Solvable Dynamical Systems in the Plane with Polynomial Interactions

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Abstract

In this paper we report a few examples of \textit{algebraically solvable} dynamical systems characterized by 2 coupled Ordinary Differential Equations which read as follows:

\[
\dot{x}_n = P^{(n)}(x_1, x_2), \quad n = 1, 2,
\]

with \(P^{(n)}(x_1, x_2)\) specific polynomials of relatively low degree in the 2 dependent variables \(x_1 \equiv x_1(t)\) and \(x_2 \equiv x_2(t)\). These findings are obtained via a new twist of a recent technique to identify dynamical systems \textit{solvable by algebraic operations}, themselves explicitly identified as corresponding to the time evolutions of the zeros of polynomials the coefficients of which evolve according to \textit{algebraically solvable} (systems of) evolution equations.

1 Introduction

It has been recently noted \cite{1} that, if the quantities \(x_n(t)\) respectively \(y_m(t)\) denote the \(N\) zeros respectively the \(N\) coefficients of a \textit{generic} time-dependent monic polynomial \(p_N(z; t)\) of degree \(N\),

\[
p_N(z; t) = z^N + \sum_{m=1}^{N} [y_m(t)] z^{N-m} = \prod_{n=1}^{N} [z - x_n(t)],
\]

there hold the following \textit{identities} relating the time evolution of these quantities:

\[
\dot{x}_n = -\left[ \prod_{\ell=1, \ell\neq n}^{N} (x_n - x_{\ell}) \right]^{-1} \sum_{m=1}^{N} [y_m(x_n)^{N-m}], \quad n = 1, 2, ..., N.
\]

\textbf{Notation 1.1.} Hereafter all quantities are \textit{a priori} assumed to be \textit{complex} numbers, with the following exceptions: \textit{indices} such as \(n, m\), take positive integer values (over ranges specified on a case-by-case basis: indeed, in most of this paper the range is limited just to the 2 values 1 and 2 for \(n\), and to 1 and 2 or 1, 2 and 3 for \(m\)); while the independent variable \(t\) ("time") is \textit{real} and it is generally assumed to run from 0 to \(+\infty\). The \(t\)-dependence of time-dependent variables such as \(x_n(t)\) and \(y_m(t)\) is often not \textit{explicitly} displayed (even, inconsistently, in the same formula: of course when this is unlikely to cause misunderstandings); and superimposed dots on these variables denote of course time-differentiations, \(\dot{x}_n \equiv dx_n(t)/dt\), \(\dot{y}_m \equiv dy_m(t)/dt\). It is of course not excluded that \textit{complex} numbers take \textit{real} or \textit{imaginary} values, as indicated below on a case-by-case basis: indeed the words "in the plane" in the title of this paper refer to the standard case in which the two coordinates \(x_1(t)\) and \(x_2(t)\) are interpreted as the 2 \textit{real} coordinates of a point moving in the Cartesian \(x_1x_2\)-plane, or as the 2 \textit{complex} coordinates of 2 points moving in the \textit{complex} plane (or, equivalently, of 2 \textit{real} two-vectors moving in a plane: see below). \hfill \Box
Analogous formulas to (2) also exist for higher time-derivatives [2] [3], and via such formulas many new algebraically solvable dynamical systems have been recently identified and discussed, especially dynamical systems characterized by second-order Ordinary Differential Equations (ODEs) of Newtonian type ("accelerations equal forces"); for an overview see [4] and references therein. But in this paper our treatment is confined to systems involving first-order time-derivatives.

In this paper we moreover confine attention to the very simplest such systems: characterized by first-order Ordinary Differential Equations involving only 2 dependent variables. Let us tersely review here—in this very simple context—how this approach works.

Systems of algebraically solvable first-order ODEs for the zeros \( x_n(t) \) are obtained from the identities (2) by assuming that the \( N \) coefficients \( y_n(t) \) satisfy themselves an algebraically solvable system of first-order ODEs. Note that in the very simple case with \( N = 2 \) the equations (2) read simply as follows:

\[
\dot{x}_n = (-1)^n \left( \frac{x_n y_1 + y_2}{x_1 - x_2} \right), \quad n = 1, 2 .
\]  

Now assume that the system of 2 ODEs

\[
\dot{y}_1 = f_1 (y_1, y_2) , \quad \dot{y}_2 = f_2 (y_1, y_2) ,
\]  

be algebraically solvable (of course, for an appropriate assignment of the 2 functions \( f_1 (y_1, y_2) \) and \( f_2 (y_1, y_2) \)). Then the system

\[
\dot{x}_n = (-1)^n \left[ x_n f_1 (-x_1 - x_2, x_1 x_2) + f_2 (-x_1 - x_2, x_1 x_2) \right], \quad n = 1, 2
\]

is as well algebraically solvable, because it clearly corresponds to (3) via the 2 identities

\[
y_1 (t) = -[x_1 (t) + x_2 (t)] , \quad y_2 (t) = x_1 (t) x_2 (t)
\]

clearly associated to the polynomial (1) with \( N = 2 \),

\[
p_2 (z; t) = z^2 + y_1 (t) z + y_2 = [z - x_1 (t)] [z - x_2 (t)] .
\]

