HECKE ALGEBRA ISOMORPHISMS AND ADELC POINTS ON ALGEBRAIC GROUPS

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Abstract. Let $G$ denote an algebraic group over $\mathbb{Q}$ and $K$ and $L$ two number fields. Assume that there is a group isomorphism $G(\mathbb{A}_K) \cong G(\mathbb{A}_L)$ of points on $G$ over the adeles of $K$ and $L$, respectively. We establish conditions on the group $G$, related to the structure and the splitting field of its Borel groups, under which $K$ and $L$ have isomorphic adele rings. Under these conditions, if $K$ or $L$ is a Galois extension of $\mathbb{Q}$ and $G(\mathbb{A}_K) \cong G(\mathbb{A}_L)$, then $K$ and $L$ are isomorphic as fields.

As a corollary, we show that an isomorphism of Hecke algebras for $GL(n)$ (for fixed $n \geq 2$), which is an isometry in the $L^1$-norm over two number fields $K$ and $L$ that are Galois over $\mathbb{Q}$, implies that the fields $K$ and $L$ are isomorphic. This can be viewed as an analogue in the theory of automorphic representations of the theorem of Neukirch that the absolute Galois group of a number field determines the field if it is Galois over $\mathbb{Q}$.

1. Introduction

Suppose that $G$ is an algebraic group over $\mathbb{Q}$, and $K$ and $L$ are two number fields such that the (adelic) Hecke algebras for $G$ over $K$ and $L$ are isomorphic. As we will see, a closely related hypothesis is: suppose that the groups of adelic points on $G$ are isomorphic for $K$ and $L$. What does this imply about the fields $K$ and $L$? Before stating the general result, let us provide two examples.

Example A. Firstly, if $G = G^r \times G^s$ for any integers $r, s$, then we have an isomorphism of topological groups $G(\mathbb{A}_K) \cong G(\mathbb{A}_L)$ while the rings $\mathbb{A}_K \not\cong \mathbb{A}_L$ are not isomorphic for any two different imaginary quadratic fields $K$ and $L$ of discriminant $\leq -8$ (cf. Section 2). To prove this, one determines separately the abstract structures of the additive and multiplicative group of the adele ring $\mathbb{A}_K$ and sees that it depends on only a few arithmetical invariants, allowing for a lot of freedom in “exchanging local factors”. This example illustrates that at least some condition on the rank, unipotent rank, and action of the torus on the unipotent part will be required to deduce that we have a ring isomorphism $\mathbb{A}_K \cong \mathbb{A}_L$.

Example B. Consider $G = GL(2)$. Then for any number fields $K$ and $L$, the existence of an isomorphism $\Phi : GL(2, \mathbb{A}_K) \cong GL(2, \mathbb{A}_L)$ of topological groups implies that $\mathbb{A}_K \cong \mathbb{A}_L$ as rings.

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Let us very briefly sketch the argument in this case, since it is essentially this reasoning that we extend to the more general case. The conjugacy classes of the diagonal and strictly upper triangular subgroups, respectively, are characterised purely group theoretically, so they are respected by $\Phi$. Hence, $\Phi$ induces two group isomorphisms $f: (A_K^*)^2 \xrightarrow{\sim} (A_L^*)^2$ (of conjugacy classes of diagonal groups) and $g: (A_K, +) \xrightarrow{\sim} (A_L, +)$ (of conjugacy classes of strictly upper triangular matrices) such that the adjoint action of diagonal matrices on upper triangular matrices is respected, yielding the adjoint relation $g(t_1 t_2^{-1} u) = f(t_1) f(t_2)^{-1} g(u)$ for all $(t_1, t_2) \in (A_K^*)^2$ and $u \in A_K$. Then we prove that $f$ and $g$ are actually local, in the sense that there is a bijection of places $\varphi: M_{K,f} \to M_{L,f}$ (where $M_{K,f}$ denotes the set of non-archimedean places of the number field $K$) such that $f$ is of the form $f = (f_v)_v$, with $f_v: (K_v^*)^2 \xrightarrow{\sim} (L_v^*)^2$, and similarly for $g$ and $\Phi$. For this, we consider maximal closed additive subgroups of $(A_K, +)$ which are invariant under the action of $(A_K^*)^2$. These are maximal ideals $m$ of $A_K$, which satisfy $A_K/m = K_v$, giving local maps. A little algebraic manipulation using the adjoint relation now shows that the local additive map $g_v/g_v(1)$ is also multiplicative. This yields an isomorphism of adele rings.

Let us now state the main technical condition, which we will elaborate on in Section 3.

**Definition C.** Let $K$ and $L$ two number fields. Let $G$ denote a smooth algebraic group over $\mathbb{Q}$. We call $G$ fertile for $K$ and $L$ if the following conditions are met: let $H$ denote the maximal closed linear algebraic subgroup of $G$ (which exists by Chevalley’s structure theorem). Suppose that $H$ contains a Borel group $B$ which is split as $B = T \ltimes U$ over a field $F$ which is Galois over $\mathbb{Q}$ and satisfies $K \cap F = L \cap F$, such that over $F$, the maximal torus $T \neq \{1\}$ acts non-trivially by conjugation on the abelianisation of the maximal unipotent group $U \neq \{0\}$.

Split tori and commutative unipotent groups are not fertile for any fields. We don’t know what happens for purely unipotent but non-commutative groups, e.g., the $3 \times 3$ strictly upper triangular matrix group. On the other hand, $GL(n)$ is fertile for any pair of number fields $K$ and $L$ if and only if $n \geq 2$. In general, fertility is slightly stronger than non-commutativity. Roughly speaking, it says that the group has semisimple elements that do not commute with some unipotent elements.

Our main result is:

**Theorem D.** Let $K$ and $L$ be two number fields, and let $G$ denote a smooth algebraic group over $\mathbb{Q}$. There is a topological group isomorphism of adelic point groups $G(A_K) \cong G(A_L)$ if and only if there is a ring isomorphism $A_K \cong A_L$.

An isomorphism of adele rings $A_K \cong A_L$ implies (but is generally stronger than) arithmetic equivalence of $K$ and $L$ (due to Komatsu [13], cf. [11], VI.2). Recall that $K$ and $L$ are said to be arithmetically equivalent if they have the same Dedekind zeta function: $\zeta_K = \zeta_L$. If $K$ or $L$ is a Galois extension of $\mathbb{Q}$, then this is known to imply that $K$ and $L$ are isomorphic as fields (due to Gaßmann [6], cf. [11], III.1).

The question whether $G(R) \cong G(S)$ for algebraic groups $G$ and rings $R, S$ implies a ring isomorphism $R \cong S$ has been considered before (following seminal work of van der Waerden and Schreier from 1928 [21]), most notably when $G = GL_n$ for $n \geq 3$ or when $G$ is a Chevalley group and $R$ and $S$ are integral domains (see, e.g., [4], [19] and the references therein). The methods employed there make extensive use of root data and Lie algebras. By contrast, our proof
of Theorem [D] uses number theory in adele rings and, by not passing to Lie algebras, applies to a more general class of (not necessarily reductive) algebraic groups. It goes via various reduction steps: first, to the case of a connected linear algebraic group; then to fields $K$ and $L$ over which a maximal torus splits; then to the case of a non-trivial extension of $G_\alpha$ by a split torus (parametrised by a non-trivial character). This mimics the situation of Example [H] except that the action of the torus is through multiplication with $n$-th powers where we do not necessarily have $n = 1$. But we can use an identity that Siegel developed in connection with his study of the Waring problem in number fields in order to emulate the $n = 1$ case. Now maximal closed additive subgroups of $G_\alpha(A_K)$ that are stable under the torus action are exactly maximal closed ideals in the adeles, and these are known by results of Iwasawa [9] and Lochter [14], describing local fields as quotients of closed maximal ideals in the adele ring. Thus, we are reduced to the local case and we can more or less argue as in Example [B].

