D-DIMENSIONAL CONFORMAL
\(\sigma\)-MODELS AND TOPOLOGICAL EXCITATIONS

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Abstract

The D-dimensional conformal nonlinear sigma-models (NSM) are constructed. It is shown that the NSM on spaces with \(\pi_{D-1} = \mathbb{Z}\) have the topological solutions of a "hedgehog" and "anti-hedgehog" type with logarithmic energies. For spaces with \(\pi_D \neq 0\) they have also the topological excitations with finite energies of instanton types.
I. Introduction.

It is known the topological excitations (TE) with logarithmically divergent energy play important role in low-dimensional systems with $D \leq 2$, where they induce the topological phase transitions (TPT) [1-4]. As can be shown for the existence of such topological excitations one needs the configurations of the boundary values of field, determining a homotopically nontrivial mapping of the physical space boundary $\partial \mathbb{R}^D = S^{D-1}$ into the vacuum manifold $\mathcal{M}$ of the internal space $\mathcal{R}^N$, and a scale invariant action [5]. An existence of such configurations depends on the nontriviality of the corresponding homotopic group $\pi_{D-1}(\mathcal{M})$. Due to this property the TPT admits another, topology changing, interpretation [3]. In the massive (high-temperature) phase the freely existing TE conserve an information about the compactness and topology of the vacuum manifold $\mathcal{M}$ and give the short range correlations. In the massless (low-temperature) phase the TE are bounded due to logarithmic interaction into neutral dipoles. This gives a lost of information about the compactness and topology of $\mathcal{M}$, at least, on large scales, and the theory behaves as a free, noncompact, conformal one. Note that under such TPT there is no real topological reconstruction of $\mathcal{M}$ at the phase transition point. The TPT gives only statistical decompactification of the theory.

After construction of the theory of low-dimensional TPTs a question has naturally arose about a possibility of such TE and TPT in systems with dimensions $D > 2$. However, in higher dimensional systems the main efforts were devoted to the discovery of the TE with finite energy [6-9]. All such excitations give finite contribution to the partition function, but cannot induce a PT similar to the TPT, since the latter is induced by TE with logarithmically divergent energy.

Unfortunately, the TE with logarithmic energies do not appear in usual higher dimensional theories with action bilinear on gradient. Instead it has been only shown that the partition function of ”particles” with logarithmic interaction can appear in some cases. For example, such partition function appears as a measure on the parameter (or moduli) space of instantons in 4-dimensional Yang-Mills theory (the so called ”merons”) [10], and in 2-dimensional nonlinear $\sigma$-model (NSM) [11]. But these ”particles” were not the TE.

Recently it was shown that the TE with logarithmic energy can exist in 3D conformal (or the van der Waals) nonlinear $\sigma$-model (NSM) [12]. In
In this paper we consider a possibility of existence of TE with logarithmical energies in higher \((D > 3)\) dimensional systems, whose behaviour at large scales can be described by generalized nonlinear \(\sigma\)-models. It will be shown that an existence of such excitations is intimately related with a conformal symmetry of the models. It appears also that these models admit another TE with finite energies similar to the instantons.

II. D-dimensional conformal invariant nonlinear \(\sigma\)-models

Here we will consider only the models of nonlinear \(\sigma\)-model type. More accurate analysis of these models has shown that an existence of topological excitations with discrete topological charges and logarithmic energy puts over the following properties on NSM:

1) their homotopical group \(\pi_{D-1}(\mathcal{M})\) must be nontrivial and abelian discrete,

2) a conformal invariance at classical level.

The first property permits some ambiguity in a dimension and a form of \(\mathcal{M}\). while the second one defines a form of the NSM action \(S\) almost uniquely in arbitrary dimensions.

