ON THE STABILITY OF THE AREA LAW FOR THE ENTANGLEMENT ENTROPY OF THE LANDAU HAMILTONIAN

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Abstract. We consider the two-dimensional ideal Fermi gas subject to a magnetic field which is perpendicular to the Euclidean plane $\mathbb{R}^2$ and whose strength $B(x)$ at $x \in \mathbb{R}^2$ converges to some $B_0 > 0$ as $\|x\| \to \infty$. Furthermore, we allow for an electric potential $V_\varepsilon(x)$ which vanishes at infinity. They define the single-particle Landau Hamiltonian of our Fermi gas (up to gauge fixing). Starting from the ground state of this Fermi gas with chemical potential $\mu \geq B_0$ we study the asymptotic growth of its bipartite entanglement entropy associated to $L\Lambda$ as $L \to \infty$ for some fixed bounded region $\Lambda \subset \mathbb{R}^2$. We show that its leading order in $L$ does not depend on the perturbations $B_\varepsilon := B_0 - B$ and $V_\varepsilon$ if they satisfy some mild decay assumptions. Our result holds for all $\alpha$-Rényi entropies $\alpha > 1/3$; for $\alpha \leq 1/3$, we have to assume in addition some differentiability of the perturbations $B_\varepsilon$ and $V_\varepsilon$. The case of a constant magnetic field $B_\varepsilon = 0$ and with $V_\varepsilon = 0$ was treated recently for general $\mu$ by Leschke, Sobolev and Spitzer. Our result thus proves the stability of that area law under the same regularity assumptions on the boundary $\partial \Lambda$.

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1. Introduction

Bipartite entanglement entropy is an important quantity that measures correlations of particles inside a given region with the particles outside that region. These non-trivial correlations are solely due to the Fermi–Dirac statistics of the particles involved. In recent years there has been considerable interest and progress in quantifying these correlations. Mathematicians and physicists alike realized fascinating connections between the large scale asymptotics of entanglement entropy and certain semi-classical asymptotic formulas of traces of certain operators, mostly Toeplitz operators in the discrete case and Wiener–Hopf operators in the continuous case.

In the discrete setting, Jin and Korepin related the Fisher–Hartwig conjecture of Toeplitz matrices to the scaling of the entanglement entropy in the XY-chain in a transverse magnetic field in [8]. More relevant to our continuous setting here is the discovery of Gioev and Klich [5] that a conjecture by Harold Widom (proved by Alexander V. Sobolev [22]) gives the precise leading asymptotic growth of the bipartite entanglement entropy in ground states of the free Fermi gas. It displays a logarithmically enhanced area law of the order $L^{d-1} \ln(L)$, where $L$ is a scaling parameter, see below. In [9], this
was finally proved by Leschke, Sobolev and Spitzer. In [14], Müller and Schulte proved that this law is stable under a perturbation by a compactly supported potential. The proof of their first paper is also important for our model here.

A ground state of a non-interacting fermions on $\mathbb{R}^2$ with single-particle Hamiltonian $H$ as in our model is given by the (Fermi) spectral projection $1_{\leq \mu}(H)$, where $\mu \in \mathbb{R}$. The function $1_{\leq \mu}$ is the indicator function of the set $(-\infty, \mu] \subset \mathbb{R}$ and the number $\mu$ is called the Fermi energy. Let $\alpha > 0$ and let $h_\alpha$ be the Rényi entropy function, see (3.3). For a given bounded region $\Lambda \subset \mathbb{R}^2$ we denote by $1_\Lambda$ the (multiplication operator associated to the) indicator function on $\Lambda$. Then we define the local entropy (or entanglement entropy) $S_\alpha(\Lambda)$ to be the (usual Hilbert space) trace of $h_\alpha$ applied to the spatially to $\Lambda$ reduced Fermi projection, that is,

$$S_\alpha(\Lambda) := \text{tr} h_\alpha(1_\Lambda 1_{\leq \mu}(H) 1_\Lambda).$$

(1.1)

At positive temperature a definition of entanglement entropy or mutual information needs to be amended, see [10].

For a fixed region $\Lambda$, it is generally hard or impossible to calculate the entropy. However, if we introduce a scaling parameter $L > 0$ and consider the leading order asymptotic expansion of $\mathbf{1}$ with $\Lambda$ replaced by $L \Lambda$ for $L \to \infty$, there are interesting results. They all assume some kind of regularity of the boundary $\partial \Lambda$, assume the Hamiltonian $H$ to be of a certain form, and may restrict to the case $\alpha = 1$. For $H = -\nabla^2 + V$, with some assumptions on $V$, there are results presented in [4, 5, 10, 15, 16, 18, 19].

In this paper, we consider the Hamiltonian $H = (-i\nabla - A)^2 + V_\epsilon$, which is a slight perturbation of the Landau Hamiltonian $H_0$ for a constant magnetic field and no electric field, see (3.2) and (3.10). Entanglement entropy of the ground state of the latter Landau Hamiltonian (for the ground state with chemical potential $\mu = B_0$) has been studied in [12, 20, 21] with some additional assumptions on the region $\Lambda$. The case of $\mu = B_0$ has been solved by Charles and Estienne in [3], and then the case of an arbitrary $\mu \geq B_0$ by Leschke, Sobolev and Spitzer in [11], both under some regularity assumptions on the boundary $\partial \Lambda$. Our main result is Corollary 3.6. It shows that the leading order asymptotic growth of the entanglement entropy for arbitrary $\alpha > 0$ does not change, if we add such a slight perturbation in both the magnetic field and the electric potential, assuming some differentiability of these perturbations in the case $\alpha \leq \frac{1}{4}$, depending on $\alpha$. Hence, we will not need to recalculate the value of the leading term, as we only estimate that this perturbation leads to an error term of smaller order in the scaling parameter $L$.

Our proof is based on a statement by Aleksandrov and Peller in [1], which is Proposition 3.4 in this paper. With the help of this and approximations of the Rényi entropy functions $h_\alpha$ (see (3.3)), we can reduce our result to some $p$-Schatten (quasi-)norm estimates, as we prove in Section 3.

Proving these $p$-Schatten (quasi-)norm estimates relies on the fact that some Sobolev embeddings on bounded subset of $\mathbb{R}^d$ are in some $p$-Schatten classes, which we specify and prove in Corollary A.11. It is based on a result by Gramsch in [6]. This allows us to estimate the $p$-Schatten (quasi-)norms of operators with sufficiently differentiable kernels. To get a representation of the kernel of the spectral projection of the perturbed Hamiltonian, we use the contour integral representation and the resolvent expansion. This has recently been done for perturbations of the free case ($H = -\nabla^2 + V$) by Müller and Schulte in [16], which inspired me to try this approach. In our case ($B_0 > 0$), we use an expanded resolvent expansion. The discrete spectrum allows us to explicitly resolve the contour integral for most terms. The general idea is explained in Section 4, while the required kernel estimates are proven in the remaining sections.

The magnetic case (with an asymptotically constant magnetic field) appears simpler and more stable than the free case with the (negative) Laplacian as its single-particle Hamiltonian. From a technical point of view this is due to the gaps in the purely essential spectrum and the exponential decay of eigenfunctions of the Landau Hamiltonian. This is also the reason for an area law growth (without any logarithmic enhancement as in the free case), see also [17].
Proposition 2.1. Let \( p \leq q \leq \infty \) and let \( S, T \) be operators on a Hilbert space. The \( p \)-Schatten norm satisfies the properties

Monotonicity I: \( \| S \|_p \geq \| S \|_q \),

Monotonicity II: If \( S \geq T \geq 0 \), then \( \| S \|_p \geq \| T \|_p \).
Triangle inequality: If \( p \geq 1 \), then \( \|S + T\|_p \leq \|S\|_p + \|T\|_p \).

\( p \)-triangle inequality: If \( p \leq 1 \), then \( \|S + T\|_p^p \leq \|S\|_p^p + \|T\|_p^p \).

Powers: If \( S \geq 0 \), then \( \|S^p\|_q = \|S\|_p^p \).

Square: \( \|S\|_{p/2}^2 = \|S^*S\|_{p/2} \), where \( S^* \) denotes the adjoint of \( S \).

Adjoint: \( \|S^*\|_p = \|S\|_p \).

Hölder I: Let \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Then \( \|ST\|_r \leq \|S\|_p \|T\|_q \).

Hölder II: Let \( \frac{1}{r} = \frac{p}{r} + \frac{1}{q} \) with \( 0 < \alpha < 1 \). Then \( \|S\|_r \leq \|S\|_p^\alpha \|S\|_q^{1-\alpha} \).

Hilbert–Schmidt kernel: If \( T: L^2(\mathbb{R}^d_1) \rightarrow L^2(\mathbb{R}^d_2) \) has an integral kernel \( t \), which is square integrable, then \( \|T\|_2 = \|t\|_{L^2(\mathbb{R}^{d_1+d_2})} \).

Orthogonality: If \( ST^* = 0 \) or \( S^*T = 0 \), then \( \|S\|_p \leq \|S + T\|_p \).

Most of these have for example been proven by McCarthy in [13]. We will now briefly prove the remaining ones.

**Proof.** “Monotonicity II” follows, as the inequality holds for the ordered sequence of singular values.

“Hölder II” is an application of “Hölder I” with the operators \( |S|^\alpha \) and \( |S|^{1-\alpha} \) and the properties “Square” and “Powers”. “Hilbert–Schmidt kernel” can be seen as a corollary of Lemma 2.2 in [13].

“Orthogonality” is based on the observation, that if \( S^*T = 0 \), we have \( (S + T)^*(S + T) = S^*S + T^*T \), “Monotonicity II”, and “Adjoint” to replace the condition \( S^*T = 0 \) by the non-equivalent condition \( ST^* = 0 \).

**Definition 2.2.** We say a densely defined operator \( T \) on \( L^2(\mathbb{R}^2) \) has the integral kernel \( t: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C} \), if for any \( f \in C^0_c(\mathbb{R}^2) \), the identity

\[
(Tf)(x) = \int_{\mathbb{R}^2} t(x,y)f(y)dy
\]

holds for almost all \( x \in \mathbb{R}^2 \). In this case, we define

\[
\text{iker}(T;x,y) := t(x,y).
\]

We say, that \( t \) is nice, or respectively, that \( T \) is a nice integral operator, if for any fixed \( x \), the functions \( t(x,\cdot) \) and \( t(\cdot,x) \) are in \( L^1(\mathbb{R}^2) \) with a norm bounded independently of \( x \).

**Corollary 2.3.** Let \( T \) be a nice integral operator. Then \( T \) is a bounded operator on \( L^2(\mathbb{R}^2) \).

**Proof.** The expression \( \|x \rightarrow \|t(x,\cdot)|\|_{L^1(\mathbb{R}^2)}\|_1 \) is finite and an upper bound for the operator norm of \( T \) as an operator on \( L^\infty(\mathbb{R}^2) \). On the other hand, the expression \( \|y \rightarrow \|t(\cdot,y)|\|_{L^1(\mathbb{R}^2)}\|_1 \) is finite and an upper bound for the operator norm of \( T \) as a bounded operator on \( L^1(\mathbb{R}^2) \). Hence, by the Riesz-Thorin interpolation theorem, the operator \( T \) is bounded on \( L^2(\mathbb{R}^2) \) with an operator norm bounded by the square root of the product of both of these expressions.

**Lemma 2.4.** Let \( S,T \) be nice integral operators on \( L^2(\mathbb{R}^2) \) with integral kernels \( s,t \). Let \( x,z \in \mathbb{R}^2 \). Then we have the identities

\[
\text{iker}(S + T)(x,z) = (s + t)(x,z),
\]

\[
\text{iker}(ST)(x,z) = \int_{\mathbb{R}^2} s(x,y)t(y,z)dy.
\]

In particular, \( S + T \) and \( ST \) are nice integral operators.

The first statement is trivial and the second follows by Fubini to interchange the integral over \( y \) with the one over \( z \), for any test function \( f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \).

**Definition 2.5.** Let \( \gamma, d \in \mathbb{N} \) and \( \lambda \in [0, \infty) \). Then we define the space \( W_{\gamma,\infty}^{(\lambda)}(\mathbb{R}^2, \mathbb{C}^d) \) as the subspace of the Sobolev space \( W^{\gamma,\infty}(\mathbb{R}^2, \mathbb{C}^d) \), where the norm

\[
\|u\|_{W_{\gamma,\infty}^{(\lambda)}(\mathbb{R}^2, \mathbb{C}^d)} := \sum_{\gamma' \leq \gamma} \sup_{x \in \mathbb{R}^2} \left\| (1 + \|x\|)^\lambda \left( \nabla^{\gamma'} u \right)(x) \right\|
\]

(2.9)
is finite. The supremum in this definition refers to the almost everywhere supremum. This is a Banach space. The limit space
\[ W^{\gamma,\infty}_{(\infty)}(\mathbb{R}^2, \mathbb{C}^d) := \bigcap_{\lambda > 0} W^{\gamma,\infty}_{(\lambda)}(\mathbb{R}^2, \mathbb{C}^d) \quad (2.10) \]
is only a vector space equipped with the inverse limit topology associated to the intersection (a set is open, if and only if it is open in each space for finite \( \lambda \)).

These spaces are motivated by Schwartz semi-norms.

### 3. Setting and main result

We introduce the Landau Hamilton operator \( H_0 \) with a constant magnetic field \( B_0 > 0 \), defined on (a suitable subspace of) \( L^2(\mathbb{R}^2) \), with magnetic gauge \( A_0 \) given by
\[ A_0(x) := \frac{B_0}{2} J x, \quad (3.1) \]
\[ H_0 := (-i \nabla - A_0)^2. \quad (3.2) \]
The spectrum of \( H_0 \), \( \sigma(H_0) \), equals \( B_0(2 \mathbb{N} + 1) \). Let \( P_l \) be the projection onto the eigenspace with eigenvalue \( B_0(2l + 1) \) for \( l \in \mathbb{N} \).

Furthermore, for any \( \alpha > 0 \), we introduce the \( \alpha \)-Rényi entropy functions \( h_\alpha : [0,1] \to [0,\ln(2)] \),
\[ h_\alpha(x) := \begin{cases} \frac{1}{\alpha - 1} \ln (x^\alpha + (1 - x)^\alpha) & \text{for } \alpha \neq 1, \\ -x \ln x - (1 - x) \ln(1 - x) & \text{for } \alpha = 1, \end{cases} \quad (3.3) \]
for \( x \in (0,1) \) and \( h_\alpha(0) = h_\alpha(1) = 0 \). Throughout this paper, let \( \Lambda \subset \mathbb{R}^2 \) be a bounded open set with Lipschitz-boundary.

Let \( \mu \in \mathbb{R} \setminus B_0(2\mathbb{N} + 1) \). We define \( 1_{\leq \mu}(H_0) \) as the spectral projection associated to \( H_0 \) and \( \mu \).

We are interested in how the leading order asymptotic expansion of the local entropy,
\[ S_\alpha(L\Lambda) := \text{tr} h_\alpha(1_{L\Lambda} 1_{\leq \mu}(H_0) 1_{L\Lambda}), \quad (3.4) \]
as \( L \to \infty \) changes under slight perturbations of \( H_0 \). The trace is defined as the usual Hilbert space trace of trace class operators on \( L^2(\mathbb{R}^2) \). This quantity is the local entropy or entanglement entropy of the ground state restricted to \( L\Lambda \). Under the assumption that \( \Lambda \) has \( C^4 \) boundary, the leading term of order \( L \) for the operator \( H = H_0 \) has been calculated by Leschke, Sobolev and Spitzer in [11]. This allows us to focus on bounding the error term that arises, as we introduce a perturbation to \( H_0 \). Our main result is Corollary 3.6 and relies on the exact calculations of the leading term for \( H = H_0 \), see [11], and the estimates we will prove in this paper.

The following condition is needed to state our main results and a lot of results along the way. Throughout this paper, we fix \( 0 < \varepsilon < 1 \).

**Definition 3.1.** Let \( \gamma \) be a natural number. We call a magnetic field \( B_\varepsilon : \mathbb{R}^2 \to \mathbb{R} \) and a potential \( V_\varepsilon : \mathbb{R}^2 \to \mathbb{R} \) \((\gamma, \varepsilon)\) tame, if \( V_\varepsilon \in W^{\gamma,\infty}(\mathbb{R}^2, \mathbb{R}) \) and \( B_\varepsilon \in W^{\gamma,\infty}_{(1+\varepsilon)}(\mathbb{R}^2, \mathbb{R}) \).

**Remark.** All the following estimates will depend on \( B_\varepsilon, V_\varepsilon \) only through \( \varepsilon, \gamma \) and the norms of \( B_\varepsilon, V_\varepsilon \) in the spaces \( W^{\gamma,\infty}_{(1+\varepsilon)}(\mathbb{R}^2, \mathbb{R}) \) and \( W^{\gamma,\infty}(\mathbb{R}^2, \mathbb{R}) \). Maybe somewhat counter-intuitively, small values of \( \varepsilon \) correspond to slowly decaying \( B_\varepsilon, V_\varepsilon \).

To define the perturbed Hamiltonian \( H_\varepsilon \), we need to choose a gauge \( A_\varepsilon \) of the magnetic field \( B_\varepsilon \).

We choose the convolution, which is given by
\[ A_\varepsilon(x) := \left( \int_{\mathbb{R}^2} B_\varepsilon(x \cdot y) \frac{J y}{2\pi \| y \|^2} dy \right) (x) = \int_{\mathbb{R}^2} B_\varepsilon(x - y) \frac{J y}{2\pi \| y \|^2} dy \quad (3.5) \]
for any \( x \in \mathbb{R}^2 \). Its relevant properties are summed up in the following Lemma.
Lemma 3.2. Let $\gamma \in \mathbb{N}$, $f \in W^{\gamma, \infty}_{(1+\varepsilon)}(\mathbb{R}^2, \mathbb{R})$ and define $g \in \text{Maps}(\mathbb{R}^2, \mathbb{R})$ as the convolution
\[
g := f * \frac{J}{2\pi \| \cdot \|^2}.
\]
Then, for any $x \in \mathbb{R}^2$, we have the identities
\[
\nabla_x \times g(x) = f(x),
\]
\[
\nabla_x \cdot g(x) = 0.
\]
Furthermore, we have $g \in W^{\gamma, \infty}_{(\varepsilon)}(\mathbb{R}^2, \mathbb{R})$.

