CORRIGENDUM TO “FROM $A_1$ TO $A_\infty$: NEW MIXED INEQUALITIES FOR CERTAIN MAXIMAL OPERATORS”

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Abstract. We devote this note to correct an estimate concerning mixed inequalities for the generalized maximal function $M_\Phi$, when certain properties of the associated Young function $\Phi$ are assumed.

Although the obtained estimates turn out to be slightly different, they are good extensions of mixed inequalities for the classical Hardy-Littlewood maximal functions $M_r$, with $r \geq 1$. They also allow us to obtain mixed estimates for the generalized fractional maximal operator $M_{\gamma, \Phi}$, when $0 < \gamma < n$ and $\Phi$ is an $L \log L$ type function.

Overview

Throughout this note we shall consider a Young function $\Phi$ with the following properties. Given $r \geq 1$ and $\delta \geq 0$, we say that a Young function $\Phi$ belongs to the family $\mathfrak{F}_{r, \delta}$ if it is submultiplicative, has lower type $r$ and satisfies the condition

$$\frac{\Phi(t)}{t^r} \leq C_0 (\log t)^\delta, \quad \text{for } t \geq t^*,$$

for some constants $C_0 > 0$ and $t^* \geq 1$.

In [2] we obtained mixed estimates for the operator $M_\Phi$, where $\Phi$ belongs to $\mathfrak{F}_{r, \delta}$. Concretely, we stated the inequality

$$uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f|}{t} \right) uv^r$$

where $u$ and $v^r$ are weights belonging to the $A_1$-Muckenhoupt class.

Later, in [1], the same kind of estimate was obtained when $v^r$ is only assumed to be an $A_\infty$ weight.

In the proofs of both results we used Claim 3.4 in [2], and Claims 1 and 3 in [1] as auxiliary tools. These claims have an error on a Hölder estimate, where a limiting argument was mistakenly used and it cannot be adapted to obtain the inequality given above.

The purpose of this note is give a proof of Theorem 1 in [1] that avoids this step on the claims and allows us to obtain a slightly different estimate, that will still be useful for our purposes. We shall only modify the results obtained in [1], since they are more general and the corresponding version of those in [2] will follow as an immediate consequence. The modified mixed estimate in [1] is the following.
Theorem 1 (Corrected version of Theorem 1 in [1]). Let $r \geq 1$, $\delta \geq 0$ and $\Phi \in \mathfrak{F}_{r, \delta}$. If $u \in A_1$ and $v^r \in A_\infty$ then there exists a positive constant $\varepsilon_0$ such that the inequality

$$uw^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \Phi) \left( \frac{|f(x)|}{t} \right) u(x)v^r(x) \, dx$$

holds for every positive $t$ and every $0 < \varepsilon < \varepsilon_0$, where $\eta_\varepsilon(z) = z(1 + \log^+ z)^{6/\varepsilon}$ and $C$ depends on $\varepsilon$.

It is not difficult to see that $M_\Phi v \gtrsim v$ when $\Phi$ belongs to $\mathfrak{F}_{r, \delta}$. So we have the following result as an immediate consequence of the theorem above.

Corollary 2 (Corrected version of Corollary 2 in [1]). Under the assumptions in Theorem 1, there exists a positive constant $\varepsilon_0$ such that

$$uw^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{M_\Phi v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \Phi) \left( \frac{|f(x)|}{t} \right) u(x)v^r(x) \, dx$$

holds for every positive $t$ and every $0 < \varepsilon < \varepsilon_0$, where $\eta_\varepsilon$ is as above and $C$ depends on $\varepsilon$.

Throughout these notes, all references, lemmas and theorems will follow the label given in [1].

1. Proof of Theorem 1

We shall first give some preliminaries in order to proceed with the proof. Recall that $w \in A_\infty$ if there exists a positive constant $C$ such that

$$\left( \frac{1}{|Q|} \int_Q w \right) \exp \left( \frac{1}{|Q|} \int_Q \log w^{-1} \right) \leq C$$

for every cube $Q$ in $\mathbb{R}^n$. The smallest constant for which the inequality above holds is denoted by $|w|_{A_\infty}$.

The following lemma will be useful in the sequel. It can be found in [3].

Lemma 3. Let $w \in A_\infty$ and let $r_w = 1 + \frac{1}{\tau_n[w]_{A_\infty}}$. Then for any cube $Q$ we have

$$\left( \frac{1}{|Q|} \int_Q w^{r_w} \right)^{1/r_w} \leq \frac{2}{|Q|} \int_Q w.$$

As a consequence, given any cube $Q$ and a measurable set $E \subseteq Q$ we have that

$$\frac{w(E)}{w(Q)} \leq 2 \left( \frac{|E|}{|Q|} \right)^{\varepsilon_w},$$

where $\varepsilon_w = 1/(1 + \tau_n[w]_{A_\infty})$. The constant $\tau_n$ is purely dimensional and can be chosen as $2^{11+n}$.

Recall that we are dealing with a function $\Phi \in \mathfrak{F}_{r, \delta}$, where $r \geq 1$ and $\delta \geq 0$ are given. Since we are assuming $v^r \in A_\infty$, there exists $\varepsilon_1 > 0$ such that $v^{r+\varepsilon} \in A_\infty$ for every $0 < \varepsilon \leq \varepsilon_1$.

Fix $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, where $\varepsilon_2 > 0$ will be chosen later. Then we have that $v^r \in RH_s$, where $s = 1 + \varepsilon/r$. We shall denote $\Psi_\varepsilon = \eta_\varepsilon \circ \Phi$. We shall follow the same sketch and steps as in [1], where the entire proof is included for the sake of clearness. Recall that it is enough to prove that

$$uw^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\Phi,D}(fv)(x)}{v(x)} > t \right\} \right) \leq C_\varepsilon \int_{\mathbb{R}^n} \Psi_\varepsilon \left( \frac{|f(x)|}{t} \right) u(x)v^r(x) \, dx,$$
where $D$ is a given dyadic grid. We can also assume that $t = 1$ and that $g = |f|v$ is a bounded function with compact support. Then, for a fixed number $a > 2^n$, we can write
\[
uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\Phi,D}(fv)(x)}{v(x)} > 1 \right\} \right) = \sum_{k \in \mathbb{Z}} uv^r \left( \left\{ x : \frac{M_{\Phi,D}g(x)}{v(x)} > 1, a^k < v \leq a^{k+1} \right\} \right)
\]
\[
= \sum_{k \in \mathbb{Z}} uv^r(E_k).
\]
For every $k \in \mathbb{Z}$ we consider the set
\[
\Omega_k = \left\{ x \in \mathbb{R}^n : M_{\Phi,D}g(x) > a^k \right\},
\]
and by virtue of the Calderón-Zygmund decomposition of the space (see [1, Lemma 6]) there exists a collection of disjoint dyadic cubes $\{Q_j^k\}_j$ that satisfies
\[
\Omega_k = \bigcup_j Q_j^k,
\]
and $\|g\|_{\Phi,Q_j^k} > a^k$ for each $j$. By maximality, we have
\[
a^k < \|g\|_{\Phi,Q_j^k} \leq 2^n a^k, \quad \text{for every } j.
\]
For every $k \in \mathbb{Z}$ we now proceed to split the obtained cubes in different classes, as in [4]. Given a nonnegative integer $\ell$, we set
\[
\Lambda_{\ell,k} = \left\{ Q_j^k : a^{(k+\ell)r} \leq \frac{1}{|Q_j^k|} \int_{Q_j^k} v^r < a^{(k+\ell+1)r} \right\},
\]
and also
\[
\Lambda_{-1,k} = \left\{ Q_j^k : \frac{1}{|Q_j^k|} \int_{Q_j^k} v^r < a^{kr} \right\}.
\]

