EINSTEIN METRICS WITH PRESCRIBED CONFORMAL INFINITY ON 4-MANIFOLDS

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Abstract. This paper considers the existence of conformally compact Einstein metrics on 4-manifolds. A reasonably complete understanding is obtained for the existence of such metrics with prescribed conformal infinity, when the conformal infinity is of positive scalar curvature. We find in particular that general solvability depends on the topology of the filling manifold. The obstruction to extending these results to arbitrary boundary values is also identified. While most of the paper concerns dimension 4, some general results on the structure of the space of such metrics hold in all dimensions.

0. Introduction.

This paper is concerned with the existence of conformally compact Einstein metrics on a given manifold $M$ with boundary $\partial M$. The main results we obtain are restricted to dimension 4, although some of the results hold in all dimensions. This existence problem was raised by Fefferman and Graham in [13] in connection with a study of conformal invariants of Riemannian manifolds. More recently, the study of such metrics has become of strong interest through the AdS/CFT correspondence, relating gravitational theories on $M$ with conformal field theories on $\partial M$, c.f. [29], [12] and references therein.

Let $M$ be a compact, oriented manifold with non-empty boundary $\partial M$. We always assume that $M$ is connected, but $\partial M$ may be connected or disconnected. A defining function $\rho$ for $\partial M$ in $M$ is a non-negative function, at least $C^1$, on the closure $\bar{M} = M \cup \partial M$ such that $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on $\partial M$. A complete Riemannian metric $g$ on $M$ is conformally compact if there is a defining function $\rho$ such that the conformally equivalent metric

$$\tilde{g} = \rho^2 \cdot g \quad (0.1)$$

extends at least continuously to a Riemannian metric $\tilde{g}$ on $\bar{M}$. The metric $\gamma = \tilde{g}|_{\partial M}$ induced on $\partial M$ is the boundary metric induced by $g$ and the compactification $\rho$. Any compact manifold with

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boundary carries many conformally compact metrics; for instance, one may let \( \tilde{g} \) be any smooth metric on \( \bar{M} \) and define \( g \) by \( g = \rho^{-2} \cdot \tilde{g} \), for any choice of defining function \( \rho \).

Defining functions for \( \partial M \) are unique only up to multiplication by positive \((C^1)\) functions on \( \bar{M} \) and hence the compactification \((0.1)\) is not uniquely determined by \((M, g)\). On the other hand, the conformal class \([\gamma]\) of a boundary metric is uniquely determined by \(g\); the class \([\gamma]\) is called the conformal infinity of \((M, g)\).

A conformal compactification is called \( C^{m,\alpha} \) or \( L^k, p \) if the metric \( \tilde{g} \) extends to a \( C^{m,\alpha} \) or \( L^k, p \) metric on the closure \( \bar{M} \); here \( C^{m,\alpha} \) and \( L^k, p \) are the usual Hölder and Sobolev function spaces.

If \((M, g)\) is conformally compact and in addition, \(w.r.t.\) the compactification \( \tilde{g} \) in \((0.1)\),
\[
|\nabla \rho| \rightarrow 1
\]
uniformly on approach to \( \partial M \), then a simple computation shows that the sectional curvatures of \( g \) tend uniformly to \(-1\) at infinity in \((M, g)\), c.f. the Appendix for example. Such a metric is naturally called asymptotically hyperbolic (AH) and again these metrics exist in abundance on any compact \( n \)-manifold. For instance, for \( \tilde{g} \) a smooth metric on \( \bar{M} \) as above, let \( \rho \) be the distance function to \( \partial M \) \( w.r.t. \) \( \tilde{g} \), smoothed outside a tubular neighborhood \( U \) of \( \partial M \). Then by definition, \(|\nabla \rho| \equiv 1\) in \( U \) and so \( g = \rho^{-2} \cdot \tilde{g} \) is AH.

In this paper, we consider complete conformally compact Einstein metrics \( g \) on \( n \)-manifolds \( M \), normalized so that
\[
\text{Ric}_g = -(n-1) \cdot g.
\]
It is easy to see that the Einstein condition \((0.3)\) necessarily forces \((0.2)\), for any smooth defining function \( \rho \), c.f. the Appendix, so that conformally compact Einstein metrics are necessarily AH.

Let \( E_{AH} = E_{AH}^{m,\alpha} \) be the space of AH Einstein metrics on \( M \) which admit a \( C^{m,\alpha} \) compactification \( \tilde{g} \) as in \((0.1)\). We require that \( m \geq 2, \alpha \in (0,1) \) but otherwise allow any value of \( m \), including \( m = \infty \) or \( m = \omega \). Let \( \mathcal{E}_{AH} = E_{AH} / \text{Diff}_1(M) \), where \( \text{Diff}_1(M) \) is the group of orientation preserving \( C^{m+1,\alpha} \) diffeomorphisms of \( \bar{M} \) inducing the identity on \( \partial M \), acting on \( E_{AH} \) in the usual way by pullback. Next, let \( \text{Met}(\partial M) = \text{Met}^{m,\alpha}(\partial M) \) be the space of \( C^{m,\alpha} \) metrics on \( \partial M \) and \( \mathcal{C} = \mathcal{C}(\partial M) \) the corresponding space of pointwise conformal classes. There is a natural boundary map, (for any fixed \((m, \alpha)\) as above),
\[
\Pi : \mathcal{E}_{AH} \rightarrow \mathcal{C},
\]
\[
\Pi[g] = [\gamma],
\]
which takes an AH Einstein metric \( g \) on \( M \) to its conformal infinity on \( \partial M \).

We then have the following result on the structure of \( \mathcal{E}_{AH} \) and the map \( \Pi \).

**Theorem A.** Suppose \( M \) is a compact, oriented 4-manifold with boundary \( \partial M \). If \( \mathcal{E}_{AH} \) is non-empty, then \( \mathcal{E}_{AH} \) is a smooth infinite dimensional Banach manifold. Further, the boundary map
\[
\Pi : \mathcal{E}_{AH} \rightarrow \mathcal{C}
\]
is a \( C^\infty \) smooth Fredholm map of index 0.

We refer to Theorem 1.2, Proposition 1.6 and §2 for a more precise statement and further details. In particular, implicit in Theorem A is the boundary regularity statement that an AH Einstein metric with \( C^{m,\alpha} \) conformal infinity has a \( C^{m,\alpha} \) compactification. A version of Theorem A also holds in arbitrary dimensions \( n > 4 \), if the definition of \( \mathcal{E}_{AH} \) is weakened to consist of the class of metrics \( g \) which have a \( C^2 \) conformal compactification and \( C^{m,\alpha} \) boundary metric \( \gamma \), c.f. Proposition 1.4 and Remark 2.4, and also [21].

Theorem A is a generalization of previous results of Graham-Lee [15] and Biquard [8], who proved a local analogue of this result, (without full boundary regularity however), in neighborhoods of metrics \( g \in \mathcal{E}_{AH} \) which are regular points of the map \( \Pi \). The proof of Theorem A strongly uses the methods introduced in [15] and [8]. Further, Theorem A is formally analogous to results on the
space of minimal surfaces, c.f. [9] and especially [27], [28] and we have also been influenced by this work.

The issues of existence and uniqueness of AH Einstein metrics with a given conformal infinity are thus equivalent to the surjectivity and injectivity of the boundary map II.

The main result of the paper which we use to approach the surjectivity question is the following. Let $\mathcal{C}^0$ be the space of non-negative conformal classes $[\gamma]$ on $\partial M$, in the sense that $[\gamma]$ has a non-flat representative $\gamma$ of non-negative scalar curvature. Let $\mathcal{E}^0_{AH} = \Pi^{-1}(\mathcal{C}^0)$ be the space of AH Einstein metrics on $M$ with conformal infinity in $\mathcal{C}^0$. Thus, we have the restricted boundary map $\Pi^0 = \Pi_{|\mathcal{E}^0_{AH}} : \mathcal{E}^0_{AH} \to \mathcal{C}^0$.

**Theorem B.** Suppose $M$ is a 4-manifold as in Theorem A, for which the inclusion $i : \partial M \to \tilde{M}$ induces a surjection

$$H_2(\partial M, \mathbb{R}) \to H_2(M, \mathbb{R}) \to 0.$$  \hspace{1cm} (0.5)

Then for any $(m, \alpha)$, $m \geq 3$, the boundary map

$$\Pi^0 : \mathcal{E}^0_{AH} \to \mathcal{C}^0$$  \hspace{1cm} (0.6)

is proper.

Unfortunately (or fortunately), the boundary map $\Pi$ is not proper in general. In fact, a sequence of AH Einstein metrics $\{g_i\}$ on $M = \mathbb{R}^2 \times T^2$ was constructed in [6], whose conformal infinity is an arbitrary fixed flat metric on $\partial M = T^2$, but which has no convergent subsequence to an AH Einstein metric on $M$. The sequence $\{g_i\}$ converges, (in a natural sense), to a complete hyperbolic cusp metric

$$g_C = dr^2 + e^{2r} g_{T^3}$$  \hspace{1cm} (0.7)

on the manifold $N = \mathbb{R} \times T^3$. Since the boundary metric on $T^3$ is flat, Theorem B is sharp, at least without further restrictions.

We will show in Theorem 5.6 that this behavior is the only way that $\Pi$ is non-proper, in that divergent sequences $\{g_i\}$ of AH Einstein metrics on a fixed 4-manifold $M$ and with a fixed conformal infinity, (or a convergent sequence of conformal infinities), necessarily converge to AH Einstein manifolds $(N, g)$ with cusps, (although the cusps are not necessarily of the simple form (0.7)). Thus, the completion $\mathcal{E}_{AH}$ of $\mathcal{E}_{AH}$ in a natural topology consists of AH Einstein metrics on $M$, together with AH Einstein metrics with cusps, c.f. Theorems 5.6 and Corollary 5.7 for further details. Using this, we show in Corollary 5.8 that the boundary map $\Pi$ is proper onto the subset $\overline{\mathcal{C}}$ of $\mathcal{C}$ consisting of conformal classes which are not the boundary metrics of AH Einstein metrics with cusps.

The topological condition (0.5) is used solely to rule out orbifold degenerations of AH Einstein sequences $\{g_i\}$. This condition is probably not necessary, but we have included it in order to not overly complicate the paper; some situations where it is not needed are described in §7, c.f. Remark 7.4.

Theorems A and B imply that at least on $\mathcal{E}^0_{AH}$, the map $\Pi^0$ is a proper Fredholm map of index 0. It follows from work of Smale [25] that the regular values of $\Pi^0$ are open and dense in $\mathcal{C}^0$ and further, $\Pi^0$ has a well-defined mod 2 degree, $\text{deg}_2 \Pi^0 \in \mathbb{Z}_2$ on each component of $\mathcal{E}^0_{AH}$. Of course the fact that $\Pi^0$ is proper implies that $\mathcal{E}^0_{AH}$ has only finitely many components mapping onto a given component of $\mathcal{C}^0$. In fact, building on the work of Tromba [26] and White [27], [28] we show in Theorem 6.1 that $\Pi^0$ has a $\mathbb{Z}$-valued degree $\text{deg} \Pi^0 \in \mathbb{Z}$, again on components of $\mathcal{E}^0_{AH}$. Tautologically, if $\text{deg} \Pi^0 \neq 0$, then $\Pi^0$ is surjective.

We compute $\text{deg} \Pi^0$ in certain cases via symmetry arguments. Namely, it is proved in Theorem 3.1 that any connected group of conformal isometries of the conformal infinity $[\gamma]$ on $\partial M$ extends to a group of isometries of any AH Einstein filling metric $(M, g)$, with $\Pi[g] = [\gamma]$. Using this, and a
straightforward classification of Einstein metrics with large symmetry groups leads to the following result.

**Theorem C.** Let \( M = B^4 \) be the 4-ball, with \( \partial M = S^3 \), and, (abusing notation slightly), let \( C^\alpha \) be the component of the non-negative conformal classes containing the round metric on \( S^3 \). Also let \( E_{AH}^\alpha \) be the component of \( \Pi^{-1}(C^\alpha) \) containing the Poincaré metric on \( B^4 \). Then

\[
\deg_B \Pi^\alpha = 1.
\]

In particular, for any \( (m, \alpha) \), \( m \geq 3 \), any conformal class \( [\gamma] \in C^\alpha \) on \( S^3 \) is the conformal infinity of an AH Einstein metric on \( B^4 \).

Explicit examples of AH Einstein metrics on \( B^4 \), whose the conformal infinity is an arbitrary Berger sphere, were constructed by Pedersen [24]. More generally, Hitchin [19] has constructed such metrics with conformal infinity an arbitrary left-invariant metric on \( S^3 \).

It remains an open question whether the boundary map \( \Pi \) is surjective onto all of \( C \) when \( M = B^4 \). Based on reasoning from the AdS/CFT correspondence, Witten [29] has remarked that this may not be the case. It appears that physically, the natural class of boundary metrics are those of positive scalar curvature; in this regard, see also the work of Witten-Yau [30] proving that \( \partial M \) is necessarily connected when the conformal infinity has a component of positive scalar curvature. In any case, as remarked above in connection with Theorem 5.6, the only obstruction to surjectivity is the possible existence of AH Einstein cusp metrics attached to \( B^4 \), c.f. also Remark 7.7.

Theorem C also holds when \( M \) is any sufficiently non-trivial disc bundle over \( S^2 \), so that \( \partial M = S^3/\mathbb{Z}_k \), \( k \geq 2 \), with again \( C^\alpha \) the component containing the standard round metric on \( S^3/\mathbb{Z}_k \), c.f. Proposition 7.6. However, for \( M = S^2 \times \mathbb{R}^2 \), we show in Proposition 7.2, that

\[
\deg_{S^2 \times \mathbb{R}^2} \Pi^\alpha = 0,
\]

and further that \( \Pi^\alpha \) is not surjective onto \( C^\alpha \). In fact, the conformal class of \( S^2(1) \times S^1(L) \), for any \( L > 2\pi/\sqrt{3} \) is not the boundary metric of any AH Einstein metric on \( S^2 \times \mathbb{R}^2 \). This result is based on the work of Hawking-Page [17] on the AdS-Schwarzschild metric. Similarly, for \( M \) the degree 1 disc bundle over \( S^2 \), i.e. \( M = \mathbb{C}P^2 \setminus B^4 \), (0.9) holds and \( \Pi^\alpha \) is not surjective. These results for \( \Pi^\alpha \) on disc bundles use properties of the AdS-Taub Bolt metrics analysed in [18],[23]. These results and others on \( \deg \Pi^\alpha \) are given in §7.

1. The Banach Manifold \( E_{AH} \).

In this section we prove that the moduli space of AH Einstein metrics on a given \( n \)-manifold is naturally an infinite dimensional Banach manifold, assuming it is non-empty. The work in this section uses the methods developed by Graham-Lee [15] and Biquard [8], as well as the work of White [27], [28].

We begin by describing some background material on the function spaces we will be working with. First, let \( \rho_0 \) be a fixed \( C^\infty \) defining function for \( \partial M \) in \( M \). Throughout §1, the defining function \( \rho_0 \) will be kept fixed and only compactifications w.r.t. \( \rho_0 \) will be considered, i.e.

\[
\bar{g} = \rho_0^2 \cdot g.
\]

Later in §2 we will discuss the situation where \( \rho_0 \) varies over the family of defining functions. Formulas relating the curvatures of \( g \) and \( \bar{g} \) are given in the Appendix. Given \( \rho_0 \), define the function \( r = r(\rho_0) \) on \( M \) by

\[
r = -\log(\rho_0/\rho_0^2).
\]

Let \( Met^{m,\rho}(\partial M) \) be the space of \( C^{m,\rho} \) Riemannian metrics on \( \partial M \), so that \( Met^{m,\rho} \) is an open cone in the Banach space \( S^{m,\rho}(\partial M) \) of symmetric bilinear forms on \( \partial M \). We give \( Met^{m,\rho}(\partial M) \) the
for a fixed \( \alpha' < \alpha \), so that bounded sequences in the \( C^{m,\alpha'} \) norm have convergent subsequences. In this topology, \( M^{m,\alpha} \) is separable, c.f. [27]. Next let \( \mathbb{S}^{k,\beta}(M) \) be the Banach space of \( C^{k,\beta} \) symmetric bilinear forms on \( M \), and let \( \mathbb{S}^{k,\beta}(\bar{M}) \) be the corresponding space of forms on the closure \( \bar{M} \), both again with the \( C^{k,\beta'} \) topology, \( \beta' < \beta \).

Observe that forms in \( \mathbb{S}^{k,\beta}(M) \) have no control or restriction on their behavior on approach to \( \partial M \), while those in \( \mathbb{S}^{k,\beta}(\bar{M}) \) of course by definition extend \( C^{k,\beta} \) up to \( \partial M \). Thus, \( (m,\alpha) \) determines the regularity of the boundary data, while \( (k,\beta) \) determines the regularity in the interior \( M \). These are not necessarily related, unless one has boundary regularity results, i.e. regularity of the data up to and including the boundary. We will always assume that \( m + \alpha > k + \beta \) and \( k \geq 2 \).

Let \( g \) be a complete Riemannian metric of bounded geometry on \( M \), i.e. \( g \) has bounded sectional curvature and injectivity radius bounded below on \( M \). As in [15] and [8], define the weighted Hölder spaces \( \mathbb{S}^{k,\beta}_{\delta}(M) = \mathbb{S}^{k,\beta}(M,g) \) to be the Banach space of symmetric bilinear forms \( h \) on \( M \) s.t.

\[
h = e^{-\delta r} h_0,
\]

for \( r \) as in (1.2), where \( h_0 \in \mathbb{S}^{k,\beta}(M) \) satisfies

\[
\| h_0 \|_{C^{k,\beta}(M)} \leq C,
\]

for some constant \( C < \infty \). Here the norm is the usual \( C^{k,\beta} \) norm w.r.t the metric \( g \), given by

\[
\| h_0 \|_{C^{k,\beta}(M)} = \sum \| \nabla^j h_0 \|_{C^{\alpha}(M)} + \| \nabla^k h_0 \|_{C^{\beta}(M)}.
\]

Thus \( h \in \mathbb{S}^{k,\beta}_{\delta}(M) \) implies that \( h \) and its derivatives up to order \( k \) w.r.t. \( g \) decay as \( e^{-\delta r} \) as \( r \to \infty \). The weighted norm of \( h \) is then defined as

\[
\| h \|_{C^{k,\beta}_{\delta}(M)} = \| h_0 \|_{C^{k,\beta}(M)}.
\]

Observe that the norms in (1.5) and (1.6) depend only on \( C^{k,\beta} \) quasi-isometry class of \( g \); two metrics \( g \) and \( g' \) are \( C^{k,\beta} \) quasi-isometric if, in a fixed local coordinate system, the linear map \( g^{-1} g' \) is bounded away from 0 and \( \infty \) in \( C^{k,\beta}(M) \). Hence the spaces \( \mathbb{S}^{k,\beta}_{\delta}(M) \) depend only on the \( C^{k,\beta} \) quasi-isometry class of \( g \).

Now suppose the metric \( g \) is conformally compact, with compactification \( \bar{g} \) as in (1.1). We may then consider the \( C^{k,\beta} \) norm of \( h \) above also w.r.t. \( \bar{g} \). Using standard formulas for conformal changes of metric gives, for any \( j, \beta \geq 0 \),

\[
\| \nabla^j h \|_{C^{\beta}(\bar{g})} = \rho_0^{-2-j-\beta} \| \nabla^j h \|_{C^{\beta}(g)} + \text{lower order terms}.
\]

Given these preliminaries, we now construct a "standard" AH metric associated to any boundary metric \( \gamma \in M^{m,\alpha}(\partial M) \). This is first done in a collar neighborhood of \( \partial M \) and then later extended to a standard metric on all of \( M \). Thus, let \( U \) be a fixed collar neighborhood of \( \partial M \) in \( M \) on which \( d\rho_0 \) is non-vanishing, and choose a fixed identification of \( U \) with \( I \times \partial M \) so that \( \rho_0 \) corresponds to the variable on \( I \). Recalling that \( \rho_0 \) is fixed, define the \( C^{m,\alpha} \) hyperbolic cone metric \( g_U = g_U(\gamma, \rho_0) \) in \( U \) by

\[
g_U = dr^2 + sinh^2 r \cdot \gamma,
\]

for \( r \) as in (1.2). Observe that the dependence of \( g_U \) is \( C^\infty \) smooth in \( \gamma \), and also in \( \rho_0 \). Observe also that if \( \gamma_1 \) and \( \gamma_2 \) are \( C^{m,\alpha} \) quasi isometric boundary metrics, then \( g_U(\gamma_1) \) and \( g_U(\gamma_2) \) are also \( C^{m,\alpha} \) quasi isometric.

If \( \{ e_i \} \) is a local orthonormal frame for \( g_U \), with \( e_1 = \partial_r \), then one easily verifies that the sectional curvatures \( K_{ij} \) of \( g_U \) in the direction \( (e_i, e_j) \) are given by

\[
K_{1i} = -1, K_{jk} = \frac{1}{sinh^2 r} (K_\gamma)_{jk} - coth^2 r,
\]
where $i, j, k$ run from 2 to $n$ and $K_{ij}$ is the sectional curvature of $\gamma$. This implies that the curvature of $g_U$ decays to that of the hyperbolic space $H^n(-1)$ at a rate of $O(e^{-2\gamma}) = O(\rho_0^2)$. The same decay holds for the covariant derivatives of the curvature, up to order $m - 2 + \alpha$. In particular by (1.3)-(1.6)

$$Ric_{g_U} + (n - 1) \cdot g_U \in \mathbb{S}_2^{m-2,\alpha}(M). \quad (1.9)$$

The metric $g_U$ is conformally compact. In fact if $\tilde{g}_U$ is the compactification (1.1) of $g_U$, then a simple computation gives

$$\tilde{g}_U = d\rho_0^2 + (1 - \frac{1}{4}\rho_0^2)^2 \cdot \gamma. \quad (1.10)$$

Hence $\tilde{g}_U$ is a $C^{m,\alpha}$ conformal compactification of $g_U$.

We will view $g_U$ as a background metric with which to compare other conformally compact metrics with the same boundary metric. Thus suppose $\tilde{g}'$ is any conformally compact metric on $M$, with compactification $\tilde{g}'$ as in (1.1). Then we may write

$$\tilde{g}' \big|_U = g_U + h, \quad (1.11)$$

and we will assume that $h \in \mathbb{S}^{k,\beta}(M)$. This implies that

$$\tilde{g}' \big|_U = d\rho_0^2 + (1 - \frac{1}{4}\rho_0^2)^2 \cdot \gamma + \rho_0^2 h, \quad (1.12)$$

so that if $\rho_0^2|h| \to 0$ on approach to $\partial M$, then $\tilde{g}'$ is a $C^\alpha$ compactification of $\tilde{g}'_U$ with boundary metric $\gamma$; here $h$ is the pointwise norm of $h$ w.r.t. any smooth metric on $\bar{M}$. The compactification $\tilde{g}'$ is $C^{k,\beta}$ when $\rho_0^2 h \in \mathbb{S}^{k,\beta}(\bar{M})$. Now using the relations (1.3)-(1.7), observe that

$$h \in \mathbb{S}^{k,\beta}(M) \text{ with } \delta \geq k + \beta \Rightarrow \rho_0^2 h \in \mathbb{S}^{k,\beta}(\bar{M}). \quad (1.13)$$

However, if $\delta < k + \beta$ and $h \in \mathbb{S}^{k,\beta}(M)$, then in general, i.e. without further restrictions, $\rho_0^2 h$ will not be in $\mathbb{S}^{k,\beta}(\bar{M})$; this is essentially the issue of boundary regularity, and will be discussed further at the end of §1.

