Trace Anomalies from Quantum Mechanics

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The one-loop anomalies of a $d$-dimensional quantum field theory can be computed by evaluating the trace of the path integral Jacobian matrix $J$, regulated by an operator $\exp(-\beta R)$ and taking the limit $\beta$ to zero [1]. In 1983, Alvarez-Gaumé and Witten [2] made the observation that one can simplify this evaluation by replacing the operators which appear in $J$ and $R$ by quantum mechanical operators with the same (anti)commutation relations. By rewriting this quantum mechanical trace as a path integral with periodic boundary conditions at time $t = 0$ and $t = \beta$ for a one-dimensional supersymmetric non-linear sigma model, they obtained the chiral anomalies for spin $\frac{1}{2}$ and $\frac{3}{2}$ fields and self-dual antisymmetric tensors in $d$ dimensions.

Some time ago it occurred to us that one can also apply these ideas to the trace anomalies. In a recent paper [3] the first author proposed a bosonic configuration space path integral for a particle moving in curved space, and found the corresponding Hamiltonian $R$ from the Schrödinger equation. He exponentiated the factors $\sqrt{g}$ in the path integral measure by using scalar ghosts, and obtained the trace anomaly for a scalar field in an external gravitational field in $d = 2$. In this article, we treat the general case of trace anomalies for external gravitational and Yang-Mills fields. We do not introduce a supersymmetric sigma model, but keep the original Dirac matrices $\gamma^\mu$ and internal symmetry generators $T^a$ in the path integral. As a result, we get a matrix-valued action $S$. Gauge covariance of the path integral then requires a definition of the exponential of the action by time-ordering. The computations are simplified by using Riemann normal coordinates. We also replace the scalar ghosts by vector ghosts in order to exhibit the cancellation of all divergences at finite $\beta$ more clearly. Finally we compute the trace anomalies in $d = 2$ and $d = 4$.

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1. Introduction

The one-loop anomalies in local symmetries of a \(d\)-dimensional quantum field theory can be computed, as first demonstrated by Fujikawa [1], by evaluating the trace of the Jacobian matrix \(J\) that arises when one varies the integration variables of the path integral under this local symmetry. This trace has to be regulated because the Jacobian is an infinite dimensional matrix, and as regulator one may take an operator of the form \(\exp(-\beta R)\). The anomaly is then obtained by letting \(\beta\) tend to zero. In general, there will be divergent terms proportional to powers of \(\beta^{-1}\), and a finite remainder. The latter is the anomaly, while the former are discarded by using several regulators whose \(\beta\)'s satisfy suitable relations as in Pauli-Villars regularization. The justification for discarding the divergences in anomalies is that the effective action is made finite by adding suitable counterterms so that its variation should also be finite. (If one regulates the quantum field theory with dimensional regularization, the counterterms which make the one-loop effective action finite produce also anomalies [4], which are finite since the one-loop pole is cancelled by a zero due to classical gauge invariance).

The choice of a regulator \(R\) is important. For some cases one knows from computations a regulator which yields consistent anomalies, or a regulator which yields a covariant anomaly, but only for consistent anomalies has a general theory been constructed [3]. This theory provides a method to construct regulators which: \(i\) preserve certain local symmetries at the quantum level, and \(ii\) produce anomalies in other symmetries which satisfy the consistency conditions. For Einstein symmetries (general coordinate invariance), Fujikawa already determined the regulators which satisfy \((i)\). For covariant anomalies, no general theory is known as yet, although in practice one usually can guess what these covariant regulators will be. In our work below, we are in the happy circumstance that the consistent regulators which preserve the Einstein symmetry are actually covariant, as we shall discuss.

The local symmetries we shall consider are local scale symmetries. The corresponding anomalies are called Weyl anomalies, or trace anomalies because they arise as a non-vanishing trace of the stress tensor \(T_{\mu\nu}\) at the quantum level. For example, consider a real scalar field \(\phi\), which couples to gravity in \(d\) dimensions with action

\[
S = \int dx \sqrt{g} \frac{1}{2} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2 \right],
\]

where \(\xi = \frac{d-2}{4(d-1)}\). We shall always work in Euclidean space so that \(g = \det g_{\mu\nu}\). This
classical action is invariant under the following Weyl rescalings

\[ g_{\mu\nu} \rightarrow g_{\mu\nu}' = \Omega g_{\mu\nu}; \quad \phi \rightarrow \phi' = \Omega^{\frac{2-d}{2}} \phi. \] (1.2)

The general theory of consistent regulators states that a consistent regulator without Einstein anomalies is obtained by using \( \tilde{\phi} = g^{\frac{1}{4}} \phi \) as quantum field. Clearly

\[ \delta W \tilde{\phi} = \frac{1}{2} \sigma \tilde{\phi}, \] (1.3)

where \( \sigma = \log \Omega \) is taken to be infinitesimal. Rewriting the action in (1.1) in term of \( \tilde{\phi} \), the following regulator is obtained

\[ R_{\text{cons}} = -g^{-\frac{1}{4}} \partial_{\mu} g^{\frac{1}{4}} g^{\mu\nu} \partial_{\nu} g^{-\frac{1}{4}} - \xi R. \] (1.4)

This is clearly not an Einstein scalar. Under a rescaling of the external metric in the path integral \( Z[g] = \int \mathcal{D}\tilde{\phi} \exp(-h^{-1} S[g, \tilde{\phi}]) \), followed by a compensating rescaling of \( \tilde{\phi} \) as in (1.3) such that \( S \) remains invariant, the following regulated Jacobian is produced:

\[ \int dx \sqrt{g} \sigma(x) g^{\mu\nu}(x) \left\langle T_{\mu\nu} \right\rangle = 2h \lim_{\beta \to 0} \left( \partial_{\tilde{\phi}} \exp(-\beta R_{\text{cons}}) \right), \] (1.5)

where we have defined the stress tensor by

\[ T_{\mu\nu} = \frac{2}{\sqrt{g}} \delta S / \delta g^{\mu\nu}. \] (1.6)

Since the Jacobian contains no derivatives, we can use the cyclicity of the regulated trace \[3\] to cycle the factor \( g^{-\frac{1}{4}} \) on the right-hand side of \( R_{\text{cons}} \) in (1.4) to the left-hand side, obtaining the trace anomaly of a scalar field in \( d \) dimensions

\[ A_{0}^{(d)} = \int dx \sqrt{g} \sigma(x) \left\langle T \right\rangle = h \lim_{\beta \to 0} \text{Tr} \sigma(x) \exp(-\beta R_{0}^{\text{cov}}) \]

\[ R_{0}^{\text{cov}} = -g^{-\frac{1}{2}} \partial_{\mu} g^{\frac{1}{2}} g^{\mu\nu} \partial_{\nu} - \xi R = -\nabla^2 - \xi R, \] (1.7)

where the subscript zero indicates that we are dealing with spin zero fields and where \( T \) denotes the trace of the stress tensor. For Einstein anomalies, one cannot replace \( R_{\text{cons}} \) by \( R_{\text{cov}} \) as the Jacobian contains derivatives \[3\]. The reason that this is possible for Weyl symmetries follows already from the well-known fact that the consistent Weyl anomaly is Einstein-covariant.
The problem is now to evaluate this trace. Fujikawa [1] and others [6] have used a complete set of plane-waves and integrated over plane-wave momenta, but the expressions become extremely unwieldy. DeWitt [7] did develop, much earlier, a heat kernel analysis, according to which an ansatz for the answer is made, which is then verified by explicit computation order by order in \( \beta \). In the ansatz, “bilocal” tensors appeared (such as the Synge world function \( \sigma(x, y) \), which measures the square of the distance between two points) and a calculus for the left and/or right derivatives of such objects was needed. Of course this approach yields much more than only the anomalies, for example one obtains in this way the complete one-loop effective action. Yet, a simpler method, which may only give the anomalies, might be welcome. Some time ago, Alvarez-Gaumé and Witten [2] observed that one can in general compute anomalies in a much simpler way by replacing the \( d \)-dimensional operators in \( J \) and \( R \), such as \( x^\mu \), \( \frac{\partial}{\partial x^\mu} \), the Dirac matrices \( \gamma^\mu \), the metric \( g_{\mu\nu}(x) \), the connection \( A_\mu(x) \) and the corresponding generator of internal symmetries \( (T^a)_i^j \), by quantum mechanical objects \( q^\mu(t) \) and \( p_\mu(t) \), \( \psi^\mu(t) \), \( g_{\mu\nu}(q) \), \( A_\mu(q) \) and \( c^*i(t)(T^a)_i^jc_j(t) \), where \( c^*i \) and \( c_j \) are anticommuting “ghosts” which contract the indices of internal symmetry generators such as Yang-Mills or Lorentz matrices. All that is needed is that they satisfy the same anticommutation rules. They then considered chiral anomalies and rewrote the trace of the quantum mechanical operator \( J \exp(-\beta R) \), with \( J = \gamma^5 \) as a one-dimensional path integral

\[
Z = \int_{\text{PBC for } q, \psi} Dq^\mu D\psi^\mu Dc_i Dc^*i \exp \left( -\int_0^\beta dt \mathcal{L}^{(\sigma)} \right), \tag{1.8}
\]

where \( \mathcal{L}^{(\sigma)} \) is the Lagrangian for a one-dimensional supersymmetric non-linear sigma model. The \( d \)-dimensional chirality operator \( \gamma^5 \) became the fermion number operator \( (-)^F \), because \( (-)^F \) anticommutes with \( \psi^\mu \) just as \( \gamma^5 \) with \( \gamma^\mu \). Since \( (-)^F \) commutes with \( q^\mu \) and the ghosts, the \( \psi^\mu \) fermions acquired periodic boundary conditions (PBC), whereas the \( c^*i \) and \( c_j \) kept their antiperiodic boundary conditions (ABC). A simplification was due to the topological nature of the chiral anomaly: the path integral itself is \( \beta \)-independent and hence could be evaluated by taking \( \beta \) small, and keeping only non-vanishing terms. In particular, no divergences proportional to \( \beta^{-1} \) could occur. For the actual evaluation, Alvarez-Gaumé and Witten expanded \( q^\mu(t) \) and \( \psi^\mu(t) \) in fluctuations about a classical constant solution \( q^\mu(t) = q_0^\mu \) and \( \psi^\mu(t) = \psi_0^\mu \), and only the terms quadratic in fluctuations did contribute. The result for the anomaly could be written in a covariant form by using normal coordinates and, since they used a covariant regulator, this covariantly looking
form was also valid in general coordinates.

