UNIVERSAL ENVELOPING POISSON CONFORMAL ALGEBRAS

P. S. KOLESNIKOV

Abstract. Lie conformal algebras are useful tools for studying vertex operator algebras and their representations. In this paper, we establish close relations between Poisson conformal algebras and representations of Lie conformal algebras. We also calculate explicitly Poisson conformal brackets on the associated graded conformal algebras of universal associative conformal envelopes of Virasoro conformal algebra and Neveu–Schwartz conformal superalgebra.

Keywords: Conformal algebra, Poisson algebra, Gröbner–Shirshov basis.

1. Introduction

This work was inspired by the following observation. Suppose $V$ is a Poisson algebra with operations $x \cdot y$ and $[x, y]$ over a field $k$. Denote by $\mathfrak{g}$ the underlying Lie algebra structure on $V$ relative to the operation $[\cdot, \cdot]$. For a formal variable $\lambda$, consider the following operation $(\cdot \circ \lambda \cdot)$:

$$(x \circ \lambda u) = [x, u] + \lambda x \cdot u, \quad x, u \in V.$$  

It is straightforward to compute that

$$x \circ \lambda (y \circ \mu u) - y \circ \mu (x \circ \lambda u) = [x, y] \circ_{\lambda + \mu} u$$

(see Proposition 2 below for more general computation). The relation obtained is known as the conformal Jacobi identity for a conformal module over the current Lie conformal algebra $\text{Cur} \mathfrak{g}$.

In this paper, we study conformal Poisson algebras. They turn to be closely related to representations of Lie conformal algebras as well as to Gel’fand–Dorfman structures introduced in [7]. The latter are known to be in one-to-one correspondence with certain class of Lie conformal algebras [19]. A series of examples of Poisson conformal algebras is given by associated graded conformal algebras $\text{gr} U$ of universal associative conformal envelopes of Lie conformal algebras corresponding to various locality bounds. We establish explicit expressions for the conformal Poisson brackets on $\text{gr} U$ for universal envelopes of the Virasoro conformal algebra $\text{Vir}$ for $N = 2, 3$. An interesting intermediate example appears as the even part of a universal associative envelope of the Neveu–Schwartz conformal superalgebra $K_1$.

A universal and effective tool for investigations related to universal envelopes is the Gröbner–Shirshov bases (GSB) theory. In Section 4, we present an approach to
the calculation of GSBs for associative conformal algebras based on the GSB theory for modules over ordinary associative algebras. Section 5 contains two examples: we compute GSBs for two particular universal envelopes for the Virasoro conformal algebra and Neveu–Schwartz conformal superalgebra. As an application, we calculate explicitly the structure of three Poisson conformal envelopes \( PV_2, PV_3, \) and \( PK_{10} \) of the Virasoro conformal algebra in Section 6.

2. Conformal algebras: preliminaries

In this section, we state definitions and examples of conformal algebras following [8]. Throughout the paper, \( k \) is a field of characteristic zero, \( H = k[\partial] \) is the algebra of polynomials, \( \mathbb{Z}_+ \) is the set of non-negative integers. We will use common notation \( x^{(s)} \) for \( \frac{1}{s!}x^{s}, s \in \mathbb{Z}_+ \).

A Lie conformal algebra \( L \) is an \( H \)-module equipped with a family of bilinear operations \( [\cdot \circ (n) \cdot], n \in \mathbb{Z}_+ \), such that for every \( x, y \in L \)

\[
[x \circ (n) y] := \sum_{s \geq 0} \lambda^{(s)} [x \circ (s) y] \in L[\lambda] \tag{1}
\]

where \( L[\lambda] \) stands for the space of polynomials over \( L \) in a formal variable \( \lambda \),

\[
[\partial x \circ (n) y] = -n[x \circ (n-1) y], \quad [x \circ (n) \partial y] = \partial[x \circ (n) y] + n[x \circ (n-1) y], \tag{2}
\]

\[
[x \circ (n) y] = -\sum_{s \geq 0} (-1)^{n+s} \partial^{(s)} [y \circ (n+s) x] \tag{3}
\]

for all \( x, y \in L, n \in \mathbb{Z}_+ \), and

\[
[x \circ (n) [y \circ (m) z]] - [y \circ (m) [x \circ (n) z]] = \sum_{s \geq 0} \binom{n}{s} [x \circ (s) y] \circ (n+m+s) z \tag{4}
\]

for all \( x, y, z \in L, n, m \in \mathbb{Z}_+ \).

Condition (1) states that for every pair \( x, y \in L \) there exist only a finite number of \( n \in \mathbb{Z}_+ \) such that \( [x \circ (n) y] \neq 0 \). In particular, one may determine locality function \( N_L : L \times L \to \mathbb{Z}_+ \) in the following way: \( N(x, y) \) is the minimal \( n \in \mathbb{Z}_+ \) such that \( [x \circ (n) y] = 0 \) for all \( m \geq n \).

An associative conformal algebra \( C \) is an \( H \)-module equipped with a series of bilinear operations \( (\cdot \circ (n) \cdot), n \in \mathbb{Z}_+ \), such that the analogues of (1), (2) hold and

\[
(x \circ (n) (y \circ (m) z)) = \sum_{s \geq 0} \binom{n}{s} ((x \circ (s) y) \circ (n+m+s) z) \tag{5}
\]

for all \( x, y, z \in C, n, m \in \mathbb{Z}_+ \).
It is convenient to write the axioms of Lie and associative conformal algebras in terms of generating functions ($\lambda$-products) given by the expression (1). For example, (2) is equivalent to
\[
\partial x \circ_{\lambda} y = -\lambda[x \circ_{\lambda} y], \quad [x \circ_{\lambda} \partial y] = (\lambda + \partial)[x \circ_{\lambda} y],
\]
and (3) and (5) are equivalent to
\[
[x \circ_{\lambda} [y \circ_{\mu} z]] - [y \circ_{\mu} [x \circ_{\lambda} z]] = [[x \circ_{\lambda} y] \circ_{\lambda+\mu} z]
\]
and
\[
(x \circ_{\lambda} (y \circ_{\mu} z)) = ((x \circ_{\lambda} y) \circ_{\lambda+\mu} z),
\]
respectively, where $\lambda$ and $\mu$ are independent commuting variables.

The expression in the right-hand side of (3) is equal to the coefficient at $\lambda^{(n)}$ in the expression $[y \circ_{-\partial-\lambda} x]$. Therefore, (3) is equivalent to
\[
[x \circ_{\lambda} y] = -[y \circ_{-\partial-\lambda} x].
\]

**Example 1.** Let $A$ be a Lie (associative) algebra. Consider the free $H$-module $C = H \otimes A$. Define
\[
[a \circ_{\lambda} b] = [a, b]
\]
for $a, b \in A$, and expand the operation $[\cdot \circ_{\lambda} \cdot]$ to the entire $C$ by (6). We obtain Lie (associative) conformal algebra structure called current conformal algebra; it is denoted by $\text{Cur}_A$.