The next step is to split every cube in the family $\Lambda_{-1,k}$. Fixed $Q_j^k \in \Lambda_{-1,k}$, we perform the Calderón-Zygmund decomposition of the function $v^rX_{Q_j^k}$ at level $a^{kr}$. Then we obtain, for each $k$, a collection $\{Q_{j,i}^k\}_i$ of maximal cubes, contained in $Q_j^k$ and which satisfy
\[
a^{kr} \leq \frac{1}{|Q_{j,i}^k|} \int_{Q_{j,i}^k} v^r \leq 2^n a^{kr}, \quad \text{for every } i.
\]

Also we define the sets
\[
\Gamma_{\ell,k} = \left\{ Q_j^k \in \Lambda_{\ell,k} : \left| Q_j^k \cap \left\{ x : a^k < v \leq a^{k+1} \right\} \right| > 0 \right\},
\]
and also
\[
\Gamma_{-1,k} = \left\{ Q_j^k : Q_j^k \in \Lambda_{-1,k} \text{ and } \left| Q_j^k \cap \left\{ x : a^k < v \leq a^{k+1} \right\} \right| > 0 \right\}.
\]

Since $E_k \subseteq \Omega_k$, we can estimate
\[
\sum_{k \in \mathbb{Z}} uv^r(E_k) = \sum_{k \in \mathbb{Z}} uv^r(\bigcup_{k \in \mathbb{Z}} E_k \cap \Omega_k)
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_j uv^r(\bigcup_{k \in \mathbb{Z}} E_k \cap Q_j^k)
\]
From the last inequality we can obtain the thesis.

\[\sum_{k \geq N} \sum_{\ell \geq 0} \sum_{Q_j^k \in \Gamma_{\ell,k}} a^{(k+1)r} u(E_k \cap Q_j^k) + \sum_{k \geq N} \sum_{i:Q_j^k \in \Gamma_{-1,k}} a^{(k+1)r} u(Q_j^k) \leq C_\epsilon \int_{\mathbb{R}^n} \Psi_\epsilon(|f|) u v^r\]

holds, then the proof would be completed by letting \(N \to -\infty\).

We shall also need the following lemma from [4]. We include an adaptation of the proof involving our parameters for the sake of clearness.

**Lemma 4.** Let \(\ell \geq 0\) and \(Q_j^k \in \Gamma_{\ell,k}\). If \(u \in A_{\infty}\) and \(v^r \in A_q\) for some \(1 < q < \infty\), then there exists positive constants \(c_1\) and \(c_2\) depending on \(u\) and \(v^r\) such that

\[u(E_k \cap Q_j^k) \leq c_1 e^{-c_2 r^q} u(Q_j^k).\]

Furthermore, we can pick \(c_1 = 2 \left([v^r]_{A_q} a^r\right)^{1/(q-1) (1+\tau_n[u]_{A_{\infty}})}\) and \(c_2 = \ln a/(q-1)(1+\tau_n[u]_{A_{\infty}})\), where \(\tau_n\) is the dimensional constant appearing in Lemma 3.

**Proof.** Since \(v^r \in A_{\infty}\), there exists \(q > 1\) such that \(v^r \in A_q\). Since \(Q_j^k \in \Gamma_{\ell,k}\), we have that

\[\left(\frac{|E_k \cap Q_j^k|}{|Q_j^k|}\right)^{q^{-1}} \leq \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} v^{r(1-q')}\right)^{q^{-1}} a^{r(k+1)} \leq \frac{[v^r]_{A_q} |Q_j^k|}{v^r(Q_j^k)} a^{r(k+1)} \leq [v^r]_{A_q} a^{(1-\ell)r}.\]

Since \(u \in A_1 \subseteq A_{\infty}\), by Lemma 3 and the estimate above we have that

\[\frac{u(E_k \cap Q_j^k)}{u(Q_j^k)} \leq 2 \left(\frac{|E_k \cap Q_j^k|}{|Q_j^k|}\right)^{1/(1+\tau_n[u]_{A_{\infty}})} \leq 2 \left([v^r]_{A_q} a^{(1-\ell)r}\right)^{1/(q-1)(1+\tau_n[u]_{A_{\infty}})).\]

From this last inequality we can obtain the thesis. \(\square\)

**Proof of Theorem 1.** Since \(u \in A_1\), we have \(u \in A_{\infty}\). Moreover, the assumption \(v^r \in A_{\infty}\) implies that there exists \(1 < q < \infty\) such that \(v^r \in A_q\). We take \(\varepsilon_2 = r/((q-1)(1+\tau_n[u]_{A_{\infty}}))\) and \(\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}\). Fixed \(0 < \varepsilon < \varepsilon_0\), recall that we have to estimate the two quantities

\[A_N := \sum_{k \geq N} \sum_{\ell \geq 0} \sum_{Q_j^k \in \Gamma_{\ell,k}} a^{(k+1)r} u(E_k \cap Q_j^k)\]

and

\[B_N := \sum_{k \geq N} \sum_{i:Q_j^k \in \Gamma_{-1,k}} a^{(k+1)r} u(Q_j^k)\]

by \(C_\varepsilon \int_{\mathbb{R}^n} \Psi_\varepsilon(|f|) u v^r\), with \(C_\varepsilon\) independent of \(N\).

We shall start with the estimate of \(A_N\). Fix \(\ell \geq 0\) and let \(\Delta_\ell = \bigcup_{k \geq N} \Gamma_{\ell,k}\). We define recursively a sequence of sets as follows:

\[P_0^\ell = \{Q : Q \text{ is maximal in } \Delta_\ell \text{ in the sense of inclusion}\}\]
and for \( m \geq 0 \) given we say that \( Q^k_j \in P^\ell_m \) if there exists a cube \( Q^s_i \) in \( P^\ell_m \) which verifies

\[
\frac{1}{|Q^k_j|} \int_{Q^k_j} u > \frac{2}{|Q^s_i|} \int_{Q^s_i} u
\]

and it is maximal in this sense, that is,

\[
\frac{1}{|Q^k_j|} \int_{Q^k_j} u \leq \frac{2}{|Q^s_i|} \int_{Q^s_i} u
\]

for every \( Q^k_j \subseteq Q^k_j' \subseteq Q^s_i \).

