Wess-Zumino term by Vacuum Overlap Formula

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Abstract

We examine the vacuum overlap formula for the two-dimensional SU(2) Wess-Zumino term in lattice regularization. Perturbatively it reproduces the Wess-Zumino term correctly in the continuum limit and yields the IR fixed point in the beta function of the chiral model. Nonperturbatively it shows a sharp Gaussian distribution for the SU(2) chiral field configurations in the scaling region, where smooth configurations dominate even in the symmetric phase due to asymptotic freedom. Crossover is sharp from the strong coupling region where the Wess-Zumino term fluctuates hard and the species doublers’ contribution is suspected to affect it.

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§1. Introduction

The vacuum overlap formula gives a well-defined method to calculate the chiral fermion determinant in lattice regularization. By this formula, one first considers two Wilson fermion fields which possess positive mass and negative mass, respectively. They are assumed to couple to the link variables of given background gauge field. We next solve the ground states of the two fields. Then the overlap of the two ground states gives the chiral determinant. There remains an ambiguity in the complex phase of the determinant. It is fixed by the convention that the inner product of each ground state with the corresponding free ground state is real and positive.

The complex phase of the vacuum overlap is able to reproduce the full complex phase of chiral determinant correctly for a smooth background gauge field. The complex phase is gauge noninvariant and its variation under a gauge transformation was shown to give the consistent anomaly. In the continuum U(1) chiral gauge theory which is defined on the two-dimensional torus, the chiral determinant with the uniform background gauge potential (toron) can be calculated exactly. The vacuum overlap reproduces this exact result.

Actually, this lattice definition of the complex phase of chiral determinant is closely related to the $\eta$ invariant definition in the continuum theory. The vacuum overlap formula can be derived from the $(2n+1)$ dimensional system of Wilson fermion with kink mass by taking infinite limit of the extra dimension, provided that appropriate Pauli-Villars (Wilson) fields are included. For this limiting procedure to be well-defined, it also needs that the background gauge field is topologically trivial. By virtue of the Pauli-Villars fields, we may take the naive continuum limit $a \to 0$ in this formula keeping the kink and Pauli-Villars masses finite. Then it reduces to the formula which defines the $\eta$ invariant. The only technical difference is that the smooth interpolation function of two gauge fields is replaced by the sharp step function at the boundary of the wave guide of gauge interaction.

For generic gauge fields on the lattice, however, several authors argued that the nature of gauge degrees of freedom may spoil the chiral structure of the formula. For the $(2n+1)$ dimensional Wilson fermion with the kink mass and the wave guide of gauge interaction, it was shown that the large fluctuation of gauge freedom induces another light fermion mode of opposite chirality at the boundary of the wave guide. Such a light mode could also give the contribution to the complex phase of chiral determinant.

Singular gauge transformations can also affect the complex phase of the vacuum overlap. In the continuum U(1) gauge theory which is defined on the two-dimensional torus, we can consider the singular gauge transformation which wipe out the toron into the singular
potential with delta functions. Then, it was shown\(^6\) that the gauge invariance breaks by the singular gauge transformation even in the anomaly free U(1) chiral gauge theory, provided that the background toron is large and the anti-periodic boundary condition is chosen for all fermions. The vacuum overlap reproduces the result as well. The authors of ref.\(^6\) claimed that it is these singular gauge transformations which spoils the chiral phase when averaged on the gauge orbit.\(^6\)

Therefore, it is a crucial problem in this approach to control and reduce the fluctuation of gauge degree of freedom. In this paper, we examine the gauge freedom of the two-dimensional SU(2) gauge field. In order to control and reduce the fluctuation, we introduce the kinetic term for the variable of gauge freedom, which consists of the nearest neighbor coupling. We calculate the Wess-Zumino term by the vacuum overlap formula and measure its distributions. The whole system we consider here is the lattice counterpart of the SU(2) Wess-Zumino-Witten model off the critical point.

This type of kinetic term for gauge freedom was examined in the U(1) case.\(^15\) It was shown that the reduction of the fluctuation is not sufficient near the critical point of the corresponding X-Y model. On the contrary, in the case of SU(2) chiral field considered here, the coupling in the critical region is weak because of the asymptotic freedom. It is expected that the fluctuation can be reduced in this region even thought the SU(2) chiral symmetry is realized linearly.

This paper is organized as follows. In section 2, the lattice counterpart of the two-dimensional SU(N) Wess-Zumino-Witten model is examined in the lattice perturbation theory. We show that the vacuum overlap formula reproduces the Wess-Zumino term correctly in the continuum limit. We also calculate the beta function of the model and show that it has the nontrivial IR fixed point, as in the continuum theory. In section 3, the Wess-Zumino term is examined numerically. The SU(2) chiral field is generated by the Monte Carlo method (Cluster algorithm\(^16\)) with the action of the nearest neighbor coupling for several values of the coupling constant. With these configurations, we calculate the complex phase of the vacuum overlap and its distribution is obtained. In section 4, we give a discussion concerning the numerical estimate of observables in the Wess-Zumino-Witten model by incorporating the imaginary Wess-Zumino action.