Remark. A gauge satisfying (3.8) is commonly referred to as a Coulomb gauge. The restriction to $\varepsilon < 1$ is necessary to get the described decay. A value of $\varepsilon > 1$ will only achieve a $(1 + \|x\|)^{-1}$ decay in $A_\varepsilon$.

The proof can be found in Appendix B.

Now we define the perturbed gauge $A$ and the perturbed Hamiltonian $H$ by
\[
A := A_0 - A_\varepsilon,
\]
\[
H := (-i\nabla - A)^2 + V_\varepsilon.
\]
As we can see, this gauge corresponds to the magnetic field $B_0 - B_\varepsilon$, that is, $\nabla_x \times A(x) = B_0 - B_\varepsilon(x)$. The operator $H$ is self-adjoint and its domain agrees with the domain of $H_0$, which we will see in Corollary 4.2.

We need the following $p$-Schatten quasi-norm estimate, which will be proven in the next section.

Theorem 3.3. Let $l \in \mathbb{N}, \gamma \in \mathbb{Z}^+$. Let $B_\varepsilon, V_\varepsilon$ be $(\gamma, \varepsilon)$ tame and let $1 \leq p > \frac{2}{\gamma + 3}$. Let $a, b \in \mathbb{R} \setminus B_0(2N + 1)$ with $a < b$. Then we have the estimates
\[
\|1_{LA^l[a,b]}(H)1_{LA^2}\|_p \leq CL,
\]
\[
\|1_{LA^l} (1_{[a,b]}(H) - 1_{[a,b]}(H_0)) 1_{LA^2}\|_p \leq CL^{1 - p\varepsilon}.
\]
The constants $C$ depend on $\gamma, a, b, \Lambda, p, \varepsilon, B_\varepsilon, V_\varepsilon$.

Finally, we need the following statement due to Aleksandrov and Peller, which is a Corollary of Theorem 5.11 in [1] and the inclusion $C^\infty_c(\mathbb{R}) \subset B^{1}_{\infty, 1}(\mathbb{R})$, where the latter refers to the Besov space as used by Aleksandrov and Peller.

Proposition 3.4 (based on Theorem 5.11 in [1]). Let $f \in C^\infty_c(\mathbb{R})$. Then there is a constant $C < \infty$, such that for any self-adjoint bounded operators $A, B$, such that $A - B$ is trace class, we have the estimate
\[
\|f(A) - f(B)\|_1 \leq C \|A - B\|_1.
\]

Now we state the key result of this paper, which is proved below.

Theorem 3.5. Let $\alpha > 0$ and choose $\beta = \min(0.5, \alpha)$. Define $\gamma$ as the smallest integer, such that $\gamma > \frac{1}{\beta} - 3$. Let $B_\varepsilon, V_\varepsilon$ be $(\gamma, \varepsilon)$ tame. Let $a, b \in \mathbb{R} \setminus B_0(2N + 1), a < b$ and $I := [a, b]$. Then we have
\[
\text{tr} (h_\alpha (1_{LA^1 I} (H) 1_{LA^1 I}) - h_\alpha (1_{LA^1 I} (H_0) 1_{LA^1 I})) = o(L),
\]
as $L \to \infty$.

Remark. The choice of $\beta = 0.5$ for $\alpha \geq 0.5$ delivers the optimal value for $\gamma$, namely 0. For $\alpha > \frac{1}{3}$, we can get away with a non-differentiable $B_\varepsilon, V_\varepsilon$.

The assumption that $a, b \not\in B_0(2N + 1)$ cannot be dropped, as the following counter example illustrates. Let $B_\varepsilon = 0, a = 0$ and $b = B_0$. By Corollary 4.2 the spectrum of $H$ has an accumulation point at $B_0$. If we assume $V_\varepsilon > 0$ pointwise, then all eigenvalues of $H$ are strictly larger than $B_0$ and hence $1_{I} (H) = 0$. But Theorem 8 in [1], which we will elaborate on shortly, states, that the leading order asymptotic expansion of $\text{tr} h_\alpha (1_{LA^1 I} (H_0) 1_{LA^1 I})$ for large $L$ is of order $O(L)$ and does
not vanish. On the other hand, if we assume that \(-B_0 < V_\varepsilon < 0\) pointwise, there is a spectral gap of the form \((B_0, 2B_0)\) in the spectrum of \(H\). Hence, we can move \(b\) to \(1.5B_0\) without changing the operators. Now we can apply our Theorem 3.5. Hence under our general assumptions, it is possible to get both one-sided limits, when \(b = B_0\). We expect similar results, whenever \(a \neq b\) are in the spectrum of \(H_0\). It is, however, a little more complicated to see, whether the leading order expansion for \(H_0\) changes, when we add or remove a single Landau level from the interval \(I\).

The following corollary is our main result. It combines Theorem 8 in [11], which can be stated as the corollary for the case \(B_\varepsilon = V_\varepsilon = 0\), with our Theorem 3.5.

**Corollary 3.6.** Let \(\alpha > 0\) and choose \(\beta = \min(\alpha, 0.5)\). Define \(\gamma\) as the smallest positive integer, such that \(\gamma > \frac{1}{\beta} - 3\). Let \(B_\varepsilon, V_\varepsilon\) be \((\gamma, \varepsilon)\) tame. Let \(\mu \notin \sigma(H_0)\) and define \(\nu\) as the largest integer, such that \(B_0(2
u + 1) < \mu\). Assume that the boundary \(\partial \Lambda\) is \(C^3\)-smooth. Then

\[
S_\alpha(L\Lambda) = \text{tr}(h_\alpha(1_{L\Lambda} 1_{\leq \nu}(H) 1_{L\Lambda})) = L\sqrt{B_0} |\partial \Lambda| |M_{\leq \nu}(h_\alpha) + o(L)|,
\]

as \(L \to \infty\) with \(0 < M_{\leq \nu}(h_\alpha) < \infty\) as described in [11] for \(\nu \geq 0\) and \(M_{\leq \nu}(h_\alpha) = 0\) for \(\nu < 0\).

In the case \(\mu < B_0\), the projection is finite dimensional and the entropy has an order at most \(O(1)\) in \(L \to \infty\).

**Proof of Theorem 3.5.** We define the function \(g_\alpha: [0, 1] \to [0, \ln(2)]\) by the identity

\[
g_\alpha(4x(1 - x)) = h_\alpha(x).
\]

The symmetry of \(h_\alpha\) guarantees the existence of \(g_\alpha\). We have

\[
g_\alpha(t) = h_\alpha \left( \frac{1 - \sqrt{1 - t}}{2} \right).
\]

Let \(\varepsilon_0 > 0\). We choose a smooth cut-off function \(\varphi: [0, 1] \to [0, 1]\) with \(\varphi(x) = 1\), if \(x \leq \varepsilon_0\), and \(\varphi(x) = 0\), if \(x \geq 2\varepsilon_0\). Now we write

\[
g_\alpha(t) = (1 - \varphi(t))g_\alpha(t) + \varphi(t)g_\alpha(t).
\]

The advantage of this decomposition is that the first summand is smooth, and the second summand is small. The second summand can be bounded using the fact, that \(h_\alpha\) is \(\beta\)-Hölder continuous on \([0, 1]\) and analytic on \((0, 1)\). As \(h_\alpha\) is symmetric around \(t = \frac{1}{2}\) and analytic on \((0, 1)\), its Taylor expansion at that point contains only even powers of \((t - \frac{1}{2})\). Thus, we see that \(g_\alpha\) is analytic at \(t = 1\). Hence \(g_\alpha \in C^\infty((0, 1])\) and it is \(\beta\)-Hölder continuous on \([0, 1]\), as \(\beta = \min(\alpha, 0.5)\).

We choose \(\beta' < \beta < \frac{1}{2}\), such that \(\gamma > \frac{1}{\beta'} - 3\). Hence, we have

\[
\varphi(t)g_\alpha(t) \leq C\varepsilon_0^{-\beta - \beta'} t^{\beta'}.
\]

We define \(P, P'\) as the spectral projections,

\[
P := 1_I(H_0),
\]

\[
P' := 1_I(H).
\]

We observe

\[
h_\alpha(1_{L\Lambda} P'(\cdot)) 1_{L\Lambda}) = g_\alpha(4|1_{L\Lambda} P'(\cdot) 1_{L\Lambda}|^2).
\]

We can now apply Proposition 3.4. Thus,

\[
\mu \left\| (1 - \varphi)g_\alpha(4|1_{L\Lambda} P'(\cdot) 1_{L\Lambda}|^2) - (1 - \varphi)g_\alpha(4|1_{L\Lambda} P(\cdot) 1_{L\Lambda}|^2) \right\|
\]

\[
\leq C\left\| 1_{L\Lambda} P'(\cdot) 1_{L\Lambda} - 1_{L\Lambda} P(\cdot) 1_{L\Lambda} \right\|_1
\]

\[
\leq C\left\| 1_{L\Lambda} P' - P \right\|_1
\]

\[
\leq CL^{1 - }.
\]

Note that the last constant \(C\) depends on \(\varepsilon_0\), but not on \(L\). In the second step we used the identity \(|A|^2 - |B|^2 = A^*(A - B) + (A^* - B^*)B\). In the last step, we used Theorem 3.3 with \(p = 1\).
We can also apply Theorem 3.3 for the remaining term, after using (3.19), $1 \geq 2\beta' > \frac{2}{\gamma+3}$ and that $H = H_0$ is admissible for Theorem 3.3.

\[
\left\| (\varphi g_0) \left( 4 |1_{LA} \delta P(\cdot) 1_{LA}|^2 \right) \right\|_1 \leq C \varepsilon_0^{\beta-\beta'} \left\| 1_{LA} \delta P(\cdot) 1_{LA} \right\|_{2\beta'} \leq C \varepsilon_0^{\beta-\beta'} L. \tag{3.27}
\]

Hence,

\[
|\text{tr} h_\alpha (1_{LA} P' 1_{LA}) - \text{tr} h_\alpha (1_{LA} P 1_{LA})| \leq C (\varepsilon_0) L^{1-\varepsilon} + C \varepsilon_0^{\beta-\beta'} L. \tag{3.28}
\]

Note that the first constant $C(\varepsilon_0)$ depends on $\varepsilon_0$ while the second one does not. This term is in $o(L)$, as for any $\varepsilon > 0$ we can choose $L$ large enough to let the first term be less than $\varepsilon_0 L$. This proves that the leading term expansion of the $\alpha$-Rényi entropy for the perturbed Landau Hamiltonian $H$ agrees with the main term in the same expansion for the Landau Hamiltonian $H_0$. This finishes the proof.

**Remark.** We can actually pick $\varepsilon_0$ dependent on $L$, which does lead to a smaller error term, if we bound the constant $C(\varepsilon_0)$ more precisely. This does however not lead to an improved error term in Corollary 3.6, as the known error term for the constant magnetic field is too large. Hence, I did not include the details here.

### 4. The Ansatz for the Proof of Theorem 3.3

The goal of this section is to explain how to prove Theorem 3.3 and, to reduce it to two more technical statements. The general approach has been inspired by [10].

We define

\[
H_\varepsilon := H - H_0, \tag{4.1}
\]

where $H$ and $H_0$ were defined in (3.10) and (3.2).

We expand $H_\varepsilon$ as

\[
H_\varepsilon = H - H_0 = (-i \nabla - A)^2 - (-i \nabla - A_0)^2 + V_\varepsilon \tag{4.2}
\]

\[
= (A_0 - A) : (-i \nabla - A - A_0 + A_0 - A_0) + (-i \nabla - A_0) : (A_0 - A) + V_\varepsilon \tag{4.3}
\]

\[
= 2 A_\varepsilon : (-i \nabla - A_0) + A_\varepsilon^2 + V_\varepsilon. \tag{4.4}
\]

We used the identity $a^2 - b^2 = (a - b)a + (a - b)b$ in the third step and (3.8), which is equivalent to $\nabla \cdot A_\varepsilon = A_\varepsilon \cdot \nabla$, in the last step. We now introduce the pseudo potential

\[
W_\varepsilon := A_\varepsilon^2 + V_\varepsilon. \tag{4.5}
\]

We introduce a few more operators. Let $I \subset \mathbb{N}$ be cofinite, $\zeta \in \mathbb{C}$ and $\zeta \notin B_0(2I + 1)$. Then we define the bounded operator

\[
M_{I, \zeta} := \sum_{l \in I} \frac{P_l}{B_0(2l + 1) - \zeta}. \tag{4.6}
\]

It satisfies $M_{I, \zeta}^* = M_{I, \zeta}^\dagger$. For $\zeta \notin \sigma(H_0)$, we have the identity

\[
M_{I, \zeta} = \frac{1}{H_0 - \zeta}. \tag{4.7}
\]

There are some results describing the kernel of the resolvent operator, but we also need the special case

\[
T_l := M_{I \setminus \{l\}, B_0(2l + 1)} = \sum_{k \neq l} \frac{P_k}{2B_0(k - l)}. \tag{4.8}
\]

Hence it is more convenient to deal with the operator $M_{I, \zeta}$ in this generality.
We define $n_0$ as the smallest integer such that

$$n_0 > \frac{1}{2\varepsilon}. \quad (4.10)$$

The following lemma will be proved in Section 6 after some preparations.

**Lemma 4.1.** Let $B_\varepsilon$ and $V_\varepsilon$ be $(0, \varepsilon)$ tame. Then for any $I \subset N$ cofinite and any $\zeta \in \mathbb{C} \setminus B_0(2I+1)$, the operator $H_\varepsilon M_{I,\zeta}$ is in the $4n_0$-Schatten class, and the $4n_0$-Schatten norm is in $L^\infty(\mathbb{C} \setminus \sigma(H_0))$ as a function of $\zeta$. The upper bound for the norm depends on $B_0$.

As $p$-Schatten class operators are compact, we now know that $H_\varepsilon$ is relatively $H_0$-compact. This implies

**Corollary 4.2.** The essential spectrum of $H$ agrees with the essential spectrum of $H_0$ which is $B_0(2N+1)$.

**Remark.** The statement is also true if $V = 0$ and $B$ is smooth and converges to $B_0$ as $\|x\| \to \infty$ (at any rate), see [7]. They state smoothness of $B$ as a condition, but I think it is not required. However, their algebraic proof does not imply that the eigenspaces of $H$ are at all related.

As $\sigma(H_0)$ is discrete, this implies, that $\sigma(H) = \sigma_p(H)$ and that the continuous part of the spectrum of $H$ vanishes. We continue with the Riesz integral representation.

**Fact 4.3.** For any path $\Gamma$ in $\mathbb{C}$ that intersects $\mathbb{R}$ in exactly two points $\lambda_1 < \lambda_2$, does not intersect $\sigma(H) \subset \mathbb{R}$ and has winding number $+1$ around $(\lambda_1 + \lambda_2)/2$, we have the identity

$$-\frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{H - \zeta} = \lambda_{1<\epsilon<\lambda_2}(H). \quad (4.11)$$

With the resolvent identity, we can write

$$\frac{1}{H - \zeta} = \frac{1}{H_0 - \zeta} - \frac{1}{H - \zeta} H_\varepsilon \frac{1}{H_0 - \zeta} \quad (4.12)$$

$$= \frac{1}{H_0 - \zeta} - \frac{1}{H_0 - \zeta} H_\varepsilon \frac{1}{H_0 - \zeta} + \frac{1}{H_0 - \zeta} H_\varepsilon \frac{1}{H_0 - \zeta}. \quad (4.13)$$

By induction, this leads to

**Corollary 4.4.** For any $n \in \mathbb{Z}^+$, $\zeta \notin \sigma(H) \cup \sigma(H_0)$, we have

$$\frac{1}{H - \zeta} = \sum_{k=0}^{2n-1} (-1)^k \left( \frac{1}{H_0 - \zeta} \right)^{k} \left( \frac{1}{H_\varepsilon} \right)^{n} \left( \frac{1}{H_0 - \zeta} \right)^{n}, \quad (4.14)$$

where $H_\varepsilon = H - H_0$, as in (4.1).

For the summands in Corollary 4.4 except the last summand, we can resolve the path integral over some paths.

**Lemma 4.5.** Let $l, k \in \mathbb{N}$ and $\Gamma$ be the path along the circle $\partial D_{B_0}(B_0(2l+1))$ that rotates in positive direction. Then we have

$$-\frac{1}{2\pi i} \int_\Gamma \frac{1}{H_0 - \zeta} \left( \frac{1}{H_\varepsilon} \right)^k d\zeta = \sum_{m=0}^{k} (T_l H_\varepsilon)^{m} P_l (H_\varepsilon T_l)^{k-m}, \quad (4.15)$$

where $H_\varepsilon = H - H_0$, as in (4.1).

**Proof.** Let $N > 2l$ and either $I = \mathbb{N}$ and $\zeta \in \Gamma$ or $I = \mathbb{N} \setminus \{l\}$ and $\zeta = B_0(2l + 1)$. We introduce $P_{\leq N} := \sum_{n \leq N} P_n$ and $P_{> N} := 1 - P_{\leq N}$. We continue with the identity

$$\sum_{j \in I, j \leq N} \frac{P_j}{B_0(2j + 1) - \zeta}. \quad (4.16)$$
There is a constant $C$, independent of $N$ and $\zeta$, such that the estimate $\|P_{>N}M_{I,\zeta}\|_\infty \leq \frac{CN}{N}$ holds (see Lemma A.7). Furthermore, by Lemma 4.1 and as the $4\theta_0$-Schatten norm is an upper bound for the operator norm, we have the estimate $\|H\|_{M_{I,\zeta}} < C$ with a constant $C$ independent of $\zeta$ (and $N$).

