BOUNDARY REPRESENTATIONS FOR OPERATOR ALGEBRAS

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ABSTRACT. All operator algebras have (not necessarily irreducible) boundary representations. A unital operator algebra has enough such boundary representations to generate its $C^*$-envelope.

1. INTRODUCTION

Concretely, an operator algebra $A$ is a subalgebra of $B(K)$, the bounded linear operators on some Hilbert space $K$. It is unital if it contains the identity operator. The algebra $M_\ell(A)$ of $\ell \times \ell$ matrices with entries from $A$ inherits a norm as a subspace of $M_\ell(B(K))$ identified canonically with $B(\oplus_1^\ell K)$. The Blecher, Ruan and Sinclair Theorem [5] characterizes unital operator algebras in terms of a matrix norm structure, while a theorem of Blecher [4] does the same for non-unital algebras assuming the algebra multiplication is completely bounded. Consequently it is possible to speak abstractly of an operator algebra without reference to an ambient $B(K)$.

A linear mapping $\phi : A \rightarrow B(H)$ induces a linear mapping $\phi_\ell : M_\ell(A) \rightarrow B(\oplus_1^\ell H)$ by applying $\phi$ entry-wise, so that $\phi_\ell(a_{jm}) = (\phi(a_{jm}))$. The map $\phi_\ell$ is completely bounded if $\phi$ is bounded and there exists $C$, independent of $\ell$, such that $\|\phi_\ell\| \leq C$, it is completely contractive if it is completely bounded with $C \leq 1$, and it is completely isometric if $\phi_\ell$ is an isometry for each $\ell$. Finally, a representation of $A$ on the Hilbert space $H$ is an algebra homomorphism $\phi : A \rightarrow B(H)$. If $A$ is unital, it is assumed that any representation of $A$ takes the unit to the identity operator.

A boundary representation ([2], [3]) of the unital operator algebra $A$ consists of a homomorphism $\phi : A \rightarrow C$, where $C$ is a $\mathbb{C}^*$-algebra and $C^*(\phi(A)) = C$, together with a representation $\pi : C \rightarrow B(H)$ such that the only completely positive map on $C$ agreeing with $\pi$ on $\phi(A)$ is $\pi$ itself. In originally defining boundary representations, Arveson also required that they be irreducible. We do not impose this condition.

The $C^*$-envelope of $A$, denoted $C^*_e(A)$, is the essentially unique smallest $C^*$-algebra amongst those $C^*$-algebras $C$ for which there is a completely isometric homomorphism $\phi : A \rightarrow C$. For instance, if $A$ is a uniform algebra, then $C_e^*(A)$ is the $C^*$-algebra of continuous functions on the Šilov boundary of $A$. In fact, in this case the irreducible boundary representations correspond to peak points of $A$. Arveson proved that $C^*_e(A)$ exists provided there are enough boundary representations for $A$. However, the existence of $C^*_e(A)$ does not imply the existence of boundary representations and Hamana [7] established the existence of $C^*_e(A)$ in general without recourse to boundary representations.

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In this note we show, by elaborating on a construction of Agler essential to his approach to model theory [11] and using a characterization of boundary representations due to Muhly and Solel [9], that boundary representations exist, and then following an argument similar to Arveson’s, we also derive the existence of $C^*_e(A)$.

Agler’s approach is to consider a family $F_A$ of representations of an algebra $A$ (which is not necessarily an operator algebra). This is a collection of representations which is

1. Closed with respect to direct sums (so if $\{\pi_\alpha\}$ is an arbitrary set of representations in the family, then $\bigoplus_\alpha \pi_\alpha$ is also a representation in the family);
2. Hereditary (that is, if $\phi$ is a representation in the family and $L$ is a subspace which is invariant for all $\phi(a), a \in A$, then $\phi|_L$, the restriction of $\phi$ to $L$, is also in the family);
3. Closed with respect to unital $*$-representations (so if $\phi : A \to B(H)$, and $\nu : B(H) \to B(K)$ is a unital $*$-representation, then $\nu \circ \phi$ is in the family).

If $A$ is non-unital, then we also require

4. $A$ is closed with respect to spanning representations with respect to the partial ordering on dilations (defined below). A consequence of (1) is that the norms of all $\pi \in F_A$ are uniformly bounded.

When $A$ is unital, (1)–(3) can be shown to imply (4), using an argument similar to that used to prove (1) of Theorem 1.1 in [6].

Special examples of families include the collection of all completely contractive representations of an operator algebra $A$ and the collection of all representations $\pi$ of the disc algebra such that $\pi(z)$ is an isometry.

