Multipartite entanglement after a quantum quench

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Abstract. We study the multipartite entanglement of a quantum many-body system undergoing a quantum quench. We quantify multipartite entanglement through the quantum Fisher information (QFI) density and we are able to express it after a quench in terms of a generalized response function. For pure state initial conditions and in the thermodynamic limit, we can express the QFI as the fluctuations of an observable computed in the so-called diagonal ensemble. We apply the formalism to the dynamics of a quantum Ising chain, after a quench in the transverse field. In this model the asymptotic state is, in almost all cases, more than two-partite entangled. Moreover, starting from the ferromagnetic phase, we find a divergence of multipartite entanglement for small quenches closely connected to a corresponding divergence of the correlation length.
1. Introduction

Over the last decade it has been established that a large body of information concerning quantum many-body systems can be extracted from the study of their entanglement properties \[1, 2, 3\]. At present, there are many examples of this connection spanning a wide spectrum of phenomena in quantum statistical mechanics and condensed matter physics. Entanglement, for example, is tightly connected to the topological properties of many-body systems \[4, 5\] and to the emergence of quantum phase transitions \[6\]. In particular, the entanglement entropy, which obeys an area law for gapped systems, is known to acquire logarithmic corrections at criticality \[7, 8\].

Understanding entanglement is also very useful for the description of the non-equilibrium dynamics of quantum many-body systems \[9\]. For example, while the block entropy \(S\) in the ground state of a one-dimensional system either saturates or grows logarithmically as a function of block length, Calabrese and Cardy \[10\] have shown that after a quench, as a result of dephasing, \(S\) obeys a volume law after a linear increase with time. The increase with time is slower in the presence of disorder \[11\], with a distinct logarithmic behaviour characterizing the many-body localised states \[12, 13, 14\]. These considerations can be generalised to the case of linear ramps \[15, 16, 17\], relevant for the Kibble-Zurek-type experiments, or to the case of periodic driving \[18, 19, 20, 21, 22\].

At present, most of the studies of entanglement in many-body systems have focused on the bipartite case (either two-site or two-block as in the entanglement entropy) while much less is known about multipartite entanglement, i.e. entanglement between multiple, \(M > 2\), subsystems \[23, 24\]. Although several important works point to the importance of this concept in the understanding of collective behaviour of many-body systems \[25, 26, 27, 28, 29, 35\], the overall picture is far less clear than for bipartite entanglement. The main reason is that the quantification and classification of multipartite entanglement is fairly more complex and full of open problems (of interest also in mathematics, see e.g. \[31, 33, 36, 34, 30, 35\]). Though a complete classification is still out of sight, very promising studies of multipartite entanglement have been performed in specific many-body systems \[39, 40\]. An overview of the field can be found in the review by Gühne and Tóth \[38\].

A very appealing quantity characterizing the degree of multipartite entanglement through a bound is the Quantum Fisher Information (QFI) \[41, 42\]. While the QFI was originally introduced to quantify phase parameter estimation, it has been shown \[41, 23\] that certain types of multipartite entanglement could be inferred from its scaling with the system size (more precisely from the coefficient of the linear dependence on the size). Most importantly, the QFI has been shown to be related to a dynamical susceptibility for a system in a thermal state \[43\]. Since susceptibilities can be experimentally measured also for quite large systems, this suggests an easy way to experimentally probe multipartite entanglement and bypass existing protocols, which exponentially scale with the system size. An important consequence of this connection is that, in critical systems, the QFI inherits the scaling properties from the susceptibilities. Therefore the QFI in many-body systems shows critical scaling at a quantum phase transition like any thermodynamic quantity \[43\].

As mentioned above, while the dynamics of bipartite entanglement has received a lot of attention \[2\], the multipartite case has been much less investigated until now. The connection between QFI and susceptibilities suggests the possibility to extend the study of multipartite entanglement not only to equilibrium situations but
also to non-equilibrium. With this goal in mind, in this paper we propose the first systematic analysis of multipartite entanglement of a quantum many-body system out-of-equilibrium. We consider an isolated quantum system subject to a quantum quench protocol: a parameter of the Hamiltonian is suddenly changed and the ensuing unitary evolution is studied. We show that in the long time limit one can relate the value of the asymptotic QFI to a generalized response function, thus generalizing the results obtained by Hauke et al. [43] to a non-equilibrium situation. The properties of the dynamics of multipartite entanglement after a quantum quench will be discussed in detail in the case of a one-dimensional Ising model in a transverse field. Depending on the nature and the type of quench, multipartite correlations may play a prominent role in the dynamics of the quantum chains. In particular, we will show that the structure of entanglement in the steady state depends crucially on whether the initial condition is ferromagnetic or paramagnetic. In the first case, there is no limitation on the degree of multipartiteness achievable and the smaller the quench the higher the entanglement. In the second, the degree of multipartiteness is both limited and maximal only close to the equilibrium critical point.

The paper is organized as follows. In Section 2 we briefly review the definition and quantification of multipartite entanglement through the quantum Fisher information. Section 3 is the core of our paper: there we discuss the behaviour of the QFI after a quantum quench. We show that, under very general conditions, it relaxes to an asymptotic value given by the fluctuations of an operator over the so-called diagonal ensemble density matrix, and we relate this quantity to a generalized response function. For a thermal state, the expression we have obtained reduces to the one found by Hauke et al. [43] at thermal equilibrium. Equipped with this formalism, in Section 4 we discuss in detail the properties of the QFI density for the quantum Ising chain in transverse field, after a quench in the external field. We find that the system almost always shows multipartite entanglement; moreover the degree of multipartiteness diverges for infinitesimal quenches applied to the system in the ferromagnetic phase. There is indeed no limitation on the degree of multipartiteness which can be achieved and this is strictly connected to a divergence of the correlation length. Finally in Section 5 we draw our conclusions and discuss perspectives for future work.

