A Relational Gradient Descent Algorithm For Support Vector Machine Training

Mahmoud Abo-Khamis
Relational AI
University of California, Merced

Sungjin Im *
∗
University of California, Merced

Benjamin Moseley †
Carnegie Mellon University

Kirk Pruhs ‡
University of Pittsburgh

Kirk Pruhs †
University of Pittsburgh

May 13, 2020

Abstract

We consider gradient descent like algorithms for Support Vector Machine (SVM) training when the data is in relational form. The gradient of the SVM objective can not be efficiently computed by known techniques as it suffers from the “subtraction problem”. We first show that the subtraction problem can not be surmounted by showing that computing any constant approximation of the gradient of the SVM objective function is \#P-hard, even for acyclic joins.

We however circumvent the subtraction problem by restricting our attention to stable instances, which intuitively are instances where a nearly optimal solution remains nearly optimal if the points are perturbed slightly. We give an efficient algorithm that computes a “pseudo-gradient” that guarantees convergence for stable instances at a rate comparable to that achieved by using the actual gradient. We believe that our results suggest that this sort of stability analysis would likely yield useful insight in context of designing algorithms on relational data for other learning problems in which the subtraction problem arises.

1 Introduction

Kaggle surveys [1] show that the majority of learning tasks faced by data scientists involve relational data. Most commonly the relational data is stored in tables in a relational database. So these data scientists want to compute something like

\[
\text{Data Science Query} = \text{Standard}\_\text{Learning}\_\text{Task(\text{Relational Tables } T_1, \ldots T_m)}
\]

However, almost all standard algorithms for standard learning problems assume that the input consists of points in Euclidean space [18], and thus are not designed to operate directly on relational data. The current standard practice for a data scientist, confronted with a learning task on relational data, is:

1. Firstly, convert any nonnumeric categorical data to numeric data. As there are standard methods to accomplish this [18], and as we do not innovate with respect to this process, we will assume that all data is a priori numerical, so we need not consider this step.

*Supported in part by NSF grants CCF-1409130, CCF-1617653, and CCF-1844939.
†Supported in part by NSF grants CCF-1725543, 173873, 1845146, a Google Research Award, a Bosch junior faculty chair and an Infor faculty award.
‡Supported in part by NSF grants CCF-1421508 and CCF-1535755, and an IBM Faculty Award.
2. Secondly, issue a feature extraction query to extract the data from the relational database by joining together the tables to materialize a design matrix \( J = T_1 \times \cdots \times T_m \) with say \( N \) rows and \((d + 1)\) columns. Each row of this design matrix is then interpreted as a point in \( d\)-dimensional Euclidean space with an associated label.

3. Finally this design matrix \( J \) is important into a standard learning algorithm to train the model.

Thus conceptually, standard practice transforms a data science query to a query of the following form:

\[
\text{Data Science Query} = \text{Standard Learning Algorithm}(\text{Design Matrix } J = T_1 \times \cdots \times T_m)
\]

where the joins are evaluated first, and the learning algorithm is then applied to the result. Note that if each table has \( n \) rows, the design matrix \( J \) can have as many as \( n^m \) entries. Thus, independent of the learning task, this standard practice necessarily has exponential worst-case time and space complexity as the design matrix can be exponentially larger than the underlying relational tables. Thus a natural research question is what we call the relational learning question:

**The Relational Learning Question:**

A. Which standard learning algorithms can be implemented as relational algorithms, which informally are algorithms that are efficient when the input is in relational form?

B. And for those standard algorithms that are not implementable by a relational algorithms, is there an alternative relational algorithm that has the same performance guarantee as the standard algorithm?

C. And if we can’t find an alternative relational algorithm that has the same performance guarantees to the standard algorithm, is there an alternative relational algorithm that has some reasonable performance guarantee (ideally similar to the performance guarantee for the standard algorithm)?

Note that a relational algorithm can not afford to explicitly join the relational tables.

One immediate difficulty that we run into is that if the tables have a sufficiently complicated structure, almost all natural problems/questions about the design matrix are NP-hard if the data is in relational form. For example, it is NP-hard to even determine whether or not the design matrix is empty or not (see for example [22, 33]). Thus, as we want to focus on the complexity of the learning problems, we conceptually want to abstract out the complexity of the tables. The simplest way to accomplish this is to primarily focus on instances where the structure of the tables is simple, with the most natural candidate for “simplicity” being that the join is acyclic. Acyclic joins are the norm in practice, and are a commonly considered special case in the database literature. For example, there are efficient algorithms to compute the size of the design matrix for acyclic joins.

Formally defining what an “relational” algorithm is problematic, as for each natural candidate definition there are plausible scenarios in which that candidate definition is not the “right” definition. But for the purposes of this paper it is sufficient to think of a “relational” algorithm as one whose runtime is polynomially bounded in \( n, m \) and \( d \) if the join is acyclic.

**Our Research Question:** In this paper we address the relational learning question within the context of gradient descent algorithms for the classic (soft-margin linear) Support Vector Machine (SVM) training problem. SVM is identified as one of the five most important learning problems
in [18], and is covered in almost all introductory machine learning textbooks. Gradient descent is probably the most commonly used computational technique for solving convex learning optimization problems [39]. So plan A is to find a relational implementation of gradient descent for the SVM objective. And if plan A fails, plan B is to find a relational descent algorithm that has the same performance guarantee as gradient descent. And finally, if both plan A fail and plan B fail, plan C is to find a relational algorithm that has some other reasonable performance guarantee.

1.1 Background

We now give the minimal background on gradient descent and SVM required to understand our results.

Gradient Descent: Gradient descent is a first-order iterative optimization method for finding an approximate minimum of a convex function \( F : \mathbb{R}^d \rightarrow \mathbb{R} \), perhaps subject to a constraint the solution lies in some convex body \( \mathcal{K} \). In the \( G \) descent algorithm, at each descent step \( t \) the current candidate solution \( \beta^{(t)} \) is updated according to the following rule:

\[
\beta^{(t)} \leftarrow \beta^{(t-1)} - \eta_t G(\beta^{(t-1)})
\]

where \( \eta_t \) is the step size. In projected \( G \) descent, the current candidate solution \( \beta^{(t)} \) is updated according to the following rule:

\[
\beta^{(t)} \leftarrow \Pi_K \left( \beta^{(t-1)} - \eta_t G(\beta^{(t-1)}) \right)
\]

where \( \Pi_K(\alpha) = \arg\min_{\beta \in \mathcal{K}} \|\alpha - \beta\|_2 \) is the projection of the point \( \alpha \) to the closest point to \( \alpha \) in \( \mathcal{K} \). In (projected) gradient descent, \( G \) is \( \nabla F(\beta^{(t)}) \), the gradient of \( F \) at \( \beta^{(t)} \). There are lots of variations of gradient descent, including variations on the step size, and variations, like stochastic gradient descent [39], in which the gradient is only approximated.

SVM training: Conceptually the input to SVM training consists of a collection \( X = \{x_1, x_2, \ldots, x_N\} \) of points in \( \mathbb{R}^d \), and a collection \( Y = \{y_1, y_2, \ldots, y_N\} \) of associated labels from \( \{-1, 1\} \). For convenience let us rescale the points so that each point in \( X \) lies within the hypercube \([-1, 1]^d\). A feasible solution is a \( d \)-dimensional vector \( \beta \), sometimes called a hypothesis. The objective is to minimize a linear combination \( F(\beta, X, Y) \) of the average “hinge” loss function of the points \( L(\beta, X, Y) = \frac{1}{N} \sum_{i \in X} \max(0, 1 - y_i \beta x_i) \) plus a regularizer \( R(\beta) \). We will take the regularizer to be the 2-norm squared of \( \beta \), as that is a standard choice [18], although this choice is not so important for our purposes. Thus the objective is to minimize:

\[
F(\beta, X, Y) = \frac{1}{N} \sum_{x_i \in X} \max(0, 1 - y_i \beta x_i) + \lambda \|\beta\|_2^2
\]

Here the loss function measures how well the hypothesis \( \beta \) explains the labels, and one of the regularizer’s purposes is to prevent overfitting. The \( \lambda \) factor intuitively specifies the amount that the loss has to decrease to justify an increase in the norm of \( \beta \). When either \( X \) and \( Y \) is understood, for notational convenience, we may drop them from the objective.

Gradient Descent for SVM: In Section A we show that by a straightforward specialization of a standard convergence analysis for projected gradient descent to SVM one obtains Theorem I which bounds the number of descent steps needed to reach a solution with a specified relative error.

