1. Introduction

Let $B_n$ be the braid group on $n$ strings. In his paper [1] Ed Formanek classified all irreducible representations of $B_n$ of dimension at most $(n - 1)$. Since then there were some attempts to classify irreducible representations of $B_n$ of dimension $n$. In particular the classification is known for very small $n$. Case $n = 3$ was done by Ed Formanek ([1], Theorem 24). Woo Lee has classified the four-dimensional irreducible representations of $B_4$ ([2]).

In this paper we solve this problem completely for $n \geq 9$. Before stating our main classification theorem let us describe the following representation of $B_n$ of dimension $n$.

Definition 1. The standard representation is the representation

$$\tau_n : B_n \rightarrow GL_n(\mathbb{Z}[t^{\pm 1}])$$

defined by

$$\rho(\sigma_i) = \begin{pmatrix}
I_{i-1} & 0 & t \\
0 & t & 0 \\
I_{n-1-i} & 1 & 0
\end{pmatrix},$$

for $i = 1, 2, \ldots, n - 1$, where $I_k$ is the $k \times k$ identity matrix.

We call the above representation standard because of its simplicity. Surprisingly, it does not seem to be well-known. In fact it looks like it was first discovered only in 1996 by Dian-Min Tong, Shan-De Yang and Zhong-Qi Ma ([3], Equation (19)).

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**Theorem 1.** Suppose that $\rho : B_n \to GL_n(\mathbb{C})$ is an irreducible representation of $B_n$ of dimension $n \geq 9$. Then it is a tensor product of a one-dimensional representation and a specialization for $u \neq 0, 1$ of the standard representation.

To explain the ideas of the proof, we need the following definition.

**Definition 2.** Suppose $\rho$ is a representation of the Artin braid group $B_n$. The corank of $\rho$ is the rank of $\rho(\sigma_i)$, where $\sigma_i$ are the standard generators of $B_n$. (This makes sense because all $\sigma_i$ are conjugate.)

If one looks at the proof of the classification theorem of Formanek in [1], it can be separated into two parts. The first is to classify all irreducible representations of braid groups of corank 1. The second is to prove that apart from a few exceptions, the irreducible representations of braid groups $B_n$ of dimension at most $(n - 1)$ can be obtained as a tensor product of a one-dimensional representation and an irreducible representation of corank 1.

Our proof follows a similar strategy. The first part of it, the classification of irreducible representations of corank 2 was carried out in [3]. In this paper we complete the proof of Theorem 1 by proving that for $n \geq 9$ every irreducible representation of $B_n$ of dimension $n$ is the tensor product of a one-dimensional representation and a representation of corank 2.

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## 2. Proof of Theorem 1

We proved in [3], Theorem 5.5 and Corollary 5.6 that for $n \geq 7$ every irreducible complex representation of $B_n$ of corank 2 is a specialization of the standard representation (see Definition 1). So to complete the proof of Theorem 1 it is enough to show that for $n \geq 9$ every irreducible representation of $B_n$ of dimension $n$ is the tensor product of a one-dimensional representation and a representation of corank 2. This will be done in Theorem 2. Before that we need some preparatory results. The key of the proof is the following theorem, which is similar to Theorem 16 of [1].

**Theorem 2.** Suppose that $\rho : B_{n+1} \to GL_{n+1}(\mathbb{C})$ is an irreducible representation of $B_{n+1}$ of dimension $n + 1$ ($n \geq 4$). Suppose that the restriction of $\rho$, $\rho|B_{n-1} \times <\sigma_n>$, stabilizes one-dimensional subspace $C_v$ of $\mathbb{C}^{n+1}$.

Then $\text{rank}(\rho(\sigma_1) - yI) = 2$ for some $y \in \mathbb{C}^*$. 
Proof. For notational simplicity we will write $\sigma$ instead of $\rho(\sigma)$ for $\sigma \in B_n$.

