The nonlinear diffusion equation of the ideal barotropic gas through a porous medium

1 Introduction

Consider the motion of the ideal barotropic gas through a porous medium, let $\rho$, $V$ and $p$ be the gas density, the velocity and the pressure respectively. Then, Antontsev-Shmarev [1] pointed out the motion is governed by the mass conservation law

$$\rho_t + \text{div} (\rho V) = 0,$$

the Darcy law

$$V = -k(x)\nabla p, k(x) \text{ is a given matrix}$$

and the equation of stage $p = P(\rho)$. It is usually assumed that $P(s) = \mu s^\alpha$ with $\mu, \alpha = \text{const.}$ The above conditions then lead to the semi-linear parabolic equation for the density $\rho$

$$\rho_t = \frac{\mu \alpha}{1 + \alpha} \text{div}(k(x)\nabla \rho^{1+\alpha}).$$

If we additionally assume that $p$ may explicitly depend on $(x,t)$ and has the form $p = \mu \rho^{\nu(x,t)}$, the equation for $\rho$ becomes

$$\rho_t = \mu \text{div}(k(x)\rho \nabla \rho^{\nu(x,t)})$$

and can be written in the form

$$\rho_t = \mu \text{div}(k(x)\rho^{\nu(x,t)} \nabla \rho + (\rho \log \rho)k(x) \cdot \nabla \gamma), \quad (x,t) \in Q_T = \Omega \times (0,T),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with appropriately smooth boundary.

Antontsev-Shmarev [1] made a simplified version of equation (1),

$$u_t - \text{div}(|u|^{\nu(x,t)} \nabla u) = f(x,t),$$

considered the usual initial boundary value conditions

$$u(x,0) = u_0(x), \quad x \in \Omega.$$
\[ u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T). \] (4)

and gave the following

**Definition 1.1.** A locally integral bounded function \( u(x, t) \) is said to be the weak solution of problem (2)-(3)-(4) if:

(i) \( u \in L^\infty(Q_T), \ |u|^{\gamma(x,t)} \nabla u \in L^2(0, T; L^2(\Omega)), \ u_t \in L^2(0, T; H^{-1}(\Omega)). \)

(ii) The boundary condition (4) is satisfied in the sense of trace.

(iii) For any test function \( \zeta(x,t) \) satisfying the conditions

\[
\zeta \in L^2(0, T; H^1(\Omega)) \bigcap L^\infty(Q_T), \ \zeta_t \in L^2(Q_T),
\]

and for \( 0 \leq t_1 \leq t_2 \leq T, \) it holds

\[
\int_{t_1}^{t_2} \int_{\Omega} (-u\zeta_t + |u|^{\gamma(x,t)}\nabla u \nabla \zeta - f \zeta) dx dt = -\int_{\Omega} u \zeta dx \bigg|_{t_1}^{t_2}.
\] (6)

Let \( \gamma(x,t) \) be a appropriately smooth function in \( Q_T. \) If

\[
\|u_0\|_{\infty; \Omega} + \int_0^T \|f(x,t)\|_{\infty; \Omega} dt = K(T) < \infty,
\] (7)

Antontsev-Shmarev [1] had proved that the problem (2)-(3)-(4) has at least one weak solution in the sense of Definition 1.1. The solution is bounded and satisfies the estimate \( \|u\|_{\infty, Q_T} \leq K(T) \) with the constant \( K(T) \) from condition (7).

In the present work, we limit ourselves to the study of the following equation

\[ u_t - \text{div}(\rho^{\alpha}|u|^{\gamma(x,t)}\nabla u) = f(x,t), \quad (x,t) \in Q_T, \] (8)

where \( \rho(x) = \text{dist}(x, \partial \Omega), \ \alpha > 0. \)

If \( \alpha = 0 \) and \( \gamma(x,t) \equiv m = \text{const}, \) equation (8) is the usual porous medium equation

\[ u_t = \text{div}(|u|^m \nabla u) + f(x,t), \quad (x,t) \in Q_T. \] (9)

Also, when \( m = 1, \ f = 0, \) equation (9) is regarded as a nonlinear heat equation

\[ u_t = \text{div}(k(u) \nabla u), \] (10)

which has been studied in many well-known monographs or textbooks, for examples one can refer to [2–7], where a wide spectrum of methods is used. The function \( k(u) \) has the meaning of nonlinear thermal conductivity, which depends on the temperature \( u = u(x,t). \) Meanwhile, if \( 0 > m > -1, \) equation (9) is called a fast diffusion equation. The name “fast diffusion” is related to the fact that since the heat conductivity is unbounded in the unperturbed (zero temperature) background, the heat propagates from warm regions into cold ones much faster than it propagates in the case of constant \( (m = 0 \text{ in } (9)) \) heat conductivity, and even faster than in the case \( m > 0, \) in which the speed of propagation of perturbations is finite.