In this article we shall consider the trace anomalies, and extend the work of Alvarez-Gaumé and Witten to this case. There are, however, important differences:

i) The trace anomaly has no topological meaning, and as a consequence its path integral representation does depend on the regulating parameter $\beta$. We must therefore expand in terms of $\beta$, which amounts to a loop expansion on the world line with coordinate $t$. For the one-loop trace anomaly in $d$ dimensions we shall need $\frac{d}{2} + 1$ loops on the world line. Hence the algebra becomes more complicated than for chiral anomalies and subtle problems which did not need to be addressed in [2], now have to be solved. For example, the naïve relation between the Hamiltonian (=regulator) and the corresponding Euclidean action in the path-integral breaks down, and extra terms proportional to the curvature appear. The occurrence of such terms was first noticed by DeWitt [7]. (In fact, in his well-known 1948 article [8], Feynman had already indicated that for models such as non-linear sigma models, it might not be correct to use the classical action and classical paths of a free particle to define the path integral.)

ii) For the case of spin $\frac{1}{2}$ fields, our Jacobian has no factor $\gamma^5$, hence no operator $(-)^F$; therefore all fermions will now acquire antiperiodic boundary conditions. Hence, there are no constant modes $\psi_0^\mu$ about which to expand. This, in itself, is no problem: we could expand $\psi^\mu$ only into antiperiodic fluctuations.

iii) As pointed out in [2], the anticommuting creation and absorption operators $c^*i$ and $c_i$ should only act on the one-particle states in the quantum-mechanical trace of the operator $J \exp(-\beta R)$ in order that $c^*iT^a_iJc_j$ acts in the quantum-mechanical problem in the same way as the matrices $T^a$ did in the original problem. Thus we would need a quantum-mechanical projection operator, which should become an extra internal kernel in the corresponding one-dimensional path integral. Although we have found this kernel, see (B.8), the boundary conditions to which it corresponds lead to complicated propagators, and therefore we decided to follow a different path. We keep at all times the Dirac matrices $\gamma^\mu$ and internal symmetry generators $T^a$ as matrices, and never introduce any $\psi^\mu(t)$, $c^*i(t)$ or $c_i(t)$ at all. Although to our regret supersymmetry leaves the stage by this approach (it could have been present but for the spin $\frac{1}{2}$ regulator, and it would anyhow have been broken by the boundary conditions), it seems the simplest way to compute the path integral. In fact, we shall rewrite the action $S$ in terms of a connection $A_\mu$ and a potential $V$, which may depend in any way on the matrices $\gamma^\mu$ and $T^a$. Note that in [2] this same approach
was followed for $c^* i$ and $c_i$, but not for the Dirac matrices $\gamma^\mu$, which were replaced by $\psi^\mu(t)$. As a result the authors of ref. [2] obtained a supersymmetric sigma model, and the zero mode $\psi_0$ was useful to write the chiral anomaly in terms of p-forms in $d$ dimensions. Since p-forms do not occur in trace anomalies, there is no advantage in replacing $\gamma^\mu$ by $\psi^\mu(t)$ in our case.

iv) If one defines the path integral by discretizing time into $t_0 = 0, t_1, \ldots, t_N, t_{N+1} = \beta$, the measure for $q^\mu(t)$ contains a factor $\sqrt{g}$, which is due to integrating out the momenta $p_\mu(t)$ in the path integral. This factor is well-known from non-linear sigma models, and it is usually exponentiated, yielding a term with $\delta(0)$ which cancels the leading-loop divergences of the non-linear sigma model [4]. In [3] these factors were exponentiated by using scalar ghosts

$$L_{gh} = b(t) \sqrt{g(q(t))} c(t).$$

Below we shall instead use vector ghosts

$$L_{gh} = \frac{1}{2} g^{\mu\nu}(q(t)) \left( b^\mu(t)c^\nu(t) + a^\mu(t)a^\nu(t) \right),$$

which give the same result, but which simplify the bookkeeping of loops because the $\dot{q}\dot{q}$ part of the action is of the same form

$$L = \frac{1}{2} g^{\mu\nu}(q(t)) \left( \dot{q}^\mu(t)\dot{q}^\nu(t) \right).$$

All divergences between the various non-ghost and ghost loops cancel for finite $\beta$, which is rather remarkable as non-linear sigma models contain interactions with derivatives, confirming the measure for $q^\mu(t)$.

It is well-known that if one has $N$ intermediate points $t_1, t_2, \ldots, t_N$ between the initial and final times $t_i$ and $t_f$, one needs $(N+1)$ intermediate sets of momentum states, leading to $(N+1)$ factors $\sqrt{\text{det} g}$ [10]. We shall exponentiate only $N$ intermediate factors $\sqrt{g}$. As a result $b^\mu$ and $c^\mu$ vanish at the boundaries, and we can expand them into $\sin(n\pi t/\beta)$. The extra factor $\sqrt{g}$ appears then in the completeness relation as $\int dq \sqrt{g(q)}|q><q| = \hat{1}$, and leads to the transition amplitude [3], which is used to evolve wave functions

$$\psi(q^\mu_f, t_f) = \int dq_1 \sqrt{g(q_1)} <q^\mu_f, t_f|q^\mu_i, t_i> \psi(q^\mu_i, t_i)$$

v) In order to determine which Euclidean action corresponds to a given Hamiltonian (=regulator), there are in principle two ways to proceed. Either one starts with the Hamiltonian operator formalism, discretizes time, introduces phase-space variables, integrates
out the momenta, and finally transforms from the discrete-time basis to, for example, a
trigonometric basis. Or one bypasses this cumbersome but fundamental approach, and
determines directly from the Schrödinger equation which Hamiltonian operator belongs to
a given path integral. In the latter approach one starts thus with the Euclidean action
and the path integral on a trigonometric basis, computes it to the required order in $\beta$, and
then finally finds the Hamiltonian to which it corresponds. We shall take a very general
Euclidean action so that we can reach all the Hamiltonians we are interested in. The main
problems with the former approach (the Hamiltonian operator approach) are the follow-
ing. a) Given our regulator, there are no ordering ambiguities in the Hamiltonian itself;
however, one has to evaluate the matrix elements $\langle x | \exp(-\epsilon H) | p \rangle$, and here we should
first expand the exponent, then move all $\hat{x}$ to the left and all $\hat{p}$ to the right, and then
re-exponentiate the c-number result. For the harmonic oscillator this program is easily
executed, but for non-linear sigma models it seems complicated. b) One has to compute
the exact Jacobian for the transition from a discrete-time basis [the variables $q(t_j)$] to
a trigonometric basis [the coefficients $q_n$ in $q(t) = \sum q_n \sin(n \pi t/\beta)$]. Usually, one first
guesses how the continuum action will look like, and then introduces a trigonometric basis
whose measure one fixes such that it gives the correct result for the harmonic oscillator.
Clearly, it would be better if no guesses or approximations were made, but we found this
problem too difficult to solve. For further details, see appendix B. We shall thus follow
Feynman’s original method of deriving the Schrödinger equation, but not use the action
which naively corresponds to the regulator, but rather a quite general action, which leaves
room for extra terms.

vi) The action we consider is of the form

$$S = \int_0^\beta dt \left( L + L_{gh} + A_\mu \dot{q}^\mu + V \right). \quad (1.13)$$

The objects $A_\mu$ and $V$ are matrices, and in order to define the path integral

$$Z = \int DqDbDcDa \ \exp(-S), \quad (1.14)$$

we must decide how to order the various terms in the expansion of $\exp(-S)$. If one uses a
definition of the configuration space path integral where time is discretized, it is well-known
that gauge invariance fixes an ambiguity with respect to the point at which we should take
the gauge field. (The midpoint rule fixes the “Ito-ambiguity”, see [10].) We shall expand
all fields on a given trigonometric basis, and hence no ambiguity seems present. Yet, for a
matrix-valued action, there is an ambiguity on how to order the terms in the expansion of \( \exp(-S) \). We shall show that gauge covariance (not invariance) of our matrix-valued path integral selects time-ordering. As a mathematical exercise, we could have considered path integrals without time-ordering, but, as already said, this would violate gauge invariance; also, from a canonical operator point of view, time-ordering seems natural.

In the following sections we shall describe these issues in detail. In sect. 2 we derive the regulators of the classical Weyl-invariant field theories, of which we shall compute the trace anomalies. In sect. 3 we define the path integral with matrix-valued connections and potentials. In sect. 4 we first expand the metric, connection and potential in terms of Riemann normal coordinates, and expand the quantum fluctuations of the coordinate and vector ghost fields in a Fourier series. We then evaluate the path integrals by making an expansion in terms of the parameter \( \beta \). We find the propagators and compute connected as well as disconnected Feynman diagrams on the world line. The result is a general formula for the trace anomalies in any dimension for abstract matrices \( A_\mu \) and \( V \) in (4.22). Finally, in sect. 5, we apply this formula to the specific case of spin 0, \( \frac{1}{2} \), and 1 in \( d = 2 \) and \( d = 4 \), coupled to external gravitational and Yang-Mills fields. In sect. 6 we present our conclusions and comments.

2. The regulators for classical Weyl-invariant theories

In this section we obtain the regulators for the fields in the dimensions we are interested in. Before starting the discussion, we present our conventions. The Riemann curvature is defined by

\[
[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma \rho \sigma} V^\sigma,
\] (2.1)

where \( V^\mu \) is an arbitrary vector and \( \nabla_\mu \) the usual covariant derivative, which commutes with the metric. The Ricci tensor is given by \( R_{\mu \sigma} = R_{\mu \nu \nu \sigma} \) and the scalar curvature by \( R = g^{\mu \sigma} R_{\mu \sigma} \).

The action of a scalar field coupled to gravity and gauge fields reads as follows

\[
S_0 = \int dx \sqrt{-g} \left[ \frac{1}{2} g^{\mu \nu} (\partial_\mu + A_\mu) \phi (\partial_\nu + A_\nu) \phi - \xi R \phi \phi \right],
\] (2.2)

where \( \xi = \frac{d^2 - 2}{4(d-1)} \). The field \( \phi \) transforms according to a given representation of the gauge group and \( \phi \phi \) according to the conjugate one, so that \( \phi \phi \) is a scalar under gauge transformations. The latter can be parametrized as follows

\[
A_\mu \rightarrow A'_\mu = U \partial_\mu U^{-1} + U A_\mu U^{-1} ; \quad \phi \rightarrow \phi' = U \phi,
\] (2.3)
where it is understood that on the scalar field $\phi$ the group element $U$ acts in the corresponding representation (whose indices are suppressed). Under Weyl transformations

$$g_{\mu\nu} \to g_{\mu\nu}', \quad A_\mu \to A'_\mu; \quad \phi \to \phi' = \Omega^{\frac{2-d}{2}} \phi$$

(2.4)

the Ricci scalar transforms as

$$R \to R' = \Omega^{-1} \left[ R + (d - 1) \nabla^\mu \partial_\mu \log \Omega + \frac{1}{4} (d - 1)(d - 2) g^{\mu\nu}(\partial_\mu \log \Omega)(\partial_\nu \log \Omega) \right];$$

(2.5)

as a consequence the action (2.2) is invariant. The choice of $\mathcal{R}$ is not arbitrary if one wants to obtain a consistent anomaly [11]. A general method to identify such a regulator is to appeal to a Pauli-Villars regularization, which guarantees the consistency of the anomaly. For further details on this procedure, we refer directly to [3]. For our purposes, we use the Fujikawa variables $\tilde{\phi} \equiv g^{\frac{1}{4}} \phi$, so that one is assured that there will be no gravitational anomalies, and we obtain the following expression for the regulator

$$\mathcal{R}_\phi = -\nabla^2_A \omega - \xi R.$$  

(2.6)

Here the gauge-covariant Laplacian $\nabla^2_A$ acts in the representation which the field $\phi$ belongs to. If $\phi$ is real, then the anomaly is given by (1.7), otherwise it is twice as large.