**Example 2.** Consider 1-generated free $H$-module $\text{Vir} = Hv$. Define
\[
[v \circ_{\lambda} v] = (\partial + 2\lambda)v
\]
and expand the operation $[\cdot \circ_{\lambda} \cdot]$ to the entire $H$-module by (6). We obtain Lie conformal algebra structure called Virasoro conformal algebra.

A general class of examples of Lie conformal algebras (quadratic conformal algebras) involving current and Virasoro conformal algebras is mentioned in Section 3, see also [19].

**Example 3.** Let $A$ be an associative algebra. Then the $H$-module $C = \mathbb{k}[\partial, x] \otimes A \simeq H \otimes A[x]$ equipped with operations
\[
(f(\partial, x) \circ_{\lambda} g(\partial, x)) = f(-\lambda, x)g(\partial + \lambda, x + \lambda), \quad f, g \in C,
\]
is an associative conformal algebra. If $A = M_n(\mathbb{k})$ then the algebra constructed in this way is denoted $\text{Cend}_n$ [8].

As in the world of ordinary algebras, an associative conformal algebra $C$ turns into a Lie one (denoted by $C^{(-)}$) relative to new operations
\[
[x \circ_{(n)} y] = (x \circ_{(n)} y) - \{y \circ_{(n)} x\}, \quad n \in \mathbb{Z}_+,
\]
where
\[
\{ y \circ_{(n)} x \} = \sum_{s \geq 0} (-1)^{n+s} \partial^{(s)} (y \circ_{(n+s)} x) .
\] (11)

However, there exist Lie conformal algebras that cannot be embedded into associative ones in this way \[17\].

Suppose \( V \) is an \( H \)-module. A con-formal linear transformation \( \varphi \) is a rule that turns every \( v \in V \) into a polynomial \( \varphi \circ \lambda v \in V[\lambda] \) in such a way that \( \varphi \circ \lambda \partial v = (\partial + \lambda)(\varphi \circ \lambda v) \). The set of all con-formal linear transformations of \( V \) is denoted \( \text{Cend} V \). The space \( \text{Cend} V \) has a natural structure of an \( H \)-module, and there is a \( \lambda \)-product \((\varphi \circ \lambda \psi) \in \text{Cend} V[[\lambda]]\) given by the rule
\[
(\varphi \circ \lambda \psi) \circ \mu v = \varphi \circ \lambda (\psi \circ_{\mu-\lambda} v).
\]

If \( V \) is a finitely generated \( H \)-module then \((\varphi \circ \lambda \psi)\) is a polynomial in \( \lambda \) and thus \( \text{Cend} V \) is an associative con-formal algebra. For a free \( H \)-module \( V \) of rank \( n \) \( \text{Cend} V \) is isomorphic to \( \text{Cend}_n \) from Example 3.

Assume \( C \) is an associative con-formal algebra. An \( H \)-module \( V \) is said to be a con-formal module over \( C \) if equipped with an \( H \)-linear map \( \rho : C \rightarrow \text{Cend} V \) preserving the \( \lambda \)-product. Alternatively, there should exist a family of bilinear maps \((\cdot \circ_{(n)} \cdot) : C \times V \rightarrow V, n \in \mathbb{Z}_+\), such that the analogues of (1), (2), and (5) hold. In a similar way, a con-formal representation of a Lie con-formal algebra \( L \) is defined as an \( H \)-linear map \( \rho : L \rightarrow (\text{Cend} V)^{(-)} \) preserving the operation \([\cdot \circ_{\lambda} \cdot]\).

Conformal algebra (or a module over a con-formal algebra) is said to be finite if it is finitely generated as an \( H \)-module. There is an open problem whether every finite Lie con-formal algebra may be embedded into an associative one. In \[13\], it was shown that if a finite Lie con-formal algebra \( L \) is a torsion-free \( H \)-module and satisfies Levi condition (i.e., if its solvable radical splits) then \( L \) has a finite faithful representation, thus may be embedded into an associative con-formal algebra \( \text{Cend}_n \) for an appropriate \( n \).

3. Poisson con-formal algebras and universal enveloping con-formal algebras

A more conceptual and general approach to the theory of con-formal algebras was proposed in \[1\]. Consider \( H \) as a Hopf algebra generated by primitive element \( \partial \). Then the class \( \mathcal{M}^*(H) \) of \( H \)-modules is a pseudo-tensor category in the sense of \[3\] relative to a natural composition rule. Then Lie (or associative) con-formal algebra may be defined as a morphism from the operad Lie (or As) to \( \mathcal{M}^*(H) \). Generator of the corresponding operad maps to an \( H \)-bilinear product (pseudo-product)
\[
*: C \otimes C \rightarrow (H \otimes H) \otimes_H C, \quad C \in \mathcal{M}^*(H),
\]
where
\[
x \ast y = \sum_{s \geq 0} ((-\partial)^{(s)} \otimes 1) \otimes_H (x \circ (s) y), \quad x, y \in C.
\]
Associativity, (anti-)commutativity, and Jacobi identity for conformal algebras turn into very natural expressions in terms of the pseudo-product (see [1]). For an arbitrary variety Var of algebras, this approach leads to the notion of a Var-conformal algebra [12]. In particular, for the variety of Poisson algebras, we obtain the following

**Definition 1.** A Poisson conformal algebra $P$ is an $H$-module equipped with two $\lambda$-products

$$(x \circ_\lambda y), [x \circ_\lambda y] \in P[\lambda], \quad x, y \in P;$$

such that (2) holds for both $\lambda$-products, $(x \circ_\lambda y)$ is associative and commutative, $[x \circ_\lambda y]$ is anti-commutative and satisfies the Jacobi identity (4), and the following conformal Leibniz rule holds:

$$[x \circ_\lambda (y \circ_\mu z)] = ([x \circ_\lambda y] \circ_{\lambda+\mu} z) + (y \circ_\mu [x \circ_\lambda z]), \quad x, y, z \in P. \quad (12)$$

**Remark 1.** Relation (12) is equivalent to

$$[(x \circ_\lambda y) \circ_\mu z] = (y \circ_{\mu-\lambda} [x \circ_\lambda z]) + (x \circ_\lambda [y \circ_\mu z]). \quad (13)$$

**Remark 2.** Note that (12) holds on every associative conformal algebra $C$ relative to $[x \circ_\lambda y]$ given by (10):

$$[x \circ_\lambda y] = (x \circ_\lambda y) - (y \circ_{-\lambda} x), \quad x, y \in C.$$ An equivalent form of (13) in the absence of commutativity is

$$[(x \circ_\lambda y) \circ_\mu z] = \{[x \circ_\lambda z] \circ_{\mu-\lambda} y\}. \quad (14)$$

Definition 1 seems close to the notion of a Poisson vertex algebra introduced in [2]. However, it is not clear what is a formal relation between them.

