Doubrov–Ferapontov general heavenly equation and the hyper-Kähler hierarchy

L V Bogdanov

L D Landau ITP RAS, Moscow, Russia

E-mail: leonid@itp.ac.ru

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Abstract
We give a description of a recently introduced Doubrov-Ferapontov general heavenly equation in terms of a closed differential Plücker two-form, rationally depending on the spectral parameter. We demonstrate that the general heavenly equation is an important generating equation in the context of Takasaki hyper-Kähler hierarchy, and it is also directly connected to hyper-Kähler geometry through the Gindikin construction. We develop a ∂-dressing scheme and introduce a formula for the potential satisfying of the general heavenly equation. Multidimensional generalization is also outlined.

Keywords: heavenly equation, hyper-Kähler hierarchy, integrable systems, self-dual Einstein equation, dressing method

1. Introduction
A general heavenly equation was introduced as a result of the classification of integrable symplectic Monge-Ampère equations in four dimensions [1]. It is one in a list of six equations, and it is remarkably simple and symmetric, having the form

\[ \alpha u_{12} u_{34} + \beta u_{13} u_{24} + \gamma u_{14} u_{23} = 0, \]  

where \( \alpha + \beta + \gamma = 0 \), subscripts denote partial derivatives. The Lax pair was also presented in [1] in terms of vector fields \( X_1, X_2 \) in involution,

\[ X_1 = u_{34} d_1 - u_{13} d_4 + \gamma \lambda (u_{34} d_1 - u_{14} d_3), \]
\[ X_2 = u_{23} d_4 - u_{34} d_2 + \beta \lambda (u_{34} d_2 - u_{24} d_3). \]

In the present work we will give a description of the Doubrov-Ferapontov general heavenly equation (1), using the construction developed in our works [2, 3], where we gave a formulation of multidimensional dispersionless integrable hierarchy in terms of differential n-form \( \Omega \) in the space of \( N \) variables \( (N \leq \infty) \) \((x_0, \ldots, x_N)\), possessing the following properties

\[ \]
1. The form $\Omega$ is decomposable, i.e.,

$$\Omega = \omega_1 \wedge ... \wedge \omega_n.$$  

In algebraic terms that means that coefficients of the form satisfy Plücker relations, so we sometimes call it a Plücker form.

2. The (decomposable) form $\Omega$ is projectively closed, i.e., there exists a function (gauge) $f(x_0, ..., x_N)$ such that

$$d(f\Omega) = 0.$$  

3. The form $\Omega$ is projectively holomorphic with respect to $x_0$, i.e., there exists a gauge $g(x_0, ..., x_N)$ such that coefficients of the form $g\Omega$ are holomorphic in some region of the complex plane of the variable $\lambda = x_0$.

The forms with these properties define multidimensional dispersionless integrable hierarchy in terms of integrable distribution of holomorphic vector fields representing Lax operators of the hierarchy. Two forms differing only by a gauge are equivalent and define the same object. The case when it is possible to introduce the form $\Omega$ simultaneously holomorphic and closed in the standard sense ($f=g$) corresponds to an important reduction (preservation of volume). Another important reduction is HCR reduction, corresponding to heavenly equations and hyper-Kähler hierarchies, for which the form $\Omega$ does not contain $d\lambda$ and $\lambda$ enters only parametrically.

To describe the Doubrov-Ferapontov general heavenly equation (1) in terms of this construction, we will not need the most general version of the technique developed in [2, 3], because this equation belongs to the HCR class, and also it corresponds to the form $\Omega$ simultaneously holomorphic and closed in the standard sense (preservation of volume reduction). We will demonstrate that general heavenly equation (1) is an important generating equation in the context of hyper-Kähler hierarchy [4, 5]. We will also show that it is directly connected to hyper-Kähler geometry and gives a solution to complex self-dual Einstein equation through the Gindikin construction [6, 7].

2. General heavenly equation through the differential two-form

Let us consider 2-form depending on the spectral parameter

$$\Omega = \sum_{i,j} \omega_{ij} dx_i \wedge dx_j,$$  

where $1 \leq i, j \leq 4$,

$$\omega_{ij}(\lambda, x) = \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) w_{ij}(x),$$  

where $w_{ij}(x)$ is symmetric.

Let $\Omega$ be a Plücker form. Plücker conditions for 2-forms are equivalent to the relation

$$\Omega \wedge \Omega = 0,$$  

which in our case gives one equation

$$\omega_{23}\omega_{14} - \omega_{13}\omega_{24} + \omega_{12}\omega_{34} = 0.$$  

For $w_\gamma(x)$ we have
\[(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)w_{23}w_{14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)w_{13}w_{24} + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)w_{12}w_{34} = 0.\] (7)

Let us also suggest that $\Omega$ is closed,
\[\omega_{\gamma,\lambda_k} = 0.\] (8)

For $w_\gamma(x)$ we have
\[\frac{1}{\lambda - \lambda_2}(\partial_1w_{23} - \partial_3w_{12}) + \frac{1}{\lambda - \lambda_3}(\partial_2w_{13} - \partial_1w_{23}) + \frac{1}{\lambda - \lambda_4}(\partial_3w_{12} - \partial_2w_{13}) = 0,\]
\[\frac{1}{\lambda - \lambda_2}(\partial_1w_{24} - \partial_3w_{12}) + \frac{1}{\lambda - \lambda_4}(\partial_2w_{14} - \partial_3w_{12}) + \frac{1}{\lambda - \lambda_2}(\partial_3w_{12} - \partial_2w_{14}) = 0,\]
\[\frac{1}{\lambda - \lambda_2}(\partial_2w_{23} - \partial_3w_{24}) + \frac{1}{\lambda - \lambda_3}(\partial_3w_{24} - \partial_2w_{23}) + \frac{1}{\lambda - \lambda_2}(\partial_2w_{23} - \partial_2w_{24}) = 0.\]

These equations imply the existence of the potential $\Theta = w_{\gamma,ij}$, and for arbitrary potential $w_\gamma = \Theta_\gamma$, satisfy the closedness equations.