By hypothesis,
$$\rho|_{B_{n-1} \times <\sigma_n>} : \mathbb{C}v \to \mathbb{C}v$$
is a one-dimensional representation of $B_{n-1} \times B_2$, so there exist $x, y \in \mathbb{C}^*$ such that
$$\sigma_1 v = \sigma_2 v = \cdots = \sigma_{n-2} v = yv, \ \sigma_n v = xv$$
Consider $\theta = \theta_{n+1} = \sigma_1 \sigma_2 \ldots \sigma_n, \ \sigma_0 = \theta \sigma_n \theta^{-1}$,
$$v_n = v, \ v_{n+1} = \theta v, \ v_1 = \theta^2 v, \ldots, v_{n-1} = \theta^n v.$$Conjugation by $\theta$ permutes $\sigma_1, \ldots, \sigma_n, \sigma_0$ cyclically.

Because $\rho$ is an irreducible representation and $\theta^{n+1}$ is central in $B_{n+1}$, $\rho(\theta^{n+1}) = dI$ for some $d \in \mathbb{C}^*$. Thus, the left action of $\theta$ permutes $\mathbb{C}v_1, \mathbb{C}v_2, \ldots, \mathbb{C}v_{n+1}$ cyclically.

We have:
$$\sigma_i v_i = xv_i,$$
$$\sigma_i v_{i+j} = yv_{i+j}$$
for
$$i = 1, \ldots, n+1, \ j = 2, \ldots, n-1,$$
where indices are taken modulo $n+1$.

The following table summarizes the above calculations:

|   | $v_1$ | $v_2$ | $v_3$ | $\ldots$ | $v_{n-1}$ | $v_n$ | $v_{n+1}$ |
|---|-------|-------|-------|-----------|-----------|-------|-----------|
| $\sigma_1$ | $xv_1$ | $yv_3$ | $\ldots$ | $yv_{n-1}$ | $yv_n$ |       |           |
| $\sigma_2$ | $xv_2$ |       | $\ldots$ | $yv_{n-1}$ | $yv_n$ | $yv_{n+1}$ |
| $\sigma_3$ | $yv_1$ | $xv_3$ | $\ldots$ | $yv_{n-1}$ | $yv_n$ | $yv_{n+1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\sigma_{n-1}$ | $yv_1$ | $yv_2$ | $yv_3$ | $\ldots$ | $xv_{n-1}$ |       | $yv_{n+1}$ |
| $\sigma_n$ | $yv_1$ | $yv_2$ | $yv_3$ | $\ldots$ |       | $xv_n$ |           |
| $\sigma_0$ | $yv_1$ | $yv_2$ | $yv_3$ | $\ldots$ | $yv_{n-1}$ |       | $xv_{n+1}$ |

Suppose that $v_1, \ldots, v_{n+1}$ are linearly dependent. Consider
$$a_1 v_1 + a_2 v_2 + \cdots + a_t v_t = a_1 v_1 + a_2 \theta v_1 + \cdots + a_t \theta^{t-1} v_1 = 0,$$a linear dependence relationship with minimal $t$.

In the equation above, $a_1 \neq 0$, since $\theta$ is invertible, and $a_t \neq 0$ by the minimality of $t$.

We claim that $t \geq n$. Indeed, suppose that $t \leq n - 1$. Then $v_{n-1}$ is a linear combination of $v_1, \ldots, v_{n-2}$, which are eigenvectors for $\sigma_n$ with $\sigma_n v_i = yv_i, \ i = 1, \ldots, n-2$. So, $\sigma_n v_{n-1} = yv_{n-1}$. Applying $\theta^3$
implies that $\sigma_2 v_1 = y v_1$, which means that $C v_1$ is $B_{n+1}$-invariant, which contradicts the irreducibility of $\rho$. So, $t \geq n$.

Thus, $v_1, \ldots, v_{n-1}$ are linearly independent.