**Example 4.** Let $V$ be an ordinary Poisson algebra. Then $P = H \otimes V$ equipped with operations $(a \circ_\lambda b) = ab, [a \circ_\lambda b] = [a, b]$ for $a, b \in V$ is a Poisson conformal algebra denoted $\text{Cur}_V$.

**Example 5.** Consider $PV_2 = k[\partial, v] \simeq H \otimes k[v]$ as a current associative commutative conformal algebra over $k[v]$ equipped with

$$[v^m \circ_\lambda v^n] = (m\partial + (n + m)\lambda)v^{n+m-1}.$$ It is straightforward to check that $P$ is a Poisson conformal algebra.

Example 5 (as a Lie conformal algebra, it is a sort of Block-type Lie conformal algebra studied in [18]) is a particular case of a more general structure.

**Proposition 1.** Given a Poisson algebra $V$ with a derivation $D$, the free $H$-module $P = H \otimes V$ is a Poisson conformal algebra relative to the following $\lambda$-products:

$$(x \circ_\lambda y) = xy,$$

$$[x \circ_\lambda y] = [x, y] + \partial(yD(x)) + \lambda D(xy),$$

$x, y \in V.$
Proof. Conformal Lie bracket \([\cdot \circ \lambda \cdot]\) turns \(P\) into a quadratic Lie conformal algebra studied in [19]. It remains to check (12) or (13) which is straightforward. 

Relation between differential Poisson algebras and conformal algebras leads to a curious structure of an ordinary Poisson algebra on the space of Laurent polynomials over a Poisson algebra.

**Corollary 1.** Suppose \(V\) is a Poisson algebra with a derivation \(D\), \(V[t, t^{-1}]\) is the commutative algebra of Laurent polynomials over \(V\). Then

\[
[at^n, bt^m] = [a, b]t^{n+m} + (naD(b) - mbD(a))t^{n+m-1}, \quad a, b \in V,
\]

is a Poisson bracket on \(V[t, t^{-1}]\).

**Proof.** Relation (15) along with the ordinary commutative multiplication on \(V[t, t^{-1}]\) represent the coefficient algebra structure on the Poisson conformal algebra \(P = H \otimes V\) from Proposition 1. 

Poisson conformal algebras, even the simplest ones from example 4, have a natural relation to representations of Lie conformal algebras.

**Proposition 2.** Let \(P\) be a Poisson conformal algebra. Suppose \(L\) is a conformal subalgebra of the underlying Lie conformal algebra \(P\) relative to \([\cdot \circ \lambda \cdot]\). Then \(P\) is a conformal module over \(L\) with respect to the following operation:

\[
\langle a \circ_\lambda u \rangle = [a \circ_\lambda u] + \lambda(a \circ_\lambda u), \quad a \in L, \ u \in P.
\]

**Proof.** It remains to check the conformal Jacobi identity

\[
\langle a \circ_\lambda \langle b \circ_\mu u \rangle \rangle - \langle b \circ_\mu \langle a \circ_\lambda u \rangle \rangle = \langle [a \circ_\lambda b] \circ_\lambda \mu u \rangle.
\]

Indeed,

\[
\langle a \circ_\lambda \langle b \circ_\mu u \rangle \rangle = \langle a \circ_\lambda [b \circ_\mu u] + \mu(b \circ_\mu u) \rangle
\]

\[
= [a \circ_\lambda [b \circ_\mu u]] + \mu([a \circ_\lambda b] \circ_\lambda \mu u) + \lambda(a \circ_\lambda [b \circ_\mu u]) + \lambda \mu(a \circ_\lambda [b \circ_\mu u])
\]

\[
= [a \circ_\lambda [b \circ_\mu u]] + \mu([a \circ_\lambda b] \circ_\lambda \mu u) + \mu(b \circ_\mu [a \circ_\lambda u])
\]

\[
+ \lambda(a \circ_\lambda [b \circ_\mu u]) + \lambda \mu(a \circ_\lambda (b \circ_\mu u)).
\]

Hence, the left-hand side of (16) is equal to

\[
[[a \circ_\lambda b] \circ_\lambda \mu + (\lambda + \mu)([a \circ_\lambda b] \circ_\lambda \mu u)
\]

since

\[
(a \circ_\lambda (b \circ_\mu u)) = (b \circ_\mu (a \circ_\lambda u))
\]

in every associative and commutative conformal algebra [17],

\[
([b \circ_\mu a] \circ_\lambda \mu u) = -(a \circ_\lambda - \vartheta b \circ_\lambda \mu u) = -(a \circ_\lambda b) \circ_\lambda \mu u)
\]

by (3) and (2). Obviously, (17) coincides with the right-hand side of (16). 

□
Corollary 2. If $V$ is an ordinary Poisson algebra then $P = \text{Cur} V$ is a conformal module over $\text{Cur} g$, where $g$ is a Lie subalgebra of $P$.

The purpose of this note is to establish more complicated Poisson conformal algebras whose commutative operation may not be reduced to a current-type structure. As in the case of ordinary algebras, it is natural to seek among universal enveloping associative algebras.

Given a Lie algebra $g$, let $P(g)$ be its symmetric algebra equipped with Poisson bracket $[\cdot, \cdot]$ induced by the commutator on $g$. As a linear space, $P(g)$ is isomorphic to the universal associative envelope $U(g)$ by the Poincaré–Birkhoff–Witt (PBW) Theorem.

For Lie conformal algebras, we have a hierarchy of universal associative envelopes \[17\]. Given a Lie conformal algebra $L$, an associative envelope of $L$ is an associative conformal algebra $C$ equipped with a homomorphism (not necessarily injective) $\varphi : L \rightarrow C^{(-)}$ such that $C$ is generated by $\varphi(L)$ as a conformal algebra. Suppose $X$ is a generating set of $L$ as of $H$-module. Fix a function $N : X \times X \rightarrow \mathbb{Z}_+$. Then the class of associative envelopes $(C, \varphi)$ of $L$, such that $N_C(\varphi(x),\varphi(y)) \leq N(x,y)$ for all $x,y \in X$ contains a unique (up to isomorphism) universal associative envelope $(U(L; X, N), \iota)$, $\iota : L \rightarrow U(L; X, N)^{(-)}$.

Associative conformal algebra $U(L; X, N)$ has a natural ascending filtration, the corresponding associated graded space carries a structure of a Poisson conformal algebra (see Section 6 for details). In order to study this structure, we need to determine a normal form of elements in $U(L; X, N)$. As in the case of ordinary algebras, $U(L; X, N)$ is determined by defining relations. In the next section, we present a general approach to the study of conformal algebras given by generators and relations, a sort of Composition-Diamond Lemma (CD-Lemma) for conformal algebras. Previous versions of the CD-Lemma for associative conformal algebras \[4, 5, 15\] work for bounded functions $N$. Our approach does not depend on $N$ and, which is more important, we reduce the problem to modules over ordinary associative algebras. Therefore, one may apply available computer algebra packages for computations in conformal algebras within this approach.

