PROOF OF CHERN’S CONJECTURE ON AFFINE MANIFOLDS

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Abstract. We prove an old conjecture of S. S. Chern that the Euler characteristic of a closed affine manifold equals to zero.

1. Introduction

An affine manifold $M$ is a smooth manifold with an atlas of charts whose coordinate transformations are affine maps. It is equivalent to that there exists a flat torsion free connection on the tangent bundle $TM$. If in addition the connection preserves a Riemannian metric on $M$ and $M$ is closed, then by the celebrated Gauss-Bonnet-Chern theorem we would get immediately the Euler characteristic of $M$ vanishes: $\chi(M) = 0$. Without this compatibility between the connection and metric, the same conclusion was conjectured by S. S. Chern around 1955 (cf. [3, 8, 10]), namely, a closed affine manifold must have zero Euler characteristic.

In dimension 2, Benzécri [1] proved the Chern conjecture affirmatively and Milnor [12] improved this result by dropping the torsion free condition. However, in higher dimensions only flatness could not ensure the vanishment of the Euler characteristic in general, due to counterexamples by Smillie [13]. Recently such generalized Chern conjecture was verified for manifolds that are locally a product of surfaces by Bucher-Gelander [2]. Hence, the torsion free condition is important essentially in higher dimensions.

Since the very beginning plenty of studies and progresses on the Chern conjecture were carried out. Several special cases were proven positively, e.g., for complete affine manifolds by Kostant-Sullivan [11] and very recently for special affine manifolds by Klingler [10]. Recall that complete affine manifolds are quotients of $\mathbb{R}^n$ by discrete subgroups of affine transformations, and special affine manifolds are those affine manifolds admitting a parallel volume form. These two special cases of affine manifolds were conjectured to be equivalent by Markus, which is still widely open so far. We refer to [10, 9] for more details and more verified special cases.

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In this paper, we prove the Chern conjecture in general.

**Theorem 1.1.** The Euler characteristic of a closed affine manifold equals to zero.

Without loss of generality, we need only consider the conjecture on oriented closed affine manifolds of even dimension since $\chi(\widetilde{M}) = k\chi(M)$ for a $k$-covering $\widetilde{M}$ of $M$ and $\chi(M^n) = 0$ if $n$ is odd. The idea originates from a result of Chern himself for which, together with a result of Lashof-Smale, we gave a unified proof using an integral formula (cf. [4, 7, 6]). Chern’s result states that the mapping degree of the normal map (generalized Gauss map)

$$f_N : N_1M \to S^{N-1}$$

from the unit normal bundle $N_1M$ of an oriented closed submanifold $f : M^n \to \mathbb{R}^N$ to the unit sphere $S^{N-1}$ equals to the Euler characteristic of $M$: $\deg f_N = \chi(M)$. When $M$ is affine, we can embed its charts as “flat” pieces into a suitable $\mathbb{R}^N$. By the affine coordinate transformations in the overlapping charts, we can compute the induced measure by $f_N$ in a clear manner. Then by moving one piece to the infinity by deforming $f$ into a one-parameter family of such embeddings $f^t$, we can show the induced measure by $f^t_N$ on this piece tends to zero with converging induced measure on all other pieces. By subsequently doing this deformation on each piece if necessary, we are able to obtain a zero limit of the normal degree, which implies $\chi(M) = 0$ by Chern’s result mentioned above.

2. Proof of the Chern conjecture

Let $M^n$ be an oriented closed affine manifold of even dimension $n \geq 2$ and let

$$\{(U_\alpha, x^i_\alpha) = \varphi_\alpha : U_\alpha \to \mathbb{R}^n) \mid \alpha = 1, \cdots, m\}$$

be its affine structure: for any $\alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$,

