Linear adjoint restriction estimates for paraboloid

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Abstract

We prove a class of modified paraboloid restriction estimates with a loss of angular derivatives for the full set of paraboloid restriction conjecture indices. This result generalizes the paraboloid restriction estimate in radial case from [Shao, Rev. Mat. Iberoam. 25(2009), 1127–1168], as well as the result from [Miao et al. Proc. AMS 140(2012), 2091–2102]. As an application, we show a local smoothing estimate for a solution of the linear Schrödinger equation under the assumption that the initial datum has additional angular regularity.

Keywords  Linear adjoint restriction estimate · Local restriction estimate · Bessel function · Spherical harmonics · Local smoothing

Mathematics Subject Classification  42B37 · 42B10 · 42B25 · 35Q55

1 Introduction

Let \( S \) be a non-empty smooth compact subset of the paraboloid,

\[
\left\{ \left( \tau, \xi \right) \in \mathbb{R} \times \mathbb{R}^n : \tau = \| \xi \|^2 \right\},
\]

where \( n \geq 1 \). We denote by \( d\sigma \) the pull-back of the \( n \)-dimensional Lebesgue measure \( d\xi \) under the projection map \( (\tau, \xi) \mapsto \xi \). Let \( f \) be a Schwartz function and define the inverse space-time Fourier transform of the measure \( f d\sigma \).
\[ (fd\sigma)^\vee(t, x) = \int_S f(\tau, \xi)e^{2\pi i (x \cdot \xi + t\tau)}d\sigma(\xi) \]
\[ = \int_{\mathbb{R}^n} f(|\xi|^2, \xi)e^{2\pi i (x \cdot \xi + t|\xi|^2)}d\xi. \]

The classical linear adjoint restriction estimate for the paraboloid reads
\[ \| (fd\sigma)^\vee \|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p,q,n,S} \| f \|_{L^p(S; d\sigma)}, \]
where \(1 \leq p, q \leq \infty\). The famous restriction problem is to find the optimal range of \(p\) and \(q\) such that the estimate (1.2) holds. It is known that the condition
\[ q > \frac{2(n + 1)}{n} \quad \text{and} \quad \frac{n + 2}{q} \leq \frac{n}{p}, \]
is necessary for (1.2), see [24,29]. Here \(p'\) denotes the conjugate exponent of \(p\). The adjoint restriction estimate conjecture on paraboloid reads as follows.

**Conjecture 1.1** The inequality (1.2) holds true if and only if inequalities (1.3) are valid.

There is a large amount of literature on this problem. For \(n = 1\), Conjecture 1.1 was proved by Fefferman-Stein [11] for the non-endpoint case and by Zygmund [36] for the endpoint case. Conjecture 1.1 in high dimension case becomes much more difficult. For \(n \geq 2\), Tomas [33] showed (1.2) for \(q > 2(n + 2)/n\), and Stein [25] fixed the limit case \(q = 2(n + 2)/n\). Bourgain [1] further proved estimate (1.2) for \(q > 2(n + 2)/n - \epsilon_n\) with some \(\epsilon_n > 0\); in particular, \(\epsilon_n = \frac{2}{15}\) when \(n = 2\). Further improvements were made by Moyua-Vargas-Vega [16] and Wolff [34]. Tao [31] used the bilinear argument to show that estimate (1.2) holds true for \(q > 2(n + 3)/(n + 1)\) with \(n \geq 2\). This result was improved by Bourgain-Guth [2] when \(n \geq 4\). This conjecture is so difficult that it remains open up to now. For more details, we refer the reader to [2,29–32,34].

On the other hand, the restriction conjecture becomes simpler (but not trivial) when a test function has some angular regularity. For example, Conjecture 1.1 is proved by Shao [22] when test functions are cylindrically symmetric and are supported on a dyadic subset of the paraboloid in the form of
\[ \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : M \leq |\xi| \leq 2M, \quad \tau = |\xi|^2, \quad M \in 2\mathbb{Z} \}. \]

Indeed, many famous conjectures in harmonic analysis (such as Fourier restriction estimates, Bochner-Riesz estimate etc.) have easier counterparts when the corresponding operators act on radial functions. Let \(S^{n-1}\) denote the unit sphere in \(\mathbb{R}^n\) and \(L^{q}_{\text{sph}} := L^q(S^{n-1})\), the intermediate situation is to replace the \(L^q(\mathbb{R}^n)\) by \(L^q_{\text{sph}} = L^q_{r=1, \ldots, d} L^2_{\text{sph}}\) in (1.2). This intermediate case has been settled for adjoint restriction estimates for a cone by the authors of [17]. More precisely, if \(S\) is a non-empty smooth compact subset of the cone:
\[ S = \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi| \}, \]
then for \(q > 2n/(n - 1)\) and \((n + 1)/q \leq (n - 1)/p'\) we have
\[ \| (fd\sigma)^\vee \|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p,q,n,S} \| f \|_{L^p(S; d\sigma)}. \]

The \(L^q_{\text{sph}}\)-norm allows us to use spherical harmonic expanding, so the problem is converted to \(L^q(\ell^2)\)-bounds for sequences of operators \(\{H_k\}\) where each \(H_k\) is an operator acting on radial...
functions. The pioneering paper using such intermediate space is the Mockenhaupt Diploma in which he proved weighted $L^p$ inequalities and then sharp $L^p_{rad}(L^2_{sph}) \to L^p_{rad}(L^2_{sph})$ estimates for the disc multiplier operator, see either Mockenhaupt [14] or Córdoba [5]. Sharp endpoint bounds for the disk multiplier were obtained by Carbery-Romera-Soria [4]. Müller-Seeger [15] established some sharp mixed space-time $L^p_{rad}(L^2_{sph}) \to L^p_{rad}(L^2_{sph})$ estimates in order to study a local smoothing of solutions for the linear wave equation. Córdoba-Latorre [9] revisited some classical conjecture including restriction estimate in harmonic analysis in this kind of mixed space-time. Gigante-Soria [12] studied a related mixed norm problem for Schrödinger maximal operators. Concerning the sphere restriction conjecture, Carli-Grafakos [7] also treated the same problem for spherically-symmetric functions and Cho-Guo-Lee [8] showed a restriction estimate for $q > 2(n + 1)/n$ and $s \geq (n + 2)/q - n/2$

$$\left\| \int_{\mathbb{S}^n} e^{2\pi i x \cdot \xi} f(\xi) d\sigma(\xi) \right\|_{L^q(\mathbb{R}^{n+1})} \leq C \| f \|_{H^s(\mathbb{S}^n)}, \quad x \in \mathbb{R}^{n+1}, \quad (1.5)$$

where $d\sigma$ is the induced Lebesgue measure on $\mathbb{S}^n$ and $H^s(\mathbb{S}^n)$ denote the $L^2$-Sobolev space of order $s$ on the sphere. An advantage of the proof consists in a fact that inequality (1.5) is based on $L^2$-spaces. The advantage of using the $L^2$-based Hilbert space also allows us to use effective the $TT^*$ arguments to obtain Strichartz estimate with a wider range of admissible indexes by compensating with extra regularity in angular direction; see Sterbenz [21] for wave equation, Cho-Lee [9] for general dispersive equations and the authors [18] for wave equation with an inverse-square potential. Concerning other results in this direction, Cho-Hwang-Kwon-Lee [10] studied profile decompositions of fractional Schrödinger equations under the angular regularity assumption.

In this paper, we prove that estimate (1.2) holds for all $p, q$ in (1.3) by compensating with some loss of angular derivatives. Our strategy is to use a spherical harmonic expanding as well as localized restriction estimates. In contrast to the radial case, e.g. [7,22], the main difficulty comes from the asymptotic behavior of the Bessel function $J_\nu(r)$ when $\nu \gg 1$. It is worth to point out that the method of treating cone restriction [17] is not valid since it can not be used to exploit the curvature property of paraboloid multiplier $e^{it|\xi|^2}$. We note that the bilinear argument used in [22], which is in spirit of Carleson-Sjölin argument or equivalently the $TT^*$ argument, can be used to deal with the oscillation of the paraboloid multiplier. To use this argument, one needs to write the Bessel function $J_\nu(r) \sim c_\nu r^{-1/2}e^{ir}$ when $r \gg 1$. This expression works well for small $\nu$ (corresponding to the radial case) but it seems complicate to write the Bessel function in that form when $\nu \gg 1$. Indeed, as in [37], one can do this when $\nu^2 \ll r$, but it will cause more loss of derivative for the case $\nu \lesssim r \lesssim \nu^2$, since it is difficult to capture simultaneously the oscillation and decay behavior of $J_\nu(r)$. Our new idea here is to establish a $L^4_{t,x}$-localized restriction estimate by directly analyzing the kernel associated with the Bessel function. The key ingredient is to explore the decay and oscillation property of $J_\nu(r)$ for $r \gg \nu$, and resonant property of paraboloid multiplier. We also have to overcome low decay shortage of $J_\nu(r)$ (when $\nu \sim r \gg 1$) by compensating a loss of angular regularity.