Different from equation (9), equation (8) reflects that the diffusion process depends on the distance function \( \rho(x) \) from the boundary. In particular, \( \rho^\alpha = 0 \) on the boundary means that the equation is degenerate on the boundary. If we want to consider the initial-boundary value problem of equation (8) with \( \gamma(x,t) \equiv \gamma = \text{const} > -1, \) the initial value (3) is always necessary. But, the boundary value condition (4) may be superfluous. To see that, we consider the equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} (\rho^{\alpha}(x) A(u) \frac{\partial u}{\partial x_i}).
\] (11)

where \( \gamma = \text{const}, \ A(u) = u^\gamma, \ a(u) = \gamma u^{\gamma-1}. \) For small \( \eta > 0, \) let

\[
S_\eta(s) = \int_0^s h_\eta(\tau) d\tau, \ h_\eta(s) = \frac{2}{\eta} (1 - \left| \frac{s}{\eta} \right|). +
\] (12)
Obviously $h_\eta(s) \in C(\mathbb{R})$, and
\[
\lim_{\eta \to 0} h_\eta(s) = 0, \quad |s h_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1; \quad \lim_{\eta \to 0} S_\eta(s) = \text{sgn}, \quad \lim_{\eta \to 0} S'_\eta(s) = 0. \quad (13)
\]
If $u$ and $v$ are two classical solutions of equation (11) with the initial values $u_0$, $v_0$ respectively, then we have
\[
\begin{align*}
\int_\Omega S_\eta(A(u) - A(v)) \frac{\partial}{\partial t} (u - v) \, dx &= -\int_\Omega \rho^\alpha(x,t) |a(u)\nabla u - a(v)\nabla v|^2 S'_\eta(A(u) - A(v)) \, dx \\
- \int_\Omega \rho^\alpha(x,t) [a(u) \frac{\partial u}{\partial x_i} - a(v) \frac{\partial v}{\partial x_i}] n_i S_\eta(A(u) - A(v)) \, d\Sigma \\
&= -\int_\Omega \rho^\alpha(x,t) [a(u)\nabla u - a(v)\nabla v]^2 S'_\eta(A(u) - A(v)) \, dx \leq 0,
\end{align*}
\]
where $n = \{n_i\}$ is the inner unit normal vector of $\Omega$. Let $\eta \to 0$. Then we have
\[
\int_\Omega |u(x,t) - v(x,t)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.
\]
It means that the classical solutions (if there are) of equation (11) are completely determined by the initial value, in other words, the solutions are free from the limitation of the boundary condition. The phenomena that the solution of a degenerate parabolic equation may be free from the limitation of the boundary condition also can be found in [8–10] et al.

In this paper, we will study the well-posedness of the solutions to equation (8) with the initial value (3) but without any boundary condition. When we study the stability, we encounter two obstacles. The first one is that the solution lacks the regularity on the boundary, the second one is how to deal with the nonlinearity of $|u|^{\gamma(x,t)}$.

We denote that
\[
\gamma^- = \text{ess inf}_{(x,t) \in Q_T} \gamma(x,t), \quad \gamma^+ = \text{ess sup}_{(x,t) \in Q_T} \gamma(x,t),
\]
and suppose that
\[
u_0 \in L^\infty(\Omega), \rho^\frac{\alpha}{2} |u_0|^{-\frac{\gamma(x,0)}{2}} \nabla u_0 \in L^2(\Omega). \quad (14)
\]

**Definition 1.2.** A function $u(x,t)$ is said to be the weak solution of equation (8) with the initial value (3), if
\[
u \in L^\infty(Q_T), u_t \in L^2(0,T; H^{-1}(\Omega)), \quad \sqrt{\rho^\alpha |u|^{\gamma(x,t)}} \nabla u \in L^2(Q_T), \quad (15)
\]
\[
\iint_{Q_T} (-\frac{\partial}{\partial t} u + \rho^\alpha |u|^{\gamma(x,t)} \nabla u \nabla \varphi) \, dx \, dt = \int_\Omega u_0 \varphi(x,0) \, dx \, + \iint_{Q_T} f(x,t) \varphi(x,t) \, dx \, dt. \quad (16)
\]
for any function $\varphi \in C^1(Q_T), \varphi |_{t=0} = 0, \varphi |_{\partial \Omega} = 0$, and the initial value (3) is satisfied in the sense of that
\[
\lim_{t \to 0} \int_\Omega |u(x,t) - u_0(x)| \, dx = 0. \quad (17)
\]
The main results in our paper are the following theorems.

**Theorem 1.3.** Suppose that $f(x,t)$ is a smooth function, $u_0(x)$ satisfies (14). If $\alpha > 0$, $\gamma(x,t) \in L^\infty(Q_T)$, $\nabla \gamma \in L^1(Q_T)$, $\gamma^- > -1$, then equation (8) with initial value (3) has a solution.

