Abstract

I consider an extension of General Relativity by an auxiliary non-dynamical dimension that enables our space-time to acquire an extrinsic curvature. Obtained gravitational equations, without or with a cosmological constant, have a selfaccelerated solution that is independent of the value of the cosmological constant, and can describe the cosmic speedup of the Universe as a geometric effect. Background evolution of the selfaccelerated solution is identical to that of ordinary de Sitter space. I show that linear perturbations on this solution describe either a massless graviton, or a massive graviton and a scalar, which are free of ghosts and tachyons for certain choices of boundary conditions. The obtained linearized expressions suggest that nonlinear interactions should, for certain boundary conditions, be strongly coupled, although this issue is not studied here. The full nonlinear Hamiltonian of the theory is shown to be positive for the selfaccelerated solution, while in general, it reduces to surface terms in our and auxiliary dimensions.
1. Extension of General Relativity

One simple way to parametrize the cosmic acceleration \[^{[1]}\] is to introduce in the Lagrangian of General relativity (GR) the cosmological constant \( \Lambda \sim (10^{-33} \text{ eV})^2 \). This is not quite satisfactory however, since the parameter \( \Lambda \) receives contributions from various scales of particle physics each of which is many orders of magnitude greater than \((10^{-33} \text{ eV})^2\). Without an underlying principle, cancellation between these contributions down to \((10^{-33} \text{ eV})^2\) seems conceptually unlikely and technically unnatural \[^{[2]}\].

Here we consider an extension of GR, such that for an arbitrary but given value of the parameter \( \Lambda \), there exists a solution – requiring adjustment of certain boundary terms – that is independent of \( \Lambda \). Furthermore, the observed cosmic acceleration will be due to a new parameter \( m \) with the dimensionality of mass, appearing in the extended GR Lagrangian. This parameter does not receive contributions from the particle physics; its value can be set to \( m \sim 10^{-33} \text{ eV} \). The present approach does not explain the smallness of \( m \); instead it gives a technically natural way of describing cosmic acceleration, with potential observational predictions that differ from those of GR with the cosmological constant. The present approach does not solve the cosmological constant problem either, but instead it reduces the problem to the choice of the boundary conditions in the classical gravitational equations, with everything else being quantized (more on this in section 2.)

The gravitational field will be described by an extended metric tensor \( \tilde{g}_{\mu\nu}(x, u) \), with \( \mu, \nu = 0, 1, 2, 3 \), which is labeled by a continuous dimensionless parameter \( u \). The extended metric varies as \( \tilde{g}_{\mu\nu}(x', u) = \omega_\mu(x)\omega_\nu(x)\tilde{g}_{\alpha\beta}(x, u) \), under the general coordinate transformations \( x'^\mu = [\omega^{-1}(x)]^\mu_\nu x^\nu \). This leaves the extended interval \( ds_u^2 \equiv \tilde{g}_{\mu\nu}(x, u)dx^\mu dx^\nu \) invariant. However, the matter fields do not depend on \( u \); they will only couple to the metric tensor

\[
g_{\mu\nu}(x) \equiv \tilde{g}_{\mu\nu}(x, u = 0),
\]

with the relevant invariant interval being \( ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \).

Consider the Lagrangian density for the gravitational field (we use the conventions of \[^{[3]}\] and also set \( M_\text{Pl}^2 = (8\pi G_N)^{-1} = 2 \), unless stated otherwise)

\[
\mathcal{L} = \sqrt{\tilde{g}}R \pm m^2 \int_{-1}^{+1} du \sqrt{\tilde{g}} \left( k_{\mu\nu}^2 - k^2 \right),
\]

where \( R \) is the Ricci scalar of the metric \( g_{\mu\nu}(x) \), while \( k_{\mu\nu} \equiv \frac{1}{2}\partial_\mu \tilde{g}_{\nu\rho} \), \( k \equiv \tilde{g}^{\mu\nu}k_{\mu\nu} \); all indexes in the Einstein-Hilbert term in \(^{[2]}\) are raised by \( g^{\mu\nu} \), while those in the second term in \(^{[2]}\) by \( \tilde{g}^{\mu\nu} \). The value of \( k_{\mu\nu} \) measures an extrinsic curvature of a \((3 + 1)\)-dimensional constant-\( u \) surface in certain coordinates in the “\( x-u \) space-time”. The Lagrangian density \(^{[2]}\) is covariant in \((3 + 1)\)-dimensions.

We impose the \( \mathbb{Z}_2 \) symmetry on the fields \( \tilde{g}_{\mu\nu}(x, u) = \tilde{g}_{\mu\nu}(x, -u) \) across the hypersurface \( u = 0 \). Then, it is enough to consider the interval \([0, 1]\) for the variable
u. Note that the “u-dimension” is not dynamical since fields have no ordinary derivative terms there. Moreover, integration boundaries in u may in general take any finite value, which can be reduced back to the interval \([-1, 1]\) by an appropriate rescaling of u and the parameter m, before specifying boundary conditions that could be sensitive to such rescaling.

We refer to the surfaces \(u = 0, \pm 1\) as fixed boundaries. Eq. (1) imposes one boundary condition on the \(u\)-dependence of the extended metric. This is not enough to determine completely the \(u\)-dependence of \(\tilde{g}_{\mu\nu}(x, u)\), the second boundary condition should also be specified. For this one can either impose the Neumann-type or Dirichlet-type condition at the either boundaries. For now we keep this condition unspecified and find various solutions that correspond to different choices of the second boundary condition.

