CONVEX CONES SPANNED BY REGULAR POLYTOPES

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Abstract. We study three families of polyhedral cones whose sections are regular simplices, cubes, and crosspolytopes. We compute solid angles and conic intrinsic volumes of these cones. We show that several quantities appearing in stochastic geometry can be expressed through these conic intrinsic volumes. A list of such quantities includes internal and external solid angles of regular simplices and crosspolytopes, the probability that a (symmetric) Gaussian random polytope or the Gaussian zonotope contains a given point, the expected number of faces of the intersection of a regular polytope with a random linear subspace passing through its centre, and the expected number of faces of the projection of a regular polytope onto a random linear subspace.

1. Definition of the cones

1.1. Introduction. In Euclidean geometry, there are three infinite series of regular polytopes: regular simplices, regular crosspolytopes and cubes. In this paper, we shall be interested in convex cones whose “sections” are these regular polytopes. It turns out that many quantities appearing in stochastic geometry can be related to the solid angles of these cones. These quantities include

(1) Internal and external angles of the regular simplex and the regular crosspolytope.

(2) Absorption probabilities for certain Gaussian random polytopes.

(3) Expected number of faces of a regular polytope intersected by a random linear subspace.

(4) Expected number of faces of a random projection of a regular polytope.

The paper is organized as follows. In the remaining part of the present Section 1 we introduce the cones we are interested in. In Section 2 we compute the solid angles and the conic intrinsic volumes of these cones. In Section 3 we relate absorption probabilities of Gaussian random polytopes to the cones we are interested in. In Section 4 we express the number of faces in an intersection of a regular polytope by a random linear subspace through the conic intrinsic volumes of our cones. Finally, the proofs are collected in Section 5.

1.2. Notation. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$ and equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. The standard orthonormal basis of $\mathbb{R}^n$ is denoted by $e_1, \ldots, e_n$. 

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For a non-empty set $A \subset \mathbb{R}^n$ its convex hull and its positive hull are defined by
\[
\text{conv } A = \{ \lambda_1 a_1 + \ldots + \lambda_k a_k : k \in \mathbb{N}, a_1, \ldots, a_k \in A, \lambda_1, \ldots, \lambda_k \geq 0, \lambda_1 + \ldots + \lambda_k = 1 \},
\]
\[
\text{pos } A = \{ \lambda_1 a_1 + \ldots + \lambda_k a_k : k \in \mathbb{N}, a_1, \ldots, a_k \in A, \lambda_1, \ldots, \lambda_k \geq 0 \}.
\]
The linear space spanned by $A$ is denoted by $\text{lin } A$. For a subset $F \subset \mathbb{R}^n$, we denote by $\text{relint}(F)$ its “relative interior”, that is the interior of $F$ with respect to its affine hull $\text{aff}(F)$. A polytope is a convex hull of finitely many points in $\mathbb{R}^n$. Similarly, a polyhedral cone (or just a cone, for the purposes of the present paper) is a positive hull of finitely many points in $\mathbb{R}^n$. Alternatively, a polytope can be defined as an intersection of finitely many closed halfspaces (provided it is bounded), whereas a cone is an intersection of finitely many closed halfspaces whose bounding hyperplanes pass through the origin. For general references on convex sets, polytopes, and stochastic geometry we refer to the monographs [13], [15], [14].

1.3. Cones associated with regular polytopes. There are three kinds of cones we are interested in. Their definitions all are motivated in the following way. Let $P \subset \mathbb{R}^n$ be a regular polytope. Identify the space $\mathbb{R}^n$ with a hyperplane in $\mathbb{R}^{n+1}$ spanned by the standard orthonormal basis vectors $e_1, \ldots, e_n$ and shift the polytope by some distance $\sigma > 0$ in direction of the last $(n+1)^{\text{th}}$ basis vector $e_{n+1}$. Our cones are the smallest cones that contain the corresponding polytope and have their apex at the origin. We are interested in the three cases when $P$ is a simplex $P_n = \text{conv}\{e_1, \ldots, e_n\}$, a crosspolytope $P_n = \text{conv}\{\pm e_1, \ldots, \pm e_n\}$ or a cube $P_n = [-1,1]^n$. The corresponding cones can be formally defined in the following way:

\[ C_n^\Box(\sigma^2) := \text{pos}(\sigma e_{n+1} + e_j : j \in \{1, \ldots, n\}), \tag{1} \]
\[ C_n^{\Delta}(\sigma^2) := \text{pos}(\sigma e_{n+1} + e_j, \sigma e_{n+1} - e_j : j \in \{1, \ldots, n\}), \tag{2} \]
\[ C_n^{\oplus}(\sigma^2) := \text{pos}\left(\sigma e_{n+1} + \sum_{j=1}^n \varepsilon_j e_j : \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n \right). \tag{3} \]

The dimension of a cone is defined as the dimension of the linear subspace it generates. Note that the dimensions of our cones are
\[ \dim(C_n^{\Delta}(\sigma^2)) = \dim(C_n^{\otimes}(\sigma^2)) = n + 1, \quad \dim(C_n^{\Box}(\sigma^2)) = n. \]

The following proposition provides convenient representations for the cones $C_n^{\Delta}(\sigma^2)$ and $C_n^{\otimes}(\sigma^2)$.

**Proposition 1.1.** For $n \in \mathbb{N}$ and $\sigma > 0$ the cones $C_n^{\Delta}(\sigma^2)$ and $C_n^{\otimes}(\sigma^2)$ given in (2) and (3) satisfy
\[ C_n^{\Delta}(\sigma^2) = \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \sigma \sum_{i=1}^n |x_i| \right\}, \tag{4} \]
\[ C_n^{\otimes}(\sigma^2) = \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \sigma \max_{1 \leq i \leq n} |x_i| \right\}. \tag{5} \]
Figure 1. The cones $C_2^{\bigtriangleup}(4)$, $C_2^{\bigtriangledown}(4)$ and $C_2^{\bigcirc}(4)$
It is possible (and will be necessary in later applications) to define the cone \( C_n(r) \) in the range slightly larger than \( r = \sigma^2 > 0 \). For \( r > -\frac{1}{n} \) let \( u_1, \ldots, u_n \) be vectors in some Euclidean space \( \mathbb{R}^n \) such that for every \( i, j \in \{1, \ldots, n\} \),

\[
\langle u_i, u_j \rangle = r + \delta_{i,j},
\]

(6)

where \( \delta_{i,j} \) denotes the Kronecker delta. Note that such vectors exist because the \( n \times n \)-matrix with entries \( r + \delta_{i,j} \) is positive definite for \( r > -\frac{1}{n} \), as can be seen from the inequality

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} (r + \delta_{i,j}) x_i x_j = r(x_1 + \ldots + x_n)^2 + (x_1^2 + \ldots + x_n^2) > 0
\]

which is valid for all real \( x_1, \ldots, x_n \) and follows from the inequality between the arithmetic and quadratic means. We define \( C_n(r) \) to be the positive hull of such \( u_1, \ldots, u_n \). For different choices of \( u_1, \ldots, u_n \) we obtain different cones, but all these cones are isometric. Thus, \( C_n(r) \) is well defined up to isometry for all \( r > -\frac{1}{n} \). The cones \( C_n(r) \) have been first introduced and studied by Vershik and Sporyshev [16] under the name contracted (\( r > 0 \)) and extended (\( r < 0 \)) orthants. For a review of their properties, we refer to [10], where these cones were denoted by \( C_n(r) \).

In the present paper, the main focus lies on the cones \( C_n^{\Phi}(\sigma^2) \) and \( C_n^{\overline{\Phi}}(\sigma^2) \). These cones also can be characterized by the scalar products of the spanning vectors. Denoting the vectors spanning \( C_n^{\Phi}(\sigma^2) \) by \( v_i^+ := \sigma e_{n+1} + e_i \) and \( v_i^- := \sigma e_{n+1} - e_i \), where \( i \in \{1, \ldots, n\} \), we have

\[
\langle v_i^+, v_j^+ \rangle = \langle v_i^-, v_j^- \rangle = \sigma^2 + \delta_{i,j}, \quad \langle v_i^+, v_j^- \rangle = \sigma^2 - \delta_{i,j}
\]

(7)

for \( i, j \in \{1, \ldots, n\} \). Analogously, denoting the vectors spanning \( C_n^{\overline{\Phi}}(\sigma^2) \) by \( v_\varepsilon = \sigma e_{n+1} + \sum_{i=1}^{n} \varepsilon_i e_i \), where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n \), we have

\[
\langle v_\varepsilon, v_\eta \rangle = \sigma^2 + \langle \varepsilon, \eta \rangle
\]

(8)

for \( \varepsilon, \eta \in \{-1, 1\}^n \). Equations (7) and (8) explain why we use \( \sigma^2 \) rather than \( \sigma \) as the argument in \( C_n^{\Phi}(\sigma^2) \) and \( C_n^{\overline{\Phi}}(\sigma^2) \).

Knowing these scalar products will be helpful when proving that a certain cone \( C \) is isometric to one of the cones above. If \( C \) is defined as a positive hull of a finite set \( A \) of vectors, to prove the isometry with one of the above cones, it suffices to show that the set \( A \) satisfies (6), (7) or (8).

2. ANGLES AND INTRINSIC VOLUMES

The aim of the present section is to state results on solid angles and conic intrinsic volumes of the cones \( C_n^{\Phi}(\sigma^2) \) and \( C_n^{\overline{\Phi}}(\sigma^2) \). The proofs will be given in Section 5.1.
2.1. **Solid angles.** The solid angle of a cone $C$ is defined as follows. Let $N$ be a random vector with some rotationally invariant distribution on the linear subspace generated by $C$. Then, the solid angle of $C$ is defined as

$$\alpha(C) := \mathbb{P}(N \in C).$$

As an example of a rotationally invariant distribution, we can take the multivariate standard normal distribution. Since the cones $C_n^\Phi(\sigma^2)$ and $C_n^\mathbb{R}(\sigma^2)$ have the full dimension $n+1$, we can take $N = (\xi_1, \ldots, \xi_{n+1})$, where $\xi_1, \ldots, \xi_{n+1}$ are independent standard normal random variables, and the representation given in Proposition 1.1 immediately yields the following

**Corollary 2.1.** The solid angles of the cones $C_n^\Phi(\sigma^2)$ and $C_n^\mathbb{R}(\sigma^2)$ are given by

$$\alpha(C_n^\Phi(\sigma^2)) = \mathbb{P}\left(\frac{1}{\sigma} \xi_{n+1} \geq \sum_{j=1}^{n} |\xi_j|\right),$$

$$\alpha(C_n^\mathbb{R}(\sigma^2)) = \mathbb{P}\left(\frac{1}{\sigma} \xi_{n+1} \geq \max_{1 \leq j \leq n} |\xi_j|\right),$$

where $\xi_1, \ldots, \xi_{n+1}$ are i.i.d. standard normal random variables.

We denote the solid angles above by

$$g_n(\sigma^2) := \mathbb{P}\left(\frac{1}{\sigma} \xi_{n+1} \geq \sum_{j=1}^{n} |\xi_j|\right),$$

$$g_n^\mathbb{R}(\sigma^2) := \mathbb{P}\left(\frac{1}{\sigma} \xi_{n+1} \geq \max_{1 \leq j \leq n} |\xi_j|\right),$$

where $n \in \mathbb{N}$ and $\sigma^2 > 0$. Similarly, we can define

$$g_n^\Delta(r) = \alpha(C_n^\Delta(r))$$

for $r \geq -\frac{1}{n}$. In [10, Proposition 1.5], it was shown that

$$g_n^\Delta(r) = \mathbb{P}[\eta_1 < 0, \ldots, \eta_n < 0],$$

(9)

where $(\eta_1, \ldots, \eta_n)$ is a Gaussian vector with zero mean and covariance matrix

$$\text{Cov}(\eta_i, \eta_j) = \delta_{i,j} - \frac{r}{1 + nr}, \quad i, j \in \{1, \ldots, n\}.$$ 

For a review of the properties of the function $g_n^\Delta(r)$ we refer to [10]. Note that $g_n^\Delta(r)$ coincides with $g_n(-r/(1 + nr))$ in the notation of [10]. We extend the above definitions to the case $n = 0$ by putting $g_0^\Phi(\sigma^2) := 1/2$, $g_0^\mathbb{R}(\sigma^2) := 1/2$, and $g_0^\Delta(r) = 1$. 

2.2. Polar cones. For a polyhedral cone $C \subset \mathbb{R}^n$, its polar cone is defined by

$$C^\circ := \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \text{ for all } y \in C \}.$$  

It is known that $C^{\circ\circ} = C$.

**Proposition 2.2.** For every $\sigma^2 > 0$, we have

$$\left( C_n^{\Delta}(\sigma^2) \right)^\circ = -C_n^{\Delta} \left( \frac{1}{\sigma^2} \right), \quad \left( C_n^{\Box}(\sigma^2) \right)^\circ = -C_n^{\Box} \left( \frac{1}{\sigma^2} \right).$$

While cones associated with the crosspolytope and the cube are polar to each other by the above proposition, the cones associated with the simplex are self-polar in the following sense: If $C \subset \mathbb{R}^n$ is an isometric copy of $C_n^{\Delta}(r)$ (so that $C$ has full dimension in $\mathbb{R}^n$), then $C^\circ$ is isometric to $C_n^{\Delta}(-r/(1+nr))$; see Proposition 2.2 in [10].

