Mimicking counterfactual outcomes to estimate causal effects

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Abstract

In observational studies, treatment may be adapted to covariates at several times without a fixed protocol, in continuous time. Treatment influences covariates, which influence treatment, which influences covariates, and so on. Then even time-dependent Cox-models cannot be used to estimate the net treatment effect. Structural nested models have been applied in this setting. Structural nested models are based on counterfactuals: the outcome a person would have had had treatment been withheld after a certain time. Previous work on continuous-time structural nested models assumes that counterfactuals depend deterministically on observed data, while conjecturing that this assumption can be relaxed. This article proves that one can mimic counterfactuals by constructing random variables, solutions to a differential equation, that have the same distribution as the counterfactuals, even given past observed data. These “mimicking” variables can be used to estimate the parameters of structural nested models without assuming the treatment effect to be deterministic.

Keywords: Causality in continuous time, Dynamic treatments, Longitudinal data, Observational studies, Panel data, Rank preservation, Stochastic differential equations, Structural nested models.

1 Introduction

Observational studies are no replacement for randomized clinical trials, but they can be used, for example, where randomization is unethical or to generate hypotheses for subsequent clinical trials. In an observational study, treatment may be adapted to patient
characteristics which predict the outcome of interest. This is called confounding by indication. If the confounding by indication only takes place at baseline, one can condition on initial person characteristics in order to get meaningful estimates of the treatment effect. However, if the confounding by indication also takes place after baseline, variables used for treatment decisions may be influenced by past treatment. Thus they may themselves be indications of the treatment effect, and in that case simply conditioning on them can lead to false conclusions.

With such time-dependent confounding by indication, even the time-dependent Cox model does not estimate the net effect of treatment (see e.g. [21], [24], or [25]). With a time-dependent Cox model, the rate of events given past treatment and covariate history can be estimated, but the true parameter(s) on treatment may not reflect the treatment effect. A consistent estimator of the effect of the treatment on the outcome of interest has to take into account the effect of treatment on intermediate covariates. This is easily understood when considering a treatment which affects the outcome only because it affects an intermediate variable \( L \). In that situation, if \( L \) and treatment are both included in the time-dependent Cox model for the event of interest, the true parameter(s) on treatment in this Cox model equal 0. However, treatment could be beneficial due to its effect on \( L \). On the other hand, not including \( L \) may also result in an inconsistent estimator, if \( L \) predicts future treatment. This follows from the same reasoning as why, in case of non-randomized point treatment, one needs to adjust for predictors of both the treatment and the outcome to consistently estimate the treatment effect: if one does not adjust for \( L \), and if persons with \( L \) indicating a bad prognosis are more likely to be treated, the treatment may seem to adversely affect the outcome, even if it has no effect on anyone. To conclude, with time-dependent confounding by indication, one needs to take confounders into account, but adding the confounders to an outcome model is not enough.

If all confounders are measured (see Assumption 5 below), structural nested models, proposed in [22, 23, 26], and marginal structural models, proposed in [30, 8], can be used to consistently estimate treatment effects in the presence of time-dependent confounding by indication. Structural nested models and marginal structural models make a distinction between the effect of the treatment and the reason why the treatment was given, by separately modeling the treatment decisions and the treatment effect. [27] compares structural nested models and marginal structural models. The current article focuses on structural nested models.

Structural nested models model relations between counterfactual outcomes. We allow for general treatment regimes. Consider a single person, who received a particular treatment regime with outcome \( Y \). For example, the particular treatment regime could be as follows: first, no treatment, then, after a certain time, initiation of treatment, then, the dosage changed some time thereafter, then treatment stopped, initiated again, etcetera. Had the person’s treatment been stopped (prematurely) at time \( t \) and not been re-initiated thereafter (or, had treatment changed to a “baseline” treatment regime from time \( t \) onwards), the person’s outcome, \( Y^{(t)} \), might have been different. Since \( Y^{(t)} \) is generally not observable, it is a counterfactual outcome. In a discrete-time setting, [6] show that existence of counterfactuals places no restrictions on the distribution of the observed variables.
No comparable proof exists for the continuous-time case.

An important controversy in the causal literature is that counterfactuals are often assumed to depend deterministically on the observed data: given the model and the parameter values, all counterfactual outcomes $Y^{(t)}$ for each person can simply be calculated from the observed data. \cite{26} calls this local rank preservation (in most cases, this implies global rank preservation), when the counterfactual outcomes are solutions to the differential equation (8) in Section 4 below. Treatment is then said *not* to affect the outcome of interest if the outcome for any particular person would have been exactly the same regardless which treatment was given. The assumption of deterministic dependence is related to the assumption of constant effect in \cite{9}: that is, the difference between counterfactual outcomes belonging to different treatments is a constant identical for all persons.

The assumption of deterministic dependence/ (local) rank preservation has frequently been attacked. This assumption does not hold if, for example, two persons with the same observed data (e.g., both receiving some prophylactic drug) could have had a different outcome had they not been treated starting from some time $t$ (e.g., one might have contacted a virus and the other might not). In addition, deterministic dependence can never be tested, with only one outcome observed for each person. For these two reasons, the assumption of (local) rank preservation should be avoided if at all possible.

In discrete time, when treatment and covariates change at fixed times which are the same for all persons, the theory of structural nested models is well developed. \cite{17} prove that it is not necessary to assume a deterministic treatment effect. In order to do so, they show that a certain “blipped down” outcome $X(t)$ mimics the outcome $Y^{(t)}$ had treatment been withheld from time $t$ onwards, in the sense that $X(t)$ has the same distribution as $Y^{(t)}$ given past treatment and covariate history. They also indicate why the resulting estimators for treatment effect are consistent and asymptotically normal.

However, in reality covariates and treatment often change in continuous time. Moreover, in discrete time the interpretation of the treatment effect (shift- or blip function) depends on the time scale chosen. In continuous time, the treatment effect (infinitesimal shift function) can often be interpreted as speed or rate. For these reasons, \cite{23, 28, 18, 11, 10} have applied continuous-time structural nested distribution models. However, because of a lack of theory for these models, the applications have relied on the assumption of (local) rank preservation. The models fitted in \cite{28, 11, 10} are described in Examples 3.1 and 3.2. Section 5 or, in greater detail, \cite{14} describes how to use the results in the current article in order to show that assuming (local) rank preservation is not necessary to estimate treatment effects with structural nested models (an example can be found in Section 9). Therefore, the main contribution of the current article is showing that the methods in \cite{23, 28, 18, 11, 10, 26} are robust to failures of the assumption of (local) rank preservation.

Structural nested models in continuous time are meant to estimate the effect of a continuous treatment, for which the effect of a small duration is small. \cite{26} conjectures that the appealing large sample properties of discrete-time structural nested models extend to continuous time; however, his proof requires the assumption of (local) rank preservation. He conjectures that (i) also without (local) rank preservation, a certain “blipped down” outcome $X(t)$ has the same distribution as $Y^{(t)}$ given past treatment and covariate history,
(ii) the resulting estimators are consistent and asymptotically normal, and (iii) for certain models, estimators and confidence intervals can be calculated with standard software, used in a non-standard way. This article proves conjecture (i), which we call mimicking counterfactual outcomes, and explains why such a subtle result is true. [14] proves conjecture (ii), using conjecture (i). [13] proves conjecture (iii), using a partial likelihood approach and conjectures (i) and (ii). Thus, the current article fills the final link in this methodology to estimate treatment effects of time-varying treatments in longitudinal observational studies without relying on (local) rank preservation. This methodology can be applied to longitudinal observational data, to study the effects of interventions affecting, for example, economic and health outcomes.

This article is organized as follows. Section 2 introduces the setting and notation of this article. Section 3 introduces the model for treatment effect, and shows some examples. Section 4 defines the mimicking variables $X(t)$ as the solution to a differential equation with a final condition. Section 4 also states the main result of this article: $X(t)$ mimics $Y(t)$ in the sense that it has the same distribution as $Y(t)$, even given past treatment- and covariate history. Section 5 formalizes the assumption of no unmeasured confounding, which as shown there is needed to use the result of the current article to estimate the treatment effect. Section 5 also indicates how, using the mimicking result, tests and estimators can be developed without assuming (local) rank preservation. Section 6 outlines the proof of the main result of this article: $X(t)$ mimics $Y(t)$ in the sense that it has the same distribution as $Y(t)$, even given past treatment- and covariate history. Section 7 proves the main result of this article for non-survival outcomes $Y$. Section 8 proves the main result for survival time outcomes $Y$. Section 9 describes a simulation study. Section 10 concludes this article with a discussion.

2 Setting and notation

The setting to which continuous-time structural nested models apply is as follows. The outcome of interest is a continuous real-valued variable $Y$. For example, $Y$ is a person’s survival time, time to clinical AIDS, the number of white blood cells, or the CD4 count. Our objective is to estimate the effect of treatment on $Y$. In this article, we consider a fixed time interval $t \in [0, \tau]$ with finite $\tau$, where $t = 0$ is the time at which follow-up of interest starts (for example, 0 could be the time of enrollment in a study, or a baseline time). During the time interval $[0, \tau]$, treatment and person characteristics are observed for each person. $Y$ is measured at or after time $\tau$, or, in the case of a survival time outcome, $Y$ could be measured before time $\tau$ if the person dies before time $\tau$. We assume that treatment starts at or after time $0$. We suppose that after time $\tau$, treatment is stopped or switched to some kind of baseline treatment regime. Most of this article assumes that there is no censoring, and $Y$ is observed for every person in the study. Section 8.4 incorporates right censoring.

The covariate process describes the course of the disease of a person, e.g. the course of the blood pressure and the white blood cell count. The covariates which must be included
are those which both (i) influence a doctor's treatment decisions and (ii) predict a person’s prognosis with respect to the outcome of interest. If such covariates are not observed the assumption of no unmeasured confounding, see Section 5, will not hold.

Denote the probability space by \((\Omega, \mathcal{F}, P)\). For the moment consider a single person.

Write \(Z(t)\) for the covariate- and treatment values at time \(t\). This article assumes that \(Z(t)\) takes values in \(\mathbb{R}^m\), and that \(Z(t) : \Omega \rightarrow \mathbb{R}^m\) is measurable for each \(t \in [0, \tau]\). Moreover, we assume that \(Z\), seen as a function on \([0, \tau]\), is continuous from the right with limits from the left (cadlag), and that with probability one this function, or “sample path”, has only finitely many jumps.

Counterfactual outcomes were already mentioned in the introduction. \(Y(t)\) is the final outcome had treatment been stopped (prematurely) at time \(t\) and not been re-initiated thereafter (or changed to some kind of baseline treatment regime \(\bar{0}\) from time \(t\) onwards). This article supposes that all counterfactual outcomes \(Y(t)\), for \(t \in [0, \tau]\) and for each person, are random variables on the probability space \((\Omega, \mathcal{F}, P)\). We assume that observations and counterfactual outcomes of different persons are independent and identically distributed, and are a random sample from a larger infinite population of interest. For notational convenience, we suppress the subscript \(i\) for person.

### 3 Model for treatment effect

Structural nested models in continuous time model distributional relations between \(Y(t)\) and \(Y^{(t+h)}\), for \(h > 0\) small, through a so-called infinitesimal shift-function \(D\). Write \(F\) for the cumulative distribution function and \(F^{-1} : (0, 1) \mapsto \mathbb{R}\) for its generalized inverse \(F^{-1}(p) = \inf \{x : F(x) \geq p\}\). Then the infinitesimal shift-function \(D\) is defined as

\[
D(y, t; Z_t) = \frac{\partial}{\partial h} \bigg|_{h=0} \left( F^{-1}_{Y^{(t)}}(Z_t) \circ F_{Y^{(t)}}(Z_t) \right)(y),
\]

the right hand derivative of the quantile-quantile transform which moves quantiles of the distribution of \(Y(t)\) to quantiles of the distribution of \(Y^{(t+h)}\) \((h \geq 0)\), given the covariate- and treatment history until time \(t\), \(Z_t\). In order to define \(D\), no assumptions are necessary about the joint distribution of the counterfactuals \(Y^{(t)}\).

**Example 3.1.** *Survival of AIDS patients.* [28] describe an AIDS clinical trial to study the effect of AZT treatment on survival in HIV-positive patients. Time 0 was the time of en-
rollment in the study. Embedded within this trial was an uncontrolled observational study of the effect of prophylaxis therapy for PCP on survival. PCP, Pneumocystis Carinii Pneumonia, is an opportunistic infection that affects HIV-positive patients. [28] use continuous-time structural nested models to study the effect of PCP prophylaxis therapy on survival of HIV-positive patients. Thus, the outcome of interest, \( Y \), is the survival time, and the treatment under study is prophylaxis for PCP. Although [28] estimate the effect of changes in the time the treatment is discontinued, we will consider estimating the effect of changes in the initiation time of the treatment. This conforms better to the clinical practice in HIV/AIDS, where PCP prophylaxis is rarely discontinued, and to the assumption in [28] that once PCP prophylaxis is started, it is never stopped. In this example, \( Y^{(t)} \) is the counterfactual outcome had PCP prophylaxis treatment been as given in reality until time \( t \) and initiated or continued thereafter. We thus define the baseline treatment regime “0” as “continuously treat with PCP prophylaxis”. In the context of this example, the local rank preservation assumption of [28] can be expressed as:

\[
Y^{(t)} - t = \int_t^Y e^{\psi_{\text{no prophylaxis at } s}} ds. \tag{2}
\]

Assumption (2) is very strong, because it requires that given the model parameter \( \psi \) and the observed outcome \( Y \), all counterfactual outcomes \( Y^{(t)} \) can be calculated from the observed data. The current article proves that it suffices to assume that

\[
D_{\psi}(y, t; Z_t) = (1 - e^{\psi}) 1_{\{\text{no prophylaxis at } t\}}. \tag{3}
\]

We show that under Assumption (3),

\[
Y^{(t)} - t \sim \int_t^Y e^{\psi_{\text{no prophylaxis at } s}} ds \tag{4}
\]

conditional on \( Z_t \) and \( Y > t \), where \( \sim \) means “has the same distribution as”. Given \( Z_t \), both \( Y^{(t)} - t \) and \( \int_t^Y e^{\psi_{\text{no prophylaxis at } s}} ds \) are random variables, depending on \( Y^{(t)} \) and \( Y \), respectively. Assumption (3) does not impose that \( Y^{(t)} - t \) is equal to \( \int_t^Y e^{\psi_{\text{no prophylaxis at } s}} ds \), but only that the distribution of these two random variables is the same conditional on \( Z_t \) and \( Y > t \). Thus, under equation (3), patients who have the exact same observed history over \([0, \tau]\), \( Z_\tau \) and \( Y \), do not necessarily have the same counterfactual outcomes \( Y^{(t)} \). This is a substantial relaxation of the assumptions previously adopted in the literature on continuous-time structural nested models. Relaxing assumption (2) is empirically relevant because in clinical practice \( Y^{(t)} \) may differ between two patients with the exact same observed history. Suppose for example that two patients with the exact same observed history were both on PCP prophylaxis. If one of the patients got in contact with pneumococcal bacteria (and therefore might have caught PCP without the preventive treatment, PCP prophylaxis), and the other did not get in contact with pneumococcal bacteria (and therefore might not have caught PCP, even without PCP prophylaxis), the outcomes for the two patients without PCP prophylaxis could be different.
In equation (4), the part of the residual survival time, \(Y - t\), that is untreated gets multiplied by \(e^{\psi}\) to attain the same distribution as \(Y^{(t)} - t\) (the residual survival time under “continuous treatment from \(t\) onwards”), conditional on \(Z_t\) and \(Y > t\). Therefore, analogous to accelerated failure time models (see e.g. Cox and Oaks, 1984), the multiplication factor \(e^{\psi}\) can be interpreted in a distributional way.

Our results do not depend on adopting the particular specification of the infinitesimal shift-function \(D\) of equation (3). For example, they also apply to an alternative specification of \(D\) from [28]. In this alternative specification, the effect of the PCP prophylaxis can depend on the AZT treatment the patient received and whether or not the patient had a history of PCP prior to the start of PCP prophylaxis. Because the data in [28] were from a clinical trial for AZT treatment, AZT treatment is described by a single variable \(R\) indicating the treatment arm the patient was randomized to (\(R\) equals 1 or 2). Let \(P(t)\) be equal to 1 if the patient had PCP before or at time \(t\) and before prophylaxis treatment started; otherwise \(P(t)\) is equal to 0. The model described in [28], but adapted to our choice of baseline treatment regime (\(\theta\) is continuous treatment with PCP prophylaxis), is

\[
D_{\psi_1,\psi_2,\psi_3}(y, t; Z_t) = \left(1 - e^{\psi_1 + \psi_2 P(t) + \psi_3 R}\right) 1_{\{\text{no prophylaxis at } t\}}.
\]

This article shows that if equation (5) holds, then

\[
Y^{(t)} - t \sim \int_t^Y e^{\psi_1 + \psi_2 P(s) + \psi_3 R} ds \quad \text{given } Z_t,
\]

for \(t < Y\).

**Example 3.2. Effect of Graft versus Host Disease (GvHD) on time to leukemic relapse.** [10] and [11] use continuous-time structural nested models to study the effect of GvHD on time to leukemic relapse in patients who had Bone Marrow Transplantation (BMT). Infection with Cytomegalovirus (CMV) is a time-dependent confounder: an independent prognostic factor for relapse that both 1. predicts the subsequent development of the exposure GvHD and 2. is predicted by past exposure GvHD. Write \(Y\) for the time until leukemic relapse. Assume that \(Y\) is observed for every patient. In [10] and [11], \(Y^{(t)}\) is the outcome had the patient been exposed (or not) to GvHD as in reality until time \(t\), and not exposed afterwards. Based on biological knowledge, [10] and [11] assume that

\[
D_{\psi}(y, t; Z_t) = \left(1 - e^{\psi}\right) 1_{\{\text{GvHD at } t\}}.
\]

This article shows that then, for \(t < Y\), preventing GvHD from \(t\) onwards leads to

\[
Y^{(t)} - t \sim \int_t^Y e^{\psi_{\text{GvHD at } s}} ds \quad \text{given } Z_t.
\]

[10] and [11] assume that (7) is true even with \(\sim\) replaced by = (although only for \(t = 0\)), hoping that could be relaxed. This article shows that indeed (6) is sufficient to estimate the effect of GvHD.
Example 3.3. *(Incorporating a-priori biological knowledge, following [26]).* Again consider survival as the outcome of interest. Suppose that it is known that treatment received at time $t$ only affects survival for patients who would die by time $t + 5$ if they would receive no further treatment. An example would be a setting in which failure is death from an infectious disease, the treatment is a preventive antibiotic treatment which is of no benefit unless the person is already infected and, if death occurs, it always does within five weeks from the time of initial unrecorded subclinical infection. In that case, the natural restriction on $D$ is that

$$D(y, t; \bar{Z}_t) = 0 \quad \text{if } y - t > 5.$$ 

As can be seen from these examples, the parameters of a continuous-time structural nested model are often rates. More biostatistical examples of models for $D$ can be found in e.g. [18, 26, 29, 23, 31].

$h \cdot D(y, t; \bar{Z}_t)$ can be interpreted as the infinitesimal effect on the outcome of the treatment actually given in the time interval $[t, t + h)$ (relative to the baseline treatment regime). To be more precise, from the definition of $D$, it follows that

$$h \cdot D(y, t; \bar{Z}_t) = \left(F_{Y_{Y(t+h)|Z_t}}^{-1}(F_{Y_{Y(t)|Z_t}}(y)) - y + o(h) \right).$$

In Figure 1 (left) this is sketched.

![Figure 1](image_url)

**Figure 1:** Left: Illustration of the infinitesimal shift-function $D$. Right: An example of a solution $X(t)$ to the differential equation $dX(t)/dt = D(X(t), t; \bar{Z}_t)$ with final condition $X(\tau) = Y$ in case the outcome is survival time.

It can be shown that $D \equiv 0$ if and only if treatment does not affect the outcome of interest, as was conjectured in [26]. To be more precise, [12] shows that, for example, $D \equiv 0$ if and only if for every $h > 0$ and $t$, $Y_{Y(t+h)}$ has the same distribution as $Y_t$ given $\bar{Z}_t$. That is, $D \equiv 0$ if and only if “at any time $t$, whatever person characteristics are selected at that time ($\bar{Z}_t)$, switching ‘treatment as given’ to ‘baseline treatment regime’ at some fixed time after $t$ would not change the distribution of the outcome in persons with these person characteristics”. To prove this one needs the mimicking result of the current article.
4 Mimicking counterfactual outcomes

Define \( X(t) \) as the continuous solution to the differential equation

\[
\frac{dX(t)}{dt} = D(X(t), t; \mathbf{Z}_t)
\]

(8)

with final condition \( X(\tau) = Y \), the observed outcome (see Figure 1, right). Then \( X(t) \) mimics \( Y(t) \) in the sense that it has the same distribution as \( Y(t) \), even given the person’s treatment- and covariate history at time \( t \), \( \mathbf{Z}_t \). To prove this main result we need the following consistency assumption.

Assumption 4.1. (Consistency). \( Y(\tau) \) has the same distribution as \( Y \) given \( \mathbf{Z}_\tau \).

Notice that because no treatment was given after time \( \tau \) and the treatment process is right continuous, there is no difference in treatment between \( Y(\tau) \) and \( Y \). Under this consistency assumption and regularity conditions only, it is proved in Sections 7 and 8 that indeed (8) has a unique solution \( X(t) \) for every \( \omega \in \Omega \), and that this solution \( X(t) \) mimics \( Y(t) \) in the sense that it has the same distribution as \( Y(t) \) given \( \mathbf{Z}_t \).

Example 4.2. Survival of AIDS patients (continuation of Example 3.1). If equation (3) holds, then

\[
X(t) = t + \int_t^Y e^{\psi_1 \{ \text{prophylaxis at } s \}} ds
\]

for \( t < Y \), and \( X(t) = Y \) for \( t \geq Y \). Alternatively, if equation (5) holds, then

\[
X(t) = t + \int_t^Y e^{\psi_1 \{ \text{prophylaxis at } s \} (\psi_2 P(s) + \psi_3 R)} ds
\]

for \( t < Y \), and \( X(t) = Y \) for \( t \geq Y \).

5 Estimators, tests, and “no unmeasured confounding”

This section contains a brief summary of [14], who shows how the result of the current article leads to testing and estimation. In addition, Appendix C provides an example of estimation in our simulation study.

The main assumption underlying structural nested models is that all information the doctors used to make treatment decisions, and which is predictive of the person’s prognosis with respect to the final outcome, is available for analysis. This assumption of no unmeasured confounding makes it possible to distinguish between treatment effect and selection bias; see e.g. [28], [26], [17] or [14].

Assume that the treatment process gives rise to a counting process \( N(t) \). For example, \( N(t) \) is the number of treatment changes until time \( t \). The assumption of no unmeasured confounding is then formalized as
Assumption 5.1. (No unmeasured confounding). The rate with which $N$ jumps given $\mathbf{Z}_{t-}$ is the same as the rate with which $N(t)$ jumps given $\mathbf{Z}_{t-}$ and $(Y^{(s)}: s < t)$, because given the observed $\mathbf{Z}_{t-}$, the (unobserved) prognosis of a person, represented by $Y^{(s)}$ for $s < t$, should not predict treatment at or after time $t$. If it does, there is no way to distinguish between the effect of the treatment and the reason why it is initiated.

Notice that if $X(t)$ mimics $Y^{(t)}$ in the sense that it has the same distribution as $Y^{(t)}$ given $\mathbf{Z}_{t-}$, it can be expected that under no unmeasured confounding, the rate with which $N(t)$ jumps at time $t$ also does not depend on $X(t)$, given $\mathbf{Z}_{t-}$. It can formally be shown that this is indeed true.

First consider how this leads to testing. If treatment does not affect the outcome of interest, $D \equiv 0$ and thus $X(t) \equiv Y$. So if treatment does not affect the outcome of interest, changes of treatment at time $t$ should be independent of $Y$, given $\mathbf{Z}_{t-}$. Thus one can test whether treatment affects the outcome of interest by testing whether, given $\mathbf{Z}_{t-}$, $Y$ adds to the prediction model for treatment changes.

Also for estimation of the infinitesimal shift-function $D$ we assume that there is no unmeasured confounding. Suppose that one has a correctly specified parametric model $D_\psi$ for $D$. Then one can calculate "$X_\psi(t)$", the solution to

$$
\frac{dX_\psi(t)}{dt} = D_\psi (X_\psi(t), t; \mathbf{Z}_t)
$$

with final condition $X(\tau) = Y$. If $X(t)$ mimics $Y^{(t)}$, then $X_\psi(t)$ has the same distribution as $Y^{(t)}$ given $\mathbf{Z}_t$ for the true $\psi$. Since $Y^{(t)}$ does not add to the prediction model for treatment changes given $\mathbf{Z}_{t-}$, $\psi$ could then be estimated by picking the $\psi$ for which, given $\mathbf{Z}_{t-}$, $X_\psi(t)$ adds the least to the prediction model for $N$, treatment changes. This can be proven to lead to the following theorem:

Theorem 5.2. Suppose that the intensity process $\lambda$ is bounded, $Y^{(t)}$ is cadlag, there is no unmeasured confounding and no instantaneous treatment effect (with probability 1, $N()$ and $Y^{(t)}$ do not jump at the same time). Suppose also that for every $t \in [0, \tau]$, $X(t)$ has the same distribution as $Y^{(t)}$ given $\mathbf{Z}_t$. Then

$$
E \int_0^\tau h_t (X(t), \mathbf{Z}_{t-}) (dN(t) - \lambda(t) dt) = 0
$$

for each $h_t$ satisfying a regularity restriction. Thus if $D_\psi$ and $\lambda_\theta$ are correctly specified parametric models for $D$ and $\lambda$, respectively,

$$
P_n \int_0^\tau h_t (X_\psi(t), \mathbf{Z}_{t-}) (dN(t) - \lambda_\theta(t) dt) = 0,
$$

with $P_n$ the empirical measure $P_n X = 1/n \sum_{i=1}^n X_i$, is an unbiased estimating equation for $(\theta_0, \psi_0)$, for each $h_t$ satisfying a regularity restriction. $h_t$ here is allowed to depend on $\psi$ and $\theta$, as long as it satisfies the regularity restriction for $(\theta_0, \psi_0)$.

In fact, these estimating equations are often martingales at the true parameter. From e.g. [32], Theorem 5.2 implies that the resulting estimators are, under regularity conditions, consistent and asymptotically normal.
6 Outline of the proof

Throughout the proof this article uses fixed versions of \( F_{Y_{\tau(t+h)}|\tau_1} \) satisfying all regularity conditions of Section 7.2. Section 7.3 shows existence of \( D \). It also derives a different expression for \( D \), which is often used in the rest of the proof. Section 7.4 shows existence and uniqueness of solutions \( X(t) \) to the differential equation with \( D \), equation (8), with final condition \( X(\tau) = Y \).

The proof that this \( X(t) \) mimics \( Y(t) \) is based on discretization. Section 7.5 therefore considers the situation where the treatment- and covariate process \( Z \) can be fully described by its values at finitely many fixed times \( 0 < \tau_1 < \tau_2 < \ldots < \tau_K \) and \( \tau \). In fact this is the discrete-time situation studied in [17], but instead of using the shift-function \( \gamma \) described there as a model this article uses the infinitesimal shift-function \( D \). Proposition 7.9 in Section 7.9 states that in this discrete-time setting with \( D \) instead of \( \gamma \), \( X(t) \) mimics \( Y(t) \) in the sense that it has the same distribution as \( Y(t) \) given the discrete-time \( Z_t \), under a regularity condition and Consistency Assumption 4.1. The proof of Proposition 7.9 is relatively easy, because in this discrete-time setting the continuous solution to the differential equation can be written down explicitly, in terms of conditional distribution functions.

Sections 7.6–7.12 consider the situation where the probability that \( Z \) jumps at \( t \) equals zero for all \( t \). We prove that also in this case, \( X(t) \) mimics \( Y(t) \), under the conditions of Section 7.2. First, Section 7.6 prepares the discretization by constructing a series \( Z^{(n)} \), containing more and more information on the covariate- and treatment history \( Z \) as \( n \) increases. \( Z^{(n)} \) depends deterministically on \( Z \), so that no extra randomness is necessary to construct \( Z^{(n)} \). The discretization does not change \( Y(t) \); just the information on the treatment- and covariate process considered is reduced. \( Z^{(n)} \) is a covariate- and treatment history as considered in Section 7.5. Therefore, \( D^{(n)} \) can be defined as

\[
D^{(n)}(y, t; Z^{(n)}_t) = \left. \frac{\partial}{\partial h} \right|_{h=0} \left( F^{-1}_{Y_{\tau(t+h)}|Z^{(n)}_t} \circ F_{Y(t)|Z^{(n)}_t}(y) \right)
\]

(10)

and we define \( X^{(n)} \) as the continuous solution to the differential equation

\[
\frac{d}{dt} X^{(n)}(t) = D^{(n)} \left( X^{(n)}(t), t; Z^{(n)}_t \right)
\]

(11)

with final condition \( X^{(n)}(\tau) = Y \). Section 7.7 shows existence of \( D^{(n)} \) and provides two expressions for \( D^{(n)} \). Section 7.8 shows that the conditions of the discrete-time result are satisfied for the discretized situation, so that Proposition 7.9 guarantees that there exists a continuous solution \( X^{(n)}(t) \) to the differential equation (11), with final condition \( X^{(n)}(\tau) = Y \) and with the same distribution as \( Y(t) \) given \( Z^{(n)}_t \).

