Sobriety of quantale-valued cotopological spaces

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Abstract

For each commutative and integral quantale, making use of the fuzzy order between closed sets, a theory of sobriety for quantale-valued cotopological spaces is established based on irreducible closed sets.

Keywords: Fuzzy topology, quantale, quantale-valued order, quantale-valued cotopological space, sobriety, irreducible closed set

1. Introduction

A topological space $X$ is sober if each of its irreducible closed subsets is the closure of exactly one point in $X$. Sobriety of topological spaces can be described via the well-known adjunction $O \dashv \text{pt}$ between the category Top of topological spaces and the opposite of the category Frm of frames [10]. Precisely, $X$ is sober if $\eta_X: X \rightarrow \text{pt}(O(X))$ is a bijection (hence a homeomorphism), where $\eta$ denotes the unit of the adjunction $O \dashv \text{pt}$.

In the classical setting, a topological space can be described in terms of open sets as well as closed sets, and we can switch between open sets and closed sets by taking complements. So, it makes no difference whether we choose to work with closed sets or with open sets. In the fuzzy setting, since the table of truth-values is usually a quantale, not a Boolean algebra, there is no natural way to switch between open sets and closed sets. So, it may make a difference whether we postulate topological spaces in terms of open sets or in terms of closed sets. An example in this regard is exhibited in [3, 4].

The frame approach to sobriety of topological spaces makes use of open sets; while the irreducible-closed-set approach makes use of closed sets. Extending the theory of sober spaces to the fuzzy setting is an interesting topic in fuzzy topology. Most of the existing works focus on the frame approach; that is, to find a fuzzy counterpart of the category Frm of frames, then establish an adjunction between the category of fuzzy topological spaces and that of fuzzy frames. Works in this regard include Rodabaugh [26, 27], Zhang and Liu [33], Kotzé [13, 14], Srivastava and Khastgir [28], Pultr and Rodabaugh [21, 22, 23, 24], Gutiérrez García, Höhle and de Prada Vicente [6], and Yao [30, 31], etc. But, the irreducible-closed-set approach to sobriety of fuzzy topological spaces is seldom touched, except in Kotzé [13, 14].

In this paper, making use of the fuzzy inclusion order between closed sets, we establish a theory of sobriety for quantale-valued topological spaces based on irreducible closed sets. Actually, this theory concerns sobriety of quantale-valued cotopological spaces. By a quantale-valued cotopological space we mean a “fuzzy topological space” postulated in terms of closed sets (see Definition 2.6). The term quantale-valued topological space is reserved for “fuzzy topological space” postulated in terms of open sets (see Definition 3.16).

It should be noted that in most works on fuzzy frames, the table of truth-values is assumed to be a complete Heyting algebra (or, a frame), even a completely distributive lattice sometimes. But, in this paper, the table of truth-values is only assumed to be a commutative and integral quantale. Complete Heyting algebra, BL-algebras and left continuous t-norms, are important examples of such quantales.
The contents are arranged as follows. Section 2 recalls basic ideas about quantale-valued ordered sets and quantale-valued \( Q \)-cotopological spaces. Section 3, making use of the quantale-valued order between closed sets in a \( Q \)-cotopological space, establishes a theory of sober \( Q \)-cotopological spaces based on irreducible closed sets. In particular, the sobrification of a stratified \( Q \)-cotopological space is constructed. The last section, Section 4, presents some interesting examples in the case that \( Q \) is the unit interval \([0,1]\) coupled with a (left) continuous t-norm.

2. Quantale-valued ordered sets and quantale-valued cotopological spaces

In this paper, \( Q = (Q, \&) \) always denotes a commutative and integral quantale, unless otherwise specified. Precisely, \( Q \) is a complete lattice with a bottom element 0 and a top element 1, \( \& \) is a binary operation on \( Q \) such that \( (Q, \& 1) \) is a commutative monoid and \( p \& \bigvee_{j \in J} q_j = \bigvee_{j \in J} p \& q_j \) for all \( p \in Q \) and \( \{q_j\}_{j \in J} \subseteq Q \).

Since the semigroup operation \( \& \) distributes over arbitrary joins, it determines a binary operation \( \to \) on \( Q \) via the adjoint property

\[
q \leq r \iff q \leq p \to r.
\]

The binary operation \( \to \) is called the implication, or the residuation, corresponding to \( \& \).

Some basic properties of the binary operations \( \& \) and \( \to \) are collected below, they can be found in many places, e.g. [2, 27].

Proposition 2.1. Let \( Q \) be a quantale. Then

1. \( 1 \to p = p \).
2. \( p \leq q \iff 1 \to p \leq q \).
3. \( p \to (q \to r) = (p \& q) \to r \).
4. \( p \& (p \to q) \leq q \).
5. \( \bigvee_{j \in J} p_j \to q = \bigwedge_{j \in J} (p_j \to q) \).
6. \( p \to \bigwedge_{j \in J} q_j = \bigvee_{j \in J} (p \to q_j) \).

We often write \( \neg p \) for \( p \to 0 \) and call it the negation of \( p \). Though it is true that \( p \leq \neg \neg p \) for all \( p \in Q \), the inequality \( \neg \neg p \leq p \) does not always hold. A quantale \( Q \) is said to satisfy the law of double negation if

\[
(p \to 0) \to 0 = p,
\]

i.e., \( \neg \neg p = p \), for all \( p \in Q \).

Proposition 2.2. [2] Suppose that \( Q \) is a quantale that satisfies the law of double negation. Then

1. \( p \to q = \neg(p \& \neg q) = \neg q \to \neg p \).
2. \( p \& q = \neg(q \to \neg p) = \neg(p \to \neg q) \).
3. \( \neg(\bigwedge_{i \in I} p_i) = \bigvee_{i \in I} \neg p_i \).

In the class of quantales, the quantales with the unit interval \([0,1]\) as underlying lattice are of particular interest in fuzzy set theory [12]. In this case, the semigroup operation \( \& \) is called a left continuous t-norm on \([0,1]\) [12]. If a left continuous t-norm \( \& \) on \([0,1]\) is a continuous function with respect to the usual topology, then it is called a continuous t-norm.

Example 2.3. [12] Some basic t-norms:

1. The minimum t-norm: \( a \& b = a \land b = \min\{a, b\} \). The corresponding implication is given by
   \[
a \to b = \begin{cases} 
1, & a \leq b; \\
b, & a > b.
\end{cases}
\]
(2) The product t-norm: \( a \& b = a \cdot b \). The corresponding implication is given by

\[
    a \rightarrow b = \begin{cases} 
        1, & a \leq b; \\
        b/a, & a > b. 
    \end{cases}
\]

(3) The Lukasiewicz t-norm: \( a \& b = \max\{a + b - 1, 0\} \). The corresponding implication is given by

\[
    a \rightarrow b = \min\{1, 1 - a + b\}. 
\]

In this case, \([0, 1], \&\) satisfies the law of double negation.

(4) The nilpotent minimum t-norm:

\[
    a \& b = \begin{cases} 
        0, & a + b \leq 1; \\
        \min\{a, b\}, & a + b > 1. 
    \end{cases}
\]

The corresponding implication is given by

\[
    a \rightarrow b = \begin{cases} 
        1, & a \leq b; \\
        \max\{1 - a, b\}, & a > b. 
    \end{cases}
\]

In this case, \([0, 1], \&\) satisfies the law of double negation.

A \( Q \)-order (or an order valued in the quantale \( Q \)) on a set \( X \) is a reflexive and transitive \( Q \)-relation on \( X \). Explicitly, a \( Q \)-order on \( X \) is a map \( R: X \times X \rightarrow Q \) such that \( 1 = R(x, x) \) and \( R(y, z) \& R(x, y) \leq R(x, z) \) for all \( x, y, z \in X \). The pair \( (X, R) \) is called a \( Q \)-ordered set. As usual, we write \( X \) for the pair \( (X, R) \) and \( x, y \) for \( R(x, y) \) if no confusion would arise.

If \( R: X \times X \rightarrow Q \) is a \( Q \)-order on \( X \), then \( R^{op}: X \times X \rightarrow Q \), given by \( R^{op}(x, y) = R(y, x) \), is also a \( Q \)-order on \( X \) (by commutativity of \( \& \)), called the opposite of \( R \).

A map \( f: X \rightarrow Y \) between \( Q \)-ordered sets is \( Q \)-order-preserving if \( X(x_1, x_2) \leq Y(f(x_1), f(x_2)) \) for all \( x_1, x_2 \in X \). We write \( Q \)-Ord

for the category of \( Q \)-ordered sets and \( Q \)-order-preserving maps.