2.3. Angles of the crosspolytope. Given a polytope $P$ and its face $F$, the tangent cone $T_F(P)$ of $P$ at $F$ is defined as the positive hull of the set $P - f_0$, where $f_0$ is some fixed point in the relative interior of $F$. The internal angle of $P$ at $F$ is defined as the angle of the tangent cone. The normal (or external) angle of $P$ at $F$ is defined as the angle of the polar of the tangent cone. The next proposition expresses the internal and the external solid angles at the faces of the regular crosspolytope through the quantities $g_n^{\Delta}(\sigma^2)$ and $g_n^{\Box}(\sigma^2)$ introduced above.

**Proposition 2.3.** For $0 \leq k \leq n - 1$ let $F$ be a $k$-dimensional face of an $n$-dimensional crosspolytope $P_n$. The internal solid angle of $P_n$ at $F$ equals

$$g_{n-k-1} \left( \frac{1}{k+1} \right) = \mathbb{P} \left( \sqrt{k+1} \xi_{n-k} \geq \sum_{j=1}^{n-k-1} |\xi_j| \right).$$

The normal solid angle of $P_n$ at $F$ equals

$$g_{n-k-1}(k+1) = \mathbb{P} \left( \xi_{n-k} \geq \sqrt{k+1} \max_{1 \leq j \leq n-k-1} |\xi_j| \right).$$

For the angles of the regular simplex, similar expressions in terms of $g_n^{\Delta}(r)$ are possible; see, e.g. [10] Proposition 1.2].

2.4. Conic intrinsic volumes. The $k^{\text{th}}$ conic intrinsic volume of an $m$-dimensional polyhedral cone $C \subset \mathbb{R}^m$ is defined by

$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(F) \alpha(N_F(C)), \quad k \in \{0, \ldots, m\},$$

where $N_F(C) = C^\circ \cap (\text{lin } F)^\perp$ is the face of $C^\circ$ corresponding to $F$ via the polar duality. We refer to [14, Section 6.5] and [3], [2] for an extensive account of the properties of conic intrinsic volumes.
Theorem 2.4. For $n \in \mathbb{N}$ and $1 \leq k \leq n + 1$, the $k^{th}$ conic intrinsic volume of the cone $C_n^\bigoplus(\sigma^2)$ is

$$
\nu_k \left( C_n^\bigoplus(\sigma^2) \right) = 2^{n-k+1} \binom{n}{k-1} g_{k-1}^\bigoplus (\sigma^2 + n - k + 1) g_{n-k+1}^\Delta \left( \frac{1}{\sigma^2} \right).
$$

(12)

For $0 \leq k \leq n$, the $k^{th}$ conic intrinsic volume of the cone $C_n^\bigoplus(\sigma^2)$ is

$$
\nu_k \left( C_n^\bigoplus(\sigma^2) \right) = 2^k \binom{n}{k} g_{n-k}^\Delta \left( \frac{1}{\sigma^2} + k \right) g_k^\bigoplus (\sigma^2).
$$

(13)

The exceptional cases are

$$
\nu_0 \left( C_n(\sigma^2) \right) = g_n^\bigoplus \left( \frac{1}{\sigma^2} \right), \quad \nu_{n+1} \left( C_n^\bigoplus(\sigma^2) \right) = g_n^\bigoplus (\sigma^2).
$$

3. Absorption probabilities

In this section we give some applications of Theorem 2.4 to the determination of absorption probabilities of certain random polytopes.

3.1. Gaussian projections of regular polytopes. Let $X_1, \ldots, X_n$ be independent standard normal random points in $\mathbb{R}^d$. The Gaussian polytope is defined as the convex hull of these random points, i.e.

$$
P_{n,d}^\bigoplus := \text{conv}\{X_1, \ldots, X_n\}.
$$

Similarly, the symmetric Gaussian polytope $P_{n,d}^\bigoplus$ is defined as the convex hull of these points along with their negatives, i.e.

$$
P_{n,d}^\bigoplus := \text{conv}\{X_1, -X_1, X_2, \ldots, X_n, -X_n\}.
$$

Finally, the Gaussian zonotope is the Minkowski sum of $n$ Gaussian intervals, i.e. with $X_1, \ldots, X_n$ as before,

$$
P_{n,d}^\bigodot := \sum_{i=1}^n \text{conv}\{X_i, -X_i\} = \left\{ \sum_{i=1}^n \lambda_i X_i : \lambda_1, \ldots, \lambda_n \in [-1, 1] \right\}.
$$

These three random polytopes are related to the three regular polytopes via the notion of Gaussian projection. Let $P$ be any (deterministic) polytope in $\mathbb{R}^n$. Let also $X$ be a $d \times n$-matrix whose entries are i.i.d. standard Gaussian random variables. The columns of $X$ can be identified with the random vectors $X_1, \ldots, X_n$ introduced above. We may consider $X : \mathbb{R}^n \to \mathbb{R}^d$ as a random linear map. Then, the Gaussian projection of $P$ (see, e.g., [17]) is defined as the random polytope $XP = \{Xp : p \in P\} \subset \mathbb{R}^d$. It is now easy to check that by taking $P$ to be the regular simplex $\text{conv}\{e_1, \ldots, e_n\}$, the regular crosspolytope $\text{conv}\{\pm e_1, \ldots, \pm e_n\}$ and the cube $[-1, 1]^n$, we recover the random polytopes $P_{n,d}^\bigoplus$, $P_{n,d}^\bigodot$, $P_{n,d}^\bigotimes$ as the corresponding Gaussian projections.

The associated absorption probability is the probability of the event that a deterministic point $x \in \mathbb{R}^d$ is contained in the polytope $P_{n,d}^\bigoplus$ (and similarly for $P_{n,d}^\bigodot$ and $P_{n,d}^\bigotimes$). Since the
standard Gaussian distribution is invariant under rotations, this probability depends on $x$ only by its Euclidean norm $|x|$. Instead of the absorption probability itself it is convenient to analyse the probabilities of non-absorption $f_{n,d}^\triangle$, $f_{n,d}^\boxplus$, $f_{n,d}^\ominus : [0, \infty) \to [0, 1]$ defined by

$$f_{n,d}^\triangle(|x|) := \mathbb{P}(x \notin P_{n,d}^\triangle), \quad f_{n,d}^\boxplus(|x|) := \mathbb{P}(x \notin P_{n,d}^\boxplus), \quad f_{n,d}^\ominus(|x|) := \mathbb{P}(x \notin P_{n,d}^\ominus).$$

An expression for the non-absorption probability $f_{n,d}^\triangle$ was provided in \cite[Theorem 1.2]{10}. The aim of this section is to give similar expressions for the non-absorption probabilities $f_{n,d}^\boxplus$ and $f_{n,d}^\ominus$ of symmetric Gaussian polytopes and Gaussian zonotopes.

### 3.2. Absorption probabilities for polytopes spanned by Gaussian points

For a $d$-dimensional standard normal random vector $X \sim \mathcal{N}^d(0, 1)$ independent of $X_1, \ldots, X_n$ and $\sigma \geq 0$ we define

$$P_{n,d}(\sigma^2) := \mathbb{P}(\sigma X \notin P_{n,d}^\triangle),$$

$$P_{n,d}(\sigma^2) := \mathbb{P}(\sigma X \notin P_{n,d}^\boxplus),$$

$$P_{n,d}(\sigma^2) := \mathbb{P}(\sigma X \notin P_{n,d}^\ominus).$$

These probabilities differ from the non-absorption probabilities, because here the point $\sigma X$ is random. Calculating them will be a first step in determining the absorption probability because, as the following proposition states, there is an connection between these two functions. The proposition is based on \cite[Corollary 1.1]{10}.
Proposition 3.1. Let $P_n \subset \mathbb{R}^d$ be a Gaussian polytope, a symmetric Gaussian polytope or a Gaussian zonotope generated by $n$ independent Gaussian points $X_1, \ldots, X_n$. Its non-absorption probability $f^{P_n} : [0, \infty) \to [0, 1]$ is defined by $f^{P_n}(|x|) := \mathbb{P}(x \notin P_n)$. Then,

\[ \int_0^\infty f^{P_n} \left( \sqrt{2u} \right) u^{d-1} e^{-\lambda u} du = \Gamma \left( \frac{d}{2} \right) \lambda^{-\frac{d}{2}} p^{P_n} \left( \frac{1}{\lambda} \right) \]

for every $\lambda > 0$. Here, $p^{P_n}(\sigma^2) := \mathbb{P}(\sigma X \notin P_n)$.  

Proof. Let $X \sim \mathcal{N}^d(0, 1)$ be independent of $X_1, \ldots, X_n$ as in the definition of $p^{P_n}$. Its Euclidean norm $|X|$ has $\chi$ distribution with $d$ degrees of freedom. Conditioning on the event $|X| = r$ and integrating over $r > 0$ we obtain

\[ p^{P_n}(\sigma^2) = \mathbb{P}[\sigma X \notin \text{conv}\{X_1, \ldots, X_n\}] = \int_0^\infty f^{P_n}(\sigma r) \frac{2^{1-\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)} r^{d-1} e^{-\frac{r^2}{2\lambda}} dr. \]

Substituting $\sigma r = \sqrt{2u}$, taking into account that $dr = \frac{1}{\sigma \sqrt{2u}} du$ and writing $\lambda = \frac{1}{\sigma^2}$, we obtain

\[ p^{P_n} \left( \frac{1}{\lambda} \right) = \frac{\lambda^\frac{d}{2}}{\Gamma \left( \frac{d}{2} \right)} \int_0^\infty f^{P_n}(\sqrt{2u}) u^{\frac{d}{2}-1} e^{-\lambda u} du, \]

thus completing the proof. \qed

3.3. Expressions for $p^{P_n}$. We will now provide explicit expressions for $p^{P_n}$.

Theorem 3.2. For all $n, d \in \mathbb{N}$ such that $n \geq d$ the probability just defined satisfies

\[ p^{\Phi}_{n,d}(\sigma^2) = \mathbb{P}(\sigma X \notin \Phi_{n,d}) = 2(b_{n,d-1}^{\Phi}(\sigma^2) + b_{n,d-3}^{\Phi}(\sigma^2) + \ldots) \]

with

\[ b_{n,k}^{\Phi}(r) := v_k(C_n^{\Phi}(r)) = \begin{cases} 2^k \binom{n}{k} g_{n-k} \left( \frac{1}{r} + k \right) g_k^{\Phi}(r), & \text{if } k \in \{0, \ldots, n\}, \\ g_n^{\Phi}(r), & \text{if } k = n + 1, \end{cases} \]

as in Theorem 2.4 and $b_{n,k}^{\Phi}(r) = 0$ for $k \notin \{0, \ldots, n + 1\}$.

Theorem 3.3. For all $n, d \in \mathbb{N}$ such that $n \geq d$ we have

\[ p^{\overline{\mathcal{P}}}_{n,d}(\sigma^2) = \mathbb{P}(\sigma X \notin \overline{\mathcal{P}}_{n,d}) = 2(b_{n,d-1}^{\overline{\mathcal{P}}}(\sigma^2) + b_{n,d-3}^{\overline{\mathcal{P}}}(\sigma^2) + \ldots), \]

where

\[ b_{n,k}^{\overline{\mathcal{P}}}(r) := v_k \left( C_n^{\overline{\mathcal{P}}}(r) \right) = 2^{n-k+1} \binom{n}{k-1} g_{k-1}(r + n - k + 1) g_{n-k+1} \left( \frac{1}{r} \right) \]

for $k \in \{0, \ldots, n + 1\}$ and $b_{n,k}^{\overline{\mathcal{P}}}(r) = 0$ for $k \notin \{0, \ldots, n + 1\}$.

Combining Theorems 3.2 and 3.3 with Proposition 3.1 we arrive at the following
Corollary 3.4. For every $\lambda > 0$ and for $n, d \in \mathbb{N}$ such that $n \geq d$ the absorption probabilities $f_{n,d}^\Phi$ and $f_{n,d}^{\overline{\Phi}}$ satisfy the equations
\[
\int_0^\infty f_{n,d}^\Phi (\sqrt{2u}) u^{\frac{d}{2}-1} e^{-\lambda u} du = 2\Gamma \left( \frac{d}{2} \right) \lambda^{-\frac{d}{2}} \left( b_{n,d-1}^\Phi \left( \frac{1}{\lambda} \right) + b_{n,d-3}^\Phi \left( \frac{1}{\lambda} \right) + \ldots \right),
\]
\[
\int_0^\infty f_{n,d}^{\overline{\Phi}} (\sqrt{2u}) u^{\frac{d}{2}-1} e^{-\lambda u} du = 2\Gamma \left( \frac{d}{2} \right) \lambda^{-\frac{d}{2}} \left( b_{n,d-1}^{\overline{\Phi}} \left( \frac{1}{\lambda} \right) + b_{n,d-3}^{\overline{\Phi}} \left( \frac{1}{\lambda} \right) + \ldots \right).
\]

3.4. Absorption probabilities in dimension $d = 2$. In dimension $d = 2$ the equalities in Corollary 3.4 simplify to
\[
\int_0^\infty f_{n,2}^\Phi (\sqrt{2u}) e^{-\lambda u} du = \frac{2}{\lambda} b_{n,1}^\Phi \left( \frac{1}{\lambda} \right),
\]
\[
\int_0^\infty f_{n,2}^{\overline{\Phi}} (\sqrt{2u}) e^{-\lambda u} du = \frac{2}{\lambda} b_{n,1}^{\overline{\Phi}} \left( \frac{1}{\lambda} \right).
\]

Using that $g_1^\Phi (r) = \frac{1}{2}$ for every $r > -\frac{1}{n}$ and $g_0^{\overline{\Phi}} (r) = \frac{1}{2}$ for every $r > 0$ we have
\[
b_{n,1}^\Phi \left( \frac{1}{\lambda} \right) = v_1 \left( C_n^\Phi \left( \frac{1}{\lambda} \right) \right) = n \cdot g_{n-1}^\Phi (\lambda + 1),
\]
\[
b_{n,1}^{\overline{\Phi}} \left( \frac{1}{\lambda} \right) = v_1 \left( C_n^{\overline{\Phi}} \left( \frac{1}{\lambda} \right) \right) = 2^{n-1} g_n^\Phi (\lambda).
\]

Thus,
\[
\int_0^\infty f_{n,2}^\Phi (\sqrt{2u}) e^{-\lambda u} du = \frac{2n}{\lambda} g_{n-1}^\Phi (\lambda + 1) = \frac{2n}{\lambda} \mathbb{P} \left( \frac{1}{\sqrt{\lambda+1}} \xi_n \geq \max_{1 \leq j \leq n-1} |\xi_j| \right),
\]
\[
\int_0^\infty f_{n,2}^{\overline{\Phi}} (\sqrt{2u}) e^{-\lambda u} du = \frac{2n}{\lambda} g_n^\Phi (\lambda).
\]

So, to calculate $f_{n,2}^\Phi$ and $f_{n,2}^{\overline{\Phi}}$ it is sufficient to invert the Laplace transforms on the right side. This can be done, for example, by using the Bromwich integral. In the case of the symmetric Gaussian polytope, a more explicit inversion is possible and stated in the following result.

