ABSTRACT: A review is given of work by Abhay Ashtekar and his colleagues Carlo Rovelli, Lee Smolin, and others, which is directed at constructing a nonperturbative quantum theory of general relativity.

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1. INTRODUCTION

I have been asked to review the current status of an approach to quantum gravity which is being developed by Abhay Ashtekar and his colleagues Carlo Rovelli, Lee Smolin, and others. I should emphasize that I have not actively worked on this approach and as a result, my knowledge of it is somewhat incomplete. However I have followed the progress in this area and would like to describe for you the main ideas involved, the current status, and open problems.

Ashtekar and colleagues are trying to quantize standard four dimensional general relativity without supersymmetry, higher derivatives, extra dimensions, extended objects, etc. The first question that probably comes to mind is why are they wasting their time on a program that is doomed to failure? Isn’t it well known that general relativity cannot be quantized? Perhaps surprisingly, the answer is no. It is, of course, known that general relativity is not perturbatively renormalizable. But unlike the case for most quantum field theories, this may not be as bad as it sounds. General relativity is qualitatively different from other field theories in that the dynamical field is the spacetime metric. One might even argue that standard field theory perturbation techniques should break down since they are based on the assumption that spacetime looks Minkowskian at arbitrarily short distances, which is not very plausible in quantum gravity. As I will describe, there are indications that quantum general relativity provides a natural cut-off at the Planck scale.

Ashtekar works in the framework of canonical quantization. Thus it is analogous to the functional Schroedinger representation for ordinary field theories. However, as we will see, the reparameterization invariance of general relativity leads to certain simplifications. Canonical quantization of general relativity has, of course, been tried before. But previous investigations have almost always used the spatial three metric and its conjugate momentum as the basic canonical variables. This leads to constraints which are difficult to solve (or even make sense of) in the quantum theory. Ashtekar instead chooses canonical variables which are analogous to those in ordinary gauge theories. The resulting constraints are simpler and more progress can be made toward constructing the quantum theory.

To motivate Ashtekar’s choice of canonical variables, we begin by considering general relativity in three dimensions. This theory can be described in terms of an SO(2,1) connection $\omega^{ab}_\mu$ and a triad of (dual) vectors $e^a_{\mu}$. (The spacetime metric is defined in terms of the triad by $g_{\mu\nu} = e^a_{\mu} e^b_{\nu} \eta_{ab}$.) The action is

$$S = \int e^a \wedge R^{bc} \epsilon_{abc}$$

where $R = d\omega + \omega \wedge \omega$ is the curvature two form or field strength of $\omega$. This action in fact describes a slight extension of general relativity. When $e^a_{\mu}$ consists of three linearly
independent vectors, one can show that (1) is equivalent to the usual Einstein action
\[ \int R \sqrt{-g}. \]
But the action (1) and the resulting field equations remain well defined even in
the limit that the triad becomes linearly dependent. Thus this theory includes degenerate
metrics. We will return to this point in Section 3.

Witten has shown\(^4\) that if one chooses the dynamical variables to be the spatial
components of the connection \( \omega_i \) and its conjugate momentum \( E^i \) (which is just the dual
of the spatial components of the triad), then the canonical quantization of this theory can
be carried out exactly. The theory has two constraints:

\[ D_i E^i = 0 \quad (2) \]
\[ R_{ij} = 0 \quad (3) \]

The first is the familiar Gauss’ law. The second says that the spatial connection \( \omega_i \) is
flat. As a result of reparameterization invariance, the Hamiltonian is proportional to the
constraints. Thus, to construct the quantum theory, one does not need to solve the time
dependent Schroedinger equation. It suffices to find states which are annihilated by the
quantum version of the constraints. One can represent states in terms of functionals of the
connection. Imposing (2) requires that \( \psi(\omega) \) be gauge invariant and imposing (3) requires
that \( \psi \) have support on just the flat connections. So physical states are functionals of
gauge inequivalent flat connections.