Given a set \( X \), a map \( A: X \rightarrow Q \) is called a fuzzy set (valued in \( Q \)), the value \( A(x) \) is interpreted as the membership degree. The map

\[
    \text{sub}_X: Q^X \times Q^X \rightarrow Q, 
\]

given by

\[
    \text{sub}_X(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x), 
\]

defines a \( Q \)-order on \( Q^X \). The value \( \text{sub}_X(A, B) \) measures the degree that \( A \) is a subset of \( B \), thus, \( \text{sub}_X \) is called the fuzzy inclusion order on \( Q^X \). In particular, if \( X \) is a singleton set then the \( Q \)-ordered set \( (Q^X, \text{sub}_X) \) reduces to the \( Q \)-ordered set \( (Q, d_L) \), where

\[
    d_L(p, q) = p \rightarrow q. 
\]

For any \( p \in Q \) and \( A \in Q^X \), write \( p \& A, p \rightarrow A \in Q^X \) for the fuzzy sets given by \( (p \& A)(x) = p \& A(x) \) and \( (p \rightarrow A)(x) = p \rightarrow A(x) \), respectively. It is easy to check that

\[
    p \leq \text{sub}_X(A, B) \iff p \& A \leq B \iff A \leq p \rightarrow B 
\]

for all \( p \in Q \) and \( A, B \in Q^X \). In particular, \( A \leq B \) if and only if \( 1 = \text{sub}_X(A, B) \). Furthermore,

\[
    p \rightarrow \text{sub}_X(A, B) = \text{sub}_X(p \& A, B) = \text{sub}_X(A, p \rightarrow B) 
\]

for all \( p \in Q \) and \( A, B \in Q^X \).
Given a map \( f: X \rightarrow Y \), as usual, define \( f^-: Q^X \rightarrow Q^Y \) and \( f^+: Q^Y \rightarrow Q^X \) by
\[
f^-(A)(y) = \bigvee_{f(x)=y} A(x), \quad f^+(B)(x) = B \circ f(x).
\]
The fuzzy set \( f^-(A) \) is called the image of \( A \) under \( f \), and \( f^+(B) \) the preimage of \( B \).

The following proposition is a special case of the enriched Kan extension in category theory
\[ 11, 17 \]. A direct verification is easy and can be found in e.g. \[ 13 \].

**Proposition 2.4.** For any map \( f: X \rightarrow Y \),

1. \( f^-: (Q^X, \text{sub}_X) \rightarrow (Q^Y, \text{sub}_Y) \) is \( Q \)-order-preserving;
2. \( f^+: (Q^Y, \text{sub}_Y) \rightarrow (Q^X, \text{sub}_X) \) is \( Q \)-order-preserving;
3. \( f^+ \) is left adjoint to \( f^- \), written \( f^+ \dashv f^- \), in the sense that

\[
\text{sub}_Y(f^-(A), B) = \text{sub}_X(A, f^+(B))
\]

for all \( A \in Q^X \) and \( B \in Q^Y \).

A fuzzy upper set \[ 16 \] in a \( Q \)-ordered set \( X \) is a map \( \psi: X \rightarrow Q \) such that
\[
X(x, y) \land \psi(x) \leq \psi(y)
\]
for all \( x, y \in X \). It is clear that \( \psi: X \rightarrow Q \) is a fuzzy upper set if and only if \( \psi: X \rightarrow (Q, d_L) \)

is \( Q \)-order-preserving.

Dually, a fuzzy lower set in a \( Q \)-ordered set \( X \) is a map \( \phi: X \rightarrow Q \) such that
\[
\phi(y) \land X(x, y) \leq \phi(x)
\]
for all \( x, y \in X \). Or equivalently, \( \phi: X^{\text{op}} \rightarrow (Q, d_L) \) is a \( Q \)-order-preserving map, where \( X^{\text{op}} \) is the opposite of the \( Q \)-ordered set \( X \).

**Definition 2.5.** A fuzzy lower set \( \phi \) in a \( Q \)-ordered set \( X \) is irreducible if \( \bigvee_{x \in X} \phi(x) = 1 \) and
\[
\text{sub}_X(\phi, \phi_1 \lor \phi_2) = \text{sub}_X(\phi, \phi_1) \lor \text{sub}_X(\phi, \phi_2)
\]
for all fuzzy lower sets \( \phi_1, \phi_2 \) in \( X \).

Irreducible fuzzy lower sets are a counterpart of directed lower sets \[ 3 \] in the quantale-valued setting. In particular, the condition \( \bigvee_{x \in X} \phi(x) = 1 \) is a \( Q \)-version of the requirement that a directed set should be non-empty.

The following definition is taken from \[ 3, 32 \].

**Definition 2.6.** A \( Q \)-cotopology on a set \( X \) is a subset \( \tau \) of \( Q^X \) subject to the following conditions:

1. \( p_X \in \tau \) for all \( p \in Q \);
2. \( A \lor B \in \tau \) for all \( A, B \in \tau \);
3. \( \bigwedge_{j \in J} A_j \in \tau \) for each subfamily \( \{A_j\}_{j \in J} \) of \( \tau \).

The pair \( (X, \tau) \) is called a \( Q \)-cotopological space; elements in \( \tau \) are called closed sets of \( (X, \tau) \). A \( Q \)-cotopology \( \tau \) is stratified if

1. \( p \rightarrow A \in \tau \) for all \( p \in Q \) and \( A \in \tau \).

A \( Q \)-cotopology \( \tau \) is co-stratified if

1. \( p \& A \in \tau \) for all \( p \in Q \) and \( A \in \tau \).

A \( Q \)-cotopology \( \tau \) is strong if it is both stratified and co-stratified.

As usual, we often write \( X \), instead of \( (X, \tau) \), for a \( Q \)-cotopological space.
Remark 2.7. If $Q$ is the quantale obtained by endowing $[0,1]$ with a continuous t-norm $T$, then $	au \subseteq [0,1]^X$ is a strong $Q$-cotopology on $X$ if and only if $(X,\tau^*)$ is a fuzzy $T$-neighborhood space in the sense of Morsi [20], where $\tau^* = \{1 - A \mid A \in \tau\}$. In particular, if $Q$ is the quantale $([0,1], \min)$, then $\tau \subseteq [0,1]^X$ is a strong $Q$-cotopology on $X$ if and only if $(X,\tau^*)$ is a fuzzy neighborhood space in the sense of Lowen [19].

A map $f : X \to Y$ between $Q$-cotopological spaces is continuous if $f^*(A) = A \circ f$ is closed in $X$ whenever $A$ is closed in $Y$. We write

$$Q\text{-CTop}$$

for the category of $Q$-cotopological spaces and continuous maps; and write

$$SQ\text{-CTop}$$

for the category of stratified $Q$-cotopological spaces and continuous maps. It is easily seen that both $Q\text{-CTop}$ and $SQ\text{-CTop}$ are well-fibred topological categories over $\text{Set}$ in the sense of [1].

Given a $Q$-cotopological space $(X,\tau)$, its closure operator $- : Q^X \to Q^X$ is defined by

$$\overline{\tau} = \bigwedge \{B \in \tau \mid A \leq B\}$$

for all $A \in Q^X$. The closure operator of a $Q$-cotopological space $(X,\tau)$ satisfies the following conditions: for all $A,B \in Q^X$,

1. $\overline{p_A} = p_X$ for all $p \in Q$;
2. $\overline{A} \leq A$;
3. $\overline{A \lor B} = \overline{A} \lor \overline{B}$;
4. $\overline{A} = \overline{A}$.

Proposition 2.8. Let $X$ be a $Q$-cotopological space. The following are equivalent:

1. $X$ is stratified.
2. $p \& \overline{A} \leq \overline{p \& A}$ for all $p \in Q$ and $A \in Q^X$.
3. The closure operator $- : (Q^X,\text{sub}_X) \to (Q^X,\text{sub}_X)$ is $Q$-order-preserving.

Proof. (1) $\Rightarrow$ (2) Since $p \to \overline{p \& A}$ is closed and $A \leq p \to \overline{p \& A}$, it follows that $\overline{\tau} \leq p \to \overline{p \& A}$, hence $p \& \overline{A} \leq \overline{p \& A}$.

(2) $\Rightarrow$ (3) For any $A,B \in Q^X$, let $p = \text{sub}_X(A,B) = \bigwedge_{x \in X} \{A(x) \to B(x)\}$. Since $p \& A \leq B$, then $p \& \overline{A} \leq \overline{p \& A} \leq \overline{B}$, hence $\text{sub}_X(A,B) = p \leq \text{sub}_X(\overline{A},\overline{B})$.

(3) $\Rightarrow$ (1) Let $A$ be a closed set and $p \in Q$. In order to see that $p \to \overline{A}$ is also closed, it suffices to check that $p \to \overline{A} \leq p \to A$, or equivalently, $p \leq \text{sub}_X(p \to \overline{A},A)$. This is easy since $p \leq \text{sub}_X(p \to A,A) \leq \text{sub}_X(p \to \overline{A},\overline{A}) = \text{sub}_X(p \to \overline{A},A)$. \qed

It follows immediately from item (3) in the above proposition that if $X$ is a stratified $Q$-cotopological space and if $B$ is a closed set in $X$, then for all $A \in Q^X$,

$$\text{sub}_X(\overline{A},B) \leq \text{sub}_X(A,B) \leq \text{sub}_X(\overline{A},\overline{B}) = \text{sub}_X(\overline{A},B),$$

hence

$$\text{sub}_X(A,B) = \text{sub}_X(\overline{A},B).$$

This equation will be useful in this paper.

Corollary 2.9. In a stratified $Q$-cotopological space $X$,

$$\overline{\tau} = \bigwedge \{\text{sub}_X(A,B) \to B \mid B \text{ is closed in } X\}$$

for all $A \in Q^X$. 

Given a \( \mathcal{Q} \)-cotopological space \((X, \tau)\), define \( \Omega(\tau): X \times X \to \mathcal{Q} \) by
\[
\Omega(\tau)(x, y) = \bigwedge_{A \in \tau} (A(y) \to A(x)).
\]
Then \( \Omega(\tau) \) is a \( \mathcal{Q} \)-order on \( X \), called the **specialization \( \mathcal{Q} \)-order** of \((X, \tau)\) \cite{10}. It is clear that each closed set in \((X, \tau)\) is a fuzzy lower set in the \( \mathcal{Q} \)-ordered set \((X, \Omega(\tau))\).

