Black Holes in de Sitter Space: 
Masses, Energies and Entropy Bounds

Alejandro Corichi\textsuperscript{1,} and Andres Gomberoff\textsuperscript{2,}\textsuperscript{†}

\textsuperscript{1}Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
A. Postal 70-543, México D.F. 04510, México

\textsuperscript{2}Centro de Estudios Científicos (CECS)
Casilla 1469, Valdivia, Chile

Abstract

In this paper we consider spacetimes in vacuum general relativity —possibly coupled to a scalar field— with a positive cosmological constant $\Lambda$. We employ the Isolated Horizons (IH) formalism where the boundary conditions imposed are that of two horizons, one of black hole type and the other, serving as outer boundary, a cosmological horizon. As particular cases, we consider the Schwarzschild-de Sitter spacetime, in both 2 + 1 and 3 + 1 dimensions. Within the IH formalism, it is useful to define two different notions of energy for the cosmological horizon, namely, the “mass” and the “energy”. Empty de Sitter space provides an striking example of such distinction: its horizon energy is zero but the horizon mass takes a finite value given by $\pi/(2\sqrt{\Lambda})$. For both horizons we study their thermodynamic properties, compare our results with those of Euclidean Hamiltonian methods and construct some generalized Bekenstein entropy bounds. We discuss these new entropy bounds and compare them with some recently proposed entropy bounds in the cosmological setting.

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\textsuperscript{\ast}Electronic address: corichi@nuclecu.unam.mx
\textsuperscript{†}Electronic address: andres@cecs.cl
I. INTRODUCTION

Black Holes are one of the most enigmatic constructs in present day physics. On the one hand they are the most simple predictions of the general theory of relativity [1], and on the other hand, they are the paradigmatic objects to test possible quantum theories of gravity; they have been shown to possess thermodynamical properties [2, 3], whose statistical origin should be explained by a quantum theory of gravity [4].

One of the aspects pertaining to gravitational physics that has gained some recent attention is the behavior of the theory in the presence of a positive cosmological constant $\Lambda$. Recent observations suggest that there is a positive cosmological constant in Nature, and this brings to the picture, among many others, some features closely related to black holes: the existence of cosmological event horizons [5]. These are causal horizons that exist even in the absence of matter, namely in de Sitter space. For each geodesic observer (all of which are equivalent given the homogeneity of the spacetime), there is a horizon that hides all the events that are inaccessible. A question that was answered by Gibbons and Hawking is whether cosmological horizons are subject to the same thermodynamical interpretation as black hole horizons. They showed that the same techniques that were applied to the BH were also useful in the cosmological case: cosmological horizons posses a temperature and entropy [6].

However, an aspect that was not uniquely defined at the time was the issue of associating a mass to the cosmological horizon, including the case of de Sitter space. The issue of finding an appropriate definition of mass in asymptotically de Sitter spacetimes is of course, not new. After the original Gibbons-Hawking construction, several new proposals have appeared in the past years, some of which involve assigning a negative mass to de Sitter spacetime [6, 7, 8, 9, 10, 11, 12, 13, 14]. The root of this particular feature, in the cases where the first law is used in the definition, is that the surface gravity $\kappa_c$ of the cosmological horizon is negative. This fact, together with the usual relation between temperature, mass and entropy implied by the first law $\delta M = \kappa_c/8\pi \delta A$, yields the mentioned result. As we shall see in what follows, this behavior is a general feature of cosmological horizons.

The purpose of this paper is threefold. Firstly, motivated by the considerations described previously, it is important to see whether some unanswered questions can find (interesting) answers: Is it possible to define an energy and mass for a cosmological horizon? can one have a consistent thermodynamical interpretation for such horizons? can one define new entropy bounds given the notion of energy contained in a bounded region? As we shall see, in this paper we will provide affirmative answers to all these questions, making use of the tools available from the Isolated Horizons formalism [15]. The second motivation of this paper, based entirely on considerations from the Isolated Horizons perspective, is that it is important to know whether the positive $\Lambda$ case can be appropriately dealt with, and whether one can learn something new pertaining static solutions (possibly hairy). Finally, we would like to compare our results with those coming from Euclidean methods.

The structure of the paper is as follows. In Sec. II we provide a brief (and incomplete) summary of the Isolated Horizons formalism and of canonical Euclidean methods needed for the remainder of the paper. However, a reader not particularly interested in the formalism can safely skip it and continue to Sec. III where we deal with the Schwarzschild-de Sitter solution in $2 + 1$ dimensions. This example is used to set the stage for the more interesting case of the Schwarzschild-de Sitter spacetime in $3 + 1$ dimensions, the subject of Sec. IV. In this section, two possible normalizations of the relevant vector field are considered, giving rise
to two possible definitions of horizon mass for the BH. In Sec. V we consider the cosmological horizon and define both its energy, and its mass. These two quantities do not coincide given that they can be interpreted to represent different objects. In this case the horizon energy can be naturally interpreted as the total energy contained in the region bounded by the cosmological horizon, while the mass is more an attribute of the horizon itself. In Sec. VI we make some thermodynamical considerations and analyze the different entropy bounds existent, from our perspective. Finally, in Sec. VII we summarize and conclude. In an appendix we re-analyze the treatment of the system in $2+1$ gravity from the perspective of Euclidean methods.

II. PRELIMINARIES: ISOLATED HORIZONS AND EUCLIDEAN METHODS

In this section, we give a brief review of the techniques used in the remaining of the sections for extracting the different dynamical and thermodynamical quantities of the systems under study. We first revise the Isolated Horizon Formalism, and then we go onto Euclidean methods. Those readers familiar with the formalisms or those interested only in the new mass formulae for the horizons can safely skip this section.

1. Isolated Horizons

In this part, we give a brief review of the Isolated Horizon Formalism, specially the notions that are used in the remaining of the sections.

In recent years, a new framework tailored to consider situations in which a black hole is in equilibrium (nothing falls in), but which allows for the exterior region to be dynamical, has been developed [15]. This Isolated Horizon (IH) formalism is now in the position of serving as starting point for several applications, from the extraction of physical quantities in numerical relativity [16] to quantum entropy calculations [17]. The basic idea is to consider space-times with an interior boundary (to represent the horizon $\Delta$, or horizons $\Delta_i$), satisfying quasi-local boundary conditions ensuring that the horizon remains ‘isolated’. Although the boundary conditions are motivated by geometric considerations, they lead to a well defined action principle and Hamiltonian framework. Furthermore, the boundary conditions imply that certain ‘quasi-local charges’ $Q_i$, defined at the horizon $\Delta$, remain constant ‘in time’, and can thus be regarded as the analogues of the global charges defined at infinity in the asymptotically flat context. The isolated horizons Hamiltonian framework allows to define the notion of Horizon Mass $M_\Delta$, as function of the ‘horizon charges’.

In the Einstein-Maxwell and Einstein-Maxwell-Dilaton systems considered originally [18, 19], the horizon mass satisfies a Smarr-type formula and a generalized first law in terms of quantities defined exclusively at the horizon (i.e. without any reference to infinity). The introduction of non-linear matter fields like the Yang-Mills field brings unexpected subtleties to the formalism [21]. However, one still is in the position of defining a Horizon Mass, and furthermore, this Horizon Mass satisfies a first law. The formalism accepts a cosmological constant without further modifications [18].