Theorem 3.5. Let $\xi, \xi_1, \ldots, \xi_n$ be independent standard normal random variables. Define $L_n := \max \{|\xi_1|, \ldots, |\xi_n|\}$. Then, for all $u > 0$, we have
\[
f_{n,2}^\Phi (\sqrt{2u}) = \mathbb{P} \left( \frac{L_n^2 + \xi^2}{2} \leq u \right) + \frac{d}{du} \mathbb{P} \left( \frac{L_n^2 + \xi^2}{2} \leq u \right).
\]

Proof. The right-hand side of (16) can be written as
\[
\frac{2n}{\lambda} \mathbb{P} \left( \frac{1}{\sqrt{\lambda+1}} \xi_n \geq \max_{1 \leq j \leq n-1} |\xi_j| \right) = \frac{2n}{\lambda} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{s^2}{2}} \left( 2\Phi \left( \frac{s}{\sqrt{\lambda+1}} \right) - 1 \right) ^{n-1} ds,
\]
where $\Phi$ is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0, 1)$. Substituting $\frac{s^2}{2} = t(\lambda + 1)$, so that $s = \sqrt{2t(\lambda + 1)}$ and $ds = \sqrt{\frac{\lambda + 1}{2t}} dt$, we rewrite the
right-hand side as
\[
\frac{2n}{\lambda \sqrt{2\pi}} \int_0^\infty e^{-\lambda t} e^{-t} (2\Phi(\sqrt{2t}) - 1)^{n-1} \frac{\sqrt{\lambda + 1}}{\sqrt{2t}} dt
\]

\[
= \frac{\sqrt{\lambda + 1}}{\lambda} \int_0^\infty e^{-\lambda t} \frac{2e^{-t}}{\sqrt{2\pi}} (2\Phi(\sqrt{2t}) - 1)^{n-1} \frac{1}{\sqrt{2t}} dt
\]

\[
= \frac{\sqrt{\lambda + 1}}{\lambda} \int_0^\infty e^{-\lambda t} \frac{d}{dt} \left( (2\Phi(\sqrt{2t}) - 1)^n \right) dt.
\]

(18)

Now, \((2\Phi(\sqrt{2t}) - 1)^n\) is the distribution function of \(\frac{1}{2}L_n^2\), where we recall that \(L_n := \max\{|\xi_1|, \ldots, |\xi_n|\}\) and the \(\xi_i\)’s are independent standard normal random variables.

On the other hand, the inverse Laplace transform of \(\sqrt{\frac{\lambda + 1}{\lambda}}\) is
\[
2\Phi(\sqrt{2t}) - 1 + \frac{e^{-t}}{\sqrt{\pi t}} = F_{\xi^2}(t) + f_{\xi^2}(t),
\]

where \(F_{\xi^2}\) is the distribution function and \(f_{\xi^2}\) the density of \(\xi^2\), with \(\xi\) being standard normal.

Thus the inverse Laplace transform of (18) is the convolution of \(F_{\xi^2} + f_{\xi^2}\) and \(f_{\frac{1}{2}L_n^2}\), where \(f_{\frac{1}{2}L_n^2}\) is the density function of \(\frac{1}{2}L_n^2\). It follows from (16) that
\[
f_{\frac{1}{n}L_n^2}(\sqrt{2u}) = \int_0^u F_{\xi^2}(t) f_{\frac{1}{2}L_n^2}(u - t) dt + \int_0^u f_{\xi^2}(t) f_{\frac{1}{2}L_n^2}(u - t) dt
\]

\[
= \mathbb{P}\left( \frac{L_n^2 + \xi^2}{2} \leq u \right) + \frac{d}{du} \mathbb{P}\left( \frac{L_n^2 + \xi^2}{2} \leq u \right),
\]

which completes the proof. \(\square\)

4. Random sections of regular polytopes

In [12] Lonke investigated the asymptotics of the expected number of \(j\)-faces of the intersection of the \(n\)-cube \([-1, 1]^n\) and a random \(k\)-dimensional linear subspace of \(\mathbb{R}^n\) chosen uniformly from the Grassmannian \(Gr(k, \mathbb{R}^n)\), i.e. the set of all \(k\)-dimensional linear subspaces of \(\mathbb{R}^n\). With the methods we used above we can give explicit expressions for these expected numbers not only for cubes, but also for crosspolytopes and simplices.

In Section 4.1 we will determine the probabilities that fixed faces of our polytopes get intersected by the random linear space. This will be an auxiliary result for Section 4.2 in which we will state the expressions we are interested in.

Let \(L\) a random \((n - l)\)-dimensional linear subspace of \(\mathbb{R}^n\) having the uniform distribution on \(Gr(n - l, \mathbb{R}^n)\). Here, \(l \in \{1, \ldots, n - 1\}\) is the codimension of \(L\). Let the random variable \(\phi^P(j, n - l, n)\) be the number of \(j\)-faces of the intersection of \(L\) and an \(n\)-dimensional convex polytope \(P \subset \mathbb{R}^n\) which contains the origin in its interior.
Proposition 4.1. For $l \in \{1, \ldots, n - 1\}$ and $j \in \{0, \ldots, n - l - 1\}$, 
\[
\phi^P(j, n - l, n) = \sum_{B \in F_{j+l}(P)} 1_{\{B \cap L \neq \emptyset\}} \text{ almost surely.}
\] (19)

In other words, almost surely the number of $j$-faces of the intersection $L \cap P$ is equal to the number of $(j + l)$-faces of $P$ that have non-empty intersection with $L$. Equation (19) looks natural because it is reasonable that with probability 1 the $j$-faces of $L \cap P$ can be obtained as the intersections of $L$ and the $(j + l)$-dimensional faces of $P$. The proof of this fact is surprisingly difficult and will be given in Section 5.3. In [9, (3.1)] a similar result is stated (without proof) for random projections instead of random sections.

An immediate consequence for regular polytopes $P$ is 
\[
\mathbb{E}\phi^P(j, n - l, n) = \mathbb{E} \sum_{B \in F_{j+l}(P)} 1_{\{B \cap L \neq \emptyset\}} = \#F_{j+l}(P) \cdot \mathbb{P}(B \cap L \neq \emptyset),
\] (20)

which is true for every $(j + l)$-face $B$.

It should be stressed that, as already observed by Lonke [12], there is a duality between the number of faces in random intersections as above and the number of faces in random projections of a dual polytope. Namely, the expected number of $j$-faces of a random $k$-dimensional projection of an $n$-dimensional polytope $P$ containing the origin in its interior coincides with the expected number of $(k - j - 1)$-faces of the intersection of the dual polytope $P^\circ$ with a random $k$-dimensional linear subspace passing through the origin. The expected face numbers of a random projection of a polytope has been expressed through its internal and external angles in the work of Affentranger and Schneider [1]. Asymptotic questions were studied in [16, 4, 5, 6, 7].

4.1. Probabilities that fixed faces get intersected. Our purpose is to apply Equation (20) to random sections of the three kinds of regular polytopes we are analysing. Essentially, we need to derive expressions for the quantity $\mathbb{P}(B \cap L \neq \emptyset)$ for a fixed face of dimension $d = j + l$, which is the second factor in (20). These expressions are given in the following three lemmas. For $l \in \{1, \ldots, n - 1\}$ we recall that $L$ is a linear subspace of $\mathbb{R}^n$ that has codimension $l$ and is chosen from the set of all such subspaces uniformly at random.

Proposition 4.2. For $d \in \{l, \ldots, n - 1\}$ the probability of the event that $L$ intersects a fixed $d$-face $B$ of the cube $[-1, 1]^n$ is 
\[
\mathbb{P}(L \cap B \neq \emptyset) = 2 \left(v_{l+1}(\mathcal{C}_d^{\mathbb{Z}}(n-d)) + v_{l+3}(\mathcal{C}_d^{\mathbb{Z}}(n-d)) + \ldots \right).
\]

Proposition 4.3. For $d \in \{l, \ldots, n - 1\}$ the probability of the event that $L$ intersects a fixed $d$-face $B$ of the crosspolytope $\text{conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n\}$ is 
\[
\mathbb{P}(L \cap B \neq \emptyset) = 2^{-d} \left(\binom{d}{l} + \binom{d}{l+1} + \ldots \right).
\]

Proposition 4.4. Let $P_n \subset \mathbb{R}^n$ be an $n$-dimensional regular simplex centred at the origin, whose edges have unit length. For $d \in \{l, \ldots, n - 1\}$ the probability of the event that $L$
Proof. Let
\[ \mathbb{P}(L \cap F \neq \emptyset) = 2 \left( v_{l+1} \left( C_{d+1} \left( -\frac{1}{n+1} \right) \right) + v_{l+3} \left( C_{d+1} \left( -\frac{1}{n+1} \right) \right) + \cdots \right). \]

4.2. Random sections of regular polytopes. We are now in position to state formulas for the expected number of faces of regular polytopes intersected by a random linear subspace.

**Theorem 4.5.** Fix \( n > k > j \geq 0 \) and let \( L \) be a random linear subspace of \( \mathbb{R}^n \) that has dimension \( k \) and is chosen from the set of all such subspaces uniformly. Denote its codimension by \( l := \text{codim}(L) = n - k \). Let \( \varphi(j, k, n) \), \( \phi^\square(j, k, n) \) and \( \phi^\bigtriangleup(j, k, n) \) be the number of \( j \)-faces of the intersection of \( L \) and respectively the cube \([-1, 1]^n\), the crosspolytope \( \text{conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n\} \) or the \( n \)-dimensional simplex in \( \mathbb{R}^n \) centred at the origin. The expectations of these random variables are given by:

\[
\mathbb{E}\varphi^\square(j, k, n) = 2^{k-j+1} \binom{n}{n-k+j} \cdot \left( v_{n-k+1}(C_{n-k+j}(k-j)) + v_{n-k+3}(C_{n-k+j}(k-j)) + \cdots \right),
\]

\[
\mathbb{E}\phi^\square(j, n-l, n) = 2 \binom{n}{j+l+1} \left( \binom{j+l}{l} + \binom{j+l}{l+1} + \cdots \right),
\]

\[
\mathbb{E}\phi^\bigtriangleup(j, n-l, n) = 2 \binom{n+1}{j+l+1} \cdot \left( v_{l+1}(C_{j+l+1}( -\frac{1}{n+1} )) + v_{l+3}(C_{j+l+1}( -\frac{1}{n+1} )) + \cdots \right).
\]

Proof. Let \( P \) be an \( n \)-dimensional polytope of any of the three types mentioned above and let \( \phi^P \) be, respectively, \( \varphi^\square \), \( \phi^\square \) or \( \phi^\bigtriangleup \).

