SPECIALIZATION MORPHISMS

ILDAR GAI SIN, JOHN WELLIAVEETIL

Abstract. We define the notion of a specialization morphism from a locally noetherian analytic adic space to a scheme. This captures the (classical) specialization morphism associated to a formal scheme. There is a well behaved theory of compactifications and it turns out that the classical specialization morphism is proper in this setup. As an application, we show that the nearby cycles functor commutes with lower shriek in great generality.

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1. Introduction

In classical algebraic geometry, given a scheme $S$, it is often convenient to pass to the associated reduced scheme $S_{\text{red}}$. The scheme $S_{\text{red}}$ has the same underlying topological space as $S$, and the structure sheaf is obtained by killing nilpotent functions. In this paper we work in the rigid setting and construct an analogous reduced space $X_{\text{red}}$ associated to any locally noetherian analytic adic space $X$. The construction of $X_{\text{red}}$ (and the theory we develop) can be extended beyond the locally noetherian case (e.g. perfectoid spaces [12]), but we make no attempt to do so. Roughly speaking, analogously to the scheme setting, $X_{\text{red}}$ is obtained from $X$ by killing the topologically nilpotent functions (cf. Definition 2.1).

Let $k$ be a nonarchimedean field and $X$ be some admissible formal scheme over $k^\circ$. Then as constructed by say Huber (cf. [8, §1.9]), associated to $X$ is an adic generic fiber $X_\eta$ and a special fiber $X_s$ (which is a scheme roughly obtained by killing topologically nilpotent functions). Relating the two spaces $X_\eta$ and $X_s$ is a specialization morphism

$$\lambda_X : X_\eta \to X_s$$

(cf. Proposition 1.9.1. in loc.cit.) of locally ringed spaces. One difficulty in studying $\lambda_X$ is that the source ($X_\eta$) and target ($X_s$) are of very different topological nature. Indeed, one is a rigid space and the other a scheme. It turns out that by considering any morphism

$$X_{\text{red}} \to S$$

of locally ringed spaces, provides sufficient flexibility to formulate what a proper or even a smooth morphism between an adic space and a scheme means. In this paper we study only proper morphisms (cf. Definition 3.18). As an application we prove that nearby cycles commutes with the lower shriek functor. Our methods are flexible enough to drop the base field $k$ (cf. Theorem 7.1).

Let us give an outline of the paper. In §2 we define the reduced space $X_{\text{red}}$ and morphisms (2), which we call specialization morphisms. The main result in this section, which allows us to construct the pushforward between the étale sites (of $X$ and $S$) along specialization morphisms is the existence of certain fibre products (cf. Proposition 2.19). In §3 we develop a theory of proper specialization morphisms and are able to compactify specialization morphisms satisfying some finiteness conditions (cf. Theorem 3.30). The proof of compactification follows similar ideas developed by Huber in [8, Theorem 5.1.5]. The short sections §4, §5 record a (smooth) base change result and introduce the lower shriek functor for specialization morphisms, respectively. In §6, we establish a proper base change result for specialization morphisms (cf. Theorem 6.3). This also recovers a version of [8, Theorem 3.5.8]. Finally in §7 we apply the results of §2–§6 together and prove that nearby cycles commutes with lower shriek (cf. Theorem 7.1).

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1For instance before writing this paper it was not even known whether nearby cycles (push-forward of $\lambda_X$) commutes with the lower shriek functor.
2. Specialization morphisms

Unless otherwise stated, for what follows $X$ will denote any analytic adic space which is locally noetherian. By [7, Proposition 1.6, Theorem 2.2], given such an $X$, we have a locally ringed space $(X, \mathcal{O}_X)$. In particular for every $x \in X$ the stalk $\mathcal{O}_{X,x}^+$ is a local ring with maximal ideal $m_x^+$. Similarly we denote the maximal ideal of $\mathcal{O}_{X,x}$ by $m_x$.

**Definition 2.1.** Let $m_{\mathcal{O}_X^+} \subseteq \mathcal{O}_X^+$ denote the vanishing sheaf of ideals defined as follows. For an open $U \subseteq X$,

$$m_{\mathcal{O}_X^+}(U) := \{ f \in \mathcal{O}_X^+(U) \mid f \text{ vanishes at all points } x \in X \} := \left\{ f \in \mathcal{O}_X^+(U) \mid f_x \in m_x^+ \subseteq \mathcal{O}_{X,x}^+ \right\},$$

where $f_x$ is the image of $f$ in $\mathcal{O}_{X,x}$. We denote by $X_{\text{red}}$ the locally ringed space $(X, \mathcal{O}_X^+/m_{\mathcal{O}_X^+})$ and we call it the reduced adic space associated to $X$.

At this point $X_{\text{red}}$ is simply a locally ringed space, but we will shortly prove that it is an object in the category of adic locally ringed spaces, $(V)$. We recall the definition.

**Definition 2.2.** The objects of the category $(V)$ are triples $(X, \mathcal{O}_X, \{\cdot(x)\}_{x \in X})$, consisting of a locally ringed topological space $(X, \mathcal{O}_X)$, and for each $x \in X$, $\cdot(x)$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$. Morphisms are given by morphisms of locally ringed topological spaces and compatible with valuations in the obvious sense.

We will also compare $m_{\mathcal{O}_X^+}$ with the presheaf of topologically nilpotent elements (cf. Proposition 2.9).

**Definition 2.3.** Let $\mathcal{O}_X^{\circ\circ} \subseteq \mathcal{O}_X^+$ denote the subpresheaf of ideals of $\mathcal{O}_X^+$, given by for an open $U \subseteq X$:

$$\mathcal{O}_X^{\circ\circ}(U) := \{ f \in \mathcal{O}_X^+(U) \mid f \in \mathcal{O}_X^+(U)^{\circ\circ} \}$$

To get things off the ground, we begin by proving some expected inclusions.

**Lemma 2.4.** The presheaf $\mathcal{O}_X^{\circ\circ}$ is a subsheaf of $m_{\mathcal{O}_X^+}$. Furthermore, for each $x \in X$,

$$m_x \subseteq \mathcal{O}_X^{\circ\circ} \subseteq m_{\mathcal{O}_X^+,x} \subseteq m_x^+ \subseteq \mathcal{O}_{X,x}^+.$$  

**Proof.** Let $U \subseteq X$ be open. If $f \in \mathcal{O}_X^{\circ\circ}(U)$ then $|f(x)|^n \to 0$ for all $x \in U$ (by continuity of the valuation $\cdot(x)$). In particular this implies the image of $f$ in $\mathcal{O}_{X,x}$, must lie in $m_x^+$. Therefore $\mathcal{O}_X^{\circ\circ}$ is a subsheaf of $m_{\mathcal{O}_X^+}$.

All inclusions are now obvious except the first. The inequality is obvious due to the existence of pseudouniformizers in $\mathcal{O}_X^{\circ\circ}$. Thus it remains to show $m_x \subseteq \mathcal{O}_X^{\circ\circ}$. We can assume $X = \text{Spa}(A, A^+)$, where $A$ is a Tate ring. Let $t$ be any pseudouniformizer in $A$. Then observe that $m_x \subseteq \mathcal{O}_{X,x}$ is uniquely $t$-divisible (as it is an ideal in a ring where $t$ is a unit). Thus $m_x = t \cdot m_x \subseteq \mathcal{O}_X^{\circ\circ}$. \hfill $\square$

A morphism $f : X \to Y$ of analytic adic spaces, induces a morphism

$$(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$$

of locally ringed spaces\(^2\) and the latter induces a morphism

$$f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$$

of locally ringed spaces. In this way we obtain:

\(^2\)This is a priori not obvious from the definition of a morphism of adic spaces but comes from the condition on compatibility of valuation rings.
Lemma 2.5. There is a functor
\[ \{ \text{locally noetherian analytic adic spaces} \} \to \{ \text{locally ringed spaces} \} \]
\[ X \mapsto X_{\text{red}} := (X, \mathcal{O}_X^+ / \mathfrak{m}_X^+) \]
\[ f \mapsto f_{\text{red}}. \]

We now describe some properties of this functor.

Lemma 2.6. Let \( x \in X \), the stalks \( \mathcal{O}_{X,\text{red},x} \) are naturally isomorphic to \( k(x)^+ \cap k(x)^\circ \) of height 1 so it suffices to show
\[ 0 \neq \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} / \mathfrak{m}_x \subset k(x)^\circ. \]

The inequality follows from Lemma 2.4, so we prove the inclusion. Let \( y \in X \) be the unique rank 1 generalization of \( x \). Let \( z \in \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} / \mathfrak{m}_x \) be any element. Then \( z \) is the image of some element \( z_0 \in \mathfrak{m}_{\mathcal{O}_x}(U) \) for some \( U \subset X \) open containing both \( x \) and \( y \). By definition the image of \( z_0 \) in \( \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} / \mathfrak{m}_x \) is topologically nilpotent in \( k(y) \).

Finally by [8, Lemma 1.1.10(iii)] the natural map \( k(x) \to k(y) \) is a homeomorphism onto its image and so in particular \( k(x)^\circ = k(y)^\circ \cap k(x) \). Therefore \( z \in k(x)^\circ \).

We compute
\[ \mathcal{O}_{X,\text{red},x} \cong \mathcal{O}_{X,x}^+ / \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} \]
\[ \cong (\mathcal{O}_{X,x}^+ / \mathfrak{m}_x) / (\mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} / \mathfrak{m}_x) \]
\[ \cong k(x)^+ / k(x)^\circ \]
where (i) is by definition, (ii) follows from the identity \( R/M = (R/I)/(M/I) \) and (iii) follows from above.

At the level of stalks we have the obvious inclusion \( \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} \subset \mathfrak{m}_x^+ \). At rank 1 points this inclusion becomes an equality.

Corollary 2.7. Suppose \( x \in X \) corresponds to a rank 1 valuation (i.e. a maximal point). Then \( \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} = \mathfrak{m}_x^+ \).

Proof. By Lemma 2.6, we have \( \mathcal{O}_{X,\text{red},x} = \mathcal{O}_{X,x}^+ / \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} = k(x)^+ / k(x)^\circ \) is a field. Therefore \( \mathfrak{m}_{\mathcal{O}_{X,\text{red},x}} \) must be the maximal ideal in \( \mathcal{O}_{X,x}^+ \).

In some sense the sheaf \( \mathfrak{m}_{\mathcal{O}_X^+} \) is a universal object which contains \( \mathfrak{m}_x \) and a pseudouniformizer at all stalks. The following lemma makes this precise.

Lemma 2.8. Let \( \mathcal{F} \) be any subpresheaf of ideals of \( \mathcal{O}_X^+ \) which is contained in \( \mathfrak{m}_{\mathcal{O}_X^+} \) such that \( \mathfrak{m}_x \subseteq \mathcal{F}_x \) at all points \( x \in X \) and \( \mathcal{F}_x / \mathfrak{m}_x \) is a prime ideal. Then the associated sheaf \( \mathcal{F}_x \) induces an isomorphism on stalks.
As a consequence we obtain the following analytic characterization of $m_{O^\circ_X}$.

**Proposition 2.9.** The sheafification of the presheaf of topologically nilpotent elements $O^\circ_X$ identifies with $m_{O^\circ_X}$.

**Proof.** Let $x \in X$. By Lemma 2.4, $O^\circ_X$ is a subpresheaf of $m_{O^+_X}$ and

$$m_x \subseteq O^\circ_{X,x}$$

Observe that for every rational subset $U$ which is a neighbourhood of $x$, $O^\circ_X(U)$ is a radical ideal.

We deduce from this that $O^\circ_{X,x}$ and $O^\circ_{X,x}/m_x$ are radical as well. Since $k(x)^+$ is a valuation ring, an ideal of $k(x)^+$ is radical if and only if it is prime. It follows that $O^\circ_{X,x}/m_x$ is prime. We can now conclude the proof by using Lemma 2.8. \hfill \square

**Remark 2.10.** In general $O^\circ_X$ is not a sheaf and sheafification is necessary in Proposition 2.9. This is due to the failure of $X$ being quasi-compact. Indeed consider $X = \bigcup_p \text{Spa}(\mathbb{C}_p \langle T^n/p \rangle, \mathcal{O}_{\mathbb{C}_p} \langle T^n/p \rangle)$, the open unit $p$-adic disk over $\mathbb{C}_p$ with coordinate in $T$. Then on each affinoid piece $\text{Spa}(\mathbb{C}_p \langle T^n/p \rangle, \mathcal{O}_{\mathbb{C}_p} \langle T^n/p \rangle)$, $T$ is topologically nilpotent, but it is not on $X$.

For analytic affinoid fields, $X_{\text{red}}$ has a simple description.

**Example 2.11.** If $X = \text{Spa}(L, L^+)$ is an analytic affinoid field, then $X_{\text{red}} \simeq \text{Spec}(L^+/L^{\infty})$ as locally ringed spaces. The fact that they are isomorphic as topological spaces follows from the fact that there is a (order-reversing) bijection of totally ordered sets

$$\{\text{prime ideals of } L^+\} \simeq \{\text{valuation rings of } L \text{ which contain } L^+\}$$

$$p \mapsto L^+_p.$$

Indeed there is an isomorphism of sets

$$\text{Spa}(L, L^+) \simeq \{\text{valuation rings of } L \text{ which contain } L^+\} \setminus \{L\},$$

$$\text{Spec}(L^+/L^{\infty}) \simeq \{\text{prime ideals of } L^+\} \setminus \{(0)\},$$

which can be extended to an isomorphism of topological spaces (after equipping the right hand sides with the obvious topology). Finally $X_{\text{red}}$ and $\text{Spec}(L^+/L^{\infty})$ are isomorphic as locally ringed spaces follows from Lemma 2.6.

Inspired by Example 2.11, the following lemma shows that the structure sheaf of $O_X$ behaves more like the structure sheaf of a scheme than an adic space (cf. [8, Lemma 1.1.10(iii)] for the analogous statement for adic spaces). In this way it is helpful to think of $X_{\text{red}}$ as the scheme part of $X$, where roughly one is killing valuation information by quotienting out the topologically nilpotent elements.

**Lemma 2.12.** Let $X$ be an analytic adic space and $x \in X$. There is a one to one correspondence between the generalizations $y$ of $x$ in $X$ and the prime ideals $p$ of $O_{X_{\text{red}},x}$. For corresponding $y$ and $p$, the natural ring homomorphism $O_{X_{\text{red}},x} \rightarrow O_{X_{\text{red}},y}$ is a localization of $O_{X,x}$ at $p$.

**Proof.** Denote $\text{Spa} \kappa(x) := \text{Spa}(k(x), k(x)^+)$. The natural morphism $\text{Spa} \kappa(x) \rightarrow X$ gives a homeomorphism from $\text{Spa} \kappa(x)$ to the set of all generalizations of $x$ in $X$ (cf. [8, (1.1.9)]). By Lemma 2.6 and Example 2.11, $(\text{Spa} \kappa(x))_{\text{red}} \simeq \text{Spec}(O_{X_{\text{red}},x})$ as locally ringed spaces. This proves the first part.

For the second part, note that the morphism $O_{X_{\text{red}},x} \rightarrow O_{X_{\text{red}},y}$ is induced by the morphism $k(x)^+ \rightarrow k(y)^+$, which by the (order-reversing) bijection of Example 2.11 is just localization at the corresponding prime ideal. \hfill \square
In general $O^+_x$ may not be henselian along its maximal ideal. However the henselian property is satisfied at stalks of the vanishing sheaf of ideals.

**Lemma 2.13.** For each $x \in X$, the pair $(O^+_x, m_{O^+_x})$ is henselian.

*Proof.* Since henselian pairs are preserved under filtered colimits of such pairs (cf. [14, Tag 0FWT]), by Proposition 2.9, it suffices to show that the pair $(A^+, A^{\circ\circ})$ is henselian for a complete Tate ring $A$. This is the content of [1, Lemma 7.2.3(5)]. □

One deduces, somewhat surprisingly, that in particular $k(x)^{\circ}$ is henselian along its maximal ideal, at rank 1 points $x \in X$.

**Corollary 2.14.** For each $x \in X$, the pair $(k(x)^+, k(x)^{\circ\circ})$ is henselian.

*Proof.* By passing to the quotient, Lemma 2.13 shows that the pair

$$(O^+_x/m_x, m_{O^+_x}/m_x) = (k(x)^+, k(x)^{\circ\circ}).$$

is henselian (we refer the reader to the proof of Lemma 2.6 for this identification). □

The next lemma shows that we do not lose cohomological information of $O^+_X$ by passing to $O_{X\text{-red}}$.

**Lemma 2.15.** Let $X$ be a topological space and $\mathcal{F}$ a sheaf of rings on $X$. Suppose $\mathcal{I} \subset \mathcal{F}$ is a subsheaf of ideals of $\mathcal{F}$, such that for each $x \in X$ and for each open subset $U \subseteq X$ containing $x$, there exists an open subset $U' \subseteq U$ containing $x$ such that the pair $(\mathcal{F}(U'), \mathcal{I}(U'))$ is henselian. Then for every open $V \subseteq X$, the pair $(\mathcal{F}(V), \mathcal{I}(V))$ is henselian.

*Proof.* We can assume without loss of generality that $V = X$. We begin by showing that $\mathcal{I}^a(X)$ is contained in the Jacobson radical of $\mathcal{F}(X)$. By [14, Tag 0AME] it suffices to show that $1 + i$ is a unit in $\mathcal{F}(X)$ for every $i \in \mathcal{I}^a(X)$. Take a covering $\{X_i\}$ of $X$ such that $i \in \mathcal{I}(X_i) \subseteq \mathcal{I}^a(X_i)$ and the pair $(\mathcal{F}(X_i), \mathcal{I}(X_i))$ is henselian. Then $1 + i$ is a unit in $\mathcal{F}(X_i)$. Therefore $1 + i$ is a unit in $\mathcal{F}(X)$.

To complete the proof we verify the criterion in [14, Tag 09XI]. Let $f(T) \in \mathcal{F}(X)[T]$ be a monic polynomial of the form

$$f(T) = T^n(T - 1) + a_nT^n + \cdots + a_1T + a_0$$

with $a_0, \ldots, a_n \in \mathcal{I}^a(X)$ with $n \geq 1$. Take a covering $\{X_i\}$ of $X$ such that $a_j \in \mathcal{I}(X_i) \subseteq \mathcal{I}^a(X_i)$ for all $0 \leq j \leq n$ and the pair $(\mathcal{F}(X_i), \mathcal{I}(X_i))$ is henselian. Then there exists a unique $a_{X_i} \in \mathcal{F}(X_i)$ with $f(a_{X_i}) = 0$ and $a_{X_i} \in 1 + \mathcal{I}(X_i)$. The $a_{X_i}$, now glue to give a unique $a_X \in 1 + \mathcal{I}^a(X)$ with $f(a_X) = 0$. □

**Corollary 2.16.** The pair $(O^+_X, m_{O^+_X}(X))$ is henselian.

*Proof.* This is a consequence of Proposition 2.9, Lemma 2.15 and [1, Lemma 7.2.3(5)]. □

Let $X$ be a type (S) formal scheme (cf. [8, §1.9]). Recall that we have a scheme $X_s$ associated to $X$ which is defined as follows. Let $I \subseteq O_X$ be the sheaf of ideals with $I(U) := \{f \in O_X(U) \mid f(x) = 0 \text{ for every } x \in U\}$ for every open subset $U \subseteq X$. Then $X_s := (X, O_X/I)$ is a reduced scheme. Furthermore, as in [8, Proposition 1.9.1], we have an analytic adic space $X_\eta$ (denoted by $d(X)$ in loc.cit.) and a morphism of locally ringed spaces (nearby cycles)

$$(3) \quad \lambda_X : (X_\eta, O^+_X) \to (X, O_X).$$

After killing functions which vanish at all points, $\lambda_X$ induces a morphism of locally ringed spaces

$$(4) \quad X_{\eta, \text{red}} \to X_s.$$
which we continue to denote by $\lambda_X$. In this way an analytic adic space specializes to a scheme. In an attempt to enlarge the class of specialization morphisms, this observation motivates the following definition.

**Definition 2.17** (Specialization morphism). Given an analytic adic space $X$ and a scheme $S$, we call a morphism of locally ringed spaces

$$X_{\text{red}} \to S$$

a **specialization morphism** and denote it by $X \to S$.

**Remark 2.18.** Specialization morphisms from analytic adic spaces to schemes can be composed with morphisms of adic spaces and morphisms of schemes. More precisely if $\alpha: X \to S$ is a specialization morphism, $f: Y \to X$ a morphism of adic spaces and $g: S \to T$ a morphism of schemes, then we get specialization morphisms $\alpha \circ f := \alpha \circ f_{\text{red}}: Y \to S$ and $g \circ \alpha: X \to T$.

We now show the existence of fibre products in certain situations.

**Proposition 2.19.** Let $S$ and $T$ be schemes and $X$ be an analytic adic space. Let $\alpha: X \to S$ be a specialization morphism and $g: T \to S$ an étale morphism of schemes. Then there exists an analytic adic space $Y$, a morphism $f: Y \to X$ of analytic adic spaces and a specialization morphism $\beta: Y \to T$ such that

$$\begin{array}{ccc}
Y & \xrightarrow{\beta} & T \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{\alpha} & S
\end{array}$$

commutes in the sense of Remark 2.18. Moreover the following universal property is satisfied. For every analytic adic space $Y'$, every morphism $f': Y' \to X$ of analytic adic spaces and every specialization morphism $\beta': Y' \to T$ such that $\alpha \circ f' = g \circ \beta'$, there exists a unique morphism $h: Y' \to Y$ of analytic adic spaces with $f' = f \circ h$ and $\beta' = \beta \circ h$. We denote $X \times_ST := Y$. Moreover the projection $f: X \times_ST \to X$ is an étale morphism of analytic adic spaces.

**Proof.** We may assume that $S$ is an affine scheme and that $X$ is an affinoid adic space of the form $\text{Spa}(A, A^+)$ where $A$ is complete. Since $X$ is analytic and as it suffices to prove the proposition locally for $X$, we can suppose without loss of generality that $A$ is Tate. We fix a topologically nilpotent unit $s \in A$ which belongs to $A^+$. Furthermore, we assume that $T$ is affine. It follows that $T$ must be of the form $\text{Spec}(E)$ where $E$ is an étale algebra over $D := \mathcal{O}_S(S)$. By [14, Tag 00U9], there exists a presentation $E = D[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\text{det}(\partial f_i/\partial x_i)$ is invertible in $E$. Since $\alpha: X_{\text{red}} \to S$ is a morphism of locally ringed spaces, we get a map $\mathcal{O}_S(S) \to (\mathcal{O}_X^+ / \mathfrak{m}_{\mathcal{O}_X^+})^\ast (X)$. It follows that if $C := (\mathcal{O}_X^+ / \mathfrak{m}_{\mathcal{O}_X^+})^\ast (X)$ then by base change, the morphism

$$C \to C' := C \otimes_D E$$

is an étale morphism of algebras. We have that $C' = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\text{det}(\partial f_i/\partial x_i)$ is invertible in $C'$ where we abuse notation and use $f_i$ to denote the image in $C[x_1, \ldots, x_n]$ of the polynomial $f_i \in D[x_1, \ldots, x_n]$ via the map $D[x_1, \ldots, x_n] \to C[x_1, \ldots, x_n]$ induced by the map $D \to C$.