Then Plücker relation (7) implies general heavenly equation (1) for the potential,
\[(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)\Theta_{23}\Theta_{14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)\Theta_{13}\Theta_{24} + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\Theta_{12}\Theta_{34} = 0.\] (9)

**Proposition 1.** Let us consider 2-form depending on the spectral parameter
\[\Omega = \sum_{\gamma,i,j} \left(\frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j}\right) w_\gamma(x)dx_i \wedge dx_j,\]
where $1 \leq i, j \leq 4$, $w_\gamma(x)$ is symmetric. The conditions
\[\Omega \wedge \Omega = 0,\]
\[\partial \Omega = 0\]
are equivalent to the existence of potential $\Theta$, $w_\gamma = \Theta_\gamma$, satisfying the general heavenly equation
\[(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)\Theta_{23}\Theta_{14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)\Theta_{13}\Theta_{24} + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\Theta_{12}\Theta_{34} = 0.\]

3. Commuting flows and the 'horizontal' hierarchy

It is possible not to restrict ourselves to the case of four variables and consider the two-form (3) for an arbitrary number of variables,
\[
\mathbf{\Omega} = \sum_{i,j=1}^{N} \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) w_{ij}(\mathbf{x}) dx_i \wedge dx_j. \tag{10}
\]

The closedness conditions for this two-form, in complete analogy with the case of four variables, imply the existence of the potential \( \mathbf{\Theta} ; w_{ij} = \mathbf{\Theta}_{ij} \), and for every four distinct indices \( 1 \leq i, j, k, l \leq N \) we have an equation

\[
\begin{aligned}
\left( \lambda_k - \lambda_j \right) (\lambda_i - \lambda_j) \Theta_{j,k} \Theta_{i,l} - (\lambda_i - \lambda_k) (\lambda_j - \lambda_l) \Theta_{j,k} \Theta_{i,l} \\
+ \left( \lambda_j - \lambda_i \right) (\lambda_k - \lambda_l) \Theta_{j,k} \Theta_{i,l} = 0.
\end{aligned} \tag{11}
\]

Thus we have a kind of ‘horizontal’ hierarchy of consistent four-dimensional equations, where all the variables \( x_i \) are on equal footing and correspond to simple poles. It is possible to obtain general two-form meromorphic in \( \lambda \) by glueing simple poles of the form (10). Moving this way, it is possible to arrive at heavenly equation hierarchy [4], where the coefficients of the form are Laurent polynomials.

It is easy to include \( \lambda_0 = \infty \) into consideration by the appropriate limit (we will denote the corresponding variable \( x_0 \)). The terms of two-form \( \mathbf{\Omega} \) containing \( dx_0 \) read

\[
\mathbf{\Omega} = \ldots + 2 \sum_{i=1}^{N} \frac{1}{\lambda_i - \lambda_i} w_{i0}(\mathbf{x}) dx_i \wedge dx_0.
\]

Equations (11) containing partial derivative over \( x_0 \) look like

\[
\begin{aligned}
\left( \lambda_k - \lambda_j \right) \Theta_{j,k} \Theta_{i,0} - (\lambda_i - \lambda_k) \Theta_{j,k} \Theta_{i,0} + \left( \lambda_j - \lambda_i \right) \Theta_{j,k} \Theta_{i,0} = 0.
\end{aligned} \tag{12}
\]

3.1. ‘Vacuum’ two-form \( \mathbf{\Omega} \) and potential \( \mathbf{\Theta} \)

Let us start from a simple case of constant two-form \( \mathbf{\Omega} \),

\[
\mathbf{\Omega}_0 = \sum \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) c_{ij} dx_i \wedge dx_j,
\]

where \( c_{ij} \) is constant and symmetric. The closedness condition is satisfied identically, and Plücker condition (7) implies that the form \( \mathbf{\Omega}_0 \) is decomposable,

\[
\mathbf{\Omega}_0 = 2dS_0^1 \wedge dS_0^2,
\]

where

\[
S_0^1 = \sum_{i} \frac{a_i x_i}{\lambda - \lambda_i}, \quad S_0^2 = \sum_{i} \frac{b_i x_i}{\lambda - \lambda_i},
\]

and the constants \( a_i, b_i \) satisfy the relations

\[
c_{ij} = \frac{a_i b_j - a_j b_i}{\lambda_i - \lambda_j}.
\]

The ‘vacuum’ potential \( \mathbf{\Theta} \) is quadratic in \( x_i \),

\[
\mathbf{\Theta}_0 = \frac{1}{2} \sum_{i \neq j} \frac{a_i b_j - a_j b_i}{\lambda_i - \lambda_j} x_i x_j. \tag{14}
\]
For the general potential of the form
\[ \Theta = \Theta_0 + \tilde{\Theta} \]  
(15)
the terms entering the ‘vacuum’ potential may be important in the limit when we glue some of the points \( \lambda_i \).

