The Lévy-flight dynamics can stem from simple random walks in a system whose operational time (number of steps $n$) typically grows superlinearly with physical time $t$. Thus, this processes is a kind of continuous-time random walks (CTRW), dual to usual Scher-Montroll model, in which $n$ grows sublinearly with $t$. The models in which Lévy-flights emerge due to a temporal subordination let easily discuss the response of a random walker to a weak outer force, which is shown to be nonlinear. On the other hand, the relaxation of an ensemble of such walkers in a harmonic potential follows a simple exponential pattern and leads to a normal Boltzmann distribution. The continuous-time random walks (CTRW) first introduced by Montroll and Weiss [12] correspond to a superdiffusive behavior can arise [11].

The mixed models, describing normal CTRW in superlinear operational time and Lévy-flights under the operational time of subdiffusive CTRW lead to paradoxical diffusive behavior, similar to the one found in transport on polymer chains. The relaxation to the Boltzmann distribution in such models is slow and asymptotically follows a power-law.

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I. INTRODUCTION

Random walk processes leading to subdiffusive or superdiffusive behavior are adequate for describing various physical situations. Thus, the continuous-time random walk (CTRW) model of Scher and Montroll [1] was a milestone in understanding of photoconductivity in strongly disordered and glassy semiconductors, while the Lévy-flight models [3] are adequate for description of transport in heterogeneous catalysis [3], self-diffusion in micelle systems [4], reactions and transport in polymer systems under conformational motion [5], transport processes in heterogeneous rocks [6], and for description of behavior of dynamical systems [7]. The closely related models appear in description of economic time series [8]. The Lévy-related statistics were observed in hydrodynamic transport [9], and in the motion of gold nanoclusters on graphite [10]. The mixed models were proposed, in which the slow temporal evolution (described by Scher-Montroll CTRW) is combined with the possibility of Lévy-jumps, so that in general both sub- or superdiffusive behavior can arise [11].

The continuous-time random walks (CTRW) first introduced by Montroll and Weiss [12] correspond to a stochastic model in which steps of a simple random walk take place at times $t_i$, following some random process with non-negative increments: $\tau_i = t_i - t_{i-1} \geq 0$. In a mathematical language one says that CTRW is a process subordinated to random walks under the operational time defined by the process \( \{ t_i \} \). It is typically thought that a CTRW-scheme alone can not describe any superdiffusive process, so that the introduction of very long jumps is an inevitable part of building a model leading to superdiffusive behavior.

Let us first discuss a typical CTRW approach. Let us consider a one-dimensional situation under which a particle from time to time makes a jump to a neighboring lattice site separated from the initial one by a distance $a$. The time $\tau$ between the two jumps is distributed according to some waiting-time distribution, represented by the probability density function (PDF) $p(\tau)$. If the mean waiting time $\overline{\tau}$ exists, the particle’s behavior is diffusive, with diffusion coefficient $D = a^2/2\overline{\tau}$. If the corresponding moment diverges, the particle’s behavior becomes subdiffusive, with diffusion coefficient $D = a^2/\tau^\alpha$, with $\alpha < 1$ depending on the PDF $p(\tau)$. The subdiffusive behavior is indicated by vanishing of the diffusion coefficient $D$. It seems impossible to obtain within this scheme any type of a superdiffusive behavior unless one allows for infinitely long jumps with $\langle a^2 \rangle \to \infty$. The superdiffusive behavior is indicated by divergence of the diffusion coefficient $D$. If $\langle a^2 \rangle$ stays finite this can be the case only if $\overline{\tau}$ vanishes. Since $\tau > 0$ and $\overline{\tau} = \int_0^\infty \tau p(\tau)d\tau$, vanishing of the mean waiting time means that $p(\tau) = \delta(\tau)$, a marginal, degenerate situation.

On the other hand the consideration presented above shows only that the waiting-time distribution is not an adequate tool for description of superdiffusive CTRW. In what follows we show that superdiffusive CTRW with bounded step lengths are just as possible as the subdiffusive ones. Our considerations will be rather formal and do not follow from any particular physical model. On the other hand, the fact that Lévy-flights can stem from a process subordinated to simple random walks has many important implications. Thus, as we proceed to show, the fast dynamics of a free process can coexist in such models with simple exponential relaxation to a normal Boltzmann equilibrium distribution, if the behavior of an ensemble of random walkers under restoring force is considered. This shows that the relation between Lévy dynamics and the nonextensive thermodynamics described by nonclassical entropy functions is much looser than typically assumed.

The combinations of the superdiffusive Lévy-flights with the typical CTRW operational time leads to paradoxical diffusion behavior, having some parallels in transport on polymer chains. Moreover, the existence of a subordination model leading to Lévy flights can be useful in...
understanding of statistical implications of the processes described by fractional generalizations of diffusion and Fokker-Planck equations [14, 15].

The article is organized as follows: In Sec. 2 we discuss general properties of subordinated random processes. In Sec. 3 and 4 the processes subordinated to symmetric and asymmetric random walks are considered, these leading to symmetric and asymmetric Lévy-flights. The dualism between the Lévy-flights and the Scher-Montroll CTRW is discussed in Sec. 5. Sections 6 and 7 discuss the models leading to paradoxical diffusion behavior. The relaxation to equilibrium is considered in Sec. 8.

II. THE SUBORDINATION OF RANDOM PROCESSES

As already mentioned, a Scher-Montroll CTRW process is a simple random walk whose steps take place at times \( t_i \), governed by a random process with nonnegative independent increments, so that

\[
P(x, t) = \sum_{n} P_{RW}(x, n)p_{n}(t),
\]

where \( P_{RW}(x, n) \) is a probability distribution to find a random walker at point \( x \) after \( n \) steps (i.e. the binomial distribution), and \( p_{n}(t) \) is the probability to make exactly \( n \) steps up to time \( t \). For both \( n \) and \( t \) large, when the binomial distribution can be approximated by a Gaussian one, and when the corresponding sum can be changed to an integral, Eq. (1) reads:

\[
P(x, t) \approx \int_{0}^{\infty} \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{x^2}{2n} \right) p_{n}(n, t)dn.
\]

In a classical Scher-Montroll CTRW \( p_{n}(n, t) \) corresponds to a random process in which \( n \) typically grows sublinearly in \( t \). Thus, the overall process is subdiffusive.