A (weakly) isolated horizon $\Delta$ is a non-expanding null surface generated by a (null) vector field $l^a$. The IH boundary conditions imply that the acceleration $\kappa$ of $l^a$ ($l^a \nabla_a l^b = \kappa l^b$) is constant on the horizon $\Delta$. However, the precise value it takes on each point of phase space (PS) is not determined a-priori. On the other hand, it is known that for each vector field
on space-time, the induced vector field $X_{t_0}$ on phase space is Hamiltonian if and only if there exists a function $E_{t_0}$ such that $\delta E_{t_0} = \Omega(\delta, X_{t_0})$, for any vector field $\delta$ on PS. This condition can be re-written as $\delta E_{t_0} = \frac{\kappa}{8\pi G} \delta a_\Delta + \text{work terms}$. Thus, the first law arises as a necessary and sufficient condition for the consistency of the Hamiltonian formulation. Thus, the allowed vector fields $t^a$ will be those for which the first law holds. Note that there are as many ‘first laws’ as allowed vector fields $t^a \equiv t^a$ on the horizon. However, one would like to have a Physical First Law, where the Hamiltonian $E_{t_0}$ be identified with the ‘physical mass’ $M_\Delta$ of the horizon. This amounts to finding the ‘right $\kappa$’. This ‘normalization problem’ can be easily overcome in the EM system [18]. In this case, one chooses the function $\kappa = \kappa(a_\Delta, Q_\Delta)$ as the corresponding function for the static solution with charges $(a_\Delta, Q_\Delta)$. However, for more complicated matter couplings such as the EYM system, this procedure is not as straightforward, given that the space of static solutions might be non-connected. In these cases, a consistent viewpoint is to abandon the notion of a globally defined horizon mass on Phase Space, and to define, for each branch labeled by $n = n_\circ$, representing a connected component of the static sector, a canonical normalization $t^a_{n_\circ}$ that yields the Horizon Mass $M_{\Delta}^{(n_\circ)}$ for the $n_\circ$ branch [20]. The horizon mass takes the form,

$$M_{\Lambda}^{(n_\circ)}(r_\Delta) = \frac{1}{2G} \int_0^{r_\Delta} \beta(n_\circ)(r) \, dr , \quad (2.1)$$

along the $n_\circ$ “branch” with $r_\Delta$ the horizon radius and where $\beta(r) = 2 \pi \kappa(r)$. Finally, for any $n$ one can relate the horizon mass $M_{\Lambda}^{(n)}$ to the ADM mass of static black holes. Recall first that general Hamiltonian considerations imply that, in the asymptotically flat context, the total Hamiltonian, consisting of a term at infinity, the ADM mass, and a term at the horizon, the Horizon Mass, is constant on every connected component of static solutions (provided the evolution vector field $t^a$ agrees with the static Killing field everywhere on this connected component) [18, 20]. In the Einstein-Yang-Mills case, since the Hamiltonian is constant on any $n$-branch, we can evaluate it at the solution with zero horizon area. This is just the soliton, for which the horizon area $a_\Delta$, and the horizon mass $M_\Delta$ vanish. Hence we have that $H^{(t_0,n)} = M_{\Lambda}^{(n)}$. Thus, we conclude:

$$M_{\Lambda}^{(n)} = M_{\Lambda, ADM}^{(n)} - M_{\Lambda}^{(n)}$$

on the entire $n$th branch [20, 21]. Thus, the ADM mass contains two contributions, one attributed to the black hole horizon and the other to the outside ‘hair’, captured by the ‘solitonic residue’. The formula (2.2), together with some energetic considerations [22], lead to the model of a colored black hole as a bound state of an ordinary, ‘bare’, black hole and a ‘solitonic residue’, where the ADM mass of the colored black hole of radius $r_\Delta$ is given by the ADM mass of the soliton plus the horizon mass of the ‘bare’ black hole plus the binding energy [22].

In pure Einstein gravity with a positive cosmological constant the formalism has to be appropriately adapted. The quasi-local geometrical conditions defining the horizon are not modified. In the case that we are interested in this paper, namely space-times with both a black hole and a cosmological horizon, one has to specify boundary conditions on both horizons (see Fig. 1). Among the assumptions one has to specify that the region of interest $M$ is bounded by a black hole type horizon $\Delta_h$ and a cosmological type horizon $\Delta_c$. Furthermore, the issue of normalization of the vector field defining horizon mass is more subtle. In the asymptotically flat context one always had, in the static sector, a properly normalized Killing
field to be used in the normalization. In the situation considered here, there is no such privileged asymptotic vector field and one has to find a new prescription. Now, there is no asymptotic region, so there is no ADM mass, but the Hamiltonian responsible will have (on shell) a term coming from each horizon. That is, the total Hamiltonian $H_t$ is given by,

$$H_t = E^c_\Delta - E^b_\Delta$$

(2.3)

where $E^c_\Delta$ is the energy associated with the boundary term at the cosmological horizon, and $E^b_\Delta$ is the corresponding one for the black hole horizon. Finally, let us remark that a positive cosmological constant with two horizons had been considered before [18], but the issue of the normalization was not fully addressed. A complete treatment of this subject is one of the objectives of this work.

2. Euclidean Methods

One may also use Euclidean methods [23] to deal with systems having more than one horizon [13, 14]. Here one is interested on the statistical mechanics of the system, which is considered to be in thermodynamical equilibrium. Because the various horizons will have, in general, different temperatures, the system may be in equilibrium with one of them only. This has the same flavor as the isolated horizon formalism in the sense that one may pick one of the horizons present in the system to proceed with the analysis.

For Kerr–de Sitter geometries, the idea is to construct an action principle appropriate to treat either of the two horizons as a boundary. One then may compute the partition function evaluating the Euclidean path integral of the system. The partition function will depend on the quantities fixed on the boundary. The other horizon is treated as any other point of the manifold, namely, no field is fixed there. In order for the action to have an extremum in a regular Euclidean geometry, one fourth of the area of that horizon must be added to the action principle [24]. There is a freedom in using either horizon as a boundary and the system will be in thermodynamical equilibrium with the temperature of the second, regular one.

The problem of normalization of the timelike killing vector will be present here as well. Normally, within this viewpoint one would like to consider the “observer” to be located
at the boundary, and normalize the time variable such that it corresponds to the proper time measured by that observer. Technically this implies requiring the lapse function to be unity at the boundary. This may be done, for instance, when dealing with asymptotically flat spacetimes with boundary at infinity, but fails when considering (anti–) de Sitter spacetimes. In the anti-de Sitter case \[25\], one may normalize the killing vector at infinity requiring that the associated canonical generators of the AdS group have the standard normalization (standard structure constants). In \[14\] the de Sitter case was treated in such a way that, on the one hand, in the limit of vanishing cosmological constant the usual flat space normalization was recovered and, on the other hand, the algebra of charges corresponds to the continuation of dS to AdS by changing the sign of the cosmological constant. In this paper we shall adopt a slightly different viewpoint.

III. DE SITTER IN 2+1 DIMENSIONS

In this section we shall consider 2 + 1 dimensional gravity with a positive cosmological constant. It is known that there are no black hole solutions in this case, but there are cosmological horizons. In the literature, it has been useful to consider a de Sitter spacetime with a point particle at the origin \[26\]. This is the situation that we shall study. Note that in this section we will consider space-times with only one horizon.

Let us recall the general form of the de Sitter space-time with a conical defect at a pole,

\[ds^2 = -\left(\alpha^2 - \frac{r^2}{l^2}\right) dt^2 + \left(\alpha^2 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2\]  

(3.1)

where \(\Lambda = 1/l^2\) is the positive cosmological constant and

\[\alpha = 1 - 4GE,\]  

(3.2)

where \(E\) is the energy of the particle sitting at the origin \(r = 0\). This energy, computed first in \(26\), may also be obtained as the conjugate of the killing vector \(\frac{1}{\alpha} \partial/\partial t\).

In appendix A a treatment within an Euclidean canonical formulation is given. The case of empty de Sitter is recovered when \(E = 0\), namely, \(\alpha = 1\). This spacetime has a horizon located at,

\[r_\Delta = l\alpha\]  

(3.3)

The surface gravity \(\kappa\) of the Killing vector field \(\ell = (\partial/\partial t)\), that generates the cosmological horizon is given by,

\[\kappa = -\frac{r_\Delta}{l^2} = -\frac{\alpha}{l}\]  

(3.4)

However, the vector field \(\ell\) is not normalized at the origin since \(\ell \cdot \ell|_{r=0} = -\alpha^2\). We would like to choose as a normalization for the vector field \(\ell\) such that it is normalized with respect to the privileged observer at \(r = 0\). From the Euclidean perspective this is quite natural. The Euclidean geometry can be arranged to be regular at the cosmological horizon (fixing the appropriate temperature) but it is always singular at \(r = 0\) if \(\alpha \neq 1\), therefore, we chose \(r = 0\) as the boundary, and we demand the killing vector to be properly normalized in there.
Thus, we define the new KVF $\tilde{\ell} := \ell / \alpha$, such that $\tilde{\ell} \cdot \tilde{\ell}|_{\tau=0} = -1$. Thus, the new surface gravity is given by,

$$\tilde{\kappa} = \frac{\kappa}{\alpha} = -\frac{1}{\ell}$$

(3.5)

We would like to argue that the temperature that is assigned to this normalized KVF is physically relevant since this corresponds to the temperature that the privileged observer at the origin measures. The temperature $|\kappa|/2\pi$ is not measured by any observer. The first observation is that the physical temperature one would assign to such family of space-times, namely

$$T = \frac{|\tilde{\kappa}|}{2\pi} = \frac{1}{2\pi l}$$

(3.6)

is independent of $E$, that is, it is insensitive to the existence of the massive particle at the origin. This is the first unexpected observation of this note.

