Image deblurring based on fractional-order total variation and total generalized variation

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Abstract. Considering the problem that the traditional method is easy to cause staircase effect and blur detail when removing image blur, we propose a model combined with fractional-order total variation (FOTV) and total generalized variation (TGV) for image deblurring. Firstly, we use the global gradient extraction method (GGES) to divide a blurred image into a smooth portion containing the image features and a detail portion containing the image details. Secondly, we use the TGV method of image deblurring and the FOTV method of image deblurring to repair the two parts of the image separately. Finally, we reconstruct the two restored images together to get the result image. In order to solve the proposed new model effectively, a new numerical algorithm is designed based on the original dual algorithm. The experimental results show that our method is superior to the traditional method in removing the staircase effect and maintaining the image detail area.

1. Introduction
In a general way, image deblurring is a technique of processing a blurred image to restore its original sharpness as much as possible, with the aim of improving the display of the image [1]. Image deblurring can usually be divided into blind deblurring and non-blind deblurring according to whether the blurred kernel is known [2]. In this paper, we mainly study the problem of non-blind deblurring by using regularization methods. As early as the last century, Rudin [3] et al. combined a total variation (TV) regularization to propose a deblurring method based on variational model. Although the method has a large improvement in deblurring and edge preservation, since the solution space of the total variation regularization is usually composed of slice constants, this method not only causes a staircase effect, but also blurs the details of the image [4]. To improve the performance of the TV-based deblurring method, a deblurring method based on weighted TV and a deblurring method based on non-local total variation are proposed [5]. Although these methods have improved the performance of the TV deblurring method to a certain extent, their changes to the TV method are too small, so they do not solve the problem that the TV deblurring method is prone to generate a staircase effect [6]. To eliminate the staircase effect, some researchers proposed a deblurring method combining fractional-order total variation (FOTV) [7]. Since the fractional differential can greatly improve the high frequency component of the image and is suitable for processing the texture part of the image, the method has better image restoration effect with rich details. However, the fractional-order differential can only preserve the medium and low frequency components of the image nonlinearly [8]. And some researchers proposed a deblurring method based...
on total generalized variational (TGV) [9]. Since the TGV regularization can effectively approximate
the polynomial function of any order and preserve the medium and low frequency information of the
image well, the method has better repair effect on the smooth region and can effectively suppress the
staircase effect. However, the TGV regular term is built on the integer-order differential, which tends to
make the high-frequency components of the restored image appear the phenomenon of excessive
smoothing [10].

In our paper, we propose a new deblurring method. In the new method, we first decompose an image
into two parts using an image decomposition algorithm. Then we use the FOTV deblurring method and
the TGV deblurring method to repair the two parts of the image with different features. The experimental
results show that compared with the traditional methods, the proposed method can not only get better
deblurring effect, but also effectively remove the staircase effect and preserve the details of the image.

2. Proposed model
In this section, we propose a new model for image deblurring. Firstly, we decompose the blurred image
into a smooth portion and a detail portion by using an image decomposition algorithm. Secondly, we
use the TGV deblurring model to repair the smooth part, then we use the FOTV deblurring model to
repair the detail part, so we can get two repaired images with different features. Finally, we reconstruct
the restored smooth portion and the detail portion to obtain the result image. In the following sections,
we will introduce the algorithms and formulas used in this model in detail.

2.1. Image decomposition algorithm
Considering the decomposition effect and decomposition speed, we use the global gradient extraction
method (GGES) to decompose the image. The GGES model is as follows

\[ \min_l \left\{ \sum_p (f_p - l_p)^2 + \lambda \cdot C(l) \right\} \]  

(1)

Where \( f \) is the image to be decomposed, \( l \) is the smooth part of image, \( f - l \) is the detail part of
image, \( p \) is the pixel point of the image, \( \lambda \) is the weight parameter used to control the smoothing
strength.

Figure 1 shows the result of decomposing a blurred image using the decomposition algorithm of this
paper. As can be seen from Figure 1, the smooth portion preserves the main structure and edges of the
blurred image, and the detail portion retains more gradient information and detail information of the
blurred image.

![Figure 1. Decomposition results of GGES](image)

(a)blurred image  (b)smooth portion  (c)detail portion

2.2. FOTV model for image deblurring
In this paper, the formula of the FOTV algorithm we used to repair the detail part of the blurred image
is as follows

\[ \arg \min_{u_{\lambda}} \left\{ \frac{1}{2} \| Hu_{\lambda} - m \|_2^2 + \lambda \| \nabla^\alpha u_{\lambda} \| \right\} \]  

(2)

Where \( m \) is the detail part of the blurred image, \( H \) is the blurred kernel, \( u_{\lambda} \) is the result image after
deblurring the detail part, \( \lambda \) is the balance parameter, \( \alpha \) is the order of the fractional differential. It can
be seen from Eq. (3) that the effect of the first term is to maintain the similarity between the result image
and the detail portion for retaining the structural features of the image. The second term is a fractional-
order total variation regular term used to protect or even enhance the detail of the image.

