A theorem on the Higgs sector of the Standard Model

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Abstract

We provide the exact solution of the classical theory for the Higgs sector of the Standard Model obtaining the exact Green function for the broken phase. This has as a consequence that higher excited states must exist for the Higgs particle representing an internal spectrum for it. These higher excited states are exponentially damped and so really difficult to observe.

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I. INTRODUCTION

After the confirmation of the existence of the Higgs particle \cite{1, 2}, it is become mandatory to improve our understanding of this sector of the Standard Model, firstly postulated by Weinberg and Salam \cite{3, 4}, both from the experimental and theoretical sides. The scalar field that describes this particle has been put forward in the sixties \cite{5-10} and has never been exploited beyond a perturbative treatment to which is supposed to be amenable. Indeed, questions like the true spectrum of such a model or the exact form of the propagators were never properly answered. Rather, it is by now a well acquired fact that this theory is plagued by triviality, that is, it cannot exist as an interacting theory. This question was recently addressed, for the four dimensional case, in \cite{11, 12} for the strong coupling limit and for the full range in \cite{13}. Also, this theory suffers from the hierarchy problem arising from the corrections to the mass term that would imply that the mass of the Higgs particle should be as large as the Planck mass due to perturbative corrections. This aspect of the theory is under scrutiny yet and on wait of experimental inputs from the just restarted LHC.

In this paper we add a further exact result to the Higgs sector of the Standard model by deriving the exact spectrum of the theory in the broken phase. The interesting point is that a finite value for the quartic coupling is enough to grant exact solutions to the classical theory and, in turn, a well definite spectrum for the quantum counterpart. This spectrum is a superimposed one on the mass of the Higgs particle and appears like substructures are working. This could be explained by a higher level theory as string theory or technicolor. It appears like a mathematical property of the Higgs field and should be taken into account when studying it.

As a by-product of this theorem, we provide a mathematical technique to solve partial differential equations like that for the Green function using Lorentz invariance and Fourier transform. This result is in agreement with our conclusions in \cite{14}. Here we show how the Higgs sector in the Standard Model just implies an extended spectrum of massive excitations.

The paper is structured as follows. In Sec. II we present the exact solutions for the Higgs sector of the Standard Model and the the Green functions of the classical theory. In Sec. III we derive the spectrum of the theory using the Feynman-Kac formula. Finally, in Sec. IV we yield the conclusions.
II. EXACT SOLUTIONS

A. Model

We consider the Higgs sector of the Standard Model given by [16]

\[ \mathcal{L}_H = \partial_{\mu} \Phi^\dagger \partial^{\mu} \Phi - \mu^2 |\Phi|^2 - \lambda |\Phi|^4 \] (1)

being

\[ \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \] (2)

so that \(|\Phi|^2 = |\phi^+|^2 + |\phi^0|^2\). From this one has the following equations of motion

\[ \partial^2 \phi^+ = -\mu^2 \phi^+ - 2\lambda (|\phi^+|^2 + |\phi^0|^2) \phi^+ \] (3)

\[ \partial^2 \phi^0 = -\mu^2 \phi^0 - 2\lambda (|\phi^+|^2 + |\phi^0|^2) \phi^0. \] (4)

We assume \(v^2/2 = -\mu^2/\lambda\) for the vacuum expectation value of the theory, being \(\mu^2 = -|\mu^2| = -m^2\) as usual in the Higgs mechanism. The coupling with the gauge fields yields the Lagrangian

\[ \mathcal{L}_H = D_{\mu} \Phi^\dagger D^{\mu} \Phi - \mu^2 |\Phi|^2 - \lambda |\Phi|^4 \] (5)

being [16]

\[ D_{\mu} = \partial_{\mu} + ig'_{2} A_{\mu} + ig_{2} \tau \cdot b_{\mu} \] (6)

and the equations of motion change accordingly.

