Fluctuations of motifs and non–self-averaging in complex networks: A self- vs. non–self-averaging phase transition scenario

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Abstract – Complex networks have been mostly characterized from the point of view of the degree distribution of their nodes and a few other motifs (or modules), with a special attention to triangles and cliques. The most exotic phenomena have been observed when the exponent $\gamma$ of the associated power-law degree distribution is sufficiently small. In particular, a zero percolation threshold takes place for $\gamma < 3$, and an anomalous critical behavior sets in for $\gamma < 5$. In this letter we prove that in sparse scale-free networks characterized by a cut-off scaling with the system size $N$, relative fluctuations are actually never negligible: given a motif $\Gamma$, we analyze the relative fluctuations $R_\Gamma$ of the associated density of $\Gamma$, and we show that there exists an interval in $\gamma$, $[\gamma_1, \gamma_2]$, where $R_\Gamma$ does not go to zero in the thermodynamic limit, where $\gamma_1 \approx k_{\min}$ and $\gamma_2 \approx 2k_{\max}$, $k_{\min}$ and $k_{\max}$ being the smallest and the largest degree of $\Gamma$, respectively. Remarkably, in $(\gamma_1, \gamma_2)$ $R_\Gamma$ diverges, implying the instability of $\Gamma$ to small perturbations.

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Introduction. – In the last decade, several complex networks models have been proposed to explain and reproduce the widespread presence of real-world networks [1–4]. Observed real-world networks are the output of certain random processes. Therefore, a complex network model should reproduce not only the same observed averages but also the same observed sample-to-sample fluctuations, if available. If data about fluctuations are not available, the model should remain maximally random around the observed averages [5] or make use of a minimal number of assumptions (a null-model approach).

In all branches of physics, fluctuations have played a crucial role in the understanding of the underlying phenomena. It is not excessive to say that any observable in physics does not have any objective meaning without the evaluation of its fluctuations. Observables in networks are not an exception. Particularly important are the motifs, also known as modules, patterns, or communities\(^1\) (fig. 1), \(i.e.\), subgraphs on which the functionality of the network largely depends [6–9]. In this letter we provide a first systematic analysis of the fluctuations of the density of motifs in an analytically treatable model of networks, the hidden-variable model [10–14], and we show that these fluctuations, in certain regions of the model parameters, are far from being negligible. In fact, we discover the existence of a phase transition scenario with regions where relative fluctuations are negligible, separated from regions where the relative fluctuations diverge. The practical consequences of this analysis are dramatic. A large class of real networks can be mapped toward suitable

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\(^{2}\)Here the term “community” is meant in a broad sense. In general a community is a motif, but not vice versa [6].

Fig. 1: Examples of motifs. The labels in $\Gamma$ may specify the number of links of the motif; for motifs made of $k$ fully connected nodes, $k$-cliques, we also use the symbol $\Gamma_{k,n}$; $d$ in $\Gamma_{k,d}$ stands for the presence of a diagonal; $\Gamma_{k\times2}$ stands for two $k$-cliques sharing a common node.
hidden-variable models. For each real network realization such a mapping requires measuring the hidden variables \( \{ h_i \} \) associated to each node. The sequence \( \{ h_i \} \), however, in the process that produces the real network, can vary, and different real network realizations in general are characterized by different sequences \( \{ h_i \} \). It is this variability of the sequence \( \{ h_i \} \) that generates extremely large fluctuations. In particular, two large network realizations will present two totally different community structures. More in general, large fluctuations translate in a effective instability of the substructures composing the network to small perturbations. For the same reason, in simulating complex networks, if we are in the high fluctuation region, in order to have a fair evaluation of the average of the density of a motif we need to make use of a very demanding statistics.