Assume that $\text{rank}(\sigma_1 - y I) > 2$. Then, as

$$\dim(Ker(\sigma_1 - y I)) + \text{rank}(\sigma_1 - y I) = n + 1,$$

$$\dim(Ker(\sigma_1 - y I)) \leq n - 2.$$

Note that $v_3, \ldots, v_n$ are $n - 2$ linearly independent elements of $L = Ker(\sigma_1 - y I)$. So, $\dim(Ker(\sigma_1 - y I)) = n - 2$, and $L = \text{span}\{v_3, \ldots, v_n\}$.

Since vectors $\{v_1, \ldots, v_{n-1}\}$ are linearly independent, $\{v_3, \ldots, v_{n+1}\}$ are also linearly independent. Therefore $v_2 \notin L$, and $v_{n+1} \notin L$.

The action of $\theta$ implies that for $i = 1, \ldots, n + 1$

$$Ker(\sigma_i - y I) = \text{span}\{v_{i+2}, \ldots, v_{i-2}\},$$

$v_{i-1} \notin L$, and $v_{i+1} \notin L$, where indices are taken modulo $n + 1$.

$\sigma_1$ commutes with $\sigma_n$, and $n \geq 4$, so

$$(\sigma_n - y I)\sigma_1 v_2 = \sigma_1(\sigma_n - y I)v_2 = 0.$$

Thus, $\sigma_1 v_2 \in Ker(\sigma_n - y I)$, so

$$\sigma_1 v_2 = b_1 v_1 + b_2 v_2 + \cdots + b_s v_s,$$

where $1 \leq s \leq n - 2$ and $b_s \neq 0$.

We claim that $s \leq 2$. Indeed, if $s \geq 3$, then

$$0 = \sigma_1(\sigma_{s+1} - y I)v_2 = (\sigma_{s+1} - y I)\sigma_1 v_2 =$$

$$= (\sigma_{s+1} - y I)(b_1 v_1 + b_2 v_2 + \cdots + b_s v_s) = (\sigma_{s+1} - y I)b_s v_s.$$

This contradicts the fact that $v_s \notin Ker(\sigma_{s+1} - y I)$.

Thus,

$$\sigma_1 v_2 = b_1 v_1 + b_2 v_2, \ b_1, b_2 \in \mathbb{C}.$$

By a symmetric argument which reverses the roles of $\sigma_1$ and $\sigma_n$, and starts with the equation

$$(\sigma_1 - y I)\sigma_n v_{n-1} = \sigma_n(\sigma_1 - y I)v_{n-1} = 0,$$

we obtain

$$\sigma_n v_{n-1} = c_1 v_{n-1} + c_2 v_n, \ c_1, c_2 \in \mathbb{C}.$$

Using the action of $\theta$, we get the following table:
If \( \rho \) are linearly dependent, then \( \rho \) are reducible. So, \( \rho \) is irreducible and \( \rho \) is reducible. So, \( \rho \) is one-dimensional. 

\[
\begin{array}{cccccc}
\sigma_1 & v_1 & v_2 & \ldots & v_n & v_{n+1} \\
\sigma_2 & c_1 v_1 + c_2 v_2 & x v_2 & \ldots & y v_n & y v_{n+1} \\
\sigma_3 & y v_1 & c_1 v_2 + c_2 v_3 & \ldots & y v_n & y v_{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{n-1} & y v_1 & y v_2 & \ldots & b_1 v_{n-1} + b_2 v_n & y v_{n+1} \\
\sigma_n & y v_1 & y v_2 & \ldots & x v_n & b_1 v_n + b_2 v_{n+1}
\end{array}
\]

\( \text{Span}\{v_1, v_2, \ldots, v_{n+1}\} \) is \( B_{n+1} \)-invariant. Thus, if \( \{v_1, v_2, \ldots, v_{n+1}\} \) are linearly independent, then \( \rho \) is reducible. So, \( \{v_1, v_2, \ldots, v_{n+1}\} \) are linearly independent, and they form a basis for \( \mathbb{C}^{n+1} \).

