Charged Particles in a 2+1 Curved Background

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Abstract

The coupling to a 2+1 background geometry of a quantized charged test particle in a strong magnetic field is analyzed. Canonical operators adapting to the fast and slow freedoms produce a natural expansion in the inverse square root of the magnetic field strength. The fast freedom is solved to the second order. At any given time, space is parameterized by a couple of conjugate operators and effectively behaves as the ‘phase space’ of the slow freedom. The slow Hamiltonian depends on the magnetic field norm, its covariant derivatives, the scalar curvature and presents a peculiar coupling with the spin-connection.

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The dynamics of a charged particle in a given electromagnetic and gravitational background is an important problem having implications in several areas of theoretical and mathematical physics—from classical gravity to condensed matter and plasma physics to quantum field theory. As a quite interdisciplinary example, it represents the first step to take in addressing the study of a plasma around a compact astrophysical object or, more in general, in space and cosmological phenomena \[1\]. Exact solutions are found when metric and electromagnetic two-form share common symmetries. Various special cases have been worked out—especially in two spatial dimensions—with particular emphasis on the underlying algebraic and analytical structures \[2\]. Beyond symmetry—in spite of the apparent simplicity—the general problem displays an extreme degree of complication. Classical motion is generally chaotic and one has to be content with approximate analysis. Even like this, however, the task to set down an appropriate perturbative expansion is not straightforward. The peculiar structure of the electromagnetic interaction makes ordinary Hamiltonian perturbation theory not directly applicable \[3\]. In the fifties and sixties, the urgency of the problem in classical applications—especially in connection with the investigation of earth’s magnetosphere and in the design of mirror machines for the confinement of hot plasma—motivated Bogoliubov, Kruskal and others to formulate adiabatic perturbation theory directly in terms of the equations of motion. This led Northrop and Teller to the familiar ‘guiding center’ picture of the effective dynamics of a charged particle in an inhomogeneous magnetic field in a flat spacetime \[4\]. Modern applications—ranging from geodesics motion around charged black holes in classical gravity to two-dimensional system of non-relativistic electrons in quantum-Hall like devices to plasma in astrophysics and cosmology to the investigation of
the coupling with matter fields in toy models for quantum gravity deal more in general with curved backgrounds and require the extension of the perturbative analysis developed in classical physics to the quantum-mechanical and field-theoretical context. To this task a whole canonical approach to the problem has to be developed. This is the aim of the present investigation.

In this paper we address the subject by discussing the effective motion of a charged particle in a 2+1 curved background. This allows to display better the peculiar canonical structure of the system, avoiding complications arising from extra dimensions. Moreover, the restriction is not just a mathematical artefact. The solution in two spatial dimensions is indeed a key ingredient in the discussion of the relativistic four-dimensional, as well as the non-relativistic three-dimensional, cases. From a rather different viewpoint the problem is also equivalent to the investigation of the effective dynamics of a test particle experiencing the ‘geometric gravitational’ force of Cangemi and Jackiw in a Wick rotated two-dimensional space-time [5].

Our analysis is based on the canonical structure of the system and is essential the same for the classical and the quantum case. For definiteness we consider the quantum case. The classical limit may be obtained straightforwardly. The topology of space-time is supposed to be trivial –the direct product of a surface Σ diffeomorphic to the plane and time– so that all the local quantities get automatically global definition; eg. Ricci rotation coefficients define a spin-connection. Under these hypothesis it is always possible to choose coordinates in such a way that the metric takes the form: $g_{00} = 1$, $g_{0\mu} = 0$, $g_{\mu\nu}$ = arbitrary functions of time and spatial coordinates; $\mu, \nu = 1, 2$. We also assume the electromagnetic field to be purely magnetic. Both relativistic and non-relativistic problems reduce then to the study of the Hamiltonian of a charged particle on the curved surface Σ.
The paper is organized as follows. In section II we discuss the canonical structure of the problem showing how the strong magnetic regime naturally produces an expansion in the inverse square root of the field strength. In section III an adapted set of canonical operators is introduced. This allows to separate the fast freedom from the slow one, identifying the adiabatic invariant of the system. The coupling with background geometry is studied in section IV. Besides contributions depending on the scalar curvature and on covariant derivatives of the magnetic field norm we find a peculiar coupling with the spin-connection. The theory is general as well as ‘Lorenz’ covariant. Our main result is the effective Hamiltonian $\text{(20)}$. The example of a particle on a conical surface in an axisymmetric magnetic field decreasing as the inverse of the distance from the vertex is presented in section V. The last section contains our conclusions. In the appendix the necessary technology for maximally simplifying the study of the adiabatic expansion is summarized.
II. CHARGED PARTICLE ON A CURVED SURFACE

We consider a charged scalar particle on a two dimensional surface $\Sigma$ in a strong magnetic background. The surface is parameterized by arbitrary coordinates $x^\mu$, $\mu = 1, 2$, and its geometry is given by the metric tensor $g_{\mu\nu}$. The magnetic field is described by a closed antisymmetric two-form $b_{\mu\nu}$. In both the non-relativistic and relativistic cases the discussion of the dynamical problem reduces to studying the Hamiltonian

$$\mathcal{H} = \frac{1}{2}g^{-1/2}\Pi_\mu g^{\mu\nu}g^{1/2}\Pi_\nu$$

The kinematical momenta $\Pi_\mu = -i\partial_\mu - l_B^{-2}a_\mu$ have been introduced, $[\Pi_\mu, \Pi_\nu] = ib_{\mu\nu}(\vec{x})$ and the physical dimension of the field is re-adsorbed in the scale factor $l_B$. The wavefunction of the system is normalized with respect to the measure $\sqrt{g}dx^1dx^2$. Our analysis is based on the smallness of the magnetic length $l_B$. Throughout our discussion we assume the background scalar curvature $R$ as well as the derivatives of the magnetic field norm $b = \sqrt{b_{\mu\nu}b^{\mu\nu}/2}$ to satisfy the conditions: $|R| \ll l_B^{-2}$, $|\Delta b/b| \ll l_B^{-2}$ and $|\nabla b/b| \ll l_B^{-1}$.