By equation \((20)\), for an arbitrary \((j+l)\)-face \( F \) we have

\[ \mathbb{E}\phi^P(j, n-l, n) = \#(\mathcal{F}_{j+l}(P)) \cdot \mathbb{P}(F \cap L \neq \emptyset). \] (21)

Applying Propositions 4.2, 4.3 or 4.4 and using the well-known numbers of \((j+l)\)-faces of regular polytopes, we directly get

\[
\mathbb{E}\varphi^\square(j, k, n) = 2^{k-j} \binom{n}{n-k+j} \cdot 2 \left( v_{n-k+1}(C_{n-k+j}(k-j)) + v_{n-k+3}(C_{n-k+j}(k-j)) + \cdots \right),
\]

\[
\mathbb{E}\phi^\square(j, n-l, n) = 2^{j+l+1} \binom{n}{j+l+1} \cdot 2^{-j-l} \left( \binom{j+l}{l} + \binom{j+l}{l+1} + \cdots \right),
\]

\[
\mathbb{E}\phi^\bigtriangleup(j, n-l, n) = \left( \binom{n+1}{j+l+1} \cdot 2 \left( v_{l+1}(C_{j+l+1}( -\frac{1}{n+1} )) + v_{l+3}(C_{j+l+1}( -\frac{1}{n+1} )) + \cdots \right) \right).
\]

\[ \square \]
4.3. Asymptotic results for the expected number of faces in random sections.  
Asymptotic behavior of the expected number of faces in sections (or, dually, projections) of regular polytopes have been studied in the works of Vershik and Sporyshev [16], Böröczky and Henk [4], Lonke [12] and Donoho and Tanner [5, 6, 7]. More specifically, Lonke [12, (2) or Corollary 3.4] proved the following asymptotic formula for \( \phi(n - m, n - l, n) \) for fixed codimensions \( 1 \leq l < m \):

\[
\mathbb{E}\phi(n - m, n - l, n) \sim \frac{(2n)^{m-l}}{(m-l)!} \quad \text{as} \quad n \to \infty.
\]

Here, \( a_n \sim b_n \) means that \( \lim_{n \to \infty} a_n / b_n = 1 \). Böröczky and Henk [4] proved an asymptotic formula for \( \phi(j, k, n) \) for fixed \( 0 \leq j < k \):

\[
\mathbb{E}\phi(j, k, n) \sim C(j, k) \cdot (\log(n))^{\frac{k-j}{2}}
\]

as \( n \to \infty \), where the constant \( C(j, k) \) can be expressed by

\[
C(j, k) = \frac{2^k \pi^{\frac{k-j}{2}} \sqrt{k(k-1)!}}{(k-j)!j!} g_j \left( \frac{1}{k-j} \right).
\]

Recalling that \( g_0 \equiv 1 \), the special case \( j = 0 \) recovers a formula Lonke has proven in [12, (5)]. Based on Theorem [4.5] we can complement these results by the following new asymptotic regimes.

**Corollary 4.6.** Fix some integers \( i > l \geq 1 \). As \( n \to \infty \), we have

\[
\left( \begin{array}{c} n+1 \\ n-l+i+1 \end{array} \right) - \mathbb{E}\phi^{\Delta}(n-i, n-l, n) \sim C(i, l) \cdot n^{-\frac{i}{2}} n^{\frac{3i-3}{2}} \left( \frac{(i-l)e}{2\pi} \right)^{\frac{n}{2}}
\]

where

\[
C(i, l) := \frac{\pi^{\frac{i-3}{2}} 2^{\frac{i-2}{2}} e^{\frac{3i-3}{2}}}{(l-1)! (i-l)! (i-l)^{\frac{i-3}{2}}}
\]

**Corollary 4.7.** Fix some integers \( j \geq 0 \) and \( l \geq 1 \). Then, as \( n \to \infty \), we have

\[
\mathbb{E}\phi^{\triangle}(j, n-l, n) \sim 2^{n-j} n^{j+(l/2)} \Gamma \left( \frac{l+1}{2} \right) \frac{\Gamma \left( \frac{l+1}{2} \right)}{\pi^{(l+1)/2}}.
\]

5. Proofs

5.1. Cones, angles, conic intrinsic volumes - Proofs.

**Proof of Proposition 1.1.** We start with the proof of identity (4). To prove it we shall show that the cone

\[
C := \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \sigma \sum_{i=1}^{n} |x_i| \right\}
\]

is the positive hull of the vectors \( v^+_i := \sigma e_{n+1} + e_i \) and \( v^-_i := \sigma e_{n+1} - e_i \), \( i = 1, \ldots, n \).
Fix any \( x = (x_1, \ldots, x_{n+1}) \in C \). It can be written in the form
\[
x = \sum_{i=1}^{n} (x_i e_i + \sigma |x_i| e_{n+1}) + \left( x_{n+1} - \sigma \sum_{i=1}^{n} |x_i| \right) e_{n+1}
\]
\[
= \sum_{i=1}^{n} |x_i| (v_i^+ 1_{\{x_i \geq 0\}} + v_i^- 1_{\{x_i < 0\}}) + \frac{x_{n+1} - \sigma \sum_{i=1}^{n} |x_i|}{2\sigma} (v_1^+ + v_1^-),
\]
where in the second line we used that \( v_i^+ + v_i^- = 2\sigma e_{n+1} \). Since \( x \in C \), we have \( x_{n+1} - \sigma \sum_{i=1}^{n} |x_i| \geq 0 \). Hence, the coefficients in the above representation are non-negative and we conclude that \( x \in \text{pos}(v_i^+, v_i^- : i = 1, \ldots, n) = C_n^\Phi(\sigma^2) \), thus proving that \( C \subseteq C_n^\Phi(\sigma^2) \).

The converse inclusion \( C_n^\Phi(\sigma^2) \subseteq C \) is obvious, since every \( v_i^+ \) and every \( v_i^- \) is in \( C \).

Now we proceed to the proof of (5). This time consider the cone

\[
C := \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \sigma \max_{1 \leq i \leq n} |x_i| \right\}.
\]

Take some \( x \in C \). To prove that \( x \in C_n^\Phi(\sigma^2) \), we shall show that \( x \) is contained in the positive hull of the vectors \( \sigma e_{n+1} + \varepsilon \), where \( \varepsilon \in \{-1, 1\}^n \subset \mathbb{R}^n \). This is evident if \( x_{n+1} = 0 \) (since then \( x = 0 \)). Therefore, let \( x_{n+1} > 0 \). Then, we can write \( x = \sigma^{-1} x_{n+1}(y_1, \ldots, y_n, y_{n+1}) \), where \( y_{n+1} = \sigma \) and \((y_1, \ldots, y_n) = \sigma x_{n+1}^{-1}(x_1, \ldots, x_n) \in [-1, 1]^n \). Any point \((y_1, \ldots, y_n)\) in the cube \( P := [-1, 1]^n \) can be represented as a convex combination of the vertices of the cube, which form the set \( \{-1, 1\}^n \). It follows that the point \( y = (y_1, \ldots, y_n, \sigma) \), which belongs to the shifted cube \( P + \sigma e_{n+1} \subset \mathbb{R}^{n+1} \), can be represented as a convex combination of the points of the form \( \sigma e_{n+1} + \varepsilon \), where \( \varepsilon \in \{-1, 1\}^n \). Hence, \( x \) can be represented as a positive combination of the same points, thus proving that \( C \subseteq C_n^\Phi(\sigma^2) \). The converse inclusion is evident since \( \sigma e_{n+1} + \varepsilon \in C \) for every \( \varepsilon \in \{-1, 1\}^n \).

**Proof of Proposition \[2.2\]** Let \( F \) be a face of \( C_n^\Phi(\sigma^2) \) of dimension \( n \). Note that \( C_n^\Phi(\sigma^2) \) has dimension \( n + 1 \). There is an index \( i \in \{1, \ldots, n\} \) and a sign \( \tau \in \{-1, 1\} \) such that
\[
F = \text{pos} \left\{ \sigma e_{n+1} + \tau e_i + \sum_{1 \leq j \leq n, j \neq i} \varepsilon_j e_j : \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \right\}.
\]
(22)

The linear space spanned by \( F \) is
\[
\text{lin} F = \left\{ \sigma e_{n+1} + \tau e_i + \sum_{1 \leq j \leq n, j \neq i} \alpha_j e_j : \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.
\]
The orthogonal complement of the vector space generated by \( F \) is therefore
\[
(\text{lin} F)^\perp = \text{lin}(-\frac{1}{\sigma} e_{n+1} + \tau e_i).
\]

Moreover, \( v := -\frac{1}{\sigma} e_{n+1} + \tau e_i \in F^\perp \) is a vector in this space that has non-negative scalar products to all vectors in \( C_n^\Phi(\sigma^2) \). Since the polar cone \( (C_n^\Phi(\sigma^2))^\circ \) is spanned by the
orthogonal complements of the faces of dimension $n$, it is given by
\[
\left( C_n^\Box (\sigma^2) \right)^\circ = \text{pos} \left( -\frac{1}{\sigma} e_{n+1} \pm e_i : i \in \{1, \ldots, n\} \right) = - C_n^\Box \left( \frac{1}{\sigma^2} \right).
\]

The second claim of the proposition follows from the first claim (with $\sigma$ replaced by $1/\sigma$) together with $C^\circ \circ = C$. \hfill \Box

**Proof of Proposition 2.3.** Fix $l \in \{1, \ldots, n\}$. We will determine the tangent cone and the normal cone of an $(l-1)$-dimensional face $F$ of the $n$-dimensional crosspolytope $P_n \subset \mathbb{R}^n$. Without loss of generality let $F$ be the face given by
\[
F = \left\{ (f_1, \ldots, f_n) \in \mathbb{R}^n : f_1, \ldots, f_l \geq 0, f_{l+1} = \ldots = f_n = 0, \sum_{i=1}^l f_i = 1 \right\}.
\]

Fix any point $f \in \text{relint} F$ meaning that $f_1, \ldots, f_l > 0$. The tangent cone $T_F := T_F(P_n)$ is the set of all $v \in \mathbb{R}^n$ satisfying $f + \varepsilon v \in P_n$ for an $\varepsilon > 0$. Since the crosspolytope is the unit ball of the 1-norm on $\mathbb{R}^n$, $P_n$ can be characterized via
\[
P_n = \left\{ u = (u_1, \ldots, u_n) \in \mathbb{R}^n : \sum_{j=1}^n |u_j| \leq 1 \right\}.
\]

Hence the tangent cone is given by
\[
T_F = \left\{ v \in \mathbb{R}^n : \text{There is } \varepsilon > 0 : f + \varepsilon v \in P_n \right\} = \left\{ \sum_{i=1}^l v_i + \sum_{i=l+1}^n |v_i| \leq 0 \right\} = \left\{ - \sum_{i=1}^l v_i \geq \sum_{i=l+1}^n |v_i| \right\}.
\]

The lineality space of the tangent cone is
\[
\text{Lineal}(T_F) = T_F \cap (-T_F) = \left\{ \sum_{j=1}^l v_j = 0 \right\} \cap \left\{ v_{l+1} = \ldots = v_n = 0 \right\}.
\]

The polytope’s internal solid angle is the angle of the cone
\[
D_{n,l} := T_F \cap (\text{Lineal}(T_F))^\perp = \left\{ v_1 = \ldots = v_l \right\} \cap \left\{ -lv_l \geq \sum_{j=l+1}^n |v_j| \right\}.
\]

Lemma 5.1 below states that $D_{n,l}$ is isometric to $C_n^\Box (1/l)$. Together with Corollary 2.1 this proves that the internal solid angle has the form (10) with $k = l - 1$.

The normal cone $N_F = N_F(P_n)$ is defined to be the polar cone of $T_F$, hence
\[
N_F \cong \left( D_{n,l} + \text{Lineal}(T_F) \right)^\circ = D_{n,l}^\circ \cap (\text{Lineal}(T_F))^\perp.
\]
where \( \cong \) denotes isometry of cones. In other words, \( N_F \) is the polar cone of \( D_{n,l} \) with respect to the ambient space \( \langle \text{Lineal}(T_F) \rangle \). Recalling that \( D_{n,l} \cong C_{n-l}^\Phi (1/l) \), \( C_{n-l}^\Phi (1/l) \subset \mathbb{R}^{n-l+1} \) and \( \dim \langle \text{Lineal}(T_F) \rangle = n-l+1 \), we have

\[
N_F \cong \left( C_{n-l}^\Phi \left( \frac{1}{l} \right) \right)^\circ \cong C_{n-l}(l),
\]

where we used Proposition 2.2 in the last step. Again, Corollary 2.1 gives (11) with \( k = l-1 \). □

To complete the proof of Proposition 2.3, it remains to establish the following

**Lemma 5.1.** For \( 1 \leq k \leq n \) the cone

\[
D_{n,k} = \left\{ (v_1, \ldots, v_n) \in \mathbb{R}^n : v_1 = \ldots = v_k, -kv_1 \geq \sum_{j=k+1}^{n} |v_j| \right\}
\]

is isometric to \( C_{n-k}^\Phi \left( \frac{1}{k} \right) \).

**Proof.** We will show that

\[
D_{n,k} = \text{pos} \{ u_1^+, \ldots, u_{n-k}^+, u_1^-, \ldots, u_{n-k}^- \},
\]

where for \( i \in \{1, \ldots, n-k\} \)

\[
u_i^+ := -\sum_{j=1}^{k} e_j + ke_{k+i}, \quad u_i^- := -\sum_{j=1}^{k} e_j - ke_{k+i}.
\]

Obviously every \( u_i^+ \) and every \( u_i^- \) is an element of \( D_{n,k} \). Hence it is sufficient to show that \( D_{n,k} \subseteq \text{pos} \{ u_1^+, \ldots, u_{n-k}^+, u_1^-, \ldots, u_{n-k}^- \} \).

