On the Rate-Memory Tradeoff of D2D Coded Caching with Three Users

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Abstract

The device-to-device (D2D) centralized coded caching problem is studied for the three-user scenario, where two models are considered. One is the 3-user D2D coded caching model proposed by Ji et al, and the other is a simpler model named the 3-user D2D coded caching with two random requesters and one sender (2RR1S), proposed in this paper, where in the delivery phase, any two of the three users will make file requests, and the user that does not make any file request is the designated sender. We allow for coded cache placement and none one-shot delivery schemes. We first find the optimal caching and delivery schemes for the model of the 3-user D2D coded caching with 2RR1S for any number of files. Next, we propose a new caching and delivery scheme for the 3-user D2D coded caching problem using the optimal scheme of the 3-user D2D coded caching with 2RR1S as a base scheme. The new caching and delivery scheme proposed employs coded cache placement and when the number of files is equal to 2 and the cache size is medium, it outperforms existing schemes which focus on uncoded cache placement. We further characterize the optimal rate-memory tradeoff for the 3-user D2D coded caching problem when the number of files is equal to 2. As a result, we show that the new caching and delivery scheme proposed is in fact optimal when the cache size is in the medium range.

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I. Introduction

The applications of wireless networks have developed from traditional real-time voice communication to multimedia transmissions such as video, virtual/augmented reality game, high definition map etc., which requires the throughput of each user to increase by nearly 1000 times [1]. Fortunately, such content can be pre-stored into the user’s storage during periods of low network utilization, thus avoiding network congestion during peak hours. This technology is known as caching [2], [3]. The caching process is typically divided into two phases [5]. The placement phase happens during the off-peak hours, where the server fills the users’ caches before the users request any content, while the delivery phase represents the transmission stage of the server when the users reveal their demands during peak hours. Caching technology has developed rapidly in recent years, and it is currently considered as one of the effective solutions to relieve the load pressure of wireless networks.

In traditional caching, the users cache the most likely requested contents, and the server transmits the uncached portions of the files requested by the users. Both the cached contents of the users and the transmitted signal of the server are uncoded. Contrary to traditional caching, Maddah-Ali and Niesen proposed an idea [5] of combining coded multi-casting and device caching to satisfy multiple uni-cast demands simultaneously through coded multi-cast transmissions, which is known as coded caching. The coded caching problem allows both coded cache contents of the users and coded transmission from the server. The goal is to design a caching and delivery scheme such that the worst-case delivery rate is the smallest, where “worst-case” refers to the largest delivery rate among all possible request demands of the users. When the optimal caching and delivery scheme that achieves the smallest worst-case delivery rate can be identified for any cache size of the users, the optimal rate-memory tradeoff is found for the system. If each user directly stores a subset of the files’ bits without coding, then the cache placement scheme is called uncoded, otherwise it is called coded. The coded caching problem studied in [5] is of a
centralized nature, where it is assumed that the set of users present during the placement phase will each request a file at the beginning of the delivery phase. Decentralized coded caching has been studied in [7], where considering the possibility that the users present during the placement phase may leave or turn off during the delivery phase, less coordinated caching strategies are studied.

To further reduce the traffic load of the server at peak hours, Ji et al [4] propose a framework for device-to-device (D2D) coded caching. During the placement phase, similar to coded caching [5], the server fills the users’ caches before the users request any content. During the delivery phase, when the users reveal their demands, the server is inactive and it is up to the users to transmit signals among themselves so that each user can decode its requested file based on the transmitted signals of the other users and its local cache content. For the centralized D2D coded caching problem, [4] used the caching strategy of [5, Algorithm 1], which is uncoded, and devised a novel delivery scheme fit for the D2D scenario. Furthermore, a widely recognized D2D caching converse was proposed in [4], and it has been shown that the proposed D2D caching and delivery scheme is order optimal within a constant factor when the memory size is large. However, the optimal caching and delivery scheme and the corresponding optimal rate-memory tradeoff for the centralized D2D coded caching problem remains open.

The optimal caching and delivery scheme for the centralized D2D coded caching problem was characterized in [8] under the assumption that the cache placement and delivery are constrained to be uncoded and one-shot, respectively. One-shot delivery schemes satisfy the condition that each user can decode any bit of its requested file from its own cache and the transmitted signal from at most one user. It has been shown in [8] that one-shot delivery schemes are optimal within a factor of 2 under the constraint of uncoded cache placement, and uncoded cache placement and one-shot delivery schemes are optimal within a factor of 4 compared to general D2D coded caching schemes.

In addition to [4] and [8], there are many other researches for the D2D coded caching problem,
such as distinct cache sizes [9], private caching [10], private caching with a trusted server [11], [12], secure coded caching [13], secure delivery [14], finite file packetizations [15], wireless multi-hop D2D networks [16], [17], partially cooperative D2D communication networks [18] and so on. Among these papers, there are few that studies the fundamental limits of centralized D2D coded caching allowing coded placement since the nature of the problem is complex and therefore difficult to solve.

In this paper, we study the centralized D2D coded caching problem for three users, i.e., coded cache placement and none one-shot delivery schemes are allowed. We would like to find better or even the optimal caching and delivery schemes and therefore understand what “coded” cache placement and “none one-shot” delivery schemes can buy us in terms of reducing the worst-case delivery rate.

Towards this goal, we first propose a new model in this paper, called the 3-user D2D coded caching with two random requesters and one sender (2RR1S), where during the delivery phase, any two out of the three users will request a file while the user that is not requesting any files is the designated sender. For this simpler model, we characterize the fundamental performance limits, i.e., the optimal rate-memory tradeoff, and the optimal caching and delivery scheme, which requires coded cache placement. Next, we identify the relationship between the 3-user D2D coded caching problem and the 3-user D2D coded caching with 2RR1S. More specifically, we show that a caching and delivery scheme for the 3-user D2D coded caching with 2RR1S can serve as a base scheme for the 3-user D2D coded caching problem. As a result, we propose a novel caching and delivery scheme which employs coded cache placement, for the 3-user D2D coded caching problem. We show that proposed scheme outperforms existing schemes when the number of files is 2 and the cache size is in the medium range. Finally, we characterize the optimal performance limits for the 3-user D2D coded caching problem for the case of 2 files. We find that the caching and delivery scheme we propose is in fact optimal for a certain range of cache sizes. Comparing our results with existing results on the 3-user D2D coded caching
problem, we characterize the amount of performance gain enabled by allowing coded cache placement. Our proposed schemes are of the one-shot nature, and therefore the performance improvement by allowing none one-shot delivery schemes are still not well understood and needs to be further studied.

A. Notations

Throughout this paper, $|\cdot|$ represents the cardinality of a set, $\oplus$ denotes finite field addition, we let $\mathcal{X} \setminus \mathcal{Y} = \{ x \in \mathcal{X} | x \not\in \mathcal{Y} \}$, $[x : y : z] = \{ x, x+y, x+2y, \ldots, z \}$, $[x : y] = [x : 1 : y]$ and $[n] = [1 : n]$. We use $H(X)$ to denote the entropy of the random variable $X$.

II. System Model

A. 3-user D2D coded caching

We first introduce the 3-user D2D coded caching model, which is the same as the ones studied in [4] and [8]. There is a database consisting of $N$ independent files, $W_1, \ldots, W_N$. Each file consists of $F$ bits, i.e.,

$$H(W_1) = H(W_2) = \cdots = H(W_N) = F,$$

$$H(W_1, W_2, \ldots, W_N) = H(W_1) + H(W_2) + \cdots + H(W_N).$$

To simplify the notation, without loss of generality, we normalize the file size of $F = 1$ for the rest of the paper. We set $K = 3$ users in the system, where each user has a cache of size $MF$ bits, $M \leq N$. The system operates in two phases: the cache placement phase, where the cache of the three users are filled without the knowledge of each user’s request, and the delivery phase, where upon receiving the three requests of the users, denote the file-request vector as $D \triangleq (d_1, d_2, d_3)$, $d_1, d_2, d_3 \in [N]$, each user sends a signal that is received without error by the other two users. The goal is to design a caching and delivery scheme so that each user can
correctly decode the message of interest based on its own cached content and the signals received from the other two users. The schematic diagram of the system model is shown in Fig. 1.

![System model](image)

**Fig. 1.** System model for the 3-user D2D coded caching problem. Solid and dotted lines indicate the placement and delivery phases, respectively.

More specifically, a caching and delivery scheme for this system consists of:

1) three caching functions

\[ \varphi_k : [2^F]^N \rightarrow [2^{MF}], \quad k = 1, 2, 3, \]

which maps the \( N \) files into the cached contents of the users, denoted by \( Z_k = \varphi_k(W_1, ..., W_N), k = 1, 2, 3. \) Thus, we have

\[ H(Z_1, Z_2, Z_3|W_{[N]}) = 0. \quad (1) \]

2) \( 3N^3 \) encoding functions

\[ \phi_k^D : [2^{MF}] \rightarrow [2^{R_k^D(M)F}], \quad k = 1, 2, 3, \]

where based on the request vector \( D = (d_1, d_2, d_3), \) User \( k \) generates a transmitted signal
\( X^D_k \overset{\Delta}{=} \phi^D_k(Z_k) \), consisting of \( FR^D_k(M) \) bits. Thus, we have

\[
H(X^D_k | Z_k) = 0. \quad (2)
\]

3) \( 3N^3 \) decoding functions

\[
\psi^D_k : [2^{MF}] \times [2^F \sum_{u \in [3] \setminus \{k\}} R^D_u(M)] \rightarrow [2^F], \quad k = 1, 2, 3,
\]

and the decoded message is denoted as \( \hat{W}_{d_k} \overset{\Delta}{=} \psi^D_k(\{X^D_u : u \in [3] \setminus \{k\}\}, Z_k) \).

It is required that the caching and delivery scheme must satisfy the constraint of correct decoding, i.e.,

\[
H \left( W_{d_1} | Z_1, X_2^{(d_1,d_2,d_3)}, X_3^{(d_1,d_2,d_3)} \right) = 0,
\]

\[
H \left( W_{d_2} | Z_2, X_1^{(d_1,d_2,d_3)}, X_3^{(d_1,d_2,d_3)} \right) = 0,
\]

\[
H \left( W_{d_3} | Z_3, X_1^{(d_1,d_2,d_3)}, X_2^{(d_1,d_2,d_3)} \right) = 0,
\]

which is called the decodability constraint. Combining (2) and (3), one can decode any file by knowing the cache of all users, which implies that we are interested in the case where \( 3M \geq N \) and

\[
H(W_{[N]} | Z_1, Z_2, Z_3) = 0.
\]

For a caching and delivery scheme, define \( R(M) \) as the worst-case delivery rate of all users, i.e.,

\[
R(M) = \max_D R^D_1(M) + R^D_2(M) + R^D_3(M).
\]

The minimum achievable worst-case rate is given by

\[
R^*(M) \overset{\Delta}{=} \inf R(M),
\]

where the infimum is taken over all possible caching and delivery schemes that satisfies the
decoding constraint (3).

**B. 3-user D2D coded caching with two random requesters and one sender (2RR1S)**

Next, we introduce a new model proposed in this paper, called the 3-user D2D coded caching with two random requesters and one sender (2RR1S). The discussion of the relationship between this model and the 3-user D2D coded caching model described in the previous subsection, will be given in Section II-C. The model of the 3-user D2D coded caching with 2RR1S is as follows:

Similar to the 3-user D2D coded caching problem described in the previous subsection, there are \( K = 3 \) users in the system, each with a cache of size \( MF \) bits, \( M \leq N \). The system operates in two phases. In the placement phase, each user’s cache is filled with a function of the \( N \) files, where we denote the content in the cache of User \( n \) as \( Z_n, n = 1, 2, 3 \). What is different from the 3-user D2D coded caching problem described in the previous subsection is that, in the delivery phase, any 2 out of the 3 users will make a file request, and the file requests are known to all 3 users. The user who does not make the file request will send a signal \( X_D \), where \( D \) denotes the request triple. The signal \( X_D \) is received correctly by the two users with file requests, and it is required that each of these two users can decode its requested file using the signal received and its own cache content. We say that the request vector \( D = (0, d_2, d_3) \) when Users 2 and 3 request Files \( d_2 \) and \( d_3 \), respectively, and User 1 does not request anything and is the designated sender. Similarly, the request vector \( D \) can take the values of \( D = (d_1, 0, d_3) \) and \( D = (d_1, d_2, 0) \), \( d_1, d_2, d_3 \in [N] \).

