A NOT operation on Majorana qubits with mobilizable solitons in an extended Su–Schrieffer–Heeger model

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Abstract

Coupling Majorana qubits with other qubits is absolutely essential for storing, manipulating and transferring information for topological quantum computing. We theoretically propose a manner to couple Majorana qubits with solitons, another kind of topological impurity, which was first studied in the spinless Su–Schrieffer–Heeger model. We present a NOT operation on the Majorana qubit by moving the soliton through a heterostructure adiabatically. Based on these two topological impurities, the operation is robust against local disorder. Furthermore, we find that the soliton may carry non-universal fractional electric charge instead of fractional charge \( \frac{1}{2} \), because of the breaking of gauge invariance induced by superconducting proximity.

1. Introduction

Topological phases (TPs) of matter are characterized by nontrivial band structures which cannot be connected to trivial band structures without closing the band gap at the Fermi energy [1–4]. Due to the holographic principle, symmetry protected boundary states will appear at the boundaries of the system. When the energy gap is opened by superconductivity, a superconductor with nontrivial band structure becomes a topological superconductor and Majorana fermions (MFs) may exist at the surfaces. The \( Z_2 \) invariant, which corresponds to the parity of the number of MF branches at each boundary, can be used to distinguish the superconductive nontrivial TP from the superconductive trivial TP [2, 5, 6]. These MFs are robust against local distortions and are considered to be suitable for the physical realization of a topological qubit [3]. There are many proposals for generating topological superconductors hosting MFs, based on systems from 1-dimensional (1D) superconducting wires with strong spin–orbital interaction [7–11] to 2D topological superconductor heterostructures [12, 13] or vortex cores [8, 14, 15].

To realize quantum computing based on Majorana qubits, one needs to transfer quantum information between different qubits. There are already some proposals to hybridize Majorana qubits with other qubits, for instance, with a fluxonium qubit [16], flux qubit [17], quantum dot qubit [18] and superconducting charge qubits [19]. In this paper, we aim to couple the Majorana qubits with the soliton qubit, another kind of topological impurity first studied in the spinless Su–Schrieffer–Heeger (SSH) model. These soliton qubits are totally different from the qubits introduced in previous proposals. As first calculated in polyacetylene, the effective mass of the soliton is about \( 6m_e \), where \( m_e \) is the mass of electron [20]. Under an electric field or thermal gradient, this light soliton can be accelerated and reach a speed as fast as the velocity of sound in polyacetylene, \( \sim 0.015 \text{ m ps}^{-1} \) [20]. So the essential difference between our model and the previous proposals is that the soliton qubit is mobilizable, as well as topologically protected because the soliton attaches to a domain wall, which cannot be destroyed or created individually. We find that moving the soliton through the SSH region of a heterostructure can induce a NOT operation on the Majorana qubit. This manipulation offers a method to operate one kind of topological qubit (on MF) with another kind of topological qubit (on soliton). So the operation is fault tolerant and robust against local disorder.
We start from a 1D extended spinless SSH model. An effective nearest neighboring $p$-wave superconducting pairing is involved so that the model can be considered as a combination of the SSH model [4, 20] and Kitaev's toy model [7]. By varying the parameters, the phases of the model can evolve from TP hosting MFs to another TP hosting solitons. Because of the presence of superconducting pairing, the electric charge carried by each soliton is no longer universally equal to 1/2 (in the units $\epsilon = 1$). We will present a topological method for calculating this charge accurately in section 3.

We begin in section 2 by introducing a 1D tight-binding model. This model has three topologically nonequivalent phases. We will also illustrate the kinds of topological impurity in these phases. In section 3, we introduce the Thouless pump to the model and calculate the movement of Wannier functions (WFs) during the pump. This can help us find the electric charge carried by each soliton. In section 4, we show how to apply NOT operation on a Majorana qubit by moving the soliton and discuss why local disorder cannot affect the operation. In section 5, we emphasize the importance of our findings and discuss new vistas of research in this field.

