Classification of integrable discrete Klein–Gordon models

Ismagil T Habibullin¹ and Elena V Gudkova²

¹ Ufa Institute of Mathematics, Russian Academy of Science, Chernyshevskii Street, 112, Ufa 450077, Russia
² Department of Applied Mathematics and Mechanics, Ufa State Petroleum Technical University, Kosmonavtov Street, 1, Ufa 450062, Russia

E-mail: habibullinismagil@gmail.com and elena.gudkova79@mail.ru

Received 3 February 2011
Accepted for publication 3 February 2011
Published 8 March 2011
Online at stacks.iop.org/PhysScr/83/045003

Abstract
The Lie algebraic integrability test is applied to the problem of classification of integrable Klein–Gordon-type equations on quad graphs. The list of equations passing the test is presented, containing several well-known integrable models. A new integrable example is found; its higher symmetry is presented.

PACS numbers: 02.20.Qs, 02.30.lk

1. Introduction
We study the integrability problem for the quad-graph equation of the form

\[ u_{1,1} = f(u, u_1, \bar{u}_1). \] (1)

Such equations have a large variety of applications in physics, biology, architecture, etc. Here the unknown \( u = u(m, n) \) is a function of two discrete variables \( m, n \). For the sake of convenience, we use the following notation: \( u_k = u(m + k, n), \bar{u}_k = u(m, n + k), u_{1,1} = u(m + 1, n + 1) \). Function \( f \) is supposed to be locally analytic, and it depends essentially on all three arguments. In other words, equation (1) can be rewritten in any of the following forms:

\[ u_{i,j} = f^{i,j}(u, u_i, \bar{u}_j), \quad \text{with} \quad i = \pm 1, \ j = \pm 1. \] (2)

Today there are various approaches for studying integrable discrete phenomena. The property of consistency around a cube [1] has been proposed as the integrability criterion for quadrilateral difference equations [2–4]. The symmetry approach to the classification of integrable systems is adopted in the discrete case [5–9]. Another characteristic property of an integrable equation is the vanishing of its algebraic entropy [10]. Alternative methods are used in [11–13]. In this paper, we study discrete phenomena from a different point of view.

Years ago it was observed that characteristic Lie algebras, introduced in [14], in the case of integrable hyperbolic-type partial differential equations (PDEs) such as the sine-Gordon and the Tzitzeica–Zhiber–Shabat equations have a very specific property: the dimensions of the linear spaces spanned by multiple commutators of the generators grow essentially slower than in the generic case. In [15], the problem of rigorous formalization of the notion of ‘slow growth’ has been discussed. A conjecture was suggested and checked to classify integrable equations of the form \( u_{x,y} = f(u, u_x) \).

In a recent paper [16], we introduced and successfully tested a classification scheme (called the algebraic test) based on investigation of multiple commutators of characteristic vector fields defined by equation (1). Now we use the algebraic test (an explanation is given in section 2 below) to classify discrete Klein–Gordon-type equations on a quad graph

\[ u_{1,1} + u = g(u_1 + \bar{u}_1). \] (3)

The list of equations that partially passed the algebraic test is studied additionally by applying the symmetry test. It is remarkable that the final list contains, in addition to well-known equations such as the discrete potential Korteweg–de Vries (PKdV) equation, the discrete Liouville equation and a discrete analogue of the sine-Gordon equation, a new integrable model (see the discussion on the corollary of theorems 2 and 3 below):

\[ u_{1,1} = \frac{\alpha u_1 \bar{u}_1 - \beta}{u(\alpha + u_1 \bar{u}_1)}. \] (4)

Recall that the problem of complete description of integrable Klein–Gordon equations

\[ u_{x,y} = g(u) \] (5)
was solved years ago by Zhiber and Shabat [17]. The authors proved that the only integrable equations in (5) are

- the linear equation;
- the Liouville equation \( u_{x,y} = e^u \);
- the sine-Gordon equation \( u_{x,y} = \sin u \);
- the Tzitzeica–Zhiber–Shabat equation \( u_{x,y} = e^u + e^{-2u} \).

This paper is organized as follows. In section 2, we define characteristic vector fields and the test Lie ring, formulate the algebraic test, and consider an illustrative example. The classification problem for the model of the form \( u_{x,y} + u = g(u_1 + \bar{u}_1) \) passing the test is investigated in section 3. The result of classification is summarized in theorems 3 and 5.

\section{The algebraic test and classification scheme}

Note that for any integers \( p \) and \( q \) the variable \( u_{p,q} \) is expressed in terms of the variables \( \{u_i, \bar{u}_j\}_{i=-\infty}^{\infty} \) in a recurrent way. Hence the variables \( u_i, \bar{u}_j \) are called dynamical variables. They are considered as independent ones. For example,

\[ u_{2,1} = f(u_1, u_2, u_{1,1}) = f(u_1, u_2, f(u_1, u_3, \bar{u}_3)) \]

and

\[ u_{-1,-2} = f^{-1,1}(\bar{u}_{-1}, u_{-1}, \bar{u}_{-2}) = f^{-1,1}(\bar{u}_{-1}, f^{-1,1}(u_{-1}, \bar{u}_{-1}), \bar{u}_{-2}). \]

