SOLUTION OF THE $\bar{\partial}$-NEUMANN PROBLEM ON A NON-SMOOTH DOMAIN

DARIUSH EHSANI

Abstract. We study the solution of the $\bar{\partial}$-Neumann problem on $(0, 1)$-forms on the product of two half-planes in $\mathbb{C}^2$. In particular, we show the solution can be decomposed into functions smooth up to the boundary and functions which are singular at the singular points of the boundary. Furthermore, we show the singular functions are log and arctan terms.

1. Introduction

The $\bar{\partial}$-Neumann problem is defined as follows. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with standard Hermitian metric. Let $\Box$ be the complex Laplacian, $\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$, $\bar{\partial}$ defined in the sense of distributions on $L^2_{(p,q)}(\Omega)$, the space of $(p,q)$-forms whose coefficients are in $L^2(\Omega)$ for $0 \leq p \leq n$, $1 \leq q \leq n$. The $\bar{\partial}$-Neumann problem is to find a solution $u$, given a function $f \in L^2_{(p,q)}$, to the equations

$$\Box u = f \text{ in } \Omega$$

and

$$u|_{\partial \Omega} = 0; \quad \bar{\partial} u|_{\partial \Omega} = 0$$

on $\partial \Omega$. Here $\rho$ is a defining function for the domain $\Omega$ ($\Omega = \{z : \rho(z) < 0\}$) whose gradient is normalized to be of length one on the boundary. The problem arose in an attempt to solve the $\bar{\partial}$ problem: find a $(p, q - 1)$-form, $u$ which solves $\bar{\partial} u = f$ in $\Omega$ and is orthogonal to the null space of $\bar{\partial}$ on $(p, q - 1)$-forms.

The $\bar{\partial}$-Neumann problem is an example of a partial differential equation which is non-coercive. Although $\Box$ is strongly elliptic, the boundary conditions are not and Gårding’s inequality breaks down at the boundary. On strictly pseudoconvex domains with smooth boundary, Kohn solved the problem and provided regularity results for the solution, $u$, with a gain of 1 derivative.

In studying the $\bar{\partial}$-Neumann problem it is natural to study subellipticity and compactness of the operator $N$, which in the case of a smooth boundary, give regularity results of the solution. Henkin and Iordan have established that $N$ is compact on piece-wise smooth strictly pseudoconvex domains, and have also obtained compactness on certain Lipschitz pseudoconvex domains. Henkin, Iordan, and Kohn in show subelliptic 1/2-estimates on relatively compact strictly pseudoconvex domains with piece-wise smooth boundary. Independently, Michel and Shaw show $N$ satisfies subelliptic 1/2-estimates on domains with piece-wise smooth strictly pseudoconvex boundary and also show in $N : H^{1/2}_{(p,q)}(\Omega) \rightarrow H^{1/2}_{(p,q)}(\Omega)$ is continuous for $\Omega$ a bounded pseudoconvex Lipschitz domain with plurisubharmonic defining function and $H^{1/2}_{(p,q)}(\Omega)$ the space of $(p,q)$-forms whose coefficients...
are in the Sobolev $1/2$-space. Straube in \[14\] proves subelliptic estimates in the case where the boundary is piece-wise smooth of finite type. Unfortunately, when the boundary of the domain is non-smooth subellipticity and compactness do not imply regularity on all Sobolev spaces. Hence, we only have estimates of the solution in terms of the data on a limited number of Sobolev spaces. Not much else is known about the solution when the boundary is non-smooth. The purpose of this paper is to aid in the study of the $\bar{\partial}$-Neumann problem on such domains. In particular, we analyze the behavior of the solution near the presence of corners of the domain. A paper, similar at least in spirit to this paper, by Harvey and Polking \[5\] gives explicitly a kernel for the operator $N$ on the model domain the ball in $\mathbb{C}^n$. And Stanton in \[13\] gives a kernel for $N$ on the strictly pseudoconvex Siegel domain
\[ D = \{ (z, w) : z \in \mathbb{C}^n, w \in \mathbb{C}, \Im w > |z|^2 \}. \]

Our paper is organized as follows. In Section 2 we prove the existence and uniqueness of a solution and also show regularity results away from the corner of two half-planes. Section 3 is devoted to determining the type of singularities of our solution near the corner. We also show, in Section 4, by operating on our solution with $\bar{\partial}^*$, we eliminate the singularities.

The results in this paper stem from the author’s Ph.D. thesis at the University of Michigan. Many of the results were achieved under the guidance of David Barrett, to whom we offer our sincere gratitude.

2. The solution on two half-planes

Let $\Omega \subset \mathbb{C}^2$ be the domain $\mathbb{H}_1 \times \mathbb{H}_2$, where $\mathbb{H}_j$ is the half-plane $\{ (x_j, y_j) : y_j > 0 \}$ for $j = 1, 2$.

The $\bar{\partial}$-Neumann problem on $\Omega$ is equivalent to solving the problems
\[ \Delta u_j = -2f_j \quad j = 1, 2, \]
where $\Delta$ is the Laplacian
\[ \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}, \]
with the boundary conditions
\[ u_j = 0 \quad \text{on} \quad y_j = 0 \quad j = 1, 2 \]
and
\[ \frac{\partial u_j}{\partial z_k} = 0 \quad \text{on} \quad y_k = 0 \quad j, k \in \{1, 2\} \quad j \neq k. \]

By $S(\overline{\Omega})$ we denote the family of Schwartz functions on $\overline{\Omega}$. Let $g \in S(\overline{\Omega})$. We look to solve
\[ (2.1) \quad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} = g \]
with the boundary conditions
\[ (2.2) \quad u = 0 \quad \text{on} \quad y_1 = 0 \]
and
\[ (2.3) \quad \frac{\partial u}{\partial z_2} = 0 \quad \text{on} \quad y_2 = 0. \]
We extend $g$ and $u$ to be odd in $y_1$, and to be 0 for $y_2 < 0$, and we denote these extended functions by $g^{o1}$ and $u^{o1}$, respectively. We shall show, after taking Fourier transforms, our solution takes the form

$$\hat{u}^{o1} = -\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, \eta_2) + \frac{i\eta_2 - \xi_2 \hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{\zeta - \xi_2} \frac{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}{\zeta - \xi_2} \frac{1}{\zeta - \xi_2}$$

where $\zeta = \sqrt{\xi_1^2 + \xi_2^2 + \eta_1^2}.$

We begin by proving estimates for the function in (2.4), and we will adopt the convention to use $\leq$ in place of $\leq c$ for $c > 0$.

**Proposition 2.1.** $\hat{u}^{o1}(\xi_1, \xi_2, \eta_1, \eta_2)$, given by equation (2.4), and $u^{o1}$, the Fourier inverse of $\hat{u}^{o1}$, have the following properties:

a) $\hat{u}^{o1} \in L^p(\mathbb{R}^4)$ for $p \in (1, 2)$.

b) $\eta_1 \hat{u}^{o1} \in L^p(\mathbb{R}^4)$ for $p \in (4/3, 2)$.

**Proof.** We prove a), the proof of b) following similar arguments. To prove a) we look at each term in (2.4) separately.

For the second term, we look at the integral

$$\int_{\mathbb{R}^4} \left| \frac{i\eta_2 - \xi_2 \hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{\zeta - \xi_2} \frac{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}{\zeta - \xi_2} \right|^p d\xi d\eta,$$

where $d\xi = d\xi_1 d\xi_2$ and $d\eta = d\eta_1 d\eta_2$. We first perform the integration over the $\eta_2$ variable,

$$\int_{-\infty}^{\infty} \left| \frac{i\eta_2 - \xi_2}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right|^p d\eta_2 = \int_{-\infty}^{\infty} \left| \frac{(\eta_2^2 + \xi_2^2)^{\frac{1}{2}}}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right|^p d\eta_2$$

$$= \int_{-\infty}^{\infty} \left| \frac{(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)_{\xi_2}^2}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right|^p d\eta_2$$

$$\leq \int_{-\infty}^{\infty} \frac{1}{(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)^{\frac{5}{2}}} d\eta_2,$$

(2.6)

Next, using the fact that

$$\int_{-\infty}^{\infty} \frac{1}{(\eta_2^2 + \zeta^2)^k} d\eta_2 = \frac{1}{\zeta^{2k-1}} \int_{-\infty}^{\infty} \frac{1}{(1 + t^2)^k} dt$$

in (2.6) we have

$$\int_{-\infty}^{\infty} \left| \frac{i\eta_2 - \xi_2}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right|^p d\eta_2 \lesssim \frac{1}{\zeta^{p-1}}.$$

Using the above estimates in (2.5), we see we have to estimate

$$\int_{\mathbb{R}^3} \left| \frac{\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{\zeta - \xi_2} \frac{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}{\zeta - \xi_2} \right|^p \frac{1}{\zeta^{p-1}} d\xi d\eta_1.$$

(2.7)

We make a change of coordinates from $(\eta_1, \xi_1, \xi_2)$ to $(r, \phi, \theta)$. (2.7) becomes

$$\int_{0}^{2\pi} \int_{0}^{\pi} \left| \frac{\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{1 - \cos \phi} \right|^p \frac{\sin \phi}{r^{2p-3}} r dr d\phi d\theta.$$

(2.8)
We estimate $|\tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)|$. We use

$$
\begin{align*}
\left| \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta) \right| \\
= \left| \int_0^\infty \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t)e^{-rt}dt \right| - \left| \int_0^\infty \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t)e^{-rt}dt \right|
\end{align*}
$$

(2.9)

$$
\leq \left| \int_0^\infty \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t)e^{-rt}dt \right| + \left| \int_0^\infty \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t)e^{-rt}dt \right|,
$$

(2.10)

where $\tilde{g}^{o1}$ is the partial Fourier transform of $g^{o1}$ in all variables except the $y_2$ variable.

$$
\left| \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| = -iy_1\tilde{g}^{o1},
$$

and

$$
\left| \frac{\partial}{\partial \eta_2} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| = \eta_1 \left( \frac{\partial}{\partial y_2}(y_1g^{o1}) \right)
$$

$$
= \left( \frac{\partial^2}{\partial y_1^2} (y_1g^{o1}) \right) + \int_{\mathbb{R}^2} \frac{\partial}{\partial y_1} (y_1g^{o1})(x_1, x_2, 0, t)e^{-ix\cdot\xi}dx
$$

$$
\leq \left( \frac{\partial^2}{\partial y_1^2} (y_1g^{o1}) \right) + \left| \int_{\mathbb{R}^2} \frac{\partial}{\partial y_1} (y_1g^{o1})(x_1, x_2, 0, t)e^{-ix\cdot\xi}dx \right|
$$

$$
\lesssim 1,
$$

where $\mathbf{x} = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, $d\mathbf{x} = dx_1dx_2$, and in the last step we use the fact that $g \in S(\Omega)$. Similarly, we have

$$
\left| \frac{\xi_2^2}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| \lesssim 1,
$$

$$
\left| \frac{\xi_2^2}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| \lesssim 1.
$$

Hence,

$$
\left| \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| \lesssim \frac{1}{1 + r^2}.
$$

Since $(1 + t^3)g(x_1, x_2, y_1, t) \in S(\Omega)$, the same reasoning shows

$$
\left| \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t) \right| \lesssim \frac{1}{(1 + r^2)(1 + t^3)},
$$

and the first term in (2.10) can be estimated by

$$
\left| \int_0^\infty \frac{\partial}{\partial \eta_1} \tilde{g}^{o1}(\xi_1, \xi_2, \eta_1, t)e^{-rt}dt \right| \lesssim \int_0^\infty \frac{1}{(1 + r^2)(1 + t^3)}e^{-rt}dt
$$

$$
\lesssim \frac{1}{(1 + r)(1 + r^2)}.
$$
whereas the second term in (2.10) can be estimated by

\[
\begin{align*}
\int_0^\infty \tilde{g}^{o1} (\xi_1, \xi_2, \eta_1, t)e^{-rt} dt & \lesssim \frac{1}{(1+r)(1+r^2)}.
\end{align*}
\]

To see these estimates hold, just consider the integrals

\[
\int_0^\infty t^j re^{-rt} \frac{dt}{1+t^3}
\]

for \( j = 0, 1 \) and integrate by parts to show they are bounded in \( r \).

Now we can finally write an estimate from (2.10) as

\[
\begin{align*}
\left| \frac{\partial}{\partial \eta_1} \hat{g}^{o1} (\xi_1, \xi_2, \eta_1, -i\zeta) \right| & \lesssim \frac{1}{1+r^3}.
\end{align*}
\]

Since, \( \hat{g}^{o1} (\xi_1, \xi_2, 0, -i\zeta) = 0 \), we have

\[
\begin{align*}
|\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)| & = \left| \int_0^{\eta_1} \frac{\partial}{\partial s} \hat{g}^{o1}(\xi_1, \xi_2, s, -i\sqrt{\xi_1^2 + \xi_2^2 + s^2}) ds \right| \\
& \lesssim \left| \frac{1}{1+r^3} \int_0^{\eta_1} ds \right| \\
& = \frac{1}{1+r^3} |\eta_1|,
\end{align*}
\]

and

\[
\begin{align*}
\int_0^{2\pi} \int_0^{\pi} \int_0^\infty \left| \frac{\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{1 - \cos \phi} \right|^p \sin \phi \frac{d\phi d\theta}{r^{p-3}} & \lesssim \int_0^{2\pi} \int_0^{\pi} \int_0^\infty \sin^{p+1} \phi \frac{1}{(1 - \cos \phi)^p r^{p-3}(1+r^3)^p} dr d\phi d\theta.
\end{align*}
\]

Now it is easy to see the integral in (2.8) converges. The integral converges near \( r = 0 \) since \( p - 3 < 1 \). Also, the integral converges near \( r = \infty \) since the integrand decays as \( \frac{1}{r^{p-3}} \). Lastly,

\[
\frac{\sin^{p+1} \phi}{(1 - \cos \phi)^p}
\]

is integrable over \( \phi \in (0, \pi) \). Hence the second term in (2.4) is in \( L^p(\mathbb{R}^4) \) for \( p \in (1, 2) \).

