Additional restrictions on quasi-exactly solvable systems

Sergey Klishevich

Institute for High Energy Physics, Protvino, Russia

Abstract

In this paper we discuss constraints on two-dimensional quantum-mechanical systems living in domains with boundaries. The constrains result from the requirement of hermicity of corresponding Hamiltonians. We construct new two-dimensional families of formally exactly solvable systems and applying such constraints show that in real the systems are quasi-exactly solvable at best. Nevertheless in the context of pseudo-Hermitian Hamiltonians some of the constructed families are exactly solvable.

*E-mail: klishevich@ihep.ru
1 Introduction

It is well known that exactly solvable systems play very important role in quantum theory. Unfortunately number of such systems is quite limited. This considerably narrows their applications. Such a situation stimulates interest to quasi-exactly solvable systems [1, 2, 3, 4]. In contrast to exactly solvable models in quasi-exactly solvable systems the spectral problem can be solved partially. Nevertheless such systems are very interesting. Besides modeling physical systems [5] they can be used as an initial point of the perturbation theory or to investigate various nonperturbative effects [6]. Furthermore, recently in the series of papers [7] (see also Refs. [8, 9]) it was revealed a connection between quasi-exactly solvable models and supersymmetric systems with polynomial superalgebras [10]. Also we can hope that progress in understanding quasi-exactly solvable quantum-mechanical systems will allow to find out methods of constructing quasi-exactly solvable models in quantum field theory.

The paper has the following structure. In section 2 a brief introduction into the Lie-algebraic approach to constructing quasi-exactly solvable systems is given. In section 3 we discuss constraints on wave functions of systems living in domains with boundaries. It is shown that for corresponding Hamiltonians to be Hermitian the wave functions must have a specific behaviour at boundaries. The role of the constraints is illustrated by an example of a known two-dimensional quasi-exactly solvable system. In section 4 we construct new families of formally exactly solvable systems. Application of the restriction on the behaviour of wave functions at boundaries leads to the conclusion that the systems are quasi-exactly solvable even if the wave functions are normalizable. Discussion of results is presented in section 5.

2 General aspects of quasi-exactly solvable systems

From the general viewpoint an operator is quasi-exactly solvable if it has a finite-dimensional invariant subspace. Bearing in mind applications to the quantum mechanics we are interested in second order differential operators, Hamiltonians. The most famous method of constructing quasi-exactly solvable differential operators is the Lie-algebraic approach [1, 2, 3, 4]. In this section we briefly discuss of main ideas of the approach.

The one-dimensional and multidimensional cases have to be treated separately. The one-dimensional case is the most elaborated one. The simplest finite-dimensional subspaces are spaces which admit the following monomial basis:

\[ \mathcal{F}_n = \text{span}\{1, z, z^2, \ldots, z^n\}. \]

For such spaces it is not difficult to find invariant (quasi-exactly solvable) differential operators of the first order:

\[ J_0 = N_z - \frac{n - 1}{2}, \quad J_+ = z (N_z - n), \quad J_- = \frac{d}{dz}, \]

where \( N_z = z \frac{d}{dz} \). These operators form finite-dimensional representations of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) for \( n \in \mathbb{N} \). Any second order operator taken as quadratic combination of the first order operators,

\[ H' = C^{ab} J_a J_b + C^a J_a = - P_4(z) \frac{d^2}{dz^2} + \ldots, \]

\( P_4(z) \) is a polynomial of order 4.

2
is automatically quasi-exactly solvable\(^1\). Here \(P_4(z)\) is a polynomial of order 4. In the one-dimensional case any second order operator can be reduced to the Schrödinger form:

\[
Ψ(x) = e^{α(z)}P(z) \quad \text{with} \quad x = ± \int \frac{dz}{\sqrt{P_4(z)}} \quad ⇒ \quad H = -\frac{d^2}{dx^2} + V(x),
\]

where \(Ψ(x)\) is a wave function of the Hamiltonian in the canonical form. This is essential difference of the one-dimensional case from multidimensional systems.

The complete classification of linear differential operators with invariant subspaces of monomials is given in Ref. \[12\] (more recent discussions see in Ref. \[13\]).

