HANDLE OPERATORS OF COSET MODELS*

Michael Crescimanno

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

Appeared in: Modern Physics Letters A8 (1993) 1877

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069, and by the Division of Applied Mathematics of the U.S. Department of Energy under contract #DE-FG02-88ER25066.
ABSTRACT

Several interesting features of coset models "without fixed points" are easily understood via Chern-Simons theory. In this paper we derive explicit formulae for the handle-squashing operator in these cosets. These operators are fixed, linear combinations of the irreducible representations of the coset. As a simple application of these curious formulae, we compute the traces of all genus-one operators for several common cosets.
1. INTRODUCTION

Although coset models are a relatively old subject\(^1-3\), they have recently come under intense scrutiny as settings in which one may explicitly study properties of string vacua which may shed some light on, for example, the nature of gravitation in string theory.\(^4-6\) Originally coset models were understood in algebraic terms only and lagrangian descriptions of classes of coset models, usually in terms of gauged Wess-Zumino-Witten (WZW) models, have been studied extensively in the last few years.\(^7-10\).

As well studied as these conformal field theories are, many structural and even philosophical questions about them remain unanswered. For example, Witten\(^11\) recognized the connection between the Hilbert space of a Chern-Simons gauge theory in three dimensions and the conformal blocks of of the \(G_k\) theory but, except for some cursory remarks\(^5,10,12\), a truly simple and self contained description of the connection between the general coset model and some topological three-dimensional theory has yet to emerge. From such a theory one would learn new things both about the three- (such as a clearer view of the associated link and three-manifold invariants) and the two- (for example, a view of the resolution of ”fixed points” under simple current identification\(^13\)) dimensional theory.

The aim of this present note is to use the the conceptually incomplete (although current) view of Chern-Simons gauge theory as it relates to coset models to answer some structural questions about their fusion ring. In particular we will derive explicit formula for the handle-squashing operator for a class of coset models. Such formulae arose naturally in studying the fusion ring of \(G_k\) theories. As the present description of the relation between Chern-Simons theory coset theories is really only applicable to coset theories ’without fixed points’ (discussed below), in this note we will restrict our attention to these theories*.

* For experts, we consider ”simple” cosets with a single numerator and denomenator, ’without fixed points’ and where the index of embedding is one.
The paper is organized as follows; In section 2 we review some generalities and methods for describing the space of conformal blocks of a coset theory. Section 3 discusses the handle-squashing operator in coset models, and section 4 describes one application of these formulae and a brief conclusion.

2. Coset Generalities

In an effort to make this note self-contained we here describe some basic notions and methodology of cosets. Unfortunately, this will be a very brief primer: those interested in a more thorough and systematic introduction are referred to Refs.[5,10]

Algebraically, coset models may be understood as a variant of the familiar Kac-Moody construction of Virasoro representations. The starting point is to consider the $G_k$ current algebra

$$[J^a(x), J^b(y)] = f^{abc} J^c(x) \delta(x - y) + k \delta^{ab} \delta'(x - y) ,$$

where, as usual, the currents carry Lie algebra indices and $k$ is some fixed integer. The representation theory for these algebras is well understood and the WZW models provide a rigorous and complete lagrangian description of these models. It was noticed long ago that one could find a representation of the Virasoro algebra on a subspace of the Hilbert space of the original model that is annihilated by a subset of the currents. The motivation for this approach arose originally from trying to understand confinement in the strong interactions from a string theory point-of-view. At any rate, for the resulting theory to be unitary it is necessary to require that the chosen subset of currents close algebraically; that is, they must form a subalgebra of Eq.(2.1). This subalgebra is itself usually a Kac-Moody algebra associated with a subgroup $H$ of the original group $G$. Requiring these chosen currents to
vanish on all the states results in a description of the coset model’s Hilbert space. The resulting theory is denoted by $G/H$. The representation theory of such coset theories is well understood$^{2,3}$ and one builds a Virasoro representation from this model via the Suguwara$^{16}$ construction, in which the stress tensor of the coset is realized as the difference of the stress tensors of the $G$ theory and the $H$ theory. These simple cosets have a lagrangian formulation in terms of gauged WZW models$^{7−9}$.