By the (finite adelic) Hecke algebra for $G$ over $K$, we mean the convolution algebra $H_G(K) := C_c^\infty(G(A_K,f), R)$ of locally constant compactly supported real-valued functions on $G(A_K,f)$. By a positive isomorphism of Hecke algebras we mean one that sends positive functions to positive functions. By an $L^1$ isomorphism we mean one that respects the $L^1$-norm.

**Theorem E.** Let $K$ and $L$ be two number fields, and let $G$ denote a smooth algebraic group over $Q$ which is fertile for $K$ and $L$. There is a positive or $L^1$ isomorphism of Hecke algebras $H_G(K) \cong H_G(L)$ if and only if there is a ring isomorphism $A_K \cong A_L$.

This follows from the previous theorem by using some density results in functional analysis and a theorem on the reconstruction of an isomorphism of groups from a positive isomorphism or isometry of $L^1$-group algebras due to Kawada [10] and Wendel [24].

Let $G_K$ denote the absolute Galois group of a number field $K$ that is Galois over $Q$. Neukirch [16] has proven that $G_K$ determines $K$ (Uchida [23] later removed the condition that $K$ is Galois over $Q$). The set of one-dimensional representations of $G_K$, i.e., the abelianisation $G_K^{ab}$, far from determines $K$ (compare [17] or [1]). Several years ago, in connection with the results in [5], Jonathan Rosenberg asked the first author whether, in a suitable sense, two-dimensional irreducible—the “lowest-dimensional non-abelian”—representations of $G_K$ determine $K$. By the philosophy of the global Langlands programme, such representations of $G_K$ in $GL(n, \mathbb{C})$ should give rise to automorphic representations, i.e., to certain modules over the Hecke algebra $H_{GL(n)}(K)$. If we consider the analogue of this question in the setting of $H_{GL(n)}(K)$-modules instead of $n$-dimensional Galois representations, our main theorem implies a kind of “automorphic anabelian theorem”:

**Corollary F.** Suppose that $K$ and $L$ are number fields that are Galois over $Q$. There is a positive or $L^1$ algebra isomorphism of Hecke algebras $H_{GL(n)}(K) \cong H_{GL(n)}(L)$ for some $n \geq 2$ if and only if there is a field isomorphism $K \cong L$.

The paper has the following structure. In Section 2 we discuss what happens if $G$ is the additive or multiplicative group or a direct product thereof. In Section 3 we introduce and discuss the notion of fertility. In Sections 4–7 we prove Theorem [D] by successive reductions to the linear case, the case of a split maximal torus, and the case of a semi-direct product of such a torus with an
additive group. In Section 8 we discuss Hecke algebras and prove Theorem E. At the end of the paper, we discuss some open problems.

2. Additive and multiplicative groups of adeles

In this section, we elaborate on Example A from the introduction. We discuss the group structure of the additive and multiplicative groups of adeles of a number field, and we recall the notions of local isomorphism of number fields and its relation to isomorphism of adele rings and arithmetic equivalence. We introduce local additive and multiplicative isomorphisms and prove that their existence implies arithmetic equivalence.

Arithmetic equivalence and local isomorphism.

2.1. Notations/Definitions. If \( K \) is a number field with ring of integers \( \mathcal{O}_K \), let \( M_K \) denote the set of all places of \( K \), and \( M_{K,f} \) the set of non-archimedean places of \( K \). If \( p \in M_{K,f} \) is a prime ideal, then \( K_p \) denotes the completion of \( K \) at \( p \), and \( \mathcal{O}_{K,p} \) its ring of integers. Let \( e(p) \) and \( f(p) \) denote the ramification and residue degrees of \( p \) over the rational prime \( p \) below \( p \), respectively.

The decomposition type of a rational prime \( p \) in a field \( K \) is the sequence \((f(p))_{p|p}\) of residue degrees of the prime ideals of \( K \) above \( p \), in increasing order, with multiplicities.

We use the notation \( \prod' (G_i, H_i) \) for the restricted product of the group \( G_i \) with respect to the subgroups \( H_i \). We denote by \( A_K = \prod' (K_p, \mathcal{O}_{K,p}) \) the adele ring of \( K \), and by \( A_{K,f} = \prod' (K_p, \mathcal{O}_{K,p}) \) its ring of finite adeles.

Two number fields \( K \) and \( L \) are arithmetically equivalent if for all but finitely many prime numbers \( p \), the decomposition types of \( p \) in \( K \) and \( L \) coincide.

Two number fields \( K \) and \( L \) are called locally isomorphic if there is a bijection \( \varphi : M_{K,f} \to M_{L,f} \) between their sets of prime ideals such that the corresponding local fields are isomorphic, i.e. \( K_p \cong L_{\varphi(p)} \) for all \( p \in M_{K,f} \).

The main properties are summarised in the following proposition (see e.g. [11], III.1 and VI.2):

2.2. Proposition. Let \( K \) and \( L \) be number fields. Then:

(i) \( K \) and \( L \) are locally isomorphic if and only if the adele rings \( A_K \) and \( A_L \) are isomorphic as topological rings, if and only if the rings of finite adeles \( A_{K,f} \) and \( A_{L,f} \) are isomorphic as topological rings.

(ii) \( K \) and \( L \) are arithmetically equivalent if and only if \( \zeta_K = \zeta_L \), if and only if there is a bijection \( \varphi : M_{K,f} \to M_{L,f} \) such that \( K_p \cong L_{\varphi(p)} \) for all but finitely many \( p \in M_{K,f} \).

(iii) We have \( K \cong L \Rightarrow A_K \cong A_L \Rightarrow \zeta_K = \zeta_L \) and none of the implications can be reversed in general, but if \( K \) or \( L \) is Galois over \( \mathbb{Q} \), then all implications can be reversed. □
The additive group of adeles.

2.3. Proposition. If $H$ is a number field, then there is a topological isomorphism of additive groups

$$(A_H, +) \cong (A_Q^{[H:Q]}, +).$$

Hence, the additive groups $(A_K, +)$ and $(A_L, +)$ are isomorphic for two number fields $K$ and $L$ if and only if $K$ and $L$ have the same degree over $Q$.

Proof. For any number field $H$, we have that

$$(A_H, +) = \prod_{p \in \mathcal{M}_H} (H_p, +) \oplus (\mathcal{O}_{H,p}, +).$$

Since there is a topological $Q_p$-algebra isomorphism $H \otimes Q Q_p \cong \bigoplus_{p \mid p} H_p$ we see that $\bigoplus_{p \mid p} H_p$ is a $Q_p$-vector space of dimension $\sum_{p \mid p} e(p)f(p) = n$, where $n$ is the degree of $H/Q$. This implies the result.

However, if the isomorphism is “local”, i.e. induced by local additive isomorphisms, then we have the following result:

2.4. Proposition. Let $K$ and $L$ be number fields such that there is a bijection $\varphi: M_K \to M_L$ with, for almost all places $p$, isomorphisms $\Phi_p: (K_v, +) \cong (L_{\varphi(v)}, +)$. Then $K$ and $L$ are arithmetically equivalent.

Proof. Additively, the group $(K_p, +)$ is a $[K_p : Q_p] = e(p)f(p)$-dimensional topological $Q_p$-vector space. At all but finitely many primes $p$, both $K$ and $L$ are unramified, so a local matching of vector spaces will match all but finitely many residue field degrees $f(p)$. By Proposition 2.2 this implies that $K$ and $L$ are arithmetically equivalent.