Before stating the main theorem, we introduce some notation. Incorporating the angular regularity, we set the infinitesimal generators of the rotations on Euclidean space:

$$\Omega_{j,k} := x_j \partial_k - x_k \partial_j$$
and define for $s \in \mathbb{R}$

$$\Delta_{\theta} := \sum_{j < k} \Omega_{j,k}^2, \quad |\Omega|^2 = (-\Delta_{\theta})^{\frac{1}{2}}.$$ 

Hence $\Delta_{\theta}$ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Define the Sobolev norm $\| \cdot \|_{H^{s}_{\text{sph}}(\mathbb{R}^n)}$ by setting

$$\|g\|_{H^{s}_{\text{sph}}(\mathbb{R}^n)}^p = \int_{\mathbb{S}^{n-1}} \int_{0}^{\infty} |(1 - \Delta_{\theta})^{s/2} g(r \theta)|^{p} d\theta \, r^{n-1} dr. \quad (1.6)$$

Given a constant $A$, we briefly write $A + \epsilon$ as $A_{+}$ or $A - \epsilon$ as $A_{-}$ for $0 < \epsilon \ll 1$.

Our main result is the following one.

**Theorem 1.1** Let $n \geq 2$. The following estimates hold for all Schwartz functions $f$

- if $q_0 = (2(n + 1)/n)_+$ and $(n + 2)/q_0 = n/p_0'$, then
  $$\| (fd\sigma)^{\vee} \|_{L^{q}_f(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p,q_0,n,r,s} \| f(\xi) \|_{H^{s}_{\text{sph}}(\mathbb{R}^n)}$$
  $$\quad \times \| (1 + |\Omega|)^{\frac{1}{2}} f \|_{L^{p}(\mathbb{S}^{n-1},d\sigma)}, \quad (1.7)$$
  where $\sigma_0 = (n - 2)(\frac{1}{2} - \frac{1}{q_0}) + \frac{2}{q_0}$;

- if $1 \leq q, p \leq \infty$ satisfy (1.3), then
  $$\| (fd\sigma)^{\vee} \|_{L^{q}_f(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p,q,n,s} \| (1 + |\Omega|)^{\frac{1}{2}} f \|_{L^{p}(\mathbb{S}^{n-1},d\sigma)}, \quad (1.8)$$

where $s = s(q,n) = \sigma_0 \alpha$ and $0 \leq \alpha \leq 1$ satisfying $1/q = \alpha/q_0 + (1 - \alpha)/q_1$. Here $q_1 = q(n)_+$ with $q(n) = 2 + 12/(4n + 1 - k)$ if $n + 1 \equiv k(\text{mod } 3), k = -1, 0, 1$ as in Bourgain-Guth [2, Theorem 1].

**Remark 1.1** Estimate (1.8) is an interpolation consequence of (1.7) and $L^p$-estimates in Bourgain-Guth [2]. Inequality (1.8) leads to the linear adjoint restriction estimate when $q \in (2(n + 1)/n, q(n))$ with some loss of angular derivatives.

**Remark 1.2** Since the sphere $\mathbb{S}^n = \{(\tau, \xi) : |\tau|^2 + |\xi|^2 = 1\}$ is closely related to the paraboloid in sense of Taylor expansion $\sqrt{1 - \rho^2} = 1 - \frac{1}{2} \rho^2 + O(\rho^4)$ near $\rho = 0$, it seems to be possible to show some modified version of (1.5) with $H^{s,p}(\mathbb{S}^n)$-norm on right hand side.

As an application of the modified restriction estimate, we show a result on the local smoothing estimate for the Schrödinger equation for initial data with additional conditions angular regularity by Rogers’ argument in [20]. Our result here extend [20, Theorem 1] from $q > 2(n + 3)/(n + 1)$ to $q > 2(n + 1)/n$ under the assumption that initial data has additional angular regularity.

More precisely, we have the following local smoothing result.

**Corollary 1.1** Let $n \geq 2, q > 2(n + 1)/n$ and $s$ be as in Theorem 1.1. Then

$$\| e^{it\Delta} u_0 \|_{L^q_t(L_r^p([0,1] \times \mathbb{R}^n))} \leq C \| (1 + |\Omega|)^{\frac{1}{2}} u_0 \|_{W^{\alpha,q}(\mathbb{R}^n)}, \quad (1.9)$$

where $\alpha > 2n(1/2 - 1/q) - 2/q$ and $W^{\alpha,q}(\mathbb{R}^n)$ is the Sobolev space.

This paper is organized as follows: In Sect. 2, we introduce notation and present some basic facts about spherical harmonics and Bessel functions. Furthermore, we use the stationary phase argument to prove some properties of Bessel functions. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we prove the key Proposition 3.1. We prove Corollary 1.1 in the final section.
2 Preliminaries

2.1 Notation

We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some large constant $C$ which may vary from line to line and depend on various parameters, and similarly employ $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. We also use $A \ll B$ to denote the statement $A \leq C^{-1}B$. If a constant $C$ depends on a special parameter other than the above, we shall write it explicitly by subscripts. For instance, $C_\epsilon$ should be understood as a positive constant not only depending on $p$, $q$, $n$ and $S$, but also on $\epsilon$. Throughout this paper, pairs of conjugate indices are written as $p, p'$, where $1/p + 1/p' = 1$ with $1 \leq p \leq \infty$. Let $R > 0$ be a dyadic number, we define the dyadic annulus in $\mathbb{R}^n$ by

$$A_R := \{ x \in \mathbb{R}^n : R/2 \leq |x| \leq R \}, \quad S_R := [R/2, R].$$

For each $M \in 2\mathbb{Z}$, we define $\mathcal{L}_M$ to be the class of Schwartz functions supported on a dyadic subset of the paraboloid in the form of

$$\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : M \leq |\xi| \leq 2M, \tau = |\xi|^2 \}.$$  (2.1)

2.2 Spherical harmonics expansions and Bessel function

We recall an expansion formula with respect to the spherical harmonics. Let

$$\xi = \rho \omega \quad \text{and} \quad x = r \theta \quad \text{with} \quad \omega, \theta \in \mathbb{S}^{n-1}. \quad (2.2)$$

For every $g \in L^2(\mathbb{R}^n)$, we have the expansion formula

$$g(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(\rho) Y_{k,\ell}(\omega),$$

where

$$\{Y_{k,1}, \ldots, Y_{k,d(k)}\}$$

is the orthogonal basis of the spherical harmonics space of degree $k$ on $\mathbb{S}^{n-1}$. This space is recorded by $\mathcal{H}^k$ and it has the dimension

$$d(k) = \frac{2k + n - 2}{k} C_{n+k-3}^{k-1} \simeq \langle k \rangle^{n-2}. \quad (2.3)$$

It is clear that we have the orthogonal decomposition of $L^2(\mathbb{S}^{n-1})$

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k. \quad (2.3)$$

It follows that

$$\|g(\xi)\|_{L^2_\omega} = \|a_{k,\ell}(\rho)\|_{\ell^2_\omega}. \quad (2.3)$$
Using the spherical harmonic expansion, as well as \cite{19,28}, we define the action of \((1-\Delta_\omega)^{s/2}\) on \(g\) as follows

\[
(1-\Delta_\omega)^{s/2} g = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1 + k(k+n-2))^{s/2} a_k,\ell(\rho) Y_{k,\ell}(\omega).
\]

(2.4)

Given \(s, s' \geq 0\) and \(p, q \geq 1\), define

\[
\|g\|_{H^s_{\mu(\rho)} H^{s'}_{\omega}} := \left\| (1-\Delta_\omega)^{s/2} \left( (1-\Delta_\omega)^{s'/2} g \right) \right\|_{L^q_{\mu}(\mathbb{R}^+; L^p_{\omega}(S^{n-1}))},
\]

where \(\mu(\rho) = \rho^{n-1} d\rho\).

For our purpose, we need the inverse Fourier transform of \(a_k,\ell(\rho) Y_{k,\ell}(\omega)\). We recall the Bochner-Hecke formula, see \cite{13} and \cite[Theorem 3.10]{26}

\[
\hat{g}(r,\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\theta) r^{-\nu_2} \int_0^{\infty} J_{\nu}(k(2\pi r \rho)) a_k,\ell(\rho) \rho^{\nu_2} d\rho.
\]

(2.5)

Here \(\nu(k) = k + \frac{n-2}{2}\) and the Bessel function \(J_\nu(r)\) of order \(\nu\) is defined by

\[
J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2) \Gamma(1/2)} \int_{-1}^{1} e^{isr}(1-s^2)^{(2\nu-1)/2} ds,
\]

where \(\nu > -1/2\) and \(r > 0\). It is easy to verify that there exists a constant \(C\) independent of \(\nu\) such that

\[
|J_\nu(r)| \leq \frac{Cr^\nu}{2\nu \Gamma(\nu + 1/2) \Gamma(1/2)} \left(1 + \frac{1}{\nu + 1/2}\right).
\]

(2.6)

To investigate a behavior of asymptotic bound on \(\nu\) and \(r\), we recall the Schl"afli integral representation \cite{35} of the Bessel function: for \(r \in \mathbb{R}^+\) and \(\nu > -\frac{1}{2}\)

\[
J_\nu(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r \sin \theta - i\nu \theta)} d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^{\infty} e^{-(r \sinh s + \nu s)} ds
\]

\[
=: \tilde{J}_\nu(r) - E_\nu(r).
\]

(2.7)

Clearly, \(E_\nu(r) = 0\) when \(\nu \in \mathbb{Z}^+\). An easy computation shows that

\[
|E_\nu(r)| \leq \left| \frac{\sin(\nu \pi)}{\pi} \int_0^{\infty} e^{-(r \sinh s + \nu s)} ds \right| \leq C(r + \nu)^{-1}.
\]

(2.8)

There is a number of references for the asymptotic behavior of a Bessel function, see e.g. \cite{9,23,25,35}. We recall some properties of a Bessel function for a convenience.

