Abstract. We announce a Godbillon-Vey index formula on foliated bundles with boundary; in particular, we define a Godbillon-Vey eta invariant. More generally, we explain a new approach to higher index theory on geometric structures with boundary. This is achieved by defining, first of all, relative and absolute Dirac index classes, denoted respectively $\text{Ind}(D,D^0)$ and $\text{Ind}(D)$, and proving that they correspond under excision. Next, for each cyclic $k$-cocycle $\tau_k$ defining a higher index in the closed case we define

- a eta cyclic cocycle $\sigma_{k+1}$, depending solely on boundary data; $\sigma_{k+1}$ is obtained by a sort of suspension procedure involving $\tau_k$ and a specific $1$-cocycle $\sigma_1$ (Roe’s $1$-cocycle);
- a relative cyclic $k$-cocycle $(\tau_k^r, \sigma_{k+1})$, with $\tau_k^r$ a cyclic cochain defined from $\tau_k$ through a regularization.

The index theorem is obtained by deforming the absolute pairing $(\text{Ind}(D), \langle \tau_k \rangle)$, which is by definition the higher index, to the relative pairing $(\text{Ind}(D, D^0), \langle \tau_k^r, \sigma_{k+1} \rangle)$, which gives the right hand side of the desired index formula. The eta-correction term is obtained through the eta cocycle $\sigma_{k+1}$.

1. Introduction

Connes’ index theorem for $G$-proper manifolds [1], with $G$ an étale groupoid, unifies under a single statement most of the existing index theorems. We shall focus on a particular case of such a theorem, that of foliated bundles. Thus, let $N$ be a closed compact manifold. Let $\Gamma \to \check{N} \to N$ be a Galois $\Gamma$-cover. Let $T$ be a smooth oriented compact manifold with an action of $\Gamma$ which is assumed to be by diffeomorphisms, orientation preserving and locally free, as in [9]. Let $Y = \check{N} \times_T T$ and let $(Y, \mathcal{F})$ be the associated foliation. (This is an example of $G$-proper manifold with $G$ equal to the groupoid $T \rtimes \Gamma$.)

If $T = \text{point}$ and $\Gamma = \{1\}$ we have a compact manifold and Connes’ index theorem reduces to the Atiyah-Singer index theorem. If $\Gamma = \{1\}$ we simply have a fibration and the theorem reduces to the Atiyah-Singer family index theorem. If $T = \text{point}$ then we have a Galois covering and Connes’ index theorem reduces to the Connes-Moscovici higher index theorem. If $\dim T > 0$ and $\Gamma \neq \{1\}$, then Connes’ index theorem is a higher foliation index theorem on the foliated manifold $(Y, \mathcal{F})$.

One particularly interesting higher index is the so-called Godbillon-Vey index; an alternative treatment of Connes’ index formula in this particular case was given by Moriyoshi-Natsume in [9]. Subsequently, Gorokhovsky and Lott [3] gave a superconnection proof of Connes’ index theorem, including an explicit formula for the Godbillon-Vey higher index. Leichtnam and Piazza [13] extended Connes’ index theorem to foliated bundles with boundary, using an extension of Melrose $b$-calculus and the Gorokhovsky-Lott superconnection approach. Unfortunately, a key assumption in [13] is that the group $\Gamma$ be of polynomial growth. This excludes many interesting examples and higher indexes; in particular it excludes the possibility of proving a Atiyah-Patodi-Singer formula for the Godbillon-Vey higher index.

One primary objective of this article is to announce such a result. In tackling the problem we also develop a new approach to index theory on manifolds with boundary. The abstract gives a short summary of the main idea; more details can be found in what follows. Full proofs will appear in [10].

2. Geometry of foliated bundles.

2.1. Manifolds with boundary. Let now $(\check{M}, g)$ be a riemannian manifold with boundary; the metric is assumed to be of product type in a collar neighborhood $U \simeq [0, 1] \times \partial M$ of the boundary. Let $\check{M}$ be a Galois $\Gamma$-cover of $M$; we let $\check{g}$ be the lifted metric. We also consider $\partial \check{M}$, the boundary of $\check{M}$. Let $X_0 = \check{M} \times_T T$; this is a manifold with boundary and the boundary $\partial X_0$ is equal to $\partial \check{M} \times_T T$. $(X_0, \mathcal{F}_0)$ denotes the associated foliated bundle. The leaves of $(X_0, \mathcal{F}_0)$ are manifolds with boundary endowed with a product-type metric. The boundary $\partial X_0$ inherits a foliation $\mathcal{F}_0$. The cylinder $\mathbb{R} \times \partial X_0$ also inherits a foliation $\mathcal{F}_{\text{cyl}}$, obtained by crossing the leaves of $\mathcal{F}_0$ with $\mathbb{R}$. Similar considerations apply to the half cylinders $(-\infty, 0] \times \partial X_0$ and $\partial X_0 \times [0, +\infty)$. 

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2.2. Manifolds with cylindrical ends. Notation. We consider \( \tilde{V} := \tilde{M} \cup_{\partial \tilde{M}} \left(-\infty, 0\right] \times \partial \tilde{M} \), endowed with the extended metric and the obviously extended \( \Gamma \) action along the cylindrical end. We consider \( X := \tilde{V} \times \Gamma \); this is a foliated bundle, with leaves manifolds with cylindrical ends. We denote by \( (X, \mathcal{F}) \) this foliation. Notice that \( X = X_0 \cup_{\partial X_0} \left(-\infty, 0\right] \times \partial X_0 \); moreover the foliation \( \mathcal{F} \) is obtained by extending \( \mathcal{F}_0 \) on \( X_0 \) to \( X \) via the product cylindrical foliation \( \mathcal{F}_{\text{cyl}} \) on \( (-\infty, 0] \times \partial X_0 \). We can write more suggestively: \( (X, \mathcal{F}) = (X_0, \mathcal{F}_0) \cup_{\partial X_0, \mathcal{F}_0} \left(((-\infty, 0]\times \partial X_0, \mathcal{F}_{\text{cyl}})\right) \). For \( \lambda > 0 \) we shall also consider the finite cylinder \( \tilde{V}_\lambda = \tilde{M} \cup_{\partial \tilde{M}} \left(-\lambda, 0\right] \times \partial \tilde{M} \) and the resulting foliated manifold \((X_\lambda, \mathcal{F}_\lambda)\). Finally, with a small abuse, we introduce the notation: \( \text{cyl}(\partial X) := \mathbb{R} \times \partial X_0 \), \( \text{cyl}^+(\partial X) := (-\infty, 0] \times \partial X_0 \) and \( \text{cyl}^+(\partial X) := \partial X_0 \times [0, +\infty) \). The foliations induced on \( \text{cyl}^+(\partial X) \), \( \text{cyl}^+(\partial X) \) by \( \mathcal{F}_0 \) will be denoted by \( \mathcal{F}_{\text{cyl}}, \mathcal{F}_{\text{cyl}}^\pm \).

2.3. Holonomy groupoid. We consider the groupoid \( G := (\tilde{V} \times \tilde{V} \times T)/\Gamma \) with \( \Gamma \) acting diagonally; the source map and the range map are defined by \( s(y, y', \theta) = [y', \theta], r(y, y', \theta) = [y, \theta] \). Since the action on \( T \) is assumed to be locally free, we know that \((G, r, s)\) is isomorphic to the holonomy groupoid of the foliation \((X_\lambda, \mathcal{F}_\lambda)\). In the sequel, we shall call \((G, r, s)\) the holonomy groupoid. If \( E \to X \) is a hermitian vector bundle on \( X \), with product structure along the cylindrical end, then we can consider the bundle over \( G \) equal to \( (s^*E)^* \otimes r^*E \).

3. Wiener-Hopf extensions

3.1. Foliation \( C^*\)-algebras. We consider \( C_c(X, \mathcal{F}) := C_c(G) \). \( C_c(X, \mathcal{F}) \) can also be defined as the space of \( \Gamma \)-invariant continuous functions on \( \tilde{V} \times \tilde{V} \times T \) with \( \Gamma \)-compact support. More generally we consider \( C_c(X, \mathcal{F}; E) := C_c(G, (s^*E)^* \otimes r^*E) \) with its well known \( * \)-algebra structure given by convolution. We shall often omit the vector bundle \( E \) from the notation.

The foliation \( C^*\)-algebra \( C^*(X, \mathcal{F}; E) \) is defined by completion of \( C_c(X, \mathcal{F}; E) \). See for example [9] where it is also proved that \( C^*(X, \mathcal{F}; E) \) is isomorphic to the \( C^*\)-algebra of compact operators of the Connes-Skandalis \( C(T) \rtimes \Gamma \)-Hilbert module \( E \) (this is also described in [9]). Summarizing: \( C^*(X, \mathcal{F}; E) \cong \mathcal{K}(E) \subseteq \mathcal{L}(E) \).

