ANOMALOUS CW-EXPANSIVE SURFACE HOMEOMORPHISMS

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Abstract. We prove that the genus two surface admits a cw-expansive homeomorphism with a fixed point whose local stable set is not locally connected.

1. Introduction. In [5, 7] Hiraide and Lewowicz proved that every expansive homeomorphism of a compact surface \( S \) is conjugate with a pseudo-Anosov diffeomorphism. Recall that a homeomorphism \( f : S \to S \) is expansive if there is \( \eta > 0 \) such that if \( \text{dist}(f^n(x), f^n(y)) \leq \eta \) for all \( n \in \mathbb{Z} \) then \( x = y \). In [6] Kato introduced a generalization of expansivity called continuum-wise expansivity. We say that \( f \) is cw-expansive if there is \( \eta > 0 \) such that if \( C \subset S \) is a continuum (i.e., compact and connected) and \( \text{diam}(f^n(C)) \leq \eta \) for all \( n \in \mathbb{Z} \) then \( C \) is a singleton. In the works of Kato on cw-expansivity we find several generalizations of results holding for expansive homeomorphisms. In this paper we investigate the possibility of extending results from [5, 7] for a cw-expansive surface homeomorphism.

A key concept in dynamical systems is that of the stable set of a point. Given a homeomorphism \( f : S \to S \) and \( \varepsilon > 0 \) we define the \( \varepsilon \)-stable set of a point \( x \in S \) as

\[
W^s_\varepsilon(x) = \{ y \in S : \text{dist}(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0 \}.
\]

For a hyperbolic set it is well known that local stable sets are embedded submanifolds (the invariant manifold theorem). In the papers [5, 7] they prove that if \( f : S \to S \) is expansive then the connected component of \( x \) in \( W^s_\varepsilon(x) \) is a locally connected set. This implies the arc-connection of these components and allows them to prove that each local stable set is a finite union of arcs. In some sense it is an invariant manifold theorem for expansive homeomorphisms of surfaces. After this, they prove the conjugacy with a pseudo-Anosov diffeomorphism, giving a complete classification of expansive surface homeomorphisms.

All the cw-expansive homeomorphisms of surfaces not being expansive known to the author are contained in [1, 2, 10, 11]. In these examples the components of local stable sets are locally connected. The purpose of this paper is to construct a cw-expansive homeomorphism of a compact surface with a point whose local stable set is connected but it is not locally connected.

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The example. The example is a variation of those in [1, 2]. We start defining a homeomorphism of \( \mathbb{R}^2 \) such that \((0, 0)\) is a fixed point and its stable set is not locally connected. Then, this anomalous saddle is inserted in a derived from Anosov diffeomorphism of the torus. Finally, this anomalous derived from Anosov system is connected via a wandering tube with a usual derived from Anosov to obtain our example.

2.1. An anomalous saddle point. First, we will construct a plane homeomorphism with a fixed point at the origin whose local stable set is connected but not locally connected. The homeomorphism will be defined as the composition of a piece-wise linear transformation \( T \) and a time-one map of a flow \( \phi \). This flow will have a non-locally connected set \( E \) of fixed points.

We start with the linear part of the construction. Let \( T_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), for \( i = 1, 2, 3 \), be the linear transformations defined by

\[
T_1(x, y) = (x^2, y), \quad T_2(x, y) = (x, 2y), \quad T_3(1, 1) = \left( \frac{1}{2}, \frac{1}{2} \right), \quad T_3(0, 1) = (0, 2).
\]

Define the piece-wise linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as

\[
T(x, y) = \begin{cases} 
T_1(x, y) & \text{if } x \geq y \geq 0, \\
T_2(x, y) & \text{if } x \leq 0 \text{ or } y \leq 0, \\
T_3(x, y) & \text{if } y \geq x \geq 0.
\end{cases}
\]

In Figure 1 we illustrate the definition of \( T \).