If we include \( x_0 \) corresponding to \( \lambda_0 = \infty \) into consideration, for \( S^1, S^2, \Theta_0 \) we will have additional terms
\[ S_0^1 = \ldots + a_0 x_0, \quad S_0^2 = \ldots + b_0 x_0, \]
\[ \Theta_0 = \ldots + \sum_i (a_i b_0 - a_0 b_i) x_i x_0. \]  
(16)

4. From the horizontal hierarchy to the standard hyper-Kähler hierarchy

General heavenly equation and the ‘horizontal hierarchy’ connected with it play the role of generating objects for the heavenly equation hierarchy, or hyper-Kähler hierarchy [4, 5], which contains illustrious Plebański first and second heavenly equations and higher equations. First, gluing simple poles of the two-form (10), it is possible to arrive at the general two-form with Laurent polynomial coefficients, which corresponds to the heavenly equation hierarchy. General heavenly equation (9) plays the role of a ‘dispersionless addition formula’ in this context. Substituting to it dispersionless vertex operators instead of partial derivatives and taking into account vacuum terms of potential \( \Theta \), we get different generating equations of the hierarchy. In this way it is possible, for example, to obtain generating equations for the second heavenly equation hierarchy introduced in [11, 12].

4.1. First heavenly equation from the general heavenly equation

First we discuss a simple example and demonstrate how to obtain the first heavenly equation from the general heavenly equation. Let us consider a limit
\[ \lambda_1, \lambda_2 \to \mu_1, \quad \lambda_3, \lambda_4 \to \mu_2 \]
for the potential \( \Theta \) with some vacuum background (15). First we pick out vacuum terms that are singular in this limit,
\[ \Theta = \frac{a_1 b_2 - a_2 b_1}{\lambda_1 - \lambda_2} x_1 x_2 + \frac{a_3 b_4 - a_4 b_3}{\lambda_3 - \lambda_4} x_3 x_4 + \Theta'. \]

Then, taking the limit, from equation (9) for the function \( \Theta' \) we get
\[ \Theta'_{13} \Theta'_{24} - \Theta'_{14} \Theta'_{23} = \frac{(a_1 b_2 - a_2 b_1)(a_3 b_4 - a_4 b_3)}{(\mu_1 - \mu_2)^2}, \]
which is (up to a scaling) the illustrious Plebanski first heavenly equation. Corresponding 2-form \( \Omega \) is also obtained in this limit,
Performing a Möbius transformation of the spectral variable \( \eta = \frac{\lambda - \mu_2}{\lambda - \mu_1} \), for \( \Omega \) (up to a factor) we get

\[
\Omega \sim \eta \left( a_1 b_2 - a_2 b_1 \right) dx_1 \wedge dx_2 + \frac{a_3 b_4 - a_4 b_3}{\eta} dx_3 \wedge dx_4 + (\mu_1 - \mu_2) \left( \Theta_{13}' dx_1 \wedge dx_3 + \Theta_{14}' dx_1 \wedge dx_4 + \Theta_{23}' dx_2 \wedge dx_3 + \Theta_{24}' dx_2 \wedge dx_4 \right).
\]

Taking in (17) \( \mu_1 = 0, \mu_2 = \infty, a_1 = a_3 = 2, b_1 = b_3 = 0 \), we get the two-form

\[
\frac{1}{2} \Omega = \frac{1}{\lambda^2} dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \frac{1}{\lambda} \left( \Theta_{13}' dx_1 \wedge dx_3 + \Theta_{14}' dx_1 \wedge dx_4 + \Theta_{23}' dx_2 \wedge dx_3 + \Theta_{24}' dx_2 \wedge dx_4 \right),
\]

which corresponds to the standard setting for the first heavenly equation. Also for this case

\[
S_0^1 = \frac{1}{\lambda} x_1 + x_3, \quad S_0^2 = \frac{1}{\lambda} x_2 + x_4.
\]

### 4.2. General heavenly equation and generating relations for the second heavenly equation hierarchy

Now we will demonstrate how to to obtain generating equations for the second heavenly equation hierarchy introduced in [11, 12] starting from the general heavenly equation. Together with the times of ‘horizontal’ hierarchy (vertex times) we will consider standard infinite sets of times of the second heavenly equation hierarchy,

\[
S_0^1 = \sum_{i} \frac{a_i x_i}{\lambda - \lambda_i}, \quad S_0^2 = \sum_{i} \frac{b_i x_i}{\lambda - \lambda_i}, \quad S_0^3 = \sum_{i} \frac{\lambda^n}{\lambda_i}.
\]

To get generating equations, we need two facts about second heavenly equation hierarchy, which we explain in more detail in the Appendix. First, Takasaki second key function [4] and the ‘\( \tau \)-function’ \( \Theta \) for the second heavenly equation hierarchy of the work [11] correspond to the function \( \widetilde{\Theta} \) (15). Second, a complete set of independent variables of the hierarchy is given by the coefficients of the series \( S_0^1, S_0^2 \), suggested to define analytic functions in the unit disk, in the neighborhood of zero,

\[
S_0^1 = \sum_{k=0}^{\infty} T_k^1 \lambda^k, \quad S_0^2 = \sum_{k=0}^{\infty} T_k^2 \lambda^k.
\]

