Boundary conditions for spin diffusion

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We develop a general scheme of deriving boundary conditions for spin-charge coupled transport in disordered systems with spin-orbit interactions. To illustrate the application of the method, we explicitly derive boundary conditions for spin diffusion in the Rashba model. Due to the surface spin precession, the boundary conditions are non-trivial and contain terms, which couple different components of the spin density. We argue that boundary conditions and the corresponding electric-field-induced spin accumulation generally depend on the nature of the boundary and therefore the spin Hall effect in a spin-orbit coupled system can be viewed as a non-universal edge phenomenon.

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I. INTRODUCTION

There has been a lot of recent interest in a phenomenon dubbed the spin Hall effect.1,2 Experimentally this effect manifests itself as an equilibrium spin accumulation near the edges, when an electric field is applied parallel to a Hall bar of a spin-orbit coupled electron system. This kind of electric-field-induced spin polarization has been observed in recent experiments.3 This phenomenon has a potential for being technologically important, as it allows one to electrically manipulate the spin degree of freedom in non-magnetic systems. There are still many controversies regarding the physical origin of the spin accumulation observed in real experiment (intrinsic vs. extrinsic).4 But even within a given theoretical model (e.g., a two-dimensional electron gas with Rashba or Dresselhaus spin-orbit interactions or Luttinger model), there is still no consensus on the correct description of the spin Hall effect.5

A common theoretical approach to the intrinsic spin Hall effect is as follows: One introduces the notion of a spin current, which is an operator intuitively related to spin transport. Using the Kubo formula, one then calculates the spin-current autocorrelator and the corresponding linear response quantity, called the “spin Hall conductivity.”6,7 In clean systems, it appears to be related to an elegant topological Berry’s phase structure.8 In a disordered system, the notion of the Fermi surface Berry’s phase is unclear, but the “spin Hall conductivity” is well-defined. It is believed that this quantity is connected to spin accumulation in the spin-Hall experiment. While this theoretical approach is mathematically well-defined and may lead to technically challenging problems, its relation to experiment remains unclear. The problem is that in spin-orbit coupled systems, the spin is not conserved and thus the definition of the spin current is ambiguous.9,10 A non-equilibrium spin density in a region relaxes not only due to the flux through the boundaries of the region, but also due to spin precession. Therefore, the spin current is not easily measurable.

To describe realistic experiment, it is preferable to use another route without involving the notion of a spin current. In disordered systems, this can be done with the help of a kinetic equation or a spin-charge coupled diffusion equation.11,12 The diffusion equation provides a complete and physically meaningful description of spin and charge diffusive transport in terms of position and time dependent spin and charge densities. However, the explicit solution of the diffusion equation (which is generally a complicated set of partial differential equations) strongly depends on boundary conditions.13 While for usual charge transport, boundary conditions are often obvious and follow from conservation laws, this is not the case in spin-orbit coupled systems. The problem is that the spin is not conserved and thus there is spin precession at the edges. This spin dynamics is very sensitive to the actual boundary (depending on the physical situation, the boundary could be modelled as a hard wall, WKB potential, rough surface, etc.). Since the bulk diffusion equation describes spin/charge dynamics only at the length-scales much larger than the mean-free path, it can not capture the behavior of the spin in the immediate vicinity of the sample edges. Due to the aforementioned boundary spin precession, the “trivial boundary conditions” (i.e., boundary conditions, which one would have, if the spin had been conserved) acquire corrections proportional to the spin-orbit coupling. These corrections are very important, as they select a unique solution of the diffusion equation and determine the observable spin accumulation.

The derivation of boundary conditions is technically a difficult task. Generally, one needs to determine the behavior of the system in the ballistic region near the boundary and match it with the diffusive behavior in the bulk. It requires the knowledge of the exact S-matrix or the boundary Green’s function. We should point out that there exists an extensive literature on the related issues in the context of the diffusion of light in a random medium and neutron diffusion.14,15 In these problems, the general form of boundary conditions is obvious due to conservation laws and only numerical coefficients...
are unknown. There exists a general method of deriving these numerical coefficients. But there are just a very few models in which exact analytical results are available (a notable example is the Milne problem, which describes a boundary separating diffusive and ballistic regions). We note that in the context of spin diffusion in spin-orbit coupled systems, even the qualitative form of boundary conditions is not known as there are no obvious conservation laws, which would provide guidance. Even though in the bulk, one has conserved quantities labeled by the band index (e.g., chirality in the Rashba model), the boundary generally does not respect these conservation laws and mixes up different bands.

