On the divergence of greedy algorithms with respect to Walsh subsystems in $L^1$

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In this paper we prove that there exists a function which $f(x)$ belongs to $L^1[0,1]$ such that a greedy algorithm with regard to the Walsh subsystem does not converge to $f(x)$ in $L^1[0,1]$ norm, i.e. the Walsh subsystem $\{W_{n_k}\}$ is not a quasi-greedy basis in its linear span in $L^1$.

1. INTRODUCTION

Let a Banach space $X$ with a norm $|| \cdot || = || \cdot ||_X$, and a basis $\Psi = \{ \psi_k \}_{k=1}^\infty$, $||\psi_k||_X = 1$, $k = 1, 2, \ldots$ be given.

Denote by $\Sigma_m$ the collection of all functions in $X$ which can be expressed as a linear combination of at most $m$ functions of $\Psi$. Thus each function $g \in \Sigma_m$ can be written in the form

$$g = \sum_{s \in \Lambda} a_s \psi_s, \quad \#\Lambda \leq m.$$ 

For a function $f \in X$ we define its approximation error

$$\sigma_m(f, \Psi) = \inf_{g \in \Sigma_m} ||f - g||_X, \quad m = 1, 2, \ldots$$

and we consider the expansion

$$f = \sum_{k=1}^{\infty} a_k(f) \psi_k.$$

**Definition 1.** Let an element $f \in X$ be given. Then the $m$-th greedy approximant of function $f$ with regard to the basis $\Psi$ is given by the formula

$$G_m(f, \Psi) = \sum_{k \in \Lambda} a_k(f) \psi_k,$$

where $\Lambda \subset \{1, 2, \ldots\}$ is a set of cardinality $m$ such that

$$|a_n(f)| \geq |a_k(f)|, \quad n \in \Lambda, \ k \notin \Lambda.$$
We'll say that the greedy approximant of \( f(t) \in L^p_{[0,1]}, \ p \geq 1 \) with regard to the basis \( \Psi \) converges, if the sequence \( G_m(x, f) \) converges to \( f(t) \) in \( L^p \) norm.

This new and very important direction invaded many mathematician’s attention (see [1]-[9]).

**Definition 2.** We call a basis \( \Psi \) greedy basis if for every \( f \in X \) there exists a subset \( \Lambda \subset \{1, 2, \ldots\} \) of cardinality \( m \), such that

\[
||f - G_m(f, \Psi)||_X \leq C \cdot \sigma_m(f, \Psi)
\]

where a constant \( C = C(X, \Psi) \) independent of \( f \) and \( m \).

In 1998 V.N.Temlyakov proved that each basis \( \Psi \) which is \( L^p \)-equivalent to the Haar basis \( H \) is Greedy basis for \( L^p(0, 1) \), \( 1 < p < \infty \) (see [4]).

**Definition 3.** We say that a basis \( \Psi \) is Quasi-Greedy basis if there exists a constant \( C \) such that for every \( f \in X \) and any finite set of indices \( \Lambda \), having the property

\[
\min_{k \in \Lambda} |a_k(f)| \geq \max_{n \notin \Lambda} |a_k(f)|
\]

we have

\[
\left\| \sum_{k \in \Lambda} a_k(f) \psi_k \right\|_X \leq C \cdot ||f||_X.
\]

In 2000 P.Wojtaszczyk [5] proved that a basis \( \Psi \) is quasi-greedy if and only if the sequence \( \{G_m(f)\} \) converges to \( f \), for all \( f \in X \). Note that in [6] S.Konyagin and V.Temlyakov constructed an example of quasi-greedy basis that is not Greedy basis.

V.Temlyakov proved that the trigonometric system \( T \) is not a Quasi-Greedy basis for \( L^p \) if \( p \neq 2 \) (see [7]).

In [8] it is proved that this result is true for Walsh system.

