Heavy Traffic Analysis of the Mean Response Time for Load Balancing Policies in the Mean Field Regime

Tim Hellemans  
Benny Van Houdt  
University Of Antwerp  
Middelheimlaan 1  
Antwerp 2000, Belgium

ABSTRACT
Mean field models are a popular tool used to analyse load balancing policies. In some exceptional cases the response time distribution of the mean field limit has an explicit form. In most cases it can be computed using either a recursion or a differential equation (for exponential job sizes with mean one). In this paper we study the value of the mean response time $\mathbb{E}[R_1]$ as the arrival rate $\lambda$ approaches 1 (i.e. the system gets close to instability). As $\mathbb{E}[R_1]$ diverges to infinity, we scale with $-\log(1 - \lambda)$ and present a method to compute the limit $\lim_{\lambda \to 1^{-}} -\frac{\mathbb{E}[R_1]}{\log(1 - \lambda)}$.

This limit has been previously determined for SQ($d$, $K$) and LL($d$), two well-known policies that assign an incoming job to a server with either the shortest queue or least work left among $d$ randomly selected servers. However the derivation of the result for SQ($d$, $K$) relied on the closed form representation of the mean response time and does not seem to generalize well, moreover the proof for LL($d$) is incomplete. In contrast, we present a general result that holds for these policies the dispatcher balances the load on the servers of view, load balancing policies can be split into two main categories. The first category exists of queue length dependent load balancing policies where the dispatcher collects some information on the queue length distribution is exponential with mean one. For these policies, jobs arrive in batches of fixed size, respectively. For the shortest queue variant, we obtain the limit $\frac{1}{\log(N)}$ while for the least loaded variant, we obtain $\frac{1}{\sum_{i=1}^{N} \frac{d_i}{N^2}}$.

1 INTRODUCTION
Load balancing plays an important role in large scale data networks, server farms, cloud and grid computing. From a mathematical point of view, load balancing policies can be split into two main categories. The first category exists of queue length dependent load balancing policies where the dispatcher collects some information on the number of jobs in some servers and assigns an incoming job using this information. A well studied example of this policy type is the SQ($d$, $K$) policy, where an incoming job is assigned to the shortest among $d$ randomly selected servers (see e.g. [1, 2]). The second category consists of workload dependent load balancing policies, for these policies the dispatcher balances the load on the servers by employing information on the amount of work that is left on some of the servers (see also [3]). This can be done explicitly if we assume the amount of work on servers is known or implicitly by employing some form of redundancy such as e.g. cancellation on start or late binding (see also [4]). A well studied policy of this type is the LL($d$) policy where each incoming job joins the server with the least amount of work left out of $d$ randomly sampled servers (see e.g. [5]).

In order to compute performance metrics such as the mean response time, the response time distribution, etc. most work relies on mean-field models [6–9]. For these models, the system behaviour is studied in a limit where the number of servers $N$ tends to infinity. For this limiting system one then assumes or proves that all servers become i.i.d. (see also [10]). The whole system can therefore be described by the behaviour of a single queue. In order to analyse this single queue, termed the queue at the cavity, one often restricts to the case of exponential job sizes of mean one. For queue length dependent load balancing policies, the state descriptor is then given by the number of jobs in the queue at the cavity. The transient behaviour of the queue length distribution is described by a system of Ordinary Differential Equations (ODEs). The equilibrium queue length distribution (as time goes to infinity) is described by a recurrence relation. For workload dependent load balancing policies, the transient workload distribution is described by a Partial Integro Differential Equation. The equilibrium workload distribution can be described by an Integro Differential Equation which can sometimes be simplified to a 1 dimensional ODE in case job sizes are exponential. Throughout this paper, we assume the job size distribution is exponential with mean one.

We relate to each system size $N$ an arrival rate $\lambda N$. To obtain the mean field limit as described earlier, one sets $\lambda N = \lambda N$ for some fixed $\lambda < 1$. One is often interested in the behaviour of the queueing system as the system approaches its critical load. To study this, one could set $\lambda N = \lambda N$ where $\lambda N \to 1^{-}$ as $N$ tends to infinity. This approach was for example used in [11, 12] to study the SQ($d$, $K$) model in heavy traffic. Another approach, which is the one we use here, is to first obtain the stationary distribution of the mean field model with a fixed $\lambda N = \lambda < 1$ and subsequently take the limit $\lambda \to 1^{-}$ of the resulting mean field models.

More specifically, in this paper we establish a general result which can be employed to obtain the limit

$$\lim_{\lambda \to 1^{-}} -\frac{\mathbb{E}[R_1]}{\log(1 - \lambda)},$$

(1)

where $R_1$ is the response time distribution of either a queue length or workload dependent load balancing policy (see Theorem 2.1 resp. 2.2). This value can be used as a reference of how well a policy behaves under a high load. As we divide by $-\log(1 - \lambda)$, we are focussing on load balancing policies where an exponential
improvement in the mean response time is expected compared to random assignment. In [1] the limit in (1) was shown to be equal to \( \varpi_{\text{SQ}(d)} \) for the SQ(d) policy. For LL(d) it is indirectly shown in [5] that the limit (1) is given by \( \varpi_{\text{LL}(d)} \), though the proof is not rigorous (c.f. Section 3.2). Both these proofs do not seem to generalize well to other load balancing policies as they rely on explicit formulas for the limiting queue or workload distribution. Our result provides a list of sufficient conditions under which the limit in (1) can be computed in a straightforward manner. Although computing the limit is easy, verifying the listed conditions may present quite a challenge, one of our main contributions is establishing these conditions for SQ(d, K) and LL(d, K).

We start by illustrating our method on SQ(d) and LL(d) and then derive a novel result for other policies. We first consider the SQ(d1, . . . , dn, p1, . . . , pn) and LL(d1, . . . , dn, p1, . . . , pn) policies, where with probability \( p_i \) we select \( d_i \) servers and assign the incoming job to the queue with the least number of jobs and the least amount of work amongst these \( d_i \) selected servers, respectively. We show that for SQ(d1, . . . , dn, p1, . . . , pn), we have

\[
\lim_{\lambda \to 1^-} \frac{\mathbb{E}[R_{\lambda}]}{\log(1 - \lambda)} = \frac{1}{\log(A)},
\]

while for LL(d1, . . . , dn, p1, . . . , pn) we have

\[
\lim_{\lambda \to 1^-} \frac{\mathbb{E}[R_{\lambda}]}{\log(1 - \lambda)} = \frac{1}{\log(1 - \lambda)} \left( \sum_{i=1}^{n} p_id_i - 1 \right).
\]

We observe that, when the system is highly loaded, the choice of \( d_i \) does not matter as long as the total amount of redundancy \( \sum_{i=1}^{n} p_id_i \) remains constant. Furthermore, we find a general method to investigate which choice of \( p_i \) and \( d_i \) yields smaller response times when \( \lambda < 1 \).

In the special case of LL(1, d, 1 − p, p), this policy applies the power of \( d \) choices only to a portion of the incoming jobs and assigns the other jobs arbitrarily. For this policy, we find that whenever \( \lambda < 1 \) the probability that an arbitrary queue has workload at least \( w \) is given by:

\[
\tilde{F}(w) = \lambda \left[ \frac{1 - (1 - p)\lambda}{p\lambda^d + (1 - (1 - p)\lambda - p\lambda^d)(1 - (1 - p)\lambda)w} \right]^\frac{1}{\pi},
\]

but no such solution appears to exist in general. This closed form expression also yields an alternative method to obtain the limiting result.

Next, we apply our method to the SQ(d, K) resp. LL(d, K) policy (see also [13, 14]). For these policies, jobs are assumed to arrive in batches of size \( K \), we then sample \( d > K \) servers and the jobs are assigned to the \( K \) queues with the least number of jobs resp. least amount of work left. We show in Theorems 5.8, 5.9 that

\[
\lim_{\lambda \to 1^-} \frac{\mathbb{E}[R_{\lambda}]}{\log(1 - \lambda)} = \frac{1}{\log(A)},
\]

for SQ(d, K) and

\[
\lim_{\lambda \to 1^-} \frac{\mathbb{E}[R_{\lambda}]}{\log(1 - \lambda)} = \frac{1}{\log(1 - \lambda)} \left( \frac{K}{d - K} \right),
\]

for LL(d, K). One of the main technical contributions of the paper, apart from establishing Theorems 2.1 and 2.2, exists in verifying the third condition of these theorems for SQ(d, K) and LL(d, K).

Note that if we denote by \( A \) the average number of queues sampled per arrival, the heavy traffic limit for the policies considered in this paper equals \( \frac{1}{\log(A)} \) for the SQ-based and \( \frac{1}{A-1} \) for the LL-based policies. As we use the same scaling \( -\log(1 - \lambda) \) for each of these policies, one can easily deduce the limit \( \lim_{\lambda \to 1^-} \frac{\mathbb{E}[R_{\lambda}]}{\log(1 - \lambda)} = \frac{1}{\log(A)} \), where \( R_{\lambda}^{\text{SQ}} \) and \( R_{\lambda}^{\text{LL}} \) are the response times for the SQ and LL variant of the same policy. We therefore observe that the gain from using the exact workload rather than the more coarse metric of the queue length increases as more queues are sampled per arrival (i.e., as \( A \) increases). Moreover, we observe that when servers are highly loaded, the only thing that matters is the average number of servers sampled per arrival and whether we use the queue length or workload information.

To obtain these results, the main insight we use is the fact that, as \( \lambda \) approaches one, all queues have more or less the same amount of work. We are able to analytically approximate this amount of work, it represents how well a policy is able to balance loads under a high arrival rate. A similar observation was made in [15], where it was noted that for Redundancy \( d \) under Processor Sharing with identical replica’s, the workload at all servers diverges to infinity at an equal rate when \( \lambda \) exceeds \( \frac{1}{d} \).

The paper is structured as follows. In Section 2 we present the two main results. We illustrate these results on SQ(d) and LL(d) in Section 3. In Section 4 we present the results for SQ(d1, . . . , dn, p1, . . . , pn) and LL(d1, . . . , dn, p1, . . . , pn), here we also consider the case where \( \lambda \) is bounded away from 1 and the special case of LL(1, d, 1 − p, p). In Section 5 we cover SQ(d, K) and LL(d, K). Conclusions are drawn in Section 6.

## 2 GENERAL RESULT

As stated before, the equilibrium queue length or workload distribution in the mean field regime is often characterized by a recurrence relation or Ordinary Differential Equation (ODE). Our two main results, Theorem 2.1 and Theorem 2.2 show how to compute the limits:

\[
\lim_{\lambda \to 1^-} \frac{\sum_{k=0}^{\infty} u_k}{\log(1 - \lambda)} = \lim_{\lambda \to 1^-} \int \frac{\tilde{F}(w) \, dw}{\log(1 - \lambda)},
\]

where \( (u_k) \) and \( \tilde{F}(w) \) satisfy some recurrence relation and ODE, respectively. As such we can use these results to study the mean field limit as \( \lambda \) tends to one.

### 2.1 Recurrence relation

Assume we have a recurrence relation of the form \( u_{k+1} = T_{\lambda}(u_k) \), where \( T_{\lambda} \) is some positive function. Before presenting our general result in a formal manner, we provide some intuition in the special case of the SQ(d) policy, for which the recurrence relation in [16] can be rewritten as \( u_{k+1} = \lambda u_k^d \) for \( k > 0 \) and \( u_0 = 1 \), yielding the well-known result that \( u_k = \lambda^{(d-1)/(d-1)} \). In Figure 1a we observe that, as we increase \( \lambda \), the value of \( u_k \) remains close to one for larger values of \( k \), but the shape of the curve as it drops to zero looks very similar for the different values of \( \lambda \). This motivates us to define \( N_{d, \lambda} \), which represents the point at which \( u_k \) drops below some threshold close to one. One would then expect that:

\[
\lim_{\lambda \to 1^-} \frac{\sum_{k=0}^{\infty} u_k}{\log(1 - \lambda)} = \lim_{\lambda \to 1^-} \frac{N_{d, \lambda}}{\log(1 - \lambda)}
\]
as \( u_k \approx 1 \) for \( k \leq N_{r, \lambda} \) and the sum of the remaining \( u_k \) values remains bounded. More specifically, we define the threshold as \( u_1 = \epsilon \), where \( u_1 \) is a solution of \( u = T_\lambda(u) \) such that \( T_\lambda \) decreases to 1 as \( \lambda \) increases to one. For \( \text{SQ}(d) \) we set \( u_1 = \lambda^{1/(1-d)} \) and one easily verifies that (with \( \lceil \cdot \rceil \) the ceiling function):

\[
N_{r, \lambda} = \frac{1}{\log(d)} \log \left( \frac{\log(\lambda^{1/(1-d)} - \epsilon)}{\log(\lambda^{1/(1-d)})} \right),
\]

from which it follows that \( \lim_{\lambda \to 1^-} - \frac{N_{r, \lambda}}{\log(1-\lambda)} = \frac{1}{\log(d)} \), as expected.

