ON SYMPLECTIC BIRATIONAL INVolutions OF MANIfolds OF $OG_{10}$ TYPE

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ABSTRACT. The classification of symplectic birational transformations of irreducible holomorphic symplectic manifolds is of great interest. In this paper we give a partial classification of symplectic birational involutions of manifolds of $OG_{10}$ type, with geometric realizations in all but one case. We approach this classification with two techniques - via involutions of the Leech lattice, and via involutions of cubic fourfolds. In particular, we obtain an exceptional involution whose coinvariant lattice is isomorphic to the lattice $D_{12}^{+}(2)$, and thus cannot be obtained from either a $K3$ surface or a cubic fourfold. We speculate our classification is complete.

1. Introduction

The classification of symplectic automorphisms of irreducible holomorphic symplectic manifolds has been widely studied. In his celebrated paper [Muk88], Mukai classified symplectic automorphisms of a $K3$ surface completely. This was further streamlined by Kondō [Kon98], who related automorphisms $K3$ surfaces to automorphisms of the Niemeier lattices. In recent years, there has been intense work on classifying symplectic automorphisms of higher dimensional irreducible holomorphic symplectic manifolds. Using a similar approach to Kondō, Mongardi obtained a classification of prime order symplectic automorphisms of manifolds of $K3^{[n]}$ type [Mon13, Mon16]. Similar results were obtained by Huybrechts [Huy16]. A classification of symplectic automorphisms of manifolds of $OG_6$ type was obtained in [GOV20]. In contrast, manifolds of $OG_{10}$ type admit no regular symplectic automorphisms, as shown recently in [GGOV22]. For a $K3$ surface, the group of automorphisms and the group of birational transformations coincide; in higher dimensions this is no longer the case. One can consider instead the more interesting group of symplectic birational transformations of an irreducible holomorphic symplectic manifold.

In this paper, we investigate symplectic birational involutions of manifolds deformation equivalent to O’Grady’s ten dimensional exceptional example ($OG_{10}$ type) [O’G99]. Our interest is motivated by our desire to study the fixed loci. In the case of involutions, the associated moduli space of the fixed locus is a type IV period domain; we obtain variations of Hodge structures of $K3$ type. This does not occur for higher order cyclic groups (see [YZ20], [LPZ18]).

Let $X$ be manifold of $OG_{10}$ type, $f \in \text{Bir}(X)$ a symplectic birational involution. We obtain an induced involution on the second cohomology, determining two
sublattices $H^2(X,\mathbb{Z})_+, H^2(X,\mathbb{Z})_-$, the invariant and the coinvariant lattice respectively. Vice versa, specifying such sublattices (subject to certain lattice theoretic conditions) determines a symplectic birational transformation of some manifold of $OG_{10}$ type via the Global Torelli Theorem (Theorem 2.6). Our main theorem is a partial classification of symplectic birational involutions of manifolds of $OG_{10}$ type.

**Theorem 1.1.** Let $X$ be a manifold of $OG_{10}$ type, $f \in \text{Bir}(X)$ a symplectic birational involution.

1. Assume that $f$ acts trivially on the discriminant group. Then the pair $H^2(X,\mathbb{Z})_-, H^2(X,\mathbb{Z})_+$ appears below:

   \[
   \begin{array}{ccc}
   H^2(X,\mathbb{Z})_- & H^2(X,\mathbb{Z})_+ \\
   E_8(2) & U^3 \oplus E_8(2) \oplus A_2 \\
   D_{12}^+(2) & E_6(2) \oplus U^3(2) \oplus A_1 \oplus A_1(-1)
   \end{array}
   \]

2. Assume that $f$ acts non-trivially on the discriminant group, and such that rank$(H^2(X,\mathbb{Z})_-) < 12$. Then the pair $H^2(X,\mathbb{Z})_-, H^2(X,\mathbb{Z})_+$ appears below:

   \[
   \begin{array}{ccc}
   H^2(X,\mathbb{Z})_- & H^2(X,\mathbb{Z})_+ \\
   E_6(2) & U^3 \oplus D_4^3 \\
   M & E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2
   \end{array}
   \]

Moreover, each involution listed above exists.

We approach this classification from two vantage points - first we use the same techniques as in [Huy16],[Mon16], relating certain involutions to involutions of the Leech lattice. This recovers the Nikulin type involution with coinvariant lattice $E_8(2)$, and more interestingly, we obtain an involution with coinvariant lattice $D_{12}^+(2)$ that cannot be realised in the case of manifolds of $K3^{[n]}$ type. Thus we obtain the two involutions listed in (1).

Secondly, we look at involutions that are obtained from cubic fourfolds via the intermediate Jacobian construction due to ([LSV17], [Sac21]). Using the observation of Saccà [Sac21, §3.1] (see also [LPZ18]), an involution of a cubic fourfold induces a birational transformation of the corresponding compactified intermediate Jacobian $X := J_V$. We use the classification of involutions of a cubic fourfold previously studied in [Mar22] to obtain a classification of symplectic birational involutions that fix a copy of $U$, the hyperbolic lattice as in (2). This method can be extended to higher order symplectic birational transformations, using the classification in [LZ22]. We also obtain geometric realisations of these involutions. We expect that this is a complete classification - there are only a short list of possibilities remaining (discussed in §5). We speculate that these do not occur.

**Outline.** We recall the relevant definitions and previous results in §2. In §3 we begin the proof of Theorem 1.1 by considering birational symplectic involutions acting trivially on the discriminant group of $\Lambda_{OG_{10}}$. We embed the lattice $(\Lambda_{OG_{10}})$ - into the Leech lattice, and use the classification of involutions [HL90]. We obtain the classification in Theorem 1.1 (1). In §4.1 we make the relationship between involutions of a cubic fourfold and that of manifolds of $OG_{10}$ type more precise.
We obtain a classification of birational symplectic involutions that fix a copy of \( U \). In §4.3, we identify a criteria for when this occurs, completing the proof of Theorem 1.1. We discuss the remaining cases in §5.

**Notations.** Throughout, all lattices are assumed to be even unless stated otherwise. Let \( L \) be a lattice and \( G \subset O(L) \) a group of isometries. We denote by \( L_G \) the \( G \)-invariant sublattice and with \( L^G := (L_G)^\perp \) the coinvariant lattice. For an involution \( \iota \in O(L) \), these are simply denoted by \( L_\iota \) and \( L_{\perp, \iota} \) respectively. For any lattice \( L \), \( L(n) \) denotes the lattice with the quadratic form scaled by \( n \). The discriminant group of a lattice is denoted \( A_L = L^*/L \). We assume all \( ADE \) lattices are negative definite unless otherwise specified. A 2-elementary lattice \( L \) is determined by its signature and the invariants \( \delta_L, \ell(A_L) \). Here \( \delta_L \in \{0,1\} \) with \( \delta_L = 0 \) if and only if \( q_{A_L} \) takes values in \( \mathbb{Z}/2\mathbb{Z} \), and \( \ell(A_L) \) is the minimum number of generators of \( A_L \).

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### 2. Preliminaries

#### 2.1. Manifolds of \( OG10 \) type.** An irreducible holomorphic symplectic manifold is a simply connected, compact, Kähler manifold \( X \) such that \( H^0(X, \Omega^2_X) \) is generated by a nondegenerate holomorphic 2-form \( \sigma \). Let \( X \) be an irreducible holomorphic symplectic manifold that is deformation equivalent to O’Grady’s 10-dimensional exceptional example \([O'G99]\). Then we say \( X \) is of \( OG10 \) type.