More specifically, a caching and delivery scheme for this system model consists of

1) three caching functions

\[
\varphi_k : [2^F]^N \rightarrow [2^{MF}], \quad k = 1, 2, 3,
\]

which maps the \( N \) files into the cached contents of the users, denoted by \( Z_k = \varphi_k(W_1, ..., W_N), \quad k = 1, 2, 3 \). Since the cached contents are deterministic functions of the files, we again have
2) $3N^2$ encoding functions

$$
\phi^D : [2^M] \rightarrow [2^{R^D(M)F}],
$$
i.e., the encoding function $\phi^D$ denotes the mapping from the cached content of the sender to the signal sent by the sender, and this mapping is a function of the file requests of the other two users. We use $X^{(0,d_2,d_3)}$ to denote the signal sent by User 1, when Users 2 and 3 are requesting files $d_2$ and $d_3$, respectively, i.e., $X^{(0,d_2,d_3)} = \phi^{(0,d_2,d_3)}(Z_1)$. Similarly, we define $X^{(d_1,0,d_3)}$ and $X^{(d_1,d_2,0)}$, $d_1, d_2, d_3 \in [N]$. The signal transmitted by the sender for file request vector $D$ consists of $R^D(M)F$ bits, where $M$ is the cache size of the three users. Thus, we have

$$
H(X^{(0,d_2,d_3)}|Z_1) = 0, \quad H(X^{(d_1,0,d_3)}|Z_2) = 0, \quad H(X^{(d_1,d_2,0)}|Z_3) = 0. \quad (4)
$$

3) $6N^2$ decoding functions

$$
\psi_k^D : [2^M] \times [2^{R^D(M)F}] \rightarrow [2^F], \quad k \in \{i| \text{ the } i\text{-th element of } D \neq 0\},
$$
which is the decoding function used at User $k$, when the request vector is $D$.

It is required that the caching and delivery scheme enables correct decoding at the users requesting files, i.e.,

$$
H(W_{d_2}|Z_2, X^{(0,d_2,d_3)}) = 0, \quad H(W_{d_3}|Z_3, X^{(0,d_2,d_3)}) = 0,
$$

$$
H(W_{d_1}|Z_1, X^{(d_1,0,d_3)}) = 0, \quad H(W_{d_3}|Z_3, X^{(d_1,0,d_3)}) = 0, \quad (5)
$$

$$
H(W_{d_1}|Z_1, X^{(d_1,d_2,0)}) = 0, \quad H(W_{d_2}|Z_2, X^{(d_1,d_2,0)}) = 0,
$$
which is called the decodability constraint. We see from (5) that for the decodability constraint to be satisfied, the contents of the cache of any two users must be able to fully recover all $N$
messages, i.e.,

$$H(W_{[1:N]}|Z_{\mathcal{I}}) = 0, \quad \mathcal{I} = \{1, 2\}, \{2, 3\}, \{3, 1\}. \quad (6)$$

which means that we must have \(2M \geq N\).

Since in the delivery phase, any two of the three users may make file requests and the delivery needs to be done by the third user through a common link between itself and the two users, we call this problem the \textit{3-user D2D coded caching with two random requesters and one sender (2RR1S)}. The schematic diagram of the system model is shown in Fig. 2.

![System Model Diagram](image)

Fig. 2. System model for the 3-user D2D coded caching with two random requesters and one sender. In this realization, User 2 does not request and therefore is the designated sender. Solid and dotted lines indicate the placement and delivery phases, respectively.

For a given caching and delivery scheme that satisfies the decodability constraint (5), the performance metric of interest is the worst-case delivery rate, i.e., \(R(M) \triangleq \max_D R^D(M)\), where \(R^D(M)\) is the number of symbols transmitted to satisfy demand \(D\). The minimum achievable worst-case rate is given by

$$R^*(M) \triangleq \inf R(M),$$

where the infimum is taken over all possible caching and delivery schemes that satisfy (5).
C. Relationship between the two models

The motivation for studying the model of the 3-user D2D coded caching with 2RR1S is: 1) it is a simpler model in terms of finding the minimum achievable worst-case rate $R^\star(M)$, as will be shown in Section III. 2) its achievable scheme can be exploited as a base scheme to come up with an achievable scheme for the 3-user D2D coded caching problem. This can be seen as follows: take a caching and delivery scheme for the 3-user D2D coded caching with 2RR1S that satisfies (5), denoted as Scheme $A$, and suppose it requires the split of each file into $L$ subfiles, i.e., $W_n = (W_{n,1}, W_{n,2}, \cdots, W_{n,L}), n \in [N]$. Then, further split each subfile into two parts of equal sizes, denoted as Part $(a)$ and Part $(b)$, respectively, i.e., $W_{n,l} = (W_{n,l}^{(a)}, W_{n,l}^{(b)}), l \in [L], n \in [N]$. The caching scheme for the 3-user D2D coded caching problem is the same as that of Scheme $A$. In terms of the delivery scheme, when the file request is $D = (d_1, d_2, d_3)$, User 1 uses the delivery scheme of scheme $A$ for the file request $(0, d_2, d_3)$, acting on Part $(a)$ of each file, User 2 uses the delivery scheme of Scheme $A$ for the file request $(d_1, 0, d_3)$, acting on Part $(a)$ of File $W_{d_1}$ and Part $(b)$ of the other files, and User 3 uses the delivery scheme of scheme $A$ for the file request $(d_1, d_2, 0)$, acting on Part $(b)$ of each file. As a result, the delivery rate for demand $D$ is $\frac{1}{2} \left( R^{(0,d_2,d_3)}(M) + R^{(d_1,0,d_3)}(M) + R^{(d_1,d_2,0)}(M) \right)$, where $R^{(0,d_2,d_3)}(M), R^{(d_1,0,d_3)}(M), R^{(d_1,d_2,0)}(M)$ denotes the delivery rates of Scheme $A$.

As an example, take a caching and delivery scheme for the 3-user D2D coded caching with 2RR1S when $M = \frac{1}{2}N$ as follows: split all files into two subfiles of equal size, denoted as $W_n = (W_{n,1}, W_{n,2})_{n=1}^N$, i.e., $L = 2$. The caching scheme is $Z_1 = (W_{n,1} \oplus W_{n,2})_{n=1}^N$, $Z_2 = (W_{n,1})_{n=1}^N$, $Z_3 = (W_{n,2})_{n=1}^N$. The delivery scheme is $X^{(0,d_2,d_3)} = \{W_{d_2,1} \oplus W_{d_2,2}, W_{d_3,1} \oplus W_{d_3,2}\}$, $X^{(d_1,0,d_3)} = \{W_{d_1,1}, W_{d_3,1}\}$, $X^{(d_1,d_2,0)} = \{W_{d_1,2}, W_{d_2,2}\}$. It can be checked that each user’s demand can be correctly decoded for the 3-user D2D coded caching with 2RR1S, and the delivery rate of $R^D(M) = 1$ for any demand vector $D$.

The above scheme can be used as a base scheme to come up with an achievable scheme for
the 3-user D2D coded caching problem as follows: further split each subfile into two equal parts, i.e., \( W_n = (W_{n,l}^{(a)}; W_{n,l}^{(b)}), n \in [N], l \in [2] \). The caching scheme is the same as the above scheme, i.e., \( Z_1 = (W_{n,1}^{(a)} \oplus W_{n,2}^{(a)}, W_{n,1}^{(b)} \oplus W_{n,2}^{(b)})_{n=1}^{N} \), \( Z_2 = (W_{n,1}^{(a)}, W_{n,1}^{(b)})_{n=1}^{N} \), \( Z_3 = (W_{n,2}^{(a)}, W_{n,2}^{(b)})_{n=1}^{N} \). The delivery scheme is as follows: \( X_1^{(d_1,d_2,d_3)} \) is the same as \( X^{(0,d_2,d_3)} \) of the above scheme acting on Part \((a)\) of the files only, i.e., \( X_1^{(d_1,d_2,d_3)} = \{W_{d_2,1}^{(a)} \oplus W_{d_2,1}^{(a)}, W_{d_3,1}^{(a)} \oplus W_{d_2,2}^{(a)}\}, \( X_2^{(d_1,d_2,d_3)} \) is the same as \( X^{(d_1,0,d_3)} \) of the above scheme acting on Part \((a)\) of File \(d_1\) and Part \((b)\) of the other files, i.e., \( X_2^{(d_1,d_2,d_3)} = \{W_{d_1,1}^{(a)}, W_{d_3,1}^{(a)}\}, \) and \( X_3^{(d_1,d_2,d_3)} \) is the same as \( X^{(d_1,d_2,0)} \) of the above scheme acting on Part \((b)\) of files, i.e., \( X_3^{(d_1,d_2,d_3)} = \{W_{d_1,2}^{(0)}, W_{d_2,2}^{(0)}\}. \) It is easy to check that each user’s demand can be correctly decoded for the 3-user D2D coded caching problem, and the delivery rate is \( \frac{3}{2} \) for any demand vector \( D \).

In Section III, we propose a new caching and delivery scheme for the 3-user D2D coded caching problem using the base scheme of an optimal caching and delivery scheme found for the 3-user D2D coded caching with 2RR1S. In addition, this new scheme achieves a lower delivery rate than currently known results for the case of 2 files and medium cache size. Furthermore, in Section IV, we show that the performance of the new scheme is optimal for the case of 2 files and certain cache size range.

D. Symmetric schemes

We observe that the characteristic of symmetry [6, Section 3] applies to both problems, i.e., the 3-user D2D coded caching problem and the 3-user D2D coded caching with 2RR1S. This observation can be used to simplify the proof of the converse.

First, look at the 3-user D2D coded caching with 2RR1S. Let \( \hat{\pi}(\cdot) \) be a permutation function on the user index set \( \{1, 2, 3\} \), and denote its inverse function as \( \hat{\pi}^{-1}(\cdot) \). Further let \( \mathcal{Z} \subseteq \{Z_1, Z_2, Z_3\}, \mathcal{X} \subseteq \{X^{(0,d_2,d_3)}, X^{(d_1,0,d_3)}, X^{(d_1,d_2,0)}\} \). The mapping \( \hat{\pi}(\mathcal{Z}) \) denotes \( \{Z_{\hat{\pi}(k)} | Z_k \in \mathcal{Z}\} \) and the mapping \( \hat{\pi}(\mathcal{X}) \) denotes \( \{X^{(d_{s-1}(1),d_{s-1}(2),d_{s-1}(3))} | X^{(d_1,d_2,d_3)} \in \mathcal{X}\} \). User-index-symmetric schemes [6, Section 3] are defined as follows.
Definition 1: A caching and delivery scheme is called user-index-symmetric if for any permutation function $\bar{\pi}(\cdot)$, any subset of files $\mathcal{W}$, any subset of caches $\mathcal{Z}$, and any subset of transmitted messages $\mathcal{X}$, we have the following relation:

$$H(\mathcal{W}, \mathcal{Z}, \mathcal{X}) = H(\mathcal{W}, \bar{\pi}(\mathcal{Z}), \bar{\pi}(\mathcal{X})).$$

For example, consider the permutation function $\bar{\pi}(1) = 2, \bar{\pi}(2) = 3, \bar{\pi}(3) = 1$. For a user-index-symmetric scheme for the 3-user D2D coded caching with 2RR1S, the entropy $H(W_1, Z_1, X^{(1,0,2)})$ under the permutation $\bar{\pi}$ is equal to $H(W_1, Z_2, X^{(2,1,0)})$.

Similar to the definition of user-index-symmetric schemes, we may define file-index-symmetric schemes. Let $\hat{\pi}(\cdot)$ be a permutation function on the file index set $\{1, 2, ..., N\}$, $\mathcal{W} \subseteq \{W_1, W_2, ..., W_N\}$, and by representing the mapping $\hat{\pi}(\mathcal{W})$ as $\{W_{\hat{\pi}(n)}, W_n \in \mathcal{W}\}$ and the mapping $\hat{\pi}(\mathcal{X})$ as $\{X^{(\hat{\pi}(d_1), \hat{\pi}(d_2), \hat{\pi}(d_3))} | X^{(d_1,d_2,d_3)} \in \mathcal{X}\}$. File-index-symmetric schemes [6, Section 3] are defined as follows.

Definition 2: A caching and delivery scheme is called file-index-symmetric if for any permutation function $\hat{\pi}(\cdot)$, any subset of files $\mathcal{W}$, any subset of caches $\mathcal{Z}$, and any subset of transmitted messages $\mathcal{X}$, we have the following relation:

$$H(\mathcal{W}, \mathcal{Z}, \mathcal{X}) = H(\hat{\pi}(\mathcal{W}), \mathcal{Z}, \hat{\pi}(\mathcal{X})).$$

For example, for the 3-user D2D coded caching with 2RR1S, if User 2 does not request, the permutation function $\hat{\pi}(0) = 0, \hat{\pi}(1) = 2, \hat{\pi}(2) = 3, \hat{\pi}(3) = 1$, will map $W_1$ to $\hat{\pi}(W_1) = W_2$, but map $X^{(1,0,2)}$ to $X^{(2,0,3)}$. For such a file-index-symmetric scheme, the entropy $H(W_1, Z_1, X^{(1,0,2)})$ under the permutation is equal to $H(W_2, Z_1, X^{(2,0,3)})$.