**Theorem 1.4.** If $0 < \alpha, \gamma^- \geq 1$, $u_0(x)$ and $v_0(x)$ satisfy (14). Let $u,v$ be two solutions of equation (8) with the initial values $u_0(x), v_0(x)$ respectively, and
\[
\int_\Omega \rho^\alpha |\nabla u|^2 \, dx < \infty, \quad \int_\Omega \rho^\alpha |\nabla v|^2 \, dx < \infty. \quad (18)
\]
Then
\[ \int_{\Omega} |u(x,t) - v(x,t)| \leq \int_{\Omega} |u_0(x) - v_0(x)|dx. \quad (19) \]

**Theorem 1.5.** If \(0 < \alpha, \gamma^- > -1, \gamma(x,t) \in C^1(Q_T), u_0(x)\) and \(v_0(x)\) satisfy (14). Let \(u, v\) be two solutions of equation (8) with the initial values \(u_0(x), v_0(x)\) respectively, and
\[ \int_{\Omega} \rho^\alpha |\ln u|^2 dx < \infty, \int_{\Omega} \rho^\alpha |\ln v|^2 dx < \infty. \quad (20) \]
Then the stability (19) is true.

**Corollary 1.6.** If \(0 < \alpha, \gamma(x,t) \equiv \gamma > -1, u_0(x)\) and \(v_0(x)\) satisfy (14). Let \(u, v\) be two solutions of equation (8) with the initial values \(u_0(x), v_0(x)\) respectively. Then the stability (19) is true.

Roughly speaking, we may conjecture only if \(\alpha > 0\), the stability (19) may be true without the condition (18) or (20). Theorem 1.4 and Theorem 1.5 have verified the fact partly. Recently, the author [12] had studied the equation
\[ u_t = \text{div}(\rho^\alpha \nabla u) + f(x,t), \quad (x,t) \in Q_T, \quad (21) \]
with \(\alpha > 0\), and had shown that the uniqueness of the solutions of equation (20) is true without any boundary value condition. Meanwhile, Jiří Benedikt et.al [13, 14] had studied the equation
\[ u_t = \text{div}(\rho^\alpha \nabla u) + q(x)|u|^{\alpha-1}u, \quad (x,t) \in Q_T, \quad (22) \]
with \(0 < \alpha < 1\), and shown that the uniqueness of the solutions of equation (22) is not true. From the short comment, one can see that the degeneracy of the coefficient \(\rho^\alpha\) plays an important role in the well-posedness of the solutions, it even can eliminate the action from the source term \(f(u,x,t)\).

The paper is arranged as follows. In the first section, we give a brief introduction and narrate the main results. In the second section, we prove the existence. In the third section, we prove the stability of the solutions.

### 2 The existence

**Proof of Theorem 1.3.** Let \(u_0 \in L^\infty(\Omega)\) satisfy (14). Consider the regularized problem of equation (8)
\[ u_t = \text{div}(a(\varepsilon, u, x, t)\nabla u) = f(x,t), \quad (x,t) \in Q_T, \quad (23) \]
with the initial-boundary value condition (3)-(4). Here \(a(\varepsilon, u, x, t) = (\rho^\alpha + \varepsilon)(\varepsilon + |u_\varepsilon|)\gamma(x,t)\). Then
\[ 0 < C'(\varepsilon) \leq a(\varepsilon, u, x, t) \leq C(\varepsilon), \]
Similar to [1], by Schauder Fixed Point Theorem, we know that the regularized problem has a solution \(u_\varepsilon\) in the sense of Definition 1.1.

By multiplying equation (23) by \(u_\varepsilon^{2k-1}\) and integrating it over \(\Omega\), we have
\[ \frac{1}{2k} \frac{d}{dt} \left( \|u_\varepsilon(\cdot,t)\|_{2k,\Omega}^{2k} \right) + (2k-1) \int_{\Omega} a|\nabla u_\varepsilon|^2 dx = \int_{\Omega} f u_\varepsilon^{2k-1} dx. \]
By Hölder inequality,
\[ \int_{\Omega} f u_\varepsilon^{2k-1} dx \leq \|u_\varepsilon(\cdot,t)\|_{2k,\Omega}^{2k-1} \|f\|_{2k,\Omega}, \quad k = 1, 2, \cdots, \]
whence
\[ \|u_\varepsilon(\cdot,t)\|_{2k,\Omega}^{2k-1} \frac{d}{dt} \left( \|u_\varepsilon(\cdot,t)\|_{2k,\Omega}^{2k} \right) + (2k-1) \int_{\Omega} a|\nabla u_\varepsilon|^2 dx \leq \|u_\varepsilon(\cdot,t)\|_{2k,\Omega}^{2k-1} \|f\|_{2k,\Omega}, \quad k = 1, 2, \cdots. \]
Simplifying and then integrating this relation in $t$, we obtain the following estimates for the solutions of equation (23)

$$
\|u_\varepsilon(\cdot, t)\|_{2k, \Omega} \leq \int_0^t \|f\|_{2k, \Omega} dt + \|u_0\|_{2k, \Omega}.
$$