Taking in expressions (18) all \( \lambda_i \) outside the unit disk, \( |\lambda_i| > 1 \), we get

\[
T_k^1 = t_k^1 - \sum_{i} a_i x_i \lambda_i^{-(n+1)}, \quad T_k^1 = t_k^1 - \sum_{i} a_i x_i \lambda_i^{-(n+1)}
\]
Thus variables $t_1^n$, $t_2^n$ and $x_i$ enter independent variables of the hierarchy in special combinations and there are some relations between derivatives. Introducing vertex operators

$$
\sum_{\mu \mu}^n = -\frac{\partial}{\partial t_{\mu}}, \quad \sum_{\mu \mu}^{n-1} = -\frac{\partial}{\partial t_{\mu}}
$$

where it is suggested that $|\mu| > 1$, we express derivatives over horizontal times $t_i$ through derivatives over times $t_1^n$, $t_2^n$,

$$
\frac{\partial}{\partial t_j} = a_i D^1(\lambda_i) + b_i D^2(\lambda_i).
$$

(19)

Rewriting the general heavenly equation (9) for the function $\Theta$ (15)

$$
\Theta = \Theta_0 + \tilde{\Theta}, \quad \Theta_0 = \frac{1}{2} \sum_{i \neq j} a_i b_j \lambda_i \lambda_j x_i x_j,
$$

and substituting vertex expressions for derivatives (19), we get a generic generating relation for the second heavenly equation hierarchy depending on four points $\lambda_1, ..., \lambda_4$ and parameters $a_i, b_i$. Generating relations introduced in [11, 12] contain three points and can be obtained by gluing a pair of points. For example, let us consider a choice

$$
S_1 = \frac{x_1}{\lambda - \lambda_1} + \frac{x_2}{\lambda - \lambda_2} + ..., \quad S_2 = \frac{x_3}{\lambda - \lambda_3} + \frac{x_4}{\lambda - \lambda_4} + ....
$$

In this case

$$
\Theta_0 = \frac{x_1 x_3}{\lambda_1 - \lambda_3} + \frac{x_1 x_4}{\lambda_1 - \lambda_4} + \frac{x_2 x_3}{\lambda_2 - \lambda_3} + \frac{x_2 x_4}{\lambda_2 - \lambda_4},
$$

and from (9) for $\tilde{\Theta}$ we get

$$
(\lambda_2 - \lambda_4)\tilde{\Theta}_{24} + (\lambda_3 - \lambda_2)\tilde{\Theta}_{23} + (\lambda_4 - \lambda_1)\tilde{\Theta}_{14} + (\lambda_1 - \lambda_3)\tilde{\Theta}_{13}
$$

$$
= (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)\tilde{\Theta}_{23}\tilde{\Theta}_{14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)\tilde{\Theta}_{13}\tilde{\Theta}_{24}
$$

$$
+ (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\tilde{\Theta}_{12}\tilde{\Theta}_{34}.
$$

(20)

Taking into account that expressions for derivatives through vertex operators in this case are

$$
\frac{\partial}{\partial x_1} = D^1(\lambda_1), \quad \frac{\partial}{\partial x_2} = D^1(\lambda_2), \quad \frac{\partial}{\partial x_3} = D^2(\lambda_3), \quad \frac{\partial}{\partial x_4} = D^2(\lambda_4),
$$

from (20) we obtain a symmetric four-point generating relation for the second heavenly equation hierarchy. Then, gluing $\lambda_1$ and $\lambda_4$, we get a three-point generating relation

$$
\frac{1}{\lambda_1 - \lambda_3} D^1(\lambda_2) (D^2(\lambda_1) - D^2(\lambda_3)) \tilde{\Theta} - \frac{1}{\lambda_1 - \lambda_2} D^2(\lambda_3) (D^1(\lambda_1) - D^1(\lambda_2)) \tilde{\Theta}
$$

$$
= D^1(\lambda_1) D^1(\lambda_2) \tilde{\Theta} \cdot D^2(\lambda_3) D^2(\lambda_1) \tilde{\Theta} - D^1(\lambda_1) D^2(\lambda_3) \tilde{\Theta} \cdot D^1(\lambda_2) D^2(\lambda_1) \tilde{\Theta},
$$

which is exactly one of the set of generating relations introduced in [11, 12]; other generating relations can be obtained in a similar way.

5. Lax pair: vector fields in involution

Here we use the technical setting described in [2, 3]. The two-form form $\Omega$ defines an associated subspace $A$ in the space of vector fields (distribution) defined by the condition that
the interior product of the vector field with the form is equal to zero,
\[ i_V \Omega = 0. \]

The Plücker property (5) (or decomposability of the form) guarantees that the dimension of this distribution is exactly \((N - 2)\), where \(N\) is the number of variables. The closedness of the Plücker form leads to involutivity of this distribution and the fact that basic vector fields can be chosen to be divergence-free.