\(^1\) For the small toron, the chiral determinant in the continuum limit is gauge invariant even under the singular gauge transformation.\(^6\) Choosing such small toron, the gauge integration of chiral determinant was performed.\(^6\) The result shows that the large gauge fluctuation affects the chiral phase even for the small toron and seems to cause it to vanish.
§2. Two-dimensional Wess-Zumino term on the Lattice

2.1. Vacuum overlap formula and Wess-Zumino term

The definition of the vacuum overlap formula for the lattice regularized chiral determinant in $2n$ dimension is given by

\[ \det(C)_{\text{reg}} \equiv \frac{1}{|\langle - | - \rangle_U \langle - |+ \rangle_U \langle + |+ \rangle_U |} \cdot \frac{1}{|\langle - | - \rangle_U \langle + |+ \rangle_U |}. \] (2.1)

$|\pm\rangle_U$ are the vacuum state of the second quantized Hamiltonian $H^\pm$, which defined by

\[ H^\pm(U) = \sum_{n} \sum_{\alpha i} \sum_{m\beta j} a_{n\alpha i}^\dagger H_{n\alpha i,m\beta j} a_{m\beta j}, \] (2.2)

\[ H^\pm(U) = \gamma_5 [\mp m_0 + C + B], \] (2.3)

\[ C_{n\alpha i,m\beta j} = \frac{1}{2} \sum_{\mu} (\gamma_\mu)_{\alpha\beta} (\delta_{m,n+\mu} U_{n,\mu} - \delta_{n,m+\mu} U_{m,\mu}^\dagger), \] (2.4)

\[ B_{n\alpha i,m\beta j} = \frac{r}{2} \sum_{\mu} \delta_{\alpha\beta} (2\delta_{n,m} - \delta_{m,n+\mu} U_{n,\mu} - \delta_{n,m+\mu} U_{m,\mu}^\dagger), \] (2.5)

where we set the lattice spacing $a$ to unity, $m_0$ is a bare mass parameter, $C$ and $B$ are the massless Dirac operator and the Wilson term respectively. $a_{n\alpha i}, a_{n\alpha i}^\dagger$ are field operators satisfying the following anti-commutation relations:

\[ \{ a_{n\alpha i}, a_{m\beta j}^\dagger \} = \delta_{nm} \delta_{\alpha\beta} \delta_{ij}, \]

\[ \{ a_{n\alpha i}, a_{m\beta j} \} = \{ a_{n\alpha i}^\dagger, a_{m\beta j}^\dagger \} = 0. \] (2.6)

The vacuum overlaps, $U \langle - |+ \rangle_U$ etc., can be expressed by the one particle wave functions $\psi^\pm_K(n,U)$, which are the eigenfunctions of $H^\pm(U)$ with positive eigenvalues labeled $K$:

\[ U \langle - |+ \rangle_U = \det_{K,K'} \left( \sum_n \psi^\pm_K(n,U) \psi^\pm_K(n,U) \right), \] (2.7)

\[ U \langle \pm |\pm \rangle_1 = \det_{K,K'} \left( \sum_n \psi^\pm_K(n,U) \psi^\pm_K(n,U) \right), \] (2.8)

where color and spin indices are suppressed.

The gauge transformation of the link variables is defined by

\[ U_{n,\mu} \rightarrow U_{n,\mu}^g = g_n^\dagger U_{n,\mu} g_{n+\mu}. \] (2.9)

where $g_n$ are elements of gauge group SU($N$) and $U^g$ denote the gauge transformation of $U$.

By this gauge transformation, $\psi^\pm_K(n,U)$ transform as

\[ \psi^\pm_K(n,U) \rightarrow \psi^\pm_K(n,U) = g_n^\dagger \psi^\pm_K(n,U), \] (2.10)
Then the vacuum overlap \( U(-|+)U \) is gauge invariant but the others are gauge dependent and are expressed as

\[
U^g(\pm|\pm)_1 = \det_{K,K'} \left( \sum_n \psi_K^\pm(n,U)^\dagger g_n \psi_{K'}^\pm(n,1) \right).
\] (2.11)

The variation of the complex phase of the chiral determinant by gauge transformation gives the gauged Wess-Zumino effective action. The corresponding quantity of the vacuum overlap formula is written by

\[
\mathcal{W}^\pm(U,g) \equiv \frac{U^g(\pm|\pm)_1}{|U^g(\pm|\pm)_1|},
\] (2.12)

\[
\exp(i \Gamma_{\text{WZ}}[U,g]) \equiv \frac{\mathcal{W}^-(U,g)^* \mathcal{W}^+(U,g)}{\mathcal{W}^-(U,\mathbb{1})^* \mathcal{W}^+(U,\mathbb{1})}.
\] (2.13)

If all the link variables are set to be unity, it is expected that Eq. (2.13) gives the Wess-Zumino effective action (we call as Wess-Zumino term in the following). In this case the vacuum overlaps (2.11) are expressed by

\[
1^g(\pm|\pm)_1 = \det_{K,K'} \left( \sum_n \psi_K^\pm(n,1)^\dagger g_n \psi_{K'}^\pm(n,1) \right),
\] (2.14)

and the Wess-Zumino term is defined by

\[
\exp(i \Gamma_{\text{WZ}}[g]) \equiv \frac{1^\langle -| - \rangle 1^\langle +|+ \rangle_1}{|1^\langle -| - \rangle 1^\langle +|+ \rangle_1|}.
\] (2.15)

For the continuum theory in two dimensions, it is well-known that the chiral determinant can be calculated exactly and the explicit expression can be obtained\(^\text{[17]}\). Next we verify that Eq. (2.15) has the correct continuum limit in two dimensions.