2. Equations, Solutions and Boundary Terms

Let us start with the action of gravity plus “everything else”:

\[
S = \int d^4x \sqrt{g} \left\{ R + \mathcal{L}(\Psi, g) \right\} \pm m^2 \int d^4x \int_{-1}^{+1} du \sqrt{\tilde{g}} \left( k_{\mu\nu}^2 - k^2 \right). \tag{3}
\]

Here \(\mathcal{L}(\Psi, g)\) is the non-gravitational Lagrangian the fields in which couple universally to the metric tensor \(g_{\mu\nu}(x)\), hence preserving the equivalence principle. We’ll be looking at very low-energy phenomena (as compared to Planck’s scale) and thus regard \(\tilde{g}_{\mu\nu}\) as an effective classical field describing large distance gravitational physics; thus, the gravitational part of the action will not be quantized (it can be regarded as the 1PI effective action in which all the quantum loop effects are encoded in the coefficients of various terms). All the other interactions encoded in \(\mathcal{L}(\Psi, g)\) will be quantized.

The Lagrangian \(\mathcal{L}(\Psi, g)\) will contain in general the cosmological constant generated by particle physics. As noted earlier, it receives contributions from the scales of electromagnetic, strong and weak interactions; we denoted it by \(\Lambda_{\text{fund}}\). Furthermore, quantum fluctuations of the non-gravitational fields in \(\mathcal{L}(\Psi, g)\) will generate higher dimensional gravitational operators, such as \(R^2, R_{\mu\nu}^2\), etc., which all are functions of \(g_{\mu\nu}\) and are suppressed by the Planck’s scale. Importantly, none of these terms, that are significant in the UV, can change the effects of the second term in (3) which switches on in the IR. Moreover, the second term in (3) does not get renormalized by the quantum loops of particle physics, since the particles couple only to \(g_{\mu\nu}(x)\) and cannot give rise to operators made of \(\tilde{g}_{\mu\nu}\). This can also be seen from the 5D representation of the model given in section 5. There, the matter fields localized on the brane cannot renormalize the bulk terms because of geometric separation in extra dimension; the bulk terms stay unchanged, as long as gravity is considered to be a classical field theory with the effective 1PI action.
The equations of motion obtained by varying the action $\delta S$ with the fixed boundary conditions in the $u$-space, $\tilde{\delta g}_{\mu\nu}(x, u)|_{x=\text{boundary}, u=0} = 0$ when the boundary is present in the $x$-space, in which case the Gibbons-Hawking (GH) boundary term [4] should also be introduced in the action) gives the following two equations for $u = 0^+$ and $0 < u \leq 1$ respectively:

$$G_{\mu\nu} \pm 2m^2 (k_{\mu\nu} - g_{\mu\nu}k) = T_{\mu\nu}/2,$$

and

$$\partial_u \left[ \sqrt{g} (k\tilde{g}^{\mu\nu} - k^{\mu\nu}) \right] = \frac{1}{2} \tilde{g}^{\mu\nu} \sqrt{\tilde{g}} \left( k^2 - k^{2\alpha\beta} \right) + 2 \sqrt{\tilde{g}} \left( k^{\mu\rho} k^{\nu}_{\rho} - k^{\mu\nu} k \right).$$

Note that the right hand side (rhs) of Eq. (5) is traceless.

Furthermore, equation (1) combined with the Bianchi identities implies that:

$$D^\mu k_{\mu\nu} = D_\nu k,$$

where $D_\mu$ denotes the covariant derivative compatible with the metric. Eq. (6) should automatically be satisfied by any solution of (1). Note that Eqs. (1) and (6) are similar to those of the DGP model [5] written in the 5D ADM [6] form (see, e.g., [7]). However, there are two significant differences: (a) What is the $\{55\}$ equation in DGP is absent here; (b) In Eq. (5) there are no derivatives w.r.t. space-time coordinates, and thus it significantly differs from its DGP counterpart (what is the bulk $\{\mu\nu\}$ equation).

Equation (5) determines the evolution of the metric $\tilde{g}_{\mu\nu}$ in the $u$-direction. This is a second order equation. One boundary conditions for it is specified by (1); pending the second boundary condition we find different dependence of the metric on $u$. The latter sets the value of the extrinsic curvature at $u = 0^+$, which by its turn determines 4D geometry via Eq. (1).

We turn now to concrete solutions. In the absence of any matter stress-tensor or cosmological constant ($T_{\mu\nu} = 0$) the above system of equations has the Minkowski solution $\tilde{g}_{\mu\nu}(x, u) = \eta_{\mu\nu} \equiv \text{diag}\{-1, 1, 1, 1\}_{\mu\nu}$. In general, for the class of extended metrics which are independent of $u$, the theory at hand reduces to GR. This would correspond to the choice of the boundary condition $\partial_u \tilde{g}_{\mu\nu}|_{u=0} = 0$, in addition to (1). Thus, for $\tilde{g}_{\mu\nu}(x, u) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ the fluctuations of the extended metric above the Minkowski solution describe a massless graviton.

There exists a choice of the boundary conditions for which the linearized fluctuations describe a Minkowski space massive graviton; for instance, the Lagrangian (2) with the minus sign in front of the second term describes the Pauli-Fierz massive graviton of $(\text{mass})^2 = 2m^2$, with $\tilde{g}_{\mu\nu}(x, u) = \eta_{\mu\nu} + (1 - |u|)h_{\mu\nu}(x)$ being a linearized solution selected by imposing the second boundary condition in the Dirichlet form: $\tilde{g}_{\mu\nu}(x, u)|_{u=1} = \eta_{\mu\nu}$.

