Superfluorescent decay in quasiballistic disordered systems.

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Abstract
Superfluorescent decay in a weakly disordered slab is considered. By solving the Maxwell-Bloch equations we show, that the cooperation number increases with increasing of the mean free path. Superfluorescent impulse is emitted at a small angle to the slab, isotropically in a plane direction.

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Introduction.
During the spontaneous emission of uncorrelated exited atoms, their dipole moments interact each other via the emitted electromagnetic field. If the lifetime of exited state is long enough, the radiation can induce the correlation between the group of large number $N_c$ (so called cooperation number) atoms. As a result of correlation, $N_c$ exited atoms emit as one macroscopic dipole. Thus, the decay of the uncorrelated exited atoms, which starts with the low level of emitted radiation during the developing of macroscopic dipole, is continued by the emission of this macroscopic dipole. At this last stage the duration of spontaneous decay decreases by the factor $N_c^{-1}$ and the maximum of intensity of emitted impulse increases by the factor $N_c^2$ compare that parameters of spontaneous decay of a single atom [1]. The total process is called superfluorescence [2].

In extended system the cooperation number depends on the geometry of the sample. For a example, in a well studied pencil-like sample it is
$N_c \sim \rho \lambda^2 L$ in one mode approximation [2], [3]. Here $\rho$ is density of the active atoms. $L$ is the length of the sample, which diameter is much larger than the wavelength of radiation $\lambda$. Maximum of impulse intensity is $\sim N_c^2 \times \frac{N_c}{N}$, where $N$ is total number of the exited atoms. The factor $\frac{N_c}{N} \gg 1$ can be interpreted as a number of macroscopic dipoles in the sample.

Correlation between the exited atoms in this case develops due to the free propagation of the electromagnetic field along the pencil-like sample. During the propagation, the field can be absorbed and re-emitted. These processes put an upper limit on the geometrical size $L$, and therefore on $N_c [2], [3]$. The case, when the superfluorescence occurs in a disordered system, was considered in [4]. It was shown that in the disordered slab of the thickness $L \gg l$ ($l$ is the mean free path of the radiation due to scattering by the static fluctuations of dielectric constant) the cooperation number is $N_c \sim \rho \lambda^2 \frac{L^2}{l}$. Here factor $\frac{L^2}{l}$ is the length of diffusive trajectory of the radiation. Scattering prevents the escape of the radiation from a system with the dimensions larger than the mean free path, therefore the cooperation number increases with the increasing of disorder up to the upper limit where absorption and re-emission starts. The maximum $N_c$ is of order of that in a pure case.

The cooperation number is a random quantity in the disordered system. Approximations adopted in [4] in order to obtain the analytical solution allows calculation only of the average value of the cooperation number.

Here we consider the superfluorescence of the weakly disordered slab with the thickness $L$ less than the radiation mean free path. We show that the cooperation number in this system is $N_c \sim \rho \lambda^2$. The correlation develops due to the propagation of the field along slab. Fluctuations of the dielectric constant scatter radiation out of volume, which contains the active atoms, thus decreasing the cooperation number. Superfluorescent impulse is emitted at small angle to the slab and isotropically in a plane direction.

**Maxwell-Bloch equations.**

We consider scalar version of problem. The coupling between the polarization density $\frac{1}{2} \left\{ e^{i\omega t} P(\vec{r}; t) + e^{-i\omega t} P^*(\vec{r}; t) \right\}$, the population difference density $\Delta N(\vec{r}; t)$, and the field $\frac{1}{2} \left\{ e^{i\omega t} E(\vec{r}; t) + e^{-i\omega t} E^*(\vec{r}; t) \right\}$ can be described by the classical Maxwell-Bloch equations. In this approach amplified spontaneous emission noise is neglected, which is a good approximation for the superfluorescence [4]. $P(\vec{r}; t)$ and $E(\vec{r}; t)$ are slowly time-varying complex quantities. $\omega = c k$ is the atomic frequency.

