Time-Fractional Optimal Control
of Initial Value Problems on Time Scales

Gaber M. Bahaa and Delfim F. M. Torres

Abstract We investigate Optimal Control Problems (OCP) for fractional systems involving fractional-time derivatives on time scales. The fractional-time derivatives and integrals are considered, on time scales, in the Riemann–Liouville sense. By using the Banach fixed point theorem, sufficient conditions for existence and uniqueness of solution to initial value problems described by fractional order differential equations on time scales are known. Here we consider a fractional OCP with a performance index given as a delta-integral function of both state and control variables, with time evolving on an arbitrarily given time scale. Interpreting the Euler–Lagrange first order optimality condition with an adjoint problem, defined by means of right Riemann–Liouville fractional delta derivatives, we obtain an optimality system for the considered fractional OCP. For that, we first prove new fractional integration by parts formulas on time scales.

Keywords: fractional derivatives and integrals on time scales, initial value problems, optimal control.

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1 Introduction

Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. We consider the following initial value problem:

$$
\begin{align*}
\mathbb{T}_{t_0}^\alpha D_t^\alpha y(t) &= f(t, y(t)), & t \in [t_0, t_0 + a] = \mathcal{J} \subseteq \mathbb{T}, & 0 < \alpha < 1, \\
\mathbb{T}_{t_0}^{1-\alpha} I_t^{1-\alpha} y(t_0) &= 0,
\end{align*}
$$

where $\mathbb{T}_{t_0}^\alpha D_t^\alpha$ is the (left) Riemann–Liouville fractional derivative operator or order $\alpha$ defined on $\mathbb{T}$ and $\mathbb{T}_{t_0}^{1-\alpha} I_t^{1-\alpha}$ is the (left) Riemann–Liouville fractional integral operator or order $1 - \alpha$ defined on $\mathbb{T}$, as introduced in [24] (see also [45, 46]), and function $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a right-dense continuous function. Necessary and sufficient conditions for the existence and uniqueness of solution to problem (1) are already discussed in [24]. Here, our goal is to prove optimality conditions for such systems.

Fractional Calculus (FC) is a generalization of classical calculus. It has been reported in the literature that systems described using fractional derivatives give a more realistic behavior. There exists many definitions of a fractional derivative. Commonly used fractional derivatives are the classical Riemann–Liouville and Caputo derivatives on continuous time scales. Fractional derivatives and integrals of Riemann–Liouville and Caputo types have a vast number of applications, across many fields of science and engineering. For example, they can be used to model controllability, viscoelastic flows, chaotic systems, Stokes problems, thermo-elasticity, several vibration and diffusion processes, bioengineering problems, and many other complex phenomena: see, e.g., [6, 11] and references therein.

Fractional optimal control problems on a continuous time scale have attracted several authors in the last two decades, and many techniques have been developed for solving such problems, involving classical fractional derivatives. Agrawal [6, 7] presented a general formulation and proposed a numerical method to solve such problems. In those papers, the fractional derivative was defined in the Riemann–Liouville sense and the formulation was obtained by means of a fractional variational principle and the Lagrange multiplier technique. Using new techniques, Frederico and Torres [32, 33] obtained Noether-like theorems for fractional optimal control problems in both Riemann–Liouville and Caputo senses. In [39, 40], Mophou and N’Guérékata studied the fractional optimal control of diffusion equations involving the classical Riemann–Liouville derivatives. In [43], Ozdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the classical Riemann–Liouville sense. For the state of the art and many generalizations, see the recent books [9, 37].

The theory of fractional differential equations, specifically the question of existence and uniqueness of solutions, is a research topic of great importance [1, 12, 34]. Another important area of study is dynamic equations on time scales, which goes back to 1988 and the work of Aulbach and Hilger, and has been used with success to unify differential and difference equations [5, 10, 26]. Starting with a linear dynamic equation, Bastos et al. have introduced the notion of fractional-order deriva-
To the best of our knowledge, the study of fractional optimal control problems for dynamical systems on time scales is under-developed, at least when compared to the continuous and discrete cases [31, 36]. Motivated by this fact, in this paper an Optimal Control Problem (OCP) for fractional initial value systems involving fractional-time derivatives on time scales is considered. The fractional-time derivative and integral are considered in the Riemann–Liouville sense on time scales, as introduced in [24].

