Planar magnetic texture on the surface of a topological insulator

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We study the planar magnetic textures in an insulating magnetic film coupled to the Dirac surface state of a topological insulator. It is shown that the radial vortex with winding number \( w = \pm 1 \) leads to the confinement of Dirac states, where an exact mapping to the Schrödinger equation of a two-dimensional hydrogen atom is found. The fully spin polarized zero energy bound state resembles the zeroth Landau level of Dirac electrons in a uniform out-of-plane magnetic field. Interestingly, when the hybrid system is proximity coupled to an \( s \)-wave superconductor, the existence of Majorana zero modes at Abrikosov vortex depends only on the relative value of the magnetic exchange coupling and the pairing strength. We conclude with a brief discussion on the physical realization with such magnetic textures.

**Introduction.** Topology has become a central theme in condensed matter physics. Interesting quantum phenomena emerge from the intricate interplay between non-trivial topology and magnetism [1–4]. Two outstanding examples are the quantum anomalous Hall effect and axion insulators discovered in magnetic topological insulators (TI) [5–19]. The exchange coupling between an out-of-plane magnetization and the TI surface state opens a gap in the surface spectrum. The gap opening is accompanied by the emergence of the surface quantum Hall effect with a half-quantized Hall conductance described by the axion electrodynamics [5], which is is the physical origin for the topological magneto-electric effect. It leads to rich phenomena such as the quantum anomalous Hall with a chiral edge state emerging at the magnetic domain wall as well as the topological magneto-optical effect [20–22].

Peculiar physics emerges when the Dirac surface states couple to spatially nonuniform magnetic textures, such as skyrmions and domain walls [4, 23–27]. In this paper, we study theoretically the two-dimensional (2D) magnetic textures, in an insulating ferromagnetic thin film proximity-coupled to TI surface states. The Dirac electron with a radial magnetic vortex with the winding number \( w = \pm 1 \) can be exactly mapped to the Schrödinger equation of a 2D hydrogen atom. The fully spin-polarized zero energy bound state resembles the zeroth Landau level of Dirac electrons in a uniform out-of-plane magnetic field. Interestingly, when the hybrid system is proximity coupled to an \( s \)-wave superconductor, the existence of Majorana zero modes at Abrikosov vortex depends only on the relative value of the magnetic exchange coupling and the pairing strength.

**Model.** The planar magnetic texture we consider exhibits the form of \( \mathbf{S}(\mathbf{r}) \equiv \mathbf{S} \mathbf{n}(\mathbf{r}) = S \mathbf{e}_{\theta}(\cos \phi(\mathbf{r}), \sin \phi(\mathbf{r}), 0) \), where \( \mathbf{n}(\mathbf{r}) \) is the unit vector describing the magnetization direction, and \( \mathbf{r} = (x, y) \). It is characterized by the topological winding number [28]

\[
w = \frac{1}{2\pi} \oint \mathbf{n} \cdot d\mathbf{r}.
\] (1)

Two typical magnetic vortices with \( w = 1 \) are illustrated in Fig. 1 (a,b).

In the presence of the magnetic vortex \( \mathbf{n}(\mathbf{r}) \), the surface states of a TI can be described by the Dirac Hamiltonian

\[
\mathcal{H} = v_F (k \times \sigma) \cdot \hat{e}_z + g_s \mathbf{n}(\mathbf{r}) \cdot \sigma.
\] (2)

Here \( \hbar = 1, v_F \) is the Fermi velocity, \( \hat{e}_z \) is the unit vector normal to the surface, and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices describing the spin. \( g_s \equiv J' \rho_s / 2, J' \) is the exchange interaction between the local spin and surface electron, and \( \rho_s \) is the sheet density of local spin. Without loss of generality, we set \( g_s / v_F > 0 \) and neglect the particle-hole asymmetry.

