An elementary purely algebraic approach to generalized functions and operational calculus

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Abstract. The space of Schwartz distributions of finite order is represented as a factor space of the space of, what we call, Mikusinski functions. The point of Mikusinski functions is that they admit a multiplication by convergent Laurent series. It is shown that this multiplication provides a natural simple basis for Heaviside’s operational calculus.

Key words. Convergent formal series, Mikusinski functions, generalized functions.

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1 Introduction

In his seminal works in electromagnetic theory, O. Heaviside developed formal rules for dealing with the differentiation operator, much of which he arrived at intuitively. A mathematical basis for his operational calculus was done by mathematicians later with the aid of the Laplace transform. A different explanation of Heaviside’s calculus, based upon the convolution ring of continuous functions, was presented by Mikusinski [1]. From this ring, which is commutative and without zero divisors, Mikusinski goes on to define the field of fractions. These fractions he calls operators.

In this note, we modify Mikusinski’s approach by starting with the module structure of continuous functions over the ring of convergent formal series (say, in $t^i$). This module is torsion free. Imitating Mikusinski, we apply the fraction space construction and obtain this way a linear space over the field of convergent Laurent series. Elements of this space, which we call Mikusinski functions, constitute a very small part of the field of Mikusinski operators. Viewing the space of these functions as a module over the polynomial ring (in $s = t^{-1}$) and taking the factor module by “derivatives” of constant functions, we get a module that is canonically isomorphic to the module $D'_{fin}$, the module of Schwartz distributions of finite order.

The goal of this note is to demonstrate that with such a representation of $D'_{fin}$ we have a natural, simple and rigorous theory for Heaviside’s operational calculus.

Throughout, $I$ is a fixed interval of real axis containing 0 and $t$ an indeterminate. The symbol 1 will stand for the unit function on $I$. Let $C(I)$ be the space of all complex-valued continuous functions defined on the interval $I$ and let $\mathbb{C}\{t\}$ be the ring of complex coefficient convergent formal series in $t$. (We remind that a formal series $\sum_{i \geq 0} b_i t^i$ is said to be convergent
if \( \sum_{i \geq 0} |b_i| \varepsilon^i < +\infty \) for some positive real \( \varepsilon \).) Let \( \mathbb{C}(\{t\}) \) denote the fraction field of \( \mathbb{C}\{t\} \), the field of convergent Laurent series. It is worth mentioning that \( \mathbb{C}(\{t\}) \) contains as a subring the polynomial ring \( \mathbb{C}[s] \), where \( s = t^{-1} \).

For every continuous function \( u \in C(I) \), let \( J(u) \) denote the function defined by

\[
J(u)(x) = \int_0^x u(\alpha) d\alpha, \quad x \in I.
\]

2 Mikusinski functions

If \( g = \sum_{i \geq 0} b_i t^i \) is a convergent formal series and \( u \) is a continuous function on \( I \), then the series

\[
\sum_{n \geq 0} b_n J^n(u)
\]
converges uniformly on every compact subinterval of \( I \) that contains 0. We therefore can define the product \( gu \) by setting

\[
gu = \sum_{n \geq 0} b_n J^n(u).
\]

This multiplication makes \( C(I) \) a module over \( \mathbb{C}\{t\} \). (A module is a "linear space" over a ring!) It is easily seen that this module is torsion free. Indeed, suppose that \( gu = 0 \), where \( g \neq 0 \). Write \( g = t^n g_0 \), where \( g_0 \) has nonzero free coefficient. By the theorem on units (see [2]), \( g_0 \) is invertible. Multiplying \( gu = 0 \) by \( g_0^{-1} \), we get that \( t^n u = 0 \). Because \( J \) is an injective operator, it follows that \( u = 0 \).

