Second-order formalism for 3D spin-3 gravity

Ippei Fujisawa and Ryuichi Nakayama

Division of Physics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan
E-mail: ifujisawa@particle.sci.hokudai.ac.jp and nakayama@particle.sci.hokudai.ac.jp

Received 10 September 2012, in final form 13 December 2012
Published 11 January 2013
Online at stacks.iop.org/CQG/30/035003

Abstract
A second-order formalism for the theory of 3D spin-3 gravity is considered. Such a formalism is obtained by solving the torsion-free condition for the spin connection $\omega^a_\mu$, and substituting the result into the action integral. In the first-order formalism of the spin-3 gravity defined in terms of $SL(3, R) \times SL(3, R)$ Chern–Simons (CS) theory, however, the generalized torsion-free condition cannot be easily solved for the spin connection, because the vielbein $e^a_\mu$ itself is not invertible. To circumvent this problem, extra vielbein-like fields $e^a_{(\mu \nu)}$ are introduced as a functional of $e^a_\mu$. New sets of affine-like connections $\Gamma^A_\mu_N$ are defined in terms of the metric-like fields, and a generalization of the Riemann curvature tensor is also presented. In terms of this generalized Riemann tensor the action integral in the second-order formalism is expressed. The transformation rules of the metric and the spin-3 gauge field under the generalized diffeomorphisms are obtained explicitly. As in Einstein gravity, the new affine-like connections are related to the spin connection by a certain gauge transformation and a gravitational CS term expressed in terms of the new connections is also presented.

PACS numbers: 11.10.Kk, 11.15.Yc, 04.50.Kd, 04.60.Rt

1. Introduction
Gravity theories coupled to massless higher spin fields have been studied extensively in recent years [1–19]. This is due to the conjectured holographic relation between these theories in AdS$_4$ and the O(N) vector model in 3D [20]. These higher spin theories contain an infinite number of fields [21–24].

Three-dimensional higher spin theories [25] are simpler to work with than those in higher dimensions, because higher spin fields can be truncated to only those with spin $s \leq N$ and the theory can be defined in terms of the Chern–Simons (CS) action [2]. By using the $SL(3, R) \times SL(3, R)$ invariant CS theory, the black hole solution with spin-3 charge in spin-3 gravity was obtained and studied [6, 10, 11, 17].
The CS theory for gravity is very efficient for obtaining solutions. The condition of flat connections is easy to implement. The asymptotic behavior of the solution was found to be not necessarily $O(\mu^2)$. One needs to perform spin-3 gauge transformations to transform the black hole solution into a form with a manifest event horizon [6, 10]. The geometry of 3D higher spin gravity, which must be a generalization of the Riemannian geometry, is not well understood. It is difficult to understand this within the CS approach. Therefore, it is necessary to understand the geometry of higher spin gravity in more details by a different approach. Additionally, general integral formulae for the spin-3 charge and the entropy are not yet derived. The explicit action integrals for matter fields coupled to higher spin gravity are also not known.

In the spin-2 gravity theory there exist first-order and second-order formalisms. In the first-order formalism, a spin connection and a vielbein field are introduced, and the action integral contains only the first-order derivatives of the fields. In the second-order formalism, the spin connection is eliminated by solving the torsion-free condition and the solution is substituted into the action integral. Then the action integral becomes quadratic in the derivatives. Both formalisms are equivalent. The CS formulation of the spin-3 gravity is the first-order one. It is expected that a second-order formalism also exists for the spin-3 gravity. In order to tackle this problem, it is necessary to rewrite the theory of higher spin gravity as a geometrical theory in terms of the metric-like fields. For this purpose, one needs to define affine-like connections, covariant derivatives and the curvature tensors in the spin-3 gravity theory by using the vielbein fields.

Apparently, this problem is difficult to solve, because in $SL(N, R) \times SL(N, R)$ CS theory (with $N \geq 3$), the dimension of the algebra $sl(N, R)$ is larger than that of spacetime. The vielbein $e^a_{\mu}$ is not a square matrix and does not have an inverse\(^1\). For example, in the $N = 3$ case which corresponds to spin-3 gravity, $a$ runs from 1 to 8 and $\mu$ from 0 to 2. In this paper, to compensate this gap of the numbers of components, auxiliary vielbein fields $e^a_{(\mu\nu)}$ will be introduced. With the traceless and symmetry conditions $g^{\mu\nu} e^a_{(\mu\nu)} = 0$, $e^a_{(\mu\nu)} = e^a_{(\nu\mu)}$, the entire vielbein field becomes an $8 \times 8$ matrix. So if this generalized vielbein field is non-degenerate, an inverse vielbein exists and the torsion-free condition can be solved.

Now, many concepts and geometrical quantities can be introduced into the spin-3 gravity in parallel with the Einstein gravity. The purpose of this paper is to pursue this possibility. For example, two connections, $\Gamma^\lambda_{\mu \nu}$ and $\Gamma^{(\lambda \rho)}_{\mu \nu}$, are obtained by generalization of the vielbein postulate, $\partial_a e^b_{\mu} + f^{b}_{\ k a} \omega^k_{\mu \nu} e^c_{\nu} = \Gamma^c_{\mu \nu} e^a_{\mu} + 1/2 \Gamma^{(\lambda \rho)}_{\mu \nu} e^a_{(\lambda \rho)}$, instead of the Christoffel symbol $\hat{\Gamma}^\lambda_{\mu \nu}$ in the Einstein gravity. These connections can be expressed purely in terms of metric-like fields, although somewhat formally. These connections are expected to be used to describe the geometry of spin-3 gravity. By using these connections, appropriate covariant derivatives $\nabla^a_{\mu}$ can be introduced in such a way that the full covariant derivatives $D^a_{\mu}$ of the vielbeins, $D^a_{\mu} e^b_{(\lambda \rho)} = \nabla^a_{\mu} e^b_{(\lambda \rho)} + f^{a}_{\ k b} \omega^k_{\mu \nu} e^c_{(\lambda \rho)}$ and $D^a_{\mu} e^a_{(\nu \lambda)} = \nabla^a_{\mu} e^a_{(\nu \lambda)} + f^{a}_{\ k b} \omega^k_{\mu \nu} e^a_{(\nu \lambda)}$, vanish. In the definition of $\nabla^a_{\mu} e^a_{(\lambda \rho)}$, two more connections, $\Gamma_{\mu, (\lambda \rho)}$ and $\Gamma^{(\lambda \rho)}_{\mu, (\lambda \rho)}$, are also defined in terms of metric-like fields. It then turns out convenient to combine $e^a_{(\lambda \rho)}$ and $e^a_{(\nu \lambda)}$ into a single vielbein field $e^a_{M}$, where $M$ takes two kinds of indices, $M = \mu$ and $M = (\nu \lambda)$. Here $(\mu \nu)$ denotes a traceless, symmetric pair of base-space indices. Similarly, the above four affine-like connections can be combined as $\Gamma^{(\lambda \rho)}_{\mu, M}$. The metric tensor can also be generalized: defining $G_{MN} = e^a_M e^a_N$, which is generalization of the metric tensor $g_{\mu\nu} = e^a_{\mu} e^a_{\nu}$, we will find that $G_{MN}$ is compatible with the covariant derivative $\nabla_{\mu}$.

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\(^1\) In [2], the torsion-free condition in spin-$s$ gravity is solved to first order of expansion around a spin-2 background. In this paper, the torsion-free condition will be solved explicitly without using perturbative expansions.
The metric-like fields in spin-3 gravity theory include the spin-3 gauge field $\phi_{\mu
u\lambda} = (1/2) d_{abc} e^a_\mu e^b_\nu e^c_\lambda [2]$ in addition to the ordinary metric field $g_{\mu\nu}$. Here $d_{abc}$ is the completely symmetric invariant tensor for $sl(3,R)$. This spin-3 gauge field is related to the generalized metric $G_{\mu\nu\lambda}$ defined above. It will be pointed out in appendix C of this paper that one can construct more metric-like fields by using invariant tensors, $d_{abc}$ and $f_{abc}$. Naturally, one expects that there will be relations among these metric-like fields. For spin-3 geometry near AdS$_3$ vacuum, it is expected that only $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$ are independent degrees of freedom and other metric-like fields can be expressed in terms of them. Indeed, by using perturbation expansions around AdS$_3$ vacuum, it is possible to convince oneself that this expectation is correct. However, closed form expressions for such relations among metric-like fields are not easy to obtain. One needs to introduce these extra fields to do all the rewriting of them in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$. It is also unclear if for any spin-3 geometry, all the metric-like fields can always be expressed in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$. These problems will not be solved in this paper. Therefore in our second-order formalism the results are not expressed just in terms of the metric-like fields, but vielbein fields and structure constants will be left out in the final expressions.

The covariant-constancy conditions for $e^a_\mu$ and $e^{(\mu\nu)}$ will be solved to yield the spin connection $\omega^a_\mu (e)$ as a functional of the vielbeins

$$\omega^a_\mu (e) \equiv f^{a}_{bc} e^b_\mu (e) = -E^c_\nu \nabla_\mu e^a_\nu - \frac{1}{2} E^{(i)k}_{c} \nabla_\mu e^{i}_{(k)}. \tag{1.1}$$

Here $E^a_\nu$ and $E^{(\mu\nu)}$ are inverse vierbeins. By substituting this into the CS action, the second-order action will be obtained. Furthermore, from the connections, $\Gamma^N_{\mu,M}$, generalized curvature tensors, $R^M_{\nu\lambda\rho}(\Gamma)$, can be defined and the action integral can be expressed in terms of these curvature tensors

$$S_{\text{second order}} = \frac{k}{12\pi} \int d^3x \left\{ -\epsilon^{\mu\nu\lambda} \left( f^{a}_{bc} e^b_\mu (e) d^a_{M} \right) R^M_{\nu\lambda\rho}(\Gamma) + 4 \epsilon^{\mu\nu\lambda} f_{abc} e^a_\mu e^b_\nu e^c_\lambda \right\}. \tag{1.2}$$

As this equation shows, the vielbein fields and the structure constants $d_{abc}$ still remain, and the action integral is not expressed purely in terms of the metric-like fields. This is due to the reason presented in the previous paragraph. However, the spin connection $\omega^a_\mu$ is eliminated and the action integral is expressed as the second-order forms of the vielbein fields. In this sense, the formulation we obtained is the second-order one. What remains to be done is to reexpress the action only in terms of the metric-like fields. This will not be attempted in this paper.

As in Einstein gravity, the connections $\Gamma^N_{\mu,M}$ and the spin connection $\omega^a_\mu$ turn out to be related by a gauge transformation. By using this fact the gravitational CS term $S_{\text{GCS}}(\Gamma)$ can be explicitly expressed in terms of the connections (\Gamma's) and the topologically massive spin-3 gravity theory is defined

$$S_{\text{GCS}}^{\text{spin-3}}(\Gamma) = \frac{k}{8\pi \mu} \int d^3x \epsilon^{\mu\nu\lambda} \left( \Gamma^\rho_{\mu,\sigma} \partial_\rho \Gamma^\sigma_{\lambda,\rho} + \frac{1}{2} \Gamma^\rho_{\mu,\sigma} \partial_\sigma \Gamma^\sigma_{\lambda,\rho} + \frac{1}{2} \Gamma^\rho_{\mu,\sigma} \partial_\rho \Gamma^\sigma_{\lambda,\rho} \right)$$

$$+ \frac{1}{4} \Gamma^\rho_{\mu,\sigma,\tau} \partial_\rho \Gamma^\tau_{\lambda,\rho} + \frac{2}{3} \Gamma^\rho_{\mu,\sigma} \Gamma^\sigma_{\lambda,\rho} + \Gamma^\rho_{\mu,\sigma} \Gamma^\sigma_{\lambda,\rho} + \Gamma^\rho_{\mu,\sigma} \Gamma^\tau_{\lambda,\rho} + \frac{1}{12} \Gamma^\rho_{\mu,\sigma} \Gamma^\tau_{\lambda,\rho} \right) \cdot \tag{1.3}$$

From the gauge transformations of the CS theory, the generalized diffeomorphism of the metric field $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ can be defined and computed. These generalized diffeomorphisms are the ordinary 3D diffeomorphism and the spin-3 gauge transformation. It will be checked that these fields appropriately transform as spin-two and spin-three fields, respectively, under the ordinary diffeomorphism. New transformation rules of these fields under the spin-3 gauge transformation will also be obtained explicitly. The results are
compact written as
\[ \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \]  
(1.4)
\[ \delta \phi_{\mu\nu\lambda} = \nabla_\mu (\xi_{\nu\lambda}) + \frac{1}{2} \rho^a \Lambda_a g_{\nu\lambda} + \nabla_\nu (\xi_{\lambda\mu}) + \frac{1}{2} \rho^a \Lambda_a g_{\lambda\mu} + \nabla_\lambda (\xi_{\mu\nu}) + \frac{1}{2} \rho^a \Lambda_a g_{\mu\nu}, \]  
(1.5)
where \( \xi_\mu \) and \( \xi_{\mu\nu} \) are parameters of the generalized diffeomorphisms, and related to a gauge function \( \Lambda^a \) in (5.2) and (5.3), \( \rho^a \) is defined by \( \rho^a = \frac{1}{2} d^a_i e^i_\mu \phi_{\mu\nu} g^\nu\rho \).

This paper is organized as follows. Section 2 is a brief review of 3D spin-3 gravity as a CS theory. In section 3, a method for an extension of the vielbein field is explained. Then two connections, \( \Gamma^a_\mu_\nu, \Gamma^a_\mu_\nu \), analogous to the Christoffel symbol are introduced and it will be shown that they can be expressed in terms of metric-like fields. In section 4, the spin connection is determined as a functional of the vielbein. The covariant derivatives of the vielbeins are defined, and two more connections, \( \Gamma^a_{\mu_\nu\rho_\lambda}, \Gamma^a_{\mu_\nu\rho_\lambda} \), are introduced and related to the old ones, \( \Gamma^a_{\mu_\nu}, \Gamma^a_{\mu_\nu} \). We obtain an invariant action integral for the spin-3 gravity in the second-order formalism. This action is surely the one in the second-order formalism, because the spin connection is eliminated, but this action still contains the vielbeins and the structure constants \( f_{abc} \), and is not expressed purely in terms of metric-like fields. We point out that the indices of tensors are shown to have a novel pairing structure, \( \mu \) and \( (\mu\nu) \). In section 5, the transformation rules of the metric field \( g_{\mu\nu} \) and the spin-3 gauge field \( \phi_{\mu\nu\rho} \) under the generalized diffeomorphism will be computed. In section 6, curvature tensors for the spin-3 geometry are defined. In section 7, the spin-3 gravity version of the gravitational CS term is derived. Finally, section 8 is reserved for the summary and discussions. There are appendices A–E. Appendix A summarizes the \( sl(3, R) \) formulae. In appendix B, it will be shown that for the AdS3 background the vielbein system \( e^a_\mu, e^{\mu}_{(\mu\nu)} \) is actually non-degenerate. The inverse vielbeins and other quantities are obtained. Killing vectors \( \xi_\mu \) and tensors \( \xi_{\mu\nu} \) for AdS3 are also presented. In appendix C, it is pointed out that in addition to the metric \( g_{\mu\nu} \) and the spin-3 gauge field \( \phi_{\mu\nu\rho} \), extra gauge fields such as \( g_{(\mu\nu)(\mu\nu)}, g_{(\mu\nu)(\mu\nu)} \), \( g_{(\mu\nu)(\mu\nu)(\sigma\rho)} \) with the number of indices up to \( 6 \) must be introduced. It is argued, by perturbation expansion to the first non-trivial order, that for spin-3 geometry near AdS3 vacuum all metric-like fields would be expressed in terms of \( g_{\mu\nu} \) and \( \phi_{\mu\nu\rho} \). In appendix D, the complicated part \( S_{\mu\nu\rho} \) of the connection \( \Gamma^a_{\mu\nu\rho} \) is obtained explicitly, if somewhat formally. In appendix E, a metric tensor \( G_{MN} \) for 8D space, which corresponds to the two types of indices \( M = \mu, (\mu\nu) \) and composed of the metric, the spin-3 gauge field and another gauge field \( g_{(\mu\nu)(\sigma\rho)} \), is introduced and some formulae for this metric tensor are derived.

Note added: while this work was being completed, we found that a work [35] appeared in the arXiv, which attempts to formulate 3D spin-3 gauge theory in terms of the metric-like fields, by means of the perturbation in powers of the spin-3 gauge field \( \phi_{\mu\nu\rho} \) up to \( O(\phi_{\mu\nu\rho})^2 \). The action integral and various transformation rules of metric-like fields obtained in this paper are not based on perturbations in \( \phi_{\mu\nu\rho} \). Further, we also generalize the geometrical notions of Einstein gravity, such as the connections and the curvature tensors, to the spin-3 geometry.

2. 3D spin-3 gravity as Chern–Simons theory

Let us start by briefly reviewing the 3D spin-3 gravity defined in terms of the CS theory.

The 3D spin-3 gravity with a negative cosmological constant is defined by the \( SL(3, R) \times SL(3, R) \) CS actions [26, 27]
\[ S_{CS} = \frac{k}{4\pi} \int \text{tr} \left( \bar{A} \wedge dA + \frac{2}{3} \bar{A} \land A \land A \right) - \frac{k}{4\pi} \int \text{tr} \left( \bar{\tilde{A}} \wedge d\tilde{A} + \frac{2}{3} \bar{\tilde{A}} \land \tilde{A} \land \tilde{A} \right). \]  
(2.1)
Here $A$ and $\tilde{A}$ are gauge-field one-forms and live in the fundamental representation of $sl(3, R)$. The constant $k$ is the level of the CS actions and is related to the three-dimensional Newton constant $G_N$. The AdS length $\ell$ is given by $k = \ell/4G_N$. The above action is invariant under $SL(3, R) \times SL(3, R)$ gauge transformations up to boundary terms

$$\delta A = dA + [A, A],$$
$$\delta \tilde{A} = d\tilde{A} + [\tilde{A}, \tilde{A}].$$

(2.2)

Here $A$ is an $sl(3, R)$ matrix. These gauge fields $A, \tilde{A}$ are related to the vielbein one-form $e = e_\mu dx^\mu$ and the spin connection one-form $\omega = \omega_\mu dx^\mu$ by the relations

$$A = \omega + \frac{1}{\ell} e, \quad \tilde{A} = \omega - \frac{1}{\ell} e.$$  

(2.3)

Here $x^\mu (\mu = 0, 1, 2)$ is the coordinate of 3D spacetime. The Greek indices $\mu, \nu, \ldots$ will be used for spacetime indices and the Roman letters $a, b, \ldots$ will be used for internal space (local frame) indices.