In heuristic terms, it is often possible to describe (a basis for) the vector space of the conformal blocks of the coset theory $G/H$ in terms of the individual conformal blocks of the $G_k$ and the $H_k$ theory. This may be understood in terms of simple considerations on the representation spaces $W_G$ and $W_H$ of the Lie algebras of $G$ and $H$. Given a representation $R$ of the Lie algebra $G$, we may decompose it with respect to the representations $r$ of the subgroup $H$,

$$R = \sum_r b_r^R r.$$  \hfill (2.2)

In decomposing a given representation $R$ into $H$ representations, the R.H.S. of Eq.(2.2) does not contain every representation. The representation theory of $G$ (and thus $G_k$) is graded with respect to the action of $Z_G$, the center of the group. Let $Z = Z_G \cap H$, the common centers of $H$ and $G$. Thus for $b_r^R \neq 0$ it is necessary that $R$ and $r$ have the same $z$ eigenvalue for each $z \in Z$. That is,

$$b_r^R \in (W_G \otimes W_H)^Z.$$  \hfill (2.3)

where by $(W_G \otimes W_H)^Z$ we mean the $Z$-invariant part of the vector space $W_G \otimes W_H$. We will see below how this ‘selection rule’ is applied in simple coset models.

The representations of Kac-Moody algebra $G_k$ above are labeled by a subset of the representations of the Lie algebra $G$. This subset is called the integrable representations
and they form a vector space that we shall, following Ref.[10], denote by $V_G$. One may also decompose Kac-Moody representations with respect to a subalgebra (i.e. $H_k$) and in simple cases an equation like Eq.(2.2) is recovered, where now the sum is over the integrable representations of the algebra $H_k$. The operation of finding the subspace of the Hilbert space of the $G_k$ which is annihilated by the $H_k$ currents is equivalent to setting all the $r$’s in the R.H.S. of Eq.(2.2) to one. That is (in simple cases atleast) the $b^R_r$ correspond to the integrable representations or ”current blocks” of the $G_k/H_k$ model.

Chern-Simons theory is a three-dimensional gauge theory that provides a complete description of the space $V_G$ and of the linear operators of interest on it. More than just a philosophical framework, Chern-Simons theory can be used for explicit computation and has indeed been useful in elucidating the structure of $G_k$ as a conformal field theory$^{11,17-20}$. However, a corresponding three-dimensional viewpoint for coset theories and for more general conformal field theories is somewhat incomplete. For example, in $G_k$, the three-dimensional theory is understood in terms of quantizing the moduli space of flat $g$-connections ($g$ is the Lie algebra of $G$); for the general conformal model no correspondingly simple picture has emerged.

For a certain simple class of coset models there is a recipe for how to proceed$^{5,10}$. Motivated by Eq.(2.2), one identifies with the coset conformal blocks the subspace of $V_G \otimes \hat{V}_H$ ($V_G$ is the space of conformal blocks of the theory $G_k$, and the $\hat{V}$ means the dual of the vector space $V$) invariant under the action of the common center $Z = Z_G \cap H$. This recipe admits a Chern-Simons interpretation: one requires the Wilson line operators associated to the common center $Z$ to be trivial (i.e. $=1$) in the coset model’s vector space of current blocks. We denote the set of Wilson line operators associated with the center action as
\( Z' = Hom(H_1(\Sigma), Z) \). It is to be stressed that this is still essentially a recipe and indeed works completely and unambiguously only when the orbits of the \( Z' \) action in \( V_G \otimes \hat{V}_H \) all have the same length. For a generic coset this is not the case, and when all the orbits are not of the same length the construction is referred to as a coset 'with fixed points'\(^{13,21-26}\). Nonetheless, cosets in which the above mentioned orbits all have the same length (called cosets 'without fixed points') do play a prominent part in conformal field theory. For example, the cosets \( SU(2)_k/U(1)_k \) and \( SU(2)_k \times SU(2)_l / SU(2)_{k+l} \) (\( l \) odd) are cosets 'without fixed points'.

3. **Handle-Squashing in Cosets**

For coset models 'without fixed points' (as discussed in the previous section) the modular transformation \( S : \tau \to \frac{-1}{\tau} \) on the torus is realized as a unitary operator on the space of blocks

\[
S_{G/H} = S_G \otimes S_H^{\dagger} \mid_{(V_G \otimes \hat{V}_H)^{Z'}}.
\] (3.1)

This \( S \) matrix is obviously unitary on \((V_G \otimes \hat{V}_H)^{Z'}\) and, for cosets "without fixed points" it leads (via the Verlinde formula) to integer fusion coefficients. Indeed, the fusion algebra of such cosets is simply given via tensoring operators in the original theories. Using the canonical map in the conformal field theory that relates states to operators (through \( \mathcal{O}_\Lambda \psi_0 = \psi_\Lambda \)

\[
V_G \rightarrow \text{End}(V_G, V_G).
\] (3.2)

we find an identification of the operators on \( V_{G/H} \)

\[
\text{End}(V_{G/H}, V_{G/H}) = \text{Image}\{(V_G \otimes \hat{V}_H)^{Z'} \rightarrow \text{End}(V_G, V_G) \otimes \text{End}(\hat{V}_H, \hat{V}_H)\}.
\] (3.3)

