ON PICARD BUNDLES OVER PRYM VARIETIES

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ABSTRACT. Let $P$ be the Prym variety associated with a covering $\pi: Y \to X$ between non-singular irreducible projective curves. If $\bar{P}$ is a principally polarized Prym-Tyurin variety associated with $P$, we prove that the induced Abel-Prym morphism $\bar{\rho}: Y \to \bar{P}$ is birational onto its image for genus $g_X > 2$ and $\deg \pi \neq 2$. We use such result to prove that the Picard bundle over the Prym variety is simple and moreover is stable when $\bar{\rho}$ is not birational. As a consequence we obtain that the Picard bundle on the moduli space $M_X(n, \xi)$ of stable vector bundles with fixed determinant and rank $n$ is simple for $g_X \geq 2$ and in the case $g_X = 2$ and $n = 2$ then we prove that the Picard bundle on $M_X(2, \xi)$ is stable.

INTRODUCTION

Let $X$ be a non-singular irreducible projective curve of genus $g_X \geq 2$. We denote by $M(n, d)$ the moduli space of stable vector bundles over $X$ of rank $n$ and degree $d$. If $n$ and $d$ are coprime there exists a universal family $U$ parametrized by $M(n, d)$, called the Poincaré bundle. The higher direct images of the Poincaré bundle on $M(n, d)$ are called Picard sheaves. If $d > 2n(g_X - 1)$, the direct image $W$ of $U$ is locally free and it is called the Picard bundle.

For $n = 1$, $M(1, d)$ is the Jacobian $J^d(X)$. Ein and Lazarsfeld in [EL] proved the stability of the Picard bundle $W_J$ when $d > 2g_X - 1$ and Kempf (see [Ka]) for $d = 2g_X - 1$. For $n \geq 2$, Li proved the stability of $W$ over $M(n, d)$ when $d > 2g_X n$ (see [Li]). Denote by $W_\xi$ the restriction of $W$ to the subvariety $M_\xi \subset M(n, d)$ determined by stable bundles with fixed determinant $\xi \in J^d(X)$. Balaji and Vishwanath proved in [BV] that for rank 2, $W_\xi$ is simple. Actually, they compute the deformations of $W_\xi$.
In this paper we prove (see Theorem 3.1) that for any rank \(W_\xi\) is simple when \(g_X \geq 2\) and \(d > n(n+1)(g_X - 1) + 6\) and in the case \(g_X = 2\) and \(n = 2\) then we prove that \(W_\xi\) is stable for \(d > 10\).

The result follows from studying the restriction of the Picard bundle over the Jacobian of a curve (which is a covering space of \(X\)) to some natural subvarieties, namely the Prym variety and the Jacobian of the base space. That is, let \(\pi : Y \to X\) be a covering between non-singular irreducible projective curves. Let \((J(Y), \Theta_Y)\) be the principally polarized Jacobian of degree zero of \(Y\) and \(\mathcal{P}_Y\) the Poincaré bundle over \(Y \times J(Y)\).

Fixing a line bundle \(L_0\) on \(Y\) of degree \(d > 2g_Y - 2\), we identify \(J(Y)\) with \(J^d(Y)\) and consider the Picard bundle \(W_J = p_2^*(p_1^*L_0 \otimes \mathcal{P}_Y)\) on \(J(Y)\). We denote the restriction of \(W_J\) to the subvarieties \(P\) and \(\pi^*(J(X))\) of \(J(Y)\) by \(W_P\) and \(W_\pi\) respectively.

We prove that if \(\pi^*(L_0)\) is stable, then \((\pi^*)^*(W_J)\) is \(\Theta_X\)-stable on \(J(X)\) where \(\pi^*: J(X) \to J(Y)\) is the pull-back morphism (see Theorem 2.2 and Remark 2.1). We deduce that \(W_\pi\) is stable with respect to the polarization \(\Theta_Y|_{\pi^*(J(X))}\).

The restriction \(\Theta_P\) of \(\Theta_Y\) to \(P\) need not be a multiple of a principal polarization. However, it is possible to construct a principally polarized abelian variety \((\tilde{P}, \Xi)\) and an isogeny \(f: \tilde{P} \to P\) with \(f^{-1}(\Theta_P) \equiv n\Xi\) such that there exists a map \(\tilde{\rho}: Y \to \tilde{P}\) with \(\tilde{\rho}(Y)\) and \(\Xi^{m-1}\) numerically equivalent where \(m = \dim P = \dim \tilde{P}\).

We prove that if \(n \neq 2\) and \(g_X > 2\), the morphism \(\tilde{\rho}: Y \to \tilde{P}\) is birational onto its image (Theorem 1.2) and we use some properties of the Fourier-Mukai transform to prove that in this case \(W_P\) is simple.

Moreover, we give explicitly the cases where \(\tilde{\rho}\) may not be birational. In this case, we prove that the Prym variety is the image in \(J(Y)\) of the Jacobian of the normalization of the curve \(\tilde{\rho}(Y)\). As a consequence of stability of \(W_\pi\), we have that the Picard bundle \(W_P\) is stable with respect to \(\Theta_Y|_P\) when \(\tilde{\rho}\) is not birational. In particular, for \(g_X \geq 2\) and \(d > 2g_Y + 4\), \(W_P\) is simple (Theorem 2.3).

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1. Prym varieties

We shall denote by $J(X)$ the Jacobian of degree zero of a non-singular projective irreducible curve $X$ and by $\Theta_X$ the natural polarization on $J(X)$ given by the Riemann theta divisor.

Let $\pi: Y \rightarrow X$ be a covering of degree $n$ between non-singular irreducible projective curves of genus $g_Y$ and $g_X$ respectively. We have the norm map $\text{Nm} \pi: J(Y) \rightarrow J(X)$ and the pull back map $\pi^*: J(X) \rightarrow J(Y)$. It is known that the map $h: J(X) \rightarrow \pi^*(J(X))$ induced by $\pi^*$ is an isogeny such that $h^{-1}(\Theta_Y|_{\pi^*(J(X))}) \equiv n\Theta_X$. We will denote by $\Theta_\pi$ the restriction of the theta divisor of $J(Y)$ to $\pi^*(J(X))$.

