Witnessing entanglement of continuous-variable systems based on their dynamics

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We introduce a family of entanglement witnesses for continuous variable (CV) systems, under the sole assumption that, at the time of the test, their dynamics is that of coupled harmonic oscillators. Entanglement is inferred from the Tsirelson nonclassicality test on one of the normal modes, without any knowledge about the state of the other mode. In each round, the protocol requires measuring only the sign of a coordinate (e.g. position) at one among several times. As such, it may be particularly suited to study entanglement of mesoscopic and macroscopic objects, where the boundary between classical and quantum behaviors are blurred. We find that our precession criterion detects non-Gaussian states, some of which are missed by previously reported CV entanglement witnesses.

I. INTRODUCTION

Once troublesome to the founders of quantum mechanics [1, 2], entanglement is now well-established as one of the defining features of quantum theory. While entanglement in discrete systems has gone through much scrutiny [3, 4], the field of continuous variable (CV) quantum entanglement has had its challenges outside the Gaussian regime [5]. The obvious testbed for CV entanglement dynamics is identical to the classical one, there exist quantum states for which \( |\Delta x|^2 + |\Delta p|^2 < c^2 + \frac{1}{2} \). This inequality surely holds for sufficiently pure classical states, though it is indeed an entanglement criterion within quantum theory thanks to Heisenberg’s uncertainty. By contrast, our criterion features a clear separation from any classical prediction. Also, exploiting the knowledge of the dynamics, our criterion is tested by measuring a single observable (we shall use a variable proportional to \( u \)) at different times, thus simplifying the required setup.

A posteriori, we find that our criterion detects states with negative Wigner functions (thus, non-Gaussian), some of which are missed by all existing criteria. Thus, besides being elegant, our criterion is also actually useful.

II. THE SINGLE-OSCILLATOR PROTOCOL

We review the protocol invented by Tsirelson [7] following the generalization given in [8]. The assumption is that the physical quantity \( A_1 \) is undergoing a uniform precession at pulsation \( \omega \), i.e. \( A_1(t) = A_1(0) \cos \omega t + A_2(0) \sin \omega t \), where \( A_2 \) is another physical quantity. For classical systems, \( A_1(t) \) is the value of \( A_1 \) at time \( t \); for quantum systems, it is the corresponding observable in the Heisenberg representation. A single round of the protocol involves measuring the value of \( A_1 \) at one of \( K \) different times given by \( t_k = (k/K)T \), where \( K > 0 \), \( k = 0, 1, ..., K - 1 \), and \( T \) is the period of oscillation. After several rounds, one estimates

\[
P_K = \frac{1}{K} \sum_{k=0}^{K-1} \left\{ \Pr[A_1(t_k) > 0] + \frac{1}{2} \Pr[A_1(t_k) = 0] \right\},
\]

where the second term in the bracket avoids singular behaviors for states with non-infinitesimal concentration on \( A_1 = 0 \). By inspection [8], the upper bound \( P_K \leq \mathbf{P}_K^c \) for a classical theory is easily derived: \( \mathbf{P}_K^c = \frac{1}{2} \) for \( K \) even, and

\[
\mathbf{P}_K^c = \frac{1}{2} \left( 1 + \frac{1}{K} \right) \text{ for } K \text{ odd.}
\]

Remarkably, in spite of the fact that the precessing dynamics is identical to the classical one, there exist quantum states for which \( P_K > \mathbf{P}_K^c \) for any odd \( K > 1 \).

For the remainder of the paper, we focus on the harmonic oscillator, i.e. a material point, whose time evolution is governed by the Hamiltonian \( H = \frac{1}{2m}p^2 + \frac{1}{2}m \omega^2x^2 \). The pair \( (\hat{x}, \hat{p}) \) clearly precesses at pulsation \( \omega \) and thus satisfies the assumption. On a given state \( \rho \), quantum theory predicts \( P_K = \text{Tr}(\rho Q_K) \) where

\[
Q_K = \frac{1}{K} \sum_{k=0}^{K-1} \text{pos}(X(t_k)),
\]

with \( \text{pos}(X) \) defined by \( \text{pos}(X) |x\rangle = \frac{1}{2} (1 + \text{sgn}(x)) |x\rangle \).

