ENDOMORPHISM ALGEBRAS AND HECKE ALGEBRAS
FOR REDUCTIVE $p$-ADIC GROUPS

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Abstract. Let $G$ be a reductive $p$-adic group and let $\text{Rep}(G)^s$ be a Bernstein block in the category of smooth complex $G$-representations. We investigate the structure of $\text{Rep}(G)^s$, by analysing the algebra of $G$-endomorphisms of a progenerator $\Pi$ of that category.

We show that $\text{Rep}(G)^s$ is ”almost” Morita equivalent with a (twisted) affine Hecke algebra. This statement is made precise in several ways, most importantly with a family of (twisted) graded algebras. It entails that, as far as finite length representations are concerned, $\text{Rep}(G)^s$ and $\text{End}_G(\Pi)\text{-Mod}$ can be treated as the module category of a twisted affine Hecke algebra.

We draw two consequences. Firstly, we show that the equivalence of categories between $\text{Rep}(G)^s$ and $\text{End}_G(\Pi)\text{-Mod}$ preserves temperedness of finite length representations. Secondly, we provide a classification of the irreducible representations in $\text{Rep}(G)^s$, in terms of the complex torus and the finite group canonically associated to $\text{Rep}(G)^s$. This proves a version of the ABPS conjecture.

Our methods are independent of the existence of types, and apply in complete generality.

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Introduction

This paper investigates the structure of Bernstein blocks in the representation theory of reductive $p$-adic groups. Let $G$ be such a group and let $M$ be a Levi subgroup. Let $(\sigma, E)$ be a supercuspidal $M$-representation (over $\mathbb{C}$), and let $s$ be its inertial equivalence class (for $G$). To these data Bernstein associated a block $\text{Rep}(G)^s$ in the category of smooth $G$-representations $\text{Rep}(G)$, see [BeDe, Ren].

Several questions about $\text{Rep}(G)^s$ have been avidly studied, for instance:

- Can one describe $\text{Rep}(G)^s$ as the module category of an algebra $H$ with an explicit presentation?
- Is there an easy description of temperedness and unitarity of $G$-representations in terms of $H$?
- How to classify the irreducible representations in $\text{Rep}(G)^s$?
- How to classify the discrete series representations in $\text{Rep}(G)^s$?

We note that all these issues have been solved already for $M = G$. In that case the real task is to obtain a supercuspidal representation, whereas we use a given $(\sigma, E)$ as starting point.

Most of the time, the above questions have been approached with types, following [BuKu2]. Given an $s$-type $(K, \lambda)$, there is always a Hecke algebra $\mathcal{H}(G, K, \lambda)$ whose module category is equivalent with $\text{Rep}(G)^s$. This has been exploited very successfully in many cases, e.g. for $GL_n(F)$ [BuKu1], for depth zero representations [Mor1, Mor2], for the principal series of split groups [Roc1], the results on unitarity from [Ciu] and on temperedness from [Sol5].

However, it is often quite difficult to find a type $(K, \lambda)$, and even if one has it, it can be hard to find generators and relations for $\mathcal{H}(G, K, \lambda)$. For instance, types have been constructed for all Bernstein components of classical groups [Ste, MiSt], but so far the Hecke algebras of most of these types have not been worked out. Already for the principal series of unitary $p$-adic groups, this is a difficult task [Bad]. At the moment, it seems unfeasible to carry out the full Bushnell–Kutzko program for arbitrary Bernstein components.

We follow another approach, which builds more directly on the seminal work of Bernstein. We consider a progenerator $\Pi$ of $\text{Rep}(G)^s$, and the algebra $\text{End}_G(\Pi)$. There is a natural equivalence from $\text{Rep}(G)^s$ to the category $\text{End}_G(\Pi)$-Mod of right $\text{End}_G(\Pi)$-modules, namely $V \mapsto \text{Hom}_G(\Pi, V)$.

Thus all the above questions can in principle be answered by studying the algebra $\text{End}_G(\Pi)$. To avoid superfluous complications, we should use a progenerator with an easy shape. Fortunately, such an object was already constructed in [BeRu]. Namely,
let $M^1$ be subgroup of $M$ generated by all compact subgroups, write $B = \mathbb{C}[M/M^1]$ and $E_B = E \otimes_\mathbb{C} B$. The latter is an algebraic version of the integral of the representations $\sigma \otimes \chi$, where $\chi$ runs through the group $X_{nr}(M)$ of unramified characters of $M$. Then the (normalized) parabolic induction $I^G_P(E_B)$ is a progenerator of $\text{Rep}(G)^\#$. In particular we have the equivalence of categories 

$$\mathcal{E} : \text{Rep}(G)^\# \longrightarrow \text{End}_G(I^G_P(E_B))-\text{Mod}$$ 

$$V \mapsto \text{Hom}_G(I^G_P(E_B), V).$$

For classical groups and inner forms of $GL_n$, the algebras $\text{End}_G(I^G_P(E_B))$ were already analysed by Heiermann [Hei1, Hei2, Hei4]. It turns out that they are isomorphic to affine Hecke algebras (sometimes extended with a finite group). These results make use of some special properties of representations of classical groups, which need not hold for other groups.

We want to study $\text{End}_G(I^G_P(E_B))$ in complete generality, for any Bernstein block of any connected reductive group over any non-archimedean local field $F$. This entails that we can only use the abstract properties of the supercuspidal representation $(\sigma, E)$, which also go into the Bernstein decomposition. A couple of observations about $\text{End}_G(I^G_P(E_B))$ can be made quickly, based on earlier work.

- The algebra $B$ acts on $E_B$ by $M$-intertwiners, and $I^G_P$ embeds $B$ as a commutative subalgebra in $\text{End}_G(I^G_P(E_B))$. As a $B$-module, $\text{End}_G(I^G_P(E_B))$ has finite rank [BeRu, Ren].
- Write $\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{nr}(M)\} \subset \text{Irr}(M)$. The group $N_G(M)/M$ acts naturally on $\text{Irr}(M)$, and we denote the stabilizer of $\mathcal{O}$ in $N_G(M)/M$ by $W(M, \mathcal{O})$. Then the centre of $\text{End}_G(I^G_P(E_B))$ is isomorphic to $\mathbb{C}[\mathcal{O}/W(M, \mathcal{O})] = \mathbb{C}[\mathcal{O}]^{W(M, \mathcal{O})}$ [BeDe].
- Consider the finite group $X_{nr}(M, \sigma) = \{\chi_c \in X_{nr}(M) : \sigma \otimes \chi_c \cong \sigma\}$. For every $\chi_c \in X_{nr}(M, \sigma)$ there exists an $M$-intertwiner $\sigma \otimes \chi \rightarrow \sigma \otimes \chi_c \chi$, which gives rise to an element $\phi_{\chi_c}$ of $\text{End}_M(E_B)$ and of $\text{End}_G(I^G_P(E_B))$ [Roc2].
- For every $w \in W(M, \mathcal{O})$ there exists an intertwining operator $I_w(\chi) : I^G_P(\sigma \otimes \chi) \rightarrow I^G_P(w(\sigma \otimes \chi))$, see [Wal]. However, it is rational as a function of $\chi \in X_{nr}(M)$ and in general has non-removable singularities, so it does not automatically yield an element of $\text{End}_G(I^G_P(E_B))$.

Based on this knowledge and on [Hei2], one can expect that $\text{End}_G(I^G_P(E_B))$ has a $B$-basis indexed by $X_{nr}(M, \sigma) \times W(M, \mathcal{O})$, and that the elements of this basis behave somewhat like a group. However, in general things are more subtle than that.

The action of any $w \in W(M, \mathcal{O})$ on $\mathcal{O} \cong X_{nr}(M)/X_{nr}(M, \sigma)$ can be lifted to a transformation $\mathfrak{w}$ of $X_{nr}(M)$. Let $W(M, \sigma, X_{nr}(M))$ be the group of permutations of $X_{nr}(M)$ generated by $X_{nr}(M, \sigma)$ and the $\mathfrak{w}$. It satisfies $W(M, \sigma, X_{nr}(M))/X_{nr}(M, \sigma) = W(M, \mathcal{O})$.

Let $K(B) = \mathbb{C}(X_{nr}(M))$ be the quotient field of $B = \mathbb{C}[X_{nr}(M)]$. In view of the rationality of the intertwining operators $I_w$, it is easier to investigate the algebra

$$\text{End}_G(I^G_P(E_B)) \otimes_B K(B) = \text{Hom}_G(I^G_P(E_B), I^G_P(E_B \otimes_B K(B))).$$
Theorem A. (see Corollary 5.8)
There exist a 2-cocycle \( \tilde{\xi} : \text{Hom}(M, \sigma, X_{\text{nr}}(M))^2 \to \mathbb{C}^\times \) and an algebra isomorphism

\[
\text{End}_G(I_G^G(E_B)) \otimes B K(B) \cong K(B) \rtimes \mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \tilde{\xi}].
\]

Here \( \mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \tilde{\xi}] \) is a twisted group algebra, it has basis elements \( T_w \) that multiply as \( T_w T_{w'} = \tilde{\xi}(w, w')T_{ww'} \). The symbol \( \rtimes \) denotes a crossed product: as vector space it just means the tensor product, and the multiplication rules on that are determined by the action of \( W(M, \sigma, X_{\text{nr}}(M)) \) on \( K(B) \).

Theorem A suggests a lot about \( \text{End}_G(I_G^G(E_B)) \), but the poles of some involved operators make it impossible to already draw many conclusions about representations. In fact the operators \( T_w \) with \( w \in W(M, \mathcal{O}) \) involve certain parameters, powers of the cardinality \( q_F \) of the residue field of \( F \). If we would manually replace \( q_F \) by 1, then \( \text{End}_G(I_G^G(E_B)) \) would become isomorphic to \( B \rtimes \mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \tilde{\xi}] \). Of course that is an outrageous thing to do, we just mention it to indicate the relation between these two algebras.

To formulate our results about \( \text{End}_G(I_G^G(E_B)) \), we introduce more objects. The set of roots of \( G \) with respect to \( M \) contains a root system \( \Sigma_{\mathcal{O}, \mu} \), namely the set of roots for which the associated Harish-Chandra \( \mu \)-function has a zero on \( \mathcal{O} \) [Hei2]. This induces a semi-direct factorization

\[
W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}, \mu}) \rtimes R(\mathcal{O}),
\]

where \( R(\mathcal{O}) \) is the stabilizer of the set of positive roots. We may and will assume throughout that \( \sigma \in \text{Irr}(M) \) is unitary and stabilized by \( W(\Sigma_{\mathcal{O}, \mu}) \).

The Harish-Chandra \( \mu \)-functions also determine parameter functions \( \lambda, \lambda^* : \Sigma_{\mathcal{O}, \mu} \rightarrow \mathbb{R}_{\geq 0} \). The values \( \lambda(\alpha) \) and \( \lambda^*(\alpha) \) encode in a simple way for which \( \chi \in X_{\text{nr}}(M) \) the normalized parabolic induction \( I_M^{\text{nr}}(\sigma \otimes \chi) \) becomes reducible, see (3.7) and [9.3]. (Here \( M_\alpha \) denotes the Levi subgroup of \( G \) generated by \( M \) and the root subgroups \( U_\alpha, U_{-\alpha} \).)

To the data \( X_{\text{nr}}(M), \Sigma_{\mathcal{O}, \mu}, \lambda, \lambda^* \) one can associate an affine Hecke algebra, which we denote in this introduction by \( H(X_{\text{nr}}(M), \Sigma_{\mathcal{O}, \mu}, \lambda, \lambda^*) \). Suppose that \( \tilde{\xi} \) descends to a 2-cocycle \( \tilde{\xi} \) of \( R(\mathcal{O}) \). Then the crossed product

\[
\tilde{H}(\mathcal{O}) := H(X_{\text{nr}}(M), \Sigma_{\mathcal{O}, \mu}, \lambda, \lambda^*) \rtimes \mathbb{C}[R(\mathcal{O}), \tilde{\xi}]
\]

is a twisted affine Hecke algebra [AMS3, §2.1]. Based in [Hei2], it is reasonable to expect that \( \text{End}_G(I_G^G(E_B)) \) is Morita equivalent with \( \tilde{H}(\mathcal{O}) \). Indeed this is "almost" true – but probably not entirely in general.

We shall need to decomposes \( \text{End}_G(I_G^G(E_B)) \)-modules according to their \( B \)-weights (which live in \( X_{\text{nr}}(M) \)). The existence of such a decomposition cannot be guaranteed for representations of infinite length, and therefore we stick to finite length in most of the paper. All the algebras we consider have a large centre, so that every finite length module actually has finite dimension. For \( \text{Rep}(G)^{\text{ad}} \), "finite length" is equivalent to "admissible".

It is known from [Lus] [AMS3] that the category of finite dimensional right modules \( H(\mathcal{O}) \)-Mod can be described with a family of (twisted) graded Hecke algebras. Write \( X_{\text{nr}}^+(M) = \text{Hom}(M/M^1, \mathbb{R}_{>0}) \) and note that its Lie algebra is \( a_M^* = \text{Hom}(M/M^1, \mathbb{R}) \). For a unitary \( u \in X_{\text{nr}}(M) \), there is a graded Hecke algebra \( \mathbb{H}_u \), built from the following data: the tangent space \( a_M^* \otimes \mathbb{R} \mathbb{C} \) of \( X_{\text{nr}}(M) \) at \( u \), a root subsystem \( \Sigma_{\sigma \otimes u} \subset \)
\(\Sigma_{\mathcal{O}, u}\) and a parameter function \(k^u_\alpha\) induced by \(\lambda\) and \(\lambda^*\). Further \(W(M, \mathcal{O})_{\sigma \otimes u}\) decomposes as \(W(\Sigma_{\sigma \otimes u}) \rtimes R(\sigma \otimes u)\), and \(\tilde{\zeta}\) induces a 2-cocycle of the local \(R\)-group \(R(\sigma \otimes u)\). This yields a twisted graded Hecke algebra \(\mathbb{H}_u \rtimes \mathbb{C}[R(\sigma \otimes u), \tilde{\zeta}_u]\) [AMS3 §1].

We remark that these algebras depend mainly on the variety \(\mathcal{O}\) and the group \(W(M, \mathcal{O})\). Only the subsidiary data \(k^u_\alpha\) and \(\tilde{\zeta}_u\) take the internal structure of the representations \(\sigma \otimes \chi \in \mathcal{O}\) into account. The parameters \(k^u_\alpha\), depend only on the poles of the Harish-Chandra \(\mu\)-function (associated to \(\alpha\)) on \(\{\sigma \otimes \chi : \chi \in \chi_{\text{nr}}(M)\}\). It is not clear to us whether, for a given \(\sigma \otimes u\), they can be effectively computed in that way, further investigations are required there.

We do not know whether a 2-cocycle \(\tilde{\zeta}\) as used in \(\tilde{\mathcal{H}}(\mathcal{O})\) always exists. Fortunately, the description of \(\text{End}_{\mathcal{G}}(I^{\tilde{\mathcal{B}}}_P(E_B))\)-fMod found via \(\tilde{\mathcal{H}}(\mathcal{O})\) turns out to be valid anyway.

**Theorem B.** (see Corollaries [8.1] and [9.3])

For every unitary \(u \in X_{\text{nr}}(M)\) there are equivalences between the following categories:

(i) finite length representations in \(\text{Rep}(G)^{\mathfrak{g}}\) with cuspidal support in \(W(M, \mathcal{O})\{\sigma \otimes u \chi : \chi \in X_{\text{nr}}^+(M)\}\);

(ii) modules in \(\text{End}_{\mathcal{G}}(I^{\mathcal{B}}_P(E_B))\)-fMod with all their \(B\)-weights in \(W(M, \sigma, X_{\text{nr}}(M))uX_{\text{nr}}(M)\);

(iii) modules in \(\mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{\zeta}_u)\)-fMod with all their \(\mathbb{C}[a^*_M \otimes \mathbb{C}]\)-weights in \(a^*_M\).

Furthermore, suppose that there exists a 2-cocycle \(\tilde{\zeta}\) on \(R(\mathcal{O}) \cong W(M, \mathcal{O})/W(\Sigma_{\mathcal{O}, u})\) which on each subgroup \(W(M, \mathcal{O})_{\sigma \otimes u}\) is cohomologous to \(\tilde{\zeta}_u\). Then the above equivalences, for all unitary \(u \in X_{\text{nr}}(M)\), combine to an equivalence of categories

\[
\text{End}_{\mathcal{G}}(I^{\mathcal{B}}_P(E_B))-\text{fMod} \longrightarrow \tilde{\mathcal{H}}(\mathcal{O})-\text{fMod}.
\]

Via \(\mathcal{E}\), the left hand side is always equivalent with the finite length subcategory of \(\text{Rep}(G)^{\mathfrak{g}}\).

We stress that Theorem [13] holds for all Bernstein blocks of all reductive \(p\)-adic groups. In particular it provides a good substitute for types, when those are not available or too complicated. Let us point out that on the Galois side of the local Langlands correspondence, analogous structures exist. Indeed, in [AMS1] [AMS2] [AMS3] twisted graded Hecke algebras and a twisted affine Hecke algebra were associated to every Bernstein component in the space of enhanced \(L\)-parameters. By comparing twisted graded Hecke algebras on both sides of the local Langlands correspondence, it might be possible to establish new cases of that correspondence.

For representations of \(\text{End}_{\mathcal{G}}(I^{\mathcal{B}}_P(E_B))\) and \(\mathbb{H}_u \rtimes \mathbb{C}[R(\sigma \otimes u), \tilde{\zeta}_u]\) there are natural notions of temperedness and essentially discrete series, which mimick those for affine Hecke algebras [Opd]. The next result generalizes [Hei3].

**Theorem C.** (see Theorem [9.4] and Proposition [9.5])

Choose the parabolic subgroup \(P\) with Levi factor \(M\) as indicated by Lemma [9.7]. Then all the equivalences of categories in Theorem [12] preserve temperedness.

Suppose that \(\Sigma_{\mathcal{O}, u}\) has full rank in the set of roots of \((G, M)\). Then these equivalences send essentially square-integrable representations in (i) to essentially discrete series representations in (ii), and the other way round.

Suppose \(\Sigma_{\sigma \otimes u}\) has full rank in the set of roots of \((G, M)\), for a fixed unitary \(u \in X_{\text{nr}}(M)\). Then the equivalences in Theorem [12] for that \(u\), send essentially
square-integrable representations in (i) to essentially discrete series representations in (iii), and conversely.

Now that we have a good understanding of $\text{End}_G(I_B^G(E_B))$, its finite dimensional representations and their properties, we turn to the remaining pressing issue from page 2: can one classify the involved irreducible representations? This is indeed possible, because graded Hecke algebras have been studied extensively, see e.g. [BaMo1, BaMo2, COT, Eve, Sol1, Sol2, Sol4]. The answer depends in a well-understood but involved and subtle way on the parameter functions $\lambda, \lambda^*, k^u$.

The classification becomes more tractable if we just want to understand $\text{Irr}(\text{End}_G(I_B^G(E_B)))$ and $\text{Irr}(\mathbb{H}_u \rtimes \mathbb{C}[R(\sigma \otimes u, \tau_u))]$ as sets, and allow ourselves to slightly adjust the weights (with respect to respectively $B$ and $\mathbb{C}[a_M^* \otimes \mathbb{C}]$) in the bookkeeping. Then we can investigate $\text{Irr}(\mathbb{H}_u \rtimes \mathbb{C}[R(\sigma \otimes u, \tau_u)])$ via the change of parameters $k^u \to 0$, like in [Sol3]. That replaces $\mathbb{H}_u \rtimes \mathbb{C}[R(\sigma \otimes u, \tau_u)]$ by $\mathbb{C}[a_M^* \otimes \mathbb{C}] \times \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \tau_u]$, for which Clifford theory classifies the irreducible representations.

**Theorem D.** (see Theorem 9.6)

There exists a (noncanonical) bijection

$$
\zeta \circ \mathcal{E} : \text{Irr}(G)^6 \longrightarrow \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M), \tau_u])
$$

such that, for $\pi \in \text{Irr}(G)^6$ and a unitary $u \in X_{nr}(M)$:

- the cuspidal support of $\zeta \circ \mathcal{E}(\pi)$ lies in $W(M, \mathcal{O})uX_{nr}(M)$ if and only if all the $\mathbb{C}[X_{nr}(M)]$-weights of $(\zeta \circ \mathcal{E}(\pi))$ lie in $W(M, \sigma, X_{nr}(M))uX_{nr}(M)$.
- $\pi$ is tempered if and only if all the $\mathbb{C}[X_{nr}(M)]$-weights of $(\zeta \circ \mathcal{E}(\pi))$ are unitary.

Notice that on the right hand side the parameter functions $\lambda, \lambda^*$ and $k^u$ are no longer involved. In view of the problems to make these parameters explicit, that is quite convenient.

In the language of [ABPS2], $\text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M), \tau_u)])$ is a twisted extended quotient $(\mathcal{O}/W(M, \mathcal{O}))_\tau$. With that interpretation, Theorem D proves a version of the ABPS conjecture (see Corollary 9.8).

We note that Theorem D is about right modules of the involved algebra. If we insist on left modules we must use the opposite algebra, which is isomorphic to $\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M), \tau_{-1})]$. Then we would get the twisted extended quotient $(\mathcal{O}/W(M, \mathcal{O}))_{\tau^{-1}}$.

Apart from the bijection, the only noncanonical object in Theorem D is the 2-cocycle $\tau$. It is trivial on $W(\Sigma_{\mathcal{O}, u})$, but apart from that it depends on some choices of $M$-isomorphisms $w(\sigma \otimes \chi) \to \sigma \otimes \chi'$ for $w \in R(\mathcal{O})$ and $\chi, \chi' \in X_{nr}(M)$. From Theorem D one sees that $\tau$, or at least its restrictions $\tau_u$, have a definite effect on the involved module categories.

Moreover, by (8.2) $\tau_{-1}$ must be cohomologous to a 2-cocycle obtained from the Hecke algebra of an $s$-type (if such a type exists). This entails that in many cases $\tau_u$ must be trivial. At the same time, this argument shows that in some examples, like [ABPS1] Example 5.5, the 2-cocycles $\tau_u$ and $\tau$ are cohomologically nontrivial. It would be interesting if $\tau$ could be related to the way $G$ is realized as an inner twist of a quasi-split $F$-group, like in [HiSa].
Structure of the paper.
All the results about endomorphism algebras of progenerators in the cuspidal case ($M = G$) are contained in Section 2. A substantial part of this was already shown in \cite{Roc2}, we push it further to describe $\text{End}_M(E_B)$ better. We also investigate what happens if we replace $E_B$ by a smaller progenerator of $\text{Rep}(M)^\mathcal{O}$, which is potentially useful to make Theorem B more canonical. Section 3 is elementary, its main use is to introduce some useful objects.

Harish-Chandra’s intertwining operators $J_{P|P'}$ play the main role in Section 4. We study their poles and devise several auxiliary operators to fit $J_w(P)$ into $\text{Hom}_G(I^G_P(E_B), I^G_P(E_B \otimes B K(B)))$. The actual analysis of that algebra is carried out in Section 5. First we express it in terms of operators $A_w$ for $w \in W(M, \mathcal{O})$, which are made by composing the $J_{P|P}$ with suitable auxiliary operators. Next we adjust the $A_w$ to $T_w$ and we prove Theorem A. Sections 2–5 are strongly influenced by \cite{Hei2}, where similar results were established in the (simpler) case of classical groups.

At this point Lemma 5.9 forces us to admit that in general $\text{End}_G(I^G_P(E_B))$ probably does not have a nice presentation. To pursue to analysis of this algebra, we localize it on relatively small subsets $U$ of $X_{nr}(M)$. In this way we get rid of $X_{nr}(M, \sigma)$ from the intertwining group $W(M, \sigma, X_{nr}(M))$, and several things become much easier. For maximal effect, we localize with analytic rather than polynomial functions on $U$ – after checking (in Section 6) that it does not make a difference as far as finite dimensional modules are concerned. We show that the localization of $\text{End}_G(I^G_P(E_B))$ at $U$, extended with the algebra $C_{me}(U)$ of meromorphic functions on $U$, is isomorphic to a crossed product $C_{me}(U) \rtimes \mathbb{C}[W(M, \mathcal{O}) \otimes \mathbb{Z}_u, \mathbb{Z}_u]$. A presentation of the analytic localization of $\text{End}_G(I^G_P(E_B))$ at $U$ is obtained in Theorem 6.7: it is almost Morita equivalent to affine Hecke algebra. The only difference is that the standard large commutative subalgebra of that affine Hecke algebra must be replaced by the algebra of analytic functions on $U$.

This presentation makes it possible to relate the localized version of $\text{End}_G(I^G_P(E_B))$ to the localized version of a suitable graded Hecke algebra. We do that in Section 7, thus proving the first half of Theorem B. In Section 8 we translate that to a classification of $\text{Irr}(G)^\sigma$ in terms of graded Hecke algebras. Next we study the change of parameters $k^u \to 0$ in graded Hecke algebras, and derive the larger part of Theorem D. All considerations about temperedness can be found in Section 9. There we finish the proofs of Theorems B, C and D.

1. Notations

We introduce some of the notations that will be used throughout the paper.

$F$: a non-archimedean local field
$
\mathcal{G}$: a connected reductive $F$-group
$
\mathcal{P}$: a parabolic $F$-subgroup of $\mathcal{G}$
$
\mathcal{M}$: a $F$-Levi factor of $\mathcal{P}$
$
\mathcal{U}$: the unipotent radical of $\mathcal{P}$
$
\overline{\mathcal{P}}$: the parabolic subgroup of $\mathcal{G}$ that is opposite to $\mathcal{P}$ with respect to $\mathcal{M}$
$G = \mathcal{G}(F)$ (and $M = \mathcal{M}(F)$ etc.): the group of $F$-rational points of $\mathcal{G}$

We often abbreviate the above situation to: $P = MU$ is a parabolic subgroup of $G$
Rep(G): the category of smooth G-representations (always on C-vector spaces)
Irr(G): the set of irreducible smooth G-representations up to isomorphism
$P^G: \text{Rep}(M) \to \text{Rep}(G)$: the normalized parabolic induction functor
$X_{nr}(M)$: the group of unramified characters of $M$, with its structure as a complex algebraic torus
$M^1 = \bigcap_{\chi \in X_{nr}(M)} \ker \chi$

$(\sigma, E)$: a supercuspidal $M$-representation (so an irreducible element of $\text{Rep}(M)$)
$O = [M, \sigma]_M$: the inertial equivalence class of $\sigma$ for $M$, that is, the subset of $\text{Irr}(M)$ consisting of the $\sigma \otimes \chi$ with $\chi \in X_{nr}(M)$
$\text{Rep}(M)^O$: the Bernstein block of $\text{Rep}(M)$ associated to $O$
$s = [M, \sigma]_G$: the inertial equivalence class of $(M, \sigma)$ for $G$
$\text{Rep}(G)^s$: the Bernstein block of $\text{Rep}(G)$ associated to $s$

$W(G, M) = N_G(M)/M$
$N_G(M)$ acts on $\text{Rep}(M)$ by $(g \cdot \pi)(m) = \pi(g^{-1}mg)$. This induces an action of $W(G, M)$ on $\text{Irr}(M)$
$N_G(M, O) = \{g \in N_G(M) : g \cdot \sigma \cong \sigma \otimes \chi \text{ for some } \chi \in X_{nr}(M)\}$
$W(M, O) = N_G(M, O)/M = \{w \in W(G, M) : w \cdot \sigma \in O\}$

$X_{nr}(M, \sigma) = \{\chi \in X_{nr}(M) : \sigma \otimes \chi \cong \sigma\}$
$B = \mathbb{C}[X_{nr}(M)]$: the ring of regular functions on the complex algebraic torus $X_{nr}(M)$
$K(B) = \mathbb{C}(X_{nr}(M))$: the quotient field of $B$, the field of rational functions on $X_{nr}(M)$
The covering map
$X_{nr}(M) \to O : \chi \mapsto \sigma \otimes \chi$
induces a bijection $X_{nr}(M)/X_{nr}(M, \sigma) \to O$. In this way we regard $O$ as a complex algebraic variety. We define $\mathbb{C}[X_{nr}(M)/X_{nr}(M, \sigma)], \mathbb{C}[O]$ and $\mathbb{C}(X_{nr}(M)/X_{nr}(M, \sigma)), \mathbb{C}(O)$ like $B$ and $K(B)$.

2. ENDOMORPHISM ALGEBRAS FOR CUSPIDAL REPRESENTATIONS

This section relies largely on [Roc2]. Let
$\text{ind}^M_M^1 : \text{Rep}(M^1) \to \text{Rep}(M)$
be the functor of smooth, compactly supported induction. We realize it as
$\text{ind}^M_M^1(\pi, V) = \{f : M \to V \mid \pi(m_1)f(m) = f(m_1m) \forall m \in M, m_1 \in M^1, \supp(f)/M^1 \text{ is compact}\},$
with the $M$-action by right translation. (Smoothness of $f$ is automatic because $M^1$ is open in $M$.)

Regard $(\sigma, E)$ as a representation of $M^1$, by restriction. Bernstein [BeRu §II.3.3] showed that $\text{ind}^M_M^1(\sigma, E)$ is a progenitor of $\text{Rep}(M)^O$. This entails that
$V \mapsto \text{Hom}_M(\text{ind}^M_M^1(\sigma, E), V)$
is an equivalence between $\text{Rep}(M)^O$ and the category $\text{End}_M(\text{ind}^M_M^1(\sigma, E))$-Mod of right modules over the $M$-endomorphism algebra of $\text{ind}^M_M^1(\sigma, E)$. We want to analyse the structure of $\text{End}_M(\text{ind}^M_M^1(\sigma, E))$. 
For \( m \in M \), let \( b_m \in \mathbb{C}[X_{nr}(M)] \) be the element given by evaluating unramified characters at \( m \). We let \( m \) act on \( \mathbb{C}[X_{nr}(M)] \) by
\[
m \cdot b = b_m b, \quad b \in \mathbb{C}[X_{nr}(M)].
\]
Then specialization/evaluation at \( \chi \in X_{nr}(M) \) is an \( M \)-homomorphism
\[
sp_{\chi} : \mathbb{C}[X_{nr}(M)] \to (\chi, \mathbb{C}).
\]
Let \( \delta_m \in \text{ind}_{M^1}^M(\mathbb{C}) \) be the function which is 1 on \( mM^1 \) and zero on the rest of \( M \). Let \( \mathbb{C}[M/M^1] \) be the group algebra of \( M/M^1 \), considered as the left regular representation of \( M/M^1 \). There are canonical isomorphisms of \( M \)-representations
\[
\begin{align*}
\mathbb{C}[X_{nr}(M)] & \to \mathbb{C}[M/M^1] \to \text{ind}_{M^1}^M(\mathbb{C}), \\
b_m & \mapsto mM^1 \mapsto \delta_{m^{-1}}.
\end{align*}
\]
We endow \( E \otimes \mathbb{C} \text{ind}_{M^1}^M(\mathbb{C}) \) with the tensor product of the \( M \)-representations \( \sigma \) and \( \text{ind}_{M^1}^M(\text{triv}) \). There is an isomorphism of \( M \)-representations
\[
E \otimes \mathbb{C} \text{ind}_{M^1}^M(\mathbb{C}) \cong \text{ind}_{M^1}^M(E),
\]
\[
\sum_{m \in M/M^1} \sigma(m^{-1}) v(m) \otimes \delta_m \leftrightarrow \left[ v_{\otimes f} : m \mapsto f(m) \sigma(m)e \right].
\]
Composing (2.1) and (2.2), we obtain an isomorphism
\[
\begin{align*}
\text{ind}_{M^1}^M(E) & \to E \otimes \mathbb{C} \mathbb{C}[X_{nr}(M)] \\
v & \mapsto \sum_{m \in M/M^1} \sigma(m)v(m) \otimes b_m.
\end{align*}
\]
With (2.3), specialization at \( \chi \in X_{nr}(M) \) becomes a \( M \)-homomorphism
\[
(2.4) \quad sp_{\chi} : \text{ind}_{M^1}^M(\sigma, E) \to (\sigma \otimes \chi, E).
\]
As \( M/M^1 \) is commutative, the \( M \)-action on \( E \otimes \mathbb{C} \mathbb{C}[X_{nr}(M)] \) is \( \mathbb{C}[X_{nr}(M)] \)-linear. Via (2.3) we obtain an embedding
\[
(2.5) \quad \mathbb{C}[X_{nr}(M)] \to \text{End}_M(\text{ind}_{M^1}^M(\sigma, E)).
\]
For a basis element \( b_m \in \mathbb{C}[X_{nr}(M)] \) and any \( v \in \text{ind}_{M^1}^M(E) \), it works out as
\[
(2.6) \quad (b_m \cdot v)(m') = \sigma(m^{-1}) v(mm').
\]
For any \( \chi_c \in X_{nr}(M) \) we can define a linear bijection
\[
(2.7) \quad \rho_{\chi_c} : \mathbb{C}[X_{nr}(M)] \to \mathbb{C}[X_{nr}(M)],
\]
\[
\begin{align*}
b & \mapsto [b_{\chi_c} : \chi \mapsto b(\chi \chi_c)].
\end{align*}
\]
This provides an \( M \)-isomorphism
\[
\text{id}_E \otimes \rho_{\chi_c} : \text{ind}_{M^1}^M(\sigma) \to \text{ind}_{M^1}^M(\sigma \otimes \chi_c).
\]
Let \( (\sigma_1, E_1) \) be an irreducible subrepresentation of \( \text{Res}_{M^1}^M(\sigma, E) \), such that the stabilizer of \( E_1 \) in \( M \) is maximal. We denote the multiplicity of \( \sigma_1 \) in \( \sigma \) by \( \mu_{\sigma_1} \). Every other irreducible \( M^1 \)-subrepresentation of \( \sigma \) is isomorphic to \( m \cdot \sigma_1 \) for some \( m \in M \), and \( \sigma(m^{-1})E_1 \) is a space that affords \( m \cdot \sigma_1 \). Hence \( \mu_{\sigma_1} \) depends only on \( \sigma \) and not on the choice of \( (\sigma_1, E_1) \).
For classical groups $\mu_{\sigma,1}$ is always 1 [Hei2 Proposition 1.16], and maybe that is the case in general. At least, we are not aware of any examples with $\mu_{\sigma,1} > 1$. Unfortunately, we cannot rule this out either, so we must work without any assumption on $\mu_{\sigma,1}$. Compared to [Hei2], that presents substantial additional complications.

