Stochastic gravitoelectromagnetic inflation

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Gravitoelectromagnetic inflation was recently introduced to describe, in an unified manner, electromagnetic, gravitatory and inflaton fields in the early (accelerated) inflationary universe from a 5D vacuum state. In this paper, we study a stochastic treatment for the gravitoelectromagnetic components $A_B = (A^\mu, \phi)$ on cosmological scales. We focus our study on the seed magnetic fields on super Hubble scales, which could play an important role in large scale structure formation of the universe.

PACS numbers:

I. INTRODUCTION

Most of the cosmologists believe that our universe has experienced an early period of accelerated expansion, called inflation. Inflation provides a mechanism that explains the origin of the large scale structure formation process. In this mechanism quantum fluctuations of the inflaton field that were stretched beyond the horizon become classical perturbations. In the stochastic approach of inflation the dynamics of the quantum to classical transition is effectively described by a classical noise. In this approach the scalar field is coarse grained. Thus, the field modes are split into a subhorizon and a superhorizon parts. The superhorizon part, which describes the fluctuations on cosmological scales, constitutes the classical and homogeneous coarse-grained field driven by the stochastic noise due to the subhorizon modes that are leaving the horizon. As the inflationary universe expands rapidly, more and more short wavelength modes are stretched beyond the horizon and thus the number of degrees of freedom of the coarse-grained field is being increased. This phenomena is viewed by this field as a noise. The dynamics of this field can be described by a second order stochastic equation, which can be rewritten as two first order (Langevin) equations. The stochastic properties of the relevant noise depend on the window function that separates the sub and superhorizon modes.

On the other hand, the possible existence, strength and structure of magnetic fields in the intergalactic plane, within the Local Superclustered, has been scrutinized recently. If the seed of these magnetic fields was originated during inflation, the study of its origin and evolution in this epoch should be very important in cosmology. The origin of the primordial magnetic fields has been subject of a great amount of research. These concepts can be extended and worked in the gravitoelectromagnetic inflation formalism recently introduced in [10]. Gravitoelectromagnetic inflation was developed with the aim to describe, in an unified manner, the inflaton, gravitatory and electromagnetic fields during inflation. The formalism has the advantage that all the 4D sources has a geometrical origin and can explain the origin of seed of magnetic fields on cosmological scales observed today. Gravitoelectromagnetic inflation was constructed from a 5D vacuum state on a $R^A_{BCD} = 0$ globally flat metric. As in all Space Time Matter (STM) models, the 4D sources are geometrically induced when we take a foliation on the fifth coordinate which is spacelike and noncompact. There is a main difference between STM and Brane-World (BW) formalisms. In the STM theory of gravity we do not need to insert any matter into the 5D manifold by hand, as is commonly done in the BW formalism. The 5D metrics used in the STM theory are exact solutions of the 5D field equations in apparent vacuum. The interesting thing here, is that matter appears in four dimensions without any dimensional compactification, but induced by the 5D vacuum conditions.

The aim of this paper is to develop a stochastic treatment for the components $A_B = (A^\mu, \phi)$ on cosmological scales, to be able to make a comparison with the results obtained in a previous paper [10].

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II. 5D FORMALISM

In order to describe a 5D vacuum, we consider the 5D canonical metric\(^1\)
\[
dS^2 = \psi^2 dN^2 - \psi^2 e^{2N} dr^2 - dw^2,
\]
where \(dr^2 = dx^2 + dy^2 + dz^2\). In this line element the coordinates \((N, r)\) are dimensionless and the fifth one \(\psi\) has spatial units. This metric describes a 5D flat manifold in apparent vacuum \(G_{AB} = 0\) and satisfies \(\mathcal{R}^{A}_{BCD} = 0\). To describe an electromagnetic field and neutral matter on this background, we consider the action
\[
I = \int d^4 x \, d\psi \sqrt{\left[ \frac{(5)g}{(5)\theta_0} \right]} \left[ \frac{(5)R}{16 \pi G} + (5) \mathcal{L}(A_B, A_C; B) \right],
\]
for a vector potential with components \(A_B = (A_\mu, \varphi)\), which are minimally coupled to gravity. Here \((5)R\) is the 5D Ricci scalar, which is zero for the metric \((1)\) and \(\bar{\Pi}(7)\) and \((8)\), we obtain
\[
Q
\]
where we define the tensor field \(A\) considering all the time that \(\bar{\Pi}(7)\) and \((8)\), we obtain
\[
Q
\]
for a vector potential with components \(A_B = (A_\mu, \varphi)\), which are minimally coupled to gravity. Here \((5)R\) is the 5D Ricci scalar, which is zero for the metric \((1)\) and \(\bar{\Pi}(7)\) and \((8)\), we obtain
\[
Q
\]
the last term being a “gauge-fixing” term. The 5D-dynamics field equations in a Lagrange formalism leads to
\[
A^B_{;D} \Delta^D - (1 - \lambda) A^C_{;C} ; B = 0.
\]
Working in the Feynman gauge \((\lambda = 1)\), equation \((5)\) yields
\[
\frac{1}{\sqrt{\left[ (5)g \right]}} \frac{\partial}{\partial x^C} \left[ \sqrt{\left[ (5)g \right]} g^{DC} A^B_{;D} \right] = 0,
\]
considering all the time that \(A^B = (A_\mu, \varphi)\). Equation \((11)\) is a massless Klein-Gordon-like equation for \(A^B\) and represents the analogous of Maxwell’s equations in a 5D manifold in an apparent vacuum. The commutators for \(A^C\) and \(\Pi^B = \frac{\partial C}{\partial A_{B,N}}\) are given by
\[
A^C(N, \vec{r}, \psi), \Pi^B(N, \vec{r}, \psi') = ig^{CB} g^{NN} \left[ (5)g \theta_0 \right] \delta^{(3)} (\vec{r} - \vec{r}') \delta(\psi - \psi')
\]
\[
A_C(N, \vec{r}, \psi), A_B(N, \vec{r}', \psi') = \left[ \bar{\Pi}_C(N, \vec{r}, \psi), \Pi_B(N, \vec{r}', \psi') \right] = 0.
\]
Here \(\bar{\Pi}^N = -g^{NN} (A^C_{;C})\) and \(\left[ (5)g \theta_0 \right] \) is the inverse of the normalized volume of the manifold \((11)\). From equations \((7)\) and \((8)\), we obtain
\[
A_C(N, \vec{r}, \psi), A_B(N, \vec{r}', \psi') = -i g^{BC} \left[ (5)g \theta_0 \right] \delta^{(3)} (\vec{r} - \vec{r}') \delta(\psi - \psi').
\]
Using equations \((11)\) and \((13)\), the equation of motion for the electromagnetic 4-vector potential \(A_\mu\), is given by (the overstar denotes derivative with respect to \(N\))
\[
\ddot{A}_\mu + 3 \dot{A}_\mu - e^{-2N} \nabla^2 A_\mu - \left[ 4\psi \frac{\partial A_\mu}{\partial \psi} + \psi^2 \frac{\partial^2 A_\mu}{\partial \psi^2} \right] = 0.
\]