Following [2], it is easy to write down vector fields belonging to the distribution \(A\) associated with two-form \(\Omega\) (10) explicitly,

\[
U_{ijk} = \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) w_{ij} \partial_k + \left( \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_k} \right) w_{jk} \partial_i + \left( \frac{1}{\lambda - \lambda_k} - \frac{1}{\lambda - \lambda_i} \right) w_{ki} \partial_j
\]

Linear span of these vector fields is \((N - 2)\)-dimensional in the tangent space (due to Plücker relations). For the (projectively) closed two-form \(\Omega\) these vector fields are in involution. The divergence-free condition, implied by the standard closedness of the form \(\Omega\), is equivalent to the existence of potential \(\Theta\): \(w_{ij} = \Theta_{ij}\).

To find the Lax pair (a pair of vector fields in involution) corresponding to the Dubrov-Ferapontov general heavenly equation, we consider a set of four indices, e.g., 1, 2, 3, 4. Any pair of vector fields \(U_{ijk}\) with distinct \(i, j, k\) belonging to our set is in involution and constitutes a Lax pair.

In equivalent form, taking \(U_{ijk} \rightarrow (\lambda - \lambda_i)(\lambda - \lambda_j)(\lambda - \lambda_k) U_{ijk}\), we get polynomial fields of the first order in spectral parameter:

\[
U_{ijk} = \left( \lambda_i - \lambda_j \right) \Theta_{ij} \left( \lambda - \lambda_k \right) \partial_k + \left( \lambda_j - \lambda_k \right) \Theta_{jk} \left( \lambda - \lambda_i \right) \partial_i + \left( \lambda_k - \lambda_i \right) \Theta_{ki} \left( \lambda - \lambda_j \right) \partial_j.
\]

After Möbius transformation of the spectral variable, it is possible to get the Lax pair exactly in Dubrov-Ferapontov form (2).

5.1. Associated system of one-forms

Associated system of one-forms for \(\Omega\) is a linear subspace \(A^*\) in the space of one-forms dual to distribution \(A\),

\[
\langle A^*, A \rangle = 0.
\]

Due to Plücker relations, this subspace is 2-dimensional (locally in cotangent space), \(\Omega\) is decomposable and can be represented as

\[ \Omega = \psi \wedge \phi, \]

where \(\psi, \phi \in A^*\). For arbitrary vector field \(V\)

\[ i_V \Omega \in A^*. \]
Taking $V_p = \partial_p$, we get the following one-forms belonging to $A^*$,

$$\phi_p = v_p, \Omega = \sum_{i \neq p} \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_p} \right) w_{ij} dx_i.$$

For the four-dimensional case ($N = 4$), it is possible to construct the basis of polynomial forms of the first order in $\lambda$, e.g.

$$\phi_{34}^1 = (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) (w_{14} \phi_3 - w_{13} \phi_4),$$

$$\phi_{34}^2 = (\lambda - \lambda_1)(\lambda - \lambda_3)(\lambda - \lambda_4) (w_{24} \phi_3 - w_{23} \phi_4).$$

(21)

This basis is important to establish a correspondence of the Doubrov-Ferapontov general heavenly equation with Gindikin construction.

### 6. Gindikin construction

The original statement from Gindikin’s work [6] reads (citation, translated from the Russian text):

Construction of complex solutions of self-dual Einstein equation is equivalent to construction of quadratic (in $t$) bundles of holomorphic two-forms $F(t) = \psi_0 + \phi_0 t F_0 + t F_1$, $t \in \mathbb{C}$, on the four-dimensional complex manifold $M$, satisfying the conditions

(i) $F(t) \wedge F(t) = 0$ for all $t$;

(ii) $dF(t) = 0$;

(iii) $F(t) \wedge F(s) \neq 0$ for $t \neq s$. (22)

Condition (iii) means nondegeneracy and it is often convenient to ignore it in the process of calculations. Due to condition (i) $F(t)$ can be represented as

$$F(t) = \left( \phi_0 + t \phi_1 \right) \wedge \left( \psi_0 + t \psi_1 \right).$$

(23)

where $\phi_i, \psi_i$ are 1-forms Then condition (iii) guarantees non-degeneracy of the metric

$$g = \phi_0 \psi_1 - \phi_1 \psi_0,$$

(24)

and condition (ii) implies that it is right-flat (satisfies self-dual Einstein equation).

Let us consider the form

$$F(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \Omega,$$

(25)

where

$$\Omega = \sum_{i \neq j \leq 4} \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) \Theta_{ij}(x) dx_i \wedge dx_j,$$

and $\Theta$ satisfies the general heavenly equation (9). The form $F(\lambda)$ is quadratic in $\lambda$ and, due to proposition 1, satisfies the conditions required by Gindikin’s statement.
The metric can be constructed explicitly, using the basis (21), which reads
\[
\phi_{34}^1 = (\lambda - \lambda_2)(\lambda_3 - \lambda_4) \Theta_{14} \Theta_{13} dx_1 + \left( (\lambda - \lambda_4)(\lambda_3 - \lambda_2) \Theta_{23} \Theta_{14} \right. \\
\left. - (\lambda - \lambda_3)(\lambda_4 - \lambda_2) \Theta_{24} \Theta_{13} \right) dx_2 + \left( (\lambda - \lambda_2)(\lambda_3 - \lambda_4) \Theta_{13} \Theta_{34} \right. \\
\left. + (\lambda - \lambda_3)(\lambda_4 - \lambda_3) \Theta_{14} \Theta_{34} dx_4 \right.
\]
\[
\phi_{34}^2 = \left( (\lambda - \lambda_3)(\lambda_3 - \lambda_1) \Theta_{13} \Theta_{24} - (\lambda - \lambda_3)(\lambda_4 - \lambda_1) \Theta_{14} \Theta_{23} \right) dx_1 \\
\left. + (\lambda - \lambda_1)(\lambda_3 - \lambda_4) \Theta_{24} \Theta_{23} dx_2 + (\lambda - \lambda_1)(\lambda_3 - \lambda_4) \Theta_{23} \Theta_{34} dx_3 \right. \\
\left. + (\lambda - \lambda_1)(\lambda_4 - \lambda_3) \Theta_{24} \Theta_{34} dx_4 \right).
\]

