Dwyer-Kan Equivalences Induce Equivalences on Topologically Enriched Presheaves

Alexander Körschgen*

This brief note elaborates on a result by Gepner and Henriques. They have shown that a Dwyer-Kan equivalence between two small, topological categories gives rise to a Quillen equivalence of the associated categories of topologically enriched presheaves. We present a more detailed account of their proof.

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1. Introduction

In [2, Lemma A.6], Gepner and Henriques show that, given a Dwyer-Kan equivalence $f: C \to D$ between topologically enriched index categories, the associated functor $f^* : \text{Pre}(D, \text{Top}) \to \text{Pre}(C, \text{Top})$ between the respective categories of topologically enriched presheaves is the right adjoint of a Quillen equivalence $f_! \dashv f^*$. We give a more detailed account of their proof, discussing the required results on point-set topology as well as the necessary transfinite techniques in depth.

One might be tempted to apply [3, Proposition 2.4] to deduce the desired result. However, a certain assumption of this proposition does not hold in our context. We will elaborate on this problem in Remark 3.6.

Section 2 introduces basic properties of the category $\text{Pre}(C, \text{Top})$ of topologically enriched presheaves on a small, topologically enriched category $C$. The following Section 3 recalls the definition of a Dwyer-Kan equivalence and proceeds by reproducing the proof of Gepner and Henriques, which we divided into two lemmas and the final Theorem 3.5.

By $\text{space}$, we mean a compactly generated weak Hausdorff space. The required point-set foundations are spelled out in the first two subsections of the appendix while the last subsection of the appendix yields a helpful characterization of cofibrant objects of $\text{Pre}(C, \text{Top})$.

These notes used to be a part of [7] until the author decided to split them off.

*Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany.
E-mail address: alex@math.uni-bonn.de
2. Basic Properties of $\text{Pre}(C, \text{Top})$

**Definition 2.1.** Let $C$ be a topologically enriched category.

(i) Denote by $\text{Pre}(C, \text{Top})$ the category of enriched functors $C \to \text{Top}$. In this paper, it will always be equipped with the projective model structure.

(ii) The category $\text{Pre}(C, \text{Top})$ is bitensored over $\text{Top}$. For $X \in \text{Pre}(C, \text{Top})$, $K \in \text{Top}$, and $c \in C$, we have

$$(X \otimes K)(c) = X(c) \times K, \quad (X^K)(c) = X(c)^K.$$ 

(iii) For every $c \in C$, write $\text{ev}_c : \text{Pre}(C, \text{Top}) \to \text{Top}$ for the functor given by evaluation at $c$.

**Proposition 2.2.** Let $C$ be a topologically enriched category.

(i) For $c \in C$, the functor $\text{ev}_c$ has a left adjoint $F_c : \text{Top} \to \text{Pre}(C, \text{Top})$, given by $(F_cK)(c') = C(c', c) \times K$. The pair $F_c \dashv \text{ev}_c$ is Quillen.

(ii) The projective model structure on $\text{Pre}(C, \text{Top})$ is cofibrantly generated by the generating cofibrations

$I_C := \{F_c(S^{n-1} \hookrightarrow D^n) \mid n \geq 0, c \in C\}$

and the generating trivial cofibrations

$J_C := \{F_c(D^n \hookrightarrow D^n \times [0; 1]) \mid n \geq 0, c \in C\}$.

The domains and codomains of the maps in $I_C \cup J_C$ are cofibrant.

(iii) $\text{Pre}(C, \text{Top})$ is a topological model category. In particular, we have natural homeomorphisms

$\text{Pre}(C, \text{Top})(X \otimes K, X') \cong \text{Pre}(C, \text{Top})(X, (X')^K)$

and the pushout-product axiom holds.

(iv) Any cofibration in $\text{Pre}(C, \text{Top})$ is a levelwise closed cofibration, i.e., a Hurewicz cofibration with closed image.

**Proof.** It is an easy exercise to verify that $F_c$ is left adjoint to $\text{ev}_c$. As $\text{ev}_c$ preserves fibrations and trivial fibrations, the pair $F_c \dashv \text{ev}_c$ is Quillen, showing (i).

The category of (CGWH) topological spaces satisfies the monoid axiom by [5, Lemma 2.3]. Therefore, we may apply [10, Theorem 24.4] and deduce (ii). Part (iii) is a straight-forward computation.

