A photonic cluster state machine gun

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We present a method to convert certain single photon sources into devices capable of emitting large strings of photonic cluster state in a controlled and pulsed “on demand” manner. Such sources would greatly reduce the resources required to achieve linear optical quantum computation. Standard spin errors, such as dephasing, are shown to affect only 1 or 2 of the emitted photons at a time. This allows for the use of standard fault tolerance techniques, and shows that the photonic machine gun can be fired for arbitrarily long times. Using realistic parameters for current quantum dot sources, we conclude high entangled-photon emission rates are achievable, with Pauli-error rates per photon of less than 0.2%. For quantum dot sources the method has the added advantage of alleviating the problematic issues of obtaining identical photons from independent, non-identical quantum dots, and of exciton dephasing.

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The primary challenge facing optical quantum computation is that of building suitable photon sources. The majority of effort has been directed at single photon sources. Four single photons can be used in an interferometer to produce a maximally entangled Bell pair of photons \(|\uparrow\rangle\leftrightarrow|\downarrow\rangle\), and given a source of Bell pairs it is in principle possible to fuse them \([2]\) into larger so-called cluster states \([3]\). These somewhat magical quantum states can be used for performing quantum computation via the simple procedure of making individual (single-qubit) measurements on the photons involved. Recently a promising new approach has been to produce Bell pairs directly \([4,5]\) via a radiative cascade in quantum dots. However, even an ideal such source would only reduce the overall resources required for a full optical quantum computation by a small factor.

We will show that with current technology it is possible to manipulate certain single photon sources, in particular quantum dots, so as to generate a continuous stream of photons entangled in long strings of (various varieties of) 1-dimensional cluster states. Using these strings, cluster states capable of running arbitrary quantum algorithms can be very efficiently generated by fusion. We analyze all error mechanisms and show that the error rates can be very low - close to fault tolerant thresholds for quantum computing - even if the source is operated for timescales much longer than the typical decoherence times.

We begin with a highly idealized description of the proposal. Consider a source with a degenerate spin 1/2 ground state manifold. The basis \(|\uparrow\rangle,|\downarrow\rangle\) denotes the spin projection along the \(z\) axis. Furthermore, imagine that optical transitions at frequency \(\omega_0\) are possible only to a doubly degenerate excited state manifold. The excited states \(|\uparrow\rangle,|\downarrow\rangle\) have \(J_z = \pm 3/2\hbar\), thus only the (single photon) transitions \(|\uparrow\rangle\leftrightarrow|\uparrow\rangle\) and \(|\downarrow\rangle\leftrightarrow|\downarrow\rangle\) are allowed. Such transitions are well known to occur, for example, in quantum dots (QDs) which emit single photons via charged-exciton decay \([6]\). We only consider the emitted photons propagating along the \(z\) axis. Therefore, if the initial state of the source is \(|\uparrow\rangle|L_0\rangle\), an excitation to the state \(|\uparrow\rangle (|\downarrow\rangle)\) followed by radiative decay, results in the emission of a single right (left)-circularly polarized photon \(|R\rangle (|L\rangle)\) and leaves the source in the state \(|\uparrow\rangle (|\downarrow\rangle)\). Now, consider the initial state \(|\uparrow\rangle + |\downarrow\rangle\), and a coherent excitation pulse with a linear polarization along the \(x\) direction. (The exciting pulse itself need not necessarily propagate along the \(z\) direction, which is useful for separation of the coherent and emitted light). Such a pulse couples equally to both transitions. Therefore, the processes described above happen in superposition, and the emitted photon will be entangled with the electron: the joint state of both systems would be the Bell pair \(|\uparrow\rangle,|\downarrow\rangle\) + \(|\downarrow\rangle,|\uparrow\rangle\). Repeating such a procedure would produce GHZ-type entangled states, which are not useful for quantum computing, and for which disentangling the photons from the electron spin is difficult. Moreover, the GHZ state is highly vulnerable to decoherence. By contrast, the cluster states suffer none of these problems.