$$x_\alpha = A_{\alpha\beta}x_\beta + b_{\alpha\beta},$$

where $x_\alpha = (x^1_\alpha, \cdots, x^n_\alpha)^T \in \mathbb{R}^n$ is the coordinate column vector on $U_\alpha$, $(\cdot)^T$ denotes the matrix transpose, and $A_{\alpha\beta} \in GL(n)$, $b_{\alpha\beta} \in \mathbb{R}^n$ are constant invertible $n \times n$ matrix and column vector respectively. Without loss of generality, we can assume $\det A_{\alpha\beta} > 0$, the image of each coordinate chart is the ball of radius 3 centered at the origin: $\varphi_\alpha(U_\alpha) = B(3) \subset \mathbb{R}^n$, and $\{W_\alpha := \varphi^{-1}_\alpha(B(1))\}$ is still an open cover of $M^n$, namely, $M = \bigcup_\alpha W_\alpha$. Set $V_\alpha := \varphi^{-1}_\alpha(B(2))$. Then $W_\alpha \subset V_\alpha \subset U_\alpha \subset M^n$ and

$$V_\alpha \backslash W_\alpha = \varphi^{-1}_\alpha(B(2) \backslash B(1)).$$
We define a smooth cut-off function \( h(x) \) on \( \mathbb{R}^n \) such that \( h|_{\overline{B(1)}} = 1, h|_{\mathbb{R}^n \setminus B(2)} = 0 \) and \( 0 < h|_{B(2) \setminus B(1)} < 1 \), by setting
\[
(2.2) \quad h(x) := \Psi(|x|^2), \quad x \in \mathbb{R}^n,
\]
where
\[
\Psi(t) := \frac{\int_{-\infty}^{t} \psi(s)ds}{\int_{-\infty}^{+\infty} \psi(s)ds}, \quad \psi(t) := \begin{cases} e^{\frac{1}{t-1}(4-t)}, & 1 < t < 4, \\ 0, & \text{otherwise}. \end{cases}
\]

Lemma 2.1. The hessian \( D^2h \) and the differential \( dh \) of \( h \) in (2.2) vanish unless in the radial direction at points of the annulus, namely
\[
D^2h(x) = \left( \frac{\partial^2 h}{\partial r^2} \right) \frac{xx^T}{|x|^2}, \quad \nabla h(x) = \left( \frac{\partial h}{\partial r} \right) \frac{x}{|x|}, \quad \text{for} \ x \in B(2) \setminus \overline{B(1)},
\]
where \( r = |x| = \sqrt{\sum_i (x_i)^2} \) is the radial distance and \( xx^T \) is regarded as the \( n \times n \) matrix by the product of the \( n \times 1 \) matrix \( x \) and the \( 1 \times n \) matrix \( x^T \).

Proof. The proof is straightforward and thus left to the readers. \( \square \)

Now we define the embedding \( f : M^n \rightarrow \mathbb{R}^N \) \((N = (n+1)m)\) in the standard way as in the proof of the Whitney embedding theorem by:
\[
(2.3) \quad f(x) := \left( h(\varphi_1(x))\varphi_1(x), \ldots, h(\varphi_m(x))\varphi_m(x), h(\varphi_1(x)), \ldots, h(\varphi_m(x)) \right),
\]
where the functions \( h(\varphi(x)) \) are regarded as global functions on \( M \) by extending to zero for \( x \in M \setminus U_\alpha \). It is easy to check that \( f \) is an embedding and we denote its normal map by \( f_N \) defined as in (1.1). The key is to prove

Lemma 2.2. The mapping degree of the normal map \( \deg f_N = 0 \).

Once this be done, we have by Chern’s result mentioned before
\[
\chi(M) = \deg f_N = 0,
\]
which completes the proof of the Chern conjecture (Theorem 1.1). However, due to the absence of some symmetry, we can only verify this by transiting to a family of deformed embeddings \( f^{(l_1, \ldots, l_m)} \). For these deformed embeddings we can show that the mapping degrees of the normal maps converge to zero.