4. A version of the Diamond Lemma for associative conformal algebras

Let $X$ be a well-ordered set, and let $N : X \times X \rightarrow \mathbb{Z}_+$ be a fixed function. Denote by $\text{Conf}(X, N)$ the free associative conformal algebra generated by $X$ with respect to locality function $N$ \[16\]. One may choose a linear basis of $\text{Conf}(X, N)$ in the form

\[ \partial^s(a_1 \circ_{(n_1)} (a_2 \circ_{(n_2)} \cdots \circ_{(n_{k-1})} (a_k \circ_{(n_k)} a_{k+1}) \cdots)), \]

\[ k, s \in \mathbb{Z}_+, \quad a_i \in X, \quad 0 \leq n_i < N(a_i, a_{i+1}). \] (18)
Consider linear operators $L^a_n$ and $R^a_n$ on $\text{Conf}(X,N)$ defined as follows:

$$L^a_n(f) = a \circ_n f, \quad R^a_n(f) = \{f \circ_n a\},$$

$a \in X$, $n \in \mathbb{Z}_+$, $f \in \text{Conf}(X,N)$, where $\{x \circ_n y\}$ is given by (11). The axioms of an associative conformal algebra imply the following relations to hold in $\text{End} \text{Conf}(X,N)$:

$$L^a_n \partial = \partial L^a_n + nL^a_{n-1}, \quad (19)$$
$$R^a_n \partial = \partial R^a_n + nR^a_{n-1}, \quad (20)$$
$$R^b_m L^a_n = L^a_n R^b_m. \quad (21)$$

Hence, $\text{Conf}(X,N)$ is a (left) module over the ordinary associative algebra $A(X)$ generated by formal variables $\partial, L^a_n, R^a_n$ relative to the relations (19)–(21).

It is not hard to find the defining relations of $\text{Conf}(X,N)$ as of $A(X)$-module.

**Theorem 1** ([14]). Let $M$ be a left $A(X)$-module generated by $X$ relative to the defining relations

$$L^a_n b = 0, \quad a, b \in X, \quad n \geq N(a,b), \quad (22)$$
$$R^b_n a = (-1)^n \sum_{s=0}^{N(a,b)-n-1} \partial^{(s)} L^a_{n+s} b, \quad a, b \in X, \quad n \in \mathbb{Z}_+. \quad (23)$$

Then $M$ is isomorphic to $\text{Conf}(X,N)$ as $A(X)$-module.

**Sketch of the proof.** Obviously, (22) and (23) hold in $\text{Conf}(X,N)$. The only problem is to show that the $A(X)$-module homomorphism $M \to \text{Conf}(X,N)$ is injective. To resolve this problem, it is natural to apply the Gröbner–Shirshov bases technique for modules [10].

Given a well order on $X$, extend it to $L^a_n$ and $R^a_n$ by the natural rule $L^a_n < L^b_m$ (or $R^a_n < R^b_m$) if $n < m$ or $n = m$ and $a < b$; assume $\partial < L^a_n < R^b_m$ for all $a, b \in X, n, m \in \mathbb{Z}_+$. Next, define the following monomial order $\prec$ on the words in the alphabet $\partial, L^a_n, R^b_m$: compare two words first by their degree in the variables $R^b_m$, then by deg-lex order.

Obviously, relations (19)–(21) form a GSB of $A(X)$, which is actually the universal enveloping algebra of some Lie algebra. Hence, the linear basis $B$ of $A(X)$ consists of all words of the form

$$\partial^s L^a_{m_1} \cdots L^a_{m_k} R^b_{m_1} \cdots R^b_{m_l}.$$

Expand the above monomial order $\prec$ to the monomials in the free $A(X)$-module generated by $X$: for $u, v \in B$ and $x, y \in X$, let $ux \prec vy$ if and only if $u \prec v$ or $u = v$ and $x < y$.

It is easy to see that (21), (22), and (23) imply a series of relations

$$L^a_n L^b_m u = \sum_{q \geq 1} (-1)^{q+1} \binom{n}{q} L^a_{n-q} L^b_{m+q} u, \quad (24)$$

where $\binom{n}{q}$ is the binomial coefficient.
where \( a, b \in X, \ n \geq N(a, b), \ m \in \mathbb{Z}_+, \) and \( u \) is of the form

\[
L_{m_1}^{c_1}L_{m_2}^{c_2}\ldots L_{m_k}^{c_k}c_{k+1}, \quad c_i \in X, \ k, m_i \in \mathbb{Z}_+.
\]

Consider the reduced words, i.e., those monomials in the free \( A(X) \)-module generated by \( X \) that do not contain a subword equal to a principal part of \( (22) \)–\( (24) \). The latter principal parts are equal to \( L_n^a b \) for \( n \geq N(a, b) \), \( R_n^a b \), and \( L_n^a L_m^b u \) for \( n \geq N(a, b) \). Therefore, \( M \) is spanned by the reduced words that are of the form

\[
\partial^s L_{n_1}^{a_1}L_{n_2}^{a_2}\ldots L_{n_k}^{a_k}a_{k+1},
\]

and their images in \( (18) \) are linearly independent. Hence, \( (22) \)–\( (24) \) is a GSB of \( \text{Conf}(X, N) \) in the sense of \([10]\). □

By the definition of an \( A(X) \)-module structure on \( \text{Conf}(X, N) \), a subspace \( I \subset \text{Conf}(X, N) \) is an ideal of the conformal algebra \( \text{Conf}(X, N) \) if and only if \( I \) is an \( A(X) \)-submodule. If \( S \subset \text{Conf}(X, N) \) is a set of conformal polynomials then the ideal generated by \( S \) in the conformal algebra \( \text{Conf}(X, N) \) coincides with the \( A(X) \)-submodule generated by \( S \). Therefore, in order to solve the word problem in an associative conformal algebra defined by generators and relations it is enough to solve that problem in the corresponding module over an ordinary associative algebra.

In general, if an associative algebra \( A \) and (left) \( A \)-module \( M \) are defined via generators and relations (say, \( A \) and \( M \) are generated by \( X \) and \( Y \), respectively) then the problem of finding normal forms in \( M \) was considered in \([10]\). However, one may apply the ordinary Composition-Diamond Lemma for associative algebras to the split null extension \( A \oplus M \) assuming obvious additional relations \( yx = 0, \ yz = 0 \) for \( x \in X, \ y, z \in Y \).