The maximum quantum score \( \mathbf{P}_K = \max_{|\psi\rangle} \langle \psi | Q_K | \psi \rangle \) (denoted \( \mathbf{P}_K^c \) in [8]) is achieved by \( |\mathbf{P}_K\rangle \), the eigenstate
of $Q_K$ with the largest eigenvalue. Tsirelson proved that $P_3 \geq 0.709 > P_5^c = 2/3$ [7]; similar violations are found for all odd $K$ [8]. The violation can be attributed to having suitable patterns in the Wigner function, in particular, suitably distributed negativities. A state with positive Wigner function cannot give any violation.

III. ENTANGLEMENT OF TWO HARMONIC OSCILLATORS

Now, the key insight is that $x$ may be the position of an effective oscillator, built out of two (or more) physical ones. We focus on the case of two oscillators, with arbitrary masses and frequencies, and an $x-x$ coupling. The standard decomposition in normal modes yields

$$H = \sum_{j=1}^{2} \left( \frac{1}{2m_j} p_j^2 + \frac{1}{2} m_j \omega_j^2 x_j^2 \right) - \frac{1}{2} g x_1 x_2$$

$$= \sum_{\sigma \in \{+, -\}} \frac{1}{2\mu^2} p_{\sigma}^2 + \frac{1}{2} \mu \omega^2 x_{\sigma}^2,$$  \hspace{1cm} (4)

where $\mu = \sqrt{m_1 m_2}$.

$$x_+(t) = \left( \frac{m_1}{m_2} \right)^{1/4} \cos \theta \ x_1(t) + \left( \frac{m_2}{m_1} \right)^{1/4} \sin \theta \ x_2(t)$$

$$x_-(t) = \left( \frac{m_1}{m_2} \right)^{1/4} \cos \theta \ x_2(t) - \left( \frac{m_2}{m_1} \right)^{1/4} \sin \theta \ x_1(t),$$

with mixing angle $\theta = \arctan(2[g, \mu(\omega_1^2 - \omega_2^2)]/2$, and normal frequencies $\omega_{\pm}^2 = \omega_1^2 \pm \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + \frac{g^2}{m_1 m_2}}$. The time evolution of the $x_\sigma(t)$ is a uniform precession around phase space with the period $T_\sigma = 2\pi/\omega_\sigma$. Therefore, the single-oscillator protocol can be performed for coupled oscillators with different frequencies by measuring the normal modes $x_\sigma(t)$ at times $t_k = (k/K) T_\sigma$ for $k = 0, 1, \ldots, K - 1$. Among the many ways to estimate $P_K$ for $x_\sigma$, one consists in measuring separately $x_1$ and $x_2$ in each round, forming the corresponding $x_\sigma(t)$, and checking its sign. Notice also that, up to a multiplicative constant, $x_\sigma(t)$ has the same form as $u$ defined in the entanglement criterion by Duan and coworkers [9].

It is important to stress that $H$ describes the dynamics during the certification protocol, not the interaction that prepared the state under study. Thus, all values of the parameters are allowed. In particular, the certification protocol can be performed when $g = 0$ and $\omega_1 = \omega_2$. In this case, $\theta$ can take on any value: indeed, for uncoupled oscillators precessing at the same frequency, all linear combinations of $x_1$ and $x_2$ are normal modes at that same frequency.

If $P_K > P_K^c$ for $x_\sigma$, the state of that mode has certainly a negative Wigner function. We want to study when one can further infer that the physical subsystems are entangled. This is not straightforward because, by performing the protocol on one of the normal modes, we learn nothing about the state of the other mode: the latter could be very mixed; or the two normal modes may be even entangled. We are going to provide the conditions under which entanglement can indeed be certified.

IV. RESULTS

For the quantum system, we denote the annihilation operators of the two physical oscillators as $\{a_1, a_2\}$: they are the subsystems whose entanglement we want to certify. As hinted, $x_\sigma(t)$ is the position of an effective oscillator denoted by the annihilation operator $a_\sigma$. Specifically, let $\{a_+, a_-\}$ be a new basis of modes, related to the original by the passive transformation

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \hspace{1cm} (6)$$

Here, $\theta$ is the mixing angle as previously defined. We are going to study the operator $Q_K$ given by Eq. (3) for the position operator $X_\sigma(t) = \sqrt{\frac{\hbar}{2i\omega_\sigma}} (a_\sigma e^{-i\omega_\sigma t} + a_\sigma^\dagger e^{i\omega_\sigma t})$.

