Optimizing Dividends and Capital Injections Limited by Bankruptcy, and Practical Approximations for the Cramér-Lundberg Process

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Abstract
The recent papers Gajek and Kucinsky (Insur Math Econ 73:1–19, 2017) and Avram et al. (Mathematics 9(9):931, 2021) cost induced dichotomy for optimal dividends in the cramér-lundberg model. Avram et al. (Mathematics 9(9):931, 2021) investigated the control problem of optimizing dividends when limiting capital injections stopped upon bankruptcy. The first paper works under the spectrally negative Lévy model; the second works under the Cramér-Lundberg model with exponential jumps, where the results are considerably more explicit. The current paper has three purposes. First, it illustrates the fact that quite reasonable approximations of the general problem may be obtained using the particular exponential case studied in Avram et al. cost induced dichotomy for optimal dividends in the Cramér-Lundberg model (Avram et al. in Mathematics 9(9):931, 2021). Secondly, it extends the results to the case when a final penalty $P$ is taken into consideration as well besides a proportional cost $k > 1$ for capital injections. This requires amending the “scale and Gerber-Shiu functions” already introduced in Gajek and Kucinsky (Insur Math Econ 73:1–19, 2017). Thirdly, in the exponential case, the results will be made even more explicit by employing the Lambert-W function. This tool has particular importance in computational aspects and can be employed in theoretical aspects such as asymptotics.

Keywords Dividend problem · Capital injections · Penalty at default · Scale functions · Lambert-W function · De Vylder-type approximations · Rational Laplace transform

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1 Introduction

This paper concerns the approximate optimization of a new type of boundary mechanism, which emerged recently in the actuarial literature (Avram et al. 2018b; Gajek and Kuciński 2017; Avram et al. 2021), in the context of the optimal control of dividends and capital injections.

The model. Consider a spectrally negative Lévy risk model $X_t$, whose Laplace exponent, defined via $E_0[e^{sX(t)}] = e^{\kappa(s)}$ is given by

$$\kappa(s) := cs + \int_0^\infty (e^{-sx} - 1)\nu(dx) + \frac{\sigma^2 s^2}{2}. \quad (1)$$

Here $\sigma$ represents a volatility (uncertainty parameter), and $c$ is the essential local drift parameter, which intervenes crucially in most applications involving Lévy processes (and restricts the class of processes one may use).

Under the assumption Eq. (1), one has the Lévy-Khinchine decomposition (Bertoin 1998; Kyprianou 2014):

$$X_t = x + ct - \xi_t,$$ \quad (2)

where $\xi_t$ is a driftless spectrally positive Lévy process, with Lévy measure $\nu(dx)$ and Brownian component $\sigma B_t$. The classic example to have in mind is that of the perturbed Cramèr-Lundberg risk model with

$$\tilde{\xi}_t = \sum_{i=1}^{N_t} \xi_i + \sigma B_t,$$ \quad (3)

where $B_t$ is a Brownian motion, $N_t$ is an independent Poisson process of intensity $\lambda > 0$, and $(\xi_i)_{i \geq 1}$ is a family of i.i.d.r.v. whose distribution, density and moments are denoted respectively by $F, f, m_i, i \in \{1, 2, \ldots\}$.

Furthermore,

- the process $X_t$ is modified by dividends and capital injection:

$$\pi := (D, I) \Rightarrow X^\pi_t := X_t - D_t + I_t,$$

where $D, I$ are adapted, non-decreasing and cg processes with $D_{0-} = I_{0-} = 0$ \footnote{In Gajek and Kuciński (2017), these processes are left-continuous. We have proceeded differently, since from an intuitive point of view, it all comes down to what happens at 0: left-continuous implies that no matter what the reserve, no dividends are to be paid ($D_{0-} = 0$, in principle). We argue differently, especially since these barrier policies say “pay the exceeding” as a lump sum... This is also valid for injections: bankruptcy is not declared when an important claim comes. Instead, injection may save the company. Finally, from a technical point of view, it all comes down to what precise Itô formula one employs when writing down the dynamic programming principle and how one constructs the admissible policies. The choice is rather important for the verification results, and we note that cg is also the standardized form in the Azcue-Muller papers.}.

Furthermore,

- the first time when we do not bail-out to positive reserves $\sigma^\pi_{0-} := \inf \{t > 0 : X^\pi_{t-} - \Delta \tilde{\xi} + \Delta I < 0\}$ is called bankruptcy/absolute ruin:
• prior to bankruptcy, dividends are limited by the available reserves: \( \triangle D_t := D_t - D_{t-} \leq X_{t^-} - \Delta \xi_t + \Delta I_t \). The set of “admissible” policies satisfying this constraint will be denoted by \( \tilde{\Pi}(x) \).

The objective is to maximize the profit:

\[
J^x := \mathbb{E}_x \left[ \int_{[0, \sigma^x]} e^{-q(t)} (dD_s - kdI_s) - Pe^{-qs^x_0} \right], \quad k \geq 1, P \geq 0,
\]

where \( q \) is the discount factor, \( k \) is the cost of capital injections, and \( P \) is a final penalty upon bankruptcy. The \textit{value function} is

\[
V(x) := \sup_{\pi \in \tilde{\Pi}(x)} J^x, \quad x \in \mathbb{R}.
\]

\textbf{Motivation.} The recent papers (Gajek and Kuciński 2017; Avram et al. 2021) investigated the above control problem of optimizing dividends and capital injections for processes with jumps, when bankruptcy is allowed as well. The second paper works under the Cramér-Lundberg model with exponential jumps, while the first works under the spectrally negative Lévy model, allowing also for the presence of Brownian motion and infinite activity jumps. It turns out that the optimal policy belongs to the class of \((-a, 0, b), \ a > 0, b \geq 0\) “bounded buffer policies”, which consist in allowing only capital injections smaller than a given \( a \) and declaring bankruptcy at the first time when the size of the overshoot below 0 exceeds \( a \), and in paying dividends when the reserve reaches an upper barrier \( b \). These will briefly be described as \((-a, 0, b)\) policies from now on. Furthermore, the optimal \((a^*, b^*)\) are the roots of one variable equations with explicit solutions related to the Lambert-W(right) function (ProductLog in Mathematica).

Below, our goal is to show numerically that exponential approximations provide quite reasonable results (as the de Vylder approximation provides for the ruin problem). We will focus in our examples on the case of matrix exponential jumps(known to be dense in the class of general nonnegative jumps, with even error bounds for completely monotone jumps being available (Vatamidou et al. 2014)), for two reasons. One is in order to highlight certain exact equations which are similar to their exponential versions, and which may at their turn be used to produce even more accurate approximations in the future, and, secondly, since numerical Laplace inversion for this class may easily tuned to have arbitrarily small errors.

\textbf{History of the problem:} The case of no capital injections (also characterized by \( k = \infty \) or absorption below 0) is the dividend problem posed by De Finetti (1957); Gerber (1969) where dividends are paid above barrier \( b^* \) and \( a^* = 0 \) is imposed. “The challenge is to find the right compromise between paying early in view of the discounting or paying late in order not to reach ruin too early and thus profit from the positive safety loading for a longer time” (Albrecher et al. 2020).

Forced injections and no bankruptcy at 0 (also characterized by a reflection at 0) is studied in Shreve et al. (1984) where dividends are paid above barrier \( b^* \) and \( a^* = \infty \) is imposed.

From Løkka and Zervos (2008) we know that in the Brownian motion case, it is optimal to either always inject, if \( k \leq k_c \), for some critical cost \( k_c \) (i.e. use Shreve), or, stop at 0 (use De Finetti). We propose to call this the \textbf{Lokka, Zervos alternative}. The “proof” of this alternative starts by largely assuming it via a heuristically justified border Ansatz [LZ08, (5.2)]: \[ \max \{-V(0), V'(0) - k\} = 0 \implies \text{either } V(0) = 0 \text{ or } V'(0) = k. \]
Extensive literature on SLG forced bailouts (no bankruptcy) can be found at Avram et al. (2007), Kulenko and Schmidli (2008), Eisenberg and Schmidli (2011), Pérez et al. (2018), Lindensjö and Lindskog (2019), Noba et al. (2020).

Articles (Gajek and Kuciński 2017; Avram et al. 2021) are the only papers which relate declaring bankruptcy to the size of jumps, with general and exponential jumps, respectively. Gajek and Kuciński (2017) deals also with the presence of Brownian motion and infinite activity jumps, by conditioning at the first draw-down time; the optimality proof is quite involved.

For claims with light tails, one may use exponential approximations, which are similar in spirit with the de Vylder-type approximations. Recall that the philosophy of the de Vylder approximation is to approximate a Cramér-Lundberg process by a simpler process with exponential jumps, with cleverly chosen exponential rate $\mu$ – see for example Avram et al. (2019b) for more details).

The efficiency of the de Vylder approximation for approximating ruin probabilities is well documented (De Vylder 1978). The natural question of whether this type of techniques may work for other objectives, like for example for optimizing dividends and/or reinsurance was already discussed in Højgaard (2002); Dickson and Drekic (2005); Beveridge et al. (2007); Gerber et al. (2008); Avram et al. (2019b). In this paper, following on previous works (Avram et al. 2011; Avram and Pistorius 2014; Avram et al. 2018a), we draw first the attention to the fact that we have not one, but three de Vylder-type approximations for the scale function $W_q(x)$ of a spectrally negative Lévy process (and for its ruin probability as well), and provide experiments on how they perform.

We end this introduction by highlighting in Fig. 1 the fact that for exponential jumps, the limited capital injections objective function $J_0$ given by (15) for arbitrary $b$ but...
optimal \( a = s(b) \) (via a complicated formula) improves the value function with respect to de Finetti and Shreve, Lehoczky and Gaver, for any \( b \).

**Contents and contributions.** Section 2 offers, in Theorem 1, a profit formula for \((-a, 0, b)\) policies, where we incorporate also a final bankruptcy penalty \( P \). This result links (Gajek and Kuciński 2017; Avram et al. 2021) together, and also goes beyond them, by showing how the penalty \( P \) affects the scale function \( G \). Its proof, along the lines of either Gajek and Kuciński (2017) or Avram et al. (2021), is provided in the Appendix when \( \sigma \equiv 0 \), and only sketched when \( \sigma > 0 \), to prevent this already lengthy paper from getting even longer. It consists in a three step argument:

1. express the cost by conditioning on the reserve \((J_x)\) starting from \(0 \leq x \leq b\) hitting either 0 or \( b \);
2. get a further relationship on costs \( J_b \) and \( J_0 \) by conditioning on the first claim;
3. finally, mix these conditions together in order to obtain the explicit formula for \( J_x \).

