ with $f_2z = f_1$. A similar argument holds for uniqueness of $D_1 \to D_2$ and $\partial(G_1) \to \partial(G_2)$, which then implies uniqueness for $A_1 = \partial(G_1) \cup D_1 \to A_2 = \partial(G_2) \cup D_2$.

Notice immediately that we have isomorphisms

\[
\partial(G_1) \xrightarrow{f_1} \partial(f_1) = \partial(f_2) \xleftarrow{f_2} \partial(G_2),
\]

which determines $z$ on $\partial(G_1)$. At this point we can assume that $\partial(G_1)$ is nonempty, as when $\partial(G_1)$ is empty, Lemma 1.26 implies that $f_1$ and $f_2$ are both isomorphisms. Further, if both $G_1$ and $G_2$ are isomorphic to the exceptional edge $\uparrow$, then $\partial(G_1) = A(G_1)$ and $\partial(G_2) = A(G_2)$, so (4) gives the isomorphism $z$.

We assume for the remainder of the proof that $G_1 \neq \uparrow 
eq G_2$ and $\partial(G_1) \neq \emptyset \neq \partial(G_2)$. By Lemma 1.22 and Construction 1.27 we have the outer commutative diagram

\[
\begin{array}{ccc}
\coprod_{\partial(G_1)} \ast_0 & \xrightarrow{z|\partial(G_1)} & \coprod_{\partial(G_2)} \ast_0 \\
\downarrow & & \downarrow \\
\text{core}(G_1) & \to & \text{core}(G_2)
\end{array}
\]

\[
\begin{array}{ccc}
G' & \xrightarrow{f'_1} & G \setminus \partial(f_1) \\
\downarrow & & \downarrow \\
G' & \xleftarrow{f'_2} & \text{core}(G_2)
\end{array}
\]

in $\text{FinSet}^\ast$. Letting $G'$ be the connected component of $G \setminus \partial(f_1) = G \setminus \partial(f_2)$ (as in Definition 1.9), which contains the vertex $f_1(\text{tia})$ for some $a \in \partial(G_1)$, we have induced diagram maps $f'_1 : \text{core}(G_1) \to G'$. These maps are étale, in fact, the bottom maps they factor are étale. We need only check this at vertices in the image of the vertical maps, but in any case we have an induced bijection

\[
\begin{array}{ccc}
t^{-1}(v) \setminus i\partial(G_j) & \xrightarrow{\simeq} & t^{-1}(f_j(v)) \setminus i\partial(f_j) \\
\downarrow & & \downarrow \\
t^{-1}(v) & \xrightarrow{\simeq} & t^{-1}(f_j(v))
\end{array}
\]
for every vertex \( v \in V_j \). Since \( f_1 \) and \( f_2 \) were embeddings, so too are the étale maps \( f_1' \) and \( f_2' \).

It follows from Lemma 1.26 that \( f_1' \) and \( f_2' \) are isomorphisms. Since [3] in Construction 1.27 is a pushout, we obtain an isomorphism \( z : G_1 \to G_2 \) making the appropriate diagram commute. \( \square \)

The collection \( \widetilde{\text{Emb}}(G) \) is rather flabby, with many uniquely isomorphic elements. Let us rectify this.

**Definition 1.28 (Small set of embeddings).** Write \( \text{Emb}(G) \) for the quotient of \( \widetilde{\text{Emb}}(G) \) by the relation \( f \sim h \) if there is an isomorphism \( z \) so that \( f = h z \).

By Proposition 1.25 the isomorphism \( z \) witnessing \( f \sim h \) is unique.

**Example 1.29.** Let \( G \) be the loop with two vertices (Example 1.4). Then

\[
A(G) = \text{nb}(1) \amalg \text{nb}(2) = \{2^1,1\} \amalg \{1^1,2\}
\]

has four elements. There exist embeddings \( f : H \to G \) if an only if \( H \) is isomorphic to \( \downarrow \cong L_0, L_1, L_2, \) or \( G \). We use the notation for arcs from Example 1.4.

- If \( H = L_n \) is a linear graph \( L_0, L_1, \) or \( L_2 \), then there are exactly four embeddings \( f : L_n \to G \). Each such embedding is determined by where it sends 0 \( \in \partial(L_n) \) (or any chosen arc in \( A(L_n) \)). The arc \( f(0^1) \) is determined since \( f \) commutes with the involution. If \( n > 0 \), then the vertex \( f(1) \) is given since \( f(1) = f(t(0^1)) = t f(0^1) \); this determines the arc \( f(1) \) since the neighborhood of the vertex \( f(1) \) is the set of arcs \( \{f(0^1), f(1)\} \), and so on.
- If \( H = G \), then there are again four embeddings \( G \to G \). Each such embedding is determined by where it sends some chosen arc, and each such embedding is an isomorphism (as in Lemma 1.26).

We have exhibited sixteen elements in the infinite set \( \widetilde{\text{Emb}}(G) \), though every other \( f : H \to G \) arises from one of these sixteen by fixing an isomorphism between \( H \) and an element of the set \( \{L_0, L_1, L_2, G\} \). All of these embeddings are injective on arcs except for those with domain \( L_2 \).

The set \( \text{Emb}(G) \) has just seven elements. Each of the embeddings \( L_n \to G \) is isomorphic to precisely one of the others. The class of \( L_0 \to G \) is determined by which edge is hit. The class of \( L_1 \to G \) is determined by which vertex is hit. The class of \( L_2 \to G \) is determined by which edge of \( G \) is hit twice. Finally, each of the four automorphisms of \( G \) are isomorphic (over \( G \)) to the identity automorphism.

**1.4. Definition of graphical maps.** In order to phrase certain ‘non-overlap’ conditions for embeddings into a fixed graph, it is convenient to work in the free commutative monoid on a vertex set \( V \). For a finite set \( S \), the free commutative monoid \( \text{NS} \) is isomorphic, as a monoid, to \( \text{NS} \) but we write elements as \( \sum_{s \in S} n_s s \) where each \( n_s \in \mathbb{N} \). We consider the power set \( \varphi(S) \) as a subset of \( \text{NS} \), consisting of those elements with \( n_s \leq 1 \) for every \( s \in S \).

**Definition 1.30 (Vertex sum \( \zeta \)).** Given any étale map \( f : G' \to G \), there is a corresponding element \( \sum_{v \in V} f(v) \in \text{NS} \) in the free commutative monoid on \( V \). The assignment of an étale map to this sum is invariant under isomorphisms in the domain. Denote by \( \zeta : \text{Emb}(G) \to \text{NS} \) the map that sends an embedding \( f : G' \to G \) to \( \sum_{v \in V} f(v) \). Since we are only working with embeddings, we have \( \zeta(f) \leq \sum_{v \in V} v \), that is, \( \zeta \) lands in the power set \( \varphi(V) \subseteq \text{NS} \).
Definition 1.31. A graphical map $\varphi : G \rightarrow G'$ consists of the following data:

- A map of involutive sets $\varphi_0 : A \rightarrow A'$
- A function $\varphi_1 : V \rightarrow \text{Emb}(G')$

These data should satisfy two conditions.

(i) The inequality $\sum_{v \in V} \varsigma((\varphi_1(v))) \leq \sum_{w \in V'} w$ holds in $NV'$.

(ii) For each $v$, we have a (necessarily unique) bijection making the diagram

$$\begin{array}{c}
\text{nb}(v) \\
\downarrow \cong \\
\varphi_1(v)
\end{array} \longrightarrow A$$

commute, where the top map $i$ is the restriction of the involution on $A$.

(iii) If the boundary of $G$ is empty, then there exists a $v$ so that $\varphi_1(v)$ is not an edge.

Remark 1.32. We will often have need to refer to a particular element of $\widehat{\text{Emb}}(G')$ representing $\varphi_1(v)$. We will always write $\varphi_v \in \widehat{\text{Emb}}(G')$ for a fixed such choice with $[\varphi_v] = \varphi_1(v) \in \text{Emb}(G')$. Typically, the domain of $\varphi_v$ will be denoted by $H_v$.

The final condition of Definition 1.31 is about avoiding collapse. It is only relevant if $G$ is of a particular form, that is, if $G$ is a single loop containing some (bivalent) vertices. For example, if $G$ is the loop with one vertex $\mathcal{G}$ and $G'$ is the exceptional edge, then there is a pair $(\varphi_0, \varphi_1)$ from $G$ to $G'$ where $\varphi_0$ is a bijection and which satisfies (1.31.i) and (1.31.ii) but not (1.31.iii).

Remark 1.33 (The graph category of Joyal & Kock). There is a related notion of morphism of connected graphs in [JK11], but based on étale maps between connected graphs, rather than embeddings. Joyal and Kock do not include the conditions (1.31.i) and (1.31.iii) in their definition. Further, condition (1.31.ii) is modified to reflect that étale maps need not be injective on boundaries. This yields a category of connected graphs $\mathbf{Gr}$, and each graphical map in the sense of Definition 1.31 is a morphism in $\mathbf{Gr}$.

We have an ample supply of graphical maps: the embeddings. Let us take a look at how this works. As a precursor to Definition 1.44, we also indicate how to compose an arbitrary graphical map with an embedding.

Definition 1.34 (Embeddings and restriction). Every embedding $f : G \rightarrow G'$ in $\widehat{\text{Emb}}(G')$ determines a graphical map via $f : A \rightarrow A'$ and the composite

$$V \xrightarrow{f} V' \longleftarrow \text{Emb}(G'),$$

that is, $v \mapsto \iota_{fv}$ as in Example 1.14. We still call this graphical map ‘$f$’.

(1) If $\varphi : G \rightarrow G'$ is a graphical map and $f \in \widehat{\text{Emb}}(G)$, then $\varphi|_f$ is the graphical map from the domain of $f$ to $G'$ defined by $(\varphi|_f)_0 = \varphi_0 f$ and $(\varphi|_f)_1 = \varphi_1 f$.

(2) Likewise, suppose $\varphi : G \rightarrow G'$ is a graphical map and $f : G' \rightarrow G''$ is an embedding. Define a new graphical map $f \circ \varphi$ with $(f \circ \varphi)_0 = f \varphi_0$ and $(f \circ \varphi)_1$ is the composite

$$V \xrightarrow{\varphi_1} \text{Emb}(G') \xrightarrow{f(\cdot)} \text{Emb}(G'').$$
It is relatively easy to see that $\varphi|_f$ is a graphical map. For Proposition 1.31.iii, note that if the domain of $\varphi|_f$ has empty boundary, then $f$ is an isomorphism by Lemma 1.26. In the next proposition we check that $f \circ \varphi$ is a graphical map. Notice if $\varphi$ comes from an embedding $h$, then $\varphi|_f$ comes from the embedding $h \circ f$ and the map $f \circ \varphi$ comes from the embedding $f \circ h$.

**Proposition 1.35.** The pair of functions $(f \circ \varphi)_0, (f \circ \varphi)_1$ from Definition 1.34 constitute a graphical map.

**Proof.** For each vertex $v$, pick a representative $(\varphi_v : H_v \to G') \in \overline{\text{Emb}}(G')$ for $\varphi_1(v)$. Notice that

$$\sum_{v \in V} \varsigma(f \circ \varphi_v) = \sum_{v \in V} \sum_{u \in H_v} f \varphi_v(u) = f \sum_{v \in V} \sum_{u \in H_v} \varphi_v(u) = f \sum_{v \in V} \varsigma(\varphi_v) \leq f \sum_{w \in V'} w.$$

Since $f$ is a injective on vertices, this last term is less than or equal to $\sum_{x \in V''} x$ in $\mathbb{N}^{V''}$, so Proposition 1.31.ii holds. Using Lemma 1.20 commutativity of the diagram:

$$\begin{array}{ccc}
\text{nb}(v) & \xrightarrow{i} & A \\
\downarrow \cong & & \downarrow \varphi_0 \\
\partial(H_v) & \xrightarrow{\varphi_v} & \partial(\varphi_v) \\
\downarrow \cong & & \downarrow f \\
\partial(f \circ \varphi_v) & \xrightarrow{f} & A''
\end{array}$$

shows that Proposition 1.31.ii holds for $f \circ \varphi$. Condition Proposition 1.31.iii for $f \circ \varphi$ follows immediately from this condition for $\varphi$. □

**Lemma 1.36.** If $\varphi : G \to G'$ is a graphical map and $\varphi_1(v)$ is an edge for some $v$, then $\varphi_0$ is not injective.

**Proof.** By Proposition 1.31.ii we know that if $\varphi_1(v)$ is an edge, then $\text{nb}(v)$ has order two. Let us first address the cases when $G$ has a single vertex. The case where $G$ is the loop with one node (Example 1.4) is disallowed by Proposition 1.31.iii, so we must be in the case when $G$ is isomorphic to the linear graph $L_1 \cong \mathbb{N}$. Then $|A| = 4$, while only two elements of $A'$ are in the image of $\varphi_0$, so the result follows.

Now suppose that $v$ and $w$ are adjacent, distinct vertices of $G$ and $\varphi_1(v)$ and $\varphi_1(w)$ are both edges. Write $\text{nb}(v) = \{a, b\}$ and $\text{nb}(w) = \{ia, c\}$. If $b = ic$ then $G$ is a loop with two vertices and the map violates Proposition 1.31.iii. Thus $b \neq ic$. By assumption that $\varphi_1(v)$ and $\varphi_1(w)$ are edges, we have $\varphi_0(a) = i\varphi_0(b)$ and $\varphi_0(ia) = i\varphi_0(c)$. Since $\varphi_0$ commutes with $i$, this implies that $\varphi_0(b) = \varphi_0(ic)$, so $\varphi_0$ is not injective.

Finally, suppose that $w$ and $v$ are adjacent vertices of $G$, $\varphi_1(v)$ is an edge, and $\varphi_1(w)$ is not an edge. Write $\text{nb}(v) = \{a, b\}$ with $ia \in \text{nb}(w)$ (that is, so that $[a, ia]$ is an edge between $v$ and $w$). We know that $ia \neq b$ since $v \neq w$, yet $\varphi_0(a) = i\varphi_0(ia) = \varphi_0(ia)$. Thus $\varphi_0$ is not injective. □

**Theorem 1.37.** Suppose that $\varphi, \psi : G \to G'$ are graphical maps with $\varphi_0 = \psi_0$. If $\varphi_0$ is injective, then $\varphi = \psi$.

**Proof.** By the second condition for graphical map, for each $v$ we have $\partial(\varphi_1(v)) = \partial(\psi_1(v))$. By the contrapositive of the previous lemma, we know that $\varphi_1(v)$ and $\psi_1(v)$ are not edges, so by Proposition 1.25 we have $\varphi_1(v) = \psi_1(v)$. □
We now show how graph substitution is related to graphical maps.

**Proposition 1.38.** Suppose that \( \varphi : G \to G' \) is a graphical map, and write \( \varphi_v : H_v \hookrightarrow G' \) for an embedding representing \( \varphi_1(v) \). Then there is an embedding \( k : G\{H_v\} \hookrightarrow G' \) which factors all of the embeddings \( \varphi_v \).