It follows from the definition of irreducible holomorphic symplectic manifolds that \( H^2(X, \mathbb{Z}) \) is a torsion free \( \mathbb{Z} \)-module; equipped with the Beauville-Bogomolov-Fujiki form \( q_X \) it becomes a lattice. By \([Rap08]\), for \( X \) of \( OG10 \) type there is an isometry \( \langle H^2(X, \mathbb{Z}), q_X \rangle \cong \Lambda_{OG10} \), where

\[
\Lambda_{OG10} := U^3 \oplus E_8^2 \oplus A_2.
\]

A *marking* is a choice of isometry \( \eta : H^2(X, \mathbb{Z}) \to \Lambda_{OG10} \). The purpose of this paper is to classify possibly symplectic birational involutions of manifolds \( X \) of \( OG10 \) type, in terms of their action on \( H^2(X, \mathbb{Z}) \cong \Lambda_{OG10} \). We always assume that a manifold of \( OG10 \) type has a fixed marking \( \eta \) throughout.

#### 2.2. Symplectic Birational Transformations.** We denote by \( \text{Aut}(X) \), \( \text{Bir}(X) \) the groups of automorphisms and birational transformations of \( X \) respectively. For an irreducible holomorphic symplectic manifold \( X \), a birational transformation \( f \in \text{Bir}(X) \) is well defined in codimension one. We thus obtain an isometry \( f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \).

**Definition 2.1.** We say a birational transformation \( f \in \text{Bir}(X) \) is **symplectic** if the induced action \( f^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \) acts trivially on \( \sigma \). Otherwise, \( f \) is non-symplectic.

Assume now that \( X \) is an irreducible holomorphic symplectic manifold of \( OG10 \) type, and consider the associated representation map

\[
\eta_* : \text{Bir}(X) \to O(\Lambda_{OG10}); \quad f \mapsto \eta \circ f^* \circ \eta^{-1}.
\]
Note that for a nontrivial birational transformation \( f \in \text{Bir}(X) \) the induced isometry \( \eta_*(f) \in O(\Lambda_{OG10}) \) is non-trivial. Indeed, if \( \eta_*(f) \) was trivial, by [Fuj81] we see that \( f \) is a regular automorphism of \( X \). By [MW17, Theorem 3.1], the associated representation \( \text{Aut}(X) \to O(\Lambda_{OG10}) \) is injective, and so \( f \) is trivial.

**Definition 2.2.** An isometry \( g \in O(\Lambda_{OG10}) \) is **induced** by a birational transformation if there exists a \( f \in \text{Bir}(X) \) such that \( \eta_*(f) = g \).

A birational transformation of \( X \) preserves the birational Kähler cone \( BK(X) \). This in turn imposes restrictions on which involutions of the lattice \( \Lambda_{OG10} \) are induced by birational involutions of such a manifold \( X \). The structure of the birational Kähler cone for a manifold of \( OG_{10} \) type is now fully understood [MO22].

**Definition 2.3.** Denote by \( W_{pex}^{OG_{10}} \) the following set of vectors:

\[
W_{pex}^{OG_{10}} = \{ v \in \Lambda_{OG10} : v^2 = -2 \} \cup \{ v \in \Lambda_{OG10} : v^2 = -6, \text{div}_{\Lambda_{OG10}}(v) = 3 \}.
\]

We call \( v \in W_{pex}^{OG_{10}} \) with \( v^2 = -2 \) (resp. \( v^2 = -6 \)) a **short root** (resp. a **long root**). The vectors in \( W_{pex}^{OG_{10}} \) correspond exactly to the stably prime exceptional divisors via the isomorphism \( \eta : H^2(X, \mathbb{Z}) \cong \Lambda_{OG10} \) [MO22, Proposition 3.1]. Note that we can define a signed hodge structure (see [GOV20, §2.5]) on the lattice \( \Lambda_{OG10} \) via this isomorphism. A chamber defined by \( W_{pex}^{OG_{10}} \) is called an **exceptional chamber**.

**Theorem 2.4.** [MO22, Theorem 3.2] Let \( X \) be a manifold of \( OG_{10} \) type. Then, the birational Kähler cone \( BK(X) \) of \( X \) is an open set inside one of the components of

\[
\mathcal{C}(X) \setminus \bigcup_{v \in W_{pex}^{OG_{10}}} v^\perp
\]

where \( \mathcal{C}(X) \) is the connected component of the positive cone containing a Kähler class.

### 2.3. Torelli Theorem.

Using this description for the birational Kähler cone, we can rephrase the Global Torelli theorem in a way that is more suited for the study of symplectic birational transformations of \( X \). We denote by \( O_{hdg}^+(\Lambda_{OG10}) \), and \( O_{sp}^+(\Lambda_{OG10}) \) the group of signed hodge isometries and signed symplectic isometries respectively.

**Theorem 2.6.** (Torelli Theorem for OG10) A subgroup of \( G \subset O(\Lambda_{OG10}) \) is induced by a group of symplectic birational transformations on a manifold of OG10-type if and only if there exist a signed hodge structure on \( \Lambda_{OG10} \) and an exceptional chamber \( C \) such that \( G \subset O_{sp}^+(\Lambda_{OG10}, C) \).

**Proof.** The proof follows almost identically to that of [GOV20, Theorem 2.15] in the case of manifolds of \( OG_6 \)-type. \( \square \)

### 2.4. Symplectic Birational Involutions.

We restrict our attention to symplectic birational involutions \( f \in \text{Bir}(X) \) for \( X \) of \( OG_{10} \)-type. A proof identical to [GOV20, Theorems 2.16 and 2.17] hold for the following results.

**Lemma 2.7.** Let \( X \) be an irreducible holomorphic symplectic manifold of \( OG_{10} \) type. Let \( f \in \text{Bir}(X) \) be a birational symplectic involution. Then let \( \iota := \eta_*(f) \in O(\Lambda_{OG10}) \) be the induced action. Then the coinvariant lattice \( (\Lambda_{OG10})_\iota \) has the following properties:
(1) \((\Lambda_{OG10})_–\) is negative definite.
(2) \((\Lambda_{OG10})_–\) does not contain any short or long roots; i.e \((\Lambda_{OG10})_–\cap W_{OG10}^{\text{per}} = \emptyset\).

**Theorem 2.8.** An involution \(\iota \subset O(\Lambda_{OG10})\) is induced by a symplectic birational transformation if and only if \((\Lambda_{OG10})_–\) is negative definite and 
\[ (\Lambda_{OG10})_– \cap W_{OG10}^{\text{per}} = \emptyset. \]

In order to classify symplectic birational involutions of manifolds of \(OG10\) type, we will consider two cases corresponding to the induced action of \(\iota \in O(\Lambda_{OG10})\) on the discriminant group 
\[ A_{\Lambda_{OG10}} := \Lambda_{OG10}^*/\Lambda_{OG10} \cong \mathbb{Z}/3\mathbb{Z}. \]

It follows that an involution acts by \(\iota|_{A_{\Lambda_{OG10}}} = \pm id_{A_{\Lambda_{OG10}}}\).

**Remark 2.9.** Note that \(A_{\Lambda_{OG10}} = A_2\); let \(A_2\) be generated by \(\alpha_1, \alpha_2\). Then \(A_2 \cong \mathbb{Z}/3\mathbb{Z}\) is generated by 
\[ \gamma := \left[ \frac{2\alpha_1 + \alpha_2}{3} \right] \]
and \(q_{A_2}(\gamma) = -\frac{2}{7}\).

**Proposition 2.10.** Let \(f \in \text{Bir}(X)\) be a symplectic birational involution, and let \(\iota = \eta_\ast(f) \in O(\Lambda_{OG10})\) the induced isometry of \(f\). Then \((\Lambda_{OG10})_–\) is a negative definite lattice of rank \(r \leq 21\), with \((\Lambda_{OG10})_–\cap W_{OG10} = \emptyset\), and the following hold:

1. If \(\iota\) acts trivially on \(A_{\Lambda_{OG10}}\), then \((\Lambda_{OG10})_–\) is a 2-elementary, negative definite even lattice determined by the invariants \((r, l(A_{\Lambda_{OG10}})_–, \delta)\).
2. If \(\iota\) acts by \(-id|_{A_{\Lambda_{OG10}}}\) on \(A_{\Lambda_{OG10}}\), then \((\Lambda_{OG10})_+\) is a 2-elementary lattice with signature \((3, 21 - r)\).