We prove necessary optimality conditions for such OCPs. The performance index of the Fractional Optimal Control Problem (FOCP) is considered as a non-autonomous delta integral of a function depending on state and control variables, and where the dynamic control system is expressed by a delta-differential system. Interpreting the Euler–Lagrange first order optimality condition with an adjoint problem, defined by means of the time-scale right fractional derivative in the sense of Riemann–Liouville, we obtain an optimality system for the FOCP on time scales.

2 Preliminaries

In this section, we collect notations, definitions, and results, which are needed in the sequel. We use $C(\mathcal{J}, \mathbb{R})$ for the Banach space of continuous functions $y$ with the norm $\|y\|_\infty = \sup \{|y(t)| : t \in \mathcal{J}\}$, where $\mathcal{J}$ is a time-scale interval.

2.1 Time-scale essentials

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The reader interested on the calculus on time scales is referred to the books [26, 27]. For a survey, see [5]. Any time scale $\mathbb{T}$ is a complete metric space with the distance $d(t, s) = |t - s|$, $t, s \in \mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, etc. In particular, for a given number $\delta > 0$, the $\delta$-neighborhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t, s) < \delta$. We also have, for functions $f : \mathbb{T} \to \mathbb{R}$, the concepts of limit, continuity, and the properties of continuous functions on a general complete metric space. Roughly speaking, the calculus on time scales begins by introducing and investigating the concept of derivative for functions $f : \mathbb{T} \to \mathbb{R}$. In the definition of derivative, an important role is played by the so-called jump operators.
Remark 1 In Definition 1, we put \( \inf = \) (See [27]). Let \( \sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \), and the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) by \( \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \).

Definition 2 (Delta derivative [4]) Assume \( \rho \) and \( T \) is defined by \( \rho(t) = \sigma(t) - t \). The function \( f : \mathbb{T} \to \mathbb{R} \), which is obtained from the time scale \( \mathbb{T} \) as follows: if \( \mathbb{T} \) has a left-scattered maximum \( m \), then \( \mathbb{T}^\kappa := \mathbb{T} \setminus \{m\} \); otherwise, \( \mathbb{T}^\kappa := \mathbb{T} \).

Definition 3 (See [27]). A function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( \mathbb{T} \) and its left-sided limits exist (finite) at right-scattered points in \( \mathbb{T} \). The set of rd-continuous functions \( f : \mathbb{T} \to \mathbb{R} \) is denoted by \( C_{rd}. \)

Definition 4 (See [27]). Let \( [a, b] \) denote a closed bounded interval in \( \mathbb{T} \). A function \( F : [a, b] \to \mathbb{R} \) is called a delta antiderivative of function \( f : [a, b] \to \mathbb{R} \) provided \( f \) is continuous on \( [a, b] \), delta differentiable on \( [a, b] \), and \( F\Delta(t) = f(t) \) for all \( t \in [a, b] \). Then, we define the \( \Delta \)-integral of \( f \) from \( a \) to \( b \) by

\[
\int_a^b f(t)\Delta t := F(b) - F(a).
\]

Proposition 1 (See [8]) Suppose \( \mathbb{T} \) is a time scale and \( f \) is an increasing continuous function on the time-scale interval \( [a, b] \). If \( F \) is the extension of \( f \) to the real interval \( [a, b] \) given by

\[
F(s) := \begin{cases} 
  f(s) & \text{if } s \in \mathbb{T}, \\
  f(t) & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T},
\end{cases}
\]

then

\[
\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.
\]
2.2 Fractional derivative and integral on time scales

We adopt a recent notion of fractional derivative on time scales introduced in [24], which is based on the notion of fractional integral on time scales $\mathbb{T}$. This is in contrast with [22, 23, 25], where first a notion of fractional differentiation on time scales is introduced and only after that, with the help of such concept, the fraction integral is defined. The classical gamma and beta functions are used.