Note that a description of CTRW-process given by Eq. (2) is an example of subordination, see Sec. X.7 of Ref. [16]: If \( \{X(T)\} \) is a Markov process with continuous transition probabilities and \( \{T(t)\} \) a process with nonnegative independent increments, then \( \{X(T(t))\} \) is said to subordiate to \( \{X(t)\} \) using the operational time \( T \). In this case

\[
P(x, t) = \int_{0}^{\infty} P_{x}(x, T)p_{T}(T, t)dT.
\]

In what follows we call the integral transform, Eq. (3) a subordination transformation, changing from time scale \( t \) to a time-scale \( T \). For example, in the Scher-Montroll case the operational time of a system is given by the number of steps of the RW, and is a random function of the physical time \( t \) whose typical value grows sublinearly in \( t \).

The operational time can also grow superlinearly with \( t \). Such a process can not be described by a waiting-time distribution, and needs a complimentary description. Let us consider a random process, where the density of events fluctuates strongly. Let us subdivide the time axis into intervals of duration \( \Delta t \) and let us consider the number \( n \) of jumping events within each interval. The value \( \rho = n/\Delta t \) defines the density of jump events. Now, if the mean density of events exists, its inverse gives us exactly the mean waiting time of a jump, and a process described by a finite density of events is a normal diffusive one. The divergence of a mean waiting time (like in Scher-Montroll CTRW) correspond to vanishing density. On the other hand, if one considers a strongly fluctuating density \( \rho(t) \) whose first moment diverges, the mean waiting time vanishes and a process that subordinates a random walk process under such operational time can be superdiffusive. At longer times, the distribution of the number of events tends to one of the Lévy-stable laws: the typical number of events can grow superlinearly in time. A simple example of such process was already known to Feller, see Chap. X.7 of Ref. [16]. He considers a process subordinated to simple random walks under the operational time governed by a fully asymmetric Lévy stable law of index 1/2. The corresponding PDF at time \( t \) is given by

\[
P(x, t) = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{x^2}{2n} \right) \frac{t}{\sqrt{n^{3/2}}} \exp \left( -\frac{t^2}{2n} \right) dn
\]

\[
= \frac{t}{\pi(t^2 + x^2)},
\]

i.e. is a Cauchy Lévy-flight.

Let us now discuss a simple analogy describing the relation between the Scher-Montroll CTRW and Lévy-flights. This analogy makes clear many of the findings we are going to discuss below. Imagine a physical clock producing ticks following with frequency 1, which govern the behavior of a random walker. Imagine a switch situated at 0, so that returning to the origin, the walker can trigger some physical process (the analogy with the Glarum model of relaxation, Ref. [17], is evident!). The times between the subsequent returns are distributed according to a fully asymmetric Lévy stable law of index 1/2 used in a previous example. Imagine now another random walker performing its motion (a step per physical unit time) independently from the first one. Imagine a movie camera, taking frame-per-frame pictures of the positions of this second random walker at the moments when the first walker is at the origin and thus triggers the switch. Watching the movie taken by the camera, we immediately recognize that the second walker performs the Cauchy Lévy-flights. Imagine, that a clock is posed in a frame and also filmed. In this case its image will show exactly the operational time of the system; the spectator’s watch measures the physical time. Imagine an opposite situation: the first walker triggers the motion of the second one, and the camera is triggered by the physical clock, as a normal movie camera is. The process we recognize at the film is then the Scher-Montroll CTRW. We can take a Scher-Montroll movie also using another trick.
(which cannot be performed in a real time, but needs a record of return times). Let us take a record of subsequent return times of a first random walker (numbers $n_1, n_2, \ldots$) and trigger our camera in such a way that it makes $n_1$ frames during the first second, $n_2$ frames during the second one, etc. If we film a normal random walker with a camera prepared in such a way, the movie will show us the Montroll-Weiss CTRW. An image of the physical clock will again show the operational time of the system, and again, looking at his watch, the spectator can measure the physical time between two events.

Let us use our camera triggered by returns of a random walker to film other processes taking place in the outer world. The film which is watched afterwards under constant speed shows us a possible world: The causality relations and thermodynamical time arrow are those of our usual world. On the other hand, a movie of a world undergoing continuous evolution, in which "natura non facit saltus" holds, will show us a revolutionary world of "great leaps" and abrupt changes (but following the same logic of development). The second camera (fed by a prescribed $n$-sequence) will show us the world of almost full stagnation seldomly interrupted by a bounded, local movement, a world developing in a slow time of old Asiatic despoty. We shall keep this analogy in mind when discussing the physical implications of subordination.

Let us consider a system which evolves according to a Markovian dynamics and whose state tends to a normal Boltzmann equilibrium under relaxation. In a system under action of outer forces, the transition probabilities between the states of the system (sites $i$ between which the random walk takes place) which are characterized by their energies $E_i$, are not independent. They are connected through the corresponding Boltzmann-factors, so that in equilibrium during any period of time $\Delta t$ the mean numbers of forwards and of backwards jumps between any two sites $i$ and $j$ fulfill the condition

$$n_{ij}(\Delta t)/n_{ji}(\Delta t) = \exp[(E_i - E_j)/kT], \quad (5)$$

where $k$ is the Boltzmann constant and $T$ is the system's temperature. The condition Eq. (5) guarantees detailed balance in equilibrium, independently of what the real dynamics of a system is. For simple RWs, where only transitions between the neighboring states are allowed, the corresponding transition rates with respect to the operational time of the system can be introduced. For a random a walker moving under the influence of a weak constant force $F$ the probabilities of the forward and backward jumps per unit time $w_+$ and $w_-$ are connected through $w_+/w_- = \exp(Fa/kT)$. The Markovian nature of RW then leads to the fact that the values of $w_+$ and $w_-$ do not depend on whether the system is in equilibrium or not. For $F$ small one can take, say, $w_+ = w_0(1 + Fa/kT)$ and $w_- = w_0(1 - Fa/kT)$ with $w_0 = 1/2\pi$.