Let us now consider The first law for Isolated Horizons that reads in this case [27],

$$\delta E_\Delta = \frac{\kappa}{8\pi G} \delta a_\Delta$$

(3.7)

Here $E_\Delta$ is the horizon contribution to the Hamiltonian that we are identifying as the horizon energy. It is valid for any choice of $\kappa$. Using our choice of surface gravity $\tilde{\kappa}$ given by Eq. (3.5), we can now integrate the first law to get,

$$E_\Delta = -\frac{r_\Delta}{4Gl} + E_0 = -\frac{1}{8\pi G l} a_\Delta + E_0 ,$$

(3.8)

where $E_0$ is an integration constant.

The horizon radius, for a point particle given by $\alpha$ is then $r_\Delta = l\alpha$. Thus, the cosmological horizon energy of such spacetime is,

$$E_\Delta = -\frac{\alpha}{4G} + E_0 .$$

(3.9)

Note that with this normalization, the mass of the point particle is equal to $E$ in (3.2) if we set $E_0 = 1/4G$, so that de Sitter is defined to have zero horizon energy. Thus we can compute the horizon energy of the cosmological horizon for the point particle spacetime.

$$E_\Delta = -\frac{\alpha}{4G} + \frac{1}{4G} = \frac{1 - \alpha}{4G} = E$$

(3.10)

That is, as expected, the surface contribution to the Hamiltonian is measuring the total energy contained within the horizon (i.e. the particle). In 2+1 gravity we don’t expect to have, in vacuum, more contribution to the energy coming from the geometry. That is, the total energy should correspond to the contribution from the point particles in the interior region. As we shall see in the following sections, such interpretation for the horizon energy of a cosmological horizon continues to be valid in 3+1 dimensions. It is interesting to note that the horizon Energy is independent of the value of the cosmological constant; it only cares about the “mass” of the point particle at the origin.

The above analysis corresponds to setting $N = 1/\alpha$ in Eq. (A5) of the Appendix A, and taking the boundary at the cosmological horizon. Nevertheless, from the Euclidean perspective, the arguments leading to (3.9) have a different origin. In this formulation one does not impose Eq. (3.7). Instead, one takes the boundary – as it is normally done
in asymptotically flat space – to be the outermost sphere in Euclidean space, that is, the cosmological horizon. Note however, that in the Euclidean formalism is much more natural to place the boundary on the particle, because doing so one can remove the conical singularity produced by it. The cosmological horizon is then a regular point, and the system may be considered to be in equilibrium at its temperature. The two choices for the boundary represent two different solutions of Euclidean gravity (see [13, 14]), and therefore it is natural they have different energies. These two energies correspond, in the terminology of the IH formalism, to the distinction between the “mass” and “energy” of the horizon.

Let us now consider the Horizon Mass $M_{\Delta}$. Intuitively, it should be closely related to the horizon energy. However, one of the main points of this paper is to argue that such quantities represent different physical objects. The 3 dimensional system we are considering is a good example for showing this fact.

Let us define the first law for the Isolated Horizons Mass as follows,

$$\delta M_{\Delta} = \frac{|\kappa|}{8\pi G} \delta a_{\Delta} \quad (3.11)$$

This form of the first law is consistent to the one adopted in the context of the Euclidean formalism, where $|\kappa|$ corresponds to the temperature of the horizon, whose inverse is a positive quantity, namely, the period in the Euclidean time.

Using our choice of surface gravity $\tilde{\kappa}$ given by Eq. (3.5), we can now integrate the first law to get,

$$M_{\Delta} = \frac{1}{8\pi G l} a_{\Delta} + M_0 \quad (3.12)$$

where $M_0$ is an integration constant. Thus, the cosmological horizon mass of such spacetime is,

$$M_{\Delta} = \frac{\alpha}{4G} + M_0 \quad (3.13)$$

This expression is exactly what is obtained in Appendix A using canonical Euclidean methods when the boundary is placed at the particle, leaving the cosmological horizon as a regular point in the Euclidean manifold satisfying. The system may be interpreted to be in thermal equilibrium at the temperature of the cosmological horizon. The first law of thermodynamics, – which takes the form (3.11) in terms of the Lorentzian parameters– then holds.

The choice of the constant $M_0$ is very important in giving physical meaning to the quantity $M_{\Delta}$ in (3.13). The normalization we shall choose is that the horizon mass vanishes when the horizon area goes to zero. We shall then chose $M_0 = 0$, so we get

$$M_{\Delta} = \frac{\alpha}{4G} \quad (3.14)$$

Again, the horizon mass only knows about the particle mass at the origin and not the cosmological constant. When such a particle is not present, that is, in the case of $2 + 1$ de Sitter space-time the cosmological horizon mass is $M_{\Delta}^{dS} = 1/4G$. Let us now consider the behavior of $M_{\Delta}$ as we increase the “energy” $E$ of the point mass. We start with $E = 0$ that corresponds to $\alpha = 1$ and, as $E$ increases, the parameter $\alpha$, the horizon radius, and therefore the mass $M_{\Delta}$ decreases. The extreme point is when $4GE$ approaches $1$. In the limit, the spacetime “opens-up”, the horizon shrinks to zero radius, and the horizon mass also vanishes (for a discussion of this spacetime see [30]). We see then that our choice of $M_0$ is justified.
Let us now compare this results with those available in the literature \cite{9, 28, 29}. The first difference is that, as already mentioned, the temperature is always positive and constant. This is to be contrasted with the value reported in \cite{28, 29} and \cite{9}: \( T_{ssv} = \alpha/2\pi l \), which seems to be the standard value in the literature. Regarding the mass, we can compare our value with that reported in \cite{9}, who report a value of \( M_{bbm} = \alpha^2/8G \). In particular, their value for pure de Sitter is \( M_{dS_{bbm}} = 1/8G \). Even when qualitatively similar, our results have quantitative different values for the horizon mass \( M_{bbm} \), due to our choice of normalization of the Killing field \( \text{(3.4)} \). There is also an important qualitative difference with \cite{9, 28}. The temperature and mass found in our case are associated to an observer, namely the particle at \( r = 0 \), whereas the quantities found in \cite{9, 28} refer to observers in an asymptotic region.

### IV. SCHWARZSCHILD-DE SITTER IN $3+1$

In this section we shall return to the situation in which the spacetime under consideration possesses two isolated horizons, one being the black hole horizon and the other the cosmological horizon.

Let us write down the metric for de Schwarzschild - de Sitter space (SdS),

\[ ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 d\Omega^2 \quad (4.1) \]

where \( f^2 = \left(1 - \frac{2\mu}{r} - \frac{r^2}{l^2}\right) \), \( l = \sqrt{3/\Lambda} \), and \( \mu \) is a “mass parameter”. If there were no cosmological constant (and therefore the solution were Schwarzschild), \( \mu \) would correspond to the ADM mass. In the case that \( \mu = 0 \) we recover (a portion of) de Sitter spacetime. In these coordinates, the spacetime possesses two horizons, given by the zeros of \( f^2 \). The smaller root is called the black hole horizon \( r_b \), and the larger one is the cosmological horizon \( r_c \).