2.2. Perturbative Calculation of Wess-Zumino term

First we examine (2.14) perturbatively. The group elements \( g_n \) can be expressed as \( e^{i\phi_n} \) by the variable \( \phi_n \) which takes the value on the Lie algebra. We take this variables as the small quantum fluctuation from the unity (not some background) so that we can expand as follows:

\[
g_n = e^{i\phi_n} \approx 1 + i \phi_n + \frac{1}{2!} (i \phi_n)^2 + \cdots.
\] (2.16)

Then, \( \Gamma_{\text{WZ}}[g] \) can also be expanded in \( \phi_n \) and the continuum limit of the terms in the expansion series can be evaluated.
Let $\Phi^\pm[g]$ denote the effective actions for Eq. (2.14). Expanding by $\phi_n$, we obtain the following result,

\[
\Phi^\pm[g] \equiv \sum_{M=1}^\infty \frac{(-1)^{M-1}}{M} \int_{k_1,\ldots,k_M} \text{tr} \left( \tilde{\delta}g(k_1) \cdots \tilde{\delta}g(k_M) \right) \delta_0, k_1, \ldots, k_M
\]

\[
\times C^\pm_{(M)}(k_1, \ldots, k_{M-1}),
\]

\[
C^\pm_{(M)}(k_1, \ldots, k_{M-1}) = \int_p O^\pm_{p,p-K_1} O^\pm_{p-K_1,p-K_2} \cdots O^\pm_{p-K_{M-1},p}
\]

\[
= \int_p \text{tr} \left( S^\pm(p) S^\pm(p-K_1) \cdots S^\pm(p-K_{M-1}) \right),
\]

where

\[
\int_k = \int_0^\pi \frac{d^2k}{(2\pi)^2},
\]

\[
K_j \equiv \sum_{i=1}^j k_i, \quad K_0 \equiv 0,
\]

\[
(\delta g_n)_{ij} \equiv (g_n)_{ij} - \delta_{ij}, \quad \tilde{\delta}g(k) \equiv \sum_n e^{-ikn} \delta g_n,
\]

\[
O^\pm_{pq} \equiv [\psi^\pm_p]^\dagger [\psi^\pm_q],
\]

\[
S^\pm(p)_{\alpha\beta} \equiv [\psi^\pm_p]_{\alpha} [\psi^\pm_p]_{\beta},
\]

\[
= \frac{1}{2\lambda_p^\pm} \left( \lambda_p^\pm \delta_{\alpha\beta} + H^\pm(p)_{\alpha\beta} \right),
\]

$H^\pm(p)$ is the momentum representation of Hamiltonians Eq. (2.3) in the free case and $\psi^\pm_p$ and $\lambda_p^\pm$ are the eigenfunctions and the eigenvalues of them. The explicit expressions are given in the appendix A. Then the Wess-Zumino term on the lattice is given by

\[
\Gamma_{WZ}[g] = \text{Im} \left( \Phi^+[g] - \Phi^-[g] \right).
\]

The order $\phi$ contribution stems from $M = 1$ term, but $C^+_{(1)}$ and $C^-_{(1)}$ are equal to each other and cancel. The order $\phi^2$ contribution also vanishes since $\phi^2$ term in $M = 2$ terms are real. Therefore the leading term is the order $\phi^3$ and is given by the following,

\[
\Gamma_{WZ}[g]^{(3)} = i \frac{1}{3} \int_{k_1,k_2,k_3} \text{tr} \left( \tilde{\phi}(k_1) \tilde{\phi}(k_3) \tilde{\phi}(k_3) \right) \delta_0, k_1, k_2, k_3
\]

\[
\times \text{Im} \left( C^+_{(3)}(k_1, k_2) - C^-_{(3)}(k_1, k_2) \right).
\]

where the explicit expressions of $C^\pm_{(3)}$ are also given in the Appendix A. In order to take the continuum limit, we introduce physical momenta $p_i$ and express the dimensionless momenta $k_i$ as $k_i = p_i a$ with the lattice spacing $a$. Then we expands the coefficient $C^\pm_{(3)}$ in $a$ and obtain

\[
\Gamma_{WZ}[g]^{(3)} \cong i \frac{1}{3} \int_{p_1,p_2,p_3} \frac{d^2p_1}{(2\pi)^2} \frac{d^2p_2}{(2\pi)^2} \frac{d^2p_3}{(2\pi)^2} \text{tr} \left( \tilde{\phi}(p_1) \tilde{\phi}(p_3) \tilde{\phi}(p_3) \right) \delta_0, p_1, p_2, p_3
\]
\[
\frac{1}{4\pi^2} \epsilon_{\mu
u} p_1 \epsilon_{p_2} (J^+ - J^-) + O(a^2),
\]
(2.26)

\[
J^\pm = \int_{-\pi}^{\pi} d^2k \frac{1}{8\lambda_k^\pm} \left[ (C_\mu^\nu(k)C_{\nu}^\mu(k) - C_{\mu}^\nu(k)) (B(k) \mp m_0) + 2 \left( C_{\mu}(k)C_{\nu}^\nu(k) - C_{\nu}(k)C_{\mu}^\nu(k) \right) B(k)^\mu \right],
\]
(2.27)

with \( f_{\mu} = \partial_\mu f \).

For the Wilson fermion, \( C \) and \( B \) are given by

\[
C_{\mu}(k) = \sin k_\mu, \quad B(k) = r \sum_\mu (1 - \cos k_\mu).
\]
(2.28)

Then the integrands of \( J^\pm \) are complex expressions of sine and cosine and include two parameters, \( m_0 \) and \( r \). It does not seem to be able to accomplish the integrals analytically. By numerical calculation we obtained \( J^+ = \pi/J^- = 0 \) for the region of \( m_0/r, 0 < m_0/r < 2 \).

Therefore, when \( a \) goes to zero Eq. (2.26) becomes as follows:

\[
\Gamma_{WZ}[g]^{(3)} \simeq \frac{i}{12\pi} \int \epsilon_{\mu\nu} \text{tr} (\phi(x) \partial_\mu \phi(x) \partial_\nu \phi(x)).
\]
(2.29)

This is identical with the leading term of the continuum Wess-Zumino term,

\[
\Gamma_{WZ}[g]^c = \frac{1}{12\pi} \int d^3y \epsilon^{ABC} \text{tr} \left( g^{-1} \partial_A g g^{-1} \partial_B g g^{-1} \partial_C g \right)
\]
(2.30)
in the \( \phi \)-expansion.