Sections 7.9–7.11 then prove that \( X^{(n)}(t) \) converges almost surely to \( X(t) \) as \( n \) tends to infinity, using a result from differential equation theory which bounds the difference between solutions to differential equations. The proof is concluded in Section 7.12 which shows that \( X(t) \) has the same distribution as \( Y(t) \) given \( Z_t \) because \( X^{(n)}(t) \) has the same distribution as \( Y(t) \) given \( Z^{(n)}_t \) and \( X^{(n)}(t) \) converges almost surely to \( X(t) \).
Section 7.13 indicates how the proof can be adapted to include situations where the probability that \( \tilde{Z} \) jumps at time \( t \) is zero except for at finitely many times \( t \).

7 Proof of main result

7.1 Introduction

The purpose of the current article is to prove that \( X(t) \) mimics \( Y(t) \). This result is proved in this section for non-survival outcomes. Section 7.2 states the assumptions and the precise statement of the result, and Sections 7.3–7.13 provide the proof.

7.2 Mimicking counterfactual non-survival outcomes: assumptions and result

This section provides precise conditions under which \( X(t) \) mimics \( Y(t) \). First, consider the definition of \( D \), equation (11). Notice that \( D \) involves an uncountable number of distribution functions \( F_{Y(t+h)|\tilde{Z}_t} \). In many cases conditioning on \( \tilde{Z}_t \) means conditioning on a null-event, so that these conditional distributions are not unique. Every single conditional distribution is almost surely unique (see Web-Appendix D), but because an uncountable number of them is used \( (t \) and \( h \) are continuous) this is not sufficient for overall almost sure uniqueness. Therefore the regularity conditions below should be read as: there exists a collection of conditional distribution functions \( F_{Y(t+h)|\tilde{Z}_t} \) such that all these regularity conditions are satisfied. These versions of \( F_{Y(t+h)|\tilde{Z}_t} \) are chosen in the definition of \( D \) as well as everywhere else in this article. We only consider \( h \geq 0 \), so the derivative with respect to \( h \) at \( h = 0 \) is always the right hand derivative.

With the support of a random variable \( X \) this article means those \( x \) such that for every open set \( U_x \) containing \( x \), \( P(X \in U_x) > 0 \). Let \( y_1 \) and \( y_2 \) be the lower- and upper limit of the support of the outcome of interest \( Y \). In this article, these are assumed to be finite, and moreover it is assumed that

**Assumption 7.1.** (support).

a) All \( F_{Y(t+h)|\tilde{Z}_t} \) for all \( t \geq 0 \) and for \( h \geq 0 \) have the same bounded support \([y_1, y_2]\).

b) All \( F_{Y(t+h)|\tilde{Z}_t}(y) \) for all \( t \geq 0 \) and for \( h \geq 0 \) have a continuous non-zero density \( f_{Y(t+h)|\tilde{Z}_t}(y) \) on \( y \in [y_1, y_2] \).

c) There exists an \( \varepsilon > 0 \) such that \( f_{Y(t)|\tilde{Z}_t}(y) \geq \varepsilon \) for all \( y \in [y_1, y_2] \), \( \omega \in \Omega \) and \( t \in [0, \tau] \).

The support condition may be restrictive for certain applications. Nevertheless, most real-life situations can be approximated this way, since \( y_1 \) and \( y_2 \) are can have arbitrary (finite) values and \( \varepsilon > 0 \) can be vary small. Although the support condition may well be stronger than necessary, it simplifies the analysis considerably and, for that reason, it is adopted here.
The remaining regularity conditions are smoothness conditions. They allow for non-smooth-ness where the covariate- and treatment process \( Z(t) \) jumps. This is important since if the covariate- and treatment process \( Z(t) \) jumps this can lead to a different prognosis for the person and thus to non-smoothness of the functions concerned.

**Assumption 7.2.** (continuous derivatives). For \( \omega \in \Omega \) fixed,

a) \( F_{Y^{(t+h)}|Z_t}(y) \) is \( C^1 \) in \((h,y)\) for \( y \in [y_1,y_2] \) and \( h \geq 0 \).

b) If \( Z \) does not jump in \((t_1,t_2)\) then both \( \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y^{(t+h)}|Z_t}(y) \) and \( \frac{\partial}{\partial y} F_{Y^{(t)}|Z_t}(y) \) are continuous in \((y,t)\) on \([y_1,y_2] \times [t_1,t_2] \) and can be continuously extended to \([y_1,y_2] \times [t_1,t_2] \).

Structural nested models in continuous time are meant to estimate the effect of a continuous treatment, for which the effect of a small duration is small. Then, Assumption 7.3 is a regularity condition:

**Assumption 7.3.** (bounded derivatives).

a) There exists a constant \( C_1 \) such that for all \( \omega \in \Omega \), \( t \), \( h \geq 0 \) and \( y \in [y_1,y_2] \),

\[
\frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t}(y) \leq C_1.
\]

b) There exists a constant \( C_2 \) such that for all \( \omega \in \Omega \), \( t \), \( h \geq 0 \) and \( y \in [y_1,y_2] \),

\[
\left| \frac{\partial}{\partial h} F_{Y^{(t+h)}|Z_t}(y) \right| \leq C_2.
\]

**Assumption 7.4.** (Lipschitz continuity).

a) There exists a constant \( L_1 \) such that for all \( \omega \in \Omega \) and \( t \) and \( y,z \in [y_1,y_2] \),

\[
\left| \frac{\partial}{\partial y} F_{Y^{(t)}|Z_t}(y) - \frac{\partial}{\partial z} F_{Y^{(t)}|Z_t}(z) \right| \leq L_1 |y-z|.
\]

b) There exists a constant \( L_2 \) such that for all \( \omega \in \Omega \) and \( t \) and \( y,z \in [y_1,y_2] \),

\[
\left| \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y^{(t+h)}|Z_t}(y) - \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y^{(t+h)}|Z_t}(z) \right| \leq L_2 |y-z|.
\]

The main theorem of this article is

**Theorem 7.5.** (mimicking counterfactual outcomes). Suppose that Regularity Conditions 7.1 and 7.4 are satisfied. Then \( D(y,t;Z_t) \) exists. Furthermore for every \( \omega \in \Omega \) there exists exactly one continuous solution \( X(t) \) to \( dX(t)/dt = D(X(t),t;Z_t) \) with final condition \( X(\tau) = Y \). If also Consistency Assumption 4.4 is satisfied, then this \( X(t) \) has the same distribution as \( Y^{(t)} \) given \( Z_t \) for all \( t \in [0,\tau] \).
7.2.1 Simpler regularity conditions

I state some more restrictive but simpler conditions implying all the conditions in Section 7.2:

**Assumption 7.6.** (regularity condition).

- (support).
  a) There exist finite numbers \( y_1 \) and \( y_2 \) such that all \( F_{Y(t+h)|Z|} \) have the same bounded support \([y_1, y_2]\).
  b) All \( F_{Y(t+h)|Z|}(y) \) have a continuous non-zero density \( f_{Y(t+h)|Z|}(y) \) on \( y \in [y_1, y_2] \).
  c) There exists an \( \varepsilon > 0 \) such that \( f_{Y(t)|Z|}(y) \geq \varepsilon \) for all \( y \in [y_1, y_2] \), \( \omega \in \Omega \) and \( t \in [0, \tau] \).

- (smoothness). For every \( \omega \in \Omega \)
  a) \((y, t, h) \rightarrow F_{Y(t+h)|Z|}(y)\) is differentiable with respect to \( t, y \) and \( h \) with continuous derivatives on \([y_1, y_2] \times [t_1, t_2] \times [0, \infty)\) if \( Z \) does not jump in \((t_1, t_2)\), with a continuous extension to \([y_1, y_2] \times [t_1, t_2] \times [0, \infty)\).
  b) The derivatives of \( F_{Y(t+h)|Z|}(y) \) with respect to \( y \) and \( h \) are bounded by constants \( C_1 \) and \( C_2 \), respectively.
  c) \( \frac{\partial}{\partial y} F_{Y(t)|Z|}(y) \) and \( \frac{\partial}{\partial h} \big|_{h=0} F_{Y(t+h)|Z|}(y) \) have derivatives with respect to \( y \) which are bounded by constants \( L_1 \) and \( L_2 \), respectively.

7.3 Existence of and a different expression for \( D \)

The lemma below can be used to prove existence of \( D \) and to find a useful formula for \( D \) (and later two useful formulas for \( D^{(n)} \) in Section 7.5):

**Lemma 7.7.** Suppose that \( F_h \) is a family of non-decreasing functions. Suppose that there exists a neighbourhood \( U_{0,y_0} \) of \((0, y_0)\) so that \( F_h(y) \) is differentiable with respect to \( y \) and \( h \) on \( U_{0,y_0} \cap \{ h \geq 0 \} \). For \( h = 0 \), the right hand derivative is meant. Suppose furthermore that these derivatives are continuous in \((h, y)\). If also \( F'_0(y_0) \) is non-zero then there exists a neighbourhood \( V_{0,y_0} \) of \((0, y_0)\) such that on the restriction of this neighbourhood to \( h \geq 0 \), \( F_h \) is invertible. Moreover, \( \left( \frac{\partial}{\partial h} F_h^{-1} \right) (F_h(y)) \) exists and satisfies

\[
\frac{\partial}{\partial h} F_h(y) + F'_h(y) \cdot \left( \frac{\partial}{\partial h} F_h^{-1} \right) (F_h(y)) = 0.
\]

**Proof.** Define an extension of \( F \) to

\[
\tilde{U}_{0,y_0} = \{(y, h) : h \geq 0 \text{ and } (y, h) \in U_{0,y_0}\} \cup \{(y, h) : h < 0 \text{ and } (y, -h) \in U_{0,y_0}\},
\]
an open neighbourhood of \((0, y_0)\), in the following way:

\[
\tilde{F}_h (y) = \begin{cases} 
F_h (y) & \text{if } h \geq 0 \\
2F_0 (y) - F_{-h} (y) & \text{if } h < 0.
\end{cases}
\]

Define \(\phi : U_{ho,y_0} \to \mathbb{R}^2\) as \(\phi (h, y) = (h, \tilde{F}_h (y))\). The result follows from the Local Inverse Function Theorem and direct calculation, after noticing that \(D (\phi \circ \phi^{-1})\) is the identity mapping; see Web-Appendix E for details.

Because of Assumptions (7.2a and 7.1c), Lemma (7.7) can be applied to \(F_h (y) = F_{Y(t+h)|\mathcal{Z}_t}(y)\) with \(y_0 = y\). Thus \(D\) as defined in equation (11) exists and

\[
D (y, t; \mathcal{Z}_t) = -\frac{\partial \phi}{\partial y} \bigg|_{y=0} F_{Y(t+h)|\mathcal{Z}_t}(y).
\]

### 7.4 Existence and uniqueness of \(X(t)\)

This section shows that the differential equation \(dX(t)/dt = D (X(t), t; \mathcal{Z}_t)\) with final condition \(X (\tau) = Y\) has a unique continuous solution. Fix \(\omega\) for the rest of Section 7.4. Since \(D\) may be discontinuous at the jump times of the covariate- and treatment process \(Z\), we consider the intervals between jumps of \(Z\) separately. It suffices to prove existence and uniqueness of \(X(t)\) with final condition on any interval between jumps of \(Z\), because with probability one \(Z\) only jumps finitely many times.

Hence suppose that \(Z\) does not jump in \((t_1, t_2)\) and that \(t_1\) is either a jump time of \(Z\) or \(0\) and that \(t_2\) is either a jump time of \(Z\) or \(\tau\). From equation (12) I conclude that \(D (y, t; \mathcal{Z}_t)\) is continuous on \([y_1, y_2] \times [t_1, t_2]\) because of Assumptions (7.2b and 7.1c). The differential equation has a final condition at the upper end of the interval \([t_1, t_2)\). Therefore we define \(\tilde{D}\) on \([y_1, y_2] \times [t_1, t_2]\) as

\[
\tilde{D} (y, t) = \begin{cases} 
D (y, t; \mathcal{Z}_t) & \text{if } t \in [t_1, t_2) \\
\lim_{t \downarrow t_2} D (y, t; \mathcal{Z}_t) & \text{if } t = t_2.
\end{cases}
\]

This limit exists because of Assumption (7.1a) and the extension-assumption in Assumption (7.2b). It makes \(\tilde{D}\) continuous on \([y_1, y_2] \times [t_1, t_2]\). When calculating the continuous solution to \(dX(t)/dt = D (X(t), t; \mathcal{Z}_t)\) on \([t_1, t_2]\), one means to use \(\tilde{D}\) on \([t_1, t_2]\) if \(D\) jumps at \(t_2\).

To prove existence and uniqueness of \(X\) on \([t_1, t_2]\), we apply Theorem (A.1) to the differential equation with \(\tilde{D}\). We check the conditions of Theorem (A.1) for \(\tilde{D}\). Continuity of \(\tilde{D}\) was shown in the previous paragraph. \(F_{Y(t+h)|\mathcal{Z}_t}^{-1} \circ F_{Y(t)|\mathcal{Z}_t}(y_1) = y_1\) for all \(h\) because of Assumption (7.1a) and b, so that \(D (y_1, t; \mathcal{Z}_t) = 0\). Similarly, \(D (y_2, t; \mathcal{Z}_t) = 0\). To show that equation (25) holds, notice that global Lipschitz continuity of \(\tilde{D}\) in \(y\) on \([y_1, y_2] \times [t_1, t_2]\) with Lipschitz constant \(C = L_2/\varepsilon + L_1C_2/\varepsilon^2\) follows from equation (12), since the numerator is bounded by \(C_2\) and is Lipschitz with Lipschitz constant \(L_2\) and also the denominator
is Lipschitz with Lipschitz constant $L_1$ and bounded away from 0 by $\varepsilon$ (Assumptions 7.3, 7.3b and 7.4); see Web-Appendix F]. This same constant works on $[y_1, y_2] \times [t_1, t_2]$ by continuity. By Theorem A.1 the differential equation (8) with $\tilde{D}$ has a unique solution, and this solution stays in $[y_1, y_2]$.

### 7.5 Mimicking counterfactual outcomes: discrete time

This section considers the situation where $\mathcal{Z}$, the available information on the treatment- and covariate process, can be fully described by its values at finitely many fixed time points $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_K < \tau_{K+1} = \tau$. At these time points, $Z(t)$ may jump with probability greater than zero. We prove that in this situation, $X(t)$ mimics $Y(t)$.

We assume that there exist conditional distribution functions ([2], [20]) $F_{Y(t)}|\mathcal{Z}_{\tau_k}$ satisfying the following regularity condition:

**Assumption 7.8. (smoothness).** Suppose that for $k = 0, \ldots, K$ and $t \in [\tau_k, \tau_{k+1}]$ there exist conditional distribution functions $F_{Y(t)}|\mathcal{Z}_{\tau_k}$ such that

a) For all $t \in [\tau_k, \tau_{k+1}]$, $F_{Y(t)}|\mathcal{Z}_{\tau_k}(y)$ is continuous in $y$.

b) For all $t \in [\tau_k, \tau_{k+1}]$, the support of $F_{Y(t)}|\mathcal{Z}_{\tau_k}(y)$ is an interval.

c) For $x \in [0, 1]$ fixed, $F_{Y(t)}^{-1}|\mathcal{Z}_{\tau_k}(x)$ is differentiable with respect to $t$ on $[\tau_k, \tau_{k+1}]$.

Throughout Section 7.5 fixed versions of $F_{Y(t)}|\mathcal{Z}_{\tau_k}(y)$ are used satisfying Assumption 7.8. Since $\mathcal{Z}_t$ contains the same information as $\mathcal{Z}_{\tau_k}$ for $t \in [\tau_k, \tau_{k+1}]$, we can and will choose the same versions when conditioning on $\mathcal{Z}_t$.

**Proposition 7.9. (mimicking counterfactual outcomes in discrete time).** Suppose that the treatment- and covariate process $Z$ can be fully described by its values at finitely many fixed points $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_K < \tau_{K+1} = \tau$, and suppose also that Smoothness Assumption 7.8 is satisfied. Then $D(y, t; \mathcal{Z}_t)$ as defined in equation (1) exists for all $t$. Furthermore if also Assumption 7.1 (consistency) is satisfied, then there exists a continuous solution $X(t)$ to $dX(t)/dt = D(X(t), t; \mathcal{Z}_t)$ with final condition $X(\tau) = Y$ for which $X(t)$ has the same distribution as $Y(t)$ given $\mathcal{Z}_t$.

**Proof.** For $t \in [\tau_k, \tau_{k+1})$, $D(y, t; \mathcal{Z}_t) = \frac{\partial}{\partial h}|_{h=0} \left( F_{Y(t+h)}^{-1}|\mathcal{Z}_{\tau_k} \circ F_{Y(t)}|\mathcal{Z}_{\tau_k} \right)(y)$, so existence of $D(y, t; \mathcal{Z}_t)$ on each interval $[\tau_k, \tau_{k+1})$ follows from Assumption 7.8.

Next, define $\tilde{X}$ as follows. $\tilde{X}(\tau) = Y$, and for $t \in [\tau_k, \tau_{k+1})$ ($k = 0, \ldots, K-1$),

$$\tilde{X}(t) = F_{Y(t)}^{-1}|\mathcal{Z}_{\tau_k} \circ F_{Y(\tau_{k+1})}|\mathcal{Z}_{\tau_k} \circ \ldots \circ F_{Y(\tau_{k-1})}|\mathcal{Z}_{\tau_k-1} \circ F_{Y(\tau_k)}|\mathcal{Z}_{\tau_k-1} \circ \ldots \circ F_{Y(\tau)}|\mathcal{Z}_{\tau_k}(Y).$$
\( \tilde{X}(t) \) is well-defined because of Assumption \ref{assumption:continuous_solution} and \( b \). First we show that \( \tilde{X} = X \): it is a continuous solution to \( \tilde{X}'(t) = D \left( \tilde{X}(t), t; \tilde{Z}_t \right) \) with \( \tilde{X}(\tau) = Y \). Next we show that \( \tilde{X}(t) \) has the same distribution as \( Y(t) \) given \( \tilde{Z}_t \).

Continuity of \( \tilde{X} \) on \([\tau_k, \tau_{k+1}]\) is clear from Assumption \ref{assumption:assumption_7.8}. Moreover,

\[
\lim_{t \uparrow \tau_{k+1}} \tilde{X}(t) = \lim_{t \uparrow \tau_{k+1}} F_{Y(t)}^{-1} \circ F_{Y(\tau_{k+1})} \left( \tilde{X}(\tau_{k+1}) \right) = F_{Y(\tau_{k+1})}^{-1} \circ F_{Y(t)} \left( \tilde{X}(\tau_{k+1}) \right)
\]

because of Assumption \ref{assumption:assumption_7.8}, which is equal to \( \tilde{X}(\tau_{k+1}) \) because of Assumption \ref{assumption:assumption_7.8}. Thus, \( \tilde{X}(t) \) is also continuous from the left at \( t = \tau_{k+1} \). For \( t \in [\tau_k, \tau_{k+1}] \), \( \tilde{X} \) satisfies the differential equation:

\[
\tilde{X}'(t) = \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y(t+h)}^{-1}(\tilde{Z}_{\tau_k}) \circ F_{Y(\tau_{k+1})} \left( \tilde{X}(\tau_{k+1}) \right) = \left( \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y(t+h)}^{-1}(\tilde{Z}_{\tau_k}) \circ F_{Y(t)}(\tilde{Z}_{\tau_k}) \right) \circ F_{Y(\tau_{k+1})} \circ F_{Y(t)} \left( \tilde{X}(\tau_{k+1}) \right) = D \left( \tilde{X}(t), t; \tilde{Z}_t \right),
\]

where in the second line it is used that conditioning on \( \tilde{Z}_t \) is the same as conditioning on \( \tilde{Z}_{\tau_k} \), so that \( F_{Y(t)}(\tilde{Z}_t) \circ F_{Y(\tau_{k+1})} \) is the identity because of Assumption \ref{assumption:assumption_7.8}. Thus indeed \( \tilde{X} \) is a continuous solution to \( \tilde{X}' = D \left( \tilde{X}(t), t; \tilde{Z}_t \right) \) with \( \tilde{X}(\tau) = Y \).

Next, we prove that \( \tilde{X}(t) \) has the same distribution as \( Y(t) \) given \( \tilde{Z}_t \) by induction, starting at \( t = \tau \), then \( t \in [\tau_K, \tau] \), etcetera. For \( t = \tau \), \( \tilde{X}(\tau) = Y \), so that \( \tilde{X}(\tau) \) has the same distribution as \( Y(\tau) \) given \( \tilde{Z}_t \) because of Assumption \ref{assumption:assumption_4.1}. For the induction step, suppose that for \( t \in [\tau_k, \tau] \) (for \( k = K + 1 \) read \( t = \tau \)), \( \tilde{X}(t) \) has the same distribution as \( Y(t) \) given \( \tilde{Z}_t \). Thus, \( \tilde{X}(\tau_k) \) has the same distribution as \( Y(\tau_k) \) given \( \tilde{Z}_{\tau_k} \), and hence \( \tilde{X}(\tau_k) \) also has the same distribution as \( Y(\tau_k) \) given \( \tilde{Z}_{\tau_k-1} \). Therefore Assumption \ref{assumption:assumption_7.8} implies that \( F_{Y(\tau_k)}(\tilde{Z}_{\tau_k-1}) \left( \tilde{X}(\tau_k) \right) \) is uniformly distributed on \([0,1]\) given \( \tilde{Z}_{\tau_k-1} \) (Lemma \ref{lemma:distribution_function} has a formal proof). Then \( \tilde{X}(t) = F_{Y(t)}^{-1}(\tilde{Z}_{\tau_k-1}) \circ F_{Y(\tau_k)}(\tilde{Z}_{\tau_k-1}) \left( \tilde{X}(\tau_k) \right) \) has distribution function \( F_{Y(t)}(\tilde{Z}_{\tau_k-1}) \) given \( \tilde{Z}_{\tau_k-1} \) (Lemma \ref{lemma:distribution_function} has a formal proof), so also given \( \tilde{Z}_t \). That finishes the induction step, so that indeed \( \tilde{X}(t) \) mimics \( Y(t) \) for all \( t \in [0, \tau] \).

\[\square\]

### 7.6 Discretization and choices of conditional distributions

We return to the continuous-time setting and define a discretization of the covariate- and treatment process \( Z \). Later, we will apply the result of the previous section to this discretized continuous-time setting. This section also chooses versions of the conditional distribution functions given this discretized process.
For $n$ fixed define $\tau_0^{(n)} = 0$, $\tau_1^{(n)} = \frac{1}{2n}\tau$, $\tau_2^{(n)} = \frac{2}{2n}\tau$, ..., $\tau_{2^n}^{(n)} = \frac{2^n}{2n}\tau = \tau$. Consider the grid at stage $n$ consisting of these points. This way the interval $[0, \tau]$ is split up into $2^n$ intervals of equal length, and when $n$ increases points are added in the middle of these intervals. For ease of notation, the superscript $^{(n)}$ in $\tau_k^{(n)}$ is dropped if it is clear which $n$ is meant. Define $Z_t^{(n)} = \left( Z(\tau_k^{(n)}) : 0 \leq \tau_k^{(n)} \leq t \right)$ if $Z$ takes values in a discrete space, 

$$Z_t^{(n)} = \left( \left[ \frac{1}{2n}, \frac{1}{n}\right) Z(\tau_k^{(n)}) : 0 \leq \tau_k^{(n)} \leq t, i \in \mathbb{Z} \right)$$ if $Z$ takes values in $\mathbb{R}$ and 

$$Z_t^{(n)} = \left( \left[ \frac{1}{2n}, \frac{1}{n}\right) Z(\tau_k^{(n)}) : 0 \leq \tau_k^{(n)} \leq t, i \in \mathbb{Z}, j = 1, \ldots, m \right)$$ if $Z$ takes values in $\mathbb{R}_m$.

With this discretization, the information about $Z_t$ contained in $Z_t^{(n)}$ increases with $n$: once a grid point is added it stays on the grid for $n$ larger, and the information about $Z$ in a fixed grid point also increases with $n$. Note also that $Z_t^{(n)}$ depends deterministically on $Z_t$, so that no extra randomness is necessary to construct $Z_t^{(n)}$. Thus $Z_t^{(n)}$ has the properties promised in the outline of the proof, Section 6.

Next, versions of conditional distributions are chosen. Recall $Z_{\tau_k}$ takes values in the space of cadlag functions on $[0, \tau_k]$ with the projection $\sigma$-algebra, which is the same as the Skorohod-$\sigma$-algebra (Theorem 14.5). This space is Polish (Chapter 3). Therefore, there exists a conditional distribution $P_{Z_{\tau_k}|Z_{\tau_k}^{(n)}}$ (Section 10.3 or 20). Moreover, 

$$P \left( Y^{(t+h)} \leq y \big| Z_{\tau_k}^{(n)} \right) = \int F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y) \, dP_{Z_{\tau_k}|Z_{\tau_k}^{(n)}}(z) \text{ a.s.}$$

This is a conditional distribution function: it is non-decreasing in $y$ since all $F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y)$ are non-decreasing because they are conditional distribution functions, and because of Lebesgue’s Dominated Convergence Theorem the limit for $y \to -\infty$ equals 0 and the limit for $y \to \infty$ equals 1. Therefore, the following choices can be made:

**Notation 7.10.** We choose fixed conditional distributions $P_{Z_{\tau_k}|Z_{\tau_k}^{(n)}}$. I also choose

$$F_{Y^{(t)}|Z_{\tau_k}^{(n)}}(y) = \int F_{Y^{(t)}|Z_{\tau_k}^{(n)}}(y) \, dP_{Z_{\tau_k}|Z_{\tau_k}^{(n)}}(z),$$

with $F_{Y^{(t)}|Z_{\tau_k}^{(n)}}$ as in Section 7.3, to be the version of the conditional distribution function of $Y^{(t)}$ given $Z_{\tau_k}^{(n)}$ which is used in the rest of the proof. If $s \in (\tau_k, \tau_{k+1})$, the same version for $F_{Y^{(t)}|Z_{\tau_k}^{(n)}}$ is chosen; this is possible since for $s \in (\tau_k, \tau_{k+1})$, $Z_{\tau_k}^{(n)} = Z_{\tau_k}^{(n)}$.

Notice that $Z^{(n)}$ has been constructed with values in a discrete space. This will assure that the two different expressions for $D^{(n)}$ in Section 7.7 below are equal except for at a null set which does not depend on $y$ and $t$.

### 7.7 Existence of and two expressions for $D^{(n)}$

This section proves existence of $D^{(n)}$ as defined in equation (10), Section 6. Moreover, two useful formulas for $D^{(n)}$ are proven. One is used to prove smoothness of $D^{(n)}$, the other formula is used to prove that $D^{(n)}$ converges to $D$. 

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First, existence of $D^{(n)}$ is shown. Fix $n$ and $t$, and choose $\tau_k^{(n)}$ such that $t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)})$. Define
\[
F_h(y) = F_{Y^{(t+h)}|Z_t^{(n)}}(y) = \int F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z).
\]

To apply Lemma 7.7 on $F_h(y)$, in $(h_0, y_0) = (0, y)$, we check the conditions. Clearly, $F_h(y)$ is non-decreasing. We show that $F_h(y)$ is differentiable with respect to $y$ with derivative $\frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y) = \int \frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z)$. For $\omega$ fixed, $P_{Z_t^{(n)}|Z_t^{(n)}}$ is a probability measure on $\mathcal{Z}_t$. Moreover, $\frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y)$ is bounded by $C_1$, which is integrable with respect to $P_{Z_t^{(n)}|Z_t^{(n)}}$, and also $F_{Y^{(t+h)}|Z_t^{(n)}}(y)$ is integrable with respect to $P_{Z_t^{(n)}|Z_t^{(n)}}$, since bounded by 1. Therefore, $F_h(y)$ is differentiable with respect to $y$ with derivative $\int \frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z)$.

With the same reasoning (but with Assumption 7.3b instead of 7.3a), $F_h(y)$ is differentiable with respect to $h$ with derivative $\int \frac{\partial}{\partial h} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z)$. That these derivatives of $F_h(y)$ with respect to $y$ and $h$ are continuous in $(y, h)$ follows from Lebesgue’s Dominated Convergence Theorem applied on the expressions we just derived (the conditions are satisfied because of Assumptions 7.2a and 7.3). Furthermore, $F_h^{(0)}(y) = \int \frac{\partial}{\partial y} F_{Y^{(t)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z)$ is non-zero (Assumption 7.11). Thus the conditions of Lemma 7.7 are satisfied for $F_h(y)$, and therefore $\frac{\partial}{\partial h} F_{Y^{(t+h)}|Z_t^{(n)}}(y)$ exists, and $D^{(n)}(y, t; \mathcal{Z}_t^{(n)})$ exists and satisfies

\[
D^{(n)}(y, t; \mathcal{Z}_t^{(n)}) = \frac{\partial}{\partial h} \bigg|_{h=0} \left( F_{Y^{(t)}}(y) \circ F_{Y^{(t+h)}|Z_t^{(n)}}(y) \right) = \int \frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z) = \int \frac{\partial}{\partial y} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z) = \frac{\partial}{\partial h} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z).
\]

Next, the second expression for $D^{(n)}$ is derived. We show that there exists an $\Omega' \subset \Omega$ with probability one such that

\[
D^{(n)}(y, t; \mathcal{Z}_t^{(n)}) = -\frac{E \left[ \frac{\partial}{\partial h} F_{Y^{(t+h)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z) \right]}{E \left[ \frac{\partial}{\partial y} F_{Y^{(t)}|Z_t^{(n)}}(y) \, dP_{Z_t^{(n)}|Z_t^{(n)}}(z) \right]} \quad \forall \omega \in \Omega' \forall y \forall t \forall n.
\]
The $Z_{\tau_k}^{(n)} = z$ are unique except for at $\omega$’s for which $Z_{\tau_k}^{(n)}(\omega)$ has probability zero, that is, except for $\omega$’s in

$$
\bigcup_{z: P(Z_{\tau_k}^{(n)} = z) = 0} \{\omega \in \Omega : Z_{\tau_k}^{(n)}(\omega) = z\}.
$$

Since, by construction, $Z_{\tau_k}^{(n)}$ takes only countably many values, this is a countable union of null sets and thus a null set. Define

$$
\Omega' = \Omega \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \{0, \ldots, 2^n\}} \bigcup_{z: P(Z_{\tau_k}^{(n)} = z) = 0} \{\omega \in \Omega : Z_{\tau_k}^{(n)}(\omega) = z\}, \tag{15}
$$

This set has probability one since its complement is a countable union of null sets: $\mathbb{N}$ is countable and for each $n$ there are only finitely many $k$. On this $\Omega'$ conditional probabilities given $Z_{\tau_k}^{(n)}$ are unique, for all $n$ and $\tau_k$.