The standard metrics $g_U$ may be naturally extended to all of $M$ as follows. Let $\eta$ be a fixed bump function on $M$, with $\eta \equiv 1$ on $U$, $\eta \equiv 0$ on $M \setminus U'$, where $U'$ is a thickening of $U$ on which $d\eta$ is also non-vanishing. If $g_C$ is any smooth Riemannian metric on the compact manifold $M \setminus U$, (so $g_C$ is incomplete), then define

$$g_\gamma = (1 - \eta)g_C + \eta g_U. \quad (1.14)$$

Thus for any $\gamma \in Met^{m,\alpha}(\partial M)$, $g_\gamma$ in (1.14) gives a standard AH metric on $M$, with boundary metric $\gamma$. The metric $g_\gamma$ on $\bar{M}$ again depends smoothly on $\gamma$ and the choices of the compact metric $g_C$ and cutoff $\eta$. As with $\rho_0$, we fix the metric $g_C$ and cutoff $\eta$ once for all. With this understood, we thus have a $C^\infty$ smooth (addition) map

$$A : Met^{m,\alpha}(\partial M) \times U_5^{k,\beta} \to Met_5^{k,\beta}(M), A(\gamma, h) = g \equiv g_\gamma + h, \quad (1.15)$$

where $U_5^{k,\beta}$ is the open subset of $\mathbb{S}_5^{k,\beta}(M)$, consisting of those $h$ such that $g_\gamma + h$ is a well-defined metric on $M$.

In view of the decay rate (1.9), the most natural choice of $\delta$ is

$$\delta = 2, \quad (1.16)$$

and we fix this choice for the remainder of the paper. The map $A$ is clearly injective and the asymptotically hyperbolic (AH) metrics (of weight $\delta = 2$) are defined to be the image of $A$;

$$Met_{AH}^{k,\beta} = Im A. \quad (1.17)$$
The inverse map to $A$, $S : \text{Met}^{k,\beta}_{\text{AH}} \to \text{Met}^{m,\beta}(\partial M) \times \mathcal{U}^{k,\beta}_2$ gives the splitting of the AH metric $g$ into its components $g_\gamma$ and $h$ in (1.15). Let

$$E^{k,\beta}_{\text{AH}} \subset \text{Met}^{k,\beta}_{\text{AH}}$$

be the subset of AH Einstein metrics, with topology induced as a subset of the product topology. Note that, as discussed in (1.13), metrics in $E^{k,\beta}_{\text{AH}}$ are $C^2$ conformally compact, but not necessarily $C^{k,\beta}$ conformally compact w.r.t. $\rho_0$, for $k + \beta > 2$.

Now following [8], for any $k \geq 2$, define a map

$$\Phi : \text{Met}^{k,\beta}_{\text{AH}} \to \mathbb{S}^{k-2,\beta}(M)$$

(1.19)

$$\Phi(g) = \Phi(g_\gamma + h) = \text{Ric}_g + (n-1)g + (\delta_g)\ast B_{g_\gamma}(h),$$

where $B_{g_\gamma}$ is the Bianchi operator, (c.f. [7]), given by

$$B_{g_\gamma}(h) = \delta_{g_\gamma} h + \frac{1}{2}d(\text{tr}_{g_\gamma} h);$$

here $\delta$ is the divergence operator and $\delta^\ast$ its $L^2$ adjoint. Observe that $\Phi$ is well-defined, by (1.9) and the choice (1.16), and is $C^\infty$ smooth.

There are several natural reasons for considering this operator. First, it is proved in [8, Lemma I.1.4] that

$$E^{k,\beta}_{\text{AH}} = \Phi^{-1}(0) \cap \{\text{Ric} < 0\},$$

(1.20)

where $\{\text{Ric} < 0\}$ is the open set of metrics with negative Ricci curvature. In particular, if $g$ is Einstein, i.e. $\text{Ric}_g = -(n-1)g$, then $\Phi(g) = 0$, and hence also

$$B_{g_\gamma}(h) = 0.$$  

(1.21)

As discussed later, the condition (1.21) defines the tangent space of a slice to the action of the diffeomorphism group on $\text{Met}_{\text{AH}}$.

Second, it is well-known that the linearization of the Einstein operator $\text{Ric}_g + (n-1)g$ at an Einstein metric $g$ is given by the operator

$$\frac{1}{2}D^\ast D - \circ \tilde{R} + \delta^\ast B,$$

(1.22)

acting on the space of symmetric 2-tensors $\mathbb{S}^2(M)$ on $M$, c.f. [7]. Here $D^\ast D$ is the (rough) Laplacian $-\text{tr}D^2$, while $\circ \tilde{R}$ is the action of the curvature tensor on $\mathbb{S}^2(M)$. The kernel of the elliptic self-adjoint linear operator

$$L = \frac{1}{2}D^\ast D - \circ \tilde{R}$$

(1.23)

gives the space of non-trivial infinitesimal Einstein deformations, analogous to the Jacobi fields for minimal submanifolds. An AH Einstein metric $g$ on $M$ will be called non-degenerate if

$$K = L^2 - \text{Ker}L = 0,$$

(1.24)

i.e. if there are no non-trivial infinitesimal Einstein deformations of $g$ in $L^2(M, g)$. Einstein metrics are critical points of the Einstein-Hilbert functional or action, and this corresponds formally to the condition that the critical point be non-degenerate, in the sense of Morse theory. Now the linearization of $\Phi$ at any $g = g_\gamma + h$, as in (1.15), with respect to variations of $h$, has the simple form

$$(D_2\Phi)_g(\dot{h}) = \frac{1}{2}D^\ast D\dot{h} + \frac{1}{2}(\text{Ric}_g \circ \dot{h} + \dot{h} \circ \text{Ric}_g + 2(n-1)\dot{h}) - \circ \tilde{R}_g(\dot{h});$$

(1.25)
this is due to cancellation of the variation of the term \(\delta g) B_{g_\gamma}(h)\) with the variation of the Ricci curvature, c.f. [8, (1.9)]. Hence, if \(g\) is Einstein, then

\[
(D_2\Phi)_g = L = \frac{1}{2} D^* D - \overset{\circ}{R}.
\]

To proceed further, we need the following Lemma.

**Lemma 1.1.** Let \(\kappa \in K\) be a non-trivial \(L^2\) infinitesimal Einstein deformation of \((M, g)\), so that

\[L(\kappa) = 0.\]

Then there is a constant \(C = C(\kappa, \rho_0) < \infty\) such that

\[
C^{-1} e^{-\frac{1}{(n-1)r}} \leq |\kappa| \leq C e^{-\frac{1}{(n-1)r}},
\]

where \(r\) is the function from (1.2). The same estimate holds for the covariant derivatives of \(\kappa\). In particular, \(\kappa \in S_2^{k, \beta}(M)\).

**Proof:** This result is essentially proved in [8, I.2.B], so we will only sketch the idea. Since the form \(\kappa\) is in the \(L^2\) kernel of \(L\), it follows that the decay of \(\kappa\) as \(r \to \infty\) is the same as the decay of the Green’s function \(G(x, y)\) of the operator \(L\) in \((M, g)\). Since \((M, g)\) is asymptotically hyperbolic, \(G(x, y)\) decays at the same rate as in the hyperbolic space \(H^n(-1)\) and it is easily verified that this decay rate is \(e^{-\frac{1}{(n-1)r}}\). Elliptic regularity gives the same decay rate for the derivatives of \(\kappa\).

The main result of this section, which leads to a version of Theorem A, is the following:

**Theorem 1.2.** At any \(g \in L_2^{k, \beta}(\text{Met}_AH)\), the map \(\Phi\) is a submersion, i.e. the derivative

\[
(D \Phi)_g : T_g \text{Met}^{k, \beta}_A \to T_{\Phi(g)} S_2^{k-2, \beta}(M)
\]

is surjective and its kernel splits in \(T_g \text{Met}^{k, \beta}_A\).

**Proof:** From (1.15) and (1.17), we have \(T_g \text{Met}_AH = T_\gamma \text{Met}^{n, \alpha}(\partial M) \oplus T_0 S_2^{k-2, \beta}(M)\). With respect to this splitting, (1.26) shows that the derivative of \(\Phi\) with respect to the second (i.e. \(h\) factor is

\[
(D_2\Phi)_g = \frac{1}{2} D^* D - \overset{\circ}{R} : S_2^{k, \beta}(M) \to S_2^{k-2, \beta}(M).
\]

Now by [8, Prop. 1.3.5], \(D_2\Phi)_g\) is a Fredholm operator whose kernel on \(S_2^{k, \beta}(M)\) equals the \(L^2\) kernel \(K\) in (1.24). Since \(D_2\Phi)_g\) is formally self-adjoint on \(L^2\), it has Fredholm index 0, and the cokernel of \(D_2\Phi)_g\) is the \(L^2\) orthogonal complement of \(K\) in \(S_2^{k-2, \beta}(M)\). Thus to prove \(D_\Phi g\) is surjective, it suffices to show that for any \(L^2\) infinitesimal Einstein deformation \(\kappa \in K\), there is a tangent vector \(X \in T_g \text{Met}_AH\) such that

\[
\int_M <(D\Phi)_g(X), \kappa> dv_g \neq 0.
\]

To do this, note that any tangent vector \(X\) may be written as \(X = g_{\gamma} + h\), where \(\gamma\) and \(h\) are variations of the boundary metric \(\gamma\) and the bulk part \(h\). For any variation \(\gamma\), there exists a variation \(\hat{h}\) such that

\[
\frac{d}{dt}(B_{g_{\gamma}(t)}(\hat{h}(l)))|_{t=0} = 0,
\]

as in (1.21), (c.f. the discussion below on diffeomorphisms and (1.35) for instance). For such variations \(X\), one thus has

\[
D_\Phi g(X) = \frac{1}{2} D^* D(X) - \overset{\circ}{R}(X) = L(X).
\]
Let $B(r)$ be the $r$-sublevel set of the function $r$ from (1.2) with $S(r) = \partial B(r)$ the $r$-level set. We apply the divergence theorem to the integral (1.29) over $B(r)$ twice. Since $\kappa \in \text{Ker}L$, it follows that the integral (1.29) reduces to an integral over the boundary, and gives

$$\int_{B(r)} <(D\Phi)_g(X)\kappa > dV_g = \int_{S(r)} <X, \nabla N\kappa > - <\nabla N X, \kappa > dV_g, \tag{1.30}\int_{B(r)}$$

where $N = \nabla r / |\nabla r|$ is the unit outward normal.

To estimate the boundary integrals, the volume growth of $S(r)$ satisfies

$$\text{vol}S(r) \sim e^{(n-1)r},$$

i.e. the ratio is bounded away from 0 and $\infty$, while from Lemma 1.1, $|k| \sim e^{-r}$. Hence, the $L^1$ integral of $\kappa$ over $S(r)$ is bounded away from 0 and $\infty$ as $r \to \infty$. The same statement holds for $\nabla N\kappa$.

Next, for $X = g_\gamma + \dot{h}$ as above, we have

$$g_\gamma = \eta(sinh^2 r)\gamma, \quad \nabla N g_\gamma = 2\eta(sinh r)(cosh r)\gamma. \tag{1.31}$$

Note that $|g_\gamma|, g \sim 1$ and $|\nabla N g_\gamma| \sim 1$, in that these quantities are bounded away from 0 and $\infty$ as $r \to \infty$. On the other hand, the term $\dot{h}$ is bounded as $O(e^{2r})$ as $r \to \infty$. Hence, in estimating (1.30) as $r \to \infty$, we may safely ignore the term $\dot{h}$, and replace $X$ by $g_\gamma$.

Substituting then (1.31) in (1.30), it follows that the boundary integral (1.30) vanishes in the limit as $r \to \infty$ for all variations $\dot{\gamma}$ if and only if

$$\frac{1}{\kappa} |\nabla N\kappa - 2\kappa| \to 0, \quad r \to \infty.$$

But this is impossible by (1.27) and hence (1.29) follows.

To prove that the kernel of $D\Phi_g$ splits, i.e. it admits a closed complement in $T_g\text{Met}_{AH}$, it suffices to exhibit a bounded linear projection $P$ mapping $T_g\text{Met}_{AH}$ onto $\text{Ker}(D\Phi_g)$. We do this following [28]. Thus, we have

$$\text{Ker}D\Phi_g = \{ (\gamma, \dot{h}) : D_1\Phi(\gamma) + D_2\Phi(\dot{h}) = 0 \}.$$

From (1.26), $D_2\Phi = L$ and $\text{Im} L = K^\perp$, for $K$ as in (1.24). Hence $D_1\Phi(\gamma) \in K^\perp$, so that $\pi_K D_1\Phi(\gamma) = 0$, i.e. $\gamma \in \text{Ker}(\pi_K D_1\Phi)$, where $\pi_K$ is orthogonal projection onto $K$. By (1.29), or more precisely its proof, $D_1\Phi$ maps onto $K$ and hence $\text{Im} \pi_K D_1 = K$. Since the finite dimensional space $K$ splits, we have $T\text{Met}_{AH} = K \oplus K^\perp = \text{Im}(\pi_K D_1\Phi) \oplus \text{Ker}(\pi_K D_1\Phi)$, so that $\text{Ker}(\pi_K D_1\Phi)$ splits. Hence, there is a bounded linear projection $P_1$ onto $\text{Ker}(\pi_K D_1\Phi)$. Now define

$$P(\gamma, \dot{h}) = (P_1\gamma, (L + \pi_K)^{-1}(-D_1\Phi(P_1\gamma) + \pi_K \dot{h})).$$

Then $P$ is the required bounded linear projection.

**Corollary 1.3.** If $E_{AH}^{k,\beta}$ is non-empty, then $E_{AH}^{k,\beta}$ is an infinite dimensional $C^\infty$ separable Banach manifold. In fact, via the splitting (1.15), $E_{AH}^{k,\beta}$ is a $C^\infty$ Banach submanifold of $\text{Me}^{m,\alpha}(\partial M) \times \mathbb{S}^{k,\beta}(M)$ and as such

$$T_gE_{AH}^{k,\beta} = \text{Ker}(D\Phi)_g. \tag{1.32}$$

**Proof:** This is an immediate consequence of Theorem 1.2 and the implicit function theorem in Banach spaces, c.f. [20]. $E_{AH}^{k,\beta}$ is separable since it is a submanifold of $\text{Me}^{m,\alpha}(\partial M) \times \mathbb{S}^{k,\beta}(M)$, each of which are separable Banach spaces in the topologies defined at the beginning of §1. \hspace{1cm} ■
Remark 1.4. Note that \( \text{Ker}(D\Phi)_g \) is the space of infinitesimal asymptotically hyperbolic Einstein deformations, (not necessarily preserving the boundary metric as is the case with the \( L^2 \) kernel \( K \) from (1.24)). Corollary 1.2 implies that any infinitesimal AH Einstein deformation may be integrated to a (local) curve of AH Einstein metrics. This is no longer true when the boundary metric is required to be fixed, i.e. an \( L^2 \) infinitesimal Einstein deformation in \( K \) may not integrate to a curve of AH Einstein metrics with the same boundary metric. This may be seen in a specific example from the analysis in Proposition 7.2.

We also point out that Theorem 1.2 and Corollary 1.3 are valid in any dimension.

The boundary map taking an AH Einstein metric \( g \) to its boundary metric \( \gamma \) w.r.t. the compactification (1.1) is given simply by projection on the first factor:

\[
\Pi : E_{AH}^{k,\beta} \to \text{Met}^{m,\alpha}(\partial M); \quad \Pi(g) = \Pi(g_\gamma + \bar{h}) = \gamma,
\]

Clearly, this map is \( C^\infty \) smooth.

The space \( E_{AH}^{k,\beta} \) is invariant under the action of suitable diffeomorphisms. In §2, we will consider larger diffeomorphism groups, but for now we restrict to the group \( \mathcal{D}_2 = \mathcal{D}_2^{k-1,\beta} \equiv \text{Diff}^{k-1,\beta}(\bar{M}) \) of \( C^{k-1,\beta} \) diffeomorphisms \( \phi \) of \( \bar{M} \) such that

\[
\phi|_{\partial M} = \text{id}|_{\partial M}, \quad \text{and } \lim_{\rho_0 \to 0} \left( \frac{\phi^* \rho_0}{\rho_0} \right) = 1,
\]

where \( \rho_0 \) is the fixed defining function. If \( g \in E_{AH}^{k,\beta} \) and \( \bar{g} = \rho_0^2 g \) is the compactification as in (1.1), then for \( \phi \in \mathcal{D}_2 \), the compactification of \( \phi^* \bar{g} \) is given by

\[
(\phi^* \bar{g}) = \rho_0^2 \phi^* g = (\phi^* \bar{g}) \left( \frac{\rho_0}{\phi^* \rho_0} \right)^2.
\]

Hence (1.34) implies that \( g \) and \( \phi^* \bar{g} \) have the same boundary metric w.r.t. \( \rho_0 \). However, the normal vectors of the compactified metrics \( \bar{g} \) and \( \phi^* \bar{g} \) at \( \partial M \) are different in general.

We observe that the action of \( \mathcal{D}_2 \) preserves the space \( \text{Met}_{AH}^{k,\beta} \), and hence the space \( E_{AH}^{k,\beta} \). This is because \( |D\phi - \text{id}|_g \) extends \( C^{k,\beta} \) smoothly up to \( \partial M \), and hence \( |D\phi - \text{id}|_g = O(e^{-r}) \), so that \( |\phi^* \bar{g} - g|_g = O(e^{-2r}) \).

The action of \( \mathcal{D}_2 \) on the spaces \( \text{Met}_{AH}^{k,\beta}, E_{AH}^{k,\beta} \) is smooth, (c.f. [8, Lemma I.3.4]). Further the action is free, since any isometry \( \phi \) of a metric inducing the identity on \( \partial M \) must itself be the identity; this is perhaps most easily seen by working on a compactification \( \bar{g} \) by a defining function which is \( \phi \)-invariant near \( \partial M \). By [8, Prop. I.4.6], the set of metrics \( g = g_\gamma + \bar{h} \in \text{Met}_{AH}^{k,\beta} \) such that

\[
B_{g_\gamma}(\bar{h}) = 0
\]

is a local slice for the action of \( \mathcal{D}_2 \) on \( \text{Met}_{AH}^{k,\beta} \). The action of \( \mathcal{D}_2 \) on these spaces is proper, c.f. [10] for example, and hence the quotient

\[
\mathcal{E}_{AH}^{(2)} = E_{AH}^{k,\beta}/\mathcal{D}_2^{k-1,\beta},
\]

is also a \( C^\infty \) separable Banach manifold. Two metrics \( g_1 \) and \( g_2 \) in \( \mathcal{E}_{AH}^{(2)} \) are equivalent if there is a \( C^{k-1,\beta} \) diffeomorphism \( \phi \) of weight 2, i.e. satisfying (1.34), such that \( \phi^* g_1 = g_2 \). In particular, \( g_1 \) and \( g_2 \) must have the same boundary metric w.r.t. \( \rho_0 \). When \( E_{AH}^{k,\beta} \) is viewed as subset of the product \( \text{Met}^{m,\alpha}(\partial M) \times S_{k,\beta}(M) \) via \( S \), since \( \mathcal{D}_2^{k-1,\beta} \) acts trivially on the first factor, we have

\[
\mathcal{E}_{AH}^{(2)} \subset \text{Met}^{m,\alpha}(\partial M) \times (S_{k,\beta}(M)/\mathcal{D}_2^{k-1,\beta}).
\]

This inclusion sends \( [g] = [g_\gamma + \bar{h}] \) to \( (\gamma, [\bar{h}]) \), and a slice representative for \( [\bar{h}] \) is that unique \( \bar{h} \in [\bar{h}] \) satisfying (1.35). Via (1.33), \( \Pi \) descends to a smooth map

\[
\Pi : \mathcal{E}_{AH}^{(2)} \to \text{Met}^{m,\alpha}(\partial M), \quad \Pi([g]) = \gamma,
\]

(1.38)
Proposition 1.5. The map $\Pi : E_{AH}^{(2)} \to \text{Met}^{m,\alpha}(\partial M)$ is a $C^\infty$ Fredholm map of index 0, with
\[
\text{Ker}(D\Pi)_{\text{g}} = K_{\text{g}},
\]
where as in (1.24), $K_{\text{g}}$ is the space of $L^2$ infinitesimal Einstein deformations at $\text{g}$. Consequently, $\text{Im}\Pi \subset \text{Met}^{m,\alpha}(\partial M)$ is a variety of finite codimension.

Proof: This is a simple consequence of the discussion above. We have
\[
\text{Ker}D\Pi = T\mathcal{E}_{AH} \cap \text{Ker}\Pi_1,
\]
where $\Pi_1$ is the linear projection on the first factor in the splitting (1.37). Since $\Pi_1$ is injective,
\[
T\mathcal{E}_{AH} \cap \text{Ker}\Pi_1 = T\mathcal{E}_{AH} \cap T(S^{k,\beta}_2/\mathcal{D}^{k+1,\beta}_2),
\]
This intersection just consists of the classes $[h]$ satisfying (1.35), and so by (1.32) and (1.26),
\[
\text{Ker}D\Pi = \text{Ker}L,
\]
where the kernel is taken in $S^{k,\beta}_2$. But this is the same as the $L^2$ kernel $K$ by [8, Prop.1.3.5].

For the cokernel, we have
\[
\text{Im}(D\Pi) = \Pi(T\mathcal{E}_{AH}) = \Pi(\text{Ker}D\Phi) = \text{Ker}(\pi_K D_1\Phi),
\]
where the second equality is from (1.32) and the last equality follows from (1.31) and the discussion following it. Again, as following (1.31), $\text{Ker} \pi_K D_1\Phi = K^\perp$ is closed, and has codimension $k = \dim K$. Hence $\Pi$ is Fredholm of index 0.

Thus we have the following dichotomy; either there exist no conformally compact Einstein metrics on $M$, or the moduli space of such metrics is at least infinite dimensional, with $\text{Im}\Pi$ a variety of finite codimension in $\text{Met}(\partial M)$.

If there exist Einstein metrics $\text{g} \in E_{AH}$ which are non-degenerate, so that $K_{\text{g}} = \{0\}$, then $\Pi$ is a local diffeomorphism in a neighborhood of $\text{g}$. This is the result of Biquard [8], extending earlier work of Graham-Lee [15]. In other words, $\Pi$ is an open map on the open submanifold $E'_{AH}$ of non-degenerate metrics.

Again, the results above are valid in any dimension.

We complete this section with a discussion of the boundary regularity of metrics in $E_{AH}$. By construction, an Einstein metric $\text{g} \in E_{AH}$ has the form $\text{g} = g_\gamma + h$. If $\gamma$ is a $C^{m,\alpha}$ metric on $\partial M$, then $g_\gamma$ is a $C^{m,\alpha}$ metric on $M$ and the compactification $\tilde{g}_\gamma$ from (1.1) is also $C^{m,\alpha}$ on $\bar{M}$, as in (1.10). However, the term $h \in S^{k,\beta}_2(M)$ and even if $(k,\beta) \geq (m,\alpha)$, the metric $\text{g}$ does not necessarily compactify to a $C^{m,\alpha}$ metric $\tilde{g}$ on $\bar{M}$. Namely, as discussed following (1.18), because of the weight 2, the relations (1.13) imply that the metric $\tilde{g}$ is a $C^2$ compactification. However, these relations do not imply that $\tilde{g}$ is $C^{2,\alpha}$, for any $\alpha > 0$, regardless of the choice of $(k,\beta)$. Thus, although the boundary metric $\gamma$ is prescribed in $C^{m,\alpha}$, the compactification $\tilde{g}$ may, apriori, be only $C^2$.

This issue of boundary regularity remains open in higher dimensions, although see [15] and very recently [21] for some partial results. In fact, in odd dimensions, boundary regularity to all orders cannot hold, due to the presence of log terms in the expansion of $\tilde{g}$ near $\partial M$, see [13], [14], [12]. However, in dimension 4, the following result is proved in [6].

Proposition 1.6. Let $\text{g}$ be an AH Einstein metric on a 4-manifold $M$, which has an $L^2$ conformal compactification $\tilde{g} = \rho^2 \cdot \text{g}$, for some $p > 4$ and defining function $\rho$, with $C^{m,\alpha}$ boundary metric $\gamma, m \geq 2$. Then there is a possibly distinct $C^{m,\alpha}$ compactification $\tilde{g} = \rho^2 \cdot \text{g}$ of $\text{g}$, with the same boundary metric $\gamma$. This result also holds if $m = \infty$ or $m = \omega$. 
Observe that since the Einstein metrics in $E_{AH}^{k,\beta}$ have $C^2$ and hence $L^{2,p}$ compactifications, they satisfy the conclusion of Proposition 1.6. The defining function $\hat{\rho}$ has the property that the scalar curvature $\tilde{s}$ of $\tilde{g}$ is constant, so that $\hat{\rho}$ satisfies a Yamabe equation, c.f. also (A.3). It then follows from standard elliptic regularity for uniformly degenerate elliptic equations, c.f. [15, Prop. 3.4], that $\tilde{\gamma} \in C^{m+2,\alpha}(\tilde{M})$.