**Theorem 1.** Let \( F(\beta) \) be the SVM objective function. Let \( \beta^* = \arg\min_{\beta} F(\beta) \) be the optimal solution. Let \( \hat{\beta}_s = \frac{1}{s} \sum_{t=s}^{s-1} \beta^{(t)} \). Let \( \eta_t = \frac{1}{8\lambda \sqrt{d}} \). Then if \( T \geq \left( \frac{4\lambda^{1/2} \|\beta^*\|_2}{\alpha \lambda F(\beta_T)} \right)^2 \) then projected gradient
descent guarantees that

\[ F(\hat{\beta}_T) \leq (1 + \epsilon)F(\beta^*) \]

Thus if the algorithm returns \( \hat{\beta} \) at the first time \( t \) where \( t \geq \left( \frac{4\beta^2}{\epsilon\lambda F(\beta)} \right)^2 \), then it achieves relative error at most \( \epsilon \).

### 1.2 Our Results

We start by making some observations about the gradient

\[ \nabla F = 2\lambda \beta - \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i \]

of the SVM objective function \( F \). First note the term \( 2\lambda \beta \) is trivial to compute, so let us focus on the term \( G = \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i \). Firstly only those points \( x_i \) that satisfy the additive constraint \( \mathcal{L} \) contribute to the gradient. Now let us focus on a particular dimension, and use \( x_{ik} \) to refer to the value of point \( x_i \) in dimension \( k \). Let \( L_k^- = \{ i \mid i \in \mathcal{L} \text{ and } y_i x_{ik} < 0 \} \) denote those points that satisfy \( \mathcal{L} \) and whose the gradient in the \( k^{th} \) coordinate has negative sign. Conceptually each point in \( L_k^- \) pushes the gradient in dimension \( k \) up with “force” proportional to its value in dimension \( k \). Let \( L_k^+ = \{ i \mid i \in \mathcal{L} \text{ and } y_i x_{ik} > 0 \} \) denote those points that satisfy \( \mathcal{L} \) and whose the gradient in the \( k^{th} \) coordinate has positive sign. And conceptually each point in \( L_k^+ \) pushes the gradient in dimension \( k \) down with “force” proportional to its value in dimension \( k \).

Next we note that \( G = \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i \) is what is called a FAQ-AI(1) query in [25, 2, 23] gives a relational approximation scheme (RAS) for certain FAQ-AI(1) queries. A RAS is a collection \( \{ A_e \} \) of relational algorithms where \( A_e \) achieve \((1 + \epsilon)-\)approximation. The results in [25] can be applied to obtain a RAS to compute a \((1 + \epsilon)-\)approximation \( \hat{G}_k^- \) to \( G_k^- = \frac{1}{N} \sum_{i \in L_k^-} y_i x_{ik} \), and a RAS to compute a \((1 + \epsilon)-\)approximation \( \hat{G}_k^+ \) to \( G_k^+ = \frac{1}{N} \sum_{i \in L_k^+} y_i x_{ik} \). However, the results in [23] cannot be applied to get a RAS for computing a \((1 + \epsilon)-\)approximation to \( G = G_k^- + G_k^+ \), as it suffers from what [25] calls the subtraction problem. Conceptually the subtraction problem is the fact that good approximations of scalars \( a \) and \( b \) are generally insufficient to deduce a good approximation of \( a - b \). This subtraction problem commonly arises in natural problems, and several examples are given in [25]. Thus an additional reason for our interest in relational algorithms to compute the (perhaps approximate) gradient of the SVM objective function is that we want to use it as test case to see if there is some way that we can surmount/circumvent the subtraction problem, and obtain a relational algorithm with a reasonable performance guarantee, ideally using techniques that are applicable to other problems in which this subtraction problem arises.

We start with a rather discouraging negative result that shows that we can not surmount the subtraction problem in the context of computing the gradient of the SVM objective problem. In particular, we show in Section [2] that computing an \( O(1) \) approximation to the partial derivative in a specified specified dimension is \( \#P \)-hard, even for acyclic joins. This kills plan A as a relational algorithm to compute the gradient would imply \( P = \#P \). This also makes it hard to imagine plan B working out since, assuming \( P \neq \#P \), a relational algorithm can’t even be sure that it is even approximately headed in the direction of the optimal solution, and thus its not reasonable to expect that we could find a relational algorithm to compute some sort of “pseudo-gradient” that would guarantee convergence on all instances.

Thus it seems we have no choice but to fall back to plan C. That is, we have to try to circumvent (not surmount) the subtraction problem. After some reflection, one reasonable interpretation of
our \#P-hardness proof is that it shows that computing the gradient is hard on unstable instances. In this context, intuitively an instance is stable if a nearly optimal solution remains nearly optimal if the points are perturbed slightly. Intuitively one would expect real world instances, where there is a hypothesis $\beta$ that explains the labels reasonably well, to be relatively stable (some discussion of the stability of SVM instances can be found in [16]). And for instances where there isn’t a hypothesis that explains the labels reasonably well, it probably doesn’t matter what hypothesis the algorithm returns, as it will likely be discarded by the data scientist anyways. Thus, our plan C will be to seek a gradient descent algorithm that has a similar convergence guarantee to gradient descent on stable instances.

Long story short, the main result of this paper is that this plan C works out. That is we give a relational algorithm that computes a “pseudo-gradient” that guarantees convergence for stable instances at a rate comparable to that achieved by using the actual gradient. The algorithm design can be found in Section 3, and the algorithm analysis can be found in Section 4. Postponing for the moment our formal definition of stability, we state our main result in Theorem 2. The reader should compare Theorem 2 to the analysis of gradient descent in Theorem 1.

**Theorem 2.** Let $X$ be an $(\alpha, \delta, \gamma)$-stable SVM instance formed by an acyclic join. Let $\beta^* = \arg\min_\beta F(\beta)$ be the optimal solution. Then there is a relational algorithm that can compute a pseudo-gradient in time $O\left(\frac{m^3}{\epsilon^2}(m^3\log^2(n))^2(m^2mn\log(n))\right)$, where $\epsilon = \min\left(\frac{\delta}{8}, \alpha\right)$. After $T = \left(\frac{256d^2}{\lambda h^2}\right)^2$ iterations of projected descent using this pseudo-gradient there is a relational algorithm that can compute in time $O\left(\frac{1}{\epsilon^2}(m^3\log^2(n))^2(m^2mn\log(n))\right)$ a hypothesis $\hat{\beta}$ such that:

$$F(\hat{\beta}, X) \leq (1 + \gamma)F(\beta^*, X)$$

**Main Takeaway Point:** In a broader context, we believe that our results suggest that this sort of stability analysis would likely yield useful insight in context of designing relational algorithms for other learning problems in which the subtraction problem arises.

### 1.3 Related Results

Relational algorithms are known for certain types of Sum of Sums (SumSum) and Sum of Products (SumProd) queries. In particular the Inside-Out algorithm [6] can evaluate a SumProd query in time $O(m^2d^2n^h\log n)$, where $m$ is the number of tables, $d$ is the number of columns, and $h$ is the fractional hypertree width [23] of the query. Note that $h = 1$ for the acyclic joins, and thus Inside-Out is a polynomial time algorithm for acyclic joins. One can reduce SumSum queries to $m$ SumProd queries [2], and thus they be solved in time $O(m^2d^2n^h\log n)$. The Inside-Out algorithm builds on several earlier papers, including [9, 21, 27, 23].

SumSum and SumProd queries with additive inequalities was first studied in [2]. [2] gave an algorithm with worst-case time complexity $O(md^2n^{m/2}\log n)$. So this is better than the standard practice of forming the design matrix, which has worst-case time complexity $\Omega(dn^m)$. Different flavors of queries with inequalities were also studied [26, 28, 5]. [25] showed that computing even very simple types of SumSum and SumProd queries with a single inequality is NP-hard. But an RAS for special types queries is introduced in [25]. The algorithm in [25] can obtain $(1 + \epsilon)$ approximation for problems such as counting the number of rows on one side of a hyperplane in time $O\left(\frac{1}{\epsilon^2}(m^3\log^2(n))^2(d^2mn^h\log(n))\right)$.