A general expression for action \(S\) of \(D\)-dimensional generalized nonlinear \(\sigma\)-models, admitting nonlocal ones, can be represented in the next form

\[
S = \frac{1}{2\alpha} \int d^Dx d^Dx' \psi^a(x) \mathcal{G}^{(D)}_{ab}(\psi|x, x') \psi^b(x'), \quad a, b = 1, ..., n
\]  

(1)

where \(\psi \in \mathcal{M}\), the manifold of degenerate vacuum states of the model, and \(n\) is its dimension. The form of the kernel \(\mathcal{G}\) depends on model. If the structures of the internal and physical spaces do not depend on each other and the latter space is homogeneous, then \(\mathcal{G}\) can be decomposed

\[
\mathcal{G}^{(D)}_{ab}(\psi|x, x') = g_{ab}(\psi(x), \psi(x')) \Box_D(x - x'),
\]  

(2)

where \(g_{ab}\) is some two-point metric function on \(\mathcal{M}\). For local models an expression for \(\mathcal{G}\) can be defined in terms of manifold \(\mathcal{M}\) only

\[
\mathcal{G}(\psi|x, x') = g_{ab}(\psi) \Box \delta(x - x')
\]  

(3)

If the manifold \(\mathcal{M}\) can be embedded in Euclidean vector space \(\mathbb{R}^{N(n)}\) with dimension \(N(n)\), depending on \(n\), then one can use instead of \(g_{ab}(\psi, \psi')\) an usual Euclidean metric

\[
g_{ab} = \delta_{ab}, \quad a, b = 1, ..., N.
\]  

(4)
and the constraints defining $\mathcal{M}$. Due to the first condition we must consider the manifolds with a discrete abelian homotopic group $\pi_{D-1}(\mathcal{M})$. The spheres $S^{D-1}$ are the simplest among them with $\pi_{D-1}(\mathcal{M}) = \mathbb{Z}$. Then $N(D - 1) = D$, and

$$\psi^a = n^a, \quad a = 1, \ldots, D, \quad (n)^2 = 1. \quad (5)$$

where $n(x)$ is a field of unit vectors in internal space $\mathcal{R}^D$. Since $N = D$ this internal space can be identified with a physical space. Below we confine ourselves by the simplest manifolds, the spheres. A possible generalizations will be shortly discussed in conclusion.

For analyzing of the possible actions (and kernels) it is more convenient to write $\mathcal{S}$ and $\Box_D$ in the momentum space

$$\mathcal{S} = \frac{1}{2\alpha} \int d^Dx d^Dx' (n(x)n(x')) \Box_D(x-x') = \frac{1}{2\alpha} \int \frac{d^Dk}{(2\pi)^D} |n(k)|^2 \Box_D(k) \quad (6)$$

For asymptotical scale invariance at large scales the kernel $\Box_D$ must have the next behaviour at small $k$

$$\Box_D(k) \simeq |k|^D(1 + a_1(ka) + \ldots) = |k|^D f(ka) \quad (7)$$

where $a$ is a UV cut-off parameter, $f(ka)$ is some regularizing function with the next asymptotics

$$f(ka) = 1 + a_1 ka + \ldots, \quad ka \to 0, \quad f(ka) \to 0, \quad ka \to \infty. \quad (8)$$

The kernel $\Box_D$ generalizes an usual local and conformal kernel of two-dimensional $\sigma$-model

$$\Box_2(k) \equiv \Box(k) = -\partial^2(k) = k^2 \quad (9)$$

For this reason $\Box_D$ can be also named as a $|\partial|^D$ kernel. From (9) it follows that in even dimensional spaces $\mathbb{R}^{2s}$ the kernel $\Box_{2s}$ is a local one

$$\Box_{2s} = (-1)^s((\partial)^2)^s. \quad (10)$$

In odd dimensions $\Box_D$ is always a nonlocal one with a large-scale asymptotics

$$\Box_D(x)|_{x \gg a} \simeq A_D/|x|^{2D},$$
where \( S_{D-2} = \frac{2^{D+1}}{\Gamma(D)} \) is a volume of the \((D - 2)\)-dimensional unit sphere. Such kernels appear often in physics. The most known and important cases correspond to the Calogero-Sutherland-Dyson [13] and to the Caldeira-Leggett kernel \((D = 1)\) [14] and to the van der Waals potential \((D = 3)\) [12].