### 2.3. Wess-Zumino-Witten model on the lattice

Next we formulate a model with the Wess-Zumino term defined in the previous subsections. The partition function of the model is defined by

\[
Z = \int \mathcal{D}g e^{-S[g]} e^{i\Gamma_{WZ}[g]},
\]
(2.32)

where \( S[g] \) is the action of \( SU(N) \) spin model:

\[
S[g] = \frac{1}{\lambda^2} \sum_{n,\mu} \text{tr} (\hat{\partial}_\mu g(n) \hat{\partial}_\mu g(n)^\dagger)
\]

\[
= -\frac{1}{\lambda^2} \sum_{n,\mu} \text{tr} \left( g(n)g(n + \hat{\mu})^\dagger + g(n)^\dagger g(n + \hat{\mu}) \right) + \text{const}.
\]
(2.33)

\(^\dagger\) For the continuum vacuum overlap formula, the same calculation can be performed. We obtain the similar expansion as Eq. (2.27) in the \((k/m)\)-expansion where \( m \) is Pauli-Villars mass.

\[
J^+ = -J^- = \int d^2p \frac{m}{4\lambda_p^2},
\]
(2.31)

\[
\lambda_p \equiv \left[ p^2 + m^2 \right]^{1/2}.
\]

It is easy to carry out this integral analytically and obtain the results \( J^+ = -J^- = \pi/2 \). It also reproduce the correct Wess-Zumino term. The fact that \( J^- = 0 \) in lattice regularization is consistent with the calculation of the Chern-Simons current by Golterman et al.\(^\dagger\)
and \( n \) is an arbitrary integer. This model is considered as the lattice counterpart of the two-dimensional Wess-Zumino-Witten model. In order to verify that the theory have the correct continuum limit, we perform the calculation of the Callan-Symanzik \( \beta \) function and show it is identical to that of the continuum model.

To evaluate the \( \beta \) function, one may use the background field method, in which the field variables are separated into two parts such as

\[
g_n = g_0^n e^{\lambda \pi_n},
\]

with \( g_0^n \) a smooth background and \( \pi_n \) as small fluctuation around \( g_0^n \). Then one calculates the correction to the coefficient of the functional of \( g_0^n \),

\[
S_{cl}[g_0] = -\sum_{n,\mu} \text{tr} \left( g_0^{\mu} \hat{\partial}_\mu g_0 \right)^2.
\]

However, in our model, it is not easy to calculate the Wess-Zumino functional defined by Eq. (2.13) keeping the nontrivial background \( g_0^n \). So we first expand the total action in \( \phi \) by which \( g_n \) is expressed as \( e^{i\phi_n} \). Then we shift \( \phi_n \) around a background \( \phi_0^n \) not linearly but nonlinearly by applying the Hausdorff's formula to Eq. (2.34) as follows:

\[
\phi_n = \phi_0^n + \lambda \pi_n + \frac{i}{2} \lambda [\phi_0^n, \pi_n] - \frac{1}{12} \left( \lambda [\phi_0^n, [\phi_0^n, \pi_n]] + \lambda^2 [\pi(n), [\pi(n), \phi_0^n]] \right) + \cdots.
\]

In this case, the lowest order of the functional (2.35) is

\[
S_{cl}[\phi_0] = \sum_{n,\mu} \text{tr} \left( \hat{\partial}_\mu \phi_0 \hat{\partial}_\mu \phi_0 \right),
\]

and it is sufficient for us to estimate the corrections to its coefficient. The one-loop order corresponds to \( O(\phi_0^2, \lambda^2) \). To obtain the propagator and vertices necessary to calculate the correction to this order, we first expand the actions \( S[g] \) and \( \Gamma_{WZ}[g] \) in \( \phi_n \) as follow,

\[
S[g] = \frac{1}{\lambda^2} \sum_{n,\mu} \text{tr} \left( \hat{\partial}_\mu \phi_n \right)^2 - \frac{1}{12} \hat{\partial}_\mu \phi_n [\phi_n, [\phi_n, \hat{\partial}_\mu \phi_n]] - \frac{1}{12} (\hat{\partial}_\mu \phi_n)^4 + \cdots,
\]

\[
\Gamma_{WZ}[g] = \int_{k_1, k_2, k_3} \tilde{\phi}^{a_1}(k_1) \tilde{\phi}^{a_2}(k_2) \tilde{\phi}^{a_3}(k_3) \delta_{0,k_1+k_2+k_3} \times V_{wz}^{(3)a_1,a_2,a_3}(k_1, k_2) + \cdots,
\]

\[
V_{wz}^{(3)a_1,a_2,a_3}(k_1, k_2) = -\frac{1}{12} f^{a_1 a_2 a_3} \text{Im}(C^+(a_1, k_1, k_2) - C^-(a_3, k_1, k_2)).
\]
Next we shift $\phi_n$ around $\phi_{0n}$ as mentioned above and obtain the followings.

\begin{align}
S_0[\phi] &= \frac{1}{\lambda^2} S_{cl}[\phi], \\
S_{cl}[\phi] &= \frac{1}{2} \int_k \bar{\phi}^a(-k) \phi^a(k) \Delta(k)^{-1},
\end{align}

where

\[ \Delta(k)^{-1} = \sum_\mu (1 - \cos k_\mu), \]