When $\theta = \pi/4$, we have an analytical proof of existence of entanglement:

**Result.** If the precession protocol is performed with $\theta = \pi/4$, all states that violate the classical bound (2) for $a_+$ (or $a_-$) are entangled in $\{a_1, a_2\}$.

**Proof.** We show this for $a_+$ with a proof by contradiction. The proof for $a_-$ proceeds in a similar way.

Take $\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}$ separable in the $\{a_1, a_2\}$ subsystems. Its Wigner function is

$$W_\rho(a_1, a_2) = \sum_k p_k W_1^{(k)}(a_1) W_2^{(k)}(a_2),$$

where $\{a_1, a_2\}$ are the phase-space coordinates in the $\{a_1, a_2\}$ modes. For $\theta = \pi/4$ in (6), the Wigner function in terms of $\{a_+, a_-\}$, the phase-space coordinates in the $\{a_+, a_-\}$ modes, can be found with a straightforward coordinate transformation:

$$W_\rho(a_+, a_-) = \sum_k p_k W_1^{(k)}(a_+ - a_- \frac{\sqrt{2}}{\sqrt{2}}) W_2^{(k)}(a_+ + a_- \frac{\sqrt{2}}{\sqrt{2}}).$$

(8)

The measurement outcome in the $a_+$ basis is determined solely by the reduced Wigner function $W_{tr_{a_+} (\rho)}(a_+) = \int d\alpha_- W_{\rho}(a_+, \alpha_-)$. By converting the passive coordinates in the arguments into active transformations of the states, we find

$$W_{tr_{a_+} (\rho)}(a_+) = 2 \sum_k p_k \int d\gamma W_1^{(k)}(\hat{\rho}_1(a_+)) W_2^{(k)}(\hat{\rho}_2(a_+))(\gamma)$$

$$= \frac{2}{\pi} \sum_k p_k \text{tr}(\hat{\rho}_1(a_+) \hat{\rho}_2(a_+)).$$

(9)
where
\[
\tilde{p}_1(a_+) = D\left(\frac{1}{\sqrt{2}} a_+\right) e^{-i\pi a_+^t} \rho_1 e^{i\pi a_+^t} D\left(\frac{1}{\sqrt{2}} a_+\right)
\]
\[
\tilde{p}_2(a_+) = D\left(-\frac{1}{\sqrt{2}} a_+\right) \rho_2 D\left(-\frac{1}{\sqrt{2}} a_+\right).
\]
and \(D(\alpha)\) is the usual displacement operator. Thus \(W_{\text{tr}-(\rho)} \geq 0\), since it is a convex sum of inner products between density operators. However, negativity in the Wigner function of \(a_+\) is necessary for a violation of the classical bound of the precession protocol [8]. Therefore, when \(\theta = \pi/4\), any violation of the classical bound witnesses entanglement of the \(\{a_1, a_2\}\) subsystems.

Next, we are going to study in detail the protocol with \(K = 3\). For any value of \(\theta\) and of \(P_3 > P_3^c = \frac{\pi}{8}\), we are going to compute a numerical lower bound on the amount of certifiable entanglement of \(\{a_1, a_2\}\). We choose the logarithmic negativity \(S_N(\rho) = \log \min \left|\text{tr} \rho F^2\right|\) as the quantifier of entanglement. Since \(\min_\rho S_N(\rho) = \log \min_\rho \left|\text{tr} \rho F^2\right|\), and
\[
\min_\rho \text{tr} \left|\rho F^2\right|
\]
subject to \(\text{tr}(\rho Q_3(\theta)) = P_3\)

is a minimization of the trace norm under convex constraints, it can be cast as a standard semidefinite program (SDP) when \(\rho\) is truncated in the basis of the excitations of \(a_+\) and \(a_-\) [10]. We run the SDP for truncation \(0 \leq n_+, n_- \leq n = 11\) (both the form of the SDP and the choice of the truncation are described in Appendix A).

The results are plotted in Fig. 1. We observe that the certifiable \(S_N\) increases with \(\theta\) and \(P_3\), as shown more explicitly by the line cuts. We already knew that every \(P_3\) certifies entanglement when \(\theta = \pi/4\), and the graph indicates that this remains true down to \(\theta \approx \pi/8\). Below this value, one needs a sufficiently large \(P_3\), a low violation of the classical bound being compatible with separable states. When \(\theta = 0\), the precession protocol is performed on the first oscillator, and so no amount of violation detects entanglement.