In Sect. 8 we provide an alternative matrix exponential form of the exact cost, in the case of matrix exponential jumps.

An explicit determination of \( a^*, b^* \) and an equity cost dichotomy when dealing with exponential jumps are given in Sect. 4, taking also advantage of properties of the Lambert-W function, which were not exploited before. The two main novelties of the section are:

- emphasizing the computations of the optimal buffer/barrier (from Avram et al. (2021)) in relation with the scale-like quantities appearing in Gajek and Kuciński (2017);
- making explicit use of the (computation-ready) Lambert-W function to describe the dependency of optimal \( a^*b \) (in Eq. (15)) and of the dichotomy-triggering cost \( k_c \) in Eq. (27).

Again, a further novelty is the presence of the bankruptcy cost \( P \).

Section 5 reviews, for completeness, the de Vylder approximation-type approximations. Section 5.1 recalls, for warm-up, some of the oldest exponential approximations for ruin probabilities. Section 5.2 recalls in Proposition 3, following (Avram and Pistorius 2014; Avram et al. 2019b) three approximations of the scale function \( W_q(x)^2 \), obtained by approximating its Laplace transform. These amount finally to replacing our process by one with exponential jumps and cleverly crafted parameters based on the first three moments of the claims.

In Sect. 6, we consider particular examples and obtain very good approximations for two fundamental objects of interest: the growth exponent \( \Phi_q \) of the scale function \( W_q(x) \), and the (last) global minimum of \( W_q''(x) \), which is fundamental in the de Finetti barrier problem. Proceeding afterwards to the problem of dividends and limited capital injections, concepts in Sect. 4 are used to compute a straightforward exponential approximation based on an exponential approximation of the claim density, and a new “correct ingredients approximation” which consists of plugging into the objective function for exponential claims the exact “non-exponential ingredients” (scale functions

\footnote{essentially, this is the “dividend function with fixed barrier”, which had been also extensively studied in previous literature before the introduction of \( W_q(x) \)}
and, survival and mean functions) of the non-exponential densities. Both methods are observed to yield reasonable values in approximating the objective.

This leads us to our conclusion that from a practical point of view, exponential approximations are typically sufficient in the problems discussed in this paper.

2 The Cost Function of \((-a, 0, b)\) Policies, for the Spectrally Negative Lévy Case

We revisit here the problem of optimizing the value of “bounded buffer \((-a, 0, b)\) policies”, following Gajek and Kuciński (2017); Avram et al. (2021) (in order to relate the results, one needs to replace \(\gamma\) in the objective of Gajek and Kuciński (2017) by \(1/k\)), while taking into account also the bankruptcy penalty \(P\).

Recall that in the first passage theory of spectrally negative Lévy processes, a crucial role is played by the scale functions (Bertoin 1998; Kyprianou 2014)

\[
W_q(x) = \frac{1}{s} Z_q(x) = 1 + q \int_0^x W_q(y) dy.
\]

In our context, an important role will also be played by the expected scale after a jump \(W_q(x)\).

The problem of limited reflection requires introducing a new “scale function \(S_a(x)\) and Gerber-Shiu function \(G_a(x)\)”– see Remark 2 for further comments on this terminology:

\[
\begin{cases}
S_a(x) = Z_q(x) + C_a(x), C_a(x) = \int_0^x W_q(x-y) \cdot \nabla(a+y) dy \\
G_a(x) = G_a(x) + k \frac{\sigma^2}{2} W_q(x)
\end{cases}
\]

where

\[
G_a(x) = \int_0^x W_q(x-y)(k m_a(y) + P\nabla(a+y)) dy := kM_a(x) + PC_a(x),
\]

\[
m_a(y) = \int_0^y z \nu(z) dz.
\]

Example 1 With exponential jumps of rate \(\mu\) and possibly \(\sigma > 0\), using the identities

\[
\nabla(y) = e^{-\lambda y}, m_a(y) = \lambda e^{-\lambda y} m(a), m(a) = \int_0^a y e^{-\mu y} dy = \frac{1 - e^{-\mu a}}{\mu} - ae^{-\mu a},
\]

we find that the functions Eq. (5) are expressible as products of \(C(x)\) and the survival or mean function of the jumps:
\[
\begin{align*}
C_a(x) &= C(x)e^{-\mu a} = C(x)\overline{F}(a), \quad \overline{F}(a) = 1 - F(a) \\
S_a(x) &= Z_q(x) + e^{-\mu a}C(x) \\
G_a(x) &= (km(a) + P\overline{F}(a))C(x)
\end{align*}
\]

(Gajek and Kuciński (2017) use \( s_c, r_c \), instead of \( M_a(x) := \int_0^x W_q(x - y) m_a(y) \, dy, C_a(x) \), respectively). When \( P = 0 = \sigma \), these reduce to quantities in Avram et al. (2021).

The formulas above will be used below as a heuristic approximation in non-exponential cases.

**Remark 1** Note that

\[
C_a(0) = 0, \quad G_a(0) = 0, \quad S_a(0) = 1, \quad C(0) = 0, \quad C'(0) = \begin{cases} \frac{\sigma}{c} & \sigma = 0 \\ 0 & \sigma > 0 \end{cases},
\]

and that \( C(x), G_a(x), S_a(x) \) are increasing functions in \( x \).

We state now a generalization of Gajek and Kuciński (2017) [Thm. 4] for the value function \( J^a_b \) of \((-a, 0, b)\) policies, in terms of \( S_a(x), G_a(x) \). In the Cramèr-Lundberg case illustrated below, the proof is straightforward, following Avram et al. (2021). In the other case, one needs to adapt the proof of Gajek and Kuciński (2017).

**Theorem 1** Cost function for \((a, b)\) policies For a spectrally negative Lévy processes, let

\[
\tau_d = \tau_{a, -} := \inf\{t \geq 0 : X_t < d\}, \quad \tau_{d, +} := \inf\{t \geq 0 : X_t > d\},
\]

and let

\[
J_x = J^{a,b}_x := \mathbb{E}_x \left[ \int_0^{T_a} e^{-qT} (dD_t - k \, dI_t) - Pe^{-qT} \right]
\]

denote the expected discounted dividends minus capital injections associated to policies consisting in paying capital injections with proportional cost \( k \geq 1 \), provided that the severity of ruin is smaller than \( a > 0 \), and paying dividends as soon as the process reaches some upper level \( b \). Put

\[
G_{a, \sigma}(x) = G_a(x) + k \frac{\sigma^2}{2} W_q(x).
\]

Then, it holds that

\[
J_x = \begin{cases} 
G_{a, \sigma}(x) + J^a_b S_a(x) = G_{a, \sigma}(x) + \frac{1 - G_{a, \sigma}(b)}{S_a(b)} S_a(x), & x \in [0, b] \\
xk + J^a_b & x \leq -a
\end{cases}
\]

**Remark 2** The first equality in Eq. (8) will be easily obtained by applying the strong Markov property at the stopping time \( T = \min[T_{a, -}, T_{b, +}] \), but it still contains the unknown \( J_0 \).

This relation suggests a definition of the scale \( S_a \) and the Gerber-Shiu function \( G_{a, \sigma} \), as the coefficient of \( J_0 \) and the part independent of \( J_0 \), respectively.

This equality is also equivalent to
which suggests another analytic definition of the scale and Gerber-Shiu function corresponding to an objective $J_x$ which involves reflection at $b$.

The functions $S_x(x), G_a(x)$ may be shown to stay the same for problems which require only modifying the boundary condition at $b$, like the problem of capital injections for the process reflected at $b$, or the problem of dividends for the process reflected at $b$, with proportional retention $k_D$ (this is in coherence with previously studied problems).

### 3 Proof of Proposition 1 in the Spectrally Negative Case

**Corollary 1** Let us consider the Cramèr-Lundberg setting without diffusion (i.e. $\sigma = 0$), For fixed $k \geq 1$, $b \geq 0$, the optimality equation $\frac{\partial}{\partial a} J_a^b = 0$ may be written as

$$
J_a^b = ka - P \iff J_a^b = -P. \tag{10}
$$

**Remark 3** The first equality in Eq. (10) provides a relation between the objective $J_0$ and the variable $a$; the second recognizes this as the smooth fit equation $J_{-a} = 0$.

**Proof:** Recalling the expressions of $J_a^b, G_a(x)$, in Eq. (9), in Eq. (5), and from Gajek and Kuciński (2017) [Lem. A.4]

$$
M_a'(x) = -aC_a'(x),
$$

where $C_a'(x), M_a'(x)$ denote derivatives with respect to the subscript $a$. Whenever $b > 0$, if $a$ achieves the maximum in $J_a^b$, it is straightforward (think of the economic interpretation) that $a$ achieves the maximum of $a \mapsto J_a^b$ for every $x \in [0,b]$. Therefore, we find

$$
\frac{\partial}{\partial a} J_0^b = 0 \iff J_0^b = \frac{-G_a'(x)}{C_a'(x)} = \frac{-kM_a'(x) - PC_a'(x)}{C_a'(x)} = ka - P
$$

$$
\iff J_{-a}^b = J_0^b - ka \iff J_{-a}^b = -P.
$$

### 4 Explicit Determination of $a^*, b^*$ when $F(x) := 1 - e^{-\mu x}, P > -\frac{c}{q}$

In this section we turn to the exponential case, where explicit formulas for the optimizers $a^*, b^*$ are available. In particular, we will take advantage of properties of the Lambert-W function, which were not exploited in Avram et al. (2021). Subsequently, in Sects. 6, 7 we will show that exponential approximations work typically excellently in the general case. Although these results have already been established in [AGLW20], the present formulations have two achievements:

1. allow an unified formulation of Avram et al. (2021) and Gajek and Kuciński (2017) (via the previously introduced scale functions);
2. make use of a numerical tool (Lambert-W function) to express the optimal quantities of interest \( a^*, b^* \).

