An Approximating Control Design for Optimal Mixing by Stokes Flows

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Abstract

We consider an approximating control design for optimal mixing of a non-dissipative scalar field $\theta$ in an unsteady Stokes flow. The objective of our approach is to achieve optimal mixing at a given final time $T > 0$, via the active control of the flow velocity $v$ through boundary inputs. Due to zero diffusivity of the scalar field $\theta$, establishing the well-posedness of its Gâteaux derivative requires $\sup_{t \in [0,T]} \|\nabla \theta\|_{L^2} < \infty$, which in turn demands the flow velocity field to satisfy the condition $\int_0^T \|\nabla v\|_{L^\infty(\Omega)} \, dt < \infty$. This condition results in the need to penalize the time derivative of the boundary control in the cost functional. Consequently, the optimality system becomes difficult to solve [21]. Our current approximating approach provides a more transparent optimality system, with the set of admissible controls square integrable in space-time. This is achieved by first introducing a small diffusivity to the scalar equation and then establishing a rigorous analysis of convergence of the approximating control problem to the original one as the diffusivity approaches to zero. Uniqueness of the optimal solution is obtained for the two dimensional case.

Key words Approximating Control, Optimal Mixing, Unsteady Stokes Flow, Navier Slip Boundary Conditions

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1. Introduction

Consider a passive scalar field advected by an unsteady Stokes flow on an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with a sufficiently smooth...
boundary $\Gamma$. The scalar field is governed by the transport equation, where the molecular diffusion is assumed to be negligible and mixing is purely driven by advection. This naturally leads to the study of optimal mixing via an active control of the flow velocity. As discussed in our previous work [21], we consider the flow velocity induced by control inputs acting tangentially on the boundary of the domain through the Navier slip boundary conditions. This is motivated by the observation that moving walls accelerate mixing compared to fixed walls (cf. [17, 18, 19, 34, 41]). We aim at designing a Navier slip boundary control that optimizes mixing at a given final time. The governing system of equations is

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad (1.1)$$

$$\frac{\partial v}{\partial t} - \Delta v + \nabla p = 0, \quad (1.2)$$

$$\nabla \cdot v = 0, \quad x \in \Omega, \quad (1.3)$$

with the Navier slip boundary conditions (cf. [23, 24, 33]),

$$v \cdot n|_{\Gamma} = 0 \quad \text{and} \quad kv + (\mathbb{T}(v) \cdot n)|_{\Gamma} = g, \quad (1.4)$$

and the initial condition is given by

$$(\theta(0), v(0)) = (\theta_0, v_0), \quad (1.5)$$

where $\theta$ is the density, $v$ is the velocity, $p$ is the pressure, and $g$ is the boundary control input, which is employed to generate the velocity field for mixing. Navier slip boundary conditions admit the fluid to slip with resistance on the boundary. Here $n$ and $\tau$ denote the outward unit normal and tangentially vectors with respect to the domain $\Omega$, $\mathbb{T}(v) = 2\mathbb{D}(v)$ with $\mathbb{D}(v) = (1/2)(\nabla v + (\nabla v)^T)$, $(\mathbb{T}(v) \cdot n)|_{\Gamma}$ denotes the tangential component of $(\mathbb{T}(v) \cdot n)$, and $g \cdot n|_{\Gamma} = 0$. The friction between the fluid and the wall is proportional to $-v$ with the positive coefficient of proportionality $k$.

Due to the divergence-free and no-penetration boundary conditions imposed on the velocity field, it can be shown that any $L^p$-norm of $\theta$ is conserved (cf. [20, 21]), i.e.,

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad t \geq 0, \quad p \in [0, \infty]. \quad (1.6)$$

To qualify mixing, the mix-norm and negative Sobolev norms $H^{-s}$, for any $s > 0$, are usually adopted, especially for the scalar field with no molecular diffusion, based on ergodic theory (cf. [29, 30, 31, 32, 40]). The bridge that connects mixing with negative
Sobolev norms is the property of weak convergence. As discussed in our previous work, we consider a general bounded domain for mixing and replace the negative Sobolev norm by the norm for the dual space \((H^s(\Omega))'\) of \(H^s(\Omega)\) with \(s > 0\). Also, we identify the space \((H^s(\Omega))'\) with \((H^s(\Omega))^*\) of \(H^s(\Omega)\) with \(s > 0\). Also, we identify the space \((H^s(\Omega))'\), where \(s > 0\), as the domain of operator \(\Lambda^{-s}\) equipped with the norm \(\| \cdot \|_{(H^s(\Omega))'}\), where \(\Lambda\) is self-adjoint, positive and unbounded in \(L^2(\Omega)\) (cf. [27, p. 9]). Thus, \(\Lambda^{2s} \in L(H^s(\Omega), (H^s(\Omega))')\). In our current work, we continue to adopt \(\| \cdot \|_{(H^1(\Omega))'}\) for qualifying mixing as in [21]. In particular, we choose \(\Lambda = A^{-1/2}\), where

\[
A\phi = (-\Delta + I)\phi, \quad \phi \in D(A) = \{ \phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n}\big|_\Gamma = 0 \}.
\]

Then \(D(\Lambda) = D(A^{1/2}) = H^1(\Omega)\) and \(D(\Lambda^{-1}) = D(A^{-1/2}) = (H^1(\Omega))'\).

Throughout this paper, we use \((\cdot, \cdot)\) and \(\langle \cdot, \cdot \rangle\) for the \(L^2\)-inner products in the interior of the domain \(\Omega\) and on the boundary \(\Gamma\), respectively. To set up the abstract formulation for the velocity field, we define

\[
V^s_{\text{in}}(\Omega) = \{ v \in H^s(\Omega) : \text{div} v = 0, \ v \cdot n|_\Gamma = 0 \}, \quad s \geq 0,
\]
\[
V^s_{\text{in}}(\Gamma) = \{ g \in H^s(\Gamma) : g \cdot n|_\Gamma = 0 \}, \quad s \geq 0.
\]

The optimal control problem is formulated as follows: For a given \(T > 0\), find a control \(g\) minimizing the cost functional

\[
J(g) = \frac{1}{2} \| \theta(T) \|^2_{(H^1(\Omega))'} + \frac{\gamma}{2} \| g \|^2_{U_{\text{ad}}}, \quad (P)
\]

subject to (1.1)–(1.5), where \(\| \theta(T) \|_{(H^1(\Omega))'} = \| \Lambda^{-1} \theta(T) \|_{L^2(\Omega)}\), \(\gamma > 0\) is the control weight parameter, and \(U_{\text{ad}}\) is the set of admissible controls, which is often determined based on the physical properties as well as the need to establish the well-posedness of the problem, i.e., the existence of an optimal solution. In fact, the existence of an optimal solution to the problem \((P)\) can be proven for \(U_{\text{ad}} = L^2(0, T; V^0_{\text{in}}(\Gamma))\). The challenge arises in deriving the first-order necessary conditions of optimality. To establish the well-posedness of the Gâteaux derivative of \(\theta\), one needs \(\sup_{t \in [0, T]} \| \nabla \theta \|_{L^2} < \infty\), which requires \(\theta_0 \in H^1(\Omega)\) and the flow velocity to satisfy

\[
\int_0^T \| \nabla v \|_{L^\infty(\Omega)} dt < \infty.
\]

Therefore, the initial condition \(v_0\) and \(U_{\text{ad}}\) were chosen in a way such that this estimate holds [21]. As a result, the time regularity of \(g\) was needed. For computational convenience, the first derivative \(\partial g/\partial t\) was adopted rather than the lower order fractional time derivative in the cost functional. Consequently, the optimality condition involved the time derivative of \(g\), and thus the optimality system became difficult to further analyze the uniqueness of the solution.
1.1. An approximating control approach

In this work, we start with investigating the approximating control problem by adding a small diffusion term $\epsilon \Delta \theta$, for $\epsilon > 0$, to the transport equation. The problem is now formulated as follows: For a given $T > 0$, find a control $g_\epsilon \in U_{\text{ad}} = L^2(0,T; V^0(\Gamma))$ minimizing the cost functional

$$ J_\epsilon(g_\epsilon) = \frac{1}{2} \| \theta_\epsilon(T) \|_{H^1(\Omega)}^2 + \frac{\gamma}{2} \| g_\epsilon \|_{U_{\text{ad}}}^2, \quad \epsilon > 0, \quad (P_\epsilon) $$

subject to an approximating system governed by

$$ \frac{\partial \theta_\epsilon}{\partial t} - \epsilon \Delta \theta_\epsilon + v_\epsilon \cdot \nabla \theta_\epsilon = 0, $$

$$ \frac{\partial v_\epsilon}{\partial t} - \Delta v_\epsilon + \nabla p_\epsilon = 0, $$

$$ \nabla \cdot v_\epsilon = 0, \quad x \in \Omega, \quad (1.7) \quad (1.8) \quad (1.9) $$

with the Neumann boundary condition for the scalar

$$ \epsilon \frac{\partial \theta_\epsilon}{\partial n} \bigg|_\Gamma = 0 $$

and the nonhomogenous Navier slip boundary conditions for the velocity

$$ v_\epsilon \cdot n \big|_\Gamma = 0 \quad \text{and} \quad (kv_\epsilon + (\mathbb{T}(v_\epsilon) \cdot n)) \big|_\Gamma = g_\epsilon. \quad (1.10) \quad (1.11) $$

The initial condition is given by

$$ (\theta_\epsilon(0), v_\epsilon(0)) = (\theta_0, v_0). \quad (1.12) $$

Note that due to one-way coupling, the flow velocity $v$ does not depend on $\epsilon$, and thus we have

$$ v_\epsilon = v. \quad (1.13) $$

However, to distinguish the approximating system from the original one, we still use the notation $v_\epsilon$.