V. COMPARISON WITH OTHER WITNESSES

Now we put our criterion in the context of entanglement witnesses for continuous variables (CVs), by comparing it to other criteria.

First of all, our witness uses quadrature measurements in the terminology of quantum optics. Other measurements than quadratures can be used to witness CV entanglement: for instance, one witness in Zhang et al. [11] uses local measurements of the generators of SU(\(N\)). In fact, any CV entanglement can in principle be witnessed by projecting the state into a finite dimensional subspace, and then applying techniques to witness entanglement of qudits [12]. Quadrature measurements have the appeal of having a classical analog, are practical in many platforms, and in some setups may be even the only available ones at this time (e.g. optomechanical systems with large masses).

The other entanglement witnesses we are aware of are open to false positives from classical theory. Specifically, the witnesses based on uncertainty relations [9, 11, 13–15] require measuring positions and momenta with a precision set by \(\hbar\). By contrast, in the case of our criterion, poor precision or wrong calibration may prevent the detection of entanglement, but will not lead to false positives. This is similar to what happens with Bell inequalities, where noise and imprecision can only decrease or cancel the violation.

Having mentioned this, it is natural to move on to the comparison in terms of characterization of the devices. Fully device-independent entanglement witnesses (Bell inequalities) that use only quadrature measurements have been hard to find: the few known examples are very specific [16–18]. Our witness is semi-device-independent: it works under the assumptions that the dynamics is a uniform precession, and that the same quadrature is measured whatever time is picked. Both assumptions are well-defined both in classical and quantum theory. In fact, the notions of “position” and “uniform precession” are operational, and their meaning immediate by everyday experience (what may not be immediate is the identification of a normal mode). Strict harmony is a very strong requirement in practice: we used it here for clarity, but it can actually be significantly relaxed. Indeed, the Tsirelson protocol can be extended to any dynamics with bounded trajectories, possibly upon adding an energy constraint [19].

Lastly, let us compare the states that are detected. Many existing criteria can witness entanglement of Gaussian states (like the two-mode squeezed state and its limiting case, the EPR state). Our criterion misses these states, since their Wigner function is positive. Conversely, there are states whose entanglement is detected by our criterion and is missed by all the previous ones. A concrete example is given in Appendix B, where we construct the class of states \(\{|\Psi_n\rangle\}\). These states are always entangled, but all are missed by [9, 11], and some by [13]. In particular, the maximal eigenstates of \(Q_K\) (with or without truncation) are of this type. More generally, our criterion with \(\theta = \pi/4\) is not a subset of any member of the family of uncertainty-based entanglement witnesses defined by Shchukin and Vogel [14] and Nha and Zubairy [15], which include [9, 13] as special cases. Indeed, these families can only detect entangled states with negative partial transpose. By contrast, our criterion with \(\theta = \pi/4\) is a nondecomposable entanglement witness (proof in Appendix C), which implies that it can detect at least one bound entangled state, although we have not been able to find an explicit example.
FIG. 1: (a) Heat map of logarithmic negativity as a function of the mixing angle $\theta$ and the score $P_3$. Only the range $0 \leq \theta \leq \pi/4$ is shown: other values of $\theta$ corresponds to this range with a sign change of $x_1$ and $x_2$, which can be effected with a local unitary on the $\{a_1, a_2\}$ basis and hence does not affect the amount of entanglement. The dimension of the Hilbert space is truncated at $n = 11$ for both modes $a_+$ and $a_-$. The dotted line separates the states with $S_N > 0$ (such that $S_N - \varepsilon > 0$, where $\varepsilon$ is the dual gap of the SDP) and those with $S_N = 0$. (b) Horizontal line cuts of the heat map. For fixed $P_3$, $S_N$ increases with $\theta$. (c) Vertical line cuts of the heat map. For fixed $\theta$, $S_N$ increases with $P_3$.

VI. CONCLUSION

We have introduced the precession criterion to witness the entanglement of two harmonic oscillators. This criterion is based on testing the quantumness of a normal mode using Tsirelon’s test for a single oscillator, and deducing the entanglement of the original oscillators without having any information about the other normal mode (obviously, having also some information about it can only tighten the lower bounds that we have obtained).