In what follows $\ell$ will be set to 1. The above action is then written in terms of $e$ and $\omega$

$$S_{CS} = \frac{k}{\pi} \int \text{tr} \bigg( de \wedge \big( d\omega + \omega \wedge \omega + \frac{1}{3} e \wedge e \big) \bigg).$$

(2.4)

The vielbein and the spin connection are expanded in terms of the $sl(3, R)$ generators $t_a$ (see appendix A):

$$e = e^a t_a = e_\mu t_\mu dx^\mu, \quad \omega = \omega^a t_a = \omega_\mu t_\mu dx^\mu.$$  

(2.5)

The gauge transformations on $e$ and $\omega$ fall into two groups.

(a) Local frame rotation (or extended Lorentz transformation)

$$\delta e = [e, A_+],$$  

(2.6)

$$\delta \omega = dA_+ + [\omega, A_+], \quad A_+ \equiv \frac{1}{2} (A + \tilde{A}).$$  

(2.7)

These transformations are Lorentz transformations and extended rotations in local frame.

(b) Generalized diffeomorphism (or extended local translation)

$$\delta e = dA_- + [\omega, A_-],$$  

(2.8)

$$\delta \omega = [e, A_-], \quad A_- \equiv \frac{1}{2} (A - \tilde{A}).$$  

(2.9)

These transformations are the ordinary spacetime diffeomorphism and the spin-3 gauge transformations.

As usual, the metric tensor $g_{\mu\nu}$ of the spacetime is defined in terms of the vielbein $e_\mu = e^a_\mu t_a$

$$g_{\mu\nu} = \frac{1}{4} \text{tr} \big( e_\mu e_\nu - h_{ab} e^b_\mu e^a_\nu \big) = e^a_\mu e^a_\nu.$$  

(2.10)

For the definition of the Killing metric $h_{ab}$ for the $sl(3, R)$ algebra, see appendix A. Throughout this paper, $g_{\mu\nu}$ is assumed to be non-degenerate. It is also assumed that its signature is $(-, +, +)$. Its inverse is denoted as $g^{\mu\nu}$. Also in the literature [2], the spin-3 gauge field $\phi_{\mu\nu\lambda}$ is defined

$$\phi_{\mu\nu\lambda} = \frac{1}{4} \text{tr} [e_\mu, e_\nu] e_\lambda.$$  

(2.11)

Here $\{ , \}$ is an anti-commutator. This tensor and $g_{\mu\nu}$ are supposed to be independent fields. The number of components of $e^a_\mu$ is 24 and there are 8 independent local frame transformations.

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2 In [2], a notation $e^a_\mu (a = 1, 2, 3)$ is used for the vielbein and the extra field $e^a_\mu (a, b = 1, 2, 3)$ is called spin-3 gauge field. We will not adopt this notation.
So there must be 16 independent degrees of freedom for the metric-like fields. The metric $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ have six and ten independent components, respectively. Thus, it is expected that $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$ are sufficient. However, in appendix C it will be shown that extra metric-like tensor fields must be introduced for describing spin-3 gravity. It is expected that these extra metric-like fields are not independent of the above fields, although explicit formulae relating them are nested and complicated.

One can resort to perturbation expansions around the AdS vacuum to the first non-trivial order, and argue that all these extra metric-like fields can be expressed in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$. See appendix C for discussion. Full-order closed expressions are, however, not available. It is also unclear if these extra metric-like fields can be described in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$, even when the vielbeins are near general backgrounds with non-vanishing spin-3 gauge field.

In this paper, it will be assumed that all the metric-like fields can be expressed in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$. However, because actually expressing them in terms of the metric fields and the spin-3 gauge fields seems very complicated, for technical reasons, we will leave the extra fields as they are and not try to represent them in terms of $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$.

3. Extension of the vielbein

The equations of motion for the CS theory (2.4) are given by the conditions of flat connections,

$$F \equiv dA + A \wedge A = 0, \quad \bar{F} \equiv d\bar{A} + \bar{A} \wedge \bar{A} = 0,$$

and in terms of $e$ and $\omega$ these equations are rewritten as a torsion-free condition

$$T \equiv \frac{1}{2} (F - \bar{F}) = de + e \wedge \omega + \omega \wedge e = 0$$

and an Einstein-like equation with negative cosmological constant

$$R \equiv \frac{1}{2} (F + \bar{F}) = d\omega + \omega \wedge \omega + e \wedge e = 0.$$

This equation describes more degrees of freedom than those of gravity.

In the ordinary (spin-2) 3D gravity described by $SL(2, R) \times SL(2, R)$ CS action, the condition (3.2) can be solved for $\omega$, if the dreibein $e^a_\mu$ is invertible. Then this solution is substituted into (3.3) and the Einstein equation for Anti-de-Sitter (AdS) space is obtained. Similarly, to formulate a second-order theory of spin-3 gravity in terms of the metric-like fields, $g_{\mu\nu}, \phi_{\mu\nu\lambda}, \ldots$, it is necessary to first solve the torsion-free condition and express the spin connection $\omega$ in terms of $e$ and then, substitute the result into the action (2.4). The vielbein $e^a_\mu$ ($a = 1, \ldots, 8; \mu = 0, 1, 2$), however, has a form of a rectangular matrix and is non-invertible, even if $g_{\mu\nu}$ is non-degenerate.

3.1. New vielbein $e^a_{(\mu\nu)}$

To resolve this difficulty, five more basis vectors in the local frame must be introduced. Let us define the $SL(3, R)$ matrices

$$\hat{e}_{\mu\nu} = \frac{1}{2} \{e_{\mu}, e_{\nu}\} - \frac{1}{2} g_{\mu\nu} I.$$

The second term on the right proportional to the identity matrix $I$ is added to ensure tracelessness of $\hat{e}$ as a matrix of $sl(3, R)$. This is symmetric in the indices $\mu$, $\nu$ and there are six independent components. We are now going to regard the set $(e^a_\mu, \hat{e}^a_{(\mu\nu)})$ as an
$8 \times 8$ matrix and define its inverse matrix. For this purpose, we need to reduce the number of components by one, and we choose to subtract the trace of $\hat{e}_{\mu \nu}$ with respect to the indices $\mu, \nu$

\[
e_{(\mu \nu)} \equiv \hat{e}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \rho = \frac{1}{2} [\varepsilon, e_{\nu}] - \frac{1}{2} g_{\mu \nu} \rho - \frac{1}{2} g_{\mu \nu} l,
\]

(3.5)

\[
\rho \equiv g^{\rho \sigma} \hat{e}_{\rho \sigma}.
\]

(3.6)

The matrix $\rho$ is the trace part of $\hat{e}_{\mu \nu}$, and $e_{(\mu \nu)}$ satisfies $g^{\mu \nu} e_{(\mu \nu)} = 0$. Although these additional matrices $e_{(\mu \nu)}$ functionally depend on $e_{\mu}$, 8D vectors $e_{(\mu \nu)} (a = 1, \ldots, 8)$ defined by $e_{(\mu \nu)} = e_{(\mu \nu)} t_a$ are assumed to span the 8D space together with $e_a^\mu$. In appendix B, the vielbeins for AdS$_3$ vacuum are examined and it is shown that this is the case. It can also be shown that extended vielbeins ($e_{a}^\mu, e_{(\mu \nu)}$) for the BTZ black hole embedded in the spin-3 gravity[2] are also non-degenerate.

However, for general $e_a^\mu$, the extended vielbeins ($e_a^\mu, e_{(\mu \nu)}$) may be degenerate at some points. Even if this is the case, we may employ the usual method of the fiber bundles to avoid the singularity. Let us consider the case where the CS gauge field one-forms take the forms, $A = b^{-1} a b + b^{-1} d b, \bar{A} = b \tilde{a} b^{-1} + b d b^{-1}$ with $b = \exp(r L_0)[2]$. $r$ is the radial coordinate, and $a$ and $\tilde{a}$ are one-forms independent of $r$. Let the extended vielbeins be degenerate only at $r = r_0$. We cover the base manifold by two open coordinate neighborhoods, $U_1 = \{(r, t, \phi) | r > \alpha\}, U_2 = \{(r, t, \phi) | r < \beta\}$ with $r_0 < \alpha < \beta$. The gauge fields on $U_1$ will be denoted as $A'$ and $\bar{A}'$, and we set $A'(r) \equiv A(r), \bar{A}'(r) \equiv \bar{A}(r)$. We choose as a transition function (gauge transformation) on the overlap $U_1 \cap U_2 = \{(r, t, \phi) | \alpha < r < \beta\}$, an extended local translation $V_r = \exp \Lambda_r = \exp(-y L_0)$, where $L_0$ is one of the $sl(3, R)$ generators (A.2) and $y$ is a constant satisfying $y > \beta - r_0 > \beta - \alpha$. The transition function $V_r$ has an effect of a translation $r \to r - y$ on the gauge fields $A'(r)$ and $\bar{A}'(r)$ in $U_1$, and the gauge fields on $U_2$ are given by $A^2(r) = A^1(r-y)$ and $\bar{A}^2(r) = \bar{A}^1(r-y)$. Then, we have $r - y < \beta - y < r_0$ in $U_2$ and the extended vielbeins ($e_a^\mu, e_{(\mu \nu)}$) computed from $A^2(r)$ and $\bar{A}^2(r)$ are non-degenerate in $U_2$. If there are more degenerate points, the extended vielbeins will also be made non-degenerate by the same procedure.

Now if $e_a^\mu$ and $e_{(\mu \nu)}$ are combined, these can be regarded as an $8 \times 8$ matrix and it has an inverse matrix. Let us define inverse vielbeins $E_{a}^\mu$ and $E_{(\mu \nu)}^a$ by

\[
E_{a}^\mu e_a^\mu = \delta^\mu_\nu, \quad E_{(\mu \nu)}^a e_{(\nu \lambda)} = 0, \quad E_{a}^\mu e_a^\mu = 0, \quad E_{(\mu \nu)}^a e_{(\nu \lambda)} = \delta_{(\mu}^\lambda \delta_{\nu)}^\rho + \delta_{\mu}^\rho \delta_{\nu}^\lambda - \frac{1}{2} g_{\mu \nu} g_{a b}.
\]

(3.7)

The right-hand side of the last equation ensures the tracelessness of $E_{a}^\mu$ (and of $e_{(\mu \nu)}$): $g_{\mu \nu} E_{(\mu \nu)}^a = 0$. They also satisfy the relation

\[
e_a^\mu E_b^\mu + \frac{1}{2} e_{(\mu \nu)} E_b^{(\mu \nu)} = \delta_b^a.
\]

(3.8)

In front of the second term on the left-hand side, a factor $1/2$ appears because of the symmetry $\mu \leftrightarrow \nu$.

3.2. New connections

Let us turn to the torsion-free condition (3.2) now expressed in terms of components

\[
\partial_\mu e_\nu - \partial_\nu e_\mu + [\omega_\mu, e_\nu] - [\omega_\nu, e_\mu] = 0.
\]

(3.9)

This relation implies that $\partial_\mu e_\nu + [\omega_\mu, e_\nu]$ is symmetric for interchange of $\mu$ and $\nu$. Because this is a traceless matrix, it can be expanded in terms of $e_\nu$ and $e_{(\mu \rho)}$

\[
\partial_\mu e_\nu + [\omega_\mu, e_\nu] = \Gamma_{\mu \nu}^{\rho} e_\rho + \frac{1}{2} \Gamma_{(\mu \nu)}^{(\rho \sigma)} e_{(\rho \sigma)}.
\]

(3.10)

\footnote{It is easy to find background vielbeins for which actually the matrix $\rho \neq 0$.}
Here $\Gamma_\mu^\lambda$ and $\Gamma_\mu^{(\rho\sigma)}$ are two connections to be determined later, as functions of the metric-like fields and their derivatives. These are symmetric in the lower indices $\mu$ and $\nu$. In (3.10), one can choose $\Gamma_\mu^{(\rho\sigma)}$ such that it satisfies $g_{\lambda \rho} \Gamma_\mu^{(\rho\sigma)} = 0$, since it is multiplied by $e_\rho^{\mu \lambda}$. Note that the existence of $\Gamma_\mu^\lambda$ and $\Gamma_\mu^{(\rho\sigma)}$ are ensured by the assumed linear independence of $e_\mu = e^\mu_\mu$ and $e_\mu = e^{\rho \mu}_\sigma$. One may represent these connections in terms of $\alpha_\mu^\sigma$ and the vielbeins. If this is the case, the introduction of the new connections will not be of much use. Certainly, this is not our purpose. We will express them only in terms of the vielbeins.

In order to derive such expressions, let us multiply (3.10) by $e_\lambda$ from the right
\begin{equation}
\partial_\mu (e_\nu e_\lambda + [\omega_\mu, e_\nu] e_\lambda) = \Gamma_\mu^\rho e_\rho e_\lambda + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda. \tag{3.11}
\end{equation}
Then by replacing $\nu$ by $\lambda$ in (3.10), multiplying it by $e_\nu$ from the left and adding the result to (3.11) the following relation is obtained
\begin{equation}
\partial_\mu (e_\nu e_\lambda + [\omega_\mu, e_\nu] e_\lambda) = \Gamma_\mu^\rho e_\rho e_\lambda + \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda. \tag{3.12}
\end{equation}
By taking the trace of this equation and using (2.10) the following relation is obtained:
\begin{equation}
\partial_\mu g_{\nu \lambda} = \Gamma_\mu^\sigma g_{\sigma \lambda} + \Gamma_\mu^{(\sigma \rho)} g_{\sigma \rho} + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} g_{(\sigma \rho)} + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} g_{(\sigma \rho).} \tag{3.13}
\end{equation}
Here the following field is defined:
\begin{equation}
g_{(\mu \nu \lambda)} = \frac{1}{2} \text{tr} (e_{(\mu \nu)} e_\lambda). \tag{3.14}
\end{equation}
The parenthesis in the subscript of $g$ on the left-hand side is put to make the permutation symmetry manifest. This field is different from, but related to, the spin-3 gauge field $\phi_{\mu \nu \lambda}$ defined in (2.11)
\begin{equation}
\phi_{\mu \nu \lambda} = g_{(\mu \nu \lambda)} + \frac{1}{2} g_{\mu \nu} \text{tr} (\rho e_\lambda). \tag{3.15}
\end{equation}
This new field satisfies $g^{\mu \nu} g_{(\mu \nu \lambda)} = 0$. The matrix $\rho$ was defined in (3.6). Note that the trace in the second term on the right can be represented in terms of the spin-three gauge field
\begin{equation}
\text{tr} (\rho e_\lambda) = 2 \phi_{\mu \nu \lambda} g_{\mu \nu} = 2 \phi_{\mu \nu}^\lambda. \tag{3.16}
\end{equation}
As shown in the above expression, spacetime indices are raised and/or lowered by $g^{\mu \nu}$ and $g_{\mu \nu}$.\footnote{In what follows, when $\phi_{\mu \nu \lambda}$ is contracted with a tensor $\psi^{\mu \nu \lambda}$ which is traceless in $\mu \nu$, $\phi_{\mu \nu \lambda}$ will be replaced by $g_{\mu \nu \lambda}$ without notice.}

Now equation (3.13) can be solved to yield the following relation by using the usual method:
\begin{equation}
\Gamma_\mu^\lambda + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} = \frac{1}{2} g^{\mu \nu} (\partial_\mu g_{\nu \lambda} + \partial_\nu g_{\mu \lambda} - \partial_\lambda g_{\mu \nu}) = \tilde{\Gamma}_\mu^\lambda, \tag{3.17}
\end{equation}
where $\tilde{\Gamma}_\mu^\lambda$ is the ordinary Christoffel symbol (connection).

3.3. Determination of $\Gamma_\mu^{(\rho \sigma)}$

Another relation is required to determine the two connections uniquely. Multiplying (3.12) by $e_\rho$ from the right and adding a similar equation obtained by multiplying (3.10) (with replacement $\nu \rightarrow \rho$) by $e_\sigma e_\lambda$ from the left, the following equation is obtained:
\begin{equation}
\partial_\mu (e_\nu e_\lambda + [\omega_\mu, e_\nu] e_\lambda) = \Gamma_\mu^\rho e_\rho e_\lambda + \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda + \frac{1}{2} \Gamma_\mu^{(\sigma \rho)} e_{(\sigma \rho)} e_\lambda. \tag{3.18}
\end{equation}
Now interchange $\lambda$ and $\rho$ in the above equation and add the result to the above. Taking the trace leading to the following equation:

$$
\partial_\mu \text{tr} e_\nu \{e_\lambda, e_\rho\} = \Gamma^\nu_{\mu \rho} \text{tr} e_\sigma \{e_\lambda, e_\rho\} + \Gamma^\sigma_{\nu \rho} \text{tr} e_\nu \{e_\lambda, e_\sigma\} + \Gamma^\sigma_{\mu \rho} \text{tr} e_\sigma \{e_\lambda, e_\rho\}
$$

$$
+ \frac{1}{2} \Gamma^\alpha_{\nu \rho} \text{tr} e_{(\sigma \lambda)} \{e_\rho, e_\nu\} + \Gamma^\alpha_{\mu \rho} \text{tr} e_{(\sigma \lambda)} \{e_\nu, e_\rho\}
$$

$$
+ \frac{1}{2} \Gamma^\alpha_{\mu \nu} \text{tr} e_{(\sigma \rho)} \{e_\lambda, e_\nu\}
$$

(3.19)

In the above equation, let us note that one can make the following substitutions:

$$
\text{tr} e_{(\sigma \lambda)} \{e_\rho, e_\nu\} = 4 \Phi_1 \phi_{\lambda \rho},
$$

(3.20)

$$
\text{tr} e_{(\sigma \lambda)} \{e_\nu, e_\rho\} = 4 g_{(\sigma \lambda) (\rho \nu)} + \frac{2}{3} g_{\lambda \rho} \text{tr} (e_{(\sigma \lambda)})\rho).
$$

(3.21)

Here $g_{(\sigma \lambda) (\rho \nu)}$ is defined in appendix C. Furthermore, the connection $\Gamma^\lambda_{\mu \nu}$ can be eliminated by use of (3.17). This yields the equation

$$
\hat{\nabla}_\mu \phi_{\lambda \rho} = \frac{1}{2} \Gamma^\alpha_{\mu \nu} (g_{(\sigma \lambda) (\rho \nu)} - g_{(\sigma \lambda)}^\tau g_{\tau (\rho \nu)} - \frac{1}{2} g_{\lambda \rho} g_{(\sigma \lambda)}^\tau \text{tr} (\rho e_{(\sigma \lambda)}) + \frac{1}{2} \text{tr} (e_{(\sigma \lambda)})\rho).
$$

(3.22)

Here $\hat{\nabla}_\mu$ is the covariant derivative which uses the Christoffel connection.