Recall that in the case of simple Kac-Moody conformal field theory \( G_k \) there is an explicit formulae\(^{19}\) for the inverse of the \( K \)-matrix\(^{27,28}\) in terms of a fixed linear combination (i.e.}

6
the coefficients were independent of the level) of the integrable representations. In Ref.[19] formulae for the handle-squashing operator, $K^{-1}$, arose by writing the inner product in the space $V_G$ in terms of the associated Gaussian model†. Philosophically, this is the strict conformal field theory analogy of the functional norm for the polynomial representation of the fusion algebra introduced in Ref.[29] (for additional background see Refs.[19,30-34].) The $K^{-1}$ matrix defines a norm on the space of operators of the conformal field theory,

$$\delta_{ij} = Tr(\mathcal{O}_i \mathcal{O}_j K^{-1}), \quad (3.4)$$

where $Tr$ denotes trace over space of conformal blocks at genus one. For a coset of the type mentioned above (‘without fixed points’) we now show that there is also a simple formula for the handle-squashing operator that descends from the constituent $G_k$ theory.

The space of conformal blocks (in genus one) for the coset is isomorphic to $(V_G \otimes \hat{V}_H)^{Z'}$ where $Z' = Hom(H_1(T^2), Z) \approx Z \times Z$ is the abelian group of the operators in the fusion algebra that are associated with the action of the common center $Z = Z_G \cap H$. As described above, these operators are always ‘simple’ currents $^{13,22-25}$ and, by construction, generate an automorphism of the fusion algebra. Let $|Z|$ be the order of the group $Z$ and let $z_l \in Z'$ be the operator corresponding to transport along cycle $l(=1,2,\text{labels of the homology basis})$ of the torus in some representation associated with the center.

Thus, if $Z'$ acts without fixed points the $S$-matrix (associated to modular transformations of the torus) is given as in Eq.(3.1). This is described in Refs.[21-26] where it is shown that projecting the matrix $S_G \otimes S_H^\dagger$ onto the $Z'$ invariant states results in a simple normalization factor $|Z|$ of the rows of the $S_G \otimes S_H^\dagger$ matrix. Now, since the eigenvalues of the $K$ matrix of

† The Gaussian model is isomorphic to the ring of theta functions from which one forms the characters of the theory.
Verlinde\textsuperscript{27} (see also Bott\textsuperscript{28}) may be written as $|S_{0i}|^{-2}$, the handle-squashing operator $K^{-1}$ of the coset is,

$$K_{G/H}^{-1} = |Z|^2 \left( K_G^{-1} \otimes K_H^{-1} \right) \bigg|_{G/H} .$$

(3.5)

Another derivation of this simple result which doesn’t make explicit use of Eq.(3.1) but proceeds from analogy with the original derivation of handle-squashing operators in the $G_k$ theory is given in appendix A. That is, the derivation relates the inner product on the space of operators of the coset theory with the inner product on the space of operators of the Gaussian model.

Actually, it is possible to make a stronger statement about the form of the handle-squashing operator. Appendix A contains a proof that for $G_k$, the handle-squashing operator $K_{G}^{-1}$ commutes with all the operators $z \in Z'$ of the center representations. Now, as shown in Ref.[19], $K_{G}^{-1}$ is proportional to a particular linear combination (independent of the level) of the integral representations of the $G_k$ theory. Since it commutes with every $z \in Z'$ then under the projection to the coset $|_{G/H}$ it follows that the handle-squashing $K_{G/H}^{-1}$ of the coset is itself proportional to a particular linear combination (independent of the level) of the representations of the coset model.

As a particularly simple example of the formula Eq.(3.5) consider the coset $U(1)_k \times U(1)_l/U(1)_{k+l}$. For $U(1)_k$ the $K^{-1} = \frac{1}{2k}O_1$, where $O_1$ is the identity representation (i.e for $U(1)_k$ there are $2k$ blocks and in this convention, we assign the label '1' to the trivial representation, the unit matrix) Now, following Ref.[5], we understand the center of $U(1)_k \times U(1)_l/U(1)_{k+l}$ to be $Z = Z_{2(k,l)}$ where $Z$ is the integers and $(k,l)$ is the greatest common divisor of $k$ and $l$. Thus using Eq.(3.5) we find that for this coset,

$$K^{-1} = \frac{(k,l)^2}{2kl(k+l)}O_{1,1,1} .$$

(3.6)
where $\mathcal{O}_{1,1,1}$ is simply the unit matrix (the trivial representation) on the coset’s space of blocks. Eq. (3.6) is precisely what one would expect from the equivalence,

$$
\frac{U(1)_k \times U(1)_l}{U(1)_{k+l}} = U(1)_{\frac{k(k+1)}{(k+1)^2}}.
$$

(3.7)