For the first term, we look at

\[
\int_{\mathbb{R}^4} \left| \frac{\hat{g}^{o1}(\xi_1, \xi_2, \eta_1, \eta_2)}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right|^p d\xi d\eta.
\]

The integral converges near the origin for \( p < 2 \). Also, we can show the decay property, in a similar manner to the arguments above,

\[
|\hat{g}^{o1}(\xi_1, \xi_2, \eta_1)| \lesssim \left( \frac{1}{1+r} \right) \left( \frac{1}{1+|\eta_2|} \right).
\]
where \( r = \sqrt{\xi_1^2 + \xi_2^2 + \eta_1^2} \). Hence,

\[
\int_{R^+} \left| \hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, \eta_2) \right|^p \frac{d\xi d\eta}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \lesssim \left( 1 + \int_{r^2 + \eta_2^2 \geq 1} \frac{1}{(r^2 + \eta_2^2)^p} \frac{1}{(1 + |\eta_2|)^p} r^2 dr d\eta_2 \right) \\
\leq \left( 1 + \int_{r^2 + \eta_2^2 \geq 1} \frac{1}{(1 + |\eta_2|)^p} dr d\eta_2 \right) < \infty.
\]

Thus a) is proved.

We can also obtain corresponding estimates on the inverse transforms of the quantities in the proposition by using a theorem of Hausdorff and Young (see Theorem 7.1.13 in Hörmander [9]). For instance, we can conclude the inverse transform of \( \hat{u}^{\alpha_1} \) is in \( L^p \) for \( 2 < p < \infty \).

It is of interest to note that \( \hat{u}^{\alpha_1} \) is not in \( L^p \) for \( p = 1, 2 \). \( \hat{u}^{\alpha_1} \) fails to be in \( L^1 \) because the second term in equation (2.4) behaves as \( 1/\eta_2 \) for large \( \eta_2 \). And \( \hat{u}^{\alpha_1} \) fails to be in \( L^p \) for \( p = 2 \) when a change of variables is made to spherical coordinates as in the proof of a), a term

\[
\frac{\sin^{p+1} \phi}{(1 - \cos \phi)^p}
\]

arises from integrating the second term in equation (2.4) which does not converge for \( p = 2 \) when integrating over \( \phi \in (0, \pi) \).

We now verify that the function \( u \) in (2.4) is an actual solution. It is convenient to invert from \( \eta_2 \) to \( y_2 \), using the residue theorem on a half-plane. The calculations which follow are to be understood in the sense of distributions.

\[
\hat{u}^{\alpha_1}(\xi_1, \xi_2, \eta_1, y_2) = -\frac{1}{2\pi} \int_0^{y_2} \int_{-\infty}^{\infty} \hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, t) e^{i\eta_2(y_2-t)} d\eta_2 dt \\
- \frac{1}{2\pi} \int_{y_2}^{\infty} \int_{-\infty}^{\infty} \hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, t) e^{-i\eta_2(t-y_2)} d\eta_2 dt - \frac{1}{\zeta - \xi_2} \frac{\eta_2 \xi_2 + \zeta \hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{2i\zeta}.
\]

Using the boundary of a semi-circle in the upper half-plane as a contour for the first integral and the boundary of a semi-circle in the lower half-plane for the contour of the second we calculate

\[
(2.14) \quad \hat{u}^{\alpha_1}(\xi_1, \xi_2, \eta_1, y_2) = \\
- \frac{1}{2} \int_0^{y_2} \frac{\hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, t)}{\zeta} e^{-\zeta(y_2-t)} dt - \frac{1}{2} \int_{y_2}^{\infty} \frac{\hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, t)}{\zeta} e^{-\zeta(t-y_2)} dt \\
- e^{-\zeta y_2} \frac{\xi_2 + \zeta \hat{g}^{\alpha_1}(\xi_1, \xi_2, \eta_1, -i\zeta)}{\zeta - \xi_2}.
\]

Now take two \( y_2 \) derivatives of equation (2.14).
We obtain

\[
\Delta u^{1}\left(\xi_1, \xi_2, \eta_1, y_2\right) = \frac{\partial^2 \tilde{u}^{1}}{\partial y_2^2}\left(\xi_1, \xi_2, \eta_1, y_2\right) - \gamma \left(\xi_1, \xi_2, \eta_1, t\right) dt + \frac{1}{2} \gamma e^{-\gamma \left(y_2 - \zeta_2\right)} \gamma^{11} \left(\xi_1, \xi_2, \eta_1\right).
\]

We rewrite equation \(2.15\) as

\[
\frac{\partial^2 \tilde{u}^{1}}{\partial y_2^2}\left(\xi_1, \xi_2, \eta_1, y_2\right) = \tilde{g}^{11}\left(\xi_1, \xi_2, \eta_1, y_2\right) + \frac{1}{2} \gamma e^{-\gamma \left(y_2 - \zeta_2\right)} \gamma^{11} \left(\xi_1, \xi_2, \eta_1\right) + \sum_{i=1}^{2} \gamma \left(y_2 - \zeta_2\right) \gamma^{11} \left(\xi_1, \xi_2, \eta_1\right).
\]

Using the fact that, as a distribution, \(\frac{\partial u^{1}}{\partial y_2}\in L^p(\mathbb{R}^4)\) for \(p \in (2, 4)\) from Proposition \(2.1\) b), it follows

\[
(\Delta_{x,y_1} u)^{1} = \Delta_{x,y_1} u^{1}
\]

in the sense of distributions (see the proof of Lemma \(2.2\) below), where

\[
\Delta_{x,y_1} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_1^2}.
\]

Then, we have

\[
(\xi_1^2 + \xi_2^2 + \eta_1^2) \tilde{u}^{1}\left(\xi_1, \xi_2, \eta_1, y_2\right) = \left(\Delta_{x,y_1} u\right)^{1}
\]

in the sense of distributions.

Thus, in the sense of distributions,

\[
(2.16) \quad \Delta u^{1} = g^{11}.
\]

Furthermore, interior ellipticity of \(\Delta\) implies that \(2.16\) holds in the classical sense in \(\Omega\). Hence \(2.1\) is satisfied. We also show condition \(2.2\) is satisfied by \(u\) in the sense that

\[
\|u^{1}(\cdot, \cdot, y_1, \cdot)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \text{as} \quad y_1 \to 0
\]

for \(q \in (2, 4)\).

From Proposition \(2.1\) b) we have

\[
\frac{\partial u^{1}}{\partial y_1} \in L^q
\]

as a distribution for \(q \in (2, 4)\). Also,

\[
(2.17) \quad \phi(y_1) u^{1} = \phi(y_1) \int_0^{y_1} \frac{\partial u^{1}}{\partial t}(x_1, x_2, t, y_2) dt
\]

in \(L^q\) for \(\phi \in C_0^\infty(\mathbb{R})\) with support near \(y_1 = 0\), and such that \(\phi = 1\) for \(y_1\) sufficiently close to 0.
Hence, using Hölder’s inequality, we have for \( y_1 \) sufficiently close to 0, for almost all \((x_1, x_2, y_2)\),

\[
|u^{o_1}| \leq \int_0^{y_1} \left| \frac{\partial u^{o_1}}{\partial t} \right| dt \\
\leq \left\{ \int_0^\infty \left| \frac{\partial u^{o_1}}{\partial t} \right|^q dt \right\}^{1/q} |y_1|^{1/p},
\]

where \( p \) is conjugate to \( q \), which implies

\[
\int_{\mathbb{R}^3} |u^{o_1}|^q dx_1 dx_2 dy_2 \leq \left\| \frac{\partial u^{o_1}}{\partial y_1} \right\|_{L^q} |y_1|^{q/p}.
\]

From this we see \( u \) satisfies condition 2.2 in the sense specified.

That condition 2.3 is satisfied (again in a certain \( L^p \) sense) is best seen when we work with the function \( \frac{\partial u}{\partial y_2} \). We have, in the sense of distributions,

\[
\frac{\partial u}{\partial y_2}^{o_1} = (i\xi_2 - \eta_2)\hat{v}_1^{o_1} \\
= -\frac{(i\xi_2 - \eta_2)\hat{v}_1^{o_1}(\xi_1, \xi_2, \eta_1, \eta_2)}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} + i\frac{\xi_2^2 + \eta_2^2 \hat{g}^{o_1}(\xi_1, \xi_2, \eta_1, -i\xi_2)}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}.
\]

If, instead of extending functions to be 0 for \( y_2 < 0 \) we extend by odd reflections across \( y_2 = 0 \), denoting such functions with a superscript \( a12 \), we obtain

\[
\frac{\partial u}{\partial y_2}^{a12} = -\frac{(i\xi_2 - \eta_2)\hat{v}_1^{a12}(\xi_1, \xi_2, \eta_1, \eta_2)}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} + \frac{(i\xi_2 + \eta_2)\hat{g}^{a12}(\xi_1, \xi_2, \eta_1, -\eta_2)}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2},
\]

(2.18)

from which we can conclude that \( \frac{\partial u}{\partial y_2}^{a12} \in L^p(\mathbb{R}^4) \) for \( p \in (1, 2) \). It is easily verified that \( \eta_2 \frac{\partial u}{\partial y_2}^{a12} \in L^p(\mathbb{R}^4) \) for \( p \in (1, 2) \), and as above this leads to the condition that

\[
\left\| \frac{\partial u^{o_1}}{\partial y_2} \right\|_{L^q(\mathbb{R}^3)} (\cdot, \cdot, y_2) \to 0 \quad \text{as} \quad y_2 \to 0
\]

for \( q \in (2, \infty) \).

We now describe regularity properties of our solution. We study the regularity locally.

**Lemma 2.2.** Let \( u \) be the solution, given by (2.4), to equation 2.4 on \( \Omega \) with boundary conditions given by (2.3) and (2.4). Then \( u \) is smooth in any neighborhood, \( V \subset \overline{\Omega} \) not intersecting \( \{ y_1 = 0 \} \cap \{ y_2 = 0 \} \).

**Proof.** We consider the three cases:

Case 1: \( V \cap \partial \Omega = \emptyset \)
Case 2: \( V \cap \partial \Omega \neq \emptyset \) and \( V \subset \overline{\Omega} \cap \{ y_2 > 0 \} \)
Case 3: \( V \cap \partial \Omega \neq \emptyset \) and \( V \subset \overline{\Omega} \cap \{ y_1 > 0 \} \).

Regularity in case 1 follows from the fact that the Laplacian is a strongly elliptic operator.
In case 2 let \( z \in V \cap \partial \Omega \) and let \( V' \) be a bounded neighborhood of \( z \), symmetric about \( y_1 = 0 \). We have \( \Delta u = g \) in \( V' \cap \Omega \). Choose an integer \( m \geq 0 \) and find an \( u' \in C^\infty(V' \cap \Omega) \) such that \( \Delta u' \) and \( g \) agree to \( m^{th} \) order on \( V' \cap \partial \Omega \) and \( u' = 0 \) on \( V' \cap \partial \Omega \). Then with \( w = u - u' \), \( \Delta w \) vanishes to \( m^{th} \) order along \( V \cap \partial \Omega \). Let \( w^{o1} \) be the extension of \( w \) across \( \partial \Omega \) in \( V' \) such that \( w^{o1} \) is odd in \( y_1 \). Similarly \( (\Delta w)^{o1} \) is an extension odd in \( y_1 \). Let \( \frac{\partial u}{\partial y_1} \) be defined as above so that \( \frac{\partial u}{\partial y_1}^{o1} \in L^p(V') \) for \( p \in (2, 4) \). Again, by Friedrichs’ Lemma we may choose \( w_{\alpha} \in C_0^\infty(\mathbb{R}^4) \) to be a sequence of functions, odd in \( y_1 \), so that \( w_{\alpha} \to w^{o1} \) in \( L^p(V') \), and \( \frac{\partial w}{\partial y_1} \to \frac{\partial w}{\partial y_1}^{o1} \) in \( L^p(V') \).

Let \( \omega_{\alpha} = w_{\alpha}|_{y_1 > 0} \). Since \( w_{\alpha} \) is odd in \( y_1 \) and smooth, \( (\Delta \omega_{\alpha})^{o1} = (\Delta \omega_{\alpha})^{o1} = \Delta w_{\alpha} \). Then by passing to limits

\[
(\Delta w)^{o1} = \Delta w^{o1}
\]

in the sense of distributions. However, since \( (\Delta w)^{o1} \in C^m(V') \), equation (2.19) holds throughout \( V' \) in the classical sense. Thus \( \Delta w^{o1} \in C^m(V') \), and the strong ellipticity of \( \Delta \) implies \( w^{o1} \in C^{m+1}(V') \), and thus \( u \in C^{m+1}(V' \cap \Omega) \). Since this can be done for all \( m \in \mathbb{N} \), we see \( u \) is smooth in a neighborhood of \( z \), hence in all of \( V \).

For case 3 define \( v = \frac{\partial u}{\partial z_2} \) and consider the related problem

\[
\Delta v = \frac{\partial g}{\partial z_2}
\]

on \( \Omega \), with the conditions

\[
\begin{align*}
v & = 0 & \text{on} & & y_1 = 0; \\
v & = 0 & \text{on} & & y_2 = 0.
\end{align*}
\]

From (2.18), we have

\[
\dot{v}^{o12} = -\frac{\partial \dot{v}}{\partial z_2}^{o12} = \frac{\xi_1 + \xi_2 + \eta_1 + \eta_2}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2},
\]

and it is easy to see \( v \in L^p(\Omega) \) for \( p \in (1, \infty) \).