and

\begin{align}
S_1 &= \int_{k_1,k_2,k_3} \bar{\phi}_0^{a_1}(k_1) \bar{\pi}^{a_2}(k_2) \bar{\pi}^{a_3}(k_3) \delta_{0,k_1+k_2+k_3} \\
&\quad \times \mathcal{V}_{1}^{a_1,a_2,a_3}(k_3), \\
S_2 &= \int_{k_1,k_2,k_3,k_4} \bar{\phi}_0^{a_1}(k_1) \bar{\phi}_0^{a_3}(k_3) \bar{\pi}^{a_2}(k_2) \bar{\pi}^{a_4}(k_4) \delta_{0,k_1+k_2+k_3+k_4} \\
&\quad \times \mathcal{V}_{2}^{a_1,a_2,a_4}(k_1,k_2), \\
S_3 &= \int_{k_1,k_2,k_3,k_4} \bar{\phi}_0^{a_1}(k_1) \bar{\phi}_0^{a_2}(k_2) \bar{\pi}^{a_3}(k_3) \bar{\pi}^{a_4}(k_4) \delta_{0,k_1+k_2+k_3+k_4} \\
&\quad \times (V_3^{a_1a_2a_3a_4}(k_3,k_4) + V_3^{a_1a_2a_4a_3}(k_1,k_2) + 4V_3^{a_1a_2a_1a_4}(k_1,k_4)), \\
S_4 &= \int_{k_1,k_2,k_3,k_4} \bar{\phi}_0^{a_1}(k_1) \bar{\phi}_0^{a_2}(k_2) \bar{\pi}^{a_3}(k_3) \bar{\pi}^{a_4}(k_4) \delta_{0,k_1+k_2+k_3+k_4} \\
&\quad \times (4V_4^{a_1a_2a_3a_4}(k_1,k_2,k_3,k_4) + 2V_4^{a_2a_1a_3a_4}(k_1,k_2,k_3,k_4)),
\end{align}

\begin{align}
\Gamma_{wz,1} &= 3\lambda^2 \int_{k_1,k_2,k_3} \bar{\phi}_0^{a_1}(k_1) \bar{\pi}^{a_2}(k_2) \bar{\pi}^{a_3}(k_3) \delta_{0,k_1+k_2+k_3} \\
&\quad \times \mathcal{V}_{wz}^{(3)a_1a_2a_3}(k_1,k_2), \\
\Gamma_{wz,2} &= -3\lambda^2 \int_{k_1,k_2,k_3,k_4} \bar{\phi}_0^{a_1}(k_1) \bar{\pi}^{a_2}(k_2) \bar{\pi}^{a_3}(k_3) \bar{\pi}^{a_4}(k_3) \delta_{0,k_1+k_2+k_3+k_4} \\
&\quad \times \mathcal{V}_{wz}^{(3)a_1a_2a_4}(k_1,k_2) f^{a_3,a_4},
\end{align}

The vertex functions $V$'s are listed in appendix B. Then the partition function is calculated as follows:

\begin{align}
Z &= e^{-S_0[\phi]} \int D\pi e^{-S_0[\pi]} \\
&\quad \times e^{-S_1 - S_2 - S_3 - S_4 + in F_{wz,1} + in F_{wz,2}} \\
&= -S_0[\phi] + \frac{1}{2!} \langle (S_1)^2 \rangle_0 - \langle S_2 \rangle_0 - \langle S_3 \rangle_0 - \langle S_4 \rangle_0 \\
&\quad - \frac{(\lambda^2 n)^2}{2!} \langle (\Gamma_{wz}^{(1,2)})^2 \rangle_0 - i \lambda^2 \langle \Gamma_{wz}^{(2,2)} \rangle_0,
\end{align}

where

\[ \langle O \rangle_0 \equiv \int D\pi O(\pi) e^{-S_{2,0}[\pi]}. \]
The contribution of the original spin model is calculated as

\[ \frac{C_A}{4} \Delta(0) S_{cl}[\phi_0] \approx \frac{1}{4 \pi} \ln a^2 m^2 \]

where \( C_A \) is the second Casimir coefficient in the adjoint representation and \( m \) is the infrared cut-off mass. On the other hand, the contribution from the Wess-Zumino term is calculated as follows. It is easy to see the second term vanishes. The first term is given by

\[ -\frac{1}{2} \int_k \Delta(k) \Delta(-k) C(k, -p)^2, \]

\[ I_1(k) = -\frac{n}{4} C_A \int_p \Delta(p) \Delta(k - p) C(k, -p)^2, \]

\[ I_2(k) = \frac{n}{4} C_A \int_p \Delta(p) \Delta(k - p) C(k, -p) C(-k, k - p), \]

\[ C(k, p) = \text{Im} \left( C_{(3)}^+(k, p) - C_{(3)}^+(k, -p) \right). \]

Toward the continuum limit we also scale the dimensionless external momenta \( k \) by \( a \) and consider the momenta \( k a \) near zero. Following Karsten and Smit, we can estimate the loop integral. The divergent contribution stems from the momenta near zero, so that we can use the continuum limit of the vertex \( C(k, p) \) obtained in the previous section. It is calculated as

\[ I_1(k) \sim I_2(k) \sim \frac{1}{(4\pi)^2} \left( \frac{\lambda^2 n}{8\pi} \right)^2 C_A \frac{k^2}{4\pi} \ln a^2 k^2. \]

Therefore the contribution from Wess-Zumino term is

\[ -\frac{1}{2} \int_k \Delta(k) \Delta(-k) C(k, -p)^2, \]

\[ -\frac{(\lambda^2 n)^2}{2} \langle (\Gamma^{(1,2)})^2 \rangle_0 \Rightarrow \frac{1}{4 \pi} \ln a^2 m^2 \]
For $n > 0$, the $\beta$ function has an IR-fixed point at $\lambda^2 = \frac{8\pi}{n}$ as expected (the factor 2 differ from Witten’s results because of the difference of the convention). For $\lambda^2 < \frac{8\pi}{n}$, this model is asymptotically free, so that it has a renormalization group invariant scale parameter, $\Lambda$ given by

$$\Lambda = \frac{1}{a} \left| \frac{\lambda^2}{\lambda_\Lambda^2} - 1 \right| \exp\left( -\frac{1}{\beta_0 \lambda^2} \right).$$