In order to compute \( \lim_{\lambda \to 1^-} - \frac{N_{r, \lambda}}{\log(1-\lambda)} \) in case we do not have an explicit expression for \( N_{r, \lambda} \), we define the sequence \( \tilde{u}_k = u_k - u_k \). Note that \( N_{r, \lambda} \) is the largest value of \( k \) for which \( \tilde{u}_k \) remains below \( \epsilon \). In Figure 1b, we plotted \( \frac{\tilde{u}_k}{u_k} \) as a function of \( k \) for \( \text{SQ}(d) \). We observe that \( \frac{\tilde{u}_k}{u_k} \approx d \) (represented by the horizontal lines) for \( k \) bounded away from 0 and \( k \leq N_{r, \lambda} \). This in turn entails that:

\[
\epsilon \approx \tilde{u}_{N_{r, \lambda}} \approx \tilde{u}_{N_{r, \lambda}-1} \approx \cdots \approx \tilde{u}_1 \approx d N_{r, \lambda} \cdot (\lambda^{1/(1-d)} - 1).
\]

Taking the log on both sides, dividing by \( -\log(1 - \lambda) \) and taking the limit of \( \lambda \to 1^- \) allows us to recover that \( \lim_{\lambda \to 1^-} - \frac{N_{r, \lambda}}{\log(1-\lambda)} \approx \frac{1}{\log(d)} \), as \( \lim_{\lambda \to 1^-} \frac{\log(\lambda^{1/(1-d)} - \epsilon)}{\log(1-\lambda)} = 1 \). To establish Theorem 2.1 we also rely on the sequence \( \tilde{u}_k \) and introduce upper and lower bounds on \( \tilde{u}_{k+1}/\tilde{u}_k \) to derive an expression for \( \lim_{\lambda \to 1^-} - \frac{N_{r, \lambda}}{\log(1-\lambda)} \).

**Theorem 2.1.** For \( \lambda \in (0, 1) \) consider the following recurrence relation:

\[
 u_{k+1} = T_\lambda(u_k),
\]

with \( u_0 = 1 \) and \( T_\lambda : [0, \infty) \to [0, \infty) \) with \( T_\lambda([0, 1]) \subseteq [0, 1] \) a function which satisfies:

(a) \( \exists l \in (0, 1) : \)

- For \( \lambda \in (l, 1) \) there exists a \( u_2 \in (1, \infty) : T_\lambda(u_2) = u_2 \).
- The function \( u : \lambda \to u_2 \) is continuous and \( \lim_{\lambda \to 1^-} u_2 \lambda = 1 \).

(b) For all \( u \in [0, 1] \), we have:

- \( T_\lambda(0) = 0, T_\lambda(u) < u \) and \( \lim_{\lambda \to 1^-} \frac{T_\lambda(u)}{u} < 1 \),

- \( \left( \frac{T_\lambda(u)}{u} \right) \geq 0 \), which implies that \( T_\lambda \) is increasing on \( [0, 1) \).

For all \( \lambda \in (l, 1) \), we define:

\[
h_\lambda(x) = \frac{\underbrace{u_2 \lambda}_{= T_\lambda(u_2)} - x}{x}.
\]

(c) There is a \( b \in \mathbb{N} \) such that for all \( \lambda \in (l, 1) \) we have \( h_\lambda(x) \) is decreasing for \( x \in [u_\lambda - \lambda^b, 1) \).

(d) If we let \( k_\lambda = \min(b \in \mathbb{N} \mid u_\lambda - \lambda^b \) then there is some \( k \) such that \( k \leq k_\lambda \lambda \).

(e) There is some \( A \in [0, \infty) \) for which \( \lim_{\lambda \to 1^-} h_\lambda(u_\lambda - \lambda^b) = A \).

(f) There is some \( B \in [0, \infty) \) for which \( \lim_{\lambda \to 1^-} - \frac{\log(u_\lambda - \lambda^b)}{\log(1-\lambda)} = B \).

It then follows that:

\[
\lim_{\lambda \to 1^-} - \frac{\sum_{k=0}^{\infty} u_k}{\log(1-\lambda)} = \frac{B}{\log(A)}.
\]

**Proof.** Throughout the proof we let \( \lambda \in (l, 1) \) and we define \( \tilde{u}_k = u_k - u_k \) for all \( k \). By definition of \( k_\lambda \) in (d) we have:

\[
\tilde{u}_{k+1} = u_{k+1} - u_k \leq u_{k} - \lambda^b \leq u_{k} - u_{k} = \tilde{u}_k.
\]

The sequence \( u_k \) decreases to zero as \( \lim_{k \to \infty} u_k = \lim_{k \to \infty} T_\lambda(u_k) \) and by the continuity of \( T_\lambda \) we have \( \lim_{k \to \infty} T_\lambda(u_k) = \lambda \lim_{k \to \infty} u_k \).

Hence \( \lim_{k \to \infty} u_k = 0 \) is a fixed point on \([0, 1]\) of \( T_\lambda \). We thus find that \( \tilde{u}_k \) increases to \( u_2 \) as \( k \) tends to infinity.

We have the following recurrence relation for \( \tilde{u}_k \):

\[
\tilde{u}_{k+1} = u_{k+1} - u_k = u_{k} - T_\lambda(u_k) = u_{k} - T_\lambda(u_{k} - u_{k}).
\]

This allows us to obtain the equality \( \frac{\tilde{u}_k}{u_k} = h_\lambda(\tilde{u}_k) \) (with \( h_\lambda(x) \) defined as in (4)). Furthermore we find from (c) and (d) that for any \( k \geq k_\lambda \):

\[
\frac{\tilde{u}_{k+1}}{\tilde{u}_k} \leq h_\lambda(\tilde{u}_k) \leq h_\lambda(u_\lambda - \lambda^b).
\]

Denote \( h_\lambda(u_\lambda - \lambda^b) \) as \( A_2 \). Let \( 0 < \epsilon < 1 \) be arbitrarily small and define \( N_{u, \lambda} = \max(k \in \mathbb{N} \mid \tilde{u}_k \leq \epsilon) \). In our proof, we will always take \( \lim_{\lambda \to 1^-} \) prior to \( \lim_{k \to \infty} \), therefore we may assume w.l.o.g. that \( \lambda \) is sufficiently close to one such that \( u_\lambda - \lambda^b \leq \epsilon \). This
Taking the logarithm on both sides and rearranging terms, we find the following inequality:

\[
\log(\epsilon) - \log(u_\lambda - \lambda^b) \leq N_{\lambda, \epsilon} - \hat{k}_\lambda + 2.
\]

As \( - \log(u_\lambda - \lambda^b) \) tends to infinity when \( \lambda \) tends to one and \( \hat{k}_\lambda \) is bounded by \( \bar{k}_\lambda \), \( N_{\lambda, \epsilon} \) must tend to infinity as well.

Dividing both sides by \( - \log(1 - \lambda) \) and taking the limit \( \lambda \to 1^- \) we find from (d), (e) and (f) that:

\[
\frac{B}{\log(A)} \leq \lim_{\lambda \to 1^-} \frac{N_{\lambda, \epsilon}}{- \log(1 - \lambda)}.
\]

For \( k \leq N_{\lambda, \epsilon} \) we have \( u_k - u_{k-1} \leq \epsilon \) and therefore \( 1 - \epsilon \leq u_k \). From this we find:

\[
(1 - \epsilon)B \leq \lim_{\lambda \to 1^-} \frac{(1 - \epsilon)N_{\lambda, \epsilon}}{- \log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{\sum_{k=0}^\infty u_k}{- \log(1 - \lambda)}.
\]

Letting \( \epsilon \to 0^+ \) we find the first inequality. For the other inequality we let \( \hat{k}_\lambda \leq k \leq N_{\lambda, \epsilon} \) be arbitrary. We find that \( (u_k - \lambda^b) \leq u_{k+1} \leq \epsilon \) and therefore we have \( \frac{u_{k+1}}{u_k} = \frac{\hat{h}_\lambda(u_k)}{h(\epsilon)} \) which implies:

\[
\epsilon \geq u_{N_{\lambda, \epsilon}} = \frac{\hat{u}_{N_{\lambda, \epsilon}+1}}{\hat{u}_{N_{\lambda, \epsilon}-1}} \cdot \cdots \cdot \frac{\hat{u}_{k+1}}{\hat{u}_k} \cdot \frac{\hat{u}_k}{\hat{h}_\lambda(u_k)} \geq \frac{\hat{h}(\epsilon)}{h(\epsilon)} \frac{\hat{u}_k}{\hat{h}_\lambda(u_k)} \leq \epsilon.
\]

Taking the logarithm on both sides and rearranging terms yields:

\[
N_{\lambda, \epsilon} - \hat{k}_\lambda \leq \log(\epsilon) - \log(u_\lambda - \lambda^b) \leq \frac{B}{\log(A)}.
\]

Dividing by \( - \log(1 - \lambda) \) and taking the limit \( \lambda \to 1^- \) on both sides allows us to find from (1):

\[
\lim_{\lambda \to 1^-} \frac{N_{\lambda, \epsilon}}{- \log(1 - \lambda)} \leq \frac{B}{\log(A)}.
\]

Note that for any \( k \geq N_{\lambda, \epsilon} + 1 \) we have \( u_k \geq \epsilon \) and therefore \( u_k \leq u_\lambda - \epsilon \leq 1 \) for \( \lambda \) large enough. It thus follows from (b) that:

\[
\frac{u_{k+1}}{u_k} = \frac{\hat{u}_{k+1}}{\hat{u}_k} \geq \frac{\hat{h}_\lambda(u_k)}{h(\epsilon)} \geq 1 - \epsilon.
\]

It follows that:

\[
\sum_{k=N_{\lambda, \epsilon}+1}^{\infty} u_k = \sum_{k=N_{\lambda, \epsilon}+1}^{\infty} u_{N_{\lambda, \epsilon}+1} \cdot \cdots \cdot u_k \leq \frac{u_{N_{\lambda, \epsilon}+1}}{1-\epsilon} \leq \frac{C_{\lambda, \epsilon}}{1-\epsilon}.
\]

Note that:

\[
C_{\lambda, \epsilon} = \lim_{\lambda \to 1^-} \frac{\hat{h}_\lambda(u_k)}{\hat{h}(\epsilon)} = \frac{T_\lambda(u_\lambda - \epsilon)}{1-\epsilon} = \frac{T_\lambda(1-\epsilon)}{1-\epsilon} < 1.
\]

Due to the continuity of \( T_\lambda \) and (b). Taking the limit \( \lambda \to 1^- \) we find that:

\[
\lim_{\lambda \to 1^-} \frac{\sum_{k=0}^{\infty} u_k}{- \log(1 - \lambda)} = \lim_{\lambda \to 1^-} \frac{\sum_{k=0}^{\infty} u_{k+1}}{- \log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{N_{\lambda, \epsilon} + 1}{- \log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{1}{1-C_{\lambda, \epsilon}} \cdot \frac{1}{- \log(1 - \lambda)} = 0.
\]

Taking the limit \( \epsilon \to 0^+ \) and applying (g) we obtain the other inequality. This completes the proof. □

Remark. Computing the value of \( A \) and \( B \) is typically quite easy, which immediately yields a possible value for the limit under consideration. For any load balancing strategy that is at least as good as random, Condition 2.1(d) follows from \( u_{k+1} \leq u_k \). Verifying that all of the conditions hold can however be challenging in some cases.

2.2 Ordinary Differential Equation

We now show Theorem 2.2, which can be seen as a continuous analogue of Theorem 2.1.