A representation $\phi$ lifts to a representation $\psi$ if $\phi$ is the restriction of $\psi$ to an invariant subspace. Following Agler [11] we will say that the representation $\phi : A \to B(H)$ is extremal, if whenever $K$ is a Hilbert space containing $H$ and $\psi : A \to B(K)$ is a representation such that $H$ is invariant for $\psi(A)$ and $\phi = \psi|_H$, then $H$ reduces $\psi(A)$. Further, if $\rho : A \to B(L)$ is a representation, then $\rho$ lifts to an extremal representation; i.e., there exists a Hilbert space $H$ containing $L$ and an extremal representation $\phi$ of $A$ such that $L$ is invariant for $\phi(A)$ and $\rho = \phi|_L$ ([11], Proposition 5.10).

Lifting induces a partial ordering on representations, with $\phi_\alpha \leq \phi_\beta$ being equivalent to $\phi_\alpha$ lifting to $\phi_\beta$. If $S$ is a totally ordered set of liftings (with respect to this partial ordering), then we define the spanning representation $\phi_s : A \to B(H_s)$ by setting $H_s$ to be the closed span of the $H_\alpha$’s over all $\alpha \in S$, and then densely defining $\phi_s$ to be $\phi_\alpha$ on $H_\alpha$ and extending to all of $H_s$ by boundedness of the representations $\phi_\alpha$. It is readily verified that $\phi_s$ is a representation which lifts each $\phi_\alpha$.

The representation $\phi : A \to B(H)$ dilates to the representation $\psi : A \to B(K)$ if $K$ contains $H$ and $\phi(a) = P_H \psi(a)|_H$ for all $a \in A$. A fundamental result of Sarason [11] says that a representation $\phi$ dilates to a representation $\psi$ if and only if $H$ is semi-invariant for $\psi$. Thus, there exists subspaces $L \subset N \subset K$ invariant for $\psi$ such that $H = N \ominus L$. Alternatively, $K = L \oplus H \oplus M$ with $L$ and $L \oplus H$ invariant for $\phi$. Just as in the case of liftings, dilating induces a partial ordering on representations in the obvious manner. We can also similarly define spanning representations of totally ordered sets of representations, and this is what is used in item (4) above. Note that liftings are also dilations (with $L = \{0\}$). Hence the partial ordering
on dilations subsumes that of liftings, and in particular, any spanning representation of liftings is one in terms of dilations as well.

As was noted above, families of representations over unital algebras contain all spanning representations formed from chains of representations in the family, though it appears that this is a needed added assumption in the non-unital setting. On the other hand, there are interesting collections of representations which are closed with respect to (1) and (4) of a family, but not necessarily (2) and (3). For example, the collection of all completely isometric representations of an algebra fall into this category. Since the theorems we prove below only depend on existence of spanning representations in the collections of representations we are considering and all representations being uniformly bounded, we define the extended family of an algebra $A$ to be a collection of representations of $A$ which is closed under the formation of direct sums and spanning representations.

For dilations, the equivalent of an extremal will be referred to as a $\partial$-representation. The representation $\phi: A \to B(H)$ is a $\partial$-representation if whenever $\psi: A \to B(K)$ dilates $\phi$, then $H$ reduces $\psi(A)$.

Muhly and Solel [9] show, in the language of Hilbert modules rather than representations, that for unital operator algebras $\partial$-representations coincide with boundary representations (forgetting the irreducibility requirement).

1.1. Theorem. Let $A$ be a unital operator algebra. Then $\rho: A \to B(H)$ is a $\partial$-representation if, and only if, given any completely isometric map $\phi: A \to C$ where $C$ is a $C^*$ algebra with $C = C^*(\phi(A))$, there exists a boundary representation $\pi: C^*(\rho(A)) \to B(H)$ such that $\pi \circ \phi = \rho$.

The proof of Muhly and Solel of this equivalence uses the existence of the $C^*$-envelope. Our main result and proof of the existence of the $C^*$-envelope do not depend on their work. However, it should be noted that a proof of the equivalence which does not already assume the existence of the $C^*$-envelope is possible, and we sketch a proof below based along a line of reasoning in [8, Theorem 1.2].

Sketch of the proof of (1.1). Suppose $\phi: A \to C = C^*(\phi(A))$ is completely isometric and $\pi: C^*(\phi(A)) \to B(H)$ is a boundary representation. Set $\rho = \pi \circ \phi$, and note that it is completely contractive. Suppose $\nu: A \to B(K)$ dilates $\rho$. The goal is to show that $H$ reduces $\nu$.