Our criterion detects a different set of states than those captured by previous ones. Also, it does not rely on other features of quantum theory (e.g., some form of uncertainty relations): it only assumes the form of the dynamics. As only straightforward coordinate measurements are used, and misidentification of entanglement in classical systems is excluded, our criterion is useful for objects which are too massive to be fully tomographed. It is timely, as recent advances in optomechanics have allowed for quantum control of objects with masses in the mesoscopic and macroscopic scales [6]. The ability to generate and detect quantum effects in such systems has proven useful in technologies that aim to exploit the effects of quantum mechanics, in particular, quantum sensing [20, 21] and metrology [22]. Moreover, it is also important in experimental studies of fundamental physics, in tests of collapse theories [23] and quantum-classical transitions [24].

Apart from what has been presented, an avenue of further research is the extension of this formalism beyond harmonic oscillators to general bound systems, where the trajectories of the system are still closed, but the period of oscillation might have an energy dependence [19].

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Appendix A: Semidefinite programming

In this appendix, we detail the numerical procedures we performed to obtain the results reported in the main text using semidefinite programming (SDP).

1. Minimizing entanglement for observed $P_k$ and given $\theta$

The entanglement of a state $\rho$ in the $\{a_1, a_2\}$ basis can be quantified with the logarithmic negativity $S_N(\rho) = \log \text{tr}[\rho^{12}]$. By bounding the logarithmic negativity from below, we can deduce the minimum entanglement of a state that violates the classical bound of the precession protocol.

To cast the minimization of $S_N(\rho)$ as an SDP, we introduce a Hermitian basis of operators $\{B_j\}_{j=0}^{n+1}$ for both $a_\sigma$ systems, for some truncation $0 \leq n_\sigma \leq n$, with the convention that $B_0 \propto 1$ and $\text{tr}(B_j B_k) = \delta_{j,k}$. In our numerical program, we used as $B_j$ the normalized form of the generalized Gell-Mann matrices. The operator basis over the truncation of $\{a_+, a_-\}$ is $\{B_j \otimes B_k\}_{j,k=0}^{n+1}$. Excluding $B_0 \otimes B_0$, we collect them into a vector as $\vec{BB} \equiv (B_0 \otimes B_1, \ldots, B_0 \otimes B_{(n+1)^2-1}, B_1 \otimes B_0, B_1 \otimes B_1, \ldots, B_1 \otimes B_{(n+1)^2-1}, \ldots, B_{(n+1)^2-1} \otimes B_{(n+1)^2-1}).$

Then $\rho = B_0 \otimes B_0 / \text{tr}(B_0^2) + \vec{x} \cdot \vec{BB}^T \vec{BB}$ and

$$\rho^{12} = \frac{(B_0 \otimes B_0)^T \vec{BB}^T}{\text{tr}(B_0^2)} + \vec{x} \cdot \vec{BB}^T \vec{BB}, \quad (A1)$$

where $\vec{x} = \text{tr}(\rho \vec{BB})$, and $\vec{BB}^T \vec{BB}$ is found by representing each operator in $\vec{BB}$ as a matrix in the $\{a_1, a_2\}$ basis (here is where $\theta$ is used) and taking the partial transpose in the usual way. Denoting its positive eigenvalues as $\lambda_+^k$ and its negative eigenvalues as $-\lambda_-^k$, so that $\lambda_{\pm}^k \geq 0$, we have the spectral decomposition

$$\rho^{12} = \sum_{\pm} \lambda_{\pm}^k |\lambda_{\pm}^k\rangle \langle \lambda_{\pm}^k| - \sum_{\pm} \lambda_{\pm}^- |\lambda_{\pm}^-\rangle \langle \lambda_{\pm}^-| \quad (A2)$$