In multidimensional case the situation is much more complicated \[2\]. Nevertheless the idea is the same. We construct finite-dimensional representation of some Lie algebra in terms of first order differential operators \[2, 14\], then one can look for Hamiltonians considering quadratic combinations of the first order operators:

\[
H' = C^{ab}J_aJ_b + C^aJ_a = -g^{μν}(\nabla_μ - A_μ)(\nabla_ν - A_ν) + V,
\]

with

\[
V = g^{μν}A_μA_ν - A^{μ;ν},
\]

where \(\nabla_μ\) and ";" stand for the covariant derivative corresponding to the metric \(g_{μν}\) and Such a Hamiltonian can be reduced to the Schrödinger form only if the self-consistency conditions are satisfied:

\[
∂_μA_ν - ∂_νA_μ = 0. \quad (3)
\]

From another point of view these equations are the necessary condition for the Hamiltonian to be Hermitian. The general solution of the constraints is unknown! Therefore in the multidimensional case the general classification of quasi-exactly solvable linear differential operators is unknown even for invariant subspaces of monomials. However as it was pointed out in Ref. \[8\] the problem with resolving the constraints (3) can be avoided if one passes from \(D\)- to \((D + 1)\)-dimensional system. Details of the procedure can be found in Ref. \[8\]). Nevertheless here we do not adopt this standpoint.

### 3 Boundary conditions

When constructing quasi-exactly solvable operators in the Lie-algebraic approach very often resulting systems live in domains with boundaries.\(^2\) If so one has to take into account behaviour of wave functions at the boundaries in addition to their normalizability.

Generally the system must evolve in a domain with a positively defined metrics

\[
S = \{ x^μ \mid g_{11} > 0 \cup \det \|g_{μν}\| > 0 \},
\]

Here we investigate situation when components of the inverse metric \(g^{μν}\) are polynomial. Therefore it is more convenient to define the boundaries of the domain \((4)\) as roots of the inverse metric determinant:

\[
\partial S \subset \{ x^μ \mid \det \|g^{μν}\| = 0 \}.
\]

\(^1\)In principle one has to proof that the finite-dimensional subspace corresponds to wave functions with finite norms. In the case of the algebra \(\mathfrak{sl}(2, \mathbb{R})\) this question was completely investigated in Ref. \[11\].

\(^2\)Here we imply the multidimensional case.
To provide hermicity of the corresponding Hamiltonian the following conditions have to be implied:

$$\sqrt{g^{\mu\nu}} \varphi \partial_{\nu} \psi \bigg|_{\partial S} = 0$$

(5)

for any \( \varphi(x) \) and \( \psi(x) \) from the domain of the Hamiltonian.

In the two-dimensional case the boundary can be locally given by the equation \( x = \xi(y) \).

Then from the conditions (5) one can infer that the wave functions from the domain of the Hamiltonian have the following behaviour at the boundary:

$$\psi \sim g^{-\frac{1}{2}} \left| x - \xi(y) \right|^\alpha \quad \text{with} \quad \alpha > \frac{1}{2}. \quad (6)$$

The normalizability of such a wave function leads to the inequality \( \alpha > -\frac{1}{2} \). So the hermicity implies the more strict inequality but often it is not taken into account. For example, consider the following Hamiltonian [2]:

$$-H' = x (1 + x) \partial_x^2 + y (1 + y) \partial_y^2 - 2xy \partial_{xy}^2 + (1 + x) (1 - cx) \partial_x + (1 + y) (1 - cy) \partial_y + 2c (jx + \tilde{j}y). \quad (7)$$

It can be represented in terms of generators of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \). In this case the inverse metric is

$$\|g^{\mu\nu}\| = \begin{pmatrix} x (1 + x) & -xy \\ -xy & y (1 + y) \end{pmatrix}. \quad (8)$$