It is easy to check that the form $F$ is expressed through this basis as
\[
F = \frac{2\phi_{34}^1 \wedge \phi_{34}^2}{(\lambda_3 - \lambda_4)(\Theta_{13} \Theta_{24} - \Theta_{14} \Theta_{23}) \Theta_{34}}
\]

The metric $g$ is then given by the formula
\[
g = \frac{2\left( \phi_{34(0)}^1 \phi_{34(1)}^2 - \phi_{34(1)}^1 \phi_{34(0)}^2 \right)}{(\lambda_3 - \lambda_4)(\Theta_{13} \Theta_{24} - \Theta_{14} \Theta_{23}) \Theta_{34}}
\]

where by the subscripts $(0)$, $(1)$ we denote the terms of the zero and first order with respect to $\lambda$. Explicit expressions for the components of the metric are
\[
g_{ij} = 2G \Theta_{ij} \Theta_{ik} \Theta_{jp}, \\
g_{kp} = 2G \Theta_{ik} \Theta_{jp} + \Theta_{ip} \Theta_{jk},
\]

(26)

where $i, j, k, p$ are pairwise distinct,
\[
G = \frac{(\lambda_3 - \lambda_4)(\lambda_k - \lambda_p)}{\Theta_{ik} \Theta_{jp} - \Theta_{ip} \Theta_{jk}}.
\]

It is easy to check that the expression for $G$ is invariant under arbitrary permutation of indices due to the general heavenly equation. Thus, starting from a solution of the general heavenly equation (9), via the Gindikin method, we have constructed a (complex) metric (26) satisfying self-dual Einstein equations.

It is interesting to note that the work [8] gives a direct recipe to calculate a self-dual conformal structure for equations of the heavenly type through the symbol of linearization of equation, which represents a symmetric bivector defining a conformal structure. For the general heavenly equation (9) this bivector reads
\[
\gamma^{ij} = e^{ijkp} (\lambda_i - \lambda_j)(\lambda_k - \lambda_p) \Theta_{kp},
\]

and it is easy to check that the inverse matrix to $\gamma^{ij}$ gives the metric $g_{ij}$ (26) (up to a factor), thus the conformal structure is the same as in Gindikin construction. This natural conjecture belongs to E V Ferapontov and it can be proved for the general case using the representation of the symbol of linearization in terms of the basis of vector fields of the first order in $\lambda$ [8], which is in some sense dual to representation (24), and results of the works [9, 10]. However, to get the self-dual metric satisfying the Einstein equation (Ricci-flat), it is important to define the normalization, because this property is not conformally-invariant, and Gindikin construction provides a direct answer to this question (26).
7. \( \bar{\partial} \)-dressing scheme

In this section we use the technique developed in [11] in the context of Plebański second heavenly equation hierarchy (see also Appendix). Due to the fact that the main results were formulated in variational form, they are applicable to our present setting with minor modifications.

The two-form \( \Omega \) can be represented as

\[
\Omega = ds^1 \wedge ds^2.
\]

Two properties of \( \Omega \) are now identically satisfied (it is Plücker and closed), now the problem is to construct \( S^1, S^2 \) to get \( \Omega \) with the necessary analytic properties.

Let us consider nonlinear vector \( \partial \) problem in some region \( G \),

\[
\partial S^1 = W_2\left( \lambda, \bar{\lambda}; S^1, S^2 \right), \quad W_2 := \frac{\partial W}{\partial S^2}, \\
\bar{\partial} S^2 = -W_1\left( \lambda, \bar{\lambda}; S^1, S^2 \right), \quad W_1 := \frac{\partial W}{\partial S^1},
\]

and \( W(\lambda, \bar{\lambda}; S^1, S^2) \) is some function defined in \( G \). This problem provides analyticity of the form \( \Omega = ds^1 \wedge ds^2 \) in \( G \).

We search for solutions of the form

\[
S^1 = S_0^1 + S^1, \quad S^2 = S_0^2 + S^2
\]

where \( S^1, S^2 \) are analytic outside \( G \) and go to zero at infinity, \( S_0^1, S_0^2 \) are analytic in \( G \) (normalization or vacuum term, compare (13), (16))

\[
S_0^1 = \sum_{i=1}^{N} a_i x_i, \quad S_0^2 = \sum_{i=1}^{N} b_i x_i.
\]

Then the form \( \Omega \) has the required analytic structure.

