LOWER QUASICONTINUITY, JOINT CONTINUITY AND RELATED CONCEPTS

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Abstract. Let \( X \) and \( Y \) be topological spaces, let \( Z \) be a metric space, and let \( f : X \times Y \to Z \) be a mapping. It is shown that when \( Y \) has a countable base \( \mathcal{B} \), then under a rather general condition on the set-valued mappings \( X \ni x \to f_x(B) \in 2^Z, B \in \mathcal{B} \), there is a residual set \( R \subset X \) such that for every \( (a,b) \in R \times Y \), \( f \) is jointly continuous at \( (a,b) \) if (and only if) \( f_a : Y \to Z \) is continuous at \( b \). Several new results are also established when the notion of continuity is replaced by that of quasicontinuity or by that of cliquishness. Our approach allows us to unify and improve various results from the literature.

1. INTRODUCTION

Let \( f \) be a mapping of the product \( X \times Y \) of two topological spaces into a metric space \( Z \). Let \( \mathcal{B} \) be a countable collection of subsets of \( Y \) and let \( B \) be the set of all \( y \in Y \) such that \( B \) includes a neighborhood base at \( y \) in \( Y \). In this note, we are mainly concerned with the following question (in the spirit of [2]): find some general assumptions on the partial mappings \( f_x, f^y \ (x \in X, y \in Y) \) ensuring the existence of a residual set \( R \subset X \) such that \( f \) is jointly continuous at each point of \( R \times B \). In [2], it is shown that if \( f \) is separately continuous, then such a set \( R \) exists. Theorem 3.3 below is the main result of this note; the first of assertions (1), (2) and (3) constituting this theorem is a much more general result than that of [2] (although the proof is hardly more difficult); the other two are of the same sort as (1), with the concept of continuity replaced by that of quasicontinuity in (2), and that of cliquishness in (3).

The paper is organized as follows. In Section 2, we introduce and examine the concept of lower quasicontinuity with respect to the variable \( x \) (at \( (a,b) \in X \times Y \)) for the mapping \( f : X \times Y \to Z \). (Here, \( Z \) may be any topological space.) The statement and the proof of the main Theorem 3.3, as well as some immediate corollaries, occupy Section 3. Finally, the results of Sections 2 and 3 are related in Section 4 to some well-known theorems from the literature.

As usual, \( \mathbb{N}, \mathbb{Q} \) and \( \mathbb{R} \) will denote, respectively, the sets of natural, rational and real numbers. For any set \( A \), \( 2^A \) will denote the set of all nonempty subsets of \( A \). For \( x \in X \), \( \mathcal{V}_X(x) \) will denote the set of all neighborhoods of \( x \) in \( X \).

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2. Lower quasicontinuity and mappings of two variables

In this section, $X$, $Y$, $Z$ are topological spaces and $f: X \times Y \to Z$ is a mapping. First, recall that a mapping $g: X \to Z$ is said to be quasicontinuous at $a \in X$ if for each neighborhood $U$ of $a$ in $X$ and each neighborhood $W$ of $g(a)$ in $Z$, there is an open set $O \subset X$ such that $\emptyset \neq O \subset U$ and $g(O) \subset W$ [16, 8] (see also [6, 1]; the terminology differs in [16, 1]). Obviously, $g$ is said to be quasicontinuous if it is quasicontinuous at each point of $X$. (From now on, obvious definitions will be omitted.) A set-valued mapping $F: X \to 2^Z$ is said to be lower quasicontinuous at $x_0 \in X$ if for each neighborhood $U$ of $x_0$ in $X$ and each open set $W \subset Z$ such that $F(x_0) \cap W \neq \emptyset$, there is an open set $O \subset X$ such that $\emptyset \neq O \subset U$ and $F(x) \cap W \neq \emptyset$ for every $x \in O$ [12, 13] (and [3] with another terminology). Given any $W \subset Z$, we put $F^-(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$.

The following concept is formulated in [10]: The mapping $f: X \times Y \to Z$ is said to be horizontally quasicontinuous at $(a, b) \in X \times Y$ if for each neighborhood $U$ of $a$ in $X$, each neighborhood $V$ of $b$ in $Y$, and each neighborhood $W$ of $f(a, b)$ in $Z$, there is an open set $O \subset X$ and $y \in V$ such that $\emptyset \neq O \subset U$ and $f(O \times \{y\}) \subset W$. Let us note that if $f^b$ is quasicontinuous at $a$, then $f$ is horizontally quasicontinuous at $(a, b)$. It is also easy to check the following:

**Proposition 2.1.** If $a \in X$ and if $f: X \times Y \to Z$ is horizontally quasicontinuous at $(a, y)$ for every $y$ in a nonempty open set $V \subset Y$, then the set-valued mapping $F^V: X \ni x \to f_x(V) \in 2^Z$ is lower quasicontinuous at $a$.