\(^1\) In our conventions, capital Latin indices run from 0 to 4 and greek indices from 0 to 3.
where

\[ A^\mu(N, \vec{r}, \psi) = e^{-3N/2} \left( \frac{\psi_0}{\psi} \right)^2 A^\mu(N, \vec{r}, \psi), \]

being

\[ A^\mu(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k \int d\psi \sum_{\lambda=0}^{3} \left[ \delta^{(\beta)}_{\mu\nu} e^{i[k_{\nu}\vec{r}+k_{\psi}\psi]} \hat{Q}_{k_{\nu}k_{\psi}}(N, \psi) + \delta^{(\beta)}_{\mu\nu} e^{-i[k_{\nu}\vec{r}+k_{\psi}\psi]} \hat{Q}_{k_{\nu}k_{\psi}}(N, \psi) \right], \]

such that \( \hat{Q}_{k_{\nu}k_{\psi}}(N, \psi) = e^{-ik_{\nu}\psi} Q_{k_{\nu}k_{\psi}}(N) \) and thus \( A^\mu(N, \vec{r}, \psi) = e^{-3N/2} (\psi_0/\psi)^2 A^\mu(N, \vec{r}) \). Similarly, for \( \varphi \), we have

\[ \varphi^* + 3 \varphi - e^{-2N} \nabla^2 \varphi - \left[ 4\psi \frac{\partial \varphi}{\partial \psi} + \psi^2 \frac{\partial^2 \varphi}{\partial \psi^2} \right] = 0. \]

According to the 5D flat geometry \( (R_{\mu\nu\lambda} = 0) \) here used in the action \( (2) \), the 5D vacuum is described by the Einstein equations \( G_{\mu\nu} = 8\pi GT_{\mu\nu} = 0 \), being \( T_{\mu\nu} \) the energy-momentum tensor. Hence, the only valid solutions for \( A^\mu \) and \( \varphi \) on the metric \( (11) \) should describe absence of matter: \( A^\nu_{\lambda\nu} = 0 \) (with \( A^0 = 0 \) and \( A^i = 0 \)). Furthermore, using \( (7) \) the commutator between \( \varphi \) and \( \varphi^* \) becomes

\[ \left[ \varphi(N, \vec{r}, \psi), \varphi^*(N, \vec{r}, \psi') \right] = i \left[ \frac{(4)g_0}{(3)g} \right] \delta^{(3)}(\vec{r}-\vec{r}') \delta(\psi-\psi'). \]

which is the same expression obtained in \( (14) \).

### III. EFFECTIVE 4D DYNAMICS.

Considering the coordinate transformations

\[ t = \psi_0 N, \quad R = r \psi_0, \quad \psi = \psi, \]

equation \( (11) \) takes the form

\[ ds^2 = \left( \frac{\psi}{\psi_0} \right)^2 \left[ dt^2 - e^{2t/\psi_0} dR^2 \right] - d\psi^2, \]

which is the Ponce Leon metric that describes a 3D spatially flat, isotropic and homogeneous extension to 5D of a Friedmann Robertson Walker (FRW) line element in a de Sitter expansion. Here \( t \) is the cosmic time and \( dR^2 = dX^2 + dY^2 + dZ^2 \).