**Lemma 2.1 (Asymptotics of Bessel functions)** Let \(\nu \gg 1\) and let \(J_\nu(r)\) be the Bessel function of order \(\nu\) defined as above. Then there exists a large constant \(C\) and small constant \(c\) independent of \(\nu\) and \(r\) such that:

- When \(r \leq \frac{\nu}{2}\), we have
  \[
  |J_\nu(r)| \leq Ce^{-(\nu+r)};
  \]

(2.9)
• When \( \nu \leq r \leq 2\nu \), we have
\[
|J_\nu(r)| \leq C\nu^{-\frac{1}{2}}(\nu^{-\frac{1}{2}}|r - \nu| + 1)^{-\frac{1}{4}}; \tag{2.10}
\]

• When \( r \geq 2\nu \), we have
\[
J_\nu(r) = r^{-\frac{1}{2}} \sum \pm a_{\pm}(v, r)e^{\pm ir} + E(v, r), \tag{2.11}
\]

where \( |a_{\pm}(v, r)| \leq C \) and \( |E(v, r)| \leq Cr^{-1} \).

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using some localized linear estimates whose proof are postpone to the next section. Since inequality (1.7) is a special case of (1.8), we aim to prove (1.8). Since (1.8) is a direct consequence of the Stein-Tomas inequality [25] for the case \( p \leq 2 \), it suffices to prove (1.8) for the case \( p \geq 2 \). More precisely, we will only establish the estimate for
\[
q > \frac{2(n + 1)}{n}, \quad \frac{n + 2}{q} = \frac{n}{p'}, \quad p' \geq 2
\]

Recall the notation \( \mathbb{L}_M \) and \( A_R \) in the Sect. 2.1. We decompose \( f \) into a sum of dyadic supported functions
\[
f = \sum_M f_M,
\]

where \( f_M = f \chi_{\tau(\xi, \tau) = |\xi|^2, M \leq |\xi| \leq 2M} \in \mathbb{L}_M \). It follows that
\[
\|(fd\sigma)^\vee\|_{L^q_t(L^q_x(R \times R^n))} = \left( \sum_R \left( \sum_M \|(f_Md\sigma)^\vee\|_{L^q_t(L^q_x(R \times A_R))} \right)^q \right)^{\frac{1}{q}} \leq \left( \sum_R \left( \sum_M \|(f_Md\sigma)^\vee\|_{L^q_t(L^q_x(R \times A_R))} \right)^q \right)^{\frac{1}{q}}. \tag{3.2}
\]

To prove (3.1), we need localized linear restriction estimates.

Proposition 3.1 Assume \( f \in \mathbb{L}_1 \) and \( R > 0 \) is a dyadic number. Then the following linear restriction estimates hold true.

• Let \( q = 2 \), then
\[
\|(fd\sigma)^\vee\|_{L^q_t(L^q_x(R \times A_R))} \lesssim \min \left\{ R^{\frac{1}{2}}, R^{\frac{2}{2}} \right\} \|f\|_{L^2(S; d\sigma)}. \tag{3.3}
\]

• Let \( q = 3p' \) with \( 2 \leq p \leq 4 \) and \( \sigma = (n - 2)(\frac{1}{2} - \frac{1}{q}) + \frac{2}{q}, \ 0 < \varepsilon \ll 1 \), then
\[
\|(fd\sigma)^\vee\|_{L^q_t(L^q_x(R \times A_R))} \lesssim \min \left\{ R^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \varepsilon}, R^{\frac{2}{q}} \right\} \|(1 + |\Omega|)^{\sigma} f\|_{L^{p'}(S; d\sigma)}. \tag{3.4}
\]
We postpone the proof of Proposition 3.1 to the next section, and we complete the proof of Theorem 1.1 by this proposition. By a scaling argument, we conclude from (3.3) that
\[
\|(fM d\sigma)^\vee\|_{L^q_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \min \left\{ (RM)^{\frac{1}{2}}, (RM)^{\frac{a}{2}} \right\} M^{n-\frac{n+2}{q} - \frac{n}{2}} \|fM\|_{L^2(S; d\sigma)}.
\]
For any \((q, p)\) satisfying
\[
q > 2(n + 1)/n, \quad (n + 2)/q = n/p' \quad \text{with} \quad p \geq 2,
\]
let \(\alpha = 2 - \frac{3}{q} - \frac{1}{p}\), then we choose \(\bar{q} = 3 \bar{p}'\) such that
\[
\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{q}, \quad \frac{1}{p} = \frac{1}{2} + \frac{\alpha}{p}.
\]
From (3.4), we have that for \(\bar{q} = 3 \bar{p}'\) with \(2 \leq \bar{p} \leq 4\) and \(\tilde{\sigma} = (n - 2)(\frac{1}{2} - \frac{1}{q}) + \frac{2}{q}\)
\[
\| (fM d\sigma)^\vee \|_{L^q_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \min \left\{ (RM)^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \epsilon}, (RM)^{\bar{q}} \right\} M^{n-\frac{n+2}{q} - \frac{n}{p} - \frac{\alpha}{q}} \left\{ (1 + |\Omega|)^{\tilde{\sigma}} fM \right\}_{L^{\bar{p}}(S; d\sigma)},
\]
where \(0 < \epsilon \ll 1\). Therefore we obtain by an interpolation theorem
\[
\| (fM d\sigma)^\vee \|_{L^q_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \min \left\{ (RM)^{\frac{a}{2}}, (RM)^{-\frac{n-1}{2} \left[ 1 - \frac{2(n+1)}{qn} \right] + \epsilon} \right\} \left\{ (1 + |\Omega|)^{\sigma} fM \right\}_{L^p(S; d\sigma)} \tag{3.5}.
\]
Here \(0 < \epsilon := \tilde{\epsilon}\alpha \ll 1\). According to (3.2), we obtain
\[
\| (fd\sigma)^\vee \|_{L^q_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \left( \sum_{R} \left( \sum_{M} \min \left\{ (RM)^{\bar{q}}, (RM)^{-\frac{n-1}{2} \left[ 1 - \frac{2(n+1)}{qn} \right] + \epsilon} \right\} \right)^q \| (1 + |\Omega|)^{\sigma} fM \|_{L^p(S; d\sigma)} \right)^{\frac{1}{q}}.
\]
Since \(q > 2(n + 1)/n, \epsilon \ll 1\), and \(R, M\) are both dyadic number, we have
\[
\sup_{R > 0} \left( \sum_{M} \min \left\{ (RM)^{\bar{q}}, (RM)^{-\frac{n-1}{2} \left[ 1 - \frac{2(n+1)}{qn} \right] + \epsilon} \right\} \right) < \infty,
\]
\[
\sup_{M > 0} \left( \sum_{R} \min \left\{ (RM)^{\bar{q}}, (RM)^{-\frac{n-1}{2} \left[ 1 - \frac{2(n+1)}{qn} \right] + \epsilon} \right\} \right) < \infty.
\]
Note that for \(q > 2(n + 1)/n > p \geq 2\), we have by the Schur lemma and embedding inequality
\[
\| (fd\sigma)^\vee \|_{L^q_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \left( \sum_{M} \left\{ (1 + |\Omega|)^{\sigma} fM \right\}_{L^p(S; d\sigma)} \right)^{\frac{1}{p}} \lesssim \| (1 + |\Omega|)^{\sigma} f \|_{L^p(S; d\sigma)}.
\]
Choosing \(q = q_0 = (2(n + 1)/n)_+\) and \((n + 2)/q_0 = n/p_0\), we have
\[
\| (fd\sigma)^\vee \|_{L^{q_0}_{1,\infty}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| (1 + |\Omega|)^{q_0} f \|_{L^{p_0}(S; d\sigma)}.
\]
This implies (1.7). Interpolating this inequality with the restriction estimate by Bourgain-Guth [2, Theorem 1], we prove (3.1). Hence, the proof of estimate (1.8) is completed.
4 Localized restriction estimate

In this section we prove Proposition 3.1. We start our proof by recalling

\[
(f(\tau, \xi)d\sigma)^\vee(t, x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i (x - \xi + t|\xi|^2)} d\xi, \quad (4.1)
\]

where \( g(\xi) = f(|\xi|^2, \xi) \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp}\ g \subset \{\xi : |\xi| \in [1, 2]\} \). We apply the spherical harmonic expansion to \( g \) to obtain

\[
g(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k, \ell}(\rho) Y_{k, \ell}(\omega).
\]

Recalling \( v(k) = k + (n - 2)/2 \), we have by (2.5)

\[
(f d\sigma)^\vee(t, x) = 2\pi r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} t^k Y_{k, \ell}(\theta) \int_0^\infty e^{-2\pi i r^2} J_v(k)(2\pi \rho) a_{k, \ell}(\rho) \rho^\frac{n}{2} \varphi(\rho) d\rho.
\]

Here we insert a harmless smooth bump function \( \varphi \) supported on the interval \((1/2, 4)\) into the above integral, since \( a_{k, \ell}(\rho) \) is supported on \([1, 2]\). Now we estimate the quantity \( \| (f d\sigma)^\vee \|_{L^q_t(L^\infty_{\mu(r)}(\mathbb{R} \times A_R))} \). To this end, we first prove the following lemma.