Fix any \( v = (v_1, \ldots, v_n) \in D_{n,k} \). By definition

\[
v_1 = \ldots = v_k,
\]

\[
-kv_1 \geq \sum_{j=k+1}^{n} |v_j|.
\]

Let \( \text{sgn}(r) = 1_{\{r \geq 0\}} - 1_{\{r < 0\}} \) be the sign function. The vector

\[
x := \sum_{j=k+1}^{n} \frac{|v_j|}{k} u_j^{\text{sgn} v_j} \in \text{pos} \{ u_1^+, \ldots, u_{n-k}^+, u_1^-, \ldots, u_{n-k}^- \}
\]

equals \( v \) in the last \( n-k \) components and the components of \( x \) satisfy \( x_1 = \ldots = x_k \) and \( -kx_1 = \sum_{j=k+1}^{n} |x_j| \). By (25), we have \( v_1 \leq x_1 \) and thus with the factor

\[
\lambda := x_1 - v_1 = -\frac{1}{k} \sum_{j=k+1}^{n} |v_j| - v_1 \geq 0
\]
we have
\[ v = x - \lambda \sum_{j=1}^{k} e_j = x + \lambda \frac{u_i^+ + u_i^-}{2} \in \text{pos}\{u_i^+, \ldots, u_{n-k}^+, u_i^-, \ldots, u_{n-k}^-\}, \]
thus proving (23). Hence,
\[ D_{n,k} = \text{pos}\left\{ \frac{u_i^+}{k}, \ldots, \frac{u_{n-k}^+}{k}, \frac{u_i^-}{k}, \ldots, \frac{u_{n-k}^-}{k} \right\}. \]
The spanning vectors satisfy
\[ \left\langle \frac{u_i^+}{k}, \frac{u_j^+}{k} \right\rangle = \left\langle \frac{u_i^-}{k}, \frac{u_j^-}{k} \right\rangle = \frac{1}{k} + \delta_{i,j}, \quad \left\langle \frac{u_i^+}{k}, \frac{u_j^-}{k} \right\rangle = \frac{1}{k} - \delta_{i,j} \]
and thus by (7) the lemma is proven. \( \square \)

To prove Theorem 2.4 we will need the following two lemmas.

**Lemma 5.2.** Fix \( k \in \{1, \ldots, n\} \). Let \( F \) be a \( k \)-face of the cone \( C_n^{\Delta}(\sigma^2) \). Then the normal cone \( N_F(C_n^{\Delta}(\sigma^2)) \) of \( C_n^{\Delta}(\sigma^2) \) at the face \( F \) is isometric to \( C_{n-k+1}^{\Delta}\left(\frac{1}{\sigma}\right) \).

**Proof.** Since \( \sum_{i=k}^{n} e_i \) is a point in the relative interior of a \((k - 1)\)-dimensional face of the \( n \)-dimensional cube \([-1, 1]^n\), it follows that
\[ f := \sigma e_{n+1} + \sum_{i=k}^{n} e_i \]
is a point in the relative interior of a \( k \)-dimensional face \( F \) of \( C_n^{\Delta}(\sigma^2) \). By the symmetry of the cone it is sufficient to show that the normal cone of \( C_n^{\Delta}(\sigma^2) \) at this face \( F \) is isometric to \( C_{n-k+1}^{\Delta}\left(\frac{1}{\sigma}\right) \).

First we describe the tangent cone of \( C_n^{\Delta}(\sigma^2) \) at \( F \). Note that for any \( v \in \mathbb{R}^{n+1} \) and any \( \delta > 0 \),
\[ f + \delta v = (\sigma + v_{n+1} \delta) e_{n+1} + \sum_{i=k}^{n} (1 + v_i \delta) e_i + \sum_{i=1}^{k-1} v_i \delta e_i. \]
Using this identity and the fact that by (5) the cone \( C_n^{\Delta}(\sigma^2) \) can be written as
\[ C_n^{\Delta}(\sigma^2) = \left\{ \sum_{i=1}^{n+1} \beta_i e_i : \max_{1 \leq i \leq n} |\beta_i| \leq \frac{\beta_{n+1}}{\sigma} \right\}, \]
we have that for every \( v \in \mathbb{R}^{n+1} \),
\[ f + \delta v \in C_n^{\Delta}(\sigma^2) \text{ for sufficiently small } \delta > 0 \iff \max_{i \in \{k, \ldots, n\}} v_i \leq \frac{v_{n+1}}{\sigma}. \]
Hence the tangent cone of $C_n^\sigma(\sigma^2)$ at $F$ is
\[ T_F(C_n^\sigma(\sigma^2)) := \{ v \in \mathbb{R}^{n+1} : f + \delta v \in C_n^\sigma(\sigma^2) \text{ for some } \delta > 0 \} \]
\[ = \left\{ v \in \mathbb{R}^{n+1} : \max_{i \in \{k, \ldots, n\}} v_i \leq \frac{v_{n+1}}{\sigma} \right\}. \]

Using that
\[ -T_F(C_n^\sigma(\sigma^2)) = \left\{ v \in \mathbb{R}^{n+1} : \min_{i \in \{k, \ldots, n\}} v_i \geq \frac{v_{n+1}}{\sigma} \right\} \]
we conclude that the lineality space of the tangent cone is given by
\[ \text{Lineal}(T_F(C_n^\sigma(\sigma^2))) = T_F(C_n^\sigma(\sigma^2)) \cap \left( -T_F(C_n^\sigma(\sigma^2)) \right) \]
\[ = \left\{ v \in \mathbb{R}^{n+1} : v_k = \ldots = v_n = \frac{v_{n+1}}{\sigma} \right\}. \]

The tangent cone $T_F(C_n^\sigma(\sigma^2))$ is the Minkowski sum of its lineality space and the cone $D$ given by
\[ D = \left\{ v \in \mathbb{R}^{n+1} : \max_{i \in \{k, \ldots, n\}} v_i \leq 0, v_{n+1} = 0 \right\}. \]

Since the smallest linear space containing $T_F(C_n^\sigma(\sigma^2))$ is the whole $\mathbb{R}^{n+1}$, the normal tangent cone is the polar cone of $T_F(C_n^\sigma(\sigma^2))$. It is
\[ \left( T_F(C_n^\sigma(\sigma^2)) \right)^\circ = \left( \text{Lineal}(T_F(C_n^\sigma(\sigma^2))) \right) \perp \cap D^\circ \]
\[ = \left\{ v \in \mathbb{R}^{n+1} : \sigma v_{n+1} = -\sum_{i=k}^{n} v_i, v_1 = \ldots = v_{k-1} = 0 \right\} \]
\[ \cap \left\{ v \in \mathbb{R}^{n+1} : \min_{i \in \{k, \ldots, n\}} v_i \geq 0, v_1 = \ldots = v_{k-1} = 0 \right\} \]
\[ = \left\{ v \in \mathbb{R}^{n+1} : v_1 = \ldots = v_{k-1} = 0, v_k, \ldots, v_n \geq 0, \sigma v_{n+1} = -\sum_{i=k}^{n} v_i \right\} \]
\[ = \text{pos} \left\{ \frac{1}{\sigma} e_{n+1} + e_j : j = k, \ldots, n \right\} \]
\[ \cong C_n^{\Delta} \left( \frac{1}{\sigma^2} \right). \]

The isometry in the last step follows from (6). \qed

**Lemma 5.3.** For $1 \leq k \leq n+1$ and $\sigma^2 > 0$ any $k$-dimensional face of the cone $C_n^\sigma(\sigma^2)$ is isometric to $C_{k-1}^\sigma(\sigma^2 + n - k + 1)$.\[ \square \]
Proof. Fix \( n \in \mathbb{N}, \sigma^2 > 0 \) and \( k \in \{1, \ldots, n + 1\} \). Since all the \( k \)-faces of \( C_n^{\mathbb{R}}(\sigma^2) \) are isometric, it is sufficient to analyse

\[
F := \text{pos} \left\{ \sigma e_{n+1} + \sum_{i=1}^{k-1} \varepsilon_i e_i + \sum_{i=k}^{n} e_i : \varepsilon_i \in \{-1, 1\} \right\}
\]

Denoting the vectors spanning \( F \) by \( v_\varepsilon := \sigma e_{n+1} + \sum_{i=1}^{k-1} \varepsilon_i e_i + \sum_{i=k}^{n} e_i, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{k-1}) \in \{-1, 1\}^{k-1} \), we have the scalar products

\[
\langle v_\varepsilon, v_\eta \rangle = \sigma^2 + \langle \varepsilon, \eta \rangle + n - k + 1
\]

for every \( \varepsilon, \eta \in \{-1, 1\}^{k-1} \). Since these coincide with the ones of the vectors spanning \( C_{k-1}^{\mathbb{R}}(\sigma^2 + n - k + 1) \), see (8), the claimed isometry holds. \( \square \)

Proof of Theorem 2.4. First let \( k \in \{1, \ldots, n\} \). Let \( F \) be a \( k \)-face of \( C_n^{\mathbb{R}}(\sigma^2) \). By Lemmas 5.2 and 5.3 we have

\[
\alpha(N_F(C_n^{\mathbb{R}}(\sigma^2))) = g_{n-k+1} \left( \frac{1}{\sigma^2} \right),
\]

\[
\alpha(F) = g_{k-1} \left( \sigma^2 + n - k + 1 \right).
\]

By definition, the \( k \)th intrinsic volume of \( C_n^{\mathbb{R}}(\sigma^2) \) is

\[
\nu_k(C_n^{\mathbb{R}}(\sigma^2)) = \sum_{F \in \mathcal{F}_k(C_n^{\mathbb{R}}(\sigma^2))} \alpha(F) \alpha(N_F(C_n^{\mathbb{R}}(\sigma^2))).
\]

All the \( k \)-faces of \( C_n^{\mathbb{R}}(\sigma^2) \) are isometric and by the construction of the cone there is a natural one-to-one-correspondence between the \( k \)-faces of \( C_n^{\mathbb{R}}(\sigma^2) \) and the \( (k-1) \)-faces of the \( n \)-dimensional cube \([ -1, 1 ]^n \). Hence the cone \( C_n^{\mathbb{R}}(\sigma^2) \) has \( 2^{n-k+1} \binom{n}{k-1} \) \( k \)-faces and its intrinsic volume is

\[
\nu_k(C_n^{\mathbb{R}}(\sigma^2)) = 2^{n-k+1} \binom{n}{k-1} \ g_{k-1} \left( \sigma^2 + n - k + 1 \right) g_{n-k+1} \left( \frac{1}{\sigma^2} \right).
\]

Now we are coming to the remaining cases. Since \( \nu_{n+1}(C_n^{\mathbb{R}}(\sigma^2)) \) is just the solid angle of \( C_n^{\mathbb{R}}(\sigma^2) \), we have

\[
\nu_{n+1}(C_n^{\mathbb{R}}(\sigma^2)) = g_{n}^{\mathbb{R}}(\sigma^2).
\]

Recalling that \( g_0^{\mathbb{R}}(\cdot) \equiv 1 \), this gives that (12) also holds for \( k = n + 1 \). In the case \( k = 0 \) we observe that the only 0-dimensional face of \( C_n^{\mathbb{R}}(\sigma^2) \) is \( \{0\} \) and hence, by Proposition 2.2

\[
\nu_0(C_n^{\mathbb{R}}(\sigma^2)) = \alpha(\{0\}) = g_{n}^{\mathbb{R}} \left( \frac{1}{\sigma^2} \right).
\]
The conic intrinsic volumes of $C_n^{\sigma^2}$ can be obtained by polarity. It is known, see [3, Fact 5.5(2)], that for every $m$-dimensional cone $C$ the $j$th intrinsic volume of its polar cone $C^\circ$ is
\[ v_j(C^\circ) = v_{m-j}(C), \quad j = 0, \ldots, m. \] (26)

Since by Theorem 2.2 we have that $C_n^{\sigma^2} \sim (C_n^{1/\sigma^2})^\circ$, it follows that
\[ v_k(C_n^{\sigma^2}) = v_{n+1-k}(C_n^{1/\sigma^2}) = 2^k \binom{n}{k} g_{n-k}(1/\sigma^2 + k) g_k(\sigma^2), \]
for all $0 \leq k \leq n$. In the remaining case $k = n + 1$ we have
\[ v_{n+1}(C_n^{\sigma^2}) = v_0(C_n^{1/\sigma^2}) = g_n(\sigma^2), \]
which completes the proof. □

5.2. Absorption probabilities - Proofs. In this section we will express the non-absorption probabilities as the probability that certain deterministic cone $C$ and a random linear space $L$ intersect trivially. The latter probability can be computed by means of the conic Crofton formula which is an important tool in this and the next section.

**Theorem 5.4** (Conic Crofton formula). Let $C \subset \mathbb{R}^n$ be a convex cone which is not a linear subspace and let $L$ be a random linear subspace that is chosen uniformly from the Grassmannian $\text{Gr}(n-l, \mathbb{R}^n)$, i.e. the set of all linear subspaces of $\mathbb{R}^n$ that have codimension $l \in \{0, \ldots, n\}$. Then
\[ \mathbb{P}(C \cap L = \{0\}) = 2(v_{l-1}(C) + v_{l-3}(C) + \ldots), \]
\[ \mathbb{P}(C \cap L \neq \{0\}) = 2(v_{l+1}(C) + v_{l+3}(C) + \ldots). \]
The probabilities $\mathbb{P}(C \cap L \neq \{0\})$ are known as the Grassmann angle $\gamma_l(C)$.

