About the impossibility to prove $P \neq NP$ or $P = NP$ and the pseudo-randomness in $NP$

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Abstract

The relationship between the complexity classes $P$ and $NP$ is an unsolved question in the field of theoretical computer science. In this paper, we look at the link between the $P - NP$ question and the "Deterministic" versus "Non Deterministic" nature of a problem, and more specifically at the temporal nature of the complexity within the $NP$ class of problems. Let us remind that the $NP$ class is called the class of "Non Deterministic Polynomial" languages. Using the meta argument that results in Mathematics should be "time independent" as they are reproducible, the paper shows that the $P \neq NP$ assertion is impossible to prove in the a-temporal framework of Mathematics. In a previous version of the report, we use a similar argument based on randomness to show that the $P = NP$ assertion was also impossible to prove, but this part of the paper was shown to be incorrect. So, this version deleted it. In fact, this paper highlights the time dependence of the complexity for any $NP$ problem, linked to some pseudo-randomness in its heart.

Index Terms

Algorithm Complexity, Non Deterministic Languages, $P - NP$ problem, 3-CNF-SAT problem

I. Introduction

A. The class $P$ of languages

A decision problem is a problem that takes as input some string, and outputs "yes" or "no". If there is an algorithm (say a Turing machine, or a computer program with unbounded memory) which is able to produce the correct answer for any input string of length $n$ in at most $c n^k$ steps, where $k$ and $c$ are constants independent of the input string, then we say...
that the problem can be solved in polynomial time and we place it in the class \( P \).

More formally, \( P \) is defined as the set of all languages which can be decided by a deterministic polynomial-time Turing machine. Here we follow the framework proposed by Stephen [1]. Let \( \Sigma \) be a finite alphabet with at least two elements, and let \( \Sigma^* \) be the set of finite strings over \( \Sigma \). Then a language over \( \Sigma \) is a subset \( L \) of \( \Sigma^* \). Each Turing Machine \( M \) has an associated input alphabet \( \Sigma \). For each string \( w \) in \( \Sigma^* \), there is a computation associated with \( M \), with input \( w \). We say that \( M \) accepts \( w \) if this computation terminates in the accepting state “Yes”. Note that \( M \) fails to accept \( w \) either if this computation ends in the rejecting state “No”, or if the computation fails to terminate.

The language accepted by \( M \), denoted \( L(M) \), has associated alphabet \( \Sigma \) and is defined by

\[
L(M) = \{ w \in \Sigma^* | M \text{ accepts } w \}
\]

We denote by \( t_M(w) \) the number of steps in the computation of \( M \) on input \( w \). If this computation never halts, then \( t_M(w) = \infty \). For \( n \in \mathbb{N} \), we denote by \( T_M(n) \) the worst case run time of \( M \); that is

\[
T_M(n) = \max \{ t_M(w) | w \in \Sigma^n \}
\]

where \( \Sigma^n \) is the set of all strings over \( \Sigma \) of length \( n \). We say that \( M \) runs in polynomial time if:

\[
\exists k \in \mathbb{N} \text{ such that } \forall n : T_M(n) \leq n^k + k
\]

**Definition I.1:** We define the class \( P \) of languages by

\[
P = \{ L | L = L(M) \text{ for a machine } M \text{ which runs in polynomial time} \}
\]

**B. The class \( NP \) of languages**

The notation \( NP \) stands for *non deterministic polynomial time*, since originally \( NP \) was defined in terms of non deterministic machines. However, it is customary to give an equivalent definition using the notion of a *checking relation*, which is simply a binary relation \( R \subseteq \Sigma^* \times \Sigma_1^* \) for some finite alphabets \( \Sigma \) and \( \Sigma_1 \). We associate with each such relation \( R \) a
language $L_R$ over $\Sigma \cup \Sigma_1 \cup \{\#\}$ defined by

$$L_R = \{w\#y | R(w, y)\}$$

where the symbol $\#$ is not in $\Sigma$. We say that $R$ is polynomial-time iff $L_R \in P$.