Due to the nature of the problem, we have the following lemma for the 3-user D2D coded caching with 2RR1S, whose proof is similar to that of [6, Proposition 3.1], and thus, omitted.

Lemma 1: For the 3-user coded caching problem with 2RR1S, for any caching and delivery
scheme, there exists a caching and delivery scheme which is both user-index-symmetric and file-index-symmetric with an equal or smaller worst-case delivery rate.

Similarly, we may define the file-index-symmetric and user-index-symmetric schemes for the 3-user D2D coded caching problem and again have the result of Lemma 1 for this model also. Hence, for both problems, it is sufficient to consider caching and delivery schemes that satisfy both user-index symmetry and file-index symmetry.

III. MAIN RESULTS ON THE 3-USER D2D CODED CACHING PROBLEM WITH 2RR1S

In this paper, we find the optimal rate-memory tradeoff for the proposed 3-user D2D coded caching problem with 2RR1S for any number of files. It turns out that the rate-memory tradeoff satisfies a uniform formula in the case of more than 4 files, and takes on distinct formulas in the case of 2 files and 3 files. More specifically, we have the following theorem when \( M \geq \frac{N}{2} \). Note that when \( M < \frac{N}{2} \), the problem is infeasible, i.e., there exists no caching and delivery scheme that can satisfy the decodability constraint (5).

**Theorem 1:** For the 3-user D2D coded caching with 2RR1S where \( M \geq \frac{N}{2} \), we have

1. For \( N \geq 4 \), the worst-case delivery rate \( R(M) \) must satisfy

\[
4M + NR(M) \geq 3N, \quad M + NR(M) \geq N, \tag{7}
\]

where the corner points are \( (M, R(M)) = \left( \frac{1}{2}N, 1 \right), \left( \frac{2}{3}N, \frac{1}{3} \right), (N, 0) \). Conversely, there exist caching and delivery schemes for any nonnegative \( R(M) \) satisfying (7).

2. For \( N = 2 \), the worst-case delivery rate \( R(M) \) must satisfy

\[
18M + 8R(M) \geq 25, \quad 3M + 3R(M) \geq 5, \quad M + 2R(M) \geq 2, \tag{8}
\]

where the corner points are \( (M, R(M)) = (1, \frac{7}{8}), \left( \frac{7}{6}, \frac{1}{2} \right), \left( \frac{4}{3}, \frac{1}{3} \right), (2, 0) \). Conversely, there exist caching and delivery schemes for any nonnegative \( R(M) \) satisfying (8).
(3) For $N = 3$, the worst-case delivery rate $R(M)$ must satisfy

$$6M + 4R(M) \geq 13, \quad 3M + 3R(M) \geq 7, \quad M + 3R(M) \geq 3, \quad (9)$$

where the corner points are $(M, R(M)) = (\frac{3}{2}, 1), \ (\frac{11}{6}, \frac{1}{2}), \ (2, \frac{1}{3}), \ (3, 0)$. Conversely, there exist caching and delivery schemes for any nonnegative $R(M)$ satisfying (9).

The proof of Theorem 1 is given in Section IV. We make the following remarks regarding of the result of Theorem 1, including comparisons with existing work.

**Remark 1:** The performance of the 3-user D2D coded caching with 2RR1S is upper bounded by the optimal performance of the original 2-user coded caching problem where the sender is the server. This is because the server knows everything and is more capable than any of the D2D sender nodes. The optimal rate-memory tradeoff for the original coded caching problem with 2 users and $N$ files was found in [6], and the converse result of $M + NR(M) \geq N$ is proved. From Theorem 1, we see that when the memory is large enough, i.e., $M \in [\frac{2}{3}N, N]$, $M + NR(M) \geq N$ is achievable for the 3-user D2D coded caching with 2RR1S, which means that when the memory is large, the random D2D sender node is as capable as the all-knowing server.

**Remark 2:** We find that the proposed optimal scheme employs coded cache placement when $M \in [\frac{1}{2}N, \frac{2}{3}N)$, while uncoded cache placement is sufficient when $M \in [\frac{2}{3}N, N]$. This observation is similar to the result of the traditional caching problem in [6] where uncoded placement is sufficient, i.e., optimal, when $M \in [\frac{K-1}{K}N, N]$.

**Remark 3:** The corner point of $(\frac{1}{2}N, 1)$ for $N \geq 2$, the corner points of $(1, \frac{7}{8}), \ (\frac{7}{6}, \frac{1}{2})$ for $N = 2$, and the corner point $(\frac{11}{6}, \frac{1}{2})$ for $N = 3$ all employ coded cache placement. More specifically, to deal with the fact that the identity of the transmitter is unknown at the time of cache placement, the cache content of the users are MDS coded across the three users. Furthermore, in the case of the corner points $(\frac{7}{6}, \frac{1}{2})$ for $N = 2$ and $(\frac{11}{6}, \frac{1}{2})$ for $N = 3$, transmitter
preprocessing is required, where the designated sender needs to compute the transmitted signal from the cached contents based on the request of the two users. We also show in the achievability proof that the two corner points \( \left( \frac{7}{6}, \frac{1}{2} \right) \) for \( N = 2 \) and \( \left( \frac{11}{6}, \frac{1}{2} \right) \) for \( N = 3 \) belong to the more general set of achievable corner points \( \left( \frac{4N-1}{6}, \frac{1}{2} \right) \) for \( N \geq 2 \). These corner points are optimal for \( N = 2, 3 \), but for \( N \geq 4 \), they are sub-optimal.

Remark 4: In the 3-user D2D coded caching with 2RR1S, the number of users \( K = 3 \). From Theorem 1, we see that the number of corner points is different for the case of \( N > K \) and the case of \( N \leq K \). This is because the corner point \( \left( \frac{4N-1}{6}, \frac{1}{2} \right) \) is below the converse line \( 4M + NR(M) \geq 3N \) only when \( N \leq K \), which means that 4 corner points exist when \( N \leq K \) and 3 corner points exist when \( N > K \). Note that in the traditional coded caching, e.g., [19], and other D2D coded caching problems where a tight converse exists, e.g., [8], the number of corner points are different for the case of \( N \geq K \) and the case of \( N < K \).

Remark 5: As mentioned in Section II-C an achievable scheme for the 3-user D2D coded caching with 2RR1S can be exploited as a base scheme to come up with an achievable scheme for the 3-user D2D coded caching problem. Thus, using the scheme that achieves the optimal performance characterized in Theorem 1 as a base scheme, we may come up with a scheme for the 3-user D2D coded caching problem, called the adapted scheme, whose performance is shown in Fig. 3 by the red solid line for \( N = 2, 3, 4 \), respectively. The achievable rates found in [4] and [8], are denoted by the black dash-dot line and the blue dashed line, respectively. Recall that the achievable scheme in [4] employs uncoded cache placement, and the achievable scheme in [8] employs both uncoded cache placement and one-shot delivery, i.e., each bit of the desired message can be reconstructed from the cache and the transmitted signal from at most one sender. It can be seen that the proposed adapted scheme does not offer better performance in the case of 3 or 4 files. So we focus on the case of 2 files, i.e., Fig. 3(a). The proposed adapted scheme outperforms both schemes of [4] and [8] when \( M \in [1.1410, \frac{4}{3}] \), for \( N = 2 \), which is due to coded cache placement. Meanwhile, the rate of the proposed scheme is the same as the
rate of the scheme in [4], when the cache capacity is large. When the cache capacity is small, the performance of the proposed adapted scheme is in general loose. This is because the proposed adapted scheme places a restriction on the number of senders being 1. As a result, first of all, the proposed adapted scheme is only possible when $M \geq \frac{N}{2}$. Secondly, some caching schemes that are feasible for the 3-user D2D coded caching problem is not included in the proposed adapted scheme as they require multiple senders to satisfy the decodability constraint. Hence, even though the base scheme is optimal for the 3-user D2D coded caching with 2RR1S, the adaptation may be sub-optimal for the 3-user D2D coded caching problem.

![Fig. 3. Comparison of the worst-case rate achieved for three schemes for the 3-user D2D coded caching problem when $N = 2$, $N = 3$, $N = 4$, respectively.](image)

IV. PROOF OF THEOREM 1

In this section, we prove the converse and achievability for Theorem 1. The proof is different for $N \geq 4$, $N = 3$ and $N = 2$, where $N$ is the number of files.

A. Achievability

1) $N \geq 4$: We show that the three corner points $(\frac{1}{2}N, 1)$, $(\frac{2}{3}N, \frac{1}{3})$ and $(N, 0)$ are achievable as long as $N \geq 2$. It will be shown via the converse proof that the three corner points $(\frac{1}{2}N, 1)$, $(\frac{2}{3}N, \frac{1}{3})$ and $(N, 0)$ are optimal only when $N \geq 4$ is satisfied.
The corner point of \((N,0)\) is trivial as all D2D nodes have enough cache to store all messages and therefore, the delivery rate is zero. As for the corner point of \((\frac{1}{2}N,1)\), its achievability scheme is as follows: split all files into two subfiles of equal sizes, denoted as \(W_n = (W_{n,1}, W_{n,2})_{n=1}^N\).

In the cache placement phase, the cache content of the three users are given as

\[ Z_1 = (W_{n,1} \oplus W_{n,2})_{n=1}^N, \quad Z_2 = (W_{n,1})_{n=1}^N, \quad Z_3 = (W_{n,2})_{n=1}^N. \]

In the delivery phase, we have

\[ X_{0,d_2,d_3} = \{W_{d_2,1} \oplus W_{d_2,2}, W_{d_3,1} \oplus W_{d_3,2}\}, \quad X_{d_1,0,d_3} = \{W_{d_1,1}, W_{d_3,1}\}, \quad X_{d_1,d_2,0} = \{W_{d_1,2}, W_{d_2,2}\}, \]

and it is easy to check that each user’s demand can be correctly decoded. Thus, the delivery rate of \(R(M) = 1\) is achieved for cache size \(M = \frac{1}{2}N\). As can be seen, coded caching is necessary to achieve the corner point of \((\frac{1}{2}N,1)\).

Lastly, we provide the achievability scheme for the corner point of \((\frac{2}{3}N,\frac{1}{3})\). The caching scheme is the same as that of the Maddah-Ali Niesen (MAN) uncoded symmetric placement in [5, Algorithm 1], more specifically, all files are split into three subfiles of equal sizes, denoted as \(W_n = (W_{n,\{1,2\}}, W_{n,\{1,3\}}, W_{n,\{2,3\}})_{n=1}^N\), and the cache placement is

\[ Z_1 = (W_{n,\{1,2\}}, W_{n,\{1,3\}})_{n=1}^N, \quad Z_2 = (W_{n,\{1,2\}}, W_{n,\{2,3\}})_{n=1}^N, \quad Z_3 = (W_{n,\{1,3\}}, W_{n,\{2,3\}})_{n=1}^N. \]

In the delivery phase, we have

\[ X_{0,d_2,d_3} = \{W_{d_2,\{1,3\}} \oplus W_{d_3,\{1,2\}}\}, \quad X_{d_1,0,d_3} = \{W_{d_1,\{2,3\}} \oplus W_{d_3,\{1,2\}}\}, \]
\[ X_{d_1,d_2,0} = \{W_{d_1,\{2,3\}} \oplus W_{d_2,\{1,3\}}\}, \]

and it is easy to check that each user’s demand can be correctly decoded. Thus, the delivery rate of \(R(M) = \frac{1}{3}\) is achieved for cache size \(M = \frac{2}{3}N\). To achieve the corner point of \((\frac{2}{3}N,\frac{1}{3})\), coded cache placement is not necessary, and the MAN symmetric uncoded placement scheme
is used.

Finally, memory sharing between the corner points \((\frac{1}{2}N, 1), (\frac{2}{3}N, \frac{1}{3})\) and \((N, 0)\) proves that \((7)\) in Theorem 1 is achievable.

2) \(N = 2\): We will prove that the four corner points \((1, \frac{7}{8}), (\frac{7}{6}, \frac{1}{2}), (\frac{4}{3}, \frac{1}{3})\), \((2, 0)\) are achievable. First, note the fact that the corner points \((\frac{4}{3}, \frac{1}{3})\), \((2, 0)\) are achievable has been proved in Subsection [IV-A1] which works for \(N = 2\). So in the following, we will prove that the remaining two points are achievable.

