Non-commutative geometry, non-associative geometry and the standard model of particle physics

Latham Boyle and Shane Farnsworth
Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

Received 1 October 2014
Accepted for publication 28 October 2014
Published 10 December 2014
New Journal of Physics 16 (2014) 123027
doi:10.1088/1367-2630/16/12/123027

Abstract
Connes’ notion of non-commutative geometry (NCG) generalizes Riemannian geometry and yields a striking reinterpretation of the standard model of particle physics, coupled to Einstein gravity. We suggest a simple reformulation with two key mathematical advantages: (i) it unifies many of the traditional NCG axioms into a single one; and (ii) it immediately generalizes from non-commutative to non-associative geometry. Remarkably, it also resolves a longstanding problem plaguing the NCG construction of the standard model, by precisely eliminating from the action the collection of seven unwanted terms that previously had to be removed by an extra, non-geometric, assumption. With this problem solved, the NCG algorithm for constructing the standard model action is tighter and more explanatory than the traditional one based on effective field theory.

Keywords: non-commutative geometry, non-associative geometry, standard model of particle physics

Introduction
Since the early 1980s, Connes and others have been developing the subject of non-commutative geometry (NCG) [1, 2]. Its mathematical interest stems from the fact that it provides a natural generalization of Riemannian geometry (much as Riemannian geometry, in turn, provides a natural generalization of Euclidean geometry). Its physical interest stems from the fact that it
suggests an elegant geometric reinterpretation of the standard model of particle physics (coupled to Einstein gravity) [3–11]. For an introduction, see [12, 13]. Here we propose a simple reformulation of the traditional NCG formalism that has fundamental advantages from both the mathematical and physical standpoint.

Our idea, in brief, is as follows. In the traditional NCG formalism, a geometry is described by a so-called real spectral triple \( \{ A, H, D, J, \gamma \} \) (again, see e.g. [12, 13] for an introduction). The essence of our reformulation is the observation that these elements naturally fuse to form a new algebra \( B \); and that many of the traditional NCG axioms may then be recovered by the single requirement that \( B \) is an associative *-algebra.

From the mathematical standpoint, this has two advantages. First, it unifies many of the traditional NCG axioms into a single one, thereby clarifying their meaning. Second, it naturally generalizes Connes’ framework from non-commutative to non-associative geometry. Remarkably, it also has an unexpected physical consequence: it solves a key problem (highlighted e.g. in [9] and the concluding section of [10]) which has plagued the NCG construction of the standard model action functional. It does so by precisely eliminating from the action the collection of seven unwanted terms that previously had to be removed by an extra (empirically-motivated, non-geometrical) assumption (called the ‘massless photon’ assumption in [10]), in order to obtain agreement between the usual standard model action and the action obtained from the NCG construction [9–11, 13].

With this problem solved, NCG gives an algorithm for constructing the standard model which is tighter and more explanatory than the traditional one based on effective field theory (EFT). In the usual EFT construction, one must give three independent inputs: (i) the symmetries of the action; (ii) the fermions and their representations; and (iii) the scalars and their representations. In the NCG construction, one only needs two inputs: (i’) the choice of algebra \( A \) (which determines the symmetry of the action); and (ii’) the representation of \( A \) (which determines the fermions and their representations). Note that the third EFT input (the scalars and their representations), is not required in the NCG approach: i.e. the number of higgs bosons and their representations are a predicted output, once the symmetries and the fermionic representations are specified! Furthermore, the NCG construction explains various aspects of the standard model fermionic representations which are unexplained by the usual EFT construction (such as why all standard model fermions transform in either the trivial or fundamental representation of the gauge groups \( SU(2) \) and \( SU(3) \)—see [10, 11, 16] for more details). This striking situation is reviewed at greater length in [15].

