Difference between minimum light numbers of sigma-game and lit-only sigma-game *

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Abstract

A configuration of a graph is an assignment of one of two states, on or off, to each vertex of it. A regular move at a vertex changes the states of the neighbors of that vertex. A valid move is a regular move at an on vertex. The following result is proved in this note: given any starting configuration $x$ of a tree, if there is a sequence of regular moves which brings $x$ to another configuration in which there are $\ell$ on vertices then there must exist a sequence of valid moves which takes $x$ to a configuration with at most $\ell + 2$ on vertices. We provide example to show that the upper bound $\ell + 2$ is sharp. Some relevant results and conjectures are also reported.

Keywords: Appropriate vertex; Sigma game; Lit-only sigma game; Tree

1 Sigma-game

We consider graphs without multiple edges but may have loops. That is, for any graph $G$ with vertex set $V(G)$, its edge set $E(G)$ is a subset of $\binom{V(G)}{2} \cup V(G)$ and we say that there is a loop at a vertex $v$ of $G$ provided $\{v\} \in E(G)$. Two vertices $u$ and $v$, possibly equal, are adjacent in $G$ if $\{u\} \cup \{v\} \in E(G)$. The neighbors of $v$ in $G$, denoted $N_G(v)$, is the set of vertices adjacent to $v$ in $G$. For any $v \in V(G)$, $\chi_v \in \mathbb{F}_2^{V(G)}$ stands for the binary function for which $\chi_v(u) = 1$ if $u = v$ and $\chi_v(u) = 0$ otherwise. For any $S \subseteq V(G)$, we set $\chi_S$ to be $\sum_{v \in S} \chi_v \in \mathbb{F}_2^{V(G)}$.

The so-called sigma-game on a graph $G$ is a solitaire combinatorial game widely studied in the literature [1, 3, 4, 5, 8, 9, 11, 12, 16, 17, 18, 19, 20].

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Let us call each element of $F_2^{V(G)}$ a configuration of $G$. We can think of a configuration $x$ as an assignment of one of two states, on or off, to the vertices of $G$ such that $v$ is on if $x(v) = 1$ and $v$ is off if $x(v) = 0$. The light number of a configuration $x$, written as $L(x)$, refers to the number of vertices which are assigned the on state by $x$. Given any configuration $x$, a move of the sigma-game is to pick a vertex $v$ and toggle the states of all its neighbors between on and off, namely to transform the configuration $x$ into $x + \chi_{N_G(v)}$. The goal of the sigma-game is to transform a given configuration by repeated moves to some configuration which is as good as possible according to certain criterion, say minimizing the light number of the configuration.

If we restrict the moves of the sigma-game at on vertices only, then we come to the lit-only sigma-game. That is, when $v$ is off, an invalid move at $v$ keeps the configuration unchanged; when $v$ is on, a valid move at $v$ transforms the configuration $x$ into $x + \chi_{N_G(v)}$. In all, a move of the lit-only sigma-game at $v$ transforms the configuration $x$ into $x + x(v)\chi_{N_G(v)} = x(I + \chi_v^\top \chi_{N_G(v)})$, where $I$ is the identity matrix. The introduction of the lit-only restriction makes the sigma-game harder to analyze and leads to an even richer mathematical structure.

For $x, y \in F_2^{V(G)}$, we write $x \rightarrow_G y$ to mean that $x$ can be transformed to $y$ by successive moves in the sigma-game on $G$. Correspondingly, we write $x \leftarrow_G y$ if we can go from $x$ to $y$ by applying a sequence of valid moves in the lit-only sigma-game on $G$. Note that we often drop the subscript $G$ from the notation if it is clear from the context. When we consider both sigma-game and lit-only sigma-game, we often call the moves in the sigma-game regular moves to distinguish it from the valid/invalid moves in the lit-only sigma-game.