2.5. Remark. By a result of Perlis (the equivalence of (b) and (c) in Theorem 1 in [18]), in the situation of the proposition, all residue field degrees of $K$ and $L$ match. This in turn implies that all ramification degrees match. So whereas two arithmetically equivalent number fields may have different ramification degrees at finitely many places, the above isomorphism excludes this possibility. However, this is still weaker than local isomorphism, since the ramification degree does not uniquely determine the ramified part of a local field extension.

The multiplicative group of adeles.

2.6. Proposition. If $H$ is a number field with $r_1$ real and $r_2$ complex places, then there is a topological group isomorphism

$$(A_H^*, \cdot) \cong (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2} \times \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \hat{\mathbb{Z}}^{[H:Q]} \times \prod_{p \in \mathcal{M}_H} (\mathbb{H}_p^* \times \mu_{p^\infty}(H_p))$$

where $\mathbb{H}_p^*$ is the multiplicative group of the residue field of $H$ at $p$ (a cyclic group of order $p^{f(p)} - 1$) and $\mu_{p^\infty}(H_p)$ is the (finite cyclic $p$-)group of $p$-th power roots of unity in $H_p$. 
Proof. We have
\[ A_H^* \cong (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2} \times A_{H,f}^* \]
and
\[ A_{H,f}^* \cong J_H \times \mathcal{O}_H^*(\hat{p}) \]
Here, \( J_H \) is the topologically discrete group of fractional ideals of \( H \), so \( J_H \cong \bigoplus \mathbb{Z} \mathbb{Z} \), where the index runs over the set of prime ideals, and the entry of \( n \in J_H \) corresponding to a prime ideal \( p \) is given by \( \text{ord}_p(n) \). Furthermore, \( \mathcal{O}_H^*(\hat{p}) = \prod_{p \in M_{H,f}} O_{H,p}^* \) is the group of finite idelic units. To determine the isomorphism type of the latter, we quote [7], Kapitel 15: let \( \pi_p \) be a local uniformizer at \( p \) and let \( T_{p} = O_{H,p} \) denote the residue field; then the unit group is
\[ \mathcal{O}_{H,p}^*(\hat{p}) \cong T_{p}^{[H_p:Q_p]} \times \mu_{p,\infty}(H_p) \]
where \( \mu_{p,\infty}(H_p) \) is the group of \( p \)-th power roots of unity in \( H_p \). □

It remains to determine the exact structure of the \( p \)-th power roots of unity, e.g.:

2.7. Example ([1], Lemma 3.1 and Lemma 3.2). If \( H \neq Q(i) \) and \( H \neq Q(\sqrt{-2}) \), then there is an isomorphism of topological groups
\[ \prod_{p \in M_{H,f}} (T_{p}^* \times \mu_{p,\infty}(H_p)) \cong \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}. \]
Hence we conclude: If \( K \) and \( L \) are two imaginary quadratic number fields different from \( Q(i) \) and \( Q(\sqrt{-2}) \), then we have a topological group isomorphism \( A_K^* \cong A_L^* \).

Combining Proposition 2.3 and Example 2.7, we obtain the claim made in Example A in the introduction:

2.8. Corollary. For any two imaginary quadratic number fields \( K \) and \( L \) different from \( Q(i) \) and \( Q(\sqrt{-2}) \) and for any integers \( r \) and \( s \), there is a topological group isomorphism
\[ (A_K^*)^r \times (A_L^*)^s \cong (A_L^*)^r \times (A_K^*)^s. \]
□

On the other hand, we again have a “local” result (and Remark 2.5 also applies in this case):

2.9. Proposition. Let \( K \) and \( L \) be number fields such that there is a bijection \( \varphi : M_K \rightarrow M_L \) with, for almost all places \( p \), topological isomorphisms \( \Phi_p : (K_p^*, \cdot) \sim (L_p^*, \cdot) \). Then \( K \) and \( L \) are arithmetically equivalent.

Proof. From (2) and (3), we find that the prime-to-\( p \) torsion subgroup (namely, \( T_p^* \)) of \( H_p^* \) has order \( p^f(p) - 1 \), where \( p \) is the rational prime below \( p \). Since a local group isomorphism \( \Phi_p \) preserves prime-to-\( p \)-torsion, we find that the bijection \( \varphi \) matches the decomposition types of all but finitely many primes. By Proposition 2.2, this implies that \( K \) and \( L \) are arithmetically equivalent. □
3. Set-up from algebraic groups and the notion of fertility

In this section, we set up notations and terminology from the theory of algebraic groups, and we elaborate on the notion of a group being fertile for a pair of number fields.

Algebraic groups.

3.1. Notations/Definitions. Let $G$ denote a connected smooth linear (viz., affine) algebraic group. We denote the multiplicative group by $\mathbb{G}_m$ and the additive group by $\mathbb{G}_a$. An $n$-dimensional torus $T$ is an algebraic subgroup of $G$ which is isomorphic, over $\overline{\mathbb{Q}}$, to $\mathbb{G}_m^n$, for some integer $n$. Every torus splits over a number field $F$, meaning that there exists an isomorphism $T \cong \mathbb{G}_m^n$ defined over $F$. All maximal $F$-split tori of $G$ are $G(F)$-conjugate and have the same dimension, called the rank of $G$. A subgroup of $G$ is unipotent if $U(\overline{\mathbb{Q}})$ consists of unipotent elements. Every unipotent subgroup of $G$ splits over $\mathbb{Q}$, meaning that it has a composition series in which every successive quotient is isomorphic to $\mathbb{G}_a$. Alternatively, it is isomorphic over $\mathbb{Q}$ to a subgroup of a group of strictly upper triangular matrices. Any group $G$ that is not unipotent contains a non-trivial torus. A Borel subgroup $B$ of $G$ is a maximal connected solvable subgroup of $G$. If all successive quotients in the composition series of $B$ over a field $F$ are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$, then $B$ is conjugate, over $F$, to a subgroup of an upper triangular matrix group, by the Lie-Kolchin theorem. Moreover, over $F$, $B \cong T \ltimes U$ is a semi-direct product induced by the adjoint representation $\rho: T \to \text{Aut}(U)$ (i.e., by the conjugation action of $T$ on $U$). Furthermore, given $U$, $B$ is the normaliser of $U$ in $G$, and $T \cong B/U$. The (rational) character group $X^*(T)$ of a torus $T$ is the group of morphisms of algebraic groups $X^*(T) = \text{Hom}(T, \mathbb{G}_m)$.

3.2. Definition. Let $K$ and $L$ denote two number fields. We call a smooth linear algebraic group $G$ fertile for $K$ and $L$ if the following conditions are met:

(i) The Borel groups $B = T \ltimes U$ of $G$ are split over a field $F$ which is Galois over $\mathbb{Q}$ with $K \cap F = L \cap F$;

(ii) $T \neq \{1\}$ acts non-trivially (by conjugation) on the abelianisation $U^{ab}$ of $U \neq \{0\}$.

A smooth algebraic group is called fertile for $K$ and $L$ if its maximal linear algebraic subgroup is so.

3.3. Remark. If $\rho: T \to GL(V)$ is an $F$-rational representation of $T$ (for example, the above adjoint representation given by $T$ acting on $U^{ab}$), then we find that $V = \bigoplus \alpha V_{\alpha}$ where $\alpha \in X^*(T)$ and $V_{\alpha} = \{v \in V: t \cdot v = \alpha(t)v \text{ for all } t \in T\}$. This is called a weight space decomposition for $V$. The characters $\alpha$ for which $V_{\alpha}$ is non-zero are the weights of $T$ in $V$. Thus, condition (ii) is the same as saying that the adjoint action of $T$ on $U^{ab}$ has a non-trivial character occurring in its weight space decomposition.