**Proof of Theorem 3.2.** By the symmetry of the standard normal distribution we have
\[ p_{n,d}(\sigma^2) = \mathbb{P}(\sigma X \notin P_n^{\sigma^2}) = \mathbb{P}(-\sigma X \notin P_n^{\sigma^2}) = \mathbb{P}(0 \notin \text{conv}\{\pm X_1 + \sigma X, \ldots, \pm X_n + \sigma X\}). \]
By the definition of convex hulls, the event $0 \notin \text{conv}\{\pm X_1 + \sigma X, \ldots, \pm X_n + \sigma X\}$ occurs if and only if
\[ 0 = \sum_{i=1}^{n} \alpha_i X_i + \alpha_{n+1} X \text{ for } \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \mathbb{R}, \text{ and } \alpha_{n+1} \geq \sigma \sum_{i=1}^{n} |\alpha_i| \]
implies that $\alpha_1 = \ldots = \alpha_n = \alpha_{n+1} = 0$. Taking everything together, we arrive at the identity
\[ p_{n,d}(\sigma^2) = \mathbb{P}(C \cap U = \{0\}) \] (27)
with
\[
U := \left\{ \left( y_1, \ldots, y_{n+1} \right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} y_i X_i + y_{n+1} X = 0 \right\},
\]
(28)
\[
C := \left\{ \left( \alpha_1, \ldots, \alpha_{n+1} \right) \in \mathbb{R}^{n+1} : \alpha_{n+1} \geq \sigma \sum_{i=1}^{n} |\alpha_i| \right\}.
\]
(29)

Note that \( U \) is a random linear subspace of \( \mathbb{R}^{n+1} \) that has codimension \( d \) a.s. Moreover, from the rotational symmetry of the standard normal distribution it follows that \( U \) is uniformly distributed on the corresponding linear Grassmannian. By (4), we have \( C \sim C_n^\Phi(\sigma^2) \) and hence by the conic Crofton formula (Theorem 5.4) the probability (27) takes the form
\[
 p_n,d(\sigma^2) = 2\left( \upsilon_d - 2 \left( C_n^\Phi(\sigma^2) \right) + \ldots \right)
\]
(30)
Plugging in the expressions from Theorem 2.4 completes the proof. \( \square \)

**Proof of Theorem 3.3.** As in the proof of Theorem 3.2 we have
\[
P_{n,d}(\sigma^2) = \mathbb{P}(\sigma X \notin F_{n,d}) = \mathbb{P}(0 \notin P_{n,d} + \sigma X).
\]
Noting that
\[
P_{n,d} + \sigma X = \left\{ \sum_{i=1}^{n} \tilde{\alpha}_i X_i + \sigma X : \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in [-1, 1] \right\}
\]
we can rewrite the event \( \{0 \notin P_{n,d} + \sigma X\} \) in the following way.
\[
0 \notin P_{n,d} + \sigma X \iff \text{for all } \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in [-1, 1] : \sum_{i=1}^{n} \tilde{\alpha}_i X_i + \sigma X \neq 0
\]
\[
\iff \text{for all } \lambda > 0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in [-1, 1] : \sum_{i=1}^{n} \lambda \tilde{\alpha}_i X_i + \lambda \sigma X \neq 0.
\]
Denoting \( \alpha_i := \lambda \tilde{\alpha}_i \) and \( \alpha_{n+1} := \lambda \sigma \) and including the case \( \alpha_{n+1} = 0 \) this event is equivalent to the statement
\[
\left( \sum_{i=1}^{n} \alpha_i X_i + \alpha_{n+1} X = 0 \text{ with } \alpha_{n+1} \geq 0 \text{ and } \max_{i=1,\ldots,n} |\alpha_i| \leq \frac{\alpha_{n+1}}{\sigma} \right) \Rightarrow \alpha_1 = \ldots = \alpha_{n+1} = 0.
\]
Thus \( P_{n,d}(\sigma^2) \) has the form
\[
P_{n,d}(\sigma^2) = \mathbb{P}(C \cap U = \{0\})
\]
(30)
with
\[
U := \left\{ \left( y_1, \ldots, y_{n+1} \right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} y_i X_i + y_{n+1} X = 0 \right\},
\]
(31)
\[
C := \left\{ \left( \alpha_1, \ldots, \alpha_{n+1} \right) \in \mathbb{R}^{n+1} : \alpha_{n+1} \geq \sigma \max_{i=1,\ldots,n} |\alpha_i| \right\}.
\]
(32)
Note that $U$ is the same as in the proof of Theorem 3.2. Thus, $U$ is a random linear subspace of $\mathbb{R}^{n+1}$ that has codimension $d$ and is uniformly distributed on the corresponding linear Grassmannian. Also, $C \cong C_{n}^{\mathbb{R}}(\sigma^2)$ by [5] and hence by the conic Crofton formula this probability equals 

$$P_{n,d}^{\mathbb{R}}(\sigma^2) = 2(v_{d-1}(C_{n}^{\mathbb{R}}(\sigma^2)) + v_{d-3}(C_{n}^{\mathbb{R}}(\sigma^2)) + \ldots).$$

Plugging in the expressions from Theorem 2.4 completes the proof. □

5.3. Proof of Proposition 4.1. The main concept we need in the proof of (19) is general position of a linear subspace with respect to a finite set of affine subspaces. This notion is defined in the following way.

Let $S_1, \ldots, S_k \subset \mathbb{R}^n$ be affine subspaces. An affine subspace $S \subset \mathbb{R}^n$ having codimension $\text{codim}(S) = l$ is said to be in general position with respect to $S_1, \ldots, S_k$ if for every choice of indices $I \subseteq \{1, \ldots, k\}$ the intersection $S \cap (\cap_{i \in I} S_i)$ either is empty if $\dim(\cap_{i \in I} S_i) < l$ or is an affine subspace of dimension $\dim(\cap_{i \in I} S_i) - l$ otherwise.

The next proposition implies that for any finite collection of affine subspaces $S_1, \ldots, S_k$ that are not linear subspaces, with probability 1 a random linear subspace $L$ which is chosen uniformly from the Grassmannian of all linear subspaces with codimension $l$ is in general position with respect to $S_1, \ldots, S_k$. It is sufficient to prove this fact for a single affine subspace $A$ because every intersection $\cap_{i \in I} S_i$ is an affine subspace.

Proposition 5.5. Let $L$ be a random linear subspace of $\mathbb{R}^n$ chosen uniformly from the set of all $(n-l)$-dimensional linear subspaces of $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be an affine linear subspace of $\mathbb{R}^n$ that is not a linear subspace. Then $L$ almost surely is in general position with respect to $A$.

Proof. The way $L$ is constructed it almost surely is in general position with respect to any fixed deterministic linear subspace $M$. Proofs for this statement are given by Goodey and Schneider in [8, Lemma 2.1] and by Schneider and Weil in [15, Lemma 13.2.1].

Thus $L$ almost surely is in general position with respect to $\text{lin}(A)$, the linear hull of $A$, i.e. almost surely the dimension of $\text{lin}(A) \cap L$ is either 0, if $\dim(\text{lin}(A)) < l$ or it is $\dim(\text{lin}(A)) - l$ otherwise. Note that this definition of general position of linear subspaces slightly differs from the one of general position of a linear subspace with respect to an affine subspace.

Unfortunately, general position of $L$ with respect to $\text{lin}(A)$ is not equivalent to general position of $L$ with respect to $A$, as one can see by analysing two parallel lines in $\mathbb{R}^3$ with one of them containing 0 and thus being a linear subspace. But to prove the proposition it is sufficient to show only one of the implications: We will prove that the general position of $L$ with respect to $\text{lin}(A)$ implies the general position of $L$ with respect to $A$.

First assume that $L$ is in general position with respect to $\text{lin}(A)$ and $\dim(A) < \text{codim}(L) = l$. Since $A$ is not a linear subspace, $0 \notin A$ and thus $\dim(\text{lin}(A)) = \dim(A) + 1$. Hence we have $\dim(\text{lin}(A)) \leq \text{codim}(L)$ and by definition we also have $\dim(\text{lin}(A) \cap L) = 0$ and thus $\text{lin}(A) \cap L = \{0\}$. Thereby we obtain $A \cap L \subset \text{lin}(A) \cap L = \{0\}$. Since $A$ is an affine subspace, $0 \notin A$ and thus $0 \notin A \cap L$, we arrive at $A \cap L = \emptyset$. By definition this means that in this case $L$ is in general position with respect to $A$. 
Now assume that $\dim(A) \geq \text{codim}(L) = l$, while $L$ is in general position with respect to $\text{lin}(A)$. Since $L$ is in general position with respect to $\text{lin}(A)$, the dimension of their intersection is $\dim(\text{lin}(A) \cap L) = \dim(\text{lin}(A)) - \text{codim}(L)$ and thus

$$
\dim(A \cap L) = \dim(\text{lin}(A \cap L)) - 1 = \dim(\text{lin}(A) \cap L) - 1 = \dim(\text{lin}(A)) - \text{codim}(L) - 1 = \dim(A) - \text{codim}(L),
$$

which gives general position. In the first step we used that $A \cap L$ is an affine subspace that is not a linear subspace. The identity $\text{lin}(A) \cap L = \text{lin}(A \cap L)$ that we used in the second step, can be proved the following way.

Since $A$ is an affine subspace that is not a linear subspace, we can write $\text{lin}(A) = \bigcup \{A \lambda : \lambda \in \mathbb{R}\}$. Thus for any $x \in \text{lin}(A) \cap L$ with $x \neq 0$ there is an $a \in A$ and a real $\lambda \neq 0$ such that $x = \lambda a$. Hence we have $\frac{x}{\lambda} \in A$ and since $L$ is a linear subspace $\frac{x}{\lambda} \in L$. As a result we have $\frac{x}{\lambda} \in A \cap L$ and thus $x \in \text{lin}(A \cap L)$. In the formally excluded case $x = 0$ we trivially have $x \in \text{lin}(A \cap L)$. The other inclusion holds trivially: $\text{lin}(A \cap L)$ is the smallest linear subspace containing $A \cap L$ and since $A \cap L \subseteq \text{lin}(A) \cap L$ we conclude that $\text{lin}(A \cap L) \subseteq \text{lin}(A) \cap L$. \qed

To prove (19) we will show the following three statements.

**Proposition 5.6.** Let $P \subseteq \mathbb{R}^n$ be an $n$-dimensional polytope that contains the origin in its interior, and let $S \subseteq \mathbb{R}^n$ be a deterministic linear subspace having codimension $l \in \{1, \ldots, n-1\}$ which is in general position with respect to $\{\text{aff}(F) : F \in \mathcal{F}(P)\}$. Here, $\mathcal{F}(P)$ is the set of all faces of $P$. Then the following three statements hold for every $j \in \{0, \ldots, n-l-1\}$.

1. Let $B_1, B_2 \in \mathcal{F}_{j+l}(P)$ be two $(j+l)$-faces of $P$ with $B_1 \cap S \neq \emptyset$ and $B_1 \neq B_2$. Then $B_1 \cap S \neq B_2 \cap S$.

2. Let $B \in \mathcal{F}_{j+l}(P)$ be a $(j+l)$-face of $P$ with $B \cap S \neq \emptyset$. Then the intersection of $B$ and $S$ is a $j$-face of $S \cap P$, i.e.

$$
B \cap S \in \mathcal{F}_j(S \cap P).
$$

3. Let $A \in \mathcal{F}_j(S \cap P)$ be a $j$-face of the intersection of $S$ and $P$. Then there is a $(j+l)$-face $B \in \mathcal{F}_{j+l}(P)$ of $P$ such that $B \cap S = A$. \hfill (33)

Proof of Proposition 4.1 assuming Proposition 5.6. In our setting of a random linear subspace $S = L$ and a deterministic polytope $P$, the subspace $L$ almost surely is in general position with respect to $\mathcal{F}(P)$ by Proposition 5.5. It follows that statements (1) to (3) of Proposition 5.6 hold almost surely.

Statement (3) gives a map that sends $A \in \mathcal{F}_j(L \cap P)$ to the face $B \in \mathcal{F}_{j+l}(P)$ satisfying (33). By statement (1) this $B$ is unique and thus the map is well-defined. Since by (33) the map is injective, we have

$$
\phi^P(j, n-l, n) = \#\mathcal{F}_j(L \cap P) \leq \#\{B \in \mathcal{F}_{j+l}(P) : B \cap L \neq \emptyset\} = \sum_{B \in \mathcal{F}_{j+l}(P)} 1_{\{B \cap L \neq \emptyset\}}.
$$
By statement \( \{2\} \) the image of the map \( B \mapsto B \cap L \) from \( \{ B \in \mathcal{F}_{j+l}(P) : B \cap L \neq \emptyset \} \) to \( \mathcal{F}_j(L \cap P) \) is a subset of \( \mathcal{F}_j(L \cap P) \), and by statement \( \{1\} \) this map in injective. Thus
\[
\phi^p(j, n - l, n) = \#\mathcal{F}_j(L \cap P) \geq \#\{ B \in \mathcal{F}_{j+l}(P) : B \cap L \neq \emptyset \} = \sum_{B \in \mathcal{F}_{j+l}(P)} 1_{\{ B \cap L \neq \emptyset \}}.
\]
So, Proposition 4.1 is a consequence of Proposition 5.6. □

The following lemma is an important step in the proof of Proposition 5.6. Its proof is inspired by and similar to the proof of \([11]\) Lemma 3.5.

**Lemma 5.7.** Let \( Q \subset \mathbb{R}^n \) be a polytope (or, more generally, an intersection of finitely many half-spaces which is allowed to be unbounded) of full dimension \( \dim Q = n \). Let the linear subspace \( S \subset \mathbb{R}^n \) of codimension \( l \in \{1, \ldots, n - 1\} \) be in general position with respect to the set of the affine hulls of its faces \( \{ \text{aff}(F) : F \in \mathcal{F}(Q) \} \). If \( S \) intersects \( Q \), then it also intersects its interior \( \text{int}(Q) \), i.e.
\[
S \cap Q \neq \emptyset \Rightarrow S \cap \text{int}(Q) \neq \emptyset.
\]

**Proof.** Assume that \( S \cap Q \neq \emptyset \) but \( S \cap \text{int}(Q) = \emptyset \). It is known that the polytope \( Q \) is the disjoint union of the relative interiors of all its faces. Because of \( S \cap Q \neq \emptyset \) there is a face \( F \) of \( Q \) with \( \text{relint}(F) \cap S \neq \emptyset \) and by \( S \cap \text{int}(Q) = \emptyset \) we have \( F \neq Q \).