For a less trivial example, consider the coset $SU(2)/U(1)$ at level $k$. The “vacuum” state of this coset model in terms of $V_{SU(2)} \otimes \hat{V}_{U(1)}$ states is

$$
|0 >_{SU(2)/U(1)} = \frac{(\mathcal{O}_1 \otimes \mathcal{O}_1 + \mathcal{O}_{k+1} \otimes \mathcal{O}_{k+1})}{\sqrt{2}} |0 > \otimes |0 > ,
$$

(3.8)

where the subscripts on the $(\mathcal{O})$ signify the dimensions of the representation and $|0 > \otimes |0 >$ is the tensor product of the vacuum of the $SU(2)_k$ and $U(1)_k$ theories respectively. Using the fact that the associated $K^{-1}$ matrices are,

$$
K^{-1}_{SU(2)} = \frac{1}{2(k+2)}(3\mathcal{O}_1 - \mathcal{O}_3)
$$

(3.9)

and

$$
K^{-1}_{U(1)} = \frac{1}{2k}\mathcal{O}_1
$$

(3.10)

with the common center $Z = \mathbb{Z}_2$ we learn,

$$
K^{-1}_{SU(2)/U(1)} = \frac{1}{k(k+2)}(3\mathcal{O}_1 \otimes \mathcal{O}_1 - \mathcal{O}_3 \otimes \mathcal{O}_1).
$$

(3.11)

Separately, each term in the above expression is an $\mathcal{O} \in End(V_{G/H}, V_{G/H})$ as may readily be seen by applying each of them to the vacuum state $|0 >$ of Eq.(3.8). We label the operators of the coset as allowed (according to the selection rule discussed above, that is, invariance with respect to the action of $Z'$) products of operators of the constituent theories and recall that this labelling (and fusion) is defined modulo the action of the center $Z'$. Thus, in general
$K^{-1}_{G/H}$ is a proportional to a fixed (independent of the level $k$) linear combination of operators of the coset, as expected from the fact that $K^{-1}$ commutes with all $z \in Z'$.

In the next section we describe the explicit formulae for the handle-squashing operator in several other simple theories and show how it may be used to compute the traces of operators in the theory.

4. Trace Formulae and Conclusion

One intriguing application of the explicit formulae for the handle-squashing operators is its use in finding the traces of operators. Using Eq.(3.4) above, we find $Tr(K^{-1}) = 1$ and $Tr(K^{-1}O_j) = 0$ for all operators $O_j$ not the identity. We now turn these relations into formulae for the traces of the individual operators of the theory.

The trace $Tr(O_j)$, where the trace is taken over the space of conformal blocks of genus one, is simply the number of one-point blocks on the torus with the label $j$. Explicit formulae for the traces are useful for, among other things, computing the handle operator ($K$-matrix)

$$K = \Sigma_l Tr(O_l)O_l.$$  \hspace{1cm} (4.1)

Thus, the traces are always integer (in general depending on the level, etc.), and usually are found combinatorially directly from the fusion rules. The point we wish to make here is that using the explicit form of the handle-squashing operator it is possible to compute these operator traces without resorting to combinatorics.

This is particularly simple for $SU(2)_k$. Note first that $Tr(O_j) = 0$ for all $j$ even integer (half-odd integer spins operator). This follows from the fact that there are $z \in Z'$ for which $zO_jz^{-1} = (-1)^{j+1}O_j$ and thus implies that $Tr(O_j) = 0$ for all even $j$. In general, for any
$G_k$, those representations which are charged under the center action (for $SU(N)$, this means all those operators with non-trivial $N$-ality) have zero trace.

The trace of the integer spin operators in $SU(2)_k$ are easy to find. Using the explicit form for the $K^{-1}$ (Eq.(3.7) above) for $SU(2)_k$ and the condition $Tr(K^{-1}) = 1$ one finds $Tr(\mathcal{O}_3) = k - 1$ and by using $Tr(K^{-1}\mathcal{O}_j) = 0, j \neq 1$ and the fusion algebra, one easily finds,

\begin{equation}
\text{for } SU(2)_k \quad Tr(\mathcal{O}_j) = k + 2 - j \quad j \text{ odd} .
\end{equation}

(4.2)

A similar analysis of the coset $SU(2)_k/U(1)_k$ yields,

\begin{equation}
\text{for } SU(2)_k/U(1)_k \quad Tr(m, 1) = \frac{k(k+2-m)}{2} \quad m \text{ odd} .
\end{equation}

(4.3)

with the notation $(m, l) = \mathcal{O}_m \otimes \mathcal{O}_l$, where the first factor corresponds to the $SU(2)_k$ label and the second to the $U(1)_k$ label. Due to the center symmetry the labels $(m, l)$ run from $m = 1, 2, 3, ..., k+1$ and $l = 1, 2, 3, ..., k$ with $m$ and $l$ either both even or both odd. All other operators have zero trace (the trace is, of course, taken over the blocks of the coset model on the torus.)