Let us introduce a $5 \times 5$ matrix $M_{(\sigma \lambda) (\rho \nu)}$ by

$$
M_{(\sigma \lambda) (\rho \nu)} \equiv g_{(\sigma \lambda) (\rho \nu)} - g_{(\sigma \lambda)}^\tau g_{\tau (\rho \nu)}.
$$

(3.23)

Its inverse matrix $J^\lambda_{\mu \nu (\sigma \lambda)}$ is assumed to exist$^7$ and defined by the following equation:

$$
\frac{1}{2} M_{(\mu \nu) (\lambda \rho)} J^\lambda_{\mu \nu (\sigma \lambda)} = \delta_\mu^\delta \delta_\nu^\sigma + \frac{1}{2} g_{\mu \nu} g^{\sigma \varepsilon}.
$$

(3.24)

Then (3.22) is rewritten as

$$
\hat{\nabla}_\mu \phi_{\lambda \rho} = \frac{1}{2} \Gamma^\alpha_{\mu \nu} M_{(\lambda \rho) (\sigma \lambda)} + \frac{1}{2} \Gamma^\alpha_{\mu \nu} M_{(\lambda \rho) (\sigma \nu)} + \frac{1}{2} \Gamma^\alpha_{\sigma \nu} M_{(\mu \rho) (\sigma \lambda)}
$$

$$
+ \frac{1}{2} \left( g_{\nu \rho} \Gamma^\alpha_{\mu \nu} + g_{\nu \rho} \Gamma^\alpha_{\nu \rho} + g_{\mu \nu} \Gamma^\alpha_{\mu \nu} \right) W_{\alpha \kappa}.
$$

(3.25)

Here $W_{\alpha \kappa}$ is defined by

$$
W_{\alpha \kappa} = \text{tr} (e_{(\sigma \lambda)}) - g_{(\sigma \lambda)}^\tau e_{(\sigma \lambda)}\rho).
$$

(3.26)

We now introduce the following field:

$$
\Phi_{\lambda \rho} = \phi_{\lambda \rho} - \frac{1}{2} (g_{\lambda \rho} \phi_{\sigma \sigma} + g_{\sigma \rho} \phi_{\sigma \sigma} + g_{\lambda \rho} \phi_{\sigma \sigma}^\tau).
$$

(3.27)

This tensor is traceless for each pair of indices, $g^{\lambda \rho} \Phi_{\lambda \rho} = 0$. In terms of this field, we can derive the following equation from (3.25):

$$
\hat{\nabla}_\mu \Phi_{\lambda \rho} = \frac{1}{2} \Gamma^\alpha_{\mu \nu} M_{(\lambda \rho) (\sigma \lambda)} - \frac{1}{2} g_{\nu \rho} g^{\gamma \mu} \Gamma^\gamma_{\mu \nu} M_{(\sigma \lambda) (\rho \nu)}
$$

$$
+ \text{cyclic permutations of (v, \lambda, \rho)}.
$$

(3.28)

This equation determines $\Gamma_{\mu \nu}$:

$$
\Gamma^\alpha_{\mu \nu} M_{(\sigma \lambda) (\rho \nu)} = \frac{1}{2} \left( \hat{\nabla}_\mu \Phi_{\rho \lambda} + \hat{\nabla}_\nu \Phi_{\mu \rho} \right) - \frac{1}{2} \left( \hat{\nabla}_\rho \Phi_{\mu \nu} + \hat{\nabla}_\nu \Phi_{\rho \mu} \right)
$$

$$
+ \frac{1}{2} g_{\nu \rho} \hat{\nabla}_\kappa \Phi_{\mu \kappa} + S_{\mu \nu \lambda \rho}.
$$

(3.29)

Here $S_{\mu \nu \lambda \rho}$ is a function which satisfies

$$
S_{\mu \nu \lambda \rho} = S_{\mu \nu \lambda \rho}, \quad g^{\lambda \rho} S_{\mu \nu \lambda \rho} = 0,
$$

$$
S_{\mu \nu \lambda \rho} - \frac{1}{2} g_{\nu \rho} g^{\tau \sigma} S_{\mu \nu \sigma \tau} + \text{cyclic permutations of (v, \lambda, \rho)} = 0.
$$

(3.30)

$^7$ This is true for AdS$_3$ solution as shown in (B.10).
By using the matrix $J (3.24), (3.29)$ gives $\Gamma_{\mu\nu}^{(\rho)}$,
\[
\Gamma_{\mu\nu}^{(\rho)} = \frac{1}{2} \left( \hat{\nabla}_\mu \Phi_{\nu\sigma} + \hat{\nabla}_\nu \Phi_{\mu\sigma} - \hat{\nabla}_\sigma \Phi_{\mu\nu} \right) J^{(\sigma)(\rho)} + \frac{1}{2} g_{\mu\nu,\sigma} J^{(\sigma)(\rho)}.
\]
Finally, by contracting (3.25) with $g^{\rho\lambda}$ one obtains another algebraic equation which $S_{\mu\nu,\lambda\rho}$ must satisfy
\[
\hat{\nabla}_\mu \Phi_{\nu\lambda} = \Gamma_{\mu\lambda}^{(\rho)} M_{(\sigma)(\nu\rho)} g^{\rho\lambda} + \frac{5}{12} \Gamma_{\mu\nu}(\sigma) W_{\sigma\lambda} = g^{\rho\lambda} S_{\mu\nu,\nu\rho} + \frac{5}{6} \hat{\nabla}_\mu \Phi_{\nu\lambda} + \frac{5}{6} \hat{\nabla}_\nu \Phi_{\mu\lambda} - \hat{\nabla}_\lambda \Phi_{\mu\nu} + \frac{5}{2} S_{\mu\nu,\nu\lambda}.
\]
The equations (3.31) and (3.17) determine $\Gamma_{\mu\nu}^\lambda$:
\[
\Gamma_{\mu\nu}^\lambda = \hat{\Gamma}_{\mu\nu}^\lambda - \frac{1}{12} \left( \hat{\nabla}_\mu \Phi_{\nu\lambda} + \hat{\nabla}_\nu \Phi_{\mu\lambda} - \hat{\nabla}_\lambda \Phi_{\mu\nu} + \frac{5}{2} S_{\mu\nu,\nu\lambda} \right) J^{(\sigma)(\rho)} \phi_{\mu\nu}^\lambda.
\]
The explicit (if formal) solution for $S_{\mu\nu,\lambda\rho}$ is much involved and is presented in appendix D.

### 4. Spin connection as a solution to the torsion-free condition

In this section, the torsion-free condition (3.10) will be solved for the spin connection $\omega_{\mu}$.
For this purpose, covariant derivative of the vielbein will be introduced. This must be done in such a way that the derivative is compatible with the metric $g_{\mu\nu}$ and other gauge fields $g_{(\mu\nu),\lambda}$, $g_{(\mu\nu)(\lambda\rho)}$.

#### 4.1. Covariant derivatives

The torsion-free condition (3.10) takes the form of covariant constancy of $e^\nu_v$:
\[
D_\mu e^\nu_v = \nabla_\mu e^\nu_v + J^{\rho\mu}_{0\nu} \omega^\rho_v e^\nu_v = 0,
\]
\[
\nabla_\mu e^\nu_v = \partial_\mu e^\nu_v - \Gamma^\lambda_{\mu\nu} e^\nu_\lambda.
\]
The first equation is defining a full covariant derivative $D_\mu$ and the second equation is defining $\nabla_\mu$. Note that $\nabla_\mu$ is a new covariant derivative which differs from $\hat{\nabla}_\mu$ associated with the Christoffel symbol $\hat{\Gamma}_{\mu\nu}^\lambda$. The last term of (4.2) can be rewritten as $-\frac{1}{4} \Gamma_{\mu\nu}^{(\rho)} d^{\rho\mu} e^\rho_v e^\nu_v$.

This definition keeps the covariance under the local frame rotations.

By using (3.13), it can be shown that the effect of $\nabla_\mu$ on $g_{\nu\lambda}$ agrees with that of $\hat{\nabla}_\mu$:
\[
\nabla_\mu g_{\nu\lambda} = \nabla_\mu (e^\nu_v e^\lambda_k) = (\nabla_\mu e^\nu_v) e^\lambda_k + e^\nu_v (\nabla_\mu e^\lambda_k) = 0.
\]
\[
\delta_{\mu\lambda} - \Gamma_{\mu\lambda}^{(\rho)} g_{\rho\lambda} - \Gamma_{\lambda\rho}^{(\mu)} g_{\rho\lambda} = \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma)} g_{(\sigma\lambda)} - \frac{1}{2} \Gamma_{\mu\lambda}^{(\rho)} g_{(\rho\nu)} = 0.
\]
Note that this can be shown by using only (3.17). The explicit expression for $\Gamma_{\mu\nu}^{(\rho)}$ is unnecessary.

Let us next start with (3.12). By interchanging $v$ and $\lambda$ and adding the result to (3.12) one obtains an equation
\[
\partial_\mu \left[ e_v, e_\lambda \right] + \left[ \omega_{\mu}, e_v, e_\lambda \right] = \Gamma_{\mu\nu}^{(\sigma)} \left[ e_\sigma, e_\lambda \right] + \Gamma_{\mu\lambda}^{(\sigma)} \left[ e_v, e_\sigma \right] + \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma)} \left[ e_{(\sigma\lambda)}, e_\lambda \right] + \frac{1}{2} \Gamma_{\mu\lambda}^{(\sigma)} \left[ e_{(\sigma\lambda)}, e_v \right]
\]
\[
\quad + \frac{1}{2} \Gamma_{\mu\lambda}^{(\sigma)} \left[ e_v, e_{(\sigma\lambda)} \right].
\]

Let us subtract the terms proportional to the identity matrix from the above equation. The trace part was already studied just after (3.12). Owing to (3.5) and (3.14) the result is
\[
\partial_\mu e_{(\sigma\lambda)} + \left[ \omega_{\mu}, e_{(\sigma\lambda)} \right]
\]
\[
= - \frac{1}{2} \partial_\mu (g_{(\sigma\lambda)} \rho) - \frac{1}{2} g_{(\sigma\lambda)} \left[ \omega_{\mu}, \rho \right] + \Gamma_{\mu\nu}^{(\sigma)} e_{(\sigma\lambda)} + \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma)} g_{\nu\lambda} \rho + \Gamma_{\mu\lambda}^{(\sigma)} e_{(\nu\lambda)}
\]
\[
+ \frac{1}{2} \Gamma_{\mu\lambda}^{(\sigma)} e_{(\nu\lambda)} + \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma)} e_{(\nu\lambda)} + \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma)} g_{(\sigma\lambda)} I.
\]
Expansion in terms of the basis $t_a$ yields the equation
\[ f^a_{bc} \omega^b_{\mu} e^c_{(\nu \lambda)} = -\partial_{\mu} e^a_{(\nu \lambda)} + \Gamma^\sigma_{\mu \nu} e^e_{(\sigma \lambda)} + \Gamma^\sigma_{\nu \lambda} e^e_{(\mu \sigma)} + \frac{1}{2} \Gamma^\sigma_{\mu \nu} d^b_{bc} e^b_{(\sigma \lambda)} e^e_{\lambda} + \frac{1}{2} \frac{1}{\Gamma_{\mu \nu}^\sigma} d^a_{bc} e^b_{(\sigma \lambda)} e^e_{\lambda} + \frac{1}{2} \delta_{\nu \lambda} \partial_{\mu} G_{\mu \nu} - \frac{1}{4} \Gamma_{\nu \lambda} \partial_{\mu} \rho^\sigma - \frac{1}{2} \delta_{\nu \lambda} f^a_{bc} \omega^b_{\mu} \rho^c + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \nu} \rho^\sigma + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \lambda} \rho^\mu. \] (4.6)

Here $\rho^\sigma$ is defined by $\rho = \rho^a t_a$. Contraction of the left-hand side with $\tilde{g}^{\lambda \kappa}$ vanishes. So the same must hold for the right-hand side. This leads to a differential equation for $\rho$. This equation can be interpreted as the covariant-constancy condition for $\rho^a$:
\[ D^a \rho^a = \nabla_{\nu} \rho^a + \frac{1}{2} f^a_{bc} \omega^b_{\mu} \rho^c = 0, \] (4.7)
\[ \nabla_{\nu} \rho^a = \tilde{\Gamma}_{\mu \nu}^\sigma \rho^\sigma + \frac{1}{2} \delta_{\nu \lambda} \partial_{\mu} G_{\mu \nu} - \frac{1}{4} \Gamma_{\nu \lambda} \partial_{\mu} \rho^\sigma - \frac{1}{2} \delta_{\nu \lambda} f^a_{bc} \omega^b_{\mu} \rho^c + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \nu} \rho^\sigma + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \lambda} \rho^\mu. \] (4.8)

The first equation is defining a full covariant derivative $D^a \rho^a$ in terms of $\nabla_{\nu} \rho^a$ and the second equation is defining $\nabla_{\nu} \rho^a$. Here (3.13) is used to relate $\Gamma_{\mu \nu}^\sigma$ to $\Gamma_{\mu \nu}^{0 \sigma}$. The expression (4.8) may look odd, because $\rho^a$ does not have any spacetime index. However, this equation can also be derived by starting with (3.6), i.e. $\rho^a = \frac{1}{2} \tilde{g}^{\mu \nu} d^a_{bc} e^b_\mu e^c_\nu$, and using (4.2), (4.3) and (3.5).

The remaining traceless part of (4.6) can also be interpreted as expressing the property of covariant-constancy of $e^a_{(\nu \lambda)}$:
\[ D^a e^a_{(\nu \lambda)} = \nabla_{\nu} e^a_{(\nu \lambda)} + \frac{1}{2} f^a_{bc} \omega^b_{\mu} e^c_{(\nu \lambda)} = 0, \] (4.9)
\[ \nabla_{\nu} e^a_{(\nu \lambda)} = \tilde{\Gamma}_{\mu \nu}^\sigma e^\sigma_{(\nu \lambda)} - \frac{1}{2} \delta_{\nu \lambda} \partial_{\mu} G_{\mu \nu} - \frac{1}{4} \Gamma_{\nu \lambda} \partial_{\mu} \rho^\sigma - \frac{1}{2} \delta_{\nu \lambda} f^a_{bc} \omega^b_{\mu} \rho^c + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \nu} \rho^\sigma + \frac{1}{2} \tilde{G}_{\nu \lambda} \Gamma^\sigma_{\mu \lambda} \rho^\mu. \] (4.10)

The right-hand side of (4.10) can be expanded in terms of $e^a_\mu$ and $e^a_{(\mu \nu)}$ as
\[ \nabla_{\nu} e^a_{(\nu \lambda)} = \partial_{\nu} e^a_{(\nu \lambda)} - \Gamma^a_{\nu,\lambda} e^a_{(\nu \lambda)} = -\frac{1}{2} \Gamma^a_{\nu,\lambda} e^a_{(\nu \lambda)} \] (4.11)
and a new set of connections are defined
\[ \Gamma^a_{\mu,\nu} = E^a_{(\mu \nu)} (\partial_{\mu} e^a_{(\nu \lambda)} - \nabla_{\nu} e^a_{(\nu \lambda)}) = \frac{1}{4} \Gamma^{(\sigma \lambda)}_{\mu \nu} d^a_{bc} E^b_{(\sigma \lambda)} e^e_{(\lambda \nu)} - \frac{1}{12} \delta_{\nu \lambda} \Gamma^{(\sigma \lambda)}_{\mu \nu} d^a_{bc} E^b_{(\sigma \lambda)} e^e_{(\lambda \nu)} + \frac{1}{18} \delta_{\nu \lambda} \rho^a \rho^c E^a_{\mu \nu} \Gamma^{(\sigma \lambda)}_{\mu \nu} e^e_{(\sigma \lambda)} + (\nu \leftrightarrow \lambda), \] (4.12)
\[ \Gamma^a_{\mu,\nu} = E^a_{(\mu \nu)} (\partial_{\mu} e^a_{(\nu \lambda)} - \nabla_{\nu} e^a_{(\nu \lambda)}) = \Gamma^a_{\mu \nu} + \frac{1}{2} \Gamma^a_{\nu,\lambda} e^e_{(\lambda \nu)} - \frac{1}{2} \Gamma^a_{\nu,\lambda} \delta^a_{\nu \lambda} \tilde{g}^{\sigma \sigma} + \frac{1}{4} \delta_{\nu \lambda} \delta^a_{\nu \lambda} e^e_{(\lambda \nu)} + \frac{1}{12} \delta_{\nu \lambda} \Gamma^{(\sigma \lambda)}_{\mu \nu} d^a_{bc} E^b_{(\sigma \lambda)} e^e_{(\lambda \nu)} + \frac{1}{18} \delta_{\nu \lambda} \rho^a \rho^c E^a_{\mu \nu} \Gamma^{(\sigma \lambda)}_{\mu \nu} e^e_{(\sigma \lambda)} + (\nu \leftrightarrow \lambda). \] (4.13)