The quotient map $\mathcal{O}_X^+ \to \mathcal{O}_X^+ / \mathfrak{m}_{\mathcal{O}_X^+}$ induces a map $A^+ \to C$. This map is not necessarily surjective. However, since $\mathcal{O}_X^+ \to \mathcal{O}_X^+ / \mathfrak{m}_{\mathcal{O}_X^+}$ is surjective as a morphism of sheaves and because it suffices to construct the adic space $Y$ from the statement of the proposition locally over $X$, we can suppose that the image of $A^+$ in $C$ contains
the coefficients of the polynomials $f_i$. For every $1 \leq i \leq n$, let $g_i$ be a lift of $f_i$ along $A^+ \to C$.

Let

$$Y := \text{Spa}(\langle A, A^+ \rangle(X_1, \ldots, X_n)/(g_1, \ldots, g_n)) =: \text{Spa}(B, B^+)$$

where we refer the reader to [7, §3] and [8, (1.4.1)] for the definition of the quotient appearing in (5). In particular

$$B = A(X_1, \ldots, X_n)/(g_1, \ldots, g_n)$$

and

$$B^+ = (A^+(X_1, \ldots, X_n)^c/(g_1, \ldots, g_n))A(X_1, \ldots, X_n) \cap A^+(X_1, \ldots, X_n)^c$$

where $A^+(X_1, \ldots, X_n)^c$ is the integral closure of $A^+(X_1, \ldots, X_n)$ in $A(X_1, \ldots, X_n)$ and $B^+$ is the integral closure of

$$A^+(X_1, \ldots, X_n)^c/(g_1, \ldots, g_n))A(X_1, \ldots, X_n) \cap A^+(X_1, \ldots, X_n)^c$$

in $B$.

Let $B_0 := A^+(X_1, \ldots, X_n)/(g_1, \ldots, g_n)$ and observe that we have a well defined map $B_0 \to B^+$. The construction thus far fits into the following commutative\(^3\)

$$\begin{array}{c}
A(X_1, \ldots, X_n)/(g_i) \leftarrow A^+(X_1, \ldots, X_n)/(g_i) \to \mathcal{O}_{X_{\text{red}}} \otimes D \leftarrow E \\
A \leftarrow A^+ \to \mathcal{O}_{X_{\text{red}}} \leftarrow D.
\end{array}$$

(8)

Let $f: Y \to X$ be the map induced by the adic morphism $A \to B$. We now construct the map $\beta: Y_{\text{red}} \to T$. By [14, Tag 0111], it suffices to construct a ring homomorphism $E \to \mathcal{O}_{Y_{\text{red}}}(Y)$ and to show that the following diagram is commutative:

$$\begin{array}{ccc}
E & \to & \mathcal{O}_{Y_{\text{red}}}(Y) \\
\uparrow & & \uparrow \\
D & \to & \mathcal{O}_{X_{\text{red}}}(X).
\end{array}$$

(9)

The following commutative diagram

$$\begin{array}{ccc}
B_0 & \to & B^+ & \to & \mathcal{O}_{Y_{\text{red}}}(Y) & \to & \mathcal{O}_{Y_{\text{red}}, Y} \\
& & \downarrow & & \uparrow & & \uparrow \\
& & A^+ & \to & \mathcal{O}_{X_{\text{red}}}(X) & \to & \mathcal{O}_{X_{\text{red}}, x}.
\end{array}$$

(10)

implies that the map $\mathcal{O}_{X_{\text{red}}}(X) \to \mathcal{O}_{Y_{\text{red}}}(Y)$ factors through a map

$$B_0 \otimes_A \mathcal{O}_{X_{\text{red}}}(X) \to \mathcal{O}_{Y_{\text{red}}}(Y).$$

(11)

By Lemma 2.20, this implies that we have a map $\mathcal{O}_{X_{\text{red}}}(X) \otimes D \to \mathcal{O}_{Y_{\text{red}}}(Y)$. Let $\beta$ denote the composition $E \to \mathcal{O}_{X_{\text{red}}}(X) \otimes D \to \mathcal{O}_{Y_{\text{red}}}(Y)$. Observe that with this choice of $\beta$ the diagram (9) commutes.

\(^3\)Since for any $h \in A^+(X_1, \ldots, X_n)$, almost all the coefficients of $h$ are topologically nilpotent in $A$, it follows that there is a well defined map $A^+(X_1, \ldots, X_n)/(g_i) \to \mathcal{O}_{X_{\text{red}}}(X) \otimes D \to E$. 

Lemma 2.20. We have that

\[ B_0 \otimes_{A^+} \mathcal{O}_{X_{\text{red}}}(X) = \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n]/(f_i)_{i} = \mathcal{O}_{X_{\text{red}}}(X) \otimes_{D} E. \]

Proof. Recall by construction, \( \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n]/(f_i)_{i} = \mathcal{O}_{X_{\text{red}}}(X) \otimes_{D} E. \) Hence, it suffices to show

\[ B_0 \otimes_{A^+} \mathcal{O}_{X_{\text{red}}}(X) = \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n]/(f_i)_{i}. \]

Recall by definition, \( B_0 = A^+[X_1, \ldots, X_n]/(g_i). \) We see that the image of \( s \) in \( \mathcal{O}_{X_{\text{red}}}(X) \) is zero. It follows that

\[ B_0 \otimes_{A^+} \mathcal{O}_{X_{\text{red}}}(X) = (B_0/sB_0) \otimes_{(A^+/sA^+)} \mathcal{O}_{X_{\text{red}}}(X). \]

Clearly, \( B_0/sB_0 = (A^+/sA^+)[X_1, \ldots, X_n]/(\tilde{g}_i) \) where \( \tilde{g}_i \) is obtained from \( g_i \in A^+[X_1, \ldots, X_n] \) by reducing each of its coefficients modulo \( sA^+ \). This follows by first observing that \( (A^+[X_1, \ldots, X_n])/(sA^+[X_1, \ldots, X_n]) = (A^+/sA^+)[X_1, \ldots, X_n] \) which is a consequence of the fact that any element of \( A^+[X_1, \ldots, X_n] \) has all but finitely many coefficients not belonging to \( sA^+ \). We hence get the equality \( B_0/sB_0 = (A^+/sA^+)[X_1, \ldots, X_n]/(\tilde{g}_i) \) by observing that if \( R = A^+[X_1, \ldots, X_n] \) then \( B_0/sB_0 = (R/sR)/(sR + 1R)/sR \) where \( I \) is the ideal generated by the \( g_i \). Now observe that the ideal \( sR + 1R \) in \( R/sR \) is generated by the elements \( \tilde{g}_i \).

Hence

\[ B_0 \otimes_{A^+} \mathcal{O}_{X_{\text{red}}}(X) = \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n]/(\tilde{g}_i). \]

By construction, the image of \( g_i \), and hence \( \tilde{g}_i \), in \( \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n] \) coincides with \( f_i \). This completes the proof. \( \square \)

We claim that \( Y \) is étale over \( X \) or equivalently that the determinant of the Jacobian \( u := \det(\partial g_i/\partial X_i) \) is invertible in \( B \) (cf. [8, Proposition 1.7.1]). Observe that \( u \in B_0 \). By construction, the image of \( u \) in \( \mathcal{O}' = \mathcal{O}_{X_{\text{red}}}(X)[X_1, \ldots, X_n]/(f_i)_{i} \) is invertible. By (11), Lemma 2.20 and diagram (10) the map \( B_0 \to \mathcal{O}_{Y_{\text{red}}}(Y) \) factors through a map \( \mathcal{O}' \to \mathcal{O}_{Y_{\text{red}}}(Y) \). It follows that the image of \( u \) in \( \mathcal{O}_{Y_{\text{red}}}(Y) \) is invertible and we deduce from this that \( u \) is invertible in \( \mathcal{O}_{Y_{\text{red}},y} \) for every \( y \in Y \). For any \( y \in Y \), since the map \( \mathcal{O}_{Y_{\text{red}},y} \to \mathcal{O}_{Y_{\text{red}},y} \) is a local morphism of rings, we must have that \( u \) is a unit in \( \mathcal{O}_{Y_{\text{red}},y} \) as well. Hence we conclude that \( u \) is a unit in \( \mathcal{O}_{Y_{\text{red}},y} = B^+ \).

We now prove the universal property. This is essentially a variant of Hensel’s Lemma. Let \( Y' \) be an analytic adic space as in the statement of the proposition. We are provided a morphism of adic spaces \( f': Y' \to X \) and a specialization morphism \( \beta': Y' \to T \) such that \( \alpha \circ f' = g \circ \beta' \) or equivalently that the following diagram commutes:

\[
\begin{array}{ccc}
Y' & \xrightarrow{\beta'} & T \\
\downarrow{f'} & & \downarrow{g} \\
X & \xrightarrow{\alpha} & S.
\end{array}
\]

We deduce that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O}_{Y_{\text{red}}}(Y') & \xleftarrow{\mathcal{O}_{Y_{\text{red}}}(f')} & D \\
\uparrow{E} & & \uparrow{\mathcal{O}_{Y_{\text{red}}}(g)} \\
\mathcal{O}_{X_{\text{red}}}(X) & \xrightarrow{\mathcal{O}_{X_{\text{red}}}(g)} & E
\end{array}
\]

Hence we have a unique map \( \mathcal{O}_{X_{\text{red}}}(X) \otimes_{D} E \to \mathcal{O}_{Y_{\text{red}}}(Y') \).

By diagram (10), we have the following commutative diagram:
such that the following diagram is commutative:

There exists a unique morphism

Lemma 2.21. There exists a unique morphism

\[ A^+(X_1, \ldots, X_n)/ (g_i, ) \rightarrow \mathcal{O}_{Y'}^+(Y') \]

such that with the addition of this morphism, the diagram above remains commutative. At this point, we simplify the situation and assume that \( n = 2, g_1 \in A^+[X_1] \) and \( g_2 \) is of the form \( X_2 \cdot h - 1 \) where \( h \in A^+[X_1] \). Note that such a simplification is possible by assuming that the morphism \( D \rightarrow E \) is a standard étale extension. We can make this assumption on the morphism \( D \rightarrow E \) since by [14, Tag 02GT], every étale morphism is locally standard étale and it suffices to prove our lemma on a cover of \( S \).

**Lemma 2.21.** There exists a unique morphism

\[ A^+(X_1, X_2)/ (g_1, g_2) \rightarrow \mathcal{O}_{Y'}^+(Y') \]

such that the following diagram is commutative:

\[
\begin{array}{c}
\mathcal{O}_{Y'}^+(Y') \rightarrow \mathcal{O}_{Y'}^+(Y') \\
A^+(X_1, X_2)/ (g_1, g_2) \rightarrow \mathcal{O}_{X_{red}}(X) \oplus_D E \rightarrow E \\
A^+ \rightarrow \mathcal{O}_{X_{red}}(X) \leftarrow D.
\end{array}
\]

**Proof.**

Observe from diagram (12) above that we have a well defined map

\[ A^+(X_1, X_2)/ (g_1, g_2) \rightarrow \mathcal{O}_{Y_{red}}^+(Y'). \]

For every \( i \), let \( \tilde{x}_i \) be the image in \( \mathcal{O}_{Y_{red}}^+(Y') \) of \( x_i \) for this map. Our goal is to show that there exists a unique set of lifts \( x_1, x_2 \in \mathcal{O}_{Y'}^+(Y') \) of \( \tilde{x}_1, \tilde{x}_2 \) such that if \( x := (x_1, x_2) \) then \( g_j(x) = 0 \) for every \( j \).

Since \( \mathcal{O}_{Y_{red}}^+ \) is the sheafification of the quotient presheaf \( U \mapsto \mathcal{O}_{Y'}^+(U)/m_{\mathcal{O}_{Y'}^+}(U) \), we can take a covering \( \{U_i\} \) of \( Y' \) such that the \( (\tilde{x}_1)_{|U_i} \) are in the image of

\[ \mathcal{O}_{Y'}^+(U_i)/m_{\mathcal{O}_{Y'}^+}(U_i) \rightarrow \mathcal{O}_{Y_{red}}^+(U_i) \]

and \( g_1(y_{1i}) = 0 \) for some \( y_{1i} \in \mathcal{O}_{Y'}^+(U_i)/m_{\mathcal{O}_{Y'}^+}(U_i) \) mapping to \( (\tilde{x}_1)_{|U_i} \). Since \( \tilde{x}_1 \) is a simple root of \( g_1 \), it follows that the choice of the \( y_{1i} \) are uniquely determined. In particular they coincide on intersections. Now \( \mathcal{O}_{Y'}^+(U_i) \) is henselian along the ideal \( m_{\mathcal{O}_{Y'}^+}(U_i) \) (cf. Corollary 2.16) and so there exists a unique lift \( x_{1i} \in \mathcal{O}_{Y'}^+(U_i) \) of \( y_{1i} \) such that \( g_1(x_{1i}) = 0 \). The \( x_{1i} \) glue to give \( x_1 \in \mathcal{O}_{Y'}^+(Y') \). Clearly \( x_1 \) does
It remains to show that \( h(x_1) \) is invertible (the value of \( x_2 \) is thereafter uniquely determined). Along the map 
\[
\mathcal{O}_Y^+(Y') \to \mathcal{O}_{\text{red}}^+(Y'),
\]
h(\( x_1 \)) is mapped to \( h(\tilde{x}_1) \). Now \( h(\tilde{x}_1) \) is invertible in \( \mathcal{O}_{\text{red}}^+(Y') \) so one can take a covering \( \{V_j\} \) of \( Y' \) such that the inverse of \( h(\tilde{x}_1) \) in \( \mathcal{O}_{\text{red}}^+(V_j) \) comes from the image of \( \mathcal{O}_Y^+(V_j) \). The kernel of the map 
\[
\mathcal{O}_Y^+(V_j) \to \mathcal{O}_{\text{red}}^+(V_j)
\]
is \( m_{\mathcal{O}_Y^+(V_j)} \), which in particular, is contained in the Jacobson radical of \( \mathcal{O}_Y^+(V_j) \) (cf. Corollary 2.16). Therefore \( h(x_1) \) is invertible in each \( \mathcal{O}_Y^+(V_j) \). One can now glue the inverses to obtain an inverse for \( h(x_1) \) in \( \mathcal{O}_Y^+(Y') \).

To conclude the proof, we must obtain a morphism \( Y' \to Y \). Such a morphism corresponds to the morphism \( A\langle X_1, X_2 \rangle/(g_1, g_2) \to \mathcal{O}_{\text{red}}^+(Y') \) which is the unique extension of the map \( A^+\langle X_1, X_2 \rangle/(g_1, g_2) \to \mathcal{O}_Y^+(Y') \) from Lemma 2.21.

**Remark 2.22.** In the proof of Proposition 2.19, we only showed that (5) is the fibre product of the morphisms \( \alpha: X \to S \) and \( g: T \to S \) when the following assumptions are satisfied (using the notation from the proof itself):

(i) \( X \) is affinoid, and \( S \) and \( T \) are affine,
(ii) \( g \) is standard étale,
(iii) the image of \( A^+ \) in \( C \) contains the coefficients of the polynomials \( f_i \), and
(iv) \( A \) is Tate.

Recall that we used \( Y \) to denote \( X \times_S T \). In general we do not know whether (i) implies \( Y \) is affinoid. One possible obstruction seems to be that there is no good notion of “affinoid morphism” between analytic adic spaces. A related question is given a specialization morphism \( Z \to \text{Spec}(D) \), does it factorize via the canonical morphism \( Z \to \text{Spa}(\mathcal{O}_Z(Z), \mathcal{O}_Z^+(Z)) \)?

**Lemma 2.23.** In the setting of Proposition 2.19, suppose \( T \) and \( S \) are affine, and \( X \) and \( Y \) are affinoid. If in addition \( g: T \to S \) is finite, then so is the projection morphism \( f: Y \to X \).

**Proof.** By [8, Proposition 1.4.6], it suffices to show that \( f \) is proper. The morphism \( f \) is in addition quasi-separated and so by Lemma 1.3.10 in loc.cit., it suffices to verify the valuative criterion for properness. This follows from the valuative criterion for properness (for schemes) and the universal property of fibre products.

We look at some fibre products (of Proposition 2.19) appearing in nature.

**Example 2.24.** The diagram

\[
\begin{array}{ccc}
\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) & \longrightarrow & \text{Spec}(\mathbb{F}_p) \\
\downarrow & & \downarrow \\
\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) & \longrightarrow & \text{Spec}(\mathbb{F}_p)
\end{array}
\]
is a fibre product, where $\mathbb{Q}_p^n$ is the unramified extension of degree $n$ of $\mathbb{Q}_p$. In this way one can also recover unramified covers of the Fargues-Fontaine curve (cf. [4]).

The diagram

\[
\begin{array}{ccc}
Y/\varphi^n \ar[r] & \text{Spec}(\mathbb{F}_p^n) \\
\downarrow & & \downarrow \\
Y/\varphi \ar[r] & \text{Spec}(\mathbb{F}_p)
\end{array}
\]

is a fibre product, where $Y/\varphi$ is the Fargues-Fontaine curve ($Y = \text{Spa}(A_{\text{int}}) \setminus V(p[p^t])$).

We now show that fibre products preserve surjective morphisms. We will need a thinning out result for an étale morphism of affine schemes:

**Lemma 2.25.** Let $g: \mathcal{T} \to \mathcal{S}$ be an étale morphism of affine schemes and $t \in \mathcal{T}$ be a point which lies over $s \in \mathcal{S}$. Then there exists a commutative diagram of affine schemes

\[
\begin{array}{ccc}
\mathcal{T}' & \ar[r]^{g'} & \mathcal{T} \\
\downarrow & & \downarrow \\
\mathcal{S}' & \ar[r] & \mathcal{S}
\end{array}
\]

such that

1. the morphism $g': \mathcal{T}' \to \mathcal{S}'$ is finite étale and $\mathcal{S}' \to \mathcal{S}$ is étale,
2. there exists a point $s' \in \mathcal{S}'$ lying over $s$ with $\kappa(s') = \kappa(s)$,
3. $\mathcal{T}'$ has exactly one point $t'$ lying over $s'$. Furthermore $t'$ lies over $t$.

**Proof.** This is a consequence of [14, Tag 00UJ].

**Proposition 2.26.** Let $S$ and $T$ be schemes and $X$ be an analytic adic space. Let $\alpha: X \to S$ be a specialization morphism and $g: T \to S$ an étale morphism of schemes. Given a point $x \in X$ and any point $t \in T$, there is a point of $X \times_S T$ mapping to $x$ and $t$ under the projections if and only if $x$ and $t$ lie above the same point of $S$.

**Proof.** The condition is obviously necessary. We prove the converse. The proof in [8, Lemma 3.5.1(i)] relies on a comparison between the surjectivity of $\text{Spec}(f)$ and $\text{Spv}(f)$ for a morphism of rings $f: \mathcal{R}_1 \to \mathcal{R}_2$, which we avoid (cf. [9, Proposition 2.1.3]). In contrast, we make use of the universal property of fibre products.

Suppose $x$ and $t$ lie above the same point $s \in S$ and assume without loss of generality that $S$ and $T$ are affine and $g$ is standard étale. The point $x$ determines a morphism $\text{Spa}(k(x), k(x)^+) \to X$, whose image of the closed point in $\text{Spa}(k(x), k(x)^+)$ is $x$. Let $k(x)^+\mathbb{h}$ denote the henselization of the local ring $k(x)^+$ and set $k(x)^h$ to be the fraction field of $k(x)^+\mathbb{h}$. Now $k(x)^+\mathbb{h}$ is again a valuation ring with the same value group as $k(x)^+$. By [14, Tag 0ASK], $k(x)^+\mathbb{h}$ is again a valuation ring with the same value group as $k(x)^+$. Now $k(x)^+\mathbb{h}$ is again a valuation ring with the same value group as $k(x)^+$. Now $k(x)^+\mathbb{h}$ is again a valuation ring with the same value group as $k(x)^+$. Now $k(x)^h$, makes the pair $(k(x)^h, k(x)^{+h})$ an analytic affinoid field. The extension $k(x)^+ \to k(x)^{+h}$ is local and so in particular the image of the closed point via $\text{Spa}(k(x)^h, k(x)^{+h}) \to \text{Spa}(k(x), k(x)^+)$ is again closed.
There exists a commutative diagram

\[
\begin{array}{ccc}
X \times_S T & \xrightarrow{P'} & T' \\
\downarrow \alpha & & \downarrow g' \\
\text{Spa}(k(x)^h, k(x)^{+h}) & \xrightarrow{S'} & S.
\end{array}
\]

We take a moment to explain its features:

1. the square

\[
\begin{array}{ccc}
T' & \xrightarrow{g'} & T \\
\downarrow g & & \downarrow s \\
S' & \xrightarrow{S} & S
\end{array}
\]

satisfies the conditions of Lemma 2.25 and hereafter we use the notation setup there,

2. since the étale morphism \(S' \to S\) induces an isomorphism on the residue fields \(\kappa(s') = \kappa(s)\), it follows that the specialization morphism \(\text{Spa}(k(x)^h, k(x)^{+h}) \to S\) factors uniquely through \(S'\) such that the image of the closed point in \(\text{Spa}(k(x)^h, k(x)^{+h})\) is \(s' \in S'\) (cf. \([14, \text{Tag 08HQ}]\)),

3. by (2), we can define the fibre product \(P'_x := \text{Spa}(k(x)^h, k(x)^{+h}) \times_{S'} T'\).

4. by the universal property, there is a unique morphism \(P'_x \to X \times_S T\) rendering the diagram commutative.

Now since \(P'_x \to \text{Spa}(k(x)^h, k(x)^{+h})\) is finite (in particular it is specializing), there is a point \(y \in P'_x\) which lies over the closed point of \(\text{Spa}(k(x)^h, k(x)^{+h})\). Since there exists a unique point \(t' \in T'\) over \(s'\), it follows that \(y\) must lie over \(t' \in T'\). Therefore the image of \(y\) in \(X \times_S T\) satisfies the conditions of the proposition.

\[\square\]

**Corollary 2.27.** Let \(S\) be a scheme, \(X\) be an analytic adic space and \(\alpha : X \to S\) a specialization morphism. The functor

\[
\alpha_{\text{ét}} : S_{\text{ét}} \to X_{\text{ét}}
\]

\[
T \mapsto X \times_S T
\]

induces a morphism of sites \(\alpha_{\text{ét}} : X_{\text{ét}} \to S_{\text{ét}}\).