**Definition 5** (Gamma function) For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:

$$
\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}dx.
$$

**Definition 6** (Beta function) The beta function, also called the Euler integral of first kind, is the special function $B(x, y)$ defined by

$$
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \quad y > 0.
$$

**Remark 2** The gamma function satisfies the following property: $\Gamma(t+1) = t\Gamma(t)$. The beta function can be expressed through the gamma function by

$$
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
$$

**Definition 7** (Fractional integral on time scales [24]) Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and $h$ is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the left fractional integral of order $\alpha$ of $h$ is defined by

$$
\mathcal{T}_a^I \alpha t h(t) := \int_a^t (t-s)^{\alpha-1} \Gamma(\alpha) h(s)\Delta s.
$$

The right fractional integral of order $\alpha$ of $h$ is defined by

$$
\mathcal{T}_t^I \alpha b h(t) := \int_t^b (s-t)^{\alpha-1} \Gamma(\alpha) h(s)\Delta s,
$$

where $\Gamma$ is the gamma function.

**Definition 8** (Riemann–Liouville fractional derivative on time scales [24]) Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The left Riemann–Liouville fractional derivative of order $\alpha$ of $h$ is defined by

$$
\mathcal{T}_a^D \alpha t h(t) := \left(\mathcal{T}_a^I \alpha t h(t)\right)^\Delta = \frac{1}{\Gamma(1-\alpha)} \left[\int_a^t (t-s)^{-\alpha} h(s)\Delta s\right]^\Delta.
$$

The right Riemann–Liouville fractional derivative of order $\alpha$ of $h$ is defined by
\[ T_\alpha^T D_t^\alpha h(t) := -\left( T_\alpha^T I_b^1 D_t^\alpha h(t) \right)^\Delta = \frac{-1}{\Gamma(1-\alpha)} \left( \int_t^b (s-t)^{-\alpha} h(s) \Delta s \right)^\Delta. \]

**Definition 9** (Caputo fractional derivative on time scales [8]) Let \( T \) be a time scale, \( t \in T, \) \( 0 < \alpha < 1, \) and \( h : T \to \mathbb{R}. \) The left Caputo fractional derivative of order \( \alpha \) of \( h \) is defined by

\[ T_\alpha^C D_t^\alpha h(t) := T_\alpha^C I_b^1 D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b (t-s)^{-\alpha} h(s) \Delta s. \]

The right Caputo fractional derivative of order \( \alpha \) of \( h \) is defined by

\[ T_\alpha^C D_t^\alpha h(t) := -T_\alpha^C I_b^1 D_t^\alpha h(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} h(s) \Delta s. \]

The relation between the left/right RLFD and the left/right CFD is as follows:

\[ T_\alpha^C D_t^\alpha x(t) = T_\alpha^C D_t^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(k+1)} (t-a)^{k-\alpha}, \]

\[ T_\alpha^C D_t^\alpha x(t) = T_\alpha^C D_t^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)}{\Gamma(k+1)} (b-t)^{k-\alpha}. \]

If \( x \) and \( x^{(i)}, i = 1, \ldots, n-1, \) vanish at \( t = a, \) then \( T_\alpha^C D_t^\alpha x(t) = T_\alpha^C D_t^\alpha x(t), \) and if they vanish at \( t = b, \) then \( T_\alpha^C D_t^\alpha x(t) = T_\alpha^C D_t^\alpha x(t). \) Furthermore, \( T_\alpha^C D_t^\alpha c = 0, \) where \( c \) is a constant, and

\[ T_\alpha^C D_t^\alpha t^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lfloor \alpha \rfloor, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lfloor \beta \rfloor, \end{cases} \]

where \( \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \)

**Remark 3** If \( T = \mathbb{R}, \) then Definition [8] gives the classical left and right Riemann–Liouville fractional derivatives [44]. Similar comment for Definition [9]. For different extensions of the fractional derivative to time scales using the Caputo approach, see [18]. For local approaches to fractional calculus on time scales, we refer the reader to [22, 23, 25]. Here we restrict ourselves to the delta approach to time scales. Analogous definitions are, however, trivially obtained for the nabla approach to time scales by using the duality theory of [3, 30].