Part (iv) uses the theory of $h$-cofibrations. see, e.g., [9, Definition A.1.18]. Every topological space is fibrant, so every object of $\text{Pre}(C, \text{Top})$ is fibrant. From [9, Corollary A.1.20.(iii)], we deduce that every cofibration in $\text{Pre}(C, \text{Top})$ is a $h$-cofibration. Picking $c \in C$, the functor $\text{ev}_c : \text{Pre}(C, \text{Top})$ commutes with colimits and tensors. Thus, it takes $h$-cofibrations to cofibrations by part (ii) of the same Corollary. So, every cofibration is a levelwise $h$-cofibration of topological spaces. An inspection of the definitions shows that $h$-cofibrations in Top are exactly the Hurewicz cofibrations.

To conclude the proof of part (iv), it remains to show that a cofibration in $\text{Pre}(C, \text{Top})$ is levelwise a closed inclusion. This is true for the generating cofibrations by their description above. The characterization of cofibrations, see Subsection A.3, together with Lemma A.2 shows that it is also true for an arbitrary cofibration.

3. Dwyer-Kan Equivalences and Induced Equivalences

Let us begin by recalling the definition of a Dwyer-Kan equivalence:

**Definition 3.1.** Let $C$ be topologically enriched category. Then the ordinary category $\pi_0C$ has $\text{ob} \pi_0C = \text{ob}C$ and $(\pi_0C)(c, c') = \pi_0(C(c, c'))$ with composition defined in the obvious way.

An enriched functor $f : C \to D$ of topologically enriched categories is a **Dwyer-Kan equivalence** if the induced functor $\pi_0f : \pi_0C \to \pi_0D$ is an equivalence of categories and $f$ is a weak equivalence on all mapping spaces.
Next, we establish the Quillen pair between topologically enriched presheaf categories induced by an enriched functor. For a Dwyer-Kan equivalence, this Quillen pair is a Quillen equivalence as we will have seen at the end of this section.

Proposition 3.2. Let \( f : \mathcal{C} \to \mathcal{D} \) be an enriched functor between topologically enriched categories. Then the restriction functor \( f^* : \text{Pre}(\mathcal{D}, \text{Top}) \to \text{Pre}(\mathcal{C}, \text{Top}) \) has a left adjoint \( f_! \). The pair \( f_! \dashv f^* \) is Quillen. Furthermore, both functors respect the tensoring over \( \text{Top} \). Finally, both functors preserve colimits.

Proof. The construction of \( f_! \) can be found in \([2, \text{Lemma A.6}]\). Alternatively, one can use the left Kan extension of the functor \( \mathcal{C} \to \text{Pre}(\mathcal{D}, \text{Top}) \) \( c \mapsto F(f)(c)^* \) along the enriched Yoneda embedding \( F(\_)(\_)^* : \mathcal{C} \to \text{Pre}(\mathcal{C}, \text{Top}) \). To show that the pair \( f_! \dashv f^* \) is Quillen, it suffices to observe that \( f^* \) clearly preserves fibrations and trivial fibrations.

The left adjoint \( f_! \) preserves colimits, and \( f^* \) preserves colimits, too, because they are computed levelwise. Moreover, it is evident from the definitions that \( f^* \) preserves the tensoring. To show that \( f_! \) preserves the tensoring, observe that the cotensoring is also compatible with \( f^* \). Let \( X \in \text{Pre}(\mathcal{C}, \text{Top}), Y \in \text{Pre}(\mathcal{D}, \text{Top}) \), and \( K \in \text{Top} \). We compute

\[
\begin{align*}
\text{Pre}(\mathcal{D}, \text{Top})(f_!(X \otimes K), Y) & \cong \text{Pre}(\mathcal{C}, \text{Top})(X \otimes K, f^*Y) \\
& \cong \text{Pre}(\mathcal{C}, \text{Top})(X, (f^*Y)^K) \\
& \cong \text{Pre}(\mathcal{C}, \text{Top})(X, f^*(Y^K)) \\
& \cong \text{Pre}(\mathcal{D}, \text{Top})(f_!X, Y^K) \\
& \cong \text{Pre}(\mathcal{D}, \text{Top})(f_!(X \otimes K), Y)
\end{align*}
\]

naturally. Hence, there is a natural isomorphism \( f_!(X \otimes K) \cong (f_!X) \otimes K \).