Next, it is shown that equation (14) holds for $\Omega'$ as defined in equation (15). As shown in Section 7.6 there exists a conditional distribution $P_{Z_t | Z_{\tau_k}^{(n)}}$. For $t \geq \tau_k$ and $h \geq 0$, $F_h(y) := P(Y^{(t+h)} \leq y | Z_{\tau_k}^{(n)}) = \int F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y) dP_{Z_t | Z_{\tau_k}^{(n)}}(z)$ a.s.. On $\Omega'$ this version is the same as the one used in the definition of $D^{(n)}$ of equation (10), since conditional probabilities given $Z_{\tau_k}^{(n)}$ are unique on $\Omega'$. Verifying the conditions of Lemma 7.7 can be done in exactly the same way as for the first expression for $D^{(n)}$. Therefore, Lemma 7.7 implies that for $\omega \in \Omega'$ and $t \in [\tau_k, \tau_{k+1})$,

$$
D^{(n)}(y, t; Z_t^{(n)}) = \left. \frac{\partial}{\partial h} \right|_{h=0} \left( \left. \frac{F^{-1}_{Y^{(t+h)}|Z_{\tau_k}^{(n)}} \circ F_{Y^{(t)}|Z_{\tau_k}^{(n)}} \right)}{y} \right|_{h=0} \int F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y) dP_{Z_t | Z_{\tau_k}^{(n)}}(z)
$$

$$
= \left. \frac{\partial}{\partial y} \right|_{h=0} \int F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y) dP_{Z_t | Z_{\tau_k}^{(n)}}(z)
$$

$$
= \left. E \left[ \frac{\partial}{\partial h} \right|_{h=0} F_{Y^{(t+h)}|Z_{\tau_k}^{(n)}}(y) | Z_{\tau_k}^{(n)} \right]
$$

$$
= \left. E \left[ \frac{\partial}{\partial y} F_{Y^{(t)}|Z_{\tau_k}^{(n)}}(y) | Z_{\tau_k}^{(n)} \right]
$$

Equation (14) follows.

### 7.8 Applying the discrete-time result

**Lemma 7.11.** Suppose that Regularity Conditions 7.1–7.4 and Consistency Assumption 4.1 are satisfied. Then for every $n$ there exists a continuous solution $X^{(n)}(t)$ to the differential equation with $D^{(n)}$ with final condition $X^{(n)}(\tau) = Y$. $X^{(n)}(t)$ is unique on $\Omega'$ of equation (15). Furthermore, $X^{(n)}(t)$ has the same conditional distribution as $Y^{(t)}$ given $Z_{\tau_k}^{(n)}$. 

20
Proof. Fix $n$. First, we show that there exists a continuous solution $X^{(n)}(t)$ for which $X^{(n)}(t)$ has the same conditional distribution as $Y(t)$ given $\bar{Z}^{(n)}_t$, using Proposition 7.9. Thus we check that the conditional distributions $F_{Y(t)|\bar{Z}^{(n)}_{\tau_k}}$ of $Y(t)$ given $\bar{Z}^{(n)}_{\tau_k}$ chosen in Notation 7.10 satisfy Assumption 7.8. In the second paragraph of Section 7.4, we showed that $F_{Y(t)|\bar{Z}^{(n)}_{\tau_k}}(y)$ is strictly increasing and differentiable with respect to $y$ on $[y_1, y_2]$, which accounts for Assumption 7.8a and b. Just before equation (13), it was concluded that for $x \in [0,1]$ fixed, $F_{Y(t)|\bar{Z}^{(n)}_{\tau_k}}(x)$ is differentiable with respect to $t$ on $[\tau_k, \tau_{k+1}]$, which accounts for Assumption 7.8c. Hence Proposition 7.9 guarantees existence of a continuous solution $X^{(n)}(t)$ to $X^{(n)}(t)' = D^{(n)}(X^{(n)}(t), t)$ with final condition $X^{(n)}(\tau) = Y$ and with $X^{(n)}(t) \sim Y(t)$ given $\bar{Z}^{(n)}_t$.

Proposition 7.9 does not imply that $X^{(n)}(t)$ is unique. Almost sure uniqueness of $X^{(n)}$ follows with Theorem A.11 in the Appendix along the same lines as uniqueness of $X$ (see Section 7.4), but using equations (13) and (14) for $D^{(n)}$ instead of equation (12) for $D$, as follows. Fix $n$ and suppose that $t \in [\tau_k, \tau_{k+1})$. First, it is proven that $D^{(n)}$ is continuous on $[y_1, y_2] \times [\tau_k, \tau_{k+1})$ with a continuous extension to $[y_1, y_2] \times [\tau_k, \tau_{k+1}]$, using equation (13). Expression (13) for $D^{(n)}$ has an obvious extension $\tilde{D}^{(n)}$ to $[\tau_k, \tau_{k+1}]$. We prove that this $\tilde{D}^{(n)}$ is continuous on $[y_1, y_2] \times [\tau_k, \tau_{k+1}]$. To show that $\int \frac{\partial}{\partial t} F_{Y(t)|\bar{Z}_{\tau_k}=z} (y) \, dP_{\bar{Z}_{\tau_k}|\bar{Z}^{(n)}_{\tau_k}}(z)$ and $\int \frac{\partial}{\partial y} F_{Y(t)|\bar{Z}_{\tau_k}=z} (y) \, dP_{\bar{Z}_{\tau_k}|\bar{Z}^{(n)}_{\tau_k}}(z)$ are continuous in $(y, t)$ Lebesgue’s Dominated Convergence Theorem can be used, as follows.

$$\frac{\partial}{\partial h} \bigg|_{h=0} F_{Y((t+h)|\bar{Z}_{\tau_k}=z} (y) = \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y((\tau_k+h)|\bar{Z}_{\tau_k}=z} (y)$$

and

$$\frac{\partial}{\partial y} F_{Y(t)|\bar{Z}_{\tau_k}=z} (y) = \frac{\partial}{\partial y} F_{Y((\tau_k+t)|\bar{Z}_{\tau_k}=z} (y)$$

are continuous in $(y, t)$ because of Assumption 7.2a. Both these derivatives are bounded because of Assumption 7.3. Therefore Lebesgue’s Dominated Convergence Theorem implies that the integrals of these derivatives with respect to the measure $\mu = P_{\bar{Z}_{\tau_k}|\bar{Z}^{(n)}_{\tau_k}}$ are continuous in $(y, t)$. Because of Assumption 7.10 the denominator of $\tilde{D}^{(n)}$ is non-zero for $y \in [y_1, y_2]$, so that indeed $\tilde{D}^{(n)}$ is continuous in $(y, t)$ on $[y_1, y_2] \times [\tau_k, \tau_{k+1}]$.

Next, it is shown that $D^{(n)}$ is Lipschitz continuous in $y$ on $[y_1, y_2] \times [\tau_k, \tau_{k+1}]$ with Lipschitz constant $L_2/\varepsilon + C_2 L_1/\varepsilon^2$ for all $\omega \in \Omega'$, with $\Omega'$ as in equation (13). Expression (13) for $D^{(n)}$ on $\Omega'$ has an obvious extension $\tilde{D}^{(n)}$ to $[\tau_k, \tau_{k+1}]$. That this $\tilde{D}^{(n)}$ is Lipschitz continuous in $y$ on $[y_1, y_2] \times [\tau_k, \tau_{k+1}]$ with Lipschitz constant $L_2/\varepsilon + C_2 L_1/\varepsilon^2$ on $\Omega'$ follows the same way as for $D$ in Section 7.4. Because of Assumption 7.11, the denominator is bounded away from 0 for $y \in [y_1, y_2]$, and because of Assumption 7.11a, the numerator is equal to zero for $y = y_1$ and for $y = y_2$. Hence, on $\Omega'$, $\tilde{D}^{(n)}(y_1, t) = \tilde{D}^{(n)}(y_2, t) = 0$. Therefore Theorem A.11 implies that, on $\Omega'$, there exists a unique solution to the differential equation with $\tilde{D}^{(n)}$ on $[\tau_k, \tau_{k+1}]$, and this solution stays in $[y_1, y_2]$. Since for $n$ fixed there are only finitely many $\tau_k$, the same is true on $[0, \tau]$. \qed
7.9 Bounding the difference between $X$ and $X^{(n)}$ in terms of $D$ and $D^{(n)}$

To bound the difference between $X$ and $X^{(n)}$ in terms of $D$ and $D^{(n)}$, Theorem A.1 is applied on $y = X(t)$ and $z = X^{(n)}(t)$. Since we need that both $D$ and $D^{(n)}$ are continuous, we apply Theorem A.1 on the intervals between the jumps of $Z$ and the grid points $\tau_k^{(n)}$. Fix $n$ and restrict $\omega$ to $\omega \in \Omega'$, with $\Omega'$ the set of probability one as defined in equation (15), so that the expression for $D^{(n)}$ of equation (14) can be used. The bound will thus hold almost surely. To focus attention on the differential equations, the $\overline{Z}_t$'s and $\overline{Z}^{(n)}_t$'s in $D$ and $D^{(n)}$ are skipped below.

Suppose that $(t_1, t_2)$ is such an interval including no jumps of $Z$ and no grid points at stage $n$. We check the conditions of Theorem A.1 for $y = X(t)$ and $z = X^{(n)}(t)$. Section 7.4 already showed that $D : [y_1, y_2] \times [t_1, t_2] \rightarrow \mathbb{R}$ has a continuous extension $\bar{D} : [y_1, y_2] \times [t_1, t_2] \rightarrow \mathbb{R}$ which satisfies the conditions of Theorem A.1, with $C$ the constant function $L_2/\varepsilon + C_2L_1/\varepsilon^2$, and in the proof of Lemma 7.11 in Section 7.8, it was shown that on $\Omega'$ the same is true for $D^{(n)}$. Therefore Theorem A.1 implies that for $t \in [t_1, t_2]$, with $C = L_2/\varepsilon + C_2L_1/\varepsilon^2$ as above,

$$|X^{(n)}(t) - X(t)| \leq e^{\int_{t_1}^{t_2} C} \int |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds \leq e^{C(t_2 - t_1)} |X^{(n)}(t_2) - X(t_2)| + \int_t^{t_2} e^{C(s-t)} |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds. \quad (16)$$

If $Z$ does not jump in $[(1 - 1/2^n)\tau, \tau]$, (16) can be applied on $[(1 - 1/2^n)\tau, \tau]$, and since $X^{(n)}(\tau) = X(\tau) = Y$ it follows that on $[(1 - 1/2^n)\tau, \tau]$,

$$|X^{(n)}(t) - X(t)| \leq \int_t^\tau e^{C(s-t)} |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds. \quad (17)$$

If $Z$ does not jump after $(1 - 2/2^n)\tau$ one can also apply (16) on $[(1 - 2/2^n)\tau, (1 - 1/2^n)\tau]$, and using equation (17) for $t = (1 - 2/2^n)\tau$, it follows that equation (17) also holds on $[(1 - 2/2^n)\tau, (1 - 1/2^n)\tau]$:

$$|X^{(n)}(t) - X(t)| \leq e^{C((1 - 1/2^n)\tau - t)} \int_{(1 - 1/2^n)\tau}^\tau e^{C(s-(1 - 1/2^n)\tau)} |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds \leq \int_t^{(1 - 1/2^n)\tau} e^{C(s-t)} |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds \leq \int_t^{(1 - 1/2^n)\tau} e^{C(s-t)} |D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds.$$
If $Z$ does not jump in $((1 - m/2^n)\tau, \tau]$ and $t \in ((1 - m/2^n)\tau, \tau]$ then, with the same reasoning, equation (17) holds on $t \in ((1 - m/2^n)\tau, \tau]$. Suppose now that $Z$ jumps in $((1 - (m + 1)/2^n)\tau, (1 - m/2^n)\tau]$. Then this interval can be split up into the part before and the part after the jump, so that, again with the same reasoning as before and since both $X^{(n)}$ and $X$ are continuous in $t$, equation (17) still holds.

With probability one there are at most finitely many jump times of $Z$, so that equation (17) holds almost surely for all $t$, and even

$$
\sup_{t \in [0, \tau]} |X^{(n)}(t) - X(t)| \leq \sup_{t \in [0, \tau]} \int_t^\tau e^{C_i(s - t)}|D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)|ds
$$

$$
= \int_0^\tau e^{C_i s}|D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)|ds \quad \text{a.s.} \quad (18)
$$

### 7.10 Convergence of $D^{(n)}$ to $D$

This section proves that $D^{(n)}(y, t; \overline{Z}_t^{(n)})$ converges almost surely to $D(y, t; \overline{Z}_t)$, for fixed $(y, t) \in [y_1, y_2] \times [0, \tau]$. From equations (12) and (14) it follows that

$$
D(y, t; \overline{Z}_t) = -E\left[\frac{\partial}{\partial h}\bigg|_{h=0} F_{Y(t+h)|\overline{Z}_t}(y) \bigg| \overline{Z}_t^{(n)} \right] \quad \text{a.s.}
$$

and

$$
D^{(n)}(y, t; \overline{Z}_t^{(n)}) = -E\left[\frac{\partial}{\partial y}\bigg|_{h=0} F_{Y(t+h)|\overline{Z}_t}(y) \bigg| \overline{Z}_t^{(n)} \right] \quad \text{a.s.}
$$

Lévy’s Upward Theorem (see e.g. [33] page 134) can be applied to the denominator and the numerator of $D^{(n)}$, since both $\frac{\partial}{\partial h}\bigg|_{h=0} F_{Y(t+h)|\overline{Z}_t}(y)$ and $\frac{\partial}{\partial y} F_{Y(t)|\overline{Z}_t}(y)$ are bounded (Assumption 7.3). Lévy’s Upward Theorem leads to

$$
E\left[\frac{\partial}{\partial h}\bigg|_{h=0} F_{Y(t+h)|\overline{Z}_t}(y) \bigg| \overline{Z}_t^{(n)} \right] \rightarrow E\left[\frac{\partial}{\partial h}\bigg|_{h=0} F_{Y(t+h)|\overline{Z}_t}(y) \bigg| \sigma \left(\bigcup_{n=1}^\infty \overline{Z}_t^{(n)}\right) \right] \quad \text{a.s.}
$$

and

$$
E\left[\frac{\partial}{\partial y} F_{Y(t)|\overline{Z}_t}(y) \bigg| \overline{Z}_t^{(n)} \right] \rightarrow E\left[\frac{\partial}{\partial y} F_{Y(t)|\overline{Z}_t}(y) \bigg| \sigma \left(\bigcup_{n=1}^\infty \overline{Z}_t^{(n)}\right) \right] \quad \text{a.s.}
$$

as $n \rightarrow \infty$. The conditioning on $\sigma \left(\bigcup_{n=1}^\infty \overline{Z}_t^{(n)}\right)$ can be replaced by conditioning on $\overline{Z}_t$ in both expressions, because of Lemma A.2 in the Appendix. Since moreover the denominators are bounded away from 0 (Assumption 7.4), the Continuous Mapping Theorem implies that, for fixed $(y, t) \in [y_1, y_2] \times [0, \tau]$,

$$
D^{(n)}(y, t; \overline{Z}_t^{(n)}) \rightarrow D(y, t; \overline{Z}_t) \quad \text{a.s..} \quad (19)
$$
7.11 \( X^{(n)}(t) \) converges to \( X(t) \) and \( X(t) \) is measurable

To show that \( X^{(n)}(t) \) converges almost surely to \( X(t) \) and that \( X(t) \) is measurable, the bound of equation (18) and almost sure convergence of \( D^{(n)}(y, t) \) to \( D(y, t) \) for \((y, t)\) fixed of equation (19) are the starting point.

First it is proven that for \( s \) fixed, \( D^{(n)} \left( X^{(n)}(s), s \right) - D \left( X^{(n)}(s), s \right) \) converges almost surely to 0. Recall from Section 7.4 that \( D^{(n)} \) implies that for fixed \( s, \) define (for details see Web-Appendix H).

\[ \int_{\Omega} D : [y_1, y_2] \times [t_1, t_2] \to \mathbb{R} \] has a continuous extension \( \tilde{D} : [y_1, y_2] \times [t_1, t_2] \to \mathbb{R} \) which is Lipschitz continuous in \( y \) with Lipschitz constant \( L_2/\varepsilon + C_2L_1/\varepsilon^2 \). Recall also that in the proof of Lemma 7.11 in Section 7.8 it was shown that on \( \Omega' \), the set of probability one of equation (15), the same is true for \( D^{(n)} \).

Therefore, the pointwise almost sure convergence of \( D^{(n)}(y, t) \) to \( D(y, t) \) of equation (19) implies that for fixed \( s \) indeed

\[ \left| D^{(n)}(X^{(n)}(s), s) - D(X^{(n)}(s), s) \right| \to 0 \quad \text{a.s.} \] (20)

(for details see Web-Appendix H).

To show that equation (20) implies that the bound of (18) converges almost surely to 0, define

\[ A = \left\{ (s, \omega) \in [0, \tau] \times \Omega : \left| D^{(n)}(X^{(n)}(s), s) - D(X^{(n)}(s), s) \right| \to 0 \right\} , \]

with \( A_s \) its section at \( s \) and \( A_\omega \) its section at \( \omega \). Then

\[ A_s = \left\{ \omega \in \Omega : \left| D^{(n)}(X^{(n)}(s), s) - D(X^{(n)}(s), s) \right| \to 0 \right\} \]

has probability one because equation (20). Therefore, using Fubini’s Theorem, with \( \lambda \) the Lebesgue-measure on \([0, \tau] \),

\[ (\lambda \times P)(A) = \int_{(0, \tau)} P(A_s) \, d\lambda (s) \]

\[ = \int_{(0, \tau)} 1 \, d\lambda (s) = \tau. \]

Also by Fubini’s Theorem,

\[ (\lambda \times P)(A) = \int \lambda (A_\omega) \, dP (\omega) ; \]

so that since \( \lambda (A_\omega) \leq \tau, \lambda (A_\omega) = \tau \) \( P \)-almost everywhere. This shows that for \( P \)-almost all \( \omega, A_\omega \) has measure \( \tau \). So for \( P \)-almost all \( \omega, \left| D \left( X^{(n)}(s), s \right) - D^{(n)} \left( X^{(n)}(s), s \right) \right| \) converges to 0 for \( \lambda \)-almost all \( s \). Moreover, because of expression (12) for \( D \) and expression (14) for \( D^{(n)} \) and Assumptions 7.3b, and 7.1c, \( e^{C \cdot s} \left| D (\cdot, s) - D^{(n)} (\cdot, s) \right| \) is bounded by \( 2e^{C \cdot s} C_2/\varepsilon \) on \( \Omega' \). Therefore for almost all \( \omega \) Lebesgue’s Dominated Convergence Theorem can be applied on the integral of \( e^{C \cdot s} \left| D \left( X^{(n)}(s), s \right) - D^{(n)} \left( X^{(n)}(s), s \right) \right| \) with respect
to \( \lambda, \int_{[0,\tau]} e^{C \cdot s} \left| D \left( X^{(n)} (s), s \right) - D^{(n)} \left( X^{(n)} (s), s \right) \right| ds \), implying that for almost all \( \omega \) this integral converges to 0 as \( n \to \infty \). With equation (18), this implies that

\[
\sup_{t \in [0,\tau]} \left| X^{(n)}(t) - X(t) \right| \to 0 \quad \text{a.s.} \tag{21}
\]

Since the almost sure limit of a sequence of random variables is measurable if the \( \sigma \)-algebra is complete, measurability of \( X(t) \) follows immediately from measurability of the \( X^{(n)} \).

### 7.12 Conclusion

This section shows that since \( X^{(n)}(t) \sim Y(t) \) given \( \overline{Z}_t^{(n)} \) (see Section 7.8) and \( X^{(n)}(t) \to X(t) \) a.s. (see Section 7.11), \( X(t) \sim Y(t) \) given \( \overline{Z}_t \). This completes the proof.

It is well-known (see e.g. [32]; Lemma D.10 provides a formal proof for this conditional version) that \( X(t) \sim Y(t) \) given \( \overline{Z}_t \) if

\[
E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] = 0 \quad \text{a.s.}
\]

for every bounded Lipschitz continuous function \( f : \mathbb{R} \to \mathbb{R} \). Suppose without loss of generality that \( f \) is bounded by 1 and has Lipschitz constant \( L \). Then, using the triangle inequality,

\[
\left| E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] \right| \leq E \left[ \left\vert f \left( X(t) \right) - f \left( X^{(n)}(t) \right) \right\vert \left\vert \overline{Z}_t^{(n)} \right. \right] - E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t \right. \right] \]

+ \left. \left| E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] - E \left[ f \left( X^{(n)}(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] \right| \right| + \left. \left| E \left[ f \left( X^{(n)}(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] \right| \right| \right|

Because of Jensen’s inequality, the second term is bounded by \( E \left[ \left\vert f \left( X(t) \right) - f \left( X^{(n)}(t) \right) \right\vert \left\vert \overline{Z}_t^{(n)} \right. \right], \) which is bounded by \( E \left[ L \left\vert X(t) - X^{(n)}(t) \right\vert \wedge 2 \left\vert \overline{Z}_t^{(n)} \right. \right] \) since \( f \) is Lipschitz continuous with Lipschitz constant \( L \) and bounded by 1. Because \( X^{(n)}(t) \sim Y(t) \) given \( \overline{Z}_t^{(n)} \), the third term is equal to \( E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] \right| \). Therefore,

\[
\left| E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] \right| \leq E \left[ \left\vert f \left( X(t) \right) - f \left( X^{(n)}(t) \right) \right\vert \left\vert \overline{Z}_t^{(n)} \right. \right] + \left. \left| E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t \right. \right] - E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] \right| \right| + \left. \left| E \left[ f \left( X^{(n)}(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] - E \left[ f \left( Y(t) \right) \left\vert \overline{Z}_t \right. \right] \right| \right| \right| \right| \right| a.s. \tag{22}
\]

We show that the right hand side converges in probability to zero. On the first and the last term, Lévy’s Upward Theorem (see e.g. [33] page 134) can be applied, since the integrands are bounded by 1. Lévy’s Upward Theorem leads to

\[
E \left[ f \left( X(t) \right) \left\vert \overline{Z}_t^{(n)} \right. \right] \to E \left[ f \left( X(t) \right) \left\vert \sigma \left( \bigcup_{n=1}^{\infty} \overline{Z}_t^{(n)} \right) \right. \right]
\]

25
and

\[ E \left[ f(Y(t)) \mid Z_t^{(n)} \right] \to E \left[ f(Y(t)) \mid \sigma \left( \bigcup_{n=1}^{\infty} Z_t^{(n)} \right) \right] \]

as \( n \to \infty \). Thus, with Lemma A.2 in the Appendix, both the first and the last term of equation (22) converge to 0 almost surely. The second term converges to 0 in probability since it is almost surely non-negative and its expectation converges to 0:

\[ E \left( E \left[ L \mid X(t) - X^{(n)}(t) \right] \wedge 2 \mid Z_t^{(n)} \right) = E \left( L \mid X(t) - X^{(n)}(t) \right) \wedge 2 \to 0 \]

because of Lebesgue’s Dominated Convergence Theorem and the fact that \( X^{(n)}(t) \) converges almost surely to \( X(t) \).

Thus \( |E \left[ f(X(t)) \mid Z_t \right] - E \left[ f(Y(t)) \mid Z_t \right] | \) is bounded by a random variable which converges in probability to 0. Hence, this first random variable is almost surely equal to 0. Therefore, indeed \( X(t) \) mimics \( Y(t) \) in the sense that \( X(t) \) has the same distribution as \( Y(t) \) given \( Z_t \).

### 7.13 Mimicking counterfactual outcomes: discrete-continuous time

In certain situations there are specific times \( t \) with \( P(t \text{ is a jump time of } Z) > 0 \). For finitely many such times \( t \), the proof in Section 7 can be adapted by adding these finitely many times to the grid, for each \( n \).

### 8 Mimicking counterfactual survival outcomes

#### 8.1 Introduction

This section indicates how to prove that \( X(t) \) mimics \( Y(t) \) in the sense that \( X(t) \) has the same distribution as \( Y(t) \) given the covariate- and treatment history \( Z_t \), under conditions aimed at survival. The conditions are similar to the ones in Section 7, but adapted to survival as the outcome of interest. The proof also follows roughly the same lines as the one for other outcomes, but some changes are necessary. A full proof can be found in Web-Appendix B.

If covariates and treatment were measured at time \( t \), it cannot be avoided to include in \( Z_t \) whether or not a person was alive at time \( t \): what are a person’s covariates if he or she is dead? Therefore we include in \( Z(t) \) an indicator for whether or not a person is alive at time \( t \). Thus if a person died at or before time \( t \), the survival time can be read from \( Z_t \).

The conditions in Section 7 usually exclude survival as the outcome of interest, since if the outcome is survival the Support Condition 7.14 saying that all \( F_{Y(t+h)}[Z_t] \) have the same bounded support \([y_1, y_2]\), will not hold: \( Z_t \) includes the covariate-measurements and treatment until time \( t \), and given that a person is dead at time \( t \) and given his or her survival time, the distribution of this survival time cannot have the fixed support \([y_1, y_2]\), independent of \( t \). Also given that a person is alive at time \( t \), the survival time often does
not have the fixed support \([y_1, y_2]\): one often expects that \(t\) is the left limit of the support, and obviously the left limit of the support should be greater than or equal to \(t\).

I make two extra assumptions. The first is a straightforward consistency assumption, stating that stopping treatment after death does not change the survival time. The second extra assumption states that there is no instantaneous effect of treatment at the time the person died (notice that the difference between \(Y(Y)\), the outcome with treatment stopped at the survival time \(Y\), and \(Y\) is in treatment at time \(Y\)).

**Assumption 8.1.** (consistency). \(Y(t) = Y\) on \(\{\omega : Y ≤ t\} \cup \{\omega : Y(t) ≤ t\}\).

**Assumption 8.2.** (no instantaneous effect of treatment at the time the person died). \(Y(t) = Y\) on \(\{\omega : Y = t\} \cup \{\omega : Y(t) = t\}\).

As can be expected, these assumptions imply that treatment in the future does not cause or prevent death at present, see Web-Appendix B.

For survival outcomes this article uses the following minor adaptation of the definition of \(D\),

\[
D(y, t; \overline{Z}_t) = \begin{cases} 
0 & \text{if } \overline{Z}_t \text{ indicates the person is dead at } t \text{ or } y < t \\
\frac{∂}{∂t} |_{h=0} \left( F_{Y(t+h)|\overline{Z}_t}^{-1} \circ F_{Y(t)|\overline{Z}_t} \right)(y) & \text{otherwise, for } y > t \\
\lim_{y\downarrow t} D(y, t; \overline{Z}_t) & \text{otherwise, for } y = t,
\end{cases}
\]

as we explain now. First remark that considering the interpretation of \(D(y, t; \overline{Z}_t)\) as the infinitesimal effect of a short duration of treatment directly after \(t\) on survival, \(D(y, t; \overline{Z}_t)\) should be zero if \(\overline{Z}_t\) indicates the person is dead at time \(t\). Although in that case indeed \(F_{Y(t+h)|\overline{Z}_t}\) and \(F_{Y(t)|\overline{Z}_t}\) are almost surely the same for every \(h ≥ 0\), since withholding treatment after death does not change the survival time, \(F_{Y(t+h)|\overline{Z}_t}^{-1}\) will often not exist. Therefore if \(\overline{Z}_t\) indicates the person is dead at time \(t\), this article just formally defines \(D(y, t; \overline{Z}_t)\) to be zero.

Next consider \(y < t\). Notice that considering the interpretation of \(D(y, t; \overline{Z}_t)\) as the infinitesimal effect of treatment directly after time \(t\) on the survival-quantile \(y\), \(D(y, t; \overline{Z}_t)\) should be zero for \(y < t\) since treatment at or after time \(t\) should not cause or prevent death at or before time \(t\), so it should not affect quantiles of the survival curve before time \(t\). Indeed if \(\overline{Z}_t\) indicates that the person is alive at time \(t\), \(F_{Y(t+h)|\overline{Z}_t}(y) = F_{Y|\overline{Z}_t}(y) = 0\) for \(y ≤ t\) for all \(h ≥ 0\), but also for these \(y\), \(F_{Y(t+h)|\overline{Z}_t}(y)\) often does not exist. Therefore, this article defines \(D(y, t; \overline{Z}_t) = 0\) for \(y < t\). In order to make \(D\) continuous on \(y ≥ t\) in between the jump times of \(Z\), we define \(D(t, t; \overline{Z}_t) = \lim_{y\downarrow t} D(y, t; \overline{Z}_t)\). This limit exists under the conditions in Section 8.2. It is not necessarily equal to zero.

Notice that the area where \(D\) is possibly non-zero is \((y, t) ∈ [0, \infty) × [0, \min\{Y, \tau\}] : y ≥ t\). Therefore if \(Y < \tau\), the solution to the differential equation \(X(t) = \text{equal to } Y\) for \(t ∈ [Y, \tau]\). An example of such \(X(t)\) is shown in Figure 1 (right).