The Prym variety $P$ associated with the covering is the abelian subvariety of $J(Y)$ defined by $P = \text{Im}(n \text{Id}_{J(Y)} - \pi^* \circ \text{Nm} \pi): J(Y) \rightarrow P$.

On $P$ there exists a natural polarization $\Theta_P$ given by the restriction of the theta divisor $\Theta_Y$ to $P$. In general $\Theta_P$ is not a multiple of principal polarization (see [LB], Theorem 3.3, pp. 376). The following theorem gathers several standard results about Prym-Tyurin varieties for which we refer to [LB], [We] and [Go].

**Theorem 1.1.** Let $\pi: Y \rightarrow X$ be a covering of degree $n$. There exists a principally polarized abelian variety $(\tilde{P}, \Xi)$, an isogeny $f: \tilde{P} \rightarrow P$ and a morphism $\tilde{\mu}: J(Y) \rightarrow \tilde{P}$ such that:

1. $f^{-1}(\Theta_Y|_{\tilde{P}}) \equiv n\Xi$.
2. The map $f \circ \tilde{\mu}: J(Y) \rightarrow P$ coincides with the morphism $\mu$ defined as above.
3. The restriction $\tilde{\mu}|_{P}$ is surjective and the composition map $f \circ \tilde{\mu}|_{P}$ coincides with the multiplication by $n$ on $P$.

Moreover, the following relation holds

$$\text{Nm}^{-1}(\Theta_X) + \tilde{\mu}^{-1}(\Xi) \equiv n\Theta_Y,$$

where $\equiv$ denotes algebraic equivalence between divisors.

From now on we shall consider a fixed isogeny $f: \tilde{P} \rightarrow P$ enjoying the properties listed in Theorem 1.1. Let $y_0$ be a fixed point on $Y$. Let $\tilde{\rho}_{y_0}: Y \rightarrow \tilde{P}$ and $\rho_{y_0}: Y \rightarrow P$ be the morphisms

$$\tilde{\rho}_{y_0} = \tilde{\mu} \circ \alpha_{y_0} \quad \text{and} \quad \rho_{y_0} = f \circ \tilde{\rho}_{y_0} = \mu \circ \alpha_{y_0}$$

where $\alpha_{y_0}$ is the morphism induced by the isogeny $f$.
where $\alpha_{y_0}: Y \to J(Y)$ is the Abel-Jacobi map defined by $y \mapsto \mathcal{O}(y - y_0)$. We shall refer to $\rho_{y_0}$ and $\tilde{\rho}_{y_0}$ as the Abel-Prym maps of $P$ and $\tilde{P}$ respectively. When there is no confusion on the point $y_0$ we shall omit it in the notations.

By the theory of Prym-Tyurin varieties (see [We] and [LB]) the morphism $\tilde{\rho}$ has the following property:

$$\tilde{\rho}_* [Y] = \frac{n}{(m-1)!} \wedge [\Xi],$$

where $m = \dim P = \dim \tilde{P}$. This last property is usually expressed by saying that the curve $Y$ (or rather the morphism $\tilde{\rho}$) is of class $n$ in $(\tilde{P}, \Xi)$.

In general the Abel-Prym map $\rho: Y \to P$ need not be birational. However, we have the following theorem.

**Theorem 1.2.** Let $\pi: Y \to X$ be a covering of degree $n$. If $g_X > 2$ and $n \neq 2$ then Abel-Prym morphism $\tilde{\rho}: Y \to \tilde{P}$ is birational onto its image.

In order to prove Theorem 1.2 we need a lemma. Let $(A, \Theta_A)$ be a principally polarized abelian variety of dimension $d$ and $C$ a non-singular projective irreducible curve. Let $\nu: C \to A$ be a morphism and let $u: J(C) \to A$ be the unique morphism (up to a translation) obtained by the universal property of the Jacobian such that $\nu = u \circ \alpha$. Using the polarizations, we define the transposed morphism $u^t: A \to J(C)$ by

$$u^t = \phi^{-1}_{\Theta_C} \circ \hat{u} \circ \phi_{\Theta_A},$$

where $\hat{u}: \hat{A} \to \hat{J}(C)$ is the dual morphism of $u$ and $\phi_{\Theta_A}: A \to \hat{A}$, $\phi_{\Theta_C}: J(C) \to \hat{J}(C)$ are the isomorphisms induced by the principal polarizations $\Theta_A$ and $\Theta_C$ respectively.

Recall that a curve $C$ is of class $n$ in $(A, \Theta_A)$, that is $\nu_* [C] = \frac{n}{(d-1)!} \wedge^{d-1}[\Theta_A]$, if and only if $u \circ u^t = n \text{Id}_A$ (see [We]).

**Lemma 1.3.** Let $(A, \Theta_A)$ be a principally polarized abelian variety and $\nu: C \to A$ a curve of class $n$ in $A$. Let $Z$ be the normalization of the image $\nu(C)$ and let $p: C \to Z$ and $\tilde{\nu}: Z \to A$ be the induced maps. If $r$ is the degree of $p$, then $r$ divides $n$ and the curve $Z$ is of class $\frac{n}{r}$ in $A$.