In the above equations, $\rho^a E^a_{\mu \nu}$ are given by
\[ \rho^a E^a_{\mu \nu} = \rho^a \delta^a_{\mu \nu} + \frac{1}{2} \tilde{g}^{\mu \nu} f^{(\sigma \lambda)}_{\mu \nu} (\psi_{\nu \lambda} \phi_{\mu \nu} - \phi_{\nu \lambda} \psi_{\mu \nu}) \tilde{g}^{\sigma \sigma}. \] (4.14)
\[ \rho^a E^a_{\mu \nu} = \frac{1}{2} f^{(\mu \nu)} \tilde{g}^{(\sigma \lambda)} (\phi_{\mu \nu} \phi_{\sigma \lambda} - \phi_{\sigma \lambda} \phi_{\mu \nu}) \tilde{g}^{\sigma \sigma}. \] (4.15)

To prove these equations, $\rho^a = \frac{1}{2} \tilde{g}^{\mu \nu} d^a_{bc} e^b_\mu e^c_\nu$ and (C.2), (E.4)–(E.6) and (E.11) must be used. Furthermore, $d^a_{bc}$ can be replaced by (C.6) in appendix C. In this way, the right-hand sides of (4.12) and (4.13) could be expressed solely in terms of the metric-like fields. This will, however, not be attempted in this paper.
Alternatively, the covariant derivative of \( e^a_{(\alpha \beta)} \) (4.10) can be derived from that for \( e^a_\mu \) (4.2) by using \( e^a_{(\alpha \beta)} = \frac{1}{2} d^a_{\beta \epsilon} e^\epsilon_{(\alpha \beta)} - \frac{1}{2} g_{\alpha \beta} \rho^a \). Therefore, we can also write
\[
\nabla_{\mu} e^a_{(\alpha \beta)} = \frac{1}{2} d^{a}_{\beta \epsilon} (\nabla_{\mu} e^\epsilon_{(\alpha \beta)} + e^\epsilon_{(\mu \alpha)} \nabla_{\beta} e^a_{\epsilon} - e^\epsilon_{(\mu \beta)} \nabla_{\alpha} e^a_{\epsilon}) - \frac{1}{2} g_{\alpha \beta} \nabla_{\mu} \rho^a. \tag{4.16}
\]
It can be shown that this equation agrees with (4.10). For \( \rho^a \), we can write
\[
\nabla_{\mu} \rho^a = d^{a}_{\beta \epsilon} \rho^a \nabla_{\mu} e^\epsilon_{(\alpha \beta)}.
\tag{4.17}
\]
Because \( D_{\mu} d^{a}_{\beta \epsilon} = 0 \beta \) and \( D_{\mu} g^{\alpha \beta} = -g^{\alpha \kappa} (\nabla_{\mu} g_{\kappa \alpha}) g^{\beta \rho} = 0 \), the covariant constancy of \( \rho^a \), \( D_{\mu} \rho^a = 0 \), is a result of that of \( e^a_{(\alpha \beta)} \).

### 4.2. Covariant derivatives for general tensors

Let \( v^a \) be a vector in the local Lorentz frame. This vector can be expanded in terms of the vielbeins in either way
\[
v^a = v^\mu e^a_{\mu} + \frac{1}{2} v^{(\mu \nu)} e^a_{(\mu \nu)}, \tag{4.18}
\]
or
\[
v^a = v_\mu E^{a \mu} + \frac{1}{2} v^{(\mu \nu)} E^a_{(\mu \nu)}. \tag{4.19}
\]
Equation (4.18) defines the contravariant components and (4.19) the covariant ones. We choose \( v^{(\mu \nu)} \) and \( v^{(\mu \nu)} \) such that they satisfy traceless conditions, \( v^{(\mu \nu)} G_{\mu \nu} = 0 \) and \( v^{(\mu \nu)} G^{\mu \nu} = 0 \), respectively. Similar decomposition can be performed for tensors with an arbitrary number of local frame indices as. A general rule for decomposition is that for each local frame index \( a \) there corresponds a pair of spacetime indices, \( \mu \) and \( (\mu \nu) \). The parenthesis in \( (\mu \nu) \) implies the symmetry under interchange of \( \mu \) and \( v \). To avoid confusion, the indices of \( v^\mu \), \( v^{(\mu \nu)} \) will not be raised or lowered by \( g_{\mu \nu} \).

In this subsection, we will restrict the discussion to the case where the covariant derivative of \( v^a \) is given by \( \nabla_{\mu} v^a = \partial_{\mu} v^a + \Gamma^\nu_{\mu \lambda} v^\lambda \). By use of (4.18) this leads to the identity
\[
(\nabla_{\mu} v^a) e^a_{\epsilon} + v^\lambda (\nabla_{\mu} e^a_{\epsilon}) + \frac{1}{2} (\nabla_{\mu} v^{(\lambda \rho)}) e^\epsilon_{(\lambda \rho)} + \frac{1}{2} v^{(\lambda \rho)} (\nabla_{\mu} e^\epsilon_{(\lambda \rho)}) = \partial_{\mu} v^a e^\epsilon_{(\alpha \beta)} + v^\lambda (\partial_{\mu} e^\epsilon_{(\alpha \beta)}) e^\epsilon_{\lambda} + \frac{1}{2} v^{(\lambda \rho)} (\partial_{\mu} e^\epsilon_{(\lambda \rho)}) + \frac{1}{2} v^{(\lambda \rho)} (\partial_{\mu} e^\epsilon_{(\lambda \rho)}).
\tag{4.20}
\]
By comparing the coefficients of \( e^\epsilon_{(\alpha \beta)} \) and \( e^\epsilon_{(\alpha \beta)} \) on both sides one obtains the definitions of the covariant derivatives of \( v^\mu \) and \( v^{(\mu \nu)} \):
\[
\nabla_{\mu} v^\nu = \partial_{\mu} v^\nu + \Gamma^\nu_{\mu \lambda} v^\lambda + \frac{1}{2} \Gamma^\nu_{\mu (\lambda \rho)} v^{(\lambda \rho)}, \tag{4.21}
\]
\[
\nabla_{\mu} v^{(\lambda \rho)} = \partial_{\mu} v^{(\lambda \rho)} + \Gamma^\nu_{\mu (\lambda \rho)} v^\nu + \frac{1}{2} \Gamma^\nu_{\mu (\lambda \rho)} v^{(\nu \rho)}. \tag{4.22}
\]
By using the expansion (4.19) and taking the similar steps, the covariant derivatives of the covariant components are also obtained
\[
\nabla_{\mu} v_\nu = \partial_{\mu} v_\nu - \Gamma^\lambda_{\mu \nu} v_\lambda - \frac{1}{2} \Gamma^\lambda_{\mu (\nu \sigma)} v^{(\lambda \sigma)}, \tag{4.23}
\]
\[
\nabla_{\mu} v_{(\lambda \rho)} = \partial_{\mu} v_{(\lambda \rho)} - \Gamma^\lambda_{\mu (\nu \rho)} v_\nu - \frac{1}{2} \Gamma^\lambda_{\mu (\nu \rho)} v^{(\nu \rho)}. \tag{4.24}
\]
An extension to the covariant derivatives for the tensors with more indices will be straightforward and clear. The rule is the same as in the spin-2 gravity. For instance, by using (4.23) \( \nabla_{\mu} g_{\nu \lambda} \) is calculated as
\[
\nabla_{\mu} g_{\nu \lambda} = \partial_{\mu} g_{\nu \lambda} - \Gamma^\rho_{\mu \gamma} g_{\nu \lambda \gamma} - \frac{1}{2} \Gamma^\rho_{\mu (\nu \sigma)} g_{\lambda (\nu \sigma)} - \Gamma^\rho_{\mu \gamma} g^{\rho \sigma} - \frac{1}{2} \Gamma^\rho_{\mu \lambda} g^{\rho \sigma} g_{(\nu \sigma)}. \tag{4.25}
\]

8 This can be proved by using Jacobi’s identity containing \( e^a_{(\alpha \beta)} \) and \( f_{abc} \).
9 Instead, this will be done in terms of \( G_{MN} \) defined in appendix E.
10 The rule of the covariant derivatives in this subsection cannot be used for \( \rho^a \) and \( \rho^{(\mu \nu)} \) defined by \( \rho^a = \rho^a e^a_{\mu} + (1/2) \rho^{(\mu \nu)} e^a_{(\mu \nu)} \), because \( \nabla_{\mu} \rho^a \) is not given by \( \partial_{\mu} \rho^a \), but by (4.8).
because \(g_{\rho\lambda\mu}\) is paired with \(g_{\rho\lambda}\). This vanishes as in (4.3). This is an important property of the metric tensor \(G_{MN}\) introduced in appendix E, which can be used to raise and lower the indices, \(\mu\) and \((\mu \nu)\). One can also explicitly check equation \(\nabla_\mu g_{\nu(\rho \lambda)} = 0\) by using (3.22), (4.12) and (4.13).

As we have seen, the indices of tensors have the structure, \(\mu a\). Multiplication of (4.1) and (4.9) by \(E^\nu_a\) and \(\frac{1}{2} E^{(\nu \lambda)}_a\), respectively, and adding the two we obtain the spin connection in the adjoint representation

\[
\omega^a_{\mu c} \equiv f^a_{bc} \omega^b_\mu = -E_c^\nu \nabla_\mu e^\nu_a - \frac{1}{2} E^a_\nu \nabla_\mu \epsilon^\mu_{(\nu \lambda)}.
\]

Then use of (A.7) yields

\[
\omega^a_\mu = \omega^a_\mu (e) \equiv \int \frac{1}{2\pi} f^{ab} \epsilon^a_{\nu \lambda} \nabla_\mu e^\nu_b + \frac{1}{2\pi} f^{ab} \epsilon^a_{(\nu \lambda)} \nabla_\mu e^\nu_b \epsilon_\mu^c.
\]

Into the full covariant derivatives for \(e^a_{\mu\nu}, e^a_{(\mu \nu)}\) and \(\rho^a\) introduced above the spin connection \(\omega^a_\mu (e)\) (4.29) is to be substituted.

4.4. The second-order action

Now we substitute the solution (4.29) into the action (2.4) and obtain the second-order action

\[
S_{\text{second order}} = \frac{k}{12\pi} \int \mathcal{M} d^3x \left[ -\epsilon^{\alpha \nu \lambda} \left( f^a_{bc} e^a_{\nu \lambda} e^b_\mu E^N_a \right) R^a_{\nu \lambda \sigma} + 4 \epsilon^{\nu \lambda \sigma} f_{abc} e^a_\mu e^b_\nu e^c_\lambda \right].
\]

We do not consider the boundary terms. To derive classical equations of motion from this action, \(\omega^a_\mu (e)\) is to be varied as a functional of \(e^a_\mu\).

In section 7, we will derive the generalized Riemann curvature tensor \(R_{\nu \lambda \sigma}^{M}\) (6.11). The above action in the second-order formalism can be reexpressed in terms of this tensor

\[
S_{\text{second order}} = \frac{k}{12\pi} \int \mathcal{M} d^3x \left[ -\epsilon^{\nu \lambda \sigma} f^a_{bc} e^a_\mu \left( e^b_\nu E^N_a R^a_{\nu \lambda \sigma} + \frac{1}{2} e^b_\nu E^N_a R^a_{\nu \lambda \sigma} + \frac{1}{2} e^b_\nu E^N_a R^a_{\nu \lambda \sigma} + \frac{1}{4} e^b_\nu E^N_a R^a_{\nu \lambda \sigma} + \frac{1}{4} e^b_\nu E^N_a R^a_{\nu \lambda \sigma} \right) \right].
\]

See section 7 for the derivation. Here, \(f^a_{bc} e^a_\mu e^b_\nu e^c_\lambda\) is a metric-like quantity which is insensitive to the local frame rotations, while \(\epsilon^{\nu \lambda \sigma} f_{abc} e^a_\mu e^b_\nu e^c_\lambda\) is the generalized cosmological
term. So the action integral is expressed in terms of the connections $\Gamma_{\mu\nu}^\lambda$ and the vielbeins. The remaining problem is to represent this action only in terms of the metric-like fields. This is not attempted in this paper.

Under the local frame rotations the vielbein transforms as (2.6), and it is easy to show that the spin connection $\omega_{\mu}^a(e)$ transforms as (2.7). Next, under the generalized diffeomorphisms, the vielbein transforms as (2.8). This can be rewritten as

$$\delta_1 e_{\mu}^a = D_\mu \Lambda^a_\nu = D_\mu \left( \tilde{\xi}^M \right) = D_\mu \left( \tilde{\xi}^v e_v^a + \frac{1}{2} \tilde{\xi}^{(v\lambda)} e_{(v\lambda)}^a \right).$$  \hfill (4.32)

Here $\tilde{\xi}^M$ represents $\tilde{\xi}^v$ and $\tilde{\xi}^{(v\lambda)}$, and they are defined by

$$\tilde{\xi}^M = \Lambda^a_\nu E^a_\mu = G^{MN} \xi_N.$$  \hfill (4.33)

These are the local parameters of the generalized diffeomorphisms. For the definition of $G^{MN}$ and the notation $M = (\mu \nu)$, see appendix E. The tilde in the notation $\tilde{\xi}^M$ means that the metric tensor $G^{MN}$ is used to raise the indices. Because $e_{\mu}^a$ and $e_{(\mu\nu)}^a$ are covariantly constant, we have

$$\delta_1 e_{\mu}^a = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \frac{1}{2} e_{(\nu\lambda)}^a \nabla_\mu \tilde{\xi}^{(v\lambda)}.$$  \hfill (4.34)

This does not look like a transformation rule for a covariant vector. However, one can perform, additionally, a local frame rotation $\delta_2 e_{\mu}^a = f^a_{bc} e_{\mu}^b \Lambda^b_\nu$ with $\Lambda^a_\nu = \alpha_{(\mu)}^a (e) \tilde{\xi}^v$. The combined transformation is

$$\delta_{\text{diffeo}} e_{\mu}^a = \delta_1 e_{\mu}^a + \delta_2 e_{\mu}^a = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a + \frac{1}{2} e_{(v\lambda)}^a \nabla_\mu \tilde{\xi}^{(v\lambda)}.$$  \hfill (4.35)

Here the torsion-free condition is used to replace $f^a_{bc} \alpha_{(\nu)}^b (e) e_{\mu}^c$ by $- \nabla_\nu e_{\mu}^a$. This is the generalization of the diffeomorphism for the vielbein to the spin-3 gravity theory. If $\tilde{\xi}^{(v\lambda)} = 0$, the above equation agrees with the transformation rule of a covariant vector.

The new transformation rule of $\omega_{\mu}^a(e)$ can also be obtained by combining the generalized diffeomorphism $\delta_1 \omega_{\mu}^a(e) = f^a_{bc} \omega_{\mu}^b \Lambda^b_\nu$ with the local frame rotation $\delta_2 \omega_{\mu}^a(e) = \partial_\mu \Lambda^a_\nu + f^a_{bc} \alpha_{(\nu)}^b (e) \Lambda^b_\nu$:

$$\delta_{\text{diffeo}} \omega_{\mu}^a(e) = \delta_1 \omega_{\mu}^a(e) + \delta_2 \omega_{\mu}^a(e) = \omega_{\nu}^a(e) \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu \omega_{\mu}^a(e) - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$$  \hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)

On the right-hand side of (4.36) a term $\tilde{\xi}^v (R_{\mu\nu}^a + f^a_{bc} \omega_{\mu}^b \delta_1 = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$\hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)

On the right-hand side of (4.36) a term $\tilde{\xi}^v (R_{\mu\nu}^a + f^a_{bc} \omega_{\mu}^b \delta_1 = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$\hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)

On the right-hand side of (4.36) a term $\tilde{\xi}^v (R_{\mu\nu}^a + f^a_{bc} \omega_{\mu}^b \delta_1 = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$\hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)

On the right-hand side of (4.36) a term $\tilde{\xi}^v (R_{\mu\nu}^a + f^a_{bc} \omega_{\mu}^b \delta_1 = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$\hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)

On the right-hand side of (4.36) a term $\tilde{\xi}^v (R_{\mu\nu}^a + f^a_{bc} \omega_{\mu}^b \delta_1 = e_{\nu}^a \nabla_\mu \tilde{\xi}^v + \tilde{\xi}^v \nabla_\nu e_{\mu}^a - \frac{1}{2} \tilde{\xi}^{(v\lambda)} \Gamma^\nu_{\mu, (\nu \lambda)} \omega_{\mu}^a(e) + \frac{1}{2} \tilde{\xi}^{(v\lambda)} f^a_{bc} \omega_{\mu}^b(e) e_{(v\lambda)}.$\hfill (4.36)

Here in computing $\nabla_\nu \omega_{\mu}^a$ we must set $\alpha_{(\mu)}^a = 0$, since this extra component does not exist.\hfill (4.36)
5. Generalized diffeomorphism for \( g_{\mu \nu} \) and \( \phi_{\mu \lambda} \)

The CS theory \( (2.4) \) has generalized diffeomorphism invariance \( (2.9) \). In this section, the transformation rules of the gauge fields \( g_{\mu \nu} \) and \( \phi_{\mu \lambda} \) will be derived.