### 4.1 The Simplified Cost Function and Optimality Equations

**Proposition 1** Cost function and optimality equations in the exponential case

\[
J_{a,b}^0 = \frac{1 - C'(b)\left(k m(a) + P\bar{F}(a)\right)}{(\bar{F}(a))C'(b) + qW_q(b)} = \frac{\gamma(b) - k m(a) - P\bar{F}(a)}{q\theta(b) + \bar{F}(a)},
\]

where we put

\[
\gamma(b) = \frac{1}{C'(b)}, \quad \theta(b) = \frac{W_q(b)}{C'(b)}.
\]

1. Put

\[
j(b) : = \frac{\gamma'(b)}{q\theta'(b)}.
\]

For fixed \( a \geq 0 \), the optimality equation \( \frac{\partial}{\partial b} J_{a,b}^0 = 0 \) may be written as

\[
J_{a,b}^0 = j(b).
\]

2. For fixed \( k \geq 1 \) and \( b \geq 0 \), at critical points with \( a(b) = a^{(k,P)}(b) \neq 0 \) satisfies \( \frac{\partial}{\partial a} J_{a,b}^{(k,P)} = 0 \) we must have

\[
\left[ J_{a,b}^0 - (ka - P) \right]_{a=a(b)} = 0.
\]

Explicitly,

\[
0 = \eta(b, a) : = \frac{\gamma(b)}{\theta(b)} - \frac{k}{\mu\theta(b)} F(a) - q(ka - P).
\]

3. When \( P \geq -\frac{c}{q} \) and \( b \geq 0 \) is fixed, the solution of Eq. (14) may be expressed in terms of the principal value of the “Lambert-W(right)” function (an inverse of \( L(z) = ze^z \))

\[
\left[ -e^{-1}, \infty \right] \ni L_0(z), \quad z \in \left[ -1, \infty \right)
\]

Corless et al. (1996); Boyd (1998); Brito et al. (2008); Pakes (2015); Vazquez-Leal et al. (2019) (this observation is missing in Avram et al. (2021)).

\[
(0, \infty) \ni a(b) = \mu^{-1}\left(-h(b) + L_0\left(\frac{h(b)}{q\theta(b)}\right)\right)
\]

\[
, h(b) = h(b, P) = \frac{1}{q\theta(b)} - \frac{\mu}{k} \left(\frac{\gamma(b)}{q\theta(b)} + P\right)
\]

(15)

It follows that
\[ J_{0}^{a(b),b} = \frac{k}{\mu} \left( -h(b) + L_0 \left( \frac{1}{q\theta(b)} e^{h(b)} \right) \right) - P. \]  

(16)

5. In the special case \( b = 0 \), Eq. (14) implies that \( a = a^{(k,P)} = a^{(k,P)}(0) \) satisfies the simpler equation

\[ 0 = \delta_{k,P}(a) := \lambda \eta(0, a) = \bar{c} - k \left( aq + \frac{\lambda}{\mu} (1 - e^{-\mu a}) \right), \quad \bar{c} = c + qP > 0, \]  

with solution

\[ \mu a^{(k,P)} = -g + L_0 \left( \frac{\lambda e^g}{q} \right) > 0, \quad g = h(0) = \frac{\lambda}{q} - \frac{\mu \bar{c}}{kq}. \]  

(18)

6. At a critical point \((a^*, b^*)\), \(a^* > 0, b^* > 0\), we must have both \( J_{0}^{a,b^*} = j(b^*) = ka^* - P \implies a^* = s(b^*), s(b) := \frac{j(b) + P}{k}, \) and

\[ 0 = \eta(b^*), \quad \eta(b) := \eta(b, s(b)) = \frac{\gamma(b)}{\theta(b)} - qj(b) - \frac{k}{\mu \theta(b)} F \left( \frac{j(b) + P}{k} \right) = 0. \]  

(20)

7. The equation \( 0 = \eta(b) \) may be solved explicitly for \( P \), yielding

\[ P = -\frac{k}{\mu} \log \left( 1 + \frac{q\theta(b)j(b) - \gamma(b)}{\frac{k}{\mu}} \right) - j(b). \]  

(21)

**Proof:** 1. Follows from Theorem 1.

2. Let \( M(b), N(b) \) denote the numerator and denominator of \( J_{0}^{a,b} := \frac{M(b)}{N(b)} \) in . The optimality equation \( \frac{\partial}{\partial b} J_{0}^{a,b} = \frac{N'(b)}{N(b)} - J_{0}^{a,b} = 0 \) simplifies to

\[ J_{0}^{a,b} \frac{M'(b)}{N'(b)} = \frac{\gamma'(b)}{q\theta'(b)} = j(b). \]

3. Eq. (14) is a consequence of 1 and of the smooth fit result Corollary 2.

4. See the proof of the particular case 5; \( a \in (0, \infty) \) holds since \( P \geq -\frac{c}{q} \implies h(b) < \frac{1}{q\theta(b)}. \)

5. Eq. (17) follows from \( W_0(0) = \frac{1}{q}, \theta(0) = \lambda^{-1}. \) To get Eq. (18), rewrite the Eq. (17) as \( ze^\frac{c}{q} = \frac{\lambda e^\frac{c}{q}}{q}, z = \mu a + g; a \in (0, \infty) \) holds since \( P \geq -\frac{c}{q} \implies g < \frac{\lambda}{q}. \)

6. follows from 2. and .3.

7. is straightforward.

**Remark 4** Note that the de Finetti and Shreve, Lehoczky and Gaver solutions \( a^* = a(b^*) = \begin{cases} 0 \\ \infty \end{cases} \) are always non-optimal, when \( P \geq -\frac{c}{q} \) (see Eq. (15)).

However, as \( k \to \infty, \ h(b) \to \frac{1}{q\theta(b)} \neq 0 \) and, \( a(b) = \mu^{-1}( -h(b) + L_0 \left( h(b)e^{h(b)} \right) ) = 0. \) This suffices to infer that you get de Finetti case.

On the other hand,
Thus, these regimes can be recovered asymptotically. Let now \( b^\ast_k, S_k, b^\ast_D \) denote the unique roots of \( u_2(b) = 0 \) in the two asymptotic cases, which coincide with the classic Shreve, Lehoczky and Gaver and de Finetti barriers.

Then, it may be checked that \( b^\ast \leq \min\{b^\ast_k, S_k, b^\ast_D\} \).

4.2 Existence of the Roots of the Equations \( \eta(b) = 0, \delta_{kp} = 0 \)

The following (new) result discusses the existence of the roots of the equations \( \eta(b) = 0, \delta_{kp} = 0 \) introduced in Proposition 3 and relates them to the Lambert-W function.

**Proposition 2**

- \( \theta \) increases from \( \theta(0) = \frac{1/c}{\lambda} = \frac{1}{\lambda} \) to \( \theta(\infty) = \frac{1}{c\Phi_q - q} \) as we see it in the Fig. 2.
  - \( \gamma \) is increasing-decreasing (from \( \frac{c}{\lambda} \) to 0), with a maximum at the unique root of \( C''(x) = 0 \) given by
  
  \[
  \tilde{b} := \frac{1}{\Phi_q - \rho_-} \log \left( \frac{\rho_- \Phi_q^2}{\Phi_q^2} \right),
  \]

  where \( \Phi_q, \rho_- \) denote the positive and negative roots of the Cramèr-Lundberg equation \( \kappa(s) = 0 \). The Fig. 3 illustrates the plot of the function \( \gamma \) and \( j(b) \) in which the \( \tilde{b} \) is represented by the black point. If \( c\mu - (q + \lambda) > 0 \), then \( b > 0 \) defined in Eq. (23) is the unique positive root of \( j(b) \) and \( \eta(\tilde{b}) = \frac{1}{W_q(\tilde{b})} > 0 \). See the Fig. 4. The function \( j(b) = \frac{\gamma'(b)}{q\delta'(b)} \) is nonnegative and decreasing to 0 on \([0, b]\), with

\[
\begin{cases}
  P \to \infty \Rightarrow h(b) \to -\infty \Rightarrow a(b) \to \infty \Rightarrow \\
  \gamma(b) - k/\mu \to -1 - kC'(b)/\mu = J_{0,SLG}(b) \\
  \eta(b) \to q\left( f_{0,k}^{SLG}(b) - j(b) \right), \quad \forall b > 0.
\end{cases}
\]

(22)

Fig. 2 Plot of \( \theta \) with \( \theta(0) = 2 \) and \( \theta(\infty) = 22.8743 \), for \( \mu = 2, c = 3/4, \lambda = 1/2, q = 1/10, P = 1 \) and \( k = 3/2 \).
See the Fig. 3.

2. Put

\[ j(0) = \frac{\lambda - C''(0)}{\mu q (C'(0))^2} = \frac{c \mu - (q + \lambda)}{\mu q}. \]  

(24)

See the Fig. 3.

\[ \delta_{k,p} := \delta_{k,p}(\mu, \lambda, q, \mu, \lambda) = \delta_{k,p}(\frac{j(0) + P}{k}) = \delta_{k,p}(\frac{c \mu - (q + \lambda)}{k \mu q}) \]

\[ = \frac{\lambda + q - \lambda k \left(1 - e^{-\frac{\lambda + q}{\mu q}}\right)}{\mu}, \]

(25)

Fig. 3 Plots of \( j(b) \) and \( \gamma(b) \) with \( b = 2.5046 \) and \( j(0) = 4.5 \), for \( \mu = 2, c = 3/4, \lambda = 1/2, q = 1/10, P = 1 \) and \( k = 3/2 \).

Fig. 4 For \( \mu = 2, c = 3/4, \lambda = 1/2, q = 1/10, P = 1 \) and \( k = 3/2 \), the root of \( \eta(b) = 0 \) is at \( b = 0.469843 \).
and assume
\[
\lim_{k \to \infty} \delta_{k,P} = \frac{\lambda + q}{\mu} - \frac{\lambda (\mu \tilde{c} - (\lambda + q))}{q \mu} = \frac{(\lambda + q)^2 - \lambda \mu \tilde{c}}{q \mu} < 0 \Leftrightarrow \tilde{c} \mu > \lambda^{-1}(\lambda + q)^2.
\] (26)

Then, \(\forall P > -\frac{\tilde{c}}{q}\), the function \(\delta_{k,P}\) is decreasing in \(k\) with \(\delta_{1,P} > 0\), and has a unique root
\[
k_c = k_c(P) := \frac{q + \lambda}{\lambda} \frac{f}{f + L_0(-fe^{-f})} > \frac{q + \lambda}{\lambda},
\] (27)

where
\[
f := \frac{\lambda}{q + \lambda} \frac{\tilde{c} \mu - (\lambda + q)}{q} > 1 \Leftrightarrow \tilde{c} \mu > \lambda^{-1}(\lambda + q)^2 \Leftrightarrow P > P_1
\] (28)

(note that the denominator \(f + L_0(-fe^{-f})\) does not equal 0 since \(f > 1\) and \(L_0\) takes always values bigger than \(-1\); or, note that \(-f = L_{-1}(L(-f))\), where \(L_{-1}\) is the other real branch of the Lambert function). Furthermore,
\[
\delta_{k,P} < 0 \Leftrightarrow k > k_c(P).
\] (29)

3. It follows that \(\eta(b) = 0\) has at least one solution of in \((0, \tilde{b})\) iff
\[
\eta(0) = \frac{c}{\lambda} - \frac{1}{\lambda} \left( c - \frac{q + \lambda}{\mu} \right) - \frac{k}{\mu} F(a^{(k,P)}) = \frac{1}{\lambda \mu} (\lambda + q - \lambda k F(a^{(k,P)})) = \frac{1}{\lambda} \delta_{k,P} < 0 \Leftrightarrow k > k_c.
\] (30)

The first such solution will be denoted by \(b^*\).

\[
\left( \frac{\mu}{k} j(b) + h \right) e^{\frac{\mu}{k} j(b) + h} = \frac{e^h}{q \theta(b)} \implies \mu \left( j(b) + P \right) = -h + L_0 \left( \frac{e^h}{q \theta(b)} \right).
\]

**Proof:** For 1. see Avram et al. (2021) [Proof of Theorem 11, A2].

2. By using the assumption \(\tilde{c} \mu > \lambda^{-1}(\lambda + q)^2\) we get \(\tilde{c} \mu \geq \lambda + q\), and \(k \in [1, \infty) \to \delta_{k,P}\) is decreasing.