The Weyl coupling of a complex (Dirac) spin $\frac{1}{2}$ field $\psi$ is described by the action

$$S_{\frac{1}{2}} = \int dx e \bar{\psi} \gamma \psi,$$

(2.7)

$$\gamma = e_{m}^{\mu} \gamma^{m} \left( \partial_\mu + \omega_\mu + A_\mu \right),$$

where $e_{\mu}^{m}$ is a vielbein for the metric $g_{\mu\nu}$, $e_{m}^{\mu}$ is its inverse, $e = \det e_{m}^{\mu}$ and $\gamma^{m}$ are the $SO(d)$ Dirac matrices. The spin connection $\omega_\mu = \frac{1}{4} \omega_{\mu mn} \gamma^{mn}$, with $\gamma^{mn} \equiv \frac{1}{2}[\gamma^m, \gamma^n]$, takes values in the spinorial representation of the $SO(d)$ Lie algebra and it is a function of the vielbein, while the gauge connection $A_\mu = A_\mu^a T_a$ takes values in an arbitrary representation of an arbitrary Lie group. The Weyl transformations now read as follows

$$e_{\mu}^{m} \to e_{\mu}^{m'}, \quad \psi \to \psi' = \Omega^{\frac{1-d}{4}} \psi.$$  

(2.8)

Contrary to the scalar case, there is no need to add a curvature-dependent term to achieve invariance. To obtain the spin $\frac{1}{2}$ regulator, we use again the fields $\tilde{\psi} = g^{\frac{1}{4}} \psi$, and find

$$\mathcal{R}_\psi = -\nabla A \omega = -\nabla^2_{A, \omega} - \frac{1}{4} R - \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu},$$

$$A_\frac{1}{2}^{(d)} = -\hbar \lim_{\beta \to 0} \text{Tr} \left( \sigma e^{-\beta \mathcal{R}_\psi} \right).$$  

(2.9)
For a real (Majorana) fermion one gets half this result.

Finally we consider the spin 1 case, which is classically Weyl-invariant only in \( d = 4 \). We use the background-field method by splitting the gauge field into a background part \( A_\mu \) and a quantum part \( A^{(qu)}_\mu \). Adding a gravitationally and background covariant gauge-fixing condition \( \nabla^{\mu} A^{(qu)}_\mu \) suitably weighted, one finds

\[
S_{gf}^f = \int d^4x \sqrt{g} \text{Tr} \left[ \frac{1}{4} g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} + \Pi \left( \nabla^{\mu} A^{(qu)}_\mu - \frac{1}{2} \Pi \right) + B \nabla^{\mu} (\nabla^{A} C + [A^{(qu)}_\mu, C]) \right],
\]

where the subscript A indicates that the derivatives are background-gauge-covariant. We contract \( A^{(qu)}_\mu \) with a vielbein field to obtain a tangent space index, and upon elimination of the auxiliary field \( \Pi \), we find for the regulator of the spin 1 field:

\[
R_A = -\delta_m^n \nabla^2_{A,\omega} - R_m^n - 2F_m^n.
\]  

In addition we need the regulator for the ghosts. Both \( B \) and \( C \) are to be considered as real, and their regulator is

\[
R_{BC} = -\nabla^2_A.
\]

These regulators act in the adjoint representation, thus the gauge connection as well as the \( F_{mn} \) term in eq. (2.11) are absent for Abelian gauge fields. The trace anomaly is given by

\[
A_1^{(4)} = \hbar \lim_{\beta \to 0} \text{Tr} \left( \sigma e^{-\beta R_A} \right) - 2\hbar \lim_{\beta \to 0} \text{Tr} \left( \sigma e^{-\beta R_{BC}} \right),
\]

where we used \( \delta \tilde{A}^{(qu)}_\mu = \frac{1}{2} \sigma \tilde{A}^{(qu)}_\mu \), \( \delta \tilde{C} = \sigma \tilde{C} \), and \( \delta \tilde{B} = 0 \), with \( \tilde{A}^{(qu)}_\mu = g^{\frac{1}{2}} e^\mu_{\mu A} A^{(qu)}_\mu \), \( \tilde{C} = g^{\frac{1}{2}} C \), and \( \tilde{B} = g^{\frac{1}{2}} B \). Perhaps we should note here that the gauge-fixing term in (2.10) naively destroys the Weyl invariance. In fact, with the following natural definition of the Weyl transformations on the \( \Pi, B, C \) fields,

\[
\Pi \to \Pi' = \Omega^{-1} \Pi \\
B \to B' = \Omega^{-1} B \\
C \to C' = C,
\]

the action is not invariant and there is no way of fixing the above transformation rules to achieve invariance. Under an infinitesimal Weyl transformation with \( \Omega = e^{\sigma} = 1 + \sigma + \ldots \), the action changes by

\[
\delta S_{gf}^f = \int d^4x \sqrt{g} \text{Tr} \left[ \Pi (g^{\mu\nu} \partial_\nu \sigma) A^{(qu)}_\mu + B (g^{\mu\nu} \partial_\nu \sigma) (\nabla^{A} C + [A^{(qu)}_\mu, C]) \right] = \delta_{BRST} \int d^4x \sqrt{g} \text{Tr} \left[ B (g^{\mu\nu} \partial_\nu \sigma) A^{(qu)}_\mu \right]
\]
In the last line we have made use of the anticommuting BRST operator defined by

\[
\delta_{\text{BRST}} A_{\mu}^{(qu)} = \nabla_{\mu} A + [A_{\mu}^{(qu)}, C]
\]

\[
\delta_{\text{BRST}} C = -\frac{1}{2} \{C, C\}
\]

\[
\delta_{\text{BRST}} B = \Pi
\]

\[
\delta_{\text{BRST}} \Pi = 0,
\]

which shows that the Weyl variation of the action is BRST-exact. For this reason it has zero expectation value between physical states \[12\].

3. Quantum mechanics with matrix-valued potentials in curved spaces

We want to find now a path integral representation of the traces which yield the Weyl anomalies. The typical trace we have to compute has the following structure

\[
I = \lim_{\beta \to 0} \text{Tr} \ e^{-\beta H}
\]

\[
H = -(\nabla^\mu + A^\mu)(\partial_\mu + A_\mu) + S,
\]

where \( H \) acts on fields which are scalars under reparametrization and transform in a fixed, but otherwise unspecified, representation of the gauge group. The matrices \( A_\mu \) and \( S \) transform as vectors and scalars under reparametrizations, respectively, but are furthermore quite general. Our strategy will be as follows. We first rewrite the trace in (3.1) using the operatorial formalism of quantum mechanics. Then we employ the path integral formulation of quantum mechanics to explicitly compute the trace. As is well-known, one can derive the path integral using the operatorial formulation of quantum mechanics and vice versa. As explained in the introduction, it is easier to start with a proper definition of the path integral on curved spaces, from which we derive unambiguously the Hamiltonian operator.

The trace in (3.1) can be taken over the Hilbert space generated by the following basis vectors:

\[
|x> \otimes |i> \equiv |x, i>
\]

where the kets \( |i> \) correspond to a basis in the representation space of the gauge group and \( |x> \) are the usual kets of configuration space. The scalar product is defined by

\[
<x, i|y, j> = \frac{\delta(x - y)}{\sqrt{g(x)}} \delta^i_j
\]
and the resolution of the identity takes the following form

\[ \hat{1} = \sum_i \int dx \sqrt{g} \ket{x, i} \bra{x, i} \]  

(3.4)

Any state \( \ket{\psi} \) and operator \( \hat{G} \) have the following components in the above basis

\[ \psi_i(x) = \bra{x, i} \psi \]  

\[ G_{ij}^a(x, y) = \bra{x, i} \hat{G} \ket{y, j} \]  

(3.5)

In particular, the usual Hermitian position operator \( \hat{q}_\mu \), Hermitian momentum operator \( \hat{p}_\mu \) and antiHermitian Lie algebra generator \( \hat{T}^a \) have the following matrix elements

\[ \bra{x, i} \hat{q}_\mu \ket{y, j} = x^\mu \delta(x - y) \sqrt{g(x)} \delta_i^j \]  

\[ \bra{x, i} \hat{p}_\mu \ket{y, j} = -i \hbar g^{-\frac{1}{2}}(x) \frac{\partial}{\partial x^\mu} g^{\frac{1}{2}}(x) \delta(x - y) \sqrt{g(x)} \delta_i^j \]  

\[ \bra{x, i} \hat{T}^a \ket{y, j} = T^a_{ij} \delta(x - y) \sqrt{g(x)} \]  

(3.6)

One may check that \( \hat{p}_\mu \) is represented by \( -i \hbar g^{-\frac{1}{2}} \partial_\mu g^{\frac{1}{2}} \) on wave functions \( \psi_i(x) \), and that it is Hermitian with respect to the inner product \( \braket{\phi | \psi} = \int dx g^{\frac{1}{2}} \phi^* \psi \).