In this basis:

\[
\sigma_1 = \begin{pmatrix}
    x & b_1 \\
    0 & b_2 \\
\end{pmatrix}
\quad \sigma_3 = \begin{pmatrix}
   y & c_1 \\
   c_2 & x & b_1 \\
\end{pmatrix}
\]

Using the \((3,2)\)-entry of the matrix \( \sigma_1 \sigma_3 = \sigma_3 \sigma_1 \), we have

\[
b_2 c_2 = y c_2.
\]

If \( c_2 = 0 \), then \( \mathbb{C} v_1 \) is invariant under \( B_{n+1} \), which contradicts the irreducibility of \( \rho \). So, \( c_2 \neq 0 \). Thus, \( b_2 = y \). Then \( \text{rank}(\sigma_1 - y I) \leq 2 \), a contradiction.

So, \( \text{rank}(\sigma_1 - y I) \leq 2 \). But by [1], Theorem 10, the case \( \text{rank}(\sigma_1 - y I) = 1 \) is impossible. Thus, \( \text{rank}(\sigma_1 - y I) = 2 \).

The following argument is due to E. Formanek. He also used it in [1], Lemma 17 and Corollary 18. My original argument was much longer.

The next Lemma 3 is a corollary of Theorem 23 of [1], which classifies the irreducible representations of \( B_n \) of dimension at most \( n - 1 \).

**Lemma 3.** If \( \rho : B_n \to GL_r(\mathbb{C}) \) is irreducible and \( r \leq n - 3 \), then \( \rho \) is one-dimensional.

**Lemma 4.** Let \( \rho : B_n \to GL_r(\mathbb{C}) \) be a representation, where \( n \geq 6 \). Suppose that \( \lambda \) is an eigenvalue of \( \rho(\sigma_{n-1}) \). Suppose that the largest Jordan block corresponding to \( \lambda \) has size \( s \) and multiplicity \( d \).

If \( d \leq n - 5 \), then \( \rho|B_{n-2} \times < \sigma_{n-1}> \) has a one-dimensional invariant subspace.
Proof. Let $f(t)$ be the minimal polynomial of $\rho_{\sigma_{n-1}}$. Set $m(t) = f(t)/(t - \lambda)$. Let $V$ be the image of $\mathbb{C}^r$ under $m(\rho_{\sigma_{n-1}})$. Then $V$ is invariant under $\rho_{\sigma_{n-1}}^2 < \sigma_{n-1} >$, and $\dim V = d$. If $d \leq n - 5$, then by Lemma 3 all composition factors of $\rho_{\sigma_{n-1}}^2 < \sigma_{n-1} > : V \to V$ are one-dimensional.

Theorem 5. For $n \geq 9$, every $n-$dimensional complex irreducible representation $\rho$ of the braid group $B_n$ is equivalent to a tensor product of a one-dimensional representation $\chi(y), y \in \mathbb{C}^*$, and an $n$-dimensional representation of corank 2.

Proof. Assume not. Then by Theorem 2 and Lemma 4, the largest Jordan block corresponding to every eigenvalue of $\rho_{\sigma_{n-1}}$ has multiplicity $\geq n - 4$.

If $\rho_{\sigma_{n-1}}$ has two or more eigenvalues, we get $(n - 4) + (n - 4) \leq n$, a contradiction, since $n \geq 9$. Similarly, if some eigenvalue has the corresponding largest Jordan block of size $s \geq 2$, we get a contradiction $2(n - 4) \leq n$.

Thus, $\rho_{\sigma_{n-1}}$ has only one eigenvalue $\lambda$ and the Jordan canonical form of $\rho_{\sigma_{n-1}}$ consists of $1 \times 1$ elementary Jordan blocks. But then $\rho_{\sigma_{n-1}} = \lambda I$, which contradicts the irreducibility of $\rho$.

This completes the proof of the theorem, and thus the proof of Theorem 1.

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