![Figure 1](image)

**Figure 1.** The action of the piece-wise linear transformation \( T \).

Now we define the non-locally connected plane continuum \( E \). Some care is needed in order to be able of relate this set with the transformation \( T \). Define the sets:

\[
C(a) = \{(a, y) \in \mathbb{R}^2 : 0 \leq y \leq a\} \text{ for } a > 0, \\
D_1 = \bigcup_{i=1}^\infty C(\frac{1}{2} + \frac{1}{2^i}), \\
D_{n+1} = T_1(D_n) \text{ for all } n \geq 1, \\
D = \bigcup_{n \geq 1} D_n.
\]

Also consider the non-locally connected continuum \( E = D \cup ([0, 1] \times \{0\}) \) shown in Figure 2.

Now we will define a flow related with the set \( E \). Consider the continuous function \( \rho : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by

\[
\rho(p) = \text{dist}(p, E) = \min\{\text{dist}(p, q) : q \in E\}
\]
and the vertical vector field $X: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$X(p) = (0, \rho(p)).$$

Since

$$|\text{dist}(p, E) - \text{dist}(q, E)| \leq \text{dist}(p, q)$$

for all $p, q \in \mathbb{R}^2$, we have that $\rho$ is Lipschitz. Therefore, by Picard’s theorem, $X$ has unique solutions and we can consider the flow $\phi: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ induced by $X$. Since $\|X(p)\| \leq \|p\|$ for all $p \in \mathbb{R}^2$ we have that every solution is defined for all $t \in \mathbb{R}$.

Let $g: [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$ be the homeomorphism

$$g(x, y) = \begin{cases} \phi_1 \circ T(x, y) & \text{if } y \geq 0, \\ T(x, y) & \text{if } y < 0, \end{cases}$$

where $\phi_1: \mathbb{R}^2 \to \mathbb{R}^2$ is the time-one homeomorphism associated to the vector field $X$. Notice that $g$ is well defined because $\phi_1(x, 0) = (x, 0)$ if $x \in [0, 1]$.

**Proposition 2.1.** The homeomorphism $g$ preserves the vertical foliation on $\mathbb{R}^2$.

**Proof.** It follows because $\phi_t$ and $T$ preserve the vertical foliation. \qed

Consider the region

$$R_1 = \{(x, y) \in [0, 1] \times [0, 1] : x \geq y\}.$$  \hfill (1)

**Lemma 2.2.** For all $p \in R_1$ it holds that $\rho(T(p)) = \frac{1}{2} \rho(p)$ and

$$\phi_t(T(p)) = T(\phi_t(p))$$

if $\phi_t(p) \in R_1$ and $t \geq 0$.

**Proof.** By the definition of $T$ we have that $T(p) = T_1(p) = \frac{1}{2} p$ for all $p \in R_1$. Given $p \in R_1$ consider $q \in E$ such that $\rho(p) = \text{dist}(p, q)$. Then $\rho(T(p)) = \text{dist}(T(p), T(q))$ and $\rho(T(p)) = \frac{1}{2} \rho(p)$.

Consider $t \geq 0$ such that $\phi_t(p) \in R_1$. Since $X$ is a vertical vector field we have that $\phi_{[0, t]}(p) \subset R_1$. For $s \in (0, t)$, if $r = \phi_s(p)$ then

$$X(T(r)) = (0, \rho(T(r))) = \left(0, \frac{1}{2} \rho(r)\right) = d_q T(X(r)).$$

Therefore, $\phi_s(T(p)) = T(\phi_s(p))$ for $s \in (0, t)$ and consequently for $s = t$. \qed

Define the stable set of the origin as usual by

$$W^s_0(0) = \{p \in \mathbb{R}^2 : \lim_{n \to +\infty} \|g^n(p)\| = 0\}.$$
Proposition 2.3. For the homeomorphism \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) defined above it holds that
\[
W^s_\ast(0) \cap ([0, 1] \times [0, 1]) = E.
\]

Proof. First notice that \( E \subset W^s_\ast(0) \) because for all \( p \in E \) and \( t \in \mathbb{R} \) we have that \( \phi_t(p) = p \) and \( T(p) = \frac{1}{2}p \). Then \( g(p) = \frac{1}{2}p \) for all \( p \in E \).