We also know, from case 2, that \( v \) is smooth on all neighborhoods not intersecting \( \{y_1 = 0\} \cap \{y_2 = 0\} \), hence in \( V \). Let \( z' = (z_1', z_2') \in V \cap \partial \Omega \). We will work in the neighborhood \( H_1 \times V_2 \), where \( V_2 \) is a bounded neighborhood of \( z_2' \) in \( \mathbb{H}_2 \) such that \( V_2 \cap \mathbb{H}_2 \) has smooth boundary. Let \( \chi \in C^\infty(\overline{V_2}) \) such that \( \chi \equiv 1 \) near \( z_2' \). Define

\[
u' = \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta_2) v(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2.
\]

\( u' \) has the properties \( \frac{\partial u'}{\partial z_2} = v \) near \( z' \) and \( u' \in C^\infty(\mathbb{H}_1 \times \overline{V_2}) \). Furthermore, since \( v(z_1, \cdot) \) is in \( L^p \) for \( p \in (2, \infty) \) in the second variable for almost all \( z_1 \), then for almost all \( z_1 \)

\[
|u'|^p \leq \frac{1}{(2\pi)^p} \left\{ \int_{V_2} |\chi(\zeta_2) v(z_1, \zeta_2)|^p |d\zeta_2 \wedge d\bar{\zeta}_2| \right\} \left\{ \int_{V_2} \frac{1}{|z_2 - z_2'|^q} |d\zeta_2 \wedge d\bar{\zeta}_2| \right\}^{p/q}.
\]
by Hölder's inequality, where $p$ is conjugate to $q$. Thus,

$$\int_{\mathbb{H}_1} |u'(\zeta_1, z_2)|^p |d\zeta_1 \wedge d\zeta_1| \lesssim \int_{\mathbb{H}_1 \times V_2} |\chi(z_2)v(z_1, z_2)|^p dx^4 < \infty,$$

and $u'(:, z_2)$ is $L^p$ in the first variable.

We compute $\Delta u'$ in $\mathbb{H}_1 \times V_2$. We will use the notation $\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$ and $\Delta_2 = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}$.

$$\Delta u' = \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta_2)\Delta_1 v(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 + \frac{1}{2\pi i} \Delta_2 \int_{V_2} \frac{\chi(\zeta_2)v(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2.$$

We extend $\chi$ and $v$ to be zero for $\zeta_2$ outside of $V_2$, and we denote the extended functions $\tilde{\chi}$ and $\tilde{v}$ respectively. Then

$$\Delta_2 \int_{V_2} \frac{\chi(\zeta_2)v(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 = \Delta_2 \int_{C} \frac{\hat{\chi}(\zeta_2)\hat{v}(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 \Delta_2 \int_{C} \frac{\hat{\chi}(\zeta_2 + z_2)\hat{v}(\zeta_1, \zeta_2 + z_2)}{\zeta_2} d\zeta_2 \wedge d\zeta_2.$$

This last integral is equal, in the sense of distributions, to

$$\int_{C} \frac{\hat{\Delta}_2 \left(\hat{\chi}(\zeta_2 + z_2)\hat{v}(\zeta_1, \zeta_2 + z_2)\right)}{\zeta_2} d\zeta_2 \wedge d\zeta_2.$$

If we set $\zeta_2 = s + it$ and let $\Delta'_2 = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$, the integral above can be written

$$\int_{C} \frac{\Delta'_2 \left(\hat{\chi}(\zeta_2 + z_2)\hat{v}(\zeta_1, \zeta_2 + z_2)\right)}{\zeta_2} d\zeta_2 \wedge d\zeta_2,$$

and changing variables once again gives

$$\int_{C} \frac{\Delta'_2 \left(\hat{\chi}(\zeta_2 + z_2)\hat{v}(\zeta_1, \zeta_2)\right)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2.$$

Because $\Delta'_2 \left(\hat{\chi}(\zeta_2)\hat{v}(\zeta_1, \zeta_2)\right)$ contributes a delta function at $\Im \zeta_2 = 0$, the last integral above equals

$$\int_{\mathbb{V}_2} \frac{\Delta'_2 \left(\hat{\chi}(\zeta_2)\hat{v}(\zeta_1, \zeta_2)\right)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 - \int_{\partial \mathbb{V}_2 \cap \{t=0\}} \frac{\partial}{\partial t} \left(\chi(s, 0)v(z_1, s, 0)\right) ds.$$

Use $\frac{\partial}{\partial t} \left(\chi(s, 0)v(z_1, s, 0)\right) = \chi(s, 0)\frac{\partial v(z_1, s, 0)}{\partial t}$ in the second integral to write the above expression as

$$\int_{\mathbb{V}_2} \frac{\Delta'_2 \left(\hat{\chi}(\zeta_2)\hat{v}(\zeta_1, \zeta_2)\right)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 - \int_{\partial \mathbb{V}_2 \cap \{t=0\}} \chi(s, 0)\frac{\partial v(z_1, s, 0)}{\partial t} ds.$$

Both integrals in (2.20) are in $C^\infty(\mathbb{H}_1 \times \mathbb{V}_2)$. We write the first integral as

$$\int_{\mathbb{V}_2} \frac{\chi(\zeta_2)\Delta_1 v(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2 + \int_{\mathbb{V}_2} \frac{(\Delta_2 \chi)v(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_2$$

$$+ \int_{\mathbb{V}_2} \frac{\partial \chi}{\partial \zeta_2} \frac{\partial v}{\partial \zeta_2} + \frac{\partial \chi}{\partial \zeta_2} \frac{\partial v}{\partial \zeta_2} d\zeta_2 \wedge d\zeta_2.$$
Therefore,

\[ \triangle u' = \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta) \triangle_1 v(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta) \triangle_1' v(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \phi(z_1, z_2), \]

where \( \phi(z_1, z_2) \) is given by

\[ \phi(z_1, z_2) = \frac{1}{2\pi i} \int_{V_2} \left( \frac{\triangle_1' \chi}{\zeta_2 - z_2} \right) d\zeta_2 \wedge d\bar{\zeta}_2 + \frac{1}{2\pi i} \int_{V_2} \left( \frac{\partial v}{\partial \zeta_2} + \frac{\partial v}{\partial \bar{\zeta}_2} \right) d\zeta_2 \wedge d\bar{\zeta}_2 \]

\[ - \frac{1}{2\pi i} \int_{\partial V_2 \cap \{t = 0\}} \chi(s, 0) \frac{\partial v(z_1, s, 0)}{s - z_2} ds. \]

The last integral in \( \phi \) was seen to be in \( \mathcal{C}^\infty(\mathbb{H}_1 \times \overline{V_2}) \). Because \( \chi(\zeta_2) \equiv 1 \) near \( z'_2 \), any derivative of \( \chi \) is 0 near \( z'_2 \), and we can differentiate under the first two integrals in \( \phi \) to conclude \( \phi \in \mathcal{C}^\infty(\mathbb{H}_1 \times \overline{V_2}) \). Next, using \( \triangle v = \frac{\partial v}{\partial \zeta_2}(z_1, \zeta_2) \), we have

\[ (2.21) \]

\[ \triangle u' = \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \phi(z_1, z_2), \]

which is also \( \mathcal{C}^\infty(\mathbb{H}_1 \times \overline{V_2}) \).

Set \( w = u - u' \). We will show \( w \in \mathcal{C}^\infty(\mathbb{H}_1 \times \overline{V_2}) \). For \( z_2 \) near \( z'_2 \), \( \frac{\partial w}{\partial \zeta_2} = 0 \), in which case

\[ (2.22) \]

\[ \triangle_1 w = \triangle w = g - \triangle u' = g - \frac{1}{2\pi i} \int_{V_2} \frac{\chi(\zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \phi(z_1, z_2). \]

We also have the boundary condition \( w = 0 \) when \( y_1 = 0 \). Hence \( w \) is a solution to a Dirichlet problem on a half-plane. We claim

\[ (2.23) \]

\[ w = \int_{H_1} G_1(z_1, \zeta_1) \Phi(\zeta_1, z_2) d\zeta_1 \wedge d\bar{\zeta}_1, \]

where \( G_1 \) is the Green’s function for \( H_1 \),

\[ G_1 = \frac{1}{4\pi} \log(|z_1 - \zeta_1|) - \frac{1}{4\pi} \log(|\bar{z}_1 - \zeta_1|) \]

and \( \Phi \) is defined to be the right hand side of (2.22). To prove the claim we shall use the \( L^p \) estimate on \( \Phi \), which we shall prove later,

**Lemma 2.3.** Let \( \Phi(z_1, z_2) \) be defined to be the right hand side of (2.22). Then for \( z_2 \) fixed, \( \Phi(\cdot, z_2) \), as a function of \( z_1 \), has the property

\[ (1 + |z_1|) \Phi(z_1, z_2) \in L^p(\mathbb{H}_1) \]

for \( p \in (4/3, 3) \).

Let

\[ b(z_1, z_2) = \int_{H_1} G_1(z_1, \zeta_1) \Phi(\zeta_1, z_2) d\zeta_1 \wedge d\bar{\zeta}_1. \]
First, $b(z_1, z_2)$ is well defined. In fact, if we choose $\alpha \in (0, 1)$ and $p \in (4/3, 3)$ such that $\alpha p' > 2$, where $p'$ is conjugate to $p$,  

$$|b(z_1, z_2)| \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\log |z_1 - \zeta_1|}{(1 + |\zeta_1|)^\alpha} (1 + |\zeta_1|)\alpha |\Phi^{\alpha}(\zeta_1, z_2)||d\zeta_1 \wedge d\zeta_1|$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\log |\zeta_1|}{(1 + |z_1 - \zeta_1|)^\alpha} (1 + |z_1 - \zeta_1|)\alpha |\Phi^{\alpha}(z_1 - \zeta_1, z_2)||d\zeta_1 \wedge d\zeta_1|$$

$$\leq \frac{1}{4\pi} (1 + |z_1|)^\alpha \int_{\mathbb{R}^2} \frac{\log |\zeta_1|}{(1 + |\zeta_1|)^\alpha} (1 + |z_1 - \zeta_1|)\alpha |\Phi^{\alpha}(z_1 - \zeta_1, z_2)||d\zeta_1 \wedge d\zeta_1|$$

$$\leq \frac{1}{4\pi} (1 + |z_1|)^\alpha \times$$

$$\left\{ \int_{\mathbb{R}^2} \frac{\log |\zeta_1|}{(1 + |\zeta_1|)^\alpha} |d\zeta_1 \wedge d\zeta_1| \right\}^{1/p'} \left\{ \int_{\mathbb{R}^2} (1 + |\zeta_1|)^{\alpha p'} |\Phi^{\alpha p'}(\zeta_1)|^{p'}|d\zeta_1 \wedge d\zeta_1| \right\}^{1/p}$$

$$\leq (1 + |z_1|)^\alpha \times$$

$$\left\{ \int_{\mathbb{R}^2} \frac{\log |\zeta_1|}{(1 + |\zeta_1|)^\alpha} |d\zeta_1 \wedge d\zeta_1| \right\}^{1/p'} \left\{ \int_{\mathbb{R}^2} (1 + |\zeta_1|)^{p} |\Phi^{\alpha p}(\zeta_1)|^{p}|d\zeta_1 \wedge d\zeta_1| \right\}^{1/p}$$

$$< (1 + |z_1|)^\alpha c_{z_2},$$

where $c_{z_2}$ is a constant depending only on $z_2$. In the third step we use  

$$\frac{1}{1 + (1 + |z_1 - \zeta_1|)^\alpha} \leq \frac{(1 + |z_1|)^\alpha}{(1 + |\zeta_1|)^\alpha}$$

(see A.1 in Folland and Kohn [3]).

Fixing $z_2$ near $z'_2$ we must have $w = b + h$, where $h(z_1, z_2)$ is harmonic in $z_1$. Since $h = 0$ on $y_1 = 0$ we may extend $h$ to be odd in $y_1$, and obtain $h^{\alpha}$ is harmonic, for fixed $z_2$ near $z'_2$, on $\mathbb{H}_1$. Hence, $h^{\alpha}$ has the mean value property. Denoting the disc of radius $r$ centered about $z_1$ in $\mathbb{C}$ by $\mathbb{D}_r(z_1)$, we write  

$$|h^{\alpha}(z_1, z_2)| = \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_1)} w^{\alpha}(\zeta_1, z_2)d\zeta_1 \wedge d\zeta_1 - \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_1)} b^{\alpha}(\zeta_1, z_2)d\zeta_1 \wedge d\zeta_1$$

$$\leq \frac{1}{\pi r^2} \left\{ \int_{\mathbb{D}_r(z_1)} |w^{\alpha}(\zeta_1, z_2)|^{p}|d\zeta_1 \wedge d\zeta_1| \right\}^{1/p} \left\{ \int_{\mathbb{D}_r(z_1)} |d\zeta_1 \wedge d\zeta_1| \right\}^{1/q}$$

$$+ \frac{c_{z_2}}{\pi r^2} \int_{|\zeta| < r} (1 + |z_1 + \zeta|)^\alpha |d\zeta \wedge d\zeta|,$$

where we use Hölder’s inequality in the first integral with $p \in (2, \infty)$ and $q \in (1, 2)$ its conjugate exponent. Note that since $w \in L^p(\mathbb{H}_1)$ as a function of $z_1$ for almost all $z_2$, we can choose arbitrarily large $|z_1|$ to obtain for fixed $z_2$ and some constants $A, B < \infty$, $|h(z_1, z_2)| \leq A + B|z_1|^\alpha$. Then a Phragmén-Lindelöf theorem (see Theorem 2.3.7 in [2]) shows $|h(z_1, z_2)| = 0$. Hence,  

$$w = \int_{\mathbb{H}_1} G_1(z_1, \zeta_1) \Phi(\zeta_1, z_2)d\zeta_1 \wedge d\zeta_1$$

as claimed. Because $\Phi \in C^\infty(\mathbb{H}_1 \times \mathbb{V}_1)$ so is $w$, and $u \in C^\infty(\mathbb{H}_1 \times \mathbb{V}_2)$ follows from the fact that $u' \in C^\infty(\mathbb{H}_1 \times \mathbb{V}_2)$. 
This proves \( u \) is smooth in a neighborhood of \( z' \) and thus in all of \( V \).