There are analogous nonlocal kernels in even-dimensional spaces

\[
\Box_{2s}(x) \sim 1/x^{2s},
\]

but their Fourier-images contain an additional logarithmic factor

\[
\Box_{2s}(k) \sim k^{2s} \ln(k/k_0)
\]

where \(k_0\) is some UV cut-off parameter, regularizing a kernel (11) at small scales. This factor breaks some important properties of the model. Further we will take in even-dimensional spaces only a local kernel.

As is well known a conformal group in the higher-dimensional \((D > 2)\) spaces is finite-dimensional [13]. Its main nontrivial transformation is an inversion transformation

\[
x^i \rightarrow x^i/r^2, \quad r = |x|.
\]

The conformal invariance of \(S\) with a kernel (11) follows from the next transformation properties of the kernel \(\Box_D\) under conformal transformation (14) (the field \(n(x)\) and a coupling constant \(\alpha\) are dimensionless)

\[
x_i \rightarrow x'_i = x_i/r^2, \quad r \rightarrow r' = 1/r, \quad x_i/r = x'_i/r',
\]

\[
d^D x \rightarrow d^D x/|x|^{2D}, \quad \frac{1}{|x_1 - x_2|^{2D}} \rightarrow \frac{|x_1|^{2D}|x_2|^{2D}}{|x_1 - x_2|^{2D}},
\]

and, consequently,

\[
S = \frac{A_D}{2\alpha} \int d^D x_1 d^D x_2 \frac{(n_1 n_2)}{|x_1 - x_2|^{2D}}
\]

is invariant and dimensionless. Thus the action (15) with a kernel (11) can be named a \(D\)-dimensional conformal NSM. Note that this invariance takes place for even dimensions also. Strictly speaking, a conformal invariance
takes place only at large scales, since a kernel (11) needs some regularization at small distances, which can break this invariance.

The corresponding Euler - Lagrange equation has a form

\[
\int \Box_D (x - x') n(x') d^D x' - n(x) \int (n(x) n(x')) \Box_D (x - x') d^D x' = 0. \tag{16}
\]

The Green function \( G^D(x) \) of the conformal kernel \( \Box_D(x) \) can be defined by next equation

\[
\int \Box_D (x - x'') G^D(x'' - x') d^D x'' = \delta(x - x') \tag{17}
\]

It has the following form

\[
G^D(x) = \Box_D^{-1} = |\partial|^{-D} = \int \frac{d^D k}{(2\pi)^D \Box_D(k)} e^{i(kx)} \tag{18}
\]

At large scales \( G^D(x) \) has a logarithmic asymptotic behaviour

\[
G^D(x) \big|_{r \gg a} \simeq -B_D \ln(r/R),
\]

\[
B_D = \frac{S_{D-2} \Gamma(\frac{1}{2}) \Gamma(\frac{D-1}{2})}{(2\pi)^D \Gamma(\frac{D}{2})} = \frac{1}{2^{D-1} \pi^{D/2} \Gamma(\frac{D}{2})}, \tag{19}
\]

where \( R \) is a radius of the space or a size of system. Thus the conformal kernels \( \Box_D \) correspond to logarithmic Green functions and for this reason are connected with kernels of the field theories equivalent to the generalized logarithmic gases \( [16] \).

An usual analysis (see for example \( [17] \)) shows that a model (15) does not have an usual phase transition with nonzero order parameter (OP) due to logarithmic divergence of the OP fluctuations

\[
(\delta n(x))^2 \sim \int d^D k G(k) \sim \int d^D k/k^D \sim \ln(R/a), \tag{20}
\]

where \( \delta n(x) \) is a deviation from some fixed value \( n_0 \). In this sense the D-dimensional conformal vector NSM (15) are analogous to 2D XY-model \( [3, 4] \).

The thermodynamic properties of the model (15) will be considered in other paper \( [18] \). Here we consider only its possible TE.