In the sequel, we’ll fix a sequence \( \{M_n\}_{n=1}^{\infty} \) so that

\[
\lim_{k \to \infty} (M_{2k} - M_{2k-1}) = +\infty
\]

and consider a subsystem of Walsh system

\[
\{W_{n_k}(x)\}_{k=1}^{\infty} = \{W_m(x) : \ M_{2s-1} \leq m \leq M_{2s}, \ s = 1, 2, \ldots\} \tag{1}
\]

In this paper we constructed a function \( f(x) \in L^1[0,1] \) such that the sequence \( \{G_m(f)\} \), with respect to Walsh system, does not converge to \( f(x) \) by \( L^1 \) norm and we can watch for spectra of ”bad” function \( f(x) \).

Moreover the following is true.

**Theorem.** There exists a function \( f(x) \) belongs to \( L^1[0,1] \) such that the approximate \( G_n(f, W_{n_k}) \) with regard to the Walsh subsystem does not converge to \( f(x) \) by \( L^1 \) norm, i.e. the Walsh subsystem \( \{W_{n_k}\} \) is not a quasi-greedy basis in its linear span in \( L^1 \).
2. PROOF OF THEOREM

First we will give a definition of Walsh-Paly system (see [10]).

\[ W_0(x) = 1, \quad W_n(x) = \prod_{s=1}^{k} r_{m_s}(x), \quad n = \sum_{s=1}^{k} 2^{m_s}, \quad m_1 > ... > m_s, \quad (2) \]

where \( \{r_k(x)\}_{k=0}^{\infty} \) is the system of Rademacher:

\[
\begin{align*}
  r_0(x) &= \begin{cases} 
    1, & x \in [0, \frac{1}{2}) \\
    -1, & x \in (\frac{1}{2}, 1]. 
  \end{cases} \\
  r_0(x+1) &= r_0(x), \quad r_k(x) = r_0(2^k x), \quad k = 1, 2, ...
\end{align*}
\]

In the proof of theorem we will used the following properties of Walsh system:

1. From (2) we have

\[ W_{2^k+j}(x) = W_{2^k}(x) \cdot W_{j}(x), \quad \text{if} \quad 0 \leq j \leq 2^k - 1. \quad (3) \]

2. The Dirichlet-Walsh kernel \( D_m(x) = \sum_{j=0}^{m-1} W_j(x) \) has the following properties (see [10] p.27)

\[ D_{2^j}(x) = \begin{cases} 
    2^j, & x \in [0, \frac{1}{2^j}], \\
    0, & x \in (\frac{1}{2^j}, 1]. 
  \end{cases} \quad (4) \]

3. There is a sequence of natural numbers \( \{m_k\}_{k=1}^{\infty} \) with \( 2^{k-1} \leq m_k < 2^k \), \( k = 1, 2, ... \) (see [10] p.47), such that

\[ \|D_{m_k}\|_1 = \int_{0}^{1} |D_{m_k}(x)| dx \geq \frac{1}{4} \log_2 m_k, \quad k = 1, 2, .... \quad (5) \]

**Proof of Theorem.** Taking into account (1)-(3) we can take the sequences of natural numbers \( \{k_\nu\}_{\nu=1}^{\infty} \) and \( \{p_\nu\}_{\nu=1}^{\infty} \) so that the following conditions are satisfied:

\[ k_\nu > (\nu - 1)^2 + 1, \quad (6) \]

\[ W_{2^{k_\nu}}(x) \cdot W_{i}(x) = W_{2^{k_\nu+i}}(x), \quad 0 \leq i < 2^{k_\nu}, \quad (7) \]

\[ 2^{k_\nu} + i \in [M_{2p_\nu-1}, M_{2p_\nu}), \quad 0 \leq i < 2^{k_\nu}, \quad (8) \]