Algorithms for linear/polynomial regression on relational data are given in [38, 3, 4, 30, 31] and an algorithm for $k$-means clustering on relational data is given in [19].
Stability analysis, similar in spirit to our results, has been considered before in clustering problems [37, 17, 32, 7, 13, 20, 29, 36, 10, 12, 14]. For example, the NP-hard $k$-means, $k$-medians and $k$-centers clustering problems are polynomially solvable for instances in which changing the distances of the points by a multiplicative factor of at most 2 does not change the optimal solution [10, 12, 14].

SVM is discussed in almost every introductory machine learning textbook, for example [18]. Optimization methods for learning problems, including variations of gradient descent, are discussed in [39]. An overview of online convex optimization, which we use in our results, can be found in [15, 24].

2 Hardness of Gradient Approximation

Lemma 3. It $\#$P hard to $O(1)$-approximate the partial derivative of the SVM objective function in a specified dimension.

Proof. We reduce the decision version of the counting knapsack problem to the problem of approximating the gradient of SVM. The input to the decision counting knapsack problem is a set of weights $W = \{w_1, w_2, \ldots, w_m\}$, a knapsack size $L$, and an integer $k$. The output of the problem is whether there are $k$ different combinations of the items that fit into the knapsack.

We create $m + 1$ tables, each with two columns. The columns of the first $m$ table are (Key, $E_i$) for $T_i$ and the rows are

$$T_i = \{(1, 0), (1, w_i/L), (0, 0)\}.$$ 

The last table has two columns (Key, Value), and it has two rows $(1, 1), (0, -k)$. Note that if we take the join of these tables, there will be $m + 2$ columns: (Key, Value, $E_1, E_2, \ldots, E_m$).

Let $\beta = (0, 0, 1, 1, \ldots, 1)$ and $\lambda = 0$, so $\beta$ is 0 on the columns Key and Value and 1 everywhere else. Then we claim, if the gradient of $F$ on the second dimension (Value) is non-negative, then the answer to the original counting knapsack is true, otherwise, it is false.

To see the reason, consider the rows in $J$: there are $2^m$ rows in the design matrix that have $(1, 1)$ in the first two dimensions and all possible combinations of the knapsack items in the other dimensions. More precisely, the concatenation of $(1, 1)$ and $w_S$ for every $S \in [m]$ where $w_S$ is the vector that has $w_i/L$ in the $i$-th entry if item $i$ is in $S$ or 0 otherwise. Further, $J$ has a single special row with values $(0, -k, 0, 0, \ldots, 0)$. Letting $G_2$ be the gradient of SVM on the second dimension (column Value), we have,

$$G_2 = \sum_{x \in J: 1 - \beta x \geq 0} x_2$$

For the row with Key = 1 for each $S \in [m]$, we have $1 - \beta x = 1 - \sum_{i \in S} w_i/L \geq 0$ if and only if the items in $S$ fits into the knapsack and $x_2 = 1$. For the single row with Key = 0, we have $1 - \beta x = 1$, and its value on the second dimension is $x_2 = k$. Therefore,

$$G_2 = C_L(w_1, \ldots, w_m) - k$$

where $C_L$ is the number of subsets of items fitting into the knapsack of size $L$. This means if we could approximate the gradient up to any constant factor, we would be able to determine if $G_2$ is positive or negative, and as a result we would be able to answer the (decision version of) counting knapsack problem, which is $\#$P-hard.\[\square\]
3 Algorithm Design

3.1 Review of Row Counting with a Single Additive Constraint

We now summarize algorithmic results from [25] for two different problems, that we will use as a black box.

In the first problem the input is a collection $T_1, \ldots, T_m$ of tables, a label $\ell \in \{-1, +1\}$, and an additive constraint $L$ of the form $\sum_{j \in [d]} g_j(x_j) \geq R$, where each function $g_j$ can be computed in constant time. The output consists of, for each $j \in [d]$ and $i \in D(j)$, where $D(j)$ is the domain of column/feature $j$, the number $C_{j,v}^\ell$ of rows in the design matrix $J = T_1 \times \cdots \times T_m$ that satisfy constraint $L$, that have label $\ell$, and that have value $v$ in column $j$. [25] gives a relational algorithm, which we will call the Row Counting Algorithm, that computes a $(1 + \epsilon)$-approximation for each such $C_{j,v}^\ell$ to each $C_j$, and that runs in time $O\left(\frac{m^3 \log^2(n)}{\epsilon^2} (d^2 mn^h \log(n))\right)$.

In the second problem the input is a collection $T_1, \ldots, T_m$ of tables, a label $\ell \in \{-1, +1\}$, and an expression in the form of $\sum_{j \in [d]} g_j(x_j)$, where the $g_j$ functions can be computed in constant time. The output consists of, for each $k \in [0, \log_{1+\epsilon} N]$, maximum value of $H_k$ such that the number of points in the design matrix $J = T_1 \times \cdots \times T_m$ with label $\ell \in \{-1, 1\}$ satisfying the additive inequality $\sum_{j \in [d]} g_j(x_j) \geq H_k$ is at least $\lceil(1 + \epsilon)^k\rceil$. [25] gives an algorithm for this problem, which we will call the Generalized Row Counting Algorithm, that runs in time $O\left(\frac{m^3 \log^2(n)}{\epsilon^2} (d^2 mn^h \log(n))\right)$. Using the result of the algorithm, for any scalar distance $H$, it is possible to obtain a row count $\hat{N}(H)$ such that $N(H)/(1 + \epsilon) \leq \hat{N}(H) \leq N(H)$, where $N(H)$ is the number of points in the design matrix with label $\ell$ satisfying the inequality $\sum_{j \in [d]} g_j(x_j) \geq H_k$.

3.2 Overview of Our Approach

Recall from the introduction that the difficulty arises when a $\hat{G}^+_k$ is approximately equal to $-\hat{G}^-_k$. In this case, it would seem that by appropriately perturbing one of $L_1^-$ or $L_1^+$ by a relatively small amount one could force $G = \hat{G}^- + \hat{G}^+$ for this perturbed instance. In which case, if we used $2\lambda \beta(t) + (\hat{G}^- + \hat{G}^+)$ as the pseudo-gradient, then it would be the true gradient for a slightly perturbed instance. However, this isn’t quite right, as there is an additional issue. If we perturb a point $x_i$, then the sign of $1 - y_i \beta(x_i)$ may change, which means this point’s contribution to the gradient may discontinuously switch between zero and $-y_i x_i$. To address this issue, when computing the pseudo-gradient, we use a new instance $X'$ that excludes points that are “close” to the separating hyperplane $1 - y_i \beta(x_i) = 0$. That is, $X'$ excludes every point that can change sides of the hyperplane in an $\epsilon$-perturbation of each coordinate. This will allow us to formally conclude that if we used $2\lambda \beta(t) + (\hat{G}^- + \hat{G}^+)$, where $\hat{G}^-$ and $\hat{G}^+$ are defined on $X'$, as the pseudo-gradient, then it would be the true gradient for a slightly perturbed instance. After the last descent step, we choose the final hypothesis to be the $\epsilon$-perturbation of any computed hypothesis $\beta(t)$, $t \in [0, T]$ that minimizes the SVM objective.

In the analysis we interpret the sequence $\beta(0), \beta(1), \ldots, \beta(T)$ as solving an online convex optimization problem, and apply known techniques from this area.

3.3 Pseudo-gradient Descent Algorithm

Firstly, in linear time it is straight-forward to determine if the points in $X$ lie in $[-1, 1]$, and if not, to rescale so that they do; This can be accomplished by, for each feature, dividing all the values of that feature in all of the input tables by maximum absolute value of that feature. The initial hypothesis $\beta(0)$ is the origin. For any vector $v$, let $u = |v|$ be a vector such that its entries are the absolute values of $v$, meaning for all $j, u_j = |v_j|$.
Algorithm to Compute the Pseudo-gradient:

1. Run the Row Counting Algorithm to compute, for each $j \in [d]$ and $v \in D(j)$, a $(1 + \epsilon)$ approximation $\hat{C}_{j,v}^-$ to $C_{j,v}$, which is the number of rows in $x \in J$ with negative label, satisfying $1 + \beta(t) \cdot x \geq \epsilon |\beta(t)| \cdot |x|$.  

2. Run the Row Counting Algorithm to compute, for each $j \in [d]$ and $v \in D(j)$, a $(1 + \epsilon)$ approximation $\hat{C}_{j,v}^+$ to $C_{j,v}^+$, which is the number of rows in $x \in J$ with positive label, satisfying $1 - \beta(t) \cdot x \geq \epsilon |\beta(t)| \cdot |x|$.  