The converse of Proposition 2.1 is false: Let $f$ be the mapping of $\mathbb{R} \times \mathbb{R}$ into the discrete space $\{0, 1\}$ defined by $f(x, y) = 1$ if and only if $x - y \in \mathbb{Q}$; then for every nonempty open set $V \subset \mathbb{R}$, the set-valued mapping $\mathbb{R} \ni x \to f_x(V) \in 2^{\{0, 1\}}$ is lower quasicontinuous; however, there is no point of $\mathbb{R} \times \mathbb{R}$ at which $f$ is horizontally quasicontinuous. (Curiously enough, both facts hold for the same reason: For every $x \in \mathbb{R}$, the sets $x + \mathbb{Q}$ and $x + (\mathbb{R} \setminus \mathbb{Q})$ are dense in $\mathbb{R}$.)

**Definition 2.2.** We say that $f: X \times Y \to Z$ is lower quasicontinuous with respect to the variable $x$ (lower $X$-quasicontinuous, for short) at the point $(a, b) \in X \times Y$ if for each neighborhood $U$ of $a$ in $X$, each neighborhood $V$ of $b$ in $Y$, and each neighborhood $W$ of $f(a, b)$ in $Z$, there is an open set $O \subset X$ such that $\emptyset \neq O \subset U$ and $f(O \times \{x\} \times V) \cap W \neq \emptyset$ for every $x \in O$. Clearly, "$f: X \times Y \to Z$ is lower $X$-quasicontinuous" means exactly that for each nonempty open set $V \subset Y$, the set-valued mapping $F^V: X \ni x \to f_x(V) \in 2^Z$ is lower quasicontinuous. On the other hand, let us point out that if $f$ is horizontally quasicontinuous at $(a, b)$, then $f$ is lower quasicontinuous with respect to the variable $x$ at $(a, b)$.

The following proposition describes a wide class of mappings for which the results in Section 3 will apply (see Section 4).

**Proposition 2.3.** Let $V \subset Y$ be a nonempty open set, and suppose that the mapping $f: X \times Y \to Z$ satisfies one of the following:
(i) \( f \) is vertically quasicontinuous at every point of \( X \times V \), and, for each \( x \in X \), there exists a dense subset \( D_x \) of the space \( V \) such that \( f \) is lower \( X \)-quasicontinuous at every point of \( \{x\} \times D_x \).

(ii) \( f \) is lower \( Y \)-quasicontinuous at every point of \( X \times V \), and there exists a dense subset \( D \) of the space \( V \) such that \( f \) is lower \( X \)-quasicontinuous at every point of \( X \times D \).

Then the set-valued mapping \( F^V : X \ni x \rightarrow f_x(V) \in 2^Z \) is lower quasicontinuous.

Proof. Let \( U \subset X \), \( W \subset Z \) be nonempty open sets such that \( U \) meets \((F^V)^-(W)\), and let us show that there is a nonempty open set in \( X \) contained in \( U \) and \((F^V)^-(W)\). Let us choose an arbitrary point \((x_0, y_0)\) in \((U \times V) \cap f^{-1}(W)\). In case (i), there is a nonempty open set \( V_0 \subset V \) and \( x_1 \in U \) such that \( f(\{x_1\} \times V_0) \subset W \); taking \( y_1 \in V_0 \cap D_{x_1} \), one can find a nonempty open set \( U_0 \subset U \) such that \( f(\{x\} \times V) \cap W \neq \emptyset \) for every \( x \in U_0 \). In case (ii), there is a nonempty open set \( V_1 \subset V \) such that \( f(U \times \{y\}) \cap W \neq \emptyset \) for every \( y \in V_1 \). Take \( y_2 \in V_1 \cap D \) and choose \( x_2 \in U \) such that \( f(x_2, y_2) \in W \); then there is a nonempty open set \( U_1 \subset U \) such that \( f(\{x\} \times V) \cap W \neq \emptyset \) for every \( x \in U_1 \). \( \square \)

The next statement in terms of the familiar concept of quasicontinuity follows from 2.3; it will be used in Sections 3 and 4 to derive several Hahn or Kempisty type results from Theorem 3.3.