**Proof.** The negative definiteness and the claim that \((\Lambda_{OG10})_–\cap W_{OG10} = \emptyset\) follows from 2.7. Claim (1) follows by [GOV20, Lemma 2.8]; for claim (2) consider \(\iota' := -\iota\); it follows that \((\Lambda_{OG10})_–\) is 2-elementary. \(\square\)

2.5. **Cubic Fourfolds.** Cubic fourfolds lead to irreducible holomorphic symplectic manifolds through various constructions, and one can study birational transformations induced by automorphisms of the cubic. This was first studied by Camere [Cam12] in their work on symplectic involutions of the Fano variety of lines, an irreducible holomorphic symplectic variety [BD85].

We will use this idea for manifolds of \(OG10\) type. Let \(V \subset \mathbb{P}^5\) be a smooth cubic fourfold, and let \(\pi_U : J_U \to U \subset (\mathbb{P}^5)^\vee\) be the Donagi-Markman fibration; i.e the family of intermediate Jacobians of the smooth hyperplane sections of \(V\). The total space \(J_U\) has a holomorphic symplectic form, by [DM96]. The main result of [LSV17] is the construction, for a general \(V\), of a smooth projective irreducible holomorphic symplectic compactification \(J_V\) of \(J_U\), with a Lagrangian fibration \(\pi : J_V \to (\mathbb{P}^5)^\vee\) extending \(\pi_U\). It was shown that \(J_V\) is an irreducible holomorphic symplectic of \(OG10\) type. This result was extended to every cubic fourfold [Sac21], to obtain the following theorem.

**Theorem 2.11.** ([LSV17],[Sac21]) Let \(V \subset \mathbb{P}^5\) be a smooth cubic fourfold, and let \(\pi_U : J_U \to U \subset (\mathbb{P}^5)^\vee\) be the Donagi-Markman fibration. Then there exists a smooth projective irreducible symplectic compactification \(J_V\) of \(J_U\) of \(OG10\) type with a morphism \(\pi : J_V \to (\mathbb{P}^5)^\vee\) extending \(\pi_U\).
We note that the same result holds for the irreducible holomorphic symplectic compactification $\mathcal{J}_V^T$ of the non trivial $J_U$-torsor $J_U^T \to U$ of [Voi18].

Recall that the primitive cohomology $H^4(V, \mathbb{Z})_{\text{prim}}$ admits a hodge structure of $K3$-type (up to a Tate twist). In particular, as a lattice $H^4(V, \mathbb{Z})_{\text{prim}} \cong U^2 \oplus E_8^2 \oplus A_2$, where here each $ADE$ lattice is taken to be positive definite. Building on the work of [LPZ18], we have the following arithmetic classification for involutions of cubic fourfolds:

**Theorem 2.12.** [Mar22, Theorem 1.1] Let $V \subset \mathbb{P}^5$ be a general cubic fourfold with $\phi_i$ an involution of $V$ fixing a linear subspace of $\mathbb{P}^5$ of codimension $i$. Then either:

1. $i = 1$, $\phi_1$ is anti-symplectic and $A(V)_{\text{prim}} \cong E_6(2)$, $T(V) \cong U^2 \oplus D_4^3$.
2. $i = 2$, $\phi_2$ is symplectic and $A(V)_{\text{prim}} \cong E_8(2)$, $T(V) \cong A_2 \oplus U^2 \oplus E_6(2)$.
3. $i = 3$, $\phi_3$ is anti-symplectic and

$$A(V)_{\text{prim}} \cong M, \quad T(V) \cong U \oplus A_1 \oplus A_1(-1) \oplus E_8(2).$$

Here $M$ is the unique rank 10 even lattice obtained as an index 2 overlattice of $D_9(2) \oplus (24)$, and every $ADE$ lattice is assumed to be positive definite.

3. **Classification of Involutions acting trivially on the discriminant**

Throughout, we let $(X, \eta)$ be a marked irreducible holomorphic symplectic manifold of $OG10$ type. We will prove the following:

**Theorem 3.1.** Let $X$ be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \text{Bir}(X)$ be a symplectic birational involution. Suppose that $\eta_*(f)$ acts trivially on the discriminant group $A_{\Lambda_{OG10}}$. Then one of the following holds:

1. $H^2(X, \mathbb{Z})_- \cong E_8(2)$ and $H^2(X, \mathbb{Z})_+ \cong U^2 \oplus E_8(2) \oplus A_2$; or
2. $H^2(X, \mathbb{Z})_- \cong D_{12}^+(2)$ and $H^2(X, \mathbb{Z})_+ \cong E_6(2) \oplus U^2(2) \oplus A_1 \oplus A_1(-1).$

The strategy to prove Theorem 3.1 is as follows: we first consider arithmetic involutions $i \in O(\Lambda_{OG10})$ such that $i$ acts trivially on $A_{\Lambda_{OG10}}$, and $(\Lambda_{OG10})_-$ is negative definite. We then use techniques of Kondo and Mongardi to embed the covariant lattice $(\Lambda_{OG10})_-$ into the Leech lattice $\Lambda$. Next, we extend the involution $i$ to one of the Leech lattice $\Lambda$, and use the classification of involutions [HL90] to obtain three candidates. We then discuss case by case and show that only $E_8(2)$ and $D_{12}^+(2)$ are realised as coinvariant lattices $(\Lambda_{OG10})_-$ for an involution of $\Lambda_{OG10}$. We then show that they contain no short or long roots, i.e $(\Lambda_{OG10})_- \cap W_{OG10}^{\text{pur}} = \emptyset$, and conclude by Theorem 2.8 that such an involution $i$ is induced by a geometric symplectic birational involution $f \in \text{Bir}(X)$ of a manifold $X$ of $OG10$ type.

### 3.1 The Leech Lattice

We reduce the classification of involutions $i \in O(\Lambda_{OG10})$ acting trivially on $A_{\Lambda_{OG10}}$ to classifications of involutions of the Leech lattice $\Lambda$. The following result is originally due to [GHV12]; the argument was then reproduced by Huybrechts [Huy16, §2.2].

**Proposition 3.2.** Let $i \in O(\Lambda_{OG10})$ be an involution acting trivially on $A_{\Lambda_{OG10}}$ and such that $(\Lambda_{OG10})_-$ is negative definite and does not contain any short roots. Then there exists a primitive embedding of $(\Lambda_{OG10})_- \hookrightarrow \text{ into the Leech lattice } \Lambda$.

**Corollary 3.3.** Assumptions as in Prop. 3.2. Then there exists an involution of the Leech lattice $\Lambda$ such that $\Lambda_- \cong (\Lambda_{OG10})_-$. 
Proof. Consider the primitive embedding \((\Lambda_{OG10})_+ \hookrightarrow \Lambda\). We apply [Nik79, Cor. 1.5.2] to extend \(\iota\); indeed, since \(\iota\) acts trivially on \(A_{(\Lambda_{OG10})_+}\), we can extend \(\iota\) to an involution of \(\Lambda\), with \(\Lambda_+ \cong (\Lambda_{OG10})_+\), acting by the identity on \((\Lambda_{OG10})_- = \Lambda_+\).

The non-trivial involutions \(\iota \in O(\Lambda)\) are classified [HL90]:

**Proposition 3.4.** There exist three conjugacy classes of non-trivial involutions of the Leech lattice \(\Lambda\). They are classified by specifying the invariant/anti-invariant sublattices:

1. \(\Lambda_- \cong E_8(2), \Lambda_+ \cong BW_{16}\);
2. \(\Lambda_- \cong BW_{16}, \Lambda_+ \cong E_8(2)\);
3. \(\Lambda_- \cong D_{12}^+(2), \Lambda_+ \cong D_{12}^+(2)\).