First of all, we focus on kinematics. In absence of magnetic interaction the essential operators appearing in the description of the system are the coordinates $x^\mu$ and the derivatives $-i\partial_\mu$. These appear as a couple of conjugate variables, $[x^\mu, -i\partial_\mu] = i\delta_\mu^\nu$. Introducing the magnetic interaction replaces the $-i\partial_\mu$s by the non-commuting $\Pi_\mu$s. In other words the magnetic background produces a twist of the canonical structure. This is made explicit by transforming to Darboux coordinate frames $\xi^\mu = \xi^\mu(x)$ in which the magnetic field strength takes the form

$$b_{\mu\nu}(\xi) = l_B^{-2}\varepsilon_{\mu\nu}$$

($\varepsilon_{\mu\nu}$ is the completely antisymmetric tensor in two dimensions). Darboux theorem ensure the existence of a well defined atlas of such frames. In the new frames $\Pi_1$ and $\Pi_2$ appear as
reciprocally conjugate while their commutators with the coordinates are still different from zero. On the other hand, $[\xi^1, \Pi_1]$ and $[\xi^2, \Pi_2]$ are order $l_B^2$ compared to $[\Pi_1, \Pi_2]$. This make clear that in the strong magnetic regime it is convenient to abandon the description in terms of the $\xi^\mu$s and $-i\partial_\mu$s introducing besides $\Pi_1$ and $\Pi_2$ a new couple of canonical variables. These turns out to be the guiding center operators $\Xi^\mu = \xi^\mu + l_B^2 \varepsilon^{\mu\nu} \Pi_\nu$ (in this paper we adopt the notation $\varepsilon^{\mu\nu} = \varepsilon_{\mu\nu}$). Rescaling for convenience the $\Pi_\mu$s by $\Pi_\mu \to l_B \Pi_\mu$ -and hence the magnetic field norm $b$ and the Hamiltonian $\mathcal{H}$ by a factor $l_B^2$- the fundamental commutation relation may finally be re-casted in the form

$$[\Pi_1, \Pi_2] = i \quad [\Xi^2, \Xi^1] = il_B^2$$

The presence of the small parameter $l_B$ in the second relation displays the guiding center operators as slow variables of the system. The physical interpretation of the new quantities emerges by considering dynamics in the semiclassical regime [6,7]: the $\Pi_\mu$s take into account the rapid rotation of the particle while the $\Xi^\mu$s the slow drift of the center of the orbit -the guiding center- on the surface.

Outlined the peculiar canonical structure we come back to the dynamical problem. This is in general of a certain complication the two freedoms of the system being coupled by the metric background $g_{\mu\nu}$ as well as by the magnetic field strength $b_{\mu\nu}$. Note that even starting from a simple geometrical context –eg. a flat one– transforming to Darboux frames produces a quite complicated form of the interaction. Nevertheless, whenever the curvature radii of the surface and the variation length scale of the magnetic field may be considered

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An exact solution of the problem is only possible when metric and magnetic field share a common symmetry. Typical examples are the Landau problem –motion on a plane in a uniform magnetic field– the motion on a sphere in a monopole field and the motion on the Poincaré half plane in a hyperbolic magnetic field [2]. The exact (degenerate) ground state of the system may be obtained whenever metric and magnetic two-form define a Kähler structure on the surface $\Sigma$ [3]. More general conditions require an approximate analysis.
larger than the magnetic length $l_B$, it turns out possible to perform an approximate analysis in quite general terms. We start, of course, from (1). As a first technical steep we rescale wavefunction and Hamiltonian by $\psi \to g^{1/4}\psi$ and $\mathcal{H} \to g^{1/4}\mathcal{H}g^{-1/4}$. This changes the integration measure from $\sqrt{g}dx^1dx^2$ to $dx^1dx^2$ making the Hamiltonian more symmetric and simplifying further manipulations. The second steep is that of adapting variables. We transform therefore to Darboux coordinate frames according to the above kinematical discussion. The transformed metric tensor is denoted by $\gamma_{\mu\nu}$. Observe that in these preferential frames the metric determinant $\gamma$ is related to the magnetic field norm by $\gamma = b^{-2}$. Moreover, all the functions of the coordinates have now to be evaluated in $\xi^\mu = \Xi^\mu - l_B\varepsilon^{\mu\nu}\Pi_\nu$ producing a natural expansion of the Hamiltonian in the small parameter $l_B$. Taking into account the rescaling of the $\Pi_\mu$s, the one of wavefunction and Hamiltonian and expanding in $l_B$, (1) takes the form

$$l_B^2\mathcal{H} = \frac{1}{2}\gamma^{\mu\nu}\Pi_\mu\Pi_\nu - \frac{l_B}{2}(\partial_\kappa\gamma^{\mu\nu})\varepsilon^{\kappa\rho}\Pi_\mu\Pi_\rho +$$

$$+ \frac{l_B^2}{4}(\partial_\kappa\partial_\lambda\gamma^{\mu\nu})\varepsilon^{\kappa\rho}\varepsilon^{\lambda\sigma}\Pi_\mu\Pi_\rho\Pi_\sigma\Pi_\nu - \frac{l_B^2}{4}\frac{\Delta b}{b} + \frac{l_B^2}{8}\frac{|\nabla b|^2}{b^2} + O(l_B^3)$$

(4)

where the inverse metric $\gamma^{\mu\nu}$, the magnetic field norm $b$ and all their derivatives are evaluated in the guiding center operators $\Xi^\mu$.