Following [Roc2 §1.6] we consider the groups

$$
M_{\sigma}^2 = \cap_{\chi \in X_{nr}(M,\sigma)} \ker \chi,
M_{\sigma}^3 = \{ m \in M : \sigma(m)E_1 = E_1 \},
M_{\sigma}^4 = \{ m \in M : m \cdot \sigma_1 \cong \sigma_1 \}.
$$

These fit in a sequence

$$
M^1 \subset M_{\sigma}^2 \subset M_{\sigma}^3 \subset M_{\sigma}^4 \subset M.
$$

Since $M^1$ is a normal subgroup of $M$ and $M/M^1$ is abelian, all these groups are normal in $M$. By this normality, for any $m' \in M$:

$$
M_{m'}^3 = \{ m \in M : \sigma(m)\sigma(m')E_1 = \sigma(m'E_1) \},
M_{m'}^4 = \{ m \in M : m \cdot (m' \cdot \sigma_1) \cong m' \cdot \sigma_1 \}.
$$

In other words, $M_{m'}^3$ consists of the $m \in M$ that stabilize the isomorphism class of one (or equivalently any) irreducible $M^1$-subrepresentation of $\sigma$. In particular $M_{\sigma}^2$ and $M_{\sigma}^4$ only depend on $\sigma$. On the other hand, it seems possible that $M_{\sigma}^3$ does depend on the choice of $E_1$.

Furthermore $[M : M_{\sigma}^3]$ equals the number of inequivalent irreducible constituents of $\text{Res}_{M^1}(\sigma)$ and, like (2.2),

$$
\ind_{M^1}^{M_{\sigma}^3}(\mathbb{C}) \cong \mathbb{C}[X_{nr}(M)/X_{nr}(M,\sigma)].
$$

By [Roc2 Lemma 1.6.3.1]

$$
[M_{\sigma}^4 : M_{\sigma}^3] = [M_{\sigma}^3 : M_{\sigma}^2] = \mu_{\sigma,1}.
$$

When $\mu_{\sigma,1} = 1$, the groups $M_{\sigma}^2, M_{\sigma}^3$ and $M_{\sigma}^4$ coincide with the group called $M^\sigma$ in [Hei2 §1.16]. Otherwise all the different $m \in M_{\sigma}^4/M_{\sigma}^3$ give rise to different subspaces $\sigma(m)E_1$ of $E$. We denote the representation of $M_{\sigma}^3$ (resp. $M_{\sigma}^2$) on $E_1$ by $\sigma_3$ (resp. $\sigma_2$). The $\sigma_1$-isotypical component of $E$ is an irreducible representation $(\sigma_4, E_4)$ of $M_{\sigma}^4$. More explicitly

$$
E_4 = \bigoplus_{m \in M_{\sigma}^4/M_{\sigma}^3} \sigma(m)E_1 \cong \ind_{M_{\sigma}^3}^{M_{\sigma}^4}(\sigma_3, E_1).
$$

From (2.11) we see that

$$
(\sigma, E) \cong \ind_{M_{\sigma}^3}^{M_{\sigma}^4}(\sigma_4, E_4) \cong \ind_{M_{\sigma}^4}^{M_{\sigma}^3}(\sigma_3, E_1).
$$

The structure of $(\sigma_4, E_4)$ can be analysed as in [GeKn, §2]:

**Lemma 2.1.** (a) In the above setting

$$
\text{Res}_{M_{\sigma}^3}^{M_{\sigma}^4}(\sigma_4) = \bigoplus_{\chi \in \text{Irr}(M_{\sigma}^3/M_{\sigma}^2)} \sigma_3 \otimes \chi.
$$

(b) All the $\sigma_3 \otimes \chi$ are inequivalent irreducible $M_{\sigma}^3$-representations.

(c) There is a group isomorphism

$$
\begin{align*}
M_{\sigma}^4/M_{\sigma}^3 & \quad \longrightarrow \quad \text{Irr}(M_{\sigma}^3/M_{\sigma}^2) \\
M_{\sigma}^4 & \quad \longmapsto \quad \chi_{3,n}
\end{align*}
$$

defined by $n \cdot \sigma_3 \cong \sigma_3 \otimes \chi_{3,n}$. 
Proof. (a) For any $\chi \in X_{nr}(M, \sigma)$ we have $\sigma \otimes \chi \cong \sigma$, so $\sigma_3 \otimes \text{Res}_{M_3}^M \chi$ is isomorphic to an $M_2^\sigma$-subrepresentation of $E$. As $M^1$-representation it is just $\sigma_1$, so $\sigma_3 \otimes \text{Res}_{M_3}^M \chi$ is even isomorphic to subrepresentation of $E_4$. As every character of $M_\sigma^3/M_\sigma^2$ can be extended to a character of $M/M_\sigma$ (that is, to an element of $X_{nr}(M, \sigma)$), all the $\sigma_3 \otimes \chi$ with $\chi \in \text{Irr}(M_3^\sigma/M_2^\sigma)$ appear in $E_4$.

Further, all the $M_3^\sigma$-subrepresentations $(n^{-1}.\sigma_3, \sigma(n)E_1)$ of $(\sigma_4, E_4)$ are extensions of the irreducible $M_2^\sigma$-representation $(\sigma_2, E_1)$. Hence they differ from each other only by characters of $M_\sigma^3/M_\sigma^2$ [GoHe Lemma 2.14]. This shows that $\text{Res}_{M_3}^M(\sigma_4, E_4)$ is a direct sum of $M_\sigma^3$-representations of the form $\sigma_3 \otimes \chi$ with $\chi \in \text{Irr}(M_3^\sigma/M_2^\sigma)$.

By Frobenius reciprocity, for any such $\chi$:

$$\text{Hom}_{M_3^\sigma}(\text{ind}_{M_3^\sigma}^M(\sigma_3 \otimes \chi), \sigma_4) \cong \text{Hom}_{M_3^\sigma}(\sigma_3 \otimes \chi, \sigma_4) \neq 0. \tag{2.13}$$

Thus there exists a nonzero $M_3^\sigma$-homomorphism $\text{ind}_{M_3^\sigma}^M(\sigma_3 \otimes \chi) \to \sigma_4$. As these two representations have the same dimension and $\sigma_4$ is irreducible, they are isomorphic. Knowing that, (2.13) also shows that $\dim \text{Hom}_{M_3^\sigma}(\sigma_3 \otimes \chi, \sigma_4) = 1$.

(b) The previous line is equivalent to: every $\sigma_3 \otimes \chi$ appears exactly once as a $M_3^\sigma$-subrepresentation of $\sigma_4$. As $\text{Res}_{M_3^\sigma}^M(\sigma_4)$ has length $[M_4^\sigma : M_3^\sigma] = [M_3^\sigma : M_2^\sigma]$, this means that they are mutually inequivalent.

(c) This is a consequence of parts (a), (b) and the Mackey decomposition of $\text{Res}_{M_3}^M(\sigma_4, E_4)$. $\square$

2.1. Description in terms of $\mathbb{C}[X_{nr}(M)]$.

For $\chi \in \text{Irr}(M/M_3^\sigma)$ we define an $M$-isomorphism

$$\phi_{\sigma, \chi}: (\sigma, E) \to (\sigma \otimes \chi, E) \quad \sigma(m)e_1 \mapsto \chi(m)\sigma(m)e_1 \quad e_1 \in E_1, m \in M. \tag{2.14}$$

This says that $\phi_{\sigma, \chi}$ acts as $\chi(m)\text{id}$ on the $M_\sigma^3$-subrepresentation $\sigma(m)E_1$ of $E$. We can extend $\phi_{\sigma, \chi}$ to an $M$-isomorphism

$$\phi_\chi = \phi_{\sigma, \chi} \otimes \rho_\chi^{-1}: \text{ind}_{M_3^\sigma}^M(\sigma, E) \to \text{ind}_{M_3}^M(\sigma, E) \quad e \otimes \delta_m \mapsto \phi_{\sigma, \chi}(e) \otimes \chi(m)\delta_m, \tag{2.15}$$

where $e \in E, m \in M$ and the elements are presented in $E \otimes_{\mathbb{C}} \text{ind}_{M_3}^M(\mathbb{C})$ using (2.2). Via (2.3), this becomes

$$\phi_\chi \in \text{Aut}_M\left(E \otimes_{\mathbb{C}} \mathbb{C}[X_{nr}(M)]\right): \quad e \otimes b \mapsto \phi_{\sigma, \chi}(e) \otimes \rho_\chi^{-1}(b), \tag{2.16}$$

where $e \in E, b \in \mathbb{C}[X_{nr}(M)]$. Given $E_1$, $\phi_\chi$ is canonical.

For an arbitrary $\chi \in \text{Irr}(M/M_2^\sigma) = X_{nr}(M, \sigma)$ we can also construct such $M$-homomorphisms, albeit not canonically. Pick $n \in M_4^\sigma$ (unique up to $M_3^\sigma$) as in Lemma 2.1c, such that $\chi_3, n = \chi|M_3^\sigma$. Choose an $M_3^\sigma$-isomorphism

$$\phi_{\sigma, \chi}: (\sigma_3, E_1) \to ((n^{-1} \cdot \sigma_3) \otimes \chi, \sigma(n)E_1).$$

For compatibility with (2.14) we may assume that

$$\phi_{\sigma_3, \chi'} = \phi_{\sigma_3, \chi} \quad \text{for all } \chi' \in \text{Irr}(M/M_2^\sigma).$$
By Schur’s lemma \( \phi_{\sigma, \chi} \) is unique up to scalars, but we do not know a canonical choice when \( M_\sigma^3 \not\subset \ker \chi \). By (2.12)

\[
\text{Hom}_M(\sigma, \sigma \otimes \chi) = \text{Hom}_M(\text{ind}_M^M(\sigma_3), \sigma \otimes \chi) \cong \text{Hom}_{M_\sigma^3}(\sigma_3, \sigma \otimes \chi),
\]

while \((n^{-1} \cdot \sigma_3) \otimes \chi, \sigma(n)E_1)\) is contained in \((\sigma \otimes \chi, E)\) as \(M_\sigma^3\)-representation. Thus \( \phi_{\sigma, \chi} \) determines a \( \phi_{\sigma, \chi} \in \text{Hom}_M(\sigma, \sigma \otimes \chi) \), which is nonzero and hence bijective. Then the formulas (2.15) and (2.16) provide

\[
(2.18) \quad \phi_\chi = \phi_{\sigma, \chi} \otimes \rho_\chi^{-1} \in \text{Aut}_M(\text{ind}_M^M(\sigma_3)) \cong \text{Aut}_M(E \otimes_{\mathbb{C}} \mathbb{C}X_{\text{irr}}(M)).
\]

For all \( \chi, \chi' \in \text{Irr}(M/M_\sigma^2) \), the uniqueness of \( \phi_{\sigma, \chi} \) up to scalars implies that there exists a \( \zeta(\chi, \chi') \in \mathbb{C}^\times \) such that

\[
(2.19) \quad \phi_\chi \phi_{\chi'} = \zeta(\chi, \chi') \phi_{\chi \chi'}.
\]

In other words, the \( \phi_\chi \) span a twisted group algebra \( \mathbb{C}X_{\text{irr}}(M, \sigma, \zeta) \). By (2.17) we have \( \zeta(\chi, \chi') = 1 \) if \( \chi \in \text{Irr}(M/M_\sigma^3) \) or \( \chi' \in \text{Irr}(M/M_\sigma^3) \). If desired, we can normalize the \( \phi_{\sigma, \chi} \) so that \( \phi_{\chi^{-1}} = \phi_{\chi^{-1}} \). In that case \( \zeta(\chi, \chi^{-1}) = 1 \) for all \( \chi \in \text{Irr}(M/M_\sigma^3) \).

From (2.18), we see that

\[
(2.20) \quad \text{sp}_\chi \circ \phi_{\chi^{-1}} = \phi_{\chi^{-1}} \circ \text{sp}_{\chi^{-1}}^{-1}.
\]

We also note that, regarding \( b \in \mathbb{C}X_{\text{irr}}(M) \) as multiplication operator:

\[
(2.21) \quad b \circ \phi_\chi = \phi_\chi \circ b_\chi \in \text{End}_M(\text{ind}_M^M(\sigma_3)) = \mathbb{C}X_{\text{irr}}(M, \sigma).
\]

The next result is a variation on [Hei2, Proposition 3.6].

**Proposition 2.2.** (a) The set \( \{ \phi_{\sigma, \chi} : \chi \in X_{\text{irr}}(M, \sigma) \} \) is a \( \mathbb{C} \)-basis of \( \text{End}_M(\sigma_3) \).

(b) With respect to the embedding (2.5):

\[
\text{End}_M(\text{ind}_M^M(\sigma_3)) = \bigoplus_{\chi \in X_{\text{irr}}(M, \sigma)} \mathbb{C}X_{\text{irr}}(M, \sigma) \phi_\chi = \bigoplus_{\chi \in X_{\text{irr}}(M, \sigma)} \phi_\chi \mathbb{C}X_{\text{irr}}(M, \sigma).
\]

**Proof.** (a) By (2.11) and Lemma 2.1

\[
\text{Res}_M^M(\sigma_3, \sigma) = \bigoplus_{m \in M/M_\sigma^3} (m^{-1} \cdot \sigma, \sigma(m)E_1),
\]

and all these summands are mutually inequivalent. Hence

\[
(2.22) \quad \text{End}_M^3(E) = \bigoplus_{m \in M/M_\sigma^3} \text{End}_M^3(\sigma(m)E_1) = \bigoplus_{m \in M/M_\sigma^3} \mathbb{C} \text{id}_{\sigma(m)E_1}.
\]

The operators \( \phi_{\sigma, \chi} \) with \( \chi \in \text{Irr}(M/M_\sigma^3) \) provide a basis of (2.22), because they are linearly independent.

For every \( \chi_3 \in \text{Irr}(M/M_\sigma^2) \) we choose an extension \( \hat{\chi}_3 \in \text{Irr}(M/M_\sigma^2) \). Then

\[
\{ \phi_{\sigma, \chi} : \chi \in \text{Irr}(M/M_\sigma^2) \} = \{ \phi_{\sigma, \hat{\chi}_3} \phi_{\sigma, \chi} : \chi \in \text{Irr}(M/M_\sigma^3), \chi_3 \in \text{Irr}(M/M_\sigma^2) \}.
\]

It follows from (2.11) that

\[
\text{Res}_M^M(\sigma, \sigma) = \bigoplus_{m \in M/M_\sigma^3} ((m \cdot \sigma_1)^{m^{-1}}, \sigma(m)E_4),
\]

\[
\text{End}_M(\sigma_3) = \bigoplus_{m \in M/M_\sigma^3} \text{End}_M(\sigma(m)E_4) = \bigoplus_{m \in M/M_\sigma^3} \sigma(m) \text{End}_M(\sigma(m)E_3) \sigma(m^{-1}).
\]

In view of the already exhibited basis of (2.22), it only remains to show that

\[
(2.23) \quad \{ \text{id}_{\sigma(m)E_4}, \phi_{\hat{\chi}_3} \}
\]
is a $\mathbb{C}$-basis of $\text{End}_{M^1}(E_4)$. Every $\phi_{\chi_3}$ permutes the irreducible $M_\sigma^3$-subrepresentations $\sigma(m)E_1$ of $E_4$ according to a unique $n \in M^4/M^3$, so the set $(2.23)$ is linearly independent. As

$$\dim \text{End}_{M^1}(E_4) = \dim \text{End}_{M^1}(\sigma^{\mu_{\sigma,1}}) = \mu_{\sigma,1}^2 = [M^4 : M^3][M^3 : M_\sigma],$$

equals the cardinality of $(2.23)$, that set also spans $\text{End}_{M^1}(E_4)$.

(b) As $M^1 \subset M$ is open and both groups are unimodular, Frobenius reciprocity for compact smooth induction holds. It gives a natural bijection

$$\text{End}_M(\text{ind}_{M^1}(E)) \cong \text{Hom}_M(E, \text{ind}_{M^1}(E)).$$

By $(2.3)$ the right hand side is isomorphic to

$$\text{Hom}_{M^1}(E, E \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)]) = \text{End}_{M^1}(E) \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)],$$

where the action of $\text{Xnr}(M)$ becomes multiplication on the second tensor factor on the right hand side. Under these bijections $\phi_\chi \in \text{Aut}_M(\text{ind}_{M^1}(E))$ corresponds to

$$\phi_{\sigma, \chi} \otimes 1 \in \text{End}_{M^1}(E) \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)].$$

We conclude by applying part (a).

We remark that $(2.19)$, $(2.21)$ and Proposition $2.2.b$ mean that

$$(2.24) \quad \text{End}_M(E \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)]) = \mathbb{C}[\text{Xnr}(M)] \times \mathbb{C}[\text{Xnr}(M, \sigma), \sharp],$$

the crossed product with respect to the multiplication action of $\text{Xnr}(M, \sigma)$ on $\text{Xnr}(M)$.

### 2.2. A smaller progenerator.

From $(2.19)$ we see that $\chi \mapsto \phi_\chi$ yields a group homomorphism

$$(2.25) \quad \text{Irr}(M/M_\sigma^3) \to \text{Aut}_M(\text{ind}_{M^1}(E)) \cong \text{Aut}_M(E \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)]).$$

In other words, the group $\text{Irr}(M/M_\sigma^3)$ acts on $\text{ind}_{M^1}(E)$ and on $E \otimes_{\mathbb{C}} \mathbb{C}[\text{Xnr}(M)]$. With the isomorphism $(2.3)$ one can easily express the space of invariants under $\text{Irr}(M/M_\sigma^3)$:

$$(2.26) \quad (\text{ind}_{M^1}(E))^{\text{Irr}(M/M_\sigma^3)} = \text{ind}_{M^1}(E_1).$$

By irreducibility of $\sigma$, every irreducible $M^1$-subrepresentation of $E$ is isomorphic to $(m^{-1} \cdot \sigma_1, \sigma(m)E_1)$ for some $m \in M$. More precisely, it follows from $(2.8)$ and $(2.10)$ that

$$\text{Res}_{M^1}^M(\sigma, E) \cong \bigoplus_{m \in M/M_\sigma^3} (m \cdot \sigma_1)^{\mu_{\sigma,1}}.$$

Since $\sigma_1$ and $m \cdot \sigma_1$ have isomorphic induction to $M$:

$$(2.27) \quad \text{ind}_{M^1}(E) \cong \bigoplus_{m \in M/M_\sigma^3} \text{ind}_{M^1}(m \cdot \sigma_1)^{\mu_{\sigma,1}} \cong \text{ind}_{M^1}(\sigma_1, E_1)^{[M:M_\sigma^3][M:M_\sigma^3]}.$$ 

Notice that

$$[M : M_\sigma^3][M : M_\sigma^3]$$

is the length of $\text{Res}_{M^1}^M(\sigma, E)$. From $(2.27)$ we see that $\text{ind}_{M^1}(\sigma_1, E_1)$ is, like $\text{ind}_{M^1}(\sigma, E)$, a progenerator of $\text{Rep}(M)^\text{ad}$ — this was already shown by Bernstein [BeRu]. Further $(2.27)$ implies

$$(2.28) \quad \text{End}_M(\text{ind}_{M^1}^M(\sigma, E)) \cong \text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1)) \otimes_{\mathbb{C}} M_{[M:M_\sigma^3]}(\mathbb{C}),$$

where $M_d(\mathbb{C})$ denotes the algebra of $d \times d$ complex matrices. For comparison with [Hei2] we analyse the Morita equivalent subalgebra $\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))$ as well.
For every $\sigma(m^{-1})$, we obtain a homomorphism
\[ M_\sigma^3/M^1 \to \text{Aut}_M(\text{ind}_{M_1}^M(\sigma_1, E_1)). \]
That extends $C$-linearly to an embedding
\[ \text{ind}_{M_1}^M(\sigma_1, E_1) \to \text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1)), \]
where $\sigma_3(m)\lambda_m(v)(m') = \sigma_3(m)v(m^{-1}m')$ for $v \in \text{ind}_{M_1}^M(\sigma_1, E_1)$. We note that $C[M_\sigma^3/M^1]$ can be regarded as the ring of regular functions on the complex torus
\[ \mathcal{O}_3 := \text{Irr}(M_\sigma^3/M^1), \]
a degree $\mu_{\sigma,1}$ cover of $X_{M}(M)/X_M(M, \sigma)$. Another way to construct the embedding (2.29) uses that $(\sigma_1, E_1)$ extends to the $M_\sigma^3$-representation $\sigma_3$. The same reasoning as in (2.23) gives an isomorphism of $M_\sigma^3$-representations
\[ \text{ind}_{M_1}^M(E_1) \to E_1 \otimes_C C[\mathcal{O}_3]. \]
Since $C[\mathcal{O}_3]$ is commutative, the $M_\sigma^3$-action on $E_1 \otimes_C C[\mathcal{O}_3]$ is $C[\mathcal{O}_3]$-linear. We find
\[ \text{ind}_{M_1}^M(E_1) \cong \text{ind}_{M_3}^M(E_1 \otimes_C C[\mathcal{O}_3]) \]
and $C[\mathcal{O}_3]$ acts on it by $M$-intertwiners, induced from the action on $E_1 \otimes_C C[\mathcal{O}_3]$.

As worked out in [Roc2] Proposition 1.6.3.2, the subalgebra
\[ C[M_\sigma^3/M^1] \cong C[X_M(M)/X_M(M, \sigma)] \]
is the centre of $\text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1))$. In view of (2.28), $C[M_\sigma^3/M^1]$ is also the centre of $\text{End}_M(\text{ind}_{M_1}^M(\sigma, E))$ – which can be derived directly from Proposition 2.2. Furthermore $\text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1))$ is free of rank $\mu_{\sigma,1}$ as a module over its centre. The commutative subalgebra
\[ C[M_\sigma^3/M^1] \cong C[\mathcal{O}_3], \]
embedded in $\text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1))$ as in (2.29) or (2.30), is free of rank $\mu_{\sigma,1}$ as a module over $C[M_\sigma^3/M^1]$. To find generators for $\text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1))$ as a module over $C[M_\sigma^3/M^1]$, we consider any $m \in M_\sigma^4$. By definition there exists an $M_1$-isomorphism
\[ \phi_{m,\sigma_1} : (\sigma_1, E_1) \to (m^{-1} \cdot \sigma_1, E_1). \]
Regarding the subspace of $\text{ind}_{M_1}^M(E_1)$ supported on $mM^1$ as the $M_1$-representation $m^{-1} \cdot \sigma_1$, $\phi_{m,\sigma_1}$ becomes an element of $\text{Hom}_{M_1}(\sigma_1, \text{ind}_{M_1}^M(\sigma_1))$. Applying Frobenius reciprocity, we obtain
\[ \phi_m \in \text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1)), \quad \phi_m(v) = \phi_{m,\sigma_1}\lambda_m(v). \]
For $m \in M_\sigma^4$ we can take $\phi_{m,\sigma_1} = \sigma_3(m)$, and then (2.32) recovers (2.29).

**Lemma 2.3.** For every $m \in M_\sigma^4/M^1$ and for every $m \in M_\sigma^4/M_\sigma^3$ we pick a representative $\tilde{m} \in M_\sigma^4$.

(a) The set $\{\phi_{\tilde{m}} : m \in M_\sigma^4/M^1\}$ is a $C$-basis of $\text{End}_M(\text{ind}_{M_1}^M(\sigma_1, E_1))$.

(b) With respect to the embedding (2.29):
\[
\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1)) = \bigoplus_{m \in M^4_\sigma / M^1_\sigma} \phi_m \mathbb{C}[M^3_\sigma / M^1] = \bigoplus_{m \in M^4_\sigma / M^2_\sigma} \phi_m \mathbb{C} \{O_3\}.
\]

**Proof.** (a) By Frobenius reciprocity

\[(2.33) \quad \text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1)) \cong \text{Hom}_{M^1}(\sigma_1, \text{ind}_{M^1}^M(\sigma_1)) \cong \text{Hom}_{M^1}(\sigma_1, \bigoplus_{m \in M/M^1} (m^{-1} \cdot \sigma_1)).\]

By the definition of \(M^4_\sigma\), this reduces to

\[\bigoplus_{m \in M^4_\sigma / M^1} \text{Hom}_{M^1}(\sigma_1, m^{-1} \cdot \sigma_1),\]

where each summand is one-dimensional. For every \(m \in M^4_\sigma\), the element \(\phi_m \in \text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\) comes by construction from the nonzero element \(\phi_{m, \sigma_1} \in \text{Hom}_{M^1}(\sigma_1, m^{-1} \cdot \sigma_1)\). It follows that, for any \(m_1 \in M^1\), \(\phi_m\) and \(\phi_{m,m_1}\) differ only by a scalar, and that the \(\phi_m\) with \(m \in M^4_\sigma / M^1\) form a basis of \(\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\).

(b) This follows directly from part (a) and \((2.29)\). \(\square\)

It is known from [Roc2] (1.6.1.1)] that the operators \(\phi_m\) with \(m \in M^4_\sigma\) commute up to scalars. However, by [Roc2] Proposition 1.6.1.2 the algebra \(\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\) is commutative if and only if \(\mu_{\sigma,1} = 1\).

**Lemma 2.4.** Let \(m \in M^3_\sigma\) and \(m \in M^4_\sigma\) and let \(\chi_{3,n} \in \text{Irr}(M^3_\sigma / M^1_\sigma)\) be as in Lemma 2.1c. Then

\[\phi_n \circ \phi_m = \chi_{3,n}(m) \phi_m \circ \phi_n.\]

For \(b \in \mathbb{C} \{O_3\}\), regarded as element of \(\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\) via \((2.29)\) and \((2.1)\),

\[b \circ \phi_n = \phi \circ b_{\chi_{3,n}}.\]

**Proof.** By Lemma 2.1c

\[\phi_{n,\sigma_1} \in \text{Hom}_{M^3}(\sigma_3, n^{-1} \cdot \sigma_3 \otimes \chi_{3,n}).\]

For \(v \in \text{ind}_{M^1}^M(\sigma_1, E_1)\) and \(m' \in M\) we compute

\[(\phi_n \phi_m v)(m') = \phi_{n,\sigma_1} \sigma_3(m) v(m^{-1} n^{-1} m') = (n^{-1} \cdot \sigma_3 \otimes \chi_{3,n})(m) \phi_{n,\sigma_1} \sigma_1(m^{-1} n^{-1} m) v(n^{-1} m^{-1} m') = \chi_{3,n}(m) \sigma_3(mmn^{-1}) \sigma_1(mm^{-1} n^{-1} m) \phi_{n,\sigma_1} v(n^{-1} m^{-1} m') = \chi_{3,n}(m) \sigma_3(m) \phi_{n,\sigma_1} v(n^{-1} m^{-1} m') = \chi_{3,n}(m)(\phi_m \phi_n v)(m').\]

This proves the first claim. The image of \(b_m \in \mathbb{C} \{O_3\}\) in \(\text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\) is \(\phi_{m^{-1}}\), so by the above

\[\phi_n b_m = \chi_{3,n}(m^{-1}) b_m \phi_n.\]

With \((b_m)_{\chi_{3,n}} = \chi_{3,n}(m) b_m\) we reformulate this as \(\phi_n (b_m)_{\chi_{3,n}} = b_m \phi_n\). Since the \(b_m\) with \(m \in M^3_\sigma / M^1\) form a basis of \(\mathbb{C} \{O_3\}\), the same equation holds with any \(b \in \mathbb{C} \{O_3\}\) instead of \(b_m\). \(\square\)

Unfortunately, we did not succeed in making the isomorphism \((2.28)\) explicit in terms of the endomorphisms of \(\text{ind}_{M^1}^M(E)\) and \(\text{ind}_{M^1}^M(E_1)\) exhibited above. Clearly

\[\mathbb{C} \{O_3\} \subset \text{End}_M(\text{ind}_{M^1}^M(\sigma_1, E_1))\]
corresponds naturally to a subalgebra of
\[ \mathbb{C}[X_{\text{nr}}(M)] \subset \text{End}_M(\text{ind}_M^1(E)). \]
From Lemmas 2.3, 2.4 and (2.21) we see that the \( \phi_m \) with \( m \in M_\sigma^4/M_\sigma^3 \) should correspond to linear combinations of the \( \phi_\chi \) with \( \chi \in X_{\text{nr}}(M, \sigma) \) and \( \chi|_{M_\sigma^3} = \chi_3 \), but we did not find a canonical choice. Thus, although the generators \( \text{ind}_M^1(E) \) and \( \text{ind}_M^1(E_1) \) of the cuspidal Bernstein component \( \text{Rep}(M) \) are equally good, they look somewhat differently. To extend the analysis to non-cuspidal Bernstein components, it is easier to work with \( \text{ind}_M^1(E) \).

Finally, we record what happens when we replace regular functions on the involved complex algebraic tori by rational functions. More generally, consider a group \( \Gamma \) and an integral domain \( R \) with quotient field \( Q \). Suppose that \( V \) is a \( \Gamma \times R \)-module, which is free over \( R \). Then \( R \subset \text{End}_\Gamma(V) \) and there is a natural isomorphism of \( R \)-modules
(2.34) \[ \text{Hom}_R(V, V \otimes_R Q) \cong \text{End}_R(V) \otimes_R Q. \]

Applying this to (2.23) and Proposition 2.2 we find
(2.35) \[ \text{Hom}_M(\text{ind}_M^1(E), \text{ind}_M^1(E) \otimes_{\mathbb{C}[X_{\text{nr}}(M)]} \mathbb{C}(X_{\text{nr}}(M))) \cong \bigoplus_{\chi \in X_{\text{nr}}(M, \sigma)} \phi_\chi \mathbb{C}(X_{\text{nr}}(M)) = \mathbb{C}(X_{\text{nr}}(M)) \times \mathbb{C}[X_{\text{nr}}(M, \sigma), \mathbb{C}]. \]

Similarly, from (2.30) and Lemma 2.3 we obtain
(2.36) \[ \text{Hom}_M(\text{ind}_M^1(E_1), \text{ind}_M^1(E_1) \otimes_{\mathbb{C}[X_{\text{nr}}(M)]} \mathbb{C}(O_3)) \cong \bigoplus_{m \in M_2^3/M_3^3} \phi_m \mathbb{C}(O_3). \]

The isomorphisms (2.35) and (2.36) generalize [Hei2] Proposition 3.6 and Lemma 4.2 from the case \( \mu_{\sigma,1} = 1 \) to any multiplicity \( \mu_{\sigma,1} \).

The action of \( \text{Irr}(M/M_2^3) \) on \( \text{ind}_M^1(E) \) from (2.25) extends naturally to an action on \( \text{ind}_M^1(E) \otimes_{\mathbb{C}[X_{\text{nr}}(M)]} \mathbb{C}(X_{\text{nr}}(M)) \). From [Hei2] Proposition 4.3, with the same proof, we obtain
(2.37) \[ (\text{ind}_M^1(E) \otimes_{\mathbb{C}[X_{\text{nr}}(M)]} \mathbb{C}(X_{\text{nr}}(M)))^{\text{Irr}(M/M_2^3)} = \text{ind}_M^1(E_1) \otimes_{\mathbb{C}[O_3]} \mathbb{C}(O_3). \]

3. Some root systems and associated groups

Let \( \mathcal{A}_M \) be the maximal \( F \)-split torus in \( Z(M) \) and let \( X_*(\mathcal{A}_M) = X_*(A_M) \) be its cocharacter lattice. We write
\[ A_M = A_M(F), \quad a_M = X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \alpha_M = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}. \]
Let \( \Sigma(G, M) \subset X^*(A_M) \) be the set of nonzero weights occurring in the adjoint representation of \( A_M \) on the Lie algebra of \( G \), and let \( \Sigma_{\text{red}}(A_M) \) be the set of indivisible elements therein.

For every \( \alpha \in \Sigma_{\text{red}}(A_M) \) there is a Levi subgroup \( M_\alpha \) of \( G \) which contains \( M \) and the root subgroup \( U_\alpha \), and whose semisimple rank is one higher than that of \( M \). Let \( \alpha^V \in a_M \) be the unique element which is orthogonal to \( X^*(A_{M_\alpha}) \) and satisfies \( \langle \alpha^V, \alpha \rangle = 2 \).

Recall the Harish-Chandra \( \mu \)-functions from [Sil2] [Wal]. The restriction of \( \mu^G \) to \( \mathcal{O} \) is a rational, \( W(M, \mathcal{O}) \)-invariant function on \( \mathcal{O} \) [Wal, Lemma V.2.1]. It determines a reduced root system [Hei2] Proposition 1.3
\[ \Sigma_{\mathcal{O}, \mu} = \{ \alpha \in \Sigma_{\text{red}}(A_M) : \mu^M_{\sigma}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O} \}. \]
For \( \alpha \in \Sigma_{\text{red}}(A_M) \) the function \( \mu^{M_\alpha} \) factors through the quotient map \( A_M \to A_M/A_{M_\alpha} \). The associated system of coroots is

\[
\Sigma^\vee_{\mathcal{O},\mu} = \{ \alpha^\vee \in a_M : \mu^{M_\alpha}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O} \}.
\]

By the aforementioned \( W(M, \mathcal{O}) \)-invariance of \( \mu^G \), \( W(M, \mathcal{O}) \) acts naturally on \( \Sigma_{\mathcal{O},\mu} \) and \( \Sigma^\vee_{\mathcal{O},\mu} \). Let \( s_\alpha \) be the unique nontrivial element of \( W(M_\alpha, M) \). By [He2, Proposition 1.3] the Weyl group \( W(\Sigma_{\mathcal{O},\mu}) \) can be identified with the subgroup of \( W(G, M) \) generated by the reflections \( s_\alpha \) with \( \alpha \in \Sigma_{\mathcal{O},\mu} \), and as such it is a normal subgroup of \( W(M, \mathcal{O}) \).

The parabolic subgroup \( P = MU \) of \( G \) determines a set of positive roots \( \Sigma_{\mathcal{O},\mu}(P) \) and a basis \( \Delta_{\mathcal{O},\mu} \) of \( \Sigma_{\mathcal{O},\mu} \). Let \( \ell_\mathcal{O} \) be the length function on \( W(\Sigma_{\mathcal{O},\mu}) \) specified by \( \Delta_{\mathcal{O},\mu} \). Since \( W(M, \mathcal{O}) \) acts on \( \Sigma_{\mathcal{O},\mu} \), \( \ell_\mathcal{O} \) extends naturally to \( W(M, \mathcal{O}) \). The set of positive roots also determines a subgroup of \( W(M, \mathcal{O}) \):

\[
R(\mathcal{O}) = \{ w \in W(M, \mathcal{O}) : w(\Sigma_{\mathcal{O},\mu}(P)) = \Sigma_{\mathcal{O},\mu}(P) \}
\]

\[
= \{ w \in W(M, \mathcal{O}) : \ell_\mathcal{O}(w) = 0 \}.
\]

As \( W(\Sigma_{\mathcal{O},\mu}) \subset W(M, \mathcal{O}) \), a well-known result from the theory of root systems says:

\[
W(M, \mathcal{O}) = R(\mathcal{O}) \ltimes W(\Sigma_{\mathcal{O},\mu}).
\]

Recall that \( X_{m}(M)/X_m(M, \sigma) \) is isomorphic to character group of the lattice \( M_\sigma^2/M_1^1 \).

Since \( M_\sigma^2 \) depends only on \( \mathcal{O} \), it is normalized by \( N_G(M, \mathcal{O}) \). In particular the conjugation action of \( N_G(M, \mathcal{O}) \) on \( M_\sigma^2/M_1^1 \) induces an action of \( W(M, \mathcal{O}) \) on \( M_\sigma^2/M_1^1 \).