Its determinant has the form

$$\det \|g^{\mu\nu}\| = xy (1 + x + y).$$

Therefore we can consider the system in the domain \( S = \{(x, y) \mid x > 0 \cup y > 0\} \). In principle there are four domains with positively defined metric but the consequence is the same for all of them. The wave functions corresponding to the quasi-exactly solvable sector have the structure

$$\psi = g^{-\frac{1}{2}} e^{c(xy+x+y)} Pol(x, y),$$

where \( Pol(x, y) \) is a polynomial in \( x \) and \( y \). One can see that such functions are normalizable in the domain \( S \) for \( c < 0 \). However functions of such a form do not belong to the domain of the Hamiltonian since they have improper behaviour at the boundaries. Therefore, actually the Hamiltonian (7) is not quasi-exactly solvable. Nevertheless it can be of interest in the context of pseudo-Hermitian Hamiltonians [15] (also see the discussion below).

4 New families of quasi-exactly solvable systems

Formally exactly-solvable systems

Exactly solvable systems form a subset of quasi-exactly solvable systems, because evidently they have (an infinite flag of) finite-dimensional invariant subspaces.

Let us start discuss of this point in the context of the Lie-algebraic approach from the one-dimensional case [11], [2]. One can select two exactly solvable operators of the first order,
\[ \frac{d}{dz} \text{ and } z \frac{d}{dz} \]. A Hamiltonian constructed in terms of these operators has the following general form:

\[ H' = -P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz}. \]

The resulting Hamiltonian is exactly-solvable, but only formally. To prove the exact solvability one has to check normalizability of corresponding wave functions. Not all systems given by the Hamiltonian are exactly solvable in this sense.

Now we pass to the two-dimensional case. The following first order operators

\[ N_y = y \partial_y, \quad N_x = x \partial_x, \quad L_0 = \partial_y, \quad L_p = y^p \partial_x, \quad (8) \]

where \( p = 0, 1, \ldots, m \), are exactly solvable because they are invariant on the spaces

\[ \mathcal{F}_{m,n} = \text{span}_{a,b \in \mathbb{Z}^+} \{ x^a y^b \} \quad (9) \]

for any \( n \in \mathbb{N} \). For any fixed \( n \) it is possible introduce yet another operator

\[ L_{m,n} = y (mN_x + N_y - mn), \]

which is quasi-exactly solvable. All these operators form a series of representations of the algebra \( \mathbb{R}^{m+1} \supset \mathfrak{gl}(2, \mathbb{R}) \) [14]. It is worth noting that the case \( m = 1 \) has to be considered separately since in this case there exist additional quasi-exactly solvable operators of the first order, e.g. see Ref [16].

The general form of the Hamiltonian constructed in terms of the first order operators (8) is

\[ -H = (p_0 x^2 + P_m(y)x + P_{2m}(y)) \partial_x^2 + (P_1(y)x + P_{m+1}(y)) \partial_{xy}^2 + P_2(y) \partial_y^2 \]

\[ + (q_0 x + Q_m(y)) \partial_x + Q_1(y) \partial_y, \quad (10) \]

where \( p_0, q_0 \in \mathbb{R}, P_q(y) \) and \( Q_p(y) \) are polynomials of degree \( q \) and \( p \) with real coefficients. The Hamiltonian (10) does not depend on the integer parameter \( n \) and, as a consequence, has an infinite flag of finite-dimensional invariant subspaces (9) parametrized by \( n \). Therefore, formally it is an exactly solvable operator [17].

From the expression (10) we infer the form of the inverse metric:

\[ \| g^{\mu \nu} \| = \left( \begin{array}{cc} p_0 x^2 + P_m(y)x + P_{2m}(y) & P_1(y)x + P_{m+1}(y) \\ P_1(y)x + P_{m+1}(y) & P_2(y) \end{array} \right) \].