Hence, the theory (2) endowed with the appropriate boundary conditions gives a nonlinear completion of massive gravity. Remarkably, the Hamiltonian of this
theory does not suffer from the problem found in Ref. [8] in 4D massive gravity, as it will be shown in Section 4. Since Minkowski space is not a subject of a primary interest here, we will not elaborate on this branch of solutions further.

Consider now a factorized expression for the extended metric

$$\tilde{g}_{\mu\nu}(x, u) = a(u)g_{\mu\nu}(x). \quad (7)$$

The rhs of Eq. (5) is identically zero for (7), and Eq. (5) reduces to $\partial^2_u a = 0$. Hence, for $u \geq 0$ we have $a(u) = c_0 + c_1 u$, where $c_0, c_1$ are integration constants. The boundary condition (1), and (7) define the value of $c_0 = 1$, while $c_1$ has to be fixed by the second boundary condition. Below we consider various solutions that differ from each other by the choice of that boundary condition.

For the second boundary condition specified in the following form

$$\partial_ug_{\mu\nu}|_{u=0}^+= \mp g_{\mu\nu}(x), \quad (8)$$

it is straightforward and not tedious to check that the system of equations (4), (5) admits a selfaccelerated solution:

$$\tilde{g}_{\mu\nu}^{cl}(x, u) \equiv (1 \mp |u|)\gamma_{\mu\nu}(x), \quad R(\gamma) = 12m^2. \quad (9)$$

Here, $\gamma_{\mu\nu}(x)$ denotes the 4D de Sitter metric with the Hubble parameter $H$ equal to $m$. This solution can describe the cosmic acceleration of the Universe, with the acceleration being due entirely to a geometric effect. In that regard, the growing solution in (9) is similar to the selfaccelerated solution [9, 10] on the DGP model, while the decaying solution to that of Refs. [11].

For the decaying solution in (9) the extended metric $\tilde{g}_{\mu\nu}$ vanishes at the boundaries $u = \pm 1$, while the inverse of $\tilde{g}_{\mu\nu}$ is singular, giving rise to a singularity of the extended Ricci tensor $\tilde{R}$ made of $\tilde{g}_{\mu\nu}$. However, since the “$u$-dimension” is nondynamical, and all the matter and their interactions are located at $u = 0$, the extended Ricci tensor $\tilde{R}$ evaluated at $u = \pm 1$ should not have a particular significance. Moreover, this singularity is easily avoidable by changing in (2) the integration interval for $u$ from $[0, 1]$ to $[0, b]$, where $b < 1$ is some positive number. This would not change the equations (4) and (5) and the solution (9), but for any $b \neq 1$ one would need to add to the Lagrangian (2) a surface term. The latter would guarantee that the effective Lagrangian obtained by integrating over the $u$-direction (i.e., by first substituting the metric (7) into the action and then varying it w.r.t. the metric $g$) gives the result consistent with the solution (9) obtained from the equations of motion (4) and (5).

The Lagrangian with the surface terms for general $b$, which gives the selfaccelerated solutions (9), reads as follows:

$$\mathcal{L}_b = \sqrt{g}R \pm m^2 \int_{-b}^{+b} du \sqrt{g} \left( k_{\mu\nu}^2 - k^2 \right) + C_b^{(\pm)} m^2 \left( \sqrt{g}|_{u=b} + \sqrt{g}|_{u=-b} \right), \quad (10)$$
where \( C_b^{(\pm)} \equiv -3/(1 \mp b) \).

For the growing solution with the positive sign in (9) the above singularity is absent, however, even when \( b = 1 \) is chosen, for this solution one has to add to the Lagrangian the surface term given in (10) in order for the effective Lagrangian (obtained by integrating out the \( u \)-direction) to give the result consistent with the solution (9) that was derived from the equations of motion (4) and (5). Moreover, for the growing solution in (9) the surface term will be crucial for calculation of its energy in Section 4.

Although we have constructed these surface terms via “inverse engineering” starting with the desired solutions, the straightforward statement is the following one: for the given boundary conditions and specified surface terms there are unique self-accelerated solutions corresponding to the two sign choices in (2).

One could of course modify the second boundary condition (8) in various ways and obtain different solutions, to some of which we’re turning now.

For a nonzero homogeneous and isotropic stress-tensor there exists a solution for which the extended metric reads \( \tilde{g}_{\mu\nu}(x, u) \equiv (1 \mp |u|)\gamma_{\mu\nu}^{FRW}(x) \), and the modified Friedmann equation in the standard notations takes the form

\[
H^2 - \zeta m^2 + \frac{\kappa}{a^2} = \frac{8\pi G N}{3} \rho ,
\]

where \( \kappa = \pm 1, 0 \) labels the 3D spatial curvature, and \( \zeta \) is an arbitrary integration constant that could be fixed only after imposing the boundary condition for e.g., \( \tilde{g}_{\mu\nu}(x, u)|_{u=\pm 1} \), or for \( \partial_u \tilde{g}_{\mu\nu}(x, u)|_{u=0, \pm 1} \).

If the stress-tensor contains the cosmological constant \( (8\pi G N T_{\mu\nu} = \Lambda_{\text{fund}} g_{\mu\nu}) \) the value of \( \zeta \) can be chosen to cancel its contribution down to zero. This can be combined with the self-accelerated solution obtained above. For instance, consider the Lagrangian with the cosmological constant and the choice of the plus sign in front of the second term in (2): \( \mathcal{L}_b \)

\[
\mathcal{L}_b = \sqrt{g}(R - 2\Lambda_{\text{fund}}) + m^2 \int_{-1}^{+1} du \sqrt{g} \left( k_{\mu\nu} - k^2 \right) + C_{-} m^2 \left( \sqrt{g}_{u=b} + \sqrt{g}_{u=-b} \right). \quad (12)
\]

The corresponding equations (11) and (5) have a consistent solution:

\[
\tilde{g}_{\mu\nu}^{cl}(x, u) \equiv (1 + \zeta |u| - |u|)\gamma_{\mu\nu}(x), \quad R(\tilde{\gamma}) = 12m^2 ,
\]

if \( C_{-} = 3(\zeta - 1)/\zeta \), where \( \zeta \equiv \Lambda_{\text{fund}}/3m^2 \gg 1 \).