3.2. Translation invariant operators. Recall \( \text{cyl}(\partial X) := \mathbb{R} \times \partial X_0 \equiv (\mathbb{R} \times \partial \tilde{M}) \times_\Gamma T \) with \( \Gamma \) acting trivially in the \( \mathbb{R} \)-direction of \( (\mathbb{R} \times \partial \tilde{M}) \). We consider the foliated cylinder \( \text{cyl}(\partial X), \mathcal{F}_{\text{cyl}} \) and its holonomy groupoid \( G_{\text{cyl}} := ((\mathbb{R} \times \partial \tilde{M}) \times (\mathbb{R} \times \partial \tilde{M}) \times T)/\Gamma \) (source and range maps are clear). Let \( \mathbb{R} \) act trivially on \( T \); then \( (\mathbb{R} \times \partial \tilde{M}) \times (\mathbb{R} \times \partial \tilde{M}) \times T \) has a \( \mathbb{R} \times \Gamma \)-action, with \( \Gamma \) acting by translation on itself. We consider the \( * \)-algebra \( B_c(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}) \equiv B_c \)

\[
(1) \quad B_c := \{ k \in C((\mathbb{R} \times \partial \tilde{M}) \times (\mathbb{R} \times \partial \tilde{M}) \times T); k \text{ is } \mathbb{R} \times \Gamma \text{-invariant, } k \text{ has } \mathbb{R} \times \Gamma \text{-compact support} \}
\]

The product is by convolution. An element \( \ell \) in \( B_c \) defines a \( \Gamma \)-equivariant family \( (\ell(\theta))_{\theta \in T} \) of translation-invariant operators. The completion of \( B_c \) with respect to the obvious \( C^* \)-norm (the sup over \( \theta \) of the operator-\( L^2 \)-norm of \( \ell(\theta) \)) gives us a \( C^* \)-algebra that will be denoted \( B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}) \) or more briefly \( B^* \).

3.3. Wiener-Hopf extensions. Recall the Hilbert \( C(T) \rtimes \Gamma \)-module \( \mathcal{E} \) and the \( C^*\)-algebras \( \mathcal{K}(\mathcal{E}) \) and \( \mathcal{L}(\mathcal{E}) \). Since the \( C(T) \rtimes \Gamma \)-compact operators \( \mathcal{K}(\mathcal{E}) \) are an ideal in \( \mathcal{L}(\mathcal{E}) \) we have the classical short exact sequence of \( C^*\)-algebras

\[
0 \to \mathcal{K}(\mathcal{E}) \to \mathcal{L}(\mathcal{E}) \xrightarrow{s} \mathcal{Q}(\mathcal{E}) \to 0
\]

with \( \mathcal{Q}(\mathcal{E}) = \mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \) the Calkin algebra. Let \( \chi_0^\mathbb{R} : \mathbb{R} \to \mathbb{R} \) be the characteristic function of \( (-\infty, 0] \); let \( \chi_\mathbb{R} : \mathbb{R} \to \mathbb{R} \) be a smooth function with values in \([0, 1]\) such that \( \chi(t) = 1 \) for \( t \leq -\epsilon \), \( \chi(t) = 0 \) for \( t \geq 0 \). Let \( \chi^0 \) and \( \chi \) be the functions induced by \( \chi_\mathbb{R}^0 \) and \( \chi_\mathbb{R} \) on \( X \). Similarly, introduce \( \chi_{\text{cyl}}^0 \) and \( \chi_{\text{cyl}} \).

Lemma 3.2. There exists a bounded linear map

\[
(3.3) \quad s : B^* \to \mathcal{L}(\mathcal{E})
\]

extending \( s_0 : B_c \to \mathcal{L}(\mathcal{E}), s_0(\ell) := \chi^0 \ell \chi^0 \). Moreover, the composition \( \rho = \pi s \) induces an injective \( C^* \)-homomorphism

\[
(3.4) \quad \rho : B^* \to \mathcal{Q}(\mathcal{E})
\]
We consider $\text{Im} \rho$ as a $C^*$-subalgebra in $\mathbb{Q}(\mathcal{E})$ and identify it with $B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}})$ via $\rho$. Set

$$A^*(X; \mathcal{F}) := \pi^{-1}(\text{Im} \rho) \quad \text{with } \pi \text{ the projection } \mathcal{L}(\mathcal{F}) \to \mathbb{Q}(\mathcal{E}).$$

Recalling the identification $C^*(X, \mathcal{F}) = \mathbb{K}(\mathcal{E})$, we thus obtain a short exact sequence of $C^*$-algebras:

$$(3.5) \quad 0 \to C^*(X, \mathcal{F}) \to A^*(X; \mathcal{F}) \xrightarrow{\pi} B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}) \to 0$$

where the quotient map is still denoted by $\pi$. Notice that (3.5) splits as a short exact sequence of Banach spaces, since we can choose $s : B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}) \cong A^*(X; \mathcal{F})$ the map in (3.5) as a section. So

$$A^*(X; \mathcal{F}) \cong C^*(X, \mathcal{F}) \oplus s(B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}))$$

as Banach spaces.

There is also a linear map $t : A^*(X; \mathcal{F}) \to C^*(X, \mathcal{F})$ which is obtained as follows: if $k \in A^*(X; \mathcal{F})$, then $k$ is uniquely expressed as $k = a + s(l)$ with $a \in C^*(X, \mathcal{F})$ and $\pi(k) = l \in B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}})$. Thus, $\pi(k) = [\chi^a \ell^\lambda] \in \mathbb{Q}(\mathcal{E})$ for one (and only one) $l \in B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}})$ since $\rho$ is injective. We set

$$(3.6) \quad t(k) := k - s(\pi(k)) = k - \chi^a \ell^\lambda$$

Then $t(k) \in C^*(X, \mathcal{F})$.

4. Relative pairings and the eta cocycle: the algebraic theory

4.1. Introductory remarks. On a closed foliated bundle $(Y, \mathcal{F})$, the Godbillon-Vey cyclic cocycle is initially defined on the "small" algebra $\mathcal{A}_c \subset C^*(Y, \mathcal{F})$ of $\Gamma$-equivariant smoothing operators of $\Gamma$-compact support (in formulae, $\mathcal{A}_c := C^\infty_c(G, (s^*E)^\ast \otimes r^*E)$). Since the index class defined using a pseudodifferential parametrix is already well defined in $K_*(\mathcal{A}_c)$, the pairing between the the Godbillon-Vey cyclic cocycle and the index class is well-defined.

In a second stage, the cocycle is continuously extended to a dense holomorphically closed subalgebra $\mathfrak{A} \subset C^*(Y, \mathcal{F})$; there are at least two reasons for doing this. First, it is only by going to the $C^*$-algebraic index that the well known properties for the signature and the spin Dirac operator of a metric of positive scalar curvature hold. The second reason for this extension rests on the structure of the index class which is employed in the proof of the higher index formula, i.e. either the graph projection or the Wassermann projection; in both cases $\mathcal{U}_c$ is too small to contain the index class and one is therefore forced to find an intermediate subalgebra $\mathfrak{A}$, $\mathcal{A}_c \subset \mathfrak{A} \subset C^*(Y, \mathcal{F})$; $\mathfrak{A}$ is big enough for the two particular index classes to belong to it but small enough for the Godbillon-Vey cyclic cocycle to extend; moreover, being dense and holomorphically closed it has the same $K$-theory as $C^*(Y, \mathcal{F})$.

Let now $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends; in this section we shall select "small" subalgebras $\mathcal{J}_c \subset C^*(X, \mathcal{F})$, $\mathcal{A}_c \subset A^*(X, \mathcal{F})$, $\mathcal{B}_c \subset B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}})$, with $\mathcal{J}_c$ an ideal in $\mathcal{A}_c$, so that there is a short exact sequence $0 \to \mathcal{J}_c \to \mathcal{A}_c \xrightarrow{s_c} \mathcal{B}_c \to 0$ which is a subsequence of $0 \to C^*(X, \mathcal{F}) \xrightarrow{\pi_c} B^*(\text{cyl}(\partial X), \mathcal{F}_{\text{cyl}}) \to 0$. We shall then proceed to define the relevant cyclic cocycles, relative and absolute, and study, algebraically, their main properties. As in the closed case, we shall eventually need to find an intermediate short exact sequence, sitting between the two, $0 \to \mathcal{J}_c \to \mathfrak{A} \to \mathfrak{B} \to 0$, with constituents big enough for the index classes (relative and absolute) to belong to them but small enough for the cyclic cocycles (relative and absolute) to extend; this is quite a delicate point and it will be explained in Section 5. We anticipate that in contrast with the closed case the ideal $\mathcal{J}_c$ in the small subsequence will be too small even for the index class defined by a pseudodifferential parametrix. This has to do with the non-locality of the parametrix on manifolds with boundary; it is a phenomenon that was explained in detail in [6].

4.2. Small dense subalgebras. Define $\mathcal{J}_c := C^\infty_c(X, \mathcal{F})$; see subsection 3.1. Redefine

$$\mathcal{B}_c := \{k \in C^\infty_c((\mathbb{R} \times \partial \bar{M}) \times (\mathbb{R} \times \partial \bar{M}) \times T); k \text{ is } \mathbb{R} \times \Gamma \text{-invariant, } k \text{ has } \mathbb{R} \times \Gamma \text{-compact support}\}$$

see subsection 3.2. We now define $\mathcal{A}_c$: consider the functions $\chi^\lambda$, $\chi^\lambda_{\text{cyl}}$ induced on $X$ and $\text{cyl}(\partial X)$ by the real function $\chi^\lambda_{\text{-inf}, -\lambda}$. We shall say that $k$ is in $\mathcal{A}_c$ if it is a smooth function on $\bar{V} \times \bar{V} \times T$ which is $\Gamma$-invariant and for which there exists $\lambda \equiv \lambda(k) > 0$, such that

- $k - \chi^\lambda k \chi^\lambda$ is of $\Gamma$-compact support
- there exists $\ell \in \mathcal{B}_c$ such that $\chi^\lambda k \chi^\lambda = \chi^\lambda_{\text{cyl}} \ell \chi^\lambda_{\text{cyl}}$ on $(-\infty, -\lambda] \times \partial \bar{M}) \times ((-\infty, -\lambda] \times \partial \bar{M}) \times T$
Lemma 4.1. $A_c$ is a *-subalgebra of $A^*(X, \mathcal{F})$. Let $\pi_c := \pi|_{A_c}$; there is a short exact sequence of *-algebras
\begin{equation}
0 \to J_c \hookrightarrow A_c \xrightarrow{\pi_c} B_c \to 0.
\end{equation}

Remark 4.3. Notice that the image of $A_c$ through $t|_{A_c}$ is not contained in $J_c$ since $\chi^0$ is not even continuous. Similarly, the image of $B_c$ through $s|_{B_c}$ is not contained in $A_c$.