The following two lemmas provide the technical details necessary for the proof of the main theorem at the end of this subsection.

Lemma 3.3. Let \( f : \mathcal{C} \to \mathcal{D} \) be as before. If the functor \( f \) is homotopically fully faithful, then the unit \( \eta_X : X \to f^* f_! X \) is a level weak equivalence in \( \text{Pre}(\mathcal{C}, \text{Top}) \) for all cofibrant \( X \in \mathcal{C} \).

Proof. Let \( \mathcal{N} \) be the class of all objects of \( \text{Pre}(\mathcal{C}, \text{Top}) \) for which the unit is a weak equivalence. Our goal is to show that the conditions of Lemma A.4 are satisfied.

The only map whose codomain is the empty presheaf \( \emptyset \) is its identity \( \text{id}_{\emptyset} \), which is an isomorphism. We proceed to verify (i)-(iii).

(i) If \( X \in \mathcal{N} \) and \( X' \) is a retract of \( X \), then the unit map \( \eta_{X'} \) is a retract of \( \eta_X \) by naturality of the unit.

A retract of a weak equivalence is a weak equivalence. Hence, \( \eta_{X'} \) is a weak equivalence.

(ii) First, let us show that \( \eta_X \) is a weak equivalence whenever \( X \) is the domain or codomain of a map in \( I \). These domains and codomains are of the form \( F_c(K) = F_c(*) \otimes K \) for \( c \in \mathcal{C} \) and \( K \) a space. The evaluation of \( \eta_{F_c(K)} \) at some \( c' \in \mathcal{C} \) fits into a diagram
using the compatibility of \( f_i \) with tensors and the formula \( f_i(F_k(\!*)) \cong F_{f_i(k)}(\!*) \) from the proof of Proposition 3.2. The bent map is a weak equivalence because \( f_{c',c} : \mathcal{C}(c',c) \to \mathcal{D}(f(c'),f(c)) \) is a weak equivalence since \( f \) is homotopically fully faithful by assumption. Hence, \( \eta_{F_k(K)} \) is a level weak equivalence.

Given a pushout diagram

\[
\begin{array}{c}
A \to X \\
\downarrow \ \\
B \to X'
\end{array}
\]

with \( X \in \mathcal{N} \), the square

\[
\begin{array}{c}
f^*f_iA \to f^*f_iX \\
\downarrow \ \\
f^*f_iB \to f^*f_iX'
\end{array}
\]

is pushout by Proposition 3.2. Evaluating at some \( c \in \mathcal{C} \), we obtain a commutative cube

\[
\begin{array}{c}
A(c) \to X(c) \\
\downarrow \ \\
B(c) \to X'(c)
\end{array}
\]

with both the front and rear faces being pushout. The maps perpendicular to the plane of projection are the respective unit maps evaluated at \( c \). Three of these are weak equivalences, namely \( \eta_X(c) \) by assumption and \( \eta_A(c) \) and \( \eta_B(c) \) by our considerations from before.

The map \( A \to B \) is a cofibration, so \( A(c) \to B(c) \) is a Hurewicz cofibration by Proposition 2.2.(iv). The left adjoint \( f_i \) preserves cofibrations, so \( f_i A \to f_i B \) is a cofibration and

\[
(f^*f_iA)(c) = (f_iA)(f(c)) \to (f_iB)(f(c)) = (f^*f_iB)(c)
\]

is a Hurewicz cofibration. In this situation, the map \( \eta_X(c) \) between the pushouts is a weak equivalence as well ([1, Proposition 4.8 (b)] or [8, Proposition 1.1]). Since \( c \) was arbitrary, \( \eta_X' \) is a level weak equivalence.
(iii) Assume that there is a $\lambda$-sequence $X : \lambda \to \text{Pre}(C, \text{Top})$ such that (a)-(c) from Lemma 3.4 hold. A case distinction is in order.

$\lambda$ is 0 A 0-sequence is just an empty diagram with colimit the initial object $\emptyset$. Also, $\beta^* f_0 = \emptyset$ and the map $\eta_0$ must be $\text{id}_0$ which is a weak equivalence.