### 2.3 Properties of the time-scale fractional operators

We recall some fundamental properties of the fractional operators on time scales.
Proposition 2 (See Proposition 15 of [24]). Let $\mathbb{T}$ be a time scale with derivative $\Delta$, and $0 < \alpha < 1$. Then, $\frac{\tau}{a}D^\alpha_{\tau} = \Delta \circ \frac{\tau}{a}I^{1-\alpha}_{\tau}$.

Proposition 3 (See Proposition 16 of [24]). For any function $h$ integrable on $[a, b]$, the Riemann–Liouville $\Delta$-fractional integral satisfies $\frac{\tau}{a}I^\alpha_{\tau} \circ \frac{\tau}{a}I^\beta_{\tau} = \frac{\tau}{a}I^{\alpha+\beta}_{\tau}$ for $\alpha > 0$ and $\beta > 0$.

Proposition 4 (See Proposition 17 of [24]). For any function $h$ integrable on $[a, b]$ one has $\frac{\tau}{a}D^\alpha_{\tau} \circ \frac{\tau}{a}I^\alpha_{\tau} h = h$.

Corollary 1 (See Corollary 18 of [24]). For $0 < \alpha < 1$, we have $\frac{\tau}{a}D^\alpha_{\tau} \circ \frac{\tau}{a}I^\alpha_{\tau} = \text{Id}$ and $\frac{\tau}{a}I^{1-\alpha}_{\tau} \circ \frac{\tau}{a}I^\alpha_{\tau} = \text{Id}$, where $\text{Id}$ denotes the identity operator.

Definition 10 (See [24]) For $\alpha > 0$, we denote by $\frac{\tau}{a}I^\alpha_{\tau}(\mathbb{J})$ the space of functions that can be represented by the Riemann–Liouville $\Delta$ integral of order $\alpha$ of some $C([a, b])$-function.

Theorem 1 (See Theorem 20 of [24]). Let $f \in C([a, b])$ and $\alpha > 0$. Function $f \in \frac{\tau}{a}I^\alpha_{\tau}(\mathbb{J})$ if and only if $\frac{\tau}{a}I^{1-\alpha}_{\tau} f \in C([a, b])$ and $\left.\left(\frac{\tau}{a}I^{1-\alpha}_{\tau} f(t)\right)\right|_{t=a} = 0$.

Theorem 2 (See Theorem 21 of [24]) Let $\alpha > 0$ and $f \in C([a, b])$ satisfy the conditions in Theorem 1. Then, $\left(\frac{\tau}{a}I^\alpha_{\tau} \circ \frac{\tau}{a}D^\alpha_{\tau}\right)(f) = f$.

2.4 Existence of solutions to fractional IVPs on time scales

Let $\mathbb{T}$ be a time scale and $\mathbb{J} = [t_0, t_0 + a] \subset \mathbb{T}$. Consider the fractional order initial value problem (1) defined on $\mathbb{T}$. Then the function $y \in C(\mathbb{J}, \mathbb{R})$ is a solution of problem (1) if $\frac{\tau}{b}D^\alpha_{\tau} y(t) = f(t, y)$ on $\mathbb{J}$ and $\left.\frac{\tau}{b}I^{1-\alpha}_{\tau} y(t)\right|_{t=a} = 0$.

Theorem 3 (See Theorem 24 of [24]) If $f : \mathbb{J} \times \mathbb{R} \rightarrow \mathbb{R}$ is a rd-continuous bounded function for which there exists $M > 0$ such that $|f(t, y)| \leq M$ for all $t \in \mathbb{J}$ and $y \in \mathbb{R}$, then problem (1) has a solution on $\mathbb{J}$.