The result of this discussion is the following: for an arbitrary value of the cosmological constant generated by particle physics \( \Lambda_{\text{fund}} \), one can choose the boundary conditions and surface term in (12) such that the background solution describes an accelerated universe with the Hubble parameter that is independent of \( \Lambda_{\text{fund}} \), but instead is defined by the UV insensitive new mass scale \( m \), introduced in the Lagrangian (2), or (12).
This scheme does not provide a dynamical mechanism for solving the cosmological constant problem, as one has to adjust the boundary terms and conditions appropriately to get rid of $\Lambda_{\text{fund}}$. However, it has an advantage over GR in the following respect: GR, as well as the present model, at the classical level can entirely be specified by their equations of motion, without any reference to the action. The GR equations with the cosmological constant have no other solutions but the (A)dS solutions with curvature set by the value of the cosmological constant. In contrast with this, the equations of motion of the present theory with the cosmological constant do have solutions with curvatures that are not related to the cosmological constant. The above properties of the equations make no reference to the boundary terms. The latter come into the play only when the action functional is invoked.

Hence, as long as gravity is treated classically while all the other interactions are quantized, the present approach reduces the cosmological constant problem to the choice of the boundary conditions in the classical gravitational equations.

The fact that $\Lambda_{\text{fund}}$ can be removed by means of the boundary conditions which specify the otherwise arbitrary integration constant, is somewhat similar to what happens in the unimodular gravity \cite{[12, 13]} where the cosmological constant can be fixed by superselection rules. However, a distinction between the two approaches is that the perturbations in the present case can be different from those of the unimodular gravity which are identical to the GR perturbations.

In the context of inflationary cosmology, the present method would remove a constant piece from the inflationary potential, while retaining all the positive aspects of the slow-roll inflationary paradigm (note a similarity in this with Ref. \cite{[14]}).

As mentioned before, the theory \cite{[2]} contains all the solutions of GR: using the factorized form \cite{[7]} with $a = 1$ one would obtain just Einstein’s equations for $g_{\mu\nu}$. For the selfaccelerated universe $a = 1 \mp |u|$, and equation \cite{[1]} for $g_{\mu\nu}$ is the ordinary Einstein equation with the cosmological constant equal to $3m^2$. Thus, for instance, the dS-Schwarzschild solution of GR is also a factorized solution on the selfaccelerated background. Similar arguments apply to any other solution of the Einstein equations. Furthermore, there may well exist other solutions, e.g., for a static source, that do not have the factorized form \cite{[7]}. The latter would be selected from the factorized solutions by the boundary conditions.

Factorized or not, the spectrum of linear and/or nonlinear perturbations about these solutions are determined by Eqs. \cite{[1]}, \cite{[3]}, which themselves may or may not have a factorized form \cite{[7]}, or depending on boundary conditions, could differ from the spectrum of GR. One example of this is the spectrum of linear perturbations on the selfaccelerated solution to which we turn in the next section.
3. Perturbations of the selfaccelerated solution

We denote the deviation from the background metric as follows:
\[ \tilde{g}_{\mu\nu}(x, u) = \tilde{g}_{\mu\nu}^{cl}(x, u) + \delta g_{\mu\nu}(x, u). \]  
(14)

Where, \( \tilde{g}^{cl} \) is defined in (9). It is straightforward to derive that
\[ k_{\mu\nu} = \bar{k}_{\mu\nu} + \delta k_{\mu\nu}, \quad \bar{k}_{\mu\nu} = \mp \frac{1}{2} \bar{\gamma}_{\mu\nu}, \quad k = \bar{k} + \delta k, \quad \bar{k} = \tilde{g}^{\mu\nu} k_{\mu\nu} = \mp \frac{2}{a}, \]  
(15)

where
\[ \delta k_{\mu\nu} = \frac{1}{2} \partial_{u} \delta g_{\mu\nu}, \quad \delta k = \frac{1}{2a} \bar{\gamma}^{\mu\nu} \partial_{u} \delta g_{\mu\nu} \pm \frac{1}{2a^{2}} \bar{\gamma}^{\mu\nu} \delta g_{\mu\nu}. \]  
(16)

An expansion of Eq. (4) in the linear approximation reads:
\[ \delta G_{\mu\nu} \pm 2m^{2} (\delta k_{\mu\nu} - \delta g_{\mu\nu} k - \bar{g}_{\mu\nu} \delta k) = \frac{T_{\mu\nu}}{2}, \]  
(17)

here \( T_{\mu\nu} \) on the r.h.s. is the stress-tensor of a probe source which has nothing to do with the background; the variation of the Einstein tensor on the dS space is
\[ \delta G_{\mu\nu} = - \frac{1}{2} (\Box \delta g_{\mu\nu} - \nabla_{\mu} \nabla^{\alpha} \delta g_{\alpha\nu} - \nabla_{\nu} \nabla^{\alpha} \delta g_{\alpha\mu} + \nabla_{\mu} \nabla_{\nu} \delta g_{\alpha}^{\alpha}) \]
\[ - \frac{1}{2} \bar{\gamma}_{\mu\nu} (\nabla^{\alpha} \nabla^{\beta} \delta g_{\alpha\beta} - \Box \delta g_{\alpha}^{\alpha}) - 2H^{2} \delta g_{\mu\nu} + \frac{1}{2} H^{2} \bar{\gamma}_{\mu\nu} \delta g_{\alpha}^{\alpha}, \]  
(18)