2. Multipartite entanglement and quantum Fisher information

In this section, in order to keep the presentation self-contained, we briefly review the classification of multipartite entanglement and we discuss in detail how to measure it by means of the Quantum Fisher Information. In the last few years, the relation between QFI and entanglement has been used to establish a strong link between quantum metrology and quantum information science; a comprehensive review on the problem can be found in Ref [44]. In discussing the relation between multipartite entanglement and the QFI we follow the original works, Refs. [41, 23].

The structure of entanglement for a partition of a system in more than two subsystems is very rich and a complete general classification and quantification of multipartite entanglement is not yet available. Therefore, in order to use a definition of multipartite entanglement as precise as possible let us follow the approach of Ref. [39] and, considering a state of $N$ particles, start with the following definitions of $k$-producible pure states and $k$-particle entangled pure states.

**Definition: $k$-producible pure states** - A pure state $|\psi_{k\text{-prod}}\rangle$ is $k$ producible (producible...
by $k$-partite entanglement) if it can be written as $|\psi_{k\text{-prod}}\rangle = \bigotimes_{i=1}^{M_{N}} |\phi_{i}\rangle$, where $|\phi_{i}\rangle$ are non-producible states of $N_{i} \leq k$ particles (such that $\sum_{i=1}^{M_{N}} N_{i} = N$ and there is at least one $N_{i} = k$). For example, for $N = 3$ particles, $1$ producible, $2$ producible and $3$ producible states can be constructed as $|\psi_{1\text{-prod}}\rangle = |\phi_{1}\rangle \otimes |\psi_{2}\rangle \otimes |\chi_{3}\rangle$, $|\psi_{2\text{-prod}}\rangle = |\phi_{1}\rangle \otimes |\psi_{1}\rangle \otimes |\chi_{2}\rangle$, and $|\psi_{3\text{-prod}}\rangle = |\phi_{1}\rangle \otimes |\phi_{2}\rangle$ respectively.

**Definition:** genuine $k$-partite entangled pure states. A pure state $|\psi_{k\text{-ent}}\rangle$ is called $k$-partite entangled if it is $k$-producible, but not $(k-1)$-producible. Therefore, a $k$-partite entangled state can be written as a product $|\psi_{k\text{-ent}}\rangle = \bigotimes_{i=1}^{M_{N}} |\phi_{i}\rangle$, which contains at least one state $|\phi_{i}\rangle$ of $N_{i} = k$ particles which does not factorise. In the previous examples, $|\psi_{2\text{-prod}}\rangle = |\phi_{1}\rangle \otimes |\chi_{1}\rangle$ is a $2$-entangled states if $|\phi_{1}\rangle \neq |\phi_{1}\rangle \otimes |\phi_{2}\rangle$.

Similarly a Greenberger-Horne-Zeilinger state is a $3$-entangled state. These definitions are extended to mixed states by the convex combination exactly as it is done for the case of bipartite entanglement [37].

Several different possible ways to quantify multipartite entanglement have been proposed in the literature. They include geometric measures [26], [48], global entanglement measures [33], [34], [25] and there are also proposals based on concurrence [31], on the distribution of purities [36] for different bi-partitions or on the construction of appropriate monotones [37]. As already mentioned, in this work we choose to focus on a specific quantity, the Quantum Fisher Information (QFI),

Let us define and discuss in detail the QFI (we closely follow the discussion of Ref. [50]). Its classical counterpart, the Fisher information, is used in information theory to evaluate how precise can be the estimate of a parameter $\theta$, upon which a probability distribution $P(\mu|\theta)$ depends, from measurements of the random variable $\mu$ only. It is defined as the variance of the score function $\partial_{\theta} \log(P(\mu|\theta))$, i.e.

$$F = \sum_{\mu} (\partial_{\theta} \log(P(\mu|\theta)))^{2} P(\mu|\theta) = \sum_{\mu} (\partial_{\theta} P(\mu|\theta))^{2}.$$  

(1)

This quantity gives a bound on the information on $\theta$ that can be obtained from an estimator $\hat{\theta}(\mu)$. In particular one can easily prove the Cramer-Rao bound, performing $m$ measurements one gets $\langle(\Delta \theta)^{2}\rangle \geq 1/(mF)$: the higher $F$ the better can in principle be our estimate of $\theta$.

It is possible to apply the same logic to quantum systems. To construct a probability distribution in the quantum case we can, for example, prepare a probe state $\hat{\rho}$, and apply to it a unitary transformation $\hat{U}(\theta) = e^{i\theta \hat{O}}$: $\theta$ is an unknown phase shift characterizing the transformation and $\hat{O}$ is the operator that generates it. The parameter $\theta$ could now be inferred by performing a measure on the shifted probe state $\hat{\rho}(\theta) = \hat{U}^{\dagger}(\theta) \hat{\rho} \hat{U}(\theta) = e^{-i\theta \hat{O}} \hat{\rho} e^{i\theta \hat{O}}$. Typically the results are given by a POVM (positive operator valued measure [32]) with $q$ independent elements $\{\hat{E}_{\mu}\}$ and corresponding outcomes $\{\mu_{i}\} = \{\mu_{1}, \ldots, \mu_{q}\}$.

For each outcome $\mu_{i}$ we can obtain an estimator $\hat{\theta}_{\text{est}}(\mu_{i})$: performing $m$ realizations of the measurement process we obtain the average $\langle\hat{\theta}_{\text{est}}\rangle$ of the estimator on the resulting probability distribution and its variance $(\Delta \hat{\theta}_{\text{est}})^{2} = \langle\hat{\theta}_{\text{est}}^{2}\rangle - \langle\hat{\theta}_{\text{est}}\rangle^{2}$. Similarly to the classical case, this variance has been proven to obey a bound [49]:

$$\Delta \hat{\theta}_{\text{est}} \geq \frac{1}{\sqrt{mF}},$$

where $F$ is the Fisher information defined as in Eq.(1) with the probability distribution $P(\mu|\theta) = \text{Tr}(\hat{\rho}(\theta) \hat{E}_{\mu})$.