2. 1D tight-binding model and its phase diagram

2.1. Hamiltonian

We use a Hamiltonian describing a 1D spinless dimerized model with the nearest neighboring $p$-wave superconducting coupling,

$$
H = \sum_i \left\{ \left[ 1 + (-1)^i \delta \right] c_i^\dagger c_{i+1} + h. c. \right\} + \hbar \sum_i c_i^\dagger c_i 
+ \left[ \Delta \sum_i \left( c_i^\dagger c_{i+1}^\dagger \right) + h. c. \right].
$$

(1)

where $c_i^\dagger$ is the creation operator for the electron on site $i$. The lattice index $i$, taking from 1 to $N$, is used to indicate the $i$th site of a 1D lattice. Here, the lattice spacing is set as the length unit. The hopping strength between the nearest neighboring sites are staggered between $1 + \delta$ and $1 - \delta$ along the chain so that each unit cell contains two sites, belonging to the two sublattices usually denoted as $A$ and $B$ in the SSH model [4]. Here we take the averaged hopping integral as the unit of energy. The parameters, $\Delta$ and $\hbar$, are for the strength of the $p$-wave superconducting pairing and the external global potential, respectively. The model will regress to the SSH model when $\Delta = 0$ and to the Kitaev model when $\delta = 0$. In this paper, we are only interested in the cases with $|\delta| < 1$. Another model has been studied recently, in which $\Delta = \Delta (1 \pm \delta)$ [21]. There is also a study on the spinful SSH model with spin–orbital interaction without superconducting coupling [22].

This model could be realized by placing a dimerized polyacetylene on the top of an $s$-wave superconductor. The effective spinless Hamiltonian equation (1) is obtained when the Fermi energy lying in the gap is opened by the staggered hoppings or by the external magnetic field. Another possible realization of the Hamiltonian has been proposed by Klinovaja et al [23], in which a helical magnetic field plays the role of staggered hoppings.

2.2. Phase diagram

The standard SSH model (equation (1) with $\Delta = 0$ and $\hbar = 0$) has two topological nonequivalent phases, a TP when $\delta > 0$ and a trivial phase when $\delta < 0$ [4]. Similarly the Kitaev model (equation (1) with $\delta = 0$) is in a TP when $|\hbar| < 2$ and in a trivial phase when $|\hbar| > 2$ [7].

For the present model with both nonzero $\Delta$ and $\delta$, the phase boundaries separating different phases are determined by the fact that the band gap closes there. The Hamiltonian in a Nambu and sublattice matrix representation is,

$$
H(k) = \left\{ \left[ (1 - \delta) + (1 + \delta) \cos(k) \right] \sigma_x + (1 + \delta) \sin(k) \sigma_y + \hbar \sigma_0 \right\} \otimes \tau_x 
- \Delta \sin(k) \sigma_x + (1 - \cos(k)) \sigma_y \otimes \tau_x,
$$

(2)

where Pauli matrices $\tau_{x,y,z}$ and $\sigma_{x,y,z}$ are operating on the particle–hole and the sublattice subspaces, respectively. $\sigma_0$ is a unit matrix. The band gap closes when the parameters obey one of the following two conditions, $\frac{\hbar^2}{4} + \Delta^2 = \delta^2$ (when $|\delta| < 1$) and $\hbar = \pm 2$. In figure 1, we sketch out the phase boundaries obtained from the first condition with the ellipsoidal cones in the parameter space spanned by $\hbar$, $\delta$, and $\Delta$. The two planes at $\hbar = \pm 2$ (obeying the second condition) are not shown for the sake of clarity.

From the phase diagrams of the SSH model and the Kitaev model, one can anticipate the TP of each region in the phase diagram. In figure 1, for the parameter region enclosed by the red cone, the system is in the TP similar to that of the SSH model. This is because the model can be regressed to the standard SSH model by decreasing both $\hbar$ and $\Delta$ to 0 while keeping $\delta > 0$. The band gap at the Fermi energy does not close during this process. So we call the TP in the red cone an SSH–like TP (SSHTP). For the parameter region in between the two ellipsoidal cones and the two planes $\hbar = \pm 2$ at distance, the system is in the Kitaev-like TP (KTP) because it can be regressed to a standard Kitaev’s toy model in TP without closing the band gap. In the upper left inset of figure 1, we schematically show that an open chain in the SSHTP (or KTP) can host soliton states (or MFs) at the ends.
For the other regions in the parameter space, including the region enclosed by the blue cone and those outside the two planes \( h = \pm 2 \), the system is in the trivial TP. Actually, the trivial phase in these regions can be separately regressed to the trivial TP of the SSH model and to the trivial TP of the Kitaev model, respectively. This phase diagram is in qualitative agreement with the phase diagram obtained by calculating the topological numbers in [21], although the model in [21] is slightly different.

MFs should appear at the interface between the region in the KTP and the region in the SSHTP or trivial TP. For a uniform chain in KTP, MFs can appear at the geometric ends because the vacuum outside the chain is topologically equivalent to a model in trivial TP (upper inset in figure 1). A soliton will appear at the interface between the regions in SSHTP and trivial TP. Therefore it can appear at the geometric ends of a SSHTP chain or at a domain wall which also separates the part of the chain in the SSHTP with the part in the trivial TP region (blue cone). A soliton on a domain wall is schematically shown in the lower inset in figure 1.