where \( \nabla \) denotes the covariant derivative w.r.t. \( \bar{\gamma} \). The constraint (6), which in the linearized approximations reads
\[ \pm \nabla_{\mu} \delta g_{\mu\nu} + \nabla^{\mu} \partial_{u} \delta g_{\mu\nu} = \pm \nabla_{\nu} \delta g_{\alpha}^{\alpha} + \nabla \nabla_{\nu} \partial_{u} \delta g_{\mu\nu}, \]  
(19)

can be satisfied by the following gauge fixing condition
\[ \nabla^{\alpha} \delta g_{\alpha\beta} = \nabla_{\beta} \delta g_{\alpha}^{\alpha}. \]  
(20)

Using the latter in equation (17), where we also substitute \( m^{2} = H^{2} \), we obtain:
\[ - \frac{1}{2} (\Box \delta g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \delta g_{\alpha}^{\alpha}) + 2H^{2} \delta g_{\mu\nu} - \frac{1}{2} \bar{\gamma}_{\mu\nu} H^{2} \delta g_{\alpha}^{\alpha} \]
\[ \pm H^{2} (\partial_{u} \delta g_{\mu\nu} - \bar{\gamma}_{\mu\nu} \partial_{u} \delta g_{\alpha}^{\alpha}) = \frac{T_{\mu\nu}}{2}. \]  
(21)

Taking trace of the above equation gives:
\[ \mp 3H^{2} \partial_{u} g_{\alpha}^{\alpha} = T/2. \]  
(22)

One needs to solve equation (5) to obtain the \( u \)-dependence of the perturbations. For this one considers variation of its left and right hand sides separately at \( u > 0 \):
\[ \delta \left\{ \partial_{u} \left[ \sqrt{g} (k \tilde{g}^{\mu\nu} - k^{\mu\nu}) \right] \right\} = \]
\[ \sqrt{\bar{g}} \partial_{u} \left\{ \frac{1}{2} \bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \partial_{u} \delta g_{\alpha\beta} \mp \frac{1}{4a} \bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \delta g_{\alpha\beta} - \frac{1}{2} \bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \partial_{u} \delta g_{\alpha\beta} \pm \frac{1}{a} \bar{\gamma}^{\mu\alpha} \bar{\gamma}^{\nu\beta} \delta g_{\alpha\beta} \right\} . \]  
(23)
Notice that all the equations presented above with the upper sign are equivalent to those with the lower sign provided that in the latter the replacement \(u \rightarrow -u\) is made. This will be reflected in our final solutions.

The variation of the rhs of (5) equals to:

\[
\sqrt{\gamma} \left\{ \frac{1}{a^2} \bar{\gamma}^{\mu\alpha} \bar{\gamma}^{\nu\beta} \delta g_{\alpha\beta} \pm \frac{1}{a} \bar{\gamma}^{\mu\alpha} \bar{\gamma}^{\nu\beta} \partial_u \delta g_{\alpha\beta} - \frac{1}{4a^2} \bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \delta g_{\alpha\beta} \mp \frac{1}{4a} \bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \partial_u \delta g_{\alpha\beta} \right\}. \tag{24}
\]

Putting Eqs. (23) and (24) together, certain cancellations occur, and we find the final equation:

\[
\bar{\gamma}^{\mu\nu} \bar{\gamma}^{\alpha\beta} \partial_u^2 \delta g_{\alpha\beta} = \bar{\gamma}^{\mu\alpha} \bar{\gamma}^{\nu\beta} \partial_u^2 \delta g_{\alpha\beta}. \tag{25}
\]

The latter has a solution

\[
\delta g_{\alpha\beta} = (1 + cu) h_{\alpha\beta}(x), \tag{26}
\]

where \(c\) is an arbitrary constant to be fixed by the boundary conditions\(^1\). The two sign choices considered above will hereafter be encoded in the value of \(c\). We’ll keep this constant unspecified till the end of our calculations.

Using the solution (26) in equation (21) we find:

\[
- \frac{1}{2} \left( \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h_\alpha^\alpha \right) + 2H^2 h_{\mu\nu} - \frac{1}{2} H^2 \bar{\gamma}_{\mu\nu} h_\alpha^\alpha \\
+ H^2 c \left( h_{\mu\nu} - \bar{\gamma}_{\mu\nu} h_\alpha^\alpha \right) = T_{\mu\nu}/2, \tag{27}
\]

with its trace equation

\[
- 3H^2 ch = T/2. \tag{28}
\]

Combining the above two equations, introducing the Lichnerowicz operator which in our case satisfies:

\[
\Delta_L h_{\mu\nu} = -\Box h_{\mu\nu} + 8H^2 h_{\mu\nu} - 2H^2 \bar{\gamma}_{\mu\nu} h_\alpha^\alpha, \tag{29}
\]

and using the standard techniques (see, [15, 16] for recent discussions), we obtain the following expression for the perturbations:

\[
h_{\mu\nu} = \frac{1}{\Delta_L - 6H^2 + 2H^2(c + 1)} T_{\mu\nu} - \frac{1}{3} \frac{\bar{\gamma}_{\mu\nu} \Box - 6H^2 + 2H^2(c + 1)}{\Delta_L - 6H^2 + 2H^2(c + 1)} T \\
+ \frac{1}{6c} \bar{\gamma}_{\mu\nu} - \frac{1}{\Delta_L - 6H^2 + 2H^2(c + 1)} T + \nabla_\mu \nabla_\nu \frac{1}{6H^2 c} - \frac{1}{\Delta_L - 6H^2 + 2H^2(c + 1)} T. \tag{30}
\]

