Fluctuation and dissipation within a deformed holographic model with backreaction

Nathan G. Caldeira\textsuperscript{1,*}, Eduardo Folco Capossoli\textsuperscript{1,2,†},
Carlos A. D. Zarro\textsuperscript{1,‡} and Henrique Boschi-Filho\textsuperscript{1,§}

\textsuperscript{1}Instituto de Física,
Universidade Federal do Rio de Janeiro,
21.941-972 - Rio de Janeiro - RJ - Brazil

\textsuperscript{2}Departamento de Física and Mestrado Profissional em Práticas de Educação Básica (MPPEB),
Colégio Pedro II,
20.921-903 - Rio de Janeiro - RJ - Brazil

Abstract

In this work we study the fluctuation and dissipation of a string in a deformed and backreated AdS-Schwarzschild spacetime. This model is a solution of Einstein-dilaton equations (backreaction) and contains a conformal exponential factor $\exp(k/r^2)$ (deformation) in the metric. Within this Lorentz invariant holographic model we have computed the admittance (linear response), the diffusion coefficient, the two-point functions and the (regularized) mean square displacement $s_{\text{reg}}^2$. From this quantity ($s_{\text{reg}}^2$) we obtain the diffuse and ballistic regimes characteristic of the Brownian motion. From the two-point functions and the admittance, we also have checked the well know fluctuation-dissipation theorem.
I. INTRODUCTION

Brownian motion [1, 2] and the fluctuation-dissipation theorem [3] stand until today as two of the most important subjects within non-equilibrium statistical mechanics. Its intersections and contributions spread over many branches of science and in particular at high energy physics, such as, matter under extreme conditions or the quark-gluon-plasma (QGP) [4, 5]. In this case, the constituents of nuclear matter, under high temperature or density, present erratic trajectories due to their interactions with each other behaving like a Brownian motion. In this sense, by studying QGP one can investigate those phenomena. A very interesting approach to deal with non-perturbative aspects of strong interactions, which appear in such processes, is based on the AdS/CFT correspondence [6] which relates a weak coupling theory in a curved spacetime (AdS\(_5\)) with a strong coupling theory in four dimensional Minkowski spacetime. An incomplete list of references which dealt with Brownian motion, dissipation, fluctuation and related topics resorting to the AdS/CFT correspondence can be found, for instance, in Refs. [7–14].

In Ref. [15] the authors studied the fluctuation and the dissipation through an AdS/QCD model based on a deformation of the AdS-Schwarzschild spacetime. This deformation is due to the introduction of a conformal factor exp(\(k/r^2\)) in the metric of such a space. Then they computed the string energy, the response function, the mean square displacement, the diffusion coefficient and checked the fluctuation-dissipation theorem. This and related deformed AdS/QCD models were used successfully in many holographic problems as can be seen in Refs. [16–21].

Here, in this work we will use a Lorentz invariant deformed AdS/QCD model taking into account the backreaction in the metric of the AdS-Schwarzschild space. This will allow us to extend the work done in Ref. [15] and investigate the contribution of the backreaction on the admittance, the diffusion coefficient, the mean square displacement and the fluctuation-dissipation theorem in this set up. In previous studies [8–14], the backreaction was not considered.

This work is organized as follows: In Sec. II we present the Einstein-dilaton action, solving the corresponding field equations. From one of these field equations we obtain the backreacted horizon function. In Sec. III we describe holographically our model and study the effects of the backreaction. In Sec. IV we compute the mean square displacement,
which is necessary to compute the ballistic and the diffusive regimes and we also check the fluctuation-dissipation theorem in this set up. Finally, in Sec. V we present our conclusions and discussions.

II. THE EINSTEIN-DILATON ACTION AND THE DEFORMED ADS-SCHWARSCHILD SPACE WITH BACKREACTION

In order to capture all features of our deformed and backreacted space, let us start with a 5-dimensional Einstein-dilaton action in Einstein frame:

\[ S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - \frac{4}{3} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right), \]

where \( G_5 \) is the 5-dimensional Newton’s constant, \( g \) is the metric determinant, \( R \) is the Ricci scalar, \( \phi \) is the dilaton field and \( V(\phi) \) its potential. From this action one obtains the field equations

\[ G_{\mu\nu} - \frac{4}{3} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right) - \frac{1}{2} g_{\mu\nu} V(\phi) = 0, \]

\[ \nabla^2 \phi + \frac{3}{8} \frac{\partial V(\phi)}{\partial \phi} = 0, \]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. For our purposes, as done in Refs. [22, 23], we will consider the ansatz

\[ ds^2 = \frac{1}{\zeta(z)^2} \left( \frac{dz^2}{f(z)} - f(z) dt^2 + d\vec{x}^2 \right), \]

where \( z \) is the holographic coordinate, \( f(z) \) is the horizon function and \( \zeta(z) \) is the metric warp factor which we choose to be

\[ \zeta(z) = z e^{-\frac{1}{2} (kz^2)}, \]

with \( k \) being the deformation parameter.