A gauge transformation is implemented by a unitary operator \( \hat{U} \) and we have the following transformation properties for the wave function and the transition amplitude

\[ \psi_i(x) = \bra{x, i} \psi \rightarrow \bra{x, i} \hat{U} \ket{\psi} = U_i^j(x) \psi_j(x) = \psi_i(x)' \]  

\[ \bra{x, i} e^{-\beta \hat{H}} \ket{y, j} \rightarrow U_i^k(x) \bra{x, k} e^{-\beta \hat{H}} \ket{y, l} (U^{-1})^l_j(y), \]  

(3.7)

where the Hamiltonian operator has the following form

\[ \hat{H} = -g^{-\frac{1}{2}}(i \hat{p}_\mu + \hat{A}_\mu) \tilde{g}^{\frac{1}{2}} g^{\mu\nu}(i \hat{p}_\nu + \hat{A}_\nu) \tilde{g}^{-\frac{1}{2}} + \hat{S}. \]  

(3.8)

From (3.4) it follows that the trace in (3.1) can be rewritten as

\[ I = \lim_{\beta \to 0} \sum_j \int dx \sqrt{g} \bra{x, j} e^{-\beta \hat{H}} \ket{x, j} \]  

(3.9)

The gauge transformation properties in (3.7) for the transition amplitude as well as its scalar character under reparametrization will be the defining principles for our construction of the path integral representation, to which we now turn.
We are going to define the path integral representation for the transition amplitude, to be considered as a matrix with indices $j_f$ and $j_i$, for a particle starting at point $q_i^\mu$ at the initial time $t_i$ and reaching point $q_f^\mu$ at the final time $t_f$:

$$
\langle q_f^\mu, j_f, t_f | q_i^\mu, j_i, t_i \rangle = \langle q_f^\mu, j_f | e^{-(t_f-t_i)\hat{H}} | q_i^\mu, j_i \rangle.
$$

(3.10)

The path integral is defined as follows

$$
\langle q_f^\mu, j_f, t_f | q_i^\mu, j_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} (\mathcal{D}q) \ T \ e^{-S[q]}
$$

$$
S[q] = \int_{t_i}^{t_f} dt \left( \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + A_\mu \dot{q}^\mu + V \right)
$$

(3.11)

$$
(\mathcal{D}q) = \prod_{t_i < \tau < t_f} \sqrt{g(q(\tau))} dq(\tau),
$$

where $T$ denotes time-ordering along the histories $q^\mu(t)$. The vector potential $A_\mu$ and the scalar potential $V$ are matrices, so that the action itself is a matrix (hence the explicit indices $j_i$ and $j_f$ shown explicitly on the left-hand side of the first equation, but omitted in the right-hand side for notational convenience). The explicit relation between the scalar potential $V$ in (1.13) and (3.11) and the scalar potential $S$ appearing in (3.1) will be given later. Note that the notation $\langle | \ldots | \rangle$ means averaging with the functional integral under the boundary conditions given in the brackets, while $< | \ldots | >$ will denote the usual bras and kets of operatorial quantum mechanics, so that (3.10) is the statement of the equivalence of the operatorial formalism and the functional integral formalism. To check the correctness of this definition of our path integral, we must check its transformation properties under reparametrizations and gauge variations. As far as reparametrizations are concerned, we can immediately see that the path integral is covariant (in fact it is a scalar), since it is built from manifestly covariant objects. Turning to the gauge symmetry, we can compute the variation of the transition amplitude under an infinitesimal gauge variation ($U = e^\Lambda$ with $\Lambda$ infinitesimal)

$$
\delta \langle q_f^\mu, j_f, t_f | q_i^\mu, j_i, t_i \rangle = \langle q_f^\mu, j_f, t_f | (\Lambda(q_f) - \Lambda(q_i)) | q_i^\mu, j_i, t_i \rangle
$$

$$
= \Lambda_j^k (q_f) \langle q_f^\mu, k, t_f | q_i^\mu, j_i, t_i \rangle - \langle q_f^\mu, j_f, t_f | q_i^\mu, l, t_i \rangle \Lambda_i^j (q_i).
$$

(3.12)

The first line is a consequence of the transformation properties of time-ordered exponentials. For the finite case it reads

$$
\langle q_f^\mu, j_f, t_f | q_i^\mu, j_i, t_i \rangle' = U_j^k (q_f) \langle q_f^\mu, k, t_f | q_i^\mu, l, t_i \rangle (U^{-1})_l^j (q_i),
$$

(3.13)
which is the correct transformation property expected from the operatorial picture.

We now turn to the path integral measure of (3.11). A key point introduced in [3] is to exponentiate the non-trivial part of the measure using ghost fields, thereby leaving a translational-invariant measure necessary for performing the perturbative expansion. We present it here in a modified form, which helps in keeping track of the cancellation of potentially divergent terms. The latter are typically present in non-linear models of quantum mechanics (see ref. [9]). Since we need to exponentiate the factor $\sqrt{g}$ present in the measure of (3.11), we introduce anticommuting ghost fields $b^\mu, c^\mu$ and commuting ghost fields $a^\mu$ coupled to the metric $g_{\mu\nu}$. The effect of path integrating over $b^\mu, c^\mu$ will be to produce a factor $g$, while the path integration over $a^\mu$ will reduce it to $\sqrt{g}$, thus recovering the correct measure. Therefore the covariant path integral which we are going to compute perturbatively in the next section reads as follows

$$\langle q^\mu_f, t_f | q^\mu_i, t_i \rangle = \int_{q(t_i) = q_i}^{q(t_f) = q_f} (Dq)(Db)(Dc)(Da) \ T e^{-S[q,b,c,a]}$$

$$S[q,b,c,a] = \int_{t_i}^{t_f} dt \left( \frac{1}{2} g_{\mu\nu} (\dot{q}^\mu \dot{q}^\nu + b^\mu c^\nu + a^\mu a^\nu) + A_\mu \dot{q}^\mu + V \right)$$

(3.14)

4. Perturbation expansion

We now present the perturbative evaluation of the covariant path integral defined in the previous section. Our strategy will be to make full use of the general coordinate invariance and pick the coordinate system best suited for carrying out the computation, namely the Riemann normal coordinates centered at the final point $q^\mu_f$. Before proceeding further, we introduce the dimensionless time variable $\tau = \frac{t - t_i}{\beta}$, with $\beta = t_f - t_i$, and rescale the ghost fields as follows $(b^\mu, c^\mu, a^\mu) \rightarrow \frac{1}{\beta}(b^\mu, c^\mu, a^\mu)$, so that $\beta$ will turn into a loop-counting parameter, nicely organizing the loop expansion. We do not keep track of a possible Jacobian for this rescaling, since the overall normalization will later be determined by other means. After such redefinitions the path integral takes the following form

$$\langle q^\mu_f, j_f, t_f | q^\mu_i, j_i, t_i \rangle = \int_{q(t_i) = q_i}^{q(t_f) = q_f} (Dq)(Db)(Dc)(Da) \ T e^{-\frac{1}{\beta} S[q,b,c,a]}$$

(4.1)

where

$$S[q,b,c,a] = \int_{-1}^{0} d\tau \left[ \frac{1}{2} g_{\mu\nu}(q)(\dot{q}^\mu \dot{q}^\nu + b^\mu c^\nu + a^\mu a^\nu) + \beta A_\mu(q) \dot{q}^\mu + \beta^2 V(q) \right].$$

(4.2)
We now go over to Riemann normal coordinates by introducing functions \( q^\mu(\tau, \lambda) \) which are geodesics in \( \lambda \) for fixed \( \tau \), and which start at \( q^\mu_f \) and end at \( q^\mu(\tau) \):

\[
q^\mu(\tau, \lambda = 0) = q^\mu_f, \quad q^\mu(\tau, \lambda = 1) = q^\mu(\tau).
\] (4.3)

The normal coordinates \( z^\mu(\tau) \) for a point with coordinates \( q^\mu(\tau) \) are then by definition

\[
z^\mu(\tau) = \frac{\partial}{\partial \lambda} q^\mu(\tau, \lambda) \bigg|_{\lambda=0}.
\] (4.4)

The action \( S[q(\tau, \lambda = 1)] \) can be Taylor expanded in \( \lambda \) around \( \lambda = 0 \), and if the action is an Einstein scalar, one can replace ordinary \( \lambda \) derivatives by covariant derivatives \( \frac{D}{\partial \lambda} = \frac{\partial q^\mu}{\partial \lambda} \nabla_\mu \):

\[
S = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{D}{\partial \lambda} \right)^n S[q(\tau, \lambda)] \bigg|_{\lambda=0}.
\] (4.5)

Since the \( q(\tau, \lambda) \) are geodesics, \( \left( \frac{D}{\partial \lambda} \right)^2 q(\tau, \lambda) = 0 \), and in the expansion either no derivatives act on \( q^\mu(\tau, \lambda) \) (yielding the \( \tau \)-independent constant \( q^\mu_f \)) or one derivative (yielding \( z^\mu(\tau) \)).

Another useful identity is

\[
\frac{D}{\partial \tau} \frac{D}{\partial \lambda} z^\mu(\tau, \lambda) = \left[ \frac{D}{\partial \lambda}, \frac{D}{\partial \tau} \right] z^\mu(\tau, \lambda) = R^\gamma_\alpha \beta^\mu \zeta^\alpha q^\beta z^\gamma,
\] (4.6)

where \( \frac{D}{\partial \tau} = \frac{\partial q^\mu}{\partial \tau} \nabla_\mu \) and \( z^\mu(\tau, \lambda) = \frac{\partial}{\partial \lambda} q^\mu(\tau, \lambda) \). Of course, \( \frac{D}{\partial \lambda} \dot{q}^\mu(\tau, \lambda) = \dot{z}^\mu + O(\lambda) \).

Using this expansion of the action into normal coordinates, one obtains

\[
S = \int_{-1}^{0} d\tau \left[ \frac{1}{2} \left( g_{\mu\nu}(0) + \frac{1}{3} R_{\alpha\mu\nu\beta}(0) z^\alpha z^\beta + \frac{1}{6} \nabla_\gamma R_{\alpha\mu\nu\beta}(0) z^\gamma z^\alpha z^\beta + R_{\gamma\delta\alpha\mu\beta}(0) z^\gamma z^\delta z^\alpha z^\beta + \cdots \right) \dot{z}^\mu \dot{z}^\nu + b^\mu c^\nu + a^\mu a^\nu \right.
\]
\[
+ \beta \left( A_\mu(0) + \nabla_\nu A_\mu(0) z^\nu \right. \left. + A_\lambda \nu \mu(0) z^\lambda z^\nu \right. \left. + A_\alpha \beta \gamma \mu(0) z^\alpha z^\beta z^\gamma \right. \left. + \cdots \right) \dot{z}^\mu
\]
\[
+ \beta^2 \left( V(0) + z^\mu \nabla_\mu V(0) + \frac{1}{2} z^\mu z^\nu \nabla_\mu \nabla_\nu V(0) + \cdots \right) \right],
\] (4.7)

where the following tensors have been defined

\[
R_{\gamma\delta\alpha\mu\beta} = \frac{1}{20} \nabla_\gamma \nabla_\delta R_{\alpha\mu\nu\beta} + \frac{2}{45} R_{\alpha\mu\sigma\beta} R_{\gamma\nu}^{\quad \sigma \delta}
\]
\[
A_{\lambda\nu\mu} = \frac{1}{2} \nabla_\lambda \nabla_\nu A_\mu + \frac{1}{6} R_{\lambda\mu}^{\quad \sigma \nu} A_\sigma
\]
\[
A_{\alpha\beta\gamma\mu} = \frac{1}{6} \nabla_\alpha \nabla_\beta \nabla_\gamma A_\mu + \frac{1}{6} R_{\beta\mu}^{\quad \rho \gamma} \nabla_\alpha A_\rho + \frac{1}{12} A_\rho \nabla_\alpha R_{\beta\mu}^{\quad \rho \gamma}.
\] (4.8)
Note the $\beta$ dependence of the vector and scalar potentials produced by the introduction of the time variable $\tau$. It implies that they will start contributing at higher loops with respect to the “interactions” originating from the metric tensor. At this stage, we could start the perturbative quantum computation by splitting the field $z^\mu$ into a classical and a quantum piece

$$z^\mu = z^\mu_{\text{cl}} + z^\mu_{\text{qu}}.$$ \hspace{1cm} (4.9)