We have three possible candidates for \((\Lambda_{OG10})_-\) as above. It remains to be seen whether there exists an involution \(\iota \in O(\Lambda_{OG10})\) whose coinvariant lattice is the given candidate. We show such an involution exists provided there exists a primitive embedding of each candidate into \(\Lambda_{OG10}\).

**Lemma 3.5.** Let \(M\) be a 2-elementary lattice with a primitive embedding \(M \hookrightarrow L\) into a lattice \(L\) such that \(N := (M)_L^+\). Then there exists an involution \(\iota \in O(L)\) such that the coinvariant lattice \(L_- = M\) and the invariant lattice \(L_+ = N\).

**Proof.** By assumption we have that

\[
M \oplus N \hookrightarrow L \hookrightarrow L \otimes \mathbb{Q} \cong (M \oplus N) \otimes \mathbb{Q}.
\]

We can define \(\iota_Q : L_Q \rightarrow \mathbb{Q}\) by \(\iota(x) = -x\) for \(x \in M\), and \(\iota(x) = x\) for \(x \in N\). We want to show that \(\iota_Q\) is defined over \(L\). By assumption \(L/(M \oplus N) \cong (\mathbb{Z}/2\mathbb{Z})^a\), and thus for all \(x \in L\), we have that \(2x \in M \oplus N\). Let \(x \in L\). By above, we can write \(x = \frac{x_+ + x_-}{2}\), with \(x_- \in M, x_+ \in N\). Thus \(\iota_Q(x) = x \mod M \oplus N\), and \([\iota_Q(x)] = [x]\) in \(L/(M \oplus N)\); thus \(\iota_Q(x) \in L\). \(\square\)

It remains to be seen whether a primitive embedding exists in each of the cases in Proposition 3.4. We consider each in turn.

3.2. Case (1). We can easily see the existence of a primitive embedding \(E_8(2) \hookrightarrow \Lambda_{OG10}\), and exhibit an involution of \(\Lambda_{OG10}\) with \((\Lambda_{OG10})_- \cong E_8(2)\). Consider the involution defined by interchanging the two copies of \(E_8\), and identity elsewhere. Then \((\Lambda_{OG10})_- \cong E_8(2)\) and we have the following result (see [Mor84]).

**Proposition 3.6.** There exists a primitive embedding \(E_8(2) \hookrightarrow \Lambda_{OG10}\). In particular, there exists an involution of \(\Lambda_{OG10}\) such that \((\Lambda_{OG10})_- = E_8(2)\).

3.3. Case (2). For the two remaining cases, we will use the following Lemma to establish whether or not there exists a primitive embedding of \(\Lambda_-\) into \(\Lambda_{OG10}\).

**Lemma 3.7.** Let \(M\) be a 2-elementary lattice negative definite lattice of rank \(r\) with invariants \((r, a, \delta)\), where \(a = l(A_M)\). Then there exists a primitive embedding \(M \hookrightarrow \Lambda_{OG10}\) if and only if there exists a lattice \(N\) of signature \((3, 21 - r)\) satisfying the following properties:

1. \(A_N = (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/3\mathbb{Z}\);
2. \(q_N|_{(\mathbb{Z}/2\mathbb{Z})^a} \cong -q_M\);
3. \(q_N|_{\mathbb{Z}/3\mathbb{Z}} \cong q_L\).
We say $\delta := \delta_N = 0$ or 1 if and only if $\delta_M = 0$ or 1.

Proof. This follows immediately from [Nik79, Prop 1.15.1]. □

Proposition 3.8. There does not exist a primitive embedding $BW_{16} \hookrightarrow \Lambda_{OG10}$. In particular, there is no involution of $\Lambda_{OG10}$ such that $(\Lambda_{OG10})_+ \cong BW_{16}$.

Proof. The Barnes–Wall lattice $BW_{16}$ is an even 2-elementary lattice of signature $(0, 16)$, $a = 8$ and $\delta = 0$. Suppose there exists such an embedding. By above, this is equivalent to the existence of an even lattice $N$ of signature $(3, 5)$, $A_N = (\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/3\mathbb{Z}$, with $q_N|_{(\mathbb{Z}/2\mathbb{Z})^8}$ taking values in $\mathbb{Z}/2\mathbb{Z}$. We also have that $q_N|_{\mathbb{Z}/3\mathbb{Z}} = q_{A_2}$.

Since $(\mathbb{Z}/2\mathbb{Z})^8 \subset A_N$, we can deduce that $K := N(1/2)$ is a well-defined integral lattice (see for example [Mar22, Lemma A.7]). Notice that $K$ has signature $(3, 5)$, and $A_K = \mathbb{Z}/3\mathbb{Z}$.

We claim that $K$ is even; indeed, suppose that $K$ was odd. Then by [Mar22, Lemma A.9], there exists an element $\xi \in \mathbb{Z}/2\mathbb{Z} \subset A_N$ such that $q(\xi) \notin \mathbb{Z}/2\mathbb{Z}$, contradicting our assumption on $N$.

Thus $K$ is an even lattice with $A_K = \mathbb{Z}/3\mathbb{Z}$, and $(A_K, q_K) \cong (A_{E_6}, q_{E_6})$. Since $q_K = q_{E_6} = -q_{A_2}$ and $A_K \cong A_{A_2}$, by [Nik79, Prop 1.15.1] there exists a primitive embedding of $K$ into some even unimodular lattice $\Gamma$ of signature $(3, 7)$. By Milnor’s theorem on unimodular forms, we see that no such even unimodular lattice $\Gamma$ exists. Thus such an $N$ cannot exist. □

3.4. Case (3). Here we show the somewhat surprising result that the lattice $D_{12}^+(2)$ does primitively embed into $\Lambda_{OG10}$. Recall that the lattice $D_{12}^+(2)$ is an even, 2-elementary lattice with signature $(0, 12)$, $a = 12$ and $\delta = 1$.

Proposition 3.9. There exists a primitive embedding $D_{12}^+(2) \hookrightarrow \Lambda_{OG10}$. In particular, there exists an involution of $\Lambda_{OG10}$ such that $(\Lambda_{OG10})_+ \cong D_{12}^+(2)$.

Proof. By Lemma 3.7, this is equivalent to the existence of a lattice $N$ with signature $(3, 9)$, satisfying the conditions of the Lemma. We will show such a lattice exists. Consider the lattice

$$N = E_6(2) \oplus U^2(2) \oplus A_1 \oplus A_1(-1).$$

This lattice satisfies condition (1) of Lemma 3.7: it remains to show the conditions (2), (3) hold. More specifically, we need to show that

1. $q_N|_{\mathbb{Z}/3\mathbb{Z}} \cong q_{A_2}$, and that
2. $q_N|_{(\mathbb{Z}/2\mathbb{Z})^{12}} \cong -q_{D_{12}^+(2)}$.

In order to do so, we will calculate the values of the discriminant form for both $A_{D_{12}^+(2)}$ and $A_N$. We will use the fact that $N$ is also isomorphic to $E_6(2) \oplus U(2) \oplus (A_1 \oplus A_1(-1))^2$, by Nikulin’s classification of 2-elementary lattices.