3. For all $k \in [d]$, compute $\hat{G}_k^- = \sum_{v \in D(k):v < 0} v \hat{C}_{k,v}^- - \sum_{v \in D(k):v \geq 0} v \hat{C}_{k,v}^+$.  

4. For all $k \in [d]$, compute $\hat{G}_k^+ = \sum_{v \in D(k):v \geq 0} v \hat{C}_{k,v}^- - \sum_{v \in D(k):v < 0} v \hat{C}_{k,v}^+$.  

5. The pseudo-gradient is then

$$\hat{G} = \frac{\hat{G}^- + \hat{G}^+}{N} + \lambda \beta(t)$$

Algorithm for a Single Descent Step: The next hypothesis $\beta^{(t+1)}$ is

$$\beta^{(t+1)} = \Pi_K(\beta^{(t)} - \eta_{t+1} \hat{G})$$

Here $\eta_t = \frac{1}{\lambda \sqrt{dt}}$ and $\Pi_K(\beta)$ is the projection of $\beta$ onto a hypersphere $K$ centered at the origin with radius $\frac{\sqrt{d}}{\lambda}$. Note that $\Pi_K(\beta)$ is $\beta$ if $\|\beta\|_2 \leq \frac{\sqrt{d}}{\lambda}$ and $\frac{\sqrt{d}}{\lambda} \beta$ otherwise.

Algorithm to Compute the Final Hypothesis: After $T - 1$ descent steps, the algorithm calls the Generalized Row Counting twice for each $t \in [0, T - 1]$, with the following inputs:

- $\ell = 1$ and additive expression $1 - \beta(t) \cdot x_i - \epsilon |\beta(t)| \cdot |x_i|$
- $\ell = -1$ and additive expression $1 + \beta(t) \cdot x_i - \epsilon |\beta(t)| \cdot |x_i|$

Note that both of these expressions are equivalent to $1 - y_i \beta(t) \cdot x_i - \epsilon |\beta(t)| \cdot |x_i|$. Let the array $H^+$ be the output for the first call and $H^-$ be the output for the second call. Note that $H^+$ and $H^-$ are monotonically decreasing by the the definition of the Generalized Row Counting algorithm. Let $L^+$ be the largest $k$ such that $H^+_k \geq 0$ and $L^-$ be the largest $k$ such that $H^-_k \geq 0$. The algorithm then returns as its final hypothesis $\hat{\beta}$, the hypothesis $\beta(t)$ where $\hat{t}$ is defined by:

$$\hat{t} = \text{argmin}_{t \in [T]} \hat{F}(\beta(t), X)$$

(5)

where

$$\hat{F}(\beta(t), X) = \frac{1}{N} \left( \sum_{k=0}^{L^+ - 1} (1 + \epsilon)^k (H^+_k - H^+_{k+1}) + (1 + \epsilon)^{L^+} H^+_{L^+} \right)$$

$$+ \frac{1}{N} \left( \sum_{k=0}^{L^- - 1} (1 + \epsilon)^k (H^-_k - H^-_{k+1}) + (1 + \epsilon)^{L^-} H^-_{L^-} \right) + \lambda \|\beta(t)\|_2^2$$

(6)

Note that the values $L^-, L^+, H^+$ and $H^-$ in the definition of $\hat{F}$, in equation (6), all depend upon $t$, which we suppressed to make the notation somewhat less ugly.
4 Algorithm Analysis

In subsection 4.1 we prove Theorem 5 which bounds the convergence of our projected pseudo-gradient descent algorithm in a rather nonstandard way by applying known results on online convex optimization [15, 24]. In subsection 4.2 we introduce our definition of stability and then prove Theorem 6.

4.1 Perturbation Analysis

Before stating Theorem 5, we need some definitions.

Definition 4.

- A point $p$ is an $\epsilon$-perturbation of point $q$ if every component of $p$ is within $(1+\epsilon)$ factor of the corresponding component of $q$. Meaning in each dimension $j$ we have $(1-\epsilon)q \leq p \leq (1+\epsilon)q$.
- A point set $X_a$ is an $\epsilon$-perturbation of a point set $X_b$ if there is a bijection between $X_a$ and $X_b$ such that every point in $X_a$ is an $\epsilon$-perturbation of its corresponding point in $X_b$.
- Let $\beta^* = \text{argmin}_\beta F(\beta, X)$ to be the optimal solution at $X$.
- For any $\epsilon$-perturbation $X_a$ of $X$, define $\beta_a^* = \text{argmin}_\beta F(\beta, X_a)$ to be the optimal solution at $X_a$.
- For a given hypothesis $\beta$, we call a point $x$ with label $y$ close if there is some $\epsilon$-perturbation $x'$ of $x$ such that $1 - y\beta x' < 0$; otherwise it is called far. In other words, a point $x$ with label $y$ is close if $1 - y\beta \cdot x < \epsilon |\beta| \cdot |x|$.

Theorem 5. Assume our projected pseudo-gradient descent algorithm ran for $T - 1$ descent steps. Then for all hypotheses $\beta \in \mathbb{R}^d$ there exist $\epsilon$-perturbations $X_a$ and $X_b$ of $X$ such that

$$F(\tilde{\beta}, X_a) \leq (1 + \epsilon)F(\beta, X_b) + \frac{32d\beta^2}{\lambda\sqrt{T}}$$

To prove Theorem 5, our main tool is a result from the online convex optimization literature [15, 24].

Theorem 6. [15, 24] Let $g_1, g_2, \ldots, g_T : \mathbb{R}^n \to \mathbb{R}$ be $G$-Lipschitz functions over a convex region $\mathcal{K}$, i.e., $||\nabla g_t(\beta)|| \leq G$ for all $\beta \in \mathcal{K}$ and all $t$. Then, starting at point $\beta^{(0)} \in \mathbb{R}^n$ and using the update rule of $\beta^{(t)} \leftarrow \Pi_{\mathcal{K}} (\beta^{(t-1)} - \eta_t \nabla g_{t-1}(\beta^{(t-1)}))$, with $\eta_t = \frac{D}{G\sqrt{t}}$ for $T - 1$ steps, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} g_t(\beta^{(t)}) \leq \frac{1}{T} \sum_{t=0}^{T-1} g_t(\beta^*) + \frac{2DG}{\sqrt{T}}$$

for all $\beta^*$ with $||\beta^{(0)} - \beta^*|| \leq D$.

To apply this Theorem 6, we set $g_t = F(\beta^{(t)}, X^{(t)}, Y)$, where $X^{(t)}$ is an $\epsilon$-perturbation of $X$, such that the pseudo-gradient at $X$ is equal to the true gradient at $X^{(t)}$. We establish the existence of $X^{(t)}$ in Lemma 7. Thus our projected pseudo-gradient descent algorithm updates the hypothesis exactly the same as stated in Theorem 5 (assuming that we use the same upper bounds on $D$ and $G$). Then in definition 8 we identify the $\epsilon$-permutation $Z$ that minimizes $F(\beta, Z)$, and then in Lemma 9 bound the relative error between $\tilde{F}(\beta, X)$ and $F(\beta, Z)$. Finally this will allow use in Lemma 10 and Lemma 11 we show the existence of $X_b$ and $X_a$, respectively, that will allow us to conclude the proof of Theorem 5.
Lemma 7. In every descent step $t$, the computed pseudo-gradient $\hat{G}$ is the exact gradient of $F(\beta(t), X(t))$ for some point set $X(t)$ that is an $\epsilon$-perturbation of $X$.

Proof. To prove the claim, we show how to find a desired $X(t)$ – this is only for the sake of the proof and the algorithm doesn’t need to know $X(t)$. We call any point $x$ with label $y$ “far” if it satisfies the inequality

$$1 - y\beta(t) \cdot x \geq \epsilon \hat{\beta(t)} \cdot |x|$$

(8)

, otherwise we call the point “close”. Note that for a far point there is no $\epsilon$-perturbation to make the derivative of the loss function $0$. That is, for any point $x$ with label $y$, if $1 - y\beta \cdot x \geq \epsilon \sum_{j \in [d]} |\beta_j| |x_j|$, then we have $1 - y\beta x' \geq 0$ for any $x'$ that is $\epsilon$-perturbation of $x$. To see this, note that we have

$$1 - y\beta x' = 1 - \sum_{k=1}^d \hat{\beta}_k x_k' \geq 1 - \sum_{k=1}^d (\beta_k x_k + |\beta_k| |x_k|) \geq 0$$

because of $x'$ being $\epsilon$-perturbation of $x$. On the other hand, for all the close points there exists a perturbation $x'$ such that $1 - y\beta(t) \cdot x' < 0$.