Let \( \nu_F : F \to \mathbb{Z} \cup \{ \infty \} \) be the valuation of \( F \). Let \( h_\alpha^\vee \) be the unique generator of \( (M_\sigma^2 \cap M^{\alpha,1})/M_1^1 \cong \mathbb{Z} \) such that \( \nu_F(\alpha(h_\alpha^\vee)) > 0 \). Recall the injective homomorphism \( H_M : M_1^1 \to a_M \) defined by

\[
(H_M(m), \gamma) = \nu_F(\gamma(m)) \quad \text{for } m \in M, \gamma \in X^*(M).
\]

In these terms \( H_M(h_\alpha^\vee) \in \mathbb{R}_{>0} \alpha^\vee \). Since \( M_\sigma^2 \) has finite index in \( M \), \( H_M(M_\sigma^2/M_1^1) \) is a lattice of full rank in \( a_M \). We write

\[
(M_\sigma^2/M_1^1)^\vee = \text{Hom}_\mathbb{Z}(M_\sigma^2/M_1^1, \mathbb{Z}).
\]

Composition with \( H_M \) and \( \mathbb{R} \)-linear extension of maps \( H_M(M_\sigma^2/M_1^1) \to \mathbb{Z} \) determines an embedding

\[
H_M^\vee : (M_\sigma^2/M_1^1)^\vee \to a_M^*.
\]

Then \( H_M^\vee(M_\sigma^2/M_1^1)^\vee \) is a lattice of full rank in \( a_M^* \).

**Proposition 3.1.** Let \( \alpha \in \Sigma_{\mathcal{O},\mu} \).

(a) For \( w \in W(M, \mathcal{O}) \): \( w(h_\alpha^\vee) = h_{w(\alpha)}^\vee \).

(b) There exists a unique \( \alpha^2 \in (M_\sigma^2/M_1^1)^\vee \) such that \( H_M^\vee(\alpha^2) \in \mathbb{R} \alpha \) and \( \langle h_\alpha^\vee, \alpha^2 \rangle = 2 \).

(c) Write

\[
\Sigma_{\mathcal{O}} = \{ \alpha^2 : \alpha \in \Sigma_{\mathcal{O},\mu} \},
\]

\[
\Sigma^\vee_{\mathcal{O}} = \{ h_\alpha^\vee : \alpha \in \Sigma_{\mathcal{O},\mu} \}.
\]

Then \( (\Sigma^\vee_{\mathcal{O}}, M_\sigma^2/M_1^1, \Sigma_{\mathcal{O}}, (M_\sigma^2/M_1^1)^\vee) \) is a root datum with Weyl group \( W(\Sigma_{\mathcal{O},\mu}) \).

(d) The group \( W(M, \mathcal{O}) \) acts naturally on this root datum, and \( R(\mathcal{O}) \) is the stabilizer of the basis determined by \( P \).
Proof. (a) Since $M_2^2/M^1$ and $\Sigma_{\sigma,\mu}$ are $W(M,\mathcal{O})$-stable, we have
\[ \tilde{w}(M_2^2 \cap M^{\alpha,1}) \tilde{w}^{-1} = M_2^2 \cap M^{w(\alpha),1}. \]
Hence $w(h^\vee_\alpha)$ is a generator of $(M_2^2 \cap M^{w(\alpha),1})/M^1$. As $w(\alpha)(w(h^\vee_\alpha)) = \alpha(h^\vee_\alpha)$, it equals $h^\vee_{w(\alpha)}$.
(b) Let $\alpha^* \in \mathbb{R}^{\alpha} \subset a_M^*$ be the unique element which satisfies
\[ \langle H_M(h^\vee_\alpha), \alpha^* \rangle = 2. \]
The group $W(\Sigma_{\sigma,\mu})$ acts naturally on $a_M$ by
\[ s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee = x - \langle x, \alpha^* \rangle H_M(h^\vee_\alpha). \]
This action stabilizes the lattice $H_M(M_\sigma/M^1)$. By construction $h^\vee_\sigma$ is indivisible in $M_\sigma^2/M^1$. It follows that for all $x \in H_M(M_\sigma^2/M^1)$ we must have $\langle x, \alpha^* \rangle \in \mathbb{Z}$. This means that $\alpha^*$ lies in $H_M^\vee(M_\sigma^2/M^1)^\vee$, say $\alpha^* = H_M^\vee(\alpha^\sharp)$. (c) By construction the lattices $M_\sigma^2/M^1$ and $(M_\sigma^2/M^1)^\vee$ are dual and $W(M,\mathcal{O})$ acts naturally on them. In view of (3.3), the map
\[ M_\sigma^2/M^1 \to M_\sigma^2/M^1 : \tilde{m} \mapsto \tilde{m} - \langle \tilde{m}, \alpha^\sharp \rangle h^\vee_\sigma \]
coincides with the action of $s_\alpha$. Hence it stabilizes $\Sigma_{\sigma,\mu}$.
Similarly, for $y \in a_M^*$:
\[ y - \langle H_M(h^\vee_\alpha), y \rangle H_M(\alpha^\sharp) = y - \langle \alpha^\sharp, y \rangle \alpha = s_\alpha(y). \]
This implies that the map
\[ (M_\sigma^2/M^1)^\vee \to (M_\sigma^2/M^1)^\vee : y \mapsto y - \langle h^\vee_\alpha, y \rangle \alpha^\sharp \]
coincides with the action of $s_\alpha$ and stabilizes $\Sigma_{\sigma,\mu}$. Thus $(\Sigma_{\sigma,\mu}^\vee, M_\sigma^2/M^1, \Sigma_{\sigma,\mu}, (M_\sigma^2/M^1)^\vee)$ is a root datum and the Weyl groups of $\Sigma_{\sigma,\mu}$ and $\Sigma_{\sigma,\mu}^\vee$ can be identified with $W(\Sigma_{\sigma,\mu})$.
(d) By part (a) $W(M,\mathcal{O})$ acts naturally on the root datum, extending the action of $W(\Sigma_{\sigma,\mu})$. The characterization of $R(\mathcal{O})$ is obvious from (3.1) and the definition of $\Sigma_{\sigma,\mu}$ and $\Sigma_{\sigma,\mu}^\vee$.

We note that $\Sigma_{\sigma,\mu}$ and $\Sigma_{\sigma,\mu}^\vee$ have almost the same type as $\Sigma_{\sigma,\mu}$. Indeed, the roots $H_M^\vee(\alpha^\sharp)$ are scalar multiples of the $\alpha \in \Sigma_{\sigma,\mu}$, the angles between the elements of $\Sigma_{\sigma,\mu}$ are the same as the angles between the corresponding elements of $\Sigma_{\sigma,\mu}$. It follows that every irreducible component of $\Sigma_{\sigma,\mu}$ has the same type as the corresponding components of $\Sigma_{\sigma,\mu}$ and $\Sigma_{\sigma,\mu}^\vee$, except that type $B_n/C_n$ might be replaced by type $C_n/B_n$.

For $\alpha \in \Sigma_{\text{red}}(M) \setminus \Sigma_{\sigma,\mu}$, the function $\mu^{M_\alpha}$ is constant on $\mathcal{O}$. In contrast, for $\alpha \in \Sigma_{\sigma,\mu}$ it has both zeros and poles on $\mathcal{O}$. By [Si3, §5.4.2]
\[ \langle \bar{s}_\alpha \cdot \sigma', \sigma' \rangle = \sigma \]
whenever $\mu^{M_\alpha}(\sigma') = 0$.

As $\Delta_{\sigma,\mu}$ is linearly independent in $X^*(A_M)$ and $\mu^{M_\alpha}$ factors through $A_M/A_{M_\alpha}$, there exists a $\sigma' \in \mathcal{O}$ such that $\mu^{M_\alpha}(\sigma') = 0$ for all $\alpha \in \Delta_{\sigma,\mu}$. In view of [Si3, §1] this can even be achieved with a unitary $\bar{\sigma}$. We replace $\sigma$ by $\bar{\sigma}$, we means that from now on we adhere to:

**Condition 3.2.** $(\sigma, E) \in \text{Rep}(M)$ is unitary supercuspidal and $\mu^{M_\alpha}(\sigma) = 0$ for all $\alpha \in \Delta_{\sigma,\mu}$. 

By (3.1) the entire Weyl group $W(\Sigma_{\mathcal{O}, \mu})$ stabilizes the isomorphism class of this $\sigma$. However, in general $R(\mathcal{O})$ need not stabilize $\sigma$. We identify $X_{nr}(M)/X_{nr}(M, \sigma)$ with $\mathcal{O}$ via $\chi \mapsto \sigma \otimes \chi$ and we define

\begin{equation}
X_\alpha = b_\alpha \in \mathbb{C}[X_{nr}(M)/X_{nr}(M, \sigma)].
\end{equation}

For any $w \in W(M, \mathcal{O})$ which stabilizes $\sigma$ in $\text{Irr}(M)$, Proposition \ref{prop:stabilizer} implies

\begin{equation}
w(X_\alpha) = X_{w(\alpha)} \quad \text{for all } \alpha \in \Sigma_{\mathcal{O}, \mu}.
\end{equation}

Let $q_F$ be the cardinality of the residue field of $F$. According to \cite{Sil} §1 there exist $a_{s_\alpha}, b_{s_\alpha} \in \mathbb{R}_{\geq 0}, c_{s_\alpha} \in \mathbb{R}_{> 0}$ for $\alpha \in \Sigma_{\mathcal{O}, \mu}$, such that

\begin{equation}
\mu^{M_\alpha}(\sigma \otimes \cdot) = \frac{c'_{s_\alpha} (1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_F^{-a_{s_\alpha}} X_\alpha)(1 - q_F^{-a_{s_\alpha}} X_\alpha^{-1})} \cdot \frac{(1 + X_\alpha)(1 + X_\alpha^{-1})}{(1 + q_F^{-b_{s_\alpha}} X_\alpha)(1 + q_F^{-b_{s_\alpha}} X_\alpha^{-1})}.
\end{equation}

We have only little explicit information about the $a_{s_\alpha}$ and the $b_{s_\alpha}$ in general ($c'_{s_\alpha}$ is not important). Obviously, knowing them is equivalent to knowing the poles of $\mu^{M_\alpha}$. These are precisely the reducibility points of the normalized parabolic induction $I_{P_{\mathcal{O}}M_\alpha}(\sigma \otimes \chi)$ \cite{Sil} §5.4. When these reducibility points are known somehow, one can recover $a_{s_\alpha}$ and $b_{s_\alpha}$ from them. In all cases that we are aware of, this method shows that $a_{s_\alpha}$ and $b_{s_\alpha}$ are integers. It would be interesting to know whether that holds in general.

We may modify the choice of $\sigma$ in Condition \ref{cond:stabilizer}, so that in addition

\begin{equation}
a_{s_\alpha} \geq b_{s_\alpha} \quad \text{for all } \alpha \in \Delta_{\mathcal{O}, \mu}.
\end{equation}

Comparing (3.7), Condition \ref{cond:stabilizer} and (3.8), we see that $a_{s_\alpha} > 0$ for all $\alpha \in \Sigma_{\mathcal{O}, \mu}$. In particular the zeros of $\mu^{M_\alpha}$ occur at

\begin{equation}
\{ X_\alpha = 1 \} = \{ \sigma' \in \mathcal{O} : X_\alpha(\sigma') = 1 \}
\end{equation}

and sometimes at

\begin{equation}
\{ X_\alpha = -1 \} = \{ \sigma' \in \mathcal{O} : X_\alpha(\sigma') = -1 \}.
\end{equation}

**Lemma 3.3.** Let $\alpha \in \Sigma_{\mathcal{O}, \mu}$ and suppose that $\mu^{M_\alpha}$ has a zero at both $\{ X_\alpha = 1 \}$ and $\{ X_\alpha = -1 \}$. Then the irreducible component of $\Sigma_{\mathcal{O}} \otimes \mathbb{F}_q$ containing $h_\alpha^\vee$ has type $B_n$ ($n \geq 1$) and $h_\alpha^\vee$ is a short root.

**Proof.** Consider any $h_\alpha^\vee \in \Sigma_{\mathcal{O}} \otimes \mathbb{F}_q$ which is not a short root in a type $B_n$ irreducible component. Then $\alpha^\vee$ is not a long root in a type $C_n$ irreducible component of $\Sigma_{\mathcal{O}}$, so there exists a $h_\beta^\vee \in \Sigma_{\mathcal{O}} \otimes \mathbb{F}_q$ with $\langle h_\beta^\vee, \alpha^\vee \rangle = -1$. Then

\begin{equation}
s_\alpha(h_\beta^\vee) = h_\beta^\vee - \langle h_\beta^\vee, \alpha^\vee \rangle h_\alpha^\vee = h_\beta^\vee + h_\alpha^\vee \in M_\alpha^2/M_\alpha^1.
\end{equation}

With (3.6) we find $s_\alpha(X_\beta) = X_\beta X_\alpha$. Assume that there exists a $\sigma' \in \mathcal{O}$ with $\mu^{M_\alpha}(\sigma') = 0$ and $X_\alpha(\sigma') = -1$. We compute

\begin{equation}
X_\beta(s_\alpha \cdot \sigma') = (s_\alpha X_\beta)(\sigma') = (X_\beta X_\alpha)(\sigma') = X_\beta(\sigma') X_\alpha(\sigma') = -X_\beta(\sigma').
\end{equation}

As $X_\beta \in \mathbb{C}[X_{nr}(M)/X_{nr}(M, \sigma)]$, this implies that $s_\alpha \cdot \sigma'$ is not isomorphic to $\sigma'$. But that contradicts (3.4), so the assumption cannot hold. \hfill \Box

Consider $r \in R(\mathcal{O})$. By the definition of $W(M, \mathcal{O})$ there exists a $\chi_r \in X_{nr}(M)$ such that

\begin{equation}
\tilde{r} \cdot \sigma \cong \sigma \otimes \chi_r.
\end{equation}
Lemma 3.4. (a) The maps $\alpha \mapsto a_{s\alpha}, \alpha \mapsto b_{s\alpha}$ and $\alpha \mapsto c'_{s\alpha}$ are constant on $W(M, O)$-orbits in $\Sigma_{O, \mu}$.
(b) For $\alpha \in \Delta_{O, \mu}$ and $r \in R(O)$, either $X_\alpha(\chi_r) = 1$ or $X_\alpha(\chi_r) = -1$ and $a_{s\alpha} = b_{s\alpha}$.

Proof. It follows directly from the definitions in [Wal] §V.2 that

$$\mu^M_w(\alpha)(\bar{w} \cdot \sigma') = \mu^M_w(\sigma') \quad \text{for all } w \in W(M, O).$$

Since every $W(S_{O, \mu})$-orbit in $\Sigma_{O, \mu}$ meets $\Delta_{O, \mu}$, (3.8) generalizes to

$$\mu^M_{\alpha}(\sigma \otimes X) = \mu^M_{\alpha}(\tilde{\sigma} \cdot (\sigma \otimes X)) = \mu^M_{\alpha}(\sigma \otimes X_{r}(\chi)) =$$

$$\sp_{X_{r}(\chi)}\left(\frac{c'_{s\alpha}(1 - X_{ra})(1 - X_{ra}^{-1})}{(1 - q_{F}^{-a_{s\alpha}} X_{r}(\chi))} \times \frac{(1 + X_{ra})(1 + X_{ra}^{-1})}{(1 + q_{F}^{a_{s\alpha}} X_{r}(\chi))} \right) =$$

Comparing the zero orders along the subvarieties $\{X_\alpha = \text{constant}\}$, we see that $X_{ra}(\chi_r) \in \{1, -1\}$. Then we look at the pole orders.

When $X_{ra}(\chi_r) = 1$, we obtain $a_{s\alpha} = a_{s\alpha}$ and $b_{s\alpha} = b_{s\alpha}$.

When $X_{ra}(\chi_r) = -1$, we find $a_{s\alpha} = b_{s\alpha}$ and $b_{s\alpha} = a_{s\alpha}$. Together with (3.11) that implies $a_{s\alpha} = b_{s\alpha} = b_{s\alpha} = a_{s\alpha}$.

Knowing all this, another glance at (3.10) reveals that $c'_{s\alpha} = c'_{s\alpha}$.

Of course, $\chi_r$ is in general not unique, only up to $X_{ir}(M, \sigma)$. If $\tilde{\sigma} \cdot \sigma \cong \sigma$, then we take $\chi_r = 1$, otherwise we just pick one of eligible $\chi_r$. We note that then

$$\tilde{\sigma}^{-1} \cdot \sigma \cong \sigma \otimes \sigma^{-1}(\chi_r^{-1}),$$

which implies

$$r^{-1}(\chi_r) \chi_{r^{-1}} \in X_{ir}(M, \sigma).$$

For $r \in R(O)$ of order larger than two, we may take $\chi_{r^{-1}} = r^{-1}(\chi_r^{-1})$.

Lemma 3.5. For all $w \in W(S_{O, \mu}), r \in R(O)$: $w(\chi_r) \chi_{r^{-1}}^{-1} \in X_{ir}(M, \sigma)$.

Proof. We abbreviate $w' = r^{-1}w^{-1}r$. Since $ww'r^{-1} = 1 \in W(M, O)$,

$$\tilde{w} \cdot \tilde{\sigma} \cdot \tilde{w} \cdot \tilde{r} \cdot \tilde{\sigma} \cdot \tilde{r}^{-1} \cdot \sigma \cong \sigma \in \text{Irr}(M).$$
We can also work out the left hand side stepwise. Recall from Condition 3.2 that $W(\Sigma_{O,\mu})$ stabilizes $\sigma \in \text{Irr}(M)$. With (3.12) we compute
\[
\tilde{w} \cdot \tilde{r} \cdot \tilde{w}' \cdot \tilde{r}^{-1} \cdot \sigma \cong \tilde{w} \cdot \tilde{r} \cdot \tilde{w}' \cdot (\sigma \otimes \chi_{r^{-1}})
\cong \tilde{w} \cdot \tilde{r} \cdot (\sigma \otimes w') (\chi_{r^{-1}})
\cong \tilde{w} \cdot (\sigma \otimes \chi_r \otimes w') (\chi_{r^{-1}})
\cong \sigma \otimes w(\chi_r) \otimes w(w') (\chi_{r^{-1}}) = \sigma \otimes w(\chi_r) (\chi_{r^{-1}}).
\]

Now we have three collections of transformations of $O$:
\[
\begin{align*}
\sigma \otimes \chi & \mapsto w(\sigma \otimes \chi) \cong \sigma \otimes w(\chi) \quad w \in W(\Sigma_{O,\mu}), \\
\sigma \otimes \chi & \mapsto r(\sigma \otimes \chi) \cong \sigma \otimes r(\chi) \quad r \in R(O), \\
\sigma \otimes \chi & \mapsto \sigma \otimes \chi \chi_c \quad \chi_c \in X_{nr}(M, \sigma).
\end{align*}
\]
These give rise to the following transformations of $X_{nr}(M)$:
\[
\begin{align*}
\text{w} : \chi & \mapsto w(\chi) \quad w \in W(\Sigma_{O,\mu}), \\
\text{r} : \chi & \mapsto r(\chi) \chi_r \quad r \in R(O), \\
\chi_c : \chi & \mapsto \chi \chi_c \quad \chi_c \in X_{nr}(M, \sigma).
\end{align*}
\]
Let $W(M, \sigma, X_{nr}(M))$ be the group of transformations of $X_{nr}(M)$ generated by the $\text{w}$, $\text{r}$ and $\phi_{\chi_c}$ from (3.13). Since $X_{nr}(M, \sigma)$ is $W(M, O)$-stable, it constitutes a normal subgroup of $W(M, \sigma, X_{nr}(M))$. Further $W(\Sigma_{O,\mu})$ embeds as a subgroup in $W(M, \sigma, X_{nr}(M))$, and $R(O)$ as the subset $\{ \text{r} : r \in R(O) \}$.

By (3.2) and Lemma 3.5 the multiplication map
\[
(4.14) \quad X_{nr}(M, \sigma) \times R(O) \times W(\Sigma_{O,\mu}) \rightarrow W(M, \sigma, X_{nr}(M))
\]
is a bijection (but usually not a group homomorphism). We note that $R(O)$ does not necessarily normalize $W(\Sigma_{O,\mu})$ in $W(M, \sigma, X_{nr}(M))$:
\[
\begin{align*}
\text{rwr}^{-1}(\chi) & = \text{rwr}(\chi^{-1} \text{rwr}^{-1} (\chi_r^{-1})) \\
& = \chi(\text{wr}^{-1}(\chi) \text{wr}^{-1} (\chi_r^{-1})) = (\text{wr}^{-1}(\chi)) \chi(\text{wr}^{-1} (\chi_r^{-1})) = \chi_r^{-1} \chi_r.
\end{align*}
\]
Rather, $W(M, \sigma, X_{nr}(M))$ is a nontrivial extension of $W(M, O)$ by $X_{nr}(M, \sigma)$.

Via the quotient maps
\[
W(M, \sigma, X_{nr}(M)) \rightarrow W(M, O) \rightarrow W(\Sigma_{O,\mu})
\]
we lift $\ell_O$ to $W(M, \sigma, X_{nr}(M))$.

4. Intertwining Operators

By [BeRu, §III.4.1] or [Ren], the parabolically induced representation $I_{P}^G(\text{ind}_{M}^{M}(E))$ is a progenerator of $\text{Rep}(G)^{\phi}$; Hence
\[
\begin{align*}
\mathcal{E} : \text{Rep}(G)^{\phi} & \rightarrow \text{End}_{G}(I_{P}^G(\text{ind}_{M}^{M}(E))) - \text{Mod} \\
V & \mapsto \text{Hom}_{G}(I_{P}^G(\text{ind}_{M}^{M}(E)), V)
\end{align*}
\]
is an equivalence of categories. We want to find elements of $\text{End}_{G}(I_{P}^G(\text{ind}_{M}^{M}(E)))$ that do not come from $\text{End}_{M}(\text{ind}_{M}^{M}(E))$. Harish-Chandra devised by now standard intertwining operators for $I_{P}^G(E)$. However, they arise as a rational functions of $\sigma \in O$, so their images lie in $I_{P}^G(E \otimes_{\mathbb{C}} \mathbb{C}(X_{nr}(M)))$ and they may have poles. We will exhibit variations which have fewer singularities.
We abbreviate
\[ E_B = E \otimes \mathbb{C} B = E \otimes \mathbb{C} \mathbb{C}[X_{nr}(M)] \cong \text{ind}^{M^1}_E (E). \]
The $M$-representation (2.3) on these spaces is denoted $\sigma_B$. Similarly we have the $M$-representation $\sigma_{K(B)}$ on
\[ E_{K(B)} = E \otimes \mathbb{C} K(B) = E_B \otimes_B K(B) = E \otimes \mathbb{C} \mathbb{C}[X_{nr}(M)]. \]
The specialization at $\chi \in X_{nr}(M)$ from (2.4) is a $M$-homomorphism
\[ \text{sp}_\chi : (\sigma_B, E_B) \to (\sigma \otimes \chi, E). \]
It extends to the subspace of $E_{K(B)}$ consisting of functions that are regular at $\chi$.

Let $\delta_P : P \to \mathbb{R}_{>0}$ be the modular function. We realize $I^G_P (E)$ as
\[ \{ f : G \to E \mid f \text{ is smooth, } f(umg) = \sigma(m)\delta_P^{-1/2}(m)f(g) \forall g \in G, m \in M, u \in U \}. \]
As usual $I^G_P (\sigma)(g)$ is right translation by $g$. With $I^G_P$, we can regard $\text{sp}_\chi$ also as a $G$-homomorphism
\[ I^G_P (\sigma_B, E_B) \to I^G_P (\sigma \otimes \chi, E). \]
Fix a maximal $F$-split torus $A_0$ in $G$, contained in $M$. Let $x_0$ be a special vertex in the apartment of the extended Bruhat–Tits building of $(G, F)$ associated to $A_0$. Its isotropy group $K = G_{x_0}$ is a good maximal compact subgroup of $G$, so it contains representatives for all elements of the Weyl group $W(G, A_0)$ and $G = PK$ by the Iwasawa decomposition.

The vector space $I^G_P (E)$ is naturally in bijection with
\[ I^K_P \cap K (E) = \{ f : K \to E \mid f \text{ is smooth, } f(umk) = \sigma(m)f(k) \forall k \in K, m \in M \cap K, u \in U \cap K \}. \]
Notice that this space is the same for $(\sigma, E)$ and $(\sigma \otimes \chi, E)$, for any $\chi \in X_{nr}(M)$.

4.1. Harish-Chandra’s operators $J_{P'|P}$.

Let $P' = MU'$ be another parabolic subgroup of $G$ with Levi factor $M$. Following [Wal, §IV.1] we consider the $G$-map
\[ J_{P'|P}(\sigma) : I^G_{P'}(E) \to I^G_P(E) \quad f \mapsto \left[ g \mapsto \int_{U' \cap U \setminus U'} f(u'g)du' \right]. \]
The integral does not always converge. Rather, $J_{P'|P}$ should be considered as a map
\[ X_{nr}(M) \times I^K_{P \cap K}(E) \to I^K_{P'|P \cap K}(E) \quad (\chi, f) \mapsto J_{P'|P}(\sigma \otimes \chi)f \quad , \]
where $I^K_{P \cap K}(E)$ is identified with $I^G_P (\sigma \otimes \chi, E)$ as above. With this interpretation $J_{P'|P}$ is rational in the variable $\chi$ [Wal, Théorème IV.1.1]. In yet other words, it defines a map
\[ I^K_{P \cap K}(E) \to I^K_{P'|P \cap K}(E) \otimes \mathbb{C} \mathbb{C}[X_{nr}(M)] \quad f \mapsto [\chi \mapsto J_{P'|P}(\sigma \otimes \chi)f] \quad . \]
For $h \in G$, let $\lambda(h)$ be the left translation operator on functions on $G$:
\[ \lambda(h) f : g \mapsto f(h^{-1}g). \]
For every $w \in W(G, M)$ we choose a representative $\tilde{w} \in N_K(M)$ (that is is possible because the maximal compact subgroup $K$ is in good position with respect to $A_0 \subset$
Proposition 4.1. \( \chi \in J \) isomorphisms parabolic induction. That makes \( J \) extend to \( G \)-isomorphisms

\[
J_{K(B),w} : I_P^G(E_B) \rightarrow I_P^G(w \cdot E_{K(B)})
\]

It follows from [Wal. Proposition IV.2.2] that \( J_{P'|P} \) and \( J_{K(B),w} \) extend to \( G \)-isomorphisms

\[
J_{P'|P} : I_P^G(E_{K(B)}) \rightarrow I_P^G(w \cdot E_{K(B)}),
J_{K(B),w} : I_P^G(E_{K(B)}) \rightarrow I_P^G(w \cdot E_{K(B)}).
\]

The algebra \( B \) embeds in \( \text{End}_G(I_P^G(E_B)) \) and in \( \text{End}_G(I_P^G(w \cdot E_{K(B)})) \) via \( (w \cdot b) \) and parabolic induction. That makes \( J_{P'|P} \) and \( J_{K(B),w} \) \( B \)-linear.

The group \( W(G, M) \) acts on \( B = \mathbb{C}[X_{nr}(M)] \) and \( K(B) = \mathbb{C}(X_{nr}(M)) \) by

\[
w \cdot b_m = b_{w(m)} = b_{\tilde{w} w^{-1}}, \quad (w \cdot b)(\chi) = b(w^{-1} \chi),
\]

for \( w \in W(G, M), m \in M, b \in K(B), \chi \in X_{nr}(M) \). This determines \( M \)-isomorphisms

\[
\tau_w : (\tilde{w} \cdot \sigma_B, w \cdot E_B) \rightarrow ((\tilde{w} \cdot \sigma)_B, (w \cdot E)_B)
\]

\[
(\tilde{w} \cdot \sigma_{K(B)}, w \cdot E_{K(B)}) \rightarrow ((\tilde{w} \cdot \sigma)_{K(B)}, (w \cdot E)_{K(B)}).
\]

With functoriality we obtain \( G \)-isomorphisms

\[
I_P^G(w(E_B)) \rightarrow I_P^G((w \cdot E)_B) \quad \text{and} \quad I_P^G(w(E_{K(B)})) \rightarrow I_P^G((w \cdot E)_{K(B)}),
\]

which we also denote by \( \tau_w \). Composition with \( J_{K(B),w} \) gives

\[
\tau_w \circ J_{K(B),w} : I_P^G(E_B) \rightarrow I_P^G((w \cdot E)_{K(B)}).
\]

In order to associate to \( w \) an element of \( \text{Hom}_G(I_P^G(E_B), I_P^G((w \cdot E)_{K(B)})) \), it remains to construct a suitable \( G \)-intertwiner from \( I_P^G((w \cdot E)_{K(B)}) \) to \( I_P^G(E_{K(B)}) \). For this we do not want to use \( \tau_{w^{-1}} \circ J_{K(B),w^{-1}} \), then we would end up with a simple-minded \( G \)-automorphism of \( I_P^G(E_{K(B)}) \) (essentially multiplication with an element of \( K(B) \)). We rather employ an idea from [Hei2]: construct a \( G \)-intertwiner \( I_P^G(w \cdot E) \rightarrow I_P^G(E) \) and extend it to \( I_P^G((w \cdot E)_B) \rightarrow I_P^G(E_B) \) by making it constant on \( X_{nr}(M) \).

With this motivation we analyse the poles of the operators \( J_{P'|P} \) and \( J_{K(B),w} \). They are closely related to zeros of the Harish-Chandra \( \mu \)-functions. Namely, for \( \alpha \in \Sigma_{\text{red}}(M) \):

\[
J_{P|s_\alpha(P)}(s_\alpha(\sigma \otimes \chi))J_{s_\alpha(P)|P}(\sigma \otimes \chi) = \frac{\text{constant}}{\mu^M_\alpha(\sigma \otimes \chi)}
\]

as rational functions of \( \chi \in X_{nr}(M) \) [SII] [§1].

**Proposition 4.1.** Let \( P' = MU' \) be a parabolic subgroup of \( G \) with Levi factor \( M \), and consider \( J_{P'|P} \) in the form \( \text{[4.3]} \).

(a) All the poles of \( J_{P'|P} \) occur at

\[
\bigcup_{\alpha \in \Sigma_{\sigma,\mu}(P') \cap \Sigma_{\sigma,\mu}(P)} \{ \chi \in X_{nr}(M) : \mu^M_\alpha(\sigma \otimes \chi) = 0 \}.
\]
(b) Suppose that $\chi_2 \in \mathcal{X}_m(M)$ satisfies $\mu^{M_\alpha}(\sigma \otimes \chi_2) = 0$ for precisely one $\alpha \in \Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(\overline{P})$. Then $J_{P|P}$ has a pole of order one at $\chi_2$ and

$$(X_\alpha(\chi) - X_\alpha(\chi_2))J_{P|P}(\sigma \otimes \chi) : I_{P}^G(\sigma \otimes \chi) \to I_{P}^G(\sigma \otimes \chi)$$

is bijective for all $\chi$ in a certain neighborhood of $\chi_2$ in $\mathcal{X}_m(M)$.

(c) There exists a neighborhood $V_1$ of 1 in $\mathcal{X}_m(M)$ on which

$$\prod_{\alpha \in \Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(\overline{P})}(X_\alpha - 1)J_{P|P} : I_{P}^K(E) \to I_{P|P}^K(E) \otimes \mathbb{C}(X_m(M))$$

has no poles. The specialization of this operator to $\chi \in V_1$ is a $G$-isomorphism $I_{P}^G(\sigma \otimes \chi) \to I_{P}^G(\sigma \otimes \chi)$.

Proof. As in \cite{Wal} p. 279] we define

$$d(P, P') = |\{\alpha \in \Sigma_{\text{red}}(A_M) : \alpha \text{ is positive with respect to both } P \text{ and } \overline{P}\}|.$$

Choose a sequence of parabolic subgroups $P_i = M U_i$ such that $d(P, P_{i+1}) = 1, P_0 = P$ and $P_d(P, P_i) = P'$. From \cite{Wal} p. 283] we know that

$$J_{P|P} = J_{P|P|P_d(P,P_i)} \circ \cdots \circ J_{P_2|P_1} \circ J_{P|P}.$$

In this way we reduce the whole proposition to the case $d(P, P') = 1$. Assume that, and let $\alpha \in \Sigma_{\text{red}}(A_M)$ be the unique element which is positive respect to both $P$ and $\overline{P}$.

When $\alpha \not\in \Sigma_{\mathcal{O}, \mu}$, \cite{Hei2} Proposition 1.10] says that the specialization of $J_{P|P}$ at any $\chi \in \mathcal{X}_m(M)$ is regular and bijective. That proves parts (a) and (c) for such a $P'$, while (b) is vacuous because $\mu^{M_\alpha}$ is constant on $\mathcal{O}$ \cite{Si} Theorem 1.6].

Suppose now that $\alpha \in \Sigma_{\mathcal{O}, \mu}$. We have

$$(U \cap U') \setminus U' \cong U_{-\alpha} \subset M_\alpha.$$

Hence $J_{P|P}$ arises by induction from $J_{P|P_{\cap M_\alpha}|P_{\cap M_\alpha}}$, and it suffices to consider the latter operator. We apply \cite{Hei2} Lemme 1.8] with $M_\alpha$ in the role of $G$, that yields parts (a) and (b) of our proposition. Part (c) follows because $X_\alpha - 1$ has a zero of order one at $\{X_\alpha = 1\}$. \hfill $\square$

4.2. The auxiliary operators $\rho_w$.

With Proposition 4.1 we define, for $w \in W(G, M)$, a $G$-homomorphism

$$\rho^{\prime}_{\sigma \otimes \chi, w} = \lambda(\tilde{w})sp_{\chi} \prod_{\alpha \in \Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(w^{-1}(\overline{P}))}(X_\alpha - 1)J_{w^{-1}(P)}(\sigma \otimes \chi) : I_{P}^G(\sigma \otimes \chi) \to I_{P}^G(\tilde{w}(\sigma \otimes \chi)).$$

We note that $\rho^{\prime}_{\sigma \otimes \chi, w}$ is not canonical, because it depends on the choice of a representative $\tilde{w} \in N_K(M)$ for $w$.

Lemma 4.2. For $w \in W(\Sigma_{\mathcal{O}, \mu})$, $\rho^{\prime}_{\sigma, w}$ arises by parabolic induction from a $M$-isomorphism $\rho^{\prime, -1}_{\sigma, w} : (\sigma, E) \to (\tilde{w} \cdot \sigma, E)$.

Proof. We compare $\rho^{\prime}_{\sigma \otimes \chi}$ with Harish-Chandra’s operator \cite{Wal} §5.3] $\circ_{c P|P}(w, \sigma \otimes \chi) \in \text{Hom}_{G \times G}(\text{End}_C(\sigma \otimes \chi), \text{End}_C(\tilde{w}(\sigma \otimes \chi))).$ By Proposition 4.1 and \cite{Wall} Lemme V.3.1] both are rational as functions of $\chi \in \mathcal{X}_m(M)$, and regular on a neighborhood of 1. For generic $\chi$ the $G$-representations $I_{P}^G(\sigma \otimes \chi)$ and $I_{P}^G(\tilde{w}(\sigma \otimes \chi))$ are irreducible, so there $\circ_{c P|P}(w, \sigma \otimes \chi)$ specializes to a
scalar times conjugation by $\rho'_{\otimes \chi, w}$. It follows that $\rho'_{\otimes \chi, w}$ equals a rational function times the intertwining operator associated by Harish-Chandra to $w$ and $\sigma$.