\(^1\)The expression in (26) is not a most general solution of (25), however, it can be selected among all the solutions by specifying appropriate boundary conditions (see below).
The expression for the physical one-particle exchange amplitude reads as follows:

\[ A \equiv \int d^4x \sqrt{\gamma} T'^{\mu\nu} h_{\mu\nu} = \int d^4x \sqrt{\gamma} T'^{\mu\nu} \frac{1}{\Delta_L - 6H^2 + 2H^2(c+1)} T_{\mu\nu} \]

\[ - \int d^4x \sqrt{\gamma} \left( \frac{1}{3} - \frac{1}{6c} \right) T' \frac{1}{-\Box - 6H^2 + 2H^2(c+1)} T \],

(31)

where \( T'^{\mu\nu} \) denotes another conserved probe source. This should be compared with the amplitude for a massless graviton on dS space

\[ A_0 \equiv \int d^4x \sqrt{\gamma} T'^{\mu\nu} h_{\mu\nu} = \int d^4x \sqrt{\gamma} \left\{ T'^{\mu\nu} \frac{1}{\Delta_L - 6H^2} T_{\mu\nu} - \frac{1}{2} T' \frac{1}{-\Box - 6H^2} T \right\} ,

(32)

or with the amplitude for a massive graviton of mass \( M \) on dS space

\[ A_M \equiv \int d^4x \sqrt{\gamma} T'^{\mu\nu} h_{\mu\nu} = \int d^4x \sqrt{\gamma} \left\{ T'^{\mu\nu} \frac{1}{\Delta_L - 6H^2 + M^2} T_{\mu\nu} - \frac{1}{3} T' \frac{1}{-\Box - 6H^2 + M^2} T \right\} .

(33)

For \( c = -1 \) the amplitude (31) is equivalent to that of a massless tensor field on dS in GR (32). The solution for the background plus its perturbation in this case reads as follows:

\[ \tilde{g}_{\mu\nu} = (1 \mp |u| \tilde{\gamma}_{\mu\nu} + (1 \mp |u|) h_{\mu\nu} ,

(34)

where \( h_{\mu\nu} \) is given in (30). Note that these solutions corresponds to choosing (8) as the second boundary condition.

For \( c > 2 \) one gets a massive graviton on the dS background [17] and a massive scalar with the graviton mass \( M^2 = 2H^2(c+1) \) and the scalar mass \( M_s^2 = 2H^2(c+1) - 6H^2 \). Moreover, the scalar couples to the stress-tensor with the \( 1/c \) suppressed strength as compared with the gravitational coupling. The metric for the solutions takes the form:

\[ \tilde{g}_{\mu\nu} = (1 \mp |u| \tilde{\gamma}_{\mu\nu} + (1 \pm c|u|) h_{\mu\nu} .

(35)

This solution corresponds to choosing in addition to (1) the following boundary condition: \( \partial_u \tilde{g}_{\mu\nu} |_{u=0^+} = \mp ((1 + c) \tilde{\gamma}_{\mu\nu} - c g_{\mu\nu}) \).

For \( c = 0 \) the solution exist only for conformal sources with \( T = 0 \), for which one gets a special massive tensor on dS background with enhanced symmetry [18].

The boundary conditions with values of \( c \) other than \( c = -1, c = 0 \) and \( c \geq 2 \) give rise to instabilities: for \( c < -1 \) one gets a tachyonic tensor field (which implies that its helicity-0 component is a ghost) and a ghost-like scalar; for \( -1 < c < 0 \) one gets a massive tensor and a tachyonic scalar ghost; For \( 0 < c < 2 \) one gets massive tensor and a tachyonic scalar.
Note that the last term in the expression for the field (30) is singular in the $H = m \to 0$ limit. This term does not enter the linearized amplitude, but as it is well known, such terms typically give rise to strongly coupled behavior of massive theories [19, 20, 21, 22, 23]. This is a welcome feature in a classical theory as it provides a way to overcome the vDVZ discontinuity [24], as was first argued by Vainshtein [19] (see also [20], and more recent works [25]). The magnitude of the strong scale should grow with $c$, as it’s suggested by (30). More detailed questions on its dependence on boundary terms and conditions are left open. The perturbative results obtained above have a limited applicability as the theory is expected to be strongly coupled. Moreover, perturbative stability is not a guarantee of a stability of the full nonlinear theory, however, it is a first and important step on the way to establish whether or not the theory could be viable.

4. Hamiltonian

In this section we derive the Hamiltonian for the theory (2). For this we use the standard ADM decomposition [6]:

\[
\begin{aligned}
\tilde{g}^{00} &= -\frac{1}{N^2}, \quad \tilde{g}_{0j} \equiv N_j, \quad \tilde{g}_{ij} \equiv \gamma_{ij}, \\
\tilde{g}^{00} &= -(N^2 - N_i \gamma^{ij} N_j), \quad \tilde{g}^{0j} = \frac{N^j}{N^2}, \quad \gamma^{ij} = \tilde{g}^{ij} + \frac{N^i N^j}{N^2}.
\end{aligned}
\] (36)

After somewhat lengthy algebra the additional term in the Lagrangian (2) can be written as:

\[
\sqrt{g} \left( k_{\mu\nu}^2 - k^2 \right) = \sqrt{\gamma} \left( N(q_{ij}^2 - q^2) - \frac{V^j \gamma_{jk} V^k}{2N} - 2q \partial_u N \right),
\] (37)

where all indexes are raised by $\gamma^{ij}; q_{ij} \equiv \frac{1}{2} \partial_u \gamma_{ij} = k_{ij}, q \equiv \gamma^{ij} q_{ij},$ and $V^j \equiv \partial_u N^j$.