The \( \bar{\partial} \) problem can be obtained by variation of the action

\[
f = \frac{1}{2\pi i} \int_G \left( S^2 \hat{\delta} S^1 - W\left( \lambda, \bar{\lambda}, S^1, S^2 \right) \right) d\lambda \wedge d\bar{\lambda},
\]

where one should consider independent variations of \( \hat{S} \), possessing required analytic properties, keeping \( S_0 \) fixed. Using the results of the work [11] in our setting, we come to the following statement:

**Proposition 2.** The function

\[
\Theta(x) = \Theta_0 + \frac{1}{2\pi i} \int_G \left( S^2(\bar{S}^1(x)) - W\left( \lambda, \bar{\lambda}, S^1(x), S^2(x) \right) \right) d\lambda \wedge d\bar{\lambda},
\]

where \( \Theta_0 \) is a vacuum term defined by formula (14),

\[
\Theta_0 = \frac{1}{2} \sum_{i \neq j} a_i b_j - a_j b_i \frac{x_i x_j}{\lambda_i - \lambda_j}.
\]

i.e., the action (28) evaluated on the solution of the \( \bar{\partial} \) problem (27) plus a term quadratic in \( x_i \) is a solution of the hierarchy of Doubrov-Ferapontov general heavenly equations (11).

A class of solutions of the general heavenly equation hierarchy (11) in terms of implicit functions (similar to [4, 6]) can be constructed using the choice
where $\delta(\lambda - \mu_i), \delta(\lambda - \nu_i)$ are two-dimensional delta functions in the complex plane, and $F_i, G_i$ are some functions of one variable. The $\tilde{\partial}$ problem (27) in this case reads

$$\tilde{\partial}S^1 = 2\pi i \sum_{i=1}^{M} \delta(\lambda - \nu_i) G_i(S^2)$$

$$\tilde{\partial}S^2 = -2\pi i \sum_{i=1}^{M} \delta(\lambda - \mu_i) F_i(S^1).$$

(31)

The solutions of the $\tilde{\partial}$ problem are then of the form

$$S^1 = \sum_{i=1}^{M} \frac{f_i}{\lambda - \nu_i}, \quad S^2 = \sum_{i=1}^{M} \frac{g_i}{\lambda - \mu_i},$$

and from (27) the functions $f_i, g_i$ are defined as implicit functions,

$$f_i(x) = G_i\left(\sum_{j=1}^{N} \frac{b_j x_j}{\nu_j - \lambda_j} + \sum_{k=1}^{M} \frac{g_k(x)}{\mu_k - \lambda_k}\right),$$

$$g_i(x) = -F_i\left(\sum_{j=1}^{N} \frac{a_j x_j}{\mu_j - \lambda_j} + \sum_{k=1}^{M} \frac{f_k(x)}{\mu_k - \nu_k}\right).$$

(32)

The potential $\Theta$ solving the general heavenly equation hierarchy is then given by the formula (29), it depends on the set of arbitrary functions of one variable $F_i, G_i$,

$$\Theta(x) = \Theta_0 + \sum_{i=1}^{M} F_i(S^1(\mu_i)) + \sum_{i=1}^{M} G_i(S^2(\nu_i)) + \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{f_i g_j}{\nu_i - \mu_j},$$

(33)

where $\Theta_0$ is given by (30),

$$S^1 = S_0^1 + S^1 = \sum_{i=1}^{N} \frac{a_i x_i}{\lambda - \lambda_i} + \sum_{i=1}^{M} \frac{f_i}{\lambda - \nu_i},$$

$$S^2 = S_0^2 + S^2 = \sum_{i=1}^{N} \frac{b_i x_i}{\lambda - \lambda_i} + \sum_{i=1}^{M} \frac{g_i}{\lambda - \mu_i},$$

and functions $f_i, g_i$ are defined as implicit functions by equations (32). Formula (33) corresponds to the special solution of hyper-Kähler hierarchy derived in [4].

8. On the multidimensional hyper-Kähler case

We will briefly outline the formulation of the multidimensional case, which is mostly similar to the four-dimensional case discussed above. Let us consider the two-form $\Omega$ of the same structure (3), but now satisfying the conditions

$$\Omega \wedge ... \wedge \Omega = 0 \quad (N \text{ times}),$$

$$d\Omega = 0,$$

(34)
that correspond to the setting for the multidimensional hyper-Kähler case considered in [4, 7].

In terms of construction of the works [2, 3], the basic decomposable (Plücker) form is

$$\Omega = \Omega \wedge \ldots \wedge \Omega (N - 1 \text{ times}),$$

and the multidimensional hyper-Kähler case is a reduction of the general case.

Closedness of $\Omega$, as in the four-dimensional case, is equivalent to the existence of potential $\Theta$. Then from relation (34) for every set of $2N$ distinct indices $i_1, \ldots, i_{2N}$ we obtain a $2N$-dimensional homogeneous equation of degree $N$, which may be considered as the ‘general hyper-Kähler equation’

$$\sum \epsilon_{i_1 \ldots i_{2N}} \left( \lambda_{i_1} - \lambda_{i_2} \right) \times \ldots \times \left( \lambda_{i_{2N-1}} - \lambda_{i_{2N}} \right) \Theta_{i_1 i_2} \times \ldots \times \Theta_{i_{2N-1} i_{2N}} = 0,$$

where summation is over permutation of indices. This equation is a generating equation for multidimensional hyper-Kähler hierarchy [4].