Complex networks studies have been mostly focused on the analysis of local averages and variances, but without a systematic study of fluctuations. Analysis of correlations in complex networks have been in fact mainly confined to the degree of adjacent nodes [3,4,11], or to the presence of loops and cliques [15–17] which are a manifestation of correlations. Particular attention has been paid to the “configuration model” [1,3], i.e. the “uncorrelated” network model" originating from all possible graphs constrained to satisfy a given degree distribution exactly (hard version) [1,3,18,19], or in average (soft version) [10–12,20]. It is well known that, in the thermodynamic limit, in the considered case the presence of loops of finite length is negligible, provided the exponent \( \gamma \) of the associated power-law degree distribution \( P(k) \sim k^{-\gamma} \) is sufficiently large [3,15]. In turn, this has led perhaps to the erroneous conclusion that, in any synthetic or real network, fluctuations are always negligible for sufficiently large \( \gamma \). In fact, exotic phenomena are believed to occur only when \( \gamma < 5 \), where an anomalous critical behavior sets in, especially when \( \gamma < 3 \), where a zero percolation threshold takes place due to a divergent second moment of \( P(k) \) [21]. In particular, fluctuations of motifs are assumed to be negligible for \( \gamma \) enough large. In this Letter, by using the framework of hidden-variable models, we prove that in sparse networks (\( \gamma > 2 \) ) characterized by a cut-off scaling with the system size \( N \), fluctuations are actually never negligible: given a motif \( \Gamma \), we analyze the relative fluctuations \( R_F \) of the density of \( \Gamma \), and we show that there exists an interval \(( \gamma_1, \gamma_2) \) where \( R_F \) diverges in the thermodynamic limit, where \( \gamma_1 \approx k_{\min} \) and \( \gamma_2 \approx 2k_{\max} \), \( k_{\min} \) and \( k_{\max} \) being the smallest and the largest degree of \( \Gamma \). As a consequence, in \(( \gamma_1, \gamma_2) \), measuring the density of \( \Gamma \) in simulations is a hard problem, and \( \Gamma \) is unstable to small perturbations, a fact that in turn provides a key to understand the stability/instability of communities [6].

The hidden-variable scheme. The choice of the cut-off. — Given \( N \) nodes, hidden-variable models are defined in the following way: i) to each node we associate a hidden variable \( h \) drawn from a given probability density function (PDF) \( \rho(h) \); ii) between any pair of nodes, we assign, or not assign, a link, according to a given probability \( p(h,h') \), where \( h \) and \( h' \) are the hidden variables associated to the two nodes. The probability \( p(h,h') \) can be any function of the \( h \)'s, the only requirement being that \( 0 \leq p(h,h') \leq 1 \). It has been shown that, when \( p(h,h') \) has the following form (or similar generalizations):

\[
p(h,h') = \left( 1 + \frac{k_s^2}{hh'} \right)^{-1}, \quad k_s = \sqrt{Nk},
\]

and \( k \) is the wanted average degree, for large \( N \), the actual degree \( k \) of the nodes of the network realized with the above scheme are distributed according to \( p \) with actual average degree equal to \( k \). In particular, if we choose the following PDF having support in \([h_{\min},h_{\max}]\):

\[
\rho(h) = a \ h^{-\gamma}, \quad h_{\max} \geq h \geq h_{\min} > 0,
\]

with \( \gamma > 2 \), the degree distribution of the resulting network will be a power law with exponent \( \gamma \) and, for \( N \) sufficiently large, the normalization constant \( a \) and the so called structural cut-off \( k_s a = \frac{(\gamma - 1)}{h_{\min}^{\gamma - 1}} \), and \( k_s = \sqrt{N h_{\min} (\gamma - 1)/(\gamma - 2)} \). When \( h_{\max} \ll k_s \), correlations of the generated network are negligible, and \( p(h,h') \approx hh'/k_s^2 \), while for \( h_{\max} \gg k_s \) correlations might be important. The choice of the cut-off \( h_{\max} \) is in principle arbitrary. However, most of real-world networks show that the maximal degree scales according to the so called natural cut-off: \( k_{\max} \sim h_{\nc} = N^{1/(\gamma - 1)} \). As a consequence, in several models of complex networks it was assumed the choice \( h_{\max} = h_{\nc} \) justified as empirical. We think however that such an approach is wrong: the fact that in most of the real-world networks \( k_{\max} \sim h_{\nc} \) is due to a probabilistic effect, is not due to a rigid upper bound \( k \leq h_{\nc} \). In fact, by using order statistics one finds that by drawing \( N \) degree values from a power law with exponent \( \gamma \), the highest degree in average scales just as \( k_{\max} \sim N^{1/(\gamma - 1)} \) [22,23]. More precisely, it is possible to prove that the PDF for the rescaled random variable \( k_{\max}/N^{1/(\gamma - 1)} \) is also a power law with exponent \( \gamma \) [24]. Power-law distributions always lead to important fluctuations. It is then clear that empirical observations of \( k_{\max} \) must be taken with care: \( k_{\max} \) is not a self-averaging variable and samples in which \( k_{\max} \gg N^{1/(\gamma - 1)} \), even if extremely rare, do exist and, as we shall see, have dramatic effects on the fluctuations of motifs. We stress that here we follow a null-model approach. We do not claim that the degree of all real networks must have a cut-off scaling with \( N \); there might be of course many other possible scalings whose value depend on the details of the system (related to physical, biological, or economical constraints). However, if the only information that we have from a given real network of size \( N \) is that \( \gamma = 1 \) the degree obeys a power-law