(2.63)

$$\beta_0 = \frac{C_A}{8\pi}, \quad \lambda_\Lambda^2 = \frac{8\pi}{n}.$$  

(2.64)

§3. Numerical Calculation of the Wess-Zumino term

In this section, we show our results of the numerical calculation of the Wess-Zumino term. We consider the SU(2) chiral field and adopt the lattice action with the nearest neighbor coupling. We generate SU(2) chiral fields by the cluster algorithm. Lattice size is set to 16 x 16. For most of the values of $\beta \equiv \frac{\lambda^2}{\lambda_\Lambda^2}$ examined, 500 configurations are generated. For $\beta = 1.1, 1.5$, we generate 4000 more configurations. With these configurations, the vacuum overlaps are calculated numerically through the LU decomposition. We measure the distribution of the Wess-Zumino term by dividing the range of the value $[-\pi, \pi]$ into bins with the width 0.1. We also calculate the standard deviation $\sigma_{WZ}$ of the distribution of the Wess-Zumino term for each $\beta$.

First of all, we show the behavior of the Wess-Zumino term for randomly generated configurations of SU(2) chiral field in Figure 1. ($\beta$ is set to 0.025.) The Wess-Zumino term distributes almost uniformly in the range $[-\pi, \pi]$. In Figure 2, we show a typical configuration of the random SU(2) chiral field in momentum space. We can find that almost all modes of momenta contribute equally. The species doublers are then suspected to contribute to the complex phase.

Next we show the distributions of the Wess-Zumino term for $\beta = 0.5, 1.1$ and 1.5 in Figure 3. We can clearly observe that for $\beta = 1.1$, the evaluated values of the Wess-Zumino term start to concentrate around zero and the distribution becomes like Gaussian. When $\beta$ increases, the distribution becomes sharp Gaussian. $\beta$ dependence of the standard deviation $\sigma_{WZ}$ of the Wess-Zumino term distribution is plotted in Figure 4. The onset of the Gaussian distribution is around $\beta = 0.9$. Crossover is sharp from the strong coupling region where the Wess-Zumino term fluctuates hard.

In Figure 5, we show a typical configuration of the SU(2) chiral field at $\beta = 1.5$ in momentum space. We can find that the modes with small momenta dominates. In this region of $\beta$, the species doublers are almost suppressed.
Fig. 1. The distribution of the two-dimensional Wess-Zumino term for randomly generated SU(2) chiral fields. The range of the value $[-\pi, \pi]$ is divided into bins by the width 0.1. The normalized number of configurations for each bin is shown. $L=16$.

Fig. 2. A typical configuration of random SU(2) chiral field in momentum space: the absolute value of the trace part is shown. $L=16$.

Fig. 3. The distribution of the two-dimensional Wess-Zumino term for $\beta = 0.5$, 1.1 and 1.5.
In Figure 4, we show the susceptibility $\chi$ of the SU(2) chiral model for $L = 16$ and $L = 64$. The scaling of the susceptibility starts around $\beta = 0.9$. This coincides with the onset of the Gaussian distribution of the Wess-Zumino term.

In summary, the two-dimensional Wess-Zumino term defined by the vacuum overlap formula shows a sharp Gaussian distribution for the configurations of SU(2) chiral field in the scaling region. In this region, the configurations are smooth and the contribution of the species doublers is suppressed. In the strong coupling region, the Wess-Zumino term fluctuates hard and the species doublers’ contribution is suspected to affect it. However the crossover is rather sharp from the strong coupling region to the weak coupling region.

A few comments are in order for our numerical calculation. In Figure 4, we find that the
finite size effect on $\chi$ is substantial for $L = 16$. We also suspect the finite size effect on the Wess-Zumino term. The onset of the Gaussian distribution of the Wess-Zumino term in the scaling region is also expected for larger lattice. However there can be much finite size effect in the values of $\sigma_{WZ}$. We need further investigation on large lattice.

![Scaling Curve](image)

Fig. 6. Scaling of susceptibility of SU(2) chiral model. L=16 and L=64.

§4. Numerical Evaluation of Observable in Wess-Zumino-Witten model

In this section, we discuss the possibility of the numerical estimate of observables in the Wess-Zumino-Witten model by incorporating the Wess-Zumino action. The lattice Wess-Zumino-Witten model we are considering is defined with the complex action,

$$S_{WZW}[g] = S[g] - in\Gamma_{WZ}[g],$$  

(4.1)

where $S[g]$ and $\Gamma_{WZ}[g]$ are given by Eq. (2.33) and by Eq. (2.15), respectively. $n$ is the integer coupling constant of the Wess-Zumino term. In order to evaluate the observables in this model, we need to perform the following integral,

$$\langle O \rangle = \int \mathcal{D}g e^{-S[g] - in\Gamma_{WZ}[g]} O[g] / Z,$$

(4.2)

$$Z = \int \mathcal{D}g e^{-S[g] - in\Gamma_{WZ}[g]}.$$

(4.3)

In general, without the positivity of the measure, the Monte Carlo method cannot be applied. One possible way out is to incorporate the imaginary Wess-Zumino term into the observable and to perform the Monte Carlo integration only with the real part of the action. That is, we consider

$$\langle O \rangle = \langle \langle e^{in\Gamma_{WZ}[g]} O[g] \rangle \rangle_{MC} / Z,$$

(4.4)

$$Z = \langle \langle e^{in\Gamma_{WZ}[g]} \rangle \rangle_{MC}.$$

(4.5)
where \( \langle \langle \rangle \rangle_{MC} \) denotes the Monte Carlo integration only with the real part of the action,

\[
\langle \langle X \rangle \rangle_{MC} \equiv \int \mathcal{D}g e^{-S[g]}X[g].
\]  

(4.6)

Another possible method is to introduce the spectral functions defined as follows:

\[
\rho(\theta) \equiv \int \mathcal{D}g e^{-S[g]}\delta(\theta - \Gamma_{WZ}[g]),
\]  

(4.7)

\[
O(\theta) \equiv \int \mathcal{D}g e^{-S[g]}\delta(\theta - \Gamma_{WZ}[g])O(g),
\]  

(4.8)

where the delta function is the periodic one. With these spectral functions, we can write the integrals Eq. (4.2) and Eq. (4.3) as

\[
\langle O \rangle = \int_{-\pi}^{\pi} d\theta e^{in\theta} O(\theta) / Z,
\]  

(4.10)

\[
Z = \int_{-\pi}^{\pi} d\theta e^{in\theta} \rho(\theta).
\]  

(4.11)

The integration of \( \theta \) is then performed as the Riemann integral.