**Corollary 3** (CD-Lemma). Let \( S \) be a set of conformal polynomials in \( \text{Conf}(X, N) \) considered as elements of the free \( A(X) \)-module generated by \( X \). Then the following conditions are equivalent:

1. \( S \) together with \( (22), (23), \) and \( (24) \) is a GSB of an \( A(X) \)-module;
2. \( S \)-reduced words of the form

\[
\partial^s L_{n_1}^{a_1}L_{n_2}^{a_2}\ldots L_{n_k}^{a_k}a_{k+1}, \quad k, s \in \mathbb{Z}_+, \ a_i \in X, \ 0 \leq n_i < N(a_i, a_{i+1}),
\]

form a linear basis of \( \text{Conf}(X, N | S) \).

To study the structure of a universal associative enveloping conformal algebra of a Lie conformal (super)algebra, it is convenient to add more defining relations to the algebra \( A(X) \). Namely, suppose \( L \) is a Lie conformal superalgebra which is a free \( H \)-module, and let \( X \) be a homogeneous basis of \( L \) over \( H \). Recall that the following identity holds in every associative conformal algebra \([17]\):

\[
x \circ_{(n)} (y \circ_{(m)} z) - (-1)^{|x||y|} y \circ_{(m)} (x \circ_{(n)} z) = \sum_{s \geq 0} \binom{n}{s} x \circ_{(s)} y \circ_{(n+m-s)} z.
\]
Therefore, $U(L; X, N)$ is a module over the associative algebra $A(X, L)$ generated by $\partial$, $L^a_n$, $R^a_n$ ($a \in X$, $n \in \mathbb{Z}_+$) relative to the defining relations (19)–(21) and

$$L_n^a L_m^b - (-1)^{|a||b|} L_m^a L_n^b = \sum_{s \geq 0} \binom{n}{s} \partial_{a(s)b} L_{n+m-s}^a,$$

(25)

Here we assume $L_n^a = -n L_{n-1}^a$ to express the right-hand side of (25).

Defining relations of $U(L; X, N)$ as of $A(X, L)$-module include (22), (23), and

$$R_n^a b - (-1)^{|a||b|} L_n^a b = -[a \circ (\lambda b)], \quad a, b \in X, \quad n \in \mathbb{Z}_+.$$

(26)

It is not hard to see that (19)–(21), (25) form a GSB of the associative algebra $A(X, L)$. In order to determine the structure of $U(L; X, N)$ it is enough to find a GSB of the $A(X, L)$-module generated by $X$ relative to (22), (23), and (26).

5. Example: Universal envelope of the Neveu–Schwartz conformal superalgebra

Consider $L = K_1$, the Neveu–Schwartz conformal superalgebra (see [9]). Then $X = \{v, g\}$, $|\lambda v| = 0$, $|\lambda g| = 1$, and the multiplication table is given by

$$[\lambda v \circ \lambda v] = \partial v + 2 \lambda v, \quad [\lambda g \circ \lambda v] = \frac{1}{2} \partial g + \frac{3}{2} \lambda g, \quad [\lambda g \circ \lambda g] = -\frac{1}{2} v.$$

Assume $v < g$. For convenience of computation, let us slightly change the order $<$ assuming $L^0_0 < L^x_1 < \partial < L^x_2 < \ldots, x \in X$ (other rules remain the same). The set of defining relations of $A(X, K_1)$ consists of

$$\partial L^0_0 = L^x_0 \partial, \quad \partial L^x_1 = L^x_1 \partial - L^x_0, \quad x \in X;$$

$$L^x_n \partial = \partial L^x_n + n L^x_{n-1}, \quad n \geq 2, \quad x \in X;$$

$$R^a_n \partial = \partial R^a_n + n R^a_{n-1}, \quad n \geq 0, \quad x \in X;$$

$$L^x_n R^a_m = L^x_m R^a_n, \quad n, m \geq 0, \quad x, y \in X;$$

$$L^v_n L^v_m = L^v_m L^v_n + (n - m) L^v_{n+m-1}, \quad n > m \geq 0;$$

(27)

$$L^g_n L^g_m = L^g_m L^g_n + \left(\frac{1}{2} n - m\right) L^g_{n+m-1}, \quad n > m \geq 0;$$

(28)

$$L^g_n L^g_m = -L^g_m L^g_n - \frac{1}{2} L^v_{n+m}, \quad n > m \geq 0;$$

$$L^g_n L^g_n = -\frac{1}{4} L^v_{2n}, \quad n \geq 0.$$

In [11], a GSB of $U(K_1; X, N)$ in the sense of [6] was found for

$$N(v, v) = 3, \quad N(v, g) = N(g, v) = N(g, g) = 2.$$
Let us show how the technique exposed in the previous section works for the same locality function $N$.

According to the general scheme described above, the following relations determine $U(K_1; X, N)$ as an $A(X, K_1)$-module:

$$L_n^xy = 0, \quad x, y \in X, \ n \geq N(x, y);$$

$$R_n^xy = 0, \quad x, y \in X, \ n \geq N(y, x);$$

$$R_0^n v = L_0^n v - \partial L_1^n v + \partial(2) L_2^n v;$$

$$R_1^n v = -L_1^n v + \partial L_2^n v, \quad R_2^n v = L_2^n v;$$

$$R_n^x y = (-1)^n(L_n^x x - \partial L_{n+1}^x x), \quad \{x, y\} = X, \ n = 0, 1;$$

$$R_0^g = L_0^g - \partial L_1^g; \quad R_1^g = -L_1^g;$$

$$R_0^v = L_0^v - \partial v; \quad R_1^v = L_1^v - 2v;$$

$$R_0 g = L_0 g - \partial g; \quad R_1^v = L_1^v - \frac{3}{2}g;$$

$$R_0^g = L_0^g - \frac{1}{2}g; \quad R_1^g = -L_1^g;$$

Calculation of a GSB of the $A(X, K_1)$-module generated by $X$ with defining relations (29) is a standard computational task: one has to add all non-trivial compositions to the set of defining relations.

**Theorem 2.** In order to obtain a GSB of $U(K_1; X, N)$ for $N$ given by (28) it is enough to enrich the system (29) with the following relations:

$$L_0^n v = -2L_1^n g; \quad L_1^n v = -2L_0^n g + \frac{1}{2}v;$$

$$L_1^n v = -L_1^n g + \frac{3}{2}g; \quad L_0^n L_1^n g = \frac{3}{2}L_1^n g - \frac{1}{2}g;$$

$$L_1^n L_1^n g = \frac{1}{2}L_1^n g; \quad L_0^n L_1^n g = \frac{1}{2}L_0^n L_1^n g + \frac{1}{2}L_0^n g;$$

$$L_0^n L_0^n g = -\frac{1}{2}L_1^n g + \frac{1}{4}g;$$

$$L_1^n \partial^s v = -2L_0^n \partial^s g + \frac{1}{2} \partial^s v + sL_0^n \partial^{s-1} v, \quad s \geq 1;$$

$$L_1^n \partial^s g = -L_0^n \partial^{s-1} g + \partial^s g + (s + 1)L_0^n \partial^{s-1} g, \quad s \geq 1;$$

$$L_1^n \partial^s g = -L_0^n \partial^{s-1} v + \frac{1}{2} \partial^s g + (s + 1)L_0^n \partial^{s-1} g, \quad s \geq 1;$$

$$L_1^n \partial^s g = (s + 2)L_0^n \partial^{s-1} g + \frac{1}{2} \partial^{s-1} v, \quad s \geq 1."
This result is agreed with the computations in [11], so we do not present the details here. However, Theorem 2 may be easily checked by means of computer algebra systems providing an opportunity of (step-by-step) computation of GSBs in non-commutative associative algebras.