**Proof of Lemma 2.3.** From above, we know \( v(z_1, z_2) \in L^p(\Omega) \) for \( p \in (1, \infty) \). With \( \phi(z_1, z_2) \) defined in Lemma 2.2, we can, holding \( z_2 \) fixed, show \( \phi(\cdot, z_2) \), as a function of \( z_1 \), is in \( L^p(\mathbb{H}_1) \) for \( p \in (1, 2) \). \( \phi \) is defined by

\[
\phi(z_1, z_2) = \frac{1}{2\pi i} \int_{\mathbb{H}_1} \left( \frac{\Delta_2^\prime \chi}{\zeta_2 - z_2} \right) d\zeta_2 \wedge d\tilde{\zeta}_2 + \frac{1}{2\pi i} \int_{V_2} \frac{\partial \chi}{\partial \zeta_2} \frac{\partial v}{\partial \zeta_2} d\zeta_2 \wedge d\tilde{\zeta}_2
\]

That the first integral is in \( L^p \) for \( p \in (1, 3) \) in the \( z_1 \) variable follows in the same way we proved \( u' \) was an \( L^p \) function of its first variable above. In fact,

\[
\int_{\mathbb{H}_1} \left| \int_{V_2} \left( \frac{\Delta_2^\prime \chi}{\zeta_2 - z_2} \right) d\zeta_2 \wedge d\tilde{\zeta}_2 \right|^p dx^2 \leq \int_{\mathbb{H}_1 \times V_2} |v(z_1, z_2)|^p d\xi^4 \left\{ \int_{V_2} \left| \frac{\partial \chi}{\partial \zeta_2} \left( \frac{\partial v}{\partial \zeta_2} \right) \right|^q \right\}^{p/q},
\]

where \( q \) is conjugate to \( p \).

It is routine to show \( \frac{\partial \phi}{\partial z_2} \in L^p(\mathbb{H}_2) \) and \( \frac{\partial v}{\partial \eta_2} \in L^p(\mathbb{H}_2) \) in the second variable for almost all \( z_1 \), for \( p \in (1, \infty) \), and then second integral is in \( L^p \) for \( p \in (1, 3) \) in the \( z_1 \) variable in the same way the first integral is.

For the third integral we use

\[
(2.24) \quad \frac{\partial \tilde{v}^{\omega_{1}}}{\partial t} (\xi_1, \xi_2, \eta_1, 0) = \frac{\partial \phi}{\partial \zeta_2} (\xi_1, \xi_2, \eta_1, -ir)
\]

in the sense of distributions, where \( r = \sqrt{\xi_1^2 + \xi_2^2 + \eta_1^2} \), and at which can be arrived by taking the Fourier transform of

\[
\Delta \tilde{v}^{\omega_{1}} = \frac{\partial \phi}{\partial \zeta_2}
\]

and setting \( \eta_2 \), the transform variable corresponding to \( y_2 \), to \( \eta_2 = -ir \). Then, we can estimate

\[
\frac{\partial \tilde{g}}{\partial \xi_2} (\xi_1, \xi_2, \eta_1, -ir)
\]

using similar methods as those applied in Proposition 2.1 to show

\[
\frac{\partial \tilde{v}^{\omega_{1}}}{\partial t} (\xi_1, \xi_2, \eta_1, 0) \in L^q(\mathbb{R}^3)
\]

for \( q \in (3/2, \infty) \), which gives for almost all \( z_1 \)

\[
\frac{\partial v}{\partial t} (z_1, \cdot, 0) \in L^p(\mathbb{R})
\]

as a function of \( s \) for \( p \in (1, 3) \). Then, that the third integral in the expression for \( \phi \) is in \( L^p(\mathbb{H}_1) \) for \( p \in (1, 3) \) follows in the same manner as the first two.

The first two terms in

\[
\Phi(z_1, z_2) = \int_{V_2} \chi (\zeta_2) \frac{\partial \phi}{\partial \zeta_2} (z_1, \zeta_2) d\zeta_2 \wedge d\tilde{\zeta}_2 - \phi(z_1, z_2)
\]
are also in $L^p$ for $p \in (1, \infty)$ as functions of $z_1$ with $z_2$ held fixed. Hence, $\Phi(\cdot, z_2) \in L^p(\mathbb{D}_1)$ for $p \in (1, 3)$.

Similar arguments show $|z_1|\Phi(z_1, z_2)$ is in $L^p(\mathbb{R}_1)$ as a function of $z_1$ for $p \in (4/3, 3)$, and the lemma is proved.

We can also prove the uniqueness of our solution in a suitable sense. We state the

**Proposition 2.4.** The inverse Fourier transform of the function

$$
-\hat{\eta}^{\alpha_1}(\xi_1, \xi_2, \eta_1, \eta_2) - \eta_2 - \xi_2 \hat{\eta}^{\alpha_1}(\xi_1, \xi_2, \eta_1, -i\zeta) + i\eta_2 - \xi_2 \hat{\eta}^{\alpha_1}(\xi_1, \xi_2, \eta_1, -i\zeta) = \zeta - \xi_2 \xi_1^2 + \eta_1^2 + \eta_2^2
$$

is the unique $L^p$ solution, extended to be odd in $y_1$, in the sense of distributions for $p \in (2, \infty)$, whose derivative with respect to $z_2$ is also in $L^p$, to equation \(2.4\) on $\Omega$ with boundary conditions given by \(2.2\) and \(2.3\), satisfied in the $L^p$ sense described above.

**Proof.** We can use the proof of Lemma 2.2 to see any solution must be smooth away from the corner. Let $u$ and $u'$ be two functions which solve the problem and exhibit the properties in the proposition. Set

$$
v = \frac{\partial u}{\partial z_2},
$$

$$v' = \frac{\partial u'}{\partial z_2}.
$$

Extend $v - v'$ to be odd in $y_1$ and $y_2$. Then $\Delta(v - v')^{12} = 0$ on $\mathbb{R}^4$, and $(v - v')^{12} \in L^p(\mathbb{R}^4)$ for $p \in (2, \infty)$. From the mean value property of harmonic functions, we can conclude $v = v'$.

$$
(v - v')^{12} = \frac{1}{r^4\omega_4} \int_{D_r} (v - v')^{12} d\mathbf{x}^4
$$

$$\lesssim \frac{1}{r^4}\|v - v'\|_{L^p(\mathbb{R}^4)}^{4/q},
$$

where $\omega_4$ denotes the volume of the unit ball in $\mathbb{R}^4$, $D_r$ the ball in $\mathbb{R}^4$ of radius $r$ centered around $z = (z_1, z_2)$, and $q \in (1, 2)$ is conjugate to $p$. Letting $r$ approach infinity, we see we must have $v = v'$. Thus,

$$
\frac{\partial u}{\partial z_2}^{\alpha_1} = \frac{\partial u'}{\partial z_2}^{\alpha_1}
$$

so $(u - u')^{\alpha_1}$ is holomorphic in $z_2$. Thus $\Delta_2(u - u')^{\alpha_1} = 0$ so $\Delta_1(u - u')^{\alpha_1} = 0$ and $(u - u')^{\alpha_1}$ is harmonic in $z_1$. Since, for almost all $z_2$, $(u - u')^{\alpha_1} \in L^p(\mathbb{R}^2)$ as a function of $z_1$ with $z_2$ held fixed

$$
(u - u')(z_1, z_2) = \frac{1}{\pi r^2} \int_{D_r(z_1)} (u - u')^{\alpha_1}(\zeta_1, z_2) d\zeta_1 \wedge d\bar{\zeta}_1
$$

$$\lesssim \frac{1}{r^2}(u - u')(\cdot, z_2)\|_{L^p(\mathbb{R})}^{2/q},
$$

where $q$ is conjugate to $p$. We can take $r$ to be arbitrarily large, and we see $u - u' = 0$ for almost all $z_2$. Since $u - u' \in C^\infty(\Omega)$ we have $u \equiv u'$.

Henceforth, when we speak of “the” solution to the $\partial$-Neumann problem on $(0, 1)$-forms on the domain $\Omega$ we mean the unique solution as described by Proposition 2.4.
In this section we prove the

**Main Theorem 1.** Let \( \Omega \subset \mathbb{C}^2 \) be the domain \( \mathbb{H}_1 \times \mathbb{H}_2 \), where \( \mathbb{H}_j \) is the half-plane \( \{ (x_j, y_j) : y_j > 0 \} \) for \( j = 1, 2 \). Let \( f = f_1 dz_1 + f_2 dz_2 \), where \( z_j = x_j + iy_j \), be a \((0,1)\)-form such that \( f \in \mathcal{S}_{(0,1)}(\Omega) \), the family of \((0,1)\)-forms whose coefficients are Schwartz functions on \( \Omega \), and \( u = u_1 dz_1 + u_2 dz_2 \) the \((0,1)\)-form which solves the \( \bar{\partial}\)-Neumann problem on \( \Omega \) with data the \((0,1)\)-form \( f \) on \( \Omega \). Then \( u_j \) can be written as

\[
u_j = \alpha_j \log(y_1^2 + y_2^2) + \beta_j + \gamma_j \arctan \left( \frac{y_1}{y_2} \right), \quad j = 1, 2,
\]

where \( \alpha_j, \beta_j, \gamma_j \) are smooth functions of \( x_1, x_2, y_1, y_2 \).

Main Theorem 1 will be a consequence of

**Theorem 3.1.** Let \( \Omega \subset \mathbb{C}^2 \) be the domain \( \mathbb{H}_1 \times \mathbb{H}_2 \) and let \( \rho \) be a defining function for \( \Omega \). Let \( f = f_1 dz_1 + f_2 dz_2 \), where \( z_j = x_j + iy_j \), be a \((0,1)\)-form such that \( f \in \mathcal{S}_{(0,1)}(\Omega) \) and \( u = u_1 dz_1 + u_2 dz_2 \) be the \((0,1)\)-form which solves the \( \bar{\partial}\)-Neumann problem on \( \Omega \) with data the \((0,1)\)-form \( f \). Then \( \forall n \in \mathbb{N} \) \( u_j \) can be written near \( y_1 = y_2 = 0 \) as

\[
u_j = \alpha_n(y_1, y_2) \log(y_1^2 + y_2^2) + \beta_n(y_1, y_2) + \gamma_n(y_1, y_2) \arctan \left( \frac{y_1}{y_2} \right) + R_n,
\]

where \( \alpha_n, \beta_n, \gamma_n \) are polynomials of degree \( n \) in \( y_1 \) and \( y_2 \), and whose coefficients are smooth functions of \( x_1 \) and \( x_2 \), and \( R_n \) is a remainder term such that \( R_n \in C^n(\Omega) \).

We shall also determine a sufficient condition for which \( u_j \in C^n(\Omega) \), that is, for which \( \alpha_n \) and \( \gamma_n \) vanish to \( n \)th order at \( y_1 = y_2 = 0 \).

We will prove the theorem for \( u_1 \) using the formula obtained in Section 2, the proof for \( u_2 \) being identical. From Section 2 we know

\[
u_1^{\text{ol}} = 2 \frac{\bar{\partial}_1 f_1^{\text{ol}}(\xi_1, \eta_1, \eta_2)}{\xi_1^2 + \eta_1^2 + \eta_2^2} + 2 \frac{\bar{\partial}_2 f_1^{\text{ol}}(\xi_1, \eta_1, -i\eta_2)}{\xi - \xi_2 \left( \xi_1^2 + \eta_1^2 + \eta_2^2 \right)}.
\]

Also, from (2.18) we know

\[
\frac{\bar{\partial} u_1^{\text{ol}}}{\bar{\partial} \zeta_2} = 2 \frac{\bar{\partial}_1 f_1^{\text{ol}}}{\xi_1^2 + \eta_1^2 + \eta_2^2}.
\]

We are only interested in the singular terms in our solution. Hence we rewrite equation (3.1) to separate terms which are transforms of functions in \( C^\infty(\mathbb{R}^4) \). Define \( \chi_\eta(\eta_1, \eta_2) \) to be a smooth function of \( \eta_1 \) and \( \eta_2 \), even in both variables, with the property \( \chi_\eta = 1 \) for \( \eta_1^2 + \eta_2^2 < a \) and \( \chi_\eta = 0 \) for \( \eta_1^2 + \eta_2^2 > b \) for some \( b > a > 0 \). We write

\[
\frac{\bar{\partial} u_1^{\text{ol}}}{\bar{\partial} \zeta_2} = 2 \frac{\chi'_\eta \bar{\partial}_1 f_1^{\text{ol}}}{\xi_1^2 + \eta_1^2 + \eta_2^2} + 2 \frac{\chi_\eta \bar{\partial}_2 f_1^{\text{ol}}}{\xi_1^2 + \eta_1^2 + \eta_2^2},
\]

where \( \chi'_\eta(\eta_1, \eta_2) = (1 - \chi_\eta) \). It is easy to see the Fourier inverse of the second term in (3.2) is \( C^\infty(\mathbb{R}^4) \). The term is \( L^1 \) and so is

\[
2\epsilon_1 \epsilon_2 \eta_1 \eta_2 \bar{\partial}_1 f_1^{\text{ol}} \frac{\chi_\eta}{\xi_1^2 + \eta_1^2 + \eta_2^2}.
\]
whenever \( j, k, l, \) and \( m \) are in \( \mathbb{N} = \{1, 2, 3 \ldots \} \). Hence the singularities in \( \frac{\partial u}{\partial \xi} \) come from the first term of (3.2).