In this paper we formulate a general method of deriving boundary conditions in spin-orbit coupled systems. Using this proposed method, we derive boundary conditions for the Rashba model in the leading order with respect to the spin-orbit coupling, which is assumed small. We show that there are corrections to the "trivial boundary conditions." In leading order, we find that at the boundary, the spin component perpendicular to the plane of the two-dimensional gas gets coupled to the in-plane spin component perpendicular to the boundary. We argue that spin accumulation in the spin Hall effect setup is determined by the combination of the bulk spin-charge coupling and boundary effects.

II. GENERAL METHOD OF DERIVING BOUNDARY CONDITIONS

In this section we formulate a general method of deriving boundary conditions applicable to any spin-orbit coupled system with a boundary. Let us consider a disordered electron system occupying a region $\mathcal{A}$. One can derive the following integral equation for the diffusion kernel, which determines the dynamics of the charge density ($\rho_0$) and the spin densities ($\rho_a$, $a=x, y, z$):

$$D_{ab}(\omega; r_1, r_2) = \sum_c \int_{r' \in \mathcal{A}} \frac{\Pi_{ab}(\omega; r_1, r')}{2\pi \nu \tau} D_{cb}(\omega; r', r_2) d^3r' + \delta_{ab} \delta(r_1 - r_2),$$

(1)

where $\nu$ is the density of states per spin at the Fermi surface, $\tau$ is the scattering time, the Latin indices label charge (0) and spin ($a$) degrees of freedom, and $D_{ab}(\omega; r_1, r_2)$ is the diffusion, which is the Green's function of the spin-charge coupled diffusion equation. The kernel in Eq. (1) reads

$$\Pi_{ab}(\omega; r, r') = \frac{1}{2} \text{Tr} \left\{ \hat{\sigma}_a \hat{G}^{R}(\varepsilon_1; r, r') \hat{\sigma}_b \hat{G}^{A}(\varepsilon_2; r', r) \right\},$$

(2)

where $\omega = \varepsilon_1 - \varepsilon_2$, $\hat{\sigma}_0$ is the unity matrix, $\hat{\sigma}_a$ are the Pauli matrices, and $\hat{G}^{R/A}(\varepsilon; r, r')$ are disorder-averaged retarded/advanced electron Green's functions, which include the effect of the boundary (here and below we denote the boundary Green's functions as $\hat{G}$ and the bulk Green's functions as $G$). They depend exponentially on distances via factors of the following types, $\propto e^{-r/(2l)}$, where $l$ is the mean-free path. Deep in the bulk, the edge effects are exponentially small and the bulk Green's functions can be used, $G^{R/A}(\varepsilon; r - r')$. Performing a gradient expansion of the bulk polarizability (i.e., taking the limit $l \to 0$), one obtains the spin-charge coupled diffusion equation. Generally, it has the following form:

$$\partial_t \rho_a = (D_a \partial^2 - \tau_a^{-1}) \rho_a + \sum_{b, \gamma} \Gamma^a_{b\gamma} \partial_t \rho_b,$$

(3)

where $\rho_a(r, t)$ are the charge/spin densities, $D_a$ are the diffusion coefficients, $\tau_a$ are the relaxation times, and $\Gamma^a_{b\gamma}$ are the spin-spin and spin-charge couplings.

Note that the integral equation (1) does not require boundary conditions (it already contains this information through the boundary Green’s function), but the differential diffusion equation (3) does. To derive these boundary conditions one needs to perform the gradient expansion near the boundary $\partial \mathcal{A}$. Let us consider the problem in half-space $\mathcal{A} = \{x \mid x > 0\}$. To find the diffusion equation near the boundary, we take the limit $x \to r \cdot \mathbf{n} \to 0$. This limit should be understood in the following sense: $p_F^{-1} \ll \ell$, i.e., we consider a point, which is near the boundary as compared to the diffusion length-scale $\ell$, but still far from it as compared to the ballistic length scale described by the Fermi wave-length. Let us first consider the case of zero spin-orbit interactions. If the boundary is a hard wall potential, then the boundary Green’s function can be obtained using the method of mirror images.