2.3. Topological impurities appearing at the boundaries

To further illustrate the topological nonequivalence of the 3 phases in the phase diagram and to show the topological impurities existing at the interfaces, in figure 2 we plot the energy spectrum of a ring, containing two semicircles which are in respective TPs. We refer to them as upper and lower semicircles. When these semicircles are in different TPs, the topological protected boundary states whose energies are inside the band gap should appear at the two joints of the semicircles. In all panels of figure 2, the parameters are fixed in the upper semicircle, while one of the parameters, referred to as \( \delta_1 \) in (a), \( h_1 \) in (b) and (d), and \( \Delta_1 \) in (c), varies in the lower semicircle. We schematically indicate the types of TP of the semicircles with the colors: red, black, and blue for the SSHTP, KTP and trivial TP, respectively. In all figures in this paper, due to the particle–hole symmetry, we only plot the positive eigenenergies, while their negative counterparts are not shown.

In figure 2(a), by varying the staggered hoppings \( \delta_1 \) in the lower semicircle, it is in the KTP when \(-0.1 < \delta_1 < 0.1\) and in the SSHTP when \( \delta_1 > 0.1 \). As a result one Majorana zero energy state, representing the emergence of one MF at each joint, appears in the former case. There are two zero energy states, demonstrating that the soliton state appears at each joint in the latter case. This result is similar to that obtained from a finite chain in [21], since now the upper semicircle is in trivial TP as the vacuum. Similarly, in figure 2(b), when \( h_1 > 2 \), a Majorana zero energy state appears in the spectrum. These results can be traced back to the conclusions from the standard SSH model or the Kitaev model.

Something different happens in the following panels. In figure 2(c), the lower semicircle is in the KTP when \( \Delta_1 > 0.2 \) and the upper semicircle is fixed in the SSHTP. Such a ring is composited by the two semicircles with the open boundary condition when the hoppings (and the superconducting pairings) between the ends of the upper and lower semicircles are considered extra couplings. At each joint, the extra couplings will couple the soliton state at the end of the upper semicircle with the MF at the end of the lower semicircle. Because the MF, a half fermion, can only be paired to a fermion with another MF but not soliton, which is a fermion, there is in total one MF left at each joint. Thus there is one zero energy state in the gap. However, the couplings lift the energy of the soliton state to the energy band, so there is no soliton state in the band gap. In figure 2(d), the upper semicircle is in the SSHTP and the lower one is in the trivial TP when \( h_1 > 2 \). This situation is the same as that in
the right part of figure 2(a). But the soliton states in (d) are lifted from zero energy and become Andreev bound states. These Andreev bound states are different from the normal ones because they take the responsibility of topological impurities which must appear at the boundaries of different TPs. As a result, as panel (d) shows, the Andreev bound states can evolve continuously to zero energy without closing the band gap.

Because of their topological nature, the evolutions to zero energy should be robust against disorder. In figure 3, disorder is introduced to the ring by replacing the staggered hopping between the \( i \)th and the \( (i+1) \)th sites with \( 1 + (-1)^i \delta + 0.15w \), where \( w \) is a random number distributed uniformly in \([-1, 1]\). Other parameters are the same as those in figure 2(d). The figure shows that disorder can remove degeneracies of the Andreev bound states but cannot destroy them.

Figure 2. The energy spectrum of a ring in which the upper half and lower semicircle are in different parameter regions. The total length is \( N = 400 \). Because of the particle–hole symmetry, we only plot the positive eigenenergies and omit their negative counterparts. The parameters are: (a) \( h = 0, \Delta = 0.1 \) and \( \delta = -0.2 \) in the upper semicircle and \( h = 0, \Delta = 0.1 \) and \( \delta = 0 \) in the lower semicircle; (b) \( h = 1.8, \Delta = 0.1 \) and \( \delta = 0.2 \) in the upper semicircle and \( h = h_1, \Delta = 0.1 \) and \( \delta = 0.2 \) in the lower semicircle; (c) \( h = 0, \Delta = 0.1 \) and \( \delta = 0.2 \) in the upper semicircle and \( h = 0, \Delta = \Delta_1 \) and \( \delta = 0.2 \) in the lower semicircle; (d) \( h = 0.6, \Delta = 0.1 \) and \( \delta = 0.4 \) in the upper semicircle and \( h = h_1, \Delta = 0.1 \) and \( \delta = 0.4 \) in the lower semicircle. The phases of the upper and lower semicircles of the ring are indicated by the colors: red for SSHTP, black for KTP and blue for trivial TP.