It is now possible to introduce a quantity $F_{Q}$ characterizing the usefulness of a quantum state for phase estimation given the operator $\hat{O}$. This can be done by maximizing $F$ over all possible POVMs measurements [50]: the result is defined as
the Quantum Fisher Information. Since $F \leq F_Q(\hat{O})$ the bound $\Delta \theta_{\text{est}} \geq \frac{1}{\sqrt{mF_Q(\hat{O})}}$ follows (quantum version of the Cramer-Rao bound). For pure states, the QFI is simply given by the variance of the operator that induces the phase shift, $F_Q(\hat{O}) = 4 \langle \Delta \hat{O}^2 \rangle$. If instead we consider a mixed state as an input $\hat{\rho} = \sum \alpha p_\alpha |\lambda_\alpha \rangle \langle \lambda_\alpha |$ (with $p_\alpha > 0$, $\sum_\alpha p_\alpha = 1$), the quantum Fisher information can be written in terms of the eigenvalues of the input state and of the matrix elements of the phase shift operator $\hat{O}$ as
\begin{equation}
F_Q(\hat{O}) = 2 \sum_{\alpha, \beta} \left( p_\alpha - p_\beta \right)^2 \frac{|\langle \lambda_\alpha | \hat{O} | \lambda_\beta \rangle|^2}{p_\alpha + p_\beta}.
\end{equation}

Let us now finally come to the connection between QFI and multipartite entanglement. This was thoroughly explored in Ref. [41], where the authors considered a system of $N$ $\frac{1}{2}$-spins subject to phase shift generated by $\hat{O}_{\text{lin}} = \frac{1}{2} \sum_{l=1}^{N} \mathbf{n}_l \cdot \hat{\sigma}_l$; for each $l = 1, \ldots, N$, $\mathbf{n}_l$ is a vector on the Bloch sphere and $\hat{\sigma}_l = (\hat{\sigma}_l^x, \hat{\sigma}_l^y, \hat{\sigma}_l^z)$ is the vector of the Pauli matrices associated to the spin $l$. The authors found an inequality relating the multipartite entanglement properties of the considered state and the QFI optimised over all the possible choices of $\hat{O}_{\text{lin}}$. For $k$-producible states, the result is $F_Q[\rho_{\text{k-prod}}] \leq sk^2 + r^2$, where $s = \lfloor \frac{s}{k} \rfloor$ (and $\lfloor x \rfloor$ is the largest integer smaller or equal $x$) and $r = N - sk$. Given a probe state $\hat{\rho}$, if the bound is violated, then the probe state contains useful $(k+1)$-partite entanglement. When $k$ is a divisor of $N$ the bound further simplifies if expressed in terms of the optimal Quantum Fisher Information density, defined as $f_Q \equiv \frac{F_Q}{N}$: if $f_Q > k$, the state is at least $(k+1)$-multipartite entangled. In the next Sections, we are going to apply this inequality to the study of the multipartite entanglement in a system subjected to a quantum quench.

3. Quantum Fisher Information out-of-equilibrium

The purpose of this work is to study multipartite entanglement in a many-body system in non-equilibrium conditions. In particular, we will consider the dynamics of thermally isolated quantum systems following a quantum quench, i.e. a rapid change of the system parameters. The system is initialised in a given (possibly mixed) many-body state, and is let free to evolve in time under the action of an Hamiltonian $\hat{H}$. In the thermodynamic limit, local observables and correlation functions are expected to attain a stationary value at long times. This eventual stationary condition is described by the diagonal ensemble, which can be obtained as the infinite-time-average of the density matrix.

In order to characterize multipartite entanglement both in the transient and in the stationary state, we now aim at studying the QFI in such conditions. It has been shown that lower bounds on the QFI can be computed in terms of few observable quantities [45]. Moreover, in the special case of thermal equilibrium, the QFI can be expressed in terms of a dynamical response function [43]: it would be highly desirable to have a similar expression for a generic non-equilibrium situation. Below, we generalize the result of Ref. [43] to a many-body system subject to a quantum quench, and show that also in this case the QFI can be expressed in terms of a generalized response function of the operator $\hat{O}$ generating the phase shift.
In order to obtain this result, let us start by choosing a basis for the initial state that diagonalizes the density matrix \( \hat{\rho} = \sum_{\alpha} p_{\alpha} |\lambda_{\alpha}\rangle\langle \lambda_{\alpha}| \). If the initial state is a thermal one relative to the initial Hamiltonian \( \hat{H}_0 \), then \( \hat{H}_0 |\lambda_{\alpha}\rangle = E_{\alpha}^0 |\lambda_{\alpha}\rangle \) and \( p_{\alpha} = e^{-\beta E_{\alpha}^0} / Z \) is the standard Gibbs weight. The state is then time evolved with the final Hamiltonian \( \hat{H} \), which leads to

\[
\hat{\rho}(t) = \sum_{\alpha} p_{\alpha} |\lambda_{\alpha}(t)\rangle\langle \lambda_{\alpha}(t)| = \sum_{i,j} a_{ij} e^{-i(E_i - E_j)t} |\psi_i\rangle\langle \psi_j| , \tag{4}
\]

where \( |\lambda_{\alpha}(t)\rangle = e^{-i\hat{H}t} |\lambda_{\alpha}\rangle \) and \{\( |\psi_i\rangle \) and \{\( E_i \)\} are the eigenvectors and eigenvalues of \( \hat{H} \). In particular, \( a_{ij} \equiv \sum_{\alpha} p_{\alpha} \langle \psi_i | \lambda_{\alpha} \rangle \langle \lambda_{\alpha} | \psi_j \rangle \). Using Eq. (2) we can write the quantum Fisher information at time \( t \) as

\[
F_Q(\hat{O}, t) = 2 \sum_{\alpha, \beta} \frac{(p_{\alpha} - p_{\beta})^2}{p_{\alpha} + p_{\beta}} |\langle \lambda_{\alpha}(t)|\hat{O}|\lambda_{\beta}(t)\rangle|^2 . \tag{5}
\]

