Random Partitions and the Quantum Benjamin-Ono Hierarchy

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Abstract

Jack measures $M_V(\varepsilon_2,\varepsilon_1)$ on partitions $\lambda$ are discrete stochastic processes occurring naturally in the study of continuum circular $\beta$-ensembles in generic background potentials $V$ and arbitrary values $\beta$ of Dyson’s inverse temperature. For analytic $V$, we prove a law of large numbers and central limit theorem in the scaling limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken at fixed inverse Jack parameter $\frac{\beta}{\varepsilon_1} = -\frac{\varepsilon_2}{\varepsilon_1} > 0$. To leading order, the random profile $f_\lambda(c|\varepsilon_2,\varepsilon_1)$ concentrates on a limit shape $f_{+v}(c)$ independent of $\beta$ whose distribution function is the push-forward of the uniform measure on the circle along $v: \mathbb{T} \to \mathbb{R}$, where $v = \Delta^{1/2} V$ (LLN). Moreover, global fluctuations of $f_\lambda(c|\varepsilon_2,\varepsilon_1)$ around $f_{+v}(c)$ converge to the push-forward along $v$ of the restriction to $\mathbb{T} \cap \mathbb{H}_+$ of a two-dimensional Gaussian free field on the upper half-plane $\mathbb{H}_+$ whose covariance is independent of $\beta$ (CLT). At $\beta = 2$, this recovers Okounkov’s LLN for Schur measures (2005) and coincides with Breuer-Duits’ CLT for biorthogonal ensembles (2013).

Our limit theorems follow from an all-order expansion of the joint cumulants of the linear statistics of $M_V(\varepsilon_2,\varepsilon_1)$ (AOE), which has the same form as the all-order $1/N$ refined topological expansion of the $\beta$-ensembles on the line in a one-cut potential $V$ at arbitrary $\beta$ obtained for formal $V$ by Chekhov-Eynard (2006) and for convergent $V$ by Borot-Guionnet (2012). To prove our AOE, we rely on the Lax operator for the quantum Benjamin-Ono hierarchy with periodic profile $v$ exhibited in collective field variables by Nazarov-Sklyanin (2013). The characterization of the limit laws as push-forwards follows from factorization formulas for resolvents of Toeplitz operators on $\mathbb{T}$ with symbol $v$ due to Krein and Calderón-Spitzer-Widom (1958).
1 Overview

This paper is devoted to the macroscopic fluctuations of Jack measures, an ensemble of random partitions at the crossroads of modern probability, representation theory, and quantum integrable systems. These discrete stochastic processes are a $\beta$-deformation of Okounkov’s Schur measures [110] and occur naturally in the study of continuum circular $\beta$-ensembles in generic background potentials $V$ and arbitrary values $\beta > 0$ of Dyson’s inverse temperature. The thermodynamic limit $N \to \infty$ of the circular $\beta$-ensembles corresponds to the scaling limit of the Jack measures. The primary goal of this article is to give an explicit description of the limit shapes, macroscopic Gaussian fluctuations, and all-order expansion of joint cumulants of linear statistics for Jack measures in this scaling limit at all $\beta > 0$ and generic $V$.

In probability theory, $\beta$-ensembles are examples of $N$-particle systems exhibiting the phenomena of phase transitions, long-range interactions, strong-weak duality, and convergence to the conformally-invariant 2D Gaussian free field [61]. $\beta$-ensembles also take center stage in other pure and applied mathematical frontiers, from the refined topological recursion in enumerative geometry [41, 42, 56] to the condensed matter theory of the fractional quantum Hall effect [1, 2, 117, 137]. It is perhaps surprising that a model so ubiquitous could satisfy such remarkable constraints: $\beta$-ensembles obey an infinite hierarchy of stochastic conservation laws due to a direct connection to the quantum Calogero-Moser-Sutherland integrable systems [53, 116, 119, 135].

In the last decade, research in continuum $\beta$-ensembles has sought techniques valid at all $\beta > 0$, generic $V$, and tractable at large $N$. We provide such a method for Jack measures by introducing a new spectral theory that links the probabilistic analysis of Jack measures to the algebraic structures underpinning their complete integrability. More precisely, our work hinges on the observation that the infinitely-many commuting Hamiltonians from the quantum Benjamin-Ono hierarchy [103] provide a method of moments to compute the joint correlation of the linear statistics of Jack measures. Our complete analysis of Jack measures at the macroscopic scale follows as an application of this spectral theory. Locating and utilizing such a dictionary in order to gain access to universal scaling limits in statistical mechanics is one of the defining characteristics of the burgeoning field of integrable probability [21, 26, 44].

In this chapter, we define Jack measures, state our limit theorems, situate our results in the literature, and outline our methods of proof.

1.1 Jack measures

1.1.1 Jack symmetric functions

The trigonometric Calogero-Sutherland model is defined by the Hamiltonian

$$\hat{\mathcal{H}} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \sum_{i<j} \frac{1}{\left( \frac{L}{\pi} \sin \frac{\pi}{L} (x_i - x_j) \right)^2}. \quad (1.1.1)$$
This describes a quantum mechanical system of \(N\) particles at positions \(x_i \in [0, L]\) with \(L\)-periodic boundary conditions [53, 116, 119, 135]. \(\hat{H}\) may also be identified with the generator of circular-\(\beta\) Dyson Brownian motion [61]. We now recall how the Jack symmetric functions, the main special functions in this paper, arise in the spectral theory of \(\hat{H}\). Throughout, \(\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}\) is the unit circle in the complex plane. Change variables to \(w_i = e^{2\pi i x_i/L}\). One checks that \(\hat{H}\Psi_\circ \equiv 0\) for

\[
\Psi_\circ(w_1, \ldots, w_N|\beta) := \prod_{i<j}(w_i - w_j)^{\beta/2}.
\]

This ground state is multi-valued for generic \(\beta > 0\): if \(w_i\) and \(w_j\) left \(\mathbb{T}\) to braid around each other in \(\mathbb{C}\), \(\Psi_\circ\) acquires a phase factor \((-1)^{\beta/2}\). To understand excitations about this ground state, conjugating \(\mathcal{H} = \Psi_\circ^{-1}\hat{H}\Psi_\circ\) one gets

\[
\mathcal{H} = \sum_{i=1}^{N} \left(w_i \frac{\partial}{\partial w_i}\right)^2 + \frac{\beta}{2} \sum_{i<j} \frac{w_i + w_j}{w_i - w_j} \left(w_i \frac{\partial}{\partial w_i} - w_j \frac{\partial}{\partial w_j}\right).
\]

up to an additive constant. For \(\vec{w} = (w_1, \ldots, w_N)\), define a pairing on symmetric polynomials \(\mathbb{C}[w_1, \ldots, w_N]^{S(N)}\):

\[
\langle F^{\text{out}}, F^{\text{in}} \rangle_{\mathbb{T},\beta,N} := \oint_{\mathbb{T}^N} F^{\text{out}}(\vec{w}) \cdot F^{\text{in}}(\vec{w}) \prod_{i<j} |w_i - w_j|^{\beta} \prod_{i=1}^{N} \frac{dw_i}{2\pi i w_i}.\]

This is simply the usual \(L^2(\mathbb{T}^N)\) pairing for \(F\Psi_\circ\). The factor \(\prod_{i<j} |w_i - w_j|^{\beta}\) mixing all variables with strength \(\beta > 0\) is \(|\Psi_\circ(\vec{w}|\beta)|^2\) the ground state amplitude.

Note: the pairing \(\langle \cdot, \cdot \rangle_{\mathbb{T},\beta,N}\) could be defined for all Laurent polynomials \(\mathbb{C}[w_1^\pm, \ldots, w_N^\pm]\). By restricting our attention to symmetric polynomials \(\mathbb{C}[w_1, \ldots, w_N]^{S(N)}\), we have made both a chirality assumption (only non-negative powers of \(w_i\) appear) and a symmetry assumption (\(F\) must be invariant under all transpositions \(w_i \leftrightarrow w_j\)).

Consider the degree (momentum) operator

\[
\mathcal{D} := \sum_{i=1}^{N} w_i \frac{\partial}{\partial w_i}.
\]

In light of three facts

1. \(\mathcal{H}\) is self-adjoint operator on \(\mathbb{C}[w_1, \ldots, w_N]^{S(N)}\) for \(\langle \cdot, \cdot \rangle_{\mathbb{T},\beta,N}\)
2. \(\mathcal{H}\) commutes with \(\mathcal{D}\)
3. Eigenspaces of \(\mathcal{D}\) on \(\mathbb{C}[w_1, \ldots, w_N]^{S(N)}\) are finite-dimensional

the spectral theorem implies that \(\mathcal{H}\) is diagonalized in \(\mathbb{C}[w_1, \ldots, w_N]^{S(N)}\) by

\[
P_\lambda(w_1, \ldots, w_N|\beta)
\]

a basis of multivariate homogeneous symmetric orthogonal polynomials for \(\langle \cdot, \cdot \rangle_{\mathbb{T},\beta,N}\).

**Definition 1.1.1.** These \(P_\lambda\) are the Jack symmetric polynomials.

Due to our chirality and symmetry assumptions, Jacks are indexed by partitions \(\lambda\) of length at most \(N\), namely the data \(0 \leq \lambda_N \leq \cdots \leq \lambda_1\) for \(\lambda_i \in \mathbb{N} = \{0, 1, 2, \ldots\}\).
An amazing property of the Jack polynomials is that they satisfy a coherency condition as the number of variables changes:

**Proposition 1.1.1.** In \( N + 1 \) variables, for \( 0 \leq \lambda_{N+1} \leq \cdots \leq \lambda_1 \),

\[
P_\lambda(w_1, \ldots, w_N, \eta|\beta) \bigg|_{\eta=0} = P_\lambda(w_1, \ldots, w_N) \cdot 1_{\lambda_{N+1}=0}.
\]

(1.1.6)

Although \( w_i \in \mathbb{T} \) were taken with unit modulus above, if we allow \( w_{N+1} = \eta \) to approach the origin, the result is essentially the same Jack symmetric polynomial. However, the limit vanishes if the partition has length \( N + 1 \), as captured by the indicator function \( 1_{\lambda_{N+1}=0} \). This is also known as the fact that Jacks form a basis of

\[
\mathcal{F} := \lim_{\leftarrow} \mathbb{C}[w_1, \ldots, w_N]^{S(N)}
\]

(1.1.7)

the ring of symmetric functions [96]. By our characterization of Jacks as eigenfunctions of \( \mathcal{H} \), one can prove this coherency condition for \( P_\lambda \) as a consequence of a similar coherency condition satisfied by the operator \( \mathcal{H} \) [87].

### 1.1.2 Power sum symmetric functions

Our strategy is to work with Jacks \( P_\lambda(w_1, \ldots, w_N|\beta) \) only via

\[
p_k(\vec{w}) = w_1^k + \cdots + w_N^k
\]

(1.1.8)

the *power sums*. All symmetric functions

\[
\mathcal{F} = \mathbb{C}[p_1, p_2, \ldots]
\]

(1.1.9)

are polynomials in the power sums, which amounts to linear combinations of

\[
p_\mu = p_1^{\#_1} p_2^{\#_2} \cdots p_k^{\#_k} \cdots
\]

(1.1.10)

for \( \# : \mathbb{N} \to \mathbb{N} \) of compact support. In light of this, we regard Jacks as functions only of the variables \( p_k \) and may write \( P_\lambda(p|\beta) \).

The power sums enjoy their own amazing property, proven implicitly in [96]:

**Proposition 1.1.2.** Unlike Jacks \( P_\lambda(w_1, \ldots, w_N|\beta) \), power sums \( p_\mu = p_{1}^{\#_1} p_{2}^{\#_2} \cdots \) are *not* orthogonal with respect to \( \langle \cdot, \cdot \rangle_{T;\beta, N} \), but *are* orthogonal in the limit

\[
\langle F_{\text{out}}, F_{\text{in}} \rangle_{T;\beta, \infty} := \lim_{N \to \infty} \frac{\langle F_{\text{out}}, F_{\text{in}} \rangle_{T;\beta, N}}{\langle 1, 1 \rangle_{T;\beta, N}}.
\]

(1.1.11)

Moreover, \( (p_\mu, p_\nu)_{T;\beta, \infty} \) decouples into 2-point functions \( (p_k, p_{k'})_{T;\beta, \infty} = \frac{2}{\beta} k \delta(k - k') \) according to Wick’s formula: we can define \( \langle \cdot, \cdot \rangle_{T;\beta, \infty} \) on \( \mathcal{F} \) by declaring

\[
p_{-k} = \frac{2}{\beta} k \frac{\partial}{\partial p_k}
\]

(1.1.12)

to be the adjoint of multiplication by \( p_k \) on \( \mathcal{F} \).

This limiting pairing \( \langle \cdot, \cdot \rangle_{T;\beta, \infty} \) on \( \mathcal{F} = \mathbb{C}[p_1, p_2, \ldots] \) is exactly the inner product for Jacks given in VI.1.4 of [96].
1.1.3 Nekrasov variables

To better understand the asymptotic orthogonality of $p_\mu$, implement a rescaling

$$p_k = \frac{V_k}{-\varepsilon_2}$$  \hspace{1cm} (1.1.13)

to variables $V_1, V_2, \ldots$ after writing $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$ via real parameters

$$\varepsilon_2 < 0 < \varepsilon_1.$$  \hspace{1cm} (1.1.14)

The pairing $\langle \cdot, \cdot \rangle_{-\varepsilon_1\varepsilon_2} := \langle \cdot, \cdot \rangle_{T;\beta,\infty}$ is now defined on $\mathcal{F} = \mathbb{C}[V_1, V_2, \ldots]$ by declaring

$$V_{-k} = -\varepsilon_1\varepsilon_2 k \frac{\partial}{\partial V_k}$$  \hspace{1cm} (1.1.15)

to be the adjoint of multiplication by $V_k$ on $\mathcal{F}$. If $\deg V_k = k$,

$$\mathcal{F} = \bigoplus_{d=0}^{\infty} \mathcal{F}_d$$  \hspace{1cm} (1.1.16)

decomposes into finite-dimensional eigenspaces of the degree operator

$$\mathcal{D} = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} V_k V_{-k}.$$  \hspace{1cm} (1.1.17)

which coincides with (1.1.5). Written via rescaled power sums $V_k$, Jacks are

$$P_\lambda(V|\varepsilon_2, \varepsilon_1) = \sum_{\mu \in \mathcal{Y}_d} \chi_\lambda^\mu(\varepsilon_2, \varepsilon_1)V_\mu$$  \hspace{1cm} (1.1.18)

strange superpositions of $V_1^{\#_1} V_2^{\#_2} \ldots$ diagonalizing $\mathcal{K}$. To recapitulate, $V_\mu \in \mathcal{F}$ are indexed by partitions $\mu = 1^{\#_1} 2^{\#_2} \ldots k^{\#_k} \ldots$, while $P_\lambda \in \mathcal{F}$ are indexed by partitions $\lambda \in \mathcal{Y}_d$ of degree $d$: $0 \leq \cdots \leq \lambda_2 \leq \lambda_1$ and $\deg(\lambda) := \sum_{i=1}^{\infty} \lambda_i$. The inner product $\langle \cdot, \cdot \rangle_{-\varepsilon_1\varepsilon_2}$ depends only on the product $-\varepsilon_1\varepsilon_2$ of $\varepsilon_2, \varepsilon_1$, while we choose to parametrize the remaining degree of freedom by the anisotropy $\varepsilon_1 + \varepsilon_2$ rather than $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$ the inverse Jack parameter. At the isotropic point $\varepsilon_1 + \varepsilon_2 = 0$

$$\varepsilon_1 \rightarrow (-\varepsilon, \varepsilon)$$  \hspace{1cm} (1.1.19)

Jacks become Schur functions. As we wrote $\beta = -\frac{\varepsilon_2}{\varepsilon_1}$ this critical point is $\beta = 2$.

1.1.4 Random partitions

Let $\mathcal{F}^{\text{out}}$ and $\mathcal{F}^{\text{in}}$ be two independent copies of $\mathcal{F}$, written in variables $V^{\text{out}}_k$ and $V^{\text{in}}_k$. Since $P_\lambda$ and $V_\mu$ are both orthogonal for $\langle \cdot, \cdot \rangle_{-\varepsilon_1\varepsilon_2}$ due to Propositions (1.1.1) and (1.1.2) respectively, we can write the resolution of the identity in two ways:

$$\prod_{k=1}^{\infty} \exp \left( \frac{V^{\text{out}}_k V^{\text{in}}_k}{-\varepsilon_1\varepsilon_2 k} \right) = \sum_{\lambda \in \mathcal{Y}} P_\lambda(V^{\text{out}}|\varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}}|\varepsilon_2, \varepsilon_1) \frac{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1\varepsilon_2}}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1\varepsilon_2}}.$$  \hspace{1cm} (1.1.20)

Either side of (1.1.20) defines the Stanley-Cauchy kernel $\Pi(V^{\text{out}}, V^{\text{in}}|\frac{1}{-\varepsilon_1\varepsilon_2})$ [132].
Definition 1.1.2. If the variables $V_{\text{out}}^1, V_{\text{out}}^2, \ldots$ and $V_{\text{in}}^1, V_{\text{in}}^2, \ldots$ are chosen in $\mathbb{C}$ so
- (non-negativity) for all $\lambda \in \mathbb{Y}$, $P_\lambda(V_{\text{out}}^| \varepsilon_2, \varepsilon_1)P_\lambda(V_{\text{in}}^| \varepsilon_2, \varepsilon_1) \geq 0$
- (regularity) the Stanley-Cauchy kernel converges $\Pi(V_{\text{out}}^, V_{\text{in}}^| \frac{1}{1-\varepsilon_2 \varepsilon_1}) < \infty$

define the Jack measures $M_V(\varepsilon_2, \varepsilon_1)$ on partitions $\lambda$ by
\[
\text{Prob}_V(\lambda| \varepsilon_2, \varepsilon_1) = \frac{1}{\Pi(V_{\text{out}}^, V_{\text{in}}^| \frac{1}{1-\varepsilon_2 \varepsilon_1})} \cdot P_\lambda(V_{\text{out}}^| \varepsilon_2, \varepsilon_1)P_\lambda(V_{\text{in}}^| \varepsilon_2, \varepsilon_1) \langle P_\lambda, P_\lambda \rangle^{-\varepsilon_1 \varepsilon_2}. 
\] (1.1.21)

In anticipation of the connection to circular $\beta$-ensembles below,

Definition 1.1.3. Collect the data $V_{\text{out}}^k, V_{\text{in}}^k \in \mathbb{C}$ defining a Jack measure $M_V(\varepsilon_2, \varepsilon_1)$
\[
V(w) = \sum_{k=1}^{\infty} \frac{V_{\text{out}}^k w^{-k}}{k} + \sum_{k=1}^{\infty} \frac{V_{\text{in}}^k w^k}{k} 
\] (1.1.22)

into a background potential $V : \mathbb{T} \to \mathbb{C}$

This formula defines a point-wise scalar function of $w = e^{i \theta}$ on the unit circle if
the Fourier modes $V_{\text{out}}^k, V_{\text{in}}^k \in \mathbb{C}$ decay rapidly enough. Indeed, the convergence
of $\Pi$ required by the regularity assumption in the Definition (1.1.2) is exactly the
assumption that $||V||_{1/2} < \infty$ where $||V||_s$ is the Sobolev norm.