First, note that $A_{E_6(2)} = \mathbb{Z}/6\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^5$. Let $\alpha_1, \ldots, \alpha_6$ be a basis for $K := E_6(2)$, with Gram matrix:

$$G_{E_6(2)} := \begin{bmatrix}
-4 & 2 & 0 & 0 & 0 & 0 \\
2 & -4 & 2 & 0 & 0 & 0 \\
0 & 2 & -4 & 2 & 0 & 2 \\
0 & 0 & 2 & -4 & 2 & 0 \\
0 & 0 & 0 & 2 & -4 & 0 \\
0 & 0 & 2 & 0 & 0 & -4
\end{bmatrix}$$
The inverse matrix below allows us to compute the dual lattice $K^*$ and the discriminant group $A_K = K^*/K$. More specifically, we consider the linear combinations of $\alpha_1, \ldots, \alpha_6$ with coefficients given by the columns of $G_{E_6(2)}^{-1}$. Denote them by $\alpha_1^*, \ldots, \alpha_6^*$, and their image in $A_K$ by $[\alpha_i^*]$.

$$G_{E_6(2)}^{-1} := \begin{bmatrix}
\frac{2}{3} & -\frac{2}{3} & -1 & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{2}{3} & -\frac{2}{3} & -2 & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-1 & -2 & -3 & -2 & -1 & -\frac{1}{3} \\
-\frac{2}{3} & -\frac{2}{3} & -2 & -\frac{2}{3} & -\frac{1}{3} & -1 \\
-\frac{3}{2} & -\frac{3}{2} & -1 & -\frac{3}{2} & -\frac{1}{2} & -1 \\
\frac{1}{2} & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & -1
\end{bmatrix}$$

Notice that $[\alpha_1^*], [\alpha_2^*], [\alpha_3^*]$ and $[\alpha_6^*]$ all have order 6. Let $\beta := [\alpha_1^*]$; then $\langle \beta \rangle \cong \mathbb{Z}/6\mathbb{Z}$. We look for generators of $(\mathbb{Z}/2\mathbb{Z})^5$; order two elements not contained in $\langle \beta \rangle$.

We find the following generators:

$$\gamma_1 := 3\alpha_2^* = \left[\frac{\alpha_2^*}{3}\right]; \quad \gamma_4 := 3\alpha_5^* - \alpha_5^* = \left[\frac{\alpha_2^*}{3}\right];$$

$$\gamma_2 := \alpha_3^* = \left[\frac{\alpha_3^*}{3}\right]; \quad \gamma_5 := \alpha_6^* - 3\alpha_2^* - 3\alpha_5^* = \left[\frac{\alpha_2^*}{3}\right].$$

$$\gamma_3 := 3\alpha_4^* = \left[\frac{\alpha_4^*}{3}\right];$$

Thus $2\beta$ is a generator of $\mathbb{Z}/3\mathbb{Z}$, and we calculate that $q_\mathbb{Z}(2\beta) = -\frac{17}{9} \equiv -\frac{4}{3} \mod 2\mathbb{Z}$. Note that $q_{A_2}(\gamma) = -\frac{4}{3}$ for a generator $\gamma$ of $A_{A_2}$, and so we have shown that $N$ satisfies (1).

Next, we need to show that $q_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z})^{12} = -q_{D_{12}^+}(2)$. We see that $q_\mathbb{Z}(\gamma_1) = 1$. Let $v, w$ be a basis for $A_{U(2)}$ with $v^2 = w^2 = 0$. Then $\left[\frac{v}{2}\right], \left[\frac{w}{2}\right]$ are generators for $U(2)$ with

$$q_\mathbb{Z}\left(\frac{v}{2}\right) = q_\mathbb{Z}\left(\frac{w}{2}\right) = 0.$$

Let $e, f$ be a basis for $A_1 \oplus A_1(-1)$ with $e^2 = -2, f^2 = 2$ and $e \cdot f = 0$. Then $\left[\frac{e}{2}\right], \left[\frac{f}{2}\right]$ generate $A_1 \oplus A_1(-1) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with

$$q_\mathbb{Z}\left(\frac{e}{2}\right) = -\frac{1}{2}, \quad q_\mathbb{Z}\left(\frac{e + f}{2}\right) = 0.$$

We now look at the discriminant form of $D_{12}^+(2)$. Let $F_1, \ldots, F_{12}$ be a basis for $D_{12}^+(2)$ with Gram matrix:

$$G_{D_{12}^+(2)} := \begin{bmatrix}
-4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6
\end{bmatrix}$$
Again, the inverse matrix will provide a basis \( \{ F_i^* \}_{i=1}^{12} \) for the dual lattice, where each column is viewed as the coefficients for \( F_i^* \) written as a linear combination of \( F_1, \ldots, F_{12} \). This allows us to find generators for \( A_{D_{12}^+} = (\mathbb{Z}/2\mathbb{Z})^{12} \). Indeed, the image of \( [F_i^*] \) is a generator, given by the \( i \)-th column of the below matrix \( (G^{-1})_{D_{12}^+} \) modulo integer coefficients, which are equivalent to 0 in the discriminant group.

\[
G^{-1}_{D_{12}^+} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

We calculate the value of the discriminant form on each generator below.

\[
q_{D_{12}^+}([F_1^*]) = \frac{1}{2}, \quad q_{D_{12}^+}([F_2^*]) = 0, \quad q_{D_{12}^+}([F_3^*]) = 0, \quad q_{D_{12}^+}([F_4^*]) = 0, \quad q_{D_{12}^+}([F_5^*]) = 0, \quad q_{D_{12}^+}([F_6^*]) = 0, \quad q_{D_{12}^+}([F_7^*]) = 0, \quad q_{D_{12}^+}([F_8^*]) = 0, \quad q_{D_{12}^+}([F_9^*]) = 0, \quad q_{D_{12}^+}([F_{10}^*]) = 0, \quad q_{D_{12}^+}([F_{11}^*]) = 0, \quad q_{D_{12}^+}([F_{12}^*]) = 0
\]

Using the generators described above we see that \( qN|_H = -q_{D_{12}^+} \), where \( H = (\mathbb{Z}/2\mathbb{Z})^{12} \). Thus \( N \) satisfies all the conditions of Lemma 3.7 and so the existence of the embedding \( D_{12}^+ \) isomorphic to \( \Lambda_{OG10} \) follows. The existence of the involution follows by Lemma 3.5.

### 3.5. Proof of Theorem 3.1

We have exhibited involutions \( \iota \in O(\Lambda_{OG10}) \) acting trivially on \( A_{\Lambda_{OG10}} \) with \( (\Lambda_{OG10})_+ \) isomorphic to either \( E_8(2) \) or \( D_{12}^+ \). In order to conclude that both involutions are induced by symplectic birational involutions of a manifold \( X \) of \( OG10 \) type we must show that neither contain long or short roots.

**Lemma 3.10.** Let \( \iota \in O(\Lambda_{OG10}) \) be an involution and suppose that \( (\Lambda_{OG10})_- \) contains a long root. Then \( A_{(\Lambda_{OG10})_-} \) contains an element of order 3.

**Proof.** Let \( v \in (\Lambda_{OG10})_- \) be a long root; i.e \( v^2 = -6 \) and \( div_{\Lambda_{OG10}}(v) = 3 \). Then 3 divides the divisibility of \( v \) in \( (\Lambda_{OG10})_- \). We can write \( div_{\Lambda_{OG10}}(v) = 3k \) for some positive integer \( k \). Then \( [v^*] = \frac{v}{3k} \) defines a non-zero element of \( A_{(\Lambda_{OG10})_-} \); in particular, \( kv^* \) is a non-trivial element of order 3. \( \square \)
Proof of Theorem 3.1. The discriminant group of both $E_8(2)$ and $D_{12}(2)$ contains no elements of order three; by Lemma 3.10 neither contains any long roots. The maximal norm of both lattices is $-4$, and so they do not contain short roots. Thus in both cases $(\Lambda_{OG10})_-$ is a negative definite lattice with $(\Lambda_{OG10})_- \cap W^\text{pex}_{OG10} = \emptyset$; we have also shown these are the only possible negative definite coinvariant lattices for an involution $i \in O(\Lambda_{OG10})$ acting trivially on the discriminant group. By Theorem 2.8 the two involutions are induced by symplectic birational involutions of a manifold of $OG10$ type. The classification of the corresponding invariant sublattices follow from the proofs of Proposition 3.6 and 3.9. □

4. Involutions via Cubic Fourfolds

We have classified in the previous section possible symplectic birational involutions that act trivially on the discriminant group $\Lambda_{OG10}$. It remains to be seen whether geometric involutions can act non-trivially. In this section we provide a partial classification in terms of the invariant and coinvariant sublattices of $\Lambda_{OG10}$ for such an involution. More precisely, we will prove the following:

**Theorem 4.1.** Let $X$ be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \text{Bir}(X)$ a symplectic birational involution, such that the induced action $\eta_*(f)$ is non-trivial on the discriminant group of $\Lambda_{OG10}$. Assume further that $\text{rank}(\Lambda_{OG10})_- < 12$. Then one of the following holds:

1. $H^2(X, \mathbb{Z})_- \cong E_6(2)$, $H^2(X, \mathbb{Z})_+ \cong U^2 \oplus D_4^+$;
2. $H^2(X, \mathbb{Z})_- \cong M$, $H^2(X, \mathbb{Z})_+ \cong E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2$ where $M$ is the unique rank 10 lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24 \rangle$.