The family of spacetimes is parameterized by two numbers, namely \( l \) and \( \mu \), or alternatively, the two numbers \( r_b, r_c \). It is convenient to use some relations between \( l \) and \( \mu \) and the two horizon radii \( r_b, r_c \):

\[ l^2 = r_b^2 + r_c^2 + r_br_c \quad ; \quad \mu = \frac{r_b r_c (r_b + r_c)}{2l^2} \quad (4.2) \]

This family of space-times is well behaved, provided that the parameter \( \mu \) is less than \( \mu < l/\sqrt{27} \). In this limit, the two parameters \( r_b \) and \( r_c \) approach each other, and \( r \) fails to be a good coordinate. In this limit, the spacetime is known as the “Nariai solution” \cite{31, 32}. The region between the two horizon becomes a “tube” with \( S^2 \) sections of constant area, bounded by the two horizons of the same area \( A_m = (4\pi/3) l^2 \).

#### A. The standard normalization

In this case, the vector field that we take in order to assign a surface gravity is \( \xi = \partial/\partial t \). This choice is motivated by the analogy with asymptotically flat solutions and AdS where this vector has some special properties. For this choice, the surface gravity is given by

\[ \kappa = \frac{1}{2} (f^2)'|_h = \frac{1}{2} \left( \frac{1}{r_h^2} - \frac{3r_h}{l^2} \right) \quad (4.3) \]
FIG. 2: The cosmological horizon (geometric) radius $r_c$ and the BH radius $r_b$ are plotted as functions of $r_b$. The $y$ axis corresponds to the de Sitter limit where there is no black hole ($r_b = 0$) and the cosmological radius $r_c = l$. The Nariai limit is when both radii coincide. In this figure, the parameter $l$ is set to $l = 1$.

where we denote by $r_h$ the horizon geometrical radius (i.e. $a_h = 4\pi r_h^2$). It turns out that in the black hole horizon $r_b$, the surface gravity is positive, and in the cosmological horizon $r_c$, it is negative.

If we now consider this normalization, the IH formalism tells us that, on each horizon, the first law is valid,

$$\delta M_\Delta = \frac{|\kappa_h|}{8\pi} \delta a_\Delta .$$  \hspace{1cm} (4.4)

This is a geometrical identity, valid on both horizons independently. Therefore, we can integrate and find the black hole horizon mass

$$M^b_\Delta (r_b) = \int_0^{r_b} r_b' \kappa (r_b') \, dr_b' = \mu (r_b)$$  \hspace{1cm} (4.5)

where we have chosen, as boundary condition that $M^b_\Delta (r_b = 0) = 0$. It is easy to see that the parameter $\mu (r_b)$ is given by:

$$\mu (r_b) = \frac{r_b}{2} \left(1 - \frac{r_b^2}{l^2}\right).$$

This was first obtained in \[13\] in the Euclidean version of the theory. The mass of the cosmological horizon, is given, by

$$M^c_\Delta = -\mu + M_0 ,$$  \hspace{1cm} (4.6)

where $M_0$ is an integration constant. In \[13, 14\] $M_0$ was taken to be equal to zero. Here we will take a different point of view. Note that in the Nariai limit the two horizons become
undistinguishable (and, in fact, the two corresponding Euclidean instantons become the same). Hence, we will set the $M_0$ so that in the Nariai limit $M^b_\Delta = M^{\Delta}_\Delta$, that is,

$$M_0 = \frac{2l}{\sqrt{27}}. \quad (4.7)$$

We therefore conclude that the horizon mass of pure de Sitter is not zero, but $M^\Delta_{DS} = M_0$.

Note that in the interval of interest, $r_b \in [0, l/\sqrt{3}]$, $\mu(r_b)$ is a monotonic function of $r_b$. It is interesting to note that in the extreme, Nariai limit, the “temperature”, defined as $T = \kappa/2\pi$, vanishes. However, one has to note that this particular normalization, even when natural from the viewpoint of the canonical coordinates, does not correspond to any observer in the region of spacetime that we are considering. Thus, if we adopt the viewpoint that the Killing field should be adapted to at least one observer inside the spacetime, then one should look for a new normalization.

### B. The Bousso-Hawking normalization

The issue of normalization of the time evolution vector field $t^a$ is a very important one within the isolated horizon perspective (and the Euclidean approach as well). Let us review very briefly how it is normally done, in the asymptotically flat context. Given an IH with charges $Q_i$, then one looks for the (unique) stationary solution with those charges and look for the surface gravity of the corresponding KVF $K^a$, that has some special property. Once one chooses the function $\kappa(Q_i)$, one extends this function to the whole IH phase space. In the case of asymptotically flat spacetimes, the standard choice is to select the Killing field that goes to a unit time translation at infinity. That is, the one that corresponds to the four-velocity of an observer at infinity. In a sense, this is the most natural choice. In our case, we do not have an asymptotic region, and therefore no preferred observer “far away”. One has to think of a new normalization criteria. Fortunately, such normalization is available, as was suggested by Bousso and Hawking, in the context of Euclidean instantons [33]. The idea is to select the preferred observer, following the integral curves of the Killing field, for which the acceleration vanishes. There is a unique value of $r_g$ for which the field $\partial/\partial t$ is geodesic. We then normalize the Killing field such that $K^a K_a|_{r_g} = -1$. It is easy to see that one has to choose,

$$K^a = \frac{1}{f(r_g)} \left( \frac{\partial}{\partial t} \right)^a \quad (4.8)$$

Now, let us denote by $\alpha(\mu, l) = 1/f(r_g)$. The normalized surface gravity is then given by,

$$\tilde{\kappa}_h := \frac{1}{2\sqrt{1 - \frac{27\mu^2}{l^2} \left[ \frac{1}{r_h} - \frac{3r_h}{l^2} \right]}} \quad (4.9)$$

This surface gravity has some nice properties. In particular, in the Nariai limit, it does not vanish but approaches a constant value given by $\kappa_{\text{nairai}} = \sqrt{3}/l$.

We are now in the position of computing the black hole horizon mass, by integrating the first law (4.4),

$$\tilde{M}_\Delta(r_b) := \int_0^{r_b} r_b' \tilde{\kappa}(r'_b) \, dr'_b$$
FIG. 3: The three possible definitions of BH Horizon Mass are drawn as function of the BH geometrical radius \( r_b \). The line in the middle is \( r_b/2 \), the lower line corresponds to \( \mu(r_b) \) and the higher line to \( M_b^\Delta \).

\[
\frac{1}{2} (l\mu)^{1/3} \sqrt{l^{2/3} - 3\mu^{2/3}} + \frac{l\sqrt{3}}{6} \arcsin \left[ \sqrt{3} \left( \frac{\mu}{l} \right)^{1/3} \right].
\]  (4.10)

This is a function of the black hole horizon \( r_b \), since the specification of the value of \( l \) and \( r_b \) fixes the parameter \( \mu(r_b) = r_b/2 (1 - r_b^2/l^2) \). This is the ‘new black hole mass’, constructed from the physically motivated condition of Bousso and Hawking; it will be the mass that we shall adopt in the remainder of this paper. We shall also drop the ‘tildes’ over the surface gravity and mass and simply refer to them as \( \kappa(r_b) \) and \( M_b^\Delta \) respectively.

Let us now compare our horizon mass \( M_b^\Delta \) with some other proposals for the black hole horizon mass in the literature. In the early papers \[6\], the mass of the black hole was found to be equal to \( \mu \), the “mass parameter”. The cosmological horizon mass was found to be equal to \( -\mu \). This is also the result found more recently using Euclidean Hamiltonian techniques \[13, 14\]. It is not difficult to see that this result is a consequence of using the standard surface gravity \( \kappa \) associated to the killing time \( t \), and of integrating the first law for both masses, with trivial integration constants. Some other papers, making use of quasi-local methods in the manner of Brown-York, find different expressions for the total mass. See \[9\] and \[10\] for two such approaches, none of which coincides with ours. In a different approach, using some thermodynamical considerations, the black hole mass is taken to be equal to \( r_b/2 \) \[12\]. A comparison of the three possible definitions for the black hole mass are shown in Fig. 3.
V. COSMOLOGICAL HORIZON ENERGY AND MASS

In this section, we shall focus on the cosmological horizon. In the first part we shall consider the horizon energy coming from our choice of normalization. Energetic consideration will allow to say something about more generic dynamical spacetimes. In the second part, we shall define the cosmological horizon mass. It is related to the energy but has different properties. Finally, We discuss the differences between them.