**Lemma 4.1** Let \( \mu(r) = r^{n-1} dr \) and \( \omega(k) \) be a weight specified below. For \( q \geq 2 \), we have

\[
\left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^\infty e^{it\rho^2} J_v(k)(r \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^\frac{n}{2} \rho^\frac{1}{q} d\rho \right)^{\frac{1}{2}} \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))} \leq \left\| \omega(k) \right\|_{L^\infty_{\mu(r)}(S_R)} \left\| \int_0^\infty e^{it\rho^2} J_v(k)(r \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^\frac{n}{2} \rho^\frac{1}{q} d\rho \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))}^{\frac{1}{2}}.
\]

**Proof** Since \( q \geq 2 \), the Minkowski inequality and the Fubini theorem show that the left hand side of (4.3) is bounded by

\[
\left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^\infty e^{it\rho^2} J_v(k)(r \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^\frac{n}{2} \rho^\frac{1}{q} d\rho \right)^{\frac{1}{2}} \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))} \leq \left\| \omega(k) \right\|_{L^\infty_{\mu(r)}(S_R)} \left\| \int_0^\infty e^{it\rho^2} J_v(k)(r \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^\frac{n}{2} \rho^\frac{1}{q} d\rho \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))}^{\frac{1}{2}}.
\]

We rewrite this by making the variable change \( \rho^2 \sim \rho \)

\[
\left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^\infty e^{it\rho} J_v(k)(r \sqrt{\rho}) a_{k, \ell}(\sqrt{\rho}) \varphi(\sqrt{\rho}) \rho^\frac{n}{2} d\rho \right)^{\frac{1}{2}} \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))} \leq \left\| \omega(k) \right\|_{L^\infty_{\mu(r)}(S_R)} \left\| \int_0^\infty e^{it\rho} J_v(k)(r \sqrt{\rho}) a_{k, \ell}(\sqrt{\rho}) \varphi(\sqrt{\rho}) \rho^\frac{n}{2} d\rho \right\|_{L^q_t(L^\infty_{\mu(r)}(S_R))}^{\frac{1}{2}}.
\]

We use the Hausdorff-Young inequality with respect to \( t \) and we change variables back to obtain
LHS of (4.3) $\lesssim \left\| \frac{d^{(k)}}{K^{(k)}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| J_{\nu(k)}(r, \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^{(n-2)/2+1/q'} \|_{L^q_p}^2 \right\|_{\| L^q_{\mu(\nu)}(S_R) \|}$.

Now we prove that the inequalities (3.3) and (3.4) with $R \lesssim 1$. For doing this, we need

**Lemma 4.2** Let $q \geq 2$ and $R \lesssim 1$, we have the following estimate

$$
\| (f \, d\sigma)^{\wedge} \|_{L_{1, x}^q(\mathbb{R} \times A_R)} \lesssim R^\frac{n}{q} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k, \ell}(\rho) \varphi(\rho) \|_{L^{q'}_{p'}} \right)^{\frac{1}{2}},
$$

where $\omega(k) = (1 + k)^{2(n-1)(1/2-1/q)}$.

We postpone the proof of this lemma for a moment. Note that for $q' \leq 2 \leq p$, we use (4.5), (2.4), the Minkowski inequality and the Hölder inequality to obtain

$$
\| (f \, d\sigma)^{\wedge} \|_{L_{1, x}^q(\mathbb{R} \times A_R)} \lesssim R^\frac{n}{q} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k, \ell}(\rho) \varphi(\rho) \|_{L^{q'}_{p'}} \right)^{\frac{1}{2}} \lesssim R^\frac{n}{q} \| g \|_{L^p_{\mu} H^m_{\mu}(\mathbb{S}^{n-1})} \lesssim R^\frac{n}{q} \| g \|_{L^p_{\mu} H^m_{\mu}(\mathbb{S}^{n-1})},
$$

where $m = (n - 1)(1/2 - 1/q)$. In particular, for $q = 2$ and $4 \leq q \leq 6$, this proves (3.3) and (3.4) when $R \lesssim 1$. Hence it suffices to consider the case $R \gg 1$ once we prove Lemma 4.2.

**Proof of Lemma 4.2** By scaling argument in variables $t$, $x$ and (4.2), we obtain

$$
\| (f \, d\sigma)^{\wedge} \|_{L_{1, x}^q(\mathbb{R} \times A_R)} \lesssim \left\| \frac{d^{(k)}}{K^{(k)}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k, \ell}(\theta) \int_0^{\infty} e^{-i t \rho^2} J_{\nu(k)}(r, \rho) a_{k, \ell}(\rho) \rho^n \varphi(\rho) \rho^2 \rho \, d\rho \right\|_{L_{1, x}^q(\mathbb{R} \times A_R)}.
$$

By Sobolev’s embedding, (2.3) and (2.4), we have

$$
\| (f \, d\sigma)^{\wedge} \|_{L_{1, x}^q(\mathbb{R} \times A_R)} \lesssim \left\| \frac{d^{(k)}}{K^{(k)}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^{\infty} e^{-i t \rho^2} J_{\nu(k)}(r, \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^2 \rho \, d\rho \right\|_{L_{1, x}^q(\mathbb{R} \times A_R)}.
$$

By Lemma 4.1, it is enough to show

$$
\left\| \frac{d^{(k)}}{K^{(k)}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| J_{\nu(k)}(r, \rho) a_{k, \ell}(\rho) \varphi(\rho) \rho^{(n-2)/2+1/q'} \|_{L^q_p}^2 \right\|_{\| L^q_{\mu(\nu)}(S_R) \|} \lesssim R^\frac{n}{q} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k, \ell}(\rho) \varphi(\rho) \|_{L^{q'}_{p'}} \right)^{\frac{1}{2}}.
$$
Writing briefly \( v = v(k) \), and noting that \( R < r < 2R \) and \( 1 < \rho < 2 \), we have by (2.6)

\[
\left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \left\| J_{v(k)}(r \rho) a_{k,\ell}(\rho) \varphi(\rho) \right\|_{L^q_{\mu,\rho}} \right) \right\|_{L^q_{\mu,v}(\{R,2R\})} \]

\[
\lesssim \left( \int_{R}^{2R} r^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \frac{(4r)^v}{2^v \Gamma(v+\frac{1}{2})} \right)^2 \left\| a_{k,\ell}(\rho) \varphi(\rho) \right\|_{L^q_{\mu,\rho}}^2 r^{-n-1} dr \right)^{\frac{1}{2}}
\]

\[
\lesssim R^{\frac{n}{q}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \left( \frac{(2R)^v - \frac{n-2}{2}}{\Gamma(v+\frac{1}{2})} \right)^2 \left\| a_{k,\ell}(\rho) \varphi(\rho) \right\|_{L^q_{\mu,\rho}}^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim R^{\frac{n}{q}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \left\| a_{k,\ell}(\rho) \varphi(\rho) \right\|_{L^q_{\mu,\rho}} \right)^{\frac{1}{2}}.
\]

In the last inequality, we use the Stirling formula \( \Gamma(v+1) \sim \sqrt{v}(v/e)^v \) and the fact that \( R \lesssim 1 \) and \( v \geq (n-2)/2 \).

Now we are in a position to prove Proposition 3.1 when \( R \gg 1 \). We first prove (3.3) by making use of (4.1). Since \( \text{supp} \ g \subset \{ \xi : |\xi| \in [1,2] \} \), we may assume \( |\xi_n| \sim 1 \). Then we freeze one spatial variable, say \( x_n \), with \( |x_n| \lesssim R \) and free other spatial variables \( x' = (x_1, \ldots, x_{n-1}) \). After making the change of variables \( \eta_j = \xi_j, \eta_n = |\xi|^2 \) with \( j = 1, \ldots, n-1 \), we use the Plancherel theorem on the spacetime Fourier transform in \((t, x')\) to obtain (3.3).

When \( R \gg 1 \), inequality (3.4) is a consequence of the interpolation theorem and the following proposition.

**Proposition 4.1** Assume \( f \in \mathbb{L}_1 \) and \( R \gg 1 \) is a dyadic number. For every small constant \( 0 < \varepsilon \ll 1 \), we have the following inequalities

- For \( q = 4 \), we have
  \[
  \| (f \, d\sigma)^\vee \|_{L^4_{\mu,\rho}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{4} + \varepsilon} \| (1 + |\Omega|) \|_{L^4_{\mu,\rho}(\mathbb{R} \times A_R)}.
  \]

- For \( q = 6 \), we have
  \[
  \| (f \, d\sigma)^\vee \|_{L^6_{\mu,\rho}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{3} + \varepsilon} \| (1 + |\Omega|) \|_{L^6_{\mu,\rho}(\mathbb{R} \times A_R)}.
  \]

**Remark 4.1** It seems to be possible to remove the \( \varepsilon \)-loss in (4.8), but we do not purchase this option here because we do not need it in this paper.

To prove this proposition, we firstly show

**Lemma 4.3** Assume \( f \in \mathbb{L}_1 \) and \( R \gg 1 \). We have the following estimate

\[
\| (f \, d\sigma)^\vee \|_{L^4_{\mu,\rho}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{4} + \varepsilon} \| g \|_{L^4_{\mu,\rho} H^\frac{n}{4,4}(\mathbb{R}^{n-1})},
\]

where \( 0 < \varepsilon \ll 1 \), and \( g(\xi) = f(|\xi|^2, \xi) \).