We intend to examine an expansion of \( \frac{\partial u}{\partial \xi} \) for large \( \eta_1 \) and \( \eta_2 \). We first define the equivalence we work with.

**Definition 3.2.** We say \( \hat{h}_1(\xi_1, \xi_2, \eta_1, \eta_2) \) is equivalent to \( \hat{h}_2(\xi_1, \xi_2, \eta_1, \eta_2) \), or \( \hat{h}_1 \sim \hat{h}_2 \), if

\[
|h_1 - h_2|_\Omega \in C^\infty(\overline{\Omega}).
\]

With slight abuse of notation we will also use the symbol \( \sim \) to define an equivalence between two functions defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \), that is, we also write \( \hat{h}_1(\eta_1, \eta_2) \sim \hat{h}_2(\eta_1, \eta_2) \) if \( h_1(y_1, y_2) - h_2(y_1, y_2)|_{\mathbb{R}_+ \times \mathbb{R}_+} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+) \).

From above,

\[
\frac{\partial u}{\partial \xi_1} \sim 2 \frac{\chi_\eta \frac{\partial \xi}{\partial \eta_1} \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}.
\]

We can simplify future calculations if we work with another equivalent form of \( \frac{\partial u}{\partial \xi_1} \). To define the equivalent function, define \( \chi_n(\eta_1) \) to be a smooth, even function of \( \eta_1 \) with the property \( \chi_n = 1 \) for \( |\eta_1| < a \) and \( \chi_n = 0 \) for \( |\eta_1| \geq b \) for some \( b > a > 0 \), and define \( \chi_\eta = 1 - \chi_n \). In the same manner define \( \chi_{\eta_2}(\eta_2) \) to be smooth and even in \( \eta_2 \) with the property \( \chi_{\eta_2} = 1 \) for \( |\eta_2| < a \) and \( \chi_{\eta_2} = 0 \) for \( |\eta_2| \geq b \), and define \( \chi_{\eta_2}' = 1 - \chi_{\eta_2} \). Now consider

(3.3)

\[
\frac{\chi_\eta \frac{\partial \xi}{\partial \eta_1} \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} - \frac{\chi_\eta \chi_{\eta_2}' \frac{\partial \xi}{\partial \eta_2} \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} = \frac{(\chi_\eta - \chi_{\eta_2} \frac{\partial \xi}{\partial \eta_2}) \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} + \frac{\chi_{\eta_2}(1 - \chi_{\eta_2} \frac{\partial \xi}{\partial \eta_2}) \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}.
\]

We show both terms on the right hand side of (3.3) are Fourier transforms of functions whose restriction to \( \Omega \) are \( C^\infty(\overline{\Omega}) \). From above, we know

\[
\frac{\chi_\eta \frac{\partial \xi}{\partial \eta_1} \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}
\]

is the transform of a function which is in \( C^\infty(\mathbb{R}^4) \), and we show

\[
\frac{\chi_{\eta_2}(1 - \chi_{\eta_2} \frac{\partial \xi}{\partial \eta_2}) \chi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}
\]

is the transform of a function whose restriction to \( \Omega \) is in \( C^\infty(\overline{\Omega}) \). We shall use here and in the future the

**Lemma 3.3.** Let \( \mathcal{G} \in C^\infty(\overline{\Omega}) \). If \( \mathcal{G} \) has a \( C^n \) extension to \( \mathbb{R}^2 \times \mathbb{R}^2 \) which is odd in \( y_1 \), and if \( \mathcal{H} \) is a solution to

\[
\Delta \mathcal{H} = \mathcal{G} \quad \text{on} \quad \Omega;
\]

\[
\mathcal{H} = 0 \quad \text{on} \quad y_1 = 0;
\]

\[
\mathcal{H} = 0 \quad \text{on} \quad y_2 = 0,
\]

then \( \mathcal{H} \in C^n(\overline{\Omega}) \). Similarly, if \( \mathcal{G} \) has a \( C^n \) extension to \( \mathbb{R}^1 \times \mathbb{R}^2 \) odd in \( y_2 \), then \( \mathcal{H} \in C^n(\overline{\Omega}) \).
Proof. In the first case, $\mathcal{H}^{o1}$ is the solution to a Dirichlet problem on the half-space $\mathbb{R}^2 \times \mathbb{H}_2$ with data which is $C^n$ up to the boundary, and as such, is itself $C^n$ on $\mathbb{R}^2 \times \mathbb{H}_2$. Hence, $\mathcal{H} \in C^n(\Omega)$. The second case is handled in a similar manner.

The terms we will be dealing with are in Fourier transform space and thus we will also need the Lemma 3.4.

**Lemma 3.4.** Let

$$\hat{\mathcal{H}}(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{\hat{\mathcal{G}}}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2},$$

where $\hat{\mathcal{G}}$ is odd in $\eta_1$ and $\eta_2$, and is the transform of a function, $\mathcal{G}$, which, when restricted to $\Omega$, is in $C^\infty(\tilde{\Omega})$. Also suppose that $\eta_1^i \hat{\mathcal{H}} \in L^p(\mathbb{R}^4)$ for $p \in (1, \infty)$ and for $i = 1, 2$ and $j = 0, 1$. Then $\mathcal{H}$, the inverse transform of $\hat{\mathcal{H}}$, solves

$$\Delta \mathcal{H} = \mathcal{G} \quad \text{on} \quad \Omega;$$

$$\mathcal{H} = 0 \quad \text{on} \quad y_1 = 0;$$

$$\mathcal{H} = 0 \quad \text{on} \quad y_2 = 0.$$

**Proof.** As in Section 2, $\Delta \mathcal{H} = \mathcal{G}$ in the sense of distributions in the interior of $\Omega$, and since $\mathcal{G}$ is $C^\infty$ in the interior the equality holds in the classical sense in the interior. The conditions $\eta_1^i \hat{\mathcal{H}} \in L^p(\mathbb{R}^4)$ for $p \in (1, \infty)$ and for $i = 1, 2$ and $j = 0, 1$ allow us to conclude as in Section 2 that the boundary values are obtained in the sense that

$$\|\mathcal{H}(\cdot, \cdot, y_1, \cdot)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \text{as} \quad y_1 \to 0$$

for $q \in (1, \infty)$ with a similar limit as $y_2 \to 0$. Combining this with the arguments in Lemma 2.2 we see the boundary values are also obtained in the classical sense.

We define

$$\mathcal{G} = F.T.^{-1} \left( \chi \frac{\partial f_1}{\partial \bar{z}_2} \right),$$

$$= (\delta(x_1)\delta(x_2)\delta(y_2)s_1) * \frac{\partial f_1}{\partial \bar{z}_2}$$

$$= \int_{-\infty}^{\infty} \frac{\partial f_1}{\partial \bar{z}_2} (x_1, x_2, t, y_2)s_1(t - y_1)dt,$$

where $F.T.$ stands for the full Fourier transform in $\mathbb{R}^4$ and $s_1$ is a Schwartz function of $y_1$. By differentiating under the integral in (3.4) we see the integral in (3.4) is smooth up to the boundary on $\mathbb{R}^2 \times \mathbb{H}_2$. Hence, $\mathcal{G}$ is an odd function of $y_1$ and $y_2$, and is smooth on $\mathbb{R}^2 \times \mathbb{H}_2$. With

$$\hat{\mathcal{H}} = \frac{\hat{\mathcal{G}}}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}$$

$$= \frac{\chi \eta_1 \delta \frac{\partial f_1}{\partial \bar{z}_2}^{o12}}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2},$$
we have $\eta_j^k \in L^p(\mathbb{R}^4)$ for $p \in (1, \infty)$ and for $i = 1, 2$ and $j = 0, 1$. Thus, combining Lemmas 3.3 and 3.4, we conclude

$$\chi_{\eta_1^j \eta_2^k} \frac{\partial^{o2}_f}{\partial \xi_1^i \partial \xi_2^j}$$

is the Fourier transform of a function whose restriction to $\Omega$ is $C^\infty(\Omega)$. We have shown the first term on the right hand side of (3.3) is the Fourier transform of a function whose restriction to $\Omega$ is $C^\infty(\Omega)$. The second term is handled in the same way. Hence,

$$\frac{\partial^{o2}_{f_1}}{\partial \xi_2} \sim 2 \frac{\chi_{\eta_1^j \eta_2^k} \frac{\partial^{o2}_{f_1}}{\partial \xi_1^i \partial \xi_2^j}}{\xi_1^i + \xi_2^j + \eta_1^i + \eta_2^j}$$

(3.5)

Since $f_1 \in \mathcal{S}(\Omega)$, $\frac{\partial f_i}{\partial y_2}$ must also be in $\mathcal{S}(\Omega)$. To determine the form of the expansion of $\frac{\partial^{o2}_{f_1}}{\partial \xi_2}$ we use the following lemma.

**Lemma 3.5.** Let $h(y_1, y_2) \in \mathcal{S}([0, \infty) \times [0, \infty))$ and define $h^{o2}$ on all of $\mathbb{R}^2$ by odd extensions of $h$ in both variables. Then $\forall n \in \mathbb{N}$ we can write

$$\hat{h}^{o2}(\eta_1, \eta_2) = \sum_{j+k \leq n} \frac{c_{j,k}}{\eta_1^j \eta_2^k} + R_n,$$

where the $c_{j,k}$ are constants and

$$R_n = \sum_{j=1}^{n} \frac{\nu_j(\eta_1, \eta_2)}{\eta_1^{3(j-1)} \eta_2^{2n-2(j+1)}} + \frac{\nu_n(\eta_1, \eta_2)}{\eta_2^{2n}},$$

where the $\nu_j$ are smooth, bounded functions of $\eta_1$ and $\eta_2$ and have the property that $\chi_{\eta_1^j \eta_2^k} R_n$ is the Fourier transform of a function whose restriction to $\mathbb{R}_+ \times \mathbb{R}_+$ is $C^\infty$ up to the boundary.

**Proof.** Upon integration by parts, we see (3.6) holds with

$$R_n = -\frac{2i}{\eta_1 \eta_2} F.T. \left( \frac{\partial^{2n} h}{\partial y_2^{2n}} \right)(0, y_2) - \frac{2i}{\eta_1 \eta_2} F.T. \left( \frac{\partial^{2n} h}{\partial y_2^{2n-1}} \right)(0, y_2)$$

$$\cdots - \frac{2i}{\eta_1 \eta_2} F.T. \left( \frac{\partial^{2n} h}{\partial y_2^{2n-2}} \right)(0, y_2) + \frac{1}{\eta_1 \eta_2} \hat{\Theta}^{o2}(\eta_2)$$

(3.7)

where $F.T.$ is the Fourier transform with respect to only the $y_2$ variable. To conclude the lemma we show each term in (3.7), when multiplied by $\chi_{\eta_1^j \eta_2^k}$, is the Fourier transform of a function whose restriction to $\mathbb{R}_+ \times \mathbb{R}_+$ is $C^\infty$ up to the boundary. We first consider the first $n$ terms in (3.7). Each term is of the form

$$\frac{1}{\eta_1^{2j-1} \eta_2^{2n-2j+2}} \hat{\Theta}^{o2}(\eta_2) \quad 1 \leq j \leq n,$$

where $\Theta(y_2) \in \mathcal{S}(\mathbb{R}_+)$ and is extended to all of $\mathbb{R}$ by an odd reflection. We have

$$\chi_{\eta_1^j \eta_2^k} = (1 - \chi_{\eta_1^j}) F.T. \left( y_1^{2j-2} \sigma(y_1) \right) + s_{n_1},$$

where
where \( \sigma(y_1) \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^+) \) and is such that \( \sigma = 1 \) when \( y_1 < a \) for some \( a > 0 \), and \( s_{\eta_1} \) is used to denote any Schwartz function of \( \eta_1 \). Thus

\[
\frac{\chi_{\eta_1}}{\eta_1^{\sigma_1}} F.T. \left( y_1^{2j-2} \sigma(y_1) \right)^{\sigma_1} + c_{\eta_1},
\]

where \( c_{\eta_1} \) is used to denote the Fourier transform of any function which is in \( C^\infty(\mathbb{R}) \).

Also, with

\[
\Phi(y_2) = F.T._2^{-1} \left( \frac{\chi_{\eta_2}}{\eta_2^{\sigma_2}} \Theta(y_2)^{\sigma_2} \right),
\]

we have

\[
\partial_{y_2}^{2n-2j+2} \Phi = i^{2n-2j+2} F.T._2^{-1} \left( \chi_{\eta_2}' \Theta(y_2)^{\sigma_2} \right) \in C^\infty(\mathbb{R}^+)
\]

when restricted to \( y_2 > 0 \) since \( \Theta(y_2) \in \mathcal{S}(\mathbb{R}^+) \). By inverting the \( y_2 \) derivatives, we can conclude that the restriction of \( \Phi \) to \( y_2 > 0 \) is in \( C^\infty(\mathbb{R}^+) \).

This shows the first \( n \) terms in (3.7), when multiplied by \( \chi_{\eta_1}' \chi_{\eta_2}' \), can be written as Fourier transforms of functions of \( y_1 \) which are in \( C^\infty(\mathbb{R}^+) \) multiplied by functions of \( y_2 \) which are in \( C^\infty(\mathbb{R}^+) \).