**Definition.** We define the Mikusinski space \( M(I) \) as the fraction space of the module \( C(I) \).

By definition, \( M(I) \) is a linear space over \( \mathbb{C}\{t\} \), and every Mikusinski function is represented by a ratio \( u/g \) with \( u \in C(I) \) and \( g \in \mathbb{C}\{t\} \), \( g \neq 0 \). Two such ratios \( u_1/g_1 \) and \( u_2/g_2 \) represent the same Mikusinski function if

\[
g_2 u_1 = g_1 u_2.
\]

Because \( C(I) \) is torsion free, the canonical map \( u \mapsto u/1 \) is injective. Therefore we can make the identification

\[
u = u/1.
\]

With this identification, a Mikusinski function \( w \) can be written in the form

\[
w = s^n u,
\]
where \( u \in C(I) \). Indeed, if \( w = u/g \), then, writing \( g = t^n g_0 \) with \( g_0 \) as above, we have

\[
u g = s^n g_0^{-1} u = s^n (g_0^{-1} u).
\]

We therefore have:

\[
C(I) \subseteq sC(I) \subseteq s^2 C(I) \subseteq s^3 C(I) \subseteq \cdots \quad \text{and} \quad M(I) = \cup_{n \geq 0} s^n C(I).
\]
Remark. If one wants, one can define the Mikusinski space $M(I)$ as the inductive limit of the sequence

$$C(I) \xrightarrow{J} C(I) \xrightarrow{J} C(I) \xrightarrow{J} C(I) \xrightarrow{J} \cdots.$$ 

Extend the operator $J$ to Mikusinski functions by setting

$$J(w) = tw.$$ 

This way, the integration operator becomes bijective. The following theorem says that the space $M(I)$ is the smallest extension of $C(I)$ in which every function is an integral.

**Theorem 1** Every continuous function is an $n$-fold integral of a Mikusinski function.

**Proof.** This is obvious. Indeed, if $u \in C(I)$, then

$$u = t^n(s^n u) = J^n(s^n u).$$

$\square$

## 3 Schwartz distributions (of finite order)

Since multiplication by $t$ is the integration operator, a natural idea is to interpret multiplication by $s = t^{-1}$ as the differentiation operator. Postulating that constant functions must have zero derivatives, we are led to the following definition.

**Definition.** Define the generalized function space $G(I)$ to be

$$G(I) = M(I)/N(I),$$

where $N(I) = \{ f \mathbf{1} \mid f \in s\mathbb{C}[s] \}$.

Since $N(I)$ is a $\mathbb{C}[s]$-submodule of $M(I)$, $G(I)$ has a structure of a module over $\mathbb{C}[s]$. In particular, we can multiply generalized functions by $s$.

**Definition.** Define the differential operator $D : G(I) \to G(I)$ by setting

$$D(\xi) = s\xi, \quad \xi \in G(I).$$

We need the following lemma.

**Lemma 1** $C(I) \cap N(I) = \{0\}$.

**Proof.** Assume there is $u \in C(I)$ that is not zero and belongs to $N(I)$. We then have

$$u = (a_0 s^n + \cdots + a_{n-1}s)\mathbf{1}$$

with $n \geq 1$, $a_i \in \mathbb{C}$ and $a_0 \neq 0$. Multiplying both of the sides by $t^{n+1}$, we get

$$t^{n+1}u = (a_0 t + \cdots + a_{n-1}t^n)\mathbf{1}.$$ 

It follows that

$$t(t^n u - (a_1 + \cdots + a_{n-1}t^{n-1})\mathbf{1}) = a_0\mathbf{1}. $$
On the left, we have a continuous function having value 0 at 0. Hence, \( a_0 = 0 \), which is a contradiction.

It follows from the previous lemma that the canonical map \( C(I) \to G(I) \) defined by

\[
u \mapsto u \pmod{N(I)}
\]

is injective. This permits us to identify \( C(I) \) with its image under this map. In other words, for every \( u \in C(I) \), we can make the identification

\[
u = u \pmod{N(I)}.
\]

Notice that the derivative \( D(u) \) of a classical continuously differentiable function \( u \) coincides with the ordinary derivative \( u' \). Indeed, by the Newton-Leibniz formula

\[
u = Ju' + u(0) \mathbf{1},
\]

and consequently

\[
su \pmod{N(I)} = (u' + su(0) \mathbf{1}) \pmod{N(I)} = u' \pmod{N(I)} = u'.
\]

The following theorem says that the space \( G(I) \) is the smallest extension of \( C(I) \) in which every function has a derivative.