Let us introduce the shift operators \( D \) and \( \tilde{D} \), shifting the first and second integer arguments, respectively: \( Dh(m, n) = h(m+1, n), \tilde{D}h(m, n) = h(m, n+1) \). We then explain the action of the operators \( D \) and \( \tilde{D} \) on the functions of dynamical variables. For the function \( h = h(u_1, u_{-1}, \ldots, u_j, \bar{u}_j, \bar{u}_{j+1}, \ldots, \bar{u}_j) \), we have

\[ h_k = D^k h = h(u_{k+1}, u_{k+1}, \ldots, u_{i+k}, \bar{u}_j, \bar{u}_{j+k}, \ldots, \bar{u}_j) \]

and similarly

\[ \tilde{h}_k = \tilde{D}^k h = h(u_{i+k}, u_{i+k}, \ldots, u_{i+k}, \bar{u}_j, \bar{u}_{j+k}, \ldots, \bar{u}_j). \]

In these two formulas one has to replace the double shifts \( u_{a,b} \) through dynamical variables, as discussed above.

Characteristic vector fields for hyperbolic-type PDEs were introduced by Goursat [18] in 1899. They provide a very effective tool for classification of Liouville-type integrable systems. For instance in [18], a list (almost complete) of hyperbolic-type PDEs was found admitting integrals in both directions by using a method based on the notion of characteristic vector fields. Interest in the subject was renewed after the paper [14], where an exhaustive description of exponential-type systems is given admitting a complete set of integrals. The concept of characteristic vector fields was adopted to the quad-graph equations in [19]. In recent papers [20–22] characteristic vector fields were used for the purpose of classification of semi-discrete Liouville-type equations. All these studies concern Liouville-type integrability. In [15] and [16], some new applications of these important notions are suggested. Let us recall some necessary definitions. Assume that equation (1) admits a nontrivial \( n \)-integral, i.e. there is a function \( I \) depending on a finite number of dynamical variables \( I = I(u_{-j}, u_{-j+1}, \ldots, u_k) \) satisfying the equation \( \tilde{D} I = I \).

In enlarged form, the last equation is

\[ I(r_{-j+1}, r_{-j+2}, \ldots, r, \bar{u}, f, f_1, \ldots, f_k) = I(u_{-j}, u_{-j+1}, \ldots, u_k), \]

(6)

where the function \( r = f^{-1,1}(u_{-1}, \bar{u}_1) \) is defined in (2). Since the right-hand side of (6) does not depend on the variable \( \bar{u}_1 \), we find \( \frac{\partial}{\partial u_1} \tilde{D} I = \frac{\partial}{\partial u_1} I = 0 \) which implies the equation \( Y I = 0 \) where the operator \( Y \) is defined as follows:

\[ Y := \tilde{D}^{-1} \frac{\partial}{\partial u_1} \tilde{D}. \]

Direct computations give (see [16, 19])

\[ Y = \frac{\partial}{\partial u_1} + x \frac{\partial}{\partial u_1} + \frac{1}{x} \frac{\partial}{\partial u_1} + x x_1 \frac{\partial}{\partial u_2} + \frac{1}{x x_1} \frac{\partial}{\partial u_2} + \ldots, \]

(7)

where

\[ x = \tilde{D}^{-1} \left( \frac{\partial f(u, u_1, \bar{u}_1)}{\partial u_1} \right) = \frac{\partial f^{-1,1}(u, u_1, \bar{u}_1) / \partial u_1}{\partial f^{-1,1}(u, u_1, \bar{u}_1) / \partial u_1}. \]

We call the operators \( X := \frac{\partial}{\partial u_1} \) and \( Y \) characteristic vector fields. It is evident that the map \( f(u, u_1, \bar{u}_1) \mapsto Y \) is correctly defined for any \( f \) due to formula (7).

We denote through \( T \) the set of vector fields obtained by taking all possible multiple commutators of the operators \( X \) and \( Y \), and take all linear combinations with coefficients depending on a finite number of dynamical variables \( \bar{u}_1, u, u_{\pm 1}, u_{\pm 2}, \ldots \). Evidently, the set \( T \) has a structure of the Lie ring. We call it the test ring of equation (1) in the direction of \( n \). In a similar way, one can define the test ring \( \tilde{T} \) in the direction of \( m \).

We note that for Liouville-type integrable equations of form (1) both rings \( T \) and \( \tilde{T} \) are of finite dimension. Actually the test ring is a subset of the characteristic Lie ring [19–22].

We denote through \( V \) the linear space over the field of locally analytic functions spanned by \( X, Y \) and all multiple commutators of \( X \) and \( Y \) of order less than or equal to \( j \) such that

\[ V_0 = [X, Y], \quad V_1 = [X, Y, [X, Y]], \ldots. \]

Such sequences of linear spaces have important applications in geometry\(^1\) (see [23, 24]). We introduce the function \( \Delta(k) := \dim V_{k+1} - \dim V_k \). The following conjecture is approved by numerous examples.

\textbf{Conjecture (algebraic test).} Any integrable model of the form (1) satisfies the following condition: there is a sequence of natural numbers \( \{k\}_{k=1}^{\infty} \) such that \( \Delta(k) \leq 1 \).