$$\hat{G}_0^{R/A}(\varepsilon; r, r') = \left[ G_0^{R/A}(\varepsilon; \Delta r) - G_0^{R/A}(\varepsilon; \Delta r_*) \right] \delta_0,$$

(4)

where $\Delta r = r - r'$, $\Delta r_* = r_0 - r'$, and $r_0$ is the mirror image with respect to the sample boundary. For the sake of concreteness, let us assume that the dimensionality $d = 2$ (all qualitative arguments are independent of dimensionality). In two dimensions, the Green’s function has the following form

$$G_0^{R/A}(\varepsilon; r) = -\frac{\pm im^2}{2\pi p_F r} \exp \left[ \mp i \left( p_F + \frac{\varepsilon}{v_F} \right) r - \frac{r}{2l} \right].$$

(5)

We note here that the frequency dependence of the diffusion kernel is irrelevant for the boundary condition problem and will be omitted from now on. When the method of mirror images applies, the polarizability kernel has the following general structure $\Pi^{(0)} = G^{RG_A} + G^{RG_A} - G^R G^A - G^{RG_A} - G^{RG_A}$ [where $G_\alpha$ implies the “reflected” Green’s function, $\alpha$, the second term in Eq. (3)]. The key observation is that due to the fast-oscillating factor $GG_\alpha \propto \exp \{ \pm i p_F (\Delta r - \Delta r_*) \}$ and our choice of the point $r$ ($\ell \gg p_F^{-1}$), the cross terms $GG_\alpha$ in the expression for $\Pi^{(0)}$ become negligible in the integral over $r'$ and can be omitted. The remaining integral can be easily calculated and we find (in the “trivial” case of no spin-orbit
is an impenetrable wall, general boundary conditions take
matrix, which determines boundary conditions
negligible due to the fast-oscillations and can be omitted,
of Eq. (1), the cross-terms
censions is equal to
as long as
tions to the spin-orbit vertex can be shown to be small
pears in the Hamiltonian. We note that disorder correc-
ions does not apply because the “reflected” Green’s
function does not satisfy the corresponding Schrödinger
equation. This is a major complication, which never oc-
curs in the problems usually studied in the literature in
this context\textsuperscript{16,17}. In addition, the matrix structure of
the boundary diffusion kernel becomes essential. There
are generally two ways to obtain the boundary Green’s
function (we have explicitly verified that both approaches
lead to identical perturbative results in the leading or-
der\textsuperscript{20}): (i) The first is to solve the Schrödinger equation
near the boundary and express the Green’s function using
the corresponding wave-functions as a basis; (ii) The sec-
ond is to find the Green’s functions perturbatively, using
the mirror-image result\textsuperscript{11} as the leading approxima-
\begin{align*}
\delta \hat{G}(\mathbf{r}, \mathbf{r}') = \sum_{k=1}^{\infty} \sum_{x_1, x_2 > 0} d^d \mathbf{r}_1 . . . d^d \mathbf{r}_k \hat{G}_0(\mathbf{r}, \mathbf{r}_1) \hat{V}_{SO}(\mathbf{r}_1) \\
\times \hat{G}_0(\mathbf{r}_1, \mathbf{r}_2) \times \ldots \times \hat{V}_{SO}(\mathbf{r}_k) \hat{G}_0(\mathbf{r}_k, \mathbf{r}'),
\end{align*}
where $\hat{V}_{SO}$ is the spin-orbit interaction operator as it ap-
pears in the Hamiltonian. We note that disorder corrections
to the spin-orbit vertex can be shown to be small
as long as $\varepsilon F \tau \gg 1$. The resulting Green’s function
\textsuperscript{7} may be quite complicated even in the simplest cases.