The outline of the rest of this paper is as follows. We first recall the basic results on Navier slip boundary control for the Stokes problem in Section 2. In Section 4, we establish the convergence of the approximating system governed by (1.7)–(1.12) to the original one governed by (1.1)–(1.5). Then in Section 5, we show the existence of
an optimal solution to the approximating control problem \((P_\epsilon)\) and derive the first-order necessary conditions of optimality by using a variational inequality. Moreover, we prove that the optimal solution \((g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*)\) to the problem \((P_\epsilon)\) strongly converges to \((g^*, v^*, \theta^*)\) as \(\epsilon \to 0\), which turns out to be the optimal solution to the original problem \((P)\). Finally, in Section 6 we prove that \((g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*)\) to the problem \((P_\epsilon)\) strongly converges to \((g^*, v^*, \theta^*)\) as \(\epsilon \to 0\), which turns out to be the optimal solution to the original problem \((P)\).

In the sequel, the symbol \(C\) denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters.

2. Preliminary

Note that boundary control of the flow velocity essentially leads to a bilinear control problem for the scalar equation. As a result of one-way coupling, it is key to understand the boundary control problem of the Stokes equations. For the convenience of the reader, we recall some results introduced in [21] on Navier slip boundary control for Stokes flows. In fact, the problems of fluid flows with Navier slip boundary conditions have been widely studied in [6, 9, 11, 12, 22, 23, 24, 26, 28].

To define the Stokes operator associated with Navier slip boundary conditions, we introduce the bilinear form

\[
a_0(v, \psi) = 2(\mathbb{D}(v), \mathbb{D}(\psi)) + k \langle v, \psi \rangle, \quad k > 0, \quad v, \psi \in V_1^2(\Omega).
\]

By Korn’s inequality and trace theorem, it is easy to check that \(c_1 \|v\|_{H^1}^2 \leq a_0(v, v) \leq c_2 \|v\|_{H^1}^2\), for some constants \(c_1, c_2 > 0\). Thus \(a_0(\cdot, \cdot)\) is \(H^1\)-coercive. Let \((V_1^1(\Omega))'\) be the dual space of \(V_1^1(\Omega)\). Define the operator \(A: V_1^1(\Omega) \to (V_1^1(\Omega))'\) by

\[
(Av, \psi) = a_0(v, \psi). \quad (2.1)
\]

The Lax-Milgram Theorem implies that \(A \in \mathcal{L}(V_1^1(\Omega), (V_1^1(\Omega))')\). This also allows us to identify \(A\) as an operator acting on \(V_0^2(\Omega)\) with the domain

\[
\mathcal{D}(A) = \{v \in V_1^1(\Omega): \psi \mapsto a_0(v, \psi) \text{ is } L^2\text{-continuous}\}.
\]

In fact, as shown in [22, (2.9)] and [24, (5.1)], for \(v \in V_1^2(\Omega)\) satisfying the homogeneous Navier slip boundary conditions in (1.4) and \(\psi \in V_1^1(\Omega)\), we have

\[
\int_{\Omega} \Delta v \cdot \psi \, dx = -2 \int_{\Omega} \mathbb{D}(v) : \mathbb{D}(\psi) \, dx - \int_{\Gamma} k(v \cdot \tau)(\psi \cdot \tau) \, dx. \quad (2.2)
\]

Thus (2.1)–(2.2) define the Stokes operator \(A = -\mathbb{P}\Delta\) with domain

\[
\mathcal{D}(A) = \{v \in V_1^2(\Omega): kv + (\mathbb{T}(v) \cdot n)_{\tau}|_{\Gamma} = 0\},
\]
where $\mathbb{P}$ is the Leray projector in $L^2(\Omega)$ on the space $V^0_n(\Omega)$. Note that $A$ is self-adjoint, strictly positive, and thus the fractal powers $A^\sigma$, for $\sigma \in \mathbb{R}$, are well-defined. By interpolation theory (cf. [23, 25, 27]), the Navier slip boundary conditions allow us to identify the domains of $A^\sigma$ for $0 \leq \sigma \leq 1$ as

\[ \mathcal{D}(A^\sigma) = V^{2\sigma}_n(\Omega), \quad 0 \leq \sigma < \frac{3}{4}, \quad \text{and} \]
\[ \mathcal{D}(A^\sigma) = \{v \in V^{2\sigma}_n(\Omega) : kv + (\mathbb{T}(v) \cdot n)_\Gamma|_\Gamma = 0\}, \quad \frac{3}{4} < \sigma \leq 1. \quad (2.3) \]

The detailed proof is given by [23, Proposition 2.4]. The Navier slip boundary operator $N: L^2(\Gamma) \to V^0_n(\Omega)$ is defined by

\[ Ng = v \iff a_0(v, \psi) = \langle g, \psi \rangle, \quad \psi \in V^1_n(\Omega). \]

Moreover,

\[ N: L^2(\Gamma) \to V^{3/2}_n(\Omega) \subset V^{3/2-\delta}_n(\Omega) = \mathcal{D}(A^{3/4-\delta/2}), \quad \delta > 0, \]

or

\[ A^{3/4-\delta/2}N \in \mathcal{L}(L^2(\Gamma), V^0_n(\Omega)), \]

and

\[ N^*A\psi = \psi|_\Gamma, \quad \psi \in \mathcal{D}(A), \quad (2.4) \]

where $N^*: V^0_n(\Omega) \to L^2(\Gamma)$ is the $L^2$-adjoint operator of $N$ (cf. [2, 21, 23, 25]). By making a change of variable, we may rewrite the nonhomogenous boundary problem (1.2)–(1.4) as a variation of parameters formula

\[ v(t) = e^{-At}v_0 + (Lg)(t), \quad (2.5) \]

where $e^{-At}$ is an analytic semigroup generated by $-A$ on $V^0_n(\Omega)$ and $L$ is given by

\[ (Lg)(t) = \int_0^t Ae^{-A(t-\tau)}Ng(\tau) \, d\tau. \quad (2.6) \]

Recall the analytic semigroup theory that (cf. [25, Proposition 0.1], [36, p. 74, Theorem 6.13])

\[ e^{-At} \in \mathcal{L}\left(V^0_n(\Omega), L^2(0, T; \mathcal{D}(A^{1/2}))\right), \quad (2.7) \]
\[ \|A^\sigma e^{-At}\| \leq Mt^{-\sigma}e^{-\omega t}, \quad \sigma \geq 0, \quad (2.8) \]
for some $M \geq 1$, $\omega > 0$, and
\[
\int_0^t e^{A(t-\tau)} \cdot d\tau : \text{continuous } L^2(0,T; V^0_n(\Omega)) \to L^2(0,T; D(A)). \tag{2.9}
\]

To understand the regularity properties of $L$, we follow the similar approaches as in (cf. [4, Theorem 3.1.4, Theorem 3.1.8], [25, Lemmas 3.2.2–3.2.3], and [35, Theorems 2.5–2.6]) for $v \cdot n|_{\Gamma} = 0$ and obtain
\[
L \in \mathcal{L}(L^2(0,T; V^2_n(\Gamma)) \cap H^s(0,T; V^0_n(\Gamma)),
L^2(0,T; V^{2s+3/2}_n(\Omega)) \cap H^{s+3/4}(0,T; V^0_n(\Omega)), \quad 0 \leq s < 1/2. \tag{2.10}
\]

For $1/2 < s \leq 1$, (2.10) holds for $g(0) = 0$. This result is the same as for classical parabolic problems with Neumann or Robin boundary condition due to (2.4) that $N^* A$ is a Dirichlet trace operator in the case of $v \cdot n|_{\Gamma} = 0$.

For $v_0 \in V^0_n(\Omega)$, using (2.5) together with (2.7)–(2.10) immediately follows
\[
\|v\|_{L^\infty(0,T;V^0_n(\Omega))} + \|v\|_{L^2(0,T;V^1_n(\Omega))} + \|d v / dt\|_{L^2(0,T; (V^1_n(\Omega))')} \leq C(\|v_0\|_{L^2} + \|g\|_{L^2(0,T; V^0_n(\Gamma))}). \tag{2.11}
\]

Moreover, if we let $L^*$ be the $L^2(0,T; \cdot)$-adjoint operator of $L$, then $L^*$ is given by
\[
(L^* \psi)(t) = \int_t^T N^* A e^{-A(t-\tau)} \psi(\tau) d\tau = \left(\int_t^T e^{-A(t-\tau)} \psi(\tau) d\tau\right)|_{\Gamma}. \tag{2.12}
\]

Slightly modifying the proof in [4, Theorem 3.1.9] yields
\[
L^* \in \mathcal{L}(L^2(0,T; V^{2s}_n(\Omega)) \cap H^s(0,T; V^0_n(\Omega)),
L^2(0,T; V^{2s+3/2}_n(\Gamma)) \cap H^{s+3/4}(0,T; V^0_n(\Gamma))), \quad 0 \leq s \leq 1. \tag{2.13}
\]

In particular, letting $s = 0$ in (2.10) we have by duality
\[
L^* \in \mathcal{L}\left(L^2(0,T; \left(V^{3/2}_n(\Omega)\right)', L^2(0,T; V^0_n(\Gamma)) \right). \tag{2.14}
\]

Next we show the existence of an optimal solution to the problem $(P)$ for $g \in U_{ad} = L^2(0,T; V^0_n(\Gamma))$. The proof follows the standard procedure addressed in [21, Theorem 3.2]. First recall the definition of a weak solution to the scalar equation (1.1) given by [21, Definition 3.1].
3. Existence of an Optimal Solution to (P)

**Definition 3.1.** For \( \theta_0 \in L^\infty(\Omega) \), \( \theta \in C([0, T], (H^1(\Omega))^\prime) \) is said to be a weak solution of (1.2) if \( \theta \) satisfies

\[
\left( \frac{\partial \theta}{\partial t}, \phi \right) - (v \theta, \nabla \phi) = 0, \quad \forall \phi \in H^1(\Omega), \tag{3.1}
\]

where \( v \in L^2(0, T; V_n^1(\Omega)) \) satisfies (2.5) for \( v_0 \in V_n^0(\Omega) \) and \( g \in V_n^0(\Gamma) \).