Actually, the distribution of the Wess-Zumino term obtained in the previous section can be regarded as the spectral function,

\[
\rho(\theta) = \langle \langle \delta(\theta - \Gamma_{WZ}) \rangle \rangle_{MC},
\]  

(4.12)

in the discrete approximation with the normalization,

\[
\sum_i \Delta \theta \rho(\theta_i) = 1.
\]  

(4.13)

We found that in the scaling region, it is given by a sharp Gaussian distribution with the width \( \sigma_{WZ} \) depending on \( \beta \),

\[
\rho(\theta_i) \sim \frac{1}{\sqrt{2\pi \sigma_{WZ}}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2\sigma_{WZ}}(\theta_i + 2\pi m)^2}.
\]  

(4.14)

For such a Gaussian distribution of the imaginary part of the action, it is known that the first method mentioned above may work only for small \( n \). To see this, we perform the following test. We assume the Gaussian distribution for the Wess-Zumino term, Eq. (4.14). We take the square of the Wess-Zumino term as an observable. Since the spectral function

\footnote{* It may be better to use the set method for the precise evaluation of the spectral function. In this case, however, since the value of the Wess-Zumino term is in the compact region \([-\pi, \pi]\), we can do without it if we have enough statistics.}
is obtained by the Monte Carlo method, we simulate the situation by performing the Monte Carlo integration on $\theta$. Thus we consider the following integrals for various $n$ at fixed $\sigma_{WZ}$,

$$
\langle (I_{WZ})^2 \rangle = \langle \langle e^{in\theta} \theta^2 \rangle \rangle / Z,
$$

$$
Z = \langle \langle e^{in\theta} \rangle \rangle,
$$

where $\langle \langle \rangle \rangle$ denotes the Monte Carlo integration of $\theta$ with the weight $\rho(\theta)$,

$$
\langle \langle X \rangle \rangle_\theta \equiv \int_{-\pi}^{\pi} d\theta \rho(\theta) X[\theta].
$$

In Figure 7, we show the result. We found that for $\sigma_{WZ} = 0.5$, up to $n = 3$ the numerical result reproduces the exact one within the acceptable error of three significant digits. For $\sigma_{WZ} = 0.2$, $n = 12$.

![Fig. 7. Monte Carlo integration of $\theta^2 \exp (i\theta)$ with the Gaussian weight. The width of the Gaussian weight $\sigma_{WZ}$ is chosen as 0.2 and 0.5. The approximated analytical result for these $\sigma_{WZ}$, $\sigma_{WZ}^2 (1 - \sigma_{WZ}^2 n^2)$, are also shown.](image)

On the other hand, it is also possible to perform the Riemann integral in Eq. (4.10) numerically for the same observable. We also found the upper bounds for $n$. In order that the numerical result reproduces the exact one within the acceptable error, we should have $n \leq 12$ for $\sigma_{WZ} = 0.5$ and $n \leq 30$ for $\sigma_{WZ} = 0.2$. We have tried several schemes of higher order integration and obtained the similar result.

Thus we obtained the upper bound for the accessible value of the coupling constant $n$ of the Wess-Zumino term for given $\sigma_{WZ}$. In other word, for a given $n$, there exists the upper bound of $\sigma_{WZ}$, with which the measurement of the observable can be performed by the direct Monte Carlo integration or by the Riemann integration of the spectral functions. In Figure 9, we show the upper bounds of $\sigma_{WZ}$ versus the coupling constant $n$ of the Wess-Zumino term which is translated to the IR fixed point $\beta_c = \frac{\mu}{4\pi}$. From this figure, it seems possible
Fig. 8. Riemann integration of $\theta^2 \exp (i n \theta)$ with the Gaussian weight. The width of the Gaussian weight $\sigma_{WZ}$ is chosen as 0.2 and 0.5. The approximated analytical result for these $\sigma_{WZ}$, $\sigma_{WZ}^2 (1 - \sigma_{WZ}^2 n^2)$, are also shown.

to simulate the model across the IR fixed point for $\beta_c \geq 1.3$, (that is $n \geq 17$), if the spectral density method is used. On the contrary, by the direct Monte Carlo calculation, only the weak coupling region can be simulated.

Fig. 9. Upper bound on $\sigma_{WZ}$ for given coupling constant $n$ of the Wess-Zumino term. $n$ is translated to the value of the IR fixed point by the relation $\beta_c = \frac{n}{4\pi}$. The values of $\sigma_{WZ}$ obtained in the previous section are also plotted.

Our discussion given here is based on the numerical result shown in the previous section. Since the systematic and statistical errors are suspected in the result, the numbers of the upper bounds given in this discussion should not be taken literary. We believe, however, that this discussion gives a correct qualitative estimate of the possibility.