The key insight in proving this classification is to utilise Theorem 2.12; the classification of involutions of a cubic fourfold ([LPZ18], [LZ22], [Mar22]). Let us briefly outline the strategy. In §4.1 we first use a lattice theoretic argument to show the existence of two involutions acting as in Theorem 4.1, using the knowledge of involutions on the smaller lattice $A_2 \oplus E_8^2 \oplus U^2$, via Theorem 2.12. Next, we approach from a geometric point of view in §4.2, and consider induced symplectic birational involutions on the compactified intermediate Jacobian starting from a cubic fourfold with an involution. Next, we note that in all of the cases considered above the invariant sublattice contains a $U$ summand, i.e $(\Lambda_{OG10})_+ = \Gamma \oplus U$ for some lattice $\Gamma$. We suspect that this is always the case for a birational symplectic involution acting non-trivially on the discriminant group. In §4.3 we investigate lattice theoretic criteria for this to be satisfied, and in particular show it is always the case assuming that $\text{rank}(\Lambda_{OG10})_- < 12$.

4.1. Existence of symplectic involutions via cubic fourfolds. Recall that all $ADE$ lattices are assumed to be negative definite. In particular for a root lattice $R$, the lattice $R(-1)$ is positive definite.

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold. Recall that $H^4(V, \mathbb{Z}) \cong L(-1)$ where $L := A_2 \oplus E_8^2 \oplus U^2$; in particular

$$\Lambda_{OG10} \cong L \oplus U.$$ 

We will use the arithmetic classification of involutions of a cubic fourfold to prove the following theorem.
Theorem 4.2. There exists symplectic birational involutions \( f \in \text{Bir}(X) \) of an irreducible holomorphic symplectic manifold \( X \) of \( OG10 \) type with either:

1. \( H^2(X, Z)_- \cong E_6(2) \), \( H^2(X, Z)_+ \cong U^2 \oplus D_4^2 \);
2. \( H^2(X, Z)_- \cong M \), \( H^2(X, Z)_+ \cong E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2 \) where \( M \) is the unique rank 10 lattice obtained as an index 2 overlattice of \( D_9(2) \oplus (24) \).

Moreover, the induced involution of \( \Lambda_{OG10} \) act non-trivially on the discriminant group in both cases.

Proof. Consider an antisymplectic involution \( \phi \) of a cubic fourfold \( V \). The action on \( H^2(V, Z)_{prim} \cong L(-1) \) has been classified by Theorem 2.12; either \( L_- \cong E_8(2) \) or \( L_- \cong M \). We can extend the involution to an involution \( \iota \) of \( \Lambda_{OG10} \cong L \oplus U \), acting by the identity on the remaining copy of \( U \). Notice now that \( (\Lambda_{OG10})_- \cong L_- \) and \( (\Lambda_{OG10})_+ \cong L_+ \oplus U \).

In both cases, the coinvariant lattice \( (\Lambda_{OG10})_- \) is negative definite, and
\[
(\Lambda_{OG10})_- \cap W_{OG10}^{\text{pos}} = \emptyset.
\]
Indeed, we know \( L_- \) contains no short or long roots since it is the coinvariant lattice for an involution of a smooth cubic [Mar22]. Hence by Theorem 2.8, \( \iota \) is induced geometrically by a symplectic birational transformation \( f \in \text{Bir}(X) \) for some manifold \( X \) of \( OG10 \) type. Further we see that such an involution necessarily acts by \( -id \) on the discriminant group \( A_{OG10} \); if \( \iota \) acted trivially, then \( (\Lambda_{OG10})_- \) would be 2-elementary, a contradiction by Proposition 2.10. The classification of \( (\Lambda_{OG10})_+ \) in both cases follows. \( \square \)

It is worth stressing that the existence of such symplectic birational involutions of a manifold of \( OG10 \) type seems to be in direct contrast with both the \( OG6 \) and the \( K3^{[n]} \) cases. For manifolds of \( OG6 \) type, symplectic automorphisms act trivially on the second cohomology, and further birational symplectic transformations of finite order act trivially on the corresponding discriminant group [GOV20]. It also seems that symplectic automorphisms of manifolds of \( K3^{[n]} \) type also act trivially on the discriminant [Mon16].

4.2. Geometric Observations. We notice in the previous section that for both examples of birational symplectic involutions \( f \in \text{Bir}(X) \), the invariant lattice \( H^2(X, Z)_+ \cong (\Lambda_{OG10})_+ \) contains a \( U \) summand. The compactified intermediate Jacobians of cubic fourfolds are examples of manifolds of \( OG10 \) type with a \( U \) polarisation; it is natural to consider involutions that appear via this construction. In particular, we show that involutions of a cubic fourfold produce symplectic birational involutions of the associated compactified intermediate Jacobian with this property.

Theorem 4.3. Let \( X \) be an irreducible holomorphic symplectic manifold of \( OG10 \) type. Let \( f \in \text{Bir}(X) \) be a symplectic birational involution of \( X \), and suppose that \( H^2(X, Z)_+ \cong \Gamma \oplus U \) for some lattice \( \Gamma \). Then \( f \) induces an involution \( \phi \) of a smooth cubic fourfold \( V \subset \mathbb{P}^5 \). Conversely, an involution \( \phi \) of a smooth cubic fourfold \( V \) induces a birational symplectic involution on the compactified associated Intermediate Jacobian \( J_V \), that leaves a copy of \( U \) invariant.

Proof. Denote by \( \iota := \eta_+(f) \in O(\Lambda_{OG10}) \) the induced involution on \( \Lambda_{OG10} \), and denote by \( U_1 := U \) the fixed copy of the lattice \( U \). Fix a primitive embedding of the invariant lattice \( (\Lambda_{OG10})_+ \cong \Gamma \oplus U_1 \hookrightarrow \Lambda_{OG10} \). Denote by \( L = (U_1^+)_{\Lambda_{OG10}} \);
then \( L \) is an even, indefinite lattice with signature \((2, 20)\) and discriminant group \( A_L \cong \mathbb{Z}/3\mathbb{Z} \cong A_2 \). By [Nik79, Cor. 1.13.3], \( L \) is unique; thus we see that
\[
L \cong U^2 \oplus E_8^2 \oplus A_2.
\]

Since \( \iota \) acts as the identity on \( \Gamma \oplus U_1 \), \( \iota \) restricts to an isometry of \( L \) with \( L_+ \cong \Gamma \) and \( L_- \cong (\Lambda_{OG10})_- \). Note that \( (\Lambda_{OG10})_- \) is negative definite of rank \( r \leq 20 \), and \( \Gamma \) has signature \((2, 20 - r)\). We can choose a hodge structure \( H \) on \( L \) of type \((0, 1, 20, 1, 0)\) such that \( H^{2,2} \cap L = (\Lambda_{OG10})_- \). By assumption, \( (\Lambda_{OG10})_- \) contains no prime exceptional vectors (Prop. 2.7); in particular it contains no long or short roots. The Global Torelli Theorem for cubic fourfolds implies that there exists a smooth cubic fourfold \( V \) with \( H^4(V, \mathbb{Z})_{prim} \cong H(-1) \) as Hodge structures.