III. SPINNING AND DRIFTING

We first focus on the zero order of expansion (4) by discussing the truncated Hamiltonian $\mathcal{H}^{(0)} = \frac{1}{2}\gamma^{\mu\nu}\Pi_\mu\Pi_\nu$. This is quadratic in the $\Pi_\mu$s with coefficients depending on the slow variables $\Xi^\mu$. It should therefore be possible to reduce the problem to an harmonic oscillator up to higher orders in $l_B$. To this task we consider the decomposition of $\gamma^{\mu\nu}$ in terms of the zwei-beinen $e_i^{\mu}; \gamma^{\mu\nu} = e_i^{\mu}\varepsilon_i^{\nu}$. We then introduce the ‘normalized zwei-beinen’ $n_i^{\mu} = b^{-1/2}e_i^{\mu}$ in such a way that
\[ H^{(0)} = \frac{1}{2} [n_i^\mu(\Xi)\Pi_\mu] b(\Xi) [n_i^\nu(\Xi)\Pi_\nu] - \frac{i}{4} n_i^\mu(\Xi) b(\Xi) n_i^\nu(\Xi) \varepsilon_{\mu\nu} \] (5)

It is clear that the new \( \bar{\Pi}_\mu \)s recasting \( H^{(0)} \) in an harmonic oscillator Hamiltonian should have the form \( \bar{\Pi}_i = n_i^\mu(\Xi)\Pi_\mu + \mathcal{O}(l_B^2) \). To obtain a genuine set of canonical variables –not a perturbative one– we produce the rotation of the \( \Pi_\mu \)s in the \( n_i^\mu \) directions by mean of the unitary transformation

\[ U = \exp \left\{ -\frac{i}{4} \varepsilon^{\mu\nu}[\lg n]_i \{\Pi_\mu, \Pi_\nu\} \right\} \] (6)

The new canonical operators are defined by \( \bar{\Pi}_i = \delta_i^\mu U\Pi_\mu U^\dagger \) and \( \bar{\Xi}^\mu = U\Xi^\mu U^\dagger \). An explicit expression as a power series in \( l_B^2 \) may now be obtained to any order. As a preparation for the next section we write the new variables to the order \( l_B^2 \). Introducing the non-covariant rotation coefficients \( \rho_{i^j k} = n_k^\mu (\partial_\mu n_i^\nu) n_j^\nu \) we have

\[ \bar{\Pi}_i = n_i^\mu \Pi_\mu - \frac{l_B^2}{8} \varepsilon^{mn} \rho_{i^k m} \rho_{j^l m} \varepsilon^{jk} n_k^\mu n_l^\nu (\Pi_\kappa \Pi_\mu \Pi_\lambda + \Pi_\lambda \Pi_\mu \Pi_\kappa) + \mathcal{O}(l_B^4) \] (7)

\[ \bar{\Xi}^\mu = \Xi^\mu - \frac{l_B^2}{4} \varepsilon^{mn} n_m^\mu \rho_{j^l m} \varepsilon^{jk} n_k^\nu (\Pi_\kappa \Pi_\lambda + \Pi_\lambda \Pi_\kappa) + \mathcal{O}(l_B^4) \] (8)

where all the functions on the right hand side are evaluated in \( \Xi \). In order to rewrite (5) in terms of the new operators these relations have to be inverted. The task is straightforward yielding

\[ n_i^\mu(\Xi)\Pi_\mu = \bar{\Pi}_i + \frac{l_B^2}{8} \varepsilon^{mn} \rho_{i^k m} \rho_{j^l m} \varepsilon^{jk} (\bar{\Pi}_k \bar{\Pi}_h \bar{\Pi}_l + \bar{\Pi}_l \bar{\Pi}_h \bar{\Pi}_k) + \mathcal{O}(l_B^4) \] (9)

and

\[ \Xi^\mu = \bar{\Xi}^\mu + \frac{l_B^2}{4} \varepsilon^{mn} n_m^\mu \rho_{j^l m} \varepsilon^{jk} (\bar{\Pi}_k \bar{\Pi}_l + \bar{\Pi}_l \bar{\Pi}_k) + \mathcal{O}(l_B^4) \] (10)

in both equations the functions on the right hand side are now evaluated in \( \bar{\Xi} \). The substitution of (9) and (10) in (5) produces the zero order Hamiltonian as a power series in \( l_B^2 \).

Introducing the harmonic oscillator \( J = \frac{1}{2}(\bar{\Pi}_1^2 + \bar{\Pi}_2^2) \) we obtain
\[ \mathcal{H}^{(0)} = b(\bar{\Xi}) J + l_B^2 \mathcal{H}^{(0,2)}(\bar{\Xi}, \bar{\Pi}) + \mathcal{O}(l_B^4) \]  

(11)

where \( \mathcal{H}^{(0,2)} \) is a quite complicated expression –quartic in the \( \bar{\Pi}_\mu \)s and depending on the \( \bar{\Xi}^\mu \)s through \( b \) and the \( \rho^{j,k}_\mu \)s– which may be evaluated by direct substitution.