3 Main Results

We begin by proving formulas of integration by parts in Section 3.1 which are then used in Section 3.2 to prove necessary optimality conditions for nonlinear Riemann–Liouville fractional optimal control problems (FOCPs) on time scales.
3.1 Fractional integration by parts on time scales

Our first result gives integration by parts formulas for fractional integrals and derivatives on time scales. For the relation between integration on time scales and Lebesgue integration we refer the reader to [29].

\textbf{Theorem 4} Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, where $p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Moreover, let

\[
\frac{\gamma}{\alpha} I^\alpha_a f (L_p) := \left\{ f : f = \frac{\gamma}{\alpha} I^\alpha_a g, \ g \in L_p(a, b) \right\}
\]

and

\[
\frac{\gamma}{\alpha} I^\alpha_b f (L_p) := \left\{ f : f = \frac{\gamma}{\alpha} I^\alpha_b g, \ g \in L_p(a, b) \right\}.
\]

The following integration by parts formulas hold.

(a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

\[
\int_a^b \varphi(t) \left( \frac{\gamma}{\alpha} I^\alpha_a \psi \right)(t) \Delta t = \int_a^b \psi(t) \left( \frac{\gamma}{\alpha} I^\alpha_b \varphi \right)(t) \Delta t. \tag{3}
\]

(b) If $g \in \frac{\gamma}{\alpha} I^\alpha_a f (L_p)$ and $f \in \frac{\gamma}{\alpha} I^\alpha_b f (L_q)$, then

\[
\int_a^b g(t) \left( \frac{\gamma}{\alpha} D^\alpha_a f \right)(t) \Delta t = \int_a^b f(t) \left( \frac{\gamma}{\alpha} D^\alpha_b g \right)(t) \Delta t. \tag{4}
\]

(c) For Caputo fractional derivatives, if $g \in \frac{\gamma}{\alpha} I^\alpha_b (L_p)$ and $f \in \frac{\gamma}{\alpha} I^\alpha_a (L_q)$, then

\[
\int_a^b g(t) \left( \frac{\gamma}{\alpha} D^\alpha_a f \right)(t) \Delta t = \left[ \frac{\gamma}{\alpha} I^{1-\alpha}_a g(t) \cdot f(t) \right]_a^b + \int_a^b f(\sigma(t)) \left( \frac{\gamma}{\alpha} D^\alpha_b g \right)(t) \Delta t
\]

and

\[
\int_a^b g(t) \left( \frac{\gamma}{\alpha} D^\alpha_b f \right)(t) \Delta t = - \left[ \frac{\gamma}{\alpha} I^{1-\alpha}_b g(t) \cdot f(t) \right]_a^b + \int_a^b f(\sigma(t)) \left( \frac{\gamma}{\alpha} D^\alpha_a g \right)(t) \Delta t.
\]

\textbf{Proof} \ (a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then, from Definition[7] we get

\[
\int_a^b \varphi(t) \left( \frac{\gamma}{\alpha} I^\alpha_a \psi \right)(t) \Delta t = \int_a^b \varphi(t) \left( \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \psi(s) \Delta s \right) \Delta t.
\]

Interchanging the order of integrals (see [24]), we reach at

\[
\int_a^b \varphi(t) \left( \frac{\gamma}{\alpha} I^\alpha_a \psi \right)(t) \Delta t = \int_a^b \psi(t) \left( \frac{\gamma}{\alpha} I^\alpha_b \varphi \right)(t) \Delta t.
\]

(b) If $g \in \frac{\gamma}{\alpha} I^\alpha_b (L_p)$ and $f \in \frac{\gamma}{\alpha} I^\alpha_a (L_q)$, then, from Definition[8] we get
\[
\int_a^b g(t) \left( \frac{\alpha}{\Gamma(1 - \alpha)} \int_t^a (t - s)^{-\alpha} f(s) \Delta s \right) \Delta t.
\]