Indeed the solution of its initial-values problem—to compute \( x_1 (t) \) and \( x_2 (t) \) via (3) from the assigned initial data \( x_1 (0) \) and \( x_2 (0) \)—can be achieved via the following 3 steps: (i) from the initial data \( x_1 (0) \) and \( x_2 (0) \) compute the corresponding initial data \( y_1 (0) \) and \( y_2 (0) \) via the simple formulas (6) (at \( t = 0 \)); (ii) compute \( y_1 (t) \) and \( y_2 (t) \) from the initial data \( y_1 (0) \) and \( y_2 (0) \) via the, assumedly algebraically solvable, system of evolution equations (1) characterizing the time-evolution of these variables; (iii) the variables \( x_1 (t) \) and \( x_2 (t) \) are then obtained as the 2 zeros of the, now known, polynomial (7) (via an algebraic operation, indeed one that in this case of a polynomial of second-degree can be performed explicitly: note however that this operation yields 2 a priori indistinguishable functions \( x_n (t) \) with \( n = 1, 2 \); to identify which is \( x_1 (t) \) and which is \( x_2 (t) \) these solutions must be followed back—by continuity in time, from the time \( t \) to the initial time \( 0 \)—to identify which one of them corresponds to the initially assigned data \( x_1 (0) \) respectively \( x_2 (0) \).

The new twist of this approach on which the findings reported in this paper are based is to assume that the two functions \( f_m (y_1, y_2) \) with \( m = 1, 2 \)—besides implying the solvability of the system (1)—feature the additional properties to be polynomial in their arguments and moreover to satisfy identically—i. e., for all values of the variable \( x \)—the relation

\[
x f_1 (-2x, x^2) + f_2 (-2x, x^2) = 0 ,
\]

which clearly implies that the 2 polynomials

\[
x_n f_1 (-x_1 - x_2, x_1 x_2) + f_2 (-x_1 - x_2, x_1 x_2) , \quad n = 1, 2
\]

contain both the factor \( x_1 - x_2 \). Therefore this condition (5) is sufficient to imply that the system of ODEs (3) in fact feature a polynomial right-hand side:

\[
\dot{x}_n = P^{(n)} (x_1, x_2) , \quad n = 1, 2 ,
\]
with \( P_n(x_1, x_2) \) polynomial in its 2 arguments.

In the following Section 2 we discuss a rather simple example manufactured in this manner (hereafter referred to as Example 1), the equations of motion of which read as follows:

Example 1:
\[
\dot{x}_n = a + b \left[ (x_n)^2 - 4x_1x_2 - (x_{n+1})^2 \right], \quad n = 1, 2 \mod 2, \tag{11}
\]

with \( a \) and \( b \) two arbitrary parameters. \( \Box \)

In Section 3 and its subsections we discuss 3 other somewhat analogous models (hereafter referred to respectively as Examples 2,3,4) obtained via a recent development of the above approach to identify algebraically solvable dynamical systems, in which the role of the generic polynomial \( \cite{1} \) is however replaced by a polynomial featuring, for all time, one double zero \( \cite{2} \). The equations of motion characterizing these 3 dynamical systems read as follows:

Example 2:
\[
\begin{align*}
\dot{x}_1 &= a + b \left[ (x_1)^2 + 7x_1x_2 + (x_2)^2 \right], \\
\dot{x}_2 &= a + b \left[ 7(x_1)^2 + 4x_1x_2 - 2(x_2)^2 \right]; \\
\end{align*}
\tag{12}
\]

Example 3:
\[
\dot{x}_n = x_n \left[ a - b (x_1)^2 x_2 \right], \quad n = 1, 2; \tag{13}
\]

Example 4:
\[
\dot{x}_n = x_n \left[ a + bx_1 (x_1 + 2x_2) \right], \quad n = 1, 2; \tag{14}
\]

again, in each of these 3 cases, with \( a \) and \( b \) two arbitrary parameters.

Remark 1.1. Of course in all these examples the presence of the 2 \textit{a priori arbitrary} parameters \( a \) and \( b \) is somewhat insignificant: indeed, both can clearly be replaced by unity by rescaling the independent variable \( (t \mapsto \alpha t) \) and the dependent variables \( (x_n \mapsto \beta x_n) \) (with obvious appropriate assignments of the parameters \( \alpha \) and \( \beta \)). Moreover all these examples with an arbitrary nonvanishing value of the parameter \( a \) can be obtained via analogous models with \( a = 0 \) via a simple change of the independent variable (see below Subsection 4.2). While models featuring more arbitrary parameters can be derived from these via a simple change of dependent variables (see below Subsection 4.1). \( \Box \)

Indeed, in Section 4 and its subsections we tersely outline some variants of the examples discussed in Sections 2 and 3, thereby enlarging the class of algebraically solvable dynamical systems identifiable via the technique introduced in this paper. These models might be of interest in applicative contexts: indeed, dynamical systems of the type discussed in this paper play a role in an ample variety of such contexts (say, from population dynamics to chemical reaction to econometric projections, etc.: you name it). But in this paper we merely focus on the presentation of the technique that subsumes the identification of this kind of algebraically solvable dynamical systems characterized by coupled systems of 2 first-order ODEs with polynomial right-hand sides, see \( \cite{10} \).