Focusing now on thermal initial states and using the identity \((p_{\alpha} - p_{\beta})/(p_{\alpha} + p_{\beta}) = \tanh(\beta(E_{\alpha}^0 - E_{\beta}^0)/2)\) it is easy to show that

\[
F_Q(\hat{O}, t) = \frac{4}{\pi} \int_0^{+\infty} d\omega \ \tanh\left[ \frac{\beta \omega}{2} \right] \tilde{\chi}''(t, \omega) , \tag{6}
\]

where

\[
\tilde{\chi}''(t, \omega) = \pi \sum_{\alpha, \beta} (p_{\alpha} - p_{\beta}) |\langle \lambda_{\alpha}(t)|\hat{O}|\lambda_{\beta}(t)\rangle|^2 \delta(\omega + E_{\alpha}^0 - E_{\beta}^0) .
\]

In particular \( \tilde{\chi}''(t, \omega) = -\text{Im}[\tilde{\chi}(t, \omega)] \) where the latter is the Fourier transform with respect to \( \tau \) of the generalized retarded correlation function

\[
\tilde{\chi}(t, \tau) = -i \delta(\tau) \text{Tr} \left[ \hat{\rho} \left[ \hat{O}(t, \tau), \hat{O}(t, 0) \right] \right] , \tag{7}
\]

where \( \hat{O}(t, \tau) = e^{i\hat{H}t} e^{i\hat{H}(\tau)} e^{-i\hat{H}t} e^{-i\hat{H}(\tau)} \).

The previous equations are one of the main results of this paper and generalize the equilibrium results obtained in Ref. [43] to the case of a quantum quench. Notice that in general Eq. (7) is the linear response function at time \( \tau \) of the operator \( \hat{O}(t, \tau) \equiv e^{i\hat{H}(\tau)} e^{-i\hat{H}t} e^{i\hat{H}(\tau)} \). In the case of thermal equilibrium (\( \hat{H}_0 = \hat{H} \)), one straightforwardly obtains that \( \tilde{\chi}(t, \tau) = -i \delta(\tau) \text{Tr} \left[ \hat{\rho} \left[ \hat{O}(t, \tau), \hat{O}(t, 0) \right] \right] \); the QFI comes from the imaginary part of the standard response function associated to the phase-shift operator \( [43] \).

In the equilibrium case, the validity of the fluctuation dissipation theorem is crucial. Out-of-equilibrium, where in general this theorem does not hold, the QFI cannot be written as a dynamical susceptibility, as discussed in Appendix A. Moreover, whenever the initial state is a pure state, as for quenches at zero temperature, we can easily obtain from Eq. (8) that

\[
F_Q(\hat{O}, t) = 4 \langle (\Delta \hat{O}(t))^2 \rangle . \tag{8}
\]

Equations (5) and (8) allow us to study the Quantum Fisher Information both in the transient and in the stationary state attained after a quantum quench. In order to find an explicit formula for the stationary state QFI, let us focus on the zero temperature case. In this case, considering the thermodynamic limit, we can show that for most systems

\[
F_Q(\hat{O}, \infty) = 4 \text{Tr}[\hat{O}^2 \hat{\rho}_0] - 4 \text{Tr}[\hat{O} \hat{\rho}_0]^2 = 4 \langle (\Delta \hat{O})^2 \rangle , \tag{9}
\]
where we have introduced the diagonal ensemble \[ \hat{\rho}_d \equiv \sum_i |\langle \lambda_0 | \psi_i \rangle|^2 |\psi_i \rangle \langle \psi_i |. \] (10)

In order to prove this, let us start from the observation that if \( F_Q(\hat{O}, t) \) attains a stationary value as \( t \to \infty \) then this is going to be given by its time average \( \overline{F_Q(\hat{O})} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T dt F_Q(\hat{O}, t) \). Taking explicitly the time average of Eq. (8), we get

\[
\langle (\hat{O}(t) - \langle \hat{O}(t) \rangle)^2 \rangle = \langle (\Delta \hat{O})^2 \rangle_d - \langle \Delta \hat{O} \rangle^2,
\] (11)

where \( \langle (\Delta \hat{O})^2 \rangle_d = \langle (\hat{O} - \langle \hat{O} \rangle^2) \rangle_d \) are the fluctuations computed with respect to the diagonal ensemble, and \( \langle \Delta \hat{O} \rangle^2 = \langle (\hat{O}(t) - \langle \hat{O}(t) \rangle)^2 \rangle \) are the temporal fluctuations of the average. Notice now that if the average \( \langle \hat{O}(t) \rangle \) attains a well defined stationary value at large times then \( \langle \Delta \hat{O} \rangle^2 \to 0 \) and therefore the equality Eq. (11) follows. In order to figure out under which conditions this happens let us focus on

\[
\langle \hat{O}(t) \rangle = \sum_{i,j} c_i c_j^* \langle \psi_i | \hat{O} | \psi_j \rangle e^{-i(E_i - E_j)t},
\] (12)

where \( c_i = \langle \lambda_0 | \psi_i \rangle \). We can split the sum on the right hand side in two parts, a diagonal one

\[
\hat{O}_d = \sum_i |c_i|^2 \langle \psi_i | \hat{O} | \psi_i \rangle
\] (13)

and an off-diagonal one which can be written as

\[
\hat{O}_{\text{off-diag}}(t) = \int_{-\infty}^{\infty} F(\Omega)e^{-i\Omega t} \quad \text{with}
\]

\[
F(\Omega) \equiv \sum_{i \neq j} c_i c_j^* \langle \psi_i | \hat{O} | \psi_j \rangle \delta (\Omega - (E_i - E_j)).
\] (14)

In analogy with Refs. [51, 52], we see that in the thermodynamic limit the delta functions contained in \( F(\Omega) \) can merge making of \( F(\Omega) \) a smooth function. Formally this occurs when the point spectrum of \( \hat{H} \) becomes a continuum, i.e. in a many-body context for clean systems in the thermodynamic limit [51, 52, 53]. Under these conditions, Riemann-Lebesgue lemma applies and we see that the off-diagonal contribution Eq. (14) vanishes in the limit \( t \to \infty \), leading to relaxation to a well defined asymptotic value given by Eq. (13).