Interchanging the order of integrals, we obtain that
\[
\int_a^b g(t) \left( \frac{\alpha}{\Gamma(1 - \alpha)} \int_t^a f(t) \Delta t \right) \Delta t.
\]

(c) If \( g \in \mathbb{T}^{\alpha}_{a} \mathcal{I}^{\alpha}_{b}(L_p) \) and \( f \in \mathbb{T}^{\alpha}_{a} \mathcal{I}^{\alpha}_{b}(L_q) \), then, from Definition 3, we get
\[
\int_a^b g(t) \left( \frac{\alpha}{\Gamma(1 - \alpha)} \int_t^a f(t) \Delta t \right) \Delta t.
\]

Interchanging the order of the integrals, and by using integration by parts on time scales, we conclude that
\[
\int_a^b g(t) \left( \frac{\alpha}{\Gamma(1 - \alpha)} \int_t^a f(t) \Delta t \right) \Delta t + \left[ \frac{\alpha}{\Gamma(1 - \alpha)} g(t) \cdot f(t) \right]_a^b.
\]

The second relation is obtained in a similar way. □

3.2 Nonlinear Riemann–Liouville FOCPs on time scales

Let \( \mathbb{T} \) be a given time scale with \( t_0, t_f \in \mathbb{T} \) and let us consider a control system given by the fractional differential equation
\[
\mathbb{T}^{\alpha}_{t_0} x(t) = f(x(t), u(t), t), \quad t \in \mathbb{T},
\]
subject to
\[
\mathbb{T}^{1-\alpha}_{t_0} x(t_0) = x_0,
\]
where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) are the state and control vectors, respectively, function \( f : \mathbb{R}^{n \times m \times 1} \to \mathbb{R}^n \) is a nonlinear vector function, and \( x_0 \in \mathbb{R}^n \) is the specified initial state vector. A similar problem is studied in [14] for problems involving AB derivatives in Caputo sense on continuous time scales. Here we study it within Riemann–Liouville derivatives on arbitrary time scales. In order to achieve a desired behavior in terms of performance requirements, we select a cost index for the dynamical system (5)–(6). In selecting the performance index, the designer attempts to define a mathematical expression that, when minimized, indicates that the system is performing in the most desirable manner. Thus, choosing a performance cost index is a translation of system’s physical requirements into mathematical terms [13]. For the fractional dynamic system (5)–(6), we choose the following performance index:
\[
J[x, u] = \int_{t_0}^{t_f} L(x(t), u(t), t) \Delta t \longrightarrow \min,
\]
where \( L : \mathbb{R}^{n \times m \times 1} \rightarrow \mathbb{R} \) is a scalar function. In the following, we derive a necessary optimality condition corresponding to the considered fractional optimal control problem (5)–(7). Under given considerations, the following theorem holds true.

**Theorem 5 (Necessary optimality conditions)** Let \((x(\cdot), u(\cdot))\) be a minimizer of problem (5)–(7). Then, there exists a function \( \lambda(\cdot) \) for which the triplet \((x(\cdot), \lambda(\cdot), u(\cdot))\) satisfies:

(i) the Hamiltonian system

\[
\begin{align*}
\tau \frac{D^\alpha}{D_t^\alpha} x(t) &= \frac{\partial H}{\partial x}(x(t), \lambda(t), u(t), t), \quad t \in T, \\
\tau \frac{D^\alpha}{D_t^\alpha} \lambda(t) &= \frac{\partial H}{\partial \lambda}(x(t), \lambda(t), u(t), t), \quad t \in T;
\end{align*}
\]

(ii) the stationary condition

\[
\frac{\partial H}{\partial u}(x(t), \lambda(t), u(t), t) = 0, \quad t \in T,
\]

where \( H \) is a scalar function, called the Hamiltonian, defined by

\[
H(x, \lambda, u, t) = L(x, u, t) + \lambda^T \frac{f(x, u)}{D_t}. \tag{10}
\]