Replacing the ansatz, Eq. (4), into the equations (2) and (3) one gets:

\[ \frac{d}{dz} \left( \zeta(z)^{-3} \frac{d}{dz} f(z) \right) = 0. \]

For simplicity, we will not show all other equations, which can be used to determine the dilaton field \( \phi(z) \) and its potential \( V(\phi) \), since they are not relevant for our analysis.
Substituting Eq. (5) into Eq. (6), satisfying \( f(0) = 1 \) and the horizon property \( f(z_h) = 0 \), one can solve it analytically, so that:

\[
f(z) = 1 - \left( \frac{3k z^2 - 2e^{\frac{3}{2}kz^2} + 2}{3k z^2 - 2e^{\frac{3}{2}kz^2} + 2} \right) e^{\frac{3}{2}k(z_h^2 - z^2)}.
\] (7)

This is the horizon function with backreaction. One can verify that the AdS-Schwarzschild space is recovered for the limit \( k \to 0 \) which is \( f_{\text{AdS-Sch}}(z) = 1 - z^4/z_h^4 \). The Eq. (7) also fulfills the condition \( f'(z_h) < 0 \). In Figure 1 we present the behavior of the horizon function in terms of the holographic coordinate \( z \) for both signs of the constant \( k \).

![Figure 1: The horizon function \( f(z) \), Eq. (7), vs the holographic coordinate \( z \) for \( k = \pm 1 \) and \( z_h = 1 \) in arbitrary units.](image)

**III. STRING IN THE BULK WITH BACKREACTION EFFECTS**

In this section we implement description of the string in a thermal bath with backreaction. For convenience, we change the coordinate \( z \) to \( r = 1/z \) so that the metric, Eq. (4), is rewritten as

\[
ds^2 = e^{\frac{k}{r^2}} \left[ -r^2 f(r) dt^2 + r^2 \left( \eta_{ij} dx^i dx^j \right) + \frac{dr^2}{r^2 f(r)} \right].
\] (8)

Also in \( r \) coordinate, the horizon function, Eq. (7), reads:

\[
f(r) = 1 - \left( \frac{3k r^2 - 2e^{\frac{3}{2}kr^2} + 2}{3k r^2 - 2e^{\frac{3}{2}kr^2} + 2} \right) e^{\frac{3}{2}k(R - \frac{1}{r})}.
\] (9)

The Nambu-Goto action is given by \( S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \), where \( \alpha' \) is the string tension, \( \gamma = \det(\gamma_{\alpha\beta}) \) and \( \gamma_{\alpha\beta} = g_{mn} \partial_{\alpha} X^m \partial_{\beta} X^n \) is the induced metric on the worldsheet.
with \( m, n = 0, 1, 2, 3, 5 \). We also choose a static gauge, where \( t = \tau, r = \sigma \) and \( X = X(\tau, \sigma) \). By using the metric, Eq. (8), and expanding the Nambu-Goto action and keeping the quadratic terms \( \dot{X}^2, X'^2 \), we get:

\[
S_{NG} \approx -\frac{1}{4\pi \alpha'} \int dt dr \left[ \dot{X}^2 \frac{e^{k_\tau}}{f(r)} - X'^2 r^4 f(r) e^{k_\tau} \right],
\]

(10)

where \( \dot{X} = \partial_t X \) and \( X' = \partial_r X \). The equation of motion from this Nambu-Goto action is

\[
\frac{\partial}{\partial r} \left( r^4 f(r) e^{k_\tau} X'(r, t) \right) - \frac{e^{k_\tau}}{f(r)} \ddot{X}(t, r) = 0.
\]

(11)