5.1. Transformation of \( g_{\mu \nu} \)

Let us first consider the metric field \( (2.10) \). This transforms as

\[
\delta g_{\mu \nu} = \frac{1}{2} \text{tr} \left( \partial_\mu \Lambda + [\omega_\mu, \Lambda] \right) e_\nu + (\mu \leftrightarrow \nu)
\]

\[
= \frac{1}{2} \partial_\mu \text{tr} \Lambda e_\nu - \frac{1}{2} \text{tr} \Lambda \left( \partial_\mu e_\nu + [\omega_\mu, e_\nu] \right) + (\mu \leftrightarrow \nu). \quad (5.1)
\]

For simplicity of notation, \( \Lambda_\nu \) in \( (2.9) \) is here denoted as \( \Lambda \). Now, two variation functions are introduced

\[
\xi_\mu = \frac{1}{2} \text{tr} \Lambda e_\mu. \quad (5.2)
\]

\[
\zeta_{(\mu \nu)} + g_{(\mu \nu)} \xi^\rho = \xi_{(\mu \nu)} = \frac{1}{2} \text{tr} \Lambda e_{(\mu \nu)}. \quad (5.3)
\]

To some extent one can regard \( \xi_\mu \) and \( \zeta_{(\mu \nu)} \) as the coordinate transformation parameter and the spin-3 gauge parameter, respectively. There are, however, some differences. Such differences can be observed at \( (5.13) \) and \( (5.14) \) below. This is because in a spin-3 gauge theory, diffeomorphisms and spin-3 gauge transformations are mixed and not easily disentangled. In equation \( (5.3) \), an extra parameter \( \xi_{(\mu \nu)} \) is also introduced. With the help of \( (3.10) \), \( \delta g_{\mu \nu} \) can be put into the form

\[
\delta g_{\mu \nu} = \partial_\mu \xi_\nu - \Gamma_{\mu \nu}^\rho \xi_\rho - \frac{1}{2} \Gamma_{\mu \nu}^{(\lambda \rho)} \left( \zeta_{(\lambda \rho)} + g_{(\lambda \rho)} \xi^\sigma \right) + (\mu \leftrightarrow \nu)
\]

\[
= \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \Gamma_{\mu \nu}^{(\lambda \rho)} \zeta_{(\lambda \rho)}. \quad (5.4)
\]

Here \( (3.17) \) is used and \( \hat{\nabla}_\mu \) is the ordinary covariant derivative that uses Christoffel symbol \( \hat{\Gamma}_{\mu \nu}^\rho \). Therefore, those parts which depend on \( \xi_\mu \) are the ordinary diffeomorphism. The remaining term, which depends on \( \zeta_{(\mu \nu)} \), is the new spin-3 gauge transformation. This term depends nonlinearly on gauge fields such as \( g_{(\mu \nu)(\lambda \rho)} \), via \( J^{(\mu \nu)(\kappa \sigma)} \), since \( \Gamma_{\mu \nu}^{(\lambda \rho)} \) also does.

Interestingly, this infinitesimal transformation can also be written as

\[
\delta g_{\mu \nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (5.5)
\]

by adopting the covariant derivative \( (4.23) \) introduced in section 4. Here \( \xi_{(\mu \nu)} \) is used as the partner of \( \xi_\mu \) in computing the derivative. The above result can also be derived by using \( g_{\mu \nu} = e^\mu_\alpha e_\nu \) and the transformation rule of \( e^\mu_\alpha \).

5.2. Transformation of \( \phi_{\mu \lambda \kappa} \)

Let us next turn to the spin-3 gauge field \( \phi_{\mu \lambda \kappa} \). In this case, the variation can be rewritten as follows:

\[
\delta \phi_{\mu \lambda \kappa} = \frac{1}{2} \text{tr} \left( \partial_\mu \Lambda + [\omega_\mu, \Lambda] \right) \left( e_\kappa, e_\lambda \right) + (2 \text{ cyclic permutations of } \mu, \nu, \lambda)
\]

\[
= \frac{1}{2} \partial_\mu \text{tr} \Lambda e_\nu + \frac{1}{2} \partial_\nu \text{tr} \rho \Lambda - \frac{1}{2} \Gamma_{\mu \nu}^\rho \text{tr} \Lambda e_\rho - \frac{1}{2} \Gamma_{\mu \nu}^{(\rho \lambda)} \text{tr} \Lambda e_{(\rho \lambda)}
\]

\[
- \frac{1}{2} \Gamma_{\mu \lambda}^{(\rho \kappa)} \text{tr} \Lambda e_{(\rho \kappa)} + \frac{1}{2} \rho \text{tr} \Lambda e_\nu g_{\rho \lambda} - \frac{1}{2} \Gamma_{\mu \nu}^{(\rho \lambda)} \text{tr} \Lambda e_{(\rho \lambda)}
\]

\[
- \frac{1}{2} \Gamma_{\mu \lambda}^{(\rho \kappa)} \text{tr} \Lambda e_{(\rho \kappa)} + (\text{permutations}). \quad (5.6)
\]

Here \( (3.5) \) and \( (4.4) \) are used.

In this expression \( \text{tr} \Lambda e_{(\rho \lambda)} \) is rewritten by \( \Lambda_\nu = t_\nu \Lambda^\nu \). To compute other terms involving \( \Lambda \), \( (5.2) \) and \( (5.3) \) must be solved for \( \Lambda = t_\nu \Lambda^\nu \). Multiplying \( \xi_\mu = \Lambda_\mu e^\mu_\alpha \) and
Here those terms which include $\xi(\mu \nu)$ can also be worked out. After certain amount of calculation the $\xi$ transformation of the spin-3 gauge field is obtained

$$
\delta \xi_{\mu \nu} = \xi^\alpha \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} + \frac{1}{2} \xi_{\beta} \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} - \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} + \frac{1}{2} \xi_{\beta} \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} + \frac{1}{2} \frac{1}{2} \xi_{\beta} \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} - \xi_{\beta} \tilde{\Gamma}_{\mu \nu}^{\alpha \beta} \xi_{\beta} + \{ \text{cyclic permutations of } \mu, \nu, \lambda \}.
$$

Therefore, except for the trace parts and the terms containing $S_{\mu \nu, \lambda \rho}$, the spin-3 gauge field $\phi_{\mu \nu \lambda}$ transforms as a spin-3 tensor under ordinary diffeomorphisms ($\xi_{\mu}$).

Those terms which depend on $\xi(\mu \nu)$ can be simplified further by using (4.14)–(4.15). The gauge fields $g(\kappa(\sigma)\tau \rho),$ $g(\kappa(\sigma)(\rho \mu),$ and $\phi_{\kappa \sigma \rho \tau}$ are defined in appendix C. Under a new spin-3 gauge
transformation \((\xi(\mu\nu))\), \(\phi_{\mu\nu\lambda}\) transforms in a complicated way which depends on higher indexed gauge fields. Transformations of these gauge fields must also be studied. However, in this paper this will not be attempted.

At the beginning of this section, it was shown that by using the new covariant derivative \(\nabla_{\mu}\), the transformation \(\delta g_{\mu\nu}\) can be compactly expressed as (5.5) just like in Einstein gravity. Then one may expect that due to relations among gauge fields, the transformation \(\delta \phi_{\mu\nu\lambda} = \delta \xi \phi_{\mu\nu\lambda} + \delta \zeta \phi_{\mu\nu\lambda}\) might also be succinctly written. Actually, using \(\delta e_{\mu}^{a} = D_{\mu} \Lambda^{c}_{a} + \frac{1}{2} g_{\mu\nu} \delta \rho^{a}\), one can show that

\[
\delta \phi_{\mu\nu\lambda} = \nabla_{\mu} \left( \xi(\nu\lambda) + \frac{1}{3} \rho^{a} \Lambda_{a} g_{\nu\lambda} \right) + \nabla_{\nu} \left( \xi(\lambda\mu) + \frac{1}{3} \rho^{a} \Lambda_{a} g_{\lambda\mu} \right) + \nabla_{\lambda} \left( \xi(\mu\nu) + \frac{1}{3} \rho^{a} \Lambda_{a} g_{\mu\nu} \right).
\]

(5.15)

Except for the trace terms this agrees with the expected transformation rule of the spin-3 gauge field.

6. Parallel transport and Curvature tensor

To investigate the spin-3 geometry, it is useful to introduce a parallel transport matrix, holonomy matrix and curvature tensor. This will be done in this section.

Let \(v^{a}(x)\) be a vector field in the local Lorentz frame. For an arbitrary curve \(x^{\mu} = x^{\mu}(s)\), this vector is said to be parallel transported along the curve [31], if it satisfies the equation

\[
\frac{dv^{a}}{ds} + \omega^{a}_{\mu b}(e) \frac{dx^{\mu}}{ds} v^{b} = 0.
\]

(6.1)

This equation can be solved in terms of the ordered exponential

\[
U^{a}_{b}(s, 0) v^{b}(0) = \left( P \exp \left\{ - \int_{0}^{s} \omega^{a}_{\mu b}(e) \frac{ds'}{ds} \right\} \right)^{a}_{b}.
\]

(6.2)

Here, as usual, the symbol \(P\) denotes path ordering

\[
P(A(s_{1}) B(s_{2})) = \begin{cases} A(s_{1}) B(s_{2}) & \text{if } s_{1} > s_{2}, \\ B(s_{2}) A(s_{1}) & \text{if } s_{2} > s_{1}. \end{cases}
\]

(6.3)

These relations can be converted into that for spacetime quantities by means of the vielbeins. Firstly, we perform the following GL(8, R) gauge transformation on the spin connection matrix \(\omega_{\mu b}(e)\): \(\omega_{\mu b}(e) \rightarrow \Upsilon_{\mu N}^{M} = E_{a}^{M} \omega_{\mu b}(e) e_{b}^{N} + E_{a}^{M} \partial_{\mu} e_{N}^{b}\).

(6.4)

Here \(M, N = \mu, (\mu\nu)\) are the indices explained in appendix E by (4.28) the new spin connection can be written as

\[
\Upsilon_{\mu N}^{M} = -E_{a}^{M} \nabla_{\mu} e_{b}^{N} + E_{a}^{M} \partial_{\mu} e_{b}^{N}.
\]

(6.5)

This agrees with the connections defined in section 4,

\[
\Upsilon_{\mu \nu}^{\lambda} = \Gamma_{\mu \nu}^{\lambda}, \quad \Upsilon_{\mu (\rho \nu)}^{\lambda} = \Gamma_{\mu (\rho \nu)}^{\lambda}, \quad \Upsilon_{\mu (\rho \nu \lambda)}^{(\sigma \rho)} = \Gamma_{\mu (\rho \nu \lambda)}^{(\sigma \rho)}.
\]

(6.6)

12 In the case of Einstein gravity a similar transformation is used [30].
Under the gauge transformation (6.5) the path-ordered exponential (6.3) transforms into a spacetime quantity,

\[ U^M_N(s, 0) = \left( P \exp \left\{ - \int_0^s \gamma_\mu \frac{d\gamma^\nu}{ds'} \right\} \right)^M_N. \]  

(6.8)

The parallel transport equation (6.1) is also rewritten as

\[ \frac{dv^M}{ds} + \gamma_\mu \gamma^M_N \frac{dv^\nu}{ds} v^N = 0. \]  

(6.9)

If the curve \( x^\mu(s), (0 \leq s \leq 1) \) is closed, the matrix \( U^M_N(1, 0) \) defines a holonomy matrix. For an infinitesimal closed curve \( \gamma \) which encloses a small surface \( S \), this holonomy can be evaluated by the expansion of the exponential. By using Stokes’s theorem this yields a generalization of the Riemann curvature tensor at the lowest order of expansion

\[ U^M_N(1, 0) = \delta^M_N + \int_S R^M_{N\mu\nu} d\Sigma^{\mu\nu} + \cdots. \]  

(6.10)

Here

\[ R^M_{N\mu\nu} \equiv \partial_\mu \Gamma^M_N - \partial_\nu \Gamma^M_N + \Gamma^M_{\rho N} \Gamma^\rho_{\mu\nu} + \Gamma^M_{\nu N} \Gamma^\nu_{\mu\rho} - \Gamma^M_{\rho \nu} \Gamma^\rho_{\mu N}. \]  

(6.11)

The action integral in the second-order formalism (4.30) can be expressed in terms of this curvature tensor. To do this, we perform the GL(8, R) gauge transformation \( \omega^a_{\mu\nu} = e^c_e \epsilon^a_{b} \Gamma_b^c M \epsilon^M_N E^N_k - E^N_k \partial_\mu \epsilon^a_k + e^k_a \partial_\nu \epsilon^a_k \) on the curvature tensor \( R^M_{\nu\mu\nu} = \partial_\mu \omega^a_{\nu\nu} - \partial_\nu \omega^a_{\nu\mu} + \omega^a_{\nu\rho} \partial_\rho \omega^a_{\mu\nu} - \omega^a_{\mu\rho} \partial_\rho \omega^a_{\nu\nu} \).

Since the curvature 2-form is gauge covariant, we obtain

\[ R^a_{b\nu\mu} = e^a_b E^b_c R^c_{M\nu\mu}. \]  

(6.12)

The identity \( R^a_{b\nu\mu} = f^a_{cde} R^c_{d\nu\mu} \), where \( R^c_{d\nu\mu} = \partial_\mu \omega^c_{\nu\nu} - \partial_\nu \omega^c_{\nu\mu} + f^{cde}_d \partial_\rho \omega^d_{\nu\mu} \omega^e_{\rho\nu} \), leads to (4.31).

In the spin-2 gravity theory, the Riemann curvature tensor also defines the commutator of the covariant derivatives, \[ [\nabla_\nu, \nabla_\mu] v^k = R^k_{\mu\nu\rho} v^\rho. \]  

In this theory, this equation can be derived by starting with the curvature 2-form \( R^M_{\nu\mu\nu} v^k = [D_\nu, D_\mu] v^k = \) by projecting onto the base space using the vielbein as \( v^k = v^i E^i_k \). In the spin-3 case, we also expect similar formulae such as

\[ [\nabla_\nu, \nabla_\mu] v^k = R^k_{\mu\nu\rho} v^\rho + \frac{1}{2} R^{(\rho\lambda)}_{(\mu\nu)\rho\lambda} v^{(\rho\lambda)}, \]  

\[ [\nabla_\nu, \nabla_\mu] v^{(\rho\lambda)} = R^{(\rho\lambda)}_{(\mu\nu)\rho\lambda} v^{(\rho\lambda)} + \frac{1}{2} R^{(\rho\lambda)}_{(\sigma\kappa)\rho\lambda} v^{(\sigma\kappa)}. \]  

(6.13)

(6.14)

However, there is an obstacle in deriving such formulae, because one does not know how to compute \( \nabla_{\mu} v^k \), and the covariant derivative does not have the component \( \nabla_\nu v^k \) in the new direction \( (\mu\nu) \). We will speculate on these formulae in the remaining part of this subsection.

If this component exists, it is possible to compute the commutators of the covariant derivatives. Actually we have \( \nabla_\nu \nabla_\mu v^k = \partial_\nu \partial_\mu v^k + (\partial_\nu \Gamma^k_{\mu\rho}) v^\rho + \Gamma^k_{\mu\nu} \Gamma^\rho_{\nu\rho} v^\rho + \Gamma^k_{\nu\nu} \nabla_\nu v^k + \Gamma^k_{\nu\rho} \Gamma^\rho_{\mu\nu} + \chi^k_{\nu\rho} \Gamma^\rho_{\mu\nu} \) and then \( [\nabla_\nu, \nabla_\mu] v^k = \partial_\nu \nabla_\mu v^k - \partial_\mu \nabla_\nu v^k + \Gamma_{\mu\rho} \nabla_\nu v^k - \Gamma_{\nu\rho} \nabla_\mu v^k + \Gamma_{\mu\nu} \nabla_\mu v^k = R^k_{\mu\nu\rho} v^\rho \). The term \( \Gamma_{\mu\nu} \nabla_\mu v^k \) cancels out in the commutator. The actual value of \( \nabla_\nu v^k \) does not matter, it is important to note that it can even be zero: \( \nabla_\nu v^k = 0 \).

In order to define \( \nabla_\nu v^k \), then, we would need to introduce new coordinates \( x^{\nu\mu} \) and set \( \nabla_\nu v^k = \partial_\nu v^k + \Gamma^k_{\nu\rho} v^\rho \). This, however, would make the spacetime to have dimension 8, and one would need to cope with a problem of integrating over the new coordinates. So one of possible prescriptions would be to avoid introducing \( x^{\nu\mu} \) and set \( \nabla_\nu v^k = \Gamma^k_{\nu\rho} v^\rho \). We would then need to introduce a new component of the spin connection, \( \omega^a_{\nu\mu} \), and impose a torsion-free condition, \( \nabla_\nu v^k = \Gamma^k_{\nu\rho} v^\rho \). This, however, compatibility of this covariant derivative \( \nabla_\nu v^k \) with \( g^a_{\nu\mu} \) would inevitably lead to the conclusion \( \Gamma^k_{\nu\mu\rho} = 0 \) and \( \omega^a_{\nu\mu} = 0 \). To define \( \nabla_\nu v^k \), introduction of extra coordinates seems unavoidable. Therefore, we will set \( \nabla_\nu v^k = 0 \) in this paper. Even in this case the rules (6.13)–(6.14) of the commutators of the covariant derivatives still apply.
7. Gravitational CS term

In 3D, there also exists a gravitational CS term [28]. It is given by

\[ S_{GCS}(\omega) = \frac{k}{8\pi \mu} \int d^3x \epsilon^{\mu\nu\lambda} \left( \omega(e)_{\mu}^{a} \partial_{\nu} \omega(e)_{\lambda}^{b} + \frac{2}{3} \omega(e)_{\mu}^{a} \omega(e)_{\nu}^{b} \omega(e)_{\lambda}^{c} \right). \] (7.1)

Here \( \mu \) is a constant. In this action, the spin connection \( \omega_{\alpha \mu}^{\beta} \) is a functional of the vielbein \( e^{\mu}_{\alpha}, e^{\alpha}_{\mu\nu} \) as defined in (4.29). This action is invariant in the bulk under both the local frame transformation and the generalized diffeomorphism. The invariance is broken at the boundary.

If this term is added to the CS action in the second-order formalism (4.31), the action of a topological massive spin-3 gravity (a generalization of the topological massive gravity [28]) is obtained. In the gravity/CFT correspondence, the gravitational action with the gravitational CS term corresponds to a left–right asymmetric (chiral) CFT. This action has derivatives of cubic order and hence the equations of motion will contain terms with cubic-order derivatives.