To achieve the corner point of \((1, \frac{7}{8})\), we split both files into 8 subfiles, i.e., \(W_1 = (A_n)_{n=1}^8\) and \(W_2 = (B_n)_{n=1}^8\). Coded cache placement is employed as shown Table I where the cache size is indeed 1.

| TABLE I |
|---|
| Cache placement for achieving the corner point \((1, \frac{7}{8})\) when \(N = 2\) |

| \(Z_1\) | \(A_1 \oplus B_2\) | \(A_2 \oplus B_1\) | \(B_4\) | \(A_4\) | \(A_5\) | \(B_5\) | \(A_7 \oplus A_8\) | \(B_7 \oplus B_8\) |
|---|---|---|---|---|---|---|---|---|
| \(Z_2\) | \(A_1\) | \(B_1\) | \(A_3 \oplus B_4\) | \(A_4 \oplus B_3\) | \(B_6\) | \(A_6\) | \(A_7\) | \(B_7\) |
| \(Z_3\) | \(B_2\) | \(A_2\) | \(A_3\) | \(B_3\) | \(A_5 \oplus B_6\) | \(A_6 \oplus B_5\) | \(A_8\) | \(B_8\) |

For the delivery phase, the transmitted signal depends on the request vector as:

\[
X_{0,d_2,d_3} = \{W_{d_2,2} \oplus W_{d_2,1}, A_4, B_4, A_5, B_5, A_7 \oplus A_8, B_7 \oplus B_8\}, \quad d_2 \neq d_3,
\]

\[
X_{0,d_2,d_3} = \{W_{d_2,7} \oplus W_{d_2,8}, A_4, B_4, A_5, B_5, A_1 \oplus B_2, A_2 \oplus B_1\}, \quad d_2 = d_3,
\]

\[
X_{d_1,0,d_3} = \{W_{d_1,3} \oplus W_{d_1,4}, A_1, B_1, A_6, B_6, A_7, B_7\}, \quad d_1 \neq d_3,
\]

\[
X_{d_1,0,d_3} = \{W_{d_1,7}, A_1, B_1, A_6, B_6, A_3 \oplus B_4, A_4 \oplus B_3\}, \quad d_1 = d_3,
\]

\[
X_{d_1,d_2,0} = \{W_{d_1,6} \oplus W_{d_1,5}, A_2, B_2, A_3, B_3, A_8, B_8\}, \quad d_1 \neq d_2,
\]

\[
X_{d_1,d_2,0} = \{W_{d_1,8}, A_2, B_2, A_3, B_3, A_5 \oplus B_6, A_6 \oplus B_5\}, \quad d_1 = d_2,
\]

where \(d_1, d_2, d_3 \in \{1, 2\}\). As can be seen, when the two random requesters request different files,
both the 7-th and the 8-th elements of the designated sender’s cache are transmitted, but one of the first six elements do not need to be transmitted, resulting in a delivery rate of \( \frac{7}{8} \). On the other hand, when the two random requesters request the same file, the first six elements of the designated sender’s cache are transmitted, and only one of the 7-th or 8-th element is transmitted, i.e., if both random requesters request file \( W_1 \), then the 7-th element of the designated sender’s cache is transmitted, and if both random requesters request file \( W_2 \), then the 8-th element of the designated sender’s cache is transmitted. This again results in a delivery rate of \( \frac{7}{8} \). It is easy to check that the decoding constraint is satisfied. Note that even though the caching and delivery scheme looks asymmetrical in user index in terms of expressions, it is in fact user-index symmetrical in terms of entropy.

To achieve the corner point of \( (\frac{7}{6}, \frac{1}{2}) \), we split both files into 6 subfiles, i.e., \( W_1 = (A_n)_{n=1}^6 \) and \( W_2 = (B_n)_{n=1}^6 \). The caching scheme is given in Table II where the cache size is indeed \( \frac{7}{6} \).

| \( Z_1 \) | \( A_1 \oplus A_2 \) | \( B_1 \oplus B_2 \) | \( A_4 \) | \( B_4 \) | \( A_5 \) | \( B_5 \) | \( A_2 \oplus B_1 \) |
|---|---|---|---|---|---|---|---|
| \( Z_2 \) | \( A_1 \) | \( B_1 \) | \( A_3 \oplus A_4 \) | \( B_3 \oplus B_4 \) | \( A_6 \) | \( B_6 \) | \( A_4 \oplus B_3 \) |
| \( Z_3 \) | \( A_2 \) | \( B_2 \) | \( A_3 \) | \( B_3 \) | \( A_5 \oplus A_6 \) | \( B_5 \oplus B_6 \) | \( A_6 \oplus B_5 \) |

For the delivery phase, the transmitted signal depends on the request vector as

\[
X_{0,d_2,d_3} = \{ W_{d_2,2} \oplus W_{d_3,1}, W_{d_3,4}, W_{d_2,5} \},
\]

\[
X_{d_1,0,d_3} = \{ W_{d_1,3} \oplus W_{d_3,4}, W_{d_3,1}, W_{d_1,6} \},
\]

\[
X_{d_1,d_2,0} = \{ W_{d_1,6} \oplus W_{d_2,5}, W_{d_2,2}, W_{d_1,3} \},
\]

where \( d_1, d_2, d_3 \in \{1, 2\} \). Note that the first six columns are MDS coded across the three users, so that any two of them can recover the entire segment, for example, the first column can recover the segment \((A_1, A_2)\) using the cache of any two users, and the second column can recover the
segment \((B_1, B_2)\) using the cache of any two users. The last column is a coded version that enables User 1 to transmit any pairwise linear combination of \((A_1, A_2, B_1, B_2)\), User 2 to transmit any pairwise linear combination of \((A_3, A_4, B_3, B_4)\), and User 3 to transmit any pairwise linear combination of \((A_5, A_6, B_5, B_6)\). This offers flexibility in the delivery signal of the designated sender based on the demand of the other two users, i.e., transmitter preprocessing is needed. For example, when the request vector is \(D = (0, 2, 1)\), i.e., User 1 does not request and is the designated sender, Users 2 and 3 requests Files \(2\) and \(1\), respectively, the transmitted signal is \((B_2 \oplus A_1, A_4, B_5)\). As can be seen, \(B_2 \oplus A_1\) is not directly stored in the cache of User 1. Preprocessing at User 1 as

\[
B_2 \oplus A_1 = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2) \oplus (A_2 \oplus B_1),
\]

is needed before \(B_2 \oplus A_1\) is transmitted. More generally, when User \(k\) needs to transmit \(A_{2k-1} \oplus B_{2k}\) which is not cached, User \(k\) does the following preprocessing:

\[
A_{2k-1} \oplus B_{2k} = (A_{2k-1} \oplus A_{2k}) \oplus (B_{2k-1} \oplus B_{2k}) \oplus (A_{2k} \oplus B_{2k-1}), \quad k = 1, 2, 3.
\]

The fact that the decoding constraint is satisfied can be checked. We mention here that sometimes several modulo-sums needs to be computed to decode, rather than just one modulo-sum. For example, in the case where the request vector is \(D = (0, 2, 1)\) as discussed above, upon receiving \((B_2 \oplus A_1, A_4, B_5)\), which is the transmitted signal of User 1, User 2 decodes \(B_3\) and \(B_4\) by first computing \((A_3 \oplus A_4) \oplus A_4\) to obtain \(A_3\), and then decode \(B_4\) as \((A_3 \oplus B_4) \oplus A_3\), and finally, decode \(B_3\) as \((B_3 \oplus B_4) \oplus B_4\). Note that the transmission rate of the proposed scheme is \(\frac{3}{6} = \frac{1}{2}\).

Memory sharing between the corner points \((1, \frac{7}{6})\), \((\frac{7}{6}, \frac{1}{2})\), \((\frac{4}{3}, \frac{1}{3})\) and \((2, 0)\) proves that \(\text{(8)}\) in Theorem \(1\) is achievable.

3) \(N = 3\): In the case of \(N = 3\), the achievability of corner points \((\frac{3}{2}, 1)\), \((2, \frac{1}{3})\) and \((3, 0)\) has been proven in Section \(\text{IV-A1}\) which works for \(N = 3\). Thus, we prove the achievability of
the corner point \((\frac{11}{6}, \frac{1}{2})\) in the following.

We split the three files \(W_1, W_2, W_3\) into 6 subfiles, which can be represented as \(W_1 = (A_n)_{n=1}^6, W_2 = (B_n)_{n=1}^6\) and \(W_3 = (C_n)_{n=1}^6\). Coded cache placement is employed as shown in Table III, where the cache size is indeed \(\frac{11}{6}\).

| Z_1 | A_1 \oplus A_2 | B_1 \oplus B_2 | C_1 \oplus C_2 | A_4 | B_4 | C_4 |
|-----|-----------------|-----------------|-----------------|-----|-----|-----|
| Z_2 | A_1             | B_1             | C_1             | A_3 \oplus A_4 | B_3 \oplus B_4 | C_3 \oplus C_4 |
| Z_3 | A_2             | B_2             | C_2             | A_3 | B_3 | C_3 |

For the delivery phase, the transmitted signal depends on the request vector as

\[
X_{0,d_2,d_3} = \{W_{d_2,2} \oplus W_{d_3,1}, W_{d_3,4}, W_{d_2,5}\},
\]

\[
X_{d_1,0,d_3} = \{W_{d_1,3} \oplus W_{d_3,4}, W_{d_3,1}, W_{d_1,6}\},
\]

\[
X_{d_1,d_2,0} = \{W_{d_1,6} \oplus W_{d_2,5}, W_{d_2,2}, W_{d_1,3}\}.
\]

It can be checked that the proposed scheme has no decoding error and the transmission rate is \(\frac{3}{6} = \frac{1}{2}\).

The scheme above is a generalization of the scheme that achieves the corner point \((\frac{7}{6}, \frac{1}{2})\) in \(N = 2\). The first nine columns are MDS coded across the three users, so that any two of them can recover the entire segment. The last two columns are a coded version that enables the designated sender to do transmitter preprocessing and send out the signal needed based on the demands of the other two users.

More generally, for any \(N \geq 2\), the corner point \((\frac{4N-1}{6}, \frac{1}{2})\) is achievable as follows: split all files \(W_1, \cdots, W_N\) into six subfiles, denoted as \(W_{n,1}, \cdots, W_{n,6}\) for File \(n = 1, \cdots, N\). The cache
placement at the three users are given as

\[
Z_1 = \{W_{n,1} \oplus W_{n,2}, W_{n,4}, W_{n,5}, W_{n,2} \oplus W_{n+1,1}\}_{n=1}^{N-1} \bigcup \{W_{N,1} \oplus W_{N,2}, W_{N,4}, W_{N,5}\},
\]

\[
Z_2 = \{W_{n,3} \oplus W_{n,4}, W_{n,1}, W_{n,6}, W_{n,4} \oplus W_{n+1,3}\}_{n=1}^{N-1} \bigcup \{W_{N,3} \oplus W_{N,4}, W_{N,1}, W_{N,6}\},
\]

\[
Z_3 = \{W_{n,5} \oplus W_{n,6}, W_{n,2}, W_{n,3}, W_{n,6} \oplus W_{n+1,5}\}_{n=1}^{N-1} \bigcup \{W_{N,5} \oplus W_{N,6}, W_{N,2}, W_{N,3}\}.
\]

The delivery scheme is given by (10). Note that for the case of \(N \geq 4\), the corner point \((\frac{4N-1}{6}, \frac{1}{2})\), though achievable, is not optimal, i.e., it lies above the time-sharing curve of the three achievable corner points \((\frac{1}{2}N, 1)\), \((\frac{2}{3}N, \frac{1}{3})\) and \((N, 0)\).

Finally, we conclude that memory sharing between the corner points \((\frac{3}{2}, 1)\), \((\frac{11}{6}, \frac{1}{2})\), \((2, \frac{1}{3})\) and \((3, 0)\) proves that (9) in Theorem [1] is achievable.

B. Converse

1) \(N \geq 4\): As mentioned in Remark [1], the performance of the 3-user D2D coded caching with 2RR1S is upper bounded by the optimal performance of the original 2-user coded caching problem where the sender is the server. Thus, the converse result of the original coded caching problem [5], more specifically,

\[
M + NR(M) \geq N, \quad N \geq 2,
\]

is also a converse result for the 3-user D2D coded caching with 2RR1S. Hence, we only need to prove \(4M + NR(M) \geq 3N\).

The main idea to prove the converse for \(N \geq 4\) is using the finding in [20, Lemma 4], which shows \(NH(Z_1|W_1) \geq (N - 1)H(Z_1)\) for any file-index-symmetric schemes. Furthermore, the property given in (6) for the model under consideration is exploited in the proof.