Proof. Recall the mapping degree can be given by
\[
\deg f_N = \frac{\int_{N_1M}(f_N)^*\Omega}{\int_{S^{N-1}} \Omega},
\]
where \( \Omega(y) = i(y)(dy^1 \wedge \cdots \wedge dy^N) = \sum_{k=1}^{N}(-1)^{k-1}y^k dy^1 \wedge \cdots \wedge \hat{dy}^k \wedge \cdots \wedge dy^N \) is the standard volume form on \( y = (y^1, \ldots, y^N)^T \in S^{N-1} \subset \mathbb{R}^N \).
Denote the lift of $W_\alpha$ to the unit normal bundle through the projection $\pi : N_1 M \to M$ by
\[ \tilde{W}_\alpha := \pi^{-1}(W_\alpha) = N_1 M|_{W_\alpha} \cong W_\alpha \times S^{N-n-1}. \]
Then $\{\tilde{W}_\alpha \mid \alpha = 1, \cdots, m\}$ is an open cover of $N_1 M$. Let $\{\psi_\alpha\}$ be a partition of unit subordinate to the open cover $\{W_\alpha\}$ of $M$ and thus $\{\tilde{\psi}_\alpha := \psi_\alpha \circ \pi\}$ is a partition of unit subordinate to the open cover $\{\tilde{W}_\alpha\}$ of $N_1 M$. Therefore, we have
\[ \int_{N_1 M} (f_N)^* \Omega = \sum_{\alpha=1}^m \int_{N_1 M} \tilde{\psi}_\alpha (f_N)^* \Omega = \sum_{\alpha=1}^m \int_{\tilde{W}_\alpha} \tilde{\psi}_\alpha (f_N)^* \Omega. \]
With respect to the embedding $f$, for each $\alpha$ we denote:
\[ D_\alpha(f) := \int_{\tilde{W}_\alpha} \tilde{\psi}_\alpha (f_N)^* \Omega. \]

We regard the unit normal bundle $N_1 M$ as a $(N - 1)$-dimensional submanifold of the product $M^n \times S^{N-1}$ with the second factor orthogonal to the tangent space $df(T_x M)$ at the first factor point $x \in M$. Hence, we have
\[ \tilde{W}_\alpha = \{(x, \nu(x)) \in W_\alpha \times S^{N-1} \mid \langle df(x), \nu(x) \rangle = 0\}. \]
Noticing the affine coordinate transformation (2.1), we calculate from (2.3) that
\[ df(x) = df \circ \varphi^{-1}_\alpha(x_\alpha) = (dh(x_1) \circ A_{1\alpha} \circ dx_\alpha)x_1 + h(x_1)A_{1\alpha} \circ dx_\alpha, \]
(2.5) \[ \cdots, dx_\alpha, \cdots, \]
\[ (dh(x_m) \circ A_{m\alpha} \circ dx_\alpha)x_m + h(x_m)A_{m\alpha} \circ dx_\alpha, \]
\[ dh(x_1) \circ A_{1\alpha} \circ dx_\alpha, \cdots, 0, \cdots, dh(x_m) \circ A_{m\alpha} \circ dx_\alpha, \]
where $x_\beta = A_{\beta\alpha}x_\alpha + b_{\beta\alpha} = \varphi_\beta \circ \varphi^{-1}_\alpha(x_\alpha) = \varphi_\beta(x) \in B(3)$ for $\beta \neq \alpha$, $x_\alpha = \varphi_\alpha(x) \in B(1)$.

**Remark 2.3.** In the components of (2.5), $h(x_\beta) \neq 0$ only if $x \in W_\alpha \cap V_\beta$, and $dh(x_\beta) \neq 0$ only if $x \in W_\alpha \cap (V_\beta \setminus \tilde{W}_\beta)$, for $\beta = 1, \cdots, \tilde{\alpha}, \cdots, m$.