B. Classical solutions

The equations of motion (3) admit an exact solution by introducing two different phases, \(U\) and \(B\), and writing down

\[ \phi^+ (x) = e^{i\theta} \varphi_{U,B}(x) \] (7)

\[ \phi^0 (x) = e^{i\theta} \varphi_{U,B}(x) \] (8)

being for the unbroken phase, \(\mu^2 > 0\),

\[ \varphi_U(x) = \Lambda \left( \frac{1}{\lambda} \right)^{\frac{1}{4}} \text{sn} \left( p \cdot x + \chi_U, -\frac{\Lambda^2 \sqrt{\lambda}}{2\mu^2 + \Lambda^2 \sqrt{\lambda}} \right) \] (9)
provided the following dispersion relation does hold

\[ p^2 = \mu^2 + \frac{\Lambda^2}{2} \sqrt{\lambda}. \]  

(10)

This solution is obtained with \( \Lambda \) and \( \chi_U \) being two integration constants and \( \text{sn} \) a Jacobi elliptic function having parameter \( k^2 = -\frac{\Lambda^2 \sqrt{\lambda}}{2\mu^2 + \Lambda^2 \sqrt{\lambda}} \). These solutions hold with the condition \( \mu^2 > 0 \). We see that, already at classical level, the mass \( \mu \) gets renormalized by the self-interaction with a coupling \( \lambda \). It is enough that the self-coupling is finite to get such a solution. We note that this solution reduces to the known case for \( \mu = 0 \) given in [15, 17].

For the broken phase one has

\[ \varphi_B(x) = \left( \frac{m^2}{3\lambda} \right)^{\frac{1}{2}} \text{dn} (p \cdot x + \chi_B, -1) \]  

(11)

being \( \text{dn} \) another Jacobi elliptic function of parameter \( k^2 = -1 \), with \( \chi_B \) the phase and \( \Lambda \) the energy scale, both integration constants. This holds provided the dispersion relation

\[ p^2 = \frac{m^2}{3}. \]  

(12)

We just note that the angle \( \chi_B \) in the broken phase must be fixed by the value of the ground value of the model. This is fixed by the zero of potential in the equations of motion to be \( \phi^0 = v/\sqrt{2} \) with \( v = \sqrt{m^2/\lambda} \) and \( \phi^+ = 0 \). This is the choice generally done in the Standard Model that we assume also here. This imply the solution of the equation

\[ \sqrt{\frac{2}{3}} \text{dn}(\chi_B, -1) = 1 \]  

(13)

so that our exact solution describes oscillations around the value \( v/\sqrt{2} \) as expected from a physical point of view. We note that the angles \( \theta_+ \) and \( \theta_0 \) are arbitrary yet.

C. Green functions

The Green function for the theory can be obtained by the functional derivative of the equations

\[ \partial^2 \phi^+ = -\mu^2 \phi^+ - 2\lambda(|\phi^0|^2 + |\phi^+|^2)\phi^+ + j_+ \]  

(14)

\[ \partial^2 \phi^0 = -\mu^2 \phi^0 - 2\lambda(|\phi^0|^2 + |\phi^+|^2)\phi^0 + j_0 \]  

(15)

\[ \partial^2 \phi^{++} = -\mu^2 \phi^{++} - 2\lambda(|\phi^0|^2 + |\phi^+|^2)\phi^{++} + j_+^* \]  

(16)

\[ \partial^2 \phi^{0*} = -\mu^2 \phi^{0*} - 2\lambda(|\phi^0|^2 + |\phi^+|^2)\phi^{0*} + j_0^* \]  

(17)
with respect to \( j_+ \) and \( j_0 \) and \( j_+^* \) and \( j_0^* \). Then, the equations for the Green functions do not imply any nonlinear term. The only physical component for the Higgs field is the real component of \( \phi^0 \). We get

\[
\frac{\partial^2 \delta \phi^0}{\partial j_0(y)} \bigg|_{j_0, j_0^* = 0} = -\mu^2 \frac{\delta \phi^0}{\delta j_0(y)} \bigg|_{j_0, j_0^* = 0} - 4\lambda |\phi^0|^2 \frac{\delta \phi^0}{\delta j_0(y)} \bigg|_{j_0, j_0^* = 0} - 2 \frac{\delta \phi^0}{\delta j_0(y)} \bigg|_{j_0, j_0^* = 0} (\phi^0)^2 + \delta^4(x - y) \quad (18)
\]

\[
\frac{\partial^2 \delta \phi^{0*}}{\partial j_0^*(y)} \bigg|_{j_0, j_0^* = 0} = -\mu^2 \frac{\delta \phi^{0*}}{\delta j_0^*(y)} \bigg|_{j_0, j_0^* = 0} - 4\lambda |\phi^{0*}|^2 \frac{\delta \phi^{0*}}{\delta j_0^*(y)} \bigg|_{j_0, j_0^* = 0} - 2 \frac{\delta \phi^{0*}}{\delta j_0^*(y)} \bigg|_{j_0, j_0^* = 0} (\phi^{0*})^2 + \delta^4(x - y). \quad (21)
\]