With regard to the compactification $\tilde{\gamma}$ in (1.1), we may write $\tilde{\gamma} = (\rho_0/\rho)^2 \cdot \tilde{g}$. Since $\tilde{\gamma} \in C^{m+2,\alpha}(\tilde{M})$, it follows that the compactification $\tilde{\gamma}$ is in fact a $C^m,\alpha$ compactification. Hence, the space $E_{AH}^{k,\beta}$ defined in (1.18) satisfies

$$E_{AH}^{k,\beta} = E_{AH}^{m,\alpha},$$

(1.40)

and $E_{AH}^{m,\alpha}$ is the space of AH Einstein metrics on $M$ which are $C^{m,\alpha}$ conformally compact w.r.t. the defining function $\rho_0$ as in (1.1). This corresponds to the definition in the Introduction.

With this understood, we have the following preliminary version of Theorem A:

**Proposition 1.7.** If $\dim M = 4$, and $m \geq 2$, then $E_{AH}^{m,\alpha}$ is the space of $C^{m,\alpha}$ conformally compact Einstein metrics on $M$ and the map $\Pi$ is a $C^\infty$ map

$$\Pi : E_{AH}^{m,\alpha} \to M e^{m,\alpha}(\partial M).$$

(1.41)

Of course, Proposition 1.7 also holds on the quotient $E_{AH}^{(2)} = E_{AH}^{(2),m,\alpha}$ of $E_{AH}^{m,\alpha}$ by $D^{n+1,\alpha}$, where $\Pi$ is then Fredholm, of index 0, by Proposition 1.5.

**Remark 1.8.** For later applications, we point out that all the results of this section hold equally well for AH Einstein orbifold metrics, i.e. Einstein metrics on a fixed orbifold $X$ in place of a smooth manifold $M$. Here an orbifold is a space which has a finite number of isolated cone singularities, each of the form $C(S^3/\Gamma)$, where $C(S^3/\Gamma)$ is the cone on a spherical space form $S^3/\Gamma$, c.f. §4.1. for further discussion.

2. **Conformal Infinity and Diffeomorphisms.**

Throughout the previous section, we have fixed the defining function $\rho_0$ as in (1.1), and used this to obtain a fixed boundary metric $\gamma$ for an AH Einstein metric $g$ on $M$. In this section, we consider the situation where $\gamma$ varies throughout its conformal class $[\gamma]$ as well as variation of $\rho$ in the class of all defining functions. These extensions are closely related to the action of the diffeomorphism group of $\tilde{M}$.

Let $\mathcal{D}_1 = \mathcal{D}_1^{k+1,\beta} = \text{Diff}^{k+1,\beta}(\tilde{M})$ be the group of orientation preserving $C^{k+1,\beta}$ diffeomorphisms of $\tilde{M}$ which restrict to the identity map on $\partial M$. Recall from §1 that $\mathcal{D}_2 \subset \mathcal{D}_1$ is the subgroup of diffeomorphisms $\phi$ satisfying $\lim_{\rho \to 0} (\phi^* \rho_0 / \rho_0) = 1$. It is easily seen that $\mathcal{D}_2$ is a normal subgroup of $\mathcal{D}_1$. With respect to $\rho_0$, we have the splitting $TM|_{\partial M} \cong T(\partial M) \oplus \mathbb{R}$, where the $\mathbb{R}$ factor is identified with the span of $\partial / \partial \rho_0$. The groups $\mathcal{D}_1$ and $\mathcal{D}_2$ act on $T(\partial M) \oplus \mathbb{R}$ by the map $\phi \to D\phi|_{\partial M}$, and so induce subgroups of $\text{Hom}(TM|_{\partial M}, TM|_{\partial M})$. Since $\mathcal{D}_2 \subset \mathcal{D}_1$ defined solely by a 1st order condition at $\partial M$, the quotient group $\mathcal{D}_1/\mathcal{D}_2$ is isomorphic to the corresponding quotient group in $\text{Hom}(TM|_{\partial M}, TM|_{\partial M})$.

**Lemma 2.1.** The quotient group $\mathcal{D}_1/\mathcal{D}_2$ is naturally isomorphic to the group of $C^{k,\beta}$ positive functions $\lambda$ on $\partial M$.

**Proof:** With respect to the splitting $TM|_{\partial M} \cong T(\partial M) \oplus \mathbb{R}$, the linear map $D\phi|_{\partial M}$, for $\phi \in \mathcal{D}_1$, has the form

$$\begin{pmatrix} I & * \\ 0 & \lambda \end{pmatrix}$$
where $\lambda = \lim_{\rho_0 \to 0} (\phi^* \rho_0 / \rho_0)$. For $\phi \in D_2, D\phi$ is the same, except that the entry $\lambda$ is 1. It follows that the quotient group is identified with the multiplicative group of functions $\lambda : \mathbb{R} \to \mathbb{R}$, acting in the $\partial/\partial \rho_0$ direction. Since $D\phi$ is non-singular, $\lambda$ cannot vanish and hence $\lambda > 0$. ■

As in §1, let $E_{A_H}^{[2]} = E_{A_H}/D_2$ be the space of isometry classes of AH Einstein metrics, among diffeomorphisms in $D_2$, and similarly, let $E_{A_H}^{(1)} = E_{A_H}/D_1$; here $E_{A_H} = E_{k_{\beta}}^{k_{\beta}}$ as in (1.18). There is a natural projection map $E_{A_H}^{[2]} \to E_{A_H}^{(1)}$ with fiber $D_1/D_2$. As in §1, $D_1$ acts freely on $E_{A_H}$, with local Bianchi slice as in (1.35) so that as in (1.36), $E_{A_H}^{(1)}$ is a $C^\infty$ separable Banach manifold.

Next, let $\mathcal{C}$ be the space of conformal classes of $C^{m,a}$ metrics on $\partial M$. Again, $\mathcal{C}$ has the structure of an infinite dimensional Banach manifold, with tangent spaces given by the space of trace-free symmetric bilinear forms. There is a natural projection map $Met^{m,a}(\partial M) \to \mathcal{C}$, with fiber the space of $C^{m,a}$ conformally equivalent metrics on $\partial M$.

**Proposition 2.2.** The boundary map $\Pi$ descends to a $C^\infty$ boundary map on the base spaces, i.e.

$$\Pi : E_{A_H}^{(1)} \to \mathcal{C}. \quad (2.1)$$

This map $\Pi$ is Fredholm, of index 0, with Ker$\Pi = K$, as in (1.39).

**Proof:** Let $g_1$ and $g_2$ be AH Einstein metrics on $M$ with $\phi^* g_2 = g_1$, for $\phi \in D_1$, and set $\lambda = \lim_{\rho_0 \to 0} (\phi^* \rho_0 / \rho_0)$. Let $\bar{g}_t$ be the compactification of $g_t, t = 1, 2$ w.r.t. $\rho_0$, as in (1.1), and let $\gamma_t$ be the induced boundary metrics. If $\bar{g}_2$ is the $\rho_0$-compactification of $\phi^* g_2$, then we have

$$\dot{g}_2 = \rho_0^2 \phi^* (\rho_0^{-2})^2 (\rho_0^2 g_2) = (\rho_0^2 \rho_0^{-2})^2 (\rho_0^2 g_2).$$

Hence, the boundary metric $\gamma_2$ of $\bar{g}_2$, which must equal $\gamma_1$, is given by

$$\gamma_1 = \gamma_2 = \lambda^2 \phi^* \gamma_2.$$ 

Since $\phi = \text{id}$ on $\partial M$, it follows that $\gamma_2 = \lambda^2 \gamma_1$, so that the boundary metrics are conformal. It follows that the boundary map $\Pi$ in (1.38) descends to the map $\Pi$ in (2.1) and is smooth.

Further, observe that Lemma 2.1 shows that the converse of the proof above also holds. Thus, suppose $g_1$ is an AH Einstein metric on $M$, with boundary metric $\gamma_1$, and let $\gamma_2 = \lambda^2 \gamma_1$, for some function $\lambda > 0$. Then there is a diffeomorphism $\phi \in D_1$, unique modulo $D_2$, such that $g_2 = \phi^* g_1$ has boundary metric $\gamma_2$. It follows that $\Pi$ maps the fibers $D_1/D_2$ diffeomorphically onto the fibers of the projection $Met^{m,a}(\partial M) \to \mathcal{C}$.

The proof that $\Pi$ is Fredholm of index 0, with Ker$\Pi = K$, is thus exactly the same as in Proposition 1.5. ■

**Remark 2.3.** Recall that the map $\Pi : E_{A_H}^{[2]} \to Met(\partial M)$ in (1.38) depends on a choice of the defining function $\rho_0$ from (1.1). The reduced map $\Pi$ in (2.1) is now independent of the choice of $\rho_0$. To see this, let $\rho_1$ be any other defining function, so that $\rho_1 = \lambda \cdot \rho_0$, for some function $\lambda > 0$ on $M$. Let

$$\bar{g} = \rho_0^2 g, \text{ and } \tilde{g} = \rho_1^2 g$$

be compactifications of $g$ w.r.t. $\rho_0$ and $\rho_1$. The boundary metrics are related by $\tilde{\gamma} = \lambda^2 \gamma$, where $\lambda = \lim_{\rho_0 \to 0} (\rho_1/\rho_0^2)$. As in the proof of Proposition 2.2, there is a diffeomorphism $\phi \in D_1$ satisfying, (along integral curves of $\partial/\partial \rho_0$), $d\phi(\rho_0)/d\rho_0 = \lambda$ at $\partial M$. Hence

$$\phi^* g = \rho_1^2 \phi^* \rho_0^2 g = \lambda^2 \rho_0^2 \phi^* \rho_0^2 g,$$

while

$$\rho_1^2 \phi^* \rho_0^2 \phi^* \rho_0^2 g = \lambda^2 \rho_0^2 \phi^* \rho_0^2 g,$$

near $\partial M$. Thus, the $\rho_1$ compactification of $\phi^* g$ is the same as the $\rho_0$ compactification of $g$, pulled back by $\phi$. 

Theorem A is now an essentially immediate consequence of the work above and in §1.

**Proof of Theorem A.**

The discussion following Lemma 2.1 shows that \( \mathcal{E}_{AH}^{(1)} = \mathcal{E}_{AH}^{(1),k,\beta} \) and \( \mathcal{C} = \mathcal{C}^{m,\alpha} \), \( m \geq 2, \alpha > 0 \), are both \( C^\infty \) smooth separable Banach manifolds and by Proposition 2.2, \( \Pi \) is a \( C^\infty \) Fredholm map of index 0. The boundary regularity result in Proposition 1.6, c.f. (1.40), shows that by setting \( (k, \beta) = (m, \alpha) \), \( \mathcal{E}_{AH}^{(1)} \) is the space of AH Einstein metrics on \( M \) which admit a \( C^{m,\alpha} \) compactification. 

**Remark 2.4.** We observe that a natural version of Theorem A also holds in dimensions \( n \geq 4 \), although one no longer has the boundary regularity given in Proposition 1.6. Nevertheless, the same arguments as above show that \( \mathcal{E}_{AH}^{(1),k,\beta} \) and \( \mathcal{C} = \mathcal{C}^{m,\alpha} \) are smooth Banach manifolds, and the boundary map \( \Pi \) is smooth and Fredholm of index 0.

We work with the space \( \mathcal{E}_{AH}^{(1)} = \mathcal{E}_{AH}^{(1),m,\alpha} \) for the rest of the paper, and to simplify notation, set
\[
\mathcal{E}_{AH} = \mathcal{E}_{AH}^{(1)}.
\] (2.2)

Although the spaces \( \mathcal{E}_{AH}^{(1),m,\alpha} \) are distinct, the results of this paper do not depend on \( (m, \alpha) \), although from §3 on we will require \( m \geq 3 \) or \( m \geq 4 \) in some instances. Obviously
\[
\mathcal{E}_{AH}^{(m',\alpha')} \subset \mathcal{E}_{AH}^{(m,\alpha)}
\] (2.3)
for \( (m', \alpha') \geq (m, \alpha) \) and the work in §5 will show that the inclusion (2.3) is dense in a reasonable sense, c.f. Remark 6.2.

**Remark 2.5.** For certain purposes, it is useful to consider quotients by larger diffeomorphism groups, and we discuss this briefly here. Thus, let \( \mathcal{D}_o \) be the group of \( (C^{m+1,\alpha}) \) orientation preserving diffeomorphisms of \( \bar{M} \) such that the induced diffeomorphism on \( \partial M \) is isotopic to the identity. Again, the group \( \mathcal{D}_1 \subset \mathcal{D}_o \) is a normal subgroup, and one may form
\[
\mathcal{E}_{AH}^{(o)} = E_{AH}/\mathcal{D}_o = \mathcal{E}_{AH}^{(1)}/(\mathcal{D}_o/\mathcal{D}_1).
\] (2.4)

Similarly, let \( \mathcal{T} \) denote the quotient space \( \mathcal{T} = \mathcal{C}/\mathcal{D}_o \). This is the space of marked conformal structures on \( \partial M \), analogous to the Teichmüller space of conformal structures on surfaces. The group \( \mathcal{D}_o \) however does not act freely on \( \mathcal{C} \). Elements in \( \mathcal{C} \) having a non-trivial isotropy group \( \mathcal{D}_o[\gamma] \) are the classes \( [\gamma] \) which have a non-trivial group of conformal diffeomorphisms, i.e. \( \mathcal{D}_o[\gamma] \) consists of diffeomorphisms \( \phi \in \mathcal{D}_o \) such that
\[
\phi^* \gamma = \lambda^2 \cdot \gamma,
\]
for some positive function \( \lambda \) on \( \partial M \). A well-known theorem of Obata [22] implies that the isotropy group \( \mathcal{D}_o[\gamma] \) of \( [\gamma] \) is always compact, with the single exception of \( (\partial M, [\gamma]) = (S^{n-1}, [\gamma_0]), \) where \( \gamma_0 \) is the round metric on \( S^{n-1}. \)

Similarly, the elements \( g \) of \( E_{AH} \) which have non-trivial isotropy groups \( \mathcal{D}_o(g) \) in \( \mathcal{D}_o \) are AH Einstein metrics which have a non-trivial group of isometries. Such isometries \( \phi \in \mathcal{D}_o \) induce a diffeomorphism \( \phi \) of \( \partial M \), which is a conformal isometry of the conformal infinity \( [\gamma] \) of \( g \). It follows that the boundary map \( \Pi \) in (2.1) descends further to a boundary map
\[
\Pi : \mathcal{E}_{AH}^{(o)} \rightarrow \mathcal{T}.
\] (2.5)

At any class \([g]\) where \( \mathcal{D}_o[g] = \text{id} \), the quotient space \( \mathcal{E}_{AH}^{(o)} \) is a smooth infinite dimensional Banach manifold, and similarly for \( \mathcal{T} \). At those classes \([g]\) or \([\gamma]\) where \( \mathcal{D}_o[g] \) or \( \mathcal{D}_o[\gamma] \) is compact, the quotients \( \mathcal{E}_{AH}^{(o)} \) and \( \mathcal{T} \) are smooth orbifolds, and \( \Pi \) is an orbifold smooth map. Only at the exceptional class \((B^m, g_{-1})\) of the Poincare metric on the ball is the quotient \( \mathcal{T} \) not well-behaved, and possibly non-Hausdorff.
Finally, one may carry out the same quotient construction with respect to the full group $\mathcal{D}_0 = \text{Diff}(\bar{M})$ of orientation preserving diffeomorphisms of $\bar{M}$ mapping $\partial M$ to itself, so that $\mathcal{E}_{AH}^{(0)} = E_{AH}/\mathcal{D}_0$, while $\mathcal{T}$ is replaced by the moduli space of conformal structures $\mathcal{M} = \mathcal{T}/\Gamma$, where $\Gamma = \mathcal{D}_0(\partial M)/\mathcal{D}_0(\partial M)$ is the subgroup of the mapping class group of $\partial M$ consisting of orientation preserving diffeomorphisms of $\partial M$ which extend to diffeomorphisms of $M$. The map $\Pi$ in (2.5) descends further to a map $\Pi : \mathcal{E}_{AH}^{(0)} \to \mathcal{M}$; see also Remark 5.9.

In the next sections, we will frequently consider compactifications $\tilde{g}$ of $g$ by a geodesic defining function $t$, for which

$$t(x) = \text{dist}_g(x, \partial M). \tag{2.6}$$

A compactification $\tilde{g} = t^2 \cdot g$ satisfying (2.6) is called a geodesic compactification and $t$ is a geodesic defining function. Such compactifications are natural from a number of viewpoints, c.f. the Appendix. It is not difficult to see that for any boundary metric $\gamma = \Pi(g)$, there is a unique geodesic compactification $\tilde{g}$ of $g$ having the given boundary metric $\gamma$. Further, by Proposition 1.6, (c.f. also [6]), if $\gamma$ is $C^{m,\alpha}$, $m \geq 3$, then the geodesic compactification $\tilde{g}$ is $C^{m-1,\alpha}$ off the cutlocus $C$ of $\partial M$ in $(M, \tilde{g})$.

Since the integral curves of $\tilde{\nabla} t$ are geodesics, the metric $\tilde{g}$ splits as

$$\tilde{g} = dt^2 + g_t, \tag{2.7}$$

within a collar neighborhood $U$ (inside the cutlocus) of $\partial M$; here $g_t$ is a curve of metrics on the boundary $\partial M$ with $g_0 = \gamma$. For $r$ related to $t$ as in (1.2), the integral curves of $\nabla r$ are also geodesics in $(M, g)$ and so the metric $g$ also splits as

$$g = dr^2 + g_r. \tag{2.8}$$

3. Renormalized Volume and Applications.

This section is divided into two subsections. In §3.1 we review the renormalized volume or action functional $V$ from the AdS/CFT correspondence, as well as the uniqueness theorem from [6]. These results are then used in §3.2 to prove that a connected group of conformal isometries of the boundary metric $\gamma$ of an AH Einstein metric $(M, g)$ extends to a subgroup of the isometry group $\text{Isom}(M, g)$, c.f. Theorem 3.1. In Theorem 3.3, we also prove that the set of non-degenerate AH Einstein metrics $\mathcal{E}_{AH}'$ is open and dense in the full space $\mathcal{E}_{AH}$.

§3.1. In this section, we summarize some results from the AdS/CFT correspondence, and results from [5], [6], needed for later purposes.

Let $g$ be an AH Einstein metric on a 4-manifold $M$, with boundary metric $\gamma$ w.r.t. a $C^{m,\alpha}$ compactification, $m \geq 4$, as in Proposition 1.6. Let $\tilde{g} = t^2 g$ be the $C^{m-1,\alpha}$ geodesic compactification determined by $\gamma$, as in (2.6)-(2.7). We may then expand $g_t$ in a Taylor series as

$$g_t = g_{(0)} + t^2 g_{(2)} + t^3 g_{(3)} + ... + t^{m-1} g_{(m-1)} + O(t^{m-1+\alpha}). \tag{3.1}$$

These coefficients may be invariantly defined by

$$g_{(j)} = (\mathcal{L}(\tilde{\nabla} t) \tilde{g})|_{\partial M}, \tag{3.2}$$

where $\mathcal{L}(\tilde{\nabla} t)$ is the $j$-fold Lie derivative. Observe that although the expression (3.2) gives symmetric bilinear forms on $TM|_{\partial M}$, the vector $\tilde{\nabla} t \in \text{Ker} g_{(j)}$ for all $j$ and so $g_{(j)}$ is uniquely determined by its restriction to $T(\partial M)$.

The term $g_{(0)} = \gamma$, while the term $g_{(1)}$, equal to the 2nd fundamental form of $\partial M$ in $(M, \tilde{g})$, vanishes. The term $g_{(2)}$ is given by

$$g_{(2)} = \frac{1}{2} \left( \text{Ric}_\gamma - \frac{s_\gamma}{4} \gamma \right), \tag{3.3}$$
while the \(g_{(3)}\) term satisfies
\[
\text{tr} \cdot g_{(3)} = 0, \quad \delta \cdot g_{(3)} = 0,
\]
i.e. \(g_{(3)}\) is transverse traceless. However, beyond the relations (3.4), the Einstein equations at \(\partial M\) do not determine the coefficients \(g_{(j)}\), for any \(j \geq 3\). These results follow from the work of Fefferman-Graham [13], c.f. also [12, 14]. Related results hold in higher dimensions, given suitable boundary regularity, up to the order \(g_{(a-1)}\).

Now the expansion (3.1) easily leads to an expansion for the volume of the region \(B(r) = \{x \in M : \rho(x) \geq 2e^{-r}\}\), of the form
\[
v_{OL}(B(r)) = v_{(0)}(r)^{3} + v_{(2)}(r)^{r} + V + o(1),
\]
c.f. again [12, 14] for instance. Although the coefficients \(v_{(0)}\) and \(v_{(2)}\) in (3.5) depend on the compactification \(\overline{\gamma}\), the constant term \(V\) is independent of \(\overline{\gamma}\) and depends only on \((M, g)\). The term \(V\) is the renormalized volume of \((M, g)\), c.f. [14, 29], and gives a smooth function
\[
V : \mathcal{E}_{AH} \rightarrow \mathbb{R}.
\]
In [5], it is proved that \(V\) is related to the \(L^2\) norm of the Weyl curvature of \((M, g)\) in the following simple way:
\[
\frac{1}{8\pi^{2}} \int_{M} |W|^{2} = \chi(M) - \frac{3}{4\pi^{2}}V.
\]
Hence, it is immediate that \(V\) depends only on \((M, g)\) and not on any choice of compactification.

Let \(dV\) be the differential of \(V\), viewed as a 1-form on \(T(E_{AH})\). In [5], it is also proved that
\[
dV(h) = \frac{1}{4} \int_{\partial M} \langle g_{(3)}, h_{(0)} \rangle \, dv_{OL},
\]
where the inner product and volume form are w.r.t. \(\gamma\) and \(h_{(0)} = \Pi_{s}(h)\) is the induced variation of the boundary metric \(\gamma\). Although (3.8) implies that \(dV\) is determined by the behavior at \(\partial M\), so that \(dV\) may be considered as a section of \(T^{*}M \otimes (\partial M)\), \(dV\) is not intrinsically determined by the boundary metric \(\gamma\). This is related to the non-uniqueness of AH Einstein metrics with a given boundary metric, and is discussed in [17, 29, 6], c.f. also § 7.

On the other hand, a uniqueness result from [6] states the following:

**Uniqueness Theorem.** Suppose \(\text{dim} M = 4\). Then the data \((\gamma, dV)\) on \(\partial M\) uniquely determine an AH Einstein metric up to local isometry, i.e. if \(g_{1}\) and \(g_{2}\) are two AH Einstein metrics on manifolds \(M_{1}\) and \(M_{2}\), with \(\partial M_{1} = \partial M_{2} = \partial M\) such that, w.r.t. the geodesic compactifications (2.7),
\[
\gamma_{1} = \gamma_{2} \quad \text{and} \quad dV_{g_{1}} = dV_{g_{2}},
\]
then \(g_{1}\) and \(g_{2}\) are locally isometric and the manifolds \(M_{1}\) and \(M_{2}\) have a diffeomorphic universal cover \(\tilde{M}\). In particular, if \(\pi_{1}(M_{1}) \simeq \pi_{1}(M_{2})\) and the isometric actions of \(\pi_{1}(M_{i})\) on \(\tilde{M}\) are conjugate, then \((M_{1}, g_{1})\) is isometric to \((M_{2}, g_{2})\).

Under the condition (3.9), if the geodesic defining functions for \(g_{1}\) and \(g_{2}\) agree near \(\partial M\), then the metrics \(g_{1}\) and \(g_{2}\) are equal in a neighborhood \(U\) of \(\partial M\), modulo a diffeomorphism of \(U\) preserving (3.9), and hence at least in \(\mathcal{D}_{2}\).