We use the telescope sum $b(ab)^k - c(ac)^k = \sum_{k'=0}^{k} (ba)^{k'} (b - c)(ac)^{k-k'}$, which holds in any ring, and the triangle inequality to get

$$\|M_{I,\zeta}(H_{\zeta}M_{I,\zeta})^k - P_{\leq N}M_{I,\zeta}(H_{\zeta}P_{\leq N}M_{I,\zeta})^k\|_\infty \leq \sum_{k'=0}^{k} \|(M_{I,\zeta}H_{\zeta})^{k'} P_{>N}M_{I,\zeta}(H_{\zeta}P_{\leq N}M_{I,\zeta})^{k-k'}\|_\infty \leq \frac{C}{N},$$

(4.17)

where $C$ is independent of $N$ and $\zeta$. The second step relies on the submultiplicativity of the norm, and the identity $M_{I,\zeta}P_{\leq N} = P_{\leq N}M_{I,\zeta}$. Thus, we have

$$-\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{H_0 - \zeta} \left(\frac{1}{H_{\zeta} - H_0 - \zeta}\right)^{k} \, d\zeta$$

(4.19)

$$= -\frac{1}{2\pi i} \lim_{N \to \infty} \int_{\Gamma} P_{\leq N}M_{I,\zeta}(H_{\zeta}P_{\leq N}M_{I,\zeta})^{k} \, d\zeta$$

(4.20)

$$= -\frac{1}{2\pi i} \lim_{N \to \infty} \int_{\Gamma} \sum_{\sigma \in \{0, \ldots, N\}^{k+1}} P_{\sigma_0} \prod_{j=1}^{k} H_{\zeta}P_{\sigma_j} \prod_{\sigma_j \neq 1} \frac{1}{2B_0(\sigma_j + 1) - \zeta} \, d\zeta$$

(4.21)

$$= \lim_{N \to \infty} \sum_{\sigma \in \{0, \ldots, N\}^{k+1}} P_{\sigma_0} \prod_{j=1}^{k} H_{\zeta}P_{\sigma_j} \prod_{\sigma_j \neq 1} \frac{1}{2B_0(\sigma_j + 1) - \zeta}$$

(4.22)

$$= \lim_{N \to \infty} \sum_{m=0}^{k} (P_{\leq N}T_{\zeta} H_{\zeta})^m P_{\leq N}T_{\zeta}^{k-m}$$

(4.23)

$$= \sum_{m=0}^{k} (T_{\zeta} H_{\zeta})^m P_{\leq N} T_{\zeta}^{k-m}.$$  

(4.24)

In the first step, we used that (4.18) holds uniformly in $\zeta \in \Gamma$ for $I = N$. In the second step, we inserted (4.10) $k+1$ times and multiplied out all terms in order to get a finite sum. We then exchanged this finite sum with the complex path integral and resolved this complex-valued integral. The fourth step uses (4.10) in reverse. The final step follows by (4.18) for $I = \mathbb{N} \setminus \{l\}$ and $\zeta = B_0(2l + 1)$. This finishes the proof.

We will prove the following theorem at the end of Section 6.

**Theorem 4.6.** Let $k, l, m, \gamma \in \mathbb{N}$ with $k \geq m$. Let $B_{\varepsilon}, V_{\varepsilon}$ be $(\gamma, \varepsilon)$ tame and let $1 \geq p > \frac{2}{\gamma + 3}$. Then there is a constant $C > 0$ and a $\lambda > 0$, such that for any $R \geq 0$, we have the upper bound

$$\left\| \int_{[0,1]^2 + x_0} (T_{\zeta} H_{\zeta})^m P_{\leq N}T_{\zeta}^{k-m} 1_{D_R^k(x_0)} \right\|_p \leq C \exp \left( -\lambda R^2 \right) \left( 1 + \|x_0\| \right)^{k\varepsilon},$$

(4.25)

for any $x_0 \in \mathbb{R}^2$. The constant $C$ depends on $B_0, l, k, m, \gamma, p, \varepsilon, B_{\varepsilon}, V_{\varepsilon}$, but is independent of $R$ and $x_0$.

**Remark.** For $k = m = 0$, this is Lemma 12 in [11].

We will now follow Theorem 13 in [11]. But we go a slightly different direction with the proof.1

**Theorem 4.7.** Let $k, l, m, \gamma \in \mathbb{N}$ with $k \geq m$, let $B_{\varepsilon}, V_{\varepsilon}$ be $(\gamma, \varepsilon)$ tame and let $1 \geq p > \frac{2}{\gamma + 3}$. Then for any $L > 1$ we have

$$\left\| 1_{L\Lambda}(T_{\zeta} H_{\zeta})^m P_{\leq N}T_{\zeta}^{k-m} 1_{L\Lambda^c} \right\|_p \leq C L^{1-p\varepsilon}.$$  

(4.26)

1We replace a sum by an integral.
The constant $C$ depends on $\Lambda, B_0, l, k, m, \gamma, p, \varepsilon, B_\varepsilon, V_\varepsilon$.

Proof. We define
\[
T := (T_1 H_\varepsilon)^m P_1 (H_\varepsilon T_1)^{k-m}.
\] (4.27)
We choose an $h_0 \in [0, 1)^2$. We will now use the $p$-Schatten norm property we called orthogonality in the first and forth step, and the $p$-triangle inequality in the second step. Hence,
\[
\|1_{LA} T_{1_{LA}}\|_p^p
\] (4.28)
\[
\leq \left\| \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} |1_{[0,1]^2 + z + h_0} T_{1_{LA}}| \right\|_p^p (4.29)
\]
\[
\leq \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} \left\| |1_{[0,1]^2 + z + h_0} T_{1_{LA}}| \right\|_p^p (4.30)
\]
\[
= \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} \left\| |1_{[0,1]^2 + z + h_0} T_{1_{DA}}| \right\|_p^p (4.31)
\]
\[
\leq \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} C \exp \left( -p\lambda \text{dist}(z + h_0, LA^0)^2 \right) \frac{(1 + \|z + h_0\|)^{pk\varepsilon}}{\Lambda} (4.32)
\]
\[
\leq \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} C \frac{\exp \left( -p\lambda \text{dist}(z + h_0, LA^0)^2 \right)}{(1 + \|z + h_0\|)^{pk\varepsilon}} (4.33)
\]
The last step follows by Theorem 4.6. The constant $C$ is independent of $z, h_0$. Now we can integrate this upper bound over $h_0 \in [0, 1)^2$. This integral can be resolved by Lemma A.2. Hence, we have
\[
\|1_{LA} T_{1_{LA}}\|_p^p \leq \int_{[0,1]^2} dh_0 \sum_{z \in \mathbb{Z}^2, z + h_0 \in D_{\mathcal{C}}(\Lambda)} \frac{\exp \left( -p\lambda \text{dist}(z + h_0, LA^0)^2 \right)}{(1 + \|z + h_0\|)^{pk\varepsilon}} (4.34)
\]
\[
= \int_{D_{\mathcal{C}}(\Lambda)} \frac{\exp \left( -p\lambda \text{dist}(x, LA^0)^2 \right)}{(1 + \|x\|)^{pk\varepsilon}} dx (4.35)
\]
\[
= L^2 \int_{\mathcal{D}^0(\Lambda)} \frac{\exp \left( -p\lambda L^2 \text{dist}(x', LA^0)^2 \right)}{(1 + L\|x'\|)^{pk\varepsilon}} dx' (4.36)
\]
\[
\leq CL^2 \left( \int_{\Lambda} \frac{\exp \left( -p\lambda L^2 \text{dist}(x', LA^0)^2 \right)}{(1 + L\|x'\|)^{pk\varepsilon}} dx' + L^{-pk\varepsilon} \left| D_{\mathcal{C}}(\Lambda) \setminus \Lambda \right| \right) (4.37)
\]
The constant $C$ does not depend on $L$. We are left to show, that the term behind $CL^2$ is bounded by $CL^{-1-p\varepsilon}$.
As $L \geq 1$, by (A.27), we have
\[
\left| D_{\mathcal{C}}(\Lambda) \setminus \Lambda \right| \leq \frac{C}{L} (4.38)
\]
because we can ignore the $\frac{1}{L}$ part. The constant depends on $\Lambda$ and this is the desired estimate.
To estimate the remaining integral, we first use Lemma A.3 and then once more Lemma A.4 to estimate the integral over the enumerator. Thus,
\[
\int_{\Lambda} \exp \left( -p\lambda L^2 \text{dist}(x', LA^0)^2 \right) dx' (4.39)
\]
\[
= \int_{\mathbb{R}} p\lambda L^2 h \exp \left( -p\lambda L^2 h^2 \right) \left| \{ x' \in \Lambda \mid \text{dist}(x', LA^0) \leq h \} \right| dh (4.40)
\]
\[
\leq \int_0^\infty p\lambda L^2 h \exp \left(-p\lambda L^2 h^2\right) Ch dh \\
= \int_0^\infty C \exp \left(-c(h')^2\right) \frac{dh'}{L} \\
= \frac{C}{L}.
\]

In the second to last step, we used the substitution \((h')^2 = p\lambda L^2 h^2\). The constant \(C\) depends on \(p, \lambda\) and in turn on \(p, l, k, m, \gamma, B_0\) and the decay of \(B_\varepsilon, V_\varepsilon\).

To deal with the denominator in (4.37), we use \(0 \leq (2\varepsilon L)^n\), such that for any \(x_0 \in \mathbb{R}^2\) and \(L > 1\):

\[
\left\| \int_{\Gamma} 1_{[0,1]^2+x_0}(\mathcal{M}_{H,\varepsilon}H_\varepsilon)^n \frac{1}{H-\zeta}(H_\varepsilon \mathcal{M}_{H,\varepsilon})^n 1_{\Lambda^\varepsilon}d\zeta \right\|_p \leq C(1 + \|x_0\|)^\gamma L^{-\nu}.
\]

The constant \(C\) depends on \(B_0, \gamma, \varepsilon, B_\varepsilon, V_\varepsilon\), but is independent of \(x_0\).

By the \(p\)-triangle inequality, the covering of \(\Lambda\) by translated unit boxes, like in the proof of Theorem 4.7 and choosing \(\nu\) sufficiently large, we arrive at

\[
\left\| \int_{\Gamma} 1_{\Lambda^\varepsilon}(\mathcal{M}_{H_\varepsilon}H_\varepsilon)^n \frac{1}{H-\zeta}(H_\varepsilon \mathcal{M}_{H_\varepsilon})^n 1_{\Lambda^\varepsilon}d\zeta \right\|_p \leq C.
\]

The constant \(C\) depends on \(\Lambda, B_0, \gamma, p, \varepsilon, B_\varepsilon, V_\varepsilon\).

We can now conclude the

**Proof of Theorem 3.3** We assume that \(a, b \notin \sigma(H)\). We begin with a fixed Landau level, meaning we even assume \(B_0(2l+1) < a < B_0(2l+3) < b < B_0(2l + 3)\) for some \(l \in \mathbb{N}\). We choose \(\Gamma\) as a path along the circle through \(a, b\) with centre \(\frac{a+b}{2}\). We choose \(n \in \mathbb{N}\), as in Corollary 4.9. Now we use Corollary 4.4. Hence, for any \(\zeta \in \text{im} \Gamma\), we have

\[
\frac{1}{H-\zeta} = \sum_{k=0}^{2n-1} (-1)^k \frac{1}{H_0-\zeta} \left(1 \frac{1}{H_0-\zeta} H_\varepsilon \right)^k \frac{1}{H-\zeta} \left(1 \frac{1}{H_0-\zeta} H_\varepsilon \right)^n.
\]

The path integral over every summand for \(0 \leq k \leq 2n - 1\) can be resolved by Lemma 4.3 and then bounded by Theorem 4.7. Hence, we have

\[
\left\| \frac{1}{2\pi i} \int_{\Gamma} (-1)^k 1_{\Lambda^\varepsilon} \frac{1}{H_0-\zeta} \left(1 \frac{1}{H_0-\zeta} H_\varepsilon \right)^k 1_{\Lambda^\varepsilon} \right\|_p \leq C L^{1-p\xi}.
\]
In particular, we realize that $P_l$ is the integral over the summand for $k = 0$ and hence this summand is cancelled in (3.12). Corollary 4.9 tells us that the path integral over the final summand is even bounded in the $p$-Schatten norm independently of $L$. Another application of the $p$-triangle inequality finishes the proof for a fixed Landau level.

For every $l \in \mathbb{N}$, such that $a < B_0(2l + 1) < b$, we choose a circle path, such that the last one hits $\mathbb{R}$ at $b$, each two neighbouring paths hit $\mathbb{R}$ at one common point not in $\sigma(H)$, the first path hits $\mathbb{R}$ at $a$ and every circle has a real-valued centre. Then we apply the estimate for a single Landau level and the $p$-triangle inequality.

If there is no Landau eigenvalue between $a$ and $b$, the associated projections are finite dimensional and will lead to an $O(1)$ term with respect to $L$. This also solves the case, where $a \in \sigma(H)$ or $b \in \sigma(H)$. Thus, it finishes the proof. $\square$

5. Kernel estimates

In this section we establish several properties of the Landau Hamilton operator $H_0$ and the operators $P_l, M_l, \zeta$ and in particular, their integral kernels. At the end of this section, we will also include an important integral bound.

We introduce the Laguerre polynomials and their generating function. For any $l \in \mathbb{N}$, the Laguerre polynomials $L_l$ is given by

$$L_l : [0, \infty) \to \mathbb{R}, \quad t \mapsto \sum_{k=0}^{l} \binom{l}{k} \frac{(-1)^k}{k!} t^k. \tag{5.1}$$

For any $s \in [0, \infty)$, $-1 < t < 1$, their generating function is given by

$$\sum_{l \in \mathbb{N}} t^l L_l(s) = \frac{1}{1 - t} \exp \left( \frac{-ts}{1 - t} \right). \tag{5.2}$$

Let $x, y \in \mathbb{R}^2$. For $l \in \mathbb{N}$, we define $p_l$ as the integral kernel of $P_l$,

$$p_l(x, y) := \frac{B_0}{2\pi} \exp \left( -\frac{B_0}{4} ||x - y||^2 + i\frac{B_0}{2} \langle x | J y \rangle \right) L_l \left( \frac{B_0}{2} \|x - y\|^2 / 2 \right). \tag{5.3}$$

Furthermore, for $0 < t < 1$, we define the operator $Q_t := \sum_l t^l P_l$. Its integral kernel is given by

$$q_t(x, y) := \sum_l t^l p_l(x, y) \tag{5.4}$$

and

$$= \frac{B_0}{2\pi(1 - t)} \exp \left( -\frac{B_0}{4} ||x - y||^2 + i\frac{B_0}{2} \langle x | J y \rangle - \frac{B_0 t}{2 - 2t} ||x - y||^2 \right) \tag{5.5}$$

$$= \frac{B_0}{2\pi(1 - t)} \exp \left( -\frac{B_0(1 + t)}{4(1 - t)} ||x - y||^2 + i\frac{B_0}{2} \langle x | J y \rangle \right). \tag{5.6}$$

We easily calculate

$$\left( -i \nabla_x - \frac{B_0}{2} J x \right) q_t(x, y) = \left( \frac{iB_0(1 + t)}{2(1 - t)} (x - y) - \frac{B_0}{2} J(x - y) \right) q_t(x, y), \tag{5.7}$$

and

$$\left( -i \nabla_x - \frac{B_0}{2} J x \right)^{\otimes 2} q_t(x, y) \tag{5.8}$$

$$= \left( \frac{iB_0(1 + t)}{2(1 - t)} (x - y) - \frac{B_0}{2} J(x - y) \right)^{\otimes 2} + \left( \frac{B_0(1 + t)}{2(1 - t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{B_0}{2} J \right) q_t(x, y). \tag{5.9}$$

Lemma 5.1. For any $j \in \mathbb{N}$, there are $C, a > 0$, independent of $l, B_0$, such that for any $x, y \in \mathbb{R}^2$

$$\| \left( -i \nabla_x - A_0(x) \right)^{\otimes j} p_l(x, y) \| \leq B_0^{1 + 0.5j} C a^j \exp \left( -\frac{B_0 ||x - y||^2}{8} \right). \tag{5.10}$$
Lemma 5.2. Let \( \mathcal{R} \subseteq C \) be cofinite, \( \zeta \in \mathbb{C}, \zeta \notin B_0(2I + 1) \) and \( l_0 \in \mathbb{N} \), such that \( l_0 \geq \max(1,2 I) \). Then we have the identity

\[
B_0 M_{I, \zeta} = \int_0^1 t^{-\zeta/B_0} \left( Q_{I, \zeta} - \sum_{t \leq t_0} t^{2l} P_l \right) dt + \sum_{l \in I, j \leq l_0} \frac{P_l}{(2l + 1 - \zeta/B_0)}.
\]

Proof. The idea of this proof is the formal identity

\[
\int_0^1 \sum_{t \in \mathbb{R}} t^{2l-\zeta/B_0} P_l dt = \sum_{l \in I} \frac{1}{1 + 2l - \zeta/B_0} P_l.
\]

Now we need to establish the precise meaning of this identity. First, we note that \( t^{-\zeta/B_0} = \exp(-\zeta/B_0 \ln(t)) \) is well defined, as \( t > 0 \). If \( \Re(\zeta)/B_0 \geq 2l + 1 \), then the integral of the summands
for $l$ will not exist, which is the reason we introduced $l_0$. We bounded the real part of $\zeta$ a little stronger than necessary to make the proof easier. Hence, we have

$$\int_0^1 \sum_{l>l_0} i^{2l-\zeta/B_0} P_l dt = \sum_{l>l_0} \frac{1}{1 + 2l - \zeta/B_0} P_l. \quad (5.23)$$

For any single $l > l_0$, the integral exists as a Bochner integral with respect to the operator norm. Lemma A.7 finishes the proof.