#### A. The effective 4D electromagnetic field \( A^\nu \).

Now, we can take the foliation \( \psi = \psi_0 \) in \( (16) \), such that we obtain the effective 4D metric for \( R = \psi_0 r \)

\[ ds^2 \rightarrow ds^2 = dt^2 - e^{2H t} dR^2, \]

which describes a 3D spatially flat, isotropic and homogeneous de Sitter expanding Universe with constant Hubble parameter \( H_0 = 1/\psi_0 \) and a 4D scalar curvature \( (4) = 12 H_0^2 \). The effective 4D action on the effective 4D metric \( (17) \) is \( (\mu, \nu = 0, 1, 2, 3) \)

\[ (4) I = \int d^4x \sqrt{\frac{(4)g}{(3)g_0}} \left[ \frac{(4)\mathcal{R}}{16\pi G} - \frac{1}{4} \left[ Q_{\mu\nu} Q^{\mu\nu} + Q_{\psi\psi} Q_{\psi} \right] \right]_{\psi_0 = H_0^{-1}}, \]

where the additional term \( (1/4)Q_{\psi\psi} Q_{\psi} \mid_{\psi_0 = H_0^{-1}} \) can be identified as an effective 4D potential. This potential has a geometrical origin and can take different representations in different frames. In our case, the observer is in the
frame described by the velocities \( U^\psi = U^r = 0 \) (and hence \( U^R = 0 \)) and \( U^t = 1 \). The effective equation of motion for \( A^\mu(t, \vec{R}, \psi = \psi_0) = A^\mu(t, \vec{R}) \) \cite{10}

\[
\dot{A}^\mu + 3H_0 A^\mu - e^{-2H_0 t} \nabla_R^2 A^\mu - H_0^2 \left[ 4\psi \frac{\partial A^\mu}{\partial \psi} + \psi^2 \frac{\partial^2 A^\mu}{\partial \psi^2} \right] \bigg|_{\psi = H_0^{-1}} = 0, \tag{19}
\]

where, using equations \( \text{(11)} \) and \( \text{(12)} \), we obtain

\[
4\psi \left( \frac{\partial A^\mu}{\partial \psi} + \psi^2 \frac{\partial^2 A^\mu}{\partial \psi^2} \right) \bigg|_{\psi = H_0^{-1}} = -2A^\mu |_{\psi = H_0^{-1}}. \]

Here, \( A^\mu = A^\mu(t, \vec{R}) \) is the effective 4D electromagnetic field induced onto the hypersurface \( \psi = H_0^{-1} \) and the overdot denotes the derivative with respect to time. Note that the last term between brackets in eq. (19) acts as an induced electromagnetic potential derived with respect \( A^\mu \). This term is the analogous to \( V'(\varphi) \) in the case of an inflationary scalar field as used in \( \text{(10)} \), and in our case the dynamics of the component \( \psi \) used in \( \text{(15)} \), and in our case the dynamics of the component \( \psi \).

The equation of motion for the effective 4D electromagnetic modes \( Q_{kR} \), becomes

\[
\ddot{Q}_{kR} + \left[ k^2 e^{-2H_0 t} - \left( \frac{9}{4} H_0^2 + \alpha \right) \right] Q_{kR} = 0, \tag{23}
\]

whose general solution is

\[
Q_{kR}(t) = F_1 \mathcal{H}^{(1)}_\nu [y(t)] + F_2 \mathcal{H}^{(2)}_\nu [y(t)], \tag{24}
\]

where \( y(t) = (kR/H_0)e^{-H_0 t}, \nu = (1/2H_0)\sqrt{9H_0^2 + \alpha} \) and \( H_0 \) remains constant in a de Sitter expansion.

The corresponding normalization condition for the modes \( Q_{kR}(t) \) becomes

\[
Q_{kR} \dot{Q}_{kR}^* - \dot{Q}_{kR} Q_{kR}^* = i. \tag{25}
\]

Therefore, taking into account the Bunch-Davies vacuum condition: \( F_1 = 0 \) and \( F_2 = i\sqrt{\pi/(4H_0)} \), we obtain the solution of (23)

\[
Q_{kR}(t) = i\sqrt{\frac{\pi}{4H_0}} \mathcal{H}^{(2)}_\nu [y(t)], \tag{26}
\]

which describes the normalized effective 4D-modes corresponding to the effective 4D electromagnetic field \( A^\mu \). Note that the solution for the 4D-modes of \( \chi(t, \vec{R}) = e^{3H_0/2}\varphi(t, \vec{R}) \) has the same solution as \( Q_{kR}(t) \) in the equation (26).