The adiabatic behavior of the system in the strong magnetic regime may now be read in the first term of expansion \([\mathbb{II}]\). The fast and slow freedoms decouple up higher order in \( l_B \). The fast freedom is frozen in one of the harmonic oscillator eigenstates of the adiabatic invariant \( J \). While ‘spinning’, the particle drifts on the surface \( \Sigma \). The drifting is Hamiltonian: the configuration space \( \Sigma \) appears now as the phase space of the slow freedom; the magnetic field norm \( b(\bar{\Xi}^1, \bar{\Xi}^2) \) evaluated in the couple of conjugate variables \( \bar{\Xi}^1 \) and \( \bar{\Xi}^2 \) is the Hamiltonian operator governing the slow motion (see ref. \([9]\)).

The situation is substantially analog to the motion on plane \([\mathbb{II}],\) the metric appearing only in the evaluation of the magnetic field norm. The crucial difference is that in a non trivial geometrical background a constant value of \( b \) does not produce in general the slow variables as exact constants of motion.

IV. COUPLING TO BACKGROUND GEOMETRY

We now study the higher order corrections to the effective motion of the charged particle. To this task we proceed by the so called averaging method (see appendix) that is by performing a series of near-identity unitary transformations separating, order by order in \( l_B \), the fast freedom from the slow one. First of all a little preparation is necessary.

We re-express all the quantities appearing in the expansion \([\mathbb{II}]\) in terms of the new canonical variables \( \bar{\Pi}_i \) and \( \bar{\Xi}^\mu \). This produces the replacements of all the curved space indices \( \mu, \nu, ... \) by the flat space indices \( i, k, ... \) . Every ‘general covariant’ index \( \mu \) is replaced by a ‘Galilei covariant’ index \( i \) according to the usual rules \( v_i = e_i^\mu v_\mu, \tilde{v}^i = e^i_\mu \tilde{v}^\mu \) etc.
As a second steep it is useful to work out a few basic geometrical identities holding in every Darboux frame. These will be precious in bringing the adiabatic expansion in an explicit covariant form. By derivating the relation between metric determinant and magnetic field norm we obtain: \((\partial_\rho \gamma^{\mu \nu}) \gamma^{\nu \lambda} \varepsilon_{\kappa \lambda} - (\partial_\rho \gamma^{\nu \kappa}) \gamma^{\mu \lambda} \varepsilon_{\kappa \lambda} = 2b(\partial_\rho b)\varepsilon^{\mu \nu}\). Contracting with \(\varepsilon_{\mu \nu}\) and rewriting in terms of flat space indices yields

\[
\Gamma^j_{ij} = -b^{-1}(\partial_i b) \tag{12}
\]

(which is the usual relation \(\Gamma^\nu_{\mu \nu} = \partial_\mu \log g^{1/2}\) evaluated in a Darboux frame). By multiplying the relation by itself, contracting and rewriting in terms of flat space indices we also obtain

\[
\Gamma^k_{ij}\Gamma^j_{ik} - \Gamma^k_{ij}\Gamma^j_{ik} - \Gamma^j_{ik}\Gamma^k_{ij} - 2\Gamma^j_{ii}\Gamma^k_{jk} - \Gamma^k_{ii}\Gamma^j_{jj} = 0 \tag{13}
\]

No other general relations hold among the various contractions of the Christoffel symbols.

We proceed now by evaluating the contributions produced by the zero, first and second order terms of \(\mathcal{H}\). Everywhere in what follows equation (13) is used to eliminate \(\Gamma^k_{ii}\Gamma^j_{jj}\) in favor of the other four possible contractions of the Christoffel symbols.

**Second order contribution from \(\mathcal{H}^{(0)}\)**

We start the averaging procedure considering the second order contribution produced by \(\mathcal{H}^{(0,2)}\). To this task it is necessary to re-express the non-geometrical quantities \(\rho^j_{i,k}\) in terms of the spin-connection \(\omega^j_{i,k} = (\nabla_{e_k} e_i) \cdot e_j\) and the Christoffel symbols \(\Gamma^k_{ij}\). A quick computation yields

\[
b^{1/2} \rho^j_{i,k} = \omega^j_{i,k} + \frac{1}{2} \delta^j_k \Gamma^l_{kl} - \Gamma^j_{ik} \tag{14}
\]

We recall that the spin-connection is completely antisymmetric in the indices \(i\) and \(j\). In two dimensions it may therefore be rewritten in terms of a \(U(1)\) gauge potential as \(\omega^j_{i,k} = \omega_k \varepsilon_{ij}\).
A point dependent rotation by an angle $\chi(\xi)$ of the zwei-beinen $e_i^\mu$ produces the gauge transformation $\omega_k \to \omega_k + \partial_k \chi$. By replacing $\rho_{ij}^k$ in $\mathcal{H}^{(0,2)}$ according to (14) the second order contribution to the perturbative expansion is readily evaluated by the formula (A1)

$$\mathcal{H}^{(0)} \longrightarrow \left( -\varepsilon^{ij} \Gamma^k_{ik} \omega_j + \frac{1}{4} \Gamma^k_{ij} \Gamma^j_{ik} + \frac{1}{2} \Gamma^j_{ij} \Gamma^k_{ik} - \frac{3}{4} \Gamma^j_{ii} \Gamma^k_{jk} \right) J^2$$

$$+ \frac{3}{16} \Gamma^k_{ij} \Gamma^j_{ik} - \frac{3}{16} \Gamma^j_{ii} \Gamma^k_{jk}$$

Quite surprisingly a term explicitly depending on $\omega_k$ survives.