The sigma-game is invertible, namely $x \rightarrow y$ if and only if $y \rightarrow x$, and the order in which we execute the moves is irrelevant. Indeed, $x \rightarrow y$ if and only if $x - y$ lies in the abelian group generated by $\chi_{N_G(v)}$, $v \in V(G)$. To study the sigma-game is just to study the action of this abelian group on $F_2^{V(G)}$.

In general, the lit-only sigma-game may be unilateral, namely $x \rightarrow y$ does not imply $y \rightarrow x$, and the order of moves is significant. In other words, the study of lit-only sigma-game is a study of the action of a semigroup, which is rarely abelian, on $F_2^{V(G)}$. As a trivial example of non-invertibility, consider the graph $G$ with $E(G) = V(G) \cup (V(G)^2)$ and we can easily find that $1 = \chi_{V(G)} \rightarrow 0$ and that $0$ cannot go anywhere else in the lit-only sigma game. Clearly, the existence of loops causes the intricate issue of non-invertibility for the lit-only sigma game. On the other hand, if $G$ has no loops, then it holds for all $v \in V(G)$ that $(I + \chi_v^\top \chi_{N_G(v)})^2 = I$, implying that the lit-only sigma-game is invertible in this case, and $\{y \in F_2^{V(G)} \mid x \rightarrow y\}$ forms an orbit under the action of the group $H$ generated by $\{I + \chi_v^\top \chi_{N_G(v)} : v \in V(G)\}$.

We will elaborate more carefully in Section 2 on the influences of the lit-only
restriction to the sigma-game and present proof details in Section 3 to confirm some of those described influences.

2 Influences of the lit-only restriction

Definition 1. Let $x \in \mathbb{F}_2^{V(G)}$.

- $ML_G(x) = \min_{x \rightarrow y} L(y)$ is called the minimum light number of $x$ for the sigma-game on $G$.
- $ML^*_G(x) = \min_{x \rightarrow y} L(y)$ is called the minimum light number of $x$ for the lit-only sigma-game on $G$.
- $ML(G) = \max_{x \in \mathbb{F}_2^{V(G)}} ML_G(x)$ is called the minimum light number of the sigma-game on $G$.
- $ML^*(G) = \max_{x \in \mathbb{F}_2^{V(G)}} ML^*_G(x)$ is called the minimum light number of the lit-only sigma-game on $G$.

To understand the influences of the lit-only restriction, a basic question to answer is the following.

Problem 2. Suppose $x, y \in \mathbb{F}_2^{V(G)}$, $x \rightarrow y$ and $L(y) = ML_G(x)$. When can we conclude that $x \rightarrow y$? How large can $ML^*_G(x) - ML_G(x)$ be? How large can $ML^*(G) - ML(G)$ be?

It is obvious that

$$ML_G(x) \leq ML^*_G(x), ML(G) \leq ML^*(G). \quad (1)$$

We flesh out a bit the above general observation by presenting some examples, which say that both $ML^*_G(x) - ML_G(x)$ and $ML^*(G) - ML(G)$ can be arbitrarily large and both equalities in display (1) hold for infinitely many graphs.

Example 3. [21] Let $G = K_{m,m,m}$ be the complete tripartite graph, namely $V(G) = \{v_{ij} : i = 1, 2, 3, j = 1, 2, \ldots, m\}$ and $E(G)$ consists of all those pairs $v_{ij}v_{k\ell}$ where $i \neq k$. Let $x$ be the configuration of $G$ such that $x(v_{ij}) = 0$ if and only if $i = 1$. Then, we have

$$ML_G(x) = 0, \quad ML^*_G(x) = 2m, \quad ML(G) = \left\lceil \frac{3m}{2} \right\rceil, \quad ML^*(G) = 2m.$$ 

Note that

$$ML^*(x) - ML(x) = 2m = \frac{2}{3}|V(G)|, \quad ML^*(G) - ML(G) = \frac{m}{2} = \frac{|V(G)|}{6}.$$
With Example 3 in mind, it was conjectured that if $G$ is a graph of order $n$ with no isolated vertices, then $ML^*(G) - ML(G) \leq \left\lceil \frac{n}{6} \right\rceil$ [21, Conjecture 4]. The next result is a counterexample to this conjecture.