4.3. Relative cyclic cocycles. Let $A$ be a $k$–algebra over $k = \mathbb{C}$. Recall the cyclic cohomology groups $HC^*(A)$ [1]. Given a second algebra $B$ together with a surjective homomorphism $\pi : A \to B$, one can define the relative cyclic complex
\[ C^*_\lambda(A, B) := \{ (\tau, \sigma) : \tau \in C^n_\lambda(A), \sigma \in C^{n+1}_\lambda(B) \} \]
with coboundary map given by
\[ (\tau, \sigma) \mapsto (\pi^* \sigma - br, b\sigma) \, . \]
A relative cochain $(\tau, \sigma)$ is thus a cocyle if $br = \pi^* \sigma$ and $b\sigma = 0$. One obtains in this way the relative cyclic cohomology groups $HC^*(A, B)$. If $A$ and $B$ are Fréchet algebra, then we can also define the topological (relative) cyclic cohomology groups. More detailed information are given, for example, in [7].

4.4. Roe’s 1-cocycle. In this subsection, and in the next two, we study a particular but important example. We assume that $T$ is a point and that $\Gamma = \{1\}$, so that we are really considering a compact manifold $X_0$ with boundary $\partial X_0$ and associated manifold with cylindrical ends $X$; we keep denoting the cylinder $\mathbb{R} \times \partial X_0$ by cyl$(\partial X)$ (thus, as before, we don’t write the subscript 0). The algebras appearing in the short exact sequence [12] are now given by $J_c = C^\infty_c(X \times X)$,
\[ B_c = \{ k \in C^\infty(\mathbb{R} \times \partial X_0) \times (\mathbb{R} \times \partial X_0) ; k \text{ is } \mathbb{R}\text{-invariant, } k \text{ has compact } \mathbb{R}\text{-support} \} \, . \]

Finally, a smooth function $k$ on $X \times X$ is in $A_c$ if there exists a $\lambda \equiv \lambda(k) > 0$ such that
(i) $k - \chi^\lambda k \chi^\lambda$ is of compact support on $X \times X$;
(ii) $\exists \ell \in B_c$ such that $\chi^\lambda k \chi^\lambda = \chi^\lambda \psi(\chi^\lambda \psi)$ on $((-\infty, -\lambda) \times \partial X_0) \times (-\infty, -\lambda) \times \partial X_0)$.

For such a $k \in A_c$ we define $\pi_c(k) = \ell$ and we have the short exact sequence of *-algebras
\[ 0 \to J_c \hookrightarrow A_c \xrightarrow{\pi_c} B_c \to 0 \, . \]
Incidentally, in the Wiener-Hopf short exact sequence [53], which now reads as $0 \to C^*(X) \xrightarrow{\pi} A^*(X) \xrightarrow{\partial X} B^*(cyl(\partial X)) \to 0$ the left term $C^*(X)$ is clearly given by the compact operators on $L^2(X)$.

We shall define below a 0-relative cyclic cocycle associated to the homomorphism $\pi_c : A_c \to B_c$. To this end we start by defining a cyclic 1-cocycle $\sigma_1$ for the algebra $B_c$; this is directly inspired from work of John Roe (indeed, a similarly defined 1-cocycle plays a fundamental role in his index theorem on partitioned manifolds [11]).

Consider the characteristic function $\chi^\lambda_{\text{cyl}}$, $\lambda > 0$, induced on the cylinder by the real function $\chi^\lambda_{\text{cyl}}((-\infty, -\lambda)]$. For notational convenience, unless absolutely necessary, we shall not distinguish between $\chi^\lambda_{\text{cyl}}$ on the cylinder $\text{cyl}(\partial X)$ and $\chi^\lambda$ on the manifold with cylindrical ends $X$.

We define $\sigma_1^\lambda : B^\mathbb{R}_c \times B_c \to \mathbb{C}$ as
\begin{equation}
\sigma_1^\lambda(\ell_0, \ell_1) := \text{Tr}(\ell_0[\chi^\lambda, \ell_1]) \, .
\end{equation}
One can check that the operators $[\chi^\lambda, \ell_0]$ and $\ell_0[\chi^\lambda, \ell_1]$ are trace class $\forall \ell_0, \ell_1 \in B_c$ (and $\text{Tr}[\chi^\lambda, \ell_0] = 0$). In particular $\sigma_1^\lambda(\ell_0, \ell_1)$ is well defined.

Proposition 4.5. The value $\text{Tr}(\ell_0[\chi^\lambda, \ell_1])$ is independent of $\lambda$ and will simply be denoted by $\sigma_1(\ell_0, \ell_1)$. The functional $\sigma_1 : B_c \times B_c \to \mathbb{C}$ is a cyclic 1-cocycle.

4.5. Melrose’ regularized integral. Recall that our immediate goal is to define a relative cyclic 0-cocycle for the homomorphism $\pi_c : A_c \to B^\mathbb{R}_c$ appearing in the short exact sequence of the previous section. Having defined a 1-cocycle $\sigma_1$ on $B^\mathbb{R}_c$ we now need to define a 0-cocycle on $A_c$. Our definition will be a simple adaptation of the definition of the $b$-trace in Melrose’ $b$-calculus [5] (but since our algebra $A_c$ is very small, we can give a somewhat simplified treatment). Recall that for $\lambda > 0$ we are denoting by $X_\lambda$ the compact manifold obtained attaching $[-\lambda, 0) \times \partial X_0$ to our manifold with boundary $X_0$.

So, let $k \in A_c$ with $\pi_c(k) = \ell \in B_c$. Since $\ell$ is $\mathbb{R}$-invariant on the cylinder $\text{cyl}(\partial X) = \mathbb{R} \times \partial X_0$ we can write $\ell(y, y', s)$ with $y, y' \in \partial X_0, s \in \mathbb{R}$. Set
\begin{equation}
\tau_0^\lambda(k) := \lim_{\lambda \to +\infty} \left( \int_{X_\lambda} k(x, x) \text{dvol}_g - \lambda \int_{\partial X_0} \ell(y, y, 0) \text{dvol}_{g_0} \right)
\end{equation}
where the superscript $r$ stands for \textit{regularized}. (The $b$-superscript would be of course more appropriate; unfortunately it gets confused with the $b$ operator in cyclic cohomology.) It is elementary to see that the limit exists; in fact, because of the very particular definition of $A_c$, the function $\varphi(\lambda) := \int_{X_\lambda} k(x, x) d\text{vol}_y - \lambda \int_{\partial X_0} \ell(y, y, 0) d\text{vol}_g$ becomes constant for large values of $\lambda$. The proof is elementary. $\tau^r_0$ defines a 0-cocycle on $A_c$.

\textbf{Remark 4.7.} Notice that (4.6) is nothing but Melrose’ \textit{regularized integral} [8], in the cylindrical language, for the restriction of $k$ to the diagonal of $X \times X$.

We shall also need the following

\textbf{Lemma 4.8.} If $k \in A_c$ then $t(k)$, which is a priori a compact operator, is in fact trace class and $\tau^r_0(k) = \text{Tr}(t(k))$.

We remark once again that $t(k)$ is not an element in $J_c$.

\section*{4.6. The regularized integral and Roe’s 1-cocycle define a relative 0-cocycle.} We finally consider the relative 0-cocycle $(\tau^r_0, \sigma_1)$ for the pair $A_c \overset{\pi_c}{\to} B_c$.

\textbf{Proposition 4.9.} The relative 0-cocycle $(\tau^r_0, \sigma_1)$ is a relative 0-cocycle. It thus defines an element $[(\tau^r_0, \sigma_1)]$ in the relative group $HC^0(A_c, B_c)$.

There are several proofs of this Proposition; we have stated that $\sigma_1$ is a cocycle and what needs to be proved now is that $b\tau^r_0 = (\pi_c)^* \sigma_1$. One proof of this equality employs Lemma 4.8; another one use the Hilbert transform and Melrose’ formula for the $b$-trace of a commutator [8], see the next Subsection.