**Second order contribution from $\mathcal{H}^{(1)}$**

The first order term of expansion (11) is cubic in the kinematical momenta, $\mathcal{H}^{(1)} = -\frac{1}{2} b^{1/2} (\partial_k \gamma^{ij}) \varepsilon^{lk} \bar{\Pi}_i \bar{\Pi}_k \bar{\Pi}_j$. As shown in the appendix, this contribute the adiabatic expansion an $l_B^2$ order term that may be directly evaluated by means of (A2). The only necessary preparation is that of re-expressing $\partial_k \gamma^{ij}$ in terms of Christoffel symbols. This is done by rewriting $\partial_k \gamma_{\mu\nu}$ in terms of $\partial_k \gamma^{\mu\nu}$ in the definition $\Gamma^\rho_{\mu\nu}$ and by symmetrizing. The contraction with the zwei-beinen produces

$$\partial_k \gamma^{ij} = -(\Gamma^j_{jk} + \Gamma^j_{ik})$$

By substituting in (A2) we obtain

$$\mathcal{H}^{(1)} \longrightarrow \left( -\frac{3}{4} \Gamma^k_{ij} \Gamma^j_{ik} - \frac{3}{4} \Gamma^k_{ij} \Gamma^j_{ik} - \frac{3}{4} \Gamma^j_{ij} \Gamma^k_{ik} + \frac{3}{2} \Gamma^j_{ii} \Gamma^k_{jk} \right) J^2$$

$$- \frac{3}{16} \Gamma^k_{ij} \Gamma^j_{ik} + \frac{7}{16} \Gamma^k_{ij} \Gamma^j_{ik} + \frac{1}{16} \Gamma^j_{ij} \Gamma^k_{ik} + \frac{3}{8} \Gamma^j_{ii} \Gamma^k_{jk}$$

**Second order contribution from $\mathcal{H}^{(2)}$**

A similar computation have to be performed for the second order term of the perturbative expansion, $\mathcal{H}^{(2)} = \frac{1}{4} (\partial_{\mu} \partial_{\nu} \gamma^{ij}) \varepsilon^{mk} \varepsilon^{nl} \Pi_i \Pi_k \Pi_j$. This time is necessary to re-express the
second order derivatives of the inverse metric in terms of the Christoffel symbols and their derivatives. This is simply obtained by derivating (16)

$$\partial_m \partial_n \gamma^{ij} = -\partial_m \Gamma^i_{nj} - \partial_m \Gamma^j_{ni} + \Gamma^k_{im} \Gamma^j_{hn} + \Gamma^i_{mh} \Gamma^j_{nh} + \Gamma^j_{mh} \Gamma^i_{nh}$$

(18)

Recalling the definition of the scalar curvature

$$R = \partial_i \Gamma^i_{jj} - \partial_j \Gamma^j_{ii} + \Gamma^k_{ii} \Gamma^k_{jk} - \Gamma^k_{ij} \Gamma^k_{ik}$$

formula (A1) yields the second order contribution produced by \( H^{(2)} \)

$$H^{(2)} \longrightarrow \left( \frac{1}{4} R - \frac{1}{4} \partial_i \Gamma^i_{ij} + \frac{3}{4} \Gamma^k_{ij} \Gamma^k_{ij} + \frac{1}{2} \Gamma^k_{ij} \Gamma^j_{ik} - \frac{1}{4} \Gamma^j_{ij} \Gamma^k_{ik} - \frac{1}{2} \Gamma^k_{ii} \Gamma^k_{jk} \right) J^2$$

$$- \frac{1}{16} R - \frac{5}{16} \partial_i \Gamma^i_{ij} + \frac{3}{16} \Gamma^k_{ij} \Gamma^k_{ij} + \frac{1}{4} \Gamma^k_{ij} \Gamma^j_{ik} + \frac{1}{16} \Gamma^j_{ij} \Gamma^k_{ik} + \frac{1}{8} \Gamma^j_{ii} \Gamma^k_{jk}$$

(19)

The two terms still containing derivatives of the Christoffel symbols may be expressed in terms of derivatives of the magnetic field norm \( b \) and contractions of the \( \Gamma^k_{ij} \)'s simply by derivating equation (12), \( \partial_i \Gamma^i_{ij} = -b^{-1} \triangle b + b^{-2} |\nabla b|^2 + \Gamma^j_{ii} \Gamma^k_{jk} \).

The effective Hamiltonian describing the motion of a charged particle to the second order in the adiabatic parameter \( l_B \) is finally obtained by adding to \( b(\bar{\Xi})J \) the contributions (15), (17), (19) as well as the term \( -\nabla b/4b + |\nabla b|^2/8b^2 \). As one have to be expect all the contractions of the \( \Gamma^k_{ij} \) but \( \Gamma^j_{ii} \Gamma^k_{jk} \) cancel. This may be rewritten in terms of \( \nabla b \) by means of (12). We obtain

$$\mathcal{H} = \frac{bJ}{l_B^2} \left( \frac{1}{4} R + \frac{\nabla b}{b} \times + \frac{1}{4} \frac{\Delta b}{b} - \frac{3}{4} \frac{|\nabla b|^2}{b^2} \right) J^2$$

$$- \frac{1}{16} R + \frac{1}{16} \frac{\Delta b}{b} - \frac{1}{16} \frac{|\nabla b|^2}{b^2} + O(l_B)$$

(20)

All the functions are evaluated in the couple if conjugate operators \( \bar{\Xi}^1 \) and \( \bar{\Xi}^2 \). As before this expression has to be interpreted as the effective Hamiltonian describing the motion of the slow freedom while the particle is frozen in one of the \( J \) eigenstates. The second term is
the correction surviving in the classical limit while the remaining ones are of a pure quantal nature.