The classical path $z^\mu_{\text{cl}}$ is taken to be a solution of the classical equation of motion in the limit $\beta \to 0$, which is the geodesic equation. So $z^\mu_{\text{cl}}(\tau) = -z^\mu_i \tau$, where $z^\mu_i$ are the Riemann normal coordinates of the initial point $q^\mu_i$. It takes into account the boundary conditions prescribed to the path integral. The quantum fluctuations $z^\mu_{\text{qu}}$ therefore vanish at $\tau = -1, 0$ and can be Fourier-expanded as follows

$$z^\mu_{\text{qu}} = \sum_{n=1}^\infty z^\mu_n \sin(\pi n \tau).$$ \hspace{1cm} (4.10)

A similar expansion holds also for the ghost fields, which are zero at the boundary

$$b^\mu = \sum_{n=1}^\infty b^\mu_n \sin(\pi n \tau); \quad c^\mu = \sum_{n=1}^\infty c^\mu_n \sin(\pi n \tau); \quad a^\mu = \sum_{n=1}^\infty a^\mu_n \sin(\pi n \tau).$$ \hspace{1cm} (4.11)

As we explained, these boundary conditions follow from the fact that the measure factor $\sqrt{g}$ is not included in (3.11) for $t_i$ and $t_f$. The evaluation of the transition amplitude in the two-loop approximation is enough to determine the quantum Hamiltonian associated with the path integral. Put differently, a two-loop computation is sufficient to verify that the transition amplitude satisfies a Schrödinger equation (diffusion equation)

$$\frac{d}{dt} \langle x^\mu, i, t | y^\mu, j, t' \rangle = -H \langle x^\mu, i, t | y^\mu, j, t' \rangle$$

$$H = -\frac{1}{2}(\nabla^\mu + A^\mu)(\partial_\mu + A_\mu) - \frac{1}{8}R + V.$$ \hspace{1cm} (4.12)

where in the first equation $H$ acts on the indices $x^\mu$ and $i$. Such a computation was presented in [3] and will not be reproduced here, since the introduction of the time ordering does not bring in any modification at this stage. The above Hamiltonian shows that the connection with the operatorial formulation presented in sect. 3 is achieved by setting the scalar potential $V = \frac{1}{2}S + \frac{1}{8}R$. An additional outcome of the above procedure is the determination of the complete normalization of the path integral measure

$$DqDbDcDa = A \prod_{m=1}^\infty \prod_{\mu=1}^d (\pi m^2)^{\frac{1}{2}} dz^\mu_m db^\mu_m dc^\mu_m da^\mu_m$$

$$A = \frac{1}{(2\pi \beta)^{\frac{d}{2}}}$$ \hspace{1cm} (4.13)

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A consequence of this result is that all Gaussian integrals over the modes and the final integration over $q_i$ in (1.12) are normalized to unity if one uses in (4.7) only the terms with $g_{\mu \nu}(0)$ and puts in (1.12) the $q_i$ in $\sqrt{g(q_i)}$ and $\psi(q_i, t_i)$ equal to $q_f$. (The kinetic terms in the action are read off from (4.7) and are given by

$$S_{(2)} = \frac{1}{2} g_{\mu \nu}(0) z_\mu^i z_\nu^j + \frac{1}{4} g_{\mu \nu}(0) \sum_{n=1}^{\infty} \left( \pi^2 n^2 z_\mu^n z_\nu^n + b_\mu^i c_\nu^i + a_\mu^i a_\nu^i \right). \quad (4.14)$$

The factors $\sqrt{g}$ due to integration over $z_m^\mu, b_m^\mu, c_m^\mu$ and $a_m^\mu$ cancel, whereas the integration over $z_i^\mu$ produces a factor $(2 \pi \beta)^{\frac{d}{2}} (g)^{-\frac{d}{2}}$, which cancels against the $(g)^{\frac{d}{2}}$ in (1.12) and $A$ in (4.13).)

Having established the equivalence (3.10) for $V = \frac{1}{2} S + \frac{1}{8} R$, we proceed to the task of computing the general trace in (3.1) using the path integral representation

$$I = \lim_{\beta \to 0} \text{Tr} e^{-\beta H} = \lim_{\beta \to 0} \sum_i \int dx \sqrt{g} \langle x^\mu, i, 0 | x^\mu, i, -\beta \rangle. \quad (4.15)$$

The evaluation of the right-hand side proceeds as follows. The classical path in (4.9) is now $z_{cl}^\mu = 0$, and we can drop the subscript in $z_{qu}^\mu$ without possibility of confusion. From the quadratic piece in (4.14) we recognize the following propagators

$$\langle z^\mu(\tau) z'^\nu(\tau') \rangle = -\beta g^{\mu \nu}(0) \Delta(\tau, \tau')$$

$$\langle b^\mu(\tau) c'^\nu(\tau') \rangle = -\beta g^{\mu \nu}(0) \left( 2 \partial_\tau^2 \Delta(\tau, \tau') \right)$$

$$\langle a^\mu(\tau) a'^\nu(\tau') \rangle = -\beta g^{\mu \nu}(0) \left( -\partial_\tau^2 \Delta(\tau, \tau') \right), \quad (4.16)$$

where

$$\Delta(\tau, \tau') = \sum_{n=1}^{\infty} \left[ -\frac{2}{\pi^2 n^2} \sin(\pi n \tau) \sin(\pi n \tau') \right]. \quad (4.17)$$

Note that we can sum the series in (4.17) and obtain

$$\Delta(\tau, \tau') = \tau (\tau' + 1) \Theta(\tau - \tau') + \tau'(\tau + 1) \Theta(\tau' - \tau), \quad (4.18)$$

where the theta function is defined as usual by

$$\Theta(\tau - \tau') = \begin{cases} 
0 & \text{for } \tau < \tau' \\
\frac{1}{2} & \text{for } \tau = \tau' \\
1 & \text{for } \tau > \tau'
\end{cases} \quad (4.19)$$

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Indeed, $\frac{d^2}{d\tau^2}\Delta(\tau, \tau') = \delta(\tau - \tau')$, while $\Delta(0, \tau') = \Delta(\tau, 0) = 0$. Note that $\Delta(\tau, \tau')$ is continuous at $\tau' \downarrow \tau$ and $\tau' \uparrow \tau$, and $\Delta(\tau, \tau) = \tau(\tau + 1)$. Also

$$
\frac{d}{d\tau}\Delta(\tau, \tau') \equiv \Delta'(\tau, \tau') = \tau' + \theta(\tau - \tau')
$$

and also $\Delta' = \Delta = \frac{1}{2} + \tau$ at $\tau' = \tau$. However, we must recall that the path integral is naturally regulated by truncating the mode expansion at a fixed mode $M$, so that the derivatives in (4.16) should be taken before the limit $M \to \infty$. At fixed $M$, all propagators are well-defined functions, with finite limits at coinciding points $\tau = \tau'$.

The interaction vertices are given by the remaining pieces of the action, $S_{\text{int}} = S - S^{(2)}$, with $S$ given in (4.17) and $S^{(2)}$ in (4.14). The final result for the basic trace is thus as follows. In $d$-dimensional Euclidean space,

$$
I^{(d)} = \lim_{\beta \to 0} \text{Tr} \ e^{-\beta H} = \lim_{\beta \to 0} \frac{1}{(2\pi \beta)^{\frac{d}{2}}} \int dx \sqrt{g(x)} \langle 0 | T e^{-\beta S_{\text{int}}} | 0 \rangle, \quad (4.22)
$$

where

$$
S_{\text{int}} = \int_{-1}^{0} d\tau \left[ \frac{1}{2} \left( \frac{1}{3} R_{\alpha\mu\nu\beta}(x) z^\alpha z^\beta + \cdots \right) (\dot{z}^\mu \dot{z}^\nu + b^\mu e^\nu + a^\mu a^\nu) + \beta \left( A_\mu(x) + \nabla_\nu A_\mu(x) z^\nu + \cdots \right) \dot{z}^\mu + \beta^2 \left( V(x) + z^\mu \nabla_\mu V(x) + \cdots \right) \right]. \quad (4.23)
$$

One must evaluate in $d$ dimensions only all “vacuum polarization” graphs that are proportional to $\beta^\frac{d}{2}$. They are obtained by expanding $e^{(-\beta^{-1} S_{\text{int}})}$, and contracting all $z^\mu, b^\mu, c^\mu$ and $a^\mu$, using the propagators in (4.16). The trace anomaly is then obtained by multiplying the result by the Weyl weights of the fields.

5. Computation of the trace anomalies in $d = 2$ and $d = 4$

To compute the trace anomalies in $d = 2$, $I^{(2)}$ in (4.22), we need the $\beta$ term in (4.22). Since propagators go like $\beta$ and vertices like $\frac{1}{\beta}$ (or $\beta^0$ for $A$, or $\beta$ for $V$), this means that we must compute 2-loop diagrams in $d = 2$. Similarly for $d = 2k$ we need the $k + 1$ loop contribution. Using the above propagators and vertices, we get for $d = 2$ only a contribution from the $\frac{1}{3} R_{\alpha\mu\nu\beta}$ vertex in (4.7). The corresponding diagrams have the
topology of the number 8, with none or one of the loops containing a ghost. The result is proportional to

$$\int_{-1}^{0} d\tau \left[ (\Delta^* + \Delta^{**}) \Delta - \Delta^* \Delta^* \right]_{\tau = \tau'}.$$  \tag{5.1}$$

The last term is clearly finite. Partially integrating the first term by using the fact that $(\Delta^* + \Delta^{**}) = \partial_{\tau} (\Delta \text{ at } \tau = \tau')$ immediately shows that (5.1) is finite. Also in higher loops finiteness is manifest thanks to using vector ghosts. The result is

$$I_{(2)} = \text{tr}_{YM} \int \frac{d^2 x}{2\pi} \sqrt{g} \left[ \frac{1}{24} R - V \right], \tag{5.2}$$

where the trace is over the finite dimensional representation space of the gauge group.