**Example 4.** Let $G = K_{2m}$ be the complete graph without loops on $2m$ vertices, and $x$ be any element of $\mathbb{F}_2^V(G)$ with $L(x) = m$. Then we have

$$ML(x) = ML(G) = 0, \quad ML^*(x) = ML^*(G) = m.$$ 

Hence

$$ML^*(G) - ML(G) = ML^*(x) - ML(x) = m = \frac{1}{2}|V(G)|.$$

**Example 5.** Let $G$ be a graph with loops everywhere. As an easy consequence of [22, Theorem 3], we know that $ML^*_G(x) = ML_G(x)$ is valid for any configuration $x$ of $G$ and hence $ML^*(G) = ML(G)$.

Besides the above result for graphs with loops everywhere, there are results for trees from which one can also see that the lit-only restriction does not matter too much. The degree of a vertex $v$ in a graph $G$ is defined to be the number of edges in $E(G) \setminus V(G)$ which contains $v$ and we will use the notation $\deg_G(v)$ for it. A vertex of degree one is said to be a leaf.

**Example 6.** [36, 37] Let $G$ be any tree, $G'$ be a graph obtained from $G$ by adding some loops, and $\ell$ the number of leaves of $G$. If $\ell \geq 2$, then $ML(G') \leq \lfloor \ell/2 \rfloor$ and $ML^*(G) \leq \lfloor \ell/2 \rfloor$. Both equalities can be attained.

Note that in Example 6 we do not directly compare the difference between the minimum light numbers of the sigma-game and the lit-only sigma-game, which is an object of interest posed in both [22, §1.3] and [21, Question 3]. Let us make the following two conjectures regarding it here, the second of which being motivated by Example 4.

**Conjecture 7.** Let $G$ be obtained from a tree by adding some loops. Then $ML^*(G) - ML(G) \in \{0, 1\}$.

**Conjecture 8.** It holds for any graph $G$ that $ML^*(G) - ML(G) \leq \frac{1}{2}|V(G)|$.

Suppose that $y, z$ are two configurations of a graph $G$ such that $ML^*(G) = ML^*_G(z)$ and $ML(G) = ML_G(y)$. Since $ML_G^*(z) - ML_G(y) \leq ML^*_G(z) - ML_G(z)$, we infer that

$$0 \leq ML^*(G) - ML(G) \leq \max_{x \in \mathbb{F}_2^V(G)} (ML^*_G(x) - ML_G(x)). \quad (2)$$

This suggests that, instead of tackling Conjecture 7 and/or Conjecture 8 directly, we may first try to find an upper bound for $ML^*_G(x) - ML_G(x)$ for any configuration $x$ of some special graph $G$. 

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Let $G$ be a graph obtained from a tree by attaching some loops and $x$ a configuration of $G$. According to Example 5, $ML^*_G(x) - ML_G(x)$ will take value 0 if $G$ has loops everywhere. We proceed to give two more examples to show that it is possible that $ML^*_G(x) - ML_G(x)$ takes value 1 or 2 as well. To demonstrate a configuration, we will draw the underlying graph and use a bullet to indicate an on vertex and use a circle for an off vertex.

![Figure 1: $ML^*(x) - ML(x) = 1$](image1)

**Example 9.** Let $x$ be the configuration depicted in Fig. 1. A computer search demonstrates that $ML_G(x) = 1$ and $ML^*_G(x) = 2$.

![Figure 2: $ML^*(x) - ML(x) = 2$](image2)

**Example 10.** Let $x$ be the configuration depicted in Fig. 2. Then, $ML(x) = 0, ML^*(x) = 2$ [7, p. 299].

![Figure 3: A graph with two pendant paths](image3)

The main idea of our current approach towards tackling Problem 2 is reflected in the result below.

