Knowledge excess duality and violation of Bell inequalities

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(Dated: October 30, 2018)

A constraint on two complementary knowledge excesses by maximal violation of Bell inequalities for a single copy of any mixed state of two qubits $S, M$ is analyzed. The complementary knowledge excesses $\Delta K(\Pi_M \rightarrow \Pi_S)$ and $\Delta K(\Pi'_M \rightarrow \Pi'_S)$ quantify an enhancement of ability to predict results of the complementary projective measurements $\Pi_S, \Pi'_S$ on the qubit $S$ from the projective measurements $\Pi_M, \Pi'_M$ performed on the qubit $M$. For any state $\rho_{SM}$ and for arbitrary $\Pi_S, \Pi'_S$ and $\Pi_M, \Pi'_M$, the knowledge excesses satisfy the following inequality $\Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi'_M \rightarrow \Pi'_S) \leq (B_{\text{max}}/2)^2$, where $B_{\text{max}}$ is maximum of violation of Bell inequalities under single-copy local operations (local filtering and unitary transformations). Particularly, for the Bell-diagonal states only an appropriate choice of the measurements $\Pi_S, \Pi'_S$ and $\Pi_M, \Pi'_M$ are sufficient to saturate the inequality.

PACS numbers: 03.65.Ud

I. INTRODUCTION

Immediately after the discovery of quantum mechanics, it was realized that it contains an interesting feature in quantum correlations between two particles. It was first discussed in seminal paper of Einstein, Podolsky and Rosen (EPR) for the coordinate and momentum of a pair of massive particles and after a time reformulated for spin-entangled systems. Assuming a pair of maximally entangled spin-1/2 particles we can perfectly predict the results of the complementary measurements on one particle from an appropriate measurements of the other one. An ability of the precise prediction of complementary variables arises from quantum nature of the correlations between the particles. From a fundamental point of view it was proved that for such the particles the measurements of correlated quantities should yield a different result in the quantum mechanical case to those expected in local realism. A condition derived in a form of Bell inequality has to be satisfied within the local realism. The predictions of quantum mechanics were satisfactorily experimentally proved using pairs of photons entangled in the polarization. From a practical point of view, such the entangled particles distributed at a distance can be used to securely distribute classical information.

An experiment demonstrating the interesting attribute of the correlations between quantum systems can be build up assuming a generally mixed state $\rho_{SM}$ of qubit $S$ and meter qubit $M$. This experiment is schematically depicted in Fig. 1. Performing two projective (ideal) measurements $\Pi_M, \Pi'_M$ on the qubit $M$, the prediction of the results of the complementary measurements $\Pi_S, \Pi'_S$ on the qubit $S$ can be improved. The complementarity of the measurements means that $\text{Tr}[\Pi_S \Pi'_S] = 1/2$. For example, having maximally entangled state $|\Psi^-(SM) = (|VH\rangle - |HV\rangle)/\sqrt{2}$, the results of arbitrary measurement $\Pi$ on the qubit $S$ can be precisely predicted performing the same measurement $\Pi$ on the qubit $M$. However, without this measurement we have vanishing knowledge since the state of $S$ is maximally random. The total ability to predict the result of measurement was quantified by a concept of knowledge defined in [6]. Generally, there are such states for which both complementary knowledge $K(\Pi_M \rightarrow \Pi_S)$ and $K(\Pi'_M \rightarrow \Pi'_S)$ obtained from the measurements $\Pi_M, \Pi'_M$ are larger than these $P(\Pi_S)$ and $P(\Pi'_S)$ without the measurements. Then corresponding knowledge excesses $\Delta K(\Pi_M \rightarrow \Pi_S) = K(\Pi_M \rightarrow \Pi_S) - P(\Pi_S)$ and $\Delta K(\Pi'_M \rightarrow \Pi'_S) = K(\Pi'_M \rightarrow \Pi'_S) - P(\Pi'_S)$ can be introduced, respectively for the complementary measurements. A duality between the knowledge excesses for any mixed state $\rho_{SM}$ can be derived, analogically as in Ref. [6]. For $\Pi_M = \Pi'_M$, the knowledge excesses satisfy the inequality $\Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi'_M \rightarrow \Pi'_S) \leq 1$. Thus performing a single measurement on $M$, the sum of the squares of knowledge excesses cannot be larger than unity.