2. TOWARD QUANTUM GENERAL RELATIVITY

Since the above approach works so well in three dimensions, it is natural to try it
in four. This leads directly to Ashtekar’s variables. (Actually, Ashtekar began his four
dimensional work several years before the three dimensional case was considered\(^5\)) In
four dimensions, general relativity can be described in terms of an SO(3,1) connection \( \omega_{\mu}^{ab} \)
and a tetrad of (dual) vectors \( e_a^{\mu} \). The action is

\[ S = \int e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd} \quad (4) \]

Using the three dimensional case as a guide, one is tempted to consider the spatial components of the connection and its conjugate momentum as the basic dynamical variables. Unfortunately, if one casts the theory into canonical form, one finds that some of the constraints are now second class. One can explicitly solve the second class constraints, but the remaining constraints become nonpolynomial\(^6\). Ashtekar’s key insight\(^5, 7\) was to replace \( \omega_{\mu} \) with its self dual part \( A_{\mu} \equiv \omega_{\mu} - i \ast \omega_{\mu} \). (One can show that the action (4) with \( \omega_{\mu} \) replaced by \( A_{\mu} \) is still equivalent to general relativity.) At first sight this appears to be a rather minor change. One is essentially replacing a connection having 24 real components

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with one having 12 complex components. However closer examination reveals that the consequences are much deeper than that.

This is most easily seen in the Euclidean context. The Euclidean Einstein equations can be obtained from the action (4) with either an SO(4) connection or its self dual part $A_\mu$ which is a real SU(2) connection. Thus one can eliminate half of the components of the connection without losing any information! (The reason is basically that the two actions differ by a term proportional to $\int R_{[\mu\nu\rho\sigma]}$ which does not contribute. Since one retains the full tetrad, one has the spacetime metric and in any solution, one can always reconstruct the complete connection and its curvature.) In addition to the obvious economy of fields, there are further advantages to working with $A_\mu$. For example one can show that Einstein’s field equation (with arbitrary cosmological constant) is equivalent to the self dual Yang-Mills equation for the connection $A_\mu$[8]. (More precisely, it is equivalent to the self dual Yang-Mills equation in a curved background where $A_\mu$ is equal to the self dual part of the spin connection.) Using this correspondence, one can find gravitational analogs of SU(2) Yang-Mills instantons: The one instanton solutions turn out to correspond to the four sphere, with the size of the instanton related to the radius of the sphere[9].

Returning to the Lorentzian context, one finds further advantages of using the self dual connection when one constructs the canonical formulation of the theory. The dynamical variables are the spatial components of the connection $A_i$ and its conjugate momentum $E^i$ which contains the information on the tetrad. The constraints are all first class and take the form

$$D_i E^i = 0$$

(5)

$$\text{Tr } F_{ij} E^i = 0$$

(6)

$$\text{Tr } F_{ij} E^i E^j = 0$$

(7)

Since $A_i$ is complex, there is also a reality condition that must be imposed*. The first constraint is the standard Gauss law constraint of Yang-Mills theory. Thus every initial data set for general relativity is also an initial data set for an SU(2) Yang-Mills theory. The only difference is that it is also subject to four additional constraints which are related to reparameterization invariance. Note that the degrees of freedom match: SU(2) Yang-Mills theory has $3 \times 2 = 6$ degrees of freedom at each point which are reduced to 2 by the four additional constraints. It should be emphasized that even though the initial data is similar, the hamiltonian for general relativity is very different than Yang-Mills theory. As in the three dimensional case, reparameterization invariance ensures that the hamiltonian for general relativity is just a multiple of the constraints (up to a surface term at infinity).

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* The momentum conjugate to $A_i$ is also complex, but it turns out that its imaginary part commutes with $A_i$. Thus one can choose $E^i$ to be real. The reality condition is simply that $E^i$ and its first time derivative - computed via Poisson brackets with the hamiltonian - be real.
Notice that all the constraints are simple polynomials in the basic fields. (The reality condition is also polynomial.) This is one of the main reasons that this approach initially attracted so much attention. But just from the form of the constraints it is difficult to tell how much of an advance this represents. Polynomial equations do not, of course, imply that the quantum theory is necessarily solvable (or even exists!) Although the standard constraints in terms of the spatial metric and conjugate momentum are not polynomial, they can be made so by simply multiplying by appropriate powers of the determinant of the metric. Furthermore, the constraint (7) is quadratic in the momenta, which means that the corresponding operator involves functional derivatives at the same point and must be regulated. This was also true in the old variables and was perhaps the main difficulty in finding solutions to the quantum constraints. To see the real advantage of this form of the constraints one must begin to construct the quantum theory.