Using the ansatz \( X(t, r) = e^{i\omega t} h(\omega)(r) \), one gets:

\[
 r^4 f(r) h''(r) + \left(-2k rf(r) + r^4 f'(r) + 4r^3 f(r)\right) h'(r) + \frac{\omega^2}{f(r)} h(r) = 0.
\]

(12)

Going to the tortoise coordinate \( r_* = \int dr / (r^2 f(r)) \) and making a Bogoliubov transformation \( h_\omega(r_*) = e^{B(r_*)} \psi(r_*) \), where \( B(r) = -k/2r^2 - \log(r) \), we obtain a Schrödinger-like equation:

\[
\frac{d^2 \psi(r_*)}{dr_*^2} + (\omega^2 - V(r)) \psi(r_*) = 0,
\]

(13)

with potential

\[
V(r) = -f(r) \left( \left(-\frac{k^2}{r^2} + k - 2r^2\right) f(r) + r \left(k - r^2\right) f'(r) \right).
\]

(14)

As the equation Eq. (13) cannot be analytically solved, we will apply the monodromy patching procedure [7, 15] and seek for approximate analytical solutions. For our purposes we will choose three regions: \( A, B, C \) and explore their solutions.

First, we consider the region \( A \) which is near the horizon \( (r \sim r_h) \). In this region one has, \( V(r) \ll \omega^2 \), so that Eq. (13) reads

\[
\frac{d^2 \psi(r_*)}{dr_*^2} + \omega^2 \psi(r_*) = 0,
\]

(15)

which has the ingoing solution \( \psi(r_*) = A_1 e^{-i\omega r_*} \). For low frequencies one can expand this ingoing solution as \( \psi(r_*) = A_1 - iA_1 \omega r_* \) allowing us to compute \( h_\omega(r_*) \) in this region:

\[
h^A_\omega(r_*) = \frac{e^{-\frac{k}{2r_h}}}{r_h} \left(A_1 - i\omega A_1 r_*\right).
\]

(16)

Close to the horizon, one can expand the tortoise coordinate as:

\[
r_* \approx \frac{2 \left(e^{\frac{3}{2}x} - 1\right) - 3x}{9x^2 r_h} \log\left(\frac{r}{r_h} - 1\right), \text{ with } x \equiv k/r_h^2.
\]

(17)
Then, the solution in region A becomes

\[ h_\omega^A(r_*) = A_1 e^{-\frac{k}{2r_h^2}} \left( 1 - i\omega \lambda \log \left( \frac{r}{r_h} - 1 \right) \right) \quad \text{with} \quad \lambda \equiv \frac{2 \left( e^{\frac{3}{4}x} - 1 \right) - 3x}{9x^2 r_h}. \]  

(18)

The next region, B, is defined as \( V(r) \gg \omega^2 \). In this region Eq. (12) becomes:

\[ \frac{d}{dr} \left( r^4 f(r)e^{\frac{k}{2r_h^2}} h'_\omega \right) = 0, \]  

(19)

whose solution is given by:

\[ h_\omega^B(r) = \int^r \frac{B_1}{r^4 f(r')} e^{-\frac{k}{2r_h^2}} dr' + B_2. \]  

(20)

In the IR limit of region B and considering the fact that \( f(r) \) has a simple pole at the horizon, one can write

\[ h_\omega^B(\text{IR})(r) \approx B_1 \frac{\lambda}{r_h^2} e^{-\frac{k}{2r_h^2}} \log \left( \frac{r}{r_h} - 1 \right) + B_2. \]  

(21)

Comparing \( h_\omega^B(\text{IR})(r) \) with Eq. (18), one gets:

\[ B_1 = -iA_1 r_h \omega e^{\frac{k}{2r_h^2}}, \quad B_2 = \frac{A_1}{r_h} e^{-\frac{k}{2r_h^2}}. \]  

(22)

In the UV location of region B, one has \( f(r) \approx 1 \) and the solution of Eq. (20) is given by:

\[ h_\omega^B(\text{UV})(r) \approx -\frac{B_1}{3r^3} + B_2. \]  

(23)

The last region, C, represents the deep UV meaning that the horizon function reduces to \( f(r) = 1 \). In this case, Eq. (12) has the solution:

\[ h_\omega^C(r) = C_1 {}_1F_1 \left( \frac{\omega^2}{4k}, -\frac{1}{2}, -\frac{k}{r_h^2} \right) + C_2 \frac{(-k)^{3/2}}{r^3} {}_1F_1 \left( \frac{3}{2} + \frac{\omega^2}{4k}, -\frac{5}{2}, -\frac{k}{r^2} \right), \]  