**Theorem 2** Any generalized function is an \( n \)-fold derivative of a continuous function.

**Proof.** This is obvious. Indeed, for every continuous function \( u \) and every nonnegative integer \( n \), we have

\[
s^n u \pmod{N(I)} = D^n(u \pmod{N(I)}) = D^n u.
\]

As an immediate consequence, we have the following corollary.

**Corollary 1** \( G(I) \) is canonically isomorphic to \( \mathcal{D}'_{\text{fin}}(I) \), the space of Schwartz distributions of finite order.

This corollary permits us to interpret generalized functions as Schwartz distributions of finite order.

**Remark.** We remind that, basically, distributions have been introduced by Schwartz in order to be able to differentiate all continuous functions (see page 72 in Schwartz [3]).

We close the section with the following important theorem.

**Theorem 3** Let \( f \in C[s] \) be a polynomial. The equation

\[
f(D)\xi = \omega
\]

has a solution \( \xi \in G(I) \) for every \( \omega \in G(I) \).

**Proof.** This is obvious. Indeed, \( M(I) \) is a divisible \( C[s] \)-module, and consequently \( G(I) \) also is divisible. To be more precise, if \( \omega \) is represented by a Mikusinski function \( w \), then

\[
\xi = \frac{1}{f} w \pmod{N(I)}
\]

is a solution. □
4 Linear constant coefficient differential equations

We already know how to find a particular solution of a nonhomogeneous linear constant coefficient differential equation. So, we only need to consider the homogeneous case.

Given $g \in \mathbb{C}(t)$, define an entire analytic function $E(g)$ by setting

$$E(g) = g1.$$

**Theorem 4** Let $f \in \mathbb{C}[s]$ be a polynomial of degree $d$. The solutions of the linear differential equation

$$f(D)\xi = 0, \quad \xi \in G(I)$$

are given by the formula

$$\xi = E(\frac{sr}{f}),$$

where $r$ runs over the polynomials in $\mathbb{C}[s]$ of degree $\leq d - 1$.

**Proof.** It is easily seen that all these functions are solutions. Indeed,

$$f(D)\xi = f \frac{sr}{f}1 = sr1 = 0.$$

To show the converse, assume that $\xi$ is a solution of our equation and assume that a Mikusinski function $s^nx$ represents it. Then

$$fs^nx \in N(I).$$

This means that there exist complex numbers $a_0, a_1, \ldots, a_k$ such that

$$fs^nx = s(a_0 + a_1s + \cdots + a_ks^k)1.$$

Multiplying both sides by $t^{d+n}$, from this we get

$$(ft^d)x = (a_0t^{d+n-1} + \cdots + a_{d+n-1})1 + (a_{d+n}s + \cdots)1.$$

Since the left side is an ordinary continuous function, we can see that $a_k = 0$ for all $k \geq d + n$.

We therefore have

$$s^n x = \frac{s(a_0 + a_1s + \cdots + a_{d+n-1}s^{d+n-1})}{f}1.$$

By the Euclidean division theorem, there exists a polynomial $r$ of degree $\leq d - 1$ such that

$$\frac{a_0 + a_1s + \cdots + a_{d+n-1}s^{d+n-1}}{f} \equiv \frac{r}{f} \mod \mathbb{C}[s].$$

So that

$$\xi = E(\frac{sr}{f}).$$

The proof is complete. □
References

[1] J. Mikusinski, Operational Calculus, Pergamon Press, London, 1959.

[2] R. Remmert, Theory of Complex Functions, Springer-Verlag, New York, 1991.

[3] L. Schwartz, Généralisation de la notion de fonction, dérivation, de transformation de Fourier et applications mathématiques et physiques, Annales Univ. Grenoble 21 (1945) 57-74.