**Proof** By the scaling argument and (4.2), it suffices to estimate the quantity

\[
\left\| r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{v(k)}(r \rho) a_{k,\ell}(\rho) \varphi(\rho) d\rho \right\|_{L^4_{\mu,\rho}(\mathbb{R} \times A_R)}.
\]
In the following, we consider the three cases. For the first two cases, we establish the estimates for general \( q \geq 4 \) so that we can use them directly for \( q = 6 \) later.

- Case 1: \( k \in \Omega_1 := \{ k : R \ll v(k) \} \). Let \( \omega(k) = (1 + k)^{2(n-1)/2-1/q} \) again. We have by a similar argument as in the proof of Lemma 4.2:

\[
\left\| \sum_{k \geq R} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{v(k)}(r \rho) a_{k,\ell}(r \rho) \rho^{n-2} \varphi(\rho) \ d\rho \right\|_{L^q_t (\mathbb{R} \times \mathcal{A}_R)} \leq e^{-c(R+v)}
\]

Recall that for \( R \gg 1 \) and \( k \in \Omega_1 \), we have \( |J_{v(k)}(r)| \lesssim e^{-c(r+v)} \) by (2.9). Using \( R < r < 2R \) and \( 1 < \rho < 2 \), we obtain

\[
\left\| \sum_{k \geq R} \sum_{\ell=1}^{d(k)} \omega(k) J_{v(k)}(r \rho) a_{k,\ell}(r \rho) \varphi(\rho) \rho^{(n-2)/2+1/q} \right\|_{L^q_t (\mathbb{R} \times \mathcal{A}_R)} \lesssim \left( \int_R^{2R} e^{-(n-2)q} \left( \sum_{k \geq R} \sum_{\ell=1}^{d(k)} \omega(k) e^{-(r+v)} \right) \left\| a_{k,\ell}(r \rho) \varphi(\rho) \right\|^2_{L^q_t} \right)^{1/2} \lesssim e^{-cR} \left( \sum_{k \geq R} \sum_{\ell=1}^{d(k)} \omega(k) \varphi(\rho) \right)^{1/2}.
\]

By Minkowski’s inequality and Hölder’s inequality, we obtain

\[
\left\| \sum_{k \geq R} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{v(k)}(r \rho) a_{k,\ell}(r \rho) \rho^{n-2} \varphi(\rho) \ d\rho \right\|_{L^q_t (\mathbb{R} \times \mathcal{A}_R)} \lesssim e^{-cR} \left( \sum_{k \geq R} \sum_{\ell=1}^{d(k)} \omega(k) \rho^{2} \right)^{1/2} \varphi(\rho) \left\| L^p_\rho \right. \quad \quad (4.11)
\]

Applying this with \( q = 4 = p \), we have

\[
\left\| \sum_{k \geq R} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{v(k)}(r \rho) a_{k,\ell}(r \rho) \rho^{n-2} \varphi(\rho) \ d\rho \right\|_{L^4_t (\mathbb{R} \times \mathcal{A}_R)} \lesssim e^{-cR} \left( \sum_{k \geq R} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/2} \left| a_{k,\ell}(\rho) \right|^2 \right)^{1/2} \varphi(\rho) \left\| L^4_\rho \right.
\]

\[
\lesssim R^{-\frac{n+1}{4} + \epsilon} \left\| g \right\|_{L^4_\rho H^{(n-1)/4,1}_\rho (\mathbb{S}^{n-1})}.
\]
Lemma 4.4 (Oscillation and asymptotic property, [3]). Let \( \nu > 1/2 \) and \( r > \nu + \nu^{1/3} \). There exists a constant number \( C \) independent of \( r \) and \( \nu \) such that

\[
J_v(r) = \sqrt{\frac{2}{\pi}} \frac{\cos \theta(r)}{(r^2 - \nu^2)^{1/4}} + h_v(r),
\]

where \( \theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos \frac{\nu}{r} - \frac{\pi}{4} \) and

\[
|h_v(r)| \leq C \left( \left( \frac{\nu^2}{(r^2 - \nu^2)^{7/4}} + \frac{1}{r} \right) 1_{[\nu + \nu^{1/3}, 2\nu]}(r) + \frac{1}{r} 1_{[2\nu, \infty)}(r) \right).
\]
Note that \( v(k) = k + (n - 2)/2 \) and \( k \in \Omega_3 \), we can write
\[
J_v(r) = I_v(r) + \tilde{I}_v(r) + h_v(r), \quad \text{where } |h_v(r)| \lesssim r^{-1},
\]
and
\[
I_v(r) = \frac{\sqrt{2/\pi} e^{i\theta(r)}}{(r^2 - v^2)^{1/4}}.
\]

A simple computation yields to
\[
\theta'(r) = (r^2 - v^2)^{1/2} r^{-1}, \quad \theta''(r) = (r^2 - v^2)^{-1/2} - (r^2 - v^2)^{1/2} r^{-2} = (r^2 - v^2)^{-1/2} v^2 r^{-2}, \quad \theta'''(r) = \frac{v^2}{r} (r^2 - v^2)^{-3/2} v^2 r^{-2} \left( -3 + \frac{2v^2}{r^2} \right).
\]

Using Sobolev embedding on sphere and Minkowski’s inequality, we estimate
\[
\left\| r^{-\frac{n-2}{2}} \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} i^\ell Y_{k, \ell}(\theta) \int_0^\infty e^{-it \rho^2} J_v(k(r, \rho) a_k, \ell(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho \right\|_{L^4_{t, \rho}(\mathbb{R} \times \Lambda_k)} \lesssim R^{-\frac{n-2}{2}} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} J_v(k(r, \rho) a_k, \ell(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho \right\|^2_{L^4_{t, \rho}(\mathbb{R} \times \Lambda_k)} \right)^{1/2} \lesssim R^{-\frac{n-1}{4}} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} J_v(k(r, \rho) a_k, \ell(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho \right\|^2_{L^4_{t, \rho}(\mathbb{R} \times \Lambda_k)} \right)^{1/2}.
\]

Since \( J_v(r) = I_v(r) + \tilde{I}_v(r) + h_v(r) \), it suffices to estimate two terms
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} h_v(k(r, \rho) a_k, \ell(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho \right\|^2_{L^4_{t, \rho}(\mathbb{R} \times \Lambda_k)} \right)^{1/2} \lesssim R^{-3/4} \left\| g \right\|_{L^4_{t, H^0_n} + \frac{n+1}{4} (\mathbb{S}^{n-1})} \tag{4.19}
\]
and
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} I_v(k(r, \rho) a_k, \ell(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho \right\|^2_{L^4_{t, \rho}(\mathbb{R} \times \Lambda_k)} \right)^{1/2} \lesssim R^{-1/2+\epsilon} \left\| g \right\|_{L^4_{t, H^0_n} + \frac{n+1}{4} (\mathbb{S}^{n-1})}. \tag{4.20}
\]

For the first purpose, we consider the operator
\[
T_v(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_0^\infty e^{-it \rho^2} h_v(r, \rho) a(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d \rho
\]
where \( |h_v(r)| \leq C/r \). By a similar argument as in the proof of Lemma 4.1, it is easy to see
\[
\left\| T_v(a)(t, r) \right\|_{L^q_{t, r}} \leq R^{-1/q'} \left\| a \varphi \right\|_{L^{q'}_p} \tag{4.21}
\]
Hence we have

\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} h_{v(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_L^2 \right)^{1/2} \leq R^{-3/4} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| a_{k,\ell}(\rho) \varphi(\rho) \right\|_L^{4/3} \right)^{1/2} \\
\leq R^{-3/4} \left\| g \right\|_L^{4} \left( \sum_{k \in \Omega_1} (1 + k)^{(n-1)/2} \left\| a_{k,\ell}(\rho) \right\|^2 \right)^{1/2} \phi_{L^4} \left( \| \omega \|_{L^\infty} \right)^{n-1},
\]

which implies (4.19).