In the case of a survival outcome, right censoring is common. For right censoring, \([26]\) proposed the artificial censoring estimator. A slight adaptation of this estimator is presented in Section 8.4.
8.2 Mimicking counterfactual survival outcomes: assumptions and result

This section presents precise conditions under which \(X(t)\) mimics \(Y(t)\), for survival outcomes, following Section 7.2.1 (Web-Appendix B provides conditions similar to Section 7.2). We choose versions of \(F_{Y(t+h)|Z_t}\) that (a) are consistent with the fact that treatment after death is irrelevant, and (b) satisfy all regularity conditions below. These versions are used in the definition of \(D\) for survival outcomes, and everywhere in the proof.

**Assumption 8.3.** (Regularity conditions).

- (support). There exists a finite number \(y_2 \geq \tau\) such that
  
  a) If \(Y > t\), all \(F_{Y(t+h)|Z_t}\), for \(h \geq 0\) and \(t \in [0, \tau]\), have support \([t, y_2]\).
  
  b) If \(Y > t\), all \(F_{Y(t+h)|Z_t}\), for \(h \geq 0\) and \(t \in [0, \tau]\), have a continuous non-zero density \(f_{Y(t+h)|Z_t}(y)\) on \(y \in [t+h, y_2]\).
  
  c) There exists an \(\varepsilon > 0\) such that for all \(\omega \in \Omega\) and \(t\) with \(Y > t\), \(f_{Y(t)|Z_t}(y) > \varepsilon\) for \(y \in [t, y_2]\).

- (smoothness). For every \(\omega \in \Omega\)
  
  a) If \(Z\) does not jump in \((t_1, t_2)\) and \(Y > t_1\), the restriction of \((y, t, h) \rightarrow F_{Y(t+h)|Z_t}(y)\) to \(\{(y, t, h) \in [t_1, y_2] \times [t_1, t_2] \times \mathbb{R}_{\geq 0} : y \geq t + h\}\) is \(C^1\) in \((y, t, h)\).
  
  b) The derivatives of \(F_{Y(t+h)|Z_t}(y)\) \((y > t + h)\) with respect to \(y\) and \(h\) are bounded by constants \(C_1\) and \(C_2\), respectively.
  
  c) \(\frac{\partial}{\partial y} F_{Y(t)|Z_t}(y)\) and \(\frac{\partial}{\partial h}\bigg|_{h=0} F_{Y(t+h)|Z_t}(y)\) \((y > t)\) have derivatives with respect to \(y\) which are bounded by constants \(L_1\) and \(L_2\), respectively.
  
 d) For all \(\omega \in \Omega\) and \(t\) with \(Y > t\), \(F_{Y|Z_t}(y)\) is continuous and strictly increasing on its support \([t, y_2]\).

**Theorem 8.4.** Suppose that Regularity Condition 8.3 is satisfied. Then \(D(y, t; Z_t)\) as defined in equation (23) exists. Furthermore, for every \(\omega \in \Omega\) there exists exactly one continuous solution \(X(t)\) to \(dX(t)/dt = D(X(t), t; Z_t)\) with final condition \(X(\tau) = Y\). If also Assumptions 4.1, 8.1 and 8.2 (consistency and no instantaneous treatment effect at time of death) are satisfied then this \(X(t)\) has the same distribution as \(Y(t)\) given \(Z_t\) for all \(t \in [0, \tau]\).

8.3 Outline of the proof

The proof of Theorem 8.4 follows the same lines as the proof of Theorem 7.5. The one essential difference between survival outcomes and non-survival outcomes is: if \(Z_t\) indicates the person is alive at time \(t\), \(X(t)\) should be greater than \(t\), since we want \(X(t)\) to have the same distribution as \(Y(t)\) given \(Z_t\) \((Y(t) > t)\) in that case because of Consistency.
Assumption 8.1). This leads to an additional problem in the proof for the continuous-time case, namely: how to prove that the solution stays above the line \( y = t \) for \( t \in [0, Y] \)? I solve this additional problem in Web-Appendix B by showing that, under the assumptions of Section 8.2, \( D(t, t; \overline{Z}_t) \leq 1 \). In addition, extra technical problems arise because the smoothness conditions have to be adapted to the survival setting; see Web-Appendix B for details.

### 8.4 Survival outcomes and right censoring

In the case of a survival outcome, right censoring is common. [26] proposed the artificial censoring estimator for administrative censoring. That is censoring due to end-of-follow-up because the study ends. The idea behind artificial censoring is that, instead of adding \( X(t) \) or \( X(0) \) to the model for predicting treatment changes (see Theorem 5.2), one could add a function \( \tilde{X}(0) \) of \( X(0) \) and the censoring time \( C \), which is observed for all patients. The artificial censoring estimator treats the censoring time \( C \) as a baseline covariate. This is justified in the case of censoring due to study closure, because in this case \( C \) only depends on the date a patient enrolled in the study. Conditional on the value of \( \overline{Z}_{t-} \), functions of \( X(t) \) and \( \overline{Z}_{t-} \) are not predictive of treatment changes (Theorem 5.2). Therefore, conditional on \( \overline{Z}_{t-} \), \( \tilde{X}(0) \) is not predictive of treatment changes either. This produces an estimation procedure for \( \psi \) analogous to that in Theorem 5.2, but that allows for right censoring.

We slightly adapt this procedure, and propose to add a function of \( X(t) \) and \( C \) to the model for the prediction of treatment changes. In particular, for \( D \) as in equation (3) and for \( \min(Y, C) \geq t \), we propose to add to the prediction model of treatment changes the function \( \tilde{X}(t, \psi) = \min(X(\psi)(t), C(t, \psi)) \), with

\[
C(t, \psi) = \begin{cases} 
C & \text{if } \psi \geq 0 \\
t + e^{\psi}(C - t) & \text{if } \psi < 0.
\end{cases}
\]

As required, \( \tilde{X}(t, \psi) \) is a function of \( X(\psi)(t) \) and \( C \). In addition, we will show that both for the case that \( \psi \geq 0 \) and for the case that \( \psi < 0 \), \( \tilde{X}(t, \psi) \) is observed for all patients. This follows from the fact that

\[
\tilde{X}(t, \psi) = \min(X^*(t, \psi), C(t, \psi)), \quad \text{with } X^*(t, \psi) = t + \int_t^{\min(Y,C)} e^{\psi\text{ no prophylaxis at } s} ds, \quad (24)
\]

which is observed for all patients. For \( \psi \geq 0 \), equation (24) follows from

\[
\tilde{X}(t, \psi) = \min(t + \int_t^Y e^{\psi\text{ no prophylaxis at } s} ds, C) \\
= \min(t + \int_t^Y e^{\psi\text{ no prophylaxis at } s} ds, t + \int_t^C e^{\psi\text{ no prophylaxis at } s} ds, C) \\
= \min(X^*(t, \psi), C(t, \psi)),
\]
where for the second equality we used that for \( \psi \geq 0 \), \( t + \int_t^C e^{\psi \text{no prophylaxis at } s \text{ ds} \geq C} \). For \( \psi < 0 \), equation (24) follows from

\[
\tilde{X}(t, \psi) = \min(t + \int_t^Y e^{\psi \text{no prophylaxis at } s \text{ ds}}, t + e^\psi (C - t))
\]

\[
= \min(\tilde{X}^*(t, \psi), C(t, \psi)),
\]

where for the second equality we used that for \( \psi < 0 \), \( t + \int_t^C e^{\psi \text{no prophylaxis at } s \text{ ds} \geq C} \). For \( \psi < 0 \), some patients are “artificially” censored, since if \( C > t \), \( C(t, \psi) = t + e^\psi (C - t) \). Artificial censoring produces a subclass of the estimators considered in Theorem 5.2 allowing \( h_t \) to depend on \( \psi \): \( h_{t, \psi}(X_\psi(t), \mathbb{Z}_{t-}) = 1_{\min(Y,C) \geq t} h_t(\min(X_\psi(t), C(t, \psi)), \mathbb{Z}_{t-}) \) (notice that \( 1_{\min(Y,C) \geq t} \) is a function of \( \mathbb{Z}_{t-} \)). In general, one could add to the prediction model for treatment changes any function of \( X_\psi(t) \) and \( C \) that is observed for all patients. \[26\] suggests to also consider adding \( \Delta(t, \psi) = 1_{\tilde{X}(t, \psi) \leq C(t, \psi)} \) to the model for the prediction of treatment changes. Since both \( \tilde{X}(t, \psi) \) and \( C(t, \psi) \) are observed for all patients, so is \( \Delta(t, \psi) \). Thus, the above reasoning shows that this procedure leads to consistent estimation of the treatment effect as well.

The procedure above can easily be adapted to for example model (5), by replacing \( C(t, \psi) \) accordingly. To be more specific, for that case one could use

\[
C(t, \psi) = t + e^{\min(\psi_1, 0) + \min(\psi_2, 0) + \min(\psi_3, 0)} (C - t).
\]

9 Simulation study

In the simulation study, we calibrated the distributions of the variables and the parameter values to HIV/AIDS data, perhaps the most salient example of application of structural nested models in the empirical literature. We focus on the first two years since HIV diagnosis. Time zero is the time of HIV diagnosis. The outcome variable is the CD4 count, a commonly used marker of the state of the immune system of HIV-positive patients. The usual treatment for HIV-positive patients is ART, antiretroviral treatment. ART is not always initiated immediately after diagnosis. ART initiation time often depends on the last measured CD4 count. When the CD4 count is at or below 350 copies/ml, HIV-positive patients are much more likely to initiate ART than when the CD4 count is above 350 copies/ml. Web-Appendix C describes how we generated the data for the simulation study in detail, including distributions and parameter values. This section provides an overview.

In this simulation study no one is treated at time zero, and once treatment is initiated, it is never stopped. \( Y^{(t)} \) is the counterfactual outcome had treatment been as given in reality until time \( t \), and continued or initiated after that. For example, if treatment was initiated by time \( t \) for a particular patient, \( Y^{(t)} \) is the observed outcome for that patient,
since he or she was already treated at time \( t \) and treatment is never stopped. On the other hand, if treatment was not initiated by time \( t \), \( Y(t) \) is the outcome had treatment been initiated at time \( t \). Thus, in the definition of \( Y(t) \) in Section \( \square \) the switch at time \( t \) to “some kind of baseline treatment regime \( \theta \)” is, in this case, “treat continuously” from time \( t \) onwards. In the simulations, we study a setting with \( t \in [0, 2] \). The subscript \( t \) indicates the treatment initiation time, so for example \( L_{1,t} \) indicates \( (\text{counterfactual covariates at time 1 under treatment started at time } t) \). Similarly, the subscript \( \infty \) indicates (counterfactual) variables under no treatment. For example, \( L_{2,\infty} \) indicates (counterfactual) covariates at time 2 under no treatment. In the simulation design, the counterfactual covariates \( L \) are as follows:

\[
\begin{align*}
L_0 &= \tilde{L}_0 + e_0, \\
L_{1,\infty} &= \tilde{L}_0 - \beta_0 + e_{1,\infty}, \\
L_{2,\infty} &= \tilde{L}_0 - 2\beta_0 + e_{2,\infty}, \\
L_{1,t} &= \tilde{L}_0 - \beta_0 + \theta(1-t) + e_{1,t} \text{ for } t \in [0, 1], \text{ and } L_{1,\infty} \text{ otherwise} \\
L_{2,t} &= \tilde{L}_0 - 2\beta_0 + \psi(2-t) + e_{2,t},
\end{align*}
\]

where \( \tilde{L}_0 \) and the \( e_{j,t} \) are random variables with values in \( \mathbb{R} \). Notice that \((1-t)\) and \((2-t)\) are simply the durations of treatment until the respective covariate measurements. We assume that the \( e_{j,t} \) (\( j = 0, 1, 2 \)) are independent of \( \tilde{L}_0 \), and that the \( e_{2,t} \) have a distribution function which does not depend on \( t \). We also assume that the \( e_{2,t} \) are independent of all previous variables (and of the treatment initiation time, \( T \), described below). In the simulations, \( \psi \geq 0 \) (a similar study could have been done for \( \psi < 0 \)). We define \( Y_t = L_{2,t} \), the counterfactual outcome with treatment initiated at time \( t \), which could potentially be observed at time 2.

We show in Web-Appendix \( \square \) that the outcome processes adopted in our simulation study are not rank preserving. This is easily seen because with probability one, two patients with the same observed data do not have the same value of \( \tilde{L}_0 \).

Suppose that the hazard of the treatment initiation time, \( T \), given the covariate history at time \( t \) and given that treatment was not initiated before time \( t \), is piecewise constant as follows:

\[
\lambda_T(t) = \begin{cases} 
\lambda_0^{(0)} & \text{if } L_0 > c_0 \text{ and } t \in [0, 1] \\
\lambda_1^{(0)} & \text{if } L_0 \leq c_0 \text{ and } t \in [0, 1] \\
\lambda_0^{(1)} & \text{if } L_{1,\infty} > c_1 \text{ and } t \in (1, 2] \\
\lambda_1^{(1)} & \text{if } L_{1,\infty} \leq c_1 \text{ and } t \in (1, 2], 
\end{cases}
\]

for constants \( c_0 \) and \( c_1 \) in \( \mathbb{R} \). Notice that \( T \) depends on \( \tilde{L}_0, e_{0,\infty} \), and \( e_{1,\infty} \), if \( \lambda_0^{(0)} \neq \lambda_1^{(0)} \) or \( \lambda_0^{(1)} \neq \lambda_1^{(1)} \).

In the simulation study, treatment can be initiated in continuous time, but the covariates are only measured at times 0, 1, and 2, so that the treatment and covariate history up to time \( t \), \( \overline{Z}_t \), consists of the treatment information up to time \( t \) and \( L_0, (L_0, L_1) \), or \((L_0, L_1, L_2)\), depending on whether \( t \in [0, 1) \), \( t \in [1, 2) \), or \( t = 2 \). In the simulations, treatment affects later outcomes, and time-dependent covariates \((L_1)\) which depend on previous
treatment also predict future treatment and the outcome of interest. This is the type of setting structural nested models were developed for.

Web-Appendix shows that for this data generating mechanism,

\[ D(y, t; Z_t) = -\psi 1_{\text{untreated at } t}. \]

Then, it follows from the definition of \( X_\psi \) that

\[ X_\psi(t) = Y + \psi (\min(T, 2) - t) 1_{T > t}, \]

where \((\min(T, 2) - t) 1_{T > t}\) is the duration of the patient not being on treatment between time \( t \) and time \( 2 \).

As shown in Web-Appendix, a consistent estimator of \( \psi \) can be defined as follows. In the first step, the nuisance parameters \( (\lambda_0(0), \lambda_1(0), \lambda_0(1), \lambda_1(1)) \) are estimated using maximum likelihood theory. In the second step, \( \psi \) is estimated as

\[
\hat{\psi} = -\sum_{i=1}^{n} A_{1i} / \sum_{i=1}^{n} A_{2i},
\]

where

\[
A_{1i} = -Y_i \left( Z_i(0) \hat{\lambda}_1^{(0)} + (1 - Z_i(0)) \hat{\lambda}_0^{(0)} \right) \min(T_i, 1) \\
- Y_i \left( Z_i(1) \hat{\lambda}_1^{(1)} + (1 - Z_i(1)) \hat{\lambda}_0^{(1)} \right) \left( 1 - \delta_i^{(0)} \right) (\min(T_i, 2) - 1) \\
+ Y_i \delta_i^{(0)} + Y_i \delta_i^{(1)},
\]

\[
A_{2i} = - \left( Z_i(0) \hat{\lambda}_1^{(0)} + (1 - Z_i(0)) \hat{\lambda}_0^{(0)} \right) \min(T_i, 1) \min(T_i, 2) \\
- \left( Z_i(1) \hat{\lambda}_1^{(1)} + (1 - Z_i(1)) \hat{\lambda}_0^{(1)} \right) \left( 1 - \delta_i^{(0)} \right) (\min(T_i, 2) - 1)^2 \\
+ \delta_i^{(0)} \min(T_i, 2) + \delta_i^{(1)} (\min(T_i, 2) - 1),
\]

\( \delta_i^{(0)} = 1_{T_i \leq 1}, \delta_i^{(1)} = 1_{1 < T_i \leq 2}, Z_i(0) = 1_{L_0 \leq c_0}, \) and \( Z_i(1) = 1_{L_1 \leq c_1}. \)

We ran a simulation study with \( n = 500, 1000, 2000, 5000, \) and \( 10000, \) with 5000 repetitions each. The results are presented in Table 1. As detailed in Web-Appendix, setting 1 has the least noise around the signals, and setting 3 the most.

In this simulation study, both for small and large samples, the bias of the estimators is small. In all three settings and for all sample sizes considered (including the small sample size \( n = 100 \)), the MSE of the estimators arises mostly from the variance, not from the bias. Also, if the true parameter \( \psi \) equals 300 as in this simulation study, for \( n = 500, \sqrt{MSE/\psi} = 0.04 \) in setting 1, and 0.08 in setting 3. Thus, the estimates are already precise in relatively small samples. Because, as shown in the Web-Appendix, the MSE in this simulation study does not depend on the true parameter, \( \psi, \) a larger sample size would be required to obtain precise estimators of small true parameter values \( \psi. \) We conclude that in this simulation study, continuous-time structural nested models perform extremely well.
Table 1: Simulations. Mean Squared Errors (MSE) and bias. 5000 repetitions each.

| n   | setting 1 | MSE | MSE × n | bias | setting 2 | MSE | MSE × n | bias | setting 3 | MSE | MSE × n | bias |
|-----|-----------|-----|---------|------|-----------|-----|---------|------|-----------|-----|---------|------|
| 100 |           | 747 | 75      | -0.28 | 1907      | 191 | -0.040  |      | 2875      | 287 | -0.39   |
| 500 |           | 146 | 73      | -0.22 | 356       | 178 | -0.29   |      | 542       | 271 | -0.61   |
| 1000|           | 72  | 72      | -0.10 | 176       | 176 | -0.068  |      | 268       | 268 | -0.23   |
| 2000|           | 35  | 70      | -0.11 | 89        | 179 | 0.051   |      | 138       | 275 | -0.08   |
| 5000|           | 14  | 69      | -0.067| 35        | 175 | -0.0040 |      | 54        | 268 | -0.05   |
| 10000|          | 6.6 | 66      | -0.066| 18        | 178 | -0.022  |      | 27        | 270 | -0.06   |

10 Discussion

Structural nested models have become a major part of statistical tools for estimation of the effect of time varying treatments, in the presence of time-dependent confounding by indication; see e.g. [34] for a discrete-time application, and e.g. [26], [28], [18], [23], [31], and [11, 10] for continuous-time applications. Structural nested models in continuous time are useful to estimate the effect of a treatment that can be initiated at any point in time, and for which a short duration of treatment has a small effect on the outcome of interest. In contrast with discrete-time structural nested models, in the case of survival outcomes, the resulting parameter estimates can often be interpreted as rates. So-far, continuous-time analyses relied on (local) rank preservation. The main result of the current article is to prove that for continuous-time structural nested models, assumptions about the joint distributions of counterfactuals or deterministic treatment effects/ (local) rank preservation are not necessary to “mimic counterfactual outcomes”, and, based on that, to consistently estimate treatment effects. This article provides a proof for outcomes that are measured at the end of the study as well as a proof for survival outcomes. Important public health decisions are based on analyses with continuous-time structural nested models, so it is important to relax unverifiable and disputable assumptions underlying these analyses.

An interesting topic for future research is to investigate whether the Support Conditions [7, 11] or 8.3 can be weakened, for example to an assumption about the support varying in a differentiable way between the jump times of the covariate- and treatment process Z. We expect that in that case one has to assume that where Z jumps, the support of Y(t) given Zt gets smaller or stays the same as t increases (see Figure 2). Otherwise, X(t) may move out of the support of Y(t) given Zt (recall that X is the solution to a differential equation with final condition). It is reasonable to assume that the support of Y(t) given Zt gets smaller or stays the same as t increases, since more information about Z should not enlarge the range of Y(t).

A problem which may occur without a support condition is that the denominator in equation (12) (the quotient expression for D) or in equation (13) or (14) (the quotient expression for D(n)) may tend to 0, which may “blow up” D or D(n). In that case it
might help to assume that there exists a constant $C$ such that (a) for all $\omega \in \Omega$, $t$ and $y$,

$$F_{Y(t+h)|Z_t}^{-1} \circ F_{Y(t)|Z_t} (y) - y \leq C \cdot h,$$

and (b) for all $t$, $y$ and $B \subset \mathbb{Z}_t$ with $P (\mathbb{Z}_t \in B) > 0$,

$$F_{Y(t+h)|Z_{t\in B}}^{-1} \circ F_{Y(t)|Z_{t\in B}} (y) - y \leq C \cdot h.$$

This assumption does not look unreasonable if there is no “instantaneous treatment effect”. It is to be expected that under this assumption both $D$ and $D^{(n)}$ are bounded by $C$.

Based on the results of the current article, [12] shows that also if a semiparametric Cox model is used to predict treatment changes in Theorem 5.2, the resulting estimating equations for the treatment effect are unbiased. However, the estimating equations are no longer of the form of an average of terms that are independent for the different persons. Thus, consistency and asymptotic normality for this situation constitute interesting topics for future research.

11 Acknowledgements

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Appendix A contains results that are frequently used in the main article. Appendix B describes Mimicking counterfactual survival outcomes. Appendix C describes details of the simulation study. Appendix D describes some facts about conditioning. Appendix E describes a corollary of the Local Inverse Function Theorem. Appendix F describes some facts about Lipschitz continuity and differentiability. Appendix G describes some theory about differential equations. Appendix H describes convergence theorems.

A Results that are frequently used

The first theorem is a corollary of a theorem in [4] Chapter 2, see Web-Appendix G.

Theorem A.1. Suppose that $I$ is a closed interval in $\mathbb{R}$, $f : I \times [y_1, y_2] \to \mathbb{R}$ is continuous with for all $t \in I$, $f(t, y_1) = f(t, y_2) = 0$ and $C : I \to [0, \infty)$ is continuous, and suppose that

$$|f(t, y) - f(t, z)| \leq C(t) |y - z|$$

for all $t \in I$ and $y, z \in [y_1, y_2]$. Then, for every $t_0 \in I$ and $y_0 \in [y_1, y_2]$, there exists a unique solution $y(t)$ of $y'(t) = f(t, y(t))$ with $y(t_0) = y_0$, and this solution is defined for all $t \in I$. Furthermore, $y(t) \in [y_1, y_2]$ for all $t \in I$. Suppose that $g : I \times [y_1, y_2] \to \mathbb{R}$ is continuous and $z : I \to [y_1, y_2]$ is a solution of $z'(t) = g(t, z(t))$. Then

$$|y(t) - z(t)| \leq e^{\int_{t_0}^t C(s) ds} |y(t_0) - z(t_0)| + \int_{t_0}^t e^{\int_{s}^{t} C(\eta) d\eta} |f(t, z(s)) - g(s, z(s))| ds$$

for all $t, t_0 \in I$ with $t \leq t_0$.

The proof of the following lemma can be found in Appendix D.

Lemma A.2. Let $X$ be a random variable with $E|X| < \infty$, and let $Z_t$ be a random variable with values in $\mathcal{Z}_t$, the space of cadlag functions on $[0, t]$ provided with the projection $\sigma$-algebra, with $P(Z$ jumps at $t) = 0$. Then any version of $E[X|\sigma(\bigcup_{n=1}^{\infty} Z^{(n)}_t)]$, with $Z^{(n)}_t$ as defined in Section 7.6, is also a version of $E[X|Z_t]$.

B Web-Appendix: Mimicking counterfactual survival outcomes

B.1 Introduction

For the definition of the infinitesimal shift function $D$ for survival outcomes see Section 8.1. Also the additional Consistency Assumption 8.1 and Assumption of No instantaneous treatment effect 8.2 can be found in Section 8.1. Assumption 8.1 implies the obvious fact that treatment in the future does not cause or prevent death at present:
Lemma B.1. Under Assumptions 8.1 (consistency),

a) For all $h \geq 0$: $Y^{(t+h)} = Y$ on $\{\omega : Y \leq t\} \cup \cup_{h \geq 0} \{\omega : Y^{(t+h)} \leq t\}$.

b) For all $(y, t, h)$ with $y \leq t + h$ and $h \geq 0$: $\{\omega : Y^{(t+h)} \leq y\} = \{\omega : Y \leq y\}$.

Proof. a): From Assumption 8.1, $Y^{(t)} = Y$ on $\{\omega : Y \leq t\} \cup \{\omega : Y^{(t)} \leq t\}$. Thus if $Y \leq t$ then $Y^{(t)} = Y$, and moreover for all $h > 0$, $Y < t + h$, so that, again from Assumption 8.1, $Y^{(t+h)} = Y$. If $Y^{(t)} \leq t$ the same reasoning can be used to prove that $Y^{(t+h)} = Y$. If for some $h > 0$, $Y^{(t+h)} \leq t$ then also $Y^{(t+h)} \leq t + h$, so that, again from Assumption 8.1, $Y^{(t+h)} = Y$.

b): For $y \leq t + h$ and $h \geq 0$: $\{\omega : Y^{(t+h)} \leq y\} = \{\omega : Y^{(t+h)} \leq y\} \cap \{Y^{(t+h)} \leq t + h\} = \{\omega : Y \leq y\}$ because of Assumption 8.1.

Henceforth, this article will only use versions of conditional distributions which are consistent with Lemma 3.1 in the sense that $F_{Y^{(t+h)}|Z_t}(y) = F_{Y|Z_t}(y)$ for all $y \leq t + h$, $h \geq 0$, and $\omega \in \Omega$.

Notice that with definition (23) of $D$, the area where $D$ is possibly non-zero is $(y, t) \in [0, \infty) \times [0, \min\{Y, \tau\}] : y \geq t$. Therefore if $Y < \tau$, the solution to the differential equation $X(t)$ is equal to $Y$ for $t \in [Y, \tau]$. An example of such $X(t)$ is shown in Figure 1 right panel.

B.2 Mimicking counterfactual survival outcomes: assumptions and result

This section provides precise conditions under which $X(t)$ mimics $Y^{(t)}$ in the sense that $X(t)$ has the same distribution as $Y^{(t)}$ given $Z_t$, for survival outcomes. We choose versions of $F_{Y^{(t+h)}|Z_t}$ which are consistent with Lemma 3.1 and which satisfy all regularity conditions below. These versions are used in the definition of $D$ of equation (28), and everywhere else in this section.

Section 8.1 indicated why it is not reasonable to assume that the conditional distribution of the survival time has the fixed support $[y_1, y_2]$ given any covariate- and treatment history $Z_t$. If a person is alive at time $t$, one often expects that $t$ is the left limit of the support. Therefore, this article assumes that

Assumption B.2. (support). There exists a finite number $y_2 \geq \tau$ such that

a) For all $\omega \in \Omega$ and $t$ with $Y > t$, all $F_{Y^{(t+h)}|Z_t}$ for $h \geq 0$ have support $[t, y_2]$.

b) For all $\omega \in \Omega$ and $t$ with $Y > t$, all $F_{Y^{(t+h)}|Z_t}$ for $h \geq 0$ have a continuous non-zero density $f_{Y^{(t+h)}|Z_t}(y)$ on $y \in [t + h, y_2]$.

c) There exists an $\varepsilon > 0$ such that for all $\omega \in \Omega$ and $t$ with $Y > t$, $f_{Y^{(t)}|Z_t}(y) > \varepsilon$ for $y \in [t, y_2]$.  

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Next consider the smoothness conditions in Section 7. It does not seem reasonable to assume that $F_{Y(t+h)}(z_t)(y)$ is continuously differentiable with respect to $h$ and $y$ on $(h, y) \in [0, \infty) \times [t, y_2]$ since for $y \leq t + h$, $F_{Y(t+h)}(z_t)(y) = F_{Y_t}(y)$ (Lemma B.11). Thus, the derivative of $F_{Y(t+h)}(z_t)(y)$ with respect to $h$ is likely not to exist at $y = t + h$ (and is equal to zero for $y < t + h$). Also the derivative of $F_{Y(t+h)}(z_t)(y)$ with respect to $y$ may not exist at $y = t + h$, because of the different treatment before and after $t + h$. For survival outcomes, the smoothness conditions are therefore replaced by:

**Assumption B.3.** (continuous derivatives). For $\omega \in \Omega$ fixed,

a) If $Y > t$ then $F_{Y(t+h)}(z_t)(y)$ restricted to $\{(h, y) \in [0, \infty) \times [t, y_2] : y \geq t + h\}$ is $C^1$ in $(h, y)$.

b) If $Z$ does not jump in $(t_1, t_2)$ and $Y > t_1$ then both $\frac{\partial}{\partial h} F_{Y(t+h)}(z_t)(y)$ and $\frac{\partial}{\partial y} F_{Y(t+h)}(z_t)(y)$ are continuous in $(y, t)$ on $\{(y, t) \in [t_1, y_2] \times [t_1, t_2] : y > t\}$ and can be continuously extended to $\{(y, t) \in [t_1, y_2] \times [t_1, t_2] : y \geq t\}$.

**Assumption B.4.** (bounded derivatives).

a) There exists a constant $C_1$ such that for all $t, h \geq 0$ and $y > t + h$, for $\omega \in \Omega$ with $Y > t$,

$$\frac{\partial}{\partial y} F_{Y(t+h)}(z_t)(y) \leq C_1.$$  

b) There exists a constant $C_2$ such that for all $t, h \geq 0$ and $y > t + h$, for $\omega \in \Omega$ with $Y > t$,

$$\left| \frac{\partial}{\partial h} F_{Y(t+h)}(z_t)(y) \right| \leq C_2.$$  

**Assumption B.5.** (Lipschitz continuity).

a) There exists a constant $L_1$ such that for all $t$ and $y, z \in (t, y_2]$, for $\omega \in \Omega$ with $Y > t$,

$$\left| \frac{\partial}{\partial y} F_{Y(t)}(z_t)(y) - \frac{\partial}{\partial z} F_{Y(t)}(z_t)(z) \right| \leq L_1 |y - z|.$$  

b) There exists a constant $L_2$ such that for all $t$ and $y, z \in (t, y_2]$, for $\omega \in \Omega$ with $Y > t$,

$$\left| \frac{\partial}{\partial h} \right|_{h=0} F_{Y(t+h)}(z_t)(y) - \left| \frac{\partial}{\partial h} \right|_{h=0} F_{Y(t+h)}(z_t)(z) \leq L_2 |y - z|.$$  

**Assumption B.6.** (smoothness). For all $\omega \in \Omega$ and $t$ with $Y > t$, $F_{Y(t)}(z_t)(y)$ is continuous in $y$. 