In general the system given by the Hamiltonian (10) is too complicated to resolve the constraints (3). For simplicity we investigate the cases of factorizable and unfactorizable metric determinants:

\[ \det \| g^{\mu \nu} \| \sim (x - \xi_1(y))(x - \xi_2(y)) \quad \text{and} \quad \det \| g^{\mu \nu} \| \sim (x - \xi_1(y))^2 + \xi_2(y)^2. \]
To this end we have to fix the coefficient functions, e.g.:

\[ P_2(y) = P_2 P_1(y)^2, \]
\[ P_{m+1}(y) = \frac{1}{2} P_1(y) \left( P_2 P_m(y) + (p_0 P_2 - 1) (\xi_1(y) + \xi_2(y)) \right), \]
\[ P_{2m}(y) = \frac{P_2}{4} \left( P_m(y) + \frac{p_0 P_2 - 1}{P_2} (\xi_1(y) + \xi_2(y)) \right)^2 + \frac{p_0 P_2 - 1}{P_2} \xi_1(y) \xi_2(y) \]

for the first case and

\[ P_2(y) = P_2 P_1(y)^2, \]
\[ P_{m+1}(y) = P_1(y) \left( \frac{P_2}{2} P_m(y) + (p_0 P_2 - 1) \xi_1(y) \right), \]
\[ P_{2m}(y) = P_2 \left( \frac{1}{2} P_m(y) + \frac{p_0 P_2 - 1}{P_2} \xi_1(y) \right)^2 + \frac{p_0 P_2 - 1}{P_2} (\xi_1(y)^2 + \xi_2(y)^2) \]

for the second case. The functions \( \xi_i(y) \) are polynomials of degrees no more then \( m \).

By shifting and rescaling the variable \( y \) we can fix the form of the linear function \( P_1(y) \). There are two different cases. Let us discuss the first one:

\[ P_1(y) = y. \]

In this case there are several solutions of equations \( 3 \) with factorizable and unfactorizable metric determinants. For all of the solutions the coefficient functions \( Q_1(y), Q_m(y) \) and \( P_m(y) \) are given by

\[
Q_1(y) = Q_1 y, \\
Q_m(y) = \frac{P_2^2 y^2 q''_m(y) + Q_1 P_2 y q'_m(y) - ((2p_0 - q_0 - 2) P_2 + 2 Q_1) q_m(y)}{1 - (2p_0 - q_0 - 1) P_2 - Q_1} + \left( p_0 - q_0 + \frac{Q_1}{P_2} - 1 \right) (\xi_1(y) + \xi_2(y)), \\
P_m(y) = 2 - \frac{P_2 y q'_m(y) - q_m(y)}{1 - (2p_0 - q_0 - 1) P_2 - Q_1} + \frac{1 - p_0 P_2}{P_2} (\xi_1(y) + \xi_2(y)).
\]

According to the first solution the polynomials \( \xi_i(y) \) are arbitrary,

\[ q_m(y) = \left( \frac{1 - Q_1}{P_2} - 2p_0 + q_0 + 1 \right) \xi_i(y) \]

and the wave functions have the form

\[ \psi = g^{-\frac{1}{4}} |y|^\alpha |x - \xi_i(y)|^\beta Pol(x, y), \]

where \( Pol(x, y) \in F_{m,n} \) \( i = 1 \) or 2 and

\[ \alpha = \frac{Q_1 p_0 + p_0 - q_0 - 1}{2 (p_0 P_2 - 1)} - 1, \quad \beta = \frac{q_0 P_2 + P_2 - Q_1 - 1}{2 (p_0 P_2 - 1)} - 1. \]
The determinant of the inverse metric has the factorizable form
\[ \det ||g^{\mu\nu}|| \sim y^2(x - \xi_1(y))(x - \xi_2(y)). \]

Possible configuration of boundaries in this system is schematically represented on the left plot of figure 1. From the structure of the wave functions (12) one can conclude that for the Hamiltonian to be Hermitian the system should live in the domain \( I_L (I_R) \) with \( x \geq \xi_2(y) \) and \( y \leq 0 \) (\( y \geq 0 \)) or \( II_L (II_R) \) with \( x \leq \xi_1(y) \) and \( y \leq 0 \) (\( y \geq 0 \)). But it is easy to check that for any choice of the parameters \( \alpha > \frac{1}{2} \) and \( \beta > \frac{1}{2} \) the wave functions (12) are not normalizable. Therefore the system is actually not quasi-exactly solvable.