However, the diffusion kernel always contains products of
the following types:
\begin{align*}
\Pi_{ab}(\mathbf{r}, \mathbf{r}') \propto \left[ T G^R(\Delta \mathbf{r}) - R G^L(\Delta \mathbf{r}_s) \right] \\
\times \left[ T' G^A(\Delta \mathbf{r}) - R' G^A(\Delta \mathbf{r}_s) \right],
\end{align*}
where $T$’s and $R$’s are some functions, which do not oscil-
late on a Fermi wavelength scale. Again, in the integral of
Eq. (11), the cross-terms $G^R/G^A(\Delta \mathbf{r}) G^A/R(\Delta \mathbf{r}_s)$ become
negligible due to the fast-oscillations and can be omitted,
but the other terms survive. One can define the following
matrix, which determines boundary conditions
\begin{equation}
B_{ab} = \lim_{\tau \to 0} \int_{x' > 0} \frac{\delta \Pi_{ab}(\mathbf{r}, \mathbf{r}')} {2 \pi \nu \tau} d^d \mathbf{r}',
\end{equation}
where the correction to polarizability is determined by
the Green’s function\textsuperscript{17} and Eq. (12) and the limit is
understood in the sense $\rho^{\text{F}} \ll |\mathbf{r}| \mathbf{n} \ll l$. If the boundary
is an impenetrable wall, general boundary conditions take
the form
\begin{equation}
\mathbf{n} \cdot \partial \rho_a(\mathbf{0}) = -\frac{1}{c_d} \sum_b B_{ab} \rho_b(\mathbf{0}),
\end{equation}
where $a, b = 0, x, y, z$, and $\mathbf{n}$ is a unit vector normal
to the boundary, $c_d$ is a number ($c_d = 2/\pi$), and $B_{ab}$
is a dimensionless matrix defined above. If an external
field is present one may need to consider another term in
the boundary conditions, which has the meaning of the
boundary spin-charge coupling\textsuperscript{21}
\begin{equation}
C_a = \lim_{\tau \to 0} \int_{x' > 0} \delta \Pi_{ab}(\mathbf{r}, \mathbf{r}') d^d \mathbf{r}'/(2 \pi \nu \tau).
\end{equation}
In this case the boundary condition becomes:
\begin{equation}
c_d \mathbf{n} \cdot \partial \rho_a(\mathbf{0}) = -\sum_b B_{ab} \rho_b(\mathbf{0}) - C_a \mathbf{n} \cdot \partial \rho_a(\mathbf{0}).
\end{equation}
We note that if the boundary is not impenetrable,
the general method does not change but the results do
even in the lowest order. For instance, if the bound-
ary is characterized by an isotropic reflection coefficient
$R \neq 1$, the zeroth order boundary condition becomes
$(1 - R^2) \rho_a(\mathbf{0}) = c_d l (1 + R^2) \mathbf{n} \cdot \partial \rho_a(\mathbf{0})$. If the reflection
coefficient is not too close to unity, the right-hand side of
the latter equation can be considered small and one can use
the following leading order boundary conditions
\begin{equation}
\rho_a(\mathbf{0}) = 0.
\end{equation}
This boundary condition implies that an excess charge or
spin density can not exist at the boundary if the bound-
ary is penetrable. The excess density leaks through the
edge (see also Ref.\textsuperscript{22}).
We note that strictly speaking the matching point at the
boundary [which we set to zero in Eqs. (12) and (13)]
can not be obtained exactly within the method described
here, but all dimensional parameters and numerical co-
efficients are robust. To determine the correct matching
point on a lengthscale of order $l$, one actually needs to
solve exactly the integral equation, which is possible only
in very special cases \textit{e.g.}, the Milne problem \textit{i.e.}, \textit{R}=0
and no spin-orbit coupling].