We will deal with a few integral kernels that have a singularity at the diagonal. To describe such a singularity, for any $s \in \mathbb{R}$, we introduce

$$b_s : \mathbb{R}^2 \to [0, \infty), \quad (x, y) \mapsto \begin{cases} -\frac{1}{\sqrt{2\pi s}} (\zeta_B) (x-y) \ln(\sqrt{B_0} ||x-y||) & s = 0, \\ \frac{1}{\sqrt{2\pi s}} (\zeta_B) (x-y) \frac{1}{||x-y||} & s \neq 0. \end{cases} \quad (5.24)$$

**Lemma 5.3.** Let $I \subset \mathbb{N}$ be cofinite. Then there is a function $F \in L^\infty_{loc}(\mathbb{C} \setminus (2I + 1))$, such that the following pointwise upper bounds hold for all $x, y \in \mathbb{R}^2, x \neq y$ and $\zeta \in \mathbb{C} \setminus B_0 (2I + 1)$:

$$\|\text{iker} M_{I, \zeta}(x, y)\| \leq F \left( \frac{\zeta}{B_0} \right) \left( b_0(x, y) + \exp \left( -\frac{B_0}{8} ||x-y||^2 \right) \right), \quad (5.25)$$

$$\|\text{iker} (-i\nabla - A_0) M_{I, \zeta}(x, y)\| \leq F \left( \frac{\zeta}{B_0} \right) \left( b_1(x, y) + \sqrt{B_0} \exp \left( -\frac{B_0}{8} ||x-y||^2 \right) \right), \quad (5.26)$$

$$\|(-i\nabla - A_0)^{\otimes 2} \text{iker} M_{I, \zeta}(x, y)\| \leq F \left( \frac{\zeta}{B_0} \right) \left( b_2(x, y) + B_0 \exp \left( -\frac{B_0}{8} ||x-y||^2 \right) \right). \quad (5.27)$$

**Remark.** The last inequality is structurally different, because the implied operator $(-i\nabla - A_0)^{\otimes 2} M_{I, \zeta}$ does not have a nice integral kernel. The differential of the integral kernel can still be considered but is not $L^1$ with respect to $y$ for any fixed $x$ and hence not a nice integral kernel. In general, this kernel does not fully describe the operator.

**Proof.** The set $I \subset \mathbb{N}$ is fixed throughout the proof.

For any $t \in [0, 1], l \in \mathbb{N}, j \in \{0, 1, 2\}$, we define

$$q_{l, j}(x, y) := (-i\nabla - A_0)^{\otimes j} q_{l}(x, y), \quad (5.28)$$

$$p_{l, j}(x, y) := (-i\nabla - A_0)^{\otimes j} p_{l}(x, y). \quad (5.29)$$

As $q_{l, j}, p_{l, j}$ are nice integral kernels, we can apply dominated convergence and see that

$$q_{l, j}(x, y) = \text{iker} \left( (-i\nabla - A_0)^{\otimes j} Q_l \right)(x, y), \quad (5.30)$$

$$p_{l, j}(x, y) = \text{iker} \left( (-i\nabla - A_0)^{\otimes j} P_l \right)(x, y). \quad (5.31)$$

We choose $l_0 \in \mathbb{N}$ minimal, such that $(2l_0 - 1)B_0 > \Re \zeta$ and $l_0 \geq \max(I^2)$. Now we use the representation established in Lemma 5.2. To prove, that for $j \in \{0, 1\},$ the operators have integral kernels, we want to use Lemma A.6. Hence, we only need to show, that the following inequality holds, in order to finish the proof for $j = 0, 1$:

$$\int_0^1 \left\| t^{-\zeta/B_0} \left( q_{2l, j}(x, y) - \sum_{l \leq l_0} t^{2l} p_{l, j}(x, y) \right) \right\| dt + \sum_{l \leq l_0} \frac{||p_{l, j}(x, y)||}{(2l + 1 - \zeta/B_0)} \quad (5.32)$$

$$\leq B_0 F \left( \frac{\zeta}{B_0} \right) \left( b_j(x, y) + B_0 \exp \left( -\frac{B_0}{8} ||x-y||^2 \right) \right). \quad (5.33)$$

For $j = 2$, however, we need to consider, that as the integrand is smooth on $(0, 1)$ and the summands at the end are smooth, we can try to exchange the integral with the differential operator $(-i\nabla - A_0)$. This will work, if the absolute value of the differential is integrable, by dominated convergence. Hence above integral bound also covers the case $j = 2$ and we will now proceed to bound all terms.
at the same time by choosing $j \in \{0, 1, 2\}$. We want to use Lemma 5.1 to bound the first integral on the interval $(0, t_0)$ and the sums. Hence,

$$\int_{t_0}^{\infty} \left\| t^{-\zeta/B_0} \left( q_{2,j}(x, y) - \sum_{l \leq t_0} t^{2l} p_{l,j}(x, y) \right) \right\| dt \leq \int_{t_0}^{\infty} \sum_{l > t_0} t^{2l-\Re(\zeta)/B_0} |p_{l,j}(x, y)| dt$$

$$\leq \int_{t_0}^{\infty} \sum_{l > t_0} t^{2l-\Re(\zeta)/B_0} C B_0^{1+0.5j} a^l \exp \left( -\frac{B_0}{8} \| x - y \|^2 \right) dt$$

$$= C B_0^{1+0.5j} \sum_{l > t_0} (2l + 1 - \Re(\zeta)/B_0)^{\Re(\zeta)/B_0} \exp \left( -\frac{B_0}{8} \| x - y \|^2 \right)$$

$$\leq F(\zeta/B_0) B_0^{1+0.5j} \exp \left( -\frac{B_0}{8} \| x - y \|^2 \right).$$

The last step holds, if $t_0^2 a < 1$, so we fix such a $t_0$ now. The function $F_0$ is in $L^\infty_{\text{loc}}(\mathbb{C} \setminus B_0(2I + 1))$, as $l_0$ is chosen locally bounded in $\zeta/B_0$. For fixed $l_0$ the function $F_0$ is continuous. The next step is bounding the remaining finite sum terms. Here, we will use, that $l \leq l_0$ and hence $a^l \leq C$. Thus,

$$\int_{t_0}^{1} \left\| \sum_{l \leq t_0} t^{2l-\zeta/B_0} p_{l,j}(x, y) \right\| dt + \sum_{l \leq t_0} \frac{\| p_{l,j}(x, y) \|}{(2l + 1 - \zeta/B_0)}$$

$$\leq C B_0^{1+0.5j} \left( \sum_{l \leq t_0} \left( \int_{t_0}^{1} t^{2l-\Re(\zeta)/B_0} dt \right) + \sum_{l \leq t_0} \left( \frac{1}{(2l + 1 - \zeta/B_0)} \right) \right) \exp \left( -\frac{B_0}{8} \| x - y \|^2 \right)$$

$$\leq F(\zeta/B_0) B_0^{1+0.5j} \exp \left( -\frac{B_0}{8} \| x - y \|^2 \right).$$

The function $F_1$ is in $L^\infty_{\text{loc}}(\mathbb{C} \setminus B_0(2I + 1))$ by the same argumentation as $F_0$. We will now turn our attention to the last remaining term. It is given by

$$\int_{t_0}^{1} \left\| q_{2,j}(x, y) t^{-\zeta/B_0} \right\| dt.$$
We want to do a change of variables to \( s := \frac{h^2}{s(t−y)} \). The interval is changed to \((h^2/5, \infty)\) and the determinant is \(h^2/(5s^2)\). In total we have

\[
\int_0^1 \left\| t^{-\zeta/B_0} q_{t^2,j}(x, y) \right\| dt
\]

(5.46)

\[
\leq F \left( \frac{\zeta}{B_0} \right) B_0^{1+0.5j} \int_{h^2/5}^{\infty} \frac{s}{h^2} \left( \frac{s}{h} \right)^j + \frac{\delta(j,2)s}{h^2} \exp(-s) \frac{h^2}{s^2} ds
\]

(5.47)

\[
\leq F \left( \frac{\zeta}{B_0} \right) B_0^{1+0.5j} \int_{h^2/5}^{\infty} \frac{1}{s} \left( \frac{s}{h} \right)^j + \frac{\delta(j,2)s}{h^2} \exp(-s) ds =: \Theta.
\]

If \( h > 1 \), we can bound the integrand by \( C \exp(-\frac{5}{8}s) \). The reduction in the exponent takes care of the factor \( s \), that appears in the case \( j = 2 \). Negative powers of \( h \) can be bounded by one. The integral can then be resolved and we have

\[
\Theta \leq CF \left( \frac{\zeta}{B_0} \right) B_0^{1+0.5j} \exp \left( \frac{-5h^2}{8S} \right)
\]

(5.49)

\[
= CF \left( \frac{\zeta}{B_0} \right) B_0^{1+0.5j} \exp \left( -\frac{B_0}{S} ||x−y||^2 \right).
\]

(5.50)

This is the desired upper bound.

If \( h \leq 1, j > 0 \), we can set the lower interval limit to 0 and get an integrable function in \( s \) multiplied by \( h^{-j} \). This gives us

\[
\Theta \leq CF \left( \frac{\zeta}{B_0} \right) B_0^{1+0.5j} h^{-j} \leq CF_2 \left( \frac{\zeta}{B_0} \right) B_0 b_j(x, y),
\]

(5.51)

which is the desired upper bound.

Finally, if \( h \leq 1, j = 0 \), we get a constant from the integral starting at \( \frac{1}{h} \). For the integral up to \( \frac{1}{h} \), we can bound the integrand by \( \frac{1}{h} \). Hence, the remaining integral is bounded by \( C(1−\ln(h^2)) \leq C(1+b_0(x,y)) \). Once again, this is the desired result.

We need one very important bound, which will have multiple uses later.

**Lemma 5.4.** Let \( u_1, u_2, u_3: \mathbb{R}^2 \to \mathbb{R}^+ \) be functions, such that \( \ln \circ u_j \) is Lipschitz with Lipschitz constant \( C_{\text{lip}} > 0 \). Let \( 0 \leq s_1, s_2 < 2 \) and \( \lambda > 0 \) be real numbers. Then there is a constant \( C > 0 \), depending only on \( B_0, s_1, s_2, \lambda \) and \( C_{\text{lip}} \), such that for all \( x, y \in \mathbb{R}^2, x \neq y \) we have the estimate

\[
\int_{\mathbb{R}^2} \left( \frac{b_{s_1}(x, y) + \exp(-B_0\lambda||x−y||^2)}{u_1(x)u_2(y)u_3(z)} \right) \left( \frac{b_{s_2}(y, z) + \exp(-B_0\lambda||y−z||^2)}{u_1(x)u_2(z)u_3(x)} \right) dy
\]

(5.52)

\[
\leq C b_{s_1+s_2-2}(x, z) + C \exp \left( -\frac{B_0\lambda||x−z||^2}{2} \right)
\]

(5.53)

If \( 1/(u_1u_2u_3) \in L^2(\mathbb{R}^2) \) and \( s_1 + s_2 < 3 \), then the integral kernel is Hilbert–Schmidt.

This is to be used together with Lemma 5.3 with \( \lambda = \frac{1}{8} \). The general \( \lambda \) is included to be able to chain more resolvents inductively. As all summands in the integral are positive, we may assume that they have the same constants in front.

**Proof.** We first need two minor results. Let \( a, b \in \mathbb{R}^2, j \in \{1, 2, 3\} \). Then for any \( \delta > 0 \), we have

\[
\frac{a_j(a)}{a_j(b)} = \exp(\ln \circ u_j(a)−\ln \circ u_j(b)) \leq \exp(C_{\text{lip}}||a−b||)
\]

(5.54)

\[
\leq \exp \left( \delta||a−b||^2 + \frac{C_{\text{lip}}^2}{4\delta} \right)
\]

(5.55)

\[
= C(C_{\text{lip}}, \delta) \exp(\delta||a−b||^2).
\]

(5.56)
We used the Young inequality. Furthermore (for any \(x, y, z \in \mathbb{R}^2\)) we have the identity
\[
\|x - y\|^2 + \|y - z\|^2 = \frac{1}{2} \|x - z\|^2 + 2 \left\|y - \frac{x + z}{2}\right\|^2.
\] (5.58)

We write \(R := \frac{b}{\sqrt{b^2}}\). Let us begin with the left-hand side of (5.59) and just write out most of the Hölder estimates. Hence,
\[
\text{LHS} \leq \frac{C}{u_1(x)u_3(z)} \left( \int_{D_R(x)} b_{s_1}(x, y)b_{s_2}(y, z)dy \left\| \frac{1}{u_2(\cdot)} \right\|_{L^\infty(D_R(x))} \right.
\]
\[
+ \exp \left( -\frac{B_0\lambda}{2} \|x - z\|^2 \right) \int_{\mathbb{R}^2} \exp \left( -\frac{B_0\lambda}{2} \|y - \frac{x + z}{2}\|^2 \right) dy
\]
\[
+ \left\|b_{s_1}(x, \cdot)\right\|_{L^1(D_R(x))} \left\| \exp \left(-\frac{B_0\lambda}{2} \|z\|^2\right) \right\|_{L^\infty(D_R(x))} \left\| \frac{1}{u_2(\cdot)} \right\|_{L^\infty(D_R(x))}
\]
\[
+ \left\|b_{s_2}(\cdot, z)\right\|_{L^1(D_R(z))} \left\| \exp \left(-\frac{B_0\lambda}{2} \|\cdot\|^2\right) \right\|_{L^\infty(D_R(z))} \left\| \frac{1}{u_2(\cdot)} \right\|_{L^\infty(D_R(z))} \right).
\] (5.60)

The \(L^\infty\) norms of the non-exponential terms can be bounded by a constant times the function evaluated at the centre, where the constant is given by (5.56), using \(a\) as the centre of the ball and \(b\) as any point in the ball. For the \(L^\infty\) norms of the exponential terms, we use Lemma A.5 with \(x_0 := y - z\). We are left to estimate the four \(L^1\) norms, some of which are written as integrals. The last two \(L^1\) norms can be bounded by a constant and that is sufficient. For the exponential integral, we first use (5.57) with \(\delta = B_0\) to replace the \(u_2(y)\) in the denominator by \(u_2((x + z)/2)\), getting a different Gaussian in the numerator, and then we can just bound its integral. With all of these, we get
\[
\text{LHS} \leq \frac{C}{u_1(x)u_3(z)} \times
\]
\[
\left( \int_{D_R(x)} b_{s_1}(x, y)b_{s_2}(y, z)dy \frac{1}{u_2(x)} + \frac{\exp \left(-\frac{B_0\lambda}{2} \|x - z\|^2\right)}{u_2((x + z)/2)} \right.
\]
\[
+ \exp \left( -\frac{B_0\lambda}{2} \|x - z\|^2 \right) \frac{1}{u_2(x)} + \exp \left( -\frac{B_0\lambda}{2} \|x - z\|^2 \right) \frac{1}{u_2(z)} \right) \right).
\] (5.61)

If we apply (5.57) again, we can get the desired bound for the last three summands. So, we only need to get the same bound for the first summand. If \(\|x - z\| > 2R\), the first summand vanishes. Otherwise, the term \(1/u_3(z)\) can be bounded by \(C/u_3(x)\) by (5.57). In the case \(2R \geq \|x - z\| \geq R/2\), we just bound the integral by a constant depending on \(R\), which can then be bounded by a constant times the Gaussian. We are left to consider the case \(\|x - z\| < R/2\). So, we are left to bound the integral
\[
\int_{D_R(x)} b_{s_1}(x, y)b_{s_2}(y, z)dy.
\] (5.62)

We have \(b_s(x, \cdot) \in L^p\) for any \(1 \leq p < 2/s\) and \(b_s\) is symmetric in \(x, y\). Hence, if \(s_1 + s_2 < 2\), we can bound this by a constant (independent of \(x, z\)) using Hölder. This can then by bounded by the Gaussian, as \(\|x - z\| \leq 2R\). We are left with the case \(s_1 + s_2 \geq 2\), where we want to bound the integral by \(b_{s_1+s_2-2}(x, z) + C\). As \(s_1, s_2 < 2\), we have \(s_1, s_2 > 0\). Let \(e_1 \in \mathbb{R}^2\) be the standard unit vector and let \(D_{r_1, r_2}(0)\) be the annulus between the two radii \(r_1 \leq r_2\). Then we have
\[
\int_{D_R(x)} b_{s_1}(x, y)b_{s_2}(y, z)dy
\] (5.63)
\begin{equation}
\leq \int_{D_0(x)} \frac{1}{\|x - y\|^{s_1} \|y - z\|^{s_2}} dy
\end{equation}

(5.69)

\begin{equation}
eq \int_{D_0(0)} \frac{1}{\|y\|^{s_1} \|y - (z - x)\|^{s_2}} dy
\end{equation}

(5.70)

\begin{equation}
= \int_{D_{R_1\|z-x\|^{-1}}(0)} \frac{\|x - z\|^{2-s_1-s_2}}{\|y\|^{s_1} \|y - e_1\|^{s_2}} dy
\end{equation}

(5.71)

\begin{equation}
\leq \int_{D_2(0)} \frac{\|y\|^{-s_1} \|y - e_1\|^{-s_2} dy}{\|x - z\|^{2-s_1-s_2}} + \int_{D_{2,R_1\|z-x\|^{-1}}(0)} \frac{\|x - z\|^{2-s_1-s_2}}{\|y\|^{s_1} \|y - e_1\|^{s_2}} dy
\end{equation}

(5.72)

\begin{equation}
\leq \frac{C}{\|x - z\|^{s_1+s_2-2}} + \int_{D_{2,R_1\|z-x\|^{-1}}(0)} \frac{C\|x - z\|^{2-s_1-s_2}}{\|y\|^{s_1+s_2}} dy
\end{equation}

(5.73)

\begin{equation}
\leq Cb_{s_1+s_2-2}(x,z).
\end{equation}

(5.74)

In the final step, we have to consider the case $s_1 + s_2 = 2$ separately. In this case, the integral at the end yields the term $b_0(x,z)$ up to a constant. In the case $s_1 + s_2 > 2$, the integral over $\|y\|^{-s_1-s_2}$ can be bounded by a constant, independent of $x, z$ and we are left with the correct singularity at the diagonal. This finishes the proof of the upper bound.