We first perturb all of the close points such that they don’t have any effect on the gradient.

Next, we need to show a perturbation of the far points for which the $\hat{G}$ is the gradient of the loss function. Let $X_f^+$ and $X_f^-$ be the set of far points with positive and negative labels. Let $X_f = X_f^+ \cup X_f^-$. We show the perturbation for each dimension $k$ separately. Based on definition of $\hat{G}_k^+$ and $\hat{G}_k^-$ we have:

$$\hat{G}_k^+ + \hat{G}_k^- = \sum_{v \in D(k)} v \hat{C}_{k,v}^- - \sum_{v \in D(k)} v \hat{C}_{k,v}^+$$

$$= \sum_{v \in D(k)} v (1 \pm \epsilon) C_{k,v}^- - \sum_{v \in D(k)} v (1 \pm \epsilon) C_{k,v}^+$$

Note that $C_{k,v}^+$ is the number of points in $X_f^+$ with value $v$ in dimension $k$. Therefore,

$$\hat{G}_k^+ + \hat{G}_k^- = \sum_{v \in D(k)} v (1 \pm \epsilon) C_{k,v}^- - \sum_{v \in D(k)} v (1 \pm \epsilon) C_{k,v}^+$$

$$= \sum_{x_i \in X_f^-} (1 \pm \epsilon) x_i,k - \sum_{x_i \in X_f^+} (1 \pm \epsilon) x_i,k$$

$$= - \sum_{x_i \in X_f} (1 \pm \epsilon) y_i x_i,k$$

where the last term is $N \frac{\partial L(\beta(t), X(t))}{\partial \beta_k}$ where $X(t)$ an $\epsilon$-perturbation of $X$. \)

Definition 8. Let $Z(t)$ be an $\epsilon$-perturbation of $X$ such that for all $z_i \in Z(t)$ and for all dimensions $k$

$$z_{i,k} = \begin{cases} (1 - \epsilon) x_{i,k} & y_i \beta_{k}^{(t)} \geq 0 \vspace{1em} \\ (1 + \epsilon) x_{i,k} & y_i \beta_{k}^{(t)} < 0 \end{cases}$$

Note that this $\epsilon$-perturbation minimizes $F(\beta(t), Z(t))$.

Lemma 9. $\frac{1}{1+\epsilon} F(\beta(t), Z(t)) \leq \hat{F}(\beta(t), X) \leq F(\beta(t), Z(t))$. 

10
Proof. Consider a value \( t \) and let \( N^+(\tau) = \{ x_i \mid y_i = +1 \text{ and } 1 - \beta(t) \cdot x_i - \epsilon \beta(t) \cdot |x_i| \geq \tau \} \), and \( N^-(\tau) = \{ x_i \mid y_i = -1 \text{ and } 1 + \beta(t) \cdot x_i - \epsilon \beta(t) \cdot |x_i| \geq \tau \} \).

Before proving the lemma we prove the following claim: \( F(\beta(t), Z(t)) = \frac{1}{N} \int_{\tau=0}^{\infty} N^+(\tau)d\tau + \frac{1}{N} \int_{\tau=0}^{\infty} N^-(\tau)d\tau + \lambda \|\beta(t)\|^2 \).

Note that based on the definition of \( Z(t) \) it is the case that \( 1 - y_i\beta(t) \cdot z_i = 1 - y_i\beta(t) \cdot x_i - \epsilon \beta(t) \cdot |x_i| \); therefore, \( N^+(\tau) = \{ y_i = +1 \in Z(t) \text{ and } 1 - y_i\beta(t) \cdot z_i \geq \tau \} \) and \( N^-(\tau) = \{ y_i = -1 \in Z(t) \text{ and } 1 - y_i\beta(t) \cdot z_i \geq \tau \} \). Hence,

\[
L(\beta(t), Z(t)) = \frac{1}{N} \sum_{i} \max(0, 1 - y_i\beta \cdot z_i) = \frac{1}{N} \sum_{i:1-y_i\beta \cdot z_i \geq 0} 1 - y_i\beta \cdot z_i
\]

\[
= \frac{1}{N} \sum_{i:1-y_i\beta \cdot z_i \geq 0} \int_{\tau=0}^{\infty} d\tau = \frac{1}{N} \int_{\tau=0}^{\infty} \sum_{i:1-y_i\beta \cdot z_i \geq \tau} d\tau
\]

\[
= \frac{1}{N} \int_{\tau=0}^{\infty} (N^+(\tau) + N^-(\tau))d\tau
\]

Therefore,

\[
F(\beta(t), Z(t)) = \frac{1}{N} \int_{\tau=0}^{\infty} N^+(\tau)d\tau + \frac{1}{N} \int_{\tau=0}^{\infty} N^-(\tau)d\tau + \lambda \|\beta(t)\|^2
\]

(9)

The number of points with label \( \ell \) satisfying \( 1 - \ell \beta(t) \cdot x_i - \epsilon \beta(t) \cdot |x_i| \geq \tau \) for any \( \tau \in [H^+_{k}, H^+_{k+1}] \) is in the range \([[(1+\epsilon)^k], [(1+\epsilon)^{(k+1)}]]\). Therefore, the claim follows by replacing \( N^+(\tau) \) in Equation (9) with \((1+\epsilon)^k\) for all the values of \( \tau \in [H^+_{k}, H^+_{k+1}] \) and replacing \( N^-(\tau) \) in (9) with \((1+\epsilon)^k\) for all the values of \( \tau \in [H^-_{k}, H^-_{k+1}] \).

\[\blacksquare\]

Lemma 10. For all hypothesis \( \beta \), there exists an \( \epsilon \)-perturbation \( X_b \) of \( X \) such that

\[
\min_s F(\beta(s), Z(s)) \leq F(\beta, X_b) + \frac{2DG}{\sqrt{T}}
\]

Proof. By Theorem 6

\[
\frac{1}{T} \sum_{t=0}^{T-1} F(\beta(t), X(t)) \leq \frac{1}{T} \sum_{t=0}^{T-1} F(\beta, X(t)) + \frac{2DG}{\sqrt{T}}
\]

(10)

Then

\[
\min_s F(\beta(s), Z(s)) \leq \frac{1}{T} \sum_{t=0}^{T-1} F(\beta(t), Z(t)) \leq \frac{1}{T} \sum_{t=0}^{T-1} F(\beta(t), X(t)).
\]

(11)

The first inequality follows since the minimum is less than the average, and the second inequality follows from the definition of \( Z(t) \). Let \( u = \arg \max_s F(\beta(s), X(t)) \), and \( X_b = X(u) \). Then

\[
\frac{1}{T} \sum_{t=0}^{T-1} F(\beta, X(t)) \leq \max_t F(\beta, X(t)) = F(\beta, X_b)
\]

(12)

Thus, combining lines (10), (11) and (12) we can conclude that:

\[
\min_s F(\beta(s), Z(s)) \leq F(\beta, X_b) + \frac{2DG}{\sqrt{T}}
\]

(13)

\[\blacksquare\]
Lemma 11. There exists an $\epsilon$-perturbation $X_a$ of $X$ such that
\[
F(\tilde{\beta}, X_a) \leq (1 + \epsilon) \min_{s} F(\beta^*(s), Z^{(s)})
\]

Proof. Let $X_a = Z^{(\tilde{t})}$ where
\[
\begin{align*}
F(\tilde{\beta}, X_a) &\leq (1 + \epsilon) \tilde{F}(\tilde{\beta}, X) & \text{By Lemma 9} \\
&= (1 + \epsilon) \min_{s} \tilde{F}(\beta^{(s)}, X) & \text{By definition of } \tilde{\beta} \\
&\leq (1 + \epsilon) \min_{s} F(\beta^{(s)}, Z^{(s)}) & \text{By Lemma 9}
\end{align*}
\]

4.2 Stability Analysis

Our formal definition of stability, which we give in Definition 12 while not unnatural, is surely not the first natural formalization that one would think of. Our formal definition is more or less forced on us, which leads to the type of non-traditional approximation achieved in Theorem 5.