Note that subordination, describing a transition from a physical time to an operational time of the system, does not change its equilibrium properties. Such subordination can be considered as random modulation of the transition rate $w_0$ by some independent process (say closing and opening the channels), and is fully irrelevant for thermodynamics (i.e. thermostatistics) of the system. On the other hand, it strongly influences its kinetics, so that a question can be posed, what kinds of kinetics are compatible with the relaxation to a normal Boltzmann distribution under arbitrary subordination transformation of time. We address this question in Sec. 8, after the free diffusion properties of superdiffusive CTRW will be discussed.

### III. Symmetric Lévy Flights from CTRW

Let us first concentrate on the symmetric random walk case. Let us consider a random process in which the number of events per given time is unbounded and follows, for example, a power-law distribution, $p_n(t) \propto t^{1-\alpha}$ with $0 < \alpha \leq 1$ (this corresponds to the typical number of events scaling as $n \propto t^{1/\alpha}$). Let us find the asymptotic behavior of $P(x, t)$ for $t$ large. Since the jumps during different intervals are uncorrelated, the PDF of $n$ for longer times converges to a fully asymmetric Lévy-stable law

$$p(n, t) \approx t^{-1/\alpha} L(n/t^{1/\alpha}; \alpha, \gamma) \quad (6)$$

with the asymmetry parameter $\gamma = -\alpha$ (here the values of $\gamma = \pm \alpha$ correspond to the strongly asymmetric PDF that vanish identically for large positive (negative) $x$ values, while $\gamma = 0$ corresponds to symmetric distributions; the notation in one of Ref. [16]). Note that the Fourier-transforms of Lévy-stable laws are known: up to the translation $P(k, t)$ is equal to

$$f(\kappa) = \exp \left[ -|\kappa|^\alpha e^{ix\gamma/2} \right] \quad (7)$$

(for $0 < \alpha < 2$, $\alpha \neq 1$). The PDF is a real function, thus $f(\kappa) = f^*(-k)$. The corresponding function is analytical everywhere except for $\kappa = 0$, so that the PDF is given by

$$L(x; \alpha, \gamma) = \frac{1}{\pi} \text{Re} \int_0^\infty e^{-ix\zeta - \zeta^{\alpha}} e^{ix\gamma/2} d\zeta. \quad (8)$$

From Eq. (8) the series expansions for $L(y; \alpha, \gamma)$ follow, see Sec. XVII.6 of Ref. [16]. In the case $\alpha < 1$ one can move the path of integration to the negative imaginary axis (since the integrand tends to zero when $\text{Im}\zeta \to -\infty$ due to the dominance of the linear term), which allows then for elementary integration after Taylor-expansion of $\exp(A\zeta^{\alpha})$. For $1 < \alpha < 2$ this dominance is no more the case, but the integrand still vanishes for $\text{Im}\zeta \to -\infty$ in the case of symmetric distributions, while $(-i\zeta^{\alpha}) = [\zeta^{\alpha} (\cos \frac{\pi}{2}\alpha - i \sin \frac{\pi}{2}\alpha)] \to -\infty$ for $1 < \alpha < 2$. Thus the series which represents Lévy distributions for $0 < \alpha < 1$ and symmetric Lévy distribution also for $1 < \alpha < 2$ reads:
\[ L(y; \alpha, \gamma) = \frac{1}{\pi y} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k\alpha + 1)}{k!} \sin \left(\frac{k\pi}{2} (\gamma - \alpha)\right) y^{-\alpha k} \]  

(9)

In general the Lévy-stable laws for \(1 < \alpha < 2\) are given by another expansion,

\[ L(y; \alpha, \gamma) = \frac{1}{\pi y} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(1 + k/\alpha)}{k!} \sin \left(\frac{k\pi}{2} (\gamma - \alpha)\right) y^{-k} \]  

(10)

which also holds for asymmetric laws.

One can easily obtain the form of \(x\)-distributions by immediate integration: using Eq.(2) and a scaling form of a Lévy-distribution

\[ p(x, t) \approx \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{x^2}{2n} \right) L(n; p^{1/\alpha}, \alpha, -\alpha) \frac{dn}{t^{1/\alpha}} \]  

(11)

Using Eq.(10) and performing a term-by-term integration, we arrive to the series of integrals of the form

\[ I_\mu(\zeta) = \int_0^\infty \frac{1}{\sqrt{2\pi} \xi} e^{-\xi^2/2} \xi^{-\mu} d\xi = \frac{1}{\sqrt{2\pi}} \left( \frac{2}{2} \right)^{1/2-\mu} \Gamma(\mu-1/2). \]  

(12)

For integral of the \(k\)-th term in Eq.(10) we have \(\mu = 1 + ak\). Let us concentrate first on the case \(0 < \alpha < 1\). Using well-known relations for \(\Gamma\)-function: \(\Gamma(z + 1) = z\Gamma(z)\) (Eq. (6.1.15) of Ref. [18]) and \(\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma(z+1/2)\) (Eq. (6.1.18) of [18]) we get

\[ p(\zeta) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(2k\alpha + 1)}{k!} \sin (-k\pi \alpha) \left( \frac{\sqrt{2}}{\zeta} \right)^{-2\alpha-1}, \]  

(13)

which represents a series expansion for a symmetric Lévy-stable law of index \(2\alpha\), Eq.(10), for the scaled variable \(\zeta/\sqrt{2}\). This corresponds to a form \(p(x, t) = t^{-1/2} L(x/\sqrt{2t^{2\alpha}}, 2\alpha, 0)\) of the \(x\)-distribution.