(24)

where \( {}_1F_1(a, b, z) \) is the confluent hypergeometric function of the first kind. Close to the boundary, keeping only terms up to \( O(\omega) \), it can be expanded as

\[ h_\omega^C(r) \approx C_1 + \frac{C_2 k^{3/2}}{r^3}. \]  

(25)

Matching \( h_\omega^B(\text{UV})(r) \) and \( h_\omega^C(r) \), one finds

\[ C_1 = B_2 = \frac{A_1}{r_h} e^{-\frac{k}{2r_h^2}}, \quad C_2 = -\frac{B_1}{3k^{3/2}} = iA_1 \frac{\omega r_h}{3} e^{\frac{k}{2r_h^2}}. \]  

(26)
Now we can write the solution close to the boundary as
\[ h^C_\omega (r) \approx \frac{A_1}{\rho_h} \left( e^{-\frac{k}{r_h}} + i \omega \frac{r_h^2}{3 \pi^3} \right) e^{\frac{k}{r_h}}. \] (27)

We are interested in computing the linear response or the admittance \( \chi(\omega) \). Such a response is due to the action of an external force in an arbitrary brane direction, \( x^i \), and can be written as \( F(t) = E e^{-i\omega t} F(\omega) \), where \( E \) is the electric field on the brane. Following Refs. [10] one can write the force as:
\[ F(t) = \frac{1}{2 \pi \alpha'} \left[ X'(t, r_b) r_b^4 f(r_b) e^{\frac{k}{r_b}} \right], \] (28)
and then
\[ F(\omega) = \frac{A_1}{2 \pi \alpha'} \left[ -i \omega r_h e^{\frac{k}{r_b}} f(r_b) e^{\frac{k}{r_b}} \right]. \] (29)

Considering the limits where the brane is far away from the horizon \( r_b \gg r_h \), and the scale of the brane is much greater than the IR scale \( r_b \gg \sqrt{k} \), then \( f(r_b) \to 1 \), therefore the admittance, is given by:
\[ \chi(\omega) \equiv \frac{h^C_\omega (\omega)}{F(\omega)} = \frac{2 \pi i \alpha'}{\omega r_h^2} e^{-x} = \frac{2 \pi i \alpha'}{\omega g_{ii}} (r_h), \text{ with } x \equiv k/r_h^2. \] (30)

Note that \( g_{ii} \) is the metric component in the \( x^i \) direction. Following Ref. [13] one can write \( \chi(\omega) \) as:
\[ \chi(\omega) = 2 \pi \alpha' \left( \frac{i}{\gamma \omega} - \frac{\Delta m}{\gamma^2} + O(\omega) \right), \] (31)
with
\[ \gamma = e^{\frac{k}{r_b^2}} r_b^2 \left( 1 + \frac{k}{r_b^2} + O \left( \frac{1}{r_b^3} \right) \right), \quad \Delta m = e^{\frac{3k}{2r_h^2}} r_b^4 \left( 1 + O \left( \frac{1}{r_b^2} \right) \right), \] (32)
where \( \gamma \) is the friction coefficient and \( \Delta m \) corresponds to the change in the bare mass \( m \) of the particle in the Langevin equation [2, 3].

The Hawking temperature associated with the black hole in our deformed AdS-Schwarzschild space, is given by:
\[ T = \frac{r^2}{4 \pi} \left| \frac{df(r)}{dr} \right|_{r=r_h} = \frac{r_h \pi}{g(x)}, \text{ where } g(x) \equiv \frac{9 x^2}{4 \left( e^{\frac{3k}{2r_h^2}} - 1 \right) - 3 x}. \] (33)
It is worthwhile to mention that in the limit \( k \to 0 \) (or equivalently \( x \to 0 \)), one recovers the AdS-Schwarzschild meaning \( T \to r_h/\pi \).
Figure 2: **Left panel:** Plot of the function $g(x)$ against $x = k/r^2$ that measures the shift from AdS-Schwarzschild Hawking temperature after backreaction. **Right panel:** Imaginary part of the admittance $\chi$ times $\pi T^2$ against $x$, for three situations: pure AdS-Schwarzschild, deformed AdS-Schwarzschild [15], and deformed AdS-Schwarzschild with backreaction. Note the asymmetry between positive and negative values of $k$ (or $x$) in both panels.