Put \(d = \frac{\tilde{c} r -(\lambda + q)}{q}\). The inequality \(\delta_{k,P} < 0\) (see ) may be reduced to
\[
e^{\frac{\mu}{k} j(b) + h} = \frac{e^h}{q \theta(b)} \implies \mu \left( j(b) + P \right) = -h + L_0 \left( \frac{e^h}{q \theta(b)} \right).
\]

Rewriting the latter as \(-f > e^z(f - z)\) we recognize, by putting \(z = y + f\), an inequality reducible to \(ye^y < -fe^{-f}\). The solution is
\[
y < L_0(-fe^{-f}),
\]
where $L_0$ is the principal branch of the Lambert-W function.

The final solution is Eq. (29), where we may note that the variables $k, P$ have been separated.

Remark 4 The function $\{1, \infty) \ni f \mapsto f + L_0(-fe^{-f}) \in (1, \infty)$ blows up at $f = 1$, and converges to 1 when $f \to \infty$ (or when either $\mu$ or $\bar{c} = c + qP$ are large enough) as may be noticed in the figure below, which blows up at the value $P_l := -4/5$. Note also that when $f$ (or one of $c, P, \mu$ are large enough), $k_c$ given by stabilizes to the equilibrium $\frac{\lambda + q}{\lambda} = 6/5$; this is related to Avram et al. (2007) [Lemma 2], Kulenko and Schmidli (2008) [Lemma 7], who obtain the same condition for $b^* = 0$ (without buffering capital injections). Intuitively, under these conditions, buffering is not crucial.

At the other end, as $f$ tends to its lower limit and to the regime B, the notion of equity expensiveness vanishes, and $k_c \to \infty$, see the Fig. 5.

Fig. 5 $k_c$ as function of $P$, for several values of $c$, with the vertical asymptote at $P_l$ fixed

Fig. 6 $k_c$ as a function of $q, \lambda$

$\square$ Springer
The next two figures illustrate how \( k_c \) blows up at the critical values \( q_l := \left( \frac{1}{2} - 2p\lambda + \sqrt{\lambda} \sqrt{\mu \sqrt{4c - 4p\lambda + 2q}} \right) \) and \( \lambda_i := (c\mu - \sqrt{(c\mu + \mu(-p)q + 2q)^2 - 4q^2 + \mu Pq - 2q}) \) (represented by red points in the Fig. 6). The dark (blue) parts correspond to the regime \( A \).

5 Which Exponential Approximation?

5.1 Three De Vylder-Type Exponential Approximations for the Ruin Probability

In this section we recall three de Vylder-type exponential approximations for the ruin probability and provide corresponding approximations for \( W_q \), and comment on their performance.

In the simplest case of exponential jumps of rate \( \mu = 0 \), the formula for the ruin probability is

\[
\Psi(x) = P_x \left[ \exists t \geq 0 : X_t < 0 \right] = \frac{1}{1 + \theta} \exp \left( - \frac{x\theta \mu}{1 + \theta} \right) = \frac{1}{1 + \theta} \exp \left( - \frac{x\theta m^{-1}}{1 + \theta} \right),
\]

where \( \theta = \frac{e^{-\lambda m}}{\lambda m} \) is the loading coefficient. By plugging the correct mean of the claims in the second formula yields the simplest approximation for processes with finite mean claims.

More sophisticated is the Renyi exponential approximation

\[
\Psi_R(x) = \frac{1}{1 + \theta} \exp \left( - \frac{x\theta \hat{m}^{-1}}{1 + \theta} \right), \hat{m} = \frac{m}{2m};
\]

This formula can be obtained as a two point Padé approximation of the Laplace transform, which conserves also the value \( \Psi(0) = (1 + \theta)^{-1} \) (Avram and Pistorius 2014). It may be also derived heuristically from the first formula in , via replacing \( \mu \) by the correct “excess mean” of the excess/severity density

\[
f_e(x) = \frac{\bar{F}(x)}{m} = \frac{1 - F(x)}{m},
\]

which is known to be \( \hat{m} \). Heuristically, it makes more sense to approximate \( f_e(x) \) instead of the original density \( f(x) \), since \( f_e(x) \) is a monotone function, and also an important component of the Pollaczek-Khinchine formula for the Laplace transform \( \Psi(s) = \int_0^\infty e^{-sx} F(dx) \) – see Ramsay (1992); Avram and Pistorius (2014).

More moments are put to work in the de Vylder approximation

\[
\Psi_{DV}(x) = \frac{1}{1 + \tilde{\theta}} \exp \left( - \frac{x\tilde{m}^{-1}}{1 + \tilde{\theta}} \right), \tilde{m} := \frac{m}{3m}, \tilde{\theta} = \frac{9m^3}{2m^2}, \tilde{\lambda} = \frac{2m_1 m_3}{3m^2}, \tilde{\theta} = \frac{m_3}{m^2} \theta.
\]

\( \tilde{c} = c - \lambda m_1 + \tilde{\lambda} \tilde{m}_3, \ \tilde{\theta} = \frac{2m_1 m_3}{3m^2} \theta = \frac{m_3}{m^2} \theta. \)
Interestingly, the result may be expressed in terms of the so-called “normalized moments”

\[ \hat{m}_i = \frac{m_i}{m_{i-1}} \]

introduced in Bobbio et al. (2005).

The de Vylder approximation parameters above may be obtained either from

1. equating the first three cumulants of our process to those of a process with exponentially distributed claim sizes of mean \( \mu \), and modified \( \lambda, c \) (De Vylder 1978) (however \( p = c - \lambda m_1 = E_0[X_1] \) must be conserved, since this is the first cumulant), or
2. a Padé approximation of the Laplace transform of the ruin probabilities (Avram et al. 2011).

The second derivation via Padé shows that higher order approximations may be easily obtained as well. They might not be admissible, due to negative values, but packages for “repairing” the non-admissibility are available – see for example Dumitrescu et al. (2016).

The first derivation of the de Vylder approximation is a process approximation (i.e., independent of the problem considered); as such, it may be applied to other functionals of interest besides ruin probabilities, dividend barriers, etc, simply by plugging the modified parameters in the exact formula for the ruin probability of the simpler process.

5.2 Three Two Point Padé Approximations of the Laplace Transform \( \hat{W}_q \) of Scale Function

The simplest approximations for the scale function \( W_q(x) \) will now be derived heuristically from the following example.

**Example 1** The Cramér-Lundberg model with exponential jumps Consider the Cramér-Lundberg model with exponential jump sizes with mean \( 1/\mu \), jump rate \( \lambda \), premium rate \( c > 0 \), and Laplace exponent \( \kappa(s) = s\left(c - \frac{\lambda}{\mu + s}\right) \). Solving \( \kappa(s) - q = 0 \Leftrightarrow c s^2 + s(\mu - \lambda - q) - q \mu = 0 \) for \( s \) yields two distinct solutions \( \gamma_2 \leq \gamma_1 = \Phi_q \) given by

\[
\gamma_1(\mu, \lambda, c) = \gamma_1 = \frac{1}{2c} \left( -(\mu c - \lambda - q) + \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c} \right),
\]

\[
\gamma_2(\mu, \lambda, c) = \gamma_2 = \frac{1}{2c} \left( -(\mu c - \lambda - q) - \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c} \right).
\]

The \( W \) scale function is:

\[
W_q(x) = \frac{A_1 e^{\gamma_1 x} - A_2 e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)} \Leftrightarrow \hat{W}_q(s) = \frac{s + \mu}{c s^2 + s(\mu c - \lambda - q) - q \mu},
\]

(35)

where \( A_1 = \mu + \gamma_1, A_2 = \mu + \gamma_2 \).

Furthermore, it is well-known and easy to check that the function \( W'_q(x) \) is in this case unimodal with global minimum at
\[ b_{DeF} = \frac{1}{\gamma_1 - \gamma_2} \begin{cases} \log \frac{(y_1)^2 A_2}{(y_1)^2} & \log \frac{(y_2)^2 (\mu + y_2)}{(y_1)^2 (\mu + y_1)} \quad \text{if} \ W_q''(0) < 0 \Leftrightarrow (q + \lambda)^2 - c\lambda \mu < 0 \\ 0 & \text{if} \ W_q''(0) \geq 0 \Leftrightarrow (q + \lambda)^2 - c\lambda \mu \geq 0 \end{cases} \]

since \( W_q''(0) = \frac{(y_1)^2 (\mu + y_2) - (y_2)^2 (\mu + y_2)}{c(y_1 - y_2)} = \frac{(q + \lambda)^2 - c\lambda \mu}{c^3} \) and that the optimal strategy for the de Finetti problem is the barrier strategy at level \( b_{DeF} \) (see for example Avram et al. (2007), Avram et al. (2019a) Sect. 4).

Plugging now the respective parameters of the de Vylder type approximations in the exact formula for the Cramèr-Lundberg process with exponential claims, we obtain three approximations for \( \tilde{W}_q \):

1. “Naive exponential” approximation obtained by plugging \( \mu^{-1} \rightarrow m_1 \) in Eq. (35) (as was done, for a different purpose) in Eq. (31)
2. Renyi\(^3\), obtained by plugging \( \mu^{-1} \rightarrow \tilde{m}_2, \lambda_R \rightarrow \lambda \frac{m_1}{\tilde{m}_2} \) (since \( c \) is unchanged, the latter equation is equivalent to the conservation of \( \rho = \frac{\tilde{m}_1}{\lambda \tilde{m}_2} \), and to the conservation of \( \theta \), so this coincides with the Renyi ruin approximation used in Eq. (32).)
3. De Vylder, obtained by plugging \( \mu^{-1} \rightarrow \tilde{m}_3, \lambda \rightarrow \lambda \frac{m_1}{\tilde{m}_3}, \tilde{c} = c - \lambda m_1 + \lambda \tilde{m}_3 \).