**Corollary 2.4.** Suppose that for each \( x \in X \), \( f_x \) is quasicontinuous and there is a dense set \( D_x \subset Y \) such that \( f^y \) is quasicontinuous at \( x \) for every \( y \in D_x \). Then \( f : X \times Y \rightarrow Z \) is lower \( X \)-quasicontinuous.

**Remark 2.5.** (1) It is easy to see that the mapping \( f : X \times Y \rightarrow Z \) is lower quasicontinuous with respect to the variable \( x \) at \((a, b) \in X \times Y\) if and only if \( f(a, b) \in f((U \cap A) \times V) \) for any neighborhood \( U \) of \( a \) in \( X \), any neighborhood \( V \) of \( b \) in \( Y \), and any dense subset \( A \) of \( X \).

(2) Let \( \tau_f \) be the topology on \( X \times Y \) generated by \( f : X \times Y \rightarrow Z \) and the two projections \( X \times Y \ni (x, y) \rightarrow x \in X \), \( X \times Y \ni (x, y) \rightarrow y \in Y \). Then, as the proof of Proposition 2.3 shows, the set \( Q \) of all \((x, y) \in X \times Y \) at which \( f \) is lower \( X \)-quasicontinuous is closed with respect to \( \tau_f \). Thus, any condition ensuring that \( Q \) is \( \tau_f \)-dense in \( X \times Y \) will imply that \( f \) is lower \( X \)-quasicontinuous; for instance, the conditions (i) and (ii) in Proposition 2.3 (when satisfied for all nonempty open subsets of \( Y \)) are of this sort.

3. JOINT CONTINUITY AND RELATED CONCEPTS

In order to state our main result in 3.3 below, let us recall some definitions concerning the concept of quasicontinuity and that, weaker, of cliquishness. Let \( X \) and \( Y \) be two topological spaces and let \((Z, d)\) be a metric space. A mapping \( g \) of \( X \) into \((Z, d)\) is said to be cliquish at \( a \in X \) if for each \( \varepsilon > 0 \) and each neighborhood \( U \) of \( a \) in \( X \) there is an open set \( O \subset X \) such that \( \emptyset \neq O \subset U \) and
diam(\(g(O)\)) \(\leq \varepsilon\) [16, 8]. (If \(W \in 2^Z\), then diam(\(W\)) = sup\{\(d(u, v) : u, v \in W\}\).) A mapping \(f\) of \(X \times Y\) into \((Z, d)\) is said to be quasicontinuous (cliquish) with respect to the variable \(x\) at \((a, b) \in X \times Y\) if for each neighborhood \(V\) of \(b\) in \(Y\) and each \(\varepsilon > 0\), there is a neighborhood \(U\) of \(a\) in \(X\) and an open set \(O \subseteq Y\) such that \(\emptyset \neq O \subseteq V\) and \(d(f(a, b), f(x, y)) \leq \varepsilon\) for all \(x \in U\), \(y \in O\) [9, 15] (respectively, and \(d(f(a, y), f(x, y)) \leq \varepsilon\) for all \(x \in U\), \(y, y' \in O\) [3]). In relation to Theorem 4.4 below, it is worth noticing that if \(a \in X\) and if \(f\) is cliquish with respect to the variable \(x\) at \((a, y) \in X \times Y\) for every \(y \in Y\), then the set of all \(y \in Y\) such that \(f\) is continuous at \((a, y)\) is residual in \(Y\) [3].

Recall also that a collection \(\mathcal{B}\) of nonempty open sets in a topological space is called a pseudobase (or \(\pi\)-base) for this space if any nonempty open set contains some member of \(\mathcal{B}\) [14].

**Lemma 3.1.** Let \(X\) be a topological space and let \((Z, d)\) be a bounded metric space. Let \(F\) be a lower quasicontinuous set-valued mapping of \(X\) into \(2^Z\). Let \(z_0 \in Z\). Then the mapping \(\phi : X \to \mathbb{R}\) defined by \(\phi(x) = \sup\{d(z_0, z) : z \in F(x)\}\) \((x \in X)\) is cliquish.