When $G$ is a reductive linear algebraic group, a central role is played by the weight space decomposition of the adjoint map on its Lie algebra $\mathfrak{g}$. The weights are then called roots. Using the Jordan-Chevalley decomposition for Lie algebras, we can recover the weight space decomposition for $U^{ab}$ by restricting to the abelian nilpotent subalgebras occurring in the root system. Thus, (ii) in the above definition can be read off from the root data for reductive linear $G$. 

3.4. Examples.

(i) Tori and unipotent groups are not fertile for any pair of number fields. Similarly, any direct product of a torus and a unipotent group is not fertile (since the adjoint representation of $T$ on $U$ is trivial).

(ii) The general linear group $\text{GL}(n)$ for $n \geq 2$ is fertile for any pair of number fields. Here, $T$ is the group of diagonal matrices, split over $F = \mathbb{Q}$, which acts non-trivially on the group of strictly upper triangular matrices. Similarly, the Borel group of (non-strictly) upper triangular matrices is fertile for any pair of number fields.

(iii) Let $G = \text{Res}_F^G(G_m \times G_a)$ denote the “$ax + b$”-group of a number field $F$, as an algebraic group over $\mathbb{Q}$. This group is fertile for any pair of number fields with $K \cap F = L \cap F$. This is not the case, for example, if $K = F \neq \mathbb{Q}, L = \mathbb{Q}$.

(iv) No semi-abelian variety is fertile for any pair of number fields.

Adelic point groups.

3.5. Definition. Let $G$ denote a smooth algebraic group over $\mathbb{Q}$ and $K$ a number field. The group of adelic points of $G$ over $K$ (also called an adelic point group) is a topological group defined as follows:

- As a set, it is given by
  $$G(A_K) := \text{Hom}_{\text{alg}}(A_K[G], A_K).$$
  Here, $A_K[G] = A_K \otimes_{\mathbb{Q}} \mathbb{Q}[G]$ is the coordinate ring of $G$ over $A_K$.

- The group structure on $G(A_K)$ arises from the $A_K$-Hopf algebra structure on $A_K[G]$, induced from the $\mathbb{Q}$-Hopf algebra structure on $\mathbb{Q}[G]$ which in turn arises from the structure of $G$ as an algebraic group over $\mathbb{Q}$. More precisely, the product of two elements of $G(A_K)$, $\phi : A_K[G] \to A_K$ and $\psi : A_K[G] \to A_K$ say, is given by $\phi \cdot \psi := m \circ (\phi \otimes \psi) \circ \Delta_G$, where $\Delta_G : A_K[G] \to A_K[G] \otimes A_K[G]$ is the co-product on the Hopf algebra $A_K[G]$ and $m : A_K \otimes A_K \to A_K$ is the multiplication map. Similarly, the (left) unit of $G(A_K)$ is defined by precomposition with the co-unit $u : A_K[G] \to A_K$ of $A_K[G]$ as $e : A_K[G] \xrightarrow{u} A_K \to A_K$. Finally, (left) inverses are defined in terms of the co-inverse (= antipode) $i : A_K[G] \to A_K[G]$ as $\phi^{-1} = \phi \circ i$. For more details, the reader may consult [12], Chapter VI, §20.

- The topology on $G(A_K)$ is the weakest for which all maps $G(A_K) \to A_K$ induced by elements of $A_K[G]$ are continuous.

We refer to [3] for a comparison of this definition to other ways of making sense of adelic points on varieties. In Section 2 we considered the group of adelic points on $G_a$ and $G_m$.

4. Proof of Theorem D: reduction to the linear part

We now turn to the proof of Theorem D. We first recall its statement:

4.1. Theorem (Theorem D). Let $K$ and $L$ be two number fields, and let $G$ denote a smooth algebraic group over $\mathbb{Q}$ which is fertile for $K$ and $L$. There is a topological group isomorphism of adelic point groups $G(A_K) \cong G(A_L)$ if and only if there is a ring isomorphism $A_K \cong A_L$. 

The proof will occupy Sections 4.7. In this section, we use the structure theorem of Chevalley to reduce to the case of a linear algebraic group.

4.2. Proposition. If Theorem D holds for any connected linear algebraic group, then it holds for any algebraic group.

Proof. First of all, it suffices to prove the theorem for connected groups. Indeed, if $G$ is fertile for $K$ and $L$, then so is its connected component of the identity (with the same field $F$), since Borel subgroups for $G^0$ are also Borel subgroups for $G$. Henceforth, we assume that $G$ is connected.

Let $K$ and $L$ denote two number fields and $G$ a smooth connected algebraic group that is fertile for $K$ and $L$. Chevalley’s structure theorem states that there exists a unique normal connected closed linear algebraic subgroup $H$ of $G$ for which the quotient $G/H$ is an abelian variety $A$, such that $H$ contains all other connected linear algebraic subgroups of $G$. Recall that there are no non-zero algebraic group morphisms from linear algebraic groups to abelian varieties. Therefore, an equivalent description of $H$ is that $H$ is the maximal algebraic subgroup of $G$ such that the only algebraic group morphism from $H$ to any abelian variety is the zero map.

By definition, $H$ is also fertile for $K$ and $L$, and we may assume that $H(A_K) \cong H(A_L)$ implies $A_K \cong A_L$. Therefore, it suffices to show that $G(A_K) \cong G(A_L)$ implies $H(A_K) \cong H(A_L)$. For this, we prove the following:

4.3. Lemma. $H(A_K)$ is the maximal subgroup $H \subseteq G(A_K)$ for which $\text{Hom}(\overline{H}, A) = \{0\}$, where $\overline{H}$ is the Zariski closure of $H$ and $\text{Hom}$ is the set of algebraic group morphisms.

Proof of Lemma 4.3. Observe that since $H$ is connected, $H(A_K)$ is Zariski dense in $H$, for $H(K)$ is already Zariski dense in $H$ (see e.g. [2], 18.3). If $\overline{H}$ is a subgroup of $G(A_K)$, then $\overline{H}$ is an algebraic subgroup of $G$. When $\overline{H}$ is linear, then $\overline{H} \subseteq H$ by definition of $H$, and hence $H \subseteq H(A_K)$. Conversely, when $H \subseteq H(A_K)$ we obtain $\overline{H} \subseteq \overline{H(A_K)} = H$ so $\overline{H}$ is linear. Hence, $H(A_K)$ is the maximal subgroup of $G(A_K)$ whose Zariski closure is linear. Now for linear algebraic groups $L$, we know that $\text{Hom}(L, A) = \{0\}$, where $A = G/H$ as above. This proves the lemma. □

The lemma allows us to finish the proof of the proposition as follows. Under a group isomorphism $G(A_K) \cong G(A_L)$, subgroups $H$ of $G(A_K)$ are mapped to isomorphic subgroups $H'$ of $G(A_L)$. In particular their Zariski closures must be isomorphic, too, so that $\overline{H}$ is linear if and only if $\overline{H}'$ is linear. Hence, $H(A_K) \cong H(A_L)$, as required. □

5. Proof of Theorem D: reduction to fields over which the torus splits

In this section, we use base change arguments to reduce to the case where $T$ splits over $F$. The condition $K \cap F = L \cap F$ is indispensable in this proof.

5.1. Proposition. If Theorem D holds for a (fixed) linear algebraic group $G$ and fields $K$ and $L$ over which the maximal tori of $G$ split, then it holds for any fields $K$ and $L$.