\section*{4.7. Melrose’ 1-cocycle and the relative cocycle condition via the $b$-trace formula.} As we have anticipated in the previous subsection, the equation $b\tau^r_0 = (\pi_c)^* \sigma_1$ is nothing but a compact way of rewriting Melrose’ formula for the $b$-trace of a commutator. We wish to explain this point here. Following now the notations of the $b$-calculus, we consider the slightly larger algebras

$$\begin{align*}
A^b_c := \Psi^{-\infty}(X, E), & \quad B^b_c := \Psi^{-\infty}(N_{+\partial X}, E|_{\partial}), & \quad J^b_c := \rho_{\Psi}\Psi^{-\infty}(X, E)
\end{align*}$$

and $0 \to J^b_c \to A^b_c \overset{\pi_c}{\to} B^b_c \to 0$, with $\pi^b_c$ equal to Melrose’ indicial operator $I(\cdot)$. Let $\tau^r_0$ be equal to the $b$-Trace: $\tau^r_0 := b\text{Tr}$. Observe that $\sigma_1$ also defines a 1-cocycle on $B^b_c$. We can thus consider the relative 0-cocycle $(\tau^r_0, \sigma_1)$ for the homomorphism $A^b_c \overset{I(\cdot)}{\to} B^b_c$; in order to prove that this is a relative 0-cocycle it remains to to show that $b\tau^r_0(k, k') = \sigma_1(I(k), I(k'))$, i.e.

$$b\text{Tr}[k, k'] = \text{Tr}(I(k)[\chi^0, I(k')]) \quad \text{(4.10)}$$

Recall here that Melrose’ formula for the $b$-trace of a commutator is

$$b\text{Tr}[k, k'] = \frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_{\partial X} (\partial_\mu I(k, \mu) \circ I(k', \mu)) d\mu \quad \text{(4.11)}$$

with $C \ni z \to I(k, z)$ denoting the indicial family of the operator $k$, i.e. the Fourier transform of the indicial operator $I(k)$.

Inspired by the right hand side of (4.11) we consider an arbitrary compact manifold $Y$, the algebra $B^b_c(\text{cyl}(Y))$ and the 1-cocycle

$$\sigma_1(\ell, \ell') := \frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_{\partial Y} (\partial_\mu \hat{\ell}(\mu) \circ \hat{\ell}'(\mu)) d\mu$$

That this is a 1-cocycle follows by integration by parts. Formula (4.12) defines what should be called Melrose’ 1-cocycle

\textbf{Proposition 4.13.} Roe’s 1-cocycle $\sigma_1$ and Melrose 1-cocycle $\sigma_1$ coincide:

$$\sigma_1(\ell, \ell') := \text{Tr}(\ell[\chi^0, \ell']) = \frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_Y (\partial_\mu \hat{\ell}(\mu) \circ \hat{\ell}'(\mu)) d\mu =: \sigma_1(\ell, \ell') \quad \text{(4.14)}$$
In order to prove \[4.14\] we employ the Hilbert transform \(\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\):
\[
\mathcal{H}(f) := \lim_{\varepsilon \to 0} \frac{i}{\pi} \int_{|x-y|>\varepsilon} \frac{f(x)}{x-y} \, dy.
\]
The crucial observation is that \((\mathcal{H}(\hat{f}))^* = (1 - 2\chi_{\mathbb{R}}^0)(\hat{f})\) (the right hand side denotes as usual the multiplication operator). Using this, we see that
\[
\text{Tr}(\ell(\chi^0, \ell')) = -\frac{1}{2} \int_{\mathbb{R}} \text{Tr}_Y (\ell(\mu) [\mathcal{H}, \ell'](\mu)) \, d\mu.
\]
Using the definition of the Hilbert transform one checks that the right hand side of this formula is equal to
\[
-\frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_Y (\ell(\mu) \circ \partial_\mu \ell'(\mu)) \, d\mu,
\]
which is equal to the right hand side of \[4.14\] once we integrate by parts.

Proposition\[4.13\] and Melrose' formula imply at once the relative 0-cocycle condition for \((\tau^0_0, \sigma_1)\). Indeed using first Proposition\[4.13\] and then Melrose’ formula we get:
\[
\sigma_1(I(k), I(k')) := \text{Tr}(I(k)\chi^0, I(k')) = \frac{i}{2\pi} \int_{\mathbb{R}} \text{Tr}_{\partial X} (\partial_\mu I(k, \mu) \circ I(k', \mu)) \, d\mu
\]
\[
= b \text{Tr}[k, k'] = b \tau^0_0(k, k').
\]
Thus \(I^*(\sigma_1) = b \tau^0_0\) as required.

**Conclusions.** We have established the following:

- the right hand side of Melrose’ formula defines a 1-cocycle \(\sigma_1\) on \(B_c(\text{cyl}(Y))\), \(Y\) any closed compact manifold;
- Melrose 1-cocycle \(\sigma_1\) equals Roe’s 1-cocycle \(\sigma_1\);
- Melrose’ formula itself can be interpreted as a relative 0-cocycle condition for the 0-cochain \((\tau^0_0, \sigma_1) \equiv (\tau^0_0, \sigma_1)\).

**4.8. Philosophical remarks.** We wish to recollect the information obtained in the last three subsections and start to explain our approach to Atiyah-Patodi-Singer higher index theory.

On a closed compact orientable riemannian smooth manifold \(Y\) let us consider the algebra of smoothing operators \(J_c(Y) := C^\infty(Y \times \mathbb{R})\). Then the functional \(J_c(Y) \ni k \to \int_Y k|\Delta| \text{dvol}\) defines a 0-cocycle \(\tau_0\) on \(J_c(Y)\); indeed by Lidski’s theorem the functional is nothing but the functional analytic trace of the integral operator corresponding to \(k\) and we know that the trace vanishes on commutators; in formulae, \(b\tau_0 = 0\). The 0-cocycle \(\tau_0\) plays a fundamental role in the proof of the Atiyah-Singer index theorem, but we leave this aside for the time being.

Let now \(X\) be a smooth orientable manifold with cylindrical ends, obtained from a manifold with boundary \(X_0\); let \(\text{cyl}(\partial X) = \mathbb{R} \times \partial X_0\). We have then defined algebras \(A_c(X), B_c(\text{cyl}(\partial X))\) and \(J_c(X)\) fitting into a short exact sequence \(0 \to J_c(X) \to A_c(X) \xrightarrow{\pi} B_c(\text{cyl}(\partial X)) \to 0\).

Corresponding to the 0-cocycle \(\tau_0\) in the closed case we can define two important 0-cocycles on a manifold with cylindrical ends \(X\):

- We can consider \(\tau_0\) on \(J_c(X) = C^\infty_c(X \times \mathbb{R})\); this is well defined and does define a 0-cocycle. We shall refer to \(\tau_0\) on \(J_c(X)\) as an *absolute* 0-cocycle.
- Starting with the absolute 0-cocycle \(\tau_0\) on \(J_c(X)\) we define a *relative* 0-cocycle \((\tau^0_0, \sigma_1)\) for \(A_c(X) \xrightarrow{\pi} B_c(\text{cyl}(\partial X))\). The relative 0-cocycle \((\tau^0_0, \sigma_1)\) is obtained through the following two fundamental steps. (1) We define a 0-cochain \(\tau^0_0\) on \(A_c(X)\) by replacing the integral with Melrose’ regularized integral. (2) We define a 1-cocycle \(\sigma_1\) on \(B_c(\text{cyl}(\partial X))\) by taking a *suspension* of \(\tau_0\) through the derivation \(\delta(\ell) := [\chi^0, \ell]\). In other words, \(\sigma_1(\ell_0, \ell_1)\) is obtained from \(\tau_0 \equiv \text{Tr}\) by considering \((\ell_0, \ell_1) \to \tau_0(\ell_0 [\chi^0, \ell_1]) \equiv \tau_0(\delta(\ell_1))\).

**Definition 1.** We shall also call Roe’s 1-cocycle \(\sigma_1\) the eta 1-cocycle corresponding to the absolute 0-cocycle \(\tau_0\).

In order to justify the wording of this definition we need to show that all this has something to do with the eta invariant and its role in the Atiyah-Patodi-Singer index formula. This will be explained in Section\[6\] and Section\[4\].
4.9. The absolute Godbillon-Vey 2-cyclic cocycle $\tau_{GV}$. Let $(Y, F), Y = \tilde{N} \times T$, be a foliated bundle without boundary. Let $E \to Y$ an hermitian complex vector bundle on $Y$. Let $(G, s : G \to Y, r : G \to Y)$ be the holonomy groupoid associated to $Y$, $G = (\tilde{N} \times \tilde{N} \times T)/\Gamma$. Consider again the convolution algebra $C^\infty_c(G, (s^*E)^* \otimes r^*E)$, of equivariant smoothing families with $\Gamma$-compact support. On $C^\infty_c(G, (s^*E)^* \otimes r^*E)$ we can define a remarkable 2-cocycle, denoted $\tau_{GV}$ and known as the Godbillon-Vey cyclic cocycle. First recall that there is a weight $\omega_\Gamma$ defined on the algebra $C^\infty_c(G, (s^*E)^* \otimes r^*E)$,

\[(4.15) \quad \omega_\Gamma(k) = \int_{Y(\Gamma)} Tr_m k(m, m, x) dm d\theta\]

where $Y(\Gamma)$ is the fundamental domain in $\tilde{N} \times T$ for the free diagonal action of $\Gamma$ on $\tilde{N} \times T$. More importantly, recall the two derivations $\delta_1 := [\phi, \dot{\phi}, \dot{\phi}]$ and $\delta_2 := [\dot{\phi}, \dot{\phi}, \dot{\phi}]$ coming from the modular automorphism group described in [2].

**Definition 2.** With $1 = \dim T$, the Godbillon-Vey cyclic 2-cocycle on $C^\infty_c(G, (s^*E)^* \otimes r^*E)$ is defined to be

\[(4.16) \quad \tau_{GV}(a_0, a_1, a_2) = \frac{1}{2} \omega_\Gamma(a_0[\phi, a_1][\dot{\phi}, a_2] - a_0[\dot{\phi}, a_1][\phi, a_2])\]

where $\dot{\phi} = d\phi/dx$. With $\delta_1(a) := [\phi, a], \delta_2(a) := [\dot{\phi}, a]$ we shall write

\[(4.17) \quad \tau_{GV}(a_0, a_1, a_2) = \frac{1}{2!} \sum_{\alpha \in \mathfrak{S}_3} \text{sign} (\alpha) \omega_\Gamma(a_0 \delta_{\alpha(1)} a_1 \delta_{\alpha(2)} a_2)\]

We now go back to a foliated bundle $(X, F)$ with cylindrical ends, with $X := \tilde{M} \times \Gamma T$, as in Section [2]. We consider the small subalgebras introduced in Subsection [12]. The weight $\omega_\Gamma$ is well defined on $J_e(X, F)$; the 2-cocycle $\tau_{GV}$ can thus be defined on $J_e(X, F)$, giving us the absolute Godbillon-Vey cyclic cocycle.