**Proof.** Let \(U\) be a nonempty open subset of the space \(X\), and let \(\varepsilon > 0\). Let us put \(r = \sup\{\phi(x) : x \in U\} - \varepsilon\) and let us choose \(x_0 \in U\) such that \(r < \phi(x_0)\). There is \(z_1 \in F(x_0)\) such that \(r < d(z_0, z_1)\); let us put \(\rho = d(z_0, z_1) - r\). Let \(O\) be a nonempty open subset of \(X\) contained in \(U \cap F^{-1}(B(z_1, \rho))\); then \(r < \phi(x) \leq r + \varepsilon\) for every \(x \in O\).

For a set-valued mapping \(F : X \to 2^Z\) and \(U \subseteq X\), let \(F(U)\) denote the set \(\bigcup_{z \in U} F(x)\).

**Lemma 3.2.** Let \(X\) be a topological space and let \((Z, d)\) be a bounded metric space. Let \(F\) be a lower quasicontinuous set-valued mapping of \(X\) into \(2^Z\). Then

\[ A = \{x \in X : \forall \varepsilon > 0, \exists U \in \mathcal{V}_X(x), \text{diam}(F(U)) \leq 2\text{diam}(F(x)) + \varepsilon\} \]

is a residual subset of \(X\).

**Proof.** Let \(\varepsilon > 0\), and let \(A_\varepsilon\) be the union of all open subsets \(U\) of \(X\) such that \(\text{diam}(F(U)) \leq 2\text{diam}(F(x)) + \varepsilon\) for every \(x \in U\). Let us show that the open subset \(A_\varepsilon\) of \(X\) is dense in \(X\); this will prove the lemma since \(\bigcap_{n \in \mathbb{N}} A_{1/n+1} \subseteq A\). Let \(O\) be a nonempty open subset of \(X\). Then, taking \(x_0 \in O\) and \(z_0 \in F(x_0)\), one can find a nonempty open subset \(O'\) of \(X\) contained in \(O \cap F^{-1}(B(z_0, \varepsilon/4))\). Let \(\phi\) be the real-valued mapping on \(X\) defined by \(\phi(x) = \sup\{d(z_0, z) : z \in F(x)\}\) for every \(x \in X\). By Lemma 3.1, there is a nonempty open set \(O'' \subseteq O'\) such that \(\text{diam}(\phi(O'')) \leq \varepsilon/4\). Let us verify that \(O'' \subseteq A_\varepsilon\), which will imply that \(O \cap A_\varepsilon \neq \emptyset\) since \(O'' \subseteq O\). Let \(z'' \in O''\). Let \(z_i \in F(O'')\), and let \(x_i \in O''\) such that \(z_i \in F(x_i)\) \((i = 1, 2)\); then

\[ d(z_1, z_2) \leq d(z_1, z_0) + d(z_0, z_2) \leq \phi(x_1) + \phi(x_2) \leq 2\phi(x'') + \varepsilon/2; \]
consequently \( \text{diam}(F(O'')) \leq 2\phi(x'') + \varepsilon/2 \). Now, let \( z'' \in F(x'') \cap B(z_0, \varepsilon/4) \); then
\[
\sup\{d(z_0, z) : z \in F(x'')\} \leq d(z_0, z'') + \sup\{d(z'', z) : z \in F(x'')\},
\]
and consequently \( \phi(x'') \leq \varepsilon/4 + \text{diam}(F(x'')) \). Since \( O'' \) is a nonempty open subset of \( X \) such that \( \text{diam}(F(O'')) \leq 2\text{diam}(F(x'')) + \varepsilon \) for every \( x'' \in O'' \), the inclusion \( O'' \subset A_\varepsilon \) holds. \( \square \)

**Theorem 3.3.** Let \( X \) and \( Y \) be topological spaces, let \( (Z, d) \) be a metric space, and let \( f : X \times Y \to Z \) be a mapping. Let \( \mathcal{B} \) be a countable collection of nonempty subsets of \( Y \). Let us suppose that, for every \( V \in \mathcal{B} \), the set-valued mapping \( F^V : X \ni x \to f_x(V) \in 2^Z \) is lower quasicontinuous. Then there is a residual set \( R \subset X \) such that for each \( a \in R \):

1. If \( f_a \) is continuous at \( b \in Y \) and \( b \) has a neighborhood base contained in \( \mathcal{B} \), then \( f \) is continuous at \( (a, b) \).
2. If \( f_a \) is quasicontinuous at \( b \in Y \) and if some neighborhood of \( b \) in \( Y \) has a pseudobase contained in \( \mathcal{B} \), then \( f \) is quasicontinuous with respect to the variable \( x \) at \( (a, b) \).
3. If \( f_a \) is cliquish at \( b \in Y \) and if some neighborhood of \( b \) in \( Y \) has a pseudobase contained in \( \mathcal{B} \), then \( f \) is cliquish with respect to the variable \( x \) at \( (a, b) \).