As said before, we often write \( X, \tau \), instead of \((X, \tau)\), for a \( \mathcal{Q} \)-cotopological space. Accordingly, we will write \( \Omega(X) \) for the \( \mathcal{Q} \)-ordered set obtained by equipping \( X \) with its specialization \( \mathcal{Q} \)-order.

The correspondence \( X \mapsto \Omega(X) \) defines a functor
\[
\Omega: \mathcal{Q} \text{-} \text{CTop} \to \mathcal{Q} \text{-} \text{Ord}.
\]
In particular, if \( f: X \to Y \) is a continuous map between \( \mathcal{Q} \)-cotopological spaces, then \( f: \Omega(X) \to \Omega(Y) \) is \( \mathcal{Q} \)-order-preserving, i.e., \( \Omega(X)(x, y) \leq \Omega(Y)(f(x), f(y)) \) for all \( x, y \in X \).

Conversely, given a \( \mathcal{Q} \)-ordered set \((X, R)\), the family \( \Gamma(R) \) of fuzzy lower sets in \((X, R)\) forms a strong \( \mathcal{Q} \)-cotopology on \( X \), called the **Alexandroff \( \mathcal{Q} \)-cotopology** on \((X, R)\). The correspondence \( (X, R) \mapsto \Gamma(X, R) = (X, \Gamma(R)) \) defines a functor
\[
\Gamma: \mathcal{Q} \text{-} \text{Ord} \to \mathcal{Q} \text{-} \text{CTop}
\]
that is left adjoint to the functor \( \Omega \) \cite{10}.

The following conclusion says that in a stratified \( \mathcal{Q} \)-cotopological space, the specialization \( \mathcal{Q} \)-order is determined by closures of singletons, as in the classical case.

**Proposition 2.10.** \cite{23} If \( X \) is a stratified \( \mathcal{Q} \)-cotopological space, then \( \Omega(X)(x, y) = \overline{\mathcal{y}}(x) \) for all \( x, y \in X \).

### 3. Sober \( \mathcal{Q} \)-cotopological spaces

Let \( X \) be a topological space. A closed set \( F \) in \( X \) is irreducible if it is non-empty and for any closed sets \( A, B \) in \( X \), \( F \subseteq A \cup B \) implies either \( F \subseteq A \) or \( F \subseteq B \). A topological space is sober if every irreducible closed set in it is the closure of exactly one point. Sobriety is an interesting property in the realm of non-Hausdorff spaces and it plays an important role in domain theory \cite{15}. In order to extend the theory of sober spaces to the fuzzy setting, the first step is to postulate irreducible closed sets in a \( \mathcal{Q} \)-cotopological space. Fortunately, this can be done in a natural way with the help of the fuzzy inclusion order between the closed sets in a \( \mathcal{Q} \)-cotopological space.

**Definition 3.1.** A closed set \( F \) in a \( \mathcal{Q} \)-cotopological space \( X \) is irreducible if \( \bigvee_{x \in X} F(x) = 1 \) and
\[
\text{sub}_X(F, A \cup B) = \text{sub}_X(F, A) \cup \text{sub}_X(F, B)
\]
for all closed sets \( A, B \) in \( X \).

This definition is clearly an extension of that of irreducible closed sets in a topological space. We haste to emphasize that the fuzzy inclusion order, not the pointwise order, between closed sets is used here. The condition \( \bigvee_{x \in X} F(x) = 1 \) is a \( \mathcal{Q} \)-version of the requirement that \( F \) is non-empty.

**Example 3.2.** (1) Let \( X \) be a stratified \( \mathcal{Q} \)-cotopological space. For any \( x \in X \), the closure \( \overline{\mathcal{y}}(x) \) of \( 1_x \) is irreducible. This follows from that \( \text{sub}_X(\overline{\mathcal{y}}(x), A) = A(x) \) for any closed set \( A \) in \( X \).

(2) A fuzzy lower set \( \phi \) in a \( \mathcal{Q} \)-ordered set \( X \) is irreducible in the sense of Definition \cite{23} if and only if \( \phi \) is an irreducible closed set in the Alexandroff \( \mathcal{Q} \)-cotopological space \( \Gamma(X) \).

Having the notion of irreducible closed sets (in a \( \mathcal{Q} \)-cotopological space) at hand, we are now able to formulate the central notion of this paper.

**Definition 3.3.** A \( \mathcal{Q} \)-cotopological space \( X \) is sober if it is stratified and each irreducible closed in \( X \) is the closure of \( 1_x \) for a unique \( x \in X \).
Write \( \text{Sob}\mathbb{Q}\text{-CTop} \) for the full subcategory of \( \mathbb{S}\mathcal{Q}\text{-CTop} \) consisting of sober \( \mathbb{Q}\)-cotopological spaces. This section concerns basic properties of sober \( \mathbb{Q}\)-cotopological spaces. First, we show that the subcategory \( \text{Sob}\mathbb{Q}\text{-CTop} \) is reflective in \( \mathbb{S}\mathcal{Q}\text{-CTop} \) and that the specialization \( \mathbb{Q}\)-order of each sober \( \mathbb{Q}\)-cotopological space \( X \) is directed complete in the sense that every irreducible fuzzy lower set in the \( \mathbb{Q}\)-ordered set \( \Omega(X) \) has a supremum. Then we will discuss the relationship between

- sobriety and Hausdorff separation in a stratified \( \mathbb{Q}\)-cotopological space;
- sober topological spaces and sober \( \mathbb{Q}\)-cotopological spaces via the Lowen functor \( \omega_\mathbb{Q} \);
- sober \( \mathbb{Q}\)-cotopological spaces and sober \( \mathbb{Q}\)-topological spaces in the case that \( \mathbb{Q} \) satisfies the law of double negation.

Given a stratified \( \mathbb{Q}\)-cotopological space \( X \), let

\[ \text{irr}(X) \]

denote the set of all irreducible closed sets in \( X \). For each closed set \( A \) in \( X \), define

\[ s(A) : \text{irr}(X) \rightarrow \mathbb{Q} \]

by

\[ s(A)(F) = \text{sub}_X(F, A). \]

**Lemma 3.4.** Let \( X \) be a stratified \( \mathbb{Q}\)-cotopological space.

1. \( s(pX)(F) = p \) for all \( p \in \mathbb{Q} \) and \( F \in \text{irr}(X) \).
2. \( s(A) = s(B) \iff A = B \) for all closed sets \( A, B \) in \( X \).
3. \( s(A \lor B) = s(A) \lor s(B) \) for all closed sets \( A, B \) in \( X \).
4. \( s(\bigwedge_{j \in J} A_j) = \bigwedge_{j \in J} s(A_j) \) for each family \( \{A_j\}_{j \in J} \) of closed sets in \( X \).
5. \( s(p \rightarrow A) = p \rightarrow s(A) \) for all \( p \in \mathbb{Q} \) and all closed sets \( A \) in \( X \).
6. \( \text{sub}_X(A,B) = \text{sub}_{\text{irr}(X)}(s(A), s(B)) \) for all closed sets \( A, B \) in \( X \).

**Proof.** We check (6) for example. On one hand,

\[
\text{sub}_X(A,B) \leq \bigwedge_{F \in \text{irr}(X)} (\text{sub}_X(F,A) \rightarrow \text{sub}_X(F,B)) = \text{sub}_{\text{irr}(X)}(s(A), s(B)).
\]

On the other hand,

\[
\text{sub}_{\text{irr}(X)}(s(A), s(B)) = \bigwedge_{F \in \text{irr}(X)} (s(A)(F) \rightarrow s(B)(F))
\]

\[
= \bigwedge_{F \in \text{irr}(X)} (\text{sub}_X(F,A) \rightarrow \text{sub}_X(F,B))
\]

\[
\leq \bigwedge_{x \in X} (\text{sub}_X(\mathcal{T}_x, A) \rightarrow \text{sub}_X(\mathcal{T}_x, B))
\]

\[
= \bigwedge_{x \in X} (A(x) \rightarrow B(x))
\]

\[
= \text{sub}_X(A,B).
\]

The proof is finished. \( \Box \)

By the above lemma, \( \{s(A) \mid A \text{ is a closed set of } X\} \) is a stratified \( \mathbb{Q}\)-cotopology on \( \text{irr}(X) \). We will write \( s(X) \), instead of \( \text{irr}(X) \), for the resulting \( \mathbb{Q}\)-cotopological space.
Proposition 3.5. \( s(X) \) is sober for each stratified \( Q \)-cotopological space \( X \).

Proof. First of all, we note that for each irreducible closed set \( F \) in \( X \), the closure of \( 1_F \) in \( s(X) \) is given by \( s(F) \). So, it suffices to show that for each closed set \( A \) in \( X \), if \( s(A) \) is irreducible in \( s(X) \), then \( A \) is irreducible in \( X \). We prove the conclusion in two steps.