**Proof.** If \( G \) is the exceptional edge, then \( \varphi \) is already an embedding from \( G = G\{\} \) to \( G' \). Suppose \( V \) is nonempty. The isomorphisms \( m_v : i(\text{nb}(v)) \to \partial(H_v) \) are defined, using \( \text{Proposition 1.38} \), and \( \varphi_v(m_v(x)) = \varphi_0(x) \). Since \( G \) and all \( H_v \) are connected, so is \( G\{H_v\} \) \( \text{Proposition 6.12} \). Consider the diagram whose top line is from Construction \( \text{Proposition 1.38} \).

\[
\begin{array}{ccc}
\prod_{e \in E_1} \begin{array}{c} \mathcal{R}_e(a) \\ \mathcal{S}_e(a) \end{array} & \xrightarrow{k} & \prod_{v \in V} H_v \\
\Pi \varphi_v & \xrightarrow{\pi} & K \\
\end{array}
\]

We have

\[
\varphi_{tx_1} \begin{pmatrix} \mathcal{R}_e(a) \\ \mathcal{S}_e(a) \end{pmatrix} = \varphi_{tx_1} \begin{pmatrix} m_{tx_1}(i x_1^1) \\ m_{tx_1}(i x_1^2) \end{pmatrix} = \varphi_0(i x_1^1) \\
\varphi_{tx_2} \begin{pmatrix} \mathcal{R}_e(a) \\ \mathcal{S}_e(a) \end{pmatrix} = \varphi_{tx_2} \begin{pmatrix} m_{tx_2}(i x_2^1) \\ m_{tx_2}(i x_2^2) \end{pmatrix} = \varphi_0(i x_2^1) = \varphi_0(x_v^1);
\]

since all maps are equivariant we have \( (\prod \varphi_v)\mathcal{R} = (\prod \varphi_v)\mathcal{S} \), thus \( k : K \to G' \) exists. The map \( k \) is automatically étale, and further we have that \( k \) is injective as a map from \( V(K) = \prod_v V(H_v) \) to \( V(G') \) by \( \text{Proposition 1.38} \). Thus \( k \) is an embedding. \( \square \)

This proof shows that the following is well-defined (that is, does not depend on the choice of \( \varphi_v \in \text{Emb}(G') \) representing \( \varphi_1(v) \in \text{Emb}(G') \)).

**Definition 1.39** (Image of a graphical map). If \( \varphi : G \to G' \) is a graphical map, then the embedding \( k : G\{H_v\} \to G' \) from Proposition \( \text{Proposition 1.38} \) represents an element \( \text{Im}(\varphi) \in \text{Emb}(G') \) called the image of \( \varphi \).

**Remark 1.40** (Image of an embedding). Notice that if \( \varphi : G \to G' \) is coming from an embedding \( f \), then we can actually take \( f \) itself as a representative for \( \text{Im}(\varphi) \). Indeed, \( \varphi_1(v) \) can be represented by \( \star_{f(v)} : \star_f(v) \to G' \) where \( \star_f(v) \cong \star_v \), and \( G\{\star_f(v)\} \cong G\{\star_v\} \cong G \).

**Lemma 1.41.** Suppose \( \varphi : G \to G' \) is a graphical map, and let \( k : G\{H_v\} \hookrightarrow G' \) represent its image. Then there exists a graphical map \( \varphi' : G \to G\{H_v\} \) so that \( \varphi' \) is a bijection on boundaries and

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow{\varphi'} & & \!
\leftarrow{k} \\
G\{H_v\} & & \!
\end{array}
\]

commutes.

**Proof.** Once again we exclude the simple case when \( G = \uparrow \) (in which case \( k = \varphi \) and \( \varphi' = \text{id} \)), and reuse the notation from Construction \( \text{Construction 1.18} \) and the proof of
Proposition 1.38. Let $\varphi'_v$ be the composite $H_v \rightarrow \bigsqcup_{v \in V} H_v \xrightarrow{\pi} K$; we have $k\varphi'_v = \varphi_v$. Write $\varphi'_1(v) = [\varphi'_v] \in \text{Emb}(K)$. Define $\varphi'_0 : A = D \cup iD \rightarrow A(K)$ by

$$
\varphi'_0(a) = \begin{cases} 
\varphi'_v(m_v(a)) & \text{if } a \in i(nb(v)) \\
i\varphi'_v(m_v(ia)) & \text{if } ia \in \partial(G) \text{ and } a \in nb(v).
\end{cases}
$$

Then if $a \in i(nb(v))$ we have $k\varphi'_0(a) = k\varphi'_v(m_v(a)) = \varphi_v(m_v(a)) = \varphi_0(a)$, while if $ia \in \partial(G)$ and $a \in nb(v)$ we have $k\varphi'_0(a) = ki\varphi'_v(m_v(ia)) = i\varphi_v(m_v(ia)) = i\varphi_0(ia) = \varphi_0(a)$. Thus we have established that $\varphi = k\varphi'$, assuming that $\varphi'$ is a graphical map.

Let us now show this. Condition (1.31.i) follows since $\varphi_0 \in \text{Emb}(H_v \rightarrow V)$. Condition (1.31.ii) follows since $\varphi_0 \in \text{Emb}(H_v \rightarrow V)$. Finally, (1.31.iii) holds for $\varphi$ by the corresponding condition for $\varphi$.

The graphical map $\varphi'$ is a bijection on boundaries by Lemma 1.19.

Suppose that $\varphi : G \rightarrow G'$ is any graphical map. By the previous lemma and Lemma 1.20, we know that $\varphi_0|\partial(G)$ is injective. We extend the definition of $\partial$, given in Definition 1.21, from embeddings to arbitrary graphical maps.

Definition 1.42 (Boundary of a graphical map). If $\varphi : G \rightarrow G'$ is any graphical map, let $\partial(\varphi) \cong \partial(G)$ be the subset $\varphi_0(\partial(G)) \subseteq A'$.

In Definition 1.44, we will explain how to compose two graphical maps. The following is key in to ensuring that composition is well-defined.

Lemma 1.43. Let $\varphi : G \rightarrow G'$ be a graphical map, and let $f : G_1 \rightarrow G$ and $h : G_2 \rightarrow G$ be embeddings. If $z$ is an isomorphism with $f = hz$, then $\text{Im}(\varphi|_f) = \text{Im}(\varphi|_h)$ in $\text{Emb}(G')$.

In other words, the function $\widehat{\text{Emb}}(G) \rightarrow \text{Emb}(G')$ which sends $f$ to $\text{Im}(\varphi|_f)$ factors through $\text{Emb}(G)$. Of course, $\varphi|_f$ need not be equal to $\varphi|_h$ (they need not even have the same domain), so $\varphi|_f$ is not defined for $f \in \text{Emb}(G)$. Despite that fact, we will still use the notation $\text{Im}(\varphi|_f)$ when $f \in \text{Emb}(G)$.

Proof. In the proof of Proposition 1.38, we represented these images as coming the universal property of coequalizers. Consider the following diagram. The map $k_1$ represents $\text{Im}(\varphi|_f)$ while the map $k_2$ represents $\text{Im}(\varphi|_h)$. The isomorphisms on the left come from $z$ applied to the indexing sets for the coproducts.
Then $K_1 \to K_2$ is an isomorphism as well, and we see that $k_1$ and $k_2$ represent the same element of $\text{Emb}(G')$.

\[\text{Definition 1.44 (Composition in U).} \quad \text{If} \quad \varphi : G \to H \quad \text{and} \quad \psi : H \to K \quad \text{are two graphical maps, define} \quad (\psi \circ \varphi)_0 \quad \text{on} \quad A \quad \text{and} \quad (\psi \circ \varphi)_1 \quad \text{on} \quad V \quad \text{by} \]

\[\begin{align*}
& (\psi \circ \varphi)_0(a) = \psi_0(\varphi_0(a)) \in A(K) \\
& (\psi \circ \varphi)_1(v) = \text{Im}(\psi|_{\varphi_1(v)}) \in \text{Emb}(K). 
\end{align*}\]

In light of Lemma 1.43, $(\psi \circ \varphi)_1$ is a well-defined function. We must still verify that $\psi \circ \varphi$ is a graphical map when both $\varphi$ and $\psi$ are; this will occur in the proof of Theorem 1.48. Before doing that, we should address the following potential inconsistency: we’ve already defined composition when one of $\varphi$ or $\psi$ is an embedding.

\[\text{Remark 1.45 (Composition with embeddings). Let} \quad \varphi : G \to G' \quad \text{be a graphical map with chosen embeddings} \quad \varphi_v : H_v \to G' \quad \text{representing} \quad \varphi_1(v). \quad \text{Let us compare the composition from Definition 1.44 with previously mentioned compositions with embeddings from Definition 1.34.}
\]

\[\begin{itemize}
\item If $f \in \text{Emb}(G)$ is an embedding, we defined $(\varphi|_f)_1(w)$ to be $\varphi_1(f(w))$, which is represented by $\varphi_{f(w)} : H_{f(w)} \to G'$. On the other hand, regarding $f$ as a graphical map, we know that $f$ sends $w$ to $[f(w)] \in \text{Emb}(G)$. But $\text{Im}(\varphi_{f(v)}) : \star v \to G'$ is represented by $\varphi_v : \star v(H_v) = H_v \to G'$, so $((\varphi \circ f)_1)(w)$ is also represented by $\varphi_{f(w)}$. Thus $f |_{\varphi} = \varphi \circ f$.
\item Suppose that $f$ is an embedding with domain $G'$. In Definition 1.34 we declared $(f \circ \varphi)_1(v)$ to be represented by $f \circ \varphi_v$. On the other hand, in Definition 1.44 we said that $(f \circ \varphi)_1(v)$ should be represented by $\text{Im}(f|_{\varphi_v})$. The graphical map $f|_{\varphi_v}$ comes from the embedding $f \circ \varphi_v$, so by Remark 1.40 we have that $\text{Im}(f|_{\varphi_v})$ is also represented by $f \circ \varphi_v$. Thus there is no ambiguity about what we mean by $f \circ \varphi$.
\end{itemize}\]

The following two lemmas will be used in verifying that $\psi \circ \varphi$ is a graphical map in the proof of Theorem 1.48.

\[\text{Lemma 1.46. If} \quad \varphi : G \to G' \quad \text{is a graphical map and} \quad \text{Im} \varphi \quad \text{is its image, then} \quad \varsigma(\text{Im} \varphi) = \sum_{v \in V} \varsigma(\varphi_1(v)).
\]

\[\text{Proof. Write} \quad \varphi_v : H_v \to G' \quad \text{for an embedding representing} \quad \varphi_1(v), \quad \text{and let} \quad k : G\{H_v\} \to G' \quad \text{be the associated embedding representing} \quad \text{Im} \varphi. \quad \text{Each} \quad \varphi_v \quad \text{factors as} \quad H_v \to G\{H_v\} \xrightarrow{\sim} G'. \quad \text{Identifying the vertex set of} \quad G\{H_v\} \quad \text{with the disjoint union of} \quad \text{the vertex sets of} \quad H_v, \quad \text{we have}
\]

\[\sum_{v \in G} \varsigma(\varphi_v) = \sum_{v \in G} \sum_{w \in H_v} \varphi_v(w) = \sum_{w \in G\{H_v\}} k(w) = \varsigma(k). \]

\[\square\]

\[\text{Lemma 1.47. If} \quad f \in \text{Emb}(G) \quad \text{is an embedding and} \quad \varphi : G \to G' \quad \text{is a graphical map, then we have a commutative diagram}
\]

\[
\begin{array}{ccc}
\varnothing(f) & \longrightarrow & A \\
\downarrow & & \downarrow \varphi_0 \\
\varnothing(\text{Im}(\varphi f)) & \longrightarrow & A'
\end{array}
\]
whose left map is a bijection.

Proof. Let $G''$ be the domain of $f$. By Lemma 1.41, we know that

$$
\partial(\text{Im}(\varphi|f)) = \partial(\varphi|f) = (\varphi|f)\circ(\partial(G'')).
$$

By definition of $\varphi|f$, this is equal to $\varphi_0(f(\partial(G''))) = \varphi_0(\partial(f))$. The composition and the first map in

$$
\partial(G'') \xrightarrow{f} \partial(f) \xrightarrow{\varphi_0} \partial(\varphi|f)
$$

are isomorphisms, hence $\varphi_0 : \partial(f) \to \partial(\varphi|f)$ is an isomorphism. □

Theorem 1.48. The graphical maps from Definition 1.31 assemble into a category $U$. The objects of $U$ are the connected graphs (excluding nodeless loops).

Proof. The graphical maps that we have defined are all maps in the category $\text{Gr}$ from [JK11, §6]. The composition is identical to that in $\text{Gr}$, so the result follows as long as we can show that $\psi \circ \varphi$ from Definition 1.44 is a graphical map. Throughout, we let $\varphi_{v} : J_{v} \to H$ be an embedding which represents $\varphi_1(v)$.

We have, using Lemma 1.46,

$$
\sum_{v \in G} \zeta((\psi \circ \varphi)_1(v)) = \sum_{v \in G} \zeta(\text{Im}(\psi|_{\varphi_1(v)})) = \sum_{v \in G} \sum_{w \in J_{v}} \zeta((\psi|_{\varphi_1(v)})(w))
$$

$$
= \sum_{v \in G} \sum_{w \in J_{v}} \zeta((\psi|_{\varphi_1(v)})(w)) \leq \sum_{w \in H} \zeta(\psi_1(u)),
$$

where the inequality is because $G\{J_{v}\} \hookrightarrow H$ is an embedding. Since (1.31.i) holds for $\psi_1$, this element is less than or equal to $\sum_{x \in K} x$. Thus (1.31.i) holds for $(\psi \circ \varphi)_1$.

To see that (1.31.ii) holds, note that we have

$$
\begin{array}{ccc}
\text{nb}(v) & \xrightarrow{i} & A(G) \\
\text{fibre} & \downarrow{=} & \varphi_0 \\
\partial(\varphi_1(v)) & \xrightarrow{\text{fibre}} & A(H) \\
\text{fibre} & \downarrow{=} & \psi_0 \\
\partial(\text{Im}(\psi|_{\varphi_1(v)})) & \xrightarrow{\text{fibre}} & A(K)
\end{array}
$$

where on the left-hand side, the bottom map is a bijection by Lemma 1.47 and the top is a bijection by (1.31.i).

For (1.31.iii), suppose that $G$ has empty boundary. By Proposition 1.38, there is an embedding $G\{J_{v}\} \hookrightarrow H$ representing $\text{Im} \varphi$; since the boundary of $G\{J_{v}\}$ is empty, Lemma 1.26 implies this embedding is an isomorphism. By (1.31.iii) applied to $\psi$, there is a vertex $w \in H$ with $\psi_1(w)$ not an edge; using the isomorphism of $H$ and $G\{J_{v}\}$, there exists a $v \in G$ so that $w \in J_{v}$. Then $(\psi \circ \varphi)_1(v) = \text{Im}(\psi|_{\varphi_1(v)})$ cannot be an edge, since $\psi_1(w)$ factors through it. □

2. Factorization of graphical maps

Now that we’ve defined the category $U$, we exhibit two (orthogonal) factorization systems on it. Recall that a factorization system on a category $C$ consists of two classes of maps $L$ and $R$ (the left class and right class, respectively), each containing all isomorphisms and closed under composition. The defining property is that every morphism $f$ of $C$ factors as $f = r \ell$ with $\ell \in L$ and $r \in R$, and this factorization is
unique up to unique isomorphism. In Theorem 2.15 we exhibit such a factorization system on $U$ whose right class is consists of all of the embeddings. The left class of this factorization system consists of ‘active’ maps, which play a major role in this paper and its companion [HRY].

The second factorization system we will deal with actually has more structure: we show that $U$ is a dualizable generalized Reedy category in the sense of Berger and Moerdijk [BM11]. This fact, established in Section 2.2, gives us Quillen model structures on categories of presheaves. We will exploit this in Section 3 when developing model categories for Segal modular operads.

2.1. Active maps and their properties. Recall that a wide subcategory of a category $C$ is a subcategory which contains all objects of $C$.

**Definition 2.1 (Active maps).** A graphical map $\varphi : G \to G'$ is called active if $\varphi_0$ is a bijection on boundaries, that is, if we have a commutative square

$$
\begin{array}{ccc}
\partial(G) & \longrightarrow & \partial(G') \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi_0} & A'
\end{array}
$$

whose top map is an isomorphism. We have two wide subcategories of $U$:

- $U_{\text{act}}$ is the subcategory of $U$ consisting of active maps
- $U_{\text{emb}}$ is the subcategory consisting of embeddings (Definition 1.34).

In this subsection, we study the interactions of these two classes of maps, with the aim of showing that they constitute an orthogonal factorization system on $U$ (Theorem 2.15). In the next subsection, we will use this fact to help establish a generalized Reedy structure on $U$.