\( \phi^0 \) and \( \phi^{0*} \) must to be computed at \( j_0 = 0 \) and \( j_0^* = 0 \) reducing at the exact solutions of the preceding section. Consistently, we can assume that

\[
\left. \frac{\delta \phi^{0*}}{\delta j_0^*(y)} \right|_{j_0, j_0^* = 0} = \left. \frac{\delta \phi^0}{\delta j_0(y)} \right|_{j_0, j_0^* = 0} = 0 \quad (22)
\]

and then, these equations for the Green functions reduce to a single one

\[
\partial^2 \Delta(x, y) = -\mu^2 \Delta(x, y) - \lambda (4|\phi^0|^2 + 2|\phi^+|^2)\Delta(x, y) + \delta^4(x - y). \quad (23)
\]

This yields the equation for the Green function of the Higgs field in the broken phase

\[
\partial^2 \Delta_H(x, y) - |\mu|^2 \Delta_H(x, y) + 6\lambda |\varphi_B(x)|^2 \Delta_H(x, y) = \delta^4(x - y). \quad (24)
\]

Our aim is to get the spectrum of the theory from it. This can be accomplished by noting that the homogeneous equation has the following solution

\[
u_1(x) = \text{sn}(p \cdot x + \chi_0, -1) \text{cn}(p \cdot x + \chi_0, -1) \quad (25)
\]

provided the dispersion relation \([12]\) holds. As we will see below, all we need is to evaluate \( \Delta_H(x, 0) \) as this distribution keeps all the information on the spectrum of the theory. In order to get this propagator we rewrite the above equation as

\[
\partial^2 \Delta_H(x, 0) - |\mu|^2 \Delta_H(x, 0) + 6\lambda |\varphi_B(x)|^2 \Delta_H(x, 0) = \delta^4(x) - \Delta_2 \Delta_H(x, 0). \quad (26)
\]

and iterate starting with the solution of the equation

\[
\partial^2 \Delta^0_H(x, 0) - |\mu|^2 \Delta^0_H(x, 0) + 6\lambda |\varphi_B(t, 0)|^2 \Delta^0_H(x, 0) = \delta^4(x) \quad (27)
\]
\[ \Delta^0_H(x, 0) = \delta^3(x)G(t) \]  
(28)

and

\[ G(t) = -\theta(t) \frac{\sqrt{3}}{\sqrt{2m^2}} u_1(t, 0) = \theta(t) \frac{\sqrt{3}}{m} \sin \left( \frac{m}{\sqrt{3}} t + \chi_0, -1 \right) \cos \left( \frac{m}{\sqrt{3}} t + \chi_0, -1 \right). \]  
(29)

The phase \( \chi_0 \) must be chosen to be \((2n+1)K(-1)\) with \( n \in \mathbb{Z} \) producing a set of propagators all having the same properties.

This can be Fourier transformed to yield

\[ G(\omega) = \frac{\sqrt{2\pi^3}}{K^3(-1)} \sum_{n=1}^{\infty} n^2 \frac{e^{-n\pi}}{1 + e^{-2n\pi}} \frac{1}{\omega^2 - m_n^2 + i\epsilon} \]  
(30)

being

\[ m_n = n \frac{\pi}{K(-1)} \frac{m}{\sqrt{3}} \]  
(31)

the mass spectrum that also entails a zero mass value, the Goldstone boson. \( K(-1) \) is a complete elliptic integral of the first kind. The next iterate takes the form

\[ \partial_t^2 \Delta^1_H(x, 0) - \mu^2 |\Delta^1_H(x, 0)|^2 \Delta^1_H(x, 0) + 6\lambda \varphi_B(t, 0) |\Delta^1_H(x, 0)|^2 \Delta^1_H(x, 0) = \delta^4(x) - \Delta_2 \Delta^0_H(x, 0) = \delta^4(x) - G(t) \Delta_2 \delta^3(x) \]  
(32)