\section{Application to the Rashba Model}

To illustrate the application of our method, let us con-
sider the Rashba model\textsuperscript{23} in the leading order in spin-
orbit coupling. In this case, the spin-orbit interaction
operator has the form $\hat{V}_{SO} = -i \alpha_R \epsilon_{\alpha \beta \gamma} \hat{\sigma}_\gamma \partial_{\alpha}$, where $\alpha_R$
is the Rashba spin-orbit coupling and $\alpha, \beta = x, y, z$.
The differential diffusion equation was derived in Ref.\textsuperscript{13} and
reads
\begin{equation}
\partial_t \rho_0 = D \partial^2 \rho_0 - \Gamma_{\alpha \beta} \epsilon_{\alpha \beta \gamma} \partial_{\alpha} \rho_0,
\end{equation}
$$\partial_t \rho_\alpha = (D\partial^2 - \tau^{-1}_\alpha) \rho_\alpha + \Gamma_{ss} \epsilon_{\alpha\beta\gamma} \epsilon_{z\beta\delta} \partial_\gamma \rho_\gamma - \Gamma_{sc} \epsilon_{z\beta\delta} \partial_\gamma \rho_\delta.$$  \hfill (15)

The spin-spin coupling in the Rashba model is $\Gamma_{sc} = 4\alpha_R E_F \tau$ and the spin relaxation times are $2\tau_x = \tau_y = \tau_z = 2\tau/(2\alpha_R E_F \tau)^2$. These parameters are universal in the sense that they do not depend on the details of the electronic spectrum and can be calculated in the $\xi$-approximation. However, the spin-charge coupling is not universal, since it strongly depends on the electronic spectrum and the high-energy cut-off (if the spectrum is quadratic, $\Gamma_{sc} = 2\alpha_R^2 E_F \tau^2$). We should point out however that this result holds only in the semiclassical diffusion approximation, i.e., only in the leading order with respect to the inverse conductance $1/(E_F \tau)$, which is an independent parameter of the model. There may exist contributions to the spin-charge coupling of order $\Gamma_{sc} \sim \Gamma_{ss}/(E_F \tau)$. They originate from diagrams with crossed impurity lines, which are omitted in the diffusion approximation (we note that there is no direct relation between $\Gamma_{sc}$ and the spin Hall conductivity, which vanishes to all orders.). These quantum corrections can be neglected only if $(E_F \tau)^{-1/2} \ll \alpha_R E_F \tau \ll 1$. This is a very strong constraint, which is not always satisfied in experimentally studied systems. In what follows, we will treat the spin-charge coupling terms in the diffusion equation \[14 \ 15\] as non-universal phenomenological parameters and assume that the above-mentioned constraint is satisfied.

The diffusion equation \[14 \ 15\] must be supplemented by boundary conditions. We consider the problem in leading approximation with respect to the spin-orbit coupling parameter $\alpha_R$ and the inverse conductance. First, we use Eqs. \[2 \ 4\] and \[7\] to derive a correction to the polarizability kernel

$$\delta \Pi^{(1)}_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = -2\alpha_R \epsilon_{\alpha\beta\gamma} \epsilon_{z\gamma}\Im \left[ \tilde{G}^R_{\alpha}(\mathbf{r}, \mathbf{r}') g_\delta(\mathbf{r}, \mathbf{r}') \right],$$  \hfill (16)

and

$$\delta \Pi^{(1)}_{0\beta}(\mathbf{r}, \mathbf{r}') = \delta \Pi^{(1)}_{\beta 0}(\mathbf{r}, \mathbf{r}') = 2\alpha_R \epsilon_{\beta\delta} \Re \left[ \tilde{G}^A_{\delta}(\mathbf{r}, \mathbf{r}') g_\beta(\mathbf{r}, \mathbf{r}') \right],$$  \hfill (17)

where the boundary Green’s functions are defined by Eqs. \[4\] and \[5\] and we introduced the following function

$$g_\alpha(\mathbf{r}, \mathbf{r}') = \int_{x, z > 0} d^2 \mathbf{r}_1 \tilde{G}^R_{\alpha}(\mathbf{r}, \mathbf{r}_1) \partial_{\alpha} \tilde{G}^R_{\delta}(\mathbf{r}_1, \mathbf{r}').$$

This function contains four terms (we can symbolically denote them as $g_{1\alpha} = \int G \partial_\alpha G$, $g_{2\alpha} = -\int G \partial_\delta G$, $g_{3\alpha} = -\int G \partial_\alpha G_\alpha$, and $g_{4\alpha} = \int G \partial_\delta G_\alpha$). Let us analyze the first one, the other terms can be calculated in a similar manner: To calculate the corresponding integral, we use a new coordinate system $(\bar{x}, \bar{y})$ in which the points $\mathbf{r}$ and $\mathbf{r}'$ have the coordinates $(0, -\Delta r/2)$ and $(0, \Delta r/2)$ respectively ($\Delta r = r - r'$). Next, we introduce the elliptic coordinates $(\eta, \bar{\phi})$, with the points $\mathbf{r}$ and $\mathbf{r}'$ in the foci of the ellipses: $|\mathbf{r}_1 - \mathbf{r}| + |\mathbf{r}' - \mathbf{r}_1| = \eta \Delta r$ and $\bar{\phi}$ being the angle between the vector $\mathbf{r}_1$ and the $\bar{x}$ axis. Using these elliptic coordinates and Eq. \[5\], we write the integral $g_{1\alpha}$ in the following form