(24)

Passing to the limit when $k \to \infty$, we have

$$
\|u_\varepsilon\|_{\infty, Q_T} \leq c.
$$

(25)

By multiplying (23) by $u_\varepsilon$, we are easily to obtain that

$$
\iint_{Q_T} a(\varepsilon, u_\varepsilon, x, t)|\nabla u_\varepsilon|^2 dx dt \leq c.
$$

(26)

If the constant $\nu \geq \gamma^* + 1$, then

$$
\iint_{Q_T} (\rho^\varepsilon + \varepsilon)|\nabla (|u_\varepsilon|^{\nu-1}u_\varepsilon)|^2 dx dt
\leq c \iint_{Q_T} \varepsilon (\rho^\varepsilon + \varepsilon)(\varepsilon + |u_\varepsilon|)^{\nu-1} |\nabla u_\varepsilon|^2 dx dt
\leq c \iint_{Q_T} \varepsilon (\rho^\varepsilon + \varepsilon)(\varepsilon + |u_\varepsilon|)^{\nu(x,t)} |\nabla u_\varepsilon|^2 dx dt.
$$

If denoting $\Omega_\lambda = \{x \in \Omega : \rho(x) > \lambda\}$, $Q_{T, \lambda} = \Omega \times (0, T)$, for any small $\lambda > 0$, by that $\rho(x) > \lambda$ when $x \in \Omega_\lambda$, we have

$$
\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^{\nu-1} u_\varepsilon|^2 dx dt \leq c(\lambda).
$$

(27)

By multiplying (23) by $v \in H^1_0(\Omega)$, $\|v\|_{H^1_0} = 1$, we easily to obtain that

$$
|v u_\varepsilon| \leq \iint_{Q_T} a(\varepsilon, u_\varepsilon, x, t)|\nabla u_\varepsilon|^2 dx dt + c \leq c,
$$

which implies that

$$
\|u_\varepsilon\|_{L^2(0, T; H^{-1}(\Omega))} \leq c.
$$

(28)

The uniform estimates (27)-(28), using the result of [11, Sec.8], yield relative compactness of the sequence $\{u_\varepsilon\}$ in $L^s(Q_{T, \lambda})$ with some $s \in (1, \infty)$. Then we can choose a subsequence $\{u_l\} \subseteq \{u_\varepsilon\}$ such that $\{u_l\}$ is compact in $L^s(Q_{T, \lambda})$. By the arbitrary of $\lambda$, we have

$$
u_l \to u, \ a.e. \ in \ Q_T.
$$

(29)

Moreover, by (25)-(28), when $l \to \infty$, we have

$$
u_l \to * u, \ weakly \ star \ in \ L^\infty(Q_T), \ nu_l \to \in L^2(0, T; H^{-1}(\Omega)),
$$

$$
\sqrt{a(l, u, x, t)u_l} \to \chi_l, \ weakly \ in \ L^2(Q_T).
$$

where $a(l, u, x, t) = a(\varepsilon, u, x, t)|u_l| = u_l, \ \chi = \{\chi_i : 1 \leq i \leq N\}$ and every $\chi_i$ is a function in $L^2(Q_T)$. In order to prove the theorem, we firstly prove