Theorem 2.2. For any \( \lambda \in (0, 1) \) let \( \hat{F} : [0, \infty) \to [0, 1) \) be a solution to the ODE:

\[
\hat{F}'(w) = \hat{T}_\lambda(\hat{F}(w)) - \hat{F}(w),
\]

with \( \hat{F}(0) = \lambda \), where we assume \( \hat{F} \) is the unique continuously differentiable solution to this ODE. Further we assume that \( T_\lambda \) satisfies all the requirements of Theorem 2.1, except that (d) is replaced by the condition:

\[
(d') \text{ If we let } \hat{w}_\lambda \text{ be such that } \hat{F}(\hat{w}_\lambda) = \lambda^b, \text{ then there is some } \hat{w} \text{ which can be chosen independently of } \lambda \text{ such that } \hat{w}_\lambda \leq \hat{w}.
\]

We then have:

\[
\lim_{\lambda \to 1^-} \int_0^\infty \hat{F}(w) dw = \frac{B}{\log(A)}.
\]

Proof. Our strategy exists in showing that \( \hat{F}(w) \) stays close to one for a long enough time and then decays sufficiently fast to zero. Throughout the proof, we assume that \( \lambda \in (\bar{1}, 1) \). Due to (b) we find that \( \hat{F}(w) \) is decreasing and the continuity of \( T_\lambda \) implies that \( \lim_{w \to -\infty} \hat{F}(w) = 0 \) (as it is a fixed point of \( T_\lambda \)).

Define \( u_\lambda \) as in (a) and let \( H(w) = u_\lambda - \hat{T}_\lambda(u_\lambda - H(w)) \). We find:

\[
H'(w) = u_\lambda - \hat{T}_\lambda(u_\lambda - H(w)) - H(w),
\]

therefore we have \( \frac{H'(w)}{H(w)} = \frac{\hat{h}_\lambda(H(w))}{1 - \epsilon} \). Due to (c) this yields for any \( w \geq \hat{w}_\lambda \):

\[
\frac{H'(w)}{H(w)} \leq \frac{\hat{h}_\lambda(u_\lambda - \lambda^b)}{1 - \epsilon} - 1.
\]

Now let \( 0 < \epsilon < 1 \) be arbitrary. As \( H(w) \) increases from \( u_\lambda - \lambda \) to \( u_\lambda \), we can define \( \hat{w}_\lambda \) such that \( H(\hat{w}_\lambda) = \epsilon \) for \( \lambda \) large enough. In fact we assume w.l.o.g. that \( \lambda \) is sufficiently close to one such that \( u_\lambda - \lambda^b \leq \epsilon \). Therefore \( \hat{w}_\lambda \leq \hat{w}_\lambda \) as \( H(\hat{w}_\lambda) = u_\lambda - \lambda^b \). By integrating (9) from \( \hat{w}_\lambda \) to \( \hat{w}_\lambda \) we find:

\[
\log \left( \frac{H(w, \lambda)}{H(\hat{w}_\lambda)} \right) = \int_{\hat{w}_\lambda}^w H'(u) = \int_{\hat{w}_\lambda}^w \left( \lambda^b - u_\lambda \right) \leq \frac{\hat{h}_\lambda(u_\lambda - \lambda^b)}{1 - \epsilon} - 1.
\]

Dividing both sides by \( - \log(1 - \lambda) \) and taking the limit \( \lambda \to 1^- \) we obtain:

\[
\lim_{\lambda \to 1^-} \frac{\log(1 - \lambda)}{- \log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{\log(\hat{h}_\lambda(u_\lambda - \lambda^b))}{- \log(1 - \lambda)} - 1.
\]

As \( \hat{w}_\lambda \) is bounded by \( \hat{w} \). Applying (d'), (f) and (e) we obtain:

\[
\frac{B}{A} \leq \lim_{\lambda \to 1^-} \frac{- \log(u_\lambda - \lambda^b)}{- \log(1 - \lambda)}.
\]
For any \( w \leq w_{r, \lambda} \) we have \( 1 - \epsilon \leq u_{\lambda} - \epsilon = F(w_{r, \lambda}) \leq F(w) \). It follows that:

\[
(1 - \epsilon) \frac{B}{\lambda - 1} \leq \lim_{\lambda \to 1^-} -\int_0^{w_{r, \lambda}} (1 - \epsilon) \frac{u_{\lambda}}{\log(1 - \lambda)} du \\
\leq \lim_{\lambda \to 1^-} -\int_0^{\infty} F(w) dw \cdot \frac{1}{\log(1 - \lambda)}.
\]

This shows one inequality by letting \( \epsilon \to 0^+ \). To show the other we first note that for any \( w \in (w_{\lambda, \lambda}, w_{r, \lambda}) \) we have \( u_{\lambda} - \lambda^b \leq H(w) \leq \epsilon \) and therefore also:

\[
H'(w) = h_{\lambda}(H(w)) - 1 \geq h_{\lambda}(\epsilon) - 1.
\]

Integrating both sides from \( w_{\lambda} \) to \( w_{r, \lambda} \) we find:

\[
\log \left( \frac{H(w_{r, \lambda})}{H(w_{\lambda})} \right) \geq (w_{r, \lambda} - w_{\lambda})(h_{\lambda}(\epsilon) - 1).
\]

Dividing both sides by \(-\log(1 - \lambda)\) and taking the limit of \( \lambda \to 1^- \), this implies that we have (also use (f)):

\[
\lim_{\lambda \to 1^-} \frac{w_{r, \lambda}}{\log(1 - \lambda)} (h_{\lambda}(\epsilon) - 1) \leq \lim_{\lambda \to 1^-} \frac{(\log(\epsilon) - \log(u_{\lambda} - \lambda^b))}{\log(1 - \lambda)} = B.
\]

Note that we have:

\[
\int_0^{\infty} F(u) du = \int_0^{w_{r, \lambda}} F(u) F'(u) dF(u) = \int_0^{u_{\lambda}} \frac{1}{1 - \frac{T(u)(x)}{\lambda}} dx,
\]

assuming that \( \lambda \) is sufficiently close to one, we find from (b) that (10) is bounded by:

\[
(u_{\lambda} - \epsilon)^2 /((u_{\lambda} - \epsilon) - (T(u)(u_{\lambda} - \epsilon)) )
\]

which can be bounded uniformly in \( \lambda \). This allows us to obtain:

\[
\lim_{\lambda \to 1^-} \int_0^{\infty} F(w) dw \frac{1}{\log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{w_{r, \lambda}}{\log(1 - \lambda)} \leq \lim_{\lambda \to 1^-} \frac{w_{r, \lambda}}{\lambda h_{\lambda}(\epsilon) - 1}.
\]

Taking the limit \( \epsilon \to 0^+ \) and applying (g), this completes the proof. \( \Box \)

For condition 2.2(d') it suffices to show that \( F(w) \leq \lambda e^{-(1-\lambda)w} \). Indeed, to have \( \lambda e^{-(1-\lambda)w} \leq \lambda^b \) it suffices to have \( b - 1 \leq w_{\lambda} \). Therefore one may pick \( w = b - 1 \). Note that \( \lambda e^{-(1-\lambda)w} \) is probability that the workload of an M/M/1 queue is at least \( w \), so again it suffices that the policy is at least as good as random.

3 POWER OF D CHOICES

In this section, we illustrate that Theorems 2.1, 2.2 can be used to compute the limit:

\[
\lim_{\lambda \to 1^-} -\frac{E[R_1]}{\log(1 - \lambda)}
\]

where \( R_1 \) corresponds to the response time of an SQ(d) or LL(d) load balancing policy with exponential job sizes of mean one and arrival rate \( \lambda \).

3.1 SQ(d)

As stated before, for SQ(d) we have \( T_1(u) = \lambda u^d \) and we clearly find that for any \( \lambda \leq 1 \) the equation \( T_1(u) = u \) has the unique solution \( u_{\lambda} = \lambda^{1/d} \in [1, \infty) \).

We verify that all requirements of Theorem 2.1 are satisfied. From (11), it is obvious that 2.1(a) is satisfied for \( \lambda = 0 \). We find that \( T_1(u) = \lambda u^d \) and also \( \{ T_1(u) \} = (d - 1)u^{d-2} \) from which 2.1(b) trivially follows. Furthermore we have (with \( h_{\lambda}(x) \) defined as in (4)):

\[
h_{\lambda}(x) = \frac{\lambda^{1/d} - \lambda (\lambda^{1/d} - x)^d}{x},
\]

differentiating \( x^2 h_{\lambda}'(x) \) once more yields:

\[
(x^2 h_{\lambda}'(x))(x')' = -\lambda(d-1)d \frac{1}{x^d} (\lambda^{1/d} - x)^{d-2} x,
\]

which is obviously negative, 2.1(c) now follows by \( b = 0 \) from the fact that \( x^2 h_{\lambda}'(x) \) equals 0 for \( x = 0 \). 2.1(d) now trivially holds as \( b = 0 \). For 2.1(e) we note that:

\[
h_{\lambda}(u_{\lambda} - 1) = \frac{\lambda^{1/d} - \lambda}{\lambda^{1/d} - 1 - \lambda^{-1}} \rightarrow \frac{d}{\epsilon} \cdot \frac{\epsilon}{\epsilon - e^{-\epsilon}}.
\]

Thus 2.1(g) also holds. Therefore Theorem 2.1 combined with Little’s law allows us to recover [1, Theorem 4.1], i.e.:

\[
\lim_{\lambda \to 1^-} \frac{E[R_1]}{\log(1 - \lambda)} = \frac{1}{\log(d)}.
\]

3.2 LL(d)

As \( b = 0 \) no additional work is required to show that 2.2(d') holds. This allows us to conclude from Theorem 2.2 that the mean response time for the LL(d) policy in the equilibrium mean field regime satisfies:

\[
\lim_{\lambda \to 1^-} \frac{E[R_1]}{\log(1 - \lambda)} = \frac{1}{d - 1}.
\]

By combining (13) and (14), we recover the result which was stated in [5, Theorem 7.2].

Remark. In the proof of Theorem 7.2 found in [5], there is an incorrect use of the Moore-Osgood Theorem, as the limit function \( U \) is not necessarily continuous, in fact, its continuity is exactly what needs to be shown.

4 PICK D SERVERS WITH PROBABILITY P_I

4.1 SQ(d_1, \ldots, d_n, p_1, \ldots, p_n)

We now consider a policy which, with probability \( p_i \), sends an incoming job to the queue with the least number of jobs amongst \( d_i \) randomly selected servers. Throughout, we assume that \( \sum_i p_i = 1 \), \( d_i \geq 1 \) and \( \sum_i p_i d_i > 1 \). We assume jobs are exponentially distributed with mean one and the arrival rate is equal to \( \lambda \in (0, 1) \).

Let \( u_i(t) \) denote the probability that a queue in the mean field limit has \( k \) or more jobs at time \( t \). First we note that the transient regime is described by the following system of ODEs:
Proposition 4.1. The sequence \((u_k(t))_k\) is the solution of the following system of ODEs:
\[
\frac{d}{dt}u_k(t) = \sum_{i=1}^{n} p_i (u_{k-1}(t)d_i - u_k(t)d_i) - (u_k(t) - u_{k+1}(t)),
\]
with boundary condition \(u_0(t) = 1\) for all \(t \in [0, \infty)\).

Proof. This is a simple generalization of the system of ODEs for SQ(d) for which \(\frac{d}{dt}u_k(t) = (u_{k-1}(t)d_i - u_k(t)d_i) - (u_k(t) - u_{k+1}(t))\).

From the transient regime in (15) we find the equilibrium queue length distribution in the mean field regime, to this end we denote \(u_k = \lim_{t \to \infty} u_k(t)\). We find:

Proposition 4.2. For the SQ(d_1, \ldots, d_n, p_1, \ldots, p_n) policy with arrival rate \(\lambda\) and exponential job sizes of mean one, we find that \((u_k)_k\) satisfies:
\[
u_{k+1} = T_k(u_k) = \lambda \sum_{i=1}^{n} p_i u_i^d_i.
\]

Proof. Taking \(t \to \infty\) we find from (15) that:
\[u_k - u_{k+1} = \lambda \sum_{i=1}^{n} p_i (u_i^{d_i - 1} - u_i^{d_i}).\]

Summing both sides from \(k + 1\) to infinity yields (16).

The solution obtained from (16) is indeed a valid stationary distribution, to this end we should show that \((u_k)_k\) is decreasing and \(\sum_{k=0}^{\infty} u_k < \infty\). Both these claims are immediate from the following result:

Proposition 4.3. The function \(T_\lambda\) defined in (16) satisfies \(T_\lambda(u) \leq \lambda u\), for all \(u \in (0, 1)\).