Next we prove (4.20). To this end, let \( \beta(\rho) = \rho^{\frac{n}{2}} \varphi(\rho) \), we see that

\[
\left\| \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \int_\mathbb{R}^2 e^{-it(\rho_1^2 - \rho_2^2)} I_{v(k)}(r\rho_1) I_{v(k)}(r\rho_2) \right)^{1/2} \right\|_L^{4} \leq \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left\| \int_\mathbb{R}^2 e^{-it(\rho_1^2 - \rho_2^2)} I_{v(k)}(r\rho_1) I_{v(k)}(r\rho_2) \right\|_L^{4} \right)^{1/2} \\
\times \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} a_{k,\ell}(\rho_1) a_{k,\ell}(\rho_2) \beta(\rho_1) \beta(\rho_2) \, d\rho_1 d\rho_2 \right)^{1/2} \\
\leq \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left\| \int_\mathbb{R}^2 e^{-it(\rho_1^2 - \rho_2^2)} I_{v(k)}(r\rho_1) I_{v(k)}(r\rho_2) \right\|_L^{4} \right)^{1/2} \\
\times \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} a_{k,\ell}(\rho_1) a_{k,\ell}(\rho_2) \beta(\rho_1) \beta(\rho_2) \, d\rho_1 d\rho_2 \right)^{1/2} \\
= \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_\mathbb{R}^4 \sum_{\ell = 1}^{d(k)} a_{k,\ell}(\rho_1) a_{k,\ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} a_{k,\ell'}(\rho_3) a_{k,\ell'}(\rho_4) \beta(\rho_1) \beta(\rho_2) \beta(\rho_3) \beta(\rho_4) \\
\int_\mathbb{R}^4 e^{-it(\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2)} \, dt K(R, v; \rho_1, \rho_2, \rho_3, \rho_4) \, d\rho_1 d\rho_2 d\rho_3 d\rho_4 \right)^{1/2} \right)^2
\]

(4.22)

where the kernel

\[
K(R, v; \rho_1, \rho_2, \rho_3, \rho_4) = \frac{1}{r} \chi(R)e^{(\theta(\rho_1 r) - \theta(\rho_2 r) + \theta(\rho_3 r) - \theta(\rho_4 r))} \\
= \int_0^\infty \frac{\chi(R)}{(r \rho_1)^2 - v^2}^{1/4} \frac{\chi(R)}{(r \rho_2)^2 - v^2}^{1/4} \frac{\chi(R)}{(r \rho_3)^2 - v^2}^{1/4} \frac{\chi(R)}{(r \rho_4)^2 - v^2}^{1/4} \, dr.
\]

(4.23)

Now we analyze the kernel \( K \). Let

\[
\phi(r; \rho_1, \rho_2, \rho_3, \rho_4) = \theta(\rho_1 r) - \theta(\rho_2 r) + \theta(\rho_3 r) - \theta(\rho_4 r).
\]
Hence if \( \rho_1^2 - \rho_2^2 = \rho_4^2 - \rho_3^2 \), we have by (4.18)

\[
\phi_r' = (\rho_1^2 - \rho_2^2)r^3 \left( \frac{1}{(r \rho_1)^2 - v^2 + \sqrt{(r \rho_2)^2 - v^2}} - \frac{1}{(r \rho_2)^2 - v^2 + \sqrt{(r \rho_4)^2 - v^2}} \right)
\]

\[
= \frac{(\rho_1^2 - \rho_2^2)(\rho_3^2 - \rho_2^2)r^3}{\left( (r \rho_1)^2 - v^2 + \sqrt{(r \rho_2)^2 - v^2} \right) \left( (r \rho_3)^2 - v^2 + \sqrt{(r \rho_4)^2 - v^2} \right)} \times \left( \frac{1}{(r \rho_3)^2 - v^2 + \sqrt{(r \rho_2)^2 - v^2}} + \frac{1}{(r \rho_4)^2 - v^2 + \sqrt{(r \rho_4)^2 - v^2}} \right).
\]

Since \( k \in \Omega_3 \), one has \( r \gg v(k) \). Therefore we have

\[
|\phi_r'| \geq |\rho_1^2 - \rho_2^2| \cdot |\rho_3^2 - \rho_2^2|.
\]

Applying integration by parts with respect to \( r \) to (4.23), we have for any \( N \geq 0 \)

\[
K(R, v; \rho_1, \rho_2, \rho_3, \rho_4) \lesssim R^{-1} \left( 1 + R |\rho_1 - \rho_2^2| \cdot |\rho_3^2 - \rho_2^2| \right)^{-N}, \quad (4.24)
\]

when \( \rho_1^2 - \rho_2^2 = \rho_4^2 - \rho_3^2 \). Let \( b_{k, \ell}(\rho) = 2a_{k, \ell}(\sqrt{\rho}) \beta(\sqrt{\rho})/\sqrt{\rho} \), from (4.22) and (4.24), it suffices to estimate

\[
\left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^4} |\delta(\rho_1 - \rho_2 + \rho_3 - \rho_4)K(R, v(k); \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}, \sqrt{\rho_4}) \right)^{1/2} \right)^2
\]

\[
\times \left( \sum_{\ell = 1}^{d(k)} b_{k, \ell}(\rho_1) b_{k, \ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} b_{k, \ell'}(\rho_3) b_{k, \ell'}(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4 \right)^{1/2} \right)^2
\]

\[
= \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^3} |K(R, v(k); \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}, \sqrt{\rho_1 - \rho_2 + \rho_3}) \right)^{1/2} \right)^2
\]

\[
\times \left( \sum_{\ell = 1}^{d(k)} b_{k, \ell}(\rho_1) b_{k, \ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} b_{k, \ell'}(\rho_3) b_{k, \ell'}(\rho_1 - \rho_2 + \rho_3) d\rho_1 d\rho_2 d\rho_3 \right)^{1/2} \right)^2
\]

\[
\leq R^{-1} \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^3} (1 + R |\rho_1 - \rho_2||\rho_3 - \rho_2|)^{-N} \right)^{1/2} \right)^2
\]

\[
\times \left( \sum_{\ell = 1}^{d(k)} b_{k, \ell}(\rho_1) b_{k, \ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} b_{k, \ell'}(\rho_3) b_{k, \ell'}(\rho_1 - \rho_2 + \rho_3) d\rho_1 d\rho_2 d\rho_3 \right)^{1/2} \right)^2
\]

\[
\lesssim R^{-1} \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^3} (1 + R |\rho_1 - \rho_2||\rho_3 - \rho_2|)^{-N} \right)^{1/2} \right)^2
\]

\[
\times b_k(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) d\rho_1 d\rho_2 d\rho_3 \right)^{1/2} \right)^2
\]

where \( b_k(\rho) = \left( \sum_{\ell = 1}^{d(k)} |b_{k, \ell}(\rho)|^2 \right)^{1/2} \). Then we aim to estimate

\[
\int_{\mathbb{R}^3} \frac{b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3)}{(1 + R |\rho_1 - \rho_2||\rho_3 - \rho_2|)^{\frac{3}{2}}} d\rho_1 d\rho_2 d\rho_3 \lesssim R^{-1+\epsilon} \|b\|^4_{L^4}. \quad (4.25)
\]
Indeed once we have proved (4.25), we show

\[
\left\| \left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/2} \left[ \int_0^\infty e^{-it\rho^2} I_{v(k)}(\tau,\rho) \varphi(\rho) d\rho \right]^2 \right) \right\|_{L^4_t(\mathbb{R}; L^4_x(S_R))}^{1/2} \leq R^{-1+\epsilon} \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} (1 + k)^{-1/2} \|b_k\|_2^2 \right)^{1/2} \]

which implies (4.20). Therefore, it remains to prove

\[
\int \left( \sum_{k \in \Omega_3} \left( \sum_{\ell=1}^{d(k)} (1 + k)^2 |a_{k,\ell}(\rho)|^2 \right)^{1/2} \right) \left( \sum_{k \in \Omega_3} \left( \sum_{\ell=1}^{d(k)} (1 + k)^2 |b_{k,\ell}(\rho)|^2 \right)^{1/2} \right) \leq R^{-1+\epsilon} \|b\|_4^4. \tag{4.26}
\]

For \( R = 2^{k_0} \gg 1 \), we decompose the integral into

\[
\int \frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1 - \rho_2 + \rho_3)}{(1 + R|\rho_1 - \rho_2||\rho_3 - \rho_2|)^N} d\rho_1 d\rho_2 d\rho_3 \leq R^{-1+\epsilon} \|b\|_4^4. \tag{4.26}
\]

To estimate it, we need the following lemma.

**Lemma 4.5** We have the following estimate for the integral

\[
\int b(\rho_2)d\rho_2 \int b(\rho_1)d\rho_1 \int b(\rho_3)b(\rho_1 - \rho_2 + \rho_3)d\rho_3 \lesssim 2^{-(i+j)} \|b\|_4^4. \tag{4.28}
\]
Proof We first have by Hölder’s inequality

\[
\int_{|\rho_3| \sim 2^{-j}} b(\rho_3) b(\rho_2 - \rho_3) d\rho_3 \leq \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int_{|\rho_3| \sim 2^{-j}} |b(\rho_1 - \rho_3)|^2 d\rho_3 \right)^{1/2}
\]

\[
\leq \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int_{|\rho| \sim 2^{-j}} |b(\rho_1 + \rho)|^2 d\rho \right)^{1/2}
\]

\[
\leq \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int_{|\rho_3| \sim 2^{-j}} |b(\rho_1 + \rho)|^2 d\rho \right)^{1/2}.
\]

(4.29)

Let \( I \) be the left hand side of (4.28). We estimate \( I \) by (4.29) and Hölder’s inequality

\[
\int_{|\rho| \sim 2^{-j}} b(\rho_2) \int_{|\rho_1 - \rho_2| \sim 2^{-j}} \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho)|^2 d\rho \right)^{1/2} b(\rho_1) d\rho_1 \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \right)^{1/2} d\rho_2
\]

\[
\leq \|b\|_{L^1} \left( \int_{|\rho_1 - \rho_2| \sim 2^{-j}} \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho)|^2 d\rho \right)^{1/2} |b(\rho_1)| d\rho_1 \right) \left( \int_{|\rho_3| \sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \right)^{1/2} \|L^2 \|
\]

\[
\leq \|b\|_{L^4} \left\| \chi_j \ast \left( (\chi_j \ast |b|^2) \frac{1}{2} |b| \right) \right\|_{L^2} \|\chi_j \ast |b|^2\|_{L^2}^{1/2},
\]

where \( \chi_j = \chi_j(\rho) = \chi(2^j \rho) \) and \( \chi \in C_c^\infty(\frac{1}{4}, 4) \). It is easy to see by the Young inequality

\[
\left\| \chi_j \ast |b|^2 \right\|_{L^2} \leq \left\| \chi_j \ast |b|^2 \right\|_{L^1} \|b\|_{L^1} \|b\|_{L^4} \leq 2^{-j/2} \|b\|_{L^4},
\]

and

\[
\left\| \chi_j \ast \left( (\chi_j \ast |b|^2) \frac{1}{2} |b| \right) \right\|_{L^2} \leq \left\| \chi_j \ast \left( (\chi_j \ast |b|^2) \frac{1}{2} |b| \right) \right\|_{L^2} \frac{1}{2} |b| \|b\|_{L^4} \leq 2^{-j} \|b\|_{L^4}^2.
\]