By using the above definition of the Hawking temperature in the expression for the admittance, Eq. (30), one can rewrite it as:

$$\chi(\omega) = \frac{2i\alpha'}{\omega \pi T^2} e^{-x} g(x)^2.$$  \hspace{1cm} (34)

At this point it is interesting to compare our result for the admittance with Ref. [15], which computes this quantity for a deformed AdS-Schwarzschild space with no backreaction ($\chi_{NBR}(\omega)$), and Refs. [10, 11, 13] where the authors compute the admittance in a geometry which includes the pure AdS-Schwarzschild ($\chi_{AdS}(\omega)$). Note that:

$$\chi(\omega) = \chi_{AdS}(\omega) e^{-x} g(x)^2 = \chi_{NBR}(\omega) g(x)^2.$$ \hspace{1cm} (35)

In Fig. 2 we present the behavior of $g(x)$ and compare the admittances presented in (35).

Using the result for the admittance found here we can calculate the diffusion coefficient, which is given by

$$D = T \lim_{\omega \to 0} (-i \omega \chi(\omega)) = \frac{2\alpha'}{\pi T} e^{-x} g(x)^2,$$ \hspace{1cm} (36)

where one can clearly see the contributions from the deformation ($e^{-x}$) of the AdS-Schwarzschild metric and the backreaction ($g(x)^2$), analogously to the admittances discussed above.
IV. MEAN SQUARE DISPLACEMENT AND FLUCTUATION-DISSIPATION
THEOREM

The Schrödinger equation (15) has as solution a linear combination between the ingoing
and outgoing modes. Considering the outgoing mode as \( \psi_{\text{out}}(r) = A_2 e^{i\omega r} \) one can follow
the above steps of the monodromy patching procedure, as done in Sec. III and obtain, for
the region \( A \) an expression given by:

\[
h_\omega(r) = A \frac{e^{-k r}}{r_h} \left[ \left( 1 + \frac{i\omega r_h^2 e^{k r_h}}{3r^3} \right) + B \left( 1 - \frac{i\omega r_h^2 e^{k r_h}}{3r^3} \right) \right].
\]  

(37)

Similarly, up to order \( \omega \), for the region \( C \) in terms of ingoing and outgoing modes, one
has:

\[
h_\omega^C = A[h_\omega^{out}(r) + Bh_\omega^{in}(r)]
\]

\[
= A \left[ F_1 \left( \frac{\omega^2}{4k}; -\frac{1}{2}; -\frac{k}{r^2} \right) - i\omega \frac{r_h^2}{3r^3} F_1 \left( \frac{\omega^2}{4k} + \frac{3}{2}; \frac{5}{2}; -\frac{k}{r^2} \right) e^{k r_h} \right.
\]

\[
+ B \left( F_1 \left( \frac{\omega^2}{4k}; -\frac{1}{2}; -\frac{k}{r^2} \right) + i\omega \frac{r_h^2}{3r^3} F_1 \left( \frac{\omega^2}{4k} + \frac{3}{2}; \frac{5}{2}; -\frac{k}{r^2} \right) e^{k r_h} \right) \right],
\]  

(38)

where \( A \) and \( B \) are constants to be determined.

On the other hand, close to the horizon one can write the general solution as:

\[
h_\omega(r) = A \frac{e^{-k r}}{r_h} \left[ e^{i\omega \lambda \log \left( \frac{r}{r_h} \right)} \right] + B e^{-i\omega \lambda \log \left( \frac{r}{r_h} \right)}, \quad \text{where } \lambda = \frac{1}{4r_h g(x)}.
\]  

(39)

Following [7], by imposing Neumann boundary conditions at the brane \((r = r_b)\) and at the
horizon where \(r/r_h = 1 + \epsilon\), with \(\epsilon \ll 1\), one can write

\[
B = \left. \frac{h_\omega^{out}(r)}{h_\omega^{in}(r)} \right|_{\frac{r}{r_h} = 1 + \epsilon} \approx e^{-2i\omega \lambda \log(1/\epsilon)},
\]  

(40)

which produces discrete frequencies, such as, \(\Delta \omega = \pi/\lambda \log (1/\epsilon)\).