To see how to create cluster states, we now imagine that before the second excitation of the system, when the state of the spin and the first photon is \(|\uparrow\rangle|R_1\rangle + |\downarrow\rangle|L_1\rangle\), the spin undergoes a \(\pi/2\)-rotation about the \(y\)-axis. Under this operation, described by \(\exp(-i\gamma \pi/4)\) the state evolves to \(|\uparrow\rangle + |\downarrow\rangle|R_1\rangle + (-|\uparrow\rangle + |\downarrow\rangle)|L_1\rangle\). A second pulse excitation, accompanied by a second photon emission, will now result in the two photons and the electron spin being in the state \(|\uparrow\rangle|R_2\rangle + |\downarrow\rangle|L_2\rangle + (-|\uparrow\rangle|R_2\rangle + |\downarrow\rangle|L_2\rangle)|L_1\rangle\). In terms of abstract (logical) qubit encodings we will take \(|R\rangle \equiv |0\rangle,|L\rangle \equiv -|1\rangle\). It can be readily verified that rotating the spin with another \(\pi/2\) rotation, now leaves the spin and two photons in the state: \(|000\rangle + |011\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle\), which is exactly the 3 qubit linear cluster state. Repeating the process of excitation followed by \(\pi/2\) rotation,
will produce a third photon such that the electron and three photons are in a 4-qubit linear cluster state. The procedure can, in principle, be repeated indefinitely, producing a continuous chain of photons in an entangled linear cluster state. Note that one advantage of producing a cluster state is that the electron can be readily disentangled from the string of entangled photons, for example by making a computational \((|R\rangle, |L\rangle)\) basis measurement on the most recently created photon. In fact, since in general the initial state of the spin will be mixed, such a detection of a photon in state \(|R\rangle\) (i.e. \(|R\rangle\)) if the electron spin is in state \(|0\rangle\) (i.e. \(|0\rangle\)), but otherwise flips it. Crucially, as depicted, a Pauli \(Y\) error on the spin localizes; i.e it is equivalent to \(Y\) and \(Z\) errors on the next two photons produced.

![Quantum Circuit Diagram](image)

**FIG. 1:** (Color online) A quantum circuit readily verified to output linear cluster state. For mapping to the cluster state machine gun, the top qubit line is the electron spin, the Hadamard gates are replaced by single qubit unitaries \(\exp(-i\pi Y/4)\) (requiring the careful tracking of certain phases), and the physical process of creating a photon with left/right circular polarization conditioned on the state of the electron spin becomes the controlled not gate which leaves the qubit (photon) in state \(|0\rangle\) (i.e. \(|R\rangle\)) if the electron spin is in state \(|0\rangle\) (i.e. \(|1\rangle\)), but otherwise flips it. Crucially, as depicted, a Pauli \(Y\) error on the spin localizes; i.e it is equivalent to \(Y\) and \(Z\) errors on the next two photons produced.

The potential imperfections to be considered are as follows: (i) The non-zero lifetime of the trion \(\tau_{\text{decay}}\) means that the magnetic field causes precession of the electrons during the emission process. This leads to errors induced on the quantum circuit of Fig.1, however we shall find that they can be understood as implementing an error model on the final output cluster state which takes the form of Pauli errors occurring with some independent probability on pairs of (photonic) qubits. (ii) Interaction of the electron spin with its environment results in a non-unitary evolution of the spin. This evolution consists of two parts: decoherence (in which we include both dephasing and spin flips) and spin relaxation. Decoherence is characterized by a \(T_2\) time. Fortunately we will see that both these processes also lead only to errors occurring independently on two (photonic) qubits at a time. Efficient cluster state quantum computation can proceed even if every qubit has a finite (though small) probability of undergoing some random error \(\mathcal{E}\). This implies that the protocol’s running time is not limited by \(T_2\), while the errors are amenable to standard quantum error correction techniques for cluster states. Spin relaxation is characterized by a \(T_1\) time, and is a process which projects the spin to the ground state. In semiconductor quantum dots \(T_1\) times are extremely long \(T_1 \gg T_2 \gg \tau_{\text{decay}}\). Therefore, we shall not discuss the effects of this process further here. We point out, however, that it can be shown this process also leads to errors of a localized form, and so in principle is no obstacle to the continuous
operation of the device even for times much longer than $T_1$. (iii) The last source of error is related to the issue of ensuring the photons are emitted into well-controlled spatial modes. In practise this technological issue of mode matching (say by placing the dot in a microcavity) results in some amount of photon loss error in the final state. Significant progress on this issue is being made for a variety of quantum dots [12 13], although we emphasize that for our proposal strong coupling to the cavity is not required. Fortunately photonic cluster state computation can proceed even in the presence of very high (up to 50%) loss [10], and we will not consider this source of error further.