**Proof.** Let $Nm p: J(C) \to J(Z)$ be the norm map of $p$ and $u: J(C) \to A$ the morphism corresponding to $\nu$ obtained by the universal property of the Jacobian. Similarly, $\tilde{u}: J(Z) \to A$ corresponds to $\tilde{\nu}$. Since $\nu: C \to A$ is of class $n$ and $N_p \circ p^* = N_p \circ N_p^t =$
Thus, for every \( x \in A_r \) in the \( r \)-torsion of \( A \), \( r \cdot (\tilde{u} \circ \tilde{u}')(x) = 0 \) and therefore \( nx = 0 \). Hence, \( A_r \subset A_n \) and \( r \) divides \( n \). Since \( A \) is connected and \( A_r \) is discrete, the morphism \( A \to A_r \) given by \( x \mapsto (\tilde{u} \circ \tilde{u}')(x) - \frac{n}{r}x \) is identically zero. Hence \( \tilde{u} \circ \tilde{u}' = \frac{n}{r} \text{Id}_A \). \( \square \)

**Proof of Theorem 1.2.** Suppose that \( \tilde{\rho} \) is not birational onto its image and \( n \neq 2 \). Let \( Z \) be the normalization of the curve \( \tilde{\rho}(Y) \) and let \( \tilde{\pi} : Y \to Z \) be the morphism induced by \( \tilde{\rho} \). Let \( \tilde{n} \) be the degree of \( \tilde{\pi} \). That is,

\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{\pi}} & Z \\
\xleftarrow{\pi} & & \\
X & & \\
\end{array}
\]

Since the morphism \( \tilde{\mu} \) is surjective, \( Z \) generates \( \tilde{\mathcal{P}} \). Therefore the map \( \tilde{u} : J(Z) \to \tilde{\mathcal{P}} \) defined by the universal property of the Jacobian is surjective. Hence \( g_Y - g_X = \dim \tilde{\mathcal{P}} \leq g_Z \), where \( g_Z \) is the genus of \( Z \).

By Riemann-Hurwitz formula for the coverings \( \pi \) and \( \tilde{\pi} \), we have the following inequality

\[
g_Y \leq \frac{g_Y - 1 - (\deg R_\pi)/2}{n} + \frac{g_Y - 1 - (\deg R_{\tilde{\pi}})/2}{\tilde{n}} + 2 \tag{1}
\]

where \( R_\pi \) and \( R_{\tilde{\pi}} \) are the ramification divisors.

From Lemma [1.3], it follows that \( \tilde{n} \) divides \( n \); let \( q = n/\tilde{n} \). Hence, inequality (1) implies that

\[
(n - q - 1)g_Y \leq 2n - q - 1 - ((\deg R_\pi)/2 + q(\deg R_{\tilde{\pi}})/2).
\]

Since \((\deg R_\pi)/2 + q(\deg R_{\tilde{\pi}})/2) \geq 0\) and \( n \neq q + 1 \) because \( n \neq 2 \), we have

\[
g_Y \leq 2 + \left[ \frac{q + 1}{n - q - 1} \right]
\]

where \([\cdot]\) denotes the integral part of a rational number. Since \( \tilde{n} > 1 \), it is easy to check that \([\frac{q + 1}{n - q - 1}] \geq 1 \) if and only if \( \tilde{n} = 2, 3, 4 \). Studying these cases we obtain that
if $\tilde{\rho}$ is not birational and $n \neq 2$, then

$$n = 3, \quad \tilde{n} = 3 \text{ and } g_Y \leq 4; \quad n = 4, \begin{cases} \text{if } \tilde{n} = 4 & \text{then } g_Y \leq 3, \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 5. \end{cases}$$

$$n = 6, \begin{cases} \text{if } \tilde{n} = 6 & \text{then } g_Y \leq 2, \\ \text{if } \tilde{n} = 3 & \text{then } g_Y \leq 3, \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 4. \end{cases}$$

$$n \neq 3, 4, 6, \begin{cases} \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 3, \\ \text{if } \tilde{n} \neq 2 & \text{then } g_Y \leq 2. \end{cases}$$

In all cases, we have that if $\tilde{\rho}$ in not birational onto its image and $n \neq 2$, then $g_Y \leq 5$.

Now if we also suppose that $g_X \geq 2$, by the Riemann-Hurwitz formula for the covering $\pi$, we deduce that if $\deg \pi \neq 2$ and $g_X \geq 2$, then $\tilde{\rho}$ is not birational onto its image if the following cases occur:

$$n = 3, \quad \tilde{n} = 3, \quad g_Y = 4, \quad g_X = 2, \quad g_Z = 2, \quad R_\pi = R_{\tilde{\pi}} = 0$$

$$n = 4, \quad \tilde{n} = 2, \quad g_Y = 5, \quad g_X = 2, \quad g_Z = 3, \quad R_\pi = R_{\tilde{\pi}} = 0$$

(2)

Therefore, if we assume that $g_X > 2$ and $n \neq 2$, the theorem is proved. \(\square\)

**Remark 1.4.** From the proof of Theorem 1.2 we deduce that if $g_X \geq 2$, then the Abel-Prym map $\tilde{\rho}$ may not be birational onto its image possibly in the cases given in (2) and when $n = 2$. For $n = 2$, it is well-known that if $Y$ is not an hyperelliptic curve then $\rho$ is birational onto its image and therefore $\tilde{\rho}$ is birational onto its image.

We finish this section with a lemma that we will use later.

**Lemma 1.5.** *For any $\xi \in \tilde{P}$, we have*

$$\tilde{\rho}^* \left( \tau^*_{\xi} \mathcal{O}_P(\Xi) \right) \cong \rho^* \mathcal{O}_\tilde{P}(\Xi) \otimes f(\tilde{\xi})^{-1},$$

*where $\tau^*_{\xi}$ is the translation by $\tilde{\xi}$.*

**Proof.** By Theorem 1.1, there exists $\xi \in P$ such that $\tilde{\mu}(\xi) = \tilde{\xi}$. Using the relation between the polarizations given in this theorem, we obtain

$$\tilde{\rho}^* \left( \tau^*_{\xi} \mathcal{O}_P(\Xi) \right) \cong \alpha^* \left( \tilde{\mu}^* \left( \tau^*_{\xi} \mathcal{O}_P(\Xi) \right) \right) \cong \alpha^* \left( \tau^*_{\xi} \left( \mu^* \mathcal{O}_P(\Xi) \right) \right) \cong \alpha^* \left( \tau^*_{\xi} \left( \mathcal{O}_{J(Y)}(n\Theta_Y) \otimes N_\pi^* \mathcal{O}_{J(X)}(\Theta_X)^\vee \otimes \mathcal{N} \right) \right),$$