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distribution with exponent $\gamma$, and ii) highest degrees scale in average as $N^{1/(\gamma-1)}$, forcing the model to have a specific cut-off other than $N$ would introduce a bias. In fact, order statistics tells us that a lower cut-off would produce $(h_{\text{max}}) \ll N^{1/(\gamma-1)}$. This observation leads us to choose $h_{\text{max}} \sim N$ for the hidden-variable scheme (1), (2). More precisely, if we consider as target degree distribution a power law $P(k) \propto k^{-\gamma}$ with finite support $k \leq N$, in order to reproduce its characteristics from the hidden-variable model, we need to use a cut-off $h_{\text{max}} = O(N^{1/\gamma})$ with $\lambda \geq 1$. In fact, any other choice implies a difference in the scaling of the moments $(k^n)$ between the hidden-variable model (1), (2) and the target distribution $P(k)$. Figure 2 shows this for the second moment. Similar plots hold for higher moments. In conclusion, the minimal cut-off of the model (1), (2) able to reproduce the correct scaling of all the moments of the target degree distribution $P(k)$ is just $h_{\text{max}} = O(N^{1/\gamma})$ with $\lambda = 1$. In this paper we set therefore $h_{\text{max}} = N$. We stress that with this choice highest degrees will be still order $N^{1/(\gamma-1)}$, but only on average.

**Fluctuations of motifs.** – Given the parameters $N$, $h_{\text{min}}$, $k$, and $\gamma$, the above hidden-variables scheme produces an ensemble of networks which, in terms of a few characteristics, like statistics of the degree and motifs, are in part representative of many real-world networks with those given parameters. In the following we will indicate the ensemble averages with the bracket symbol $\langle \cdot \rangle$. The averages are built by following the above steps i) and ii) of the hidden-variables scheme. Notice that each time we generate a network realization, we need to draw $N$ hidden variables from the PDF $\rho(h)$, and $N(N-1)/2$ numbers to sample $\rho(h, h')$. In terms of the adjacency matrix $a_{i,j}$, taking value 0 or 1 for the presence or not of a link between nodes $i$ and $j$, steps i) and ii) give

$$\langle a_{i,j} \rangle = \int dh_i dh_j \rho(h_i)\rho(h_j)p(h_i, h_j)$$

(3)

We will indicate by $n_r$ the density of the motif $\Gamma$ in a network realization. As is known [11,25], for $\gamma > 2$, the hidden-variable model defined through eqs. (1), (2) leads to a small clustering coefficient $C = \langle n_{\text{tr}} \rangle / \langle 3n_{\text{r}} \rangle$. For example, for $\gamma \gg 3$ we have $\langle n_{\text{tr}} \rangle = O(1)$, while $\langle n_{\text{r}} \rangle = O(1/N)$. More in general, the more the motif is clustered, the smaller is its density. Yet, for finite $N$, and for any motif $\Gamma$, clustered or not, by tuning the parameters $h_{\text{min}}$, $k$, and $\gamma$, one can set, within some freedom, a desired value of $\langle n_r \rangle$. However, as we shall see, the sample-to-sample fluctuations of $n_r$ can be unexpectedly large. Fluctuations of $n_r$ must be compared with the corresponding average of $n_r$, therefore we are going to analyze the following standard ratio

$$R_r = \frac{\langle n_{r}^2 \rangle - \langle n_{r} \rangle^2}{\langle n_{r} \rangle^2}.$$  