The models $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ are an interesting class of cosets related to the supersymmetric and non-supersymmetric minimal models. In order for this to be a coset without fixed points it is necessary that one of either $k$ or $l$ be an odd integer thus, without loss of generality, in what follows we assume that $l$ is odd. To fix notation, we will label each operator (and thus each state) by a triple of integers $(n, m, p)$ each number being the dimension of the representation of the corresponding $SU(2)_k$, $SU(2)_l$ and $SU(2)_{k+l}$ factor respectively. Obviously, the action of the center furthermore implies that the labeling is either $(\text{odd, odd, odd})$ and $(\text{odd, even, even})$ for $k$ odd, or $(\text{odd, odd, odd})$ and $(\text{even, even, odd})$
for $k$ even. It is relatively straightforward using the ideas discussed above and the explicit form of the handle operator in these models to compute the traces of all the operators. A trivial selection rule (the traceslessness of the spinor representations) implies that only the trace of the operators associated with the $(odd, odd, odd)$ sector are nonzero. Finally, using the ideas discussed in the body of the paper, we compute these traces without resorting to combinatorics to find,

$$\text{Tr}(m,n,p) = (k+2-m)(l+2-n)(k+l+2-p)/4 \quad m, n, p \text{ odd}. \quad (4.4)$$

It is relatively straightforward, however tedious, to compute the traces of operators for many other models at any level with these explicit formulae of the handle operators.

In conclusion, in this note we have shown that many common coset models have handle-squashing operators that are simply related to the constituent theory’s. It would be clearly of much interest to extend this derivation (and the methodology of the quantization of moduli space) to cosets ‘with fixed points’ and other, more general, models although first a more thorough understanding the ‘fixed point resolutions’ of ref.[13] in the context of Chern-Simons theory is necessary.

6. Acknowledgements

The authors gratefully acknowledge conversations with S. Axelrod, K. Bardakci, D. Freed, S. Elitzur, S.A. Hotes and I.M. Singer.
Figure 1 A diagramatic view of why $[K, z] = 0$. 
REFERENCES

1. K. Bardakci and M. B. Halpern, *Phys. Rev.* D3 (1971) 2493.

2. P. Goddard, A. Kent and D. Olive, *Phys. Lett.* B152 (1985)

3. K. Bardakci, E. Rabinovici and B. Säring, *Nucl. Phys.* B299 (1988), 151.

4. D. Gepner and E. Witten, *Nucl. Phys.* B278 (1986), 493.

5. G. Moore and N. Seiberg, *Phys. Lett.* B220 (1989), 422.

6. E. Witten, *Phys. Rev* D44 (1991), 314.

7. D. Karabali, Q.-H. Park, H. J. Schnitzer and Z. Yang, *Phys. Lett.* B162 (1989) 307; H. J. Schnitzer, *Nucl. Phys.* B324 (1989) 412; D. Karabali and H. J. Schnitzer, *Nucl. Phys.* B329 (1990) 649.

8. K. Bardakci, M. Crescimanno and E. Rabinovici, *Nucl. Phys.* B344 (1990), 344.

9. K. Gawedzki and A. Kupiainen, *Phys. Lett.* B215 (1988), 119; *Nucl. Phys.* B320 (1989) 625.

10. E. Witten, *Commun. Math. Phys.* 144 (1992), 189.

11. E. Witten, *Commun. Math. Phys.* 121 (1989), 351.

12. J.M. Isidro, J.M.F. Lambastida and A.V. Ramallo, *Phys. Lett.* B282 (1992), 63.

13. A.N. Schellekens and S. Yankielowicz, *Int. J. Mod. Phys.* A5 (1990), 2903.

14. E. Witten, *Commun. Math. Phys.* 92 (1984), 455.

15. V.G. Kac and M. Wakimoto, *Adv. in Math.* 70 (1988), 156.
16. H. Suguwara, *Phys. Rev.* **170** (1968), 1659.

17. S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326** (1989), 108.

18. M. Crescimanno and S.A. Hotes, *Nucl. Phys.* **B372** (1992), 683.

19. M. Crescimanno, *Nucl. Phys.* **B393** (1993), 361.

20. C. Imbimbo, ”$Sl(2,\mathbb{R})$ Chern-Simons Theories with Rational Charges and 2-Dimensional Conformal Field Theories,” Genoa Preprint GEF-TH 5/1992, Hepth/9208016.