1.1.5 Assumptions

In this paper, to satisfy the non-negativity and regularity conditions, we assume
- (Reality) $V_{\text{out}}^k = V_{\text{in}}^k$ for all $k \in \mathbb{Z}_+$
- (Analyticity) $V(w)$ is analytic in an open neighborhood of $\mathbb{T}$

The reality assumption is equivalent to the condition that $V : \mathbb{T} \to \mathbb{R}$ takes real
values on the unit circle. Indeed, as $P_\lambda(V| \varepsilon_2, \varepsilon_1)$ are polynomials in $V_k$ with real
coefficients if $\varepsilon_2 < 0 < \varepsilon_1$, the random $\lambda$ sampled from such a Jack measure appears
with likelihood proportional to
\[
\text{Prob}_V(\lambda| \varepsilon_2, \varepsilon_1) \propto \left| \psi_\lambda(V| \varepsilon_2, \varepsilon_1) \right|^2
\] (1.1.23)

the amplitude of the normalized Jack
\[
\psi_\lambda(V| \varepsilon_2, \varepsilon_1) = \frac{P_\lambda(V| \varepsilon_2, \varepsilon_1)}{\sqrt{\langle P_\lambda, P_\lambda \rangle^{-\varepsilon_1 \varepsilon_2}}}.
\] (1.1.24)

evaluated at $V$. We emphasize that the reality assumption taken here is not the
only way to achieve the non-negativity condition in Definition (1.1.2), and that our
techniques and results are easily adapted to any such exceptional measures [13, 27, 100]. Also, we do not pursue the greatest analytic generality in this paper: our use of
the analyticity assumption in the proof of Proposition (4.5.1) could be considerably
relaxed. Nevertheless, we work transparently in terms of $V$ to clarify comparison to
potentials $V$ of low regularity that appear in applications, which we now describe.
1.1.6 Motivation

We introduce Jack measures to unify $\beta$ and $V$ deformations of the Poissonized Plancherel measures $M_\bullet(\varepsilon, \varepsilon)$ in the representation theory of symmetric groups $S(d)$:

\[
\begin{align*}
M_V(\varepsilon_2, \varepsilon_1) & \quad \Rightarrow \quad M_\bullet(\varepsilon_2, \varepsilon_1) \\
M_V(-\varepsilon, \varepsilon) & \quad \Rightarrow \quad M_\bullet(-\varepsilon, \varepsilon) \\
M_\bullet(-\varepsilon, \varepsilon) & \quad \Rightarrow \quad M_V(\varepsilon_2, \varepsilon_1)
\end{align*}
\]

At the isotropic point $(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$, Jack measures degenerate to the Schur measures introduced in [110]. Across the Robinson-Schensted-Knuth correspondence, certain Schur measures are mapped to models of last passage percolation [71] whose “finite-temperature” generalization to directed polymers led to exact solutions of the Kardar-Parisi-Zhang equation [20, 43, 44]. Alternatively, at the simplest background

\[
V_\bullet(w) = w + \frac{1}{w}
\]  

(1.1.25)

the Jack measures degenerate to the mixture of Jack-Plancherel measures [81] by a Poisson distribution of frequency $\frac{1}{\varepsilon_1 \varepsilon_2}$. These measures $M_\bullet(\varepsilon_2, \varepsilon_1)$ appear most notably as the Nekrasov-Okounkov ensembles of random partitions in the context of abelian pure $N=2$ SUSY gauge theories on $\mathbb{R}^4$ [106].

One can also encounter these discrete Jack measures in the study of stochastic processes in the continuum. The circular $\beta$-ensembles, denoted $LG_{T,V}(N|\beta,t)$, are the Gibbs measures associated to a two-dimensional Coulomb gas of $N$ electrostatic particles $w_1, \ldots, w_N$ of identical charge $\frac{1}{N}$ in a degenerate domain $T \subset \mathbb{C}$ in a background potential $V$ at inverse temperature $\beta > 0$. Mathematically, this defines a random $N$-point configuration $\vec{w} \in T^N$ with law

\[
\text{Prob}_V(w_1, \ldots, w_N|\beta, t) := \frac{1}{Z_{T,V}(N|\beta,t)} \int_{T^N} e^{-\frac{N}{t} \sum_{i=1}^{N} V(w_i) \prod_{i<j} |w_i - w_j|^\beta \prod_{i=1}^{N} \frac{dw_i}{2\pi i w_i}}
\]

These ensembles have two famous degenerations mirroring those of the Jack measures:

\[
\begin{align*}
LG_{T,V}(N|\beta) & \quad \Rightarrow \quad LG_{T,\bullet}(N|\beta) \\
LG_{T,V}(N|2) & \quad \Rightarrow \quad LG_{T,\bullet}(N|2)
\end{align*}
\]

At $\beta = 2$, such ensembles come from random unitary matrices sampled uniformly from Haar measure. For a comparison with $\beta$-ensembles on the line $\mathbb{R}$, see [6, 45, 46, 61].
A parallelism between random partitions and $\beta$-ensembles on the line was first noted at $\beta = 2$ due to the “striking similarity” of the limit theorems for the Poissonized Plancherel measures $M_\bullet(-\varepsilon, \varepsilon)$ as $\varepsilon \to 0$ and Gaussian Unitary Ensembles $L_{\mathbb{G}_{\mathbb{B}, L^2}(N)}$ at large $N \to \infty$ at both the macroscopic [68, 80] and microscopic scales [8, 9, 24, 70, 112]. Our study of random partitions is motivated not just by analogy but by a structural connection between Jack measures and circular $\beta$-ensembles:

**Proposition 1.1.3.** After the change of variables

$$-\varepsilon_1 \varepsilon_2 = \frac{2}{\beta} \cdot \frac{t^2}{N^2} \quad \varepsilon_1 + \varepsilon_2 = \left( \frac{2}{\beta} - 1 \right) \frac{t}{N}$$

(1.1.26)

up to a factor of Stanley-Cauchy kernel $\Pi = \Pi(\mathbb{V}, \mathbb{V} | -\varepsilon_1 \varepsilon_2)$ can determine

$$Z_{T; \mathbb{V}}(N | \beta, t) = \Pi \cdot E \left[ 1_{\lambda'_1 \leq N} \cdot \frac{\langle P_\lambda, P_\lambda \rangle_{T; \beta, N}}{\langle P_\lambda, P_\lambda \rangle_{T; \beta, \infty}} \right]$$

(1.1.27)

the partition function of the circular $\beta$-ensemble in background $\mathbb{V}$ as the expected value of an explicit random variable against the Jack measure $M_\mathbb{V}(\varepsilon_2, \varepsilon_1)$ with same $\mathbb{V}$.

Here $\lambda'_1$ is the length of the partition $\lambda$. In light of formula (10.37) in [96], what is special about this perspective on circular $\beta$-ensembles is that from the vantage point of random partitions, $N$ is a formal variable! The thermodynamic limit $N \to \infty$ becomes the scaling limit of Jack measures $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken at rate $\frac{\beta}{2} = -\frac{\varepsilon_2}{\varepsilon_1}$.

At $\beta = 2$, this relation simplifies to Gessel’s identification of a Toeplitz determinant

$$\det T_N\left( e^{-\frac{N}{T} V(w)} \right) = Z_{T; \mathbb{V}}(N | 2, t) = \Pi \cdot \mathbb{P}\left( \lambda'_1 \leq N \right).$$

(1.1.28)

with the law of the first column of $\lambda$ sampled from a Schur measure $M_\mathbb{V}(-\varepsilon, \varepsilon)$ [63].

Note that an important family of Jack measures not represented in the diagram above are Borodin-Olshanski’s anisotropic $z$-measures [25, 84, 114], which we can recover at

$$V(w) = \varepsilon_2 z^\text{out} \log \left( 1 - \eta^\text{out} w \right) + \varepsilon_2 z^\text{in} \log \left( 1 - \eta^\text{in} w \right)$$

(1.1.29)

for $w \in \mathbb{T}$ and $\eta^\text{out}, \eta^\text{in} \in \mathbb{D}_+$ in the open unit disk. To satisfy our reality assumption, pass to the principal series $z^\text{out} = z^\text{in}$, $\eta^\text{out} = \eta^\text{in}$. Across the bridge provided by Proposition (1.1.3), the corresponding circular $\beta$-ensemble has partition function

$$Z_{T; \mathbb{V}}(N | \beta, t) = \oint T^N \prod_{i=1}^N (1 - \eta w_i) z (1 - \overline{\eta w_i}) \prod_{i<j} |w_i - w_j|^\beta \prod_{i=1}^N \frac{dw_i}{2\pi i w_i}$$

(1.1.30)

When $\eta \to 1$, this is the circular Jacobi ensemble of Bourgade-Nikeghbali-Roualt [33]. At $\beta = 2$, these are Toeplitz determinants with one Fisher-Hartwig singularity [47].
1.2 Macroscopic limits

Let us perform our first computation in a Jack measure $M_{V}(\varepsilon_2, \varepsilon_1)$:

$$E[\deg(\lambda)] = \Pi^{-1}(\mathcal{D} \otimes 1)\Pi = \frac{1}{-\varepsilon_1\varepsilon_2} \sum_{k=1}^{\infty} |V_k|^2$$

(1.2.1)

by formulas (1.1.17), (1.1.20), and Lemma (4.1.1). As either $\varepsilon_1 \to 0$ or $\varepsilon_2 \to 0$, the expected degree of $\lambda$ will diverge. In the figure below, we depict the typical $\lambda$ sampled from the Poissonized Plancherel measures $M_{\bullet}(-\varepsilon, \varepsilon)$ with $V_{\bullet}(w) = w + \frac{1}{w}$ as $\varepsilon \to 0$:

For analytic $V$, we will examine the behavior of the random partition sampled from $M_{V}(\varepsilon_2, \varepsilon_1)$ in the scaling limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken so that $\frac{\beta_2}{2} = \frac{-\varepsilon_2}{\varepsilon_1} > 0$ is fixed. To state our results, we need a precise description the anisotropic profile of a partition, first drawn by Kerov in [81]. Although our limit comes from $N \to \infty$ limit of circular $\beta$-ensembles in a background $V$, our theorems are stated more naturally in terms of:

**Definition 1.2.1.** Collect the data $V_{\text{out}}^k, V_{\text{in}}^k \in \mathbb{C}$ defining a Jack measure $M_{V}(\varepsilon_2, \varepsilon_1)$

$$v(w) = \sum_{k=1}^{\infty} V_{\text{out}}^k w^{-k} + \sum_{k=1}^{\infty} V_{\text{in}}^k w^k$$

(1.2.2)

into a scalar symbol $v : \mathbb{T} \to \mathbb{C}$ on the circle. Notice that $v = \Delta^{1/2}V$.

### 1.2.1 Anisotropic profiles

Let $\varepsilon_2 < 0 < \varepsilon_1$ determine side lengths $-\varepsilon_2 \sqrt{2} \times \varepsilon_1 \sqrt{2}$ of an anisotropic (rectangular) box $\square_{\varepsilon_2, \varepsilon_1}$ with area $\square_{\varepsilon_2, \varepsilon_1} = 2(-\varepsilon_1\varepsilon_2)$ by vectors $(\varepsilon_1, \varepsilon_1), (\varepsilon_2, -\varepsilon_2)$ in $\mathbb{R}^2$:
An anisotropic partition centered at $a$ is the data of a partition $\lambda$ together with $a \in \mathbb{R}$ and $\varepsilon_2 < 0 < \varepsilon_1$. With this extra data, we may represent a partition as a pile of $|\lambda|$ identical anisotropic boxes $\square_{\varepsilon_2, \varepsilon_1}$ in the two-dimensional corner $|c - a|$. The $i$th row consists of $\lambda_i$ boxes stacked in the direction of positive slope. In the isotropic case $(\varepsilon_2, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ of squares, one recognizes the usual presentation of $\lambda$ as a Young diagram. Denote the set of all anisotropic partitions centered at $a$ by $\mathbb{Y}(a; \varepsilon_2, \varepsilon_1)$.

![Diagram](image)

Given an anisotropic partition $\lambda \in \mathbb{Y}(a; \varepsilon_2, \varepsilon_1)$, define the shifted coordinates

$$
\lambda_i^* = a + \varepsilon_2 (i - 1) + \varepsilon_1 \lambda_i.
$$

Unlike $\lambda_i$, these $\lambda_i^*$ are all distinct for $\varepsilon_2 < 0 < \varepsilon_1$, and are separated by at least $-\varepsilon_2 > 0$. Let $\mathcal{S}(\lambda|a; \varepsilon_2, \varepsilon_1) = \{\lambda_i^*\}_{i=1}^\infty$ denote this infinite point configuration. Given a probability measure $M$ on $\mathbb{Y}(a; \varepsilon_2, \varepsilon_1)$, the map $\lambda \mapsto \mathcal{S}(\lambda|a; \varepsilon_2, \varepsilon_1)$ gives a random infinite configuration of points on the real line. We regard this stochastic point process as a microscopic configuration. In this paper, we are concerned only with macroscopic features of the random partition $\lambda \in \mathbb{Y}(a; \varepsilon_2, \varepsilon_1)$, the collective behavior of the random swarm of points $\mathcal{S}(\lambda|\varepsilon_2, \varepsilon_1)$.

![Diagram](image)

The anisotropic profile $f_\lambda(c|a; \varepsilon_2, \varepsilon_1)$ of $\lambda \in \mathbb{Y}(a; \varepsilon_2, \varepsilon_1)$ is the piecewise-linear function outlined by the outermost boxes of the anisotropic partition. One may recover the profile from its weak second derivative $f''_\lambda(c|a; \varepsilon_2, \varepsilon_1)$, i.e. from the interlacing extrema $c^\dagger_{e+1} < c^\dagger_e < c^\dagger_{e-1} < \cdots < c^\dagger_2 < c^\dagger_1 < c^\dagger_0$ of the profile depicted in the figure.
A continuous profile \( f \) centered at \( a \in \mathbb{R} \) is a function \( f : \mathbb{R} \to \mathbb{R}_{\geq 0} \) of a real variable \( c \in \mathbb{R} \) which is 1-Lipschitz and eventually agrees with \( |c - a| \):

\[
|f(c_1) - f(c_2)| \leq |c_1 - c_2| \quad \text{for all} \quad c_1, c_2 \in \mathbb{R}
\] (1.2.4)

\[
\text{area}(f) := \int_{-\infty}^{\infty} \left( f(c) - |c - a| \right) dc < \infty
\] (1.2.5)

These assumptions ensure that continuous profiles are bounded below by the empty profile \( \emptyset_a(c) = |c - a| \) centered at \( a \in \mathbb{R} \). Let \( \mathcal{Y}(a) \) denote the convex space of continuous profiles. At each center \( a \), for all \( \varepsilon_2 < 0 < \varepsilon_1 \), taking the profile of an anisotropic partition provides embeddings \( \mathcal{Y}(a; \varepsilon_2, \varepsilon_1) \subset \mathcal{Y}(a) \). Thus, any probability measure \( M \) on anisotropic partitions is a measure on continuous profiles. Conversely, we may regard \( \mathcal{Y}(a; \varepsilon_2, \varepsilon_1) \) as a lattice approximation of \( \mathcal{Y}(a) \).

In what follows, fix \( a = 0 \). For \( f \in \mathcal{Y}(0) \), consider the \( l \)th linear statistic

\[
\text{ch}_l[f] = \frac{1}{2} \int_{-\infty}^{\infty} c^lf^{(l)}(c) dc.
\] (1.2.6)

This notation defers to the appearance of \( \text{ch}_l[f_{\lambda}(c|\varepsilon_2, \varepsilon_1)] \) as Chern classes in [98].

### 1.2.2 Law of large numbers

The random partition concentrates around a deterministic interface:

**Theorem 1.2.1.** (LLN) For random \( \lambda \) sampled from \( M_{\nu}(\varepsilon_2, \varepsilon_1) \) with analytic symbol \( \nu \), in the limit \( \varepsilon_2 \to 0 \leftarrow \varepsilon_1 \) taken so that \( \frac{\beta}{2} = \frac{\varepsilon_2}{\varepsilon_1} > 0 \) is fixed, the joint moments of the linear statistics \( \text{ch}_l[f] \) of the random profile

\[
f_{\lambda}(c|\varepsilon_2, \varepsilon_1) \to f_{*|\nu}(c)
\] (1.2.7)

converge to those of a limit shape \( f_{*|\nu}(c) \in \mathcal{Y} \), independent of \( \beta \):

\[
2\pi \cdot \frac{1 + f_{*|\nu}(c)}{2} = (\nu_*(d\theta))\left((-\infty, c)\right)
\] (1.2.8)

is the distribution function of the push-forward along \( \nu : \mathbb{T} \to \mathbb{R} \) of the uniform measure on the circle. [This appears as Theorem 5.1.1 below.]

At \( \beta = 2 \), the description of this limit shape in this “interesting asymptotic regime” is due to Okounkov [111] by a map to free fermions and the method of steepest descent. To recover the Vershik-Kerov-Logan-Shepp limit shape [85, 95], set \( \nu_*(w) = w + 1/\nu \) and derive \( f_{*|\nu}(c) = \frac{2}{\pi} \arcsin \frac{c}{\nu} \). Note that limit shapes are known for Schur measures with different non-negativity and regularity assumptions [11, 13, 19]. Despite possibly many valleys in the graph of the symbol \( \nu \), we are in one-cut regime because the range of \( \nu \) is connected (due to its regularity). The Nekrasov-Okounkov ensembles \( M_{\nu}(a_1, \ldots, a_r; \varepsilon_2, \varepsilon_1) \) of random partitions are discrete analogs of \( \beta \)-ensembles in a multi-cut regime [106, 112] and are at the heart of Nekrasov’s partition function for the gas of \( U(r) \) instantons on \( \mathbb{R}^4 \) in the Omega background.
By variational principles available at $\varepsilon_1 + \varepsilon_2 = 0$, the exact calculation of the limit shape around which the additive superposition of profiles $f_{\lambda(\cdot)}(c-a_-|\cdot,\varepsilon,\varepsilon)$ concentrates as $\varepsilon \to 0$ is the main result of [106]. Identifying the emergent limit shape verifies Seiberg-Witten’s proposal that the low-energy dynamics of the gauge theory are encoded in a pre-potential $F_v(a_1, \ldots, a_r)$, which can be reconstructed from a hidden family of hyper-elliptic curves with a distinguished differential [125, 126]. Considering the role played by the curve $u = v(w)$ in our work, our results for the simpler case $r = 1$ of abelian gauge group $U(1)$ yet at arbitrary $v$ and arbitrary $\varepsilon_2, \varepsilon_1$ are in harmony with what is known in the multi-cut case, and we hope that the techniques in this paper can find further use in this rapidly developing subject.

Any limit transition in which variance tends to zero is affectionately referred to as a law of large numbers, even if the mechanism behind this concentration is not necessarily that of sums of independent random variables. Similarly, if a random height function converges to a Gaussian process, it is dubbed a central limit theorem.

1.2.3 Central limit theorem

Almost twenty years after [85, 95], for the Plancherel measures Kerov showed that macroscopic fluctuations of the height function around the limit shape $f_{\lambda(\cdot)}(c)$ converge to an explicit generalized Gaussian process [68, 77]. For central limit theorems at $\beta = 2$ but at other $v(w)$, see [131]. At any $\beta > 0$ but still $v_*(w) = w + 1/w$, Kerov’s CLT receives only a deterministic mean shift at $\beta \neq 2$, the covariance being the same [50, 62, 65]. Our second theorem is an exact description of the Gaussian asymptotics for arbitrary Jack measures, giving new proofs of all aforementioned results.
Theorem 1.2.2. (CLT) For random \( \lambda \) sampled from \( M_V(\varepsilon_2, \varepsilon_1) \) with analytic symbol \( v \), in the limit \( \varepsilon_2 \to 0 \leftarrow \varepsilon_1 \) taken so that \( \frac{\beta}{2} = \frac{-\varepsilon_2}{\varepsilon_1} > 0 \) is fixed, the joint moments of linear statistics \( \chi[l] \) of profile fluctuations

\[
\phi_{\lambda}(c|\varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1 \varepsilon_2}} \left( f_{\lambda}(c|\varepsilon_2, \varepsilon_1) - f_{\lambda|v}(c) \right)
\]  

converge to that of a Gaussian field

\[
\phi_{v}(c) - \left( \sqrt{\frac{\beta}{2}} - \sqrt{\frac{\beta}{3}} \right) X_v(c),
\]

where \( X_v(c) \) is a deterministic mean shift determined by formula (6.4.3) and \( \phi_{v}(c) \) is the push-forward along \( v : T \to \mathbb{R} \) of the restriction to \( T_+ = T \cap \mathbb{H}_+ \) of \( \Phi^H \), the Gaussian free field on \( \mathbb{H}_+ \).

\[\text{Cov}\left[ \Phi^H(w_1), \Phi^H(w_2) \right] = \frac{1}{4\pi} \log \left| \frac{w_1 - w_2}{w_1 - w_2} \right|^2 \]  

with zero boundary conditions.  [This appears as Theorem 6.0.2 below.]