In fact, according to [28, Corollary II.1], there exists a unique solution \( \theta \in L^\infty(0, T; L^\infty(\Omega)) \) to (1.1) for \( \theta_0 \in L^\infty(\Omega) \) and \( v \in L^2(0, T; V_n^1(\Omega)) \). Moreover, it is straightforward to check that

\[
\left\| \frac{\partial \theta}{\partial t} \right\|_{L^2(0, T; (H^1(\Omega))^\prime)}^2 = \left\| v \cdot \nabla \theta \right\|_{L^2(0, T; (H^1(\Omega))^\prime)}^2 = \int_0^T \left( \sup_{\phi \in H^1(\Omega)} \left| \int_\Omega v \cdot \nabla \phi \, dx \right| \right. \left. \left/ \left\| \phi \right\|_{H^1} \right. \right)^2 \, dt \tag{3.2}
\]

\[
\leq C \int_0^T \left( \sup_{\phi \in H^1(\Omega)} \left\| v \right\|_{L^2} \left\| \theta \right\|_{L^\infty} \left\| \phi \right\|_{H^1} \right)^2 \, dt \tag{3.3}
\]

where from (3.2) to (3.3) we used the divergence formula that

\[
\int_\Omega v \cdot \nabla (\theta \phi) \, dx = \int_\Gamma (v \cdot n) \theta \phi \, dx - \int_\Omega (\nabla \cdot v) \theta \phi \, dx = 0.
\]

Thus by Aubin-Lions-Simon Lemma (cf. [14]), we get \( \theta \in C([0, T], (H^1(\Omega))^\prime) \).

**Theorem 3.2.** Assume that \( \theta_0 \in L^\infty(\Omega) \) and \( v_0 \in V_n^0(\Omega) \). There exists an optimal solution \( g^* \in U_{ad} \) to the problem (P).

**Proof.** The proof follows the same approach as in [21, Theorem 3.2]. We provide the complete proof for the convenience of the reader. Since \( J \) is bounded from below, we may choose a minimizing sequence \( \{g_m\} \subset U_{ad} \) such that

\[
\lim_{m \to \infty} J(g_m) = \inf_{g \in U_{ad}} J(g). \tag{3.4}
\]
This also indicates that \( \{g_m\} \) is uniformly bounded in \( U_{ad} \), and hence there exists a weakly convergent subsequence, still denoted by \( \{g_m\} \), such that

\[
g_m \rightharpoonup g^* \quad \text{weakly in} \quad L^2(0,T;V_0^0(\Gamma)), \quad \text{as} \quad m \to \infty.
\]  

(3.5)

With the help of (2.11) we can extract a subsequence \( \{v_m\} \) corresponding to \( \{g_m\} \), such that

\[
v_m \rightharpoonup v^* \quad \text{weakly in} \quad L^2(0,T;V_1^1(\Omega)),
\]

\[
\frac{\partial v_m}{\partial t} \rightharpoonup \frac{\partial v^*}{\partial t} \quad \text{weakly in} \quad L^2(0,T;(V_1^1(\Omega))').
\]

Thus

\[
v_m \rightharpoonup v^* \quad \text{strongly in} \quad L^2(0,T;V_0^0(\Omega)) \quad (3.6)
\]

(cf. [39, Theorem 2.1]). Let \( \{\theta_m\} \) be the solutions corresponding to \( \{v_m\} \) with \( \theta_m(0) = \theta_0 \in L^\infty(\Omega) \). Then \( \theta_m \in C([0,T];(H^1(\Omega))') \) and by (1.6) \( \|\theta_m(t)\|_{L^\infty} = \|\theta_0\|_{L^\infty} \) for any \( t \geq 0 \). Thus there exists a subsequence, still denoted by \( \{\theta_m\} \), satisfying

\[
\theta_m \rightharpoonup \theta^* \quad \text{weak* in} \quad L^\infty(0,T;L^\infty(\Omega)). \quad (3.7)
\]

Next we show that \( \theta^* \) is the solution corresponding to \( v^* \) by Definition 3.1. Recall that \( v_m \) and \( \theta_m \) satisfy

\[
\begin{aligned}
\left( \frac{\partial \theta_m}{\partial t}, \phi \right) - (v_m \theta_m, \nabla \phi) &= 0, \quad \phi \in H^1(\Omega), \\
\theta_m(0) &= \theta_0.
\end{aligned}
\]  

(3.8)

Let \( \psi \in C^1[0,T] \). For each \( \phi \in H^1(\Omega) \), multiplying (3.8) by \( \psi \) and integrating the first term by parts yields

\[
(\theta_m(T), \phi \psi(T)) - \int_0^T (\theta_m, \phi \dot{\psi}) \, dt - \int_0^T (v_m \theta_m, \nabla \phi \psi) \, dt = (\theta_0, \phi \psi(0)).
\]  

(3.9)

With the help of (3.7) and \( \phi \dot{\psi} \in L^1(0,T;L^1(\Omega)) \), it is easy to pass to the limit in the second term of (3.9). Next we show that applying (3.6)–(3.7) makes passing to the
limit in the nonlinear term \( v_m \theta_m \to v^* \theta^* \) possible. In fact, we have
\[
\left| \int_0^T \int_\Omega (v_m \theta_m) \cdot \nabla (\phi \psi) \, dx \, dt - \int_0^T \int_\Omega (v^* \theta^*) \cdot \nabla (\phi \psi) \, dx \, dt \right|
\leq \left| \int_0^T \int_\Omega (v_m \theta_m) \cdot \nabla (\phi \psi) - (v^* \theta_m) \cdot \nabla (\phi \psi) \, dx \, dt \right|
+ \left| \int_0^T \int_\Omega (v^* \theta_m) \cdot \nabla (\phi \psi) - (v^* \theta^*) \cdot \nabla (\phi \psi) \, dx \, dt \right|
\leq \int_0^T \| v_m - v^* \|_{L^2} \| \theta_m \|_{L^\infty} \| \nabla \phi \|_{L^2} \| \psi \| \, dt + \int_0^T \int_\Omega (\theta_m - \theta^*) v^* \cdot \nabla (\phi \psi) \, dx \, dt
\]
where
\[
\int_0^T \| v_m - v^* \|_{L^2} \| \theta_m \|_{L^\infty} \| \nabla \phi \|_{L^2} \| \psi \| \, dt
\leq \| v_m - v^* \|_{L^2(0,T,V^0_\Omega)} \| \theta_0 \|_{L^\infty} \| \nabla \phi \|_{L^2} \| \psi \|_{L^2(0,T)} \to 0. \tag{3.10}
\]
Further note that \( v^* \cdot \nabla (\phi \psi) \in L^1(0,T; L^1(\Omega)) \). In light of (3.7) we get
\[
\left| \int_0^T \int_\Omega (\theta_m - \theta^*) v^* \cdot \nabla (\phi \psi) \, dx \, dt \right| \to 0. \tag{3.11}
\]
From (3.10)–(3.11) we have established that
\[
v_m \theta_m \to v^* \theta^* \quad \text{weakly in} \quad L^2(0,T; (H^1(\Omega))^\prime). \tag{3.12}
\]
Furthermore, as proven in [21, Theorem 3.2] choosing \( \psi \in C^1(0,T] \) such that \( \psi(0) = 1 \) and \( \psi(T) = 0 \), we obtain \( \theta^*(0) = \theta_0 \). Similarly, choosing \( \psi \in C^1(0,T] \) such that \( \psi(T) = 1 \) and letting \( m \to \infty \) in (3.9), we get
\[
\theta_m(T) \to \theta^*(T) \quad \text{weakly in} \quad (H^1(\Omega))^\prime.
\]
Clearly, \( \theta^* \in C([0,T]; (H^1(\Omega))^\prime) \) is the solution corresponding to \( v^* \) based on Definition 3.1.

Lastly, using the weakly lower semicontinuity property of norms yields
\[
\| g^* \|_{U_{ad}} \leq \lim_{m \to \infty} \| g_m \|_{U_{ad}} \quad \text{and} \quad \| \theta^*(T) \|_{(H^1(\Omega))^\prime} \leq \lim_{m \to \infty} \| \theta_m(T) \|_{(H^1(\Omega))^\prime}.
\]
In other words,
\[
J(g^*) \leq \lim_{m \to \infty} J(g_m) = \inf_{g \in U_{ad}} J(g),
\]
which indicates that \( g^* \) is an optimal solution to the problem \( (P) \).
\[
\square
\]
4. Convergence of the Approximating System

Let \((\theta, v)\) and \((\theta_\epsilon, v_\epsilon)\) be solutions of (1.1)–(1.5) and (1.7)–(1.12), respectively, with the same initial condition \((v_0, \theta_0)\) and boundary condition \(g\). The well-posedness and regularity of the approximating system follow the results from parabolic boundary value problems (cf. [7]) and the details can be found in [5, Theorem 1].