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Theorem B.7. Suppose that Regularity Conditions B.2–B.5 are satisfied. Then $D(y, t; Z_t)$ as defined in equation (23) exists. Furthermore for every $\omega \in \Omega$ there exists exactly one continuous solution $X(t)$ to $dX(t)/dt = D(X(t), t; Z_t)$ with final condition $X(\tau) = Y$. If also Assumptions 4.1, 8.1 and 8.2 (consistency and no instantaneous treatment effect at time of death) are satisfied then this $X(t)$ has the same distribution as $Y(t)$ given $Z_t$ for all $t \in [0, \tau]$.

The simpler regularity conditions, comparable with Section 7.2.1 for non-survival outcomes, can be found in Section 8.2.

B.3 Existence of and a different expression for $D$

If $Y \leq t$, $D(y, t; Z_t) = 0$ by definition (23). Thus we can concentrate on $\omega \in \Omega$ with $Y > t$. If $y > t$, Corollary 7.7 can be applied on $F_h(y) = F_{Y(t+h)|Z_t}(y)$ with $y_0 = y$ and $U_{0,y_0} \cap \{h \geq 0\} = [0, y - t] \times (t, y_2]$, because of Assumptions B.3a and B.2c. Thus for $y > t$, $D$ as defined in equation (23) exists, and it is equal to

$$D(y, t; Z_t) = \left( \frac{\partial}{\partial h} \bigg|_{h=0} F_{Y(t+h)|Z_t}^{-1}(y) \right) \left( F_{Y(t)|Z_t}(y) \right) = -\frac{\partial}{\partial h} \bigg|_{h=0} F_{Y(t+h)|Z_t}(y).$$

(26)

$D(t, t; Z_t)$ is by definition (23) equal to the limit as $y \downarrow t$ of this $D(y, t; Z_t)$, which exists because of Assumptions B.2c and B.3a.

B.4 Existence and uniqueness of $X(t)$

If $Z_t$ indicates the person is alive at time $t$ and $X(t)$ has the same distribution as $Y(t)$ given $Z_t$, we should have that $X(t)$ stays above $t$ ($Y(t) > t$ in that case because of Consistency Assumption 8.1). In order to prove that $X(t)$ stays indeed above $t$ if the person is alive at time $t$, we prove that $D(t, t; Z_t) \leq 1$.

Lemma B.8. Under Assumptions B.2 and B.3a, if $Y > t$ then $D(t, t; Z_t) \leq 1$.

Proof. We start with some ideas, which are made precise below. Intuition says that $D$ not only measures the increase of quantiles when treatment is prolonged but also the decrease of quantiles when treatment is withheld. Thus quantiles $y$ seem to approximately move to $y - h D(y, t; Z_t)$ when treatment is withheld between $t$ and $t + h$. If quantiles near $t$ move down to $t$ with speed greater than 1 when treatment is withheld starting from $h$ it seems like these quantiles will end up below $t$ when treatment is withheld at time $t$. However, if treatment is withheld starting from $t$ this does not cause death at or before time $t$, so the quantiles above $t$ should stay above $t$. This leads to a contradiction. The following makes this precise.
Fix $\omega \in \Omega$ and fix $t$ for which $Y > t$. Recall from Section B.3 that $a := \lim_{y \to t} D (y, t; \overline{Z}_t)$ exists. We need to prove that $a \leq 1$. Suppose that $a > 1$. It is shown that this leads to a contradiction. Notice that because of the chain rule, for $y > t + h$,

$$\frac{\partial}{\partial h} \left( F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h)Z_t}(y) \right) = \left( F^{-1}_{Y(t)Z_t} \right)' \left( F_{Y(t+h)Z_t}(y) \right) \frac{\partial}{\partial h} F_{Y(t+h)Z_t}(y)$$

$$= \frac{\partial}{\partial h} F_{Y(t+h)Z_t}(y) \left( F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h)Z_t}(y) \right)$$

exists and is continuous in $(h, y)$ for $y > t + h$ with a continuous extension to \{(h, y) \in [0, \infty) \times [t, y_2]: y \geq t + h\} because of Assumptions B.3a and B.2c. Notice that for $h = 0$ this expression is equal to $-D (y, t; \overline{Z}_t)$ because of expression (26) for $D$. Thus the limit of (27) for $h = 0$ and $y \downarrow t$ is equal to $-a$. This can be compared with the intuitive idea that quantiles $y$ approximately move to $y - h D (y, t; \overline{Z}_t)$ when treatment is withheld between $t$ and $t + h$.

Now choose $\delta = \frac{a-1}{2}$, which is greater than 0 since we assumed that $a > 1$. By continuity of (27) in $(h, y)$ there exists an open neighbourhood $U_{(0, t)}$ of $(0, t)$ such that on \{(h, y) \in U_{(0, t)}: y > t + h and h \geq 0\}, the expression (27) above is not further than $\delta$ away from $-a$. Thus there also exist $h_0 > 0$ and $y_0 > t$ such that for $h \in [0, h_0]$, $y \leq y_0$ and $y > t + h$, (27) is not further than $\delta$ away from $-a$. Choose $h_1 \in [0, h_0]$ with $t + (1 + \delta) h_1 \leq y_0$, and define $y_1 = t + (1 + \delta) h_1$.

Notice that since $y_1 > t + h_1$,

$$t < F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h_1)Z_t}(y_1)$$

(informally this is about withholding treatment in the future not causing death at present, which we wanted to use; formally this follows e.g. from Assumption B.2b and b). Moreover, for $y = y_1$, the derivative (27) exists on $h \in [0, h_1]$, since for $h \in [0, h_1]$, $y_1 = t + (1 + \delta) h_1 > t + h_1 \geq t + h$. Thus by Taylor expansion there exist an $\tilde{h}_1 \in [0, h_1]$ with

$$F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h_1)Z_t}(y_1) = y_1 + h_1 \frac{\partial}{\partial h} \bigg|_{h=\tilde{h}_1} F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h)Z_t}(y_1).$$

Combining this it follows that

$$t < y_1 + h_1 \frac{\partial}{\partial h} \bigg|_{h=\tilde{h}_1} F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h)Z_t}(y_1)$$

for some $\tilde{h}_1 \in [0, h_1]$. Rewriting this leads to

$$- h_1 \frac{\partial}{\partial h} \bigg|_{h=\tilde{h}_1} F^{-1}_{Y(t)Z_t} \circ F_{Y(t+h)Z_t}(y_1) < y_1 - t = (1 + \delta) h_1.$$
For \((\hat{h}_1, y_1)\), \((27)\) is not further than \(\delta\) away from \(-a\), since \(\hat{h}_1 \in [0, h_0]\), \(y_1 \leq y_0\) and \(y_1 > t + h_1 \geq t + \hat{h}_1\), so that

\[
\frac{\partial}{\partial h} F_{Y(t)}^{-1}(\hat{Z}_t) \circ F_{Y(t+h)}(y_1) \in (-a - \delta, -a + \delta).
\]

Therefore the expression on the left hand side of equation \((28)\) lies in the interval \(((a - \delta) h_1, (a + \delta) h_1)\). Equation \((28)\) thus implies that \((a - \delta) h_1 < (1 + \delta) h_1\), so \(a < 1 + 2\delta\), so \(\frac{a-1}{2} < \delta\). This is in contradiction with our choice of \(\delta\), which was \(\delta = \frac{a-1}{2}\). \(\square\)

Fix \(\omega\) for the rest of this section. Just as in Section 7.4 it suffices to prove existence and uniqueness of \(X(t)\) with final condition on any interval between jumps of \(Z\), because with probability one \(Z\) jumps only finitely many times. Hence suppose that \(Z\) does not jump in \((t_1, t_2)\) and that \(t_1\) is either a jump time of \(Z\) or 0 and that \(t_2\) is either a jump time of \(Z\) or \(\tau\).

If \(\bar{Z}_{t_1}\) indicates that the person is dead at \(t_1\), \(D\) is identically 0 on \([t_1, \tau]\) and \(X(t)\) exists, \(X(t)\) is unique, and \(X(t)\) is identically \(Y\) on \([t_1, \tau]\).

If \(\bar{Z}_{t_1}\) indicates that the person is alive at \(t_1\), I use Corollary G.4 to prove existence and uniqueness of \(X(t)\). Notice that \(D\) \((y, t; \bar{Z}_t)\) is continuous on \(\{(y, t) \in [t_1, y_2] \times [t_1, t_2] : y \geq t\}\) because of equation \((26)\) and Assumptions \([B.3], [B.2]c\). However, the differential equation with \(X(t)\) has a final condition at the upper end of the interval \([t_1, t_2]\). Just as in Section 7.4 we define the continuous extension \(\tilde{D}\) of \(D\) on \([t_1, y_2] \times [t_1, t_2] : y \geq t\), which exists because of Assumption \([B.2]\); and the extension-assumption in Assumption \([B.3]\). Just as in Section 7.4 \(\tilde{D}\) is Lipschitz continuous in \(y\) on \(\{(y, t) \in [t_1, y_2] \times [t_1, t_2] : y > t\}\) with Lipschitz constant \(L_2/\varepsilon + C_2 L_1/\varepsilon^2\). The same constant works on \(\{(y, t) \in [t_1, y_2] \times [t_1, t_2] : y \geq t\}\) by continuity. Because of Lemma \([B.8]\) above, \(D(t, t; \bar{Z}_t) \leq 1\) for all \(t\). Thus Corollary G.4 gives existence and uniqueness of a continuous solution \(X(t)\) to the differential equation with \(\tilde{D}\), with \(X(t) \geq t\) if \(Y > t\).

B.5 Mimicking counterfactual survival: discrete time

This section considers the situation where \(Z\), the available information on the treatment- and covariate process, can be fully described by its values at finitely many fixed points \(0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_K < \tau_{K+1} = \tau\). At any time at which a person’s covariates are measured, one has to include in the covariates whether or not a person was alive at that time (otherwise, the covariates would be ill-defined). Hence we assume that \(\bar{Z}_t\) includes whether or not a person was alive at \(\tau_1, \ldots, \tau_{p(t)}\), with \(\tau_{p(t)}\) the last \(\tau_k\) before or at time \(t\).

For simplicity we pose differentiability conditions and restrictions on the support of \(Y(t)\) given \(\bar{Z}_{\tau_k}\) for \(t \geq \tau_k\) that are similar to the continuous-time case. Notice that if \(\tau_k\) is the lower support limit of \(F_{Y[\bar{Z}_{\tau_k}]}\), then \(\tau_k\) is also the lower support limit of \(F_{Y(t)[\bar{Z}_{\tau_k}]}\) for all \(t > \tau_k\): if \(\tau_k\) is the lower support limit of \(F_{Y[\bar{Z}_{\tau_k}]}\), then, because of Lemma \([B.1]a\), for all \(h > 0\) and all \(\delta > 0\), \(P(Y(\tau_k+h) \leq \tau_k - \delta) = P(Y \leq \tau_k - \delta) = 0\), and, again because of Lemma \([B.1]b\), for all \(h > 0\) and all \(0 < \delta < h\), \(P(Y(\tau_k+h) \leq \tau_k + \delta) = P(Y \leq \tau_k + \delta) > 0\).
In most cases \( \tau_k \) will then also be the lower support limit of \( F_{Y(\tau_k)|Z_{\tau_k}} \), unless by stopping treatment at time \( \tau_k \) the person stays alive with probability one for a certain period of time, while if treatment is not stopped at time \( \tau_k \) the hazard of dying is non-zero immediately after time \( \tau_k \). For the same reasons as in the continuous-time case, differentiability is only assumed for \( y \geq t \). Assumption [7.8] is replaced for survival outcomes by:

**Assumption B.9.** (smoothness). Suppose that there exists a \( y_2 > \tau \) such that for \( k = 0, \ldots, K \) and \( t \in [\tau_k, \tau_{k+1}] \) there exist conditional distribution functions \( F_{Y(t)|Z_{\tau_k}} \) which are consistent with Lemma B.7 and such that if \( Z_{\tau_k} \) indicates that the person is alive at time \( \tau_k \):

a) For every \( t \in [\tau_k, \tau_{k+1}] \), \( F_{Y(t)|Z_{\tau_k}} \) has support \( [\tau_k, y_2] \).

b) \( F_{Y(t)|Z_{\tau_k}}(y) \) is continuous in \((y, t)\) on \((y, t) \in \mathbb{R} \times [\tau_k, \tau_{k+1}]\).

c) \( F_{Y(t)|Z_{\tau_k}}(y) \) is \( C^1 \) in \((y, t)\) on \([\tau_k, y_2] \times [\tau_k, \tau_{k+1}] : y > t \) with a \( C^1 \) extension to \((y, t) \in [\tau_k, y_2] \times [\tau_k, \tau_{k+1}] : y \geq t \).

d) For \( t \in [\tau_k, \tau_{k+1}] \), \( \frac{\partial}{\partial y} F_{Y(t)|Z_{\tau_k}}(y) \) is strictly positive on \( y \in [t, y_2] \).

Throughout Section B.5 fixed versions of \( F_{Y(t)|Z_{\tau_k}}(y) \) are used satisfying Assumption [B.9]. Since \( Z_t \) contains the same information as \( Z_{\tau_k} \) for \( t \in [\tau_k, \tau_{k+1}] \), we can and will choose the same versions when conditioning on \( Z_t \).

In this discrete-time case \( Z_t \) contains no indicator for death or alive at time \( t \) except for if \( t \) is one of the \( \tau_k \)'s. However, also in this case \( X(t) \) should be above \( t \) for \( t < Y \): for such \( Y \), \( X(t) \) should not play the role of \( Y(t) \)'s less than \( t \). The reason for this is, intuitively, that if \( Y(t) < t \), \( Y(t) = Y < t \), and if also \( Y \)'s greater than \( t \) would play this role there would be too many of them. It will be shown explicitly that there exists a solution \( X(t) \) with \( X(t) > t \) for \( Y > t \). Hence the following analogue of Theorem [7.9] for survival outcomes:

**Proposition B.10.** Suppose that the treatment- and covariate process \( Z \) can be fully described by its values at finitely many fixed points \( 0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_K < \tau_{K+1} = \tau \), and suppose also that Assumption [B.9] is satisfied. Then \( D(y, t; Z_t) \) exists for all \( t \). Furthermore if also Assumptions 4.1, 8.1, and 8.2 (consistency and no instantaneous treatment effect at time of death) are satisfied, then there exists a continuous solution \( X(t) \) to \( dX(t)/dt = D(X(t), t; Z_t) \) with final condition \( X(\tau) = Y \) and with \( X(t) > t \) if \( Y > t \), for which \( X(t) \) has the same distribution as \( Y(t) \) given \( Z_t \).

**Proof.** For \( t \in [\tau_k, \tau_{k+1}] \) and \( y > t \), Lemma [7.7] can be applied on \( F_t(y) = F_{Y(t)|Z_{\tau_k}}(y) \) with \( y_0 = y \), because of Assumptions [B.9] c and d. Thus \( D(y, t; Z_t) \) as defined in equation (23) exists for \( y > t \) and

\[
D(y, t; Z_t) = -\frac{\partial}{\partial y} F_{Y(t)|Z_{\tau_k}}(y).
\] (29)
By definition, \( D(t, t; \overline{Z}_t) \) is equal to limit of (29) for \( y \downarrow t \), which exists because of Assumptions B.9 c and d.

Under Assumption B.9 one can explicitly write down a solution to the differential equation \( dX(t)/dt = D(X(t), t; \overline{Z}_t) \) with final condition \( X(\tau) = Y \), as follows. For the moment consider \( \omega \in \Omega \) fixed. Define \( \tau_{p(\omega)} \) as the last \( \tau_k \) before the survival time \( Y \). In the following, \( F_{Y(\tau_{p(\omega)})|\overline{Z}_{\tau_{p(\omega)}}} \) will denote the distribution function \( F_{Y(\tau_{k})|\overline{Z}_{\tau_{k}}} \) for which \( k = p(\omega) \).

It will \textit{not} denote a distribution function conditional on \( p(\omega) \). Define \( \tilde{X}(t) \) as follows. For \( t \geq \tau_{p(\omega)+1} \), define \( \tilde{X}(t) = Y \). For \( t < \tau_{p(\omega)+1} \), \( t \in [\tau_k, \tau_{k+1}) \), define

\[
\tilde{X}(t) = F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{k}}}^{-1}(\overline{Z}_{\tau_{k}}(t)) \circ \ldots \circ F_{Y(\tau_{p(\omega)})|\overline{Z}_{\tau_{p(\omega)}}}^{-1}(\overline{Z}_{\tau_{p(\omega)}}(t)) \circ F_{Y(\tau_{p(\omega)+1})|\overline{Z}_{\tau_{p(\omega)}}}(t) .
\]

This \( \tilde{X}(t) \) is well-defined under Assumption B.9 a and b.

It is first shown that if \( Y > t \) then also \( \tilde{X}(t) \) is greater than \( t \). First, consider \( t \in [\tau_{p(\omega)}, Y) \cap [0, \tau] \). For such \( t, Y > t \), so

\[
F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{p(\omega)}}}(Y) > F_{Y(\tau_{p(\omega)+1})|\overline{Z}_{\tau_{p(\omega)}}}(t) ,
\]

since because of Assumption B.9 a, \( F_{Y(\tau_{p(\omega)+1})|\overline{Z}_{\tau_{p(\omega)}}} \) is strictly increasing on its support \([\tau_{p(\omega)}, y_2]\), which includes both \( t \) and \( Y \). Because of Lemma B.1b, the right hand side of this expression is equal to \( F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}}(t) \). Hence

\[
F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{p(\omega)}}}(Y) > F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}}(t) ,
\]

so that since \( F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}}^{-1} \) is strictly increasing on \([0, 1]\) (Assumption B.9 a),

\[
\tilde{X}(t) = F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}}^{-1}(\overline{Z}_{\tau_{p(\omega)}}(t)) \circ \ldots \circ F_{Y(\tau_{p(\omega)+1})|\overline{Z}_{\tau_{p(\omega)}}}^{-1}(\overline{Z}_{\tau_{p(\omega)}}(t)) \circ F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}}(t) .
\]

The right hand side is equal to \( t \), since \( t \in [\tau_{p(\omega)}, y_2] \) and \( F_{Y(t)|\overline{Z}_{\tau_{p(\omega)}}} \) is strictly increasing on \([\tau_{p(\omega)}, y_2]\) (Assumption B.9 a). Thus indeed \( \tilde{X}(t) > t \) for \( t \in [\tau_{p(\omega)}, Y) \).

To show that \( \tilde{X}(t) \) is also greater than \( t \) for other \( t < Y \), I use induction, starting with \( k = p(\omega) - 1 \) and ending with \( k = 0 \). It thus needs to be proven that if \( t \in [\tau_k, \tau_{k+1}) \) and \( \tilde{X}(\tau_{k+1}) > \tau_{k+1} \) then \( \tilde{X}(t) > t \). So suppose that \( t \in [\tau_k, \tau_{k+1}) \) and that \( \tilde{X}(\tau_{k+1}) > \tau_{k+1} \). Notice that \( \tilde{X}(\tau_{k+1}) > \tau_k \geq t \), so that

\[
F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{k}}} \left( \tilde{X}(\tau_{k+1}) \right) > F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{k}}}(t) ,
\]

since because of Assumption B.9 a, \( F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{k}}} \) is strictly increasing on its support \([\tau_k, y_2]\), which includes both \( t \) and \( \tilde{X}(\tau_{k+1}) \). Because of Lemma B.1d, the right hand side of this expression is equal to \( F_{Y(t)|\overline{Z}_{\tau_{k}}}(t) \). Therefore

\[
F_{Y(\tau_{k+1})|\overline{Z}_{\tau_{k}}} \left( \tilde{X}(\tau_{k+1}) \right) > F_{Y(t)|\overline{Z}_{\tau_{k}}}(t) ,
\]

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so that, because $F_{Y(t)|Z_k}^{-1}$ is strictly increasing on $[0, 1]$ (Assumption B.9 a),

$$
X(t) = F_{Y(t)|Z_k}^{-1} \circ F_{Y(t)|Z_k} \left( X(t+1) \right) > F_{Y(t)|Z_k}^{-1} \circ F_{Y(t)|Z_k} \left( X(t) \right).
$$

The right hand side of this expression is equal to $t$, since $t \in \left[ \tau_k, y_2 \right]$ and $F_{Y(t)|Z_k}$ is strictly increasing on $[\tau_k, y_2]$ (Assumption B.9 a). It follows that indeed $X(t) > t$ if $Y > t$.

Next, it is shown that $\tilde{X}(t)$ is a continuous solution to $\tilde{X}'(t) = D \left( \tilde{X}(t), t; Z_t \right)$ with final condition $\tilde{X}(\tau) = Y$. First consider $t \geq \tau_{p(\omega) + 1}$. For these $t$, $D \left( y, t; Z_t \right) = 0$, so $\tilde{X}(t)$ should be equal to $Y$; and indeed $\tilde{X}(t)$ is equal to $Y$. For $t \in \left[ \tau_{p(\omega) + 1}, \tau_{p(\omega) + 1} \right]$ it is also true that $D \left( y, t; Z_t \right) = 0$, so $\tilde{X}(t)$ should be equal to $Y$. Because of Lemma B.1b,

$$
F_{Y(t)|Z_{p(\omega)}} (Y) = F_{Y(t)|Z_{p(\omega)}^2} (Y),
$$

so $\tilde{X}(t)$ is indeed equal to $Y$.

To show that $\tilde{X}(t)$ satisfies $\tilde{X}'(t) = D \left( \tilde{X}(t), t; Z_t \right)$ on $\left[ \tau_{p(\omega)}, Y \right)$, notice that for $h \geq 0$ small, since $F_{Y(t)|Z_{p(\omega)}}$ is continuous (Assumption B.9 b),

$$
\tilde{X}(t + h) = F_{Y(t+h)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( \tilde{X}(t) \right) = F_{Y(t+h)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \circ F_{Y(t)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( X(t) \right).
$$

This expression is differentiable at $h = 0$ with derivative $D \left( \tilde{X}(t), t; Z_t \right)$, since, as shown before, $\tilde{X}(t) > t$. Thus indeed $\tilde{X}(t)$ satisfies $\tilde{X}'(t) = D \left( \tilde{X}(t), t; Z_t \right)$ on $\left[ \tau_{p(\omega)}, Y \right)$.

I still need to prove continuity of $\tilde{X}(t)$ at $t = Y$, but it is easier to show continuity on $\left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right]$, so we show continuity of $\tilde{X}(t)$ on $\left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right]$. From Van der Vaart [32] Lemma 21.2, $F_n$ converges weakly to $F$ if and only if $F_n^{-1}(t) \rightarrow F^{-1}(t)$ at every $t$ where $F^{-1}$ is continuous. Notice that because of Assumption B.9 b, $F_{Y(t)|Z_{p(\omega)}}$ converges weakly to $F_{Y(t)|Z_{p(\omega)}}$ as $t \in \left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right] \rightarrow t_0$ for any $t_0 \in \left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right]$. Moreover, because of Assumption B.9 a, $F_{Y(t)|Z_{p(\omega)}}^{-1}$ is continuous on $[0, 1]$. Therefore, $F_{Y(t)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( X(t+1) \right) \rightarrow F_{Y(t)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( X(t) \right)$ as $t \in \left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right] \rightarrow t_0$, for every $x \in (0, 1]$. Thus also

$$
\tilde{X}(t) = F_{Y(t)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( \tilde{X}(t+1) \right) \rightarrow F_{Y(t)|Z_{p(\omega)}}^{-1} \circ F_{Y(t)|Z_{p(\omega)}} \left( \tilde{X}(t) \right)
$$

as $t \in \left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right] \rightarrow t_0$ (notice that $F^{-1}_{Y(t)|Z_{p(\omega)}} \circ F_{Y(t)|Z_{p(\omega)}} (Y) > 0$ since $Y > H$ given $Z_{p(\omega)}$ has support $[\tau_{p(\omega)}, y_2]$ because of Assumption B.9 b). For $t_0 \in \left[ \tau_{p(\omega)}, \tau_{p(\omega) + 1} \right]$, the right hand

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side of this expression is equal to $\tilde{X}(t_0)$, which implies continuity of $\tilde{X}(t)$ on $[\tau_p(\omega), \tau_p(\omega)+1)$. For $t_0 = \tau_p(\omega)+1$, the right hand side of this expression is equal to

$$F^{-1}_{Y(\tau(\omega)+1)\mid \mathcal{Z}_{\tau(\omega)}} \circ F_{Y(\tau(\omega)+1)\mid \mathcal{Z}_{\tau(\omega)}}(Y),$$

which is equal to $Y$ since $Y$ is in the support of $F_{Y(\tau(\omega)+1)\mid \mathcal{Z}_{\tau(\omega)}}$ (Assumption B.9 a) and $F_{Y(\tau(\omega)+1)\mid \mathcal{Z}_{\tau(\omega)}}$ is strictly increasing on its support (Assumption B.9 a). That implies continuity of $\tilde{X}(t)$ at $\tau_p(\omega)+1$.

That also for $k < p(\omega)$, $\tilde{X}(t)$ satisfies $\tilde{X}'(t) = D\left(\tilde{X}(t), t; \mathcal{Z}_t\right)$ on $[\tau_k, \tau_{k+1})$ and that $\tilde{X}(t)$ is continuous on $[\tau_k, \tau_{k+1}]$ follows the same way as in the previous paragraph, starting from the fact that for such $k$ and $t \in [\tau_k, \tau_{k+1})$, $\tilde{X}(t) = F^{-1}_{Y(t)\mid \mathcal{Z}_{\tau_k}} \circ F_{Y(t)\mid \mathcal{Z}_{\tau_k}}\left(\tilde{X}(\tau_{k+1})\right)$. For the induction step, suppose that for $t \in [\tau_k, \tau]$ (for $k = K + 1$ read $t = \tau$), $\tilde{X}(t)$ has the same distribution as $Y(t)$ given $\mathcal{Z}_t$, To show: for $t \in [\tau_{k-1}, \tau_k)$, $\tilde{X}(t)$ has distribution function $F_{Y(t)\mid \mathcal{Z}_t} = F_{Y(t)\mid \mathcal{Z}_{\tau_{k-1}}}$. If $\mathcal{Z}_{\tau_k}$ indicates that the person is dead at $\tau_k$ then $\tilde{X}(t) = Y = Y(t)$ because of Lemma B.1, so certainly $\tilde{X}(t) \sim Y(t)$ given $\mathcal{Z}_{\tau_k}$. If $\mathcal{Z}_{\tau_k}$ indicates that the person is alive at $\tau_k$, then

$$\tilde{X}(t) = F^{-1}_{Y(t)\mid \mathcal{Z}_{\tau_{k-1}}} \circ F_{Y(t)\mid \mathcal{Z}_{\tau_{k-1}}}\left(\tilde{X}(\tau_k)\right)$$

and the rest of the proof can be copied from the proof of Theorem 7.39.

\[\text{\[}\] B.6 Discretization and choices of conditional distributions\]

The construction of the $\mathcal{Z}^{(n)}_{\tau_k}$ can be copied from Section 7.6. Notice that, by construction, $\mathcal{Z}^{(n)}_{\tau_k}$ includes whether or not a person is alive at $\tau_k$.

**Notation B.11.** At this point we choose conditional distributions $P_{\mathcal{Z}_{\tau_k} \mid \mathcal{Z}^{(n)}_{\tau_k}}$. In addition, we choose

$$F_{Y(t)\mid \mathcal{Z}_{\tau_k}^{(n)}}(y) = \int F_{Y(t)\mid \mathcal{Z}_{\tau_k}^{(n)} = z}(y) \, dP_{\mathcal{Z}_{\tau_k}^{(n)} \mid \mathcal{Z}_{\tau_k}^{(n)}}(z)$$

(30)

to be the version of the conditional distribution function of $Y(t)$ given $\mathcal{Z}_{\tau_k}^{(n)}$ which is used in the rest of the proof, and similarly for $Y$ instead of $Y(t)$. If $s \in (\tau_k, \tau_{k+1})$ we take the same version for $F_{Y(t)\mid \mathcal{Z}_{s}^{(n)}}$; this is possible because in that case $\mathcal{Z}_{s}^{(n)} = \mathcal{Z}_{\tau_k}^{(n)}$.

These distributions are consistent with Lemma B.1 in the sense that for $y \leq t$ and for all $\omega \in \Omega$, $F_{Y(t)\mid \mathcal{Z}_{\tau_k}^{(n)}}(y) = F_{Y\mid \mathcal{Z}_{\tau_k}^{(n)}}(y)$. This follows immediately from the fact that all $F_{Y(t)\mid \mathcal{Z}_{\tau_k}^{(n)}}$ are consistent with Lemma B.1 in this sense.
B.7 Existence of and two expressions for $D^{(n)}$

We prove existence of $D^{(n)}$ as defined in equation (23) for the discretized situation of Section 7.6. Moreover, as in Section 7.7, two useful formula’s for $D^{(n)}$ are derived.