Let \( \eta_V \in H^4(V, \mathbb{Z}) \) be the square of the hyperplane class. We wish to extend \( \iota \) to an isometry of \( H^4(V, \mathbb{Z}) \) fixing \( \eta_V \). We have \( L(-1) \oplus \langle \eta_V \rangle \subset H^4(V, \mathbb{Z}) \); in order to do this, \( \iota \) must act trivially on \( A_{L(-1)} \cong \mathbb{Z}/3\mathbb{Z} \cong A_{(\eta_V)} \). Note that \( O(A_{L(-1)}) \cong \mathbb{Z}/2\mathbb{Z} \); thus \( \iota \) can act as \( \pm id_{A_{L(-1)}} \).

Set \( \sigma := -\iota \) if \( \iota \) acts by \( -id_{A_{L(-1)}} \), and \( \sigma = \iota \) otherwise. Then \( \sigma \oplus id_{(\eta_V)} \) extends to an isometry of \( H^4(V, \mathbb{Z}) \); let us denote this also by \( \sigma \). Thus \( \sigma \in \text{Aut}_{HS}(V, \eta_V) \), and by the Strong Global Torelli theorem there exists a unique automorphism \( \phi \in \text{Aut}(V) \) such that \( \sigma = \phi^* \).

Conversely, suppose we have an involution \( \phi \in \text{Aut}(V) \) of a smooth cubic fourfold \( V \subset \mathbb{P}^5 \); let \( \sigma \) := \( \phi^* \) be the induced involution on \( H^4(V, \mathbb{Z})_{prim} \). By ([LSV17], [Sac21]), we can associate to \( V \) an irreducible holomorphic symplectic manifold \( \mathcal{J}_V \) of OG10 type, with a Lagrangian fibration \( \pi : \mathcal{J}_V \to \mathbb{P}^5 \) that compactifies the intermediate Jacobian fibration of \( V \). Note that the compactification \( \mathcal{J}_V \) is not unique; the cubic fourfold \( V \) is a special cubic fourfold containing either a plane or a cubic scroll [Mar22], and so by [Sac21] may have many birational compactifications. Let \( \Theta \) denote the relative theta-divisor of \( \mathcal{J}_V \); then the sublattice \( \langle \Theta, \pi^*\mathcal{O}(1) \rangle \) is isomorphic to the hyperbolic lattice \( U \) [Sac21, Lemma 3.5, K.Hulek, R.Laza].

To obtain an involution of \( \mathcal{J}_V \), we follow [Sac21, Sect. 3.1]. The automorphism \( \phi \in \text{Aut}(V) \) acts on the universal family of hyperplane sections of \( V \), and thus on the Donagi-Markman fibration \( \mathcal{J}_U \to U \), where \( U \subset (\mathbb{P}^5)^* \) parametrises smooth hyperplane sections of \( V \) [Sac21, Section 3.1]. We thus obtain in this way a birational transformation \( f : \mathcal{J}_V \to \mathcal{J}_U \), that leaves the sublattice \( \langle \theta, \pi^*\mathcal{O}(1) \rangle \cong U \) invariant. If \( \phi \in \text{Aut}(V) \) is symplectic, then the induced birational involution \( f \in \text{Bir}(\mathcal{J}_V) \) is symplectic, by [Sac21, Lemma 3.2]. If not, there exists a regular anti-symplectic involution \( \tau \in \text{Aut}(\mathcal{J}_V) \) given geometrically by sending \( x \mapsto -x \) on the fibers of \( \mathcal{J}_V \to \mathbb{P}^5 \). Then \( \tau \circ f \) is a non-trivial birational symplectic involution of \( \mathcal{J}_V \). Set \( \tilde{f} := f \) if \( f \) is symplectic, \( \tilde{f} := \tau \circ f \) otherwise. Note that \( \tilde{f} \) leaves \( \langle \theta, \pi^*\mathcal{O}(1) \rangle \) invariant in both cases.

Finally, note that if \( \phi \) and thus \( f \) is anti-symplectic, then the birational symplectic involution \( f \) acts by \( -id_{\Lambda_{OG10}} \) on the discriminant group of \( \Lambda_{OG10} \).

\[ \square \]

Remark 4.4. The proof of the previous theorem classifies the invariant and coinvariant lattices for a symplectic birational involution such that \( H^2(X, \mathbb{Z})_+ \cong \Gamma \oplus U \); we see that \( \Gamma \) necessarily is either the coinvariant or invariant sublattice for the induced involution \( \phi \) on \( H^4(V, \mathbb{Z}) \). Moreover, such involutions exist by Theorem 4.2. To complete the proof of Theorem 4.1, it remains to show the assumption \( \text{rank}(\Lambda_{OG10})_- < 12 \) implies that \( (\Lambda_{OG10})_+ \) contains a \( U \) summand.
4.3. Criteria for splitting a $U$ summand. The aim of this subsection is to identify a lattice theoretic criteria to complete the proof of Theorem 4.1.

Assume that $f \in \text{Bir}(X)$ is a symplectic birational involution of a manifold $X$ of $OG10$-type, such that $\iota := \eta_*(f) \in O(\Lambda_{OG10})$ is an involution that acts by $-\text{id}$ on $A_{\Lambda_{OG10}}$. By Lemma 2.7, $(\Lambda_{OG10})_-$ is negative definite of rank $1 \leq r \leq 21$, and $(\Lambda_{OG10})_- \cap W_M = \emptyset$. By Proposition 2.10, $(\Lambda_{OG10})_+$ is a 2-elementary lattice. Thus $A_{\Lambda_{OG10}} = (\mathbb{Z}/2\mathbb{Z})^a$, and $(\Lambda_{OG10})_+$ is determined by the invariants $((3, 21 - r), a, \delta)$. Notice that since $(\Lambda_{OG10})_+$ is indefinite, its isomorphism class is unique in its genus. By Nikulin’s classification of 2-elementary lattices (Nikulin, see [Dol83]), the following hold:

1. $a \leq \min\{24 - r, r\}$
2. $a \equiv r \pmod{2}$,
3. $r \equiv 2 \pmod{4}$ if $\delta = 0$,
4. $\delta = 0, r \equiv 2 \pmod{8}$ if $a = 0$,
5. $r \equiv 3 \pmod{8}$ if $a = 1$,
6. $\delta = 0$ if $a = 2, r \equiv 6 \pmod{8}$,
7. $r \equiv 2 \pmod{8}$ if $\delta = 0, a = 24 - r$.

Using the above consequence, we establish a numerical criteria for $(\Lambda_{OG10})_+$ to split of a $U$ summand.

Lemma 4.5. Let $r = \text{rank}(\Lambda_{OG10})_-$, $a, \delta$ as above. Then $(\Lambda_{OG10})_+$ splits of a $U$ summand if and only if:

1. $r \leq 20$, and $a \leq 22 - r$,
2. If $a = 22 - r$ and $\delta = 0$, then $r \equiv 2 \pmod{8}$

Proof. Assume $(\Lambda_{OG10})_+$ splits of a $U$ summand, i.e $(\Lambda_{OG10})_N \oplus U$. Applying Nikulin’s classification of 2-elementary lattices to the lattice $N$ with invariants $(2, 20 - r), a, \delta$, we see the above conditions are necessary for the existence of such a lattice $N$. Conversely, assume the conditions in the theorem hold. Then again by the classification, there exists a 2-elementary lattice $N$ with invariants $(2, 20 - r), a, \delta$. Then $N \oplus U$ has the same invariants as $(\Lambda_{OG10})_+$, and thus are in the same genus. Since $(\Lambda_{OG10})_+$ is indefinite, it is unique and the claim holds.