**Theorem 11.** Let $G_1$ be a connected graph with $v \in V(G_1)$. Let $G$ be the graph obtained from $G_1$ by adding a set of new vertices $S = \{v_{11}, \ldots, v_{1n_1}, v_{21}, \ldots, v_{2n_2}\}$, $n_1, n_2 \geq 1$, and adding a set of new edges
as well as some loops inside $S$; see Fig. 3. For any $x \in \mathbb{F}_2^{V(G)}$, we will have $ML^*_G(x) - ML_G(x) \leq 2$ provided either $\max(n_1, n_2) \geq 2$ or $G$ has a loop at either $v_{11}$ or $v_{21}$, or $x(v_{11}) \neq x(v_{21})$.

In light of (2), what comes next may be viewed as a partial support to Conjecture 7.

**Theorem 12.** If $G$ is obtained from a tree by adding some loops, then it holds $ML^*_G(x) - ML_G(x) \leq 2$ for any $x \in \mathbb{F}_2^{V(G)}$ and the upper bound 2 is sharp.

Let $G$ be obtained from a tree by adding some loops and let $x \in \mathbb{F}_2^{V(G)}$. We point out that Mu Li designed a polynomial algorithm to derive $ML^*_G(x)$ and some relevant complexity results for determining $ML^*_G(x)$ for tree-like graphs $G$ can be found in [16, 32]. For our present purposes, it is pertinent to ask what is the complexity of calculating $ML^*_G(x)$.

For any graph $G$ and $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted $G[S]$. We say that $v$ is a branch vertex of $G$ if $\deg_G(v) \geq 3$. Going through the procedure of establishing Theorems 11 and 12, it will be not hard to realize the following result about a graph obtained by ‘planting’ a tree with at least 3 branch vertices on a connected graph. Note that the graph $G$ treated in Theorem 11 can be viewed as a graph obtained by ‘planting’ a path on the connected graph $G_1$.

**Theorem 13.** Let $G$ be a graph with $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \{v\}$. Suppose that $G[V_1]$ is a tree with some loops attached and contains at least three branched vertices of itself and $G[V_2]$ is connected. If there is no edge in $G$ which intersects both $V_1 \setminus \{v\}$ and $V_2 \setminus \{v\}$, then it holds $ML^*_G(x) - ML_G(x) \leq 2$ for any $x \in \mathbb{F}_2^{V(G)}$.

Employing similar idea as in the proof of preceding theorems but with more technical details, we can carry out the proof of the following two results, which will be reported in a subsequent paper [38].

**Theorem 14.** If $G$ is unicyclic, then it holds $ML^*_G(x) - ML_G(x) \leq 3$ for any $x \in \mathbb{F}_2^{V(G)}$. This bound is tight.

**Theorem 15.** If $G$ is obtained from a grid graph by adding some loops, then $ML^*_G(x) - ML_G(x) \leq 3$ for any configuration $x$.

Theorems 12, 14 and 15 stimulate us to set forth the following conjecture.

**Conjecture 16.** For any graph $G$ and any $x \in \mathbb{F}_2^{V(G)}$, it holds $ML^*_G(x) - ML_G(x) \leq \max_{v \in V(G)} \deg_G(v)$. 
3 Proofs

This concluding section is devoted to proofs of Theorems 11, 12 and 13.

Lemma 17. Let $G$ be a connected graph and $0 \neq x \in \mathbb{F}_2^{V(G)}$. Suppose $a$ and $b$ are two vertices of $G$ satisfying $N_G(a) \neq N_G(b)$. Then, there is $y \in \mathbb{F}_2^{V(G)}$ such that $y(a) \neq y(b)$ and $x \xrightarrow{*} G y$.

Proof. Without loss of generality, assume that there is $c \in N(a) \setminus N(b)$. If $x(c) = 1$, then a valid move at $c$, if necessary, will bring us to a configuration where the states of $a$ and $b$ are different. Consequently, it remains to show that there is $z \in \mathbb{F}_2^{V(G)}$ such that $z(c) = 1$ and $x \xrightarrow{*} G z$.