Collecting the above estimates, we obtain

\[
I \leq 2^{-(i+j)} \|b\|_{L^4}^4.
\]

This completes the proof of Lemma 4.5. \( \square \)

Now we return to prove (4.26). Applying Lemma 4.5 to (4.27), we have

\[
\int_{\mathbb{R}^3} \frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1 - \rho_2 + \rho_3)}{(1 + R|\rho_1 - \rho_2||\rho_3 - \rho_2|)^N} d\rho_1 d\rho_2 d\rho_3
\]

\[
\lesssim \left( \sum_{(i, j) \in \mathbb{N}^2 : i + j \geq 0} 2^{-(i+j)} + R^{-N} \sum_{(i, j) \in \mathbb{N}^2 : i + j \leq 0} 2^{(N-1)(i+j)} \right) \|b\|_{L^4}^4
\]

\[
\lesssim R^{-1+\epsilon} \|b\|_{L^4}^4.
\]

(4.30)
Hence we prove (4.26), and so, we finish the proof of (4.7).

We next prove (4.8) in Proposition 4.1. We need to prove the following lemma.

Lemma 4.6 Let $R \gg 1$ and $f \in L_1$, we have the following estimate for every $0 < \epsilon \ll 1$

\[
\| (f \, d\sigma)^\vee \|_{L^p_{t,x}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{2} + \epsilon} \| g \|_{L^2_{R,\omega} H_{\omega}^{\frac{n-1}{2}}(\mathbb{S}^{n-1})},
\]

(4.31)

where $g(\xi) = f(|\xi|^2, \xi)$.

Proof It suffices to estimate, by a scaling argument, the following quantity

\[
\left\| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \phi(\rho) \, d\rho \right\|_{L^6_{t,x}(\mathbb{R} \times A_R)}. \tag{4.32}
\]

We divide the above integral into three cases.

- Case 1: $k \in \Omega_1 := \{ k : R \ll v(k) \}$. Using (4.11) with $q = 6$, we prove

\[
\left\| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \phi(\rho) \, d\rho \right\|_{L^6_{t,x}(\mathbb{R} \times A_R)} \lesssim e^{-cR} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1 + k)^{2(n-1)/3} \| a_{k,\ell}(\rho) \|^2 \right)^{1/2} \| \phi(\rho) \|_{L^2_{\rho}} \lesssim e^{-cR} \| g \|_{L^2_{R,\omega} H_{\omega}^{\frac{n-1}{2}}(\mathbb{S}^{n-1})}.
\]

- Case 2: $k \in \Omega_2 := \{ k : v(k) \sim R \}$. Applying (4.14) with $q = 6$ and $p = 2$, we show

\[
\left\| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \phi(\rho) \, d\rho \right\|_{L^6_{t,x}(\mathbb{R} \times A_R)} \lesssim R^{-(n-1)/3} \| g \|_{L^2_{R,\omega} H_{\omega}^{\frac{n-1}{2}}(\mathbb{S}^{n-1})}. \tag{4.33}
\]

- Case 3: $k \in \Omega_3 := \{ k : v(k) \ll R \}$. We introduce the operator

\[
T_v(a)(t, r) = \chi\left( \frac{r}{R} \right) \int_0^\infty e^{-it\rho^2} h_v(r\rho) a(\rho) \rho^{\frac{n}{2}} \phi(\rho) \, d\rho.
\]

where $|h_v(r)| \leq C/r$ and the operator

\[
H_v(a)(t, r) = \chi\left( \frac{r}{R} \right) \int_0^\infty e^{-it\rho^2} I_v(r\rho) a(\rho) \rho^{\frac{n}{2}} \phi(\rho) \, d\rho,
\]

where $v = v(k) = k + (n - 2)/2$. Since

\[
J_v(r) = I_v(r) + \tilde{I}_v(r) + h_v(r),
\]
our aim here is to estimate
\[
\left\| r^{-\frac{n+2}{2}} \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r, \rho) a_{k,\ell}(\rho) \rho^2 \varphi(\rho) \, d\rho \right\|_{L_{t,r}^6(\mathbb{R} \times A_R)} \\
\lesssim R^{-\frac{n+1}{2} + \varepsilon} \left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/3} \left( \left\| T_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L_t^2(\mathbb{R} ; L_r^p(S_R))}^2 + \left\| H_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L_t^6(\mathbb{R} ; L_r^p(S_R))}^2 \right)^{1/2} \right)^{1/2}.
\]
By making use of (4.21) with \( q = 6 \), we have
\[
\left\| T_{\nu}(a)(t, r) \right\|_{L_{t,r}^6} \leq R^{-5/6} \| a \varphi \|_{L^{6/5}}.
\]
This implies that
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/3} \left\| T_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L_t^2(\mathbb{R} ; L_r^p(S_R))}^2 \right)^{1/2} \lesssim R^{-5/6} \left\| \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/3} |a_{k,\ell}(\rho)|^2 \right\|_{L^{6/5}} \lesssim R^{-5/6} \| g \|_{L_{t}^2 H_{w}^{\frac{n-2}{2}}(\mathbb{S}^{n-1})}. \tag{4.34}
\]
On the other hand, by (2.11), one has \( |I_{\nu}(r) | \lesssim r^{-1/2} \) when \( k \in \Omega_3 \). Consider the operator
\[
H_{\nu}(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_0^\infty \int_{\mathbb{R}^n} e^{-it\rho^2} I_{\nu}(r, \rho) a(\rho) \rho^2 \varphi(\rho) \, d\rho.
\]
where \( \nu = \nu(k) = k + (n - 2)/2 \) with \( k \in \Omega_3 \).
On the one hand, it is easy to see
\[
\left\| H_{\nu}(a)(t, r) \right\|_{L_{t,r}^\infty(\mathbb{R} \times \mathbb{R}^n)} \lesssim R^{-1/2} \| a \varphi \|_{L^1}.
\]
On the other hand, we have the claim that for any \( \epsilon > 0 \)
\[
\left\| H_{\nu}(a)(t, r) \right\|_{L_{t,r}^4(\mathbb{R} \times \mathbb{R}^n)} \lesssim R^{-1/2 + \epsilon} \| a \varphi \|_{L_{t}^{4}}. \tag{4.35}
\]
We postpone the proof of this claim to the end of this section. Hence, by the interpolation of the above two estimates, for any \( \epsilon > 0 \), we obtain that
\[
\left\| H_{\nu}(a)(t, r) \right\|_{L_{t,r}^6(\mathbb{R} \times \mathbb{R}^n)} \lesssim R^{-1/2 + \epsilon} \| a \varphi \|_{L_{t}^{4}}.
\]
This shows
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/3} \left\| H_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L_t^6(\mathbb{R} ; L_r^p(S_R))}^2 \right)^{1/2} \lesssim R^{-1/2 + \epsilon} \left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/3} \| a_{k,\ell}(\rho) \varphi(\rho) \|_{L^2}^2 \right)^{1/2} \lesssim R^{-1/2 + \epsilon} \| g \|_{L_{t}^2 H_{w}^{\frac{n-2}{2}}(\mathbb{S}^{n-1})}. \tag{4.36}
\]

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K. M. Rogers [20] developed an argument showing that a restriction estimate implies a local smoothing estimate

\[ \left\| r^{-\frac{n-1}{2}} \sum_{k \in \Omega_1} \sum_{l=1}^{d(k)} i^k Y_{k,l}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r, \rho) a_{k,l}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L^6_t L^{\frac{n}{2}}_\rho} \lesssim R^{-\frac{n-1}{3} + \epsilon} \| g \|_{L^2_\rho H^{\frac{n-2}{2}}_\rho}.
\]

This implies (4.31), which completes the proof of Lemma 4.6.

**The proof of claim (4.35)** The same argument in the proof the (4.20) shows the claim (4.35). Recall the kernel (4.23), it is enough to estimate the integral

\[ \left\| H_v(a)(t, r) \right\|_{L^q_t L^\infty_r(\mathbb{R}^4 \times \mathbb{R}^n)}^4 = \int_\mathbb{R}^4 \int_\mathbb{R} e^{-it(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2)} K(R, v; \rho_1, \rho_2, \rho_3, \rho_4) a(\rho_1) a(\rho_2) a(\rho_3) a(\rho_4)
\]

where \( \beta(\rho) = \rho^\frac{n}{2} \varphi(\rho). \) For \( b(\rho) = 2a(\sqrt{\rho}) \beta(\sqrt{\rho}) \), therefore we obtain

\[ \left\| H_v(a)(t, r) \right\|_{L^q_t L^\infty_r(\mathbb{R}^4 \times \mathbb{R}^n)}^4 = \int_\mathbb{R}^4 \delta(\rho_1 - \rho_2 + \rho_3 - \rho_4) K(R, v; \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}, \sqrt{\rho_4}) b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4 = \int_\mathbb{R} K(R, v; \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}, \sqrt{\rho_4}) b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4 \lesssim R^{-2+\epsilon} \| b \|_{L^4_r}^4 \lesssim R^{-2+\epsilon} \| a \varphi \|_{L^4_r}^4.
\]

where we use the kernel estimate (4.24) and (4.26) in the first inequality.