Let us conclude this section re-expressing the QFI in the stationary state in terms of the Keldysh component of the response function

\[
\chi_K(\tau, \hat{O}) = \frac{1}{2} \text{Tr} \left[ \hat{\rho}_d \{ \delta \hat{O}(\tau), \delta \hat{O} \} \right]
\] (15)

where \( \delta \hat{O} = \hat{O} - \langle \hat{O} \rangle_d \). Considering the Fourier transform and using a Lehmann representation we find

\[
\chi_K(\omega, \hat{O}) = \frac{1}{2} \sum_{i,j} |c_i|^2 \left| \delta \hat{O}_{ij} \right|^2 \left[ \delta (\omega + E_i - E_j) \right]
+ \delta (\omega + E_j - E_i),
\] (16)

Integrating over \( \omega \) it is then easy to check that

\[
F_Q(\hat{O}, \infty) = 4(\Delta \hat{O}^2)_d = 4 \int_{-\infty}^{\infty} \chi_K(\omega, \hat{O}) d\omega.
\] (17)
4. Multipartite entanglement in the Ising model after a quantum quench

After having introduced the necessary formalism we now move to the discussion of multipartite entanglement in the specific case of a quantum quench in a one-dimensional Ising model \([46, 54]\) described by the Hamiltonian

\[
\hat{H}(t) = -\frac{1}{2} \sum_{j=1}^{L} \left( J \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + g(t) \hat{\sigma}_j^z \right).
\]

(18)

Here \(\hat{\sigma}_j^x, z\) are spin operators at site \(j\) of a chain of length \(L\) with boundary conditions which can be periodic (PBC) \(\hat{\sigma}_{L+1}^x = \hat{\sigma}_1^x\) or open (OBC) \(\hat{\sigma}_{L+1}^z = 0\), and \(J\) is a longitudinal coupling (\(J = 1\) from now on). This model has two gapped phases: a ferromagnet (\(|g_0| < 1\)) and a paramagnet (\(|g_0| > 1\)), separated by a quantum phase transition at \(g_c = 1\).

Thanks to the Jordan-Wigner mapping \([46, 54]\), this system can be shown to be equivalent to a non-interacting fermion model. Integrability allows to easily address not only the statics, but also the dynamics after a quantum quench, as elucidated, for instance, in Ref. \([10]\). Consistently with integrability, after a quantum quench in the thermodynamic limit, all the local observables have been demonstrated to asymptotically relax \(\dagger\) to a condition described by a generalized Gibbs ensemble (GGE) \([55, 56, 57, 58, 59, 60, 61, 62]\) (i.e. the density matrix maximizing the entropy provided all the constants of motion of the integrable system are conserved). The GGE is the form acquired in this case by the diagonal ensemble we are interested in (Eq. (10)): using the results of Refs. \([55, 56]\), we can explicitly use it and evaluate the asymptotic QFI through Eq. (11) and following.

In order to estimate the multipartite entanglement \([41, 23]\), we optimize the QFI density over operators of the form

\[
\hat{O}_{\text{lin}} = \frac{1}{2} \sum_{l=1}^{N} n_l \cdot \hat{\sigma}_l,
\]

(19)

where \(n_l\) are vectors with unit norm and \(\hat{\sigma}_l = (\hat{\sigma}_l^x, \hat{\sigma}_l^y, \hat{\sigma}_l^z)\) is the vector of the Pauli matrices associated to the spin \(l\). We focus on a translationally invariant system without antiferromagnetic order and our dynamics starts with uniform initial conditions, therefore in the following we will assume \(n_l = n\) and optimize over its direction. Substituting the expression for \(\hat{O}_{\text{lin}}\) in the expression for the QFI, we get

\[
F_Q(\hat{O}_{\text{lin}}) = 4 \sum_{\alpha=x,y,z} (n^\alpha)^2 \langle \Delta(\hat{S}^\alpha)^2 \rangle
\]

(20)

where we have defined the operators \(\hat{S}^\alpha = \frac{1}{2} \sum_{j=1}^{N} \hat{\sigma}_j^\alpha\). We note that in Eq. (20) terms of the form \(n_\alpha n_\beta \left( \left\langle \hat{S}^\alpha \hat{S}^\beta \right\rangle - \left\langle \hat{S}^\alpha \right\rangle \left\langle \hat{S}^\beta \right\rangle \right)\) with \(\alpha \neq \beta\) are always vanishing if they are not present in the initial state (the symmetry of the Hamiltonian that governs the evolution prevents the build up of such correlations). We finally optimize Eq. (20) over all the possible directions of the unit norm vector \(n\) and obtain that the optimal

\(\dagger\) In order to get this result it is enough to show that all the two-point fermionic correlation functions undergo such a relaxation. Being the Hamiltonian quadratic and the state Gaussian, this implies asymptotic relaxation for all the local observables.

\(\S\) This can be done, for instance, by representing \(n\) in polar coordinates. Another possible way is to look at Eq. (20) as the expectation over an unit vector of a \(3 \times 3\) Hermitian matrix. This is maximized taking \(n\) as the eigenvector with maximum eigenvalue and the maximum is given by this eigenvalue.
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QFI can be written as

\[ F_Q = 4 \max_{\alpha=x,y,z} \left[ \left\langle \Delta (\hat{S}^\alpha)^2 \right\rangle \right] = \max_{\alpha=x,y,z} F_Q(\hat{S}^\alpha). \] (21)

In particular, \( \hat{S}^x / L \) is the order parameter of the ferromagnetic transition: when the thermodynamic limit is considered, in the phase \( g_0 < 1 \), its expectation over the ground state is non-vanishing \[54\].