§5. Discussion and Conclusion
The vacuum overlap formula for the two-dimensional Wess-Zumino term was examined both perturbatively and nonperturbatively. We showed by the lattice perturbation theory that the formula correctly reproduces the Wess-Zumino term in the continuum limit and yields the IR fixed point in the beta function of the SU(2) chiral model. We calculated the vacuum overlaps numerically and found that the complex phase shows a sharp Gaussian distribution for the SU(2) chiral fields in the scaling region. In this region, the smooth configurations were obtained even in the symmetric phase due to asymptotic freedom. We also found that the crossover is rather sharp from the strong coupling region where the Wess-Zumino term fluctuates hard and the species doublers’ contribution is suspected to affect it.

It was also argued that, if we use the spectral density method, it seems possible to examine the region of the IR fixed point of the Wess-Zumino-Witten model numerically.

Also from this study, we found that the asymptotically free coupling for gauge degree of freedom is able to reduce the gauge fluctuation. In two dimensions, the nearest neighbor coupling of the SU(2) chiral field plays such a role. In four dimensions, it is known in the continuum theory that the four-derivative coupling which is induced from the gauge fixing term is asymptotically free. It may be interesting to examine the gauge degree of freedom of the four-dimensional nonabelian gauge field with the gauge fixing term from this point of view.

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Appendix A

Eigenvalues and Eigenfunctions of $H^\pm(p)$

In this appendix we give the detail expression of the functions used in subsection 2.1. The momentum representation of Hamiltonians and the Wave functions are

$$H^\pm(q)\psi^\pm_p(q) = \lambda^\pm_p \psi^\pm_p(q), \quad (A\cdot1)$$
\[ H_{\alpha \beta}^\pm (p) = \begin{pmatrix} B(p) \mp m_0 & C(p) \\ C(p)^\dagger & -B(p) \pm m_0 \end{pmatrix} , \quad (A.2) \]
\[ \psi_{\alpha, \beta}^\pm (q) = \delta_{\alpha, \beta} \psi_{\alpha, \beta}^\pm 
\]
\[ \psi_{\alpha, \beta}^\pm (q) = \frac{1}{N_p^\pm} \begin{pmatrix} \lambda_p^\pm + B(p) \mp m_0 \\ C(p) \end{pmatrix} , \quad (A.4) \]
\[ \lambda_p^\pm = [C_p(p)^2 + (B(p) \mp m_0)^2]^{\frac{1}{2}} , \quad (A.5) \]
\[ N_p^\pm = [2\lambda_p^\pm (\lambda_p^\pm + B(p) \mp m_0)]^{\frac{1}{2}} . \quad (A.6) \]

For Wilson fermion,
\[ C_p(p) = \sin p_\mu, B(p) = r \sum_\mu (1 - \cos k_\mu) . \quad (A.7) \]

But one may take any regularization function as \( C(p) \) and \( B(p) \). Imaginary part of \( C_{(3)}^\pm \) are
\[ \text{Im} C_{(3)}^\pm (k_1, k_2) = \int \frac{B_{(3)}^\pm (p; k_1, k_2)}{4\lambda_p^\pm \lambda_{p-k_1}^\pm \lambda_{p-k_1-k_2}^\pm} . \quad (A.8) \]
\[ B_{(3)}^\pm (p; k_1, k_2) = -\epsilon_{\mu\nu}[C_p(p)C_\nu(p - k_1)(B(p - k_1 - k_2) \mp m_0) + C_p(p - k_1)C_\nu(p - k_1 - k_2)(B(p) \mp m_0) + C_p(p - k_1 - k_2)C_\nu(p)(B(p - k_1) \mp m_0)] . \quad (A.9) \]

Expanding on \( k_1, k_2 \), one obtain the expression of the integral \( J^\pm \).

### Appendix B

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**Vertices for One-Loop Calculation**

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Here we give the vertex functions used in subsection 2.3.
\[ V_1^{a_1, a_2, a_3} (k_3) = -\frac{1}{2} f^{a_1 a_2 a_3} \Delta(k_3)^{-1} , \quad (B.1) \]
\[ V_2^{a_1, a_2, a_3, a_4} (k_1, k_2) = f^{a_1 a_2 e} f^{a_3 a_4 e} \]
\[ \times \left( \frac{1}{8} \Delta(k_1 + k_2)^{-1} - \frac{1}{12} \left( \Delta(k_1)^{-1} + \Delta(k_2)^{-1} \right) \right) , \quad (B.2) \]
\[ V_3^{a_1, a_2, a_3} (k, k') = -\frac{1}{12} f^{a_1 a_3 e} f^{a_2 a_4 e} \sum_\mu (e^{ik_\mu} - 1)(e^{ik'_\mu} - 1) , \quad (B.3) \]
\[ V_4^{a_1, a_2, a_3, a_4} (k_1, k_2, k_3, k_4) = -\frac{1}{12} \text{tr} (T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \]
\[ \times \sum_\mu (e^{ik_{1, \mu}} - 1)(e^{ik_{2, \mu}} - 1)(e^{ik_{3, \mu}} - 1)(e^{ik_{4, \mu}} - 1) , \quad (B.4) \]
\[ V_5^{a_1, a_2, a_3} (k_1, k_2) = -\frac{1}{12} f^{a_1 a_2 a_3} \text{Im} \left( C_{(3)}^+(k_1, k_2) - C_{(3)}^+(k_1, k_2) \right) . \quad (B.5) \]
Appendix C

--- Convention for SU(N) Lie Algebra ---

\[ \text{tr} \left( T^a T^b \right) = \frac{1}{2} \delta_{ab}, \quad (C.1) \]
\[ [T^a, T^b] = i f^{abc} T^c, \quad (C.2) \]
\[ \text{tr} \left( T^a_{\text{Ad}} T^b_{\text{Ad}} \right) = f^{acd} f^{bcd} = C_A \delta_{ab}, \quad (C.3) \]
\[ \phi(x) = \phi^a(x) T^a. \quad (C.4) \]

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