We note that taking Fourier-transform of the both parts of for symmetric RWs,

\[ L(ax, 2\alpha, 0) = \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{x^2}{2n} \right) L(n; \alpha, -\alpha) dn, \]  

(14)

where \(a\) is an unimportant scaling factor, we get:

\[ \exp(-A |k|^{2\alpha} t) = \int_0^\infty \exp \left( -k^2 n \right) L(n; \alpha, -\alpha) dn, \]  

(15)

which holds for any real \(k\) (i.e. for any positive \(k^2\)), where \(A\) is a number factor. This gives us a general expression for a Laplace-transform of an asymmetric Lévy-distribution with \(\alpha < 1\): A Laplace-transform of \(L(n, \alpha, -\alpha)\) is \(\exp(-A |k|^{\alpha} t)\). From this fact an important result follows:

\[ L(ax; \alpha, \beta, 0) = \int_0^\infty n^{1/\beta} L(x/n^{1/\beta}; \beta, 0) L(n; \alpha, -\alpha) dn : \]  

(16)

A Lévy distribution with index \(\alpha \beta\) is subordinated to a Lévy-distribution with index \(\beta < \alpha\) under the operational time given by an asymmetric Lévy law of index \(\alpha < 1\). To see this, consider the characteristic functions of both sides of Eq.(16) and use Eq.(15).

\[ \exp(-A |k|^{\alpha \beta}) = \int_0^\infty e^{-|k|^{\beta} n} L(n; \alpha, -\alpha) dn, \]  

(17)

see Sec.X.7 of Ref. [19]. Eq.(15) corresponds to a special case of \(\beta = 2\) of Eq.(16). The distributions \(L(n; \alpha, -\alpha)\) thus coincide with inverse Laplace transforms of stretched-exponentials. For example for \(L(n; 1/2, -1/2)\) one readily gets:

\[ p(n, t) = L^{-1}(\exp(-tu^{1/2})) = \frac{t}{2\sqrt{\pi n^{3/2}}} \exp \left( -\frac{t^2}{4n} \right), \]  

(18)

which differs only by a scale for the time-unit from a distribution used in the example Eq.(15).

**IV. ASYMMETRIC LÉVY-FLIGHTS**

Imagine a random walker moving under the influence of a weak constant force \(F\). Such force introduces an asymmetry into the walker’s motion, since the probabilities of the forward and backward jumps, \(w_+\) and \(w_-\) are now weighted with the corresponding Boltzmann-factors, \(w_+ / w_- = \exp(Fa/kT)\). For \(F\) small one can take \(w_+ = 1/2 + Fa/2\pi kT\) and \(w_- = 1/2 + Fa/\pi kT\). For \(t\) large such random walks lead to the Gaussian distribution of the particles’ positions

\[ P_{RW}(x, t) = \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{(x-vn)^2}{2n} \right) \]  

(19)

whose center moves with a constant velocity \(v = \mu F = Fa^2/2\pi kT\). Note that our RW fulfill the Einstein’s relation between the mobility \(\mu\) and diffusion coefficient \(D: \mu = D/kT\). The PDF of a random process which subordinates biased RW under an operational time following the asymmetric Lévy-law is given by:

\[ P(x, t) \approx \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{(x-vn)^2}{2n} \right) \times L\left( n; \alpha, -\alpha \right) \frac{dn}{t^{1/\alpha}}. \]  

(20)
Using the series expansion, Eq.(1) and performing the term-by-term integration leads to the series of the integrals of the type:

\[ I_\mu(\zeta, \omega) = \int_0^\infty \frac{1}{\sqrt{2\pi \xi}} e^{-\frac{(\zeta-\xi t)^2}{2\xi}} \xi^{-\mu} d\xi \]

\[ = 2 \exp(\zeta \omega) \left( \frac{\zeta}{\omega} \right)^{1/4-\mu/2} K_{1/2-\mu}(\zeta \omega) \]  

(21)

for \( \omega \neq 0 \). For integral of the \( k \)-th term in Eq.(1) we again have \( \mu = 1 + ak \). Let us concentrate first on the case \( 0 < \alpha < 1 \). For \( \zeta \omega \) small, \( v \) cancels (see the expansion 9.6.9 of Ref. [18]), \( K_\nu(z) \approx \frac{1}{2} \Gamma(\nu)(\frac{\zeta}{2})^{-\nu} (\nu > 0) \), note that \( K_{-\nu}(z) = K_\nu(z) \), so that the corresponding distribution tends to be a function of \( \zeta \) only, it coincides with one for \( \omega = 0 \), Eq.(16), so that a symmetric Lévy-stable law of index \( 2\alpha \), Eq.(9) emerges. On the other hand, for \( \omega \neq 0 \) and \( x \) large the overall distributions follow from the expansion of \( K_\nu \) for large values of the argument which reads: \( K_\nu(z) \approx \frac{1}{2} e^{-\zeta/2} \) (Eq. 9.7.2 of Ref. [18]). The corresponding integral then tends to \( \frac{1}{2} (\zeta/\omega)^{-\mu} \), so that the corresponding PDF reproduces the PDF of the density of events (up to rescaling). This last form is also the asymptotic from corresponding to the behavior of Eq.(20) for large \( t \).

Hence, the distribution \( P(x,t) \) tends to a fully asymmetric one of index \( \alpha \) for \( x \) and \( t \) large. In this case the distribution shows scaling with a scaling parameter \( \xi = x/(vt)^{\alpha} \). We see that in this case the motion under the influence of a constant force is superdiffusive, so that \( x \approx (Ft)^{1/\alpha} \), and its dependence on the outer force is nonlinear. Thus, the model shows a behavior that differs considerably from a linear-response assumption of Refs. [11,19,20]. This absence of a linear response regime is parallel to the CTRW-findings [1] (see Ref. [21] for a review) and mirrors the fact that only for normal diffusion a sweep with constant velocity and a drift under a constant force result in the same pattern of motion, see Ref. [14].

The case \( \alpha = 1/2 \) again results in a closed expression:

\[ P(x,t) = \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{(x-\nu n)^2}{2n} \right) \]

\[ \times \frac{t}{\sqrt{2\pi n}} \exp \left( -\frac{n^2}{2n} \right) dn \]

\[ \approx \frac{1}{\pi \sqrt{vt}} e^{\psi(v)} K_1 \sqrt{\nu^2 (x^2 + t^2)} \]

(2.3.16.1 of Ref. [22]). For \( \nu, x \) and \( t \) small, the corresponding distribution tends to a Cauchy-law. On the other hand, for \( t \) large we can take approximately:

\[ P(x,t) \approx \frac{1}{\sqrt{2\pi (x^2 + t^2)^3/4}} e^{\psi(x-\sqrt{x^2 + t^2})} \]  

(23)

The second moment of this distribution diverges, but the position of the maximum of \( P(x,t) \), determining the typical particle position at time \( t \), tends to grow as \( x_{\text{max}} = \frac{2}{3} t^2 \) for \( t \) large. Thus, the typical behavior of \( x(t) \) under constant force is superlinear.