Below, we restrict our attention to the behaviour of the QFI density in the diagonal ensemble; we will give later some hint about the evolution towards this stationary condition. It is now possible to evaluate the Quantum Fisher Information density

\[ f_Q(\hat{S}^\alpha, t) = 1 + 2 \sum_{n=1}^{L-1} G_n^\alpha(t) \]

\[ \xrightarrow{t \to \infty} f_Q(\hat{S}^\alpha, \infty) = 1 + 2 \sum_{n=1}^{\infty} G_n^\alpha(\infty). \] (22)

where we have defined \( G_n^\alpha(t) = \langle \sigma_j^\alpha \sigma_{j+n}^\alpha \rangle_t \) as the connected spin correlation function at time \( t \) and we have and exploited the translational invariance of the problem and the inversion symmetry \( G_n^\alpha(t) = G_{-n}^\alpha(t) \). The asymptotic condition is reached only in the thermodynamic limit (see Appendix B for a detailed discussion) and it is given in terms of the diagonal-ensemble GGE density matrix Eq. (13): \( G_n^\alpha(\infty) = \langle \sigma_j^\alpha \sigma_{j+n}^\alpha \rangle_d \).

It is possible to explicitly evaluate this correlator: we need to use the Jordan-Wigner transformation in order to write the \( \hat{\sigma}_j^\alpha \) operators in terms of the fermionic ones; then we have to apply the Wick theorem to the resulting expectation of a string of fermionic operators on the Gaussian asymptotic GGE state \( \hat{\rho}_d \) \[55\]. The details of the calculation can be found in Refs. \[63, 65\].

As we will show in the rest of this Section, multipartite entanglement has a prominent role in the steady state after a quantum quench. Depending on the value of the final external field, it may involve a macroscopic number of spins. In Fig.1 we summarize our results for the asymptotic state. We find that the attained state after a quantum quench is never separable. Moreover, we notice a strong dependence on the initial conditions: this is not surprising given the integrability of the system. For quenches starting from the ferromagnetic phase the QFI density diverges as \( \delta g \sim \delta g_f \) for the so-called small quenches, i.e. \( \delta g = g_f - g_0 \to 0 \). In the paramagnetic phase the multipartiteness is limited and maximal at the equilibrium critical point.

In the next sections we will present the details on the calculation of \( f_Q(\hat{S}^\alpha, \infty) \) with \( \alpha = x, y, z \) and comment the behaviour of the QFI density as a function of time. Furthermore we discuss the entanglement divergence in the ferromagnetic phase, through a perturbative expansion in \( g_f - g_0 \).

4.1. Exact results for the asymptotic QFI for \( g_0 = 0 \) and \( g_0 = \infty \)

In the two limiting cases of quenches starting from \( g_0 = \infty \) (fully polarised initial state) or \( g_0 = 0 \) (maximally ordered classical point), the asymptotic correlation function after the quench can be expressed in very nice and simple forms (see Ref. \[66\] for the details of the calculation) leading, for \( g_0 = \infty \), to
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Figure 1. We plot the maximum over the three total spin components $\hat{S}^x, \hat{S}^y$ and $\hat{S}^z$ from Eq. (21) for different values of $g_0$. It is evident a strong dependence on the initial conditions: $f_Q(\infty)$ it is limited when $g_0$ is in the paramagnetic phase, whereas it diverges as $g_0^{-2}$ for $g_0$ in the ferromagnetic one (see Secs. 4.2-4.3). In the inset we plot $f_Q$ in a larger scale, in order to emphasize this divergent behavior in the ferromagnetic phase.

$$f_Q(\hat{S}^x, \infty) = \begin{cases} 
\frac{3}{5-4g_f} & \text{for } g_f \leq 1, \\
\frac{2g_f+1}{2g_f-1} & \text{for } g_f \geq 1;
\end{cases} \tag{23}$$

while in the opposite limit, when $g_0 = 0$, we find

$$f_Q(\hat{S}^x, \infty) = \begin{cases} 
\frac{5-5g_f^2}{g_f} & \text{for } 0 < g_f \leq 1, \\
3 & \text{for } g_f \geq 1.
\end{cases} \tag{24}$$

It is possible to analytically compute also the expression for the QFI relative to the magnetization $\hat{S}^z$, obtaining for $g_0 = 0$

$$f_Q(\hat{S}^z, \infty) = \begin{cases} 
3 + g_f^4 - \frac{5}{2}g_f^2 & \text{for } g_f \leq 1, \\
\frac{3}{2} & \text{for } g_f \geq 1;
\end{cases} \tag{25}$$
Figure 2. Exact results for the the asymptotic $f_Q$ obtained for the three total spin components $\hat{S}^x$, $\hat{S}^y$ and $\hat{S}^z$. The plot shows the behaviour of $f_Q$ as a function of the final field $g_f$ starting from the totally ordered state $g_0 = 0$ (upper panel) and the fully polarised one $g_0 = \infty$ (lower panel).

on the opposite side, for $g_0 = \infty$, the QFI reads

$$
\begin{align*}
\hat{S}^z, \infty \quad \Rightarrow \quad f_Q(\hat{S}^z, \infty) &= \begin{cases} 
\frac{1}{2} & \text{for } g_f \leq 1 \\
1 + \frac{1}{2g_f} & \text{for } g_f \geq 1.
\end{cases}
\end{align*}
$$

(26)

These functions, together with the corresponding $f_Q(\hat{S}^y, \infty)$ with the same initial conditions, are plotted in Fig. 2. Information about the degree of multipartiteness of
the entanglement can be obtained from the maximum of these three functions, which is \( f_Q(S^e, \infty) \) in almost all cases, with the exception of \( g_f < 3/4 \) for \( g_0 = +\infty \), where \( f_Q(S^z, \infty) \) dominates.