References for the algebraic and probabilistic aspects of the two-dimensional Gaussian free field are [48, 72, 73, 129]. Our description of the macroscopic fluctuations, a 1D slice of a conformally-invariant 2D object warped by a non-conformal transformation, is identical to that encountered in \( \beta \)-Jacobi ensembles [22] and in Wigner matrices [18]. For \( v \) arbitrary and \( \beta = 2 \), the Gaussian process appearing in our CLT is identical to that appearing in the CLT recently established in [34] for biorthogonal ensembles [17].

Finally, let us compare our CLT to that for circular \( \beta \)-ensemble empirical measures \( \rho(w|N, \beta) \) at \( V \equiv 0 \). Seen with a probabilistic eye, Proposition (1.1.2) is exactly the statement that the linear statistics \( p_k(\bar{w}) = w_1^k + \cdots + w_N^k \) for \( LG_{T;0}(N|\beta) \) are asymptotically independent complex Gaussians with variance \( \langle p_k, p_k \rangle_{T;\beta,\infty} = \frac{2}{\beta} k \). At \( \beta = 2 \), by formula (1.1.28) this is the strong Szegö theorem for Toeplitz determinants [130], see also [49]. The precious factorization of the Stanely-Cauchy kernel (1.1.20) is due to the convergence of circular \( \beta \)-ensembles to a Gaussian field! In light of this, our CLT in the spectral variable \( \lambda \) relies on a CLT in the spatial variable \( w \in T \).

### 1.3 All-order expansions

We deduce Theorems 1.2.1 and 1.2.2 from an all-order expansion for the joint cumulants of the moments of the transition measures \( \tau^\lambda_{\varepsilon_1}(c|\varepsilon_2, \varepsilon_1) \) [11, 78, 80, 81, 92, 114].

Theorem 1.3.1. (AOE) For random \( \lambda \) sampled from \( M_V(\varepsilon_2, \varepsilon_1) \) with analytic symbol \( v \), the joint cumulants of transformed linear statistics \( \chi[l] \) [\( f_{\lambda}(c|\varepsilon_2, \varepsilon_1) \)] extracted from random transition measure \( \tau^\lambda_{\varepsilon_1}(c|\varepsilon_2, \varepsilon_1) \) have the convergent expansion

\[
\hat{W}_n^V(\ell_1, \ldots, \ell_n|\varepsilon_2, \varepsilon_1) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1 \varepsilon_2)^{(n-1)+g} (\varepsilon_1 + \varepsilon_2)^m \hat{W}_{n,g,m}^v(\ell_1, \ldots, \ell_n)
\]  

14
where $\hat{W}_{n,g,m}(\ell_1, \ldots, \ell_n)$ is the $v$-weighted enumeration of connected “ribbon paths” on $n$ sites of lengths $\ell_1, \ldots, \ell_n$ with $(n-1)+g$ pairings and $m$ slides [see section 4.3 for definitions]. The quantities $\hat{W}_{n,g,m}(\ell_1, \ldots, \ell_n)$ are expressed solely through matrix elements of the Toeplitz operator $T(v)$ on the circle with scalar symbol $v(w)$. [This appears as Theorem 4.2.1 below.]

At $v_o(w) = w + 1/w$, our result may be compared with [54]. Identifying

$$-\varepsilon_1\varepsilon_2 \leftrightarrow \frac{2}{\beta} \cdot \frac{1}{N^2} \text{ and } \varepsilon_1 + \varepsilon_2 \leftrightarrow \left( \frac{2}{\beta} - 1 \right) \cdot \frac{1}{N},$$

(1.3.2)

Theorem 1.3.1 is identical in form to the $1/N$ refined topological expansion of joint cumulants of linear statistics for the one-cut $\beta$-ensembles on $\mathbb{R}$. Assuming existence of a $1/N$ expansion, for one-cut polynomial potentials $V$, Chekhov-Eynard proved [40, 41] that the quantities $\hat{W}_{n,g,m}(\ell_1, \ldots, \ell_n)$ enumerate the number of ribbon graphs of genus $g$ with $m$ Möbius strips built from $n$ vertices of degree $\ell_i$ with vertex weights depending on $V$. For details, see chapter 10 in [42]. The existence of this expansion was verified for the one-cut regime by Borot-Guionnet in [30]. A modification of [41] to accommodate the case of $V$ with $V'$ a rational function is given independently in [36, 39]. For further computations and applications, see [28, 36, 97].

At the special value $\beta = 2$, only orientable surfaces appear in the $1/N$ expansion [5, 52, 16, 35, 136, 91]. The Eynard-Orantin theory reinterprets these quantities as symplectic invariants of a hidden algebraic curve [59]. The data $W^V_{1,0,0}(u)$ and $W^V_{2,0,0}(u_1, u_2)$ specifies a spectral curve $\Sigma_V$, and the refined topological recursion gives a means of computing $W^V_{n,g,m}(u_1, \ldots, u_n)$ by residue calculus on $\Sigma_V$. The topological recursion appears in a stunning variety of moduli problems in geometry [56]. For formulations of the $\beta$-deformation of this recursion, see [41, 42, 58].

Our derivations of Theorems 1.2.1 and 1.2.2 from Theorem 1.3.1 proceed in exactly the same way as Frostmann’s equilibrium [123] and Johansson’s CLT [69] are deduced from the $1/N$ expansion, simply by computing the unstable correlators $W^V_{1,0,0}(u)$, $W^V_{2,0,0}(u_1, u_2)$, and $W^V_{1,0,1}(u)$ which determine the LLN, CLT covariance, and CLT mean, respectively. For accounts of the oscillatory nature of the CLT in the multi-cut regime, see [29, 55, 57, 118, 128]. Although we do not rely on any structural connection to the $\beta$-ensembles in our analysis of $M_V(\varepsilon_2, \varepsilon_1)$, this modern approach to $\beta$-ensembles was a major influence in the formation of this project.

The Nekrasov variables $\varepsilon_2, \varepsilon_1$ are specializations of equivariant parameters

$$H^*_T(\bullet) = H^*(BT) = H^*(\mathbb{C} \mathbb{P}^\infty \times \mathbb{C} \mathbb{P}^\infty) \cong \mathbb{Z}[\varepsilon_2, \varepsilon_1].$$

(1.3.3)

for the $T = \mathbb{C}^\times \times \mathbb{C}^\times$ action on the Hilbert scheme $\overline{M}(1, d)$ of $d$ points in $\mathbb{C}^2$, where Jacks $P_\lambda$ with $|\lambda| = d$ are identified with torus fixed points [102]. In [99], the parameter $\varepsilon_1 + \varepsilon_2$ is the weight of the symplectic form on $\overline{M}(1, d)$, while $-\varepsilon_1\varepsilon_2$ is the handle-gluing element in the Frobenius algebra $H^*_T(\mathbb{C}^2)[\frac{1}{\varepsilon_1\varepsilon_2}]$. Considering also the relation between Schur measures and double Hurwitz numbers [109], we anticipate that our “ribbon paths” could gain a more suitable geometric description in the future.
1.4 Benjamin-Ono waves

To work with the $\beta$-ensembles at general $V$, a first approach is to derive relations between the joint correlators of linear statistics by a single integration by parts known as the loop equations, Schwinger-Dyson equations, Pastur equations, or Virasoro constraints \[7\]. These give a recursive means of determining an all-order asymptotic expansion of joint cumulants $W^V_n(u_1, \ldots, u_n|N, \beta)$ of the linear statistics. Note that a discrete analog of loop equations which emerge in non-abelian gauge theory \[105, 107, 108\] has recently led to the Gaussian asymptotics of a different integrable discretization of the $\beta$-ensembles \[23\]. To establish Theorem 1.3.1, we work directly, not recursively, by making a connection with quantum integrable systems.

A second approach to $\beta$-ensembles is to realize their law as the eigenvalue process of a random matrix model \[6, 51, 61, 86, 90\]. The extra angular degrees of freedom coming from the random eigenvectors provides a larger framework in which to determine the statistical properties of the ensembles $LG_{R,V}(N|\beta)$ and are an alternative to the loop equations. In this paper, the Nazarov-Sklyanin Lax operator $L$ for the quantum Benjamin-Ono hierarchy with periodic boundary conditions \[103\] plays the role of the matrix model for the Jack measures $M_{V}(\varepsilon_2, \varepsilon_1)$. In other words, in lieu of variational principles, mappings to free fermions at $\beta = 2$, or any discrete analogs of the loop equations, we determine the law of the Jack measures $M_{V}(\varepsilon_2, \varepsilon_1)$ through the spectral theory of an operator. This provides a new parallelism between random partitions and random matrices beyond the theory of determinantal point processes.

Let $(x, t) \in \mathbb{R} \times [0, \infty)$ and fix $\varepsilon \in \mathbb{R}$. The classical Benjamin-Ono equation

$$v_t + 2vv_x + \varepsilon J v_{xx} = 0 \quad (1.4.1)$$

is a model for internal waves in a stratified fluid of infinite depth in (1+1)-dimensions \[10, 115\]. Despite the non-local Hilbert transform $J$, this $v$ obeys infinitely many conservation laws, which can be used to solve the Cauchy problem for localized or periodic initial data \[4, 74, 75, 76\]. By encoding the evolution in the form of a Lax pair, the time evolution of the disturbance $v(x, t)$ can be determined by the inverse scattering method \[3\]. For recent global well-posedness results, see \[66, 101\].

As a Hamiltonian system, the Benjamin-Ono equation is classical field theory that can be canonically quantized \[2\]. In a space of period $L$, the quantization is best known as the hydrodynamic limit $N \to \infty$ of right-moving solutions of the trigonometric Calogero-Sutherland quantum $N$-body system on a circle at high density \[1\]. Recall, this system is defined by the Hamiltonian $\hat{H}$ in formula (1.1.1). Almost half a century after Calogero and Sutherland proved that the integrability of this interacting quantum system \[38, 134\], recent work in representation theory has sought to clarify the stable algebraic structure controlling this integrability in the limit $N \to \infty$ \[99, 103, 104, 124, 127\]. In this paper, we draw only on the work of Nazarov-Sklyanin \[103\], which gives an explicit quantization of the Lax pair for the classical Benjamin-Ono equation with periodic boundary conditions.
The classical Lax operator for Benjamin-Ono equation on a circle is

\[ T(v) + \varepsilon D \]  

(1.4.2)

the sum of a Toeplitz operator \( T(v) \) on \( T = \{ |w| = 1 \} \) with symbol \( v(w) \) and \( D = w \frac{\partial}{\partial w} \). For Laurent \( v \), \( T(v) \) acts on the pre-Hardy space \( \mathbb{C}[w] \) with basis \( |h\rangle := w^h \) for \( h \in \mathbb{N} \). In [103], Nazarov-Sklyanin show that the quantized Lax operator

\[ \mathcal{L}(\varepsilon_2, \varepsilon_1) : \mathcal{F} \otimes \mathbb{C}[w] \rightarrow \mathcal{F} \otimes \mathbb{C}[w] \]  

(1.4.3)

is \( \mathcal{L}(\varepsilon_2, \varepsilon_1) = T(v | -\varepsilon_1 \varepsilon_2) + (\varepsilon_1 + \varepsilon_2) D \), featuring a Toeplitz operator whose symbol is the \( \mathfrak{gl}_1 \) current at level \( -\varepsilon_1 \varepsilon_2 \) and \( V_0 = 0 \) in its Fock space representation \( \mathcal{F} \). For \( \ell = 1, 2, \ldots \) the VEVs \( \langle 0|\mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell|0 \rangle \) commute and are simultaneously diagonalized on Jacks \( P_\lambda \), which we recall in Theorem 3.4.1. The dictionary between probability and integrability is Corollary 3.4.1: the transition measures \( \tau^\lambda(c|\varepsilon_2, \varepsilon_1) \) [11, 78, 80, 81, 92, 114] are the spectral measures of \( \mathcal{L}(\varepsilon_2, \varepsilon_1) \) at the vector \( P_\lambda(v|\varepsilon_2, \varepsilon_1) \otimes |0\rangle \).

One might compare this to Biane’s realization of the isotropic transition measures \( \tau^\lambda(c|\varepsilon, \varepsilon) \) as the spectral measures of the Jucy-Murphy elements [11]. If one considers the problem of explaining why free probability theory emerges in the description of large symmetric group modules, we may remember Biane’s suggestion that

“the problem resembles that of statistical mechanics where one has to find the relevant macroscopic parameters of a system whose microscopic description is known.” [12].

Despite the fact that a large deviation principle is not known for Jack measures with arbitrary symbols \( v \), we hope that this work is a first step towards elucidating the problem posed by Biane, and that our results might one day find their true home in the exact dynamics of the Benjamin-Ono quantum liquid [1, 2, 117, 137].

### 1.5 Outline

We recall the Nazarov-Sklyanin Lax operator \( \mathcal{L} \) in chapter 3 as a quantization of the classical theory of Toeplitz operators on the circle with scalar symbol, which we review selectively in chapter 2. We prove our all-order expansion, law of large numbers, and central limit theorem in chapters 4, 5, and 6, respectively. Finally, we gather specializations of our results to Jack-Plancherel measures in appendix A.

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2 Spectral theory

In this chapter, we compile a select sequence of results from the vast theory of Toeplitz operators. For reviews, see [31, 32]. We clearly state assumptions on the symbols \( v: \mathbb{T} \to \mathbb{C} \) of Toeplitz operators so as to mark when a result is applicable to those

\[
v(w) := \sum_{k=1}^{\infty} V_k w^{-k} + \sum_{k=1}^{\infty} V_k w^k
\]

satisfying the regularity and non-negativity assumptions for our Jack measures from section 1.1.5. Our *analyticity* assumption on \( v(w) \) is a drastic one in this theory, as the primary emphasis in the early history of Toeplitz operators was on the spectral theory for symbols at low regularity. Our *reality* assumption \( v: \mathbb{T} \to \mathbb{R} \) ensures that the associated Toeplitz operator \( T(v)^* = T(v) \) is self-adjoint.

2.1 Toeplitz operators

Let \( T = \{|w| = 1\} \) denote the unit circle in \( \mathbb{C} \). Equip the ring of Laurent polynomials \( \mathbb{C}[w, w^{-1}] \subset L^2(\mathbb{T}) \) with the usual pairing

\[
\langle v^{(2)}, v^{(1)} \rangle := \frac{1}{2\pi i} \int_{|w|=1} \frac{v^{(2)}(w)v^{(1)}(w)}{w} \, dw
\]

Abbreviate \(|h\rangle := w^h\) for \( h \in \mathbb{Z} \) and matrix elements of an operator \( A \) by

\[
A_{h_+, h_-} := \langle w^{h_+}, Aw^{h_-} \rangle
\]

where \( h_- \) and \( h_+ \) stand for the initial and final “heights”, respectively.

Define the projection

\[
\pi_\bullet: \mathbb{C}[w, w^{-1}] \to \mathbb{C}[w]
\]

which eliminates negative powers of \( w \), and write \( \pi_0 \) for \(|0\rangle \langle 0|\) projection onto the component of \(|0\rangle = w^0\), so that

\[
\pi_+ := \pi_\bullet - \pi_0 \quad \text{and} \quad \pi_- := 1 - \pi_\bullet
\]

project on to strictly positive and negative powers of \( w \). That is,

\[
1 = \pi_- + \pi_0 + \pi_+
\]

Given \( v \in \mathbb{C}[w, w^{-1}] \), the *Toeplitz operator*

\[
T(v) = \pi_\bullet v(w) \pi_\bullet
\]

is the compression of point-wise multiplication by \( v(w) \), and is a well-defined map

\[
T(v): \mathbb{C}[w] \to \mathbb{C}[w]
\]

from the polynomial ring to itself.

One may care to work with Toeplitz operators \( T(v) \) whose symbols \( v \) are less regular than Laurent polynomials. In this case, one may complete \( \mathbb{C}[w, w^{-1}] \) with the norm above to \( L^2(\mathbb{T}) \) and see if \( T(v): H_\bullet \to H_\bullet \) where \( H_\bullet \) is the *Hardy space*, the closure of \( \mathbb{C}[w] \) in \( L^2(\mathbb{T}) \).
2.2 Wiener-Hopf factorization

Given \( \gamma : \mathbb{T} \rightarrow GL(1) = \mathbb{C}^\times \) an invertible function with winding number \( a \in \mathbb{Z} \), have

\[
\gamma(w) = \exp \left( a \log w + L(w) \right)
\]  
(2.2.1)

for \( L(w) \) a single-valued function. Taking the additive factorization \( 1 = \pi_+ + \pi_0 + \pi_- \) of \( L(w) \) gives us the multiplicative Wiener-Hopf factorization of \( \gamma \):

\[
\gamma(w) = w^a \cdot \gamma_-(w) \gamma_0(w) \gamma_+(w).
\]  
(2.2.2)

The factors

\[
\gamma_{\pm}(w) := \exp \left( L_{\pm}(w) \right)
\]  
(2.2.3)

extend to non-vanishing holomorphic functions \( \gamma_{\pm} : \mathbb{D}_{\pm} \rightarrow \mathbb{C} \) on

\[
\mathbb{D}_{\pm} = \{|w|^{\pm} < 1\}
\]  
(2.2.4)

in \( \mathbb{P}^1 \) and take the value 1 at 0 and \( \infty \), respectively. We isolate the zero mode

\[
\gamma_0(w) := \exp \left( L_0 \right)
\]  
(2.2.5)

in our notation, as it will be crucial for us in what follows.

In [93] one checks that a Toeplitz operator \( T(\gamma) : H_\bullet \rightarrow H_\bullet \) is an invertible operator on Hardy space if and only if \( \gamma : \mathbb{T} \rightarrow GL(1) \) is invertible and the winding number of \( \gamma \) vanishes, that is \( a = 0 \). In this case, a two-sided inverse of \( T(\gamma) \) is

\[
T^{-1}(\gamma) = T(\gamma_+^{-1})T(\gamma_0^{-1})T(\gamma_-^{-1})
\]  
(2.2.6)

the factor \( T(\gamma_0^{-1}) \) being simply \( \gamma_0^{-1} \cdot 1 \).