Reformulation of Connes’ Framework

In this section, we present our formulation in two steps. In the first step, we explain how to extend \( A \) (a *-algebra) to \( \Omega A \) (the differential graded *-algebra of forms over \( A \)), even when \( A \) is non-associative. In the second step, we explain how \( H \) may be promoted to a bimodule over \( A \) by defining a new algebra \( B_0 = A \oplus H \); and similarly, \( H \) may be promoted to a bimodule over \( \Omega A \) by defining a new algebra \( B = \Omega A \oplus H \). In the process, we obtain a new view of the operator \( J \), and Connes’ ‘order-zero’ and ‘order-one’ axioms. Again, a key virtue of our reformulation is that it naturally extends to the case where \( A \) is non-
associative. (The mathematical and physical motivations for generalizing NCG to non-associative geometry are explained in [14]; for earlier work related to non-associative geometry, see [17–21].)

**Step 1: Promoting** $A \to \Omega A$. Let $A$ be a unital *-algebra over a field $F$. ($A$ may be non-commutative, or even non-associative.) We introduce, for each element $a \in A$, a corresponding formal symbol $\delta[a]$. $\Omega A$ is the algebra generated by $A$ and these differentials $\delta[a]$, modulo the relations $\delta[fa] = f\delta[a]$, $\delta[a + a'] = \delta[a] + \delta[a']$, $\delta[aa'] = \delta[a]a' + a\delta[a']$ (with $f \in F$, and $a$, $a' \in A$) and, in addition, modulo appropriate associativity relations. For example, in the usual case where $A$ is associative, we take $\Omega A$ to be associative as well, and impose relations like $(a\delta[b])c = a(\delta[b]c)$, etc; in this way, we recover the usual algebra $\Omega A$ defined, e.g. in section 6.1 of [22]. More generally, when $A$ is non-associative, $\Omega A$ must be equipped with compatible associativity relations (for example, if $A$ is an alternative algebra, then $\Omega A$ could be taken to be alternative as well; other examples are given in [25]).

We can write $\Omega A = \Omega^0 A \oplus \Omega^1 A \oplus \Omega^2 A \oplus \cdots$ where $\Omega^m A$ is the subspace of $\Omega A$ consisting of linear combinations of terms containing $m$ differentials $\delta[a]$; for example, $(a_1\delta[a_2])(\delta[a_3]\delta[a_4]) + (\delta[a_5]\delta[a_6]\delta[a_7])a_8$ is an element of $\Omega^3 A$. In particular, $\Omega^0 A = A$; and if $\omega_m \in \Omega^m A$ and $\omega_n \in \Omega^n A$, then $\omega_m \omega_n \in \Omega^{m+n} A$, so $\Omega A$ is graded. If we define $\delta[a^*] = -\delta[a]^*$, then $\Omega A$ is a *-algebra, with its *-operation naturally inherited from $A$. Note that we can interpret $\delta$ as a linear map from $\Omega^0 A \to \Omega A$; and this may, in turn, be promoted to a linear map $d: \Omega^m A \to \Omega^{m+1} A$ which (even in the non-associative case) may be defined recursively by requiring it to satisfy a graded Leibniz rule $d[\omega_m \omega_n] = d[\omega_m] \omega_n + (-1)^m \omega_m d[\omega_n]$, along with the conditions $d[a] = \delta[a]$ and $d^2[a] = 0$ ($\omega_m \in \Omega^m A$, $\omega_n \in \Omega^n A$, $a \in A$). It follows that $d^2[\omega] = 0$ ($\omega \in \Omega A$), so that $(\Omega A, d)$ is a differential graded *-algebra.

**Step 2: Promoting** $\Omega^0 A \to B_0$ and $\Omega A \to B$. Usually, in NCG, one starts by defining (in two steps) a bi-representation of $A$ on $H$, so that $H$ becomes a bi-module over $A$. In the first step, one defines a left action of $A$ on $H$—i.e. a bilinear product $ah = L_ah \in H$ between elements $a \in A$ and $h \in H$. In the second step one uses $J$, an anti-unitary operator on $H$, to define a corresponding right-action of $A$ on $H$—i.e. another bilinear product $ha \equiv R_ah \in H$ given by $R_a \equiv JL_{a^*}J^*$. The left and right action are required to satisfy the so-called order-zero condition $(ah)b = a(hb)$ or, equivalently, $[L_a, R_b] = 0$ ($\forall a, b \in A$).