The same way as in Section 7.7 it follows that for $t \geq \tau_k$ and $y \in (t, y_2]$, if $\overline{Z}_t^{(n)}$ indicates the person is alive at $\tau_k$,

$$
\frac{\partial}{\partial h} \bigg|_{h=0} \left( F_{Y(t+h)}^{-1}(\overline{Z}_t^{(n)}) \circ F_Y(0) \overline{Z}_t^{(n)} \right) (y) = -\frac{\int \frac{\partial}{\partial h} |_{h=0} F_{Y(t+h)} |_{Z_{\tau_k} = z} (y) \, dP_{Z_{\tau_k}} |_{Z_{\tau_k} = z} (z)}{\int \frac{\partial}{\partial y} F_Y |_{Z_{\tau_k} = z} (y) \, dP_{Z_{\tau_k}} |_{Z_{\tau_k} = z} (z)}.
$$

The limit for $y \downarrow t$ exists because of Assumption B.3a and Assumptions B.4 and B.2b (the proof is the same as the proof for continuity of this expression in $(y, t)$ in Section 7.8). Hence with the versions of $F_{Y(t)} |_{Z_t^{(n)}}$ chosen in Notation B.11

$$
D^{(n)} \left( y, t; \overline{Z}_t^{(n)} \right) = \begin{cases} 
0 & \text{if } \overline{Z}_t^{(n)} \text{ indicates the person is dead at } t \text{ or } y < t \\
\lim_{y \downarrow t} D \left( y, t; \overline{Z}_t \right) & \text{otherwise for } y > t \\
\frac{\partial}{\partial h} \bigg|_{h=0} \left( F_{Y(t+h)}^{-1} |_{Z_{\tau_k} = z} \circ F_Y |_{Z_{\tau_k} = z} \right) (y) & \text{otherwise for } y = t
\end{cases}
$$

exists for every $t$. Moreover, for $t \in [\tau_k, \tau_{k+1})$ and $y > t$,

$$
D^{(n)} \left( y, t; \overline{Z}_t^{(n)} \right) = -\frac{\int \frac{\partial}{\partial h} |_{h=0} F_{Y(t+h)} |_{Z_{\tau_k} = z} (y) \, dP_{Z_{\tau_k}} |_{Z_{\tau_k} = z} (z)}{\int \frac{\partial}{\partial y} F_Y |_{Z_{\tau_k} = z} (y) \, dP_{Z_{\tau_k}} |_{Z_{\tau_k} = z} (z)}.
$$

This expression is similar to expression (13) for $D^{(n)}$ for non-survival outcomes.

Expression (14) for non-survival outcomes takes a different form for survival outcomes. Just as for expression (14), I restrict to the $\Omega'$ defined in equation (15) in Section 7.7 a set of probability one on which conditional probabilities given $\overline{Z}_t^{(n)}$ are uniquely defined. For $n$ and $\omega \in \Omega'$ such that $\overline{Z}_t^{(n)}$ indicates the person is alive at the last $\tau_k$ at or before time $t$, it will be shown that for $y > t$,

$$
D^{(n)} \left( y, t; \overline{Z}_t^{(n)} \right) = -\frac{E \left[ 1_{\{ \text{alive at } t \}} \frac{\partial}{\partial h} |_{h=0} F_{Y(t+h)} |_{Z_{\tau_k} = z} (y) \mid \overline{Z}_t^{(n)} \right]}{E \left[ 1_{\{ \text{alive at } t \}} \frac{\partial}{\partial y} F_Y |_{Z_{\tau_k} = z} (y) \mid \overline{Z}_t^{(n)} \right]}.
$$

The indicator of being alive at time $t$ is new as compared to the non-survival case of Section 7

To prove equation (33), first restrict to $\omega \in \Omega'$. Suppose that $\overline{Z}_t^{(n)}$ indicates the person is alive at $\tau_k$ and suppose that $t \geq \tau_k$. Then for $h \geq 0$ and $y \geq t + h$,

$$
P \left( Y(t+h) \leq y \mid \overline{Z}_t^{(n)} \right) = P \left( Y(t+h) \leq y \mid \overline{Z}_{\tau_k}^{(n)}, Y \in (\tau_k, t] \right) P \left( Y \in (\tau_k, t] \mid \overline{Z}_{\tau_k}^{(n)} \right) + \int_{z_t: \text{ alive at } t} F_{Y(t+h)} |_{Z_{\tau_k} = z_t} (y) \, dP_{Z_{\tau_k}^{(n)}} |_{Z_{\tau_k} = z_t} (z_t).
$$
Given that \( Y \leq t \), Lemma B.11 gives that \( Y^{(t+h)} = Y \leq t \), and, since \( y \geq t \), also \( Y^{(t+h)} \leq t \leq y \). It follows that

\[
P \left( Y^{(t+h)} \leq y \mid \mathcal{Z}_{\tau_k}^{(h)} \right) = P \left( Y \in (\tau_k, t] \mid \mathcal{Z}_{\tau_k}^{(h)} \right) + \int_{zt} 1_{\{\text{alive at } t\}} F_{Y^{(t+h)}} (y) dP_{\mathcal{Z}_t \mid \mathcal{Z}_{\tau_k}^{(h)}} (zt). \tag{34}
\]

To derive equation (33) from equation (34), Corollary 7.7 is applied on

\[
F_h (y) = P \left( Y \in (\tau_k, t] \mid \mathcal{Z}_{\tau_k}^{(h)} \right) + \int_{zt} 1_{\{\text{alive at } t\}} F_{Y^{(t+h)}} (y) dP_{\mathcal{Z}_t \mid \mathcal{Z}_{\tau_k}^{(h)}} (zt),
\]

for \( y > t \), with \( y_0 = y \). We check the conditions of Corollary 7.7. Just as in Section 7.7 on \( y > t + h \), \( h \geq 0 \), \( F_h (y) \) is differentiable with respect to \( y \) and \( h \) with derivatives

\[
\int_{zt} 1_{\{\text{alive at } t\}} F'_{Y^{(t+h)}} (y) dP_{\mathcal{Z}_t \mid \mathcal{Z}_{\tau_k}^{(h)}} (zt) \quad \text{and} \quad \int_{zt} 1_{\{\text{alive at } t\}} \frac{\partial}{\partial h} F_{Y^{(t+h)}} (y) dP_{\mathcal{Z}_t \mid \mathcal{Z}_{\tau_k}^{(h)}} (zt),
\]

respectively. Also the same way as in Section 7.7 it follows that these derivatives are continuous in \((y, h)\). That \( F_0 (y) \) is non-zero follows from Assumption B.2c, if the probability that the person is alive at time \( t \) given \( \mathcal{Z}_{\tau_k}^{(h)} \) is non-zero. Indeed the probability that the person is alive at time \( t \) given \( \mathcal{Z}_{\tau_k}^{(h)} \) is non-zero, since given any \( \mathcal{Z}_{\tau_k}^{(h)} \) indicating that the person is not dead at \( \tau_k \), \( Y \), which has the same distribution as \( Y^{(t)} \) given \( \mathcal{Z}_{\tau_k}^{(h)} \) because of Assumption 4.1 has support \([\tau_k, y_2]\) (Assumption B.2b). Thus the conditions of Corollary 7.7 are satisfied, and equation (33) follows from equation (34).

### B.8 Applying the discrete-time result

**Lemma B.12.** Suppose that Regularity Conditions B.2 B.3 and Assumptions 4.1, 8.1 and 8.3 (consistency and no instantaneous treatment effect at time of death) are satisfied. Then for every \( n \) there exists a continuous solution \( X^{(n)} (t) \) to the differential equation in the discretised setting,

\[
\frac{d}{dt} X^{(n)} (t) = D^{(n)} \left( X^{(n)} (t), t; \mathcal{Z}_t^{(n)} \right),
\]

with final condition \( X^{(n)} (\tau) = Y \). \( X^{(n)} (t) \) is almost surely unique. Moreover, \( X^{(n)} (t) > t \) if \( Y > t \). Furthermore, \( X^{(n)} (t) \) has the same conditional distribution as \( Y^{(t)} \) given \( \mathcal{Z}_t^{(n)} \).

The proof of this lemma is different from the proof in Section 7.8 because of the different assumptions for the discrete-time case if the outcome is survival.

**Proof.** Fix \( n \). First, it is shown that there exists a continuous solution \( X^{(n)} \) with \( X^{(n)} (t) > t \) if \( Y > t \) for which \( X^{(n)} (t) \) has the same conditional distribution as \( Y^{(t)} \) given \( \mathcal{Z}_t^{(n)} \), using Proposition B.10. We thus need to check that the versions of the conditional distributions \( F_{Y^{(t)}} (\mathcal{Z}_t^{(n)}) \) of \( Y^{(t)} \) given \( \mathcal{Z}_t^{(n)} \) chosen in Notation B.11 satisfy Assumption B.9.

Section B.6 already showed that the conditional distributions \( F_{Y^{(t)}} (\mathcal{Z}_t^{(n)}) (y) \) chosen in Notation B.11 are consistent with Lemma B.11. We check Assumption B.9 a–d. If \( \mathcal{Z}_t^{(n)} \) indicates that the person is alive at time \( \tau_k \) then:
a) $F_{Y(t)|Z^{(n)}_{t_k}}$ has support $[\tau_k, y_2]$ since all $F_{Y(t)|Z_{t_k}}$ have support $[\tau_k, y_2]$ (Assumption [B.2] a).

b) $F_{Y(t)|Z^{(n)}_{t_k}}(y) = F_{Y(\tau_k+(t-\tau_k))|Z^{(n)}_{t_k}}(y)$ is continuous in $(y, t)$ on $(y, t) \in \mathbb{R} \times [\tau_k, \tau_{k+1}] : y \geq t$ because of Assumption [B.3] a and Lebesgue’s dominated convergence theorem, since all $F_{Y(t)|Z_{t_k}}$ are bounded by 1. For $y \leq t$, $F_{Y(t)|Z^{(n)}_{t_k}}(y) = F_{Y(t)|Z_{t_k}}(y)$ because of Lemma [B.1], which is continuous in $(y, t)$ because of Assumption [B.6]. Therefore, the same is true for the version of $F_{Y(t)|Z^{(n)}_{t_k}}(y)$ chosen in Notation [B.11].

c) $F_{Y(t)|Z^{(n)}_{t_k}}(y) = F_{Y(\tau_k+(t-\tau_k))|Z^{(n)}_{t_k}}(y)$ is $C^1$ in $(y, t)$ on $(y, t) \in [\tau_k, y_2] \times [\tau_k, \tau_{k+1}] : y \geq t$ because all $F_{Y(t)|Z^{(n)}_{t_k}}(y)$ are $C^1$ there (Assumption [B.3] a), and all derivatives are bounded there (Assumption [B.4]), which follows with the same reasoning as for $F_{Y(t+h)|Z^{(n)}_{t_k}}$ as in Section 7.7, integration and differentiation can be interchanged here.

d) Under c) it was shown that for $y \geq t$,

$$\frac{\partial}{\partial y} F_{Y(t)|Z^{(n)}_{t_k}}(y) = \int \frac{\partial}{\partial y} F_{Y(t)|Z_{t_k}}(y) dP_{Z_{t_k}|Z^{(n)}_{t_k}}(z).$$

Because of Assumption [B.2] b this is greater than 0. \hfill \Box

Continuity of $D^{(n)}$ on $\{(y, t) \in (\tau_k, y_2] \times (\tau_k, \tau_{k+1}) : y > t\}$ with a continuous extension $\tilde{D}^{(n)}$ to $\{(y, t) \in [\tau_k, y_2] \times [\tau_k, \tau_{k+1}] : y \geq t\}$ follows from equation (32), similar to Section 7.8. Also similar to Section 7.8, $D^{(n)}(y, t)$ is Lipschitz continuous in $y$ on $\{(y, t) \in [\tau_k, y_2] \times [\tau_k, \tau_{k+1}] : y \geq t\}$ for $\omega \in \Omega'$ of equation (15), and uniqueness of $X^{(n)}$ on $\Omega'$ follows from Corollary [C.4].

**B.9 Bounding the difference between $X$ and $X^{(n)}$ in terms of $D$ and $D^{(n)}$**

$X^{(n)}$ and $X$ satisfy the differential equations with the continuous extensions of $D^{(n)}$ and $D$, $\tilde{D}^{(n)}$ and $\tilde{D}$, respectively, on the closed intervals $[t_1, t_2]$ as in Section 7.9 because if $Y \geq t$, both $X(t) \geq t$ (Section [B.4]) and $X^{(n)}(t) \geq t$ (Section [B.8]). $\tilde{D}$ is Lipschitz continuous in $y$ on these intervals, as shown at the end of Section [B.4]. Therefore it follows in a similar way as in Section 7.9 but with Corollary [C.4] instead of Theorem [A.1] that almost surely

$$|X^{(n)}(t) - X(t)| \leq \int_t^T e^{C(s-t)}|D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds \quad (35)$$

and

$$\sup_{t \in [0, T]} |X^{(n)}(t) - X(t)| = \int_0^T e^{C(s)}|D(X^{(n)}(s), s) - D^{(n)}(X^{(n)}(s), s)| ds, \quad (36)$$

with $C = L_2/\varepsilon + C_2 L_1/\varepsilon^2$. 

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B.10 Convergence of $D^{(n)}$ to $D$

This section proves that for all $(y, t)$ fixed, $D^{(n)}(y, t; \overline{Z}_t^{(n)})$ converges almost surely to $D(y, t; \overline{Z}_t)$. First, consider $\overline{Z}_t$ indicating the person is dead at time $t$. Then, $D(y, t; \overline{Z}_t) = 0$. Note that the probability the person is dead at exactly time $t$ is 0. Therefore, almost surely, for $n$ large enough $\overline{Z}_t^{(n)}$ indicates the person is dead at the last $\tau_k$ at or before time $t$. Thus, also $D^{(n)}(y, t; \overline{Z}_t^{(n)}) = 0$, so $D^{(n)}$ converges to $D$. For $y < t$, $D(y, t; \overline{Z}_t)$ and all $D^{(n)}(y, t; \overline{Z}_t^{(n)})$ are 0. Therefore it suffices to consider $y \geq t$ and $\omega$ and $t$ for which $\overline{Z}_t$ indicates the person is alive at time $t$. We start by proving that for $y > t$, $D^{(n)}(y, t; \overline{Z}_t^{(n)})$ converges almost surely to $D(y, t; \overline{Z}_t)$. Equation (26) implies that if $\overline{Z}_t$ indicates the person is alive at time $t$, for $y > t$

$$D(y, t; \overline{Z}_t) = -\frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y),$$

and equation (33) implies that, since the person is alive at the last $\tau_k$ at or before time $t$, for $y > t$

$$D^{(n)}(y, t; \overline{Z}_t^{(n)}) = -\frac{E\left[1_{\{\text{alive at } t}\} \frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}}{E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}},$$

We will apply Lévy’s Upward Theorem (see e.g. Williams [33] page 134), which is allowed since \(\frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y)\) and \(\frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y)\) are bounded because of Assumption 7.3. Lévy’s Upward Theorem leads to

$$E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}] \rightarrow E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}] \sigma \left(\bigcup_{n=1}^{\infty} \overline{Z}_t^{(n)}\right) \text{ a.s.} \quad (37)$$

and

$$E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}] \rightarrow E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t^{(n)}}] \sigma \left(\bigcup_{n=1}^{\infty} \overline{Z}_t^{(n)}\right) \text{ a.s.} \quad (38)$$

Replacing the conditioning on $\sigma \left(\bigcup_{n=1}^{\infty} \overline{Z}_t^{(n)}\right)$ by conditioning on $\overline{Z}_t$ in (37) and (38) is allowed because of Lemma A.2. Since for these $\overline{Z}_t$,

$$E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t}] = 1_{\{\text{alive at } t\} E\left[\frac{\partial}{\partial h} |_{h=0} F_{Y^{(t+h)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t}]$$

and

$$E\left[1_{\{\text{alive at } t\} \frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t}] = 1_{\{\text{alive at } t\} E\left[\frac{\partial}{\partial y} F_{Y^{(t)}|\overline{Z}_t}(y) \right] |_{\overline{Z}_t}]$$
this implies that for \( y > t \) fixed

\[
D^{(n)}(y, t; Z_t^{(n)}) \rightarrow D(y, t; Z_t) \text{ a.s..}
\]  

(39)

To prove that also

\[
D^{(n)}(t, t; Z_t^{(n)}) \rightarrow D(t, t; Z_t) \text{ a.s.,}
\]  

(40)

We will apply Lemma H.1. To do that it suffices to have Lipschitz continuity of all \( D^{(n)} \) and \( D \) in \( y \) with the same Lipschitz constant, and if that is the case equation (40) follows. That \( D \) is Lipschitz continuous in \( y \) with Lipschitz constant \( \frac{1}{\varepsilon} L_2 + \frac{C_2}{\varepsilon} L_1 \) was shown in Section B.4. That \( D^{(n)} \) is Lipschitz continuous in \( y \) with Lipschitz constant \( \frac{1}{\varepsilon} L_2 + \frac{C_2}{\varepsilon^2} L_1 \) on \( \Omega' \) was shown in Section B.8.

B.11 \( X^{(n)}(t) \) converges to \( X(t) \) and \( X(t) \) is measurable

Equations (36), (39) and (40) are the starting point here. If \( Y \geq t \), both \( X(t) \geq t \) (Section B.4) and \( X^{(n)}(t) \geq t \) (Section B.8), so that the rest of the proof can be copied from Section 7.11.

B.12 Conclusion

This section can be copied from Section 7.12.

B.13 Mimicking counterfactual outcomes: discrete-continuous time

This section can be copied from Section 7.13.

C Web-Appendix: A simulation study

C.1 Introduction

This appendix provides further details on the design of the simulation study of Section 9. In the simulation study, we calibrated the distributions of the variables and the parameter values to HIV/AIDS data, perhaps the most salient example of application of structural nested models in the empirical literature. For details, see Section C.4.

Web-Appendix C is organized as follows. Section C.2 presents the counterfactual outcomes and the treatment initiation process. The outcomes and treatment initiation are designed so that treatment predicts intermediate covariates which in turn predict future treatment: the type of setting structural nested models were designed for. Continuous-time structural nested models can be adopted when the treatment is initiated in continuous time. Consequently, in the simulations we impose that treatment decisions are adopted
continuously in time. Section C.3 shows that our setting does not impose (local) rank preservation. In Section C.4 the distributions and parameters are calibrated to real data on HIV/AIDS. Section C.5 derives the parametric form of the infinitesimal shift-function \( D \). Section C.6 derives the solution to the differential equation (8): the “mimicking” variable \( X_\psi(t) \) of Section 4. Section C.7 describes the estimators. Section C.8 describes the results of the simulation study.

C.2 An additive model for treatment effect, and the treatment initiation process

In this simulation study no one is treated at time zero, and once treatment is initiated, it is never stopped. \( Y(t) \) is the counterfactual outcome had treatment been as given in reality until time \( t \), and continued or initiated after that. For example, if treatment was initiated by time \( t \) for a particular patient, \( Y(t) \) is the observed outcome for that patient, since he or she was already treated at time \( t \) and treatment is never stopped. On the other hand, if treatment was not initiated by time \( t \), \( Y(t) \) is the outcome had treatment been initiated at time \( t \). Thus, in the definition of \( Y(t) \) in Section 2 the switch at time \( t \) to “some kind of baseline treatment regime \( \theta \)’ is, in this case, “treat continuously” from time \( t \) onwards. In the simulations, we study a setting with \( t \in [0, 2] \). The subscript \( t \) indicates the treatment initiation time, so for example \( L_{1,t} \) indicates (counterfactual) covariates at time 1 under “treatment started at time \( t \)”. Similarly, the subscript \( \infty \) indicates (counterfactual) variables under no treatment. For example, \( L_{2,\infty} \) indicates (counterfactual) covariates at time 2 under no treatment. In the simulation design, the counterfactual covariates \( L \) are as follows:

\[
L_0 = \tilde{L}_0 + e_0,
L_{1,\infty} = \tilde{L}_0 - \beta_0 + e_{1,\infty},
L_{2,\infty} = \tilde{L}_0 - 2\beta_0 + e_{2,\infty},
L_{1,t} = \tilde{L}_0 - \beta_0 + \theta(1 - t) + e_{1,t} \text{ for } t \in [0, 1], \text{ and } L_{1,\infty} \text{ otherwise}
L_{2,t} = \tilde{L}_0 - 2\beta_0 + \psi(2 - t) + e_{2,t},
\]

where \( \tilde{L}_0 \) and the \( e_{j,t} \) are random variables with values in \( \mathbb{R} \). Notice that \((1 - t)\) and \((2 - t)\) are simply the durations of treatment until the respective covariate measurements. We will assume that the \( e_{j,t} \) \((j = 0, 1, 2)\) are independent of \( \tilde{L}_0 \), and that the \( e_{2,t} \) have a distribution function which does not depend on \( t \). We will also assume that the \( e_{2,t} \) are independent of all previous variables (and of the treatment initiation time, \( T \), described below). In the simulations, \( \psi \geq 0 \) (a similar study could have been done for \( \psi < 0 \)). We define \( Y_t = L_{2,t} \), the counterfactual outcome with treatment initiated at time \( t \), which could potentially be observed at time 2.

\( T \) will be the treatment initiation time, with \( T = \infty \) if treatment was not initiated in the time interval \([0, 2]\). The treatment initiation time \( T \) determines which of the above variables is observed. \( Y = L_2 = Y_T \) no matter when treatment is started. \( L_1 = L_{1,T} \) if
\( T \leq 1 \) and \( L_1 = L_{1,\infty} \) if \( T > 1 \). \( T \) also determines what are the \( Y^{(t)} \), with \( Y^{(t)} = Y_t \) if \( T > t \) and \( Y^{(t)} = Y_T \) if \( T \leq t \).

Suppose that the hazard of the treatment initiation time \( T \), given the covariate history at time \( t \) and given that treatment was not initiated before time \( t \), is piecewise constant as follows:

\[
\lambda_T(t) = \begin{cases} 
\lambda_0^{(0)} & \text{if } L_0 > c_0 \text{ and } t \in [0, 1] \\
\lambda_1^{(0)} & \text{if } L_0 \leq c_0 \text{ and } t \in [0, 1] \\
\lambda_0^{(1)} & \text{if } L_{1,\infty} > c_1 \text{ and } t \in (1, 2] \\
\lambda_1^{(1)} & \text{if } L_{1,\infty} \leq c_1 \text{ and } t \in (1, 2],
\end{cases}
\]

for constants \( c_0 \) and \( c_1 \) in \( \mathbb{R} \). Notice that \( T \) depends on \( \tilde{L}_0, e_{0,\infty}, \) and \( e_{1,\infty} \), if \( \lambda_0^{(0)} \neq \lambda_1^{(0)} \) or \( \lambda_0^{(1)} \neq \lambda_1^{(1)} \).

In the simulation study, treatment can be initiated in continuous time, but the covariates are only measured at times 0, 1, and 2, so that the treatment and covariate history up to time \( t \), \( \mathbf{Z}_t \), consists of the treatment information up to time \( t \) and \( L_0, (L_0, L_1) \), or \( (L_0, L_1, L_2) \), depending on whether \( t \in [0, 1) \), \( t \in [1, 2) \), or \( t = 2 \).

In the simulations, treatment affects later outcomes, and time-dependent covariates \( (L_1) \) which depend on previous treatment also predict future treatment and the outcome of interest. This is the type of setting structural nested models were developed for.

C.3 No rank preservation in the simulations

The outcomes described in Section C.2 are not rank preserving. For two patients with the same treatment history, the complete observed data are the same if the sum of \( \tilde{L}_0 \) and the \( e_{j,T} \) (for \( j = 0, 1, 2 \)) are all three the same. However, under an alternative treatment, the outcomes for two patients with the same observed data can be different. In fact, in this simulation study, they are different with probability one. This is easily seen because with probability one, the value of \( \tilde{L}_0 \) is not the same for these two patients. Under rank preservation, two patients with the same observed data \( ((L_0, L_1, L_2, T, Y) \) the same for both patients) also would have had the same outcomes had they both followed the same alternative treatment \( (Y_t \) the same for both patients). Thus, rank preservation does not hold in this simulation study.

C.4 Choice of parameter values in the simulation study

In the simulation study, we calibrate the distributions of the variables and the parameter values to HIV/AIDS data. We focus on the first two years since HIV diagnosis. Time zero is the time of HIV diagnosis. The outcome variable is the CD4 count, a commonly used marker of the state of the immune system of HIV-positive patients. The usual treatment for HIV-positive patients is ART, antiretroviral treatment. ART is not always initiated immediately after diagnosis. ART initiation time often depends on the last measured CD4 count. When the CD4 count is at or below 350 copies/ml, HIV-positive patients are much more likely to initiate ART than when the CD4 count is above 350 copies/ml. Thus, in
the simulation study we choose $c_1 = c_0 = 350$. Intermediate CD4 counts are affected by previous treatment and predict both future treatment and the final outcome $Y$, the CD4 count at year two.

Based on a histogram of the first measured CD4 count in the AIEDRP data (Acute Infection and Early Disease Research Program, see [7]), and based on the median and IQR estimates of the first measured CD4 count in [1], we choose to simulate $L_0$ so that the square root of $L_0$ is approximately normal. According to [1], “the median CD4 count at presentation increased from 256 cells/mm$^3$ (interquartile range, 96 – 455 cells/mm$^3$) to 317 cells/mm$^3$ interquartile range (IQR), 135 – 517 cells/mm$^3$) from 1997 to 2007.” For our first scenario, we choose $\sqrt{L_0} \sim \mathcal{N}(17, 8^2)$. We simulate $e_k \sim \mathcal{N}(0, 20^2)$, so it has a relatively small standard deviation. Based on a preliminary simulation with one million observations, the median $L_0$ in this simulation scenario is 294, IQR 135 – 501; these values are close to the empirical values.

In our simulations, the probability of treatment initiation in the first year is 0.70 for patients with a baseline CD4 count below 350 and 0.30 for patients with a baseline CD4 count above 350. While in clinical practice, patients with a higher CD4 count are less likely to be treated, with 350 often used as a cut-off, the 0.70 and 0.30 values are not chosen based on data, because treatment guidelines have been changing considerably over time in the past few years and differ by country ([19, 35]). The simulation values of the treatment initiation parameters ensure that all patients have a considerable probability of being untreated and also a considerable probability of being treated. If patients with specific covariates are either always treated or always untreated, we cannot estimate the effect of treatment for these patients (because it is impossible to distinguish the treatment effect from the reason why the treatment was given). Thus, in the simulations we choose $\lambda_0^{(0)} = -\log(1 - 0.3)$ and $\lambda_1^{(0)} = -\log(1 - 0.7)$. For treatment initiation during the second year, we choose the same parameter values: $\lambda_0^{(1)} = -\log(1 - 0.3)$ and $\lambda_1^{(1)} = -\log(1 - 0.7)$. This implies that for any untreated covariate history, the probability of ever initiating treatment is $.3 + (.7 \times .3)$ to $.7 + (.3 \times .7)$, or 0.51 to 0.91. Based on the estimates in [15] and [16], the median CD4 count could increase by about 200 between ART initiation and one year later, and the effect of one year of ART is about $12 \times 25 = 300$. Therefore, we choose: $\beta_0 = 100$, and $\theta_0 = \psi_0 = 300$. Table 1, setting 1 describes the results of this simulation scenario.

In a second simulation scenario, setting 2 in Table 1, we introduce more variation around the signals. In the second scenario, we increase the variance of $e_k$ to $200^2$, with $\sqrt{L_0} \sim \mathcal{N}(17, 6^2)$. Based on a preliminary simulation with one million observations, the median $L_0$ in scenario 2 is 313, IQR 125 – 501. The other parameters are as in scenario 1. In the third simulation study, setting 3 in Table 1, we introduce even more variation around the signals. In the third scenario, we increase the variance of $e_k$ to $300^2$, with $\sqrt{L_0} \sim \mathcal{N}(18, 3^2)$. Based on a preliminary simulation with one million observations, the median $L_0$ in scenario 3 is 333, IQR 119 – 545. The other parameters are as in scenario 1.
C.5 Calculating the infinitesimal shift function \( D \)

This section calculates \( D \) for the simulation study. First note that if treatment had already started by time \( t \), there is no difference between \( Y(t) \) and \( Y(t+h) \), so that \( D(y, t; \tilde{Z}_t) = 0 \) for all \( y \). Therefore, we focus on calculating \( D \) for \( \tilde{Z}_t \) such that \( T > t \). Let \( t \) be given. We only need to derive \( F_{Y(t+h)|\tilde{Z}_t}(y) \) for \( h > 0 \) small. Therefore, we restrict calculations to \( h \in [t, t+h_0] \) such that \([t+h_0]\) lies within either \([0, 1)\) or \([1, 2)\), depending on which of these two intervals contains \( t \). Let \( e_2 \) be any random variable which is independent of \( \tilde{Z}_t \) and the treatment process and which has the same distribution as the \( e_{2,t} \). Denote the actual duration of treatment until time \( t+h \) by the random variable \( \text{cum}(t+h) \). We derive:

\[
F_{Y(t+h)|\tilde{Z}_t}(y) = P\left(Y(t+h) \leq y|\tilde{Z}_t\right) = P\left(\tilde{L}_0 - 2\beta_0 + \psi\left(\text{cum}(t+h) + 2 - (t+h)\right) + e_{2,\min(T,t+h)} \leq y|\tilde{Z}_t\right) = P\left(\tilde{L}_0 - 2\beta_0 + \psi\left(\text{cum}(t+h) + 2 - (t+h)\right) + e_2 \leq y|\tilde{Z}_t\right) = E\left[P(\tilde{L}_0 - 2\beta_0 + \psi\left(\text{cum}(t+h) + 2 - (t+h)\right) + e_2 \leq y|\tilde{Z}_t, e_2, \tilde{L}_0)|\tilde{Z}_t\right] = E\left[P(\text{cum}(t+h) \leq \frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t-h)|\tilde{Z}_t, e_2, \tilde{L}_0)|\tilde{Z}_t, e_2, \tilde{L}_0\right]
\]

Next, since we restrict to \( \tilde{Z}_t \) such that \( T > t \), we have that \( 0 \leq \text{cum}(t+h) \leq h \), and because of the way the treatment initiation process was simulated, for \( x \in [0, h] \),

\[
P\left(\text{cum}(t+h) \leq x|\tilde{Z}_t, e_2, \tilde{L}_0\right) = P\left(\text{cum}(t+h) \leq x|\tilde{Z}_t\right) = P\left((t+h) - T \leq x|\tilde{Z}_t\right) = P\left(T - t \geq h - x|\tilde{Z}_t\right) = \begin{cases} e^{-\lambda_{[t]}(h-x)} & \text{if } L_{[t]} > c_{[t]} \\ e^{-\lambda_{[t]}(h-x)} & \text{if } L_{[t]} \leq c_{[t]} \end{cases}
\]

(42)

where \([t]\) is the floor of \( t \), the largest integer less than or equal to \( t \). In the first line of (42) we use that, in our simulation design, the rate of treatment initiation does not depend on \((e_2, L_0)\), given \( \tilde{Z}_t \), which includes \( L_{[t]} \).