**5 Local smoothing estimate**

K. M. Rogers [20] developed an argument showing that a restriction estimate implies a local smoothing estimate under some suitable conditions. For the sake of convenience, we closely follow this argument to prove Corollary 1.1. In fact, by making use of the standard Littlewood-Paley argument, it can be reduced to prove the claim

\[ \| e^{it\Delta} (1 - \Delta_\theta)^{-s/2} u_0 \|_{L^q_t L^\infty_r([0,1] \times \mathbb{R}^n)} \lesssim N^{(2n(1/2 - 1/q) - 2/q) +} \| u_0 \|_{L^q_t}, \quad \forall N \gg 1 \]  

where

\[ \text{supp } \mathcal{F}((1 - \Delta_\theta)^{-s/2} u_0) \subset \{ \xi : |\xi| \leq N \}.
\]

Here we denote by \( \mathcal{F} \) the Fourier transform. We also use the notation \( \hat{h} \) to express the Fourier transform of \( h \). Let \( h = (1 - \Delta_\theta)^{-s/2} u_0 \). Denote by \( P_N \) the Littlewood-Paley projector, i.e.

\[ P_N h = \mathcal{F}^{-1} \left( \ell \left( \frac{|\xi|}{N} \right) \hat{h} \right), \quad \chi \in \mathbb{C}_c^\infty([1/2, 1]).
\]
By the Littlewood-Paley theory and the claim (5.1), one has for $\alpha > 2n(1/2 - 1/q) - 2/q$
\[
\|e^{it\Delta} h\|_{L^q_t(L^\infty_x([0,1] \times \mathbb{R}^n))} \lesssim \|e^{it\Delta} P_{\leq 1} h\|_{L^q_t(L^\infty_x([0,1] \times \mathbb{R}^n))} + \sum_{N \gg 1} \|e^{it\Delta} P_N h\|_{L^q_t(L^\infty_x([0,1] \times \mathbb{R}^n))}^2
\]
\[
\lesssim \|u_0\|_{L^2_t(\mathbb{R}^n)}^2 + \sum_{N \gg 1} N^{2(2n(1/2 - 1/q) - 2/q)} + \|P_N u_0\|_{L^q_t}^2
\]
\[
\lesssim \|u_0\|_{L^2_t(\mathbb{R}^n)}^2 + \left( \sum_{N \gg 1} N^{2\alpha} |P_N u_0|^q \right)^{1/q} \|L^q_t\|
\]
\[
\lesssim \|u_0\|_{L^2_t(\mathbb{R}^n)}^2 + \left( \sum_{N \gg 1} N^{2\alpha} |P_N u_0|^2 \right)^{1/2} \|L^q_t\|
\]
\[
\simeq \|u_0\|_{W^{\alpha,q}_x(\mathbb{R}^n)}^2.
\]
Here we use Hölder’s inequality for the third inequality, Sobolev imbedding for the fourth one. Hence we have
\[
\|e^{it\Delta} u_0\|_{L^q_t(L^\infty_x([0,1] \times \mathbb{R}^n))} \lesssim \|(1 - \Delta_\theta)^{t/2} u_0\|_{W^{\alpha,q}_x(\mathbb{R}^n)}.
\]
Now we are left to prove claim (5.1). Assume supp $\hat{f} \subset [0,1]$. Note that
\[
e^{it\Delta} f = \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/it} f(y) dy, \quad \forall t \in \mathbb{R}\setminus\{0\}.
\]
On the other hand, we have for $t \neq 0$
\[
e^{it\Delta} f = \int_{\mathbb{R}^n} e^{it|\xi|^2 + x \cdot \xi} \hat{f}(\xi) d\xi = e^{-\frac{i|x|^2}{4t}} \int_{\mathbb{R}^n} e^{it|\xi|^2 + \frac{x \cdot \xi}{t}} \hat{f}(\xi) d\xi
\]
\[
= \frac{1}{(it)^{n/2}} e^{-\frac{|x|^2}{4t}} \left( e^{i \frac{x}{t} \hat{f}} \right) \left( -\frac{x}{2t} \right).
\]
So we have for every dyadic number $N$
\[
\|e^{it\Delta} f\|_{L^q_t(L^\infty_x([|t| < N^2; |x| \leq N]) \leq N^{-n}} \left\| \left( e^{i \frac{x}{t} \hat{f}} \right) \left( -\frac{x}{2t} \right) \right\|_{L^q_t(L^\infty_x([|t| < N^2; |x| \leq N]) \leq N^{-n}})
\]
\[
\lesssim N^{-n + \frac{2n+4}{q}} \|e^{it\Delta} \hat{f}\|_{L^q_t([|t| < N^2; |x| \leq 1])}.
\]
By making use of Theorem 1.1, we obtain for $q > 2(n + 1)/n$ and $\frac{n+2}{q} = \frac{n}{p}$
\[
\left\| e^{it\Delta} \hat{f} \right\|_{L^q_t(L^\infty_x([|t| < N^2; |x| \leq 1]) \leq N^{-n}} \lesssim \|f\|_{L^p_{\mu(\theta)}(\mathbb{R}^+; H^{\alpha}_x(\mathbb{R}^{n-1}))}.
\]
This yields
\[
\|e^{it\Delta} f\|_{L^q_t(L^\infty_x([|t| < N^2; |x| \leq N^2]) \leq N^{-n - \frac{2n}{q} + \frac{2n+4}{q}}} \|f\|_{L^p_{\mu(\theta)}(\mathbb{R}^+; H^{\alpha}_x(\mathbb{R}^{n-1}))}.
\]
This implies that
\[
\|e^{it(1 - \Delta_\theta)^{-s/2} f}\|_{L^q_t(L^\infty_x([|t| < N^2; |x| \leq N^2]) \leq N^{-n - \frac{2n+4}{q}} \|f\|_{L^q_t}}.
\]
For the sake of convenience, we recall [20, Lemma 8]
Lemma 5.1 Let $q \geq p_1 \geq p_0$, $r \geq 1$ and $I \subset [0, R^2]$. If one has
\[
\|e^{it\Delta} f\|_{L^q_t(B_{R^2}; L^p_L(I))} \leq CR^s \|f\|_{L^{p_0}(\mathbb{R}^n)}
\]
where $R \gg 1$, and $f$ is frequency supported in unite ball $\mathbb{B}^n$. Then for all $\epsilon > 0$
\[
\|e^{it\Delta} f\|_{L^q_t(\mathbb{R}^n; L^p_L(I))} \leq C\epsilon R^{s + 2(n(\frac{1}{p_0} - \frac{1}{p_1}) + \epsilon)} \|f\|_{L^{p_1}(\mathbb{R}^n)}.
\]

Since $q > p$ when $q > 2(n + 1)/n$, for any $0 < \epsilon \ll 1$, we have by this lemma
\[
\|e^{it\Delta} (1 - \Delta_0)^{-s/2} f\|_{L^q_t(\mathbb{R}^n; L^p_L(I))} \lesssim N^{-n + 2n + 4n(1 - \frac{1}{p} - \frac{1}{q}) + \epsilon} \|f\|_{L^p_L(I)}
\]
\[
\lesssim N^{n(1 - \frac{2}{q}) + \epsilon} \|f\|_{L^q_L(I)}.
\]

Using the scaling argument, if
\[
\text{supp}\hat{f}_{k,N} \subset B_{2^{k/2}N} := \{\xi : |\xi| \in [0, 2^{k/2}N]\}, \quad \forall k \geq 0,
\]
then
\[
\|e^{it\Delta} (1 - \Delta_0)^{-s/2} f_{k,N}\|_{L^q_t(2^{k-1}, 2^{k+1} \times \mathbb{R}^n)} \lesssim N^{n(1 - \frac{3}{q}) + \epsilon} \left(2^{\frac{k}{2}} N\right)^{-\frac{3}{q}} \|f_{k,N}\|_{L^q_L(I)}. \quad (5.4)
\]

Since
\[
\text{supp}\hat{h} \subset \{\xi : |\xi| \in [N/2, N]\} \subset B_{2^{k/2}N}, \quad \forall k \geq 2,
\]
we replace $(1 - \Delta_0)^{-s/2} f_{k,N}$ by $h$ to obtain
\[
\|e^{it\Delta} h\|_{L^q_t([0,1] \times \mathbb{R}^n)} = \left(\sum_{k \geq 0} \|e^{it\Delta} (1 - \Delta_0)^{-s/2} u_0\|_{L^q_t(2^{k-1}, 2^{k+1} \times \mathbb{R}^n)}^q\right)^{1/q}
\]
\[
\lesssim \left(\sum_{k \geq 0} 2^{-k}\right)^{1/q} N^{(2n(1/2 - 1/q) - 2/q) + \epsilon} \|u_0\|_{L^q_L(I)}. \quad (5.5)
\]

This proves inequality (5.1).

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