First two Maxwell-Bloch equations have the form [4]
\[ \frac{d}{dt} P (\vec{r}; t) = \frac{i |\mu|^2}{\hbar} \Delta N (\vec{r}; t) E (\vec{r}; t) \]  \quad (1)

\[ \frac{d}{dt} \Delta N (\vec{r}; t) = -\frac{i}{2\hbar} \{ P^* (\vec{r}; t) E (\vec{r}; t) - P (\vec{r}; t) E^* (\vec{r}; t) \} . \quad (2) \]

Here \( \mu \) is the electric dipole moment.

It is assumed that the population inversion relaxation time and the dephasing time are larger than the delay time of the superfluorescent impulse \( t_0 \), which will be defined later in the equation (9). We also neglect inhomogeneous broadening.

The population difference \( \Delta N (\vec{r}; t) \) and the polarization \( \frac{P(\vec{r}; t)}{\mu} \) in the equations (1) and (2) are the components of the local Bloch vector averaged over scales smaller than the wavelength of the radiation \( \lambda \). We choose the initial conditions \( P (\vec{r}, t = 0) = 0 \) and \( \frac{P (\vec{r}, t = 0)}{\mu} P^* (\vec{r}'; t = 0) = \rho |\mu|^2 \delta (\vec{r} - \vec{r}') \), which describe the initially uncorrelated atoms. The line means averaging over the initial state. \( \vec{r} = (\vec{p}, z) \), \( \vec{p} \) is in-plane coordinate. The population difference and the polarization are confined in a region of the slab \( |z| \leq \frac{L}{2} \).

The field wave equation for the slow time-varying component \( E (\vec{r}; t) \) has the form

\[ i \epsilon (\vec{r}) \frac{d}{dt} E (\vec{r}; t) - \left\{ -\frac{c^2}{2\omega} \Delta - \frac{\omega \epsilon (\vec{r})}{2} \right\} E (\vec{r}; t) = 2\pi \omega P (\vec{r}; t) . \quad (3) \]

Here \( \epsilon (\vec{r}) = 1 + \delta \epsilon (\vec{r}) \) is the dielectric function of the medium which contains active atoms. \( \delta \epsilon (\vec{r}) \) causes scattering.

**Calculation of cooperation number.**

We consider scattering by the small fraction of the particles of the size less than \( \lambda \) embedded into the active medium. This is experimentally relevant case and for the theoretical treatment this case has an advantage that \( l \) is equal to the transport mean free path.

Averaged over random positions of the scattering particles equation for the field has the form of the equation (3) with substitution of dumping \( \frac{1}{kt} \) for \( \delta \epsilon (\vec{r}) \).  

To consider superfluorescence we neglect time derivative in equation for the field at the very beginning. This usual approximation means that the time of the escape of radiation \( \frac{t}{c} \) from the system is smaller, than the time of exchange energy between field and atoms.
Solving equation (1) we obtain equation for the field, averaged over the disorder $\langle E(\vec{r}; t) \rangle$

$$\left\{ \Delta + k^2 + \frac{ik}{l} \right\} \langle E(\vec{r}; t) \rangle = 4\pi k^2 P(\vec{r}; t = 0) + \frac{4\pi i k |\mu|^2}{\hbar} \int_0^t dt \Delta N(\vec{r}, t) \langle E(\vec{r}; t) \rangle$$

(4)

To find population difference from equation (2) we need to solve equation (4) inside the slab. Below we consider the case when $l > kL^2$. This allows to neglect dependence of $\Delta N(\vec{r}, t)$ on coordinate $\vec{r}$. We assume that at initial moment $\Delta N(\vec{r}, t = 0) = \rho$, where $\rho$ is density of the active atoms.

The quantity $\langle E(\vec{r}; t) \rangle \langle P^*(\vec{r}; t) \rangle$ weakly depends on coordinate when mean free path is larger than $l > kL^2$. This allows to neglect the dependence of $\Delta N(\vec{r}, t)$ during the decay.

From equation (4) it follows that the time dependence of field is determined by the quantity

$$\chi \equiv \frac{2}{\rho \omega \tau_0^2} \int_0^t dt \Delta N(t)$$

(5)

where $\tau_0 \equiv \sqrt{\frac{\hbar}{2\pi \rho \omega |\mu|}} = \sqrt{\frac{1\pi \tau_{rad}}{3\omega \rho \lambda^2}}$ is the characteristic time of energy exchange between the field and the atomic system \[. \tau_{rad}^{-1} \equiv \frac{8\pi^2 |\mu|^2}{3\bar{\hbar} \lambda^3} \] is the radiative decay time of a single atom.