Since $0 \neq x$, we can take $d \in V(G)$ such that $x(d) = 1$. Choose a shortest path connecting $d$ and $c$ in $G$, say $w_0w_1 \cdots w_t$, where $w_0 = d$ and $w_t = c$. If $x(c) = 1$, it suffices to put $x = z$. Otherwise, let $q$ be the largest integer less than $t$ such that $x(w_q) = 1$. A sequence of valid moves at $w_q, w_{q+1}, \ldots, w_{t-1}$ transforms $x$ to a configuration $z$ satisfying $z(c) = 1$, finishing the proof.

Lemma 18. Let $G$ be a graph, $a, b \in V(G)$, $ab \notin E(G)$, $c \in N_G(a) \cap N_G(b)$. Let $S \subseteq V(G) \setminus (N_G(a) \cup N_G(b))$ such that $G[S \cup \{c]\} is connected. Assume that $x$ and $y$ are two configurations of $G$ such that $x \xrightarrow{G} y$. If $x(a) \neq x(b)$, then there exists $R \subseteq V(G) \setminus (S \cup \{c\}) such that $x \xrightarrow{*} G y + \sum_{v \in R} \chi_N(v)$.

Proof. Our goal is to show that for any finite set $T \subseteq S \cup \{c\}$, there exists $R' \subseteq V(G) \setminus (S \cup \{c\}) such that $x \xrightarrow{*} G x + \sum_{v \in T \cup R'} \chi_N(v)$. This can be accomplished by inductively appealing to the following claim: let $d \in S \cup \{c\}$ be the vertex whose distance to $c$ in the graph $G[S \cup \{c\}] is largest, say $D$, and let $S'$ be the set of those vertices in $S \cup \{c\}$ which have a distance less than $D$ to $c$ in $G[S \cup \{c\}]$, then we can find $U \subseteq S' \cup \{a, b\}$ such that $x \xrightarrow{*} G w$ where $w = x + \chi_N(d) + \sum_{v \in U} \chi_N(v)$ and $w(a) \neq w(b)$. To establish this claim, we choose a shortest path in $G[S \cup \{c\}] which connects $c$ and $d$, say $v_1v_2 \cdots v_{D+1}$ where $v_1 = c$ and $v_{D+1} = d$. Let us refer to the only on vertex among $\{a, b\}$ in the configuration $x$ as $v_0$ and let $t$ be the largest integer no greater than $D + 1$ such that $x(v_t) = 1$. What is left to do is to distinguish two cases.

CASE 1: Either $t > 0$ or there is no loop at $v_0 in G$. It is easy to check that the valid moves at $v_t, v_{t+1}, \ldots, v_{D+1}$ in that order transforms $x$ to the required $w$.

CASE 2: There is a loop at $v_0 in G$ and $t = 0$. The sequence of valid moves at $v_0, v_1, \ldots, v_{D+1}, v_0 successively is what we want.

We now arrive at the key ingredient in our proofs of Theorems 11, 12, 13, 14 and 15.
Lemma 19. Let $G$ be a graph, $a, b \in V(G)$, $ab \notin E(G)$, $c \in N_G(a) \cap N_G(b)$ and $N_G(a) \neq N_G(b)$. Let $S \subseteq V(G) \setminus (N_G(a) \cup N_G(b))$ such that $G[S \cup \{c\}]$ is connected. Assume that $x$ and $y$ are two configurations of $G$ such that $x \rightarrow_G y$. If $x \neq 0$, then there exists $R \subseteq V(G) \setminus (S \cup \{c\})$ such that $x \rightarrow_G y + \sum_{v \in R} \chi_{N(v)}$.

Proof. Lemma 17 in conjunction with Lemma 18 gives this result.

Lemma 20. If $G$ is obtained from a path $v_1v_2\ldots v_n$ by adding some loops, then any configuration $x$ of $G$ can be transformed to a configuration with light number at most one by a series of valid moves inside $\{v_2, \ldots , v_n\}$.