## B. Coarse-graining in 4D

In this section we study the induced effective 4D dynamics of the fields \( \chi(t, \vec{R}) \) and \( A^\beta(t, \vec{R}) \) in a stochastic framework. With this aim we introduce the corresponding coarse-graining components of 4D redefined scalar and electromagnetic fields

\[
\chi_L(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int d\psi \Theta(\psi_0 - k_R \psi) \left[ a_{k_R k_\psi} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_R}(t) + cc \right] \delta(k_\psi - \kappa H_0), \tag{27}
\]

\[
A^\beta_L(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int d\psi \sum_{\lambda=0}^3 c^\beta_{\lambda}(\psi) \Theta(\psi_0 - k_R \psi) \left[ b^\lambda_{k_R k_\psi} e^{i\vec{k}_R \cdot \vec{R}} Q_{k_R}(t) + cc \right] \delta(k_\psi - \kappa H_0). \tag{28}
\]
where \( k \) are wavelengths larger than 10^3 times the size of the horizon during inflation. Analogously, the dynamics in the short wavelength sector \((k_R \gg k_0)\) is described by the fields

\[
\chi_S(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \int dk_\psi \Theta(k_R - \partial k_0) \left[ a_{k_R k_0} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_R}(t) + cc \right] \delta(k_\psi - \kappa H_0),
\]

\[
A_{\beta}^S(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \int dk_\psi \sum_{\lambda=0}^{3} \epsilon_{(\lambda)}^\beta \Theta(k_R - \partial k_0) \left[ b_{k_R k_0} e^{i\vec{k}_R \cdot \vec{R}} Q_{k_R}(t) + cc \right] \delta(k_\psi - \kappa H_0),
\]

being \( k_0^3 = e^{2H_0 t} \left( \frac{9}{4} H_0^2 + \alpha \right) \). As in the previous sections the relations \( A^\beta = A_L^\beta + A_S^\beta \) and \( \chi = \chi_L + \chi_S \) are satisfied. Thus, its dynamics is governed by

\[
\dot{\chi}_L - \omega_{k_R}^2(t) \chi_L = \nu \left[ \dot{k}_0 \eta_3(t, \vec{R}) + k_0 \kappa_3(t, \vec{R}) + 2k_0 \gamma_3(t, \vec{R}) \right],
\]

\[
\dot{A}_L^\beta - \omega_{k_R}^2(t) A_L^\beta = \nu \left[ \dot{k}_0 \eta_4^\beta(t, \vec{R}) + k_0 \kappa_4^\beta(t, \vec{R}) + 2k_0 \gamma_4^\beta(t, \vec{R}) \right],
\]

where \( \omega_{k_R}^2 = e^{-2H_0 t} \left( k_R^2 - k_0^2 \right) \) and

\[
\eta_3(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \delta(\partial k_0 - k_R) \left[ a_{k_R} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_R}(t) + cc \right],
\]

\[
\kappa_3(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \delta(\partial k_0 - k_R) \left[ a_{k_R} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_R}(t) + cc \right],
\]

\[
\gamma_3(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \delta(\partial k_0 - k_R) \left[ a_{k_R} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_R}(t) + cc \right],
\]

\[
\eta_4^\beta(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{\lambda=0}^{3} \epsilon_{(\lambda)}^\beta \delta(\partial k_0 - k_R) \left[ b_{k_R}^{(\lambda)} e^{i\vec{k}_R \cdot \vec{R}} Q_{k_R}(t) + cc \right],
\]

\[
\kappa_4^\beta(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{\lambda=0}^{3} \epsilon_{(\lambda)}^\beta \delta(\partial k_0 - k_R) \left[ b_{k_R}^{(\lambda)} e^{i\vec{k}_R \cdot \vec{R}} Q_{k_R}(t) + cc \right],
\]

\[
\gamma_4^\beta(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{\lambda=0}^{3} \epsilon_{(\lambda)}^\beta \delta(\partial k_0 - k_R) \left[ b_{k_R}^{(\lambda)} e^{i\vec{k}_R \cdot \vec{R}} Q_{k_R}(t) + cc \right].
\]

The second-order stochastic system \[ (31) \] and \[ (32) \] can be written as

\[
\dot{\nu}^\beta = \omega_{k_R}^2(t) A_L^\beta + \partial \dot{k}_0 \gamma_4^\beta, \quad A_L^\beta = \nu^\beta + \partial \dot{k}_0 \eta_4^\beta,
\]

\[
\dot{u} = \omega_{k_R}^2(t) \chi_L + \partial \dot{k}_0 \gamma_3, \quad \chi_L = u + \partial \dot{k}_0 \eta_3,
\]

with \( \nu^\beta = \dot{A}^\beta - \partial \dot{k}_0 \eta_4^\beta \) and \( u = \dot{\chi}_L - \partial \dot{k}_0 \eta_3 \). The conditions to neglect the noise quantities \( \gamma_4^\beta \) and \( \gamma_3 \) compared with \( \eta_4^\beta \) and \( \eta_3 \) respectively, now become

\[
\frac{\dot{Q}_{k_R}}{Q_{k_R} Q_{k_R}^*} \ll \left( \frac{\dot{k}_0}{k_0} \right)^2, \quad \frac{\dot{\xi}_{k_R}}{\xi_{k_R} \xi_{k_R}^*} \ll \left( \frac{\dot{k}_0}{k_0} \right)^2,
\]

which are valid on super Hubble scales. The corresponding Fokker-Planck equations that describe the dynamics of transition probabilities \( P_1 \left[ A_L^\beta(0), v^\beta(0) \right| A_L^\beta, v^\beta \right] \) and \( P_2 \left[ \chi_L(0), u(0) \right| \chi_L, u \right] \), are