For \( T > t \), \( L_{[t]} \leq c_{[t]} \), and \( \lambda_1 = \lambda_{[t]} \), since \( \text{cum}(t+h) \in [0, h] \) for patients with \( T > t \),
it follows from equations (41) and (42) that

\[
F_{Y^{(t+h)|Z_t}}(y) = 0 \cdot P\left(\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) < 0 | \bar{Z}_t\right)
\]

\[+ 1 \cdot P\left(\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) > h | \bar{Z}_t\right)
\]

\[+ E\left[e^{-\lambda_1\left(h - \frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h)\right)}\right]
\]

\[\left|\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) \in (0, h), \bar{Z}_t\right|
\]

\[\cdot P\left(\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) \in (0, h) | \bar{Z}_t\right)
\]

\[= P\left((y - e_2 - \tilde{L}_0 + 2\beta_0) - \psi(2 - t) > 0 | \bar{Z}_t\right)
\]

\[+ E\left[e^{-\lambda_1\left(-\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) + (2 - t)\right)}\right]
\]

\[\left|\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) \in (0, h), \bar{Z}_t\right|
\]

\[\cdot P\left((y - e_2 - \tilde{L}_0 + 2\beta_0) - \psi(2 - t) \in (-\psi h, 0) | \bar{Z}_t\right)
\]

\[= P\left(Y^{(t)} < y | \bar{Z}_t\right)
\]

\[+ E\left[e^{-\lambda_1\left(-\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) + (2 - t)\right)}\right]
\]

\[\left|\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) \in (0, h), \bar{Z}_t\right|
\]

\[\cdot P\left((y - Y^{(t)}) \in (-\psi h, 0) | \bar{Z}_t\right)
\]

\[= P\left(Y^{(t)} \leq y | \bar{Z}_t\right)
\]

\[+ E\left[e^{-\lambda_1\left(-\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) + (2 - t)\right)}\right]
\]

\[\left|\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t - h) \in (0, h), \bar{Z}_t\right|
\]

\[\cdot P\left(Y^{(t)} \in (y, y + \psi h) | \bar{Z}_t\right)
\]

\[= F_{Y^{(t)}|Z_t}(y)
\]

\[+ E\left[e^{-\lambda_1\left(-\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) + (2 - t)\right)}\right]
\]

\[\left|\frac{1}{\psi}(y - e_2 - \tilde{L}_0 + 2\beta_0) - (2 - t) \in (-h, 0), \bar{Z}_t\right|
\]

\[\cdot \left(F_{Y^{(t)}|Z_t}(y + \psi h) - F_{Y^{(t)}|Z_t}(y)\right).
\]  

(43)

In our simulation study, \((\tilde{L}_0, e_2)\) has a continuous conditional distribution \(f_{(\tilde{L}_0, e_2)|Z_t}\). There-
fore, conditional on a value of \( \mathcal{Z}_t \) such that \( T > t \), we have that
\[
E \left[ e^{-\lambda_t \left( -\frac{1}{\psi}(y-e_2-L_0+2\beta_0)+(2-t) \right)} \frac{1}{\psi} (y - e_2 - L_0 + 2\beta_0) - (2 - t) \in (-h, 0), \mathcal{Z}_t \right] 
= \int_{-\infty}^{\infty} d\tilde{l}_0 \int_{-\infty}^{(-2-t+h)\psi+y-L_0+2\beta_0} d\lambda \frac{\psi}{2} f_{(\tilde{l}_0,e_2)|\mathcal{Z}_t} (\tilde{l}_0, e_2) e^{-\lambda_t \left( -\frac{1}{\psi}(y-e_2-L_0+2\beta_0)+(2-t) \right)},
\]
which is continuously differentiable in \( h \geq 0 \) for \((y, t, \mathcal{Z}_t)\) fixed, with some derivative, \( g(y, t, h, \mathcal{Z}_t) \). Therefore, equation (43) implies that
\[
\frac{\partial}{\partial h} F_{Y(t+h)|\mathcal{Z}_t}(y) = g(y, t, h, \mathcal{Z}_t) \left( F_{Y(t)|\mathcal{Z}_t}(y + \psi h) - F_{Y(t)|\mathcal{Z}_t}(y) \right) + E \left[ e^{-\lambda_t \left( -\frac{1}{\psi}(y-e_2-L_0+2\beta_0)+(2-t) \right)} \frac{1}{\psi} (y - e_2 - L_0 + 2\beta_0) - (2 - t) \in (-h, 0), \mathcal{Z}_t \right] \cdot \psi f_{Y(t)|\mathcal{Z}_t}(y + \psi h),
\]
where \( f_{Y(t)|\mathcal{Z}_t} \) is the density of \( Y(t) \) given \( \mathcal{Z}_t \). Notice that since \( T > t \), \( Y(t) = Y_t \) given \( \mathcal{Z}_t \), and the density \( f_{Y(t)|\mathcal{Z}_t} = f_{Y_t|\mathcal{Z}_t} \) exists and is continuous. Letting \( h \downarrow 0 \), it follows that
\[
\frac{\partial}{\partial h} \bigg|_{h=0} F_{Y(t+h)|\mathcal{Z}_t}(y) = \psi f_{Y(t)|\mathcal{Z}_t}(y).
\]
Clearly,
\[
\frac{\partial}{\partial y} F_{Y(t)|\mathcal{Z}_t}(y) = f_{Y(t)|\mathcal{Z}_t}(y).
\]
Hence, because of equation (12),
\[
D(y, t; \mathcal{Z}_t) = -\psi 1_{\text{untreated at } t}.
\]
(44)
The same derivation can be used for \( L_{[t]} > c_{[t]} \).

In fact, it can be shown that if the counterfactual covariates and the counterfactual outcomes are as in this simulation study, \( D = -\psi 1_{\text{untreated at } t} \) if \( T \) has a continuous conditional density \( f_{Y(t)|\mathcal{Z}_t}(y) \) given \( \mathcal{Z}_t \) for \( y \in [t, t+h_0] \) for some \( h_0 > 0 \). This is beyond the scope of the current article. In future work, we also plan to address multidimensional \( \psi \), as well as parameterizing \( D \) for survival outcomes.

### C.6 Calculating \( X_\psi(t) \)

As a consequence of Section 4 and equation (44), it follows that \( X_\psi(t) \) is the solution to
\[
dX_\psi(t)/dt = -\psi 1_{\text{untreated at } t}
\]
with end condition \( X_\psi(2) = Y \) (recall that time 2 is the time the outcome is measured). Therefore,

\[
X_\psi(t) = Y + \psi(\min(T, 2) - t) 1_{T > t},
\]

where \((\min(T, 2) - t) 1_{T > t}\) is the duration of the patient not being on treatment between time \(t\) and time 2.

### C.7 Estimating equations when treatment initiation follows a piecewise exponential model

Suppose that we know that treatment initiation follows a piecewise exponential model, with parameter depending on a discretized covariate, measured at time 0 and time 1. In the simulation, we assume

\[
\lambda_T(t | \mathbf{Z}_t) = \begin{cases} 
\lambda_0^{(0)} & \text{if } L_0 > c_0 \text{ and } t \in [0, 1] \\
\lambda_0^{(0)} & \text{if } L_0 \leq c_0 \text{ and } t \in [0, 1] \\
\lambda_0^{(1)} & \text{if } L_1 > c_1 \text{ and } t \in (1, 2] \\
\lambda_1^{(1)} & \text{if } L_1 \leq c_1 \text{ and } t \in (1, 2]
\end{cases}
\]

for known constants \(c_0\) and \(c_1\) in \(\mathbb{R}\), and for \(\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \text{ and } \lambda_1^{(1)}\) unknown values in \(\mathbb{R}\). To select from the many possible estimating equations for \(\psi\) provided in Theorem 5.2, we follow the approach of [26], which was proved to lead to consistent estimation in [13] provided the main result of the current article holds true. Below we explain why this approach works in the context of this simulation study. [26] proposed to add \(\alpha\) times a function of \(X_\psi(t)\) and \(\mathbf{Z}_{t-}\) to the model for treatment initiation \(\lambda_T\), and find the parameter \(\psi\) such that adding this function has no effect on the estimated hazard (that is, the \(\psi\) that leads to \(\hat{\alpha} = 0\)); that particular \(\psi\) will be the estimate \(\hat{\psi}\). The underlying observation for this procedure is that given \(\mathbf{Z}_{t-}\), for the true \(\psi\), \(X_\psi(t)\) should not help to predict treatment changes (Section 5). Adding \(\alpha\) times a function of \(X_\psi(t)\) and \(\mathbf{Z}_{t-}\) to \(\lambda_T\) can be done in many different ways. For simplicity of calculations, we choose to add \(\alpha X_\psi(0)\), a function of \(X_\psi(t)\) and \(\mathbf{Z}_{t-}\), to the model for treatment initiation \(\lambda_T\) in the time interval \([0, 1]\), and \(\alpha X_\psi(1)\) to the model for treatment initiation \(\lambda_T\) in the time interval \([1, 2]\), both in a way similar to a Cox proportional hazards component: as a factor \(e^{\alpha X_\psi(0)}\) and \(e^{\alpha X_\psi(1)}\), respectively. Let \(\delta_i^{(0)} = 1_{T_i \leq 1}, \delta_i^{(1)} = 1_{1 < T_i \leq 2}, \mathbf{Z}_i(0) = 1_{L_0 < c_0}, \text{ and } \mathbf{Z}_i(1) = 1_{L_1 \leq c_1}.\) The partial likelihood for the model extended with \(X_\psi\) as described above is

\[
L\left(\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \alpha\right) = \prod_{i=1}^{n} \left(\lambda_0^{(0)} e^{\alpha X_\psi(i)(0)}\right)^{\delta_i^{(0)}} e^{-\lambda_0^{(0)} e^{\alpha X_\psi(i)(0)} \min(T_i, 1)} \\
\prod_{i=1}^{n} \left(\lambda_1^{(1)} e^{\alpha X_\psi(i)(1)}\right)^{\delta_i^{(1)}} e^{-\lambda_1^{(1)} (1-\delta_i^{(0)}) e^{\alpha X_\psi(i)(1)} (\min(T_i, 2) - 1)}.
\]
The log likelihood is:
\[
\log L \left( \lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \alpha \right) = \sum_{i=1}^{n} \delta_i^{(0)} \log \left( \lambda_{Z_i(0)}^{(0)} e^{\alpha X_{\psi,i}(0)} \right) - \lambda_{Z_i(0)}^{(0)} \min(T_i, 1) e^{\alpha X_{\psi,i}(0)}
\]
\[
+ \delta_i^{(1)} \log \left( \lambda_{Z_i(1)}^{(1)} e^{\alpha X_{\psi,i}(1)} \right) - \left( 1 - \delta_i^{(0)} \right) \lambda_{Z_i(1)}^{(1)} \min(T_i, 2) - 1 \right) e^{\alpha X_{\psi,i}(1)}
\]
\[
= \sum_{i=1}^{n} \delta_i^{(0)} Z_i(0) \log \lambda_1^{(0)} - Z_i(0) \lambda_1^{(0)} \left( \min(T_i, 1) \right) e^{\alpha X_{\psi,i}(0)}
\]
\[
+ \sum_{i=1}^{n} \delta_i^{(0)} \left( 1 - Z_i(0) \right) \log \lambda_0^{(0)} - \left( 1 - Z_i(0) \right) \lambda_0^{(0)} \min(T_i, 1) e^{\alpha X_{\psi,i}(0)}
\]
\[
+ \sum_{i=1}^{n} \delta_i^{(1)} Z_i(1) \log \lambda_1^{(1)} - \left( 1 - \delta_i^{(0)} \right) Z_i(1) \lambda_1^{(1)} \left( \min(T_i, 2) - 1 \right) e^{\alpha X_{\psi,i}(1)}
\]
\[
+ \sum_{i=1}^{n} \delta_i^{(1)} \left( 1 - Z_i(1) \right) \log \lambda_0^{(1)} - \left( 1 - \delta_i^{(0)} \right) \left( 1 - Z_i(1) \right) \lambda_0^{(1)} \left( \min(T_i, 2) - 1 \right) e^{\alpha X_{\psi,i}(1)}
\]
\[
+ \sum_{i=1}^{n} \delta_i^{(0)} X_{\psi,i}(0) + \delta_i^{(1)} X_{\psi,i}(1).
\]

Following \cite{13}, to calculate \( \hat{\psi} \), we take the derivative of this expression with respect to \( (\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \alpha) \), then set \( \alpha = 0 \), and solve for \( (\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \hat{\psi}) \); as indicated below, consistency of the estimator will follow from Theorem 5.2. We obtain the estimating equations:

\[
0 = \left. \frac{\partial}{\partial \lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \alpha} \log L \left( \lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_0^{(1)}, \lambda_1^{(1)}, \alpha \right) \right|_{\alpha = 0}
\]
\[
= \sum_{i=1}^{n} \begin{pmatrix}
\delta_i^{(0)} (1 - Z_i(0)) - (1 - Z_i(0)) \min(T_i, 1) \\
\delta_i^{(1)} (1 - Z_i(1)) - (1 - \delta_i^{(0)}) (1 - Z_i(1)) \min(T_i, 2) - 1 \\
\delta_i^{(0)} - Z_i(0) \min(T_i, 1) \\
\delta_i^{(1)} Z_i(1) - (1 - \delta_i^{(0)}) Z_i(1) \min(T_i, 2) - 1 \\
\end{pmatrix}
\]
\[
G_i(\psi)
\]

with
\[
G_i(\psi) = - \left( Z_i(0) \lambda_1^{(0)} + (1 - Z_i(0)) \lambda_0^{(0)} \right) \min(T_i, 1) \lambda_{\psi,i}(0)
\]
\[
- \left( Z_i(1) \lambda_1^{(1)} + (1 - Z_i(1)) \lambda_0^{(1)} \right) \left( 1 - \delta_i^{(0)} \right) \min(T_i, 2) - 1 \lambda_{\psi,i}(1)
\]
\[
+ \delta_i^{(0)} \lambda_{\psi,i}(0) + \delta_i^{(1)} \lambda_{\psi,i}(1).
\]
According to Theorem 5.2, with \( h_t(X(t), \mathbf{Z}_{t-}) = X(0) \) for \( t \in [0, 1] \) and \( h_t(X(t), \mathbf{Z}_{t-}) = X(1) \) for \( t \in (1, 2] \), these are indeed unbiased estimating equations. Solving the estimating equations, in the first step, the \( \lambda \)'s are estimated by their maximum likelihood estimates without adding \( X_\psi \) to the model:

\[
\begin{pmatrix}
\hat{\lambda}_0^{(0)} \\
\hat{\lambda}_1^{(0)} \\
\hat{\lambda}_0^{(1)} \\
\hat{\lambda}_1^{(1)}
\end{pmatrix} = \begin{pmatrix}
\sum_{i=1}^n \delta_i^{(0)} (1-Z_i(0)) \\
\sum_{i=1}^n \delta_i^{(0)} Z_i(0) \\
\sum_{i=1}^n \delta_i^{(0)} (1-Z_i(1)) \\
\sum_{i=1}^n \delta_i^{(1)} (1-Z_i(1)) \\
\sum_{i=1}^n \delta_i^{(0)} Z_i(0) \\
\sum_{i=1}^n \delta_i^{(0)} Z_i(0) \min(T_i, 1) \\
\sum_{i=1}^n \delta_i^{(1)} Z_i(1) \\
\sum_{i=1}^n \delta_i^{(1)} Z_i(1) \min(T_i, 2)-1)
\end{pmatrix}.
\]

(47)

General theory, see e.g. [32], says these estimates for the hazard are consistent and asymptotically normal. In the second step, \( \psi \) is then estimated by plugging these estimates for the hazard in \( \sum_{i=1}^n G_i(\psi) \), with \( G_i(\psi) \) as in equation (46), and solving for \( \psi \). In the simulations, \( X_\psi \) is linear in \( \psi \), see Section C.6, and thus \( \sum_{i=1}^n G_i(\psi) \) is also linear in \( \psi \). Therefore, solving for \( \psi \) requires solving a linear, in our case even one-dimensional, equation. If the coefficient before \( \psi \) is non-zero, there is a unique solution \( \hat{\psi} \). In addition, the expectation of \( G_i(\psi) \) at the true \( \lambda \) is linear in \( \psi \), so \( EG_i(\psi) = 0 \) has a unique solution if the coefficient on \( \psi \) in the linear equation \( EG_i(\psi) \) is non-zero, which guarantees consistency and asymptotic normality ([32] Chapter 5). We conclude from Section C.6 and equation (46) that a consistent estimator of \( \psi \) can be defined as \( \hat{\psi} = -\sum_{i=1}^n A_{1i} / \sum_{i=1}^n A_{2i} \), where

\[
A_{1i} = -Y_i \left( Z_i(0) \hat{\lambda}_1^{(0)} + (1-Z_i(0)) \hat{\lambda}_0^{(0)} \right) \min(T_i, 1)
- Y_i \left( Z_i(1) \hat{\lambda}_1^{(1)} + (1-Z_i(1)) \hat{\lambda}_0^{(1)} \right) \left( 1 - \delta_i^{(0)} \right) \left( \min(T_i, 2) - 1 \right)
+ Y_i \delta_i^{(0)} + Y_i \delta_i^{(1)}
\]

and

\[
A_{2i} = - \left( Z_i(0) \hat{\lambda}_1^{(0)} + (1-Z_i(0)) \hat{\lambda}_0^{(0)} \right) \min(T_i, 1) \min(T_i, 2)
- \left( Z_i(1) \hat{\lambda}_1^{(1)} + (1-Z_i(1)) \hat{\lambda}_0^{(1)} \right) \left( 1 - \delta_i^{(0)} \right) \left( \min(T_i, 2) - 1 \right)^2
+ \delta_i^{(0)} \min(T_i, 2) + \delta_i^{(1)} \left( \min(T_i, 2) - 1 \right).
\]

We choose to add \( X_\psi(0) \) to the prediction model for treatment changes in the time interval \([0, 1]\) and to add \( X_\psi(1) \) in the time interval \((1, 2]\). Optimally choosing the function of \( X_\psi(t) \) and \( \mathbf{Z}_{t-} \) to add to the prediction model for treatment changes is an interesting topic for future research.

Algebra shows that in this simulation study, the bias and the MSE of \( \hat{\psi} \) do not depend on the values of \( \psi_0 \) or \( \theta_0 \) (for \( \theta_0 \) this is easily seen by noticing that the estimators depend only on the \( Y_i \), the \( T_i \) (which depend only on pre-treatment variables), and pre-treatment variables). These algebraic calculations were confirmed by simulating scenario 2 with \( \theta_0 = \psi_0 = 300 \) replaced by \( \theta_0 = \psi_0 = 100 \), which lead to the same bias and MSE as scenario 2 itself. Therefore, we did not vary \( \psi_0 \) or \( \theta_0 \) in the simulation study.
C.8 Results of the simulation study

We ran a simulation study with \( n = 500, 1000, 2000, 5000, \) and \( 10000, \) and \( 5000 \) repetitions each. The results for the three settings described in Section C.4 are presented in Section 9, Table 1.

In this simulation study, both for small and large samples, the bias of the estimators is small. In all three settings and for all sample sizes considered (including the small sample size \( n = 100 \)), the MSE of the estimators arises mostly from the variance, not from the bias. Also, if the true parameter \( \psi \) equals 300 as in this simulation study, for \( n = 500, \sqrt{MSE}/\psi = 0.04 \) in setting 1, and 0.08 in setting 3. Thus, the estimates are already precise in relatively small samples. Because, as we noted before, the MSE in this simulation study does not depend on the true parameter \( \psi \), a larger sample size would be required to obtain precise estimators of small true parameter values \( \psi \).

Table 1 also shows that the mean squared error decreases appropriately as the sample size increases. In all three settings, the MSE times the sample size is roughly constant. This indicates that the large-sample theory in Section C.7 (which follows from the fact that the estimating equations are unbiased) provides a reasonable approximation for the rate of convergence of the estimator in finite samples.

We conclude that adding \( X_\psi(0) \) to the prediction model for treatment changes in the time interval \([0, 1]\) and adding \( X_\psi(1) \) in the time interval \((1, 2]\), as described in Section C.7, provides estimators with good finite-sample properties in this simulation study.

Finally, to confirm consistency of the estimators, we ran a simulation study with one dataset and \( n \) equal to one million, in all three settings. This resulted in \( \hat{\psi} = 299.545 \) for setting 1, \( \hat{\psi} = 299.863 \) for setting 2, and \( \hat{\psi} = 299.673 \) for setting 3, all very close to the true value of \( \psi = 300 \) in the simulation study.

We conclude that in this simulation study, continuous-time structural nested models perform extremely well.

D Web-Appendix: Some facts about conditioning

The following definition and two theorems on existence and uniqueness of conditional distributions can be found in Bauer [2] Section 10.3, in a different formulation. The first is a definition of conditional distributions. A conditional distribution of a random variable \( X \) with values in \( \mathbb{R} \) is more than just a set of conditional probabilities \( P(X \leq x | G) \) for \( x \in \mathbb{R} \); it is also a probability measure on \( \mathbb{R} \). Conditional probabilities always exist; a conditional distribution always exists e.g. if \( X \) takes values in \( \mathbb{R} \), but not in general. Conditional probabilities are almost surely unique; under conditions the same is true for conditional distributions.

**Definition D.1.** Let \( X : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{A}) \) be a random variable on a probability space \((\Omega, \mathcal{F}, P)\) with values in a measurable space \((\mathcal{X}, \mathcal{A})\). Let \( \mathcal{G} \subset \mathcal{F} \) be a sub-\( \sigma \)-algebra. Then \( P_{X|G} : \Omega \times \mathcal{A} \rightarrow \mathbb{R} \) is a conditional distribution of \( X \) given \( \mathcal{G} \) if
a) \( \forall A \in \mathcal{A}: \omega \rightarrow P_{X|G}(\omega, A) \) is a version of \( P(\{X \in A\}|G) \), i.e. it is \( \mathcal{G} \)-measurable and for all \( G \in \mathcal{G} \):
\[
\int_G P_{X|G}(\omega, A) \, dP(\omega) = \int_G 1_A(X) \, dP = P(G \cap X^{-1}(A)).
\]
b) \( P_{X|G}(\omega, \cdot) \) is a probability measure on \( (\mathcal{X}, \mathcal{A}) \).

If \( X \) takes values in \( \mathbb{R} \) the distribution function belonging to the probability measure \( P_{X|G} \) is often denoted by \( F_{X|G} \).

**Theorem D.2.** Let \( X : (\Omega, \mathcal{F}) \rightarrow (\mathcal{X}, \mathcal{A}) \) be a random variable on a probability space \( (\Omega, \mathcal{F}, P) \) with values in a measurable space \( (\mathcal{X}, \mathcal{A}) \). Suppose that \( \mathcal{A} \) is a countably generated \( \sigma \)-algebra and \( \mathcal{G} \subseteq \mathcal{F} \) is a sub-\( \sigma \)-algebra. If \( P_{X|G} \) and \( P_{X|G}^* \) are two conditional distributions of \( X \) given \( \mathcal{G} \) then they are almost surely the same in the sense that there exists a \( P \)-null set \( N \in \mathcal{F} \) such that for all \( \omega \in \Omega \setminus N \) and all \( A \in \mathcal{A} \),
\[
P_{X|G}(\omega, A) = P_{X|G}^*(\omega, A).
\]

A topological space \( E \) is called Polish if it has a countable dense subset and there exists a metric that generates the topology and for which the space is complete. An example of a Polish space is \( \mathbb{R}^k \) with the usual topology.

**Theorem D.3.** Let \( X : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{B}(E)) \) be a random variable on a probability space \( (\Omega, \mathcal{F}, P) \) with values in a Polish space \( E \) with its Borel-\( \sigma \)-algebra. Then for every \( \sigma \)-algebra \( \mathcal{G} \subseteq \mathcal{F} \) there exists a conditional distribution \( P_{X|G} \).

The next theorem is very useful in combination with Theorem D.3. Suppose that \( Z \) is a random variable on \( (\Omega, \mathcal{F}) \) with values in the space of cadlag functions on \( [a,b] \), \( D[a,b] \), equipped with the \( \sigma \)-algebra generated by the coordinate projections. Then Theorems D.3 and D.4 imply that for any \( \sigma \)-algebra \( \mathcal{G} \subseteq \mathcal{F} \), \( Z \) has a conditional distribution given \( \mathcal{G} \).

**Theorem D.4.** Suppose that \( a, b \in \mathbb{R} \) are finite. Then \( D[a,b] \) with the Skorohod topology is a Polish space. Furthermore, the \( \sigma \)-algebra on \( D[a,b] \) generated by the Skorohod topology is the same as the \( \sigma \)-algebra on \( D[a,b] \) generated by the coordinate projections.

The first statement of this theorem can be found in Billingsley [3], Chapter 3, the second statement is Theorem 14.5 in the same book.

**Lemma D.5.** Let \( X \) and \( Y \) be random variables on a probability space \( (\Omega, \mathcal{F}, P) \) with values in \( (\mathbb{R}, \mathcal{B}) \), with \( \mathcal{B} \) the Borel-\( \sigma \)-algebra on \( \mathbb{R} \). Suppose that \( \mathcal{G} \subseteq \mathcal{F} \) is a sub-\( \sigma \)-algebra. Then there exist conditional distributions \( P_{X|G} \) and \( P_{Y|G} \). If moreover for every \( x \in \mathbb{Q} \), \( P(X \leq x|\mathcal{G}) = P(Y \leq x|\mathcal{G}) \) a.s., then \( P_{X|G} = P_{Y|G} \) a.s. in the sense that there exists a \( P \)-null set \( N \in \mathcal{F} \) such that for all \( \omega \in \Omega \setminus N \) and all \( B \in \mathcal{B} \),
\[
P_{X|G}(\omega, B) = P_{Y|G}(\omega, B).
\]
Proof. Existence of conditional distributions follows from Theorem [D.3] since ($\mathbb{R}, \mathcal{B}$) is a Polish space. Furthermore a probability measure on ($\mathbb{R}, \mathcal{B}$) is completely determined by its values on $(-\infty, x]$ for $x \in \mathbb{Q}$. So it is enough to prove that there exists a $P$-null set $N \in \mathcal{F}$ such that

$$\forall \omega \in \Omega \setminus N \quad \forall x \in \mathbb{Q} \quad P_{X|G}(\omega, (-\infty, x]) = P_{Y|G}(\omega, (-\infty, x]).$$

(48)

But for every $x \in \mathbb{Q}$,

$$P_{X|G}(\omega, (-\infty, x]) = P(X \leq x | G) \text{ a.s.}$$

$$= P(Y \leq x | G) \text{ a.s.}$$

$$= P_{Y|G}(\omega, (-\infty, x]) \text{ a.s.}$$

Define

$$N = \bigcup_{x \in \mathbb{Q}} \{ \omega : P_{X|G}(\omega, (-\infty, x]) \neq P_{Y|G}(\omega, (-\infty, x]) \}.$$ 

This is a countable union of null sets, so a null set, and it satisfies (48).

For the proof of Lemma [D.8] and Lemma [D.9] the following two lemma’s are used. The first is well-known.

Lemma D.6. If $P_{X|G}$ is a conditional distribution of $X$ given $G$ then

$$E[f(X) | G] = \int f(x) dP_{X|G}(x) \quad \text{a.s.}$$

Lemma D.7. Suppose that $Z$ and $Y$ are random variables on a probability space $(\Omega, \mathcal{F}, P)$ with values in Polish spaces $(\mathcal{Y}, \mathcal{A}_1)$ and $(\mathcal{Z}, \mathcal{A}_2)$, respectively. Then

$$(\omega, A) \mapsto \int_{A} \delta_{z', z(\omega)} P_{Y|Z=z(\omega)}(dy) dz'$$

: $\Omega \times \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \to \mathbb{R}$ is a version of $P_{(Y,Z)|Z}$, i.e. it is a conditional distribution function of $(Y, Z)$ given $Z$.

Proof. Define $\tilde{P}(\omega, A) = \int_{A} \delta_{z', z(\omega)} P_{Y|Z=z(\omega)}(dy) dz'$. Condition a and b of Definition [D.1] have to be checked for $\tilde{P}$. Condition b: for $\omega$ fixed it is indeed a probability measure on $((\mathcal{Y} \times \mathcal{Z}), \sigma(\mathcal{A}_1 \times \mathcal{A}_2))$ (concentrated on $z = z(\omega)$).