Even if our computation has been carried out in a Darboux coordinate frame, Eq. 20 is explicitly covariant so that we are free to transform back to the original –arbitrary– coordinates $x^\mu$. The price to pay is that of dealing with non-canonical operators, the Hamiltonian getting evaluated in $X^\mu = x^\mu(\bar{\Xi})$. These ‘guiding center variables’ satisfy in fact the non-canonical commutation relations $[X^2, X^1] = il_B^2 b^{-1}(X)$ (compare references [6,7]).

The effective dynamics is sensitive to the background scalar curvature. This coupling is particularly relevant when the magnetic two-form is proportional –in arbitrary coordinates– to the volume two-form, $b_{\mu\nu} = l_B^{-2} \sqrt{g} \varepsilon_{\mu\nu}$. $g_{\mu\nu}$ and $b_{\mu\nu}$ define then a Kähler structure on $\Sigma$. The particle interacts only with the surface. The magnetic force becomes the ‘geometric gravitational’ force of Cangemi and Jackiw [3]. In the strong magnetic regime the effective Hamiltonian driving the slow motion is proportional to the scalar curvature. In the semiclassical regime test particles drift along the line of constant curvature of the surface $\Sigma$.

The effective dynamics is coupled to the background spin-connection as well. The coupling is not explicitly gauge invariant. A gauge transformation $\omega_k \rightarrow \omega_k + \partial_k \chi$ adds the term $l_B^2 b^{-1} \varepsilon^{ij}(\partial_i b)(\partial_j \chi) J^2$ to the Hamiltonian. Gauge invariance may nevertheless be restored by the unitary transformation $U = e^{ib^{-1}J\chi}$. The second order term $-ib^{-1}[b(\bar{\Xi}), \chi(\bar{\Xi})] J^2$ produced in this way brings (20) in the original form.

Last but not the least, it is worth to mention that expansion (20) yields the correct flat limit [6,7] supplying a full canonical derivation of it.
A typical situation of interest in 2+1 gravity is the motion around a conical singularity. As an example we consider therefore a charged particle on a conical surface subjected to an axisymmetric magnetic field decreasing as the inverse of the distance from the vertex. The problem is explicitly solvable allowing a check of our strong magnetic field expansion.

The cone is parameterized by the distance $\rho$ from the vertex, ranging $0 \leq \rho \leq +\infty$, and the angle $\phi$, ranging $0 \leq \phi \leq 2\pi$; the points $\phi = 0$ and $\phi = 2\pi$ are identified. In these coordinates metric and magnetic two-form take the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \rho^2 \end{pmatrix}, \quad b_{\mu\nu} = \frac{1}{l_B^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(21)

where $\alpha$ is the conical angle; setting $\alpha = 1$ brings the cone in the Euclidean plane. Though the curvature of $\gamma_{\mu\nu}$ vanishes, the geometry of the space is non trivial. The spin-connection of the surface reads $\omega_{\mu} = (0, \alpha)$ and can not be gauged away for non-integer values of $\alpha$.

We first consider the exact solution. In order to have a deeper insight into the problem we focus on the classical motion, the discussion of quantum problem proceeding essentially along the same lines. Choosing the vector potential as $a_{\mu} = (0, l_B^{-2} \rho)$ the Hamiltonian of the system writes

$$\mathcal{H} = \frac{1}{2} p_\rho^2 + \frac{1}{2 \alpha^2 \rho^2} \left( p_\phi - \frac{\rho}{l_B^2} \right)^2$$

(22)

Given to the axial symmetry the momentum $p_\phi$ is conserved, $\{\mathcal{H}, p_\phi\} = 0$, and can be replaced by its constant value $L$. The radial motion of the system takes place in the effective Keplerian potential

$$V_{eff}(\rho) = \frac{L^2}{2 \alpha^2 \rho^2} - \frac{L}{\alpha^2 l_B^2 \rho} + \frac{1}{\alpha^2 l_B^4}$$

(23)
where $L/\alpha^2 l_B^2$ appears as an attractive Newton constant, $L/\alpha$ as the angular momentum and the whole spectrum in shifted by the energy $1/\alpha^2 l_B^4$. The presence of the magnetic field produces bound states in the system. There is no need to go through the well known solution of this problem, we focus instead on the qualitative behavior of the system in the strong magnetic regime. For small values of $l_B$ the minimum $\bar{\rho} = L l_B^2$ of the effective potential becomes extremely deep and narrow. $V_{eff}$ is very well approximated by a harmonic oscillator centered in $\bar{\rho}$ and with frequency $\omega = 1/\alpha L l_B^4$. While rotating around the axis of the cone at a distance $\bar{\rho}$ the particle performs therefore very rapid oscillations. The result is that of a very thin and dense spiral wrapping around an orbit of constant radius. Neglecting the rapid oscillation, the effective angular velocity may be evaluated by eliminating $L$ in favor of $\bar{\rho}$ in the relation $L = p_\phi = \alpha^2 \rho^2 \dot{\phi}$. This yields

$$\dot{\phi} \approx \frac{1}{\alpha^2 l_B^2 \bar{\rho}}$$ (24)

The angular velocity distribution gives informations on the conical angle $\alpha$.