**Proof** To deduce the necessary optimality conditions that the optimal pair \((x(\cdot), u(\cdot))\) must satisfy, we use the Lagrange multiplier technique to adjoin the dynamic constraint (5) to the performance index (7). Thus, we form the augmented functional

\[
J_a[x, \lambda, u] = \int_{t_0}^{t_f} \left( H(x(t), \lambda(t), u(t), t) - \lambda^T(t) \tau \frac{D^\alpha}{D_t^\alpha} x(t) \right) \Delta t, \tag{11}
\]

where \( \lambda(t) \in \mathbb{R}^n \) is the Lagrange multiplier, also known as the costate or adjoint variable. Taking the first variation of the augmented performance index \( J_a[x, \lambda, u] \) given by (11), we obtain that

\[
\delta J_a[x, \lambda, u] = \int_{t_0}^{t_f} \left( \left[ \frac{\partial H}{\partial x} \right]^T \delta x(t) + \left[ \frac{\partial H}{\partial \lambda} - \tau \frac{D^\alpha}{D_t^\alpha} x(t) \right]^T \delta \lambda(t) \right.
\]

\[
+ \left. \left[ \frac{\partial H}{\partial u} \right]^T \delta u(t) - \lambda^T(t) \tau \frac{D^\alpha}{D_t^\alpha} \delta x(t) \right) \Delta t. \tag{12}
\]

Using the fractional integration by parts formula (4), the last integral in (12) can be written as

\[
\int_{t_0}^{t_f} \lambda^T(t) \tau \frac{D^\alpha}{D_t^\alpha} \delta x(t) \Delta t = \int_{t_0}^{t_f} \left( \tau \frac{D^\alpha}{D_t^\alpha} \lambda(t) \right)^T \delta x(t) \Delta t. \tag{13}
\]

Using (13) in (12), we deduce that
The necessary condition for an extremal asserts that the first variation of $J_a[x, \lambda, u]$ must vanish along the extremal for all independent variations $\delta x(t)$, $\delta \lambda(t)$, and $\delta u(t)$. Because of this, all factors multiplying a variation in Eq. (14) must vanish. We obtain conditions (8)–(9).

Equations (8)–(9) represent the Euler–Lagrange equations of the FOCP (5)–(7). Note that Theorem 5 covers fractional optimal control problems defined on isolated time scales with a non-constant graininess, as well as variational problems on time scales that are partially continuous and partially discrete, i.e., on hybrid time scales.

### 3.3 An illustrative example

Let $\mathbb{T}$ be a time scale with $0, T \in \mathbb{T}$. Consider the control system

$$\frac{\tau}{0} D^\alpha_{\tau} x(t) = u(t), \quad t \in [0, T]_{\mathbb{T}},$$

subject to the initial condition

$$\frac{\tau}{0} I^{1-\alpha}_{\tau} x(0) = x_0,$$

where the control $u$ belongs to $L^2$. Consider the problem of minimizing

$$J[x, u] = \frac{1}{2} \left( ||x - z||_{L^2}^2 + N||u||_{L^2}^2 \right)$$

subject to (15)–(16), where $z \in L^2$ and $N > 0$ are fixed/given. In agreement with Theorem 5, the optimal control $u$ is characterized by (15)–(16) with the adjoint system

$$\frac{\tau}{0} D^\alpha_{\tau} \lambda(t) = x(t) - z(t), \quad t \in [0, T]_{\mathbb{T}},$$

and with the optimality condition

$$u(t) = -\frac{\lambda(t)}{N}.$$
4 Conclusion

We studied optimal control problems for fractional initial values systems involving fractional-time derivatives on time scales. As a main result, a necessary optimality condition is proved. In the formulation of the optimal control problem, the control $u$ takes values in $\mathbb{R}^m$. As future work, it would be interesting to consider the case where the control takes values on a closed subset of $\mathbb{R}^m$. This question is far from being trivial \[28, 35\] and needs further developments.

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