\[
\frac{\partial P_1}{\partial t} = -v^\beta \frac{\partial P_1}{\partial A_L^\beta} - \mu^2(t) A_L^\beta \frac{\partial P_1}{\partial v^\beta} + \frac{1}{2} \mathcal{D}_{11}(t) \frac{\partial^2 P_1}{\partial (A_L^\beta)^2},
\]

\[
\frac{\partial P_2}{\partial t} = -v^\beta \frac{\partial P_2}{\partial \chi_L} - \mu^2(t) \chi_L \frac{\partial P_2}{\partial u} + \frac{1}{2} \mathcal{D}_{11}(t) \frac{\partial^2 P_2}{\partial \chi_L^2},
\]
where $\mu^2(t) = e^{-2H_0t}k_0^2(t)$ and the diffusion coefficients due to stochastic effect of the noises, $D_{11}(t)$ and $D_{11}(t)$ related to the variables $A_L^\beta$ and $\chi_L$, respectively, are

$$D_{11}(t) = \frac{\partial^3}{\pi^2}k_0^2k_0^2|Q_{0k}|^2,$$

$$D_{11}(t) = \frac{\partial^3}{4\pi^2}k_0^2k_0^2|\xi_{0k}|^2.$$

Hence, the equations of motion for $\langle A_L^\beta \rangle = \int dA_L^{\alpha}dv^\beta(A_L^\alpha A_L^\beta)\mathcal{P}_1[A_L^\alpha, v^\sigma]$ and $\langle \chi_L^2 \rangle = \int d\chi_L d\chi_L^2 \mathcal{P}_2[\chi_L, u]$ are

$$\frac{d}{dt} \langle A_L^\beta \rangle = \frac{1}{2}D_{11}(t) \simeq \frac{\partial^3}{8\pi^3}2\nu^2H_0^{2\nu} \Gamma^2(\nu) \left( \frac{9}{4}H_0^2 + \alpha \right)^{\frac{2}{2-\nu}},$$

$$\frac{d}{dt} \langle \chi_L^2 \rangle = \frac{1}{2}D_{11}(t) \simeq \frac{\partial^3}{32\pi^3}2\nu^2H_0^{2\nu} \Gamma^2(\nu) \left( \frac{9}{4}H_0^2 + \alpha \right)^{\frac{2}{2-\nu}}.$$ 

Rewriting expressions (46) and (47) in terms of the original fields $A_L^\beta(t, \tilde{R}) = e^{-\frac{1}{2}H_0t}A_L^\beta(t, \tilde{R})$ and $\varphi_L = e^{-\frac{1}{2}H_0t}\chi_L$, we obtain

$$\frac{d}{dt} < A_L^\beta > = -3H_0 < A_L^\beta > + \frac{\partial^3}{8\pi^3}2\nu^2H_0^{2\nu} \Gamma^2(\nu) \left( \frac{9}{4}H_0^2 + \alpha \right)^{\frac{2}{2-\nu}},$$

$$\frac{d}{dt} < \varphi_L^2 > = -3H_0 < \varphi_L^2 > + \frac{\partial^3}{32\pi^3}2\nu^2H_0^{2\nu} \Gamma^2(\nu) \left( \frac{9}{4}H_0^2 + \alpha \right)^{\frac{2}{2-\nu}}.$$ 

Equation (48) gives information about the dynamics of the electromagnetic field $A^\beta$ on large-scale. However, although $< A_L^\beta >$ is not an observable, allows to explain the appearance of large-scale magnetic fields. In view of this fact it is convenient to calculate the amplitude of the seed magnetic field induced from $< A_L^\beta >$. On the other hand, equation (49) describes the dynamic of $< \varphi_L^2 >$, which has been studied in more detail in [14]. The integration of eqs. (48) and (49) give us

$$\langle \varphi_L^2 \rangle = \langle A_L^\beta \rangle = \frac{\partial^3}{2\pi^2}2\nu^2H_0^{2\nu-1} \Gamma^2(\nu) \left( \frac{9}{4}H_0^2 + \alpha \right)^{\frac{2}{2-\nu}} \frac{C}{6\pi^2} + e^{-3H_0t},$$

where $C$ is an integration constant. The power spectrum $\mathcal{P}(k_R)$ for $\langle \varphi_L^2 \rangle = \frac{1}{4} \langle A_L^\beta \rangle = \frac{e^{-3H_0t}}{2\pi^2} \int_0^{\infty} dk_R \mathcal{P}(k_R) \xi_{k_R}(t) \xi_{k_R}^*(t) \sim \int_0^{\infty} \frac{d\xi_{k_R}}{k_R} \mathcal{P}(k_R)$, is

$$\mathcal{P}(k_R) \sim k_R^{3-2\nu},$$

where $k_R$ is the wavenumber related to $R$. This spectrum is nearly scale invariant for $\nu \approx 3/2$ (i.e., for $|\alpha|/H_0^2 \ll 1$).

Note that $\mathcal{P}(k_R)$ is the power spectrum for both, $\langle \varphi_L^2 \rangle$ and $\langle A_L^\beta \rangle$. This is an interesting result, because the spectrum of $\langle \varphi_L^2 \rangle$ is determinant for the structure formation on cosmological scales after inflation.