where $\mathcal{N} \in \text{Pic}^0(J(Y))$. In particular, $\mathcal{N}$ is invariant under translations. Moreover, we have $\tau^*_{\xi} \left( N_\pi^* \mathcal{O}_{J(X)}(\Theta_X)^\vee \right) \cong N_\pi^* \left( \mathcal{O}_{J(X)}(\Theta_X)^\vee \right)$ because $\xi \in P$ and the isomorphism

$$\alpha^* \left( \tau^*_{\xi} \mathcal{O}_{J(Y)}(n\Theta_Y) \right) \cong \alpha^* \left( \mathcal{O}_{J(Y)}(n\Theta_Y) \right) \otimes \xi^{-n}.$$
Therefore, it follows that \( \bar{\rho}^* \left( \tau_\xi^* \mathcal{O}_\tilde{\mathcal{P}}(\Xi) \right) \cong \bar{\rho}^* \mathcal{O}_\tilde{\mathcal{P}}(\Xi) \otimes \xi^{-n} \). Since \( \xi^{-n} = f(\bar{\mu}|_{\mathcal{P}}(\xi^{-1})) = f(\bar{\xi})^{-1} \), the lemma is proved. \( \square \)

2. Stability and restrictions of Picard bundles

Let \( Y \) be a non singular irreducible projective curve of genus \( g_Y \geq 2 \). We fix a line bundle \( L_0 \) on \( Y \) of degree \( d \) and a point \( y_0 \in Y \). Let \( \mathcal{P}_Y \) be the Poincaré bundle over \( Y \times J(Y) \) normalized with respect to the point \( y_0 \), i.e. \( \mathcal{P}_Y|_{\{y_0\} \times J(Y)} \cong \mathcal{O}_{J(Y)} \). The Picard sheaves (relative to \( L_0 \)) on \( J(Y) \) are defined as the higher direct images \( R^i p_2^* (p_1^* L_0 \otimes \mathcal{P}_Y) \) where \( p_j \) are the canonical projections of \( Y \times J(Y) \) in the \( j = 1, 2 \) factor. If \( d > 2g_Y - 2 \), then \( \mathcal{W}_j := p_2^* (p_1^* L_0 \otimes \mathcal{P}_Y) \) is a vector bundle known as the Picard bundle. We shall consider the restriction of \( \mathcal{W}_j \) to some subvarieties when the curve is a covering space.

Let \( \pi: Y \to X \) be a covering of degree \( n \) between non-singular projective irreducible curves of genus \( g_Y \) and \( g_X \) respectively and let \( \pi^*: J(X) \to J(Y) \) be the pull-back morphism. As in §1 consider the Prym variety \( (P, \Theta_P) \) and \( (\pi^*(J(X)), \Theta_\pi) \) in \( J(Y) \).

From now on we fix a line bundle \( L_0 \) on \( Y \) of degree \( d > 2g_Y - 2 \). Denote by \( \mathcal{W}_P \) and \( \mathcal{W}_n \) the restrictions of the Picard bundle \( \mathcal{W}_j \) (relative to \( L_0 \)) to \( P \) and \( \pi^*(J(X)) \) respectively.

**Remark 2.1.** We can take a line bundle \( L_0 \in J^d(Y) \) such that \( \pi_*(L_0) \) is stable and \( \deg \pi_*(L_0) > 2ng_X \) without loss of generality. Indeed: for a generic line bundle \( L \) on \( Y \), A. Beauville in [Be] has proved that \( \pi_* L \) is stable on \( X \) if \( |\chi(L)| \leq g_X + \frac{g_X^2}{n} \) or \( \deg \pi < \max \{g_X(1 + \sqrt{3}) - 1, 2g_X + 2\} \). Therefore, if \( L \in J^d(Y) \) with

\[
g_Y - g_X - 1 - \frac{g_X^2}{n} \leq d' \leq g_Y + g_X - 1 + \frac{g_X^2}{n} ,
\]

then \( \pi_* L \) is stable on \( X \) and \( \pi_*(L \otimes \pi^* M) \) is stable for any line bundle \( M \) on \( X \). Thus, for a generic line bundle \( L_0 \in J^d(Y) \) with \( d \) such that:

(i) \( d > \max \{2g_Y - 2 + 2n - (\deg R_\pi)/2 , 2g_Y - 2\} \),

(ii) \( d \) is equivalent (modulo \( n \)) to a number \( d' \) such that \( d' \) fulfills the condition (3),

we have that \( \pi_*(L_0) \) is stable and \( \deg \pi_*(L_0) = \deg L_0 - (\deg R_\pi)/2 > 2ng_X \). In the case \( \deg \pi < \max \{g_X(1 + \sqrt{3}) - 1, 2g_X + 2\} \), it is sufficient that \( \deg L_0 \) fulfills the condition (i).
Theorem 2.2. If \( \pi_*(L_0) = E_0 \) is stable and \( \deg(E_0) > 2ng_X \), then \( (\pi^*)^*(W_J) \) is \( \Theta_X \)-stable on \( J(X) \).

**Proof.** For convenience of writing, we set up the following commutative diagrams:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
p_x & & p_1 \\
X \times J(X) & \xrightarrow{\pi \times id} & Y \times J(X) \\
p_{J(X)} & & p_2 \\
J(X) & \xrightarrow{\pi^*} & J(Y)
\end{array}
\]

where the vertical morphisms are the natural projections.