$$g_{1\alpha}(\mathbf{r}, \mathbf{r}') = -\frac{im^2}{4\pi p_F} \frac{\partial_{\alpha}^2}{\partial_r^2} \int_1^\infty d\eta \int_{x, z > 0} d\bar{\phi} \sqrt{\eta^2 - \sin^2 \bar{\phi}} - 1 \times e^{i(p_F \eta/\bar{r} + \tau/2) \eta \Delta r}.$$  \hfill (18)

We note that the limits of integration over $\bar{\phi}$ are generally non-trivial due to the constraint $x_1 > 0$. It is possible to evaluate analytically the correction to the diffusion kernel [see Eqs. \[2 \ 19\], \[10\], and \[11\] and the boundary matrix \[9\]. Indeed, from Eqs. \[2 \ 19\], \[10\], and \[11\], we find that due to the fast-oscillating exponents, only a few terms survive the integration over $\mathbf{r}'$. These remaining terms contain combinations of the following types $\exp[ip_F \Delta r (\eta - 1)]$, which constrain the parameter $\eta$ in (18) to be close to unity. We find in the leading order the following results for the boundary matrix \[9\]

$$B_{\alpha\beta} = \frac{4\alpha_R m l}{\tau} \epsilon_{\alpha\beta y} \quad \text{and} \quad B_{0\beta} = B_{\beta 0} = \delta_{0y} \mathcal{O} \left( \frac{\alpha_R m l}{E_F \tau} \right),$$  \hfill (19)

where the last equation implies that the coupling between the spin and charge densities is small with respect to the inverse conductance and vanishes in the framework of the diffusion approximation and the accuracy of the method.

From Eqs. \[10\] and \[11\], we find the following boundary conditions in the leading approximation with respect to the spin-orbit coupling and the inverse conductance:

$$\partial_x \rho_0(0) = \partial_x \rho_y(0) = 0,$$  \hfill (20)

$$\partial_x \rho_z(0) = -\frac{\Gamma_{ss}}{2D} \rho_z(0), \quad \text{and} \quad \partial_x \rho_z(0) = \frac{\Gamma_{ss}}{2D} \rho_z(0),$$  \hfill (21)

where $\Gamma_{ss}$ is defined after Eq. \[15\]. These boundary conditions can be generalized to describe any form of a hard-wall boundary. To find such a general form of boundary conditions, one should do the following replacements: $\partial_x \rightarrow n \cdot \partial$ and $\rho_z \rightarrow \rho_n$, where $n$ is the unit vector normal to the segment of the boundary and $\rho_n$ is the corresponding component of the spin density. However, if the boundary is a smooth potential with the length-scale in the $n$-direction being $a$, then boundary conditions would be different and strongly dependent on the ratios of $a/l$ and $ap_F$. Also, the boundary potential would lead to an additional boundary spin-orbit coupling, which would strongly affect the boundary conditions.

We note here that in the next-to-leading order with respect to the spin-orbit interaction, two new effects appear: Boundary spin-charge coupling and boundary spin relaxation (the former is parametrically larger than the
latter due to an additional smallness of the spin density). To this order boundary conditions take the form:

$$\mathbf{n} \cdot \partial_{\mathbf{r}} \rho_z(0) = \frac{\Gamma_{ss}}{2D} \rho_n(0) + \gamma \rho_z(0) - C_z \nabla \rho_0 (0),$$

where $\gamma$ and $C_z$ are constants $\propto a_0^2$. We should note that the boundary spin-charge coupling term can be shown to be strongly dependent on the electronic spectrum and the physics far from the Fermi line and as such is non-universal and can not be calculated in the $\xi$-approximation used in this paper.