(4)

In general $R_r$ will strongly depends on $\Gamma$, $N$ and $\gamma$. When $\lim_{N \to \infty} R_r = 0$ the network is said to be self-averaging with respect to the motif density $n_r$. In practical terms, when this occurs, even one single sample is enough to get by simulations an accurate estimation of the average $\langle n_r \rangle$, provided $N$ is large enough. The behavior of $R_r$ with respect to the network size $N$ is therefore of crucial importance: if the network is not self-averaging with respect to some motif $\Gamma$, the number of samples necessary to get a good estimation of $\langle n_r \rangle$ in simulations will have to grow with $N$ or, from another perspective, it is hard to generate only those samples whose density is close to a target value, and a kind of hard searching problem emerges. This aspect is in fact connected with spin-glass and NP-complete problems; we will see in fact that $R_r$ can be read as a susceptibility of a homogeneous system.

**Analysis of $R_r$.** – Given a motif $\Gamma$, the density of $\Gamma$ in a graph realization is

$$n_r = \frac{c}{N} \sum_i k_r(i),$$

(5)

where $k_r(i)$ counts the number of motifs $\Gamma$ passing through the node $i$. The coefficient $c$ depends on the definition of the motif considered and serves to avoid over-counting when the motif is symmetric. For example, if the motif $\Gamma$ is the triangle, we set $c = 1/3$. If the motif is not symmetric, we can establish to count only those motifs that pass through a specific node of $\Gamma$. For example, if $\Gamma$ is a triple (two consecutive links), we can set $c = 1$, but a motif contributes only when the center of the triple
coincides with $i$. However, since we are interested only in the relative fluctuations $R_{i}$, we do not need to specify it since $c$, as well as any constant, does not play any role for $R_{i}$. Let us consider now the numerator of eq. (4). Note that the hidden-variable scheme does not distinguish nodes, therefore we can make use of the fact that nodes are all statistically equivalent. By using this property, from eq. (5) we get the following susceptibility:

$$\langle n_{i}^{2} \rangle - \langle n_{i} \rangle^{2} = \frac{c^{2}}{N} \left\{ \langle k_{l}^{2} (i) \rangle - \langle k_{l} (i) \rangle^{2} \right\} + c^{2} \left\{ \langle k_{l} (i) k_{l} (j) \rangle_{i \neq j} - \langle k_{l} (i) \rangle^{2} \right\}, \quad (6)$$

where $i$ and $j$ represent two arbitrary distinct indices. In the rhs of eq. (6) we have a self-term proportional to the motif variance, rescaled by the factor $1/N$, and a mixed term that accounts for correlations between two motifs centered at two different nodes. Note that, in general, the self-term, despite appears to be order $1/N$, cannot be neglected. In fact, due to exact cancellations in the mixed term, the mixed and self-term give contributions of the same order of magnitude.

For what follows, we find it convenient to introduce another symbol for the averages with respect to the PDF $\rho(h)$ given by eq. (2): if $f(\cdot)$ is any function of the hidden variables $h_{1}, \ldots, h_{N}$ we define

$$[f] = \int \prod_{i=1}^{N} dh_{i} \rho(h_{i}) f(\cdot). \quad (7)$$

In particular, from eq. (3) we have $\langle a_{i,j} \rangle = [p(h_{i}, h_{j})]$. For $\gamma > 2$, $k_{a} \sim N^{1/2}$, therefore eqs. (1) and (2) imply

$$\langle a_{i,j} \rangle = [p(h_{i}, h_{j})] = O \left( N^{-1} \right). \quad (8)$$

Next, we analyze eq. (6) in a few crucial motifs.