Let \( P^\ell = \bigcup_{m \geq 0} P^\ell_m \), the set of principal cubes in \( \Delta_\ell \). By applying Lemma 4 and the definition of \( \Lambda_{\ell,k} \), we have that

\[
\sum \sum \sum_{k \geq N \ell \geq 0} a^{(k+1)r} u(E_k \cap Q^k_j) \leq \sum \sum \sum_{k \geq N \ell \geq 0} c_1 a^{(k+1)r} e^{-c_2 \ell r} u(Q^k_j)
\]

\[
\leq \sum_{\ell \geq 0} c_1 e^{-c_2 \ell r} a^r (1-\ell) \sum_{k \geq N} Q^k_j \sum_{\ell,k} \frac{v^r(Q^k_j)}{|Q^k_j|} u(Q^s_i).
\]

Let us sort the inner double sum in a more convenient way. We define

\[
A^t_{(t,s)} = \left\{ Q^k_j \in \bigcup_{k \geq N} \Gamma_{\ell,k} : Q^k_j \subseteq Q^s_i \text{ and } Q^k_j \text{ is the smallest cube in } P^\ell \text{ that contains it} \right\}.
\]

That is, every \( Q^k_j \in A^t_{(t,s)} \) is not a principal cube, unless \( Q^k_j = Q^s_i \). Recall that \( v^r \in A_\infty \) implies that there exist two positive constants \( C \) and \( \theta \) verifying

\[
v^r(E) \leq C \left( \frac{|E|}{|Q|} \right)^\theta,
\]

for every cube \( Q \) and every measurable set \( E \) of \( Q \).

By using (1.5) and Lemma 12 in [1] we have that

\[
\sum \sum \sum_{k \geq N} Q^k_j \in \Gamma_{\ell,k} \frac{v^r(Q^k_j)}{|Q^k_j|} u(Q^k_j) = \sum_{Q^s_i \in P^\ell} \sum_{(k,j):Q^k_j \in A^t_{(t,s)}} \frac{u(Q^k_j)}{|Q^s_i|} v^r(Q^k_j)
\]

\[
\leq 2 \sum_{Q^s_i \in P^\ell} \frac{u(Q^s_i)}{|Q^s_i|} \sum_{(k,j):Q^k_j \in A^t_{(t,s)}} v^r(Q^k_j)
\]

\[
\leq C \sum_{Q^s_i \in P^\ell} \frac{u(Q^s_i)}{|Q^s_i|} v^r(Q^s_i) \left( \frac{Q^k_j}{|Q^s_i|} Q^k_j \right)^\theta
\]

\[
\leq C \sum_{Q^s_i \in P^\ell} \frac{u(Q^s_i)}{|Q^s_i|} v^r(Q^s_i).
\]

Therefore,

\[
\sum \sum \sum_{k \geq N \ell \geq 0} a^{(k+1)r} u(E_k \cap Q^k_j) \leq C \sum_{\ell \geq 0} e^{-c_2 \ell r} a^{-r} \sum_{Q^s_i \in P^\ell} \frac{v^r(Q^s_i)}{|Q^s_i|} u(Q^s_i).
\]
\[ e^{k\tau} \leq C \frac{\varepsilon^{\delta/\varepsilon} a^{\ell}}{|Q_j^k|} \int_{Q_j^k} \Psi_\varepsilon(|f(x)|) v^r(x) \, dx, \] where \( C \) depends on \( \varepsilon \).

By applying the estimate above we obtain
\[
\sum_{k \geq N} \sum_{\ell \geq 0} \sum_{Q_j^k \in \Gamma_{\ell,k}} a^{(k+1)r} u(E_k \cap Q_j^k) \leq C \sum_{\ell \geq 0} e^{-c_2 e^{\delta/\varepsilon} a^{\ell}} \sum_{Q_j^k \in \Gamma_{\ell,k}} \frac{u(Q_j^k)}{|Q_j^k|} \int_{Q_j^k} \Psi_\varepsilon(|f(x)|) v^r \left( \sum_{Q_s^l \in \Gamma_{\ell,k}} u(Q_s^l) \right) \]
\[
= C \sum_{\ell \geq 0} e^{-c_2 e^{\delta/\varepsilon} a^{\ell}} \int_{\mathbb{R}^n} \Psi_\varepsilon(|f(x)|) v^r \left( \sum_{Q_s^l \in \Gamma_{\ell,k}} \frac{u(Q_s^l)}{|Q_s^l|} X_{Q_s^l} \right) \]
\[
= C \sum_{\ell \geq 0} e^{-c_2 e^{\delta/\varepsilon} a^{\ell}} \int_{\mathbb{R}^n} \Psi_\varepsilon(|f(x)|) v^r(x) h_1(x) \, dx \]
\[
\leq C \int_{\mathbb{R}^n} \Psi_\varepsilon(|f(x)|) v^r(x) u(x) \, dx,
\]
by virtue of Claim 2 in [1]. Notice that the sum is finite since we are assuming \( \varepsilon < \varepsilon_2 \). Indeed, we have that
\[
e^{-c_2 e^{\delta/\varepsilon} a^{\ell}} = e^{-c_2 e^{\delta/\varepsilon} a^{\ell + \varepsilon \ln a}} = e^{\ell(-c_2 r + \varepsilon \ln a)},\]
and this exponent is negative by the election of \( \varepsilon \). This completes the estimate of \( A_N \).

Let us center our attention on the estimate of \( B_N \). Fix \( 0 < \beta < \theta \), where \( \theta \) is the number appearing in (1.6). We shall build the set of principal cubes in \( \Delta_{-1} = \bigcup_{k \geq N} \Gamma_{-1,k} \). Let
\[
P_{m-1}^{-1} = \{ Q : Q \text{ is a maximal cube in } \Delta_{-1} \text{ in the sense of inclusion} \}
\]
and, recursively, we say that \( Q_j^{k} \in P_{m+1}^{-1}, m \geq 0, \) if there exists a cube \( Q_{s,l}^{t} \in P_{m}^{-1} \) such that
\[
\frac{1}{|Q_j^{k}|} \int_{Q_j^{k}} u \geq \frac{a^{(k-t)\beta r}}{|Q_s^{l}|} \int_{Q_s^{l}} u
\]
and it is the biggest subcube of \( Q_{s,l}^{t} \) that verifies this condition, that is
\[
\frac{1}{|Q_j^{k'}|} \int_{Q_j^{k'}} u \leq \frac{a^{(k-t)\beta r}}{|Q_s^{l}|} \int_{Q_s^{l}} u
\]
if \( Q_j^{k} \subseteq Q_j^{k'} \subseteq Q_s^{l} \). Let \( P^{-1} = \bigcup_{m \geq 0} P_{m}^{-1} \), the set of principal cubes in \( \Delta_{-1} \). Similarly as before, we define the set
\[
A_{(t,s,l)}^{-1} = \left\{ Q_j^{k} \in \bigcup_{k \geq N} \Gamma_{-1,k} : Q_j^{k} \subseteq Q_{s,l}^{t} \text{ and } Q_{s,l}^{t} \text{ is the smallest cube in } P^{-1} \text{ that contains it} \right\}.
\]
We can therefore estimate $B_N$ as follows