2.3 Krein-Calderón-Spitzer-Widom factorization

Suppose \( \gamma \) is invertible and does not wind around the origin. What are \( T_{h_+ h_-}^{-1}(\gamma) := \langle w^{h_+}, T^{-1}(\gamma) w^{h_-} \rangle \)? To answer this, we’ll derive a formula for the generating series

\[
\sum_{h_+=0}^{\infty} \sum_{h_-=0}^{\infty} w_+^{h_+} T^{-1}_{h_+, h_-}(\gamma) w_-^{h_- - 1}
\]  
(2.3.2)

for variables \( w_{\pm} \in \mathbb{D}_{\pm} \) in disjoint open disks. Keep in mind that \( w_{\pm} \) can get quite close to \( w \in \mathbb{T} \) of norm 1, but these \( w_{\pm} \) are not the variable \( w \) we are integrating over when we take matrix elements via \( \langle \cdot, \cdot \rangle \). The generating series is equal to

\[
\frac{1}{2\pi i} \oint_{\mathbb{T}} dw \frac{1}{w - w_+} T^{-1}(\gamma) \frac{1}{w_- - w}.
\]  
(2.3.3)
Plugging in
\[ T^{-1}(\gamma) = \gamma_+^{-1} \gamma_0^{-1} \pi_\bullet \gamma_-^{-1}, \]  
(2.3.4)
to simplify we’ll need to write the lone Hardy projection \( \pi_\bullet \) as an integral operator:

\[
\pi_\bullet \left( (\gamma^-)^{-1}(w_--w)^{-1} \right) = \sum_{h=0}^{\infty} \left( \frac{1}{2\pi i} \oint_{|z|=1} \frac{\gamma_-(z)^{-1}(w_--z)^{-1}z^{-h-1}dz}{w} \right) w^h
\]

\[
= \sum_{h=0}^{\infty} w_+^{-1} \left( \frac{1}{2\pi i} \oint_{|z|=1} \frac{\gamma_-(1/z)^{-1}}{z-1/w_+} z^h dz \right) w^h
\]

\[
= \sum_{h=0}^{\infty} \gamma_-(w_-)^{-1} w_-^{-h-1} w^h
\]

\[
= \gamma_-(w_-)^{-1}(w_- - w)^{-1}
\]

(2.3.5)
after the change of variables \( z \mapsto 1/z \) in the second line. Formula 2.3.2 is now

\[
\frac{\gamma^-}{\gamma_0} \cdot \frac{1}{2\pi i} \oint_T \frac{\gamma_+(w)^{-1} dw}{(w-w_+)(w_--w)} = \frac{\gamma_-(w_-)^{-1} \gamma_+(w_+)^{-1}}{w_- - w_+}.
\]

(2.3.6)

We have proven

**Theorem 2.3.1.** Let \( \gamma : \mathbb{T} \to \mathbb{C}^\times \) be an invertible analytic function on the circle with winding number zero and Wiener-Hopf factorization \( \gamma = \gamma_- \gamma_0 \gamma_+ \). For \( w \in \mathbb{D}_\pm \),

\[
\sum_{h=0}^{\infty} \sum_{h_-=0}^{\infty} w_+^{h} \mathcal{T}_{h+}^{-1}(\gamma) w_-^{-h_-} = \frac{1}{\gamma_0} \cdot \frac{\gamma_-(w_-)^{-1} \gamma_+(w_+)^{-1}}{w_- - w_+}.
\]

(2.3.7)

Let’s gather some corollaries of this factorization theorem. Multiplying both sides by \( w_- \) and send \( w_- \to \infty \), get

\[
\sum_{h_+=0}^{\infty} w_+^{h_+} \mathcal{T}_{h_+}^{-1}(\gamma)^{-1} = \frac{1}{\gamma_0} \cdot \frac{1}{\gamma_+(w_+)}.
\]

(2.3.8)

In particular,

\[
\mathcal{T}_{0,0}(\gamma)^{-1} = \frac{1}{\gamma_0} = \exp \left( \frac{1}{2\pi i} \oint_{|w|=1} \log \left[ \frac{1}{\gamma(w)} \right] \frac{dw}{w} \right).
\]

(2.3.9)

In the opposite limit \( w_+ \to 0 \), get

\[
\sum_{h_-=0}^{\infty} \mathcal{T}_{0,h_-}(\gamma)^{-1} w_-^{-h_-} = \frac{1}{\gamma_0} \cdot \frac{1}{\gamma_-(w_-)}.
\]

(2.3.10)

In the original papers of Krein and Calderón-Spitzer-Widom [37, 89], one encounters a version of Theorem 2.3.1 tailored for the description of *resolvents* of Toeplitz operators, which we now describe.
2.4 Ismagilov-Rosenblum spectral resolutions

Assume \( v : \mathbb{T} \to \mathbb{R} \) is real-valued and analytic (which covers our assumptions \(1.1.2\)). The Toeplitz operator \( T(v) \) with symbol \( v \) is a bounded self-adjoint operator on the Hardy space \( H_\bullet \). Can we write its spectral resolution

\[
T(v) = \int_{-\infty}^{\infty} c \, dE^{T(v)}(c)
\]  

(2.4.1)

explicitly in terms of the symbol \( v \)? In other words, can we find \( E^{T(v)}(c) \) the left-continuous resolution of the identity of \( T(v) \)?

The on-diagonal elements of the resolution of the identity

\[
\langle \psi, E^{T(v)}(c) \psi \rangle = \tau^{T(v)}_{\psi}((-\infty, c])
\]  

(2.4.2)

are the spectral measures \( \tau^{T(v)}_{\psi} \) of \( T(v) \) at the vector \( \psi \in H_\bullet \). For \( \langle \psi, \psi \rangle = 1 \), this is the unique probability measure on \( \mathbb{R} \) such that for all \( \ell \geq 0 \) we have

\[
\langle \psi, T(v)^{\ell} \psi \rangle = \int_{-\infty}^{\infty} c^{\ell} d\tau^{T(v)}_{\psi}(c).
\]  

(2.4.3)

Form the resolvent

\[
R(u) := (u - T(v))^{-1} = \sum_{\ell=0}^{\infty} u^{-\ell-1} T(v)^{\ell}
\]  

(2.4.4)

of the operator \( T(v) \). The series expansion is valid for \( |u| > ||T(v)|| = ||v||_\infty \) while the operator is truly defined for all \( u \) outside the spectrum. One can prove that \( \text{Spec}(T(v)) \subset v(\mathbb{T}) \subset \mathbb{R} \). The spectral measure is equivalently characterized

\[
\langle \psi, R(u) \psi \rangle = \int_{-\infty}^{\infty} \frac{d\tau^{T(v)}_{\psi}(c)}{u - c}
\]  

(2.4.5)

by the fact that its Stieltjes transform is a diagonal matrix element of the resolvent.

To analyze the resolvent, and hence learn something about the spectrum of \( T(v) \), can try to use Theorem 2.3.1. Indeed, fix \( u \in \mathbb{C} \setminus \mathbb{R} \) in either \( \mathbb{H}_\pm \) the upper or lower half-planes, and consider the family of maps

\[
\gamma(w; u) := u - v(w).
\]  

(2.4.6)

indexed by \( u \). Since \( v \) takes real values, for each \( u \notin \mathbb{R} \) we know \( \gamma(w; u) \) is an invertible analytic function on the circle with winding number zero! Moreover, using

\[
u - T(v) = T(u - v)
\]  

(2.4.7)

where here \((u - v)(w) = u - v(w) = \gamma(w; u)\) we treat \( u = u1 \) as a scalar multiple of the identity operator on the Hardy space, the resolvent of our Toeplitz operator is

\[
R(u) = T^{-1}(\gamma).
\]  

(2.4.8)
By formula 2.4.8, we can determine the matrix elements $R_{h_+,h_-}(u)$ of the resolvent through Theorem 2.3.1:

$$
\sum_{h_+,h_-=0}^{\infty} w_+^{h_+} R_{h_+,h_-}(u) w_-^{h_-} = \frac{1}{\gamma_0(u)} \gamma_-(w_-;u)^{-1} \gamma_+(w_+;u)^{-1}. \tag{2.4.9}
$$

Combining formulas 2.3.9 and 2.4.8, the vacuum expectation value of the resolvent is

$$
R_{0,0}(u) = \frac{1}{\gamma_0(u)} = \exp \left( \frac{1}{2\pi i} \int_T \log \left[ \frac{1}{u - v(w)} \right] \frac{dw}{w} \right). \tag{2.4.10}
$$

This gives an analytic continuation of the series development of the Stieltjes transform $R_{0,0}(u)$.

Theorem 2.5.1. There exists a distribution function $\xi_{T+}(c)$ so that for all $l \in \mathbb{N}$

$$
\text{Tr} \left( T^l - T_+^l \right) = \int_{-\infty}^{\infty} c^l d\xi_{T+}(c). \tag{2.5.1}
$$

This $\xi_{T+}$ is related to the spectral measure $d\tau_{\langle 0 \rangle}(c)$ of $T$ at $|0 \rangle \in H_\star$ by

$$
\int_{-\infty}^{\infty} \frac{d\tau_{\langle 0 \rangle}(c)}{u - c} = \exp \left( \int_{-\infty}^{\infty} \log \left[ \frac{1}{u - c} \right] d\xi_{T+}(c) \right). \tag{2.5.2}
$$

The right-hand side is a multiplicative analog of the Stieltjes transform appearing on the left-hand side. This function $\xi$ is known as the spectral shift function. The trace formula above first appeared in the work of I. M. Lifshitz [94], and the second relation is due to M. G. Krein [88]. For a survey of spectral shift functions $\xi_{\widetilde{A}/A}(c)$ for trace-class perturbations $\widetilde{A}$ of self-adjoint operators $A$, see [14, 15].
Let us now specialize Theorem 2.5.1 to the Toeplitz operator $T(v)$ at the vacuum $|0\rangle \in H_\bullet$ in Hardy space. The relation becomes

$$R_{0,0}(u) = \exp \left( \int_{-\infty}^{\infty} \log \left[ \frac{1}{u-c} \right] d\xi_{T_+(v)/T(v)}(c) \right). \quad (2.5.4)$$

where

$$T(v) = \pi_\bullet v(w) \pi_\bullet$$

and

$$T_+(v) = \pi_+ v(w) \pi_+ \quad (2.5.5)$$

and we recall $\pi_\bullet = \pi_0 + \pi_+$. Comparing formulas 2.5.4 and 2.4.10, we have shown:

**Corollary 2.5.1.** Given an analytic real-valued function $v(w)$ on $\mathbb{T}$, the spectral shift function of the Toeplitz operator $T(v)$ at the vacuum vector $|0\rangle \in H_\bullet$ is

$$2\pi \cdot \xi_{T_+(v)/T(v)}(c) = (v, d\theta)((-\infty, c)) \quad (2.5.6)$$

the distribution function of the push-forward of the normalized Haar measure on $\mathbb{T}$ along the symbol $v : \mathbb{T} \to \mathbb{R}$.

Note that this is a non-negative measure, i.e. $\xi_{T_+(v)/T(v)}(c)$ is monotonically increasing from 0 to 1. Moreover, it is of compact support, since $v(\mathbb{T})$ is compact. By chapter 5, the reader will see why this particular spectral shift function is exactly what appears in the description of the limit shape $f_{\ast|v}(c)$ in Theorem 1.2.1.

### 2.6 Kerov’s Markov-Krein correspondence

The correspondence between spectral shift functions and spectral measures is a particular incarnation of Kerov’s Markov-Krein correspondence:

**Theorem 2.6.1.** [79] The relation

$$\int_{-\infty}^{\infty} \frac{d\tau(c)}{u-c} = \exp \left( \int_{-\infty}^{\infty} \log \left[ \frac{1}{u-c} \right] d\xi(c) \right) \quad (2.6.1)$$

defines a non-local invertible transformation $\xi \to \tau$ between the space of probability measures $\tau$ on $\mathbb{R}$ and the space of differences $d\xi(c)$ of “interlacing measures” on $\mathbb{R}$.

This correspondence emerges from Nevanlinna’s work on integral representations of functions of negative imaginary type.

We say that $\tau(c)$ is the KMK transform of $d\xi(c)$ and that $f(c)$, defined by

$$f(c) = \int_{-\infty}^{c} \xi(\bar{c})d\bar{c} + \int_{c}^{\infty} \left( 1 - \xi(\bar{c}) \right) d\bar{c} \quad (2.6.2)$$

is the profile of $d\xi(c)$. Recall the linear statistics $\text{ch}[f]$ of continuous profiles $f \in Y(0)$ defined in formula 1.2.6, and define

$$\text{ch}_\ell^\gamma[f] = \int_{-\infty}^{\infty} e^\ell \, d\tau_f(c) \quad (2.6.3)$$

the transformed linear statistics.
Corollary 2.6.1. The linear statistics \( ch_\ell[f] \) are finite polynomial combinations of the transformed linear statistics \( ch_\ell^\tau[f] \) for \( 0 \leq \ell \leq l \).

One of the bright strands connecting Kerov’s diverse work is his realization that the KMK transform \( \tau_f(c) \) of the profile \( f = f_\lambda(c|\varepsilon,\varepsilon) \) of an isotropic Young diagram \( \lambda \in \mathcal{Y}(-\varepsilon,\varepsilon) \) is actually the transition measure \( \tau_\lambda^\uparrow(c|\varepsilon,\varepsilon) \) of \( \lambda \) with respect to the Plancherel growth process, a Gibbs measure the space of infinite Young tableaux corresponding to the regular representation of the infinite symmetric group \( S(\infty) \) [82]. This is a model for a growing discrete profile \( f_\lambda(c|\varepsilon,\varepsilon) \) whose marginal at time \( d \) is the Plancherel measure \( M_\bullet(d|2) \).

In [81], Kerov extended this observation: the KMK transform of the profile of the anisotropic partition \( \lambda \in \mathcal{Y}(0;\varepsilon_2,\varepsilon_1) \) is yet again a transition measure \( \tau_\lambda^\uparrow(c|\varepsilon_2,\varepsilon_1) \), this time for the Jack-Plancherel growth process on \( \mathcal{Y}(\varepsilon_2,\varepsilon_1) \) Young’s lattice with Jack edge multiplicities [81, 83]. In simple terms, the algebraic theory of Jack symmetric functions is incarnation of a larger, analytic theory of interlacing sequences that is specialized to the case \( c_i^+,c_j^- \in \varepsilon_2 \mathbb{N} + \varepsilon_1 \mathbb{N} \) of interlacing extrema taking values in a degenerate quarter lattice.

It seems like the two instances of the KMK correspondence we have mentioned so far

1. spectral theory of Toeplitz operators
2. function theory of Jack symmetric functions

have nothing to do with each other. In the next chapter, we will present a framework that encompasses both examples. This is made possible by an extension of scalars, i.e. through Toeplitz operators with symbols taking values in \( \text{End}(\mathcal{F}) \).

3 Quantum integrability

In this chapter, we realize Jacks as simultaneous eigenfunctions of the quantum Benjamin-Ono hierarchy following [103]. Reinterpreting [103], we identify the transition measures of anisotropic profiles as the spectral measures of the Nazarov-Sklyanin Lax operator \( \mathcal{L}(\varepsilon_2,\varepsilon_1) \) at the vector \( P_\lambda(v|\varepsilon_2,\varepsilon_1) \otimes |0\rangle \in \mathcal{F} \otimes \mathbb{C}[w] \). These operators provide a companion to the generators of Olshanski’s infinite-dimensional diffusions [114].

3.1 Collective field variables

A crucial distinction between the Nazarov-Sklyanin operators and other explicit infinite families of operators in the literature (such as the Sekiguchi-Debiard or trigonometric Dunkl-Cherednik operators) is that they are written via \( V_k \) and

\[
V_{-k} = -\varepsilon_1\varepsilon_2 k \frac{\partial}{\partial V_k}.
\]

For a clear account of this change of variables in the hydrodynamic limit \( N \to \infty \) of the Calogero-Sutherland models, see [133].
3.2 Auxiliary Hardy spaces

We seek families of operators $\mathcal{O}(u; \varepsilon_2, \varepsilon_1) : \mathcal{F} \to \mathcal{F}$ indexed by $u$ on our Fock space acting diagonally on Jacks $P_\lambda(v \mid \varepsilon_2, \varepsilon_1)$. In the next section, we will realize such

$$\mathcal{O}(u; \varepsilon_2, \varepsilon_1) P_\lambda(v \mid \varepsilon_2, \varepsilon_1) = o_\lambda(u \mid \varepsilon_2, \varepsilon_1) P_\lambda(v \mid \varepsilon_2, \varepsilon_1)$$  \hspace{1cm} (3.2.1)

as matrix elements of a larger operator

$$\mathcal{R}(u; \varepsilon_2, \varepsilon_1) : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w]$$  \hspace{1cm} (3.2.2)

acting on the tensor product of $\mathcal{F}$ with an auxiliary space $\mathbb{C}[w]$. Elements of $\mathcal{F} \otimes \mathbb{C}[w]$ are linear combinations of $|\Psi \otimes h\rangle$ where $\Psi \in \mathcal{F}$ and $|h\rangle = w^h \in \mathbb{C}[w]$. We refer to $|0\rangle = 1 \in \mathbb{C}[w]$ as the auxiliary vacuum vector. The inner products on each induce

$$\langle \Psi_1 \otimes h_1 | \Psi_2 \otimes h_2 \rangle := \langle \Psi_1 | \Psi_2 \rangle_{-\varepsilon_1 \varepsilon_2} \delta(h_1 - h_2)$$  \hspace{1cm} (3.2.3)

a pairing on the tensor product space. With this in play, it becomes possible to use orthogonal projections to extract the matrix elements

$$\mathcal{R}_{h_+, h_-}(u; \varepsilon_2, \varepsilon_1) : \mathcal{F} \to \mathcal{F}$$  \hspace{1cm} (3.2.4)

of the larger operator $\mathcal{R}(u; \varepsilon_2, \varepsilon_1)$.

Just as we studied the resolvent $R(u)$ of Toeplitz operators via Theorem 2.3.1, in the next section we will define $\mathcal{R}(u; \varepsilon_2, \varepsilon_1)$ as the resolvent of the Nazarov-Sklyanin Lax operator $L(\varepsilon_2, \varepsilon_1)$ [103]. From its matrix elements, we find the desired $\mathcal{O}(u; \varepsilon_2, \varepsilon_1)$.

3.3 Nazarov-Sklyanin’s Lax operator

Introduce the $\hat{\mathfrak{gl}}_1$ current

$$v(w \mid -\varepsilon_1 \varepsilon_2) = \sum_k V_k \otimes w^{-k}$$  \hspace{1cm} (3.3.1)

with zero mode $V_0 = 0$ at level $-\varepsilon_1 \varepsilon_2$. The Fourier modes of this current are actually operators on $\mathcal{F}$. To clarify the failure of $v(w \mid -\varepsilon_1 \varepsilon_2)$ to define point-wise in $w$

$$v : \mathbb{T} \to \text{End}(\mathcal{F}),$$  \hspace{1cm} (3.3.2)

an operator valued function on the unit circle, we refer to the theory of vertex algebras [72]. Recall $\pi_* : \mathbb{C}[w, w^{-1}] \to \mathbb{C}[w]$ the Hardy projection from Laurent polynomials to ordinary polynomials. The Toeplitz operator with symbol $v(w)$ is defined by

$$T(v \mid -\varepsilon_1 \varepsilon_2) := (1 \otimes \pi_*) v(w \mid -\varepsilon_1 \varepsilon_2) (1 \otimes \pi_*).$$  \hspace{1cm} (3.3.3)
Proposition 3.3.1. \( T(v \mid -\varepsilon_1 \varepsilon_2) : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w] \) is well-defined.

\( \triangleright \) Proof: It is enough to show that each basis element \( |V_\mu \otimes h\rangle \) is sent to a finite linear combination of basis elements. Well

\[
T(v \mid -\varepsilon_1 \varepsilon_2) |V_\mu \otimes h\rangle = (1 \otimes \pi_\bullet) \left( \sum_{k=-\infty}^{\infty} V_k \otimes w^{-k} \right) |V_\mu \otimes w^h\rangle = \sum_{k=-\infty}^{\infty} V_k V_\mu \otimes \pi_\bullet w^{h-k}
\]

and so we need to argue that this sum is finite in both directions. For \( k > 0 \),

- \( \pi_\bullet w^{h-k} = 0 \) if \( k > h \). Since \( h \) is fixed, the sum terminates before \( k \to +\infty \)
- \( V_k V_\mu = 0 \) if \( \#_k[\mu] = 0 \). Since \( \mu \) is fixed, the sum terminates before \( k \to -\infty \).