We would like to reformulate this construction in a way that makes sense even when $A$ is non-associative. Fortunately, the natural definition of a ‘bi-representation’ of a non-associative algebra (or, equivalently, a ‘bimodule’ over a non-associative algebra) was found long ago (perhaps by Samuel Eilenberg [23]), and is explained simply and succinctly in chapter II.4 of [24]. The idea is that a bimodule $H$ over $A$ is nothing but a new algebra

$$B_0 = A \oplus H,$$

with the product between two elements of $B_0$ ($b_0 = a + h$ and $b_0' = a' + h'$) given by

$$b_0b_0' = aa' + ah' + ha',$$

where $aa' \in A$ is the product inherited from $A$, while $ah' \in H$ and $ha' \in H$ are precisely the left- and right-actions defined above. In this language, four familiar axioms of NCG—namely
(i) the associativity of $A$, (ii) the fact that $A$ is left-represented on $H$, (iii) the fact that $A$ is right-represented on $H$, and (iv) the order-zero condition—are condensed into the single assumption that $B_0$ is an associative algebra. Furthermore, the familiar definition of right-action in terms of left action, $R_a = JL_a J^*$, is reinterpreted as the statement that the map

$$b_0 = a + h \rightarrow b_0^* = a^* + Jh$$

(3)

is an anti-automorphism of $B_0$, with period 2 when the KO dimension is 0, 1, 6 or 7 mod 8 (i.e. when $J^2 = 1$) and period 4 when the KO dimension is 2, 3, 4 or 5 mod 8 (i.e. when $J^2 = -1$). In particular, when $J^2 = 1$, $B_0$ is a *-algebra, with * operation given by (3). The advantage of this reformulation is that it continues to make sense when $A$ is non-associative: in this case, $B_0$ is non-associative, too, and the familiar order zero condition is replaced by a compatible restriction on the associativity properties of $B_0$. For example, if $A$ is an alternative algebra, like the octonions, we can require $B_0$ to be an alternative algebra, too. The interpretation of $J$ in terms of the anti-automorphism (3) is unaffected.

Just as we give a bi-representation of $A$ on $H$ by defining a new algebra $B_0 = A \oplus H$, we give a bi-representation of $\Omega A$ on $H$ by defining a new algebra

$$B = \Omega A \oplus H$$

(4)

with the product between two elements of $B$ ($b = \omega + h$ and $b' = \omega' + h'$) given by

$$bb' = \omega\omega' + \omega h' + h\omega'$$

(5)

where $\omega\omega' \in \Omega A$ is the product inherited from $\Omega A$, while $\omega h' \in H$ and $h\omega' \in H$ are bilinear products that define the left-action and right-action of $\Omega A$ on $H^1$. Thus, just as $A = \Omega^0 A$ is a subalgebra of $\Omega A$, $B_0$ is a corresponding subalgebra of $B$. The elements $\omega \in \Omega A$ are linear combinations of products of $a'$s and $\delta[a]'s$: having already introduced the left- and right-action of $a$ on $H$ ($ah = L_a h$ and $ha = R_a h$), we now obtain the left- and right-action of $\delta[a]$ on $H$ by regarding $D$, a hermitian operator on $H$, as the representation on $H$ of the map $\delta: \Omega^0 A \rightarrow \Omega^1 A$, and requiring that it satisfies the corresponding Leibniz rule: $D(ah) = \delta[a]h + a(Dh)$. This gives $\delta[a]h = [D, L_a]h$ and $h\delta[a] = J[D, L_a]^* J^* h$. We see that Connes’ order-zero and order-one conditions are reinterpreted here as associativity conditions on $B$: in particular, the order zero condition $[L_{a_0}, R_b] = 0$ is the requirement that the associator $[\omega_0, h, \omega_0']$ vanishes, while the order one conditions $[L_{a_0}, [D, R_b]] = 0$ and $[[D, L_{a_0}], R_b] = 0$ are the requirements that the associators $[\omega_0, h, \omega_1]$ and $[\omega_1, h, \omega_0]$ vanish. In the case where $A$ is non-associative, the familiar order-zero and order-one conditions are replaced by compatible associativity constraints on $B$.