In this paper, we analyze a duality between knowledge excesses beyond the condition $\Pi_M = \Pi'_S$ and derive the following restriction of the complementary knowledge excesses $\Delta K(\Pi_M \rightarrow \Pi_S)$ and $\Delta K(\Pi'_M \rightarrow \Pi'_S)$, namely

$$\Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi'_M \rightarrow \Pi'_S) \leq \left[\frac{B_{\text{max}}}{2}\right]^2,$$

where $B_{\text{max}}$ is maximal violation of Bell inequalities after optimal local filtering operations on a single copy of the state $\rho_{SM}$. Thus $B_{\text{max}}$ restricts the ability to enhance both the knowledge excesses by the different measurements on $M$. To overcome the unit value of sum of squares of the knowledge excesses we need a state violating the Bell inequality. For any state having vanishing the a priori knowledge in any basis (for example, Bell-diagonal state) we achieve the equality only by choosing appropriate measurements $\Pi_S, \Pi'_S$ and $\Pi_M, \Pi'_M$. Such the state, exhibiting the maximal accessible $B_{\text{max}}$ under
local filtering on a single copy and unitary operations, can be probabilistically but uniquely obtained from an arbitrary two-qubit state using local single-copy filtrations. Thus the inequality (1) with \( B_{\text{max}} \) can be always saturated assuming an appropriate local filtration before the suitable measurements \( \Pi_S, \Pi'_S \) and \( \Pi_M, \Pi'_M \). We analyze simple and experimentally feasible examples covering such the cases in which noise prevents an maximal extraction of knowledge.

II. KNOWLEDGE EXCESS DUALITY

In this Section we define the complementary knowledge excesses \( \Delta K(\Pi_M \rightarrow \Pi_S) \) and \( \Delta K(\Pi'_M \rightarrow \Pi'_S) \) and discuss a duality between them which arises from a single measurement on \( M \). Assume two-component projective measurement \( \Pi_S = \{ |\Psi\rangle_S, |\Psi\perp\rangle_S |\Psi\rangle_S \} \) giving results either 0 or 1, respectively. We expand the state \( \rho_{SM} \) as

\[
\rho_{SM} = w|\Psi\rangle_S\langle\Psi| + w^\perp|\Psi\perp\rangle_S\langle\Psi\perp| + \sqrt{ww^\perp} |\psi\rangle_S\langle\psi|_{\chi_M + \text{h.c.}},
\]

where \( 0 \leq w, w^\perp \leq 1 \), \( w + w^\perp = 1 \) and the meter operators \( \rho_M, \rho_M^\perp, \chi_M \) depend on a choice of the measurement \( \Pi_S \). To predict a result of \( \Pi_S \) on \( S \) we can unambiguously discriminate the mixed states \( \rho_M, \rho_M^\perp \) by a projective two-component measurement \( \Pi_M = \{ \Pi_{M0}, \Pi_{M1} \} \) \((\Pi_{M0} + \Pi_{M1} = 1, \Pi_{M0}\Pi_{M1} = 0)\) on the qubit \( M \). After projection \( \Pi_{M_i}, \) the local state of the qubit \( S \) collapses to

\[
\rho_{Si} = \frac{1}{\pi_i} \left[ |\psi\rangle_S\langle\psi|_{\rho_i} + w^\perp|\psi\perp\rangle_S\langle\psi\perp|_{\rho_i} + \sqrt{ww^\perp} |\psi\rangle_S\langle\psi|_{c^i + \text{h.c.}} \right],
\]

where \( \pi_i = \text{Tr}\rho_{SM} = w_{pi} + w^\perp_{pi} \) is the probability of the projection. Thus after the meter measurement we obtain two sub-ensembles of the states \( \rho_{S0} \) and \( \rho_{S1} \) weighted with probabilities \( \pi_0 \) and \( \pi_1 \). We denote \( w_i = w_{pi} / \pi_i \) and \( w_i^\perp = w^\perp_{pi} / \pi_i \).