This confirms that \( T(v \mid -\varepsilon_1 \varepsilon_2) \) preserves the pre-Hilbert space \( \mathcal{F} \otimes \mathbb{C}[w] \). \( \square \)

Next, consider the unbounded operator \( \mathcal{D}_{\text{aux}} = w \frac{\partial}{\partial w} \). This auxiliary degree operator acts on \( \mathbb{C}[w] \) by \( \mathcal{D}_{\text{aux}} |h\rangle = h |h\rangle \) and extends to the tensor product \( \mathcal{F} \otimes \mathbb{C}[w] \) by \( 1 \otimes \mathcal{D}_{\text{aux}} \).

The Nazarov-Sklyanin Lax operator is defined by

\[
\mathcal{L}(\varepsilon_2, \varepsilon_1) = T(v \mid -\varepsilon_1 \varepsilon_2) + (\varepsilon_1 + \varepsilon_2)(1 \otimes \mathcal{D}_{\text{aux}}).
\]  

By Proposition 3.3.1, this is a well-defined operator \( \mathcal{L}(\varepsilon_2, \varepsilon_1) : \mathcal{F} \otimes \mathbb{C}[w] \to \mathcal{F} \otimes \mathbb{C}[w] \).

The matrix elements of \( \mathcal{L}(\varepsilon_2, \varepsilon_1) \) are maps \( \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_+, h_-} : \mathcal{F} \to \mathcal{F} \) defined by

\[
\mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_+, h_-} = V_{h_- h_+} + (\varepsilon_1 + \varepsilon_2) h \delta(h_+ - h_-).
\]

Written as an operator in \( \mathbb{C}[w] \) with coefficients in \( \text{End}(\mathcal{F}) \),

\[
\mathcal{L}(\varepsilon_2, \varepsilon_1) = \begin{bmatrix}
0 & V_1 & V_2 & V_3 & \cdots & V_h & \cdots \\
V_1 & (\varepsilon_1 + \varepsilon_2) & V_1 & V_2 & \cdots & V_{h-1} & \cdots \\
V_2 & V_1 & 2(\varepsilon_1 + \varepsilon_2) & V_1 & \cdots & \cdots & \cdots \\
V_3 & V_2 & V_1 & 3(\varepsilon_1 + \varepsilon_2) & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
V_{h-1} & V_{h-1} & \cdots & \cdots & \cdots & h(\varepsilon_1 + \varepsilon_2) & \cdots \\
V_h & V_{(h-1)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

\[
\mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_+, h_-} = \begin{bmatrix}
0 & V_1 & V_2 & V_3 & \cdots & V_h & \cdots \\
V_1 & (\varepsilon_1 + \varepsilon_2) & V_1 & V_2 & \cdots & V_{h-1} & \cdots \\
V_2 & V_1 & 2(\varepsilon_1 + \varepsilon_2) & V_1 & \cdots & \cdots & \cdots \\
V_3 & V_2 & V_1 & 3(\varepsilon_1 + \varepsilon_2) & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
V_{h-1} & V_{h-1} & \cdots & \cdots & \cdots & h(\varepsilon_1 + \varepsilon_2) & \cdots \\
V_h & V_{(h-1)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

Notice that the first term \( V_{h_- h_+} \) in \( \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_+, h_-} \) is either creation operator \( V_k \) or annihilation operator \( V_{-k} = -\varepsilon_1 \varepsilon_2 k \frac{\partial}{\partial w} \), depending on the sign of \( h_+ - h_- \). Also, the second term in \( \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_+, h_-} \) only appears at \( h_+ = h_- = h \) with a weight \( h \), unless we are in the isotropic case \( \varepsilon_1 + \varepsilon_2 = 0 \). In this case, \( \mathcal{L}(-\varepsilon, \varepsilon)_{h, h} \equiv 0 \) since \( V_0 = 0 \).
3.4 Commuting Hamiltonians

As $\mathcal{L}(\varepsilon_2, \varepsilon_1)$ is self-adjoint, $\langle h_+ | \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell | h_- \rangle$ is unambiguous. Observe that

$$
\langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^3 | 0 \rangle = \sum_{h_1, h_2=0}^\infty \mathcal{L}(\varepsilon_2, \varepsilon_1)_{0, h_1, h_2} \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_1, h_2, 0} = \sum_{k=1}^\infty V_k V_{-k} \tag{3.4.1}
$$

is the degree operator $\mathcal{D}$ we met in formula (1.1.17) scaled by $-\varepsilon_1 \varepsilon_2$, and

$$
\langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1) | 0 \rangle = \sum_{h_1, h_2=0}^\infty \mathcal{L}(\varepsilon_2, \varepsilon_1)_{0, h_1, h_2} \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_1, h_2, 0}
= \sum_{h_1, h_2=0}^\infty V_{h_1} V_{h_2-h_1} V_{-h_2} + \langle \varepsilon_1 + \varepsilon_2 \rangle \sum_{h=0}^\infty h V_h V_{-h} \tag{3.4.2}
$$

is the Hamiltonian of the quantum Benjamin-Ono equation [103], the original $\mathcal{H}$ from formula (1.1.3) in collective field variables. For $\ell = 1, 2, 3, \ldots$, actually all $\langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle$ mutually commute on $\mathcal{F}$. Form the resolvent

$$
\mathcal{R}(u; \varepsilon_2, \varepsilon_1) = (u - \mathcal{L}(\varepsilon_2, \varepsilon_1))^{-1} := \sum_{\ell=0}^\infty u^{-\ell-1} \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell. \tag{3.4.3}
$$

Given $(h_+, h_-) \in \mathbb{N}^2$, have matrix element

$$
\mathcal{R}_{h_+, h_-}(u; \varepsilon_2, \varepsilon_1) : \mathcal{F} \to \mathcal{F}[u^{-1}]. \tag{3.4.4}
$$

**Theorem 3.4.1.** [103] Given an anisotropic partition $\lambda \in \mathcal{Y}(0; \varepsilon_2, \varepsilon_1)$ whose profile $f_\lambda(c|\varepsilon_2, \varepsilon_1)$ has interlacing extrema

$$
c^\uparrow_{e+1} < c^\uparrow_e < c^\downarrow_e < \cdots < c^\downarrow_2 < c^\downarrow_1 < c^\uparrow_1 \tag{3.4.5}
$$

define

$$
T^\uparrow_\lambda(u; \varepsilon_2, \varepsilon_1) = \prod_{j=1}^\infty (u - c^\uparrow_j) = \int_{-\infty}^\infty \frac{\tau_\lambda(c|\varepsilon_2, \varepsilon_1) dc}{u - c} \tag{3.4.6}
$$

the Stieltjes transform of transition measure. The auxiliary vacuum expectation value $\mathcal{T}^\uparrow(u; \varepsilon_2, \varepsilon_1) = \mathcal{R}_{0,0}(u; \varepsilon_2, \varepsilon_1)$ of the Nazarov-Sklyanin Lax operator resolvent is an operator $\mathcal{T}^\uparrow(u; \varepsilon_2, \varepsilon_1) : \mathcal{F} \to \mathcal{F}$ which acts diagonally

$$
\mathcal{T}^\uparrow(u; \varepsilon_2, \varepsilon_1) P_\lambda(v|\varepsilon_2, \varepsilon_1) = T^\uparrow_\lambda(u; \varepsilon_2, \varepsilon_1) P_\lambda(v|\varepsilon_2, \varepsilon_1) \tag{3.4.7}
$$
on Jacks with eigenvalue $T^\uparrow_\lambda(u; \varepsilon_2, \varepsilon_1)$. Explicitly, the infinite family of operators $\langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle$ act diagonally on Jacks with eigenvalues $c^\uparrow_\lambda\left[ f_\lambda(c|\varepsilon_2, \varepsilon_1) \right]$ and comprise the quantum Benjamin-Ono hierarchy with periodic boundary conditions.

As $\langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle$ are commuting self-adjoint operators which commute with degree operator, apply the spectral theorem to their restriction to the finite-dimensional spaces $\mathcal{F}_d$. This gives a new characterization of Jacks, with transparent dependence.
on the Jack parameter: Jacks differ from the isotropic Schur case $\varepsilon_1 + \varepsilon_2 = 0$ only by the presence of the purely auxiliary degree operator $\mathbb{1} \otimes D_{\text{aux}}$.

The auxiliary VEV of the Lax operator yields the conserved quantities of the classical Benjamin-Ono equation as well. As discussed in [103], this is a consequence of the fact that this integrable system has rational spectral curve $\Sigma = \mathbb{P}^1$.

**Proof:** To verify that our formulation is equivalent to that in [103], one must note that their current is a rescaling by $-\varepsilon_2$ of $T^\downarrow(u; \varepsilon_2, \varepsilon_1)$ satisfying relation

$$T^\uparrow(u; \varepsilon_2, \varepsilon_1) = T^\downarrow(u; \varepsilon_2, \varepsilon_1) = 1.$$ (3.4.8)

and computing Stieltjes transform of cotransition measures $\tau^\uparrow_\lambda(c|\varepsilon_2, \varepsilon_1)$ [81, 114]. □

In light of Kerov’s Markov-Krein correspondence, Theorem 2.6.1 above, we draw two immediate corollaries of Theorem 3.4.1:

**Corollary 3.4.1.** The spectral measure of the Nazarov-Sklyanin Lax operator $L$ at the vector $[P_\lambda(v|\varepsilon_2, \varepsilon_1) \otimes 0] \in \mathcal{F} \otimes \mathbb{C}[w]$ is the transition measure $\tau^\uparrow_\lambda(c|\varepsilon_2, \varepsilon_1)$ of the profile of the anisotropic partition $\lambda \in Y(0; \varepsilon_2, \varepsilon_1)$.

Compare this to Biane’s realization of transition measures $\tau^\uparrow_\lambda(c|\varepsilon_2, \varepsilon_1)$ in the isotropic case via spectrum of Jucy-Murphy elements in irreducible symmetric group modules, in which $\mathbb{C} \left[ S(d+1)/S(d) \right]$ is the $(d+1)$-dimensional auxiliary space [11].

Next, using the relation

$$\int_{-\infty}^{\infty} d\tau^\uparrow_\lambda(c|\varepsilon_2, \varepsilon_1) \frac{u - c}{u - c} = T^\uparrow_\lambda(u; \varepsilon_2, \varepsilon_1) = \exp \left( \frac{1}{2} \int_{-\infty}^{\infty} \log \left[ \frac{1}{u - c} \right] f''_\lambda(c|\varepsilon_2, \varepsilon_1) dc \right)$$ (3.4.9)

we can conclude:

**Corollary 3.4.2.** The logarithmic derivative

$$-\frac{\partial}{\partial u} \log T^\uparrow(u; \varepsilon_2, \varepsilon_1)$$ (3.4.10)

generates local commuting Hamiltonians: its $u^{-l-1}$ coefficient acts diagonally on Jacks $P_\lambda(v|\varepsilon_2, \varepsilon_1)$ with eigenvalue

$$\text{ch}_l[f_\lambda(c|\varepsilon_2, \varepsilon_1)] = \frac{1}{2} \int_{-\infty}^{\infty} dc f''_\lambda(c|\varepsilon_2, \varepsilon_1) dc.$$ (3.4.11)

the linear statistic introduced in formula 1.2.6.

This second corollary is of a form frequently encountered in the both the classical and quantum inverse scattering method. For example, the logarithmic derivative of transfer matrix of quantum spin chains generates local Hamiltonians [60], while the logarithmic derivative of the transmission coefficient of the Lax operator for the KdV equation generates conserved densities [64].

We are finally ready to turn to the analysis of the Jack measures $M_V(\varepsilon_2, \varepsilon_1)$. 28
4 All-order expansions

In this chapter, we use the commuting Hamiltonians \( \langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^f | 0 \rangle \) from Theorem 3.4.1 to compute the joint correlation of transformed linear statistics \( \text{ch}_f^c [ f_\lambda(c | \varepsilon_2, \varepsilon_1) ] \) for the Jack measures \( M_V(\varepsilon_2, \varepsilon_1) \) as convergent power series in the coupling constants \( \varepsilon_2, \varepsilon_1, V_{k}^{\text{out}}, \) and \( V_{k}^{\text{in}}. \)

4.1 Observables

Let \( O \) denote an operator on the pre-Hilbert space \( O : \mathcal{F} \to \mathcal{F} \) acting diagonally on Jacks \( P_\lambda \) with eigenvalue \( o_\lambda. \) Write \( \mathcal{F}_O \subset \mathcal{F} \) for the maximal domain of definition \( O : \mathcal{F}_O \to \mathcal{F} \) in the Hilbert space completion \( \mathcal{F} \) of \( \mathcal{F}. \) Recall Stanley-Cauchy kernel

\[
\sum_\lambda \frac{P_\lambda(V^{\text{out}} | \varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}} | \varepsilon_2, \varepsilon_1)}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1 \varepsilon_2}} = \Pi(V^{\text{out}}, V^{\text{in}} \frac{1}{-\varepsilon_1 \varepsilon_2}) = \prod_{k=1}^\infty \exp \left( \frac{V_{k}^{\text{out}} V_{k}^{\text{in}}}{\varepsilon_1 \varepsilon_2} \right) \quad (4.4.1)
\]

If we regard \( V_{k}^{\text{in}} \in \mathbb{C} \) while \( V_{k}^{\text{out}} \) are live variables, this partition function \( \Pi \) of our Jack measure is an infinite linear combination of Jacks \( P_\lambda(V^{\text{out}} | \varepsilon_2, \varepsilon_1) \in \mathcal{F}^{\text{out}}. \) Indeed, \( \Pi \in \mathcal{F}_{\text{out}} \) lives in the Hilbert space completion of \( \mathcal{F}^{\text{out}}, \) since the \( V_{k}^{\text{in}} \) are chosen so that \( ||V||_1 \leq \infty. \) However, we will need more regularity from the background potential \( V : \mathbb{T} \to \mathbb{R} \) if we want \( \Pi \) to be in the domain of definition of a given operator \( O. \)

Lemma 4.1.1. Suppose that \( V(w) \) is regular enough so that \( \Pi = \Pi(V^{\text{out}}, V^{\text{in}} \frac{1}{-\varepsilon_1 \varepsilon_2}) \) is in \( \mathcal{F}_{\text{out}} \), the maximal domain of definition of an operator \( O = \mathcal{F}^{\text{out}} \to \mathcal{F}^{\text{out}} \) acting diagonally on Jacks with eigenvalue \( o_\lambda. \) Then the expectation of the random variable \( o_\lambda \) with respect to the Jack measure \( M_V(\varepsilon_2, \varepsilon_1) \) can be computed by

\[
\mathbb{E}[o_\lambda] = \left( \Pi^{-1}(O \otimes 1) \Pi \right)_{V^{\text{out}} = V^{\text{in}}}. \quad (4.4.2)
\]

Proof: This simple swindle

\[
\mathbb{E}[o_\lambda] := \sum_\lambda o_\lambda \text{Prob}_V(\lambda | \varepsilon_2, \varepsilon_1)
\]

\[
= \Pi^{-1} \sum_\lambda o_\lambda P_\lambda(V^{\text{out}} | \varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}} | \varepsilon_2, \varepsilon_1)
\]

\[
= \Pi^{-1} \sum_\lambda (O \otimes 1) P_\lambda(V^{\text{out}} | \varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}} | \varepsilon_2, \varepsilon_1)
\]

\[
= \Pi^{-1} (O \otimes 1) \sum_\lambda P_\lambda(V^{\text{out}} | \varepsilon_2, \varepsilon_1) P_\lambda(V^{\text{in}} | \varepsilon_2, \varepsilon_1)
\]

\[
= \Pi^{-1} (O \otimes 1) \Pi \quad (4.4.3)
\]

is for us the pivot upon which everything turns. \( \square \)

Keep in mind that Jack measures \( M_V(\varepsilon_2, \varepsilon_1) \) are invariant under \( V_{k}^{\text{in}} \rightleftharpoons V_{k}^{\text{out}}, \) which for the symbol \( v \) amounts to the exchange \( v(w) \rightleftharpoons v(1/w). \) This symmetry is manifest if we compute \( \mathbb{E}[o_\lambda] \) not by \( O \otimes 1 \) but rather by \( O \otimes 1 + 1 \otimes O. \)
4.2 Cumulants

We would like to apply Lemma 4.1.1 to the infinite family of operators

\[ O_\ell = \langle 0 | L(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle \]  

(4.2.1)

which act diagonally on Jacks by Theorem 3.4.1. This would compute joint moments

\[ \mathbb{E} \left[ \text{ch}_{\ell_1}^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \cdots \text{ch}_{\ell_n}^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \right] = \Pi_{\ell_1}^{-1} O_{\ell_1} \cdots O_{\ell_n} \Pi \]  

(4.2.2)

of the transformed linear statistics \( \text{ch}_\ell^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] = \int c^\ell \tau_\uparrow^\lambda (c|\varepsilon_2,\varepsilon_1) dc \). Let

\[ \hat{W}_n^V(\ell_1, \ldots, \ell_n|\varepsilon_2,\varepsilon_1) = \mathbb{E} \left[ \text{ch}_{\ell_1}^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \cdots \text{ch}_{\ell_n}^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \right] \]  

(4.2.3)

be the associated joint cumulants.

**Theorem 4.2.1.** (AOE) For random \( \lambda \) sampled from \( M_V(\varepsilon_2,\varepsilon_1) \) with analytic symbol \( v \), the joint cumulants of transformed linear statistics \( \text{ch}_\ell^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \) extracted from random transition measure \( \tau_\uparrow^\lambda (c|\varepsilon_2,\varepsilon_1) \) have the convergent expansion

\[ \hat{W}_n^V(\ell_1, \ldots, \ell_n|\varepsilon_2,\varepsilon_1) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1\varepsilon_2)^{(n-1)+g} (\varepsilon_1 + \varepsilon_2)^m \hat{W}_{n,g,m}^V(\ell_1, \ldots, \ell_n) \]  

(4.2.4)

where \( \hat{W}_{n,g,m}^V(\ell_1, \ldots, \ell_n) \) is the \( V \)-weighted enumeration of connected “ribbon paths” on \( n \) sites of lengths \( \ell_1, \ldots, \ell_n \) with \( (n-1) + g \) pairings and \( m \) slides. The quantities \( \hat{W}_{n,g,m}^V(\ell_1, \ldots, \ell_n) \) are expressed solely through matrix elements of the Toeplitz operator \( T(v) \) on the circle with scalar symbol \( v(w) \).

We prove Theorem 4.2.1 in two steps. In section 4.4, we use generating functions

\[ R_{0,0}(u_j;\varepsilon_2,\varepsilon_1) = \sum_{\ell_j=0}^{\infty} u_j^{-\ell_j-1} O_{\ell_j}. \]  

(4.2.5)

of operators and joint cumulants to derive the identity in Theorem 4.2.1. This “algebraic step” relies on a combinatorial description of ribbon paths which we give in section 4.3. In section 4.5, we will check that the analyticity assumption on the symbol \( v \) of our Jack measure ensures that the assumptions of Lemma 4.1.1 are satisfied by all \( O_{\ell} \), completing the second “analytic step” of the all-order expansion.

Note: as the linear statistics

\[ \text{ch}_\ell^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] = \frac{1}{2} \int_{-\infty}^{\infty} c^\ell f_\lambda''(c;\varepsilon_2,\varepsilon_1) \]  

(4.2.6)

are finite polynomial combinations of \( \text{ch}_\ell^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \) by Corollary 2.6.1, Theorem 4.2.1 implies an all-order expansion for \( \text{ch}_\ell^\lambda [f_\lambda(c|\varepsilon_2,\varepsilon_1)] \) as well.
### 4.3 Ribbon paths

Consider a fixed sequence

\[ 0 \to h_1 \to \cdots \to h_\ell \to 0 \quad (4.3.1) \]

of non-negative integers \( h_i \in \mathbb{N} = \{0, 1, 2, \ldots\} \) of length \( \ell \in \mathbb{N} \) which starts and ends at the origin. Plot \((i, h_i)\), a discrete path staying at or above the horizontal axis.

