A sharp threshold for rainbow connection in small-world networks

Y. Shang
A SHARP THRESHOLD FOR RAINBOW CONNECTION IN SMALL-WORLD NETWORKS

Y. SHANG

Received 30 March, 2011

Abstract. An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. We prove that $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n, p, H)) \leq 2$ in the small-world networks. As by-products, our extension of the concept of independence in graph theory and generalized small-world network models are of independent interest.

2000 Mathematics Subject Classification: 05C82; 05C15; 05C40

Keywords: rainbow connection, edge coloring, small world, networks

1. INTRODUCTION

We utilize the terminology and notation of [19] in this letter. An interesting connectivity concept of a graph was recently introduced in [3] and has attracted attention of some researchers. An edge-colored graph $G$ is referred to as rainbow connected if any two vertices are connected by a path whose edges have distinct colors. A rainbow connected graph must be connected, and conversely, any connected graph has a trivial edge coloring that makes it rainbow connected. The rainbow connection of a connected graph $G$, denoted $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected.

An easy observation is that if $G$ has $n$ vertices then $rc(G) \leq n - 1$, since one may color the edges of a given spanning tree of $G$ with distinct colors, and color the remaining edges with one of the already used colors. It is also known that $rc(G) = 1$ if and only if $G$ is a complete graph, and that $rc(G) = n - 1$ if and only if $G$ is a tree. Note that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of $G$. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been dealt with in the work [2,13], of which a primary result is $rc(G) \leq 20n/\delta(G)$. Related concepts such as rainbow path [6], rainbow tree [5] and rainbow $k$-connectivity [4] have also been investigated recently.

A natural and intriguing direction to explore is the random graph scenarios [12,17]. Let $G(n, p)$ be the classical random graph with $n$ vertices and edge probability $p$. For
a graph property \( \mathcal{A} \), we say that \( G(n, p) \) satisfies \( \mathcal{A} \) almost surely if the probability that \( G(n, p) \) satisfies \( \mathcal{A} \) tends to 1 as \( n \) tends to infinity. A function \( f(n) \) is called a sharp threshold function for the property \( \mathcal{A} \) if there are two positive constants \( C \) and \( c \) such that \( G(n, p) \) satisfies \( \mathcal{A} \) almost surely for \( p \geq Cf(n) \) and \( G(n, p) \) almost surely does not satisfy \( \mathcal{A} \) for \( p \leq cf(n) \). A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. [1, 10, 11]).

The parameter \( rc(G) \) is monotone non-increasing in the sense that if we add an edge to \( G \) we cannot increase its rainbow connection. The authors of [2] show that \( p = \sqrt{\ln n}/n \) is a sharp threshold function for the property \( rc(G(n, p)) \leq 2 \). In this note, we propose a generalized small-world network model and explore the threshold of rainbow connection of it. The small-world network is a model with two important characteristics: the clustering effect and the small-world phenomenon, which was originally introduced by Watts and Strogatz [18] as a model of real world complex networks. It has since been the subject of considerable research interest within the physics community, see e.g. [7, 14–16] and references therein.

The rest of the note is organized as follows. In Section 2, we present some necessary notions including the generalized small-world model and state our sharp threshold result. The proofs are given in Section 3.

2. NOTIONS AND MAIN RESULT

Watts-Strogatz (WS) rewiring model [18] and its variant Newman-Watts (NW) model [16] are classical small-world network models. The NW model can be regarded as the union of an Erdős-Rényi random graph \( G(n, p) \) and a \( 2k \)-regular lattice. It is known that the NW model and the WS model are, essentially, the same. A natural extension would be to use a general sparse graph to replace the low-dimensional regular lattices.

Let \( S(n, p, H) \) be a small-world network that is the union of a random graph \( G(n, p) \) and a graph \( H \) on \( n \) vertices. Note that \( S(n, p, H) \) is not necessarily connected when \( H \) is not connected. When \( H \) is a regular lattice, we then obtain the NW model.