For any natural \( \nu \) we set

\[ f_\nu(x) = \sum_{N_\nu \leq n_k < N_\nu - 1} c_{n_k}^{(\nu)} W_{n_k}(x) = \]
\[
\begin{align*}
&= \sum_{i=0}^{2^{k\nu} - 1} \left( \frac{1}{\nu^2} + 2^{-(2^{k\nu} + i)} \right) \cdot W_{2^{k\nu} + 1}(x) = \\
&= W_{2^{k\nu}}(x) \cdot \sum_{i=0}^{2^{k\nu} - 1} \left( \frac{1}{\nu^2} + 2^{-(2^{k\nu} + i)} \right) \cdot W_i(x) = \\
&= W_{2^{k\nu}}(x) \cdot \left[ \frac{1}{\nu^2} \sum_{i=0}^{2^{k\nu} - 1} W_i(x) + \frac{1}{2^{k\nu}} \sum_{i=0}^{2^{k\nu} - 1} 2^{-i} W_i(x) \right] = \\
&= W_{2^{k\nu}}(x) \cdot \left[ \frac{1}{\nu^2} D_{2^{k\nu}}(x) + \frac{1}{2^{k\nu}} \sum_{i=0}^{2^{k\nu} - 1} 2^{-i} W_i(x) \right], \quad (9)
\end{align*}
\]

where
\[
\begin{align*}
&c_{n_k}^{(\nu)} = \begin{cases} 
\frac{1}{\nu^2} + 2^{-n_k}, & N_{\nu} \leq n_k = N_{\nu} + i < N_{\nu+1}, \quad 0 \leq i < N_{\nu}, \\
0, & n_k < N_{\nu}, \quad \nu \geq 1,
\end{cases} \\
&\quad N_{\nu} = 2^{k\nu}, \quad N_{\nu+1} = 2^{k\nu+1}. \quad (10)
\end{align*}
\]

We set
\[
\begin{align*}
f(x) &= \sum_{k=1}^{\infty} c_{n_k}(f) W_{n_k}(x) = \\
&= \sum_{\nu=1}^{\infty} f_{\nu}(x) = \sum_{\nu=1}^{\infty} \left[ \sum_{N_{\nu} \leq n_k < N_{\nu+1}} c_{n_k}^{(\nu)} W_{n_k}(x) \right], \quad (12)
\end{align*}
\]

where
\[
\begin{align*}
c_{n_k}(f) = c_{n_k}^{(\nu)} \text{ for } N_{\nu} \leq n_k < N_{\nu} - 1, \quad \nu = 1, 2, ... \quad (13)
\end{align*}
\]

Now we will show that \( f(x) \in L^1[0, 1] \)

Taking into account (9)-(11) we get
\[
\begin{align*}
f(x) &= \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} W_{2^{k\nu}}(x) D_{2^{k\nu}}(x) + \sum_{\nu=1}^{\infty} \left[ \sum_{i=0}^{2^{k\nu} - 1} 2^{-i} W_{2^{k\nu} + i}(x) \right] = \\
&= G(x) + H(x). \quad (14)
\end{align*}
\]

For function \( G(x) \) from (4) and definition of Walsh system we have
\[
\int_0^1 |G(x)| \, dx \leq \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} < \infty
\]

and we get that \( G(x) \in L^1[0, 1] \).

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Analogously
\[ \int_0^1 |H(x)| \, dx \leq \sum_{\nu=1}^{\infty} 2^{-\nu} < \infty \]
i.e. \( H(x) \in L^1[0,1] \).
Hence and from (14) it follows that \( f(x) \in L^1[0,1] \).

For any natural \( \nu \) we choose numbers \( k, j \) so that
\[ N_{\nu} \leq n_k < N_{\nu+1} \leq n_j < N_{\nu+2} \]
Then from (10) we have
\[ c_{n_j}(f) = c_{n_j}^{(\nu+1)} = \frac{1}{(\nu + 1)^2} + 2^{-n_j} < \]
\[ < \frac{1}{\nu^2} + 2^{-n_k} = c_{n_k}^{(\nu)} = c_{n_k}(f) \]
i.e. \( c_{n_j}(f) < c_{n_k}(f) \).
Analogously for any number \( n_k, N_{\nu} \leq n_k < N_{\nu+1}, \nu \geq 1 \) we have
\[ c_{n_{k+1}}^{(\nu)} = \frac{1}{\nu^2} + 2^{(-n_k+1)} < \frac{1}{\nu^2} + 2^{-n_k} = c_{n_k}^{(\nu)} \]
Thus we get
\[ c_{n_{k+1}}(f) < c_{n_k}(f). \]
In other hand if \( k \to \infty \) then \( n_k \to \infty \) and \( \nu \to \infty \) (see (10), (11)).
From (13) we get \( \lim_{k \to \infty} c_{n_k}(f) = 0 \) and consequently \( c_{n_k}(f) \downarrow 0 \).
For any numbers \( m_{\nu} \) so that
\[ 2^{k_{\nu}} \leq m_{\nu} < 2^{k_{\nu}+1}. \tag{15} \]