**Corollary 4.** The following words form a linear basis of $U(K_1; X, N)$:

$$\left(L^v_0\right)^n\partial^sx, \quad \left(L^v_0\right)^nL^g_0\partial^sx, \quad \left(L^v_0\right)^nL^z_1g, \quad n \geq 0, \quad x \in X. \quad (31)$$

**Proof.** Let $S$ be the set of defining relations (27), (29), (30). Then (31) is exactly the set of $S$-reduced words in the free $A(X, K_1)$-module generated by $X$. \quad \Box

To make sure that the results of Theorem 2 and Corollary 4 are correct, one may recall the following presentation of $K_1$. Consider the associative conformal algebra $C\text{end}_2$ with the natural $\mathbb{Z}_2$-grading as $k[x, \partial] \otimes M_{11}(k)$. Then $v = \begin{pmatrix} x & 0 \\ 0 & x - \frac{1}{2}\partial \end{pmatrix}$, $g = \frac{1}{2} \begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}$

span a Lie conformal superalgebra $L$ in $C\text{end}_2$ isomorphic to $K_1$. Associative envelope $C$ of $L$ in $C\text{end}_2$ coincides with the set of all matrices

$$\begin{pmatrix} xf_{11}(x, \partial) & xf_{12}(x, \partial) \\ f_{21}(x, \partial) & f_{22}(x, \partial) \end{pmatrix}, \quad f_{ij} \in k[x, \partial]. \quad (32)$$

Straightforward computation shows that the images of (31) in $C\text{end}_2$ exactly form a linear basis of $C$.

**Corollary 5.** For the Virasoro conformal algebra $\text{Vir}$, a linear basis of $U(\text{Vir}; v, 3)$ consists of the words

$$L^s_0\partial^sL^m_1v, \quad L^s_0L^m_1L^2_2v, \quad n, m, s \geq 0, \quad (33)$$

where $L^n_k$ stands for $(L^v_k)^n$. In particular, $U(L; v, 3)$ is a free $H$-module generated by

$$x_{n+1} = L^n_0v, \quad y_{n+1, m+1} = L^n_0L^m_1L^2_2v, \quad n, m \geq 0.$$ 

**Proof.** To find a GSB of $U(\text{Vir}; v, 3)$ it is enough to add the following to the initial set of defining relations:

$$L_2L_2v = 0, \quad \partial L_2v - 2L_1v = 0,$$

see [14] for details. The set of reduced words coincides with (33). \quad \Box

6. **Conformal Poisson brackets on associative envelopes of the Virasoro conformal algebra**

Let $L$ be a Lie conformal algebra generated by a set $X$ as an $H$-module. For a fixed function $N : X \times X \to \mathbb{Z}_+$, consider the universal associative conformal
envelope \( U = U(L; X, N) \). The latter is a homomorphic image of \( \text{Conf}(X, N) \), so there is an ascending filtration

\[
U = \bigcup_{n \geq 1} F_n U, \quad (F_n U \circ_\lambda F_m U) \subseteq F_{n+m} U[\lambda],
\]

where \( F_n U \) consists of images of all words \([18]\) of degree \( k + 1 \leq n \) in \( X \). Assume \( F_0 U = \{0\} \).

Consider the associated graded linear space

\[
\text{gr} U(L; X, N) = \bigoplus_{n \geq 1} F_n U / F_{n-1} U
\]
equipped with well-defined operations

\[
\partial \bar{u} = \overline{\partial u}, \quad u \in F_n U,
\]

and

\[
(\bar{u} \circ_\lambda \bar{v}) = (u \circ_\lambda v) \in F_{n+m} U / F_{n+m-1} U, \quad u \in F_n U, \ v \in F_m U.
\]
The associative and commutative conformal algebra obtained is a Poisson conformal algebra relative to

\[
[u \circ_\lambda v] = (u \circ_\lambda v) - \{v \circ_\lambda u\} \in F_{n+m-1} U / F_{n+m-2} U \quad (34)
\]
for \( u \in F_n U, \ v \in F_m U \). The operation (34) is well-defined since (12) and (13) imply

\[
[u \circ_\lambda v] = (u \circ_\lambda v) - \{v \circ_\lambda u\} \in F_{n+m-1} U.
\]

It follows from the same relations that (3), (4), and (12) hold for the operations \([ \cdot \circ_\lambda \cdot ]\) and \(( \cdot \circ_\lambda \cdot )\) on \( \text{gr} U(L; X, N) \). Therefore, \( \text{gr} U(L; X, N) \) carries a natural structure of a Poisson conformal algebra, let us denote it by \( P(L; X, N) \).

Obviously, \( \mathbb{Z}_2 \)-graded version of this construction leads to a Poisson conformal superalgebra structure on the associated graded universal associative conformal envelope of a Lie conformal superalgebra.

**Example 6.** For the Virasoro Lie conformal algebra, \( P(\text{Vir}; \{v\}, 2) \) is isomorphic to the Poisson conformal algebra \( PV_2 \) from Example 5.

It is easy to find a GSB of \( U = U(\text{Vir}; \{v\}, 2) \) (see [6]), the corresponding set of reduced words is \( \partial^s (L_0^s)^n v \in F_{n+1} U, \ n, s \geq 0 \). Since \( L_1^0 v = v \), we have the isomorphism of conformal algebras \( \text{gr} U \simeq \text{Curk}[v], \ (L_0^s)^n v \mapsto v^{n+1} \). It is straightforward to evaluate conformal Poisson bracket using (12) and (13) to get the formula from Example 5.