4.10. The eta 3-cocycle $\sigma_{GV}$ corresponding to $\tau_{GV}$. Now we apply the general philosophy explained at the end of the previous Section. Let $\chi^0$ be the usual characteristic function of $(-\infty, 0] \times \partial X_0$ in cyl($\partial X$) $= \mathbb{R} \times \partial X_0$. Write cyl($\partial X$) $= (\mathbb{R} \times \partial \tilde{M}) \times \Gamma T$ with $\Gamma$ acting trivially on the $\mathbb{R}$ factor. Let cyl($\Gamma$) be a fundamental domain for the action of $\Gamma$ on $(\mathbb{R} \times \partial \tilde{M}) \times T$; finally, let $\omega_\Gamma^{\text{cyl}}$ be the corresponding weight. We keep denoting this weight by $\omega_\Gamma$. Recall the derivation $\delta(\ell) := [\chi^0, \ell]$; recall that we passed from the absolute 0-cocycle $\tau_0 \equiv \text{Tr}$ to the 1-eta cocycle on the cylindrical algebra $B_e^{\text{cyl}}$ by considering $(\ell_0, \ell_1) \to \tau_0(\ell_0 \delta(\ell_1))$. We referred to this operation as a suspension.

We are thus led to suspend Definition [2] thus defining the following 3-cyclic cochain on the algebra $B_e$.

**Definition 3.** The eta cochain $\sigma_{GV}$ associated to the absolute Godbillon-Vey 2-cocycle $\tau_{GV}(a_0, a_1, a_2)$ is the 3-cochain given by

\[(4.18) \quad \sigma_{GV}(\ell_0, \ell_1, \ell_2, \ell_3) = \frac{1}{3!} \sum_{\alpha \in \mathfrak{S}_3} \text{sign} (\alpha) \omega_\Gamma(\ell_0 \delta_{\alpha(1)} \ell_1 \delta_{\alpha(2)} \ell_2 \delta_{\alpha(3)} \ell_3)\]

with $\delta_3(\ell) := [\chi^0, \ell]$. The eta cochain is an element in $C^3_e(B_e(\text{cyl}(\partial X), F_{\text{cyl}}))$

In fact, we can define, as we did for $\sigma_1$, the 3-cocycle $\sigma_{GV}^\lambda$ by employing the characteristic function $\chi^\lambda$. However, one checks easily that the value of $\sigma_{GV}^\lambda$ does not depend on $\lambda$.

One can prove that this definition is well posed, namely that each term $(\ell_0 \delta_{\alpha(1)} \ell_1 \delta_{\alpha(2)} \ell_2 \delta_{\alpha(3)} \ell_3)$ is of finite weight. The fact that $\sigma_{GV}$ is cyclic is elementary. Next we have the important

**Proposition 4.19.** The eta cochain $\sigma_{GV}$ is a 3-cocycle: $b\sigma_{GV} = 0$.

4.11. The relative Godbillon-Vey cyclic cocycle $(\tau^r_{GV}, \sigma_{GV})$. We now apply our strategy as in Subsection [12]. Thus starting with the absolute cyclic cocycle $\tau_{GV}$ on $J_e(X, F)$ we first consider the 3-linear functional on $A_e(X, F)$ given by $\psi^r_{GV}(k_0, k_1, k_2) := \frac{1}{3} \sum_{\alpha \in \mathfrak{S}_3} \text{sign} (\alpha) \omega^r_\Gamma(k_0 \delta_{\alpha(1)} k_1 \delta_{\alpha(2)} k_2)$ with $\omega^r_\Gamma$ the regularized weight corresponding to $\omega^r_\Gamma$.

Next we consider the cyclic cochain associated to $\psi^r_{GV}$:

\[(4.20) \quad \tau^r_{GV}(k_0, k_1, k_2) := \frac{1}{3} (\psi^r_{GV}(k_0, k_1, k_2) + \psi^r_{GV}(k_1, k_2, k_0) + \psi^r_{GV}(k_2, k_0, k_1))\]

The next Proposition is crucial:
Proposition 4.21. The relative cyclic cochain $(\tau_{GV}^c, \sigma_{GV}) \in C^2_c(A_c, B_c)$ is a relative 2-cocycle: thus $b\sigma_{GV} = 0$ (which we already know) and $b\tau_{GV}^c = (\pi_c)^*\sigma_{GV}$.

For later use we also state the analogue of Lemma 4.3.

Proposition 4.22. Let $t : A^*(X, F) \to C^*(X, F)$ be the section introduced in Subsection 3.6. If $k \in A_c \subset A^*(X, F)$ then $t(k)$ has finite weight. Moreover, for the regularized weight $\omega^c_t : A_c \to \mathbb{C}$ we have

$$\omega^c_t = \omega_t \circ t$$

5. Shatten extensions

In this section we select important subsequences of $0 \to C^*(X, F) \to A^*(X, F) \to B^*(\text{cyl}(\partial X), \mathcal{F}_{cyl}) \to 0$.

5.1. Shatten ideals. Let $\chi^\Gamma$ be a characteristic function for a fundamental domain of $\Gamma \to \hat{M} \to M$. Consider $C^\infty_c(G) =: J_c(X, F) \equiv J_c$ (we omit the bundle $E$ from the notation).

Definition 4. Let $k \in J_c$ be positive and self-adjoint. The Shatten norm $|||k|||_m$ of $k$ is defined as

$$|||k|||_m := \sup_{\theta \in T} |||\chi^\Gamma(k(\theta))^m\chi^\Gamma||_1$$

with the $||| \ |||_1$ denoting the usual trace-norm on the Hilbert space $\mathcal{H}_\theta = L^2(\hat{\mathcal{V}} \times \{\theta\})$. Equivalently

$$|||k|||_m := \sup_{\theta \in T} |||\chi^\Gamma(k(\theta))^m/2||^2_{HS}$$

with $||| \ |||_{HS}$ denoting the usual Hilbert-Schmidt norm. In general, we set $|||k||| := |||(kk^*)^{1/2}|||_m$. The Shatten norm of $k \in J_c$ is easily seen to be finite for any $m \geq 1$; we define $J_m(X, F) \equiv J_m$ as the completion of $J_c$ with respect to $||| \ |||_m$.

Proposition 5.3. $J_m$ is a Banach algebra and an ideal inside $C^*(X, F)$.

Proposition 5.4. The weight $\omega^c_t$ extends continuously from $J_c \equiv C^\infty_c(G)$ to $J_1$.

5.2. Shatten extensions. Recall that $\chi^\text{cyl}_0$ (often just $\chi^0$) is the function on the cylinder induced by the characteristic function of $(-\infty, 0]$ in $\mathbb{R}$. Notice that the definition of Shatten norm also applies to $\text{cyl}(\partial X)$, viewed as a manifold with cylindrical ends. Consider an element $\ell \in B_c$; one proves that $[\chi^0_{\text{cyl}}, \ell]$ is of $\Gamma$-compact support on $\text{cyl}(\partial X)$; thus for any $m \geq 1$ its Shatten norm $|||[\chi^0_{\text{cyl}}, \ell]||_m$ is finite.

Definition 5. We define $B_m(\text{cyl}(\partial X), \mathcal{F}_{cyl}) \equiv B_m$ as the completion of $B_c$ with respect to the norm

$$|||\ell||| := |||\ell|| + |||[\chi^0_{\text{cyl}}, \ell]||_m$$

Notice that $|||\ell||$ here denotes, as usual, the $C^*$-norm; so this is not a graph-norm (and in fact the Shatten $m$-norm $|||\ell|||_m$ would be infinite since $\ell$ is translation invariant).

Proposition 5.6. $B_m$ is a Banach algebra with respect to $||| \ |||_m$ and a subalgebra of $B^*(\text{cyl}(\partial X), \mathcal{F}_{cyl})$. Moreover, $B_m$ is holomorphically closed in $B^*(\text{cyl}(\partial X), \mathcal{F}_{cyl})$.

We now define

$$A_m(X, F) := \{ k \in A^*(X, F); \pi(k) \in B_{m+1}(\text{cyl}(\partial X), \mathcal{F}_{cyl}), t(k) \in I_m(X, F) \}$$

Lemma 5.8. $A_m(X, F)$ is a subalgebra of $A^*(X, F)$.

Now we observe that, as vector spaces,

$$A_m \cong J_m \oplus s(B_{m+1}).$$

Granted this result, we endow $A_m$ with the direct-sum norm:

$$|||k|||_{A_m} := |||t(k)|||_m + |||\pi(k)|||_{B_{m+1}}$$

Obviously $s$ induces a bounded linear map $B_{m+1} \to A_m$ of Banach spaces.