We now turn to detailed calculations of the error rate inflicted by imperfections (i) and (ii) discussed above. We first calculate the effect of a finite ratio of the trion decay time $\tau_{\text{decay}}$ to the spin precession time. We denote by $p_0(\tau + t_n)$ the state of the system (the quantum dot and photons) at time $\tau$ after the $n^{\text{th}}$ excitation pulse, $t_n = nT_{\text{cycle}}$. By $p_0(t_n^\ast)$ we mean the state of the system just before the $n^{\text{th}}$ excitation pulse (we assume the excitation is instantaneous). Following the excitation, the trion state decays, emitting a photon and leaving an electron in the QD, the spin of which then precesses in the magnetic field. These lead to an evolution of the quantum state described by the following map (see [11] for details):

$$\rho(t_n + \tau) = U^{\dagger}(\tau)(G + F)\rho(t_n)(G + F)U(\tau) \tag{1}$$

The unitary operator $U = \exp(iY\omega_B\tau)\exp(iH_0\tau)$ describes the precession of the electron spin and the free propagation of the photons. The generalized creation operators $G^\dagger = G_0^\dagger(\downarrow)\langle\uparrow | + G_1^\dagger(\uparrow)\langle\downarrow |$, $F^\dagger = F_0^\dagger(\downarrow)\langle\uparrow | - F_1^\dagger(\uparrow)\langle\downarrow |$, describe the excitation and decay process, adding a photon to the state. The trion states decay exponentially with $\tau$, therefore we have omitted them from Eq. (1) (which describes the state of the system at times greater than the trion decay time, i.e. $\tau \gg \tau_{\text{decay}}$). Note that the photons created in each cycle are well separated from the ones created in the previous cycles (formally, this is taken into account by the free propagation of the photons).

Equation (1) describes a circuit isomorphic to the one in Fig. 1. The operator $G^\dagger$ corresponds to a correct application of a CNOT gate. This happens with an amplitude $g(\kappa)$, which depends on the photon’s energy $k$:

$$\langle k\varepsilon | G^\dagger | 0 \rangle \equiv g(\kappa) = \sqrt{\frac{pB}{B^2 - (g_0\mu B)^2}/T}. \quad \text{Here} \quad | k\varepsilon \rangle = a_{k,\varepsilon}^\dagger | 0 \rangle \text{ and } \varepsilon = L, R.$$

The complex energy of the trion states is denoted by $Z = \omega_0 - i\Gamma/2$, where $\tau_{\text{decay}} = 1/\Gamma$ is their lifetime. The operator $F^\dagger$ corresponds to a CNOT gate followed by a $Y$ error on the spin qubit. This errored gate is applied with amplitude $f(\kappa)$, where

$$\langle k\varepsilon | F^\dagger | 0 \rangle \equiv f(\kappa) = \sqrt{\frac{pB}{B^2 - (g_0\mu B)^2'}}/T.$$

Let us for the moment treat the processes described by $G^\dagger$ and $F^\dagger$ as incoherent with each other. Then the resulting state is described by the circuit of Fig. 1 with a probability $p_B = |\langle f | f \rangle|^2 = \frac{(g_0\mu B)^2}{2(g_0\mu B)^2 + 2\epsilon^2}$, that each CNOT gate is followed by $Y$ error on the spin qubit. As noted in Fig. 1, a state with a $Y$ error on the spin after generation of the $n^{\text{th}}$ photon (i.e. after the $n^{\text{th}}$ CNOT), is equivalent to a state $Y$ and a $Z$ error on the $n^{\text{th}} + 1$ and $n^{\text{th}} + 2$ photons, with no error on the spin. Note that the error probability increases with magnetic field strength, because the spin can precess more during the lifetime of the trion $15$. Therefore it is advantageous to consider relatively low magnetic fields, for which $g_0\mu B \ll 1$. Taking the coherence between $G^\dagger$ and $F^\dagger$ into account, it can be seen that a unitary correction $e^{i\gamma\theta}$ with tan $\phi = |\langle g | f \rangle/\langle g | g \rangle|$ yields a further improvement of the error rate. We also point out that as $g(\kappa)$ is more localized around $\omega_0$ then $f(\kappa)$ (inset of Fig. 2), selection of photons with energy $|k - \omega_0| > \Delta$ would yield a lower error rate at the expense of (heralded) loss.

The calculation above ignores the possibility of the exciton dephasing [10] during the decay process. Pure dephasing, in which both excited levels evolve the same (random) phase, will have no affect on the entanglement in polarization with which we are concerned. Cross dephasing (experimentally seen to be very small [16]) will lead to Z-errors on the qubits, which also localize (See [14] for a detailed discussion).