Proof. The proposition follows from two lemmas, one on base change (tensoring by $F$) and one on descent (taking Galois invariants). Although the proofs appear rather standard, we could not find a suitable reference in the literature.
5.2. **Lemma.** Suppose that $G(\mathcal{A}_K) \cong G(\mathcal{A}_L)$ as topological groups. Then, for $F/\mathbb{Q}$ Galois such that $K \cap F = L \cap F$, we have that $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_{LF})$ as topological groups.

**Remark.** The lemma certainly needs some condition of the form $K \cap F = L \cap F$. For example, if $G = G_\alpha$, then $G(\mathcal{A}_K) \cong G(\mathcal{A}_L)$ is equivalent to $[K : \mathbb{Q}] = [L : \mathbb{Q}]$ and $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_{LF})$ is equivalent to $[KF : \mathbb{Q}] = [LF : \mathbb{Q}]$, hence the conclusion is wrong for, e.g., $F = K$ and $F \neq L$.

**Proof.** The proof consists of several steps, each adding a layer of structure. Note that for $\mathcal{A}_K$-modules $M$,

$$M \otimes_{\mathcal{A}_K} \mathcal{A}_{KF} = M \otimes_{\mathcal{A}_K} \mathcal{A}_K \otimes_{F \cap K} F = M \otimes_{F \cap K} F.$$

For ease of notation, we will write both $\otimes_{\mathcal{A}_K} \mathcal{A}_{KF}$ and $\otimes_{F \cap K} F$ simply as $\otimes F$.

**Step 1:** $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_{LF})$ as sets. This will follow from $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_K) \otimes F$. We have

$$G(\mathcal{A}_{KF}) = \text{Hom}_{\mathcal{A}_{KF} - \text{alg}}(\mathcal{A}_{KF}[G], \mathcal{A}_{KF}) \cong \text{Hom}_{(\mathcal{A}_K \otimes F) - \text{alg}}(\mathcal{A}_K[G] \otimes F, \mathcal{A}_K \otimes F) \cong \text{Hom}_{\mathcal{A}_K - \text{alg}}(\mathcal{A}_K[G], \mathcal{A}_K) \otimes F = G(\mathcal{A}_K) \otimes F.$$

The last isomorphism is given by $g \mapsto [a \mapsto g(a \otimes 1)]$ for any homomorphism $g : \mathcal{A}_K[G] \otimes F \to \mathcal{A}_K \otimes F$ and $a \in \mathcal{A}_K[G]$. Indeed, for any $r \in \mathcal{A}_K$ and $f \in F$, we have that

$$g((r \otimes f)(a \otimes f')) = (r \otimes f)(1 \otimes f')g(a \otimes 1) = (r \otimes f f')g(a \otimes 1),$$

i.e., $g$ is completely determined by its values on $\mathcal{A}_K[G]$. Now $\mathcal{A}_{KF}$ is locally free, hence flat, over $\mathcal{A}_K$, since $F$ is flat over $F \cap K$. Also, the coordinate ring $\mathcal{A}_K[G]$ is finitely presented. Hence (cf. e.g. [20], Lemma 4.86)

$$\text{Hom}_{\mathcal{A}_K - \text{alg}}(\mathcal{A}_K[G], \mathcal{A}_{KF}) \cong \text{Hom}_{\mathcal{A}_K - \text{alg}}(\mathcal{A}_K[G], \mathcal{A}_K) \otimes F = G(\mathcal{A}_K) \otimes F.$$

**Step 2:** $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_{LF})$ as groups. The group structure on $G(\mathcal{A}_K)$ is induced from the Hopf algebra structure on the coordinate ring $\mathcal{A}_K[G]$, so we need to prove that $\mathcal{A}_{KF}[G]$ and $\mathcal{A}_{LF}[G]$ are isomorphic as Hopf algebras. Since by definition, $G(\mathcal{A}_K) = \mathcal{A}_K[G]^\vee$ is the dual of the $\mathcal{A}_K$-module $\mathcal{A}_K[G]$, the group isomorphism $G(\mathcal{A}_K) \cong G(\mathcal{A}_L)$ induces an isomorphism $\mathcal{A}_K[G] \cong \mathcal{A}_L[G]$ of group algebras. Now $\mathcal{A}_K[G]$ is a Hopf algebra over $K$. Under base extension $KF/K$, $\mathcal{A}_K[G] \otimes F$ becomes a $KF$-Hopf algebra by base changing the structure morphisms of $\mathcal{A}_K[G]$, namely co-multiplication, co-inverse and co-unit ([12], Chapter VI, §20). Thus, the Hopf algebras $\mathcal{A}_{KF}[G]$ and $\mathcal{A}_{LF}[G]$ are isomorphic, too.

**Step 3:** $G(\mathcal{A}_{KF}) \cong G(\mathcal{A}_{LF})$ as topological groups. Recall that the topology of $G(\mathcal{A}_K)$ is the weakest topology relative to which all maps $G(\mathcal{A}_K) \to \mathcal{A}_K$ induced by elements of $\mathcal{A}_K[G]$ are continuous. Now maps $G(\mathcal{A}_{KF}) \to \mathcal{A}_{KF}$ induced by $\mathcal{A}_{KF}[G]$ are, by construction, maps $G(\mathcal{A}_K) \otimes F \to \mathcal{A}_K \otimes F$ induced by $\mathcal{A}_K[G] \otimes F$, so the topology on $G(\mathcal{A}_K)$ determines that of $G(\mathcal{A}_{KF})$. Therefore, when $G(\mathcal{A}_K)$ and $G(\mathcal{A}_L)$ are homeomorphic, so are $G(\mathcal{A}_{KF})$ and $G(\mathcal{A}_{LF})$.

This concludes the proof of Lemma 5.2. \qed
Now suppose that $G(A_K) \cong G(A_L)$ for an algebraic group $G$ over $\mathbb{Q}$ and fields $K$ and $L$ such that $G$ is fertile for $K$ and $L$. We conclude from the lemma that $G(A_{KF}) \cong G(A_{LF})$, and $G$ is fertile for $KF$ and $LF$, since $F$ splits the maximal tori (which are also split over $KF$ and $LF$, and $KF \cap F = LF \cap F (= F)$). Our hypothesis is that the main theorem is proven in this case, so we conclude that $A_{KF} \cong A_{LF}$ as topological rings. To finish the proof of the proposition, we use the following lemma:

**5.3. Lemma.** Suppose that $F/\mathbb{Q}$ is Galois with $K \cap F = L \cap F$ and $A_{KF} \cong A_{LF}$, then $A_K \cong A_L$.

**Proof.** Let $\Gamma = \text{Gal}(KF/K)$ and $\Delta = \text{Gal}(LF/L)$. By our assumption that $K \cap F = L \cap F$, we know that $\Gamma \cong \Delta$. By the canonical embeddings $A_K \hookrightarrow A_{KF}$ and $A_L \hookrightarrow A_{LF}$, we see that the elements of $A_L$ are precisely the $\Gamma$-invariant elements of $A_{KF}$, and similarly the elements of $A_L$ are the $\Delta$-invariant elements of $A_{LF}$. Thus,

$$A_K \cong A_{KF}^\Gamma \cong A_{LF}^\Delta \cong A_L,$$

as required. Note that all the isomorphisms are isomorphisms of topological rings. \qed

### 6. Proof of Theorem D: reduction to a non-trivial extension of $T$ by $G_a$

In this section, we show how to pass from a general group $G$ to a semi-direct product:

**6.1. Proposition.** If Theorem D holds for a linear algebraic group $G$ that splits over $F$ as $G \cong T \ltimes G_a$ for some non-trivial character $\chi: T \to \text{Aut}(G_a)$, then it holds for any $G$ and fields $K$ and $L$ containing $F$.