We notice that the in-plane Zeeman term is equivalent to a vector potential, thus the Hamiltonian becomes

\[
\mathcal{H} = v_F \left[(\mathbf{k} - \mathbf{A}) \times \sigma\right] \cdot \hat{e}_z,
\] (3)

where \( \mathbf{A}(\mathbf{r}) \equiv (g_s / v_F)(-\sin \phi(\mathbf{r}), \cos \phi(\mathbf{r})) \). For a generic \( \mathbf{A}(\mathbf{r}) \), Eq. (3) is difficult to solve analytically. In the following, we assume \( \mathbf{n}(\mathbf{r}) \) is rotational invariant along \( z \) axis. The Hamiltonian conserves total angular momentum, and the energy spectrum can be obtained analytically by squaring \( \mathcal{H} \) to solve \( \mathcal{H}^2 \psi = E^2 \psi \), which is

\[
\mathcal{H}^2 / v_F^2 = -\nabla^2 + \mathbf{A}^2 + 2i [\mathbf{A} \cdot \nabla + i (\nabla \cdot \mathbf{A}) - \nabla \times \mathbf{A} \cdot \sigma].
\] (4)

In terms of the 2D polar coordinates \((r, \phi)\), Eq. (4) is separable into the radial and angular parts. Two typical magnetic textures are radial and curling vortices.
function satisfies the equation

\[ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{m^2}{\rho^2} + \left( 2E' + \frac{2}{r^2} \right) f(r') = 0. \]  

Eq. (9) can be solved algebraically due to the hidden dynamical symmetry of hydrogen atom [30], which is related to the conserved Runge–Lenz operator \( \mathbf{K} = -(i\mathbf{\nabla} \times \mathbf{L}_z - \mathbf{L}_z \times i\mathbf{\nabla}) - 2\mathbf{r}_e \), satisfying \( [\mathcal{H}^2, \mathbf{K}] = 0 \), \( [\mathbf{L}_z, \mathbf{K}_x] = iK_y \), \( [\mathbf{L}_z, \mathbf{K}_y] = -iK_x \) and \( [\mathbf{K}_x, \mathbf{K}_y] = -4i\mathbf{L}_z\mathcal{H}_z^2 \).

Now \( r' > 0 \) corresponds to an attractive Coulomb potential, which contains both bound discrete states and unbound continuous states. For \( m \geq 0 \), the eigenvalue of the bound states in Eq. (7) is \( E' = -1/2n_2^2 \), where the principle quantum number \( n_2 \equiv |m| + 1/2 \), with \( n_r = 0, 1, 2, \ldots \). The radial wave function \( r_{m,n}(r') = \frac{e^{im\phi}}{n_2} F(-n_r,2|m|+1,2r'/n_2) \), with \( F(\alpha, \gamma; x) \) the confluent hypergeometric function. The degeneracy between \( |\uparrow\rangle \) and \( |\downarrow\rangle \) exists when \( n_r,\uparrow = n_r,\downarrow + 1 \) and \( n_{r,z,\uparrow} = n_{r,z,\downarrow} - 1 \), namely, \( E_{+,n_{r,m}} = E_{-,n_{r,-1,m+1}} \). Explicitly, the energy spectrum and (un-normalized) spinor wavefunction of bound states are

\[
E_{\pm} = \pm \left( 1 - \left( \frac{m + 1}{n_2^2} \right)^2 \right)^{1/2},
\]

\[
\psi_{\pm} = \left( \frac{e^{im\phi}}{n_2} f_{m,n}(r'), \right),
\]

The bound state energies are within the interval between \( [-g_s, g_s] \). The maximum of radial probability density is located at \( r_{max} \approx n_2^2 a/(m + 1/2) \), with Bohr radius \( a \equiv \hbar v_F/g_s \). For \( m < 0 \) (and \( r' < 0 \), Eq. (9) corresponds to a repulsive Coulomb potential, there is no bound states and the continuous energy spectrum satisfies \( E' \geq 0 \), namely \( |E| \geq g_s \).