**Theorem 4.1.** For given \(\epsilon > 0, \theta_0 \in L^\infty(\Omega)\) and \(v_\epsilon \in L^2(0, T; V^0_n(\Omega))\), there exists a unique weak solution \(\theta_\epsilon \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\) to (1.7), (1.10) and (1.12). Moreover,

\[
\|\theta_\epsilon\|_{L^\infty(0, T; L^\infty(\Omega))} + \sqrt{\epsilon} \|\theta_\epsilon\|_{L^2(0, T; H^1(\Omega))} + \|d\theta_\epsilon/dt\|_{L^2(0, T; (H^1(\Omega))^\prime)} + \|v_\epsilon \cdot \nabla \theta_\epsilon\|_{L^2(0, T; (H^1(\Omega))^\prime)} \leq C(\theta_0, v_\epsilon). \tag{4.1}
\]

To show the convergence of \(\theta_\epsilon\) to \(\theta\) as \(\epsilon \to 0\), we shall need an a priori estimate on \(\theta\), that is,

\[
\int_0^T \|\nabla \theta\|_{L^2}^2 dt < \infty. \tag{4.2}
\]

Note that applying \(H^1\)-estimate to scalar equation (1.1) and the Gronwall inequality, we obtain

\[
\sup_{t \in [0, T]} \|\nabla \theta\|_{L^2} \leq C \|\nabla \theta_0\|_{L^2} e^{\int_0^T \|\nabla v\|_{L^\infty} dt}. \tag{4.3}
\]

The detailed proof can be found in [21, Lemma 2.1] and the references therein. For \(\theta_0 \in H^1(\Omega)\), if

\[
\int_0^T \|\nabla v\|_{L^\infty} dt < \infty, \tag{4.4}
\]

then (4.3) holds, and hence (1.2) follows. It remains to identify the initial and boundary data of the velocity field such that (4.4) holds. Using Agmon inequality (cf. [38, (2.21), p. 11])

\[
\|\nabla v\|_{L^\infty} \leq C \|v\|_{H^{1+d/2+\eta}}, \quad d = 2, 3, \quad \forall \eta > 0, \tag{4.5}
\]

and the variation of parameters formula

\[
v(t) = e^{-At}(v_0 - Ng_0) + Ng(t) - \int_0^t e^{-A(t-\tau)} Ng(\tau) d\tau, \tag{4.6}
\]

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we have shown in [21] that if
\[ v_0 \in V_n^{d/2-1+2\eta}(\Omega) \quad \text{and} \quad g \in L^2(0,T;V_n^{d/2-1/2+\eta}(\Gamma)) \cap H^1(0,T;V_n^{0}(\Gamma)), \quad d = 2, 3, \]
then (4.4) follows. In fact, the first order time derivative on \( g \) can be relaxed, which will be proven in the following lemma.

**Lemma 4.2.** Let
\[ S = L^2(0,T;V_n^{d/2-1/2+2\eta}(\Gamma)) \cap H^{d/4-1/4+\eta}(0,T;V_n^{0}(\Gamma)), \quad d = 2, 3, \]
equipped with the norm
\[ \|g\|_S = \|g\|_{L^2(0,T;V_n^{d/2-1/2+2\eta}(\Gamma))} + \|g\|_{H^{d/4-1/4+\eta}(0,T;V_n^{0}(\Gamma))}, \]
where \( 0 < \eta < 1/4 \) for \( d = 2 \) and \( 0 < \eta < 1/8 \) for \( d = 3 \). If \( v_0 \in V_n^{d/2-1+2\eta}(\Omega) \) and \( g \in S \), then (4.4) holds.

**Proof.** We first consider \( d = 2 \) and let \( 0 < \eta < 1/4 \). In this case, \( g \in S = L^2(0,T;V_1^{1/2+2\eta}(\Gamma)) \cap H^{1/4+\eta}(0,T;V_0^{0}(\Gamma)) \), where \( 1/4 + \eta < 1/2 \). According to (2.5), (2.10), and (4.5), we have
\[
\int_0^T \|\nabla v\|_{L^\infty} \, dt \leq C \int_0^T \|v\|_{H^{1+d/2+\eta}} \, dt \leq C \left( \int_0^T \|e^{-At}v_0\|_{H^{1+d/2+\eta}} \, dt + \int_0^T \|Lg\|_{H^{1+d/2+\eta}} \, dt \right) \leq C \left( \int_0^T \|A^{1-\eta/2}e^{-At}A^{1+\eta/2}A^{1/2+d/4+\eta/2}v_0\|_{L^2} \, dt + \sqrt{T}\|Lg\|_{L^2(0,T;H^{1+d/2+\eta}(\Omega))} \right) \leq C \left( \int_0^T e^{-\frac{\omega t}{4}} \frac{A^{1-\eta/2}A^{1+\eta/2}A^{1/2+d/4+\eta/2}}{t^{1-\eta/2}} \|v_0\|_{H^{d/2-1+2\eta}} + \sqrt{T}\|g\|_S \right) \leq C \left( \|v_0\|_{H^{d/2-1+2\eta}} + \sqrt{T}\|g\|_S \right).
\]
(4.7)
From (4.7) to (4.8) we used (2.8)–(2.10) and Young’s inequality for convolution.

For \( d = 3 \), we shall need \( g \in L^2(0,T;V_n^{1+2\eta}(\Gamma)) \cap H^{1/2+\eta}(0,T;V_n^{0}(\Gamma)) \), which indicates that \( g \in C([0,T];V_n^{2\eta}(\Gamma)) \), and hence \( g(0) \) comes into play in deriving the regularity of \( L \) when \( s \geq 1/2 \) in (2.10). In fact, applying integration by parts gives
\[
(Lg)(t) = Ng(t) - e^{-At}Ng(0) - \int_0^t e^{-A(t-\tau)}Ng(\tau) \, d\tau.
\]
However, \( g(0) \) does not interfere \( Lg \in L^1(0, T; V_n^{5/2+\eta}(\Omega)) \). First of all, it is clear that \( Ng \in L^2(0, T; V_n^{5/2+\eta}(\Omega)) \subset L^1(0, T; V_n^{5/2+\eta}(\Omega)) \) for \( T < \infty \). Moreover, since \( \dot{g} \in L^2(0, T; V_n^{-1+2\eta}(\Gamma)) \), we have \( N\dot{g} \in L^2(0, T; V_n^{1/2+2\eta}(\Omega)) = L^2(0, T; D(A^{1/4+\eta})) \) for \( 0 < \eta < 1/8 \). Thus

\[
\int_0^t e^{-\Lambda(t-\tau)} N\dot{g}(\tau) \, d\tau \in L^2(0, T; D(A^{5/4+\eta/2})) = L^2(0, T; D(A^{1/4+\eta/2})).
\]

Lastly, using the same estimate as for \( v_0 \) from (4.7) to (4.8), we get for \( 0 < \eta < 1/8 \),

\[
\int_0^T \| e^{-At} Ng(0) \|_{H^{5/2+\eta}} \, dt \leq C \| Ng(0) \|_{H^{1/2+2\eta}} \leq C \| g(0) \|_{L^2} \leq C \| g \|_{S}.
\]

Therefore, (4.9) also holds for \( d = 3 \). This completes the proof.

In the rest of our discussion, \( \eta \) always satisfies the assumptions in Lemma 4.2.

The following Theorem establishes the convergence of \( \theta_\epsilon \) to \( \theta \) as \( \epsilon \to 0 \). To this end, we shall need \( \theta_0 \in H^1(\Omega) \).

**Theorem 4.3.** Assume that \( \theta_0 \in H^1(\Omega) \), \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), d = 2, 3 \), and \( g \in S \). We have

\[
\sup_{t \in [0, T]} \| \theta_\epsilon - \theta \|_{L^2} \leq C(\theta_0, v_0, g, T) \epsilon^{1/2}
\]

and

\[
\| \frac{\partial \theta_\epsilon}{\partial n} |_\Gamma - \frac{\partial \theta}{\partial n} |_\Gamma \|_{L^2(0, T; H^{-1}(\Gamma))} \leq C(\theta_0, v_0, g, T) \epsilon^{1/4}.
\]

**Proof.** Let \( \Theta_\epsilon = \theta_\epsilon - \theta \) and recall \( v_\epsilon = v \) for given \( v_0 \) and \( g \). Then based on (1.1)–(1.5) and (1.7)–(1.12), \( \Theta_\epsilon \) satisfies

\[
\frac{\partial \Theta_\epsilon}{\partial t} - \epsilon \Delta \Theta_\epsilon + v \cdot \nabla \Theta_\epsilon = \Delta \theta,
\]

with the boundary condition

\[
\epsilon \frac{\partial \Theta_\epsilon}{\partial n} |_\Gamma = -\epsilon \frac{\partial \theta}{\partial n} |_\Gamma
\]

and the initial condition

\[
\Theta_\epsilon(0) = 0.
\]
Taking the inner product of (4.13) with $\Theta$ and using (4.14), we get
\[
\frac{1}{2} \frac{d}{dt} \|\Theta_\epsilon\|_{L^2}^2 + \epsilon \|\nabla \Theta_\epsilon\|_{L^2}^2 = \langle \epsilon \frac{\partial \Theta_\epsilon}{\partial n}, \Theta_\epsilon \rangle + \langle \epsilon \frac{\partial \theta}{\partial n}, \Theta_\epsilon \rangle - \epsilon \langle \nabla \theta, \nabla \Theta_\epsilon \rangle,
\]
\[
\leq \epsilon \|\nabla \theta\|_{L^2} \|\nabla \Theta_\epsilon\|_{L^2} \leq \frac{\epsilon}{2} \|\nabla \theta\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla \Theta_\epsilon\|_{L^2}^2,
\]
which yields
\[
\frac{d}{dt} \|\Theta_\epsilon\|_{L^2}^2 + \epsilon \|\nabla \Theta_\epsilon\|_{L^2}^2 \leq \epsilon \|\nabla \theta\|_{L^2}^2.
\] (4.17)

By the Gronwall inequality and the initial condition (4.15), we have
\[
\|\Theta_\epsilon\|_{L^2} \leq \epsilon \int_0^T e^{-\epsilon(t-\tau)} \|\nabla \theta\|_{L^2}^2 d\tau \leq \epsilon \int_0^T \|\nabla \theta\|_{L^2}^2 d\tau \leq C(\theta_0, v_0, g, T) \epsilon,
\]
(4.18)

where we used (4.3) and Lemma 4.2 for deriving the last inequality. Moreover, from (4.17),
\[
\int_0^T \|\nabla \Theta_\epsilon\|_{L^2}^2 d\tau \leq C(\theta_0, v_0, g, T).
\] (4.19)

To establish (4.12), using the trace theorem together with (4.18)–(4.19) we obtain
\[
\|\frac{\partial \Theta_\epsilon}{\partial n}|_{r}\|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma))} \leq C\|\Theta_\epsilon\|_{L^2(0,T;H^\frac{1}{2}(\Omega))}
\]
\[
\leq C( \int_0^T \|\Theta_\epsilon\|_{L^2} \|\Theta_\epsilon\|_{H^\frac{1}{2}} dt)^{1/2}
\]
\[
\leq C( \sup_{t \in [0,T]} \|\Theta_\epsilon\|_{L^2})^{1/2} (\int_0^T \|\Theta_\epsilon\|_{H^\frac{1}{2}} dt)^{1/2} \leq C(\theta_0, v_0, g, T) \epsilon^{1/4}.
\]

This completes the proof. □

Note that since there is no boundary condition imposed on $\theta$, $\frac{\partial \theta}{\partial n}|_{r}$ is not defined. In addition, under the assumptions of Theorem 4.3 we can further verify that $\frac{\partial \theta}{\partial t} \in L^2(0,T; L^2(\Omega))$. Again by the Aubin–Lions–Simon Lemma, we have
\[
\theta \in L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)) \subset C([0,T]; H^{1/2}(\Omega)).
\]
Combining this with Theorem 4.1 and (4.11) yields
\[
\|\theta_\epsilon(T) - \theta(T)\|_{L^2} \leq C(\theta_0, v_0, g, T) \epsilon^{1/2}.
\] (4.20)
5. Existence of an Optimal Solution to \((P_\epsilon)\) and its Conditions of Optimality

Note that the existence of an optimal controller to the problem \((P)\) is independent of \(\epsilon\). With the help of (1.13) and (4.1), the existence of an optimal controller to the problem \((P_\epsilon)\) follows immediately.