**Definition I.2:** We define the class $NP$ of languages by the condition that a language $L$ over $\Sigma$ is in $NP$ iff there is $k \in \mathbb{N}$ and a polynomial-time checking relation $R$ such that for all $w \in \Sigma^*$,

$$w \in L \Leftrightarrow \exists y(|y| \leq |w|^k \text{ and } R(w, y))$$

where $|w|$ and $|y|$ denote the lengths of $w$ and $y$, respectively. We say that $y$ is a certificate associated to $w$.

**C. The $P$ - $NP$ question**

The “$P$ versus $NP$ problem”, i.e. the question whether $P = NP$ or $P \neq NP$, is an open question and is the core of this paper. See [4] for the history of the question. Here, we show that neither $P = NP$ nor $P \neq NP$ can be proved in the “a-temporal” framework of Mathematics where results should always be reproducible. We link this assertion to the existence of some pseudo-random part in the heart of any $NP$ problem.

**D. An example of $NP$ problem : the 3-CNF-satisfiability problem**

Boolean formulae are built in the usual way from propositional variables $x_i$ and the logical connectives $\land$, $\lor$ and $\neg$, which are interpreted as conjunction, disjunction, and negation, respectively. A literal is a propositional variable or the negation of a propositional variable, and a clause is a disjunction of literals. A Boolean formula is in conjunctive normal form iff it is a conjunction of clauses.

A 3-CNF formula $\varphi$ is a Boolean formula in conjunctive normal form with exactly three literals per clause, like $\varphi := (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3 \lor \neg x_4) := \psi_1 \land \psi_2$. The 3-CNF-satisfiability or 3-CNF-SAT problem is to decide whether there exists or not logical values for the literals so that $\varphi$ can be true (on the previous example, $\varphi = 1(True)$ if $x_1 = \neg x_2 = 1$).

Until now, nobody knows whether or not it is possible to check the satisfiability of any given
3-CNF formula $\varphi$ in a polynomial time, as the 3-CNF-SAT problem is known to belong to the class $\text{NP}$ of problems. See [2] for details.

Let us give some general properties of the 3-CNF formulae.

The size $s$ of a 3-CNF formula $\varphi$ is defined as the size of the corresponding Boolean circuit, i.e. the number of logical connectives in $\varphi$. Let us note the following property of the size $s$:

$$s = \mathcal{O}(m) = \mathcal{O}(n^3)$$ (1)

where $n$ is the number of propositional variables $x_i$ and $m$ the number of clauses in $\varphi$.

Indeed,

$$\frac{n}{3} \leq m \leq 2^3 \frac{n(n-1)(n-2)}{3 \times 2}$$

and

$$(3m - 1) \leq s \leq (6m - 1)$$

as there is a maximum of $2^3 \times C_n^3$ possible clauses which corresponds to the choice of 3 different variables among $n$, each of them being in an affirmative or negative state. Note that $s = 3m - 1$ when there is no “$\neg$” in $\varphi$ [m × 2 logical connectives “$\lor$” for the $\psi_i$ and $m - 1$ “$\land$” as conjunctions] and $s = 6m - 1$ when all the literals in $\varphi$ are in a negative form.

In this paper, we define the dimension $d$ of a 3-CNF formula as $(n, m)$. And we represent any 3-CNF formula by a matrix $A$ of size $2n \times m$. The signature $u_i$ of a clause $\psi_i$ is defined as the value of the binary number corresponding to the row in the matrix. The signature of a formula is the ordered vector of these clause’s signatures: $\varphi_{n,m} \approx (u_1, u_2, \ldots, u_m)$ with $21 \leq u_i \leq 21 \cdot 2^{2n-5}$ and $u_i > u_j$ for $i < j$. See Table I.