Proof. This trivially follows from the fact that \(u_i^d_i \leq u\) for all \(d_i \geq 1\) and \(u \in [0, 1]\).

We now show that the requirements to apply Theorem 2.1 are satisfied with \(\lambda = 0\) in 2.1(a), \(b = 0\) in 2.1(c) and \(\hat{k} = 0\) in 2.1(d).

Lemma 4.4. For any \(\lambda \in (0, 1)\) the equation \(u = T_\lambda(u)\) with \(T_\lambda\) as in (16) has exactly one solution \(u_\lambda\) on \((1, \infty)\) moreover this solution satisfies:
(a) \(\lim_{\lambda \to 1^-} u_\lambda = 1\), in particular condition 2.1(a) holds.
(b) \(T_\lambda(u) < u\), \((T_\lambda(u))/u'^{\gamma} \geq 0\) and \(\lim_{\lambda \to 1^-} T_\lambda(u)/u \leq \lambda < 1\) in particular condition 2.1(b) holds.
(c) The function \(\xi_\lambda(x) = \frac{u_i^d_i - (u_i - x)^{d_i}}{x}\) is decreasing on \([u_i - 1, u_i]\).

In particular condition 2.1(c) holds with \(b = 0\). Moreover condition 2.1(d) holds with \(\hat{k} = 0\).

(d) \(\lim_{\lambda \to 1^-} u_\lambda^{d_i} = \frac{1}{\sum_{i=1}^{n} p_i d_i}\).

(e) For any \(d \in \mathbb{N}\), we have \(\lim_{\lambda \to 1^-} u_\lambda^{d_i - 1} \frac{u_\lambda}{\sum_{i=1}^{n} p_i d_i} = d\), in particular condition 2.1(e) holds with \(A = \sum_{i=1}^{n} p_i d_i\).

(f) \(\lim_{\lambda \to 1^-} \log(u_\lambda - 1) \log(1 - \lambda) = 1\), in particular condition 2.1(f) holds.

(g) We have \(\lim_{\epsilon \to 0^+} h_\lambda(\epsilon) = \sum_{i=1}^{n} p_i d_i\), in particular condition 2.1(g) holds.

Proof. Define the function \(r(u) = \lambda \sum_{i=1}^{n} p_i u_i^d_i - u\). We find that \(r(1) = \lambda - 1 < 0\) and it is obvious that \(r(u)\) tends to infinity as \(u\) tends to infinity. This shows that there certainly is an \(u \in (1, \infty)\) for which \(r(u) = 0\). Now let:
\[u_\lambda = \min\{u \in (1, \infty) \mid r(u) = 0\}\.

We find that for all \(u > u_\lambda\):
\[r'(u) = \lambda \sum_{i=1}^{n} p_i d_i u_i^{d_i - 1} - 1 > \left(\frac{\lambda}{\sum_{i=1}^{n} p_i d_i}\right)^{d_i - 1} u_\lambda \leq 0,
\]
this shows that \(r(u) > 0\) for all \(u > u_\lambda\) and hence uniqueness follows. For the other claims we have:
(a) As \(\lim_{\lambda \to 1^-} r(1) = 0\).
(b) Due to Proposition 4.3.
(c) For the function \(\xi_\lambda(x)\) defined in 4.4(c) we have:
\[\xi_\lambda(x) = -(u_\lambda)^{d_i} + (u_\lambda - x)^{d_i - 1}(u_\lambda - (d_i - 1)x).
\]
Computing the derivative of this, we find:
\[\xi_\lambda'(x) = -(d_i - 1)(u_\lambda - x)^{d_i - 2} x \leq 0,
\]
for \(x \in [u_\lambda - 1, u_\lambda]\). From the fact that \((u_\lambda - 1)^2 \xi_\lambda'(u_\lambda - 1) = 1 + (u_\lambda - 1)d_i - (u_\lambda - x)^{d_i} \leq 0\), we can now incur that \(\xi_\lambda'(x)\) is negative on \([u_\lambda - 1, u_\lambda]\) and therefore \(\xi_\lambda(x)\) is indeed decreasing on \([u_\lambda - 1, u_\lambda]\). This suffices to show 2.1(c) with \(b = 0\) and 2.1(d) with \(\hat{k} = 0\) as we have:
\[h_\lambda(x) = \frac{\tau(\lambda) - \lambda \sum_{i=1}^{n} p_i u_i^d_i}{x} = \frac{\lambda}{\sum_{i=1}^{n} \xi_\lambda(x)}.
\]
(d) From the fixed point equation of \(u_\lambda\) we have:
\[u_\lambda^{d_i} = \frac{u_\lambda}{\lambda} + \lambda \sum_{i=1}^{n} p_i d_i u_i^{d_i - 1} - u_\lambda^{d_i - 1}.
\]
The result now follows as \(\lim_{\lambda \to 1^-} u_\lambda = 1\).
(e) This is immediate from l’Hospital’s rule.
(f) This follows by applying l’Hospital’s rule twice.
(g) First one may compute the limit:
\[\lim_{\lambda \to 1^-} h_\lambda(\epsilon) = \frac{\lambda}{\sum_{i=1}^{n} p_i (1 - \epsilon)^{d_i}},
\]
taking the limit of \(\epsilon \to 0^+\) we obtain:
\[\lim_{\epsilon \to 0^+} \lim_{\lambda \to 1^-} h_\lambda(\epsilon) = \sum_{i=1}^{n} p_i d_i.
\]

Let \(R^{(i)}\) denote the response time for a job which is assigned to the server with the shortest queue amongst \(d_i\) randomly selected servers, \(R_\lambda = \sum_{i=1}^{n} p_i R^{(i)}\) the response time for SQ(d_1, \ldots, d_n, p_1, \ldots, p_n) and \(U_\lambda\) the number of jobs in a queue for this policy (note that \(P(U_\lambda = u) = u - u_{k+1}\)). We obtain the following limits:

Theorem 4.5. We have for all i:
\[\lim_{\lambda \to 1^-} \frac{\mathbb{E}[U_\lambda]}{\lambda} = \frac{\beta(1 - \lambda)}{\log(1 - \lambda)} = \frac{\mathbb{E}[R_\lambda]}{\mathbb{E}[R^{(i)}]} = \frac{1}{\log(\sum_{i=1}^{n} p_i d_i)}.
\]

Proof. From Little’s law it follows that:
\[\frac{\mathbb{E}[U_\lambda]}{\lambda} = \mathbb{E}[R_\lambda] = \sum_{i=1}^{n} p_i \mathbb{E}[R^{(i)}].\]
Proposition 4.6 combines the main idea of the proofs of Theorem 4.1 and Theorem 5.1 in [5] and the result of Theorem 5.2 in [3].

Proposition 4.6. The cdf of the workload distribution for LL(d1, . . . , dn, p1, . . . , pn) satisfies the following ODE:

\[
\tilde{F}'(w) = -\lambda \sum_{i=1}^{n} p_i d_i \mathbb{P} \{ U \leq w, Q_i(U) > w \},
\]

with \( \tilde{T}_i = T_i \), where \( \tilde{T}_i \) is as defined in (16).

Proof. The most direct method to show that (17) indeed holds is to set \( d = \max_{i=1}^{n} (d_i) \) and note that from Theorem 5.2 in [3] it follows that (for any \( w \in \mathbb{R} \)):

\[
\tilde{F}'(w) = -\lambda \sum_{i=1}^{n} p_i d_i \mathbb{P} \{ U \leq w, Q_i(U) > w \},
\]

where \( Q_i(U) \) represents the workload at an arbitrary queue with workload \( U \) after it was one of \( d_i \) selected servers for a job arrival.

Note that we have:

\[
\mathbb{P} \{ U \leq w, Q_i(U) > w \} = 0 \quad \text{if} \quad 0 < U \leq w, Q_i(U) > w \}
\]

\[
\mathbb{P} \{ 0 < U \leq w, Q_i(U) > w \} = \text{to apply integration by parts}:
\]

\[
\int_{0}^{w} f(u)\tilde{F}(u)d_{i-1}e^{u-w}du = \frac{1}{d_i} \left( \lambda e^{-w} - \tilde{F}(w)d_i \right) + \int_{0}^{w} \tilde{F}(u)d_i e^{u-w}du
\]

with \( f(u) \) the density of the workload distribution.

For \( \mathbb{P} \{ U = 0, Q_i(U) > w \} \) we compute:

\[
e^{-w}(1 - \tilde{F}(0)) \sum_{j=1}^{d_i-1} \binom{d_i-1}{j} \frac{(1 - \tilde{F}(0))^{j}\tilde{F}(0)^{d_i-1-j}}{j+1} = e^{-w} 1 - \lambda \frac{d_i}{d_i} = e^{-w} 1 - \lambda d_i d_i^{-1}.
\]

This allows us to conclude, using (18) that:

\[
\tilde{F}'(w) = -\lambda \sum_{i=1}^{n} p_i \left( e^{-w} - \tilde{F}(w)d_i \right) + \int_{0}^{w} \tilde{F}(u)d_i e^{u-w}du.
\]  

Integrating both sides of (19) we obtain:

\[
\tilde{F}(w) - \tilde{F}(0) = -\lambda \sum_{i=1}^{n} p_i \left( 1 - e^{-w} - \int_{0}^{w} \tilde{F}(u)d_i du \right).
\]

Proposition 4.7. We have for the LL(d1, . . . , dn, p1, . . . , pn) policy with arrival rate \( \lambda \) and exponential job sizes with mean one:

\[
\lim_{\lambda \to 1^-} \frac{\mathbb{E}[W_i]}{\log(1 - \lambda)} = \lim_{\lambda \to 1^-} \frac{\mathbb{E}[-W_i]}{\log(1 - \lambda)} = \lim_{\lambda \to 1^-} \frac{\mathbb{E}[\tilde{F}(i)]}{\log(1 - \lambda)} = \frac{1}{\sum_{i=1}^{n} p_i d_i - 1}
\]

Proof. The first two equalities follow in the same way as in Theorem 4.5. The remaining equality follows from Proposition 4.6, Lemma 4.4 and Theorem 2.2. No extra work is required to show 2.2(d1) as it is immediate for \( b = 1 \).

Let \( p_1, p_2, . . . , p_n \) and \( d_1, . . . , d_n \) be arbitrary. As the function \( \phi_a(x) = xu^a \) is a convex function for any \( u \in [0, 1] \), we find that for all \( u \in [0, 1] \) we have \( u\sum_{i=1}^{n} p_i d_i \leq \sum_{i=1}^{n} p_i d_i \). From (16) resp. (17) it follows that for any arrival rate \( \lambda \in (0, 1) \) and fixed \( \sum_{i=1}^{n} p_i d_i \in \mathbb{N} \) the optimal policy is SQ(\( \sum_{i=1}^{n} p_i d_i \)) resp. LL(\( \sum_{i=1}^{n} p_i d_i \)). On the other hand it follows from Theorems 4.5 and 4.7 that in the heavy traffic limit, the choice of \( p_i, d_i \) does not affect the mean response time.

4.3 The impact of \( d_i \) and \( p_i \)

One may wonder whether SQ(a1, . . . , an, p1, . . . , pn) (resp. LL) outperforms any other policy SQ(b1, . . . , bn, q1, . . . , qn) (resp. LL). Moreover, given a maximal amount of average choice \( \sum_{i=1}^{n} p_i d_i \) on job arrival, what is the optimal choice of \( p_i \in (0, 1) \) and \( d_i \in \mathbb{N} \)? To answer these questions we introduce the concept of Majorization with weights which is presented in [17] (Chapter IV, Section 14A). Specifically the following result is shown (originally introduced in [18], but a more comprehensive proof can be found in [19]):

Proposition 4.8. Let \( p = (p_1, . . . , p_n) \) and \( q = (q_1, . . . , q_m) \) be fixed vectors with nonnegative components such that \( \sum_{i=1}^{n} p_i = \sum_{i=1}^{m} q_i = 1 \). For \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \) the following are equivalent:

(1) For all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \) we have \( \sum_{i=1}^{n} p_i \varphi(x_i) \leq \sum_{i=1}^{m} q_i \varphi(y_i) \).

(2) There exists an \( m \times n \) matrix \( A = (a_{ij}) \) which satisfies \( a_{ij} \geq 0, eA = e \) (with \( e = (1, . . . , 1) \)) and \( p^T A = q^T (\text{with } p^T \text{ the transpose of } p \) and \( x = A y \).