Let us make this more precise. By Condition 3.2 there exists an $M$-isomorphism

$$\phi_{\tilde{w}} : (\tilde{w} \cdot \sigma, E) \to (\sigma, E).$$

For any $\chi \in X_w(M)$, it gives an $M$-isomorphism $\tilde{w} \cdot \sigma \otimes w\chi \to \sigma \otimes w\chi$. Consider the $G$-homomorphism

$$I_P^G(\phi_{\tilde{w}}) \circ \rho'_{\otimes \chi, w} : I_P^G(\sigma \otimes \chi, E) \to I_P^G(\sigma \otimes w\chi, E).$$

By the above, this is equal to a rational function times the operator

$$c_{P'|P}(w, \sigma \otimes \chi) \in \text{Hom}_G(I_P^G(\sigma \otimes \chi), I_P^G(\sigma \otimes w\chi))$$

considered in [Sil1]. By the Knapp–Stein theorem for $p$-adic groups [Sil1], (4.8) specializes at $\chi = 1$ to the identity operator, while by Proposition 4.1 the operator (4.7) specializes at $\chi = 1$ to an isomorphism. Hence (4.7) for $\chi = 1$ is a nonzero scalar multiple of the identity operator and

$$\rho'_{\sigma, w} = zI_P^G(\phi_{\tilde{w}})^{-1} = I_P^G(z\phi_{\tilde{w}}^{-1})$$

for some $z \in \mathbb{C}^\times$. \hfill \qed

From $\rho'_{\sigma, w}$ and Lemma 4.2 we obtain an isomorphism of $M$-$B$-representations

$$\rho_{\sigma, w}^{-1} \otimes \text{id}_B : (\sigma, E \otimes B) \to (\tilde{w} \cdot \sigma, E \otimes B).$$

Applying $I_P^G$ with $P' = MU'$, this yields an isomorphism of $G$-$B$-representations

$$I_P^G(\rho_{\sigma, w}^{-1} \otimes \text{id}_B) : I_P^G(\sigma \otimes B) \to I_P^G(\tilde{w} \cdot \sigma\otimes B)$$

whose specialization at $P' = P, \chi = 1$ is $\rho'_{\sigma, w}$. (However, its specialization at other $\chi \in X_w(M)$ need not be equal to $\rho'_{\otimes \chi, w}$.) To comply with the notation from [Hei2] we define

$$\rho_{P', w} = I_P^G(\rho_{\sigma, w} \otimes \text{id}_K(B)) : I_P^G(\tilde{w} \cdot E) \to I_P^G(E_K(B)).$$

Following the same procedure with $K(B)$ instead of $B$, we can also regard $\rho_{P', w}$ as an isomorphism of $G$-$B$-representations

$$I_P^G(\rho_{\sigma, w} \otimes \text{id}_{K(B)}) : I_P^G((\tilde{w} \cdot E)_{K(B)}) \to I_P^G(E_{K(B)}).$$

When $P' = P$, we often suppress it from the notation. We need a few calculation rules for the operators $\rho_{P', w}$.

**Lemma 4.3.** Let $w, w_1, w_2 \in W(\Sigma_{\mathcal{O}, \mu})$.

(a) $J_{P'|P}(\sigma \otimes \cdot) \circ \rho_{P, w} = \rho_{P', w} \circ J_{P'|P}(\tilde{w} \sigma \otimes \cdot) : I_P^G((\tilde{w} \cdot E)_{K(B)}) \to I_P^G(E_{K(B)}).$

(b) As operators $I_{w_2^{-1}w_1^{-1}(P)}^G(E_{K(B)}) \to I_P^G(E_{K(B)})$:

$$\rho_{w_1^{-1}w_2} \lambda(\tilde{w}_1) \rho_{w_1^{-1}w_2} \lambda(\tilde{w}_2) = \prod_{\alpha} \left( sp_{\chi=1} \frac{\mu M_{\alpha}(\chi \otimes \cdot)}{(X_{\alpha} - 1)(X_{\alpha} - 1)} \right) \rho_{w_1w_2} \lambda(\tilde{w}_1\tilde{w}_2)$$

where the product runs over $\Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(w_2^{-1}(P)) \cap \Sigma_{\mathcal{O}, \mu}(w_2^{-1}w_1^{-1}(P))$.

(c) For $r \in R(\mathcal{O})$:

$$\lambda(\tilde{r}) \rho_{r^{-1}(P), w} \lambda(\tilde{r})^{-1} = \rho_{P_{r^{-1}(P), w}, w} \lambda(\tilde{r}^{-1}w r^{-1} \tilde{r}^{-1}).$$
Proof. (a) In this setting $J_{P^1} | P$ is invertible \cite{[4.5]}, so we can reformulate the claim as

$$J_{P^1} | P (\tilde{w} \cdot \sigma \otimes \cdot)^{-1} \circ \rho_{P^1,w}^{-1} \circ J_{P^1} | P (\sigma \otimes \cdot) = \rho_{P^1,w}^{-1}.$$

The left hand side one first transfers everything from $I_P^G$ to $I_P$ by means of $\int_{(U \cap U')} | U'$, then we apply $\rho_{P^1,w}^{-1} = I_P^G(\rho^\mu_w \otimes \text{id}_B)$ and finally we transfer back from $I_P^G$ to $I_P$ (in the opposite fashion). In view of (4.9), this is just a complicated way to express $\rho_{P^1,w}^{-1}$. (b) The map

$$\tau_{w_1} \lambda(\tilde{w_1}) \rho_{P^1,w_2} \lambda(\tilde{w_2})^{-1} \tau_{w_1}^{-1} : I_{w_1(P')} ((\tilde{w_1} \tilde{w_2} \cdot E)_{K(B)}) \rightarrow I_{w_1(P')} ((\tilde{w_1} \cdot E)_{K(B)})$$

is denoted simply $\rho_{w_2}$ in \cite{H2}. Knowing that and part (a), the claim can shown in the same way as in \cite{Hei2 p. 729}. (c) By definition

$$\rho_{P^1}^{-1} \rho_{P^1,w}^{-1} = \lambda(\tilde{s}_o) \sigma_{X_{\alpha}}(X_{\alpha}(\chi) - 1) J_{w_1(P')} (\sigma \otimes \chi),$$

$$\rho_{P,F \sigma w}^{-1} = \lambda(\tilde{s}_o) \sigma_{X_{\alpha}}(X_{\alpha}(\chi) - 1) J_{w_1(P')} (\sigma \otimes \chi).$$

Here $\alpha$ runs over $\Sigma_{O,\mu}(w^{-1}P)$ and $\beta$ over $\Sigma_{O,\mu}(P) \cap \Sigma_{O,\mu}(w^{-1}P)$.

It follows that

$$\lambda(\tilde{s}_o) \sigma_{X_{\alpha}}(X_{\alpha}(\chi) - 1) J_{w_1(P')} (\sigma \otimes \chi) = \lambda(\tilde{s}_o) \sigma_{X_{\alpha}}(X_{\alpha}(\chi) - 1) J_{w_1(P')} (\sigma \otimes \chi).$$

Taking inverses yields the claim. \hfill \square

Now we associate similar operators to elements of the group $R(O)$ from \cite{[3.1]} and \cite{[3.2]}. We may assume that the representatives $\tilde{w} \in N_K(M)$ are chosen so that

(4.11) \quad $\tilde{w} \in R(O)$ for all $r \in R(O), w \in W(\Sigma_{O,\mu}).$

For $r \in R(O)$, Proposition 2.2 and \cite{Hei2} Proposition 1.10 say that $J_{r(P')} (\sigma \otimes \chi)$ is rational and regular on $X_{\mu}(M)$, and that its specialization at any $\chi$ is a $G$-isomorphism $I_P^G (\sigma \otimes \chi) \rightarrow I_{r(P')}^G (\sigma \otimes \chi)$. For such $r$ we construct an analogue of $\rho_{w}$ in a simpler way. Let $\chi_r$ be as in \cite{[3.2]} and pick an $M$-isomorphism

(4.12) \quad $\rho_{\sigma, r} : \tilde{r} \cdot \sigma \rightarrow \sigma \otimes \chi_r.$

Recall $\rho_{\chi_r}$ from \cite{[2.7]}. It combines with $\rho_{\sigma, r}$ to an $M$-isomorphism

$$\rho_{\sigma, r} \otimes \rho_{\chi_r}^{-1} : ((\tilde{r} \cdot \sigma)_{B}, E_B) \rightarrow (\sigma_B, E_B),$$

which is not $B$-linear when $\chi_r \neq 1$. With parabolic induction we obtain a $G$-isomorphism

$$\rho_{P^1} = I_P^G (\rho_{\sigma, r} \otimes \rho_{\chi_r}^{-1}) : I_P^G (\tilde{r} \cdot \sigma)_{B}, E_B) \rightarrow I_P^G (\sigma_B, E_B).$$

The same works with $K(B)$ instead of $B$.

We note that Lemma 4.3a also applies to $\rho_{r}$, with the same proof:

(4.13) \quad $J_{P^1} (\sigma \otimes \cdot) \circ \rho_{\sigma, r} = \rho_{\sigma, r} \circ J_{P^1} (\tilde{r} \sigma \otimes \chi_r^{-1} \cdot) : I_P^G ((\tilde{r} \cdot E)_{K(B)}) \rightarrow I_P^G (E_{K(B)}).$
For an arbitrary \( w \in W(M, \mathcal{O}) \), we use (3.2) and (4.11) to write \( \tilde{w} = r \tilde{w} \) with \( r \in R(\mathcal{O}) \) and \( w \in W(\Sigma_{\mathcal{O}, \mu}) \). Then we put \( \chi_{w} = \chi_{r} \) and

\[
\rho_{\sigma, w} = \rho_{\sigma, r \rho_{\sigma, w}} : \tilde{w} \cdot \sigma \otimes \chi_{r} \to \sigma \otimes \chi_{r},
\]

\[
\rho_{\rho_{\sigma, w}} : \rho_{\rho_{\sigma, (w \in \mathcal{O})}, \mu} \lambda(\tilde{r}) = (\rho_{\rho_{\sigma, w}}(w \in \mathcal{O})) \to I_{E_{B}}^{G}(E_{B}).
\]

Let us discuss the multiplication relations between all the operators constructed in this section and the \( \phi_{\chi} \) with \( \chi \in \text{Xnr}(M, \sigma) \) from (2.18). Via \( I_{P}^{G} \), we regard \( \phi_{\chi} \) also as an element of \( \text{Hom}_{G}(I_{E_{B}}^{G}(E_{B})) \). We note that

\[
(4.14) \quad \text{sp}_{\chi'} \circ \phi_{\chi} = \text{sp}_{\chi'} \circ (\phi_{\sigma, \chi} \otimes \rho_{\chi}^{-1}) = \phi_{\sigma, \chi} \otimes \text{sp}_{\chi'}^{-1} \chi' \in \text{Xnr}(M).
\]

From the very definition of \( J_{P}^{G}_{|P} \) in (4.2) we see that

\[
(4.15) \quad J_{P}^{G}_{|P} \circ \phi_{\chi} = (\phi_{\sigma, \chi} \otimes \rho_{\chi}^{-1}) \circ \phi_{\sigma, \chi} = J_{P}^{G}_{|P}(\sigma \otimes \chi'),
\]

which quickly implies

\[
(4.16) \quad \phi_{\tilde{w}, \sigma, w}(\chi) = \text{Hom}_{M}(\tilde{w} \cdot \sigma, \tilde{w} \cdot \sigma \otimes w(\chi))
\]
equal to \( \phi_{\sigma, \chi} \) as \( \mathbb{C} \)-linear map. Next we define

\[
\phi_{\sigma, \chi}^{w}(\chi) = \phi_{\tilde{w}, \sigma, w}(\chi) \otimes \rho_{\sigma, w(\chi)}^{-1} \in \text{End}_{M}(w \cdot E_{B}),
\]

and we tacitly extend to an element of \( \text{End}_{G}(I_{E_{B}}^{G}(w \cdot E_{B})) \) by functoriality. Then

\[
(4.17) \quad \tau_{w} \lambda(\tilde{w}) \lambda_{\chi} = (\phi_{w \cdot \sigma, w(\chi)} \otimes \rho_{w(\chi)}^{-1}) \tau_{w} \lambda(\tilde{w}) = \phi_{w \cdot \sigma, w(\chi)} \tau_{w} \lambda(\tilde{w}).
\]

By the irreducibility of \( \sigma \) there exists a \( z \in \mathbb{C}^{\times} \) such that

\[
(4.18) \quad \rho_{\sigma, w} \phi_{\tilde{w}, \sigma, w}(\chi) = z \phi_{w, \sigma, w}(\chi) \rho_{\sigma, w} : \tilde{w} \cdot \sigma \to \sigma \otimes w(\chi).
\]

With that we compute

\[
(4.19) \quad \rho_{w} \circ \phi_{\sigma, w}(\chi) = I_{E_{B}}^{G}(\rho_{\sigma, w}(\chi) \otimes \rho_{\sigma, w(\chi)}^{-1})
\]

\[
= I_{E_{B}}^{G}(z \phi_{w, \sigma, w}(\chi) \rho_{\sigma, w(\chi)}^{-1}) = z I_{E_{B}}^{G}(\phi_{w, \sigma, w}(\chi) \otimes \rho_{w(\chi)}^{-1}) I_{E_{B}}^{G}(\rho_{\sigma, w(\chi)} \otimes \rho_{w(\chi)}^{-1}) = z \phi_{w(\chi)} \rho_{w}.
\]

From (4.15) – (4.19) we deduce that

\[
\rho_{w} \tau_{w} \lambda(\tilde{w}) J_{w^{-1}(P)|P} \circ \phi_{\chi} = z \phi_{w, \sigma, w}(\chi) \tau_{w} \lambda(\tilde{w}) J_{w^{-1}(P)|P} \in \text{Hom}_{G}(I_{E_{B}}^{G}(E_{B}), I_{E_{B}}^{G}(E_{B}))
\]

5. ENDOMORPHISM ALGEBRAS WITH RATIONAL FUNCTIONS

5.1. The operators \( A_{w} \).

Let \( B \) and the \( \phi_{\chi} \) with \( \chi \in \text{Xnr}(M, \sigma) \) from (2.18) act on \( I_{E_{B}}^{G}(E_{B}) \) and \( I_{E_{B}}^{G}(E_{K(B)}) \) by parabolically inducing their actions on \( E_{B} \) and \( E_{K(B)} \). For \( w \in W(M, \mathcal{O}) \) we combine the operators \( J_{K(B), w}, \tau_{w} \) and \( \rho_{w} \) from Section 4 to a \( G \)-homomorphism

\[
A_{w} = \rho_{w} \circ \tau_{w} J_{K(B), w} : I_{E_{B}}^{G}(E_{B}) \to I_{E_{B}}^{G}(E_{K(B)}).
\]

With (4.5) we can also regard \( A_{w} \) as an invertible element of \( \text{End}_{G}(I_{E_{B}}^{G}(E_{K(B)})) \).

According to [He2, Proposition 3.1], \( A_{w} \) does not depend on the choice of the representative \( \tilde{w} \in N_{K}(M) \) of \( w \). Hence \( A_{w} \) is canonical for \( w \in W(\Sigma_{\mathcal{O}, \mu}) \), while for \( w \in R(\mathcal{O}) \) it depends on the choices of \( \chi_{r} \) and \( \rho_{r}^{-1} \in \text{Hom}_{M}(\sigma, \tilde{r} \cdot \sigma \otimes \chi_{r}^{-1}) \). Further
\[\text{Lemma 3.2} \] says that, for every \( \chi \in X_{nr}(M) \) such that \( J_{w^{-1}(P), P}^1(\sigma \otimes \chi) \) is regular:
\[
sp_{(w \chi)\chi_w} A_w(v) = \rho_w \lambda(\tilde{w}) J_{w^{-1}(P), P}^1(\sigma \otimes \chi) \sp\chi(v) \quad v \in \mathcal{I}_{R}^G(E_B).
\]
Consequently, for any \( b \in B = \mathbb{C}[X_{nr}(M)] \):
\[
(5.1) \quad \sp_{(w \chi)\chi_w} A_w(bv) = \rho_w \circ \lambda(\tilde{w}) \circ J_{w^{-1}(P), P}(b(\chi) \sp\chi(v))
\]
\[
= b(\chi) \sp_{(w \chi)\chi_w} A_w(v) = \sp_{(w \chi)\chi_w} ((w \cdot b)_{\chi_w}^{-1} A_w(v)).
\]
In view of Proposition \([\text{Hei2}, \text{Proposition 2.4}]\), this holds for \( \chi \) in a nonempty Zariski-open subset of \( X_{nr}(M) \). Thus
\[
(5.2) \quad A_w \circ b = (w \cdot b)_{\chi_w}^{-1} \circ A_w \quad \in \text{Hom}_G(I_{F}^G(E_B), I_{F}^G(E_{K(B)})).
\]
From (4.15)–(4.19) we see that for all \( w \in W(M, \mathcal{O}), \chi \in X_{nr}(M, \sigma) \) there exists a \( z(w, \chi) \in \mathbb{C}^\times \) such that
\[
(5.3) \quad A_w \circ \phi_{\chi} = z(w, \chi) \phi_{w(\chi)} \circ A_w \quad \in \text{Hom}_G(I_{F}^G(E_B), I_{F}^G(E_{K(B)})).
\]
Compositions of the operators \( A_w \) are not as straightforward as one could expect.

**Proposition 5.1.** Let \( w_1, w_2 \in W(\Sigma_{\mathcal{O}, \mu}) \).
(a) As \( G \)-endomorphisms of \( I_{F}^G(E_{K(B)}) \):
\[
A_{w_1} \circ A_{w_2} = \prod_{\alpha} \left( \sp_{\chi=1} \mu_{Ma}(\sigma \otimes \cdot) \right) \mu_{Ma}(\sigma \otimes w_2^{-1} w_1^{-1} \cdot)^{-1} A_{w_1 w_2}
\]
\[
= A_{w_1 w_2} \prod_{\alpha} \left( \sp_{\chi=1} \mu_{Ma}(\sigma \otimes \cdot) \right) \mu_{Ma}(\sigma \otimes \cdot)^{-1}
\]
where the products run over \( \Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(w_2^{-1}(P)) \cap \Sigma_{\mathcal{O}, \mu}(w_2^{-1} w_1^{-1}(P)) \).
(b) If \( \ell_{\mathcal{O}}(w_1 w_2) = \ell_{\mathcal{O}}(w_1) + \ell_{\mathcal{O}}(w_2) \), then \( A_{w_1 w_2} = A_{w_1} \circ A_{w_2} \).
(c) For \( \alpha \in \Delta_{\mathcal{O}, \mu} \):
\[
A_{s_{\alpha}}^2 = \frac{4c_{s_{\alpha}}}{(1 - q_F^{-a_{s_{\alpha}}})^2 (1 + q_F^{-b_{s_{\alpha}}})^2 \mu_{Ma}(\sigma \otimes \cdot)}.
\]

**Proof.** The second equality in part (a) is an instance of (5.2).

Lemma \([\text{I.3}]\) is equivalent to two formulas established in \([\text{Hei2}, \text{Proposition 2.4}]\) for classical groups. With those at hand, the parts (a) and (b) can be shown in the same way as \([\text{Hei2}, \text{Proposition 3.3} \text{ and Corollaire 3.4}]\). Part (c) is a special case of part (a), made explicit with \([3.7]\).

For \( r \in R(\mathcal{O}) \), Proposition \([\text{I.1}]\) implies that \( J_{r^{-1}(P), P} \) does not have any poles on \( \mathcal{O} \). Hence it maps \( I_{F}^G(E_B) \) to itself, and
\[
(5.4) \quad A_r = \rho_{P, r, \tau_r} \lambda(\tilde{r}) J_{r^{-1}(P), P} \in \text{End}_G(I_{F}^G(E_B)).
\]
The maps \( A_r \) with \( r \in R(\mathcal{O}) \) behave more multiplicatively than in Proposition \([5.1] \) but still they do not form a group homomorphism in general.

**Proposition 5.2.** Let \( r, r_1, r_2 \in R(\mathcal{O}) \) and \( w, w' \in W(\Sigma_{\mathcal{O}, \mu}) \).
(a) Write \( \chi(r_1, r_2) = \chi_{r_1 r_2}(\chi_{r_2}) \chi_{r_1}^{-1} \in X_{nr}(M, \sigma) \) and recall \( \phi_{\chi(r_1, r_2)} \) from (2.7).
There exists a \( \bar{z}(r_1, r_2) \in \mathbb{C}^\times \) such that
\[
A_{r_1} \circ A_{r_2} = \bar{z}(r_1, r_2) \phi_{\chi(r_1, r_2)} \circ A_{r_1 r_2}.
\]
(b) $A_r \circ A_w = A_{rw}$.

(c) There exists a $\xi(w', r) \in \mathbb{C}^\times$ such that

$$A_{w'} \circ A_r = \xi(w', r) \phi_{w'(\chi r)^{-1}} A_{w'r}.$$  

If $w'(\chi_r) = \chi_r$, then $\xi(w', r) = 1$ and $A_{w'} \circ A_r = A_{w'r}$.

Proof. (a) By (4.12)

$$\sigma \otimes \chi_{r_{1}r_{2}} \cong \tilde{r_{1}} \tilde{r_{2}} \cdot \sigma \cong \tilde{r_{1}} \tilde{r_{2}} \cdot (\sigma \otimes \chi_{r_{2}}) \cong \tilde{r_{1}} \cdot (\sigma \otimes \chi_{r_{2}}) \cong \sigma \otimes \chi_{r_{1}}(\chi_{r_{2}}).$$

Hence the unramified characters $\chi_{r_{1}r_{2}}$ and $\chi_{r_{1}}(\chi_{r_{2}})$ differ only by an element $\chi_{e} \in X_{nr}(M, \sigma)$ (as already used in the statement). With (4.13) we compute

$$A_{r_{1}} \circ A_{r_{2}} = \rho_{r_{1}} \tau_{r_{1}} \lambda(\tilde{r_{1}}) \rho_{r_{1}}^{-1}(\tilde{P}) \rho(\sigma \otimes \cdot) \rho_{r_{2}} \tau_{r_{2}} \lambda(\tilde{r_{2}})$$

$$\cdots$$

$$= \rho_{r_{1}} \tau_{r_{2}} \lambda(\tilde{r_{1}}) \rho_{r_{1}}^{-1}(\tilde{P}) \rho(\sigma \otimes \cdot) \rho_{r_{2}} \tau_{r_{2}} \lambda(\tilde{r_{2}})$$

Now we use that $r_{1}, r_{2} \in R(\mathcal{O})$, which by [He22 Proposition 1.9] or [Wal IV.3.4)] implies that the J-operators in the previous line compose in the expected way. Hence

$$A_{r} \circ A_{r} = \rho_{r_{1}} \tau_{r_{1}} \lambda(\tilde{r_{1}}) \rho_{r_{1}}^{-1}(\tilde{P}) \rho(\sigma \otimes \cdot).$$

Comparing (5.5) with the definition of $A_{r_{1}r_{2}}$, we see that it remains to relate

$$\rho_{r_{1}} \tau_{r_{1}} \lambda(\tilde{r_{1}}) \rho_{r_{1}}^{-1}(\tilde{P}) \rho(\sigma \otimes \cdot)$$

to $\rho_{r_{1}r_{2}} \tau_{r_{1}r_{2}} \lambda(\tilde{r_{1}} \tilde{r_{2}})$. Both (5.6) and

$$\phi_{(r_{1}, r_{2})} \rho_{r_{1}r_{2}} \tau_{r_{1}r_{2}} \lambda(\tilde{r_{1}} \tilde{r_{2}})$$

give $G$-homomorphisms

$$J_{r_{1}^{-1}} \lambda_{r_{1}^{-1}}(\tilde{P}) (\sigma \otimes \chi) \to J_{r_{1}^{-1}} \lambda_{r_{1}^{-1}}(\tilde{P}) (\sigma \otimes \chi_{r_{1}}(\chi_{r_{2}})) \otimes \chi_{r_{1}}(\chi_{r_{2}})^{-1} \chi_{r_{1}}(\chi_{r_{2}})^{-1} \chi \lambda$$

that are constant in $\chi \in X_{nr}(M)$. For generic $\chi$ the involved $G$-representations are irreducible, so then

$$\sp_{\chi} \rho_{r_{1}} \tau_{r_{1}} \lambda(\tilde{r_{1}}) \rho_{r_{1}}^{-1}(\tilde{P}) \rho(\sigma \otimes \cdot) \rho_{r_{2}} \tau_{r_{2}} \lambda(\tilde{r_{2}}) = \sp_{\chi} \phi_{(r_{1}, r_{2})} \rho_{r_{1}r_{2}} \tau_{r_{1}r_{2}} \lambda(\tilde{r_{1}} \tilde{r_{2}})$$

for some $\xi(\chi) \in \mathbb{C}^\times$. But then $\xi(\chi)$ does not depend on $\chi$ (for generic $\chi$), so it is a constant $\xi(r_{1}, r_{2})$ and in fact (5.7) already holds without specializing at $\chi$. With (b) Pick any $\chi \in X_{nr}(M, \sigma)$. With Lemma 4.3.a one easily computes

$$\sp_{\chi} A_{r} \circ A_{w} = \sp_{\chi} A_{rw} =$$

$$\rho_{r} \lambda(\tilde{r}) \rho_{r}^{-1}(\tilde{P}) \rho_{w} \lambda(\tilde{w}) J_{w^{-1}}(\tilde{P}) (\sigma \otimes w^{-1} r^{-1}(\chi r^{-1})) \sp_{w^{-1}} r^{-1}(\chi r^{-1}).$$

(c) We relate this to part (b) by setting $w = r^{-1} w' r$. By Lemma 4.3, (5.8) becomes

$$\rho_{r} \rho_{r} \rho_{w} \lambda(\tilde{w}) \lambda(\tilde{r}) J_{w^{-1}}(\tilde{P}) (\sigma \otimes w^{-1} r^{-1}(\chi r^{-1})) \sp_{w^{-1}} r^{-1}(\chi r^{-1}).$$

A similar computation yields

$$\rho_{r} \rho_{r} \rho_{w} \lambda(\tilde{w}) \rho_{w^{-1}}(\tilde{P}) \rho_{w^{-1}}(\tilde{P}) (\sigma \otimes w^{-1} r^{-1}(\chi r^{-1})) \sp_{w^{-1}} r^{-1}(\chi r^{-1}).$$

Thus it remains to compare

$$\rho_{r} \rho_{r} \rho_{w} \lambda(\tilde{w}) \rho_{w^{-1}}(\tilde{P}) \rho(\tilde{P}) (\sigma \otimes w^{-1} r^{-1}(\chi r^{-1})) \sp_{w^{-1}} r^{-1}(\chi r^{-1}).$$
Lemma 3.5 guarantees that \( w'(\chi_r^{-1})\chi_r \in X_{\mathfrak{u}}(M, \sigma) \). Recall the convention \( \text{4.16} \).

By Schur’s lemma there exists a \( \tilde{\zeta}(w', r) \in \mathbb{C}^\times \) such that
\[
(5.12) \quad \rho_{\sigma, r} \rho_{\tilde{\zeta}, w'} = \tilde{\zeta}(w', r) \phi_{\sigma, w'}(\chi_r^{-1})_{\chi_r} \cdot \rho_{\sigma, w'} \rho_{w', r} : w' \tilde{\zeta} \sigma \otimes \chi \chi_r^{-1} \rightarrow \sigma \otimes \chi.
\]

Instead of \( A_{\tilde{\zeta}w^{-1}} \circ A_r \) we consider \( \phi_{w'(\chi_r^{-1})\chi_r} \circ A_{w'} \circ A_r \). Set \( \chi' = \chi w'(\chi_r)^{-1} \), compose \( (5.10) \) on the left with \( \tilde{\zeta}(w', r) \phi_{\sigma, w'}(\chi_r^{-1})_{\chi_r} \) and recall \( (2.20) \). With \( (5.12) \) we find
\[
\text{sp}_{\chi} \zeta(w', r) \phi_{w'(\chi_r^{-1})_{\chi_r}} A_{w'} A_r = \tilde{\zeta}(w', r) \phi_{\sigma, w'}(\chi_r^{-1})_{\chi_r} \cdot \text{sp}_{\chi'} A_{w'} A_r = \rho_{\sigma, r} \rho_{\tilde{\zeta}, w'} \lambda(w') \lambda(\tilde{\zeta}) \text{J}_{w^{-1}r^{-1}(P)\text{I}_{P}(\sigma \otimes w^{-1}r^{-1}(\chi \chi_r^{-1})))sp_{w^{-1}r^{-1}(\chi \chi_r^{-1})}.
\]

The last line equals \( (5.9) \) and \( (5.10) \). This holds for every \( \chi \in X_{\mathfrak{u}}(M) \), so we obtain the desired expression for \( A_{w'} \circ A_r \).

If in addition \( w'(\chi_r) = \chi_r \), then
\[
\rho_{\sigma, r}^{-1} \circ \rho_{\sigma, w'} \circ \rho_{\tilde{\zeta}, w'} = \rho_{\tilde{\zeta}, w'}.
\]

In that case the two sides of \( (5.11) \) are equal (with \( \chi' = \chi \)). \( \square \)

With Bernstein’s geometric lemma we can determine the rank of \( \text{End}_G(I^G_P(E_B)) \) as \( B \)-module:

**Lemma 5.3.** The \( B \)-module \( \text{End}_G(I^G_P(E_B)) \) admits a filtration with successive subquotients isomorphic to \( \text{Hom}_M(w \cdot E_B, E_B) \), where \( w \in W(M, \mathcal{O}) \). This same holds for \( \text{Hom}_G(I^G_P(E_B), I^G_P(E_{K(B)})) \), with subquotients \( \text{Hom}_M(w \cdot E_B, E_{K(B)}) \).

**Proof.** Let \( r^G_P : \text{Rep}(G) \rightarrow \text{Rep}(M) \) be the normalized Jacquet restriction functor associated to \( P = M \). By Frobenius reciprocity
\[
(5.13) \quad \text{Hom}_G(I^G_P(E_B), I^G_P(E_B)) \cong \text{Hom}_M(r^G_P I^G_P(E_B), E_B).
\]

According to Bernstein’s geometric lemma \( \text{[Ren Théorème VI.5.1]} \), \( r^G_P I^G_P(E_B) \) has a filtration whose successive subquotients are
\[
I^M_{(M \cap w^{-1}Mw)(M \cap P)} \circ w \circ r^M_{M \cap w^{-1}Mw^{-1}(M \cap P)} E_B
\]
with \( w \in W(M, A_0) \setminus W(G, A_0)/W(M, A_0) \). That induces a filtration of \( (5.13) \) with subquotients isomorphic to
\[
(5.14) \quad \text{Hom}_M \left( I^M_{(M \cap w^{-1}Mw)(M \cap P)} \circ w \circ r^M_{M \cap w^{-1}Mw^{-1}(M \cap P)} E_B, E_B \right).
\]

By the Bernstein decomposition and the definition of \( W(M, \mathcal{O}) \), \( (5.14) \) is zero unless \( w \in W(M, \mathcal{O}) \). For \( w \in W(M, \mathcal{O}) \), \( (5.14) \) simplifies to \( \text{Hom}_M(w \cdot E_B, E_B) \), which we can analyse further with \( (2.35) \). Thus \( (5.13) \) has a filtration with subquotients
\[
(5.15) \quad \text{Hom}_M(w \cdot E_B, E_B) \cong \bigoplus_{\chi \in X_{\mathfrak{u}}(M, \sigma)} \phi_{\chi} B = \bigoplus_{\chi \in X_{\mathfrak{u}}(M, \sigma)} B \phi_{\chi}
\]
where \( w \) runs through \( W(M, \mathcal{O}) \). The same considerations apply to \( \text{Hom}_G(I^G_P(E_B), I^G_P(E_{K(B)})) \). \( \square \)

Now we can describe the space of \( G \)-homomorphisms that we are after in this subsection:
Theorem 5.4. As vector spaces over $K(B) = \mathbb{C}(X_{nr}(M))$:

$$\text{Hom}_G(I_p^G(E_B), I_p^G(E_{K(B)})) = \bigoplus_{w \in W(M, \mathcal{O})} \bigoplus_{\chi \in X_{nr}(M, \sigma)} K(B)A_w \phi_\chi.$$

Proof. We need Proposition [2.2] and [5.2]. With those, the proof (for classical groups) in [Hei2, Proposition 3.7] applies and shows that the operators $\phi_\chi A_w$ with $w \in W(M, \mathcal{O})$ and $\chi \in X_{nr}(M, \sigma)$ are linearly independent over $K(B)$. Further by (5.15) (with the second $E_B$ replaced by $E_{K(B)}$), the dimension of $\text{Hom}_G(I_p^G(E_B), I_p^G(E_{K(B)}))$ over $K(B)$ is exactly $|X_{nr}(M, \sigma)| |W(M, \mathcal{O})|$. 

Since all elements of $K(B)A_w \phi_\chi$ extend naturally to $G$-endomorphisms of $I_p^G(E_{K(B)})$. Theorem 5.4 shows that $\text{Hom}_G(I_p^G(E_B), I_p^G(E_{K(B)}))$ is a subalgebra of $\text{End}_G(I_p^G(E_{K(B)}))$. The multiplication relations from Proposition 5.2 become more transparent if we work with the group $W(M, \sigma, X_{nr}(M))$ from (3.13). For $\chi_e \in X_{nr}(M, \sigma), r \in R(\mathcal{O})$ and $w \in W(\Sigma_{O, \mu})$ we define

$$A_{\chi_e, rw} = \phi_{\chi_e} A_r A_w \in \text{End}_G(I_p^G(E_{K(B)})).$$

By (2.19) and Propositions 5.1 and 5.2 all the $A_{\chi_e, rw}$ are invertible in $\text{End}_G(I_p^G(E_{K(B)}))$. By (2.21) and (5.2), for $b \in \mathbb{C}(X_{nr}(M))$:

$$A_{\chi_e, rw} b A_{\chi_e, rw}^{-1} = (ru \cdot b)b_{\chi_e}^{-1} = b \circ (\chi_e w)^{-1} \in \mathbb{C}(X_{nr}(M)).$$

This implies that we may change the order of the factors in Theorem 5.4 for any $w \in W(M, \mathcal{O}), \chi_e \in X_{nr}(M, \sigma)$:

$$BA_w \phi_\chi = A_w \phi_\chi B \quad \text{and} \quad K(B)A_w \phi_\chi = A_w \phi_\chi K(B).$$

5.2. The operators $T_w$.

To simplify the multiplication relations between the $A_w$, we will introduce a variation. For any $\alpha \in \Sigma_{O, \mu}$ we write

$$g_\alpha = \frac{(1 - X_\alpha)(1 + X_\alpha)(1 - q_F^{a_{ls}})(1 + q_F^{-b_{ls}})}{2(1 - q_F^{-a_{ls}} X_\alpha)(1 + q_F^{-b_{ls}} X_\alpha)} \in \mathbb{C}(X_{nr}(M)).$$

By (3.6) and Lemma 3.4a

$$w \cdot g_\alpha = g_{w(\alpha)} \quad \alpha \in \Sigma_{O, \mu}, w \in W(M, \mathcal{O}).$$

Our alternative version of $A_{s_\alpha}$ ($\alpha \in \Delta_{O, \mu}$) is

$$T_{s_\alpha} = g_\alpha A_{s_\alpha} = \frac{(1 - X_\alpha)(1 + X_\alpha)(1 - q_F^{a_{ls}})(1 + q_F^{-b_{ls}})}{2(1 - q_F^{-a_{ls}} X_\alpha)(1 + q_F^{-b_{ls}} X_\alpha)} A_{s_\alpha}.$$

By Proposition 5.1 the only poles of $A_{s_\alpha}$ are those of $((\mu M_\alpha)^{-1}$, and by Proposition 4.4 they are simple. A glance at (5.19) then reveals that

$$the \ poles \ of \ T_{s_\alpha} \ are \ at \ \{X_\alpha = q_F^{a_{ls}}\} \ and, \ if \ b_{s_\alpha} > 0, \ at \ \{X_\alpha = -q_F^{-b_{ls}}\}.$$

Proposition 5.5. The map $s_\alpha \mapsto T_{s_\alpha}$ extends to a group homomorphism $w \mapsto T_w$ from $W(\Sigma_{O, \mu})$ to the multiplicative group of $\text{End}_G(I_p^G(E_{K(B)}))$. 