$$= \rho_+ - \rho_-,$$  where $\rho_\pm > 0.$
Considering that the restriction \(0 \leq n_+, n_- \leq n\) allows for states in the \(\{a_1, a_2\}\) basis with \(0 \leq n_1, n_2 \leq 2n\), the Hermitian operator basis for each \(a_i\) is \(\{A_j\}_{j=0}^{(2n+1)^2-1}\). The full space of \(\varrho_0\) is spanned by \(\{A_j \otimes A_k\}_{j,k=0}^{(2n+1)^2-1}\), and we similarly define \(\overline{AA} \equiv (A_0 \otimes A_1, \ldots, A_{(2n+1)^2-1} \otimes A_j)\). Then,\[
\begin{align*}
\varrho_+ &= z \frac{A_0 \otimes A_0}{\text{tr}(A_0)^2} + \bar{\varrho}_+ \cdot \overline{AA}, \\
\varrho_- &= (z - 1) \frac{A_0 \otimes A_0}{\text{tr}(A_0)^2} + \bar{\varrho}_- \cdot \overline{AA},
\end{align*}
\tag{A3}
\]
where \(\bar{\varrho}_\pm = \text{tr}(\varrho_\pm \overline{AA})\), \(z = \text{tr}(\varrho_+), \) and \(\text{tr}(\varrho_-)\) is fixed by \(\text{tr}(\varrho_+ - \varrho_-) = 1\). This means that the logarithmic negativity of \(\rho\) in the \(\{a_1, a_2\}\) basis is\[
S_N(\rho) = \log \text{tr}[\varrho_+ - \varrho_-] = \log(2z - 1).
\tag{A4}
\]
Finally, defining \(\bar{q} \equiv \text{tr}(Q_K \bar{B} \bar{B})\), we have\[
P_K = \text{tr}(\rho Q_K) = \frac{1}{2} + \bar{x} \cdot \bar{q}.
\tag{A5}
\]

\textbf{SDP Procedure}

Define \(A_j\) and \(B_j\) according to the preceding discussion. Then,\[
\begin{align*}
\min_{\bar{x} \in R^{(n+1)^2-1},} & \quad z \quad \text{subject to} \quad P_K = \frac{1}{2} + \bar{x} \cdot \bar{q} \\
& \quad \frac{(B_0 \otimes B_0)\varphi_2}{\text{tr}(B_0)^2} + \bar{x} \cdot \overline{BB} = A_0 \otimes A_0 + (\bar{\varrho}_+ - \bar{\varrho}_-) \cdot \overline{AA},
\end{align*}
\tag{A6}
\]
\[
\begin{pmatrix}
\frac{(B_0 \otimes B_0)\varphi_2}{\text{tr}(B_0)^2} + \bar{x} \cdot \overline{BB} & 0 & 0 \\
0 & z \frac{A_0 \otimes A_0}{\text{tr}(A_0)^2} + \bar{\varrho}_+ \cdot \overline{AA} & 0 \\
0 & 0 & (z - 1) \frac{A_0 \otimes A_0}{\text{tr}(A_0)^2} + \bar{\varrho}_- \cdot \overline{AA}
\end{pmatrix} \succeq 0.
\]

With the minimum \(z\), \(S_N(\rho) = \log(2z - 1)\). For the results in the main text, we used \(K = 3\) and \(n = 11\). The choice of \(n\) is justified in the following appendix.

\section{Choice of truncation \(n\)}

We consider how our choice of truncation \(n\) affects the entanglement of states that maximally violate the classical bound. For each \(n\), we perform an SDP optimization as laid out in the preceding appendix with \(P_3 = P_3\), where \(P_3\) is the largest eigenvalue of \(Q_3\) restricted to the truncated subspace \(0 \leq n_+, n_- \leq n\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sn.png}
\caption{Log negativity of maximally-violating states against truncation \(n\) for various values of \(\theta\), found with SDP optimization. In all cases, the minimum \(S_N\) occurs at \(n = 11\).}
\end{figure}

In Fig. 2, we plot the logarithmic negativity against the truncation \(n\) for various values of \(\theta\). We find that the log negativity increases with \(n\) in steps of \(\Delta n = 6\). This reflects how \(P_K\) increases in steps of \(\Delta n = 2K\), which is due to the symmetry of the protocol and the relationship between the odd and even number states of \(|\psi_N\rangle\) [8]. Importantly, the minimum value of \(S_N\) is found for \(n = 11\) for all values of \(\theta\). As such, choosing \(n = 11\) for the numerical results in our main text provides us with a conservative estimate of the amount of entanglement witnessed for a given violation \(P_3\).
\[ \langle \Psi_n | a^\dagger_1 a_1 | \Psi_n \rangle = (\cos \theta)^2 \langle \psi_n | a^\dagger_+ a_+ | \psi_n \rangle, \]
\[ \langle \Psi_n | a^\dagger_2 a_2 | \Psi_n \rangle = (\sin \theta)^2 \langle n \rangle_{\psi_n}, \]
\[ \langle \Psi_n | a^\dagger_1 a_2 | \Psi_n \rangle = \sin \theta \cos \theta \langle n \rangle_{\psi_n}. \]