**Application to the standard model**

In this section, we first review the traditional formulation of the standard model in NCG, and then explain how our reformulation naturally yields a new constraint that resolves a well-known puzzle that arises in the traditional formulation. For clarity, we will deal in this section with a

---

1 For brevity, we are skipping over the issue of junk forms, and the corresponding distinction between $\Omega A$ and $\Omega A_{D} = \Omega A/J$ (see section 6.2 of [22]), since this nuance is not important for our present purposes; but we note that they may be readily incorporated in our approach.
single-generation of standard model fermions; the extension to the full set of three generations is straightforward.

The standard model is described by a finite-dimensional real spectral triple \( \{ A, H, D, J, \gamma \} \) of \( K_0 \) dimension 6. \( A \) is a \(*\)-algebra given by \( \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \), where \( \mathbb{C} \) is the algebra of complex numbers, \( \mathbb{H} \) is the algebra of quaternions, and \( M_3(\mathbb{C}) \) is the algebra of \( 3 \times 3 \) complex matrices. \( H \) is a 32-dimensional complex Hilbert space (32 is the number of fermionic degrees of freedom in a standard model generation, including the right-handed neutrino). To describe the action of \( \gamma \) and \( J \) on \( H \), it is convenient to split \( H \) into four 8-dimensional subspaces \( H = H_R \oplus H_L \oplus \bar{H}_R \oplus \bar{H}_L \). Here \( H_R \) and \( H_L \) contain the right-handed and left-handed particles, while \( \bar{H}_R \) and \( \bar{H}_L \) contain the corresponding anti-particles. If \( h_R \in H_R \) is a right-handed particle (with \( \bar{h}_R \in \bar{H}_R \) the corresponding anti-particle) and \( h_L \in H_L \) is a left-handed particle (with \( \bar{h}_L \in \bar{H}_L \) the corresponding anti-particle), then the helicity operator \( \gamma \) and the antilinear charge conjugation operator \( J \) act as follows:

\[
\begin{align*}
\gamma h_R &= -h_R, & \gamma h_L &= h_L, & \gamma \bar{h}_R &= \bar{h}_R, & \gamma \bar{h}_L &= -\bar{h}_L, \\
J h_R &= \bar{h}_R, & J h_L &= \bar{h}_L, & J \bar{h}_R &= h_R, & J \bar{h}_L &= h_L.
\end{align*}
\]

(6)

To describe the action of \( A \) on \( H \), it is convenient to further split each of the four spaces \( (H_R, H_L, \bar{H}_R, \bar{H}_L) \) into a lepton and quark subspace: \( H_R = L_R \oplus Q_R \), \( H_L = L_L \oplus Q_L \), \( \bar{H}_R = \bar{L}_R \oplus \bar{Q}_R \), and \( \bar{H}_L = \bar{L}_L \oplus \bar{Q}_L \). Each of the four lepton spaces \( \{ L_R, L_L, \bar{L}_R, \bar{L}_L \} \) is a copy of \( \mathbb{C}^2 \); an element of any of these four spaces correspondingly carries a doublet (neutrino versus electron) index. Each of the four quark spaces \( \{ Q_R, Q_L, \bar{Q}_R, \bar{Q}_L \} \) is a copy of \( \mathbb{C}^2 \otimes \mathbb{C}^3 \); an element of any one of these four spaces correspondingly carries two indices: a doublet (up quark versus down quark) index and a triplet (color) index. Now consider an element \( a = (\lambda, q, \mu) \in \mathcal{A}_F \), where \( \lambda \in \mathbb{C} \) is a complex number, \( q \in \mathbb{H} \) is a quaternion, and \( \mu \in M_3(\mathbb{C}) \) is a \( 3 \times 3 \) complex matrix, and write

\[
q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}.
\]