We now turn to the issue of decoherence of the spin as a result of its interaction with the nuclei in the quantum dot. Assuming Markovian dynamics (discussed further
in [14]), it is well known [17] that the resulting dephasing and spin flip dynamics are equivalent to the action of random Pauli operations $X, Y, Z$ with some probabilities $p_x, p_y, p_z$. The probabilities $p_x, p_z$ are suppressed due to the presence of the magnetic field, while $p_y$ is characterized by $T_2$, the dephasing time. This can readily be shown to give $p_y = \frac{1}{2}(1 - e^{-T_{\text{sync}}/T_2})$ as the probability of a given spin error in 1 cycle. We already noted that a $Y$ error on the spin becomes a pauli error on the next 2 photons. Similarly a $Z$ error at the end of the $n$th cycle is equivalent to a $Z$ error on the $n^{th} + 1$ photon, as can be seen from Fig. 1 and the fact that the operator $C_{\text{NOT}}(Z_{\text{spin}} \otimes I_{\text{photon}})$ and $(I_{\text{spin}} \otimes Z_{\text{photon}})C_{\text{NOT}}$ have similar actions on the states $|00\rangle$ and $|10\rangle$. As $X = iZY$, an error on $X$ again affects only the next two photons to be generated.

In Fig. 2 we plot the total probability of error on any given qubit, $1 - (1 - p_B)(1 - p_y)$, as a function of the two dimensionless parameters $g_e \mu_B/\Gamma$ and $(\Gamma T_2)^{-1}$. We include the aforementioned easily implemented unitary correction. Although free induction decay $T_2$ may be relatively short in low magnetic fields, using a spin echo pulse at half the cycle time can extend $T_2$ considerably, and remove the dephasing caused by a wide distribution of nuclear (Overhauser) magnetic fields (often termed inhomogeneous broadening and characterized by a $T_2^*$ time). To estimate an achievable error rate, we consider a decay time of $1/\Gamma = 100\text{ ps}$, and a dephasing time of $T_2 = 1\mu s$ with the addition of the spin echo pulses (a lower bound of 3 $\mu$s have been measured in high magnetic fields [18]). This gives $(\Gamma T_2)^{-1} = 10^{-4}$. From Fig. 2 one can deduce that a probability of error less than 0.2% can be achieved by applying a magnetic field of $15\text{mT}$ (we take $g_e = 0.5$). We note that even without the spin echo pulses, error rates of about 1% are achievable, which enables the production of considerable longer and higher quality optical cluster states than those produced by current methods.

So far we have considered pulse excitations that are timed to coincide with (integer multiples of) $\pi/2$ rotations of the spin. In fact it can be advantageous to sometimes wait for a full $\pi$ rotation to occur. This has the effect of emitting subsequent photons which are redundantly encoded [2]. Fusing together such qubits gives a highly efficient method for producing higher-dimensional cluster states which are universal for quantum computing. Photons which undergo fusion can be spectrally filtered (via a suitable prism), such that if they fail to pass the filter they can still be measured and removed from the cluster state. This filtering does not lead to an increase in loss error rates, but simply decreases the overall success probability of the fusion gates.

Current experiments produce photonic cluster states via spontaneous parametric downconversion [19], and would seem to be limited to producing 6 to 8 photon cluster states. Our proposal in principle can produce strings of thousands of photons; however initial experiments will be limited by collection efficiency. With the parameters above a simple analysis shows that we would need a collection+photodetection efficiency of about 18% for a demonstration of on-demand 12-photon cluster states, where the full 12 qubits are expected to be detected about once every 10 seconds.

Finally, our proposal is suggestive of an efficient mechanism for entangling matter qubits [5, 20], and we feel this is a topic worthy of further investigation.

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SUPPLEMENTARY MATERIAL FOR THE PHOTONIC CLUSTER STATE MACHINE GUN

Note: In the published version this document accompanies the above paper as supplementary material in the form of an EPAPS archive - i.e it is the same material as can be found at reference [15] above. The references in this section are self contained and can be found at the end of this document.

CALCULATION OF EMITTED PHOTON WAVEPACKETS

In this section we compute the emitted photon wavepacket, taking into account the fact that the electron spin is precessing during the emission. This effect and the finite $T_2$ time of the electron, are the dominant source of imperfect cluster state production in the machine gun. Any notation not defined here is defined in the paper.

The Hamiltonian we consider is

$$H = H_0 + H_{\text{int}}$$

where $H_0 = \omega_0 P + H_B + H_{EM}$. Here $P$ is a projector on the dot excited state (trion) manifold and $H_{EM}$ is the free EM field Hamiltonian.