Finally Section 5 mentions possible future developments of these findings.

2 Example 1

In this Section 2 we demonstrate the algebraically solvable character of the dynamical system \( \cite{3} \).

The starting point of our treatment is the dynamical system \( \cite{3} \) with
\[
\begin{align*}
\dot{y}_1 &= \alpha_0 + \alpha_1 y_2, \quad \dot{y}_2 = \beta_0 y_1 + \beta_1 (y_1)^3, \\
\end{align*}
\tag{15}
\]
corresponding to \( \cite{4} \) with
\[
\begin{align*}
\hat{f}_1 (y_1, y_2) &= \alpha_0 + \alpha_1 y_2, \quad \hat{f}_2 (y_1, y_2) = \beta_0 y_1 + \beta_1 (y_1)^3, \\
\end{align*}
\tag{16}
\]
where \( \alpha_0, \alpha_1, \beta_0, \beta_1 \) are \textit{a priori arbitrary} parameters.

These equations of motion clearly imply that the condition \( \cite{5} \) is satisfied provided
\[ \alpha_0 = 2\beta_0, \quad \alpha_1 = 8\beta_1; \] (17)

and it is as well easily seen that there then obtains the system (11) with \( a = -\beta_0, \ b = \beta_1, \) via the insertion of (15) in (3) (of course with \( y_1 = -(x_1 + x_2) \) and \( y_2 = x_1 x_2 \); see (3)).

On the other hand it is easily seen that the system (15) is explicitly solvable: indeed the equations of motion (15) clearly imply the second-order ODE (of Newtonian type: “acceleration equal force”)

\[ \ddot{y}_1 = \alpha_1 \left[ \beta_0 y_1 + \beta_1 (y_1)^3 \right], \] (18)\n
namely, via (17),

\[ \ddot{y}_1 = 8\beta_1 \left[ \beta_0 y_1 + \beta_1 (y_1)^3 \right]. \] (19)\n
This second-order ODE—which is of course integrable via two quadratures—is clearly the Newtonian equation of motion of the simplest anharmonic oscillator (although, in the real domain, with a force that at large distance pushes the solution away from the origin). The most direct way to demonstrate the algebraically solvable character of this equation of motion is to exhibit its solution which—as the interested reader will easily verify—reads, in terms (for instance) of the first Jacobian elliptic function \( sn(z) \) (see for instance [6]), as follows:

\[ y_1(t) = \mu \ sn(\lambda t + \rho, k), \] (20)\n
where \( \mu \) and \( \lambda \) are determined in terms of the parameter \( k \) as follows:

\[ \lambda^2 = -\frac{8\beta_0 \beta_1}{1 + k^2}, \quad \mu^2 = -\frac{2\beta_0 k^2}{\beta_1 (1 + k^2)}, \] (21)\n
\( \rho \) is determined in terms of the initial datum \( y_1(0) \) as follows,

\[ y_1(0) = \mu \ sn(\rho, k), \] (22)\n
and the parameter \( k \) is determined in terms of the initial data \( y_1(0) \) and \( y_2(0) \) as the solution of the following algebraic equation

\[ \left\{ \left[ y_1(0) \right]^2 - 8\beta_0 \beta_1 \left[ y_1(0) \right]^2 - (2\beta_1)^2 \left[ y_1(0) \right]^4 \right\} (1 + k^2)^2 = (4\beta_0)^2 k^2, \] (23)\n
where of course (see the first (15) with (17))

\[ \dot{y}_1(0) = 2\beta_0 + 8\beta_1 y_2(0). \] (24)\n
And of course, once \( y_1(t) \) is known, \( y_2(t) \) is given directly by the first (15) with (17).

Remark 2.1. For given assigned values of \( y_1(0) \) and \( y_2(0) \) (hence \( \dot{y}_1(0) \), see (21), (23) is a quadratic equation for \( k^2 \); the choice of the appropriate solution for \( k^2 \) among the 2 solutions of this elementary equation must of course be made cum grano salis. \( \Box \)

3 Examples 2, 3 and 4

In the 3 subsections of this Section 3 we demonstrate the algebraically solvable character of the 3 dynamical systems (12), (13) and (14).

But let us first summarize some relevant findings of [5].