Figure 3. The energy spectrum of a disordered ring. Most of the parameters are the same as those in figure 2(d) except the randomized staggered hoppings \( t_1 = 1 + (-1)^i \delta + 0.15w \), where \( w \) is a random number in \([-1, 1]\).
3. Thouless pump and the nonuniversal fractional charge carried by each soliton

Similar to the standard SSH model, solitons are mobilizable. To electrically control its movement, the electric charge carried by each soliton should first be determined. In this section, with the help of the evolution of WFs during the Thouless pump, we can determine this charge accurately.

3.1. Wannier functions for the occupied bands

The most localized WFs for the occupied bands are defined as the eigenvectors of the tilde position operator $\tilde{R} = \hat{R}\hat{P}$, where $\hat{R}$ is the position operator and $\hat{P}$ is the projection on the occupied states \[24, 25\]. Here $\hat{P}$ can be written explicitly as $\hat{P} = \sum_{\alpha \in \text{occupied states}} | \alpha \rangle \langle \alpha |$ and in the lattice representation the position operator $\hat{R} = \tau_0$ is a diagonal matrix with the diagonal elements running through lattice sites from 1 to $N$. The eigenvalues of $\tilde{R}$, denoted as $R_s$, are the central positions of the WFs. One should recognize that these $R_s$ are meaningful only when the WFs are localized in the real space, which has been ensured when the band gap keeps open \[25\]. Here equation (2) has been extended to the particle and hole subspaces so that each unit cell contributes 2 WFs while in the standard SSH model, each unit cell gives only 1 WF. The additional set of WFs, as a result of extending the WF definition to the Nambu representation, is for the unoccupied bands, which are absent in the usual particle representation. The usual WFs obtained from the standard SSH model (not in the Nambu representation) during the Thouless pump are plotted in the panel (c) in figure 4 for comparison.

3.2. Thouless pump

The Thouless pump is introduced to the Hamiltonian by varying the staggered hopping and the on-site energy slowly with an extra parameter $\phi$ \[4, 24\]. Then the Hamiltonian becomes

$$H(\phi) = \sum_i \left[ 1 + (-1)^i \delta \cos(\phi) \right] (c_i^\dagger c_{i+1} + h. c.) + \sum_i (-1)^i h_{it} \sin(\phi) c_i^\dagger c_i$$

$$+ \hbar \sum_i c_i^\dagger c_i + \Delta \sum_i (c_i^\dagger c_{i+1}^\dagger + h. c.).$$

Figure 4. The energy spectrum (a) and the WFs centers (b) during the Thouless pump of $\phi$. The parameters are $h = 0, h_{st} = 0.3$, $\delta = -0.2$ and $\Delta = 0.1$. The colors of the points in panel (b) represent the particle weights of the corresponding WFs in the particle–hole subspace with scale given by the right palette, where 1 (blue) means that the WF contains only one particle while 0 (red) indicates that the WF is completely hole like. (c) For comparison, the conventional WFs for a standard SSH model during the Thouless pump are shown. (d) A schematic shows the two kinds of layouts of $\phi(x)$ along chains in the arguments in the next subsection.

This pump extends the 1D momentum space, $k_x$, of the 1D model to an artificial 2D momentum space, spanned by $k_x$ and $\phi$. The bulk topological nature as well as the boundary states can be understood within this new scenario in considering the evolution of the center of WFs with $\phi$. These evolutions can reveal the Chern
number or the $Z_2$ invariant in the 2D space like that in the quantum Hall or spin Hall systems[26–29]. In this paper, we raise a new application of these evolutions in determining the electric charge carried by each soliton.

### 3.3. Evolution of the centers of WFs during the Thouless pump

In figure 4, we plot the energy spectrum (a) and the centers of WFs (b) as functions of $\phi$ for an $N = 200$ chain with open boundary condition. As figure 4(a) shows, the chain is in the trivial TP with no boundary state around $\phi = 0$. As $\phi$ is varies, the chain is pushed to the SSHTP near $\phi = \pi$ and is then pulled back to the trivial TP at the end of a cycle. During this pump, the band gap at the Fermi energy remains open so that the WFs we calculated are localized and their centers shown in (b) are reliable. One should not confuse the above statement about the band gap with the argument that topology can only be changed by closing a gap. We have extended the model with an extra parameter $\phi$ by the Thouless pump, so that the restriction on the original model about band closing accompanied with topological change can be bypassed in the extended model.