Now take a point \( p \in [0, 1] \times [0, 1] \). For \( p \notin R_1 \), the set defined in (1), we have that \( g(p) = \phi_1 \circ T = \phi_1 \circ T_3 \) (recall the transformation \( T_3 \) in Figure 1). Then, it is easy to see that \( f^n(p) \to \infty \) as \( n \to +\infty \).

Assume that \( p \in R_1 \setminus E \). The velocity of \( \phi_t(p) \) is \( \rho(\phi_t(p)) > 0 \) and this velocity increases with \( t \). Then there is \( n \geq 0 \) such that \( \phi_n(p) \in R_1 \) and \( \phi_{n+1}(p) \notin R_1 \). By Lemma 2.2 we have that
\[
g^n(p) = (\phi_1 \circ T)^n(p) = T^n(\phi_n(p)).
\]
Let \( q = g^n(p) \) and take \( s \in (0, 1) \) such that \( \phi_s(q) \in \partial R_1 \) (in the line \( y = x \)). Again by Lemma 2.2 we have that
\[
g(q) = \phi_1(T(q)) = \phi_1(\phi_s(T(q))) = \phi_{1-s}(T(\phi_s(q))).
\]
Since \( \phi_s(q) \in \partial R_1 \) we have that \( T(\phi_s(q)) \in \partial R_1 \). Then \( \phi_{1-s}(T(\phi_s(q))) \notin R_1 \) because \( 1 - s > 0 \). Therefore \( g^{n+1}(p) \notin R_1 \) and the previous argument finishes the proof. \( \square \)

2.2. A variation of a derived from Anosov. We start recalling some properties of what is known as a derived from Anosov diffeomorphisms. The interested reader should consult [13] (Section 8.8) for a construction of such map and detailed proofs of its properties. A derived from Anosov is a \( C^\infty \) diffeomorphism \( f_{DA} : T^2 \to T^2 \) of the two-dimensional torus such that: it satisfies Smale’s axiom A and its non-wandering set consists of an expanding attractor and a repeller fixed point \( p \in T^2 \). The expanding attractor is locally a Cantor set times an arc, and it has two hyperbolic fixed points of saddle type \( q \) and \( q' \) as in Figure 3.

![Figure 3](image-url)

Figure 3. The derived from Anosov diffeomorphism on the two-dimensional torus.

We will assume that there is a local chart \( \varphi : D \to T^2 \), defined on the disc \( D = \{ x \in \mathbb{R}^2 : \| x \| \leq 2 \} \), such that
1. \( \varphi(0) = p \),
2. the pull-back of the stable foliation by \( \varphi \) is the vertical foliation on \( D \) and
3. \( \varphi^{-1} \circ f_{DA} \circ \varphi(x) = 4x \) for all \( x \in D \) with \( \| x \| \leq 1/2 \).

Now we will insert the anomalous saddle in the derived from Anosov. Let \( q \) be the hyperbolic fixed point shown in Figure 3. Consider a topological rectangle \( R_q \) covering a half-neighborhood of \( q \) as in Figure 4. Consider the homeomorphism \( g \) with an anomalous saddle fixed point defined in Section 2.1. Denote by \( o \) its fixed
point (the origin of $\mathbb{R}^2$) and take a topological rectangle $Q_o \subset \mathbb{R}^2$, similar to $R_p$, as illustrated in Figure 4.

The idea now is to replace $R_q$ with $Q_o$. For this purpose, consider a homeomorphism $h: R_q \cup f_{DA}(R_q) \to Q_o \cup g(Q_o)$ such that $f_{DA}(x) = h^{-1} \circ g \circ h(x)$ for all $x \in \partial R_q$. Define the homeomorphism $f_1: T^2 \to T^2$ by

$$f_1(x) = \begin{cases} f_{DA}(x) & \text{if } x \notin R_q, \\ h^{-1} \circ g \circ h(x) & \text{if } x \in R_q. \end{cases}$$

In this way we obtain a homeomorphism $f_1$ that we call derived from Anosov with an anomalous saddle as in Figure 5.