Step 1. \( \bigvee_{x \in X} A(x) = 1 \). Since \( s(A) \) is an irreducible closed set in \( s(X) \), it holds that
\[
\bigvee_{F \in s(X)} s(A)(F) = \bigvee_{F \in s(X)} \text{sub}_X(F, A) = 1.
\]
Since for each \( F \in s(X) \) and \( x \in X \),
\[
F(x) \& \text{sub}_X(F, A) = F(x) \& \bigvee_{z \in X} (F(z) \to A(z)) \leq A(x),
\]
it follows that
\[
\bigvee_{x \in X} A(x) \geq \bigvee_{x \in X} \bigvee_{F \in s(X)} F(x) \& \text{sub}_X(F, A) = \bigvee_{F \in s(X)} \bigvee_{x \in X} F(x) \& \text{sub}_X(F, A) = \bigvee_{F \in s(X)} \text{sub}_X(F, A) = 1.
\]

Step 2. \( \text{sub}_X(A, B \lor C) = (\text{sub}_X(A, B)) \lor (\text{sub}_X(A, C)) \) for all closed sets \( B, C \) in \( X \). Since \( s(A) \) is irreducible in \( s(X) \),
\[
\text{sub}_X(A, B \lor C) = \text{sub}_{s(X)}(s(A), s(B \lor C)) = \text{sub}_{s(X)}(s(A), s(B) \lor s(C)) = \text{sub}_{s(X)}(s(A), s(B)) \lor \text{sub}_{s(X)}(s(A), s(C)) = \text{sub}_X(A, B) \lor \text{sub}_X(A, C).
\]
Therefore, \( A \) is an irreducible closed set in \( X \).

Proposition 3.6. For a stratified \( Q \)-cotopological space \( X \), define
\[
\eta_X : X \to s(X)
\]
by \( \eta_X(x) = \overline{1_x} \). Then

1. \( \eta_X : X \to s(X) \) is continuous.
2. \( X \) is sober if and only if \( \eta_X \) is a homeomorphism.

Proof. (1) For any closed set \( A \) in \( X \) and \( x \in X \),
\[
\eta_X^{-1}(s(A))(x) = s(A)(\eta_X(x)) = A(x),
\]
hence \( \eta_X^{-1}(s(A)) = A \). This shows that \( f \) is continuous.

(2) If \( X \) is sober then each irreducible closed set in \( X \) is of the form \( \overline{1_x} = \eta_X(x) \) for a unique \( x \in X \), hence \( \eta_X : X \to s(X) \) is a bijection. Since for each closed set \( A \) in \( X \) and \( x \in X \),
\[
\eta_X^{-1}(A)(\overline{1_x}) = \bigvee \{ A(z) \mid \eta_X(z) = \overline{1_x} \} = A(x) = s(A)(\overline{1_x}),
\]
it follows that \( \eta_X^{-1}(A) = s(A) \). This shows that \( \eta_X \) is a continuous closed bijection, hence a homeomorphism. The converse conclusion is trivial, since \( s(X) \) is sober by Proposition 3.5. \( \square \)
Lemma 3.8. Let $f : X \to Y$ be a continuous map between stratified $\mathcal{Q}$-cotopological spaces. Then for each irreducible closed set $F$ of $X$, the closure $\overline{f^{-1}(F)}$ of the image of $F$ under $f$ is an irreducible closed set of $Y$.

**Proof.** For any closed sets $A, B$ in $Y$,

$$
\text{sub}_Y(\overline{f^{-1}(F)}, A \lor B) = \text{sub}_Y(f^{-1}(F), A \lor B) \quad (A \lor B \text{ is closed})
$$

$$
= \text{sub}_X(F, f^{\lor-}(A \lor B)) \quad (f \to f^{\lor-})
$$

$$
= \text{sub}_X(F, f^{\lor-}(A)) \lor \text{sub}_X(F, f^{\lor-}(B)) \quad (F \text{ is irreducible})
$$

$$
= \text{sub}_Y(f^{-1}(F), A) \lor \text{sub}_Y(\overline{f^{-1}(F)}, B)
$$

$$
= \text{sub}_Y(\overline{f^{-1}(F)}, A) \lor \text{sub}_Y(\overline{f^{-1}(F)}, B),
$$

hence $\overline{f^{-1}(F)}$ is irreducible. \hfill \Box

**Proof of Theorem 3.7** Existence. For each $F \in s(X)$, since $\overline{f^{-1}(F)}$ is an irreducible closed set of $Y$ and $Y$ is sober, there is a unique $y \in Y$ such that $\overline{f^{-1}(F)}$ equals the closure of $1_y$. Define $f^*(F)$ to be this $y$. We claim that $f^* : s(X) \to Y$ satisfies the conditions.

First, we show that $B \circ f^*$ is a closed set in $s(X)$ for any closed set $B$ in $Y$, hence $f^*$ is continuous. Let $A$ be the closed set $f^{\lor-}(B) = B \circ f$ in $X$. Then for any $F \in s(X)$,

$$
s(A)(F) = \text{sub}_X(F, f^{\lor-}(B)) = \text{sub}_Y(f^{-1}(F), B)
$$

$$
= \text{sub}_Y(\overline{f^{-1}(F)}, B) = \text{sub}_Y(\overline{f^{-1}(F), B})
$$

$$
= B(f^*(F)) = B \circ f^*(F),
$$

thus, $B \circ f^* = s(A)$ and is closed in $s(X)$.

Second, for any $x \in X$, since

$$
1_{f(x)} \leq f^*(1_x) \leq \overline{f^{-1}(1_x)} = \overline{f_{f(x)}},
$$

it follows that

$$
1_{f(x)} = \overline{f^{-1}(1_x)} = \overline{f^{-1}(\eta_X(x))},
$$

hence $f^*(\eta_X(x)) = f(x)$, showing that $f^* \circ \eta_X = f$.

Therefore, $f^* : s(X) \to Y$ satisfies the conditions.

**Uniqueness.** Since $Y$ is sober, it suffices to show that if $g : s(X) \to Y$ is a continuous map such that $f = g \circ \eta_X$, then $\overline{g(F)} = \overline{f^{-1}(F)}$ for all $F \in s(X)$.

Since for any $F \in s(X)$,

$$
\Omega(s(X))(E, F) = \overline{f^{-1}(F)} = \text{sub}_X(E, F),
$$

it follows that for any $x \in X$,

$$
F(x) = \text{sub}_X(1_x, E)
$$

$$
= \Omega(s(X))(\eta_X(x), F) \quad \leq \Omega(Y)(g(\eta_X(x)), g(F))
$$

$$
= \Omega(Y)(f(x), g(F)) = \overline{g(F)}(f(x)),
$$

showing that $f^{-1}(F) \leq \overline{g(F)}$, hence $\overline{f^{-1}(F)} \leq \overline{g(F)}$.

Conversely, since $\overline{\eta_X(F)}$ is closed in $s(X)$, there is some closed set $A$ in $X$ such that $\overline{\eta_X(F)} = s(A)$. For any $x \in X$, since

$$
F(x) \leq \overline{\eta_X(F)}(\eta_X(x)) \leq s(A)(\eta_X(x)) = A(x),
$$
it follows that \( F \leq A \), then
\[
g^{-1}(s(A)) = g^{-1}(\eta_X^*(F)) \leq g^{-1} \circ \eta_X^*(F) = f^{-1}(F),
\]

hence
\[
\overline{f^{-1}(F)}(g(F)) \geq g^{-1}(s(A))(g(F)) \geq s(A)(F) = 1.
\]

Therefore, \( \overline{g(F)} \leq f^{-1}(F) \).

Theorem 3.7 shows that the full subcategory of sober \( \mathbb{Q} \)-cotopological spaces is reflective in \( \mathbb{S} \mathbb{Q} \text{-CTop} \). For any \( \mathbb{Q} \)-cotopological space \( X \), the sober space \( s(X) \) is called the sobrification of \( X \).

An important property of sober spaces is that the specialization order of a sober space is directed complete \([5, 10]\). The following Proposition 3.10 says this is also true in the quantale-valued setting if we treat irreducible fuzzy lower sets as “directed fuzzy lower sets”.

**Definition 3.9.** \([29, 15]\) A supremum of a fuzzy lower set \( \phi \) in a \( \mathbb{Q} \)-ordered set \( X \) is an element \( \text{sup} \phi \) in \( X \) such that
\[
X(\text{sup} \phi, x) = \bigwedge_{z \in X} (\phi(z) \to X(z, x)) = \text{sub}_X(\phi, X(\_, x))
\]
for all \( x \in X \).

The notion of supremum of a fuzzy lower set in a \( \mathbb{Q} \)-ordered set is a special case of that of weighted colimit in category theory \([11]\).

**Proposition 3.10.** Let \( X \) be a sober \( \mathbb{Q} \)-cotopological space. Then each irreducible fuzzy lower set in the specialization \( \mathbb{Q} \)-order of \( X \) has a supremum.

**Proof.** Let \( \phi \) be an irreducible fuzzy lower set in the \( \mathbb{Q} \)-ordered set \( \Omega(X) \). First, we show that the closure \( \overline{\phi} \) of \( \phi \) in \( X \) is an irreducible closed set. Let \( A, B \) be closed sets in \( X \). By definition of the specialization \( \mathbb{Q} \)-order, both \( A \) and \( B \) are fuzzy lower sets in \( \Omega(X) \). Hence
\[
\text{sub}_X(\overline{\phi}, A \lor B) = \text{sub}_X(\phi, A \lor B) = \text{sub}_X(\phi, A) \lor \text{sub}_X(\phi, B) = \text{sub}_X(\overline{\phi}, A) \lor \text{sub}_X(\overline{\phi}, B),
\]
showing that \( \overline{\phi} \) is an irreducible closed set in \( X \). Since \( X \) is sober, there is a unique \( a \in X \) such that \( \overline{\phi} = \overline{1_a} \). We claim that \( a \) is a supremum of \( \phi \) in \( \Omega(X) \). That is, for all \( x \in X \),
\[
\Omega(X)(a, x) = \bigwedge_{z \in X} (\phi(z) \to \Omega(X)(z, x)).
\]
On one hand, for each \( z \in X \), since \( \phi(z) \leq \overline{\phi}(z) = \overline{1_a}(z) = \Omega(X)(z, a) \), it follows that
\[
\phi(z) \to \Omega(X)(z, x) \geq \Omega(X)(z, a) \to \Omega(X)(z, a) \geq \Omega(X)(a, x),
\]
hence
\[
\Omega(X)(a, x) \leq \bigwedge_{z \in X} (\phi(z) \to \Omega(X)(z, x)).
\]
On the other hand,
\[
\bigwedge_{z \in X} (\phi(z) \to \Omega(X)(z, x)) = \text{sub}_X(\phi, \overline{1_x}) = \text{sub}_X(\overline{\phi}, \overline{1_x}) \leq \overline{\phi}(a) \to \overline{1_x}(a) = \Omega(X)(a, x).
\]
This completes the proof. \( \square \)
It is well-known that a Hausdorff topological space is always sober \([8, 10]\). The following proposition says this is also true for \(Q\)-cotopological spaces if \(Q\) is linearly ordered.