As the model is represented in the Nambu representation, an extra new set of WFs corresponding to the bands of holes emerges. In figure 4(b) we use the colors blue and red to illustrate the weights of the WFs on the particle and hole subspaces, respectively. Blue (red) points in the panel illustrate that WFs centered at the positions are particle (hole) like functions. Panel (b) shows that the particle-like WFs move right accompanied with the hole-like WFs that are moving left during the pump.

### 3.4. Electric charge carried by each soliton

For the standard SSH model, there is a counting formula describing the total number $M$ of the unoccupied zero energy states in term of electric charge $Q$ carried by each topological impurity (including both domain wall and geometric end), $M = -2Q \mod 2$ [30]. The fractional charge carried by each soliton is a direct consequence of this equation (when $M = 1, Q = \frac{1}{2}$) [4, 20, 31]. Turning on the superconducting coupling will not break the charge conjugation symmetry and the counting formula should survive. But the electric charge $Q$ must be replaced by the conserved quasiparticle charge $Q_{\text{BdG}}$ [30] because the global electromagnetic gauge invariance is broken in the BdG mean field Hamiltonian. It should be emphasized that $Q_{\text{BdG}}$ is a topological character in the present case and has nothing to do with the actual electric charge $Q$ anymore. So we need to figure out the actual charge $Q$ carried by each soliton in order to electrically control their motion.

We use a thought experiment to figure out the actual electric charge $Q$ as well as the conserved BdG charge $Q_{\text{BdG}}$ carried by each soliton. Suppose that there is an infinite chain with the Hamiltonian of equation (3). The parameter $\phi$ is not fixed, but varies slowly along the chain as $\phi(x)$. The total variation of $\phi$ along the chain is $2\pi$. Without loss of generality, we let $\phi(-\infty) = 0$ and $\phi(+\infty) = 2\pi$. As $\phi(x)$ is varying slowly (shown in the upper panel in figure 4(d)), the positions of WFs given by figure 4(b) are still valid in each microscopic large but macroscopic small region in which $\phi(x)$ is almost uniform. Now we want to compare the positions of WFs in this chain with those in a uniform chain in which $\phi(x)$ is fixed at 0 (the extended Hamiltonian equation (3) regresses to equation (1)). At the far left segments of the chains, the positions of WFs in the two chains are the same because $\phi(x) = 0$ in both regions. As $x$ increases, the WFs are moved slightly away from the positions for the uniform chain because $\phi(x)$ increases. As figure 4(b) shows, particle-like WFs are misaligned in the $x$ direction accompanied with hole-like WFs misaligned in the inverse direction. These deviations keep increasing with $x$ and reach the length of one unit cell in the far right region in which $\phi(x) = 2\pi$. We conclude that, compared with the uniform chain, the chain with varying $\phi(x)$ loses a particle-like WF and gains a hole-like WF. This can also be regarded as $\phi(x)$ pushing out a particle-like WF and pulling in a hole-like WF at the right end of the chain at $x = \infty$. Now we relax the restriction that $\phi(x)$ is varying slowly with $x$. This relaxation does not alter the above conclusion because the local variation at finite $x$ will not affect the physics at $x = \infty$. We choose a new layout of $\phi(x)$ as $\phi(x) = 0$ when $x < -L, \phi(x) = \pi$ when $L > x > -L$ and $\phi(x) = 2\pi$ when $x > L$, where $L$ is a large but finite number (shown in the lower panel in figure 4(d)). The new layout represents a chain with a pair of domain wall and anti-domain wall at $\pm L$. We know that each domain wall (anti-domain wall) has a soliton state on it. So these two soliton states must take the responsibility for the lost and gained WFs. As a result, each soliton takes half of the total lost charges, $Q = -\frac{1}{2}\left[\langle \Psi_{\text{Particle-like WF}} | \hat{\rho} | \Psi_{\text{Particle-like WF}} \rangle - \langle \Psi_{\text{Hole-like WF}} | \hat{\rho} | \Psi_{\text{Hole-like WF}} \rangle \right]$, where $\hat{\rho}$ is the single particle density operator $\hat{\rho} = \sum_i c_i^\dagger c_i$ and $\langle \Psi_{\text{Particle (Hole-)like WF}} \rangle$ is a wave function of the particle (hole-) like WF. For a chain with the parameters shown in figure 4(a), this charge $Q$ is still $\frac{1}{2}$. As to the conserved BdG charge $Q_{\text{BdG}}$, this conserved charge counts that how many WFs have been pushed out from the chain with one domain wall. From the above argument, we see it is universal $\frac{1}{2}$.