Note that in the case \( 1 < \alpha < 2 \) the distribution of the particle’s displacement for the case \( v = 0 \) will tend to a Gaussian, but in the case \( v > 0 \) it still tends to a fully asymmetric Lévy one. On the other hand, in this case the distribution of the particle’s position possesses the first moment which grows linearly with time, thus the situation under \( \alpha > 1 \) shows the linear response behavior. Since the second moment of the distribution is absent, the fluctuations are strong, and the width of such distribution is of the order of the typical value of \( x \) itself.

V. THE DUALISM BETWEEN THE SUB- AND THE SUPERDIFFUSIVE CTRW

There exists a clear dualism between the normal, subdiffusive CTRW and a superdiffusive one. The corresponding concepts are illustrated in discrete time by Fig.1, where we return to a situation discussed in Sec.2. Imagine a clock producing ticks following with frequency \( 1 \), marking the physical time of a system. Imagine a system which is triggered not by each tick of a physical clock, but follows some waiting-time distribution, \( \psi(n) \). This means that after our random walker has jumped, the next jump will take place after \( n \) ticks of a clock, where the number \( n \) is chosen according to a power-law distribution, say \( \psi(n) \propto (n+1)^{-1+\gamma} \). The number \( n \) fluctuates strongly, so that the sequence of jumps (corresponding to a randomly decimated sequence of ticks) shows lacunae of different duration. Fig. 1a) shows a realization of such a sequence for the case \( \gamma = 0.75 \). The lacuna starting in the middle of Fig. 1a) at \( t = 54 \) ends at \( t = 161 \). The mean number of jumps during the time \( t \) grows sublinearly with \( t \), namely as \( t^{\frac{3}{4}} \). Let us denote the corresponding subordination transformation as time-expanding transformation (TET) of index \( \gamma \). According to the procedure described above, the corresponding sequence does not have any intervals where the density of events is larger than one. The process subordinated to random walks under such operational time (normal CTRW) is subdiffusive.

Let us now consider the sequence of jumps of a walker as ticks marking relevant time epochs of a system (i.e. associate each jump with a tick of a physical clock). From this point of view, the ticks of initial clocks follow extremely inhomogeneously, so that the number of such ticks within a physical time unit varies according to \( p(n) \propto (n+1)^{-1+\gamma} \). Fig.1b) illustrates this situation: Here we took 100 jumps from the realization shown in Fig.1a) and rescaled each of the corresponding time intervals to the unit length. The ticks of initial clock (shown as bars) follow inhomogeneously and show the intervals of high concentration (but no lacunae). The number of such events grows superlinearly in time. The corresponding subordination transformation will be called a
"time-squeezing transformation" (TST) of index $\gamma$. The process subordinated to random walks under such operational time is superdiffusive and corresponds to Lévy-flights. Note that both TST and TET are the probability distributions $P(n, t)$ of the operational time $n$ for a given physical time $t$, i.e. are positive, integrable functions of $n$.

\[ L(ax; \gamma, 0) = \int_0^\infty n^{-1/\beta} L(x/n^{1/\beta}; \beta, 0) S(n, t) dn, \]  

where $S(n, t)$ is supposed to be a probability distribution of the number of steps $n$ done up to time $t$. Taking Fourier-transform of both parts of Eq.(24) and changing to a variable $u = |k|^\beta$ we get:

\[ \exp(-A|u|^\alpha) = \int_0^\infty e^{-un} S(n, t) dn \]  

with $\alpha = \gamma/\beta$. From Eq.(23) it follows that $S(n, t)$ are the inverse Laplace transforms of stretched-exponentials $\exp(-Au^n)$. Note that according to the Bernstein’s theorem, a function $f(x)$ is a Laplace-transform of a probability distribution if and only if it is completely monotone (i.e. it is infinitely differentiable and $(-1)^nf^{(n)}(x) \geq 0$ for all derivatives $f^{(n)}$) and $f(0) = 1$. The last condition is always fulfilled. Note that according to Criterion 2 discussed on p.441 of vol.III of Ref. [16] a function $f(x) = e^{-\psi(x)}$ is a completely monotone function if and only if $\psi$ is a positive function with a completely monotone derivative. In our case $\psi(x) = A\alpha x$. For $0 < \alpha < 1$ one has: $g(x) = \psi'(x) = A\alpha x^{-1} > 0$, and the higher derivatives (defined on the interval $0 < x < \infty$) are: $g'(x) = A\alpha(\alpha - 1)x^{-2} < 0$, $g''(x) = A\alpha(\alpha - 1)(\alpha - 2)x^{-3} > 0$, etc., so that $(-1)^n g^{(n)}(x) \geq 0$, and thus the function $g$ is completely monotone. Thus $S(n, t)$ is a probability distribution (namely the one which we have found above by explicit calculation). On the other hand, for $\alpha > 1$ the function $g(x)$ is not completely monotone, so that $S(n, t)$ is not a probability distribution. Thus, there is no random process which defines the operational time in such a way that the Lévy-flight of index $\alpha_1$ will be transformed into a Lévy-flight with index $\alpha_2 > \alpha_1$. The absence of an inverse of a TST belonging to a class of subordination transformations has a deep physical interpretation: a TST is a coarse-graining procedure (see Fig.1): the information about the internal steps of the process gets lost. One can not anticipate that the transformation inverse to a coarse-graining belongs to the same class as the direct one.