**Remark 6** In the case of exponential claims, these three approximations are exact, by definition (or check that for exponential claims all the normalized moments are equal to \( \mu^{-1} \)).

**Remark 7** The conditions for the non-negativity of the barrier is \( W_q''(0_+) < 0 \Leftrightarrow \left( \frac{\lambda + q}{c} \right)^2 < \frac{\lambda}{c^2} f(0) \). Here, this condition is satisfied for the exact when \( \theta > \frac{(\lambda + q)^2 (1 - \frac{1}{c^2})}{c^2 f(0)m_1} \).

It is shown in Avram et al. (2019b) [Prop. 1] that the three de Vylder type approximations are two-point Padé approximations of the Laplace transform (hence higher order generalizations are immediately available).

We recall that two-point Padé approximations incorporate into the Padé approximation two initial values of the function (which can be derived easily via the initial value theorem, from the Pollaczek-Khinchine Laplace transform):

\[
\begin{align*}
W_q(0_+) &= \lim_{s \to \infty} s \tilde{W}_q(s) = \frac{1}{c}, \\
W_q'(0_+) &= \lim_{s \to \infty} s \left( \frac{s}{\kappa(s) - q} - W_q(0_+) \right) = \frac{q + \lambda}{c^2}.
\end{align*}
\]

In our case, incorporating both \( W_q(0_+), W_q'(0_+) \) leads to the natural exponential approximation which is therefore the best near \( x = 0 \). Incorporating none of them yields the de Vylder approximation, which is the best asymptotically. Incorporating only \( W_q(0_+) \) leads to Renyi, which is expected to be the best in an intermediate regime.

Note that when the jump distribution has a density \( f \), it holds that :\(^4\)

---

\(^3\) This is called DeVylder B) method in Gerber et al. (2008) [(5.6-5.7)], since it is the result of fitting the first two cumulants of the risk process.

\(^4\) This equation is important in establishing the nonnegativity of the optimal dividends barrier.
Thus, $W''(0)$ already requires knowing $f_c(0)$ (which is a rather delicate task starting from real data); therefore we will not incorporate into the Padé approximation more than two initial values of the function.

We recall below in Proposition 3 three types of two-point Padé approximations (Avram et al. 2019b) [Prop. 1], and particularize them to the case when the denominator degree is $n = 2$ (which are further illustrated below).

**Proposition 3** Three matrix exponential approximations for the scale function.

1. To secure both the values of $W_q(0)$ and $W'_q(0)$, take into account Eqs. (37) and (38), i.e. use the Padé approximation

$$
\hat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, \ \ a_{n-1} = 1, \ b_{n-1} = ca_{n-2} - \lambda - q.
$$

For $n = 2$ we recover the “natural exponential” approximation of plugging $\mu \to \frac{1}{m_1}$ in Eq. (35):

$$
\hat{W}_q(s) \sim \frac{\frac{1}{m_1} + s}{cs^2 + s\left(\frac{c}{m_1} - \lambda - q\right) - \frac{q}{m_1}},
$$

used also (for a different purpose) in Eq. (31).

2. To ensure only $W_q(0) = \frac{1}{e}$, we must use the Padé approximation

$$
\hat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, \ \ a_{n-1} = 1.
$$

For $n = 2$, we find

$$
\hat{W}_q(s) \sim \frac{\frac{2m_1}{m_2} + s}{cs^2 + s\left(\frac{2cm_1 - 2m_2^2 - m_1 q}{m_2}\right) - \frac{2m_1 q}{m_2}} = \frac{\frac{1}{\hat{m}_2} + s}{cs^2 + s\left(\frac{c}{\hat{m}_2} - \frac{\lambda m_1}{\hat{m}_2} - q\right) - \frac{q}{\hat{m}_2}},
$$

where $\hat{m}_2 = \frac{m_2}{2m_1}$ is the first moment of the excess density $f_c(x)$. Note that it equals the scale function of a process with exponential claims of rate $\hat{m}_2^{-1}$ and with $\lambda$ modified to $\lambda = \frac{m_1}{\hat{m}_2}$. Since $c$ is unchanged, the latter equation is equivalent to the conservation of $\rho = \frac{c}{\lambda m_1}$, and to the conservation of $\theta$, so this coincides with the Renyi approximation$^5$ used in Eq. (37).

$^5$ This is called DeVylder B) method in Gerber et al. (2008) [(5.6-5.7)], since it is the result of fitting the first two cumulants of the risk process.
3. The pure Padé approximation yields for \( n = 2 \)

\[
\hat{W}_q(s) \sim \frac{s + \frac{3m_2}{m_3}}{s^2 \left(c - \lambda m_1 + \lambda \frac{3m_2^2}{2m_3}\right) + s \left(\frac{3m_2}{m_3} \frac{3m_1 m_2}{m_3} \lambda - q\right) - \frac{3m_2}{m_3} q} = \frac{s + \frac{1}{m_3}}{\tilde{c}s^2 + s \left(\tilde{c} \frac{1}{m_3} - \lambda - q\right) - \frac{1}{m_3} q}, \quad \tilde{c} = c - \lambda m_1 + \tilde{\lambda} m_3, \quad \tilde{\lambda} = \frac{9m_2^2}{2m_3^2}. \tag{41}
\]

Note that both the coefficient of \( s^2 \) in the denominator coincides with the coefficient \( \tilde{c} \) in the classic de Vylder approximation, since \( \tilde{\lambda} m_3 = \frac{9m_2^2}{2m_3^2} \), and so does the coefficient of \( s \), since

\[
c = \frac{3m_2}{m_3} - \frac{3m_1 m_2}{m_3} \lambda = \tilde{c} \frac{1}{m_3} - \tilde{\lambda} = \left(c - \lambda m_1 + \tilde{\lambda} m_3\right) \frac{1}{m_3} - \tilde{\lambda}.
\]

6 Examples of Computations Involving Scale Function and Dividend Value Approximations

Our goal in this section is to investigate whether exponential approximations are precise enough to yield reasonable estimates for quantities important in control like

1. the dominant exponent \( \Phi_q(x) \) of \( W_q(x) \)
2. the last local minimum of \( W'_q(x), \ b_{\text{DeF}}, \) which yields, when being the global minimum, the optimal De Finetti barrier
3. \( W''_q(0), \) which determines if \( b_{\text{DeF}} = 0 \)
4. the functional \( J_0 \) yielding the maximum dividends with capital injections.

We found out that when the loading coefficient \( \theta \) is large, the best approximation turns out to be the classic de Vylder approximation (which replaces \( \mu^{-1} \) in the exact exponential formula by \( \frac{m_2}{3m_3} \), and both \( \lambda, c \) are modified as well). However, for approximating near the origin, the two point Padé approximation which fixes both the values \( W_q(0) = \frac{1}{c}, \ W'_q(0) = \frac{q + \lambda}{c^2} \) works better. In between \( x = 0 \) and \( x \to \infty \), the winner is sometimes the “Renyi approximation” (which replaces the inverse exponential rate by \( \frac{m_2}{3m_1} \), and modifies \( \lambda \) as well).

All the examples considered involve a Cramèr-Lundberg model with rational Laplace transform \( W_q(s) \) (since in this case, the computation of \( W_q, Z_q \) is fast and in principle arbitrarily large precision may be achieved with symbolic algebra systems).

1. For the first three problems, we will use de Vylder type approximations. Graphs of \( W''_q \), \( W'_q \) and some tables summarizing the simulation results will be presented. We note that in most of the cases that we observed, the de Vylder approximation of \( \Phi_q \) deviates from the exact value the least – see for example Table 2. For the De Finetti barrier, the “winner” depends on the size of \( b_{\text{DeF}} \). Unsurprisingly, when near 0, the natural exponential approximation wins, and as \( b_{\text{DeF}} \) increases, Renyi and subsequently the de Vylder approximation take the upper hand – see for example Table 3.
2. For the computation of \( J_0 \), we provide, besides the exact value, also two approximations:
1. For a given density of claims \( f \) one computes an exponential density approximation \( f_e(x) = \frac{1}{m_1} \exp(-\frac{x}{m_1}) \) where \( m_1 \) is the first moment of \( f \). Subsequently, \( W, Z, J_0 \) and \( a, b \) are obtained using the exponential approximation \( f_e \). Quantities obtained by this method would be referred to with an affix ‘expo pure’.

2. For a given density of claims \( f \), the value function is computed via the formula which assumes exponential claims in Eq. (11), but the “ingredients” \( W, Z, F \) and the mean function \( m \) are the correct ingredients corresponding to our original density \( f \). Quantities obtained by this method would be referred to with an affix ‘expo CI’.

It turns out that the pure expo approximation works better for large \( \theta \), and the correct ingredients approximation works better for small \( \theta \). Note that we only included tables illustrating approximating \( J_0 \) for the first two examples, to keep the length of the paper under control, but similar results were obtained for the other examples.

### 6.1 A Cramér-Lundberg Process with Hyperexponential Claims of Order 2

We take a look at a Cramér-Lundberg process with density function \( f(x) = \frac{2}{3} e^{-x} + \frac{2}{3} e^{-2x} \) with \( \lambda = 1, \theta = 1 \) and \( q = \frac{1}{10} \). Then we get the Fig. 7a.

Tables 1, 2, 3, 4, 5 and 6 provide the results of the different approximations and a comparison is established.