1. Application to criterion by Duan et al.

In the criterion laid out by Duan et al. [9], a state is entangled if for some \( c \in \mathbb{R} \),
\[ \left( \langle \Delta u \rangle^2 + \langle \Delta v \rangle^2 \right) < c^2 + \frac{1}{c^2}, \tag{B2} \]
where
\[ u = |c| \sqrt{\frac{m_1 \omega_1}{\hbar}} x_1 + \frac{1}{c} \sqrt{\frac{m_2 \omega_2}{\hbar}} x_2 \]
\[ = \frac{|c| \sqrt{2}}{1} \langle a_1 + a_\dagger \rangle + \frac{1}{c \sqrt{2}} \langle a_2 + a_\dagger_2 \rangle, \]
\[ v = |c| \sqrt{\frac{m_1 \omega_1}{\hbar} p_1 - \frac{1}{c \sqrt{2}}} \langle a_2 - a_\dagger_2 \rangle. \]

For the class of states in (B1), we have
\[ \langle \Delta u \rangle^2 + \langle \Delta v \rangle^2 = \sin(2\theta) \langle n \rangle_{\psi_n} \left( \frac{c^2}{\tan \theta} + \frac{\tan \theta}{c^2} + \text{sgn}(c) \right) + c^2 + \frac{1}{c^2}. \]

Note that \( \sin \theta, \cos \theta > 0 \) for the valid range of \( \theta \). Using \( \text{sgn}(x) \geq -1 \) and \( x + 1/x \geq 2 \) for \( x \geq 0 \),
\[ \langle \Delta u \rangle^2 + \langle \Delta v \rangle^2 \geq \sin(2\theta) \langle n \rangle_{\psi_n} + c^2 + \frac{1}{c^2} \tag{B3} \]
\[ > c^2 + \frac{1}{c^2}, \]
since \( |\psi_0 \rangle \neq 1 \implies \langle n \rangle_{\psi_n} > 0. \)

Therefore, all states in this family fails the Duan criterion for any choice of \( c \), while some—in particular, the maximally-violating state with or without truncation—are detected by our criterion.

2. Application to criterion by Zhang et al.

For states with \( \langle x_j \rangle = \langle p_j \rangle = 0 \), as is the case for the family of states given in (B1), the entanglement criteria given in (6) of Zhang et al. [11] becomes
\[ 4 \langle a^\dagger_1 a_1 \rangle \langle a^\dagger_2 a_2 \rangle < 4 \left| \langle a_1 a_2 \rangle \right|^2; \tag{B4} \]
\[ 4 \langle a^\dagger_1 a_1 \rangle \langle a^\dagger_2 a_2 \rangle < 4 \left| \langle a_1 a_2 \rangle \right|^2, \]
where the state is entangled if any of the two inequalities are met. For our family of states, we have
\[ 4 \langle a^\dagger_1 a_1 \rangle \langle a^\dagger_2 a_2 \rangle = 4 \left| \langle a_1 a_2 \rangle \right|^2 = (\sin(2\theta))^2 \langle n \rangle_{\psi_n}^2, \]
\[ 4 \left| \langle a_1 a_2 \rangle \right|^2 = 0. \tag{B5} \]

Hence, neither of the inequalities in (B4) are satisfied, so no member of our family of states will be found to be entangled by Zhang et al.’s criteria.

3. Application to criterion by Hillery and Zubairy

In the criterion by Hillery and Zubairy [13], a state is entangled if
\[ \left| \langle a_1 a_\dagger_2 \rangle \right|^2 > \left| \langle a^\dagger_1 a_1 a^\dagger_2 a_2 \rangle \right| \tag{B6} \]
As before, for our class of states in (B1),
\[ \left| \langle a^\dagger_1 a_1 a^\dagger_2 a_2 \rangle \right| = (\sin \theta \cos \theta)^2 \left( \langle n^2 \rangle_{\psi_n} - \langle n \rangle_{\psi_n} \right) \tag{B7} \]
\[ \left| \langle a^\dagger_1 a^\dagger_2 \rangle \right|^2 = (\sin \theta \cos \theta)^2 \langle n \rangle_{\psi_n}^2. \tag{B8} \]

Therefore, any state with \( \langle n^2 \rangle_{\psi_n} - \langle n \rangle_{\psi_n} \geq \langle n \rangle_{\psi_n} \) will not be found to be entangled by Hillery and Zubairy’s criteria. In particular, this includes maximally-violating states of \( K_W \) with truncation \( n \).