A. Energetics

In this subsection, we shall consider the Hamiltonian, and as a result, the energy of the space-time. We shall return to the issue of the cosmological horizon mass later on.

Recall from Sec.II that the form of the Hamiltonian is given by Eq.(2.3), with two contributions, one from each horizon. From the general formalism of isolated horizons, each of the surface terms (accounting for the horizon “energy”) will satisfy a first law of the form,

$$\delta E^i = \frac{\kappa^i}{8\pi} \delta a\Delta . \quad (5.1)$$

where \(i = b, c\) is a label for the horizons. Now, let us recall that the sign of the surface gravity \(\kappa^i\) depends on the nature of the horizon; it is positive for the BH horizon and negative for the cosmological horizon. With this convention, the change of the total Hamiltonian of the system is such that

$$\delta H_t = \delta E^c - \delta E^b \quad (5.2)$$

Let us now discuss the physical motivation for these choices. Let us assume that we have, as initial condition, a Schwarzschild de Sitter spacetime, and an inertial observer in the region between the two horizons with a test mass \(m\) (much smaller than the BH mass), and we throw the test mass across the cosmological horizon. Then, for a positive change in area of the cosmological horizon (i.e. when it grows), the change in the horizon energy \(E^c\) is negative \(\delta E^c < 0\). The change in the total Hamiltonian \(\delta H_t\) will also be negative. Let us now suppose that the same test mass crosses the BH horizon. In this case, the BH horizon also grows and therefore \(\delta E^b > 0\). However, the change in the total Hamiltonian \(\delta H_t\) is again negative. Thus we see that it is natural to regard the Total Hamiltonian \(H_t\) as a measure of the total energy contained between the horizons. This is very similar to the asymptotically flat case where the difference in the ADM energy and the BH energy is given by the total energy radiated through infinity \(18\). If we follow this reasoning, we should then state that the quantity,

$$E^c - E^b = \delta E^c \quad (5.3)$$

is equal to the total available energy to be radiated across both horizons. Even when we know some important properties for the cosmological horizon energy \(E^c\), we do not have a functional form on the full IH phase space. The first law for the cosmological horizon tells us that \(E^c\) is a function of \(r\) only, just as the BH horizon mass \(11\).

In order to find the functional form for the cosmological horizon energy, it is convenient to consider the case of static space-times. We know from the general theory of symplectic dynamics that the change in the value of the total Hamiltonian on a static solution is zero. Thus when considering the relation \(5.2\) connecting static solutions we have

$$\delta H_t = \delta E^c(r_c) - \delta E^b(r_b) = 0 \quad (5.4)$$
FIG. 4: The total energy contained in the $SdS$ space-time is plotted as function of the black hole radius $r_b$. The lower limit $r_b = 0$ corresponds to empty de-Sitter space-time where the total energy is zero.

Now, it is important to recall that in the static case, namely for the Schwarzschild-de Sitter family, the cosmological radius $r_c$ and the black hole radius are related by (4.2). This means that in Eq. (5.4) one can parameterize the connected component of the static sector by one parameter, for instance $r_b$. We then have that for the SdS family, the total Hamiltonian is a constant,

$$H_t = E^c_{\Delta}(r_c) - E^b_{\Delta}(r_b) = C$$

(5.5)

Since $C$ is a constant, we can try to evaluate the LHS of the equation in the limit when $r_b = 0$. In this case, there is no Black Hole and therefore the value of the total Hamiltonian corresponds to the total energy contained inside the (vacuum) de Sitter horizon. We shall assume that this energy is zero $^1$. Therefore, we have that as functions of $r_b$, both horizon energies have the same value. This is true in the static family where the value of the BH horizon radius uniquely determines the value of the cosmological horizon radius (See Fig. 4). Using this relation valid for the static family, we arrive at the cosmological horizon energy $E^c_{\Delta}(r_c)$ given by,

$$E^c_{\Delta}(r_c) = -\frac{1}{2}(l\mu)^{1/3} \sqrt{l^{2/3} - 3\mu^{2/3}}$$

$$+ \frac{l\sqrt{3}}{6} \arcsin \left[ \sqrt{3} \left( \frac{\mu}{l} \right)^{1/3} \right].$$

(5.6)

with $\mu(r_c) = r_c^2 / (1 - r_c^2 / l^2)$. Thus, as a function of its area, the cosmological horizon energy is a decreasing function; It takes its maximum value when the horizon area is the smallest,

$^1$ This choice is justified, by analogy with the case of empty anti-de Sitter spacetime where the total ADM energy is zero [34], even when there might be some “vacuum energy” due to the cosmological constant.
FIG. 5: The total energy $E_{\Delta}^c(r_c)$ contained in a space-time with a cosmological isolated horizon as an outer boundary is plotted as function of the cosmological horizon radius $r_c$. The $r_c = 1$ limit corresponds to empty de-Sitter where the total energy is zero.

namely in the Nariai limit, and it decreases monotonically until it reaches zero at the de Sitter limit (see Fig. 5). It is important to note that the expression (5.6), as a function of the cosmological horizon radius $r_c$ will be the total horizon energy on the entire phase space. That is, on a generic spacetime containing a cosmological isolated horizon of radius $r_c$ —independently of whether a black hole is present or not— the total energy contained within will be given by the expression (5.6).

Two remarks are in order. i) Throughout this paper, we are assuming that, just as in the static SdS case where $r$ is a good coordinate in the region of interest, in the full IH phase space the cosmological horizon is larger than the black hole horizon. Assuming the validity of our energy formula (5.6) then implies that a $\Lambda > 0$ Penrose conjecture is valid. ii) Given a cosmological constant $\Lambda$, there is a maximum bound on the amount of energy that can be contained within a cosmological horizon given by the Nariai limit of the cosmological horizon energy, namely $E_{\text{max}} = l\pi\sqrt{3}/12$.

B. Cosmological Horizon Mass

In this part we shall consider the cosmological horizon mass. The first thing one needs to specify are “boundary conditions” for integrating the first law, which is valid for both types of horizons. The other input, namely the choice of the preferred normalization of the surface gravity, is already at our disposal, so the only remaining input comes from the choice of “initial data” when integrating the first law. Let us be more specific. In the case of black hole horizons (for a given theory, which means fixing $l$), we have a one parameter family of such spacetimes, labelled by $\mu$, such that for $\mu = 0$ there is no black hole, and therefore, no horizon. At the same time, the point $\mu = 0$ represents the de Sitter universe
with a cosmological horizon $r_{dS} = l$. For small values of $\mu$ (in $l$ units), one expects that the horizon mass behaves as the horizon mass of a Schwarzschild BH would, given that in the vicinity of such a horizon, the geometry “looks like” a Schwarzschild BH. This expectation is indeed satisfied since the horizon mass behaves as $\bar{M}(r_b) \approx \mu \approx r_b/2$, when $\mu \ll l$. As we increase the value of $\mu$, the black hole horizon grows, and the cosmological horizon “shrinks”, up to the Nariai point in which both horizons have the same size.

What is then the value of the Mass that we shall assign to the cosmological horizon? The first observation is that, from the first law, the variation of the cosmological energy and mass should be related. Thus, let us define the cosmological horizon mass $M_{\Delta}^c$ to be such that $\delta M_{\Delta}^c = -\delta E_{\Delta}^c = |\kappa|/8\pi \delta a_{\Delta}$. This proposal is motivated by the following consideration. Let us recall the situation considered before where an observer in the region between both horizons, would through a test mass $\Delta m$ into the cosmological horizon. We know that the horizon will grow, so $\delta a_{\Delta} > 0$. The change in cosmological horizon energy will be negative, since this quantity measures the total amount of energy contained in the partial Cauchy surface “inside the horizon”. However, one could expect the cosmological horizon mass to increase by such process, just as it happens for BH horizons where a test mass falling in decreases the energy outside the BH but increases the BH mass by the same amount.