The computation for obtaining the $d = 4$ result is more involved, but nevertheless straightforward. Let us first quote the result and then briefly comment on its derivation.

$$I_{(4)} = \text{tr}_{YM} \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \left[ \frac{1}{720} (R_{\alpha\mu\nu\beta} R^{\alpha\mu\nu\beta} - R_{\mu\nu} R_{\mu\nu}) + \frac{1}{480} \nabla^2 R - \frac{1}{12} \nabla_\alpha^2 V \right. \right.$$  \tag{5.3}\left. \left.+ \frac{1}{2} \left( \frac{1}{24} R - V \right)^2 + \frac{1}{48} F_{\mu\nu} F^{\mu\nu} \right].$

The $A_\mu$ contribution due to one vertex $-\frac{1}{\beta} S_{int}$ vanishes, since $\int_{-1}^{0} d\tau \Delta(\tau, \tau') \Delta(\tau, \tau)$ as well as $\int_{-1}^{0} d\tau \Delta(\tau, \tau') \Delta(\tau, \tau)$ are zero. The $A_\mu$ contribution coming from two vertices $-\frac{1}{\beta} S_{int}$ produce the kinetic terms in $\frac{1}{48} F_{\mu\nu}^2$. This comes about because $\int \int d\tau d\tau' \langle z^\nu \dot{z}^\mu \rangle \langle \Delta(\tau, \tau') \rangle$ is proportional to $\int \int d\tau d\tau' [g^{\mu\nu} g^{\rho\sigma} \Delta(\tau, \tau') \Delta^* (\tau, \tau') + g^{\nu\sigma} g^{\mu\rho} \Delta^* (\tau, \tau') \Delta(\tau, \tau')]$, and partially integrating $\Delta^*$, using that $\Delta$ vanishes at the boundary, gives the curl structure of the Maxwell action. The $(DA)AA$ terms in the Yang-Mills action are a good test that our time-ordering prescription is required by gauge invariance. They are treated like the $AV$ and $AAV$ terms, see below. The $V$ terms in (5.3) come from the vertex $-\beta [V + \frac{1}{2} z^\mu z^\nu \nabla_\mu \nabla_\nu V]$. First there is the term $\frac{1}{(2\pi)^2} \frac{1}{2} [\beta V]^2 = \frac{1}{8\pi^2} V^2$. The term with $\nabla^2 V$ comes of course from $\langle z^\nu \dot{z}^\mu \rangle \nabla_\mu \nabla_\nu V$. It is instructive to see how the covariantizations in $-\frac{1}{12} \nabla_\alpha^2 V = -\frac{1}{12} \partial^\mu (\partial_\mu V + [A_\mu, V]) - \frac{1}{12} \left[ A^\mu, \partial_\mu V + [A_\mu, V] \right]$ arise, because here we see the crucial role of our time-ordering prescription. For example, from

$$T\left( -\frac{1}{\beta} \right)^2 \frac{1}{2} \int_{-1}^{0} \beta (A_\mu \dot{z}^\mu + \nabla_\nu A_\mu z^\nu \dot{z}^\mu) d\tau \int_{-1}^{0} \beta^2 (V + z^\rho \partial_\rho V) d\tau'$$

$$= \beta A_\mu \partial_\rho V \int_{-1}^{0} \dot{z}^\mu d\tau \int_{-1}^{\tau} z^\rho d\tau' + \beta \partial_\rho V A_\mu \int_{-1}^{0} z^\rho d\tau \int_{-1}^{\tau} \dot{z}^\mu d\tau'. \tag{5.4}$$
Partially integrating and using $z^\mu(0) = z^\mu(-1) = 0$ yields indeed $-\frac{q^2}{\pi} [\partial^\mu, V]$. Similarly the $[\partial^\mu A_\mu, V]$ is recovered. The most interesting term is $[A_\mu, [A_\mu, V]]$; one now has three ways of ordering the time coordinates $\tau, \tau', \tau''$, and one indeed obtains the combination $A_\mu A^\mu V - 2A_\mu V A^\mu + V A_\mu A^\mu$, and with the correct coefficient.

We have been explicit in these calculations, because they show the importance of time ordering in order to obtain gauge-invariant results.

Having computed the general traces (5.2) and (5.3), we can specialize them to deduce the Weyl anomalies of the various fields discussed in sect. 2. We need to use $V = \frac{i}{2} R - \frac{i}{2} \xi R$ for the spin 0 regulator, and $V = -\frac{1}{4} F_{\mu\nu} \gamma^{\mu\nu}$ and $A_\mu \to A_\mu + \omega_\mu$ for the spin $\frac{1}{2}$ regulator, obtaining the following result for the trace anomalies of a real scalar and a complex spin $\frac{1}{2}$ field in two dimensions:

$$A^{(2)}_0 = A^{(2)}_{\frac{1}{2}} = -h \text{tr}_{YM} \int \frac{d^2x}{2\pi} \sqrt{g} \frac{1}{12} R \sigma. \quad (5.5)$$

In four dimensions we have instead

$$A^{(4)} = h \text{tr}_{YM} \int \frac{d^4x}{(2\pi)^2} \sqrt{g} \left[ a R_{\alpha\mu\nu\beta} R^{\alpha\mu\nu\beta} + b R_{\mu\nu} R^{\mu\nu} + c R^2 + d \nabla^2 R + e F_{\mu\nu} F^{\mu\nu} \right] \sigma. \quad (5.6)$$

with the various fields contributing as follows

- spin 0: $a = \frac{1}{240}$, $b = -\frac{1}{240}$, $c = 0$, $d = -\frac{1}{720}$, $e = \frac{1}{18}$
- spin $\frac{1}{2}$ (Dirac spinor): $a = \frac{7}{1440}$, $b = \frac{1}{180}$, $c = -\frac{1}{288}$, $d = -\frac{1}{120}$, $e = \frac{1}{6}$
- spin 1 (Abelian): $a = -\frac{13}{720}$, $b = \frac{11}{90}$, $c = -\frac{5}{144}$, $d = \frac{1}{40}$, $e = 0$
- spin 1 (non-Abelian): $a = -\frac{13}{720}$, $b = \frac{11}{90}$, $c = -\frac{5}{144}$, $d = \frac{1}{40}$, $e = -\frac{11}{24}$

where in addition we have used for the ghost regulator $V = \frac{i}{2} R$ and for the gauge regulator $V = \frac{i}{8} R \delta_m^m - \frac{1}{2} R_m^m - \frac{1}{2} F_m^m$ and $A_\mu \to A_\mu + \omega_\mu$, with $\omega_\mu$ acting in the vector representation. These results agree with the literature [4]. For completeness, we recall that eq. (5.6) can also be written in the following form

$$A^{(4)} = h \int \frac{d^4x}{(2\pi)^2} \sqrt{g} \left[ \alpha F + \beta G + \gamma \nabla^2 R + e (\text{tr}_{YM} F_{\mu\nu} F^{\mu\nu}) \right] \sigma, \quad (5.8)$$

where $F = R_{\alpha\mu\nu\beta} R^{\alpha\mu\nu\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$ is the square of the Weyl conformal tensor and $G = R_{\alpha\mu\nu\beta} R^{\alpha\mu\nu\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$ is the topological Euler density. The coefficients $\alpha, \beta, \gamma$ are given by

$$\alpha = \frac{1}{480} (N_0 + 6 N_{\frac{1}{2}} + 12 N_1)$$
$$\beta = -\frac{1}{1440} (N_0 + 11 N_{\frac{1}{2}} + 62 N_1)$$
$$\gamma = -\frac{1}{720} (N_0 + 6 N_{\frac{1}{2}} - 18 N_1). \quad (5.9)$$
where $N_0$, $N_{\frac{1}{2}}$, and $N_1$ are the number of real spin 0, complex (Dirac) spin $\frac{1}{2}$, and real spin 1 fields, respectively. This form of presenting the trace anomaly clearly shows that the coefficients in eq. (5.4) are not all independent [4]. This is a consequence of the Wess-Zumino consistency conditions, which guarantee the absence of a single $R^2$ term in the anomaly [13]. Moreover, the coefficient $\gamma$ in (5.8) can be modified at will (in particular can be set to zero) by adding to the effective action a finite local counterterm proportional to $R^2$. For spin 1, only the sum of the contributions of the non-ghost sector and the ghost sector satisfies the consistency conditions. The gauge-fixed action is not Weyl-invariant, but its Weyl variation is BRST exact and has vanishing expectation value [12].

6. Conclusions

In this article we have computed trace anomalies in quantum field theory by using path integrals in quantum mechanics. Our work extends the analysis of Alvarez-Gaumé and Witten of chiral anomalies, but because trace anomalies are not of a topological nature, certain fundamental and interesting aspects of path integrals had to be taken into account, which could be neglected in the case of chiral anomalies. As already indicated in [14], this makes the evaluation of trace anomalies more complicated than that of chiral anomalies. Of interest is the method itself, rather than the results. The quantum mechanical approach is an excellent tool for computing anomalies. It avoids the painful Baker-Campbell-Hausdorf expansion of a direct application of Fujikawa’s method. It is clearly related to the DeWitt’s heat-kernel approach [15], but it avoids the bilocal tensor calculus of the latter.

We have given a general formula, eq. (4.22), from which trace anomalies in any dimension can be obtained. We then computed the trace anomalies in $d = 2$ and $d = 4$ for spin 0, $\frac{1}{2}$ and 1 at one loop with external gravitational and Yang-Mills fields. Our results agree, of course, with the literature [4]. In the literature also the trace anomalies for $d = 6$ and $d = 8$ have been obtained (by Gilkey [16] and Avramidi [17], respectively, who both used the heat-kernel approach). It would be interesting to find an explicit closed formula for the trace anomalies in any $d$, similar to the result of [2] based on the “Dirac genus” for the chiral anomalies. Perhaps our eq. (4.22) could be a starting point.

In the case of the chiral anomaly, one needs in $d$ dimensions $d$ powers of the fermionic zero modes $\psi_0^\mu$ in order that the Berezin integration is non-vanishing. This introduces a factor $\beta^{\frac{d}{2}}$, which cancels the factor $\beta^{-\frac{d}{2}}$ in eq. (4.13). For the trace anomaly, no $\psi_0^\mu$ are present, and hence one must make a loop expansion on the world line.
In the transition from phase space to configuration space path integrals, a measure $\sqrt{g}$ is produced, familiar from non-linear sigma models. We have treated this measure in a way that is different from the usual $\delta(0)$ approach \cite{9}. Namely, rather than add extra vertices to the action, which are proportional to $\ln \det g$ times $\delta(0)$ and which are supposed to cancel leading divergences, we exponentiated this $\sqrt{g}$ to a perfectly regular term in the action with new ghost fields \cite{3}, similar to the familiar Faddeev-Popov ghost action. It was then clear, diagram by diagram, how all the divergences (not only the leading ones) cancel in all loops for finite $\beta$. An example of such a cancellation was given in (5.1). Of course, we need to exponentiate the factor $\sqrt{g}$ in order to be left with a translationally invariant measure from which propagators can be defined. Historically, these factors $\delta(0)$ first occurred in the proof of Matthews’s theorem \cite{18}, which states that the Hamiltonian and the Lagrangian canonical quantization are equivalent if one adds in the latter extra vertices proportional to $\delta(0)$, and in the analysis of Lee and Yang \cite{19} of massive vector bosons. We believe that our approach is simpler and clearer.