On the basis of these data, we can therefore infer that for a maximally ordered initial condition (\( g_0 = 0 \)), when the final field is larger than the critical value \( g_f > 1 \), all the final states display at least tripartite entanglement \( f_Q(\hat{S}^z, \infty) = 3 \forall g_f \). When \( 0 < g_f \leq 1 \), the QFI density is greater than three and the smaller the quench is, the higher the degree of entanglement. This suggests that the best achievable multipartiteness of entanglement is obtained for infinitesimal quenches (\( g_f = g_0 + \epsilon \)).

We are going to see in the next subsection that this is a general feature for \( g_0 \leq 1 \).

For ferromagnetic initial conditions, indeed, there is no limitation on the degree of multipartiteness achievable.

When the initial condition is fully polarised (\( g_0 = +\infty \)), instead, the structure of entanglement in the steady state is completely different and the multipartiteness is limited and reaches a maximum when the quench ends at the equilibrium critical point. In particular, while for \( g_f < 3/4 \) the entanglement is at least bipartite, if we get closer to the critical point the degree of entanglement grows as well; in particular there is a region (\( 7/8 < g_f < 3/2 \), Fig. 2) where the entanglement is at least tripartite.

For even larger \( g_f (g_f > 3/2) \) we have \( f_Q > 1 \): in this interval the entanglement finally lapses back to be at least bipartite.

4.2. Numerical results for generic \( g_0 \)

Let us now discuss to what extent the features observed above for quenches starting in maximally ordered/disordered states are generic. In order to do so, let us consider the case of generic \( g_0 \) and present the results of a numerical evaluation of the QFI. The asymptotic condition is reached only in the thermodynamic limit, which is numerically attained along the lines described in Appendix B.

Let us start by focusing on \( S^e \). As in the extreme cases described above, the results are significantly different depending on the nature of the initial state. In particular, whenever \( g_0 > 1 \), i.e. the initial condition is paramagnetic, there is a peak at \( g_f = 1 \) which tends to become a divergence in the limit \( g_0 \to 1 \) (see Fig. 3). Therefore, for final \( g_f \) close to criticality, the multipartite structure of entanglement is maximised. This is in sharp contrast with the case of ferromagnetic initial conditions: here the entanglement is the more multipartite the smaller the quench. Indeed, we can see a divergence of the QFI density for \( g_f \to g_0 \) (the so-called small quench regime \( \delta g \to 0 \)), which behaves as \( \sim \delta g^{-2} \) (see Fig. 3). This divergence is linked to the behaviour of the correlation length of the spin correlation function as discussed below.

Therefore the two main observed features are robust: we see high degree of multipartiteness close to criticality for \( 0 \leq g_0 < 1 \) (system initially in the paramagnetic phase) and diverging multipartiteness for small quenches when \( g_0 \geq 1 \) (system initially in the ferromagnetic phase). This statement is further corroborated by comparing the graphs for \( f_Q(S^e, \infty) \) (Figs. 3 and 4) to those in Fig. 5 and 6, where we show the numerical results for the QFI related to \( \hat{S}^e \) and \( \hat{S}^z \), for generic \( g_0 \). We see that, for \( g_0 > 1 \), there is an interval of \( g_f \) (one of the extrema is 0) where the optimal asymptotic QFI density is given by the \( \hat{S}^z \) QFI density, exactly as happens for \( g_0 = \infty \) (lower panel of Fig. 2).
4.3. Perturbative approach to the small-\(\delta g\) divergence for quenches with \(g_0 < 1\)

The divergence for \(g_0 < 1\) can be understood in terms of the corresponding divergence of the correlation length. Consider a ferromagnetic initial condition: for a whatsoever small quench with \(g_f \neq g_0\) the system will not be able to sustain a finite order parameter in the stationary state, i.e. \(\langle \hat{S}_x \rangle = 0\), while the correlations of the order parameter will exponentially decay according to \(G_n(\infty) \simeq e^{-n/\xi}\). Therefore, using Eq.\,(22), we can estimate for large \(\xi\) the asymptotic QFI as

\[
f_Q(\hat{S}_x, \infty) \sim 1 + 2\xi. \tag{27}\]

For \(g_f \rightarrow g_0\), the correlation length diverges, as we can perturbatively find: following Ref.\,[55], the correlator behaves as \(G_n(\infty) \simeq e^{-n/\tilde{\xi}}\) with

\[
\frac{1}{\tilde{\xi}} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \log \left( \frac{g_0 g_f - (g_f + g_0) \cos k + 1}{E_k(g_0) E_k(g_f)} \right), \tag{28}\]

where \(E_k(g) = \sqrt{1 + g^2 - 2g \cos(k)}\) is the dispersion of the Ising quasiparticles \([54]\).

Expanding this expression up to second order in \(\delta g\) we find

\[
\frac{1}{\tilde{\xi}} = \delta g^2 \frac{1}{2\pi} \int_{0}^{\pi} dk \frac{\sin^2 k}{E_k^2(g_0)} + O(\delta g^3). \tag{29}\]

Here we can define \(\tilde{\xi}(g_0)^{-1} = \frac{1}{2\pi} \int_{0}^{\pi} dk \frac{\sin^2 k}{E_k^2(g_0)} = \frac{1}{4(1-g_0)^2}\) for \(g_0 \neq 1\). This proves therefore that the correlation length for small quenches within the ferromagnetic phase
Figure 4. $\hat{S}_x$ QFI density for the asymptotic state as a function of the final field for different values of the initial field in the ferromagnetic phase. We consider $g_0 = 0.5$ (blue plot with stars) evaluated for $L = 1200$ and $g_0 = 0.8$ (red plot with boxes) evaluated with $L = 1200$. The numerical results are compared with the exact evaluation in the case of $g_0 = 0$ (dashed black line in the plot).