2.3. Anomalous cw-expansive surface homeomorphism. In this section we finish the construction with ideas from [1, 2]. Consider $S_1$ and $S_2$ two disjoint copies of the torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $f_i: S_i \to S_i$, $i = 1, 2$, be two homeomorphisms such that:

- $f_1$ is the derived from Anosov with an anomalous saddle from Section 2.2, denote by $p_1 \in S_1$ the source fixed point of $f_1$,
- $f_2$ is (conjugate to) the inverse of the derived from Anosov $f_{DA}$ with a sink fixed point at $p_2 \in S_2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.jpg}
\caption{Topological rectangles on the derived from Anosov (left) and on the anomalous saddle (right).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.jpg}
\caption{Derived from Anosov with an anomalous saddle fixed point $q$.}
\end{figure}
Consider local charts \( \varphi_i : D_2 \to S_i, \ i = 1, 2 \), where \( D_2 \) is the compact disk
\[
D_2 = \{ x \in \mathbb{R}^2 : \| x \| \leq 2 \},
\]
such that:
1. \( \varphi_i(0) = p_i \),
2. the pull-back of the unstable foliation by \( \varphi_2 \) is the vertical foliation on \( D_2 \) and
3. \( \varphi_1^{-1} \circ f_1^{-1} \circ \varphi_1(x) = \varphi_2^{-1} \circ f_2 \circ \varphi_2(x) = x/4 \) for all \( x \in D \).

Consider the open disk
\[
D_{1/2} = \{ x \in \mathbb{R}^2 : \| x \| < 1/2 \}
\]
and the compact annulus
\[
A = D_2 \setminus D_{1/2}.
\]
Define \( \psi : A \to A \) as the inversion \( \psi(x) = x/\| x \|^2 \). The pull-back of the unstable foliation on \( S_2 \) by \( \varphi_2 \circ \psi \) on the annulus \( A \) is shown in Figure 6.

![Figure 6. Unstable foliation of \( f_2 \) on the annulus, in the local chart \( \varphi_2 \circ \psi \).](image)

On the disjoint union \( S_3 = [S_1 \setminus \varphi_1(D_{1/2})] \cup [S_2 \setminus \varphi_2(D_{1/2})] \) consider the equivalence relation generated by
\[
\varphi_1(x) \sim \varphi_2 \circ \psi(x)
\]
for all \( x \in A \). Denote by \([x]\) the equivalence class of \( x \). The surface \( S = S_3/\sim \) is the genus two surface if equipped with the quotient topology. Consider the homeomorphism \( f : S \to S \) defined by
\[
f([x]) = \begin{cases} 
[f_1(x)] & \text{if } x \in S_1 \setminus \varphi_1(D_{1/2}) \\
[f_2(x)] & \text{if } x \in S_2 \setminus \varphi_2(D_2)
\end{cases}
\]

For \( x \in S \) and \( \eta > 0 \) define the set
\[
\Gamma_\eta(x) = W^s_\eta(x) \cap W^u_\eta(x).
\]

**Remark 2.4.** In order to prove that a homeomorphism \( f \) is cw-expansive it is equivalent to find \( \eta > 0 \) such that \( \Gamma_\eta(x) \) is totally disconnected for all \( x \in S \).

**Theorem 2.5.** There are cw-expansive homeomorphisms of the genus two surface having a fixed point whose local stable set is connected but it is not locally connected.
**Proof.** Define $A_S = [\varphi_1(A)]$ the annulus on $S$ corresponding to $A$. First note that the non-wandering set of $f$ is expansive and dynamically isolated, i.e. there is a neighborhood $U$ of the non-wandering set $\Omega$ such that if $f^n(x) \in U$ for all $n \in \mathbb{Z}$ then $x \in \Omega$. Also note that for every wandering point $x \in S$ there is $n \in \mathbb{Z}$ such that $f^n(x) \in A_S$. In Figure 6 we have the picture of the unstable foliation on $A_S$. Stable sets do not make a foliation because there is an anomalous saddle. Then, it is convenient to consider the stable partition, i.e., the partition defined by the equivalence relation of being positively asymptotic. This partition is illustrated in Figure 7. We know that the unstable leaves are circle arcs, as