(7)

where \( \alpha \) and \( \beta \) are complex numbers. Here \( q \) is the standard \( 2 \times 2 \) complex matrix representation of a quaternion, and \( q_\lambda \) is the corresponding diagonal embedding of \( \mathbb{C} \) in \( \mathbb{H} \). Then \( L_a \) (the left action of \( a \) on \( H \)) is given by

\[
\begin{align*}
L_a L_R &= q_\lambda L_R, & L_a L_L &= qL_L, \\
L_a Q_R &= q_\lambda Q_R, & L_a Q_L &= qQ_L, \\
L_a \bar{L}_R &= \lambda \mathbb{1}_{2 \times 2} \bar{L}_R, & L_a \bar{L}_L &= \lambda \mathbb{1}_{2 \times 2} \bar{L}_L, \\
L_a \bar{Q}_R &= \mu \bar{Q}_R, & L_a \bar{Q}_L &= \mu \bar{Q}_L.
\end{align*}
\]

(8)

Where \( q, q_\lambda \) and \( \lambda \mathbb{1}_{2 \times 2} \) act on the doublet index, while \( \mu \) acts on the color index.

\( D \) obeys the following four geometric constraints: \( D^* = D, \{ D, \gamma \} = 0, \{ D, J \} = 0 \) and \([\{ D, L_a \}, R_b] = 0\). In the basis \( \{ L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L \} \), these imply
\[
D = \begin{pmatrix}
0 & 0 & y_l^+ & 0 & m^+ & n^+
0 & 0 & 0 & y_q^+ & 0 & 0
y_i & 0 & 0 & 0 & 0 & 0
0 & y_q & 0 & 0 & 0 & 0
m & n^T & 0 & 0 & 0 & y_l^T
n & 0 & 0 & 0 & 0 & y_q^T
0 & 0 & 0 & 0 & \tilde{y}_i & 0
0 & 0 & 0 & 0 & \tilde{y}_q & 0
\end{pmatrix}
\]

where
\[
y_l = \begin{pmatrix} y_{l,11} & y_{l,12} \\ y_{l,21} & y_{l,22} \end{pmatrix}
\] and
\[
y_q = \begin{pmatrix} y_{q,11} & y_{q,12} \\ y_{q,21} & y_{q,22} \end{pmatrix}
\]

are arbitrary 2 \times 2 matrices that act on the doublet indices in the lepton and quark sectors, respectively, while
\[
m = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}
\] and
\[
n = \begin{pmatrix} \vec{c} & \vec{d} \\ \vec{0} & \vec{0} \end{pmatrix}
\]

are 2 \times 2 and 6 \times 2 matrices, respectively; and in \( n \) we have used vector notation to emphasize that \( \vec{c}, \vec{d} \) and \( \vec{0} \) are 3 \times 1 columns. Of the eight complex parameters \( b, \vec{c}, \vec{d} \), only \( a \) is present in the standard model (where it corresponds to the right-handed neutrino’s majorana mass). The remaining seven parameters \( \{b, \vec{c}, \vec{d}\} \) present a puzzle—they are an unwanted blemish that must be removed in order to match observations. Traditionally, they are removed by introducing an extra assumption (namely, that \( D \) commutes with \( L_a \) for \( \lambda \in A \) \{0, q_0, \lambda\}) [9]; but, as emphasized by Chamseddine and Connes (see e.g. section 5 of [10]), this ad hoc solution is unsatisfying, and cries out for a better understanding.