Classically, the Gauss law constraint (5) generates gauge transformations, just as in any gauge theory. One can show that the vector constraint (6) generates reparameterizations of the three dimensional surface and the scalar constraint (7) is related to reparameterizations of time, or motions of the spatial surface in the four dimensional solution. To construct the quantum theory, we begin by representing states by functionals of \( A^i \). We wish to turn the classical constraints into quantum operators by replacing \( E^i \) by \(-i\delta/\delta A^i\) and define physical states to be those annihilated by the quantum constraints. This of course requires a choice of factor ordering. For the constraints linear in the momenta, the ordering given in (5) and (6) ensures that the quantum constraints have a similar action as their classical counterparts. However the quantum scalar constraint (7) has no direct interpretation since, as we have already mentioned, it must be regulated. Jacobson and Smolin have shown \( C_{\delta} \) and a class of states \( \psi_{\gamma} \) (parameterized by a loop \( \gamma \)) such that

\[
\lim_{\delta \to 0} C_{\delta} \psi_{\gamma} = 0
\tag{8}
\]

The regulator is a type of point splitting in which the functional derivatives are evaluated at different points separated by a distance \( \delta \). The states are just the familiar Wilson loops. Given a non-self-intersecting smooth closed curve \( \gamma \), set

\[
\psi_{\gamma}(A) = Tr Pe^{\oint_{\gamma} A} \tag{9}
\]

Roughly speaking, the reason this satisfies the constraint is that each \( \delta/\delta A_i \) brings down a term proportional to the tangent vector to the curve. Both of these tangent vectors are contracted into the antisymmetric \( F_{ij} \) and hence vanish. This type of solution is possible only if one uses variables like Ashtekar’s in which the momentum \( E^i \) has two type of indices (the tangent space index \( i \) and an internal index which we have suppressed). Although the constraint (7) is symmetric under interchange of the two momentum (as it must be), it is anti-symmetric under interchanging each type of index separately.
There are several reasons why one might feel uneasy about this result. First, since one must introduce a notion of distance to regulate the constraint, the regulator breaks three dimensional reparameterization invariance. Formally, this invariance is restored as the regulator goes to zero, but there is always the possibility of anomalies. A related difficulty is that \( C_\delta \psi_\gamma \neq 0 \) for \( \delta \neq 0 \). Thus in some sense the regulator breaks four dimensional reparameterization invariance as well. Finally, the regulated constraint is not unique. At the moment, there are several proposals for the regulated constraint which appear to be inequivalent\textsuperscript{11}.

Nevertheless, this is a significant achievement. Despite extensive work on the old canonical formalism for general relativity, no one has ever achieved an analogous result. The analog of the scalar constraint in the old variables is known as the Wheeler-DeWitt equation. Because of the difficulty in regulating and solving this equation, extensive work was done on simpler “minisuperspace” models in which one freezes out all but a finite number of degrees of freedom of the metric*. The full Wheeler-DeWitt equation has never been solved.

Even more remarkable is the fact that the solutions to the (analog of the) Wheeler-DeWitt equation are just the simple Wilson loops. These have long been considered as natural gauge invariant variables for describing Yang-Mills theory both classically and quantum mechanically\textsuperscript{12}. The fact that these same objects solve the scalar constraint of quantum general relativity is quite surprising.

Although \( \psi_\gamma \) solves the scalar and Gauss law constraint, it does not solve the vector constraint. In a key development, Rovelli and Smolin showed that one can obtain solutions to all quantum constraints by passing to a new representation in which states are functionals of loops\textsuperscript{13}. This can be obtained formally by the integral transformation

\[
\psi(\gamma) = \int \mathcal{D}A \, W(\gamma, A) \, \psi(A)
\]

where the kernel is again the Wilson loop

\[
W(\gamma, A) = \text{Tr} \, P e^{\oint_{\gamma} A}
\]

This transforms functionals of \( A \) into functionals of loops. Alternatively, the loop representation can be introduced directly by starting with an algebra of loop observables, computing their Poisson bracket, and introducing operators on functionals of loops with the same commutation relations. One then expresses the constraints as operators in the loop representation. Since gauge invariance is automatic, there is no analog of the Gauss constraint. From the above discussion, it might appear that there should be no analog of the scalar constraint either. This would indeed be the case if one could restrict to only