As a consequence of Proposition 4.8 we say that \( (p, a) \) is majorized by \( (q, b) \) and write \( (p, a) \preceq (q, b) \) if and only if (1) or (2) in Proposition 4.8 holds. The interpretation is that \( (q, b) \) is more
scattered than \((p, a)\). This yields a method for comparing policies as \((p, a) \leq (q, b)\) also implies that \(SQ(p, a) \text{ resp. } LL(p, a)\) stochastically has less jobs resp. work than \(SQ(q, b) \text{ resp. } LL(q, b)\).

Despite the fact that given a budget \(\bar{d} = \sum_i p_i \bar{d}_i\), the optimal policy is simply \(SQ(d)\) resp. \(LL(d)\) we may have \(d \not\in \mathbb{N}\). In this case we simply use \(SQ(p_1 - p), ([d], [\bar{d}]) \text{ resp. } LL(p_1 - p), ([d], [\bar{d}])\) for an appropriate \(p \in [0, 1]\). We show that this is indeed the optimal choice (here \([d]\) denotes the floor and \([\bar{d}]\) denotes the ceiling of \(d\)).

**Theorem 4.9**. Let \(p = (p_1, \ldots, p_n)\) with \(\sum_{i=1}^n p_i = 1, p_i \geq 0\) and \(d = (d_1, \ldots, d_n)\) with \(d_i \in \mathbb{N}\). If \(\bar{d} = \sum_{i=1}^n p_i \bar{d}_i\) and \(q = (q_1, q_2)\) s.t. \(q_1 + q_2 = 1\) and \(q_1 [\hat{d}] + q_2 [\bar{d}] = \bar{d}\) then \((q, ([\hat{d}], [\bar{d}])) \leq (p, d)\).

**Proof.** We show that Proposition 4.8, (2) holds to this end we let \(A = (a_{ij}) \in \mathbb{R}^{n \times 2}\). From \(A^{d+1} = p\) it follows that for all \(j\):

\[
a_{ij} = \frac{p_j - q_2 a_{ij}}{q_1}
\]

(21)

It is not hard to see that one can indeed choose \(0 \leq a_{ij} \leq \frac{p_j}{q_1}\) such that \(\sum_{d=1}^n a_{ij}d_i = [\bar{d}]\) and \(\sum a_{ij} = 1\). Moreover it follows that:

\[
\sum_{j=1}^n a_{ij}d_i = \frac{\bar{d}}{q_1} - \frac{q_2}{q_1}[\bar{d}] = [\bar{d}].
\]

This completes the proof. \(\square\)

### 4.4 \(LL(d, p)\)

We take a closer look at the particular case where \(n = 2\) and \(d_2 = 1\), we denote \(p = p_1\) and thus \(p_2 = 1 - p\). We write \(LL(d, p)\) as a shorthand for \(LL(d_1, d_2, p_1, p_2)\). In practice this policy can be viewed as having two arrival streams: one at each server individually, at rate \(\lambda(1 - p)\) for which there is no load balancing and a second at rate \(\lambda p N\) which is distributed using the \(LL(d)\) load balancing policy. It turns out that (as for \(LL(d)\) in [5]), this policy has a closed form solution for the cdf of the workload distribution:

**Proposition 4.10.** The equilibrium workload distribution for the \(LL(d, p)\) policy with exponential job sizes of mean one is given by (2).

**Proof.** In this case, the ODE defined in (17) reduces to:

\[
\ddot{F}(w) = \lambda \left( p\dot{F}(w)^2 + (1 - p)\dot{F}(w) - \frac{F(w)}{\lambda} \right).
\]

(22)

This is an autonomous ODE, we find that it can be solved explicitly by writing it as:

\[
d\dot{F}(w) = \frac{d\dot{F}(w)}{\lambda} = \frac{p\dot{F}(w)^2 + (1 - p)\dot{F}(w) - F(w)}{\lambda - d\dot{F}(w)}
\]

integrating and rewriting in function of \(\dot{F}(w)\) yields (2).

**Proposition 4.11.** We find that the mean workload for the \(LL(d, p)\) policy with exponential job sizes of mean one is given by:

\[
E[W_\lambda] = \frac{\lambda}{1 - (1 - p)\lambda} \sum_{n=0}^\infty \frac{1}{(n + (d - 1))} \left( \frac{\lambda}{1 - (1 - p)\lambda} \right)^n.
\]

(23)

**Proof.** This proof goes along the same lines as the proof of Theorem 5.2 in [5] and relies on the Hypergeometric function \(\, _2F_1(a, b, c; z)\) for which the following two properties hold:

\[
\, _2F_1(a, b, c; z) = (1 - z)^{-a} \cdot \, _2F_1\left(a, c - b; \frac{z}{1 - z}\right)
\]

(24)

\[
z \, _2F_1(a, b, c; z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n z^n}{(c)_n n!} \quad \text{if } |z| < 1.
\]

(25)

Here \((a)_n\) is the Pochhammer symbol (or falling factorial) we have \((a)_n = \Gamma(a + n)/\Gamma(a)\). We apply (24) to ensure that \(z \in (0, 1)\) which in turn allows us to apply the sum formula (25).

The mean workload is given by \(\int_0^\infty F(w) \, dw\). Using \(y = e^{-w}\) we find that it equals:

\[
- \lambda \int_0^1 \left( \frac{b}{p\lambda d + (b - p\lambda d)} \right) y^{(d - 1)} \, dy
\]

(26)

with \(b = 1 - (1 - p)\lambda\). By definition of the Hypergeometric function (26) is equal to:

\[
\frac{\lambda}{b} \left( 1 + \frac{p\lambda d}{b - p\lambda d} \right) \sum_{n=0}^\infty \left( \frac{b}{b - p\lambda d} \right)^n (n + p\lambda d)^n.
\]

Equality (24) allows us to rewrite the mean workload as:

\[
\frac{\lambda}{b} \sum_{n=0}^\infty \left( \frac{b}{b - p\lambda d} \right)^n (n + p\lambda d)^n.
\]

As \(p\lambda d/b \in (0, 1), (25)\) implies that the mean workload is given by:

\[
\frac{\lambda}{b} \sum_{n=0}^\infty \left( \frac{b}{b - p\lambda d} \right)^n (n + p\lambda d)^n.
\]

Using this and the fact that \((1)_n = n!\), we obtain the result. \(\square\)

We find a simple lower and upper bound for the mean workload:

**Proposition 4.12.** We have:

\[
\tilde{W}_\lambda = \frac{\lambda}{b} \sum_{n=0}^\infty \left( \frac{b}{b - p\lambda d} \right)^n \frac{n^2}{6} \leq E[W_\lambda] \leq \tilde{W}_\lambda,
\]

with:

\[
\tilde{W}_\lambda = \frac{\lambda}{1 - (1 - p)\lambda} \left( 1 + \frac{1}{d - 1} \log \left( \frac{1 - (1 - p)\lambda}{1 - (1 - p)\lambda - p\lambda d} \right) \right).
\]

(27)

**Proof.** Throughout this proof we denote \(z = \frac{p\lambda d}{1 - (1 - p)\lambda}\), where \(z < 1, \lambda < 1\). We first note that as \(\log(1/(1 - z)) = \sum_{n=1}^\infty z^n/n\), \(\tilde{W}_\lambda\) can be written as:

\[
\tilde{W}_\lambda = \frac{\lambda}{1 - (1 - p)\lambda} \left( 1 + \frac{1}{d - 1} \sum_{n=1}^\infty \frac{z^n}{n(d - 1)} \right).
\]

From this it is obvious that \(E[W_\lambda] \leq \tilde{W}_\lambda\). Furthermore we find:

\[
\tilde{W}_\lambda - E[W_\lambda] \leq \frac{\lambda}{1 - (1 - p)\lambda} \sum_{n=1}^\infty \frac{z^n}{n} \leq \frac{6(d - 1)^2(1 - (1 - p)\lambda)}{d^2(1 - (1 - p)\lambda)}.
\]

As \(\sum_{n=1}^\infty 1/n^2 = \pi^2/6\). This concludes the proof. \(\square\)

Similar bounds for \(LL(d)\), i.e. when \(p = 1\) were not presented in [5]. Using these bounds we obtain an alternative proof for the result in Theorem 4.7 for the special case where \(n = 2\) and \(d_2 = 1\). Indeed, a simple application of \(\Gamma\)’s Hospital’s rule yields that:

\[
\lim_{\lambda \to 1} - \frac{\tilde{W}_\lambda}{\log(1 - \lambda)} = \frac{1}{p(d - 1)}.
\]
5 SQ(D, K) AND LL(D, K)

5.1 SQ(d, K)

We consider the SQ(d, K) policy, where at rate \lambda/K batches of K i.i.d. exponentially distributed jobs with mean one arrive which are then routed to the K servers with the shortest queues amongst d randomly selected servers. Let \( u_k(t) \) denote the probability that at time \( t \) an arbitrary server has \( k \) or more jobs in its queue. We find from [13] that (u_k(t))_k satisfies:

\[
\frac{d}{dt} u_k(t) = \frac{\lambda}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (1 - u_{k-1}(t))j u_k(t)j (1 - u_k(t))j - (1 - u_k(t))j u_k(t)j - (u_k(t) - u_{k+1}(t)).
\]

(28)

In the limit \( t \) to infinity we find that \( (u_k)_k \) (with \( u_k = \lim_{t \to \infty} u_k(t) \)) satisfies:

**Proposition 5.1.** For the SQ(d, K) policy with arrival rate \( \lambda/K \), we find that \( (u_k)_k \) satisfies:

\[
u_{k+1} = T_\lambda u_k = \frac{\lambda}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (1 - u_k)^j u_k.
\]

(29)

**Proof.** From (28), we obviously have:

\[
(u_k - u_{k+1}) = \frac{\lambda}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (1 - u_{k-1})j u_k j - (1 - u_k)j u_k j.
\]

(30)

Summing both sides in (30) from \( k = 1 \) to infinity yields the result.

\( \square \)

The fact that \( (u_k)_k \) is decreasing and \( \sum_{k=0}^{\infty} u_k < \infty \) is a consequence of the following result:

**Proposition 5.2.** For any \( \lambda, u \in (0, 1) \) with \( T_\lambda \) defined as in (29) we have \( T_\lambda(u) \leq \lambda u \). In particular it follows that condition 2.1(d) holds with \( k = b \).

**Proof.** We may compute:

\[
T_\lambda(u) = \lambda \sum_{j=0}^{K-1} (d-j) \left( \frac{d}{j} \right) u^{d-j}(1 - u)^j
\]

\[
\leq \lambda \sum_{j=0}^{K-1} \frac{d-j}{d} \left( \frac{d}{j} \right) u^{d-j}(1 - u)^j = \lambda u \sum_{j=0}^{K-1} \frac{d-j}{d} \left( \frac{d}{j} \right) u^{d-j}(1 - u)^j,
\]

(31)

and this last sum is bounded by one, from which the result follows.

\( \square \)

We shall see that condition 2.1(b) holds, that the first bullet is immediate from the previous result.

**Lemma 5.3.** Let \( T_\lambda \) be defined as in (29) and let \( u \in (0, 1) \), the following inequality holds:

\[
\left( \frac{T_\lambda(u)}{u} \right)^r > 0.
\]

(31)

**Proof.** We may divide both sides in (31) by \( u^{d-2} \), we compute:

\[
\frac{1}{u^{d-2}} \left( \frac{T_\lambda(u)}{u} \right)^r = \frac{\lambda}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (d-j) \left( \frac{1 - u}{u} \right)^j = 0.
\]

(31)

Now let \( \xi = \frac{u}{\lambda} \) and note that \( \xi \in (0, \infty) \) for \( u \in (0, 1) \), we find that \( \frac{1}{u^{d-2}} \left( \frac{T_\lambda(u)}{u} \right)^r \) can be further simplified as:

\[
\lambda \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) u^{d-j}(1 - u)^j (1 - u)^j = (d-K) \lambda \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (d-j) (d-j-1) \xi^j.
\]

This shows that (31) holds.