$$B_N \leq a^r \sum_{k \geq N} \sum_{i,j,i \in \Gamma_{-1,k}} \frac{v^r(Q^k_{j,i})}{|Q^k_{j,i}|} u(Q^k_{j,i})$$

$$\leq a^r \sum_{Q^k_s,t \in P-1} \sum_{k,j,i \in A^{-1}_{(s,t)}} \frac{u(Q^k_{j,i})}{|Q^k_{j,i}|} v^r(Q^k_{j,i})$$

$$\leq a^r \sum_{Q^k_s,t \in P-1} \sum_{k \geq t} a^{(k-t)\beta r} \sum_{j,i \in Q^k_s,t} v^r(Q^k_{j,i}).$$

Fixed $k \geq t$, observe that

$$\sum_{j,i \in Q^k_s,t \in A^{-1}_{(s,t)}} |Q^k_{j,i}| < \sum_{j,i \in Q^k_s,t \in A^{-1}_{(s,t)}} a^{-kr} v^r(Q^k_{j,i}) \leq a^{-kr} v^r(Q^k_{s,t}) \leq 2^n a^{(t-k)r} |Q^k_{s,t}|.$$

Combining this inequality with the $A_{\infty}$ condition of $v^r$ we have, for every $k \geq t$, that

$$\sum_{j,i \in Q^k_s,t \in A^{-1}_{(s,t)}} a^{(k-t)\beta r} v^r(Q^k_{j,i}) \leq C v^r(Q^k_{s,t}) \left( \sum_{j,i \in Q^k_s,t \in A^{-1}_{(s,t)}} |Q^k_{j,i}| \right)^\theta \leq C a^{(t-k)r} \beta.$$

Thus,

$$B_N \leq C \sum_{Q^k_s,t \in P-1} \frac{u(Q^k_{s,t})}{|Q^k_{s,t}|} v^r(Q^k_{s,t}) \sum_{k \geq t} a^{(t-k)r} \beta$$

$$= C \sum_{Q^k_s,t \in P-1} \frac{v^r(Q^k_{s,t})}{|Q^k_{s,t}|} u(Q^k_{s,t})$$

$$\leq C \sum_{Q^k_s,t \in P-1} a^r u(Q^k_{s,t}).$$

Claim 2 (Corrected version of Claim 3 in [1]). If $Q^k_j \in A_{-1,k}$ then there exists a positive constant $C_\varepsilon$ such that

$$a^{kr} \leq C_\varepsilon \int_{Q^k_j} \Psi_\varepsilon(|f(x)|) v^r(x) dx.$$

By using this estimate we can proceed as follows

$$\sum_{k \geq N} \sum_{i,j,i \in \Gamma_{-1,k}} a^{(k+1)r} u(Q^k_{j,i}) \leq C \sum_{Q^k_s,t \in P-1} a^r u(Q^k_{s,t})$$

$$\leq C_\varepsilon \sum_{Q^k_s,t \in P-1} \frac{u(Q^k_{s,t})}{|Q^k_{s,t}|} \int_{Q^k_s} \Psi_\varepsilon(|f(x)|) v^r(x) dx$$

$$\leq C_\varepsilon \int_{\mathbb{R}^n} \Psi_\varepsilon(|f(x)|) v^r(x) \left[ \sum_{Q^k_s,t \in P-1} \frac{u(Q^k_{s,t})}{|Q^k_{s,t}|} \chi_{Q^k_s}(x) \right] dx$$

$$= C_\varepsilon \int_{\mathbb{R}^n} \Psi_\varepsilon(|f(x)|) v^r(x) h_2(x) dx.$$
by virtue of Claim 4 in [1]. This concludes the proof. \hfill \Box

We proceed with the proofs of the claims, in order to complete the argument above.

**Proof of Claim 1.** Fix \( \ell \geq 0 \) and a cube \( Q_j^k \in \bigcup_{k \geq N} \Gamma_{\ell,k} \). We know that \( \|g\|_{\Phi,Q_j^k} > a^k \) or, equivalently, \( \|\frac{4}{a^k}\|_{\Phi,Q_j^k} > 1 \). Denote with \( A = \{ x \in Q_j^k : v(x) \leq t^*a^k \} \) and \( B = Q_j^k \setminus A \), where \( t^* \) is the number verifying that if \( z \geq t^* \), then

\[
\frac{\Phi(z)}{z^r} \leq C_0 (\log z)^{\delta}.
\]

Then,

\[
1 < \left\| \frac{g}{a^k} \right\|_{\Phi,Q_j^k} \leq \left\| \frac{g}{a^k} \chi_A \right\|_{\Phi,Q_j^k} + \left\| \frac{g}{a^k} \chi_B \right\|_{\Phi,Q_j^k} = I + II.
\]

This inequality implies that either \( I > 1/2 \) or \( II > 1/2 \). Since \( \Phi \in \mathfrak{F}_{r,\delta} \) we can easily see that \( I > 1/2 \) implies that

\[
a^{kr} < 2^r C_0 (\log(2t^*))^{\delta} \int_{Q_j^k} \Phi (|f|) v^r \leq 2^r C_0 (\log(2t^*))^{\delta} \int_{Q_j^k} (\eta_v \circ \Phi) (|f|) v^r,
\]

because \( \eta_v(z) \geq z \).

On the other hand, if \( II > 1/2 \) then again

\[
1 < \frac{1}{|Q_j^k|} \int_B \Phi \left( \frac{2|f|v}{a^k} \right) \leq \frac{\Phi(2) C_0}{|Q_j^k|} \int_B \Phi (|f|) \frac{v^r}{a^{kr}} \left( \log \left( \frac{v}{a^k} \right) \right)^{\delta},
\]

since \( \Phi \in \mathfrak{F}_{r,\delta} \). This implies that

\[
a^{kr} \leq \frac{\Phi(2) C_0}{|Q_j^k|} \int_{Q_j^k} \Phi (|f|) v^r w_k,
\]

where \( w_k(x) = \left( \log \left( \frac{v(x)}{a^k} \right) \right)^{\delta} \chi_B(x) \). We shall now perform a generalized Hölder inequality with the Young functions

\[
\eta_v(z) = z (1 + \log^+ z)^{\delta/\epsilon} \quad \text{and} \quad \tilde{\eta}_v(z) \approx (e^{z/\epsilon} - z) \chi_{[1,\infty)}(z),
\]

with respect to the measure \( d\mu(x) = v^r(x) \, dx \). Thus we have

\[
(1.10) \quad \frac{1}{|Q_j^k|} \int_{Q_j^k} \Phi(|f|) w_k v^r \leq \frac{v^r(Q_j^k)}{|Q_j^k|} ||\Phi(|f|)||_{\eta_v,Q_j^k,v^r} w_k ||\tilde{\eta}_v,Q_j^k,v^r||.
\]