A \(Q\)-cotopological space \(X\) is Hausdorff if the diagonal \(\Delta: X \times X \to Q\), given by

\[
\Delta(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}
\]

is a closed set in the product space \(X \times X\).

**Proposition 3.11.** Let \(Q = (Q, \&)\) be a linearly ordered quantale. Then each stratified Hausdorff \(Q\)-cotopological space is sober.

**Proof.** Let \(X\) be a stratified Hausdorff \(Q\)-cotopological space. In order to see that \(X\) is sober, it suffices to show that if \(F\) is an irreducible closed set in \(X\), then \(F(x) \neq 0\) for at most one point \(x\) in \(X\). Suppose on the contrary that there exist different \(x, y \in X\) such that \(F(x) > 0\) and \(F(y) > 0\). Let \(b = \min\{F(x), F(y)\}\). Then \(b > 0\) by linearity of \(Q\). Since \(X\) is Hausdorff, there exist two families of closed sets in \(X\), say, \(\{A_j\}_{j \in J}\) and \(\{B_j\}_{j \in J}\), such that

\[
\Delta(x, y) = \bigwedge_{j \in J} A_j(x) \lor B_j(y).
\]

Since \(\Delta(x, y) = 0\), there exists some \(i \in J\) such that \(A_i(x) \lor B_i(y) < b\). Since \(A_i(z) \lor B_i(z) = 1\) for all \(z \in X\), we have either \(F \leq A_i\) or \(F \leq B_i\), hence either \(F(x) < b\) or \(F(y) < b\), a contradiction. \(\square\)

**Note 3.12.** The assumption that \(Q\) is linearly ordered is indispensable in the above proposition. To see this, let \(Q = \{0, a, b, 1\}\) be the Boolean algebra with four elements; let \(X\) be the discrete \(Q\)-cotopological space with two points \(x\) and \(y\). It is clear that \(X\) is Hausdorff. One can verify by enumerating all possibilities that the map \(\lambda\) given by \(\lambda(x) = a\) and \(\lambda(y) = b\), is an irreducible closed set in \(X\), but it is neither the closure of \(1_x\) nor that of \(1_y\).

An element \(a\) in a lattice \(L\) is a coprime if for all \(b, c \in L\), \(a \leq b \lor c\) implies that either \(a \leq b\) or \(a \leq c\). A complete lattice \(L\) is said to have enough coprimes if every element in \(L\) can be written as the join of a family of coprimes. It is clear that every linearly ordered quantale has enough coprimes and the complete lattice of closed sets in a topological space has enough coprimes.

We say that an element in a quantale \(Q = (Q, \&)\) is a coprime if it is a coprime in the underlying lattice \(Q\); and \(Q\) has enough coprimes if every element in \(Q\) can be written as the join of a family of coprimes. It is easily seen that if \(1 \in Q\) is a coprime and if \(F\) is an irreducible closed set in a \(Q\)-cotopological space \(X\), then for any closed sets \(A, B\) in \(X\), \(F \leq A \lor B\) implies either \(F \leq A\) or \(F \leq B\). Said differently, in this case, an irreducible closed set in a \(Q\)-cotopological space is a coprime in the lattice of its closed sets.

Let \(Q\) be a quantale and \(X\) be a (crisp) topological space. We say that a map \(\lambda: X \to Q\) is upper semicontinuous if for all \(p \in Q\),

\[
\lambda_{[p]} = \{x \in X \mid \lambda(x) \geq p\}
\]

is a closed set in \(X\).

**Lemma 3.13.** Let \(Q\) be a quantale with enough coprimes and \(X\) be a topological space.

1. \(\lambda: X \to Q\) is upper semicontinuous if and only if \(\lambda_{[p]}\) is a closed set in \(X\) for each coprime \(p \in Q\).
2. If both \(\lambda, \mu: X \to Q\) are upper semicontinuous then so is \(\lambda \lor \mu\).
3. The meet of any family of upper semicontinuous maps is upper semicontinuous.
4. If \(\lambda: X \to Q\) is upper semicontinuous then so is \(p \to \lambda\) for all \(p \in Q\).
Proof. (1) follows from the fact that $Q$ has enough coprimes and (2) is an immediate consequence of (1). The verification of (3) is straightforward. And (4) follows from

$$(p \rightarrow \lambda)_{[q]} = \{x \in X \mid q \leq p \rightarrow \lambda(x)\} = \{x \in X \mid p \& q \leq \lambda(x)\} = \lambda_{[p \& q]}$$

for all $p, q \in Q$.

The above lemma shows that if $Q$ is a quantale with enough coprimes, then for each topological space $X$, the family of upper semicontinuous maps $X \rightarrow Q$ forms a stratified $Q$-cotopology on $X$. We write $\omega_Q(X)$ for the resulting stratified $Q$-cotopological space.

For each closed set $K$ in $X$, $1_K : X \rightarrow Q$ is obviously upper semicontinuous, hence every closed set in $X$ is also a closed in $\omega_Q(X)$. Moreover, for any $A \subseteq X$, the closure of $1_A$ in $\omega_Q(X)$ equals $1_{\overline{A}}$, where $\overline{A}$ is the closure of $A$ in $X$.

The correspondence $X \mapsto \omega_Q(X)$ defines an embedding functor

$$\omega_Q : \text{Top} \rightarrow \text{SQ-CTop}.$$  

This functor is one of the well-known Lowen functors in fuzzy topology [18].

The following conclusion says that for a linearly ordered quantale $Q$, the notion of sobriety for $Q$-cotopological spaces is a good extension in the sense of Lowen [18].

**Proposition 3.14.** If $Q$ is a linearly ordered quantale, then a topological space $X$ is sober if and only if the $Q$-cotopological space $\omega_Q(X)$ is sober.

**Proof.** **Necessity.** Let $\lambda$ be an irreducible closed in $\omega_Q(X)$. Firstly, we show that for each $x \in X$, the value $\lambda(x)$ is either 0 or 1. Suppose on the contrary that there is some $x \in X$ such that $\lambda(x)$ is neither 0 nor 1. We proceed with two cases. If there is no element $y$ in $X$ such that $\lambda(y)$ is strictly between $\lambda(x)$ and 1, let $\phi = 1_{\lambda(y)}$ and $\psi$ be the constant map $X \rightarrow Q$ with value $\lambda(x)$. Then both $\phi$ and $\psi$ are closed in $\omega_Q(X)$ and $\lambda \leq \phi \lor \psi$, but neither $\lambda \leq \phi$ nor $\lambda \leq \psi$, contradictory to that $\lambda$ is irreducible. If there is some $y \in X$ such that $\lambda(x) < \lambda(y) < 1$, let $\phi = 1_{\lambda(y)}$ and $\psi$ be the constant map $X \rightarrow Q$ with value $\lambda(y)$. Then both $\phi$ and $\psi$ are closed in $\omega_Q(X)$ and $\lambda \leq \phi \lor \psi$, but neither $\lambda \leq \phi$ nor $\lambda \leq \psi$, contradictory to that $\lambda$ is irreducible. Therefore, $\lambda = 1_K$ for some closed set $K$ in $X$. Since $\lambda$ is irreducible in $\omega_Q(X)$, $K$ must be an irreducible closed set in $X$. Thus, in the topological space $X$, $K$ is the closure of $\{x\}$ for a unique $x$, i.e., $K = \{x\}$. This shows that in $\omega_Q(X)$, $\lambda$ is the closure of $1_x$ for a unique $x$. Therefore, $\omega_Q(X)$ is sober.

**Sufficiency.** We prove a bit more, that is, if $Q$ has enough coprimes and $\omega_Q(X)$ is sober, then $X$ is sober.

Let $K$ be an irreducible closed set in $X$. Firstly, we show that $1_K$ is an irreducible closed set in the $Q$-cotopological space $\omega_Q(X)$. That $1_K : X \rightarrow Q$ is upper semicontinuous is trivial. For any closed sets $\lambda, \mu$ in $\omega_Q(X)$, one has by definition that

$$\text{sub}_X(1_K, \lambda \lor \mu) = \bigwedge_{x \in K} (\lambda(x) \lor \mu(x)).$$

For any coprime $p \leq \text{sub}_X(1_K, \lambda \lor \mu)$, $K$ is clearly a subset of $(\lambda \lor \mu)_{[p]} = \lambda_{[p]} \cup \mu_{[p]}$, hence either $K \subseteq \lambda_{[p]}$ or $K \subseteq \mu_{[p]}$, and then either $p \leq \text{sub}_X(1_K, \lambda)$ or $p \leq \text{sub}_X(1_K, \mu)$. Therefore,

$$\text{sub}_X(1_K, \lambda \lor \mu) \leq \text{sub}_X(1_K, \lambda) \lor \text{sub}_X(1_K, \mu).$$

The converse inequality

$$\text{sub}_X(1_K, \lambda \lor \mu) \geq \text{sub}_X(1_K, \lambda) \lor \text{sub}_X(1_K, \mu)$$

is trivial. Thus, $1_K$ is an irreducible closed set in $\omega_Q(X)$.