Let us note that if the solution for the spin connection is not substituted into the action integral, and the vielbein and the spin connection were treated independently, the torsion-free equation would be modified. In order to avoid this, the torsion-free condition may be imposed by means of a Lagrange multiplier field [29, 12]. However, the generalized diffeomorphism invariance (2.9) will be broken by the multiplier term13.

It is known that in the case of ordinary 3D spin-2 gravity, the gravitational CS term can also be expressed in terms of the Christoffel connection up to a winding number term:

\[ S_{GCS}(\omega) = \frac{k}{8\pi \mu} \int d^3x \epsilon^{\mu\nu\lambda} \left( \Gamma_{\mu,\nu}^{\lambda} \partial_{\sigma} \Gamma_{\lambda,\rho}^{\sigma} + \frac{2}{3} \Gamma_{\mu,\nu}^{\lambda} \Gamma_{\nu,\rho}^{\kappa} \Gamma_{\lambda,\kappa}^{\sigma} \right). \] (7.2)

Actually, this last form of the gravitational CS term must be used in the second-order formalism.

In the case of spin-3 gravity, a similar expression for the action can be derived by using the gauge transformation (6.5). After substitution we have, up to winding number terms,

\[ S_{GCS}^{spin-3}(\Gamma) = \frac{k}{8\pi \mu} \int d^3x \epsilon^{\mu\nu\lambda} \left( \Gamma_{\mu,\nu}^{\lambda} \partial_{\sigma} \Gamma_{\lambda,\rho}^{\sigma} + \frac{1}{2} \Gamma_{\mu,\nu}^{\lambda} \Gamma_{\lambda,\rho}^{\sigma} + \frac{1}{2} \Gamma_{\lambda,\nu}^{\kappa} \Gamma_{\kappa,\rho}^{\sigma} \right) + \frac{1}{8} \Gamma_{\mu,\nu}^{\lambda} \Gamma_{\lambda,\rho}^{\sigma} \Gamma_{\rho,\nu}^{\kappa} \Gamma_{\kappa,\rho}^{\lambda} \right). \] (7.3)

In the spin-2 gravity theory, solutions such as BTZ black hole [34] in the theory without the gravitational CS term are known to be also solutions of the equations of motion of the topologically massive gravity theory. Therefore, the natural questions to ask are: do the solutions in the spin-3 gravity without the gravitational CS term, such as the spin-3 black hole [6], also solve the equations of motion in the spin-3 topologically massive gravity? If it is the case, how the central charges of the W3 algebras in the boundary CFT and the value of the entropy will be modified in the presence of the gravitational CS term?

The black hole solution with spin-3 charge is asymptotically AdS3 with AdS radius \( \lambda \) [6]. Therefore, it may be interesting to study the existence of propagating gravitons with this asymptotic boundary condition. These problems are left for the future studies.

13 A linearized action in the topological massive higher spin gravity is studied in [32]. Topological massive higher spin gravity with a multiplier field is studied in [33].
8. Summary and discussion

In this paper, a second-order formalism of the 3D spin-3 gravity is addressed and it is shown that many of the notions and geometrical quantities of Einstein gravity theory can be introduced into this theory. Extra vielbeins $e_{(μν)}$ (3.5) are introduced in order to eliminate the spin connection from the CS formulation of the 3D spin-3 gravity in a way covariant under the local frame rotations. It is shown that new connections $\Gamma_{\mu}^{N}$ can be expressed in terms of the metric-like fields and that a covariant derivative $\nabla_{\mu}$ (4.2), (4.10) can be defined. The torsion-free condition is solved for the spin connection $\omega_{μ}^{a}$ as (4.29) in terms of the generalized vielbein and its inverse. In terms of this solution, the action integral in the second-order formalism (4.30) is presented, although in a somewhat implicit form. Many metric-like fields other than $g_{μν}$ and $φ_{μνλ}$ are shown to exist. Although they are expected to be expressed in terms of $g_{μν}$ and $φ_{μνλ}$ at least in the case of fluctuations around $\text{AdS}_3$ vacuum, a precise relation among these fields needs to be worked out in the future study. Then a generalized Riemann curvature tensor for the spin-3 gravity is also defined. The explicit form of the generalized diffeomorphism of the metric $g_{μν}$ and the spin-3 gauge field $φ_{μνλ}$ is presented. Finally, the action integral for topologically massive spin-3 gravity is presented explicitly.

In this paper, the transformation rules of the connections $\Gamma_{\mu}^{N}$ under the generalized diffeomorphisms are not studied explicitly. This is because the expression for $S_{μν,λρ}$ in $\Gamma_{\mu}^{(λρ)}$ is complicated. This problem must be studied in the future. However, by assuming the transformation rule of $ω_{μ}^{a}(e)$ as (4.36) and using the relation between $Γ$'s and $ω_{μ}^{a}$ it is possible to derive the transformation rule of $Γ$'s.

For other future work, we would like to consider the coupling of matter fields to the spin-3 gravity. For this purpose, it is necessary to define density and tensors which transform appropriately under the generalized diffeomorphisms. Then it must be shown that the covariant derivatives of the general tensors also transform as tensors. At present, this remains an unsolved problem.

Finally, there will be several directions for future investigations. To enumerate a few, the geometry of the 3D spin-3 gravity is still not well understood. This must be studied further and the spin-3 gravity must be formulated from scratch without relying on CS theory. In the case of supergravity, where gravity theory is likewise extended by supersymmetry transformations, one can understand the theory geometrically by introducing supercoordinates, a superspace and superfields. Likewise, it might be possible to better understand the spin-3 gravity analogously by introducing a 'spin-3 space'.

A generalization of the work in this paper to spin-N($≥$ 4) gravity theories will be straightforward. For example, in the spin-4 gravity theory, extended vielbeins $e_{μ}$, $e_{(μν)}$, and $e_{(μνλ)}$, which are completely symmetric in the indices and satisfy traceless conditions, will provide $3+5+7=15$ basis vectors. This number agrees with the dimension of $\mathfrak{sl}(4, R)$. The case of spin-N gravity works similarly.

Appendix A. $\mathfrak{sl}(3, R)$ algebra

Let the generators $L_{i}$ ($i = −1, 0, 1$), $W_{n}$ ($n = −2, \ldots, 2$) satisfy an $\mathfrak{sl}(3, R)$ algebra

\[
[L_{i}, L_{j}] = (i − j) L_{i+j}, \quad [L_{i}, W_{n}] = (2i − n) W_{i+n},
\]

\[
[W_{m}, W_{n}] = -\frac{1}{3}(m − n)[2m^{2} + 2n^{2} − mn − 8] L_{m+n}. \quad (A.1)
\]
We use the same three-dimensional representation as in [2] with the parameter \( \sigma = -1 \)

\[
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad
L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad
L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad
W_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad
W_0 = \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad
W_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (A.2)

Nonvanishing norms of these matrices are given by

\[
\text{tr}(L_0)^2 = 2, \quad \text{tr}(L_{-1} L_1) = -4, \quad \text{tr}(W_0)^2 = \frac{8}{3}, \quad \text{tr}(W_1 W_{-1}) = -4, \quad \text{tr}(W_2 W_{-2}) = 16.
\] (A.3)

These generators will also be collectively denoted as \( t_a \), \((a = 1, \ldots, 8)\),

\[
t_1 = L_1, \quad t_2 = L_0, \quad t_3 = L_{-1},
\]

\[
t_4 = W_2, \quad t_5 = W_1, \quad t_6 = W_0, \quad t_7 = W_{-1}, \quad t_8 = W_{-2}.
\] (A.4)

The structure constants \( f_{ab}^c \) are defined by

\[
[t_a, t_b] = f_{ab}^c t_c.
\] (A.5)

The Killing metric \( h_{ab} \) for the local frame is defined by

\[
h_{ab} = \frac{1}{2} \text{tr} (t_a t_b).
\] (A.6)

Its nonzero components are given by \( h_{22} = 1, \ h_{13} = h_{31} = -2, \ h_{48} = h_{84} = 8, \ h_{57} = h_{75} = -2, \ h_{66} = 4/3 \). This metric tensor has a signature \((3, 5)\). Indices of the local frame are raised and lowered by \( h_{ab} \) and its inverse \( h^{ab} \). Then \( f_{abc} = f_{ab}^d h_{dc} \) is completely anti-symmetric in the three indices. It can be shown that \( f_{abc} \) and \( h_{ab} \) are related by

\[
h_{ab} = -\frac{1}{12} f_{a}^{\ \cd} f_{bcd}.
\] (A.7)

The structure constants are given by

\[
f_{123} = -2, \ f_{558} = 8, \ f_{167} = -4, \ f_{248} = -16,
\]

\[
f_{257} = 2, \ f_{347} = 8, \ f_{356} = -4.
\] (A.8)

The invariant tensor \( d_{ab}^c \) is defined by

\[
[t_a, t_b] = d_{ab}^c t_c + d_{ab}^0 t_0.
\] (A.9)

where \( t_0 = I \) is an identity matrix. The constant with the lowered index \( d_{abc} = d_{ab}^d h_{dc} \) is completely symmetric in all the indices. These constants are given by

\[
d_{127} = d_{235} = -2, \quad d_{136} = d_{226} = d_{567} = \frac{4}{3}, \quad d_{118} = d_{334} = 8,
\]

\[
d_{468} = \frac{32}{3}, \quad d_{477} = d_{558} = -8, \quad d_{666} = -\frac{16}{9}, \quad d_{ab}^0 = \frac{2}{3} h_{ab}.
\] (A.10)
Appendix B. AdS$_{3}$

The flat connections which yield AdS$_{3}$ spacetime are given by

\[
A = e^{a}L_{a}dx^{a} + L_{0}dr,
\]
\[
\tilde{A} = -e^{a}L_{-a}dx^{a} - L_{0}dr.
\]

In this appendix, to avoid confusion of $\mu = 0, 1, 2$ with $a = \{1, 2, \ldots\}$ a different notation $\mu = t, \phi, r$ will be used, and $dx^{\pm} \equiv dt \pm d\phi$. The corresponding vielbein and spin connection are given by

\[
e_{t} = \omega_{t} = \frac{1}{2}e^{r}(L_{1} + L_{-1}),
\]
\[
e_{\phi} = \omega_{\phi} = \frac{1}{2}e^{\phi}(L_{1} - L_{-1}),
\]
\[
e_{r} = L_{0}, \quad \omega_{r} = 0.
\]

The metric is $d\mathbf{s}^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu} = dr^{2} + e^{2r}(d\phi^{2} + d\phi^{2})$.

By (B.2) one obtains

\[
(e_{t})^{2} = \frac{1}{2}W_{0} + \frac{2}{3}I \quad \rightarrow \quad \tilde{e}_{tt} = \frac{1}{2}W_{0},
\]
\[
(e_{\phi})^{2} = \frac{1}{8}e^{2\phi}(W_{2} + W_{-2} + 2W_{0} - \frac{2}{3}I) \quad \rightarrow \quad \tilde{e}_{\phi\phi} = \frac{1}{8}e^{2\phi}(W_{2} + W_{-2} + 2W_{0}),
\]
\[
(e_{r})^{2} = \frac{1}{8}e^{2r}(W_{2} + W_{-2} - 2W_{0} + \frac{2}{3}I) \quad \rightarrow \quad \tilde{e}_{rr} = \frac{1}{8}e^{2r}(W_{2} + W_{-2} - 2W_{0}).
\]

Since $\tilde{e}_{\mu\nu}$ satisfies

\[
\rho = g^{\mu\nu}\tilde{e}_{\mu\nu} = 0,
\]

one has $e_{(\mu\nu)} = \tilde{e}_{\mu\nu}$ for this special geometry. The other components are given by

\[
e_{(tr)} = \tilde{e}_{t\phi} = \frac{1}{4}e^{r}(W_{1} + W_{-1}),
\]
\[
e_{(\phi\phi)} = \tilde{e}_{r\phi} = \frac{1}{8}e^{2r}(W_{2} - W_{-2}).
\]

The non-vanishing components of the vielbein in terms of the basis $t_{a}$ are

\[
e_{1}^{2} = 1, \quad e_{1}^{1} = \frac{1}{2}e^{r}, \quad e_{1}^{\phi} = \frac{1}{2}e^{\phi}, \quad e_{1}^{3} = \frac{1}{2}e^{r}, \quad e_{1}^{3} = \frac{1}{2}e^{r}.
\]
\[
e_{2}^{2} = 1, \quad e_{2}^{1} = \frac{1}{2}e^{r}, \quad e_{2}^{\phi} = \frac{1}{2}e^{\phi}, \quad e_{2}^{3} = \frac{1}{2}e^{r}, \quad e_{2}^{3} = \frac{1}{2}e^{r}.
\]
\[
e_{3}^{3} = 1, \quad e_{3}^{t} = \frac{1}{2}e^{r}, \quad e_{3}^{\phi} = \frac{1}{2}e^{\phi}, \quad e_{3}^{1} = \frac{1}{2}e^{r}, \quad e_{3}^{1} = \frac{1}{2}e^{r}.
\]
\[
e_{(tr)}^{2} = 4e^{-2r}, \quad E_{1}^{(tr)} = 4e^{-2r}, \quad E_{2}^{(tr)} = 4e^{-2r}, \quad E_{3}^{(tr)} = 2e^{-r}, \quad E_{4}^{(tr)} = 2e^{-r}, \quad E_{5}^{(tr)} = 2e^{-r}, \quad E_{6}^{(tr)} = 2e^{-r}, \quad E_{7}^{(tr)} = 2e^{-r}, \quad E_{8}^{(tr)} = 2e^{-r}.
\]

Then the inverse vielbein exists. An explicit calculation shows that

\[
E_{1}^{c} = 1, \quad E_{1}^{c} = e^{-r}, \quad E_{3}^{c} = e^{-r}, \quad E_{1}^{c} = e^{-r}, \quad E_{3}^{c} = -e^{-r},
\]
\[
E_{4}^{c} = 4e^{-2r}, \quad E_{4}^{c} = 4e^{-2r}, \quad E_{2}^{c} = 4e^{-2r}, \quad E_{2}^{c} = 2e^{-r}, \quad E_{5}^{c} = 2e^{-r}, \quad E_{6}^{c} = 2e^{-r}, \quad E_{7}^{c} = 2e^{-r}, \quad E_{8}^{c} = 2e^{-r}.
\]

Therefore, the 8D local frame spanned by $e_{\mu}^{t}$ and $e_{(\mu\nu)}$ is actually non-degenerate.

The spin-3 gauge field vanishes

\[
\phi_{(\mu\nu)} = g_{(\mu\nu)} = 0.
\]

$M_{(\mu\nu)(\lambda)} = g_{(\mu\nu)(\lambda)}$ (3.23) have the following non-vanishing components:

\[
g_{(tr)(tr)} = \frac{1}{2}, \quad g_{(tr)(tr)} = -\frac{1}{2}e^{2r}, \quad g_{(tr)(\phi)} = \frac{1}{2}e^{2r}, \quad g_{(tr)(\phi)} = \frac{1}{2}e^{2r}, \quad g_{(tr)(\phi)} = \frac{1}{2}e^{2r}.
\]

\[
g_{(\phi\phi)(\phi\phi)} = \frac{1}{2}e^{2r}, \quad g_{(tr)(tr)} = \frac{1}{2}e^{2r}, \quad g_{(tr)(\phi)} = \frac{1}{2}e^{2r}, \quad g_{(tr)(\phi)} = \frac{1}{2}e^{2r}.
\]
So the tensor $J^{(\mu\nu)(\rho\sigma)}$ \((3.24)\) is given by
\[
J^{(rr)(rr)} = \frac{16}{3}, \quad J^{(rr)(rt)} = -4e^{-2r}, \quad J^{(\phi\phi)(\phi\phi)} = 4e^{-2r}, \quad J^{(\phi\phi)(rr)} = -4e^{-4r},
\]
\[
J^{(rt)(rt)} = \frac{16}{3}e^{-4r}, \quad J^{(\phi\phi)(\phi\phi)} = \frac{16}{3}e^{-4r}, \quad J^{(rr)(rr)} = \frac{8}{3}e^{-2r}, \quad J^{(rr)(\phi\phi)} = \frac{8}{3}e^{-2r}.
\]
(B.10)

The killing vectors determine the generalized diffeomorphisms which do not change the metric-like quantities, $g_{\mu\nu}$, $g_{\mu(\nu\alpha\lambda\kappa)}$ and $g_{\mu(\nu)(\rho)(\sigma)}$. In the spin–3 geometry, there exist two types: Killing vectors $\xi_\mu$ and Killing tensors $\xi_{(\mu\nu)}$. They are determined by equations $\delta g_{\mu\nu} = 0$ and $\delta \phi_{\mu\nu\lambda\kappa} = 0$. By the results (5.5) and (5.15), they are determined by the following set of equations:
\[
\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0,
\]
(B.11)
\[
\nabla_\mu (\xi_\nu(\alpha) + \frac{1}{2} \rho^\alpha \Lambda_\sigma g_{\nu\alpha}) + \nabla_\nu (\xi_\mu(\alpha) + \frac{1}{2} \rho^{\alpha} \Lambda_\sigma g_{\mu\alpha}) + \nabla_\lambda (\xi_{(\mu\nu)} + \frac{1}{2} \rho^{\alpha \beta} \Lambda_\sigma g_{\mu\nu\sigma}) = 0.
\]
Generally, these are coupled equations for $\delta g_{\mu\nu} = 0$ and $\delta \phi_{\mu\nu\lambda\kappa} = 0$. However, if the background geometry is AdS$_3$, $\delta \phi_{\mu\nu\lambda\kappa}$ and $\rho^{\alpha\beta}$ vanish, and the equations are decoupled. Then $\xi_\mu$ and $\xi_{(\mu\nu)}$ are determined by
\[
\hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu = 0,
\]
(B.13)
\[
\hat{\nabla}_\mu (\xi_\nu(\alpha) + \frac{1}{2} \rho^\alpha \Lambda_\sigma g_{\nu\alpha}) + \hat{\nabla}_\nu (\xi_\mu(\alpha) + \frac{1}{2} \rho^{\alpha} \Lambda_\sigma g_{\mu\alpha}) + \hat{\nabla}_\lambda (\xi_{(\mu\nu)} + \frac{1}{2} \rho^{\alpha \beta} \Lambda_\sigma g_{\mu\nu\sigma}) = 0.
\]
(B.14)

Here $\hat{\nabla}_\mu$ is the Christoffel symbol for AdS$_3$ background. The Killing tensors cannot be expressed in terms of the Killing vectors as $\xi_{(\mu\nu)} = \xi_\mu \xi_\nu - \frac{1}{2} g_{\mu\nu} \hat{g}^{\alpha\beta} \xi_\alpha \xi_\beta$.