Regard this path as a *live path* if it parametrizes a sequence

\[ \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{0,h_1} \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h_1,h_2} \cdots \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h_{\ell-1},0} \quad (4.3.2) \]

of \( \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h_+,h_-} : \mathcal{F}^{\text{out}} \to \mathcal{F}^{\text{out}} \), matrix elements of the Nazarov-Sklyanin Lax operator

\[ \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h_+,h_-} = \frac{\mathcal{V}^{\text{out}}_{h_--h_+} + (\varepsilon_1 + \varepsilon_2)h_\delta(h_- - h_+)}{} \quad (4.3.3) \]

made only of multiplication by \( \mathcal{V}^{\text{out}}_k \) and differentiation \( \mathcal{V}^{\text{out}}_{-k} := -\varepsilon_1 \varepsilon_2 k \frac{\partial}{\partial \mathcal{V}^{\text{out}}_k} \).

To compute

\[ \Pi^{-1} \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{0,h_1} \cdots \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h_{\ell-1},0} \Pi \quad (4.3.4) \]

all we need are the following rules:

1. **(slides)** at diagonal terms \( \mathcal{L}(\varepsilon_2, \varepsilon_1)^{\text{out}}_{h,h} \) record \((\varepsilon_1 + \varepsilon_2)h\)

2. **(pairings)** for \( k > 0 \), send annihilation operators \( \mathcal{V}^{\text{out}}_{-k} \) to the right, recording

\[ [\mathcal{V}^{\text{out}}_{-k}, \mathcal{V}^{\text{out}}_{k}] = -\varepsilon_1 \varepsilon_2 k \delta(k_- - k_+) \quad (4.3.5) \]

when \( \mathcal{V}^{\text{out}}_{-k} \) meets a creation operator \( \mathcal{V}^{\text{out}}_{k} \) of the same frequency \( k \in \mathbb{Z}_+ \).

3. **(exchanges)** for \( k > 0 \), we have

\[ [\mathcal{V}^{\text{out}}_k, \Pi] = 0 \quad (4.3.6) \]
\[ [\mathcal{V}^{\text{out}}_{-k}, \Pi] = \mathcal{V}^{\text{in}}_k \Pi \quad (4.3.7) \]

which follows from the decoupled exponential form of \( \Pi \), see formula 1.1.20.
To expand $\Pi^{-1}L(\varepsilon_2,\varepsilon_1)_{0,h_1}^{\text{out}} \cdots L(\varepsilon_2,\varepsilon_1)_{h_{\ell-1},0}^{\text{out}}\Pi$, keep track of diagonal terms (rule 1), pass all derivatives (annihilation operators) to the right (rule 2), and swap variables once these derivatives get to the right and hit $\Pi$ (rule 3). We describe the result as a sum over ribbon paths with multiplicative edge weights depending on $V_{k}^{\text{out}}$ and $V_{k}^{\text{in}}$.

For non-trivial $\varepsilon_2 < 0 < \varepsilon_1$, both pairings and slides occur, but at the isotropic point $\varepsilon_1 + \varepsilon_2 = 0$ no slides occur. A ribbon path on a single site is the data of a path $0 \to h_1 \to \cdots \to h_\ell \to 0$, with a certain number $m$ of slides, together with the data of $g$ pairings of edges $h_{i-1} - h_i = k = h_{i+1} - h_{i+1}$ for $k \in \mathbb{Z}_+$ and $i_- \leq i_+$. These objects parametrize the summands in the expansion of a single $\Pi^{-1}L(\varepsilon_2,\varepsilon_1)_{0,h_1}^{\text{out}} \cdots L(\varepsilon_2,\varepsilon_1)_{h_{\ell-1},0}^{\text{out}}\Pi$ according to the rules above. Here we represent a ribbon path with a single slide.

![Diagram of a ribbon path with a single slide](image)

and here a ribbon path with a single pairing.

![Diagram of a ribbon path with a single pairing](image)

Consider the most drastic degeneration $\varepsilon_2 = \varepsilon_1 = 0$. No slides nor pairings are allowed: $V_{-k}^{\text{out}}$ commute past $V_{k}^{\text{out}}$, but they do hit $\Pi$ and return $V_{k}^{\text{in}}$. The correlator in formula 4.3.4 consists of a single term, the product of all edge weights of the underlying path $\prod_{i=1}^{\ell} V_{h_{i} - h_{i-1}}^{\text{sign}(h_{i} - h_{i-1})}$ where $\text{sign}(h_{i} - h_{i-1}) \in \{\text{out}, \text{in}\}$ when $h_{i} - h_{i-1} > 0$ or $< 0$, respectively. This is the contribution of the classical path underlying a given live path.
Ribbon paths on a single site of length $\ell$ with $m$ slides and $g$ pairings must obey

$$m + 2g \leq \ell. \quad (4.3.8)$$

By rule 3, we see that those edges $h_i \rightarrow h_{i+1}$ of the underlying path not involved in a slide nor a pairing contribute a multiplicative weight $V_{\text{sign}(h_i-h_{i-1})}^{\text{sign}(h_i-h_{i-1})}$. Given $n$ paths $0 \rightarrow h_{1,j} \rightarrow \cdots \rightarrow h_{\ell,j} \rightarrow 0$ indexed by $1 \leq j \leq n$, the expansion of $\Pi^{-1}\left(\mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{0,h_{1,1}} \cdots \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{h_{1,1},0} \cdots \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{h_{n-1,n},0} \cdots \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{0,h_{\ell,n}} \cdots \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{h_{\ell,n-1},0}\right) \Pi$ according to rules 1,2,3 results in ribbon paths on $n$ sites. Most importantly, one may have pairings between the $n$ sites, as depicted below for $n=2, g=1, m=0$:

A ribbon path on $n$ sites is connected if the data of its pairings corresponds to a connected graph on $n$ vertices. Note that connectivity requires at least $n-1$ pairings.

### 4.4 Analytic continuations

In the language of previous section, the VEV of resolvent of $\mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}$ is

$$\mathcal{R}_{0,0}^{\text{out}}(u;\varepsilon_2,\varepsilon_1) = \sum_{\ell=0}^{\infty} u^{-\ell-1} \sum_{h_1,\ldots,h_{\ell}=0}^{\infty} \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{0,h_{1,1}} \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{h_{1,1},h_{2,2}} \cdots \mathcal{L}(\varepsilon_2,\varepsilon_1)^{\text{out}}_{h_{n-1,n},0}.$$ 

This is a generating function of live paths. Introducing

$$W^V_n(u_1,\ldots,u_n|\varepsilon_2,\varepsilon_1) := \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} u_1^{-\ell_1-1} \cdots u_n^{-\ell_n-1} W^V_n(\ell_1,\ldots,\ell_n|\varepsilon_2,\varepsilon_1) \quad (4.4.1)$$

and

$$W^V_{n,g,m}(u_1,\ldots,u_n) := \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} u_1^{-\ell_1-1} \cdots u_n^{-\ell_n-1} W^V_{n,g,m}(\ell_1,\ldots,\ell_n) \quad (4.4.2)$$
we will derive an identity of formal power series

\[ W_n^V(u_1, \ldots, u_n | \varepsilon_2, \varepsilon_1) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1 \varepsilon_2)^{(m-1)+g} (\varepsilon_1 + \varepsilon_2)^m W_{n,g,m}^V(u_1, \ldots, u_n) \]  

(4.4.3)

whose \( u_1^{-\ell_1} \cdots u_n^{-\ell_n} \) coefficient is the identity in Theorem 4.2.1.

**Proof of Theorem 4.2.1, algebraic part:** to evaluate \( n = 1 \) cumulant,

\[ W_1^V(u | \varepsilon_2, \varepsilon_1) := \mathbb{E}[T_1^V(u)] = \Pi^{-1}(\mathcal{R}_{0,0}^\text{out}(u) \otimes \mathbb{1})\Pi \]  

(4.4.4)

we just have to compute the commutator

\[ [\mathcal{R}_{0,0}^\text{out}(u; \varepsilon_2, \varepsilon_1) \otimes \mathbb{1}, \Pi(V^\text{out}, V^\text{in}|_{\varepsilon_1^{-1}})]. \]  

(4.4.5)

By linearity

\[ \sum_{\ell=0}^{\infty} u^{-\ell} \sum_{h_1, \ldots, h_{\ell-1}=0}^{\infty} \Pi^{-1} \mathcal{L}(\varepsilon_2, \varepsilon_1)_{0,h_1}^{\text{out}} \cdots \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_{\ell-1},0}^{\text{out}} \Pi \]  

(4.4.6)

we can work with one term \( 0 \rightarrow h_1 \rightarrow \cdots \rightarrow h_{\ell-1} \rightarrow 0 \) at a time. By section 4.3, \( \Pi^{-1} \mathcal{L}(\varepsilon_2, \varepsilon_1)_{0,h_1}^{\text{out}} \cdots \mathcal{L}(\varepsilon_2, \varepsilon_1)_{h_{\ell-1},0}^{\text{out}} \Pi \) is a sum over ribbon paths of length \( \ell \) with this fixed underlying path (having \( m \) slides), together with the data of \( g \) pairings, and \( V \)-dependent weights. Sort \( \Pi^{-1}(\mathcal{R}_{0,0}^\text{out}(u; \varepsilon_2, \varepsilon_1) \otimes \mathbb{1})\Pi \) according to \( g \) and \( m \):

\[ W_1^V(u | \varepsilon_2, \varepsilon_1) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1 \varepsilon_2)^g (\varepsilon_1 + \varepsilon_2)^m W_{1,g,m}^V(u) \]  

(4.4.7)

where \( W_{1,g,m}^V(u) \) are functions of \( V \) but not \( \varepsilon_2, \varepsilon_1 \). For example,

\[ W_{1,0,0}^V(u) = R_{0,0}(u) \]  

(4.4.8)

is the VEV of the resolvent of \( T(v) \). Indeed, as \( \varepsilon_2 \rightarrow 0 \leftrightarrow \varepsilon_1 \), the \( \hat{g}_{\varepsilon_1} \) current

\[ v^\text{out}(w | -\varepsilon_1 \varepsilon_2) = \sum_{k=1}^{\infty} V_k^\text{out} w^{-k} + \sum_{k=1}^{\infty} V_k^- w^k \]  

(4.4.9)

against the Stanley-Cauchy kernel \( \Pi \) becomes the classical symbol

\[ v(w) = \sum_{k=1}^{\infty} V_k^\text{out} w^{-k} + \sum_{k=1}^{\infty} V_k^- w^k \]  

(4.4.10)

and so \( \mathcal{L}(\varepsilon_2, \varepsilon_1) \) degenerates to the classical Toeplitz operator. Next are \( W_{1,0,1}^V(u) \) or \( W_{1,1,0}^V(u) \), the contributions with one slide or one pairing (respectively):

\[ W_{1,0,1}^V(u) = \sum_{h=0}^{\infty} h R_{0,h}(u) R_{h,0}(u) \]  

(4.4.11)

\[ W_{1,1,0}^V(u) = \sum_{k=0}^{\infty} k \sum_{h_h, h_s=0}^{\infty} R_{0,h_s+k}(u) R_{h_h, h_s}(u) R_{h_s+k,0}(u). \]  

(4.4.12)
Given a specified location for \( g \) pairings and \( m \) slides, \( W^V_{1,g,m}(u) \) is filled in via matrix elements of the classical Toeplitz operator resolvent \( R_{h+,h-}(u) \). Via Theorem 2.3.1, we gain access to an analytic continuation of these series in \( u^{-1} \) beyond a small neighborhood of infinity. This covers the \( n = 1 \) point functions \( \Pi^{-1} R_{1,0}(u; \varepsilon_2, \varepsilon_1) \Pi \). The joint moments \( \Pi^{-1} \mathcal{R}^\text{out}_{0,0}(u_1; \varepsilon_2, \varepsilon_1) \cdots \mathcal{R}^\text{out}_{0,0}(u_n; \varepsilon_2, \varepsilon_1) \Pi \) are generating functions of ribbon paths on \( n \) sites. If we define \( W_n(u_1, \ldots, u_n | \varepsilon_2, \varepsilon_1) \) to be the associated joint cumulants, we gain the desired expansion over connected ribbon paths, as \( n - 1 \) pairings are required to connect the correlator. \( \square \)

### 4.5 Estimates

Due to \([V_{-k}, V_k] = -\varepsilon_1 \varepsilon_2 k\), \( \mathcal{O}_\ell := \langle 0 | \mathcal{L}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle \) is an unbounded self-adjoint operator \( \mathcal{F} \to \mathcal{F} \) even if \( \varepsilon_1 + \varepsilon_2 = 0 \), so care must be taken regarding its domain of definition.

**Proposition 4.5.1.** If \( V \) satisfies our analyticity assumption, \( \Pi = \Pi(\mathcal{V}^\text{out}, \mathcal{V}^\text{in}|\varepsilon_1\varepsilon_2) \) is in the domain of definition of \( \langle 0 | \mathcal{L}^\text{out}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle \) for all \( \ell \in \mathbb{N} \).

**Proof:** If \( V \) defines an analytic function of \( w \) in an open neighborhood of the unit circle \( \mathbb{T} \), then so does the symbol \( v(w) = \sum_{k=1}^\infty \mathcal{V}^\text{out}_k w^{-k} + \sum_{k=1}^\infty \mathcal{V}^\text{in}_k w^k \), hence we can find a radius \( r_V < 1 \) so that \( \max\{|\mathcal{V}^\text{out}_k|, |\mathcal{V}^\text{in}_k|\} < r_V^k \) for all \( k \in \mathbb{Z} \). Expand

\[
\Pi^{-1}\langle 0 | \mathcal{L}^\text{out}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle = \sum_{h_1,\ldots,h_\ell=1}^{\infty} \Pi^{-1} \mathcal{R}^\text{out}_{0,h_1} \cdots \mathcal{R}^\text{out}_{h_\ell,0} \Pi
\]

over live paths of length \( \ell \), and sort them according to the maximal height \( h^* \in \mathbb{N} \). There are a finite number \( K_{\ell,h^*} \) of ribbon paths of length \( \ell \) with maximal height \( h^* \). To achieve this height, the path must have made jumps up \(+k_i\) with \( \sum_i k_i = h^* \), these jumps indexing \( \mathcal{V}^\text{out}_{+k_i} \) which could not have been hit by an annihilation operator from its left, hence contribute to the sum a weight \( \mathcal{V}^\text{out}_{k_i} \). Similarly, the path must have made jumps down \(-k_j\) with \( \sum_j k_j = h^* \), these jumps indexing \( \mathcal{V}^\text{out}_{-k_j} \) which could not have paired with a creation operator to its right, hence contributing \( \mathcal{V}^\text{in}_{k_j} \). By our assumption, each contribution has modulus less than \( r_V < 1 \). All pairings or slides that could have occurred in the \( \ell - 2 \) intermediate steps can contribute at most \( h^* \) per step, since the live path has maximal height \( h^* \). This crude consideration bounds

\[
|\Pi^{-1}\langle 0 | \mathcal{L}^\text{out}(\varepsilon_2, \varepsilon_1)^\ell | 0 \rangle| \leq \sum_{h^*=0}^{\infty} K_{\ell,h^*}(h^*)^{\ell-2} r_V^{2h^*} \leq \ell! \sum_{h^*=0}^{\infty} (h^*)^{2\ell-1} r_V^{2h^*} < \infty
\]

since \( K_{\ell,h^*} < \ell!(h^*)^{\ell-1} \) and \( r_V < 1 \). \( \square \)

**Proof of Theorem 4.2.1, analytic part:** if \( V(w) \) is analytic, by Proposition 4.5.1, all mixed moments of transformed linear statistics \( \mathcal{L}_V(f\lambda|\varepsilon_2, \varepsilon_1) \)

\[
\mathbb{E} \left[ \left( \int_{-\infty}^{\infty} c_{1}^{\ell_1} \tau^+_\lambda(c_{1}|\varepsilon_2, \varepsilon_1) dc_{1} \right) \cdots \left( \int_{-\infty}^{\infty} c_{n}^{\ell_n} \tau^+_\lambda(c_{n}|\varepsilon_2, \varepsilon_1) dc_{n} \right) \right] < \infty
\]

exist and are computable via Lemma 4.1.1. This implies that the algebraic result in section 4.4 is actually the convergent analytic statement in Theorem 4.2.1. \( \square \)
5 Limit shapes

In this chapter, we argue that Theorem 4.2.1 and Corollary 2.5.1 implies Theorem 1.2.1. This limit shape was first observed for Schur measures $\beta = 2$ in [111].

5.1 Concentration of profiles

**Theorem 5.1.1. (LLN)** For random $\lambda$ sampled from $M_V(\varepsilon_2, \varepsilon_1)$ with analytic symbol $v$, in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken so that $\frac{\beta}{2} = \frac{-\varepsilon_2}{\varepsilon_1} > 0$ is fixed, the joint moments of the linear statistics $ch_\lambda[f]$ of the random profile

$$f_\lambda(c|\varepsilon_2; \varepsilon_1) \to f_{*|v}(c)$$  \hspace{1cm} (5.1.1)

converge to those of a limit shape $f_{*|v}(c) \in Y$, independent of $\beta$:

$$2\pi \cdot \frac{1 + f_{*|v}(c)}{2} = (v d\theta)((-\infty, c))$$  \hspace{1cm} (5.1.2)

is the distribution function of the push-forward along $v : \mathbb{T} \to \mathbb{R}$ of the uniform measure on the circle.

**Proof:** To determine that this weak LLN occurs, it is enough to show that the variance of the random $ch_\lambda[f_\lambda(c|\varepsilon_2; \varepsilon_1)]$ goes to zero in this limit. As these quantities are finite polynomial combinations of transformed linear statistics $ch_\lambda^V[f_\lambda(c|\varepsilon_2; \varepsilon_1)]$ by Corollary 2.6.1, it is enough to show that $ch_\lambda^V[f_\lambda(c|\varepsilon_2; \varepsilon_1)]$ have variance going to zero. Working with all $\ell$ at once, we want to show

$$\mathbb{E}[T_\lambda^\lambda(u)^2] - \mathbb{E}[T_\lambda^\lambda(u)]^2 \to 0.$$  \hspace{1cm} (5.1.3)

More generally, we will check that the covariance

$$\mathbb{E}[T_\lambda^\lambda(u_1; \varepsilon_2, \varepsilon_1)T_\lambda^\lambda(u_2; \varepsilon_2, \varepsilon_1)] - \mathbb{E}[T_\lambda^\lambda(u_1; \varepsilon_2, \varepsilon_1)]\mathbb{E}[T_\lambda^\lambda(u_2; \varepsilon_2, \varepsilon_1)] \to 0$$  \hspace{1cm} (5.1.4)

and we can then take the coefficient of $u_1^{-\ell_1 - 1} \cdot u_2^{-\ell_2 - 1}$. This difference is exactly

$$W_2^V(u_1, u_2; \varepsilon_2, \varepsilon_1) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} (-\varepsilon_1 \varepsilon_2)^{(2-1)+g(\varepsilon_1 + \varepsilon_2)^m}W_1^V(u_1, u_2)$$  \hspace{1cm} (5.1.5)

by Theorem 4.2.1, whose dominant power of $\varepsilon$ is $(\varepsilon^2)^{(2-1)+0\varepsilon^0} = \varepsilon^2$ occurring at $g = m = 0$, so the covariance does indeed vanish at this scale.