Now we treat \( \chi_{\eta_1}' \chi_{\eta_2}' \) multiplied by the last term in (3.7), which is of the form

\[
\frac{\chi_{\eta_1}' \chi_{\eta_2}' \Theta_1^{\sigma_1} \Theta_2^{\sigma_2}}{\eta_1^{\sigma_1} \eta_2^{\sigma_2}}
\]

where \( \Theta_1 \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^+) \). We write from above,

\[
(3.8) \quad \frac{\chi_{\eta_1}'}{\eta_1^{\sigma_1}} = F.T. \left( y_1^{2n} \sigma(y_1) \right)^{\sigma_1} + c_{\eta_1},
\]

Hence,

\[
\frac{\chi_{\eta_1}' \chi_{\eta_2}' \Theta_1^{\sigma_1} \Theta_2^{\sigma_2}}{\eta_1^{\sigma_1} \eta_2^{\sigma_2}} = (1 - \chi_{\eta_2} F.T. \left( y_1^{2n-1} \sigma^1(y_1) *_1 \Theta_1^{\sigma_1} \right)
\]

\[
\quad + (1 - \chi_{\eta_2}) F.T. \left( c(y_1) *_1 \Theta_2^{\sigma_2} \right)
\]

\[
= F.T. \left( y_1^{2n-1} \sigma^1(y_1) *_1 \Theta_1^{\sigma_1} \right) + F.T. \left( c(y_1) *_1 \Theta_2^{\sigma_2} \right)
\]

\[
- F.T. \left( (c(y_1) s_2(y_2)) *_1 \Theta_1^{\sigma_1} \right)
\]

where the superscript \( *_1 \) is used to denote an even extension in the \( y_1 \) variable, \( c(y_1) \) is \( c_{\eta_1} \), \( *_1 \) denotes convolution in the first variable, and \( s_2 \) is a Schwartz function of \( y_2 \). Each term is easily seen to be the Fourier transform of a function which is in \( C^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \).  

Applying Lemma 3.3 to \( \frac{\partial_{y_2}^{\sigma_2}}{\partial \xi_2} \) in (3.3) we see

\[
(3.9) \quad \frac{\partial_{y_1}}{\partial \xi_2} \sim \sum_{j+k \leq n+1} c_{j,k} \left( \frac{\chi_{\eta_1}' \chi_{\eta_2}'}{\eta_1^{\sigma_1} \eta_2^{\sigma_2}} \right) \frac{1}{\xi_1 + \xi_2 + \eta_1 + \eta_2} + \chi_{\eta_1}' \chi_{\eta_2}' R_1',
\]
where the $c_{jk}(\xi_1, \xi_2) \in \mathcal{S}(\mathbb{R}^2)$ and the remainder terms, $R'_n$, are given by
\begin{equation}
R'_n = \sum_{j=1}^{n} \frac{c_j(\xi_1, \xi_2, \eta_1, \eta_2)}{\eta_1^{2j-1} \eta_2^{2n-(2j-2)}} \frac{1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} + \frac{c_{n+1}(\xi_1, \xi_2, \eta_1, \eta_2)}{\eta_1^n} \frac{1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2}.
\end{equation}
Here the functions $c_j$ are smooth and bounded and decay faster than any power of $\xi_1$ or $\xi_2$.

Using the expansion
\begin{equation}
\frac{1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} = \frac{1}{\eta_1^2 + \eta_2^2} - \frac{\xi_1^2}{(\eta_1^2 + \eta_2^2)^2} + \frac{\xi_4^4}{(\eta_1^2 + \eta_2^2)^3} - \cdots
\end{equation}
and
\begin{equation}
\frac{1}{\eta_1^n} = (-1)^n \frac{\xi_2^{2n}}{(\eta_1^2 + \eta_2^2)^{n+1}} + (-1)^{n+1} \frac{\xi_2^{2n+1}}{(\eta_1^2 + \eta_2^2)^{n+1}} + \cdots
\end{equation}
in (3.9), with $\xi_2^2 = \xi_1^2 + \xi_2^2$, we have
\begin{equation}
\frac{\partial^{[n/2]}}{\partial \xi_2} \sim \sum_{j+k \leq n+1, j, k \geq 1} \chi'_{n-1,2j-1,2k-1} \frac{c_{jkl}}{\eta_1^2 + \eta_2^2} \frac{1}{\eta_1^{2j-1} \eta_2^{2k-1}} + \chi'_{n,2j-1} R'_n + \chi'_{n,2j} R''_n,
\end{equation}
where $[n/2]$ is the least integer greater than or equal to $n/2$, and where the $c_{jkl}(\xi_1, \xi_2) \in \mathcal{S}(\mathbb{R}^2)$ and
\begin{equation}
R''_n = \sum_{j+k \leq n+1, j, k \geq 1} \frac{c'_{jk}}{\eta_1^{2j-1} \eta_2^{2k-1}} \frac{1}{(\eta_1^2 + \eta_2^2)^{[n/2]+1}} \frac{1}{(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)}
\end{equation}
with $c'_{jk}(\xi_1, \xi_2) \in \mathcal{S}(\mathbb{R}^2)$.

We first handle the remainder terms $R'_n$ and $R''_n$ in (3.11) and show they are the Fourier transforms of functions which are in $C^{n-2}(\Omega)$.

**Lemma 3.6.** Let $R'_n$ and $R''_n$ be the functions defined in (3.11) and (3.12), respectively. Then $\chi'_{n-1,2j-1} R'_{n+2}$ and $\chi'_{n-1,2j} R''_{n+2}$ are the Fourier transforms of functions which are in $C^{n}(\Omega)$.

**Proof.** To prove the lemma for $R''_n$ consider the term
\begin{equation}
\hat{\delta}_l = \chi'_{n-1,2j-1} \frac{c_j(\xi_1, \xi_2, \eta_1, \eta_2)}{\eta_1^{2j-1} \eta_2^{2n+4-(2j-2)}} \frac{1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2},
\end{equation}
where $1 \leq j \leq n+2$, from (3.11), and define
\begin{equation}
\Phi_1 = F.T.^{-1} \chi'_{n-1,2j-1} \frac{c_j(\xi_1, \xi_2, \eta_1, \eta_2)}{\eta_1^{2j-1} \eta_2^{2n+4-(2j-2)}}.
\end{equation}
From the proof of Lemma 3.3 we know $\Phi_1 \in C^\infty(\Omega)$ and it is also clear that
\begin{equation}
\eta_1^s \frac{\hat{\delta}_l}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \in L^p(\mathbb{R}^4)
\end{equation}
for $p \in (1, \infty)$ and for $i = 1, 2$ and $s = 0, 1$. By Lemma 3.4
\begin{equation}
\delta_l = F.T.^{-1} \left( \frac{\Phi_1}{\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2} \right)
\end{equation}
solves
\[ \triangle \mathcal{H}_1 = \mathcal{G}_1 \big|_{\Omega} \quad \text{on} \quad \Omega; \]
\[ \mathcal{H}_1 = 0 \quad \text{on} \quad y_1 = 0; \]
\[ \mathcal{H}_1 = 0 \quad \text{on} \quad y_2 = 0. \]

Now, either \( 2j - 1 \geq n + 2 \) or \( 2n + 4 - (2j - 2) \geq n + 2 \). Assume \( 2j - 1 \geq n + 2 \), while the other case is to be handled in the same manner. In this case \( \mathcal{G}_1 \) is \( n \)-times differentiable in \( y_1 \). Furthermore, by definition, \( \mathcal{G}_1 \) is odd in \( y_1 \) and \( y_2 \), and since \( \mathcal{G}_1 \) is \( n \)-times differentiable in \( y_1 \), \( \mathcal{G}_1 \mid_{\Omega} \in C^n(\mathbb{R}^2 \times \mathbb{R}^2) \). Thus, by Lemma 3.3 \( \mathcal{H}_1 \in C^n(\Omega) \) when restricted to \( \Omega \).

The last term in \( R'_{n+2} \) is handled in the same manner.

To prove the lemma for \( R''_{n+2} \) consider the term
\[ \hat{\mathcal{H}}_2 = \chi'_{\eta_1} \chi'_{\eta_2} c_{jk}(\xi_1, \xi_2) \frac{1}{\eta_1^{2j-1} \eta_2^{2k-1} (\eta_1^2 + \eta_2^2)^{(n+2)/2} (\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)}. \]
From the factor \( \frac{1}{(\eta_1^2 + \eta_2^2)^{(n+2)/2}} \) we see
\[ \eta_1^l \eta_2^m \hat{\mathcal{H}}_2 \in L^1(\mathbb{R}^4) \]
whenever \( l + m \leq n \) which implies \( \hat{\mathcal{H}}_2 \) is the Fourier transform of a function in \( C^n(\mathbb{R}^4) \).

Returning to equation (3.11), we have to determine which functions, when transformed, give the summation of terms of the form
\[ \chi'_{\eta_1} \chi'_{\eta_2} c_{jk} \frac{1}{\eta_1^{2j-1} \eta_2^{2k-1} (\eta_1^2 + \eta_2^2)^l} \]

We will need a few lemmas to help interpret such terms.

In what follows, the function \( \chi(y_1, y_2) \) is a smooth function on \( \mathbb{R}^2 \) with the property \( \chi = 1 \) for \( y_1^2 + y_2^2 < a \) and \( \chi = 0 \) for \( y_1^2 + y_2^2 > b \) for some \( b > a > 0 \).

**Lemma 3.7.** Let
\[ \Phi_1(y_1, y_2) = -\frac{i}{2} \log(y_1^2 + y_2^2) \]
and define \( \Phi_{j+1} \) to be the unique solution of the form
\[ (3.13) \quad p_1(y_1, y_2) \log(y_1^2 + y_2^2) + p_2(y_1, y_2), \]
where \( p_1 \) and \( p_2 \) are homogeneous polynomials of degree \( 2j - 2 \) in \( y_1 \) and \( y_2 \) such that \( p_2(y_1, 0) = 0 \), to the equation
\[ \frac{\partial \Phi_{j+1}}{\partial y_2} = \frac{1}{2j} y_2 \Phi_j \]
for \( j \geq 1 \). Then
\[ \chi'_{\eta_1} \chi'_{\eta_2} (\chi \Phi_j) \sim \frac{\chi'_{\eta_1} \chi'_{\eta_2}}{(\eta_1^2 + \eta_2^2)^l} \quad j \geq 1, \]
where \( \sim \) is defined in Definition 3.2.
Proof. We first notice the lemma is true when \( j = 1 \) as a straight-forward calculation shows. When \( j = 1 \)

\[
\hat{\chi}_j \Phi_1 = -\frac{i}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log(y_1^2 + y_2^2) \chi e^{-i\gamma \eta} dy_2 dy_1
\]

\[
= \frac{1}{2\eta_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log(y_1^2 + y_2^2) \frac{\partial}{\partial y_2} e^{-i\gamma \eta} dy_2 dy_1
\]

\[
= -\frac{1}{\eta_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y_2}{y_1^2 + y_2^2} \chi e^{-i\gamma \eta} dy_2 dy_1 + s_1,
\]

(3.14)

where \( s_1 \) is \( \frac{1}{\eta_2} \) multiplied by a function in \( \mathcal{S}(\mathbb{R}^2) \), and thus has the property \( \chi_{\eta_1} \chi_{\eta_2} s_1 \sim 0 \).

Now write

\[
-\frac{1}{\eta_2} \frac{\hat{\chi}_{y_2}}{y_1^2 + y_2^2} = -\frac{1}{\eta_2} \frac{y_2}{y_1^2 + y_2^2} + \frac{1}{\eta_2} \frac{(1 - \chi) y_2}{y_1^2 + y_2^2}
\]

(3.15)

so that

\[
\chi_{\eta_1} \chi_{\eta_2} (\chi \Phi_1) \sim \frac{\chi_{\eta_1} \chi_{\eta_2}}{\eta_1^2 + \eta_2^2} - \frac{\chi_{\eta_1} \chi_{\eta_2} (1 - \chi) y_2}{\eta_2} - \frac{1}{\eta_2} \frac{y_2}{y_1^2 + y_2^2}
\]

We show

\[
\frac{\chi_{\eta_1} \chi_{\eta_2} (1 - \chi) y_2}{\eta_2} \sim 0.
\]

If we define \( \hat{\psi} \) by

\[
\hat{\psi} = \frac{\chi_{\eta_1} \chi_{\eta_2} (1 - \chi) y_2}{\eta_2},
\]

then

\[
\eta_1^m \eta_2^m \hat{\psi} = \frac{\chi_{\eta_1} \chi_{\eta_2} \eta_1^m \eta_2^m (1 - \chi) y_2}{\eta_1^2 + \eta_2^2}
\]

\[
= \frac{\chi_{\eta_1} \chi_{\eta_2}}{\eta_2^2} \text{F.T.} \left( \frac{\partial^l \partial^m (1 - \chi) y_2}{\partial y_1^l \partial y_2^m + y_1^2 + y_2^2} \right) \in L^2(\mathbb{R}^2) \quad \forall l, m \geq 0
\]

which shows \( \hat{\psi} \) is the transform of a function in \( C^\infty(\mathbb{R}^2) \). Thus \( \hat{\psi} \sim 0 \) and \( \chi_{\eta_1} \chi_{\eta_2} (\chi \Phi_1) \sim \frac{\chi_{\eta_1} \chi_{\eta_2}}{\eta_1^2 + \eta_2^2} \) from (3.16) which proves the lemma in the case \( j = 1 \).

In order to show the lemma is true for higher \( j \), we will use the recursive equation

\[
(\chi \Phi_{j+1}) = -\frac{1}{\eta_2} \frac{1}{2j} \frac{\partial}{\partial \eta_2} (\chi \Phi_j) + s_{j+1},
\]

(3.17)

where \( s_{j+1} \) has the form \( \frac{1}{\eta_2} \) multiplied by a function in \( \mathcal{S}(\mathbb{R}^2) \).
Lastly, let \( \Phi \) be the function \( \chi \) in (3.11) extended to all of \( \mathbb{R}^2 \) by an odd reflection in the \( y_2 \) variable. Then

\[
\chi'_{\eta_1} \chi'_{\eta_2} \left( \Phi_{j+1}^{(2)} \right) \sim \chi'_{\eta_1} \chi'_{\eta_2} \frac{1}{(\eta_1^2 + \eta_2^2)^{j+1}}.
\]

The proof is by integration by parts. Without loss of generality, we can consider \( \chi(y_1, y_2) \) to be of the form \( \chi_1(y_1) \chi_2(y_2) \), where, for \( j = 1, 2 \), \( \chi_j(y_j) \in C_0^\infty \) with the property that \( \chi_j = 1 \) for \( y_j < a \) and \( \chi_j = 0 \) for \( y_j > b \) for some \( 0 < a < b \).