Proposition 5.11. $(A_m, ||| \ |||_{A_m})$ is a Banach algebra. Moreover, $J_m$ and $A_m$ define a short exact sequence of Banach algebras:

$$0 \to J_m(X, F) \to A_m(X, F) \xrightarrow{\pi} B_{m+1}(\text{cyl}(\partial X), \mathcal{F}_{cyl}) \to 0.$$

Finally, $t : A^*(X, F) \to C^*(X, F)$ restricts to a bounded section $t : A_m(X, F) \to J_m(X, F)$.
5.3. Derivations. In order to extend continuously the cyclic cocycles \( \tau_{GV} \) and \( (\tau'_{GV}, \sigma_{GV}) \) we need to take into account the modular automorphism group, thus decreasing further the size of the short exact sequence \( 0 \rightarrow J_m \rightarrow A_m \xrightarrow{\pi} B_{m+1} \rightarrow 0 \). Consider the two unbounded derivations \( \delta_1 \) and \( \delta_2 \) introduced in Subsection 4.2. Recall that \( \delta_1 \) and \( \delta_2 \) are closable unbounded derivations on \( A^*(X,F) \) with core domain \( J_c \). Consider the domain of their closure: we denote these two domains by \( \text{Dom}(\delta_1) \subset A^*(X,F) \) and \( \text{Dom}(\delta_2) \subset A^*(X,F) \). Similarly, consider the analogous derivations \( \delta_1^\mathbb{R} \), \( \delta_2^\mathbb{R} \) on \( \mathbb{R} \times \partial X_0 \), defined with respect to the product volume-forms on \( \mathbb{R} \times \partial X_0 \) (and notice incidentally that the corresponding modular function will be translation invariant). Set

\[
J_m := J_m \cap \text{Dom}(\delta_1) \cap \text{Dom}(\delta_2) , \quad A_m := A_m \cap \text{Dom}(\delta_1) \cap \text{Dom}(\delta_2)
\]

We anticipate that we shall choose \( 2n \tau \). We obtain in this way \( \text{Excision isomorphism, we obtain} \)

\( K \)-groups we refer, for example, to [4], [7]. Recall that a relative \( K \)-isomorphism of \( K \)-groups.

The section \( s \) restricts to bounded sections \( s : B_{m+1} \rightarrow A_m \) and \( t : A_m \rightarrow J_m \).

5.4. Isomorphism of \( K \)-groups. Let \( 0 \rightarrow J \rightarrow A \xrightarrow{\pi} B \rightarrow 0 \) a short exact sequence of Banach algebras. Recall that \( K_0(J) := K_0(J^+, J) \approx \text{Ker}(K_0(J^+) \rightarrow \mathbb{Z}) \) and that \( K(A^+, B^+) = K(A, B) \). For the definition of relative \( K \)-groups we refer, for example, to [4], [7]. Recall that a relative \( K \)-element for \( A \) is \( B \). Represented by a triple \( (P, Q, p_t) \) with \( P \) and \( Q \) idempotents in \( M_{k \times k}(A) \) and \( p_t \in M_{k \times k}(B) \) a path of idempotents connecting \( \pi(P) \) to \( \pi(Q) \). The excision isomorphism

\[
\alpha_{ex} : K_0(J) \longrightarrow K_0(A, B)
\]

is given by

\[
\alpha_{ex}([[(P, Q)]]) = [[P, Q, c]]
\]

with \( c \) denoting the constant path (this is not necessarily the 0-path, given that we are taking \( J^+ \)).

Consider in particular \( J_m := J_m \cap \text{Dom}(\delta_1) \cap \text{Dom}(\delta_2) \). Applying the arguments of [9] one shows that \( \text{Dom}(\delta_1) \cap \text{Dom}(\delta_2) \) is dense holomorphically closed in \( A^*(X,F) \); since \( J_m \) is an ideal in \( A^*(X,F) \) and is dense in \( C^*(X,F) \), we conclude that \( J_m \) is dense and holomorphically closed in \( C^*(X,F) \). In particular, using also the excision isomorphism, we obtain

\[
K_0(A^*(X,F), B^*(\text{cyl}(\partial X), F_{\text{cyl}})) \approx K_0(C^*(X,F)) \approx K_0(J_m) \approx K_0(A_m, B_{m+1}).
\]

5.5. Extended cocycles. Recall, from general theory, that \( [\tau_{GV}] \in HC^2(J_c) \) and \( [[\tau'_{GV}, \sigma_{GV}]] \in HC^2(A_c, B_c) \) can be paired with elements in \( K_0(J_c) \) and \( K_0(A_c, B_c) \) respectively. Introduce now the \( S^{n-1} \)-operation and

\[
S^{n-1} \tau_{GV} =: \tau_{2n, GV} \quad \text{and} \quad (S^{n-1} \tau_{GV}, S^{n-1} \sigma_{GV}) =: ([\tau'_{2n, GV}, \sigma_{(2n+1), GV}]).
\]

We obtain in this way \( [\tau_{2n, GV}] \in HC^{2n}(J_c) \) and \( [[\tau_{2n, GV}, \sigma_{(2n+1), GV}]] \in HC^{2n}(A_c, B_c) \).

Proposition 5.19. The absolute cocycle \( \tau_{2n, GV} \) extends to a continuous cyclic cocycle on \( J_{2n} \). The relative cyclic cocycle \( (\tau'_{2n, GV}, \sigma_{(2n+1), GV}) \) extends to a continuous relative cyclic cocycle for \( \mathfrak{A}_{2n} \).

Notation. We keep the simpler notation \( \tau_{GV} \) and \( (\tau'_{GV}, \sigma_{GV}) \) for the two cyclic cocycles \( \tau_{2n, GV} \) and \( (\tau'_{2n, GV}, \sigma_{(2n+1), GV}) \). Notice that from the previous Proposition we have

\[
[\tau_{GV}] \in HC^{2n}(J_{2n}) \quad \text{and} \quad [[\tau'_{GV}, \sigma_{GV}]] \in HC^{2n}(\mathfrak{A}_{2n}, \mathfrak{B}_{2n+1}).
\]

We anticipate that we shall choose \( 2n \) greater or equal to the dimension of the leaves in \( X = \tilde{M} \times_T S^1 \).

6. Relative and absolute \( C^* \)-index classes. Excision

6.1. Dirac operators. We begin with a closed foliated bundle \( (Y,F) \), with \( Y = \tilde{N} \times_T T \). We are also given a \( \Gamma \)-equivariant complex vector bundle \( \tilde{E} \) on \( \tilde{N} \times T \), or, equivalently, a complex vector bundle on \( Y \). We assume that \( \tilde{E} \) has a \( \Gamma \)-equivariant vertical Clifford structure. We obtain in this way a \( \Gamma \)-equivariant family of Dirac operators \( (D_\theta)_{\theta \in T} \) that will be simply denoted by \( D \). If \( (X_0, \mathcal{F}_0) \), \( X_0 = \tilde{M} \times_T T \), is a foliated bundle with boundary, as in the previous sections, then we shall assume the relevant geometric structures to be of product-type near the boundary. If \( (X,F) \) is the associated foliated bundle with cylindrical ends, then we shall extend
all the structure in a constant way along the cylindrical ends. We shall eventually assume $\tilde{M}$ to be of even dimension, the bundle $\tilde{E}$ to be $\mathbb{Z}_2$-graded and the Dirac operator to be odd and formally self-adjoint. We denote by $D^\theta = (D^\theta_\theta)_{\theta \in T}$ the boundary family defined by $D^\theta$. This is a $\Gamma$-equivariant family of formally self-adjoint first order differential operators on a complete manifold. We denote by $D^cyl$ the operator induced by $D^\theta$ on the cylindrical foliated manifold $(\text{cyl}(\partial X), \mathcal{F}_{cyl})$; $D^cyl$ is $\mathbb{R} \times \Gamma$-equivariant. We refer to [6] [2] for precise definitions. In all of this section we shall make the following fundamental

**Assumption.** There exists $\epsilon > 0$ such that $\forall \theta \in T$

\[(6.1) \quad L^2 - \text{spec}(D^\theta_\theta) \cap (-\epsilon, \epsilon) = \emptyset \]

For specific examples where this assumption is satisfied, see [6]. We shall concentrate on the spin-Dirac case, but it will be clear how to extend the results to general Dirac-type operators.

### 6.2. Index classes in the closed case.

Let $(Y, \mathcal{F})$ be a closed foliated bundle. First, we need to recall how in the closed case we can define an index class $\text{Ind}(D) \in K_\ast (C^\ast (Y, \mathcal{F}))$. There are in fact three equivalent description of $\text{Ind}(D)$, each one with its own interesting features:

- the Connes-Skandalis index class, defined by the Connes-Skandalis projector $e_Q$ associated to a pseudodifferential parametrix $Q$ for $D$; $Q$ can be chosen of $\Gamma$-compact support;
- the Wassermann index class, defined by the Wassermann projector $W_D$;
- the index class of the graph projection, defined by the graph projection $e_D$.

It is well known that the three classes introduced above are equal in $K_0 (C^\ast (Y, \mathcal{F}))$.