Corresponding to the representation of $df(x)$ in (2.5), we decompose $\nu(x)$ as
\[ \nu(x) = (\nu^1, \cdots, \nu^N) = (\nu_1(x), \cdots, \nu_m(x), \mu(x)), \]
where $\nu_\alpha(x) = (\nu^1_\alpha, \cdots, \nu^m_\alpha)^T \in \mathbb{R}^n$ and $\mu(x) = (\mu^1, \cdots, \mu^m) \in \mathbb{R}^m$. A straightforward calculation from (2.4) and (2.5) shows that
\[ -\nu_\alpha(x) = \sum_{\beta \neq \alpha} \left( (\mu^\beta + \langle x_\beta, \nu_\beta \rangle)A^T_{\beta\alpha} \nabla h(x_\beta) + h(x_\beta)A^T_{\beta\alpha} \nu_\beta \right). \]
Hence the unit normal sphere $\pi^{-1}(x)$ for each $x \in W_\alpha$ is the round sphere obtained by the intersection of $\mathbb{S}^{N-1}$ with the $(N - n)$-plane which is determined by the linear equation (2.6) and thus can be parameterized only by $(\nu_1, \cdots, \tilde{\nu}_\alpha, \cdots, \nu_m, \mu) =: \tilde{\nu}_\alpha$. 


This gives in fact a local trivialization of the (unit) normal bundle $NM(N_1M)$ over $W_\alpha$'s by $W_\alpha \times \mathbb{R}^{N-n}$ ($S^{N-n-1}$) with coordinates $(x_\alpha, \nu_\alpha)$. Let $L_x : \mathbb{R}^{N-n} \to \mathbb{R}^n$ be the linear map defined by the right hand of the linear equation (2.6) such that $-\nu_\alpha(x) = L_x(\nu_\alpha)$. Then the unit normal sphere $\pi^{-1}(x)$ can be parameterized by (2.6) as the following ellipsoid:

$$\Sigma_x := \left\{ \nu_\alpha \in \mathbb{R}^{N-n} \mid |L_x(\nu_\alpha)|^2 + |\nu_\alpha|^2 = 1 \right\}.$$

Observing that

$$(f_N)^*\Omega = i(\nu)(d\nu^1 \wedge \cdots \wedge d\nu^N) = \sum_{k=1}^N (-1)^{k-1} \nu^k d\nu^1 \wedge \cdots \wedge d\nu^k \wedge \cdots \wedge d\nu^N,$$

we need only compute the derivative of $\nu_\alpha$ module that of $\mu$ and $\nu_\beta$ for $\beta \neq \alpha$:

$$-d\nu_\alpha = \sum_{\beta \neq \alpha} A^T_{\beta\alpha} \left( \left( \mu_\beta + \langle x_\beta, \nu_\beta \rangle \right) \partial^2 h(x_\beta) + \nabla h(x_\beta) \nu_\beta + \nu_\beta \nabla h^T(x_\beta) \right) A_{\beta\alpha} \circ dx_\alpha,$$

$$(2.8) \quad =: \sum_{\beta \neq \alpha} B_{\beta\alpha} \circ dx_\alpha, \quad \text{mod} \{ d\mu^1, d\nu^i_\beta | \beta \neq \alpha, 1 \leq \gamma \leq m, 1 \leq i \leq n \}.$$

Then it is not hard to prove from (2.7) [2.8] that on $\tilde{W}_\alpha$:

$$(f_N)^*\Omega = \det \left( \sum_{\beta \neq \alpha} B_{\beta\alpha} \left( \sum_{i=1}^{N-n} (-1)^{i-1} \nu^i_\alpha d\nu^1_\alpha \wedge \cdots \wedge d\nu^i_\alpha \wedge \cdots \wedge d\nu^{N-n}_\alpha \right) \right) \wedge dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha.$$

As $\tilde{\psi}_\alpha = \psi_\alpha \circ \pi$ is constant on $\pi^{-1}(x)$, by our choice of orientation of the coordinates and the formula above, we have

$$D_\alpha(f) = \int_{\tilde{W}_\alpha} \tilde{\psi}_\alpha(f_N)^*\Omega$$

$$= \int_{W_\alpha} \psi_\alpha \left( \int_{\pi^{-1}(x)} \Phi \right) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \int_{B(1)} \psi_\alpha \circ \varphi_\alpha^{-1} \left( \int_{\Sigma_x} \Phi \right) dV_{B(1)}.$$

Here we have identified the $(N-n-1)$-form $\Phi$ under the parametrization $\Sigma_x \to \pi^{-1}(x)$,

$$\Phi := \det \left( \sum_{\beta \neq \alpha} B_{\beta\alpha} \right) \vartheta_\alpha,$$

and $\vartheta_\alpha := \left( \sum_{i=1}^{N-n} (-1)^{i-1} \nu^i_\alpha d\nu^1_\alpha \wedge \cdots \wedge d\nu^i_\alpha \wedge \cdots \wedge d\nu^{N-n}_\alpha \right)$ is a positive density form on $\Sigma_x$ under the natural orientation.