**Link (Γ1).** If Γ is the link $k_{l} (i)$ coincides with the standard definition of degree of the node $i$. For a given graph realization, corresponding to a given realization of the $h$’s, in terms of adjacency matrix we have

$$k_{r_{1}} (i) = \sum_{l \neq i} a_{i,l}. \quad (9)$$

By using eq. (3), eqs. (7)–(9), and the statistical equivalence of nodes, we have

$$\langle k_{r_{1}} (i) \rangle = \sum_{l \neq i} \langle a_{i,l} \rangle = (N - 1) [p(h_{1}, h_{2})]. \quad (10)$$

Let us now consider the product $k_{r_{1}} (i) k_{r_{1}} (j)$. Notice that $a_{i,j}^{2} = a_{i,j}$. We have to distinguish the cases $i = j$ and $i \neq j$, see fig. 3. For $i = j$ we have

$$\langle k_{r_{1}}^{2} (i) \rangle = (N - 1)(N - 2) [p(h_{1}, h_{2})p(h_{2}, h_{3})] + (N - 1) [p(h_{1}, h_{2})], \quad (11)$$

while for $i \neq j$ we have

$$\langle k_{r_{1}} (i) k_{r_{1}} (j) \rangle = (N - 2)(N - 3) [p(h_{1}, h_{2})]^{2} + 3(N - 1) [p(h_{1}, h_{2})p(h_{2}, h_{3})], \quad (12)$$

where the factor 3 comes from the fact that two links emanating from nodes $i$ and $j$ can share a same node in 3 topologically equivalent ways. On plugging eqs. (10)–(12) into eq. (4) via eq. (6) and keeping only terms in $N^{2}$, which cancel exactly, and terms in $N$, we obtain

$$R_{r_{1}} = \frac{4}{N} \frac{[p(h_{1}, h_{2})p(h_{2}, h_{3})]}{[p(h_{1}, h_{2})]^{2}} - \frac{4}{N^{2}} \frac{1}{N^{2}} \frac{[p(h_{1}, h_{2})]^{2}}{[p(h_{1}, h_{2})]}, \quad (13)$$

Due to eq. (8) and its generalizations, for $\gamma > 3$, each term present in the rhs of eq. (13) is of order $1/N$, therefore we have $R_{r_{1}} = O(1/N)$ and the network is self-averaging with respect to the link density, while for $\gamma < 3$ we have still self-averaging but $R_{r_{1}}$ decays slower as $N^{2-\gamma}$. It is interesting, however, to observe the general behavior of $R_{r_{1}}$ with respect to $\gamma$ for finite $N$. As a general rule, $[p(h_{1}, h_{2})p(h_{2}, h_{3}) \ldots p(h_{m}, h_{m+1})]$ for large $\gamma$ tends to factorize: $[p(h_{1}, h_{2})p(h_{2}, h_{3}) \ldots p(h_{m}, h_{m+1})] \rightarrow [p(h_{1}, h_{2})][p(h_{2}, h_{3})] \ldots [p(h_{m}, h_{m+1})]$. Therefore, the first two terms in the rhs of eq. (13) tend to cancel for large $\gamma$. However, the last term does not cancel for large $\gamma$ (this issue will be discussed elsewhere).

**Diagrammatic calculus.** From eq. (13) we see that the main term is given by the ratio between a $[\cdot]$-average of two links sharing a common node, and the square of the $[\cdot]$-average of a single link, i.e., our motif. From this example is clear that a correspondence between formulas and diagrams can be established to avoid unnecessary simulations and to improve our understanding about the main contributions to $R_{r_{i}}$, especially those that can generate non-self-averaging. In this sense, we find it convenient to make use of the compact notation $[\Gamma]$, where $\Gamma$ can be any motif. For example, by referring to fig. 1, we have $[\Gamma_{1}] = [p(h_{1}, h_{2})]$, $[\Gamma_{2}] = [p(h_{1}, h_{2})p(h_{2}, h_{3})]$, $[\Gamma_{3}] = [p(h_{1}, h_{2})p(h_{2}, h_{3})p(h_{3}, h_{4})]$, and so on. The role played by these $[\cdot]$-averages, is similar to the role played
by Green functions in statistical field theory. Moreover, since $R_T$ is defined in terms of connected correlation functions (6), we need to work only with Green functions of connected motifs, as the contributions of disconnected motifs always cancel. Next, by using this diagrammatic tool, we evaluate $R_T$ in the crucial case of $k$-cliques. We first analyze the case $k = 3$ in detail, and then we look at the general behavior $R_{T_{k,c}}$, omitting contributions which are not essential here. Further details will be given elsewhere.