Since $\omega_Q(X)$ is sober, there is a unique $x \in X$ such that $1_K$ is the closure of $1_x$ in $\omega_Q(X)$. Because the closure of $1_x$ in $\omega_Q(X)$ equals $1_{\overline{x}}$, one gets $K = \{x\}$. 

\[12\]
Note 3.15. The assumption in Proposition 3.14 that \( Q \) is linearly ordered is indispensable. To see this, let \( Q = \{0, a, b, 1\} \) be the Boolean algebra with four elements; let \( X \) be the discrete (hence sober) topological space with two points \( x \) and \( y \). It is clear that \( \omega_Q(X) \) is the discrete \( Q \)-cotopological space in Note 3.12 hence it is not sober.

At the end of this section, we discuss the relationship between sober \( Q \)-cotopological spaces and sober \( Q \)-topological spaces in the case that the quantale \( Q \) satisfies the law of double negation.

**Definition 3.16.** A \( Q \)-topology on a set \( X \) is a subset \( \tau \) of \( Q^X \) subject to the following conditions:

1. \( p_X \in \tau \) for all \( p \in Q \);
2. \( U \land V \in \tau \) for all \( U, V \in \tau \);
3. \( \bigvee_{j \in J} U_j \in \tau \) for each subfamily \( \{U_j\}_{j \in J} \) of \( \tau \).

The pair \((X, \tau)\) is called a \( Q \)-topological space; elements in \( \tau \) are called open sets of \((X, \tau)\).

A \( Q \)-topological space in the above definition is also called a weakly stratified \( Q \)-topological space in the literature, see e.g. [8, 9]. A \( Q \)-topology \( \tau \) is stratified [9] if

\[ p \land U \in \tau \text{ for all } p \in Q \text{ and } U \in \tau. \]

It is clear that if \( Q = (Q, \&) \) is a frame, i.e., if \( \& = \land \), then every \( Q \)-topology is stratified.

Let \( Q \) be a quantale that satisfies the law of double negation. If \( \tau \) is a (stratified) \( Q \)-cotopology on a set \( X \), then

\[ \neg(\tau) = \{\neg A \mid A \in \tau\} \]

is a (stratified) \( Q \)-topology on \( X \), where \( \neg A(x) = \neg(A(x)) \) for all \( x \in X \). Conversely, if \( \tau \) is a (stratified) \( Q \)-cotopology on \( X \), then

\[ \neg(\tau) = \{\neg A \mid A \in \tau\} \]

is a (stratified) \( Q \)-cotopology on \( X \). So, for a quantale \( Q \) that satisfies the law of double negation, we can switch freely between (stratified) \( Q \)-topologies and (stratified) \( Q \)-cotopologies, hence between open sets and closed sets.

If \( Q = (Q, \& \land) \) satisfies the law of double negation, then for any \( A, B \in Q^X \),

\[ \text{sub}_X(A, B) = \text{sub}_X(\neg B, \neg A) \]

and

\[ \bigvee_{x \in X} A(x) \& B(x) = \neg \text{sub}_X(A, \neg B) = \neg \text{sub}_X(B, \neg A). \]

These equations are clearly extensions of the properties listed in Proposition 3.16.

**Proposition 3.17.** Let \( Q \) be a quantale that satisfies the law of double negation; and let \((X, \tau)\) be a stratified \( Q \)-cotopological space. Then for each irreducible closed set \( F \) in \((X, \tau)\), the map

\[ f_F : \neg(\tau) \to Q, \quad f_F(U) = \bigvee_{x \in X} F(x) \& U(x) \]

satisfies the following conditions:

1. \( f_F(p_X) = p \).
2. \( f_F(U \land V) = f_F(U) \land f_F(V) \).
3. \( f_F(\bigvee_{i \in I} U_i) = \bigvee_{i \in I} f_F(U_i) \).
4. \( f_F(p \& U) = p \& f_F(U) \).

Conversely, if \( g : \neg(\tau) \to Q \) is a map satisfying (Fr1)–(Fr4), then there is a unique irreducible closed set \( F \) in \((X, \tau)\) such that \( g = f_F \).
Therefore, $g$ by definition of $F$.

Definition of sober.

Let $U$ all.

Proof. We check (Fr2) for example.

\[
    f_F(U \land V) = \lnot(\text{sub}_X(U \land V, \lnot F)) = \lnot(\text{sub}_X(F, \lnot(U \land V))) = \lnot(\text{sub}_X(F, \lnot U \lor \lnot V)) = \lnot(\text{sub}_X(F, \lnot U) \lor \text{sub}_X(F, \lnot V)) = \lnot(\text{sub}_X(U, \lnot F)) \land \lnot(\text{sub}_X(V, \lnot F)) = f_F(U) \land f_F(V).
\]

Conversely, suppose $g: (\tau) \to Q$ is a map that satisfies (Fr1)–(Fr4). Let $F = \bigwedge \{ A \in \tau | g(\lnot A) = 0 \}$.

We show that $F$ is an irreducible closed set in $(X, \tau)$ and $g = f_F$.

**Step 1.** $g(\lnot F) = 0$. This follows from (Fr3) and Proposition \([22, 13]\).

**Step 2.** $g(U) = \bigvee_{x \in X} F(x) \land U(x)$ for all $U \in (\tau)$. On one hand, if we let $p = \text{sub}_X(U, \lnot F)$, then $p \land U \leq \lnot F$, hence

\[
    p \land g(U) = g(p \land U) \leq g(\lnot F) = 0.
\]

Therefore,

\[
    g(U) \leq \lnot \text{sub}_X(U, \lnot F) = \bigvee_{x \in X} F(x) \land U(x).
\]

On the other hand, since $g(\lnot(g(U)) \land U) = g(U) \land g(U) = 0$, it follows that $\lnot(g(U)) \land U \leq \lnot F$ by definition of $F$. Therefore, $\lnot(g(U)) \leq \text{sub}_X(U, \lnot F)$, hence

\[
    g(U) \geq \lnot \text{sub}_X(U, \lnot F) = \bigvee_{x \in X} F(x) \land U(x).
\]

**Step 3.** $\bigvee_{x \in X} F(x) = 1$. Otherwise, let $\bigwedge_{x \in X} \lnot F(x) = p$. Then $p \neq 0$ and $p_X \leq \lnot F$.

Therefore, $g(\lnot F) \geq g(p_X) = p$, contradictory to that $g(\lnot F) = 0$.

**Step 4.** $\text{sub}_X(F, A \lor B) = \text{sub}_X(F, A) \lor \text{sub}_X(F, B)$ for all closed sets $A, B$ in $(X, \tau)$. In fact,

\[
    \text{sub}_X(F, A \lor B) = \text{sub}_X(\lnot A \land \lnot B, \lnot F) = \lnot(\text{sub}_X(\lnot A \land \lnot B)) = \lnot(\lnot A) \lor \lnot(\lnot B) = \text{sub}_X(\lnot A, \lnot F) \lor \text{sub}_X(\lnot B, \lnot F) = \text{sub}_X(F, A) \lor \text{sub}_X(F, B).
\]

The proof is completed. \(\square\)

For any stratified $\mathcal{Q}$-topological space $(X, \tau)$ and $x \in X$, the map

\[
    f_x: \tau \to Q, \quad f_x(U) = U(x)
\]

clearly satisfies (FR1)–(FR4) in Proposition \([8, 14]\). This fact leads to the following:

**Definition 3.18.** A stratified $\mathcal{Q}$-topological space $(X, \tau)$ is sober if for each map $f: \tau \to Q$ satisfying (FR1)–(FR4) in Proposition \([8, 14]\) there is a unique $x \in X$ such that $f(U) = U(x)$ for all $U \in \tau$.

We leave it to the reader to check that if $\mathcal{Q} = (Q, \&)$ is a frame, i.e., $\& = \land$, then the above definition of sober $\mathcal{Q}$-topological spaces coincides with that in \([3]\).

**Proposition 3.19.** Let $\mathcal{Q}$ be a quantale that satisfies the law of double negation. Then a $\mathcal{Q}$-topological space $(X, \tau)$ is sober if and only if the $\mathcal{Q}$-cotopological space $(X, \lnot(\tau))$ is sober.

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Proof. **Necessity:** Let $F$ be an irreducible closed set in the $Q$-cotopological space $(X, \neg(\tau))$. By Proposition 3.17, the map

$$f_F : \tau \to Q, \quad f_F(U) = \bigvee_{x \in X} F(x) \& U(x)$$

satisfies (Fr1)–(Fr4), hence there is a unique $a \in X$ such that $f_F(U) = U(a)$ for all $U \in \tau$. We claim that the closure of $1_a$ in $(X, \neg(\tau))$ is $F$. Since

$$\neg F(a) = f_F(\neg F) = \bigvee_{x \in X} F(x) \& (\neg F(x)) = 0,$$

then $F(a) = 1$. Therefore, $1_a \leq F$. Conversely, since

$$\bigvee_{x \in X} F(x) \& (\neg (1_a)(x)) = f_F(\neg (1_a)) = \neg (1_a)(a) = 0,$$

it follows that $F(x) \leq (\neg (1_a)(x)) = 1_a(x)$ for all $x \in X$, hence $F \leq 1_a$.