Figure 5. $\hat{S}_y$ QFI density for the asymptotic state as a function of the final field $g_f$ for different initial $g_0$ and $L = 400$. 
diverges like
\[ \xi \sim \frac{\tilde{\xi}(g_0)}{\delta g^2}. \] (30)

Using now Eq. (27) we obtain \( f_Q(\hat{S}_x, \infty) \sim 1/(\delta g)^2 \), as expected.

4.4. Time dependence of the QFI density

We conclude this Section briefly commenting on the behaviour of the QFI density for the order parameter as a function of time. Here, the correlator is no more given by a Toeplitz determinant as in the asymptotic case and we need to evaluate a Pfaffian of a more complex correlation matrix, as elucidated in Ref. [63] (from a numerical point of view we use the algorithms and the routines introduced in Ref. [64]).

From the exact expression of the correlation functions we can also extract how the QFI attains its asymptotic value. Since most of the phase diagram is dominated by \( \hat{S}_x \) let us discuss its Fisher information only (the other cases are qualitatively very similar). We report here in Fig. 7 some results for \( g_0 = 0 \) and in Fig. 8 some for \( g_0 = \infty \). In the first case we see a peak in the entanglement at short times whose height increases with \( g_f \). We find that the QFI density tends to the asymptotic value oscillating around it with an amplitude decreasing as a power law. From the Fourier transform of the signal it can be clearly seen that the frequency of these oscillations equals the quasi-particle gap \[ \Delta E_0(g_f) = 2|1 - g_f| \] of the final Hamiltonian.

5. Conclusions and perspectives

In conclusion, we have studied the multipartite entanglement in a quantum system subjected to a quantum quench. Probing multipartite entanglement through the
Figure 7. $S^x$ QFI density dynamics for a quench from $g_0 = 0$ to $g_f = 0.9$, $g_f = 2$. The size of the chain is set to $L = 100000$.

Figure 8. $S^x$ QFI density dynamics for a quench from $g_0 = \infty$ to $g_f = 2$, $g_f = 1.5$ and $g_f = 0.6$. The size of the chain is set to $L = 10^6$. 
Quantum Fisher Information, we have found an expression of the latter in terms of a generalized correlation function. Considering a quench starting from a pure state and taking the system in the thermodynamic limit, we have demonstrated that the QFI relaxes to an asymptotic value given by the fluctuations of an operator in the diagonal ensemble. We have then discussed in detail the structure of entanglement in the stationary condition attained by a specific model: the quantum Ising chain after a quench in the transverse field. We have found two different scenarios, depending on the initial condition being ferromagnetic or paramagnetic. In the first case, there is no limitation on the degree of multipartiteness achievable: the smaller the quench the higher the entanglement. In the second, the degree of multipartiteness is limited (while it tends to diverge as \( g_0 \to 1 \)) and attains a maximum only close to the equilibrium critical point \( g_f \simeq 1 \).

A possibility of future work will be to study multipartite entanglement in periodically driven systems, both in the asymptotic condition [53] and in the Floquet ground state [67, 68] (the latter is an eigenstate of the stroboscopic dynamics which can undergo quantum phase transitions). In addition, it would be interesting to address the quantum Fisher information in disordered systems, both in connection with quantum phase transitions at equilibrium [90] and many body localization [70]. Especially in the latter case, there are detailed analyses of bipartite entanglement [12, 71, 72] but an analysis of the multipartite case, with the notable exception of Refs. [73, 74], is still missing.

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Appendix A. Fisher information and susceptibility out of equilibrium

We have found that after a quantum quench the QFI can be expressed in terms of a generalized response function of the operator \( \hat{O} \) generating the shift, Eq. (1). In Ref. [43], considering a quantum many body system at equilibrium, the authors show that QFI can be written as an integral of the linear response susceptibility of the operator \( \hat{O} \). This is a very important result: it allows to measure the multipartite entanglement in the laboratory (there are well established experimental methods to measure susceptibilities independently of the size of the considered system). So one would be tempted to directly generalize this result, at least for the stationary state, with a susceptibility averaged over the diagonal ensemble [9]. Nonetheless, far from equilibrium, this would be wrong for a general system even for a pure initial state. This follows from the fact that the fluctuation-dissipation theorem [75], which at equilibrium relates the Keldysh component of the response function to the linear susceptibility as \( \chi''(\omega) = \tanh \left( \frac{\omega}{2T} \right) \chi^K(\omega) \), is not valid in this general form in the non-equilibrium case [76]. The susceptibility on the diagonal ensemble, written in Lehmann representation

\[
\chi''(\omega) = \pi \sum_{ij} (|c_i|^2 - |c_j|^2)|\mathcal{O}_{ij}|^2 \delta(\omega - (E_j - E_i))
\]
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can not be related in general to the Keldysh component of Eq. (17).

Appendix B. Numerics and the thermodynamic limit

![Figure B1. Convergence on the \( f_Q(S^x, \infty) \) as increasing the length of the chain \( L \). We consider quenches within the ferromagnetic phase, where the correlation length diverges with \( g_f - g_0 \to 0 \). We take \( g_0 = 0.5 \) and \( g_f = 0.67 \) (purple line) and \( g_f = 0.7 \) (green line).](image)

As we have discussed, the stationary state is well defined only in the thermodynamic limit. So, in the numerical evaluation of the asymptotic QFI (see Eq. (22)), we have to be sure that this limit is reached \( \parallel \). For each \( g_0 \) and \( g_f \), we choose \( L \) in order to satisfy \( L \gg \xi(g_0, g_f) \). When this condition is met, the thermodynamic limit is reached and the asymptotic QFI is well defined.

As an example, in Fig. B1 we show the convergence of \( f_Q(S^x, \infty) \) to its well-defined thermodynamic limit value, by increasing the length of the chain \( L \). The value becomes constant above a certain value of \( L \). Notice that in the ferromagnetic phase, since \( \xi \sim \delta g^{-2} \), for smaller \( \delta g \) one should consider longer chains.

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\( \parallel \) For the results of Section 4.1, this limit is exact since the correlators are analytic and the series converges.
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