21. D. Gepner, *Phys. Lett.* **B222** (1989), 207.

22. K. Intriligator, *Nucl. Phys.* **B332** (1990), 541.

23. R. Brustein, S. Yankielowicz and J.B. Zuber, *Nucl. Phys.* **B313** (1989), 321.

24. B. Blok and S. Yankielowicz, *Nucl. Phys.* **B315** (1989), 25.

25. A.N. Schellekens and S. Yankielowicz, *Nucl. Phys.* **B327** (1989), 673; **B334** (1990), 67.

26. C. Ahn and M.A. Walton, *Phys. Rev.* **D41** (1990), 2558.

27. E. Verlinde and H. Verlinde, ”Conformal Field Theory and Geometric Quantization,” published in *Trieste Superstrings* (1989), 422.

28. R. Bott, *Surveys in Diff. Geom.* **1** (1991), 1.

29. D. Gepner, *Commun. Math. Phys.* **141** (1991), 381.

30. K. Intriligator, *Mod. Phys. Lett.* **A6** (1991), 3543.

31. M. Bourdeau, E. Mlawer, H. Riggs, and H. Schnitzer, *Mod. Phys. Lett.* **A7** (1992), 689.

32. D. Nemeschansky and N. P. Warner, *Nucl. Phys.* **B380** (1992), 241.
33. D. Gepner and A. Schwimmer, "Symplectic Fusion Rings and their Metric," Weizmann Preprint WIS-92/34, Hepth/9204020.

34. P. Di Francesco and J.-B. Zuber, "Fusion Potentials I," Saclay preprint SPhT 92/138, Hepth/9211138.
APPENDIX A. Derivation of handle-squashing operator for cosets

Here we give another derivation of Eq.(3.5) by studying the inner product on the space of operators of the theory. This inner product is induced from the canonical inner product on the space of states \( V_G \) by the 1-to-1 correspondence with the operators in the theory.

We proceed as in the \( G_k \) case by first relating the vacuum state of the coset model to the vacuum of the Gaussian model

\[
\psi_0 = \tilde{\Gamma}|0> \quad \tilde{\Gamma} = \sqrt{N} P_0 \Gamma ,
\]

where \(|0> = |0 >_G \oplus_H H < 0|\) is the vacuum state of the Gaussian model associated to \( G_k \otimes H_k \) and \( \Gamma = \Gamma_G \otimes \Gamma_H^\dagger \) where \( \Gamma_G \) is the operator that relates the vacuum of the Gaussian model to the vacuum of \( G_k \) (we use the notation and convention of Ref.[18,19] throughout.) \( P_0 \) is a projection operator that annihilates any state that is not invariant under the action of every \( z_2 \in Z' \sim Z \times Z \) where by \( z_2 \) we mean the center action written in terms of raising operators of the quantization (as is \( \Gamma \) in Eq.(A.1).)* For example, if \( Z \) is \( \mathbb{Z}_N \) then \( P_0 = \frac{1}{N} \sum_{j=0}^{N-1} z_2^j \) where \( z_2 \) is a generator of \( \mathbb{Z}_N \). In general \( Z \) is simply abelian and finite and not simply \( \mathbb{Z}_N \) for some \( N \), but it is always simple to construct \( P_0 \) and the discussion here won’t rely on any particular form of \( P_0 \) or \( Z \).

Let \( \mathcal{O}_l \) be the operator in the coset theory associated to the coset state labeled by the (multi-index) \( l \), that is, \( \psi_l = \mathcal{O}_l \psi_0 \). Then we may write the inner product on the coset states as

\[
(\psi_l, \psi_m) = \delta_{lm} = \frac{1}{\lambda_G \lambda_H} Tr_{Gauss}(\mathcal{O}_l \mathcal{O}_m \tilde{\Gamma}^\dagger \tilde{\Gamma}) ,
\]

* By the subscript \( j \) on \( z_j \) we mean a label of the homology basis of the torus.
where $Tr_{\text{Gauss}}$ means that the trace is to be taken over the Gaussian model’s state space (of the $G_k \otimes H_k$ theory) and $\lambda_G \equiv \left| \frac{\Lambda_w}{(k+c)\Lambda_r} \right|$ where $\Lambda_w$ and $\Lambda_r$ are, respectively, the weight (actually the co-root) lattice and root lattice of the Lie-algebra of $G$. Let $|W_G|$ be the order of the Weyl group of $G$. Using the definition of $\tilde{\Gamma}$ in Eq.(A.1) we have

$$\delta_{lm} = \frac{|Z| |W_G||W_H|}{\lambda_G \lambda_H} Tr_{(V_G \otimes \hat{V}_H)^{P_0}} (\mathcal{O}_l \mathcal{O}_m \Gamma^\dagger \Gamma), \quad (A.3)$$

where $(V_G \otimes \hat{V}_H)^{P_0}$ stands for the subspace of $V_G \otimes \hat{V}_H$ stabilized by $P_0$.