**Sufficiency.** Let $f : \tau \to Q$ be a map satisfying (FR1)–(FR4). By Proposition 3.17 there is an irreducible closed set $F$ in the $Q$-cotopological space $(X, \neg(\tau))$ such that $f(U) = \bigvee_{x \in X} F(x) \& U(x)$ for all $U \in \tau$. Since $(X, \neg(\tau))$ is sober, there is a unique $a \in X$ such that

$$f(U) = \bigvee_{x \in X} F(x) \& U(x) = \neg (1_a)(a) = 0,$$

completing the proof.

4. Examples

This section discusses the sobriety of some natural $Q$-cotopological spaces in the case that $Q$ is the unit interval $[0,1]$ endowed with a (left) continuous t-norm. In this section, we will write $a$, instead of $a_{[0,1]}$, for the constant map $[0,1] \to [0,1]$ with value $a$.

Let $Q = ([0,1], \&)$ with $\&$ being a (left) continuous t-norm on $[0,1]$. We consider three $Q$-cotopologies on $[0,1]$:

- $\tau_{CK}$: the stratified $Q$-cotopology on $[0,1]$ generated by $\{id\}$ as a subbasis;
- $\tau_{SK}$: the strong $Q$-cotopology on $[0,1]$ generated by $\{id\}$ as a subbasis;
- $\tau_{AK}$: the Alexandroff $Q$-cotopology on the $Q$-ordered set $([0,1], d_R)$, where $d_R(x, y) = y \to x$.

First of all, we list some facts about these $Q$-cotopologies.

(F1) A closed set in $([0,1], \tau_{AK})$ is, by definition, a $Q$-order-preserving map $\phi : ([0,1], d_L) \to ([0,1], d_L)$, where $d_L(x, y) = x \to y$. So, it is easy to verify that for all $\phi \in \tau_{AK}$:

- $\phi$ is increasing, i.e., $\phi(x) \leq \phi(y)$ whenever $x \leq y$;
- $\phi(1) = 1 \iff \phi \geq id$. 

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(F2) For each \( x \in [0, 1] \), the closure of \( 1_x \) in \( ([0, 1], \tau_{AK}) \) is \( x \to \text{id} \), i.e.,

\[ \overline{1_x} = x \to \text{id}. \]

On one hand, \( x \to \text{id} \) is a closed set in \( ([0, 1], \tau_{AK}) \) and \( (x \to \text{id})(x) = 1 \), hence \( \overline{1_x} \leq x \to \text{id} \).

On the other hand, for any \( \phi \in \tau_{AK} \), if \( \phi(x) = 1 \), then \( \phi(t) = \phi(x) \to \phi(t) \geq x \to t \) for all \( t \leq x \), hence \( \phi \geq x \to \text{id} \).

(F3) Since every Alexandroff \( \mathcal{Q} \)-cotopology is a strong \( \mathcal{Q} \)-cotopology and \( \text{id} \in \tau_{AK} \), it follows that

\[ \tau_{CK} \subseteq \tau_{SK} \subseteq \tau_{AK}. \]

Moreover, since \( x \to \text{id} \in \tau_{CK} \) for all \( x \in X \), the closure of \( 1_x \) in both \( ([0, 1], \tau_{CK}) \) and \( ([0, 1], \tau_{SK}) \) is \( x \to \text{id} \).

(F4) Since finite joins and arbitrary meets of right continuous maps \( [0, 1] \to [0, 1] \) are right continuous, and \( x \to \text{id} \) is right continuous for all \( x \in [0, 1] \), it follows that every closed set in \( ([0, 1], \tau_{CK}) \), as a map from \( [0, 1] \) to itself, is right continuous.

(F5) The space \( ([0, 1], \tau_{CK}) \) is initially dense in the category of stratified \( \mathcal{Q} \)-cotopological spaces. Indeed, for each stratified \( \mathcal{Q} \)-cotopological space \( (X, \tau) \),

\[ \{(X, \tau) \xrightarrow{A} ([0, 1], \tau_{CK})\}_{A \in \mathcal{T}} \]

is an initial source in the topological category \( \mathcal{SQ-CTop} \).

**Proposition 4.1.** Let \( \mathcal{Q} = ([0, 1], \&) \) with \( \& \) being a left continuous t-norm on \( [0, 1] \). Then the stratified \( \mathcal{Q} \)-cotopological space \( ([0, 1], \tau_{CK}) \) is sober.

**Proof.** It suffices to show that if \( \phi \) is an irreducible closed set in \( ([0, 1], \tau_{CK}) \), then \( \phi = x \to \text{id} = \overline{1_x} \) for some \( x \in [0, 1] \). Since \( \phi \) is increasing and \( \bigvee_{t \in [0, 1]} \phi(t) = 1 \), one obtains that \( \phi(1) = 1 \). Let

\[ x = \inf\{t \in [0, 1] \mid \phi(t) = 1\}. \]

Since \( \phi \) is right continuous by (F4), then \( \phi(x) = 1 \), hence \( \phi \geq \overline{1_x} = x \to \text{id} \). We claim that \( \phi = x \to \text{id} \). Otherwise, there is some \( t < x \) such that \( \phi(t) > x \to t \). Since \( x \to t = \bigwedge_{y < x} f(y \to t) \), there is some \( y \in (t, x) \) such that \( \phi(t) > y \to t \). It is clear that both \( y \to \text{id} \) and \( \phi(y) \) belong to \( \tau_{CK} \) and \( \phi \leq (y \to \text{id}) \lor \phi(y) \), but neither \( \phi \leq (y \to \text{id}) \) nor \( \phi \leq \phi(y) \), contradicting the assumption that \( \phi \) is irreducible. \( \square \)

In the following we consider sobriety of the \( \mathcal{Q} \)-cotopologies \( \tau_{SK} \) and \( \tau_{AK} \) in the case that \& is the minimum t-norm, the product t-norm and the Lukasiewicz t-norm.

**Proposition 4.2.** Let \( \mathcal{Q} = ([0, 1], \&) \) with \( \& \) being the product t-norm. Then the Alexandroff \( \mathcal{Q} \)-cotopology \( \tau_{AP} \) on \( ([0, 1], d_R) \) is not sober; but the strong \( \mathcal{Q} \)-cotopology \( \tau_{SP} \) on \( [0, 1] \) generated by \{id\} is sober.

**Proof.** By Example 3.11 in [10], the Alexandroff \( \mathcal{Q} \)-cotopology \( \tau_{AP} \) consists of maps \( \phi : [0, 1] \to [0, 1] \) subject to the following conditions:

(i) \( \phi \) is increasing; and

(ii) \( y/x \leq \phi(y)/\phi(x) \) whenever \( x > y \), where we agree by convention that \( 0/0 = 1 \).

We note that each \( \phi \) in \( \tau_{AP} \) is continuous on \( [0, 1] \), but it may be discontinuous at \( 0 \).

It is easy to verify that the map \( \phi : [0, 1] \to [0, 1] \), given by \( \phi(0) = 0 \) and \( \phi(t) = 1 \) for all \( t > 0 \), is an irreducible closed set in \( ([0, 1], \tau_{AP}) \), but it is not the closure of \( 1_x \) for any \( x \in [0, 1] \). So, \( ([0, 1], \tau_{AP}) \) is not sober.

In the following we prove in three steps that \( \tau_{SP} \) on \( [0, 1] \) is sober.

**Step 1.** We show that the stratified \( \mathcal{Q} \)-cotopology \( \tau_{CP} \) on \( [0, 1] \) generated by \{id\} is given by

\[ \tau_{CP} = \{ \phi \land a \mid a \in [0, 1], \phi \in \mathcal{B} \}, \]
Then $B \subseteq \text{it suffices to check that}$ $S$.

It is routine to verify that

\[ \phi(a) = \phi(a) \land (a \in [0, 1], \phi \in B) \] is a stratified $\mathcal{Q}$-cotopology on $[0, 1]$ that contains the identity $\text{id}: [0, 1] \to [0, 1]$, hence $\tau_{\mathcal{C}P} \subseteq \mathcal{C}$. To see that $\mathcal{C}$ is contained in $\tau_{\mathcal{C}P}$, it suffices to check that $B \subseteq \tau_{\mathcal{C}P}$. Let $\phi \in B$. For each $x \in (0, 1]$, define $g_x: [0, 1] \to [0, 1]$ by

\[ g_x = \phi(x) \lor ((\phi(x) \to x) \to t) = \phi(x) \lor \left( \frac{x}{\phi(x)} \to \text{id} \right) \]

Then $g_x \in \tau_{\mathcal{C}P}$. We leave it to the reader to check that

\[ \phi = \bigwedge_{x \in [0, 1]} g_x = \bigwedge_{x \in [0, 1]} \left( \phi(x) \lor \left( \frac{x}{\phi(x)} \to \text{id} \right) \right) \]

hence $\phi \in \tau_{\mathcal{C}P}$. Therefore, $\mathcal{C} \subseteq \tau_{\mathcal{C}P}$.