Ring \( T \) admits an automorphism, generated by the shift operator \( D \),

\[ T \ni Z \xrightarrow{Aut} DZD^{-1} \in T, \]

(8)

which plays a crucial role in our further considerations. It is important that \( X \) and \( Y \) considered as operators on the set

\(^1\) The authors thank E. Ferapontov for drawing their attention to this fact.
of functions depending on the variables \(\tilde{u}_{-1}, u, u_{+1}, u_{+2}, \ldots\)

satisfy the following conjugation relations [16]:

\[
DXD^{-1} = pX \quad \text{and} \quad DYD^{-1} = \frac{1}{x} Y,
\]

where

\[
p = D \left( \frac{\partial f^{-1} - (u, u_{-1}, \tilde{u}_{-1})}{\partial u_{-1}} \right) = \frac{1}{\partial f^{-1} - (u, u_{1}, \tilde{u}_{1})/\partial \tilde{u}_{-1}}.
\]

Indeed, one can specify the coefficients of the operator \(DXD^{-1} = \sum a_i \partial^i u + p \frac{\partial^i}{\partial u_{-1}}\) by applying it to the dynamical variables and find that \(a_i = DXD^{-1} u_i = 0\) for any integer \(i\). Moreover, \(p = DXD^{-1} u_{-1} = DXf^{-1} - (u, u_{-1}, \tilde{u}_{-1}) = D(\partial f^{-1} - \tilde{u}_{-1}/u_{-1})\). In a similar way, one can prove the second formula. Apply the operator \(DYD^{-1} = \sum c_i \partial^i u + p \frac{\partial^i}{\partial u_{-1}}\) to \(u_i\) and find \(c_j = D(Y_{u_{j-1}}) = Y u_j\).

Then evaluate \(d = DYD^{-1} u_{-1} = f_{u_{-1}} - 1 + \frac{1}{x} f_{u_{-1}}\). Since \(u_{-1} = f^{-1} - 1(u, u_{-1}, \tilde{u}_{-1})\), one obtains the equation \(u = f^{-1} - 1(f(u, u_{1}, \tilde{u}_{1}), u_{1}, \tilde{u}_{1})\). Let us differentiate it with respect to \(\tilde{u}_{1}\) and find \(D(\partial f^{-1} - \tilde{u}_{-1}/u_{-1}) + D(\partial f^{-1} - \tilde{u}_{-1}/u_{-1}) = 0\) or the same \(\frac{\partial f^{-1} - \tilde{u}_{-1}}{\partial u_{-1}} + \frac{1}{x} \frac{\partial f^{-1} - \tilde{u}_{-1}}{\partial u_{-1}} = 0\). Now due to the equation \(x = D^{-1} \left( \frac{\partial f^{-1} - \tilde{u}_{-1}}{\partial u_{-1}} \right)\), we conclude that \(d = 0\).

**Lemma 1.** Suppose that \(Z = \sum_{b} b_j \frac{\partial}{\partial u_{b}} \in T\) satisfies the following two conditions:

1. \(DZD^{-1} = cZ\) for some function \(c\);
2. \(b_j \equiv 0\) for some fixed value of \(j\).

Then \(Z = 0\).

The proof of the lemma can be found in [16].

**Example.** As an illustrative example we consider the following discrete Liouville-type equation (see [25]):

\[
u_{1,1} = \frac{1}{u}(u_{1} - 1)(\tilde{u}_{1} - 1).
\]

We first find an explicit form of the characteristic vector field \(Y\). Since \(f = \frac{1}{u}(u_{1} - 1)(\tilde{u}_{1} - 1)\) and \(\frac{\partial}{\partial u_{1}} = \frac{1}{u}(u_{1} - 1)\),

\[
x = D^{-1} \left( \frac{1}{u}(u_{1} - 1) - \frac{1}{u_{-1}}(u_{-1} - 1) \right).
\]

We express \(u_{1,1}\) through the dynamical variables due to equation (10). Applying the operator \(DYD^{-1}\) to both sides of (10), we find \(u_{1,1} = \frac{1}{u_{1}}(u_{1-1} - 1)(u-1)\). Comparison of the last equation with (11) yields \(x = u_{1}^{-1}\), therefore

\[
y = \frac{\partial}{\partial u_{1}} + \frac{u_{1}}{u} \frac{\partial}{\partial u_{1}} - \frac{1}{u} \frac{\partial}{\partial u_{1}} + \frac{u_{1} - 1}{u} \frac{\partial}{\partial u_{1}} + \frac{u_{1} - 1}{u} \frac{\partial}{\partial u_{1}} + \cdots,
\]

For this equation \(p = \frac{\partial f^{-1} - \tilde{u}_{1}/u_{1}}{\partial u_{1}} = \frac{1}{u_{1}}\); hence \(DXD^{-1} = pX\), \(DYD^{-1} = pY\). We evaluate \(D[X, Y]D^{-1} = p^2[X, Y] + pX(p)\) by \(pY(p)X\). Since \(X(p) = Y(p) = 0\) we get \(D[X, Y]D^{-1} = p^2[X, Y]\). Moreover \([X, Y] = \sum_{b} b_{b} \frac{\partial}{\partial u_{b}} + a_{a} \frac{\partial}{\partial u_{a}}\). Now due to lemma 1, we obtain \([X, Y] = 0\). Hence the dimension of the ring \(T\) for equation (10) equals two and \(\Delta(k) = 0\) for all \(k \geq 0\); therefore equation (10) passes the test.

3. **Equations of the form \(u_{1,1} + u = g(u_{1,1} + \tilde{u}_{1})\)**

In this section, we apply the conjecture to the following Klein–Gordon-type particular class of discrete model (1),

\[
u_{1,1} + u = g(u_{1} + \tilde{u}_{1}).
\]

where the function \(g\) is to be determined.