**Proof.** One can assume that \((Z, d)\) is bounded. By Lemma 3.2, for every \( V \in \mathcal{B} \), the set
\[
R_V = \{x \in X : \forall \varepsilon > 0, \exists U \in \mathcal{V}_X(x), \text{diam}(F^V(U)) \leq 2\text{diam}(F^V(x)) + \varepsilon\}
\]
is a residual subset of \( X \). Let us put \( R = \bigcap_{V \in \mathcal{B}} R_V \); since \( R \) is the intersection of a countable family of residual subsets of \( X \), \( R \) is a residual subset of \( X \). Let us consider \( a \in R \).

(1) Let \( b \in Y \) such that \( f_a \) is continuous at \( b \) and \( \mathcal{B} \) contains a neighborhood base at \( b \). Let \( \varepsilon > 0 \), and let \( V \in \mathcal{B} \) be a neighborhood of \( b \) in \( Y \) such that \( \text{diam}(f_a(V)) \leq \varepsilon/4 \). Since \( a \) belongs to \( R \), there exists \( U \in \mathcal{V}_X(a) \) such that \( \text{diam}(F^V(U)) \leq 2\text{diam}(F^V(a)) + \varepsilon/2 \). Since \( U \times V \) is a neighborhood of \( (a, b) \) in \( X \times Y \), and since
\[
\text{diam}(f(U \times V)) = \text{diam}(F^V(U)) \leq 2\text{diam}(f_a(V)) + \varepsilon/2 \leq \varepsilon,
\]
f is continuous at \( (a, b) \).

(2) Let us suppose that \( f_a \) is quasicontinuous at \( b \in Y \), and let us suppose that \( \mathcal{B} \) contains a pseudobase for some neighborhood \( V' \) of \( b \) in \( Y \). Let \( V \in \mathcal{V}_Y(b) \) and \( \varepsilon > 0 \). There is a nonempty open set \( O' \) in \( Y \) such that \( O' \subset V \cap V' \) and \( f_a(O') \subset B(f_a(b), \varepsilon/6) \). Let \( O \in \mathcal{B} \) contained in \( O' \) and open in \( V' \); \( O \) is a nonempty open set in \( Y \) contained in \( V \). Since \( a \in R \), there exists \( U \in \mathcal{V}_X(a) \) such that \( \text{diam}(F^O(U)) \leq 2\text{diam}(F^O(a)) + \varepsilon/6 \). Now, since for any \( (x, y) \in U \times O \),
\[
d(f(a, b), f(x, y)) \leq d(f_a(b), f_a(y)) + d(f_a(y), f_x(y)) \leq \varepsilon/6 + \text{diam}(F^O(U)) \leq \varepsilon,
\]
...
statement (2) is proved.

(3) Suppose that $f_a$ is cliquish at $b \in Y$ and that $B$ contains a pseudobase for some neighborhood $V'$ of $b$ in $Y$. Let $V \in \mathcal{V}_Y(b)$ and $\varepsilon > 0$. There is a nonempty open set $O'$ in $Y$ such that $O' \subset V \cap V'$ and $\text{diam}(f_a(O')) \leq \varepsilon/4$. Let $O \in B$ contained in $O'$ and open in $V'$; $O$ is a nonempty open set in $Y$ contained in $V$. Since $a \in R$, there is $U \in \mathcal{V}_X(a)$ such that $\text{diam}(f_a(U)) \leq 2\text{diam}(f_a(O)) + \varepsilon/2$. It follows from what precedes that

$$\text{diam}(f(U \times O)) = \text{diam}(F^O(U)) \leq 2\text{diam}(f_a(O)) + \varepsilon/2 \leq \varepsilon,$$

which establishes statement (3). □

**Remark 3.4.** In point (1) of Theorem 3.3, the topology on $Y$ is quite irrelevant inasmuch as the neighborhood of $b$ used to establish the continuity of $f$ at $(a, b)$ belongs to the collection $B$. Similar remarks hold for point (2) and point (3) of that same theorem.