$G$:

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    v1 --v2 --v3 --v4 --v5 --v6
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Proof. Let $t_y = \infty$ if $y = 0$ and $t_y = \min\{t : y(v_t) = 1\}$ for any $y \in \mathbb{F}_2^{V(G)} \setminus \{0\}$. Let $\mathcal{C}$ be the set of configurations for which we can reach from $x$ by applying a series of valid moves inside $\{v_2, \ldots , v_n\}$. Choose a configuration $y$ from $\mathcal{C}$ whose $t_y$ is as large as possible. It suffices to deduce that $L(y) \leq 1$. Assuming otherwise, let $L(y) > 1$, then there is $t > t_y$ such that $y(v_t) = 1$ and $y(v_q) = 0$ for any $t_y < q < t$. Now a series of valid moves at $v_t, v_{t-1}, \ldots , v_{t+1}$ transforms $y$ into another member $y'$ of $\mathcal{C}$ with $t_y' > t_y$, yielding a contradiction.

Our proof of Theorem 11 rests on Lemmas 18, 19 and 20.

Proof of Theorem 11. If $x = 0$, then $ML^*_G(x) = ML_G(x) = 0$ and hence we are done. Now consider $x \neq 0$. Choose $y$ such that $x \rightarrow_G y$ and $L(y) = ML_G(x)$. Taking $a = v_{11}, b = v_{21}$ and $c = v$, we deduce from Lemmas 19 (Lemma 18) that there exists $z \in \mathbb{F}_2^{V(G)}$ such that $x \rightarrow_G z = y + \sum_{v \in R} \chi_{N(v)}$ where $R$ is a subset of $S$. This means that $z(u) = y(u)$ for all $u \in \mathbb{F}_2^{V(G)} \setminus (S \cup \{v\})$. Therefore, the result will follow if we can show that there exists a series of valid moves inside $S$ which transforms $z$ to $z'$ where $z'$ is a configuration with at most two on vertices among $S \cup \{v\}$. This last step is completed by making use of Lemma 20 on $G[\{v, v_{11}, \ldots , v_{1n_1}\}]$ and $G[\{v, v_{21}, \ldots , v_{2n_2}\}]$, respectively.

Remark 21. In much the same vein as the above proof of Theorem 11, we can prove the following extra claim where all undefined parameters are as described in Theorem 11. Putting $x_1$ to be the restriction of $x$ on $V(G_1)$, we have

$$ML^*_G(x) \leq ML_{G_1}(x_1) + 2$$

provided either $\max(n_1, n_2) \geq 2$ or $G$ has a loop at either $v_{11}$ or $v_{21}$, or $x(v_{11}) \neq x(v_{21})$.

We are going to establish Theorem 12 in the sequel. To do that, we need to develop some special facts on trees.

Let $G$ be any graph and $v \in V(G)$. We call $v$ an end vertex of $G$ if $\deg_G(v) \leq 1$. A subset of $V(G)$ is good for $G$ if it does not contain any branch
vertex of $G$. The vertex $v$ of $G$ is appropriate\[2, 33\] provided there are at least two (connected) components of $G[V(G) \setminus \{v\}]$ which are good for $G$. Here is a very intuitive result, which may be traced back to a paper of Nylen \[33\].

**Lemma 22.**\[33, Lemma 3.1\] Let $H$ be obtained from a tree by attaching some loops. Suppose $H$ has at least two vertices. Then there exists a path in $H$ such that the path contains two different leaves of $H$ and contains at most one branch vertex of $H$.

**Proof.** The result is trivial if $H$ does not contain any branch vertex. For the remaining case, consider the graph $G = H[S]$ where $S$ is the inclusion-wise smallest subset of $V(H)$ which contains all branch vertices of $H$ and makes $H[S]$ connected. Note that $G$ definitely has some end vertex $v$. It is easy to see that all components of $H[V(H) \setminus \{v\}]$ other than that which contains $S$ are good for $H$ and there are at least two such components. This says that $v$ is an appropriate vertex of $H$ and the required path can be obtained by combining $v$ with any two components of $H[V(H) \setminus \{v\}]$ which are good for $H$.