We have calculated the energy spectrum as well as the motions of WFs during the Thouless pump for a chain with nonzero external potential, $h = 0.3$, for comparison. Other parameters are the same as those in figure 4. We find that $h$ lifts the energies of the soliton states at $\phi = \pi$ but the motions of WFs are same as those shown with
$h = 0$. But these WFs have been altered from the pure particle-like or hole-like functions to the mixed functions.

We can count the total charge being pumped out and find that it is 0.97 instead of 1. As a result, each soliton in this model carries nonuniversal fractional charge 0.485. So we can conclude that the charge carried by each soliton is not a universal fractional number but depends on $h$ when $h \neq h_0$.

4. Manipulating the Majorana qubit by moving the soliton

In the present model, there are two kinds of topological impurity, solitons and MFs. We will show that when coupling these two kinds of topological impurity together, we can change the state of the Majorana qubit by adiabatically moving a soliton along the chain. In the following we concentrate on the two MFs taking part in the coupling and ignoring the two uncoupling MFs at the other far ends.

4.1. The heterostructure

As the inset in figure 5(a) shows, only a part of a circular polyacetylene is placed on the surface of a superconductor so that one part of the circle is in the KTP (the black arc) and the rest is in the SSH phase (the red arc). We suppose that there is a domain wall (sketched by the blue point) in the circle. In our numerical calculation, this domain wall is simulated by two adjacent stronger bonds with hoppings $t + \delta$ (the domain wall can also be simulated with two adjacent weaker bonds). It will induce a soliton state when the domain wall is in the SSH region. It should be noted that owing to the existence of the domain wall the total length of circle is an odd number instead of an even one. There are two MFs (indicated by the green points) at the joints of the arcs.

4.2. Level crossing between Majorana zero energy state and soliton state

Figure 5(a) shows the energy spectrum of a heterostructure ring shown in the left inset. $r$ is the position of the domain wall (soliton) anticlockwise measured from the initial position shown in the inset. The length of the ring that hosts the domain wall is $N = 201$. The part of the ring, from the sites $i = 100$ to $i = 120$ is in the SSH phase (red) and the rest is in the KTP (black). MFs appearing at the joints are indicated by the green points. The parameters are $h = 0, \delta = 0.2, \Delta = 0$ for the part in the SSH phase and $h = 0, \delta = 0.2, \Delta = 0.4$ in KTP. The length of the SSH region is 20. The right inset shows the wavefunctions (only the parts in the particle subspace) of the two in-gap states when $r = 105$. The SSH region, from $i = 80$ to $i = 120$, has been enlarged for the sake of clarity. (b) The robust level crossing for a disordered chain. The parameters are the same as those in (a) except for the randomized staggered hoppings. (c) The level crossing becomes anti-crossed when $h = 0.1$. 

$h = 0$. But these WFs have been altered from the pure particle-like or hole-like functions to the mixed functions. We can count the total charge being pumped out and find that it is 0.97 instead of 1. As a result, each soliton in this model carries nonuniversal fractional charge 0.485. So we can conclude that the charge carried by each soliton is not a universal fractional number but depends on $h$ when $h \neq 0$. 

Figure 5. (a) The energy spectrum of a heterostructure ring shown in the left inset. $r$ is the position of the domain wall (soliton) anticlockwise measured from the initial position shown in the inset. The length of the ring that hosts the domain wall is $N = 201$. The part of the ring, from the sites $i = 100$ to $i = 120$ is in the SSH phase (red) and the rest is in the KTP (black). MFs appearing at the joints are indicated by the green points. The parameters are $h = 0, \delta = 0.2, \Delta = 0$ for the part in the SSH phase and $h = 0, \delta = 0.2, \Delta = 0.4$ in KTP. The length of the SSH region is 20. The right inset shows the wavefunctions (only the parts in the particle subspace) of the two in-gap states when $r = 105$. The SSH region, from $i = 80$ to $i = 120$, has been enlarged for the sake of clarity. (b) The robust level crossing for a disordered chain. The parameters are the same as those in (a) except for the randomized staggered hoppings. (c) The level crossing becomes anti-crossed when $h = 0.1$. 