In order to compute the constant \( A \) one can use the normalized Klein-Gordon inner
product

\[
(X_\omega(r,t), X_\omega(r,t)) = \frac{-i}{2\pi \alpha'} \int_{r_{h}}^{r_{b}} dr \sqrt{g_{rr} - g_{rt}} \left( \partial_t h_\omega^*(r,t) \partial_t h_\omega(r,t) - (\partial_t h_\omega(r,t)) h_\omega^*(r,t) \right)
\]

\[
= \frac{\omega}{\pi \alpha'} \int_{r_{h}}^{r_{b}} dr \frac{e^{kr}}{f(r)} |h_\omega(r)|^2 = 1.
\]  

(41)
This integral is dominated by the near horizon region:

\[
\frac{2\omega |A|^2}{\pi \alpha'} \int_{r_h + \epsilon}^{r_h} \frac{1}{r^2 f(r)} \approx \frac{2\omega |A|^2}{\pi \alpha'} \int_{r_h + \epsilon}^{r_h} \frac{r_h \left(-2r_h^2 e^{3k} + 3k + 2r_h^2\right)}{9k^2(r' - r_h)} = \frac{2\omega \lambda |A|^2}{\pi \alpha'} \log \left(\frac{1}{\epsilon}\right)
\]

so that

\[
A = \sqrt{\frac{\pi \alpha'}{2\omega \lambda \log (1/\epsilon)}}.
\] (42)

To compute the mean square displacement of the end point of the string located at the brane one has to write the thermal two-point function as a Fourier integral:

\[
X(t, r) = \sum_{\omega > 0} (h^C_{\omega}(r)e^{-i\omega t}a_\omega + h^{C\ast}_{\omega}(r)e^{i\omega t}a_\omega^\dagger),
\] (43)

where the frequencies \(\omega\) are discrete since \(\Delta \omega = \pi / \lambda \log (1/\epsilon)\), while \(a_\omega\) and \(a_{\omega}^\dagger\) are the annihilation and creation operators, respectively. Then, disregarding terms of the order \(1/r_b\) or less, one gets

\[
\langle x(t)x(0) \rangle \equiv \langle X(t, r_b)X(0, r_b) \rangle = \frac{2\alpha' e^{-\frac{k}{r_h^2}}}{\pi \alpha'} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{2\cos(\omega t) e^{\beta \omega} - 1 + e^{-i\omega t}}{e^{\beta \omega} - 1},
\] (44)

where we have approximated the sum by an integral considering \(d\omega \sim \Delta \omega\). Analogously one has

\[
\langle x(0)x(t) \rangle = \frac{2\alpha' e^{-\frac{k}{r_h^2}}}{\pi \alpha'} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{2\cos(\omega t) e^{\beta \omega} - 1 + 1}{e^{\beta \omega} - 1} = \langle x(t)x(0) \rangle^*,
\] (45)

and

\[
\langle x(t)x(t) \rangle = \frac{2\alpha' e^{-\frac{k}{r_h^2}}}{\pi \alpha'} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{2}{e^{\beta \omega} - 1} = \langle x(0)x(0) \rangle,
\] (46)

where these integrals are divergent, as well as the mean square displacement. Using the normal ordering prescription, the regularized mean square displacement can be written as:

\[
s_{reg}^2(t) \equiv \langle : [x(t) - x(0)]^2 : \rangle
\]

\[
= \langle x(t)x(t) \rangle^2 + \langle x(0)x(0) \rangle^2 - \langle x(t)x(0) \rangle - \langle x(0)x(t) \rangle
\] (47)

Then, one gets:

\[
s_{reg}^2(t) = \frac{16\alpha' e^{-\frac{k}{r_h^2}}}{r_h^2} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{\sinh^2 \left(\frac{\omega t}{2}\right)}{e^{\beta \omega} - 1} = \frac{4\alpha' e^{-\frac{k}{r_h^2}}}{r_h^2} \log \left(\frac{\sin(\frac{tn}{\beta})}{\frac{tn}{\beta}}\right).
\] (48)
Considering the late time approximation $t \gg \beta/\pi$, we get:

$$s_{\text{reg}}^2(t) \approx \frac{4\alpha' e^{-x}}{\pi T} g(x)^2 t = 2Dt. \quad (49)$$

which is identified with the diffusive regime since $s_{\text{reg}}^2 \sim 2Dt$. The diffusion coefficient $D$ obtained here coincides with the one given by Eq. (36). The factor 2 in this equation is a characteristic of a one dimensional problem. On the other hand, for the short time approximation $t \ll \beta/\pi$, one finds

$$s_{\text{reg}}^2 \approx \frac{2\alpha' e^{-x} g(x)^2}{3} t^2. \quad (50)$$

which corresponds to the ballistic regime since $s_{\text{reg}}^2 \sim t^2$.