Performing Laplace transformation of equation (4) over $\chi$ and Fourier transformation over in plane coordinate $\vec{p}$ we obtain solution inside the slab

$$\langle E(\vec{q}; s) \rangle = \frac{2\pi k^2}{is \left[ p + \frac{KL}{2l} \left( 1 - \frac{kL}{s} \right) \right]} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz' P(\vec{q}, z'; t = 0),$$

(6)

where $\langle E(\vec{q}; s) \rangle = \int d^2 \vec{p} \exp(ip\vec{q}) \int_0^\infty d\chi \exp(-s\chi) \langle E(\vec{r}; \chi(t)) \rangle$,

$p = \sqrt{k^2 - q^2}, \ Im(p) > 0$.

The inverse Laplace transformation of (6) gives an expression

$$\langle E(\vec{q}; \chi) \rangle = \frac{2\pi k^2}{i \left[ p + \frac{KL}{2l} \right]} \exp \left[ \frac{Lk^2 \chi}{2 \left( p + \frac{KL}{2l} \right)} \right] \int_{-\frac{L}{2}}^{\frac{L}{2}} dz' P(\vec{q}, z'; t = 0).$$

(7)
Performing the inverse Fourier transformation and taking into account the initial conditions for the polarization density, we obtain

$$\langle E(\vec{r}; \chi) \rangle \langle P^*(\vec{r}; \chi) \rangle \sim - \frac{L}{8k\lambda^2} \exp(2k\lambda). \quad (8)$$

Expression (8) gives asymptotic for $k\lambda > 1$.

Equation (2) and expression (8) allow to solve for the population difference. Neglecting time-dependence of slow varying pre-factor in expression (8) we obtain for the population difference

$$\Delta N(\vec{r}, t) = \rho \tanh \frac{t_0 - t}{2\tau_N}, \quad (9)$$

where $\tau_N \equiv \frac{2\tau_{\text{rad}}}{NC}$ and cooperation number is equal

$$NC = 6 \rho \lambda^2 l. \quad (10)$$

It is of order of number of active atoms in a tube with cross section $\lambda^2$ and length $l$.

The delay time $t_0 \approx \tau_N ln \frac{\rho \lambda^2 l}{L} \gg \tau_N$ is calculated by matching (8) and (9) with initial case when $\chi = \frac{2t_0}{\omega_0}$ and assuming that at the beginning of collective decay ($k\lambda \sim 1$) population difference does not change.

Let us note that expression (8) coincides with the standard expression for the population difference in Markovian theory of superfluorescence.

To obtain the expression for emitted radiation we must solve equation (4) for $|z| > \frac{L}{2}$ and given time-dependent population difference (9).

Intensity of radiation from a unite volume of slab can be represented as

$$I(\vec{r}; t) = I_0(\vec{r}) \left[ \frac{2l}{L} \sin \theta \frac{1}{1 + \frac{2l}{L} \sin \theta} \right]^2 \left[ \frac{\cosh \frac{t_0}{\tau_N}}{\cosh \frac{t - t_0}{\tau_N}} \right]^{1 + \frac{2}{L} \sin \theta}. \quad (11)$$

Here $I_0(\vec{r})$ is the intensity of the ordinary fluorescence of a unite volume with exited atoms density $\rho$ at distance $\vec{r}$. It is isotropic function $I_0(\vec{r}) \sim r^{-2}$.

The second factor in (11) accounts for the mean free path of radiation in the medium, $\sin \theta = \frac{L}{r}$.

The last factor in (11) describes the angular and time dependences of superfluorescent impulse.
Conclusion.
We calculated the cooperation number and the intensity of cooperative decay in weakly disordered slab. Expressions (10) and (11) are the main results of the paper.
It is shown that volume of the cooperating regions is proportional to volume of a tube with length, which is equal mean free path, and of cross-section dimension of order of the wavelength. The maximum of radiation is emitted in a small angle to the surface of slab isotropically in a plane direction.
While calculation of (10) and (11) was performed under the condition \( l > kL^2 \) it is reasonable to assume, that the scattering limits the cooperation number until \( l \approx L \).

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