\* There is a striking similarity between the motivation that used to be given for working on minisuperspace models and the motivation one currently hears for two dimensional gravity.
smooth non-self-intersecting loops. On the one hand this sounds reasonable since all gauge invariant information in the connection is contained in the Wilson loops for this class of $\gamma$. On the other hand, to obtain a closed Poisson bracket algebra for the loop observables, it seems necessary to work with the larger space of all piecewise smooth loops. Fortunately, even in this larger space, one can satisfy the scalar constraint by simply restricting the functionals to have support on just the smooth non-self-intersecting loops. We can now impose the vector constraint. This says that the states are invariant under diffeomorphisms of the three surface. By definition, the diffeomorphism class of a smooth non-self-intersecting loop is called a “knot”. Thus one is led to the remarkable result that \textit{functions of knot classes satisfy all the constraints of quantum general relativity}\textsuperscript{13}. This result probably represents the main achievement of Ashtekar’s program so far\textsuperscript{*}.

In order to be sure that these knot states are physical we must check that they are normalizable. This is nontrivial since the inner product cannot be chosen arbitrarily but must be chosen so that physical observables are hermitian. (The classical reality condition will enter here.) Unfortunately, at the present time, very few observables are known and hence the inner product has not yet been determined. One might worry that since one starts with functions on the infinite dimensional space of loops (or connections) the inner product will necessarily be a functional integral which could only be evaluated perturbatively. This would violate the whole spirit of this nonperturbative approach to quantization. However, one only needs the inner product on the solutions to the constraints which, like the knot states above, might well have a countable basis. In simpler models such as three dimensional gravity\textsuperscript{14} and the weak field limit\textsuperscript{15}, the loop representation and the inner product have been constructed with the result that the loop states are normalizable. This lends support to the idea that they will be normalizable in the full theory as well.

We have not yet discussed the algebra of the quantum constraints. If one ignores regularization and formally calculates the commutator of the quantum constraints, one finds that there exists a choice of factor ordering such that the algebra closes\textsuperscript{5}. However before the discovery of the knot states, there was little reason to trust this result since it was shown\textsuperscript{16} that regularization has an important effect on the operator algebra. The calculation of the regulated constraint algebra has not yet been completed. But the existence of solutions to all the constraints shows that there can be no c-number central extension. Either the constraints will close, or the nonclosure will be a term which annihilates all the knot states.

How might one physically interpret these knot states? Work on this question is currently in progress. One possibility is the following. Consider a collection of knots defined as follows. Take a flat metric on $R^3$ and draw three families of parallel nonintersecting

\textsuperscript{*} There have also been applications of these variables to problems in classical general relativity which we will not review here.
lines separated by a distance $a$ as shown in Fig. 1. Now connect the ends at infinity to form a knot. (This can be done in many inequivalent ways.) This collection of knots are called *weaves*. A state which is one on these weaves and zero for all other knots might be interpreted as representing the flat Euclidean metric on $\mathbb{R}^3$. (One should state this more precisely since a knot is diffeomorphism invariant while a particular flat metric is not. The correct statement is that given a flat metric, one constructs a particular representative of a knot class to describe it. A diffeomorphism acting on the knot representative describes the new flat metric obtained by applying the same diffeomorphism to the original metric.) Preliminary calculations indicate that the physical spacing between the lines is determined by the theory to be the Planck length: If one considers the operator representing the metric and averages it over scales much larger than $a$, then the weave state gives the best approximation to the flat metric for $a$ equal to the Planck length. It is tempting to conjecture\footnote{Similar two dimensional weaves were considered by Witten\cite{Witten} in his discussion of the relation between Chern-Simons theory and integrable models in statistical mechanics.} that other background metrics might correspond to topologically different weaves. Roughly speaking, given any metric on a three manifold, one might associate a weave consisting of fibers whose tangent vectors form an orthonormal basis for the given metric. It is intriguing to see discrete structure at the Planck scale emerge from the theory. In the past, many people have referred to the “fabric of spacetime”. If these ideas are correct, this phrase may have literal and not just literary meaning!