The next question is then about the constant $M_0$ that relates both functions as

$$M_{\Delta}^c = M_0 - E_{\Delta}^c.$$ 

The prescription we shall put forward in this paper is the following: let us consider the plot of $M_{\Delta}^b$ and $M_{\Delta}^c$ as functions of $r_b$ (or alternatively, $\mu$). We know that the black hole mass $M_{\Delta}^b(\mu)$, is a monotonically increasing function of $\mu$, starting at zero, and reaching its maximum value at the Nariai point $r_{\text{max}} = l/\sqrt{3}$, of $M_{\Delta}^b(r_{\text{max}}) = l\pi \sqrt{3}/12$. We shall now assume that the cosmological horizon mass is also positive, and reaches its maximum value when the horizon is largest, that is, in the pure de Sitter case. Then, the “continuity proposal” is that both masses coincide at the Nariai limit. That is, $M_{\Delta}^c(r_c) = M_{\Delta}^b(r_b)$ when $r_b = r_c$. As already mentioned in Section IV A, this is consistent with the Euclidean formulation, where the solution with the boundary at the cosmological horizon is not distinguishable from the solution with the boundary taken at the black hole horizon in the Nariai limit.

Thus, given the symmetries of the surface gravity $\tilde{\kappa}$, the plot of the masses, as functions of $r_b$ is symmetric with respect to the horizontal line at $l\pi \sqrt{3}/12$. It is then straightforward to arrive at the conclusion that the integration constant $M_0$ is precisely the horizon mass of pure de Sitter and is given by,

$$M_0 = M_{\Delta}^{dS} := \frac{\pi l}{2\sqrt{3}}. \quad (5.7)$$

This is the second observation of this paper. Note that with this proposal, in the static case the sum of the masses –as functions of $r_b$– is a constant:

$$M_{\Delta}^b(\mu) + M_{\Delta}^c(\mu) = \text{const.} = M_0 = \frac{\pi l}{2\sqrt{3}} \quad (5.8)$$

Let us now consider the general dynamical case, where there might be radiation escaping through the horizons. We know from the previous part that the total energy to be radiated is given by the difference in horizon energies, which in terms of horizon masses is,

$$E_{\text{rad}} = E_{\Delta}^c - E_{\Delta}^b = M_{\Delta}^{dS} - (M_{\Delta}^b + M_{\Delta}^c) \quad (5.9)$$

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VI. SOME CONSIDERATIONS

This section has three parts. In the first one, we discuss the thermodynamics of the horizons, in view of our previous discussion. In the second part we review different entropy bounds that have been considered in the literature and propose new ones, motivated by our definitions of energy and mass for black holes and cosmological horizons. In the last part, we discuss the possibility of applying the 'bound state' model of hairy black holes when a positive cosmological constant is present.

A. Thermodynamics

The first law of horizon mechanics (4.4), applied to each horizon suggests that the area can be regarded as entropy and surface gravity as temperature (with the standard coefficients). This expectation is realized both by the semiclassical considerations of Gibbons and Hawking [6], by the Euclidean methods of [23], and more recently by statistical computations within different approaches to quantum gravity [17, 36]. For spacetimes with two horizons, recent results based on Euclidean methods have shown that indeed the BH and cosmological horizons are subject to thermodynamic considerations [13, 14]. Let us now see that contrary with the earlier treatments of the subject [6], we have consistent relations (i.e. signs) relating all the parameters. Let us focus on the cosmological horizon given that the black hole horizon can be treated in the standard manner. Let us recall that we made a distinction between the horizon mass and energy: the energy $E_\Delta$ was associated to the total energy content in the (region of the) spacetime, whereas the mass $M_\Delta$ is a quantity associated the horizon itself. If we want to make some thermodynamical considerations regarding the horizon, it is natural to regard the Mass as the quantity entering the first law

![Graph showing the relationship between $M_\Delta$ and $r_c$.]
of thermodynamics,
\[ \delta M^c_\Delta = \frac{|\kappa|}{8\pi} \delta a_\Delta \] (6.1)

The temperature we would associate to such a horizon would then be
\[ T_c = \frac{\hbar |\kappa_c|}{2\pi} \]

which is a positive quantity. It is important to note that this relation is valid, independently of the choice of \( \kappa(r_h) \). Had we chosen the usual normalization \[6, 13, 14\], we would have found different values for temperature and mass, but the same qualitative behavior, namely, positive temperature and mass. Note that the fact that we get positive masses for both horizons is a consequence of the basic requirements that we have imposed, namely that the mass of the horizon grows when the area increases (positive temperature), that the Horizon mass vanishes in the zero area limit and that in the Nairai limit both masses coincide.

### B. Entropy bounds

In recent years, there has been an increasing interest in the so called “entropy bounds” in the presence of a cosmological constant \[37, 38\]. In particular, as consequence of the generalized second law applied to cosmological horizons, there is the entropy D-bound that states that the entropy of matter \( S_m \) allowed inside the cosmological horizon, at a given initial time \( t_i \), is bounded by \[38\],
\[ S_m \leq \frac{1}{4}(A_0 - A_c) \] (6.2)

where \( A_0 = 4\pi l^2 \) is the area of the cosmological horizon of empty de Sitter, and \( A_c \) is the area of the cosmological horizon when the matter is present at time \( t_i \) (which is smaller than \( A_0 \)). In particular, when a BH is the “matter” considered, the bound is satisfied, and it is saturated in empty de Sitter space, that is, the matter entropy has to vanish in that case \[38\]. Bousso has also compared the D-bound to the holographic entropy bound applied to the cosmological horizon area,
\[ S \leq \frac{A_c}{4} \] (6.3)

There is also the original Bekenstein bound \[39\] that puts some limits on the entropy of a system of energy \( E \) contained in a region of “size \( R \),
\[ S_m \leq 2\pi E R \] (6.4)

In the case of gravitational systems with a positive cosmological constant, Bousso has suggested that, instead of the energy \( E \) of the system (for which he does not have an expression), one can replace it by its “gravitational radius” \( R_g \), in such a way that the entropy bound looks like,
\[ S_m \leq \pi R_g R \] (6.5)

Bousso has argued that these two entropy bounds agree for large dilute systems in de Sitter space \[37\], and that it coincides with the D-bound in the de Sitter limit.
In our case, we do have a notion of energy for a gravitational system with an outer boundary given by a cosmological horizon, namely $E_{\Delta}(r_c)$. In particular, when the “matter” is a BH in de Sitter, one can ask whether an entropy bound like (6.4) is valid, without the necessity of introducing a radius. The expression we would like to propose when there is a BH present is given by,

$$S_{bh} \leq 2\pi E_{\Delta}^b(r_b) r_c$$

(6.6)

where the matter system to be considered inside de Sitter is a black hole of area $A_b = 4\pi r_b^2$, with energy equal to $E_{\Delta}^b(r_b)$, and the maximum size of the system is assumed to be $r_c$, the cosmological radius (which is smaller than $r_0 = l$, the de Sitter radius). Then, in order to test the validity of (6.6), let us rewrite it as,

$$r_b \leq \frac{M_b^h(r_b)}{(r_b/2)} r_c$$

Now, it is clear from Fig. 2 that the term on the RHS multiplying $r_c$ is always larger than one, and $r_c \geq r_b$. Therefore, the inequality is always satisfied and is saturated in the $r_b \to 0$, de Sitter limit. In the general case of a unique cosmological horizon of radius $r_c$, the modified Bekenstein bound reads

$$S_m \leq 2\pi E_{\Delta}^c(r_c) r_c$$

(6.7)

Let us now consider a more stringent condition, where we assume that we have a black hole and some matter in the interior region. We have to consider the entropy of both the BH and the matter. The condition now reads,

$$S_{bh} + S_m \leq 2\pi E_{\Delta}^c(r_c) r_c$$

(6.8)

which can be written as,

$$S_m \leq \pi(2E_{\Delta}^c(r_c) r_c - r_b^2)$$

(6.9)

Now recall that $E_{\Delta}^c(r_c)$ is equal to the Horizon mass $M_b^h(r_b)$ of the corresponding BH in SdS and that $2M_b^h(r_b) > r_b$. Thus, the new entropy bound (6.9) is well defined. For the case of pure de Sitter, $r_b = 0$, $E_{\Delta}^c(r_c) = 0$ and therefore the bound is saturated by $S_m = 0$ (just as the D-bound).