In the canonical formalism, Heisenberg fields satisfy the field equations, and fields must be expanded in a complete set of solutions. In the path integral, fields do not need to satisfy any field equations, and one is free to expand in any basis. We found it convenient to decompose the coordinates $q^\mu(t)$ into a background piece, which satisfies the field equation (a geodesic), and fluctuations. We expanded all fluctuations in the trigonometric basis $\sin(\pi m \tau)$ with $m = 1, 2, 3, \ldots$. This raises the question whether our results depend on the choice of basis. Since an ambiguity is known to exist if one uses time discretization, and is fixed by imposing gauge invariance, we expect our results to be basis-independent as long as they are Einstein- and gauge-invariant.

We end with a comment on Feynman diagrams, which provide the oldest method of computing anomalies and which have been used with success for chiral anomalies \cite{4}. The simplest diagram which one might consider to yield a trace anomaly is the two-point function in $d$ dimensions, with candidate trace anomaly $(\nabla^2)^{d-1} R$. In $d = 2$ it is a genuine anomaly, but for $2d > 2$ it can be removed by a counterterm with two factors $R$ (as observed by Adam Schwimmer (private communication) the most general two-point function which is both transversal and traceless is proportional to $\frac{1}{2}(d-1)(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \theta_{\mu\nu}\theta_{\rho\sigma}$, where $\theta_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu k^{-2}$, but in $d = 2$ this expression identically vanishes). The triangle diagram cannot reproduce all $h^2$ terms in $d = 4$, because the $h^2$ terms of the Euler invariant vanish in all dimensions. Thus, in $d = 4$, one would need to compute
the 4-point function, which is tedious to evaluate. Moreover, in higher $d$ there are more invariants, which constitute the trace anomaly, and hence further diagrams would have to be computed. (For the chiral anomaly, only one diagram needs to be calculated in a given dimension.) We conclude that the quantum mechanical approach is a much simpler way to compute the trace anomalies.

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Appendix A.

In the computation of loops on the world line, we have used the explicit form of the propagator function $\Delta(\tau, \tau')$. As a result, all loop integrals became trivial, typically powers of $\tau$ and $\tau'$ from $-1$ to $0$. However, if one is interested in truncated results one needs to use the representation of $\Delta(\tau, \tau')$ in terms of sine functions, and then the loop integrals for $d=4$ become (when the truncation is removed at the end of the calculation) proportional to the following double sums

$$
\sum_{n,m=1}^{\infty} \frac{1}{(m+n)^2 mn} = \frac{\pi^4}{180}; \quad \sum_{n,m=1}^{\infty} \frac{1}{(m+n)m^2 n} = \frac{\pi^4}{72}; \quad \sum_{n,m=1}^{\infty} \frac{1}{(m+n)^2 m^2} = \frac{\pi^4}{120}. \tag{A.1}
$$

The sums $\sum_{n=1}^{\infty} n^{-p}$ are given by the polylogarithms $\text{Li}_{2p-2}(1)$; for example $\text{Li}_2(1) = \frac{\pi^2}{6}$, $\text{Li}_4(1) = \frac{\pi^4}{90}$, $\text{Li}_6(1) = \frac{\pi^6}{945}$ and $\text{Li}_8(1) = \frac{\pi^8}{9450}$, see \[20\]. Since $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$, the factor $\pi^4$ in (A.1) is plausible, and one easily finds the coefficients $\frac{1}{180}$, $\frac{1}{72}$ and $\frac{1}{120}$ from a numerical approximation. However, since (to our surprise) the results for these double sums are not found in the standard references \[21\], we give an analytical derivation, which can also be generalized to more complicated sums.

First, consider $I(p, q, r) = \sum_{n,m=1}^{\infty} (m+n)^{-p} m^{-q} n^{-r}$. By decomposing the summands for fixed $n$ into partial sums with $m$-dependent singularities and $n$-dependent coefficients, one derives the following identities

$$
I(1, 2, 1) = I(1, 0, 3) + I(0, 2, 2) - I(0, 1, 3)
$$

$$
I(2, 1, 1) = -I(2, 0, 2) - I(1, 0, 3) + I(0, 1, 3)
$$

$$
I(3, 1, 0) = -I(3, 0, 1) - I(2, 0, 2) - I(1, 0, 3) + I(0, 1, 3)
$$

$$
I(1, 3, 0) = -I(1, 0, 3) + I(0, 3, 1) - I(0, 2, 2) + I(0, 1, 3).
$$

(From $I(2, 2, 0)$ one obtains the same result as from $I(1, 3, 0)$.) From these relations, one finds $I(1, 2, 1) + I(2, 1, 1) + I(2, 2, 0)$ and $2I(3, 1, 0) + I(2, 2, 0)$, which agree with (A.1).

To obtain also $I(2, 1, 1)$, A. Martin gave us the following derivation

$$
I(2, 1, 1) = \sum_{n,m=1}^{\infty} \frac{1}{(m+n)^2 mn} = \sum_{n,m=1}^{\infty} \int_0^1 dx \int_0^x dy y^m y^n
$$

$$
= \int_0^1 dx \int_0^x dy \left[ -\ln(1-y) \right]^2 = \int_0^1 dy \frac{\ln^2(1-y)}{y} \int_0^1 dx \frac{1}{x} = -\int_0^1 dx \frac{\ln^2(1-x) \ln x}{x}. \tag{A.3}
$$
Next we replace \( \ln(1-x) \) by \( \frac{(1-x)^{\epsilon}-1}{\epsilon} \), but for \( \ln x \) we use \( \frac{2^x-x^x}{e} \) to cancel the \( \frac{1}{x} \) singularity. One then obtains a series of beta functions \( B(x, y) = \int_0^1 dt \, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \), \( \Gamma \) being the gamma function:

\[
I(2, 1, 1) = -\frac{1}{\epsilon^4} \left[ B(2\epsilon, 1 + 2\epsilon) - B(\epsilon, 1 + 2\epsilon) - 2B(2\epsilon, 1 + \epsilon) + 2B(\epsilon, 1 + \epsilon) + \frac{1}{2\epsilon} - \frac{1}{\epsilon} \right]. \tag{A.4}
\]

Next we use \( \Gamma(2\epsilon) = \frac{\Gamma(1+2\epsilon)}{2\epsilon} \) and obtain

\[
I(2, 1, 1) = -\frac{1}{2\epsilon^4} \left[ \left( \frac{\Gamma^2(1 + 2\epsilon)}{\Gamma(1 + 4\epsilon)} - 1 \right) + 4 \left( \frac{\Gamma^2(1 + \epsilon)}{\Gamma(1 + 2\epsilon)} - \frac{\Gamma(1 + \epsilon)\Gamma(1 + 2\epsilon)}{\Gamma(1 + 3\epsilon)} \right) \right]. \tag{A.5}
\]

Writing \( \Gamma(1+z) \) as \( \exp(z\psi(1) + \frac{1}{2}z^2\psi'(1) + \ldots) \) where \( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) is the psi (digamma) function, given also by \( \psi'(z) = \sum_{n=0}^{\infty} (n + z)^{-2} \), we see that all \( \psi(1) \) and all even derivatives of \( \psi(z) \) at \( z = 1 \) cancel. The \( 2n+1 \) derivatives of \( \psi(z) \) at \( z = 1 \) are proportional to \( \zeta(2n) \), which are known: \( \psi'(1) = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6} \), \( \psi'''(1) = 6 \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{15} \). Substitution of these expansions into \( I(2, 1, 1) \) then reveals that all \( \epsilon \) poles cancel, and the finite result is \( I(2, 1, 1) = \frac{\pi^4}{180} \).

We thank A. Schellekens for discussions, and especially A. Martin for a derivation of \( I(2, 1, 1) \).

Appendix B.

In this appendix we comment on some problems that are one encountered in the operator approach to anomalies. We start from \( N \) operators \( \hat{A}_1, \ldots, \hat{A}_N \) depending on canonical variables \( \hat{a} = \frac{(\hat{a} + \hat{p})}{\sqrt{2}} \) and \( \hat{a}^\dagger = \frac{(\hat{a} - \hat{p})}{\sqrt{2}} \), satisfying \( [\hat{a}, \hat{a}^\dagger] = 1 \). If the Hamiltonian is

\[ H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2Q^2, \]

then \( q = (m\omega)^{\frac{3}{2}}Q \) and \( p = (m\omega)^{-\frac{3}{2}}P \), so that \( dPdQ = dpdq \) and \( H = (\hat{a}^\dagger \hat{a} + \frac{1}{2})\omega \). Using the so-called holomorphic representation \([22],[23]\), the trace of the product of the \( \hat{A}_i \) can be written as a multiple integral (where \( dwd\bar{w} = idpdq, \) etc.)

\[
\text{Tr}(\hat{A}_1, \ldots, \hat{A}_N) = \int \frac{dwd\bar{w}}{2\pi i} \prod_{j=1}^{N-1} \frac{d\bar{v}_jdv_j}{2\pi i} \exp \left( -w\bar{w} - \sum_{i=1}^{N-1} \bar{v}_iv_i \right) A_1(\bar{w}, v_1) \ldots A_N(\bar{v}_{N-1}, w). \tag{B.1}
\]

The kernels \( A(\bar{z}, z) \) are equal to \( \exp(\bar{z}z)K(\bar{z}, z) \), where \( K(\bar{z}, z) = \sum_{m,n \geq 0} \bar{z}^m K_{mn} z^n \) if \( \hat{A} = \sum_{m,n \geq 0} \hat{a}^\dagger m K_{mn} \hat{a}^n \). Note that this last expression has to be normal ordered with respect to \( \hat{a}^\dagger \) and \( \hat{a} \).

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Consider now $\exp(-\beta \hat{H}) = (\exp(-\epsilon \hat{H}))^N$, where $N \epsilon = \beta$. Let $\exp(-\epsilon \hat{H}(\hat{a}^\dagger, \hat{a})) = \exp(-\epsilon h(\hat{a}^\dagger, \hat{a}))$; where $\hat{h}$ will in general depend on $\epsilon$, then the trace can be written as

$$\text{Tr} e^{-\beta \hat{H}} = \int \frac{dw \bar{w}}{2\pi i} \prod_{j=1}^{N-1} \frac{d\bar{v}_j dv_j}{2\pi i} \exp(-L)$$

$$-L = -\bar{w}(w - v_1) - \bar{v}_1(v_1 - v_2) + \ldots - \bar{v}_{N-1}(v_{N-1} - w) - \epsilon[h(\bar{w}, v_1) + h(\bar{v}_1, v_2) + \ldots h(\bar{v}_{N-1}, w)].$$

By identifying $w = v_N$ and $\bar{w} = \bar{v}_0$, this result can be written in a suggestive form as a path integral with periodic boundary conditions (PBC)

$$\text{Tr} e^{-\beta \hat{H}} = \int_{PBC} \prod_{\tau} \frac{dv(\tau) d\bar{v}(\tau)}{2\pi i} \exp \left( \int_0^\beta d\tau \left( \bar{v} \frac{dv}{d\tau} - h(\bar{v}, v) \right) \right).$$

but the exact meaning (exact even for finite $N$) is given by (B.2).