This uniqueness theorem assumed, for technical reasons, that \(g_{1}\) and \(g_{2}\) have \(C^{6, \alpha}\) compactifications \(g_{1}, g_{2}\). However, using Theorem A and the work in § 4–§ 5, this may be relaxed to the assumption that \(g_{1}\) and \(g_{2}\) have \(C^{3, \alpha}\) geodesic compactifications \(\tilde{g}_{1}, \tilde{g}_{2}\), by taking sequences \(g_{1}^{j}, g_{2}^{j}\) such that the \(C^{5, \alpha}\) geodesic compactifications \(\tilde{g}_{1}^{j}, \tilde{g}_{2}^{j}\) associated to \(g_{1}^{j}, g_{2}^{j}\) converge to \(\tilde{g}_{1}\) and \(\tilde{g}_{2}\) in the \(C^{3, \alpha}\) topology, c.f. also Remark 6.2.
§3.2. Next, we prove that continuous groups of conformal isometries of the conformal infinity extend to groups of isometries of any AH Einstein filling metric.

**Theorem 3.1.** Let $g$ be an AH Einstein metric on a 4-manifold $M$, with $C^{m,\alpha}$ boundary metric $\gamma, m \geq 4$, (in some compactification), and let $G$ be a connected Lie group of conformal isometries of the boundary metric $(\partial M, \gamma)$. Then $G$ extends to an action by isometries on $(M, g)$.

**Proof:** Let $\phi_s$ be a local 1-parameter group of conformal isometries of $\gamma = \Pi(g)$, with $\phi_0 = \text{id}$, so that

$$\phi_s^* \gamma = \lambda_s^2 \cdot \gamma, \quad (3.10)$$

where $\lambda_s$ is a 1-parameter family of positive smooth functions on $\partial M$. We may extend the diffeomorphisms $\phi_s$, in numerous ways, to diffeomorphisms of $M$ to obtain a curve $\tilde{g}_s = \phi_s^* g$ of AH Einstein metrics on $M$. Before choosing such an extension, first note that by (the proof of) Proposition 2.2, there is a curve of diffeomorphisms $\psi_s \in D_1$ such that the boundary metric of $\psi_s^* g$ is given by $\lambda_s^{-2} \cdot \gamma$. Let $t_s$ be the geodesic defining function for $\psi_s^* g$ w.r.t. the boundary metric $\lambda_s^{-2} \cdot g$.

We then choose the extension of $\phi_s$ into $M$ so that $\phi_s = \text{id}$ on $M$ and for $s \neq 0$,

$$\phi_s^* (t_s) = t, \quad (3.11)$$

in a collar neighborhood $U$ of $\partial M$, where $t = t_0$ is the geodesic defining function determined by $g$ and $\gamma$. Now consider the curve of AH Einstein metrics $g_s = \phi_s^* \psi_s^* g$. By construction, the curve $g_s$ thus has a fixed boundary metric $\gamma$ and fixed geodesic defining function $t_0$.

It then suffices to prove that there exists a curve of diffeomorphisms $F_s \in D_2$ such that

$$(F_s^* \phi_s^* \psi_s^* g) = F_s^* (g_s) = g, \quad (3.12)$$

so that the metric $g$ has a 1-parameter group of isometries inducing the given group at infinity.

We first prove this in a simple special case. Namely, suppose that $\dim K = 0$ at $(M, g)$, for $K$ as in (1.24). Then by Proposition 1.5, the map $\Pi : \mathcal{L}^{(2)}_{AH} \to \text{Met}(\partial M)$ is a local diffeomorphism near $g$, and hence $g$ is the unique AH Einstein metric with boundary metric $\gamma$ in a neighborhood of $g$ in $\mathcal{L}^{(2)}_{AH}$. Hence, (3.12) follows in this case, since the maps $\phi_s, \psi_s$ are close to the identity map on $M$, and $g_s$ has fixed boundary metric $\gamma$.

To prove (3.12) in general, by (3.11), the uniqueness theorem and the remark following it, it suffices to prove that the coefficient $g_{(3),s}$ remains constant among the family of metrics $g_s$, i.e. for small $s$,

$$g_{(3),s} = g_{(3)}, \quad (3.13)$$

Let $\mathcal{W}$ be the square of the $L^2$ norm of the Weyl curvature, as in (3.7). By (3.8), we have

$$d \mathcal{W}_{g_s}(h) = \frac{3}{2} \int_{\partial M} < g_{(3),s}, h_{(0)} > dvol_{\gamma},$$

for any infinitesimal AH Einstein deformation $h$. To calculate $\frac{d}{dh}(d \mathcal{W}_{g_s})$, consider the 2-parameter family of metrics $g_{s,u} = g_s + uh_s$, where $h_s$ is an infinitesimal AH Einstein deformation of $g_s$, with $h_{(0)} = h_s$, such that for the induced variation of the boundary metric $h_{(0)}$, we have $h_{(0),s} = h_{(0),0} = h_{(0)}$. From this and the fact that $g_s$ has fixed boundary metric, it follows that

$$\int_{\partial M} \frac{d}{ds} g_{(3),s}, h_{(0)} > dvol_{\gamma} = \frac{d}{ds} \int_{\partial M} < g_{(3),s}, h_{(0),s} > dvol_{\gamma} = \frac{2}{3} \frac{d}{ds} d \mathcal{W}_{g_s}(h_s).$$

Thus, we need to show that

$$\frac{d}{ds} \frac{d}{du} \mathcal{W}(g_s + uh_s) = 0, \quad (3.14)$$
at \( t = u = 0 \). We may interchange the derivatives and, for any fixed \( u \) and varying \( s \), write

\[
\frac{1}{s} \left[ \mathcal{W}(g_s + uh_s) - \mathcal{W}(g_0 + uh_0) \right] = \frac{1}{s} \left[ \mathcal{W}(g_s + uh_s) - \mathcal{W}(g_0 + u_0) \right] + \frac{1}{s} \left[ \mathcal{W}(g_0 + uh_s) - \mathcal{W}(g_0 + u_0) \right].
\]

For the first term, we have \( \mathcal{W}(g_s) = \mathcal{W}(g_0) \), since \( \mathcal{W} \) is invariant under diffeomorphisms. Hence, 
\( \mathcal{W}(g_s + uh_s) - \mathcal{W}(g_0 + u_0) = o(s) \), and so the first term vanishes in the limit as \( s \to 0 \). Similarly, letting \( s \to 0 \), the second term becomes

\[
d\mathcal{W}_{g_0,uh_s} \left( \frac{dh_s}{ds} \right),
\]

which vanishes since the metrics \( g_0 + uh_s \) all have the same boundary metric to 1st order.

This shows that (3.13) holds, which proves the result. \( \blacksquare \)

Remark 3.2. (i). Theorem 3.1 is not true in general for isometries of the boundary metric which are not isotopic to the identity on \( \partial M \). This follows from [6, Prop. 4.4], which exhibits infinitely many isometrically distinct AH Einstein metrics \( g_i \) on \( \mathbb{R}^2 \times T^2 \) with a fixed flat conformal infinity \( T^3 \), and \( g(3,i) = 0 \), for all \( i \). These metrics are all locally isometric. They are also the examples discussed following Theorem B in \( \$9 \).

(ii). Theorem 3.1 does not require that \( \partial M \) is connected. In fact, suppose \( \partial M \) is disconnected and \( g \) is an AH Einstein metric on \( M \), with boundary metric \( \gamma \) admitting an (effective) isometric \( S^1 \) action on one boundary component. Theorem 3.1 implies that the metric \( g \) has an induced isometric \( S^1 \) action, and hence this induces an effective \( S^1 \) action of the boundary metric on any other component of \( \partial M \). This gives non-trivial topological restrictions, since many 3-manifolds do not admit effective \( S^1 \) actions.

(iii). The proof of Theorem 3.1 in the situation where \( \dim K_g = 0 \) holds in all dimensions.

We complete this section with the following application. First, recall that \( E'_{AH} \) is the set of non-degenerate AH Einstein metrics on \( M \), i.e. those for which the \( L^2 \) kernel \( K \) is trivial.

Theorem 3.3. Suppose \( \dim M = 4 \), and that the inclusion map induces a surjection

\[
\pi_1(\partial M) \to \pi_1(M) \to 0.
\]

Then for \( m \geq 4 \), \( E'_{AH} \) is open and dense in \( E_{AH} \) and so generically, all metrics in \( E_{AH} \) are non-degenerate. In particular, \( \text{Im} \Pi \) has non-empty interior in \( \text{Met}(\partial M) \) if \( E_{AH} \neq \emptyset \).

Proof: It is clear that \( E'_{AH} \) is open in \( E_{AH} \). To prove that \( E'_{AH} \) is dense, suppose there is an open set \( U \subset E_{AH} \) such that \( E'_{AH} \cap U = \emptyset \). This means that all points in \( U \) are critical points for the map \( \Pi \), and so for every \( g \in U \), the \( L^2 \) kernel \( K_g \) of \( (M,g) \), c.f. (1.24), satisfies \( \dim K_g = k(g) \geq 1 \). The function \( \dim K \) is upper semi-continuous on \( E_{AH} \) and hence there exists a possibly smaller open set, also called \( U \), in \( E_{AH} \) such that \( \dim K = k = \text{const.} \) on \( U \). By the implicit function theorem, it follows that \( \Pi(U) \) is a submanifold of \( C(\partial M) \), of codimension \( k \), and, for \( \gamma \in \Pi(U) \), the fibers \( \Pi^{-1}(\gamma) \) of \( \gamma \) are \( k \)-dimensional submanifolds \( K_\gamma \) of \( E_{AH} \). Further, for any metric \( g \in K^{-1}(\gamma) \), there is a local slice to \( \Pi \) in \( U \), i.e. a map \( I_g : \Pi(U) \to E_{AH} \) such that \( \Pi \circ I_g = \text{id} \), with \( I_g(\gamma_0) = g \) and \( I_g|_{\gamma} \) a local diffeomorphism. As \( g \) ranges over the fiber \( K_\gamma = \Pi^{-1}(\gamma) \), the submanifolds \( \text{Im} I_g \) are disjoint, and give a local foliation of \( U \), transverse to the fiber \( K_\gamma \).

Now choose a pair of distinct metrics \( g_1 \) and \( g_2 \) in a connected component of \( K_\gamma \), whose conformal compactifications are close in the \( C^{4,\alpha'} \) topology. Consider the function

\[
F : \Pi(U) \to \mathbb{R}, \quad F(\gamma) = V(I_{g_1}(\gamma)) - V(I_{g_2}(\gamma)),
\]

where \( V \) is the renormalized volume. By the chain rule, we have

\[
dF_{g_0}(\gamma) = dV_{g_1}(\gamma) - dV_{g_2}(\gamma).
\]
We claim that
\[ dF_{\gamma_0} \neq 0. \]
For if \( dF_{\gamma_0} = 0 \), then we have \( dV_{g_1}(\gamma) = dV_{g_2}(\gamma) \) for all \( \gamma \), and \( g_1 \) and \( g_2 \) have the same boundary metric. The uniqueness theorem implies that \( g_1 \) is locally isometric to \( g_2 \). This means that on the universal cover \( \tilde{M} \), the representations of \( \pi_1(M) \) as subgroups of \( \text{Isom}(\tilde{M}, g_1) \) and \( \text{Isom}(\tilde{M}, g_2) \) are (small) deformations of each other. By (3.15), any representation of \( \pi_1(M) \) induces a representation of \( \pi_1(\partial M) \). Since the boundary metrics are fixed, so is the induced representation of \( \pi_1(\partial M) \), and hence so is the representation of \( \pi_1(M) \). Hence, \( g_1 \) and \( g_2 \) are isometric, a contradiction.

On the other hand, we claim that \( F \equiv 0 \) on \( \Pi(U) \). For, given any \( \gamma_0 \in \Pi(U), \) the metrics \( I_{g_1}(\gamma_0) \) and \( I_{g_2}(\gamma_0) \) are in the fiber \( \mathcal{K}_{\gamma_0} \). The function \( V \) however is constant on connected components of \( \mathcal{K}_{\gamma_0} \). To see this, if \( \sigma(t) \) is any curve in \( \mathcal{K}_{\gamma_0} \), then the tangent vector \( \sigma'(t) \in K_{\gamma_0} \sigma(t) \). However, since the boundary variation corresponding to any \( k \in K \) vanishes, (3.8) gives
\[
 dV(k) = 0, \quad \text{for all } k \in K.
\]
This contradiction implies that there is no open set of critical values of \( \Pi \) and hence \( \mathcal{E}_{AH}' \) is dense, and in particular non-empty, in \( \mathcal{E}_{AH}. \)

\[ \blacksquare \]

4. Compactness I: Interior Behavior.

The purpose of the next two sections is to prove Theorem B; this section deals with the interior behavior, while §5 is mostly concerned with the behavior at the boundary. The property that the boundary map \( \Pi \) is proper is a compactness issue. Thus, given a sequence of boundary metrics \( \gamma_i \) converging to a limit metric \( \gamma \), one needs to prove that a sequence of AH Einstein metrics \( (M, g_i) \) with boundary metrics \( \gamma_i \) has a subsequence converging, modulo diffeomorphisms, to a limit AH Einstein metric \( g \) on \( M \), with boundary metric \( \gamma \).

In §4.1, we summarize background material on convergence and degeneration of sequences of metrics in general, as well as sequences of Einstein metrics. This section may be glanced over and then referred to as necessary. The following section §4.2 then applies these results to the interior behavior of AH Einstein metrics.

§4.1. In this section, we discuss \( L^p \) Cheeger-Gromov theory as well as the convergence and degeneration results of Einstein metrics on 4-manifolds from [1], [2].

We begin with the \( L^p \) Cheeger-Gromov theory. The \( L^\infty \) Cheeger-Gromov theory [11], [16], describes the (moduli) space of metrics on a manifold, (or sequence of manifolds), with uniformly bounded curvature in \( L^\infty \), i.e.
\[
 R_g(x) \leq \Lambda < \infty, \quad (4.1)
\]
in that it describes the convergence or possible degenerations of sequences of metrics satisfying the bound (4.1). The space \( L^\infty \) is not a good space on which to carry out analysis, and so we replace (4.1) by a corresponding \( L^p \) bound, i.e.
\[
 \int_M |R_g|^p dV \leq \Lambda < \infty. \quad (4.2)
\]
The curvature involves 2 derivatives of the metric, and so (4.2) is analogous to an \( L^{2,p} \) bound on the metric. The critical exponent \( p \) w.r.t. Sobolev embedding \( L^{2,p} \subseteq C^0 \) is \( n/2 \), where \( n = \dim M \) and hence we will always assume that
\[
 p > n/2. \quad (4.3)
\]
In order to obtain local results, we need the following definitions of local invariants of Riemannian metrics, c.f. [3].
Definition 4.1. If \((M, g)\) is a Riemannian \(n\)-manifold, the \(L^p\) curvature radius \(\zeta(x) \equiv \zeta^p(x)\) at \(x\) is the radius of the largest geodesic ball \(B_\varepsilon(\zeta(x))\) such that, for all \(B_y(s) \subset B_\varepsilon(\zeta(x)), s \leq \text{dist}(y, \partial B_\varepsilon(\zeta(x)))\), one has

\[
\frac{s^{2p}}{\text{vol}B_y(s)} \int_{B_y(s)} |R|^pdV \leq c_0, \tag{4.4}
\]

where \(c_0\) is a fixed sufficiently small constant. Although \(c_0\) is an essentially free parameter, we will fix \(c_0 = 10^{-2}\) throughout the paper. The left-side of (4.4) is a scale-invariant local average of the curvature in \(L^p\).

The volume radius \(\nu(x)\) of \((M, g)\) at \(x\) is given by

\[
\nu(x) = \sup \{ r : \frac{\text{vol}B_y(s)}{\omega_n s^n} \geq \mu_0, \forall B_y(s) \subset B_r(x) \}, \tag{4.5}
\]

where \(\omega_n\) is the volume of the Euclidean unit \(n\)-ball and again \(\mu_0 > 0\) is a free small parameter, which will be fixed in any given discussion, e.g., \(\mu_0 = 10^{-2}\).

Observe that \(\zeta(x)\) and \(\nu(x)\) scale as distances, i.e., if \(g' = \lambda^2 \cdot g\), for some constant \(\lambda\), then \(\zeta'(s) = \lambda \cdot \zeta(x)\) and \(\nu'(x) = \lambda \cdot \nu(x)\). By definition, \(\zeta\) and \(\nu\) are Lipschitz functions with Lipschitz constant 1; in fact for \(y \in B_\varepsilon(\zeta(x))\), it is immediate from the definition that

\[
\zeta(y) \geq \text{dist}(y, \partial B_\varepsilon(\zeta(x))), \tag{4.6}
\]

and similarly for \(\nu\).

A sequence of Riemannian metrics \((\Omega_i, g_i, x_i)\) is said to converge in the \(L^{k,p}\) topology to a limit \(L^{k,p}\) metric \(g\) on \(\Omega\) if there is an atlas \(\mathcal{A}\) for \(\Omega\) and diffeomorphisms \(F_i : \Omega \to \Omega_i\) such that \(F_i^* (g_i)\) converges to \(g\) in the \(L^{k,p}\) topology in local coordinates w.r.t. the atlas \(\mathcal{A}\). Thus, the local components \((F_i^* (g_i))_{\alpha\beta} \to g_{\alpha\beta}\) in the usual \(L^{k,p}\) Sobolev norm on functions on \(\mathbb{R}^n\). Similar definitions hold for \(C^{m,\alpha}\) convergence, and convergence in the weak \(L^{k,p}\) topology. Any bounded sequence in \(L^{k,p}\) has a weakly convergent subsequence, and similarly any bounded sequence in \(C^{m,\alpha}\) has a convergent subsequence in the \(C^{m,\alpha'}\) topology, for any \(\alpha' < \alpha\). By Sobolev embedding,

\[
L^{k,p} \subset C^{m,\alpha}, \tag{4.7}
\]

for \(m + \alpha < k - \frac{n}{p}\). In particular, in dimension 4, for \(p \in (2,4), L^{2,p} \subset C^\alpha, \alpha = 2 - \frac{4}{p}\), and for \(p > 4, L^{2,p} \subset C^{1,\alpha}, \alpha = 1 - \frac{4}{p}\).

We then have the following result on the convergence and degeneration of metrics with bounds on \(\zeta\), c.f. [3].

Theorem 4.2. Let \((\Omega_i, g_i, x_i)\) be a pointed sequence of connected Riemannian \(n\)-manifolds and suppose there are constants \(\zeta_0 > 0, d_0 > 0\) and \(D < \infty\) such that, for a fixed \(p \geq n/2\),

\[
\zeta^p (y) \geq \zeta_0, \quad \text{diam} S_i \leq D, \quad \text{dist}(x_i, \partial \Omega_i) \geq d_0, \tag{4.8}
\]

Then for any \(0 < \epsilon < d_0\), there are domains \(U_i \subset \Omega_i\), with \(\epsilon/2 \leq \text{dist}(\partial U_i, \partial \Omega_i) \leq \epsilon\), for which one of the following alternatives holds.

(I). (Convergence) If there is constant \(\nu_0 > 0\) such that,

\[

\nu_i (x_i) \geq \nu_0 > 0,
\]

then a subsequence of \(\{(U_i, g_i, x_i)\}\) converges, in the weak \(L^{2,p}\) topology, to a limit \(L^{2,p}\) Riemannian manifold \((U, g, x), x = \lim x_i\). In particular, \(U_i\) is diffeomorphic to \(U\), for \(i\) sufficiently large.

(II). (Collapse) If instead,

\[
\nu (x_i) \to 0,
\]

then \(U_i\) has an \(F\)-structure in the sense of Cheeger-Gromov, c.f. [11]. The metrics \(g_i\) are collapsing everywhere in \(U_i\), i.e., \(\nu_i (y_i) \to 0\) for all \(y_i \in U_i\), and so in particular the injectivity radius
in \( j_h(y_i) \to 0 \). Any limit of \( \{g_i\} \) in the Gromov-Hausdorff topology \([16]\) is a lower dimensional length space.

**Remark 4.3.** (i). The local hypothesis on \( \zeta^p \) in (4.8) can be replaced by the global hypothesis

\[
\int_{\Omega_i} |R|^p \, dV \leq \Lambda, \tag{4.9}
\]

in case the volume radius satisfies \( \nu_i(y_i) \geq \nu_0 \), for some \( \nu_0 > 0 \) and all \( y_i \in \Omega_i \); in fact under this condition (4.8) and (4.9) are then equivalent, with \( \Lambda = \Lambda(\zeta_0, D, \nu_0) \), c.f. \([4\), Lemma 1.4].

(ii). It is an easy consequence of the definitions that the \( L^p \) curvature radius is \( \zeta^p \) continuous under convergence in the (strong) \( L^{2,p} \) topology, i.e. if \( g_i \to g \) in the \( L^{2,p} \) topology, then

\[
\zeta_i(x_i) \to \zeta(x), \tag{4.10}
\]

whenever \( x_i \to x \), c.f. \([3\) and references therein. However, (4.10) does not hold if the convergence is only in the weak \( L^{2,p} \) topology.

If in addition to a bound on \( \zeta^p \) one has \( L^p \) bounds the covariant derivatives of the Ricci curvature up to order \( k \), then in Case (I) one obtains convergence to the limit in the \( L^{k+2,p} \) topology. Analogous statements hold for convergence in the \( C^{m,\alpha'} \) topology.

Next we discuss the convergence and degeneration of Einstein metrics. If \( M \) is a closed 4-manifold and \( g \) an Einstein metric on \( M \), then the Chern-Gauss-Bonnet theorem gives

\[
\frac{1}{8\pi^2} \int_M |R|^2 \, dV = \chi(M), \tag{4.11}
\]

where \( \chi(M) \) is the Euler characteristic of \( M \). This gives apriori control on the \( L^2 \) norm of the curvature of Einstein metrics on \( M \). However the \( L^2 \) norm is critical in dimension 4 w.r.t. Sobolev embedding, c.f. \((4.3) \) and so one may not expect Theorem 4.2 to hold for sequences of Einstein metrics on \( M \). In fact, there is a further possible behavior of such sequences.

**Definition 4.4.** An Einstein orbifold \((X, g)\) is a 4-dimensional orbifold with a finite number of cone singularities \( \{q_j\}, j = 1, \ldots, k \). On \( X_\circ = X \setminus \cup \{q_j\} \), \( g \) is a smooth Einstein metric while each singular point \( q \in \{q_j\} \) has a neighborhood \( U \) such that \( U \setminus q \) is diffeomorphic to \( C(S^3/\Gamma) \setminus \{0\} \), where \( \Gamma \) is a finite subgroup of \( O(4) \) and \( C \) denotes the cone with vertex \( \{0\} \). Further \( \Gamma \neq \{e\} \) and when lifted to the universal cover \( \tilde{B}^4 \setminus \{0\} \) of \( C(S^3/\Gamma) \setminus \{0\} \), the metric \( g \) extends smoothly over \( \{0\} \) to a smooth Einstein metric on the 4-ball \( B^4 \).

There are numerous examples, at least on compact manifolds of non-negative scalar curvature, where sequences of smooth Einstein metrics \( g_i \) on \( M \) converge, in the Gromov-Hausdorff topology, to an Einstein orbifold limit \((X, g)\). Such orbifold metrics are not smooth metrics on the manifold \( M \), but may be viewed as singular metrics on \( M \), in that \( M \) is a resolution of \( X \), c.f. \([2\)\).

We then have the following result describing the convergence and degeneration of Einstein metrics on 4-manifolds, c.f. \([2\).

**Theorem 4.5.** Let \((\Omega_i, g_i, x_i)\) be a pointed sequence of connected Einstein 4-manifolds satisfying

\[
diam \Omega_i \leq D, \quad dist(x_i, \partial \Omega_i) \geq d_0, \tag{4.12}
\]

and

\[
\int_{\Omega_i} |\nabla g_i|^2 \, dV_{g_i} \leq \Lambda_0, \tag{4.13}
\]

for some constants \( d_0 > 0 \), \( D, \Lambda_0 < \infty \). Then for any \( 0 < \epsilon < d_0 \), there are domains \( U_i \subset \Omega_i \), with \( \epsilon/2 \leq \text{dist}(\partial U_i, \partial \Omega_i) \leq \epsilon \), for which exactly one of the following alternatives holds.