The Zeeman interaction for the electron is

$$H_B = g_e \mu_B \vec{S_e} \cdot \vec{B},$$

where $\mu_B$ is the Bohr magneton, and $g_e$ is the effective gyromagnetic ratio of the electron in the QD. For the low magnetic fields we are dealing with, we can assume $g_h = 0$ for the heavy hole [1]. We abbreviate $\frac{1}{2}g_e \mu_B \rightarrow g_e$. We take $\vec{B} = B \hat{y}$. The interaction Hamiltonian of the QD with the photon field, $H_{\text{int}}$ is given in the rotating wave and dipole approximation by

$$H_{\text{int}} = \sum_k V_k \left( |\uparrow\rangle \langle \uparrow| a_{R,k}^\dagger + |\downarrow\rangle \langle \downarrow| a_{L,k}^\dagger \right)$$

where $V_k$ are the coupling constants which depend on the details of the specific quantum dot in question, and the (heavy) trion states are denoted by $|\uparrow\rangle$, $|\downarrow\rangle$ respectively.

We want to calculate matrix elements of the form:

$$\langle \uparrow, R_k | U(t) | \uparrow \rangle = \frac{1}{2\pi i} \int dE \exp(-iEt) \langle \uparrow | G(E) | \uparrow \rangle \tag{2}$$

Here $|R_k\rangle$ and $|L_k\rangle$ are left and right circularly polarised single photon states created by $a_{R,k}^\dagger$, $a_{L,k}^\dagger$, and

$$G(E) = \frac{1}{E - H}$$

is the resolvent of the hamiltonian.

Denote by $Q$ the projector on ground state manifold. Then the matrix block inversion formula gives

$$QG(E)P = \frac{Q}{E - QHQ} V' \frac{P}{E - PH_0P - PR(E)P}$$

with $R(E) = V \frac{1}{E - QHQ} V$.

The operator $R(E)$ can only couple the state $|\uparrow\rangle$ (or $|\downarrow\rangle$) to itself (recalling that the hole is assumed to not precess). Therefore

$$G^h(E) \equiv \left( \frac{P}{E - PH_0P + PR(E)P} \right)_{ss'} = \frac{1}{E - \omega_0 - r(E)} \delta_{ss'}$$

where $r(E) = \int dk \frac{|V_k|^2}{E - \omega_0}$, and $s, s' = |\uparrow\rangle, |\downarrow\rangle$. When performing the contour integral (1), we can replace $r(E)$ with the lamb shift $\Delta$ and the inverse decay rate $\Gamma$. In the following we shall take $V_k = \sqrt{\Gamma/2\pi}$ for simplicity, and absorb $\Delta$ into the definition of $\omega_0$. This gives

$$G^h(E) = \frac{1}{E - \omega_0 + i\Gamma/2}$$
Next, we consider

\[ G^r(E) \equiv \left( \frac{Q}{E - QHQ} \right)_{\sigma'\sigma}^{-1} = \left( \begin{array}{cc} E - h\kappa & i\gamma_e B \\ -i\gamma_e B & E - h\kappa \end{array} \right) \]

where \( D^r(E) = (E - h\kappa)^2 - g_e^2 B^2 \). We now have the final result:

\[ \langle \sigma, \epsilon_k | G(E) | s \rangle = V_k^r G^r(E)_{\sigma\sigma'\epsilon_k} G(E)_{s'\epsilon_k} \]

Writing the matrix elements explicitly:

\[
\begin{align*}
\langle 1, R_k | G(E) | \uparrow \rangle &= V_k \frac{(E - h\kappa)}{D^r(E)(E - \omega_0 + i\Gamma/2)} \\
\langle \downarrow, R_k | G(E) | \uparrow \rangle &= V_k \frac{i\gamma_e B}{D^r(E)(E - \omega_0 + i\Gamma/2)} \\
\langle 1, L_k | G(E) | \downarrow \rangle &= V_k \frac{-i\gamma_e B}{D^r(E)(E - \omega_0 + i\Gamma/2)} = -\langle \downarrow, R_k | G(E) | \uparrow \rangle \\
\langle \downarrow, L_k | G(E) | \downarrow \rangle &= V_k \frac{(E - h\kappa)}{D^r(E)(E - \omega_0 + i\Gamma/2)} = \langle \uparrow, R_k | G(E) | \uparrow \rangle
\end{align*}
\]

We now consider the contour integral. The two poles contributing in the limit \( t \to \infty \) are \( E = k - g_e B \) and \( E = k + g_e B \). We denote \( Z = \omega_0 - i\Gamma/2 \) and a standard complex integration gives

\[
\langle 1, R_k | U(t) | \uparrow \rangle = [\exp(-i\gamma_e B t)f_1 + \exp(i\gamma_e B t)f_2]
\]

where \( f_1 = \frac{\sqrt{\Gamma/2\pi}}{(k + g_e B - Z)^2} \), \( f_2 = \frac{\sqrt{\Gamma/2\pi}}{(k - g_e B - Z)^2} \). Above and in the following we omit the phase factors \( e^{-ikt} \). Using the notation

\[
g(k) = f_1(k) + f_2(k) = \frac{\sqrt{\Gamma/2\pi}(k - Z)}{(k - Z)^2 - (g_e B)^2}
\]

\[
f(k) = f_1(k) - f_2(k) = \frac{\sqrt{\Gamma/2\pi}g_e B}{(k - Z)^2 - (g_e B)^2}
\]

this simplifies to

\[
\langle 1, R_k | U(t) | \uparrow \rangle = \cos(g_e B t)g(k) - i \sin(g_e B t)f(k).
\]