![Fig. 2. The operational time stemming from subordination of the two processes depicted in Fig.1. Note that the bar-code-like set shows both the intervals of high condensation and the long lacunae.](image)

Note also that the fact that the TET and TST are not inverse of each other is mirrored by the fact that...
Let us first discuss the situation mentioned in the beginning of the section: a RW subordinated to Lévy-distributed operational time, driven by a sublinear one. The PDF of the corresponding random walks has power-law tails, namely, exactly those of a Lévy-distribution of index $\gamma$. On the other hand, the overall width of the corresponding curve grows as $\Lambda \simeq \sqrt{t}$. Moreover, the whole distribution scales as a function of dimensionless displacement $\xi = x/\Lambda$: the overall behavior is somewhat similar to one found on a polymer chain with bridges. The overall form of the function can be found using the well-known expression for $p(n, u)$, the Laplace-transform of the probability $p(n, t)$ to make exactly $n$ steps up to time $t$. Such a process corresponds to a directed motion under the same operational time as CTRW. For the ordinary renewal process one has $p(n, u) = \frac{1}{u}[1 - \psi(u)] \psi^n(u)$, with $\psi(u) \simeq 1 - u^\gamma$ [22]. For $u$ small ($t$ large) this form corresponds to

$$p(n, u) \simeq u^{\gamma-1} \exp(-nu^\gamma).$$

(26)

Considering paradoxical diffusion as a process subordinated to Lévy-flights of index $2\gamma$ under operational time given by $p(n, t)$, we get for $P(k, u)$, the Fourier-Laplace transform of $P(x, t)$,

$$P_\gamma(k, u) = \int_0^\infty e^{-|k|^2nu}p(n, u)dn \simeq \frac{u^{\gamma-1}}{|k|^{2\gamma} + u^\gamma}.$$ 

(27)

The scaling nature of the distribution is immediately evident, the nature of its power-law tails follows from the asymptotic analysis for $k$ small: The tail of $P_\gamma(\xi)$ stems from those of $L(x, 2\gamma, 0)$ and has a power-law asymptotics $P_\gamma(\xi) \propto \xi^{-1-2\gamma}$ ($\gamma < 1$). Note that such a distribution was obtained in Ref. [11] as a solution of a fractional diffusion equation, describing a random process incorporating Lévy-jumps taking place under sublinear operational time. As an example let us consider the distribution $P_{1/2}(x, t)$, i.e. one for $\gamma = 1/2$. This distribution has a simple analytical form, which can be obtained by an inverse Laplace-Fourier transformation of Eq. (27). The inverse Laplace transform of Eq. (27) is one given in 3.21 of Ref. [23] and reads: $P_{1/2}(k, t) = \exp(k^2t)\text{erfc}(|k|t^{1/2})$. The inverse (cosine)-Fourier transform of this function is given by No. 10.6 of Ref. [24] and reads:

$$P_{1/2}(x, t) = -\frac{1}{2\sqrt{t}} \pi^{-3/2} \exp(x^2/4t)\text{Ei}(-x^2/4t),$$

(28)

where Ei($x$) is the exponential integral, see Eq. 5.1.2 of Ref. [18]. The corresponding function is a scaling function of $\xi = x/t^{1/2}$: its behavior for $\xi$ large follows from asymptotic expansion of $-\text{Ei}(-x) = \text{Ei}(x) = x^{-1}e^x[1 - 1/x + ...]$, so that asymptotically $P_{1/2}(\xi)$ shows the $\xi^{-2}$-like tail, similar to one of Cauchy-distribution. For $\xi \to \infty$ the distribution $P_{1/2}(k, t)$ shows weak (logarithmic) singularity (following from Eq.(5.1.11) of Ref. [18]), a sign of strong lacunarity of the corresponding operational time. The asymptotic analysis of Eq.(27) shows
that such integrable singularities appear in the center of distribution for $0 < \gamma \leq 1/2$: the behavior for $\xi \to 0$

is given by $P_\gamma(\xi) \propto \xi^{2\gamma - 1}$, for $\gamma = 1/2$, $P_\gamma(\xi)$ diverges logarithmically, as we have already seen in Eq. (28).

The distribution $P_{1/2}(\xi)$ is plotted in Fig.3 together with the Gaussian distribution (i.e. the distribution $P_1(\xi)$ of the same class, the one corresponding to a normal diffusion) and with the distribution stemming from a simple diffusion, which is discussed in detail in the next section. All distributions are normalized in such a way that their quartiles coincide. Note that the quartiles of $P_{1/2}(\xi)$ are situated at ±0.841.

VII. NON-COMMUTATIVITY OF TIME-SUBORDINATION

Applying the transformations other way around, i.e. considering a process subordinated to Scher-Montroll CTRW under Lévy-time, we get a process which is different from one discussed above. Let us start from a simple example.

Let us note that the TET of index 1/2 (corresponding to an inverse Laplace-transform of the function $e^{-\sqrt{n\pi}/\sqrt{t}}$) is given by

$$Q_{1/2}(n, t) = \frac{1}{\pi t} e^{-n^2/4t},$$

(29)

i.e. corresponds to a part of a Gaussian distribution for $n > 0$, so that $n$ typically grows as $t^{1/2}$. The corresponding TST is given by a distribution, Eq. (18), $R_{1/2}(T, n) = \frac{n}{2\sqrt{\pi T^2}} e^{-n^2/4T}$. The subordination of these two processes is described by a function

$$S_{1/2}(T, t) = \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-n^2/4t} \cdot \frac{n}{2\sqrt{\pi T^3/2}} e^{-n^2/4T} dn$$

$$= \frac{2}{\pi} \sqrt{\pi} \frac{T}{t} \left( \frac{T}{t} + 1 \right)^{-1},$$

(30)

which is a probability distribution with the tail decaying as $T^{-3/2}$ (as a tail of a stable distribution of index 1/2) and with the square-root singularity at zero. Note that this distribution is just a solution of a fractional Liouville equation describing directed motion under such an operational time, just like Eq. (23) is the solution of a fractional diffusion equation. This is a process subordinated to a Lévy one under sublinear time growth.

We now show that the $Q$- and $R$- distributions leading to the paradoxical diffusion are not commutative: An operational time resulting from a $R \ast Q$ transformation has a different distribution from one stemming from a $Q \ast R$-one. For example, the distribution $S_{1/2}(T, t)$ given by Eq. (30) is $S_{1/2}(T, t) = Q \ast R = \int Q(n, t) R(T, n) dn$. Let us calculate a conjugated distribution, $S^*_{1/2}(T, t) = R \ast Q = \int R(n, t) Q(T, n) dn$, one describing a process subordinated to a sublinear growth under the operational time growing according to a Lévy distribution. The distribution $S^*_{1/2}(T, t)$ is given by:

$$S^*_{1/2}(T, t) = \int_0^\infty \frac{1}{\sqrt{\pi n}} e^{-t^2/4n} \cdot \frac{t}{2\sqrt{\pi n^3/2}} e^{-t^2/4n} dn$$

$$= \frac{2t}{\pi t^2 + T^2},$$

(31)

i.e. corresponds to a positive part of a Cauchy-distribution. Note that even such a robust scaling property of a probability distribution as a nature of its power-law tail is different from one of the conjugated counterpart.