Then

\[
\chi'_{\eta_1} \chi'_{\eta_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \Phi_{(j+1)(2k-1)}^{(2)} \right) e^{-iy'y} dy_2 dy_1 \sim
\]

\[
\frac{\chi'_{\eta_1}}{i(\eta_1)^{2j-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^{2j-1}}{\partial y_1^{2j-1}} \frac{\partial^{2k-1}}{\partial y_2^{2k-1}} \left( \Phi_{(j+1)(2k-1)}^{(2)} \right) e^{-iy'y} dy_2 dy_1.
\]

again using the fact that derivatives of \( \chi \) vanish to infinite order near \( y_1 = 0 \) or \( y_2 = 0 \). By definition,

\[
\frac{\partial^{2j-1}}{\partial y_1^{2j-1}} \frac{\partial^{2k-1}}{\partial y_2^{2k-1}} \left( \Phi_{(j+1)(2k-1)}^{(2)} \right) = \Phi_l,
\]
where \( \Phi_t \) is defined as in Lemma 3.7. Thus, equation 3.21 gives
\[
\frac{\chi'_n}{\chi'_{n_2}} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right) \sim \chi'_n \chi'_{n_2} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_t e^{-i\eta y} dy_1 dy_2
\]
which, by Lemma 3.7 is equivalent to
\[
\frac{\chi'_n}{\chi'_{n_2}} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right) \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \left( \eta_1^2 + \eta_2^2 \right).
\]

For \( y_2 \geq 0 \),
\[
(\Phi_t)_{jk} = p_1 \log(y_1^2 + y_2^2) + p_2 + p_3 \arctan \left( \frac{y_1}{y_2} \right) + p_4 \log y_1,
\]
where the \( p_m \) are homogeneous polynomials of degree \((2l - 2) + (2j - 1) + (2k - 1)\) in \( y_1 \) and \( y_2 \) for \( m = 1, 2, 3, 4 \).

**Proposition 3.9.** \( \forall n \in \mathbb{N}, \exists \) polynomials, \( A_n, B_n, \) and \( C_n, \) of degree \( n \) in \( y_1 \) and \( y_2, \) and whose coefficients are Schwartz functions of \( \xi_1 \) and \( \xi_2, \) and \( D_n, \) the partial transform in the \( x \) variables of a function which belongs to \( C^n(\bar{\Omega}) \), such that near \( y_1, y_2 = 0 \)
\[
F.T. \left( \frac{\partial u_{12}^{12}}{\partial \xi_2} \right) (\xi_1, \xi_2, y_1, y_2) = A_n(y_1, y_2) \log(y_1^2 + y_2^2) + B_n(y_1, y_2) + C_n(y_1, y_2) \arctan \left( \frac{y_1}{y_2} \right) + D_n,
\]
where \( F.T. \) stands for the partial Fourier transform in the \( x \) variables.

**Proof.** Write an expansion of \( \frac{\partial u_{12}^{12}}{\partial \xi_2} \) as in (3.1). By Lemma 3.6 we know the remainder terms in (3.1) are the transforms of functions which are in \( C^n(\bar{\Omega}) \). The other terms in (3.1) are equivalent to Schwartz functions of \( \xi_1 \) and \( \xi_2 \) multiplied by terms of the form
\[
\frac{\chi'_n}{\chi'_{n_2}} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right)
\]
as shown in Lemma 3.8. Here we show
\[
\chi'_n \chi'_{n_2} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right) \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \left( \eta_1^2 + \eta_2^2 \right)
\]
is the Fourier transform of a function which, when restricted to \( \mathbb{R}_+ \times \mathbb{R}_+ \), is in \( C^\infty(\mathbb{R}_+ \times \mathbb{R}_+) \) plus terms which are polynomials of \( y_1 \) and \( y_2 \) multiplied by functions of only one of \( y_1 \) or \( y_2 \). From the proofs of Lemmas 3.7 and 3.8 we see the only singularities of \( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \) are poles of the form \( \frac{a}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \) for \( a, b, c \in \mathbb{N} \). Hence for large enough \( l, s, \) and \( t, \)
\[
(\eta_1^2 + \eta_2^2)_s \eta_1^t \eta_2^t \left( \chi'_n \chi'_{n_2} \left( \frac{1}{\eta_1^{2j+2k-2} \eta_2^{2k-1}} \right) \right) \in \mathcal{S}(\mathbb{R}^2).
\]
Since \((\eta_1^2 + \eta_2^2)^l\) corresponds to the symbol of an elliptic operator, we can conclude that
\[
\eta_1^l \eta_2^l \left( \chi'_{\eta_1} \chi'_{\eta_2} \left( \hat{\chi}^{(2)}_{\Phi_1} (2j-1) (2k-1) \right) - \hat{\chi}^{(2)}_{\Phi_1} (2j-1) (2k-1) \right)
\]
is the Fourier transform of a function in \(C^\infty (\mathbb{R}^2)\). Thus,
\[
\chi'_{\eta_1} \chi'_{\eta_2} \left( \hat{\chi}^{(2)}_{\Phi_1} (2j-1) (2k-1) \right) - \hat{\chi}^{(2)}_{\Phi_1} (2j-1) (2k-1)
\]
is the transform of a function, call it \(\tau(y_1, y_2)\), which satisfies
\[
\frac{\partial^{n+t} \tau}{\partial y_1 \partial y_2} \in C^\infty (\mathbb{R}_+ \times \mathbb{R}_+).
\]
Integrating with respect to the first variable from 0 to \(y_1\) and in the second from 0 to \(y_2\), we see \(\tau\) is a function which, when restricted to \(\mathbb{R}_+ \times \mathbb{R}_+\) is in \(C^\infty (\mathbb{R}_+ \times \mathbb{R}_+)\) plus terms which are polynomials of \(y_1\) and \(y_2\) multiplied by functions of only one of \(y_1\) or \(y_2\), as claimed.

Then the proposition is proved by Lemma 3.8 which shows the structure of \((\Phi_1) (2j-1) (2k-1)\) and by the use of Lemma 2.2 to ignore the terms which may be singular along \(y_1 = 0\) or \(y_2 = 0\).

**Proof of Theorem 3.1.** We will prove Theorem 3.1 locally. Pick any \((x_1', x_2') \in \mathbb{R}^2\) and let \(\phi \in C^\infty_0 (\mathbb{R}^4)\) be a cutoff function such that \(\phi \equiv 1\) in a neighborhood of \((x_1', x_2', 0, 0)\). Without loss of generality we can assume \(\phi\) is of the form \(\phi_x (x_1, x_2) \phi_1 (y_1) \phi_2 (y_2)\), where \(\phi_x \in C^\infty_0 (\mathbb{R}^2)\), \(\phi_1 \in C^\infty_0 (\mathbb{R})\) with the property that \(\phi_1 = 1\) near \(y_1 = 0\) and \(\phi_2 \in C^\infty_0 (\mathbb{R})\) with the property that \(\phi_2 = 1\) near \(y_2 = 0\).

Since,
\[
\frac{\partial}{\partial \bar{z}_2} (\phi u^{(1)}}\ = \phi \frac{\partial u^{(1)}}{\partial \bar{z}_2} + \frac{\partial \phi}{\partial \bar{z}_2} u^{(1)}
\]
Proposition 3.9 gives for \(y_2 > 0\), using equation 3.22.

\begin{equation}
(3.23) \quad F.T.\_x \left( \frac{\partial}{\partial \bar{z}_2} (\phi u^{(1)}}\ \right) = e^{-\xi_2 y_2} \frac{\partial}{\partial y_2} (e^{\xi_2 y_2} F.T.\_x (\phi u^{(1)}}) =
\end{equation}
\[
a_n (y_1, y_2) \log(y_1^2 + y_2^2) + b_n (y_1, y_2) + c_n (y_1, y_2) \arctan \left( \frac{y_1}{y_2} \right) + r_n,
\]
where here, \(a_n, b_n,\) and \(c_n\) are just \(F.T.\_x (\phi) \ast \xi A_n, F.T.\_x (\phi) \ast \xi B_n,\) and \(F.T.\_x (\phi) \ast \xi C_n\) respectively, \(\ast \xi\) denoting convolution with respect to \((\xi_1, \xi_2)\), and hence are Schwartz functions of \(\xi_1\) and \(\xi_2\), and are polynomials in \(y_1\) and \(y_2\) near \(y_1 = y_2 = 0\), and \(r_n\) is a remainder term which is the partial transform in the \(x\) variables of a function which, when restricted to a neighborhood \(V \subset \Omega\) of \((x_1', x_2', 0, 0)\), is in \(C^n(\Omega)\).

Then we invert the operator \(e^{-\xi_2 y_2} \frac{\partial}{\partial y_2} e^{\xi_2 y_2}\) in (3.23) to obtain

\begin{equation}
(3.24) \quad F.T.\_x (\phi u^{(1)}}) =
\end{equation}
\[
e^{-\xi_2 y_2} \int_0^{y_2} e^{\xi_2 t} \left( a_n (y_1, t) \log(y_1^2 + t^2) + b_n (y_1, t) + c_n (y_1, t) \arctan \left( \frac{y_1}{t} \right) \right) dt
\]
\[+ e^{-\xi_2 y_2} \int_0^{y_2} e^{\xi_2 t} r_n (\xi_1, y_1, \xi_2, t) dt + e^{-\xi_2 y_2} v (\xi_1, y_1, \xi_2),
\]
where \( v \) is a function resulting from the lower limit of integration. By considering \( y_2 \to \infty \), we can see, in the case \( \xi_2 < 0 \), \( v \) is forced to be

\[
v(\xi_1, y_1, \xi_2) =
- \int_0^{\xi_2} e^{\xi t} \left( a_n(y_1, t) \log(y_1^2 + t^2) + b_n(y_1, t) + c_n(y_1, t) \arctan \left( \frac{y_1}{t} \right) \right) dt
- \int_0^{\xi_2} e^{\xi t} r_n(\xi_1, y_1, \xi_2, t) dt
\]

in which case

\[(3.25) \quad F.T.(\varphi u_1^{(2)}) =
- e^{-\xi y_2} \int_{y_2}^{\infty} e^{\xi t} \left( a_n(y_1, t) \log(y_1^2 + t^2) + b_n(y_1, t) + c_n(y_1, t) \arctan \left( \frac{y_1}{t} \right) \right) dt
- e^{-\xi y_2} \int_{y_2}^{\infty} e^{\xi t} r_n(\xi_1, y_1, \xi_2, t) dt.
\]

If we consider \( \xi_2 > 0 \), it suffices to use

\[
v(\xi_1, y_1, \xi_2) = F.T.(\varphi u_1^{(2)}) (\xi_1, y_1, \xi_2, 0)
\]

so that, for \( \xi_2 > 0 \) we write

\[(3.26) \quad F.T.(\varphi u_1^{(2)}) =
\]

\[
e^{-\xi y_2} \int_0^{y_2} e^{\xi t} \left( a_n(y_1, t) \log(y_1^2 + t^2) + b_n(y_1, t) + c_n(y_1, t) \arctan \left( \frac{y_1}{t} \right) \right) dt
+ e^{-\xi y_2} \int_0^{y_2} e^{\xi t} r_n(\xi_1, y_1, \xi_2, t) dt + e^{-\xi y_2} F.T.(\varphi u_1^{(2)}) (\xi_1, y_1, \xi_2, 0).
\]

We Taylor expand the exponential factors in the integrals of equations (3.25) and \( 3.26 \) using

\[
e^{\xi (t-y_2)} = 1 + \xi_2 (t-y_2) + \cdots + \frac{\xi_2^n}{n!} (t-y_2)^n + \frac{1}{n!} \int_{y_2}^{t} (t-s)^n \xi_2^n e^{\xi s} (s-y_2) ds
\]

in (3.25) and

\[
e^{\xi (t-y_2)} = e^{-\xi y_2} \left( 1 + \xi_2 t + \frac{\xi_2^2}{2} t^2 + \cdots + \frac{\xi_2^n}{n!} t^n \right)
+ \frac{1}{n!} \int_0^{t} (t-s)^n \xi_2^n e^{\xi s} (s-y_2) ds
\]

in (3.26).

We first concentrate on terms arising from the remainders in the Taylor expansions. Consider, for \( \xi_2 < 0 \),

\[(3.27) \quad \int_{y_2}^{\infty} \left( \frac{1}{n!} \int_{y_2}^{t} (t-s)^n \xi_2^n e^{\xi s} (s-y_2) ds \right) \left( a_n \log(y_1^2 + t^2) + b_n + c_n \arctan \left( \frac{y_1}{t} \right) \right) dt.
\]

Changing the order of integration gives

\[
\frac{\xi_2^n}{n!} \int_{y_2}^{\infty} e^{\xi (s-y_2)} \left( \int_{y_2}^{\infty} (t-s)^n \left( a_n \log(y_1^2 + t^2) + b_n + c_n \arctan \left( \frac{y_1}{t} \right) \right) dt \right) ds.
\]
Expanding the factor \((t - s)^n\) and integrating shows
\[
\int_0^\infty (t - s)^n \left( a_n(y_1, t) \log(y_1^2 + t^2) + b_n(y_1, t) + c_n(y_1, t) \arctan \left( \frac{y_1}{t} \right) \right) dt
\]
can be written in the form
\[
p_n(y_1, s) \log(y_1^2 + s^2) + q_n(y_1, s) \arctan \left( \frac{y_1}{s} \right),
\]
where \(p_n\) and \(q_n\) are polynomials in \(y_1\) and \(s\), each term being of degree greater than or equal to \(n + 1\) near \(y_1 = s = 0\), plus terms which are in \(C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)\), and hence a function which is \(C^n(\mathbb{R}_+ \times \mathbb{R}_+)\) in the \(y_1\) and \(s\) variables multiplied by Schwartz functions of \(\xi_1\) and \(\xi_2\). Thus, the integral in (3.27) is in \(C^n(\mathbb{R}_+ \times \mathbb{R}_+)\).