Let \( p_3(z; t) \) be a time-dependent polynomial of third degree in its argument \( z \) which, for all time, features a double pole:

\[ p_3(z; t) = z^3 + \sum_{m=1}^{3} y_m(t) z^{3-m} = [z - x_1(t)]^2 [z - x_2(t)]. \] (25)\n
This of course implies that its 3 coefficients \( y_m(t) \) are expressed as follows in terms of the double zero \( x_1(t) \) and the zero (of unit multiplicity) \( x_2(t) \):

\[ y_1 = -(2x_1 + x_2), \quad y_2 = x_1 (x_1 + 2x_2), \quad y_3 = -(x_1)^2 x_2; \] (26)
and correspondingly that the 3 coefficients \( y_m (t) \) are, for all time, related to each other by the (single) condition implied by the simultaneous vanishing at \( z = x_1 (t) \) of both \( p_3 (z; t) \) and its \( z \)-derivative \( p_{3,z} (z; t) \):

\[
p_3 (x_1; t) = [x_1 (t)]^3 + \sum_{m=1}^{3} \left\{ y_m (t) [x_1 (t)]^{3-m} \right\} = 0 ,
\]

\[
p_{3,z} (x_1; t) = 3 [x_1 (t)]^2 + 2 y_1 (t) x_1 (t) + y_2 (t) = 0 .
\]

In an analogous manner (see the treatment in Section 2, and if need be [5]) it is possible to obtain the following 3 pairs of formulas (analogous to, but of course somewhat different from, the formulas (3)):

\[
\dot{x}_1 = - \frac{2 x_1 \dot{y}_1 + \dot{y}_2}{2 (x_1 - x_2)} , \quad \dot{x}_2 = \frac{(x_1 + x_2) \dot{y}_1 + \dot{y}_2}{x_1 - x_2} ;
\]

\[
\dot{x}_1 = \frac{(x_1)^2 \dot{y}_1 - \dot{y}_3}{2 x_1 (x_1 - x_2)} , \quad \dot{x}_2 = \frac{x_1 x_2 \dot{y}_1 - \dot{y}_3}{x_1 (x_1 - x_2)} ;
\]

\[
\dot{x}_1 = \frac{x_1 \dot{y}_2 + 2 \dot{y}_3}{2 x_1 (x_1 - x_2)} , \quad \dot{x}_2 = - \frac{x_1 x_2 \dot{y}_2 + (x_1 + x_2) \dot{y}_3}{(x_1)^2 (x_1 - x_2)} .
\]

It is then clear—in close analogy to the treatment described above (see Section 1)—that each of these 3 pairs of formulas opens the way to the identification of \( \text{algebraically solvable} \) dynamical systems involving the 2 dependent variables \( x_1 (t) \) and \( x_2 (t) \): as separately discussed in the following 3 subsections.

### 3.1 Example 2

In this Subsection 3.1 we demonstrate the \( \text{algebraically solvable} \) character of the dynamical systems (12).

Now the starting point of our treatment is—instead of the system (4)—the slightly different system (11). Clearly this system is \( \text{solvable by algebraic operations} \) if the quantities \( y_1 (t) \) and \( y_2 (t) \) satisfy the system (4) and this system is \( \text{itself solvable} \). Then the system satisfied by the variables \( x_1 (t) \) and \( x_2 (t) \)—obtained by replacing, in the right hand side of (12), \( \dot{y}_1 \) and \( \dot{y}_2 \) via the equations of motion (4)—reads

\[
\dot{x}_1 = \frac{2 x_1 f_1 \left( -2 x_1 - x_2, (x_1)^2 + 2 x_1 x_2 \right) + f_2 \left( -2 x_1 - x_2, (x_1)^2 + 2 x_1 x_2 \right)}{2 (x_1 - x_2)} ,
\]

\[
\dot{x}_2 = \frac{(x_1 - x_2)^{-1} \left( (x_1 + x_2) f_1 \left( -2 x_1 - x_2, (x_1)^2 + 2 x_1 x_2 \right) + f_2 \left( -2 x_1 - x_2, (x_1)^2 + 2 x_1 x_2 \right) \right)}{(x_1 - x_2)^{3}} ,
\]

(32)

corresponding now to the assignment (26) (instead of (13)) of \( y_1 (t) \) and \( y_2 (t) \) in terms of \( x_1 (t) \) and \( x_2 (t) \).

It is now clear that the conditions on the 2 functions \( f_1 (y_1, y_2) \) and \( f_2 (y_1, y_2) \) which are sufficient to guarantee that the right-hand side of the equations of motion (32) be \( \text{polynomial} \) in the 2 dependent variables \( x_1 (t) \) and \( x_2 (t) \) are that these 2 functions \( f_1 (y_1, y_2) \) and \( f_2 (y_1, y_2) \) be themselves \( \text{polynomial} \) in their 2 variables \( y_1 \) and \( y_2 \) and moreover that there hold identically—i. e., for all values of \( x \)—the relation

\[
2 x_1 f_1 \left( -3 x, 3 x^2 \right) + f_2 \left( -3 x, 3 x^2 \right) = 0 .
\]

(33)

We now assume that the time-evolution of the 2 quantities \( y_1 (t) \) and \( y_2 (t) \) be again characterized by the equations of motion (15)—the \( \text{solvable} \) character of which has been pointed out in Section 2—hence by the assignments (10) of the two functions \( f_1 (y_1, y_2) \) and \( f_2 (y_1, y_2) \). It is then easily seen that the condition (33) entails now the relations

\[
\alpha_0 = \frac{3 \beta_0}{2} , \quad \alpha_1 = \frac{9 \beta_1}{2} .
\]

(34)

(instead of (17)).