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The soliton state will appear in the gap and its energy decreases when the domain wall is moved into the SSH region, while the zero energy state is increased in energy. These two energy levels cross when the domain wall passes the center of the SSH region. Because of this level crossing, particle occupations on the zero energy state and on the soliton state will exchange after the wall passing the SSH region. This mechanism can modify the Majorana zero energy state with a complementary soliton,

\[ n_{ZE}^f(n_{ZE}^s) \leftrightarrow n_{ZE}^s(n_{ZE}^f) \]

where \( n_{if} \) is the number of particles on the MF zero energy (soliton) state before (after) the movement of the domain wall.

### 4.3. Robust level crossing against disorder

In figure 5(b), we plot the energy spectrum for a disordered ring. The parameters are the same as those in (a) except that the staggered hoppings are randomized, \( t_i = 1 + (-1)^i 5 + 0.1 w \), where \( w \) is a random number in \([-1, 1]\). The figure shows that the level crossing is preserved even when the lattice disorder is introduced. In figure 5(c), we show that the level crossing can be destroyed by a nonzero external potential \( h \). The parameters are same as those in (a) except \( h = 0.1 \).

The above crossing and anti-crossing effects can be understood by expressing the effective Hamiltonian for the low energy states (inside the band gap) in the Majorana representation. The MFs at the joints are denoted by \( \nu_1 \) and \( \nu_4 \) and the soliton state is regarded as a combination of two MFs, denoted by \( \nu_1 \) and \( \nu_2 \). For the effective Hamiltonian spanned by these 4 Majorana states, the coupling between \( \nu_1 \) and \( \nu_2 \) is proportional to the energy of the soliton state \( E_s \), while the coupling between \( \nu_1 \) and \( \nu_3 \) (or \( \nu_2 \) and \( \nu_4 \)) is proportional to \( e^{-\alpha d_{13}} \) (or \( e^{-\alpha d_{24}} \)), where \( \alpha \) is proportional to the band gap of the SSH phase and \( d_{13} \) (or \( d_{24} \)) is the distance between the MFs \( \nu_1 \) and \( \nu_3 \) (or \( \nu_2 \) and \( \nu_4 \)). When \( h = 0 \) and the domain wall is moved deeply to the SSH region, the finite size effect can only lift up the energy of the soliton state by \( e^{-\alpha L} \), where \( L \) is the length of the SSH region. This is much smaller than the coupling between \( \nu_1 \) and \( \nu_3 \) (or \( \nu_2 \) and \( \nu_4 \)) because \( d_{13} \) and \( d_{24} \) are in the scale of \( \frac{L}{2} \). So the level crossing must be protected by the leading couplings in the effective Hamiltonian. But when \( h \neq 0 \), the energy of the soliton state in the SSH region is nonzero intrinsically and the coupling between \( \nu_1 \) and \( \nu_2 \) must be considered. So the crossing is no longer protected.

### 4.4. NOT operator on the Majorana qubit

The above level crossing implies a mechanism to manipulate MF qubits by moving the soliton. A MF qubit needs four MFs. Let us denote their MF operators as \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \). The heterostructure extended from that has been discussed in figure 6 with the SSH region (red) and the KTP region (black). The four MFs are illustrated by 1, 2, 3 and 4 in the figure. Our discussion is in fermion representation with the complex fermion creation operator.

\[ \text{Figure 6. How the NOT operation works for a MF qubit. The red region is in the SSH phase and the black region is in the KTP. The domain wall is the blue point. The quantum state in each case is written at the center of each circle. The four rows indicate the four kinds of initial state of soliton and MF qubit.} \]

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\[ ^5 \] When we use soliton state to mention the domain wall, the wall is in the SSH region by default.
\[ \psi^+_{\alpha} = (\mu_1 + i\mu_2)/2 \text{ and } \psi^+_{\beta} = (\mu_3 + i\mu_4)/2. \]
The quantum states of the MF qubit are illustrated as \(|n_{\alpha} n_{\beta}\rangle\), where \(n_{\alpha}\) and \(n_{\beta}\) can take 0 (empty) and 1 (occupied). One can define the two states of a MF qubit as ‘0’ state: \(|10\rangle\) and ‘1’ state: \(|01\rangle\) [3, 14, 16].