**Example 7.** On the 1-generated free commutative conformal algebra one may define a Poisson conformal bracket induced by the Virasoro \( \lambda \)-bracket (9). Let us denote this Poisson algebra \( PV_3 \).
Corollary 5 and Section 9.3 show that \( \text{gr} U(\text{Vir}; \{ v \}, 3) \) is isomorphic to the 1-generated commutative conformal algebra \( \text{ComConf}(\{ v \}, 3) \). Let us evaluate, for example, \((x_n \bullet_{\lambda} x_m)\). By definition,

\[
(x_n \bullet_{\lambda} x_m) = L_0^{n-1}v \bullet_{\lambda} L_0^{m-1}v
\]

\[
= L_0^{n+m-2}(v \bullet_{\lambda} v) = L_0^{n+m-2}(L_0v + \lambda L_1v + \lambda(2)L_2v)
\]

\[
= L_0^{n+m-1}v + \frac{1}{2} \lambda \partial L_0^{n+m-2}L_1v + \lambda(2)L_0^{n+m-2}L_2v
\]

\[
= x_{n+m} + \frac{1}{2}(\lambda \partial + \lambda^2)y_{n+m-1,1}.
\]

Similarly,

\[
(x_n \bullet_{\lambda} y_{m,k}) = y_{n+m,k} + \lambda y_{n+m-1,k+1},
\]

\[
(y_{n,m} \bullet_{\lambda} y_{k,l}) = 0,
\]

Explicit formulas for the Poisson conformal bracket on \( PV_3 \) may be deduced from (12) and (13). For example,

\[
[x_1 \bullet_{\lambda} x_m] = [x_1 \bullet_{\lambda} L_0 x_{m-1}] = ([v \bullet_{\lambda} v] \bullet_{\lambda} x_{m-1}) + L_0[x_1 \bullet_{\lambda} x_{m-1}]
\]

\[
= \lambda(x_m + \frac{1}{2}(\lambda \partial + \lambda^2)y_{m-1,1}) + L_0[x_1 \bullet_{\lambda} x_{m-1}]
\]

\[
= (m-1)\lambda x_m + \frac{m-1}{2}\lambda^2(\lambda + \partial)y_{m-1,1} + L_0^{m-1}(\partial + 2\lambda)v
\]

\[
= (\partial + (m+1)\lambda)x_m + \frac{m-1}{2}\lambda^2(\lambda + \partial)y_{m-1,1}.
\]

In a similar way,

\[
[x_n \bullet_{\lambda} x_m] = (n\partial + (n+m)\lambda)x_{n+m-1} + \frac{1}{2}\lambda(\partial + \lambda)((n-1)\partial + (n+m-2)\lambda)y_{n+m-2,1}.
\]

To compute \([x_n \bullet_{\lambda} y_{m,k}]\), let us start with \([x_1 \bullet_{\lambda} y_{1,1}] = [v \bullet_{\lambda} L_2v]\) which is equal to the coefficient of \([v \bullet_{\lambda} (v \bullet_{\mu} v)]\) at \(\mu(2)\):

\[
[v \bullet_{\lambda} (v \bullet_{\mu} v)] = ((\partial + 2\lambda)v \bullet_{\lambda+\mu} v) + (v \bullet_{\mu} (\partial + 2\lambda)v)
\]

\[
= (\lambda - \mu)(L_0v + (\lambda + \mu)L_1v + (\lambda + \mu)(2)L_2v) + (2\lambda + \partial + \mu)(L_0v + \mu L_1v + \mu(2)L_2v)
\]

\[
= (\lambda + \partial)\mu(2)L_2v + \ldots.
\]

Hence,

\[
[x_1 \bullet_{\lambda} y_{1,1}] = (\lambda + \partial)y_{1,1}.
\]

In a similar way,

\[
[x_1 \bullet_{\lambda} y_{m,1}] = (m\lambda + \partial)y_{m,1} + (m-1)\lambda^2y_{m-1,2}, \quad m \geq 2.
\]

For \(k \geq 2\), we may represent \(y_{m,k} = L_1y_{m,k-1}\) and compute

\[
[x_1 \bullet_{\lambda} y_{m,k}] = -y_{m+1,k-1} + L_1[v \bullet_{\lambda} y_{m,k-1}].
\]
Therefore, 
\[
[x_1 \cdot \lambda y_{m,k}] = (1 - k)y_{m+1,k-1} + L_{1}^{k-1}[x_1 \cdot \lambda y_{m,1}]
\]
\[
= (\partial + \lambda)y_{m,k} + (m - 1)\lambda^2 y_{m-1,k+1} - (k - 1)y_{m+1,k-1}.
\]
Finally,
\[
[x_n \cdot \lambda y_{m,k}] = n(1 - k)y_{n+m,k-1} + (n(\partial + \lambda) - (n - 1)(k - 1)\lambda)y_{n+m-1,k}
\]
\[
+ \lambda((n + m - 2)\lambda + (n - 1)\partial)y_{n+m-2,k+1}
\]
by induction in \(n \geq 1\). It remains to note that
\[
[y_{n,m} \cdot \lambda y_{k,l}] = 0.
\]
Therefore, \(PV_3\) is a central extension of \(PV_2\) by means of the conformal module spanned by \(y_{n,m}, n, m \geq 1\).

An interesting example of a Poisson conformal envelope of the Virasoro conformal algebra appears from the associative envelope of the Neveu–Schwartz conformal superalgebra \(K_1\).

**Example 8.** Suppose \(L = K_1\) is the Neveu–Schwartz conformal superalgebra generated by \(X = \{v, g\}\). Then \(U = U(K_1; X, N)\) for \(N\) given by (28) is isomorphic to the conformal subalgebra of \(\text{Cend}_2\) that consists of matrices (32). Although \(P = \text{gr}U\) is not isomorphic to the supercommutative conformal algebra generated by \(X\) relative to the locality function \(N\), the conformal Lie bracket on \(K_1\) induces Poisson conformal superalgebra structure on \(P\) denoted \(PK_1\).

According to Corollary \(4\), every \(F_n U/F_{n-1} U \subset PK_1, n > 1,\) is a 4-dimensional free \(H\)-module with a basis
\[
\bar{a}_n = a_n + F_{n-1} U, \quad \bar{b}_n = b_n + F_{n-1} U, \quad \bar{e}_n = e_n + F_{n-1} U, \quad \bar{f}_n = f_n + F_{n-1} U,
\]
where
\[
a_n = \begin{pmatrix} x^n & 0 \\ 0 & x^n - \frac{1}{2}\partial x^{n-1} \end{pmatrix}, \quad b_n = \begin{pmatrix} 0 & 0 \\ 0 & x^{n-2} \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ -x^{n-1} \end{pmatrix}, \quad f_n = \begin{pmatrix} 0 & 0 \\ x^{n-2} & 0 \end{pmatrix}.
\]
Here \(a_n\) and \(b_n\) are even elements of the \(\mathbb{Z}_2\)-graded associative conformal algebra \(U\), \(e_n\) and \(f_n\) are odd elements. Let us evaluate explicitly the structure of the even part \(PK_10\) of the Poisson conformal superalgebra \(PK_1\).