Proof. It suffices to check that the relations in the standard presentation of the Coxeter group $W(\Sigma_{\mathcal{O},\mu})$ are respected. For the quadratic relations, consider any $\alpha \in \Delta_{\mathcal{O},\mu}$. With (5.2) and Proposition 5.1c we compute

$$T_{s_{\alpha}}^2 = g_{\alpha} A_{s_{\alpha}} g_{\alpha} A_{s_{\alpha}} = g_{\alpha} g_{-\alpha} A_{s_{\alpha}}^2 = \frac{(1 - X_{\alpha})(1 + X_{\alpha})(1 - X_{\alpha}^{-1})(1 - X_{\alpha}^{-1}) c_{s_{\alpha}}}{(1 - q_F^{-a_{s_{\alpha}}} X_{\alpha})(1 + q_F^{-a_{s_{\alpha}}} X_{\alpha})(1 - q_F^{-a_{s_{\alpha}}} X_{\alpha}^{-1})(1 + q_F^{-a_{s_{\alpha}}} X_{\alpha}^{-1}) M_{\alpha}} = 1$$

For the braid relations, let $\alpha, \beta \in \Delta_{\mathcal{O},\mu}$ with $s_{\alpha} s_{\beta}$ of order $m_{\alpha\beta} \geq 2$. Then

$$s_{\alpha} s_{\beta} s_{\alpha} \cdots = s_{\beta} s_{\alpha} s_{\beta} \cdots$$  (with $m_{\alpha\beta}$ factors on both sides),

and this is an element of $W(\Sigma_{\mathcal{O},\mu})$ of length $m_{\alpha\beta}$. We know from Proposition 5.1b that

(5.21)  $$A_{s_{\alpha}} A_{s_{\beta}} A_{s_{\alpha}} \cdots = A_{s_{\beta}} A_{s_{\alpha}} A_{s_{\beta}} \cdots$$  (with $m_{\alpha\beta}$ factors on both sides).

Applying (5.18) repeatedly, we find

(5.22)  $$T_{s_{\alpha}} T_{s_{\beta}} T_{s_{\alpha}} \cdots = g_{\alpha}(s_{\alpha} \cdot g_{\beta})(s_{\alpha} s_{\beta} \cdot g_{\alpha}) \cdots A_{s_{\alpha}} A_{s_{\beta}} A_{s_{\alpha}} \cdots = \left( \prod_{\gamma} g_{\gamma} \right) A_{s_{\alpha}} A_{s_{\beta}} A_{s_{\alpha}} \cdots,$$

where the product runs over $\{\alpha, s_{\alpha}(\beta), s_{\alpha} s_{\beta}(\alpha), \ldots\}$. Similarly

(5.23)  $$T_{s_{\beta}} T_{s_{\alpha}} T_{s_{\beta}} \cdots = \left( \prod_{\gamma'} g_{\gamma'} \right) A_{s_{\beta}} A_{s_{\alpha}} A_{s_{\beta}} \cdots,$$

where $\gamma'$ runs through $\{\beta, s_{\beta}(\alpha), s_{\beta} s_{\alpha}(\beta), \ldots\}$.

We claim that $\{\alpha, s_{\alpha}(\beta), s_{\alpha} s_{\beta}(\alpha), \ldots\}$ is precisely the set of positive roots in the root system spanned by $\{\alpha, \beta\}$. To see this, one has to check it for each of the four reduced root systems of rank 2 ($A_1 \times A_1, A_2, B_2, G_2$). In every case, it is an easy calculation.

Of course this applies also to $\{\beta, s_{\beta}(\alpha), s_{\beta} s_{\alpha}(\beta), \ldots\}$. Hence the products in (5.22) and (5.23) run over the same set. In combination with (5.21) that implies

$$T_{s_{\alpha}} T_{s_{\beta}} T_{s_{\alpha}} \cdots = T_{s_{\beta}} T_{s_{\alpha}} T_{s_{\beta}} \cdots,$$

as required. □

Since $T_w$ is the product of $A_w$ with an element of $K(B)$, the relation (5.2) remains valid:

(5.24)  $$T_w \circ b = (w \cdot b) \circ T_w \quad b \in K(B), w \in W(\Sigma_{\mathcal{O},\mu}).$$

The $T_w$ also satisfy analogues of (5.3) and Proposition 5.2c:

Lemma 5.6. Let $w \in W(\Sigma_{\mathcal{O},\mu}), r \in R(\mathcal{O})$ and $\chi_c \in X_m(\mathcal{M}, \sigma)$.

(a)  $$A_r T_w \phi_{\chi_c} = z(r w, \chi_c) \phi_{\tau_{rw}(\chi_c)} A_r T_w,$$

(b)  $$T_w A_r = z(w, r) \phi_{\tau_{rw}^{-1}(\chi_c)} A_r T_{r^{-1}w}.$$

If $w(\chi_r) = \chi_r$, then $z(w, r) = 1$ and $A_r^{-1} T_w A_r = T_{r^{-1}w}$. Proof. (a) In view of (5.3), it suffices to consider $r = 1$ and $w = s_{\alpha}$ with $\alpha \in \Delta_{\mathcal{O},\mu}$. The element $X_{\alpha} \in \mathbb{C}[X_m(\mathcal{M})]$ is $X_m(\mathcal{M}, \sigma)$-invariant, so $g_{\alpha} \in \mathbb{C}(X_m(\mathcal{M}))$ is also $X_m(\mathcal{M}, \sigma)$-invariant. Then (5.3) implies

$$T_{s_{\alpha}} \phi_{\chi_c} = g_{\alpha} A_{s_{\alpha}} \phi_{\chi_c} = g_{\alpha} z(s_{\alpha}, \chi_c) \phi_{s_{\alpha}(\chi_c)} A_{s_{\alpha}} = z(s_{\alpha}, \chi_c) \phi_{s_{\alpha}(\chi_c)} T_{s_{\alpha}}.$$
(b) First we consider the case \( w = s_\alpha \) with \( \Delta_{\alpha_\beta} \). By Proposition 5.2c
\[
\mathcal{T}_{s_\alpha} A_r = g_\alpha A_{s_\alpha} A_r = g_\alpha \hat{z}(s_\alpha, r) \phi_{s_\alpha(\chi^{-1}_r)\chi_r} A_r A_{r^{-1} s_\alpha r}
\]
With (5.18) we obtain
\[
(5.25) \quad \mathcal{T}_{s_\alpha} A_r = \hat{z}(s_\alpha, r) \phi_{s_\alpha(\chi^{-1}_r)\chi_r} A_r g_{r^{-1}(\alpha)} A_{r^{-1} s_\alpha r} = \hat{z}(s_\alpha, r) \phi_{s_\alpha(\chi^{-1}_r)\chi_r} A_r T_{r^{-1} s_\alpha r}.
\]
For a general \( w \in W(\Sigma_{\alpha_\beta}) \) we pick a reduced expression \( w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} \). Then part (a) enables us to apply (5.25) repeatedly. Each time we move one \( \mathcal{T}_{s_{\alpha_i}} \) over \( A_r \), we pick up the same correction factors as we would with \( A \)'s instead of \( \mathcal{T} \)‘s. As the desired formula with just \( A \)'s is known from Proposition 5.2c, this procedure yields the correct formula.

If \( w(\chi_r) = \chi_r \), then the special case of Proposition 5.2c applies. \[ \square \]

Let \( \chi_c \in X_{\text{fr}}(M, \sigma), r, r \in R(O), w \in W(\Sigma_{\alpha_\beta}) \) we write, like we did for \( A_{\chi_c, \text{fr}} \):
\[
(5.26) \quad \mathcal{T}_{\chi_c, \text{fr}} = \phi_{\chi_c} A_r T_w \in \text{End}_G(I_f(E_{K(B)})).
\]
Recall that the \( \phi_{\chi_c} \) can be normalized so that \( \phi_{\chi_c}^{-1} = \phi_{\chi_c^{-1}} \). Similarly, we can normalize the \( A_r \) so that \( A_r^{-1} = A_{r^{-1}} \). Then
\[
\hat{z}(\chi_c, \chi_c^{-1}) = 1 \quad \text{and} \quad \hat{z}(r, r^{-1}) = 1.
\]

**Lemma 5.7.** Let \( \chi_c, \chi'_c \in X_{\text{fr}}(M, \sigma), r, r' \in R(O), w, w' \in W(\Sigma_{\alpha_\beta}) \).
(a) There exists a \( \hat{z}(\chi_c, w, \chi'_c, w') \in \mathbb{C}^\times \) such that
\[
\mathcal{T}_{\chi_c, \text{fr}} \circ \mathcal{T}_{\chi'_c, \text{fr}} = \hat{z}(\chi_c, w, \chi'_c, w') \mathcal{T}_{\chi_c, \text{fr}} \mathcal{T}_{\chi'_c, \text{fr}}.
\]
(b) If in addition \( r = \chi'_c = 1 \) and \( w(\chi_{r'}) = \chi_{r'} \), then \( \hat{z}(\chi_c, w, \chi'_c, w') = 1 \).
(c) The map \( \hat{z} : W(M, \sigma, X_{\text{fr}}(M))^2 \to \mathbb{C}^\times \) is a 2-cocycle.

**Proof.** (a) In the setting of Proposition 5.2 we write \( r_3 = rr' \) and \( w_3 = r^{-1} wr \). That gives the following equalities in \( W(M, \sigma, X_{\text{fr}}(M)) \):
\[
(5.27) \quad w r' = \chi(r, r') r_3 \quad \text{and} \quad w r' = (w(\chi_{r'}^{-1}) \chi_{r'}) r' r_3.
\]
Thus the already established Propositions 5.2 and 5.5 as well as (2.19) and Lemma 5.6 can be regarded as instances of the statement.

We denote equality up to nonzero scalar factors by \( \dot{=} \). With aforementioned available instances we compute
\[
\mathcal{T}_{\chi_c, \text{fr}} \circ \mathcal{T}_{\chi'_c, \text{fr}} = \phi_{\chi_c} A_r T_w \phi_{\chi'_c} A_{r'} T_{w'}
\]
\[
= \phi_{\chi_c} \phi_{r w(\chi'_c)} A_r T_w A_{r'} T_{w'}
\]
\[
= \phi_{\chi_c} \phi_{r w(\chi'_c)} \phi_{w(\chi_{r'}^{-1}) \chi_{r'}} A_r A_{r'} T_{r^{-1} w r', w'}
\]
\[
= \phi_{\chi_c} \phi_{r w(\chi'_c)} \phi_{w(\chi_{r'}^{-1}) \chi_{r'}} \phi_{r(r') A_{r'} T_{r^{-1} w r', w'}}
\]
\[
= \phi_{\chi_c} \phi_{r w(\chi'_c)} \phi_{w(\chi_{r'}^{-1}) \chi_{r'}} \phi_{r(r') A_{r'} T_{r^{-1} w r', w'}}
\]
\[
(5.28) \quad \mathcal{T}_{\chi_c, \text{fr}} \circ \mathcal{T}_{\chi'_c, \text{fr}} = \phi_{\chi_c} A_r T_w \phi_{\chi'_c} A_{r'} T_{w'}
\]
In each of the above steps we preserved the underlying element of \( W(M, \sigma, X_{\text{fr}}(M)) \), so in the notation from (5.27)
\[
\chi_c w \chi'_c r' w' = \chi_c w \chi'_c r' \chi(\chi_{r'}^{-1}) \chi(\chi_{r'}^{-1}) \chi(r, r') r_3 w_3 r_3 w'.
\]
(b) When \( r = \chi'_c = 1 \) and \( w(\chi_{r'}) = \chi_{r'} \), the second, fifth and sixth steps of (5.28) become trivial. Thanks to Propositions 5.1b and 5.2c the third and fourth steps
become equalities, so the entire calculation consists of equalities.

(c) This follows from the associativity of $\text{End}_G(I_P^G(E_{K(B)}))$. \hfill \Box

By (5.16) and (5.24)

$$T_{\chi_{\text{rev}}} b T_{\chi_{\text{rev}}}^{-1} = (r w \cdot b)_{\chi_{\text{rev}}}^{-1} = b \circ (\chi_{\text{rev}})^{-1} \in \mathbb{C}(X_{\text{nr}}(M)).$$

We embed the twisted group algebra $\mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \zeta]$ in $\text{Hom}_G(I_P^G(E_B), I_P^G(E_K(B)))$ with the operators $T_{\chi_{\text{rev}}}$.

Then Theorem 5.4 and Lemma 5.7 show that the multiplication map

$$K(B) \otimes \mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \zeta] \to \text{Hom}_G(I_P^G(E_B), I_P^G(E_K(B)))$$

is bijective. That and (5.29) can be formulated as:

**Corollary 5.8.** The algebra $\text{Hom}_G(I_P^G(E_B), I_P^G(E_K(B)))$ is the crossed product

$$\mathbb{C}(X_{\text{nr}}(M)) \rtimes \mathbb{C}[W(M, \sigma, X_{\text{nr}}(M)), \zeta]$$

with respect to the canonical action of $W(M, \sigma, X_{\text{nr}}(M))$ on $\mathbb{C}(X_{\text{nr}}(M)) = K(B)$.

We end this section with some investigations of the structure of $\text{End}_G(I_P^G(E_B))$.

By the theory of the Bernstein centre [BeDe, Théorème 2.13], its centre is

$$Z(\text{End}_G(I_P^G(E_B))) = \mathbb{C}^{[O]}_{W(M, \sigma)} = \mathbb{C}[X_{\text{nr}}(M)]^{W(M, \sigma, X_{\text{nr}}(M))}.$$

From Lemma 5.3 and (5.15) we know that $\text{End}_G(I_P^G(E_B))$ is a free $B$-module of rank $[W(M, \sigma, X_{\text{nr}}(M))]$. The $\phi_{\chi_{\text{rev}}}$ with $\chi_{\text{rev}} \in X_{\text{nr}}(M, \sigma)$ and the $A_r$ with $r \in R(\sigma)$ belong to $\text{End}_G(I_P^G(E_B))$, but the $A_w$ with $w \in W(\Sigma_{\sigma, \mu}) \setminus \{1\}$ do not, because they have poles. To see whether these poles can be removed in a simple way, we analyse their residues:

**Lemma 5.9.** Let $\alpha \in \Delta_{\sigma, \mu}$ and $\chi \in X_{\text{nr}}(M)$ with $X_\alpha(\chi) = \pm 1$.

(a) If $s_\alpha(\chi_{\text{rev}}) = \chi_{\text{rev}}$, then $\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha} = \pm \text{sp}_{\chi_{\text{rev}}}$. If $\chi_{\text{rev}} \in X_{\text{nr}}(M_\alpha)$, then $\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha} = \text{sp}_{\chi_{\text{rev}}}$. If $\chi_{\text{rev}} \in X_{\text{nr}}(M_\alpha)$, then

$$\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha} = \pm \left(1 + q_F^{-a_{s_\alpha}}(1 - q_F^{-b_{s_\alpha}})(1 + q_F^{-b_{s_\alpha}})\text{sp}_{\chi_{\text{rev}}} \right).$$

**Remark.** With a closer analysis of the operators $J_{s_\alpha(P)}(\sigma \otimes \chi)$, as in [Wal, §IV.1], it should be possible to prove that the signs $\pm$ in this lemma are always $+1$.

**Proof.** (a) By Proposition 4.11 $\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha}$ defines a $G$-isomorphism

$I_P^G(\sigma \otimes \chi_{\text{rev}}) \to I_P^G(\sigma \otimes s_\alpha(\chi_{\text{rev}}))$, parabolically induced from an $M_\alpha$-isomorphism

$$I_{P \cap M_\alpha}^M(\sigma \otimes \chi_{\text{rev}}) \to I_{P \cap M_\alpha}^M(\sigma \otimes s_\alpha(\chi_{\text{rev}})).$$

The same holds for $\phi_{s_\alpha(\chi_{\text{rev}})}$. Further, $I_{P \cap M_\alpha}^M(\sigma \otimes \chi_{\text{rev}})$ is irreducible [Sl2] §4.2.

By Schur’s lemma, these two operators are scalar multiples of each other.

Suppose now that $s_\alpha(\chi_{\text{rev}}) = \chi_{\text{rev}}$. By the above $\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha} \in \mathbb{C} \text{sp}_{\chi_{\text{rev}}}$. From Proposition 5.3 and (5.19) we see that in fact $\text{sp}_{\chi_{\text{rev}}}(1 - X_\alpha)A_{s_\alpha} = \pm \text{sp}_{\chi_{\text{rev}}}$.\hfill \Box


The variety \( X_{\text{nr}}(M_\alpha) \subset X_{\text{nr}}(M) \) is connected and fixed pointwise by \( s_\alpha \). The sign in \( \pm \text{sp}_{\chi_+} \) established above depends algebraically on \( \chi_+ \), so it is constant on \( X_{\text{nr}}(M_\alpha) \). Therefore it suffices to consider \( \chi_+ = 1 \). Let us unravel the definitions:

\[
\text{sp}_{\chi=1}(1-X_\alpha)A_{s_\alpha} = \text{sp}_{\chi=1}(1-X_\alpha)\rho_{s_\alpha}\tau_{s_\alpha}J_{K(B),s_\alpha} \\
(5.31)
\]

By construction \( I_P^G(\rho_{\sigma,s_\alpha}) \) is the inverse of

\[
\text{sp}_{\chi=1}\tau_{s_\alpha}\lambda(s_\alpha)(X_\alpha - 1)J_{s_\alpha(P)}|_P(\sigma \otimes \cdot) = \text{sp}_{\chi=1}\lambda(s_\alpha)(X_\alpha - 1)J_{s_\alpha(P)}|_P(\sigma \otimes \cdot),
\]

see (4.9) and (4.10). We find \( \text{sp}_{\chi}(s_\alpha - 1)A_{s_\alpha} = \text{sp}_{\chi=1} \).

(b) This is analogous to part (a). For the second claim we use that \( \text{sp}_{\chi} \) fixed by \( X \) and \( A \) in \( U \).

We note that by [Opd, Lemma 4.4] we can consider the algebra \( \text{End} \) to find a nice presentation of \( \text{End} \).

We have that \( G \) is not finite dimensional right \( \text{End} \)-module, so we can consider the algebra \( \text{End} \).

The variety \( W(M,\sigma,X_{\text{nr}}(M)) \) does not provide enough control over the poles of \( \text{sp}_{\chi} \). Similar considerations apply to \( T_{s_\alpha} \) in particular the method from [Hei2 §5] does not apply in our generality.

Although we expect that there are \( W(M,\sigma,X_{\text{nr}}(M)) \) elements that generate \( \text{End} \) as \( \text{B}-\text{module} \), we do not have good candidates. That renders it hard to find a nice presentation of \( \text{End} \).

6. Analytic Localization on Subsets on \( X_{\text{nr}}(M) \)

Lemma 5.9 does not provide enough control over the poles of \( A_{s_\alpha} \) to deal with all of them in one stroke. Therefore we approach \( \text{End} \) via localization on suitable subsets of \( X_{\text{nr}}(M) \). Let \( U \) be a \( W(M,\sigma,X_{\text{nr}}(M)) \)-stable subset of \( X_{\text{nr}}(M) \), open with respect to the analytic topology Then \( U \) is a complex submanifold of \( X_{\text{nr}}(M) \), so we can consider the algebra \( C^\text{an}(U) \) of complex analytic functions on \( U \). The natural map \( \mathbb{C}[X_{\text{nr}}(M)] \to C^\text{an}(U) \) is injective because \( U \) is Zariski dense in \( X_{\text{nr}}(M) \). This and (5.20) enable us to construct the algebra

\[
\text{End} \cap \text{End}^G_I(E_B) \cap \text{End}^{\text{an}}_I(E_B) := \text{End} \cap \text{End}^G_I(E_B) \cap \text{End}^{\text{an}}_I(E_B).
\]

Its centre is

\[
Z(\text{End} \cap \text{End}^G_I(E_B) \cap \text{End}^{\text{an}}_I(E_B)) = C^\text{an}(U)^{W(M,\sigma,X_{\text{nr}}(M))}.
\]

We note that by [Opd, Lemma 4.4]

\[
\mathbb{C}[X_{\text{nr}}(M)] \cap \mathbb{C}[X_{\text{nr}}(M)]^{W(M,\sigma,X_{\text{nr}}(M))} \cong C^\text{an}(U).
\]

The subalgebra \( C^\text{an}(U) \) of \( \text{End} \) plays the same role as \( B = \mathbb{C}[X_{\text{nr}}(M)] \) in \( \text{End} \).

Lemma 6.1. There are natural equivalences between the following categories:

(i) finite dimensional right \( \text{End} \)-modules;
(ii) finite dimensional right $\text{End}_G(I_P^G(E_B))$-modules all whose $B$-weights lie in $U$, or equivalently with all weights of the centre in $U/W(M,\sigma,X_{\text{nr}}(M))$;

(iii) finite length representations in $\text{Rep}(G)^d$, whose cuspidal support is contained in $\{\sigma \otimes \chi : \chi \in U\}$.

**Proof.** The equivalence between (i) and (ii) is analogous to [Opd, Proposition 4.3]. The equivalence between (ii) and (iii) follows from (4.1) and the way the $B$-action on $I_P^G(E_B)$ is constructed in (2.5). $\square$

Now we specialize to very specific submanifolds of $X_{\text{nr}}(M)$. Let

$$X_{\text{nr}}(M) = X_{\text{unr}}(M) \times X_{\text{nr}}^+(M) = \text{Hom}(M/M^1, S^1) \times \text{Hom}(M/M^1, \mathbb{R}_{>0})$$

be the polar decomposition of the complex torus $X_{\text{nr}}(M)$. Fix a unitary unramified character $u \in X_{\text{unr}}(M)$.

**Condition 6.2.** Let $U_u$ be a (small) connected open neighborhood of $u$ in $X_{\text{nr}}(M)$, such that

- $U_u$ is stable under $W(M,\sigma,X_{\text{nr}}(M))$ and $X_{\text{nr}}^+(M)$;
- $W(M,\sigma,X_{\text{nr}}(M))u \cap U_u = \{u\}$;
- $\Re(X_u(\chi u^{-1})) > 0$ for all $\alpha \in \Sigma_{\text{red}}(A_M)$, then $\mu^{M_u}(\alpha \otimes u) = 0$.

The first two conditions entail that $W(M,\sigma,X_{\text{nr}}(M))U_u$ is homeomorphic to $W(M,\sigma,X_{\text{nr}}(M))u \times U_u$. The last condition implies that, if $\mu^{M_u}(\alpha \otimes \chi) = 0$ for some $\chi \in U_u$ and $\alpha \in \Sigma_{\text{red}}(A_M)$, then $\mu^{M_u}(\alpha \otimes u) = 0$.

In the remainder of this section we consider

$$U := W(M,\sigma,X_{\text{nr}}(M))U_u,$$

an open neighborhood of $W(M,\sigma,X_{\text{nr}}(M))u X_{\text{nr}}^+(M)$. By Lemma 6.1 the family of algebras $\text{End}_G(I_P^G(E_B))_{U_u}^{an}$, for all possible $u \in X_{\text{unr}}(M)$, suffices to study the entire category of finite length representations in $\text{Rep}(G)^d$.

We want to find a presentation of $\text{End}_G(I_P^G(E_B))_{U_u}^{an}$, as explicit as possible. For $w \in W(M,\sigma,X_{\text{nr}}(M))$ we write $U_{wu} = w(U_u)$. By (6.2) $\text{End}_G(I_P^G(E_B))_{U_u}^{an}$ contains the element $1_{wu} \in C^{an}(U)$ defined by

$$1_{wu}(\chi) = \begin{cases} 
1 & \chi \in U_{wu} \\
0 & \chi \in U \setminus U_{wu} 
\end{cases}.$$ 

The $1_{wu}$ with $wu \in W(M,\sigma,X_{\text{nr}}(M))u$ form a system of mutually orthogonal idempotents in $C^{an}(U)$ and

$$1_U = \sum_{wu \in W(M,\sigma,X_{\text{nr}}(M))u} 1_{wu}.$$ 

This yields a decomposition of $C^{an}(U)$-modules

$$\text{End}_G(I_P^G(E_B))_{U_u}^{an} = \bigoplus_{wu,vu \in W(M,\sigma,X_{\text{nr}}(M))u} 1_{wu}\text{End}_G(I_P^G(E_B))_{U_u}^{an}1_{vu} = \bigoplus_{wu,vu \in W(M,\sigma,X_{\text{nr}}(M))u} C^{an}(U_{wu})\text{End}_G(I_P^G(E_B))_{U_u}^{an}C^{an}(U_{vu}).$$ 

Here the submodules with $wu = vu$ are algebras, while those with $wu \neq vu$ are not.

**Lemma 6.3.** The inclusion $1_u\text{End}_G(I_P^G(E_B))_{U_u}^{an}1_u \to \text{End}_G(I_P^G(E_B))_{U_u}^{an}$ is a Morita equivalence.
Proof. The Morita bimodules are \( \text{End}_G(I_P^G(E_B))^{\text{an}}_U \) and \( 1_u \text{End}_G(I_P^G(E_B))^{\text{an}}_U \). Most of the required properties are automatically fulfilled, it only remains to verify that \( 1_u \) is a full idempotent in \( Y := \text{End}_G(I_P^G(E_B))^{\text{an}}_U \):

\[
Y 1_u Y \text{ should equal } Y
\]

In view of (6.8), it suffices to show that \( 1_w u \in Y 1_u Y \) for all \( w \in W(M, \sigma, X_{\text{nr}}(M)) \).

By (3.14) there exist \( \chi_c \in X_{\text{nr}}(M, \sigma), r \in R(\mathcal{O}) \) and \( w' \in W(\Sigma_{\mathcal{O}, \mu}) \) such that \( w = \chi_c r w' \). Choose an element \( v \in W(\Sigma_{\mathcal{O}, \mu}) \) which satisfies \( vu = w'u \) and whose length is minimal for that property. By (5.24)

\[
\mathcal{T}_v 1_u \mathcal{T}_v^{-1} = 1_{vu} \mathcal{T}_v \mathcal{T}_v^{-1} = 1_{vu}.
\]

We claim that \( \text{sp}_u \mathcal{T}_v \) is regular for all \( \chi \in U_{vu} \), or equivalently

\[
\mathcal{T}_v \text{ does not have poles on } U_{vu}.
\]

We will prove this with induction to \( \ell_{\mathcal{O}}(v) \). The case \( \ell_{\mathcal{O}}(v) = 0 \) is trivial. For the induction step, write \( v = s_a v' \) with \( a \in \Delta_{\mathcal{O}, \mu} \) and \( \ell_{\mathcal{O}}(v') = \ell_{\mathcal{O}}(v) - 1 \). If \( \mathcal{T}_{s_a} \) has a pole on \( U_{vu} \), then \( U_{vu} = X_{\text{nr}}(M) U_{vu} \) and (5.20) entail that

\[
\exists \chi \in U_{vu} : X_{\alpha}(\chi) = 1 \text{ or } b_{s_a} > 0 \text{ and } \exists \chi \in U_{vu} : X_{\alpha}(\chi) = -1.
\]

By the minimality of \( \ell_{\mathcal{O}}(v), v'u \neq vu \). Then

\[
s_a(U_{vu}) \cap U_{vu} = U_{v'U} \cap U_{vu} = \emptyset.
\]

By (Lus, Lemma 3.15) \( X_{\alpha}(\chi) \neq 1 \) for all \( \chi \in U_{vu} \) and, if \( X_{\alpha} \in \mathcal{X}^{\ast}(X_{\text{nr}}(M)) \), also \( X_{\alpha}(\chi) \neq -1 \) for all \( \chi \in U_{vu} \). Comparing with (6.7), we conclude that \( \mathcal{T}_{s_a} \) is regular on \( U_{vu} \). By the induction hypothesis \( \mathcal{T}_{v'} \) is regular on \( U_{v'u} \), so \( \mathcal{T}_v = \mathcal{T}_{s_a} \mathcal{T}_{v'} \) is regular on \( U_{vu} \), affirming (6.6).

From (6.6) and (5.24) we obtain \( \mathcal{T}_v 1_u = 1_{vu} \mathcal{T}_v \in Y \) and \( \mathcal{T}_v^{-1} 1_{vu} = 1_u \mathcal{T}_v \in Y \).

Then (5.29) says

\[
\mathcal{T}_{\chi_c rv}^{-1} u \mathcal{T}_{vr}^{-1} \mathcal{T}_{rv}^{-1} = 1_{\chi_c rvu} = 1_{vu},
\]

where by (5.4) and (2.15):

\[
\mathcal{T}_{\chi_c rv}^{-1} u = \phi_{\chi} A_r \mathcal{T}_v 1_u \in Y \quad \text{and} \quad 1_u \mathcal{T}_v^{-1} v^{-1} \mathcal{T}_{rv}^{-1} = 1_u \mathcal{T}_v^{-1} A_r^{-1} \phi_{\chi}^{-1} \in Y.
\]

This confirms (6.5).

Lemma 6.3 tells us that we have to understand the subalgebra \( 1_u \text{End}_G(I_P^G(E_B))^{\text{an}}_U 1_u \) of \( \text{End}_G(I_P^G(E_B))^{\text{an}}_U \). Let \( C^{\text{me}}(U) \) be the ring of meromorphic functions on \( U \). We proceed via

\[
\text{Hom}_G(I_P^G(E_B), I_P^G(E_K(B))) \subset \text{End}_G(I_P^G(E_B))^{\text{an}}_U \otimes_{B^{W(M, \sigma, X_{\text{nr}}(M))}} C^{\text{me}}(U)^{W(M, \sigma, X_{\text{nr}}(M))} =: \text{End}_G(I_P^G(E_B))^{\text{me}}_U.
\]

For the same reasons as in (6.2), \( C^{\text{me}}(U) \) embeds in \( \text{End}_G(I_P^G(E_B))^{\text{me}}_U \).

For \( \chi_c \in X_{\text{nr}}(M, \sigma) \) and \( w \in W(M, \mathcal{O}) \), (5.16) says that

\[
1_u \phi_{\chi_c} A_w 1_u = \begin{cases} 1_u \phi_{\chi_c} A_w = \phi_{\chi_c} A_w 1_u & \chi_c \mathcal{O}(u) = u \\ 0 & \text{otherwise} \end{cases}
\]
Since $X_{nr}(M, \sigma)$ acts freely on $X_{nr}(M)$, for a given $w \in W(M, \mathcal{O})$ there exists at most one $\chi_c = \chi_c(w) \in X_{nr}(M, \sigma)$ such that $\chi_c w$ fixes $u$. Let $W(M, \mathcal{O})_{\sigma \otimes u}$ be the $W(M, \mathcal{O})$-stabilizer of $\sigma \otimes u \in \text{Irr}(M)$. Then

$$\Omega_u : W(M, \mathcal{O})_{\sigma \otimes u} \to W(M, \sigma, X_{nr}(M))_u$$

is a group isomorphism. With Theorem 5.4, 5.19 and (6.9) this yields

$$1_u \text{End}_G(I_E^G(E_B)) \mu_{me} 1_u = \bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{me}(U)A_{\Omega_u(w)} =$$

$$\bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{me}(U)A_{\Omega_u(w)} = \bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{me}(U)\mathcal{T}_{\Omega_u(w)}.$$

### 6.1. Localized endomorphism algebras with meromorphic functions.

Consider the set of roots

$$\Sigma_{\sigma \otimes u} := \{\alpha \in \Sigma_{\text{red}}(A_M) : \mu^{M_{\alpha}}(\sigma \otimes u) = 0\}.$$

This is a root system [24, §1] and the parabolic subgroup $P = MU$ of $G$ determines a positive system $\Sigma_{\sigma \otimes u}(P)$ and a basis $\Delta_{\sigma \otimes u}$ of $\Sigma_{\sigma \otimes u}$. The relevant $R$-group (the Knapp–Stein $R$-group) is

$$R(\sigma \otimes u) = \{w \in W(M, \mathcal{O})_{\sigma \otimes u} : w(\Sigma_{\sigma \otimes u}(P)) = \Sigma_{\sigma \otimes u}(P)\}$$

Like in (3.2)

$$W(M, \mathcal{O})_{\sigma \otimes u} = W(\Sigma_{\sigma \otimes u}) \rtimes R(\sigma \otimes u).$$

We note that $R(\sigma \otimes u)$ need not be contained in $R(\mathcal{O})$, even though $W(\Sigma_{\sigma \otimes u}) \subset W(\Sigma_{\mathcal{O}, \mu}).$

To obtain generators of $1_u \text{End}_G(I_E^G(E_B)) \mu_{me} 1_u$ with nice and simple relations, we vary on the previous constructions. We follow the setup from Sections 3.5 but now with base point $\sigma \otimes u$ of $\mathcal{O}$, root system $\Sigma_{\sigma \otimes u}$, Weyl group $W(\Sigma_{\sigma \otimes u})$ and $R$-group $R(\sigma \otimes u)$. On $X_{nr}(M)$ we have new functions $X_{\alpha}^u(\chi) := X_{\alpha}(u)^{-1}X_{\alpha}(\chi)$. We recall that by (6.10) $X_{\alpha}(u) \in \{1, -1\}$ for all $\alpha \in \Sigma_{\sigma \otimes u}$.