remembering that should be interpreted in the sense of distribution just noting that \( \int dx f(x) \delta''(x) = f''(0) \). Therefore we will get

\[ \Delta^1_H(x, 0) = \int d^4 x' \Delta^0_H(x - x', 0) \delta^4(x') - \int d^4 x' \Delta^0_H(x - x', 0) G(t') \Delta_2 \delta^3(x') \]  
(33)

that is

\[ \Delta^1_H(x, 0) = \Delta^0_H(x, 0) - \int d^4 x' \Delta^0_H(x, x') G(t') \Delta_2 \delta^3(x') \]  
(34)

that we can Fourier transform to give

\[ \Delta^1_H(p, 0) = \Delta^0_H(p, 0) + p^2 \Delta^0_H(p, 0) G(\omega). \]  
(35)

This procedure can be iterated how far we want and we recover the fact that we are just obtaining the correction due to momentum \( p \) in the denominator of \( G(t) \) producing the full propagator. This result implies that we have immediately the exact propagator for the Higgs field in the classical theory but, assuming that translation invariance is a property of the theory, we would have got it just moving from the rest frame with a boost obtaining

\[ \Delta_H(p) = \frac{\sqrt{2\pi^3}}{K^3(-1)} \sum_{n=1}^{\infty} n^2 \frac{e^{-n\pi}}{1 + e^{-2n\pi}} \frac{1}{p^2 - m_n^2 + i\epsilon}. \]  
(36)
Thus, we have provided in this way a general mathematical technique to solve equations like (26) reducing them to ordinary differential equations. The linearity of the partial differential equation played a relevant role. This represents the key result of our paper. We will show in the following section the way the mass spectrum $m_n$ enters into the theory. Also, it is important to note that higher excited states appear to be exponentially depressed and so, really difficult to observe.

Given eq.(36), the classical theory is completely solved as already shown in [15]. We note that, notwithstanding the “wrong” sign in the mass term, the solutions satisfy a correct dispersion relation and so, we obtain a proper spectrum for the quantum theory.

### III. SPECTRUM OF THE THEORY

In order to get a consistent quantum theory we have to prove that the set of solutions we obtained are indeed unique. This can be seen by introducing the new variable $\xi = p \cdot x = p_0 t - p \cdot x$ into the more general equation

$$\partial^2 \varphi(x) + \mu^2 \varphi(x) + \lambda \varphi^3(x) = 0 \quad (37)$$

yielding

$$p^2 \varphi''(\xi) + \mu^2 \varphi(\xi) + \lambda \varphi^3(\xi) = 0 \quad (38)$$

that can be stated in the form

$$\varphi''(\xi) = -\frac{\mu^2}{p^2} \varphi(\xi) - \frac{\lambda}{p^2} \varphi^3(\xi). \quad (39)$$

Now, we can rescale it writing $\varphi = \zeta \tilde{\varphi}$ being $\zeta$ a constant. One has

$$\tilde{\varphi}''(\xi) = -\frac{\mu^2}{p^2} \tilde{\varphi}(\xi) - \zeta^2 \frac{\lambda}{p^2} \tilde{\varphi}^3(\xi). \quad (40)$$

This is the definition of the sn Jacobi function [18] provided the parameter is given by $k^2 = -\zeta^2 \lambda / 2p^2$ and the relation of dispersion (10) holds for a proper choice of $\zeta$. Similarly, for the broken phase one obtain the differential equation for the dn Jacobi function provided the dispersion relation (12) holds and the parameter is $k^2 = -1$. So, these solutions represent unique traveling waves for the equations of the scalar field and, as such, are amenable to quantization.
In order to obtain the spectrum of the theory we use the Feynman-Kac formula. This can be justified in the following way. Let us consider the propagator as defined by

\[ \langle x | e^{-itH} | y \rangle = \sum_n e^{-iE_n t} \phi_n(x) \phi^*_n(y) \] (41)

after the introduction of the complete set of eigenstates that diagonalizes the Hamiltonian, \( H|n\rangle = E_n|n\rangle \), such that \( \langle x | n \rangle = \phi_n(x) \). Then,

\[ \langle x | e^{-itH} | 0 \rangle = \sum_n e^{-iE_n t} \phi_n(x) \phi^*_n(0) \] (42)

and the ground state is obtained as usual after a Wick rotation \( \tau \rightarrow it \) and taking the limit \( \tau \rightarrow \infty \). We will get

\[ E_0 = - \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log \left( \sum_n e^{-E_n \tau} \phi_n(x) \phi^*_n(0) \right). \] (43)