### 6.3. The relative index class $\text{Ind}(D, D^\theta)$. Let now $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends. Let $(\text{cyl}(\partial X), \mathcal{F}_{cyl})$ the associated foliated cylinder. Recall $0 \rightarrow C^\ast (X, \mathcal{F}) \rightarrow A^\ast (X; \mathcal{F}) \xrightarrow{\pi} B^\ast (\text{cyl}(\partial X), \mathcal{F}_{cyl}) \rightarrow 0$, the Wiener-Hopf extension of the $C^\ast$-algebra of translation invariant operators $B^\ast (\text{cyl}(\partial X), \mathcal{F}_{cyl})$; see Subsection 6.3. We shall be concerned with the K-theory group $K_\ast (C^\ast (X, \mathcal{F}))$ and with the relative group $K_\ast (A^\ast (X; \mathcal{F}), B^\ast (\text{cyl}(\partial X), \mathcal{F}_{cyl}))$. We shall write more briefly $0 \rightarrow C^\ast \rightarrow A^\ast \xrightarrow{\pi} B^\ast \rightarrow 0$, and $K_\ast (A^\ast, B^\ast)$. Recall that a relative $K_0$-cycle for $A^\ast \xrightarrow{\pi} B^\ast$ is a triple $(P, Q, p_t)$ with $P$ and $Q$ idempotents in $M_{k \times k} (A^\ast)$ and $p_t \in M_{k \times k} (B^\ast)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$.

**Proposition 6.2.** Let $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends, as above. Consider the Dirac operator on $X$, $D = (D_\theta)_{\theta \in T}$. Assume (6.1). Then the graph projection $e_D$ and the Wassermann projection $W_D$ define two relative classes in $K_0(A^\ast, B^\ast)$. These two classes are equal and fix the relative index class $\text{Ind}(D, D^\theta)$.

The relative classes of Proposition 6.2 are more precisely given by the triples

\[(6.3) \quad (e_D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, p_t) \text{ with } p_t := e_{D^cyl} \text{ and } (W_D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, q_t) \text{ with } q_t := W_{D^cyl}; t \in [1, +\infty].\]

The content of the Proposition is that these two triples do define elements in $K_0(A^\ast, B^\ast)$ and that these two elements are equal.

### 6.4. The absolute index class $\text{Ind}(D)$. Let the results in [6] where it is proved that there is a well defined parametrix $Q$ for $D^\theta$, $QD^\theta = \text{Id} - S_{-\sigma}$, $D^\theta Q = \text{Id} - S_{-\sigma}$, with remainders $S_{-\sigma}$ in $\mathbb{K}(\mathcal{E}) \equiv C^\ast (X, \mathcal{F})$. Consequently, there is a well defined Connes-Skandalis projector $e_Q$. The construction explained in [6] is an extension to the foliated case of the parametrix construction of Melrose, with particular care devoted to the non-compactness of the leaves.

**Definition 6.** The absolute index class associated to a Dirac operator on $(X, \mathcal{F})$ satisfying assumption (6.1) is the Connes-Skandalis index class associated to the Connes-Skandalis projector $e_Q$. It is denoted by $\text{Ind}(D) \in K_0(C^\ast (X, \mathcal{F}))$.

### 6.5. Excision for index classes. The following Proposition plays a fundamental role in our approach to higher APS index theory:

**Proposition 6.4.** Let $D = (D_\theta)_{\theta \in T}$ be a $\Gamma$-equivariant family of Dirac operators on a foliated manifold with cylindrical ends $X = M \times \Gamma T$. Assume that $M$ is even dimensional. Assume (6.1). Then

\[(6.5) \quad \alpha_{ex} (\text{Ind}(D)) = \text{Ind}(D, D^\theta)\]
7. INDEX THEOREMS

7.1. Smooth index classes. In the previous Section we stated the existence of $C^*$-algebraic indices. We have also proved in Subsection 6.5 that the absolute and relative cyclic cocycles $\tau_{GV}$ and $(\tau_{GV}^+, \sigma_{GV})$ extend from $J_\varepsilon$ and $A_\varepsilon \xrightarrow{p} B_\varepsilon$ to the smooth (:= dense and holomorphically closed) subalgebras $\mathfrak{A}_m$ and $\mathfrak{B}_m \xrightarrow{p} \mathfrak{B}_{m+1}$. Having extended the cocycles to a smooth subsequence $0 \rightarrow \mathfrak{A}_m \xrightarrow{\tau} \mathfrak{B}_{m+1} \rightarrow 0$ of the $C^*$-algebraic short exact sequence, we now want to show that we can simultaneously smooth-out our index classes and define them directly in $0 \rightarrow \mathfrak{A}_m \xrightarrow{\tau} \mathfrak{B}_{m+1} \rightarrow 0$. Once this will be achieved, we will be able to pair directly $[\tau_{GV}]$ with $\text{Ind}(D)$ and $[\tau_{GV}^+, \sigma_{GV}]$ will $\text{Ind}(D, D^\partial)$ (see the proof of our main result below for the definition of the relative pairing). This is, as often happens in higher index theory, a rather crucial point.

Proposition 7.1.  
1) Let $D = (D_\theta)_{\theta \in T}$ and $X = \tilde{M} \times_T T$ as above; then we can choose a parametrix $Q$ for $D^+$ in such a way that the Connes-Skandalis projection $e_Q$ belong to $\mathfrak{A}_m$.

2) Let $e_{D^\partial}$ be the graph projection for the translation invariant Dirac family $D^\partial = (D_\theta^\partial)_{\theta \in T}$ on the cylinder. Let $m \geq \dim \partial \tilde{M}$. Then $e_{D^\partial} \in \mathfrak{B}_{m+1}$. More generally, $\forall s \geq 1$ we have $e_{s(D^\partial)} \in \mathfrak{B}_{m+1}$.

3) Let $e_D$ be the graph projection on $X$. Let $t : A^*(X, \mathcal{F}) \rightarrow C^*(X, \mathcal{F})$ be the section introduced in (3.6). Let $m \geq \dim \tilde{M}$ Then $t(e_D) \in \mathfrak{A}_m$. Since $\pi(e_D) = e_{D^\partial}$ belongs to $\mathfrak{B}_{m+1}$ by 2), we conclude that $e_D \in \mathfrak{A}_m$.

The proof of 1) employs the $b$-calculus. The proof of 2) and 3) employs finite propagation speed techniques.

As a consequence of this rather long statement we obtain that for $m$ larger than the dimension of the leaves in $(X, \mathcal{F})$

$\text{Ind}(D) \in K_0(\mathfrak{A}_m), \quad \text{Ind}(D, D^\partial) \in K_0(\mathfrak{A}_m, \mathfrak{B}_{m+1})$.

7.2. The higher APS index formula for the Godbillon-Vey cocycle. We now have all the ingredients to state and prove a APS formula for the Godbillon-Vey cocycle. We first treat the simplified case in which the dimension of $X$ is equal to 3. The general case is easily obtained using the $S$ operation. Fix $m$ greater or equal to 2, where we remark that 2 equals the dimension of the leaves and set

$\tilde{M} := \mathfrak{A}_m, \quad \mathfrak{A} := \mathfrak{A}_m, \quad \mathfrak{B} := \mathfrak{B}_{m+1}$.

Then we know from Proposition 7.1 that there are well defined absolute and relative index classes

$\text{Ind}(D) \in K_0(\mathfrak{A}), \quad \text{Ind}(D, D^\partial) \in K_0(\mathfrak{A}, \mathfrak{B})$,

the first given in terms of a parametrix and the second given in term of the graph projection. Proposition 5.19 implies the existence of the following additive maps:

$\langle \cdot, \tau_{GV} \rangle : K_0(\mathfrak{A}) \rightarrow \mathbb{C}, \quad \langle \cdot, [\tau_{GV}^+, \sigma_{GV}] \rangle : K_0(\mathfrak{A}, \mathfrak{B}) \rightarrow \mathbb{C}$.

Also, it can be checked that the proof of the excision correspondence between relative and absolute index class, see (6.5), carries over to the smooth representativ.

Definition 7. Let $X_0 = \tilde{M} \times_T S^1$ as above and assume (6.1). The Godbillon-Vey higher index is the number $\text{Ind}_{GV}(D) := \langle \text{Ind}(D), [\tau_{GV}] \rangle$.

The following theorem is one of the main results of this paper in the 3-dimensional case:

Theorem 7.4. Let $X_0 = \tilde{M} \times_T S^1$ be a foliated bundle with boundary, with $\tilde{M}$ of dimension 2. Let $D := (D_\theta)_{\theta \in S^1}$ be an equivariant family of Dirac operators. Assume (6.1) on the boundary family. Then,

$\text{Ind}_{GV}(D) = \int_{X_0} \omega_{GV} - \eta_{GV}, \quad \text{with} \quad \eta_{GV} := \int_{0}^{\infty} \sigma_{GV}([p_t, p_t], p_t, p_t, p_t) dt,$

with $\omega_{GV}$ equal to (a representative of) the Godbillon-Vey class [7] and with $p_t := \text{e}_{1(D^\partial)}$.

Notice that by inverse Fourier transform the Godbillon-Vey eta invariant $\eta_{GV}$ only depends on the boundary family $D^\partial \equiv (D_\theta^\partial)_{\theta \in S^1}$. 