(1). (Convergence). A subsequence of \((U_i, g_i, x_i)\) converges in the \( C^\infty \) topology to a limit smooth Einstein metric \((U, g, x)\), \( x = \lim x_i \).
(II). (Orbifolds). A subsequence of \((U_i, g_i, x_i)\) converges to an Einstein orbifold metric \((X, g, x)\) in the Gromov-Hausdorff topology. Away from the singular variety, the convergence \(g_i \to g\) is \(C^\infty\), while the curvature of \(g_i\) blows up in \(L^\infty\) at the singular variety.

(III). (Collapse). A subsequence of \((U_i, g_i, x_i)\) collapses, in that \(\nu(y_k) \to 0\) for all \(y_k \in U_i\). The collapse is with uniformly bounded curvature metrically away from a finite number of singular points; however such singularities might be more complicated than orbifold cone singularities.

The cases (I) and (II) occur if and only if

\[
\nu_i(x_i) \geq \nu_0, \tag{4.14}
\]

for some \(\nu_0 > 0\), while (III) occurs if (4.14) fails. One obtains \(C^\infty\) convergence in (I), and (II) away from the singularities, since Einstein metrics satisfy an elliptic system of PDE, (in harmonic coordinates).

When \(\Omega_i\) is a closed 4-manifold \(M\), the bound (4.13) follows immediately from (4.11). When \((\Omega_i, g_i)\) are complete AHE metrics on a fixed 4-manifold \(M\), then of course (4.13) does not hold with \(\Omega_i = M\). However, in the normalization (0.3), we have \(W^2 = |R|^2 - 6\) where \(W\) is the Weyl tensor of \((M, g)\) and so by (3.7)

\[
\frac{1}{8\pi^2} \int_M (|R_g|^2 - 6) dV_g = \chi(M) - \frac{3}{4\pi^2} V. \tag{4.15}
\]

Thus, a lower bound on the renormalized volume \(V\) and upper bound on \(\chi(M)\) give a global bound on the \(L^2\) norm of \(W\). In particular, under these bounds (4.13) holds for \(\Omega_i\) a geodesic ball \(B_{x_i}(R)\) of any fixed radius \(R\) about the base point \(x_i \in (M, g_i)\).

We point out that Theorem 4.5 is special to dimension 4. A similar result holds in higher dimensions only if one has a uniform bound on the \(L^{n/2}\) norm of curvature in place of (4.13); however there is no analogue of (4.15) for (general) Einstein metrics in higher dimensions.

**Remark 4.6.** Theorems 4.2 and 4.5 are local results. However, we will often apply them globally, to sequences of complete manifolds, and with complete limits. This is done by a standard procedure as follows, in the situation of Theorem 4.2 for example. Suppose \(\{g_i\}\) is a sequence of complete metrics on \(M\), with base points \(x_i\), and satisfying \(\zeta(y_i) \geq \zeta_o\), for all \(y_i \in (M, g_i)\). One may then apply Theorem 4.2 to the domains \((B_{x_i}(R), g_i, x_i)\), where \(B_{x_i}(R)\) is the geodesic \(R\)-ball about \(x_i\) to obtain a limit manifold \((U(R), g_\infty, x_\infty)\), in the non-collapse case. Now take a divergent sequence \(R_j \to \infty\) and carry out this process for each \(j\). There is then a diagonal subsequence \(B_{x_i}(R_i)\) of \(B_{x_i}(R_j)\) which converges to a complete limit \((N, g_\infty, x_\infty)\). Similar arguments apply in the case of collapse and the cases of Theorem 4.5.

The convergence in these situations is also convergence in the pointed Gromov-Hausdorff topology, [16].

§4.2. In this section, we study the behavior of sequences of AHE metrics in the interior, i.e. away from \(\partial M\). We first discuss the hypotheses, and then state and prove the main result, Theorem 4.7.

Essentially for the rest of the paper, we will assume the topological condition on \(M = M^4\) that

\[
H_2(\partial M, \mathbb{R}) \to H_2(M, \mathbb{R}) \to 0, \tag{4.16}
\]

where the map is induced by inclusion \(i : \partial M \to \overline{M}\). As will be seen, this serves to rule out any orbifold degenerations of AH Einstein sequences. While many of the results of this paper carry over to the orbifold setting, c.f. Remark 1.7 for example, we prefer for simplicity to exclude this possibility here. (We intend to analyse the situation where (4.16) is not assumed in a separate paper).
Next, let \( \bar{g}_i \in \mathcal{E}_{AH} \) be a sequence of AH Einstein metrics on \( M \) with \( C^{m,\alpha} \) boundary metrics \( \gamma_i \), \( m \geq 3 \). As noted at the end of \( \S 2 \), the geodesic compactification \( \bar{g}_i \) of \( g_i \) with boundary metric \( \gamma_i \) is a \( C^{m-1,\alpha} \) compactification. We assume the boundary behavior of \( \{ g_i \} \) is controlled, in that
\[
\gamma_i \to \gamma, \tag{4.17}
\]
in the \( C^{m,\alpha'} \) topology on \( \partial M \), for some \( \alpha' \leq \alpha \). Next we assume that the inradius of \( (M, \bar{g}_i) \) has a uniform lower bound, i.e.
\[
\text{In}_{\bar{g}_i}(\partial M) = \text{dist}_{\bar{g}_i}(\bar{C}_i, \partial M) \geq \tau_o, \tag{4.18}
\]
for some constant \( \tau_o > 0 \), where \( \bar{C}_i \) is the cut locus of \( \partial M \) in \( (\bar{M}, \bar{g}_i) \) and also an upper diameter bound
\[
diam_{\bar{g}_i} S(t_1) \leq T, \tag{4.19}
\]
where \( t_1 = \tau_o/2 \) and \( S(t_1) = \{ x \in (M, \bar{g}_i) : t_i(x) = t_1 \} \) is the \( t_1 \)-level set of the geodesic defining function \( t_i \) for \( (g_i, \gamma_i) \).
Finally, we assume that the Weyl curvature of \( g_i \) is uniformly bounded in \( L^2 \), i.e.
\[
\int_M W_{g_i} \, dv_{g_i} \leq \Lambda_o < \infty. \tag{4.20}
\]
Most of these assumptions, namely (4.18)-(4.20), will be removed in \( \S 5 \). The main result of this subsection is the following

**Theorem 4.7.** Let \( \{ g_i \} \) be a sequence of metrics in \( \mathcal{E}_{AH} \) on \( M \), satisfying (4.16)-(4.20) and let \( x_i \) be a base point of \( (M, \bar{g}_i) \) satisfying
\[
d \leq \text{dist}_{g_i}(x_i, \partial M) \leq D, \tag{4.21}
\]
for some constants \( d > 0 \) and \( D < \infty \), where \( \bar{g}_i \) is the geodesic compactification associated to \( \gamma_i \).
Then a subsequence of \( (M, g_i, x_i) \), converges to a complete Einstein metric \( (N, g, x_\infty) \), \( x_\infty = \lim x_i \). The convergence is in the \( C^\infty \) topology, uniformly on compact subsets, and the manifold \( N \) weakly embeds in \( M \),
\[
N \subset M, \tag{4.22}
\]
in the sense that smooth bounded domains in \( N \) embed as such in \( M \).

**Proof:** By Theorem 4.5 and the discussion following (4.15), together with Remark 4.6, a subsequence of \( \{ (M, g_i) \} \) based at \( x_i \) either (i): converges smoothly to a complete limit Einstein manifold \( (N, g, x_\infty) \), (ii): converges to an Einstein orbifold, smoothly away from the singular variety, or (iii): collapses everywhere uniformly on compact subsets.

The first (and most important) task is to rule out the possibility of collapse. To do this, we first prove a useful volume monotonicity formula in the Lemma below; this result will also be important in \( \S 5 \).
Let \( (M, g) \) be any AH Einstein manifold, with \( C^{2,\alpha} \) geodesic compactification
\[
\bar{g} = l^2 \cdot g \tag{4.23}
\]
and boundary metric \( \gamma \). Let \( \bar{E} \) be the inward normal exponential map of \( (M, \bar{g}) \) at \( \partial M \), so that \( \sigma_x(t) = \bar{E}(x, t) \) is a geodesic in \( t \), for each fixed \( x \in \partial M \). Note from (2.7f) that \( \sigma_x(t) \) is also a geodesic in the Einstein manifold \( (M, g) \). Let \( \bar{J}(x, t) \) be the Jacobian of \( \bar{E}(x, t) \), so that \( \bar{J}(x, t) = \bar{dV}(x, t)/\bar{dV}(x, 0) \), where \( \bar{dV} \) is the volume form of the ‘geodesic sphere’ \( S(t) \), i.e. the \( t \)-level set of \( t \) as following (4.19). Finally, let \( t_o = t_o(x) \) be the distance to the cut locus of \( \bar{E} \) at \( x \in \partial M \).
Lemma 4.8. In the notation above, the function
\[ \frac{J(x,t)}{t_o^3(1 - \frac{t}{t_o})^3} \uparrow \] (4.24) is monotone non-decreasing in \( t \), for any fixed \( x \).

Proof: Let \( r = \log(\frac{T}{t}) \) as in (1.2) and, for any fixed \( x \), set \( r_o = r_o(x) = \log(\frac{T}{t_o}) \). Then the curve \( \sigma_x(r) \) is a geodesic in \( (M,g) \) and \( S(r) \) is the \( r \)-level set of the distance function \( r \). Let \( J(x,r) \) be the corresponding Jacobian for \( S(r) \) along \( \sigma_x(r) \), so that \( J(x,r) = t^{-3}J(x,t) \) by (4.33). Since \( Ric_g = -3g \), the infinitesimal form of the Bishop-Gromov volume comparison theorem \([16]\) implies
\[ \frac{J(x,r)}{\sinh^3(r - r_o)} \downarrow \]
is monotone non-increasing in \( r \), as \( r \to \infty \). Converting this back to \( (M,\bar{g}) \), it follows that
\[ \frac{J(x,t)}{\sinh^3(\log(\frac{T}{t}))} \uparrow , \]
since as \( r \) increases to \( \infty \), \( t \) decreases to 0. Since \( \sinh^3(\log(\frac{T}{t})) = \frac{1}{8}\log^3(1 - (\frac{t}{t_o})^2)^3 \), (4.24) follows. \( \blacksquare \)

Let \( q \) be any point in \( \partial M \) and consider the metric \( s \)-ball \( B_q(s) = \{ y \in \bar{M} : dist_g(y,q) \leq s \} \) in \( (\bar{M},\bar{g}) \). Let \( D_q(s) = B_q(s) \cap \partial M \), so that \( D_q(s) \) is the metric \( s \)-ball in \( (\partial M,\gamma) \), since \( \partial M \) is totally geodesic. Observe that there are constants \( \mu_0 > 0 \) and \( \mu_1 < \infty \), which depend only on the \( C^0 \) geometry of the boundary metric \( \gamma \), such that
\[ \mu_0 \leq \frac{\text{vol}_\gamma(D_q(s))}{s^3} \leq \mu_1. \] (4.25)

Let \( D_q(s,t) = \{ x \in M : x = \bar{E}(y,t), \text{for some } y \in D_q(s) \} \), so that \( D_q(s,t) \subset S(t) \). It follows immediately from (4.24) by integration over (a domain in) \( \partial M \) that
\[ \frac{\text{vol}_\gamma D(s,t)}{(s\tau_o)^3(1 - \frac{t}{\tau_o})^3} \uparrow, \text{ and } \frac{\text{vol}_\gamma S(t)}{\tau_o^3(1 - \frac{t}{\tau_o})^3} \uparrow, \] (4.26)
for \( \tau_o \) as in (4.18). Now let \( t_1 = \tau_o/2 \), so that (4.26) implies in particular that
\[ \text{vol}_\gamma S(t_1) \geq (\frac{3}{4})^3\text{vol}_\gamma \partial M, \] (4.27)
and hence, w.r.t. \( (M,g) \),
\[ \text{vol}_\gamma S(t_1) \geq (\frac{3}{4})^3t_1^3\text{vol}_\gamma \partial M. \] (4.28)

This leads easily to the following lower bound on volumes of balls.

Corollary 4.9. Let \( (M,g) \) be an AH Einstein metric, satisfying (4.18)-(4.19), and (4.21), so that \( d = dist_g(x, \partial M) \leq D \). Then
\[ \text{vol}_\gamma B_x(1) \geq \nu_o > 0, \] (4.29)
where \( \nu_o \) depends only on \( d, D, \tau_o, T \) and the \( C^0 \) geometry of the boundary metric \( \gamma \).

Proof: By the Bishop-Gromov volume comparison theorem on \( (M,g) \) again, we have for any \( R \geq 1 \),
\[ \frac{\text{vol} B_x(1)}{\sinh^3(1)} \geq \frac{\text{vol} B_x(R)}{\sinh^3(R)}. \]
Suppose \( r(x) = D_1 \), so that \( D_1 \in [-\log \frac{D}{2}, -\log \frac{d}{2}] \). Then for any \( y \in S(t_1) \), the triangle inequality and (4.19) imply that
\[
dist_g(x, y) \leq |D_1| + t_1^{-1} \cdot T \equiv D_2.
\]
Hence, choose \( R \) so that \( R = D_2 + 1 \), which implies that \( S(t_1) \subset B_x(R - 1) \). But \( \text{vol} B_x(R) \geq \text{vol} A(R - 2, R) \geq \frac{1}{2} \text{vol} S(t_1) \), where the second inequality follows from the coarea formula, (changing \( t_1 \) slightly if necessary). Combining this with the comparison estimate above implies
\[
\text{vol} B_x(1) \geq c_2 \text{vol} S(t_1) \sinh^{-3}(D_2) \geq c_3 t_1^{-3} \sinh^{-3}(D_2) \text{vol}_g \partial M,
\]
where the last inequality follows from (4.28). This gives (4.29).

Hence there is a uniform lower bound on the volume radius of each \( g_i \) at \( x_i \) satisfying (4.21) and so there is no possibility of collapse. Theorem 4.5 and Remark 4.6 then imply that \( (M, g_i, x_i) \) has a subsequence converging in the Gromov-Hausdorff topology either to a complete Einstein manifold \( (N, g, x) \) or to a complete Einstein orbifold \( (V, g, x) \).

Next, we use the hypothesis (4.16) to rule out orbifold limits. Thus, suppose the second alternative above holds, so that \( (M, g_i) \) converges in the Gromov-Hausdorff topology to a complete Einstein orbifold \((X, g)\). With each orbifold singularity \( q \in X \) with neighborhood of the form \( C(S^3/\Gamma) \), there is associated a (preferred) smooth complete Ricci-flat 4-manifold \((E, g_0)\), which is asymptotically locally Euclidean (ALE), in that \((E, g)\) is asymptotic to a flat cone \( C(S^3/\Gamma), \Gamma \neq \{e\}, \text{ c.f. } [1,5] \), [2,83]. The manifold \( E \) is embedded in the ambient manifold \( M \); in fact for any \( \delta > 0 \) and points \( y_i \rightarrow q \), \( E \) is topologically embedded in \((B_y(\delta), g_i)\). (The complete manifold \((E, g_0)\) is obtained as a limit of blow-ups or rescalings of the metrics \( g_i \) restricted to \( B_y(\delta) \)). The Einstein metrics \( g_i \) crush the topology of \( E \) to a point in that \( E \subset (B_y(\delta), g_i) \) and \( B_y(\delta) \) converges to the cone \( C(S^3/\Gamma) \), in the Gromov-Hausdorff topology for any fixed \( \delta > 0 \) sufficiently small.

Any such ALE space \( E \) has non-trivial topology; in fact by [1, Lemma 6.3]
\[
H_2(E, \mathbb{R}) \neq 0. \tag{4.30}
\]
(In all examples, \( H_2 \) is represented by essential 2-spheres in \( E \)). As in [1, p.487], we observe that there is an injection
\[
0 \rightarrow H_2(E, \mathbb{R}) \rightarrow H_2(M, \mathbb{R}). \tag{4.31}
\]
This follows from the Mayer-Vietoris sequence for (a thickening of) the pair \((E, M \setminus E)\):
\[
H_2(E \cap (M \setminus E), \mathbb{R}) \rightarrow H_2(E, \mathbb{R}) \oplus H_2(M \setminus E, \mathbb{R}) \rightarrow H_2(M, \mathbb{R}).
\]
Since \( E \cap (M \setminus E) = S^3/\Gamma \) , \( H_2(E \cap (M \setminus E), \mathbb{R}) = 0 \), which gives (4.31).

Let \( \Sigma \) be a non-zero 2-cycle in \( H_2(E, \mathbb{R}) \). By (4.16), there is a 2-cycle \( \Sigma_0 \) in \( H_2(\partial M, \mathbb{R}) \) homologous to \( \Sigma \), so that there is a 3-chain \( W \) in \( M \) such that \( \partial W = \Sigma - \Sigma_0 \). But \( W \cap (S^3/\Gamma) \neq \emptyset \) and since \( H_2(S^3/\Gamma, \mathbb{R}) = 0 \), we have \( W \cap (S^3/\Gamma) = \partial D \), for some domain \( D \subset S^3/\Gamma \). It follows that \( W \cap E \) is a 3-chain in \( E \), with boundary \( \Sigma \), i.e. \( [\Sigma] = 0 \) in \( H_2(E, \mathbb{R}) \), a contradiction.

Thus, the hypothesis (4.16) rules out any possible orbifold degeneration of the sequence \((M, g_i)\). Having ruled out the possibilities (ii) and (iii), it follows that (i) must hold. The fact that \( N \) is weakly embedded in \( M \) follows immediately from Remark 4.6 and the definition of smooth convergence following (4.6). This completes the proof.

\textbf{Remark 4.10.} (i). Note that Theorem 4.7 does not imply that the limit manifold \( N \) topologically equals \( M \). For instance, it is possible at this stage that some of the topology of \( M \) escapes to infinity, and is lost in the limit \( N \).

(ii). We also observe that the proof shows that there is a uniform bound \( \Lambda = \Lambda(\Lambda_o, d, D, T, \{\gamma_i\}) < \infty \) such that, for \( x \) satisfying (4.21),
\[
|R_{g_i}(x)| \leq \Lambda, \tag{4.32}
\]
i.e. there is a uniform bound on the sectional curvatures of \( \{g_i\} \) in this region.

(iii). The existence and behavior of \((M, g_i)\) at base points \(x_i\) where \(\text{dist}_{g_i}(x_i, \partial M) \to \infty\) will be discussed at the end of §5.

5. Compactness II: Boundary Behavior.

In this section, we extend the analysis in §4 to the boundary, and at the same time remove the hypotheses (4.18)-(4.20) from Theorem 4.7. Once this is done, the proof of Theorem B follows quite easily. In §5.2, we analyse the behavior when \(\Pi\) is not proper and prove that this is solely due to the formation of AH Einstein cusp metrics, c.f. Theorem 5.6 and the discussion following it.

§5.1. Theorem 4.7 essentially corresponds to a uniform interior regularity result in that we have uniform control of the AHE metrics \(g_i\) in the interior of \(M\), i.e. on compact subsets of \(M\). In this subsection, we extend this to similar control on the boundary.

We begin with the following result.

\textbf{Proposition 5.1.} Let \(\{g_i\}\) be a sequence in \(\mathcal{E}_{AH}\) with \(C^{m,\alpha}\) boundary metrics \(\{\gamma_i\}, m \geq 3, w.r.t.\) some compactifications \(\tilde{g}_i\), and suppose \(\gamma_i \to \gamma\) in the \(C^{m,\alpha'}\) topology on \(\partial M\), for some \(\alpha' < \alpha\). Let \(U_\delta = \{x \in \tilde{M} : \text{dist}_{\tilde{g}_i}(x, \partial M) \leq \delta\}\) and suppose the \(L^p\) curvature radius satisfies

\[\zeta^p(x) \geq \zeta_0,\]  

(5.1)

for some constant \(\zeta_0 > 0\), for all \(x \in U_{\delta}\).

Then there is a \(\delta_1 = \delta_1(\zeta_0, \gamma) > 0\) such that the \(C^{m-1,\alpha'}\) geometry of the geodesic compactifications \(\tilde{g}_i\) is uniformly controlled in \(U_{\delta_1}\), in that a subsequence of \(\{\tilde{g}_i\}\) converges, modulo diffeomorphisms in \(\mathcal{D}_1\), to a limit \(C^{m-1,\alpha}\) metric on \(U_{\delta_1}\). The convergence is in the \(C^{m-1,\alpha'}\) topology on \(U_{\delta_1}\). The same statement holds if \(m = \infty\) or \(m = \omega\).

\textbf{Proof:} By Proposition 1.6 and the discussion at the end of §2, the geodesic compactification \(\tilde{g}_i\) of \(g_i\) with boundary metric \(\gamma_i\) exists and is a \(C^{m-1,\alpha}\) compactification. Further, the proof of Proposition 1.6, as detailed in [6, Remark 2.5], implies that the \(C^{m-1,\alpha'}\) geometry of the metrics \(\tilde{g}_i\) is uniformly controlled in some \(U_{\delta_1}\), provided the \(C^{m,\alpha}\) geometry of the boundary metrics \(\gamma_i\) is uniformly bounded, and provided there is a uniform lower bound on the volume radius of \(\tilde{g}_i\) in \(U_{\delta_1}\).

The control on the boundary metrics is given by hypothesis. To obtain a lower bound on the volume radius \(v\), observe that Lemma 4.8, c.f. also (4.26), gives a lower bound on the volume ratios of balls, \(vol B_q(s)/s^4\) centered at points \(q \in \partial M\), depending only on the \(C^0\) geometry of the boundary metric. In particular, there is a constant \(v_\alpha > 0\) such that \(vol B_q(\zeta_0)/\zeta_0^4 \geq v_\alpha, q \in \partial M\), for all \((M, \tilde{g}_i)\). Within the \(L^p\) curvature radius, i.e. for balls \(B_x(r) \subset B_q(\zeta_0)\), standard volume comparison results imply then a lower bound \(vol B_x(r)/r^4 \geq v_1\), where \(v_1\) depends only on \(v_\alpha\), c.f. [3,§3]. This gives a uniform lower bound on the volume radius within \(U_{\zeta_0}\), and hence the result follows. \(\blacksquare\)

Proposition 5.1 implies in particular that the inradius of \(\tilde{g}_i\) is uniformly bounded below in terms of \(\zeta_0\), c.f. (4.18). It also gives the existence of the upper diameter bound \(T\) in (4.19), depending only on the boundary metric \(\gamma\) and \(\zeta_0\). We next show that there is a global \(L^2\) bound on \(W\).

\textbf{Proposition 5.2.} Under the assumptions of Proposition 5.1, there is a constant \(\Lambda = \Lambda(M, \gamma_i)\) such that

\[\int_M |W_{g_i}|^2dV_{g_i} \leq \Lambda.\]  

(5.2)

\textbf{Proof:} The integral (5.2) is conformally invariant. In the region \(U = U_{\delta_1}\) about \(\partial M\), we compute the \(L^2\) integral (5.2) w.r.t. the compactification \(\tilde{g}_i\). Thus, by Proposition 5.1, the curvature \(\tilde{R}_i\) is uniformly bounded in \(U\). Since \(vol_{\gamma_i}\partial M\) is also uniformly bounded, the curvature bound also gives a uniform upper bound on the volume of \((U, \tilde{g}_i)\). Hence, the integral over \(U\) in (5.2) is uniformly
bounded. For the integral over \( M \setminus U \), we use the Chern-Gauss-Bonnet theorem for manifolds with boundary, as in [5]. Let \( S(t_1) = \partial U_{\delta_1} \), viewed as a boundary in \((M, g_i)\), and let \( \Omega = M \setminus U \). Since \((\Omega, g)\) is Einstein, we have, dropping the subscript \( i \),
\[
\frac{1}{8\pi^2} \int_\Omega |W|^2 dV - \frac{1}{8\pi^2} \int_\Omega (R^2 - 6) = \chi(\Omega) - \frac{3}{4\pi^2} \text{vol} \Omega + \int_{S(t_1)} B(R, A) \leq \chi(M) + \int_{S(t_1)} B(R, A);
\]
compare also with (4.15). Here \( B(R, A) \) is a boundary term, depending on the curvature \( R \) and second fundamental form \( A \) of \( S(t_1) \) in \((M, g)\). But again by Proposition 5.1, \( R \) and \( A \) are uniformly controlled on \( S(t_1) \subset (M, g_i) \), as is \( \text{vol} g S(t_1) \). Hence, the \( L^2 \) norm of \( W \) over \( \Omega \) is uniformly bounded in \( i \) and so (5.2) follows.