Based on Lemma [1], we may, without loss of generality, consider only user-index-symmetric and file-index-symmetric caching and delivery schemes. For any user-index-symmetric and file-
index-symmetric caching and delivery scheme with achievable rate $R(M)$, it must satisfy

\[(N - 1)H(X_{1,0,2}, Z_1, W_1)\]

\[= (N - 1)[H(Z_1, W_1) + H(X_{1,0,2}|Z_1, W_1)] \]

\[= (N - 1)H(Z_1, W_1) + \sum_{i=2}^{N} H(X_{1,0,i}|Z_1, W_1) \quad (12)\]

\[\geq (N - 1)H(Z_1, W_1) + H(X_{1,0,|2:N]|Z_1, W_1) \]

\[= (N - 2)H(Z_1, W_1) + H(X_{1,0,|2:N}], Z_1, W_1) \]

\[= H(Z_3, W_1) + H(Z_2, W_1) + H(X_{1,0,|2:N}], Z_1, W_1) + (N - 4)H(Z_1, W_1) \quad (13)\]

\[= H(Z_3, W_1) + H(Z_2, X_{1,0,|2:N}], W_1) + H(X_{1,0,|2:N}], Z_1, W_1) + (N - 4)H(Z_1, W_1) \quad (14)\]

\[\geq H(Z_3, W_1) + H(Z_1, Z_2, X_{1,0,|2:N}], W_1) + H(X_{1,0,|2:N}], W_1) + (N - 4)H(Z_1, W_1) \quad (15)\]

\[= H(Z_3, W_1) + H(W_{[1:N]}[2:N]) + H(X_{1,0,|2:N}], W_1) + (N - 4)H(Z_1, W_1) \quad (16)\]

\[\geq N + H(Z_3, X_{1,0,|2:N}], W_1) + H(W_1) + (N - 4)H(Z_1, W_1)) \quad (17)\]

\[= 2N + 1 + (N - 4)H(Z_1, W_1), \quad (18)\]

where (12) follows from the property of file-index-symmetric schemes, (13) follows from the property of user-index-symmetric schemes, (14) follows from (4), (15) and (17) both follow from the sub-modular property of the entropy function, i.e.,

\[H(Y_A) + H(Y_B) \geq H(Y_{A\cup B}) + H(Y_{A\cap B}),\]

(16) follows from (1) and (6), and (18) follows from the decodability constraint in (5), more specifically, $H(W_{[2:N]}|Z_3, X_{1,0,|2:N]) = 0$. Using the result of (18), we have

\[(N - 1)H(X_{1,0,2}|Z_1, W_1)\]

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\[(N - 1)H(X_{1,0,2}, Z_1, W_1) - (N - 1)H(Z_1, W_1)\]
\[\geq 2N + 1 + (N - 4)H(Z_1, W_1) - (N - 1)H(Z_1, W_1)\]
\[= 2(N - 1) - 3H(Z_1|W_1),\]  \hspace{1em} (19)

where (19) follows from (18). Finally, we have

\[M + R(M) \geq H(Z_1) + H(X_{1,0,2}) \geq H(Z_1, X_{1,0,2}, W_1)\]
\[= H(W_1) + H(Z_1|W_1) + H(X_{1,0,2}|Z_1, W_1)\]
\[\geq 1 + H(Z_1|W_1) + 2 - \frac{3}{N - 1}H(Z_1|W_1)\]
\[= 3 + \frac{N - 4}{N - 1}H(Z_1|W_1)\]
\[\geq 3 + \frac{N - 4}{N}H(Z_1),\]  \hspace{1em} (21)

where (21) follows from (20), (22) follows [20, Lemma 4] that shows \(NH(Z_1|W_1) \geq (N - 1)H(Z_1)\) for any file-index-symmetric schemes. From (22), we have

\[4M + NR(M) \geq 3N,\]

which completes the proof of (7) in Theorem 1. \(\blacksquare\)

2) \(N = 2\): The bound \(M + 2R(M) \geq 2\) has been proved in (11), which is valid for any \(N \geq 2\). We now proceed to first prove the bound \(18M + 8R(M) \geq 25\). First, we notice the following proposition for the problem under consideration.

**Proposition 1**: The entropy of users’ caches must satisfy

\[H(Z_f, A) + H(Z_g, A) \geq H(A) + H(W_{[N]}), \hspace{1em} \forall f \neq g, \hspace{1em} f, g \in [K],\]  \hspace{1em} (23)

where \(A\) can be any set of random variables in our system model.
Proof: For $\forall f, g \in [K]$, $f \neq g$, due to the sub-modular property of the entropy function, we can get that

$$H(Z_f, A) + H(Z_g, A) \geq H(A) + H(Z_f, Z_g, A)$$

$$= H(A) + H(W_{[N]}),$$

(24)

where (24) follows from (1), (4) and (6).

To prove the bound $18M + 8R(M) \geq 25$, we have the following lemma, whose proof is given in Appendix A.

Lemma 2: For $N = 2$, we have the following results:

$$8M + 6R(M) + 2H(X_{1,0,1}) - 4H(W_1, X_{0,1,2}, X_{1,0,1}) \geq 8,$$

(25)

$$4M + 2R(M) - 2H(X_{0,1,1}) + 2H(W_1, X_{0,1,1}) - H(W_2, X_{0,1,2}, X_{1,0,2}) \geq 5,$$

(26)

$$6M - 3H(X_{1,0,2}, W_1, X_{0,1,2}) + 3H(X_{0,1,2}, X_{1,0,2}) \geq 6.$$  

(27)

Adding (25)-(27) together, we have

$$18M + 8R(M) - 3H(X_{1,0,2}, W_1, X_{0,1,2}) + 3H(X_{0,1,2}, X_{1,0,2})$$

$$\geq 19 + 4H(W_1, X_{0,1,2}, X_{1,0,1}) + H(W_2, X_{0,1,2}, X_{1,0,2}) - 2H(W_1, X_{0,1,1})$$

$$= 19 + 2H(W_1, X_{0,2,1}, X_{1,1,0}) + 2H(W_1, X_{2,0,1}, X_{1,1,0}) - 2H(W_1, X_{1,1,0})$$

$$+ H(W_2, X_{0,1,2}, X_{1,0,2})$$

$$\geq 19 + 2H(W_1, X_{0,2,1}, X_{1,1,0}, X_{2,0,1}) + H(W_2, X_{0,1,2}, X_{1,0,2})$$

$$= 19 + 2(W_2, X_{0,1,2}, X_{2,0,1}, X_{1,0,2}) + H(W_2, X_{0,1,2}, X_{1,0,2})$$

$$\geq 19 + 3H(W_2, X_{0,1,2}, X_{1,0,2}),$$

(29)
where (28) and (29) both follow from the property of file-index-symmetric schemes. Hence, we have

\[ 18M + 8R(M) \geq 19 + 3H(W_2, X_{0,1,2}, X_{1,0,2}) + 3H(X_{1,0,2}, W_1, X_{0,1,2}) - 3H(X_{0,1,2}, X_{1,0,2}) \]

\[ \geq 19 + 3H(W_1, W_2) = 25, \]

which completes the proof of the bound \( 18M + 8R(M) \geq 25 \).

Next, we proceed to prove the bound \( 3M + 3R(M) \geq 5 \). We have the following lemma, whose proof is given in Appendix B.

**Lemma 3:** For \( N = 2 \), we have the following results:

\[ M + R(M) + H(W_1, X_{2,0,1}) \geq 3, \] (30)
\[ 2M + 2R(M) - H(W_1, X_{1,0,2}) \geq 2. \] (31)

Then, adding (30) and (31) together, we have

\[ 3M + 3R(M) + H(W_1, X_{2,0,1}) - H(W_1, X_{1,0,2}) = 3M + 3R(M) \geq 5, \] (32)

where the equality in (32) follows from the property of user-index-symmetric schemes. Thus, (8) in Theorem 1 is proved.

3) \( N = 3 \): The bound \( M + 3R(M) \geq 3 \) has been proved in (11), which is valid for any \( N \geq 2 \). We now proceed to first prove the bound \( 6M + 4R(M) \geq 13 \) by using the following lemma, whose proof is given in Appendix C.

**Lemma 4:** For \( N = 3 \), we have the following results:

\[ 2M + 2R(M) - H(W_1, X_{0,1,3}, X_{1,0,2}) \geq 3, \] (33)
\[ 4M + 2R(M) + H(W_1, X_{0,1,3}, X_{1,0,2}) \geq 10. \] (34)
Adding (33) and (34) together, we have

\[ 6M + 4R(M) - H(W_1, X_{0,1,3}, X_{1,0,2}) + H(W_1, X_{0,1,3}, X_{1,0,2}) = 6M + 4R(M) \geq 13. \]

Next, we proceed to prove the bound \(3M + 3R(M) \geq 7\). We have the following lemma, whose proof is given in Appendix D.

**Lemma 5:** For \(N = 3\), we have the following results:

\[ M + R(M) + H(W_1, X_{0,1,3}, X_{1,0,2}) \geq 4, \quad (35) \]
\[ 2M + 2R(M) - H(W_1, X_{0,1,3}, X_{1,0,2}) \geq 3. \quad (36) \]

Adding (35) and (36) together, we have

\[ 3M + 3R(M) + H(W_1, X_{0,1,3}, X_{1,0,2}) - H(W_1, X_{0,1,3}, X_{1,0,2}) = 3M + 3R(M) \geq 7. \]

Thus, (9) in Theorem I is proved.

**V. Main Result on the 3-user D2D Coded Caching Problem**

As mentioned before, the 3-user D2D coded caching with 2RR1S is a simpler model, and as a result, the minimum achievable worst-case rate is found in Theorem I. The minimum achievable worst-case rate for the 3-user D2D coded caching problem is more difficult to find and thus, remains open.

In this section, we find the minimum achievable worst-case rate for the 3-user D2D coded caching problem when the number of files is 2, i.e., \(N = 2\). More specifically, we have the following theorem.

**Theorem 2:** For the 3-user D2D coded caching problem, when \(M \geq \frac{1}{3}N\), we have

\[ 2M + R(M) \geq 3, \quad 3M + 2R(M) \geq 5, \quad 3M + 4R(M) \geq 6, \quad (37) \]
where the corner points are \((\frac{2}{3}, \frac{5}{3}), (1,1), (\frac{4}{3}, \frac{1}{2}), (2,0)\). Conversely, there exist caching and delivery schemes for any nonnegative \(R(M)\) satisfying (37).

The proof of Theorem 2 is in Section VI.

Remark 6: Recall that [8] has found the optimal performance of the \(K\)-user D2D coded caching problem for any number of files under the assumption of uncoded cache placement and one-shot delivery. Comparing [8, Corollary 2] with Theorem 2 above, for 3 users and 2 files, we see that while the corner points \((\frac{2}{3}, \frac{5}{3}), (\frac{4}{3}, \frac{1}{2}), (2,0)\) are the same, the corner point \((1,1)\) exists and is optimal when we remove the constraint of uncoded cache placement and one-shot delivery. We will see in Section VI-A that the achievability scheme of the corner point \((1,1)\) employs coded cache placement, but is still a one-shot delivery scheme.

Remark 7: Now that we have found the optimal performance of the 3-user D2D coded caching problem for \(N = 2\), i.e., Theorem 2, we may plot it in Fig. 4, denoted by the cyan dotted dashed line. We further compare it to the performance of several achievability schemes and existing converse results. In terms of achievable schemes, the performance of the proposed adapted scheme discussed in Remark 5 is denoted by the dotted red solid line, the scheme of [4] is denoted by the black dash-dot line, and the scheme of [8] is denoted by blue dashed line. As can be seen, the proposed adapted scheme is in fact optimal for \(M \in \left[\frac{7}{6}, \frac{4}{3}\right]\) in the case of \(N = 2\). Hence, when the cache size is in the range of \(M \in \left[\frac{7}{6}, \frac{4}{3}\right]\), using the scheme adapted from the base scheme of the optimal scheme for the 3-user D2D coded caching with 2RR1S does not cause any performance loss. We further observe that neither the existing schemes of [4] and [8], nor the proposed adapted scheme is optimal for the cache size of \(M \in \left[\frac{2}{3}, \frac{7}{6}\right]\). For this cache size range, the scheme used in proving Theorem 2 which will be given in Section VI-A is optimal.

In terms of existing converse results, the optimal result for the traditional coded caching problem in the case of 3 users and 2 files was found in [6]. This also serves as a converse result for the 3-user D2D coded caching problem and is plotted by the yellow solid line. The converse
results derived in [4] is denoted by the purple dotted line. From Fig. 4, we can see that both converse results are rather loose, compared to the optimal performance found in Theorem 2.

![Fig. 4. Achievable performance and converse results for the 3-user D2D coded caching problem when N = 2.](image)

VI. PROOF OF THEOREM 2

A. Achievability

In this subsection, we prove that the four corner points \((\frac{2}{3}, \frac{5}{3}), (1, 1), (\frac{4}{3}, \frac{1}{2}), (2, 0)\), as stated in Theorem 2, are achievable. The achievability proof of the three corner points \((\frac{2}{3}, \frac{5}{3}), (\frac{4}{3}, \frac{1}{2}), (2, 0)\) is given in [8], and it has been shown that uncoded cache placement is sufficient to achieve these three corner points. Thus, we only need to prove that the corner point \((1, 1)\) is achievable.