$\lambda$ a successor ordinal If $\lambda = \mu + 1$, we have $\text{colim}_{\beta<\lambda} X_\beta = X_\mu \in N$ by assumption.

$\lambda$ a limit ordinal Some categorical yoga shows that $\text{colim}_{\beta<\lambda} \eta_{X_\beta} = \eta_{\text{colim}_{\beta<\lambda} X_\beta}$. Therefore, it is sufficient to show that $\text{colim}_{\beta<\lambda} \eta_{X_\beta}$ is a weak equivalence.

By condition (b), the maps $X_\beta \to X_{\beta+1}$ are cofibrations, hence levelwise closed inclusions by Proposition 2.2. The maps $f_1 X_\beta \to f_1 X_{\beta+1}$ are cofibrations and levelwise closed inclusions as well, and we derive that $\beta^* f_1 X_\beta \to \beta^* f_1 X_{\beta+1}$ are levelwise closed inclusions. Hence, both $\text{colim}_{\beta<\lambda} X_\beta(c)$ and $\text{colim}_{\beta<\lambda} \beta^* f_1 X_\beta(c)$ are taken along closed inclusions, and the maps $X_\beta \to \beta^* f_1 X_\beta$ are weak equivalences because $X_\beta \in N$ by assumption (c):

\[
\begin{align*}
X_\beta(c) \xrightarrow{\approx} & \quad (\beta^* f_1 X_\beta)(c) \\
\downarrow \quad & \\
X_{\beta+1}(c) \xrightarrow{\approx} & \quad (\beta^* f_1 X_{\beta+1})(c)
\end{align*}
\]

\[
\begin{align*}
\text{colim} X_\beta(c) \quad & \xrightarrow{\approx} \quad \text{colim}(\beta^* f_1 X_\beta(c)) \\
\downarrow \quad & \\
\eta_{X_\beta} \quad & \text{colim}(\beta^* f_1 X_\beta(c))
\end{align*}
\]

By Lemma A.3, $\pi_k \text{colim} X_\beta(c) \cong \pi_k X_\beta(c)$ and $\pi_k \text{colim}(\beta^* f_1 X_\beta)(c) \cong \pi_k (\beta^* f_1 X_\beta)(c)$. Thus, $\pi_k(\text{colim} \eta_{X_\beta})$ is a colimit of isomorphisms and therefore an isomorphism itself. In conclusion, $\eta_{\text{colim} X_\beta} \cong \text{colim} \eta_{X_\beta}$ is a weak equivalence as desired. Note that we did not need condition (a) in our specific situation.

\[\square\]

**Lemma 3.4.** Let $D$ be a topologically enriched category and $Y \in \text{Pre}(D, \text{Top})$. If $a : d \to d'$ is a homotopy equivalence in $D$, then so is $Y(a) : Y(d') \to Y(d)$.

**Proof.** If $a$ is such a homotopy equivalence, then there is $b : d' \to d \in D$ such that $[b \circ a] = [\text{id}_d] \in \pi_0(D(d, d))$ and $[a \circ b] = [\text{id}_{d'}] \in \pi_0(D(d', d'))$.

The enriched functor $Y$ induces a map

\[
\pi_0(D(d, d)) \to \pi_0(\text{Top}(Y(d), Y(d))).
\]

As $[b \circ a]$ and $[\text{id}_d]$ are the same on the left hand side, we obtain that $[Y(\text{id}_d)] = [Y(b) \circ Y(a)] = [Y(a) \circ Y(b)]$ agree, too. Two maps represent the same path component in $\pi_0(\text{Top}(Y(d), Y(d)))$ if and only if they are homotopic. Therefore, $Y(a) \circ Y(b) \simeq Y(\text{id}_d)$. By an analogous argument, $Y(b) \circ Y(a) \simeq Y(\text{id}_{d'})$. We conclude that $Y(a)$ is a homotopy equivalence. \[\square\]

**Theorem 3.5.** Let $f : C \to D$ be a Dwyer-Kan equivalence between topologically enriched categories. Then the induced Quillen pair $f_\ast : f^\ast$ is a Quillen equivalence.