Next, we need to extend the classical notion of independence in graph theory to distant \( l \)-independence. A subset \( X \) of vertices in a graph \( G \) is called distant \( l \)-independent for some \( l \in \mathbb{N} \), if the distance between any two vertices in \( V \) is larger than \( l \). Thus, a distant \( 1 \)-independent set is independent in the classical sense. Recall that there is another generalization of independence, called \( k \)-independence [8, 9], which requires the induced subgraph has maximum degree less than \( k \). The relative strength relationship of these three concepts can be described as follows:

\[
\text{\( k \)-independence} < \text{independence} < \text{distant \( l \)-independence}.
\]

Now we are on the stage to state our main result.
Theorem 1. Let $H$ be a graph on $n$ vertices, which contains a distant 2-independent set of order $\Theta(n^\varepsilon)$ for some $\varepsilon > 0$. For the small-world network $S(n, p, H)$, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n, p, H)) \leq 2$.

Clearly, a $2k$-regular lattice with $k \ll n^\alpha$ for some $\alpha \in (0, 1)$ serves as an eligible graph $H$ in Theorem 1. Therefore, the above result holds for both WS and NW models.

3. Proof of Theorem 1

In this section, we will provide a proof of Theorem 1 as per the reasoning in [2]. As mentioned in Section 1, $rc(G) \geq 2$ for any non-complete graph $G$. The following lemma gives a sufficient condition for $rc(G) = 2$.

Lemma 1. ([2]) If $G$ is a non-complete graph on $n$ vertices and any two vertices of $G$ have at least $2\ln n$ common neighbors, then $rc(G) = 2$.

Proof of Theorem 1. For the first part of the theorem, we need to prove that for a sufficiently large constant $C$, the small-world network $S(n, p, H)$ with $p = C\sqrt{\ln n/n}$ almost surely has $rc(G) = 2$. Recall that $rc(G)$ is monotone non-increasing, we need only to prove this for the random graph $G(n, p)$. By Lemma 1, it suffices to show that almost surely any two vertices of $G(n, p)$ have at least $2\ln n$ common neighbors.

Fix a pair of vertices $x, y$, and the probability that $z$ is a common neighbor of them is $C\ln n/n$. Let random variable $X$ represents the number of common neighbors of $x$ and $y$. Accordingly, we get $EX = (n-2)(C\ln n/n)$. By using the Chernoff bound (e.g. [12] pp.26), for large enough $C$, we have

$$P(X < 2\ln n) \leq P\left(X < EX - \frac{C^2 \ln n}{4}\right) \leq e^{-\frac{C^2 \ln n}{16}} = o(n^{-2}).$$

Since there are $\binom{n}{2}$ pairs of vertices in $G(n, p)$, the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant $c$, the small-world network $S(n, p, H)$ with $p = c\sqrt{\ln n/n}$ almost surely has $diam(S(n, p, H)) \geq 3$. By the assumption in Theorem 1, fix a distant 2-independent set $X$ of order $\Theta(n^\varepsilon)$ for some $\varepsilon < 1/4$ in $H$, and let $Y$ be the remaining $n - \Theta(n^\varepsilon)$ vertices. Let $A$ be the event that $X$ induces an independent set in the small-world network $S(n, p, H)$. Let $B$ be the event that there exists a pair of vertices $x, y \in X$ with no common neighbor in $Y$. Consequently, it suffices to prove that (i) $P(A) \to 1$; and (ii) $P(B) \to 1$, as $n \to \infty$.

For (i): For $c$ sufficiently small we obtain

$$P(A) = (1 - p)^{\binom{\Theta(n^\varepsilon)}{2}} = (1 - c\sqrt{\ln n/n})^{\binom{\Theta(n^\varepsilon)}{2}}$$

$$\sim e^{-\frac{c\sqrt{\ln n/n}}{2\ln n^\varepsilon}} \to 1,$$
as $n \to \infty$, since $0 < \varepsilon < 1/4$.