![Figure 7. Stable partition on the annulus $A_S$.](image)

in Figure 6. Also, we can assume that in local charts the non-locally connected stable set associated to $E$ is a countable union of straight lines. Therefore, the intersection of $E$ with each of these circles is at most countable, and consequently, totally disconnected. In the complement of $E$ stable sets are arcs forming a regular $C^0$ foliation. We have the problem that a stable arc in $A_S \setminus E$ may contain a circle arc of the unstable foliation.\(^1\) To solve this problem we define an equivalence relation on $S$ as: $x \sim y$ if there are $n \in \mathbb{Z}$ and an arc $\gamma \subset A_S$ such that $f^n(x), f^n(y) \in \gamma$ and $\gamma \subset W^s(f^n(x)) \cap W^u(f^n(y))$. By construction the equivalence classes are singletons or compact arcs. Moreover, for every non-trivial class $\gamma$ there is $n \in \mathbb{Z}$ such that $f^n(\gamma) \subset A_S$. Denote by $g: S/ \sim \to S/ \sim$ the quotient homeomorphism induced by $f$. By [12] we have that $S/ \sim$ is homeomorphic to $S$. Since $E$ cuts each unstable arc in a totally disconnected set, $E/ \sim$ is homeomorphic to $E$. In this way $g$ is a cw-expansive homeomorphism of the genus two surface $S/ \sim$ and there is a fixed point with a stable set that is connected but not locally connected.

The example constructed for the proof of Theorem 2.5 has further properties that we wish to remark. Following [8], given $N \geq 1$ we say that $f$ is $N$-expansive if there is $\eta > 0$ such that $|\Gamma_\eta(x)| \leq N$ for all $x \in S$, where $|A|$ stands for the cardinality of the set $A$. We have that the example of the previous proof is not $N$-expansive for all $N \geq 1$ because there are points with $|\Gamma_\eta(x)| = \infty$ for arbitrarily small values of $\eta$.

We say that a probability measure $\mu$ on $S$ is an expansive measure [9] if there is $\eta > 0$ such that $\mu(\Gamma_\eta(x)) = 0$ for all $x \in S$. Obviously, if $\mu$ is an expansive measure

\(^1\)It seems that with standard techniques of surface foliations as in [4] we can perturb $f$ in $A_S$ in order to destroy these non-trivial arcs that contradict cw-expansivity. However, we give another argument.
then $\mu(x) = 0$ for all $x \in S$, i.e., $\mu$ is non-atomic. In [3] it is shown that every non-atomic probability measure is expansive if and only if there is $\eta > 0$ such that $|\Gamma_\eta(x)| \leq |Z|$ for all $x \in S$. This property is called countable-expansivity. It seems that our example, or a small $C^0$ perturbation, is countable-expansive.

In the generalized pseudo-Anosov shown in [10, 11] there is a finite number of spines (or 1-prongs), i.e., points whose local stable sets do not separate arbitrarily small neighborhoods. This is a cw-expansive homeomorphism on the two-sphere that is not $N$-expansive for all $N \geq 1$. Our example has a countable set of spines, namely, the points in the set $E$ of Figure 2 in the line $y = x$ give rise to spines in the example. As explained in [10] the generalized pseudo-Anosov of the two-sphere has points with its local stable set non-locally connected. But the components are arcs. Our example has connected components not being locally connected. It seems that if we consider the graph of $\sin(1/x)$ to construct a non locally connected set as the set $E$, we can obtain an anomalous saddle with no arc-connected stable set. Notice that our set $E$ is arc-connected.

Let us finally give some questions. May an example as in Theorem 2.5 be smooth? Assuming that $f$ is a transitive cw-expansive homeomorphism of a compact surface, is it true that the connected components of local stable sets are locally connected? Transitivity means that there is a dense orbit. Does every compact surface admit a cw-expansive homeomorphism with a connected but non-locally connected stable set?

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