**Proof.** Let $X \in \text{Pre}(C, \text{Top})$ be cofibrant, $Y \in \text{Pre}(D, \text{Top})$, and $\alpha : X \to f^\ast Y$ be a map with adjoint $\beta : f_\ast X \to Y$. By one of the many equivalent characterizations of a Quillen equivalence, we need to show that $\alpha$ is a (level) weak equivalence if and only if $\beta$ is (note that any $Y$ is fibrant). The map $\alpha$ factors through the unit $\eta_X$, which is a weak equivalence by Lemma 3.3:

\[
\begin{align*}
X \xrightarrow{\eta_X} & \quad f^\ast f_\ast X \\
\alpha \quad & \xrightarrow{\approx} \\
f^\ast & \quad f^\ast Y
\end{align*}
\]

Therefore, $\alpha$ is a weak equivalence if and only if $f^\ast(\beta)$ is. It remains to show that this is the case if and only if $\beta$ is a weak equivalence.
Assume that \( f^*(\beta) \) is a weak equivalence and choose \( d \in D \). As \( f \) is homotopically essentially surjective, there is \( c \in C \) and a map \( a : fc \to d \) in \( D \) that is a homotopy equivalence. We obtain a commutative diagram

\[
\begin{array}{ccc}
(f_!X)(fc) & \xrightarrow{\beta_d} & Y(fc) \\
\uparrow a^* & & \uparrow a^* \\
(f_!X)(d) & \xrightarrow{\beta_d} & Y(d)
\end{array}
\]

The vertical arrows are homotopy equivalences by Lemma 3.4, therefore, \( \beta_d \) is a weak equivalence and \( \beta \) is a level weak equivalence.

Vice versa, let \( \beta \) be a level weak equivalence. Obviously, this implies that \( f^*(\beta) \) is a level weak equivalence, concluding the proof. 

This concludes the main proof of this note. Let us end this section with a remark promised in the introduction.

**Remark 3.6.** As already mentioned in the introduction, there is an issue preventing us from applying [3, Proposition 2.4] to deduce the previous theorem. Namely, it does not hold in general that cofibrations in \( \text{Pre}(C, \text{Top}) \) are level cofibrations of topological spaces.

To circumvent this issue, we have to work with Hurewicz cofibrations in Lemma 3.3, which still interact nicely with weak equivalences of topological spaces under pushouts and transfinite composition. The remaining parts of the proof are independent of this issue.

One of the assumptions of [3, Proposition 2.4], hidden in [3, Theorem 4.31], is that the functors \( C(c,c') \otimes - \) preserve cofibrations. If this is the case, then cofibrations in \( \text{Pre}(C, \text{Top}) \) are indeed level cofibrations, and the arguments in Lemma 3.3 become much simpler. However, it is usually not the case that \( C(c,c') \otimes - \) preserves cofibrations unless the \( C(c,c') \) happen to be cofibrant themselves.

### A. Appendix

#### A.1. Compactly Generated Weak Hausdorff Spaces

The main body of this paper takes place in the category of compactly generated weak Hausdorff spaces, also referred to CGWH spaces. Before we deal with the necessary point-set arguments, let us make the used terminology precise.

A space is **compact** if every open cover admits a finite subcover. This is also being referred to as **quasi-compact** in other sources which include the Hausdorff property into the definition of compactness.

Moreover, a space \( X \) is **compactly generated** if, for any subset \( Y \subseteq X \), \( Y \) is closed if and only if \( u^{-1}(Y) \) is closed for every compact Hausdorff \( K \) and every continuous \( u : K \to X \). The space \( X \) is weak Hausdorff if for every such \( u \) and \( K \), the image \( u(K) \) is closed in \( X \).

These definitions are taken from [11]. Note that this terminology varies within the literature, and some sources refer to compactly generated spaces as **\( k \)-spaces** while they take compactly generated spaces to be compactly generated weak Hausdorff spaces in our sense.

Note that the property of being compactly generated is a local property, i.e., a space is compactly generated if and only if each point has a compactly generated neighborhood. The property of being weak Hausdorff is not local, though.

In this paper, we refer to CGWH spaces as **(topological) spaces** and denote the corresponding category by \( \text{Top} \). Within the next subsections, we will have to deal with their point-set subtleties and cite statements about not necessarily CGWH spaces. To this end, we will use the term **general topological space** for a space that is not necessarily CGWH.