To this end, define a map $\gamma: \phi(A) \to B(K)$ by $\gamma(\phi(a)) = \nu(a), a \in A$. This map is completely contractive, and so by the Arveson extension theorem extends to a completely positive unital map $\gamma: C^*(\phi(A)) \to B(K)$ with $\gamma \circ \phi = \nu$. Observe that the map which takes $b \mapsto P_H\gamma(\phi(b))|_H$, $b \in C^*(\phi(A))$ is completely positive, and by definition, $P_H\gamma(\phi(a))|_H = \rho(a) = \pi(\phi(a))$ for all $a \in A$. We have assumed that $\pi$ is a boundary representation, so in fact
\[ P_H \gamma(\phi(b))|_H = \pi(b) \text{ for all } b \in C^*(\phi(A)). \]

From this we have for all \( a \in A, \)
\[
\rho(a)\rho(a)^* = \pi(\phi(a))\pi(\phi(a))^*
= \pi(\phi(a)\phi(a)^*)
= P_H \gamma(\phi(a)\phi(a)^*)|_H
\geq P_H \gamma(\phi(a))^*|_H = P_H \nu(a)\nu(a)^*|_H
\geq P_H \gamma(\phi(a))P_H \gamma(\phi(a)^*)|_H = P_H \nu(a)P_H \nu(a)^*|_H
= \rho(a)\rho(a)^*,
\]

where the first inequality is the Cauchy Schwarz inequality for completely positive maps [10]. From this we see that \( \nu(a)^*H \subseteq H. \) An identical argument gives \( \nu(a)H \subseteq H, \) proving that \( H \) reduces \( \nu. \)

The converse is a straightforward exercise and is left to the reader. \( \square \)

In the course of this paper, we will prove the following.

1.2. Theorem. If \( \rho : A \to B(H) \) is a representation in an extended family \( \mathcal{F}_A, \) then there exists a Hilbert space \( K \) containing \( H \) and a \( \partial \)-representation \( \phi : A \to B(K) \) also in \( \mathcal{F}_A \) such that \( \rho \) dilates to \( \phi. \)

As mentioned above, Arveson’s original definition of boundary representation required \( \pi \) to be irreducible. Note that [1,2] does not imply the existence of irreducible boundary representations.

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The remainder of the paper is organized as follows. Section 2 establishes (2.1) giving the existence of extremals, which form the core of any model in Agler’s approach to model theory [1]. Although versions of this result are quite old, no proofs yet appear in the literature. In Section 3 we prove (1.2) and in Section 4 we explain how to obtain the existence of \( C^*_e(A) \) from Theorem 1.2.

2. LIFTINGS AND EXTREMALS

In Agler’s approach to model theory a family is a collection of representations of a unital algebra satisfying the first three canonical axioms listed in the last section. A key result in his model theory is that an arbitrary member of a family \( \mathcal{F}_A \) lifts to an extremal member of the family \( \mathcal{F}_A \) ([1], Proposition 5.10). We establish this result for extended families.

2.1. Theorem. If \( \rho : A \to B(H) \) is a representation in an extended family \( \mathcal{F}_A, \) then \( \rho \) lifts to an extremal representation in \( \mathcal{F}_A. \)

In this section we establish a preliminary version of Theorem 2.1 in Lemma 2.2 below, and then indicate a proof of the theorem based on the lemma.

Suppose the representation \( \phi_\alpha : A \to B(H_\alpha) \) lifts to the representation \( \phi_\beta : A \to B(H_\beta). \) Then the lifting is trivial if \( H_\alpha \) is reducing, not just invariant, for \( \phi_\beta(A). \) If the only liftings of \( \phi_\alpha \) are trivial, then \( \phi_\alpha \) is extremal.
If \( \phi_\beta \) lifts \( \phi_\alpha \), then we define a lifting \( \phi_\delta : A \rightarrow B(H_\delta) \) of \( \phi_\beta \) to be strongly non-trivial with respect to \( \phi_\alpha \) if there exists an \( a \in A \) such that

\[
P_{H_\delta \ominus H_\delta} \phi_\delta(a)^* |_{H_\alpha} \neq 0.
\]

Otherwise, the lifting is weakly trivial relative to \( \phi_\alpha \). Finally, \( \phi_\beta \) is weakly extremal relative to \( \phi_\alpha \) if every lifting of \( \phi_\beta \) is weakly trivial relative to \( \phi_\alpha \).