**Theorem 5.1.** Assume that \(\theta_0 \in L^\infty(\Omega)\) and \(v_0 \in V_n(\Omega)\). There exists an optimal solution \(g_\epsilon^* \in U_{\epsilon ad}\) to the problem \((P_\epsilon)\).

We now derive the first-order necessary optimality conditions for the problem \((P_\epsilon)\) by using a variational inequality (cf. [27]), that is, if \(g_\epsilon\) is an optimal solution of the problem \((P_\epsilon)\), then

\[
J'_\epsilon(g_\epsilon) \cdot (f_\epsilon - g_\epsilon) \geq 0, \quad f_\epsilon \in U_{\epsilon ad}. \tag{5.1}
\]

Let \(w_\epsilon = v'(g_\epsilon) \cdot h_\epsilon\) be the Gâteaux derivative of \(v_\epsilon\) with respect to \(g_\epsilon\) in every direction \(h\) in \(U_{\epsilon ad}\). Then by (2.10), we have

\[
w_\epsilon = (Lh_\epsilon)(t) \in L^2(0,T;V_n^{3/2}(\Omega)) \cap H^{3/4}(0,T;V_n^0(\Omega)) \subset C([0,T];V_n^{1/2}(\Omega)), \tag{5.2}
\]

which is the solution to the Stokes equations (1.2)–(1.5) with the boundary condition \(g = h_\epsilon\) and the initial condition is zero.

Now denote by \(z_\epsilon = \theta'_\epsilon(g) \cdot h_\epsilon\) the Gâteaux derivative of \(\theta_\epsilon\) with respect to \(g_\epsilon\). Then \(z_\epsilon\) satisfies the equation

\[
\frac{\partial z_\epsilon}{\partial t} - \epsilon \Delta z_\epsilon + w_\epsilon \cdot \nabla \theta_\epsilon + v_\epsilon \cdot \nabla z_\epsilon = 0, \tag{5.3}
\]

with the boundary condition

\[
\epsilon \frac{\partial z_\epsilon}{\partial n}|_\Gamma = 0 \tag{5.4}
\]

and the initial condition

\[
z_\epsilon(0) = 0. \tag{5.5}
\]

To show that (5.3)–(5.5) is well-posed, we first establish an *a priori* estimate for \(z_\epsilon\).
Taking the inner product of (5.3) with \( z \) gives
\[
\frac{1}{2} \frac{d \| z \|_{L^2}^2}{dt} + \epsilon \| \nabla z \|_{L^2}^2 = - \int_{\Omega} (w_e \cdot \nabla \theta_e) z_e \, dx - \int_{\Omega} (v_e \cdot \nabla \theta_e) z_e \, dx
\]
\[
= - \int_{\Omega} (w_e \cdot \nabla (\theta_e z_e)) \, dx + \int_{\Omega} (w_e \theta_e) \cdot \nabla z_e \, dx - 1/2 \int_{\Omega} v_e \cdot \nabla z_e^2 \, dx
\]
\[
= \int_{\Omega} (w_e \theta_e) \cdot \nabla z_e \, dx \leq \|w_e\|_{L^4} \|\theta_e\|_{L^4} \|\nabla z_e\|_{L^2}
\]
\[
\leq C \|w_e\|_{H^{d/4}} \|\theta_e\|_{H^{d/4}} \|\nabla z_e\|_{L^2}, \quad d = 2, 3,
\]
\[
\leq C \|w_e\|_{H^{d/4}}^2 \|\theta_e\|_{H^{d/4}}^2 + \frac{\epsilon}{2} \|\nabla z_e\|_{L^2}^2,
\]
which follows
\[
\frac{d \| z \|_{L^2}^2}{dt} + \epsilon \| \nabla z \|_{L^2}^2 \leq C \|w_e\|_{H^{d/4}}^2 \|\theta_e\|_{H^{d/4}}^2. \tag{5.6}
\]
To complete the estimate, it suffices to show the right hand side of (5.6) is integrable. Note that
\[
\int_0^T \|w_e\|_{H^{d/4}}^2 \|\theta_e\|_{H^{d/4}}^2 \, dt
\]
\[
\leq \begin{cases} 
C \int_0^T \|w_e\|_{L^2} \|\nabla w\|_{L^2} \|\theta_e\|_{L^2} \|\nabla \theta_e\|_{L^2} \, dt & \text{if } d = 2, \\
C \int_0^T \|w_e\|_{H^{3/2}}^{1/2} \|w_e\|_{H^{3/2}}^{1/2} \|\theta_e\|_{L^2} \|\nabla \theta_e\|_{L^2}^{3/2} \, dt & \text{if } d = 3.
\end{cases}
\]
\[
\leq \begin{cases} 
C \|w_e\|_{L^\infty(0,T;L^2(\Omega))} \|\theta_e\|_{L^\infty(0,T;L^2(\Omega))} \|w_e\|_{L^2(0,T;H^1(\Omega))} \|\theta_e\|_{L^2(0,T;H^1(\Omega))} & \text{if } d = 2, \\
C \|w_e\|_{L^\infty(0,T;H^{1/2}(\Omega))} \|\theta_e\|_{L^\infty(0,T;L^2(\Omega))} \|w_e\|_{L^2(0,T;H^{1/2}(\Omega))} \|\theta_e\|_{L^2(0,T;H^{1/2}(\Omega))} & \text{if } d = 3.
\end{cases}
\]
Applying (4.1) and (5.2) gives
\[
\| z \|_{L^2}^2 + \epsilon \int_0^T \| \nabla z_e \|_{L^2}^2 \, dt < \infty.
\]
The rest of the proof follows the standard approaches for parabolic problems (cf. [7, p. 342]).

In order to apply the variational inequality (5.1) to derive the optimality system, we first rewrite the cost functional \( J_e \) as
\[
J_e(g_e) = \frac{1}{2} \langle \Phi_e(T), \theta_e(T) \rangle + \gamma \int_0^T \langle g_e, g_e \rangle \, dt, \quad (P'_e)
\]
where $\Phi_\epsilon$ satisfies
\begin{align}
A\Phi_\epsilon(T) &= \theta_\epsilon(T) \quad (5.7) \\
\frac{\partial \Phi_\epsilon(T)}{\partial n} &= 0. \quad (5.8)
\end{align}

The Neumann boundary value problem (5.7)–(5.8) has a unique solution $\Phi_\epsilon(T) = A^{-1}\theta_\epsilon(T)$ (cf. [26], [42]) and $\Phi_\epsilon(T) \in H^2(\Omega)$ due to $\theta_\epsilon(T) \in L^2(\Omega)$ by Theorem 4.1. The variational inequality (5.1) becomes
\begin{align}
J'(g_\epsilon) \cdot h_\epsilon &= (\Phi_\epsilon(T), z_\epsilon(T)) + \gamma \int_0^T \langle g_\epsilon, h_\epsilon \rangle dt \geq 0, \quad h_\epsilon \in U_{cad}. \quad (5.9)
\end{align}

For given $\epsilon > 0$, the adjoint system associated with the cost functional $(P'_\epsilon)$ is defined by
\begin{align}
-\frac{\partial \rho_\epsilon}{\partial t} - \epsilon \Delta \rho_\epsilon - v_\epsilon \cdot \nabla \rho_\epsilon &= 0, \quad (5.10)
\end{align}

with the boundary condition
\begin{align}
\epsilon \frac{\partial \rho_\epsilon}{\partial n} |_{\Gamma} &= 0 \quad (5.11)
\end{align}

and the final time condition
\begin{align}
\rho_\epsilon(T) &= \Phi_\epsilon(T) \in H^2(\Omega), \quad (5.12)
\end{align}

where $v_\epsilon = v$ satisfies the Stokes equations (1.8)–(1.9) and (1.11)–(1.12). Since $\rho_\epsilon(T) \in H^2(\Omega)$, the compatibility condition for final and boundary data need to be satisfied, i.e., $\epsilon \frac{\partial \rho_\epsilon(T)}{\partial n} |_{\Gamma} = 0$. This is indeed true by (5.8). However, the compatibility condition will not get in the way as $\epsilon \to 0$.