### III. Topological excitations with logarithmic energy.
As was noted in [12] an equation (16) can be represented in a "linear" form, introducing new function $g(x)$:

$$\int \Box_D (x - x') n(x') d^D x' = g(x) n(x),$$

(21)

$$g(x) = \int (n(x) n(x')) \Box_D (x - x') d^D x'.$$

(22)

As all equations for NSM on spheres, it means that the action of the operator $\Box_D$ on vector $n(x)$ must be proportional to this vector, i.e. the vector field $n(x)$ must be in some sense an "eigenvector" of the operator $\Box_D$ with the "eigenvalue" $g(x)$, functionally depending on $n(x)$. For each solution of equation (21) a value of the $S[n]$ can be expressed through the "eigenvalue"

$$S = \frac{1}{2\alpha} \int g(x) d^D x$$

(23)

Since $\pi_2(S^2) = \mathbb{Z}$, there are the TE with topological charge $Q \in \mathbb{Z}$. The simplest TE with charge $Q = 1$, corresponding to the identical map of spheres $S^2$, must have the next asymptotic form

$$n^i(x)_{r \gg a} \simeq \frac{x^i}{r}$$

(24)

Substituting (24) into equation (16), passing to the momentum space and using the Fourier-image of the corresponding function

$$n^i(k) = -i C_D \frac{k^i}{k^D k},$$

(25)

where

$$C_D = 2^{D-1} \pi^{\frac{D-1}{2}} (D - 1) \Gamma((D - 1)/2),$$

(26)

we see, after returning to the $x$-space, that the field (24) is an "eigenvector" of $\Box_D$ and, consequently, a solution of equation (16). The corresponding "eigenvalue" is

$$g(x) = \frac{C_D^2}{(2\pi)^{D/2} r^D}.$$ 

(27)

The action $S$ of this solution can be found by two ways. In the first case one can use (6,25) to obtain

$$S \simeq \frac{C_D}{\alpha} \int \frac{dk}{k} f(ka)$$

(28)
The integral in (28) is logarithmically divergent as it should be. With a logarithmic accuracy one obtains

\[ S = \frac{C_D}{\alpha} \ln(R/a), \quad C_D = \frac{C_D^D S_{D-1}}{2(2\pi)^D} = \frac{2^{D-2} - \frac{D-2}{2}(D-1)^2 \Gamma^2(\frac{D-1}{2})}{\Gamma(\frac{D}{2})}. \tag{29} \]

The same result follows also from (23,27). One can show that the interaction of two different TE with charges \( Q_1 \) and \( Q_2 \) on large distances has a form of the Green function \( G^D(r) \)

\[ H_{12}(r) = Q_1 Q_2 G(r) \simeq -Q_1 Q_2 B_D \ln(r/R) \tag{30} \]

Analogously one can show that a field (24) is also a solution of corresponding equations in even dimensional spaces. For local kernel the equation is

\[ \Box^{2s} n(x) - n(x) (n(x) \Box^{2s} n(x)) = 0, \quad D = 2s, \tag{31} \]

and

\[ \Box^{2s} n(x) = b_s/r^{2s} n(x), \quad b_s = \prod_{k=1}^{s} k(D-k) = \frac{\Gamma(D)\Gamma(\frac{D+2}{2})}{\Gamma(D/2)}. \tag{32} \]

Its energy is

\[ S = \frac{b_s S_{D-1}}{2\alpha} \ln(R/a) = \frac{\pi^{D/2}\Gamma(D+1)}{2\alpha\Gamma(D/2)}. \tag{33} \]

For nonlocal kernel (11,12) the corresponding equation has the same form as (16), but the "eigenvalue" \( g(r) \) will now contain a part with logarithmic factor, which gives for the energy more complicated expression than simple logarithm. For this reason the nonlocal kernels in even dimensional spaces are not so interesting ones.

Note that in the usual local D-dimensional NS-model with action

\[ S = \frac{1}{2A} \int d^Dx (\partial n)^2 \tag{34} \]

such TE have the energy

\[ E \simeq \frac{S_{D-1}(D-1)}{2A(D-2)} (R^{D-2} - a^{D-2}). \tag{35} \]
It is interesting that if we consider the mixed action $S_{\text{mix}}$, containing a sum of kernels $\Box_d$, $d = 2, D$, then the corresponding equation has again the "hedgehog" solution (24) with total energy

$$E = \frac{S_{D-1}(D-1)}{2A(D-2)}(R^{D-2} - a^{D-2}) + \frac{C_D}{\alpha} \ln(R/a).$$

(36)

It means that a logarithmic part of the "hedgehog" energy can be observed also in mixed models at scales

$$(A/\alpha)^{1/(D-2)} > l > a.$$ 

It is clear that analogous "anti-hedgehog" solutions with the same logarithmic energies also exist.