In order to write Eq.(A.3) as a trace over just the states in the coset we must understand the action of the remaining elements of the group $Z'$, that is, the $z_1$ on $V_G \otimes \hat{V}_H$. These are the operators that assign a phase to each of the states of $V_G \otimes \hat{V}_H$. Any $\mathcal{O}_{l,2}$ (not necessarily associated with a state in the coset) is homogeneous with respect to the action of the $z_1 \in Z'$ because it is associated with a irreducible representation and so must commute up to a phase (the center is abelian, and these are the one-dimensional representations) with each $z_1 \in Z'$. However each particular $\mathcal{O}_{l,2}$ associated with a state in the coset commutes with each $z_1 \in Z'$ (without phases) by construction.

Finally if each $z_1 \in Z'$ commuted with the operator $\Gamma^\dagger \Gamma$ then the trace in Eq.(A.3) would break up into sums of traces over individual eigenspaces labeled by different $z_1$ eigenvalues. We now show that this is indeed the case. Because $\Gamma^\dagger \Gamma$ is $\Gamma_G^\dagger \Gamma_G \otimes \Gamma_H \Gamma_H^\dagger$ it will be enough to show that each $z_1$ commutes with $\Gamma_G^\dagger \Gamma_G$. Now note that $\Gamma_G^\dagger \Gamma_G$ is proportional to the $K^{-1}$ of the $G_k$ theory. Each $z_1$ is a grading (indeed a one-dimensional representation of an automorphism) of the fusion algebra. As such it is easy to see diagramatically (see figure 1) that $z_1$ and $K$ must commute. Thus so do $z_1$ and $K^{-1}$. For the reader who is unsatisfied with
this diagramatic proof, appendix B contains an explicit Lie-algebraic proof of the commutation of \( z_1 \) and \( K^{-1} \). At any rate, it is clear from these arguments that

\[
\delta_{lm} = \frac{|Z| |W_G||W_H|}{\lambda_G \lambda_H} \sum_s Tr_{H_s}(O_l O_m \Gamma^\dagger \Gamma) ,
\]

(A.4)

where \( H_s \) are the eigenspaces of the various \( z_1 \in Z' \) (\( s \) is a multi-index distinguishing the the various eigenvalues of the \( z_1 \)'s) in \((V_G \otimes \hat{V}_H)^{P_0}\). Note that since the \( z_1 \)'s are 1-dimensional representations of an automorphism of the algebra their various eigenspace are distinct and orthogonal. Furthermore, there are exactly \(|Z|\) of these spaces. The coset \((V_G \otimes \hat{V}_H)^{Z'}\) corresponds to the \( H_1 \), i.e. the subspace on which all the eigenvalues of the \( z_1 \)'s are 1.

Let us now assume that the automorphism group \( Z' \) acts ‘without fixed points’ as described earlier. Then each orbit under the \( z_2 \in Z' \) in of length \(|Z|\). Since the operators inside the trace of Eq.(A.3) commute with all the \( z \in Z' \) and each state occurs once on the orbit under the automorphism group we learn that each term in the above sum is the same. Thus

\[
\delta_{lm} = \frac{|Z|^2 |W_G||W_H|}{\lambda_G \lambda_H} Tr_{G/H}(O_l O_m \Gamma^\dagger \Gamma) ,
\]

(A.5)

and so

\[
K_{G/H}^{-1} = |Z|^2 (K_G^{-1} \otimes K_H^{-1})|G/H\ ,
\]

(A.6)

is the handle-squashing operator of the coset in terms of the handle-squashing operators of the constituent \( G_k \) theories. Note that this derivation made critical use of the assumption that \( G/H \) is a coset ‘without fixed points’. This result is consistent with Eq.(3.1) as the operators \( K^{-1} \) have a simple interpretation in terms of \(|S_{0l}|^2\).
APPENDIX B. Lie Algebraic proof of $[\Gamma^\dagger \Gamma, z_1] = 0$

It is relatively straightforward, using ideas from the quantization of Chern-Simons theory to show directly that the center action always commutes with the $K^{-1}$ matrix. In this appendix we first show that there will be a unique set of fields in the $G_k$ that will carry the full center symmetry for any $G$ and level $k$. Having demonstrated existence, we then directly show that the operators generating the center commute with $K^{-1}$. This approach has the advantage that one never explicitly uses the fact that this symmetry generates an automorphism of the fusion rules.