The expression in (37) does not contain any time derivatives. Therefore, the canonical momenta in the extended theory (2) are the same as in GR. The Hamiltonian density can straightforwardly be calculated

\[
\mathcal{H}_u = \sqrt{\gamma}(NR^0 + N_j R^j) \delta(u) \mp m^2 \sqrt{\gamma} \left( N(q_{ij}^2 + q^2) - \frac{V^j \gamma_{jk} V^k}{2N} + 2N \partial_u q \right) + \Sigma. \] (38)

Here $\Sigma$ denotes the surface terms for both, the possible spatial boundaries, as well as the boundaries in the \textquotedblright u-dimension\textquotedblright

\[
\Sigma \equiv 2\nabla_j (\gamma^{-1/2} N_k \pi^{kj}) \delta(u) \pm 2m^2 \partial_u \left( \sqrt{\gamma}q N \right). \] (39)

\footnote{Note that in this section $\gamma$ refers to the 3D metric, as defined in (36), while $\tilde{\gamma}$ denotes, as before, the 4D de Sitter metric.}
The first two terms in (38) (the ones that are multiplied by $\delta(u)$) are those of GR with $R^0 \equiv -R^{(3)} + \gamma^{-1}(\pi^2 - \frac{1}{2}\pi^2)$, and $R^i \equiv -2\nabla_k(\gamma^{-1/2}\pi^k)$, with $\pi_{ij}$ being the canonical momenta of GR (see, e.g. [6], [3]).

Since the additional terms in the Lagrangian (2) have no time derivatives (37), the primary constraints of GR are preserved; the conjugate momenta for the lapse $N$ and shift $N_j$ are zero, $P_N = P_{N_j} = 0$. Hence, variation of the Hamiltonian $\delta H$ under the variations $\tilde{\delta}N$ (such that $\tilde{\delta}N(x,u)|_{u=\pm1} = 0$, and vanishing variation at $u = 0$ and $x = boundary$) and $\tilde{\delta}N_j$ (such that $\tilde{\delta}N_j(x,u)|_{u=\pm1} = 0$ and vanishing variation at $u = 0$ at $x = boundary$) leads respectively to the following relations:

$$R^0\delta(u) \pm 2m^2\sqrt{\gamma}\partial_u q \mp m^2\sqrt{\gamma}\left(\pi^2 + q^2 + \frac{V^j\gamma_{jk}V^k}{2N^2}\right) = 0,$$ (40)

$$\sqrt{\gamma}R^j\delta(u) \mp m^2\gamma^{jk}\partial_u \left[\sqrt{\gamma}\frac{\gamma^{ki}V^i}{N}\right] = 0.$$ (41)

Substituting these into the expression for the Hamiltonian (38), one finds that the “bulk” terms all cancel and what is left is just the boundary terms:

$$H(t) = \int d^3x \int_{-1}^{+1} du H_u(x) = \pm 4m^2 \int d^3x \sqrt{\tilde{g}}(\tilde{g}^{ij}k_{ij} + \tilde{g}^{0i}k_{0i})|_{0}^{+1},$$ (42)

where we used the relation $\sqrt{\gamma}(Nq + (N_j\partial_u N^j/2N)) = \sqrt{g}(\tilde{g}^{ij}k_{ij} + \tilde{g}^{0i}k_{0i})$, and dropped the surface term that appears in the GR Hamiltonian (that is the first term in (39)). Note that this is a Hamiltonian that follows from the Lagrangian (2). If one adds additional surface terms as in (10), those terms should simply be subtracted from (42) to get the right Hamiltonian.

For illustration we calculate the energy for the selfaccelerated solution (9) with $a = 1 - |u|$. The result is positive:

$$H(t) = 6m^2 \int d^3x \sqrt{\tilde{g}}.$$ (43)

For the selfaccelerated solution with the growing $a(u)$ in (9) the calculation of energy gives the same result (43) only after inclusion of the boundary term given in (10).

As we see, the positive semi-definiteness of the Hamiltonian (42), in which the constraints (or algebraically determined relations) were used, depends on the boundary conditions in the $u$-direction. However, making these boundary terms positive semi-definite does not in general guarantee absence of instabilities, since the latter can be “hidden” in the constraint equations. One example of this is GR with a minimally coupled scalar of a negative kinetic term. The GR constraints put the Hamiltonian of this system to be zero, however, there are instabilities in the theory already at the classical level.

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3If the boundary is present in the $x$-space the GH boundary term should also be introduced.
In our case, the above derived results can be used to deduce the following important observation: In the \(m \to 0\) limit the Hamiltonian \((42)\) goes to zero. This is in contrast with the \(1/m^2\) term in the Hamiltonian for 4D massive gravity found by Boulware and Deser in [8]. Moreover, the expression \((42)\) has no singular behavior in the field fluctuations, that was found in [8] as a source of various instabilities in massive gravity. Hence, even though there is no complete proof of the absence of instabilities in the full non-linear theory, the absence of the Boulware-Deser singular term in the Hamiltonian is a promising step forward.