**IV. Other topological excitations.**

Besides TE with logarithmic energy the TE of instantons type with finite energy can exist in the conformal NSM. They correspond to the configurations with trivial boundary condition

$$\mathbf{n}(x) \to \mathbf{n}_0, \quad r \to \infty,$$

(37)

where $\mathbf{n}_0$ is some constant unit vector. A necessary condition for their existence is a nontriviality of other relevant homotopic group $\pi_D(S^{D-1})$. As is known from a general theory of homotopic groups of spheres $\pi_D(S^{D-1})$ is defined through the suspension construction by homotopic group $\pi_1(SO(D-1))$ [15]. Since $\pi_1(SO(k)) = \mathbb{Z}_2$, $k > 2$, it follows from this that the conformal NSM on spheres must have the instanton-like TE with topological charges

$$Q \in \pi_1(SO(D-1)) = \mathbb{Z}_2 = \mathbb{Z}(\text{mod}2).$$

All above topological charges are scalar (one-component).

The natural interesting generalization of the conformal NSM on spheres is the D-dimensional conformal NS-models on simple compact groups $G$ or on homogeneous spaces $G/H$. As is known the nontrivial homotopic groups of $G$ correspond to the so-called characteristic classes of this groups [15].

$$\pi_k(G) \neq 0, \quad k = 2k_i(G) - 1, \quad 1 \leq i \leq r(G)$$

(38)

where $k_i(G)$ are the Weyl indices of group $G$, and $r(G)$ is a rank of group $G$. It follows from (3) that only odd homotopic groups can be nontrivial. All
Weyl indices, the degree of the Weyl invariant polynomials on the maximal abelian Cartan subalgebra are known [19]. Then the TE with logarithmic energy are possible only for $D-1 = 2k_i(G)-1$, i.e. only for even dimensional spaces with $D = 2k_i(G)$, and the instanton-like TE for odd $D = 2k_i(G)-1$. Since $\pi_k(G) = 0$ or $\mathbb{Z}$, the corresponding topological charges will also be scalar ones.

In some cases it is very important to have a vectorial topological charges [20]. A simple generalization of the sphere $S^{D-1}$, analogous to the torus $T^n$ in 2D case, is a bouquet of $n$ spheres 

$$B_n^{D-1} = S_1^{D-1} \lor \ldots \lor S_n^{D-1}$$

and all spaces $\mathcal{M}$ with this first nontrivial topological cell complex. Then the topological charges will have a vector form

$$Q = (q_1, \ldots, q_n), \quad q_i \in \mathbb{Z}.$$ 

But, in this case the vector topological charges, corresponding to different spheres, will not interact between themselves due to their orthogonality as in a case of torus [20]. For obtaining an interaction of topological charges in 3D case one needs to consider NS-models on deformed $B_n^{D-1}$. For example, the maximal flag spaces $F_G = G/T_G$ of the simple compact groups $G$ have $\pi_3(F_G) = \mathbb{L}_v$, where $\mathbb{L}_v$ is a dual root lattice of $G$. Note that a sphere $S^2$ is a particular case of $F_G$: $S^2 = SU(2)/U(1)$. Then the 3D conformal NS-models on $F_G$ will have topological excitations with a logarithmic energy and interacting vector topological charges $Q \in \mathbb{L}_v$. Since $\pi_3(F_G) = \pi_3(G) = \mathbb{Z}$, in this case the "neutral" configurations will also have different topological structure described by group $\pi_3(F_G)$.

But for higher homotopic groups $\pi_i(F_G) = \pi_i(G), \quad i > 3$, thus in higher dimensions ($D > 3$) only scalar charges are possible in these models.

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