**Proof.** By \([14, \text{Tag 00X1}]\), we must show the following.

1. The functor \(\alpha_{\text{ét}}\) preserves fibre products.
2. The functor \(\alpha_{\text{ét}}\) preserves coverings i.e. if \(T \in S_{\text{ét}}\) and \(\{T_i \to T\}_i\) is an étale cover of \(T\) then \(\{X \times_S T_i \to X \times_S T\}_i\) is an étale cover of \(X \times_S T\).
(3) The pullback functor $\alpha^*_\text{ét}$ is exact.

Property (1) can be deduced from the universal property of fibre products in Proposition 2.19. Property (2) is a consequence of Proposition 2.26. Property (3) is a consequence of [14, Tag 00X6] and (1) above.

**Proposition 2.28.** Let $\alpha : X \to S$ be a specialization morphism, $f : Y \to X$ a morphism of analytic adic spaces and $g : S \to T$ a morphism of schemes. Then $(\alpha \circ f)_\text{ét} = \alpha_\text{ét} \circ f_\text{ét}$ and $(g \circ \alpha)_\text{ét} = g_\text{ét} \circ \alpha_\text{ét}$.

**Proof.** This is a consequence of the universal property of fibre products. We first show $(\alpha \circ f)_\text{ét} = \alpha_\text{ét} \circ f_\text{ét}$. Let $h : R \to S$ be an étale morphism of schemes and consider a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{Y \times_X (X \times_S R)} & X \times_S R \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X & \xrightarrow{\alpha} & S \\
\end{array}
\]

where $Z$ is an analytic adic space, $Z \to Y$ a morphism of analytic adic spaces and $Z \to R$ a specialization morphism. Then by the universal property of $X \times_S R$, there exists a unique morphism of analytic adic spaces $Z \to X \times_S R$, which results in a commutative diagram. Finally one applies the universal property of $Y \times_X (X \times_S R)$ to conclude.

In a similar fashion we show $(g \circ \alpha)_\text{ét} = g_\text{ét} \circ \alpha_\text{ét}$. Let $h : R \to T$ be an étale morphism of schemes and consider a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{X \times_S (S \times_T R)} & S \times_T R \\
\downarrow & & \downarrow \\
X & \xrightarrow{\alpha} & S & \xrightarrow{g} & T \\
\end{array}
\]

where $Z$ is a analytic adic space, $Z \to X$ a morphism of analytic adic spaces and $Z \to R$ a specialization morphism. By [14, Tag 01JN], the fibre product $S \times_T R$ is actually a fibre product in the category of locally ringed spaces. Thus by the universal property of $S \times_T R$ (in the category of locally ringed spaces), there exists a unique specialization morphism $Z \to S \times_T R$ which results in a commutative diagram. Finally one applies the universal property of $X \times_S (S \times_T R)$ to conclude.

We close this section by relating the nearby cycles functor (the pushforward of (3)) and the pushforward of specialization morphisms (cf. (4)).
Lemma 2.29. Let \( f : \mathcal{Y} \to \mathcal{X} \) be an étale morphism of type (S) formal schemes. Then the diagram

\[
\begin{array}{ccc}
\mathcal{Y}_s & \xrightarrow{\lambda_s} & \mathcal{Y} \\
\downarrow f_s & & \downarrow f \\
\mathcal{X}_s & \xrightarrow{\lambda_x} & \mathcal{X}
\end{array}
\]

is cartesian in the sense of Proposition 2.19.

Proof. We can assume \( \mathcal{X} = \text{Spf}(A) \) and \( \mathcal{Y} = \text{Spf}(B) \) and \( f_s : \mathcal{Y}_s \to \mathcal{X}_s \) is standard étale. Then

\[
\mathcal{Y}_s = \text{Spec}(\mathcal{O}_{\mathcal{X}_s}(\mathcal{X}_s)[x_1, x_2]/(f_1, f_2)).
\]

Let \( g_i \) be any lifts of \( f_i \) along \( A(X_1, X_2) \to \mathcal{O}_{\mathcal{X}_s}(\mathcal{X}_s)[x_1, x_2] \). Then

\[
\text{Spf}(B) \simeq \text{Spf}(A(X_1, X_2)/(g_1, g_2)).
\]

In the case \( \mathcal{X} \) is of type (S)(b), the lemma is now clear from the construction of the fibre product in Proposition 2.19 (in this case \( \mathcal{X}_s \) and \( \mathcal{Y}_s \) are both affinoid and the ring of global functions coincides with the one constructed in Proposition 2.19). Suppose now \( \mathcal{X} \) is of type (S)(a) (in this case we do not know if \( \mathcal{X}_s = \text{Spa}(A, A) \) is affinoid). This means \( \mathcal{Y} \) is also of type (S)(a). In this case we check universality directly. Let \( Z \) be an analytic adic space making the following diagram commutative

\[
\begin{array}{ccc}
Z & \xrightarrow{f_s} & \mathcal{Y}_s \\
\downarrow & & \downarrow \\
\mathcal{X}_s & \xrightarrow{\lambda_x} & \mathcal{X}_s
\end{array}
\]

It suffices to show that there exists a unique morphism \( Z \to \text{Spa}(B, B) \) making the following diagram commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{f_s} & \mathcal{Y}_s & \xrightarrow{f_s} & \mathcal{Y}_s \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_s & \xrightarrow{\lambda_x} & \text{Spa}(A, A) & \xrightarrow{f_s} & \text{Spa}(A, A)
\end{array}
\]

This is because such a morphism \( Z \to \text{Spa}(B, B) \) must be adic (since the composition \( Z \to \text{Spa}(B, B) \to \text{Spa}(A, A) \) is adic) and therefore factors through the open subspace \( \mathcal{Y}_s \subseteq \text{Spa}(B, B) \). Let us remark that the morphisms \( \text{Spa}(A, A) \to \mathcal{X}_s \) and \( \text{Spa}(B, B) \to \mathcal{Y}_s \) are defined in the same way as in Definition 2.17 (the only difference in the current situation being that \( \text{Spa}(A, A) \) and \( \text{Spa}(B, B) \) are not necessarily analytic). To construct a morphism \( Z \to \text{Spa}(B, B) \), it suffices to construct a continuous morphism \( B \to \mathcal{O}_{\mathcal{Y}^x}(Z) \). For this one proceeds as in Lemma 2.21. □

3. Compactifications of specialization morphisms

In this section we construct compactifications of specialization morphisms which satisfy some finiteness conditions. Therefore we need to develop notions of specialization morphisms, which are of finite type, separated and partially proper. The definition of a valuation ring of a scheme/adic space and of a center of a valuation
ring of a scheme/adic space extend immediately to arbitrary locally ringed spaces. This will be the starting point for us.

**Definition 3.1** (valuation ring of a locally ringed space). Let \((X, \mathcal{O}_X)\) be a locally ringed space. A valuation ring of \((X, \mathcal{O}_X)\) is a pair \((x, A)\), where \(x\) is a point of \(X\) and \(A\) is a valuation ring of the residue class field \(k(x)\) of \(x\).

**Definition 3.2** (center of a valuation ring). Let \((X, \mathcal{O}_X)\) be a locally ringed space with a valuation ring \((x, A)\). A point \(y \in X\) is called a center of \((x, A)\) if there exists a morphism \(\text{Spec}(A) \to (X, \mathcal{O}_X)\) of locally ringed spaces, such that the image of the generic point (resp. closed point) of \(\text{Spec}(A)\) in \(X\) is \(x\) (resp. \(y\)). Moreover we demand that the composition \(\text{Spec}(k(x)) \to \text{Spec}(A) \to (X, \mathcal{O}_X)\) induces \(\text{id}: \mathcal{O}_{X,x}/m_x \to k(x)\), where \(m_x \subset \mathcal{O}_{X,x}\) is the maximal ideal.

**Lemma 3.3.** Let \((s, A)\) be a valuation ring of an affine scheme \(S\). Then \((s, A)\) has at most one center on \(S\).

**Proof.** Let \(S = \text{Spec}(R)\) and let \(p \subseteq R\) correspond to \(s \in S\). Then morphisms \(\text{Spec}(A) \to \text{Spec}(R)\) such that \((0)\) is sent to \(p\) correspond to injections \(R/p \hookrightarrow A\). There is at most one such morphism over \(k(s)\). \(\square\)

**Remark 3.4.** For a locally ringed space \((X, \mathcal{O}_X)\), let \(X_v\) denote the set of all valuation rings of \(X\). Every morphism \(\alpha: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) induces a mapping \(\alpha_v: X_v \to Y_v\) given by \((x, A) \mapsto (\alpha(x), A \cap k(\alpha(x)))\). We equip \(X_v\) with the topology which is generated by the sets

\[
D(U, f) := \{ (x, A) \in X_v \mid x \in U \text{ and } f(x) \in A \}
\]

where \(U \subset X\) is open and \(f \in \mathcal{O}_X(U)\).

In case of overlap, the following lemma says that the notion of a valuation ring/center of a valuation ring introduced in Definitions 3.1/3.2 coincide with that of Huber’s definition in [8].

**Lemma 3.5.** Let \(X\) be an analytic adic space. Let \(X_v^h\) be the \(X_v\) defined in [8, page 52]. We equip \(X_v^h\) with the topology defined in (1.3.11) loc.cit. Then there is an isomorphism as topological spaces \((X_{\text{red}})_v \cong X_v^h\).

**Proof.** It is enough to note that the canonical morphism \(X_{\text{red}} \to (X, \mathcal{O}_X^+\) induces an isomorphism on residue fields (for \(x \in X\), the residue field corresponds to the residue field of \(k(x)^+\)) and that there is a 1-1 correspondence between the set of all valuation rings of the residue field of \(k(x)^+\) and valuation rings of \(k(x)\) which are contained in \(k(x)^+\). \(\square\)

**Definition 3.6** (locally of finite type). Let \(\alpha: X \to S\) be a specialization morphism. Then \(\alpha\) is called locally of finite type if for every \(x \in X\), there exists an affinoid open neighbourhood \(U \ni x\) in \(X\) and an affine open subscheme \(W \subseteq S\) satisfying the following conditions.

1. We have that \(\alpha(U) \subseteq W\) and if
   \[
a: \mathcal{O}_S(W) \to \mathcal{O}_{X_{\text{red}}}(U) \\
b: \mathcal{O}_X(U) \to \mathcal{O}_{X_{\text{red}}}(U)
\]
   denote the natural homomorphisms then \(\text{im}(a) \subseteq \text{im}(b)\).

2. There is a finite subset \(E \subseteq \text{im}(b)\) with \(\text{im}(b)\) integral over \(\text{im}(a)[E]\).

We will sometimes call such a pair \((U, W)\) (and the induced specialization morphism \(\alpha_U: U \to W\)) good.
SPECIALIZATION MORPHISMS

The reader will note (despite the choice of terminology) that Definition 3.6 resembles the notion of locally of \(^{+}\) weakly finite type morphisms between adic spaces modulo topologically nilpotent elements.

**Remark 3.7.** It is easy to check that if the pair \((U, W)\) satisfies the conditions in Definition 3.6 then so does \((V, W)\) for any affinoid open neighbourhood \(V \ni x\) in \(X\) with \(V \subseteq U\). Indeed the modified morphisms \(a'\) and \(b'\) sit in commutative diagrams

\[
a': \mathcal{O}_S(W) \xrightarrow{a} \mathcal{O}_{X_{\text{red}}}(U) \xrightarrow{\tilde{\rho}_{UV}} \mathcal{O}_{X_{\text{red}}}(V)
\]

and

\[
\begin{array}{ccc}
\mathcal{O}^+_X(U) & \xrightarrow{b} & \mathcal{O}_{X_{\text{red}}}(U) \\
\downarrow{\tilde{\rho}_{UV}} & & \downarrow{\tilde{\rho}_{UV}} \\
\mathcal{O}^+_X(V) & \xrightarrow{b'} & \mathcal{O}_{X_{\text{red}}}(V).
\end{array}
\]

Thus \(\text{im}(a') = \text{im}(\tilde{\rho}_{UV} \circ a) \subseteq \text{im}(\tilde{\rho}_{UV} \circ b) \subseteq \text{im}(b')\). Since \(V \to U\) is of finite type, there exists a finite subset \(E_V\) of \(\mathcal{O}_X(V)\) such that \(\mathcal{O}^+_X(V)\) is the integral closure of \(\mathcal{O}^+_X(U)[E_V \cup \mathcal{O}_X(V)^{\infty}]\). Thus \(\text{im}(b')\) is integral over \(\text{im}(a')E_V \cup \tilde{\rho}_{UV}(E)\).

Similarly, it is easy to check that if the pair \((U, W)\) satisfies the above conditions then so does \((U, Z)\) for any affine open subscheme \(W \subseteq Z \subseteq S\).

We now show that the notion of good pair is stable under certain base change.

**Lemma 3.8.** Let \(\alpha: X \to S\) be a good specialization morphism such that \(\mathcal{O}_X(X)\) is a Tate ring. Let \(T\) be an affine scheme and \(g: T \to S\) be a standard étale morphism. Let \(Y := X \times_S T\) be the fibre product as constructed in Proposition 2.19 and \(\beta: Y \to T\) be the projection. We then have that \(Y\) is affinoid and \(\beta\) is a good specialization morphism.

**Proof.** We make use of the notation introduced in Proposition 2.19. Hence, \(S = \text{Spec}(D)\) and \(T = \text{Spec}(E)\) where \(E = D[x_1, x_2]/(f_1, f_2)\) where \(f_1 \in D[x_1]\) and \(f_2\) is of the form \(x_2h - 1\) for some \(h \in D[x_1]\). This is due to the fact that we assumed \(E\) is standard étale over \(D\). Furthermore, let \((A, \mathcal{O}^+_X) = (\mathcal{O}_X(X), \mathcal{O}^+_X(X))\).

Since the specialization \(\alpha\) is good, we deduce that we have a morphism \(D \to \mathcal{O}^+_X/\mathfrak{m}_{\mathcal{O}^+_X}(X)\). Indeed, we have an injection \(\mathcal{O}^+_X/\mathfrak{m}_{\mathcal{O}^+_X}(X) \hookrightarrow \mathcal{O}_{X_{\text{red}}}(X)\). Furthermore, the image of \(D\) for the morphism \(D \to \mathcal{O}_{X_{\text{red}}}(X)\) is contained in \(\mathcal{O}^+_X/\mathfrak{m}_{\mathcal{O}^+_X}(X)\).

Hence, we have a morphism \(D \to \mathcal{O}^+_X/\mathfrak{m}_{\mathcal{O}^+_X}(X)\). It follows that we have a morphism \(D[x_1, x_2] \to \mathcal{O}^+_X/\mathfrak{m}_{\mathcal{O}^+_X}(X)[x_1, x_2]\). Let \(g_1, g_2 \in \mathcal{O}^+_X[X_1, X_2]\) be some lifts of the images of \(f_1, f_2\) respectively.

Remark 2.22 and our construction in Proposition 2.19 shows that \(Y\) is of the form \(\text{Spa}(B, B^+)|\mathcal{O}_X(X))\) where \(B = \mathcal{A}(X_1, X_2)/(g_1, g_2)\). Let

\[
B'_0 := B^+/(g_1, g_2)\mathcal{A}(X_1, X_2) \cap B^+/(g_1, g_2).
\]
The preceding discussion gives the following diagram where we add labels to the natural morphisms referred to in Definition 3.6.

\[
\begin{array}{cccc}
B^+ & B'_0 & A^+[X_1, X_2] & A^+ \\
\downarrow b' & \downarrow & \downarrow & \downarrow b \\
\mathcal{O}_{Y_{red}}(Y) & A^+/\mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X)[x_1, x_2] & A^+/\mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X) & \mathcal{O}_{X_{red}}(X) \\
\uparrow a' & \uparrow & \uparrow & \uparrow a \\
E & D[x_1, x_2] & D & \\
\end{array}
\]

In the diagram above, the morphism \( A^+[X_1, X_2] \to A^+/\mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X)[x_1, x_2] \) sends \( X_1, X_2 \) to \( x_1, x_2 \) respectively. The map \( A^+/\mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X)[x_1, x_2] \to \mathcal{O}_{Y_{red}}(Y) \) is induced by the universal property of the tensor product. To check the commutativity of the diagram, we must verify that the top left square is commutative since commutativity is clear everywhere else. This reduces to checking that the images of \( X_1, X_2 \) for the composition \( A^+[X_1, X_2] \to B'_0 \to \mathcal{O}_{Y_{red}}(Y) \) coincides with the images of \( x_1, x_2 \) for the map \( A^+/\mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X)[x_1, x_2] \to \mathcal{O}_{Y_{red}}(Y) \) respectively. Equivalently, we must show that the images of \( X_1, X_2 \) for the composition \( A^+[X_1, X_2] \to B'_0 \to \mathcal{O}_{Y_{red}}(Y) \) coincides with the images of \( x_1, x_2 \) for the map \( a' \) respectively. This is a consequence of the construction of the morphism \( a' \) in Proposition 2.19 with the key point being Lemma 2.20.

We claim that \( a'(E) \subseteq b'(B'_0) \). This can be verified by a simple diagram chase. This verifies condition (1) in Definition 3.6.

We now verify condition (2) of Definition 3.6. Observe that it suffices to show that there exists a finite set \( \mathcal{E} \subseteq b'(B'_0) \) such that \( b'(B'_0) \) is integral over \( a'(E)[\mathcal{E}] \). This is a consequence of the fact that \( B^+ \) is integral over \( B'_0 \). By definition, we have that

\[
B_1 := A^+[X_1, X_2]/((g_1, g_2)A(X_1, X_2) \cap A^+[X_1, X_2]) \subseteq B'_0.
\]

We claim \( b'(B_1) = b'(B'_0) \). Indeed, let \( A_0 \) be a ring of definition such that \( A_0 \subseteq A^+ \) and \( I \subseteq A_0 \) be an ideal of definition. Recall that \( I \subseteq \mathcal{O}_{\hat{X}}^* \subseteq \mathfrak{m}_{\mathcal{O}_{\hat{X}}}(X) \) where the first inclusion is by definition and the second is due to Lemma 2.4. Observe that every element \( f \in A^+[X_1, X_2] \) can be realized as \( f_1 + f_2 \) where \( f_1 \in A^+[X_1, X_2] \) and \( f_2 \in IA^+[X_1, X_2] \). The claim is now easily deduced from this.

The goodness of the morphism \( \alpha \) implies that there exists a finite set \( \mathcal{D} \subseteq b(A^+) \) such that \( b(A^+) \) is integral over \( a(D)[\mathcal{D}] \). Let \( \mathcal{E} \) denote the image of \( \mathcal{D} \) in \( \mathcal{O}_{Y_{red}}(Y) \) for the map \( \mathcal{O}_{X_{red}}(X) \to \mathcal{O}_{Y_{red}}(Y) \). Observe that \( \mathcal{E} \subseteq b'(B'_0) \). A diagram chase then shows that the subring generated by the image of \( A^+ \) in \( \mathcal{O}_{Y_{red}}(Y) \) and \( a'(E) \) is integral over the ring \( a'(E)[\mathcal{E}] \).

Furthermore, since the diagram above commutes, the images of \( X_1, X_2 \) for the map \( b'_1 \) are contained in \( a'(E) \). It follows that \( b'(B_1) \) is in fact integral over \( a'(E)[\mathcal{E}] \). This completes the proof. \( \square \)

We now use Lemma 3.8 to deduce that locally finite type specializations are stable under base change.

**Proposition 3.9.** Let \( \alpha \colon X \to S \) be a specialization morphism which is locally of finite type. Let \( g \colon T \to S \) be an étale morphism. Let \( Y := X \times_S T \) be the fibre
product as constructed in Proposition 2.19 and \( \beta: Y \to T \) be the projection. Then \( \beta \) is a specialization morphism which is locally of finite type.

**Proof.** Let \( p \in Y \) and let \( h: Y \to X \) denote the projection. Since \( \alpha \) is locally of finite type, there exists an affine open neighbourhood \( S' \subseteq S \) of \( \alpha(h(p)) \) and an affinoid open neighbourhood \( X' \subseteq X \) of \( h(p) \) such that the restriction \( \alpha|_{X'}: X' \to S' \) is good. By Remark 3.7, we can shrink \( X' \) to be such that \( \mathcal{O}_{X'}(X') \) is Tate. Let \( T' \subseteq T \) be an affine open neighbourhood of \( \beta(p) \) such that the restriction \( g_{T'}: T' \to S' \) is standard étale. By Lemma 3.8, the map \( X' \times_{S'} T' \to T' \) is good. Now observe that by the universal property of base change, \( X' \times_{S'} T' \) is isomorphic as an adic space to \( \beta^{-1}(T') \cap h^{-1}(X') \) which is open in \( X \times_{S} T \). □

In the following lemma, given a site \( C \), we abuse notation and use \( C \) itself to denote the associated topos \( C^{-} \).

**Lemma 3.10.** Let \( X \) be an analytic adic space and \( S \) be a scheme.

1. Then the topoi \( X_{\text{ét}} \) and \( S_{\text{ét}} \) are algebraic (cf. [11, Exposé VI, Définition 2.3]).

2. If \( f: X \to S \) is a good specialization morphism such that \( \mathcal{O}_X(X) \) is Tate then the induced morphism of topoi \( f_{\text{ét}}: X_{\text{ét}} \to S_{\text{ét}} \) is coherent (cf. [11, Exposé VI, Définition 3.1]).

**Proof.** We only show that \( X_{\text{ét}} \) is an algebraic topos. The proof for \( S_{\text{ét}} \) is similar. Let \( \mathcal{F} \) be the full subcategory of objects \( Y \in X_{\text{ét}} \) such that \( Y \) is affinoid and in addition the morphism \( Y \to X \) factors through an open embedding \( Z \to X \) where \( Z \) is affinoid. Note that in this case, \( Y \times_X Y = Y \times_Z Y \) and \( Y \times_Z Y \) is affinoid. Observe that \( \mathcal{F} \) is a generating family. By [11, Exposé VI, Proposition 2.1], every object in \( \mathcal{F} \) is coherent. By the criterion given in [11, Exposé VI, Proposition 2.2(i ter)], we see that \( X_{\text{ét}} \) is an algebraic topos.

Lastly, we check that the induced morphism of topoi \( X_{\text{ét}} \to S_{\text{ét}} \) is coherent. This is a direct consequence of [11, Exposé VI, Proposition 3.2] applied to the generating family of affines which are standard étale over \( S \) and Lemma 3.8. □

**Definition 3.11** (finite type). Let \( \alpha: X \to S \) be a specialization morphism. Then \( \alpha \) is called **finite type** if \( \alpha \) is locally of finite type and quasi-compact.

**Definition 3.12** (quasi-separated).  
1. Let \( (X, \mathcal{O}_X) \) be a locally ringed space. Then \( (X, \mathcal{O}_X) \) is called **quasi-separated** if the intersection of two quasi-compact open subsets of \( X \) is quasi-compact.