Definition 12. An SVM instance $X$ is $(\alpha, \delta, \gamma)$-stable for $\delta \leq 1$ if for all $X_a$ and $X_b$ that are $\alpha$-perturbations of $X$ it is the case that:

- $\beta_a^*$ is a $(1 + \delta)$ approximation to the optimal objective value at $X_b$, that is, $F(\beta_a^*, X_b) \leq (1 + \delta) \min_{\beta} F(\beta, X_b)$.
- If $\beta_a$ is $(1 + 2\delta)$ approximation to the optimal SVM objective value at $X_a$ then $\beta_a$ is a $(1 + \gamma)$ approximation to the optimal SVM objective value at $X_b$. That is if $F(\beta_a, X_a) \leq (1 + 2\delta) \min_{\beta} F(\beta, X_a)$ then $F(\beta_a, X_b) \leq (1 + \gamma) \min_{\beta} F(\beta, X_b)$

Proof of Theorem 2. Let $\epsilon \leq \min(\delta/8, \alpha)$.
\[
F(\tilde{\beta}, X_a) \leq (1 + \epsilon) F(\beta_a^*, X_b) + \frac{32d^{3/2}}{\lambda \sqrt{T}}
\]
\[
\begin{align*}
&= (1 + \epsilon)(1 + \delta) F(\beta_a^*, X_a) + \frac{32d^{3/2}}{\lambda \sqrt{T}} & \text{By definition of stability} \\
&= (1 + \epsilon)(1 + \delta) F(\beta_a^*, X_a) + \frac{\delta}{8} F(\tilde{\beta}, X_a) & \text{By definition of } T \\
&\leq \frac{(1 + \delta)(1 + \epsilon)}{1 - \delta/8} F(\beta_a^*, X_a) & \text{By algebra} \\
&\leq (1 + 2\delta) F(\beta_a^*, X_a) & \text{by definition of } \epsilon
\end{align*}
\]

Finally since $\tilde{\beta}$ is $(1 + 2\delta)$ approximate solution at $X_a$, by the definition of stability, $\tilde{\beta}$ is a $(1 + \gamma)$ approximate solution at $X$.

References

[1] Kaggle machine learning and data science survey. [https://www.kaggle.com/kaggle/kaggle-survey-2018](https://www.kaggle.com/kaggle/kaggle-survey-2018), 2018.
[2] Mahmoud Abo Khamis, Ryan R Curtin, Benjamin Moseley, Hung Q Ngo, XuanLong Nguyen, Dan Olteanu, and Maximilian Schleich. On functional aggregate queries with additive inequalities. In Proceedings of the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, pages 414–431. ACM, 2019.

[3] Mahmoud Abo Khamis, Hung Q Ngo, XuanLong Nguyen, Dan Olteanu, and Maximilian Schleich. Ac/dc: in-database learning thunderstruck. In Proceedings of the Second Workshop on Data Management for End-To-End Machine Learning, page 8. ACM, 2018.

[4] Mahmoud Abo Khamis, Hung Q. Ngo, XuanLong Nguyen, Dan Olteanu, and Maximilian Schleich. In-database learning with sparse tensors. In Proceedings of the 37th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, SIGMOD/PODS ’18, pages 325–340, New York, NY, USA, 2018. ACM.

[5] Mahmoud Abo Khamis, Hung Q. Ngo, Dan Olteanu, and Dan Suciu. Boolean tensor decomposition for conjunctive queries with negation. In ICDT, pages 21:1–21:19, 2019.

[6] Mahmoud Abo Khamis, Hung Q. Ngo, and Atri Rudra. Faq: Questions asked frequently. In Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS ’16, pages 13–28, New York, NY, USA, 2016. ACM. URL: http://doi.acm.org/10.1145/2902251.2902280, doi:10.1145/2902251.2902280.

[7] Margareta Ackerman and Shai Ben-David. Clusterability: A theoretical study. In Artificial intelligence and statistics, pages 1–8, 2009.

[8] Isolde Adler. Width functions for hypertree decompositions. 2006. Ph.D. Dissertation, Albert-Ludwigs-Universität Freiburg. 2006.

[9] S. M. Aji and R. J. McEliece. The generalized distributive law. IEEE Trans. Inf. Theor., 46(2):325–343, 2006.

[10] Haris Angelidakis, Konstantin Makarychev, and Yury Makarychev. Algorithms for stable and perturbation-resilient problems. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 438–451, 2017.

[11] Albert Atserias, Martin Grohe, and Dániel Marx. Size bounds and query plans for relational joins. In 2008 49th Annual IEEE Symposium on Foundations of Computer Science, pages 739–748. IEEE, 2008.

[12] Pranjal Awasthi, Avrim Blum, and Or Sheffet. Center-based clustering under perturbation stability. Information Processing Letters, 112(1-2):49–54, 2012.

[13] Maria-Florina Balcan, Avrim Blum, and Anupam Gupta. Clustering under approximation stability. Journal of the ACM (JACM), 60(2):1–34, 2013.

[14] Maria Florina Balcan and Yingyu Liang. Clustering under perturbation resilience. SIAM Journal on Computing, 45(1):102–155, 2016.

[15] Nikhil Bansal and Anupam Gupta. Potential-function proofs for first-order methods. CoRR, abs/1712.04581, 2017. URL: http://arxiv.org/abs/1712.04581 arXiv:1712.04581.

[16] Jinbo Bi and Tong Zhang. Support vector classification with input data uncertainty. In Advances in neural information processing systems, pages 161–168, 2005.
[17] Yonatan Bilu and Nathan Linial. Are stable instances easy? *Combinatorics, Probability and Computing*, 21(5):643–660, 2012.

[18] Andriy Burkov. *The hundred-page machine learning book*. Andriy Burkov Quebec City, Can., 2019.

[19] Ryan R. Curtin, Benjamin Moseley, Hung Q. Ngo, XuanLong Nguyen, Dan Olteanu, and Maximilian Schleich. Rk-means: Fast clustering for relational data. *CoRR*, abs/1910.04939, 2019. URL: [http://arxiv.org/abs/1910.04939](http://arxiv.org/abs/1910.04939) [arXiv:1910.04939](http://arxiv.org/abs/1910.04939).

[20] Amit Daniely, Nati Linial, and Michael Saks. Clustering is difficult only when it does not matter. *arXiv preprint arXiv:1205.4891*, 2012.

[21] Rina Dechter. Bucket elimination: A unifying framework for probabilistic inference. In *Proceedings of the Twelfth International Conference on Uncertainty in Artificial Intelligence*, 1996.

[22] Martin Grohe. The structure of tractable constraint satisfaction problems. In *International Symposium on Mathematical Foundations of Computer Science*, pages 58–72. Springer, 2006.

[23] Martin Grohe and Dániel Marx. Constraint solving via fractional edge covers. In *SODA*, pages 289–298, 2006.

[24] Elad Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3-4):157–325, 2016. doi:10.1561/2400000013.

[25] Mahmoud Abo Khamis, Sungjin Im, Benjamin Moseley, Kirk Pruhs, and Alireza Samadian. Approximate aggregate queries under additive inequalities. *CoRR*, abs/2003.10588, 2020. [arXiv:2003.10588](http://arxiv.org/abs/2003.10588).

[26] Anthony Klug. On conjunctive queries containing inequalities. *J. ACM*, 35(1):146–160, January 1988. URL: [http://doi.acm.org/10.1145/42267.42273](http://doi.acm.org/10.1145/42267.42273) doi:10.1145/42267.42273.

[27] J. Kohlas and N. Wilson. Semiring induced valuation algebras: Exact and approximate local computation algorithms. *Artif. Intell.*, 172(11):1360–1399, 2008.

[28] Paraschos Koutris, Tova Milo, Sudeepa Roy, and Dan Suciu. Answering conjunctive queries with inequalities. *Theory of Computing Systems*, 61(1):2–30, Jul 2017.

[29] Amit Kumar and Ravindran Kannan. Clustering with spectral norm and the k-means algorithm. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 299–308. IEEE, 2010.

[30] Arun Kumar, Jeffrey Naughton, and Jignesh M. Patel. Learning generalized linear models over normalized data. In *Proceedings of the 2015 ACM SIGMOD International Conference on Management of Data*, SIGMOD ’15, pages 1969–1984. ACM, 2015.

[31] Arun Kumar, Jeffrey Naughton, Jignesh M. Patel, and Xiaojin Zhu. To join or not to join?: Thinking twice about joins before feature selection. In *Proceedings of the 2016 International Conference on Management of Data*, SIGMOD ’16, pages 19–34, New York, NY, USA, 2016. ACM. URL: [http://doi.acm.org/10.1145/2882903.2882952](http://doi.acm.org/10.1145/2882903.2882952) doi:10.1145/2882903.2882952.
[32] Konstantin Makarychev, Yury Makarychev, and Aravindan Vijayaraghavan. Bilu–linial stable instances of max cut and minimum multiway cut. In Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms, pages 890–906. SIAM, 2014.