Let us now compare the different entropy bounds that we have introduced. In Fig. 7 we have plotted four of these bounds as functions of the cosmological horizon radius $r_c$. Three of them are given by a decreasing function and only one, the holographic bound (6.3), is given by an increasing function of $r_c$. Of the remaining bounds, the less restrictive is the D-bound, and the most restrictive is the ‘generalized Bekenstein bound’ as written by Bousso (6.5). The modified Bekenstein bound proposed in this paper (6.7) corresponds to the intermediate line in Fig. 7. In [38] Bousso has argued that (6.5) represents an interpolation between the Holographic bound and the D-bound and can be regarded a valid generalization of the (flat) Bekenstein bound to cosmological settings. We would like to argue against this interpretation. First, one should notice that the Bousso bound (6.5) was derived in the small $r_b$ limit, and therefore there is no contradiction with it not being valid near the large BH radius, “Nairai limit”. In fact we can understand the departure between both bounds for ‘large’ $r_b$ from the fact that in this region, the Horizon Energy $E_{\Delta}^c(r_b)$ and (one half of) the “gravitational radius” $r_b$ depart from each other. That is, the ‘gravitational radius’ fails to be a good measure of the horizon mass for large black holes.

Second, we would like to argue against the validity of the holographic entropy bound when applied to the cosmological horizon, as presently interpreted. For that, let us recall
the argument for assigning entropy to a black hole. If we have some matter outside the BH, with some entropy $S_m$, and throw the matter across the horizon, the entropy in the exterior region, where the ‘observer’ is, will decrease. The area of the BH increases and the total balance of entropy is saved by associating entropy to the area of the BH horizon. The standard interpretation is that the quantity $A/4$ is the entropy that the external observer assigns to the horizon and in a sense could be though as providing some information about the causally disconnected region inside the horizon. In the case where there is a cosmological horizon, the observer is in the interior region, that is, in the region $r < r_c$. If one throws some matter with a certain amount of entropy across the cosmological horizon, the entropy in the observer’s region will decrease, the horizon will grow and therefore the quantity $A/4$ will increase. Thus, in analogy with the BH case, one is forced to assign the entropy $S_{\text{hor}} = A/4$ to the cosmological horizon as seen from the inside. This also means that this quantity is giving a measure of the information contained “everywhere else” outside the observer’s region. Thus, the holographic bound can not be a bound for the entropy within the observer’s region $r < r_c$. This can also explain why the Holographic bound \([6.3]\) has a different behavior as the rest of the bounds; it is bounding the entropy of a different, disconnected region. What one could do with the holographic bound, in spirit of the holographic principle \([40]\), is to combine it with the generalized Bekenstein bound to have a “total” bound on the entropy everywhere:

$$S_{\text{tot}} \leq S_m + S_{\text{out}} \leq 2\pi E^c_{\Delta}(r_c) r_c + A_c/4 \quad (6.10)$$

That is, there is a contribution to the total entropy, as seen by an observer in the region $r_b < r < r_c$, coming from the matter in the region (including the BH entropy $A_b/4$) and a contribution from the ‘outside’ matter, bounded by $A_c/4$. From Fig. 8 we see that the total bound \([6.10]\) grows with the size of the cosmological horizon, but reaches a maximum before the de Sitter limit.
Let us end this section with a remark. We have generalized the Bekenstein bound \[6.4\] using our measure of energy contained inside the cosmological horizon, but we have not intended (nor claim) to prove that such a bound is valid in the strong gravity regime. As far as we know the validity of such bound has only been proven in the weak gravity limit \[41\]. On the other hand, the bound has been shown to be violated when NUT charges are coupled to gravity and the physical quantities are computed in the asymptotic region \[42\].

C. Black Holes as Bound States

One of the most successful applications of the isolated horizons formalism is the physical model of a static, hairy black hole as a bound state of a soliton and a “bare” black hole. This model was motivated by considerations of spherically symmetric static solutions to the EYM equations \[22\], but it has proved to be valid also for axisymmetric solutions of EYM \[43\], EYMH \[44\] and Born-Infeld \[45\] black holes. A natural question is whether static solutions with a cosmological constant can be given such an interpretation. Let us recall that in the asymptotically flat case, the total (ADM) energy of the spacetime is written as a sum of the energy of a bare black hole (the Schwarzschild BH in the non-rotating case), the energy of the soliton and a “binding energy”. The first observation is that the analog of the ADM energy corresponds to the energy of the spacetime inside de cosmological horizon, and the bare black hole will be the Schwarzschild de Sitter solution. The total energy can be written as,

\[ E^\Delta_{\Delta}(r_b) = M_0(r_b) + M_{\text{sol}} + E_{\text{bin}}(r_b) \]  \hspace{1cm} (6.11)

where the binding energy \( E_{\text{bin}} \) is given by

\[ E_{\text{bin}}(r_b) = -[M_0(r_b) - M_\Delta(r_b)] \]
As expected, the binding energy should always be negative, and its absolute value is a decreasing function of $r_b$. That is, larger black holes are “less bounded”. This is a general feature observed in other gravity-matter systems [22].

Now, let us recall that in the asymptotically flat case for say, Einstein Maxwell fields, there are no solitons other than Minkowski spacetime (by solitons it is normally understood regular stationary solutions, even when unstable). In that case the horizon and ADM masses coincide, the soliton mass is zero and therefore there is no bound state and the binding energy in (6.11) vanishes. Our simple vacuum $\Lambda > 0$ case has the same behavior. When one couples matter and finds non-trivial, hairy, solutions one is then led to the ‘bound state’ model [22]. The form of the first law we have considered (without a ‘work term’) will be valid for theories with no ‘gauge couplings’ like a minimally coupled scalar field (with an arbitrary potential) or a YM field in the pure magnetic sector. Non-minimal coupling can also be incorporated to the formalism, both classically and quantum mechanically [46], where the main modification is that the entropy also has a contribution from the scalar field at the horizon. It would be interesting to explore gravity-matter systems to find such hairy solutions, and check whether the ‘bound state’ model is satisfied also for theories in the presence of a cosmological constant. These investigations will be reported elsewhere.

VII. DISCUSSION AND OUTLOOK

Let us summarize our results. For static solutions with a cosmological constant in 2+1 and 3+1 dimensions, we have found mass formulae for both black hole and cosmological horizons. Furthermore, the powerful isolated horizons formalism based on a Hamiltonian formulation allowed us to extend these results and establish a formula for the total energy contained inside a cosmological horizon in the full IH phase space. This formula in turn led us to propose a generalized Bekenstein bound for the entropy of the matter content. This entropy bound was compared with some other bounds available in the literature. The existence of a first law of horizon dynamics made it possible to infer some thermodynamical properties for both horizons. These properties are qualitatively similar to previous treatments, but differ in the quantitative value for the thermodynamical parameters. In particular, the choice of proper normalization for the Killing field on static solutions given by the Bousso-Hawking prescription provides a physically acceptable definition of the temperature to be used in the IH formalism and, upon integration, defines the (isolated) horizon energy and mass, for appropriate choices of integration constants.