Appendix C: The precession criterion is an optimal, nondecomposable entanglement witness

The canonical form of an entanglement witness is an operator \( W = W^\dagger \) such that \( \text{tr}(W \rho) \geq 0 \) for all \( \rho \) separable and \( \text{tr}(W \rho) < 0 \) for at least one \( \rho \) entangled. We can rewrite our criterion for \( \theta = \pi/4 \) into the canonical form of an entanglement witness with
\[ W = \frac{1}{2} \left( 1 + \frac{1}{K} \right) \mathbb{1} - Q_K, \tag{C1} \]
as in this case, any violation of the classical bound \( (1 + 1/K)/2 \) witnesses entanglement. Following this, we shall prove a few properties of \( W \).

1. \( W \) is an optimal entanglement witness

From Theorem 1 of Lewenstein et al. [26], we call \( W \) optimal iff for all \( P \gg 0 \) and \( \epsilon > 0 \), \( W' = (1 + \epsilon)W - \epsilon P \) is not an entanglement witness. Let us consider the state
\[ |r \rangle \equiv |\alpha = -r \rangle_1 \otimes |\alpha = -r \rangle_2 \]
\[ = |\alpha = -\sqrt{2r} \rangle_+ \otimes |\alpha = 0 \rangle_- . \tag{C2} \]
The Wigner function of this state in \( a_+ \) is a Gaussian centered around \( x_+ = -2r/\sqrt{m\hbar\omega}, \ p_+ = 0 \). A direct calculation yields
\[
\langle r | W | r \rangle = \frac{1}{6} (1 - 2 \text{erf}(r) + \text{erf}(2r)) \leq \frac{1}{6} (1 - \text{erf}(r)).
\]
(C3)

Notice that \( \langle r | W | r \rangle \) can be chosen to be arbitrarily small by choosing \( r \) sufficiently large. In particular, by choosing \( r > \text{erf}^{-1} \left( \frac{6\epsilon}{1 + \epsilon} \langle r | P | r \rangle \right) \), we have
\[
\langle r | W | r \rangle < \frac{\epsilon}{1 + \epsilon} \langle r | P | r \rangle \implies \langle r | W' | r \rangle < 0. \quad (C4)
\]
Since \( |r\rangle \) is separable, \( W' \) is not an entanglement witness. As an appropriate \( r \) can be chosen for any \( P \gg 0 \) and \( \epsilon > 0 \), \( W \) is an optimal entanglement witness.

2. \( W \) is a nondecomposable entanglement witness

We can now proceed to show that \( W \) is not decomposable—that is, it cannot be written as \( W = cP + (1-c)Q^{T_2} \) for some \( c \in [0, 1] \) and \( P, Q \gg 0 \). Theorem 2 of Lewenstein et al. [26] states that if a decomposable entanglement witness is optimal, then it can be written as \( W = Q^{T_2} \), where \( Q \gg 0 \) contains no product vector in its range.

As we have shown that \( W \) is optimal, this means that \( W^{T_2} \gg 0 \) if \( W \) is decomposable. Here, we consider the smallest eigenvalue of \( W'' = \Pi_{\{K\}} W^{T_2} \Pi_{\{K\}} \), where
\[
\Pi_{\{K\}} = \left( \sum_{n=0}^{[K]} |n\rangle\langle n|_1 \right) \otimes \left( \sum_{m=0}^{[K]} |m\rangle\langle m|_2 \right)
\]
(C5)
is a projection of \( W^{T_2} \) onto a truncated space. The matrix elements of \( Q_K \) are known analytically, so \( W'' \) can be constructed explicitly. By diagonalizing \( W'' \) with standard techniques, the smallest eigenvalue of \( W'' \) is found to be negative. Hence, \( W'' \neq 0 \Rightarrow W^{T_2} \neq 0 \).

Therefore, since \( W \) is an optimal entanglement witness and \( W \neq Q^{T_2} \) for some \( Q \gg 0 \), \( W \) is not decomposable.

3. \( W \) can detect positive partial transposed entangled states

Theorem 3 of Lewenstein et al. [26] states that an entanglement witness is nondecomposable iff it detects positive partial transposed entangled states. Having established in the preceding discussion that \( W \) is both optimal and nondecomposable, \( W \) detects entangled positive partial transpose states.