In view of 3.3 (and 2.4, for Corollary 3.7), we can state:

**Corollary 3.5.** Let $X$ and $Y$ be topological spaces, let $Z$ be a metric space, and let $f : X \times Y \to Z$ be a mapping. Let us suppose that $Y$ has a countable base and that for every $y \in Y$, the mapping $f^y$ is quasicontinuous. Then there is a residual set $R \subset X$ such that for every $(a, b) \in R \times Y$, $f$ is continuous at $(a, b)$ provided that $f_a$ is continuous at $b$.

**Corollary 3.6.** Let $X$ and $Y$ be topological spaces, let $Z$ be a metric space, and let $f : X \times Y \to Z$ be a mapping. Let us suppose that $Y$ has a countable pseudobase and that for every $y \in Y$, the mapping $f^y$ is quasicontinuous. Then there is a residual set $R \subset X$ such that for every $(a, b) \in R \times Y$, $f$ is quasicontinuous (cliquish) with respect to the variable $x$ at $(a, b)$ provided that $f_a$ is quasicontinuous (respectively, cliquish) at $b$.

**Corollary 3.7.** Let $X$ be a topological space, let $Y$ be a topological space with a countable base (pseudobase), and let $f : X \times Y \to Z$ be a mapping of $X \times Y$ into a metric space $Z$. Suppose that for each $x \in X$, the mapping $f_x$ is continuous (quasicontinuous) and there is a dense set $D_x \subset Y$ such that $f^y$ is quasicontinuous at $x$ for every $y \in D_x$. Then there is a residual set $R \subset X$ such that $f$ is continuous (respectively, quasicontinuous with respect to the variable $x$) at every point of $R \times Y$.

**Remark 3.8.** One can also obtain Theorem 3.3 by using the following lemma established in [7]: Let $X$ be a topological space, let $(Z, d)$ be a metric space, and let $F : X \to 2^Z$ be a lower quasicontinuous set-valued mapping. Then for every $r > 0$ there exists an open dense subset $U$ of $X$ and a continuous mapping $f : U \to Z$ such that $d(f(u), F(u)) < r$ for all $u \in U$. 
4. New light on some well-known results

The aim of this section is to relate the results in Section 3 to some well-known results from the literature. In what follows, $X$ is a Baire space, $Y$ is a topological space, and $f$ is a mapping of $X \times Y$ into a metric space $(Z, d)$.

In [11, Theorem 1], Mibu proves that the set of continuity points of $f$ is residual in $X \times Y$ assuming $f_x$ to be continuous for all $x \in X$, $f^y$ continuous for all $y$ in a given dense subset of $Y$, and $Y$ first countable. Theorem 3.3 and Corollary 2.4 above immediately give Mibu’s result (in a somewhat more general form).

Theorem 4.1. Suppose that $Y$ is first countable and that, for each $x \in X$, $f_x$ is continuous and $f^y$ is quasicontinuous at $x$ for every $y$ in a dense set $D_x \subset Y$. Then for each $y \in Y$, there exists a residual set $R_y \subset X$ such that $f$ is continuous at every point of $R_y \times \{y\}$.

In [3, Theorem 3], Fudali concludes that $f$ is cliquish assuming $f_x$ to be cliquish for each $x \in X$, $f^y$ quasicontinuous for each $y \in Y$, and $Y$ locally second countable (a result stronger than Mibu’s Theorem 2 in [11]). The following (slightly more general) statement is deduced from 3.3.

Theorem 4.2. Suppose that $f^y$ is quasicontinuous for every $y \in Y$, and suppose that there is a dense set $D \subset Y$ such that for every $y \in D$:

(i) There is a dense Baire subspace $Q_y$ of $X$ such that $f_x$ is cliquish at $y$ for every $x \in Q_y$, and

(ii) some neighborhood of $y$ in $Y$ has a countable pseudobase.

Then $f$ is cliquish.

Proof. Let $U$ be a nonempty open set in $X$, $V$ a nonempty open set in $Y$, and $\varepsilon > 0$. Let us choose $b \in V \cap D$. By Theorem 3.3, there is a residual set $R \subset X$ such that for any $a \in R$, if $f_a$ is cliquish at $b$, then $f$ is cliquish with respect to the variable $x$ at $(a, b)$. Since $Q_b$ is a dense Baire subspace of $X$, $R \cap Q_b$ is dense in $X$; therefore there is $a \in U$ such that $f$ is cliquish with respect to the variable $x$ at $(a, b)$. Let us choose $U_1 \subset U \cap Q_y(a)$ contained in $U$ and a nonempty open set $V_1$ in $Y$ contained in $V$ such that $d(f(a, y), f(x, y')) < \varepsilon/2$ for all $x \in U_1$ and $y, y' \in V_1$; then $d(f(x, y), f(x', y')) < \varepsilon$ for all $x, x' \in U_1$ and $y, y' \in V_1$. \(\square\)

In [9, Theorem 1], it is proved by Martin that if $f_x$ is quasicontinuous for every $x \in X$, $f^y$ quasicontinuous for every $y \in Y$, and $Y$ second countable, then $f$ is quasicontinuous. Using 3.3 and 2.4 under condition (i), and 3.3 under condition (ii), Martin’s result can be improved as follows.