**IV. LARGE-SCALE SEED MAGNETIC FIELDS**

Once we know the effective 3D spatial components of the electromagnetic potential and their evolution on cosmological scales, we can calculate the components of the magnetic field. In this section we develop a stochastic treatment for this field on cosmological scales.

**A. Induced Seed Magnetic Fields.**

By means of the use of the equations (19) and (21) we obtain that the dynamics of the electromagnetic potential $A^i$ satisfies

$$\ddot{A}^i + 3H_0 \dot{A}^i - e^{-2H_0t} \nabla_R A^i - \alpha A^i = 0.$$ 

(52)
Considering the physical components of $\tilde{A}$ and $\tilde{B}$ measured in a comoving frame. Using $\tilde{\nabla}_R \cdot \tilde{B} = 0$ and $\tilde{B} = \tilde{\nabla}_R \times \tilde{A}$, in spherical coordinates, equation (52) becomes

$$\tilde{B}^i_{com} + H_0 \tilde{B}^i_{com} - e^{-2H_0 t} \tilde{\nabla}^2_R B^i_{com} - (\alpha + 2H_0^2) B^i_{com} = 0.$$  

(53)

This expression describes the dynamics of the comoving components of the seed magnetic field. Transforming $B^i$ according to $B^i_{com}(t, \vec{R}) = e^{-\frac{3}{4}H_0 t} \tilde{B}^i_{com}$, from equation (53) we have

$$\tilde{B}^i_{com} - e^{-\frac{3}{4}H_0 t} \tilde{\nabla}_R^2 B^i_{com} - \left( \alpha + \frac{9}{4}H_0^2 \right) B^i_{com} = 0.$$  

(54)

As in the case of $A^\mu$, we can express these components as a Fourier expansion

$$B^i_{com}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + b^{(l)}_{k_R} \right],$$

(55)

where $b^{(l)}_{k_R}$ and $b^{(l)}_{k_R}$ are the creation and annihilation operators and $\epsilon^i_{(l)}(k_R)$ are the 3-polarisation vectors which satisfy $\epsilon^{(i)} \cdot \epsilon^{(j)} = g^{ij}$. Therefore, the equation of motion for $G_{k_R}(t)$ obtained from (53), acquires the form

$$\ddot{G}_{k_R} + \left[ k^2_R e^{-2H_0 t} - \left( \frac{9}{4}H_0^2 + \alpha \right) \right] G_{k_R} = 0.$$  

(56)

The normalized solution of (56) is

$$G_{k_R}(t) = i \sqrt{\frac{2H_0 \pi}{\lambda}} e^{-\frac{\omega^2}{2}} w(t),$$

(57)

where $\lambda = \frac{1}{2H_0 \sqrt{9H_0^2 + 4\alpha}}$ and $\omega(t) = \frac{k_R e^{-H_0 t}}{H_0^2}.$

### B. Coarse-graining treatment for a seed magnetic field

Now we are able to obtain the induced seed magnetic field on large-scale related with the electromagnetic potential $A^\mu_L$ in an analogous manner as we proceeded in the previous sections. Therefore, we introduce the 3D coarse-graining field associated with the redefined components of the magnetic field $B^i_{com}(t, \vec{R})$ as

$$B^i_{L}\big|_{com}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \Theta(\vartheta k_0 - k_R) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + \right].$$

(58)

where the modes with $k_R/k_0 \ll \vartheta$ are referred as the outside of the horizon. The short wave length modes are described by the field

$$B^i_{S}\big|_{com}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \Theta(k_R - \vartheta k_0) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + \right],$$

(59)

such that the relation $B^i_{com} = B^i_{L}\big|_{com} + B^i_{S}\big|_{com}$ is satisfied. The stochastic equation of motion for $B^i_{L}\big|_{com}$ is given by

$$\tilde{B}^i_{L}\big|_{com}(t, \vec{R}) - \omega^2_{k_R}(t) \tilde{B}^i_{L}\big|_{com}(t, \vec{R}) = \delta \left[ \hat{k}_0 \eta^i_{k_R}(t, \vec{R}) + \hat{k}_0 \kappa^i_{k_R}(t, \vec{R}) + 2\hat{k}_0 \gamma^i_{k_R}(t, \vec{R}) \right],$$

(60)

being the stochastic operators

$$\eta^i_{k_R}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \delta(\vartheta k_0 - k_R) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + \right],$$

(61)

$$\kappa^i_{k_R}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \delta(\vartheta k_0 - k_R) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + \right],$$

(62)

$$\gamma^i_{k_R}(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \sum_{l=1}^{3} \epsilon^i_{(l)}(k_R) \delta(\vartheta k_0 - k_R) \left[ b^{(l)}_{k_R} e^{i\vec{k}_R \cdot \vec{R}} G_{k_R}(t) + \right].$$