Finally, we are going to verify explicitly the fluctuation-dissipation theorem, which can be stated as:

$$\frac{1}{2} [\langle x(\omega)x(0) \rangle + \langle x(0)x(\omega) \rangle] = (2n_B + 1) \text{Im}(\chi(\omega)), \quad (51)$$

where $n_B = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein distribution associated to thermal noise effects.

To do this, we define a symmetric Green’s function in Fourier space as:

$$G_{\text{sym}}(\omega) \equiv \frac{1}{2} [\langle x(\omega)x(0) \rangle + \langle x(0)x(\omega) \rangle]. \quad (52)$$

Then, one can write the corresponding symmetric time dependent Green’s function, using Eqs. (44) and (45), as:

$$G_{\text{sym}}(t) = \frac{2\pi \alpha' e^{-\frac{k}{r_H^2}}}{r_H^2} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{2e^{-i\omega t}}{|\omega| (e^{\beta|\omega|} - 1)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{|\omega|} \right). \quad (53)$$

So, the symmetric Green’s function in Fourier space is found to be:

$$G_{\text{sym}}(\omega) = (2n_B(\omega) + 1) \frac{2\pi \alpha' e^{-\frac{k}{r_H^2}}}{r_H^2|\omega|} \quad (54)$$

Furthermore, the imaginary part of the admittance $\chi(\omega)$, given by Eq. (30), is

$$\text{Im}(\chi(\omega)) = \frac{2\pi \alpha' e^{-\frac{k}{r_H^2}}}{r_H^2|\omega|}, \quad (55)$$

so that our deformed and backreacted model satisfies the well known fluctuation-dissipation theorem defined in Eq. (51).
V. CONCLUSIONS

Here, in this work, taking into account a conformal exponential factor \( \exp(k/r^2) \) and the horizon function obtained from the solutions of Einstein-dilaton equations we have constructed a deformed and backreacted Lorentz invariant holographic model. By using our model we could investigate the fluctuation and dissipation of a string in this set up. In particular, we computed the response function (admittance), the diffusion coefficient, the relevant two-point functions and the regularized mean square displacement. From this last result we obtained the diffuse and the ballistic regimes characteristic of the Brownian motion. We also verified the fluctuation-dissipation theorem within our model from the two-point functions and the imaginary part of the admittance. This analysis can be sought as an extension of the one described in Ref. [15], where the horizon function is just the usual AdS-Schwarzschild one.

The backreacted horizon function, Eq. (7), is displayed in Fig. 1 for \( k \pm 1 \) and \( z_h = 1 \), where we clearly see the difference between these two choices, although they merge for low values of the holographic coordinate \( z \) and also meet at \( z = z_h \), satisfying the condition \( f(z_h) = 0 \). Remember that \( z = 1/r \), where \( r \) is the radial holographic coordinate pointing outwards the black hole, so that the interval \( 0 < z < z_h \) represents the region outside the horizon.

The backreaction effects on the fluctuation and dissipation of the string are encoded in the function \( g(x) \), defined in Eq. (33), where \( x = k/r_h^2 \) and \( k \) is the IR scale. This function corresponds to the deviation from the Hawking temperature due to the deformation \( \exp(k/r^2) \) and the backreaction in our model with respect to the pure AdS-Schwarzschild case. In the left panel of Fig. 2, we show the shape of this function where one notes the asymmetry between the two branches identified with \( k < 0 \) and \( k > 0 \). At \( k = 0 \) and finite \( r_h \), \( g(x)|_{x=0} = 1 \), there is no deformation or backreaction and the Hawking temperature reduces to its usual form, \( T = r_h/\pi \). For the branch \( k < 0 \) the function \( g(x) \) grows exponentially with \( |x| \to \infty \), so the deviation from the AdS-Schwarzschild becomes larger with larger \( |x| \). On the other hand, for \( k > 0 \), \( g(x) \) decreases exponentially with \( x \to \infty \), vanishing for very large \( x \).