**Lemma 2.2.** If $\varphi : \downarrow \to G$ is an active map, then $\varphi$ is an isomorphism.

**Proof.** Consider the exceptional edge $\downarrow$ from Example 1.4 and write $\{\sharp, \flat\}$ for the set of arcs. Suppose that $\varphi$ is not an isomorphism. Then since $\varphi$ is active and $G$ is connected, the graph $G$ has a vertex $v$ together with an arc $b \in \text{nb}(v)$ so that $ib \in \partial(G) = \varphi_0(\{\sharp, \flat\})$. Without loss of generality, we may assume that $ib = \varphi_0(\sharp)$. Then $b = \varphi_0(\flat)$ is in $\partial(G)$. This contradicts our assumption that $b \in \text{nb}(v)$ and thus $\varphi$ must be an isomorphism. □

**Proposition 2.3.** A map $\varphi : G \to G'$ is active if and only if $\text{Im} \varphi$ is an isomorphism.

**Proof.** The reverse implication follows because $G$ and $G\{H_{\varphi}\}$ share a boundary. For the forward implication, if $G = \downarrow$, then Lemma 2.2 implies $\varphi$ is an isomorphism hence $\text{Im} \varphi$ is as well. If $G \neq \downarrow$, this is a special case of Proposition 1.25 using $f = \text{Im} \varphi$ and $h = \text{id}_{G'}$. □

The following proposition follows immediately from the definition of active maps.

**Proposition 2.4.** If $\varphi$ is an active map and $\psi$ is any other map graphical map so that $\psi \circ \varphi$ is defined, then $\partial(\psi) = \partial(\psi \circ \varphi)$. □

**Definition 2.5 (Wide subcategories of $U_{\text{act}}$).** Consider the following two subcategories:

1. A map $\varphi$ is in $U^{-}$ if and only if $\varphi_0$ is surjective and $\varphi$ is active.
(2) $U^+_{\text{act}}$ is the subcategory of $U_{\text{act}}$ consisting of those active maps with $\varphi_0$ injective.

We will later have need for another wide subcategory $U^+ \subseteq U$ (see Definition 2.16), but we postpone its introduction until we’ve done some preliminary work. The following remark is not essential in what follows, but it describes a collection of generators for the category $U$ as well as particular subcategories.

**Remark 2.6 (Cofaces and codegeneracies).** One can consider, for each of the subcategories $U_{\text{emb}}, U^+_{\text{act}},$ and $U^-$, those non-isomorphisms $\varphi$ so that if $\varphi = \varphi^1 \circ \varphi^2$ (with all three morphisms in the subcategory), then either $\varphi^1$ or $\varphi^2$ is an isomorphism. Such maps are called outer coface maps, inner coface maps, and codegeneracy maps, respectively. To allow us to be a little more concrete about what these maps are, for a generic map $\varphi: G \to G'$ we will write $\varphi_v: H_v \to G'$ for an embedding representing $\varphi_1(v)$. There are two kinds of embeddings which are outer cofaces: those embeddings where $G'$ has exactly one more internal edge than $G$, and maps from the exceptional edge into a star. Inner cofaces and codegeneracies, which are all active, come with a distinguished vertex $w \in V(G)$ and have the property that $H_v \sim \text{whitestar} \cdot v$ whenever $v \neq w$. Such a map is an inner coface just when $H_w$ has exactly one internal edge, and is a codegeneracy just when $H_w$ is the exceptional edge. The outer cofaces generate $U_{\text{emb}},$ the inner cofaces generate $U^+_{\text{act}},$ and the codegeneracies generate $U^-$. A coface map is just a map which is either an inner or outer coface, and the coface maps generate the category $U^+$ from Definition 2.16.

**Theorem 2.7.** Given a graphical map $\varphi: G \to G'$, there is a factorization

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow & & \downarrow \text{Im} \varphi \\
G_1 & \xrightarrow{\delta} & G_2
\end{array}
$$

in which $\varphi^-$ is in $U^-$ and $\delta$ is in $U^+_{\text{act}}$.

**Proof.** We’ve already constructed the factorization $\varphi = (\text{Im} \varphi)\alpha$ in Lemma 1.41. It remains to factor $\alpha$.

Let $I = \{ v \in V \mid \varphi_1(v) \text{ is an edge} \}$ and let $I'$ be its complement in $V$. Define $G_1 = G\{v\}_{v \in I}$ and let $\varphi^-$ be the evident map; note that $\varphi^-_0$ is surjective. The set $I'$ may be identified with the set of vertices of $G_1$ via $\varphi^-$. For each $v \in I'$, let $\delta_v$ be the embedding guaranteed by Proposition 1.38, that is, so that

$$
H_v \xrightarrow{\delta_v} G_2 = G\{H_v\}_{v \in V} \xrightarrow{\text{Im} \varphi} G'
$$

is equal to $\varphi_v$. Note that $G_1\{v\}_{v \in I'} = G\{v\}_{v \in I}\{H_v\}_{v \in V} = G\{H_v\}_{v \in V} = G_2$ and this determines the map $\delta: G_1 \to G_2$. The graphical map $\delta$ is active and $\delta_1(v) = [\delta_v]$ is not an edge for any $v \in G_1$. It follows that $\delta_0$ is injective. \qed

We now employ a technical construction which supports the proof of Lemma 2.10. On a first reading of that lemma, we recommend focusing on the case when $\varphi_0$ is already injective, so that $m(a) = a = x(a)$ for all $a$ and Lemma 2.9 is trivial.
Construction 2.8. Let \( \varphi : G \to G' \) be a graphical map. Define \( m = m^{\varphi} : A \to A \) by

\[
ma = \begin{cases} 
  a & \text{if } a \in \partial(G), \\
  a' & \text{if } a \in D \text{ and } \varphi_1(ta) \text{ is not an edge} \\
  a'' & \text{if } a \in D, \varphi_1(ta) \text{ is an edge, and } nb(ta) = \{ a, ia' \}.
\end{cases}
\]

For each \( a \), there exists a \( k > 0 \) so that \( m^k(a) = m^{k+1}(a) \). This occurs since \( A \) is finite and (1.31.iii) holds. Write \( x = x^{\varphi} : A \to A \) for the stabilization of \( m \), that is, \( xa = m^k a \) for some \( k \) and \( mxa = xa \). One can actually choose \( k \) uniformly by taking, for instance, \( k = |A| \).

Suppose that \( \psi : G \to G' \) is another graphical map so that \( \varphi_1(v) \) is an edge if and only if \( \psi_1(v) \) is an edge. Then \( m^{\psi} = m^{\varphi} \), hence \( x^{\varphi} = x^{\psi} \).

Suppose that \( \varphi^- \) is the graphical map appearing in Theorem 2.7. Notice that if \( (\varphi^-)_0(a) = (\varphi^-)_0(a') \), then \( xa = xa' \). One interpretation of Construction 2.8 is that it provides a preferred section to the surjective function \( (\varphi^-)_0 \).

Lemma 2.9. If \( \varphi : G \to G' \) is active, then the restriction

\[ \varphi_0 |_{xA} : xA \to A \xrightarrow{\varphi_0} A' \]

is a monomorphism.

Proof. Suppose that \( a_1, a_2 \) are elements of \( xA \) such that \( \varphi_0(a_1) = \varphi_0(a_2) \). If \( a_1 \) and \( a_2 \) are both elements of \( \partial(G) \), then \( a_1 = a_2 \) since \( \varphi \) is active.

If \( a_k \in D \cap xA \) write \( f_k : H_k \to G' \) for an embedding representing \( \varphi_1(tak) \); we know that \( H_k \) is not the exceptional edge. Letting \( il \in \partial(H_k) \) be the unique element with \( f_k(il) = \varphi_0(iak) \) (by (1.31.iii)), we know \( tl \in V(H_k) \) is defined since \( H_k \neq \emptyset \). We thus have \( \varphi_0(a_k) \in D' \) with \( v_k = t\varphi_0(a_k) = tf_k(l) = f_k(tl) \) in the image of \( f_k \).

Now, if \( a_1 \) is an element of \( D \cap xA \), then \( a_2 \) is as well. If \( a_2 \) was in \( \partial(G) \), then \( \varphi_0(a_2) \) would be in \( \partial(G') \), but in the previous paragraph we showed that \( \varphi_0(a_1) \) is an element of \( D' \). We are thus left to address the situation where \( a_1, a_2 \in D \cap xA \). We have \( v_1 = v_2 \in f_1(V(H_1)) \cap f_2(V(H_2)) \), so by (1.31.ii) we have \( ta_1 = ta_2 \). Now \( a_1, a_2 \in nb(ta_1) = nb(ta_2) \), so by (1.31.ii), \( \varphi_0(a_1) = \varphi_0(a_2) \) implies \( a_1 = a_2 \). \( \square \)

Lemma 2.10. Suppose \( \varphi, \psi : G \to G' \) are in \( U_{\text{set}} \) and \( f : G' \to G'' \) is in \( U_{\text{emb}} \) with \( f\varphi = f\psi \). Then \( \varphi = \psi \).

Proof. If \( G \) is the exceptional edge, then the active maps \( \varphi \) and \( \psi \) are isomorphisms by Lemma 2.2. Any embedding \( \downarrow : \to G'' \) is a monomorphism (in \( \text{FinSet}^\xi \)), so the fact that \( f\varphi = f\psi \) implies that the embeddings \( \varphi \) and \( \psi \) are equal. For the remainder of the proof we only consider the case when \( G \neq \emptyset \).

Since \( f \) is an embedding, for each vertex \( v \), we have \( \varphi_1(v) \) is an edge if and only if \( (f\varphi)_1(v) \) is an edge, and similarly for \( \psi \). Since \( f\varphi = f\psi \), we thus have \( \varphi_1(v) \) is an edge if and only if \( \psi_1(v) \) is an edge. This implies that the functions \( x^{\varphi} \) and \( x^{\psi} \) from Construction 2.8 are equal; we simply write \( x \) for this function.

We wish to show that \( \varphi_0 = \psi_0 \). Suppose that \( a \) is an arc of \( G \) with \( \varphi_0(a) \neq \psi_0(a) \); we will show this leads to a contradiction. Since \( \varphi_0(a) = \varphi_0(xa) \) and \( \psi_0(a) = \psi_0(xa) \), we may replace \( a \) by \( xa \) and assume \( a \in xA \). Since \( f\varphi_0(a) = f\psi_0(a) \), we apply Lemma 2.8 to the arcs \( \varphi_0(a) \) and \( \psi_0(a) \) of \( G' \). One of these arcs is in \( \partial(G') \) while the other is in \( D' \). Without loss of generality, assume that \( \varphi_0(a) \in \partial(G') \) and \( \psi_0(a) \in D' \), the other situation is symmetric. Since \( \varphi_0 |_{xA} : xA \to A' \) is a
monomorphism (Lemma 2.9), \(\partial(G)\) is a subset of \(\mathbf{x}A\), and \(\varphi_0|_{\partial(G)}: \partial(G) \to \partial(G')\) is a bijection, we know that \(a \in \partial(G)\). But then \(\psi_0(a) \in \partial(G')\) since \(\psi\) is active. Since \(D' \cap \partial(G') = \emptyset\), this is impossible. Thus \(\varphi_0(a) = \psi_0(a)\) for every \(a \in A\).

We now turn to vertices. Since \(\varphi_0 = \psi_0\), we know by (1.31.iii) that \(\partial(\varphi_1(v)) = \partial(\psi_1(v))\) for every vertex \(v\). But \(\varphi_1(v)\) is an edge if and only if \(\psi_1(v)\) is an edge, so we can apply Proposition 1.25 to deduce that \(\varphi_1(v) = \psi_1(v)\) for every vertex \(v\). Thus \(\varphi = \psi\).

We next show that it may be the case that two graphical maps with common domain and codomain are distinct despite being identical on arcs. This is similar to behavior that occurs for the wheeled properadic graphical category \(\Gamma \ast \) from \[HRY15, HRY18\], where we only have determination by edge maps for the isomorphisms \[HRY18\] Lemma 3.9]. This behavior is either not present or can be avoided for other types of graphs, for example in the situation of the dendroidal category \(\Omega\) of \[MW07\], the unrooted tree category \(\Xi\) of \[HRY19\], and the properadic graphical category \(\Gamma\) (see \[HRY15\] Corollary 6.62).

Example 2.11. Active maps are not necessarily determined by what they do on arcs. As an example, consider graphical maps from the loop with two vertices to the loop with one vertex, \(G\). Any graphical map between graphs without boundary is automatically active.

```
1
a
<--->
\text{ia} \quad \text{ib}
```

The set \(\text{Emb}(G)\) has only three elements: each of the graphs \(L_0 \cong \uparrow\), \(L_1 \cong \star_2\), and \(G\) admit exactly two embeddings into \(G\). All other embeddings into \(G\) are isomorphic to one of these six. For each of the source graphs \(H \in \{L_0, L_1, G\}\), the two embeddings \(H \to G\) are isomorphic using the unique nontrivial automorphism of \(H\), hence give rise to a single element \([H \to G]\) in \(\text{Emb}(G)\).

Let us exhibit two maps \(\varphi, \psi\) where \(\varphi_0 = \psi_0\) is the function sending \(a, ib\) to \(c\) and \(ia, b\) to \(ic\). On vertices, we can declare that

\[
\varphi_1(v_1) = [\iota_w : \star_w \to G] = \psi_1(v_2)
\]

\[
\varphi_1(v_2) = [\iota \to G] = \psi_1(v_1).
\]

Thus \(\varphi_0 = \psi_0\) but \(\varphi \neq \psi\).

Proposition 2.12. Suppose that \(\varphi, \psi: G \to G'\) are active maps and \(\varphi_0 = \psi_0\). Further, assume that one of the following two conditions holds:

- the boundary of \(G\) is nonempty, or
- there exists a vertex \(v\) so that neither \(\varphi_1(v)\) nor \(\psi_1(v)\) is an edge.

Then \(\varphi = \psi\).

Since only bivalent vertices may be sent to edges, the second condition is automatic whenever \(G\) has a vertex which is not bivalent. At least one of the two conditions is guaranteed to be satisfied unless \(G\) is the loop with \(n\) vertices (Definition 1.4). At least one of the two conditions holds whenever \(\varphi, \psi \in \mathbf{U}_{\text{act}}^+\).
Proof. Notice that for each \( v \in G \), we have \( \partial(\varphi_1(v)) = \partial(\psi_1(v)) \subseteq A' \) by (1.31,iii). We would like to apply Proposition 1.25 to infer that \( \varphi_1(v) = \psi_1(v) \), but it might be the case that one of these is an edge while the other isn’t. We will show that the given conditions imply this never happens, whence the statement follows. Let \( S_1 = \{ v \mid \varphi_1(v) \text{ is an edge} \} \), \( S_2 = \{ v \mid \psi_1(v) \text{ is an edge} \} \) and \( S = (S_1 \setminus S_2) \cup (S_2 \setminus S_1) \). Our goal is to show that \( S \) is empty.

To this end, let \( T = V \setminus (S_1 \cup S_2) \) be the set of vertices \( v \) so that neither \( \varphi_1(v) \) nor \( \psi_1(v) \) is an edge; in particular, every vertex which is not in \( T \) must be bivalent.

Let \( \gamma : V \to \mathbb{N} \cup \{-1\} \) be the function that sends a vertex to its distance from the boundary of \( T \) (so \( \gamma(v) = -1 \) if and only if \( v \) is in \( T \)). By hypothesis, at least one of \( T \) and \( \partial(G) \) is non-empty, so \( \gamma \) is well-defined. Let \( \gamma|_S : S \to \mathbb{N} \) be the restriction.

Let \( v \in S \) be minimal with respect to \( \gamma|_S \), say \( \gamma(v) = n \). Without loss of generality, assume \( \varphi_1(v) \) is an edge and \( \psi_1(v) \) is not an edge. Consider a path \( v = w_n, w_{n-1}, \ldots, w_0 \) with \( \gamma(w_k) = k \) and \( \text{nb}(w_k) = \{ i a_{k+1}, a_k \} \). If \( i a_0 \notin \partial(G) \), write \( w_{-1} \in T \) for the vertex with \( i a_0 \in \text{nb}(w_{-1}) \). Write \( f_k : H_k \to G' \) for an embedding representing \( \psi_1(w_k) \).