(63)
The equation (60) can be expressed by the system
\[ B^i_L|_{\text{com}} = W^i + \partial \bar{k}_0 \eta^i, \quad \dot{W}^i = \omega^2_{kR}(t) B^i_L|_{\text{com}} + \bar{k}_0 \gamma^i, \] (64)
where \( W^i \) is an auxiliary field defined by \( W^i = B^i_L|_{\text{com}} - \partial \bar{k}_0 \eta^i \). In this system the effect of the noise \( \gamma^i \) can be minimized if \((\bar{k}_0)^2 \left< \left( \gamma^i \right)^2 \right> \ll (\bar{k}_0)^2 \left< \left( \eta^i \right)^2 \right>\), which is valid if the condition
\[ \frac{\dot{G}_{kR} G_{kR}}{G_{kR}^2} \ll \left( \frac{\bar{k}_0}{\bar{k}_0} \right)^2 = H_0^2, \] (65)
is satisfied. For a de Sitter expansion this condition means that the noise \( \gamma^i \) can be neglected on scales \( k R \ll e^{H_0 t} \sqrt{(9/4) H_0^2 + \alpha} \), i.e. on super Hubble scales. In such a case the system (64) can be approximated by
\[ \dot{W}^i = \mu^2(t) B^i_L|_{\text{com}}, \] (66)
\[ B^i_L|_{\text{com}} = W^i + \partial \bar{k}_0 \eta^i. \] (67)
These are two Langevin equations where the noise \( \eta^i \) satisfies
\[ \left< \eta^i \right> = 0, \quad \left< \left( \eta^i \right)^2 \right> = \frac{3}{2\pi^2} \frac{\partial \bar{k}_0^2}{\bar{k}_0} G_{\partial k_0} G_{\partial k_0} \delta(t - t'), \] (68)
which means that the noise \( \eta^i \) is Gaussian and white in nature. As in the case of \( \mathcal{A}_L^0 \), the dynamics of the transition probability \( P_3 \left[ (B^i_L|_{\text{com}})^{(0)}, (W^k)^{(0)} \right| \bar{B}^i_L|_{\text{com}}, W^k \] is given through the Fokker-Planck equation
\[ \frac{\partial P_3}{\partial t} = -W^k \frac{\partial P_3}{\partial B^k_L|_{\text{com}}} - \mu^2(t) B^i_L|_{\text{com}} \frac{\partial P_3}{\partial W^i} + \frac{1}{2} \bar{D}_{11}(t) \frac{\partial^2 P_3}{\partial \left( B^i_L|_{\text{com}} \right)^2}, \] (69)
with \( \bar{D}_{11}(t) = \frac{3\mu^2}{2\pi^2} \bar{k}_0 \bar{k}_0^3 \left| G_{\partial k_0} \right|^2 \) being the diffusion coefficient related to \( B^i_L|_{\text{com}} \). Thus the equation of motion for
\[ \left< (B^i_L|_{\text{com}})^2 \right> = \int d^3 B^i_L|_{\text{com}} dW^j (B^j_L|_{\text{com}}) P_3 \left[ (B^i_L|_{\text{com}})^{(0)}, (W^k)^{(0)} \right| \bar{B}^i_L|_{\text{com}}, W^k \] is
\[ \frac{d}{dt} \left< (B^i_L|_{\text{com}})^2 \right> = \frac{1}{2} \bar{D}_{11}(t) \simeq \frac{3 \mu^3}{16\pi^3} 2^{2\nu} \Gamma^2(\nu) H_0^{2\nu+2} e^{3H_0 t} \left( \frac{9}{4} H_0^2 + \alpha \right)^{3/2-\nu}. \] (70)
In terms of the original field \( B^i_L|_{\text{com}}(t, \bar{R}) = e^{-\frac{t}{2} H_0 t} B^i_L|_{\text{com}}(t, \bar{R}) \), equation (70) becomes
\[ \frac{d}{dt} \left< (B^i_L|_{\text{com}})^2 \right> = -H_0 \left< (B^i_L|_{\text{com}})^2 \right> + \frac{3 \mu^3}{16\pi^3} 2^{2\nu} \Gamma^2(\nu) H_0^{2\nu+2} e^{2H_0 t} \left( \frac{9}{4} H_0^2 + \alpha \right)^{3/2-\nu}. \] (71)
The general solution of this equation is
\[ \left< (B^i_L|_{\text{com}})^2 \right> = \frac{\mu^3}{16\pi^3} 2^{2\nu} \Gamma^2(\nu) H_0^{2\nu+2} \left( \frac{9}{4} H_0^2 + \alpha \right)^{\frac{\nu}{2}} e^{2H_0 t} + C e^{-H_0 t}, \] (72)
where \( C \) is an integration constant. For \( \nu = \frac{3}{2} \), this expression is reduced to
\[ \left< (B^i_L|_{\text{com}})^2 \right> = \frac{\Gamma^2(\frac{3}{2})}{2\pi^3} H_0^4 e^{2H_0 t} + C e^{-H_0 t}. \] (73)
On the other hand the physical magnetic field \( B_{\text{phys}} \) and the comoving one are related by
\[ B_{\text{phys}} \sim a^{-2} B_{\text{com}}. \] (74)
After inflation, $B_{\text{phys}}$ decreases as $a^{-2}$. Therefore, we could make an estimation of the actual strength of the cosmological magnetic field $B_{\text{phys}}^{(a)}$. Hence, we can use the expression