In particular, the backreaction effect on the imaginary part of the admittance is shown in the right panel of Fig. 2. From this picture, we see that for \( k < 0 \) the deviation from the pure
AdS-Schwarzschild and the deformed with no backreaction cases increases with increasing $x$. On the other side, for $k > 0$ we note that the deviation from the pure AdS-Schwarzschild is limited and the two deformed solutions with or without backreaction vanish for high $x$.

Note that the admittance found here within this model $\chi(\omega)$ could be compared with the ones computed from a deformed AdS space model without backreaction $\chi_{NBR}(\omega)$ [15], and the one in a geometry which includes the pure AdS-Schwarzschild case $\chi_{AdS}(\omega)$ [10, 11, 13], as given by Eq. (35).

Analogously, the diffusion coefficient $D$, Eq. (36), obtained from the admittance is also modified by the deformation exponential and the backreaction effects by a factor $e^{-xg(x)^2}$. This result was checked in the calculation of the regularized mean square displacement $s_{reg}^2(t)$ from the two-point functions in the limit of late times, Eq. (49). This result can be interpreted as a check of the fluctuation-dissipation theorem, Eq. (51), which we verify explicitly in Eqs. (52)-(55).

So, in general, the effects of the deformation with the exponential factor $\exp(k/r^2)$ presented in Ref. [15] without backreaction on the physical quantities related to the fluctuation and dissipation of the string are enhanced by the backreaction considered here.

Acknowledgments

The authors would like to thank Diego M. Rodrigues and Alfonso Ballon-Bayona for discussions. N.G.C. is supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES). H.B.-F. and C.A.D.Z. are partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) under grants No. 311079/2019-9 and No.309982/2018-9, respectively.

[1] R. Brown, Philos. Mag. 4, 161 (1828).
[2] P. Langevin, C. R. Acad. Sci. Paris 146, 530-533 (1908).
[3] R. Kubo, Reports on Progress in Physics 29, 255 (1966).
[4] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87, 081601 (2001)
[5] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, “Gauge/String
Duality, Hot QCD and Heavy Ion Collisions,”
[6] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183
(2000)
[7] J. de Boer, V. E. Hubeny, M. Rangamani and M. Shigemori, JHEP 07, 094 (2009)
[8] D. T. Son and D. Teaney, JHEP 07, 021 (2009)
[9] A. N. Atmaja, J. de Boer and M. Shigemori, Nucl. Phys. B 880, 23-75 (2014)
[10] D. Tong and K. Wong, Phys. Rev. Lett. 110, no.6, 061602 (2013)
[11] M. Edalati, J. F. Pedraza and W. Tangarife Garcia, Phys. Rev. D 87, no.4, 046001 (2013)
[12] W. Fischler, P. H. Nguyen, J. F. Pedraza and W. Tangarife, JHEP 08, 028 (2014)
[13] D. Giataganas, D. S. Lee and C. P. Yeh, JHEP 08, 110 (2018)
[14] D. Giataganas and H. Soltanpanahi, Phys. Rev. D 89, no.2, 026011 (2014)
[15] N. G. Caldeira, E. Folco Capossoli, C. A. D. Zarro and H. Boschi-Filho, Phys. Rev. D 102,
086005 (2020)
[16] O. Andreev, Phys. Rev. D 73, 107901 (2006)
[17] M. Rinaldi and V. Vento, Eur. Phys. J. A 54, 151 (2018)
[18] R. C. L. Bruni, E. Folco Capossoli and H. Boschi-Filho, Adv. High Energy Phys. 2019,
1901659 (2019)
[19] S. S. Afonin and A. D. Katanaeva, Phys. Rev. D 98 (2018) no.11, 114027
[20] E. Folco Capossoli, M. A. Martín Contreras, D. Li, A. Vega and H. Boschi-Filho, Chin. Phys.
C 44, no.6, 064104 (2020)
[21] E. Folco Capossoli, M. A. Martín Contreras, D. Li, A. Vega and H. Boschi-Filho, Phys. Rev.
D 102, no.8, 086004 (2020)
[22] A. Ballon-Bayona, H. Boschi-Filho, L. A. H. Mamani, A. S. Miranda and V. T. Zanchin, Phys.
Rev. D 97, no.4, 046001 (2018)
[23] A. Ballon-Bayona, H. Boschi-Filho, E. Folco Capossoli and D. M. Rodrigues,
[arXiv:2006.08810 [hep-th]].
[24] H. Grabert, P. Schramm and G. L. Ingold, Phys. Rept. 168, 115-207 (1988)