\( \square \)

We have the following elementary Lemma:

**Lemma 5.4.** Let \( f : [0, 1] \to \mathbb{R} \) and \( g : [0, 1] \to [0, \infty) \) be continuous differentiable functions and let \( h_0(x) = f(x) + ag(x) \) for \( a \in [0, 1] \). If \( f(0) = 0 \) and \( f'(0) < 0 \) then there exists a value \( a_0 > 0 \) such that for all \( a \in [0, a_0] \) the function \( h_0(x) \) has a root in \( [0, 1] \). Moreover if we let \( x_0 = \min\{x \in [0, 1] \mid h_0(x) = 0\} \) we have \( \lim_{a \to a_0^-} x_0 = 0 \).

**Proof.** As \( [0, 1] \) is compact and \( g \) is continuous we have \( \Lambda = \max_{x \in [0, 1]} g(x) < \infty \). Moreover as \( f \) is continuous and decreasing in \( 0 \), we find a \( \delta, y \geq 0 \) such that \( f(\delta) = -y \) and \( f'(x) < 0 \) for all \( x \in [0, \delta] \). If we now let \( a_0 = \frac{y}{\Lambda} \), the result easily follows as \( h_0(0) \geq 0 \) and \( h_0(\delta) \leq 0 \).

\( \square \)

The most difficult condition to verify for the SQ(d, K) policy is 2.1(c) therefore we first verify the other remaining conditions. We have:

**Lemma 5.5.** Let \( 1 \leq K < d \) be fixed. There exists a \( \lambda_0 < 1 \) such that for all \( \lambda \in (\lambda_0, 1) \), the equation \( T_\lambda(u) = u \) with \( T_\lambda \) defined as in (29) has a solution on \([1, \infty) \). Moreover, if we let \( u_\lambda \) denote the minimal solution in \([1, \infty) \) for \( \lambda \in (\lambda_0, 1) \) we have:

(a) \( \lim_{\lambda \to 1^-} u_\lambda \equiv 1 \) (therefore 2.1(a) holds).

(b) \( \lim_{\lambda \to 1^-} \frac{1 - u_\lambda}{u_\lambda} = \frac{d-K}{K} \).

(c) For any \( b \in \mathbb{N} \) we have: \( \lim_{\lambda \to 1^-} \frac{\log(u_\lambda - \lambda^b)}{\log(1-\lambda)} = 1 \).

Therefore 2.1(f) holds with \( B = 1 \) for any \( b \in \mathbb{N} \).

(d) For \( \lambda \in (\lambda_0, 1) \) and any \( b \in \mathbb{N} \) we have:

\[
\lim_{\lambda \to 1^-} h_0(u_\lambda - \lambda^b) = \frac{d-K}{K}.
\]

With \( h_0(x) \) defined as in (4). Therefore 2.1(e) follows with \( A = \frac{d}{K} \).

(e) We have \( \lim_{\lambda \to 1^-} \lim_{\lambda \to 1^-} h_0(x) \equiv A. \) Therefore 2.1(g) holds.

**Proof.** Dividing both sides of \( u - T_\lambda(u) = 0 \) by \( \lambda \cdot u^d \) we find this equation to be equivalent to:

\[
\frac{1}{\lambda} \frac{1}{u^{d-1}} - \frac{1}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) \left( \frac{1-u}{u} \right)^j = 0.
\]

Let \( \xi = \frac{u-1}{u} \) for \( u \geq 1 \), we obtain:

\[
(1 - \xi)^{d-1} - \frac{1}{K} \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} \right) (-1)^j \xi^j = 0.
\]
adding and subtracting \((1 - \xi)^d - 1\), we further find this to be equivalent to

\[
(1 - \alpha) - 1 \cdot (1 - \xi)^d - 1 = \frac{1}{K} \sum_{j=0}^{K-1} (K - j) \left(-1\right)^j \xi^j = 0.
\]

If we let \(a = \frac{1}{\lambda} - 1\) we find a value \(a_0 > 0\) from Lemma 5.4 such that there exists a root at \(\xi_0\) for all \(a \in [0, a_0]\) (as \(f'(0) = 1 - d/K < 0\)). It now suffices to take \(\lambda \geq \frac{1}{\lambda_0}\) from which the existence of a root for \(T_\lambda(u) = u\) follows, moreover \((a)\) trivially follows from Lemma 5.4 if we define \(u_\lambda = \min(u \in [1, \infty]) | T_\lambda(u_\lambda) = u_\lambda\).

Using \(u_\lambda = T_\lambda(u_\lambda)\) one can show that \(\lim_{\lambda \to \lambda_0^-} u_\lambda = -K/(d-K)\), where the derivative is taken with respect to \(\lambda\). In fact in Lemma 5.6 an expression for the \(n\)-th derivative of \(u_\lambda\) is established (for \(n \leq K\)). Using this expression, the proofs of \((b)\) and \((c)\) are immediate applications of l’Hôpital’s rule. To show \((d)\) we first note that:

\[
\lim_{\lambda \to \lambda_0^-} h_\lambda(u_\lambda - \lambda^0) = \lim_{\lambda \to \lambda_0^-} \frac{T_\lambda(u_\lambda) - T_\lambda(\lambda^b)}{u_\lambda - \lambda^b}.
\]

Furthermore one can see that both \(T_\lambda(u_\lambda) / u_\lambda - \lambda^b\) and \(T_\lambda(\lambda^b) / u_\lambda - \lambda^b\) the second up to \((K-1)\)'st term disappear in the limit of \(\lambda \to 1^-\). Therefore we find that:

\[
\lim_{\lambda \to \lambda_0^-} h_\lambda(u_\lambda - \lambda^b) = \lim_{\lambda \to \lambda_0^-} \lambda \left| \frac{\lambda^0}{K} u_\lambda^0 - \lambda^b \right| + (K-1)d \left| \frac{\lambda^0 - 1}{K} u_\lambda^0 - \lambda^b \right|.
\]

The result now follows by applying l’Hôpital’s rule to conclude that:

\[
\lim_{\lambda \to \lambda_0^-} h_\lambda(u_\lambda - \lambda^b) = \frac{1}{K} (Kd + (K-1)d) (1 - \epsilon) = \frac{d}{K}.
\]

To show \((e)\) we note that:

\[
\lim_{\lambda \to \lambda_0^-} h_\lambda(\epsilon) = \frac{1 - (1 - \epsilon)^d}{\epsilon} = \frac{1}{K} \sum_{j=0}^{K-1} (K - j) \left(-1\right)^j \epsilon^{d-j} e^{j-1}.
\]

Taking the limit \(\epsilon \to 0^+\) of this expression yields the sought result.

To show 2.1(c) we first need to do some extra work, in particular we compute \(\lim_{\lambda \to \lambda_0^-} u_\lambda^{(n)}\) for \(n = 1, \ldots, K+1\) (cf. Lemma 5.6). To show this result, we employ the Faà di Bruno formula which states that for functions \(f\) and \(g\) we have:

\[
(f \circ g)^{(n)}(x) = \sum_{k=1}^{n} \frac{n!}{k!} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)),
\]

where \(B_{n,k}\) denotes the exponential Bell polynomial defined as:

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{j_1 \geq 1} \cdots \sum_{j_n \geq 1} \frac{n!}{j_1! \cdots j_n!} x_{n-k+1}^{j_1} \cdots x_{n-k+1}^{j_n}.
\]

Here the sum is taken over all non-negative integers \(j_1, \ldots, j_n\) which satisfy:

\[
k = j_1 + \cdots + j_n
\]

\[
n = j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1}.
\]

Furthermore we employ the fact that:

\[
B_{n,k}(1, \ldots, (n-k+1)) = \frac{n!}{k! (n-k)!},
\]

which are known as the Lah numbers. We are now able to show:

**Lemma 5.6.** For any \(d, K\) we have:

\[
\lim_{\lambda \to \lambda_0^-} u_\lambda^{(n)} = (-1)^n n! \frac{d^{n-1} K}{(d-K)^n}\]

for \(1 \leq n \leq K\) and

\[
\lim_{\lambda \to \lambda_0^-} u_\lambda^{(K+1)} = (-1)^{K+1} (K+1)! \frac{d^K K}{(d-K)^{K+1}} - \frac{d!}{(d-K)!} \left(\frac{K}{d-K}\right)^{K+1}.
\]

**Proof.** We first show that for \(\Theta(u) = \frac{1}{d} T_\lambda(u)\):

\[
\Theta^{(n)}(1) = (-1)^n n! \frac{d}{(d-K)!}.
\]

To show this result, we employ the Faà di Bruno formula which states that:

\[
\Theta^{(n)}(1) = (-1)^n n! \sum_{j=0}^{K-n-1} j! (1 - u)^j + \sum_{j=1}^{n-1} E_{j}^{(n-j-1)}.
\]

where we denote:

\[
E_{j} = (-1)^{j+1} \frac{d}{(d-K)} (d-K-1) u^{d-K-2} (1-u)^{K-j}.
\]

Indeed, one finds:

\[
\frac{d}{d-K} \Theta^{(n)}(u) = \frac{d}{d-K} \frac{\partial}{\partial a} \left(-1\right)^n (n-1)! \sum_{j=0}^{K-n-1} j! (1 - u)^j + \sum_{j=1}^{n-1} E_{j}^{(n-j-1)}
\]

The result now follows by induction applying the equality:

\[
\left(\frac{d}{d-K}\right)^j = \sum_{j=1}^{n} E_{j}^{(n-j-1)}.
\]

Noting that for any \(n \leq K\) we have \(\lim_{u \to 1^-} \sum_{j=1}^{n} E_{j}^{(n-j-1)} = 0\), we find that (35) indeed holds. Furthermore we have:

\[
\frac{d}{d-K} \Theta^{(K+1)}(u) = (-1)^{K+1} K! (d-K-1) u^{d-K-2} + \sum_{j=1}^{K-1} E_{j}^{(K-j)}.
\]

Moreover, it is not hard to see that:

\[
\lim_{u \to 1^-} E_{j}^{(K-j)} = (-1)^{K+1} \left(\frac{d}{K-j}\right) (d-K-1)! (K-j)!.
\]

This allows us to compute:

\[
\frac{d}{d-K} \Theta^{(K+1)}(1) = (-1)^{K+1} K! + \sum_{j=1}^{K-1} \left(\frac{d}{K-j}\right) (K-j)! (d-K-1)
\]

10
where we used identity (4.1) in [20, p.46] with \( B/y. \) Analogously, one may compute:

\[
\begin{aligned}
&\lim_{\lambda \to 1^-} u^{(n)}_\lambda = 0 = n! \sum_{k=1}^{n-1} \binom{n-1}{k} \left( \frac{d}{d-K} \right)^{n-k} u^{(k)}_\lambda \\
&+ n! \sum_{k=2}^{n} \binom{n-1}{k-1} (-1)^{n-k+1} \frac{d^{n-k}K}{(d-K)^{n-k+1}}.
\end{aligned}
\]

Taking the limit \( \lambda \to 1^- \) we obtain \( \lim_{\lambda \to 1^-} u^{(n)}_\lambda = 1 + \frac{d}{d-K} \lim_{\lambda \to 1^-} u^{(n)}_\lambda \) yielding (33) with \( n = 1 \). Let \( 2 \leq n \leq K + 1 \) note that \( u^{(n)}_\lambda = T_\lambda (u^{(n)}_\lambda) \) and therefore also \( 1 = \lambda (u^{(1)}_\lambda) \). By differentiating both sides \( n \geq 2 \) times, it follows that we have:

\[
0 = n \left( \frac{d}{d-K} \right)^n u^{(n-1)}_\lambda + n! \left( \frac{d}{d-K} \right)^n \Theta(u^{(n)}_\lambda).
\]