The plausible scaling consideration here is as follows. The distribution $Q(T, n)$ has all moments, so that for $n$ large the value of $T$ is well-defined and is of the order of $n^\alpha$, $\alpha < 1$. On the other hand, the distribution of $n$ as a function of $t$ is broad and shows a power-law tail $P(n, t) \propto t^{-1/\alpha(n/t^{1/\alpha})^{-1-\alpha}} \propto tn^{-1-\alpha}$. Changing now variable from $n$ to $T \propto n^\alpha$ we get the asymptotics of the PDF of $T$ in a form: $P(T, t) \propto T^{-2}$, independently on $\alpha$. We note thus that the probability distribution subordinating a subliner continuous-time directed motion under the Lévy-distributed operational time of the same index has a power-law tail decaying as $T^{-2}$, i.e. is similar to a Cauchy-distribution.

The process subordinated to a Gaussian RW under operational time defined by $S_{1/2}(T, t)$ is also not a normal diffusion, but represents a marginal situation of a distribution whose second moment diverges logarithmically. The corresponding PDF shows power-law tails of a $x^{-3}$ type. This PDF is given by:

$$P^*_{1/2}(x, t) = \int_0^\infty \frac{1}{\sqrt{2\pi T}} e^{-x^2/2T} \frac{2t}{\pi t^2 + T^2} dT.$$  (32)

Changing to a new variable $\zeta = x^2/2T$ and then introducing a scaling variable $\xi = \sqrt{T}$ we get the PDF $P(x, t)$ as a scaling function of $\xi$:

$$P^*_{1/2}(\xi) = \frac{1}{\pi^{3/2}} |\xi| \int_0^\infty \frac{1}{\zeta^{1/2} e^{-\zeta}} d\zeta.$$  (33)

For $\xi$ large the corresponding integral decays as $(2/\pi)\xi^{-3}$. Note that Eq. (33) can be expressed in terms of Fresnel sine- and cosine integrals, $S(\xi)$ and $C(\xi)$, so that $P(\xi)$ can be obtained in a closed form:

$$P^*_{1/2}(\xi) = \frac{1}{\sqrt{\pi}} \left\{ \sin \left( \frac{\xi^2}{2} \right) \left[ 1 - 2S \left( \frac{\xi}{\sqrt{\pi}} \right) \right] + \cos \left( \frac{\xi^2}{2} \right) \left[ 1 - 2C \left( \frac{\xi}{\sqrt{\pi}} \right) \right] \right\},$$  (34)

see Eq.(2.3.7.10) of Ref. [23]. The corresponding distribution is also plotted in Fig.3 as a dashed line. Note that the distribution shows a cusp-singularity at $\xi = 0$. The value of $P(\xi)$ in this point is $1/\sqrt{\pi} = 0.564...$. The quartiles of this distribution are situated at ±0.621.
VIII. RELAXATION PHENOMENA UNDER TEMPORAL SUBORDINATION

The fact that the Lévy dynamics can follow from a temporal subordination is important if one wants to analyze the possible thermodynamical implications of the Lévy-flight transport. Imagine an ensemble of thermodynamical systems (say, Brownian particles in a harmonic potential) which was put out of equilibrium and then let relax. As discussed in Sec. 2, such relaxation will lead to a stationary state corresponding to a normal equilibrium Boltzmann distribution. Since this distribution is time-independent, it would not change under temporal subordination, so that the systems with Lévy dynamics may have very ordinary thermodynamical equilibrium states and thus be described by normal Gibbs-Boltzmann entropy. The non-Boltzmann nature of the equilibrium found in Ref. [29] was connected with the fact that the linear response was considered, as proposed by Ref. [20], an assumption at variance with the findings of Sec. 4.

Let us now discuss the relaxation to this equilibrium. A system slightly outside of the equilibrium can be described by a Fokker-Planck equation. For an overdamped particle in a harmonic potential we get, for example:

\[
\frac{\partial P}{\partial n} = \frac{\partial}{\partial x} \left( \gamma k x P + D \frac{\partial}{\partial x} P \right) \tag{35}
\]

Note that the values of \( \gamma \) and \( D \) fulfill the Einstein’s relation, \( \gamma = D/kT \). The Green’s function of Eq. (35) has a form of a Gaussian distribution and reads:

\[
G(x, n | x_0, n_0) = \sqrt{\frac{\gamma}{2\pi D(1 - e^{-2\gamma(n-n_0)})}} \times \exp \left( \frac{\gamma k(x - e^{-\gamma(n-n_0)}x_0)^2}{2D(1 - e^{-2\gamma(n-n_0)})} \right), \tag{36}
\]

see Sec. 5.4 of Ref. [26]. This equation gives us e.g. the PDF at time \( n \) in a system, in which the particles were all situated at \( x = x_0 \) at \( t = 0 \). It is easy to see that the first two central moments \( M_1 = \langle x \rangle \) and \( M_2 = \langle (x - \langle x \rangle)^2 \rangle \) relax exponentially to their equilibrium values, so that

\[
\langle x(n) \rangle = x_0 \exp(-\tau^{-1}n) \tag{37}
\]

and

\[
\sigma^2(n) = \frac{D}{k\gamma}(1 - \exp(-2\tau^{-1}n)), \tag{38}
\]

being a typical pattern of relaxation of a system with only one relaxation time \( \tau = (k\gamma)^{-1} \). Since all higher moments of a Gaussian distribution are the combinations of the lower two, they also relax to their equilibrium values in a (multi-)exponential fashion. Let us start from the Fourier-transform of Eq. (36) and to note that under subordination

\[
P(k, t) = \int \exp \left( -ikx e^{-\gamma n} - Dk^2(1 - e^{-2\gamma n})/2\gamma \right) \times t^{-1/\alpha} L(n/\tau^{1/\alpha}, \alpha, -\alpha)dn. \tag{39}
\]