2. A morphism \( \alpha: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) between locally ringed spaces is called **quasi-separated** if for every quasi-separated open subspace \( U \) of \( Y \), the inverse image \( \alpha^{-1}(U) \) is quasi-separated.

We next show (cf. Lemma 3.15) that quasi-separated is stable under a standard étale base change.

**Lemma 3.13.** Let \( X \) be a locally noetherian adic space. Let \( Y_1, Y_2 \) be affinoid open subsets of \( X \) such that \( Y_1 \cap Y_2 \) is quasi-compact. Let \( U_1 \subseteq Y_1 \) and \( U_2 \subseteq Y_2 \) be affinoid open subsets. Then \( U_1 \cap U_2 \) is quasi-compact.

**Proof.** Let \( \mathcal{V} \) be a finite family of affinoid open subsets of \( Y_1 \cap Y_2 \) such that \( Y_1 \cap Y_2 = \bigcup_{V \in \mathcal{V}} V \). It suffices to verify that \( U_1 \cap U_2 \cap V \) is affinoid for every \( V \in \mathcal{V} \). Given \( V \in \mathcal{V} \), observe that since \( U_1 \) and \( V \) are both affinoid open subsets of the affinoid adic space \( Y_1 \), \( U_1 \cap V \) is an affinoid open subset of \( V \). Indeed, one checks using the universal property of the fibre product that \( U_1 \cap V = U_1 \times_{Y_1} V \). Similarly, \( U_2 \cap V \) is an affinoid open subset of \( V \). Applying the same principle again, we get that \( (U_1 \cap V) \cap (U_2 \cap V) = U_1 \cap U_2 \cap V \) is an affinoid open subset of \( V \). □
Lemma 3.14. Let $X$ be a locally noetherian adic space. The following statements are equivalent.

1. The space $X$ is quasi-separated.
2. There exists a family $V$ of affinoid open subsets such that
   \[ X = \bigcup_{V \in V} V \]
   and for every $V_1, V_2 \in V$, $V_1 \cap V_2$ is quasi-compact.

Proof. We prove (2) implies (1) as the other implication is clear. Let $U_1, U_2$ be quasi-compact open subsets of $X$. For $i \in \{1, 2\}$, let $W_i$ be a finite family of affinoid open subsets which cover $U_i$ such that for every $W \in W_i$, there exists $V \in V$ and $W \subseteq V$. Such a cover exists by observing that for every point $p \in U_i$, there exists $V \in V$ containing $p$ and we can take an affinoid open neighbourhood of $p$ contained in $U_i \cap V$. Note that
   \[ U_1 \cap U_2 = \bigcup_{W_1 \in W_1, W_2 \in W_2} W_1 \cap W_2. \]
By Lemma 3.13 and our assumption on the family $V$, for every $W_1 \in W_1$ and $W_2 \in W_2$, $W_1 \cap W_2$ is quasi-compact. This concludes the proof. \qed

Lemma 3.15. Let $f: X \to S$ be a specialization morphism where $X$ is an analytic adic space and $S$ is an affine scheme. Let $g: T \to S$ be a standard étale morphism of schemes. If $X$ is quasi-separated then $X \times_S T$ is quasi-separated.

Proof. There exists a family $V$ of affinoid open subsets covering $X$ and satisfying the following properties.

1. For every $V_1, V_2 \in V$, $V_1 \cap V_2$ is quasi-compact.
2. If $V \in V$ then for any affinoid open subset $V' \subseteq V$, $V' \times_S T$ is affinoid.

Property (1) can be obtained using that $X$ is quasi-separated. Observe that by Remark 2.22, for every point $x \in X$, there exists an affinoid open neighbourhood $U$ of $x$ such that for every affinoid open set $U' \subseteq U$, $U' \times_S T$ is affinoid. Hence we have property (2).

It follows that there exists a family $U$ of affinoid open subsets that covers $V_1 \cap V_2$ and for every $U \in U$, $U \times_T T$ is an affinoid open in $X \times_S T$. Since $V_1 \cap V_2$ is quasi-compact, we can take $U$ to be a finite family. Note that for every $V_1, V_2 \in V$,
   \[ (V_1 \times_S T) \cap (V_2 \times_S T) = \bigcup_{U \in U} U \times_T T. \]
This implies that $(V_1 \times_S T) \cap (V_2 \times_S T)$ is quasi-compact. The lemma now follows from Lemma 3.14 applied to $X \times_S T = \bigcup_{V \in V} V \times_S T$. \qed

Definition 3.16 (separated). Let $\alpha: X \to S$ be a specialization morphism. Then $\alpha$ is called separated if $\alpha$ is quasi-separated and for every valuation ring $(x, A)$ of $X_{\text{red}}$ and for every center $s \in S$ of $\alpha_\ast((x, A))$, there exists at most one center $z \in X$ of $(x, A)$ with $\alpha(z) = s$.

Definition 3.17 (partially proper). Let $\alpha: X \to S$ be a specialization morphism. Then $\alpha$ is called partially proper if $\alpha$ is locally of finite type, quasi-separated and for every valuation ring $(x, A)$ of $X_{\text{red}}$ and for every center $s \in S$ of $\alpha_\ast((x, A))$, there exists a unique center $z \in X$ of $(x, A)$ with $\alpha(z) = s$.

Definition 3.18 (proper). Let $\alpha: X \to S$ be a specialization morphism. Then $\alpha$ is called proper if $\alpha$ is partially proper and quasi-compact.

As expected the definitions of a separated (resp. partially proper) specialization morphism admits the following reformulation.
Lemma 3.19. Let $\alpha: X \to S$ be a specialization morphism. The condition on centers of valuation rings in Definition 3.16 (resp. 3.17) is equivalent to the following condition: For every commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow j & & \downarrow \alpha \\
V & \xrightarrow{\beta} & S
\end{array}
$$

with $U = \text{Spa}(K,K^+)$ and $V = \text{Spa}(L,L^+)$ spectra of analytic affinoid fields, $j$ a morphism of analytic adic spaces with $\mathcal{O}_V(V) \to \mathcal{O}_U(U)$ an isomorphism, $i$ a morphism of analytic adic spaces and $\beta$ a specialization morphism, then there is at most one (resp. unique) morphism of analytic adic spaces $f: V \to X$ such that the following diagram commutes

$$
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow j & \xrightarrow{f} & \downarrow \alpha \\
V & \xrightarrow{\beta} & S.
\end{array}
$$

Proof. This is an immediate consequence of Example 2.11. □

Remark 3.20. If $f: X \to S$ is a proper specialization morphism and $U \subseteq S$ is open then the induced morphism $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is proper. This is an easy consequence of Proposition 3.9.

Lemma 3.21. Let $f: X \to S$ be a specialization morphism where $X$ is an analytic adic space and $S$ is an affine scheme. Let $g: T \to S$ be a standard étale morphism of schemes. If $f$ is (partially) proper then $f_T$ is (partially) proper where $f_T: X \times S T \to T$ is the base change morphism.

Proof. Suppose $f$ is partially proper. By Lemma 3.9, the morphism $f_T$ is locally of finite type. Since $S$ is quasi-separated and $f$ is quasi-separated, $X$ is quasi-separated. By Lemma 3.15, $X_T$ is quasi-separated. In particular, the morphism $f_T$ is quasi-separated. The condition on centers of valuation rings in Definition 3.17 is easily deduced from Lemma 3.19. Hence $f_T$ is partially proper.

Suppose in addition that $f$ is quasi-compact. We claim that $f_T$ is quasi-compact. We show first that $X_T$ is quasi-compact. Indeed, $X$ is quasi-compact and hence there exists a finite family $U$ of affinoid open subsets $U$ of $X$ such that $\mathcal{O}_U(U)$ is Tate, the restriction $f|_U$ is good and $X = \bigcup_{U \in U} U$. By Lemma 3.8, for every $U \in U$, $U_T := U \times S T$ is affinoid as well. Since $X_T = \bigcup_{U \in U} U_T$, we deduce that $X_T$ is quasi-compact. By using the same argument, we deduce that if $W \subseteq T$ is standard open then $f_T^{-1}(W) = X_T \times T W$ is quasi-compact. This verifies the claim and concludes the proof. □

For the notion of taut, we mean [8, Definition 5.1.2].

Proposition 3.22. The properties of locally of ($^+$ weakly) finite type, finite type, quasi-separated, separated, partially proper, proper and taut are stable under compositions of specialization morphisms with morphisms of analytic adic spaces or morphisms of schemes, when they make sense.

Proof. Let us fix $\alpha: X \to S$ a specialization morphism, $f: Y \to X$ a morphism of analytic adic spaces and $g: S \to T$ a morphism of schemes.

(1) (locally of finite type) Suppose $\alpha$ is locally of finite type and $f$ is locally of $^+$ weakly finite type. We show $\alpha \circ f$ is locally of finite type. Let $y \in Y$. 

For the notion of taut, we mean [8, Definition 5.1.2].

Proposition 3.22. The properties of locally of ($^+$ weakly) finite type, finite type, quasi-separated, separated, partially proper, proper and taut are stable under compositions of specialization morphisms with morphisms of analytic adic spaces or morphisms of schemes, when they make sense.

Proof. Let us fix $\alpha: X \to S$ a specialization morphism, $f: Y \to X$ a morphism of analytic adic spaces and $g: S \to T$ a morphism of schemes.

(1) (locally of finite type) Suppose $\alpha$ is locally of finite type and $f$ is locally of $^+$ weakly finite type. We show $\alpha \circ f$ is locally of finite type. Let $y \in Y$. 

Since $\alpha$ is locally of finite type, there exists an affinoid open neighbourhood $U \ni f(y)$ in $X$ and an affine open subscheme $W \subseteq S$ such that $\alpha(U) \subseteq W$ and for the natural homomorphisms
\[ a: O_S(W) \to O_{X_{red}}(U) \]
\[ b: O_X^+(U) \to O_{X_{red}}(U) \]
we have that $\text{im}(a) \subseteq \text{im}(b)$ and there is a finite subset $E \subseteq \text{im}(b)$ with $\text{im}(b)$ integral over $\text{im}(a)[E]$. Shrinking $U$ if necessary (cf. Remark 3.7), since $f$ is locally of weakly finite type we can assume that there exists an open affinoid $V \ni y$ of $Y$ such that $f(V) \subseteq U$ and $O_Y^+(V)$ is integral over $O_X^+(U)[E' \cup O_Y(V)^{\infty}]$ for some finite subset $E' \subseteq O_Y^+(V)$.

We claim that the composition $V \to U \to W$ is good. Indeed, we have the following commutative diagram where we add appropriate labels.

\[ \begin{array}{ccc}
O_X^+(U) & \to & O_Y^+(V) \\
\downarrow b & & \downarrow b' \\
O_{X_{red}}(U) & \to & O_{Y_{red}}(V) \\
\downarrow a & & \downarrow a' \\
O_S(W) & & \\
\end{array} \]

A diagram chase shows that $\text{im}(a') \subseteq \text{im}(b')$. We deduce from Lemma 2.4 that $\text{im}(b')$ is integral over the image of $O_X^+(U)[E']$ in $O_{Y_{red}}(V)$ . Let $E'$ denote the image in $O_{Y_{red}}(V)$ of $E'$. Similarly, let $E_V$ denote the image of $E$ in $O_{Y_{red}}(V)$. We see that the image of $O_X^+(U)$ in $O_{Y_{red}}(V)$ is integral over $\text{im}(a')[E_V]$. Hence, $\text{im}(b')$ is integral over $\text{im}(a')[E_V \cup E']$. This verifies the claim.

(2) (finite type) We just need to check the following case: $\alpha$ and $g$ are locally of finite type then so is $g \circ \alpha$. But this is clear from the definitions. The remaining cases follow from (1), since the condition of quasi-compactness is a topological one.

(3) (quasi-separated) This is obvious.

(4) (separated) This follows immediately from Lemma 3.19 and the analogous statement for schemes/adic spaces.

(5) (partially proper) This follows immediately from Lemma 3.19 and the analogous statement for adic spaces.

(6) (proper) This follows from (5).

(7) (taut) This is obvious.

\[ \square \]

**Lemma 3.23.** Let $f: X \to S$ be a specialization morphism where $X$ is affinoid and $S$ is affine. Then $f$ is separated. If in addition $f$ is good and $O_X(X)$ is Tate then $f$ is taut.

**Proof.** By [10, Theorem 3.5(i)], the space $X$ is spectral. It follows that $X$ is quasi-separated. We deduce from this that the morphism $f$ is quasi-separated as well. The fact that the morphism $f$ is separated can be deduced from [8, Lemma 1.3.6(ii)] and Lemma 3.5.
We now prove the second assertion of the lemma assuming in addition that \( f \) is good and \( \mathcal{O}_X(X) \) is Tate. We claim the morphism \( f \) is quasi-compact. Indeed, let \( W \subseteq S \) be a quasi-compact open subset. We must verify that \( f^{-1}(W) \) is quasi-compact. Let \( R \) be a ring such that \( S = \text{Spec}(R) \). Since every quasi-compact open subset of \( S \) can be covered by finitely many affine open subsets of the form \( \text{Spec}(R_f) \) for \( f \in R \), we can assume without loss of generality that \( W \) is of the form \( \text{Spec}(R_f) \) where \( f \in R \). By Remark 2.22, \( X \times_S W \) is affinoid. One checks easily that \( f^{-1}(W) \) is isomorphic as an adic space to \( X \times_S W \). In particular, it must be quasi-compact. Observe that as \( f \) is quasi-compact and \( X \) is quasi-separated, \( f \) must be spectral. By [8, Lemma 5.1.3(i),(iii)], we see that \( f \) is taut.

\[ \square \]

The next proposition gives a crucial example of a proper specialization morphism.

**Proposition 3.24.** Let \( \mathfrak{X} \) be a type \((S)\) formal scheme. The specialization morphism (cf. (4))

\[ \lambda_{\mathfrak{X}} : \mathfrak{X}_\eta \to \mathfrak{X}_s \]

is proper in the sense of Definition 3.18.

**Proof.** Topologically the morphism \( \lambda_{\mathfrak{X}} \) is constructed in [8, Proposition 1.9.1]. By (1.9.3) in loc.cit., \( \lambda_{\mathfrak{X}} \) is quasi-compact and quasi-separated (for quasi-separated one also uses Lemma 3.14). Thus it remains to show that \( \lambda_{\mathfrak{X}} \) is locally of finite type and satisfies the valuative criterion for a partially proper morphism (cf. Definition 3.17).

We begin by proving \( \lambda_{\mathfrak{X}} \) is locally of finite type. We can assume \( \mathfrak{X} = \text{Spf}(A) \). For this we treat the type \((S)(a)\) and \((S)(b)\) cases separately. Suppose first \( \mathfrak{X} \) is of type \((S)(a)\) (in particular this means \( \mathfrak{X}_\eta = \text{Spa}(A,A)_s \) is an open subspace of \( \text{Spa}(A,A) \)). In this case it suffices to prove the specialization morphism \( \text{Spa}(A,A) \to \mathfrak{X}_s \) is locally of finite type. We have a commutative diagram

\[ \begin{array}{ccc}
\mathcal{O}_X(\mathfrak{X}) & \xrightarrow{i} & \mathcal{O}^+_{\text{Spa}(A,A)}(\text{Spa}(A,A)) \\
\downarrow{q} & & \downarrow{b} \\
\mathcal{O}_{\mathfrak{X}_s}(\mathfrak{X}_s) & \xrightarrow{a} & \mathcal{O}^+_{\text{Spa}(A,A),\text{red}}(\text{Spa}(A,A))
\end{array} \]

where \( i \) is the identity map and \( q \) is a surjection by vanishing of higher (coherent) cohomology on affine spaces. Thus \( \text{im}(a) = \text{im}(b) \) and the map \( \text{Spa}(A,A) \to \mathfrak{X}_s \) is clearly locally of finite type. Suppose now \( \mathfrak{X} \) is of type \((S)(b)\). In this case \( \mathfrak{X}_\eta = \text{Spa}(A[1/s],B) \), where \( s \in A \) is such that \( sA \) is an ideal of definition of \( A \) and \( B \) is the integral closure of \( A \) in \( A[1/s] \). We have a commutative diagram

\[ \begin{array}{ccc}
\mathcal{O}_X(\mathfrak{X}) & \xrightarrow{i} & \mathcal{O}^+_{\mathfrak{X}_\eta}(\mathfrak{X}_\eta) \\
\downarrow{q} & & \downarrow{b} \\
\mathcal{O}_{\mathfrak{X}_s}(\mathfrak{X}_s) & \xrightarrow{a} & \mathcal{O}^+_{\mathfrak{X}_\eta}(\mathfrak{X}_\eta)
\end{array} \]

\[ ^4 \text{Strictly speaking, as } \text{Spa}(A,A) \text{ is not necessarily analytic, we did not define what it means for such a morphism to be locally of finite type, but the same definition as in 3.6 works and as in Proposition 3.22, compositions (whenever they make sense) of locally of finite type morphisms are again locally of finite type. So once } \text{Spa}(A,A) \to \mathfrak{X}_s \text{ is proven to be locally of finite type, then so is the composition } \mathfrak{X}_\eta \to \text{Spa}(A,A) \to \mathfrak{X}_s, \text{ as desired.} \]
where again by vanishing of higher cohomology \( q \) is a surjection. Thus \( \text{im}(a) = \text{im}(a \circ q) \subseteq \text{im}(b) \). Moreover the map \( i: A \to B \) is integral (as remarked above). Thus \( \text{im}(b) \) is integral over \( \text{im}(a) \) and this shows \( \lambda_X \) is locally of finite type.

Finally we show \( \lambda_X \) satisfies the valuative criterion for partially proper morphisms. In fact this is an immediate consequence of the universal property satisfied by \( \text{sp} \). Indeed given a commutative diagram (as in Lemma 3.19)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\alpha_X & \xrightarrow{\lambda_X} & \alpha_Y
\end{array}
\]

by [8, Proposition 1.9.1(c)], there exists a unique morphism \( f: V \to X_{\eta} \), which makes the relevant diagram commute. \( \square \)

For later use we remark the following results.

**Lemma 3.25.** Let \( \alpha: X \to S \) be a partially proper specialization morphism. Then for every quasi-compact subset \( T \) of \( X \), the closure \( \overline{T} \) of \( T \) in \( X \) is quasi-compact.

**Proof.** It is enough to prove the lemma for \( S \) affine. We may assume that \( T \) is open in \( X \). Put

\[
Y := \{(x, A) \in (X_{\text{red}})_v \mid x \in T \text{ and } \alpha_v((x, A)) \text{ has a center on } S\}.
\]

By Lemma 3.3 and Definition 3.17, every \( (x, A) \in Y \) has a unique center on \( X \). Let \( c: Y \to X \) be the map which assigns to each \( (x, A) \in Y \) the center of \( (x, A) \) on \( X_{\text{red}} \). We equip \( Y \) with the subspace topology of \( (X_{\text{red}})_v \) (cf. Remark 3.4). By Lemma 3.5, just as in the proof of [8, Lemma 1.3.13], we obtain that \( Y \) is quasi-compact.

We now show that \( c \) is continuous. Let \( U \) be an affinoid open subset of \( X \). We have to show \( c^{-1}(U) \) is open. Since \( \alpha \) is in particular locally of finite type, by Remark 3.7 the natural homomorphisms

\[
a: O_S(S) \to O_{X_{\text{red}}}(U),
b: O_X(U) \to O_{X_{\text{red}}}(U)
\]

are such that \( \text{im}(a) \subseteq \text{im}(b) \) and there is a finite subset \( F \subseteq \text{im}(b) \) with \( \text{im}(b) \) integral over \( \text{im}(a)[F] \). Let \( E \subseteq O_X(U) \) be any finite subset such that \( b(E) = F \). Then by By Lemma 3.5 and [8, Lemma 1.3.6(ii)]

\[
c^{-1}(U) = \{(x, A) \in Y \mid x \in U \text{ and } e(x, A) \in A \text{ for every } e \in E\}.
\]

Hence \( c^{-1}(U) \) is open. The rest of the proof is the same as [8, Lemma 1.3.13]. \( \square \)

**Lemma 3.26.** Let \( S \) be a scheme, \( X' \) and \( Y \) be analytic adic spaces equipped with specialization morphisms \( \alpha: X' \to S \) and \( \beta: Y \to S \), \( X \) an open subspace of \( X' \) and \( f: X \to Y \) a morphism of analytic adic spaces sitting in a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \alpha_X & & \downarrow \beta \\
S & &
\end{array}
\]

(1) If every point of \( X' \) is a specialization point of \( X \) and \( \beta \) is separated, then there is at most one morphism \( g: X' \to Y \) (over \( S \)) such that \( g|_X = f \).

(2) Suppose \( \beta \) is locally of finite type, the inclusion \( X \to X' \) is quasi-compact and for every open subset \( V \) of \( X' \), the restriction mapping \( O_{X'}(V) \to O_X(V \cap X) \) is an isomorphism of topological rings. Let \( x' \in X' \), \( y \in Y \) such that there exists a valuation ring \( (x, A) \) of \( (X, O_X) \) such that \( x' \) is a
Proof. (1) By Lemma 3.5, this is the same proof as the proof of [8, Lemma 1.3.14(i)].

(2) Since \( \beta \) is locally of finite type, there exists an affinoid open neighbourhood \( L \ni y \in Y \) and an affine open subscheme \( M \subseteq S \) such that \( \beta(L) \subseteq M \) and for the natural homomorphisms

\[
a: \mathcal{O}_S(M) \to \mathcal{O}_{Y_{red}}(L)
\]

\[
b: \mathcal{O}_Y^+(L) \to \mathcal{O}_{Y_{red}}(L)
\]

we have that \( \text{im}(a) \subseteq \text{im}(b) \) and there is a finite subset \( F \subseteq \text{im}(b) \) with \( \text{im}(b) \) integral over \( \text{im}(a)[F] \). Let \( E' \subseteq \mathcal{O}_Y^+(L) \) be any finite set such that \( b(E') = F \). Then the rest of the proof is the same as the proof of [8, Lemma 1.3.14(ii)] with \( E' \) playing the role of \( E \) in loc.cit. Indeed employing the notation of loc.cit. the only thing we need to check is \( \psi(\mathcal{O}_Y^+(L)) \subseteq \mathcal{O}_{X_{red}}(U) \).

Indeed take \( y \in \mathcal{O}_Y^+(L) \). Since \( \text{im}(b) \) is integral over \( \text{im}(a)[F] \), we see that there exists \( m \in m_{\mathcal{O}_Y^+(L)} \) and \( a_0, a_1, \ldots, a_{n-1} \in b^{-1}(\text{im}(a)[F]) \) such that

\[
m = y^n + a_{n-1}y^{n-1} + \cdots + a_0.
\]

It is easy to conclude that if \( |\psi(a_i)(x)| \leq 1 \) for all \( 0 \leq i \leq n \) and \( x \in U \), then this implies \( |\psi(y)(x)| \leq 1 \).