If $1/(u_1 u_2 u_3) \in L^2$ and $s_1 + s_2 < 3$, we get

\begin{equation}
C \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dz \left( \frac{b_{s_1+s_2-2}(x,z) + \exp \left( -\frac{B\lambda}{3} \|x-z\|^2 \right)}{u_1(x)u_2(x)u_3(x)} \right)^2
\end{equation}

(5.75)

\begin{equation}
=C \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} d(x,z) \left( \frac{b_{s_1+s_2-2}(x,z) + \exp \left( -\frac{B\lambda}{3} \|x-z\|^2 \right)}{u_1(x)u_2(x)u_3(x)} \right)^2
\end{equation}

(5.76)

\begin{equation}
\leq \int_{\mathbb{R}^2} dx \left( \frac{C}{(u_1(x)u_2(x)u_3(x))^2} \right) \leq C.
\end{equation}

(5.77)

Hence the integral kernel is Hilbert–Schmidt.

\textbf{Corollary 5.5.} Let $n \in \mathbb{N}$ and for any $0 \leq i \leq n$, let there be an operator $K_i$ with integral kernel $k_i$ on $L^2(\mathbb{R}^2)$, log-Lipschitz functions $u_i, v_i : \mathbb{R}^2 \to \mathbb{R}^+$, $\lambda_i > 0$, and $0 \leq s_i < 2$. Assume the integral kernels $k_i$ satisfy the upper bound

\begin{equation}
|u_i(x)||k_i(x,y)||v_i(y)| \leq Cb_{s_i}(x,y) + C \exp \left( -\lambda_i \|x-y\|^2 \right),
\end{equation}

(5.78)

for any $x \neq y$. Define $K := \prod_{i=0}^{n} K_i$ and let

\begin{equation}
s = -2n + \sum_{i=0}^{n} s_i.
\end{equation}

(5.79)

Then $K$ has an integral kernel $k$ and there are $\lambda > 0, C > 0$, such that for any $x \neq y$, we have the inequalities

\begin{equation}
|k(x,y)| \leq \frac{Cb_{s}(x,y) + C \exp \left( -\lambda \|x-y\|^2 \right)}{\prod_{i=0}^{n} u_i(x)v_i(x)}.
\end{equation}

(5.80)

\begin{equation}
|k(x,y)| \leq \frac{Cb_{s}(x,y) + C \exp \left( -\lambda \|x-y\|^2 \right)}{\prod_{i=0}^{n} u_i(y)v_i(y)}.
\end{equation}

(5.81)

For $s < 0$, we can replace $b_s$ by $0$ in (5.80) and (5.81), as $b_s$ is bounded and can be absorbed in the Gaussian.
Proof. The case \( n = 0 \) follows by (5.57). We continue with the case \( n = 1 \). By Lemma 5.4, we only have to show that \( K_0K_1 \) has is an integral operator and that for any \( x, z \in \mathbb{R}^2 \) with \( x \neq z \), we have
\[
\text{iker } K_0K_1(x, z) = \int_{\mathbb{R}^2} dy \, \text{iker } K_0(x, y) \text{iker } K_1(y, z). \tag{5.82}
\]
To do so, it is sufficient to find a function space \( \mathcal{Y} \supset C^0([0, 1]^2) \), on which \( K_0 \) and \( K_1 \) are continuous. We claim the topological vector space
\[
\mathcal{Y} := \bigcap_{\lambda \in \mathbb{R}} \{ f(\cdot) \exp(\langle \lambda \cdot \cdot \rangle) \in L^\infty(\mathbb{R}^2) \}
\]
does the trick.

We observe that any log-Lipschitz function \( u : \mathbb{R}^2 \to \mathbb{R}^+ \) with log-Lipschitz constant \( C_{Lip} \) satisfies for any \( x \in \mathbb{R}^2 \) that
\[
u(0) \exp(-C_{Lip}\|x\|) \leq u(x) \leq \nu(0) \exp(C_{Lip}\|x\|). \tag{5.84}
\]
Hence, such a function defines a continuous multiplication operator on \( \mathcal{Y} \). By the assumption (5.78), the operators \( K_i \) can each be written as a product of two such multiplication operators and a nice integral operator \( K'_i \) satisfying the kernel estimate
\[
\text{iker } K'_i(x, y) \leq b_n(x, y) + \exp\left(\frac{-\lambda_i\|x-y\|^2}{2}\right)
\]
for any \( x, y \in \mathbb{R}^2 \) with \( x \neq y \). We observe that such an integral operator is bounded on \( \mathcal{Y} \). Now, by Fubini we can conclude (5.82). This finishes the case \( n = 1 \) with \( \lambda = \frac{1}{4} \min\{\lambda_1, \lambda_2\} \).

As the resulting estimate for \( K_0K_1 \) is of the same form as the required estimate in (5.78), the induction over \( n \) follows trivially. \( \square \)

6. Proof of Theorem 4.6 and Theorem 4.8

We will first briefly summarize the approach for both proofs. We will start by conjugating with the unitary operator \( U_{x_0} \), as defined in Lemma A.1. Then, we can show that the operators we produce this way are Hilbert–Schmidt operators from \( L^2(\mathbb{R}^2) \) to \( H^{\gamma+2}([0, 1]^2) \) using the quasi isometry \( D_{\gamma+2} \), that we have constructed in Lemma A.9 and some commutator relations to move the differentials around. The proofs will conclude with Corollary A.11.

We denote by \( t_i \) the integral kernel of \( T_i \). By Lemma 5.3, we can only apply one full differential in \( x \) or \( y \) to \( t_i \), before we get a function, that is not a nice integral kernel anymore. However, the operator \( P_1 \) has a smooth integral kernel, which is why we would like to move differentials over to it. We will see that we can apply two differentials after \( M_{x, \epsilon} \) and still remain with a bounded operators, that is (in general) not an integral operator in Lemma 6.2.

We also still need to prove Lemma 4.1. However, it is convenient to prove a more general integral kernel bound along with it. For that, we need to introduce some new notation.

For any \( x_0 \in \mathbb{R}^2 \), \( j \in \mathbb{N} \), we introduce the three multiplication operators, which are given for any \( x \in \mathbb{R}^2 \) by
\[
B_{\epsilon, x_0}^{(j)}(x) := (-i)^j \left( \prod_{h=1}^{j} \partial_{\theta_h} \right) B_{\epsilon}(x + x_0) \tag{6.1}
\]
\[
A_{\epsilon, x_0}^{(j)}(x) := B_{\epsilon}^{(j)}(\cdot + x_0) * \frac{J}{2\pi \|\cdot\|^2}(x), \tag{6.2}
\]
\[
W_{\epsilon, x_0}^{(j)}(x) := (-i)^j \left( \prod_{h=1}^{j} \partial_{\theta_h} \right) W_{\epsilon}(x + x_0) \tag{6.3}
\]
\[
H_{\epsilon, x_0}^{(j)} := A_{\epsilon, x_0}^{(j)} - (-i\nabla - A_0) + W_{\epsilon, x_0}^{(j)}. \tag{6.4}
\]

The last equation defines a non-multiplication operator. [We have not defined \( A_0 \) and \( H_0 \), as this may lead to confusion with \( A_0 \) and \( H_0 \), if we set \( x_0 = 0 \).] The scalar product in the definition of
Lemma 6.1. Let $\gamma \in \mathbb{N}$, $V_\varepsilon$ be $(\gamma, \varepsilon)$ tame, $I \subset \mathbb{N}$ cofinite, and let $\zeta \in \mathbb{C} \setminus (2I + 1)B_0$. Furthermore, let $x_0 \in \mathbb{R}^2$ and $\mathbb{N} \ni d \leq \gamma$. Then there is a function $F \in L^\infty_{loc}(\mathbb{C} \setminus (2I + 1)B_0)$ and a real number $\lambda > 0$, such that for any $x, y \in \mathbb{R}^2$ with $x \neq y$, we have the upper bound

$$\|\text{ker} \, H_{x,x_0}^{(d)} M_{I, \zeta}(x, y)\| \leq F(\zeta) \frac{b_1(x, y) + \exp(-\lambda \|x - y\|^2)}{(1 + \|x + x_0\|^\varepsilon)}.$$ (6.8)

In particular, this is a nice integral kernel and the operator norm of $H_{x,x_0}^{(d)} M_{I, \zeta}$ is bounded independently of $x_0$. Furthermore, we have the estimate

$$\|H_{x} M_{I, \zeta}\|_{1n_0} \leq F(\zeta),$$ (6.9)

where $n_0$ is the smallest integer such that $2n_0 \varepsilon > 1$.

This lemma generalizes Lemma 4.1. The operators $H_{x,x_0}^{(d)}$ and $M_{I, \zeta}$ have been defined in (4.4) and (4.7). The function $b_0$ has been defined in (4.24).

Proof. We can estimate pointwise for $x \neq y$, using the assumption that $B_\varepsilon, V_\varepsilon$ are $(\gamma, \varepsilon)$ tame, Lemma 3.2 and Lemma 5.3. Thus, we have

$$\|\text{ker} \, A_{x,x_0}^{(d)} (-i\nabla - A_0) M_{I, \zeta}(x, y)\| \leq \frac{C}{(1 + \|x + x_0\|^\varepsilon)}\|\text{ker}(-i\nabla - A_0) M_{I, \zeta}(x, y)\|$$ (6.10)

$$\leq F\left(\frac{\zeta}{B_0}\right) \frac{b_1(x, y) + \sqrt{B_0} \exp\left(-\frac{B_0}{\varepsilon} \|x - y\|^2\right)}{(1 + \|x + x_0\|^\varepsilon)}.$$ (6.11)

And now for the other part, we observe that $W_\varepsilon = A_\varepsilon^2 + V_\varepsilon \in W^{1,\infty}_{(\varepsilon)}(\mathbb{R}^2, \mathbb{R})$ and can then use Lemma 5.3 to see

$$\|\text{ker} \, W_{x,x_0}^{(d)} M_{I, \zeta}(x, y)\| \leq \frac{C}{(1 + \|x + x_0\|^\varepsilon)}\|\text{ker} \, M_{I, \zeta}(x, y)\|$$ (6.12)

$$\leq F\left(\frac{\zeta}{B_0}\right) \frac{b_0(x, y) + \exp\left(-\frac{B_0}{\varepsilon} \|x - y\|^2\right)}{(1 + \|x + x_0\|^\varepsilon)}.$$ (6.13)
\begin{equation}
\leq F\left(\frac{\zeta}{B_0}\right) \frac{b_1(x, y) + \sqrt{B_0} \exp \left(-\frac{B_0}{2} \|x - y\|^2\right)}{(1 + \|x + x_0\|)^\xi}. \tag{6.16}
\end{equation}

In the last step we used \(b_0 \leq Cb_1\) and \(1 = C\sqrt{B_0}\). This shows the first claim.

We use properties we denoted as powers and Hilbert–Schmidt kernel of the \(p\)-Schatten norms. Hence the \(4n_0\)-Schatten norm of \(T\) can be calculated as the \(4n_0\)th root of the square integral of the integral kernel of \((TT^*)^{n_0}\). We note, that \(u(x) := (1 + \|x\|)^\xi\) is log-Lipschitz. We want to use \textbf{Corollary 5.5} Hence, we define for \(0 \leq i \leq 2n_0 - 1\)
\begin{equation}
K_i := \begin{cases} H_{\xi} M_{1, \xi} & i \text{ even}, \\ (H_{\xi} M_{1, \xi})^* & i \text{ odd}. \end{cases} \tag{6.17}
\end{equation}

For even \(i\), we choose \(u_i(x) = u(x), v_i(x) = 1\) and for odd \(i\), we choose \(v_i(x) = u(x), u_i(x) = 1\). We always have \(s_i = 1\). Now we can apply \textbf{Corollary 5.5} and get for any \(x \neq y\) that
\begin{equation}
|\ker ((H_{\xi} M_{1, \xi}) (H_{\xi} M_{1, \xi})^*)^{n_0} (x, y)| \leq F(\zeta) \frac{b_0(x, y) + \exp \left(-\lambda B_0 \|x - y\|^2\right)}{(1 + \|x\|)^{2n_0\xi}}. \tag{6.18}
\end{equation}

\begin{equation}
F(\zeta) \frac{b_0(x, y) + \exp \left(-\lambda B_0 \|x - y\|^2\right)}{(1 + \|x\|)^{2n_0\xi}}. \tag{6.19}
\end{equation}

The function \(F\) is in \(L^\infty(C \setminus (2I + 1)B_0)\). This integral kernel is in \(L^2\), as \(2n_0\xi > 1\). The \(b_0\) term only appears for \(n_0 = 1\), as for \(n_0 > 1\), we get \(s < 0\), which corresponds to a bounded \(b_s\). \(\square\)

We will now prove some useful methods to deal with the differentials we will have to apply in order to use \textbf{Corollary A.11}. We will first see that, in a way, \(M_{1, \xi}\) can take two differentials, and then we will see how to move further differentials past \(M_{1, \xi}\) and \(H_{\xi, x_0}\).

Let \(j_1, j_2 \in \{1, 2\}\) and \(h \in \{\pm 1\}\). Then we observe the commutator relation
\begin{equation}
\left[(-i\nabla - hA_0)_{j_1}, (-i\nabla - A_0)_{j_2}\right] = \frac{B_0}{2} \left[\left(J_{j_1}, (JX)_{j_2}\right) + h\left((JX)_{j_1}, \nabla_{j_2}\right)\right] \tag{6.20}
\end{equation}
\begin{equation}
= \frac{B_0}{2} \left(J_{j_2j_1} - hJ_{j_1j_2}\right). \tag{6.21}
\end{equation}

Here, \(X\) refers to the multiplication operator associated to the identity on \(\mathbb{R}^2\). As the matrix \(J\) is skew-symmetric, this states that the so called covariant derivative \(-i\nabla + A_0\) commutes with \(-i\nabla - A_0\) and hence it commutes with the operators \(H_0, P_l, M_{1, \xi}, T_l\) for any \(l \in \mathbb{N}\), cofinite subset \(I \subset \mathbb{N}\) and any \(\zeta \in C \setminus B_0(2I + 1)\).

For \(h = +1\), however, it motivates the definition of the annihilation and construction operators. They are defined by
\begin{equation}
a_\pm := \frac{1}{\sqrt{B_0}} ((-i\nabla - A_0)_{j_1} \pm i (-i\nabla - A_0)_{j_2}). \tag{6.22}
\end{equation}

Using \textbf{6.21} for \(j_1 = 1, j_2 = 2, h = 1\) and \textbf{2.29}, we observe
\begin{equation}
B_0 a_- a_+ = H_0 + B_0, \quad B_0 a_+ a_- = H_0 - B_0, \quad a_+^* = a_- \tag{6.23}
\end{equation}

This implies that \(a_-\) is surjective and \(a_+\) is injective. Let \(l \in \mathbb{N}\). Then we have
\begin{equation}
(H_0 + B_0) a_- P_l = B_0 a_- a_+ a_- P_l = a_- (H_0 - B_0) P_l = 2l B_0 a_- P_l. \tag{6.24}
\end{equation}

This states that \(a_- P_l\) maps into the eigenspace of \(H_0\) with eigenvalue \((2l - 1)B_0\), which is the image of \(P_{l-1}\). If \(l > 0\), as \(a_-\) is surjective, it has to map the image of \(P_l\) onto the image of \(P_{l-1}\). With an analogous computation for \(a_+^* = a_+\), we arrive at
\begin{equation}
P_{l-1} a_- = P_{l-1} a_- P_l = a_- P_l. \tag{6.25}
\end{equation}

We recall that the operator \(M_{1, \xi}\) has been defined in \textbf{4.7}. 22
Lemma 6.2. For any $I \subset \mathbb{N}$ cofinite and $j_1, j_2 \in \{1, 2\}$, there is an $F \in L^\infty(\mathbb{C} \setminus B_0(2I + 1))$, such that for any $\zeta \in \mathbb{C} \setminus B_0(2I + 1)$, we have the estimates
\[
\|(-i\nabla - A_0)_{j_1} M_{I, \zeta} (-i\nabla - A_0)_{j_2}\|_\infty \leq F(\zeta), \tag{6.26}
\]
\[
\|(-i\nabla - A_0)_{j_1} (-i\nabla - A_0)_{j_2} M_{I, \zeta}\|_\infty \leq F(\zeta). \tag{6.27}
\]

Proof. We will only prove the first claim, as the second follows completely analogous. As both components of $(-i\nabla - A_0)$ are linear combinations of $a_+, a_-$, it suffices to show that for any $h_1, h_2 \in \{+, -\}$, we have the required estimate for the operator $a_h M_{I, \zeta} a_{h_2}$. Let $l \in \mathbb{N}$. We consider the operator
\[
S_l := a_{h_1} M_{I, \zeta} a_{h_2} P_l. \tag{6.28}
\]
For any $k \in \mathbb{N}$, we define $k + + = k + 1$ and $k + - = k - 1$. Using (6.25), we see
\[
a_{h_1} M_{I, \zeta} a_{h_2} P_l = a_{h_1} M_{I, \zeta} P_{l+h_2} a_{h_2} \tag{6.29}
\]
\[
= \frac{1}{(2(l + h_2) + 1)B_0} \frac{a_{h_1} P_{l+h_2} a_{h_2}}{a_{h_1}} \tag{6.30}
\]
\[
= \frac{1}{(2(l + h_2) + 1)B_0} a_{h_1} P_{l+h_2} a_{h_2} \tag{6.31}
\]
We use the convention $\frac{a}{b} = 0$ in this proof. Hence, the family of operators $S_l$ satisfy the conditions of Lemma A.7. So we just need to bound the norm of $S_l$. Using (6.23), we observe that for any $h \in \{+, -\}$ and $k \in \mathbb{N}$,
\[
\|a_h P_k\|^2 = \|P_k a_h^* a_{h} P_k\| = 2k + 1 + h. \tag{6.32}
\]
Using (6.30), this leads to
\[
\|S_l\| = \frac{1}{(2(l + h_2) + 1)B_0} \frac{\|a_{h_1} P_{l+h_2}\| \|P_{l+h_2} a_{h_2}\|}{\|a_{h_1}\|} \tag{6.33}
\]
\[
= \frac{1}{(2(l + h_2) + 1)B_0} \sqrt{(2(l + h_2) + 1 + h_1)(2l + 1 + h_2)} \leq F(\zeta). \tag{6.34}
\]
This finishes the proof. \hfill \Box

Lemma 6.3. Let $\gamma, n \in \mathbb{N}$ and assume that $(B_{\varepsilon}, V_{\varepsilon})$ are $((\gamma, \varepsilon))$ tame. Let $\mathbb{N} \ni \gamma' \leq \gamma$. Then there is a set of matrices $\left(N_{\mu} \in \text{Lin}\left((\mathbb{C}^{2\gamma}, \mathbb{C}^{2\gamma'})\right)\right)_{\mu \in \mathbb{N}^{n+1}, |\mu| = \gamma'}$, such that for any admissible $I, \zeta, x_0$, the identity
\[
(-i\nabla + A_0)^{\otimes\gamma'}(M_{I, \zeta} H_{x_0})^n = \sum_{\mu \in \mathbb{N}^{n+1}, |\mu| = \gamma'} N_{\mu} \left(\bigotimes_{j=1}^{n} M_{I, \zeta} H_{x_0}^{[\mu_j]}\right) \otimes (-i\nabla + A_0)^{\otimes|\mu|+1} \tag{6.35}
\]
holds in the sense that both operators agree as continuous operators from the space $W_{(\infty)}^{\gamma, \infty}(\mathbb{R}^2, \mathbb{C})$ to the space $W_{(\infty)}^{0, \infty}(\mathbb{R}^2, \mathbb{C}^{2\gamma'})$.