We first note that \( \varphi_1(w_k) \) is an edge for \( 0 \leq k < n \). If not, we have that \( w_k \notin S_1 \), so by minimality of \( n \) we have \( w_k \notin S_2 \). Thus \( w_k \notin S_1 \cup S_2 \), that is, \( w_k \in T \). Hence \( \gamma(k) = -1 < k \), a contradiction.

Since \( \varphi_1(w_k) \) is an edge for \( 0 \leq k < n \) (that is, \( w_0, \ldots, w_n \in S_1 \)), we have \( \varphi_0(i a_{k+1}) = \varphi_0(a_k) = \varphi_0(i a_k) \) for \( 0 \leq k \leq n \). So we get \( \varphi_0(i a_{n+1}) = \varphi_0(i a_n) = \cdots = \varphi_0(i a_0) \).

If \( w_{-1} \in T \) exists, that is, if \( i a_0 \notin \partial(G) \), then we are in a situation where both \( \psi_1(v) = \psi_1(w_n) \) and \( \psi_1(w_{-1}) \) are not edges, that is, that \( H_n \neq \emptyset \neq H_{-1} \). It follows that \( \psi_0(i a_{n+1}) = \psi_0(i a_n) \) for some vertex \( u_n \in H_n \) and \( \psi_0(i a_0) \in \text{nb}(f_{n} u_n) \) for some vertex \( u_{-1} \in H_{-1} \). But of course \( \psi_0(i a_{n+1}) = \psi_0(i a_n) = \psi_0(i a_0) \), so \( f_n u_n = f_{n} u_{-1} \), contradicting (1.31).

We are now in the situation where \( \psi_1(w_k) \) is an edge for \( 0 \leq k < n \) and \( i a_0 \notin \partial(G) \). But then \( \psi_0(i a_0) = \psi_0(i a_{n+1}) \in \text{nb}(f_n u_n) \), so \( \psi_0(i a_0) \notin \partial(G') \). This contradicts the assumption that \( \psi \) is active. \( \square \)

**Lemma 2.13.** Suppose \( \varphi : G \to G' \) is a graphical map in \( U_{\text{act}} \). If \( \varphi \) has two decompositions \( \varphi = \gamma \alpha = \delta \beta \) where \( \gamma, \delta \in U^+_{\text{act}} \) and \( \alpha, \beta \in U^- \) then there is an isomorphism \( z : H \to K \) making the following diagram commute:

\[
\begin{array}{ccc}
H & \xrightarrow{\gamma} & G' \\
\downarrow{\alpha} & & \downarrow{\beta} \\
G & \xrightarrow{z} & K
\end{array}
\]

**Proof.** If \( v \) is a vertex of \( G \), then the following are equivalent:

- \( \varphi_1(v) \) is an edge,
- \( \alpha_1(v) \) is an edge,
- \( \beta_1(v) \) is an edge.

Thus \( H \) is an edge if and only if \( K \) is an edge. In this case we know that \( \gamma \) and \( \delta \) are isomorphisms by Lemma 2.2, and we can take \( z = \delta^{-1} \gamma \). From now on, suppose that \( H \) and \( K \) are not edges.

Since \( \alpha_0 \) is surjective, \( \alpha_1(v) \) is either an edge or is \( \{ i_w : \star_w \to H \} \) for some \( w \in V(H) \) (using the notation from Example 1.14) for every vertex \( v \) (and likewise
for $\beta$). Let us first define $z$ on vertices. If $v$ is a vertex of $H$ let $\tilde{v}$ be the unique vertex of $G$ so that $\alpha_1(\tilde{v})$ contains $v$. Let $\gamma(v)$ be the unique vertex in $\beta_1(\tilde{v})$. On $\text{nb}(v)$, define $z$ via the following bijection

$$\text{nb}(v) \overset{\sim}{\xrightarrow{1}} \partial(\alpha_1(\tilde{v})) \overset{\sim}{\xrightarrow{\alpha_0}} \text{nb}(\tilde{v}) \overset{\sim}{\xrightarrow{\beta_0}} \partial(\beta_1(\tilde{v})) \overset{\sim}{\xrightarrow{1}} \text{nb}(z(v)).$$

Since $\alpha$ and $\beta$ are bijections on the boundary, we can extend $z$ to the boundary so that $z\alpha = \beta$.

At this point we know $z\alpha_0 = \beta_0$ and $\gamma_0\alpha_0 = \delta_0\beta_0$. Since $\alpha_0$ is surjective, the fact that $\gamma_0\alpha_0 = \delta_0z\alpha_0$ implies that $\gamma_0 = \delta_0z$. By assumption, we know that $H$ contains at least one vertex. For each vertex $v$ of $H$ we have $\gamma_1(v)$ and $(\delta z)_1(v)$ are not edges, so $\gamma = \delta z$ by Proposition 2.12.

**Proposition 2.14.** Every map in $U$ factors uniquely (up to unique isomorphism) as a map in $U_{\text{act}}$ followed by a map in $U_{\text{emb}}$.

**Proof.** Let $\varphi: G \to G'$ be a graphical map. The existence of such a factorization is guaranteed by Lemma 1.41. Consider two such factorizations $\varphi = h\alpha = f\beta$ with $\alpha, \beta$ active maps and $f, h$ embeddings.

$$\begin{array}{ccc}
G & \xrightarrow{\beta} & G_1 \\
\downarrow^\alpha & & \downarrow^f \\
G_2 & \xrightarrow{h} & G'
\end{array}$$

By Proposition 2.4, $\partial(f) = \partial(f\beta) = \partial(\varphi) = \partial(h\alpha) = \partial(h)$. Since $f$ is an embedding and $v$ is a vertex of $G$, we have $\beta_1(v)$ is an edge if and only if $\varphi_1(v)$ is an edge and similarly for $\alpha$. Thus $\beta_1(v)$ is an edge if and only if $\alpha_1(v)$ is an edge. Since $\beta$ is active, the graph $G_1$ is the exceptional edge if and only if $\beta_1(v)$ is an edge for every $v$ and similarly for $G_2$. Thus $G_1 = \downarrow$ if and only if $G_2 = \downarrow$.

By Proposition 1.25 there is a unique isomorphism $z$ with $fz = h$. Now we have a diagram

$$\begin{array}{ccc}
G & \xrightarrow{\beta} & G_1 \\
\downarrow^\alpha & & \downarrow^f \\
G_2 & \xrightarrow{h} & G'
\end{array}$$

where the outer square commutes, as does the lower triangle. But then $fz\alpha = h\alpha = f\beta$. Since $z\alpha$ and $\beta$ are active maps and $f$ is an embedding, we have $z\alpha = \beta$ by Lemma 2.10. \qed

**Theorem 2.15.** The category $U$ admits a factorization system with left class $U_{\text{act}}$ and right class $U_{\text{emb}}$.

**Proof.** By Proposition 2.14 we know that any graphical map factors uniquely, up to unique isomorphism, as an active map followed by an embedding. Further, the classes $U_{\text{act}}$ and $U_{\text{emb}}$ are closed under composition and contain all isomorphisms. Thus the conditions of [AHS06 Proposition 14.7] are satisfied, and $(U_{\text{act}}, U_{\text{emb}})$ is an orthogonal factorization system on $U$. \qed
2.2. **Reedy structure.** A dualizable generalized Reedy structure on a small category $R$ consists of
   - wide subcategories $R^+$ and $R^-$, and
   - a degree function $\deg: \text{Ob}(R) \to \mathbb{N}$
   satisfying five axioms from [BM11, Definition 1.1]. Our goal for the remainder of the section is to prove Theorem 2.22 which asserts that the structure from Definition 2.18 constitutes a dualizable generalized Reedy structure on $U$.

**Definition 2.16.** We say a graphical map $\varphi: G \to G'$ is almost injective on edges if in the decomposition $\varphi = f \varphi'$ (from Proposition 2.14), with $f$ an embedding and $\varphi'$ an active map, we have that the function $\varphi'_0$ is injective. Write $U^+$ for the class of maps which are almost injective on edges.

It is clear that $U^+$ contains all isomorphisms, hence all identities. This class is also closed under composition, so $U^+$ is a wide subcategory of $U$:

**Remark 2.17 (U+ is subcategory).** Suppose that $f \alpha = \beta h$ with $f, h \in U_{\text{emb}}$ and $\alpha, \beta \in U_{\text{act}}$. Further, assume that $\beta \in U^+_{\text{act}}$, that is, that $\beta_0$ is injective. This implies that $\beta_1(v)$ is never an edge by Lemma 1.36, hence $(\beta h)_1(v)$ is never an edge. Since $(\beta h)_1(v) = (f \alpha)_1(v)$ is not an edge for any vertex $v$ and $f$ is an embedding, we know that $\alpha_1(v)$ is never an edge. This implies that the function $\chi^\alpha$ from Construction 2.8 is the identity, so by Lemma 2.9 we know $\alpha_0$ is a monomorphism. It follows that $U^+$ is closed under composition, since if $\varphi = h \gamma$ and $\psi = k \beta$ are two composable morphisms (with specified factorizations), then $\psi \varphi = k \beta h \gamma = k f \alpha \gamma$.

In Theorem 2.22 we will show that the following structure on $U$ constitutes a generalized Reedy structure in the sense of [BM11].

**Definition 2.18 (Generalized Reedy structure).** The categories $U^+$ and $U^-$ are as given in Definition 2.16 and Definition 2.5 respectively. For the latter, recall that a map $\varphi$ is in $U^-$ if and only if $\varphi_0$ is surjective and $\varphi$ is active.

Recall that an internal edge $e = [a, ia]$ is one in which $a, ia \in D$, and that $E_i \subseteq E$ denotes the set of internal edges. If $G$ is a graph, then the degree of $G$ is the sum of the number of vertices and the number of internal edges, that is $\deg(G) = |V| + |E_i|$.

**Proposition 2.19.** Non-invertible morphisms in $U^+$ (respectively, $U^-$) raise (respectively, lower) the degree. Isomorphisms in $U$ preserve the degree.

**Proof.** We first prove the statement about $U^+$. In light of Proposition 2.14, it is enough to show that maps in $U_{\text{emb}}$ and $U^+_{\text{act}}$ are nondecreasing on degree, and that non-isomorphisms in these wide subcategories are strictly increasing in degree.

Embeddings are injective on vertices and on the set of internal edges, so in particular are non-decreasing in degree. If $f: G \to G'$ is an embedding which is not a bijection on vertices, then because $f$ is nondecreasing on internal edges, $f$ is strictly increasing in degree. Suppose that $f$ is an embedding which is not an isomorphism but is a bijection on vertices. This cannot be the case if $G' = \emptyset$. Then $D \to D'$ is also a bijection (since $f$ is etale), hence $A \to A'$ is not a bijection. The map $f: A \to A'$ is automatically a surjection, since if $a' \in A' \setminus f(A) \subseteq \partial(G')$, then $ia' = f(d)$ for some $d \in D \cong D'$. But then $a' = iia = if(d) = f(ii(d)) \in f(A)$, which is a contradiction. Hence there exist $a_1 \neq a_2 \in A$ with $f(a_1) = f(a_2)$. Since $D \to D'$ is injective, either $a_1 \in \partial(G)$ or $a_2 \in \partial(G)$. Without loss of generality,
assume \( a_1 \in \partial(G) \); by Lemma 1.20 \( a_2 \notin \partial(G) \). Since \( D \to D' \) is injective and \( f(ia_1) = f(ia_2) \), we know \( ia_2 \notin \partial(G) \). Thus in \( G' \) we have created a new internal edge \([f(a_1), i(f(a_1))] = [f(a_2), i(f(a_2))] \) which does not come from an internal edge of \( G \). Thus \( \deg(G) < \deg(G') \).

Below we write \( \varphi : G \to G' \) for a map in \( \mathbf{U} \) and \( \varphi_v : H_v \hookrightarrow G' \) for a representative of \( \varphi_1(v) \).

Suppose \( \varphi \) is in \( \mathbf{U}_+^+ \). Since \( \varphi_0 \) is injective, \( \varphi_1(v) \) is not an edge for any \( v \) by Lemma 1.36. Thus \( \varphi \) is nondecreasing on number of vertices. Also, \( \varphi \) takes internal edges to internal edges. If \( \varphi \) is not an isomorphism, then there exists a vertex \( v \) so that \( H_v \) contains an internal edge. Thus \( \varphi \) is strictly increasing in internal edges, hence degree.

Now suppose \( \varphi \) is in \( \mathbf{U}^- \). Since \( \varphi_0 \) is surjective, for each \( v \) we know that \( H_v \) has no internal edges. Thus \( H_v \) has at most one vertex and \( \varphi \) is nonincreasing in degree. Suppose \( \varphi \) is not invertible. Then there exists a \( v \) with \( H_v \neq \ast \); since \( \varphi_0 \) is surjective, \( H_v \) has no internal edges. It follows that \( \varphi_1(v) \) is an edge, and \( \varphi \) is strictly decreasing on number of vertices, hence on degree.

**Proposition 2.20.** Every morphism \( \varphi \) of \( \mathbf{U} \) factors as \( \varphi = \varphi^+ \varphi^- \) with \( \varphi^+ \) in \( \mathbf{U}^+ \) and \( \varphi^- \) in \( \mathbf{U}^- \), and this factorization is unique up to isomorphism.

**Proof.** It follows from Theorem 2.7 that we can factor every graphical map \( \varphi = f \delta \varphi^- \) where \( \varphi^- \in \mathbf{U}^- \), \( f \) is an embedding (hence in \( \mathbf{U}^+ \)), and \( \delta \in \mathbf{U}_+^+ \). Setting \( \varphi^+ = f \delta \), it remains to check that the decomposition is unique up to isomorphism.

Suppose that \( \varphi = \alpha \beta \) with \( \alpha \in \mathbf{U}^+ \) and \( \beta \in \mathbf{U}^- \). Factor \( \alpha = h \gamma \) with \( \gamma \) active and \( h \) an embedding. Since \( \varphi = f(\delta \varphi^-) = h(\gamma \beta) \) are two factorizations into active followed by embedding, by Proposition 2.14 there is an isomorphism \( z \) so that \( fz = h \) and \( \delta \varphi^- = z \gamma \beta \). By Lemma 2.13 there is an isomorphism \( z' \) with \( \delta z' = z \gamma \) and \( z' \beta = \varphi^- \). Further, \( \varphi^+ z' = f \delta z' = f \gamma \gamma \beta = h \gamma = \alpha \).

**Proposition 2.21.** If \( \varphi : G \to G' \) in \( \mathbf{U}^- \), \( \theta \in \text{Iso}(\mathbf{U}) \), and \( \theta \varphi = \varphi \), then \( \theta = \text{id}_{G'} \).

Likewise, if \( \varphi : G \to G' \) in \( \mathbf{U}^+ \), \( \theta \in \text{Iso}(\mathbf{U}) \), and \( \varphi \theta = \varphi \), then \( \theta = \text{id}_G \).

**Proof.** For the first statement, note that the source and target of \( \theta \) are both \( G' \). The assumption implies that \( \theta_0 \varphi_0 = (\text{id}_{G'})_0 \varphi_0 \), with \( \varphi_0 \) surjective. Thus \( \theta_0 = (\text{id}_{G'})_0 \).

By Theorem 1.37 \( \theta = \text{id}_{G'} \).

For the second statement, note that the source and target of \( \theta \) are both \( G \). Write \( \varphi = f \varphi' \) with \( f \) an embedding and \( \varphi' \in \mathbf{U}_+^+ \). Then \( f \varphi' \theta = \varphi \theta = \varphi = f \varphi' \). By Lemma 1.24 we have \( \varphi' \theta = \varphi' \). But \( \varphi_0 \) is injective, hence \( \theta_0 = (\text{id}_{G})_0 \).

By Theorem 1.37 \( \theta = \text{id}_G \).

**Theorem 2.22.** With the structure from Definition 2.18, the graphical category \( \mathbf{U} \) is a dualizable generalized Reedy category.

**Proof.** In Proposition 2.19 Proposition 2.20 and Proposition 2.21 we have shown all of the conditions of [BM11, Definition 1.1] except for \( \mathbf{U}^+ \cap \mathbf{U}^- = \text{Iso}(\mathbf{U}) \). The uniqueness of the decomposition in Proposition 2.20 implies the inclusion from left to right. It is also clear that any isomorphism is in both \( \mathbf{U}^- \) and \( \mathbf{U}^+ \), concluding the proof.