**Step 2.** We show that

\[ \tau_{\mathcal{S}P} = \{ \phi \in \tau_{\mathcal{A}P} \mid \phi \text{ is continuous} \} \]

It is easily verified that $\mathcal{S} = \{ \phi \in \tau_{\mathcal{A}P} \mid \phi \text{ is continuous} \}$ is a strong $\mathcal{Q}$-cotopology on $[0, 1]$ that contains the identity $\text{id}: [0, 1] \to [0, 1]$, hence $\tau_{\mathcal{S}P} \subseteq \mathcal{S}$. Conversely, for any $\phi \in \mathcal{S}$ with $\phi(1) > 0$, let $\psi = \phi(1) \to \phi = \phi/\phi(1)$. Then $\psi \in \tau_{\mathcal{A}P}$, $\psi(1) = 1$, and $\psi$ is continuous. Thus, $\psi \in \mathcal{S} \subseteq \tau_{\mathcal{S}P}$. Since $\tau_{\mathcal{S}P}$ is strong and $\phi = \phi(1) \land \psi$, then $\phi \in \tau_{\mathcal{S}P}$, therefore $\mathcal{S} \subseteq \tau_{\mathcal{S}P}$.

**Step 3.** $([0, 1], \tau_{\mathcal{S}P})$ is sober. Suppose $\phi$ is an irreducible closed set in $([0, 1], \tau_{\mathcal{S}P})$. Since $\phi$ is increasing and $\bigwedge_{t \in [0, 1]} \phi(t) = 1$, one has $\phi(1) = 1$, hence $\phi \in \mathcal{S} \subseteq \tau_{\mathcal{S}P}$. Since $\tau_{\mathcal{S}P}$ is coarser than $\tau_{\mathcal{S}P}$, then $\phi$ is an irreducible closed set in the sober space $([0, 1], \tau_{\mathcal{S}P})$, and consequently, $\phi = x \to \text{id}$ for a unique $x \in [0, 1]$. This shows that $\phi$ is the closure of $1_x$ for a unique $x \in [0, 1]$ in $([0, 1], \tau_{\mathcal{S}P})$, hence $([0, 1], \tau_{\mathcal{S}P})$ is sober.

**Proposition 4.3.** Let $\mathcal{Q} = ([0, 1], \&)$ with $\&$ being the Lukasiewicz t-norm. Then the strong $\mathcal{Q}$-cotopology $\tau_{\mathcal{Q}L}$ on $[0, 1]$ generated by $\{ \text{id} \}$ is sober and coincides with the Alexandroff $\mathcal{Q}$-cotopology $\tau_{\mathcal{A}L}$ on the $\mathcal{Q}$-ordered set $([0, 1], d_R)$.

**Proof.** By Example 3.10 in [13], the Alexandroff $\mathcal{Q}$-cotopology $\tau_{\mathcal{A}L}$ on $[0, 1]$ consists of maps $\phi: [0, 1] \to [0, 1]$ that satisfy the following conditions:

(i) $\phi$ is increasing; and
(ii) $\phi$ is $1$-Lipschitz, i.e., $\phi(x) - \phi(y) \leq x - y$ for all $x \geq y$.

Firstly, we show that the stratified $\mathcal{Q}$-cotopology $\tau_{\mathcal{Q}L}$ on $[0, 1]$ generated by $\{ \text{id} \}$ is given by

\[ \tau_{\mathcal{Q}L} = \{ \phi \land \phi(a) \mid a \in [0, 1], \phi \in \tau_{\mathcal{A}L}, \phi \geq \text{id} \} \]

It is routine to verify that

$\mathcal{C} = \{ \phi \land \phi(a) \mid a \in [0, 1], \phi \in \tau_{\mathcal{A}L}, \phi \geq \text{id} \}$ is a stratified $\mathcal{Q}$-cotopology on $[0, 1]$ that contains the identity $\text{id}: [0, 1] \to [0, 1]$, hence $\tau_{\mathcal{Q}L} \subseteq \mathcal{C}$. To see that $\mathcal{C}$ is contained in $\tau_{\mathcal{Q}L}$, it suffices to check that for any $\phi \in \tau_{\mathcal{A}L}$, if $\phi \geq \text{id}$ then $\phi \in \tau_{\mathcal{Q}L}$. For each $x \in [0, 1]$, define $g_x: [0, 1] \to [0, 1]$ by

\[ g_x = \phi(x) \lor ((\phi(x) \to x) \to \text{id}), \]

i.e.,

\[ g_x(t) = \max\{\phi(x), \min\{\phi(x) + t - x, 1\}\} \]

Then $g_x \in \tau_{\mathcal{Q}L}$. We leave it to the reader to check that

\[ \phi = \bigwedge_{x \in [0, 1]} g_x = \bigwedge_{x \in [0, 1]} ((\phi(x) \to x) \to \text{id}), \]

hence $\phi \in \tau_{\mathcal{Q}L}$. Therefore, $\mathcal{C} \subseteq \tau_{\mathcal{Q}L}$.

Secondly, we show that $\tau_{\mathcal{S}L} = \tau_{\mathcal{Q}L}$. For any $\phi \in \tau_{\mathcal{Q}L}$ with $\phi(1) > 0$, it is clear that $\phi(1) \to \phi \in \tau_{\mathcal{Q}L}$ and $(\phi(1) \to \phi)(1) = 1$. Thus, $\phi(1) \to \phi \in \tau_{\mathcal{Q}L} \subseteq \tau_{\mathcal{S}L}$. Since $\tau_{\mathcal{S}L}$ is strong and
\( \phi = \phi(1) \wedge (\phi(1) \to \phi) \), it follows that \( \phi \in \tau_{SL} \), hence \( \tau_{AL} \subseteq \tau_{SL} \). The converse inclusion \( \tau_{SL} \subseteq \tau_{AL} \) is trivial.

Finally, we show that \( ([0, 1], \tau_{SL}) \) is sober. Let \( \phi \) be an irreducible closed set in \( ([0, 1], \tau_{SL}) \).

Since \( \phi \) is increasing and \( \bigvee_{t \in [0, 1]} \phi(t) = 1 \), then \( \phi(1) = 1 \), hence \( \phi \in \tau_{CL} \). Since \( \tau_{CL} \) is coarser than \( \tau_{SL} \), it follows that \( \phi \) is an irreducible closed set in the sober space \( ([0, 1], \tau_{CL}) \), hence \( \phi = x \to \text{id} \) for a unique \( x \in [0, 1] \). This shows that \( \phi \) is the closure of \( 1_x \) for a unique \( x \in [0, 1] \) in \( ([0, 1], \tau_{SL}) \). Therefore, \( ([0, 1], \tau_{SL}) \) is sober. \( \square \)

**Proposition 4.4.** Let \( Q = ([0, 1], \& \rangle \) with \( \& \rangle \) being the \( t \)-norm min. Then the strong \( Q \)-cotopology \( \tau_{SM} \) on \([0, 1]\) generated by \{id\} is sober; but the Alexandroff \( Q \)-cotopology \( \tau_{AM} \) on the \( Q \)-ordered set \( ([0, 1], d_R) \) is not sober.

**Proof.** First of all, since \( \& = \min \), every stratified \( Q \)-cotopological space is obviously a strong \( Q \)-cotopological space, then the strong \( Q \)-cotopology \( \tau_{SM} \) on \([0, 1]\) generated by \{id\} coincides with the stratified \( Q \)-cotopology \( \tau_{CM} \) on \([0, 1]\) generated by \{id\}, hence it is sober by Proposition 4.1.

With the help of Example 3.12 in [10], it can be verified that

\[
\tau_{AM} = \{ \phi \wedge a \mid a \in [0, 1], \phi: [0, 1] \to [0, 1] \text{ is increasing, } \phi \geq \text{id} \}.
\]

For any \( a \in (0, 1) \), the map \( \phi: [0, 1] \to [0, 1] \) given by

\[
\phi(t) = \begin{cases} 
1, & t > a, \\
 t, & t \leq a,
\end{cases}
\]

is an irreducible closed set in \( ([0, 1], \tau_{AM}) \), and it is not the closure of \( 1_x \) for any \( x \in [0, 1] \), so, \( ([0, 1], \tau_{AM}) \) is not sober. \( \square \)

We would like to record here that

\[
\tau_{CM} = \tau_{SM} = \{ \phi \in \tau_{AM} \mid \phi \text{ is right continuous} \}.
\]

It is easily verified that \( \mathcal{C} = \{ \phi \in \tau_{AM} \mid \phi \text{ is right continuous} \} \) is a stratified \( Q \)-cotopology on \([0, 1]\) that contains the identity \( \text{id}: [0, 1] \to [0, 1] \), hence \( \tau_{CM} \subseteq \mathcal{C} \). To see the converse inclusion, we show firstly that for all \( \phi \in \mathcal{C} \), if \( \phi(1) = 1 \) then \( \phi \in \tau_{CM} \). For each \( x \in [0, 1] \), define \( g_x: [0, 1] \to [0, 1] \) by \( g_x = \phi(x) \vee (x \to \text{id}) \). Then \( g_x \in \tau_{CM} \). Since

\[
\phi = \bigvee_{x \in [0, 1]} g_x = \bigvee_{x \in [0, 1]} (\phi(x) \vee (x \to \text{id})),
\]

then \( \phi \in \tau_{CM} \). Secondly, for any \( \phi \in \mathcal{C} \), since \( \phi(1) \to \phi \in \mathcal{C} \), \( (\phi(1) \to \phi)(1) = 1 \), and \( \phi = \phi(1) \wedge (\phi(1) \to \phi) \), it follows that \( \phi \in \tau_{CM} \). Therefore, \( \mathcal{C} \subseteq \tau_{CM} \).

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