From Eq. (40) we find two equations for \(u = 0\) and \(u > 0\) respectively:

\[
\sqrt{\gamma} R^0|_{u=0} = \pm 2m^2 \sqrt{\gamma} q^0_{00}^{0+}, \quad \sqrt{\gamma} \partial_u q = -\frac{1}{2} \sqrt{\gamma} \left( q^2_{ij} + q^2 + \frac{V}{2 N^2} \right), \tag{44}
\]

where the rhs of the last equation is positive semi-definite. Similarly, we obtain from (41) the following equations for \(u = 0\) and \(u > 0\) respectively:

\[
\sqrt{\gamma} R^j|_{u=0} = \pm \frac{m^2}{N} \sqrt{\gamma} V^j|_{00}^{00+}, \quad \partial_u \left( \sqrt{\gamma} \frac{\gamma_{ij} V^i}{N} \right) = 0. \tag{45}
\]

Let us count the degrees of freedom. The variables \(N|_{u=0}\) and \(N_j|_{u=0}\) can be fixed by gauge transformations. The variables \(\partial_u N|_{u=0}\) and \(\partial_u N_j|_{u=0}\) can also be fixed after choosing the boundary conditions, for instance as \(N|_{u=\pm 1}\) and \(N_j|_{u=\pm 1}\), and using the equations (44) and (45). After fixing the boundary conditions what is left undetermined is the 12 variables \(\gamma_{ij}|_{u=0}\), \(\pi_{ij}|_{u=0}\). Hence, in general this theory described 6 degrees of freedom, as we found it already in linearized calculations on the selfaccelerated background. For appropriate choice of boundary conditions these could be a massive graviton plus an additional scalar, which have no ghosts or tachyons, as it was shown in Section 3. For some particular boundary conditions though, due to enhanced symmetries of the linearized perturbations, the number of linear degrees of freedom gets reduced. In this case some of the equations in (44) and (45) should appear as constraints in the linearized theory.

5. Discussions

The extension of GR considered in this work is a convenient way of putting various theories of massive gravity in a single framework. All these theories, known in the linearized level, emerge as a consequence of choosing different boundary conditions in the auxiliary dimension. Moreover, the present framework provides a non-linear completion to these theories with the Hamiltonian that does not suffer from the problems found in Ref. [8]. The auxiliary dimension is just a convenient technical tool; it can in principle be “integrated out” entirely, and this should lead to GR amended by new terms in 4D.

Most importantly, the extended theory admits the selfaccelerated solution with the spectrum of linear perturbations that has no ghosts or tachyons. In a general
case one obtains massive graviton and a scalar. This may have some cosmological signatures along the lines of Refs. \[26, 27\]. The vDVZ discontinuity of the linearized theory has to be overcome through the strong dynamics via the Vainshtein mechanism \[19\] (see also \[20\]). If this is the case, then the theory is likely to have also short distance signatures \[28\], \[29\].

We end this section by a few comments.

The auxiliary dimension discussed so far had a finite extent in the $u$-direction. It is straightforward to present a Lagrangian in which the $u$-direction is infinite:

$$\sqrt{g} R + m^2 \int_{-\infty}^{+\infty} du \sqrt{\tilde{g}} \left( k_{\mu \nu}^2 - k^2 - 3 \right).$$

The equations of motion of this Lagrangian have a selfaccelerated solution $\tilde{g}_{\mu \nu} = a(u) \tilde{\gamma}(x)$, where $a(u) = e^{-u}$, and as before, $\tilde{\gamma}$ denotes the 4D de Sitter metric with curvature $R = 12m^2$.

The Lagrangians \(2\) and \(46\), can be obtained by a certain truncation of a 5D theory. The 5D theory giving \(46\) can be defined as follows:

$$\sqrt{g} R + m_c \int dy \sqrt{g^{(5)}} \left( R(\tilde{g}) - R_5(g^{(5)}) - 3\tilde{m}_c^2 \right) |_{g_{55} = 1, g_{\mu 5} = 0},$$

where $R_5$ is the 5D Ricci scalar, $g^{(5)}_{AB} = \{\tilde{g}_{\mu \nu}, g_{\mu 5}, g_{55}\}$, $A, B = 0, 1, 2, 3, 5$, $y = u/\tilde{m}_c$, $m^2 = m_c \tilde{m}_c$, and the substitutions in the last term are taken before the equations of motion are obtained, i.e., there is no variation w.r.t. $g_{55}$ and $g_{\mu 5}$. To get the analogous expression for \(2\) one would have to drop the last term in the parenthesis, and set the integration w.r.t. $y$ from $-1/\tilde{m}_c$ to $+1/\tilde{m}_c$. The expression \(47\) is somewhat similar to the DGP Lagrangian \[5\], or its sign-flipped counterpart \[11\], with two crucial differences: (1) There is a subtraction of the $R$ term from the $R_5$ term in the bulk action; (2) There are no $\{55\}$ or $\{\mu 5\}$ equations.\[4\]

Similar constructions with an auxiliary dimension can be considered for a scalar or vector, by adding the term $-m^2 \int du [(\partial_u \phi)^2 + \phi^2 + ...]$, to the conventional scalar field Lagrangian, or the term $-m^2 \int du [(\partial_u A_{\mu})^2 + ...]$ to the Maxwell Lagrangian (with a finite or an infinite range of integration).

**Acknowledgments**

I’d like to thank Massimo Porrati for useful discussions. The work was partially supported by the NSF grant PHY-0758032.

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\[4\] One gets back the analog of the $\{\mu 5\}$ equation in Eq. \(6\) due to the Bianchi identities. It is really the absence of the $\{55\}$ equation that makes a crucial difference.
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