The category of CGWH spaces is cocomplete. Limits and colimits may, however, differ from those computed in the category of general topological spaces. Our convention is that limits and colimits are computed in CGWH unless it is explicitly declared that the diagram in consideration lives in the category of general topological spaces. In this case, limits and colimits are to be taken in the category of general topological spaces. The latter situation does only occur in Subsection A.2.

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1This subsection and the following one have been copied from [7] as of version 2.
For the special case of products, we adopt the following notation from [11]: Given two spaces \( X \) and \( Y \), we denote by \( X \times_0 Y \) the product taken in the category of general topological spaces, which is not necessarily compactly generated. In contrast, \( X \times Y \) shall denote the product in the category of CGWH spaces.

### A.2. Closed Inclusions and CGWH Colimits

We will now shed some light on situations where specific colimits agree regardless of whether they are computed in CGWH or in the category of general topological spaces.

**Lemma A.1** ([6, Section 2.4, p. 59]). The category of topological spaces is cocomplete. In the case of pushouts along closed inclusions or transfinite compositions of injections, colimits may be computed in the category of general topological spaces since they are already weak Hausdorff.

**Lemma A.2.** In the category of topological spaces, closed inclusions are closed under pushouts, transfinite compositions, and retracts.

**Proof.** As weak Hausdorff spaces are automatically \( T_1 \), a closed inclusion in the category of topological spaces is a closed \( T_1 \) inclusion in the category of general topological spaces. Retracts of maps of topological spaces are also retracts of maps of general topological spaces. Also, the relevant pushouts and transfinite compositions can be computed in the category of general topological spaces.

The claim follows from the proof of [6, Lemma 2.4.5] for the cases of pushouts and transfinite compositions and from the proof of [6, Corollary 2.4.6] for the case of retracts. \( \square \)

Let us end this subsection by noting that sequential colimits along closed inclusions commute with \( \pi_k \).

**Lemma A.3.** Let \( \lambda \) be a limit ordinal and \( X : \lambda \to \text{Top} \) be a \( \lambda \)-sequence along closed inclusions. Furthermore, let \( K \) be a compact space. Then any map \( K \to \text{colim}_{\beta<\lambda} X_\beta \) factors through some \( X_\mu, \mu<\lambda \). In particular, the canonical map

\[
\text{colim}_{\beta<\lambda} \pi_k(X_\beta) \to \pi_k(\text{colim}_{\beta<\lambda} X_\beta)
\]

is an isomorphism.

**Proof.** The colimit can be computed in the category of general topological spaces. As in the proof of the previous lemma, a closed inclusion is a closed \( T_1 \) inclusion of general topological spaces. Also, note that the cofinality \( \text{cf} \lambda \) of \( \lambda \) is infinite because \( \lambda \) is a limit ordinal. In particular, \( \lambda \) is \( \gamma \)-filtered for each finite cardinal \( \gamma \) in the sense of [6, Definition 2.1.2]. Therefore, [6, Proposition 2.4.2] tells us that the canonical map

\[
\text{colim}_{\beta<\lambda} \text{Top}(K, X_\beta) \to \text{Top}(K, \text{colim}_{\beta<\lambda} X_\beta)
\]

is an isomorphism, proving the first claim. The second claim follows by a standard argument. \( \square \)

### A.3. Cofibrant Objects in Cofibrantly Generated Model Categories

Recall that cofibrations in a cofibrantly generated model category \( \mathcal{M} \) with generating cofibrations \( I \) are precisely the retracts of transfinite compositions of pushouts of elements of \( I \) [4, Corollary 10.5.22]. An additional assumption, which is satisfied by \( \text{Pre}(\mathcal{C}, \text{Top}) \), allows us to simplify the characterization of cofibrant objects. It shall be noted that this extra assumption does usually not apply in pointed contexts.

**Lemma A.4.** Let \( \mathcal{M} \) be as above and suppose that any map \( X \to \emptyset \) is an isomorphism. Let \( \mathcal{N} \) be a class of objects of \( \mathcal{M} \). Assume that \( \mathcal{N} \) satisfies the following properties:

(i) \( \mathcal{N} \) is closed under retracts.