$$
\chi = \sqrt{\rho^\varepsilon |u|^{\nu(x,t)}} \nabla u, \ in \ L^2(Q_T).
$$

(30)
For any $\varphi(x,t) \in C^1_0(Q_T)$, noticing that $u_l \to u$, a.e. in $Q_T$, then
\[
\iint_{Q_T} \chi_i \varphi dx dt = \lim_{l \to \infty} \iint_{Q_T} \varphi(x,t)\sqrt{a(l,u,x,t)u_{lx_i}} dx dt
\]
\[
= \lim_{l \to \infty} \left[ \iint_{Q_T} \varphi(x,t)\left( \int_0^{u_l} \sqrt{a(l,s,x,t)} ds \right) dx dt - \iint_{Q_T} \varphi(x,t)\left( \int_0^{u_l} \sqrt{a(l,s,x,t)} s dx \right) dx dt \right]
\]
\[
= - \lim_{l \to \infty} \iint_{Q_T} \varphi(x,t)\left( \int_0^{u_l} \sqrt{\rho|\gamma(x,t)| s} dx \right) ds dx dt
\]
\[
= \iint_{Q_T} \varphi(x,t)\rho|\gamma(x,t)| u_{lx_i} dx dt.
\]
Here, we have used the fact $|\nabla \rho| = 1$, and
\[
\left( \sqrt{\rho|\gamma(x,t)|} \right)_{x_i} = \frac{1}{2\sqrt{\rho|\gamma(x,t)|}} [a\rho^{-1}\rho_{x_i}|\gamma(x,t)| + \rho|\gamma(x,t)| \gamma(x,t) \rho_{x_i} \log |s|].
\]
the function $\int_0^u (\sqrt{\rho|\gamma(x,t)|} s)_{x_i} ds \in L^1(Q_T)$, due to that $\gamma(x,t) \in L^\infty(Q_T)$, $\nabla \gamma \in L^1(Q_T)$, $\gamma^\rho > -1$, and
\[
\int_0^1 s^{\gamma^\rho} \log s ds < \infty.
\]
Secondly, we prove (17). For any given small $\lambda > 0$, large enough $k, l$, we declare that
\[
\int_{\Omega_{2\lambda}} |u_k(x,t) - u_l(x,t)| dx \leq \int_{\Omega_{\lambda}} |u_k(x,0) - u_l(x,0)| dx + c_{\lambda}(t),
\]
where $c_{\lambda}(t)$ is independent of $k, l$, and $\lim_{t \to 0} c_{\lambda}(t) = 0$. By (16), for any $\varphi \in C^1_0(Q_T)$,
\[
\int_0^t \int_{\Omega_{2\lambda}} \varphi(u_k - u_l) dx dt + \int_0^t \int_{\Omega_{\lambda}} \nabla \varphi \rho^\rho [u_k u_l] \nabla [u_k - u_l] u_l dx dt = 0.
\]
Supposing that $\xi(x) \in C^1_0(\Omega_{\lambda})$ such that
\[
0 \leq \xi \leq 1; \xi\mid_{\Omega_{2\lambda}} = 1,
\]
and choosing $\varphi = \xi S_{\rho}(u_k - u_l)$ in (34), we get
\[
\int_0^t \int_{\Omega_{2\lambda}} \xi S_{\rho}(u_k - u_l)(u_k - u_l) dx dt
\]
\[
+ \int_0^t \int_{\Omega_{\lambda}} \rho^\rho [u_k u_l] \nabla u_k - [u_l]^{\rho^\rho} \nabla u_l] \nabla \xi S_{\rho}(u_k - u_l) dx dt = 0.
\]
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\[ \int_0^t \int_{\Omega_\lambda} \rho^\alpha \| u_k - u_l \|^\alpha x \nabla u_k - |u_l|^\alpha x \nabla u_l \nabla (u_k - u_l) \xi S'_{n}(u_k - u_l) \, dx \, d\tau \]

\[ = \int_0^t \int_{\Omega_\lambda} \rho^\alpha |u_k|^\alpha x \nabla (u_k - u_l) \nabla S'_{n}(u_k - u_l) \, dx \, d\tau \]

\[ + \int_0^t \int_{\Omega_\lambda} \rho^\alpha (|u_k|^\alpha - |u_l|^\alpha x \nabla u_k \nabla S'_{n}(u_k - u_l) \, dx \, d\tau \]

By (13), \( \lim_{\eta \to 0} S'_{n}(s) = 0 \), and using the fact

\[ \int_0^t \int_{\Omega_\lambda} \rho^\alpha |u_k|^\alpha x \nabla (u_k - u_l) \nabla S'_{n}(u_k - u_l) \, dx \, d\tau \geq 0, \]

then we have

\[ \lim_{\eta \to 0} \int_0^t \int_{\Omega_\lambda} \xi S_{n}(u_k - u_l)(u_k - u_l) \, dx \, d\tau + \lim_{\eta \to 0} \int_0^t \int_{\Omega_\lambda} \rho^\alpha \| u_k - u_l \|^\alpha x \nabla u_k - |u_l|^\alpha x \nabla u_l \nabla \xi S_{n}(u_k - u_l) \, dx \, d\tau \leq 0. \] (36)

At the same time,

\[ \lim_{\eta \to 0} \int_0^t \int_{\Omega_\lambda} \xi S_{n}(u_k - u_l)(u_k - u_l) \, dx \, d\tau = \lim_{\eta \to 0} \int \int_{\Omega_\lambda} \xi( \int S_{n}(s) \, ds) \, dx \, d\tau \]

\[ = \lim_{\eta \to 0} \int_0^t \int_{\Omega_\lambda} \xi \int S_{n}(s) \, ds \, dx = \int_0^t \int_{\Omega_\lambda} \xi |u_k - u_l| \, dx - \int_0^t \int_{\Omega_\lambda} \xi |u_{0k} - u_{0l}| \, dx. \] (37)

By (36), (37), we have

\[ \int_{\Omega_{2\lambda}} |u_k - u_l| \, dx \leq \int_{\Omega_{\lambda}} |u_{0k} - u_{0l}| \, dx + \frac{c}{\lambda} \int_0^t \int_{\Omega_\lambda} \rho^\alpha |u_k|^\alpha x \nabla u_k - |u_l|^\alpha x \nabla u_l \| \, dx \, d\tau, \]

which means (33) is true.