[33] Dániel Marx. Tractable hypergraph properties for constraint satisfaction and conjunctive queries. Journal of the ACM (JACM), 60(6):42, 2013.

[34] Hung Q. Ngo, Ely Porat, Christopher Ré, and Atri Rudra. Worst-case optimal join algorithms: [extended abstract]. In Proceedings of the 31st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS ’12, pages 37–48, New York, NY, USA, 2012. ACM. URL: [http://doi.acm.org/10.1145/2213556.2213565](http://doi.acm.org/10.1145/2213556.2213565), doi:10.1145/2213556.2213565.

[35] Hung Q. Ngo, Christopher Ré, and Atri Rudra. Skew strikes back: New developments in the theory of join algorithms. In SIGMOD Rec., pages 5–16, 2013.

[36] Rafail Ostrovsky, Yuval Rabani, Leonard J Schulman, and Chaitanya Swamy. The effectiveness of lloyd-type methods for the k-means problem. Journal of the ACM (JACM), 59(6):1–22, 2013.

[37] Tim Roughgarden. Beyond worst-case analysis. Communications of the ACM, 62(3):88–96, 2019.

[38] Maximilian Schleich, Dan Olteanu, and Radu Ciucanu. Learning linear regression models over factorized joins. In Proceedings of the 2016 International Conference on Management of Data, SIGMOD ’16, pages 3–18. ACM, 2016.

[39] Suvrit Sra, Sebastian Nowozin, and Stephen J Wright. Optimization for machine learning. Mit Press, 2012.

[40] Todd L. Veldhuizen. Triejoin: A simple, worst-case optimal join algorithm. In ICDT, pages 96–106, 2014.
A Analysis of Gradient Descent for SVM

Theorem 13 and Corollary 14 give bounds on the number of iterations on projected gradient descent to reach solutions with bounded absolute error and bounded relative error, respectively.

**Theorem 13.** [15, 24] Let $\mathcal{K}$ be a convex body and $F$ be a function such that $\|\nabla F(\beta)\|_2 \leq G$ for $\beta \in \mathcal{K}$. Let $\beta^* = \arg\min_{\beta \in \mathcal{K}} F(\beta)$ be the optimal solution. Let $D$ be an upper bound on $\|\beta^{(0)} - \beta^*\|_2$, the 2-norm distance from the initial candidate solution to the optimal solution. Let $\hat{\beta}_t = \frac{1}{s} \sum_{t=0}^{s-1} \beta^{(t)}$. Let $\eta_t = \frac{D}{G \sqrt{t}}$. Then after $T - 1$ iterations of projected gradient descent, it must be the case that

$$F(\hat{\beta}_T) - F(\beta^*) \leq \frac{2DG}{\sqrt{T}}$$

**Corollary 14.** Adopting the assumptions from Theorem 13, if $T \geq \left(\frac{4DG}{\epsilon F(\beta_T)}\right)^2$ then $F(\hat{\beta}_T) \leq (1 + \epsilon)F(\beta^*)$

That is, projected gradient descent achieves relative error $\epsilon$.

The gradient of SVM objective $F$ is

$$\nabla F = 2\lambda \beta - \frac{1}{N} \sum_{i \in \mathcal{L}} x_i$$

where $\mathcal{L}$ is the collection $\{i \mid \beta x_i \leq 1\}$ of indices $i$ where $x_i$ is currently contributing to the objective. Note that in this hinge loss function, the gradient for the points on the hyperplane $1 - \beta x = 0$ does not exist, since the gradient is not continuous at this point. In our formulation we have used the sub-gradient for the points on $1 - \beta x = 0$, meaning for a $\beta$ on the hyperplane $1 - \beta x = 0$, we have used the limit of the gradient of the points that $1 - \beta' x > 0$ when $\beta'$ goes to $\beta$. For all the points that $1 - \beta' x > 0$, the gradient is $x$; therefore, the limit is also $x$.

Assume $\beta^{(0)}$ is the origin and adopt the assumptions of Theorem 13. Then $\nabla F(\beta^*) = 0$ implies for any dimension $j$

$$|\beta^*_j| = \left| \frac{1}{2N\lambda} \sum_{i \in \mathcal{L}} x_{ij} \right| \leq \frac{1}{2\lambda}$$

where the additional subscript of $j$ refers to dimension $j$. And thus

$$\|\beta^{(0)} - \beta^*\|_2 \leq \|\beta^*\|_2 \leq \sqrt{d} \max_{j \in [d]} |\beta^*_j| \leq \frac{\sqrt{d}}{2\lambda}$$

Thus let us define our convex body $\mathcal{K}$ to be the hypersphere with radius $\frac{\sqrt{d}}{2\lambda}$ centered at the origin.
Thus for $\beta \in K$, 
\[
\|\nabla F(\beta)\|_2 = \sqrt{\sum_{j \in [d]} \left(2\lambda \beta_j - \frac{1}{N} \sum_{i \in L} x_{ij}\right)^2} 
\leq \sqrt{\sum_{j \in [d]} 4(\lambda \beta_j)^2 + 2 \left(\frac{1}{N} \sum_{i \in L} x_{ij}\right)^2} 
\text{Since } (a - b)^2 \leq 2a^2 + 2b^2
\leq 2\lambda \sqrt{\sum_{j \in [d]} \beta_j^2 + \frac{\sqrt{2}}{N} \sum_{j \in [d]} \sum_{i \in L} |x_{ij}|} 
\text{Since } \sqrt{\sum_i a_i^2} \leq \sum_i |a_i|
\leq \sqrt{d} + \sqrt{2d}
\leq 4d
\]

**Theorem 15.** Let the convex body $K$ be the hypersphere with radius $\frac{\sqrt{d}}{2\lambda}$ centered at the origin. Let $F(\beta)$ be the SVM objective function. Let $\beta^* = \arg\min_{\beta} F(\beta)$ be the optimal solution. Let $\hat{\beta}_s = \frac{1}{s} \sum_{t=0}^{s-1} \beta(t)$. Let $\eta_t = \frac{1}{8\sqrt{dt}}$. Then after $T - 1$ iterations of projected gradient descent, it must be the case that 
\[
F(\hat{\beta}_T) - F(\beta^*) \leq \frac{4d^{3/2}}{\lambda \sqrt{T}}
\]

Theorem 1 then follows by a straightforward application of Theorem 15.

### B Background

#### B.1 Fractional edge cover number and output size bounds

In what follows, we consider a conjunctive query $Q$ over a relational database instance $I$. We use $n$ to denote the size of the largest input relation in $Q$. We also use $Q(I)$ to denote the output and $|Q(I)|$ to denote its size. We use the query $Q$ and its hypergraph $\mathcal{H}$ interchangeably.

**Definition 1** (Fractional edge cover number $\rho^*$). Let $\mathcal{H} = (V, E)$ be a hypergraph (of some query $Q$). Let $B \subseteq V$ be any subset of vertices. A fractional edge cover of $B$ using edges in $\mathcal{H}$ is a feasible solution $\bar{\lambda} = (\lambda_S)_{S \in E}$ to the following linear program:

\[
\min \sum_{S \in E} \lambda_S \\
\text{s.t. } \sum_{S : v \in S} \lambda_S \geq 1, \quad \forall v \in B \\
\lambda_S \geq 0, \quad \forall S \in E.
\]

The optimal objective value of the above linear program is called the fractional edge cover number of $B$ in $\mathcal{H}$ and is denoted by $\rho_\mathcal{H}^*(B)$. When $\mathcal{H}$ is clear from the context, we drop the subscript $\mathcal{H}$ and use $\rho^*(B)$.

Given a conjunctive query $Q$, the fractional edge cover number of $Q$ is $\rho_\mathcal{H}^*(\mathcal{V})$ where $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the hypergraph of $Q$. 

17
Theorem 16 (AGM-bound [11, 23]). Given a full conjunctive query $Q$ over a relational database instance $I$, the output size is bounded by

$$|Q(I)| \leq n^{\rho^*},$$

where $\rho^*$ is the fractional edge cover number of $Q$.