| 3-CNF formula $\varphi$ (dimension $d = (4,3)$) |
|-------------------------------------------------|
| $\psi_1$ : $(x_1 \lor x_2 \lor \neg x_3)$ | $x_1$ | $\neg x_1$ | $x_2$ | $\neg x_2$ | $x_3$ | $\neg x_3$ | $x_4$ | $\neg x_4$ | $u_1$ |
| $\psi_2$ : $(\neg x_2 \lor x_3 \lor \neg x_4)$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 164 |
| $\psi_3$ : $(\neg x_1 \lor \neg x_3 \lor x_4)$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 70 |

**Table I**

**Example of matrix representation and signatures of a 3-CNF formula.**
There are $2^3 \times C^3_n$ possible clauses with $n$ variables. A 3-CNF formula with dimension $(k, m)$ with $k \leq n$ is composed of $m$ different clauses drawn from the $2^3 \times C^3_n$ possible clauses. So, the total number of such formulae is

$$C_m^{2^3 \times C^3_n} = \frac{(2^3 \times C^3_n)!}{m! \times (2^3 \times C^3_n - m)!} = \mathcal{O}(n^{3m})$$

(2)

Let $\Phi_{n,m}$ denote the set of all these formulae:

$$\Phi_{n,m} = \{ \varphi : \varphi \text{ is a 3-CNF formula of dimension } (k, m) \text{ with } k \leq n \}$$

The 3-CNF-Satisfiability problem is to find a function $\Xi$:

$$\Xi : \Phi_{n,m} \rightarrow \{0, 1\}$$

(3)

$$\varphi \leadsto 0 \text{ if } \varphi \text{ is non satisfiable and } 1 \text{ otherwise}$$

The 3-CNF-Satisfiability problem is known to belong to the $NP$ class.

II. A “Meta Mathematical” proof that $P \neq NP$ is impossible to prove

One way to prove that $P \neq NP$ is to show that the complexity measure $T_M(n)$ for some $NP$ problem, like the 3-CNF-SAT problem, cannot be reduced to a polynomial time. We will show that the 3-CNF-SAT problem behaves as a common safe problem and that its complexity is time dependent. In fact, at some specific time $t_0 + \Delta t$, the 3-CNF-SAT problem will be of polynomial complexity. So, $P \neq NP$ will not be provable, as $T_M(n)$ is not “always” supra-polynomial.

A. The analogy with the safe problem and the time dependent nature of complexity

Finding whether or not a given 3-CNF formula $\varphi$ is satisfiable is like being in front of a safe, trying to find the opening combination. One has to try any possible value (0 or 1) for the variable $x_i$ in $\varphi$ to see whether some combination satisfies $\varphi$, in the same way as one tries any combination to get the one, if it exists, that opens the safe.

Let us consider more deeply the analogy between the 3-CNF-SAT problem and the safe
problem, especially by looking to the time dependent nature of the complexity involved here.

It is clear that when you are in front of a safe for the first time, it is a very hard problem, as you do not have any information about the correct opening combination. In fact, in the worst case, it takes an exponential time to find it. But as soon as you have succeeded in opening the safe (or in finding that there is no solution), the problem becomes trivial. It takes only one operation to open the safe or to declare it impossible to open.

Let us denote by $t_0$ the first time you try to open the safe, and by $\Delta t$ the time needed to find the solution. Let us remark that $\Delta t$ can be huge but it is always finite as the number of possible combinations is finite. Now we compute the complexity measure $T_{safe}(n)$ for the safe problem at $t_0$ and $t_0 + \Delta t$.

In $t_0$, one has to test all possible combinations. If the safe has $n$ buttons with only two positions (0 or 1), there will be $2^n$ possibilities. Because no information is available about the solution, there is no way to reduce the number of cases to be tested. The exponential complexity of the problem comes from the total lack of information about the solution. This absence of information is strictly related to the random nature of the problem: the finding of the opening combination is a random search process for anyone in front of the safe, at least in $t_0$. So, we get

$$T_{safe, t_0}(n) = 2^n$$

But after $\Delta t$, the correct opening combination is known forever, and the complexity measure is now

$$T_{safe, t_0 + \Delta t}(n) = 1$$

As one can see, the complexity measure $T_{safe}(n)$ for the safe problem is time dependent.