![Fig. 7](image_url) Exact and approximate plots of \( W^\alpha_q(x) \) and \( W''_q(x) \) for \( f(x) = \frac{2}{3} e^{-x} + \frac{2}{3} e^{-2x}, \theta = 1, q = \frac{1}{10} \)

| Dominant exponent \( \Phi_q \) | Percent relative error (\( \Phi_q \)) | Optimal barrier \( b_{DeF} \) | Percent relative error (\( b_{DeF} \)) |
|-----------------------------|-------------------------------|---------------------|-------------------------------|
| Exact                       | 0.110113                      | 3.45398             | 0                             |
| Expo                        | 0.110657                      | 0.494313            | 3.51173                       | 1.67191                      |
| Dev                         | 0.110115                      | 0.00195933          | 3.48756                       | 0.972251                     |
| Renyi                       | 0.110078                      | 0.0321413           | 3.5323                        | 2.26744                      |

Table 1 Exact and approximate values of \( \Phi_q \) and \( b_{DeF} \) for \( f(x) = \frac{2}{3} e^{-x} + \frac{2}{3} e^{-2x}, \theta = 1, q = \frac{1}{10} \), as well as percent relative errors, computed as the absolute value of the difference between the approximation and the exact, divided by the exact, times 100. Relative errors for \( \Phi_q \) are less than 0.5%, with the pure exponential approximation proving to be the worst and the DeVylnder the best approximations, respectively. The optimal barrier \( b_{DeF} \) is also best approximated by DeVylnder, with Renyi being the worst at 2.26%
6.2 A Cramér-Lundberg Process with Hyperexponential Claims of Order 3

Consider a Cramér-Lundberg process with density function

\[ f(x) = \frac{12}{83} e^{-x} + \frac{42}{83} e^{-2x} + \frac{150}{83} e^{-3x}, \]

and \( c = 1, \lambda = \frac{83}{48}, \theta = \frac{263}{235}, p = \frac{263}{498}, q = \frac{5}{48}. \)

The Laplace exponent of this process is

\[ \kappa(s) = s - \frac{12s}{83(s+1)} - \frac{21s}{83(s+2)} - \frac{50s}{83(s+3)}, \]

and from this one can invert \( \frac{1}{\kappa(s) - q} = W_q(s) \) to obtain the scale function \( W_q(x) = -0.0813294 e^{-2.60997x} - 0.179472 e^{-1.68854x} + 0.373887 e^{-0.779311x} + 1.63469 e^{0.18198x}. \)

From this, we see that the dominant exponent is \( \Phi_q = 0.18198. \)

Figure 8 shows the exact and approximate plots of the first two derivatives of \( W_q. \) The exact plots are labelled \( W_{\text{exact}}, \) and coloured as the darkest. The plots of \( W' \) exhibit noticeable unique minima around \( x = 2, \) with the exact one being at \( b_{\text{DeF}} = 1.89732, \) which is the optimal barrier that maximizes dividends here. Note that the approximations are practically indistinguishable from the exact around this point (which is our main object of interest here).

Table 7 provides an exact and approximate values of \( \Phi_q \) and \( b_{\text{DeF}}. \)

Table 8 gives exact and the winning DeVylder approximate values of \( \Phi_q. \)

Table 9 provides results for the exact and approximate values of \( b_{\text{DeF}}. \)

We move now to the dividend problem with capital injections with cost \( k \geq 1 \) as in Theorem 1. One can compute the value function \( J_0 \) at \( x = 0 \) in terms of \( W, Z, C, S, \) and \( G - \) see Eq. (8).

To provide a more concrete example, fixing \( q = \frac{5}{48}, P = 0, k = 3/2 \) as input parameters we compute for values of \( J_0 \) as a function of \( \theta, \) with results summarized in the Tables 9, 10, 11 and 12. The tables provide comparisons of the computed optimal quantities \( J_0, a, \) and \( b \) to an approximation using all exponential inputs (referred to as \( J_0, a, \) and \( b \) expo pure) and to an approximation which uses actual inputs but computed using the exponential formula as described in Eq. (11) (referred to as \( J_0, a, \) and \( b \) expo CI).

\footnote{Laplace inversion done via Mathematica; coefficients and exponents are decimal approximations of the real values.}
To provide a point of comparison, we fix \( q = \frac{5}{48} \), and compute the de Finetti barrier to be \( b_{\text{DeF}} = 1.89732 \) and the corresponding dividend value function when starting at \( x = 0 \) to be \( J_{\text{DeF}} = 1.99847 \). See the Table 13 for a comparaison between values of \( J_0 \) and \( b \) in presence and absence of capital injections.

### 6.3 A Cramér-Lundberg Process with Oscillating Density and Scale Function

In the following example, we study a Cramèr-Lundberg model with density of claims given by

\[
f(x) = u e^{-ax} \cos^2 \left( \frac{\alpha x + \phi}{2} \right) = u e^{-ax} \left( 1 + \cos(\alpha x + \phi) \right) =
\]

\[
e^{-ax} \left( u + u \cos(\phi) \cos(\alpha x) - u \sin(\phi) \sin(\alpha x) \right)
\]

Table 4 Values of \( J_0 \) compared with approximations using all exponential inputs (\( J_0 \) expo pure) and actual inputs but computed using the exponential formula (\( J_0 \) expo CI). The pure exponential approximation does a good job of approximating \( J_0 \) for higher values of \( \theta \) considered, while the exponential CI approximation seemed to fair better for lower \( \theta \) values.
Assuming further that \( a = 1 \), \( \phi = 2 \), \( \omega = 20 \), and that \( \theta = 1 \), \( q = 1/10 \), the Laplace exponent for this process is \( \kappa(s) = \frac{s(2.09898s^3 + 5.29695s^2 + 843.502s + 420.846)(s + 1)}{(s^2 + 2s + 401)} \) and the scale function is

\[
W_q(x) = 0.824723e^{0.0881484x} - 0.348141e^{-0.540677x} + e^{-1.01173x}\cos(19.9957x) \left( - (0.0000285494 + 0.0000804151i)\sin(39.9914x) \\
- (0.0000804151 + 0.0000285494i) + (-0.00000804151 + 0.0000285494i)\cos(39.9914x) \right) \\
+ e^{-1.01173x}\sin(19.9957x) \left( - (0.0000804151 - 0.00000804151i)\sin(39.9914x) \\
+ (0.0000285494 + 0.0000804151i)\cos(39.9914x) - (0.0000285494 - 0.00000804151i) \right). 
\]

### Table 5

| \( \theta \) | \( a \) exact | \( a \) expo pure | \( a \) expo pure error | \( a \) expo CI | \( a \) expo CI error |
|--------------|--------------|------------------|------------------------|---------------|---------------------|
| 1            | 3.9669       | 3.99434          | 0.0691861              | 4.17339       | 0.20551             |
| 0.9          | 3.4372       | 3.45049          | 0.386665               | 3.63512       | 0.5784              |
| 0.8          | 2.92922      | 2.9233           | 0.202204               | 3.11361       | 0.629489            |
| 0.7          | 2.45533      | 2.42622          | 1.18555                | 2.61958       | 0.668958            |
| 0.6          | 2.03051      | 1.97818          | 2.57704                | 2.16741       | 0.67423             |
| 0.5          | 1.66888      | 1.59961          | 4.15022                | 1.77268       | 0.621974            |
| 0.4          | 1.37888      | 1.30566          | 5.31006                | 1.44696       | 0.493725            |
| 0.3          | 1.16063      | 1.10411          | 4.86983                | 1.19323       | 0.280878            |
| 0.2          | 1.00293      | 0.961612         | 4.11969                | 1.0058        | 0.286672            |
| 0.1          | 0.868476     | 0.835496         | 3.79748                | 0.868476      | 0                   |

### Table 6

| \( \theta \) | \( b \) exact | \( b \) expo pure | \( b \) expo pure error | \( b \) expo CI | \( b \) expo CI error |
|--------------|--------------|------------------|------------------------|---------------|---------------------|
| 1            | 1.41036      | 1.46188          | 3.65293                | 1.25374       | 11.045              |
| 0.9          | 1.37645      | 1.44439          | 4.93621                | 1.23362       | 10.3761             |
| 0.8          | 1.31492      | 1.40417          | 6.78809                | 1.19529       | 9.09781             |
| 0.7          | 1.21057      | 1.32258          | 9.25207                | 1.12775       | 6.84178             |
| 0.6          | 1.04634      | 1.17215          | 12.0245                | 1.01753       | 2.7529              |
| 0.5          | 0.810767     | 0.920406         | 13.5229                | 0.853397      | 5.25805             |
| 0.4          | 0.510085     | 0.538725         | 5.61475                | 0.634716      | 24.4335             |
| 0.3          | 0.17425      | 0.0105496        | 93.9457                | 0.376872      | 116.282             |
| 0.2          | 0            | 0                | 0                      | 0.105322      | 100                 |
| 0.1          | 0            | 0                | 0                      | 0             | 0                   |
Figure 9a shows the exact and approximate plots of the first two derivatives of $W_q$. Table 14 provides an exact and approximate values of $\Phi_q$ and $b_{DeF}$.

Clearly, our completely monotone approximation cannot fully reproduce functions like $W_q'(x)$, $W_q''(x)$ in examples like this where oscillations occur (note however that the de Finetti optimal barrier is well approximated here). If a more exact reproduction is necessary, higher order approximations should be used.

7 The Maximal Error of Exponential Approximations $J_0$ Along One Parameter Families of Cramér-Lundberg Processes

In this section, we provide the two approximations for the dividend value with capital injections $J_0$, and the dividend barrier $b$, for two one parameter families of Cramér-Lundberg processes, with densities given respectively by:

$$f(x) = k_\epsilon \left[ e^{-x} + \epsilon e^{-2x} \right]$$  

(42)

$$f(x) = k_\epsilon \left[ \frac{12}{83} e^{-x} + \frac{42}{83} e^{-2x} + \frac{150}{83} e^{-3x} \right]$$  

(43)

where $k_\epsilon$ is the normalization constant, and compute the maximal error of approximation when $\epsilon \in (0, \infty)$ and $\theta \approx 1$. For this choice, the pure exponential approximation works

| Dominant exponent $\Phi_q$ | Percent relative error ($\Phi_q$) | Optimal barrier $b_{DeF}$ | Percent relative error ($b_{DeF}$) |
|---------------------------|---------------------------------|---------------------------|---------------------------------|
| Exact                     | 0.18198                         | 0.189732                  | 0                               |
| Expo                      | 0.184095                        | 1.162215628              | 2.04608                         | 7.840532962 |
| Renyi                     | 0.181708                        | 0.149466974              | 2.08136                         | 9.699997892 |
| Dev                       | 0.182011                        | 0.017034839              | 1.91233                         | 0.79111589  |
considerably better, ∀ε. Table 15 provides values of \( J_0 \) compared with approximations using all exponential inputs (\( J_0 \) expo pure) and actual inputs but computed using the exponential formula (\( J_0 \) expo CI) ∀ε.

Figure 10 depicts the plots of \( J_0 \) values and errors plotted against \( \theta \).

We do the same thing for the family of densities given by \( f(x) = k \left[ \frac{12}{83} e^{-x} + \frac{42}{83} e^{-2x} + \frac{150}{83} e^{-3x} \right] \).

See Table 16 for values of \( J_0 \) compared with approximations using all exponential inputs (\( J_0 \) expo pure) and actual inputs but computed using the exponential formula (\( J_0 \) expo CI) ∀ε, and see Fig. 11 for the plots of \( J_0 \) values and errors as functions of \( \epsilon \).