**Classification scheme.** Obviously, equation (1) passing the algebraic test above should satisfy one of the following conditions:

(i) \(\Delta(0) < \Delta_{max}(0) = 1\);
(ii) \(\Delta(0) = \Delta_{max}(0), \Delta(1) < \Delta_{max}(1) = 2\);
(iii) \(\Delta(0) = \Delta_{max}(0), \Delta(1) = \Delta_{max}(1), \Delta(2) < \Delta_{max}(2) = 3\);
(iv) \(\Delta(0) = \Delta_{max}(0), \Delta(1) = \Delta_{max}(1), \Delta(2) = \Delta_{max}(2)\) and \(\Delta(k) \leq 1\) for some \(k > 2\);

where \(\Delta_{max}(k)\) stands for the greatest value of \(\Delta(k)\) for equation (1) when \(f(u, u_{1}, \tilde{u}_{1})\) spans the class of arbitrary functions.

Note that in the case of equation (13) investigation of the first three particular cases (i)–(iii) allows one to extract a very short list of equations expected to be integrable. The list is exhaustive because case (iv) is never realized (see the corollaries of theorems 2 and 4 below).

Introducing vector fields \(R_{1} = [X, Y], P_{1} = [X, R_{1}], Q_{1} = [Y, R_{1}], R_{2} = [X, Q_{1}], W = [Y, Q_{1}]\) and \(Z = [X, P_{1}]\), we can span in addition to \(V_{0}\) and \(V_{1}\) two more linear spaces:

\[
V_{2} = V_{1} + [P_{1}, Q_{1}], \quad V_{3} = V_{2} + [W, Z, R_{2}].
\]

In order to evaluate \(\Delta(k)\), we will use the automorphism (8). At first evaluate the factors \(x\) and \(p\) in formula (9) for the case (13). We have \(x = p = g^{r}(u_{1} + \tilde{u}_{1})\), where the function \(\beta = g^{-1}(\alpha)\) is the inverse to the function \(\alpha = g(\beta)\). Conversely, knowing \(x = x(u_{1} + \tilde{u}_{1})\) one can recover \(g(\beta)\) by using the equation

\[
\beta = g^{-1}(\alpha) = \int (g^{-1}(\alpha))d\alpha = \int \frac{d\alpha}{g^{r}(\alpha)} = \int \frac{d\alpha}{x(\alpha)}.
\]

We specify the action of the characteristic operators \(X\) and \(Y\) on the variable \(x\). Evidently, \(Xx = x', Yx = xx'\). It is found by direct calculation that

\[
DR_{1}D^{-1} = R_{1} - x^{\prime}X - x^{\prime\prime}Y,
\]

\[
DP_{1}D^{-1} = xP_{1} - x^{\prime}X - x^{\prime\prime}Y - x^{\prime\prime\prime}Y,
\]

\[
DQ_{1}D^{-1} = \frac{1}{x} Q_{1} + x^{\prime}R_{1} - x^{\prime\prime}Y - x^{\prime\prime\prime}X,
\]

\[
DW_{1}D^{-1} = \frac{1}{x^{2}} W + \left(\frac{2x^{\prime\prime}}{x} - \frac{2X}{x^2}\right) R_{1} - x^{\prime\prime}Y - x^{\prime\prime\prime}X,
\]

\[
DZD_{1} = x^{2}Z + (x^{\prime\prime} - 2x^{\prime}\prime\prime)x + (x^{\prime\prime\prime} - 2x^{\prime\prime\prime}X) R_{1} - qY - x^{\prime\prime\prime}X,
\]

\[
q = x^{\prime\prime\prime} + 2x^{\prime\prime} + x^{\prime\prime\prime},
\]

\[
DR_{2}D^{-1} = R_{2} - x^{\prime}Q_{1} + x^{\prime}P_{1} - \frac{1}{x} R_{1} - sY - x^{\prime\prime\prime}X
\]

\[
s = x^{\prime\prime\prime} + x^{\prime\prime\prime\prime}.\]
Let us study the set $G$ of all multiple commutators of $X$ and $Y$.

**Lemma 2.** The coefficients of any operator in $G$ are functions of a finite number of the dynamical variables $x, x_{±1}, x_{±2}, \ldots$.

**Proof.** Here, $x(\alpha)$ is a fixed function of the argument $\alpha = u_1 + \bar{u}_1$. Therefore, one can write $x' = \phi(x)$ for some function $\phi$. Then $X(x) = -\phi(x)$ and $Y(x) = x(x) = \psi(x)$. By using the conjugation relations $DXD^{-1} = xX$ and $DYD^{-1} = \frac{1}{2}Y$, one derives that $X(x_j) = \phi'(x_j, x_{j+1}, \ldots)$ and $Y(x_j) = \psi'(x_j, x_{j+1}, \ldots)$. Similarly, $X(x_{j+1}) = \phi'(x, x_{j+1}, \ldots)$ and $Y(x_{j+1}) = \psi'(x, x_{j+1}, \ldots)$. Now evidently $R_1 = X(x) \frac{\partial}{\partial u} + X(x) \frac{\partial}{\partial x}$ and $X(x_{j+1}) = \phi'(x, x_{j+1}, \ldots)$ satisfies the statement of the lemma. Due to the formulae $R_1(x_j) = X(x_j) x_j$ and $D R_1 D^{-1} = R_1 = D Y - x X$, one obtains $R_1(x_j) = \rho(x_j, x_{j+1}, \ldots)$. Obviously the proof can be completed by using induction. □

**Theorem 2.** Suppose that an equation of the form (13) satisfies one of the conditions (i)–(iii) of the classification scheme. Then function $x = x(\alpha)$ solves the following ordinary differential equation

$$x'^2 = (x^2 + 1)\gamma + xv$$

with the constant coefficients $\gamma$, $v$.