The experiment found that when the fractional order is chosen to be 1.5, the result image can retain
more original image details without noise spots. Therefore, the final choice of the fractional order is 1.5.

2.3. TGV model for image deblurring

In this part, the formula of the TGV algorithm we used to repair the smooth part of the blurred image is
as follows

\[
\min_{u} \{ \frac{\beta}{2} \|Hu - l\|_2^2 + TGV_\alpha^2(u_t) \} \tag{3}
\]

In Eq. (3), \( l \) is the smooth part of the blurred image, \( H \) is the blurred kernel, \( u_t \) is the result image
after deblurring the smooth part, \( \beta \) is the balance parameter. From Eq. (3), the effect of the first term is
to maintain the similarity between the result image and the detail portion, and the second term (second-
order TGV regular term) can effectively protect the medium and low frequency components of the image,
this item is expressed as follows

\[
TGV_\alpha^2(u_t) = \min_{\omega \in BD(\Omega)} \alpha_1 \|\nabla u_t - \omega\|_2 + \alpha_2 \|\varepsilon(\omega)\|_2 \tag{4}
\]

Where \( BD(\Omega) \) is a bounded and distorted vector space, \( \varepsilon(\omega) \) is a weak symmetric derivative, the
expression of \( \varepsilon(\omega) \) is \( \varepsilon(\omega) = \frac{1}{2}(\nabla \omega + \nabla \omega^T) \).

3. Numerical algorithm

3.1. Numerical algorithm of FOTV model

As shown below, we give the original dual description of Eq. (2)

\[
\min_{u \in X} \max_{s \in \mathcal{T}} \left\{ \langle -1 \rangle_s \text{div}^\alpha s, u \rangle \right\}_X + \frac{\lambda}{2} \|u - m\|_2^2 - \delta_s(s) \tag{5}
\]

In Eq. (5), \( X \) and \( Y \) represent finite dimensional vector spaces, \( s = (s^1, s^2) \) is a two-dimensional
vector, \( S = \{s \in Y \|s\|_\infty = \max_{i,j} |s_{ij}| \leq 1 \} \) is the dual space, \( \delta_s(\cdot) \) is an indication function in the dual space,
which is used to represent the topology of the fractional regularization term. It is expressed as follows

\[
\delta_s(s) = \begin{cases} 0, & s \in S \\ +\infty, & s \notin S \end{cases} \tag{6}
\]

Therefore, combining Eqs. (5, 6), the numerical solution algorithm for FOTV model is given in
Algorithm 1 at the end of Section 3.

3.2. Numerical algorithm of TGV model

Combined with the description of FOTV original dual deblurring algorithm, the original dual problem
of Eq. (3) is given below

\[
\min_{u_t, \omega} \max_{p \in P, q \in Q} \left\{ \langle \nabla u_t - \omega, p \rangle - T_1^*(p) + \langle \varepsilon(\omega), q \rangle - T_2^*(q) + \frac{\beta}{2} \int_{\Omega} (Hu_t - l)^2 dx \right\} \tag{7}
\]

\( P \) and \( Q \) in the above equation are respectively expressed as follows

\[
P = \{p = (p_1, p_2)^T \|p(x)\| \leq \alpha_1 \}
\]

\[
Q = \{q = (q_{11}, q_{12}, q_{21}, q_{22}) \|q\|_\infty \leq \alpha_0 \}
\]

Therefore, combining Eqs. (7, 8), the numerical solution algorithm for the TGV model is given in
Algorithm 2.
Algorithm 1: Primal dual algorithm for FOTV deblurring

(1) Initialization: $\tau_0 > 0 \, , \, \sigma_0 > 0 \, , \, \tau_0 \sigma_0 L^2 \leq 1 \, , \, (u^0_m, s^0) \in X \times Y \, , \, u^0_m = u^0_m$

(2) Iteration:

$$
\begin{align*}
    s^{(n+1)} &= (s^n + \sigma_s A u^n_m) / \max(1, |s^n + \sigma_s A u^n_m|) \\
    u^{(n+1)}_m &= (u^n_m - \tau_s A^* s^{(n+1)} + \tau_s A H^T m) / (1 + \tau_s A H^T H) \\
    \theta^{(n+1)}_m &= 1 / \sqrt{1 + 2 \tau_s \tau_{s1} + \tau_s \tau_{s1}} \, , \, \sigma_{n+1} = \sigma_n / \theta^{(n+1)}_m \\
    u_{m}^{(n+1)} &= u_{m}^{(n+1)} + (u_{m}^{(n+1)} - u_m^{(n+1)})
\end{align*}
$$

(3) Verification:

$$
\begin{align*}
    \ell(u_m, s) &= \text{max}(\langle s, A u_m \rangle - F^*(s) + G(u_m)) \\
    -\text{min}(\langle s, A u_m \rangle - F^*(s) + G(u_m))
\end{align*}
$$

When the above formula satisfies the iteration termination condition, the iteration is terminated, otherwise let $n = n + 1$ continue to perform step 2.