Table 8

| \( \theta \) | Closest approximation | \( \Phi_q \) exact | \( \Phi_q \) approximation | % error \( \Phi_q \) |
|-----------|----------------------|-------------------|---------------------------|------------------|
| 263/235   | Dev                  | 0.18198           | 0.182011                   | 0.0168217        |
| 243/235   | Dev                  | 0.194712          | 0.194754                   | 0.0213671        |
| 223/235   | Dev                  | 0.209221          | 0.209279                   | 0.0274827        |
| 203/235   | Dev                  | 0.225876          | 0.225957                   | 0.0358309        |
| 183/235   | Dev                  | 0.245146          | 0.245262                   | 0.0474032        |
| 163/235   | Dev                  | 0.267635          | 0.267806                   | 0.0637063        |
| 143/235   | Dev                  | 0.294126          | 0.294382                   | 0.0870647        |
| 123/235   | Dev                  | 0.325643          | 0.326038                   | 0.121115         |
| 103/235   | Dev                  | 0.363539          | 0.364163                   | 0.171618         |
| 83/235    | Dev                  | 0.40961           | 0.410625                   | 0.247788         |
| 63/235    | Dev                  | 0.466261          | 0.46796                    | 0.364457         |
| 43/235    | Dev                  | 0.536719          | 0.539647                   | 0.545532         |
| 23/235    | Dev                  | 0.62533           | 0.630516                   | 0.829419         |
| 3/235     | Dev                  | 0.737962          | 0.747389                   | 1.27736          |

Table 9

| \( \theta \) | Closest approximation | Barrier exact | Barrier approx | % error Barrier |
|-----------|----------------------|---------------|----------------|----------------|
| 263/235   | Dev                  | 1.89732       | 1.91233        | 0.791183       |
| 243/235   | Dev                  | 1.79954       | 1.78002        | 1.08482        |
| 183/235   | Ren                  | 1.45224       | 1.52484        | 4.9989         |
| 163/235   | Ren                  | 1.31579       | 1.33691        | 1.60463        |
| 143/235   | Ren                  | 1.16804       | 1.12368        | 3.79796        |
| 123/235   | Expo                 | 1.00898       | 1.04123        | 3.19653        |
| 103/235   | Expo                 | 0.839228      | 0.794964       | 5.27444        |
| 83/235    | Expo                 | 0.660338      | 0.513179       | 22.2854        |
| 63/235    | Expo                 | 0.474896      | 0.196234       | 58.6785        |
| 43/235    | Expo                 | 0.286563      | 0              | 100            |
| 23/235    | Expo                 | 0.0998863     | 0              | 100            |
| 3/235     | All                  | 0             | 0              | 0              |
8 The Profit Function when the Claims are Distributed According to a Matrix Exponential Jumps Density

Consider now the more general case when the claims are distributed according to a matrix exponential density generated by a row vector $\beta$ and by an invertible matrix $B$.

| $\theta$ | $J_0$ | $J_0$ expo pure | $J_0$ expo pure error | $J_0$ expo CI | $J_0$ expo CI error |
|---------|-------|-----------------|-----------------------|---------------|---------------------|
| 263/235 | 3.7747 | 3.76883 | 0.155556 | 4.11784 | 9.09041 |
| 243/235 | 3.41491 | 3.38603 | 0.845606 | 3.74156 | 9.5654 |
| 223/235 | 3.0636 | 3.00802 | 1.81444 | 3.36985 | 9.99637 |
| 203/235 | 2.72335 | 2.63828 | 3.1238 | 3.00466 | 10.3296 |
| 183/235 | 2.39737 | 2.28225 | 4.80185 | 2.64879 | 10.4871 |
| 163/235 | 2.08958 | 1.94765 | 6.79226 | 2.3062 | 10.3665 |
| 143/235 | 1.80446 | 1.64396 | 8.8946 | 1.9823 | 9.85516 |
| 123/235 | 1.54668 | 1.38072 | 10.73 | 1.68379 | 8.86472 |
| 103/235 | 1.32041 | 1.16526 | 11.7499 | 1.4178 | 7.37587 |
| 83/235 | 1.12864 | 1.00194 | 11.2528 | 1.19022 | 5.45555 |
| 63/235 | 0.972835 | 0.88785 | 8.73585 | 1.00404 | 3.20798 |
| 43/235 | 0.852739 | 0.789923 | 7.36635 | 0.859039 | 0.738837 |
| 23/235 | 0.751597 | 0.701299 | 6.69218 | 0.751597 | 0 |
| 3/235 | 0.660372 | 0.620567 | 6.02761 | 0.660372 | 0 |

| $\theta$ | $a$ | $a$ expo pure | $a$ expo pure error | $a$ expo CI | $a$ expo CI error |
|---------|-----|---------------|---------------------|-------------|------------------|
| 263/235 | 2.51647 | 2.74523 | 0.155536 | 2.51255 | 9.09042 |
| 243/235 | 2.27661 | 2.49437 | 0.845956 | 2.25735 | 9.56541 |
| 223/235 | 2.0424 | 2.24657 | 1.81443 | 2.00535 | 9.99638 |
| 203/235 | 1.81557 | 2.00311 | 3.1238 | 1.75885 | 10.3296 |
| 183/235 | 1.59825 | 1.76586 | 4.80185 | 1.5215 | 10.4871 |
| 163/235 | 1.39306 | 1.53747 | 6.79226 | 1.29844 | 10.3665 |
| 143/235 | 1.20298 | 1.32153 | 8.8946 | 1.09598 | 9.85516 |
| 123/235 | 1.03112 | 1.12252 | 10.73 | 0.920479 | 8.86472 |
| 103/235 | 0.880271 | 0.945198 | 11.7499 | 0.77684 | 7.37587 |
| 83/235 | 0.752428 | 0.793477 | 11.2258 | 0.667962 | 5.45555 |
| 63/235 | 0.648557 | 0.669362 | 8.73585 | 0.5919 | 3.20798 |
| 43/235 | 0.568493 | 0.572693 | 7.36635 | 0.526616 | 0.738838 |
| 23/235 | 0.501065 | 0.501065 | 6.69218 | 0.467532 | 0 |
| 3/235 | 0.440248 | 0.440248 | 6.02761 | 0.413711 | 0 |
of order \(n\), which are such that the vector \(\beta e^{\mathbf{t}^B}\) is decreasing componentwise to 0, and \(\beta \mathbf{1} \neq 0\), with \(\mathbf{1}\) a column vector. As customary, we restrict w.l.o.g. to the case when \(\beta\) is a probability vector, and \(\beta \mathbf{1} = 1\), so that
\[
\bar{F}(x) = \beta e^{\mathbf{t}^B} \mathbf{1}
\]

| \(\theta\) | \(b\) | \(b\) expo pure | \(b\) expo pure error | \(b\) expo CI | \(b\) expo CI error |
|----------|------|----------------|----------------------|--------------|-----------------|
| 263/235 | 0.709355 | 0.805116 | 13.4997 | 0.677918 | 4.43179 |
| 243/235 | 0.695874 | 0.801936 | 15.2416 | 0.671779 | 3.46259 |
| 223/235 | 0.677601 | 0.794377 | 17.2337 | 0.662801 | 2.18425 |
| 203/235 | 0.653005 | 0.779265 | 19.3352 | 0.649805 | 0.490147 |
| 183/235 | 0.620126 | 0.751601 | 21.2012 | 0.631097 | 1.76912 |
| 163/235 | 0.576553 | 0.704104 | 22.123 | 0.604293 | 4.81128 |
| 143/235 | 0.519526 | 0.627369 | 20.7579 | 0.566198 | 8.98366 |
| 123/235 | 0.446259 | 0.511076 | 14.5246 | 0.519261 | 14.947 |
| 103/235 | 0.354524 | 0.346046 | 2.39143 | 0.440755 | 24.3231 |
| 83/235  | 0.243362 | 0.126054 | 48.2032 | 0.347059 | 42.6099 |
| 63/235  | 0.113593 | 0 | 100 | 0.231975 | 104.216 |
| 43/235  | 0 | 0 | 0 | 0.098748 | 0 |
| 23/235  | 0 | 0 | 0 | 0 | 0 |
| 3/235   | 0 | 0 | 0 | 0 | 0 |

Table 13 Values of \(J_0\) and \(b\) in presence of capital injections compared to the case where capital injections are non-existent, \(J_{DeF} = 1.99847\) and \(b_{DeF} = 1.89732\). As \(k\) is increased one can see that \(J_0\) and \(b\) approaches \(J_{DeF}\) and \(b_{DeF}\). This is expected since higher costs of injecting capital makes it less viable, hence it is treated like the concept does not exist

| \(k\) | \(J_0\) % deviation | \(J_0 - J_{DeF}\) | \(J_0\) | \(b\) % deviation | \(b\) |
|-------|---------------------|----------------|-------|-----------------|------|
| 1     | 154.123             | 3.0801         | 5.07857 | 100            | 0    |
| 2     | 65.714              | 1.31337        | 3.31174 | 43.0208        | 1.08108 |
| 3     | 42.863              | 0.856604       | 2.85507 | 25.4792        | 1.4139 |
| 4     | 31.4465             | 0.628448       | 2.62692 | 17.6853        | 1.56178 |
| 5     | 24.6995             | 0.493611       | 2.49208 | 13.4234        | 1.64264 |
| 6     | 20.2855             | 0.4054         | 2.40387 | 10.7759        | 1.69287 |
| 7     | 17.1884             | 0.343504       | 2.34197 | 8.98441        | 1.72686 |
| 8     | 14.9014             | 0.2978         | 2.29627 | 7.69619        | 1.7513 |
| 9     | 13.1463             | 0.262724       | 2.26119 | 6.72732        | 1.76968 |
| 10    | 11.758              | 0.23498        | 2.23345 | 5.97306        | 1.78399 |
| 100   | 1.11095             | 0.022019       | 2.02067 | 0.533448       | 1.8872 |
| 1000  | 0.110409            | 0.0020648      | 2.00067 | 0.0527257      | 1.89632 |
| 10000 | 0.0109536           | 0.000218903    | 1.99869 | 0.00533526     | 1.89722 |

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is a valid survival function.

The matrix versions of our functions are:

\[
\begin{align*}
C_q(x) &= \lambda \int_0^x W_q(x - y) \, \bar{F}(y + \alpha) \, dy = \lambda \tilde{\beta} \int_0^x W_q(x - y) \, e^{yB} \, dy \, e^{aB} \mathbf{1} = C(x)e^{aB} \mathbf{1} \\
m_a(y) &= \int_0^a \frac{y}{z}f(y + z) \, dz = \tilde{\beta} \, e^{aB} \int_0^a \frac{e^{zB}(-B)}{z} \, dz = \tilde{\beta} \, e^{aB}M(a) \mathbf{1} \\
G_a(x) &= \lambda \int_0^x W_q(x - y) \, m_a(y) \, dy = C(x)M(a) \mathbf{1} \\
\end{align*}
\]

where

\[
\begin{align*}
C(x) &= \lambda \int_0^x W_q(x - y) \, e^{yB} \, dy \\
C(x) &= \lambda \tilde{\beta} \int_0^x W_q(x - y) \, e^{yB} \, dy \\
\end{align*}
\]

The product formulas may also be established directly in the phase-type case, using the conditional independence of the ruin probability of the overshoot size.