Adopting maximum likelihood estimation strategy, we guess for each event that the measurement \( \Pi_S \) gave the most likely results either 0 or 1. Our strategy is maximize the likelihood function \( L = \text{Max}\{w_0, w_1\} \). The knowledge in a binary decision problem is the fractional excess of right guesses over wrong guesses in many experiments repeated under identical conditions. If we have 70% of right guesses and 30% of wrong guesses than our knowledge is 0.4. In our task a priori knowledge without the measurement \( \Pi_M \) is \( \mathbf{P}(\Pi_S) = |w - w^\perp| \). A sub-ensemble knowledge after the particular projection \( \Pi_{M_i} \) is \( \mathbf{P}(\Pi_{M_i} \rightarrow \Pi_S) = |w_i - w_i^\perp| \). Then after the meter measurement, an amount of the knowledge \( K(\Pi_M \rightarrow \Pi_S) \) is the \( \pi_i \)-weighted sum of sub-ensembles knowledge

\[
K(\Pi_M \rightarrow \Pi_S) = \sum_i \pi_i \mathbf{P}(\Pi_{M_i} \rightarrow \Pi_S) = \sum_i \pi_i |w_i - w_i^\perp|,
\]

where \( \mathbf{P}(\Pi_S) \leq K(\Pi_M \rightarrow \Pi_S) \leq 1 \). The knowledge excess \( \Delta K(\Pi_M \rightarrow \Pi_S) \) is that amount of knowledge which exceeds the apriori knowledge \( \mathbf{P}(\Pi_S) = |w - w^\perp| \), explicitly

\[
\Delta K(\Pi_M \rightarrow \Pi_S) = K(\Pi_M \rightarrow \Pi_S) - \mathbf{P}(\Pi_S),
\]

where \( 0 \leq \Delta K(\Pi_M \rightarrow \Pi_S) \leq 1 \). The knowledge excess quantifies only a part of knowledge which is gained from the measurement on \( M \). If we are not able to extract any extra knowledge from the measurement on \( M \) then the knowledge excess is vanishing. Using the expansion (2) we have

\[
\Delta K(\Pi_M \rightarrow \Pi_S) = \sum_i |\text{Tr}_M \rho_i (w_{\rho_M} - w_{\rho_M}^\perp)| - |w - w^\perp|.
\]

The largest \( \Delta K(\Pi_M \rightarrow \Pi_S) \) over all \( \Pi_M \) is the distinguishibility excess \( \Delta D(\Pi_S) \)

\[
\Delta D(\Pi_S) = \text{Max}_{\Pi_M} \Delta K(\Pi_M \rightarrow \Pi_S) = |\text{Tr}_M (w_{\rho_M} - w_{\rho_M}^\perp)| - |w - w^\perp|,
\]

where \( A = |A^\dagger A| \), and thus \( 0 \leq \Delta K(\Pi_M \rightarrow \Pi_S) \leq \Delta D(\Pi_S) \). The analogous quantities \( \Delta K(\Pi'_M \rightarrow \Pi'_S) \) and \( \Delta D(\Pi'_S) \) can be defined for the complementary measurement \( \Pi'_S \).

Now we shortly prove a relation between the knowledge excesses for \( \Pi_M = \Pi'_M \). The derivation is inspired by a similar one in Ref. \( 2 \). The sub-ensemble knowledge about prediction of the complementary projection along the state \( |\Psi\rangle_S = (|\Psi\rangle_S + e^{i\theta} |\Psi\perp\rangle_S) / \sqrt{2} \), after a particular projection \( \Pi_{M_i} \), is \( \mathbf{P}(\Pi_{M_i} \rightarrow \Pi_S) = 2 \sqrt{w_i w_i^\perp} \text{Re} |c_i| \text{exp}(i\theta) \leq 2 \sqrt{w_i w_i^\perp} |c_i| \). Since \( |c_i|^2 \leq 1 \) we have

\[
\mathbf{P}^2(\Pi_{M_i} \rightarrow \Pi_S) + \mathbf{P}^2(\Pi_{M_j} \rightarrow \Pi_S) \leq 1
\]

and using Schwarz inequality \( \sum_k a_k b_k^2 \leq \sum_k a_k^2 \sum_l b_l^2 \), we obtain

\[
\mathbf{P}(\Pi_{M_i} \rightarrow \Pi_S) \mathbf{P}(\Pi_{M_j} \rightarrow \Pi_S) + \mathbf{P}(\Pi_{M_i} \rightarrow \Pi'_S) \mathbf{P}(\Pi_{M_j} \rightarrow \Pi'_S) \leq 1.
\]
Then using (4) and (9) we can straightforwardly derive that for \( \Pi_M' = \Pi_M \) the knowledge excesses satisfy
\[
\Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi_M \rightarrow \Pi'_S) \leq 1.
\]
Using the same measurement on \( M \), the sum of squares of the knowledge excesses can never overcome unity. It is a duality between the knowledge excesses from a single meter measurement.