The same argument applied when \(\xi_2 > 0\) gives
\[
\int_0^{y_2} \left( \frac{1}{n!} \int_0^t (t - s)^n \xi_2^n e^{\xi_2(s - y_2)} ds \right) \left( a_n \log(y_1^2 + t^2) + b_n + c_n \arctan \left( \frac{y_1}{t} \right) \right) dt
\]
is a function which is \(C^n(\mathbb{R}_+ \times \mathbb{R}_+)\) in the \(y_1\) and \(y_2\) variables multiplied by Schwartz functions of \(\xi_1\) and \(\xi_2\).

We may now invert from the \(\xi\) variables to the \(x\) variables. As was shown above the terms from the Taylor remainder, when inverted, gives a function which is in \(C^n(\Omega)\).

We make use of the fact that the factor of \(e^{-\xi_2 y_2}\) in (3.26) leads to convolving with
\[
\int_0^\infty e^{-\xi_2 y_1} e^{\xi_2 x_2} d\xi_2 = \frac{i}{x_2 + iy_2}
\]
in the \(x_2\) variable. Thus, in inverting the integrals which contain the Taylor polynomials we first carry out the integration with respect to \(t\) and use
\[
\int_0^\infty \kappa(x_1, \alpha, y_1, y_2) d\alpha
\]
is in \(C^\infty(\Omega)\) (Theorem 3.1[2]) whenever \(\kappa \in S(\Omega)\) to obtain
\[
(3.28) \quad \alpha_n(y_1, y_2) \log(y_1^2 + y_2^2) + \beta_n(y_1, y_2) + \gamma_n(y_1, y_2) \arctan \left( \frac{y_1}{y_2} \right) + \Phi,
\]
where \(\alpha_n, \beta_n,\) and \(\gamma_n\) are polynomials of degree \(n\) in \(y_1\) and \(y_2\) near \(y_1 = y_2 = 0\), and whose coefficients are smooth functions of \(x_1\) and \(x_2\), and where \(\Phi\) is a function which has a singularity at \(y_1 = 0\).

Now, let us examine the term
\[
(3.29) \quad \begin{cases} 0, & \text{if } \xi_2 < 0; \\ e^{-\xi_2 y_2} F.T.\, \left( \varphi u_1^1 \right) (\xi_1, y_1, \xi_2, 0), & \text{if } \xi_2 > 0 \end{cases}
\]
from (3.26). We can, without loss of generality, work in the case in which
\[
F.T.\, \left( \varphi u_1^1 \right) (\xi_1, y_1, \xi_2, 0)
\]
is a smooth function of \(y_1\) up to \(y_1 = 0\), since, by Lemma 2.2 we see that \(u_1\) is a \(C^\infty\) function of \(y_1\) up to \(y_1 = 0\) when \(y_2 > 0\) is held constant. Thus, if there are singularities in \(e^{-\xi_2 y_2} F.T.\, \left( \varphi u_1^1 \right) (\xi_1, y_1, \xi_2, 0)\) for \(y_2 > 0\), they will cancel out with the singular terms in the function \(\Phi\) in (3.28).

When (3.29) is inverted, we get
\[ 2\pi i \int_{-\infty}^{\infty} \frac{\varphi u_1^{11}|_{y_2=0}}{(x_2 - t) + iy_2} \, dt, \]

where \( F.T. x_2 \) refers to the partial Fourier transform with respect to \( x_2 \). In the case \( \varphi u_1^{11}|_{y_2=0} \) is smooth up to \( y_1 = 0 \), we see the integral in (3.30) is in \( C^\infty(\Omega) \) (see Theorem 3.1 [2]). Thus, from the discussion above, the integral in (3.30), when combined with \( \Phi \) in (3.28), gives a term in \( C^\infty(\Omega) \).

We are left to invert
\[ \left\{ \begin{array}{ll}
- \int_0^{y_2} e^{\xi_2(t-y_2)r_n(\xi_1, y_1, \xi_2, t)}dt, & \text{if } \xi_2 < 0; \\
\int_{y_2}^{\infty} e^{\xi_2(t-y_2)r_n(\xi_1, y_1, \xi_2, t)}dt, & \text{if } \xi_2 > 0.
\end{array} \right. \]

We denote \( F.T. x_2(r_n) \) by \( \tilde{r}_n \), and since \( \tilde{r}_n \) has compact support, in the sense of distributions, (3.32)
\[ \frac{-\partial f_1}{\partial \bar{z}_2} \bigg|_{y_1=0} = 0 \] (3.33)

∀ \( j, k \geq 0 \) such that \( j + k \leq n + 2 \), then \( u_1 \in C^n(\Omega) \).

**Proposition 3.10.** If
\[ \frac{\partial^{2j}}{\partial y_1^{j}} \frac{\partial^{2k}}{\partial y_2^{k}} \left( \frac{\partial f_1}{\partial \bar{z}_2} \right) \bigg|_{y_1=y_2=0} = 0 \]

\[ e^{-\xi_2y_2} \frac{\partial}{\partial y_2} \left( e^{\xi_2y_2 F.T. x_2 (\varphi u_1^{11})} \right) = r_n, \]

At this point it is easy to determine a sufficient condition for our function \( u_1 \) to be in \( C^n(\Omega) \).
where \( \varphi \) is as in the proof of Theorem 3.1 and \( r_n \) is the partial Fourier transform in the \( x \) variables of a function which is \( C^\infty \) in a neighborhood of a boundary point, \((x'_1, x'_2, 0, 0)\). Upon inverting the operator \( e^{-\xi_2 y_2} \partial_y e^{\xi_2 y_2} \), we obtain

\[
F.T_x (\varphi u^{(1)}_1) (\xi_1, y_1, \xi_2, y_2) = e^{-\xi_2 y_2} \int_0^{y_2} e^{\xi_2 t} r_n (\xi_1, \xi_2, 0, 0) e^{\xi_2 t} r_n (\xi_1, \xi_2, y_1, t) dt + e^{-\xi_2 y_2} v(\xi_1, y_1, \xi_2),
\]

where \( v(\xi_1, y_1, \xi_2) \) is determined as in the proof of Theorem 3.1. We have

\[
F.T_x (\varphi u^{(1)}_1) (\xi_1, y_1, \xi_2, y_2) = \begin{cases} 
- \int_{y_2}^{\infty} e^{\xi_2 (t - y_2)} r_n (\xi_1, \xi_2, t) dt, & \text{if } \xi_2 < 0; \\
\int_0^{y_2} e^{\xi_2 (t - y_2)} r_n (\xi_1, \xi_2, t) dt + e^{-\xi_2 y_2} F.T_x (\varphi u^{(1)}_1) (\xi_1, y_1, \xi_2, 0), & \text{if } \xi_2 > 0.
\end{cases}
\]

As in the proof of Theorem 3.1, we can take the inverse Fourier transform of \((\ref{3.34})\) with respect to the \( \xi \) variables to conclude \( u_1 \in C^\infty (\Omega) \), and the proposition is proved.

**Corollary 3.11.** If \( f \) vanishes to infinite order at \( y_1 = y_2 = 0 \) then \( u \in C^\infty (\Omega(0, 1)) \).

**Remark 3.12.** It is relevant to note that there are \( f \in S(\Omega(0, 1)) \) such that the \( \alpha_j \) and \( \beta_j \) are not zero in Main Thereom 1. One can see this by choosing \( f_1, f_2 \) to be of compact support and equivalently equal to 1 in a neighborhood of \((x'_1, x'_2, 0, 0)\) for some \((x'_1, x'_2) \in \mathbb{R}^2\) and following the analysis above.

### 4. Regularity of \( \bar{\partial}^* N \)

With \( \Omega = \mathbb{H}_1 \times \mathbb{H}_2 \), Let \( N \) be the operator defined on \( S(\Omega(0, 1)) \) such that, for \( f \in S(\Omega(0, 1)) \), \( Nf = u \), the solution we found in Section 3. We show here the

**Proposition 4.1.** Let \( f \in S(\Omega(0, 1)) \) with the property \( \bar{\partial} f = 0 \). Then \( \bar{\partial}^* N f \in C^\infty (\Omega) \).

**Proof.** Let \( u \) be the solution found in Section 3 to the \( \bar{\partial} \)-Neumann problem with data \( f \), that is, \( u = Nf \). We will show \( \bar{\partial}^* u \in C^\infty (\Omega) \).

We work with

\[
\frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} = -\frac{1}{2} \bar{\partial}^* u
\]

and, as in Section 3, we find the calculations easier when we apply the operator \( \frac{\partial}{\partial z_2} \) to \((\ref{4.1})\). Using

\[
\frac{\partial}{\partial z_2} \left( \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} \right) = \frac{\partial}{\partial z_1} \left( \frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1} \right) + f_2.
\]

we obtain

\[
(\ref{4.2}) \quad \frac{\partial}{\partial z_2} \left( \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} \right) = \frac{\partial}{\partial z_1} \left( \frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1} \right) + f_2.
\]
Now, the quantity in parentheses on the right hand side of (4.2) is 0. This follows from the formula (3.1) for $\frac{\partial u_1}{\partial z_2}$ and its counterpart for $\frac{\partial u_2}{\partial z_2}$. We have

$$\frac{\partial u_1}{\partial z_2} - \frac{\partial u_2}{\partial z_2} = \frac{\partial f_2}{\partial z_2} = \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 = 0,$$

since $\partial f = 0$. Thus (4.2) becomes

$$\frac{\partial}{\partial z_2} \left( \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} \right) = f_2.$$  

We can recover $\frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2}$ from equation (4.4) by taking the partial Fourier transform of (4.4) in the $x_2$ variable. Let

$$v = \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2}.$$  

We work with $\varphi v$, where $\varphi \in C_0^\infty(\Omega)$ is a cutoff function such that $\varphi \equiv 1$ in a neighborhood of $(x_1', x_2', 0, 0)$ for any chosen $(x_1', x_2') \in \mathbb{R}^2$ as we did in the proof of Theorem 3.1. We show $\varphi v \in C^\infty$ in a neighborhood of $(x_1', x_2', 0, 0)$, and this together with Lemma 2.2 implies $v \in C^\infty(\Omega)$.

From

$$\frac{\partial (\varphi v)}{\partial z_2} = \varphi \frac{\partial v}{\partial z_2},$$

it is easy to see $\frac{\partial (\varphi v)}{\partial z_2}$ restricted to some neighborhood $V$ of $(x_1', x_2', 0, 0)$ is in $C^\infty(\Omega)$. Let $i\tilde{\Phi}$ denote the function $\frac{\partial (\varphi v)}{\partial z_2}$. Following the proof of Theorem 3.1 we take partial Fourier transforms with respect to the $x_2$ variable.

$$e^{-\xi_2 y_2} \frac{\partial}{\partial y_2} \left( e^{\xi_2 y_2} F.T._{x_2} (\varphi v) \right) = F.T._{x_2} \tilde{\Phi},$$

and inverting the operator $e^{-\xi_2 y_2} \frac{\partial}{\partial y_2} e^{\xi_2 y_2}$ we get

$$F.T._{x_2} (\varphi v) = \int_0^{y_2} e^{\xi_2 (t-y_2)} F.T._{x_2} \tilde{\Phi}(x_1, \xi_2, y_1, t) dt + e^{-\xi_2 y_2} \Phi(x_1, \xi_2, y_1).$$

We can determine $\Phi(x_1, \xi_2, y_1)$, and write

$$F.T._{x_2} (\varphi v) = \begin{cases} - \int_0^{y_2} e^{\xi_2 (t-y_2)} F.T._{x_2} \tilde{\Phi}(x_1, \xi_2, y_1, t) dt, & \text{if } \xi_2 < 0; \\ \int_0^{y_2} e^{\xi_2 (t-y_2)} F.T._{x_2} \tilde{\Phi}(x_1, \xi_2, y_1, t) dt + e^{-\xi_2 y_2} \varphi v|_{y_2=0}, & \text{if } \xi_2 > 0. \end{cases}$$

As before, when we take inverse transforms, we see

$$\varphi v = - 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{(t-y_2) + i(s-x_2)} \tilde{\Phi}(x_1, y_1, s, t) ds dt + 2\pi i \int_{-\infty}^{\infty} \frac{\varphi v|_{y_2=0}}{(x_2 - t) + iy_2} dt$$

in the sense of distributions (see equations (3.30) and (3.32)). And it follows as in the proof of Theorem 3.1 that $\varphi v \in C^\infty(\Omega)$. \hfill \Box

As a consequence of Proposition 4.1 we have the
Corollary 4.2. Let $f \in S(\overline{\Omega})$ satisfy $\partial\bar{\partial}f = 0$. Then $\exists u \in C^\infty(\overline{\Omega})$ which solves $\partial\bar{\partial}u = f$.

Proof. Let $u = \partial^* Nf$. From Proposition 4.1, $u \in C^\infty(\overline{\Omega})$. Also,

$$\partial\bar{\partial}u = \partial\bar{\partial}\partial^* Nf = (\partial\bar{\partial} + \partial^* \partial)Nf - \partial^* \partial Nf = f$$

using the fact that $N$ is the inverse of $\partial\bar{\partial} + \partial^* \partial$, and $\partial Nf = 0$ from (4.3).

On smoothly bounded strictly pseudoconvex domains, $\partial^* Nf$ is commonly referred to in the literature as the Kohn solution for $\partial$.

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E-mail address: ehsani@math.tamu.edu