3. Simplicial presheaves on \( \mathbf{U} \)

The purpose of this section is to describe categories of presheaves over the graphical category \( \mathbf{U} \) and give circumstances under which a presheaf over \( \mathbf{U} \) models
an up-to-homotopy modular operad. To do so we use the language of Quillen model categories and take [Hir03] as our standard reference. We are mostly concerned with categories of presheaves over the graphical category $U$ into a cofibrantly-generated model category $M$. Such categories always admit a projective model structure where weak equivalences and fibrations are defined entry-wise in $M$. Since our graphical category is a dualizable generalized Reedy category, the category of presheaves over $U$ also admits a Reedy model structure in the sense of [BM11]. We will study certain (left) Bousfield localizations of these model categories in Section 3.2.

Notation 3.1. Let $C$ be a category. The category of $U$-presheaves in $C$ is the category of contravariant functors from $U$ to $C$. We denote this category by $C^{U\text{op}}$.

(1) If $X$ is a $U$-presheaf in $C$ write the evaluation of a presheaf $X$ at a graph $G \in U$ as $X_G$.

(2) We write $U[G] \in \text{Set}^{U\text{op}}$ for the representable presheaf at a graph $G$, that is, $U[G]_H = U(H, G)$ when $H$ is in $U$.

The Yoneda Lemma says that a map $x : U[G] \to X$ in $\text{Set}^{U\text{op}}$ is equivalent to an element $x \in X_G$. Every $X \in \text{Set}^{U\text{op}}$ is, up to isomorphism, a colimit of representables

$$X \cong \text{colim} \ U[G],$$

where the colimit is indexed by the maps $U[G] \to X$.

In Section 2.2 we showed that $U$ admits the structure of a dualizable generalized Reedy category. We now recall the basic definitions of the Berger–Moerdijk Reedy model structure on a diagram category $M^R$ when $R$ is a generalized Reedy category, all of which can be found just before [BM11, Theorem 1.6]. Recall that a model category $M$ is $R$-projective if for every $r \in R$, the category $M^{\text{Aut}(r)}$ admits a model structure whose weak equivalences and fibrations are created in $M$ (that is, by forgetting the $\text{Aut}(r)$ actions). This is the case, for instance, if $M$ is cofibrantly generated [Hir03 11.6.1].

Definition 3.2. Suppose that $R$ is a generalized Reedy category and $M$ is an $R$-projective category (for instance, if $M$ is cofibrantly generated).

(1) If $r$ is an object of $R$, then $R^-(r)$ is the category whose objects are maps of $R^- \setminus \text{Iso}(R)$ with domain $r$.

(2) If $r$ is an object of $R$, then $R^+(r)$ is the category whose objects are maps of $R^+ \setminus \text{Iso}(R)$ with codomain $r$.

Let $X : R \to M$ be a functor, and let $r$ be an object of $R$.

(1) The $r$-th matching map is defined to be the map

$$X_r \to \lim_{R^-(r)} X = M_r X$$

whose codomain is the $r$-th matching object.

(2) The $r$-th latching map is defined to be the map

$$L_r X = \text{colim}_{R^+(r)} X \to X_r$$

whose domain is the $r$-th latching object.

Finally, let $f : X \to Y$ be a map in $M^R$.

(1) If the relative matching map $X_r \to M_r X \times_{M_r Y} Y_r$ is a fibration in $M$ for every $r$, then $f$ is called a Reedy fibration.
(2) If the relative latching map $\cup_{i \leq n} L_i Y \to Y$ is a cofibration in $\mathcal{M}^{\text{Aut}(r)}$ for every $r$, then $f$ is called a \textit{Reedy cofibration}.

The theorem of Berger and Moerdijk asserts that $\mathcal{M}^R$ admits a model structure with these (co)fibrations and with levelwise weak equivalences. If $R$ happens to be ‘dualizable’, then $\mathcal{M}^{\text{op}}$ is also a generalized Reedy category, so $\mathcal{M}^{\text{op}}$ will inherit such a Berger–Moerdijk Reedy model structure. For concision, we refer to this model structure as the \textit{Reedy model structure} in what follows. The most important special case for us is $\mathbf{sSet}^\text{Uop}$ where $\mathbf{sSet}$ has the Kan–Quillen model structure.

3.1. The Segal core of a graph. We begin by following Construction \[1.18\] where we exhibit graphs as coequalizers. Suppose that $G$ is a graph containing at least one vertex, and let $E_i$ be its set of internal edges. For each internal edge $e \in E_i$, choose an ordering $e = [x_1^e, x_2^e]$ for the two-element equivalence class of arcs comprising $e$. We can exhibit $G$ as a coequalizer (in the diagram category $\text{FinSet}^\text{op}$)

$$\prod_{e \in E_i} \uparrow e \overset{\mathbb{R}}{\longrightarrow} \prod_{v \in V} \star_v \longrightarrow G,$$

where the map on the right is $\prod_v t_v$. Explicitly, we have

- $\mathbb{R}$ is the coproduct of maps $\mathbb{R}_e : \uparrow x_1^e \to \star x_1^e$ with $\mathbb{R}_e(\sharp) = (x_1^e)^\uparrow \in D(\star x_1^e)$ and $\mathbb{R}_e(\checkmark) = x_1^e \in D(\star x_1^e)$;
- $\mathbb{S}$ is the coproduct of maps $\mathbb{S}_e : \uparrow x_2^e \to \star x_2^e$ with $\mathbb{S}_e(\sharp) = (x_2^e)^\uparrow \in D(\star x_2^e)$ and $\mathbb{S}_e(\checkmark) = x_2^e \in D(\star x_2^e)$.

We likewise can form corresponding coequalizer in $\mathbf{Set}^\text{Uop}$,

$$\prod_{e \in E_i} U[\uparrow e] \overset{\mathbb{R}}{\longrightarrow} \prod_{v \in V} U[\star_v] \longrightarrow \mathcal{S}\mathcal{c}[G],$$

and we call the target the \textit{Segal core} of $G$ (which should not be confused with the graph core$(G)$ from Definition \[1.9\]). It comes with a map $\mathcal{S}\mathcal{c}[G] \to U[G]$ induced by $\prod_t v_\cdot \cup_{v \in V} U[\star_v] \to U[G]$. In the case when $G = \uparrow$, we declare the map $\mathcal{S}\mathcal{c}[G] \to U[G]$ to be the identity map on $U[G]$.

Notice that the object $\mathcal{S}\mathcal{c}[G]$ does not depend upon the choices we made for the orderings of the internal edges of $G$. Indeed, any two such choices yield isomorphic results via a unique isomorphism of coequalizer diagrams utilizing only the involution on $\uparrow$.

Remark 3.3 (Alternative description). Suppose that $G$ is a graph with at least one vertex, where we’ve made choices about orderings of each internal edge as above.

- Write $\mathbb{R}_e : \uparrow x_1^e \to \star x_1^e$ for the embedding that sends $\downarrow$ to $(x_1^e)^\uparrow \in D(\star x_1^e)$.
- Write $\mathbb{S}_e : \uparrow x_2^e \to \star x_2^e$ for the embedding that sends $\downarrow$ to $x_2^e \in D(\star x_2^e)$.

That is, $\mathbb{R}$ is the coproduct of the $\mathbb{R}_e$ and $\mathbb{S}$ is the coproduct of the $\mathbb{S}_e$ in the category $\mathbf{FinSet}^\text{op}$. Define a new category $G^\mathcal{S}\mathcal{c}$ with object set $E_i \amalg V$. The non-identity morphisms in this category are precisely the set of arcs comprising the internal edges (that is, the set of arcs of core$(G)$), with so that an internal arc $x$ goes from the internal edge $[x] \in E_i$ associated to $x$ to the vertex $tx \in V$. There is a functor
\( C^G \to \text{Set}^{U^\text{op}} \) such that

- \( x^1_e : e \to tx^1_e \) maps to \( \mathcal{R}_e : U[\emptyset] \to U[\star_{tx^1_e}] \)
- \( x^2_e : e \to tx^2_e \) maps to \( \mathcal{I}_e : U[\emptyset] \to U[\star_{tx^2_e}] \).

The colimit of this functor is \( \mathcal{S}c[G] \).

There is an inclusion \( \text{Set}^{U^\text{op}} \hookrightarrow \text{sSet}^{U^\text{op}} \) coming from the inclusion \( \text{Set} \to \text{sSet} \), and we use this to consider \( \mathcal{S}c[G] \) and \( U[G] \) as objects in \( \text{sSet}^{U^\text{op}} \).

**Lemma 3.4.** As in Example 1.4, let \( \downarrow \) have arc set \( \{a, b\} \) and \( \star_n \) have \( D(\star_n) = \{1, \ldots, n\} \), \( \mathcal{D}(\star_n) = \{1^!, \ldots, n^!\} \). Let \( h_k : \downarrow \to \star_n \) be the embedding sending \( a \) to \( k \).

If \( S \) is any subset of \( \{1, \ldots, n\} \), then the map

\[
\prod_{k \in S} h_k : \prod_{k \in S} U[\emptyset] \to U[\star_n]
\]

is a cofibration in the generalized Reedy model structure on \( \text{sSet}^{U^\text{op}} \).

Note that the same statement then holds when \( h_k \) is replaced by the embedding \( h'_k = h_k z : \downarrow \to \star_n \) sending \( a \) to \( k^! \) (where \( z \) is the unique nontrivial automorphism of \( \downarrow \)).

**Proof.** For the purposes of this proof, we can take \( S = \{1, \ldots, n\} \) since \( U[\emptyset] \) is Reedy cofibrant. Let \( Z \) be any object of \( \text{sSet}^{U^\text{op}} \). Then the map

\[
\prod_{k=1}^n Z_{h_k} : Z_{\star_n} \to \prod_{k=1}^n Z_{\star_n}
\]

is a concrete realization of the map \( Z_{\star_n} \to M_{\star_n}(Z) \) to the matching object. This occurs because \( \star_n \) has degree 1, and the only degree 0 object in \( U \) is \( \downarrow \). We then have a splitting of categories (see Definition 3.2)

\[
(U^{\text{op}})^-(\star_n) = (U^+(\star_n))^{\text{op}} = \prod_{k=1}^n \{h_k \leftrightarrow h'_k\} \simeq \prod_{k=1}^n \{h_k\}
\]

where each groupoid \( \{h_k \leftrightarrow h'_k\} \) has two objects together with a unique isomorphism between them. As we can use the discrete category on the right to compute the limit expressing the matching object, we see (5) does indeed model this matching map.

Suppose that \( X \to Y \) is an acyclic Reedy fibration in \( \text{sSet}^{U^\text{op}} \). Diagrams

\[
\prod_{k=1}^n U[\emptyset] \to X \\
\downarrow \downarrow \\
U[\star_n] \to Y
\]
correspond to vertices \( y \in Y_{\mathcal{G}_n} \) and \( \ell \in \prod_{k=1}^n X^+_k \) mapping to the same vertex of \( \prod_{k=1}^n Y^+_k \) in the diagram

\[
\begin{array}{ccc}
X_{\mathcal{G}_n} & \longrightarrow & Y_{\mathcal{G}_n} \\
\downarrow & & \downarrow \\
\prod_{k=1}^n X^+_k & \longrightarrow & \prod_{k=1}^n Y^+_k
\end{array}
\]

whose vertical maps are induced by the \( h_k \). A lift for \( \emptyset \) is the same thing as a vertex \( x \in X_{\mathcal{G}_n} \) which maps to both \( y \) and to \( \ell \).

Since the only objects of \( U \) of degree less than or equal to 1 are edges and stars, Proposition 5.7 of [BM11] implies that

\[ X_{\mathcal{G}_n} \to M_{\mathcal{G}_n}(X) \times M_{\mathcal{G}_n}(Y) Y_{\mathcal{G}_n} \]

is an acyclic fibration of simplicial sets, hence surjective (in particular, on vertices). By our calculation of \( M_{\mathcal{G}_n}(Z) \) as \([3]\), we then know that every diagram \( \emptyset \) admits a lift \( U[\mathcal{G}_n] \to X \). Since \( \prod_{k=1}^n h_k \) lifts against all acyclic Reedy fibrations, it is a Reedy cofibration. \( \square \)

As one can see from the proof of the preceding lemma, one does not expect these maps to be projective cofibrations (that is, when acyclic fibrations are the levelwise acyclic fibrations). Therefore, we expect the projective version of the following proposition to be false in general.

**Proposition 3.5.** If \( G \in U \), then \( \mathsf{Sc}[G] \) is Reedy cofibrant in \( \mathbf{sSet}^{U^{op}} \).

**Proof.** The map

\[ \exists : \coprod_{e \in E_i} U[\sharp] \to \coprod_{v \in V} U[\ast_v] \]

is the coproduct (over \( V \)) of maps, each of which isomorphic to one from Lemma 3.4. This is because \( \exists \) restricts to a monomorphism

\[ \prod_{E_i} \{\sharp\} \to \prod_{v \in V} \text{nb}(v), \]

and for each \( v \) we can consider \( S = \exists(\coprod_{E_i} \{\sharp\}) \cap \text{nb}(v) \subseteq D(\ast_v) \). A similar argument (using \( \flat \) instead of \( \sharp \)) shows that \( \mathcal{R} \) is isomorphic to a coproduct of maps from Lemma 3.4. Hence both \( \mathcal{R} \) and \( \exists \) are cofibrations. As the pushout of a cofibration is a cofibration, all of the maps in the defining diagram for \( \mathsf{Sc}[G] \)

\[
\begin{array}{ccc}
\prod_{e \in E_i} U[\sharp] & \xrightarrow{\mathcal{R}} & \prod_{v \in V} U[\ast_v] \\
\downarrow \exists & & \downarrow \\
\prod_{v \in V} U[\ast_v] & \longrightarrow & \mathsf{Sc}[G]
\end{array}
\]

are cofibrations, so \( \emptyset \to \prod_{E_i} U[\sharp] \to \mathsf{Sc}[G] \) is a composition of cofibrations. \( \square \)
3.2. A Segal model for up-to-homotopy modular operads. Recall the dualizable generalized Reedy structure on $U$ from Section 2.2. We will say that a presheaf $X \in sSet^{U^{op}}$ is ‘Reedy fibrant’ if it is fibrant in the Reedy model structure on $sSet^{U^{op}}$ (from [BM11, Theorem 1.6]) discussed earlier in this section. The Reedy model structure is simplicial with mapping objects $\text{map}(X,Y) \in sSet$. We will also utilize homotopy function complexes, denoted by $\text{map}^h(X,Y) \in sSet$, and do not insist upon a particular model for these.

**Definition 3.6** (Segal modular operads). Suppose that $X$ is an object of $sSet^{U^{op}}$.

- The presheaf $X$ will be called (weakly-) monochrome if $X_\uparrow$ is weakly contractible.
- The presheaf $X$ is said to satisfy the (weak) Segal condition if for each $G \in U$, the Segal map $X_G = \text{map}(U[G], X) \to \text{map}(\mathcal{S}[G], X)$ is a weak equivalence of simplicial sets.

If $X$ is Reedy fibrant, monochrome, and satisfies the Segal condition, then we call $X$ a (monochrome) Segal modular operad.

The purpose of this subsection is to point out that there is a model category whose fibrant objects are precisely the Segal modular operads. In the companion paper [HRY], we give a precise definition of (colored) modular operads (called compact symmetric multicategories in [JK11]) and prove Theorem B. Segal modular operads should be thought of as one-colored modular operads where all of the structure is only defined up to coherent homotopy. At the end of Section 4.2, we provide potential examples which should be adaptable to give non-strict examples of Definition 3.6.

**Remark 3.7.** If $X_\uparrow$ is a point, instead of just being weakly equivalent to a point, then $\text{map}(\mathcal{S}[G], X)$ is isomorphic to the product $\prod_{v \in G} X_{\star_v} \cong \text{map}(\mathcal{S}[G], X)$, and the $v$th projection of the map from $X_G$ to this product is induced from $\iota_v : \star_v \to G$. In this case, the Segal map being a weak equivalence tells us $X$ should be determined by its value on vertices.