Fix the Poincaré bundle \( \mathcal{P}_X \) on \( X \times J(X) \) normalized with respect to the point \( \pi(y_0) \in X \). Then, the vector bundle \( (id \times \pi^*)^*\mathcal{P}_Y \) on \( Y \times J(X) \) is isomorphic to the bundle \( (\pi \times id)^*\mathcal{P}_X \). From diagram (4) and the base change formula, we have

\[
(\pi^*)(W_J) \cong q_2*(q_1^*L_0 \otimes (id \times \pi^*)^*\mathcal{P}_Y) \cong p_{J(X)}*(q_1^*L_0 \otimes (\pi \times id)^*\mathcal{P}_X) \\
\cong p_{J(X)}*((\pi \times id)_*(q_1^*L_0) \otimes \mathcal{P}_X) \cong p_{J(X)}*(p_X^*(E_0) \otimes \mathcal{P}_X).
\]

Since \( E_0 \) is stable of degree \( d > 2ng_X \), \( p_X^*(E_0) \otimes \mathcal{P}_X \) is a family of stable bundles parametrized by \( J(X) \). Such family corresponds to an embedding of the Jacobian in the moduli space \( \mathcal{M}(n, d) \) of stable vector bundles of rank \( n \) and degree \( d \). As in the proof of Theorem 2.5 in [L], it follows that \( p_{J(X)}*(p_X^*(E_0) \otimes \mathcal{P}_X) \) is \( \Theta_X \)-stable. \( \square \)

With the hypothesis of the previous theorem, we obtain

**Corollary 2.3.** The restriction \( W_\pi \) of \( W_J \) to \( \pi^*(J(X)) \) is \( \Theta_\pi \)-stable.

**Proof.** The map \( h : J(X) \to \pi^*(J(X)) \) is an isogeny such that \( h^{-1}(\Theta_\pi) \equiv n\Theta_X \). Since \( h^*(W_\pi) \cong (\pi^*)^*(W_J) \), we have from Lemma 2.1 in [BBN] that \( W_\pi \) is \( \Theta_\pi \)-stable. \( \square \)

To study the restriction \( W_P \) of \( W_J \) to the Prym variety \( P \) we consider two cases, namely when the Abel-Prym map \( \tilde{\rho} \) is not birational and when it is. In both cases we can reduce our study to consider the corresponding vector bundle over \( (\tilde{P}, \Xi) \) where \( f : \tilde{P} \to P \) is a fixed isogeny as in Theorem 1.1. That is, if \( j_P : P \hookrightarrow J(Y) \) is the natural inclusion, then we will denote by \( \beta : \tilde{P} \to J(Y) \) the composition map \( f \circ j_P \) and by \( W_{\tilde{P}} \) the vector bundle \( \beta^*(W_J) \). Actually, \( f^*(W_P) \cong W_{\tilde{P}} \).

**Proposition 2.4.** If \( W_{\tilde{P}} \) is \( \Xi \)-stable (resp. simple), then \( W_P \) is \( \Theta_P \)-stable (resp. simple).
Proof. The stability again follows from Lemma 2.1 in [BBN]. Suppose $\mathcal{W}_P$ is simple. Since $\mathcal{O}_P \hookrightarrow f_*\mathcal{O}_P$ is injective, the map

$$\text{Hom}(\mathcal{W}_P, \mathcal{W}_P) \hookrightarrow \text{Hom}(\mathcal{W}_P, \mathcal{W}_P \otimes f_*\mathcal{O}_P) \cong \text{Hom}(\mathcal{W}_P, f_*f^*\mathcal{W}_P) \cong \text{Hom}(\mathcal{W}_P, \mathcal{W}_P) \cong \mathbb{C}$$

is injective. Hence, $\text{Hom}(\mathcal{W}_P, \mathcal{W}_P) \cong \mathbb{C}$ and $\mathcal{W}_P$ is simple.

Suppose $\tilde{\rho}$ is not birational. As in section §1, let $Z$ be the normalization of $\tilde{\rho}(Y)$ and $\tilde{\pi} : Y \to Z$ the morphism induced by $\tilde{\rho}$ of degree $\tilde{n}$.

**Proposition 2.5.** If the Abel-Prym map $\tilde{\rho} : Y \to \tilde{P}$ is not birational onto its image and $\deg L_0 > 2g_Y + 4$, then the Picard bundle $\mathcal{W}_P$ is $\Theta_P$-stable.

Proof. We consider separately each one of the cases where $\tilde{\rho}$ is not birational (see Remark 1.4).

1) Suppose $n = 2$.

By lemma 1.3, $\deg \tilde{\pi} = 2$ and $\tilde{\rho}(Y)$ is of class one in $(\tilde{P}, \Xi)$. Then, by Matsusaka’s Criterion, $Z = \tilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\tilde{P}, \Xi)$.

From the construction of the map $\tilde{\rho}$, we have $\text{Nm} \tilde{\pi} = \tilde{\mu}$. Moreover, $\tilde{\mu}^t = j_P \circ f = \beta$ (see section §1 in [We]). Therefore the map $\tilde{\pi}^* : J(Z) \to J(Y)$ coincides with the map $\beta : \tilde{P} \to J(Y)$ via the isomorphism $J(Z) \cong \tilde{P}$. By definition of $\mathcal{W}_P$, we have

$$\mathcal{W}_P \cong (\tilde{\pi}^*)(\mathcal{W}_J).$$

In this case, $\deg \tilde{\pi} = 2 < \max\{g_Z(1 + \sqrt{3}) - 1, 2g_Z + 2\}$ because $g_Z \geq 1$. Therefore, by Remark 2.1 and Theorem 2.2, if $\deg L_0 = d > 2g_Y + 2 \geq \max\{2g_Y + 2 - \deg(R_\pi)/2, 2g_Y - 2\}$, then $(\tilde{\pi}^*)(\mathcal{W}_J) \cong \mathcal{W}_P$ is $\Xi$-stable and from Proposition 2.4, $\mathcal{W}_P$ is $\Theta_P$-stable.

2) Suppose $n = 3$, $\tilde{n} = 3$, $g_Y = 4$, $g_X = 2$, $g_Z = 2$, $R_\pi = R_\tilde{\pi} = 0$.

Since $\tilde{\rho}(Y)$ is of class one in $(\tilde{P}, \Xi)$, $Z = \tilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\tilde{P}, \Xi)$. In this case, $\deg \tilde{\pi} = 3 < \max\{g_Z(1 + \sqrt{3}) - 1, 2g_Z + 2\}$. Hence, if $\deg L_0 = d > 2g_Y + 4 = 12$, then $\mathcal{W}_P$ is $\Xi$-stable and $\mathcal{W}_P$ is $\Theta_P$-stable.