It is then easily seen that the corresponding dynamical system satisfied by the coordinates \( x_1 (t) \) and \( x_2 (t) \) is just the system of 2 ODEs (12) (with \( a = -\beta_0/2, \ b = -\beta_1/2 \)).
There remains to report—from [5]—how to obtain from the variables \( y_1(t) \) and \( y_2(t) \) the variables \( x_1(t) \) and \( x_2(t) \). The variable \( x_1(t) \) is one of the 2 roots of the—of course explicitly solvable—second-degree polynomial equation in \( x \):

\[
3x^2 + 2y_1(t)x + y_2(t) = 0
\]  
(35)

(see (28)) which, by continuity in \( t \), corresponds at \( t = 0 \) to the initially assigned datum \( x_1(0) \). While \( x_2(t) \) is then given by the formula

\[
x_2(t) = -y_1(t) - 2x_1(t)
\]  
(36)

(see the first of the 3 formulas (26)).

### 3.2 Example 3

In this Subsection 3.2 we demonstrate the algebraically solvable character of the dynamical systems [13].

Now the starting point of our treatment is the system of 2 coupled ODEs [30]. Clearly this system is solvable by algebraic operations if the quantities \( y_1(t) \) and \( y_3(t) \) satisfy the system

\[
\dot{y}_1 = f_1(y_1, y_3) \quad \text{and} \quad \dot{y}_3 = f_3(y_1, y_3),
\]  
(37)

and this system is itself solvable (of course for an appropriate assignment of the 2 functions \( f_1(y_1, y_3) \) and \( f_3(y_1, y_3) \)). Then the system satisfied by the variables \( x_1(t) \) and \( x_2(t) \) —obtained by replacing, in the right-hand side of (30), \( \dot{y}_1 \) and \( \dot{y}_3 \) via these equations of motion (37)—reads

\[
\begin{align*}
\dot{x}_1 &= - (x_1)^2 f_1 \left( -2x_1 - x_2, - (x_1)^2 x_2 \right) + f_3 \left( -2x_1 - x_2, - (x_1)^2 x_2 \right) \\
\dot{x}_2 &= [x_1(x_1 - x_2)]^{-1} \left[ x_1 x_2 f_1 \left( -2x_1 - x_2, (x_1)^2 + 2x_1 x_2 \right) \right. \\
&\quad \left. - f_3 \left( -2x_1 - x_2, (x_1)^2 + 2x_1 x_2 \right) \right]
\end{align*}
\]  
(38)

corresponding to the assignment (26) of \( y_1(t) \) and \( y_3(t) \) in terms of \( x_1(t) \) and \( x_2(t) \); and it is easily seen that sufficient conditions to guarantee that this become a system of 2 ODEs featuring in their right-hand sides a polynomial dependence on the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) are that these 2 functions \( f_1(y_1, y_2) \) and \( f_3(y_1, y_2) \) be themselves polynomial in their 2 variables \( y_1 \) and \( y_3 \) and moreover that there hold identically—i.e., for all values of \( x \)—the relation

\[
x^2 f_1 (-3x, -x^3) - f_3 (-3x, -x^3) = 0.
\]  
(39)

Let us now assume that the two functions \( f_1(y_1, y_3) \) and \( f_3(y_1, y_3) \) read as follows:

\[
f_1(y_1, y_3) = y_1 (\alpha_1 + \alpha_2 y_3), \quad f_3(y_1, y_3) = y_3 (\beta_1 + \beta_2 y_3),
\]  
(40)

so that the system (37) reads

\[
\dot{y}_1 = y_1 (\alpha_1 + \alpha_2 y_3), \quad \dot{y}_3 = y_3 (\beta_1 + \beta_2 y_3).
\]  
(41)

Here the 4 parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are a priori arbitrary, but clearly to satisfy (39) it is necessary and sufficient that (as we hereafter assume, in this Subsection 3.2)

\[
\beta_1 = 3\alpha_1, \quad \beta_2 = 3\alpha_2.
\]  
(42)

It is then a matter of trivial algebra to verify that the corresponding system of 2 ODEs for the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) is just (33), with \( a = \alpha_1, b = -\alpha_2 \).