Without loss of generality let the origin \( 0 \in \text{relint}(F) \cap S \). Then the affine hull of \( F \) is its linear hull \( \text{aff}(F) = \text{lin}(F) \). By assumption of the lemma, \( S \) is in general position with respect to \( \text{lin}(F) \). Thus \( S \cap \text{lin}(F) \) is the empty set if \( \dim(\text{lin}(F)) < l \) or it is a linear subspace of dimension \( \dim(F) - l \) else. Since by assumption \( \text{relint}(F) \cap S \neq \emptyset \), the first case is impossible, thus we must have \( \dim(F) \geq l \) and \( \dim(\text{lin}(F) \cap S) = \dim(F) - l \). Since \( S \) and \( \text{lin}(F) \) are linear subspaces, this even implies
\[
\dim(S + \text{lin}(F)) = \dim(S) + \dim(\text{lin}(F)) - \dim(S \cap \text{lin}(F)) = n - l + \dim(F) - (\dim(F) - l) = n,
\]
and thus \( S + \text{lin}(F) = \mathbb{R}^n \). Denoting \( V_0 = \text{lin}(F) \cap S \) this implies the existence of two linear subspaces \( V_1, V_2 \subset \mathbb{R}^n \) satisfying \( V_0 \perp V_1, V_0 \perp V_2, \text{lin}(F) = V_0 + V_1, S = V_0 + V_2 \) and \( V_0 + V_1 + V_2 = \mathbb{R}^n \). We will show that this implies a contradiction.

Let \( T_F(Q) = \{ y \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ such that } \varepsilon y \in Q \} \) be the tangent cone of \( Q \) at \( F \). Fix any \( z \in \text{int}(T_F(Q)) \). By our result above there is a decomposition \( z = v_0 + v_1 + v_2 \) with \( v_i \in V_i, i = 0, 1, 2 \). We will prove that there is an \( \varepsilon > 0 \) such that \( \varepsilon v_2 \in \text{int}(Q) \), which is a contradiction to \( \varepsilon v_2 \in V_2 \subset S \) and \( S \cap \text{int}(Q) = \emptyset \).

Since \( v_0 + v_1 \in \text{lin}(F) \subset T_F(Q) \), we have \( -(v_0 + v_1) \in \text{lin}(F) \subset T_F(Q) \) and thus
\[
v_2 = z - v_0 - v_1 \in T_F(Q).
\]

Note that \( v_2 \) is the projection of \( z \) onto \( V_2 \) along \( V_0 + V_1 \). The above argument showing that \( v_2 \in T_F(Q) \) applies to every point in a sufficiently small ball around \( z \). The projection of this ball onto \( V_2 \) along \( V_0 + V_1 \) covers some set \( B_r(v_2) \cap V_2 \), where \( B_r(v_2) \) is a ball of radius \( r' > 0 \) around \( v_2 \). Thus \( B_r(v_2) \cap V_2 \subset T_F(Q) \). Since \( V_0 + V_1 = \text{lin}(F) \subset T_F(Q) \), by the convex cone property of \( T_F(Q) \) there is an \( r \in (0, r'] \) such that \( B_r(v_2) \subset T_F(Q) \).
Again by the convex cone property of $T_F(Q)$, for every $\varepsilon > 0$ we have $B_{cr}(\varepsilon v_2) \subset T_F(Q)$. By the definition of the tangent cone $T_F(Q)$, for sufficiently small $\varepsilon$ we have $B_{cr}(\varepsilon v_2) \subset Q$ and thus $\varepsilon v_2 \in \text{int}(Q)$, which is a contradiction as explained above. \hfill \Box

In the proof of statement \((3)\) in Proposition \ref{prop:tancone} we will need the following corollary from the Hyperplane Separation Theorem. To state it, we need the following definition. Let $H$ be an affine hyperplane given by the equation $H = \{x \in \mathbb{R}^n : \langle x, v \rangle = r\}$, $v \in \mathbb{R}^n \setminus \{0\}$, $r \in \mathbb{R}$. Then we define the two half spaces that $H$ divides $\mathbb{R}^n$ into by $H^+ := \{x \in \mathbb{R}^n : \langle x, v \rangle \geq r\}$, $H^- := \{x \in \mathbb{R}^n : \langle x, v \rangle \leq r\}$. Note that these spaces swap positions, if $v$ is replaced by $-v$ and $r$ by $-r$. Thus these spaces are not well-defined, if only $H$, but not the exact form of its defining equation is given. So when we speak of $H^+$ it can be any of the two half spaces, but it will always contain $H$.

**Lemma 5.8.** Let $Q \subset \mathbb{R}^n$ be a convex set with non-empty interior and let $H_0 \subset \mathbb{R}^n$ be an affine subspace with $H_0 \cap Q \neq \emptyset$, but $H_0 \cap \text{int}(Q) = \emptyset$. Then there is an affine hyperplane $H \subset \mathbb{R}^n$ with $H_0 \subset H$ and $Q \subset H^+$.

**Proof.** In the situation of the Lemma $\text{int}(Q)$ and $H_0$ are two disjoint convex sets. By the Hyperplane Separation Theorem, see Theorem 1.3.7 in \cite{13}, there is a hyperplane $H$ such that $H_0 \subset H^-$ and $\text{int}(Q) \subset H^+$. Since $H_0$ is a linear subspace of $\mathbb{R}^n$ which is contained in the half space $H^-$ it must be parallel to $H$. By $H_0 \cap Q \neq \emptyset$ there is a point $x \in Q \cap H_0$ and by the construction of $H$ we have $x \in H$. Thus $H$ and $H_0$ both contain the point $x$ and hence we have $H_0 \subset H$. Recalling that $\text{int}(Q) \subset \text{int}(H^+)$ and taking the closure we conclude that $Q \subset H^+$. \hfill \Box

Now we can prove Proposition \ref{prop:tancone}.

**Proof of Proposition \ref{prop:tancone}** We first prove statement \((1)\)

Let $B_1, B_2, S$ be defined as in the statement. We can think of $B_1$ as a polytope in its affine hull $\text{aff}(B_1)$. In this setting $B_1 \subset \text{aff}(B_1)$ is a polytope of full dimension and $S \cap \text{aff}(B_1)$ is an affine linear subspace of $\text{aff}(B_1)$ which is in general position with respect to $\{\text{aff}(F) : F \in \mathcal{F}(B_1)\}$. Lemma \ref{lem:intersection} gives that

\[(\text{relint } B_1) \cap S = (\text{relint } B_1) \cap (S \cap \text{aff}(B_1)) \neq \emptyset. \tag{34} \]

Since the dimensions of $B_1$ and $B_2$ are equal, $\text{relint}(B_1)$ and $B_2$ are disjoint. By (34) there is an $x \in (\text{relint } B_1) \cap S \subset B_1 \cap S$ and since $x \in \text{relint}(B_1)$, we have $x \notin B_2 \supseteq B_2 \cap S$. Thus we have $B_1 \cap S \neq B_2 \cap S$.

To prove statement \((2)\) fix $B \in \mathcal{F}_{j+l}(P)$. As a first step we prove that $S \cap B$ is a face of $S \cap P$. Since $B$ is a face of $P$, there is an affine hyperplane $H \subset \mathbb{R}^n$ such that $H \cap P = B$ and $P \subset H^+$. Thus we have $B \cap S = (H \cap P) \cap S = H \cap (P \cap S)$ and $P \cap S \subset P \subset H^+$, which implies by definition that $S \cap B$ is a face of $S \cap P$. 
To prove statement (2) we show that the dimension of $S \cap B$ is $j$. First note that since $S$ is in general position with respect to $\text{aff}(B)$, we have $\dim(\text{aff}(B) \cap S) = j$ and thus $\dim(B \cap S) \leq j$.

By the same argument as in the beginning of the proof of statement (1) we have $\text{relint}(B) \cap S \neq \emptyset$. Thus there is an $x \in \text{relint}(B) \cap S$ and since $\text{relint}(B)$ is a relatively open subset of $\text{aff}(B)$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \cap \text{aff}(B) \subset \text{relint}(B)$. Here $B_\epsilon(x)$ denotes the $n$-dimensional ball with radius $\epsilon > 0$ and centre $x$. Thus we have

$$B_\epsilon(x) \cap \text{aff}(B) \cap S \subset \text{relint}(B) \cap S \subset B \cap S.$$ 

Since $S$ is in general position with respect to $\text{aff}(B)$, the intersection $S \cap \text{aff}(B)$ has dimension $j$ and since $B_\epsilon(x)$ is an $n$-dimensional ball centred at $x \in S \cap \text{aff}(B)$, intersecting with it does not change the dimension. Thus we have

$$j = \dim(B_\epsilon(x) \cap \text{aff}(B) \cap S) \leq \dim(B \cap S).$$

To prove statement (3), fix $A \in \mathcal{F}_j(S \cap P)$. We first show that there is a face $B$ of $P$ such that $A = B \cap S$. To do this we will construct an affine hyperplane $H \subset \mathbb{R}^n$ such that $H \cap S \cap P = A$ and $P \subset H^+$.

In this setting $B := H \cap P$ is a face of $P$ by definition and $B$ suffices $B \cap S = A$.

Since $A$ is a face of $S \cap P$, there is a hyperplane $H_0 \subset S$ such that $H_0 \cap (S \cap P) = A$ and $S \cap P \subset H_0^+$. Note that $H_0$ is a hyperplane in $S$, but not in $\mathbb{R}^n$.

By Lemma 5.8 there is a hyperplane $H \subset \mathbb{R}^n$ such that $H_0 \subset H$ and $P \subset H^+$.

If $H \cap S = H_0$, the hyperplane $H$ obviously suffices

$$H \cap P \cap S = (H \cap S) \cap (S \cap P) = H_0 \cap (S \cap P) = A.$$ 

To prove the identity $H \cap S = H_0$, first note that $H_0 = H_0 \cap S \subset H \cap S$. To prove the equality, we assume by contraposition that there exists an $x \in (H \cap S) \setminus H_0$. Then $H \cap S$ is an affine linear subspace of $S$ that contains the hyperplane $H_0$ and the point $x \notin H_0$. Thus $H \cap S = S$ and, since $S$ is a linear subspace, $0 \in S = S \cap H \subset H$. On the other hand, the point 0 belongs to the interior of $P$ by the assumption of the proposition. This is a contradiction to $P \subset H^+$.

It remains to show that the face $B := H \cap P$ has the right dimension $j + l$. Recall that $A \in \mathcal{F}_j(S \cap P)$. By the first part of the proof of statement (3), there is $k \geq j$ and a face $B \in \mathcal{F}_k(P)$ such that $B \cap S = A$.

We first assume that the dimension $k$ is smaller than the codimension $l$ of $S$. Since $S$ is in general position with respect to $\text{aff}(B)$, the intersection $S \cap \text{aff}(B)$ is the empty set, which is a contradiction to $A = B \cap S \subset S \cap \text{aff}(B)$. Thus we have $k \geq l$.

In this case we have $B \in \mathcal{F}_{l+k-l}(P)$ with $k-l \geq 0$. Hence statement (2) immediately gives

$$k - l = \dim(B \cap S) = \dim(A) = j$$

and thus $k = j + l$. This completes the proof of statement (3) and thus of the proposition. □
5.4. Probabilities that fixed faces get intersected - Proofs.

Proof of Proposition 4.2. The subspace $L$ intersects the face $B$ if and only if it intersects the cone $C$ spanned by $B$ not only at the origin. Hence, by the conic Crofton formula (see Theorem 5.4) we have
\[
P(L \cap B \neq \emptyset) = P(L \cap C \neq \emptyset) = 2(v_{l+1}(C) + v_{l+3}(C) + \ldots).
\] (35)
Since $F$ is a face of $[-1, 1]^n$, it is a cube itself and $C$ is a convex cone isometric to $C_d^{\sigma^2}$, where the parameter $\sigma$ is the distance $\text{dist}(\{0\}, B)$ between the origin and the cube $B$. It is easy to see that this distance is $\sqrt{n-d}$ and hence $C$ is isometric to $C_d^{\sigma^2}(n-d)$, which completes the proof. □

Proof of Proposition 4.3. As in the proof of Proposition 4.2 let $C$ be the convex cone that is spanned by $B$. As before $L$ and $B$ are disjoint if and only if the intersection of $L$ and $C$ is only the origin. It follows that (35) holds.

By the symmetry of the problem it is obvious that the probability $P(L \cap B \neq \emptyset)$ depends on the face $B$ only by its dimension. Hence, without loss of generality we can assume that $B$ has the form $B = \text{conv}\{e_1, \ldots, e_{d+1}\}$, which by definition is a regular simplex. In this case $C$ is isometric to $C_{d+1}^{0}$ and thus by Theorem 5.4
\[
P(L \cap B \neq \emptyset) = 2(v_{l+1}(C) + v_{l+3}(C) + \ldots) = 2 \left( v_{l+1}(C_{d+1}^{0}) + v_{l+3}(C_{d+1}^{0}) + \ldots \right).
\]
In [10, Propositions 1.3 and 1.4(d)], it was shown that for $k \in \{0, \ldots, d+1\}$ the $k$-th intrinsic volume of $C_{d+1}^{0}$ is
\[
v_k(C_{d+1}^{0}) = \binom{d+1}{k} 2^{-(d+1)},
\]
which gives
\[
P(L \cap B \neq \emptyset) = 2^{-d} \left( \binom{d+1}{l+1} + \binom{d+1}{l+3} + \ldots \right) = 2^{-d} \left( \binom{d}{l} + \binom{d}{l+1} + \ldots \right).
\]
□

Proof of Proposition 4.4. As is in the proofs of Propositions 4.2 and 4.3, $L$ intersects the face $B$ if and only if it intersects the cone $C$ spanned by $B$ not only in the origin. Hence by the conic Crofton formula stated in Theorem 5.4 we have (35).