Proof. The left hand side of the formula (7.4) is, by definition, the pairing \( \langle [e_Q, e_1], \tau_{GV} \rangle \) with \( e_Q \) the Connes-Skandalis projection and \( e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Now, we know that \( \alpha_{ex}([e_Q, e_1]) \), which is by definition \([e_Q, e_1, c]\), with \( c \) the constant path, is equal as a relative class, to \([e_D, e_1, p_t]\) with \( p_t := e_{tD^2} \). Notice that, in fact, the same is true if we replace \( D \) by \( sD, s > 0 \). In formulae

\[
\alpha_{ex}([e_Q, e_1]) = [e_{sD}, e_1, p_t(s)] \quad \text{with} \quad p_t(s) = e_{t(sD^2)}
\]

Moreover, since the derivative of the constant path is equal to zero and since \( \tau^r_{GV}|_\partial = \tau_{GV} \) (this follows from \( \langle \text{an extension of} \ (4.23) \rangle \)), we obtain at once the crucial relation

\[
\langle \alpha_{ex}([e_Q, e_1]), ([\tau^r_{GV}, \sigma_{GV}]) \rangle = \langle [e_Q, e_1], [\tau_{GV}] \rangle.
\]

Summarizing, using first (7.7) and then (4.6) (for \( s = 1 \)) we get

\[
\text{Ind}_{GV}(D) := \langle [e_Q, e_1], [\tau_{GV}] \rangle = \langle \alpha_{ex}([e_Q, e_1]), ([\tau^r_{GV}, \sigma_{GV}]) \rangle
\]

\[
= \langle [e_D, e_1, p_t], ([\tau^r_{GV}, \sigma_{GV}]) \rangle := \tau^r_{GV}(e_D - e_1) + \int_1^{+\infty} \sigma_{GV}([\hat{p}_t, p_t, p_t, p_t])dt
\]

Now replace \( D \) by \( sD, s > 0 \) and take the limit as \( s \downarrow 0 \). The first summand in the last term can be proved to converge to \( \int_{X_0} \omega_{GV} \) using Getzler rescaling as in [9]. The second summand gives \( \eta_{GV} \) as in the statement of the theorem, using a simple change of variable. Notice that the convergence at infinity of \( \int_0^{+\infty} \sigma_{GV}([\hat{p}_t, p_t, p_t, p_t])dt \) follows from the fact that the pairing is well defined. The convergence at \( 0 \) follows from the \( s \)-independence of the absolute index pairing together with the fact that \( \tau^r_{GV}(e_{sD} - e_1) \) does convergence as \( s \downarrow 0 \). The situation here is similar to the one for the eta invariant in the seminal paper of Atiyah-Patodi-Singer; its regularity was a consequence of their index theorem. The theorem is proved. \( \square \)

In the general case, when the leaves have dimension \( 2n \), we proceed similarly and obtain the following formula

\[
\text{Ind}_{GV}(D) = \int_{X_0} \widehat{A}(\tilde{M}, \nabla^M) \wedge \omega_{GV} - \eta_{GV},
\]

with \( \eta_{GV} := \int_0^{\infty} \sigma_{(2n+1),GV}([\hat{p}_t, p_t, p_t, \ldots, p_t])dt \).

The number \( \eta_{GV} \) is, by definition, the Godbillon-Vey eta invariant of the boundary foliation.

Remark 7.9. The classic Atiyah-Patodi-Singer index theorem in obtained proceeding as above, but pairing the absolute index class with the absolute 0-cocycle \( \tau_0 \) and the relative index class with the relative 0-cocycle \( (\tau^r_0, \sigma_1) \). (If we use the Wassermann projector we don’t need to use the \( S \) operation; if we use the graph projection then we need to consider \( \tau_{2n} := S^*\tau_0 \) and \( \sigma_{2n+1} := S^*\sigma_1 \) with \( 2n \) equal to the dimension of the manifold.) Equating the absolute and the relative pairing, as above, we obtain an index theorem. It can be proved that this is precisely the APS index theorem on manifolds with cylindrical ends; in other words, the eta-term we obtain is precisely the APS eta invariant for the boundary operator.

7.3. General theory. The ideas explained in the previous sections can be extended to general cocycles \( \tau_k \in H^{rk}(C^{\infty}_c(G, (s^*E)^* \otimes r^*E)) \); we simply need to require that these cocycles are in the image of a suitable Alexander-Spanier homomorphism. In particular, for Galois coverings, the above techniques give an alternate approach to the higher index theory developed in [5], much more in line with the original treatment of Connes and Moscovici [2]. We briefly illustrate this important example. Let \( \Gamma \to \tilde{M} \to M \) be a Galois covering with boundary and let \( \Gamma \to \tilde{V} \to V \) be the associated covering with cylindrical ends. In the closed case higher indices for a \( \Gamma \)-equivariant Dirac operator on \( \tilde{M} \) are obtained through Alexander-Spanier cocycles, so we concentrate directly on these. Let \( \phi \) be an Alexander-Spanier \( p \)-cocycle; for simplicity we assume that \( \phi \) is the sum of decomposable elements given by the cup product of Alexander-Spanier 1-cochains:

\[
\phi = \sum_i \delta f^{(i)}_1 \cup \delta f^{(i)}_2 \cup \cdots \cup \delta f^{(i)}_p \quad \text{where} \quad f^{(i)}_j : \tilde{M} \to \mathbb{C} \quad \text{is continuous}.
\]

Notice that \( \delta f^{(i)}_j, \delta f^{(i)}_j(\tilde{m}, \tilde{m}') := (f^{(i)}_j(\tilde{m}') - f^{(i)}_j(\tilde{m})) \), is indeed \( \Gamma \)-invariant with respect to the diagonal action of \( \Gamma \) on \( \tilde{M} \times \tilde{M} \). We shall omit \( \cup \) from the notation. The cochain \( \phi \) is a cocycle (where we recall that for an Alexander-Spanier \( p \)-cochain given by a continuous function \( \phi : \tilde{M}^{p+1} \to \mathbb{C} \) invariant under the diagonal \( \Gamma \)-action,
one sets \( \delta \phi(x_0, x_1, \ldots, x_{p+1}) := \sum_{0}^{p+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) \). Always in the closed case we obtain a cyclic \( p \)-cocycle for the convolution algebra \( C_c^\infty(M \times_G \hat{M}) \) by setting
\[
\tau^\phi_k(k_0, \ldots, k_p) = \frac{1}{p!} \sum_{\alpha \in \mathcal{S}_p} \sum_i \text{sign}(\alpha) \omega_T(k_0 \delta^{(\alpha)}_{(1)} k_1 \cdots \delta^{(\alpha)}_{(p)} k_p) \quad \text{with} \quad \delta^{(i)} k := [k, f^{(i)}].
\]
Notice that \([k, f^{(i)}]\) is the \( \Gamma \)-invariant kernel whose value at \((\tilde{m}, \tilde{n})\) is given by \( k(\tilde{m}, \tilde{n})' \delta f^{(i)}(\tilde{m}, \tilde{n}) \) which is by definition \( k(\tilde{m}, \tilde{n})' (f^{(i)}(\tilde{m}')) - f^{(i)}(\tilde{m})) \); \( \omega_T \) is as usual given by \( \omega_T(k) = \int_{\hat{M}} \text{Tr}_\tilde{m} k(\tilde{m}, \tilde{n}) \), with \( F \) a fundamental domain for the \( \Gamma \)-action.

Pass now to manifolds with boundary and associated manifolds with cylindrical ends. Consider the small subalgebras \( J_c(\tilde{V}), A_c(\tilde{V}), B_c(\partial \tilde{V} \times \mathbb{R}) \) of the Wiener-Hopf extension constructed in Section 4 (just take \( T = \text{point there} \)). We write briefly \( J_c, A_c, B_c \) and \( 0 \to J_c \to A_c \xrightarrow{\partial} B_c \to 0 \). We adopt the notation of the previous sections. Given \( \phi \) as above, we can clearly define an \textit{absolute} cyclic \( p \)-cocycle \( \tau^\phi \) on \( J_c \). Next, define the \((p+1)\)-linear functional \( \psi^\tau_\phi \) on \( A_c \) by replacing the integral in \( \omega_T \) with Melrose’ regularized integral. Consider next
\[
\tau^\phi_k(k_0, \ldots, k_p) := \frac{1}{p!+1} \left( \psi^\tau_\phi(k_0, k_1, \ldots, k_p) + \psi^\tau_\phi(k_1, \ldots, k_p, k_0) + \cdots + \psi^\tau_\phi(k_p, k_0, \ldots, k_{p-1}) \right).
\]
This is a cyclic \( p \)-cochain on \( A_c \). Finally, introduce the new derivation \( \delta_{(p+1)}(\ell) := [\ell, \chi^0] \) with \( \chi^0 \) the function on \( \partial \tilde{V} \times \mathbb{R} \) induced by the characteristic function of \( (\tilde{m}, \tilde{n}) \). Then the eta cocycle associated to \( \tau^\phi \) is given by
\[
\sigma^\phi_\ell(\ell_0, \ldots, \ell_{p+1}) = \frac{1}{(p+1)!} \sum_{\alpha \in \mathcal{S}_{p+1}} \sum_i \text{sign}(\alpha) \omega_T(\ell_0 \delta^{(\alpha)}_{(1)} \ell_1 \cdots \delta^{(\alpha)}_{(p+1)} \ell_{p+1})
\]
One can prove that this is a cyclic \((p+1)\)-cocycle for \( B_c \) and that \((\tau^\phi_\ell, \sigma^\phi_\ell)\) is a \textit{relative} cyclic \( p \)-cocycle for the pair \((A_c, B_c)\). Proceeding exactly as above, thus equating the absolute pairing \( \langle \text{Ind}(\hat{D}), \tau^\phi_\ell \rangle \) with the relative pairing \( \langle \text{Ind}(\hat{D}, D_0), [\tau^\phi_\ell, \sigma^\phi_\ell] \rangle \) one obtains a higher (Atiyah-Patodi-Singer)-(Connes-Moscovici) index formula, with boundary correction term given by
\[
\eta^\phi := \int_0^\infty \sigma^\phi([\hat{p}_\ell, p_1], p_{t_1}, \ldots, p_t) dt \quad \text{with} \quad p_t := e_{t \delta \phi^\ell}
\]
A full treatment of the general theory on foliated bundles, together with this important particular case will appear in [14].

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