**Proof of Theorem 2.** Begin with case (i). Suppose that $\Delta(0) = 0$; then we have $R_1 = \lambda X + \mu Y$. It is evident that $R_1 = X(x) \frac{\partial}{\partial u} + \cdots, X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x} + \cdots$; hence $\lambda = \mu = 0$ and therefore $R_1 = 0$. By applying the automorphism above to both sides of the last equation,

$$-\frac{x'}{x} = Y - x' = 0.$$

Since $X$ and $Y$ are linearly independent we obtain equation $x' = 0$, which is a particular case of (16). Evidently its solution is $x = c$, and due to (14) it can be found that $\beta = g^{-1}(\alpha) = \frac{1}{2}(u_1 + c_1).$ Thus our equation $\alpha = g(\beta)$ (see (13)) is linear, $u_1 + u = c(1 + 1 + \bar{c})$. For this case, dim $T = 2$ so that $\Delta(k) = 0$ for $k \geq 0$. In a similar way, one checks that condition (ii) leads to (16). Indeed suppose that $\Delta(0) = 1$ and $\Delta(1) < 2$. Then we have

$$P_1 = \nu Q_1 + \epsilon R_1.$$  

Due to lemma 2, functions $\nu$ and $\epsilon$ might depend only on $x, x_{±1}, x_{±2}, \ldots$. Applying the automorphism (8) to both sides of (17) and then simplifying due to equations (15) gives

$$x(\nu Q_1 + \epsilon R_1) - x' R_1 - r Y - x R_1 = D(v) \times \left( \frac{1}{x} Q_1 + \frac{x'}{x} R_1 - \frac{x''}{x} Y - x' X \right) + D(\epsilon) \left( R_1 - \frac{x'}{x} Y - x' X \right).$$

Comparison of the coefficients before linearly independent operators gives rise to the conditions

$$Q_1 : xv = \frac{1}{x} D(v),$$

$$R_1 : -x' + \epsilon = \frac{x'}{x} D(v) + D(\epsilon),$$

$$Y : -r = \frac{x'}{x} D(\epsilon) + \frac{x'}{x} D(v),$$

$$X : -x r = -x' D(\epsilon) - x' D(v).$$

Simple analysis of these equations implies that equation (17) holds if and only if the following three conditions are valid: $\nu = 0$, $\epsilon = \text{const}$, $x' = e(x - 1)$. Actually, under these conditions the last two equations above are satisfied automatically. In a similar way one can check that $Q_1 = v P_1 + \epsilon R_1$ is equivalent to the same three conditions. Hence, if $\Delta(1) < 2$ then $\Delta(1) = 0$ and consequently $\Delta(0) = 0$ for any natural $k \geq 1$. Thus in this case, dim $T = 3$.

Let us suppose now that $\Delta(0) = 1$, $\Delta(1) = 2$ and $\Delta(2) < 2$, which corresponds to case (iii). First consider the case when $Z$ is linearly expressed through the other vector fields in subspace $V_3$:

$$Z = \gamma R_1 + \delta P_1 + \epsilon Q_1 + \phi R_2 + \psi W.$$  

Applying the automorphism (8) to both sides of equation (18) and comparing the coefficients before linearly independent operators yields

$$W : x^2 \psi = D(\psi) \frac{1}{x^2},$$

$$R_2 : x^2 \phi = D(\phi),$$

$$Q_1 : x^2 R_1 = D(e) \frac{1}{x} + x' \phi,$$

$$P_1 : x^2 \delta = D(\delta) x + x' \phi,$$

$$R_1 : x^2 \gamma + x^2 - 2x x'' = D(\delta) x' + D(\gamma).$$

Since $x = x(u_1 + \bar{u}_1)$, we have $\psi = 0$, $\phi = 0$, $\epsilon = 0$, $\delta = 0$, $\gamma = \text{const}$. Comparison of the coefficients of $X$ and $Y$ gives one more equation $x q = y x'$. Finally, we obtain two ordinary differential equations for $x$ that are absolutely the same as in the case of the equation $u_{1,1} - u = g(u_1 + \bar{u}_1)$ (see [16]):

$$x^2 x''' - 2x x'' x' + x'' = y x', \quad (x^2 - 1) y + x^2 - 2x x'' = 0.$$  

The compatibility condition of these equations is equivalent to equation (16). In this case, we have $Z = \gamma R_1$. It is remarkable that $x$ solves equation (16) if and only if $W$ is linearly expressed through the other elements of $V_3$ and then $W = \gamma R_1$. The last possibility is when $R_2$ is linearly expressed through $Z, X, Y, R_1, P_1, Q_1, W$ and $Z$. In this case, $x$ solves the equation $x' = 0$. The proof of the theorem is completed. □

In order to find $x = x(\alpha)$, we evaluate the integral

$$H(x) := \int \frac{dx}{(x^2 + 1)y + xv} = \alpha - \alpha_0.$$  

For the case $\gamma \neq 0$, the answer is given by the formula

$$H(x) = \frac{1}{\sqrt{\gamma}} \ln(2\sqrt{x^2 + 1 + \xi b + 2x b}) = \frac{\nu}{\beta}.$$  