We come now to the strong magnetic expansion (20). Observe that the coordinates $\rho$ and $\phi$ are already of Darboux’s type. The rapid oscillation of the particle have obviously to be identified with the freedom $\Pi_\rho - \Pi_\phi$ while the drift on the cone with the motion of the guiding center variables $R = \rho + l_B \Pi_\phi$ and $\Phi = \phi - l_B \Pi_\rho$. The coordinates $\rho$, $\phi$ and the couple of conjugate variables $R$, $\Phi$ parameterize two phase-space surfaces very close to each other and may be confused when order higher than $l_B$ are neglected. The Hamiltonian driving the effective motion is immediately obtained from (20) by evaluating gradient and Laplacian of the magnetic field norm $b(\rho) = 1/(\alpha l_B^2 \rho)$; the Galilei covariant components of the spin-connection are given by $\omega_i = (0, 1/\rho)$;

$$\mathcal{H}_{eff} = \frac{1}{\alpha l_B^2 R} J - \frac{1}{2R^2} j^2 + ...$$ (25)
The angle $\Phi$ does not appear in the Hamiltonian so that $R$ is a second constant of motion besides $J$. The particle moves around the axis at the constant value of the radius $\bar{R}$. The angle $\Phi$ evolves linearly in time according to Hamilton equations $\dot{\Phi} = J/\alpha \bar{R} + \mathcal{O}(l_B^2)$. The adiabatic invariants $J$ and $R$ are directly related by the conservation of energy. Recalling that the classical system is in the adiabatic regime for small values of the total energy we re-obtain an angular velocity distribution with the behavior $(24)$.

**VI. CONCLUSIONS**

The purpose of this paper was that of showing how is possible to set down a systematic canonical perturbative analysis for the motion of charged particles in a curved background geometry. This bridges the gap between the classical canonical theory and the non-canonical averaging methods traditionally employed in the classical analysis. Most important, the method allows a direct discussion of the quantum case extending to this realm the whole classical ‘guiding center’ picture. The aim is essentially achieved by means of Darboux transformations, standard averaging methods and elementary differential geometry. For the shake of simplicity we restricted our attention to 2+1 dimensions. Aside from its importance in the discussion of the whole 3+1 dimensional problem, the 2+1 dimensional system is already of a certain applicative importance in itself. An immediate application concerns the investigation of the non-minimal coupling of Cangemi and Jackiw in a Wick-rotated two dimensional gravity. More in general, Hamiltonian $(24)$ gives us immediate informations on how wave-functions and eigenvalues of an electron in a quantum-Hall like device are modified when a small inhomogeneity of the magnetic field or of the thin film geometry are introduced. The electron behaves like a one degree of freedom system having the thin film –the spatial surface $\Sigma$– as ‘phase space’. The fast freedom is still frozen in a harmonic oscillator eigenstate.
and the discussion of section III indicates how the harmonic oscillator eigenfunctions have to be constructed. The peculiar way the slow freedom couples to the ‘phase space’ scalar curvature and spin-connection is particularly intriguing and deserve further investigation. Another important issue concern the convergence of the perturbative expansion, which is expected to be in general an asymptotic series. We conclude by pointing out that considering more spatial dimensions produces other interesting phenomena –the coupling of the effective dynamics of the new freedoms with geometry induced gauge structures– that can be described essentially by the same formalism. The inclusion of spin is also quite immediate. The restriction to 2+1 dimensions allowed us to single out the effective coupling with the background geometry without mixing it with phenomena of a different nature. The effective motion in 3+1 dimensions and the inclusion of spin will be considered in future publications.

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APPENDIX: AVERAGING AROUND A HARMONIC OSCILLATOR

The introduction of a suitable set of variables reduces the study of the effective motion of a charged particle to the discussion of a Hamiltonian of the form

\[ \mathcal{H} = \alpha J + \epsilon \alpha^{ijk} \Pi_i \Pi_k \Pi_j + \epsilon^2 \alpha^{ijkl} \Pi_i \Pi_k \Pi_l \Pi_j + \ldots \]

As in the body of the paper, \( J = \frac{1}{2}(\Pi_1^2 + \Pi_2^2) \), and \( \Pi_1, \Pi_2 \) are a couple of conjugate variables, \([\Pi_1, \Pi_2] = i\). \( \epsilon \) is a small parameter. The coefficients appearing in the expansion are allowed to depend on slow variables. The self-adjointness of \( \mathcal{H} \) requires \( \alpha^{ijk} = \alpha^{jik} \) and \( \alpha^{ijkl} = \alpha^{jilk} \). For a charged particle in a strong magnetic field the coefficients \( \alpha, \alpha^{ijk}, \alpha^{ijkl}, \ldots \), are quite complicated expressions involving the spin-connection, the metric tensor and their derivatives evaluated in the slow guiding center variables \( \Xi^1 \) and \( \Xi^2 \), \([\Xi^2, \Xi^1] = i\epsilon^2 \). Very handy formulas will be worked out in this appendix in order to maximally simplify the manipulation of this expressions.