The branch of states with \( n_r = 0 \) are zero energy states and fully spin-polarized as shown in their wave functions \( \psi = (e^{im\phi} f_{0,m}(r'), 0)^T \). They form a flat band resembling the spin polarized zeroth Landau level of Dirac fermions in a uniform magnetic field. However, the magnetic translation symmetry is absent here since \( \mathbf{\nabla} \times \mathbf{A}_1 \) scales as \( 1/r \). Consequently, the density of states of the zero energy states scales as \( 1/r \) as away from the center consistent with the classic radius \( r_{max} = (m + 1/2)a \). This situation is similar to the Landau level formation of a 2D Rashba system subject to a harmonic potential [31, 32].

The analytic solutions of energy spectrum is further confirmed by the numerical calculation in Fig. 1(c). Here we choose a disc geometry, and the radial wave function is expanded in terms of the Bessel function [33], the expanding order and disc radius \( R \) is large enough to ensure convergence and capture the low energy states [34].

Curling vortex.- Next we consider a curling magnetic vortex shown in Fig. 1(b)

\[
\mathbf{n}_2(r) = \hat{e}_\phi = \frac{1}{r}(y\hat{e}_x - x\hat{e}_y).
\]
Then $A_2(r) = (g_s/v_F)(x/r, y/r)$, with a zero emergent magnetic field $\nabla \times A_2 = 0$ and $\nabla \cdot A_2 = (g_s/v_F)/(1/r)$. Interestingly, the curling texture can be gauged away, i.e. $A_2 \to A_2 + (g/v_F)\nabla r$ and $\psi \to \psi \exp(i gr/v_F)$. There are no bound state but only continuous state solutions [34]. The energy spectrum is numerically calculated in Fig. 1(d).

Majorana bound state. It is well known that the localized Majorana zero modes (MZM) arise in the Abrikosov vortex core when the Dirac fermion surface states of a TI is proximity coupled to an s-wave superconductor [35–40]. Now we study the fate of MZM in the presence of magnetic texture on TI surface.

Now the pairing term $V = \Delta(r)\psi_+^\dagger \psi_+^\dagger + \text{H.c.}$ is added to $H$, where $\Delta(r)$ is the pairing potential. The system can be diagonalized with the Bogoliubov transformation

$$
\psi_{s}(r) = \sum_{n} \left[ u_{n,\sigma}(r) \gamma_n + v_{n,\sigma}(r) \gamma_n^\dagger \right],
$$

where $\gamma_n^\dagger$ create a Bogoliubov quasiparticle. Then the resulting Bogoliubov-de Gennes (BdG) equation is

$$
\begin{align*}
\left( \frac{\mathcal{H}}{\Delta(r)} - \sigma_g \mathcal{H}^* \sigma_g + \mu \right) \Phi_n(r) &= E_n \Phi_n(r),
\end{align*}
$$

with the BdG energy spectrum $E_n$ and Nambu wave function $\Phi_n(r) = (u_{n\uparrow}(r), u_{n\downarrow}(r), v_{n\downarrow}(r), -v_{n\uparrow}(r))^T$.

The BdG equation can be solved numerically for a general pairing potential. Here we study the Abrikosov vortex, where the magnetic flux is contributed by the radial magnetic vortex. Thus the origin of vortex core coincides with the center of magnetic texture as in Fig. 3(a), then Eq. (13) has a cylindrical symmetry. In the polar coordinate, the pairing potential with vortex is $\Delta(r) = \Delta_0(r)e^{i\theta} = \Delta_0 \tanh(\ell_0 r) e^{i\theta}$, where $\ell_0$ characterizes the size of vortex core. The wave function can be factorized into

$$
\Phi_{n, \ell} = e^{i\theta} \left( u_{n, \ell \uparrow}, u_{n, \ell+1 \downarrow} e^{i\theta}, v_{n, \ell \downarrow}, -v_{n, \ell+1 \uparrow} e^{-i\theta} \right)^T,
$$

where the principle quantum number $n$ is determined by solving the radial equation in the basis of Bessel function. The details of numerical calculations are given in Supplemental Material [34]. The calculation is performed on disc of radius $\mathcal{R} = 4\ell_0$.