Observe that the hypotheses (4.18)-(4.20) of Theorem 4.7 thus hold, provided (5.1) holds. Thus, the main remaining task is to prove the bound (5.1) holds.

**Theorem 5.3.** Let \((M, g)\) be an AH Einstein metric with a \( L^{2,p} \) conformal compactification, \( p > 4 \), and boundary metric \( \gamma \). Suppose further that the boundary metric \( \gamma \in C^{3,\alpha} \) and that, in a fixed coordinate system for \( \partial M, \| \gamma \|_{C^{3,\alpha}} \leq K \), for some \( \alpha > 0 \). If \( \tilde{g} \) is the associated \( C^{2,\alpha} \) geodesic compactification of \( g \) determined by \( \gamma \), then there are constants \( t_0 > 0 \) and \( \zeta_0 > 0 \), depending only on \( K, \alpha \), such that
\[
\zeta^p(x) \geq \zeta_0, \text{ for all } x \text{ with } t(x) \leq t_0. \tag{5.3}
\]

**Proof:** The proof is rather long, and so is broken into several steps. Overall, the estimate (5.3) is proved by contradiction and so we assume (5.3) is false, and show this leads to a contradiction. Before beginning, it is worth pointing out that the curvature of \( \tilde{g} \) at \( \partial M \) is uniformly bounded by the \( C^2 \) geometry of the boundary metric \( \gamma \), c.f. (A.8)-(A.10) in the Appendix.

If (5.3) is false, then there must exist a sequence of AHE metrics \( g_i \) on \( M \), with \( C^{2,\alpha} \) geodesic compactifications \( \tilde{g}_i \), for which the boundary metrics \( \gamma_i \) satisfy
\[
\| \gamma_i \|_{C^{3,\alpha}} \leq K, \tag{5.4}
\]
but for which the \( L^p \) curvature radius \( \zeta^p = \zeta^p(i) \) of \( \tilde{g}_i \) tends to 0 on some sequence \( x_i \) approaching \( \partial M \). The bound (5.4) implies that a subsequence of \( \gamma_i \) converges in the \( C^{3,\alpha} \) topology, \( \alpha' < \alpha \), to a limit \( C^{3,\alpha} \) metric \( \gamma \) on \( \partial M \).

For any \( x \in \partial M \), we have \( \zeta^p(x) > 0 \), and hence ratio
\[
\frac{\zeta^p(x)}{t(x)} \to \infty, \text{ as } x \to \partial M, \tag{5.5}
\]
again for each \( \tilde{g}_i, i \) fixed; here \( t = t_i \) is the geodesic defining function w.r.t. \( \tilde{g}_i \). Let
\[
W_i = \{ x \in (M, \tilde{g}_i) : \frac{\zeta^p(x)}{t(x)} \geq 1 \}; \tag{5.6}
\]
more precisely, we let \( W_i \) be the component of this set containing \( \partial M \). By assumption, there is a sequence \( x_i \in (M, \tilde{g}_i) \) such that \( \zeta^p(x_i) \to 0 \) and \( t(x_i) \to 0 \). The Lipschitz property of \( \zeta^p \), c.f. (4.6), then implies that there are points \( p_i \in \partial M \) such that \( \zeta^p(p_i) \to 0 \). In particular, there are points \( y_i \in W_i \) such that
\[
\zeta^p(y_i) \to 0, \text{ and } t(y_i) \to 0. \tag{5.7}
\]

**Step I. (Construction of a Blow-up Limit).**

We choose base points \( y_i \in W_i \) which realize the minimal value of \( \zeta = \zeta^p \) on the closed set \( W_i \); henceforth we drop the \( p \) from the notation. Of course \( y_i \) may be in the interior of \( W_i \), or on the boundary, and \( y_i \) satisfies (5.7). The boundary of \( W_i \) consists of two parts \( \partial W = \partial_i W_i \cup \partial_0 W_i \), where the inner boundary \( \partial_i W_i = \partial M \) and the outer boundary \( \partial_0 W_i = \{ x \in (M, \tilde{g}_i) : \frac{\zeta(x)}{t(x)} = 1 \}. \)
Let $U_i$ be the $\zeta$-tubular neighborhood of $W_i$, in that

$$U_i = \{ z \in (M, \tilde{g}_i) : z \in B_q(\zeta(q)) \}, \text{ for some } q \in W_i \}.$$  (5.8)

Then as before, $\partial U_i = \partial_1 U_i \cup \partial_2 U_i$, with $\partial_1 U_i = \partial M$. Note that $\partial_2 W_i$ is strictly contained in $U_i$.

Now blow-up or rescale the metrics $\tilde{g}_i$ at $y_k$ to make $\zeta'(y_k) = 1$, i.e. set

$$g'_i = \lambda^2 \cdot \tilde{g}_i,$$  (5.9)

where $\lambda = (\zeta(x_i))^{-1} \to \infty$. By the minimality property and the scaling of $\zeta$, we then have

$$\zeta'(x_i) \geq \zeta'(y_k) = 1,$$  (5.10)

for $x_i \in W_i$, where $\zeta'$ is the $L^p$ curvature radius w.r.t. $g'_i$. This means that the metrics $g'_i$ within $U_i$, and at any definite distance to $\partial_2 U_i$, have uniformly bounded curvature, locally and on the average, in $L^p$. Let $t'_i = \lambda_i t_i$, so that $t'_i$ is the geodesic defining function for $g'_i$. By definition of $\partial_2 W_i$ and (5.10), we have

$$t'_i(x_i) \geq 1, \text{ for all } x_i \in \partial_2 W_i,$$  (5.11)

so that $W_i$ contains the $g'_i$-tubular neighborhood of radius 1 about $\partial_2 W_i = \partial M$. For the same reason, $U_i$ contains the $g'_i$-tubular neighborhood of radius 2 about $\partial M$.

We recall from the proof of Proposition 5.1 that, within $U_i$, the volume radius of $\tilde{g}_i$ is bounded below by the curvature radius; $\nu(x) \geq \nu_0 \cdot \zeta(x)$ for some $\nu_0$ depending only on the boundary geometry. The volume radius also scales as a distance, and hence the metrics $g'_i$ have a uniform lower bound on their volume radius in $U_i$. Thus, the metrics $g'_i$ do not collapse anywhere on $U_i$.

By Theorem 4.2 a subsequence of $\{(U_i, g'_i, y_k)\}$ converges in the weak $L^{2,p}$ and $C^{1,\alpha}$ topologies, to an $L^{2,p}$ limit manifold $(U, g', y) = \lim y_k$. Observe that (5.10) and (5.6) imply that the $g'_i$-distance of $y_k$ to $\partial M$ remains uniformly bounded.

Now the boundary metrics $\gamma_i$ are uniformly controlled in $C^{3,\alpha}$ and so in particular have curvature uniformly controlled in $C^{1,\alpha}$. Hence by scaling, the $C^{1,\alpha}$ norm of the curvature of $\gamma'_i$ converges to 0. This means that the boundary metrics $\gamma'_i$ converge in $C^{3,\alpha'}$, $\alpha' < \alpha$, to a limit $C^{3,\alpha}$ metric $\gamma'$ on the inner boundary $\partial_2 U_i$ and the limit $\gamma'$ is flat, c.f. also Remark 4.3(ii). It then follows from the uniform $C^0$ control on $\{\gamma_i\}$ that $\partial_1 U = \mathbb{R}^3$.

Further, as explained in the proof of Proposition 5.1, Proposition 1.6 and [6, Remark 2.5] imply that the $C^{3,\alpha'}$ convergence of the boundary metrics extends to $C^{2,\alpha'}$ convergence of the geodesic compactifications $g'_i$ within $U_i$ to the $C^{2,\alpha}$ limit $g'$ in $U$. Thus, we have a stronger convergence to the limit than that nominally given by the $L^p$ curvature bound. (This stronger control of course comes about, indirectly, from the Einstein equations).

Since $t_i$ is a geodesic defining function, the 2nd fundamental form $A$ of $\partial M$ in $(M, \tilde{g}_i)$ satisfies $A \equiv 0$, i.e. the boundary is totally geodesic, c.f. also the Appendix. This property is preserved under blow-ups and since the convergence to the limit is in $C^{2,\alpha'}$, it follows that $\partial_1 U$ is also totally geodesic in $U$. We have

$$t'(x) = \text{dist}_{g'}(x, \partial_1 U) = \lim_{t \to \infty} t'_i(x).$$  (5.12)

**Step II.** $(U, g')$ is flat.

In this step, we prove that the blow-up limit $(U, g')$ is flat. Before beginning, since the boundary metric $\gamma'_i$ on $\mathbb{R}^3 = \partial W$ is real-analytic, (since it is flat), Proposition 1.6 again and the discussion following it imply that $g'$ is a $C^{\omega}$ metric on $U$, which extends real-analytically up to $\partial_1 U$. (Although useful, we will not actually use this degree of smoothness).

First, we derive some basic properties of the limit $(U, g', y)$. The equations (A.1)-(A.3) for the curvatures $R'$, $Ric'$ and $s'$ also hold on $(U, g')$, with $t'$ in place of $t$. We note that the Hessian $D^2 t' = A$, where $A$ is the 2nd fundamental form of the level sets of $t'$, and $\Delta t' = H$, the mean curvature of the level sets.
The following elementary result plays a crucial role in the proof.

**Lemma 5.4.** The function $t'$ is subharmonic on $(U, g')$, i.e.

$$\Delta t' = H \geq 0.$$  \hfill (5.13)

**Proof:** This essentially follows from Lemma 4.8. Namely, (5.13) is equivalent to the statement that the gradient flow of $t'$ is volume non-decreasing. Lemma 4.8, c.f. also (4.26), implies that

$$(s(1 - t^2))^{-3}vol_{g}(s, t)$$  \hfill (5.14)

is monotone increasing in $t$, for any fixed $s$, and $i$. To translate this statement to the blow-ups $g'_i$, let $\lambda_i$ be as in (5.9), and let $t = \lambda_i^{-1} \cdot t'$, $s = \lambda_i^{-1} s'$, where the parameters $t'$, $s'$ run over any fixed interval $[0, T']$, $[0, S']$. Then (5.14) is equivalent to the statement that

$$(s'(1 - (\lambda_i^{-1} t')^2))^{-3}vol_{g'}(s', t')$$

is monotone increasing in $t'$, for any fixed $s'$, $i$. Since $\lambda_i^{-1} \to 0$, it follows that on the limit $(U, g')$,

$$(s')^{-3}vol_{g'}D(s', t')$$

is monotone increasing in $t'$, $s'$ fixed. This gives the result. \hfill \blacksquare

We now use some of the curvature properties of geodesic compactifications given in the Appendix. First, by (A.3), the scalar curvature $s'$ of $(U, g')$ is given by

$$s' = -\frac{1}{6} \frac{\Delta t'}{t'} \leq 0,$$  \hfill (5.15)

where the inequality follows from (5.13). The equality in (5.15) holds off the cutlocus of $t'$ in $(U, g')$, and so in particular in a collar neighborhood of $\partial_t U$. However, the scalar curvature $s'_0$ of the boundary metric $g'_0$ on $\partial_t U = \mathbb{R}^3$ vanishes since $g'_0$ is flat. By (A.8), it follows that the ambient scalar curvature $s'$ of $g'$ also vanishes at $\partial_t U$. From (A.11), it follows that

$$s' \geq 0,$$  \hfill (5.16)

in the collar neighborhood $U'$ of $\partial_t U$ where $t'$ is smooth. Thus, (5.15) and (5.16) imply that in this region

$$s' \equiv 0.$$  \hfill (5.17)

From (5.13) and (5.15), we then have

$$\Delta t' = H = 0,$$  \hfill (5.18)

so that $t'$ is a smooth harmonic function, with $|\nabla t'| = 1$, in $U'$; the level sets of $t'$ are all minimal hypersurfaces. In the following, we generally drop the prime from the notation.

Now the Ricatti equation (A.7) for the $t'$ geodesics on $(U', g')$ gives

$$|A|^2 + \text{Ric}(\nabla t, \nabla t) = 0,$$  \hfill (5.19)

where $A$ is the 2nd fundamental form of the levels of $t = t'$. On the other hand, as noted above, the formula (A.2) for the Ricci curvature holds, so that on $(U', g')$,

$$\text{Ric} = -2t^{-1}D^2 t - t^{-1} \Delta t = -2t^{-1}D^2 t.$$  \hfill (5.20)

Clearly $D^2 t(\nabla t, \nabla t) \equiv 0$ in $U'$, and so (5.19) and (5.20) imply $|A|^2 = 0$, i.e. all the level sets are totally geodesic. Since $A = D^2 t$, (5.20) implies that $\text{Ric} \equiv 0$, so that $(U', g')$ is Ricci-flat. The vector field $\nabla t$ is thus a parallel vector field, and so $(U', g')$ splits as a product along the flow lines of $\nabla t$. It follows that $(U', g')$ is flat, which implies that all of $(U, g')$ is flat, as claimed.

It follows that all of the curvature in the interior of $(U, g'_i)$ goes to 0, uniformly on compact sets and at least in $C^0$ as $i \to \infty$. In fact, and this will be used in Step III, the discussion from Lemma 5.4 on can be done effectively on $(U, g'_i)$ in place of on the limit $(U, g')$. Thus, for any given $\varepsilon_0 > \ldots$
0, and for any $i \geq i_0 = i_0(\varepsilon_0)$, the estimate (5.13) translates to $H_{g_i^1} \geq -\varepsilon_o$ within $U_i$ and hence, arguing as in (5.15)-(5.18),

$$|\varepsilon_{g_i^1}| \leq \varepsilon_o, |\Delta t_{g_i^1}| \leq \varepsilon_o,$$

(5.21)

within $(U_i, g_i^1)$, (at any fixed distance away from $\partial_o U_i$). It follows as in (5.19)-(5.20) and following that

$$|R_{g_i^1}| \leq \varepsilon_o,$$

(5.22)

within $(U_i, g_i^1)$, (again away from $\partial_o U_i$).

**Step III. (Induction to Contradiction).**

Consider the geodesic balls $B_{y_i^1}(1)$ and recall that $\zeta'(y_i) = 1$. If $B_{y_i^1}(1)$ is strictly contained in $U_i$, in the sense that

$$\text{dist}_{g_i^1}(\partial_B y_i^1(1), \partial_o U_i) \geq \delta_o,$$

(5.23)

for some fixed $\delta_o > 0$, then one immediately has a contradiction, since the convergence of $(U_i, g_i^1)$ to $(U, g^1)$ is in the $C^{2,\alpha'}$ topology, and the curvature radius $\zeta$ is continuous in this topology. Note that for a flat manifold $U$ with boundary $\partial_o U$, $\zeta(x) = \text{dist}(x, \partial_o U)$.

By (5.11) and the line following it, we have $\text{dist}_{g_i^1}(\partial_o U_i, \partial M) \geq 2$. Since again $\zeta'(y_i) = 1$, it follows that we must have

$$\text{dist}_{g_i^1}(y_i, \partial_o W_i) \to 0, \text{ as } i \to \infty,$$

(5.24)

Further, the curvature of $g_i^1$ must blow up at some $q_i \in \partial_o U_i$, with $\text{dist}_{g_i^1}(q_i, y_i) \to 1$ to $y_i$, in that $\zeta'(q_i) \to 0$. If for not, then

$$\zeta'(q_i) \geq \zeta_o,$$

for some $\zeta_o > 0$ and all $q_i \in \partial B_{y_i^1}(1)$, which gives the same contradiction as following (5.23). (Sequences satisfying the condition above are called strongly buffered in [4] where it is shown that strong limits of strongly buffered sequences cannot be flat). Note also that necessarily

$$\text{dist}_{g_i^1}(q_i, \partial_o U_i) \to 2, \text{ as } i \to \infty.$$

(5.25)

It is worth emphasizing again that $\zeta_i(q_i) \ll \zeta_i(y_i)$ and so $\zeta_i(q_i) \ll \zeta_i(p_i)$, for any $p_i \in W_i$, so that the curvature blows up at $q_i$ much faster than anywhere in $W_i$.

We now carry out an induction argument leading to the contradiction. The steps in the proof above, in particular the hypothesis (5.7), describe the first level of the induction, with the points $q_i$ leading to the second level. (This induction argument is very similar, although much easier, than the main induction argument in [4, Thm.3.10]).

First, we relabel the data, changing $y_i$ to $y_i^1$, $g_i^1$ to $g_i^{2,1}$, $\zeta$ to $\zeta$, $t_i^{1,1}$ to $t_i^{1,2}$, $\partial M = \partial_i U_i$ to $L_i^{1,2}$ and $q_i$ to $q_i^2$. Choose an $\varepsilon_o > 0$ sufficiently small, and choose $i_o = i_o(\varepsilon_o) < \infty$ such that $g_i^{1,2}$ is $\varepsilon_o$-close to the flat metric in $B_{y_i^{1,2}}(\tfrac{9}{10})$, for all $i \geq i_o$. In the following, we work with any given $i \geq i_o$. Observe that the minimality property (5.10) of $y_i^{1,2}$ and (5.11ff) then implies that $\zeta_i^{1,2}$ is $\varepsilon_o$-close to the flat metric everywhere, in the sense of (5.22), in the $\tfrac{9}{10}$ band about $L_i^{1,2}$.

Now by the discussion above, the point $q_i^2 \in \partial B_{y_i^{1,2}}(1)$ with $t_i^{1,2}(q_i^2) \to 2$, satisfies for $i \geq i_o$, (i.o sufficiently large),

$$\zeta_i(q_i^2) \leq 10^{-1} \zeta_i(y_i^1) = 10^{-1}.$$  

(5.26)

Define the next level hypersurface $L_i^{2,2}$ to be

$$L_i^{2,2} = (t_i^{1,2})^{-1}(2 - \frac{9}{10}).$$

Note that $L_i^{2,2}$ is contained in the interior of $U_i$ and within the region where the metric $g_i^{1,2}$ is $\varepsilon_o$-flat. Define $t_i^{2,2}$ to be the $g_i^{1,2}$ distance to $L_i^{2,2}$: $t_i^{2,2}(x) = \text{dist}_{g_i^{1,2}}(x, L_i^{2,2})$. 

With this in place, we now repeat the blow-up construction in Steps I and II with the data 
\((g_i^1, L_i^2)\) in place of \((g_i, L_i)\), \(L_i = L_i^1\). Thus, form \(W_i^2 = \{ x \in M : \zeta_i^1(x)/t_i^2(x) \geq 1 \}\) and construct the next level of base point \(y_i^2\) as before, with rescaled metric
\[
g_i^2 = \zeta_i^1(y_i^2)^{-2}g_i^1,
\]
(5.27)
based at the point \(y_i^2\). Set also \(t_i^2 = \zeta_i^1(y_i^2)^{-1}t_i^2\); this is the distance function to \(L_i^2\) in the \(g_i^2\) metric. Define also \(U_i^2\) as before in (5.8) w.r.t. \(W_i^2\) in place of \(W_i^1\).

In Step II, the metric was proved to be \(\varepsilon_0\)-flat in \(U_i^1\) by examining the mean curvature \(H\) and scalar curvature \(s\) from monotonicity and the behavior at the boundary \(\partial M\). But these estimates still hold at the new boundary \(L_i^2\) - and in fact the terms are even smaller, since we are rescaling, (or blowing up), even further.

Thus, \(g_i^2\) is also \(\varepsilon_0\)-flat in \(U_i^2\), and \(U_i^2\) contains the \(t_i^2\)-band of size 2 about \(L_i^2\). This shows that exactly the same situation holds at the \(g_i^2\) scale as previously at the \(g_i^1\) scale. It follows that this process can be repeated indefinitely, at any fixed \(i \geq i_0\), since the \(\varepsilon_0\)-flat condition and scaling to \(\zeta = 1\) always implies the existence of boundary points satisfying (5.26) at the previously defined scale.

However, for any fixed \(i\), the inductive procedure gives base points \(y_i^k\) satisfying the scale-invariant condition
\[
\zeta(y_i^{k+1}) \leq 10^{-1}\zeta(y_i^k),
\]
and hence
\[
\zeta(y_i^{k+1}) \leq 10^{-k}\zeta(y_i^k).
\]
(5.28)
This is impossible for a fixed \(i\) for \(k\) sufficiently large, since the fixed manifold \((M, \bar{g}_i)\) has a fixed lower bound on its curvature radius \(\zeta\), (depending on \(i\)).

This contradiction shows that (5.7) cannot hold, which proves the result. \(\square\)

We now assemble the results above, and in §4 to obtain the following result, which represents a major part of Theorem B. We recall from [6] that if \((M, g)\) is an AH Einstein metric and \(\bar{g}\) is a geodesic compactification, then its width \(\text{Wid}_{\bar{g}}M\) is defined by
\[
\text{Wid}_{\bar{g}}M = \sup\{t(x) : x \in M\}.
\]
(5.29)
The width depends on a choice of the boundary metric \(\gamma\) for the conformal infinity. However, if \(\gamma\) and \(\gamma'\) are representatives in \([\gamma]\), then
\[
C^{-1}\text{Wid}_{\gamma}M \leq \text{Wid}_{\gamma'}M \leq C\text{Wid}_{\gamma}M,
\]
where \(C\) depends only on the \(C^1\) norm of \(\gamma^{-1}\gamma'\) and \((\gamma')^{-1}\gamma\) in a fixed coordinate system on \(\partial M\).

**Theorem 5.5.** Let \(\{g_i\}\) be a sequence of AH Einstein metrics on \(M\), with boundary metrics \(\gamma_i\) and suppose \(\gamma_i \to \gamma\) in the \(C^{m,\alpha'}\) topology on \(\partial M\), \(m \geq 3\). Suppose further that
\[
H_2(\partial M, \mathbb{R}) \to H_2(M, \mathbb{R}) \to 0,
\]
(5.30)
and there is a constant \(D < \infty\) such that
\[
\text{Wid}_{\gamma_i}M \leq D.
\]
(5.31)

Then a subsequence of \(\{g_i\}\) converges smoothly and uniformly on compact subsets to an AH Einstein metric \(g\) on \(M\) with boundary metric \(\gamma\). The geodesic compactifications \(\bar{g}_i\) converge in the \(C^{m-1,\alpha'}\) topology to the geodesic compactification \(\bar{g}\) of \(g\), within a fixed collar neighborhood \(U\) of \(\partial M\).
Proof: Theorem 5.3 and Proposition 5.1 together prove the last statement. From this, together with Proposition 5.2, it follows that the all the hypotheses (4.16)-(4.20) of Theorem 4.7 are satisfied. Thus if we choose base points \(x_i\) satisfying (4.21), then a subsequence of \((M, g_i, x_i)\) converges smoothly and uniformly on compact sets, to a limit AH Einstein metric \((N, g, x)\).

For a fixed \(d > 0\) small, let

\[
(M_i)_d = \{ x \in (M, g_i) : t_i(x) \leq d \},
\]

and define \(N_d\) in the same way. Then the smooth convergence of the compactifications \(\bar{g}_i\) implies that \((M_i)_d\) is diffeomorphic to \(N_d\) and each is a collar neighborhood of \(\partial M\). Further, the limit metric \(g\) on \(N_d\) is AH, with conformal infinity \([\gamma]\).

On the other hand, (5.31) implies that the complementary domains

\[
(M_i)^d = \{ x \in (M, g_i) : t_i(x) \geq d \}
\]

have uniformly bounded diameter w.r.t. \(g_i\), and so Theorem 4.7 or Theorem 4.2 again imply that \((M_i)^{d/2}\) is diffeomorphic to the limit domain \(N^{d/2}\) in \((N, g)\). It follows that \(N = M\) and so \(g\) is an AH Einstein metric on \(M\), with conformal infinity \([\gamma]\).

We are now in position to prove Theorem B.