Given the operator $\hat{H}$ (the regulators in our case), it is very difficult to find $h(\bar{z}, z)$ in closed form. One has to expand $\exp(-\epsilon \hat{H})$, commute all $\hat{a}^\dagger$ to the left and $\hat{a}$ to the right, replace $\hat{a}^\dagger$ by $z$ and $\hat{a}$ by $z$, and finally re-exponentiate. For the harmonic oscillator this has been done but not, to our knowledge, for non-linear sigma models. For the harmonic oscillator with $\hat{H} = (\hat{a}^\dagger \hat{a} + \frac{1}{2})\omega$ one obtains [24]

$$\exp(-\epsilon h(\bar{z}, z)) = \exp \left[ -\frac{1}{2} \epsilon \omega - \exp(1 - e^{-\epsilon \omega}) \bar{z} z \right].$$

To obtain the configuration space equivalent of (B.2), we must integrate out the $N$ momentum variables $p, p_1, \ldots, p_{N-1}$. In the continuum limit (B.3) we find from the term $\int_0^\beta d\tau \frac{dv}{d\tau}$, upon re-discretization, a sum $\sum_{j=0}^{n-1} \frac{1}{2}(q_j - ip_j)(\dot{q}_j + ip_j)\epsilon$. Owing to PBC, this can be rewritten as a sum of $\frac{1}{4} \frac{d}{d\tau} (q_j^2 + p_j^2) - ip_j \dot{q}_j$, and if $h(p, q)$ is quadratic in $p$ [so $h(p, q) = \frac{1}{2} g^{ij}(q)p_ip_j + V(q)$] one obtains the Euclidean action $L = \int_0^\beta d\tau \left( \frac{1}{2} g^{ij}(q) \dot{q}_i \dot{q}_j + V \right)$ together with the measure $\sqrt{g}$ discussed in sect. 1.

However, if one starts from the exact result in (B.2), the computation becomes considerably more complicated. Consider first the harmonic oscillator. One obtains then in the exponent an expression of the form $-\frac{1}{2} \sum (q_j - ip_j)M_{jk}(q_k + ip_k)\epsilon - \frac{1}{2} N \epsilon \omega$, where the non-symmetric matrix $M$ has entries +1 along the diagonal, and $-e^{-\epsilon \omega}$ just above the diagonal and in the lower-left position. The integration over the $N$ variables $p_j$ yields

$$(2\pi)^{\frac{N}{2}} (\det M_S)^{-\frac{1}{2}} \int_{PBC} \prod_{j=0}^{N-1} \frac{dq_j}{2\pi} \exp \left( -\frac{1}{2} q (M_S - M_A M_S^{-1} M_A) q \right)$$

(B.5)
and further integration over the $q$ variables yields the expected result $\det M = 1 - \exp(-\epsilon \omega N)$. Together with the factor $(\exp(\frac{1}{2} \epsilon \omega))^N$, one finds $(2\sinh \frac{\beta \omega}{2})^{-1}$, which is indeed the partition function for the harmonic oscillator. To see $\det M$ coming out, note that

$$\det M_S \det (M_S - M_A M^{-1}_S M_A) = \det M_S (M_S - M_A M^{-1}_S (M_S + M_A),$$

(B.6)

where $M_S = \frac{1}{2} (M + M^T)$ and $M_A = \frac{1}{2} (M - M^T)$. For a system more general than the harmonic oscillator, one would have to compute $\det M_S$ in order to obtain the exact corresponding configuration space path integral, and this is in general difficult. (Even for the harmonic oscillator, we have not found $\det M_S$. In [10] the determinant is given of a symmetric matrix, which is equal to our $M_S$ except that the lower left entry and upper right entry are zero.)

For the chiral anomaly, one must evaluate $\text{Tr} \exp(i\theta \hat{F}) \exp(-\beta \hat{H})$, where $\hat{F} = \hat{c}^\dagger \hat{c}$ is the fermion number operator. The integral kernel for $\hat{A}_1 = \exp(i\theta \hat{F})$ can be found in closed form, because the variables $c$ and $\bar{c}$ are anticommuting: $\hat{A}_1 (\bar{c}, c) = \exp(e^{i\theta} \bar{c} c)$. The path integral corresponding to (B.3) is obtained by replacing the first factor $\exp(-\epsilon \hat{H})$ by $\exp(i\theta \hat{F})$, and one finds

$$\text{Tr} e^{i\theta \hat{F}} e^{-\beta \hat{H}} = \int \frac{d\xi d\bar{\xi}}{2\pi i} \prod_{j=1}^{N-1} \frac{d\xi_j d\bar{\xi}_j}{2\pi i} \exp(-L)$$

$$-L = \xi_{N-1} (\xi - \xi_{N-1}) + \ldots + \xi_1 (\xi_2 - \xi_1) + \xi e^{i\theta} (\xi_1 + e^{-i\theta} \xi)$$

$$- \epsilon [h(\xi_1, \xi_2) + \ldots h(\xi_{N-1}, \xi)].$$

(B.7)

We have taken care to define (B.1) such that it also holds for anticommuting variables, as in (B.7). By identifying $\xi_N = \xi$ and $\xi_0 = -e^{-i\theta} \xi$ we obtain a path integral as in (B.3), but now with antiperiodic boundary conditions ABC for $\theta = 0$, and PBC for $\theta = \pi$. Hence, for the trace anomaly we would need fermions with ABC, see sect. 1.

Finally, we consider the ghosts $\hat{c}^\dagger$ and $\hat{c}$ for internal symmetry, discussed in sect. 1. We have found the projection operator $P_1$ onto the one-particle states $\hat{c}^\dagger |0\rangle$ in closed form. It is given by

$$P_1 =: \exp(\hat{c}^\dagger \hat{c}) : -1.$$ 

(B.8)

Written out, $P_1 = \hat{c}^\dagger \hat{c} + \frac{1}{2} \hat{c}^\dagger \hat{c}^\dagger \hat{c} \hat{c} + \ldots$, and it is clear that $P_1 |0\rangle = 0$ and $P_1 \hat{c}^\dagger |0\rangle = \hat{c}^\dagger |0\rangle$. One can check that $P_1$ annihilates the states with $2, \ldots, M$ particles (where $j = 1, \ldots, M$). It follows that the analysis is very similar to that of the chiral anomaly:
one must only replace the integral kernel \( \exp(e^{i\theta \bar{c}c}) \) by \( \exp(\bar{c}c) - 1 \). However, we have not been able to interpret this result in terms of suitable boundary conditions.

Our conclusion is that the operator approach, which starts from the quantum Hamiltonian, is much more complicated than the Feynman approach, which produces the Hamiltonian at the end. For this reason we have followed in the main text the Feynman approach (but we have not made the error of omitting quantum fluctuations from the path integral \([25]\)). The approach of the authors in \([2]\) was based on an operator approach, but they made various approximations: they assumed that \( h = H \), and used \((B.3)\) instead of \((B.2)\), to obtain the Euclidean action. For the chiral anomalies, their approximate path integral yields the correct result because chiral anomalies depend on very few details of the path integral (they are topological). For the trace anomaly these approximations would be incorrect, as we discussed in the introduction.
References

[1] K. Fujikawa, Phys. Rev. Lett. 44 (1980) 1733; Phys. Rev. D21 (1980) 2848 and D23 (1981) 2262.
[2] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1984) 269.
[3] F. Bastianelli, Nucl. Phys. B376 (1992) 113.
[4] M. Duff, Nucl. Phys. B125 (1977) 334.
[5] A. Diaz, W. Troost, A. Van Proeyen and P. van Nieuwenhuizen, Int. J. Mod. Phys. A4 (1989) 3959;
    M. Hatsuda, W. Troost, A. Van Proeyen and P. van Nieuwenhuizen, Nucl. Phys. B335 (1990) 166;
    F. Bastianelli, A. Van Proeyen and P. van Nieuwenhuizen, Phys. Lett. B253 (1991) 67.
[6] A. Ceresole, P. Pizzocchero and P. van Nieuwenhuizen, Phys. Rev. D39 (1989) 1567.
[7] B.S. DeWitt, in “Relativity, Group and Topology”, eds. B.S. DeWitt and C. DeWitt (Gordon Breach, New York, 1964); “Relativity, Group and Topology II” eds. B.S. DeWitt and R. Stora (North Holland, Amsterdam, 1984).
[8] R.P. Feynman, Rev. Mod. Phys. 20 (1948) 367.
[9] E.S. Abers and B.W. Lee, Phys. Rep. 9 (1973) 1.
[10] L.S. Schulman, “Techniques and applications of path integration”, (John Wiley and Sons, New York 1981).
[11] W.A. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421.
[12] N.K. Nielsen and P. van Nieuwenhuizen, Phys. Rev. D38 (1988) 318.
[13] L. Bonora, P. Cotta-Ramusino and C. Reina, Phys. Lett. B126 (1983) 305.
[14] K. Fujikawa, S. Ojima and S. Yajima, Phys. Rev. D34 (1986) 3223.
[15] G.A. Vilkovisky, preprint CERN-TH.6392/92.
[16] P.B. Gilkey, J. Diff. Geom. 10 (1975) 601;
    For the case $d = 4$ see Proc. of Symposia in Pure Math. 27 (1975) 265 (published by Amer. Math. Soc.).
[17] I.G. Avramidi, Theor. Math. Phys. 79 (1989) 219.
[18] P.T. Matthews, Phys. Rev. 76 (1949) 684;
    T. Duncan and C. Bernard, Phys. Rev. D11 (1975) 848.
[19] T.D. Lee and C.N. Yang, Phys. Rev. 128 (1962) 885.
[20] L. Levin, “Polylogarithms and associated functions” (Elsevier, North-Holland, 1981).
[21] M. Abramowitz and I.A. Stegun, “Handbook of mathematical functions”, 1968.
[22] L. Faddeev, in “Methods in Field Theory”, eds. R. Balian and J. Zinn-Justin (North Holland, Amsterdam, 1976);
    C. Itzykson and J.B. Zuber, “Quantum Field Theory” (Mc Graw-Hill, New York, 1980).
[23] L. Alvarez-Gaumé, in “Supersymmetry”, Nato-ISA series vol.N125, eds. K. Dietz, R. Flume, G.C. von Gehlen and V. Rittenberg (Plenum Press, New York, 1984).

[24] W.H. Louisell, “Radiation and Noise in Quantum Electronics” (Mc Graw-Hill, New York, 1964);
J. Katriel, J. Phys. A16 (1983) 4171.

[25] K.S. Cheng, J. Math. Phys. 13 (1972) 1723.