We derive first these extensions from scratch for \((\tilde{\beta}, B)\) phase-type densities, in order to highlight their probabilistic interpretation. Later, we will show that the matrix exponential jumps case follows as a particular case of Gajek and Kučiński (2017).

Recall first (Albrecher and Asmussen 2010) that \(\Psi_q(x) = \bar{\Psi}_q(x) \mathbf{1}\), where \(\bar{\Psi}_q(x)\) is a vector whose components represent the probability that ruin occurs during a certain phase, and that the conditional independence of ruin and overshoots translates into the product formula

\[
\Psi_q(x, y) := P_x[T_{0-} < \infty, X_{T_{0-}} < -y] = \bar{\Psi}_q(x)e^{yB} \mathbf{1}.
\]

### Table 14

| Methodology and Computing in Applied Probability (2022) 24:2339–2371 |
|---|---|---|---|---|
| Dominant exponent \(\Phi_q\) | Percent relative error (\(\Phi_q\)) | Optimal barrier \(b_{DeF}\) | Percent relative error (\(b_{DeF}\)) |
| Exact | 0.0881484 | 0 | 4.38201 | 0 |
| Expo | 0.0878658 | 0.32053 | 4.42263 | 0.927122 |
| Renyi | 0.0881481 | 0.000314617 | 4.39788 | 0.362284 |
| Dev | 0.0881484 | 6.11743*10^{-6} | 4.39745 | 0.352331 |

The DeVylder approximation wins on both fronts.
Table 15 \( \lambda = 1, \theta = 1, q = \frac{1}{10}, k = 3/2 \) and \( P = 0 \). As expected, the errors decrease both as \( \epsilon \) goes to zero and infinity since the densities approach an exponential density

| \( \epsilon \)  | J0 exact | J0 expo pure | J0 expo pure error | J0 expo CI | J0 expo CI error |
|------|----------|-------------|------------------|-----------|------------------|
| 0.001 | 7.1879   | 7.18802     | 0.0016603        | 7.18849   | 0.00827967       |
| 0.01  | 7.17075  | 7.17193     | 0.0164663        | 7.17666   | 0.0824782        |
| 0.1   | 7.008    | 7.01863     | 0.151653         | 7.06358   | 0.793041         |
| 1     | 5.95034  | 5.99151     | 0.691856         | 6.26009   | 5.20551          |
| 10    | 4.20175  | 4.19406     | 0.183122         | 4.40089   | 4.73941          |
| 100   | 3.66909  | 3.6654      | 0.100631         | 3.69555   | 0.721228         |
| 1000  | 3.6025   | 3.60208     | 0.0117065        | 3.60523   | 0.075585         |

To take advantage of this, it is convenient to replace from the beginning \( Z_q(x) \) by \( \Psi_q(x) \), taking advantage of the formula (Avram et al. 2004; Kyprianou 2014)

\[
Z_q(x) = \Psi_q(x) + W_q(x) \frac{q}{\Phi_q} \implies C(x) = (c - \frac{q}{\Phi_q})W_q(x) - \Psi_q(x).
\] (47)

Alternatively, one may introduce a vector function

\[
\overline{Z}_q(x) := \Psi_q(x) + W_q(x) \frac{q}{\Phi_q} 1.
\] (48)

On the other hand, the mean function may be written as

\[
m_a = \int_0^a y F(dy) \approx -a F(a) + \int_0^a \overline{F}(x)dx = \overline{\beta} M(a)1, M(a)
\]

\[
= -B^{-1} - e^{a\overline{B}}(aI_n - B^{-1}).
\]

The following result follows in the phase-type case just as in the exponential case (Avram et al. 2021).

**Proposition 4** For a Cramèr-Lundberg process (compound Poisson) with matrix exponential jumps of type \((\overline{\beta}, B)\), it holds that

Fig. 10 \( J_0 \) values and errors plotted against \( \epsilon \). Errors peak at \( \epsilon = 1 \)
Table 16 \( \lambda = 1, c = 1, q = \frac{5}{4}, k = 3/2 \) and \( P = 0 \). As \( \varepsilon \) goes to zero, the density becomes exponential hence the decrease in errors. As \( \varepsilon \) goes to infinity, the density approaches a hyper exponential density of order 2, but still both methods of approximating \( J_0 \) yield reasonable results

| \( \varepsilon \) | \( J_0 \) exact | \( J_0 \) expo pure | \( J_0 \) expo pure error | \( J_0 \) expo CI | \( J_0 \) expo CI error |
|---|---|---|---|---|---|
| 0.001 | 7.95508 | 7.95771 | 0.0330111 | 7.96565 | 0.132796 |
| 0.01 | 7.68765 | 7.71127 | 0.307292 | 7.78772 | 1.30166 |
| 0.1 | 6.06176 | 6.15381 | 1.5186 | 6.62641 | 9.31507 |
| 1 | 3.7747 | 3.76883 | 0.155556 | 4.11784 | 9.09041 |
| 10 | 3.1382 | 3.1379 | 0.00959692 | 3.23354 | 3.03813 |
| 100 | 3.05894 | 3.06427 | 0.174306 | 3.12284 | 2.08921 |
| 1000 | 3.0508 | 3.05678 | 0.196219 | 3.11149 | 1.98956 |

\[
J_0 = \begin{cases} 
  kG_a(x) + J_0 S_a(x) = kG_a(x) + \frac{1-kG'(b)}{S'(b)} S_a(x), & x \in [0, b] \\
  kx + J_0, & x \in [-a, 0] \\
  0, & x \leq -a
\end{cases}
\]

(49)

1. where

\[
\begin{align*}
  C(x) &= \lambda \int_{0}^{x} W_q(x-y) e^{\beta y} dy \\
  C(x) &= \lambda \beta \int_{0}^{x} W_q(x-y) e^{\beta y} dy \\
  G_a(x) &= C_q(x) M(a) 1 \\
  R_a(x) &= S_a(x) - Z_q(x) = C_q(x) e^{\alpha B} 1
\end{align*}
\]

and

\[
J_0 = \frac{1 - kC_q'(b) M(a) 1}{qW_q(b) + C_q'(b) e^{\alpha B} 1}.
\]

(51)

2. For fixed \( a \), the optimality equation \( \frac{\partial}{\partial b} J_0^{a,b} = 0 \) simplifies to

Fig. 11 \( J_0 \) values and errors plotted against \( \varepsilon \). Errors peak at \( \varepsilon = 0.1 \).
\[ J_0 = \frac{k\overline{C}'(b) M(a)1}{qW'(b) + \overline{C}'(b)e^{ab}1}. \] (52)

**Remark 8** The additive separation of \(a; b\) which was the basis of proving optimality in the exponential case does not seem possible anymore, but Eq. (51) allows the numeric computation of the optimum.

**Appendix: The Proof of Theorem 1 when \(\sigma \equiv 0\)**

To simplify our readers’ journey, we sketch here the main elements of proof, in the case \(\sigma \equiv 0\), generalizing Avram et al. (2021). Please note that the main modification when diffusion is present applies to the computation of the term \(I_y\) below, in which \(\sigma^2\) will appear accompanying a Dirac mass in the Gerber-Shiu measure.

We begin by applying the strong Markov property at the stopping time \(\tau^x := \tau_{0^-}^x \wedge \tau^x_{b^+} = \inf \{ t \geq 0 : X_t^x < 0 \} \wedge \inf \{ t \geq 0 : X_t^x > b \}\). It follows that, for \(0 \leq x \leq b\),

\[
J_x = \mathbb{E}_x \left[ e^{-q\tau_{0^-}^x} 1_{\tau_{0^-}^x < \tau^x_{b^+}} J_b + \mathbb{E}_x \left[ e^{-q\tau^x_{b^-}} 1_{\tau^x_{b^-} > \tau_{0^-}^x} \left( J_0 + kX_{\tau_{0^-}^x} \right) 1_{X_{\tau_{0^-}^x} \geq -a} \right] \right]
- P\mathbb{E}_x \left[ e^{-q\tau_{0^-}^x} 1_{\tau_{0^-}^x > \tau^x_{b^-}} 1_{X_{\tau^x_{b^-}} < -a} \right] = \frac{W_a(x)}{W_q(b)} \left( J_b - I_b \right) + I_x,
\] (53)

where

\[
I_y := \mathbb{E}_y \left[ e^{-q\tau_{0^-}^0} \left( (J_0 + kX_{\tau_{0^-}^0}) 1_{X_{\tau_{0^-}^0} \geq -a} - P1_{X_{\tau_{0^-}^0} < -a} \right) \right].
\]

The term \(I_y\) can be explicitly computed (using the Gerber-Shiu measure).

\[
I_y = \int_{\mathbb{R}_+} \left[ (J_0 - ku) 1_{u \leq \sigma \leq -P} 1_{u > -a} \right] \int_{\mathbb{R}_+} (e^{-\Phi(q)v} W_q(y) - W_q(y - v)) \nu(du + dv)
+ \int_{\mathbb{R}_+} (J_0(\bar{v}(v) - \bar{v}(a + v)) - km_a(v) - P\bar{v}(a + v))(e^{-\Phi(q)v} W_q(y) - W_q(y - v)) dv
= W_q(y) \int_{\mathbb{R}_+} (J_0(\bar{v}(v) - \bar{v}(a + v)) - km_a(v) - P\bar{v}(a + v)) e^{-\Phi(q)v} dv
+ G_a(y) - J_0(C(y) - C_a(y)).
\]

Since the term accompanying \(W_q(y)\) is a constant, by replacing this in Eq. (53), it follows that

\[
J_x - \left( G_a(x) - J_0(C(x) - C_a(x)) \right) = \frac{W_a(x)}{W_q(b)} \left( J_b - \left( G_a(b) - J_0(C(b) - C_a(b)) \right) \right). \] (54)

With the particular choice of \(x = 0\), by recalling that \(C_a(0) = G_a(0) = C(0) = 0\), the last equation yields
which, combined with \( W_q(0) = \frac{1}{\epsilon} \) and \( C(x) = cW_q(x) - Z_q(x) \), leads to

\[
J_x = G_a(x) - J_0 \left( C(x) - C_a(x) - cW_q(x) \right) = G_a(x) + J_0 \left( C_a(x) + Z_q(x) \right).
\]

Theorem 1 is now proven, in the case \( \sigma \equiv 0 \).

Data Availability The datasets generated during the current study are available from the corresponding author upon request.

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