To show that there will always be a unique set of fields that generate the full center symmetry of $G$ in $G_k$, we begin by recalling that, from the point of view of quantizing the moduli space of flat $g$-connections over $T^2$, the states of $G_k$ are odd under the generators of the Weyl group whereas the operators are even under Weyl. It is always possible to find a monomial in the raising operators of the Gaussian model that is Weyl invariant. Let $s_l$ be the components of a rank($G$) vector that correspond to such a monomial. That is, consider the monomial

$$z_2 = \Pi_j B_j^{s_j} .$$

Requiring this monomial to be Weyl invariant gives a condition on the $s_l$,

$$exp\left(\frac{2\pi i}{k+c} A_{jm} C^{-1}_{ml} s_l\right) = 1 \quad \forall j .$$

where $A$ is the Cartan matrix and $C$ is the matrix associated to the Lie-algebraic part of the symplectic form of the quantization. As described in Refs.[18,19], for simply laced groups, $C$ is simply the Cartan matrix (more generally, $C$ is the matrix of inner products of the
root vectors and so is always symmetric, even for non-simply laced $G$.) For clarity of this exposition we discuss the simply laced case, and note here that it is straightforward to extend this discussion to include the non-simply laced case. Thus the condition that $z_2$ be Weyl invariant is simply that

$$s_l = 0 \mod (k + c).$$  \hfill (B.3)

We call the primitive $z_2$ those for which $s_l$’s are zero except for a single component equal to $k + c$. Some of these primitive $z_2$ will not be 1 in the Gaussian model. It is easy to see that there will be precisely one non-trivial primitive monomial for each element in the center of $G$.

Now, since these primitive $z_2$ are Weyl invariant, they must take Weyl-odd states to other Weyl-odd states (by Weyl-odd, we mean those states odd under each of the generating reflections of the Weyl group.) Now, the Weyl-odd states are precisely the states that represent the current blocks of the $G_k$ theory and since each of these Weyl-odd states are composed of distinct linear combination of Gaussian states, the monomial operator $z_2$ can take each Weyl-odd state to precisely one other Weyl-odd state, up to an overall phase. Furthermore, such a $z_2$ may be interpreted as the fusion operator associated to the state proportional to $z_2 \psi_0$, where $\psi_0$ is the vacuum state of the Weyl-odd states. Thus, there are, for every $k$, states whose fusion realizes the center action are simple currents in the sense of Refs.[13,21-25].

Having shown that there is always a realization of the center of the group in terms of a particular set of operators of the theory, we now show that these operators must commute with $K^{-1}$, the handle-squashing operator. As advertised we show this in complete generality with a simple computation.

We proceed as follows; We first show that $[\Gamma, z_1]$ has a simple form and then use this to compute $[\Gamma^\dagger \Gamma, z_1]$. From the point of view of the quantization of Chern-Simons theory
we describe this entirely in Lie algebraic terms. Recall that $\Gamma$, is the raising operator in the Gaussian model that applied to $|0\rangle$ of the Gaussian model yields the state associated to the “vacuum” of the conformal field theory $G_k$. Explicitly,

$$\Gamma = \Sigma_{w \in W} (-1)^w \prod_{\alpha_j} B_j^{w(\rho)_j} . \quad (B.4)$$

where $W$ is the Weyl group, $B_j$ are, as before, the raising operator associated with the co-root to $\alpha_j$ and $\rho$ is one-half the sum of the positive weights. Note that the exponent $w(\rho)_j$ is the $j$-th component of the Weyl transform of $\rho$ but written in the co-root basis (for further details consult Refs.[18,19].)

Trivially, each $\Gamma$ (and therefore $K^{-1}$) commutes with any polynomial in the $B_j$’s. What we need to show is that $\Gamma^\dagger \Gamma$ commutes with all $z_1 = \prod_j A_{s_j}^j$ for the $s_j$ given in Eq.(B.3). Then the full group of Wilson lines on the torus associated to the center action of $G$ will commute with $K^{-1}$. In view of the commutation relations of the Gaussian model, application of Schur’s lemma yields

$$[z_1, \prod_j B_j^{w(\rho)_j}] = \exp \left( \frac{2\pi i}{k + c} s_l(C^{-1})_{lm} w(\rho)_m \right) , \quad (B.5)$$

where $w(\rho)_m$ are the components of $w(\rho)$ in the co-root basis. Note that for any simple root $\alpha$, $w_\alpha(\rho) = \rho - \alpha$. Now since we go from the root basis to the co-root basis by the application of the matrix $C$, then by Eq.(B.3) we see that the right hand side of Eq.(B.5) is independent of which Weyl transformation $w$ appears on the left hand side. Thus,

$$[z_1, \Gamma] = \exp \left( \frac{2\pi i}{k + c} s_l(C^{-1})_{lm} \rho_m \right) . \quad (B.6)$$

Finally, since $\rho = \frac{1}{2} \Sigma_{\alpha > 0} \alpha$ then as a vector in the root basis $\rho$ has components that are integer and half-integer. Thus, the right hand side of Eq.(B.6) is either +1 or −1, depending essentially on the group and the particular $z_1$. Therefore

$$[z_1, \Gamma^\dagger \Gamma] = 1 . \quad (B.7)$$
and so all Wilson lines on the torus that represent the center action in $G_k$ commute with $K^{-1}$. 