Triangle ($\Gamma_3$). This is the simplest $k$-clique. We have

$$R_{\Gamma_3} = \frac{9}{N} \left[ \frac{\Gamma_{3\times 2}}{\Gamma_3} \right]^2 - \frac{9}{N^2} \left[ \frac{\Gamma_{4d}}{\Gamma_3} \right]^2 + \frac{6}{N^3} \left[ \frac{1}{\Gamma_3} \right],$$

(14)

The factor 9 comes from: 8 ways to build $\Gamma_{3\times 2}$ from the mixed term with $i \neq j$, and an extra contribution from the last term in $i = j$. Similarly to the last diagrammatic tool, the last two terms of eq. (13), the last two terms of eq. (14) do not cancel for large $\gamma$.

$k$-clique ($\Gamma_{k,c}$). Given $\Gamma_{k,c}$ ($(k - 1)$ is the degree of each node), if $\Gamma_{k,c \times 2}$ indicates the motif in which two $k$-cliques $\Gamma_{k,c}$ share a common node, we have

$$R_{\Gamma_{k,c}} = \frac{b_k}{N} \left[ \frac{\Gamma_{k,c \times 2}}{\Gamma_{k,c}} \right]^2 - \frac{b_k}{N} + O \left( \frac{1}{N} \right),$$

(15)

where $b_k$ is a combinatorial term which depends only on $k$, and the last term is positive and plays a role similar to the last two terms of eq. (14).

In fig. 4 we show the behavior of $R_{\Gamma_1}$ and $R_{\Gamma_3}$ vs. $N$ for $\gamma = 4.2$, and show the matching simulations vs. diagrammatic analysis. Notice that the theory (see next paragraph) predicts $R_{\Gamma_3} \to 0$ for $\gamma > 4$, and $R_{\Gamma_3} \to \infty$ for $2.5 < \gamma < 4$, however, $\gamma = 4.2$ is quite close to 4 so that $R_{\Gamma_3}$ decays very slowly with $N$.

Singular terms. The main message of this letter is that, if the maximal degree $k_{\text{max}}$ of the motif $\Gamma$ is greater than 1, there are contributions which make $R_T$ divergent for $N \to \infty$. Let us analyze the Green function $[\Gamma_{k,c \times 2}]$ which appears in eq. (15). From eqs. (1), (2), by enumerating the $2k - 1$ nodes of $\Gamma_{k,c \times 2}$ with $1, \ldots, 2k - 1, 2k - 1$ being the central node, we have

$$[\Gamma_{k,c \times 2}] = a^{2k-1} k_s \frac{4 \gamma \gamma}{\gamma^2} \int \prod_{i=1}^{2k-1} dh_i \prod_{i=1}^{2k-2} h_i^{k-1-\gamma} h_{2k-1-\gamma} \times \prod_{i=1}^{2k-2} \left( 1 + \frac{h_{2k-1} h_i}{k_s^2} \right)^{-1} \prod_{i<j, i,j \neq 2k-1} \left( 1 + h_i h_j h_{2k-1} \right)^{-1}. \frac{1}{}$$