$$\left\langle \left( \frac{B_{L}^{(a)}_{\text{phys}}}{B_{L}^{(a)}} \right)^{2} \right\rangle_{IR} \approx \left( \frac{a(t)}{a(t_{0})} \right)^{-4} \left\langle B_{L}^{2}_{\text{phys}}(t) \right\rangle_{IR}^{1/2},$$

being $t_{i}$ and $t_{0}$ the time at the end of inflation and the actual, respectively. We estimate the factor

$$\left( \frac{a(t_{0})}{a(t_{i})} \right)^{-4} \simeq 10^{-136},$$

where we have used $H_{0} = 0.5 \times 10^{-9} M_{\odot}$ that takes into account $N_{e} = 63$ at the end of inflation for $t_{i} = 1.26 \times 10^{11} G^{1/2}$. Note that the eq. (76) accounts for the actual (at $t = t_{0}$) size of the observable horizon ($\sim 10^{28}$ cm.) and the size of the horizon ($\sim 3.6 \times 10^{-9}$ cm.) at the end of inflation (at $t = t_{i}$). In order to obtain an estimation of the actual strength of the magnetic field, we substitute equation (73) into (75) such that the strength of the magnetic field is given by

$$\left\langle \left( B_{L}^{(a)_{\text{phys}}} \right)^{2} \right\rangle_{IR}^{1/2} = \left( 4.9448 \times 10^{-16} \right) \left[ \frac{1^{2}(3/2)}{2\pi^{3}} H_{0}^{4} e^{2H_{0}t} + C e^{-H_{0}t} \right]^{1/2} \text{ Gauss.}$$

For an integration constant $C \leq 10^{45}$, we obtain

$$\left\langle \left( B_{L}^{(a)_{\text{phys}}} \right)^{2} \right\rangle_{IR}^{1/2} \leq 10^{-9} \text{ Gauss, that agrees with some other calculations of cosmological magnetic fields strengths made in [10].}$$

Values considered for $\theta$ correspond to actual scales from $3 \times 10^{5}$ to $3 \times 10^{6}$ Mpc and a nearly scale invariant power spectrum.

V. FINAL REMARKS

In this work we have studied the emergence of a classical behavior from the quantum dynamics of the components of the potential vector $A_{B} = (A_{\mu}, \varphi)$, in gravitoelectromagnetic inflation, by considering a coarse-grained average. Our approach leads to a suitable formalism for studying the temporal evolution of $A_{B}$ beyond the slow-roll approximation by assuming spatial homogeneity. Thus, we neglect the gradient term in the equation of motion when we consider a coarse-grained representation on cosmological scales. This assumption allows us to develop a rather simple description of its temporal evolution in terms of two-dimensional Fokker-Planck equations (one related to $\chi_{L}$, which is the redefined field of $\varphi_{L}$, and the other related to $A^{\mu}$, which is the redefined field of $A^{\mu}$), but at the same time, it restricts the cases where the formalism is applicable. The approach is based on a consistent semiclassical expansion for $A_{B}$. In this framework the inflation in a 4D de Sitter expansion is driven by the vacuum mean value of the components $A_{B}$ (whose only non-null part is $\langle A_{5} \rangle = \varphi_{0}$), whereas the long-wavelength modes of the quantum fluctuations reduce to a quantum system subject to quantum noises originated by the short-wavelength sector. In other words, $\langle A^{\mu} \rangle$ are considered as null in the universe on an effective de Sitter expansion, being $A^{\mu}(t, \vec{R})$ their space-time fluctuations. On the other hand, $\langle A_{4} \rangle = \varphi_{0}$ (which is a constant of $t$ in a 4D de Sitter expansion), is the solution of $\varphi(t, \vec{R})$ on the background (spatially isotropic and homogeneous) 4D metric (17). The range of applicability of this assumption must be carefully considered because the regime of temporal evolution and the development of spatial inhomogeneities are related. For an effective 4D de Sitter expansion here studied, the scales of viability of our approach is $k_{R} \ll e^{H_{0}t_{i}} \sqrt{(9/4)H^{2}_{0} + a}$, which describes super Hubble wavelengths during the inflationary expansion. Hence, the system can be considered as classical on cosmological scales due to the contribution of the noises $\gamma_{5}^{\nu}$ can be neglected with respect to $\eta_{5}^{\nu}$.

Finally, we have made an estimation of

$$\left\langle \left( B_{L}^{(a)_{\text{phys}}} \right)^{2} \right\rangle_{IR}^{1/2}$$

for $\nu \simeq 3/2$ and we obtained values of the order of $\leq 10^{-9}$ Gauss for scales no much smaller than the actual horizon. Our results agree with the WMAP CMB data constrains for cosmological magnetic fields [10]. It is remarkable that the results here obtained also agree with other recently obtained by a different method [10]. However, the advantage of our stochastic method is that the problem of the divergence for a scale invariant power spectrum (with $\nu = 3/2$) in

$$\left\langle \left( B_{L}^{(a)_{\text{phys}}} \right)^{2} \right\rangle_{IR}^{1/2}$$

now is avoided.
Acknowledgements
JEMA acknowledges CNPq-CLAF and UFPB and MB acknowledges CONICET and UNMdP (Argentina) for financial support.

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