Let us moreover expand the exponential term in a Taylor series in \( k \): the coefficients of this series give the moments of the corresponding distribution. From Eq. (39) it follows then that the \( i \)-th moment is a combination of integrals of the type

\[
\Phi(t) = \int_0^\infty \exp(-\lambda n)t^{-1/\alpha} L(n/\tau^{1/\alpha}, \alpha, -\alpha)dn \tag{40}
\]

with \( \lambda = m\gamma, 0 \leq m \leq i \). Using the fact that a Laplace transform of a fully asymmetric Lévy-distribution is a stretched exponential function, we get:

\[
\Phi(t) = \exp(-A(\lambda t^{1/\alpha})^\alpha) = \exp(-AX^{\alpha}t). \tag{41}
\]

This means that the exponential relaxation under Lévy dynamics stays a simple exponential relaxation (only the corresponding relaxation time changes). For example, the first moment of the distribution (the particle’s position) still relaxes exponentially to its equilibrium value of zero. On the other hand, the dependence of the relaxation time on the outer parameters (say, temperature) entering through the values of \( \gamma \) and \( D \) can change considerably. Thus, the superdiffusive Lévy-flights dynamics in the force-free case can coexist with standard thermodynamics and with very simple relaxation patterns as soon as the case of a harmonic force is concerned.

Let us consider the relaxation in a harmonic potential under ”paradoxical” diffusion. Here again we can use the moment expansion, Eq. (39), and put down the expression for the characteristic function of the overall distribution:

\[
P(k, t) = \int \exp \left( -ikx e^{-\gamma n} - Dk^2(1 - e^{-2\gamma n})/2\gamma \right) \times S_\alpha(n, t)dn. \tag{42}
\]

Note that the moments of the corresponding distribution are the combinations of the functions:

\[
\Phi(t) = \int_0^\infty \exp(-\lambda T)S_\alpha(T, t)dT. \tag{43}
\]

Note that \( S_\alpha(n, t) \) is a PDF of a process subordinated to a Lévy distribution under TET:

\[
S_\alpha(T, t) = \int d\tau \tau^{-1/\alpha} L_\alpha(T/\tau^{1/\alpha}, \alpha, -\alpha)Q_\alpha(\tau, t)d\tau \tag{44}
\]

Thus, a Laplace transform of \( S \) according to its outer time-variable is a stretched-exponential, so that

\[
\Phi(t) = \int_0^\infty p(\tau, t) \exp(-AX^{\alpha}t)d\tau. \tag{45}
\]
Let us take a Laplace-transform of this expression. Using Eq. (26) we get:

$$\Phi(u) = \int_0^\infty u^{-1} \exp(-\tau u) \exp(-A\lambda^\alpha \tau) d\tau = \frac{u^{\alpha-1}}{u^{\alpha} + A\lambda^\alpha},$$

(46)

For small $u$ (long times) this corresponds to a power-law decay of $\Phi(t)$ of a form $\Phi(t) \propto t^{-\alpha}$ for $t \gg \lambda^{-1}$. Thus, the relaxation in the case of paradoxical diffusion resembles those in normal CTRW and is dominated by large lacunae. In the case when the processes are subordinated other way around, i.e. according to $S_0^\alpha(T,t)$, the decay at longer times follows the universal $t^{-1}$-law: for example for $\alpha = 1/2$ we get:

$$\Phi(t) = \frac{2t}{\pi} \int_0^\infty \frac{\exp(-\lambda T)}{T^2 + t^2} dT =$$

$$= \frac{2A}{\pi} \left[ \sin(\lambda t) \text{ci}(\lambda t) - \cos(\lambda t) \text{si}(\lambda t) \right],$$

(47)

see Eq.(2.3.7.11) of Ref. [2] (here the integral sine and cosine-functions, si($x$) $= -\int_x^\infty \frac{\sin(t)}{t} dt$ and ci($x$) $= -\int_x^\infty \frac{\cos(t)}{t} dt$, are used). For $\lambda t \gg 1$ we get:

$$\Phi(t) \simeq \frac{2}{\pi} (\lambda t)^{-1},$$

(48)

which asymptotic behavior is universal for all Lévy-driven CTRWs of the same index.

IX. CONCLUSIONS

A broad range of physical processes can be described as processes subordinated to a random walk under some operational time. In particular, such subordination leads to anomalous transport properties, the well-known example being the Scher-Montroll continuous-time random walks, a process in which the operational time (given by the number of steps) is sublinear in the physical time $t$. Here we have considered the processes subordinated to a diffusive process under operational time governed by a Lévy-distribution with index $0 < \alpha < 1$, namely the operational time superlinear in physical one. We have shown that in the absence of outer forces this subordination leads exactly to Lévy-flights. The response of such a system to weak outer force is strongly nonlinear. Interestingly enough the relaxation patterns in such systems are simpler than expected. Thus, we show that the behavior in the presence of a weak harmonic force corresponds to a simple exponential relaxation to a normal Boltzmann distribution. The combination of super- and sublinear operational times (i.e. Lévy-flights under sublinear operational time or the Scher-Montroll CTRW under Lévy-time) correspond to the “paradoxical” diffusion, a random process which in a force-free case leads to the probability-distributions of the particle’s displacements, which show the power-law tails and lack the second moment. The width of the distribution, on the other hand, grows proportionally to the square-root of time, showing a typically diffusive behavior. Some physical implications of these findings have been discussed.

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[22] A. Blumen, J. Klafter and G. Zumofen, in *Optical Spectroscopy of Glasses*, ed. I. Zschokke (Reidel, Dordrecht, 1986)
[23] F.Oberhettinger and L.Badii, *Tables of Laplace Transforms*, Springer, NY, 1973, p. 229
[24] F.Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer, Berlin, 1980, p. 49
[25] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series* (Russ.), Moscow, Nauka, 1981
[26] Risken, *The Fokker-Planck Equation*, Springer, Berlin (1984)