Corollary 4.6. Let $X$ be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \text{Bir}(X)$ a birational involution of $X$ acting by $-\text{id}_{\Lambda_{OG10}}$ and such that the covariant lattice $(\Lambda_{OG10})_-$ has rank $r < 12$. Then $f$ induces an involution of a smooth cubic fourfold as in Theorem 4.3

Proof. Since $r < 12$, then by assumption $a \leq r \leq 22 - r$ and the conditions of Lemma 4.5 are satisfied. Thus the invariant sublattice $(\Lambda_{OG10})_+ \cong M \oplus U$ for some even lattice $M$, and the conditions of Theorem 4.3 are satisfied.

This completes the proof of Theorem 4.1. To get a complete classification, it remains to be seen whether the condition rank$(\Lambda_{OG10})_- < 12$ always holds for a birational symplectic involution acting non-trivially on $A_{\Lambda_{OG10}}$.

5. Remaining Cases

Let $\iota \in O(\Lambda_{OG10})$ be an involution acting non-trivially on the discriminant such that $(\Lambda_{OG10})_-$ is negative definite. In this section we will assume $(\Lambda_{OG10})_+$ fails to split a $U$ summand, we shall see that this occurs in only 3 cases. We believe that
these cases do not occur geometrically; we speculate that the coinvariant lattice for
these actions contains either a short or long root. This remains to be seen - the
difficulty lies in the fact that the lattice \((\Lambda_{OG10})_-\) is no longer unique in its genus.
Due to the high rank and discriminant, classification of the lattices in the genera
that appear seems to be a difficult task.

**Proposition 5.1.** Let \( \nu \in O(\Lambda_{OG10}) \) acting non-trivially on \( A_{OG10} \). Assume that
\((\Lambda_{OG10})_+\) does not split a \(U\) summand. Let \( r := \text{rank}(\Lambda_{OG10})_-\). Then one of the
following holds:

1. \((\Lambda_{OG10})_+ \cong U(2)^3 \) and \( r = 18\);
2. \((\Lambda_{OG10})_+ \cong U(2)^3 \oplus D_4\) and \( r = 14\);
3. \((\Lambda_{OG10})_+ \cong (2)^3 \oplus (-2)^{21-r}\) and \( r \geq 12\).

**Proof.** For ease of notation, let \( M := (\Lambda_{OG10})_+\). Since \( M \) does not split of a \(U\)
summand, we have that \( r \geq 12\). Assume first that \( r \not\equiv 2 \mod 8\), then \( M \) is an indefinite,
2-elementary lattice and is classified uniquely by the invariants \((r, a, \delta)\). There are
two cases to consider by Lemma 4.5: either \(22 - r < a\), or \(a = 22 - r\), \(\delta = 0\) and \(r \not\equiv 2 \mod 8\).

**Case 1:** Assume that \(22 - r < a\); we necessarily have that \(22 - r < a \leq 24 - r\).
Since \( M \) is 2-elementary, we have that \(a \equiv r \mod 2\); we can exclude \(a = 23 - r\).
Thus \(a = 24 - r = rk(M)\). The lattice \(N := M(1/2)\) is well defined [Mar22,Lemma
A.7]. Further, \(A_N = \{1\}\), and so \(N\) is unimodular.

Assume that \(\delta = 0\); this implies that \(N\) is an even unimodular lattice (see for example [Mar22, Lemma A.9]). By Milnor’s theorem on unimodular forms (see
[Nik79, Thm 0.2.1] for a precise statement), \(N\) exists if and only if

\[
3 + r - 21 \equiv 0 \mod 8;
\]
\[r \equiv 2 \mod 8.
\]

Since \(r \geq 12\), we have that \(r = 18\). Thus \(N\) has signature \((3, 3)\), and hence \(N \cong U^3\).
Thus \(M \cong U(2)^3\).

Now assume that \(\delta = 1\). It follows that \(N\) is an odd indefinite unimodular lattice
(see for example [Mar22, Lemma A.8]). By Milnor’s theorem again, \(N\) exists and
is isomorphic to \(\langle 1 \rangle^3 \oplus \langle -1 \rangle^{21-r}\), thus

\[
M \cong \langle 2 \rangle^3 \oplus \langle -2 \rangle^{21-r}.
\]

**Case 2:** Assume that \(a = 22 - r\), with \(\delta = 0\) and \(r \not\equiv 2 \mod 8\). Note again that
\(r \geq 12\); if \(r \leq 11\), since \(22 - r = a \leq r \leq 22 - r\), we must have that \(r = 11\). But
since \(\delta = 0\), for \(M\) to exist \(r \equiv 2 \mod 4\), a contradiction.

So \(r \geq 12\), and since \(r \equiv 2 \mod 4\), \(r \in \{14, 18\}\). By assumption, \(r \not\equiv 2 \mod 8\),
thus \(r = 14\). Hence \(M\) has signature \((3, 7)\) with \(a = 8, \delta = 0\). Consider the lattice
\(U(2)^3 \oplus D_4\); it has the same signature and invariants. Since indefinite 2-elementary
lattices are unique up to isomorphism, we necessarily have \(M \cong U(2)^3 \oplus D_4\).

Finally, assume that \(r = 21\). In this case, \(M\) has signature \((3, 0)\). Since \(a \equiv r \mod 2\), \(a = 1\) or \(3\). If \(a = 1\), no such lattice exists by (Nikulin, see [Dol83]) ;
thus \(a = 3\). Again, the lattice \(N := M(1/2)\) is well defined. Further, \(A_N = \{1\}\), so
\(N\) is unimodular. If \(\delta = 0\), \(N\) is an even unimodular lattice: once more, Milnor’s
Theorem on unimodular forms gives an immediate contradiction with the rank of \(N\).
Thus \(\delta = 1\) and \(N\) is an odd unimodular lattice, thus \(N \cong \langle 1 \rangle^3\), and \(M \cong \langle 2 \rangle^3\). \(\square\)
In order to conclude our classification of symplectic involutions of \( OG_{10} \) type, it remains to be seen whether an involution \( \iota \in O(\Lambda_{OG_{10}}) \) as in Proposition 5.1 is induced by a birational symplectic involution. By Theorem 2.8, we need to see whether \( (\Lambda_{OG_{10}})^+ = (\Lambda_{OG_{10}})^\perp_+ \) contains any short or long roots. One possible strategy is to classify the possibilities for the lattices \( (\Lambda_{OG_{10}})^- \). Unfortunately, these lattices are no longer unique in their genus, and have both large rank and discriminant (the methods of Conway-Sloane have not been extended [CS88]). We illustrate this difficulty with an example.

**Example 5.2.** Consider \( (\Lambda_{OG_{10}})^+ \cong U(2)^3 \). Then \( (\Lambda_{OG_{10}})^- \) has rank 18, and discriminant group
\[
A_{(\Lambda_{OG_{10}})^-} \cong \mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6,
\]
and \( q(\Lambda_{OG_{10}}^-)|_{\mathbb{Z}/3\mathbb{Z}} = q_{A_2} \). There are at least two possibilities for \( (\Lambda_{OG_{10}})^- \):
\[
\begin{align*}
A_2 & \oplus K; \\
A_2 & \oplus E_8 \oplus N,
\end{align*}
\]
where \( K \) is the Kummer lattice and \( N \) is the Nikulin lattice (see [Mor84] for a description of these lattices). Both of these embed into the lattice \( \Lambda_{OG_{10}} \) and are orthogonal to \( U(2)^3 \). Although both examples contain short roots and thus cannot be realised by a geometric birational involution, there may be other lattices in the same genus without short or long roots.

**Example 5.3.** Consider \( (\Lambda_{OG_{10}})^+ \cong U(2)^3 \oplus D_4 \). Then \( (\Lambda_{OG_{10}})^- \) has rank 14, and discriminant group
\[
A_{(\Lambda_{OG_{10}})^-} \cong \mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^8.
\]
Thus \( (\Lambda_{OG_{10}})^- \) is in the same genera as the lattice \( A_2 \oplus N \oplus D_4 \). Again this example contains short roots.

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