The loop representation has also been explored for electromagnetism\cite{Faddeev} and linearized gravity\cite{Deser}. It can be constructed either by transforming the connection representation or directly in terms of loop observables. In this case it suffices to consider simple, unknotted loops. The result is that for electromagnetism, a one photon state with momentum $k$ and polarization $\epsilon$ is described by the following functional of loops:

$$\psi(\gamma) = \oint d\sigma \epsilon^j \gamma^j e^{ik \cdot \gamma}$$

This is simply $\psi(\gamma) = \oint \gamma A$ where $A$ is the wave function for the one photon state. Notice that gauge invariance is automatically enforced by the line integral around the loop. The states of linear gravity are similar. The main difference is that for linearized gravity, there are essentially three copies of the electromagnetic states (since the self dual connection has three complex internal components).

How does one incorporate these linear states into the full theory? One possibility is the following. For each simple loop $\gamma$, one considers a knot consisting of the weave together with the loop $\gamma$ attached (in a manner analogous to ordinary embroidery on fabric). One now defines a state by the condition that it equal $\psi(\gamma)$ on this knot and similarly for all possible positions of the loop in the weave. Notice that in this picture, gravitons do not make sense on scales less than the background scale $a$. Any loop smaller than this will be topologically disconnected.
3. OPEN QUESTIONS AND NEW DIRECTIONS

Although the results that have been obtained so far are promising, there is much that remains to be done before one can claim to have a consistent quantum theory of gravity. This section is divided into three parts. In the first, I consider some open questions in the main program described above. The second includes a short discussion of other approaches to quantum gravity using Ashtekar variables. In the third I consider the question of whether there exists yet another set of canonical variables for general relativity (or a theory containing general relativity) such that the constraints are simplified even further.

3.1 Open questions in the main program

We have already discussed two important unresolved issues in Ashtekar’s approach to quantum gravity. One is to determine the physical inner product and show that the knot states are normalizable. The other is to understand better the regularization procedure. Are there principles which determine it uniquely? Does it lead to anomalies?

One also needs to improve the physical interpretation of the knot states. For example, can a black hole be described in terms of functionals of knots? Since there is some indication how to interpret flat space and linearized gravitons in terms of knot states, one can now begin to consider graviton-graviton scattering. This will be an important test of this approach to quantum gravity. Uncontrollable divergences will show that this approach suffers from the same difficulties of standard quantum field theory methods. On the other hand, finite answers will be an important confirmation of the basic principles underlying this approach. A related issue is whether there exist other solutions to the quantum constraints besides the knot states. If one can work with a loop representation consisting only of smooth non-self-intersecting loops, it would appear that the answer is no. If one works with the larger space of piecewise smooth loops, then additional solutions can be found\textsuperscript{19}.

So far, I have only considered pure general relativity without matter. The first step toward including matter is to show that the combined gravity matter system can be written in terms of the self dual connection in such a way that the constraints are still polynomial in the basic canonical variables. This requires that the metric and its inverse cannot both appear. This step has been carried out for scalar, spinor and gauge fields\textsuperscript{20}. In particular, the action for supergravity has been written in terms of Ashtekar variables\textsuperscript{21}. The next step is to find solutions to the quantum constraints. In the presence of matter, this is not well understood. It should be kept in mind that unlike superstring theory, this approach does not at present provide a unified picture of all forces and matter. Its main advantage (assuming it is successful) is in staying as close as possible to the experimentally tested
If one considers general relativity with a cosmological constant $\Lambda$, then one solution to all of the quantum constraints turns out to be

$$\psi(A) = e^{-S_{CS}/\Lambda}$$

(13)

where $S_{CS}$ is the Chern-Simons action for the self dual connection $A_i$. The calculation of the transform of this state into the loop representation is similar to the calculation performed by Witten which reproduced knot invariants. One might worry that this indicates that the knot states will not be normalizable. If one considers the state (13) as a state in ordinary Yang-Mills theory, then for some choice of $\Lambda$, it turns out to be a zero energy eigenstate. But it is outside of the physical Hilbert space and hence appears to have no physical significance. Why should the situation for gravity be any better? The key point is that for gravity one is using self dual connections rather than real connections. Even for electromagnetism, one can show that if one uses the self dual representation (and $E^i$ real) then the Chern-Simon’s state is just the vacuum for one helicity of the photon! (The vacuum for the other helicity is a constant.) These states are normalizable with respect to the standard Poincare invariant inner product. This gives further evidence that the knot states are physical.