Proof. Let $d \in \mathbb{N}, h \in \{1, 2\}$ with $0 \leq d < \gamma$. We recall (6.1) to (6.4), and (6.6). We have $A_{(\varepsilon, x_0)} \in W_{(\varepsilon)}^{\gamma-d, \infty}(\mathbb{R}^2, \mathbb{C}^{2\gamma'})$ and $W_{(\varepsilon)}^{(d), \infty}(\mathbb{R}^2, \mathbb{C}^{2\gamma'})$ by the assumptions and Lemma 3.2. Hence, by the product rule, we have for any $\mathbb{N} \ni \gamma' \leq \gamma - d$ that the multiplication operators $A_{(\varepsilon, x_0)}$ and $W_{(\varepsilon)}^{(d), \infty}$ are continuous operators from $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C})$ to the spaces $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C}^{2\gamma'})$ respectively $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C}^{2\gamma'})$. Furthermore, the operator $(-i\nabla + A_0)$ obviously maps $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C})$ to $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C})$ continuously for any $\gamma' \in \mathbb{N}$. Finally, by Lemma 5.3 and the fact that the covariant derivative $-i\nabla + A_0$ commutes with $M_{I, \zeta}$ by (6.21), for any $\gamma' \in \mathbb{N}$, the operators $M_{I, \zeta}$ and $(-i\nabla - A_0)M_{I, \zeta}$ are continuous from $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C})$ to the spaces $W_{(\infty)}^{\gamma', \infty}(\mathbb{R}^2, \mathbb{C})$, respectively
In the claimed sense. We can now prove Theorem 4.8.

Proof of Theorem 4.8. We begin by conjugating with the unitary operator $U_{x_0}$, that we have defined in Lemma A.1. Hence, as the $p$-Schatten quasi norm is unitarily equivalent, we have

$$
\left\| \int_{\Gamma} 1_{[0,1]^2-x_0} (M_{\mu_1,\zeta} H_{x_0})^n \frac{1}{H - \zeta} (H_x M_{\mu_1,\zeta})^n 1_{\Lambda \zeta} d\zeta \right\|_p
$$

$$
= \left\| \int_{\Gamma} 1_{[0,1]^2} (M_{\mu_1,\zeta} H_{x_0})^n U_{x_0} \frac{1}{H - \zeta} (H_x M_{\mu_1,\zeta})^n 1_{\Lambda \zeta} U_{x_0}^{-1} d\zeta \right\|_p
$$
Let $q$ satisfy $\frac{1}{q} + \frac{1}{2} = \frac{1}{p}$. As $p > \frac{2}{\gamma + 2}$, we have $q > \frac{2}{\gamma + 2}$. Hence, we can apply [Corollary A.11] with $\gamma + 2$ and the property Hölder I (see [Proposition 2.1]) to get the upper bound

\begin{equation}
(6.50) \leq C \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right) \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right)
\end{equation}

\begin{equation}
(6.51) \leq C \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right) \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right)
\end{equation}

The last step relies on the fact that the Hilbert–Schmidt norm (2-Schatten norm) is a norm and not just a quasi-norm. Now it suffices to bound the integrand uniformly on the integration path. For this, we first use the quasi-isometry $D_{\gamma + 2}$ constructed in [Lemma A.9]. Hence, we have

\begin{equation}
(6.53) \leq C \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right) \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right)
\end{equation}

\begin{equation}
(6.54) \leq C \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right) \left( \int |(H \gamma M_n)^{\gamma} 1_{L^6} U_{x_0}^{\gamma + 2} \rangle \langle 1_{L^6} U_{x_0}^{\gamma + 2} \rangle | \right)
\end{equation}

The third step relies on applications of Hölder I (see [Proposition 2.1]). The last step uses that $U_{x_0}$ is unitary on $L^2(\mathbb{R}^2)$ and another application of Hölder I. The conjugation with $U_{x_0}$ was only needed for the first term. It does make a difference there, as $U_{x_0}$ is not unitary on $H^{\gamma + 2}([0, 1]^2)$ and does not commute with $D_{\gamma + 2}$.

We begin with the last factor in (6.57). As we are still free to choose $n \in \mathbb{N}$, we can assume $n > 2$. We use the kernel estimate in [Lemma 6.1] and [Corollary 5.5] similar to the proof of the second result of [Lemma 6.1] to arrive at the following estimate for any $x, y \in \mathbb{R}^2$:

\begin{equation}
|\text{ker} (H \gamma M_n)^{\gamma} (x, y)| \leq \frac{F(\gamma)}{(1 + \|y\|)^{n+\nu}}.
\end{equation}

Now we let $n \varepsilon > 1 + \nu$. Then using the Hilbert–Schmidt kernel identity, we have

\begin{equation}
(6.59) \leq F(\gamma) \int \frac{1}{(1 + \|y\|)^{2+2\nu}}
\end{equation}

In the second step, we use that the Gauss kernel is integrable over $x$, that the integral is independent of $y$, and that $n \varepsilon > 1 + \nu$. The third step uses that there is some $r > 0$ such that $D_r(0) \subset A$ and that $\nu > 0$.

For the second factor in (6.57), we observe

\begin{equation}
\left( \int \frac{1}{H - \zeta} \right) \leq \frac{1}{\text{dist}(\zeta, \sigma(H))}
\end{equation}

which is bounded along the path $\Gamma$. 

For the first factor in (6.57), we first consider the case \( \gamma' \geq 0 \). Here, we start by using \textbf{Lemma 6.3} with the parameters \( \gamma' \) and 2. Hence, we have

\[
\left\| 1_{[0,1]^2} \left( -i \nabla + A_0 \right)^{\otimes (\gamma' + 2)} \left( M_{N,\zeta} H_{\varepsilon,x_0} \right)^n \right\|_{\infty} \leq C \left\| M_{N,\zeta} \right\|_{S_{\infty}(L^2(\mathbb{R}^2),H^2([0,1]^2))} \leq C \sum_{j=0}^{2} \left\| ( -i \nabla - A_0 )^{\otimes j} M_{N,\zeta} \right\|_{\infty} \leq F(\zeta). \tag{6.67}
\]

For any \( N \geq d \leq \gamma \), the multiplication operators \( A^{(d)}_{\varepsilon}, W^{(d)}_{\varepsilon} \) are bounded operators with a norm not depending on \( x_0 \). Furthermore, by \textbf{Lemma 6.3} the operators \( M_{N,\zeta} (-i \nabla - A_0) M_{N,\zeta} \) and \( M_{N,\zeta} (-i \nabla - A_0) M_{N,\zeta} \) are bounded, and the operator \( ( -i \nabla - A_0 )^{\otimes 2} M_{N,\zeta} \) is bounded by \textbf{Lemma 6.2}. Now, we use (6.6) and (6.6) to conclude

\[
\left\| H^{(\mu_1)}_{\varepsilon,x_0} \otimes M_{N,\zeta} H^{(\mu_2)}_{\varepsilon,x_0} \right\|_{\infty} = \left\| A^{(\mu_1)}_{\varepsilon,x_0} \cdot (-i \nabla - A_0) + W^{(\mu_1)}_{\varepsilon,x_0} \right\|_{\infty} \leq F(\zeta). \tag{6.69}
\]

We are left to estimate the expression in (6.66). We rename \( \mu_3 \) to \( d \) and do an induction over \( d \) for \( 0 \leq d \leq \gamma \). Let \( e \in \mathbb{N} \) be minimal with \( e \varepsilon \geq 1 \). The claim of our induction is that for \( n \geq d(e + 2) + 3 \), we have the estimate

\[
\left\| ( -i \nabla + A_0 )^{\otimes d} ( M_{N,\zeta} H_{\varepsilon,x_0} )^{n-2} \right\|_{\infty} \leq F(\zeta)(1 + \| x_0 \|)^d, \tag{6.71}
\]

for some \( F \in L_{\infty}(\mathbb{C} \setminus \sigma(H)) \) depending on \( n,d \). The induction start at \( d = 0 \) only uses that \( \| M_{N,\zeta} H_{\varepsilon,x_0} \|_{\infty} = \left\| \left( H_{\varepsilon,x_0} M_{\zeta} \right)^* \right\|_{\infty} \leq F(\zeta) \) by \textbf{Lemma 6.1} and that the product of bounded operators is bounded. For the step \( d \rightarrow d + 1 \leq \gamma \), we first use \textbf{Lemma 6.3} with the parameters \( d \) and \( e + 2 \). Hence, we have

\[
\left\| ( -i \nabla + A_0 )^{\otimes (d+1)} ( M_{N,\zeta} H_{\varepsilon,x_0} )^{n-2} \right\|_{\infty} \leq F(\zeta)(1 + \| x_0 \|)^d. \tag{6.72}
\]

for some \( F \in L_{\infty}(\mathbb{C} \setminus \sigma(H)) \) depending on \( n,d \). The induction start at \( d = 0 \) only uses that \( \| M_{N,\zeta} H_{\varepsilon,x_0} \|_{\infty} = \left\| \left( H_{\varepsilon,x_0} M_{\zeta} \right)^* \right\|_{\infty} \leq F(\zeta) \) by \textbf{Lemma 6.1} and that the product of bounded operators is bounded. For the step \( d \rightarrow d + 1 \leq \gamma \), we first use \textbf{Lemma 6.3} with the parameters \( d \) and \( e + 2 \). Hence, we have

\[
\left\| ( -i \nabla + A_0 )^{\otimes (d+1)} ( M_{N,\zeta} H_{\varepsilon,x_0} )^{n-2} \right\|_{\infty} \leq F(\zeta)(1 + \| x_0 \|)^d. \tag{6.72}
\]
This finishes the induction over \(d\). Hence, we have

\[
\left\| \text{iker}(-i\nabla + A_0) M_{N,L}(x,y) \right\| \leq \left\| \text{iker}(-i\nabla - A_0) M_{N,L}(x,y) \right\| + C\|x\| \left\| \text{iker} M_{N,L}(x,y) \right\|
\]

(6.78)

\[
\leq F(\zeta) \left( 1 + \|x\| \right) \left( b_1(x,y) + \exp(-\lambda \|x - y\|^2) \right).
\]

(6.79)

The final step relies on the fact \(\varepsilon \geq 1\). Using Corollary 2.3, we can conclude

\[
\left\| \left(-i\nabla + A_0\right) M_{N,L} \otimes \bigotimes_{j=1}^{e} \left(H_{\varepsilon,x_0}^{(\mu_j)} M_{N,L}\right)(x,y) \right\| \leq F(\zeta)(1 + \|x_0\|).
\]

(6.83)

This finishes the induction over \(d\). Hence, we have proven (6.71) and can continue the estimate in (6.69). Using (6.68) and (6.70), we observe that for \(0 \leq \gamma' \leq \gamma\), we have

\[
\left\| 1_{[0,1]^2} (-i\nabla + A_0)^{\otimes (\gamma' + 2)} (M_{N,L} H_{\varepsilon,x_0})^n \right\|_{\infty} \leq F(\zeta)(1 + \|x_0\|)^{\gamma}. \tag{6.84}
\]

Now we need to consider the case \(\gamma' \in \{-2,-1\}\). For these, we estimate

\[
\left\| 1_{[0,1]^2} (-i\nabla + A_0)^{\otimes (\gamma' + 2)} (M_{N,L} H_{\varepsilon,x_0})^n \right\|_{\infty} \leq F(\zeta)(1 + \|x_0\|)^{\gamma}. \tag{6.85}
\]

The operator \(M_{N,L} H_{\varepsilon,x_0} = H_{\varepsilon,x_0} M_{N,L}^\dagger\) has an operator norm \(\leq F(\zeta)\) by Lemma 6.1, the middle factor is bounded by (6.70), and the first factor is bounded by Lemma 5.3 for \(\gamma' = -2\) and by (6.79) for \(\gamma' = -1\), in both cases the operator norm is \(\leq F(\zeta)\).

Now we have suitable upper bounds for the all factors in (6.57). The other factors are bounded by (6.61) and (6.62). Thus, we conclude

\[
\left\| (M_{N,L} H_{\varepsilon,x_0})^n U_{x_0} \frac{1}{H - \zeta} (H_{\varepsilon} M_{N,L})^n 1_{L^2(\mathbb{R}^2)} U_{x_0}^{-1} \right\|_{S_2(L^2(\mathbb{R}^2), H^{2+\delta(0,1)}^\dagger)^2} \leq F(\zeta)(1 + \|x_0\|)^{\gamma} L^{-\nu}. \tag{6.88}
\]
Using (6.52), we have now finished this proof.\(\square\)

We need one more technical lemma to prove Theorem 4.6.

**Lemma 6.4.** Let \(d \in \mathbb{N}, \kappa \in [0, \infty),\) and let \(S\) be an integral operator on \(L^2(\mathbb{R}^2)\) satisfying for any \(x, y \in \mathbb{R}^2\)
\[
|\text{ker} S(x, y)| \leq C \frac{(1 + \|x\|^d)}{(1 + \|x + x_0\|^\kappa)} \exp \left(-\lambda \|x - y\|^2\right).
\]

Furthermore, let \(\Omega \subset \mathbb{R}^2\) be bounded. Then there are constants \(C, \lambda'\) such that for any \(R \in [0, \infty),\) we have the estimate
\[
\left\|1_{\Omega}S1_D^R(0)\right\|_2 \leq C\exp(-\lambda' R^2)
\]

**Proof.** We use the Hilbert–Schmidt kernel property (see Proposition 2.1). Hence, by the unitary equivalence of the \(p\)-Schatten norms, we have
\[
\left\|1_{\Omega}S1_D^R(0)\right\|_2^2 = \int_\Omega dx \int_{D^R(0)} dy |\text{ker} S(x, y)|^2
\]
\[
\leq C \int_\Omega dx \int_{D^R(0)} dy \frac{(1 + \|x\|)^{2d}}{(1 + \|x + x_0\|)^{2\kappa}} \exp \left(-2\lambda \|x - y\|^2\right)
\]
\[
\leq C \int_\Omega dx \int_{D^R(0)} dy \frac{1}{(1 + \|x_0\|)^{2\kappa}} \exp(-\lambda\|y\|^2)
\]
\[
\leq C\exp(-2\lambda' R^2)
\]

The second step uses \(x \in \Omega\) and Lemma A.3 Then we used Lemma A.3 again. This finishes the proof.\(\square\)

**Proof of Theorem 4.6.** We start off similarly to the proof of Theorem 4.8. In particular, we begin by conjugating with the unitary operator \(U_{x_0}\), as defined in Lemma A.1. Hence, we have
\[
\left\|1_{[0,1]^2-x_0}(T_iH_{x,0})^m P_l(H_{x,0}T_i)^{k-1}1_D^R(0)\right\|_p
\]
\[
\leq C \left\|1_{[0,1]^2}(T_iH_{x,0})^m P_l(H_{x,0}T_i)^{k-1}1_D^R(0)\right\|_{L^2(\mathbb{R}^2, H^\gamma)}
\]
\[
\leq C \gamma \sum_{\gamma'=-2}^\gamma \left\|1_{[0,1]^2}(\gamma' + 2)(T_iH_{x,0})^m P_l(H_{x,0}T_i)^{k-1}1_D^R(0)\right\|_2.
\]

We used the quasi-isometry \(D_{\gamma+2}\) as constructed in Lemma A.9. We will now establish two kernel estimates that will be needed to finish this proof.

Looking at (5.3), we observe that for any \(d \in \mathbb{N}\) and \(h \in \{0, 1\},\) there are \(\lambda, C \in \mathbb{R}^+\), depending on \(B_0, l, d, h,\) such that for any \(x, y \in \mathbb{R}^2,\) we have the upper bound
\[
\left\|(-i\nabla - A_0)^\otimes h \otimes (-i\nabla_x + A_0(x))\otimes d P_l(x, y)\right\| \leq C(1 + \|x\|)^{d+h} \exp \left(-\lambda \|x - y\|^2\right).
\]

Let \(\mathbb{N} \ni j \geq \gamma.\) Then, using (6.4), we observe
\[
|\text{ker} H_{x,0}^{(j)}(-i\nabla + A_0)^\otimes d P_l(x, y)\| \leq C \frac{(1 + \|x\|)^{d+1}}{(1 + \|x + x_0\|)^e} \exp \left(-\lambda \|x - y\|^2\right).
\]
Now we consider the case $m = 0$. Here, we can use (6.98), the kernel estimate for $H_{x,x_0}T_l$, that is provided by Lemma 6.1 and Corollary 5.5 to arrive at

$$\|\ker(-i\nabla + A_0)\otimes\rho_+^2 P_l(H_{x,x_0})^k(x,y)\| \leq C\left(\frac{1 + ||x||^2}{1 + ||x + x_0||}\right)^{ke} \exp(-\lambda ||x - y||^2).$$

(6.100)

As the kernel of $(-i\nabla + A_0)\otimes d P_l$ has no singularity on the diagonal, the term $b_s$ can be ignored. By Lemma 6.4 we have now finished the case $m = 0$.