Similarly

\[
\langle \downarrow, R_k | U(t) | \uparrow \rangle = i \exp(-i\gamma_e B t)f_1 - i \exp(i\gamma_e B t)f_2
\]

which yields

\[
\langle 1, R_k | U(t) | \uparrow \rangle = i [-i \sin(g_e B t)g(k) + \cos(g_e B t)f(k)]
\]

\[
= \sin(g_e B t)g(k) + i \cos(g_e B t)f(k)
\]

An analogous calculation shows that the amplitudes for starting with the down trion are

\[
\langle 1, L_k | U(t) | \downarrow \rangle = -\langle \downarrow, R_k | U(t) | \uparrow \rangle = -\sin(g_e B t)g(k) - i \cos(g_e B t)f(k)
\]

and

\[
\langle 1, L_k | U(t) | \downarrow \rangle = \langle 1, R_k | U(t) | \uparrow \rangle = \langle 1, R_k | U(t) | \uparrow \rangle = \cos(g_e B t)g(k) - i \sin(g_e B t)f(k)
\]

Starting from the ground state manifold, the excitation and subsequent photon emission and spin prescission can be broken into "good" and "bad" parts, by introducing the operators:
\[ G^\dagger = G_R^\dagger |\uparrow\rangle \langle |\uparrow| + G_L^\dagger |\downarrow\rangle \langle |\downarrow|, \]
\[ F^\dagger = F_R^\dagger |\uparrow\rangle \langle |\uparrow| - F_L^\dagger |\downarrow\rangle \langle |\downarrow|. \]

The operator \( G^\dagger \) corresponds to a correct application of a CNOT gate. This happens with an amplitude \( g(k) \), which depends on the photon’s energy \( k \): \( \langle k | G^\dagger_1 |0\rangle \rangle \equiv g(k) \), the operator \( F^\dagger \) corresponds to an errored gate which is applied with amplitude \( f(k) \), where \( \langle k | F^\dagger_2 |0\rangle \rangle \equiv f(k) \).

Starting from the state \(| \uparrow \rangle \rangle \), we get

\[ |\psi\rangle = U_{\uparrow,R_d,\uparrow} |\uparrow, R_d \rangle + U_{\uparrow,R_d,\downarrow} |\downarrow, R_d \rangle \]

which we break \(|\psi\rangle\) into the good and bad states: \(|\psi\rangle\) = \(|\psi_{\text{good}}\rangle\) + \(|\psi_{\text{bad}}\rangle\) as follows:

\[ \langle \uparrow, R_d | \psi_{\text{good}} \rangle = \cos (g_c B t) g(k) \]
\[ \langle \downarrow, R_d | \psi_{\text{good}} \rangle = \sin (g_c B t) g(k) \]

and

\[ \langle \uparrow, R_d | \psi_{\text{bad}} \rangle = -i \sin (g_c B t) f(k) \]
\[ \langle \downarrow, R_d | \psi_{\text{bad}} \rangle = i \cos (g_c B t) f(k) \]

Note that \(|\psi_{\text{bad}}\rangle = \frac{L - L'}{f_1 + f_2} Y_{\text{spin}} |\psi_{\text{good}}\rangle\) where \( Y_{\text{spin}} |\uparrow\rangle \rangle = i |\downarrow\rangle \rangle \) and \( Y_{\text{spin}} |\downarrow\rangle \rangle = -i |\uparrow\rangle \rangle \). The relation to the operators \( G \) and \( F \) is

\[ \langle \uparrow, R_d | \psi_{\text{good}} \rangle = \langle \uparrow, R_d | U(t) G |\uparrow\rangle \]
\[ \langle \downarrow, R_d | \psi_{\text{good}} \rangle = \langle \downarrow, R_d | U(t) G |\uparrow\rangle \]

and

\[ \langle \uparrow, R_d | \psi_{\text{bad}} \rangle = \langle \uparrow, R_d | U(t) F |\uparrow\rangle \]
\[ \langle \downarrow, R_d | \psi_{\text{bad}} \rangle = \langle \downarrow, R_d | U(t) F |\uparrow\rangle \]

where \( U(t) \) denotes the free propagation.