III. KNOWLEDGE EXCESSES AND BELL-INEQUALITY VIOLATION

Accomplishing generally different \( \Pi_M', \Pi_M \), the sum of squares of the knowledge excesses can be larger than unity if the state violates the Bell inequalities. Let us discuss in this case a limitation of the knowledge excesses for any mixed state of the two-qubit system
\[
\rho_{SM} = \frac{1}{4}(1 \otimes 1 + 1 \otimes \sum_{i=1}^{3} m_i \sigma_i + \sum_{i=1}^{3} n_i \sigma_i \otimes 1 + \sum_{k,l=1}^{3} t_{kl} \sigma_k \otimes \sigma_l),
\]
where \( 1 \) stands for the identity operator, \( m_i, n_i \) are vectors in \( R^3 \), \( \sigma_i \) are the standard Pauli operators and \( |H \rangle_S \) are the eigenstates of \( \sigma_3 \). The coefficients \( t_{kl} \) form a real correlation matrix \( T \) and vectors \( m_i \) and \( n_i \) determine the local states \( \rho_S = \frac{1}{2}(1 + m_i \sigma_i) \), \( \rho_M = \frac{1}{2}(1 + n_i \sigma_i) \). We assume a subset of states \( \tilde{\rho}_{SM} \) with the diagonal correlation tensor \( \tilde{T} = \text{diag}(\tilde{t}_{33}, \tilde{t}_{11}, \tilde{t}_{22}) \), \( \tilde{t}_{33} \geq \tilde{t}_{11}, \tilde{t}_{22} \) and with vectors \( \tilde{m}_i \) and \( \tilde{n}_i \). Any mixed state \( \rho_{SM} \) can be uniquely converted to some \( \tilde{\rho}_{SM} \) using appropriate local unitary operations. Further we have the following ordering of the diagonal elements \( \tilde{t}_{11} \geq \tilde{t}_{22} \) or \( \tilde{t}_{11} \leq \tilde{t}_{22} \). Since the prove for \( \tilde{t}_{11} \leq \tilde{t}_{22} \) is only analogical we will shortly discuss afterward.

According to the strongest correlation \( |\tilde{t}_{33}| \), one measurement on \( S \) is naturally chosen as \( \Pi_S = \{ |V \rangle_S \langle V|, |H \rangle_S \langle H| \} \) and from the expansion \( |H \rangle_S \) we obtain using (4) we have \( \tilde{P} = [n_3] \) and either \( \Delta \tilde{D} = 0 \) or \( \Delta \tilde{D} = |\tilde{t}_{33}| - |\tilde{n}_3| > 0. \) And according to the second strongest correlation \( |\tilde{t}_{11}| \), the complementary measurement is chosen as \( \Pi'_S = \{ |+\rangle_S \langle +|, |-\rangle_S \langle -| \} \) and then we analogically have \( \tilde{P}' = |n_1| \) and either \( \Delta \tilde{D}' = 0 \) or \( \Delta \tilde{D}' = |\tilde{t}_{11}| - |\tilde{n}_1| > 0. \) We can also express a violation of Bell inequalities in a simple way for such states with a diagonal \( T \). Adopting criterion in Ref. (4,5), a state \( \rho_{SM} \) violates a Bell inequality if its maximal Bell factor is
\[
2 < B_{\text{max}} = 2 \sqrt{\tilde{t}_{11}^2 + \tilde{t}_{33}^2} \leq 2 \sqrt{2}
\]
which factor \( B_{\text{max}} \in (0, 2\sqrt{2}) \) is invariant under local unitary transformations \( U_S \otimes U_M \) of the state. Similarly, we can derive analogetical result for the case when \( \tilde{t}_{11} \leq \tilde{t}_{22} \), either we have \( \Delta \tilde{D} = 0 \) or \( \Delta \tilde{D} = |\tilde{t}_{33}| - |\tilde{n}_3| > 0 \) and for the complementary measurement, either \( \Delta \tilde{D}' = 0 \) or \( \Delta \tilde{D}' = |\tilde{t}_{22}| - |\tilde{n}_2| > 0 \) and maximum of the Bell factor is \( B_{\text{max}} = 2 \sqrt{\tilde{t}_{22}^2 + \tilde{t}_{33}^2} \). Thus we can generally obtain that
\[
\Delta \tilde{D}^2 + \Delta \tilde{D}'^2 \leq \left( \frac{B_{\text{max}}}{2} \right)^2 \tag{13}
\]
for arbitrary state \( \rho_{SM} \) with diagonal \( T \), where \( \Delta \tilde{D}, \Delta \tilde{D}' \) corresponds to the two largest correlations. The equality is obtained for the states having vanishing prior knowledge in any basis.