Now we find $x$ by solving the equation $H(x) = \alpha - \alpha_0$:

$$x(\alpha) = \frac{1}{4} e^{\sqrt{\gamma}(\alpha - \alpha_0)} - \frac{\nu}{2 \beta} - \left( 1 - \frac{\nu^2}{4 \beta^2} \right) e^{-\sqrt{\gamma}(\alpha - \alpha_0)}.$$  

In order to obtain the corresponding quad-graph equation (13), we integrate again:

$$\beta = g^{-1}(\alpha) = \int \frac{dx}{x(\alpha)}.$$
Integration gives

\[ \beta = \frac{1}{\sqrt{V}} \ln \left| \frac{e^{\sqrt{V}(\alpha - u)}}{e^{\sqrt{V}(\alpha - a_0)}} - b - 2 \right| + b_0. \]  

(22)

Then the equation searched is given by the formula (take \( \sqrt{V} = 1 \))

\[ a_1 e^u + a_2 e^u + a_3 e^u + a_4 = 0, \]

where \( a_1 = e^{-\alpha - \beta_0}, a_2 = -e^{-\alpha}, a_3 = (2 - b)e^{-\beta_0}, a_4 = 2 + b \). By setting \( u = \ln v \), one reduces it to a quadrilinear form

\[ a_1 v_1, u, u_1, v + a_2 v_1, v + a_3 u_1, v + a_4 = 0, \]  

(23)

with arbitrary constants \( a_j \).

If \( y = 0 \), then \( x = x(\alpha) \) solves the equation \( x' = \sqrt{vx} \). Thus obviously \( x = c(\alpha - \alpha_0)^2 \) and therefore

\[ \beta = g^{-1}(\alpha) = \int \frac{d\alpha}{c(\alpha - \alpha_0)^2} = \frac{d}{\alpha - \alpha_0} - \beta_0, \quad d = -1/c. \]

(24)

Corollary of theorem 2. If for a nonlinear chain (13) one of the conditions (i)–(iii) is satisfied, then the chain is of one of the following forms:

1. \( u_{1,1} + u = c(u_1 + u_1 + c_1) \),
2. \( a_{1,1} + u_1 + u_1 + a_2 + u_3 + a_4 = 0 \),
3. \( (u_{1,1} + u - \alpha_0)(u_1 + u_1 - \beta_0) = 0. \)  

(25)

Let us study in detail the list obtained. It is remarkable that it contains a new integrable example of equation (1).

**Theorem 3.** Chain (2) in the corollary of theorem 2 admits a symmetry of the form

\[ u_t = g(u_1, u_{-1}, u, u_1, u_1) \]  

(26)

if and only if the condition \( k := \frac{a_0}{\alpha_0} = \pm 1 \) holds. Under this condition, symmetry is of the form

\[ u_t = \lambda u + \mu u + \mu^{\pm 1}, \]

Chain (3) admits symmetry of the form (25) if and only if \( a_0 = \beta_0 \) and then it is reduced to the discrete potential KdV equation.

Theorem can be proved by using the technique developed in [5]. Actually it claims that chain (2) with \( a_3 = \pm a_2 \) passes the symmetry test. For the case \( k = 1 \) (or \( a_3 = a_2 \), the chain is a particular case of the Viiallet equation (see [10]). This means that by a point transformation it can be reduced to one of the equations of the Adler–Bobenko–Suris list [4]. Higher symmetries for this case are found in [6]. For some special choice of the parameters, chain (2) with \( k = -1 \) is reduced to the discrete sine-Gordon equation

\[ 3(u_{1,1}u_1u_1 + u_{1,1} - u_1) = 0. \]

(27)

found by Hernandez Heredero, Levi and Scimiterna [26]. Recently, in [5] it was proved that equation (26) admits higher symmetries. To the best of our knowledge, the general case of equation (2) with \( k = -1 \) has never been identified before as an integrable one.

When \( a_1 = 0 \) and \( a_3 = -a_2 \), it is reduced to the well-known discrete Liouville equation

\[ e^{\int u dt} = e^{\int u_1 dt} + 1 \]

having nontrivial integrals, i.e. functions solving the equations \( DF = F \) and \( DF = F \), with \( F = e^{-u_1} + e^{u_1} + 1 \) and \( F = e^{u_1} - 1 + e^{u_1} - 1 \).

The second part of the theorem claims that case (3) admits a symmetry of small order only if it coincides up to a point transformation with the discrete PKdV equation. The question is still open of whether it admits any symmetry of a more complicated form, say \( u_t = g(u_2, u_3, u_4, u_5, u_6, \bar{u}_1, \bar{u}_2) \). There are some technical difficulties with applications of the symmetry approach to discrete models (see the discussion in [5]).