We split the two files \(W_1\) and \(W_2\) into 6 subfiles, which can be represented as \(W_1 = (A_n)_{n=1}^6\), \(W_2 = (B_n)_{n=1}^6\). For \(M = 1\), the caching scheme at the three users are

\[ Z_1 = (A_1 \oplus B_1, A_2 \oplus B_2, A_3, A_4, B_3, B_4), \]
\[ Z_2 = (A_3 \oplus B_3, A_4 \oplus B_4, A_5, A_6, B_5, B_6), \]
\[ Z_3 = (A_5 \oplus B_5, A_6 \oplus B_6, A_1, A_2, B_1, B_2). \]

Hence, the cache size is 1. Also, note that coded cache placement is employed. During the delivery phase, the transmitted signals are

\[ X_1^{(d_1, d_2, d_3)} = \{ W_{d_3, 3}, W_{d_3, 4} \}, \quad X_2^{(d_1, d_2, d_3)} = \{ W_{d_1, 5}, W_{d_1, 6} \}, \quad X_3^{(d_1, d_2, d_3)} = \{ W_{d_2, 1}, W_{d_2, 2} \}. \]

It can be checked that the decodability constraint is satisfied for the above scheme and the delivery rate is 1.

Memory sharing between the corner points \( (\frac{2}{3}, \frac{5}{3}), (1, 1), (\frac{4}{3}, \frac{1}{2}) \) and \( (2, 0) \) shows that (37) in Theorem 2 is achievable.

**B. Converse**

We first derive the following proposition for the 3-user D2D coded caching problem.

**Proposition 2:** For the 3-user D2D coded caching problem, when \( N \geq 2 \), the following equation must be satisfied for any caching and delivery scheme:

\[ 3M + 2NR(M) \geq 3N. \]

**Proof:** We write the chain of inequalities as

\[ 3M + 2NR(M) \geq H(Z_1) + H(Z_2) + H(Z_3) + 2N[H(X_{1,2,2}^1) + H(X_{1,2,2}^2)]\quad (38) \]

Observe that

\[ H(Z_1) + NH(X_{1,2,2}^2) + NH(X_{1,2,2}^3) \]
\[ = H(Z_1) + H(X_{1,2,2}^2) + H(X_{1,2,2}^3) + \sum_{i=2}^{N} [H(X_{i,1,1}^2) + H(X_{i,1,1}^3)]\quad (39) \]
\[
\geq H(Z_1) + H(X^2_{1,2,2}) + H(X^3_{1,2,2}) + H(X^2_{[2:N],1,1}) + H(X^3_{[2:N],1,1}) \\
\geq H(Z_1, X^2_{1,2,2}, X^3_{1,2,2}, X^2_{[2:N],1,1}, X^3_{[2:N],1,1}) \\
= H(Z_1, X^2_{1,2,2}, X^3_{1,2,2}, X^2_{[2:N],1,1}, X^3_{[2:N],1,1}, W_{[N]}) \\
= N, \quad (40)
\]

where (39) follows from the property of file-index-symmetric schemes, and (40) follows from (1), (2) and (3).

Similarly, we have
\[
H(Z_2) + NH(X^1_{1,2,2}) + NH(X^3_{1,2,2}) \geq N, \quad (41)
\]
\[
H(Z_3) + NH(X^1_{1,2,2}) + NH(X^2_{1,2,2}) \geq N. \quad (42)
\]

From (38), (40), (41) and (42), we have proved Proposition 2. 

The bound $3M + 4R(M) \geq 6$ is a special case of Proposition 2 when $N = 2$. Hence, we are left to prove the bounds $2M + R(M) \geq 3$ and $3M + 2R(M) \geq 5$.

We first prove the bound $2M + R(M) \geq 3$. Due to the property of file-index-symmetric and user-index-symmetric schemes for the 3-user D2D coded caching model, we have
\[
H(X^1_{1,1,2}) = H(X^1_{1,2,1}) = H(X^2_{1,1,2}) = H(X^2_{1,2,2}) = H(X^3_{1,1,2}) = H(X^3_{1,2,2}), \\
H(X^1_{1,2,2}) = H(X^2_{1,2,1}) = H(X^3_{1,2,2}). \quad (43)
\]

Then, we have the following lemma, whose proof is given in Appendix E.

**Lemma 6:** In the 3-user D2D coded caching problem, when $N = 2$, we have the following results:
\[
M + R(M) \geq H(X^1_{1,2,2}) - H(W_t, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - 2H(W_t, Z_1, Z_2) \\
+ 3H(W_t, W_2) + H(W_t), \quad (44)
\]
\[2M + R(M) \geq H(X_{1,2,1}^1, X_{1,2,1}^2) + H(Z_1, X_{1,2,1}^2, X_{1,2,1}^3) - H(W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3) \] (45)
\[+ H(W_1, W_2), \]
\[3M + R(M) \geq 2H(Z_1) + H(Z_1, X_{1,1,2}^2, X_{1,1,2}^3) + H(X_{1,1,2}^1). \] (46)

Using (43) to combine (44), (45) and (46), we have
\[6M + 3R(M) \]
\[\geq 2[H(Z_1, X_{1,1,2}^2, X_{1,1,2}^3) + H(Z_2, X_{1,1,2}^1, X_{1,1,2}^3) - H(W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3)] \] (47)
\[- 2H(W_1, Z_1, Z_2) + 4H(W_1, W_2) + H(W_1) \]
\[= 2[H(Z_1, W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3) + H(Z_2, W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3)] \] (48)
\[- H(W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3)] - 2H(W_1, Z_1, Z_2) + 4H(W_1, W_2) + H(W_1) \]
\[\geq 2H(Z_1, Z_2, W_1, X_{1,2,1}^1, X_{1,2,1}^2, X_{1,2,1}^3) - 2H(W_1, Z_1, Z_2) + 4H(W_1, W_2) + H(W_1) \] (49)
\[\geq 4H(W_1, W_2) + H(W_1) = 9, \] (50)

where (47) follows from the property of user-index-symmetric and file-index-symmetric schemes, (48) follows from (2) and (3), (49) follows from the sub-modular property of the entropy function. The chain of inequalities (50) proves the bound \(2M + R(M) \geq 3\).

Next, we proceed to prove the bound \(3M + 2R(M) \geq 5\). We have the following lemma, whose proof is given in Appendix F.

**Lemma 7:** In the 3-user D2D coded caching problem, when \(N = 2\), we have the following results:
\[3M + 2R(M) - H(W_1, Z_1) - H(W_1, X_{1,1,2}^1, X_{1,1,2}^2) \geq 2, \] (51)
\[H(W_1, Z_1) + H(W_1, X_{1,1,2}^1, X_{1,1,2}^2) \geq 3. \] (52)
Adding (51) and (52) together, we have $3M + 2R(M) \geq 5$.

Thus, (37) in Theorem 2 is proved.

VII. CONCLUSIONS

In this paper, we studied the 3-user D2D coded caching problem and a related simpler problem, called the 3-user D2D coded caching with 2RR1S, where during the delivery phase, any two out of the three users will make file requests and the user who does not make any file request will be the designated sender. We characterized the optimal rate-memory tradeoff of the 3-user D2D coded caching with 2RR1S for any number of files. Using the optimal achievable scheme for the 3-user D2D coded caching with 2RR1S as a base scheme, we proposed a new achievable scheme for the 3-user D2D coded caching problem which involves coded cache placement. The new achievable scheme outperforms existing schemes when the number of files is 2 and the cache size is medium. We further characterized the optimal rate-memory tradeoff of the 3-user D2D coded caching when the number of files is 2. In doing so, we showed that the proposed new achievable scheme is in fact optimal when the cache size is medium. Comparing to existing works which focus on schemes of uncoded cache placement, we characterized the amount of performance gain enabled by allowing coded cache placement. Our proposed schemes are of the one-shot nature, and therefore the performance improvement by allowing none one-shot delivery schemes is still not well understood and needs to be further studied.

APPENDIX A

PROOF OF LEMMA 2

We will first prove (25).

$$8M + 6R(M) + 2H(X_{1,0,1}) \geq 4H(Z_1) + 4H(Z_2) + 4H(X_{0,1,2}) + 2H(X_{1,0,1}) + 2H(X_{1,0,1})$$

$$\geq 4H(Z_1, X_{1,0,1}) + 4H(Z_2, X_{0,1,2})$$
\[= 4H(Z_1, X_{1,0,1}, X_{0,1,2}) + 4H(Z_2, X_{0,1,2}, X_{1,0,1}) \quad (53)\]
\[= 4H(Z_1, X_{1,0,1}, X_{0,1,2}, W_1) + 4H(Z_2, X_{0,1,2}, X_{1,0,1}, W_1) \quad (54)\]
\[\geq 4H(W_1, X_{0,1,2}, X_{1,0,1}) + 4H(W_1, W_2), \quad (55)\]

where (53) follows from the fact that \(X_{0,1,2}\) is a deterministic function of \(Z_1\) and \(X_{1,0,1}\) is a deterministic function of \(Z_2\), (54) follows from the fact that knowing \((Z_1, X_{1,0,1})\) can decode \(W_1\) and knowing \((Z_2, X_{0,1,2})\) can decode \(W_1\), and (55) follows from (23). Thus, (25) is proved.

Next, we will prove (26). We have

\[4M + 2R(M) - 2H(X_{0,1,1}) \geq 2H(Z_1) + 2H(Z_2) + 2H(X_{0,1,2}) - 2H(X_{0,1,1})\]
\[= 2H(Z_1, X_{0,1,1}) + 2H(Z_2) + 2H(X_{0,1,2}) - 2H(X_{0,1,1}) \quad (56)\]
\[\geq 2H(Z_1, X_{0,1,1}) + 2H(Z_2, X_{0,1,2}) - 2H(X_{0,1,1})\]
\[\geq 2H(W_1, Z_1, X_{0,1,1}) - 2H(W_1, X_{0,1,1}) + 2H(Z_2, X_{0,1,2}), \quad (57)\]

where (56) follows from the fact that \(X_{0,1,1}\) is a deterministic function of \(Z_1\), (57) follows the sub-modular property of the entropy function. Hence, we further have

\[4M + 2R(M) + 2H(W_1, X_{0,1,1}) - 2H(X_{0,1,1})\]
\[\geq 2H(W_1, Z_1, X_{0,1,1}) + 2H(Z_2, X_{0,1,2})\]
\[\geq 2H(W_1, Z_1) + 2H(Z_2, X_{0,1,2})\]
\[= 2H(W_1, Z_1, X_{0,1,2}) + 2H(Z_2, X_{0,1,2}, W_1) \quad (58)\]
\[\geq 2H(W_1, X_{0,1,2}) + 2H(W_1, W_2) \quad (59)\]
\[= H(W_2, X_{0,1,2}) + H(W_2, X_{1,0,2}) + 2H(W_1, W_2) \quad (60)\]
\[\geq H(W_2, X_{0,1,2}, X_{1,0,2}) + H(W_2) + 2H(W_1, W_2), \quad (61)\]
where (58) follows from the fact that $X_{0,1,2}$ is a deterministic function of $Z_1$ and knowing $(Z_2, X_{0,1,2})$ can decode $W_1$, (59) follows from (23), (60) follows from 

\[ H(W_1, X_{0,1,2}) \overset{(a)}{=} H(W_1, X_{2,0,1}) \overset{(b)}{=} H(W_2, X_{1,0,2}), \]

\[ H(W_1, X_{0,1,2}) \overset{(c)}{=} H(W_2, X_{0,2,1}) \overset{(d)}{=} H(W_2, X_{0,1,2}), \]

where (a) and (d) follow from the property of user-index-symmetric schemes, and (b) and (c) follow from the property of file-index-symmetric schemes, and (61) follows again from the sub-modular function of the entropy function. Hence, (26) is proved.

Finally, to prove (27), we have

\[ 6M \geq 6H(Z_1) \]
\[ = 6H(Z_1) + 3H(X_{1,0,2}) - 3H(X_{0,1,2}) \]
\[ \geq 3H(Z_1) + 3H(Z_1, X_{1,0,2}) - 3H(X_{0,1,2}) \]
\[ = 3H(Z_1, X_{0,1,2}) + 3H(Z_1, X_{1,0,2}) - 3H(X_{0,1,2}) \] (62)
\[ \geq 3H(Z_1, X_{0,1,2}, X_{1,0,2}) + 3H(Z_1, X_{1,0,2}) - 3H(X_{0,1,2}, X_{1,0,2}) \] (63)
\[ \geq 6H(Z_1, X_{1,0,2}) - 3H(X_{0,1,2}, X_{1,0,2}) \]
\[ = 3H(Z_1, X_{1,0,2}) + 3H(Z_2, X_{0,1,2}) - 3H(X_{0,1,2}, X_{1,0,2}) \] (64)
\[ = 3H(Z_1, X_{1,0,2}, W_1, X_{0,1,2}) + 3H(Z_2, X_{0,1,2}, W_1, X_{1,0,2}) - 3H(X_{0,1,2}, X_{1,0,2}) \] (65)
\[ \geq 3H(X_{1,0,2}, W_1, X_{0,1,2}) + 3H(W_1, W_2) - 3H(X_{0,1,2}, X_{1,0,2}), \] (66)

where (62) follows from the fact that $X_{0,1,2}$ is a deterministic function of $Z_1$, (63) follows from the sub-modular property of the entropy function, (64) follows from the property of user-index-symmetric schemes, (65) follows from the fact that knowing $(Z_1, X_{1,0,2})$ can decode $W_1$, $X_{0,1,2}$ is a deterministic function of $Z_1$, knowing $(Z_2, X_{0,1,2})$ can decode $W_1$ and $X_{1,0,2}$ is a deterministic
function of $Z_2$, and (66) follows from (23). Thus, (27) is proved.