In this case of AdS$_3$ geometry, there exist six Killing vectors $\xi_{(\mu\nu)}^i (i = 1, \ldots, 6)$ and ten Killing tensors $\xi_{(\alpha\mu\nu)}^i (\alpha = 1, \ldots, 10)$. The Killing vectors correspond to the isometry SO(2, 2), and are the same as those in the spin-2 gravity: $\xi^{(i)} = \xi_{(\mu\nu)}^i g^{\mu\nu} \partial_\nu$.
\[
\begin{align*}
\xi_1 &= \partial_r, \\
\xi_2 &= \partial_\phi, \\
\xi_3 &= \partial_r - t \partial_\phi, \\
\xi_4 &= \partial_r + t \partial_\phi, \\
\xi_5 &= t \partial_r - \frac{1}{2} (t^2 + \phi^2) \partial_\phi - t \phi \partial_\theta, \\
\xi_6 &= \phi \partial_\phi + \frac{1}{2} (t^2 + \phi^2) \partial_r - \frac{1}{2} e^{-2r} \partial_\theta.
\end{align*}
\]
(B.15)

The Killing tensors are given by
\[
\begin{align*}
\xi_{(rr)} &= 1, \\
\xi_{(rt)} &= \frac{1}{2} (t^2 + \phi^2) e^{2r} + \frac{1}{2} e^{2r}, \\
\xi_{(r\phi)} &= \frac{1}{2} (t^2 + \phi^2) e^{2r} - \frac{1}{2} e^{2r}, \\
\xi_{(t\phi)} &= \frac{3}{2} t e^{2r}, \\
\xi_{(\phi\phi)} &= -3 t \phi e^{2r}, \\
\xi_{(rr)} &= \frac{1}{2} \left( t^2 + 3 \phi^2 \right) e^{2r} + \frac{1}{2} t e^{2r}, \\
\xi_{(rt)} &= \frac{3}{2} t \left( t^2 + \phi^2 \right) e^{2r} + \frac{1}{2} \phi e^{2r}, \\
\xi_{(r\phi)} &= \frac{3}{2} \left( t^2 + \phi^2 \right) e^{2r} + \frac{1}{2} t \phi e^{2r}, \\
\xi_{(t\phi)} &= \frac{3}{2} \left( t^2 + \phi^2 \right) e^{2r} - \frac{1}{2} \phi e^{2r}, \\
\xi_{(\phi\phi)} &= -\frac{3}{2} \left( t^2 + \phi^2 \right) e^{2r} - \frac{1}{2} t e^{2r}, \\
\xi_{(tt)} &= \frac{1}{4} \phi (3 t^2 + 2 \phi^2) e^{2r}, \\
\xi_{(t\phi)} &= \frac{1}{2} \phi (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} t \phi e^{2r}, \\
\xi_{(\phi\phi)} &= \frac{1}{2} \phi (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} \phi e^{2r}, \\
\xi_{(rr)} &= \frac{1}{2} \phi (2 \phi^2 + 4 t^2) e^{2r} + \frac{1}{2} t \phi e^{2r}, \\
\xi_{(rt)} &= \frac{1}{2} \phi (2 \phi^2 + 4 t^2) e^{2r} - \frac{1}{2} \phi e^{2r}, \\
\xi_{(r\phi)} &= \frac{1}{4} t^2 + 6 t^2 \phi^2 + \frac{1}{2} e^{2r}, \\
\xi_{(t\phi)} &= -\frac{1}{4} t^2 + 6 t^2 \phi^2 + \frac{1}{2} e^{2r}, \\
\xi_{(\phi\phi)} &= -\frac{1}{4} \phi (3 t^2 + 2 \phi^2) e^{2r} - \frac{1}{2} t \phi e^{2r}, \\
\xi_{(rr)} &= -\frac{1}{2} t (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} t, \\
\xi_{(rt)} &= -\frac{1}{2} t (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} t, \\
\xi_{(r\phi)} &= -\frac{1}{2} \phi (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} \phi, \\
\xi_{(t\phi)} &= -\frac{1}{2} \phi (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} \phi, \\
\xi_{(\phi\phi)} &= -\frac{1}{2} \phi (3 t^2 + 2 \phi^2) e^{2r} + \frac{1}{4} \phi.
\end{align*}
\]
\[ \xi_{(ir)}^{(16)} = e^{\varphi}, \quad \xi_{(r\varphi)}^{(6)} = e^{\rho} \]
\[ \xi_{(ir)}^{(7)} = e^{\vartheta}, \quad \xi_{(r\varphi)}^{(8)} = \xi_{(r\varphi)}^{(8)} = 2t\phi e^{\varphi}, \quad \xi_{(r\varphi)}^{(8)} = \xi_{(r\varphi)}^{(8)} = -t e^{\varphi}, \quad \xi_{(r\varphi)}^{(8)} = -(r^2 + \phi^2) e^{\varphi} \]
\[ \xi_{(ir)}^{(9)} = 2t \phi e^{4r}, \quad \xi_{(r\varphi)}^{(9)} = 2t e^{4r}, \quad \xi_{(r\varphi)}^{(9)} = e^{2r}, \quad \xi_{(r\varphi)}^{(9)} = -2 \phi e^{4r} \]
\[ \xi_{(ir)}^{(10)} = -2 \phi e^{4r}, \quad \xi_{(r\varphi)}^{(10)} = e^{2r}, \quad \xi_{(r\varphi)}^{(10)} = 2r e^{4r}, \quad \xi_{(r\varphi)}^{(10)} = -2 \phi e^{4r}. \]

(B.16)

Those components which are not present vanish.

The Killing vectors (tensors) are related to the $SL(3, R)$ matrices $\Lambda_a t^a$, which generate the generalized diffeomorphisms, by $\xi_{ir} = \Lambda_a t^a$ and $\xi_{r\varphi} = \Lambda_a e^a_{\varphi}$. The Killing vectors can also be obtained by solving equations

\[ \delta e^a_\mu = \delta a \Lambda^a + f^a_\beta b^a_\mu \Lambda^c = f^a_\beta c^b = \delta e^a_\mu. \]

Here $\Sigma^b$ are some functions to be determined by $\Lambda^a$.

### Appendix C. Metric-like fields

In this appendix, various metric-like fields are defined.

Let us recall that a product of two generators of $sl(3, R)$, $t_a$ and $t_b$, can be reduced to terms which are linear in $t_a$ or proportional to an identity matrix by using the structure constants

\[ t_a t_b = \frac{1}{2} t_a t_b + \frac{1}{2} t_a t_b = \frac{1}{2} f_{a b}^c t_c + \frac{1}{2} (d_a b c + d_b a) t. \]

This is easy to expand $t_a$ in terms of $e_a$ and $e_{(\mu\nu)}$.

Now it is easy to expand $t_b$ in terms of $e_a$ and $e_{(\mu\nu)}$.

By using this equation, then the vielbein $e_{(\mu\nu)}$ can be expressed in terms of $E_a$.

\[ e_{(\mu\nu)} = \frac{1}{2} t^{(\alpha\beta)} e_{(\mu\nu)} = \frac{1}{2} \text{tr} (e_a E_{a}^{(\mu\nu)} + \frac{1}{2} e_{(\mu\nu)} E_{a}^{(\mu\nu)} + \frac{1}{2} e_{(\mu\nu)} E_{a}^{(\mu\nu)}). \]

(C.2)

Now, by using this formula (C.2) $h_{ab}$, $d_{abc}$ and $f_{abc}$ are expressed in terms of $E$, $e$ and gauge fields. For $h_{ab}$ one obtains

\[ h_{ab} = \frac{1}{2} t^{(a b)} = \frac{1}{2} \text{tr} (e_a E_{a}^{(\mu\nu)} + \frac{1}{2} e_{(\mu\nu)} E_{a}^{(\mu\nu)} + \frac{1}{2} e_{(\mu\nu)} E_{a}^{(\mu\nu)}) + \frac{1}{2} g_{(\mu\nu)} E_{a}^{(\mu\nu)} + \frac{1}{2} g_{(\mu\nu)} E_{a}^{(\mu\nu)} + \frac{1}{2} g_{(\mu\nu)} E_{a}^{(\mu\nu)} E_{a}^{(\mu\nu)}. \]

(C.5)

For $d_{abc}$ one obtains

\[ d_{abc} = \frac{1}{2} t^{(a b)} t^{(c)} \]

\[ = 2 E_{a}^{(\mu)} E_{b}^{(\nu)} E_{c}^{(\rho)} E_{a}^{(\mu)} E_{b}^{(\nu)} E_{c}^{(\rho)} (g_{(\mu\nu)}(\nu\rho) + \frac{1}{2} g_{(\mu\nu)} E_{a}^{(\mu\nu)} E_{b}^{(\mu\nu)} E_{c}^{(\mu\nu)}) + \frac{1}{2} g_{(\mu\nu)} E_{b}^{(\nu)} E_{c}^{(\rho)} (g_{(\mu\nu)}(\nu\rho) + \frac{1}{2} g_{(\mu\nu)} E_{a}^{(\mu\nu)} E_{b}^{(\mu\nu)} E_{c}^{(\mu\nu)}) \]

(C.6)
Here the following manipulation is used:
\[
\text{tr} [e_{\mu}, \epsilon_{\nu}, e_{(\lambda \rho)}] = \text{tr} (2 e_{(\mu \nu)} + \frac{4}{3} g_{\mu \nu} I + \frac{2}{3} g_{\mu \nu} \rho) e_{(\lambda \rho)} = 4 g_{(\mu \nu)(\lambda \rho)} + \frac{2}{3} g_{\mu \nu} \text{tr} \rho e_{(\lambda \rho)}.
\]  
(C.7)

Extra gauge fields \( g_{(\mu \nu)(\lambda \rho)} \ldots \) are defined as follows:
\[
g_{(\mu \nu)(\lambda \rho)} = \frac{1}{2} \text{tr} e_{(\mu \nu)} e_{(\lambda \rho)},
\]
(C.8)
\[
g_{(\mu \nu)(\lambda \rho)\sigma} = \frac{1}{2} \text{tr} [e_{(\mu \nu)}, e_{(\lambda \rho)}] e_{\sigma},
\]
(C.9)
\[
g_{(\mu \nu)(\lambda \rho)(\sigma \kappa)} = \frac{1}{4} \text{tr} [e_{(\mu \nu)}, e_{(\lambda \rho)}] e_{(\sigma \kappa)}.
\]
(C.10)

So for the spin-3 gravity gauge fields with up to six indices must be introduced. Note that one can also define a gauge field such as
\[
\phi_{\mu \nu \lambda \rho} = \frac{1}{2} \epsilon_{\mu \nu \lambda \rho} \hat{e}_{\kappa}.
\]

Similarly, for \( f_{abc} \) one has
\[
f_{abc} = \frac{1}{2} \text{tr} [I_{a}, I_{b}, I_{c}]
\]
\[
= \frac{1}{2} E_{a}^{\mu} E_{b}^{\nu} E_{c}^{\sigma} \text{tr} [e_{\mu}, e_{\nu}, e_{\sigma}]
\]
\[
+ \frac{1}{4} E_{a}^{\mu} E_{b}^{\nu} E_{c}^{\sigma} E_{d}^{\rho} \text{tr} [e_{\mu}, e_{\nu}, e_{\sigma}, e_{(\rho \lambda \kappa)}] + \text{permutations}
\]
\[
+ \frac{1}{8} E_{a}^{\mu} E_{b}^{(\nu \rho)} E_{c}^{(\sigma \lambda \kappa)} \text{tr} [e_{\mu}, e_{(\rho \lambda \kappa)}] e_{(\sigma \lambda \kappa)} + \text{permutations}
\]
\[
+ \frac{1}{16} E_{a}^{(\nu \rho \sigma \lambda \kappa)} E_{b}^{(\nu \rho \tau) \lambda \kappa} E_{c}^{(\sigma \lambda \kappa)} \text{tr} [e_{(\nu \rho \tau)}, e_{(\lambda \kappa)}] e_{(\sigma \lambda \kappa)}.
\]
(C.11)

Here some terms which can be obtained by permutation of indices are not written explicitly.

Therefore for spin-3 gravity, partly anti-symmetric gauge fields with up to six indices such as
\[
F_{\mu \nu \sigma} = \frac{1}{2} \text{tr} [e_{\mu}, e_{\nu}, e_{\sigma}], \quad F_{\mu \lambda (\sigma \kappa)} = \frac{1}{2} \text{tr} [e_{\mu}, e_{\lambda}, e_{(\sigma \kappa)}], \ldots
\]
(C.13)

must also be introduced. To remove the local frame indices \( a, b, \ldots \) from the action integral (4.31) and the various relations obtained in this paper it is necessary to use the matrix-like fields defined in this appendix. However, not all these fields will be independent. They will be expressed in terms of fewer fields. At present, explicit relations among these fields are not known and we cannot carry out this program.

In the remaining part of this appendix, it will be argued that these metric-like fields can be expressed in terms of \( g_{\mu \nu} \) and \( \phi_{\mu \nu \lambda \rho} \), when the spin-3 geometry is in the neighborhood of AdS3 vacuum.

This is performed by using perturbation expansion around the AdS3 vacuum. Let us expand the vielbein as
\[
e_{\mu}^{a} = \tilde{e}_{\mu}^{a} + \psi_{\mu}^{a},
\]
where \( \tilde{e}_{\mu}^{a} \) is the AdS3 vacuum (B.2) and \( \psi_{\mu}^{a} \) is a small fluctuation around it. By computing the metric \( g_{\mu \nu} = \tilde{g}_{\mu \nu} + \tilde{\phi}_{\mu \nu} \) and the spin-3 gauge field \( \phi_{\mu \nu \lambda} = 0 + \tilde{\phi}_{\mu \nu \lambda} \) up to first order in \( \psi_{\mu}^{a} \), and gauge fixing the local Lorentz rotation by imposing eight conditions on \( \psi_{\mu}^{a} \), one can express \( \psi_{\mu}^{a} \) in terms of the fluctuations \( \tilde{g}_{\mu \nu} \) and \( \tilde{\phi}_{\mu \nu \lambda} \):
\[
\psi_{\mu}^{a} = \frac{1}{2} \tilde{g}_{\mu \nu}, \quad \psi_{\nu}^{a} = \frac{1}{2} \tilde{g}_{\nu \mu},
\]
\[
\psi_{\mu}^{1} = \epsilon_{\mu} e^{-\tilde{\phi}_{\mu \nu}}, \quad \psi_{\nu}^{1} = \frac{1}{2} \tilde{\phi}_{\mu \nu},
\]
\[
\psi_{\mu}^{3} = e^{\tilde{\phi}_{\mu \nu} - \frac{1}{2} \tilde{g}_{\mu \nu} - \frac{1}{2} \tilde{g}_{\phi} - \frac{1}{2} \tilde{g}_{\phi}},
\]
\[
\psi_{\mu}^{5} = e^{-\tilde{\phi}_{\mu \nu} - \frac{1}{2} \tilde{g}_{\mu \nu} + \frac{1}{2} \tilde{g}_{\phi} + \frac{1}{2} \tilde{g}_{\phi}},
\]
\[
\psi_{\mu}^{7} = -\frac{1}{2} \epsilon_{\mu} \tilde{\phi}_{\nu \rho} - \frac{1}{2} \epsilon_{\nu} \tilde{\phi}_{\mu \rho} + \frac{1}{2} \epsilon_{\rho} \tilde{\phi}_{\mu \nu} - \frac{1}{4} \epsilon^{\nu \rho} \tilde{\phi}_{\mu \nu} - \frac{1}{4} \epsilon^{\rho \nu} \tilde{\phi}_{\mu \nu} + \frac{1}{12} \epsilon^{\nu \rho} \tilde{\phi}_{\mu \nu} + \frac{1}{12} \epsilon^{\rho \nu} \tilde{\phi}_{\mu \nu}.
\]
(C.14)
Here the index \( i \) takes two values, \( i = t, \phi \) and \( \epsilon_i = +1, \epsilon_\phi = -1 \). The remaining components vanish; \( \psi_1, 3, 4, 5, 7, 8 = 0 \). Then by substituting the result into the other metric-like fields one obtains them in terms of \( \tilde{g}_{\mu\nu} \) and \( \phi_{\mu\nu} \). For example, the case of \( g_{\mu\nu}(\theta_\rho) = \tilde{g}_{\mu\nu}(\theta_\rho) + \phi_{\mu\nu}(\theta_\rho) \) is presented below. \( \tilde{g}_{\mu\nu}(\theta_\rho) \) is the background (B.9) and \( \tilde{g}_{\mu\nu}(\theta_\rho) \) is the fluctuation:

\[
\begin{align*}
\tilde{g}_{(rr)(rr)} &= \frac{1}{2} \tilde{g}_{rr}, \quad \tilde{g}_{(rr)(\theta r)} = \frac{1}{2} \tilde{g}_{r\theta}, \quad \tilde{g}_{(rr)(\phi)} = -\frac{1}{2} \tilde{g}_{r\phi}, \\
\tilde{g}_{(\theta r)(\theta r)} &= -\frac{1}{2} \tilde{g}_{t\phi} - \frac{1}{2} \tilde{g}_{\theta\phi} + \frac{1}{4} \tilde{g}_{\theta t}, \\
\tilde{g}_{(\theta r)(\phi)} &= -\frac{3}{2} \tilde{g}_{r\phi} + \frac{1}{2} \tilde{g}_{\theta\phi}, \\
\tilde{g}_{(\phi r)(\phi)} &= -\frac{3}{2} \tilde{g}_{r\phi} - \frac{1}{2} \tilde{g}_{\phi t}, \\
\tilde{g}_{(\theta r)(\theta r)} &= \frac{1}{2} \tilde{g}_{r\phi} + \frac{1}{2} \tilde{g}_{\theta\phi} + \frac{1}{4} \tilde{g}_{\theta t}, \\
\tilde{g}_{(\phi r)(\phi)} &= \frac{1}{2} \tilde{g}_{r\phi} - \frac{1}{2} \tilde{g}_{\phi t}.
\end{align*}
\]

Other extra metric-like fields can also be worked out similarly.