We can extend the above notation of quantum states to include the occupation on the soliton state as \(|n_{\alpha} n_{\beta} n_{\text{soliton}}\rangle\). The states shown in figure 6 are in this extended form. Now we show explicitly how the NOT operation works. Suppose that the initial state is \(|01\rangle\) with the domain wall at the point in the ‘3 o’clock direction’. After moving it half a circle to ‘9 o’clock’, \(n_{\beta}\) and \(n_{\text{soliton}}\) exchange due to the level crossing, so we get the state \(|00\rangle\). After moving the domain wall a half circle back to ‘3 o’clock’, \(n_{\alpha}\) and \(n_{\text{soliton}}\) exchange and we get \(|10\rangle\). Finally the domain wall is moved an extra one half circle to the point in the ‘9 o’clock direction’ and the state remains at \(|10\rangle\). This is a NOT operation that changes from a ‘1’ state to ‘0’ state for a MF qubit. In the next three rows in the figure, we show explicitly that this NOT operation is independent of the initial state of \(n_{\text{soliton}}\) and works well for the case, ‘0’ to ‘1’. In this way, the above NOT operation also applies when the Majorana qubit is in a superposition state. One can suppose an initial superposition Majorana qubit \(a|01\rangle + b|10\rangle\). After moving the soliton one and a half circles, one will get \(a|10\rangle + b|01\rangle\) which is the NOT state of the initial state.

There is a mechanism that introduces errors to the NOT operation: the quasi-particle on the soliton state may spontaneously jump to the empty zero energy state. When the soliton moves into the KTP region, its state does not stay in the gap but merges into the band states. This extended state in the KTP region, whose length is \(L\), will turn on the spontaneous jumps with a rate of the possibility in the scale of \(1/L\) because the MFs are still localized at the joints. So the movement of soliton in the KTP region should be fast to decrease the total possibility of jumping. We know that the effective mass of the soliton in polyacetylene is only six times larger than the electron mass. The light soliton makes it possible to increase its speed. Incidentally, a fast soliton can be in favor of the NOT operation because the Landau–Zener effect induced by a fast soliton favors the occupation exchange during the level crossing even when the level crossing is anti-crossed by a small \(\hbar\) [32, 33]. In the fast moving case, one can replace the \(r\) axis in figure 5 with time \(t\). This refers to the fact that the Hamiltonian becomes time-dependent when the adiabatic approximation is not good. As the soliton state crosses with the Majorana zero energy state in the SSH region and with many band states in the KTP region, The Landau–Zener effect will cause it to remain in its state as long as the Hamiltonian is varying faster than \((\Delta E)^2/\hbar\), where \(\Delta E\) are the anticrossing gaps opened by, for instance, disorders, at the crossing points.

The above manipulation of the Majorana qubit is a complementary operator to the braiding operators. For a system in a network with \(2\pi\) MFs (with at least \(2\pi\) geometric ends), braiding MFs will introduce a non-Abelian unitary transformation in the Hilbert space spanned by the degenerated ground states. But because the braiding operators conserve the parity of the number of fermions, they should operate in two independent \(2^{n-1}\) dimensional sub-spaces (corresponding to even and odd parities, respectively) [3, 14]. But moving with the soliton on the domain walls to the proper positions in the network and controlling the quantum state, one is able to entangle the qubits on the solitons with the qubits on the MFs. Many desired operations, such as the NOT operator, can be achieved with this process. And the dimension of the achievable Hilbert space of the ground states for the MF qubits has been enlarged from \(2^{n-1}\) to \(2^n\) without breaking the total parity conservation.

5. Discussion

We propose a NOT operation on Majorana qubits through moving the complementary soliton. The two ingredients of the operation are both topological impurities so that local disorder cannot influence it. The advantage of the present proposal is that the desired operators can be realized by moving the solitons to the proper positions in a network hosting several MF qubits, differently from other schemes where the nontopological qubits are fixed. This could make the setup more flexible and scalable.

We theoretically propose that this kind of network can be fabricated by etching the superconducting metal plane to a desired pattern and placing the prepared polyacetylene network on it. Thus at low temperature, the parts on the superconducting metal are in the KTP while the other impending parts are in the SSHTP. The finite-size induced lift of Majorana zero energy restricts the typical length of each part to longer than several \(\mu m\). The maximal speed of soliton in polyacetylene is on the same scale as that of sound, which is about 0.015 \(\mu m\) \(ps^{-1}\) [20]; so the typical speed of the operator is in the scale of \(10^7\)\(s^{-1}\). Although our discussion is based on a ring, the key requirement to entangle the soliton with MFs is that the soliton should be moved through the SSHTP region in between the two KTP regions. The local properties near the joints, such as a stronger or weaker bond at the joints, cannot affect the crossing of the two topological impurities shown in figure 5. As the electric charge carried by the soliton may flip the sign when its state changes from empty to occupied, we propose the use of thermal gradient [34], instead of electric field, to control its motion. In experiment, one may use a laser facula, which would induce a local thermal gradient, to guide the motion of the soliton.
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