It is on the other hand easily seen that the system (41) is explicitly solvable: by firstly integrating by a quadrature the ODE satisfied by the dependent variable \( y_3(t) \), and by then integrating the linear ODE satisfied by the dependent variable \( y_1(t) \). There results the following neat expressions of the 2 variables \( y_1(t) \) and \( y_3(t) \):

\[
y_1(t) = y_1(0) \varphi(t), \quad y_3(t) = y_3(0) [\varphi(t)]^3,
\]  
(43)
\varphi(t) = \exp(at) \left\{ 1 - \left( \frac{b}{a} \right) y_3(0) \left[ 1 - \exp(3at) \right] \right\}^{-1/3}.

(44)

The subsequent computation of the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) from the 2 quantities \( y_1(t) \) and \( y_3(t) \) can then be easily performed: it involves the algebraic operation of solving a cubic equation (a task which can actually be performed explicitly), as the interested reader will easily ascertain (or, if need be, see [5]).

Remark 3.2.1. If the parameter \( a \) is imaginary—\( a = \omega i \) (with, here and hereafter, \( i \) the imaginary unit, so that \( i^2 = -1 \)) and \( \omega \) is real and nonvanishing, \( \omega \neq 0 \)—both coefficients \( y_1(t) \) and \( y_3(t) \) are clearly periodic with period \( T = 2\pi/|\omega| \), see (43) with (44); actually \( y_3(t) \) is clearly periodic with period \( T/3 \); while \( y_1(t) \) is certainly periodic with period \( T \) but—depending on the value of the initial datum \( y_3(0) \)—it might also be periodic with period \( T/3 \). Hence the 2 coordinates \( x_1(t) \) and \( x_2(t) \) are themselves periodic with period \( T \) (or possibly a small integer multiple of \( T \); see [2] [8]).

3.3 Example 4

In this Subsection 3.3 we demonstrate the algebraically solvable character of the dynamical systems [14].

Now the starting point of our treatment is system [11]. Clearly this system is solvable by algebraic operations if the quantities \( y_2(t) \) and \( y_3(t) \) satisfy the system

\[ \dot{y}_2 = f_2(y_2, y_3), \quad \dot{y}_3 = f_3(y_2, y_3), \]

(45)

and this system is itself solvable (of course for an appropriate assignment of the 2 functions \( f_2(y_1, y_3) \) and \( f_3(y_1, y_3) \)). Then the system satisfied by the variables \( x_1(t) \) and \( x_2(t) \)—obtained by replacing, in the right-hand side of (41), \( \dot{y}_2 \) and \( \dot{y}_3 \) via the equations of motion (45)—reads

\[ \dot{x}_1 = \frac{x_1 f_2(x_1 (x_1 + x_2) + 2 f_3(x_1 (x_1 + x_2) + x_1 (x_1 + x_2), -(x_1)^2 x_2))}{2 x_1 (x_1 - x_2)}, \]

\[ \dot{x}_2 = -(x_1)^2 (x_1 - x_2)^{-1} \left\{ x_1 x_2 f_2(x_1 (x_1 + x_2) + x_1 (x_1 + x_2), -(x_1)^2 x_2) \right\} + (x_1 + x_2) f_3(x_1 (x_1 + x_2), -(x_1)^2 x_2), \]

(46)

corresponding to the assignment (46) of \( y_2(t) \) and \( y_3(t) \) in terms of \( x_1(t) \) and \( x_2(t) \); and it is easily seen that sufficient conditions to guarantee that this become a system of 2 ODEs featuring in their right-hand sides a polynomial dependence on the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) are that these 2 functions \( f_2(y_1, y_3) \) and \( f_3(y_1, y_2) \) be themselves polynomial in their 2 variables \( y_2 \) and \( y_3 \) and moreover that there hold identically—i. e., for all values of \( x \)—the relation

\[ \frac{xf_2(3(x)^2, -(x)^3)}{x} + 2 f_3(3(x^2), -(x)^3) = 0. \]

(47)

Let us now assume that the two functions \( f_2(y_1, y_3) \) and \( f_3(y_1, y_3) \) read as follows:

\[ f_2(y_2, y_3) = y_2(\alpha_1 + \alpha_2 y_2), \quad f_3(y_2, y_3) = y_3(\beta_1 + \beta_2 y_2), \]

(48)

so that the system (37) reads

\[ \dot{y}_2 = y_2(\alpha_1 + \alpha_2 y_2), \quad \dot{y}_3 = y_3(\beta_1 + \beta_2 y_2). \]

(49)

Here the 4 parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are a priori arbitrary, but clearly to satisfy (17) it is necessary and sufficient that (as we hereafter assume, in this Subsection 3.3)

\[ \beta_1 = \frac{3 \alpha_1}{2}, \quad \beta_2 = \frac{3 \alpha_2}{2}. \]

(50)

It is then a matter of trivial algebra to verify that the corresponding system of 2 ODEs for the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) is just [14], with \( a = \alpha_1/2 \), \( b = \alpha_2/2 \).

It is on the other hand easily seen that the system (19) is explicitly solvable: it is indeed, up to simple notational changes, identical to the system (11) discussed in the preceding Subsection 3.2.
And the subsequent computation of the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) from the 2 quantities \( y_2(t) \) and \( y_3(t) \) can as well be easily performed: it involves again the algebraic operation of solving a cubic equation (a task which can actually be performed explicitly), as the interested reader will easily ascertain (or, if need be, see [5]).