**APPENDIX B**

**PROOF OF LEMMA 3**

We will first prove (30).

$$M + R(M) + H(W_1, X_{2,0,1}) \geq H(Z_1) + H(X_{1,0,2}) + H(W_1, X_{2,0,1})$$

$$\geq H(Z_1, X_{1,0,2}) + H(W_1, X_{2,0,1})$$

$$= H(Z_1, X_{1,0,2}, W_1) + H(W_1, X_{2,0,1})$$

$$\geq H(Z_1, W_1) + H(W_1, X_{2,0,1})$$

$$\geq H(Z_1, W_1, X_{2,0,1}) + H(W_1)$$

$$= H(Z_1, W_1, X_{2,0,1}, W_2) + H(W_1)$$

$$= H(W_1, W_2) + H(W_1) = 3,$$

(67)

where (67) follows from the fact that knowing $(Z_1, X_{1,0,2})$ can decode $W_1$, (68) follows from the sub-modular property of the entropy function, (69) follows from the fact that knowing $(Z_1, X_{2,0,1})$ can decode $W_2$, and (70) follows from (1), (4). Hence, (30) is proved.

Next, we will prove (31).

$$2M + 2R(M) - H(W_1, X_{1,0,2})$$

$$\geq H(Z_1) + H(Z_2) + H(X_{1,0,2}) + H(X_{0,1,2}) - H(W_1, X_{1,0,2})$$

$$\geq H(Z_1, X_{1,0,2}) + H(Z_2, X_{0,1,2}) - H(W_1, X_{1,0,2})$$

$$= H(Z_1, X_{1,0,2}, W_1, X_{0,2,1}) + H(Z_2, X_{0,1,2}, W_1, X_{1,0,2}) - H(W_1, X_{1,0,2})$$

$$\geq H(Z_1, X_{1,0,2}, W_1, X_{0,2,1}) + H(Z_2, X_{0,1,2}, W_1, X_{1,0,2}, X_{0,2,1})$$

(71)

(72)
\[- H(W_1, X_{1,0,2}, X_{0,2,1}) \]
\[ \geq H(Z_2, X_{0,1,2}, W_1, X_{1,0,2}, X_{0,2,1}, W_2) \] (73)
\[ = H(W_1, W_2) = 2, \] (74)

where (71) follows from the fact that knowing \((Z_1, X_{1,0,2})\) can decode \(W_1, X_{0,2,1}\) is a deterministic function of \(Z_1\), knowing \((Z_2, X_{0,1,2})\) can decode \(W_1\) and \(X_{1,0,2}\) is a deterministic function of \(Z_2\), (72) follows from the sub-modular property of the entropy function, (73) follows from the fact that knowing \((Z_2, X_{0,2,1})\) can decode \(W_2\), and (74) follows from (1) and (4). Hence, (31) is proved.

**APPENDIX C**

**PROOF OF LEMMA 4**

We will first prove (33).

\[ 2M + 2R(M) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1) + H(Z_2) + H(X_{1,0,2}) + H(X_{0,1,3}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1, X_{1,0,2}) + H(Z_2, X_{0,1,3}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ = H(Z_1, X_{1,0,2}, W_1, X_{0,1,3}) + H(Z_2, X_{0,1,3}, W_1, X_{1,0,2}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \] (75)
\[ \geq H(W_1, W_2, W_3) + H(W_1, X_{0,1,3}, X_{1,0,2}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \] (76)
\[ = 3, \]

where (75) follows from the fact that knowing \((Z_1, X_{1,0,2})\) can decode \(W_1, X_{0,1,3}\) is a deterministic function of \(Z_1\), knowing \((Z_2, X_{0,1,3})\) can decode \(W_1\) and \(X_{1,0,2}\) is a deterministic function of \(Z_2\), (76) follows from (23). Hence, (33) is proved.
Next, we will prove (34). Firstly, we have

\[ 3M + 2R(M) + H(X_{1,0,3}) \]

\[ \geq 3H(Z_1) + 2H(X_{1,0,2}) + H(X_{1,0,3}) \]

\[ \geq 2H(Z_1, X_{1,0,2}) + H(Z_1, X_{1,0,3}) \]

\[ = H(Z_1, X_{1,0,2}) + H(Z_1, X_{1,0,2}, W_1) + H(Z_1, X_{1,0,3}, W_1) \]  \hspace{1cm} (77)

\[ \geq H(Z_1, X_{1,0,2}) + H(Z_1, X_{1,0,2}, W_1, X_{1,0,3}) + H(Z_1, W_1) \]  \hspace{1cm} (78)

\[ = H(Z_1, X_{1,0,2}) + H(Z_2, X_{0,1,2}, W_1, X_{0,1,3}) + H(Z_1, W_1) \]  \hspace{1cm} (79)

\[ = H(Z_1, X_{1,0,2}, W_1, X_{0,1,2}, X_{0,1,3}) + H(Z_2, X_{0,1,2}, W_1, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) \]  \hspace{1cm} (80)

\[ \geq H(W_1, W_2, W_3) + H(W_1, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) \]  \hspace{1cm} (81)

\[ = 3 + H(W_1, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1), \]  \hspace{1cm} (82)

where (77) follows from the fact that knowing \((Z_1, X_{1,0,2})\) or \((Z_1, X_{1,0,3})\) both can decode \(W_1\), (78) follows from the sub-modular property of the entropy function, (79) follows from the property of user-index-symmetric schemes, (80) follows from the fact that \(X_{0,1,2}, X_{0,1,3}\) is a deterministic function of \(Z_1\) and \(X_{1,0,2}\) is a deterministic function of \(Z_2\), (81) follows from (23). Then, we notice that

\[ M - H(X_{1,0,2}) + H(X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) \]

\[ \geq H(Z_1) - H(X_{1,0,2}) + H(X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) \]

\[ = H(Z_1, X_{0,1,2}, X_{0,1,3}) - H(X_{1,0,2}) + H(X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) \]  \hspace{1cm} (83)

\[ \geq H(Z_1, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) + H(X_{0,1,2}, X_{0,1,3}) - H(X_{1,0,2}) \]  \hspace{1cm} (84)

\[ = H(Z_1, X_{1,0,2}) + H(X_{0,1,2}, X_{0,1,3}) - H(X_{1,0,3}) \]  \hspace{1cm} (85)
\[ H(Z_1, X_{1,0,3}) + H(X_{1,0,3}, X_{2,0,3}) - H(X_{1,0,3}) \]  
\[ \geq H(Z_1, X_{1,0,3}, X_{2,0,3}), \]  
(86)  

where (83) and (85) both follow from the fact that \( X_{0,1,2}, X_{0,1,3} \) is a deterministic function of \( Z_1 \), (84) and (87) both follow from the sub-modular property of the entropy function, (86) follows from

\[ H(Z_1, X_{1,0,2}) \overset{(a)}{=} H(Z_1, X_{1,0,3}), \]  
\[ H(X_{0,1,2}, X_{0,1,3}) \overset{(d)}{=} H(X_{2,0,1}, X_{3,0,1}), \]

where (a), (b) and (d) all follow from the property of file-index-symmetric schemes, and (c) follows from the property of user-index-symmetric schemes.

Using the property of user-index-symmetric schemes to combine (82) and (87), we have

\[ 4M + 2R(M) \]
\[ \geq 3 + H(W_1, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) - H(X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq 3 + H(W_1, W_2, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \]
\[ - H(W_2, X_{0,1,2}, X_{0,1,3}, X_{1,0,2}) \]
\[ = 3 + H(W_1, W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \]
\[ - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}) \]
\[ \geq 3 + H(W_1, W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \]
\[ - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]
\[ = 3 + H(W_3, W_2, X_{0,2,3}, X_{0,1,3}, X_{3,2,0}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \]
\[ - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]
\[3 + H(W_3, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \] 

\[- H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}),\]

where (88) and (90) both follow from the sub-modular property of the entropy function, (89) follows from the property of user-index-symmetric schemes, (91) follows from the property of file-index-symmetric schemes.

Adding \( H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) \) to both sides of (92), we have

\[4M + 2R + H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) \]

\[\geq 3 + H(W_3, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) \]

\[+ H(Z_1, X_{1,0,3}, X_{2,0,3}, X_{1,0,2}) \]

\[\geq 3 + H(W_1, W_2, W_3, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(W_2, W_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) \] 

\[+ H(Z_1, X_{1,0,3}, X_{2,0,3}) - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]

\[= 3 + H(W_1, W_2, W_3) + H(W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) \] 

\[+ H(Z_1, X_{1,0,3}, X_{2,0,3}) - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]

\[= 3 + H(W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_1, X_{1,0,3}, X_{2,0,3}) \]

\[- H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]

\[= 3 + H(W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_2, X_{0,1,3}, X_{0,2,3}) \]

\[- H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]

\[= 3 + H(W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1) + H(Z_2, X_{0,1,3}, X_{0,2,3}, X_{1,0,2}, W_1, W_2) \]

\[- H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2}) \]

where (93) follows from the sub-modular property of the entropy function, (94) follows from
(1) and (4), (95) follows from the property of user-index-symmetric schemes, (96) follows from the fact that $X_{1,0,2}$ is a deterministic function of $Z_2$ and knowing $(Z_2, X_{0,1,3}, X_{0,2,3})$ can decode $W_1, W_2$.

Through further calculations, we find that

$$H(W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) - H(W_2, X_{0,2,1}, X_{0,3,1}, X_{1,2,0}, X_{3,0,2})$$

$$= H(X_{0,1,3}, X_{0,2,3}, W_2, X_{1,0,2}) - H(X_{0,1,3}, X_{0,2,3}, X_{3,2,0}, W_2, X_{1,0,2})$$

$$\geq H(W_2, X_{1,0,2}) - H(X_{3,2,0}, W_2, X_{1,0,2})$$

$$= H(W_1, X_{2,0,1}) - H(X_{1,0,2}, W_1, X_{0,1,3}),$$

(97)

(98)

(99)

where (97) follows from the property of user-index-symmetric schemes, (98) follows from the sub-modular property of the entropy function, (99) follows from

$$H(W_2, X_{1,0,2}) \overset{(a)}{=} H(W_1, X_{2,0,1}),$$

$$H(X_{3,2,0}, W_2, X_{1,0,2}) \overset{(b)}{=} H(X_{2,1,0}, W_1, X_{3,0,1}) \overset{(c)}{=} H(X_{1,0,2}, W_1, X_{0,1,3}),$$

where (a) and (b) both follow from the property of file-index-symmetric schemes, and (c) follows from the property of user-index-symmetric schemes.

Again, adding $H(Z_1, W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2})$ to both sides of (96) and applying (99), we have

$$4M + 2R + H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(Z_1, W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2})$$

$$\geq 6 + H(Z_1, W_1) + H(Z_2, X_{0,1,3}, X_{0,2,3}, X_{1,0,2}, W_1, W_2)$$

$$+ H(Z_1, W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(W_1, W_2, X_{1,0,1}) - H(X_{1,0,2}, W_1, X_{0,1,3})$$

$$\geq 6 + H(Z_1, W_1) + H(W_1, W_2, W_3) + H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2})$$

(100)
\[ + H(W_1, X_{2,0,1}) - H(X_{1,0,2}, W_1, X_{0,1,3}) \]
\[ = 9 + H(Z_1, W_1) + H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(W_1, X_{2,0,1}) - H(X_{1,0,2}, W_1, X_{0,1,3}), \]

(101)

where (100) follows from (23).

Then, adding \( H(W_1, X_{0,1,3}, X_{1,0,2}) \) to both sides of (101), and removing \( H(W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) \) and \( H(Z_1, W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) \) from both sides of (101), we have

\[ 4M + 2R + H(W_1, X_{0,1,3}, X_{1,0,2}) \geq 9 + H(Z_1, W_1) - H(Z_1, W_1, W_2, X_{0,2,3}, X_{0,1,3}, X_{1,0,2}) + H(W_1, X_{2,0,1}) \]
\[ = 9 + H(Z_1, W_1) - H(Z_1, W_2, X_{1,0,2}) + H(W_1, X_{2,0,1}) \]
\[ = 9 + H(Z_1, W_1) - H(Z_1, W_1, X_{2,0,1}) + H(W_1, X_{2,0,1}) \]
\[ \geq 9 + H(W_1) - H(X_{2,0,1}, W_1) + H(W_1, X_{2,0,1}) \]
\[ = 10, \]

where (102) follows from the fact that \( X_{0,2,3}, X_{0,1,3} \) is a deterministic function of \( Z_1 \) and knowing \( (Z_1, X_{1,0,2}) \) can decode \( W_1 \). (103) follows from the property of file-index-symmetric schemes, (104) follows from the sub-modular property of the entropy function. Hence, (34) is proved.