At this point, we know $f_\lambda(c) \to f_{*|v}(c)$ concentrates on a profile determined by

$$\exp \left( + \int_{-\infty}^{\infty} \log \left[ \frac{1}{u - c} \right] d\xi_{*|v}(c) \right) = W_{1,0,0}^V(u) = R_{0,0}(u).$$  \hspace{1cm} (5.1.6)

for

$$\xi_{*|v}(c) = \frac{1 + f_{*|v}(c)}{2}. \hspace{1cm} (5.1.7)$$

By Corollary 2.5.1, we recognize $\xi_{*|v}$ as the spectral shift function of the pair of Toeplitz operators $T(v) = \pi_0 v(w)\pi_0$ and $T_+(v) = \pi_+ v(w)\pi_+$, which we characterized as a push-forward along $v$ as desired. □
6 Global fluctuations

In this chapter, we derive Theorem 1.2.2 from Theorems 4.2.1, 5.1.1, and 2.3.1.

**Theorem 6.0.2. (CLT)** For random \( \lambda \) sampled from \( M_v(\varepsilon_2, \varepsilon_1) \) with analytic symbol \( v \), in the limit \( \varepsilon_2 \to 0 \leftarrow \varepsilon_1 \) taken so that \( \frac{\beta}{2} = \frac{\varepsilon_2}{\varepsilon_1} > 0 \) is fixed, the joint moments of linear statistics \( ch[\phi] \) of profile fluctuations

\[
\phi_\lambda(c|\varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1 \varepsilon_2}} (f_\lambda(c|\varepsilon_2, \varepsilon_1) - f_{\star|v}(c))
\]

(6.0.8)

converge to that of a Gaussian field

\[
\phi_v(c) - \left( \sqrt{\frac{\beta}{2}} - \sqrt{\frac{2}{\beta}} \right) X_v(c).
\]

(6.0.9)

where \( X_v(c) \) is a deterministic mean shift determined by formula (6.4.3) and \( \phi_v(c) \) is the push-forward along \( v : T \to \mathbb{R} \) of the restriction to \( T_+ = T \cap H_+ \) of \( \Phi^{H+} \), the Gaussian free field on \( H_+ \).

\[
\text{Cov}\left[ \Phi^{H+}(w_1), \Phi^{H+}(w_2) \right] = \frac{1}{4\pi} \log \left| \frac{w_1 - w_2}{w_1 - w_2} \right|^2
\]

(6.0.10)

with zero boundary conditions.

In section 6.1, we show that the convergence of \( \phi_\lambda \) to some Gaussian process must occur. We compute the covariance and \( \beta \neq 2 \) mean shift of the process explicitly in terms of the symbol \( v \) in sections 6.3 and 6.4, respectively. The key ingredient is the construction of a welding operator \( W \) in section 6.2, an analog of the loop insertion operator \( K \) well-known in theory of loop equations, matrix models, and the topological recursion. This allows us to determine the covariance of the macroscopic fluctuations (CLT) directly from two independent copies limit shape, i.e.

\[
W : \text{LLN} \times \text{LLN} \Rightarrow \text{CLT}.
\]

6.1 Wick’s formula

A generalized Gaussian process is a random distribution whose \( n \)-point functions break up into a product of 2-point functions in a specific way. This relation is known as Wick’s formula, and may be expressed by saying that the higher cumulants \( n \geq 3 \) of the process vanish. Thus, one way to prove convergence to a Gaussian process is to show that the higher cumulants vanish in the limit of consideration.

6.1.1 CLT for transition measures

Looking at the expansion of cumulants \( W_n^V(\ell_1, \ldots, \ell_n|\varepsilon_2, \varepsilon_1) \) in powers of \( \varepsilon_2 \) and \( \varepsilon_1 \) in Theorem 4.2.1, we can see that the centered transition measures

\[
\varphi_\lambda^\uparrow(c; \varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1 \varepsilon_2}} \left( \tau_\lambda^\uparrow(c|\varepsilon_2, \varepsilon_1) - \tau_{\star|v}^\uparrow(c) \right)
\]

(6.1.1)
converge weakly to a Gaussian process $\varphi_\lambda^\dagger(c|\beta)$ in the limit $\varepsilon_2 \to 0 \leftarrow \varepsilon_1$ taken so that $\frac{\beta}{2} = \frac{-\varepsilon_2}{\varepsilon_1} > 0$ is fixed. Since cumulants don’t notice deterministic shifts, it is enough to observe that $(-\varepsilon_1\varepsilon_2)^{-n/2}W_n^V(u_1, \ldots, u_n|\varepsilon_2, \varepsilon_1)$ has lowest power of $\varepsilon$ equal to $-n+2(n-1) = n-2 \geq 1$ for $n \geq 3$, so the higher cumulants will vanish in the limit.

The scaling factor $1/\sqrt{-\varepsilon_1\varepsilon_2}$ is chosen so that covariance

$$
\mathbb{E} \left[ \int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c_1|\beta)}{u_1-c_1} \, dc_1, \int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c_2|\beta)}{u_2-c_2} \, dc_2 \right] = W_{2,0,0}^V(u_1, u_2) \quad (6.1.2)
$$

is independent of $\beta$. The mean shift for the process $\varphi_\lambda^\dagger(c|\beta)$ in the limit is

$$
\mathbb{E} \left[ \int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c|\beta)}{u-c} \right] = -\left( \sqrt{\frac{2}{\beta}} - \sqrt{\frac{2}{\beta}} \right) W_{1,0,1}^V(u) \quad (6.1.3)
$$

proportional to a factor $\frac{1}{\sqrt{-\varepsilon_1\varepsilon_2}} (\varepsilon_1 + \varepsilon_2) \to -\sqrt{\frac{2}{\beta}} + \sqrt{\frac{2}{\beta}}$.

### 6.1.2 CLT for profiles

We now argue that a CLT for transition measures implies a CLT for the slopes $f_\lambda(c|\varepsilon_2, \varepsilon_1)$. This in turn will imply a CLT for the profiles $f_\lambda(c|\varepsilon_2, \varepsilon_1)$ after integration by parts, since slopes are weak derivatives of profiles.

**Lemma 6.1.1.** Consider an analytic function $L(y) = \sum_{b=0}^{\infty} L_b y^b$ of the Stieltjes transform of $\varphi_\lambda^\dagger(c|\varepsilon_2, \varepsilon_1)$. In the formal variable $u$, this $L\left( \int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c|\varepsilon_2, \varepsilon_1)}{u-c} \, dc \right)$ converges to a Gaussian process with mean

$$
-(\partial L)(W_{1,0,0}^V(u)) \cdot \left( \sqrt{\frac{2}{\beta}} - \sqrt{\frac{2}{\beta}} \right) W_{1,0,1}^V(u) \quad (6.1.4)
$$

and covariance

$$(\partial L)(W_{1,0,0}^V(u_1)) \cdot (\partial L)(W_{1,0,0}^V(u_2)) \cdot W_{2,0,0}^V(u_1, u_2). \quad (6.1.5)$$

**Proof:** For $L(y) = y$, we have seen that cumulants of $\int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c|\varepsilon_2, \varepsilon_1)}{u-c} \, dc$, denoted $W_{1,0,1}^V(u_1, \ldots, u_n|\varepsilon_2, \varepsilon_1)$, have all-order expansion in Theorem 4.2.1. In 6.1.1, we have argued that to attain the mean and covariance of the limiting Gaussian process for $L(y) = y$ we must seek sums over ribbon paths with exactly one slide or pairing, which are $W_{1,0,1}^V(u)$ and $W_{2,0,0}^V(u)$, respectively. The same is true for the process defined by arbitrary $L$. By linearity, consider

$$
\mathbb{E} \left[ \left( \int_{-\infty}^{\infty} \frac{\varphi_\lambda^\dagger(c|\varepsilon_2, \varepsilon_1)}{u-c} \, dc \right)^b \right] = \frac{1}{\sqrt{-\varepsilon_1\varepsilon_2}} \left( \Pi^{-1}R_{0,0}(u; \varepsilon_2, \varepsilon_1)^b \Pi - R_{0,0}(u)^b \right) \quad (6.1.6)
$$

Here, the leading order term consists of certain ribbon paths on $b$ sites with 0 pairings and 1 slide. This slide can only occur in one of the $b$ sites, and it does so in $b$ different
ways, while the remaining \(b-1\) terms must have no slides or pairings, hence contribute only leading order quantities. This proves the formula for mean shift. Similarly, in

\[
\mathbb{E} \left[ \left( \int_{-\infty}^{\infty} \varphi^*_\lambda(c, \varepsilon_2, \varepsilon_1) dc \right)^{b_1} \left( \int_{-\infty}^{\infty} \varphi^*_\lambda(c_2, \varepsilon_2, \varepsilon_1) dc \right)^{b_2} \right] \circ \quad (6.1.7)
\]

which we compute via \(\frac{1}{-\varepsilon_1 \varepsilon_2}\) times

\[
\Pi^{-1} \mathcal{R}_{0,0}(u; \varepsilon_2, \varepsilon_1)^{b_1} \mathcal{R}_{0,0}(u_2; \varepsilon_2, \varepsilon_1)^{b_2} \Pi - \left( \Pi^{-1} \mathcal{R}_{0,0}(u; \varepsilon_2, \varepsilon_1)^{b_1} \Pi \right) \left( \Pi^{-1} \mathcal{R}_{0,0}(u_2; \varepsilon_2, \varepsilon_1)^{b_2} \Pi \right)
\]

we count 1 pairing and 0 slides at leading order. This pairing must involve only one of the \(b_1\) sites and one of the \(b_2\) sites, which can happen in \(b_1 b_2\) different ways, while the remaining terms must have no slides or pairings, hence contribute only leading order quantities. This proves the formula for the covariance. □

**Proof of Theorem 6.0.2, implicit part:** Applying Lemma 6.1.1 for general \(L\) to the logarithm function \(L(y) = \log y\), we get the CLT for the derivatives of profiles:

\[
\phi'_\lambda(c | \varepsilon_2, \varepsilon_1) := \frac{1}{\sqrt{-\varepsilon_1 \varepsilon_2}} \left( f'_\lambda(c | \varepsilon_2, \varepsilon_1) - f'_\lambda | v(c) \right) \quad (6.1.8)
\]

converges to generalized Gaussian process \(\phi'_v(c) - \left( \sqrt{\frac{\beta}{2}} - \sqrt{\frac{\beta}{3}} \right) X'(c)\), where \(\phi_v(c)\) has mean zero and covariance determined by

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Cov}[\phi'_v(c_1) dc_1, \phi'_v(c_2) dc_2] \frac{1}{(u_1 - c_1)(u_2 - c_2)} = \frac{1}{W_{1,0,0}(u_1)} \cdot \frac{1}{W_{1,0,0}(u_2)} \cdot W_{2,0,0}(u_1, u_2) \quad (6.1.9)
\]

while \(X_v(c)\) is a mean shift determined by

\[
\int_{-\infty}^{\infty} X'_v(c) dc \frac{1}{u - c} = \frac{1}{W_{1,0,0}(u)} \cdot W_{1,0,1}(u). \quad (6.1.10)
\]

This proves that convergence to a Gaussian process in Theorem 6.0.2 must occur. □

Now that we know a CLT occurs, it remains to compute the mean shift and covariance explicitly in terms of \(v\) by analytic continuation of the quantities expressed here.

### 6.2 Welding operator

In the previous section, we argued that the CLT covariance only involves one pairing, which we know is some \(V_{\infty}^m\) meeting up with a \(V_{\infty}^n\) and contributing \(-\varepsilon_1 \varepsilon_2 \kappa\). Instead of searching for such a pairing in a fixed path \(0 \rightarrow h_1 \rightarrow \cdots \rightarrow h_{\ell-1} \rightarrow 0\) and then summing over paths, we can simulate the contribution of a pairing given two independent sums over classical paths.
Given independent symbols \( v^{(1)}, v^{(2)} \) with modes 
\[
v^{(j)}(w) := \sum_{k=1}^{\infty} V_{k,j}^{\text{out}} w^{-k} + \sum_{k=1}^{\infty} V_{k,j}^{\text{in}} w^{k}
\]  
(6.2.1)
form the welding operator
\[
W := \sum_{k=0}^{\infty} k \frac{\partial^2}{\partial V_{k,1}^{\text{out}} \partial V_{k,2}^{\text{in}}}
\]  
(6.2.2)
Then
\[
W_{2,0,0}(u_1, u_2) = W \left( W_{1,0,0}^{(1)}(u_1) \cdot W_{1,0,0}^{(2)}(u_2) \right) \Bigg|_{v^{(1)} = v^{(2)} = v}.
\]  
(6.2.3)
This computes the CLT covariance of \( \varphi_\|^1(c|\beta) \) the limiting behavior of the random transition measure. To get the CLT covariance for the slopes (derivatives of profiles), by Lemma 6.1.1 it is enough to take
\[
W \left( \log W_{1,0,0}^{(1)}(u_1) \cdot \log W_{1,0,0}^{(2)}(u_2) \right) \Bigg|_{v^{(1)} = v^{(2)} = v}.
\]  
(6.2.4)

### 6.3 Gaussian free fields

Let’s use the welding operator \( W \) compute the covariance in Theorem 6.0.2:

**Proof of Theorem 6.0.2, explicit covariance:** By Theorem 2.3.1, have
\[
\log W_{1,0,0}^V(u) = \frac{1}{2\pi i} \oint_{|w|=1} \log \left[ \frac{1}{u - v(w)} \right] \frac{dw}{w}.
\]  
(6.3.1)
By our regularity assumptions on the symbol \( v(w) \), know that \( v(w) \) extends to a holomorphic function on an open neighborhood of the unit circle \( \mathbb{T} \). This means we may deform the contour in the formula for \( \log W_{1,0,0}^V(u) \) to lie either in \( r_v < |w| < 1 \) or \( 1 < |w| < 1/r_v \) for some \( r_v < 1 \). Using the relations
\[
\frac{\partial}{\partial V_{k}^{\text{out}}} \log \left[ \frac{1}{u - v(w)} \right] = \frac{w^k}{u - v(w)}
\]  
(6.3.2)
\[
\frac{\partial}{\partial V_{k}^{\text{in}}} \log \left[ \frac{1}{u - v(w)} \right] = \frac{w^{-k}}{u - v(w)}
\]  
(6.3.3)
the covariance is
\[
\frac{1}{(2\pi i)^2} \oint_{|w_1| < |w_2|} \frac{1}{u_1 - v(w_1)} \cdot \frac{1}{u_2 - v(w_2)} \cdot \frac{dw_1 dw_2}{(w_1 - w_2)^2}.
\]  
(6.3.4)
after using the formula
\[ \sum_{k=0}^{\infty} kw_k^{k-1}w_1^{k-1}dw_1dw_2 = \frac{dw_1dw_2}{(w_2 - w_1)^2} \] (6.3.5)
valid in the sector \(|w_1| < |w_2|\). In the theory of the topological recursion, the right-hand side is the genus 0 Bergman kernel.

Using the relation
\[ \frac{\partial^2}{\partial w_1 \partial w_2} \log(w_1 - w_2) = \frac{1}{(w_1 - w_2)^2} \] (6.3.6)
and integrating by parts, the integral 6.3.4 is
\[ \frac{1}{(2\pi i)^2} \oint_{\mathbb{T} \times \mathbb{T}} \frac{v'(w_1)dw_1v'(w_2)dw_2}{(u_1 - v(w_1))^2(u_2 - v(w_2))^2} \cdot \log(w_1 - w_2) \] (6.3.7)
Although we required \(|w_1| < |w_2|\) in the previous integral, the kernel \(\log(w_1 - w_2)\) is integrable in two-dimensions, and so we may allow the contours to approach each other and lie exactly on \(T = \{|w| = 1\}\).

The quantities \(\frac{1}{u - c}\) in Stieltjes transforms are simply generating functions of test functions \(c^\ell\) for \(\ell \in \mathbb{N}\). For polynomial test functions \(g(c)\), so far we have shown
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g'_1(c_1)g'_2(c_2)\text{Cov}[\phi_v(c_1)dc_1, \phi_v(c_2)dc_2] \] (6.3.8)
is
\[ \frac{1}{(2\pi i)^2} \oint_{\mathbb{T} \times \mathbb{T}} g'_1(v(w_1))g'_2(v(w_2))\log(w_1 - w_2)dw_1dw_2 \] (6.3.9)
Chop up this integral into four integrals over \(T_+ \times T_+\), where
\[ T_\pm := T \cap \mathbb{H}_\pm \] (6.3.10)
are the semi-circles in the upper and lower half-planes, respectively. Using the assumption \(v(w) = v(1/w)\), which is \(v(w) = v(\overline{w})\) on \(T\), this is
\[ \frac{1}{(2\pi i)^2} \oint_{T_+ \times T_+} g'_1(v(w_1))g'_2(v(w_2))\log \left| \frac{w_1 - w_2}{w_1w_2} \right|^2 dw_1dw_2 \] (6.3.11)
and we recognize the Green’s function of the Laplacian \(\Delta\) in \(\mathbb{H}_+\). To remove the assumption \(v(w) = v(1/w)\), observe that we did not have to compute the covariance with VEVs of \((u_1 - \mathcal{L}^{\text{out}})^{-1}\) and \((u_2 - \mathcal{L}^{\text{out}})^{-1}\) but could have applied operators to Jacks in ingoing variables as well. \(\square\)

Let us stress that the last step, in which the GFF on \(\mathbb{H}_+\) emerges from Bergmann kernel, is also the last step in Borodin’s derivation of CLT for Wigner matrices [18].
6.4 Mean shifts

To complete Theorem 6.0.2, we give an expression for the mean-shift \( X_v(c) \) at the macroscopic scale which only occurs at \( \beta \neq 2 \).

**Proof of Theorem 6.0.2, explicit mean-shift:** combining formulas 6.1.10, 4.4.11, and

\[
\begin{align*}
\gamma_0(u)^{-1} &= R_{0,0}(u) \\
\gamma_+(w_+, u)^{-1}\gamma_0(u)^{-1} &= \sum_{h_+ = 0}^{\infty} w_+^{h_+} R_{h_+, 0}(u) \\
\gamma_-(w_-, u)^{-1}\gamma_0(u)^{-1} &= \sum_{h_- = 0}^{\infty} R_{h_-, 0}(u) w_-^{h_-}
\end{align*}
\]

which are corollaries of Theorem 2.3.1, the mean shift \( X_v(c) \) is determined by

\[
\int_{-\infty}^{\infty} \frac{X_v'(c)dc}{u - c} = \frac{1}{R_{0,0}(u)} \sum_{h = 0}^{\infty} h R_{0,h}(u) R_{h,0}(u)
\]

\[
= \frac{1}{\gamma_0(u)} \sum_{h = 0}^{\infty} h \left( \frac{1}{2\pi i} \oint_{|w_-| = 1} w_-^{-1}dw_- \right) \left( \frac{1}{2\pi i} \oint_{|w_+| = 1} w_+^{-1}dw_+ \right)
\]

\[
= \frac{1}{\gamma_0(u)} \frac{1}{(2\pi i)^2} \oint_{|w_-| < |w_+|} \frac{1}{\gamma_-(w_-, u)\gamma_+(w_+, u) (w_+ - w_-)^2}
\]

In the last line, we used the regularity assumption on \( v(w) \), which imply that \( \gamma_\pm(w_\pm, u) \) can be extended to non-vanishing holomorphic functions beyond \( D_\pm \) and to open disks \( D_\pm \supset D_\pm \). This requires keeping \( u \in \mathbb{H}_+ \) fixed. Indeed, we then move the contours to \( |w_-| < 1 \) and \( |w_+| > 1 \), so that we may use the convergence

\[
\sum_{h = 0}^{\infty} h w_-^{h-1} w_+^{h-1}dw_-dw_+ = \frac{dw_-dw_+}{(w_+ - w_-)^2}.
\]

By Cauchy’s formula, the outer integral is

\[
\oint_{|w_+| > 1} \frac{\gamma_+(w_+, u)^{-1}dw_+}{(w_+ - w_-)^2} = (-1) \cdot \frac{1}{\gamma_+(w_+, u)} \frac{\partial}{\partial w_+} \log \gamma_+(w_+, u) \bigg|_{w_+ = w_-}
\]

Calling the remaining variable \( w_- \) simply by \( w \), this leaves us with

\[
\int_{-\infty}^{\infty} \frac{X_v'(c)dc}{u - c} = (-1) \cdot \frac{1}{2\pi i} \oint_{|w| = 1} \frac{1}{u - v(w)} \frac{\partial}{\partial w} \log \gamma_+(w, u)dw
\]

This completes the computation of the last remaining unstable correlator, and the proof of Theorem 6.0.2. \( \square \)
A Airfoils

In this appendix, we gather the explicit degenerations of our results for arbitrary Jack measures \(M_V(\varepsilon_2,\varepsilon_1)\) to \(M_\star(\varepsilon_2,\varepsilon_1)\), the Poissonized Jack-Plancherel measures. We also carry out a necessary dePoissonization in order to demonstrate coherence of our LLN and CLT with the literature. The law of large numbers and central limit theorem for the Plancherel measures \(M_\star(d|\beta)\) are among the first and most prominent results which reveal the probabilistic nature of the representation theory of the infinite symmetric group \(S(\infty)\) [82, 113]. Our work provides new proofs of these results, though at no point have we invoked the character theory of symmetric groups. All we need are the analytic properties of the infamous Zhukovsky transform

\[
v_\star(w) = w + \frac{1}{w}
\]  

(A.0.4)
in aerodynamics.