Proof of Theorem B.

Let \(\gamma_i\) be a sequence of boundary metrics in \(C^0\), with \(\gamma_i \to \gamma \in C^0\) in the \(C^{m,\alpha'}\) topology on \(\partial M\), with \(\Pi(g_i) = [\gamma_i]\). Since only the conformal classes are uniquely determined, one may choose for instance \(\gamma_i\) to be metrics of constant scalar curvature. To prove \(\Pi^0\) is proper, we need to show that \(\{g_i\}\) has a convergent subsequence in \(E_{\text{AH}}\) to a limit metric \(g \in E_{\text{AH}}\) with \(\Pi[g] = [\gamma]\).

Suppose first there is a constant \(s_0 > 0\) such that

\[
s_{\gamma_i} \geq s_0,
\]

where \(s_{\gamma_i}\) is the (intrinsic) scalar curvature of the boundary metric \(\gamma_i\). It is then proved in [6, Prop.5.1] that

\[
\text{Width}_{3\gamma_i} M \leq \sqrt{3\pi / \sqrt{s_0}}.
\]

Hence, in this case Theorem B follows directly from Theorem 5.5.

Next, suppose only \(s_{\gamma_i} \geq 0\). If there is some constant \(D < \infty\) such that \(\text{Width}_{3\gamma_i} M \leq D\), then again Theorem 5.5 proves the result. Suppose instead

\[
\text{Width}_{3\gamma_i} M \to \infty.
\]

In this case, there is a rigidity result associated with the limiting case of (5.34) as \(s_0 \to 0\), proved in [6, Rmk.5.2, Lem.5.5]. Namely, \(s_{\gamma_i} \geq 0\) and (5.35) imply that the sequence \((M, g_i, x_i)\), for \(x_i\) as in (4.21), converges in the Gromov-Hausdorff topology to a hyperbolic cusp metric

\[
g_C = dr^2 + r^2 g_F,
\]

where \(g_F\) is a flat metric on \(\partial M\). It follows from the proof of Theorem 5.5, as in (5.32)ff, that \(\gamma = g_F, \) so that, by definition, \(\gamma \not\in C^0\). This implies that necessarily \(\text{Width}_{3\gamma_i} \leq D\), for some \(D < \infty\), which completes the proof.

\(\S 5.2\). In this section, we characterize the possible degenerations when \(\Pi\) is not proper. Observe first that Theorem 5.5 implies that if the manifold \(M\) satisfies (5.30), then the "enhanced" boundary map

\[
\Psi : E_{\text{AH}} \to \text{Met}(\partial M) \times \mathbb{R},
\]

\[
\Psi(g) = (\Pi(g), \text{Width}_{\Pi(g)} M)
\]
is proper. Thus, degenerations of AH Einstein metrics with controlled conformal infinity can only occur when the width diverges to $\infty$. On the other hand, as indicated in the Introduction, in general the boundary map $\Pi$ is not proper.

Define an AH Einstein metric with cusps $(N, g)$ to be a complete Einstein metric $g$ on a 4-manifold $N$ which has two types of ends, namely AH ends and cusp ends. A cusp end of $(N, g)$ is an end $E$ such that $\text{vol}_E E < \infty$. Thus, $N$ has a compact (possibly disconnected) hypersurface $H$, disconnecting $N$ into two non-compact connected components $N = N_1 \cup N_2$ where $(N_1, g)$ is an AH Einstein metric with boundary $H$ and $(N_2, g)$ has finite volume, so that each end of $N_2$ is a cusp end. A natural choice for $H$ is the level set $t^{-1}(1)$, where $t$ is a geodesic defining function for the conformally compact boundary $\partial_{AH} N$ of $N$. Then $N_1 = \{ x \in N : t(x) \leq 1 \}$, $N_2 = \{ x \in N : t(x) > 1 \}$.

Note that any cusp end $E$ is not conformally compact. As one diverges to infinity in $E$, the metric $g$ is collapsing, in that $\text{vol}_E B_x(1) \to 0$ as $x \to \infty$ in $E$, and hence the injectivity radius satisfies $\text{inj}_g(x) \to 0$, as $x \to \infty$ in $E$, see §4.1.

**Theorem 5.6.** Let $M$ be a 4-manifold satisfying (0.5), and let $g_i \in E_{AH}$ be AH Einstein metrics with boundary metrics $\gamma_i \in C^{m,\alpha}$, with $\gamma_i \to \gamma$ in the $C^{m,\alpha}$ topology on $\partial M, m \geq 3$. Let $t_i$ be the geodesic defining function associated with $\gamma_i$ and choose base points $x_i \in H_i = t_i^{-1}(1)$.

Then a subsequence of $\{g_i\}$ converges, modulo diffeomorphisms in $\mathcal{D}_1$, either to an AH Einstein metric $g$ on $M$, or to an AH Einstein metric with cusps $(N, g, x)$, $x = \lim x_i$. The convergence is smooth and uniform on compact subsets of $M, N$ respectively. In both cases, the conformal infinity is given by $(\partial M, [\gamma])$. Further, the manifold $N$ weakly embeds in $M$, as in (4.22).

**Proof:** For any $D < \infty$, let $(M_i)_D = \{ x \in (M, g_i) : t_i(x) \leq D \}$, as in (5.32). Theorem 5.5 implies that a subsequence of $(M_i)_D, g_i, x_i$ converges smoothly to a limit AH Einstein metric $g$ on a domain $N_D$, with $N_D$ diffeomorphic to $(M_i)_D$, and with conformal infinity of $N_D$ given by $(\partial M, [\gamma])$.

If there is a fixed $D < \infty$ such that (5.31) holds, then the result follows from the proof of Theorem 5.5 or Theorem B. Thus, we may suppose

$$Wid_{\gamma_i} M \to \infty.$$ (5.38)

In this case, it follows from [6, Lemma 5.4] that there is a constant $V_0 < \infty$, depending only on $\{\gamma_i\}$ and the Euler characteristic $\chi(M)$, such that

$$\text{vol}_{g_i} (M_i)^1 \leq V_0,$$ (5.39)

where $(M_i)^1 = \{ x \in (M, g_i) : t_i(x) \geq 1 \}$ is the complementary domain to $(M_i)_D$. (The estimate (5.39) is a straightforward consequence of (3.7), or more precisely the bound $V \leq \frac{4V_0^2}{3} \chi(M)$, given uniform control of the metrics $\bar{g}_i$ and $\bar{g}_i$ on $(M_i)_D$).

By Theorem 4.5 and Remark 4.6, the pointed manifolds $(M, g_i, x_i)$, for $x_i$ base points in $S_i(1) = t_i^{-1}(1)$, converge in a subsequence and in the Gromov-Hausdorff topology to a complete Einstein manifold $(N, g, x)$, $x = \lim x_i$. The convergence is also in the $C^\infty$ topology, uniform on compact sets. The domain $N^1 = \{ x \in (N, g) : t(x) \geq 1 \}$, with $t(x) = \lim t_i(x)$, is the limit of the domains $(M_i)^1$.

The bound (5.39) implies that $N^1$ is of finite volume while (5.38) implies that $N^1$ is non-compact. It follows that the full limit $N = N^1 \cup N_1$ with limit metric $g$ is a complete AH Einstein manifold with conformal infinity $(\partial M, [\gamma])$ and with a non-empty collection of cusp ends. The fact that $N$ weakly embeds in $M$ follows exactly as in the proof of Theorem 4.7.

The fact that the AH cusp metric $(N, g)$ has conformal infinity $(\partial M, [\gamma])$ and that it weakly embeds in $M$ implies that there is a sequence $t_j \to \infty$ such that $N_j \subset N$ embeds in $M$, for $N_j$ as
above. Hence, for any $j$ large, the manifold $M$ may be decomposed as

$$M = N^{t_j} \cup (M \setminus N^{t_j}).$$

(5.40)

With respect to a suitable diagonal subsequence $j = j_i$, the metrics $g_i$ on $M$ push the region $M \setminus N^{t_j}$ off to infinity as $i \to \infty$ and $t_{j_i} \to \infty$, giving rise to cusp ends in the limit $(N, g)$.

The main examples of AH Einstein metrics with cusps are complete hyperbolic 4-manifolds, which have a smooth conformal infinity, (possibly disconnected), together with a finite number of rank 3 hyperbolic cusp metrics of the form (5.36). Note that the conformal infinity of any hyperbolic manifold is necessarily conformally flat on $\partial M$, and thus very restricted; there is at most a finite dimensional space of such metrics on $\partial M$. In this context, we mention the following rigidity theorem from [6, Theorem 5.3]; if $(N, g)$ is an AH Einstein manifold with cusps, and one cusp end of $(N, g)$ is weakly hyperbolic, in the sense that

$$|K + 1|(x) \to 0 \text{ as } x \to \infty$$

in $E$, where $K$ denotes the sectional curvature, then $(N, g)$ is a complete globally hyperbolic manifold with cusps. It is an interesting open question whether there exist non-hyperbolic AH Einstein metrics with cusps.

Theorem 5.6 suggests the construction of a natural completion $\tilde{\mathcal{E}}_{AH}$ of $E_{AH}$. Thus, for any $R < \infty$ large, let $\mathcal{C}(R)$ denote the space of conformal classes $[\gamma]$ on $\partial M$ which contain a representative $\gamma$ satisfying $||\gamma||_{C^{0, \alpha}} \leq R$, w.r.t. some fixed coordinate system for $\partial M$. Let $E_{AH}(R) = \Pi^{-1}(R)$ and for any $g \in E_{AH}(R)$, choose a base point $x \in t^{-1}(1) \equiv H_g$, where $t$ is the geodesic defining function associated to the boundary metric $\gamma$.

Then Theorem 5.6 implies that the completion $\tilde{\mathcal{E}}_{AH}(R)$ of $E_{AH}(R)$ in the pointed Gromov-Hausdorff topology based at points $x \in H_g$ is the set of AH Einstein metrics on $M$, together with AH Einstein metrics with cusps $(N, g)$. If $\gamma_i$ are metrics in $\mathcal{C}(R)$, then a subsequence of $\gamma_i$ converges in $C^{m, \alpha'}$ to a limit metric $\gamma \in \mathcal{C}(R)$. If $g_i \in E_{AH}(R)$ satisfy $\Pi(g_i) = \gamma_i$, then the corresponding subsequence of $g_i$ converges in the Gromov-Hausdorff topology based at $x_i$ to $(N, g) \in \tilde{\mathcal{E}}_{AH}(R)$, and the conformal infinity of $(N, g)$ is $\gamma = \lim \gamma_i$. The convergence is also in the $C^\infty$ topology, uniform on compact sets. Of course one may have $N = M$, in which case the limit is an AH Einstein metric on $M$.

For $R < R'$, $E_{AH}(R) \subset E_{AH}(R')$ and so we may form the union $\tilde{\mathcal{E}}_{AH} = \cup_R \tilde{E}_{AH}(R)$ with the induced topology. It is clear that in this topology, the boundary map $\Pi$ extends to a continuous map

$$\tilde{\Pi} : \tilde{\mathcal{E}}_{AH} \to \mathcal{C}.$$  

(5.41)

Further, the following corollary is an immediate consequence of Theorem 5.6.

**Corollary 5.7.** Let $M$ be a 4-manifold satisfying (0.5). Then the extended map $\tilde{\Pi} : \mathcal{E}_{AH} \to \mathcal{C}$ is proper.

It follows in particular that the image

$$\tilde{\Pi}(\tilde{\mathcal{E}}_{AH}) \subset \mathcal{C}$$

is a closed subset of $\mathcal{C}$. However, it is not known if $\mathcal{E}_{AH}$ is a Banach manifold, as in Theorem A. Even if $\tilde{\mathcal{E}}_{AH}$ is not a Banach manifold, it would be interesting to understand the 'size' of the space of AH Einstein metrics with cusps $\partial \tilde{\mathcal{E}}_{AH} = \tilde{\mathcal{E}}_{AH} \setminus \mathcal{E}_{AH}$. In particular, with regard to the work to follow in §6-§7, one would like to know if the image $\tilde{\Pi}(\partial E_{AH})$ disconnects $\mathcal{C}$ or not, c.f. also Remark 7.7 below. Again the fact that $\tilde{\Pi}$ is proper implies that $\tilde{\Pi}(\partial E_{AH})$ is a closed subset of $\mathcal{C}$.

Let

$$\hat{\mathcal{C}} = \mathcal{C} \setminus \tilde{\Pi}(\partial E_{AH});$$  

(5.42)
this is the space of conformal classes on \( \partial M \) which are not the boundaries of AH Einstein metrics with cusps associated to the manifold \( M \). Also let \( \mathcal{E}_{AH} = \Pi^{-1}(\mathcal{C}) \), so that \( \mathcal{E}_{AH} \) is the class of AH Einstein metrics on \( M \) whose conformal infinity is not the conformal infinity of any AH Einstein metric with cusps associated to \( M \). If we give \( \mathcal{C} \) the relative topology, as a subset of \( \mathcal{C} \), then the following result is also an immediate consequence of the results above.

**Corollary 5.8.** Let \( M \) be a 4-manifold satisfying (0.5). Then the map

\[
\Pi : \mathcal{E}_{AH} \to \mathcal{C}
\]

is proper.

**Remark 5.9.** At present, the only known example where \( \Pi \) is not proper, for example \( \Pi^{-1}(pt) \) is non-compact in \( \mathcal{E}_{AH} \), is the sequence of AH Einstein metrics \( g_i \) on \( \mathbb{R}^2 \times T^2 \), converging to the hyperbolic cusp metric on \( \mathbb{R} \times T^3 \), see [6, Prop.4.4] for the explicit construction of \( \{g_i\} \).

However, these metrics lie in distinct components of the moduli space \( \mathcal{E}_{AH} = \mathcal{E}_{AH}^{(1)} \) on \( \mathbb{R}^2 \times T^2 \), and so cannot be connected by a curve of metrics on \( \mathbb{R}^2 \times T^2 \). Thus \( \mathcal{E}_{AH}^{(1)} \) has infinitely many components, and this is the case of \( \Pi \) being non-proper. It is possible that when \( \Pi \) is restricted to a component of \( \mathcal{E}_{AH}^{(1)} \), then \( \Pi \) is proper. On the other hand, if one passes to the quotient \( \mathcal{E}_{AH}^{(0)} \) by the full diffeomorphism group, as in Remark 2.4, then these metrics can be joined by a curve in \( \mathcal{E}_{AH}^{(0)} \).

6. Degree of the Boundary Map.

In this section, we prove that the boundary map \( \Pi^0 \) has a well-defined degree in \( \mathbb{Z} \) and \( \mathbb{Z} \), following Smale [25] and White [28], c.f. also [26]. Throughout this section, we work componentwise on \( \mathcal{E}_{AH}^{(0)} \) and \( \mathcal{C}^0 \), but we will not distinguish components with extra notation. Thus, we assume \( \Pi^0 : \mathcal{E}_{AH}^{(0)} \to \mathcal{C}^0 \) where \( \mathcal{E}_{AH}^{(0)} \) and \( \mathcal{C}^0 \) are connected, that is connected components of the full spaces. The results of this section also hold for the restricted boundary map \( \Pi : \mathcal{E}_{AH} \to \mathcal{C} \), but since there is no intrinsic characterization of \( \mathcal{C} \), we work with \( \mathcal{C}^0 \); see also Remark 7.7.

Since \( \Pi^0 \) is a proper Fredholm map of index 0, the Sard-Smale theorem [25] implies that the regular values of \( \Pi^0 \) are open and dense in \( \mathcal{C}^0 \). For \( \gamma \) a regular value, the fiber \( (\Pi^0)^{-1}(\gamma) \) thus consists of a finite number of points, i.e. (equivalence classes of) AH Einstein metrics on \( M \). By [25]

\[
\text{deg}_{\mathbb{Z}} \Pi^0 = \#(\Pi^0)^{-1}([\gamma]) \mod 2, \tag{6.1}
\]

is well-defined, for any regular value \([\gamma]\) in \( \mathcal{C}^0 \). We recall that if \([\gamma] \notin \text{Im}\Pi^0 \), then \([\gamma]\) is tautologically a regular value of \( \Pi^0 \).

Next, we show that \( \Pi^0 \) has a well-defined degree in \( \mathbb{Z} \), essentially following [28]. Thus, for \([\gamma] \in \text{Im}\Pi^0 \), let \( g \) be any AH Einstein metric on \( M \), with \( \Pi^0(g) = [\gamma] \). Consider the linearization of the Einstein equations, i.e. as in (1.23), the elliptic operator

\[
L = \frac{1}{2} D^* D - \bar{R},
\]

acting on \( L^2(M,g) \). The operator \( L \) is bounded below on \( L^2 \) and as in the theory of geodesics or minimal surfaces, let

\[
\text{ind}_g \in \mathbb{Z}, \tag{6.2}
\]

be the \( L^2 \) index of the operator \( L \) at \((M,g)\), i.e. the maximal dimension of the subspace of \( L^2(M,g) \) on which \( L \) is a negative definite bilinear form, w.r.t. the \( L^2 \) inner product. The nullity of \((M,g)\) is the dimension of the \( L^2 \) kernel \( K \).
The main result of this section is the following:

**Theorem 6.1.** Let \( \gamma \) be a regular value of \( \Pi^0 \) on \((M, g)\) and define

\[
\deg \Pi^0 = \sum_{g_i \in \Pi^0 \cap \{\gamma\}} (-1)^{\text{ind} g_i}.
\]  

(6.3)

Then \( \deg \Pi^0 \) is well-defined, i.e. independent of the choice of \([\gamma]\) among regular values of \( \Pi^0 \).

**Proof:** In [28], White presents general results guaranteeing the existence of a \( \mathbb{Z} \)-valued degree, and we will show that the current situation is covered by these results. Thus, we refer to [28] for some further details.

Let \([\gamma_1]\) and \([\gamma_2]\) be regular values of \( \Pi^0 \) and let \([\tilde{\sigma}(t)], t \in [0, 1]\) be an oriented curve in \( C^0 \) joining them. We choose representatives \( \gamma_1 \in [\gamma_1] \) and \( \gamma_2 \in [\gamma_2] \) and let \( \tilde{\sigma}(t) \) be a curve in \( Met(\partial M) \) joining \( \gamma_1 \) to \( \gamma_2 \). By [25], we may assume that \( \tilde{\sigma} \) is transverse to \( \Pi \), so that the lift \( \sigma = \Pi^{-1}(\tilde{\sigma}) \) is a collection of curves in \( \mathcal{E}_{AH} \), with boundary in the fibers over \( \gamma_1 \) and \( \gamma_2 \). Define an orientation on \( \sigma \) by declaring that \( \Pi \) is orientation preserving at any regular point of \( \sigma \) which has even index, while \( \Pi \) is orientation reversing at regular points of \( \sigma \) of odd index. Thus, provided this orientation is well-defined, the map \( \Pi|_\sigma : \sigma \rightarrow \tilde{\sigma} \) has a well-defined mapping degree, as a map of 1-manifolds. By construction, this 1-dimensional degree is given by (6.3) at any regular point of \( \sigma \) and hence it follows that (6.3) is well-defined. Thus, it suffices to prove that the orientation constructed above is well-defined.

If \( \Pi|_\sigma(d\sigma/dt) \neq 0 \) for all \( t \), so that all points of \( \sigma \) are regular, then the index of \( \sigma(t) \) is constant, and so there is nothing more to prove. Suppose instead that \( \Pi|_\sigma(\sigma'(t_0)) = 0 \), so that \( \sigma(t_0) \) is a critical point of \( \Pi \); (w.l.o.g. from here on assume \( \sigma \) is connected). Hence \( \sigma'(t_0) = \kappa_0 \in K \). Let \( K_0 = \langle \kappa_0 \rangle \) be the span of \( \kappa_0 \) in \( K \), and let \( \pi : S^m_{2, \mathcal{A}}(M) \rightarrow K_0 \) be the \( L^2 \) orthogonal projection onto \( K_0 \). Let \( g_0 = \sigma(t_0), \gamma_0 = \Pi(g_0) \) and as in (1.15), write \( \sigma(t) = g_t = g_{\gamma(t)} + h_t, \) where \( h_t \in S^m_{2, \mathcal{A}}(M) \); hence \( g_t = g_{\gamma(t)} + h_0 \) and \( \gamma(t) = \tilde{\sigma}(t) \). Also, let \( \kappa_1 = \pi(h_0) \).

By the implicit function theorem, c.f. [20], there exist neighborhoods \( I \) about \( t_0 \) and \( J \) about \( \kappa_1 \) in \( K_0 \) and a map

\[
\phi : I \times J \rightarrow S^m_{2, \mathcal{A}}(M), \quad \phi(t, \kappa) = h,
\]  

(6.4)

where \( h \) is the unique symmetric bilinear form such that

\[
\Phi(\gamma(t), h) \in K_0 \quad \text{and} \quad \pi(h) = \kappa \in K_0,
\]  

(6.5)

for \( \Phi \) as in (1.19), i.e. \( \Phi(\gamma, h) = \text{Ric}_g + h + 3\sigma g + h + \delta^s(B_{g}(h)) \). This defines a 2-parameter family of metrics \( g_{(t, \kappa)} = g_{\gamma(t)} + \phi(t, \kappa) \) in \( Met^m_{2, \mathcal{A}} \), where \( (t, \kappa) \) vary over a domain \( G \) in the \((I, J)\) plane \( P \). Recall that \( \Phi^{-1}(0) \) defines the set of AH Einstein metrics, and hence \( \Phi^{-1}(0) \) intersected with \( G \) is the curve \( \sigma(t) = g_t, t \in I \).

Now Einstein metrics, normalized so that \( \text{Ric}_g = -3g \), are critical points of the Einstein-Hilbert action defined (modulo physical constants) by

\[
I(g) = \int_M (s_g + 6)dvol_g + 2\int_{\partial M} H dvol_g,
\]  

(6.6)

c.f. [17], [29]. In the context of AH metrics, the expression (6.6) is meaningless, since both integrals are infinite. However, as described in §3, the renormalization procedure of the AdS/CFT correspondence shows that \( I \) may be renormalized to give a well-defined finite value \( \bar{I} \) when \( g \) is an AH Einstein metric. In fact \( \bar{I}(g) = -6\text{Vol}(g) \), where \( \text{Vol}(g) \) is the renormalized volume, since (3.4)-(3.5) imply that the boundary integral does not contribute to the renormalization. Exactly the same reasoning, (detailed further in [5, Remark 1.2]), shows that \( \bar{I}(g) \) is well-defined for metrics \( g' \in Met^m_{2, \mathcal{A}}(M) \) for which there is an AH Einstein metric \( g \) satisfying

\[
g' - g \in S^m_{3, \mathcal{A}}(M),
\]  

(6.7)
i.e. which differ from an Einstein metric by a term of order $\rho^3$. For such metrics $g' = g_\gamma + h'$, standard computation of the variation of $\bar I$ shows that

$$(D_2\bar I)_{(\gamma, h')} = (D_2\bar I)_{(\gamma, h)} + 3g'.$$  \hfill (6.8)

To apply this to the setting above, recall from Lemma 1.1 that any element $K$ in the $L^2$ kernel $K$ of an AH Einstein metric $g$ satisfies $\kappa \in S^m_{3, \alpha}(M)$. Hence, for sufficiently small neighborhoods $\bar I, J$ above, the map $\phi$ in (6.4) satisfies $\phi(t, \kappa) - h_t \in S^m_{3, \alpha}(M)$. Hence, the metrics $g(t, \kappa)$ above differ from an Einstein metric by a term of weight 3, for sufficiently small intervals $I, J$.