Condition a: first it is shown that for any $A$ of the form $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $\omega \to \tilde{P}(\omega, A_1 \times A_2)$ is a version of $P((Y, Z) \in (A_1 \times A_2) | Z)$. Equivalently, for $A$ of the form $A_1 \times A_2$ and $G \in \sigma(Z)$, so $G$ of the form $Z^{-1}(B)$ with $B \in \mathcal{A}_2$,

$$\int_{G} \tilde{P}(\omega, A) dP(\omega) = \int_{G} 1_A((Y, Z)) dP = P(G \cap (Y, Z)^{-1}(A)).$$
This can be shown as follows:

\[
\int_{Z^{-1}(B)} \tilde{P}(\omega, A_1 \times A_2) \, dP(\omega) = \int_B \left( \int_{A_1 \times A_2} \delta_{\omega, z} P_{Y|Z=z} (dy') \, dz' \right) \, dP_Z(z)
\]

\[
= \int_{B \cap A_2} \left( \int_{A_1} P_{Y|Z=z} (dy') \right) \, dP_Z(z)
\]

\[
= \int_{B \cap A_2} P(Y \in A_1 | Z = z) \, dP_Z(z)
\]

\[
= P(Y^{-1}(A_1) \cap Z^{-1}(B \cap A_2))
\]

\[
= P(Z^{-1}(B) \cap (Y, Z)^{-1}(A_1 \times A_2)).
\]

Next it is shown that this is sufficient. Notice first that since \((Y \times Z, \sigma(A_1 \times A_2))\) is a Polish space, there exists a conditional distribution \(P_{(Y,Z)|Z}\). We show that \(\tilde{P}\) and \(P_{(Y,Z)|Z}\) are almost surely equal, using the Uniqueness Theorem on page 27 of Bauer [2]. Remark which is intersection-stable (i.e., finite intersections of elements in \(A\) it stays countable. Notice that these finite intersections are still of the form \(A\) countable intersection-stable generator \(P\) so \(\tilde{P}\) and \(P_{(Y,Z)|Z}\) are equal except for on this null set. Hence because of the countability

\[
\cup_{A_1 \times A_2 : A_1 \in A_1, A_2 \in A_2} \left\{ \omega : \tilde{P}(\omega, A_1 \times A_2) \neq P_{(Y,Z)|Z}(\omega, A_1 \times A_2) \right\}
\]

is a null set. Thus the Uniqueness Theorem on page 27 of Bauer [2] implies that indeed \(\tilde{P}\) and \(P_{(Y,Z)|Z}\) are equal except for on this null set. \(\square\)

Lemma D.8. Suppose that \(Y\) has a continuous conditional distribution function \(F_{Y|Z}\) given \(Z\). Then \(F_{Y|Z}(Y)\) is uniformly distributed on \([0, 1]\) and independent of \(Z\).

Proof. Because of Lemma [D.5] it suffices to prove that for all \(x \in [0, 1]\),

\[
P(F_{Y|Z}(Y) \leq x | Z) = x \text{ a.s.}
\]

This can be done as follows. Define

\[
F_{Y|Z}^{-1}(x+) = \sup \{ y : F_{Y|Z}(y) \leq x \}.
\]

Then

\[
P(F_{Y|Z}(Y) \leq x | Z = z) = P(Y \leq F_{Y|Z}^{-1}(x+) | Z = z),
\]

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since $F_{Y|Z}(Y) \leq x$ implies that $Y \leq \sup \{y : F_{Y|Z}(y) \leq x\} = F_{Y|Z}^{-1}(x+)$ and since $Y \leq F_{Y|Z}^{-1}(x+) = \sup \{y : F_{Y|Z}(y) \leq x\}$ implies that $F_{Y|Z}(Y) \leq x$ by continuity of $F_{Y|Z}$. Hence

$$
P(F_{Y|Z}(Y) \leq x | Z = z) = E\left[1\{y \leq F_{Y|Z}^{-1}(x+)\} | Z = z\right]
$$

$$
= \int_{(y,z') : y \leq F_{Y|Z}^{-1}(x+)} P(Y,Z|Z=z) (dy, dz')
$$

$$
= \int_{(y,z') : y \leq F_{Y|Z}^{-1}(x+)} \delta_{z,z'} F_{Y|Z=z} (dy) dz'
$$

$$
= \int_{y : y \leq F_{Y|Z}^{-1}(x+)} F_{Y|Z}(dy)
$$

$$
= F_{Y|Z}(F_{Y|Z}^{-1}(x+)) = x \quad \text{a.s.,}
$$

where Lemma D.6 is used in the second line, Lemma D.7 in the third line, and continuity of $F_{Y|Z}$ in the last line. \hfill \Box

**Lemma D.9.** Suppose that $X$ is uniformly distributed on $[0,1]$ and independent of $Z$ and that $F_{Y|Z}$ is a conditional distribution function of $Y$ given $Z$. Then $F_{Y|Z}^{-1}(X)$ has conditional distribution function $F_{Y|Z}$ given $Z$.

**Proof.** Because of Lemma D.5 it suffices to prove that for all $s$, $P(F_{Y|Z}^{-1}(X) \leq s | Z = z) = F_{Y|Z}(s)$ a.s. This can be done as follows:

$$
P(F_{Y|Z}^{-1}(X) \leq s | Z = z) = P(F_{Y|Z}^{-1}(X) \leq F_{Y|Z}^{-1} \circ F_{Y|Z}(s) | Z = z)
$$

$$
= P(X \leq F_{Y|Z}(s) | Z = z)
$$

$$
= E\left[1\{x \leq F_{Y|Z}(s)\} | Z = z\right]
$$

$$
= \int_{(x',z') : x' \leq F_{Y|Z}(z') \circ F_{Y|Z}(s)} P(X,Z|Z=z) (dx', dz')
$$

$$
= \int_{(x',z') : x' \leq F_{Y|Z}(z') \circ F_{Y|Z}(s)} \delta_{z,z'} F_{X|Z=z} (dx') dz'
$$

$$
= \int_{x' : x' \leq F_{Y|Z}(z) \circ F_{Y|Z}(s)} F_{X|Z=z} (dx')
$$

$$
= F_{Y|Z}(s) \quad \text{a.s.}
$$

In the second line it is used that if $X \leq F_{Y|Z}(s)$ then also $F_{Y|Z}^{-1}(X) \leq F_{Y|Z}^{-1} \circ F_{Y|Z}(s)$, and moreover that if $X > F_{Y|Z}(s)$ then also, since $F_{Y|Z}(s)$ is in the range of $F_{Y|Z}$ and conditional distribution functions are right continuous, $F_{Y|Z}^{-1}(X) > F_{Y|Z}^{-1} \circ F_{Y|Z}(s)$. In the fourth line I use Lemma D.6, in the fifth line Lemma D.7 is used, and in the last line it is used that $X$ is uniformly distributed on $[0,1]$ given $Z$. \hfill \Box
Lemma D.10. If $X$ is a random variable taking values in $\mathbb{R}$ and for every bounded Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$

$$E[f(X) | Z] = E[f(Y) | Z] \text{ a.s.}$$

then $X$ has the same conditional distribution as $Y$ given $Z$.

**Proof.** Because of Lemma D.5 it suffices to show that for every $x \in \mathbb{R}$, $P(X \leq x | Z) = P(Y \leq x | Z)$ a.s.

Analogously to a proof of the Portmanteau Lemma, define

$$f_m(y) = m \, d(y, (-\infty, x]) \wedge 1$$

for $m = 1, 2, \ldots$. Then $0 \leq f_m \uparrow 1_{(x, \infty)}$ as $m \to \infty$ and $f_m$ is bounded and Lipschitz, so that $E[f_m(X) | Z] = E[f_m(Y) | Z]$ a.s. The remaining part is straightforward:

$$P(X \leq x | Z) = E[1_{(-\infty, x]}(X) | Z]$$
$$= E[1 - 1_{(x, \infty)}(X) | Z]$$
$$= 1 - E[1_{(x, \infty)}(X) | Z] \text{ a.s.}$$
$$= 1 - \lim_{m \to \infty} E[f_m(X) | Z] \text{ a.s.}$$
$$= 1 - \lim_{m \to \infty} E[f_m(Y) | Z] \text{ a.s.}$$
$$= P(Y \leq x | Z) \text{ a.s.},$$

where in the fourth line the conditional Monotone Convergence Theorem (see e.g. [5]) is used.

**Proof of Lemma A.2.** Remark that $\sigma(\overline{Z}^{(n)}_t)$ is increasing in $n$, and that for $t$ on the infinite grid $\sigma(\cup_{n=1}^{\infty} \overline{Z}^{(n)}_t) = \sigma(\overline{Z}_t)$ and for $t$ not on the infinite grid $\sigma(\cup_{n=1}^{\infty} \overline{Z}^{(n)}_t) = \sigma(\overline{Z}_{t-})$, where $\overline{Z}_{t-} = (Z(s) : s < t)$. But the probability that $Z$ jumps at time $t$ is equal to zero. Therefore $E[X|\overline{Z}_t] = E[X|\overline{Z}_{t-}]$ a.s.: any version of $E[X|\overline{Z}_{t-}]$ is a version of $E[X|\overline{Z}_t]$. This can be seen as follows. $E[X|\overline{Z}_{t-}]$ is trivially $\sigma(\overline{Z}_t)$-measurable. So it still has to be checked that for any measurable $f : \overline{Z}_t \to \mathbb{R}$ for which $E(|Xf(\overline{Z}_t)|) < \infty$, $E(Xf(\overline{Z}_t)) = E(E[X|\overline{Z}_{t-}] f(\overline{Z}_t))$. So let $f$ with $E(|Xf(\overline{Z}_t)|) < \infty$ be given. Define $g : \overline{Z}_{t-} \to \overline{Z}_t$ as the “continuous” extension:

$$g(\overline{Z}_{t-})(s) = \begin{cases} z(s) & \text{if } s < t \\ \lim_{u \uparrow t} z(u) & \text{if } s = t. \end{cases}$$

Then

$$E(Xf(\overline{Z}_t)) = E(E(Xf(g(\overline{Z}_{t-}))))$$
$$= E(E[X|\overline{Z}_{t-}] f(g(\overline{Z}_{t-})))$$
$$= E(E[X|\overline{Z}_{t-}] f(\overline{Z}_t)),$$

where in the first and the last line it is used that the probability that $Z$ jumps at time $t$ is equal to zero. Therefore the conditional expectation of $X$ given $\overline{Z}_t$ is almost surely equal to the conditional expectation of $X$ given $\sigma(\cup_{n=1}^{\infty} \overline{Z}^{(n)}_t)$. □
E Web-Appendix: A corollary of the Local Inverse Function Theorem

Continuation of the proof of Lemma 7.7.

It is easy to see that \( \phi \) is differentiable at \((0, y_0)\) with non-singular derivative. Therefore, the Local Inverse Function Theorem implies that there exists an open neighbourhood \( V_{ho,y_0} \) of \((h_0, y_0)\) such that \( W = \phi(V_{ho,y_0}) \) is open and \( \phi|_{V_{ho,y_0}} : V_{ho,y_0} \to W \) is a \( C^1 \)-diffeomorphism. Hence \( \phi^{-1} \) exists and is \( C^1 \).

Notice that \( \phi^{-1}(h, x) \) must have the form \((h, y)\) with \( y \) satisfying \( F_h(y) = x \). For \((h, x) \in W\) such \( y \) is unique, since all \( F_h \) are non-decreasing by assumption and \( F'_h(y) \) is non-zero on \( V_{ho,y_0} \). Thus \( F_h^{-1}(x) \) is well-defined on \( W \), and it follows that

\[
\phi^{-1}(h, x) = (h, F_h^{-1}(x)).
\]

Both \( \phi \) and \( \phi^{-1} \) are \( C^1 \), so the chain rule can be applied to calculate

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= D (\phi \phi^{-1}) (h, x)
= (D\phi)(\phi^{-1}(h, x)) \cdot (D\phi^{-1})(h, x)
= \begin{pmatrix}
1 & 0 \\
\frac{\partial}{\partial h} F_h(y) & F_h'(y)
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 \\
\frac{\partial}{\partial x} F_h^{-1}(x) & (F_h^{-1})'(x)
\end{pmatrix},
\]

with \( y = F_h^{-1}(x) \). Lemma 7.7 follows by comparing the bottom left entries of the matrices on the left- and right hand side of this equation. \( \square \)

F Web-Appendix: Lipschitz continuity and differentiability

The following lemma can be useful for proving Lipschitz continuity of quotients of functions.

Lemma F.1. Suppose that \( f \) and \( g \) are functions from \( \mathbb{R} \) to \( \mathbb{R} \) which are Lipschitz continuous with Lipschitz constants \( L_f \) resp. \( L_g \). Suppose furthermore that \( g \geq \varepsilon > 0 \) and \(|f| \leq C \) for some \( C > 0 \). Then \( f/g \) is Lipschitz continuous with Lipschitz constant e.g. \( L_f/\varepsilon + C L_g/\varepsilon^2 \).

Proof.

\[
\left| \frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_2)} \right|
\leq \left| \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} \right| + \left| \frac{f(x_2) - f(x_1)}{g(x_1) - g(x_2)} \right|
= \left| \frac{1}{g(x_1)} \right| |f(x_1) - f(x_2)| + \left| \frac{f(x_2)}{g(x_1) g(x_2)} \right| |g(x_1) - g(x_2)|
\leq \frac{1}{\varepsilon} L_f |x_1 - x_2| + \frac{C}{\varepsilon^2} L_g |x_1 - x_2|. \quad \square
\]
The next lemma deals with a continuous function \( f \) on a closed interval which is continuously differentiable on the interior of that interval. If \( f' \) can be continuously extended to the closed interval, then \( f \) is continuously differentiable on the closed interval.

**Lemma F.2.** Suppose that \( f \) is continuous on \( [t_1, t_2] \) and \( f \) is continuously differentiable on \((t_1, t_2)\). Suppose furthermore that \( f' \) has a continuous extension to \( [t_1, t_2] \). Then \( f \) is differentiable from the right at \( t_1 \) with derivative \( \lim_{t \uparrow t_1} f'(t) \) and differentiable from the left at \( t_2 \) with derivative \( \lim_{t \downarrow t_2} f'(t) \).

**Proof.** I just prove the statements for \( t_1 \); the proof for \( t_2 \) is similar. Define \( g(t) = f(t_1) + \int_{t_1}^t f'(x)dx \). Then \( g \) is continuous and continuously differentiable on \( (t_1, t_2) \) with derivative \( f'(t) \) on \( (t_1, t_2) \) and \( \lim_{t \uparrow t_1} f'(t) \) at \( t_1 \). It suffices to show that \( f - g \) on \( [t_1, t_2) \). \( f - g \) is also constant on \( [t_1, t_2) \). \( (f - g)(t_1) = 0 \). Thus \( f = g \) on \( [t_1, t_2) \). \( \square \)

**G Web-Appendix: Some theory about differential equations**

**Theorem G.1.** Suppose that a function \( D(y, t; Z_t) \) satisfies

a) **(continuity between the jump times of \( Z \).)** If \( Z \) does not jump in \( (t_1, t_2) \) then \( D(y, t; Z_t) \) is continuous in \( (y, t) \) on \( [t_1, t_2) \) and can be continuously extended to \( [t_1, t_2] \).

b) **(Lipschitz continuity).** For each \( \omega \in \Omega \) there exists a constant \( L(\omega) \) such that

\[
|D(y, t; Z_t) - D(z, t; Z_t)| \leq L(\omega) |y - z|
\]

for all \( t \in [0, \tau] \) and all \( y, z \).

Suppose furthermore that for each \( \omega \in \Omega \) there are no more than finitely many jump times of \( Z \). Then, for each \( t_0 \in [0, \tau] \) and \( y_0 \in \mathbb{R} \), there is a unique continuous solution \( x(t; t_0, y_0) \) to

\[
\frac{dx(t)}{dt} = D(x(t), t; Z_t)
\]

with boundary condition \( x(t_0) = y_0 \) and this solution is defined for all \( t \in [0, \tau] \).

This theorem follows from well-known results about differential equations, see e.g. Duistermaat and Eckhaus [4] Chapter 2.

For the next theorem we also refer to Duistermaat and Eckhaus [4] Chapter 2. It is a consequence of Gronwall’s lemma.
Theorem G.2. Suppose that I is an open or closed interval in $\mathbb{R}$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $C : I \to [0, \infty)$ is continuous, and suppose that

$$\| f(x, y) - f(x, z) \| \leq C(x) \| y - z \|$$

for all $x \in I$ and $y, z \in \mathbb{R}^n$. Then, for every $x_0 \in I$ and $y_0 \in \mathbb{R}$, there is a unique solution $y(x) = f(x, y(x))$ with $y(x_0) = y_0$, and this solution is defined for all $x \in I$. If $g : I \times \mathbb{R} \to \mathbb{R}^n$ is continuous and $z : I \to \mathbb{R}^n$ is a solution of $z'(x) = g(x, z(x))$ then

$$\| y(x) - z(x) \| \leq e^{\int_{x_0}^x C(\xi) d\xi} \| y(x_0) - z(x_0) \| + \int_{x_0}^x e^{\int_{\xi}^x C(\eta) d\eta} \| f(\xi, z(\xi)) - g(\xi, z(\xi)) \| d\xi$$

for all $x, x_0 \in I$ with $x_0 \leq x$.

In Duistermaat and Eckhaus [4] the interval is always an open interval, but as is generally known this can be overcome by extending both $f$ and $g$ outside the closed interval $I$ by taking the values at the boundary of $I$. This preserves the Lipschitz- and continuity conditions. Existence and uniqueness on all of finitely many intervals implies global existence and uniqueness; this is the way one often applies this theorem.

This article is about a differential equation with end condition at $\tau$, so interested lies in $x, x_0$ with $x \leq x_0$. The following corollary can be used.

Corollary G.3. Suppose that the conditions of Theorem G.2 are satisfied. Then, for every $x_0 \in I$ and $y_0 \in \mathbb{R}^n$, there is a unique solution $y(x)$ of $y'(x) = f(x, y(x))$ with $y(x_0) = y_0$, and this solution is defined for all $x \in I$. If $g : I \times \mathbb{R} \to \mathbb{R}^n$ is continuous and $z : I \to \mathbb{R}^n$ is a solution of $z'(x) = g(x, z(x))$ then

$$\| y(x) - z(x) \| \leq e^{\int_{x_0}^x C(s) ds} \| y(x_0) - z(x_0) \| + \int_{x_0}^x e^{\int_{\xi}^x C(\eta) d\eta} \| f(s, z(s)) - g(s, z(s)) \| ds$$

for all $x, x_0$ with $x \leq x_0$.

Proof. Define $\tilde{y}(t) = y(x_0 - t)$. Then

$$\tilde{y}'(t) = \frac{\partial}{\partial t} y(x_0 - t)$$

$$= -y'(x_0 - t)$$

$$= -f(x_0 - t, y(x_0 - t))$$

$$= -f(x_0 - t, \tilde{y}(t))$$

$$= \tilde{f}(t, \tilde{y}(t))$$

where $\tilde{f}(t, y) = -f(x_0 - t, y)$. So $\tilde{y}(t) = y(x_0 - t)$ is a solution of the differential equation $\tilde{y}'(t) = \tilde{f}(t, y(t))$ with boundary condition $\tilde{y}(0) = y(x_0) = y_0$. Applying Theorem G.2 on
\( \tilde{y} \) concludes the proof, as follows.

\[
\| y(x) - z(x) \| = \| y(x - x_0 + x_0) - z(x - x_0 + x_0) \|
\]
\[
= \| y(x_0 - (x_0 - x)) - z(x_0 - (x_0 - x)) \|
\]
\[
= \| \tilde{y}(x_0 - x) - \tilde{z}(x_0 - x) \|
\]
\[
= \| \tilde{y}(t) - \tilde{z}(t) \|
\]

with \( t = x_0 - x \geq 0 \). Notice that since because of equation (49),

\[
\| \tilde{f}(t, y) - \tilde{f}(t, z) \| \leq C(\tau) - t \| y - z \| =: \tilde{C}(t) \| y - z \|
\]

with \( \tilde{C}(t) = C(x_0 - t) \). Hence Theorem [G.2] implies that

\[
\| y(x) - z(x) \| \leq e^{\int_0^t C(\xi)\,d\xi} \| \tilde{y}(0) - \tilde{z}(0) \| + \int_0^t e^{\int_0^s C(\eta)\,d\eta} \| \tilde{f}(\xi, \tilde{z}(\xi)) - \tilde{g}(\xi, \tilde{z}(\xi)) \| \, d\xi
\]
\[
= e^{\int_0^t C(x_0 - \xi)\,d\xi} \| y(x_0 - 0) - z(x_0 - 0) \|
\]
\[
+ \int_0^t e^{\int_0^s C(x_0 - \eta)\,d\eta} \| \tilde{f}(\xi, \tilde{z}(\xi)) - \tilde{g}(\xi, \tilde{z}(\xi)) \| \, d\xi.
\]

For the first term a change of variables is done; \( \xi \) from 0 to \( t \), define \( s = x_0 - \xi; d\xi = -ds \).

\[ 0 \leq \xi \leq t; s \] from \( x_0 - 0 \) to \( x_0 - t = x_0 - (x_0 - x) = x \). Therefore, the first term is equal to

\[
e^{-\int_0^t C(s)\,ds} \| y(x_0) - z(x_0) \| = e^{\int_0^s C(s)\,ds} \| y(x_0) - z(x_0) \|.
\]

For the second term similar changes of variables can be done, resulting in Corollary [G.3] \( \Box \)

**Proof of Theorem [A.1]** Write \( I = [x_1, x_2] \). In order to apply Theorem [G.2] define an extension \( \tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) of \( f \) as follows:

\[
\tilde{f}(x, y) = \begin{cases} 
  f(x, y) & \text{if } (x, y) \in I \times [y_1, y_2], \\
  f(x_1, y) & \text{if } (x, y) \in (-\infty, x_1) \times [y_1, y_2], \\
  f(x_2, y) & \text{if } (x, y) \in (x_2, \infty) \times [y_1, y_2], \\
  \tilde{f}(x, y_1) & \text{if } y < y_1, \\
  \tilde{f}(x, y_2) & \text{if } y > y_2.
\end{cases}
\]

If there exists a unique solution of the differential equation with \( \tilde{f} \) and this solution stays in \([y_1, y_2]\), then this solution is also the unique continuous solution of the differential equation with \( f \).

On the differential equation with \( \tilde{f} \), Theorem [G.2] will be applied. \( \tilde{f} \) is continuous on \( \mathbb{R} \times \mathbb{R} \) because \( f \) is continuous on \( I \times [y_1, y_2] \) and \( \tilde{f}(x, y_1) = 0 = \tilde{f}(x, y_2) \) for every \( x \in I \). Also there exists a continuous \( \tilde{C} \) satisfying equation (49): define \( \tilde{C} \) as an extension of \( C \) as follows:

\[
\tilde{C}(x) = \begin{cases} 
  C(x) & \text{if } x \in I, \\
  C(x_1) & \text{if } x \in (-\infty, x_1), \\
  C(x_2) & \text{if } x \in (x_2, \infty).
\end{cases}
\]

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That this $\tilde{C}$ satisfies equation (49) can easily be checked by first considering $x \in [x_1, x_2]$ and reducing different $x$ to $x_1$ and $x_2$.

Thus Theorem \text{[G.2]} implies that there is a unique solution of the differential equation with $\tilde{f}$. That the solution stays in $[y_1, y_2]$ is clear from the fact that $\tilde{f} = 0$ for $y \in \{y_1, y_2\}$ and the fact that the solution is unique.

Since $g$ can be extended the same way as $f$ and $z$ stays in $[y_1, y_2]$ by assumption, the bound for $|y(x) - z(x)|$ given by Theorem \text{[G.2]} also holds here. The bound of Theorem \text{[A.1]} follows with the same reasoning from Corollary \text{[G.3]}. This finishes the proof. \hfill \square

Corollary G.4. Suppose that $I = [x_1, x_2] \subset [0, y_2]$ is a closed interval in $\mathbb{R}$, $f : \{(x, y) \in I \times [0, y_2] : y \geq x\}$ \mathbb{R} is continuous with for all $x \in I$, $f(x, y_2) = 0$ and $f(x, x) \leq 1$ and $C : I \to [0, \infty)$ is continuous, and suppose that

$$|f(x, y) - f(x, z)| \leq C(x)|y - z|$$

for all $x \in I$ and $y, z \in [x, y_2]$. Then for every $y_0 \in [x_2, y_2]$ there is a unique solution $y(x)$ of $y'(x) = f(x, y(x))$ with final condition $y(x_2) = y_0$, and this solution is defined for all $x \in I$. Furthermore $y(x) \in [x, y_2]$ for all $x \in I$. Suppose that $g : \{(x, y) \in I \times [0, y_2] : y \geq x\} \to \mathbb{R}$ is continuous and $z : I \to [0, y_2]$ is a solution of $z'(x) = g(x, z(x))$ with $z(x) \geq x$ then

$$|y(x) - z(x)| \leq e^{\int_{x}^{x_2} C(s) ds} |y(x_2) - z(x_2)| + \int_{x}^{x_2} e^{\int_{s}^{x_2} C(\eta) d\eta} |f(s, z(s)) - g(s, z(s))| ds$$

for all $x \in I$.

Proof. This can be proved the same way as Corollary \text{[A.1]} if one defines

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in I \times [0, y_2] : y \geq x \\ f(x, y_2) & \text{if } x \in I \text{ and } y > y_2 \\ f(x, x) & \text{if } x \in I \text{ and } y < x \\ \tilde{f}(x_1, y) & \text{if } x < x_1 \\ \tilde{f}(x_2, y) & \text{if } x > x_2. \end{cases}$$

Remark that the solution $y(x)$ stays in $(x, y) \in I \times [0, y_2] : y \geq x$ for $x \in I$ since $f(x, x) \leq 1$ and $f(x, y_2) = 0$. \hfill \square

Remark that if it is not known whether $f(x, x) \leq 1$ but it is known that $f$ is continuous in $(x, y)$ and Lipschitz continuous in $y$ on the set mentioned in Corollary \text{[G.4]} then the proof above shows that if a solution $y(x)$ exists for which $y(x) \geq x$ for $x \leq x_2$ then this solution is unique.

H Web-Appendix: Convergence Theorems

A lemma with a corollary:
Lemma H.1. Suppose that the random functions \( f^\omega : [y_1, y_2] \to \mathbb{R} \) and \( f_n^\omega : [y_1, y_2] \to \mathbb{R} \) (\( n = 1, 2, \ldots \)) are ‘asymptotically uniformly equicontinuous with probability one’, i.e. there exists \( \Omega' \subset \Omega \) with \( P(\Omega') = 1 \) such that for all \( \omega \in \Omega' \): \( \forall \varepsilon > 0 \ \exists \delta > 0 \ \exists N : \forall n \geq N : \)

\[
|y - z| < \delta \Rightarrow \begin{cases} 
|f_n^\omega (y) - f_n^\omega (z)| < \varepsilon \\
|f^\omega (y) - f^\omega (z)| < \varepsilon.
\end{cases}
\]

Suppose furthermore that for all \( y \in (y_1, y_2) \cap \mathbb{Q} \), \( f_n^\omega (y) \to f^\omega (y) \) a.s. Then

\[
\sup_{y \in [y_1, y_2]} |f_n^\omega (y) - f^\omega (y)| \to 0 \quad \text{a.s.}
\]

Remark: the regularity condition for Lemma H.1 is e.g. satisfied if there is a Lipschitz constant \( L \) such that all \( f^\omega \) and \( f_n^\omega \) are Lipschitz continuous in \( y \) with Lipschitz constant \( L \) (define \( \delta = \varepsilon / L \)).

Proof. Define \( \Omega'' = \{ \omega : f_n^\omega (y) \to f^\omega (y) \ \forall y \in \mathbb{Q} \cap (y_1, y_2) \} \). Then \( \Omega'' \) has probability one (\( \Omega := \Omega \backslash \text{countably many null sets} \)). Define \( \Omega_0 = \Omega' \cap \Omega'' \). Then also \( \Omega_0 \) has probability one. We show that for all \( \omega \in \Omega_0 \): \( \sup_{y \in [y_1, y_2]} |f_n^\omega (y) - f^\omega (y)| \to 0 \).

Let \( \omega \in \Omega_0 \) and \( \varepsilon > 0 \) be given. To show: there exists an \( N \) such that \( \forall n \geq N \): \( \sup_{y \in [y_1, y_2]} |f_n^\omega (y) - f^\omega (y)| < \varepsilon \). Choose \( N_1 \) and \( \delta > 0 \) such that for all \( n \geq N_1 \):

\[
|y - z| < \delta \Rightarrow \begin{cases} 
|f_n^\omega (y) - f_n^\omega (z)| < \varepsilon / 3 \\
|f^\omega (y) - f^\omega (z)| < \varepsilon / 3.
\end{cases}
\]

This is possible because \( \omega \in \Omega' \). Next choose \( y^{(i)}_1, \ldots, y^{(N_2)} \in \mathbb{Q} \cap (y_1, y_2) \) such that for all \( y \in [y_1, y_2] \) there is a \( y^{(i)} \) with \( |y - y^{(i)}| < \delta \). After this choose \( N_3 \) such that for all \( n \geq N_3 \):

\[
\max_{1 \leq i \leq N_2} |f_n^\omega (y^{(i)}) - f^\omega (y^{(i)})| < \varepsilon / 3.
\]

This is possible because \( \omega \in \Omega'' \) and the number of \( y^{(i)} \)'s is finite. Then for \( n \geq N = \max\{N_3, N_1\} \):

\[
|f_n^\omega (y) - f^\omega (y)| \leq \min_{1 \leq i \leq N_2} \left( |f_n^\omega (y) - f_n^\omega (y^{(i)})| + |f^\omega (y^{(i)}) - f^\omega (y)| \right)
\]

\[
+ \max_{1 \leq i \leq N_2} \left( |f_n^\omega (y^{(i)}) - f^\omega (y^{(i)})| \right)
\]

\[
< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon.
\]

\[\square\]

Corollary H.2. Under the conditions of Lemma H.1, if \( X_n \) is a series of random variables with values in \([y_1, y_2] \), then

\[
|f_n^\omega (X_n) - f^\omega (X_n)| \to 0 \quad \text{a.s.}
\]
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