The form $\Omega$ can be represented as

$$\Omega = S^1 \wedge S^2 \wedge \ldots,$$

Similar to the four-dimensional case, it is possible to formulate a $\partial$-dressing scheme and find a formula for $\Theta$ completely analogous to (29).

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Appendix: Second heavenly equation hierarchy and vertex variables

We will briefly outline the basic facts concerning the second heavenly equation hierarchy necessary to obtain the generating relations, see [4, 5] and [11, 12] for more detail. In the framework of the second heavenly equation hierarchy the form $\Omega$ can be represented as

$$\Omega = dS^1 \wedge dS^2,$$

(35)

where $S^1, S^2$ are the series

$$S^1 = \sum_{-\infty}^{\infty} u_k^1 \lambda^k, \quad S^2 = \sum_{-\infty}^{\infty} u_k^2 \lambda^k.$$

The coefficients for nonnegative $k$ are considered as independent variables (times) of the hierarchy,

$$T^1_k := u_k^1, \quad T^2_k := u_k^2, \quad 0 \leq k < \infty,$$

and for negative $k$ as dependent variables (fields). The form $\Omega$ is evidently decomposable and closed, and the hierarchy is defined by the relation connected with holomorphic properties of the form,

$$\left( dS^1 \wedge dS^2 \right)_- = 0,$$

(36)

where minus means the projection to negative powers. This relation implies that coefficients of the form $\Omega$ are polynomial and generates the hierarchy in Lax-Sato form. It also implies that one-form $\text{Res}(S^2 dS^1 - S^1 dS^2)$ is closed and defines a function $\Theta$. 

13
\[ \theta = \text{Res} \left( S^2 \text{d}S^2 - S^1 \text{d}S^1 \right). \]  

(37)

In particular, from (37) we get
\[ \frac{\partial \theta}{\partial T^k_1} = u^1_{-k-1}, \quad \frac{\partial \theta}{\partial T^2_k} = -u^2_{-k-1}, \]

which defines coefficients of the form \( \Omega \) through \( \theta \).

To introduce vertex variables \( x_i \) to the hierarchy, leading to the appearance of rational terms in the form \( \Omega \), we suggest that the series \( S^1, S^2 \) and \( S^1, S^2 \) define certain holomorphic functions, namely that the functions \( S^1_0 = S^1_0, S^2_0 = S^2_0 \) are analytic inside the unit disk, and \( S^1 = S^1_0, S^2 = S^2_0 \) are analytic outside the unit disk and decrease at infinity. The functions \( S^1 = S^1_0 + S^1_2, S^2 = S^2_0 + S^2_2 \) in this case can be considered as functions on the unit circle. The projection "outside" in relation (36) can be considered as the standard "out" projector to the space of functions analytic outside the unit disk, and the function \( \theta \) can be defined as a functional on the space of functions analytic in the unit disk by the variational one-form (see [11, 12])

\[ \delta \theta = \frac{1}{2\pi i} \oint \left( \tilde{S}^2 \delta S^1_0 - \tilde{S}^1 \delta S^2_0 \right) d\lambda, \]

(38)

which is closed due to the variational version of relation (36). Variables of the hierarchy appear as we introduce some parameterization of \( S^1_0, S^2_0 \). Considering parameterization (18),

\[ S^1_0 = \sum_i \frac{a_i x_i}{\lambda - \lambda_i} + \sum_{n=0}^{\infty} t^n_n \lambda^n, \quad S^2_0 = \sum_i \frac{b_i x_i}{\lambda - \lambda_i} + \sum_{n=0}^{\infty} t^n_n \lambda^n, \]

where \(|\lambda_i| > 1\), from (38) we get
\[ \frac{\partial \theta}{\partial t^1_k} = u^1_{-k-1}, \quad \frac{\partial \theta}{\partial t^2_k} = -u^2_{-k-1}, \quad \frac{\partial \theta}{\partial x_i} = a_i S^1(\lambda_i) - b_i S^2(\lambda_i). \]  

(39)

The form (35) corresponding to this set of variables contains rational and polynomial in \( \lambda \) terms, its coefficients can be expressed through \( \theta \) using relations (39). Comparing the part of \( \Omega \) containing \( dx_i \wedge dx_j \) to (10), (15), we come to the conclusion that \( \theta \) defined by formula (38) corresponds to \( \tilde{\Theta} \) of relation (15). The identity (19) for vertex derivatives,

\[ \frac{\partial}{\partial x_i} = a_i D^1(\lambda_i) + b_i D^2(\lambda_i), \]

follows from the initial setting of the hierarchy, it is also easy to check it using formula (38), taking into account that

\[ \begin{align*}
    \left( \frac{\partial}{\partial x_i} - a_i D^1(\lambda_i) - b_i D^2(\lambda_i) \right) S^1_0 &= 0, \\
    \left( \frac{\partial}{\partial x_i} - a_i D^1(\lambda_i) - b_i D^2(\lambda_i) \right) S^2_0 &= 0.
\end{align*} \]

Thus we have all the ingredients to consider the general heavenly equation as a generating equation for the second heavenly equation hierarchy.

In a similar way it is possible to consider a more general hyper-Kähler hierarchy containing first and second heavenly equations introduced by Takasaki [4]. In this case we will have two types of vertex operators: corresponding to infinity (which we used for the second
heavenly) and corresponding to zero, and both types may be substituted to the general heavenly equation to obtain some generating relations.

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