Up to a factor, $P_n$ is isometric to the $n$-dimensional standard simplex
\[
S_n = \text{conv}\{e_1, \ldots, e_{n+1}\} \subset \mathbb{R}^{n+1}.
\]
Thus, denoting the centre of $S_n$ by $m = \frac{e_1+\ldots+e_{n+1}}{n+1}$, the cone $C$ is isometric to the cone $D$ spanned by the vectors $e_1-m, \ldots, e_{d+1}-m$. To determine the isometry type of $D$, and thus $C$, we just calculate the scalar products of the vectors spanning it. These are
\[
\langle e_i - m, e_j - m \rangle = -\frac{1}{n+1} + \delta_{i,j},
\]
where $\delta_{i,j} = 1_{\{i=j\}}$ is the Kronecker delta. It follows that $D \cong C_{d+1}^{\Delta} \left( -\frac{1}{n+1} \right)$ and thus
\[
C \cong C_{d+1}^{\Delta} \left( -\frac{1}{n+1} \right).
\]
This completes the proof of the proposition. \qed

5.5. Asymptotics.

Proof of Corollary 4.6. We start by analysing the term $g_{n-i+s+1} \left( -\frac{1}{n-l+s+1} \right)$, where
\[
g_k(r) := \mathbb{P}[\eta_1 < 0, \ldots, \eta_k < 0],
\]
and $(\eta_1, \ldots, \eta_k)$ is a zero-mean Gaussian vector with
\[
\text{Cov}(\eta_i, \eta_j) = r + \delta_{i,j}.
\]
Note that $g_k(r)$ has the same meaning as in [10] and $g_k(-r/(1+kr)) = g_k^{\Delta}(r)$. Recalling that by [10, Proposition 1.2(b)] the internal solid angle $\beta(F, P_n)$ of an $(n-1)$-dimensional regular simplex $P_n$ at a $(k-1)$-dimensional face $F$ equals
\[
\beta(F, P_n) = g_{n-k} \left( -\frac{1}{n} \right)
\]
we can express our term by
\[
g_{n-i+s+1} \left( -\frac{1}{n-l+s+1} \right) = g_{n-i+s+1} \left( -\frac{1}{n-i+s+1+(i-l)} \right) = \beta(F, T),
\]
where $T := \text{conv}\{e_1, \ldots, e_{n-l+s+1}\}$ is an $(n-l+s)$-dimensional simplex and $F \in \mathcal{F}_{i-l-1}(T)$ is one of its $(i-l-1)$-faces. In other words, we need the asymptotic behaviour of the internal solid angle of a simplex with increasing dimension at a face of fixed dimension. Such a formula has been derived in [4, Corollary 2.1]. It states that for $n \to \infty$
\[
\beta(F, T) = \frac{(i-l)^{\frac{n-i-l}{2}}}{2^\frac{n-i-l}{2}} \exp \left( \frac{2\pi}{i-l} \right) \left( \frac{2\pi}{i-l} \right)^{\frac{i-l}{2}} \cdot n^{-\frac{n}{2}} \left( \frac{i-l}{2\pi} \right)^{\frac{n}{2}} \cdot \left( 1 + O \left( \frac{(i-l-1)^2 + 1}{n-l+s+1} \right) \right)
\]
\[
\sim (2\sqrt{\pi})^{-1} e^{\frac{2\pi}{i-l}} \left( \frac{2\pi}{i-l} \right)^{\frac{i-l}{2}} \cdot n^{-\frac{n}{2}} \left( \frac{i-l}{2\pi} \right)^{\frac{n}{2}} \left( 1 + O \left( \frac{(i-l-1)^2 + 1}{n-l+s+1} \right) \right). \quad (36)
\]

In the last step the only non-trivial formula we used is
\[
(n-l+s+1)^{\frac{n-i-l}{2}} = (n-l+s+1)^{\frac{n}{2}} \cdot (n-l+s+1)^{\frac{i-l}{2}} \sim e^{\frac{s+i}{2}} n^{\frac{n}{2}} \cdot n^{\frac{s-i}{2}}.
\]

Having (36) we can prove the corollary’s actual statement.

Recalling that by Theorem 4.5
\[
\mathbb{E} \varphi^n_{\Delta} (n-i, n-l, n) = 2 \left( \frac{n+1}{n-i+l+1} \right) \sum_{s=1,3,5,\ldots} \psi_{l+s} \left( C_{n-i+l+1}^{\Delta} \left( -\frac{1}{n+1} \right) \right)
\]
and that the sum over all odd intrinsic volumes as well as the sum over all even intrinsic volumes equals 1/2, we have
\[
\left( \frac{n+1}{n-i+l+1} \right) - \phi^D(n-i, n-l, n) = 2 \left( \frac{n+1}{n-i+l+1} \right) \sum_{s=1,3,5,\ldots} v_{l-s} \left( C_{n-i+l+1}^A \left( -\frac{1}{n+1} \right) \right).
\]

(37)

With the formula \( v_k(C_n^\phi(r)) = \binom{n}{k} g_k \left( -\frac{r}{1+kr} \right) g_{n-k} \left( \frac{r}{1+kr} \right) \) from [10] Proposition 1.3 we can express the summands by
\[
2 \left( \frac{n+1}{n-i+l+1} \right) v_{l-s} \left( C_{n-i+l+1}^A \left( -\frac{1}{n+1} \right) \right)
\]
\[
= 2 \left( \frac{n+1}{n-i+l+1} \right) \left( n-i+l+1 \right) g_{l-s} \left( \frac{1}{n+1-l+s} \right) g_{n-i+s+1} \left( -\frac{1}{n+1-l+s} \right).
\]

With \( g_{l-s} \left( \frac{1}{n+1-l+s} \right) \xrightarrow[n\to\infty]{\sim} g_{l-s}(0) = 2^{s-l} \) by [10] Proposition 1.4(d)] and using the asymptotics [36],
\[
\left( \frac{n+1}{n-i+l+1} \right) \sim \frac{n^l}{(i-l)!} \quad \text{and} \quad \left( \frac{n-i+l+1}{l-s} \right) \sim \frac{n^{l-s}}{(l-s)!}
\]
this is asymptotically equivalent to
\[
\frac{2^{s-l} e^{\frac{3l-3i}{2}}}{(i-l)!(l-s)!\sqrt{\pi}} \left( \frac{2\pi}{i-l} \right)^{\frac{i-l}{2}} \cdot n^{\frac{n}{2} - \frac{3l-3i}{2}} \left( \frac{(i-l)e}{2\pi} \right)^{\frac{n}{2}}
\]

In view of (37) we are actually interested in the sum of these formulas over \( s = 1, 3, 5, \ldots \) and \( s \leq l \). Note that the summand with \( s = 1 \) is of a larger order than every other summand. Since the number of summands is finite and does not depend on \( n \), the sum is dominated by its largest summand. Thus
\[
\left( \frac{n+1}{n-i+l+1} \right) - \phi^D(n-i, n-l, n) \sim 2 \left( \frac{n+1}{n-i+l+1} \right) v_{l-1} \left( C_{n-i+l+1}^A \left( -\frac{1}{n+1} \right) \right)
\]
\[
\sim \frac{n^{l-2} 2^{\frac{l-2}{2}+1} e^{\frac{3l-3i}{2}}}{(l-1)!(i-l)!\sqrt{\pi}} \cdot n^{\frac{n}{2} - \frac{3l-3i}{2}} \left( \frac{(i-l)e}{2\pi} \right)^{\frac{n}{2}}.
\]

Proof of Corollary 4.7: Recall from Theorem 4.5 that
\[
\phi^D(j, n-l, n) = 2^{n-l-j+1} \left( \frac{n}{l+j} \right) \sum_{s=1,3,5,\ldots} v_{l+s} \left( C_{l+j}^D(n-l-j) \right).
\]

The intrinsic volumes on the right-hand side are given by Theorem 2.4 as follows:
\[
v_{l+s} \left( C_{l+j}^D(n-l-j) \right) = 2^{l-s+1} \left( \frac{l+j}{l+s-1} \right) g_{l+s-1}^D(n-l-s+1) g_{l-s+1}^D \left( -\frac{1}{n-l-j} \right),
\]
where we assume that \( s \leq j + 1 \) because otherwise the intrinsic volumes vanish. We start by analysing the asymptotic behavior of the term

\[
g_{l+s-1}(n - l - s + 1) = \mathbb{P} \left[ \xi_{l+s} \geq \sqrt{n - l - s + 1} \max_{1 \leq i \leq l+s-1} |\xi_i| \right],
\]

where \( \xi_1, \xi_2, \ldots \) are i.i.d. standard normal distributed random variables. Expressed as an integral it has the form

\[
g_{l+s-1}(n - l - s + 1) = \int_0^\infty \varphi(t) F^{l+s-1} \left( \frac{t}{\sqrt{n - l - s + 1}} \right) dt,
\]

where \( \varphi \) is the density of the standard normal distribution and \( F \) is the cumulative distribution function of \( |\xi_1| \).

Since as \( n \to \infty \) while \( t > 0 \) stays constant we have

\[
F \left( \frac{t}{\sqrt{n - l - s + 1}} \right) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{t}{\sqrt{n - l - s + 1}}} e^{-x^2} dx \sim \sqrt{\frac{2}{\pi}} \frac{t}{\sqrt{n - l - s + 1}} \sim \sqrt{\frac{2}{\pi}} \frac{t}{\sqrt{n}},
\]

the integrand satisfies

\[
\varphi(t) F^{l+s-1} \left( \frac{t}{\sqrt{n - l - s + 1}} \right) \sim \frac{1}{n^{l+s-1\cdot \frac{1}{2}}} \varphi(t) \left( \frac{2\sqrt{n}}{\pi} \right)^{l+s-1}
\]

for every fixed \( t > 0 \). The natural hypothesis, i.e. that as \( n \to \infty \)

\[
g_{l+s-1}(n - l - s + 1) \sim \int_0^\infty \frac{1}{n^{l+s-1\cdot \frac{1}{2}}} \varphi(t) \left( \frac{2\sqrt{n}}{\pi} \right)^{l+s-1} dt
\]

(39)

follows from the dominated convergence theorem. Expressing the \((l + s - 1)^{th}\) moment of \( |\xi_1| \) as

\[
\sqrt{\frac{2}{\pi}} \int_0^\infty t^{l+s-1} e^{-t^2} dt = \frac{2^{l+s-1}}{\sqrt{\pi}} \Gamma \left( \frac{l+s}{2} \right),
\]

we can simplify (39) to

\[
g_{l+s-1}(n - l - s + 1) \sim \frac{1}{n^{l+s-1\cdot \frac{1}{2}}} \left( \frac{2}{\sqrt{\pi}} \right)^{l+s-1} \Gamma \left( \frac{l+s}{2} \right). \frac{1}{n^{l+s-1}}
\]

Using the continuity of \( g_{j-s+1}^\Delta \) and [10, Proposition 1.4(d)] we have \( \lim_{n \to \infty} g_{j-s+1}^\Delta \left( \frac{1}{n-l-j} \right) = 2^{s-j-1} \) and thus the intrinsic volumes satisfy

\[
v_{l+s} \left( C_{l+j}^{l+s}(n - l - j) \right) = 2^{j-s+1} \binom{l+j}{l+s-1} g_{l+s-1}(n - l - s + 1) g_{j-s+1}^\Delta \left( \frac{1}{n-l-j} \right)
\]

\[
\sim \binom{l+j}{l+s-1} \left( \frac{2}{\sqrt{\pi}} \right)^{l+s-1} \frac{\Gamma \left( \frac{l+s}{2} \right)}{2\sqrt{\pi}} \cdot \frac{1}{n^{l+s-1}}
\]
as $n \to \infty$. Recalling that

$$\phi(n, l, n) = 2^{n-l-j+1} \left( \begin{array}{c} n \\ l+j \end{array} \right) \sum_{s=1,3,5,...} v_{l+s} \left( C_{l+j}(n-l-j) \right)$$

we conclude that the summand with $s = 1$ is of a higher order than any other one. Observe also that the number of non-zero summands is bounded by a term not depending on $n$. Thus, the whole sum is dominated by the first summand, i.e.

$$\phi(n, l, n) \sim 2^{n-l-j+1} \left( \begin{array}{c} n \\ l+j \end{array} \right) v_{l+1} \left( C_{l+j}(n-l-j) \right)$$

$$\sim 2^{n-j} \left( \begin{array}{c} n \\ l+j \end{array} \right) \left( \begin{array}{c} l+j \\ l \end{array} \right) \left( \frac{1}{\sqrt{\pi}} \right)^{l+1} \Gamma \left( \frac{l+1}{2} \right) \frac{1}{n^{l+1}}$$

$$\sim 2^{n-j} \frac{n^{j+(l/2)} \Gamma \left( \frac{l+1}{2} \right)}{l! j!} \frac{\Gamma \left( \frac{l+1}{2} \right)}{\pi^{(l+1)/2}}$$

as $n \to \infty$.

□

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