Similarly, if we start from the state \(| \downarrow \rangle \rangle \), the resulting state would be

\[ |\psi\rangle = U_{\downarrow,L_d,\downarrow} |\downarrow, L_d \rangle + U_{\downarrow,L_d,\uparrow} |\uparrow, L_d \rangle \]

Lets break \(|\psi\rangle\) into good and bad states: \(|\psi\rangle\) = \(|\psi_{\text{good}}\rangle\) + \(|\psi_{\text{bad}}\rangle\) as follows:

\[ \langle \uparrow, L_d | \psi_{\text{good}} \rangle = - \sin (g_c B t) g(k) \]
\[ \langle \downarrow, L_d | \psi_{\text{good}} \rangle = \cos (g_c B t) g(k) \]

and

\[ \langle \uparrow, L_d | \psi_{\text{bad}} \rangle = -i \cos (g_c B t) f(k) \]
\[ \langle \downarrow, L_d | \psi_{\text{bad}} \rangle = -i \sin (g_c B t) f(k) \]

The relation to the operator \( G \) and \( F \) is similar to the one described above.

Note that again \(|\psi_{\text{bad}}\rangle = \frac{L - L'}{f_1 + f_2} Y_{\text{spin}} |\psi_{\text{good}}\rangle\).

If we start from the state \(|\uparrow\rangle + |\downarrow\rangle\), the resulting state would be

\[ |\psi\rangle = U(g_c B t)(g(k) + f(k) Y_{\text{spin}})(|\uparrow, R_d \rangle + |\downarrow, L_d \rangle) \]
where $U_y$ denotes spin rotation around the $y$ axis. The bad part of the wavefunction is therefore equivalent to a Pauli $Y$ error on the ideal cluster state - and as discussed in the paper this error on the spin can be turned into errors on the either the two photons emitted the error occurs on the spin, or as $Z$ errors on the photon which has already been emitted and the first emitted photon. This latter possibility shows that this “quantum scapegoat” effect can act backwards in time! Note that while the error is coherent, the coherence between $|\psi_{\text{good}}\rangle$ and $|\psi_{\text{bad}}\rangle$ can be removed by applying stabilizers randomly to the photons. In fact the good and bad wavepackets have non-trivial overlap, and in the next section we consider how our knowledge of this can allow us to apply a simple unitary correction which increases the amplitude of ideal cluster state produced.

**UNITARY CORRECTION**

Considering the “bad” amplitude $f(k)$ we can clearly write it as

$$f(k) = (f(k) - \alpha g(k)) + \alpha g(k)$$

with $\alpha = \langle g|f\rangle / \langle g|g\rangle$. Note that first term is orthogonal to $g(k)$. Then

$$|\psi\rangle = \frac{1}{\sqrt{N}} U_y(g,Bt)[(1 + \alpha Y_{\text{spin}})g(k)(|\uparrow, R_k\rangle + |\downarrow, L_k\rangle) + (f(k) - \alpha g(k))Y_{\text{spin}} (|\uparrow, R_k\rangle + |\downarrow, L_k\rangle)]$$

(Note that $\alpha$ is purely imaginary.)

We can now make the correction $\exp(-i\phi Y_{\text{spin}})$ where $\tan \phi = \alpha$. Figure 2 of the paper is plotted assuming this simple form of unitary correction has been carried out.

**EXCITON DEPHASING**

In the paper we only briefly mentioned the spectral dephasing which will occur while the system is excited. Our intuition contrasted with that of others, namely we felt that this process would not affect the entanglement of the state - in particular with respect to the polarization degrees of freedom we are interested in - but would only lead to the emitted photon wavepackets being in a mixture of different frequencies. This in turn would only affect the (small fraction) of photons which have to go through fusion gates, and such photons can be filtered before entering the gates in a way which will only lead to a change in the success probability of the (non-deterministic) gate. That is, such
filtering need not even lead to a loss error (as explained below). As such the only effect will be that we need to use more photons - but the overhead is some constant factor.