Before giving the proof, we state a general lemma:

**6.2. Lemma.** Let $U$ denote a maximal unipotent subgroup of $G$, $B$ a Borel subgroup of $G$ that contains $U$, and $T \cong B/U$ the corresponding torus, which acts on $U$ through conjugation. If $G(A_K) \cong G(A_L)$ is a topological isomorphism of point groups, then we also have induced topological isomorphisms of point groups

$$g: U(A_K) \simto U(A_L)$$

and

$$f: T(A_K) \simto T(A_L),$$

which are equivariant for the action of $T$ on $U$ by conjugation in the sense that

$$g(tut^{-1}) = f(t)g(u)f(t)^{-1}$$

for all $u \in U(A_K)$ and $t \in T(A_K)$.

**Proof.** We use that (conjugacy classes of) point groups of $U, B$ and $T$ have an intrinsic definition inside the adelic points of $G$, together with the Zariski density of point groups in the corresponding algebraic groups.

Let $U$ denote a maximal unipotent (i.e., consisting of unipotent elements) subgroup of $G(A_K)$. By [2], Corollary 18.3, taking the Zariski closure yields a unipotent (hence connected) algebraic group:

$$U = U.$$
In fact, we prove that $U$ is a maximal unipotent algebraic group. Otherwise, there would be a strictly larger $U \subset U'$, for which we must have

$$(5) \quad U(A_K) \subseteq U'(A_K) \subseteq U.$$  

We take Zariski closures everywhere, to find

$$\overline{U(A_K)} \subseteq \overline{U'(A_K)} \subseteq U,$$

since the adelic point groups of both $U$ and $U'$ are dense in $U$ and $U'$ respectively. Therefore, the second line in the diagram shows that $U = U'$. We conclude that any maximal unipotent group $U \subseteq G(A_K)$ is the point group of a maximal unipotent algebraic group $U$, that is, $U = U(A_K)$. In particular, these are all conjugate over $\mathbb{Q}$, since the maximal unipotent algebraic subgroups of $G$ are so.

Since both this construction and the notion of being unipotent are group theoretic in nature, we may do the same thing for $G(A_L)$ and obtain the isomorphism

$$(6) \quad g: U(A_K) \rightarrow U(A_L)$$

(up to $\mathbb{Q}$-conjugation). Next we consider the normaliser of $U$ in $G(A_K)$. The normaliser of $U$ in $G$ as an algebraic group is a Borel group $B$ in $G$ (Theorem of Chevalley, e.g. [2], 11.16). Since taking points and taking normalisers commute ([15], Proposition 6.3), we obtain that

$$N_{G(A_K)}U = N_{G(A_K)}U(A_K) = (N_GU)(A_K) = B(A_K).$$

As taking normalisers is a group theoretic operation, we find (again, up to $\mathbb{Q}$-conjugation) that

$$(7) \quad B(A_K) \cong B(A_L).$$

Since $T \cong B/U$ is a maximal torus, at the level of point groups, taking quotients $B(A_K)/U(A_K)$ and $B(A_L)/U(A_L)$ and using (6) and (7) yields an isomorphism

$$(8) \quad f: T(A_K) \rightarrow T(A_L).$$

All these maps are induced from a group isomorphism $G(A_K) \cong G(A_L)$, hence the isomorphisms are automatically equivariant for the action of $T$ on $U$ by conjugation. This finishes the proof of the lemma.

**Proof of Proposition 6.1.** Before we turn to the adelic point groups, we analyse the structure of the fertile algebraic group $G$ a bit further, without considering the fields $K$ and $L$. The hypothesis that $T$ splits over $F$ implies that $T \cong G_m^\ell$ for some $\ell$. The adjoint action of $T$ by conjugation on $U$ maps commutators to commutators, so it factors through the abelianisation $U^{ab}$, and we can consider the linear adjoint $F$-action $\rho: T \rightarrow \text{Aut}(U^{ab})$. Note that $U^{ab} \cong G_a^k$ for some integer $k$, so we have an action

$$(9) \quad \rho: T(\cong G_m^\ell) \rightarrow \text{Aut}(G_a^k) = \text{GL}(k),$$
which is diagonalisable over $F$ as a direct sum $\rho = \oplus \chi_i$ of $k$ algebraic characters $\chi_i \in \text{Hom}(T, G_m)$. In coordinates $t = (t_1, \ldots, t_\ell) \in G_m^\ell = T$, any such character is of the form

$$\chi(t) = \chi(t_1, \ldots, t_\ell) = t_1^{n_1} \cdots t_\ell^{n_\ell}$$

for some $n_1, \ldots, n_\ell \in \mathbb{Z}$. Let $\chi$ be a non-trivial character chosen from among the $\chi_1, \ldots, \chi_k$ (which exists by the assumption of fertility). Without loss of generality, $n_1 \neq 0$. Since we have split $\rho = \oplus \chi_i$ over $F$, we can choose coordinates $(u_1, \ldots, u_k)$ on the vector group $U^{ab}$ such that $(t_1, \ldots, t_\ell) \in T$ acts by

$$(u_1, \ldots, u_k) \mapsto (\chi(1) \cdot u_1, u_2, \ldots, u_k).$$

We now apply this analysis to the adelic point groups over $K$ and $L$. Since the group isomorphism $G(A_K) \cong G(A_L)$ respects commutators, the map $g: U(A_K) \xrightarrow{\sim} U(A_L)$ from the previous lemma gives an isomorphism of abelianisations (also denoted by $g$)

$$g: U^{ab}(A_K) = U(A_K)^{ab} \cong U(A_L)^{ab} = U^{ab}(A_L).$$

Since the isomorphism $G(A_K) \cong G(A_L)$ maps $T(A_K)$ to $T(A_L)$ and the action of $T$ on $U^{ab}$ is given by conjugation inside $G$ (which is preserved by a point group isomorphism), the adjoint action of $T(A_K)$ on $U^{ab}(A_K)$ (and similarly for $L$) is given by the adelic specialisation of $\chi$.

Let $V_K$ denote the maximal subgroup consisting of $u \in U^{ab}(A_K)$ on which every element of $\chi(T(A_K))$ acts trivially. Obviously, $V_K = \{ (0, u_2, \ldots, u_k) \in A_K : u_i \in A_K \}$, and it contains no more elements, since for $t = (2, 1, \ldots, 1) \in T(Q) \subseteq T(A_K)$, $2^{n_1} u_1 = u_1$ only for $u_1 = 0$.

From the definition of $V_K$, it is clear that $g(V_K) = V_L$. Hence we find an isomorphism

$$G_a(A_K) \cong U^{ab}(A_K)/V_K \cong U^{ab}(A_L)/V_L \cong G_a(A_L),$$

equivalent for the respective action of $T(A_K)$ and $T(A_L)$ by conjugation. Thus, we find an isomorphism of point groups $\tilde{B}(A_K) \cong \tilde{B}(A_L)$ for $\tilde{B} = T \rtimes \chi G_a$ and we are reduced to proving the theorem for $\tilde{B}$. □

7. Proof of Theorem D: reduction to the local case and end of proof

In this section, we use the description of maximal ideals in the adele ring due to Iwasawa [9] and Lochter [15] to conclude that the isomorphism of point groups for $T \rtimes \chi G_a$ is local. For this, we describe ideals as additive subgroups of $G_a$ which are stable for the $T$-action, using a trick of Siegel to write every adele as a sum of a fixed number of $n$-th powers. Finally, since we are now working over a field, we can use some algebraic manipulation to conclude that we have a local isomorphism of adele rings.