As we have mentioned, Ashtekar’s approach to quantum gravity is similar in spirit to the functional Schroedinger approach to ordinary field theory. However the reparameterization invariance leads to the technical simplification that one does not have to solve the time dependent Schroedinger equation since the Hamiltonian is proportional to the constraints. However this raises a conceptual problem: How does one recover time and make physical predictions? This is one of the deep issues that every (nonperturbative) approach to quantum gravity must address. It has been discussed extensively, but there is still no clear answer. Simple models of reparameterization invariant systems suggest that one part of the argument of the wave function should play the role of time. In Ashtekar’s approach, there has been some progress in identifying “time” in the connection representation but not much is known yet in the loop representation. Another possibility is that time will arise only when gravity is coupled to matter and “physical clocks” can be constructed.

3.2 Other approaches to quantization with Ashtekar variables

Although canonical quantization with constraint operators has been the main focus of work in this area, it may be worthwhile to examine other approaches. One alternative is to

* In the standard treatment one works with positive frequency fields. Then self dual configurations describe one helicity and anti-self dual the other. Here one works only with self dual fields but allows both positive and negative frequency. This explains how both helicities can be obtained and why there is an asymmetry.
solve the constraints classically and then quantize the resulting “true degrees of freedom”. (This was in fact the way Witten first quantized the 2+1 theory.) Remarkably enough, the general solution of the vector and scalar constraint can be expressed in terms of an arbitrary symmetric, invertible, traceless $3 \times 3$ matrix $\phi^{ab}(x)$\textsuperscript{27}. Given an arbitrary self dual connection $A_i^a$, define $E^i_a$ so that

$$F^a_{ij} = \epsilon_{ijk} E^k_b \phi^{ab}$$

Substituting this into the constraints, one sees immediately that (6) is satisfied since $\phi^{ab}$ is symmetric, and (7) is satisfied since $\phi^{ab}$ is tracefree. One can argue that this is the general solution since $\phi^{ab}$ has five independent components which is the number one expects after solving four equations for the nine components of $E^i_a$. Gauss’ law is the only remaining constraint on $A_i^a$ and $\phi^{ab}$. Unfortunately, a simple solution to this equation is not yet available.

Another possibility is to consider covariant approaches to quantization. This should be more conducive to answering a certain class of questions such as whether the topology of space can change in quantum gravity. Even classically, one has the following result. Both the action (4) and the one obtained by replacing $R$ by the curvature of the self dual connection $A$ do not involve the inverse of the tetrad. Thus the action and the resulting field equations remain well defined even in the limit that the metric becomes degenerate. In general relativity, it has been shown that any solution to the vacuum Einstein equation which interpolates between spaces of different topology must be singular. But the only “singularity” that is required is for the metric to become degenerate at one moment of time: There exist smooth solutions to the equations derived from (4) which change topology and have an invertible metric almost everywhere\textsuperscript{28}.

Since Ashtekar’s approach and the tetrad approach to general relativity both naturally include degenerate metrics, one is faced with the question of why the metric we see is invertible. In fact, it is not even clear how to formulate this question precisely. It is tempting to consider the expectation value of the metric $\langle g_{\mu\nu} \rangle$, and one often hears speculation that $\langle g_{\mu\nu} \rangle = 0$ may correspond to a diffeomorphism invariant phase of quantum gravity while $\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}$ corresponds to a state of broken symmetry. However, it is clear from the quantum constraints that the physical states of quantum general relativity are always diffeomorphism invariant. Moreover, the expectation value of any non-gauge invariant operator (such as the metric) must always be gauge invariant: If $U$ denotes a general gauge transformation, then

$$\langle \psi | g_{\mu\nu} | \psi \rangle = \langle \psi | U^{-1} g_{\mu\nu} U | \psi \rangle$$

since physical states are gauge invariant. As there are no nonvanishing diffeomorphism invariant tensor fields, this shows $\langle \psi | g_{\mu\nu} | \psi \rangle = 0$ for all physical states*. Analogous