Suppose that $P$ is a modular operad in $sSet$ (in the sense of [HRY]) whose color set has just one element. If $X = NP$ is the nerve of $P$, then $X_\uparrow$ is a point and the Segal map $X_G \to \prod X_{\star_v}$ is an isomorphism for every $G$. Both a precise construction of $N$ and a proof of this fact are provided in the companion paper [HRY]. Thus every one-colored modular operad gives rise to a monochrome presheaf that satisfies the Segal condition. Note, however, that Reedy fibrancy requires more assumptions on the modular operad $P$.

**Theorem 3.8.** The category $sSet^{U^{op}}$ admits a cofibrantly generated model structure whose fibrant objects are the Segal modular operads.

**Proof.** The Reedy model structure on $sSet^{U^{op}}$ is left proper and cellular by [HRY19, Theorem 7.2 & Proposition 7.4]. Thus if $S$ is any set of maps, we may apply left Bousfield localization [Hir03, Theorem 4.1.1] to obtain a localized model structure $L_S sSet^{U^{op}}$ with the same underlying category. We specialize to the case when $S$
is the set of maps consisting of the Segal core inclusions $\text{Sc}[G] \rightarrow U[G]$ as well as the unique map $\emptyset \rightarrow U[\emptyset]$. Here, we are using the inclusion $\text{Set}^{\text{op}} \rightarrow \text{sSet}^{\text{op}}$ to regard these set-valued presheaves as simplicial set-valued presheaves.

To complete the proof, we only need to characterize the fibrant objects in this localized model structure. As with any left Bousfield localization, these are the objects $X$ so that $X$ is fibrant in the original model structure and $\map^b(s, X)$ is a weak equivalence of simplicial sets for every $s \in S$. In other words, we must characterize those Reedy fibrant $X$ so that (for all $G$)

$$
\map^b(U[G], X) \rightarrow \map^b(\text{Sc}[G], X)
$$

$$
\map^b(U[\emptyset], X) \rightarrow \map^b(\emptyset, X)
$$

are weak equivalences of simplicial sets.

In any simplicial model category, if $A$ is cofibrant and $Z$ is fibrant, then $\map^b(A, Z)$ is weakly equivalent to $\map(A, Z)$ by [DK80, Corollary 4.7]. Note that $U[G]$, $\emptyset$, and $\text{Sc}[G]$ are all cofibrant in $\text{sSet}^{\text{op}}$ (the last of these by Proposition 3.5), which is a simplicial model category. Further, $\map(U[G], X) = X_G$ for any presheaf $X$. Rephrasing the condition for fibrancy in $L_S\text{sSet}^{\text{op}}$ gives that $X$ is fibrant if and only if

- $X$ is Reedy fibrant,
- $X_G \rightarrow \map(\text{Sc}[G], X)$ is a weak equivalence of simplicial sets for all $G$, and
- $X_{\emptyset} \rightarrow \map(\emptyset, X) = \Delta[0]$ is a weak equivalence.

Thus $X$ is fibrant if and only if it is a Segal modular operad. \qed

Lemma 3.9. Suppose that $R$ is a generalized Reedy category and $M$ is a cofibrantly generated model category. Write $M^R_{\text{re}}$ for the diagram category with the Berger–Moerdijk Reedy model structure [BM11] and $M^R_{\text{pr}}$ for the same category with the projective model structure [Hir03, Theorem 11.6.1]. Then the identity functor

$$
M^R_{\text{pr}} \longrightarrow M^R_{\text{re}}
$$

is a Quillen equivalence.

Assume further that $M$ is left proper and cellular. If $S$ is any set of maps in $M^R$ and $L_S$ denotes left Bousfield localization at $S$ [Hir03, Definition 3.3.1], then $L_S M^R_{\text{re}}$ and $L_S M^R_{\text{pr}}$ have the same class of weak equivalences. In particular, (7) remains a Quillen equivalence after left Bousfield localization at $S$.

Proof. It is immediate that (7) is a Quillen adjunction since each cofibration in the projective model structure is a cofibration in the Reedy model structure, and each fibration in the Reedy model structure is a fibration in the projective model structure. Since the two model structures have the same class of weak equivalences, (7) is a Quillen equivalence.

It remains to show that the localized model structures have the same class of weak equivalences. Suppose that $W$ is any object, $W \rightarrow \tilde{W}$ is a Reedy fibrant replacement of $W$, and $f : A \rightarrow B$ is any morphism in $M^R$. We then have the
following commutative diagram of homotopy function complexes.

\[
\begin{array}{ccc}
\text{map}^h(B, W) & \longrightarrow & \text{map}^h(A, W) \\
\downarrow \cong & & \downarrow \cong \\
\text{map}^h(B, \widehat{W}) & \longrightarrow & \text{map}^h(A, \widehat{W})
\end{array}
\]

The vertical maps in this diagram are weak equivalences using [Hir03, 17.6.3].

If \(W\) is a projective \(S\)-local object, then \(\widehat{W}\) is a Reedy \(S\)-local object. To see this, notice that if \(f : A \to B\) is any element of \(S\), then the top map of (8) is a weak equivalence by assumption, which implies that the bottom map is as well.

Now suppose that \(f : A \to B\) is a Reedy local equivalence and \(W\) is a projective \(S\)-local object. Since we know that \(\widehat{W}\) is a Reedy \(S\)-local object, we have that the bottom map of (8) is an equivalence, hence the top map is as well. Since \(W\) was an arbitrary projective \(S\)-local object, this implies that \(f\) is a projective \(S\)-local equivalence.

On the other hand, suppose that \(f : A \to B\) is a projective \(S\)-local equivalence. Any Reedy \(S\)-local object \(W\) is automatically a projective \(S\)-local object (since every Reedy fibrant object is also projectively fibrant). Hence \(\text{map}^h(A, W) \leftarrow \text{map}^h(B, W)\) is an equivalence. Since the Reedy \(S\)-local object \(W\) was arbitrary, this implies that \(f\) is a Reedy \(S\)-local equivalence.

Proposition 3.10. There exists a model category structure on \(\text{sSet}^{U^{op}}\) so that an object \(X\) is fibrant if and only if

- \(X_G\) is fibrant for all graphs \(G\),
- \(X_+ \simeq *\), and
- for all graphs \(G\), the Segal map
  \[
  X_G = \text{map}(U[G], X) \simeq \text{map}^h(U[G], X) \to \text{map}^h(\text{Sc}[G], X)
  \]
  is a weak equivalence of simplicial sets.

Furthermore, this model structure is Quillen equivalent (via the identity functor) to the model structure from Theorem 3.8.

Proof. The proof of the first part is the same as in Theorem 3.8 except we start with the projective model structure on \(\text{sSet}^{U^{op}}\) instead of the Reedy model structure. The second statement is a direct application of Lemma 3.9. □

4. Variations on the modular graphical category

We now discuss two variations on the graphical category \(U\). The first of these essentially just adds in a single object, the nodeless loop. We’ve postponed the introduction of the nodeless loop until now partly because it allows us to use a cleaner definition of graph in the early parts of the paper, and because we could avoid addressing many special cases throughout. Further, from the point of view of Segal modular operads, the value of a presheaf at the nodeless loop should be indistinguishable (up to homotopy) from the value at the exceptional edge. That said, the extended graphical category does come in handy for one construction in [HRY].

The second variation we address is related to the original definition [GK98] of modular operads, where the underlying collections had an additional genus grading. Modular operads in this sense satisfied a geometric condition called stability. In
Section 4.2 we modify \( U \) to have objects graphs which have a genus labeling on each vertex which satisfies a stability condition.

4.1. The extended graphical category. Inspired by Remark 1.1, we drop the assumption that graphs have boundary exactly equal to \( A \setminus D \). The following extension allows us to express the nodeless loop from Example 1.4.

**Definition 4.1.** A graph \( G \) consists of

- a diagram of finite sets

\[
i \subset A \xleftarrow{s} D \xrightarrow{t} V
\]

where \( i \) is a fixedpoint-free involution and \( s \) is a monomorphism, and

- a subset \( \partial(G) \subseteq A \) so that
  1. \( iD \setminus D \subseteq \partial(G) \subseteq A \setminus D \), and
  2. \( \partial(G) \setminus iD \) is an \( i \)-closed subset of \( A \).

The subset \( \partial(G) \) is called the **boundary** of \( G \). If \( G \) is such a graph where the boundary \( \partial(G) \) is maximal, that is, \( \partial(G) = A \setminus D \), then we say that \( G \) is **safe**.

For the rest of this subsection, the graphs from Definition 1.3 are the safe graphs, while other graphs may be referred to as **unsafe**. But what are these unsafe graphs? Before answering this question fully, let us give an example that we couldn’t quite include in Example 1.4.

**Definition 4.2 (Nodeless loop).** The **loop with zero vertices** is the graph with \( A = 2\{0\} = \{0, 0\} \) and \( D = V = \partial = \emptyset \). Any graph isomorphic to this one will be called a **nodeless loop**.

Let us return to the question at hand. Given two graphs \( G \) and \( H \), we can form a new graph \( G \coprod H \) by taking the coproduct of the underlying functors in \( \text{FinSet}^\partial \) (see Definition 1.6) and declaring that \( \partial(G \coprod H) = \partial(G) \coprod \partial(H) \). A graph will be called **connected** if it is nonempty and cannot be decomposed non-trivially via \( \coprod \) (equivalently, if the underlying object in \( \text{FinSet}^\partial \) is connected). A graph is safe if, and only if, all of its connected components are safe. In other words, unsafe graphs are precisely those graphs that have at least one unsafe connected component. The conditions in Definition 4.1 imply that \( A \setminus (\partial(G) \coprod D) \) is \( i \)-closed for any graph \( G \). If \( G \) is unsafe, then \( A \setminus (\partial(G) \coprod D) \) contains at least one element \( x \), whence it also contains \( ix \). Thus \( G \) contains the nodeless loop with arc set \( \{x, ix\} \) as a summand.

**Remark 4.3.** The only unsafe, connected graphs are nodeless loops. A graph is unsafe precisely when it contains at least one nodeless loop as a summand.

**Proposition 4.4.** Isomorphism classes of graphs from Definition 4.1 are in one-to-one correspondence with Yau–Johnson graphs \([YJ15]\). □

**Proof.** Couple \([BB17, Proposition 15.6]\) with a minor variation of \([BB17, Proposition 15.2]\). □

We now adapt étale maps (Definition 1.11) and embeddings (Definition 1.13) to the present context.

**Definition 4.5.** Suppose that \( G \) and \( G' \) are (possibly unsafe) graphs.
• An étale map $G \to G'$ is a morphism of underlying objects in $\text{FinSet}^r$:

\[
\begin{array}{ccc}
A & \xleftarrow{\varphi_0} & D \\
\downarrow & & \downarrow \\
A' & \xleftarrow{\varphi_0'} & D'
\end{array}
\]

so that

– the right-hand square is a pullback, and
– the set $A \setminus (\partial(G) \cap D)$ maps into $A' \setminus (\partial(G') \cap D')$.

• An embedding $G \to G'$ is an étale map where $V \to V'$ is a monomorphism.

If $G$ is safe, then $\partial(G) \cap D = A$, so the second condition for étale maps is automatically satisfied. We have not added too many embeddings:

• If $G$ is a nodeless loop and $G \to G'$ is an embedding, then $G'$ is also a nodeless loop.
• If $G'$ is a nodeless loop and $G \to G'$ is an embedding, then $G$ is either an exceptional edge or a nodeless loop.

We now adapt Definition 1.31 to our more general class of connected graphs (that includes nodeless loops). A more hands-on description follows in Remark 4.7.

**Definition 4.6.** The extended graphical category $\tilde{U}$ has objects the connected graphs from Definition 4.1. A morphism $\varphi : G \to G'$ consists of

• A map of involutive sets $\varphi_0 : A \to A'$
• A function $\varphi_1 : V \to \text{Emb}(G')$

satisfying (1.31.i), (1.31.ii), and

(iii') If the boundary of $G$ is empty and $\varphi_1(v)$ is an edge for every $v$, then $G'$ is a nodeless loop.

Composition is defined essentially as in Definition 1.44.

Condition (4.6.iii') implies that if $G$ is a nodeless loop and $G \to G'$ is a map, then $G'$ is also a nodeless loop. On the other hand, if $G'$ is a nodeless loop then the set $\text{Emb}(G')$ has precisely two elements. In this case, a map $\varphi : G \to G'$ is entirely determined by $\varphi_0$. Associativity of composition in $\tilde{U}$ then follows from Theorem 1.48 and associativity of composition in the category of involutive sets.

**Remark 4.7.** By comparing (1.31.iii) and (4.6.iii'), we see that $U$ is a full subcategory of $\tilde{U}$. Further, if $G \to G'$ is a map and $G' \in U$, then $G \in U$, i.e. $U$ is a sieve on $\tilde{U}$ (as in Proposition 5.2). Let $K$ denote a nodeless loop. We then have

\[
|\tilde{U}(K,G)| = \begin{cases} 
0 & \text{if } G \in U \\
2 & \text{if } G \text{ is a nodeless loop} 
\end{cases}
\]

\[
|\tilde{U}(G,K)| = \begin{cases} 
2 & \text{if each vertex of } G \text{ has valence two,} \\
1 & \text{if } A(G) \text{ is empty,} \\
0 & \text{if } G \text{ contains a vertex of valence different from 0 or 2.} 
\end{cases}
\]

In the cases where these sets are nonempty, they are identified with $\text{hom}(A(K), A(G))$, respectively $\text{hom}(A(G), A(K))$. Essentially only the linear graphs $L_n$, the isolated vertex $\star_0$, and the loops with $n$ vertices (including nodeless loops) admit maps to a nodeless loop.
Theorem 4.8. The category $\tilde{U}$ admits a factorization system extending that on $U$ from Theorem 2.13.

Sketch of Proof. Let $K$ be a fixed nodeless loop. The right class $\tilde{U}_{\text{emb}}$ consists of embeddings. It contains two maps $\uparrow \rightarrow K$, two maps $K \rightarrow K$, as well as all maps isomorphic to these and all maps in $U_{\text{emb}}$. The left class $\tilde{U}_{\text{act}}$ is obtained from $U_{\text{act}}$ by adding in the unique map $\star_0 \rightarrow K$, the maps from loops with $n$ vertices to $K$, and all maps isomorphic to these. Since $U$ is a sieve, we need only check factorizations and uniqueness of such on maps whose codomain is $K$. There are only a few such cases and this is routine.

Likewise, a version of Theorem 2.22 is true for $\tilde{U}$.

Theorem 4.9. The category $\tilde{U}$ admits the structure of a dualizable generalized Reedy category.

Sketch of Proof. The degree function must be modified from that in Definition 2.18 and is essentially given in [HRY13, Definition 3.2]. The exceptional edge $\uparrow \rightarrow K$ has degree 0, while the isolated vertex $\star_0$ has degree 1. For all other graphs, the degree is given by the formula $|V| + |E_1| + 1$, i.e., an increase of one from the usual degree. We emphasize that the nodeless loop has degree 2.

Let $K$ be a nodeless loop. We describe $\tilde{U}^-$ and $\tilde{U}^+$ up to isomorphisms. The inverse category $\tilde{U}^-$ consists of maps in $U^-$ and maps from loops with $n$ vertices to $K$. The direct category $\tilde{U}^+$ consists of maps in $U^+$, $\uparrow \rightarrow K$, $K \rightarrow K$, and $\star_0 \rightarrow K$. As in Theorem 4.8, the analogue of Proposition 2.20 may be proved by factoring only those maps with codomain $K$.

Definition 4.10. Let $\iota : U \rightarrow \tilde{U}$ denote the inclusion functor.