(ii) If \( X \in \mathcal{N} \) and \( X' \) is a pushout of \( X \) along a generating cofibration,

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
B & \relbar unreversed pushout_{I} & \longleftarrow \ \\
& \downarrow {I_\exists} & \ \\
& B & \longrightarrow X'
\end{array}
\]

then \( X' \in \mathcal{N} \).
(iii) If \( \lambda \) is an ordinal and \( X : \lambda \to M \) is a \( \lambda \)-sequence such that

\[ a) \ X_0 = \emptyset \text{ if } \lambda > 0, \]

\[ b) \ X_\beta \in N \text{ for } \beta < \lambda, \text{ and} \]

\[ c) \text{ for } \beta < \lambda \text{ such that } \beta + 1 < \lambda, \text{ there is a pushout} \]

\[
\begin{array}{ccc}
A_\beta & \to & X_\beta \\
\downarrow & & \downarrow \\
B_\beta & \to & X_{\beta+1}
\end{array}
\]

then \( \operatorname{colim}_{\beta<\lambda} X_\beta \in N \).

Then \( N \) contains all cofibrant objects. Moreover, the class of cofibrant objects is the smallest class satisfying these properties.

Proof. First, note that \( \emptyset \in N \) because the empty 0-sequence, whose colimit is \( \emptyset \), satisfies all the conditions (iii).(a)-(c).

Let \( X' \in M \) be cofibrant, i.e., the map \( \emptyset \to X' \) is a cofibration. We wish to show that \( X' \in N \). As \( M \) is cofibrantly generated, this means that there is an ordinal \( \lambda \) and a \( \lambda \)-sequence \( X : \lambda \to M \) such that

\[ \text{• there is a retract diagram} \]

\[
\begin{array}{ccc}
\emptyset & \to & X_0 & \to & \emptyset \\
\downarrow & & \downarrow & & \downarrow \\
X' & \to & \operatorname{colim}_{\beta<\lambda} X_\beta & \to & X'
\end{array}
\]

\[ \text{• for } \beta < \lambda \text{ such that } \beta + 1 < \lambda, \text{ there is a pushout} \]

\[
\begin{array}{ccc}
A_\beta & \to & X_\beta \\
\downarrow & & \downarrow \\
B_\beta & \to & X_{\beta+1}
\end{array}
\]

As the map \( X_0 \to \emptyset \) is an isomorphism by assumption, we can assume without loss of generality that \( X_0 = \emptyset \). Furthermore, \( N \) is closed under retracts, therefore, it suffices to show that \( \operatorname{colim}_{\beta<\lambda} X_\beta \in N \).

Conditions (iii).(a) and (iii).(c) hold for the \( \lambda \)-sequence \( X \), and we wish to verify the remaining condition (iii).(b), i.e., \( X_\beta \in N \) for \( \beta < \lambda \). If this is not the case, let \( \mu < \lambda \) be the smallest ordinal such that \( X_\mu \notin N \). Since \( \emptyset \in N \) and \( X_0 = \emptyset \), we must have \( \mu > 0 \). The truncated diagram \( X_{|\mu} \) is a \( \mu \)-sequence and satisfies (iii).(a)-(c). Hence, \( \operatorname{colim}_{\beta<\mu} X_\beta \in N \).

If \( \mu = \nu + 1 \) is a successor ordinal, we obtain \( \operatorname{colim}_{\beta<\mu} X_\beta = X_\nu \in N \). As \( X_\mu \) is a pushout of \( X_\nu \) along a generating cofibration, \( X_\mu \in N \) by (ii), a contradiction.

If \( \mu \) is a limit ordinal, we immediately obtain \( \operatorname{colim}_{\beta<\mu} X_\beta = X_\mu \) because \( X \) is a \( \lambda \)-sequence. Therefore, \( X_\mu \in N \), a contradiction as well. In conclusion, \( \operatorname{colim}_{\beta<\lambda} X_\beta \in N \) by (iii), and we have shown that \( N \) contains all cofibrant objects.

For the second claim, we need to check that the class \( M^{\operatorname{cof}} \) of cofibrant objects of \( M \) satisfies (i)-(iii). This is obvious for (i) and (ii). If \( X \) is a \( \lambda \)-sequence such that (iii).(a)-(c) hold (note that (b) actually follows from (a) and (c) in this case), then the map \( \emptyset = X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta \) is a cofibration, and the object \( \operatorname{colim}_{\beta<\lambda} X_\beta \) is cofibrant. \( \square \)
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