Let us first estimate the last factor. Since \( e^{(\log z)^{\epsilon}} \leq z^\epsilon \) when \( z \geq e^{(\epsilon/(-1))} \), we proceed as follows

\[
(1.11) \quad \frac{1}{v^r(Q_j^k)} \int_{Q_j^k} \tilde{\eta}_v(w_k)v^r \leq \tilde{\eta}_v \left( \frac{\epsilon}{(\epsilon-1)} \right) + \frac{1}{v^r(Q_j^k)} \int_{Q_j^k \cap \{ v/a^k \geq e^{(\epsilon/(-1))} \}} \frac{v^{r+\epsilon}}{a^{k\epsilon}}.
\]

Since \( \epsilon^{-\epsilon} \leq \epsilon \) we have that

\[
\tilde{\eta}_v \left( \frac{\epsilon}{(\epsilon-1)} \right) \leq e^{(\epsilon)/(1-\epsilon)}.
\]
On the other hand, our hypothesis on \( v \) implies that \( v^r \in \text{RH}_s \), where \( s = 1 + \varepsilon/r \). Since \( v^r \in \Lambda_{t,k} \) we obtain

\[
\frac{1}{v^r(Q_j^k)} \int_{Q_j^k \cap \{v^r > e^{1/(r-1)} \}} v^{r+\varepsilon} a^{k\varepsilon} \leq \frac{a^{-k\varepsilon}}{v^r(Q_j^k)} \int_{Q_j^k} v^{r+\varepsilon} \\
\leq [v^r]_{\text{RH}} a^{-k\varepsilon} \left( \frac{1}{|Q_j^k|} \int_{Q_j^k} v^r \right)^{s-1} \\
= [v^r]_{\text{RH}} a^{-k\varepsilon} \left( \frac{1}{|Q_j^k|} \int_{Q_j^k} v^r \right)^{s-1} \\
\leq [v^r]_{\text{RH}} a^{-k\varepsilon} a^{(k+\ell+1)\varepsilon} \\
= [v^r]_{\text{RH}} a^{(\ell+1)\varepsilon}.
\]

By using these two estimates in (1.11), we get

\[
\|w_k\|_{\tilde{\nu}^r, Q_j^k, v^r} \leq e^{e^{1/(r-1)}} + [v^r]_{\text{RH}} a^{(\ell+1)\varepsilon} \leq (e^{e^{1/(r-1)}} + [v^r]_{\text{RH}}) a^{(\ell+1)\varepsilon}.
\]

We also observe that

\[
(1.12) \quad \|\Phi([f])\|_{\eta^r, Q_j^k, v^r} \approx \inf_{\tau > 0} \left\{ \tau + \frac{\tau}{v^r(Q_j^k)} \int_{Q_j^k} \eta_\varepsilon \left( \frac{\Phi([f])}{\tau} \right) v^r \right\}.
\]

If we choose \( \tau = (2a^{(\ell+1)(r+\varepsilon)}(e^{e^{1/(r-1)}} + [v^r]_{\text{RH}}))^{-1} \) then we can estimate the right-hand side of (1.10) as follows

\[
\frac{v^r(Q_j^k)}{|Q_j^k|} \|\Phi([f])\|_{\eta^r, Q_j^k, v^r} |w_k|_{\tilde{\nu}^r, Q_j^k, v^r} \leq \frac{a^{kr}}{2} + (e^{e^{1/(r-1)}} + [v^r]_{\text{RH}}) a^{(\ell+1)\varepsilon} \tau \eta_\varepsilon \left( \frac{1}{\tau} \right) \frac{1}{|Q_j^k|} \int_{Q_j^k} \Psi_\varepsilon([f]) v^r.
\]

Notice that

\[
\tau \eta_\varepsilon \left( \frac{1}{\tau} \right) = \left( 1 + \log \left( \frac{1}{\tau} \right) \right)^{\delta/\varepsilon} \leq 2^{\delta/\varepsilon} \left( \log(2(e^{e^{1/(r-1)}} + [v^r]_{\text{RH}})) + (\ell + 1)(r+\varepsilon) \log a \right)^{\delta/\varepsilon} \leq C_{\varepsilon} \delta^{\delta/\varepsilon}.
\]

By plugging these two estimates in (1.10) we arrive to

\[
a^{kr} \leq C_{\varepsilon} \delta^{\delta/\varepsilon} \frac{1}{|Q_j^k|} \int_{Q_j^k} \Psi_\varepsilon([f]) v^r,
\]

and we are done. \( \square \)

**Proof of Claim 2.** The proof follows similar arguments as the previous one. By adopting the same notation, we have that \( \|\Phi\|_{\nu^r, Q_j^k} > 1 \), and this implies that either \( I > 1/2 \) or \( II > 1/2 \). If \( I > 1/2 \), we obtain

\[
a^{kr} < C_0 (\log(2t^*))^\delta \frac{1}{|Q_j^k|} \int_{Q_j^k} \Phi([f]) v^r \leq C_0 (\log(2t^*))^\delta \frac{1}{|Q_j^k|} \int_{Q_j^k} \Psi_\varepsilon([f]) v^r.
\]

We now assume that \( II > 1/2 \). By performing the same Hölder inequality as in Claim 1, we get

\[
(1.13) \quad \frac{1}{|Q_j^k|} \int_{Q_j^k} \Phi([f]) w_k v^r \leq \frac{v^r(Q_j^k)}{|Q_j^k|} \|\Phi([f])\|_{\eta^r, Q_j^k, v^r} |w_k|_{\tilde{\nu}^r, Q_j^k, v^r}.
\]
In order to estimate the factor \( \|w_k\|_{\tilde{v}_\delta, Q_j^r, v^r} \) we proceed as before. Since

\[
\frac{1}{v^r(Q_j^r)} \int_{Q_j^r \cap \{ v_{\tilde{v}} > e^{r-(1-\epsilon)} \}} \frac{v^{r+\epsilon}}{a^{k\epsilon}} \leq \frac{a^{-k\epsilon}}{v^r(Q_j^r)} \int_{Q_j^r} v^{r+\epsilon} \\
\leq [v^r]_{\text{RH}_s} \frac{a^{-k\epsilon}|Q_j^r|}{v^r(Q_j^r)} \left( \frac{1}{|Q_j^r|} \int_{Q_j^r} v^s \right)^{s-1} \\
= [v^r]_{\text{RH}_s} a^{-k\epsilon} \left( \frac{1}{|Q_j^r|} \int_{Q_j^r} v^s \right)^{s-1} \\
\leq [v^r]_{\text{RH}_s} a^{-k\epsilon} a^{k\epsilon} \\
= [v^r]_{\text{RH}_s}
\]

we obtain that

\[
\|w_k\|_{\tilde{v}_\delta, Q_j^r, v^r} \leq e^{(1/\epsilon)} + [v^r]_{\text{RH}_s} = C_\epsilon.
\]