- If $G$ is a safe graph, then the Segal core inclusion is just the left Kan extension
  \[
  \iota! (\text{Sc}[G] \rightarrow U[G])
  \]
  of the usual Segal core inclusion (Section 3.1). We use the same notation for the domain, writing this as $\text{Sc}[G] \rightarrow U[G]$.
- The Segal core inclusion for a nodeless loop $K$ is
  \[
  \tilde{U}[^{\iota}]
  \]
  \[
  = \text{Sc}[K] \rightarrow \tilde{U}[K].
  \]
- A $\tilde{U}$-presheaf $X$ is said to satisfy the strict Segal condition if $\hom(-, X)$ sends every Segal core inclusion to a bijection of sets.

Theorem 4.11. If $X \in \text{Set}^{U^\text{op}}$ is Segal, then its right Kan extension $\iota_* X \in \text{Set}^{\tilde{U}^\text{op}}$ is also Segal.

Proof. As $U$ is a full subcategory of $\tilde{U}$, we have that $\iota_* X_G = X_G$ for every safe graph $G$, so the Segal condition holds at safe graphs $G$. Thus we must show that $\iota_* X_K \rightarrow (\iota_* X)_K$ is a bijection when $K$ is a nodeless loop.

Write $C_n$ for the loop with $n$ vertices ($n \geq 1$) from Example 1.4, all of which are safe graphs. We also write $C_0$ for the loop with zero vertices from Definition 4.2. We restrict $U$ and $\tilde{U}$ to skeletal full subcategories $\mathcal{A} \subseteq U$ and $\tilde{A} \subseteq \tilde{U}$ whose objects are $\star_0$, $L_n$ for $n \geq 0$, and $C_m$ for $m \geq 1$ (resp. $m \geq 0$). Every map in $\tilde{U}$ with source or target $C_0$ is isomorphic to a map in $\tilde{A}$, so it is sufficient to restrict $X$ to $\mathcal{A}^\text{op}$ and examine its right Kan extension along $\iota : \mathcal{A}^\text{op} \rightarrow \tilde{A}^\text{op}$. 
The arc set of every object in \( \tilde{A} \) is of the form \( 2\{0, \ldots, n\} \) or \( 2\{1, \ldots, m\} \), and we say that a morphism \( \varphi \in \tilde{A} \) is oriented if \( \varphi_0 \) is of the form \( 2f \). That is, a map \( \varphi \) is oriented if it satisfies the condition that if \( j \) is an integer then \( \varphi_0(j) \) is not of the form \( k^1 \) for an integer \( k \). Let \( B \subseteq A \) denote the category with objects \( L_n \) \((n \geq 0) \) and \( C_m \) \((m \geq 1) \) with maps the oriented maps. Each object in \( B \) admits a unique oriented map to \( C_0 \) and there is a functor

\[
F : B^{op} \to C_0 \downarrow (\iota : A^{op} \to \tilde{A}^{op})
\]

taking a graph \( G \) to the opposite of the oriented map \( G \to C_0 \). One can check that the functor \( F \) is initial, so the pointwise formula for right Kan extension (Theorem 1 of [ML98], X.3]) gives

\[
(\iota_* X)_{C_0} \cong \lim_{C_0 \downarrow B^{op}} X.
\]

It remains to show that \( \lim_{B^{op}} X \) is isomorphic to \( X_{L_0} \). Let \( p : L_0 \to L_1 \) be defined by \( p(0) = 1 \) and \( q : L_0 \to L_1 \) be defined by \( q(0) = 0 \). For \( n \geq 1 \) we have a diagram

\[
\begin{array}{ccc}
X_{L_0} & \overset{\cong}{\rightarrow} & X_{L_1} \\
\downarrow & & \downarrow \\
X_{L_n} & \overset{\text{diagonal}}{\rightarrow} & X_{L_1} \times_p X_{L_1} \times_p \cdots \times_p X_{L_1} \\
& & \rightarrow X_{L_1}^{\times n}
\end{array}
\]

coming from the unique oriented maps \( L_1 \to L_0 \) and \( L_n \to L_0 \) and the \( n \) oriented embeddings \( L_1 \to L_n \). The bottom left map is an isomorphism by the Segal condition, that is, elements in \( X_{L_n} \) are lists \( (x_1, \ldots, x_n) \) with \( p^* x_j = q^* x_{j+1} \) for \( 1 \leq j < n \).

There is no map in \( B \) from \( C_m \) to \( L_0 \). However, the Segal condition implies the oriented embedding \( L_m \to C_m \) which is the identity on vertices induces an inclusion \( X_{C_m} \to X_{L_m} \). That is, \( X_{C_m} \) may be regarded as the subset of \( X_{L_m} \) consisting of those lists \( (x_1, \ldots, x_m) \) satisfying the additional condition \( q^* x_1 = p^* x_m \). The oriented rotation \( r_m : C_m \to C_m \) acts on \( X_{C_m} \) by rotating these lists. We see that the map \( X_{L_0} \to X_{L_m} \) from \([4]\), which lands in the diagonal, actually factors through \( X_{C_m} \); write \( \kappa_m : X_{L_0} \to X_{C_m} \) for this special function not coming from \( B \). As \( \kappa_m \) lands in a diagonal, we have \( r_m^* \kappa_m = \kappa_m \).

One now checks that the special functions \( \kappa_m : X_{L_0} \to X_{C_m} \) and the natural maps \( X_{L_0} \to X_{L_n} \) determine a function \( X_{L_0} \to \lim_{B^{op}} X \) which is both left and right inverse to the projection \( \lim_{B^{op}} X \to X_{L_0} \). This is tedious but straightforward. \( \square \)

4.2. Genus grading and stable maps. The original definition [GK98] of modular operad had an additional `genus' grading. In this case, the underlying objects satisfy a stability condition. One can certainly import these notions directly into the setting of colored modular operads studied in [HRY]. In this section, we discuss the presheaf side, and propose, in Theorem 4.17, a stable version of the Segal modular operads of Definition 3.6.

**Definition 4.12.** Let \( G \) be a graph.

- A genus function for \( G \) is just a function \( g : V(G) \to \mathbb{N} \).
- The total genus of a pair \((G, g : V \to \mathbb{N})\) is given by

\[
g(G) = \beta_1(G) + \sum_{v \in V} g(v)
\]
where $\beta_1(G)$ is the first Betti number of $G$. More generally, if $f : H \to G$ is an embedding, then we can define

$$g(f) = \beta_1(H) + \sum_{v \in V(H)} g(f(v))$$

which descends to a function $g : \text{Emb}(G) \to \mathbb{N}$.

- A pair $(G, g)$ is called stable if $G$ is connected and for every vertex $v$,

$$2g(v) + |\text{nb}(v)| - 2 > 0.$$  

If $G \neq \uparrow$, then the first Betti number of $G$ is given by

$$\beta_1(G) = |E_i| - |V| + 1.$$  

Using this fact, or the long exact sequence for relative homology, one sees that

$$\beta_1(G \{ H_v \}) = \beta_1(G) + \sum_v \beta_1(H_v)$$

(which should be proved working one vertex at a time) whenever $G$ and all of the $H_v$ are connected.

- The exceptional edge admits only one genus function $g$, and $g(\uparrow) = \beta_1(\uparrow) = 0$. This graph trivially satisfies the stability condition.
- Note that if $(G, g)$ is a stable graph, then $G$ has no bivalent vertices with genus 0. Moreover, if $g(v) = 1$, then $|\text{nb}(v)| > 0$.
- The function $g : \text{Emb}(G) \to \mathbb{N}$ sends $\iota_v : \star_v \to G$ to $g(v)$ and $\text{id}_G : G \to G$ to the total genus $g(G)$.

Suppose that $\varphi : H \to G$ is any graphical map and $g$ is a genus function on $G$. Then the composition

$$g_{\varphi} : V(H) \xrightarrow{g_1} \text{Emb}(G) \xrightarrow{g} \mathbb{N}$$

is a genus function for $H$. If $(G, g)$ happens to be stable, it is not necessarily true that $(H, g_{\varphi})$ is also stable. However, if $\varphi$ is an embedding then $(H, g_{\varphi})$ is stable since stability is just checked at each vertex.

**Example 4.13.** Suppose that $(G, g)$ is stable and $\varphi : H \to G$ is a graphical map. If there is a vertex $v$ with $\varphi_1(v)$ an edge, then $(H, g_{\varphi})$ is not stable. This is because $g_{\varphi}(v) = g(\uparrow) = 0$, so $2g_{\varphi}(v) + |\text{nb}(v)| - 2 = |\text{nb}(v)| - 2 = 0$.

**Definition 4.14 (Stable graphical category).** The stable graphical category $U_{st}$ has:

- Objects those pairs $(G, g)$ where $G \in U$ is a graph and $g$ is a genus function so that $(G, g)$ is stable.
- Morphisms $(G, g) \to (G', g')$ are precisely those graphical maps $\varphi : G \to G'$ so that the diagram

$$
\begin{array}{ccc}
V(G) & \xrightarrow{g} & \mathbb{N} \\
\downarrow & & \downarrow \\
\text{Emb}(G') & & \text{Emb}(G')
\end{array}
$$

commutes. One has such a morphism just when $g = g_{\varphi}$.

One defines composition using the composition in $U$. We let $R : U_{st} \to U$ be the functor which forgets the genus function.

**Proposition 4.15.** The morphisms defined above for $U_{st}$ are closed under composition.
We wish to show that Theorem 3.8. The category \( \mathcal{U}_{st}^+ \) is fibrant. Let us compute: since \( g^{(1)} = g^{(2)} \), we have for each vertex \( v \) of \( G^{(1)} \) that \( g^{(1)}(v) = g^{(2)}(\varphi_v : H_v \to G^{(2)}) \). Writing \( \psi_{v,w} : K_{v,w} \to G^{(3)} \) for a representative of \( \psi_1(\varphi_v(w)) \) and using \( g^{(2)} = g^{(3)} \), we have

\[
g^{(1)}(v) = g^{(2)}(\varphi_v) = \beta_1(H_v) + \sum_{v \in H_v} g^{(2)}(\varphi_v(w)) = \beta_1(H_v) + \sum_{v \in H_v} g^{(3)}(\psi_1(\varphi_v(w))) = \beta_1(H_v) + \sum_{v \in H_v} g^{(3)}(\psi_{v,w} : K_{v,w} \to G^{(3)})
\]

The summand for a given \( w \) is \( \beta_1(K_{v,w}) + \sum_{u \in K_{v,w}} g^{(3)}(\psi_{v,w}(u)) \), so rearranging we have

\[
g^{(1)}(v) = \beta_1(H_v) + \sum_{v \in H_v} \beta_1(K_{v,w}) + \sum_{v \in H_v} \sum_{u \in K_{v,w}} g^{(3)}(\psi_{v,w}(u)) = \beta_1(H_v \{K_{v,w}\}) + \sum_{v \in H_v} \sum_{u \in K_{v,w}} g^{(3)}(\psi_{v,w}(u)),
\]

which is exactly

\[
g^{(3)}(\Im(\psi|_{\varphi_v}) : H_v \{K_{v,w}\} \to G^{(3)}).
\]

Thus \( g^{(1)}(v) = g^{(3)}((\psi \varphi)_1(v)) \), so \( g^{(1)} = g^{(3)} \).

\[\square\]

**Remark 4.16.** Example 4.13 tells us that the functor \( R : \mathcal{U}_{st} \to \mathcal{U} \) factors through \( \mathcal{U}^+ \). In particular, \( \{1.3.1.3\} \) is automatic for a map between stable graphs, and if we were starting from scratch in this section we would omit this condition entirely. In any case, combining \( R \) with the degree function from Definition 2.18 yields a generalized Reedy structure on \( \mathcal{U}_{st} \) where \( \mathcal{U}_{st}^+ = \mathcal{U}_{st} \) and \( \mathcal{U}_{st}^{-} = \text{Iso}(\mathcal{U}_{st}) \). Further, there is an orthogonal factorization system \((R^{-1}\mathcal{U}_{act}, R^{-1}\mathcal{U}_{emb})\) on \( \mathcal{U}_{st} \) coming from Theorem 2.13.

If \( (G,g) \) is a stable graph, then one can define the Segal core \( \mathcal{S}_{st}(G,g) \) as a subobject of the representable presheaf \( \mathcal{U}_{st}[G,g] \) just as we did in the previous section. Imitating the other proofs from Section 3 yields the following analogue of Theorem 3.8.

**Theorem 4.17.** The category \( \text{sSet}^{\mathcal{U}_{st}^+} \) admits a cofibrantly generated model structure whose fibrant objects are those presheaves \( X \) satisfying the following three conditions:

- \( X \) is Reedy fibrant,
- \( X^+ \) is weakly contractible, and
- for each \( (G,g) \in \mathcal{U}_{st} \), the Segal map

\[
X_{G,g} = \text{map}(\mathcal{U}_{st}[G,g], X) \to \text{map}(\mathcal{S}_{st}[G,g], X)
\]

is a weak equivalence of simplicial sets.

\[\square\]
As in the unstable setting, there is a nerve functor landing in $U_{st}$-presheaves. The analogue of Theorem \[3.7\] also holds in the stable setting. This produces a large collection of fibrant objects in the model structure from Theorem \[4.17\] in a similar manner to Remark \[3.7\]. We now propose an example, related to surfaces, that is not of this form. Our example is written by considering $U_{st}$-presheaves in the category of groupoids. It can be transferred to simplicial sets by using the classifying space functor that goes from the category of groupoids to the category of simplicial sets.

**Example 4.18.** We consider compact, orientable surfaces $S$ where the set of boundary components is given an ordering and each boundary component is equipped with a specified collar. Define a collection of groupoids $\{S_{g,n}\}$ where $S_{g,n}$ has objects $S$ where $S$ is a surface of genus $g$ with $n$ boundary components constructed by gluing atomic surfaces as in \[Til00, 2.2\]. Morphisms in $S_{g,n}$ are given by isotopy classes of homeomorphisms which fix the boundary components pointwise and preserves the orderings (modulo the identifications imposed in \[Til00\]). Notice that the automorphism group of an object $S$ in $S_{g,n}$ is the mapping class group $\Gamma_{g,n}$. Tillmann \[Til00, 2.1\], and later Giansiracusa–Salvatore \[GS10\], show that by gluing along boundaries the collection $\{S_{g,n}\}$ constitutes a modular operad. In forthcoming work of the second author, there is a need to not just understand how that surfaces are built up from atomic pieces, but how these atomic surfaces are assembled. She will show that there is a groupoid-valued $U_{st}$-presheaf $X$ which satisfies a weak Segal condition and is related to the nerve of the above modular operad. If $(G, g)$ is stable, then an object of $X_{G,g}$ consists of one surface for each vertex of $G$ as well as gluing data for the collars connected by the edges.

5. Simply-connected graphs

We now introduce two full subcategories of $U$ which are related to notions of cyclic operad \[GK95\].

**Definition 5.1.** Denote by $U_{cyc} \subseteq U_0 \subseteq U$ the following full subcategories:

- $\text{Ob}(U_0)$ is the set of connected acyclic graphs.
- Graphs in $U_{cyc}$ additionally have nonempty boundary.

In the language of \[DCH18\], $U_0$ is related to cyclic operads and $U_{cyc}$ is related to positive cyclic operads; see \[HRY, Remark 3.20\].

**Proposition 5.2.** The full subcategories $U_0$ and $U_{cyc}$ are sieves (in the sense of \[Lur09, Definition 6.2.2.1\]) on $U$. That is, if $\varphi : G \to T$ is a morphism in $U$ with $T \in U_0$, then $G \in U_0$ (and similarly for $U_{cyc}$).

**Proof.** As in Proposition \[1.38\] the map $\varphi$ factors as

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi} & T \\
\downarrow & & \uparrow \\
G\{H_v\} & \xrightarrow{k} &
\end{array}
$$

with $k$ an embedding. Embeddings into simply-connected graphs must have simply-connected sources, hence $G\{H_v\} \in U_0$. On the other hand, any loop in $G$ (that is, a path with no repeated entries except for the ends) may be extended to a loop in $G\{H_v\}$ since each $H_v$ is connected. Since such a loop cannot exist in $G\{H_v\}$, we see there was no loop in $G$. 
For the second statement, note that any map $\varphi : G \to G'$ in $U$ with $\delta(G) = \emptyset$ is automatically active, which implies that $\delta(G') = \emptyset$. This implies that $U_{\text{cyc}}$ is a sieve on $U_0$, hence on $U$.