In the regions \( III_L \) and \( III_R \) the wave functions (12) do not belong to the domain of the Hamiltonian. To consider these domains it is necessary to discard the hermicity of the Hamiltonian. Nevertheless in the quasi-exactly solvable sector matrix elements are real hence corresponding eigenvalues are real or form complex conjugated pairs. But this pattern is specific to pseudo-Hermitian Hamiltonians [15]. Therefore the system in the domains \( III_L \) and \( III_R \) can be interesting from this point of view. Moreover some choice of the parameter \( \alpha \) can provide power-like falloff and normalizability of the wave functions (12) (at least part of them). In this sense the system is quasi-exactly solvable. Besides if we discard the hermicity then it is possible to consider another configurations of boundaries when the corresponding system is quasi-exactly solvable or even exactly solvable. Such a configuration is represented on the right plot of the figure (1). Indeed in this case the system living the finite region \( I \) is exactly solvable because wave functions are normalizable for any values of the parameters \( \alpha > -\frac{1}{2} \) and \( \beta > -\frac{1}{2} \).

According to the second solution
\[ \xi_2(y) = \xi_1(y) + \xi_2 y^k, \quad q_m(y) = \left( \frac{1 - Q_1}{P_2} - 2p_0 + q_0 + 1 \right) \xi_1(y) + q_m y^k \]

and the wave functions have the following structure
\[ \psi = y^{-\frac{1}{2}} |y|^\alpha |x - \xi_1(y)|^\beta |x - \xi_1(y) - \xi_2 y^k|^\gamma \text{Pol}(x,y), \]

where \( \text{Pol}(x,y) \in \mathcal{F}_{m,n}, k = P_2^{-1} \in \mathbb{N} \) and
\[ \alpha = \frac{Q_1 p_0 + p_0 - q_0 - 1}{2(p_0 P_2 - 1)} - 1, \quad \beta = \frac{(q_0 P_2 + P_2 - Q_1 - 1) \xi_2 - P_2 q_m}{2(p_0 P_2 - 1) \xi_2} - 1, \quad \gamma = \frac{P_2 q_m}{2(p_0 P_2 - 1) \xi_2}. \]
The determinant of the inverse metric is factorizable
\[
\det \|g^{\mu\nu}\| \sim y^2 (x - \xi_1(y)) \left( x - \xi_1(y) - \xi_2 y^k \right).
\]

Possible configuration of boundaries in this system is schematically represented on the central plot of the figure. From the structure of the wave functions one can conclude that for some choice of the parameters the wave functions are normalizable with power-like falloff and the system is quasi-exactly solvable in any of the domains.

According to the third solution the polynomials \( \xi_i(y) \) and \( q_m(y) \) are not fixed, \( Q_1 = 1 - P_2 (2p_0 - q_0 - 1) \) and the wave functions have the following form
\[
\psi = g^{-\frac{1}{4}} |y|^{\frac{2n-1}{2} p_0} Pol(x, y),
\]
where \( Pol(x, y) \in \mathcal{F}_{m,n} \). Such functions do not belong to the domain of the Hamiltonian or are not normalizable for any choice of the parameters. Therefore the corresponding system is not quasi-exactly solvable. Nevertheless if one discards the hermicity of the Hamiltonian then for the situation represented on the right plot of figure the system can be quasi-exactly solvable (in the regions \( II_L \) and \( II_R \)) or even exactly solvable (in the region \( I \)) for some choice of the parameters \( q_0 \) and \( p_0 \).

Now we pass to the metric with the unfactorizable determinant, i.e. it has no roots in the variable \( x \). In this case there is the following nontrivial solution:
\[
\xi_2(y) = \xi_1(y) + \xi_2 y^k, \quad q_m(y) = q_m \xi_2 y^k - (Q_1 k - k + 2p_0 - q_0 - 1) \xi_1(y)
\]
and the wave functions have the following structure
\[
\psi = g^{-\frac{1}{4}} |y|^\alpha \left( (x - \xi_1(y))^2 + \xi_2^2 y^{2k} \right)^\beta e^{\gamma \arctan \frac{\xi_2 y^k}{x - \xi_1(y)}} Pol(x, y), \tag{14}
\]
where \( Pol(x, y) \in \mathcal{F}_{m,n}, k = P_2^{-1} \in \mathbb{N} \) and
\[
\alpha = \frac{k (Q_1 p_0 + p_0 - q_0 - 1)}{2 (p_0 - k)} - 1, \quad \beta = \frac{Q_1 k + k - q_0 - 1}{4 (k - p_0)} - \frac{1}{2}, \quad \gamma = \frac{q_m}{2 (k - p_0)}.
\]