Now, for any given small \( r \), if \( k, l \) are large enough, by (33), we have

\[ \int_{\Omega_{2\lambda}} |u(x, t) - u_0(x)| \, dx \leq \int_{\Omega_{\lambda}} |u(x, t) - u_k(x, t)| \, dx + \int_{\Omega_{2\lambda}} |u_{0k}(x) - u_{0l}(x)| \, dx \]

\[ + \int_{\Omega_{2\lambda}} |u_{0l}(x) - u_0(x)| \, dx + \int_{\Omega_{2\lambda}} |u_l(x) - u_0(x)| \, dx \]

letting \( \lambda \to 0 \), we get (17). Theorem 1.3 is proved.

\[ \square \]

3 The stability

Proof of Theorem 1.4. For a small positive constant \( \lambda > 0 \), let

\[ \phi_{\lambda}(x) = \begin{cases} 1, & x \in \Omega_{\lambda} \setminus \Omega_{2\lambda}, \\
\frac{1}{\lambda} (\rho(x) - \lambda), & x \in \Omega_{\lambda} \setminus \Omega_{2\lambda}, \\
0, & x \in \Omega \setminus \Omega_{\lambda}. \end{cases} \]
Now, if \( u_0 \) and \( v_0 \) only satisfy (14), let \( u, v \) be two solutions of equation (8) with the initial values \( u_0, v_0 \) respectively. Then
\[
\frac{\partial (u - v)}{\partial t} = \text{div}[\rho^\alpha (|u|^\gamma(x,t)\nabla u - |v|^\gamma(x,t)\nabla v)].
\] (38)

By multiplying (38) by \( \phi_\lambda(x)S_\eta(u - v) \), and integrating it over \( \Omega \), we have
\[
\int_\Omega \phi_\lambda(x)S_\eta(u - v) \frac{\partial (u - v)}{\partial t} \, dx + \int_\Omega \phi_\lambda(x)\rho^\alpha (|u|^\gamma(x,t)\nabla u - |v|^\gamma(x,t)\nabla v) \cdot \nabla (u - v) S_\eta'(u - v) \, dx
\]
\[
+ \int_\Omega \rho^\alpha (|u|^\gamma(x,t)\nabla u - |v|^\gamma(x,t)\nabla v) \cdot \nabla \phi_\lambda(x) S_\eta(u - v) \, dx = 0.
\] (39)

We now calculate the terms of (39) as follows.
\[
\int_\Omega \phi_\lambda(x)\rho^\alpha (|u|^\gamma(x,t)\nabla u - |v|^\gamma(x,t)\nabla v) \cdot \nabla (u - v) S_\eta'(u - v) \, dx
\]
\[
= \int_\Omega \phi_\lambda(x)\rho^\alpha |u|^\gamma(x,t)\nabla u \cdot \nabla (u - v) S_\eta'(u - v) \, dx
\]
\[
+ \int_\Omega \phi_\lambda(x)\rho^\alpha \nabla v \cdot \nabla (u - v) (|u|^\gamma(x,t) - |v|^\gamma(x,t)) S_\eta'(u - v) \, dx.
\] (40)

Since
\[
|(|u|^\gamma(x,t) - |v|^\gamma(x,t))| = \left| \int \gamma(x,t) s^{\gamma(x,t) - 1} ds \right| \leq c |u - v| \gamma(x,t) \xi^{\gamma(x,t) - 1},
\] (42)

where \( \min\{|v|, |u|\} \leq \xi \leq \max\{|v|, |u|\} \), by \( |S_\eta'(u - v)| \leq \frac{c}{|u - v|}, \gamma^{-} \geq 1 \),
\[
\left| \int \phi_\lambda(x)\rho^\alpha \nabla v \cdot \nabla (u - v) (|u|^\gamma(x,t) - |v|^\gamma(x,t)) S_\eta'(u - v) \, dx \right|
\]
\[
\leq \int \phi_\lambda(x)\rho^\alpha (|\nabla u|^2 + |\nabla v|^2) \gamma(x,t) \xi^{\gamma(x,t) - 1} \, dx \, dt < \infty,
\] (43)
due to the condition (18)
\[
\int \rho^\alpha |\nabla u|^2 \, dx < \infty, \int \rho^\alpha |\nabla v|^2 \, dx < \infty.
\]