Nevertheless, to be sure of how the device behaves and of the potential magnitude of this effect, we turn now to the necessary calculations. We begin with a description of the results and a heuristic discussion. There are two excited state (exciton) energy levels to consider, which in the paper we denote $|⇑\rangle$, $|⇓\rangle$. These levels are degenerate. We consider both pure dephasing and cross dephasing. In pure dephasing of these levels, their couplings to the environment are identical - in practise this means that both states could evolve the same (random) phase. Note that this will not affect the relative phase between the two states (such a relative phase would ultimately become a relative phase between the two terms in the entangled photonic cluster state). Therefore, it will have no bearing on the quality of the entanglement in the polarization degrees of freedom. It will, however, cause the emitted photons to have a broader range of frequencies (broadened linewidth). In [2] such pure dephasing was measured, and from their results one can infer that the new linewidth would be approximately double the natural linewidth. However, we stress that this broadening is not of particular importance to the operation of our protocol, for two reasons. Firstly the majority of the emitted photons will never go through any optical element where (for instance) they may be required to interfere with other photons. Rather they will simply be measured directly, and this polarization measurement will not care about their frequency. For the photons that do need to undergo Type-II fusion gates (to cross link with
other qubit lines) the incoherence over the emitted frequencies can be removed (if necessary) by suitable spectral filtering. This filtering can be done in an “active” manner with a suitable prism wherein any photon which does not pass through the filter is simply diverted into another path and measured in an appropriate basis - in terms of the cluster state this simply removes the qubit. As such the only effect of such imperfections is to change the probability of success of the Type-II fusion gate. However efficient quantum computation is obviously possible regardless how small this probability is, as long as it is finite.

Let us now show some calculations to bolster the above discussion. We treat the pure exciton dephasing as a Markovian process and describe it using a Lindblad operator \( L = \sqrt{\gamma_d/2} (|\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow|) \) where we use the same notations as in the paper. The density matrix of the system evolves according to a standard master equation. We compare the probabilities described by this density matrix to the probabilities \(|g(k)|^2\) corresponding to no error, and \(|f(k)|^2\), corresponding to a pauli error, which were calculated in the paper. In the presence of pure dephasing, the probabilities corresponding to no pauli error are

\[
|g_{\text{dephase}}(k)|^2 = \frac{\Gamma}{4\pi} \int_0^\infty dt_1 \int_0^{t_1} dt_2 e^{-\Gamma/2\gamma_d/2}t_1 e^{-\gamma_d/2}t_2 \cos(b t_1) \cos(b t_2) e^{i k (t_2 - t_1)} + \frac{\Gamma}{4\pi} \int_0^\infty dt_2 \int_0^{t_2} dt_1 e^{-\Gamma/2\gamma_d/2}t_2 e^{-\gamma_d/2}t_1 \cos(b t_1) \cos(b t_2) e^{i k (t_2 - t_1)}
\]

In the above \( b = g \mu_B k \) is the zeeman splitting, and we have shifted \( k \rightarrow k - \omega_0 \). The off-diagonal matrix elements corresponding to pauli errors have the same form as Eq. but with sin functions replacing the cos functions. Although the function \(|g_{\text{dephase}}(k)|^2\) is wider then \(|g(k)|^2\), the total probability is equal to the total probability without dephasing

\[
\int dk |g_{\text{dephase}}(k)|^2 = \int dk |g(k)|^2,
\]

for any value \( \gamma_d \).

In Figure 1 we plot the probabilities \(|g_{\text{dephase}}(k)|^2\) versus \(|g(k)|^2\). The parameters chosen for the plot are \( b/\Gamma = 0.15 \) (as in the inset of Fig. 2 of our manuscript) and \( \gamma_d/\Gamma = 1 \) [1], which of course enters only into \(|g_{\text{dephase}}(k)|^2\). Likewise, Figure 2 plots \(|f_{\text{dephase}}(k)|^2\) vs. \(|f(k)|^2\). In Figure 3, we compare \(|g_{\text{dephase}}(k)|^2\) and \(|f_{\text{dephase}}(k)|^2\) showing that spectral filtering is still possible also in the presence of pure exciton dephasing.

A potentially more serious source of error would be cross dephasing. Experimentally such dephasing has not been observed, despite an attempt to measure it in [2], where they could only lower bound the dephasing to be at least 20 times smaller than the pure dephasing. Even if this cross dephasing were exactly equal to the lower bound obtained in [2], the effect on the emitted photons would ultimately lead to Z-errors on the qubits, which also localize. This potential increase in the pauli error rate is negligible compared to the other sources of pauli error we considered in the paper.

**NOTE ON SPIN-BATH DYNAMICS**

To calculate the error probability on the photonic qubits we have assumed Markovian dynamics for the interaction of the spin with its environment. Previous studies [3-7] suggest a non-Markovian decay in time of the coherence in the reduced density matrix of a spin coupled to a nuclear spin bath. We note that for the short times we are interested in, the Markovian assumption actually overestimates the single qubit error rate. The resulting non-markovian correlations of the errors on the different qubits will be studied in a forthcoming publication. However, we note that these correlations can be dealt with, see for example the discussion in [4].

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