By (11) - (13) and Definition 1 we have
\[ G_{2^{k_{\nu}} + m_{\nu}}(f, W_{n_k}) - G_{2^{k_{\nu}}}(f, W_{n_k}) = \]
\[ = \sum_{i=2^{k_{\nu}}}^{2^{k_{\nu}+m_{\nu}-1}} \left( \frac{1}{\nu^2} + 2^{-(2^{k_{\nu}+i})} \right) \cdot W_{2^{k_{\nu}+i}}(x) = \]
\[ = \frac{1}{\nu^2} \cdot W_{2^{k_{\nu}}}(x) \cdot \sum_{i=2^{k_{\nu}}}^{2^{k_{\nu}+m_{\nu}-1}} W_i(x) + \]
\[ + \frac{1}{2^{k_{\nu}}} \cdot W_{2^{k_{\nu}}}(x) \cdot \sum_{i=2^{k_{\nu}}}^{2^{k_{\nu}+m_{\nu}-1}} \frac{1}{2^i} W_i(x) = \]
\[ = J_1 + J_2. \tag{16} \]
By (6) we get

\[ J_1 = \frac{1}{\nu^2} \cdot W_{2\nu}(x) \cdot \sum_{i=0}^{m_\nu-1} W_{2\nu+i}(x) = \]

\[ = \frac{1}{\nu^2} \cdot W_{2\nu}^2(x) \cdot D_{m\nu}(x). \]

\[ |J_2| \leq \sum_{i=2^{k\nu}}^{2^{k\nu}+m_\nu-1} \frac{1}{2^i} |W_i(x)| \leq \sum_{i=2^{k\nu}}^{\infty} \frac{1}{2^i} \leq 2^{2-k\nu}. \]

From this and (16) we obtain

\[ |G_{2\nu+m\nu}(f, W_{n\nu}) - G_{2\nu}(f, W_{n\nu})| \geq \]

\[ \geq \frac{1}{\nu^2} \cdot |D_{m\nu}(x)| - 2^{2-k\nu+1}. \]  

(17)

Now we take the sequence of natural numbers \( m_\nu \) defined as (5) such that \( 2^{k\nu} \leq m_\nu < 2^{k\nu+1} \). Then from (6), (17) we have

\[ \int_0^1 |G_{2\nu+m\nu}(f, W_{n\nu}) - G_{2\nu}(f, W_{n\nu})| \, dx > \]

\[ > \frac{1}{\nu^2} \cdot \int_0^1 |D_{m\nu}(x)| \, dx - 2^{2-k\nu+1} \geq \]

\[ \geq \frac{1}{4 \cdot \nu^2} \cdot \log_2 m_\nu - 2^{2-k\nu+1} \geq \frac{k_\nu}{4 \cdot \nu^2} - 2^{2-k\nu+1} \geq \]

\[ \geq \frac{(\nu - 1)^2 + 1}{4 \cdot \nu^2} - 2^{2-k\nu+1} \geq \frac{1}{8} - 2^{2-k\nu+1} \geq C_1, \quad \nu \geq 2 \]

Thus the sequence \( \{G_n(f, W)\} \) does not converge by \( L^1[0,1] \) norm, i.e. the Walsh subsystem \( \{W_{n\nu}\}_{k=1}^{\infty} \) is not a quasi-greedy basis in its linear span in \( L^1 \).

The Theorem is proved.

**Remark.** As we well known (see [10] p.149) if the \( c_i \downarrow 0 \), then the series \( \sum_{n=1}^{\infty} c_n W_n(x) \) converges on \((0,1)\). In the proof of Theorem we constructed the series (11) so that the coefficients strongly decreasing, but the series diverges by \( L^1 \)-norm.
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