For (ii): For a pair $x, y \in X$, the probability that $x, y$ have a common neighbor in $Y$ is shown to be given by

$$1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n-\Theta(n^\varepsilon)} \approx \left(1 - n^{-c^2}\right).$$

Since the vertex set $X$ can be divided into $\Theta(n^\varepsilon)/2 = \Theta(n^\varepsilon)$ pairs, the probability that all $\Theta(n^\varepsilon)$ pairs have a common neighbor is

$$1 - P(\mathcal{B}) = \left(1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n-\Theta(n^\varepsilon)}\right)^{\Theta(n^\varepsilon)} \approx \left(1 - n^{-c^2}\right)^{\Theta(n^\varepsilon)} \approx e^{-\frac{\Theta(n^\varepsilon)}{n^{c^2}}}.$$  (3.1)

For sufficiently small $c$, the right hand side of (3.1) tends to zero, which thus completes the proof.  ■

REFERENCES

[1] B. Bollobás and A. Thomason, “Threshold functions,” Combinatorica, vol. 7, pp. 35–38, 1987.
[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, “On rainbow connection,” Electron. J. Comb., vol. 15, no. 1, pp. 13, Research Paper R57, 2008.
[3] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, “Rainbow connection in graphs,” Math. Bohem., vol. 133, no. 1, pp. 85–98, 2008.
[4] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, “The rainbow connectivity of a graph,” Networks, vol. 54, no. 2, pp. 75–81, 2009.
[5] G. Chartrand, F. Okamoto, and P. Zhang, “Rainbow trees in graphs and generalized connectivity,” Networks, vol. 55, no. 4, pp. 360–367, 2010.
[6] D. j. Dellamonica, C. Magnant, and D. M. Martin, “Rainbow paths,” Discrete Math., vol. 310, no. 4, pp. 774–781, 2010.
[7] R. Durrett, Random graph dynamics, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press, 2007, vol. 20.
[8] J. F. Fink and M. S. Jacobson, “n-domination in graphs,” in Graph theory with applications to algorithms and computer science, ser. Proc. 5th Int. Conf., Kalamazoo/Mich., 1984. New York: John Wiley & Sons, 1985, pp. 283–300.
[9] J. F. Fink and M. S. Jacobson, “On n-domination, n-dependence and forbidden subgraphs,” in Graph theory with applications to algorithms and computer science, ser. Proc. 5th Int. Conf., Kalamazoo/Mich. 1984, New York: John Wiley & Sons, 1985, pp. 301–311.
[10] E. Friedgut, “Hunting for sharp thresholds,” Random Struct. Algorithms, vol. 26, no. 1-2, pp. 37–51, 2005.
[11] E. Friedgut and G. Kalai, “Every monotone graph property has a sharp threshold,” Proc. Am. Math. Soc., vol. 124, no. 10, pp. 2993–3002, 1996.
[12] S. Janson, T. Łuczak, and A. Ruciński, Random graphs, ser. Wiley-Interscience Series in Discrete Mathematics and Optimization. New York: Wiley, 2000.
[13] M. Krivelevich and R. Yuster, “The rainbow connection of a graph is (at most) reciprocal to its minimum degree,” J. Graph Theory, vol. 63, no. 3, pp. 185–191, 2010.
[14] M. E. Newman, “Models of the small world,” J. Stat. Phys., vol. 101, no. 3-4, pp. 819–841, 2000.
[15] M. E. J. Newman, C. Moore, and D. J. Watts, “Mean-field solution of the small-world network model,” Phys. Rev. Lett., vol. 84, pp. 3201–3204, 2000.
[16] M. E. J. Newman and D. J. Watts, “Renormalization group analysis of the small-world network model,” Phys. Lett., A, vol. 263, no. 4-6, pp. 341–346, 1999.
[17] Y. Shang, “Sharp concentration of the rainbow connection of random graphs,” Notes Number Theory Discrete Math., vol. 16, no. 4, pp. 25–28, 2010.
[18] D. J. Watts and S. H. Strogatz, “Collective dynamics of ”small-world” networks,” Nature, vol. 393, pp. 440–442, 1998.
[19] D. B. West, Introduction to graph theory. Prentice Hall, 2000.

Author’s address

Y. Shang
University of Texas at San Antonio, Institute for Cyber Security, San Antonio, TX 78249, USA
E-mail address: shylmath@hotmail.com