It follows from the Faà di Bruno formula that:

\[
\left( \frac{d}{d\lambda} \right)^n \Theta(u^{(n)}_\lambda) = n! \sum_{k=0}^{n} \frac{d}{d-K}^{k} \left( \frac{d}{d-K} \right)^{n-k} \Theta^{(k)}(u^{(n-k)}_\lambda),
\]

where \( B_{n,k} \) denotes the exponential Bell polynomial. We have:

\[
B_{n,1}(u^{(1)}_\lambda, \ldots, u^{(n-k)}_\lambda) = u^{(n)}_\lambda
\]

and for \( k > 1 \) induction allows us to state for \( n \leq K + 1 \):

\[
\lim_{\lambda \to 1^-} B_{n,k}(u^{(1)}_\lambda, \ldots, u^{(n-k)}_\lambda) = B_{n,k}(\frac{(-1)(n-1)}{d-K}, \ldots, \frac{(-1)}{d-K}, \frac{\frac{d}{d-K}^{n-k}K}{(d-K)^n}),
\]

we used the simple identities

\[
B_{n,k}(x_1, y_1, \ldots, x_{n-k+1}, y_1) = B_{n,k}(x_1, \ldots, x_{n-k+1}, y_1),
\]

\[
B_{n,k}(x, y, z, \ldots, x_{n-k+1}) = B_{n,k}(x, \ldots, x_{n-k+1})z^n,
\]

with \( y = K/d \) and \( z = -d/(d-K) \). Using (32) we have:

\[
\lim_{\lambda \to 1^-} B_{n,k}(u^{(1)}_\lambda, \ldots, u^{(n-k)}_\lambda) = n! \sum_{k=1}^{n} \binom{n-1}{k} \left( \frac{(-1)^{n-k+1}}{d-K} \right)^n \left( \frac{d^{n-k}K}{(d-K)^n} \right).
\]

Analogously, one may compute:

\[
\lim_{\lambda \to 1^-} \left( \frac{d}{d-K} \right)^{k} \left( \frac{d}{d-K} \right)^{n-k} \Theta^{(k)}(u^{(n-k)}_\lambda).
\]

Therefore (40), (41) and (35) imply for \( n \leq K + 1 \):

\[
0 = n \sum_{k=1}^{n-1} \binom{n-1}{k} \left( \frac{(-1)^{n-k+1}}{d-K} \right)^n \left( \frac{d^{n-k+1}K}{(d-K)^{n-k+1}} \right)
\]

\[
+ \sum_{k=2}^{n} \binom{n}{k} \left( \frac{(-1)^{n-k+1}K}{(d-K)^{n-k+1}} \right) \Theta^{(k)}(1)
\]

\[
+ \Theta^{(1)}(1) \lim_{\lambda \to 1^-} u^{(n)}_\lambda.
\]

We thus find:

\[
Kx^{2}h'_\lambda(x) = \left[ -\frac{1}{x^2} (u^{(1)}_\lambda + \frac{d}{x} (u^{(1)}_\lambda - x)) + \frac{1}{x} \Theta'(u^{(1)}_\lambda - x) \right] \cdot Kx^2
\]

\[
- K \mu_1 + \lambda \sum_{j=0}^{K-1} (K-j) \left( \frac{d}{j} (u^{(1)}_\lambda - x)^{j} - (1 - u^{(1)}_\lambda + x)^{j} \right)
\]

\[
+ \lambda x \sum_{j=0}^{K-1} (d-j) \left( u^{(1)}_\lambda - x)^{j} - (1 - u^{(1)}_\lambda + x)^{j} \right)
\]

We are now able to show that 2.2(c) indeed holds:

**Lemma 5.7.** Let \( 1 \leq K < d \) be fixed, there exists a \( \dot{\lambda} < 1 \) and \( b \in \mathbb{N} \) (independent of \( \lambda \)) such that the function \( h_\lambda(x) \) defined as in (8) with \( T_\lambda \) as in (29) is decreasing as a function of \( x \in [u^{(1)}_\lambda - \lambda, u^{(1)}_\lambda] \) for all \( \dot{\lambda} < \lambda < 1 \).}

**Proof.** Throughout, we assume that \( \dot{\lambda} = \lambda - \frac{1}{9} < 1 \) as in Lemma 5.5. We show there is some \( \dot{\lambda} \geq \lambda \) and \( b \in \mathbb{N} \) which does not depend on the value of \( \lambda \) for which \( h_\lambda(x) \) is decreasing on \( [u^{(1)}_\lambda - \dot{\lambda}, u^{(1)}_\lambda] \) for all \( \lambda \).
\[ \zeta(\lambda) = -Ku_\lambda - \lambda(d - K) \sum_{j=0}^{K-1} \left( \frac{d}{j} \right) (u_\lambda - x)^{d-j}(1 - u_\lambda + x)^j \\
+ \lambda u_\lambda \sum_{j=0}^{K-1} \left( d - j \right) \left( \frac{d}{j} \right) (u_\lambda - x)^{d-j-1}(1 - u_\lambda + x)^j. \]

If we now define \( \zeta_\lambda(x) = Kx^2k_\lambda(x) \) we obtain:

\[ \zeta_\lambda(x) = -Ku_\lambda - \lambda(d - K) \sum_{j=0}^{K-1} \left( \frac{d}{j} \right) (u_\lambda - x)^{d-j}(1 - u_\lambda + x)^j \\
+ \lambda u_\lambda \sum_{j=0}^{K-1} \left( d - j \right) \left( \frac{d}{j} \right) (u_\lambda - x)^{d-j-1}(1 - u_\lambda + x)^j. \]

It therefore suffices to show that \( \zeta_\lambda(x) \leq 0 \) for \( \lambda \) sufficiently close to one. To this end we compute:

\[ \zeta_\lambda'(x) = \lambda(d - K) \left( \frac{d}{K-1} \right) (u_\lambda - x)^{d-K}(1 - u_\lambda + x)^{K-1} \\
+ \lambda(d - K) \sum_{j=0}^{K-2} \left( \frac{d}{j} \right) (u_\lambda - x)^{d-j}(1 - u_\lambda + x)^j \\
- \lambda(u_\lambda - K)(d + 1) \left( \frac{d}{K-1} \right) (u_\lambda - x)^{d-K-1}(1 - u_\lambda + x)^{K-1} \\
- \lambda u_\lambda \sum_{j=0}^{K-2} \left( d - j \right) (u_\lambda - x)^{d-j-2}(1 - u_\lambda + x)^j \\
+ \lambda u_\lambda \sum_{j=0}^{K-2} \left( d - j \right) (u_\lambda - x)^{d-j}(1 - u_\lambda + x)^j, \]

which simplifies to:

\[ \zeta_\lambda'(x) = -\lambda(d - K) \left( \frac{d}{K-1} \right) (d - K + 1)(1 - u_\lambda + x)^{K-1} \]

This is obviously negative for all \( x \in [u_\lambda - 1, u_\lambda] \). It thus suffices to show that we can find a value \( b \in \mathbb{N} \) such that \( \zeta_\lambda(u_\lambda - \lambda b) \leq 0 \). To this end, we find:

\[ \zeta_\lambda(u_\lambda - \lambda b) = -Ku_\lambda - \lambda(d - K) \sum_{j=0}^{K-1} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j \\
+ \lambda u_\lambda \sum_{j=0}^{K-1} \left( d - j \right) \left( \frac{d}{j} \right) \lambda^{b(d-j)-1}(1 - \lambda b)^j \\
= -Ku_\lambda - \lambda(d - K) \sum_{j=0}^{K-1} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j \\
+ \lambda u_\lambda \sum_{j=0}^{K-1} \left( d - j \right) \left( \frac{d}{j} \right) \lambda^{b(d-j)-1}(1 - \lambda b)^j \\
+ \frac{\lambda u_\lambda}{\lambda b} \sum_{j=0}^{K-1} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j. \]

Now let us denote \( \Theta(u) = \frac{1}{\lambda} \sum_{j=0}^{n-1} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j. \)

For now let us focus on the case \( K = d - 1 \). By (37) and (38) we have for \( n \leq K = d - 1 \) that

\[ \Theta^{(n)}(u) = (-1)^{n+1} \frac{1}{n!} \sum_{j=0}^{d-n-1} \left( \frac{d}{j} \right) u^{d-j-n-1}(1 - u)^j. \]

It is easy to show by applying l'Hopital's rule and using the fact that \( \Theta^{(n)}(u) = 0 \) for \( n > K = d - 1 \) is constant and therefore \( \Theta^{(n)}(u) = 0 \) for \( n > K = d - 1 \).

Employing the Taylor expansion of \( \Theta(u) \) at \( \lambda b \), we find that (44) can be written as:

\[ \lambda b \zeta_\lambda(u_\lambda - \lambda b) = \lambda \left( u_\lambda - \lambda b \right)^{d-1} \sum_{j=0}^{d-2} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j \\
- u_\lambda \lambda b \sum_{n=1}^{d-1} \Theta^{(n)}(\lambda b) \left( \frac{u_\lambda - \lambda b}{n!} \right)^n. \]

Due to (45)

\[ u_\lambda \lambda b \sum_{n=1}^{d-1} \Theta^{(n)}(\lambda b) \left( \frac{u_\lambda - \lambda b}{n!} \right)^n = u_\lambda \sum_{n=1}^{d-1} \Theta^{(n)}(\lambda b) \left( \frac{u_\lambda - \lambda b}{n!} \right)^n = \]

\[ = -u_\lambda \sum_{n=1}^{d-1} \Theta^{(n)}(\lambda b) \left( \frac{u_\lambda - \lambda b}{n!} \right)^n \sum_{n=1}^{d-j-1} \left( \frac{1 - u_\lambda}{\lambda b} \right)^{d-j} = \]

\[ = \sum_{j=0}^{d-2} \left( \frac{d}{j} \right) \lambda^{b(d-j)}(1 - \lambda b)^j \left( \lambda^{b(d-j)}(u_\lambda - \lambda b) + \lambda b(\lambda b - u_\lambda) \right)^{d-j}. \]

Combined with (46) this yields:

\[ \lambda b \zeta_\lambda(u_\lambda - \lambda b) = \lambda^{b+1} \sum_{j=0}^{d-2} \left( -1 \right)^{d-j-1} \left( \frac{d}{j} \right) (u_\lambda - \lambda b)^{d-j}(1 - \lambda b)^j. \]

Dividing by \( (u_\lambda - \lambda b)^d \) we find that:

\[ \frac{\lambda b}{(u_\lambda - \lambda b)^d} \zeta_\lambda(u_\lambda - \lambda b) = \lambda^{b+1} \sum_{j=0}^{d-2} \left( -1 \right)^{d-j-1} \left( \frac{d}{j} \right) \left( 1 - \frac{\lambda b}{u_\lambda - \lambda b} \right)^j. \]

It is easy to show by applying l'Hopital's rule and using the fact that \( \lim_{\lambda \to 1} u'_\lambda = -d + 1 \) for \( K = d - 1 \) that:

\[ \lim_{\lambda \to 1} \frac{1 - \lambda b}{u_\lambda - \lambda b} = \frac{b}{b + d - 1}. \]
Therefore we find
\[
\lim_{\lambda \to 1^-} \frac{\lambda^b}{(u_1 - \lambda b)^d} \xi_{\lambda}(u_1 - \lambda^b) = \sum_{j=0}^{d-2} (-1)^{d-j+1} \binom{d}{j} \frac{b}{b+j+1}^{d-1-j}
\]
which converges to 1 if \(d \geq 0\) as \(b\) tends to infinity. This proves Lemma 5.7 for \(K = d\).

Fix \(K\) and let \(d \geq K + 1\) be variable, we find (42-43) that:
\[
\xi_{\lambda}(x) = -\lambda(d-K)(K+1)(1-u_1+x)^{K-1}(u_1-x)^{d-K-1}x
\]
and
\[
\xi_{\lambda}(u_1 - 1) = (d-K)u_1 - \lambda(d-K).
\]
(47)
Now let (\(K_1, d_1\)) and (\(K_2, d_2\)) be arbitrary (with \(K_1 < d_1\)), denote by \(u_{\lambda}\) the fixed point associated to (\(K_1, d_1\)) and \(\xi_{\lambda}\) the associated \(\xi_{\lambda}\) function. We show the following inequalities :
\[
2\xi_{\lambda}(x) \leq 1\xi_{\lambda}(x + 1 \cdot u_1 - 2u_1)\xi_{\lambda}(2u_1 - 1) \leq 1\xi_{\lambda}(u_1 - 1),
\]
(48)
for \(x \in [2u_1 - 1, 2u_1 - 2 \lambda^b]\)
and
\[
2\xi_{\lambda}(2u_1 - \lambda^b) = \xi_{\lambda}(2u_1 - 1) = \int_{u_1-1}^{2u_1-3} 2\xi_{\lambda}(x) \, dx
\]
\[
\leq 1\xi_{\lambda}(u_1 - 1) = \int_{u_1-1}^{2u_1-3} 1\xi_{\lambda}(x + 1 \cdot u_1 - 2u_1) \, dx
\]
\[
= 1\xi_{\lambda}(1 \cdot u_1 - 1).
\]
(49)
This shows that if \(1\xi_{\lambda}(1 \cdot u_1 - 1) \leq 0\), then also \(2\xi_{\lambda}(2u_1 - \lambda^b) \leq 0\). Applying (i) would then conclude the proof for \(K\) even as we already established the result for \(K = d - 1\). Having shown the result for \(K\) even then implies that the result also holds for \(K\) odd by applying (ii).