Our reformulation yields a simple and satisfying solution to this puzzle. We have seen that the associativity of \( B = \Omega A \bigoplus H \) implies the usual order zero and order one constraints ([\( L_a, R_b \]) = 0 and \([D, [D, L_a]], R_b\]) = 0); but notice that it also implies a new constraint: \([D, [D, R_b]]\) = 0. This constraint may be satisfied in four different ways, by setting (i) \( b = \vec{c} = \vec{d} = 0 \); (ii) \( y_{q,11} = y_{q,21} = b = 0 \); (iii) \( y_{l,11} = y_{l,21} = \vec{c} = \vec{d} = 0 \); or (iv) \( y_{l,11} = y_{l,21} = y_{q,11} = y_{q,21} = \vec{c} = 0 \). Note, in particular, that solution (i) precisely corresponds to setting the seven unwanted parameters \( \{b, \vec{c}, \vec{d}\} \) to zero, without the additional ad hoc assumption described above!

We can go further by noting that the general embedding of \( C \) in \( H \) is given by \( q_2(\vec{n}) = \text{Re}(\lambda) I_{2 \times 2} + \text{Im}(\lambda) \hat{n} \cdot \hat{\sigma} \), where \( \hat{\sigma} \) are the three Pauli matrices, and \( \hat{n} \) is a unit 3-vector specifying the embedding direction. Since all of these embeddings are equivalent, the diagonal embedding \( q_2(\vec{n}) = q_2(\vec{c}) \) in equation (8) was arbitrary, and may be replaced by the more general possibility \( L_a L_R = q_2(\hat{n}) L_R \) and \( L_a Q_R = q_2(\hat{n}) Q_R \). If we redo the preceding analysis with this modification, the four solutions for \( D \) are modified accordingly: in particular, in solution (i), \( D \) is given by equation (9), where the 2 \times 2 matrices \( y_l \) and \( y_q \) are arbitrary, the 6 \times 2 matrix \( n \)
vanishes, and the $2 \times 2$ matrix $m$ is given by $m = P^T M P$, with $M$ an arbitrary $2 \times 2$ symmetric matrix and $P = (\mathbb{I}_{2 \times 2} + \hat{n}_1 \cdot \hat{\sigma})/\sqrt{2}$ a projection operator. Then, one can check the following result: given the arbitrary $2 \times 2$ matrices $y_p$, $y_q$ and $M$, there is a preferred choice for the embedding directions $\hat{n}_p$ and $\hat{n}_q$ such that, after a change of basis on $H$, $L_a$ is given by equation (8) (with the diagonal embedding $q_a = q_a(\hat{\xi})$), $D$ is given by equation (9), $m$ and $n$ are given by equation (11) with $b = \vec{c} = \vec{d} = 0$, while $y_l$ and $y_q$ are given by:

$$y_l = \begin{pmatrix} y_e \varphi_1 - y_e \bar{\varphi}_2 \\ y_e \varphi_2 + y_e \bar{\varphi}_1 \end{pmatrix}, \quad y_q = \begin{pmatrix} y_d \varphi_1 - y_d \bar{\varphi}_2 \\ y_d \varphi_2 + y_d \bar{\varphi}_1 \end{pmatrix},$$

(12)

with $\{y_e, y_e, y_d, \varphi_1, \varphi_2\} \in \mathbb{C}$. This is precisely the finite geometry that (after fluctuation and substitution into the spectral action) generates the standard model of particle physics (see [9–11, 13]).

This strikingly successful match between the geometric structure on the one hand, and the standard model Lagrangian on the other, appears to provide significant additional evidence: (i) for the suitability of Connes’ NCG framework for describing the standard model, and (ii) for the appropriateness of the reformulation presented here.