Proof of Theorem D. Recall that we are reduced to proving the main theorem when $G = T \rtimes \chi G_a$, where we know that the point group isomorphism $\Phi: G(A_K) \xrightarrow{\sim} G(A_L)$ is given by two group isomorphisms

$$f: T(A_K) \xrightarrow{\sim} T(A_L)$$

and

$$g: G_a(A_K) = (A_K, +) \xrightarrow{\sim} G_a(A_L) = (A_L, +)$$

which respect the action, in the sense that
\[ g(\chi(t)u) = \chi(f(t))g(u) \quad \text{for all } t \in T(A_K) \text{ and } u \in A_K. \]

We first reinterpret a formula of Siegel ([22], p. 134) as the following lemma:

**7.1. Lemma.** Let \( R \) denote a ring and \( n \) a positive integer such that \( n! \) is invertible in \( R \). Then any element of \( R \) belongs to the \( \mathbb{Z} \)-linear span of the \( n \)-th powers in \( R \). In particular, we have the following explicit formula for any \( z \in R \):

\[
z = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} \left\{ \left( \frac{z}{n!} + k \right)^n - k^n \right\}. \quad \square
\]

Consider the collection \( \mathcal{I}_K \) of subgroups \( I \) of \( (A_K, +) \) that are stable for the action of \( T(A_K) \), i.e., such that \( \chi(t)I \subseteq I \) for all \( t \in T(A_K) \). We claim:

**7.2. Lemma.** \( \mathcal{I}_K \) is the collection of ideals of the ring \( A_K \).

**Proof.** Recall that \( \chi \) is of the form \( \chi(t) = \chi(t_1, \ldots, t_\ell) = t_1^{n_1} \cdots t_\ell^{n_\ell} \) for some \( n_1, \ldots, n_\ell \in \mathbb{Z} \). Let \( n \) be a non-zero integer chosen from among the \( n_i \), which exists as \( \chi \) is non-trivial. The integer \( n! \) is invertible in the adeles. Hence by Siegel’s formula, any element \( z \) of \( A_K \) is a \( \mathbb{Z} \)-linear combination of \( n \)-th powers in \( A_K \). Let \( I \in \mathcal{I}_K \). Since \( I \) is closed under addition and multiplication by \( n \)-th powers from \( A_K^* \), it follows that \( A_K^* I \subseteq I \). Hence, \( I \) is an ideal. \( \square \)

Let \( \mathcal{M}_K \) denote the collection of elements from \( \mathcal{I}_K \) which are proper maximal topologically closed subgroups of \( A_K \). The lemma implies that \( \mathcal{M}_K \) is precisely the collection \( M_K \) of closed maximal ideals of \( K \). Since \( \mathcal{M}_K \) is defined inside \( A_K \) in purely topological group theoretic terms, the map \( \Phi \) respects its structure, and \( g \) induces a bijection \( g^*: \mathcal{M}_K \to \mathcal{M}_L \).

Now recall the description of the closed maximal ideals in an adele ring \( A_K \) as given by Iwasawa and Lochter ([14], Satz 8.6 and [9], p. 340–342, cf. [11], VI.2.4):

\[ \mathcal{M}_K = \{ m_p = \prod_{q \neq p} K_q : p \in M_K \}. \]

It follows that \( g^* \) gives a bijection
\[
\varphi: M_K \xrightarrow{\sim} M_L \text{ with } g(m_p) = m_{\varphi(p)},
\]
and it induces local maps
\[
g_p: (K_p, +) \cong (A_K, +)/m_p \xrightarrow{\sim} (A_L, +)/m_{\varphi(p)} \cong (L_{\varphi(p)}, +)
\]
for every prime \( p \in M_K \).

Consider the group
\[
T_{K,p} := \{ t \in T(A_K) : \chi(t)u = u, \forall u \in K_p \}.
\]

Then, since the adjoint action of \( T(A_K) \) on \( G_\alpha(A_K) \) is component-wise in the adeles, we find that
\[
T_{K,p} \cap T(K_q) = T(K_q) \text{ for } q \neq p.
\]
On the other hand, \( T_{K,p} \cap T(K_p) \) consists of those \( t_p \in K_p^* \) for which \((\chi(t_p) - 1)K_p = 0\), so
\[
T_{K,p} \cap T(K_p) = T(K_p) \cap \ker \chi.
\]

The group theoretic description of \( T_{K,p} \) implies that \( f(T_{K,p}) = T_{L,\varphi(p)} \), and, taking quotients, we obtain local multiplicative isomorphisms
\[
f_p : T(K_p)/(T(K_p) \cap \ker \chi) = T(A_K)/T_{K,p} \overset{\sim}{\longrightarrow} T(A_L)/T_{L,\varphi(p)} = T(L,\varphi(p))/(T(L,\varphi(p)) \cap \ker \chi).
\]
Furthermore, the adjoint relation yields
\[
g_p(\chi(t)u) = \chi(f_p(t))g_p(u) \text{ for all } t \in T(K_p) \text{ and } u \in K_p.
\]

We now show that the existence of this structure implies that the maps
\[
g'_p := g_p/g_p(1) : K_p \overset{\sim}{\longrightarrow} L,\varphi(p)
\]
are isomorphisms of local fields, and this will finish the proof of the main theorem. Let \( a, b \in K_p^* \) and use Lemma 7.1 to write \( a = \sum_i a_i^a \) for some \( a_i \in K_p^* \). By slight abuse of notation, we will denote by \( a_i \) both \( a_i \in K_p^* \) and \( a_i = (1, 1, \ldots, a_i, \ldots, 1) \in T(K_p) \). Then we compute
\[
g_p(ab) = g_p(\sum_i a_i^{a^a}b) = \sum_i g_p(a_i^{a_i^a})b = \sum_i \chi(f_p(a_i))g_p(b) = \left( \sum_i \chi(f_p(a_i)) \right) g_p(b),
\]
while at the same time,
\[
g_p(a) = g_p(\sum_i a_i^{a_i^a}) = \sum_i g_p(a_i^{a_i^a}) = \sum_i \chi(f_p(a_i))g_p(1) = \left( \sum_i \chi(f_p(a_i)) \right) g_p(1),
\]
so we obtain \( g_p(ab) = g_p(a)g_p(b)/g_p(1) \). Note that \( g_p(1) \neq 0 \) because otherwise, equation (11) would imply that \( g_p \) is identically zero. It follows that \( g'_p(ab) = g'_p(a)g'_p(b) \) and \( g'_p(a + b) = g'_p(a) + g'_p(b) \), as desired.

**7.3. Remark.** Lochter has shown that \( \mathcal{M}_K \) is also precisely the set of principal maximal ideals in the adele ring \( A_K \). Thus, topology is not needed in this part of the proof.

The proof goes through word for word with the finite adeles instead of the adeles, so we find that if \( G \) is fertile for \( K \) and \( L \) and \( G(A_{K,f}) \cong G(A_{L,f}) \), then \( A_{K,f} \cong A_{L,f} \). But now Proposition 2.2(i) implies:

**7.4. Theorem.** Let \( K \) and \( L \) two number fields, and let \( G \) denote a smooth algebraic group over \( \mathbb{Q} \) that is fertile for \( K \) and \( L \). Then a topological group isomorphism of finite adelic point groups \( G(A_{K,f}) \cong G(A_{L,f}) \) induces an isomorphism of full adele rings \( A_K \cong A_L \).