3) Suppose $n = 4$, $\tilde{n} = 2$, $g_Y = 5$, $g_X = 2$, $g_Z = 3$, $R_\pi = R_\tilde{\pi} = 0$. 


We have the following commutative diagrams:

\[ \begin{array}{cccccc}
Y & \xrightarrow{\pi} & Z & \xrightarrow{\tilde{\rho}} & \tilde{P} \\
\downarrow{\alpha_g} & & \downarrow{\alpha_{\tilde{\pi}(g_0)}} & & \\
J(Y) & \xrightarrow{\text{Nm}_{\tilde{\pi}}} & J(Z) & \xrightarrow{u} & \tilde{P}
\end{array} \]

where \( u \) is the map induced by \( Z \to \tilde{P} \). Therefore,

\[ \beta = j_P \circ f = \tilde{\mu}^t = \tilde{\pi}^* \circ u^t: \tilde{P} \to J(Y) \]

and \( \text{Im} \tilde{\mu}^t = P \subset \text{Im} \tilde{\pi}^* \). Since \( \dim P = 3 = \dim J(Z) = \dim \tilde{\pi}^*(J(Z)) \), it follows that \( \tilde{\pi}^*(J(Z)) = P \subset J(Y) \). Since \( \deg \tilde{\pi} = 2 < \max\{g_Z(1 + \sqrt{3}) - 1, 2g_Z + 2\} \), if \( \deg L_0 = d > 2g_Y + 2 = 12 \) then, from Corollary 2.3, the restriction of \( \mathcal{W}_f \) to \( \tilde{\pi}^*(J(Z)) \) is \( \Theta_{\tilde{Y}|\tilde{\pi}^*(J(Z))} \)-stable, i.e. \( W_P \) is \( \Theta_P \)-stable.

**Remark 2.6.** Observe that, in the previous proposition, the result holds for any line bundle \( L_0 \) (of degree \( d > 2g_Y + 4 \)) whereas in Remark 2.1 the condition “\( \pi^*(L_0) \) is stable” holds for a generic line bundle \( L_0 \). If the Picard bundle \( W_{L_0,P} \) corresponding to \( L_0 \) is stable, then \( W_{L_0,P} = p_{2*}(p_1^*L \otimes \mathcal{P}_Y|_Y \otimes P) \) is stable for any \( L \) (of degree \( d \)). In fact, since \( L \otimes L^{-1} \in J(Y) \), one can write \( L \cong L_0 \otimes \pi^*N \otimes M \) where \( M \in P \) and \( N \in J(X) \), therefore the two Picard bundles are related as follows \( W_{L,P} \cong \tau_{M}^*(W_{L_0 \otimes \pi^*N,P}) \), where \( \tau_M: P \to P \) is the translation by \( M \). Since \( \pi^*(L_0) \) is stable, \( \pi^*(L_0 \otimes \pi^*N) \) is stable. Hence \( W_{L,P} \) is stable.

Before we consider the case when the map \( \tilde{\rho} \) is birational we describe the bundle \( W_{\tilde{P}} \) in different way.

The abelian variety \( (\tilde{P}, \Xi) \) is principally polarized, so we can identify \( \tilde{P} \) with its dual abelian variety. The Poincaré bundle on \( \tilde{P} \times \tilde{P} \) is given by

\[ \mathcal{Q} \cong m^*O_{\tilde{P}}(\Xi) \otimes q_1^*O_{\tilde{P}}(\Xi)^\vee \otimes q_2^*O_{\tilde{P}}(\Xi)^\vee, \]

where \( m \) is the multiplication law on \( \tilde{P} \) and \( q_i \) the canonical projections of \( \tilde{P} \times \tilde{P} \) in the \( i = 1, 2 \) factor.

**Proposition 2.7.** The vector bundle \( W_{\tilde{P}} \) is isomorphic to the bundle \( q_{2*}(q_1^*(\rho_*(L_0)) \otimes \mathcal{Q}^\vee) \), where \( q_i \) are the projection of \( \tilde{P} \times \tilde{P} \) in the \( i = 1, 2 \) factor.
Proof. Consider the commutative diagrams:

\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{\rho}} & \tilde{P} \\
\downarrow{p_1} & & \downarrow{q_1} \\
Y \times P & \xrightarrow{id \times f} & Y \times \tilde{P} \xrightarrow{\tilde{\rho} \times id} \tilde{P} \times \tilde{P} \\
\downarrow{p_2} & & \downarrow{q_2} \\
 & & \tilde{P}
\end{array}
\]

Let \( \mathcal{P}_{\tilde{P}} \) be the line bundle \( \mathcal{P}_{\tilde{P}} = (id \times (f \circ j_P))^*(\mathcal{P}_Y) \) on \( Y \times \tilde{P} \), where \( j_p : P \to J(X) \) is the natural inclusion. From the normalization of \( \mathcal{P}_Y \), the restrictions of the line bundles \( \mathcal{P}_{\tilde{P}} \) and \( (\tilde{\rho} \times id)^* \mathcal{Q}' \) to \( \{y_0\} \times \tilde{P} \) are trivial and for every \( \tilde{\xi} \in \tilde{P} \), the restrictions of the both bundles to \( Y \times \{ \tilde{\xi} \} \) are isomorphic to \( f(\tilde{\xi}) \) by Lemma 1.5. Therefore, \( \mathcal{P}_{\tilde{P}} \cong (\tilde{\rho} \times id)^* \mathcal{Q}' \).

From the projection formula and the base-change formula, we have

\[
\mathcal{W}_{\tilde{P}} \cong p_{2*} \left( p_1^* L_0 \otimes \mathcal{P}_{\tilde{P}} \right) \cong q_{2*} \left( (\tilde{\rho} \times id)_* (p_1^* L_0 \otimes (\tilde{\rho} \times id)^* \mathcal{Q}') \right) \\
\cong q_{2*} \left( (\tilde{\rho} \times id)_* (p_1^* L_0) \otimes \mathcal{Q}' \right) \cong q_{2*} \left( q_1^* (\tilde{\rho}_*(L_0)) \otimes \mathcal{Q}' \right).
\]

Proposition 2.8. If the Abel-Prym map \( \tilde{\rho} \) is birational onto its image then the restriction \( \mathcal{W}_P \) of the Picard bundle \( \mathcal{W}_J \) to the Prym variety is simple.