**APPENDIX D**

**PROOF OF LEMMA 5**

We will first prove (35).

\[ M + R(M) + H(W_1, X_{0,1,3}, X_{1,0,2}) \geq H(Z_1) + H(X_{1,0,2}) + H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1, X_{1,0,2}) + H(W_1, X_{0,1,3}, X_{1,0,2}) = H(Z_1, X_{1,0,2}, W_1) + H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1, W_1) + H(W_1, X_{0,1,3}, X_{1,0,2}) \]

DRAFT
\[ H(Z_1, W_1) + H(W_1, X_{1,0,2}) + H(W_1, X_{0,1,3}, X_{1,0,2}) = H(W_1, X_{1,0,2}) \]
\[ H(Z_1, W_1) + H(W_1, X_{2,0,1}) + H(W_1, X_{0,1,3}, X_{1,0,2}) = H(W_1, X_{1,0,2}) \] (106)
\[ \geq H(Z_1, X_{2,0,1}, W_1) + H(W_1) + H(Z_3, W_1, X_{0,1,3}, X_{1,0,2}) = H(Z_3, W_1, X_{1,0,2}) \] (107)
\[ = H(Z_3, X_{1,0,2}, W_1) + H(W_1) + H(Z_3, W_1, X_{0,1,3}, X_{1,0,2}) = H(Z_3, W_1, X_{1,0,2}) \] (108)
\[ = H(W_1) + H(Z_3, W_1, X_{0,1,3}, X_{1,0,2}, W_2, W_3) \] (109)
\[ = H(W_1) + H(W_1, W_2, W_3) = 4, \] (110)
where (105) follows from the fact that knowing \((Z_1, X_{1,0,2})\) can decode \(W_1\), (106) and (108) both follow from the property of user-index-symmetric schemes, (107) follows from the sub-modular property of the entropy function, (109) follows from the fact that knowing \((Z_3, X_{0,1,3}, X_{1,0,2})\) can decode \(W_2, W_3\), (110) follows from (1) and (4). Hence, (35) is proved.

Next, we will prove (36).

\[ 2M + 2R(M) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1) + H(Z_2) + H(X_{1,0,2}) + H(X_{0,1,3}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ \geq H(Z_1, X_{1,0,2}) + H(Z_2, X_{0,1,3}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \]
\[ = H(Z_1, X_{1,0,2}, W_1, X_{0,1,3}) + H(Z_2, X_{0,1,3}, W_1, X_{1,0,2}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \] (111)
\[ \geq H(W_1, W_2, W_3) + H(X_{1,0,2}, W_1, X_{0,1,3}) - H(W_1, X_{0,1,3}, X_{1,0,2}) \] (112)
\[ = 3, \]
where (111) follows from the fact that knowing \((Z_1, X_{1,0,2})\) can decode \(W_1, X_{0,1,3}\) is a deterministic function of \(Z_1\), knowing \((Z_2, X_{0,1,3})\) can decode \(W_1\), and \(X_{1,0,2}\) is a deterministic function of \(Z_2\), (112) follows from (23). Hence, (36) is proved.
APPENDIX E

PROOF OF LEMMA 6

We will first prove (44). Applying (1), (2), (3) and (43), the following chains of inequalities can be written as

\[ M + R(M) \]

\[ \geq H(Z_1) + H(X^1_{1,2,2}) + H(X^2_{1,2,2}) + H(X^3_{1,2,2}) \geq H(Z_1, X^2_{1,2,2}, X^3_{1,2,2}, X^1_{1,2,2}) \]

\[ = H(Z_1, X^2_{1,2,2}, X^3_{1,2,2}, W_1) + H(Z_2, Z_1, X^2_{1,2,2}, W_1) - H(Z_2, Z_1, X^2_{1,2,2}, W_1) + H(X^1_{1,2,2}) \]  (113)

\[ \geq H(X^3_{1,2,2}, Z_2, Z_1, X^2_{1,2,2}, W_1) + H(W_1, Z_1, X^2_{1,2,2}) - H(W_1, Z_1, Z_2, X^2_{1,2,2}) + H(X^1_{1,2,2}) \]  (114)

\[ = H(W_1, W_2) + H(W_1, Z_1, X^2_{1,2,2}) - H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) \]  (115)

\[ = H(W_1, Z_1, X^3_{1,2,2}) - H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + H(W_1, W_2) \]  (116)

\[ \geq H(W_1, Z_1, Z_2, X^2_{1,2,2}) - H(W_1, Z_1) + H(X^1_{1,2,2}) + H(W_1, W_2) \]  (117)

\[ = H(W_1, W_2) - H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + H(W_1, W_2) \]  (118)

\[ \geq H(W_1, Z_1) - 2H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + 2H(W_1, W_2) \]  (119)

\[ = H(W_1, Z_2) - 2H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + 2H(W_1, W_2) + H(W_1, X^1_{1,2,1}) - H(W_1, X^1_{1,2,1}) \]

\[ \geq H(W_1, Z_2, X^1_{1,2,1}) - H(W_1, X^1_{1,2,1}) - 2H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + 2H(W_1, W_2) + H(W_1) \]

\[ \geq H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, Z_2) + H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) \]

\[ - H(W_1, X^1_{1,2,1}) - 2H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + 2H(W_1, W_2) + H(W_1) \]  (121)

\[ \geq H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}, Z_2) + H(W_1, X^1_{1,2,1}, X^2_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) \]

\[ - H(W_1, X^1_{1,2,1}) - 2H(W_1, Z_1, Z_2) + H(X^1_{1,2,2}) + 2H(W_1, W_2) + H(W_1) \]  (122)

\[ \geq H(X^1_{1,2,2}) + 3H(W_1, W_2) + H(W_1) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - 2H(W_1, Z_1, Z_2), \]  (123)
where (113) follows from the fact that knowing \((Z_1, X^2_{1,2,2}, X^3_{1,2,2})\) can decode \(W_1\), (114), (117), (120) and (122) all follow from the sub-modular property of the entropy function, (115) and (118) both follow from the fact that \(X^1_{1,2,2}\) is a deterministic function of \(Z_1\), knowing \((Z_2, X^1_{1,2,2}, X^3_{1,2,2})\) can decode \(W_2\), (1) and (2), (116) follows from the property of user-index-symmetric schemes, (119) follows from the fact that \(H(W_1, Z_1) \leq H(W_1, Z_1, Z_2)\), (121) follows from the fact that \(X^2_{1,2,1}\) is a deterministic function of \(Z_2\), (123) follows from the fact that knowing \((Z_2, X^1_{1,2,1}, X^3_{1,2,1})\) can decode \(W_2\), (1) and (2) and \(H(W_1, X^1_{1,2,1}, X^2_{1,2,1}) \geq H(W_1, X^1_{1,2,1})\). Hence, (44) is proved.

Next, we will prove (45).

\[
2M + R(M) \geq H(Z_2, X^1_{1,2,1}, X^3_{1,2,1}) + H(Z_1) + H(X^2_{1,2,1})
\]

\[
\geq H(Z_2, X^1_{1,2,1}, X^3_{1,2,1}) + H(Z_1, X^2_{1,2,1})
\]

\[
= H(W_2, Z_2, X^1_{1,2,1}, X^3_{1,2,1}, X^2_{1,2,1}) + H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1})
\]

\[
- H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) + H(Z_1, X^2_{1,2,1})
\]

\[
\geq H(X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) + H(Z_1, X^2_{1,2,1}) + H(W_1, W_2)
\]

\[
= H(X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) + H(Z_1, X^2_{1,2,1}, X^1_{1,2,1}) + H(W_1, W_2)
\]

\[
\geq H(X^1_{1,2,1}, X^2_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) + H(W_1, W_2)
\]

\[
= H(Z_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) - H(W_1, X^1_{1,2,1}, X^2_{1,2,1}, X^3_{1,2,1}) + H(W_1, W_2)
\]

where (124) follows from the fact that \(X^1_{1,2,1}\) is a deterministic function of \(Z_2\) and knowing \((Z_2, X^1_{1,2,1}, X^3_{1,2,1})\) can decode \(W_2\), (125) follows from the sub-modular property of the entropy function, (1) and (2), (126) and (128) both follow from the fact that \(X^1_{1,2,1}\) is a deterministic function of \(Z_1\), (127) follows from the sub-modular property of the entropy function. Hence,
is proved.

Moreover, we directly have

\[
3M + R(M) \geq 2H(Z_1) + H(Z_1) + H(X^1_{1,1,2}) + H(X^2_{1,1,1}) + H(X^3_{1,1,1}) \\
\geq 2H(Z_1) + H(Z_1, X^2_{1,1,2}, X^3_{1,1,1}) + H(X^1_{1,1,1}),
\]

Hence, (46) is proved.

APPENDIX F

PROOF OF LEMMA 7

We will first prove (51).

\[
3M + 2R(M) \geq 3H(Z_1) + 2[H(X^1_{1,1,1}) + H(X^2_{1,1,1}) + H(X^3_{1,1,1})] \\
\geq 2H(Z_1, X^2_{1,1,2}, X^3_{1,1,1}) + H(Z_1, X^2_{1,1,2}, X^3_{1,1,1}) \\
\geq 2H(W_1, Z_1, X^2_{1,1,1}) + H(W_1, Z_1, X^3_{1,1,1}) \\
= H(W_1, Z_1, X^2_{1,1,1}) + H(W_1, Z_2, X^1_{1,1,1}) + H(W_1, Z_1, X^3_{1,1,1}) \\
= H(W_1, Z_1, Z_2, X^1_{1,1,1}, X^2_{1,1,1}) + H(W_1, Z_2, X^1_{1,1,1}, X^2_{1,1,1}) + H(W_1, Z_1, X^3_{1,1,1}) \\
\geq H(W_1, Z_1, Z_2, X^1_{1,1,1}, X^2_{1,1,1}) + H(W_1, X^2_{1,1,1}, X^3_{1,1,1}) + H(W_1, Z_1, X^3_{1,1,1}) \\
\geq H(W_1, Z_1) + H(W_1, Z_1, Z_2, X^1_{1,1,1}, X^2_{1,1,1}) + H(W_1, Z_1, Z_2, X^1_{1,1,1}, X^2_{1,1,1}, X^3_{1,1,1}) \\
= H(W_1, Z_1) + H(W_1, X^1_{1,1,1}, X^2_{1,1,1}) + H(W_1, W_2) \\
= 2 + H(W_1, Z_1) + H(W_1, X^1_{1,1,1}, X^2_{1,1,1}),
\]

where (129) follows from (43), (130) follows from the fact that knowing \((Z_1, X^2_{1,1,2}, X^3_{1,1,1})\) can decode \(W_1\) and knowing \((Z_1, X^2_{1,1,2}, X^3_{1,1,1})\) can decode \(W_1\), (131) follows from the property of user-index-symmetric schemes, (132) follows from the fact that \(X^1_{1,1,1}\) is a deterministic function
of $Z_1$ and $X^2_{1,1,2}$ is a deterministic function of $Z_2$, (133) and (134) both follow from the submodular property of the entropy function, (135) follows from the fact that $X^1_{1,2,2}$ is a deterministic function of $Z_1$, knowing $(Z_2, X^1_{1,2,2}, X^2_{1,2,2})$ can decode $W_2$, (1) and (2). Hence, (51) is proved.

Next, we will prove (52). We notice that

\[
H(W_1, Z_1) + H(W_1, X^2_{2,1,1}) + H(W_1, X^1_{1,1,2}, X^2_{1,1,2}) + H(W_1, X^1_{1,1,2}, Z_3)
\geq H(W_1, Z_1, X^2_{2,1,1}) + H(W_1) + H(W_1, X^1_{1,1,2}) + H(W_1, X^1_{1,1,2}, X^2_{1,1,2}, Z_3)
\]

(136)

\[
= H(W_1, Z_1, X^2_{2,1,1}) + H(W_1) + H(W_1, X^1_{1,1,2}) + H(W_1, X^1_{1,1,2}, X^2_{1,1,2}, Z_3, W_2)
\]

(137)

\[
= H(W_1, Z_1, X^2_{2,1,1}) + H(W_1) + H(W_1, X^1_{1,1,2}) + H(W_1, W_2)
\]

(138)

\[
= 3 + H(W_1, Z_1, X^2_{2,1,1}) + H(W_1, X^1_{1,1,2})
\]

(139)

where (136) follows from the sub-modular property of the entropy function, (137) follows from the fact that knowing $(Z_3, X^1_{1,1,2}, X^2_{1,1,2})$ can decode $W_2$, (138) follows from (1) and (2), (139) follows from the property of user-index-symmetric schemes. Removing $H(W_1, Z_3, X^1_{1,1,2}) + H(W_1, X^2_{2,1,1})$ from both sides of (139), (52) is proved.

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