Discussion

We end with a few brief remarks. (i) In this paper, although we have developed a formalism suited to non-associative geometry, our main application has been to the associative finite geometry that describes the standard model of particle physics. In a forthcoming paper [25] we present a family of geometries (and their associated spectral actions) that provide a nice illustration of our formalism in the fully non-associative case. (ii) References [10, 11] observe that the standard model algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ analyzed above may be understood more deeply as a subalgebra of $\mathcal{A}' = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ (see also [26]). What new light does our formalism shed on this observation? (iii) We have seen that the finite geometry $T$ that encodes the standard model corresponds to an algebra $B$ that is associative. But, to evaluate the spectral action, one then tensors this finite geometry with a continuous geometry to form a new geometry $T'$, and one can check that the corresponding algebra $B'$ is not associative (when one goes beyond the order one associators). In this sense, non-associativity already appears in the traditional NCG embedding of the standard model. It is interesting to consider whether this non-associativity might be connected to the generalized inner fluctuations considered in [27, 28], which bear a striking resemblance to the inner derivations of a non-associative (and in particular, an alternative) algebra [14]. (iv) In a follow-up paper [29], we have shown how the formalism presented here yields a new perspective on the symmetries of a non-commutative geometry and, in particular, when we apply this new understanding to the spectral triple traditionally used to describe the standard model (i.e. the spectral triple discussed in the previous section), we find that it actually predicts a slight extension of the standard model, with two new particles: a $U(1)_{B-L}$ gauge boson, and a complex scalar field which carries $B-L$ charge and is responsible for "higgsing" the new $U(1)_{B-L}$ gauge symmetry. These two new particles have important phenomenological and cosmological consequences that will be analyzed in subsequent work.
Acknowledgments

Part of this work was carried out at the ‘Noncommutative Geometry and Particle Physics’ Workshop at the Lorentz Center, in Leiden; we thank the organizers and participants, and particularly Ali Chamseddine and Alain Connes for helpful input. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. LB also acknowledges support from an NSERC Discovery Grant.

References

[1] Connes A 1996 Non-Commutative Geometry (Boston, MA: Academic)
[2] Connes A 1985 *Publ. Math. de l’IHES* [0073-8301] 62 41
[3] Connes A and Lott J 1991 *Nucl. Phys. Proc. Suppl.* 18B 29
[4] Connes A 1996 *Commun. Math. Phys.* 182 155
[5] Chamseddine A H and Connes A 1996 *Phys. Rev. Lett.* 77 4868
[6] Chamseddine A H and Connes A 1997 *Commun. Math. Phys.* 186 731
[7] Barrett J W 2007 *J. Math. Phys.* 48 012303
[8] Connes A 2006 *J. High Energy Phys.* JHEP11(2006)081
[9] Chamseddine A H, Connes A and Marcolli M 2007 *Adv. Theor. Math. Phys.* 11 991
[10] Chamseddine A H and Connes A 2008 *J. Geom. Phys.* 58 38
[11] Chamseddine A H and Connes A 2007 *Phys. Rev. Lett.* 99 191601
[12] Chamseddine A H and Connes A 2010 arXiv:1008.0985
[13] van den Dungen K and van Suijlekom W D 2012 *Rev. Math. Phys.* 24 1230004
[14] Farnsworth S and Boyle L 2013 arXiv:1303.1782
[15] Boyle L and Farnsworth S in preparation
[16] Krajewski T 1998 *J. Geom. Phys.* 28 1
[17] Wulkenhaar R 1998 *J. Geom. Phys.* 25 305
[18] Wulkenhaar R 1997 *Phys. Lett.* B 390 119
[19] Wulkenhaar R 1997 *J. Math. Phys.* 38 3358
[20] Wulkenhaar R 1998 *Int. J. Mod. Phys.* A 13 2627
[21] Akrami S E and Majid S 2004 *J. Math. Phys.* 45 3883
[22] Landi G 1997 arXiv:9701078
[23] Eilenberg S 1948 *Ann. Soc. Pol. Math.* 21 125
[24] Schafer R D 1966 An Introduction to Nonassociative Algebras (New York: Academic)
[25] Farnsworth S and Boyle L in preparation
[26] Devastato A, Lizzì F and Martinetti P 2014 *J. High Energy Phys.* JHEP01(2014)042
[27] Chamseddine A H, Connes A and van Suijlekom W D 2013 *J. Geom. Phys.* 73 222
[28] Chamseddine A H, Connes A and van Suijlekom W D 2013 *J. High Energy Phys.* JHEP11(2013)132
[29] Farnsworth S and Boyle L 2014 arXiv:1408.5367