**Remark 23.** Keep the assumption on $H$ as in Lemma 22 and let $u$ be a leaf of $H$. The above proof of Lemma 22 could be extended a bit more to yield an inductive proof of the following still ‘obvious’ fact: if $H$ has at least three leaves, then the path asserted in Lemma 22 can be further required to avoid $u$.

For any two integers $n, k \geq 1$, let $P_{n,k}$ be the graph with vertex set \{v_1, v_2, \ldots, v_n, w_1, \ldots, w_k\} and edge set \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nw_1, \ldots, v_nw_k\}; see Fig. 4. We refer to any graph obtained from $P_{n,k}$ by attaching some loops as a rake with $k$ teeth $w_1, \ldots, w_k$ and an $n$-handle $v_1, \ldots, v_n$. We call the vertex $v_1$ the top of the rake and the other vertices the common vertices. When $k = 1$, $P_{n,k}$ is just a path of length $n$ one of whose end vertices is specified as the top. In preparation for our proof of Theorem 12, we prove the following simple fact, whose role will be similar to that of Lemma 20 in proving Theorem 11.

**Lemma 24.** Let $G$ be a rake with $k$ teeth and an $n$-handle; see Fig. 4. For any configurations $x, y$ of $G$, if there is a sequence of regular moves inside the common vertices of $G$ which brings $x$ to $y$, then there exists $z \in \mathbb{F}_2^V(G)$ satisfying $L(z) \leq L(y) + 1$ and a series of valid moves inside the common vertices which transforms $x$ to $z$.

![Figure 4: A rake with $k$ teeth and an $n$-handle](image-url)
Proof. For any set $S \subseteq V(G)$, denote by $d(S)$ the minimum one among all distances in $G$ between the top $v_1$ and elements of $S$. For any $u \in V(G) \setminus \{v_1\}$, put $\overrightarrow{u}$ to be the set of vertices other than $u$ in the shortest path connecting $u$ and $v_1$, $f(u)$ be the unique vertex adjacent to $u$ falling in $\overrightarrow{u}$, and set $\overrightarrow{u}$ to be $V(G) \setminus \overrightarrow{u}$.

Suppose that by valid moves inside $V(G) \setminus \{v_1\}$ we get to $x' = y + \sum_{v \in S_{x'}} \chi_{N(v)}$ from $x$ for some set $S_{x'} \subseteq V(G) \setminus \{v_1\}$. There are three cases to consider. If $S_{x'}$ is empty, the proof is completed by setting $z = x'$. Otherwise, we pick a vertex from $S_{x'}$, say $u$, whose distance to $v_1$ equals $d(S_{x'})$. Note that $x'$ coincides with $y$ when restricted on $f(u)$. If $x'$ vanishes on $\overrightarrow{u}$, then the only possible on vertex for $x'$ inside $V(G) \setminus f(u)$ is $f(u)$ and this demonstrates that $L(x') \leq L(y) + 1$ and so the required $z$ can still be taken as $x'$. For the moment, it is sufficient to consider the third case that vertices in $\overrightarrow{u}$ are not all off in the assignment $x'$. We can thus find $u' \in \overrightarrow{u}$ such that $x'(u') = 1$ and $x'(u_1) = x'(u_2) = \cdots = x'(u_t) = 0$ where $u_1 u_2 \cdots u_t u'$ is the shortest path connecting $u$ and $u'$ (Note that $t = 0$ when $u = u'$). A sequence of valid moves along $u', u_1, \ldots, u_t$ turns $x'$ into $x''$ for which we have the following:

(i) $x \xrightarrow{G} x''$;

(ii) There exists $S_{x''} \subseteq V(G) \setminus \{v_1\}$ such that $y = x'' + \sum_{v \in S_{x''}} \chi_{N(v)}$ and either $d(S_{x''}) > d(S_{x'})$ or $d(S_{x''}) = d(S_{x'}) = d$ but $\{v \in S_{x'} : d(\{v\}) = d\}$ is a proper subset of $\{v \in S_{x'} : d(\{v\}) = d\}$. Note that the former case happens if $S_{x'} \setminus \{w_1, \ldots, w_k\} \neq \emptyset$.