Proof. From Proposition 2.4 it is enough to prove that \( \mathcal{W}_{\tilde{P}} \) is simple. The bundle \( \mathcal{W}_{\tilde{P}} \) is the Fourier-Mukai transform of the sheaf \( \tilde{\rho}_*(L_0) \) by Proposition 2.7. Since \( \deg(L_0) > 2g_Y - 2 \), the sheaf \( q_1^* (\tilde{\rho}_*(L_0)) \otimes \mathcal{Q}' \) has only one non-null direct image. Then the Fourier-Mukai transform is a complex concentrated in degree zero and, therefore, \( \mathcal{W}_{\tilde{P}} \) is simple if \( \tilde{\rho}_*(L_0) \) is simple (see [M], Corollary 2.5).

To prove that \( \tilde{\rho}_*(L_0) \) is simple write \( \tilde{\rho} \) as a composite \( j \circ \tilde{\nu} \) where \( \tilde{\nu} : Y \to \tilde{\rho}(Y) \) is a birational morphism between curves and \( j : \tilde{\rho}(Y) \to \tilde{P} \) is a closed embedding. Since \( \tilde{\rho} \) is birational onto its image, \( \tilde{\nu}_*(L_0) \) is of rank 1. Therefore the sheaf \( K \) defined by

\[
0 \to K \to \tilde{\nu}^* \tilde{\nu}_*(L_0) \to L_0 \to 0
\]

is a torsion sheaf. From the exact sequence

\[
0 \to \text{Hom}(L_0, L_0) \to \text{Hom}(\tilde{\nu}^* \tilde{\nu}_*(L_0), L_0) \to \text{Hom}(K, L_0).
\]
and the fact that $\text{Hom}(K, L_0) = 0$, we obtain

$$\text{Hom}(\tilde{\nu}^*L_0, \tilde{\nu}^*L_0) \cong \text{Hom}(\tilde{\nu}^*\tilde{\nu}^*L_0, L_0) \cong \text{Hom}(L_0, L_0) \cong \mathbb{C}.$$ 

Since $j$ is a closed embedding, $j^*j_*E \cong E$ for any sheaf $E$ and applying the adjunction formula again, we obtain that $j_*\tilde{\nu}_*(L_0) \cong \tilde{\rho}_*L_0$ is simple. \hfill $\square$

From Proposition 2.3 and Proposition 2.8 we have that:

**Theorem 2.9.** The restriction $W_P$ of the Picard bundle $W_J$ to the Prym variety is simple if $g_X \geq 2$ and $\deg L_0 > 2g_Y + 4$.

### 3. Picard bundles over moduli spaces

We shall recall the construction and properties of spectral covering given by Beauville, Narasimhan and Ramanan in [BNR].

Let $X$ be a non-singular projective irreducible curve of genus $g_X \geq 2$. Denote by $\mathcal{M}(n, d)$ the moduli space of stable vector bundles over $X$ of rank $n$ and degree $d$. If $n$ and $d$ are coprime there exists a universal family $U$ parametrized by $\mathcal{M}(n, d)$ called the Poincaré bundle. If $d > 2n(g_X - 1)$ the Picard bundle $W$ is the direct image of $U$ and it is locally free. Denote by $W_\xi$ the restriction of $W$ to the subvariety $\mathcal{M}_\xi \subset \mathcal{M}(n, d)$ determined by stable bundles with fixed determinant $\xi \in J^d(X)$.

Let $K$ be the canonical bundle over $X$ and $W = \bigoplus_{i=1}^n H^0(X, K^i)$. For every element $s = (s_1, \ldots, s_n) \in W$, denote by $Y_s$ the associated spectral curve (see [BNR]). For a general $s \in W$, $Y_s$ is non-singular of genus $g_Y = n^2(g_X - 1) + 1$ and the morphism

$$\pi_s: Y_s \to X$$

is of degree $n$.

In [BNR] it is proved that if $\delta = d + n(n-1)(g_X - 1)$ then there is an open subvariety $T_\delta$ of $J^\delta(Y_s)$ such that the morphism $T_\delta \to \mathcal{M}(n, d)$ defined by $L \mapsto \pi_{ss}(L)$ is dominant. Moreover, the direct image induces a dominant rational map

$$f: P' \longrightarrow \mathcal{M}_\xi$$

defined on an open subvariety $T' \subseteq P'$, where $P'$ is a translate of the Prym variety $P_s$ of $\pi_s$ (see [BNR], Proposition 5.7). The complement of the open subvariety $T' \subseteq P'$ is of codimension at least 2. Actually, $f: T' \longrightarrow \mathcal{M}_\xi$ is generically finite.
Consider the following commutative diagram:

\[
\begin{array}{c}
Y_s \times T' \xrightarrow{\pi_s \times id} X \times T' \xrightarrow{id \times f} X \times M_\xi \\
q_2 \downarrow \quad \downarrow p_2 \\
T' \xrightarrow{f} M_\xi
\end{array}
\]

where \(q_2', q_2, p_2\) are the projections to the second factor.

If \(P_T'\) is the restriction of the Poincaré bundle over \(Y_s \times P'\) to \(Y_s \times T'\) and \(U_\xi\) is the restriction of the universal bundle \(U\) to \(X \times M_\xi\), then, by the definition of \(f\),

\[
(id \times f)^*(U_\xi) \cong (\pi_s \times id)_*(P_T') \otimes q_2^*(M)
\]

for some line bundle \(M\) over \(T'\), which depends on the choice of \(U\). Therefore,

\[
f^*(W_\xi) \cong (q_2)_*(id \times f)^*(U_\xi) \cong (q_2)_*((\pi_s \times id)_*(P_T')) \otimes M \cong (q_2')_*(P_{T'}) \otimes M.
\]

Actually, \((q_2')_*(P_{T'})\) is just the restriction of the Picard bundle \(W_{P'}\) over \(P'\) to \(T'\).