Now we generalize the discussion for any state \( \rho_{SM} \) assuming arbitrary measurements \( \Pi_S, \Pi'_S, \Pi_M, \Pi'_M \). Any mixed two-qubit state can be uniquely prepared from the set of states \( \tilde{\rho}_{SM} \) by appropriate local unitary transformations \( U_S, U_M \) on the qubits \( S \) and \( M \). Further, the transformation of the measurements \( \Pi_S \) and \( \Pi'_S \) to the arbitrary but still complementary ones effectively corresponds the extra local unitary transformation \( U(\Pi_S) \) of the state qubit \( S \). Since the distinguishabilities \( \Delta \tilde{D}(\Pi_S), \Delta \tilde{D}(\Pi'_S) \) are generally invariant under any local unitary transformation on the meter \( M \), it is sufficient to implement a joint unitary transformation \( U \otimes U_M \) transformation its correlation matrix transforms as follows (10)
\[
T = O_S T O_M^\dagger.
\]
Thus a joint unitary transformation \( U \otimes U_M \) can be represented as a transformation of the correlation tensor \( T = O(\alpha, \beta, \gamma)T \), where \( O(\alpha, \beta, \gamma) \) is the matrix of rotation in \( R^3 \) space
\[
\begin{pmatrix}
o_{11} & o_{12} & o_{13} 
o_{21} & o_{22} & o_{23} 
o_{31} & o_{32} & o_{33}
\end{pmatrix}
\]
(15)
with the elements
\[
\begin{align*}
o_{11} &= \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma, 
o_{12} &= - \cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma, 
o_{13} &= \cos \alpha \sin \beta, 
o_{21} &= \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma, 
o_{22} &= - \sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma, 
o_{23} &= \sin \alpha \sin \beta, 
o_{31} &= - \sin \beta \cos \gamma, 
o_{32} &= \sin \beta \sin \gamma, 
o_{33} &= \cos \beta,
\end{align*}
\]
where \( \alpha \in (0, 2\pi), \beta \in (0, \pi), \gamma \in (0, 2\pi) \).

We explicitly calculate \( \Delta \tilde{D}(\Pi_S) \) and \( \Delta \tilde{D}(\Pi'_S) \) for the state after the previous transformation. First we assume that \( \tilde{t}_{11} \geq \tilde{t}_{22} \). For the first measurement from the complementary measurements we obtain either \( \Delta \tilde{D}(\Pi_S) = 0 \)
or
\[ \Delta D (\Pi) = \sqrt{t_{11}^2 + t_{12}^2 + t_{13}^2 - |n_1|} > 0. \] (16)

For the second complementary measurement is either \( \Delta D (\Pi') = 0 \) or
\[ \Delta D (\Pi') = \sqrt{t_{11}' + t_{12}' + t_{13}' - |n_1|} > 0. \] (17)

and using the transformation \( T = O(\alpha, \beta, \gamma)^T \) we can derive that
\[ \Delta D^2 (\Pi) \leq t_{33}^2 \cos^2 \beta + \sin^2 \beta (t_{11}^2 \cos^2 \alpha + t_{22}^2 \sin^2 \alpha), \]
\[ \Delta D^2 (\Pi') \leq t_{11}' (\cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma)^2 + t_{22}' (\cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma)^2 + t_{33}' \cos^2 \alpha \sin^2 \beta. \] (18)

Consequently, assuming \( t_{33}^2 > t_{11}^2 > t_{22}^2 \) we can prove the following inequality
\[ \Delta D^2 (\Pi) + \Delta D^2 (\Pi') \leq t_{11}^2 + t_{33}' = \left( \frac{B_{\text{max}}}{2} \right)^2. \] (19)