Theorem 4.3. Suppose that for each $y \in Y$, some neighborhood of $y$ in $Y$ has a countable pseudobase, and suppose that one of the following holds:

(i) For each $x \in X$, $f_x$ is quasicontinuous and there is a dense set $D_x \subset Y$ such that $f^y$ is quasicontinuous at $x$ for every $y \in D_x$. 

(ii) For each $y \in Y$, $f^y$ is quasicontinuous and there is a dense Baire subspace $Q_y \subset X$ such that $f_x$ is quasicontinuous at $y$ for every $x \in Q_y$.

Then $f$ is quasicontinuous.

**Proof.** Let $(a, b)$ in $X \times Y$. Let $U$ be an open neighborhood of $a$ in $X$, $V$ an open neighborhood of $b$ in $Y$, and $\varepsilon > 0$.

Case (i): By Theorem 3.3 and Corollary 2.4, for each $y \in Y$ there exists a residual set $R_y \subset X$ such that $f$ is quasicontinuous with respect to the variable $x$ at every point of $R_y \times \{y\}$. Let $V_1 \subset V$ be a nonempty open set such that $d(f(a, y), f(a, b)) < \varepsilon$ for every $y \in V_1$. Choose $y_1 \in V_1 \cap D_a$ and let $U_1 \subset U$ be a nonempty open set such that $d(f(x, y_1), f(a, b)) < \varepsilon$ for every $x \in U_1$. Now, choosing $x_1 \in U_1 \cap R_{y_1}$ gives an open neighborhood $U_2$ of $x_1$ contained in $U_1$ and a nonempty open set $V_2 \subset V_1$ such that $d(f(x, y), f(a, b)) < \varepsilon$ for every $(x, y) \in U_2 \times V_2$. Hence the mapping $f$ is quasicontinuous at $(a, b)$.

Case (ii): By Theorem 3.3, for each $y \in Y$ there is a residual set $R_y \subset X$ such that for any $a' \in R_y$, if $f_{a'}$ is quasicontinuous at $y$, then $f$ is quasicontinuous with respect to the variable $x$ at $(a', y)$. Choose a nonempty open set $U_1 \subset U$ such that $d(f(x, b), f(a, b)) < \varepsilon$ for each $x \in U_1$. Since $Q_b$ is a dense Baire subspace of $X$, $R_b \cap Q_b$ is dense in $X$. Choosing $x_1 \in U_1 \cap R_b \cap Q_b$ gives an open neighborhood $U_2$ of $x_1$ contained in $U_1$ and a nonempty open set $V_1 \subset V$ such that $d(f(x, y), f(a, b)) < \varepsilon$ for every $(x, y) \in U_2 \times V_1$. Hence the mapping $f$ is quasicontinuous at $(a, b)$. \qed

In [9, Theorem 4], Martin also proves that if $f_x$ is continuous for every $x \in X$, $f^y$ quasicontinuous for every $y \in Y$, and $Y$ first countable, then $f$ is quasicontinuous with respect to the variable $y$ (cf. also [15]). The following variant of this theorem is an easy application of 3.3.

**Theorem 4.4.** Let $b \in Y$ be a point with a countable neighborhood base in $Y$. Suppose that $f^y$ is quasicontinuous for every $y \in Y$, and suppose that $f_x$ is continuous at $b$ for every $x$ in a given dense Baire subspace $Q$ of $X$. Then:

1. The mapping $f$ is quasicontinuous with respect to the variable $y$ at each point of $X \times \{b\}$.

2. The set of all $x \in X$ such that the mapping $f$ is continuous at $(x, b)$ is residual in $X$.

**Proof.** Let $A$ be the set of all $x \in X$ such that the mapping $f$ is continuous at $(x, b)$. By Theorem 3.3, there is a residual set $R \subset X$ such that $f$ is continuous at each point $(x, b) \in R \times \{b\}$ provided that $f_x$ is continuous at $b$. Remark that $Q$ being a dense Baire subspace of $X$, the subset $R \cap Q$ of $A$ is dense in $X$, and consequently, $A$ is dense in $X$.