By using (1.12) and choosing \( \tau = (2C_\epsilon)^{-1} \), we can estimate the right-hand side in (1.13) as follows

\[
\frac{v^r(Q_j^r)}{|Q_j^r|} \|\Phi(\{f\})\|_{\tilde{v}_\delta, Q_j^r, v^r} \|w_k\|_{\tilde{v}_\delta, Q_j^r, v^r} \leq \frac{a^{kr}}{2} + \tau \eta_\epsilon \left( \frac{1}{\tau} \right) \frac{1}{|Q_j^r|} \int_{Q_j^r} \Psi_\epsilon (|f|) v^r \\
\leq \frac{a^{kr}}{2} + (1 + \log(2C_\epsilon))^{\delta/\epsilon} \frac{1}{|Q_j^r|} \int_{Q_j^r} \Psi_\epsilon (|f|) v^r.
\]

This yields

\[
a^{kr} \leq \frac{C_\epsilon}{|Q_j^r|} \int_{Q_j^r} \Psi_\epsilon (|f|) v^r.
\]

This concludes the proof. \( \square \)

2. Applications: Mixed Estimates for the Generalized Fractional Integral Operator

Mixed inequalities for the generalized fractional maximal operator \( M_{\Phi} \) were also given in [1]. One of the key properties in order to establish the following result was to define an auxiliary operator that is bounded in \( L^\infty(\tilde{v}v^r) \) when \( v^r \in A_\infty \). This operator is given by

\[
\mathcal{T}_\Phi f(x) = \frac{M_\Phi(fv)(x)}{M_\Phi(v)(x)}.
\]

It is not difficult to see that \( M_\Phi v \approx v \) when \( \Phi \in \mathcal{F}_{\tau, \delta} \) and \( v^r \) is an \( A_1 \)-weight, so this operator is an extension of the Sawyer operator \( S_\Phi f = M_\Phi(fv)/v \) considered in the main theorem.

**Corollary 5** (Corrected version of Corollary 3 in [1]). Let \( r \geq 1, \delta \geq 0 \) and \( \Phi \in \mathcal{F}_{\tau, \delta} \). Let \( u \in A_1 \), \( v^r \in A_\infty \) and \( \Psi \) be a Young function that verifies \( \Psi(t) \approx \Phi(t) \), for every \( t \geq t_0 \geq 0 \). Then, there exists \( \varepsilon_0 > 0 \) such that the inequality

\[
uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{M_\Phi(v)(x)} > t \right\} \right) \leq C_1 \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \Psi) \left( \frac{C_2 f(x)}{t} \right) u(x) v^r(x) dx
\]

holds for every \( t > 0 \) and every \( 0 < \varepsilon < \varepsilon_0 \), where \( C_1 \) depends on \( \varepsilon \) and \( \eta_\varepsilon(z) = z(1 + \log^+ z)^{\delta/\epsilon} \).
Proof. By combining the equivalence between $\Phi$ and $\Psi$ and Proposition 8 in [1], we obtain that there exist positive constants $A$ and $B$ such that

$$A\| \cdot \|_{\Phi, Q} \leq \| \cdot \|_{\Psi, Q} \leq B\| \cdot \|_{\Phi, Q},$$

for every cube $Q$. By setting $c_1 = B/A$, we have that

$$\frac{M_{\Psi}(f v)(x)}{M_{\Psi}v(x)} \leq c_1 \frac{M_{\Phi}(f v)(x)}{M_{\Phi}v(x)}$$

for almost every $x$. By applying Corollary 2, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have

$$uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\Psi}(f v)(x)}{M_{\Psi}v(x)} > t \right\} \right) \leq uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\Phi}(f v)(x)}{M_{\Phi}v(x)} > \frac{t}{c_1} \right\} \right) \leq C \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \Phi) \left( \frac{c_1|f|}{t} \right) uv^r.$$

Observe that

$$\| \mathcal{T}_{\Psi} f \|_{L^\infty} = \left\| \frac{M_{\Psi}(f v)}{M_{\Psi}v} \right\|_{L^\infty} \leq \| f \|_{L^\infty},$$

which directly implies $\| \mathcal{T}_{\Psi} f \|_{L^\infty(w^r)} \leq \| f \|_{L^\infty(w^r)}$ since the measure given by $d\mu(x) = u(x)v^r(x)\,dx$ is absolutely continuous with respect to the Lebesgue measure. We now apply Lemma 13 in [1] to obtain

$$uv^r \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\Phi}(f v)(x)}{M_{\Phi}v(x)} > t \right\} \right) \leq C \int_{\{x : |f(x)| > t/2\}} (\eta_\varepsilon \circ \Phi) \left( \frac{2c_1|f(x)|}{t} \right) u(x)v^r(x)\,dx \leq C \int_{\{x : |f(x)| > t/2\}} (\eta_\varepsilon \circ \Phi) \left( \frac{2t_0|f(x)|}{t} \right) u(x)v^r(x)\,dx \leq C \int_{\{x : |f(x)| > t/2\}} (\eta_\varepsilon \circ \Psi) \left( \frac{2t_0|f(x)|}{t} \right) u(x)v^r(x)\,dx \leq C \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \Psi) \left( \frac{C_2|f(x)|}{t} \right) u(x)v^r(x)\,dx. \qedhere$$

The corollary above is key in order to obtain mixed inequalities for the generalized fractional maximal operator defined, for $0 < \gamma < n$ and a Young function $\Phi$, by the expression

$$M_{\gamma, \Phi} f(x) = \sup_{Q \ni x} |Q|^\gamma/n \| f \|_{\Phi, Q}.$$ Mixed estimates for this operator are contained in the following theorems.

Theorem 6. Let $\Phi(z) = z^r(1 + \log^+ z)^\delta$, with $r \geq 1$ and $\delta \geq 0$. Let $0 < \gamma < n/r$, $r < p < n/\gamma$ and $1/q = 1/p - \gamma/n$. If $u \in A_1$ and $v^{(1/p+1/r')}$ \in $A_{\infty}$, then the inequality

$$uv^{q(1/p+1/r')} \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\gamma, \Phi}(f v)(x)}{M_{\varphi}v(x)} > t \right\} \right)^{1/q} \leq C \left[ \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{t} \right)^p u^{p/q}(x)v(x)^{1+p/r'}\,dx \right]^{1/p},$$

holds for every positive $t$, where $\varphi(z) = z^{q/p+1/r'}(1 + \log^+ z)^{n\delta/(n-\gamma)}$. 