Although this argument is far from the all-order proof, this at least supports the conjecture that all metric-like fields can be expressed in terms of \( g_{\mu\nu} \) and \( \phi_{\mu\nu} \) in the neighborhood of AdS\(_3\) vacuum.

**Appendix D. Solution for \( S_{\mu\nu,\lambda\rho} \)**

In this appendix, a solution to the equations for \( S_{\mu\nu,\lambda\rho} \), (3.30), (3.32) are presented. Let us define matrices \( A_{\mu\nu} \) and \( B^{\mu\nu} \) by

\[
B^{\mu\nu} = \frac{5}{38} f^{(\mu\nu)(\lambda\rho)} W_{\lambda\rho},
\]

\[
A_{\mu\nu} = \tilde{\nabla}_\mu \phi_{\lambda\rho} - \frac{5}{4} \tilde{\nabla}_\lambda \phi_{\mu\rho} - \frac{5}{4} \left( \tilde{\nabla}_\mu \phi_{\nu\phi} + \tilde{\nabla}_\nu \phi_{\mu\phi} - \tilde{\nabla}_\phi \phi_{\mu\nu} \right) f^{(\alpha\beta)(\sigma\xi)} W_{\alpha\beta}. \tag{D.1}
\]

The matrix \( B^{\mu\nu} \) is symmetric traceless but \( A_{\mu\nu} \) is not symmetric. In terms of these matrices, equation (3.32) is written as

\[
g^{\rho\sigma} S_{\mu\sigma,\phi\lambda} B^{\lambda\rho} = A_{\mu\nu}. \tag{D.2}
\]

If \( \phi_{\mu\lambda\rho} = 0 \), then the connections reduce to the Christoffel symbol in Einstein gravity. In this case, \( A_{\mu\nu} = 0 \) and the solution will be given by \( S_{\mu\nu,\lambda\rho} = 0 \). In what follows the above matrices are considered to be small and \( S_{\mu\nu,\lambda\rho} \) will be obtained as a power series in these matrices. Then the second term on the left-hand side of (D.2) is second order in \( A, B \).

Let us first consider the equation

\[
g^{\rho\sigma} S^{(0)}_{\mu\sigma,\phi\lambda} = A_{\mu\nu}. \tag{D.3}
\]

The solution to this equation and (3.30) is obtained as

\[
S^{(0)}_{\mu\nu,\lambda\rho} = -\frac{1}{3} g_{\mu\nu} \left( A_{\lambda\rho} + A_{\rho\lambda} \right) - \frac{2}{3} g_{\lambda\rho} \left( A_{\mu\nu} + A_{\nu\mu} \right) - \frac{1}{3} \left( g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda} \right) A_{\alpha\beta}
- \frac{1}{3} \left( g_{\lambda\mu} g_{\rho\nu} + g_{\lambda\rho} g_{\mu\nu} \right) A_{\alpha\beta}
+ \frac{1}{3} g_{\mu\rho} \left( 2 A_{\nu\phi} + A_{\nu\phi} \right) + \frac{1}{3} g_{\rho\nu} \left( 2 A_{\lambda\phi} + A_{\lambda\phi} \right)
- \frac{1}{3} g_{\lambda\rho} \left( 2 A_{\mu\phi} + A_{\mu\phi} \right). \tag{D.4}
\]

Now we split \( S_{\mu\nu,\lambda\rho} \) as \( S^{(0)}_{\mu\nu,\lambda\rho} + \tilde{S}_{\mu\nu,\lambda\rho} \). Owing to (D.4) this remaining part \( \tilde{S}_{\mu\nu,\lambda\rho} \) satisfies

\[
g^{\rho\sigma} \tilde{S}_{\mu\sigma,\phi\lambda} B^{\lambda\rho} = A^{(1)}_{\mu\nu}. \tag{D.5}
\]

where \( A^{(1)}_{\mu\nu} = -S^{(0)}_{\mu\nu,\lambda\rho} B^{\lambda\rho} \). This equation has the same structure as (D.3). Because the term \( \tilde{S}_{\mu\nu,\lambda\rho} B^{\rho\sigma} \) is subleading, we can drop this term to leading order, replace \( g^{\rho\sigma} \tilde{S}_{\mu\sigma,\phi\lambda} \) by \( g^{\rho\sigma} S^{(1)}_{\mu\sigma,\phi\lambda} \), and solve equation \( g^{\rho\sigma} S^{(1)}_{\mu\sigma,\phi\lambda} = A^{(1)}_{\mu\nu} \), which has the same form as (D.4).
We will repeat this procedure, obtain \( S^{(n)}_{\mu \nu \lambda \rho} \) at each step and by assuming convergence finally sum up \( S^{(n)}_{\mu \nu \lambda \rho} = \sum_{n=0}^{\infty} S^{(n)}_{\mu \nu \lambda \rho} \). \( A^{(n)}_{\mu \nu} \) is defined by

\[
A^{(n)}_{\mu \nu} = -S^{(n-1)}_{\mu \nu \lambda \rho} B^{\lambda \rho} \quad (A^{(0)}_{\mu \nu} = A_{\mu \nu}).
\] (D.7)

The equation for \( S^{(n)}_{\mu \nu \lambda \rho} \) is given by

\[
g^{(n)}_{\mu \nu} S^{(n)}_{\mu \nu \lambda \rho} = A^{(n)}_{\lambda \rho}
\] (D.8)

and the solution is given by (D.5) with \( S^{(0)}_{\mu \nu \lambda \rho} \) and \( A_{\mu \nu} \) replaced by \( S^{(n)}_{\mu \nu \lambda \rho} \) and \( A^{(n)}_{\mu \nu} \), respectively.

The above procedure yields recursion relations between \( A^{(n)}_{\mu \nu} \) and \( A^{(n+1)}_{\mu \nu} \), and their solution for \( A^{(n)}_{\mu \nu} \) takes the following form:

\[
A^{(n)}_{\mu \nu} = g_{\mu \nu} F^{(n)} + \sum_{m=1}^{n-1} (B^{m})_{\mu \nu} X^{(n)}_{m} + \sum_{m=0}^{n} \{(B^{m} A B^{m-m})_{\mu \nu} + (B^{m} A B^{m-m})_{\mu \nu}\} Y^{(n)}_{m}.
\] (D.9)

Here \( g_{\mu \nu} \) is inserted to construct the powers of \( B^{\lambda \rho} \). By computing \( S^{(n)}_{\mu \nu \lambda \rho} \) by using (D.5) with suitable replacements and obtaining \( A^{(n+1)}_{\mu \nu} \) by using (D.7), the recursion relations for \( F^{(n)} \), \( X^{(n)}_{m} \) and \( Y^{(n)}_{m} \) are obtained

\[
F^{(n+1)} = \frac{6}{5} \sum_{m=1}^{n-1} \text{tr}(B^{m+1}) X^{(n)}_{m} + \frac{12}{5} \sum_{m=0}^{n} Y^{(n)}_{m} \text{tr}(A B^{m+1}),
\] (D.10)

\[
X^{(n+1)}_{m+1} = -\frac{3}{5} F^{(n)} + \frac{6}{5} \text{tr}(A B^{m}) \sum_{m=0}^{n} Y^{(n)}_{m} + \frac{3}{5} \sum_{m=1}^{n-1} (\text{tr} B^{m}) X^{(n)}_{m},
\] (D.11)

\[
Y^{(n+1)}_{m} = -\frac{12}{5} X^{(n)}_{m}, \quad Y^{(n+1)}_{m} = -\frac{6}{5} Y^{(n)}_{m} - \frac{6}{5} Y^{(n-1)}_{m}.
\] (D.12)

The initial condition at \( n = 1 \) is

\[
Y^{(1)}_{0} = -\frac{4}{5}, \quad Y^{(1)}_{1} = -\frac{7}{5}, \quad F^{(1)} = \frac{6}{5} \text{tr}(B A), \quad X^{(1)}_{1} = \frac{3}{5} \text{tr} A.
\] (D.13)

The solution for \( Y^{(n)}_{m} \) is given by

\[
Y^{(n)}_{m} = \frac{1}{3} \left( -\frac{6}{5} \right)^{m} \frac{(n-1)!}{m! (n-m)!} (2n-m).
\] (D.14)

\( X^{(n)}_{m} \) is determined in terms of \( F^{(n)} \) by \( X^{(n)}_{m} = (3/4) (-12/5)^{m} F^{(n-m)}, \) and \( F^{(n)} \) is the solution to the recursion relation

\[
F^{(n+1)} = \frac{6}{5} \sum_{m=1}^{n} (-12/5)^{m} \text{tr}(B^{m+1}) F^{(n-m)} - \frac{1}{2} (-12/5)^{n+1} \text{tr}(A B^{n+1}).
\] (D.15)

This last equation can be solved by iterations

\[
F^{(n)} = -\frac{1}{2} (-12/5)^{n} \text{tr}(A B^{n}) + \frac{1}{4} \sum_{m=1}^{n-2} (-12/5)^{m} \text{tr}(B^{m-n}) \text{tr}(A B^{m})
\]
\[+(25/576) \sum_{m=1}^{n-2} \sum_{k=1}^{m-2} (-12/5)^{m-k} \text{tr}(B^{m-k}) F^{(k)} = \ldots.
\] (D.16)

For example, \( F^{(2)} = -(-72/25) \text{tr}(A B^{2}), \quad F^{(3)} = -(432/125) \text{tr}(B^{2}) \text{tr}(A B^{2}) + (864/125) \text{tr}(A B^{2}), \quad F^{(4)} = (5184/625) \text{tr}(B^{2}) \text{tr}(A B^{2}) + (5184/625) \text{tr}(B^{2}) \text{tr}(A B^{2}) - (10368/625) \text{tr}(A B^{2}) \) and then we obtain \( X^{(2)}_{1} = -54/25 \text{tr}(B A) \), \( X^{(3)}_{1} = (648/125) \text{tr}(B^{2} A), \) \( X^{(4)}_{1} = (648/125) \text{tr}(B A) \). Then, \( A^{(n)}_{\mu \nu} \) can be computed using (D.9) to any desired larger value of \( n \).
Finally, $S_{\mu \nu, \lambda \rho}$ is given by

$$S_{\mu \nu, \lambda \rho} = \sum_{n=0}^{\infty} S^{(n)}_{\mu \nu, \lambda \rho}$$

$$= \sum_{n=0}^{\infty} \left\{ -\frac{3}{2} g_{\mu \nu} \left( A^{(n)}_{\lambda \rho} + A^{(n)}_{\rho \lambda} \right) - \frac{2}{5} g_{\lambda \rho} \left( A^{(n)}_{\mu \nu} + A^{(n)}_{\nu \mu} \right) \\
+ \frac{3}{5} g_{\mu \lambda} g_{\nu \rho} A^{(n) a}_a \\
+ \frac{1}{5} g_{\mu \lambda} \left( 2 A^{(n)}_{\nu \rho} + A^{(n)}_{\rho \nu} \right) + \frac{1}{5} g_{\nu \rho} \left( 2 A^{(n)}_{\mu \lambda} + A^{(n)}_{\lambda \mu} \right) \right\}. \quad (D.17)$$

### Appendix E. Metric $G_{MN}$ for ‘8D space’

The metric tensor $g_{\mu \nu}$ is constructed in terms of $e^a_{\mu}$. By a similar construction, one can extend the metric tensor to that for an extended 8D space by combining $e^a_{\mu}$ and $e^{a(\mu \nu)}$. Let us define a new metric

$$
\begin{pmatrix}
G_{\mu \nu} & G_{\mu(\nu \lambda)} \\
G_{(\nu \lambda) \mu} & G_{(\nu \lambda)(\rho \sigma)}
\end{pmatrix}
$$

by the equations

$$G_{\mu \nu} = g_{\mu \nu} = e^a_{\mu} e_{a \nu}, \quad (E.1)$$

$$G_{\mu(\nu \lambda)} = G_{(\nu \lambda) \mu} = g_{\mu(\nu \lambda)} = e^a_{\mu} e_{a(\nu \lambda)}, \quad (E.2)$$

$$G_{(\mu \nu)(\lambda \rho)} = g_{(\mu \nu)(\lambda \rho)} = e^a_{(\mu \nu)} e_{a(\lambda \rho)}. \quad (E.3)$$

This tensor is a metric tensor in a fictitious 8D space which contains the ordinary spacetime. The inverse metric is easily obtained

$$G_{\mu \nu} = g_{\mu \nu} + \frac{1}{4} g_{\mu(\rho \sigma)} J_{(\rho \sigma)(\nu \lambda)} g_{\nu \lambda}. \quad (E.4)$$

$$G_{\mu(\nu \lambda)} = g_{\mu(\nu \lambda)} = -\frac{1}{2} g_{(\rho \sigma)} J_{(\rho \sigma)(\nu \lambda)}. \quad (E.5)$$

$$G_{(\mu \nu)(\lambda \rho)} = g_{(\mu \nu)(\lambda \rho)} = J_{(\mu \nu)(\lambda \rho)}. \quad (E.6)$$

$J^{(\rho \sigma)(\nu \lambda)}$ is defined in (3.24). They satisfy the following relations:

$$G_{\mu \nu} G_{\nu \lambda} + \frac{1}{2} G_{\mu(\nu \sigma)} G_{(\nu \sigma) \lambda} = \delta^\mu_\lambda, \quad (E.7)$$

$$G_{\mu \nu} G_{\nu(\lambda \rho)} + \frac{1}{2} G_{\mu(\nu \sigma)} G_{(\nu \sigma)(\lambda \rho)} = 0, \quad (E.8)$$

$$G_{(\mu \nu) \lambda} G_{\lambda \rho} + \frac{1}{2} G_{(\mu \nu)(\sigma \rho)} G_{(\sigma \rho) \lambda} = 0, \quad (E.9)$$

$$G_{(\mu \nu) \sigma} G_{(\lambda \rho) \sigma} + \frac{1}{2} G_{(\mu \nu)(\rho \sigma)} G_{(\lambda \sigma)(\rho \sigma)} = \delta^\mu_\lambda \delta^\nu_\rho + \delta^\nu_\lambda \delta^\mu_\rho - \frac{2}{5} g_{\mu \nu} g_{\lambda \rho}. \quad (E.10)$$

This inverse metric can be expressed in terms of the inverse vielbein. One can show that

$$G_{\mu \nu} = E^a_{\mu} E_{a \nu},$$

$$G_{\mu(\nu \lambda)} = E^a_{\mu} E_{a(\nu \lambda)},$$

$$G_{(\mu \nu)(\lambda \rho)} = E^a_{(\mu \nu)} E_{a(\lambda \rho)}. \quad (E.11)$$
By using the above formulae, one can show that $E_s$ can be expanded in terms of $e$ as follows:

$$E^\mu = G^\mu e^\mu + \frac{1}{2} G^{(\nu\lambda)} e^{\nu\lambda},$$  \hspace{1cm} (E.12)

$$E^{\mu(\nu\lambda)} = G^{\mu(\nu\lambda)} e^{\nu\lambda} + \frac{1}{2} G^{(\nu\lambda)(\rho\sigma)} e^{\nu\lambda \rho\sigma}.$$  \hspace{1cm} (E.13)

Let us denote the above metric as $G_{MN}$, where $M$ and $N$ take two types of indices $(\nu\lambda)$. Then, the relation (E.7)–(E.10) can be succinctly written as

$$G_{MN} G^{NL} = \delta^M_L,$$  \hspace{1cm} (E.14)

where $\delta^M_L$ is the ordinary Kronecker’s $\delta$ symbol for $M = \mu$ and $L = \nu$. Otherwise, $\delta_{\mu(\nu\lambda)} = \delta_{(\mu\nu)} = 0$, $\delta_{(\mu\nu)(\rho\sigma)} = \delta_{\mu\nu} \delta^\rho_\sigma + \delta_{\mu\rho} \delta^\nu_\sigma - \frac{3}{2} g_{\mu\nu} \delta^\rho_\sigma$.

Finally, the covariant derivative $\nabla_\mu$ is compatible with this metric tensor. As was shown in (4.25), $G_{\mu\nu} = g_{\mu\nu}$ satisfies $\nabla_\lambda g_{\mu\nu} = 0$. This property is true for all components

$$\nabla_\mu G_{MN} = \partial_\mu G_{MN} - \Gamma^K_{\mu M} G_{KN} - \Gamma^K_{\mu N} G_{KM} = 0.$$  \hspace{1cm} (E.16)

This is because $G_{MN}$ is given by $e^\mu M e^\nu N$ and the vielbeins are covariantly constant, $D_\mu e^\nu_N = 0$. The above relation is not sufficient to determine $\Gamma^K_{\mu M}$ completely, since this is not symmetric under interchange of the lower indices.

When $\nabla_\mu G_{(\nu\lambda)(\rho\sigma)}$ is computed explicitly, this does not seemingly vanish. The result contains $g_{(\mu\nu)(\lambda\rho)}$. As was discussed at the end of section 2, however, not all the gauge fields are independent. Those relations among these fields will be such that these covariant derivatives of $G_{MN}$ actually vanish. Thus, by using the condition of metric-compatibility (E.16), some of these relations may be obtained.

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