Further, $E$ is by default endowed with the $M$-representation $\sigma \otimes u$, and we get a slightly different version $J_{P_{\sigma \otimes u}}^u$ of $J_{P_{\sigma \otimes u}}$. Instead of $\rho_{\sigma, w}$ we use

$$\rho_{\sigma \otimes u, w} := \lambda(w)\text{sp}_{\chi = 1} \prod_{\alpha \in \Sigma_{\sigma \otimes u}(P) \cap \Sigma_{\sigma \otimes u}(w^{-1}P)} (X_{\alpha}^u - 1)J_{w^{-1}(P)\sigma \otimes u}^u.$$  

Then Lemmas 4.2 and 4.3 remain true with obvious small modifications. In particular, in Lemma 4.3 we have to replace the product over

$$\alpha \in \Sigma_{\mathcal{O}, \mu}(P) \cap \Sigma_{\mathcal{O}, \mu}(w_2^{-1}(P)) \cap \Sigma_{\mathcal{O}, \mu}(w_1^{-1}(P))$$

by the analogous product (with $\sigma \otimes u$ instead of $\sigma$) over

$$\alpha \in \Sigma_{\sigma \otimes u}(P) \cap \Sigma_{\sigma \otimes u}(w_2^{-1}(P)) \cap \Sigma_{\sigma \otimes u}(w_1^{-1}(P)).$$

For $r \in R(\sigma \otimes u)$ we can now take $\chi_r = 1$ and

$$\rho_{\sigma \otimes u, r} : \tilde{r}(\sigma \otimes u) \to \sigma \otimes u.$$  

With these we define $\rho_{\sigma \otimes u, r, w}$ and

$$A_{rw}^u = \rho_{rw}^u \circ \tau_{rw} \circ J_{K(B), rw}^u \quad rw \in W(M, \mathcal{O})_{\sigma \otimes u}.$$
as before. The superscripts $u$ are meant to distinguish these operators from their ancestors without $u$ (or rather with $u = 1$). Then (6.2) becomes

$$A_{rw}^u \circ b = (rw \cdot b) \circ A_{rw}^u \quad b \in \mathbb{C}(X_{\text{nr}}(M)).$$

Let $w_1, w_2 \in W(\Sigma_{\sigma \otimes u})$ and $r_1, r_2 \in R(\sigma \otimes u)$. By Proposition 5.1

$$A_{w_1}^u \circ A_{w_2}^u = \prod_\alpha \left(\text{sp}_{y=1} \left(\frac{M_{\alpha}(\sigma \otimes u \otimes \cdot)}{(X^\alpha - 1)((X^\alpha)^{-1} - 1)}\right)\mu^M_{\alpha}(\sigma \otimes u \otimes w_1^{-1}w_1^{-1})^{-1}A_{w_1w_2}^u, \right.$$  

where the product runs over

$$\alpha \in \Sigma_{\sigma \otimes u}(P) \cap \Sigma_{\sigma \otimes u}(w_1^{-1}(P)) \cap \Sigma_{\sigma \otimes u}(w_2^{-1}w_1^{-1}(P)).$$

In particular for $\alpha \in \Delta_{\sigma \otimes u}$:

$$\left( A_{s_{n_0}}^u \right)^2 = \frac{4c_{s_{n_0}}^u}{(1 - X_{\alpha}(u)q^{-a_{s_{n_0}}})^2(1 + X_{\alpha}(u)q^{-b_{s_{n_0}}})^2\mu^M_{\alpha}(\sigma \otimes u \otimes \cdot)}.$$

Similarly Proposition 5.2 yields the following multiplication rules:

$$A_{r_1}^u \circ A_{w_1}^u = A_{r_1w_1}^u,$$

$$A_{w_2}^u \circ A_{r_2}^u = A_{w_2r_2}^u,$$

$$A_{r_1}^u \circ A_{r_2}^u = \tau_u(r_1, r_2)A_{r_1r_2}^u.$$  

Here $\tau_u$ is a two-cocycle $R(\sigma \otimes u)^2 \rightarrow \mathbb{C}^\times$. By appropriate normalizations of the $p_{\sigma \otimes u,r}$ we can achieve that

$$\tau_u(1, r) = \tau_u(r, 1) = 1 \quad \text{and} \quad \tau_u(r, r^{-1}) = 1$$

for all $r \in R(\sigma \otimes u)$. In other words, we may assume that $A_1^u = 1$ and $(A_r^u)^{-1} = A_{r^{-1}}^u$.

The arguments for Theorem 5.4 apply only partially in the current situation, because we may have fewer than $|W(M, O)|$ operators $A_{rw}^u$. Rather, [Hei2, Proposition 3.7] shows that in $\text{End}_G(I_P^G(E_B)) \otimes_B K(B)$

$$\{A_{rw}^u : rw \in W(M, O)_{\sigma \otimes u}\} \text{ is } K(B)\text{-linearly independent.}$$

We note that (6.15) and (6.16) mean that, when $X_{\alpha}(u) = -1$, in effect the roles of $a_{s_{n_0}}$ and $b_{s_{n_0}}$ are exchanged. With that in mind we define

$$g_{s_{n_0}}^u = \frac{(1 - X_{\alpha})^2(1 - X_{\alpha}(u)q^{-a_{s_{n_0}}})(1 + X_{\alpha}(u)q^{-b_{s_{n_0}}})}{2(1 - X_{\alpha}(u)q^{-a_{s_{n_0}}}X_{\alpha})(1 + X_{\alpha}(u)q^{-b_{s_{n_0}}}X_{\alpha})} \in \mathbb{C}(X_{\text{nr}}(M)),$$

and $T_{s_{n_0}}^u := g_{s_{n_0}}^u A_{s_{n_0}}^u$. This gives rise to elements $T_w^u$ for $w \in W(\Sigma_{\sigma \otimes u})$, which satisfy the analogues of Proposition 5.5 Lemma 5.6 and Lemma 5.7 — with $W(M, O)_{\sigma \otimes u}$ instead of $W(M, \sigma, X_{\text{nr}}(M))$.

To show that the set (6.19) spans (6.11) as $C^{\text{me}}(U_u)$-module, we vary on the proof of [Hei2 Théorème 3.8].

**Lemma 6.4.** Regard $C^{\text{me}}(U_u)^{\text{Hom}}\left(I_P^G(E_B), I_P^G_{\text{End}}(E_{\text{K}}(B))\right)$ and $C^{\text{me}}(U)$ as subsets of, respectively, $\text{End}_G(I_P^G(E \otimes C^{\text{can}}(U)))$, $\text{Hom}_G(I_P^G(E \otimes C^{\text{me}}(U)))$ and $\text{Hom}_G(I_P^G(E \otimes C^{\text{me}}(U)))$. Then $\text{End}_G(I_P^G(E_B))^{\text{me}}1_u$ equals

$$\text{span}\left(C^{\text{me}}(U_u)^{\text{Hom}}\left(I_P^G(E_B), I_P^G_{\text{End}}(E_{\text{K}}(B))\right)C^{\text{can}}(U_u)\right) = \bigoplus_{w \in W(M, O)_{\sigma \otimes u}} C^{\text{me}}(U_u)A_w^u.$$

Furthermore the elements $A_{r_w}^u T_w^u (rw \in W(M, O)_{\sigma \otimes u})$ span a subalgebra isomorphic to $\mathbb{C}[W(M, O)_{\sigma \otimes u}; \tau_u]$, where $\tau_u$ is the 2-cocycle from Lemma 5.7c. This makes
1_u\text{End}_G(I^G \mathcal{J}(EB))^{\text{me}} 1_u isomorphic to the crossed product of \( C^{\text{me}}(U) \) and \( \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, 1_u] \), with respect to the canonical action of \( W(M, \mathcal{O})_{\sigma \otimes u} \) on \( C^{\text{me}}(U) \).

**Proof.** Recall from Lemma 5.3 and (5.15) that \( \text{Hom}_G(I^G \mathcal{J}(EB), I^G \mathcal{J}(E_K(B))) \) has a filtration with successive subquotients isomorphic to

\[
\text{Hom}_M(w \cdot E_B, E_K(B)) \cong \bigoplus_{\chi \in X_{\text{nr}}(M, \sigma)} \phi_{\chi, K}(B),
\]

where \( w \) runs through \( W(M, \mathcal{O})_{\sigma \otimes u} \). Considering the left hand side as a subset of \( \text{Hom}_M(w \cdot (E \otimes_{\mathcal{C}} C^{\text{an}}(U)), E \otimes_{\mathcal{C}} C^{\text{me}}(U)) \), we can compose it on the left with \( C^{\text{me}}(U) \) and on the right with \( C^{\text{an}}(U) \). Using (2.21) we find

\[
\text{span}(C^{\text{me}}(U) \text{Hom}_M(w \cdot E_B, E_K(B)) C^{\text{an}}(U)) \cong \begin{cases} \bigoplus_{\chi \in X_{\text{nr}}(M, \sigma)} C^{\text{me}}(U) \phi_{\chi, C^{\text{an}}(U)} \quad \text{if } w X_{\text{nr}}(M, \sigma) U_u \cap U_u \neq \emptyset \\ 0 \quad \text{otherwise} \end{cases}.
\]

In view of the construction of \( U_u \), the above condition on \( w \) is equivalent to \( w \in W(M, \mathcal{O})_{\sigma \otimes u} \). Furthermore \( C^{\text{me}}(U_u) \phi_{\chi, C^{\text{an}}(U_u)} = 0 \) for all \( \chi \in X_{\text{nr}}(M, \sigma) \setminus \{1\} \). We conclude that

\[
\text{span}(C^{\text{me}}(U_u) \text{Hom}_G(I^G \mathcal{J}(EB), I^G \mathcal{J}(E_K(B))) C^{\text{an}}(U_u))
\]

has a filtration with subquotients isomorphic to

\[
\text{span}(C^{\text{me}}(U_u) \text{Hom}_M(w \cdot E_B, E_K(B)) C^{\text{an}}(U_u)) \cong C^{\text{me}}(U_u) \quad w \in W(M, \mathcal{O})_{\sigma \otimes u}.
\]

In particular (6.20) has dimension \( |W(M, \mathcal{O})_{\sigma \otimes u}| \) over \( C^{\text{me}}(U_u) \) (notice that \( C^{\text{me}}(U_u) \) is a field because \( U_u \) is connected). By (6.19) the \( A^u_w \) are \( C^{\text{me}}(U_u) \)-linearly independent, so \( \bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{\text{me}}(U_u) A^u_w \) is the whole of (6.20).

The properties involving the \( T_w \) can be shown in the same way as at the end of Section 5. \( \square \)

### 6.2. Localized endomorphism algebras with analytic functions.

We set out to find a \( C^{\text{an}}(U_u) \)-basis of \( 1_u \text{End}_G(I^G \mathcal{J}(EB))^{\text{an}} 1_u \). An element of \( 1_u \text{End}_G(I^G \mathcal{J}(EB))^{\text{an}} 1_u \) lies in \( 1_u \text{End}_G(I^G \mathcal{J}(EB))^{\text{me}} 1_u \) precisely when it does not have any poles on \( U_u \). By (6.15) the poles of \( A^u_{\alpha} \) \( (\alpha \in \Delta_{\sigma \otimes u}) \) are precisely the zeros of \( t_{\alpha}^{\text{me}}(\sigma \otimes u \otimes \cdot) \). In view of Condition 6.2 the only poles on \( U_u \) are those at \( \{X^u_{\alpha} = 1\} = \{X_{\alpha} = X_{\alpha}(u)\} \). The intersection of this set with \( U_u \) is connected and equals

\[
U^u_{\alpha} = uX_{\text{nr}}(M_{\alpha}) \cap U_u.
\]

By Lemma 5.9a, for \( \alpha \in \Delta_{\sigma \otimes u}, \chi \in U^u_{\alpha}; \)

\[
\text{sp}(1 - X^u_{\alpha}) A^u_{\alpha} = \text{sp}(\chi).
\]

For \( \alpha \in \Sigma_{\sigma, u} \) we define

\[
f_{\alpha} = \frac{X^2_{\alpha}(q^{a_{\alpha} + b_{\alpha}}_F - 1) + X_{\alpha}(q^{a_{\alpha} - b_{\alpha}}_F - q^{b_{\alpha}}_F)}{X^2_{\alpha} - 1} = \frac{X_{\alpha}(q^{a_{\alpha} + b_{\alpha}}_F - 1) + X_{\alpha}(u)(q^{a_{\alpha}}_F - q^{b_{\alpha}}_F)}{X^2_{\alpha} - (X^u_{\alpha})^{-1}}.
\]

When \( b_{\alpha} = 0 \) (which happens for most roots), \( f_{\alpha} \) reduces to

\[
\frac{(X_{\alpha} + 1)(q^{a_{\alpha}}_F - 1)}{X_{\alpha} - X_{\alpha}^{-1}} = \frac{q^{a_{\alpha}}_F - 1}{1 - X_{\alpha}^{-1}}.
\]

By (3.6) and Lemma 3.3a

\[
w \cdot f_{\alpha} = f_{w(\alpha)} \quad \alpha \in \Sigma_{\sigma, u}, w \in W(M, \mathcal{O}).
\]
One checks that, for \( \alpha \in \Delta_{\sigma \otimes u}, \chi \in U^{s_{\alpha}}_u, \):

\[
s_{\chi}((1 - X^u_\alpha)f_\alpha) = s_{\chi}\left(\frac{q^{a_{\alpha} + b_{\alpha}}_F - 1 + X_\alpha(u)(q^{a_{\alpha}}_F - q^{b_{\alpha}}_F)}{-1 - (X^u_\alpha)^{-1}}\right)
= -(q^{a_{\alpha}}_F - X_\alpha(u))(q^{b_{\alpha}}_F + X_\alpha(u))/2.
\]

By (6.21) and (6.23), the element

\[
\left(\frac{(q^{a_{\alpha}}_F - X_\alpha(u))(q^{b_{\alpha}}_F + X_\alpha(u))}{2}\right)A_{s_{\alpha}} + f_{\alpha} \in \text{Hom}_G(I_{P}^{G}(E_B), I_{P}^{G}(E_{K(B)}))
\]
does not have any poles on \( U_{u} \). Therefore

\[
T^u_{s_{\alpha}} := 1_u\left(\frac{q^{a_{\alpha}}_F - X_\alpha(u))(q^{b_{\alpha}}_F + X_\alpha(u))}{(X^u_\alpha)^2 - 1}\right)A_{s_{\alpha}} + 1_u f_{\alpha}
\]
belongs to \( 1_u\text{End}_G(I_{P}^{G}(E_B))_{\otimes}^{1}1_u \). We note that

\[
1 + f_{\alpha} = \frac{X^2 q^{a_{\alpha} + b_{\alpha}}_F - 1 + X_\alpha(q^{a_{\alpha}}_F - q^{b_{\alpha}}_F)}{X^2_\alpha - 1} = \frac{(X^u_\alpha q^{a_{\alpha}}_F - X_\alpha(u))(X^u_\alpha q^{b_{\alpha}}_F + X_\alpha(u))}{(X^u_\alpha)^2 - 1},
\]

(6.24)

\[
T^u_{s_{\alpha}} = 1_u(1 + f_{\alpha})T^u_{s_{\alpha}} + 1_u f_{\alpha} = T^u_{s_{\alpha}}(1 + f_{\alpha})1_u + f_{\alpha}1_u \in 1_u\text{End}_G(I_{P}^{G}(E_B))_{\otimes}^{1}1_u.
\]

The quadratic relations for the operators \( T^u_{s_{\alpha}} \) read:

**Lemma 6.5.** \((T^u_{s_{\alpha}} + 1_u)(T^u_{s_{\alpha}} - q^{a_{\alpha} + b_{\alpha}}_F 1_u) = 0 \) for \( \alpha \in \Delta_{\sigma \otimes u}. \)

**Proof.** With (6.24) and the multiplication rules for \( T^u_{s_{\alpha}} \) in (5.24) we compute

\[
(T^u_{s_{\alpha}} + 1_u)(T^u_{s_{\alpha}} - q^{a_{\alpha} + b_{\alpha}}_F 1_u) = 1_u((1 + f_{\alpha})T^u_{s_{\alpha}} + f_{\alpha}1_u + 1_u)((1 + f_{\alpha})T^u_{s_{\alpha}} + f_{\alpha} - q^{a_{\alpha} + b_{\alpha}}_F) = 1_u(1 + f_{\alpha})(1 + f_{\alpha})T^u_{s_{\alpha}} + (1 + f_{\alpha})T^u_{s_{\alpha}}(f_{\alpha} - q^{a_{\alpha} + b_{\alpha}}_F + 1 + f_{\alpha} + 1)(f_{\alpha} - q^{a_{\alpha} + b_{\alpha}}_F) = 1_u((1 + f_{\alpha})T^u_{s_{\alpha}}f_{\alpha} + f_{\alpha} + 1 - q^{a_{\alpha} + b_{\alpha}}_F) + (1 + f_{\alpha})(f_{\alpha} + f_{\alpha} + 1 - q^{a_{\alpha} + b_{\alpha}}_F).
\]

By direct calculation \( f_{\alpha} + f_{\alpha} + 1 - q^{a_{\alpha} + b_{\alpha}}_F = 0, \) so the last line of (6.25) reduces to 0. \( \square \)

With (5.2) we compute, for \( b \in C_{\text{me}}(U_{u}) \):

\[
b T^u_{s_{\alpha}} = \frac{(q^{a_{\alpha}}_F - X_\alpha(u))(q^{b_{\alpha}}_F + X_\alpha(u))}{2}A^{u}_{s_{\alpha}}(s_{\alpha} \cdot b) + f_{\alpha}b
\]

(6.26)

\[
= T^u_{s_{\alpha}}(s_{\alpha} \cdot b) + f_{\alpha}(b - s_{\alpha} \cdot b)
= T^u_{s_{\alpha}}(s_{\alpha} \cdot b) + \left(q^{a_{\alpha} + b_{\alpha}}_F - 1 + X^{-1}_\alpha(q^{a_{\alpha}}_F - q^{b_{\alpha}}_F))(1 - X^{-2}_\alpha)^{-1}(b - s_{\alpha} \cdot b).\right.
\]

Any \( w \in W(\Sigma_{\sigma \otimes u}) \) can be written as a reduced word \( s_{1}s_{2} \cdots s_{I_{O}(w)} \) in the generators \( s_{\alpha} \) with \( \alpha \in \Delta_{\sigma \otimes u}. \) We pick such a reduced expression and we define

\[
T^u_{w} = T^u_{s_{1}}T^u_{s_{2}} \cdots T^u_{s_{I_{O}(w)}} \in 1_u\text{End}_G(I_{P}^{G}(E_B))_{\otimes}^{1}1_u.
\]

**Lemma 6.6.** Let \( w \in W(\Sigma_{\sigma \otimes u}) \) and \( r \in R(\sigma \otimes u). \)

(a) The operator (6.27) does not depend on the choice of the reduced expression \( w = s_{1}s_{2} \cdots s_{I_{O}(w)}. \)
(b) \((A^u_r)^{-1}T^u_w A^u_r = T^u_{r^{-1}wr}\).

Proof. (a) In view of the defining relations in the Coxeter group \(W(\Sigma_{\sigma\otimes u})\), it suffices to show the following statement. Let \(\alpha, \beta \in \Delta_{\sigma\otimes u}\) with \(s_\alpha s_\beta\) of order \(m_{\alpha\beta} > 1\). Then

\[
T^u_s a T^u_s b \cdots T^u_s \alpha / \beta = T^u_s b T^u_s a T^u_s b \cdots T^u_s (m_{\alpha\beta} \text{ factors on both sides})
\]

Consider the affine Hecke algebra \(\mathcal{H}\) with root system \(\{h^\vee_\alpha : \alpha \in \Sigma_{\sigma\otimes u}\}\), torus \(X_{nr}(M)\) and parameters

\[
\lambda(h^\vee_\alpha) = a_{s_\alpha} + b_{s_\alpha}, \quad \lambda^*(h^\vee_\alpha) = a_{s_\alpha} - b_{s_\alpha}.
\]

By definition \(\mathcal{H}\) is generated by a subalgebra \(\mathbb{C}[X_{nr}(M)]\) and elements \(T_{s_\alpha}\) \((\alpha \in \Delta_{\sigma\otimes u})\) that satisfy:

- the braid relations (6.28) from \(W(\Sigma_{\sigma\otimes u})\);
- \((T_{s_\alpha} + 1)(T_{s_\alpha} - q^a_{s_\alpha} + b_{s_\alpha}) = 0\);
- \(bT_{s_\alpha} = T_{s_\alpha}(s_\alpha \cdot b) + f_\alpha(b - s_\alpha \cdot b)\) \(b \in \mathbb{C}[X_{nr}(M)]\).

Alternatively, \(\mathcal{H} \otimes \mathbb{C}[X_{nr}(M)]^W(\Sigma_{\sigma\otimes u}) \mathbb{C}(X_{nr}(M))^W(\Sigma_{\sigma\otimes u})\) can be generated by \(\mathbb{C}(X_{nr}(M))\) and the elements

\[
\tau_{s_\alpha} := (T_{s_\alpha} + 1)(1 + f_\alpha)^{-1} - 1.
\]

These elements stem from [Lus] [5.1], where \(1 + f_\alpha\) is denoted \(G(\alpha)\). By [Lus] Proposition 5.2 they satisfy the same relations as our \(T_{s_\alpha}\), namely Proposition 5.5 and (5.24). That gives a new presentation of \(\mathcal{H} \otimes \mathbb{C}[X_{nr}(M)]^W(\Sigma_{\sigma\otimes u}) \mathbb{C}(X_{nr}(M))^W(\Sigma_{\sigma\otimes u})\), with defining relations

- the braid relations from \(W(\Sigma_{\sigma\otimes u})\) (but now for the \(\tau_{s_\alpha}\));
- \(s_\alpha^2 = 1\);
- \(b\tau_{s_\alpha} = \tau_{s_\alpha}(s_\alpha \cdot b)\) \(b \in \mathbb{C}(X_{nr}(M))\).

We map \(U_u\) to \(X_{nr}(M)\) by considering \(u \in U_u\) and \(1 \in X_{nr}(M)\) as basepoints, so \(\chi \mapsto u^{-1}\chi\). That gives an injection

\[
\mathbb{C}(X_{nr}(M)) \to \mathcal{C}^{me}(U_u).
\]

We checked that the \(\tau_{s_\alpha}\) and the \(T_{s_\alpha}\) satisfy the same relations. Hence there is a unique algebra homomorphism

\[
\mathcal{H} \otimes \mathbb{C}[X_{nr}(M)]^W(\Sigma_{\sigma\otimes u}) \mathbb{C}(X_{nr}(M))^W(\Sigma_{\sigma\otimes u}) \to 1_u \text{End}_G(I^\sigma_H/E_B)^{me} 1_u
\]

that extends (6.30) and sends \(\tau_w\) to \(T_{w}\) for \(w \in W(\Sigma_{\sigma\otimes u})\). From (6.24) and (6.29) we see that \(T_{s_\alpha}\) is mapped \(T_{s_\alpha}\) for \(\alpha \in \Delta_{\sigma\otimes u}\). As the \(T_{s_\alpha}\) satisfy the braid relations (6.28), so do the \(T_{s_\alpha}^u\).

(b) From the definition of \(T_{s_\alpha}^u\) and Proposition 5.2c we obtain

\[
T_{s_\alpha}^u A^u_r = 1_u \left(q^a_{s_\alpha} - X_{\alpha}(u))q^b_{s_\alpha} + X_{\alpha}(u)\right) A^u_r A^u_{r^{-1}a} + A^u_{r^{-1}a} f_{r^{-1}\alpha} = A^u_{r^{-1}a} T_{s_\alpha}^u.
\]

Recall from (6.18) that \((A^u_r)^{-1} = A^u_{r^{-1}}\). Applying that and (6.31) repeatedly, we find

\[
(A^u_r)^{-1}T^u_w A^u_r = (A^u_r)^{-1}T^u_{s_1} T^u_{s_2} \cdots T^u_{s_{\sigma\otimes u}(w)} A^u_r = T^u_{r^{-1}s_1} T^u_{r^{-1}s_2} \cdots T^u_{r^{-1}s_{\sigma\otimes u}(w)} r.
\]

Since conjugation by \(r \in R(\sigma \otimes u)\) preserves the lengths of elements of \(W(\Sigma_{\sigma\otimes u})\),

\[
r^{-1}wr = (r^{-1}s_1 r)(r^{-1}s_2 r) \cdots (r^{-1}s_{\sigma\otimes u}(w)) r
\]
is a reduced expression. Now part (a) guarantees that the right hand side of

\[ T_{r^{-1}w} \]

equals \( T_{r^{-1}w} \).

The arguments from [Hei2, §5] apply to the operators \( A^u_r \) and \( T^u_w \) in \( \text{Hom}_G(\mathcal{I}^G(E_B), \mathcal{I}^G(E_{K(B)})) \), provided that we only look at \( U_u \subset X_u(M) \). In particular [Hei2, Proposition 5.9] proves that, for any \( \chi \in U_u \):

\[
\{ \text{sp}_{\chi}A^u_r T^u_w : rw \in W(M, \mathcal{O})_{\sigma \otimes u} \}
\]

is \( \mathbb{C} \)-linearly independent in \( \text{Hom}_G(\mathcal{I}^G(E_B), \mathcal{I}^G(E, \sigma \otimes \chi)) \).

**Theorem 6.7.** The algebra

\[
1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u = \text{span}(C^{an}(U_u) \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B)) C^{an}(U_u))
\]

can be expressed as

\[
\bigoplus_{r \in R(\sigma \otimes u)} \bigoplus_{w \in W(\Sigma_{\sigma \otimes u})} C^{an}(U_u) A^u_r T^u_w = \bigoplus_{w \in W(\Sigma_{\sigma \otimes u})} \bigoplus_{r \in R(\sigma \otimes u)} T^u_w A^u_r C^{an}(U_u).
\]

**Proof.** By (6.14), for all \( rw \in W(M, \mathcal{O})_{\sigma \otimes u} \),

\[
1_u A^u_r T^u_w = 1_u A^u_r T^u_w 1_u = A^u_r T^u_w 1_u
\]

and it is a nonzero element of \( 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u \). Lemma 6.5 tells us that

\[
\bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{an}(U_u) A^u_r T^u_w \subset 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u \subset \bigoplus_{w \in W(M, \mathcal{O})_{\sigma \otimes u}} C^{an}(U_u) A^u_r T^u_w.
\]

With (6.33) that entails, for any \( \chi \in U_u \), the \( \mathbb{C} \)-vector space

\[
\{ \text{sp}_{\chi}(A) : A \in 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u \}
\]

has a basis \( \{ \text{sp}_{\chi}A^u_r T^u_w : rw \in W(M, \mathcal{O})_{\sigma \otimes u} \} \). Suppose that \( f_{rw} \in C^{an}(U_u) \) and

\[
A := \sum_{rw \in W(M, \mathcal{O})_{\sigma \otimes u}} f_{rw} A^u_r T^u_w \in 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u.
\]

It remains to show that all the \( f_{rw} \) belong to \( C^{an}(U_u) \). Consider any \( \chi \in U_u \). By the above there are unique \( z_{rw} \in \mathbb{C} \) such that

\[
\text{sp}_{\chi}(A) = \sum_{rw \in W(M, \mathcal{O})_{\sigma \otimes u}} z_{rw} \text{sp}_{\chi}A^u_r T^u_w.
\]

Then \( f_{rw}(\chi) = z_{rw} \). Hence none of the \( f_{rw} \) has a pole at any \( \chi \in U_u \). In other words, they are analytic.

Theorem 6.7 and the multiplication rules (6.14), (6.17), Lemma 6.5, (6.26), (6.27), Lemma 6.6 provide a presentation of \( 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u \). We note that it is quite similar to an affine Hecke algebra (when \( R(\sigma \otimes u) = 1 \)) or to a twisted affine Hecke algebra [AMS3, Proposition 2.2]. The only difference is that the complex torus \( T \) in the definition of an affine Hecke algebra has been replaced by the complex manifold \( U_u \), and \( \mathbb{C}[T] \) by \( C^{an}(U_u) \).

This observation enables us to compute the centre of \( 1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u \) with the methods from [Lus, Proposition 3.11] and [Sol3, §1.2]:

\[
Z(1_u \text{End}_G(\mathcal{I}^G_{\mathcal{P}}(E_B))_{U}^{an} 1_u) = C^{an}(U_u) W(M, \mathcal{O})_{\sigma \otimes u}.
\]
7. Link with graded Hecke algebras

We will provide an easier presentation of $1_u \text{End}_G(I_P^G(E_B))_U 1_u$, which comes from a graded Hecke algebra $[\text{Lus}$. Let us recall the construction of the graded Hecke algebras that we want to use, starting from the affine Hecke algebra $\mathcal{H}$ in the proof of Lemma 6.6. We replace the complex torus $X_{nr}(M) = \text{Hom}(M/M^1, \mathbb{C}^*)$ by its Lie algebra

$$\text{Lie}(X_{nr}(M)) = \text{Hom}(M/M^1, \mathbb{C}) = a_M^* \otimes \mathbb{R} \mathbb{C},$$

The algebra of regular functions $\mathbb{C}[X_{nr}(M)] = \mathbb{C}[M/M^1]$ is replaced by the algebra of polynomial functions

$$\mathbb{C}[\text{Lie}(X_{nr}(M))] = \mathbb{C}[a_M^* \otimes \mathbb{R} \mathbb{C}] = S(a_M \otimes \mathbb{R} \mathbb{C}),$$

where $S$ denotes the symmetric algebra of a vector space. The group $W(M, \mathcal{O})$ acts naturally on $\text{Lie}(X_{nr}(M))$ and on $\mathbb{C}[a_M^* \otimes \mathbb{R} \mathbb{C}]$.

Recall that in Proposition 3.1 we associated to every $\alpha \in \Sigma_{\mathcal{O}, u}$ elements $h^\vee_\alpha \in M/M^1 \subset \sigma^* a_M$, $\alpha^\sharp \in a_M^*$. These elements form root systems $\Sigma^\vee$ and $\Sigma_{\mathcal{O}}$, respectively. The quadruple

$$(a_M, \{h^\vee_\alpha, \alpha \in \Sigma_{\mathcal{O}, u}, a_M^*, \{\alpha^\sharp : \alpha \in \Sigma_{\mathcal{O}, u}, \{h^\vee_\alpha, \alpha \in \Delta_{\mathcal{O}, u}\})$$

will be denoted $\mathcal{R}_u$, and is sometimes called a degenerate root datum. Let $k^u : \Sigma_{\mathcal{O}, u} \to \mathbb{R}_{\geq 0}$ be the $W(M, \mathcal{O})_{\sigma \otimes u}$-invariant parameter function

$$k^u_\alpha = \begin{cases} \log(q_F) a_{\alpha \sigma} & \text{if } X_\alpha(u) = 1 \\ \log(q_F) b_{\alpha \sigma} & \text{if } X_\alpha(u) = -1 \end{cases} .$$

We also need the 2-cocycle

$$\Omega_u : \log(q_F) \to (W(M, \mathcal{O})_{\sigma \otimes u}/W(\Sigma_{\mathcal{O}, u}))^2 \to \mathbb{C}^\times$$

from Lemma 6.4. It gives rise to a twisted group algebra $\mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \Omega_u]$ with basis $\{N_w : w \in W(M, \mathcal{O})_{\sigma \otimes u}\}$ and multiplication rules

$$N_{w_1} N_{w_2} = \Omega_u(w_1, w_2) N_{w_1 w_2} .$$

**Lemma 7.1.** Recall the bijection $\Omega_u : W(M, \mathcal{O})_{\sigma \otimes u} \to W(M, \sigma, X_{nr}(M))_u$ from (6.10). The 2-cocycles $\Omega_u$ and $\Omega_u \circ \Omega_u$ of $W(M, \mathcal{O})_{\sigma \otimes u}$ are cohomologous.

**Proof.** By definition

$$A^u_\sigma T^u_w \circ A^u_\sigma T^u_w = \Omega_u(r', r) A^u_\sigma T^u_{r-w'}$$

for $r, r' \in W(M, \mathcal{O})_{\sigma \otimes u}$. For a generic $\chi \in X_{nr}(M)$, $I_P^G(\sigma \otimes \chi)$ and $I_P^G(\sigma \otimes \Omega_u(rw)^{-1} \chi)$ are irreducible. Both $sp_\chi A^u_\sigma T^u_w$ and $sp_\chi T^u_{\Omega_u(rw)}$ are $G$-homomorphisms

$$I_P^G(\sigma \otimes \chi) \to I_P^G(\sigma \otimes \Omega_u(rw)^{-1} \chi),$$

so they differ only by a scalar factor (at least away from their poles). Furthermore $sp_\chi A^u_\sigma T^u_w$ and $sp_\chi T^u_{\Omega_u(rw)}$ are rational functions of $\chi \in X_{nr}(M)$, so there exists a $f^u_{rw} \in \mathbb{C}[X_{nr}(M)]$ such that

$$A^u_\sigma T^u_w = f^u_{rw} T^u_{\Omega_u(rw)} .$$
Comparing the expressions with $C$-cocycles. □

On the other hand, by Lemma 5.6

This algebra is denoted $G$. An advantage of this algebra over $\text{End}(\Omega_u)$ is that it is a module for the Hecke algebra $\mathcal{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)$. The following categories are naturally equivalent:

- finite dimensional right $\mathcal{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)$-modules;
- finite dimensional right $\mathcal{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)$-modules, all whose $\mathbb{C}[a_M \otimes \mathbb{R} \mathbb{C}]$-weights lie in $V$.

**Lemma 7.2.** The following categories are naturally equivalent:

- finite dimensional right $\mathcal{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)$-modules;
- finite dimensional right $\mathcal{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)$-modules, all whose $\mathbb{C}[a_M \otimes \mathbb{R} \mathbb{C}]$-weights lie in $V$.

**Proof.** This can be shown in the same way as Lemma 5.3 and [Opd, Proposition 4.3]. □
We can also involve meromorphic functions on $V$, in an algebra

\[(7.5) \quad \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u)_V^\text{me} := \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u) \otimes_{C[a_M^* \otimes \mathbb{C}]^W(M, \mathcal{O})_{\sigma \otimes u}} C^\text{me}(V)^W(M, \mathcal{O})_{\sigma \otimes u}, \]

which as vector space equals

\[C^\text{me}(V) \otimes _\mathbb{C} \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \tilde{z}_u].\]

As in [Lus §5.1] we define, for $\alpha \in \Delta_{\sigma \otimes u}$, the following element of (7.5):

\[(7.6) \quad \tilde{T}_{s_\alpha} = -1 + (N_{s_\alpha} + 1) \frac{h_\alpha^\text{v}}{k_\alpha^u + h_\alpha^u}.\]

According to [Lus Proposition 5.2], $s_\alpha \mapsto \tilde{T}_{s_\alpha}$ extends uniquely to a group homomorphism $w \mapsto \tilde{T}_w$ from $W(\Sigma_{\sigma \otimes u})$ to the multiplicative group of (7.5), and

\[\tilde{T}_w f = (w \cdot f) \tilde{T}_w \quad f \in C^\text{me}(V), w \in W(\Sigma_{\sigma \otimes u}).\]

An argument analogous to Lemma 6.6b shows that

\[N_r \tilde{T}_w N_r^{-1} = \tilde{T}_{r w^{-1}} \quad r \in R(\sigma \otimes u), w \in W(\Sigma_{\sigma \otimes u}).\]

It is easy to see from (7.6) that the $\mathbb{C}(a_M^* \otimes \mathbb{C})$-span of the $\tilde{T}_w$ coincides with the $\mathbb{C}(a_M^* \otimes \mathbb{C})$-span of the $N_w$ ($w \in W(\Sigma_{\sigma \otimes u})$). With (7.4) that yields

\[(7.7) \quad \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u)_V^\text{me} = \text{span}(C^\text{me}(V) \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u)) = \bigoplus_{r w \in W(M, \mathcal{O})_{\sigma \otimes u}} N_r \tilde{T}_w C^\text{me}(V).\]

In view of the above multiplication relations, (7.7) means that the algebra (7.5) is a crossed product of $C^\text{me}(V)$ and $\mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \tilde{z}_u]$, where the latter factor is spanned by the $N_r \tilde{T}_w$.