We see that the choice \( y = 0 \) is irrelevant to determine the spectrum of the theory. The procedure can be iterated to compute the energy levels of the higher states.

Then, we notice that the action entering into the generating functional of the theory of the theory admits a series in the currents. One has

\[ A[j_0, j^*_0, j^+_+, j^*_+] = \int d^4x \left[ \partial \phi^+ \partial \phi^+ \partial \phi^0 \partial \phi^0 - \mu^2(|\phi^+|^2 + |\phi^0|^2) - \lambda(|\phi^+|^2 + |\phi^0|^2)^2 + j^*_+ \phi^+ + \phi^+_+ j^*_+ + j^*_0 \phi^0_0 + \phi^0_0 j^*_0 \right]. \] (44)

So, the Taylor series takes the form

\[ A[j_0, j^*_0, j^+_+, j^*_+] = A[0, 0, 0, 0] + \int d^4x \phi^+ j^*_+ |j_0, j^*_0, j^+_0, j^*_0 = 0 j_+^* + \phi^+ j^*_+ |j_0, j^*_0, j^+_0, j^*_0 = 0 j_+^* + \phi^0 j^*_0 |j_0, j^*_0, j^+_0, j^*_0 = 0 j^*_0 + \phi^0 j^*_0 |j_0, j^*_0, j^+_0, j^*_0 = 0 j^*_0 \] + \int d^4x d^4y j^*_0(x) \Delta_H(x - y) j_0(y) + O(j^2) + O(j^3) \] (45)

being \( \phi^+ |j_0, j^*_0, j^+_0, j^*_0 = 0 \) and \( \phi^0 |j_0, j^*_0, j^+_0, j^*_0 = 0 \) the exact solutions given in eq. (7) and the Higgs propagator will be different depending on the phase being broken or unbroken. This represents the first few terms of the generating functional of the theory and so, we can extract the spectrum from the Higgs propagator using the Feynman-Kac formula. One has, using
\[ \Delta_H(t, p) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip_0 t} \frac{\sqrt{2\pi^3}}{K^3(-1)} \sum_{n=1}^{\infty} n^2 \frac{e^{-n\pi}}{1 + e^{-2n\pi}} \frac{1}{p^2 - m_n^2 + i\epsilon} \]

where we used contour integration. The application of the Feynman-Kac formula at this stage is straightforward and we can conclude that the Higgs potential in the Standard Model entails a continuous spectrum of free particles having a superimposed discrete spectrum of masses. The argument runs similarly for the unbroken phase. This means that higher excited states of the observed particle should be expected even if these are exponentially damped and so even more difficult to be seen than the ground state.

IV. CONCLUSIONS

We have shown how the classical theory of Higgs sector of the Standard Model has exact solutions with the correct behavior for the dispersion relation, notwithstanding the possible “wrong” sign in the mass term. These solutions are propagating nonlinear waves that share the property of being unique. In this way, a quantum theory can be easily developed that, in the simplest case, displays a spectrum for the mass of the Higgs particle representing some kind of internal degrees of freedom. In the broken phase, the theory admits a classical solution describing oscillations around a constant value that is never zero as expected for the spontaneous breaking of symmetry. Green functions can be exactly computed and we provide a general technique to do so.

This mathematical proof implies that higher excited states of the current observed Higgs particles should be expected and that the observed one is just the ground state of an extended spectrum. This internal excited states could be explained by a higher level theory like string theory or technicolor where bound states display such a behavior in a natural way. With the restart of the LHC these states should be easier to see.

In a near future we will provide a more general derivation of the behavior of the Standard Model, particularly for the gauge bosons, in presence of this general formulation of the Higgs
sector.

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