Fig. 2 shows the numerical results of BdG energy spectrum for a radial vortex. The MZM is denoted as red dot. As shown in Fig. 2(a,b) for $\mu = 0$, the existence of MZM depends only on the relative value of magnetic coupling $g_s$ and pairing strength $\Delta_0$, namely MZM exists for $g_s < \Delta_0$ and disappears for $g_s > \Delta_0$. The zero energy states in Fig. 2(b) are not MZM, but are originated from the zero energy bound states in Fig. 1(c), which deviate from zero by adding a small Zeeman term $\Delta_z \sigma_z$ into $\mathcal{H}$. These results can be simply understood that for $\mu = 0$, the chemical potential resides in the bound state spectrum. Only when the pairing potential exceeds the bound state energy and reaches the continuous state, then MZM from superconducting Dirac fermion proposed by Fu-Kane applies [35]. This is further confirmed in Fig. 2(c,d), where a finite $\mu > g_s$ crosses the continuous states, MZM always exists in the vortex core. However, $\mu > g_s$ case, the system is gapless for finite $\ell$ when $g_s > \Delta_0$ shown in Fig. 2(d). We further calculate the vortex case $\Delta_0(r)e^{i\theta}$ and $g_s < 0$ [34], where MZM exists $|g_s| < |\Delta_0|$ and disappears $|g_s| > |\Delta_0|$. The subtle difference between positive and negative $g_s$ in the vortex case can be seen from the analytic solution. In the limit of $\ell_0 \to 0$, the vortex core (where $\Delta_0$ vanishes) can be taken to have negligible size and the boundary condition at $r \to 0$ is unimportant, the analytic zero energy MZM solution is

$$
\gamma_{0_{-\uparrow}}^\dagger = (1, 0, 0, 1)^T \exp \left( -\int_0^r \frac{dr'}{v_F} \Delta_0(r') + g_s \right).
$$

The solution is unphysical at $r \to \infty$ when $g_s < -\Delta_0 < 0$. For an anti-vortex $\Delta_0(r)e^{-i\theta}$, the MZM solution is

$$
\gamma_{0_{-\downarrow}}^\dagger = (0, 1, -1, 0)^T \exp \left( -\int_0^r \frac{dr'}{v_F} \Delta_0(r') - g_s \right).
$$

This represents MZM is from the consistent phase winding between $\Delta(r)$ and magnetic texture $A_1$.

Now we understand that the radial magnetic texture effectively act like an Abrikosov vortex, where the Abrikosov vortex core is at the center of magnetic texture. We put the Dirac surface state on a sphere, and assume magnetic vortex ($g_s > 0$) and anti-vortex ($g_s < 0$) pairs are located at the north and south poles, respectively. In Fig. 3(a), when the flux line of the effective Abrikosov vortex penetrates the two poles, the pairing
a vortex with spiralling magnetization configuration could have a singularity at the origin, while in ferromagnetism, there is gapless with a vortex-free pairing potential. The adiabatical continuity fails when the BdG spectrum for the existence of MZMs in the vortex core. However, the pairing potential is \( \Delta_0(r) e^{i\theta} \) and \( \Delta_0(r) e^{-i\theta} \) locally at the north and south poles, respectively. The solutions for MZM at north and south poles are exactly the same, for they are the time-reversal partner of each other. Now in the case of weak exchange interaction \( g_s \), where the radial magnetic texture could not contribute to a flux quantum. We need add external magnetic field to generate Abrikosov vortex. If the external generated Abrikosov vortex core coincides with the center of the magnetic texture, the MZM locates at the Abrikosov vortex. If the external generated Abrikosov vortex core coincides with the center of the magnetic texture, the MZM, if exist, can only be localized at the Abrikosov vortex center. The vortex center coincides with the north pole \((\xi_0, \phi_0, g_s) \approx (5 \sim 10 \text{ nm})\) as long as \( a > \xi_0 \), the above study applies to the coupling between Dirac electron and such a realistic magnetic vortex. Here \( a \) is the size of the magnetic vortex within which a flux quantum is obtained. The bound state located at \( r_{\text{max}} \) may be probed by the scanning tunneling microscopy. For an estimation, take \( v_F \approx 5 \times 10^5 \text{ m/s} \) in Bi\(_2\)Te\(_3\), then \( a \approx 30 \text{ nm} \) when \( g_s = 11 \text{ meV} \).