Now we specialize to a particular $V$. The analytic map

\[
\exp_u : a_M^* \otimes \mathbb{C} = \text{Hom}(M/M^1, \mathbb{C}) \twoheadrightarrow \text{Hom}(M/M^1, \mathbb{C}^\times) = X^+_{\text{me}}(M) \quad \lambda \mapsto u \exp(\lambda)
\]

is a $W(M, \mathcal{O})_{\sigma \otimes u}$-equivariant covering. Notice that

\[\exp_u(a_M^*) = uX^+_{\text{me}}(M).\]

Let $V$ be the connected component of $\exp_u^{-1}(U_u)$ that contains 0. By Condition 6.2, $\exp_u : V \rightarrow U_u$ is an isomorphism of analytic varieties. In particular $f \mapsto f \circ \exp_u$ provides $W(M, \mathcal{O})_{\sigma \otimes u}$-equivariant algebra isomorphisms

\[C^\text{an}(U_u) \rightarrow C^\text{an}(V) \quad \text{and} \quad C^\text{me}(U_u) \rightarrow C^\text{me}(V).\]

From Lemma 6.4, (7.7) and the multiplication relations in these algebras, we see that $\exp_u$ induces an algebra isomorphism

\[\Phi_u : 1_u \text{End}_G(I_F^G(EB))_{U_u}^\text{me} 1_u \rightarrow \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u)_V^\text{me}.\]

**Proposition 7.3.** $\Phi_u$ restricts to an algebra isomorphism

\[1_u \text{End}_G(I_F^G(EB))_{U_u}^\text{an} 1_u \rightarrow \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{z}_u)_V^\text{an}.\]
Proof. By construction $\Phi_u(C^a(\mathbb{U}_u)) = C^a(V)$ and $\Phi_u(1_u A^u_r) = N_r$, where $1_u A^u_r \in 1_u \text{End}_G(I^u_F(E_B))_{\mathbb{F}_\ell}^{an}$. Hence, by Theorem 6.7 it suffices to show that

$$\Phi_u(T^u_w) \in \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z^u)^{an}_{\mathbb{V}}$$

and $\Phi^{-1}_u(N_w) \in 1_u \text{End}_G(I^u_F(E_B))_{\mathbb{F}_\ell}^{an}$. Hence, for all $w \in W(\Sigma_{\sigma \otimes u})$. The argument for that is a variation on [Lus, Theorem 9.3] and [Sol3, Theorem 2.1.4]. By (6.27) for the $T^u_w$, it suffices to consider $w = s_\alpha$ with $\alpha \in \Delta_{\sigma \otimes u}$. We compute

$$\Phi_u(1_u + T^u_{s_\alpha}) = \Phi_u((1_u + 1_u T^u_{s_\alpha})(1 + f_\alpha))$$

$$= (1 + T^u_{s_\alpha})(1 + f_\alpha \circ \exp_u)$$

$$= (N_{s_\alpha} + 1) \left( \frac{h^\vee}{k^u_{\alpha} + h^\vee} \right) \left( \frac{X_\alpha q_{a_s} + X_\alpha q_{b_{s_\alpha}} - q_{a_{s_\alpha}} - 1}{X_\alpha^2 - 1} \right) \circ \exp_u$$

$$= (N_{s_\alpha} + 1) \left( \frac{h^\vee}{k^u_{\alpha} + h^\vee} \right) \left( \frac{e^{2h^\vee} q_{a_s} + X_\alpha(u) e^{h^\vee} q_{b_{s_\alpha}} - q_{a_{s_\alpha}} - 1}{e^{2h^\vee} - 1} \right)$$

$$= (N_{s_\alpha} + 1) \left( \frac{h^\vee}{k^u_{\alpha} + h^\vee} \right) \left( \frac{(e^{h^\vee} q_{a_s} - X_\alpha(u))(e^{h^\vee} q_{b_{s_\alpha}} + X_\alpha(u))}{k^u_{\alpha} + h^\vee} \right)$$

By Condition 6.2 for all $v \in V$

$$(7.8) \quad h^\vee_\alpha(v) = \log(X_\alpha(u^{-1} \exp_u(v)))$$

has imaginary part in $(-\pi/2, \pi/2)$. Hence $\frac{h^\vee}{e^{2h^\vee} - 1}$ is an invertible analytic function on $V$. When $X_\alpha(u) = 1$, (7.8) entails that $e^{h^\vee} q_{a_s} + X_\alpha(u)$ is an invertible and analytic on $V$, and by l'Hopital's rule, so is

$$\frac{e^{h^\vee} q_{a_s} - X_\alpha(u)}{k^u_{\alpha} + h^\vee_\alpha} = \frac{e^{h^\vee} q_{a_s} - 1}{\log(qF_{a_s} + h^\vee_\alpha)}.$$ 

Similarly, when $X_\alpha(u) = -1$, $e^{h^\vee} q_{a_s} - X_\alpha(u)$ and

$$\frac{e^{h^\vee} q_{b_{s_\alpha}} + X_\alpha(u)}{k^u_{\alpha} + h^\vee_\alpha} = \frac{e^{h^\vee} q_{b_{s_\alpha}} - 1}{\log(qF_{b_{s_\alpha}} + h^\vee_\alpha)}$$

are invertible analytic functions on $V$. The above computation and these considerations about invertibility allow us to conclude that

$$\Phi_u(1_u + T^u_{s_\alpha}) = 1_u + \Phi_u(T^u_{s_\alpha}) \in \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z^u)^{an}_{\mathbb{V}}.$$

Applying $\Phi^{-1}_u$ to the entire computation and rearranging, we obtain

$$\Phi^{-1}_u(N_{s_\alpha} + 1) = (1_u + 1_u T^u_{s_\alpha}) \left( \frac{h^\vee_\alpha(e^{h^\vee} q_{a_s} - X_\alpha(u))(e^{h^\vee} q_{b_{s_\alpha}} + X_\alpha(u))}{(e^{2h^\vee} - 1)(k^u_{\alpha} + h^\vee_\alpha)} \right) \circ \exp_u^{-1}.$$ 

We just argued that the function between the large brackets is invertible and analytic on $V$. So its composition with $\exp_u^{-1}$ is invertible and analytic on $U_u$. In particular $\Phi^{-1}_u(N_{s_\alpha} + 1) = \Phi^{-1}_u(N_{s_\alpha}) + 1_u$ lies in $1_u \text{End}_G(I^u_F(E_B))_{\mathbb{F}_\ell}^{an}$. ∎
8. Classification of irreducible representations

8.1. Description in terms of graded Hecke algebras.

In this and the next section, when we talk about modules for an algebra, we tacitly mean right modules. Each of these algebras $H$ has a large commutative subalgebra $A$ ( $\mathbb{C}[X^\infty_M]$ or $\mathbb{C}[a_M^* \otimes \mathbb{R} \mathbb{C}]$), such that $H$ has finite rank as $A$-module. For an $H$-module $V$, we denote the set of $A$-weights by $\text{Wt}(V)$.

Every finite dimensional $H$-module $V$ decomposes canonically, as $A$-module, as the direct sum of the subspaces

$$V_\chi := \{ v \in V : (a - a(\chi))^{\dim V}(v) = 0 \}$$

for $\chi \in \text{Wt}(V)$. For this reason it is much easier to work with representations of finite length. We denote the category of finite dimensional right $H$-modules by $H/-\text{Mod}$.

For $\text{Rep}(G)^{\sigma}$, the role of weights is played by the cuspidal support. When $\pi \in \text{Rep}(G)^{\sigma}$ has finite length, we define $\text{Sc}(\pi)$ as the set of $\sigma' \in \mathcal{O}$ which appear in the Jacquet restriction $J^G_B(\pi)$.

In Sections 6.3 and 7 we investigated the following algebra homomorphisms:

$$\text{End}_G(I_P^G(E_B)) \hookrightarrow \text{End}_G(I_P^G(E_B))^\text{an}_U \leftarrow \text{End}_G(I_P^G(E_B))^\text{ad}_U 1_u \cong \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)^\text{an} \leftarrow \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u).$$

**Corollary 8.1.** There are equivalences between the following categories:

(i) modules $V_u \in \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u) - \text{fMod}$ with $W(V_u) \subset a_M^u$;

(ii) modules $V \in \text{End}_G(I_P^G(E_B)) - \text{fMod}$ with $W(V) \subset W(M, \mathcal{O}, X^\infty_M) u X^+_m(M)$;

(iii) finite length representations $\pi \in \text{Rep}(G)^{\sigma}$ with $\text{Sc}(\pi) \subset W(M, \mathcal{O}) \{ \sigma \otimes u \chi : \chi \in X^+_m(M) \}$.

**Proof.** By Lemmas 6.3, 6.4, 7.2 and Proposition 7.3 the homomorphisms (8.1) provide equivalences between the categories of finite dimensional modules of the respective algebras, with the restriction that we only consider modules all whose central weights lie in, respectively $U/W(M, \mathcal{O}, X^\infty_M)$ (twice), $U_u/W(M, \mathcal{O})_{\sigma \otimes u}$ and $V/W(M, \mathcal{O})_{\sigma \otimes u}$ (twice). Restricting from $V$ to $a_M^u$ and from $U_u$ to $uX^+_m(M)$, we obtain the equivalence between (i) and (ii).

The equivalence between (ii) and (iii) can be shown in the same way as in Lemma 6.3.

Sometimes it is more convenient to use left modules instead of right modules. That could have been achieved by considering the $G$-endomorphisms of $I_P^G(E_B)$ as acting from the right. Then we would get the opposite algebra $\text{End}_G(I_P^G(E_B))^{\text{op}}$, and item (ii) of Corollary 8.1 would involve left modules of $\text{End}_G(I_P^G(E_B))^{\text{op}}$. The constructions summarised in (8.1) relate those to left modules of $\mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)^{\text{op}}$.

Fortunately, with the multiplication rules (i)–(iii) before (7.3) it is easy to identify the opposite algebra of a (twisted) graded Hecke algebra. Namely, there is an algebra isomorphism

$$\mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u)^{\text{op}} \rightarrow \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tau_u^{-1})$$

The only subtlety to check in (8.2) is that the 2-cocycles match up – for that one needs (6.18).
Thus the categories in Corollary 8.1 are also equivalent with the category of finite dimensional left $H$-modules, all whose $\mathbb{C}[a_M^* \otimes \mathbb{R}]$-weights lie in $a_M^*$. In other words, for the algebras that we consider it does not make too much difference whether we use left or right modules.

For $\chi_c \in X_{\text{irr}}(M, \sigma)$, the $M$-representations $\sigma \otimes u$ and $\sigma \otimes u\chi_c$ are equivalent, and sometimes it is hard to distinguish them. Fortunately, the equivalences of categories from Corollary 8.1 are essentially the same for $\sigma \otimes u$ and $\sigma \otimes u\chi_c$. To make that precise, we assume that in (6.13) the choices are made so that

$$\rho_{\sigma, \chi_c} = \phi_{\sigma, \chi_c} \rho_{\sigma \otimes u} \phi_{\sigma, \chi_c}^{-1}.$$  

(8.3)

Lemma 8.2. Let $\chi_c \in X_{\text{irr}}(M)$.

(a) The algebras $\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{\tau}_u)$ and $\mathbb{H}(\tilde{\mathcal{R}}_{\chi_c u}, W(M, \mathcal{O})_{\sigma \otimes \chi_c u}, k^{\chi_c u}, \tilde{\tau}_{\chi_c u})$ are equal.

(b) Let $V \in \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{\tau}_u)_{\text{fMod}}$ with $W(V) \subset a_M^*$. Then its image in $\text{End}_G(I_P^G(E_B))_{\text{fMod}}$ via Corollary 8.1 coincides with the image of $V$ as $\mathbb{H}(\tilde{\mathcal{R}}_{\chi_c u}, W(M, \mathcal{O})_{\sigma \otimes \chi_c u}, k^{\chi_c u}, \tilde{\tau}_{\chi_c u})_{\text{module}}$ in $\text{End}_G(I_P^G(E_B))_{\text{fMod}}$, obtained from Corollary 8.1 for $\chi_c u$.

Proof. (a) The $M$-representations $\sigma \otimes u$ and $\sigma \otimes u\chi_c$ are the same in $\text{Irr}(M)$ and $W(M, \mathcal{O})$ acts on that set, so $W(M, \mathcal{O})_{\sigma \otimes u} = W(M, \mathcal{O})_{\sigma \otimes \chi_c u}$. By the $X_{\text{irr}}(M, \sigma)$-invariance of $\mu^M$, $\tilde{\mathcal{R}}_u = \tilde{\mathcal{R}}_{\chi_c u}$ and $k^u = k^{\chi_c u}$.

Conjugation with $\phi_{\chi_c}$ provides an algebra isomorphism

$$\text{Ad}(\phi_{\chi_c}) : 1_u \text{End}_G(I_P^G(E_B))_{\tilde{U}} \rightarrow 1_{\chi_c u} \text{End}_G(I_P^G(E_B))_{\tilde{U}} 1_{\chi_c u},$$

and similarly with meromorphic functions on $U$. By (8.3) this isomorphism sends $T_w^u (w \in W(M, \mathcal{O})_{\sigma \otimes u})$ to $T_{\chi_c u}^w$, so the 2-cocycles $\tilde{\tau}_u$ and $\tilde{\tau}_{\chi_c u}$ of $W(M, \mathcal{O})_{\sigma \otimes u}$ coincide. Furthermore $\text{Ad}(\phi_{\chi_c})$ sends $f \in C^{\text{an}}(U_u)$ to $f(\chi_c^{-1})$, which equals $(\exp u \exp^{-1})^* f$. Thus Proposition 6.3 gives a commutative diagram

$$\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{\tau}_u) \rightarrow 1_u \text{End}_G(I_P^G(E_B))_{\tilde{U}} 1_u \downarrow \text{Ad}(\phi_{\chi_c}).$$  

(8.4)

(b) Let us retrace what happens in (8.1). First we translate $V$ to a module for $1_u \text{End}_G(I_P^G(E_B))_{\tilde{U}} 1_u$ (on the same vector space). Then we apply the Morita equivalence in Lemma 6.3. That yields a module

$$V' := \text{End}_G(I_P^G(E_B))_{\tilde{U}} \otimes_{1_u \text{End}_G(I_P^G(E_B))_{\tilde{U}}} V = \bigoplus_{u' \in W(M, \sigma, X_{\text{irr}}(M))} V_{u'},$$

where $V_{u'} \in 1_u \text{End}_G(I_P^G(E_B))_{\tilde{U}} 1_{\chi_c u}$-fMod has the same dimension as $V$. This $V'$ is also the $\text{End}_G(I_P^G(E_B))$-module that results from (8.1). From the proof of Lemma 6.3 we see that $V_{u'} = V_{u''}$ for a $u'' \in W(M, \sigma, X_{\text{irr}}(M))$ which satisfies $u'u'' = u$ and whose length is minimal for that property. In particular $V_{u\chi_c} = V_{\chi_c}^*$.

Hence the module of $1_{\chi_c u} \text{End}_G(I_P^G(E_B))_{\tilde{U}} 1_{\chi_c u}$ obtained from $V'$ via Lemma 6.3 is $\text{Ad}(\phi_{\chi_c}^*)$. In view of the commutative diagram (8.4), this procedure recovers $V$ as module of $\mathbb{H}(\tilde{\mathcal{R}}_{\chi_c u}, W(M, \mathcal{O})_{\sigma \otimes \chi_c u}, k^{\chi_c u}, \tilde{\tau}_{\chi_c u})$. Then Corollary 8.1 for $\chi_c u$ implies that $V$ in the latter sense has the same image in $\text{End}_G(I_P^G(E_B))_{\text{fMod}}$ as $V$ in the former sense.

$\square$
A weaker version of Lemma 8.2 holds for all points in $W(M, \sigma, X_{\text{nr}}(M))u$.

Lemma 8.3. Let $w$ be an element of $W(M, \sigma, X_{\text{nr}}(M))$ which is of minimal length in the coset $wW(M, \sigma, X_{\text{nr}}(M))u$.

(a) Conjugation by $T_w$ gives rise to an algebra isomorphism

$$\text{Ad}(T_w) : \mathbb{H}(\tilde{\mathcal{R}}_w, W(M, \mathcal{O})_{\sigma \otimes u}, k^w(u), z_{w(u)}) \rightarrow \mathbb{H}(\tilde{\mathcal{R}}_{w(u)}, W(M, \mathcal{O})_{\sigma \otimes w(u)}, k^{w(u)}, z_{w(u)})$$

with $\text{Ad}(T_w)((CN_v)) = CN_{wvw^{-1}}$ for $v \in W(M, \mathcal{O})_{\sigma \otimes u}$ and $\text{Ad}(T_w)(N_v) = N_{wvw^{-1}}$ for $v \in W(\Sigma_{\sigma \otimes u})$.

(b) Let $V \in \mathbb{H}(\tilde{\mathcal{R}}_w, W(M, \mathcal{O})_{\sigma \otimes u}, k^w(u), z_{w(u)}) - \text{fMod}$ with $Wt(V) \subset a_M^\ast$. Then $V$ and

$$\text{Ad}(T_w^{-1})V \in \mathbb{H}(\tilde{\mathcal{R}}_{w(u)}, W(M, \mathcal{O})_{\sigma \otimes w(u)}, k^{w(u)}, z_{w(u)}) - \text{fMod}$$

have the same image in $\text{End}_G(I_B^G(E_B)) - \text{fMod}$ (via Corollary 8.2 for, respectively, $\sigma \otimes u$ and $\sigma \otimes w(u)$).

Proof. (a) From the proof of Lemma 6.3 we see that $T_w1_u \in \text{End}_G(I_B^G(E_B))_{\mathbb{C}}$ and $T_w1_u T_w^{-1} = 1_{w(u)}$. Therefore conjugation by $T_w$ gives an algebra isomorphism

$$\text{Ad}(T_w) : 1_u \text{End}_G(I_B^G(E_B))_{\mathbb{C}} 1_u \rightarrow 1_{w(u)} \text{End}_G(I_B^G(E_B))_{\mathbb{C}} 1_{w(u)},$$

which on $C^{\text{can}}(U_u)$ is just $f \mapsto f \circ w^{-1}$. The elements $T_wT_w^{-1}$ with $v \in W(\Sigma_{\sigma \otimes u})$ satisfy the same multiplication relations as the elements $T_wT_w^{-1}$ and they have the same specialization at $w(u)$ (namely the identity on $I_B^G(E)$, by Lemma 5.9a). Therefore $\text{Ad}(T_w)(T_w^{-1}) = T_{w(u)}^{-1}$. The same applies to the $A_v^u$ with $v \in R(\sigma \otimes u)$, but for those we can only say that $sp_{w(u)}(T_wA_v^uT_w^{-1}$ and $sp_{w(u)}A_v^uT_w^{-1}$ are equal up to a scalar factor. Hence $\text{Ad}(T_w)(A_v^u)$ is a scalar multiple of $A_{w(u)}^{w(u)}$

Via Proposition 7.3, $\text{Ad}(T_w)$ becomes an algebra isomorphism

$$\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z_u)_{\mathbb{C}} \rightarrow \mathbb{H}(\tilde{\mathcal{R}}_{w(u)}, W(M, \mathcal{O})_{\sigma \otimes w(u)}, k^{w(u)}, z_{w(u)})_{\mathbb{C}}.$$

It restricts to $f \mapsto f \circ w$ on $C^{\text{can}}(V)$ and sends $N_v$ to $N_{wvw^{-1}}$ for $v \in W(\Sigma_{\sigma \otimes u})$, and to a scalar multiple of that for $v \in W(M, \mathcal{O})_{\sigma \otimes u}$. Now it is clear that $\text{Ad}(T_w)$ restricts to the required isomorphism between (twisted) graded Hecke algebras.

(b) This can be shown in the same way as Lemma 8.2b. \(\Box\)

Corollary 8.3 tells us that there is a surjection from the union of the sets

$$\{ \pi \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z_u)) : Wt(\pi) \subset a_M^\ast \}$$

with $u \in X_{\text{nr}}(M)$ to $\text{Irr}(\text{End}_G(I_B^G(E_B)))$. For $u$ and $u'$ in different $W(M, \sigma, X_{\text{nr}}(M))$-orbits, the images in $\text{Irr}(\text{End}_G(I_B^G(E_B)))$ are disjoint. For $u$ and $u'$ in the same $W(M, \sigma, X_{\text{nr}}(M))$-orbit, Lemma 8.3b tells us precisely which modules of

$$\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z_u) \text{ and } \mathbb{H}(\tilde{\mathcal{R}}_{u'}, W(M, \mathcal{O})_{\sigma \otimes u'}, k^{u'}, z_{u'})$$

have the same image – the relation between them comes from an element $w \in W(M, \sigma, X_{\text{nr}}(M))$ with $w(u) = u'$. If we agree that $W(M, \sigma, X_{\text{nr}}(M))$ acts trivially on $\text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z_u))$, it does not matter which $w$ with $w(u) = u'$ we pick. Thus we obtain a bijection

$$\bigcup_{u \in X_{\text{nr}}(M)} \{ \pi \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, z_u)) : Wt(\pi) \subset a_M^\ast \}/W(M, \sigma, X_{\text{nr}}(M))$$

(8.5) $\rightarrow \text{Irr}(\text{End}_G(I_B^G(E_B)))$. 

Here the group action of $W(M, \sigma, X_{nr}(M))$ on the disjoint union comes from the relations described in Lemma 8.3.

8.2. **Comparison by setting the $q$-parameters to 1.**

It is interesting to investigate what happens when in Corollary 8.1 we replace the parameter function $k^u$ by 0. It is known that the analogous operation for affine Hecke algebras gives rise to a bijection on the level of irreducible representations [Sol3]. Replacing all the $k^u$ by 0 corresponds to manually setting all the parameters $q^{a_{ns}}$ and $q^{b_{ns}}$ to 1. In view of Corollary 8.8 that transforms $\text{End}_G(I^G_F(E_B))$ into $\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \bar{z}]$. Therefore we start by analysing the irreducible representations of that simpler crossed product algebra.

**Lemma 8.4.** There is a canonical bijection

$$\bigcup_{u \in X_{nr}(M)} \left\{ \pi \in \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M))_u, \bar{z}]) : \text{Wt}(\pi) \subset uX_{nr}^+(M) \right\} / W(M, \sigma, X_{nr}(M)) \longrightarrow \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \bar{z}]).$$

Here $w \in W(M, \sigma, X_{nr}(M))$ acts on the disjoint union by pullback along the algebra isomorphism $\text{Ad}(N_w^{-1})$:

$$\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M))_w, \bar{z}] \rightarrow \mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M))_u, \bar{z}].$$

**Proof.** Choose a central extension $\Gamma$ of $W(M, \sigma, X_{nr}(M))$ such that $\bar{z}$ becomes trivial in $H^2(\Gamma, \mathbb{C}^\times)$. Then there exists a central idempotent $p_\bar{z} \in \mathbb{C}[\ker(\Gamma \rightarrow W(M, \sigma, X_{nr}(M)))$ such that

$$\mathbb{C}[W(M, \sigma, X_{nr}(M)), \bar{z}] \cong p_\bar{z} \mathbb{C}[\Gamma].$$

The isomorphism sends $\mathbb{C} N_w$ to $\mathbb{C} p_\bar{z} N_\bar{w}$, for any lift $\bar{w} \in \Gamma$ of $w \in W(M, \sigma, X_{nr}(M))$.

Lift the $W(M, \sigma, X_{nr}(M))$-action on $X_{nr}(M)$ to $\Gamma$ and note that (8.6) gives rise to a bijection

$$\text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M))], \bar{z}) \leftrightarrow \{ V \in (\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma]) : p_\bar{z} V \neq 0 \}. $$

By Clifford theory every irreducible representation $\pi$ of $\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma]$ is of the form

$$\text{ind}_{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma]}^{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma]}(\chi \otimes \rho),$$

where $\chi \in X_{nr}(M)$ and $\rho \in \text{Irr}(\Gamma_\chi)$. Moreover the pair $(\chi, \rho)$ is determined by $\pi$, uniquely up to the $\Gamma$-action

$$\gamma(\chi, \rho) = (\gamma(\chi), \text{Ad}(N^{-1}_{\gamma}) \rho).$$

When $u$ is the unitary part of $\chi$, $\Gamma_u \supset \Gamma_\chi$. Again by Clifford theory, every irreducible representation of $\mathbb{C}[X_{nr}(M)] \rtimes \Gamma_u$ is of the form

$$\text{ind}_{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma_u]}^{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma_u]}(\chi \otimes \rho),$$

where $(\chi, \rho)$ is unique up to the action of $\Gamma_u$. Hence there is a canonical bijection

$$\bigcup_{\chi \in uX_{nr}(M)} \text{Irr}(\Gamma_\chi) / \Gamma_u \longrightarrow \{ \pi \in \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \Gamma_u) : \text{Wt}(\pi) \subset uX_{nr}^+(M) \},$$

where $\gamma(\chi, \rho)$ is determined by $\pi$.
Comparing this with Clifford theory for \( \mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma] \), we deduce a canonical bijection
\[
\bigsqcup_{u \in X_{nr}(M)} \{ \pi \in \text{Irr}((\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma_u]) : Wt(\pi) \subset uX_{nr}^+(M)) / \Gamma \rightarrow \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[\Gamma]).
\]

Now we restrict on both sides to the subsets that are not annihilated by \( p^2 \) and we use (8.7).

\[\square\]

It is possible to vary on Lemma [8.3] by taking on the left hand side a \( \alpha \in \mathbb{R} \). To formulate that properly, we recall some results from the representation theory of graded Hecke algebras.

Let \((a_M^* \otimes \mathbb{R}) \subset a_M^* \) be the obtuse negative cone with respect to the root system \( \Sigma_{\sigma \otimes u} \) and the basis \( \Delta_{\sigma \otimes u} \). Let \( k \) be either \( k^u \) or 0. We say that a \( \mathbb{H}(\tilde{R}_u, W(M, \sigma, X_{nr}(M)), \mathbb{R}) \)-module \( V \) is tempered if \( Wt(V) \subset i a_M^* + (a_M^* \otimes u) \). For \( k = 0 \), temperedness is equivalent to \( Wt(V) \subset i a_M^* \).

For \( P \subset \Delta_{\sigma \otimes u} \), we denote the Weyl group generated by the reflections \( s_\alpha \) with \( \alpha \in P \) by \( W_P \). The set \( \mathbb{C}[a_M^* \otimes \mathbb{R}] \mathbb{C}[W_P] \) constitutes a parabolic subalgebra \( \mathbb{H}(P, k) \) of \( \mathbb{H}(\tilde{R}_u, W(\Sigma_{\sigma \otimes u}), k) \). As algebra, it decomposes as a tensor product
\[
\mathbb{C}[\text{span}_C(P)] \mathbb{C}[W_P] \otimes \mathbb{C}[(a_M^* \otimes \mathbb{R}) \mathbb{C}^{\perp P}],
\]
where the subscript \( \perp P \) denotes the subspace orthogonal to the set of coroots \( P^\vee \).

The Langlands classification, proven for graded Hecke algebras in [Eve], expresses irreducible representations in terms of parabolic subalgebras, tempered representations and parabolic induction. We need an extension that includes R-groups like \( R(\sigma \otimes u) \). Such a version was proven for affine Hecke algebras in [Sol3, §2.2]. In view of Lusztig’s second reduction theorem [Lus, §9], generalized in [Sol3, Corollary 2.1.5], that extended Langlands classification also applies to graded Hecke algebras.

**Proposition 8.5.** [Sol3, Corollary 2.2.5]

Let \( \Gamma \) be a finite group acting linearly in \( a_M^* \), stabilizing \( \Sigma_{\sigma \otimes u} \) and \( \Delta_{\sigma \otimes u} \).
(a) Suppose that the following data are given: $P \subset \Delta_{\sigma \otimes u}$, $t \in (a_M^*)^P$ which is strictly positive with respect to $\Delta_{\sigma \otimes u} \setminus P$, a tempered $\tau \in \text{Irr}(\mathbb{H}(P, k))$, an irreducible representation $\rho$ of $\mathbb{C}[\Gamma_{P, \tau, t}, \kappa]$ (where the 2-cocycle $\kappa$ is determined by the action of $\Gamma_{P, \tau, t}$ on $\tau$). Then the $\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)$-representation
\[ \text{ind}_{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}^{\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)}((\tau \otimes t) \otimes \rho) \]
has a unique irreducible quotient. It is called the Langlands quotient and we denote it by an $L$.

(b) For every $\pi \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k))$ there exist data as in part (a), unique up to the action of $\Gamma$, such that
\[ \pi \cong L\left(\text{ind}_{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}^{\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)}((\sigma \otimes t) \otimes \rho)\right). \]

In Proposition 8.5, we can combine $\tau$ and $\rho$ in
\[ (8.11) \quad \tau' := \text{ind}_{\text{H}(P, k) \rtimes \Gamma_{P, \tau, t}}^{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}((\tau \otimes t) \otimes \rho), \]
an irreducible tempered representation such that
\[ \text{ind}_{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}^{\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)}((\tau' \otimes t) \otimes \rho) \cong \text{ind}_{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}^{\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)}((\tau \otimes t) \otimes \rho). \]

Then Proposition 8.5 holds also with the alternative data $P, \tau', t$.

**Theorem 8.6.** There exists a bijection
\[ (8.12) \quad \zeta_u : \{ V \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})\sigma \otimes u, k^u, \xi_u)) : \text{Wt}(V) \subset a_M^* \} \rightarrow \{ V \in \text{Irr}(\mathbb{C}[a_M^* \otimes_{\mathbb{R}} \mathbb{C}] \times \mathbb{C}[W(M, \mathcal{O})\sigma \otimes u, \xi_u]) : \text{Wt}(V) \subset a_M^* \} \]
such that
- $\pi$ is tempered if and only if $\zeta_u(\pi)$ is tempered,
- $\zeta_u$ is compatible with the Langlands classification from Proposition 8.5.

**Proof.** First we get rid of the 2-cocycle $\xi_u$. Choose a central extension
\[ 1 \rightarrow Z(\sigma \otimes u) \rightarrow \Gamma \rightarrow R(\sigma \otimes u) \rightarrow 1 \]
such that $\xi_u$ becomes trivial in $H^2(\Gamma, \mathbb{C}^*)$. Let $p_{\xi_u} \in \mathbb{C}[Z(\sigma \otimes u)]$ be a central idempotent such that
\[ p_{\xi_u} \mathbb{C}[Z(\sigma \otimes u)] \cong \mathbb{C}[R(\sigma \otimes u), \xi_u]. \]

For both $k = k^u$ and $k = u$ that gives a bijection
\[ \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})\sigma \otimes u, k, \xi_u)) \rightarrow \{ V \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)) : p_{\xi_u}V \neq 0 \}. \]

Hence it suffices to construct the required bijection with $\Gamma$ instead of $R(\sigma \otimes u)$, provided that it does not change the $Z(\sigma \otimes u)$-characters of representations.

Consider $\pi \in \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k))$ with $\text{Wt}(\pi) \subset a_M^*$. By Proposition 8.5 with the modified data from (8.11), we have
\[ (8.13) \quad \pi \cong L\left(\text{ind}_{\text{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, \tau, t}]}^{\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u})\Gamma, k)}((\tau' \otimes t))\right), \]
for data $(P, \tau', t)$ that are unique up to the $\Gamma$-action. Since both $\text{Wt}(\pi)$ and $t$ lie in $a_M^*$ and $\text{Wt}(\tau') + t$ consists of weights of $\pi$ [EVE], we must have $\text{Wt}(\tau') \subset a_M^*$. By [SoII, Theorem 6.5.c] the restrictions to $\mathbb{C}[W_P \Gamma_{P, t}]$ of the set
\[ (8.14) \quad \{ V \in \text{Irr}(\mathbb{H}(P, k) \rtimes \mathbb{C}[\Gamma_{P, t}]) : \text{Wt}(V) \subset a_M^* \} \]
form a $\mathbb{Q}$-basis of the representation ring of $W_P \Gamma_{P,t}$. As $Z(\sigma \otimes u) \subset \Gamma_{P,t}$, we can find a bijection $\zeta_{P,t}$ from (8.14) to $\text{Irr}(W_P \Gamma_{P,t})$, such that $\zeta_{P,t}(V)$ occurs in $V|_{W_P \Gamma_{P,t}}$. We regard $\zeta_{P,t}(V)$ as a $\mathbb{C}[a^{*_M}_M \otimes \mathbb{R} \mathbb{C}] \rtimes \mathbb{C}[W_P \Gamma_{P,t}]$-representation on which $\mathbb{C}[a^{*_M}_M \otimes \mathbb{R} \mathbb{C}]$ acts via evaluation at $0 \in a^{*_M}_M$.

Now we define

\[(8.15) \quad \zeta_u(\pi) := L\left(\text{ind}_{\mathbb{H} (\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u}))^0}^{\mathbb{H}(\Gamma_{P,t}) \rtimes \mathbb{C}[W_P \Gamma_{P,t}]}(\zeta_{P,t}(\tau') \otimes t)\right).\]

By Proposition 8.5 this is a well-defined irreducible representation of $\mathbb{H}(\tilde{\mathcal{R}}_u, W(\Sigma_{\sigma \otimes u}^\circ) \Gamma, 0)$. The only weight of $\zeta_{P,t}(\tau')$ is 0, so by [BaMo2, Theorem 6.4]

\[\text{Wt}(\zeta_u(\pi)) \subset W(\Sigma_{\sigma \otimes u}^\circ) \Gamma t \subset a^{*_M}_M.\]

The analogy between (8.13) and (8.15) is our compatibility with the extended Langlands classification. The construction of $\zeta_u$ also works in the other direction (with $\zeta_{P,t}^{-1}$), so it is bijective. Since $\zeta_u$ is built from operations that do not change anything in $Z(\sigma \otimes u)$, it preserves the $Z(\sigma \otimes u)$-characters of representations. \hfill \square

Unfortunately we do not know how to make $\zeta_u$ canonical, because for some parameters $k$ we do not have canonical bijections $\zeta_{P,t}$ as above.