Replacing $t$ by $T-t$ and using the similar approach as in the proof of Theorem 4.1, we obtain that there exists a unique solution $\rho_\epsilon \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$ to (5.10)–(5.12), which satisfies
\begin{align}
\|\rho_\epsilon\|_{L^\infty(0,T; H^1(\Omega))} + \sqrt{\epsilon}\|\rho_\epsilon\|_{L^2(0,T; H^2(\Omega))} + \|\frac{d\rho_\epsilon}{dt}\|_{L^2(0,T; L^2(\Omega))} + \|v_\epsilon \cdot \nabla \rho_\epsilon\|_{L^2(0,T; L^2(\Omega))} \leq C(\theta_\epsilon(T), v_\epsilon). \quad (5.13)
\end{align}

If $v_\epsilon$ further satisfies (1.4), then using the same argument as in the proof of Theorem 4.3 and the relation between the final conditions given by (4.20), we have
\begin{align}
\sup_{t\in[0,T]} \|\rho_\epsilon - \rho\|_{L^2} \to 0, \quad \text{as} \quad \epsilon \to 0, \quad (5.14)
\end{align}
where $\rho$ satisfies
\begin{align}
-\frac{\partial}{\partial t} \rho - v \cdot \nabla \rho &= 0, \quad v = v_\epsilon, \\
\rho(T) &= \Lambda^{-2} \theta(T),
\end{align}
and
\[ \sup_{t \in [0,T]} \|\nabla \rho\|_{L^2} < \infty. \tag{5.17} \]

In fact, (5.15)–(5.16) define the adjoint system of (1.1)–(1.5) associated with the cost functional $J$. We now establish the optimality system of the approximating problem \( (P_\epsilon) \) and its convergence to the optimality system of the problem \( (P) \).

**Theorem 5.2.** Let $\theta_0 \in L^\infty(\Omega)$ and $v_0 \in V^0_n(\Omega)$. Assume that $g^*_\epsilon$ is an optimal controller of the problem \( (P_\epsilon) \). If \( (v_\epsilon, \theta_\epsilon) \) is the corresponding solution of (1.7)–(1.12) and $\rho_\epsilon$ is the solution of the adjoint equations \( (5.10) \)–\( (5.12) \) associated with \( (v_\epsilon, \theta_\epsilon) \) then
\[ g^*_\epsilon = -\frac{1}{\gamma} L^*(\mathbb{P}(\theta_\epsilon \nabla \rho_\epsilon)) \in L^2(0,T;V^{3/2}_n(\Gamma)) \cap H^{3/4}(0,T;V^0_n(\Gamma)). \tag{5.18} \]

**Proof.** First multiplying (5.3) by $\rho_\epsilon$, we have
\[ \int_0^T \left( \frac{\partial}{\partial t} z_\epsilon, \rho_\epsilon \right) dt + \int_0^T ((Lh_\epsilon) \cdot \nabla \theta_\epsilon, \rho_\epsilon) dt + \int_0^T (v_\epsilon \cdot \nabla z_\epsilon, \rho_\epsilon) dt = \int_0^T (\epsilon z_\epsilon, \Delta \rho_\epsilon) dt. \]

Integrating the first term with respect to $t$ and the third term with respect to $x$ yield
\[ -\int_0^T \left( \frac{\partial}{\partial t} \rho_\epsilon, z_\epsilon \right) dt + (\rho_\epsilon(T), z_\epsilon(T)) + \int_0^T ((Lh_\epsilon) \cdot \nabla \theta_\epsilon, \rho_\epsilon) dt \]
\[ -\int_0^T (z_\epsilon, v_\epsilon \cdot \nabla \rho_\epsilon) dt = \int_0^T (\epsilon z_\epsilon, \Delta \rho_\epsilon) dt, \]
where we used \( (\Delta z_\epsilon, \rho_\epsilon) = (\epsilon z_\epsilon, \Delta \rho_\epsilon) \) due to (5.4) and (5.11). In light of the adjoint equation (5.10) and the final condition (5.12), we have
\[ (\Phi_\epsilon(T), z_\epsilon(T)) = (\rho_\epsilon(T), z_\epsilon(T)) = -\int_0^T ((Lh_\epsilon) \cdot \nabla \theta_\epsilon, \rho_\epsilon) dt. \tag{5.19} \]
Combining (5.9) with (5.19) yields
\[ J'_\epsilon(g_\epsilon) \cdot h_\epsilon = -\int_0^T ((Lh_\epsilon) \cdot \nabla \theta_\epsilon, \rho_\epsilon) dt + \gamma \int_0^T (g_\epsilon, h_\epsilon) dt \geq 0. \] (5.20)

Note that \( \nabla \cdot (Lh_\epsilon) = 0 \) and \((Lh_\epsilon) \cdot n|_\Gamma = 0\). Thus
\[ \int_0^T ((Lh_\epsilon) \cdot \nabla \theta_\epsilon, \rho_\epsilon) dt = \int_0^T \int_{\Omega} (\mathbb{P}(Lh_\epsilon)) \cdot \nabla (\theta_\epsilon \rho_\epsilon) dx dt - \int_0^T (\mathbb{P}(Lh_\epsilon), \theta_\epsilon \nabla \rho_\epsilon) dt \]
\[ = -\int_0^T (h_\epsilon, L^* \mathbb{P}(\theta_\epsilon \nabla \rho_\epsilon)) dt, \] (5.21)

where \( \theta_\epsilon \nabla \rho_\epsilon \in L^2(0,T;L^2(\Omega)) \). Therefore, based on (5.20)–(5.22) if \( g_\epsilon^* \) is an optimal solution, then
\[ J'_\epsilon(g_\epsilon^*) \cdot h_\epsilon = \int_0^T (h_\epsilon, L^* \mathbb{P}(\theta_\epsilon \nabla \rho_\epsilon)) dt + \gamma \int_0^T (g_\epsilon, h_\epsilon) dt \geq 0, \quad h_\epsilon \in U_{ead}, \]
which gives
\[ g_\epsilon^* = -\frac{1}{\gamma} L^* (\mathbb{P}(\theta_\epsilon \nabla \rho_\epsilon)). \]

Moreover, by the continuity of \( \mathbb{P} \) on \( L^2(\Omega) \) (cf. [39, p. 13], (4.1), and (5.13), we have
\[ \| \mathbb{P}(\theta, \nabla \rho_\epsilon) \|_{L^2} \leq C \| \theta \|_{H^s(\Omega)} \leq C \| \theta \|_{L^\infty(0,T;L^\infty(\Omega))} \| \rho_\epsilon \|_{L^2(0,T;H^1(\Omega))} \leq \infty. \] (5.23)

Lastly, combining (5.23) with the regularity property of \( L^* \) given by (2.13) yields (5.18). This completes the proof.

**Remark 5.3.** As mentioned in [2, Remark 6], since the Leray projector \( \mathbb{P}: L^2(\Omega) \to V^0_n(\Omega) \) can be extended from \( H^s(\Omega), s > 0, \) to \( V^s_n(\Omega), \) its adjoint \( \mathbb{P}^*: V^0_n(\Omega) \to L^2(\Omega) \) can be extended as a bounded operator from \( (V^s_n(\Omega))' \) to \((H^s(\Omega))'\) by
\[ (\mathbb{P}^* \psi, \varphi)_{(H^s(\Omega))', H^s(\Omega)} = (\psi, \mathbb{P} \varphi)_{(V^s_n(\Omega))', V^s_n(\Omega)}, \quad \psi \in (V^s_n(\Omega))', \ \varphi \in V^s_n(\Omega). \] (5.24)

Therefore, if \( \theta \nabla \rho_\epsilon \in L^2(0,T;H^s(\Omega)) \), where \( s < 0, \) then we use duality from (5.21) to (5.22) and replace \( \mathbb{P} \) by \( \mathbb{P}^* \). In this case,
\[ g_\epsilon^* = -\frac{1}{\gamma} L^* (\mathbb{P}^*(\theta_\epsilon \nabla \rho_\epsilon)). \] (5.25)
In order to address the convergence of the optimality conditions for the approximating problem, we shall assume \( \theta_0 \in L^\infty(\Omega) \cap H^1(\Omega) \) and \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), \ d = 2, 3, \) in the rest of our discussion.

**Theorem 5.4.** Assume \( \theta_0 \in L^\infty(\Omega) \cap H^1(\Omega) \) and \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), \ d = 2, 3. \) Let \((g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*)\) be an optimal solution for \((P_\epsilon)\). Then, there exists \((g^*, v^*, \theta^*)\) such that a subsequence of \((g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*)\) in terms of \( \epsilon \), still denoted by \( \{g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*\} \), satisfying

\[
g_\epsilon^* \to g^* \text{ strongly in } L^2(0, T; V_n^{3/2-\delta}(\Gamma)),
\]

\[
v_\epsilon^* \to v^* \text{ strongly in } L^2(0, T; V_n^{d/2+2\eta}(\Omega)),
\]

for \( 0 < \delta \leq 3 - d/2 - 2\eta \), and

\[
\theta_\epsilon^* \to \theta^* \text{ strongly in } L^2(\Omega), \text{ uniformly in } t \in [0, T],
\]

as \( \epsilon \to 0 \). Moreover, \((g^*, v^*, \theta^*)\) is an optimal solution to the problem \((P)\), which can be solved from

\[
g^* = -\frac{1}{\gamma}L^*(\mathbb{P}(\theta^* \nabla \rho^*)) \in L^2(0, T; V_n^{3/2}(\Gamma)) \cap H^{3/4}(0, T; V_n(\Gamma)),
\]

where \( \rho^* \) is the solution to the dual problem \((5.15)-(5.16)\) corresponding to \((v^*, \theta^*)\).