We propose the artificial two bilayer ferromagnets heterostructure to create magnetic radial vortex, where the dipolar energy is minimized by an antiferromagnetic coupling between them and the radial vortex is obtained by the DMI gradient along the radial direction, namely the DMI parameter monotonically decreases as \( r \) increases.

The incorporation of the magnetic proximity effect into a TI has been exemplified in heterostructures with magnetic insulators [45–47]. Recently a van der Waals magnetic TI MnBi\(_2\)Te\(_4\) and its descendents have been discovered [48–53], which is compatible with the Bi\(_2\)Te\(_3\) family materials. Such experimental progress on the material growth and rich material choice of TI and magnetic insulators makes it possible to realize the magnetic vortex in TI heterostructures.

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SUPPLEMENTARY MATERIAL FOR "PLANAR MAGNETIC TEXTURE ON THE SURFACE OF A TOPOLOGICAL INSULATOR"

S1. NUMERICAL METHOD TO DIRAC FERMION COUPLED TO MAGNETIC TEXTURES

In this section, we provide the detail of numerical calculation method for Dirac fermion with Hamiltonian given by:

\[ H_0 = v_F ((k - \mathbf{A}) \times \sigma) \cdot \mathbf{e}_z \]
\[ = v_F(k_x - A_x)\sigma_y - v_f(k_y - A_y)\sigma_x \]
\[ = \begin{pmatrix} 0 & -v_F(\partial_x - i\partial_y) + v_F A(r, \phi) \\ v_F(\partial_x + i\partial_y) + v_F A^*(r) & 0 \end{pmatrix} \]
\[ (15) \]

with \( A(r, \phi) = (g_s/v_F)(-iy/r + x/r) = (g_s/v_F)e^{-i\phi} \) for radial vortex and \( A(r, \phi) = (g_s/v_F)(ix/r + y/r) = i(g_s/v_F)e^{-i\phi} \) for curling vortex. We numerically solve this equation in a disk geometry. Cause \( \partial_x \pm i\partial_y = e^{\pm i\phi}(\partial_r \pm i\partial_\phi/r) \), the wave function can be factorized as

\[ \psi_{n,l} = e^{i\phi}(u_{n,l,\uparrow}, u_{n,l+1,\downarrow}e^{i\phi})^T \]
\[ (16) \]

and the radial wave function can be expanded in terms of Bessel function of the first kind

\[ u_{n,l} = \sum_{j=1}^{N} c_{n,j,l} \frac{\sqrt{2}}{R J_{l+1}(\beta_{j,l})} J_l(\beta_{j,l} R) \equiv \sum_{j=1}^{N} c_{n,j,l} \phi_{j,l} \]
\[ (17) \]

where \( J_l \) is m-th order Bessel function of the first kind, \( \beta_{j,l} \) is its j-th zeros, \( N \) is cutoff for the expansion and \( R \) is the radius of our disk. Then the Hamiltonian is reduced to \( 2N \times 2N \) matrix as

\[ H_{BdG} = \begin{pmatrix} 0 & V_{\mu,\nu+1} + A_{\mu,\nu+1} \\ V_{\mu,\nu+1} + A_{\mu,\nu+1}^T & 0 \end{pmatrix} \]
\[ (18) \]

with

\[ (V_{l,l'})_{i,j} = \frac{2v_F}{R} \frac{\beta_{l,l'}\beta_{l,l'}}{\beta_{l,l'}^2 - \beta_{l,l'}^2} (A_{l,l'})_{i,j} = \int_0^R rdr A(r)\phi_{i,l}\phi_{j,l'} \]
\[ (19) \]

In the following calculations we choose \( v_F = 10, R = 400, N = 400 \) which is enough to ensure the convergence of the low energy spectrum.