This means that the functional $\bar I$ is well-defined in a neighborhood of $\sigma(t)$ in the $(I, J)$ plane $P$ and (6.8) holds, so that the curve $\sigma(t) \subset P$ is the set in $P$ where $D_2\bar I = 0$. (The functional $\bar I$ plays the role of $g$ in [28].) It follows that $\sigma(t)$ is the boundary of the open set $\{D_2\bar I > 0\} \cap P$ in $P$.

$$\sigma(t) = \partial\{D_2\bar I > 0\}. \hfill (6.9)$$

If $\text{ind}_{\sigma(t)}$ is even, for $t$ near $t_0$, $t \neq t_0$, give $\sigma(t)$ the boundary orientation induced by this open domain, while if $\text{ind}_{\sigma(t)}$ is odd for $t$ near $t_0$, $t \neq t_0$, give $\sigma(t)$ the reverse orientation. The result then follows exactly as in [28]. Briefly, the point $t_0 \to \sigma(t_0) = (\gamma(t_0), \kappa_1)$ is a critical point for the map $\pi_1: I \times J \to I$ is projection onto the first factor. If this critical point is a folding singularity for $\pi_1 \circ \sigma$, then the index of $\sigma(t)$ changes by 1 in passing through $\sigma(t_0)$ and reverses the orientation of $\gamma(t) = \pi_1 \circ \sigma(t)$, (exactly as is the case with the standard folding singularity $x \to x^2$). On the other hand, if $\pi_1 \circ \sigma$ does not fold w.r.t. $\pi_1$, (so that one has an inflection point), then the index of $\sigma(t)$ does not change through $t_0$ and $\pi_1$ maps $\sigma(t)$ to $\gamma(t)$ in an orientation preserving way. We refer to [28] for further details, and also to §7 for concrete examples of folding behaviors.

\begin{remark}
As in (2.3), we have the inclusion $\mathcal{E}^{(m', \alpha')}_{AH} \subset \mathcal{E}^{(m, \alpha)}_{AH}$, for $(m', \alpha') \geq (m, \alpha)$. The work in §5 shows that the same inclusion holds for the completions $\hat{\mathcal{E}}_{AH}$, as in (5.40). Further, Theorem 5.6 implies that the inclusion of the completion $\mathcal{E}^{(m', \alpha')}_{AH}$ is dense in $\mathcal{E}^{(m, \alpha)}_{AH}$; hence of course $\mathcal{E}^{(m', \alpha')}_{AH}$ is at least dense in $\mathcal{E}^{(m, \alpha)}_{AH}$. It follows that the degree, both on $\mathcal{E}^0_{AH}$ and on $\hat{\mathcal{E}}_{AH}$ as in (5.42ff), is independent of $(m, \alpha)$, for $m \geq 3$.
\end{remark}

7. Computations of the Degree.

We conclude the paper with some computations of $\text{deg}\Pi^0$ for several interesting examples of 4-manifolds. In all cases, the evaluation of the degree is made possible by symmetry arguments, using Theorem 3.1.

Recall that if $\text{deg}\Pi^0 \neq 0$, then $\Pi$ is surjective onto $\mathcal{C}^0$, and so any conformal class $[\gamma]$ in $\mathcal{C}^0$ on $\partial M$ may be filled in to an AH Einstein metric $g$ with conformal infinity $[\gamma]$. Of course, $\text{deg}\Pi^0 = 0$ does not imply that $\Pi$ cannot be surjective.

Observe also that if $M$ has a “seed” AH Einstein metric $g_0$, then Theorem 3.3 implies that $\Pi(\mathcal{E}_{AH}) \subset \mathcal{C}$ has non-empty interior, at least when $\partial^1_1(\partial M)$ surjects onto $\pi_1(M)$. By the work of Witten-Yau [30], this is always the case when $\partial M$ admits a metric of positive scalar curvature.

We begin with the proof of Theorem C.

\textbf{Proof of Theorem C.}

As seed metric, we take the hyperbolic (Poincaré) metric $g_0$ on $B^4$. This has conformal infinity $[\gamma_0]$, where $\gamma_0$ is the round metric on $S^3 = \partial B^4$. Now the boundary metric $\gamma_0$ admits a large connected group $\text{Conf}(S^3)$ of conformal isometries. Theorem 3.1 implies that for any AH Einstein metric $g$ on $B^4$, $\text{Conf}(S^3)$ acts effectively by isometries on $(B^4, g)$. It is then standard that $g$ must be the Poincaré metric on $B^4$.

It follows that up to isometry, $g_0$ is the unique metric with conformal infinity $[\gamma_0]$. Further, it is well-known that the $L^2$ kernel $K$ of $g_0$ is trivial, i.e. $K = \{0\}$; in fact, this holds for any Einstein
metric of negative sectional curvature, c.f. [7], [8]. Thus, \( g_0 \) is a regular point of \( \Pi \) and since \( \Pi^{-1}[\gamma_0] = g_0 \), the point \( \{\gamma_0\} \) is a regular value of \( \Pi \). The result then follows from (6.3).

We point out the following immediate consequence of the proof.

**Corollary 7.1.** Let \( M \) be any 4-manifold satisfying (0.5), with \( \partial M = S^3 \) and \( M \neq B^4 \). Then on any component of \( \mathcal{E}^0_{AH} \),

\[
\deg \Pi^0 = 0, \tag{7.1}
\]

and \( \Pi \) is not surjective.

**Proof:** Suppose the class \([\gamma_0] \in \text{Im} \Pi\), so that there is an AH Einstein metric \( g \) on \( M \), with boundary metric \( \gamma_0 \). The proof of Theorem C above implies that necessarily \( g = g_0 \), where \( g_0 \) is the Poincaré metric on \( B^4 \). Hence, \( M = B^4 \), a contradiction.

Next, we have:

**Proposition 7.2.** Let \( M = \mathbb{R}^2 \times S^2 \), so that \( \partial M = S^1 \times S^2 \). Then

\[
\deg \Pi^0 = 0, \tag{7.2}
\]

and \( \Pi \) is not surjective.

**Proof:** As seed metric(s) in this case, we take the remarkable 1-parameter family of AdS Schwarzschild metrics, discussed in detail in [17], c.f. also [29]. Thus, on \( \mathbb{R}^2 \times S^2 \), consider the metric

\[
g_m = F^{-1} dr^2 + F d\theta^2 + r^2 g_{S^2(1)}, \tag{7.3}
\]

where \( F = F(r) = 1 + r^2 - 2m \). The mass parameter \( m > 0 \) and \( r \in [r_+, \infty) \), where \( r_+ \) is the largest root of the equation \( F(r) = 0 \). The locus \( \{r_+ = 0\} \) is a totally geodesic round 2-sphere \( S^2 \), of radius \( r_+ \). Smoothness of the metric at \( \{r_+ = 0\} \) requires that the circle parameter \( \theta \) run over the interval \([0, \beta]\), where \( \beta \) is given by

\[
\beta = \frac{4\pi r_+}{1 + 3r_+^2}. \tag{7.4}
\]

It is easily seen that as \( m \) varies from 0 to \( \infty \), \( r_+ \) varies monotonically from 0 to \( \infty \).

The metrics \( g_m \) are isometrically distinct, for distinct values of \( m \), and form a smooth curve in \( \mathcal{E}_{AH} \), with conformal infinity given by the conformal class of the product metric \( \gamma_m = S^1(\beta) \times S^2(1) \). Notice however that the length \( \beta \) has a maximum value as \( m \) ranges over \((0, \infty)\), namely

\[
\beta \leq \beta_{\max} = 2\pi / \sqrt{3}, \tag{7.5}
\]

achieved at \( r_+ = 1 / \sqrt{3} \), \( m = m_0 = 2 / \sqrt{3} \). As \( m \to 0 \) or \( m \to \infty \), one has \( \beta \to 0 \).

Thus, the boundary map \( \Pi \) on the curve \( g_m \) is a fold map, folding the ray \( m \in (0, \infty) \) onto the \( \beta \)-interval \((0, \beta_{\max})\). Hence \( \Pi \) restricted to the curve \( g_m \) is a 2-1 map, except at the point \( g_{m_0} \). By means of symmetry arguments as below, it is not difficult to verify that the metrics \( g_m, m \neq m_0 \) are regular points of \( \Pi \), while \( g_{m_0} \) is critical point of \( \Pi \); the tangent vector \((d g_m / d m)_{m = m_0}\) spans the \( L^2 \) kernel \( K_{g_{m_0}} \).

Next, we claim that the metrics \( g_m \) are the only AH Einstein metrics on \( \mathbb{R}^2 \times S^2 \) with conformal infinity given by a product \( \gamma_L = S^1(L) \times S^2(1) \). The isometry group \( \text{Isom}(\gamma_L) \) is \( O(2) \times SU(2) \) and by Theorem 3.1, any AH Einstein metric \( g_L \) on \( M \) with boundary metric \( \gamma_L \) has \( O(2) \times SU(2) \) acting effectively by isometries. As in (2.7), we may choose a geodesic defining function for \( g_L \) so that \( g_L \) has the form

\[
g_L = ds^2 + g_s, \tag{7.6}
\]

where \( g_s \) is a curve of metrics on \( S^1 \times S^2 \) invariant under the \( O(2) \times SU(2) \) action. Thus, the Einstein metric \( g_L \) has cohomogeneity 1. It then follows from the classification given in [23] for instance that the metric \( g_L \) on \( \mathbb{R}^2 \times S^2 \) is isometric to \( g_m \), for some \( m = m(L) \).
We may thus compute $\text{deg}\Pi^o$ by evaluating the formula (6.3) on a pair of distinct metrics $g_{m_1}$ and $g_{m_2}$ with $\Pi(g_{m_1}) = \Pi(g_{m_2})$. From [17, §3], one has
\begin{equation}
\text{ind}_{g_{m_1}} = +1, \text{ind}_{g_{m_2}} = 0,
\end{equation}
for $m_1 < m_2 < m_2$, which proves (7.2). Alternately, since $\Pi$ is a 2-1 fold map on $g_m$, the proof of Theorem 6.1 shows directly that (7.2) holds.

Further, the symmetry argument above implies that the metrics $S^1(L) \times S^2(1)$ are not in $\text{Im}\Pi$, whenever
\begin{equation}
L > \beta_{\text{max}},
\end{equation}
and hence $\Pi$ is not surjective.

This result should be compared with the following:

**Proposition 7.3.** Let $M = S^1 \times \mathbb{R}^3$, so that $\partial M = S^1 \times S^2$. Then
\begin{equation}
\text{deg}\Pi^o = 1,
\end{equation}
and $\Pi^o$ is surjective.

**Proof:** As seed metrics in this situation, we take a family of hyperbolic metrics, namely the metrics $\mathbb{H}^1(-1)/\mathbb{Z}$, where the $\mathbb{Z}$-quotient is obtained by a hyperbolic or loxodromic translation of length $L$ along a geodesic in $\mathbb{H}^1(-1)$. The conformal infinity is the product metric $S^1(L) \times S^2(1)$ in the case of a hyperbolic translation, and the bent product metric $S^1(L) \times_\alpha S^2(1)$ on the same space when the translation is loxodromic; the angle $\alpha$ between the factors $S^1(L)$ and $S^2(1)$ corresponds to the loxodromic rotation.

As in Proposition 7.2, any AH Einstein metric on $S^1 \times \mathbb{R}^3$ with boundary metric $S^1(L) \times S^2(1)$ has $O(2) \times SU(2)$ acting effectively by isometries. Again, the classification in [23] implies that, on this manifold, the only such metrics are hyperbolic. Hence the result follows as in the proof of Theorem C.

Next, we turn to non-trivial disc bundles over $S^2$. For the disc bundle of degree 1 over $S^2$, i.e. $M = \mathbb{CP}^2 \setminus B^4$, with $\partial M = S^3$, Corollary 7.1 implies that
\begin{equation}
\text{deg}\Pi^o = 0,
\end{equation}
and $\Pi$ is not surjective.

**Remark 7.4.** Actually, to justify this statement we need to remove the hypothesis (0.5), since in this situation $H_2(\partial M)$ does not surject onto $H_2(M)$. Recall that (0.5) was only used to rule out orbifold degenerations in the proof of Theorem 4.7. However, it is possible to rule out orbifold degenerations in the case of sufficiently low Euler characteristic directly, without the use of (0.5).

This situation was treated in [1] in the case of orbifold degenerations on compact manifolds, and the argument for AH metrics is almost exactly the same. Thus, we sketch the argument here, and refer to [1] for some further details.

Let $M$ be a disc bundle over $S^2$, so that $\chi(M) = 2$, and let $\{g_k\}$ be a sequence of AH Einstein metrics on $M$ which converge to an AH Einstein orbifold $(X, g)$, with boundary metrics $\gamma_k$ converging in $C^{m,\alpha}$ to the boundary metric $\gamma$ for $(X, g)$. For simplicity, we assume that $X$ has only one orbifold singularity; the proof for more than one is the same. By (3.7), we have
\begin{equation}
\frac{1}{8\pi^2} \int_M |W_{g_k}|^2 = 2 - \frac{3}{4\pi^2} V(g_k).
\end{equation}
The results of Theorem 4.7, Theorem 5.3 and Proposition 5.2 show that $V(g_k) \to V(g)$, where $V(g)$ is the renormalized volume of the Einstein orbifold $(X, g)$. On the other hand, by [1, Prop.6.1],
\begin{equation}
\lim_{i \to \infty} \int_M |W_{g_k}|^2 \geq \int_X |W_g|^2 + \int_E |W_{g_\infty}|^2,
\end{equation}
where the second integral is over the Ricci-flat ALE manifold \((E, g_\infty)\) attached to the orbifold singularity of \(X\), c.f. also the proof of Theorem 4.7. The formula (3.7) for Einstein orbifolds \((X, g)\) gives
\[
\frac{1}{8\pi^2} \int_X |W_g|^2 = \chi(X_o) + \frac{1}{|\Gamma|} - \frac{3}{4\pi^2} V(g),
\]
where the orbifold singularity of \(X\) is \(C(S^3/\Gamma)\), c.f [1, (6.2)]. A simple Mayer-Vietoris argument as following (4.31) implies \(\chi(X_o) \leq 1\). Combining these estimates, one obtains
\[
\frac{1}{8\pi^2} \int_E |W_{g_\infty}|^2 \leq 1 - \frac{1}{|\Gamma|} < 1.
\]
However, this contradicts [1, (6.5)].

It follows that orbifold degenerations cannot occur when \(\chi(M) \leq 2\), and so the hypothesis (0.5) is not necessary when \(M\) is a disc bundle over \(S^2\).

**Example 7.5.** It is interesting to compare the result (7.9) with an explicit family of AH Einstein metrics on \(M = \mathbb{CP}^2 \setminus B^4\), namely the AdS Taub-Bolt family, c.f. [18], [23]. This is a 1-parameter family of metrics given by
\[
g_s = E_s \left\{(r^2 - 1)F^{-1}(r)dr^2 + (r^2 - 1)^{-1}F(r)(d\tau + \cos \theta d\phi)^2 + (r^2 - 1)g_{S^2(1)}\right\},
\]
(7.10)
where the parameters \(r, s\) satisfy \(r \geq s\) and \(s > 2\), the constant \(E_s\) is given by
\[
E_s = \frac{2}{3} \frac{s - 2}{s^2 - 1},
\]
(7.11)
and the function \(F(r) = F_s(r)\) is
\[
F_s(r) = Er^4 + (4 - 6E)r^2 + \left\{-Es^3 + (6E - 4)s + \frac{1}{s}(3E - 4)\right\}r + (4 - 3E).
\]
(7.12)
The parameter \(\tau\) runs over the vertical \(S^1\), and \(\tau \in [0, \beta]\), with
\[
\beta = 2\pi.
\]
(7.13)
The locus \(\{r = s\}\) is a round, totally geodesic 2-sphere, of area \(A_s = \frac{2}{3\pi}(s - 2)\). These metrics are AH, with conformal infinity given by a Berger (or squashed) \(S^3\), with base \(S^2(1)\), and Hopf fiber \(S^1 = S^1(L)\) of length \(L = 2\pi E_s\).

In analogy to (7.4)-(7.5), notice that \(E \to 0\) as \(s \to 2\) or \(s \to \infty\), and has a maximal value \(E_{\text{max}} = (2 - \sqrt{3})/3\) at \(s = s_0 = 2 + \sqrt{3}\). Note in particular that since
\[
E_{\text{max}} < 1,
\]
the round metric \(S^3(1)\) is not in \(\text{Im } \Pi(g_s)\) for any \(s\).

We see that on the curve \(g_s, s \in (2, \infty)\), the boundary map \(\Pi\) has exactly the same 2-1 fold behavior as for the AdS Schwarzschild metric. The use of Theorem 3.1 as in Proposition 7.2 implies that \(\Pi\) is not surjective, and in fact the Berger spheres with Hopf fiber length \(L\), for \(L > 2\pi E_{\text{max}}\) are not in \(\text{Im } \Pi\).

In contrast, we have the following behavior on more twisted disc bundles over \(S^2\).

**Proposition 7.6.** Let \(M = M_k\) be the disc bundle over \(S^2\), with Chern number \(k\), \(|k| \geq 2\), so that \(\partial M = S^3/\mathbb{Z}_k\). Then
\[
\deg \Pi^0 = 1,
\]
(7.14)
and \(\Pi^0\) is surjective.
**Proof:** As seed metrics, we again use the AdS-Taub bolt metrics on $M_k$, c.f. again [18], [23]. These have exactly the same form as (7.10)-(7.13), except that the parameter $s$ satisfies $s > 1$, $E_s$ in (7.11) is replaced by $E_{s,k}$ of the form

$$E_{s,k} = \frac{2ks - 4}{3(s^2 - 1)}, \quad (7.15)$$

and the period $\beta$ for $\tau$ is given by $\beta = 2\pi/k$. For each $k$, these metrics are AH, and the conformal infinity on $S^3/Z_k$ is given by the Berger metric, with, as before, Hopf circle fibers of length $L = E_{s,k}/k$.

Note however from (7.15) that now the function $E_{s,k}$ is a monotone decreasing function of $s$, as $s$ increases from 1 to $\infty$. Hence, $\Pi$ is 1-1 on this curve, and in particular, the metric $g_{s,u} = k + (k^2 - 3)^{1/2}$ has conformal infinity the constant curvature metric on $S^3/Z_k$. The symmetry argument using Theorem 3.1 as before implies these metrics are the unique metrics with these boundary values. The metrics $g_s$ are regular points for $\Pi$, and so (7.14) follows. \[\blacksquare\]

**Remark 7.7.** We close the paper with some observations on whether the full boundary map $\Pi : \mathcal{E}_{AH} \to \mathcal{C}$ might be surjective, or almost surjective.

First, recall from Corollaries 5.7 and 5.8 that both the extended map $\hat{\Pi} : \hat{\mathcal{E}}_{AH} \to \mathcal{C}$ and the restricted map $\Pi : \mathcal{E}_{AH} \to \hat{\mathcal{C}}$ are proper. In this regard, it would be very interesting to know if the set of boundary values of AH Einstein metrics with cusps disconnects $\hat{\mathcal{C}}$ or not, that is whether

$$\Pi(\partial \mathcal{E}_{AH}) \subset \mathcal{C}$$

disconnects $\mathcal{C}$, or whether $\hat{\mathcal{C}} = \mathcal{C} \setminus \hat{\Pi}(\partial \hat{\mathcal{E}}_{AH})$ is path connected, (for instance if $\hat{\Pi}(\partial \hat{\mathcal{E}}_{AH})$ is of codimension at least 2). If $\hat{\mathcal{C}}$ is path connected, then the degree theory arguments of §6 and §7 hold without any change and give a well-defined degree $deg \Pi$ on each component of $\mathcal{E}_{AH}$. In particular, if this holds and $deg \Pi \neq 0$, then $\Pi$ is almost surjective, in that $\Pi$ surjects onto $\hat{\mathcal{C}}$.

On the other hand, if $\hat{\Pi}(\partial \hat{\mathcal{E}}_{AH})$ disconnects $\mathcal{C}$, then $\hat{\Pi}(\partial \hat{\mathcal{E}}_{AH})$ represents a "wall", past which it may not be possible to fill in boundary metrics with AH Einstein metrics. This would be the case for instance if $\mathcal{E}_{AH}$ is a Banach manifold with boundary $\partial \mathcal{E}_{AH}$ and $\hat{\Pi}$ maps $\partial \hat{\mathcal{E}}_{AH}$ onto a set of codimension 1 in $\mathcal{C}$.

Finally, it would also be interesting to know if there are topological obstructions to the possible formation of cusps, as is the case with the formation of orbifold singularities as in (0.5). Thus, with respect to the decomposition (5.39), one would like to know for instance if some of the homology of one of the two factors injects into that of the union $M$.

**Appendix**

In this appendix, we collect several formulas for the curvature of conformal compactifications of AH Einstein metrics. The formulas below are given for dimension 4, although the discussion holds in all dimensions, with slight changes in the coefficients depending on dimension.

Let $\bar{g} = \rho^2 \cdot g$, where $\rho$ is a defining function w.r.t. $\bar{g}$. The curvatures of the metrics $g$ and $\bar{g}$ are related by the following formulas:

$$K_{ab} = \frac{K_{ab} + |\bar{\nabla} \rho|^2}{\rho^2} - \frac{1}{\rho} \{ D^2 \rho(\bar{e}_a, \bar{e}_b) + D^2 \rho(\bar{e}_b, \bar{e}_a) \}. \quad (A.1)$$

$$\bar{R}_{uc} = -2 \bar{D}^2 \rho + (3 \rho^{-2} |\bar{\nabla} \rho|^2 - 1) - \frac{\bar{\Delta} \rho}{\rho} \bar{g}, \quad (A.2)$$

$$\bar{z} = -6 \frac{\bar{\Delta} \rho}{\rho} + 12 \rho^{-2} (|\bar{\nabla} \rho|^2 - 1). \quad (A.3)$$
The equation (A.2) is equivalent to the Einstein equation (0.3), with \( n = 4 \). Observe that (A.2) implies that if the compactification \( \tilde{g} \) is \( C^2 \), then

\[
|\tilde{\nabla} \rho| \to 1 \quad \text{at} \quad \partial M,
\]

and so by (A.1), \(|K_{ab} + 1| = O(\rho^2)\). For \( r = \log \rho \), as in (1.2), a simple calculation gives

\[
|\nabla \rho| = |\nabla r|,
\]

where the norm and gradient on the left are w.r.t. \( \tilde{g} \), and on the right are w.r.t. \( g \).

A defining function \( \rho = t \) is a geodesic defining function if

\[
|\nabla t| = 1,
\]

in a collar neighborhood \( U \) of \( \partial M \). Clearly, the formulas (A.1)-(A.3) simplify considerably in this situation. The function \( t \) is the distance function to \( \partial M \) on \( (M, \tilde{g}) \), and similarly by (A.5), the function \( r \) is a (signed) distance function on \( (M, g) \). The integral curves of \( \nabla t \) and \( \nabla r \) are geodesics in \( (M, \tilde{g}) \) and \( (M, g) \) respectively.

The 2\textsuperscript{nd} fundamental form \( \tilde{A} \) of the level sets \( S(t) \) of \( t \) in \( (M, \tilde{g}) \) is given by \( \tilde{A} = D^2 t \), with \( \tilde{H} = \tilde{\Delta} t \) giving the mean curvature of \( S(t) \). Along \( t \)-geodesics of \( (M, \tilde{g}) \), one has the standard Riccati equation

\[
\tilde{H}' + |\tilde{A}|^2 + \tilde{Ric}(\nabla t, \nabla t) = 0,
\]

where \( \tilde{H}' = d\tilde{H} / dt \).

The following formulas relating the curvatures of \( (M, \tilde{g}) \) at \( \partial M \) to the intrinsic curvatures of \( (\partial M, \gamma) \) may be found in [6, §1].

Let \( \tilde{g} \) be a \( C^2 \) geodesic compactification of \( (M, g) \), with \( C^2 \) boundary metric \( \gamma \). Then at \( \partial M \),

\[
\bar{s} = 6\tilde{Ric}(N, N) = \frac{3}{2}s_{\gamma},
\]

where \( N = \nabla t \) is the unit normal to \( \partial M \) w.r.t. \( \tilde{g} \). If \( X \) is tangent to \( \partial M \), then

\[
\tilde{Ric}(N, X) = 0,
\]

while if \( T \) denotes the projection onto \( T(\partial M) \), then

\[
(\tilde{Ric})^T = 2\tilde{Ric} - \frac{1}{4}s_{\gamma} \cdot \gamma.
\]

In particular, the full curvature of ambient metric \( \tilde{g} \) at \( \partial M \) is determined by the curvature of the boundary metric \( \gamma \).

Finally, we have the following formula for \( \bar{s}' = d\bar{s} / dt \):

\[
\bar{s}' = 6t^{-1}D^2 t^2 \geq 0.
\]

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