Algorithm 2: Primal dual algorithm for TGV deblurring

(1) Initialization: $u^0_i, \omega^0, \tilde{u}_i = u_{i0}, \omega^0 = 0, \rho^0, q^0 = 0, \sigma, \tau, \beta > 0, k = 0$

(2) Iteration:

$$
\begin{align*}
    p^{(k+1)} &= \text{proj}_p[p^k + \sigma_p (\nabla u^k - \bar{w}^k)] \\
    q^{(k+1)} &= \text{proj}_q[q^k + \sigma_q (\omega^k - \bar{\omega}^k)] \\
    \omega^{(k+1)} &= \omega^k + \tau_s (p^{(k+1)} + \text{div} u^{(k+1)}) \\
    u^{(k+1)}_i &= u_i^k + \tau_s \text{div} p^{(k+1)} + \tau_s \beta H^T l / (1 + \tau_s \beta H^T H) \\
    \bar{u}^{(k+1)} &= \omega^{(k+1)} + \theta (\omega^{(k+1)} - \bar{w}^k) \\
    u_i^{(k+1)} &= u_i^{(k+1)} + \theta (u_i^{(k+1)} - u_i^k)
\end{align*}
$$

get $(p^{(k+1)}, q^{(k+1)}, u^{(k+1)}, \omega^{(k+1)}, \bar{u}^{(k+1)}, \bar{\omega}^{(k+1)})$

(3) Verification:

If $u^{(k+1)}, \omega^{(k+1)}$ are astringed, the iteration is terminated, otherwise let $n = n + 1$ continue to perform step 2.

4. Experimental results and discussion

As shown in Figure 2, we select two images with different characteristics as experimental subjects. They are all of the size of $512 \times 512$. At the same time, the three algorithms of TV, FOTV and TGV are selected as the comparison algorithm to verify the effectiveness of our deblurring algorithm. All experiments were programmed on MATLAB R2016a. The experimental platform hardware parameters: Intel Core i5-6440HQ, CPU 2.6GHz, memory 8GB.

In the experiment, we choose two common blurred kernels: (1) Gaussian blur with a blurred kernel size of $24 \times 24$ and a standard deviation of $2$; (2) motion blur with angle of $20$ and length of $50$. In our experiments, $\alpha = 1.5, \lambda = 8, \beta = 0.2, \alpha_i = 5, \alpha_0 = 10$.

![Figure 2. Experimental images](image-url)
Figure 3 and Figure 4 show the results of two different experiments. In Figure 3, the first row of result images are obtained by using different algorithms to repair the motion blur, and the second row of result images are obtained by using different algorithms to repair the Gaussian blur. In Figure 4, result images of the first row are partial detail maps obtained by using different algorithms to repair motion blur, and result images of the second row are partial detail maps obtained by using different algorithms to repair Gaussian blur.

Observing all the experimental results, we can find that result images repaired by the TV model show a significant staircase effect. Smooth parts of result images repaired by the FOTV model show a block-like blur phenomenon, and result images repaired by the TGV model show the loss of detail. The results repaired by our model does not exhibit the above-mentioned undesirable phenomenon.

Table 1 shows PSNR values of different deblurring algorithms. It can be seen from Table 1 that, under the same experimental conditions, the PSNR value obtained by using TV model has the lowest value, the PSNR value obtained by using our algorithm has a certain degree of improvement compared with the other three algorithms. Therefore, from the objective point of view of experimental data, the new model and the new algorithm in the paper have a good effect of removing blur.

| Image   | Blurred kernel | TV     | FOTV   | TGV     | Our    |
|---------|----------------|--------|--------|---------|--------|
| cameraman | motion         | 30.06  | 36.54  | 38.79   | 39.92  |
|          | Gauss          | 28.91  | 33.83  | 35.56   | 37.03  |
| house    | motion         | 25.71  | 30.44  | 32.09   | 33.85  |
|          | Gauss          | 21.69  | 25.73  | 27.87   | 28.99  |
5. Conclusion
In this paper, we consider that deblurring algorithms constrained by only one regular term tend to cause staircase effect and detail blur when blurring is removed. Therefore, we combine the advantages of the FOTV model and the TGV model to propose a new model. The experimental data shows that the newly proposed model is better than the traditional model in removing the blur. In the following days, the focus of our experiment will be on improving the speed of algorithm operation and the accuracy of calculation.

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