**Theorem 3.1.** The Picard bundle \(W_\xi\) on \(M_\xi\) is simple for \(d > n(n + 1)(g_X - 1) + 6\) and \(g_X \geq 2\).

**Proof.** Since \(\text{codim}(T') \geq 2\), we have

\[
\text{End}(W_{P'}) \cong H^0(P', \mathcal{E}nd(W_{P'})) \cong H^0(T', \mathcal{E}nd(W_{P'}|_{T'})) \cong \text{End}(f^*W_\xi).
\]

The Abel-Prym map for the spectral cover is birational onto its image if \(n \neq 2\) or \(g_X > 2\). In this case, from Proposition 2.8, the bundle \(W_{P'}\) is simple if \(\delta = d + n(n - 1)(g_X - 1) > 2g_{Y_s} - 2\). Since the map \(f: T' \to M_\xi\) is dominant and generically finite, as in the proof of Proposition 2.4, we deduce that the Picard bundle \(W_\xi\) on \(M_\xi\) is simple for degree \(d > n(n + 1)(g_X - 1)\).

For \(n = 2\) and \(g_X = 2\), if the map \(\tilde{\rho}\) is not birational onto its image, then by the proof of Proposition 2.3, \(W_\xi\) is simple for \(d > n(n + 1)(g_X - 1) + 6\).

**Remark 3.2.** Denote by \(\Theta_\xi\) a generalized theta divisor on \(M_\xi\). By [L], Theorem 4.3, we have that

\[
f^*(\mathcal{O}(\Theta_\xi)) \cong \mathcal{O}(\Theta_{P'})|_{T'}
\]

where \(\Theta_{P'}\) is the restriction of the theta divisor of \(J^d(Y_s)\) to \(P'\). Since \(W_{P'}|_{T'} \cong f^*W_\xi\), by Lemma 2.1 in [BBN], it follows that if \(W_{P'}\) is \(\Theta_{P'}\)-stable, then \(W_\xi\) is \(\Theta_\xi\)-stable. Moreover, from Theorem 2.2 in [BHM], \(W_{P'}\) is stable if \(\tilde{\rho}_L(W_{P'})\) is stable on \(Y\) for a general line bundle \(L\).
Now we will focus on the case \( n = 2 \) and \( g_X = 2 \). \( \mathcal{M}(2, \xi) \) will denote the moduli space of stable rank 2 vector bundles with fixed determinant \( \xi \) (with \( \deg \xi \) odd). We shall prove that it is possible to construct a spectral covering \( \pi: Y_s \to X \) of degree 2 in such way that the curve \( Y_s \) is hyperelliptic. Let \( p: X \to \mathbb{P}^1 \) be the covering of degree 2 given by the canonical bundle \( K_X \). Let \( s \) be a section of \( H^0(X, K_X^2) \) such that the spectral covering \( \pi: Y_s \to X \) of degree 2 corresponding to \((0, s)\) is smooth, integral and such that the induced map from the Prym variety associated to \( Y_s \to X \) to the moduli space \( \mathcal{M}(2, \xi) \) is dominant. Observe that since \( g_X = 2 \) we can write the section \( s \) as a product \( s = s_1 \cdot s_2 \) where \( s_i \in H^0(X, K_X^2) \) for \( i = 1, 2 \). If \( P_i + Q_i \) is the divisor associated to the section \( s_i \), then \( p(P_i) = p(Q_i) = a_i \in \mathbb{P}^1 \).

Following the construction of cyclic coverings of curves given in [Go1], the morphism \( \tilde{X} = \mathbb{P}^1 \to \mathbb{P}^1 \) of degree 2, ramified at the points \( a_1, a_2 \) is the cyclic covering defined by the construction data \((D = a_1 + a_2, L = \mathcal{O}_{\mathbb{P}^1}(2))\). Let \( C \) be the desingularization of the curve \( X \times_{\mathbb{P}^1} \tilde{X} \). From Theorem 2.13 in [Go1] the map \( C \to X \) is the cyclic covering associated to the data \((p^{-1}(D), p^*L)\), but since \( p^{-1}(D) = \{P_1, Q_1, P_2, Q_2\} \) and \( p^*L \cong K_X^2 \), then \( Y_s \cong C \) and the map \( Y_s \cong C \to \tilde{X} = \mathbb{P}^1 \) is of degree 2. Therefore the curve \( Y_s \) is hyperelliptic.

Let \( P \) be the Prym variety associated to the degree 2 covering \( Y_s \to X \). Since the spectral curve \( Y_s \) is hyperelliptic then the Abel-Prym morphism \( \rho: Y_s \to P \) is not birational onto its image. As in the case 3) of the proof of Proposition 2.13 it follows that, in this case, if \( \delta = d + n(n - 1)(g_X - 1) > 2g_Y + 2 = 12 \), then the Picard bundle on the Prym variety \( P \) is stable with respect to the restriction of the theta divisor. From Remark 3.2 the stability of Picard bundles on \( \mathcal{M}_X(2, \xi) \) is ensured.

**Theorem 3.3.** Let \( X \) be a smooth projective irreducible curve of genus 2 and let \( \mathcal{M}_X(2, \xi) \) be the moduli space of of rank 2 stable vector bundles on \( X \) with fixed determinant \( \xi \). If \( d = \deg(\xi) > 10 \) and \( d \) odd, then the Picard bundle \( W_\xi \) on \( \mathcal{M}_X(2, \xi) \) is stable with respect to the theta divisor.

**Remark 3.4.** After this paper was finished, I. Biswas and T. Gómez informed us that they have obtained the stability of the Picard bundle in the case of the moduli space \( \mathcal{M}_X(2, \xi) \) of rank 2 stable bundles over a smooth curve \( X \) of genus \( g_X \geq 3 \) such that \( d = \deg \xi \geq 4g_X - 3 \) and odd ([BG]).
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