Now we analyze the integral function:

$$G_x(\nu_\alpha) := \det \left( \sum_{\beta \neq \alpha} B_{\beta\alpha} \right).$$

It is clear that $G_x(\nu_\alpha)$ is a homogeneous polynomial of degree $n$ about $\nu_\alpha$. To simplify the computation, we rewrite the matrices $B_{\beta\alpha}$ of (2.8) with the help of Lemma [2.1]. Set

$$\lambda^\beta := \left( \mu_\beta + \langle x_\beta, \nu_\beta \rangle \right) \frac{\partial^2 h(x_\beta)}{\partial r^2} / |x_\beta|^2, \quad c^\beta := \frac{\partial h(x_\beta)}{\partial r} / |x_\beta|,$$
which are nonzero only if \( x \in W_\alpha \cap (V_\beta \setminus W_\beta) \) as mentioned in Remark 2.3. Then each \( \lambda^\beta \) is linear about \( \nu^\beta \) and \( \mu^\beta \), and each \( c^\beta \) is constant for fixed \( x \). It follows that

\[
B_{\beta \alpha} = A_{\beta \alpha}^T \left( \lambda^\beta x^\beta x^T + c^\beta x^\beta \nu^T + c^\beta \nu x^T \right) A_{\beta \alpha}.
\]

Let \( 0 \leq p \leq m - 1 \) be the number of the annuluses, say \( V_{\beta_k} \setminus W_{\beta_k} \) (\( k = 1, \ldots, p \)), such that \( x \) lies in their common intersection with \( W_\alpha \). When \( p = 0 \), all \( B_{\beta \alpha} \)'s are zero and thus \( G_x(\tilde{\nu}_\alpha) = 0 \). When \( 1 \leq p < \frac{n}{2} \), we can show also \( G_x(\tilde{\nu}_\alpha) = 0 \) by analyzing the matrix rank of the following clearer representation:

(2.9) \[
G_x(\tilde{\nu}_\alpha) = \det \left( A_\alpha C_\alpha A^T_\alpha \right),
\]

where

\[
C_\alpha := \begin{pmatrix}
\lambda^{\beta_1} & c^{\beta_1} & \cdots & c^{\beta_p} \\
& \lambda^{\beta_p} & \cdots & c^{\beta_p} \\
& c^{\beta_1} & \cdots & c^{\beta_p}
\end{pmatrix} \in M(2p)
\]

with zeros in the blank, and

\[
A_\alpha := \left( A_{\beta_1 \alpha}^T x_{\beta_1}, \ldots, A_{\beta_p \alpha}^T x_{\beta_p}, A_{\beta_1 \alpha}^T \nu_{\beta_1}, \ldots, A_{\beta_p \alpha}^T \nu_{\beta_p} \right)
\]

is a \((n \times 2p)\) matrix. When \( p \geq \frac{n}{2} \), it follows from (2.9) that

\[
G_x(\tilde{\nu}_\alpha) = \sum_{\substack{i_1 < \cdots < i_n \\
j_1 < \cdots < j_n}} A_\alpha \left( \begin{array}{cccc}
1 & \cdots & n \\
i_1 & \cdots & i_n
\end{array} \right) C_\alpha \left( \begin{array}{cccc}
1 & \cdots & n \\
j_1 & \cdots & j_n
\end{array} \right) A_\alpha \left( \begin{array}{cccc}
1 & \cdots & n \\
i_1 & \cdots & i_n
\end{array} \right),
\]