The same occurs for the 3-CNF-SAT problem as well as for any NP problem. Their complexity measure changes in time. The idea of this section about the impossibility to prove $P \neq NP$ is to show that, even if $T_{3-CNF-SAT, t_0}(n)$ is not known (exponential or polynomial ?), there exists some $\Delta t$, even huge, such that the complexity measure is polynomial in $t_0 + \Delta t$. 

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B. The Computation of $T_{3-CNF-SAT, t_0 + \Delta t(n)}$

Let us take $\Delta t$ large enough so that $\Xi$ [the 3-CNF-SAT decision function, see equation (3)] is known for all the 3-CNF formulae in $\Phi_{n,m}$. $\Delta t$ exists and is finite. In the analogy with the safe problem, it corresponds to the time needed to find the solution for all safe equipments of dimension $n$. Until now, we do not know whether $\Xi$ can be computed in polynomial time or not, but this only changes the size of $\Delta t$.

The output of $\Xi$ is the set $\mathcal{S}_{n,m}$ of all satisfiable 3-CNF formulae of $\Phi_{n,m}$, or equivalently $\mathcal{F}_{n,m} = \Phi_{n,m} \setminus \mathcal{S}_{n,m}$, the set of all non satisfiable 3-CNF formulae. As equation (2) shows, $\mathcal{S}_{n,m}$ contains at most $O(n^{3m})$ elements. The worst case occurs when $m = (2^3 \times C_n^3)/2 = O(n^3)$. As $\mathcal{S}_{n,m} \subseteq \Phi_{n,m}$, the equation (2) gives us the following result:

$$\#\{\mathcal{S}_{n,m}\} < \#\{\Phi_{n,m}\} = O(n^{3(n^3)}) \Rightarrow \#\{\mathcal{S}_{n,m}\} = O(2^{n^3}) \text{ as } n^3 > 2$$

(4)

See Figure 1 for an example of $\#\{\Phi_{n,m}\}$ and $\#\{\mathcal{S}_{n,m}\}$ with $n = 4$. The figure shows that $\#\{\Phi_{n,m}\}$ and $\#\{\mathcal{S}_{n,m}\}$ behaves similarly.

So, one can now calculate $T_{3-CNF-SAT, t_0 + \Delta t(n)}$: it is the time required to check whether a specific 3-CNF formula belongs or not in $\mathcal{S}_{n,m}$, after $\Delta t$ large enough for the entire set $\mathcal{S}_{n,m}$ to be computed. If one can allocate an exponential space for memory to save the elements of $\mathcal{S}_{n,m}$ (as accepted in Turing machines), then a hash algorithm, based on the clause’s signatures, can be used to see whether a 3-CNF formula $\varphi$ belongs or not to the set $\mathcal{S}_{n,m}$.

For instance, one can use $u_i$, the $i^{th}$ ordered signature of clauses, as the $i^{th}$ successive hash function $h_i(\varphi)$. It takes $O(2m)$ operations to compute each of these $m$ clause’s signatures of $\varphi$ and $O(m \log m)$ computations to sort them. We need then $O(2^3 \times C_n^3)$ operations, which corresponds to the maximum number of possible values for the signatures, to find whether the signature belongs or not to the corresponding section of $\mathcal{S}_{n,m}$ where the formulae are also ordered, in a lexical ordering, following their clause’s signatures. Using equation (1) [i.e. $O(m) = O(n^3)$],

$$T_{3-CNF-SAT, t_0 + \Delta t(n)} = O(m(2n) + (m \log m) + m(2^3 C_n^3))$$

$$= O(m^2) = O(n^k) \text{ for some } k \in \mathbb{N}$$

(5)

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Logarithmic scale: the upper curve represents the total number of all possible 3-CNF in $\Phi_{4,m}$; the second one, the total of non-satisfiable 3-CNF, i.e. $\#\{\mathcal{S}_{4,m}\}$, and the lower one, the total of Irreducible Non-Satisfiable 3-CNF, i.e. $\#\{\mathcal{S}^{INS}_{4,m}\}$ (i.e. 3-CNF satisfiable with m-1 clauses).