The determinant of inverse metric has the form
\[
\det \|g^{\mu\nu}\| \sim y^2 \left( (x - \xi_1(y))^2 + \xi_2^2 y^{2k} \right).
\]

The determinant vanishes only at the line \( y = 0 \), therefore the system can live on the left or right semiplane of \( Oyx \). One can see that negative values of the parameter \( \beta \) with large enough modulus can provide power-like falloff and normalizability of a finite number of the wave functions (14). Therefore the system is quasi-exactly solvable.

Now let us consider the second possibility of fixing the linear function \( P_1(y) \):
\[
P_1(y) = P_1.
\]
In this case for all of the solutions the coefficient functions $Q_1(y), Q_m(y)$ and $P_m(y)$ are given by

\[ Q_1(y) = P_1 P_2 Q_1 y, \]
\[ Q_m(y) = \frac{P_2^2 Q_m''(y) + P_1 P_2^2 Q_1 q_m'(y) - P_2 (2p_0 - q_0 + 2Q_1) q_m(y)}{1 - (2p_0 - q_0 + Q_1) P_2} \]
\[ + (p_0 - q_0 + Q_1) \left( \xi_1(y) + \xi_2(y) \right), \]
\[ P_m(y) = 2 \frac{P_1 P_2 q_m'(y) - q_m(y)}{1 - (2p_0 - q_0 + Q_1) P_2} + \frac{1 - p_0 P_2}{P_2} \left( \xi_1(y) + \xi_2(y) \right). \]

(15)

According to the first solution the polynomials $\xi_i(y)$ are arbitrary,

\[ q_m(y) = \left( \frac{1}{P_2} - 2p_0 + q_0 - Q_1 \right) \xi_i(y) \]

and the wave functions have the form

\[ \psi = g^{-\frac{1}{4}} |x - \xi_i(y)|^\alpha e^{-\beta y} Pol(x, y), \]

(16)

where $Pol(x, y) \in F_{m,n}, i = 1 \text{ or } 2$ and

\[ \alpha = \frac{P_2 (q_0 - Q_1) - 1}{2 (p_0 P_2 - 1)} - 1, \quad \beta = \frac{q_0 - p_0 (P_2 Q_1 + 1)}{2 P_1 (p_0 P_2 - 1)}. \]

The determinant of the inverse metric has the form

\[ \text{det} \parallel g^{\mu\nu} \parallel \sim (x - \xi_1(y))(x - \xi_2(y)). \]

(17)

For the Hamiltonian to be Hermitian the functions $\xi_i(y)$ have to obey the inequality $\xi_1(y) \geq \xi_2(y)$ or $\xi_1(y) \leq \xi_2(y)$. Such a configuration is represented on the left side of figure 1. One can check that for any choice of the parameters $\alpha$ and $\beta$ the wave functions (16) are not normalizable. Therefore the system is actually not quasi-exactly solvable. Nevertheless if non-Hermitian Hamiltonians are allowed then for configurations of boundaries represented on the right plot of figure 1 the systems are exactly solvable in the regions $I, II_L$ and $II_R$.

Also there is a solution with the determinant of the form (17) with arbitrary polynomials $\xi_i(y), q_m(y)$ and wave functions with the structure $\psi \sim g^{-\frac{1}{4}} e^{-\beta y} Pol(x, y)$. In this case hermicity of the Hamiltonian or normalizability cannot be provided. Therefore the system is not quasi-exactly solvable. But if we discard the hermicity of the Hamiltonian then for the configuration represented on the right plot of figure 1 the system is exactly solvable if it lives in the domains $I, II_L$ and $II_R$.