Now we consider the case $m > 0$ and $\gamma' \in \{-2, -1\}$. Here, we can use Lemma 5.3 to get (compare (6.79))

$$\|\ker(-i\nabla + A_0)\otimes\rho_+^2 T_l(x,y)\| \leq C(1 + ||x||^{\gamma'+2} \{b_1(x,y) + \exp(-\lambda ||x - y||^2).$$

(6.101)

With this kernel estimate, the one in Lemma 6.1, and (6.99) with $d = j = 0$, we can employ Corollary 5.5 to get

$$\|\ker(-i\nabla + A_0)\otimes\rho_+^2 (T_l H_{x,x_0})^m P_l(H_{x,x_0})^k \gamma - m(x,y)\| \leq C\left(\frac{1 + ||x||^3}{1 + ||x + x_0||}\right)^{ke} \exp(-\lambda ||x - y||^2).$$

(6.102)

Once again, as $P_l$ has no singularity at the diagonal, the term $b_s$ can be ignored and by Lemma 6.4 we have finished this case as well.

We are left with the case $m > 0$ and $\gamma' \geq 0$. Here, we first apply Lemma 6.3 with the parameters $\gamma'$ and $m$. Hence, we have

$$\left\|\sum_{\mu \in \mathbb{N}^{\mu + 1}_{\geq \gamma'}} \Bigg(\sum_{j=1}^{m} (T_l H_{x,x_0})^j \otimes (-i\nabla + A_0)\otimes\rho_+^{m+1} P_l(H_{x,x_0})^k \right\|_{2}$$

$$= C \sup_{\mu \in \mathbb{N}^{\mu + 1}_{\geq \gamma'}} \left\|\sum_{j=1}^{m} (T_l H_{x,x_0})^j \otimes (-i\nabla + A_0)\otimes\rho_+^{m+1} P_l(H_{x,x_0})^k \right\|_{2}$$

$$\leq C \sup_{\mu \in \mathbb{N}^{\mu + 1}_{\geq \gamma'}} \left\|\sum_{j=1}^{m} (T_l H_{x,x_0})^j \otimes (-i\nabla + A_0)\otimes\rho_+^{m+1} P_l(H_{x,x_0})^k \right\|_{2}$$

$$= C \sup_{\mu \in \mathbb{N}^{\mu + 1}_{\geq \gamma'}} \left\|\sum_{j=1}^{m} (T_l H_{x,x_0})^j \otimes (-i\nabla + A_0)\otimes\rho_+^{m+1} P_l(H_{x,x_0})^k \right\|_{2}$$

(6.104)

(6.105)

(6.106)

(6.107)

The operator $1_{[0,1]}(-i\nabla + A_0)\otimes\rho_+^2 T_l$ does not have a nice integral kernel. This is why we cannot directly get a kernel bound from this representation. Let $\varphi \in C^\infty_c(\mathbb{R}^2)$ be a smooth cutoff function satisfying $\varphi(x) = 1$ for $x \in D_l(0)$, $\varphi(x) = 0$ for $x \in D_2^2(0)$, and $\varphi(x) \leq 1$ everywhere. We introduce the operators $T_{l,n}$ and $T_{l,f}$, which are defined by the integral kernels given for any $x, y \in \mathbb{R}^2$ with $x \neq y$ by

$$t_{l,n}(x,y) := \varphi(x - y) t_l(x,y),$$

(6.108)

$$t_{l,f}(x,y) := (1 - \varphi(x - y)) t_l(x,y).$$

(6.109)

Obviously, $T_{l,n} + T_{l,f} = T_l$. Furthermore, for any $d \in \{0, 1, 2\}$, the operator $(-i\nabla - A_0)\otimes d T_{l,f}$ has a nice integral kernel satisfying

$$\|\ker(-i\nabla - A_0)\otimes d T_{l,f}(x,y)\| \leq C \exp(-\lambda ||x - y||^2)$$

(6.110)
by Lemma 5.3. This implies the kernel estimate
\begin{equation}
\|\text{ker}(-(i\nabla + A_0)^{\otimes 2} T_{l,f}(x,y))\| \leq C(1 + \|x\|)^2 \exp(-\lambda \|x - y\|^2).
\end{equation}
(6.111)
Hence, the operator $1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,f}$ is bounded. The operator $(-i\nabla - A_0)^{\otimes 2} T_l$ is bounded by Lemma 5.3 for $d = 0, 1$ and by Lemma 6.2 for $d = 2$. Hence, the operator $1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_l$ is bounded. By the triangle inequality, we can conclude that the operator $1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n}$ is bounded. Furthermore, we have the identity
\begin{equation}
1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} = 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} = 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n}.
\end{equation}
(6.112)
The value at $x \in [0,1]^2$ of $(i\nabla + A_0)f$ only depends on $f$ in an arbitrary small neighbourhood of $x$, which proves the first identity. The second identity follows by the construction of $T_{l,n}$ as an integral operator with a kernel that vanishes if $\|x - y\| \geq 2$.

We will now estimate the kernel of the operator in (6.107), where we replace the first $T_l$ by $T_{l,f}$. The kernels of the operators $H_{\varepsilon,x_0}(T_l$ and $H_{\varepsilon,x_0}T_l$ can be bounded by Lemma 6.1 the kernel of $H_{\varepsilon,x_0}((-i\nabla + A_0)^{\otimes \mu + 1} P_l$ has been bounded in (6.99), and the kernel of $(-i\nabla + A_0)^{\otimes 2} T_{l,f}$ has been bounded in (6.111). Hence, we can apply Corollary 5.5 to arrive at
\begin{equation}
\left\| \text{ker}(-(i\nabla + A_0)^{\otimes 2} T_{l,f} \otimes \bigotimes_{j=1}^{m-1} (H_{\varepsilon,x_0}^{(\mu_j)}T_l) \otimes H_{\varepsilon,x_0}^{(\mu_m)}(-i\nabla + A_0)^{\otimes \mu + 1} P_l(H_{\varepsilon,x_0}T_l)^{k-m}1_{D_{\varepsilon}(0)} \right\| \leq C \left\| (1 + \|x\|)^{1+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.113)
\begin{equation}
\left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.114)
Once more, the $b_2$ term can be ignored as the operator $H_{\varepsilon,x_0}^{(\mu_m)}(-i\nabla + A_0)^{\otimes \mu + 1} P_l$ has no singularity at the diagonal and by Lemma 6.4 this establishes the required estimate.

We are only left with the term in (6.107), where we replace the first $T_l$ by $T_{l,n}$. Here, we can use (6.112) to see
\begin{equation}
\left\| 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} \otimes \bigotimes_{j=1}^{m-1} (H_{\varepsilon,x_0}^{(\mu_j)}T_l) \otimes H_{\varepsilon,x_0}^{(\mu_m)}(-i\nabla + A_0)^{\otimes \mu + 1} P_l(H_{\varepsilon,x_0}T_l)^{k-m}1_{D_{\varepsilon}(0)} \right\| \leq C \left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.115)
\begin{equation}
\left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.116)
\begin{equation}
\left\| 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} \right\| \leq C \left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.117)
\begin{equation}
\left\| 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} \right\| \leq C \left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.118)
\begin{equation}
\left\| 1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n} \right\| \leq C \left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.119)
The operator $1_{[0,1]^2}(-(i\nabla + A_0)^{\otimes 2} T_{l,n}$ is bounded. For the remaining part, we estimate the kernel. This is incredibly similar to (6.114). The kernels of the operators $H_{\varepsilon,x_0}^{(\mu_j)}T_l$ and $H_{\varepsilon,x_0}T_l$ can be bounded by Lemma 6.1 and the kernel of $H_{\varepsilon,x_0}^{(\mu_m)}(-i\nabla + A_0)^{\otimes \mu + 1} P_l$ has been bounded in (6.99). Hence, we can apply Corollary 5.5 to arrive at
\begin{equation}
\left\| \text{ker} \bigotimes_{j=1}^{m-1} (H_{\varepsilon,x_0}^{(\mu_j)}T_l) \otimes H_{\varepsilon,x_0}^{(\mu_m)}(-i\nabla + A_0)^{\otimes \mu + 1} P_l(H_{\varepsilon,x_0}T_l)^{k-m}1_{D_{\varepsilon}(0)} \right\| \leq C \left\| (1 + \|x\|)^{2+3} \exp(-\lambda \|x - y\|^2) \right\|^2.
\end{equation}
(6.120)
Here, $X$

\[ \Pr. \quad \text{For any Lemma A.1.} \]

\[ \text{Let } \mathcal{C} \text{ calculus and hence the multiplication operator with the function } f \]

\[ \text{derivative is then resolved in the next step.} \]

\[ \text{In the second step, we used the product and chain rule and the exponentials cancel. The interior} \]

\[ \text{As we can see, these operators and their inverses preserve } C^\infty \text{. Hence, it is sufficient to show} \]

\[ \text{that the claimed operator identities hold, when evaluated at a test function } \psi \in C^\infty \text{.} \]

\[ \text{We have} \]

\[ U_{x_0} := U_{x_01} U_{x_02}. \]

\[ \text{As we can see, these operators and their inverses preserve } C^\infty \text{. Hence, it is sufficient to show} \]

\[ \text{that the claimed operator identities hold, when evaluated at a test function } \psi \in C^\infty \text{.} \]

\[ \text{We have} \]

\[ U_{x_01} U_{x_02} f(X) U_{x_02}^{-1} U_{x_01}^{-1} = U_{x_01} f(X) U_{x_01}^{-1} \]

\[ = f(X + x_0). \]

\[ \text{Now, we need to check how } (-i \nabla - \frac{B_0}{2} J X) \text{ behaves under conjugation with } U_{x_0}. \text{ Hence, we get} \]

\[ \left( U_{x_02} \left( -i \nabla - \frac{B_0}{2} J X \right) U_{x_02}^{-1} U_{x_01} \psi \right)(x) \]

\[ = \exp \left( -i \frac{B_0}{2} (x \mid J x_0) \right) \left( -i \nabla_x - \frac{B_0}{2} J x \right) \exp \left( i \frac{B_0}{2} (x \mid J x_0) \right) \psi(x - x_0) \]

\[ = \left( -i \nabla_x - \frac{B_0}{2} J x \right) \psi(x - x_0) + \psi(x - x_0) \left( -i \nabla_x \left( i \frac{B_0}{2} (x \mid J x_0) \right) \right) \]

\[ = \left( U_{x_01}^{-1} \left( -i \nabla - \frac{B}{2} J X \right) \psi \right)(x). \]

\[ \text{In the second step, we used the product and chain rule and the exponentials cancel. The interior} \]

\[ \text{derivative is then resolved in the next step.} \]

\[ \text{In conclusion, we have} \]

\[ U_{x_0} (-i \nabla - A_0) U_{x_0}^{-1} = (-i \nabla - A_0). \]
This implies
\[ U_{x_0}T_iU_{-x_0}^{-1} = T_i. \] (A.16)

Together with (A.9), this implies the identity
\[ U_{x_0}H_{\varepsilon}U_{-x_0}^{-1} = H_{\varepsilon,x_0}. \] (A.17)

This finishes the proof. \( \square \)

**Lemma A.2.** Let \( \Omega \subset \mathbb{R}^n \) be measurable and let \( f: \Omega \to \mathbb{C} \) be integrable. Then we have the identity
\[ \int_{[0,1)^n} \sum_{z \in \mathbb{Z}^n, z + h_0 \in \Omega} f(z + h_0)dh_0 = \int_{\Omega} f(x)dx. \] (A.18)

**Proof.** We observe
\[ \int_{[0,1)^n} \sum_{z \in \mathbb{Z}^n, z + h_0 \in \Omega} f(z + h_0)dh_0 = \int_{[0,1)^n} \sum_{z \in \mathbb{Z}^n} 1_{\Omega(z + h_0)} f(z + h_0)dh_0 \] (A.19)
\[ = \int_{\mathbb{R}^n} 1_{\Omega}(x)f(x)dx \] (A.20)
\[ = \int_{\Omega} f(x)dx. \] (A.21)

In the second step we used Fubini with \([0,1)^n \times \mathbb{Z}^n = \mathbb{R}^n\).

**Lemma A.3.** Let \( f \in C^1(\mathbb{R}) \) with \( f' \leq 0 \) and \( \lim_{t \to \infty} f(t) = 0 \), \( h \in C^0(\mathbb{R}^2, \mathbb{R}) \) and \( \Lambda \subset \mathbb{R}^2 \) be measurable.

Then we have
\[ \int_{\Lambda} f(h(x))dx = \int_{\mathbb{R}} -f'(t)|\{x \in \Lambda \mid h(x) \leq t\}|dt. \] (A.22)

As both integrands are positive, we do not need to require the existence of the integral, both sides being \( \infty \) is an option.

**Proof.** We use the fundamental theorem of calculus and Fubini. As everything is positive, we can apply both theorems. Thus,
\[ \int_{\Lambda} f(h(x))dx = \int_{\Lambda} dx \int_{h(x)}^{\infty} (-f'(t))dt \] (A.23)
\[ = \int_{\mathbb{R}^2} dx \int_{\mathbb{R}} dt 1_{\Lambda}(x)1_{(h(x),\infty)}(t)(-f'(t)) \] (A.24)
\[ = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx 1_{\Lambda}(x)1_{(h(x),\infty)}(t)(-f'(t)) \] (A.25)
\[ = \int_{\mathbb{R}} -f'(t)|\{x \in \Lambda \mid h(x) \leq t\}|dt. \] \( \square \)

**Lemma A.4.** Let \( \Lambda \subset \mathbb{R}^2 \) be a bounded Lipschitz region. Then there is a constant \( C > 0 \), such that for any \( r > 0 \)
\[ \{x \in \Lambda \mid \text{dist}(x, \Lambda^0) \leq r\} \leq Cr, \] (A.26)
\[ \{x \in \Lambda^0 \mid \text{dist}(x, \Lambda) \leq r\} \leq C(r + r^2). \] (A.27)

In both cases, for small \( r \) we have an approximately linear dependency. In the first case, it is bounded by \( |\Lambda| < \infty \) and in the second case it is contained in a ball of radius \( r + r_0 \), which explains the \( r^2 \) term.
Lemma A.5. Let $R, \lambda > 0$ be real numbers and $x_0, x \in \mathbb{R}^2$ with $\|x - x_0\| \leq R$. Then we have
\[
\exp(-\lambda \|x\|^2) \leq e^{\lambda R^2} \exp\left(-\frac{\lambda}{2} \|x_0\|^2\right),
\]
where $\Phi$ is assumed to be an operator with a nice integral kernel $\kappa : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C}$. Assume, that for every $x \in \mathbb{R}^2$, the function $[0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{C}$; $(t, y) \mapsto \kappa_i(x, y)$ is integrable, its integral is bounded independently of $x$, and the same holds for $x$ and $y$ reversed. Then we have
\[
\text{ker} \left( \int_0^1 K_t dt \right)(x, y) = \int_0^1 \kappa_t(x, y) dt.
\]
Proof. The integral $\int_0^1 K_t dt$ exists as a Bochner integral with respect to the operator norm from $L^\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2)$ by the integrability assumptions on the kernel. Let $f \in C^0_c(\mathbb{R}^2)$. Then, for every $x \in \mathbb{R}^2$, we have
\[
\left( \int_0^1 K_t dt \right) f(x) = \left( \int_0^1 K_t f dt \right)(x)
\]
\[
= \int_0^1 \left( \int \kappa_t(x, y) f(y) dy \right) dt
\]
\[
= \int \left( \int_0^1 \kappa_t(x, y) dt \right) f(y) dy.
\]
The first step holds, as the Bochner integral commutes with the (linear, bounded) evaluation operator. The second step is the definition of $\kappa_t$ and the last step is Fubini, as $f$ is bounded and we assumed $\kappa \cdot (x, \cdot)$ to be integrable for any $x \in \mathbb{R}^2$. The same holds, if $x$ and $y$ are reversed, hence this is a nice integral kernel again.

Lemma A.6. For every $t \in (0, 1)$, let $K_t : L^\infty(\mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}^2)$ be an operator with a nice integral kernel $\kappa : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C}$. Assume, that for every $x \in \mathbb{R}^2$, the function $[0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{C}$; $(t, x) \mapsto \kappa(x, y)$ is integrable, its integral is bounded independently of $x$, and the same holds for $x$ and $y$ reversed. Then we have
\[
\text{ker} \left( \int_0^1 K_t dt \right)(x, y) = \int_0^1 \kappa_t(x, y) dt.
\]
Proof. The integral $\int_0^1 K_t dt$ exists as a Bochner integral with respect to the operator norm from $L^\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2)$ by the integrability assumptions on the kernel. Let $f \in C^0_c(\mathbb{R}^2)$. Then, for every $x \in \mathbb{R}^2$, we have
\[
\left( \int_0^1 K_t dt \right) f(x) = \left( \int_0^1 K_t f dt \right)(x)
\]
\[
= \int_0^1 \left( \int \kappa_t(x, y) f(y) dy \right) dt
\]
\[
= \int \left( \int_0^1 \kappa_t(x, y) dt \right) f(y) dy.
\]
The first step holds, as the Bochner integral commutes with the (linear, bounded) evaluation operator. The second step is the definition of $\kappa_t$ and the last step is Fubini, as $f$ is bounded and we assumed $\kappa \cdot (x, \cdot)$ to be integrable for any $x \in \mathbb{R}^2$. The same holds, if $x$ and $y$ are reversed, hence this is a nice integral kernel again.