(3) This is the same proof as [8, Lemma 1.3.14(ii)]. The key point is that since \( \alpha \) is separated, then \( g \) satisfies the valuative criterion for separatedness\(^5\) by Lemma 3.19.

\[ \square \]

Remark 3.27. Let us remark that in the statement of Lemma 3.26(2) if \( f \) is in addition locally of \( + \) weakly finite type, then so is \( g \). This follows from Lemma [7, Lemma 3.3(ii), (iv)].

Now we construct compactifications of specialization morphisms. Just as in the case of adic spaces (or more generally morphisms of \( e \)-stacks, cf. [13]), there will exist canonical compactifications.

Definition 3.28 (compactification). Let \( \alpha: X \to S \) be a specialization morphism.

(1) A compactification of \( \alpha \) is a commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
\alpha \downarrow & & \downarrow \\
S & \xrightarrow{\beta} & Y
\end{array}
\]

where \( Y \) is an analytic adic space, \( \beta \) a partially proper specialization morphism and \( j \) a quasi-compact open embedding of analytic adic spaces.

\[^5\text{One can prove that } g \text{ is quasi-separated, however this is not needed.}\]
(2) A universal compactification of $\alpha$ is a compactification $(Y, \beta, j)$ of $\alpha$ such that if
\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
S & \xleftarrow{\gamma} &
\end{array}
\]
is a commutative triangle where $Z$ is an analytic adic space, $\gamma$ a partially proper specialization morphism and $h$ a morphism of analytic adic spaces, then there exists a unique morphism of analytic adic spaces $i : Y \to Z$ such that the diagram commutes
\[
\begin{array}{ccc}
X & \xleftarrow{i} & Y \\
\downarrow{h} & & \downarrow{\beta} \\
Z & \xrightarrow{\gamma} &
\end{array}
\]

**Lemma 3.29.** Let
\[
\begin{array}{ccc}
X & \xleftarrow{i} & Y \\
\downarrow{\alpha} & & \downarrow{\beta} \\
S & \xleftarrow{\alpha} &
\end{array}
\]
be a commutative diagram where $\alpha$ is a specialization morphism, $\beta$ is a partially proper specialization, and $j$ is a quasi-compact open embedding such that every point of $Y$ is a specialization of a point of $j(X)$ and $O_Y \to j_* O_X$ is an isomorphism of sheaves of topological rings. Then for every open subset $U \subseteq S$, the induced diagram
\[
\begin{array}{ccc}
\alpha^{-1}(U) & \xleftarrow{j_{|\alpha^{-1}(U)}} & \beta^{-1}(U) \\
\downarrow{\alpha_{|\alpha^{-1}(U)}} & & \downarrow{\beta_{|\beta^{-1}(U)}} \\
U & \xleftarrow{i_{|\alpha^{-1}(U)}} &
\end{array}
\]
is a universal compactification of $\alpha_{|\alpha^{-1}(U)}$.

**Proof.** Let $\gamma : Z \to U$ be a partially proper specialization morphism and $h : \alpha^{-1}(U) \to Z$ a morphism of analytic adic spaces (over $U$). We have to show that there exists a unique morphism $i : \beta^{-1}(U) \to Z$ (over $U$), which extends $h$. Uniqueness is a consequence of Lemma 3.26(1). By Lemma 3.5, the rest of the proof is the same as the proof of [8, Lemma 5.1.7] with Lemma 3.26(2) playing the role of Lemma 1.3.14(iii) in loc.cit. \qed

**Theorem 3.30.** Let $\alpha : X \to S$ be a separated, taut and locally of finite type specialization morphism. Then $\alpha$ has a universal compactification
\[
\begin{array}{ccc}
X & \xleftarrow{j} & Y \\
\downarrow{\alpha} & & \downarrow{\beta} \\
S & \xleftarrow{\beta} &
\end{array}
\]
Every point of $Y$ is a specialization of a point of $j(X)$ and $O_Y \to j_* O_X$ is an isomorphism of sheaves of topological rings.
Proof. We follow closely the proof of [8, Theorem 5.1.5]. By Lemma 3.29, we can assume \( S \) is affine. Let \( \mathcal{F} \) be the set of all open affinoid subsets \( U \) of \( X \) such that the natural homomorphisms

\[
a_U : \mathcal{O}_S(S) \rightarrow \mathcal{O}_{X_{\text{red}}}(U) \\
b_U : \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X_{\text{red}}}(U)
\]
satisfy \( \text{im}(a_U) \subseteq \text{im}(b_U) \) and there is a finite subset \( E_U \subseteq \text{im}(b_U) \) with \( \text{im}(b_U) \) integral over \( \text{im}(a_U)[E_U] \). For every \( U \in \mathcal{F} \), put \( U_c := \text{Spa}(\mathcal{O}_X(U), I(U)) \), where \( I(U) \) is the smallest ring of integral elements containing \( b_U^{-1}(\text{im}(a_U)) \) (i.e. \( I(U) \) is the integral closure\(^\ast\) of \( b_U^{-1}(\text{im}(a_U)) \) in \( \mathcal{O}_X(U) \)). We have a diagram (the squares being commutative)

\[
\begin{array}{ccc}
0 & \rightarrow & \mathfrak{m}_{\mathcal{O}^+_U(U_c)} & \rightarrow & \mathcal{O}^+_U(U_c) & \rightarrow & \mathcal{O}_{U_c,\text{red}}(U_c) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathfrak{m}_{\mathcal{O}^+_X(U)} & \rightarrow & \mathcal{O}^+_X(U) & \rightarrow & \mathcal{O}_{X_{\text{red}}}(U) & \rightarrow & \mathfrak{m}_{\mathcal{O}^+_S(U)}
\end{array}
\]

We explain the equality

\[
\mathfrak{m}_{\mathcal{O}^+_X(U)} = \mathfrak{m}_{\mathcal{O}^+_U(U_c)}.
\]

Note that the identity \( \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U) \) induces an injective morphism \( \varphi_U : U \rightarrow U_c \). We claim that every point of \( U_c \) is a specialization point of \( \varphi_U \). Indeed this follows by the same proof as in [8, Proposition 4.4.3(5)(ii)]. This implies the equality \( \mathfrak{m}_{\mathcal{O}^+_X(U)} = \mathfrak{m}_{\mathcal{O}^+_U(U_c)} \).

We need to show that there is a morphism \( a_U' : \mathcal{O}_S(S) \rightarrow \mathcal{O}_{U_{\text{red}}}(U_c) \) (indicated by the dashed line in the above diagram), which makes the relevant triangle commutative. This is a simple diagram chase. Let \( s \in \mathcal{O}_S(S) \). Since \( \text{im}(a_U) \subseteq \text{im}(b_U) \), there exists \( y \in \mathcal{O}^+_X(U) \) such that \( b_U(y) = a_U(s) \). By construction \( y \in \mathcal{O}^+_{U_c}(U_c) \) and is unique up to a choice of element \( x \in \mathfrak{m}_{\mathcal{O}^+_U(U_c)} \). Thus \( b'_U(y) \) depends only on \( x \) and we can set \( a'_U(s) := b'_U(y) \). By [14, Tag 0111] the morphism \( a'_U \) corresponds to a specialization morphism \( \gamma_U : U_c \rightarrow S \). In summary we have a commutative triangle

\[
\begin{array}{ccc}
\mathcal{O}^+_S(S) & \xrightarrow[\gamma_U]{\varphi_U} & \mathcal{O}^+_U(U_c) \\
\downarrow & & \downarrow \\
S & \xrightarrow{\gamma_U} & U_c
\end{array}
\]

We claim that

**Lemma 3.31.**

(1) \( \varphi_U \) gives an isomorphism from \( U \) onto a rational subspace of \( U_c \).

(2) \( \mathcal{O}_{U_c} \rightarrow \varphi_U_* \mathcal{O}_U \) is an isomorphism of sheaves of topological rings.

(3) The specialization morphism \( \gamma_U \) is good (in fact it is tight, cf. Definition 6.4).

**Proof.** Let us prove the first point. Let \( E_U^+ \subset \mathcal{O}^+_X(U) \) be any finite subset such that \( b_U(E_U^+) = E_U \). Then we claim that \( \varphi_U \) is an isomorphism from \( U \) onto the rational subspace

\[
\{ x \in U_c \mid e^+(x) \leq 1 \ orall e^+ \in E^+_U \}.
\]

\(^\ast\)Note that \( \ker(b_U) = \mathfrak{m}_{\mathcal{O}^+_X(U)} \) which contains \( \mathcal{O}_X(U)^{\geq 0} \) and so \( I(U) \) is indeed open in \( \mathcal{O}_X(U) \).
Indeed take \( y \in \mathcal{O}_U^+(U) \). Since \( \text{im}(b_U) \) is integral over \( \text{im}(a_U)[E_U] \), we see that there exists \( m \in \mathfrak{m}_{\mathcal{O}_X^+(U)} \) and \( a_0, a_1, \ldots, a_{n-1} \in b_U^{-1}(\text{im}(a_U)[E_U]) \) such that
\[
m = y^n + a_{n-1}y^{n-1} + \cdots + a_0.
\]
It is easy to conclude that if \( |a_i(x)| \leq 1 \) for all \( 0 \leq i \leq n \) and \( x \in U_c \), then this implies \( |y(x)| \leq 1 \) (the point is that \( \mathfrak{m}_{\mathcal{O}_X^+(U_c)} = \mathfrak{m}_{\mathcal{O}_X^+(U)} \)). The second point follows the same proof as [8, Proposition 4.4.3(5)(iii)].

We now proceed to verify assertion (3) of the lemma. Recall the natural homomorphisms
\[
da'_U : \mathcal{O}_U(S) \to \mathcal{O}_{U_c,\text{red}}(U_c) \\
b'_U : \mathcal{O}_{U_c}^+(U_c) \to \mathcal{O}_{U_c,\text{red}}(U_c)
\]
associated to \( \gamma_U \). By construction of the morphism \( a'_U \), we see that \( \text{im}(a'_U) \subseteq \text{im}(b'_U) \). Equivalently, the image of \( \mathcal{O}_{U_c}^+(U_c)/\mathfrak{m}_{\mathcal{O}_{U_c}^+(U_c)} \) in \( \mathcal{O}_{U_c,\text{red}}(U_c) \) contains \( a'_U(\mathcal{O}_U(S)) \). By definition, the image of \( \mathcal{O}_{U_c}^+(U_c) \) in \( \mathcal{O}_{X,\text{red}}(U) \) is integral over \( a_U(\mathcal{O}_U(S)) \) where \( a_U \) is the morphism \( \mathcal{O}_U(S) \to \mathcal{O}_{X,\text{red}}(U) \) associated to the specialization \( U \to S \). Observe from diagram (13) that we have an injection
\[
\mathcal{O}_{U_c}^+(U_c)/\mathfrak{m}_{\mathcal{O}_{U_c}^+(U_c)} \hookrightarrow \mathcal{O}_{X,\text{red}}(U).
\]
We deduce from this and the commutativity of diagram (13) that \( \mathcal{O}_{U_c}^+(U_c)/\mathfrak{m}_{\mathcal{O}_{U_c}^+(U_c)} \) is integral over \( a'_U(\mathcal{O}_U(S)) \). \( \square \)

Via the open embedding \( \varphi_U \), we consider \( U \) as an open subspace of \( U_c \). For every \( U \in \mathcal{F} \), we define subsets of the space \((X_{\text{red}})_v\) as follows:
1. \( R(U) := \{(x, A) \in (U_{\text{red}})_V \mid \varphi_{U,v}((x, A)) \text{ has a center on } U_{c,\text{red}} \} \)
2. \( S(U) := \{(x, A) \in U_v \mid (x, A) \text{ has a center on } X \} \)
3. \( T(U) \subseteq R(U) \) denotes the set \( (x, A) \in R(U) \) such that for every valuation \( v \) of \( A \subseteq B \subseteq k(x)^+/m_x \) (here \( m_x \subseteq k(x)^+ \) is the maximal ideal of \( k(x)^+ \)), either \( (x, B) \in (U_{\text{red}})_v \) has no center on \( X_{\text{red}} \) or \( (x, B) \) has a center on \( X_{\text{red}} \).

Let \( c_U : R(U) \to U_c \) be the mapping which assigns to \( (x, A) \in R(U) \) the center of \( (x, A) \) on \( U_{c,\text{red}} \) (since \( U_c \) is affinoid, \( c_U \) is well defined). By Lemma 3.5, the rest of the proof follows almost verbatim as the proof of [8, Theorem 5.1.5]. We provide a brief summary of the proof as follows:

1. We put \( U_d := c_U(T(U)) \subseteq U_c \) and one shows that \( U_d \) is open in \( U_c \) and \( \mathcal{U} \subseteq U_d \). For \( U, V \in \mathcal{F} \) with \( U \subseteq V \), let \( \psi_{V,U} : U_c \to V_c \) be the morphism of adic spaces which is induced by the restriction mapping \( \mathcal{O}_X(V) \to \mathcal{O}_X(U) \).
2. We proceed to glue the family \( U_d \) for \( U \in \mathcal{F} \) in two steps.

(a) Suppose we are given a sub-family \( \mathcal{F}_0 \subset \mathcal{F} \) such that if \( U, V \in \mathcal{F}_0 \) then \( U \cap V \) belongs to \( \mathcal{F}_0 \). Given, \( U, V \in \mathcal{F}_0 \), the set \( Q_{U,V} := \psi_{U,V:U \cap V}(U \cap V d) \subseteq U_d \) is open in \( U_d \) and the induced morphism \( \psi_{U,V:U \cap V} : (U \cap V) \to Q_{U,V} \) is an isomorphism. For \( U, V \in \mathcal{F}_0 \), we put \( \lambda_{V,U} := \psi_{V,U:V} \circ \psi_{U,V:U}^{-1} : Q_{U,V} \to Q_{V,U} \). Then the \( \{U_d\}_{U \in \mathcal{F}_0} \) glue along the \( \lambda_{V,U} \) (i.e., the \( \lambda_{V,U} \) satisfy the cocycle condition) to give an analytic adic space over \( S \).

(b) In the general situation, we proceed as follows. Let \( U, V \in \mathcal{F} \) and \( \{W_1, \ldots, W_m\} \) be a finite family of affinoid open subsets of \( U \cap V \) such that \( U \cap V = \bigcup_i W_i \). Let \( \mathcal{F}_0 \) denote the family obtained by taking intersections of the elements of \( \{W_1, \ldots, W_m\} \). Observe that these intersections are affinoid spaces and hence by Remark 3.7 \( \mathcal{F}_0 \subset \mathcal{F} \).
Hence, by (2a), we get that the $W_{i,d}$ glue together to give $(U \cap V)_d$. An important point to note is that this construction is independent of the choice of the family $\mathcal{F}_0$. We deduce that the set $Q_{U,V} := \psi_{U,U \cap V}((U \cap V)_d) \subseteq U_d$ is open in $U_d$ and the induced map $\psi_{U,U \cap V}: (U \cap V)_d \to Q_{U,V}$ is an isomorphism. We put $\lambda_{V,U} := \psi_{V,U \cap V} \circ \psi_{U,U \cap V}^{-1}: Q_{U,V} \to Q_{V,U}$. Then the $\{U_d\} \in \mathcal{F}$ glue along the $\lambda_{V,U}$ (i.e. the $\lambda_{V,U}$ satisfy the cocycle condition) to give the analytic adic space $Y$ over $S$.

(3) We denote $\beta: Y \to X$ and $j: X \hookrightarrow Y$ the natural inclusion. Then every point of $Y$ is a specialization point of $j(X)$ and $O_Y \to j_* O_X$ is an isomorphism of sheaves of topological rings. Moreover $j$ is a quasi-compact open embedding and $\beta$ is a partially proper specialization morphism.

$\square$

**Remark 3.32.** Observe from the proof of Theorem 3.30 that if $f: U \to S$ is a good specialization morphism with $U$ affinoid and $S$ affine then $O_U(U) = \mathcal{O}_U(U)$ and there exists $e_1, \ldots, e_m \in O_U(U)$ for some $m \in \mathbb{N}$ such that $U = \{x \in U \mid ||e_i(x)|| \leq 1, 1 \leq i \leq m\}$.

**Corollary 3.33.** *In the notations of Theorem 3.30 if $\alpha$ is quasi-compact, then $\beta$ is proper.*

**Proof.** We need to show $\beta$ is quasi-compact (by Theorem 3.30 it is already partially proper). Let $U$ be a quasi-compact open subset of $S$. It suffices to show $\beta^{-1}(U)$ is quasi-compact. By Theorem 3.30 every point of $\beta^{-1}(U)$ is a specialization of a point of $j(\alpha^{-1}(U))$. By assumption $\alpha^{-1}(U)$ is quasi-compact and therefore $j(\alpha^{-1}(U))$ is quasi-compact. The result now follows from Lemma 3.25. $\square$

We introduce notation regarding compactifications of specialization morphisms. Let $\alpha: X \to S$ be a specialization morphism which is separated, taut and locally of finite type. Recall from Theorem 3.30 that we have a universal compactification, for which we use the following notation.

$\begin{array}{ccc}
X & \xrightarrow{j} & X^S/S \\
\downarrow \alpha & & \downarrow \pi^S/S \\
S & & \\
\end{array}$

For the purposes of proving proper base change (cf. Theorem 6.3), we record that universal compactifications behave well with respect to base change.

**Lemma 3.34.** Let $f: U \to S$ be a good specialization morphism. We assume in addition that $\mathcal{O}_U(U)$ is Tate. Let $W$ be an affine scheme and $g: W \to S$ be a standard étale morphism. We then have that

$$U \times^S W/W \simeq (U^S/S \times^S W).$$

Note that by Lemma 3.23, the morphism $f$ satisfies the conditions of Theorem 3.30. Hence, $U^S/S$ exists. By Lemma 3.8, $U \times^S W \to W$ is a good specialization morphism. Hence, for the same reason, $U \times^S W/W$ exists.
Proof. Let $j$ denote the quasi-compact open embedding $U \to \mathcal{U}^S$. It suffices to show that the diagram

\[\begin{array}{ccc}
U \times S W & \xrightarrow{j_W} & \mathcal{U}^S \times_S W \\
\downarrow & \downarrow & \downarrow \\
W & \xrightarrow{j_W} & W
\end{array}\]

where the subscript $W$ is used to indicate the morphisms induced by base change to $W$, satisfies the hypotheses of Lemma 3.29.

We have the following cartesian diagram:

\[\begin{array}{ccc}
U \times S W & \xrightarrow{j_W} & \mathcal{U}^S \times_S W \\
\downarrow & \downarrow & \downarrow \\
U & \xrightarrow{j} & \mathcal{U}^S.
\end{array}\]

By [8, Corollary 1.2.3(iii)(b)] and the fact that open embeddings are stable under base change, the morphism $j_W$ is a quasi-compact open embedding. By Lemma 3.31(3), the specialization $\mathcal{U}^S \to S$ is good. Hence by Lemma 3.8, $\mathcal{U}^S \times_S W$ is affinoid and the projection morphism $\mathcal{U}^S \times_S W \to W$ is good. By Lemma 3.23, the projection $\mathcal{U}^S \times_S W \to W$ is separated. We deduce easily using Lemma 3.19 that the specialization morphism $\mathcal{U}^S \times_S W \to W$ is partially proper.

By Remark 3.32, there exists $e_1, \ldots, e_m \in \mathcal{O}_U(U)$ such that

$$U = \{x \in \mathcal{U}^S | |e_i(x)| \leq 1, 1 \leq i \leq m\}.$$ 

It follows that $U \times S W = \{x \in \mathcal{U}^S \times_S W | |e_i(x)| \leq 1, 1 \leq i \leq m\}$ where we have abused notation and denoted by $e_i$ their images in $\mathcal{O}_{U \times S W}(U \times S W)$. We deduce that

$$\mathcal{O}_{\mathcal{U}^S \times_S W} \simeq j_W^* \mathcal{O}_{U \times S W}.$$

It remains to verify that every point in $\mathcal{U}^S \times_S W$ is the specialization of a point in the image of the morphism $j_W$. Let $x \in \mathcal{U}^S \times_S W$. Let $y$ denote the image of $x$ for the projection $\mathcal{U}^S \times_S W \to \mathcal{U}^S$. Since $\mathcal{U}^S$ is a universal compactification of the map $U \to S$, there exists a point $y' \in U$ which specializes to $y$. By [8, Lemma 1.1.10(v)], there exists a point $x' \in \mathcal{U}^S \times_S W$ such that $x'$ specializes to $x$ and maps to $y'$. Note that by the universal property of fibre products, the preimage of $U$ for the projection $\mathcal{U}^S \times_S W \to \mathcal{U}^S$ is $U \times S W$. Hence $x' \in U \times S W$. \qed

4. Smooth base change

In order to develop a well behaved theory of cohomology with compact support for specialization morphisms, we need some preparation. In this section we establish a variant of a smooth base change result for specialization morphisms. This is a formal consequence of the commutativity of fiber products of specialization morphisms and morphisms between schemes, cf. Proposition 2.28. For the remainder of the paper, we fix a torsion ring $A$. 

Lemma 4.1. Let

\[
\begin{array}{ccc}
X & \xrightarrow{j_1} & Y \\
\downarrow & & \downarrow j \\
S & \xrightarrow{j_2} & T
\end{array}
\]

be a cartesian diagram where \(\alpha\) and \(\beta\) are specialization morphisms, and \(j_2\) (and hence also \(j_1\) by Proposition 2.19) an étale morphism. Then there is a natural equivalence from \(D^+(Y_{\text{ét}}, A) \to D^+(S_{\text{ét}}, A)\)

\[
j_2^* \circ R^+\beta_* \xrightarrow{\sim} R^+\alpha_* \circ j_1^*.
\]

Proof. It’s enough to show that for an étale sheaf \(\mathcal{F}\) on \(Y\) in \(A\)-modules, the base change morphism

\[
(j_2^* \circ R^n\beta_* \mathcal{F}) \xrightarrow{\sim} R^n\alpha_* \circ j_1^* \mathcal{F}
\]

is bijective. Let \(\overline{s} \to S\) be a geometric point of \(S\). We compute

\[
(j_2^* \circ R^n\beta_* \mathcal{F})|_{\overline{s}} = \lim_{\xleftarrow{(U, u)}} R^n\beta_* \mathcal{F}(U)
\]

\[
= \lim_{\xleftarrow{(U, u)}} H^n(U \times_T Y, \mathcal{F})
\]

\[
= \lim_{\xleftarrow{(U, u)}} H^n(U \times_S X, j_1^* \mathcal{F})
\]

\[
= (R^n\alpha_* \circ j_1^* \mathcal{F})|_{\overline{s}}
\]

where the (direct) limit in (i) is over all étale neighbourhoods of \(\overline{s}\) over \(S\), (ii) and (iv) follow by definition of higher direct image, and (iii) follows from Proposition 2.28.