5 Conclusion

In this paper we have considered quantum-mechanical systems in domains with boundaries. Such a situation is usual for the Lie-algebraic approach \cite{1, 2} to construction of multidimensional quasi-exactly solvable Hamiltonians. The hermicity of the operators prescribes the
specific behaviour of wave functions which belong to domains of Hamiltonians. By an example it was shown that not all of known quasi-exactly solvable systems are in fact quasi-exactly solvable even if corresponding wave functions are normalizable.

Besides we have constructed new two-dimensional families of formally exactly solvable systems. Application of the restrictions the behaviour on wave functions in a domain with boundaries leads to the conclusion that the constructed systems are quasi-exactly solvable at best.

If one discards the hermicity then the behaviour of the wave functions is governed only by the normalizability. It is worth noting that the corresponding matrix elements are real. In the quasi-exactly solvable sector the corresponding matrix is finite dimensional therefore quasi-exactly solvable eigenvalues are defined by an algebraic equation with real coefficients. This means that the eigenvalues are represented by real values and complex conjugated pairs of numbers but this pattern corresponds exactly to the spectrum of a pseudo-Hermitian Hamiltonian. Therefore the formally quasi-exactly solvable and exactly solvable systems can be interesting in this context. Of course, the exact relationship between such quasi-exactly solvable operators and pseudo-Hermitian Hamiltonians requires more detailed investigation.

References

[1] A. Turbiner, Commun. Math. Phys. 118 (1988) 467.

[2] M. Shifman, Int. J. Mod. Phys. A4 (1989) 2897.

[3] A. Ushveridze, Sov. J. Part. Nucl 20 (1989) 504; 23 (1992) 25.

[4] A. Ushveridze, Quasi-exactly solvable models in quantum mechanics, (IOP Publishing, Bristol, 1994).

[5] S. Klishevich and M. Plyushchay, Nucl. Phys. B616 (2001) 403, hep-th/0105135. Nucl. Phys. B640 (2002) 481, hep-th/0202077.

[6] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, Nucl. Phys. B553 (1999) 644, hep-th/9808034.

[7] S. Klishevich and M. Plyushchay, Nucl. Phys. B606 (2001) 583, hep-th/0012023. Nucl. Phys. B628 (2002) 217, hep-th/0112158. H. Aoyama, M. Sato and T. Tanaka, Nucl. Phys. B619 (2001) 105, quant-ph/0106037. H. Aoyama, N. Nakayama, M. Sato, and T. Tanaka, Phys. Lett. B519 (2001) 260, hep-th/0107048.

[8] A. Andrianov and A. Sokolov, Nucl. Phys. B660 (2003) 25, hep-th/0301062. A. Andrianov and F. Cannata, J. Phys. A37 (2004) 10297, hep-th/0407077.

[9] S. Klishevich, Quasi-exact solvability and intertwining relations, hep-th/0410064.

[10] A. Andrianov, M. Ioffe, and V. Spiridonov, Phys. Lett. A174 (1993) 273, hep-th/9303005.

[11] A. González-López, N. Kamran and P. Olver, Commun. Math. Phys. 153 (1993) 117.
[12] G. Post and A. Turbiner, Russ. Journ. Math. Phys. 3 (1995) 113, func-an/9307001.

[13] D. Gomez-Ullate, N. Kamran, and R. Milson, J. Phys. A38 (2005) 2005, nlin.si/0401030

[14] A. González-López, N. Kamran and P. Olver, J. Phys. A24 (1991) 3995.

[15] C. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243. A. Mostafazadeh, J. Math. Phys. 43 (2002) 205, math-ph/0107001. C. Bender, S. Boettcher and P. Meisinger, J. Math. Phys. 40 (1999) 2201, quant-ph/9809072. C. Bender, D. Brody and H. Jones, Phys. Rev. Lett. 89 (2002) 270401, Erratum-ibid. 92 (2004) 119902, quant-ph/0208076.

[16] N. Kamran, R. Milson, and P. Olver, Adv. Math. 156 (2000) 286, solv-int/9904014

[17] A. Turbiner, Lie algebraic approach to the theory of polynomial solutions. 1. Ordinary differential equations and finite difference equations in one variable, hep-th/9209079