**Proof.** Step 1: We first show the strong convergence of the optimal solution to the problem \((P_\epsilon)\). With the help of \((4.1)\) and \((5.13)\) we get

\[
\|\theta_\epsilon^* \nabla \rho_\epsilon^*\|_{L^2(0, T; L^2(\Omega))} \leq \|\theta_\epsilon^*\|_{L^\infty(0, T; L^\infty(\Omega))} \|\nabla \rho_\epsilon^*\|_{L^2(0, T; L^2(\Omega))} \leq C,
\]

independent of \( \epsilon \). Thus there exist subsequences, still denoted by \( \{\theta_\epsilon^* \nabla \rho_\epsilon^*\} \) and \( \{\nabla \rho_\epsilon^*\} \), such that

\[
\theta_\epsilon^* \nabla \rho_\epsilon^* \to \xi \text{ weakly in } L^2(0, T; L^2(\Omega)) \text{ as } \epsilon \to 0,
\]

for some \( \xi \in L^2(0, T; L^2(\Omega)) \), and

\[
\nabla \rho_\epsilon^* \to \nabla \rho^* \text{ weakly in } L^2(0, T; L^2(\Omega)) \text{ as } \epsilon \to 0.
\]

Based on the property of \( L^* \) given by \((2.10)\) and the continuity of \( \mathbb{P} \), there exists a subsequence \( \{g_\epsilon^* = L^*(\mathbb{P}(\theta_\epsilon^* \nabla \rho_\epsilon^*)))\} \) in terms of \( \epsilon \), such that

\[
g_\epsilon^* \to g^* = L^*(\mathbb{P}\xi) \text{ weakly in } L^2(0, T; V_n^{3/2}(\Gamma)) \cap H^{3/4}(0, T; V_n^0(\Gamma)), \text{ as } \epsilon \to 0,
\]

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Thus in light of \[39, \text{Theorem 2.2, p. 186}],

\[ g_\epsilon^* \rightarrow g^* \quad \text{strongly in} \quad L^2(0,T;V_0^{3/2-\delta}(\Gamma)), \quad \forall 0 < \delta < \frac{3}{2}. \]

Correspondingly, by (2.5), (2.7), and (2.10), we have for \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), d = 2, 3, \)
that

\[ v_\epsilon^* - v^* = L(g_\epsilon^* - g^*) \rightarrow 0 \quad \text{strongly in} \quad L^2(0,T;V_0^{\text{min}(d/2+2\eta,3-\delta)}(\Omega)) = L^2(0,T;V_0^{d/2+2\eta}(\Omega)). \]  

(5.30)

for \( 0 < \delta \leq 3 - d/2 - 2\eta \). Let \( \theta^* \) be the solution of (1.1) associated with \( v^* \) and initial condition \( \theta_0 \). Next we prove

\[ \sup_{t \in [0,T]} \| \nabla \theta^* \|_{L^2} < \infty. \]  

(5.31)

According to Lemma 4.2, we know that \( \int_0^T \| \nabla v^* \|_{L^\infty} \, dt < \infty \) for \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), d = 2, 3, \) and \( g^* \in L^2(0,T;V_0^{3/2}(\Gamma)) \cap H^{3/4}(0,T;V_0^0(\Gamma)). \) Thus (5.31) follows immediately from (4.3).

To establish (5.26), we let \( \Theta_\epsilon^* = \theta_\epsilon^* - \theta^* \) and \( V_\epsilon^* = v_\epsilon^* - v^*. \) Then \( \Theta_\epsilon^* \) satisfies

\[ \frac{\partial \Theta_\epsilon^*}{\partial t} - \epsilon \Delta \Theta_\epsilon^* + v_\epsilon^* \cdot \nabla \Theta_\epsilon^* + V_\epsilon^* \cdot \nabla \theta^* = \Delta \theta^*, \]  

(5.32)

with the boundary condition

\[ \epsilon \frac{\partial \Theta_\epsilon^*}{\partial n}|_{\Gamma} = -\epsilon \frac{\partial \theta^*}{\partial n}|_{\Gamma} \]  

(5.33)

and the initial condition \( \Theta_\epsilon^*(0) = V_\epsilon^*(0) = 0. \) Recall by (5.30) that

\[ V_\epsilon^* \rightarrow 0 \quad \text{strongly in} \quad L^2(0,T;V_0^{d/2+2\eta}(\Omega)), \quad \text{as} \quad \epsilon \rightarrow 0. \]  

(5.34)

As shown in the proof of Theorem 4.3 applying \( L^2 \)-estimate for \( \Theta_\epsilon^* \) follows

\[ \frac{d\|\Theta_\epsilon^*\|_{L^2}^2}{dt} + \epsilon \| \nabla \Theta_\epsilon^* \|_{L^2}^2 \leq \epsilon \| \nabla \theta^* \|_{L^2}^2 + \| V_\epsilon^* \cdot \nabla \theta^* \|_{L^2}^2 + \| \Theta_\epsilon^* \|_{L^2}^2. \]  

(5.35)

Using the Gronwall inequality and \( \Theta_\epsilon^*(0) = 0, \) we get

\[ \| \Theta_\epsilon^* \|_{L^2}^2 \leq \int_0^t e^{(t-\tau)} (\epsilon \| \nabla \theta^* \|_{L^2}^2 + \| V_\epsilon^* \cdot \nabla \theta^* \|_{L^2}^2) \, d\tau, \]  

(5.36)
Theorem 5.5. Assume \( \theta_0 \in L^\infty(\Omega) \cap H^1(\Omega) \) and \( v_0 \in V_n^{d/2-1+2\eta}(\Omega), d = 2, 3 \). If \((g^*, v^*, \theta^*)\) is an optimal solution to the problem \((P)\), then \(g^*\) can be solved from \((5.1) - (5.5), (5.11) - (5.16)\), and the optimality condition \((5.24)\).
Proof. Let \((g^*, v^*, \theta^*)\) be any optimal solution to the problem \((P)\). We first employ the idea as in [5, Theorem 5] to impose a penalization on the cost functional \(J_\epsilon\) as to establish the relation between \((g^*, v^*, \theta^*)\) and the optimal solution to the new defined cost functional. Consider the minimization problem

\[
\min \{ J_\epsilon(g) + \frac{1}{2} \int_0^T \| g - g^* \|^2_{L^2(\Gamma)} dt \}. \quad (\hat{P}_\epsilon)
\]

If we let \((\hat{g}_\epsilon, \hat{v}_\epsilon, \hat{\theta}_\epsilon)\) be the optimal solution to the problem \((\hat{P}_\epsilon)\), then

\[
J_\epsilon(\hat{g}_\epsilon) + \frac{1}{2} \int_0^T \| \hat{g}_\epsilon - g^* \|^2_{L^2} dt \leq J_\epsilon(g) + \frac{1}{2} \int_0^T \| g - g^* \|^2_{L^2} dt, \quad (5.39)
\]

for any \(g \in L^2(0, T; V^0_n(\Gamma))\). As proven in Theorem 5.4, there exists a subsequence, still denoted by \(\{(\hat{g}_\epsilon, \hat{v}_\epsilon, \hat{\theta}_\epsilon)\}\), satisfying

\[
\hat{g}_\epsilon \to \hat{g}^* \text{ strongly in } L^2(0, T; V^0_n(\Gamma)), \quad \text{as } \epsilon \to 0, \quad (5.40)
\]

\[
\hat{v}_\epsilon \to \hat{v}^* \text{ strongly in } L^2(0, T; V^0_0(\Omega)), \quad \text{as } \epsilon \to 0, \quad (5.40)
\]

\[
\hat{\theta}_\epsilon \to \hat{\theta}^* \text{ strongly in } L^2(\Omega), \text{ uniformly in } [0, T], \quad \text{as } \epsilon \to 0. \quad (5.40)
\]

By the weakly lower semicontinuity of norms, we can pass to the limit in (5.39) and obtain

\[
J(\hat{g}^*) + \frac{1}{2} \int_0^T \| \hat{g}^* - g^* \|^2_{L^2(\Gamma)} dt \leq J(g) + \frac{1}{2} \int_0^T \| g - g^* \|^2_{L^2(\Gamma)} dt, \quad (5.41)
\]

for all \(g \in L^2(0, T; V^0_n(\Gamma))\). In particular, setting \(g = g^*\) yields

\[
J(\hat{g}^*) + \frac{1}{2} \int_0^T \| \hat{g}^* - g^* \|^2_{L^2(\Gamma)} dt \leq J(g^*), \quad (5.42)
\]

which indicates

\[
\int_0^T \| \hat{g}^* - g^* \|^2_{L^2} dt = 0.
\]

Therefore, \(g^* = \hat{g}^*\), and hence \(v^* = \hat{v}^*\) and \(\theta^* = \hat{\theta}^*\). Moreover, according to (5.40) we get

\[
\hat{g}_\epsilon \to g^* \text{ strongly in } L^2(0, T; V^0_n(\Gamma)), \quad \text{as } \epsilon \to 0. \quad (5.43)
\]
Following the proof of Theorem 5.2, we have the optimality condition for the problem \((\hat{P}_\epsilon)\) given by

\[
\gamma \hat{g}_\epsilon^* + \hat{g}_\epsilon^* - g^* = -L^* \mathbb{P}(\hat{\theta}_\epsilon \nabla \hat{\rho}_\epsilon),
\]

where \(\hat{\rho}_\epsilon\) satisfies (5.10)–(5.12). Letting \(\epsilon \to 0\), we obtain from (5.38) and (5.43) that

\[
g^* = -\frac{1}{\gamma} L^* \mathbb{P}(\theta^* \nabla \rho^*),
\]

which completes the proof.

6. Uniqueness of the optimal controller to \((P)\) for \(d = 2\)

In this section, we present the uniqueness of the optimal controller to the problem \((P)\) for \(d = 2\) and \(\gamma\) sufficiently large. In this case we set \(0 < \eta < 1/4\). The main result is given by the following theorem.

**Theorem 6.1.** Assume \(\theta_0 \in L^\infty(\Omega) \cap H^1(\Omega), v_0 \in V^{2n}_n(\Omega)\) for \(d = 2\), and \(\gamma\) sufficiently large, there exists at most one optimal controller \(g \in \mathcal{U}_{ad}\) to the problem \((P)\), which is given by (5.27).