By the definition of \(\text{Cend}_2\),
\[
[a_n \circ \lambda a_m] = a_n(-\lambda, x)a_m(\partial + \lambda, x + \lambda) - a_m(\lambda + \partial, x)a_n(-\lambda, x - \partial - \lambda)
\]
\[
= \begin{pmatrix} x f(\partial, \lambda, x) & 0 \\ 0 & g(\partial, \lambda, x) \end{pmatrix}.
\]
To find the component from $F_{n+m-1}U/F_{n+m-2}U$, find the principal (relative to $x$) term of
\[
xf(\partial, \lambda, x) = (m\lambda + n(\partial + \lambda))x^{n+m-1} + \ldots
\]
and of
\[
g(\partial, \lambda, x) - \left(x - \frac{1}{2}\partial\right)f(\partial, \lambda, x) = \frac{\lambda^2 + \partial\lambda}{4}((m + n - 2)\lambda + (n - 1)\partial)x^{n+m-3} + \ldots.
\]
Therefore,
\[
[a_n \cdot \lambda \ b] = (n\partial + (m + n)\lambda)\bar{a}_{n+m-1} + \frac{\lambda^2 + \partial\lambda}{4}((m + n - 2)\lambda + (n - 1)\partial)\bar{b}_{n+m-1}. \quad (35)
\]
In a similar way, we may evaluate
\[
[a_n \cdot \lambda \ b] = ((n + m - 2)\lambda + n\partial)\bar{b}_{n+m-1}, \quad (36)
\]
\[
[\bar{b}_n \cdot \lambda \ b] = 0, \quad (37)
\]
\[
(a_n \cdot \lambda \ a_m) = \bar{a}_{n+m} + \frac{\lambda^2 + \partial\lambda}{4}\bar{b}_{n+m}, \quad (38)
\]
\[
(a_n \cdot \lambda \ b_m) = \bar{b}_{n+m}, \quad (39)
\]
\[
(\bar{b}_n \cdot \lambda \ b_m) = 0. \quad (40)
\]
Therefore, $PK_{10}$ as an $H$-module is generated by $\bar{a}_n$, $\bar{b}_m$, $n \geq 1$, $m \geq 2$, and the multiplication table is given by (35)–(40). It is easy to see that $PK_{10}$ is a central extension of $PV_2$ via the submodule generated by $b_m$, $m \geq 2$. The extension is not split since the 1st component of the grading does not intersect with the $H$-submodule spanned by $\bar{b}_n$, $n \geq 2$. To simplify the multiplication table, let us introduce
\[
\hat{a}_1 = \bar{a}_1, \quad \hat{a}_n = \bar{a}_n - \frac{1}{8}\partial^2\bar{b}_n, \quad n \geq 2.
\]
Then
\[
(\hat{a}_1 \cdot \lambda \ \hat{a}_m) = \hat{a}_{m+1} + \frac{1}{8}\lambda^2\hat{b}_{m+1}, \quad m > 1,
\]
\[
(\hat{a}_n \cdot \lambda \ \hat{a}_m) = \hat{a}_{n+m}, \quad n, m > 1,
\]
\[
[\hat{a}_1 \cdot \lambda \ \hat{a}_m] = (\partial + (m + 1)\lambda)\bar{a}_m + \frac{1}{8}((m - 1)\lambda^3 - \partial\lambda^2 - \partial^2\lambda - \partial^3)\bar{b}_m, \quad m > 1,
\]
\[
[\hat{a}_n \cdot \lambda \ \hat{a}_m] = (n\partial + (n + m)\lambda)\bar{a}_{n+m-1} - \frac{n}{8}(4\lambda\partial^2 + \lambda^2\partial)\bar{b}_{n+m-1}, \quad n, m > 1.
\]
Remark 3. For $N > 3$, the associated graded Poisson conformal algebra $PV_N = \text{gr} U(\text{Vir}; \{v\}, N)$ would not be a null extension of $PV_2$. However, it is easy to see that $PV_{N+1}$ is a null extension of $PV_N$. It is interesting problem to find the corresponding conformal modules and cocycles. This problem is closely related with finding a linear basis of the free commutative conformal algebra.
References

[1] B. Bakalov, A. D’Andrea, V. G. Kac, Theory of finite pseudoalgebras, *Adv. Math.* **162** (2001) 1–140.
[2] A. Barakat, A. De Sole, V. G. Kac, Poisson vertex algebras in the theory of Hamiltonian equations *Jpn. J. Math.* **4** (2) (2009) 141–252.
[3] A. A. Beilinson, V. G. Drinfeld, *Chiral algebras*, Amer. Math. Soc. Colloquium Publications **51** (AMS, Providence, RI, 2004).
[4] L. A. Bokut, Y. Fong, W.-F. Ke, Gröbner—Shirshov bases and composition lemma for associative conformal algebras: an example, *Contemp. Math.* **264** (2000) 63–90.
[5] L. A. Bokut, Y. Fong, W.-F. Ke, Composition-Diamond lemma for associative conformal algebras, *J. Algebra* **272** (2004) 739–774.
[6] L. A. Bokut', Yu. Fong, V.-F. Ke, P. S. Kolesnikov, Gröbner and Gröbner—Shirshov bases in algebra, and conformal algebras [Russian], *Fundam. Prikl. Mat.* **6** (3) (2000) 669–706.
[7] I. M. Gel’fand, I. Ja. Dorfman, Hamiltonian operators and algebraic structures associated with them [Russian], *Funktsional. Anal. i Prilozhen.* **13** (4) (1979) 13–30.
[8] V. G. Kac, *Vertex algebras for beginners*, second ed., University Lecture Series **10** (AMS, Providence, RI, 1998).
[9] V. G. Kac, Classification of infinite dimensional simple linearly compact Lie superalgebras, *Adv. Math.* **139** (1) (1998) 1–55.
[10] S.-J. Kang, K.-H. Lee, Gröbner—Shirshov bases for representation theory, *J. Korean Math. Soc.* **37** (2000) 55–72.
[11] P. Kolesnikov, Universally defined representations of conformal Lie superalgebras, *J. Symbolic Computation* **43** (6–7) (2008), 406–421.
[12] P. Kolesnikov, Identities of conformal algebras and pseudoalgebras, *Comm. Algebra* **34** (6) (2006) 1965–1979.
[13] P. Kolesnikov, The Ado theorem for finite Lie conformal algebras with Levi decomposition, *J. Algebra Appl.* **15** (7) (2016), 1650130 (13 pages).
[14] P. Kolesnikov, Gröbner—Shirshov bases for associative conformal algebras with arbitrary locality function, [arXiv:1807.08428](https://arxiv.org/abs/1807.08428).
[15] L. Ni, Y.-Q. Chen, A new Composition-Diamond lemma for associative conformal algebras, *J. Algebra Appl.* **16** (2017) 1750094-1–1750094-28.
[16] M. Roitman, On free conformal and vertex algebras, *J. Algebra* **217** (1999) 496–527.
[17] M. Roitman, Universal enveloping conformal algebras, *Sel. Math., New Ser.* **6** (2000) 319–345.
[18] Y. Su, C. Xia, Y. Xu, Classification of quasifinite representations of a Lie algebra related to Block type, *J. Algebra* **393** (2013) 71–78.
[19] X. Xu, Quadratic conformal superalgebras, *J. Algebra* **231** (1) (2000) 1–38.

Sobolev Institute of Mathematics
E-mail address: pavelsk@math.nsc.ru