* This assumes that the inner product is defined not just on physical states, but also on states such as $g_{\mu\nu} | \psi \rangle$ which are unphysical. Otherwise, $\langle \psi | g_{\mu\nu} | \psi \rangle$ is simply not defined.
arguments can be made for spontaneous symmetry breaking in ordinary gauge theory|29. However in that case, one can argue that even though the local symmetry is not spontaneously broken, the corresponding global symmetry is. It may be possible to extend this argument to gravity with asymptotically flat boundary conditions. But it certainly cannot apply to closed universes where there is no way to disentangle local and global diffeomorphisms. What is the appropriate gauge invariant operator which captures the notion of nondegenerate metrics?

3.3 Newer variables?

Although the constraints (5-7) are considerably simpler than the usual form in terms of the old canonical variables, it is reasonable to ask whether this is the best one can do. Does there exist an even more clever choice of variables which will lead to further simplifications? As we have discussed, one of the constraints in Ashtekar variables is quadratic in momenta and must be regulated. Are there canonical variables for which all constraints are linear in momenta? To see that one’s choice of variables can, in principle, change the structure of the constraints in this way, consider again three dimensional general relativity. In terms of the standard canonical variables (the spatial metric and extrinsic curvature) the constraints are very similar to the four dimensional case. In particular, they are quadratic in momenta (and nonpolynomial in the spatial metric). However in gauge theory variables, although the constraints are quadratic in the connection, they are linear in the triad which is its conjugate momentum. Thus there is a natural representation in which all constraints are linear in momentum.

Comparing the actions for general relativity in three (1) and four (4) dimensions there are two obvious differences: the group is changed from SO(2,1) to SO(3,1) and there is an extra $e^\mu_\nu$ in the action. One can actually separate these two effects. There are three dimensional theories which generalize (1) to any gauge group including SO(3,1). Let $A$ be the gauge field for an arbitrary Lie group, $F = dA + A \wedge A$ the field strength, and $e$ be a Lie algebra valued one form. Then one can consider the action|30

$$S = \int Tr e \wedge F$$

(16)

The constraints are identical to (2) and (3) with $R$ replaced by $F$. In particular, they are linear in the momentum conjugate to $A_i$. In four dimensions, there are theories with actions similar to (4) for any gauge group:

$$S = \int Tr [e, e] \wedge F$$

(17)

where $e$ is again a Lie algebra valued one form. For the case of SO(3) the canonical quantization of this theory has been carried out|31. Unlike the three dimensional examples,
this theory has an infinite number of degrees of freedom. Nevertheless, once again all constraints are linear in momentum.

As a final example, consider supergravity. In this theory, the scalar constraint can in fact be replaced by its “square root” - the supersymmetry constraints. Since the original constraint is quadratic in momentum, one might hope that the supersymmetry constraints would be at most linear in the momentum. Unfortunately, this is not the case. Although the supersymmetry constraints are linear in the momentum conjugate to the metric, they contain a term which is the product of the momentum conjugate to the metric and the momentum conjugate to the spin $3/2$ field $|^{32}_{21}$. This is again the product of functional derivatives at the same point and must be regulated. In retrospect, it is clear that the supersymmetry constraints cannot be linear in all momenta: The (anti) commutator of two constraints linear in momenta is always linear in momenta and cannot yield the scalar constraint of general relativity.

It is perhaps worth mentioning that going to higher dimensions does not seem very promising. In higher dimensions, general relativity can still be expressed in terms of a Lorentz connection $\omega_{\mu}^{ab}$ and collection of one forms $e_{\mu}^{a}$ with the action
\[
S = \int e^{a} \wedge \cdots \wedge e^{b} \wedge R^{cd} \epsilon_{a \cdots bcd}
\] (18)
However, there is no obvious analog of using the self-dual part of the connection. Thus it is not even clear how to mimic the simplification obtained by Ashtekar: Ashtekar’s variables do not have a natural generalization to higher dimensions.

In conclusion, I would say that I find the general ideas of Ashtekar’s approach to quantum gravity attractive, and the results obtained so far intriguing. It is still far from clear whether this program (or some variation of it) can be completed, but it certainly seems worth pursuing.

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A weave may be interpreted as representing a flat metric on $R^3$. 

FIGURE CAPTION