Remark 5.3. The category $U_{\text{cyc}}$ is related to other categories in the literature.

(1) Tashi Walde’s category $\Omega_{\text{cyc}}$ from [Wal17] can be considered as a non-symmetric version of $U_{\text{cyc}}$. To be precise, $U_{\text{cyc}}$ is equivalent to a category $U_{\text{cyc}}$ whose objects have cyclic orderings on each set $\text{nb}(v)$. Then $\Omega_{\text{cyc}}$ is the wide subcategory of $U_{\text{cyc}}$ consisting of maps which preserve the cyclic orderings.

(2) In [HRY19], the authors developed a category $\Xi$ with objects the unrooted trees with non-empty boundary. Let $U''_{\text{cyc}}$ denote the category (equivalent to $U_{\text{cyc}}$) where the boundary and each $\text{nb}(v)$ comes equipped with a total ordering and where morphisms can disregard those orderings. There is a functor $U''_{\text{cyc}} \to \Xi$ which is the identity on objects. For a non-linear tree $T$, we have

$$U''_{\text{cyc}}(T, S) = \Xi(T, S),$$

while for linear trees $T$ this map is just surjective.

Proposition 5.4. The categories $U_{\text{cyc}}$ and $U_0$ are both dualizable generalized Reedy categories, with structure induced from the ambient category $U$.

Proof. This follows from the fact that these are sieves on $U$ (Proposition 5.2), the fact that $U$ is Reedy (Theorem 2.22), and the fact that sieves in a Reedy category are again Reedy categories (Lemma 5.5).

Lemma 5.5. Suppose that $R$ is a (dualizable) generalized Reedy category, and $S \subseteq R$ is a full subcategory. If $S$ is a sieve on $R$, then $S$ is also a (dualizable) generalized Reedy category with $S^\pm = R^\pm \cap S$ and $\text{deg}_S = \text{deg}_R|_{\text{Ob}(S)}$.

Proof. Most of the axioms are immediate. The only place we use the sieve property is in verifying that the factorizations in [BM11, Definition 1.1(iii)] for $R$ are actually factorizations in $S$: given a diagram

\[
s \xrightarrow{f} s' \quad \xleftarrow{f^-} \quad r \xrightarrow{f^+} s'
\]

in $R$ with $s, s' \in S$, the existence of the arrow $f^+: r \to s'$ guarantees that $r$ is an object of $S$ also.

5.1. Comparison of Reedy model structures. The functor $J : U''_{\text{cyc}} \to \Xi$ from Remark 5.3 preserves the Reedy factorization system, but does not commute with the degree function. However, we have the following:

Theorem 5.6. The functor

$$J^* : \text{sSet}\text{^{\Xi^{op}}} \to \text{sSet}^{(U''_{\text{cyc}})^{op}}$$

preserves and reflects fibrations and weak equivalences. It detects cofibrations but does not preserve them.
To avoid clutter, we will omit the ′′ from $U_{cyc}''$ and just write $U_{cyc}$ for the remainder of the paper.

We first show that the matching objects for $X$ and for $J^*X$ coincide.

**Lemma 5.7.** The functor

$$F: U_{cyc}^+(T) \to \Xi^+(T)$$

is an equivalence of categories. Therefore, if $T \in \textbf{sSet}^{\Xi_{op}}$, then the $T$-th matching map $X_T \to M_TX$ is isomorphic to the $T$-th matching map $(J^*X)_T \to M_T(J^*X)$.

**Proof.** The functor $F$ is surjective on objects. We would like to show that $F$ is fully-faithful, which will imply that $F$ is an equivalence of categories. Then so is $(U_{cyc})_{op}\sim (\Xi_{op})_{op}$ and the coincidence of the matching maps (see Definition 3.2) follows.

Suppose that $\alpha: G \to T$ and $\beta: H \to T$ are two objects of $U_{cyc}^+(T)$, that is, $\alpha, \beta \in U_{cyc}^+ \backslash \text{Iso}(U_{cyc})$. Since $T$ is simply-connected, $\beta$ must be a monomorphism on arcs (this essentially follow from Lemma 1.22 after factoring $\beta$ as in Proposition 2.14). If $\gamma, \gamma' \in U_{cyc}^+(G,H)$ are two morphisms from $\alpha$ to $\beta$, that is, if $\beta\gamma = \alpha = \beta\gamma'$, then we have $\gamma_0 = \gamma_0'$. For each vertex $v$ of $G$, we know that $\gamma_1(v)$ and $\gamma'_1(v)$ are not edges. Thus, by Proposition 1.25, $\gamma_1(v) = \gamma'_1(v)$ since they have the same boundaries. We’ve shown that $\text{hom}(\alpha, \beta)$ has at most one element, so

$$\text{hom}(\alpha, \beta) \to \text{hom}(F\alpha, F\beta)$$

is injective.

Now suppose we have a map $F\alpha \to F\beta$ in $\Xi^+(T)$, that is, suppose we have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\gamma} & H \\
\downarrow^{F\alpha} & & \downarrow^{F\beta} \\
T & \xrightarrow{\beta} & T
\end{array}
$$

in $\Xi^+$. We have a commutative square

$$
\begin{array}{ccc}
U_{cyc}^+(H,T) \times U_{cyc}^+(G,H) & \xrightarrow{\circ} & U_{cyc}^+(G,T) \\
\downarrow & & \downarrow \\
\Xi^+(H,T) \times \Xi^+(G,H) & \xrightarrow{\circ} & \Xi^+(G,T)
\end{array}
$$

whose vertical maps are isomorphisms as long as $G$ has at least one vertex. So if $G$ has at least one vertex we know there is a $\tilde{\gamma}$ so that $\beta\tilde{\gamma} = \alpha$ and $J(\tilde{\gamma}) = \gamma$. Hence, in this case, (10) is surjective. On the other hand, if $G$ is the exceptional edge, then $\gamma$ hits a single edge $[x, ix]$ in $H$. Since (11) commutes, we have (using the notation for the arcs of $\uparrow$ from Example 1.4)

$$[\alpha_0(\uparrow), \alpha_0(\downarrow)] = (F\alpha)_0[\uparrow, \downarrow] = (F\beta)_0[\uparrow, \downarrow] = (F\beta)_0[x, ix] = [\beta_0(x), \beta_0(ix)].$$

If $\alpha_0(\uparrow) = \beta_0(x)$, define $\tilde{\gamma}$ by $\tilde{\gamma}(\uparrow) = x$, while if $\alpha_0(\uparrow) = \beta_0(ix)$, let $\tilde{\gamma}(\uparrow) = ix$. We’ve thus established that (10) is also surjective when $G$ is the exceptional edge. $\square$

We also have the following lemma.
Lemma 5.8. The functor

\[ F : \U_{\text{cyc}}^{-}(T) \to \Xi^{-}(T) \]

is an equivalence of categories. Therefore, if \( X \in \text{sSet}^{\Xi^p} \), then the \( T \)-th latching map \( L_T X \to X_T \) is isomorphic to the \( T \)-th latching map \( L_T (J^*X) \to (J^*X)_T \).

Proof. For the most part, the proof follows that of Lemma 5.7 with strictly formal changes. The one exception is in the second paragraph, where we applied Proposition 1.25 to show that if the two maps \( \gamma, \gamma' \) are the same on arcs, then they are the same. In the present situation, this follows from Proposition 2.12 using that all maps in \( \U_{\text{cyc}}^{-} \) are active and that all objects in \( \U_{\text{cyc}}^{-} \) have non-empty boundary.

In light of the preceding lemma, it may seem strange that Theorem 5.6 asserts that \( J^* \) does not preserve cofibrations. After all, \( X \) and \( J^*X \) have the same latching maps! The following proposition addresses the underlying reason, while Remark 5.10 allows us to produce concrete examples of cofibrations which are not preserved by \( J^* \).

Proposition 5.9. Suppose that \( f : X \to Y \) is a morphism in \( \text{sSet}^{\Xi^p} \), with \( J^*f : J^*X \to J^*Y \) its image in \( \text{sSet}^{\text{cyc}}_U \).

- Suppose that \( f \) is a Reedy cofibration. Then \( J^*f \) is a Reedy cofibration if and only if \( f \) is an isomorphism when evaluated at \( \uparrow \).
- If \( J^*f \) is a Reedy cofibration, then \( f \) is a Reedy cofibration.

Proof. If \( T \) contains at least one vertex, then \( J \) induces an identity between the two automorphism groups \( \text{Aut}_U(T) \) and \( \text{Aut}_\Xi(T) \) of \( T \). By Lemma 5.8, it follows that the relative latching map

\[ X_T \cup_{L_T X} L_T Y \to Y_T \]

is a cofibration in \( \text{sSet}^{\text{Aut}_\Xi(T)^{\text{op}}} \) if and only if

\[ (J^*X)_T \cup_{L_T(J^*X)} L_T(J^*Y) \to (J^*Y)_T \]

is a cofibration in \( \text{sSet}^{\text{Aut}_U(T)^{\text{op}}} \).

On the other hand, if \( T \) is the exceptional edge, then \( \text{Aut}_U(\downarrow)^{\text{op}} \) is the cyclic group of order two, \( C_2 \), while \( \text{Aut}_\Xi(\downarrow)^{\text{op}} \) is the trivial group. Further, \( L_T Z = \emptyset \). Thus the relative latching map just becomes \( X_\uparrow \to Y_\uparrow \). Now \( \text{Aut}_U(\downarrow)^{\text{op}} \) acts trivially on \( (J^*X)_\uparrow \) and \( (J^*Y)_\uparrow \). So if \( J^*(f) \) is a Reedy cofibration, then \( (J^*X)_\uparrow \to (J^*Y)_\uparrow \) is a cofibration in \( \text{sSet}^{C_2} \), hence in \( \text{sSet} \). By the previous paragraph, we know that \( f \) is then a Reedy cofibration. If \( f \) is a Reedy cofibration, then \( J^*(f) \) is a Reedy cofibration if and only if \( X_\uparrow \to Y_\uparrow \) is a cofibration in \( \text{sSet}^{C_2} \). Since both sides have trivial \( C_2 \) actions, this happens if and only if \( X_\uparrow \to Y_\uparrow \) is an isomorphism (Lemma 2.4 of [GJ99, Chapter V]).

Remark 5.10. Suppose that \( Z \) is an object of \( \text{sSet}^{\text{cyc}}_U \) and let \( J_1 Z \) in \( \text{sSet}^{\Xi^p} \) be its left Kan extension along \( J^0 : \U_{\text{cyc}}^{\text{op}} \to \Xi^p \). The following three sets of morphisms coincide

\[ \text{hom}(Z, \emptyset) = \text{hom}(Z, J^*\emptyset) \cong \text{hom}(J_1 Z, \emptyset), \]

so \( Z \) is non-empty if and only if \( J_1 Z \) is non-empty.
Proof of Theorem \[5.6\]. As Reedy weak equivalences are levelwise and \( J \) is the identity on objects, a map \( f \) in \( \text{sSet}^{\Xi^{op}} \) is a weak equivalence if and only if \( J^* f \) is a weak equivalence. By Lemma \[5.7\], \( f \) is a fibration if and only if \( J^* f \) is a fibration. The statement about detecting cofibrations follows from Proposition \[5.9\].

Finally, let us show that \( J^* \) does not preserve cofibrations. Since \( J_1 \) is left Quillen, it preserves cofibrations. Both \( J^* \) and \( J_1 \) are left adjoints, so preserve initial objects. Suppose that \( Z \) is any cofibrant object in \( \text{sSet}^{\Xi^{op}} \) other than the initial object. Then \( J_1 Z \) is non-empty by Remark \[5.10\] and is also cofibrant. But \( C_2 \) acts trivially on \((J^* J_1 Z)_2 \neq \emptyset \), hence \( J^* J_1 Z \) is not cofibrant. \( \square \)

Remark 5.11. Barwick, in \[Bar10\] Theorem 3.22, gave a characterization of those functors \( F \) between strict Reedy categories so that restriction \( F^* \) is left or right Quillen for every model category \( M \); see \[HV19\] for an alternate presentation of this theorem. It follows from Theorem \[5.6\] that the naïve generalization of this characterization does not hold for functors between generalized Reedy categories, as \( J^{op} : U^{op}_{\text{cyc}} \to \Xi^{op} \) satisfies a strong analogue of the appropriate condition (called ‘cofibering’ in \[HV19\]) that would imply \( J^* \) is left Quillen (which we know to not be true). For simplicity of notation, we study whether or not \( J : U_{\text{cyc}} \to \Xi \) is some kind of ‘fibering’ functor using \[HV19\] Proposition 8, p. 32.

Suppose that \( \sigma : S \to T \) is in \( \Xi^- \), and write \( C_{\sigma} \) for the category that would be the evident analogue of Fact \(\Xi^- (S, \sigma) \) in \[HV19\] Definition 7, p. 27 and \( \partial(S/(J^* /T)) \) in \[Bar10\] Theorem 3.22. Concretely, the category \( C_{\sigma} \) has objects those pairs \((\nu, \mu)\) where \( \nu \in U^- \setminus \text{Isoc}(U_{\text{cyc}}) \), \( \mu \in \Xi^- \), and \( \mu J(\nu) = \sigma \). A morphism \((\nu, \mu) \to (\nu', \mu')\) consists of a morphism \( \tau \in U_{\text{cyc}} \) making the diagrams

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\downarrow \tau \\
\bullet
\end{array} \\
\begin{array}{c}
S \\
\downarrow \nu \\
\bullet \\
\downarrow \nu'
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \mu \\
\bullet \\
\downarrow \mu'
\end{array} \\
\begin{array}{c}
T
\end{array}
\end{align*}
\]

commute. If \( \sigma \) is an isomorphism, then \( C_{\sigma} \) is empty. If \( T \) contains a vertex, then \( U_{\text{cyc}}(S, T) = \Xi^-(S, T) \), so there is a unique lift \( \hat{\sigma} \in U_{\text{cyc}} \) of \( \sigma \) and the object \((\hat{\sigma}, \text{id}_T)\) is a terminal object of \( C_{\sigma} \). If \( T \) is isomorphic to the exceptional edge, then \( S \) must be isomorphic to a linear graph \( L_n \); we suppose that \( \sigma \) is not an isomorphism. Let \( \hat{\sigma} \in U_{\text{cyc}} \) be either of the two lifts of \( \sigma \). Then \((\hat{\sigma}, \text{id}_T)\) is again a terminal object of \( C_{\sigma} \). Indeed, if \( \nu : S \to R \) is any map in \( U_{\text{cyc}} \), then there is a unique morphism \( \tau : \hat{R} \to T \) so that \( \tau \nu = \sigma \). Further, any diagram that looks like the right hand triangle of (12) commutes when \( T \cong \hat{T} \).

Thus we have shown that \( C_{\sigma} \) is either empty or contractible for any \( \sigma \in \Xi^- \). In the strict case, the conditions of Barwick and Hirschhorn–Volic simply request that each \( C_{\sigma} \) is either empty or connected in order to infer that

\[
J^* : \text{sSet}^{\Xi^{op}} \to \text{sSet}^{U_{\text{cyc}}^{op}}
\]

is left Quillen. We presume that there is a much simpler counterexample to \( F^* \) being left Quillen than this one. Further, we anticipate that the characterization for when \( F^* \) is right Quillen should be similar to that in the strict case.

\[9\] Here we use the following fact: if \( Y \) is nonempty, then \( Y_2 \) is nonempty. We know that \( Y_T \) is nonempty for some tree \( T \). Since the edge set of \( T \) is not empty, there is at least one map \( \hat{T} \to T \) which implies there is a map \( Y_T \to Y_{\hat{T}} \). Thus \( Y_2 \) is not empty.
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Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-3568 USA
E-mail address: philip@phck.net
URL: http://phck.net

School of Mathematics and Statistics, The University of Melbourne, Melbourne, Victoria, Australia
E-mail address: marcy.robertson@unimelb.edu.au

Department of Mathematics, The Ohio State University at Newark, Newark, OH, USA
E-mail address: dyau@math.ohio-state.edu