**Proof.** Assume that there are two pair of optimal solutions to the problem \((P)\), denoted by \((g_i, \theta_i, v_i), i = 1, 2\). Then from (5.27), Lemma 4.2, and (4.3) we have

\[
g_i \in L^2(0, T; V^{3/2}_n(\Gamma)) \cap H^{3/4}(0, T; V^0_n(\Gamma)),
\]

\[
\int_0^T \|\nabla v_i\|_{L^\infty} \leq C(v_0, g_i, T), \quad \text{and} \quad \sup_{t \in [0, T]} \|\nabla \theta_i\|_{L^2} \leq C(\theta_0, v_0, g_i, T).
\]

(6.1)

The corresponding solutions to the adjoint problem (5.10)–(5.12) are denoted by \(\rho_i, i = 1, 2\). Then \(G = g_1 - g_2, \vartheta = \theta_1 - \theta_2, LG = v_1 - v_2,\) and \(\varrho = \rho_1 - \rho_2\) satisfy the system

\[
\frac{\partial \vartheta}{\partial t} + (LG) \cdot \nabla \theta_1 + v_2 \cdot \nabla \vartheta = 0, \quad \vartheta(0) = 0,
\]

(6.2)

\[
-\frac{\partial \varrho}{\partial t} - (LG) \cdot \nabla \rho_1 - v_2 \nabla \varrho = 0, \quad \varrho(T) = \Lambda^{-2} \vartheta(T),
\]

(6.3)

and

\[
G = -\frac{1}{\gamma} L^* (\mathbb{P}(\vartheta \nabla \rho_1 + \theta_2 \nabla \varrho)).
\]

(6.4)
Applying $L^2$-estimate on $\vartheta$ gives

$$
\frac{d\|\vartheta\|_{L^2}^2}{2dt} = \int_{\Omega} (LG) \cdot \nabla \vartheta \, dx \leq \|LG\|_{L^\infty} \|\nabla \vartheta_1\|_{L^2} \|\vartheta\|_{L^2},
$$

from where

$$
\sup_{t \in [0,T]} \|\vartheta\|_{L^2} \leq \int_0^T \|LG\|_{L^\infty} \|\nabla \vartheta_1\|_{L^2} \, dt \leq \sup_{t \in [0,T]} \|\nabla \vartheta_1\|_{L^2} \int_0^T \|LG\|_{L^\infty} \, dt. \quad (6.5)
$$

By (6.4), (2.10) and (2.13), we get

$$
\int_0^T \|LG\|_{L^\infty} \, dt \leq C \int_0^T \|L^{-1}L^*P(\partial \nabla \rho_1 + \theta_2 \nabla \vartheta)\|_{H^{1+\varepsilon}(\Omega)} \, dt, \quad 0 < \varepsilon \leq 1/2,
$$

$$
\leq C \frac{1}{\gamma} \int_0^T \|L^*P(\partial \nabla \rho_1 + \theta_2 \nabla \vartheta)\|_{L^2(0,T;H^{1+\varepsilon}(\Omega))} \, dt
\leq C \frac{1}{\gamma} \int_0^T \|L^*P(\partial \nabla \rho_1 + \theta_2 \nabla \vartheta)\|_{L^2(0,T;L^2(\Gamma))} \, dt
\leq C \frac{1}{\gamma} \int_0^T \|P^*(\partial \nabla \rho_1 + \theta_2 \nabla \vartheta)\|_{L^2(0,T;H^{1+\varepsilon}(\Omega)^\prime)} \, dt
\leq C \frac{1}{\gamma} \int_0^T (\|P^*(\partial \nabla \rho_1)\|_{L^2(0,T;H^{1+\varepsilon}(\Omega)^\prime)} + \|P^*(\theta_2 \nabla \vartheta)\|_{L^2(0,T;H^{1+\varepsilon}(\Omega)^\prime)}) \, dt. \quad (6.6)
$$

From (6.6) to (6.7) we used the property of $L^*$ given by (2.14) and replaced $P$ by $P^*$ due to Remark 5.3. For the first term on the right hand side of (6.8) we used duality (5.24) and obtain

$$
\int_0^T \|P^*(\partial \nabla \rho_1)\|^2_{(H^{1+\varepsilon}(\Omega))} \, dt = \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\int_{\Omega} (\partial \nabla \rho_1) \cdot (P^*\psi) \, dx}{\|\psi\|_{H^{1+\varepsilon}}} \right)^2 \, dt,
$$

$$
\leq C \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\|\partial \nabla \rho_1\|_{L^2} \|\nabla \psi\|_{L^\infty} \|\psi\|_{H^{1+\varepsilon}}}{\|\psi\|_{H^{1+\varepsilon}}} \right)^2 \, dt. \quad (6.9)
$$

$$
\leq C \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\|\partial \nabla \rho_1\|_{L^2} \|\nabla \psi\|_{H^{1+\varepsilon}}}{\|\psi\|_{H^{1+\varepsilon}}} \right)^2 \, dt \quad (6.10)
$$

$$
\leq CT (\sup_{t \in [0,T]} \|\vartheta\|_{L^2})^2 (\sup_{t \in [0,T]} \|\nabla \rho_1\|_{L^2})^2. \quad (6.11)
$$
To estimate the second term on the right hand of (6.15), we have
\[
\int_0^T \| \mathbb{P}^* (\theta \nabla \varrho) \|^2_{H^{1+\varepsilon}(\Omega)} \, dt = \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{| \int_{\Omega} (\theta \nabla \varrho) \cdot (\mathbb{P} \psi) \, dx |}{\| \psi \|_{H^{1+\varepsilon}}} \right)^2 \, dt,
\]
\[
= \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\int_{\Omega} \theta \nabla \cdot (\varrho \mathbb{P} \psi) \, dx - \int_{\Omega} \theta \nabla \cdot (\mathbb{P} \psi) \, dx}{\| \psi \|_{H^{1+\varepsilon}}} \right)^2 \, dt
\]
\[
\leq C \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\| \nabla \theta \|_{L^2} \| \varrho \|_{L^2} \| \psi \|_{L^\infty}}{\| \psi \|_{H^{1+\varepsilon}}} \right)^2 \, dt \tag{6.12}
\]
\[
\leq C \int_0^T \left( \sup_{\psi \in H^{1+\varepsilon}(\Omega)} \frac{\| \nabla \theta \|_{L^2} \| \varrho \|_{L^2} \| \psi \|_{H^{1+\varepsilon}}}{\| \psi \|_{H^{1+\varepsilon}}} \right)^2 \, dt \tag{6.13}
\]
\[
\leq C T \left( \sup_{t \in [0,T]} \| \nabla \theta \|_{L^2} \right)^2 \left( \sup_{t \in [0,T]} \| \varrho \|_{L^2} \right)^2. \tag{6.14}
\]
Combining (6.8) with (6.11)–(6.14) yields
\[
\int_0^T \| \mathbb{L} \|_{L^\infty} \, dt \leq C \frac{1}{\gamma} \left( \sup_{t \in [0,T]} \| \varrho \|_{L^2} \sup_{t \in [0,T]} \| \nabla \rho_1 \|_{L^2} + \sup_{t \in [0,T]} \| \nabla \theta \|_{L^2} \sup_{t \in [0,T]} \| \varrho \|_{L^2} \right). \tag{6.15}
\]

It remains to estimate \( \| \varrho \|_{L^2(\Omega)} \). Let \( \tilde{\varrho}(t) = \varrho(T - t), \, t \in [0,T] \). Then \( \tilde{\varrho}(t) \) satisfies
\[
\frac{d}{dt} \tilde{\varrho} - (\mathbb{L}(T - t)) \cdot \nabla \rho_1(T - t) - v_2(T - t) \nabla \tilde{\varrho} = 0, \tag{6.16}
\]
\[
\tilde{\varrho}(0) = \Lambda^{-2} \varrho(T). \tag{6.17}
\]

Applying \( L^2 \)-estimate for \( \tilde{\varrho} \) yields
\[
\sup_{t \in [0,T]} \| \tilde{\varrho} \|_{L^2} \leq \int_0^T \| \mathbb{L}(T - t) \|_{L^\infty} \| \nabla \rho_1(T - t) \|_{L^2} \, dt + \| \tilde{\varrho}(0) \|_{L^2}
\]
\[
\leq \sup_{t \in [0,T]} \| \nabla \rho_1 \|_{L^2} \int_0^T \| \mathbb{L} \|_{L^\infty} \, dt + C \| \varrho(T) \|_{L^2}. \tag{6.18}
\]
Now combining (6.13) with (6.1), (6.5), and (6.18) gives

\[
\int_0^T \|LG\|_{L^\infty} \leq C(\theta_0, v_0, g_1, g_2, T) \frac{1}{\gamma} \left( \sup_{t \in [0,T]} \|v\|_{L^2} + \sup_{t \in [0,T]} \|\vartheta\|_{L^2} \right)
\]

\[
\leq C(\theta_0, v_0, g_1, g_2, T) \frac{1}{\gamma} \left[ \sup_{t \in [0,T]} \|\nabla \vartheta_1\|_{L^2} \int_0^T \|LG\|_{L^\infty} \, dt + \left( \sup_{t \in [0,T]} \|\nabla \rho_1\|_{L^2} \int_0^T \|LG\|_{L^\infty} \, d\tau + \sup_{t \in [0,T]} \|\nabla \vartheta_1\|_{L^2} \int_0^T \|LG\|_{L^\infty} \, dt \right) \right]
\]

\[
\leq C(\theta_0, v_0, g_1, g_2, T) \frac{1}{\gamma} \int_0^T \|LG\|_{L^\infty} \, d\tau. \tag{6.19}
\]

If we let \(\gamma\) be sufficiently large so that

\[
C(\theta_0, v_0, g_1, g_2, T) \frac{1}{\gamma} < 1 \quad \text{or} \quad \gamma > C(\theta_0, v_0, g_1, g_2, T),
\]

then

\[
\int_0^T \|LG\|_{L^\infty} \, dt = 0. \tag{6.20}
\]

Lastly, by the linearity of \(L\), we derive that \(G = 0\). Uniqueness of the optimal solution for large \(\gamma\) and \(d = 2\) is established.

\[
\square
\]

**Remark 6.2.** The uniqueness for \(d = 3\) cannot be carried out by the current approach due to the failure from (6.9) to (6.10) and from (6.12) to (6.13). This is because when \(d = 3\), the regularity of the test function \(\psi\) cannot go beyond \(H^{3/2-\varepsilon}\), where \(\varepsilon\) is arbitrarily small. Therefore, the \(L^\infty\)-norm of \(\psi\) in the numerators of (6.9) and (6.12) cannot be bounded.

### 7. Conclusions

Compared to the optimality system presented in [21, Theorem 4.1], the current approach of constructing an approximating control problem provides a much more transparent result. In addition, uniqueness of the optimal controller can be derived for \(d = 2\). These will greatly contribute to implementing the solution by employing the gradient based iterative schemes in our future work.
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