For \( t_{11}^2 \leq t_{22}^2 \), by repeating our calculation we have for the first measurement the result \( \Delta D (\Pi) \) the second complementary measurement is either \( \Delta D (\Pi') = 0 \) or
\[ \Delta D (\Pi') = \sqrt{t_{11}' + t_{22}^2 + t_{33}' - |n_2|} > 0. \] (20)

Using \( \Delta D (\Pi) \) we derive
\[ \Delta D^2 (\Pi') \leq t_{11}' (\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma)^2 + t_{22}' (-\sin \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma)^2 + t_{33}' \sin^2 \alpha \sin^2 \beta \] (21)

and subsequently,
\[ \Delta D^2 (\Pi) + \Delta D^2 (\Pi') \leq t_{22}^2 + t_{33}' = \left( \frac{B_{\text{max}}}{2} \right)^2. \] (22)

Finally, since \( \Delta K (\Pi_M \rightarrow \Pi_S) \leq \Delta D (\Pi_S) \) and \( \Delta K (\Pi_M' \rightarrow \Pi'_S) \leq \Delta D (\Pi'_S) \) we prove that
\[ \Delta K^2 (\Pi_M \rightarrow \Pi_S) + \Delta K^2 (\Pi_M' \rightarrow \Pi'_S) \leq \left( \frac{B_{\text{max}}}{2} \right)^2 \] (23)

is generally satisfied. Thus the maximum of Bell factor represents an important bound on the squares of the excess of knowledge which can be extracted from the meter measurements. For class of the states with vanishing priori knowledge for any measurements \( \Pi_S, \Pi'_S, \Pi_M, \Pi'_M \), i.e. \( n_1 = n_2 = n_3 = 0 \), this inequality can be saturated only by an appropriate choice of the measurements \( \Pi_S, \Pi'_S, \Pi_M, \Pi'_M \). For a mixture of Bell states \( \Psi \) we can find that \( \Delta D = |p_1 - p_2 + p_3 - p_4| \), \( \Delta D' = |p_1 + p_2 - p_3 - p_4| \)

and \( B_{\text{max}} = 2\sqrt{2}(p_1 - p_4)^2 + (p_2 - p_3)^2 \) which saturates the relation \( \Psi \).

It was shown that local filtering operations on single copy of the state can increase the degree of violation of Bell inequalities \( \mathcal{B} \). There is a unique local (stochastically reversible) filtering operation \( F_S \) and \( F_M \) \( (F_S^2 F_S \leq 1_S \) and \( F_M^2 F_M \leq 1_M) \) on single copy of the state
\[ \rho_{SM} = \frac{F_S \otimes F_M \rho_{SM} F_S^\dagger \otimes F_M^\dagger}{\text{Tr}(F_S \otimes F_M \rho_{SM} F_S^\dagger \otimes F_M^\dagger)} \] (24)

which transforms with some non-zero probability any two-qubit mixed state into a state which is diagonal in Bell basis
\[ \rho_{SM}' = p_1 |\Psi_-' \rangle \langle \Psi_-' + p_2 |\Phi_-' \rangle \langle \Phi_-' + p_3 |\Psi_+ \rangle \langle \Psi_+ + p_4 |\Phi_+ \rangle \langle \Phi_+ | \] (25)

having the largest \( B_{\text{max}}' \geq B_{\text{max}} \). A two-qubit mixed state can be uniquely bring to this Bell diagonal form with the maximal violation either with finite probability or asymptotically. For the Bell diagonal states, the Bell violation cannot be increased by any local filtering on a single copy. Naturally, after the local filtering the inequality \( \Psi \) is still satisfied also for the remaining state \( \rho_{SM}' \). The Bell diagonal states have both the local states maximally disordered, both the apriori knowledges vanish and we can always saturate the inequality \( \Psi \). Thus assuming that \( B_{\text{max}} \) in the inequality \( \Psi \) is then maximum under local filtering and unitary operations on a single copy of the state, the inequality is satisfied for any mixed state and can be for any mixed state saturated if we use an appropriate local filtering.