We now turn back to the set \( G \) consisting of \( X, Y \) and all their multiple commutators. We assign two integers, order and degree, to each element in \( G \) and define the order of an element \( Z \in G \) as a number of its factors \( X \) and \( Y \) minus 1. For instance, \( \text{ord}[X, Y] = 1, \text{ord}[X, X, Y] = 2 \) and so on. We define the degree \( \text{deg}(Z) \) of \( Z \) as the exponent \( k \) in the expansion of the operator obtained by applying the automorphism (8): \( DZD^{-1} = x^kZ + \cdots \), where the tail is a linear combination of the elements with order less than \( \text{ord}(Z) \), and denote through \( G_{i,j} \) a subset of \( G \) containing elements with order \( i \) and degree \( j \). Let \( G_i = \bigcup G_{i,j} \) be the union of all \( G_{i,j} \) with one and the same \( i \). Evidently the set \( G_{i-1} \) (as well as \( G_{i-1} \)) contains the only element \( Z_{i-1} = ad_x(Y) \) up to the factor \(-1\) (correspondingly, the only element \( Z_{i+1} = ad_x(Y) \) up to the factor \(-1\)). Here the operator \( ad_x \) is defined as \( ad_x(Y) = [X, Y] \).

**Theorem 4.** Suppose that \( Z_{k+1} \) (or \( Z_{k-1} \)) is in the basis of the linear space \( V_k \supseteq G_k \) for any integer \( k : m \leq k < N \), but \( Z_{N-1} \) (respectively \( Z_{N+1} \)) is linearly expressed through the other operators in \( V_N \) and then function \( x = x(u_1 + u_1) \) solves an equation of the form \( x' = \epsilon(x - 1) \) with the constant coefficient \( \epsilon \).

The proof of the theorem is done exactly in the same way as the proof of theorem 3 from [16], but presented in the article for the readers’ convenience.

**Lemma 3.** For any integer \( k \geq 3 \), we have

\[ DZ_{i-k-1}D^{-1} = x^kZ_{i-k-1} + c_1x^{k+1}Z_{i-k-1} + \cdots \]

(27)

where \( c_1 \geq 0 \) (but \( c_1 > 0 \) for \( k > 3 \)) and the tail contains a linear combination of operators with order less than \( k \).

We prove lemma 3 by induction. From the list of equations (15), one obtains for \( Z = Z_{3,2} \) and \( R_{1} = R_{1,0} \) the following representation:

\[ DZ_{3,2}D^{-1} = x^2Z_{3,2} + (x^2 - 2xx'x')Z_{1,0} - qY - xqX, \]

\[ q = xx'' - 2x'x'' + \frac{x^3}{x} \]

(28)
showing that the statement is true for the case \( k = 3 \). We suppose now that \( DZ_{k, k-1}D^{-1} = x^{k-1}Z_{k, k-1} + c_{k-1}x^kZ_{k-1, k-2} + \cdots \) and evaluate \( DZ_{k+1, k}D^{-1} \):

\[
DZ_{k+1, k}D^{-1} = [x X, x^{k-1}Z_{k, k-1} + c_{k-1}x^kZ_{k-1, k-2} + \cdots ] = x^kZ_{k+1, k} + c_{k-1}x^kZ_{k-1, k-1} + \cdots ,
\]

where \( c_k = c_{k-1} + (k - 1) > c_{k-1} \geq 0 \). The proof is completed.

**Proof of theorem 4.** Suppose that

\[
Z_{N, N-1} = \sum_{\text{ord}(Z_\nu) = N} a_\nu Z_\nu + \sum_{\text{ord}(Z_\mu) = N-1} b_\mu Z_\mu + \cdots , (29)
\]

where \( Z_\nu \) and \( Z_\mu \) span the basis of \( V_N \) and the tail contains a linear combination of operators of less order. We apply the automorphism (8) to both sides of (29):

\[
x^{N-1}\left( \sum_{\text{ord}(Z_\nu) = N} a_\nu Z_\nu + \sum_{\text{ord}(Z_\mu) = N-1} b_\mu Z_\mu + \cdots \right) - x^{N-2}\sum_{\text{ord}(Z_\nu) = N-1} a_\nu Z_\nu + \cdots = \sum_{\text{ord}(Z_\mu) = N} D(a_\nu)(x^\mu Z_\mu) + \cdots + \sum_{\text{ord}(Z_\mu) = N-1} D(b_\mu)(x^\mu Z_\mu) + \cdots .
\]

We collect the coefficients before \( Z_\nu \) and obtain

\[
x^{N-1}a_\nu = x^\mu D(a_\nu), \quad k_\mu \neq N - 1 . \quad (30)
\]

Due to lemma 2, functions \( a_\nu \) and \( b_\mu \) depend on \( x \) and its shifts. Moreover, it follows from (30) that \( a_\nu \) cannot depend on \( x \), \( x_{\pm 1} \), \( x_{\pm 2} \), \ldots at all. Therefore, the only possibility is \( a_\nu = 0 \). If we compare now the coefficients before \( Z_{N-1, N-2} \) we find

\[
x^{N-1}b - c_{N-1}x^{N-2}x' = x^{N-2}D(b), \quad (31)
\]

where \( b \) is the coefficient of \( Z_{N-1, N-2} \) in the expansion (29).

A simple analysis of equation (31) shows that \( b \) is constant. Thus, (31) is equivalent to the equation \( x' = \epsilon(x - 1) \) with \( \epsilon = b/c_{N-1} \).

**Corollary of theorem 4.** Case (iv) of the classification scheme is never realized.

**Proof.** Suppose, on the contrary, that such a case is realized. Then at least one of the vector fields \( Z_{k, k-1} \) or \( Z_{k-k+1} \) should be linearly expressed through other elements of \( V_k \), otherwise \( \Delta(k) \geq 2 \). Therefore, due to theorem 4 we have \( x' = \epsilon(x - 1) \), which corresponds to case (ii), \( \Delta(0) = 1 \), \( \Delta(1) < 2 \) and \( \dim T = 3 \). The contradiction shows that our assumption is not true. The proof is completed.