At this stage it is not difficult to see that we can apply the above procedure repeatedly and finally terminate at one of the first two cases. This ends the proof.

Remark 25. Mimicking the above proof of Lemma 24, it is easy to show that $ML_{G}^*(x) \leq ML_{G}(x) + 1$ for any rack $G$ and $x \in \mathbb{F}_{2}^{V(G)}$.

Having derived Lemmas 19, 22 and 24, we are ready to give a proof of Theorem 12.

Proof of Theorem 12. The tightness of the bound follows from Example 10 and hence we just need to establish that bound.

By virtue of Theorem 11 we could and will make the following assumption from now on: for any appropriate vertex $v$ of $G$, every component of $G[V(G) \setminus \{v\}]$ that is good for $G$ contains exactly one vertex and this vertex has no loop attached and all these good components (for the same $v$) are in the same states in the assignment $x$. We further rule out the trivial case that $x = 0$. To complete the proof, let us distinguish two cases.

If there is at most one branch vertex in $G$, we can infer that $G$ must be a rake and therefore Remark 25 yields the result. (Indeed, under the current assumption, we can prove directly a stronger result that $ML_{G}^*(x) \leq 1$.)
We continue to dwell on the case that $G$ contains at least two branch vertices. For each appropriate vertex $v$ of $G$, we assume that $G[V(G) \setminus \{v\}]$ has $k_v$ components which are good for $G$, each of which should be a singleton set by our assumption. Now, for each appropriate vertex $v$ of $G$, delete $k_v - 1$ good components corresponding to $v$ as well as their incident edges and call the resulting graph $H$. By Lemma 22 applied to $H$, we conclude that there is a path $P$ in $H$ which contains two different leaves of $H$ and at most one branch vertex of $H$. We consider two subcases.

**Subcase 1:** $P = u_1u_2 \cdots u_p$ and $P$ does not contain any branch vertex of $H$. Since $G$ has at least two branch vertices, the only possibility is that $u_2$ and $u_{p-1}$ are the two branch vertices of $G$. Let $H'$ be the rake with top $u_1$ obtained from $G$ by removing $S = N_G(u_2) \cap (V(G) \setminus V(H))$. Note that any regular move inside $S$ can only affect the state of $u_2$. This says that there is a set of regular moves inside $V(H')$ which takes us from $x$ to a configuration $y$ fulfilling $L(y) \leq ML_G(x) + 1$. Thanks to Lemma 23 we can now assert that there is $z$ with $L(z) \leq ML_G(x) + 2$ and $x \xrightarrow{z} G z$, implying $ML^*_G(x) \leq ML_G(x) + 2$, as claimed.

**Subcase 2:** $P = u_1u_2 \cdots u_p uv_q \cdots v_1$, $p \geq q \geq 1$, and $u$ is a branch vertex of $H$. Denote by $U$ the component of $G[V(G) \setminus \{u\}]$ containing $u_p$ and by $V$ the component of $G[V(G) \setminus \{u\}]$ containing $v_q$. Assume that $x \xrightarrow{a} G y$ and $L(y) = ML_G(x)$. Take $a = u_p$, $b = v_q$, $c = u$ and $S = V(G) \setminus (\{u\} \cup U \cup V)$. Due to our construction of $H$, we see that $p > 1$. This then allows us to utilize Lemma 19 to get that there exists $R \subseteq U \cup V$ such that $x \xrightarrow{a} G y + \sum_{v \in R} \chi_{N(v)}$. Observe that both $G[U \cup \{u\}]$ and $G[V \cup \{u\}]$ are rakes with top $u$. Henceforth, an application of Lemma 24 to $G[U \cup \{u\}]$ and $G[V \cup \{u\}]$ completes the proof.

**Proof of Theorem 13.** This can be done analogous to the previous proof of Theorem 12 with Remark 23 in place of Lemma 22. We omit the details.

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