Using the dominated convergence theorem, by \( \lim_{\eta \to 0} S_\eta'(s) = 0 \), we have
\[
\lim_{\eta \to 0} \lim_{\lambda \to 0} \int_\Omega \rho^\alpha \nabla v \cdot \nabla (u - v) (|u|^\gamma(x,t) - |v|^\gamma(x,t)) S_\eta'(u - v) \phi_\lambda(x) \, dx
\]
\[
= \lim_{\eta \to 0} \int_\Omega \rho^\alpha \nabla v \cdot \nabla (u - v) (|u|^\gamma(x,t) - |v|^\gamma(x,t)) S_\eta'(u - v) \, dx = 0.
\] (44)
At the same time,
\[
\int_{\Omega} \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) \cdot \nabla \phi_\alpha (x) S_\eta(u - v) \, dx \\
\leq \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v \cdot |\nabla \phi_\alpha (x)| S_\eta(u - v) \, dx \\
\leq \frac{c}{\lambda} \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v \, dx \\
\leq \frac{c}{\lambda} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |u|^{\gamma(x,t)} \frac{\partial u}{\partial t} \, dx \right) - \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |v|^{\gamma(x,t)} \frac{\partial v}{\partial t} \, dx \right) \right) \frac{1}{2} \\
+ \frac{c}{\lambda} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |u|^{\gamma(x,t)} \frac{\partial u}{\partial t} \, dx \right) - \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \left( \int_{\Omega_\lambda \setminus \Omega_{2\lambda}} \rho^\alpha |v|^{\gamma(x,t)} \frac{\partial v}{\partial t} \, dx \right) \right) \frac{1}{2}.
\]
then
\[
\lim_{\lambda \to 0} \int_{\Omega} \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) \cdot \nabla \phi_\alpha (x) S_\eta(u - v) \, dx = 0. \tag{45}
\]
By
\[
\lim_{\eta \to 0} \lim_{\lambda \to 0} \int_{\Omega} \phi_\alpha (x) S_\eta(u - v) \frac{\partial (u - v)}{\partial t} \, dx = \frac{d}{dt} \|u - v\|_{L^1(\Omega)}, \tag{46}
\]
after letting \( \lambda \to 0 \), let \( \eta \to 0 \) in (39). Then
\[
\int_{\Omega} |u(x,t) - v(x,t)| \, dx \leq \int_{\Omega} |u_0 - v_0| \, dx.
\]
Theorem 1.4 is proved. \(\square\)

**Proof of Theorem 1.5.** If \( u_0 \) and \( v_0 \) satisfy (14), let \( u, v \) be two solutions of equation (8) with the initial values \( u_0, v_0 \) respectively. Multiplying (38) by \( \phi_\alpha (x) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \), and integrating it over \( \Omega \), we have
\[
\int_{\Omega} \phi_\alpha (x) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \frac{\partial (u - v)}{\partial t} \, dx \\
+ \int_{\Omega} \phi_\alpha (x) \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) \cdot \nabla (\int_{\Omega} |s|^{\gamma(x,t)} \, ds) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \, dx \\
+ \int_{\Omega} \phi_\alpha (x) \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) \cdot \nabla \phi_\alpha (x) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \, dx = 0. \tag{47}
\]
We now calculate the terms of (47) as follows.
\[
\int_{\Omega} \phi_\alpha (x) \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) \cdot \nabla (\int_{\Omega} |s|^{\gamma(x,t)} \, ds) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \, dx \\
= \int_{\Omega} \phi_\alpha (x) \rho^\alpha (|u|^{\gamma(x,t)} \nabla u - |v|^{\gamma(x,t)} \nabla v) S_\eta(\int_\Omega |s|^{\gamma(x,t)} \, ds) \, dx \, dx \\
\]
\begin{equation}
\int_\Omega \phi_\lambda(x) \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v) \cdot \nabla \ln \xi \int_\Omega |s|^{\gamma(x,t)} ds' \int_\Omega |s|^{\gamma(x,t)} ds dx,
\end{equation}

where \( \xi \in (|v|^\gamma, |u|^\gamma) \).

\begin{equation}
\int_\Omega \phi_\lambda(x) \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v)^2 S'_h \int_\Omega |s|^{\gamma(x,t)} ds dx dt \geq 0.
\end{equation}

Since \(|s S'_h(s)| \leq c\), then

\begin{align*}
& \left| \int_\Omega \phi_\lambda(x) \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v) \cdot \nabla \ln \xi \int_\Omega |s|^{\gamma(x,t)} ds' \int_\Omega |s|^{\gamma(x,t)} ds dx \right| \\
& \leq \int_\Omega \rho^\alpha (|u|^\gamma + |v|^\gamma) \nabla |\nabla \xi| \ln \xi \int_\Omega |s|^{\gamma(x,t)} ds dx \\
& \leq \left( \int_\Omega \rho^\alpha |u|^\gamma |\nabla u|^2 dx \right)^{1/2} \left( \int_\Omega \rho^\alpha |v|^\gamma |\nabla v|^2 |\ln \xi|^2 dx \right)^{1/2} \\
& + \left( \int_\Omega \rho^\alpha |v|^\gamma |\nabla v|^2 dx \right)^{1/2} \left( \int_\Omega \rho^\alpha |u|^\gamma |\nabla u|^2 |\ln \xi|^2 dx \right)^{1/2} \\
& \leq c \int_\Omega \rho^\alpha [(\ln |u|^2 + \ln |v|^2)] dx \leq c,
\end{align*}

due to the condition (20). Using the dominated convergence theorem, by \( \lim_{\eta \to 0} S'_h(s) = 0 \), we have

\begin{align*}
\lim_{\eta \to 0} \lim_{\lambda \to 0} \int_\Omega \phi_\lambda(x) \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v) \cdot \nabla \ln \xi \int_\Omega |s|^{\gamma(x,t)} ds' \int_\Omega |s|^{\gamma(x,t)} ds dx & = \int_\Omega \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v) \cdot \nabla \phi_\lambda(x) S'_h \int_\Omega |s|^{\gamma(x,t)} ds dx | = 0. \tag{50}
\end{align*}