As far as \( \epsilon \) is set equal to zero dynamics is described by \( h^{(0)} = \alpha J \). The system behaves as a harmonic oscillator with frequency depending on the non-dynamical parameters \( \Xi^i \). A nonzero value of \( \epsilon \) turns the perturbation on, making, on the same time, the guiding center operators in a couple of conjugate dynamical variables. In order to extract the effective dynamical content of the theory to the various order in the perturbative parameter \( \epsilon \) we will subject the system to a series of near-identity unitary transformations. These are chosen in such a way that the various terms of the perturbative expansion depend on \( \Pi_1 \) and \( \Pi_2 \) only though \( J \) and it powers. This makes \( J \) into an adiabatic invariant—a quantity conserved up to higher order of some power of \( \epsilon \)— and allows to identify the Hamiltonian driving the effective motion of the slow variables in correspondence of every value taken by \( J \). The technique is based essentially on the identity \( e^{ia}\mathcal{H}e^{-ia} = \mathcal{H} + i[a, \mathcal{H}] - \frac{1}{2!}[a, [a, \mathcal{H}]] + \ldots \), where
\( a = 1 + \epsilon a^{(1)} + \epsilon^2 a^{(2)} + \ldots \) is the generator of a nearly-identity unitary transformation. The self-adjoint operators \( a^{(1)}, a^{(2)}, \) etc. have to be chosen order by order in such a way that the desired conditions are matched.

We start by the order \( \epsilon \) of the expansion: \( h^{(1)} \equiv \alpha^{ijk} \Pi_i \Pi_k \Pi_j \). Note that since \( \Pi_i \Pi_k \Pi_j + \Pi_j \Pi_k \Pi_i \) is completely symmetric in the indices \( i, j \) and \( k \) only the completely symmetric part of \( \alpha^{ijk} \) matters. We can therefore assume the complete symmetry of the \( \alpha^{ijk} \). It is then easy to verify that the choice

\[
a^{(1)} = -\frac{1}{3} \alpha^{-1} \left( \alpha^{ij} + 2\delta^{ij} \alpha^{hhl} \right) \varepsilon_{lk} \Pi_i \Pi_k \Pi_j
\]

produces the counterterm \( i[a^{(1)}, h^{(0)}] = -h^{(1)} \). The first order term of the transformed expansion vanishes identically. The operation is nevertheless not painless. The transformation contribute in fact the second order term \( h^{(1,2)} = \frac{i}{2} [a^{(1)}, h^{(1)}] \). This can be evaluated in

\[
h^{(1,2)} = \frac{3}{2} \left( \frac{\alpha^{ihh} \alpha^{jkl}}{\alpha} + \frac{\alpha^{jhh} \alpha^{ikl}}{\alpha} - \frac{\alpha^{ijk} \alpha^{hhl}}{\alpha^2} - 2 \frac{\delta^{ij} \alpha^{hlg} \alpha^{klg}}{\alpha} \right) \Pi_i \Pi_k \Pi_j
\]

The problem is re-conduced to the discussion of the second order term.

Focus therefore on \( h^{(2)} \equiv \alpha^{ijkl} \Pi_i \Pi_k \Pi_l \Pi_j \). The symmetrization in the various couples of indices can still be performed producing contributions of the form \( \alpha^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \) etc., not depending on \( \Pi_1 \) and \( \Pi_2 \). Nevertheless a quite handy expression can already be obtained by assuming only the symmetrization of the first and second couples of indices; the case we have to deal with. A few computation show then that the right choice to make the second order of the perturbative expansion to depend only on powers of \( J \) is

\[
a^{(2)} = -\frac{1}{8} \alpha^{-1} \left[ \alpha^{ijkh} + \delta^{ij} \left( \alpha^{kghg} + \frac{1}{2} \alpha^{khgg} \right) \right] \varepsilon^{hl} \Pi_i \{ \Pi_k, \Pi_l \} \Pi_j
\]

The second order term \( i[a^{(2)}, h^{(0)}] \) produced by this transformation combines with \( h^{(2)} \) in such a way to give the final contribution to the perturbative expansion.
\[ \alpha^{ijkl} \Pi_i \Pi_k \Pi_j \rightarrow \left( \alpha^{ijij} + \frac{1}{2} \alpha^{iijj} \right) J^2 - \frac{1}{4} \alpha^{ijij} + \frac{5}{8} \alpha^{iijj} \]  \hspace{1cm} (A1)

No matter how complicated is \( h^{(2)} \), formula \( A1 \) allows to immediately write down the contribution to the effective dynamics by evaluating a few contractions of the coefficients \( \alpha^{ijkl} \).

The first application of \( A1 \) is the second order contribution produced by \( h^{(1)} \) through \( h^{(1,2)} \).

A brief computation yields the quite compact formula

\[ \alpha^{ijk} \Pi_i \Pi_k \Pi_j \rightarrow \left( \frac{3}{2} \frac{\alpha^{ijk} \alpha^{ijk}}{\alpha} + \frac{9}{4} \frac{\alpha^{iik} \alpha^{jjk}}{\alpha} \right) J^2 - \frac{5}{8} \frac{\alpha^{ijk} \alpha^{ijk}}{\alpha} + \frac{3}{16} \frac{\alpha^{ijk} \alpha^{ijk}}{\alpha} \]  \hspace{1cm} (A2)

Again, the contribution to the effective dynamics produced by \( h^{(1)} \) may be obtained through \( A2 \) by evaluating a few contraction on the square of the coefficients \( \alpha^{ijk} \).
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