Theorem 17 (AGM-bound is tight [11, 23]). Given a full conjunctive query $Q$ and a non-negative number $n$, there exists a database instance $I$ whose relation sizes are upper-bounded by $n$ and satisfies

$$|Q(I)| = \Theta(n^{\rho^*}).$$

Worst-case optimal join algorithms [40, 34, 35] can be used to answer any full conjunctive query $Q$ in time

$$O(|V| \cdot |E| \cdot n^{\rho^*} \cdot \log n).$$

(14)

B.2 Tree decompositions, acyclicity, and width parameters

Definition 2 (Tree decomposition). Let $H = (V, E)$ be a hypergraph. A tree decomposition of $H$ is a pair $(T, \chi)$ where $T = (V(T), E(T))$ is a tree and $\chi : V(T) \rightarrow 2^V$ assigns to each node of the tree $T$ a subset of vertices of $H$. The sets $\chi(t), t \in V(T)$, are called the bags of the tree decomposition. There are two properties the bags must satisfy

(a) For any hyperedge $F \in E$, there is a bag $\chi(t), t \in V(T)$, such that $F \subseteq \chi(t)$.

(b) For any vertex $v \in V$, the set $\{t | t \in V(T), v \in \chi(t)\}$ is not empty and forms a connected subtree of $T$.

Definition 3 (acyclicity). A hypergraph $H = (V, E)$ is acyclic iff there exists a tree decomposition $(T, \chi)$ in which every bag $\chi(t)$ is a hyperedge of $H$.

When $H$ represents a join query, the tree $T$ in the above definition is also called the join tree of the query. A query is acyclic if and only if its hypergraph is acyclic.

For non-acyclic queries, we often need a measure of how “close” a query is to being acyclic. To that end, we use width notions of a query.

Definition 4 ($g$-width of a hypergraph: a generic width notion [8]). Let $H = (V, E)$ be a hypergraph, and $g : 2^V \rightarrow \mathbb{R}^+$ be a function that assigns a non-negative real number to each subset of $V$. The $g$-width of a tree decomposition $(T, \chi)$ of $H$ is $\max_{t \in V(T)} g(\chi(t))$. The $g$-width of $H$ is the minimum $g$-width over all tree decompositions of $H$. (Note that the $g$-width of a hypergraph is a Minimax function.)

Definition 5 (Treewidth and fractional hypertree width are special cases of $g$-width). Let $s$ be the following function: $s(B) = |B| - 1, \forall V \subseteq V$. Then the treewidth of a hypergraph $H$, denoted by $tw(H)$, is exactly its $s$-width, and the fractional hypertree width of a hypergraph $H$, denoted by $fhtw(H)$, is the $\rho^*$-width of $H$.

From the above definitions, $fhtw(H) \geq 1$ for any hypergraph $H$. Moreover, $fhtw(H) = 1$ if and only if $H$ is acyclic.
B.3 Algebraic Structures

In this section, we define some of the algebraic structures used in the paper. First, we discuss the definition of a monoid. A monoid is a semi-group with an identity element. Formally, it is the following.

**Definition 18.** Fix a set $S$ and let $\oplus$ be a binary operator $S \times S \rightarrow S$. The set $S$ with $\oplus$ is a monoid if (1) the operator satisfies associativity; that is, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in S$ and (2) there is identity element $e \in S$ such that for all $a \in S$, it is the case that $e \oplus a = a \oplus e = e$.

A commutative monoid is a monoid where the operator $\oplus$ is commutative. That is $a \oplus b = b \oplus a$ for all $a, b \in S$.

Next, we define a semiring.

**Definition 19.** A semiring is a set $R$ with two operators $\oplus$ and $\otimes$. The $\oplus$ operator is referred to as addition and the $\otimes$ is referred to as multiplication. This is a semiring if,

1. it is the case that $R$ and $\oplus$ are a commutative monoid with $0$ as the identity.
2. $R$ and $\otimes$ is a monoid with identity $1$.
3. the multiplication distributes over addition. That is for all $a, b, c \in R$ it is the case that $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$.
4. the $0$ element annihilates $R$. That is, $a \otimes 0 = 0$ and $0 \otimes a = 0$ for all $a \in R$.

A commutative semiring is a semiring where the multiplication is commutative. That is, $a \otimes b = b \otimes a$ for all $a, b \in S$.

B.4 FAQ-AI Query

The input to FAQ-AI problem consists of three components:

- A collection of relational tables $T_1, \ldots, T_m$ with real-valued entries. Let $J = T_1 \times T_2 \times \cdots \times T_m$ be the design matrix that arises from the inner join of the tables. Let $n$ be an upper bound on the number of rows in any table $T_i$, let $N$ be the number of rows in $J$, and let $d$ be the number of columns in $J$.

- An FAQ $Q(J)$ that is either a SumProd query or a SumSum query. We define a SumSum query to be a query of the form:

$$Q(J) = \bigoplus_{x \in J} \bigoplus_{i=1}^{d} F_i(x_i)$$

where $(R, \oplus, I_0)$ is a commutative monoid over the arbitrary set $R$ with identity $I_0$. We define a SumProd query to be a query of the form:

$$Q(J) = \bigoplus_{x \in J} \bigotimes_{i=1}^{d} F_i(x_i)$$

where $(R, \oplus, \otimes, I_0, I_1)$ is a commutative semiring over the arbitrary set $R$ with additive identity $I_0$ and multiplicative identity $I_1$. In each case, $x_i$ is the entry in column $i$ of $x$, and $F_i$ is an arbitrary function with range $R$. 

19
A collection \( \mathcal{L} = \{(G_1, L_1), \ldots, (G_b, L_b)\} \) where \( G_i \) is a collection \( \{g_{i,1}, g_{i,2}, \ldots g_{i,d}\} \) of \( d \) functions that map the column domains to the reals, and each \( L_i \) is a scalar.

FAQ-AI(\( k \)) is a special case of FAQ-AI when the cardinality of \( \mathcal{L} \) is at most \( k \).

The output for the FAQ-AI problem is the result of the query on the subset of the design matrix that satisfies the additive inequalities. That is, the output for the FAQ-AI instance with a SumSum query is:

\[
Q(\mathcal{L}(J)) = \bigoplus_{x \in \mathcal{L}(J)} \bigoplus_{i=1}^{d} F_i(x_i)
\]  

(15)

And the output for the FAQ-AI instance with a SumProd query is:

\[
Q(\mathcal{L}(J)) = \bigoplus_{x \in \mathcal{L}(J)} \bigotimes_{i=1}^{d} F_i(x_i)
\]  

(16)

Here \( \mathcal{L}(J) \) is the set of tuples \( x \in J \) that satisfy all the additive inequalities in \( \mathcal{L} \), that is for all \( i \in [1, b] \), \( \sum_{j=1}^{d} g_{i,j}(x_j) \leq L_i \), where \( x_j \) is the value of coordinate \( j \) of \( x \).

We now illustrate how some of the SVM related problems can be reduced to FAQ-AI(1). First consider the problem of counting the number of negatively labeled points correctly classified by a linear separator. Here each row \( x \) of the design matrix \( J \) conceptually consists of a point in \( \mathbb{R}^{d-1} \), whose coordinates are specified by the first \( d-1 \) columns in \( J \), and a label in \( \{1, -1\} \) in column \( d \). Let the linear separator be defined by \( \beta \in \mathbb{R}^{d-1} \). A negatively labeled point \( x \) is correctly classified if \( \sum_{i=1}^{d-1} \beta_i x_i \leq 0 \). The number of such points can be counted using SumProd query with one additive inequality as follows: \( \oplus \) is addition, \( \otimes \) is multiplication, \( F_i(x_i) = 1 \) for all \( i \in [d-1], F_d(x_d) = 1 \) if \( x_d = -1 \), and \( F_d(x_d) = 0 \) otherwise, \( g_{1,j}(x_j) = \beta_j x_j \) for \( j \in [d-1], g_{1,d}(x_d) = 0 \), and \( L_1 = 0 \). Next, consider the problem of finding the minimum distance to the linear separator of a correctly classified negatively labeled point. This distance can be computed using a SumProd query with one additive inequality as follows: \( \oplus \) is the binary minimum operator, \( \otimes \) is addition, \( F_i(x_i) = \beta_i x_i \) for all \( i \in [d-1], F_d(x_d) = 1 \) if \( x_d = -1 \), and \( F_d(x_d) = 0 \) otherwise, \( g_{1,j}(x_j) = \beta_j x_j \) for \( j \in [d-1], g_{1,d}(x_d) = 0 \), and \( L_1 = 0 \).