Meanwhile, similarly to the proof of the (45), we can prove that

\begin{equation}
\lim_{\lambda \to 0} \int_\Omega \rho^\alpha (|u|^\gamma(x,t) \nabla u - |v|^\gamma(x,t) \nabla v) \cdot \nabla \phi_\lambda(x) S'_h \int_\Omega |s|^{\gamma(x,t)} ds dx = 0. \tag{51}
\end{equation}

By

\begin{equation}
\lim_{\eta \to 0} \lim_{\lambda \to 0} \int_\Omega \phi_\lambda(x) S'_h \int_\Omega |s|^{\gamma(x,t)} ds \frac{\partial (u - v)}{\partial t} dx = \frac{d}{dt} \|u - v\|_{L^1(\Omega)}, \tag{52}
\end{equation}

after letting \( \lambda \to 0 \), let \( \eta \to 0 \) in (54). Then

\begin{equation}
\int_\Omega |u(x, t) - v(x, t)| dx \leq \int_\Omega |u_0 - v_0| dx.
\end{equation}

Theorem 1.5 is proved.
Proof of Corollary 1.6. The proof is similar as that of Theorem 1.5, but simpler. Clearly,
\[
\frac{\partial(u-v)}{\partial t} = \text{div}[\rho^\alpha (u^\gamma \nabla u - v^\gamma \nabla v)] = \frac{1}{\gamma + 1} \text{div}[\rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1})].
\] (53)

By multiplying (53) by \( \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \), and integrating it over \( \Omega \), we have
\[
\int_\Omega \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \frac{\partial(u-v)}{\partial t} \, dx
+ \frac{1}{\gamma + 1} \int_\Omega \rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla (u^{\gamma+1} - v^{\gamma+1}) S_\eta'(u^{\gamma+1} - v^{\gamma+1}) \, dx
+ \int_\Omega \rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \, dx = 0.
\] (54)

\[
\int_\Omega \phi_\lambda(x) \rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla (u-v) S_\eta'(u^{\gamma+1} - v^{\gamma+1}) \, dx \geq 0.
\] (55)

At the same time, since
\[
\int_\Omega \rho^\alpha u^\gamma \| \nabla u \|^2 \, dx < \infty,
\]
then
\[
| \int_\Omega \rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \, dx |
\leq \int_\Omega \rho^\alpha |(\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1})| \, dx
\leq \int_\Omega \rho^\alpha |(\nabla u^{\gamma+1} - \nabla v^{\gamma+1})| |\nabla \phi_\lambda(x)| \, dx
\leq \frac{c}{\lambda} \left[ \int_\Omega \rho^\alpha |\nabla u^{\gamma+1}| \, dx + \int_\Omega \rho^\alpha |\nabla v^{\gamma+1}| \, dx \right]
= \frac{c}{\lambda} \left( \int_\Omega (\rho^\alpha u^{\gamma+1})^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega (\rho^\alpha v^{\gamma+1})^2 \, dx \right)^{\frac{1}{2}}
+ \frac{c}{\lambda} \left( \int_\Omega (\rho^\alpha v^{\gamma+1})^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega (\rho^\alpha u^{\gamma+1})^2 \, dx \right)^{\frac{1}{2}},
\]
so
\[
\lim_{\lambda \to 0} \int_\Omega \int_\Omega \rho^\alpha (\nabla u^{\gamma+1} - \nabla v^{\gamma+1}) : \nabla \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \, dx = 0.
\] (56)

By
\[
\lim_{\eta \to 0} \lim_{\lambda \to 0} \int_\Omega \phi_\lambda(x) S_\eta(u^{\gamma+1} - v^{\gamma+1}) \frac{\partial(u-v)}{\partial t} \, dx
= \lim_{\eta \to 0} \lim_{\lambda \to 0} \int_\Omega \phi_\lambda(x) S_\eta(u-v) \frac{\partial(u-v)}{\partial t} \, dx
= \frac{d}{dx} \| u-v \|_{L^1(\Omega)}.
\]
Now, after letting $\lambda \to 0$, let $\eta \to 0$ in (53). Then

$$\int_{\Omega} |u(x,t) - v(x,t)| \, dx \leq \int_{\Omega} |u_0 - v_0| \, dx.$$ 

Corollary 1.6 is proved.

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