We analyze two interesting and experimentally feasible examples of the Bell-diagonal states. In both cases, we can use as a source state the maximally entangled state \( |\Psi_-\rangle_{SM} = \frac{1}{\sqrt{2}} (|V\rangle_{SM} - |H\rangle_{SM}) \) produced by the SPDC process. Evidently, the source state maximally violates Bell inequalities and has \( \Delta D = \Delta D' = 1 \). In the first example, we simultaneously perform a depolarization by (i) random flip of linear polarizations \( |V\rangle_M \leftrightarrow |H\rangle_M \) with the probability \( (1 - R_1)/2 \) and simultaneously, by (ii) phase-shift \( \pi \) between linear polarizations \( |V\rangle_M \leftrightarrow |H\rangle_M \) with the probability \( (1 - R_2)/2 \). As a result of the depolarizing procedure we prepare a mixture of Bell states
\[ \rho_1 = \frac{1 + R_2}{2} (\frac{1 + R_1}{2} |\Psi_-\rangle \langle \Psi_-| + \frac{1 - R_1}{2} |\Phi_-\rangle \langle \Phi_-|) + \frac{1 - R_2}{2} (\frac{1 + R_1}{2} |\Psi_+\rangle \langle \Psi_+| + \frac{1 - R_1}{2} |\Phi_+\rangle \langle \Phi_+|). \] (26)

For this case, the distinguishability excess is \( \Delta D = R_1 \) and vanishes only if the random polarization flip has
larger probability, whereas the complementary distinguishability excess is $\Delta D' = R_2$ and decreases only with an increasing probability of the random polarization phase-shift. The maximal value of Bell factor is $B_{\text{max}} = 2\sqrt{R_1^2 + R_2^2}$ and the inequality \[ \Delta D' \leq 1 \] is saturated. Thus we can independently control both the complementary distinguishability excesses. For $R_1 = 1$ ($R_2 = 1$) and changing $R_2$ ($R_1$) we are able to extract the maximal unit distinguishability excess $\Delta D$ ($\Delta D'$) irrespective of the complementary one $\Delta D' - \Delta D$. For $R_1 = 0$ ($R_2 = 0$) and controlling $R_2$ ($R_1$), the depolarization prevents us from extracting any knowledge excess since $\Delta D = 0$ ($\Delta D' = 0$) however we are still able to obtain the complementary knowledge excess $\Delta D' > 0$ ($\Delta D > 0$).

In the second example, we show as both $\Delta D$ and $\Delta D'$ can be gradually enhanced sharing separable state, entangled state which satisfies the Bell inequalities and consequently, sharing entangled state violating Bell inequalities. We assume a loss of both the knowledge excesses from the state $|\Psi_-\rangle_{SM}$ by an extraction of the photon in the meter beam with probability $1 - R$ and its substitution by another photon with a completely random polarization. Thus we detect the results produced by a mixture of the Bell states which is known as Werner state \[ \rho_2 = R|\Psi_-\rangle\langle\Psi_-| + \frac{1 - R}{4} 1 \otimes 1. \] In this case, a loss of the distinguishability excesses with decreasing $R$ are identical $\Delta D = \Delta D' = R$. We know that the Werner’s state is entangled only for $R > 1/3$ and violates Bell inequalities, having maximum of Bell factor $B_{\text{max}} = 2\sqrt{2}R$, only if $R > 1/\sqrt{2}$. Also in this case, the inequality \[ \Delta D \leq 1 \] can be saturated. Since we have an entangled state non-violating Bell inequalities for $1/3 \leq R \leq 1/\sqrt{2}$ and non-entangled state for $R \leq 1/3$ it means that we can observe both non-vanishing distinguishability excesses $\Delta D < 1$ and $\Delta D' < 1$ even in a case of classical correlated states.

Acknowledgments We would like to thank J. Fiurášek, L. Mišta Jr. for the fruitful discussions. The work was supported by the projects 202/03/D239 of GACR and LN00A015 and CEZ: J14/98 of the Ministry of Education of Czech Republic.

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[12] Theoretically, for a given state $\rho$ maximal value of the Bell factor $B$ can be theoretically calculated using the following formula $B_{\text{max}} = 2\sqrt{M(\rho)}$, where $M(\rho)$ is real-valued function of the density matrix $\rho$. To define $M(\rho)$, one needs a $3 \times 3$ matrix $T$ with elements $T_{ij} = \text{Tr}\{\rho(\sigma_i \otimes \sigma_j)\}$. Then, $M(\rho)$ is the sum of the two largest eigenvalues of the Hermitian matrix $T^\dagger T$. If $B_{\text{max}} > 2$ then Bell inequalities are violated and all experimental outcomes which state $\rho$ generate cannot be explained only by local realism and quantum theory must be used.

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