\(\square\)

5. Cohomology with compact support

If \(\alpha: X \to S\) is a specialization morphism which admits a compactification in the sense of Definition 3.28, one can define the functor \(R^+\alpha_!\). In this section we will do so for the class of morphisms that interests us. Armed with Theorem 3.30 and Corollary 3.33, we make the following definition.

Definition 5.1. Let \(\alpha: X \to S\) be a separated and finite type specialization morphism. Write \(\alpha\) as the composite of the open immersion \(j: X \hookrightarrow X^{/S}\) and the proper map \(\beta: X^{/S} \to S\) (here the triple \((X^{/S}, \beta, j)\) is the universal compactification of \(\alpha\)). Then we define

\[
R^+\alpha_! := R^+\beta_* \circ j_! : D^+(X_{\text{ét}}, A) \to D^+(S_{\text{ét}}, A).
\]

We now investigate the behaviour of \(R^+\alpha_!\) with respect to composition of morphisms in the sense of Remark 2.18. First we show that \(R^+\alpha_!\) can be defined on any compactification.

Lemma 5.2. Let \(\alpha: X \to S\) be a separated and finite type specialization morphism. Let

\[
\begin{array}{ccc}
X & \xleftarrow{j'} & Y \\
\downarrow & & \downarrow j \\
S & \xleftarrow{\beta'} & T
\end{array}
\]

be a compactification of \(\alpha\). Then

\[
R^+\alpha_! = R^+\beta'_* \circ j_!.
\]
Proof. By the universal property there exists a unique morphism $i: \overline{X}/S \rightarrow Y$ such that the diagram commutes

![Diagram](image)

We claim that $i$ is proper. By Remark 3.27, it is locally of $\mathbb{C}$-weakly finite type. Let $T \subseteq S$ be an affine open subset. By Corollary 3.33, $\beta$ is proper and hence $\beta^{-1}(T)$ is quasi-compact. Since $\beta'$ is quasi-separated, we get that $\beta'^{-1}(T)$ is quasi-separated. It follows that the morphism $i|_{\beta^{-1}(T)}$ is quasi-compact. We deduce from this that there exists a cover $\{W_1\}$ of open subsets of $Y$ such that $i|_{i^{-1}(W_1)}$ is quasi-compact and hence $i$ is quasi-compact. Since $\beta' \circ i = \beta$ is proper (and $\beta'$ is in particular separated), it follows from applying Lemma 3.19, that $i$ satisfies the valuative criterion for partially properness. It remains to show that $i$ is quasi-separated. Let $V \subseteq Y$ be an affinoid open which maps into an affine open of $S$. Let $U_1, U_2 \subseteq \overline{X}/S$ be affinoid opens which map into $V$. Then $U_1 \cap U_2$ is a finite union of affinoid opens because $U_1, U_2$ map into a common affine open of $S$. Covering $Y$ by affinoid opens like $V$ gives that $i$ is quasi-separated. This proves the claim.

We now compute

$$R^+(\beta' \circ j') \equiv R^+ \beta'_* \circ (i \circ j)_!$$

where (i) follows from commutativity, (ii) follows from $[8, \text{Theorem 5.4.3}]$, (iii) because $i$ is proper, and (iv) follows from Proposition 2.28. □

Lemma 5.3. Let $\alpha: X \rightarrow S$ be a separated and finite type specialization morphism. Let $f: Y \rightarrow X$ be a separated and $\mathbb{C}$-weakly finite type morphism of analytic adic spaces. Then $\alpha \circ f$ is a separated and finite type specialization morphism. Moreover there is a natural equivalence

$$R^+(\alpha \circ f)_! \simeq R^+ \alpha_! \circ R^+ f_!$$

of functors $D^+(Y_{\text{ét}}, A) \rightarrow D^+(S_{\text{ét}}, A)$.

Proof. The fact that the composition $\alpha \circ f$ is a separated and finite type specialization morphism follows from Proposition 3.22. For the second part consider the following diagram

![Diagram](image)
where the top left triangle is the universal compactification of $f$ via [8, Theorem 5.1.5], the bottom right triangle is the universal compactification of $\alpha$ via Theorem 3.30. Let us explain the top right square. The morphism $j_2 \circ f_1 \colon \overline{Y}^X \rightarrow \overline{X}^S$ is separated and \*weakly of finite type. Thus again by [8, Theorem 5.1.5], it admits a universal compactification which we denote by the triple $(Z, f_2, j_2)$. In particular all the horizontal arrows (in the diagram) are quasi-compact open embeddings and the vertical arrows are proper. Therefore the composition $j_2 \circ j_1$ is a quasi-compact open embedding of analytic adic spaces and by Proposition 3.22, the composition $\beta \circ f_2$ is a proper specialization morphism. Thus the outer triangle is a compactification of $\alpha \circ f$.

Then

$$R^+ \alpha_! \circ R^+ f_1 = R^+ \beta_* \circ j_{3!} \circ R^+ f_{1*} \circ j_{1!}$$

and

$$R^+ (\alpha \circ f)_! (i) \equiv R^+ (\beta \circ f_2)_* \circ (j_2 \circ j_1)_!$$

$$\equiv R^+ \beta_* \circ R^+ f_{2*} \circ j_{3!} \circ j_{1!}$$

$$\equiv R^+ \beta_* \circ j_{3!} \circ R^+ f_{1*} \circ j_{1!}$$

where (i) follows from Lemma 5.2, (ii) follows from Proposition 2.28 and [8, Theorem 5.4.3], and (iii) follows from [8, Lemma 5.4.2]. This completes the proof. \hfill \Box

6. Proper base change

Our goal in this section is to prove a base change theorem for proper specialization morphisms. Owing to the fact that we only construct fibre products for specialization morphisms when the base change map is \*étale and the fact that arbitrary projective limits of adic spaces do not exist, we formulate a version of proper base change in Theorem 6.3 that takes into account these challenges. In the case when the target of a proper specialization map is strictly local then we have the familiar statement as in Corollary 6.17.

We then employ Theorem 6.3 to deduce the crucial identity in Corollary 7.5 from which can be easily deduced the commutativity of the nearby cycles and lower shriek functors.

Remark 6.1. In what follows, we make use of the notion of (pre)pseudo-adic space. Recall from [8, §1.10] that a prepseudo-adic space $Z$ is given by a pair $(Z, |Z|)$ where $Z$ is an adic space and $|Z|$ is a subset of $Z$. When $|Z|$ is convex and locally pro-constructible in $Z$, it is said to be a pseudo-adic space. The conditions are satisfied for instance when $|Z|$ is closed in $Z$. Given a specialization morphism $f : X \rightarrow W$ and a point $w \in W$, we write $f^{-1}(w)$ for the prepseudo-adic space with $f^{-1}(w) = X$ and $|f^{-1}(w)| = \{ x \in X | f(x) = w \}$.

We also require the notion of \*étale neighbourhood of a geometric point in the context of schemes. The relevant definitions and proofs can be found in [14, Tag 03PN]. We calculate all cohomology groups in the relevant \*étale topos.

Situation 6.2. Let $f : X \rightarrow S$ be a specialization morphism. Let $s \in S$ be a closed point and $\overline{s} \rightarrow S$ be a geometric point over $s$. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\text{et}}$. For every $n \in \mathbb{N}$, we have a natural map

$$[R^n f_*(\mathcal{F})]_{\overline{s}} \rightarrow \lim_{\substack{\longrightarrow \\\{(W, \overline{w})\}}} H^n(f_{W*}(w), \mathcal{F})$$

where the colimit on the right runs over all \*étale neighbourhoods $(W, \overline{w})$ of $(S, \overline{s})$, $w \in W$ is the image of $\overline{w}$, $f_{W} : X_{W} \rightarrow W$ is the morphism induced by base change

$$\alpha \circ f$$

and

$$\alpha \circ f$$

is an adic space and
and \( f_W^{-1}(w) \) is the pseudo-adic space as defined in Remark 6.1. Indeed, for every \( \text{étale neighbourhood} \ (W, \overline{\eta}) \) of \((S, \overline{\eta})\), we have a natural morphism \( H^n(X_W, \mathcal{F}) \to H^n(f_W^{-1}(w), \mathcal{F}) \) and hence we get a natural map

\[
[R^n f_*(\mathcal{F})]_T = \lim_{(W, \overline{\eta})} H^n(X_W, \mathcal{F}) \to \lim_{(W, \overline{\eta})} H^n(f_W^{-1}(w), \mathcal{F}).
\]

**Theorem 6.3.** In Situation 6.2, if in addition \( f \) is proper then the natural morphism (14) is an isomorphism.

Our strategy to prove Theorem 6.3 is to reduce to when \( X \) is affinoid, \( S \) is affine and the morphism \( f \) is tight. This situation can then be dealt with by Proposition 6.12 which makes crucial use of [8, Theorem 3.2.1].

We introduce the following definition which plays a crucial role in Proposition 6.12.

**Definition 6.4.** Let \( Y \) be an analytic affinoid adic space and \( T \) be an affine scheme. A specialization morphism \( f: Y \to T \) induces a morphism

\[
a: \mathcal{O}_T(T) \to \mathcal{O}_{Y_{red}}(Y).
\]

Furthermore, we have the morphism

\[
b: \mathcal{O}_Y(Y) \to \mathcal{O}_{Y_{red}}(Y).
\]

We say that the specialization morphism \( f \) is tight if the following conditions are satisfied.

1. \( \text{im}(a) \subseteq \text{im}(b) \).
2. \( \text{im}(b) \) is integral over \( \text{im}(a) \).

**Lemma 6.5.** Let \( f: X \to S \) be a tight specialization morphism. We suppose that \( \mathcal{O}_X(X) \) is Tate. Then \( f \) is partially proper.

**Proof.** Since \( f \) is tight, it is in particular good. By Lemma 3.23, \( f \) satisfies the conditions of Theorem 3.30. Hence, \( X^S \) and \( T^S \) exist. We claim that \( X = X^S \) and \( f = T^S \). Let \( a : \mathcal{O}_S(S) \to \mathcal{O}_{X_{red}}(X) \) be the morphism induced by \( f \) and let \( b \) denote the map \( \mathcal{O}_Y(Y) \to \mathcal{O}_{X_{red}}(X) \). We follow the construction of \( X^S \) in the proof of Theorem 3.30. Let \( X_c := \text{Spa}(\mathcal{O}_X(X), I(X)) \) where \( I(X) \) is the integral closure of \( b^{-1}(\text{im}(a)) \) in \( \mathcal{O}_X(X) \). We show \( X_c = X \). Indeed, suppose \( x \in \mathcal{O}_X(X) \). Since \( f \) is tight, there exists a monic polynomial \( g \) with coefficients in \( \text{im}(a) \) such that \( g(b(x)) = 0 \). Since \( f \) is tight, \( \text{im}(a) \subseteq \text{im}(b) \). It follows that we can choose a monic lift \( g' \) of \( g \) with coefficients in \( b^{-1}(\text{im}(a)) \). Note that \( g'(x) \in m_{\mathcal{O}_X^S}(X) \). Let \( r := g'(x) \). Hence, the monic polynomial \( g' := g' - r \) has coefficients in \( b^{-1}(\text{im}(a)) \) and is such that \( g''(x) = 0 \). Since \( I(X) \) is integrally closed, we get that \( x \in I(X) \).

Let \( \mathcal{F} \) be as defined in the beginning of the proof of Theorem 3.30. Recall that we obtain \( X^S \) by glueing together the \( U_i \) for \( U \in \mathcal{F} \) where \( U_d \) is as defined in the end of the proof of Theorem 3.30. Observe that \( X \in \mathcal{F} \) and \( X_d = X \). Furthermore, for every \( U \in \mathcal{F}, U_d \subseteq X_d = X \). Since \( X^S \) is obtained by glueing \( U_d \) as \( U \) varies along \( \mathcal{F} \), it follows that \( X^S = X \).

**Lemma 6.6.** Let \( f: X \to S \) be a tight specialization morphism. Let \( p: Y \to X \) be a finite morphism of adic spaces. Then the specialization \( f \circ p \) is tight.

**Proof.** Note that since \( Y \) is finite over \( X \) and \( X \) is analytic affinoid, \( Y \) is analytic affinoid as well. Let \( (A, A^+) := (\mathcal{O}_X(X), \mathcal{O}_X^S(X)) \) and \( (B, B^+) := (\mathcal{O}_Y(Y), \mathcal{O}_Y^S(Y)) \).
Let $R := \mathcal{O}_S(S)$. We have the following commutative diagram where we add appropriate labels.

\[
\begin{array}{ccc}
A^+ & \to & B^+ \\
\downarrow b & & \downarrow b' \\
\mathcal{O}_{X_{\text{red}}} (X) & \to & \mathcal{O}_{Y_{\text{red}}} (Y) \\
\downarrow a & & \downarrow a' \\
R & \to & \mathcal{O}_{Y_{\text{red}}} (x) \\
\end{array}
\]

A diagram chase shows that $\text{im}(a') \subseteq \text{im}(b')$. Since $Y$ is finite over $X$, $B^+$ is integral over $A^+$. It follows that $\text{im}(b')$ is integral over the image of $A^+$ in $\mathcal{O}_{Y_{\text{red}}} (Y)$. Since the diagram is commutative and the specialization morphism $f$ is tight, we see that the image of $A^+$ in $\mathcal{O}_{Y_{\text{red}}} (Y)$ is integral over $\text{im}(a')$. Thus, $\text{im}(b')$ is integral over $\text{im}(a')$. This concludes the proof. □

**Lemma 6.7.** Let $\alpha: X \to S$ be a tight specialization morphism such that $\mathcal{O}_X (X)$ is a Tate ring. Let $T$ be an affine scheme and $g: T \to S$ be a standard étale morphism. Let $Y := X \times_S T$ be the fibre product as constructed in Proposition 2.19 and $\beta: Y \to T$ be the projection. We then have that $Y$ is affinoid and $\beta$ is a tight specialization morphism.

**Proof.** The proof is the same as that of Lemma 3.8, with $\mathfrak{D}$ (and hence $\mathfrak{C}$) being the empty sets there. □

**Lemma 6.8.** Let $f: X \to S$ be a specialization morphism such that the following properties are satisfied.

1. $X$ is affinoid.
2. $S$ is affine.
3. With respect to the natural morphisms

\[
a: \mathcal{O}_S(S) \to \mathcal{O}_{X_{\text{red}}} (X) \\
b: \mathcal{O}_X^+ (X) \to \mathcal{O}_{X_{\text{red}}} (X)
\]

we have $\text{im}(a) \subseteq \text{im}(b)$.

Let $s \in S$ be a Zariski closed point. Then $f^{-1}(s)$ is a pro-special subset of $X$.

**Proof.** Let $(A, A^+) := (\mathcal{O}_X (X), \mathcal{O}_X^+ (X))$ and $R$ be such that $S = \text{Spec}(R)$. Let $\mathfrak{p}$ be the maximal ideal corresponding to the point $s$. Let $x \in X$. We then have the following commutative diagram.

\[
\begin{array}{ccc}
A^+ & \to & k(x)^+ \\
\downarrow b & & \downarrow \\
R & \to & \mathcal{O}_{X_{\text{red}}} (X) \\
\downarrow a & & \downarrow \\
\mathcal{O}_{X_{\text{red}}, x} & \to & \mathcal{O}_{X_{\text{red}}, x}.
\end{array}
\]

Recall that we have an injection $A^+ / \mathfrak{m}_{\mathcal{O}_X^+ (X)} \hookrightarrow \mathcal{O}_{X_{\text{red}}} (X)$. By condition (3), we have that the image of $a$ is contained in $A^+ / \mathfrak{m}_{\mathcal{O}_X^+ (X)}$. We can hence remake the
Let $\text{Situation 6.9.}$ which confirms that $f$ is affine. Let $S$ be Tate. By Lemma 6.10, the composition $R \to A^+ / m_{O_X}^+ (X) \to O_{X_{red}, x}$. This can be deduced from the commutative diagram above. Hence we see that $x \in f^{-1}(s)$ if and only if $e(p) \subseteq e_x^{-1}(p_x/k(x)^\infty)$. By construction, $e_x^{-1}(p_x/k(x)^\infty)$ is an ideal in $A^+ / m_{O_X}^+ (X)$. It follows that $f(x) = s$ if and only if $e(p) \cdot A^+ / m_{O_X}^+ (X) \subseteq e_x^{-1}(p_x/k(x)^\infty)$. Equivalently, $x \in f^{-1}(s)$ if and only if for every $g \in e(p) \cdot A^+ / m_{O_X}^+ (X)$, $e_x(g) \in p_x/k(x)^\infty$.

Let $g' \in A^+$. We then have that $e_x(d(g')) \in p_x/k(x)^\infty$ if and only if $|g'(x)| < 1$. This can be deduced from the commutative diagram above. Hence we see that $x \in f^{-1}(s)$ if and only if for every $g' \in d^{-1}(e(p) \cdot A^+ / m_{O_X}^+ (X))$, $|g'(x)| < 1$. Thus $f^{-1}(s) = \{ x \in X \mid |g'(x)| < 1 \forall g' \in d^{-1}(e(p) \cdot A^+ / m_{O_X}^+ (X)) \}$ which confirms that $f^{-1}(s)$ is a pro-special subset of $X$.

**Situation 6.9.** Let $f : X \to S$ be a partially proper specialization morphism with $S$ affine. Let $U$ be an affinoid open subset of $X$ such that $f|_U$ is good and $O_X(U)$ is Tate. By Lemma 3.23, the morphism $f|_U : U \to S$ satisfies the conditions of Theorem 3.30. Hence the universal compactification $f' : \overline{U}^S \to S$ exists and $\overline{U}^S$ is affinoid. Let $g : \overline{U}^S \to X$ denote the unique map that satisfies the universal property associated to $\overline{U}^S$.

**Lemma 6.10.** In Situation 6.9, the following hold.

1. The morphism $g$ induces a homeomorphism from $[\overline{U}^S]$ onto $[U]$ where $[U]$ is the closure of $[U]$ in $[X]$.
2. Let $v \in \overline{U}^S$. There exists affinoid open neighbourhoods $P \subseteq \overline{U}^S$ and $Q \subseteq X$ of $v$ and $g(v)$ respectively such that $g(P) \subseteq Q$ and the induced morphism $O_X(Q) \to O_{\overline{U}^S}(P)$ has dense image.

**Proof.** Since $\overline{U}^S$ is affinoid, it is quasi-compact. Furthermore, since $f'$ is partially proper, it is separated and by Theorem 3.30, every point of $\overline{U}^S$ is a specialization of a point of $U$. Therefore, by Lemma 3.26(3), $g$ is a homeomorphism from $[\overline{U}^S]$ onto its image in $X$. Observe that this image must be contained in $[U]$.

We show that the image of $g$ coincides with $[U]$. It suffices to verify that $\text{im}(g)$ is closed. Observe firstly that $g$ is quasi-compact. This is a consequence of the fact
that \( \overline{U}^S \) is quasi-compact and \( X \) is quasi-separated. Hence \( \text{im}(g) \) is closed if \( g \) is specializing. Let \( y \in X \) be a point which generalizes to a point \( g(x) \) for some \( x \in U \). Let \( A \subseteq k(x)^+ \) be a valuation ring such that \( y \) is the center of \( (x, A) \). Note that \( g \) lifts centers uniquely because both \( f \) and \( f' \) lift centers uniquely. It follows that there exists a unique center \( y' \) of \( (x, A) \) in \( \overline{U}^S \) and \( g(y') = y \). This verifies that \( g \) is specializing.

The proof of assertion (2) in the statement of the lemma is identical to the proof of the similar statement in (7) of [8, Proposition 4.4.3] where we use Lemma 3.31(1)-(2) in place of 5(i) and 5(iii) of loc.cit., respectively. □

**Lemma 6.11.** We work within the context of Situation 6.9. Let \( \mathcal{F} \) be a torsion abelian sheaf on \( X_{\text{et}} \). Let \( \mathcal{J} \) denote the category of standard étale neighbourhoods of \((S, \pi)\) and given \((W, \pi) \in \mathcal{J}\), let \( g_W : \overline{U}^S \times_S W \to X_W \) denote the induced morphism. The natural morphism

\[
\lim_{(W, \pi) \in \mathcal{J}} H^q((X_W, \overline{U_W}), \mathcal{F}) \to \lim_{(W, \pi) \in \mathcal{J}} H^q((X_W, \overline{U_W} \cap f_W^{-1}(w)), \mathcal{F})
\]

is an isomorphism if and only if the natural morphism

\[
\lim_{(W, \pi) \in \mathcal{J}} H^q(\overline{U}^S \times_S W, \mathcal{F}) \to \lim_{(W, \pi) \in \mathcal{J}} H^q(f_W^{-1}(w), \mathcal{F})
\]

is an isomorphism where \( f_W : \overline{U}^S \times_S W \to W \) is the projection to \( W \).

**Proof.** Let \( j \) denote the open embedding \( U \hookrightarrow \overline{U}^S \). By Lemma 3.34, \( \overline{U}^S \times_S W, f_W, jw \) is a universal compactification of \( f_W |_{U_W} : U_W \to W \) where \( jw \) is induced by base change. The map \( g_W : \overline{U}^S \times_S W \to X_W \) which is induced from \( g : \overline{U}^S \to X \) is the unique map such that the following diagram

\[
\begin{array}{ccc}
U_W & \xrightarrow{jw} & \overline{U}^S \times_S W & \xrightarrow{gw} & X_W \\
& & f_W \downarrow & \downarrow f_W & \\
& & W & \to & W \\
\end{array}
\]

commutes.

By Lemma 6.10 and [8, Proposition 2.3.7], we get compatible isomorphisms

\[
H^q((X_W, \overline{U_W}), \mathcal{F}) \simeq H^q(\overline{U}^S \times_S W, \mathcal{F})
\]

and

\[
H^q((X_W, \overline{U_W} \cap f_W^{-1}(w)), \mathcal{F}) \simeq H^q(f_W^{-1}(w), \mathcal{F}).
\]

This is sufficient to conclude the proof. □

**Proposition 6.12.** In Situation 6.2, if in addition we assume

1. The morphism \( f \) is tight and \( \mathcal{O}_X(X) \) is a Tate ring.
2. Let \( Y \) be the pseudo-adic space \((X, |Y|)\) where

\[
|Y| := \{ x \in X | e_i(x) < |e_0(x)| \text{ for } 1 \leq i \leq n \}
\]

and \( e_0, \ldots, e_n \in \mathcal{O}_X(X) \) are such that \( \mathcal{O}_X(X) = \sum \epsilon_i \mathcal{O}_X(X) \). Let \( i : Y \hookrightarrow X \) be the natural closed embedding. The sheaf \( \mathcal{F} \) is of the form \( i^*(G_Y) \) where \( G_Y \) is the constant sheaf on \( Y_{\text{et}} \) associated to an abelian torsion group \( G \).

Then (14) is an isomorphism.