A.1 Jack-Plancherel measures

Set \(V_k^{\text{in}} = 0\) for \(k \geq 2\), so that

\[
\exp \left( \frac{V_1^{\text{out}} V_1^{\text{in}}}{-\varepsilon_1 \varepsilon_2} \right) = \Pi(V^{\text{out}}, V_1^{\text{in}} \mid \frac{1}{-\varepsilon_1 \varepsilon_2}) = \sum_{\lambda} \frac{P_\lambda(V^{\text{out}} \mid \varepsilon_2, \varepsilon_1) P_\lambda(V_1^{\text{in}}, 0, \ldots, \varepsilon_2, \varepsilon_1)}{\langle P_\lambda, P_\lambda \rangle_{-\varepsilon_1 \varepsilon_2}}
\]  

(A.1.1)

Even though on the right-hand side \(P_\lambda(V^{\text{out}} \mid \varepsilon_2, \varepsilon_1)\) depends on infinitely-many \(V_k^{\text{out}}\), the sum actually depends only on \(V_1^{\text{out}}\)! Further specializing \(V_k^{\text{out}} = 0\) for \(k \geq 2\) and setting \(V_1^{\text{out}} = V_1^{\text{in}} = 1\), the formula above defines the Jack-Plancherel measures \(M_\star(\varepsilon_2,\varepsilon_1)\). Actually, we can write the likelihood of sampling \(\lambda\) from \(M_\star(\varepsilon_2,\varepsilon_1)\) explicitly in terms of \(\lambda\). Combining the \(N \to \infty\) limit of the explicit formula for the principal specialization \(P_\lambda(1/N, \ldots, 1/N; \beta)\) of Jack symmetric polynomials together with knowledge of the norm \(\|P_\lambda\|_{T;\beta,\infty}^2\) in [96], if we remember \(V_k = (-\varepsilon_2)p_k\), we have

\[
\text{Prob}_\star(\lambda \mid d) = \frac{1}{\Pi} \prod_{\Box \in \lambda} \frac{\varepsilon_1^2/(-\varepsilon_1 \varepsilon_2)}{(-\varepsilon_2 l(\Box) + 1) + \varepsilon_1 a(\Box))(-\varepsilon_2 l(\Box) + \varepsilon_1 (a(\Box) + 1))}
\]  

(A.1.2)

where

\[
a(\Box) = \lambda_i - i'
\]  

(A.1.3)

\[
l(\Box) = \lambda'_i - i
\]  

(A.1.4)

are the arm and leg lengths of the box \(\Box = (i, i') \in \lambda\). To define \(l(\Box)\), we wrote \(\lambda'\) for the transposition of \(\lambda\). Besides the factors of \(-\varepsilon_1 \varepsilon_2\), the product depends only on the ratio \(\frac{1}{\alpha} = \frac{\beta}{2} = \frac{-\varepsilon_2}{\varepsilon_1}\). Gather terms so that

\[
\Pi =: \sum_{d=0}^{\infty} \frac{(-\varepsilon_1 \varepsilon_2)^{-d}}{d!} \Pi_d
\]  

(A.1.5)

our Jack measure \(M_\star(\varepsilon_2,\varepsilon_1)\) is a mixture of the \textit{micro-canonical ensembles} \(M_\star(d|\beta)\) by a Poisson distribution of frequency \(1/(-\varepsilon_1 \varepsilon_2)\).
Specialize to the isotropic point $\varepsilon_1 + \varepsilon_2 = 0$, or $\beta = 2$, from Jacks to Schur functions:

$$s_\lambda(1,0,\ldots)s_\lambda(1,0,\ldots) = \prod_{\square \in \lambda} \frac{1}{\varepsilon_2} \cdot \frac{1}{(a(\square) + l(\square) + 1)^2} \quad \text{(A.1.6)}$$

$$= \frac{\varepsilon^{-2d}}{(d!)^2} \left( \frac{d!}{\prod_{\square \in \lambda} h(\square)} \right)^2 \quad \text{(A.1.7)}$$

$$=: \frac{\varepsilon^{-2d} \dim^2 \lambda}{d!} \quad \text{(A.1.8)}$$

Using the identity

$$d! = \sum_{|\lambda|=d} \dim^2 \lambda \quad \text{(A.1.9)}$$

we can describe $M_\bullet(-\varepsilon,\varepsilon)$ as the mixture of micro-canonical ensembles $M_\bullet(d|2)$ by a Poisson distribution with frequency $\varepsilon^{-2}$. Indeed,

$$\text{Prob}_\bullet(\lambda|-\varepsilon,\varepsilon) = \frac{1}{\exp(\varepsilon^{-2})} \cdot \frac{\varepsilon^{-2d}}{d!} \cdot \text{Prob}_\bullet(\lambda|d) \quad \text{(A.1.10)}$$

where

$$\text{Prob}_\bullet(\lambda|d) = \frac{\dim^2 \lambda}{d!} \quad \text{(A.1.11)}$$

is the Plancherel measure on Young diagrams of degree $d$. This name comes from the fact that the formula comes from taking dimensions

$$\dim V^\lambda = \frac{d!}{\prod_{\square \in \lambda} h(\square)} \quad \text{(A.1.12)}$$

of irreducible $S(d)$-modules $V^\lambda$ in the decomposition

$$L^2(S(d)) = \bigoplus_{|\lambda|=d} V_L^\lambda \otimes V_R^\lambda \quad \text{(A.1.13)}$$

of the action of $S_L(d) \times S_R(d)$ from the left and right on the regular representation.

### A.2 dePoissonization

Lemma 4.1.1 may be modified to suit this context: one can apply $\mathcal{R}_{0,0}(u)$ the VEV of resolvent of $\mathcal{L}(-\varepsilon,\varepsilon)$ to the micro-canonical partition function $\Pi_d$ to compute joint moments of linear statistics for random partitions $\lambda$ sampled from a $|\lambda| = d$ conditioned Jack measure. Computations change slightly, since the third exchange relation $[V_{\text{out}}^\lambda, \Pi] = V_{\text{in}}^\lambda \Pi$ for the full Stanley-Cauchy kernel must be modified: one needs to determine $[V_{\text{out}}^\lambda, \Pi_d]$.

A benefit of conditioning $|\lambda| = d$ is that the estimates in section 4.5 now hold for arbitrary $V$, as $\Pi_d$ is concentrated in degree $d$, which means that the only live paths from $(0|\mathcal{L}^\ell|0)$ which contribute are those which stay at or above 0 and also at or below $d$, hence are finite in number for a fixed $\ell < \infty$. 

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Fortunately, we can determine $[V_{-k}^{\text{out}}, \Pi_d]$ for $v_{\bullet}(w) = w + \frac{1}{w}$ the micro-canonical Jack-Plancherel measures $M_{\bullet}(d|\beta)$. Keeping $V_{1\text{out}}$ and $V_{1\text{in}}$ arbitrary, we have

$$\Pi_d = \frac{(V_{1\text{out}} V_{1\text{in}})^d}{d!}$$

and so

$$[V_{-1\text{out}}^{\text{out}}, \Pi_d] = -\varepsilon_1 \varepsilon_2 V_{1\text{out}}^{\text{in}} \Pi_{d-1}$$

we experience a shift $d \rightarrow d - 1$. This means that when we compute

$$\Pi_{d-1}^{-1} \mathcal{O}_{\ell_1} \cdots \mathcal{O}_{\ell_n} \Pi_d$$

in addition to pairings and slides, we need to keep track of the attack number $e$ of annihilation operators $V_{-1\text{out}}^{\text{out}}$ that make it to the micro-canonical partition function $\Pi_d$, as in the end we will have a ratio

$$\left. \frac{\Pi_{d-e}}{\Pi_d} \right|_{V_{1\text{out}}^{\text{out}} = V_{1\text{in}}^{\text{in}} = 1} = \frac{d!}{(d-e)!} = d(d-1)(d-2) \cdots (d-e+1)$$

a polynomial in $d$ of degree $e - 1$. Note: we will take $d \rightarrow \infty$ at fixed $\ell < \infty$.

In this paper, the parameters $\varepsilon_2, \varepsilon_1$ played two roles: first, they were the coupling constants of the Jack measure, and secondly we chose to take these constants as the mesh parameters for our anisotropic partitions $Y(\varepsilon_2, \varepsilon_1)$. In the micro-canonical theory, the mesh is taken to be of order $1/\sqrt{d}$ in each direction, to compensate for the growth of rows and columns at rate $\sqrt{d}$. This ensures that all factors $-\varepsilon_1 \varepsilon_2 d$ are of order 1.

### A.3 Vershik-Kerov-Logan-Shepp limit shape

The classical paths of length $\ell$ counted by $\langle 0 | T(v)^\ell | 0 \rangle$ for $v_{\bullet}(w) = w + w^{-1}$ are the Catalan paths $C_{\ell}$, which are non-zero only for even $\ell$. Indeed, we have $V_{+1\text{out}}^{\text{out}} = V_{+1\text{in}}^{\text{in}} = 1$ and all other $V_{-k\text{out}}^{\text{out}} = V_{-k\text{in}}^{\text{in}} = 0$. In the course of the proof of Theorem 5.1.1, we have determined that the Stieltjes transform of the limiting transition measure

$$W_{1,0,0}^{\bullet}(u) = \int_{-\infty}^{\infty} \frac{\tau_{\bullet}^{\uparrow}(c) dc}{u-c}$$

is

$$W_{1,0,0}^{\bullet}(u) = R_{0,0}(u) = \sum_{\ell=0}^{\infty} u^{-\ell-1} C_{2\ell} = \frac{u - \sqrt{u^2 - 4}}{2}$$

which is well-known to be the Stieltjes transform of Wigner’s semi-circle. As this is known to be the transition measure of the profile $f_{\bullet}(c)$, we agree with $[85, 95]$. 

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Alternatively, can arrive at
\[ f'_\star(c) = \frac{2}{\pi} \arcsin \frac{c}{2} \]  
(A.3.3)
in \([-2, 2]\) from our form of the limit shape in Theorem 5.1.1. This follows from our presentation of the distribution function
\[ \xi_{\star\star}(c) = \frac{1 + f'_\star(c)}{2} = \frac{1}{2\pi} \int_0^{2\pi} 1_{\{2\cos \theta < c\}}(c) dc \]  
(A.3.4)
after using the relation \(\arccos \frac{c}{2} + \arcsin \frac{c}{2} = \frac{\pi}{2}\).

### A.4 Kerov’s central limit theorem

In Plancherel case \(v_\star(w) = w + w^{-1}\), observe
\[ \frac{1}{u - (w + w^{-1})} = \sum_{k=1}^{\infty} U_{k-1}(\frac{w}{2}) w^k \]  
(A.4.1)
where \(U_k\) are Chebyshev polynomials of the second kind in complex variable \(\frac{u}{2}\). These series expansions have two different domains of convergence, either by expansion around 0 or \(\infty\), avoiding the poles at the two solutions
\[ w = C^\pm(u) := \frac{u \pm \sqrt{u^2 - 4}}{2} \]  
(A.4.2)
of the quadratic equation \(u = w + w^{-1}\).

Combine formulas 6.3.4 and A.4.1 to express covariance of macroscopic fluctuations:
\[ \mathbb{E}\left[ \int_{-\infty}^{\infty} \phi'_\nu(c_1) dc_1, \int_{-\infty}^{\infty} \phi'_\nu(c_2) dc_2 \right] \circ = \sum_{k=1}^{\infty} k U_{k-1}(\frac{u_1}{2}) U_{k-1}(\frac{u_2}{2}). \]  
(A.4.3)
Next, using the integral representation
\[ U_{k-1}(\frac{u}{2}) = \int_{-\infty}^{\infty} \frac{T_k(\xi) dc}{u - c} \]  
(A.4.4)
via Chebyschev polynomial of the first kind, collect terms and account for derivatives, remove \(\partial/\partial u_i\) to go from slopes to profiles, get
\[ \text{Cov}[\phi_v(c_1), \phi_v(c_2)] = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta) \cdot 1_{[-2,2]}(c) \]  
(A.4.5)
in variables \(c = 2 \cos \theta = w + w^{-1}\) for \(c = 2 \cos \theta\) and \(0 \leq \theta \leq \pi\). This series will only converge when averaged against a smooth test function on \(\mathbb{R}\), giving a Gaussian random variable. The result is Gaussian measure on the space of distributions with support on \([-2, 2]\). Its trajectories are not ordinary functions but generalized functions.
In Ivanov-Olshanski’s account of Kerov’s CLT [68], one encounters almost the same result for the micro-canonical Plancherel measures $M_{\bullet}(d|2)$:

$$f_\lambda\left(c|\frac{1}{\sqrt{d}}, -\frac{1}{\sqrt{d}}\right) \sim f_{\lambda,\bullet}(c) + \frac{2}{\sqrt{d}}\Delta_{\bullet}(c)$$

(A.4.6)

where

$$2\Delta_{\bullet}(c) = \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta)1_{[-2,2]}(c).$$

(A.4.7)

Beware: formula A.4.7 is almost identical to formula A.4.5 except for the missing $k=1$ term! To leading order, Poissonization of $d$ will agree with its mean. However, lower-order terms in our all-order expansion Theorem 4.2.1 will receive larger and larger corrections due to conditioning $|\lambda| = d$, since we have to account for the attack numbers in the method of moments. Informally, this new term $k=1$ will appear due to the classical central limit theorem for the Poisson random variable. To check this discrepancy, we can re-derive the micro-canonical result in [68] directly by adapting the operator formalism used in the bulk of the paper. This takes two steps:

1. First, check that higher cumulants of the process $\phi_\lambda$ vanish. This reduces to a combinatorial argument verifying that the sum over set partitions

$$\kappa_{e_1,\ldots,e_n}(d) = \sum_\pi (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} (d)e_B$$

(A.4.8)

defined by

$$(d)e_B = d(d-1)(d-2)\cdots(d-e_B + 1)$$

(A.4.9)

for

$$e_B = \sum_{i \in B} e_i$$

(A.4.10)

satisfies the estimate

$$\kappa_{e_1,\ldots,e_n}(d) \in O(d^{1-n})$$

(A.4.11)

for all $e_1,\ldots,e_n \geq 1$.

2. Second, it remains to compute the limiting covariance. In the grand-canonical case, only had to account for one pairing. This is still possible, but now we can lose order from the fact that in computing

$$\Pi_{d-1}^{\ell_1} \Pi_{d-1}^{\ell_2} \Pi_d - (\Pi_{d-1}^{\ell_1} \Pi_{d-1}^{\ell_2}) (\Pi_{d-1}^{\ell_1} \Pi_{d-1}^{\ell_2} \Pi_d)$$

(A.4.12)

even when all derivatives $V_{-1}^{in}$ do not pair up with any $V_{+1}^{in}$ and go all the way to the right, the $e_2$ copies of $V_{-1}^{in}$ in the term $J_{\ell_2}^{\lambda}$ make it so that the $V_{-1}^{in}$ in $J_{\ell_1}^{\lambda}$ start to hit $\Pi_{d-e_2}$, not $\Pi_d$ as happens in the separated term on the right. In light of the connection to Catalan numbers $W_{1,0,0}(\ell) = \hat{C}\ell$ which vanish unless $\ell$ is even, the micro-canonical covariance differs from the one above by

$$- \sum_{e_1,e_2=0}^{\infty} (e_1 + 1)(e_2 + 1)u_1^{-2e_1 - 1}u_2^{-2e_2 - 1}\hat{C}_{e_1}\hat{C}_{e_2}. $$

(A.4.13)
Using formula A.4.2 and recognizing the holomorphic derivative

$$\partial C^+(u) = \frac{C^+(u)}{C^+(u) - C^-(u)}$$

(A.4.14)

this explains the missing independent random amount of Wigner’s semi-circle in Kerov’s CLT. □

A.5 Hora-Obata-Dołęga-Féray mean shift

Although conditioning on $|\lambda|$ gave a correction to covariance, we will not get such a correction to CLT mean. This is because CLT mean appears at order $\varepsilon_1 \approx \sqrt{d}$, whereas the corrections to replacing $\Pi$ by $\Pi(d)$ appear with whole powers of $d$. At $\beta \neq 2$, specializing our formula

$$\int_{-\infty}^{\infty} X'(c) dc = (-1) \cdot \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{u-v(w)} \frac{\partial}{\partial w} \log \gamma_+(w,u) dw$$

(A.5.1)

from section 6.4 to the case $v_*(w) = w + w^{-1}$, using $\gamma_+(w;u)\gamma_0(w;u) = C_-(u) - w$ the right-hand side becomes

$$\frac{1}{2\pi i} \int_{|w|=1} \frac{1}{u-w-w^{-1}} \cdot \frac{dw}{C_-(u) - w} = \frac{C_+(u)}{u^2 - 4}$$

(A.5.2)

after residue theorem and the relation $C_-(u) - C_+(u) = \sqrt{u^2 - 4}$. This agrees with

$$X_*(c) = -\frac{1}{2\pi} \arcsin \left( \frac{c}{2} \right) \cdot 1_{[-2,2]}(c)$$

(A.5.3)

as derived in [50]. Indeed, take weak derivative of function discontinuous at $\pm 2$,

$$X'(c) = -\frac{1}{2\pi} \frac{1}{\sqrt{4-c^2}} \cdot 1_{[-2,2]}(c) + \frac{1}{4} \left( \delta(c-2) + \delta(c+2) \right)$$

(A.5.4)

and so

$$\int_{-\infty}^{\infty} X'(c) dc = -\frac{1}{2\pi} \int_{-2}^{2} \frac{1}{u-c} \frac{dc}{\sqrt{4-c^2}} + \frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{u-c} \left[ \delta(c-2) + \delta(c+2) \right] dc$$

$$= \frac{1}{\sqrt{u^2 - 4}} + \frac{1}{4} \left[ \frac{1}{u-2} + \frac{1}{u+2} \right]$$

$$= \frac{C_+(u)}{u^2 - 4}$$

(A.5.5)

which completes the check. □

Determination of $X_*(c)$ for Jack-Plancherel measures appears implicitly in chapter 12.6 of [65], as the CLT for anisotropic profiles $f_\lambda(c|\varepsilon_2,\varepsilon_1)$ is equivalent to the CLT for Jack characters $\chi^\lambda_{\mu}(\varepsilon_2,\varepsilon_1)$. 

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