First, we show (48) for (i). To this end we let \(x \in [2u_1 - 1, 2u_1 - 2 \lambda^b]\) be arbitrary, we find that \(2\xi_{\lambda}(x) \leq 1\xi_{\lambda}(x + 1 \cdot u_1 - 2u_1)\) is equivalent to:
\[
\left(1 + \frac{1u_1 - 2u_1}{x} \right) \frac{1}{2u_1 - x} \leq \left(\frac{d-K}{d-K} \cdot \frac{d-1}{d-1} \right) \frac{d-K}{d-K-1} \frac{d-K}{d-K-1}
\]
This can be shown to hold for \(\lambda\) sufficiently close to 1 by noting that for \(x \in [2u_1 - 1, 2u_1 - 2 \lambda^b]\) we have:
\[
\lim_{\lambda \to 1^-} \frac{1 + \frac{1u_1 - 2u_1}{x}}{2u_1 - x} \leq \lim_{\lambda \to 1^-} \left(1 + \frac{1u_1 - 2u_1}{2u_1 - 1} \right) \frac{1}{2u_1 - x} \leq \left(\frac{d-K}{d-K} \right) \frac{d-K}{d-K-1} \frac{d-K}{d-K-1},
\]
(50)
from this we find that (48) indeed holds in case (i) for any \(K\) and thus certainly for \(K\) even.

We now consider (49) for case (i). Due to (47) one finds for any \(n \geq 1\) that:
\[
\left(\frac{\partial}{\partial \lambda}\right)^n \xi_{\lambda}(u_1 - 1) = n d u_{\lambda}^{(n-1)} + (d-K) u_{\lambda}^{(n)} - \delta_{\{n=1\}}(d-K).
\]
(51)
We employ (33) to conclude that for \(n \leq K\):
\[
\lim_{\lambda \to 1^-} \left(\frac{n}{\partial \lambda}\right)^K \xi_{\lambda}(u_1 - 1) = 0,
\]
while for \(n = K + 1\) we find from (33-34):
\[
\lim_{\lambda \to 1^-} \left(\frac{n}{\partial \lambda}\right)^{K+1} \xi_{\lambda}(u_1 - 1) = \frac{K}{d-K} \frac{d-K}{d-K} ^{K+1}.
\]
(52)
We now denote
\[
H_n = \lim_{\lambda \to 1^-} \left(\frac{n}{\partial \lambda}\right)^n (2\xi_{\lambda}(2u_1 - 1) - 1i_{\lambda}(1u_1 - 1)),
\]
(53)
From (51) we clearly have \(H_n = 0\) for \(0 \leq n \leq K\). For \(n = K + 1\) we find:
\[
H_{K+1} = \frac{d-K}{d-K} \left(\frac{K}{d-K} \frac{d-K}{d-K} ^{K+1} \right).
\]
which is positive if and only if:
\[
(1 + d-K)K+1 - (d+1)(d-K)K > 0
\]
Letting \(d = K + y\) for \(y \geq 0\) we find that this is equivalent to:
\[
(1 + y)K + 1 - (1 + K + y)K > 0
\]
As \((1 + y)K + 1 - (1 + K + y)K = \sum_{j=0}^{K-1} (K+1) j^y\), which is positive for \(K \geq 2\) and \(y \geq 0\), we conclude that \(H_{K+1}\) is positive.

By looking at the Taylor series expansion of \(2\xi_{\lambda}(2u_1 - 1) - 1i_{\lambda}(1u_1 - 1)\) in \(\lambda = 1\) and noting that \(H_0 = \ldots = H_K = 0\), we note that for \(\lambda\) sufficiently close to one:
\[
2\xi_{\lambda}(2u_1 - 1) - 1i_{\lambda}(1u_1 - 1) \approx H_{K+1}(\lambda - 1)K+1/(K+1)!,
\]
which is negative for \(K\) even. This shows that (49) indeed holds for case (i).

We now consider (48) for (ii), using simple computations we find that this is equivalent to:
\[
\left(\frac{d+1}{K}\right) (1 - 2u_1 + x) (x + 1 \cdot u_1 - 2u_1) \leq \left(\frac{d}{K} \right) \frac{d}{d-K} - x,
\]
for \(x \in [2u_1 - 1, 2u_1 - 2 \lambda^b]\). It therefore suffices to show that for \(\lambda\) close to one, we have:
\[
\left(\frac{d+1}{K}\right) (1 + \frac{1u_1 - 2u_1}{2u_1 - 1}) \leq \frac{K}{d-K}\frac{d-K}{d-K}.
\]
This holds as \((1 - 2u_1 + x)\) converges to zero as \(\lambda \to 1\) and \(\lim_{\lambda \to 1^-} \left(\frac{1u_1 - 2u_1}{2u_1 - 1} \frac{d-K}{d-K} \right) = (1-K)/K\).
The final step is to show (49) for (ii). If we define \(H_n\) as in (53) and make use of (51) and (52), we find that \(H_n = 0\) for \(0 \leq n \leq K\) while for \(n = K + 1\) we have:
\[
H_{K+1} = \lim_{\lambda \to 1^-} \left(\frac{\partial}{\partial \lambda}\right)^{K+1} 2\xi_{\lambda}(2u_1 - 1) = -\left(\frac{K}{d-K}\right) \frac{d-K}{d-K} \frac{d-K}{d-K} < 0.
\]
As \(K\) is odd, \(H_{K+1}(\lambda - 1)K+1/(K+1)!\) is negative, which completes the proof.

We conclude with our main result for \(SQ(d, K)\):

**Theorem 5.8.** Let \(R_d\) denote the response time distribution for the \(SQ(d, K)\) policy in equilibrium. We find:
\[
\lim_{\lambda \to 1^-} \frac{-E[R_d]}{\log(1-\lambda)} = \frac{1}{\log \left(\frac{d}{K}\right)}.
\]
(54)

**Proof.** By Little’s law we have \(E[R_d] = \sum_{i=0}^{K} u_i\), the result now follows from Theorem 2.1 using the other results in this section. □

### 5.2 LL(d, K)

We consider the LL(d, K) model where we assume that with rate \(\lambda/K\) a group of \(K\) i.i.d. jobs which have an exponential size with mean 1
arrive to the $K$ least loaded servers amongst $d$ randomly selected servers. It is shown in [21] that the cdf of the equilibrium workload distribution $\bar{F}(w)$ satisfies the ODE $\ddot{F}(w) = T_3(\dot{F}(w)) - \dot{F}(w)$ with $T_3$ defined as in (29). We have the following result:

**Theorem 5.9.** Let $R_3$ denote the response time distribution for the LL($d$, $K$) policy in equilibrium. We find:

$$\lim_{\lambda \to 1^{-}} \frac{\mathbb{E}[R_3]}{\log(1-\lambda)} = \lim_{\lambda \to 1^{-}} \frac{\mathbb{E}[W_3]}{\log(1-\lambda)} = \frac{K}{d-K}. \quad (55)$$

**Proof.** The first equality follows by Little’s Law. The second equality follows from Theorem 2.2. □

6 CONCLUSIONS AND FUTURE WORK

In this paper we studied the heavy traffic behaviour of the expected response time $\mathbb{E}[R_3]$ for a variety of load balancing policies in the mean field regime. We present a set of sufficient conditions such that the limit $\lim_{\lambda \to 1^{-}} -\mathbb{E}[R_3]/\log(1-\lambda)$ can be derived without much effort. For some load balancing policies (such as LL($d$)) these conditions are easy to verify, while for other policies (such as SQ($d$,$K$)) this turned out to be much more challenging. Even if it is unclear how to verify these conditions, our result yields a natural conjecture on the limiting value. The resulting limiting value is also surprisingly elegant for the policies studied in this paper. As our main theorems apply to any recurrence relation or ODE for which $T_3$ satisfies the sufficient conditions, our main results may also find applications outside the area of load balancing.

Numerical experiments (not reported in the paper) suggest that the observations made in Figure 3a also hold for non-exponential job size distributions, which suggests that the main ideas presented in this paper may also be applicable to other job size distributions.

REFERENCES

[1] M. Mitzenmacher, "The power of two choices in randomized load balancing," IEEE Transactions on Parallel and Distributed Systems, vol. 12, no. 10, pp. 1094–1104, 2001.

[2] N. Vvedenskaya, R. Dobrushin, and F. Karpelevich, "Queueing system with selection of the shortest of two queues: an asymptotic approach," Problemy Peredachi Informatsii, vol. 32, pp. 15–27, 1996.

[3] T. Hellemans, T. Bodas, and B. Van Houdt, "Performance analysis of workload dependent load balancing policies," Proceedings of the ACM on Measurement and Analysis of Computing Systems, vol. 3, no. 2, p. 23, 2019.

[4] K. Ousterhout, P. Wendell, M. Zaharia, and I. Stoica, "Sparrow: distributed, low latency scheduling," in Proceedings of the Twenty-Fourth ACM Symposium on Operating Systems Principles. ACM, 2013, pp. 69–84.

[5] T. Hellemans and B. Van Houdt, "On the power-of-$d$-choices with least loaded server selection," arXiv preprint arXiv:1802.05420, 2018.

[6] T. Kurtz, Approximation of population processes. Society for Industrial and Applied Mathematics, 1981.

[7] M. Benaim and J. Le Boudec, "A class of mean field interaction models for computer and communication systems," Performance Evaluation, vol. 65, no. 11-12, pp. 823–838, 2008.

[8] L. Bortolussi, T. Hillston, D. Latella, and M. Massink, "Continuous approximation of collective system behaviour: A tutorial," Performance Evaluation, vol. 70, pp. 317–349, 2012.

[9] N. Gast, "Expected values estimated via mean-field approximation are 1/n-accurate," Proc. ACM Meas. Anal. Comput. Syst., vol. 1, no. 1, pp. 17:1–17:26, Jun. 2017. [Online]. Available: http://doi.acm.org/10.1145/3084454

[10] M. Bramson, "Stability of join the shortest queue networks," Ann. Appl. Probab., vol. 21, no. 4, pp. 1568–1625, 2011. [Online]. Available: http://dx.doi.org/10.1214/10-AAP726

[11] G. Brightwell and M. Luczak, "The supermarket model with arrival rate tending to one," arXiv preprint arXiv:1201.5523, 2012.

[12] P. Eschenfeldt and D. Gamarnik, "Join the shortest queue with many servers. the heavy-traffic asymptotics," Mathematics of Operations Research, vol. 43, no. 3, pp. 867–886, 2018.

[13] B. Van Houdt, "Global attraction of ode-based mean field models with hyper-exponential job sizes," Proceedings of the ACM on Measurement and Analysis of Computing Systems, vol. 3, no. 2, p. 23, 2019.

[14] L. Ying, R. Srikant, and X. Kang, "The power of slightly more than one sample in randomized load balancing," in Proc. of IEEE INFOCOM, 2015.

[15] E. Anton, U. Ayesta, M. Jonckheere, and I. M. Verloop, "On the stability of redundancy models," arXiv preprint arXiv:1903.04414, 2019.

[16] M. Mitzenmacher, "Analyses of load stealing models based on families of differential equations," Theory of Computing Systems, vol. 34, pp. 77–98, 2001.

[17] A. W. Marshall, I. Olkin, and B. C. Arnold, Inequalities: theory of majorization and its applications. Springer, 1979, vol. 143.

[18] D. Blackwell, "The range of certain vector integrals," Proceedings of the American Mathematical Society, vol. 2, no. 3, pp. 390–395, 1951.

[19] J. Borcea, "Equilibrium points of logarithmic potentials induced by positive charge distributions: i. generalized de bruijn-springer relations," Transactions of the American Mathematical Society, vol. 359, no. 7, pp. 3209–3237, 2007.

[20] H. Wadsworth Gould, Combinatorial Identities: A standardized set of tables listing 500 binomial coefficient summations. Morgantown, W Va, 1972.

[21] T. Hellemans and B. Van Houdt, "Performance of redundancy (d) with identical/independent replicas," ACM Transactions on Modeling and Performance Evaluation of Computing Systems (TOMPECS), vol. 4, no. 2, p. 9, 2019.