Proof. We shall follow the scheme given in the proof of Theorem 4 in [1]. The difference lies when we apply Corollary 5, but the function controlling the right-hand side turns out to be auxiliary. We define
\[ \sigma = \frac{nr}{n - r\gamma}, \quad \nu = \frac{n\delta}{n - r\gamma}, \quad \beta = \frac{q}{\sigma} \left( \frac{1}{p} + \frac{1}{r'} \right), \]
and let \( \xi \) be the auxiliary function given by
\[ \xi(z) = \begin{cases} 
  z^{q/\beta}, & \text{if } 0 \leq z \leq 1, \\
  z^\sigma (1 + \log^+ z)^\nu, & \text{if } z > 1.
\end{cases} \]
Observe that
\[ \xi^{-1}(z)z^{\gamma/n} \approx \frac{z^{1/\sigma + \gamma/n}}{(1 + \log^+ z)^{\nu/\sigma}} = \frac{z^{1/r}}{(1 + \log^+ z)^{\delta/r}} \lesssim \Phi^{-1}(z), \]
for every \( z \geq 1 \). Observe that \( \beta > 1 \): indeed, since \( p > r \) we have \( q > \sigma \) and thus \( q/(\sigma r') > 1/r' \).

On the other hand, \( q/(\sigma \rho) > 1/r \). By combining these two inequalities we have \( \beta > 1 \). Applying Proposition 10 and Lemma 9 with \( \beta \) from [1], we can conclude that

\[ M_{\gamma,\Phi} \left( \frac{f_0}{w} \right)(x) \leq C \left[ M_{\xi} \left( \frac{f_0^{p\beta/q}}{w^\beta} \right)(x) \right]^{1/\beta} \left( \int_{\mathbb{R}^n} f_0^p(y) \, dy \right)^{\gamma/n}. \]

Also observe that
\[ \left( M_{\xi} v^\beta(x) \right)^{1/\beta} \lesssim M_{\nu} v(x), \quad \text{a.e. } x. \]

Notice that \( \xi \) is equivalent to a Young function in \( \mathcal{F}_{\sigma,\nu} \) for \( t \geq 1 \). Since \( q(1/p + 1/r') = \beta\sigma \), if we set \( f_0 = |f|wv \), then we can use inequalities (2.1) and (2.2) to estimate
\[ \frac{2 + r}{uv^\beta} \left( \left\{ x : M_{\gamma,\Phi}(f)(x) > t \right\} \right) \lesssim uv^\beta \left( \left\{ x : M_{\gamma,\Phi}(f)(x) > t \right\} \right) \]
\[ \leq uv^\beta \left( \left\{ x : \frac{M_{\xi} \left( f_0^{p\beta/q} w^{-\beta} \right)(x)}{M_{\xi} v^\beta(x)} > \frac{t^\beta}{ \left( \int_{\mathbb{R}^n} |f_0|^p \right) \delta/\gamma} \right\} \right). \]

Since \( v^{\beta\sigma} \in A_{\infty} \), by Corollary 5 there exists \( \varepsilon_0 > 0 \) such that the inequality
\[ uv^\beta \left( \left\{ x : \frac{M_{\xi} \left( f_0^{p\beta/q} w^{-\beta} \right)(x)}{M_{\xi} v^\beta(x)} > t_0 \right\} \right) \leq C_\varepsilon \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \xi) \left( c \frac{\left( f_0^{p\beta/q} w^{-q} \right) t_0}{\int_{\mathbb{R}^n} |f_0|^p} \right) uv^\beta \]
holds for every \( 0 < \varepsilon < \varepsilon_0 \), with \( t_0 = t^\beta \| f_0 \|^{-\beta\gamma/n} \). Notice that \( \eta_\varepsilon(z) = z(1 + \log^+ z)^{\nu/\varepsilon} \) in this case.

Fixed \( \varepsilon \), we write
\[ \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \xi) \left( c \frac{\left( f_0^{p\beta/q} w^{-q} \right)^{\beta(p-1)/p} t_0^{\beta \gamma/n}}{t^\beta} \right) uv^\beta = \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \xi)(\lambda) uv^\sigma \]
where
\[ \lambda = c \frac{\left( f_0^{p\beta/q} w^{-q} \right)^{\beta(p-1)/p} t_0^{\beta \gamma/n}}{t^\beta} \left( \int_{\mathbb{R}^n} |f|^p(wv)^p \right)^{\gamma/n}. \]
We further split $\mathbb{R}^n$ into the sets $A = \{x \in \mathbb{R}^n : \lambda(x) \leq 1\}$ and $B = \mathbb{R}^n \setminus A$. Since $(\eta \circ \xi)(z) = z^{q/\beta}$ for $0 \leq z \leq 1$, we have that

$$\int_A (\eta \circ \xi)(\lambda(x))u(x)[v(x)]^{\sigma \beta} \, dx = \int_A [\lambda(x)]^{q/\beta}u(x)[v(x)]^{\sigma \beta} \, dx.$$ 

If we set $w = u^{1/q}v^{1/p+1/r'-1}$, then

$$\lambda^{q/\beta}uv^{\sigma \beta} = c^{q/\beta} \int_{\mathbb{R}^n} |f|^p (uw) \int_{\mathbb{R}^n} |f|^p (uw) \, dx \, dw = c^{q/\beta} \int_{\mathbb{R}^n} |f|^p (uw) \, dx \, dw \int_{\mathbb{R}^n} |f|^p (uw) \, dx \, dw \int_{\mathbb{R}^n} w^{q/p} \, dx \, dw.$$ 

Observe that

$$\sigma \beta + (p - q) \left( \frac{1}{p} + \frac{1}{r'} \right) = q \left( \frac{1}{p} + \frac{1}{r'} \right) + (p - q) \left( \frac{1}{p} + \frac{1}{r'} \right) = 1 + \frac{p}{r'}.$$ 

Also, notice that

$$(uw)^p = w^{p/q}u^{1+p/p'-p} = u^{p/q}u^{1+p/p'}.$$ 

Therefore,

$$\int_A (\eta \circ \xi)(\lambda)uv^{\sigma \beta} \leq c^{q/\beta} \int_{\mathbb{R}^n} |f|^p u^{p/q}u^{1+p/p'} \int_{\mathbb{R}^n} |f|^p u^{p/q}u^{1+p/p'} \int_{\mathbb{R}^n} w^{q/p} \, dx \, dw.$$ 

On the other hand, $\lambda(x) > 1$ over $B$ and

$$(\eta \circ \xi)(z) \leq z^\sigma (1 + \log z)^{\nu(1+1/c)},$$

and this function has an upper type $q/\beta$. Therefore we can estimate the integrand by $\lambda^{q/\beta}uv^{\sigma \beta}$ and proceed as we did with the set $A$. Thus, we obtain

$$u^{q(1/p+1/r')} \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\gamma \Phi}(f)(x)}{M_{\gamma \Phi}(v)(x)} > t \right\} \right)^{1/q} \leq C \left[ \int_{\mathbb{R}^n} \left( \frac{|f|^p}{t} \right) u^{p/q}u^{1+p/p'} \right]^{1/p}. \quad \square$$