Let us summarize the results of the reasoning above in the following theorem.

**Theorem 5.** Suppose that an equation of the form (13) passes the algebraic test. Then it is one of the forms given in the corollary of theorem 2.

**4. Conclusions**

A classification scheme introduced recently in [16] is applied to the quad-graph Klein–Gordon-type equation. The list of equations passing the test contains, along with well-known integrable models, a new example, which also passes the symmetry test.

**Acknowledgments**

We thank Professor A V Zhiber and Dr R N Garifullin for valuable advice. This work was partially supported by the Russian Foundation for Basic Research (RFFR), grants nos 10-01-91222-CT-a, 11-01-00732-a, 11-01-97005-r-povoljje-a and 10-01-00088-a.

**References**

[1] Nijhoff F W and Walker A J 2001 The discrete and continuous Painlevé hierarchy and the Garnier system *Glasgow Math. J.* 43 A 109–123

[2] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* 11 573–611

[3] Nijhoff F W 2002 Lax pair for the Adler (lattice *J. Nonlinear Math. Phys.* 9 297–311

[4] Tongas A, Tsoubelis D and Xenitidis P 2001 A family of integrable models, a new example, which also passes the symmetry test *Commun. Math. Phys.* 233 513–43

[5] Levi D and Yamlov R 2011 Generalized symmetry integrability test for discrete equations on the square lattice *J. Phys. A: Math. Theor.* accepted (arXiv:1011.0070v2 [nlin.SI])

[6] Xenitidis P 2009 Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case *Proc. 4th Workshop Group Analysis of Differential Equations and Integrable Systems* arXiv:0902.3954

[7] Rasin O G and Hydon P E 2007 Conservation laws for integrable difference equations *J. Phys. A: Math. Theor.* 40 12763–75

[8] Mikhailov A V, Wang J P and Xenitidis P 2005 Recursion operators, conservation laws and integrability conditions for difference equations arXiv:1004.5346v1 [nlin.SI]

[9] Tongas A, Tsoubelis D and Xenitidis P 2001 A family of integrable nonlinear equations of hyperbolic type *J. Math. Phys.* 42 5762

[10] Bellon M P and Viallet C M 1999 Algebraic entropy *Commun. Math. Phys.* 204 425–37

[11] Nijhoff F W, Ramani A, Grammaticos B and Ohta Y 2001 On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation *Stud. Appl. Math.* 106 261–314

[12] Grammaticos B, Karra G, Papageorgiou V and Ramani A 1992 Integrability of Discrete-Time Systems, Chaotic Dynamics (Patras, 1991) (NATO Adv. Sci. Inst. Ser. B Phys. vol 298) (New York: Plenum) pp 75–90

[13] Hietarinta J Searching for CAC-maps *J. Nonlinear Math. Phys.* (Suppl. 2) 12 223–30

[14] Leznov A N, Smirnov V G and Shabat A B 1982 Gruppa vnutrennikh simmetrii i usloviya integriruemosti dvumenykh dinamicheskikh system *Teor. Mat. Fiz.* 51 10–21

[15] Zhiber A V and Murtazina R D 2006 On the characteristic Lie algebras for the equations *Fundam. Prikl. Mat.* 12 65–78
Zhiber A V and Murtazina R D 2008 J. Math. Sci. 151 3112–22 (Engl. Transl.)

[16] Habibullin I T and Gudkova E V 2010 An algebraic method of classification of S-integrable discrete models Theor. Math. Phys. accepted (arXiv:1006.3423v2 [nlin.SI])

[17] Zhiber A V and Shabat A B 1979 Klein–Gordon equations with a nontrivial group Dokl. Akad. Nauk SSSR 247 1103–7
Zhiber A V and Shabat A B 1979 Sov. Phys. Dokl. 24 607–9 (Engl. Transl.)

[18] Goursat E 1899 Recherches sur quelques équations aux dérivées partielles du second ordre Ann. Fac. Sci. Toulouse (Ser. 2) I 31–78

[19] Habibullin I T 2005 Characteristic algebras of fully discrete hyperbolic type equations Symmetry Integrability Geom. Methods Appl. 1 23

[20] Habibullin I and Pekcan A 2007 Characteristic Lie algebra and classification of semi-discrete models Theor. Math. Phys. 151 781–90

[21] Habibullin I, Zheltukhina N and Pekcan A 2008 On the classification of Darboux integrable chains J. Math. Phys. 49 102702

[22] Habibullin I, Zheltukhina N and Pekcan A 2009 Complete list of Darboux integrable chains of the form \( t_1 = t + d(t, t_1) \) J. Math. Phys. 50 102710

[23] Cartan E 1910 Les systemes de Pfaff a cinque variables et les equations aux derivees partielles du second ordre Ann. Sci. Ecole Norm. Sup. 27 109–92

[24] Doubrov B and Zelenko I 2007 On local geometry of nonholonomic rank 2 distributions arXiv:math/0703662v1 [math.DG]

[25] Adler V E and Startsev S Ya 1999 On discrete analogues of the Liouville equation Teor. Mat. Fiz. 121 271–84
Adler V E and Startsev S Ya 1999 Theor. Math. Phys. 121 1484–95 (Engl. Transl.)

[26] Hernandez Heredero R, Levi D and Scimiterna C 2011 A discrete integrability test based on multiple scaled analysis, in preparation