Note that in what follows, when there is no ambiguity, we simplify notation and write \( G \) for the constant sheaf \( G_X \) on \( X_{\text{et}} \) for \( X \) a pseudo-adic space or scheme.
Proof. Recall that
\[ [R^n f_*(\mathcal{F})]_{\mathfrak{m}} = \lim_{(W, \mathfrak{m})} H^n(X_W, \mathcal{F}) \]
where the colimit runs over étale neighbourhoods of \((S, \mathfrak{m})\) and we must show that the natural morphism
\[ \lim_{(W, \mathfrak{m})} H^n(X_W, \mathcal{F}) \to \lim_{(W, \mathfrak{m})} H^n(f^{-1}_W(w), \mathcal{F}) \]
is an isomorphism. Note that we can restrict to when the colimits above run over affine étale neighbourhoods of \((S, \mathfrak{m})\). By [14, Tag 02GT], we can further restrict so that the colimits are over standard étale neighbourhoods of \((S, \mathfrak{m})\). Let \(\mathcal{F}\) denote the category of standard étale neighbourhoods of \((S, \mathfrak{m})\) and let \((W, \mathfrak{m}) \in \mathcal{F}\). Observe that by Lemma 6.7, \(X_W\) is affinoid. For the remainder of the proof, all limits under consideration will run over pairs \((W, \mathfrak{m}) \in \mathcal{F}\).

Lemma 6.13. (1) We have a well defined pseudo-adic space \(Y_W := Y \times_X X_W\) such that the morphism \(i_W : Y_W \to X_W\) induced by base change is a closed embedding. If for every \(0 \leq i \leq n\), \(e_i^1\) denotes the image of \(e_i\) for the morphism \(O_X(X) \to O_{X_W}(X_W)\) then \(|Y_W| = \{ x \in X_W | e_i^1(x) | < |e_0(x)| \} \) for \(1 \leq i \leq n\). 

(2) We have that \(i_W, G = \mathcal{F}|_{X_W}\).

(3) If \(e_W\) denotes the closed embedding \((X_W, (f_W^{-1}(w) \cap |Y_W|)) \hookrightarrow f_W^{-1}(w)\) of pseudo-adic spaces then
\[ \mathcal{F}|_{f_W^{-1}(w)} = e_W(G). \]

Proof. The morphism \(i : Y \to X\) is locally of finite type. By [8, Lemma 1.10.6(a)], \(Y_W\) exists and from the proof of loc.cit., it must be of the form described in part (1) of the lemma. Similarly, \((X_W, (f_W^{-1}(w) \cap |Y_W|)) = f_W^{-1}(w) \times_X Y\). By [8, Lemma 1.10.17(i)], \(i\) is proper. Parts (2) and (3) follow from [8, Proposition 4.1.2(b)]. \(\square\)

We first reduce to when \(e_0 = 1\). Let
\[ U := \{ x \in X | |e_i(x)| \leq |e_0(x)| \} \text{ for } i = 1, \ldots, n. \]
Then \(U\) is an affinoid open subset of \(X\). Observe that it suffices to show that the natural morphism
\[ \lim_{(W, \mathfrak{m}) \in \mathcal{F}} H^n((X_W, [U_W]), \mathcal{F}) \to \lim_{(W, \mathfrak{m}) \in \mathcal{F}} H^n((X_W, [U_W] \cap f_W^{-1}(w)), \mathcal{F}) \]
is an isomorphism where \(U_W := U \times_S W\). Indeed, by Lemma 6.13(1,2), \(i_W : Y_W \hookrightarrow X_W\) is a closed embedding and \(\mathcal{F}|_{X_W} = i_W^*(G)\). Hence,
\[ H^n(X_W, \mathcal{F}) = H^n(Y_W, G) = H^n((X_W, [U_W]), \mathcal{F}). \]

Similarly, by Lemma 6.13(3), we deduce that
\[ H^n(f_W^{-1}(w), \mathcal{F}) = H^n((X_W, |Y_W| \cap f_W^{-1}(w)), G) = H^n((X_W, [U_W] \cap f_W^{-1}(w)), \mathcal{F}). \]

By Lemma 6.5, \(f\) is partially proper and by Remark 3.7, \(f_W\) is good. We are hence in the context of Situation 6.9. By Lemma 6.11, it suffices to verify the proposition after replacing \(X\) with \(U^S\). Since \(O_X(U) = O_{U^S}(U^S)\), \(e_0\) is invertible on \(U^S\) and after replacing \(e_i\) with \(e_i/e_0\) for every \(1 \leq i \leq n\), we can suppose \(e_0 = 1\) in the definition of \(|Y|\).
Let \((A,A^+) := (\mathcal{O}_X(X), \mathcal{O}_X^{\dagger}(X))\). We begin by simplifying the left hand side of (14). By Lemma 6.7, \(X_W\) is affinoid and let \((A_W,A_W^+) := (\mathcal{O}_{X_W}(X_W), \mathcal{O}_{X_W}^{\dagger}(X_W))\). Associated to the special subset \(Y_W \subseteq X_W\), we define the algebra \(A_W(Y_W)\) as follows. Let \(A_W^+(Y_W)\) denote the henselization of \(A_W^+[e_1,\ldots,e_n]\) with respect to the ideal generated by \(\{e_1,\ldots,e_n\} \cup A_W^{\dagger}\) where we abuse notation and use \(e_i\) to denote the image of \(e_i\) for the morphism \(A \to A_W\). We define \(A_W(Y_W) := A_W^+(Y_W) \otimes_{A_W^+[e_1,\ldots,e_n]} A_W\). By (15) and [8, Theorem 3.2.1], we have a natural isomorphism

\[
H^n(X_W, \mathcal{F}) \simeq H^n(\text{Spec}(A_W(Y_W)), G).
\]

Hence we have that

\[
\lim_{\longrightarrow (W, \mathcal{F})} H^n(X_W, \mathcal{F}) \overset{(i)}{\simeq} \lim_{\longrightarrow (W, \mathcal{F})} H^n(\text{Spec}(A_W(Y_W)), G)
\]

\[
\overset{(ii)}{=} H^n(\text{Spec}(\varinjlim_{(W, \mathcal{F})} A_W(Y_W)), G)
\]

\[
\overset{(iii)}{=} H^n(\text{Spec}(A_W(Y_W)), G)
\]

(18)

where the colimit and limit runs over all standard étale neighbourhoods of \((S, \mathcal{F})\). Observe that (i) is a consequence of the naturality of the isomorphism (17). Note that (ii) follows from [14, Tag 09YQ] and (iii) is due to [14, Tag 01YW].

We now focus on the right hand side of (14). Observe that by Lemma 6.7, the morphism \(f_W\) is tight. Then by Lemma 6.8, \(f_W^{-1}(w)\) is a pro-special subset of \(X_W\). Furthermore, if \(R\) and \(R_W\) are such that \(S = \text{Spec}(R)\) and \(W = \text{Spec}(R_W)\), and \(w \in W\) corresponds to a maximal ideal \(p_W\) in \(R_W\) then we have the following diagram

\[
\begin{array}{ccc}
A_W^+ & \longrightarrow & A_W^+/m_{\mathcal{O}_{X_W}^{\dagger}}(X_W). \\
\downarrow d_W & & \\
R_W & \longrightarrow & A_W^+/m_{\mathcal{O}_{X_W}^{\dagger}}(X_W).
\end{array}
\]

We abuse notation and write \(p_W\) for \(e_W(p_W) \subset A_W^+/m_{\mathcal{O}_{X_W}^{\dagger}}(X_W)\). As in the proof of Lemma 6.8,

\[
f_W^{-1}(w) = \{x \in X_W \mid |a(x)| < 1 \forall a \in d_W^{-1}(p_W \cdot A_W^+/m_{\mathcal{O}_{X_W}^{\dagger}}(X_W))\}.
\]

Let \(I_W\) denote the ideal generated by \(d_W^{-1}(p_W \cdot A_W^+/m_{\mathcal{O}_{X_W}^{\dagger}}(X_W))\) \(\bigcup\{e_1,\ldots,e_n\}\) in \(A_W^+[e_1,\ldots,e_n]\) and let \(A_W^{h(I_W)}\) denote the henselization of \(A_W^+[e_1,\ldots,e_n]\) along \(I_W\).

Define \(A_W^{h(I_W)} := A_W^{h(I_W)} \otimes_{A_W^+[e_1,\ldots,e_n]} A\).

Note that \(A_W^{h(I_W)} = A_W^{h(I_W)} \otimes_{A^+[e_1,\ldots,e_n]} A\).

The equality above is a consequence of the fact that \(A\) is Tate and hence \(A = A^+[1/\pi]\) for some pseudo-uniformizer \(\pi \in A\). By (16) and [8, Theorem 3.2.1], we have a natural isomorphism

\[
H^n(f_W^{-1}(w), \mathcal{F}) \simeq H^n(\text{Spec}(A_W^{h(I_W)}), G).
\]
By a similar calculation as made for (18),

\[
\lim_{(W, \pi)} H^n(W^+_{\pi}(w), \mathcal{F}) \simeq \lim_{(W, \pi)} H^n(\text{Spec}(A^{b(iw)}_W), G)
\]

\[
= H^n(\lim_{(W, \pi)} \text{Spec}(A^{b(iw)}_W), G)
\]

\[
(19)
\]

We claim that

\[
\lim_{(W, \pi)} A^{b(iw)}_W = \lim_{(W, \pi)} A_W(W_Y).
\]

Let \( J_W \subset A_W(W_Y) \) denote the ideal generated by the image of \( I_W \) for the map \( A^+_W[e_1, \ldots, e_n] \rightarrow A^+_W(W_Y) \). A diagram chase implies that if \( (V, \pi) \) is a standard étale neighbourhood of \( (S, \pi) \) such that the map \( (V, \pi) \rightarrow (S, \pi) \) factors through \( (W, \pi) \rightarrow (S, \pi) \) then

\[
J_W \cdot A^+_W(Y_V) \subseteq J_V
\]

where \( J_W \) on the left is the image of the ideal \( J_W \subset A_W(W_Y) \) for the morphism \( A_W^+(W_Y) \rightarrow A_Y^+(Y_V) \).

We set \( B := \lim_{(W, \pi)} A_W^+(W_Y) \). Let \( J \subset B \) denote the ideal \( \lim_{(W, \pi)} J_W \). Let \( J_W^0 \subset A_W^+(Y_W) \) be the ideal generated by the image of \( m_{\mathcal{O}^+_{X_W}}(Y_W) \cup \{e_1, \ldots, e_n\} \) for the morphism \( A_W^+[e_1, \ldots, e_n] \rightarrow A_W^+(Y_W) \) and set \( J' := \lim_{(W, \pi)} J'_W \subset B \). Finally, let \( J_W'' \subset A_W^+(Y_W) \) be the ideal generated by the image of \( A_W^+[e_1, \ldots, e_n] \) for the morphism \( A_W^+[e_1, \ldots, e_n] \rightarrow A_W^+(Y_W) \) and set \( J'' := \lim_{(W, \pi)} J''_W \subset B \).

**Lemma 6.14.**

(1) The pair \((B, J')\) is henselian.

(2) The pair \((B, J', J''_W)\) is henselian.

(3) The pair \((B, J)\) is henselian.

**Proof.** Let us verify part (1) of the lemma. By construction, \( A_W^+(Y_W) \) is henselian along \( J''_W \). Using [14, Tag 0FWT], we deduce that \( B \) is henselian along \( J''_W \). We check that the pair \((B, J', J''_W)\) is henselian. Since filtered colimits are exact,

\[
(B/J'', J'/J'') = (\lim_{(W, \pi)} A_W^+(Y_W)/J''_W, \lim_{(W, \pi)} J_W/J''_W).
\]

Since \( A_W^+(Y_W) \) is the henselization of \( A_W^+[e_1, \ldots, e_n] \) along the ideal \( A_W^+[\{e_1, \ldots, e_n\}] \), we get that

\[
(20) \quad A_W^+(Y_W)/J''_W = A_W^+/A_W^+[\{e_1, \ldots, e_n\}].
\]

Furthermore, \( J''_W/J''_W = m_{\mathcal{O}^+_{X_W}}(X_W)/A_W^+[\{e_1, \ldots, e_n\}] \).

\[
(B/J'', J'/J'') = (\lim_{(W, \pi)} A_W^+/A_W^+[\{e_1, \ldots, e_n\}], \lim_{(W, \pi)} m_{\mathcal{O}^+_{X_W}}(X_W)/A_W^+[\{e_1, \ldots, e_n\}]).
\]

By Corollary 2.16 and [14, Tag 0DYD and Tag 0FWT], we deduce that \((B, J', J''_W)\) is a henselian pair. Since \((B, J'')\) is a henselian pair, applying [14, Tag 0DYD] once again implies that \((B, J', J'')\) is henselian.

We now verify part (2) of the lemma. Since filtered colimits are exact,

\[
B/J' = \lim_{(W, \pi)} A_W^+(Y_W)/J''_W.
\]

Similarly,

\[
\lim_{(W, \pi)} A_W^+/m_{\mathcal{O}^+_{X_W}}(X_W) = \lim_{(W, \pi)} A_W^+/m_{\mathcal{O}^+_{X_W}}(X_W).
\]
We have the following commutative diagram.

\[
\begin{array}{ccc}
\lim_{\to h(W,\varpi)} A^+_W & \to & B \\
\downarrow^{\lim_{\to h(W,\varpi)} R_W} & & \\
\lim_{\to h(W,\varpi)} \lim_{\to h(W,\varpi)} A^+_W/m_{\sigma W}(X_W) & \to & \lim_{\to h(W,\varpi)} A^+_W(Y_W)/J'_W.
\end{array}
\]

The equality in the above diagram is because by construction, \( A^+_W(Y_W)/J'_W = A^+_W/m_{\sigma W}(X_W) \).

Observe that \( \lim_{\to h(W,\varpi)} R_W \) is a strictly henselian local ring whose maximal ideal is \( p_0 := \lim_{\to h(W,\varpi)} p_W \). By Lemma 6.7, the morphism

\[ e_W : R_W \to A^+_W/m_{\sigma W}(X_W) \]

is integral and hence by the equality in the diagram, the induced map

\[ R_W \to A^+_W(Y_W)/J'_W \]

is integral as well. By [14, Tag 09XK], \( \lim_{\to h(W,\varpi)} A^+_W(Y_W)/J'_W \) is henselian along the ideal \( p_0 \cdot \lim_{\to h(W,\varpi)} A^+_W(Y_W)/J'_W = \lim_{\to h(W,\varpi)} p_W \cdot A^+_W(Y_W)/J'_W \). Using that filtered colimits are exact, we deduce that

\[ J/J' = \lim_{\to h(W,\varpi)} J_W/J'_W. \]

A diagram chase shows that \( J_W/J'_W = p_W \cdot A^+_W(Y_W)/J'_W \). The equality (21) now implies that \( B/J' \) is henselian along the ideal \( J/J' \).

Part (3) of the lemma is a direct consequence of Parts (1)-(2) and [14, Tag 0DYD].

Note that

\[ \lim_{\to h(W,\varpi)} A^+_W h(I_W) = ( \lim_{\to h(W,\varpi)} A^+_W) \otimes_{A^+[e_1, \ldots, e_n]} A. \]

and similarly,

\[ \lim_{\to h(W,\varpi)} A_W(Y_W) = ( \lim_{\to h(W,\varpi)} A^+_W(Y_W)) \otimes_{A^+[e_1, \ldots, e_n]} A. \]

Hence, to show that \( \lim_{\to h(W,\varpi)} A^+_W h(I_W) \simeq \lim_{\to h(W,\varpi)} A_W(Y_W) \), it suffices to verify that

\[ \lim_{\to h(W,\varpi)} A^+_W h(I_W) \simeq \lim_{\to h(W,\varpi)} A_W(Y_W). \]

To complete the proof, we apply [2, Remark 3.19] with \( S = \lim_{\to h(W,\varpi)} A^+_W[e_1, \ldots, e_n] \), \( I = \lim_{\to h(W,\varpi)} I_W \) and \( \tilde{S} = B \). Note that by [14, Tag 0A04], \( B \) is the henselization of \( \lim_{\to h(W,\varpi)} A^+_W[e_1, \ldots, e_n] \) along the ideal \( \lim_{\to h(W,\varpi)} (A^+_W \cup \{ e_1, \ldots, e_n \}) \). It follows that \( B \) is a filtered colimit of étale \( S \)-algebras. Since \( (B, J) \) is henselian by Lemma 6.14(3), we can apply [2, Remark 3.19] to get that \( (B, J) \) is the henselization of \( \lim_{\to h(W,\varpi)} A^+_W[e_1, \ldots, e_n], \lim_{\to h(W,\varpi)} I_W \). By [14, Tag 0A04], \( \lim_{\to h(W,\varpi)} A^+_W h(I_W) \) is the henselization of \( \lim_{\to h(W,\varpi)} A^+_W[e_1, \ldots, e_n] \) along \( \lim_{\to h(W,\varpi)} I_W \). It follows that the natural morphism \( B \to \lim_{\to h(W,\varpi)} A^+_W h(I_W) \) is an isomorphism. Equations (18), (19) and the naturality of the isomorphism in [8, Theorem 3.2.1] allow us to conclude the proof.
Lemma 6.15. Let \( \mathcal{W} \) be a cofiltered category. Let \( A \) be the category of pseudo-adic spaces. Let \( p: \mathcal{W} \to A \) and \( q: \mathcal{W} \to A \) be functors such for any \( W \in \mathcal{W} \), \( q(W) = p(W) \) and \( |q(W)| \subseteq |p(W)| \). Suppose \( \mathcal{W} \) has a final object 0 and let \( X_0 := p(0) \). Let \( \mathcal{F}_0 \) be a sheaf of abelian groups on \( X_{0,et} \) and for every \( W \in \mathcal{W} \), let \( \mathcal{F}_W \) denote the pullback of \( \mathcal{F}_0 \) to \( p(W) \). Assume that we have a finite index set \( \mathcal{I} \) such that \( |X_0| = \bigcup_{j \in \mathcal{I}} U_j \) where for every \( j \in \mathcal{I} \), \( U_j \) is a subspace of \( |X_0| \).

For every \( j \in \mathcal{I} \), let \( U_{j,W} \) be the preimage of \( U_j \) for the morphism \( |p(W)| \to |X_0| \). Then the natural map

\[
\lim_{W \in \mathcal{W}} H^n(p(W), \mathcal{F}_W) \to \lim_{W \in \mathcal{W}} H^n(q(W), \mathcal{F}_W)
\]

is an isomorphism if for every non-empty subset \( J \subseteq \mathcal{I} \), the natural morphism

\[
\lim_{W \in \mathcal{W}} H^n(\bigcap_{j \in J} U_{j,W}, \mathcal{F}_W) \to \lim_{W \in \mathcal{W}} H^n(\bigcap_{j \in J} U_{j,W} \cap q(W), \mathcal{F}_W).
\]

is an isomorphism where by \( \overline{U_{j,W}} \), we mean the closure of \( U_{j,W} \) in \( |p(W)| \).

Proof. By [8, Corollary 2.6.10], we have a spectral sequence

\[
\prod_{\alpha \in \mathcal{J}^{r+1}} H^s(\bigcap_{0 \leq i \leq r} U_{\alpha_i,W}, \mathcal{F}_W) \Rightarrow H^{r+s}(p(W), \mathcal{F}_W)
\]

where \( \alpha = (\alpha_i)_{0 \leq i \leq r} \) and for every \( i \), \( \alpha_i \in \mathcal{J} \). Note that \( \mathcal{J}^{r+1} \) is a finite set.

Since \( \lim_{W \in \mathcal{W}} \) is a filtered colimit, it is exact by [14, Tag 04B0]. Note that finite products are coproducts in the category of abelian groups. It follows that we have a spectral sequence

\[
\prod_{\alpha \in \mathcal{J}^{r+1}} \lim_{W \in \mathcal{W}} H^s(\bigcap_{0 \leq i \leq r} U_{\alpha_i,W}, \mathcal{F}_W) \Rightarrow \lim_{W \in \mathcal{W}} H^{r+s}(p(W), \mathcal{F}_W).
\]

Similar arguments then show that we have a spectral sequence

\[
\prod_{\alpha \in \mathcal{J}^{r+1}} \lim_{W \in \mathcal{W}} H^s(\bigcap_{0 \leq i \leq r} U_{\alpha_i,W} \cap q(W), \mathcal{F}_W) \Rightarrow \lim_{W \in \mathcal{W}} H^{r+s}(q(W), \mathcal{F}_W).
\]

Note that the natural morphisms

\[
\prod_{\alpha \in \mathcal{J}^{r+1}} H^s(\bigcap_{0 \leq i \leq r} U_{\alpha_i,W}, \mathcal{F}_W) \to \prod_{\alpha \in \mathcal{J}^{r+1}} H^s(\bigcap_{0 \leq i \leq r} U_{\alpha_i,W} \cap q(W), \mathcal{F}_W)
\]

are compatible with the differentials and hence they induce the natural map

\[
\lim_{W \in \mathcal{W}} H^{r+s}(p(W), \mathcal{F}_W) \to \lim_{W \in \mathcal{W}} H^{r+s}(q(W), \mathcal{F}_W).
\]

The hypothesis (22) shows that this is sufficient to conclude the proof. \( \square \)

Proof. (of Theorem 6.3) We can reduce to when \( S \) is an affine scheme. Indeed, this is a consequence of Lemma 4.1 and Remark 3.20. Let \( \mathcal{J} \) denote the category of standard étale neighbourhoods of \( (S, \mathfrak{m}) \) and let \( (W, \mathfrak{m}) \in \mathcal{J} \). Note that the category \( \mathcal{J} \) is cofiltered. Observe that it suffices to prove (14) when the colimit on the right runs over \( \mathcal{J} \).

We proceed as in [8, Proposition 4.4.3]. We first reduce to the case when \( X \) is affinoid and \( \mathcal{O}_X(X) \) is a Tate ring. Since \( f \) is a quasi-compact morphism and \( S \) is quasi-compact, we get that \( X \) is quasi-compact. Let \( \{U_i\}_{i \in I} \) be a finite covering of \( X \) by affinoid open subsets such that for every \( i \in I \), the morphism \( U_i \to S \) is good. By Remark 3.7, we can assume in addition that \( \mathcal{O}_{U_i}(U_i) \) is a Tate ring. By Lemma 3.8, \( \{U_{i,W} := U_i \times_S W\}_{i \in I} \) is a finite covering of \( X_W \) by affinoid open subsets.
By Lemma 6.15, we deduce that to verify (14), it suffices to prove that for every non-empty subset $J \subseteq I$ and every $q \in \mathbb{N}$, the natural morphism

$$\lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, \bigcap_{j \in J} |U_j, W|), \mathcal{F}) \to \lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, \bigcap_{j \in J} |U_j, W| \cap f_W^{-1}(w)), \mathcal{F})$$

is an isomorphism.