In our treatment, there was a key input that allowed us to construct different physical quantities. The general theme of this construction is that the role of particular observers is crucial to identify certain quantities. More precisely, by requiring that a preferred observer be the one to assign a temperature to both horizons lead us to the Bousso-Hawking normalization. Similarly, by requiring that observers in the region between the horizons be the ones that assign energy, mass and entropy to both horizons, allowed us to propose generalized entropy bounds, even for the strong gravity regime. Again, in the spirit of the holographic principle, where different observers might have a different perception of the world, these two quantities, the mass and energy for the cosmological horizon are different. In one case, the energy is associated to the total energy contained within the horizon (i.e. in the region where the observer is), whereas the horizon mass is a quantity associated to the horizon itself, that contains some information about the radiation and matter that has fallen into it. Furthermore, we were able to pose some conjectures regarding a maximum value of energy
Let us end with some remarks:

1. The results obtained in most of the sections of this paper were based on the isolated horizons formalism. However, we have made, when possible, some comparisons with the results obtained with Euclidean canonical methods. Let us now compare the similarities and differences of both approaches. Perhaps the most important feature that these approaches share is the fact that both are formulated considering the spacetime region between the two horizons. That is, they do not consider nor have anything to say about the interior region of the black hole, nor the asymptotic region. Second, the physical quantities that both formalisms produce are tied to a particular observer: in the Euclidean approach one has the liberty of choosing which horizon is left ‘free’ and which is taken as a boundary. Both formalisms are subject to the issue of normalization of the Killing field (or alternatively ‘time function’). Finally, in both cases one is free to add a constant when integrating the first law to define an energy or mass. As we have repeatedly seen, the choice of such constants involves a good part of the ‘physics’ of the problem. The main difference between both approaches are obvious but we shall enumerate them: The IH formalism is intrinsically Lorentzian since the (null) boundaries are at the forefront of the formalism. Furthermore, it allows for generic dynamical situations; static spacetimes are one of an –infinite dimensional– realm of possibilities. On the other hand the Euclidean approach is limited to static spacetimes (admitting Wick rotations), and the boundary conditions of the null horizon are difficult to track down.

2. The Isolated Horizons formalism as originally formulated for asymptotically flat spacetimes has been very useful in many applications. One of its main virtues was the possibility of implementing boundary conditions and extracting physical predictions at the horizon, where no preferred background structure is present –such as Poincaré invariance at infinity–, and thus for the strong gravity regime. There was however some shadow of the (flat) asymptotic conditions through the normalization of the evolution vector field at infinity, whose choice affects the values of both the ADM energy at infinity and the horizon mass. In our situation with two horizons and no asymptotic region, one is forced to implement a consistent choice of normalization that does not make use of an asymptotic region. In this paper the proposal that we have put forward was based on a preferred geodesic observer, namely the particle sitting at the origin in the 3D case and the static observer (not accelerating to any of the horizons) in the 4D case. Whether these choices are the physically most adequate still needs to be explored.

3. We have based our treatment on the isolated horizons formalism. this is well defined in a situation like the one depicted in Fig. 1 where initial data for SdS is specified, together with a small perturbation $\Delta m$ in the intermediate region (a scalar field for instance). The spacetime will evolve dynamically, the matter field will be radiated and eventually will cross the horizons, thus violating the ‘isolated’ character of the horizons. Thus, the region of interest in spacetime is the one before the radiation reaches either of the horizons. In order to treat the full dynamical situation one would have to apply the ‘dynamical horizons’ formalism [47]. However, one should note that the formalism of [47] would have to be modified slightly to accommodate for the normalization chosen in this paper.
4. In this paper we have avoided mentioning the dS/CFT correspondence \cite{48}, and will continue to do so. This is because we have not considered, from the very beginning, the asymptotic region of spacetime. On the contrary, the Isolated horizon formulation is restricted to the region of spacetime contained in-between the horizons (see Fig.\ref{fig:boundaries}), a region very far from the asymptotic future where the correspondence is conjectured to be valid. In this respect, our mass formulas are conceptually very different to those of \cite{9,10} that were computed using the Brown-York approach in the asymptotic future.

5. We have not attempted to make a connection between our mass formulae and entropy with the approach of the so called Cardy-Verlinde formula \cite{49}, where the mass definition that is taken is the one given in \cite{9} and the mass formulae of \cite{7}. It would be of interest to see whether that formalism can be made consistent with our choice of horizon mass.

The mass and energy formulas that we have proposed in this paper differ from the standard expressions found elsewhere. In order to settle the validity of our formulas, it is necessary to have quantitative dynamical evolutions (with a spherically symmetric scalar field for instance), for which control on the total amount of initial and radiated energy is possible. We shall report those findings in a future communication.

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APPENDIX A: EUCLIDEAN 2 + 1

In this appendix we shall consider the 2 + 1 system considered in Sec.\ref{sec:2+1} from the perspective of a canonical Euclidean approach. Further details on the methods used here may be found in \cite{14}, where an analogous treatment for (3+1)-dimensional Schwarzschild–de Sitter is given in its appendix.

We assume that the metric becomes axially symmetric as it approaches the boundary, with the form

\[ ds^2 = N^2(\gamma,\rho)^2dt^2 + d\rho^2 + \gamma^2d\phi^2 \ , \]

The re-scaled lapse $N$ and the coefficient $\gamma$ are functions of $\rho$ only. The coordinate $\rho$ is defined so that $\gamma,\rho$ is negative as one approaches the boundary. The boundary is a surface of constant $\rho$. We will be interested in two possible of such choices: a cosmological horizon or a conical singularity generated by a point particle. It is convenient to redefine the hamiltonian generator,

\[ \tilde{\mathcal{H}}_\perp = -\mathcal{H}_\perp \gamma,\rho \ , \]
so that the associated Lagrange multiplier is $N$, and the corresponding term in the Hamiltonian is

$$
\int d^3x \left( N \dot{H}_\perp \right) = \int d^3x (N \dot{H}_\perp) \ ,
$$

where

$$
\dot{H}_\perp = \frac{1}{8\pi G} \gamma_{,\rho} \left( \gamma_{,\rho\rho} + \gamma \right) \ .
$$

The boundary term in the variation of the Hamiltonian (A3) reads,

$$
\frac{1}{16\pi G} \int d\phi \ N \delta \left( \gamma_{,\rho}^2 + \gamma_{,l}^2 \right) \bigg|_{\rho_1}^{\rho_2} \ .
$$

Here $\rho_1, \rho_2$ are the bounds of the coordinate $\rho$. Note, however, that in the case at hand only one of them will be considered to be the boundary, where we will set $N = 1$, so that the surface integral at the boundary may be written as

$$
- \delta U \ ,
$$

where the energy $U$ is given by

$$
U = \pm \frac{1}{8G} \left( \gamma_{,\rho}^2 + \gamma_{,l}^2 \right) \ .
$$

Here the minus sign is taken when the boundary is at $\rho_1$ and the plus sign if the boundary is at $\rho_2$. To evaluate expression (A7) for the de Sitter geometry with a point particle, we first make the change of coordinates $r = \alpha l \cos(\rho/l)$ to rewrite it in the form (A1),

$$
ds^2 = \alpha^2 \sin^2 \left( \frac{\rho}{l} \right) dt^2 + d\rho^2 + \alpha^2 l^2 \cos^2 \left( \frac{\rho}{l} \right) d\phi^2 \ .
$$

[Note that our prescription requires the metric to be cast in the form (A1) at the boundary only, and not everywhere as it is possible in this case.] Here $\rho \in [0, l\pi/2]$, so that $\rho_1 = 0$ is the cosmological horizon $r = \alpha l$ and $\rho_2 = l\pi/2$ is the location of the particle, $r = 0$. If we choose the cosmological horizon as the boundary, then a delta function must be added at the location of the point particle, $r = 0$, in order for the Einstein equations to be satisfied [26], and the energy reads

$$
U_h = -\frac{\alpha^2}{8G} + K_h \ .
$$

Now, if the boundary is taken at the particle, then the action principle must be supplemented with one quarter of the area of the cosmological horizon [24], and

$$
U_p = \frac{\alpha^2}{8G} + K_p \ .
$$

The constants $K_h, K_p$ represent the arbitrariness in fixing the zero point of the energy. Usually one demands that in the absence of a conical singularity, that is, when $\alpha^2 = 1$, the energy vanishes, so that $K_h = -K_p = 1/8G$. An alternative choice is to ask that the energy that the particle assigns to a zero area horizon be zero. In that case, one should choose $K_h = 0$. 

25
The energy $U$ considered in this appendix is conjugated to the Killing time $t$. We may also use a different normalization, and set $N = 1/\alpha$ on the boundary. In that case we will get the expressions (3.11), (3.8), (3.9) in the main text.

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