1. Let $a \in X$. Let $U \in \mathcal{V}_X(a)$ and $\varepsilon > 0$. Let $U'$ a nonempty open set in $X$ such that $U'' \subset U$ and $f^b(U'') \subset B(f^b(a), \varepsilon/2)$; taking $x_0 \in U' \cap A$ gives a
nonempty open set \( O \subset U' \) and \( V \in \mathcal{V}_Y(b) \) such that, for every \((x, y) \in O \times V, d(f(x, y), f(x_0, b)) < \varepsilon / 2\), and hence such that \( d(f(x, y), f(a, b)) < \varepsilon \).

(2) The \( G_\delta \)-set \( A \) in \( X \) being dense in \( X \), it is residual in \( X \).

As our last application of Theorem 3.3, we will now prove a variant of a recent result [5] Theorem 2.5. Following [5], a sequence \((f_n)_{n \in \mathbb{N}}\) of mappings of \( X \) into \((Z, d)\) is said to be equi-quasicontinuous at \( x \in X \) if, for each neighborhood \( U \) of \( x \) in \( X \) and each \( \varepsilon > 0 \), there exists an open set \( O \subset X \) and \( n_0 \in \mathbb{N} \) such that \( \emptyset \neq O \subset U \) and \( d(f_n(x), f_n(y)) < \varepsilon \) for every \( n \geq n_0 \) and \( y \in O \).

**Theorem 4.5.** Let \( f_n : X \to Z, n \in \mathbb{N}, \) be a sequence of quasicontinuous mappings, and let \( f : X \to Z \) be a mapping. Suppose that for each \( x \in X \), the sequence \((f_n(x))_{n \in \mathbb{N}}\) clusters to \( f(x) \), and suppose that for each \( x \) in a given dense Baire subspace \( A \) of \( X \), the sequence \((f_n(x))_{n \in \mathbb{N}}\) converges to \( f(x) \). Then, for each \( x \in A \), the following are equivalent:

1. The sequence \((f_n)_{n \in \mathbb{N}}\) is equi-quasicontinuous at \( x \).

2. The mapping \( f : X \to Z \) is quasicontinuous at \( x \).

**Proof.** We refer to [5] for the implication (1) \( \Rightarrow \) (2). To prove that (2) implies (1), let \( Y = \mathbb{N} \cup \{\infty\} \) equipped with the topology whose nonempty open sets are the subsets \( \{m \in \mathbb{N} : m \geq n\} \cup \{\infty\} \) of \( Y \) \((n \in \mathbb{N})\). Let \( g : X \times Y \to Z \) be the mapping defined by \( g(x, n) = f_n(x) \) and \( g(x, \infty) = f(x) \). It is easy to check that for each \( n \in \mathbb{N} \), the set-valued mapping \( X \ni x \to \{f_m(x) : m \geq n\} \cup \{f(x)\} \in 2^Z \) is lower quasicontinuous. Let \( a \in A \) such that \( f \) is quasicontinuous at \( a \). Let \( U \in \mathcal{V}_X(a) \) and \( \varepsilon > 0 \). There is \( n_0 \in \mathbb{N} \) such that \( d(f_n(a), f(a)) < \varepsilon / 3 \) for every \( n \geq n_0 \). Let \( V \) be a nonempty open set in \( X \) such that \( V \subset U \) and \( d(f(a), f(x)) < \varepsilon / 3 \) for every \( x \in V \). The mapping \( g_x \) being continuous at \( \infty \) for any \( x \in A \), and \( A \cap R \) being dense in \( X \) for any residual subset \( R \) of \( X \), it follows from Theorem 3.3 that there is \( b \in V \) such that \( g \) is continuous at \((b, \infty)\); in particular, there is a nonempty open set \( O \subset V \) and \( n_1 \geq n_0 \) such that \( d(f(b), f_n(x)) < \varepsilon / 3 \) for every \( x \in O \) and \( n \geq n_1 \). For every \( x \in O \) and \( n \geq n_1 \), we have

\[
d(f_n(x), f_n(a)) \leq d(f_n(x), f(b)) + d(f(b), f(a)) + d(f(a), f_n(a)) < \varepsilon.
\]

\[\square\]

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