**Theorem 7** (Corrected version of Theorem 5 in [1]). Let $\Phi(z) = z^\sigma (1 + \log^+ z)^{\delta}$, with $r \geq 1$ and $\delta \geq 0$. Let $0 < \gamma < n/r$ and $1/q = 1/r - \gamma/n$. If $u \in A_1$ and $v^q \in A_\infty$, then there exists a positive constant $\varepsilon_0$ such that the inequality

$$\Phi^q \left( \left\{ x \in \mathbb{R}^n : \frac{M_{\gamma \Phi}(f)(x)}{M_{\gamma \Phi}(v)(x)} > t \right\} \right)^{1/q} \leq C \Phi \Phi_{\gamma, \varepsilon} \left( \left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{t} \right\} \Psi ^{1/q} u^{1/q}v(x) \, dx \right),$$

holds for $0 < \varepsilon < \varepsilon_0$, where $\Phi_{\gamma, \varepsilon}(z) = [z(1 + \log^+ z)^{\delta(1+1/c)/r} \Psi_{\varepsilon}(z) = z^\gamma (1 + \log^+ (z^{-\gamma/r}))^{\delta(1+1/c)/r}$, $\Phi_{\gamma, \varepsilon}(z) = \Phi(z)(1 + \log^+ z)^{\delta(1+1/c)/r}$, and $C$ depends on $\varepsilon$.

**Proof.** Set $\xi(z) = z^\gamma (1 + \log^+ z)^{\nu}$, where $\nu = \delta q/r$. Thus $z^{\gamma/n} \xi^{-1}(z) \leq \Phi^{-1} \xi^{-1}(z)$. By applying Proposition 10 in [1] with $p = r$ we have that

$$M_{\gamma, \xi} \left( \int f_{1/w}^{1/q} \right) (x) \leq C \left( \int_{\mathbb{R}^n} f_{1/w}^{1/q} \, dy \right)^{\gamma/n}.$$
By setting \( f_0 = |f| w v \) we can write

\[
uv^q \left( \left\{ x : \frac{M_\gamma, \Phi(fv)(x)}{v(x)} > t \right\} \right) = uv^q \left( \left\{ x : \frac{M_\gamma, \Phi(f_0/w)(x)}{v(x)} > t \right\} \right)
\]

\[
\leq uv^q \left( \left\{ x : \frac{M_\xi (f_0^{r/q}/w)(x)}{M_\xi v(x)} > \frac{t}{(f_0^{r})^{\gamma/n}} \right\} \right).
\]

Since \( \xi \in \mathfrak{F}_{q, q'} \), by Corollary 2 there exists \( \varepsilon_0 > 0 \) such that

\[
\text{(2.3)} \quad uv^q \left( \left\{ x : \frac{M_\gamma, \Phi(fv)(x)}{v(x)} > t \right\} \right) \leq C_{\varepsilon} \int_{\mathbb{R}^n} (\eta_\varepsilon \circ \xi) \left( \frac{f_0^{r/q} (f f_0^{r})^{\gamma/n}}{w v t} \right) uv^q,
\]

for \( 0 < \varepsilon < \varepsilon_0 \) and being \( \eta_\varepsilon(z) = z(1 + \log^+ z)^{\nu/\varepsilon} \). Fixed \( \varepsilon \), the argument of \( \eta_\varepsilon \circ \xi \) above can be written as

\[
\frac{f_0^{r/q} (f f_0^{r})^{\gamma/n}}{w v t} = \left( \frac{|f|}{t} \right)^{r/q} (w v)^{r-1} \left( \int_{\mathbb{R}^n} \frac{|f|}{t} (w v)^r \right)^{\gamma/n}
\]

\[
= \left[ \left( \frac{|f|}{t} \right) (w v)^{1-q/r} \left( \int_{\mathbb{R}^n} \frac{|f|}{t} (w v)^r \right)^{\gamma q/(nr)} \right]^{r/q}.
\]

Observe that for \( 0 \leq z \leq 1 \), \( (\eta_\varepsilon \circ \xi)(z^{r/q}) = z^r \), and for \( z > 1 \) we have

\[
(\eta_\varepsilon \circ \xi)(z^{r/q}) \lesssim z^r (1 + \log z)^{\nu(1+1/\varepsilon)},
\]

which implies that \( (\eta_\varepsilon \circ \xi)(z^{r/q}) \lesssim \Phi_{\eta, \varepsilon}(z) = z^r (1 + \log^+ z)^{\nu(1+1/\varepsilon)} \), for every \( z \geq 0 \). Since \( \Phi_{\gamma, \varepsilon} \) is submultiplicative, we can estimate as follows

\[
(\eta_\varepsilon \circ \xi) \left( \frac{f_0^{r/q} (f f_0^{r})^{\gamma/n}}{w v t} \right) \leq \Phi_{\gamma, \varepsilon} \left( \left( \frac{|f|}{t} \right) (w v)^{1-q/r} \left( \int_{\mathbb{R}^n} \frac{|f|}{t} (w v)^r \right)^{\gamma q/(nr)} \right)
\]

\[
\leq \Phi_{\gamma, \varepsilon} \left( \int_{\mathbb{R}^n} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} \right) (w v)^r \right)^{\gamma q/(nr)} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} (w v)^{1-q/r} \right)
\]

Returning to (2.3) and setting \( w = u^{1/q} \), the right hand side is bounded by

\[
\Phi_{\gamma, \varepsilon} \left( \int_{\mathbb{R}^n} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} \right) (w v)^r \right)^{\gamma q/(nr)} \int_{\mathbb{R}^n} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} \right) (w v)^{1-q/r} (w v)^q.
\]

Notice that \( \Phi_{\gamma, \varepsilon}(z^{1-q/r}) z^q \leq \Psi_\varepsilon(z) \). Therefore, the expression above is bounded by

\[
\Phi_{\gamma, \varepsilon} \left( \int_{\mathbb{R}^n} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} \right) \Psi_\varepsilon(u^{1/q} v) \right)^{\gamma q/(nr)} \int_{\mathbb{R}^n} \Phi_{\gamma, \varepsilon} \left( \frac{|f|}{t} \right) \Psi_\varepsilon(u^{1/q} v).
\]

To finish, observe that

\[
z \Phi_{\gamma, \varepsilon}(z^{\gamma q/(nr)}) \lesssim z^{1+\gamma q/u}(1 + \log^+ z)^{\nu(1+1/\varepsilon)} = z^{q/r}(1 + \log^+ z)^{\delta q(1+1/\varepsilon)/r} = \varphi_\varepsilon(z).
\]
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