**IV. SOLUTION OF THE DIFFUSION EQUATION**

We now solve the diffusion equation for a two-dimensional gas of Rashba electrons occupying a half-space $x > 0$ and in the presence of an electric field parallel to the edge. We consider two types of edges: a partially or totally transparent interface and an impenetrable boundary. It is convenient to rewrite the diffusion equation in terms of the rescaled variables: $\tilde{\rho}_s(\tilde{x}) = [\rho_s(\tilde{x}) - \rho_\infty] / \rho_\infty$ and $\tilde{\rho}_z(\tilde{x}) = \rho_z(\tilde{x}) / \rho_\infty$, where $\tilde{x} = x / \sqrt{D\tau_0}$ is the dimensionless distance and $\rho_\infty = 2\Gamma_{ss}\nu\tau_0 eE$ is the uniform spin density due to the spin-charge coupling. In these variables, the diffusion equation takes the following form

$$\tilde{\rho}_s''(\tilde{x}) - \tilde{\rho}_s + 2\tilde{\rho}_z = 0; \quad (22)$$

$$\tilde{\rho}_z''(\tilde{x}) - 2\tilde{\rho}_z - 2\tilde{\rho}_s = 0. \quad (23)$$

Boundary conditions for a transparent boundary are $\tilde{\rho}_s(0) = 0$ and $\tilde{\rho}_z(0) = -1$ and for a hard-wall boundary $\tilde{\rho}_s'(0) = -\tilde{\rho}_s(0)$ and $\tilde{\rho}_z'(0) = \tilde{\rho}_s(0) + \zeta$, where $\zeta = 1 - C_{zy} \partial_y \rho_0 / \rho_\infty$ is a non-universal spectrum-dependent parameter. We note that in the latter case, one has to take into account the next-to-leading order boundary spin relaxation terms, which should determine the observable spin density. The solution of the differential equation (22, 23) is straightforward and we find for the transparent boundary (e.g., ballistic contacts)

$$\tilde{\rho}_s(\tilde{x}) = \frac{-1}{\text{Re} \left( \frac{k e^{ik\tilde{x}}}{k^2 + 1} \right)} \text{Re} \left( \frac{k e^{ik\tilde{x}}}{k^2 + 1} \right) \quad (24)$$

and

$$\tilde{\rho}_z(\tilde{x}) = \frac{\text{Im} \left( e^{ik\tilde{x}} \right)}{2\text{Re} \left( \frac{k}{k^2 + 1} \right)} \quad (25)$$

and for the hard-wall boundary

$$\tilde{\rho}_s(\tilde{x}) = \frac{2\zeta}{\text{Re} k} \text{Re} \left( \frac{k e^{ik\tilde{x}}}{k^2 - 1} \right) \quad (26)$$

and

$$\tilde{\rho}_z(\tilde{x}) = \frac{\zeta}{\text{Re} k} \text{Im} \left( \frac{k^2 + 1}{k^2 - 1} e^{ik\tilde{x}} \right). \quad (27)$$

where $k = \sqrt{(1 + i\sqrt{7})/2} \approx 0.978 + 0.676i$ is an eigenvalue of the diffusion equation (22, 23). Since the corresponding differential operator is non-Hermitian, the eigenvalues are complex. This leads to oscillations of the spin density near the boundary with the period $l_{\text{osc}} \approx 6.422\sqrt{D\tau_0}$. We should point out that there definitely exists other types of small-lengthscale oscillations (with the period $\pi/p_F$), which are superimposed onto the regular behavior (24 – 27). However, these Friedel oscillations are beyond the reach of the diffusion approximation. From Eqs. (24 – 27), we see that the spin density rapidly decays as we move away from the boundary. The corresponding lengthscale is $l_s \approx 1.479\sqrt{D\tau_0}$.

**V. SUMMARY**

In this paper, we have developed a general method of deriving boundary conditions for spin diffusion equation. The method involves a gradient expansion of the boundary Green’s function, which takes into account the behavior of the electron wave-functions on ballistic length-scales near the edges. Using the proposed method, we have found a general form of boundary conditions in the Rashba model for two types of edges: a transparent interface and a hard wall.

We have found an exact solution of the diffusion equation satisfying these two types of boundary conditions. The spin accumulation (24 – 27) is shown to oscillate and decay away from the boundary with the corresponding length scales being determined by the eigenvalues of the bulk diffusion equation. However, we argue that even small changes in the boundary potential would lead to a different solution of the diffusion equation: The eigenvalues, which determine the density profile, are robust and universal, but the overall amplitude and phase shifts are boundary-specific. Therefore, the spin Hall effect is generally a non-universal phenomenon, which depends on the structure of the sample edges.

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