Let $J \subseteq I$ be non-empty. Let $\{V_i\}_{i \in I}$ be a finite affine cover of $\bigcap_{j \in J} U_j$. Hence, $\{\overline{V_i, W}\}_{i \in I}$ is a finite cover of $\bigcap_{j \in J} |U_j, W|$ because $\bigcap_{j \in J} |U_j, W| = \bigcap_{j \in J} |U_j, W|$. This is a consequence of [8, Lemma 1.1.10(i)]. Then, as before, it suffices to show that for every $J' \subseteq I$, the natural morphism

$$\lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, \bigcap_{j' \in J'} |V_{j'}, W|), \mathcal{F}) \to \lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, \bigcap_{j' \in J'} |V_{j'}, W| \cap f_W^{-1}(w)), \mathcal{F})$$

is an isomorphism.

We choose an index $j_0 \in J$. Let $J' \subseteq I_{J'}$. Observe that for every $j' \in J'$, $V_{j'} \subseteq U_{j_0}$. It follows that $\bigcap_{j' \in J'} V_{j'}$ is an affinoid space. Furthermore, by Remark 3.7, the composition $\bigcap_{j' \in J'} V_{j'} \to U_{j_0} \to S$ is good. In summary, it suffices to prove that the natural map

$$\lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, |\overline{U_{j_0}}|), \mathcal{F}) \to \lim_{(W, \overline{w}) \in \mathcal{F}} H^q((X_W, |\overline{U_{j_0}}| \cap f_W^{-1}(s)), \mathcal{F})$$

is an isomorphism where $U$ is an affinoid open subset of $X$ and $U \to S$ is good. In addition, we have that $\mathcal{O}_U(U)$ is a Tate ring.

Observe that $f_U$ is good. We are hence in the context of Situation 6.9. By Lemma 6.11, we can suppose $X$ is affinoid and is the universal compactification of $U \to S$ with $\mathcal{O}_U(U)$ Tate. By Lemma 3.31(3), $f : X \to S$ is tight and $\mathcal{O}_X(X) = \mathcal{O}_U(U)$ is Tate.

Suppose $\mathcal{F} \simeq \lim_i \mathcal{F}_i$ is a filtered colimit of torsion abelian sheaves. By Lemma 3.10 and [11, Exposé VI, Theorem 5.1], we deduce that

$$(R^n f_* \mathcal{F})_{|\mathcal{F}} \simeq \lim_i (R^n f_* \mathcal{F}_i)_{|\mathcal{F}}.$$ 

By Lemma 3.8, $X_W$ is affinoid. Since $w$ is a closed point, $f_W^{-1}(w)$ is a closed subspace of $X_W$. Observe that $f_W^{-1}(w)$ is a closed subspace of a spectral space. It is hence quasi-separated and quasi-compact. By [8, Lemma 2.3.13(i)],

$$H^n(f_W^{-1}(w), \mathcal{F}) \simeq \lim_i H^n(f_W^{-1}(w), \mathcal{F}_i).$$

Just as in (8) of the proof of [8, Proposition 4.4.3], we can reduce to when $\mathcal{F} = p_* G$ where $p : Y \to X$ is a finite morphism of pseudo-adic spaces and $G$ is a constant sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules of finite type for some $m \in \mathbb{N}$.

**Lemma 6.16.** We can reduce to the case where there exists $e_0, \ldots, e_n \in \mathcal{O}_Y(Y)$ such that $\mathcal{O}_Y(Y) = \sum_i e_i \mathcal{O}_Y(Y)$ and $|Y| = \{y \in Y \mid e_i(y) < |e_0(y)| \text{ for } i = 1, \ldots, n\}$.

**Proof.** Since the morphism $p$ is finite, $|Y|$ is closed in $Y$. Hence $|Y| = \bigcap_{i \in \mathcal{I}} C_i$, where $C_i \subseteq Y$ is a closed constructible set where we use the notion of constructible set as introduced in [6, Chapitre 0, Définition 2.3.10]. We claim that we can reduce to when $|Y|$ is closed and constructible. Note that, Indeed, for every $(W, \overline{w}) \in \mathcal{F}$, let $Y_W := Y \times_X X_W$ where we take the fibre product in the category of pseudo-adic spaces. We then have that $|Y_W| = \bigcap_{i \in \mathcal{I}} C_{i,W}$ where for every $i \in \mathcal{I}$, $C_{i,W}$ is the...
preimage of $C_i$ in $Y_W$ for the projection $Y_W \to \underline{Y}$. Since $p$ is finite, we deduce that to show (14), it suffices to show that for every $q \in \mathbb{N}$, the natural morphism
\[
\lim_{(W,[m]) \in \mathcal{F}} H^q((\bigcap_{i \in \mathcal{I}} C_{i,W}), G) \to \lim_{(W,[m]) \in \mathcal{F}} H^q((\bigcap_{i \in \mathcal{I}} C_{i,W} \cap f_{W}^{-1}(w)), G)
\]
is an isomorphism. Since colimits commute with colimits, we deduce from [8, Corollary 2.4.6] that it suffices to verify that for every finite subset $J \subseteq \mathcal{I}$ and $q \in \mathbb{N}$, the natural morphism
\[
\lim_{(W,[m]) \in \mathcal{F}} H^q((\bigcap_{j \in J} C_{j,W} \cap f_{W}^{-1}(w)), G) \to \lim_{(W,[m]) \in \mathcal{F}} H^q((\bigcap_{j \in J} C_{j,W} \cap f_{W}^{-1}(w)), G)
\]
is an isomorphism. We may hence suppose that $[Y]$ is closed and constructible. By [8, Lemma 3.1.10(ii)] and Lemma 6.15, we can conclude a proof of the lemma. □

Note that since $X$ is affinoid, $\underline{Y}$ is affinoid as well. Let $Y' := \underline{Y}$. Let $i$ denote the closed embedding $Y \hookrightarrow Y'$ and set $\mathcal{G} := i_*(G)$. Let $p'$ denote the finite morphism $Y' \to X$.

We claim that it suffices to prove the theorem for the specialization morphism $f \circ p' : Y' \to S$ and the sheaf $\mathcal{G}$. Indeed, observe firstly that by [8, Lemma 1.10.17(i)] the morphisms $i$ and $p'$ are proper. Hence,
\[
(23) \quad \mathcal{F}|_{X_W} \xrightarrow{\sim} p'_W_*(\mathcal{G}|_{Y'_W})
\]
where $p'_W : Y'_W \to X_W$ is the morphism induced by base change. This implies an isomorphism
\[
(24) \quad H^n(X_W, \mathcal{F}|_{X_W}) \xrightarrow{\sim} H^n(Y'_W, \mathcal{G}|_{Y'_W}).
\]
On the other hand, we have a sequence of isomorphisms,
\[
(25) \quad \mathcal{F}|_{f_{W}^{-1}(w)} \xrightarrow{(i)} (p'_W_*(\mathcal{G}|_{Y'_W}))|_{f_{W}^{-1}(w)} \xrightarrow{(ii)} p'_0 |_{f_{W} \circ p'_W} \mathcal{G}|_{(f_{W} \circ p'_W)^{-1}(w)}
\]
where $p'_0 |_{W}$ denotes the finite morphism $(f_{W} \circ p'_W)^{-1}(w) \to f_{W}^{-1}(w)$. The isomorphism (i) is due to equation (23). Note that $p'_W$ is finite and hence proper. Hence, [8, Theorem 4.1.2(b)] implies the isomorphism (ii). Equation (25) gives an isomorphism
\[
(26) \quad H^n(f_{W}^{-1}(w), \mathcal{F}) \xrightarrow{\sim} H^n((f_{W} \circ p'_W)^{-1}(w), \mathcal{G}).
\]
One deduces the claim using equations (24) and (26).

By Lemma 6.6, since $Y'$ is finite over $X$ and $X$ is a universal compactification, the composition $Y' \to X \to S$ is tight. We now apply Proposition 6.12 to conclude a proof of the theorem. □

**Corollary 6.17.** Let $f : X \to S$ be a proper specialization morphism where $S$ is strictly local i.e. $S = \text{Spec}(R)$ where $R$ is a strictly henselian local ring. Let $s \in S$ denote the closed point and $X_s := f^{-1}(s)$ be the fibre over $s$. Let $\mathcal{F}$ be a torsion abelian sheaf. We then have that, for every $n \in \mathbb{N}$,
\[
[R^n f_*(\mathcal{F})]_s \simeq H^n(X_s, \mathcal{F}|_{X_s}).
\]

**Proof.** This is a direct consequence of Theorem 6.3. □

**Corollary 6.18.** Let $f : X \to S$ be a proper specialization morphism. Let $s \in S$ be a closed point and $\pi : S \to \text{Spec}(k)$ be a geometric point over $s$. Let $\mathcal{F}$ be a complex in
$$D^+(X_{\text{et}}, \mathbb{Z})$$ whose cohomology sheaves are torsion. For every $$n \in \mathbb{Z}$$, we have a natural isomorphism

$$[R^n f_*(\mathcal{F})]_\varpi \to \lim_{(W,y)} R^n \Gamma(f_W^{-1}(w), \mathcal{F})$$

where the colimit on the right runs over all étale neighbourhoods $$(W,y)$$ of $$(S,\varpi)$$, $$w \in W$$ is the image of $$\varpi$$, $$X_W := X \times_S W$$ and $$f_W : X_W \to W$$ is the morphism induced by base change.

Proof. Our proof is similar to that of [14, Tag 0DDE]. Let $$(W,y)$$ be an étale neighbourhood of $$(S,\varpi)$$. Observe that we have spectral sequences

$$R^p f_* H^q(F) \Rightarrow R^{p+q} f_*(F)$$

and

$$H^p(f_W^{-1}(w), H^q(F)) \Rightarrow R^{p+q} \Gamma(f_W^{-1}(w), \mathcal{F}).$$

Since $$\lim_{(W,y)}$$ is a filtered colimit, it is exact by [14, Tag 04B0]. Moreover, the functor of taking the stalk at $$\varpi$$ is exact as well. It follows that we have spectral sequences

$$[R^p f_* H^q(F)]_\varpi \Rightarrow [R^{p+q} f_*(F)]_\varpi$$

and

$$\lim_{(W,y)} H^p(f_W^{-1}(w), H^q(F)) \Rightarrow \lim_{(W,y)} R^{p+q} \Gamma(f_W^{-1}(w), \mathcal{F}).$$

We now apply Theorem 6.3 to conclude the proof. □

7. An application

Recall from §4 that we fixed a torsion ring $$A$$. As an application of our work we prove the following theorem concerning type (S) formal schemes (cf. [8, §1.9]). We remind the reader that given a type (S) formal scheme $$X$$, there is the associated reduced scheme $$X_s$$ and the analytic adic generic fiber $$X_\eta$$ (cf. the paragraph immediately preceding Definition 2.17). Moreover there is the specialization morphism $$\lambda_X : X_\eta \to X_s$$ and by Lemma 2.29, there is no ambiguity whether one considers the pushforward along $$(3)$$ or $$(4)$$.

**Theorem 7.1.** Let $$f : X \to \mathfrak{M}$$ be a finite type and separated morphism of type (S) formal schemes. Suppose in addition that $$\mathfrak{M}$$ is quasi-compact and quasi-separated. Then there is a canonical equivalence

$$R^+ f_! \circ R^+ \lambda_X_* \sim \circ R^+ \lambda_\mathfrak{M}_* \circ R^+ f_!$$

of functors $$D^+(X_\eta, A) \to D^+(\mathfrak{M}_s, A)$$.

The idea for proving Theorem 7.1 is simple. One takes the compactification of the morphism $$f_\eta$$ in the sense of [8, Definition 5.1.8] and reduces Theorem 7.1 to the case of an open immersion, where the result is known by Corollary 3.5.11(ii) in loc.cit. However the compactification of $$f_\eta$$ may no longer be of finite type (in particular may not possess a formal model). Thus the strategy is use the enlarged class of specialization morphisms we have constructed and their compactifications (cf. Theorem 3.30). First we need some preparation.

**Situation 7.2.** Let

$$\begin{array}{cccc}
X & \xrightarrow{j_1} & Y \\
\alpha & & \beta \\
S & \xrightarrow{j_2} & T
\end{array}$$
be a commutative diagram where $\alpha$ and $\beta$ are proper specialization morphisms, and $j_1$ and $j_2$ are open embeddings.

**Lemma 7.3.** We work within the context of Situation 7.2. There is a natural equivalence

$$j_2 j_*^* R^+ \beta_* j_1! \to R^+ \beta_* j_1!.$$  

**Proof.** By adjointness, we have a natural morphism

$$\Phi: j_2 j_*^* R^+ \beta_* j_1! \to R^+ \beta_* j_1!.$$  

Observe that it suffices to show that if $F \in D^+(X_{\text{ét}}, A)$, $t \in T \smallsetminus S$ and $n \in \mathbb{Z}$ then

$$[R^n \beta_* j_1!(F)]_t = 0.$$  

By Corollary 6.18, for every $n \in \mathbb{Z}$, we have an isomorphism

$$(27) \quad [R^n \beta_* j_1!(F)]_t \sim \llim_{(W, w)} R^n \Gamma(\beta^{-1}_W(w), j_1!(F))$$

where the colimit on the right runs over all étale neighbourhoods $(W, w)$ of $(T, t)$. Let $(W, w)$ be an étale neighbourhood of $(T, t)$ and let $X_W$ be the preimage of $X$ for the morphism $Y_W \to Y$. If $j_1 W$ denotes the open embedding $X_W \to Y_W$ then by [8, Theorem 5.2.2(iv)], we have an isomorphism

$$(j_1!(F))_{Y_W} \sim j_1 W!(F_{X_W}).$$

Since $\beta^{-1}_W(w) \cap X_W = \emptyset$, we see that $R^n \Gamma(\beta^{-1}_W(w), j_1!(F)) = 0$. This concludes the proof. \[\square\]

**Corollary 7.4.** We work within the context of Situation 7.2.

1. Let $I$ be an injective $A$-module on $X_{\text{ét}}$. Then, for every $n \geq 1$,

$$R^n \beta_* j_1!(I) = 0.$$  

2. There is a natural equivalence

$$R^+ \beta_* j_1! \sim j_2 \circ R^+ \alpha_*.$$  

**Proof.** We verify part (1) as follows. Let $Z$ be the fibre product $Y \times_T S$, which exists by Proposition 2.19. Consider the commutative diagram coming from the universal property of fibre products

$$\begin{array}{ccc}
X & \xrightarrow{j_3} & Z & \xrightarrow{j_4} & Y \\
\alpha & & \gamma & & \beta \\
S & \xrightarrow{j_2} & T.
\end{array}$$

Note that since $j_1$ and $j_4$ are étale, so is $j_3$ by [8, Proposition 1.6.7(iii)]. By Remark 3.20, since $\beta$ is proper, $\gamma$ is proper. Since $\alpha$ is proper as well, it follows that $j_3$ is both quasi-compact and separated (by similar arguments appearing in the proof of Lemma 5.2).

Then

$$j_2^* \circ R^+ \beta_* \circ j_1! \overset{(i)}{=} R^+ \gamma_* \circ j_1! \overset{(ii)}{=} R^+ \gamma_* \circ j_3! \overset{(iii)}{=} R^+ \alpha_*.$$  

(28)
where (i) follows from Lemma 4.1, (ii) follows from \( j_1 = j_4 \circ j_3 \) and (iii) follows from Lemma 5.3 after noting that \( j_3 \) is quasi-compact because \( \alpha \) is quasi-compact and \( \gamma \) is quasi-separated. Hence, we get that for every \( n \geq 1 \),

\[
j_2^* R^n \beta_* j_{1!} (I) = R^n \alpha_*(I) = 0.
\]

It remains to show that \( R^n \beta_* j_{1!} (I) \) vanishes outside of \( S \). This is an immediate consequence of Lemma 7.3. Hence, \( R^n \beta_* j_{1!} (I) = 0 \) everywhere.

We now verify part (2). Recall that we have a natural morphism \( j_! \to j_* \) which implies a natural map \( \beta_* j_1^! \to \beta_* j_1 \). Since \( \beta \circ j_1 = j_2 \circ \alpha, \beta_* j_1 = j_2 \circ \alpha_* \) and hence we have a natural morphism

\[
\beta_* j_{11} \to j_2^* \alpha_*. 
\]

Applying \( j_2^! j_2^* \), we get a morphism

\[
j_2^! j_2^* \beta_* j_{1!} \to j_2^! j_2^* j_2^* \alpha_*. 
\]

By Lemma 7.3, we get a natural morphism

\[
\epsilon: \beta_* j_{1!} \to j_2^* \alpha_*. 
\]

We claim that \( \epsilon \) is an equivalence. Indeed, observe that Equation (28) shows that this is true over \( S \) while Lemma 7.3 shows that this true outside of \( S \). Part (1) of the lemma implies that \( R^+ (\beta_* j_{1!}) = R^+ \beta_* j_{11} \). Since \( \epsilon \) is a natural equivalence, we have that the induced morphism

\[
R^+ \beta_* j_{11} \to j_2^! R^+ \alpha_* 
\]

is an isomorphism.

\[\square\]

**Corollary 7.5.** Let \( \alpha: X \to S \) be a separated and finite type specialization morphism. Let \( g: S \to T \) be a separated and finite type morphism of schemes. Then \( g \circ \alpha \) is a separated and finite type specialization morphism. Moreover, if \( T \) is quasi-compact and quasi-separated, there is a natural equivalence

\[
R^+ (g \circ \alpha)_! \simeq R^+ g_! \circ R^+ \alpha_! 
\]

of functors \( D^+ (X_{\text{ét}}, A) \to D^+ (T_{\text{ét}}, A) \).

**Proof.** The strategy is very similar to the proof of Lemma 5.3. The fact that the composition \( g \circ \alpha \) is a separated and finite type specialization morphism follows from Proposition 3.22. For the second part consider the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j_1} & X^S \\
\downarrow \alpha & & \downarrow j_2 \\
S & \xrightarrow{\beta_1} & Z \\
\downarrow j_3 & & \downarrow \beta_2 \\
U & \xrightarrow{g} & T \\
\end{array}
\]

where the top left triangle is the universal compactification of \( \alpha \) via Theorem 3.30, the bottom right triangle is a compactification of \( g \) via [3, Theorem 4.1]. Let us explain the top right square. The specialization morphism \( j_3 \circ \beta_1: X^S \to U \) is separated and of finite type by Proposition 3.22. Thus again by Theorem 3.30, it admits a universal compactification which we denote by the triple \((Z, \beta_2, j_2)\). In particular all the horizontal arrows (in the diagram) are quasi-compact open embeddings and the vertical arrows are proper. Therefore the composition \( j_2 \circ j_1 \) is a quasi-compact open embedding of analytic adic spaces and by Proposition 3.22,
the composition \( h \circ \beta_2 \) is a proper specialization morphism. Thus the outer triangle is a compactification of \( g \circ \alpha \).

Then
\[
R^+ g_! \circ R^+ \alpha_! = R^+ h_* \circ j_{3!} \circ R^+ \beta_1^* \circ j_{1!}
\]
and
\[
R^+ (g \circ \alpha)! = (i) \Rightarrow R^+ (h \circ \beta_2)_* \circ (j_2 \circ j_1)! \\
(ii) \Rightarrow R^+ h_* \circ R^+ \beta_2^* \circ j_{3!} \circ j_{1!} \\
(iii) \Rightarrow R^+ h_* \circ j_{3!} \circ R^+ \beta_1^* \circ j_{1!}
\]
where (i) follows from Lemma 5.2, (ii) follows from Proposition 2.28 and [8, Theorem 5.4.3], and (iii) follows from Corollary 7.4. This completes the proof. \(\square\)

We now arrive at the promised application of the theory developed thus far.

**Proof.** (of Theorem 7.1) We have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{X}_\eta & \xrightarrow{\lambda_X} & \mathcal{X}_s \\
\downarrow^f & & \downarrow^{f_*} \\
\mathcal{Y}_\eta & \xrightarrow{\lambda_Y} & \mathcal{Y}_s
\end{array}
\]
where we remind the reader that \( \lambda_X \) (resp. \( \lambda_Y \)) are the specialization morphisms, in the sense of Definition 2.17 induced by the morphisms of locally ringed spaces \( (\mathcal{X}_\eta, \mathcal{O}_{\mathcal{X}_\eta}^+) \rightarrow (\mathcal{X}, \mathcal{O}_X) \) (resp. \( (\mathcal{Y}_\eta, \mathcal{O}_{\mathcal{Y}_\eta}^+) \rightarrow (\mathcal{Y}, \mathcal{O}_Y) \)), cf. §2. The composition \( f_\ast \circ \lambda_X = \lambda_Y \circ f_\ast \) is a separated and finite type specialization morphism (because both \( \lambda_X \) and \( \lambda_Y \) are proper by Proposition 3.24, and both \( f_\ast \) and \( f_\ast \) are separated and of finite type by assumption on \( f \)). Thus by Theorem 3.30 it admits a (universal) compactification. By Definition 5.1/Lemma 5.2, we have a well defined \( R^+ (f_\ast \circ \lambda_X)! = R^+ (\lambda_Y \circ f_\ast)! \) right derived lower shriek functor. We compute
\[
R^+ (f_\ast \circ \lambda_X)! = (i) \Rightarrow R^+ f_{3!} \circ R^+ \lambda_X! \\
(ii) \Rightarrow R^+ f_{1!} \circ R^+ \lambda_X!
\]
where (i) follows from Corollary 7.5, (ii) follows from the fact that \( \lambda_X \) is proper (cf. Proposition 3.24).

Similarly one obtains \( R^+ (\lambda_Y \circ f_\ast)! = R^+ \lambda_Y^* \circ R^+ f_{1!} \) where one uses Lemma 5.3 in place of Corollary 7.5. The result now follows. \(\square\)

We end this section with the following remark.

**Remark 7.6.** Let \( \alpha: X \rightarrow S \) be a separated, taut and locally of finite type specialization morphism. Then as in Definition 5.1, one can define a functor \( R^+ \alpha_! \) with the expected properties. Moreover if \( X \) satisfies some finiteness conditions (e.g. \( X \) is locally of finite type over an algebraically closed valued field \( k \)), then the functor
\[
R^+ \alpha_! : \mathcal{D}^+(X_{\text{ét}}, A) \rightarrow \mathcal{D}^+(S_{\text{ét}}, A)
\]
admits a right adjoint functor
\[
R^+ \alpha^* : \mathcal{D}^+(S_{\text{ét}}, A) \rightarrow \mathcal{D}^+(X_{\text{ét}}, A).
\]
This places [5, Corollary 4.3(1)] into a larger framework.
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Ildar Gaisin
Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro, Tokyo, 153-0041, Japan.
*email : ildar@ms.u-tokyo.ac.jp*

John Welliaveetil
Kavli Institute for the Physics and Mathematics of the Universe, The University of Tokyo,
5-1-5 Kashiwanoha Kashiwa, 277-8583, Japan
*email : welliaveetil@gmail.com*