### 4 Variants

In this Section 4 and its subsections we tersely outline some interesting variants of the algebraically solvable models discussed above, which might be of interest for possible utilizations of these findings in applicative contexts.

#### 4.1 First variant

Each of the 4 dynamical systems identified above as algebraically solvable—see [11], [12], [13], [14]—features only 2 arbitrary parameters, \( a \) and \( b \). Systems featuring more free parameters can of course be obtained from these via the trivial change of dependent variables

\[
x_1(t) = u_{10} + u_{11}\xi_1(t) + u_{12}\xi_2(t), \quad x_2(t) = u_{20} + u_{21}\xi_1(t) + u_{22}\xi_2(t),
\]

featuring the 6 parameters \( u_{n\ell}, \, n = 1, 2, \, \ell = 0, 1, 2 \). This change of variables is easily inverted:

\[
\xi_1(t) = u^{-1} \left\{ u_{22} [x_1(t) - u_{10}] - u_{12} [x_2(t) - u_{20}] \right\},
\]

\[
\xi_2(t) = u^{-1} \left\{ u_{21} [x_1(t) - u_{10}] - u_{11} [x_2(t) - u_{20}] \right\},
\]

where, here and hereafter,

\[
u = u_{11}u_{22} - u_{12}u_{21}.
\]

Clearly the properties of algebraic solvability are not affected, although the relevant formulas become marginally more complicated, requiring the solution of some (purely algebraic) equations. On the other hand the new systems of 2 ODEs satisfied by the new dependent variables \( \xi_1(t) \) and \( \xi_2(t) \) feature now several more free parameters. For instance for Example 1 the equations that replace (11) read as follows:

\[
\dot{\xi}_n = A_n + B_n\xi_1 + B_{n2}\xi_2 + C_{n1} (\xi_1)^2 + C_{n2} (\xi_2)^2 + C_{n3}\xi_1\xi_2,
\]

\[
A_1 = u^{-1} \left\{ (u_{22} - u_{12}) (a - 4bu_{10}u_{20}) + b (u_{22} + u_{12}) \left[ (u_{10})^2 - (u_{20})^2 \right] \right\},
\]

\[
A_2 = u^{-1} \left\{ (u_{21} - u_{11}) (a - 4bu_{10}u_{20}) + b (u_{21} + u_{11}) \left[ (u_{10})^2 - (u_{20})^2 \right] \right\},
\]

\[
B_{1n} = 2bu^{-1} \left\{ 2 (u_{12} - u_{22}) (u_{10}u_{2n} + u_{20}u_{1n}) 
\right.
\]

\[
+ (u_{22} + u_{12}) (u_{10}u_{1n} - u_{20}u_{2n}) \right\}, \, n = 1, 2,
\]

\[
B_{2n} = 2bu^{-1} \left\{ 2 (u_{11} - u_{21}) (u_{10}u_{2n} + u_{20}u_{1n}) 
\right.
\]

\[
+ (u_{21} + u_{11}) (u_{10}u_{1n} - u_{20}u_{2n}) \right\}, \, n = 1, 2,
\]

\[
C_{1n} = bu^{-1} \left\{ 4 (u_{12} - u_{22}) u_{1n}u_{2n} + (u_{22} + u_{12}) \left[ (u_{1n})^2 - (u_{2n})^2 \right] \right\}, \, n = 1, 2,
\]

\[
C_{2n} = bu^{-1} \left\{ 4 (u_{11} - u_{21}) u_{1n}u_{2n} + (u_{21} + u_{11}) \left[ (u_{1n})^2 - (u_{2n})^2 \right] \right\}, \, n = 1, 2,
\]

\[
C_{13} = 2bu^{-1} \left\{ 2 (u_{12} - u_{22}) (u_{11}u_{22} + u_{12}u_{21}) + (u_{22} + u_{12}) (u_{11}u_{21} - u_{21}u_{22}) \right\},
\]

\[
C_{23} = 2bu^{-1} \left\{ 2 (u_{11} - u_{21}) (u_{11}u_{22} + u_{12}u_{21}) + (u_{21} + u_{11}) (u_{11}u_{21} - u_{21}u_{22}) \right\}.
\]

Note that if \( a = u_{10} = u_{20} = 0 \) then \( A_n = B_{nm} = 0 \) and the equations of motion (54) have homogeneous right-hand sides (of degree 2) featuring only the 6 coefficients \( C_{n\ell} \) with \( n = 1, 2 \) and \( \ell = 1, 2, 3 \), expressed in terms of the 5 arbitrary parameters \( b \) and \( u_{nm} \) with \( n \) and \( m \) taking the values 1 and 2.