We see that $h_{2k-1}$ plays a “privileged” role with respect to the other variables $h_1, \ldots, h_{2k-2}$. When $h_1, \ldots, h_{2k-2} \ll k_s$, all the factors in the second row of this equation remain finite, independently of the value of $h_{2k-1}$, which must be integrated from $h_{\text{min}}$ to $h_{\text{max}} = N$. Such a region of integration, when $(k - 1) < \gamma < 2(k - 1)$, gives the leading contribution to $[\Gamma_{k,c \times 2}]$, and taking into account that $[\Gamma_k] = O(1/k_s^{2(\gamma)})$, the net result is $R_{T_{k,c}} = O(N^{2k-2-\gamma})$. This preliminary analysis is approximate (the actual exponent is smaller) but consistent, and shows a general mechanism that does not instead apply for $h_{\text{max}} = N^{1/(\gamma - 1)}$. It holds for any motif $\Gamma$, both sparse (e.g., cliques) and dense (e.g., open chains of links): $\lim_{N \to \infty} R_T = \infty$ for $\gamma \in (\gamma_1, \gamma_2)$, where $\gamma_1 \approx \max\{2, k_{\text{min}}\}$, and $\gamma_2 \approx 2k_{\text{max}}$, $k_{\text{min}}$ and $k_{\text{max}}$ being the smallest and the largest degrees of $\Gamma$. Numerical integrations confirm this phase transition scenario when $h_{\text{max}} \sim N$, while we do not see any divergent behavior for $h_{\text{max}} \sim N^{1/(\gamma - 1)}$ (compatibly with
ref. [16,17,25]). In fig. 5 we show the $k$-clique cases $k = 3$ and $k = 4$. This general result is expected to hold for any probability $p(h, h')$ as a function of $hh'/k_2^2$ when $h_{\text{max}} \sim N$. An urgent question concerns the possibility to single out ensembles where the non-self-averaging terms in $R_\ell$ are absent. Related results will be given elsewhere.

Conclusions. – Hidden-variables models provide a powerful tool to investigate complex networks analytically. In this framework we present a first systematic analysis of the fluctuations of the density of motifs. Surprisingly, if the cut-off $h_{\text{max}}$ of the model is properly chosen as to reproduce a power-law distribution $P(k) \propto k^{−\gamma}$ having support $k \leq N$, i.e., $h_{\text{max}} = O(N^\lambda)$, with $\lambda \geq 1$, a phase transition scenario emerges, with self-averaging regions separated from non-self-averaging regions\(^3\). Under the simple null-model assumption that the system obeys a power law, the potential practical consequences of such a picture are dramatic. The existence of non-self-averaging regions implies the instability of the substructures composing a network to even small perturbations: patterns and communities structures observed in a given network realization, will have a totally different configuration in another network realization. For the same reason, in simulating complex networks, the presence of large fluctuations implies a very demanding statistics in order to have a fair evaluation of the averages of the observables of interest: given a motif $\Gamma$, if we are in a non-self-averaging region, the number of samples that we need to properly evaluate the average $\langle n_\Gamma \rangle$ will be a growing function of the system size $N$. Hence, since the evaluation of $n_\Gamma$ already requires $N$ operations per each sample, the total number of operation to evaluate $\langle n_\Gamma \rangle$ scales as $N^{1+\alpha}$, with $\alpha > 0$.

Our analysis could also explain why most of the observed networks have a small value of $\gamma$ [26]. In fact, with such small values, even small motifs and communities are guaranteed to be stable as soon as they have $k_{\text{min}} \lesssim \gamma$. Whereas, for larger $\gamma$, communities for which $k_{\text{min}} \lesssim \gamma \lesssim 2k_{\text{max}}$ will be unstable to small perturbations. In practical terms this means that, between two networks, one with say, $\gamma = 2.3$, and another with say, $\gamma = 2.1$, the latter is more stable or, in other words, it has a larger probability to exist. Moreover, our analysis also shows that fluctuations of motifs become negligible for $\gamma \to 2$. This latter observation is consistent with the fact that the entropy of networks for $\gamma \to 2$ goes to zero [19,27].

We have specialized the analysis to the hidden-variable model defined by eqs. (1), (2). A similar self- vs. non-self-averaging phase transition scenario is expected to hold for any hidden-variable model characterized by power laws.

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\(^3\)We postpone the issue of the minimal value of $\lambda$, $\lambda_\ell$, above which such a picture still applies. However, for $\gamma > 2$, $1 \geq \lambda_\ell > 1/(\gamma - 1)$.

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