S2. NUMERICAL METHOD TO DIRAC FERMION BDG HAMILTONIAN

In presence of pairing vortex \( \Delta(r) = \Delta_0 \tanh(r/\xi_0) e^{i\phi} \), the wave function is given by

\[ \Phi_{n,l} = e^{i\phi}(u_{n,l,\uparrow}, u_{n,l+1,\downarrow}e^{i\phi}, v_{n,l-1,\downarrow}e^{-i\phi}, -v_{n,l,l})^T \]
\[ (20) \]

Following the same method, we expand the radial part by Bessel functions, then the BdG matrix is

\[ H_{BdG} = \begin{pmatrix} \Delta_x - \mu & V_{l,l+1} + A_{l,l+1} & \Delta_{l,l-1}^* & 0 \\ V_{l,l+1}^* + A_{l,l+1}^T & -\Delta_x - \mu & 0 & \Delta_{l-1,l}^* \\ \Delta_{l-1,l} & 0 & \Delta_x + \mu & -V_{l-1,l} + A_{l-1,l}^T \\ 0 & \Delta_{l+1,l}^* & -V_{l+1,l}^* + A_{l+1,l} & -\Delta_x + \mu \end{pmatrix} \]
\[ (21) \]

with \( \mu \) as the Chemical potential and \( \Delta_x \) as the z direction Zeeman field.

As we can see from Fig. 4, the zero mode doesn’t split with a nonzero Zeeman field when \( g_s < \Delta_0 \). But zero modes in Fig. 4(b) deviate from zero which indicates they are originated from bound states in Fig. 1(c) of the main text.
FIG. 4. The BdG spectrum of the Abrikosov vortex $\Delta_0(r)e^{i\phi}$ in presence of the z-direction Zeeman field for the radial vortex and $g_s=0.5/2$ for (a,b). Other parameters are $v_F = 10$, $\Delta_0 = 1$ and $\Delta_z = 0.1$.

Next, lets consider the the case with $g_s < 0$. The phenomena is similar with $g_s > 0$ as the majorana zero modes exists only when $|g_s| > |\Delta_0|$. And nonzero chemical potential would make the system gapless when $|g_s| > |\Delta_0|$.

S3. VORTEX FREE BDG SPECTRUM

Finally, we present the BdG spectrum for vortex free superconductor pairing. In this case, the superconductor pairing is $\Delta(r) = \Delta_0 \tanh(r/\epsilon_0)$. Due to absence of winding number, the wave should factorized as

$$\Phi_{n,l} = e^{il\phi}(u_{n,l,\uparrow}, u_{n,l+1,\uparrow}e^{i\phi}, v_{n,l,\downarrow}e^{-i\phi}, -v_{n,l+1,\downarrow})^T$$

such that the angular momentum is $j = l + 1/2$ which implies the absence of the majorana zero mode. Fig. 6 shows the BdG spectrums for radial. Similar to the Abrikosov vortex case, the system is always gapped when $g_s < \Delta_0$.

FIG. 5. The BdG spectrum of the Abrikosov vortex $\Delta_0(r)e^{i\phi}$ with $g_s < 0$ for the radial vortex. (a) $g_s = -0.5, \mu = 0$. (b) $g_s = -2, \mu = 0$. (c) $g_s = -0.5, \mu = 5$. (d) $g_s = -1.5, \mu = 5$. Other parameters are $v_F = 10, \Delta_0 = 1$.

FIG. 6. The BdG spectrum of the Abrikosov vortex $\Delta_0(r)$ for the radial vortex. (a) $g_s = 0.5, \mu = 0$. (b) $g_s = 2, \mu = 0$. (c) $g_s = 0.5, \mu = 5$. (d) $g_s = 1.5, \mu = 5$. Other parameters are $v_F = 10, \Delta_0 = 1$. 