where \( C \left( \begin{array}{cccc}
i_1 & \cdots & i_n \\
j_1 & \cdots & j_n
\end{array} \right) \) denotes the determinant of the sub-matrix of \( C \) with rows \( i_1, \ldots, i_n \) and columns \( j_1, \ldots, j_n \). Observe that each \( \lambda^\beta \) of \( \{\lambda^{\beta_1}, \ldots, \lambda^{\beta_p}\} \) appears at most linearly in the components of \( C_\alpha \left( \begin{array}{cccc}
i_1 & \cdots & i_n \\
j_1 & \cdots & j_n
\end{array} \right) \) and correspondingly no \( \nu^\beta \) or \( \mu^\beta \) appears in the factor \( A_\alpha \left( \begin{array}{cccc}
i_1 & \cdots & i_n \\
j_1 & \cdots & j_n
\end{array} \right) \) in the meantime. Therefore \( G_x(\tilde{\nu}_\alpha) \) is a polynomial of degree at most 2 about each \( \nu^\beta \) and \( \mu^\beta \).

Now we are ready to deform the embedding \( f \). We define a new one-parameter family of embeddings \( f^t \) of \( M \) into \( \mathbb{R}^N \) by deforming \( f \) of (2.3) in the domain \( V_\alpha \) with
a factor \( t > 0 \), explicitly,

\[
(2.10) \quad f^t(x) := \left( h(\varphi_1(x))\varphi_1(x), \ldots, t \left( h(\varphi_\alpha(x))\varphi_\alpha(x) \right), \ldots, h(\varphi_m(x))\varphi_m(x) \right).
\]

As the calculation for the embedding \( f \), we proceed the same calculations for the embeddings \( f^t \) with respect to a superscript \( t \). The key point is still Chern’s result:

\[
\chi(M) = \deg f^t_N, \quad \text{for any } t > 0.
\]

Notice that for \( x \in M \setminus V_\alpha \), \( f^t(x) = f(x) \) and \( \int_{\Sigma_x^t} \Phi^t = f^t \Phi \). So for the domains \( W_\beta \) with no intersection with \( V_\alpha \), we have

\[
D_\beta(f^t) = \int_{B(1)} \psi_\beta \circ \varphi_\beta^{-1} \left( \int_{\Sigma_x^t} \Phi^t \right) dV_{B(1)} = D_\beta(f).
\]

Then we need only consider on the points \( x \in V_\alpha = W_\alpha \cup (V_\alpha \setminus \overline{W_\alpha}) \).

For \( x \in W_\alpha \), the equations \((2.6, 2.7)\) of the normal space and the normal sphere at \( f^t(x) \) are replaced by

\[
-\nu_\alpha(x) = \frac{1}{t} \sum_{\beta \neq \alpha} \left( \left( \mu^\beta + \langle x_\beta, \nu_\beta \rangle \right) A^T_{\beta\alpha} \nabla h(x_\beta) + h(x_\beta) A^T_{\beta\alpha} \nu_\beta \right),
\]

\[
\Sigma_x^t := \left\{ \hat{\nu}_\alpha \in \mathbb{R}^{N-n} \mid \frac{|L_x(\hat{\nu}_\alpha)|^2}{t^2} + |\hat{\nu}_\alpha|^2 = 1 \right\}.
\]

Then \( G_x^t(\hat{\nu}_\alpha) = \frac{1}{\nu_\alpha} G_x(\hat{\nu}_\alpha) \) and thus

\[
\lim_{t \to +\infty} \int_{\Sigma_x^t} \Phi^t = 0,
\]

\[
\lim_{t \to +\infty} D_\alpha(f^t) = \lim_{t \to +\infty} \int_{B(1)} \psi_\alpha \circ \varphi_\alpha^{-1} \left( \int_{\Sigma_x^t} \Phi^t \right) dV_{B(1)} = 0.
\]