For any $w \in W(M, \sigma, X_{nr}(M))$, conjugation with $T_w$ in $\text{End}_G(I^G_P(E_B)) \otimes_B K(B)$ defines an isomorphism

\[\text{Ad}(T_w) : \mathbb{C}[W(M, \sigma, X_{nr}(M))_u, \sharp] \to \mathbb{C}[W(M, \sigma, X_{nr}(M))_{w(u)}, \sharp].\]

Recall from Lemma 7.1 that $\mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \sharp_u]$ is embedded in $\mathbb{C}[W(M, \sigma, X_{nr}(M))_u, \sharp]$ as the span of $W(M, \sigma, X_{nr}(M))_u$. Thus $\text{Ad}(T_w)$ can be transferred to an algebra isomorphism

\[\text{Ad}(N_w) : \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes w(u)}, \sharp_u] \to \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes w(u)}, \sharp_w(u)],\]

which sends $\mathbb{C}N_{\Omega_w(u)}$ to $\mathbb{C}N_{\Omega_w(u)}\cdot w^{-1}$. We denote the differential of $w : U_u \to U_{w(u)}$ also by $w$, but now from $a^{*_M}_M \otimes \mathbb{R} \mathbb{C}$ to itself. For $f \in \mathbb{C}[a^{*_M}_M \otimes \mathbb{R} \mathbb{C}]$ we define $\text{Ad}(N_w)f = f \circ w^{-1}$. These instances of $\text{Ad}(N_w)$ combine to an algebra isomorphism

\[\text{Ad}(N_w) : \mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \sharp_u) \to \mathbb{H}(\tilde{\mathcal{R}}_{w(u)}, W(M, \mathcal{O})_{\sigma \otimes w(u)}, k^{w(u)}, \sharp_{w(u)}).\]

For $w \in W(M, \sigma, X_{nr}(M))_u$, this is just the inner automorphism $\text{Ad}(N_{\Omega_w(u)})$ of $\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k, \sharp_u)$. For other $w \in W(M, \sigma, X_{nr}(M))$ the notation $\text{Ad}(N_w)$ is only suggestive, because we have not defined an element $N_w$.

With all that set, we define a bijection

\[(8.16) \quad \text{Ad}(N_w)^{-1} : \text{Irr}(\mathbb{H}(\tilde{\mathcal{R}}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \sharp_u)) \to \mathbb{H}(\tilde{\mathcal{R}}_{u'}, W(M, \mathcal{O})_{\sigma \otimes u'}, k^{u'}, \sharp_{u'}),\]

for any $w \in W(M, \sigma, X_{nr}(M))$ such that $w(u) = u'$. Since inner automorphisms act trivially on the set of irreducible representation of an algebra, (8.16) does not depend on the choice of $w$ with $w(u) = u'$. Clearly, the construction of $\text{Ad}(N_w)$ also works with $k = 0$ instead of $k^u$ and $k^{w(u)}$.

However, because of the lack of canonicity of $\zeta_u$ it is not clear whether

\[\zeta_{w(u)} \circ \text{Ad}(N_w^{-1})^* = \text{Ad}(N_w^{-1})^* \circ \zeta_u.\]
To achieve that desirable equality we can enforce it in the following way. For every \( W(M, \sigma, X_{nr}(M)) \)-orbit in \( X_{nr}(M) \) we fix one representative \( u \). Then we define

\[
(8.17) \quad \zeta_{w(u)} := \text{Ad}(N_w^{-1})^* \circ \zeta_u \circ \text{Ad}(N_w)^*: \\
\{ V \in \text{Irr}(\mathcal{H}(\mathcal{N}_w^{(u)}, W(M, \sigma) \otimes w(u), \mathcal{B}_{w(u)})) : \text{Wt}(V) \subset a_M^* \} \\
\rightarrow \{ V \in \text{Irr}(\mathcal{C}[a_M^* \otimes \mathbb{C}] \times \mathbb{C}[W(M, \sigma) \otimes w(u), \mathcal{B}_{w(u)}) : \text{Wt}(V) \subset a_M^* \}.
\]

When \( w \) is of minimal length in \( wW(M, \sigma, X_{nr}(M))u \), it sends \( \Delta_{\sigma \otimes w(u)} \) to \( \Delta_{\sigma \otimes w(u)} \). Then \( w(a_M^*)_w = (a_M^*)_{w(u)} \), so \( \text{Ad}(N_w^{-1})^* \) preserves temperedness. That particular \( \text{Ad}(N_w^{-1})^* \) also maps a Langlands datum \( (P, \tau, t') \) (as in Proposition 8.5) to another Langlands datum, so it respects the compatibility with the Langlands classification from (8.13) and (8.15).

As \( \zeta_{u'} \), as defined in (8.17), does not depend on the choice of \( w \) with \( w(u) = u' \), this means that \( \zeta_{u'} \) always satisfies the requirements of Theorem 8.6.

**Corollary 8.7.** There exists a bijection

\[
\zeta : \text{Irr}(\text{End}_G(I_F^G(E_B))) \rightarrow \text{Irr}(\mathcal{C}[X_{nr}(M)] \times \mathbb{C}[W(M, \sigma, X_{nr}(M))]), \]

such that \( \text{Wt}(\pi) \subset W(M, \sigma, X_{nr}(M))uX_{nr}^+(M) \) if and only if \( \text{Wt}(\zeta(\pi)) \subset W(M, \sigma, X_{nr}(M))uX_{nr}^+(M) \).

**Proof.** With (8.5) we decompose \( \text{Irr}(\text{End}_G(I_F^G(E_B))) \) as a disjoint over \( X_{nr}(M) \), modulo an action of \( W(M, \sigma, X_{nr}(M)) \). Notice that the \( W(M, \sigma, X_{nr}(M)) \)-actions in (8.5) and (8.16) agree, because both are induced by \( \text{Ad}(\mathcal{T}_w) \). By Theorem 8.6 the terms in the disjoint union in (8.5) are in bijection with

\[
\{ V \in \text{Irr}(\mathcal{C}[a_M^* \otimes \mathbb{C}] \times \mathbb{C}[W(M, \sigma) \otimes w(u), \mathcal{B}_{w(u)}) : \text{Wt}(V) \subset a_M^* \}.
\]

By (8.17) the bijections from Theorem 8.6 are \( W(M, \sigma, X_{nr}(M)) \)-equivariant. That brings us to the left hand side of Lemma 8.3. Applying that lemma, we finally obtain the required bijection. \( \square \)

9. Temperedness

Like in [He3], [So5], we want to show that the equivalence of categories

\[
\mathcal{E} : \text{Rep}(G)^G \rightarrow \text{End}_G(I_F^G(E_B))-\text{Mod}
\]

preserves temperedness. At the moment we have not even defined temperedness for representations of \( \text{End}_G(I_F^G(E_B)) \), so we address that first.

Our definition will mimick that for affine Hecke algebras [Opd] §2. It depends on the choice of the parabolic subgroup \( P \) with Levi factor \( M \). Before we just picked one, in this section we have to be more careful.

Recall that \( A_0 \) is a maximal \( F \)-split torus of \( G \), contained in \( M \). By the standard theory of reductive groups [Spr] there are (non-reduced) root systems \( \Sigma(M, A_0) \) and \( \Sigma(G, A_0) \) in \( X^*(A_0) \). Further \( \Sigma(G, M) \cup \{ 0 \} \) is the image of \( \Sigma(G, A_0) \cup \{ 0 \} \) in the quotient \( X^*(A_0) \otimes \mathbb{R}/\mathbb{R} \Sigma(M, A_0) \).

The root system \( \Sigma_{M, \mu} \) is contained in \( \Sigma_{\text{red}}(A_M) \subset \Sigma(G, M) \). We write

\[
\Sigma_{M, \mu} = \Sigma(M, A_0) = \{ \alpha \in \Sigma(G, A_0) : \alpha + \mathbb{R} \Sigma(M, A_0) \in \mathbb{R} \Sigma_{M, \mu} \},
\]

a parabolic root subsystem of \( \Sigma(G, A_0) \).
Lemma 9.1. There exists a basis $\Delta$ of $\Sigma(G, A_0)$ which contains a basis $\Delta_M$ of $\Sigma(M, A_0)$ and a basis of $\Sigma_{\delta,\mu} \Sigma(M, A_0)$.

Proof. Choose a linear function $t$ on $X^*(A_0) \otimes \mathbb{Z} \mathbb{R}$ such that, for all $\alpha \in \Sigma(M, A_0), \beta \in \Sigma_{\delta,\mu} \Sigma(M, A_0)$ and $\gamma \in \Sigma(G, A_0) \setminus \Sigma(M, A_0)$ and $\gamma \in \Sigma(G, A_0) \setminus \Sigma_{\mu} \Sigma(M, A_0)$:

$$0 < |t(\alpha)| < |t(\beta)| < |t(\gamma)|.$$ 

Now take the system of positive roots

$$\Sigma(G, A_0)^+ := \{\alpha \in \Sigma(G, A_0) : t(\alpha) > 0\}$$

and let $\Delta$ be the unique basis of $\Sigma(G, A_0)$ contained therein. Then $\Delta$ consists of the positive roots that cannot be written as sums of positive roots with smaller $t$-values. Hence $\Delta$ consists of a basis of $\Sigma(M, A_0)$, added to that some roots to create a basis of $\Sigma_{\delta,\mu} \Sigma(M, A_0)$ and completed with other roots (all with $t$-values as small as possible) to a basis of $\Sigma(G, A_0)$. □

Let $P_0$ be the "standard" minimal parabolic $F$-subgroup of $G$ determined by $A_0$ and $\Delta$ and put $P = P_0 M$. Then $\Sigma(G, M)$ is spanned by $\Delta \setminus \Delta_M$ and a subset of $\Delta \setminus \Delta_M$ spans $\mathbb{R} \Sigma_{\delta,\mu}$. We note that

$$(a_M^\pm)^{-\Delta} := \{x \in a_M^\pm : \langle \alpha^V, x \rangle = 0 \forall \alpha \in \Delta\}$$

always contains $a^\pm_G = X^*(A_G) \otimes \mathbb{Z} \mathbb{R}$, but can be larger (if $\Sigma_{\delta,\mu}$ has smaller rank than $\Sigma(G, M)$). Consider the obtuse negative cones with respect to $\Delta_{\delta,\mu}$:

$$a^{*-}_{M} = \{\sum_{\alpha \in \Delta_{\delta,\mu}} x_{\alpha} \alpha : x_{\alpha} \in \mathbb{R}_{\leq 0}\},$$

$$a^{*-\bot}_{M} = \{\sum_{\alpha \in \Delta_{\delta,\mu}} x_{\alpha} \alpha : x_{\alpha} \in \mathbb{R}_{< 0}\}.$$

Definition 9.2. Let $\pi$ be a finite dimensional $\text{End}_G(I_B^G)$-representation. Then $\pi$ tempered if $Wt(\pi) \subset X_{\text{vm}}(M) \exp(a^{*-\bot}_M)$.

We say that $\pi$ is discrete series if $(a^*_M)^{-\Delta} = 0$ and $Wt(\pi) \subset X_{\text{vm}}(M) \exp(a^{*-\bot}_M)$. We call $\pi$ essentially discrete series if $Wt(\pi) \subset \exp((a^*_M)^{-\Delta} \otimes \mathbb{R} \mathbb{C}) X_{\text{vm}}(M) \exp(a^{*-\bot}_M)$.

We will see later that these essentially discrete series representations correspond (under an extra condition) to essentially square-integrable representations in $\text{Rep}(G)^\circ$.

Now we will translate $\text{End}_G(I_B^G)$ to the module category of an affine Hecke algebra, as far as possible. Every finite dimensional $\text{End}_G(I_B^G)$-module decomposes canonically as a direct sum of submodules, each of which has weights in just one set $W(M, \sigma, X_{\text{vm}}(M)) u X_{\text{vm}}(M)$. Combining that with Corollary 8.1 and Lemma 8.2 we obtain an equivalence of categories between $\text{End}_G(I_B^G)$-fMod and

$$\bigoplus_{u \in X_{\text{vm}}(M)} \mathbb{H}(\tilde{R}_u, W(M, \sigma \otimes u, k^u, \mathbb{Z}_u) \text{-fMod} / W(M, \sigma, X_{\text{vm}}(M))).$$

To make sense of this as category, the action of $w \in W(M, \sigma, X_{\text{vm}}(M))$ on the summands indexed by $u$ with $w(u) = u$ is supposed to be trivial. Hence the quotient operation only takes place in the index set $X_{\text{vm}}(M)$, and the result can be considered as a direct sum of module categories, indexed by $X_{\text{vm}}(M) / W(M, \sigma, X_{\text{vm}}(M))$. Unfortunately this is not canonical, it depends on the choice of a set of representatives for the action of $X_{\text{vm}}(M, \sigma)$ on $X_{\text{vm}}(M)$.

With Lemma 8.2 we can rewrite (9.1) as

$$\bigoplus_{u \in X_{\text{vm}}(M) / X_{\text{vm}}(M, \sigma)} \mathbb{H}(\tilde{R}_u, W(M, \sigma \otimes u, k^u, \mathbb{Z}_u) \text{-fMod} / W(M, \sigma)).$$
This is very similar to the module category of an affine Hecke algebra with torus $X_{unr}(M)/X_{nr}(M, \sigma) = \text{Irr}(M^2/\mathcal{M})$. More precisely, recall the root datum

$$(\Sigma^v, M^2/\mathcal{M}, \Sigma_\mathcal{O}, (M^2/\mathcal{M})^v)$$

from Proposition 3.1. Endow it with the basis determined by $P$ and the parameters

$$(9.3) \quad \lambda(h^v) = a_{sa} + b_{sa}, \quad \lambda^*(h^v) = a_{sa} - b_{sa}.$$  

The group $R(\mathcal{O})$ acts on this root datum, preserving all the structure. Suppose that

$$(9.4) \quad \tilde{\zeta} : (W(M, \mathcal{O})/W(\Sigma_\mathcal{O}, \mu))^2 \rightarrow \mathbb{C}^\times$$

is a 2-cocycle which on each subgroup $W(M, \mathcal{O})_{\sigma \otimes u}$ is cohomologous to $\zeta_u$. To these data we associate a twisted affine Hecke algebra

$$(9.5) \quad \tilde{\mathcal{H}}(\mathcal{O}) := \mathcal{H}(\Sigma^v, M^2/\mathcal{M}, \Sigma_\mathcal{O}, (M^2/\mathcal{M})^v), \lambda, \lambda^*, q_{F^{1/2}}^\times \mathcal{O}[R(\mathcal{O}), \tilde{\zeta}],$$

as in [AMS3] Proposition 2.2. From Lusztig’s reduction theorems, in the form [AMS3] Theorems 2.5 and 2.9), we see that $\tilde{\mathcal{H}}(\mathcal{O})$-fMod is also equivalent with $(9.2)$. Notice that here we do not need the entire 2-cocycle $\tilde{\zeta}$, only its restrictions to the subgroups $W(M, \mathcal{O})_{\sigma \otimes u}$.

**Corollary 9.3.** Assume that a $\tilde{\zeta}$ as above exists. Then the categories $\tilde{\mathcal{H}}(\mathcal{O})$-fMod and $\text{End}_G(I^P(F)(E_B))$-fMod are equivalent.

This looks like a Morita equivalence, although it is not quite. Let us describe it more explicitly. Start with $V \in \text{End}_G(I^P(F)(E_B))$-fMod. Decompose it as

$$V = \bigoplus_{u \in X_{unr}(M)} V_u \quad \text{where } \text{Wt}(V_u) \subset uX_{nr}^+(M).$$

Pick a fundamental domain $\tilde{X}$ for the action of $X_{nr}(M, \sigma)$ on $X_{unr}(M)$. Then put $\tilde{V} = \bigoplus_{u \in \tilde{X}} V_u$, this is the associated $\tilde{\mathcal{H}}(\mathcal{O})$-module. The $\mathbb{C}[X_{nr}(M)/X_{nr}(M, \sigma)]$-action can be read of directly, to reconstruct how the rest of $\tilde{\mathcal{H}}(\mathcal{O})$ acts one needs Lemmas 8.2 and 8.3.

The effect of this equivalence on weights is simple. Whenever a module $\tilde{V}$ of $\tilde{\mathcal{H}}(\mathcal{O})$ has a weight $\chi_{X_{nr}(M, \sigma)} \in X_{nr}(M)/X_{nr}(M, \sigma)$, all elements of $\chi_{X_{nr}(M, \sigma)} \subset X_{nr}(M)$ are weights of $V \in \text{End}_G(I^P(F)(E_B))$-fMod, and conversely. Definition 9.2 of tempered and (essentially) discrete series representations also applies to $\tilde{\mathcal{H}}(\mathcal{O})$, we only have to replace $X_{nr}(M)$ by $X_{nr}(M)/X_{nr}(M, \sigma)$. (In fact that recovers the standard definitions for affine Hecke algebras.) This means in particular that the equivalence between $\tilde{\mathcal{H}}(\mathcal{O})$-fMod and $\text{End}_G(I^P(F)(E_B))$-fMod preserves temperedness and (essentially) discrete series.

The problem with $\tilde{\mathcal{H}}(\mathcal{O})$ lies in the existence of a 2-cocycle $\tilde{\zeta}$ with the properties mentioned after (9.4). We do not know whether such a 2-cocycle exists in general. Of course it is easy to fulfill the condition for one given $u \in X_{unr}(M)$, but then it could fail for different $u' \in X_{unr}(M)$.

Nevertheless, even if we cannot find such $\tilde{\zeta}$, we can still regard $(9.2)$ and $\text{End}_G(I^P(F)(E_B))$-fMod as equivalent the category of finite dimensional modules of a hypothetical algebra $\tilde{\mathcal{H}}(\mathcal{O})$. By this we mean that $(9.2)$ and $\text{End}_G(I^P(F)(E_B))$-fMod have all the properties of category of finite dimensional modules of a twisted affine Hecke algebra with data as in $(9.5)$, for instance with respect to parabolic induction, weights, temperedness, discrete series and the Langlands classification. The reason
is that all these properties can be expressed entirely in terms of (9.2), see [AMS3 §2.1–2.2].

For many purposes, that suffices to transfer knowledge about affine Hecke algebras to $\text{End}_G(I_P^G(E_B))$. We will do that with the results of [Sol5]. In that paper it is assumed that $\text{Rep}(G)^s$ is equivalent with the module category of an extended affine Hecke algebra. We replace that assumption by the above interpretation of $\text{End}_G(I_P^G(E_B))-\text{fMod}$ in terms of $\hat{\mathcal{H}}(O)-\text{fMod}$ and (9.2).

For every $F$-Levi subgroup $L$ of $G$ containing $M$, $\text{End}_L(I_P^L(E_B))$ embeds naturally in $\text{End}_G(I_P^G(E_B))$, by the functoriality of $I_P^G$. It can be identified with the set of regular (i.e. without poles) operators in the subalgebra of $\text{End}_G(I_P^G(E_B)) \otimes_B K(B)$ generated by $K(B)$, the $\phi_{\chi_c}$ with $\chi_c \in X_{\text{ir}}(M, \sigma)$ and the $A_w$ with $w \in N_L(M, O)/M$.

Let $\Sigma_{O,L}$ be the parabolic root subsystem of $\Sigma_{O,\mu}$ consisting of roots that come from the action of $A_M$ on the Lie algebra of $L$. Then

$$N_L(M, O)/M = W(\Sigma_{O,L}) \times R(O, L),$$

where $R(O, L) = R(O) \cap N_L(M, O)/M$.

With these notions, we can interpret $\text{End}_L(I_P^L(E_B))-\text{fMod}$ as the category of finite dimensional modules of

$$\hat{\mathcal{H}}(O, L) := \mathcal{H}(\Sigma_{O,L}^+, M_\sigma^2/M_1^1, \Sigma_{O,L}, (M_\sigma^2/M_1^1)^{\vee}, \lambda, \lambda^{s, q^{1/2}}) \times \mathbb{C}[R(O, L), \tilde{z}],$$

a parabolic subalgebra of $\hat{\mathcal{H}}(O)$. (Again, $\hat{\mathcal{H}}(O, L)$ is conditional on the existence of $\tilde{z}$.) With this interpretation of the "parabolic subalgebras" $\text{End}_L(I_P^L(E_B))$ of $\text{End}_G(I_P^G(E_B))$, all parts of [Sol5] that involve only finite length representations apply to $\text{End}_L(I_P^L(E_B))$.

The relevant results from [Sol5 §4.2] depend on some conditions, which we have to check. Condition [Sol5 4.1] is about compatibility of the equivalence $\text{Rep}(G)^s \to \text{End}_G(I_P^G(E_B))$ with parabolic induction and Jacquet restriction. This condition was already verified in [Sol5 Lemma 6.1].

Condition [Sol5 4.2.i] is replaced by our interpretation of (9.2) and $\hat{\mathcal{H}}(O)-\text{fMod}$. Condition [Sol5 4.2.ii] agrees with that, because $\text{End}_L(I_P^L(E_B))$ is related in the same way to the parabolic subalgebra $\hat{\mathcal{H}}(O, L)$ of $\hat{\mathcal{H}}(O)$.

In Condition [Sol5 4.2.iii] it is required that $\Sigma_{O,\mu}^+$ lies in the cone $\mathbb{Q}G_{>0}\Sigma(G, M)^+$ and that $\mathbb{Q}\Sigma_{O,\mu}$ has a $\mathbb{Q}$-basis consisting of simple roots of $\Sigma(G, M)$. Both are guaranteed by Lemma 9.1.

In Condition [Sol5 4.2.iv] it is required firstly that, for every $F$-Levi subgroup $L \subset G$ containing $M$, $R(O, L)$ stabilizes $\Sigma_{O,L}$ – which is clear. Secondly, when $\Sigma_{O,L}$ has full rank in $\Sigma(L, M)$, Condition [Sol5 2.1] has to be fulfilled. That says

- $R(O, L) \subset R(O, L')$ if $L \subset L'$;
- the action of $R(O, L)$ on $X_{\text{ir}}(M)$ stabilizes the subsets $\exp(\mathbb{C}\Sigma_{O,L})$ and $X_{\text{ir}}(M)^L := \exp((a_M^s)^{+L} \otimes_{\mathbb{R}} \mathbb{C})$, where

$$\{a_M^s\}^{+L} = \{x \in a_M^s : \langle \alpha^\vee, x \rangle = 0 \ \forall \alpha \in \Sigma_{O,L}\};$$

- $R(O, L)$ acts on $X_{\text{ir}}(M)^L$ by multiplication with elements of $X_{\text{ir}}(M)^L \cap \exp(\mathbb{C}\Sigma_{O,L})$.

The first of these bullets is obvious. By the full rank assumption

$$a_M^s = X^*(A_L) \otimes_{\mathbb{R}} \mathbb{R}.$$
Recall that the action of \( r \in R(O, L) \) on \( X_{\text{nr}}(M) \) consists of a part which is linear on the Lie algebra and a translation by \( \chi_r \). By the \( R(O, L) \)-stability of \( \Sigma_{O, L} \), the linear part stabilizes \( \exp(\mathbb{C}\Sigma_{O, L}) \). Further the linear part of the action of \( r \) fixes \( a'_r \) pointwise, so by (9.6) it fixes \( X_{\text{nr}}(M)^L \) pointwise. The definition of \( \chi_r \) in (9.4) shows that it is an unramified character of \( L \) which is trivial on \( Z(L) \). This means that \( \chi_r \in X_{\text{nr}}(M)^L \cap \exp(\mathbb{C}\Sigma_{O, L}) \). Hence the second and third bullets hold.

We have verified everything needed to make the arguments in [Sol5 §4.2] about finite length representations work. Recall that a \( G \)-representation (of finite length) is called essentially square-integrable if its restriction to the derived group of \( G \) is square-integrable.

**Theorem 9.4.** [Sol5 Theorem 4.9 and Proposition 4.10]

(a) The equivalence \( E : \text{Rep}(G)^\# \to \text{End}_G(I_P^\varepsilon(E_B))\text{-Mod} \) restricts to an equivalence between the subcategories of finite length tempered representations on both sides.

(b) If \( \Sigma_{O, \mu} \) has smaller rank than \( \Sigma(G, \mathcal{M}) \), then \( \text{Rep}(G)^\# \) contains no essentially square-integrable representations.

(c) Suppose that \( \Sigma_{O, \mu} \) has full rank in \( \Sigma(G, \mathcal{M}) \). Then \( E \) provides a bijection between:

- finite length essentially square-integrable representations in \( \text{Rep}(G)^\# \),
- finite length essentially discrete series representations of \( \text{End}_G(I_P^\varepsilon(E_B)) \).

This remains valid if we add "tempered" and/or "irreducible" on both sides.

(d) When \( \Sigma_{O, \mu} \) has the same rank as \( \Sigma(G, \mathcal{M}) \) and \( Z(G) \) is compact, part (c) also holds without "essentially".

We can also include (9.2) and part (i) of Corollary 8.1 in our considerations about temperedness. We already defined temperedness for representations of \( \mathbb{H}(\mathcal{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k, \zeta_u) \), before Proposition 8.3.

**Proposition 9.5.** Let \( u \in X_{\text{nnr}}(M) \). Let \( V \in \text{End}_G(I_P^\varepsilon(E_B))\text{-Mod} \) with \( W(V) \subset W(M, \sigma, X_{\text{nr}}(M)^L u X_{\text{nr}}^+(M) \) and let \( V_u \in \mathbb{H}(\mathcal{R}_u W(M, \mathcal{O})_{\sigma \otimes u}, k, \zeta_u)\text{-Mod} \) be its image via Corollary 8.4.

(a) \( V \) is tempered if and only if \( V_u \) is tempered.

(b) \( V \) is essentially discrete series if and only if \( V_u \) is essentially discrete series and \( \Sigma_{\sigma \otimes u} \) has full rank in \( \Sigma_{O, \mu} \).

**Proof.** (a) Recall from the discussion following Corollary 9.3 that \( V \) is tempered if and only if the associated \( \mathcal{H}(O) \)-module \( \tilde{V} \) is tempered. To make things a little easier, we note that \( \tilde{V} \) is tempered if and only if its restriction to

\[ \mathcal{H}(\Sigma'_{O}, M_L^2/M_1, \Sigma_O, (M_L^2/M_1)^{\vee}, \lambda, \lambda^*, q_F^{1/2}) \]

is tempered. Indeed, the condition of temperedness for the latter algebra is exactly the same as for \( \mathcal{H}(O) \), because the underlying complex tori and root system are the same. Now we do not have to worry about the existence of \( \tilde{\xi} \) anymore. By [AMS3 Theorems 2.5 and 2.11], \( \mathcal{H}(\Sigma'_{O}, M_L^2/M_1, \Sigma_O, (M_L^2/M_1)^{\vee}, \lambda, \lambda^*, q_F^{1/2})\)-Mod is equivalent with

\[ \bigoplus_{u \in X_{\text{nnr}}(M)/X_{\text{nr}}(M, \sigma)} \mathbb{H}(\mathcal{R}_u, W(\Sigma_{\sigma \otimes u}) k^u)\text{-Mod} / W(\Sigma_{O, \mu}), \]

interpreted like (9.1) and (9.2). Moreover, by [AMS3 Proposition 2.7] that equivalence of categories respects temperedness (in both directions).
Thus for any \( u \in X_{\text{irr}}(M) \):

\[ \text{Wt}(V) \subset W(M, \sigma, X_{\text{irr}}(M))uX^+_\text{irr}(M) \]

with \( \text{Wt}(V) \subset W(M, \sigma, X_{\text{irr}}(M))uX^+_\text{irr}(M) \) if and only if the irreducible representation \( V_1 \) of \( \mathbb{C}[X_{\text{irr}}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{\text{irr}}(M)), \tilde{\varepsilon}] \) associated to it by Lemma 8.4 has \( u \) as its only weight. This is the case if and only if the irreducible representation \( V_2 \) of \( \mathbb{C}[a^*_M \otimes \mathbb{C}] \rtimes \mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes u}, \tilde{\varepsilon}_u] \) obtained from \( V_1 \) as in (8.10) has \( 0 \in a^*_M \) as its only weight.

Now \( V_2 \), a representation of a twisted graded Hecke algebra \( \mathbb{H}(\tilde{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, 0, \tilde{\varepsilon}_u) \) with \( \text{Wt}(V_2) \subset a^*_M \), is tempered if and only if \( \text{Wt}(V_2) = \{0\} \). To see that, notice that the weights of \( V_2 \) form full \( W(\Sigma_{\sigma \otimes u}) \)-orbits. Every \( W(\Sigma_{\sigma \otimes u}) \)-orbit in \( a^*_M \), except \( \{0\} \), contains elements outside the cone \( (a^*_M)_u \).

By Theorem 8.6, \( V_2 \) is tempered if and only if

\[ \zeta^{-1}_{\sigma}(V_2) \in \text{Irr}(\mathbb{H}(\tilde{R}_u, W(M, \mathcal{O})_{\sigma \otimes u}, k^u, \tilde{\varepsilon}_u)) \]

is tempered. Next Proposition 9.5 says that \( \zeta^{-1}_{\sigma}(V_2) \) is tempered if and only if its image \( V_3 \) in \( \text{Irr}(\text{End}_G(I^G_{EB})) \) is so. Comparing the above with the proof of Corollary 8.7, we see that \( V_3 \) equals \( \zeta^{-1}(V_0) \). Finally, in Theorem 9.4.a we showed that \( \zeta^{-1}(V_0) \) is tempered if and only if \( \mathcal{E}^{-1}(\zeta^{-1}(V_0)) \) is tempered.

The space \( \text{Irr}(\mathbb{C}[X_{\text{irr}}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{\text{irr}}(M)), \tilde{\varepsilon}]) \) admits an alternative description, which clarifies the geometric structure in Theorem 9.6.

**Lemma 9.7.** There is a canonical bijection

\[
\bigcup_{\chi \in X_{\text{irr}}(M)/X_{\text{irr}}(M, \sigma)} \text{Irr}(\mathbb{C}[W(M, \mathcal{O})_{\sigma \otimes \chi}, \tilde{\varepsilon}_\chi]) \bigg/ W(M, \mathcal{O}) \longrightarrow \text{Irr}(\mathbb{C}[X_{\text{irr}}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{\text{irr}}(M)), \tilde{\varepsilon}]). 
\]
Proof. Let the central extension \( \Gamma \) of \( \chi \) be as in the proof of Lemma \ref{lem:central-extension}, so that
\[
p_\chi \mathbb{C}[\Gamma] \cong \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural].
\]
By \eqref{eq:central-extension} and \eqref{eq:central-extension2}, every irreducible representation \( \pi \) of \( \mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \) is of the form
\[
\text{ind}_{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}^{\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}(\chi \otimes \rho),
\]
where \( \chi \in X_{nr}(M) \) and \( \rho \in \text{Irr}(\mathbb{C}[W(M, \sigma, X_{nr}(M))]), \natural] \). The pair \((\chi, \rho)\) is determined by \( \pi \), uniquely up to the action of \( \Gamma \) given by
\[
\gamma(\chi, \rho) = (\gamma(\chi), \text{Ad}(\gamma^{-1})^*) \rho.
\]
Since \( \Gamma \) is a central extension of \( W(M, \sigma, X_{nr}(M)) \), this action descends to an action of \( W(M, \sigma, X_{nr}(M)) \) on the collection of such pairs. This yields a bijection
\[
\bigcup_{\chi \in X_{nr}(M)} \text{Irr}(\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]) / W(M, \sigma, X_{nr}(M)) \longrightarrow \text{Irr}(\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]).
\]
For \( \chi \in X_{nr}(M, \sigma) \) there are group isomorphisms
\[
\text{Ind}_{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}^{\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}(\chi \otimes \rho) \longrightarrow \text{Ind}_{\mathbb{C}[X_{nr}(M)] \rtimes \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}^{\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]}(\chi \otimes \rho).
\]
It follows from \eqref{eq:central-extension} that conjugation with \( \phi_{\chi} \) induces an algebra isomorphism
\[
\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \cong \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \]
which via \eqref{eq:isomorphism} translates to the identity map
\[
\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \longrightarrow \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural).
\]
Hence, in \eqref{eq:central-extension}, we can canonically identify all the terms associated to \( \chi \)'s in one \( X_{nr}(M, \sigma) \)-orbit. If we do that, the action of \( W(M, \sigma, X_{nr}(M)) \) descends to an action of
\[
W(M, \sigma) = W(M, \sigma, X_{nr}(M))/X_{nr}(M, \sigma)
\]
and the left hand side of \eqref{eq:central-extension} becomes
\[
\bigcup_{\chi \in X_{nr}(M, \sigma)} \text{Irr}(\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural]) / W(M, \sigma).
\]
We note that in Lemma \ref{lem:central-extension}, \( \natural \) is not necessarily equal to \( \natural \otimes u \). These 2-cocycles are merely cohomologous (by Lemma \ref{lem:2-cocycle}) with \( u \) the unitary part of \( \chi \). An advantage of \( \natural \) is that it factors via
\[
(W(M, \sigma, X_{nr}(M))/X_{nr}(M, \sigma))^2 \cong R(\sigma \otimes u)^2.
\]
The action of \( w \in W(M, \sigma) \) on the left hand side of Lemma \ref{lem:central-extension} comes from isomorphisms
\[
\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \cong \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural],
\]
and
\[
\mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural] \cong \mathbb{C}[W(M, \sigma, X_{nr}(M)), \natural].
\]
Here the outer automorphisms are described in the proof of Lemma 7.1, while the middle isomorphism is computed in $\mathbb{C}[W(M,\sigma,X_{\text{nr}}(M)),\tilde{z}]$. In particular we still make use of the entire 2-cocycle $\tilde{z}$, not just of the $\tilde{z}_\chi$.

Define the root system $\Sigma_{\sigma\otimes\chi}$ like $\Sigma_{\sigma\otimes u}$. By Lemma 8.3 the composed isomorphism (9.9) sends $N_v$ to $N_{wvw^{-1}}$ for $v \in W(\Sigma_{\sigma\otimes\chi})$, and to a scalar multiple of that for $v \in W(M,\mathcal{O})_{\sigma\otimes\chi}$. Since $X_{\text{nr}}(M)/X_{\text{nr}}(M,\sigma) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi$ is bijective, we can rewrite the left hand side of Lemma 9.7 as

$$\bigcup_{\sigma'\in\mathcal{O}} \text{Irr}(\mathbb{C}[W(M,\mathcal{O})_{\sigma'},\tilde{z}_{\sigma'}]) / W(M,\mathcal{O}).$$

In the terminology of [ABPS2, §2.1], (9.10) is the twisted extended quotient $(\mathcal{O}/W(M,\mathcal{O}))_{\tilde{z}}^\sigma$.

From Theorem 9.6, Lemma 9.7 and (9.10) we conclude

**Corollary 9.8.** There exists a bijection

$$\tilde{\zeta} \circ \mathcal{E} : \text{Irr}(G)^{\sigma} \to (\mathcal{O}/W(M,\mathcal{O}))_{\tilde{z}},$$

such that, for $\pi \in \text{Irr}(G)^{\sigma}$ and $u \in X_{\text{unr}}(M)$:

- the cuspidal support $\text{Sc}(\pi)$ lies in $W(M,\mathcal{O})\{\sigma \otimes u\chi : \chi \in X_{\text{nr}}^+(M)\}$ if and only if $\tilde{\zeta} \circ \mathcal{E}(\pi)$ has $\mathcal{O}$-coordinate in $W(M,\mathcal{O})\{\sigma \otimes u\chi : \chi \in X_{\text{nr}}^+(M)\}$;
- $\pi$ is tempered if and only if $\tilde{\zeta} \circ \mathcal{E}(\pi)$ has a unitary (or equivalently tempered) $\mathcal{O}$-coordinate.

This proves a version of the ABPS conjecture, namely [ABPS2, Conjecture 2].

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