**8. Hecke algebras and proof of Theorem**

**8.1. Definition.** Let \( G \) denote an algebraic group over \( \mathbb{Q} \). Then \( \mathbf{G}_K := G(A_{K,f}) \) is a locally compact topological group (since \( A_{K,f} \) is locally compact) for the topology described in Definition 3.5 equipped with a (left) invariant Haar measure \( \mu_{\mathbf{G}_K} \). The finite (or non-archimedean) real Hecke
algebra $\mathcal{H}_G(K) = C_c^\infty(G_K, \mathbb{R})$ of $G$ over $K$ is the algebra of all real-valued locally constant compactly supported continuous functions $\Phi : G_K \to \mathbb{R}$ with the convolution product:

$$\Phi_1 * \Phi_2 : g \mapsto \int_{G_K} \Phi_1(gh^{-1})\Phi_2(h) d\mu_{G_K}(h).$$

Every element of $\mathcal{H}_G(K)$ is a finite linear combination of characteristic functions on double cosets $KhK$, for $h \in G_K$ and $K$ a compact open subgroup of $G_K$. Alternatively, we may write

$$\mathcal{H}_G(K) = \lim_{K} \mathcal{H}(G_K//K),$$

where $\mathcal{H}(G_K//K)$ is the Hecke algebra of $K$-biinvariant smooth functions on $G_K$ (for example, if $K$ is maximally compact, this is the spherical Hecke algebra).

8.2. Definition. If $G$ is a locally compact topological group equipped with a Haar measure $\mu_G$, let $L^1(G)$ denote its group algebra, i.e., the algebra of real-valued $L^1$ functions with respect to the Haar measure $\mu_G$, under convolution.

A map of group algebras $\Psi : L^1(G) \to L^1(H)$ is positive if $f \geq 0$ almost everywhere in $G$ implies that $\Psi(f) \geq 0$ almost everywhere in $H$. A similar definition is used for a map of Hecke algebras $\Psi : \mathcal{H}_G(K) \to \mathcal{H}_G(L)$, but (by local constancy) this is the same as requiring that if $f \geq 0$, then $\Psi(f) \geq 0$.

An isomorphism of Hecke algebras $\Psi : \mathcal{H}_G(K) \to \mathcal{H}_G(L)$ which is an isometry for the $L^1$ norms arising from the Haar measures (i.e., $||\Psi(f)||_1 = ||f||_1$ for all $f \in \mathcal{H}_G(K)$) is called an $L^1$ isomorphism.

Before we give its proof, let us recall the statement of Theorem [E]

8.3. Theorem (Theorem [E]). Let $K$ and $L$ be two number fields, and let $G$ denote a smooth algebraic group over $\mathbb{Q}$ which is fertile for $K$ and $L$. There is a positive or $L^1$ isomorphism of Hecke algebras $\mathcal{H}_G(K) \cong \mathcal{H}_G(L)$ if and only if there is a ring isomorphism $A_K \cong A_L$.

Proof. The proof consists of two steps: first we show, using the Stone-Weierstrass theorem, that the Hecke algebras are dense in the group algebras, and then we use results on reconstructing a locally compact group from its group algebra due to Kawada (for positive maps) and Wendel (for $L^1$ isometries).

Step 1: $\mathcal{H}_G(K) \cong \mathcal{H}_G(L)$ implies $L^1(G_K) \cong L^1(G_L)$ as algebras. The locally compact real version of the Stone-Weierstrass theorem implies that $\mathcal{H}_G(K)$ is dense in $C_0(G_K)$ for the sup-norm, where $C_0(G_K)$ denotes the functions that vanish at infinity, i.e., such that $|f(x)| < \varepsilon$ outside a compact subset of $G_K$. Indeed, one needs to check the nowhere vanishing and point separation properties of the algebra ([E], 7.37.b). Since $\mathcal{H}_G(K)$ contains the characteristic function of any compact subset $K \subseteq G_K$, the algebra vanishes nowhere, and the point separating property follows since $G_K$ is Hausdorff.

A fortiori, $\mathcal{H}_G(K)$ is dense in the compactly supported functions $C_c(G_K)$ for the sup-norm, and hence also in the $L^1$ norm. Now $C_c(G_K)$ is dense in $L^1(G_K)$, and the claim follows.
Step 2: A positive isomorphism or an $L^1$ isometry $L^1(G_K) \cong L^1(G_L)$ implies a group isomorphism $G_K \cong G_L$. Indeed, a positive isomorphism $\mathcal{H}_G(K) \cong \mathcal{H}_G(L)$ implies a positive isomorphism of group algebras $L^1(G_K) \cong L^1(G_L)$, and an $L^1$ isometry $\mathcal{H}_G(K) \cong \mathcal{H}_G(L)$ implies an $L^1$ isometry of group algebras $L^1(G_K) \cong L^1(G_L)$. Hence the result follows from the theorem of Kawada [10] that a positive algebra isomorphism of group algebras of locally compact topological groups is always induced by an isomorphism of topological groups, respectively the theorem of Wendel [24] that reaches the same conclusion for an $L^1$ isometry of group algebras. (Actually, Wendel proves that the existence of an $L^1$ isometric isomorphism implies that of a positive isomorphism and uses the result of Kawada.)

Finally, Theorem 7.4 says that if $G$ is fertile for $K$ and $L$, then $G_K \cong G_L$ implies that $A_K \cong A_L$. □

8.4. Corollary. If $G$ is a smooth connected algebraic group over $\mathbb{Q}$ which is fertile for two number fields $K$ and $L$ that are Galois over $\mathbb{Q}$, then a positive or $L^1$ isomorphism of Hecke algebras $\mathcal{H}_G(K) \cong \mathcal{H}_G(L)$ implies that the fields $K$ and $L$ are isomorphic.

Proof. Since the hypotheses imply that the fields are arithmetically equivalent, the result follows from Proposition 2.2.(iii). □

Since $\text{GL}(n)$ is fertile for any pair of number fields $K, L$ if $n \geq 2$, we obtain Corollary [1].

Variations on Theorem [1]

(1) The theorem is also true if the real-valued Hecke algebra is replaced by the complex-valued Hecke algebra (using the complex versions of Stone-Weierstrass and Kawada/Wendel).

(2) It seems that the theorem also holds for the full Hecke algebra $\mathcal{H}_G \otimes \mathcal{H}_G^\infty$, where $\mathcal{H}_G^\infty$ is the archimedean Hecke algebra for $G$, viz., the convolution algebra of distributions on $G(\mathbb{R} \otimes \mathbb{Q} K)$ supported on a maximal compact subgroup of $G(\mathbb{R} \otimes \mathbb{Q} K)$, but we have not checked the analytic details.

9. Discussion

One may wonder in what exact generality it holds true that $G(A_K) \cong G(A_L)$ implies $A_K \cong A_L$.

(1) What happens if $G$ is an abelian variety, e.g., an elliptic curve? For every number field, is there a sufficiently interesting elliptic curve $E/\mathbb{Q}$ such that $E(A_K)$ determines all localisations of $K$?

(2) The theorem does not hold for all linear algebraic groups; see, e.g., Example [A]. Is it possible to characterise precisely the linear algebraic groups for which $G(A_K) \cong G(A_L)$ implies $A_K \cong A_L$? For example, it is unclear to us at the moment what happens if $G$ is non-commutative unipotent.

(3) Is the theorem true without imposing a relation between $K$ and $L$ and the splitting field of the maximal torus $F$? The base change technique from Lemma 5.2 needs some such condition, but this does not exclude the possibility that the main theorem holds without such a condition (but with another proof).

(4) What happens over global fields of positive characteristic?
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