It is left to the interested reader to obtain analogous formulas for Examples 2, 3, 4.
4.2 Second variant

If the \((\text{autonomous})\) system of 2 coupled ODEs

\[
\dot{x}_n = f_n(x_1, x_2), \quad n = 1, 2, \tag{63}
\]

features \textit{homogeneous} functions \(f_n(x_1, x_2)\) satisfying the scaling property

\[
f_n(cx_1, cx_2) = c^p f_n(x_1, x_2), \quad n = 1, 2, \quad p \neq 1 \tag{64}
\]

(where \(c\) is an arbitrary parameter), then by setting

\[
w_n(t) = \exp\left(\frac{\alpha t}{p - 1}\right)x_n(\tau(t)), \quad \tau(t) = \frac{\exp(\alpha t) - 1}{\alpha}, \quad n = 1, 2, \tag{65}
\]

one gets for the new dependent variables \(w_n(t)\) the new \((\text{autonomous})!\) system

\[
\dot{w}_n = \frac{\alpha}{p - 1}w_n + f_n(w_1, w_2), \quad n = 1, 2. \tag{66}
\]

Then—if the original system \((63)\) is \textit{algebraically solvable}—the solutions \(w_n(t)\) of this system satisfy interesting properties: in particular, if \(\alpha = i\omega\) is \textit{imaginary}—with \(\omega\) a nonvanishing real parameter and \(p\) a real rational number—then \textit{all} solutions of these systems \((66)\) are \textit{completely periodic} with some rational integer multiple of the basic period \(T = 2\pi/|\omega|\) \textit{(isochrony)!} For more details on the transformation \((65)\) and its implications see \([8]\) and references therein.

Note that the 4 dynamical systems of \textit{Examples 1, 2, 3, 4} belong to the class \((64)\) if the parameter \(a\) vanishes, \(a = 0\): with \(p = 2\) in the cases of \textit{Examples 1} and \textit{2} (see \((11)\) and \((12)\)), with \(p = 4\) in the case of \textit{Example 3} and \(p = 3\) in the case of \textit{Example 4} (see \((13)\) and \((14)\)); and these properties continue to hold after the generalization described in the preceding \textit{Subsection 4.1}, provided the parameters \(u_{10}\) and \(u_{20}\) vanish, \(u_{10} = u_{20} = 0\) (see \((51)\)).

4.3 Third variant

Let us note that the dynamical systems detailed in the \textit{Examples} reported above can be reformulated as describing the evolution of \textit{real} 2-vectors \(\vec{r}_n(t)\) lying in a \textit{real} plane. Indeed set

\[
\vec{r}_n(t) \equiv (\text{Re}[x_n(t)], \text{Im}[x_n(t)]), \quad n = 1, 2, \tag{67}
\]

\[
\vec{a} \equiv (\text{Re}[a], \text{Im}[a]), \quad \vec{b} \equiv (\text{Re}[b], -\text{Im}[b]). \tag{68}
\]

Then—as the diligent reader will easily verify—the version of \((11)\) yielded by this notational change reads as follows:

\[
\dot{\vec{r}}_n = \vec{a} + 2\vec{r}_n \left[\vec{b} \cdot (\vec{r}_n - 2\vec{r}_{n+1})\right] - 2\vec{r}_{n+1} \left[\vec{b} \cdot (\vec{r}_{n+1} + 2\vec{r}_n)\right] + 5 \left[(\vec{r}_{n+1})^2 - (\vec{r}_n)^2 + 4(\vec{r}_n \cdot \vec{r}_{n+1})\right], \quad n = 1, 2 \mod [2]. \tag{69}
\]

Here of course the dot among two vectors denotes the standard scalar product, and \((\vec{r}_n)^2 \equiv \vec{r}_n \cdot \vec{r}_n\). Note the \textit{covariant} character of these equations.

The interested reader will have no difficulty to reformulate in an analogous manner the equations of motion \((12)\) of \textit{Example 2}; and analogous reformulations of the equations of motion of \textit{Examples 3} and \textit{4} are also possible (hint: before applying the same procedure as indicated above, see \((67)\) and \((68)\), replace \(b\) with \(b^3\) in \((13)\), and \(b\) with \(b^2\) in \((14)\)).
5 Outlook

The literature on the simple kind of dynamical systems treated in this paper is of course vast; see for instance [9], [10] and standard compilations of solvable ODEs such as [11]. But it seems to us that—in spite of their simplicity—the findings reported in this paper (including their variants mentioned in Section 4) are new.

Further applications of the approach described in this paper are of course also possible: for instance by exploiting the extension of the results of [5] to time-dependent polynomials featuring zeros of arbitrary multiplicity (see some progress made in this direction by Oksana Bihun’s recent paper [12]; we report additional progress in [13]); or by exploiting the extensions of the fundamental results—such as (2)—on which the findings reported in this paper are based, from polynomials to rational functions [14].

And of course extensions of the approach of this paper to systems of higher-order ODEs (including in particular second-order ODEs of Newtonian type: “accelerations equal forces”), to PDEs, to discrete-time evolutions (see [4] and [15]) deserve further investigations.

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