Lemma A.7. For any $k \in \mathbb{Z}^+$, let $S_k$ be an operator on the Hilbert space $\mathcal{H}$ and assume that for any $k \neq l$, the conditions $S_k^* S_l = 0$ and $S_k S_l^* = 0$ hold. Then we have
\[
\left\| \sum_{k \in \mathbb{Z}^+} S_k \right\|_\infty = \sup_{k \in \mathbb{Z}^+} \|S_k\|_\infty.
\]
Proof. For $l \in \mathbb{Z}^+$, let $\mathcal{H}_l$ be the orthogonal complement of the kernel of $S_l$ and define $\mathcal{H}_0 := \cap_{l \in \mathbb{Z}^+} \ker(S_l)$. The condition $S_k S_l^* = 0$ tells us that the spaces $\mathcal{H}_l$ and $\mathcal{H}_k$ are orthogonal. Hence, we have $\mathcal{H} = \bigcup_{l \in \mathbb{N}} \mathcal{H}_l$. Let $\Psi \in \mathcal{H}$. Then we can consider the expansion along this direct sum and get a sequence $(\Psi_l \in \mathcal{H}_l)_{l \in \mathbb{N}}$. We consider
\[
\left\| \left( \sum_{k \in \mathbb{Z}^+} S_k \right) \Psi \right\|^2 = \left\| \sum_{k \in \mathbb{Z}^+} S_k \Psi_k \right\|^2
\]
\[
= \sum_{k \in \mathbb{Z}^+} \|S_k \Psi_k \|^2
\]
\[
\leq \sum_{k \in \mathbb{Z}^+} \|S_k\|_\infty^2 \|\Psi_k \|^2
\]
\[
\leq \sup_{k \in \mathbb{Z}^+} \|S_k\|_\infty^2 \sum_{k \in \mathbb{Z}^+} \|\Psi_k \|^2
\]
\[
= \sup_{k \in \mathbb{Z}^+} \|S_k\|_\infty^2 \|\Psi \|^2.
\]
The condition $S^*_l S_l = 0$ implies that the images of $S_k$ and $S_l$ are orthogonal. We used this in the second step. For the other inequality, for any $l \in \mathbb{Z}^+$, we observe

$$\|S_l \Psi\|^2 = \|S_l \Psi_l\|^2 \leq \sum_{k \in \mathbb{Z}^+} \|S_k \Psi_k\|^2 = \left(\sum_{k \in \mathbb{Z}^+} S_k\right) \Psi \|^2. \quad (A.40)$$

This finishes the proof. \hfill \square

**Definition A.8.** Let $\gamma \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^2$ be open with Lipschitz-boundary. Then we define the Hilbert space $H^\gamma(\Omega)$ as the closure of $C^\infty(\Omega, \mathbb{C})$ under the norm

$$\|u\|_{H^\gamma(\Omega)}^2 := \sum_{0 \leq \gamma' \leq \gamma} \|\nabla^{\gamma'} u\|_{L^2(\Omega)}^2. \quad (A.41)$$

We also write $H^\gamma(\Omega)$ for $H^\gamma(\Omega)$.

The more commonly used norm

$$u \mapsto \sqrt{\sum_{\alpha \in \mathbb{N}^N, |\alpha| \leq \gamma} \|\partial^{\alpha} u\|_{L^2(\Omega)}^2} \quad (A.42)$$

is equivalent to $(A.41)$.

**Lemma A.9.** Let $\gamma \in \mathbb{Z}^+$. Then the map $D_{\gamma} : H^\gamma([0,1]^2) \to L^2\left([0,1]^2, C^{2^{\gamma+1}-1}\right)$ given by

$$u \mapsto \left((-i \nabla + A_0)^{\otimes \gamma'} u\right)_{\gamma' = 0} \quad (A.43)$$

is a quasi-isometry, meaning that there is a constant $1 < C < \infty$ such that for any $u \in H^\gamma([0,1]^2)$, we have

$$\frac{1}{C} \|u\|_{H^\gamma([0,1]^2)} \leq \|D_{\gamma} u\|_{L^2\left([0,1]^2, C^{2^{\gamma+1}-1}\right)} \leq C \|u\|_{H^\gamma([0,1]^2)}. \quad (A.44)$$

The multiplication operator $A_0$ has been defined in (3.1).

**Proof.** Let $0 \leq \gamma' \leq \gamma$ be a natural number and let $\kappa \in \{1, 2\}^{\gamma'}$ be a multiindex. Now we can multiply out and simplify:

$$\left((-i \nabla + A_0)^{\otimes \gamma'} u(x) - (-i)^{\gamma'} \nabla^{\otimes \gamma'} u(x)\right) = \sum_{k,l \in \mathbb{N}, k+l \leq \gamma'} r_{k,l,\kappa}(x) \partial^k_1 \partial^l_2 u(x), \quad (A.45)$$

where $r_{k,l,\kappa}$ is a polynomial of degree at most $\gamma' - k - l$ that does not depend on $u$. As it is a polynomial, it is bounded on $[0,1]^2$. This leads to the upper bound

$$\left\|\left((-i \nabla + A_0)^{\otimes \gamma'} u - (-i)^{\gamma'} \nabla^{\otimes \gamma'} u\right)\right\|_{L^2\left([0,1]^2, C^{2^{\gamma+1}-1}\right)} \leq C \|u\|_{H^\gamma([0,1]^2)} \quad (A.46)$$

for any $0 < \gamma' \leq \gamma$. This specific estimate is needed for the lower bound. For the upper bound, we can just put the $\nabla^{\otimes \gamma'} u$ on the other side and get

$$\left\|\left((-i \nabla + A_0)^{\otimes \gamma'} u\right)\right\|_{L^2\left([0,1]^2, C^{2^{\gamma'+1}}\right)} \leq C \|u\|_{H^\gamma([0,1]^2)} \quad (A.47)$$

The claimed upper bound now follows by the triangle inequality.

For the lower bound, we let $C_0 \geq 1$ be a constant that is sufficiently large to be the constant $C$ in (A.46) for any $1 \leq \gamma' \leq \gamma$. If there is a $0 < \gamma' \leq \gamma$ such that

$$\|\nabla^{\otimes \gamma'} u\|_{L^2\left([0,1]^2, C^{2^{\gamma+1}-1}\right)} \geq 2C_0 \|u\|_{H^{\gamma'-1}([0,1]^2)}, \quad (A.48)$$

we choose $\gamma'$ maximal with this property. Otherwise, we set $\gamma' = 0$. Now we observe that for any $\gamma \geq r > \gamma'$, we have

$$\|u\|^2_{H^r([0,1]^2)} = \|u\|^2_{H^{r-1}([0,1]^2)} + \|\nabla^{\otimes r} u\|^2_{L^2([0,1]^2, C^{2^r})} \leq (4C_0^2 + 1) \|u\|^2_{H^{r-1}([0,1]^2)}. \quad (A.49)$$
In conclusion, we have the estimate
\[ \|u\|_{H^\gamma([0,1]^2)}^2 \leq (4C_0^2 + 1)J^{-\gamma'}\|u\|_{H^\gamma([0,1]^2)}^2 \leq 2(4C_0^2 + 1)J^{-\gamma'}\|\nabla^{\otimes\gamma'} u\|_{L^2([0,1]^2,\mathbb{C}^{2\gamma'})}^2. \] (A.50)

The last estimate relies on \((A.48)\) and \(C_0 \geq 1\), if \(\gamma' > 0\). If \(\gamma' = 0\), then without the factor 2, equality holds in the second inequality. By the triangle inequality, \((A.46)\), and \((A.48)\) or trivially, if \(\gamma' = 0\), we get
\[ \|(-i\nabla + A_0)^{\otimes\gamma'} u\|_{L^2([0,1]^2,\mathbb{C}^{2\gamma'})} \geq \frac{1}{2}\|\nabla^{\otimes\gamma'} u\|_{L^2([0,1]^2,\mathbb{C}^{2\gamma'})}. \] (A.51)

This finishes the lower bound and thus, the proof.

The following proposition is a special case of Theorem 1 in [6] by Gramsch.

**Proposition A.10.** Let \(\gamma \in \mathbb{Z}^+, \Omega \subset \mathbb{R}^2\) open, bounded and with \(C^\infty\)-boundary, and let \(\infty > q > \frac{2}{\gamma}\). Then the embedding
\[ \iota': H_0^\gamma(\Omega) \to L^2(\Omega) \] (A.52)
is in the \(q\)-Schatten class. Here, \(H_0^\gamma(\Omega)\) is the closure of \(C_c^\infty(\Omega)\) under the norm of \(H^\gamma(\Omega)\).

For the reader’s convenience, we provide a different proof of this statement. This proof requires no regularity of \(\partial \Omega\). [It can also be expanded to fractional exponent Hilbert spaces \(H_0^\gamma(\Omega)\).]

**Proof.** Let \(-\Delta\) be the Dirichlet Laplacian on \(\Omega\). Then the operator
\[ \iota: H_0^\gamma(\Omega) \to L^2(\Omega), \quad u \mapsto (1 - \Delta)^{-\frac{q}{2}} u \] (A.53)
is bounded and its inverse is bounded as well. This is because the pullback of the norm on \(L^2(\Omega)\) via \(U\) is equivalent to the norm on \(H_0^\gamma(\Omega)\). To be precise, we have for any \(u \in H_0^\gamma(\Omega)\)
\[ \|U u\|_{L^2(\Omega)} = \sum_{k=0}^\infty \gamma_k \|\nabla^{\otimes k} u\|_{L^2(\Omega,\mathbb{C}^{2k})}^2. \] (A.54)

This can be verified on the dense subset \(C_c^\infty(\Omega)\) by partially integrating.

Now we consider the operator \(\iota: L^2(\Omega) \to H_0^\gamma(\Omega)\), given by \(u \mapsto U^{-1} u \in H_0^\gamma(\Omega) \subset L^2(\Omega)\). We want to estimate the \(q\)-Schatten norm of \(\iota\). We define
\[ N(\lambda) = \#\{\lambda' \leq \lambda: \lambda\text{ is an eigenvalue of } -\Delta\}. \] (A.55)

By Weyl’s law, we conclude that there is a constant \(C\), depending on \(\Omega\), such that
\[ N(\lambda) \leq C(1 + \lambda), \] (A.56)
for any \(\lambda \in \mathbb{R}^+\). Now we can write
\[ \|\iota\|_q = \int_{\mathbb{R}^+} (1 + \lambda)^{-q} dN(\lambda) \] (A.57)
\[ = \lim_{R \to \infty} \int_0^R (1 + \lambda)^{-q} dN(\lambda) \] (A.58)
\[ = \lim_{R \to \infty} \left( N(R)(1 + R)^{-q} + q \frac{\gamma}{2} \int_0^R (1 + \lambda)^{-q - 1} dN(\lambda) \right) \] (A.59)
\[ \leq C \lim_{R \to \infty} \left( (1 + R)^{1-q} + \int_0^R (1 + \lambda)^{-q} d\lambda \right) \] (A.60)
\[ \leq C \lim_{R \to \infty} \left( 1 + (1 + R)^{1-q} \right) \leq C. \] (A.61)

The final estimate relies on the condition \(q > \frac{2}{\gamma}\). Now, we just use that \(U\) and \(U^{-1}\) are bounded operators to get
\[ \|\iota\|_q = \|V U\|_q \leq \|V\|_q \|U\|_\infty. \] (A.62)
This finishes the proof. □

We want to apply the statement for the space $H^\gamma([0,1]^2)$. Neither Gramsch’s result nor our proof is sufficient for that application. Hence, we need a slight extension.

**Corollary A.11.** Let $\gamma \in \mathbb{Z}^+$, $\Omega \subset \mathbb{R}^2$ open with Lipschitz-boundary, and $\infty > q > \frac{2}{\gamma}$. Then the embedding

$$\iota: H^\gamma(\Omega) \to L^2(\Omega) \quad \text{(A.63)}$$

is in the $q$-Schatten class.

**Remark.** In Proposition 2.1 in [2], Birman and Solomyak have shown an estimate of the singular values depending on the differentiability of the kernel. From that, one can see that for any Hilbert–Schmidt operator $S: L^2(\mathbb{R}^2) \to H^\gamma([0,1]^2)$, the operator $S$ is in the $p$-Schatten class for any $p > \frac{2}{\gamma + 1}$.

This statement also follows from our corollary here.

We decided not to use Birman and Solomyak’s result directly, as it is convenient for us to have this statement in the operator setting. Furthermore, we can directly use the quasi-isometry $D_\gamma$, that we constructed in Lemma 6.3.

**Proof of Corollary A.11** Let $\Omega' \supset \overline{\Omega}$ be an open ball. As $\Omega$ has Lipschitz-boundary, there is a continuous extension operator,

$$E: H^\gamma(\Omega) \to H^\gamma_0(\Omega'). \quad \text{(A.64)}$$

One such operator can be constructed as a composition of a multiplication operator with a smooth cutoff function and the extension operator constructed by Stein in Theorem 5 in [23]. Furthermore, there obviously is the continuous restriction operator

$$R: L^2(\Omega') \to L^2(\Omega). \quad \text{(A.65)}$$

Hence, the operator

$$\iota = Ri' E \quad \text{(A.66)}$$

is in the $q$-Schatten class by Proposition A.10. □

**Appendix B. Proof of Lemma 3.2**

**Proof of Lemma 3.2** We recall

$$g(x) := \int_{\mathbb{R}^2} \frac{Jy}{2\pi \|y\|^2} f(x - y) dy. \quad \text{(B.1)}$$

The last property will be seen by bounding this integral. $C_\varepsilon$ will be a constant depending only on $\varepsilon$, that may change from line to line. To begin with we have the bound

$$\|g(x)\| \leq \int_{\mathbb{R}^2} \frac{C}{\|y\|^2 (1 + \|y - x\|)^{1+\varepsilon}} dy \quad \text{(B.2)}$$

$$\leq \int_{D_{2\|x\|}(0)} \frac{C}{\|y\| (1 + \|y - x\|)^{1+\varepsilon}} dy + \int_{D_{\|x\|}(0)} \frac{C}{\|y\| (1 + \|y - x\|)^{1+\varepsilon}} dy \quad \text{(B.3)}$$

$$\leq \int_{D_{\|x\|}(0)} \frac{C}{\|y\| (1 + \|y - e_1\|)^{1+\varepsilon}}\|x\|^{2-1-\varepsilon} dy + \int_{D_{\|x\|}(0)} \frac{C}{\|y\| (1 + \|y/2\|)^{1+\varepsilon}} dy \quad \text{(B.4)}$$

$$\leq C \min \{\|x\|^{-\varepsilon}, \|x\|\} + C \max \{\|x\|, 1\}^{-\varepsilon} + C_\varepsilon 1_{D_{1}(0)}(x). \quad \text{(B.7)}$$
Hence, we have
\begin{equation}
\frac{C}{(1 + \|x\|^2)}.
\end{equation}
In the second to last step, we got the first minimum by ignoring either of the summands in the denominator of the bounded domain integral and for the second part we just did a different bound on the annulus from \(|x|\) to 1, if \(|x| < 1\). This directly shows that \(g \in W^{0,\infty}(\mathbb{R}^2, \mathbb{R}^2)\). For \(\gamma > 0\), we can first use the result for \(\partial_j f\) for \(j = \{1, 2\}\) and then use dominated convergence to see that \(\partial_j g = (\partial_j f) * \frac{1}{2\pi \|\eta\|^2}\). Hence, by an induction on \(\gamma' \leq \gamma\), we see that \(g \in W^{\gamma'}(\mathbb{R}^2, \mathbb{R}^2)\).

For the first two properties, we use the Fourier transform,
\begin{equation}
\mathcal{F}(h)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x) \exp(-ix \cdot \xi) dx, \quad \xi \in \mathbb{R}^2
\end{equation}
for any \(n \in \mathbb{N}\) and \(h \in L^1 \cap L^2(\mathbb{R}^2, \mathbb{C}^n)\). It can be expanded to tempered distributions and has the following properties for any \(\xi \in \mathbb{R}^2\), tempered distributions \(h, h_1, h_2\):
\begin{align*}
\mathcal{F}(\cdot h(\cdot))(\xi) &= i\nabla \mathcal{F}(h)(\xi), \\
\mathcal{F}((\nabla h)(\cdot))(\xi) &= -i\xi \mathcal{F}(h)(\xi), \\
\mathcal{F}(1)(\xi) &= 2\pi \delta_0(\xi), \\
\mathcal{F}(h_1 * h_2)(\xi) &= 2\pi \mathcal{F}(h_1)(\xi) \mathcal{F}(h_2)(\xi).
\end{align*}
Here \(\delta_0\) refers to the \(\delta\)-distribution at 0. Furthermore, the Fourier transform is linear and invertible. As \(f\) and \(g\) are bounded, they are both tempered distributions. Now we can apply the Fourier transform to our first two claimed equations and are left to show
\begin{align*}
-2\pi iJ \cdot \mathcal{F} \left( \frac{J}{2\pi \|\cdot\|^2} \right)(\xi) \mathcal{F}(f)(\xi) &= \mathcal{F}(f)(\xi), \\
-2\pi i\xi \cdot \mathcal{F} \left( \frac{J}{2\pi \|\cdot\|^2} \right)(\xi) \mathcal{F}(f)(\xi) &= 0.
\end{align*}
Basically, this equation does not depend on \(f\). Now we have to compute the Fourier transform of \(\frac{J}{2\pi \|x\|^2}\),
\begin{equation}
\mathcal{F} \left( \frac{J}{2\pi \|\cdot\|^2} \right)(\xi) = \frac{1}{2\pi} (J \nabla (\nabla) \mathcal{F}(1))(\xi)
\end{equation}
\begin{align*}
&= iJ \nabla (\nabla) \mathcal{F}(1) \\
&= iJ \nabla \frac{1}{2\pi} \ln(\|\xi\|) \\
&= iJ \frac{\xi}{2\pi \|\xi\|^2}.
\end{align*}
Hence, we have
\begin{align*}
-2\pi iJ \xi \cdot \mathcal{F} \left( \frac{J}{2\pi \|\cdot\|^2} \right)(\xi) &= -2\pi iJ \xi \cdot iJ \frac{\xi}{2\pi \|\xi\|^2} = 1, \\
-2\pi i\xi \cdot \mathcal{F} \left( \frac{J}{2\pi \|\cdot\|^2} \right)(\xi) &= -2\pi i\xi \cdot iJ \frac{\xi}{2\pi \|\xi\|^2} = 0.
\end{align*}
This finishes the proof. \(\square\)

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