For \( x \in V_\alpha \setminus \overline{W_\alpha} \), let \( \alpha' \) be any subscript such that \( x \in W_{\alpha'} \). Then the equations \((2.6, 2.7)\) of the normal space and the normal sphere at \( f^t(x) \) are replaced by

\[
-\nu_{\alpha'}(x) = t \left( \left( \mu^\alpha + \langle x_\alpha, \nu_\alpha \rangle \right) A^T_{\alpha\alpha'} \nabla h(x_\alpha) + h(x_\alpha) A^T_{\alpha\alpha'} \nu_\alpha \right) + \\
\sum_{\beta \neq \alpha, \alpha'} \left( \left( \mu^\beta + \langle x_\beta, \nu_\beta \rangle \right) A^T_{\beta\alpha'} \nabla h(x_\beta) + h(x_\beta) A^T_{\beta\alpha'} \nu_\beta \right),
\]

\[
\Sigma_x^t := \left\{ \hat{\nu}_{\alpha'} \in \mathbb{R}^{N-n} \mid |L_x(\hat{\nu}_{\alpha'})|^2 + |\hat{\nu}_{\alpha'}|^2 = 1 \right\}.
\]

Notice that \( h(x_\alpha) > 0, \epsilon^\alpha < 0, A^T_{\alpha\alpha'} \in \text{GL}(n) \) and \( L_x^t \) acts on the \( \alpha \)-component as:

\[
L_x^t \left( \begin{array}{c} \nu_\alpha \\ \mu_\alpha \end{array} \right) = t \left( A^T_{\alpha\alpha'} (h(x_\alpha) I_n + \epsilon^\alpha x_\alpha^T x_\alpha^T), \epsilon^\alpha A^T_{\alpha\alpha'} x_\alpha \right) \left( \begin{array}{c} \nu_\alpha \\ \mu_\alpha \end{array} \right).
\]

Thus \( L_x^t \) has full rank \( n \) with singular values in \( O(t) \) as \( t \) tends to the infinity. Then it is not hard to show that the measure (or density) of \( \Sigma_x^t \) is in \( O(\frac{1}{t}) \) as \( t \) tends to the
infinity. Meanwhile, $G_x^t(\tilde{\nu}_\alpha)$ is now in $O(t^2)$ since it is a polynomial of degree at most 2 about $t$ as we analyzed for $G_x(\tilde{\nu}_\alpha)$ before. Therefore, we obtain

$$\lim_{t \to +\infty} \int_{\Sigma_2^t} \Phi^t = \lim_{t \to +\infty} O(t^{2-n}) = \begin{cases} 0, & \text{for } n \geq 4, \\ \text{finite}, & \text{for } n = 2. \end{cases}$$

Together with the former two cases when $x \in W_{\alpha'} \setminus W_\alpha$ and $x \in W_{\alpha'} \cap W_\alpha$, we get

$$\lim_{t \to +\infty} D_{\alpha'}(f^t) = \text{finite}.$$ 

Then we can subsequently deform the embeddings $f^t$ by a factor $t'$ in the domain $V_{\alpha'}$ into embeddings $f^{(t,t')}$ as (2.11), even deform in each domain into embeddings $f^{(t_1,\ldots,t_m)}$ (if necessary). By taking limit as $t, t'$ tend to the infinity and changing the order of the limits (it is applicable because the limits exist) with respect to $t$ and $t'$ if necessary, we still get

$$\lim_{t,t' \to +\infty} D_{\alpha'}(f^{(t,t')}) = 0 \quad \text{and} \quad \lim_{t,t' \to +\infty} D_\alpha(f^{(t,t')}) = 0.$$ 

In conclusion, we have shown for all $\alpha = 1, \ldots, m$,

$$\lim_{t_1,\ldots,t_m \to +\infty} D_\alpha(f^{(t_1,\ldots,t_m)}) = 0.$$ 

This implies

$$\chi(M) = \lim_{t_1,\ldots,t_m \to +\infty} \deg(f^{(t_1,\ldots,t_m)}) = \lim_{t_1,\ldots,t_m \to +\infty} \sum_{\alpha=1}^m \frac{D_\alpha(f^{(t_1,\ldots,t_m)})}{\int_{\Omega} f_{N-1}^t \Omega} = 0.$$ 

The proof is complete. \qed

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