C. The “unprovability” of $P \neq NP$

Theorem II.1: It is impossible to prove that $P \neq NP$ in the deterministic or time independent framework of Mathematics.

Proof: The solution of the 3-CNF-SAT problem is equivalent to the setting of these two functions $\Xi'$ and $\Xi''$:

$$\Xi': \Phi_{n,m} \xrightarrow{O(t_0)} \{0,1\} \quad \text{(the construction of } \mathcal{S}_{n,m})$$

$$\varphi \mapsto 0 \text{ if } \varphi \in \mathcal{S}_{n,m} \text{ and } 1 \text{ otherwise} \quad \text{(6)}$$

$$\Xi'': \Phi_{n,m} \xrightarrow{O(t_0+\Delta t)} \{0,1\} \quad \text{(for } \varphi \in \mathcal{S}_{n,m} \text{ when } \mathcal{S}_{n,m} \text{ is known)}$$

$$\varphi \mapsto 0 \text{ if } \varphi \in \mathcal{S}_{n,m} \text{ and } 1 \text{ otherwise} \quad \text{(7)}$$
The meta mathematical argument lies in the fact that any operation done by $\Xi'$ in $t_0$ can be reduced to a polynomial time operation by $\Xi''$ in $t_0 + \Delta t$.

Mathematically speaking, it is impossible to make a formal or mathematical distinction between both functions $\Xi'$ and $\Xi''$, as time does not interfere with proofs in mathematics. More precisely, if someone proves that the 3-CNF-SAT problem $\Xi$ (or $\Xi'$) is non polynomial, this assertion, as well as the steps for the demonstration, should be true at any time, independently of $t$, even in $t_0 + \Delta t$. The proof could not introduce time in the demonstration. But people will only be able to prove the non polynomial nature of 3-CNF-SAT for time $t_0$, certainly not for time $t_0 + \Delta t$ as shown in equation (5). And this argument holds for all $NP$ problems because all of them are equivalent, in term of complexity, to the 3-CNF-SAT problem.

This is exactly the same situation as with the safe problem: the complexity measure of the problem is changing over time, becoming polynomial after some large $\Delta t$. But the $P \neq NP$ question does not consider time as far as complexity is concerned: if we do not consider the time dependent nature of complexity, one should conclude that $P = NP$.

III. Conclusions

This paper tries to show that the $P \neq NP$ problem is impossible to solve within the time independent framework of Mathematics, as $P \neq NP$ can be proved without reference to time. The key concept of the paper is the temporal nature of the complexity measure for the $NP$—hard problems. This time dependence is closely related to some (pseudo) randomness in the heart of these problems. Some analogy can be found with the Chaos theory, when pseudo randomness arises from deterministic processes.

For the author, $NP$ is really different from $P$ but the difference lies in the distinction between true randomness and mathematical pseudo-randomness, and this frontier is situated

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To make it easier to understand, let us think of the version of 3-CNF-SAT with $n = 4$: it took us several months to build $S_{n,m}$, but now it only takes seconds to solve the 3-CNF-SAT problem with 4 variables. And this is done forever. A similar reasoning can be done for the $i^{th}$ decimal of $\pi$, or for the list of the $n$ first prime numbers.
on the limit border of Mathematics (which is deterministic).

The impossibility to prove that $P \neq NP$ gives a new perspective on the pseudo non deterministic (or random) nature of the most difficult problems, the $NP$—hard problems: we can see these problems as so inextricable that we are in front of them like someone facing some random search problem (as the safe problem), even if they are deterministic (not random) in their very essential nature, i.e. as quasi chaotic problems.

Therefore, the $P \neq NP$ “unprovability” can be seen as the expression of the incapacity for Mathematics to give a time independent definition of randomness.

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