2.2. Lemma. Each representation \( \phi_0 : A \rightarrow B(H_0) \) in an extended family \( F_A \) lifts to a representation in \( F_A \) which is weakly extremal relative to \( \phi_0 \).

Proof. The proof is by contradiction. Accordingly, suppose \( \phi_0 \) does not lift to a weakly extremal representation relative to \( \phi_0 \).

Let \( \kappa_0 \) be the cardinality of the the set of points in the unit sphere of \( H_0 \), \( \kappa_1 \) the cardinality of the set of elements in the unit ball of \( A \). Set \( \kappa = 2^{\kappa_0 \cdot \kappa_1} > \kappa_0 \cdot \kappa_1 \). Let \( \lambda \) be the smallest ordinal greater than or equal to \( \kappa \). Note that there is a \( C > 0 \) so that for \( \pi \in F_A \), \( \|\pi(a)h\| \leq C\|a\|\|h\| \) for all \( h \in H_0 \) and \( a \in A \).

Construct a chain of liftings in \( F_A \) by transfinite recursion on the ordinal \( \lambda \) as follows: if \( \alpha \leq \lambda \), and \( \alpha \) has a predecessor, let \( \phi_\alpha \) denote a strong (with respect to \( \phi_0 \)) nontrivial lifting of \( \phi_{\alpha-1} \). Such an lifting exists by the assumption that \( \phi_0 \) does not lift to a weak extremal. If \( \alpha \) is a limit ordinal, set \( \phi_\alpha \) to the spanning representation of \( \{ \phi_\delta \} \delta < \alpha \). For any \( h \) in the unit sphere of \( H_0 \) and \( a \) in the unit ball of \( A \), there are at most countably many \( \alpha \)'s with predecessors where \( P_{H_\alpha \ominus H_\alpha-1} \phi_\alpha(a)^* h \neq 0 \). Since the cardinality of the set of ordinal numbers less than or equal to \( \lambda \) and having a predecessor is \( \kappa \), there must be an ordinal \( \beta < \lambda \) with predecessor where \( P_{H_\beta \ominus H_\beta-1} \phi_\beta(a)^* h = 0 \) for all \( h \) in the unit sphere of \( H_0 \) and \( a \) in the unit ball of \( A \), so that \( \phi_\beta \) is a lifting of \( \phi_{\beta-1} \) which is weakly trivial with respect to \( \phi_0 \); a contradiction, ending the proof.

Proof of (2.1). We use (2.2) to prove (2.1). Let \( \phi_0 : A \rightarrow B(H_0) \) denote a given representation. Lift \( \phi_0 \) to a representation \( \phi_1 : A \rightarrow B(H_1) \) which is weakly extremal relative to \( \phi_0 \). Lift \( \phi_1 \) to a representation \( \phi_2 \) which is weakly extremal relative to \( \phi_1 \). Continuing in this manner, constructs a chain \( \phi_j, j \in \mathbb{N}, \) with respect to the partial order on liftings with the property that \( \phi_j \) is weakly extremal relative to \( \phi_{j-1} \). The resultant spanning representation \( \phi_\infty : A \rightarrow B(H_\infty) \) lifts \( \phi_0 \) and it is easily checked to be extremal, since it is weakly extremal relative to \( \phi_j \) for all \( j \in \mathbb{N} \). □

It is not difficult to see that the restriction of an extremal to a reducing subspace is an extremal. Also, in (2.1) if we were to take the intersection of all reducing subspaces of \( \phi_\infty \) containing \( H_0 \), we end up with the smallest reducing subspace for \( \phi_\infty \) containing \( H_0 \). Restricting to this gives a minimal extremal \( \phi_e \) lifting \( \phi_0 \), in the sense that if \( \psi \) lifts \( \phi_0 \) and \( \psi \leq \phi_e \), then \( \psi = \phi_e \). Of course \( \phi_e \) may still be reducible even if \( \phi_0 \) is irreducible. In addition, there may be non-isomorphic minimal extremal liftings of \( \phi_0 \).

3. Dilations and Boundary Representations

Let \( \phi_\alpha : A \rightarrow B(H_\alpha) \) be a representation. In parallel with the theory of liftings, a dilation \( \phi_\beta : A \rightarrow B(H_\beta) \) is termed trivial if \( H_\alpha \) is reducing for \( \phi_\beta(A) \). If the only dilations of \( \phi_\alpha \) are trivial ones, then \( \phi_\alpha \) is a \( \partial \)-representation.

Likewise, suppose \( \phi_\delta \geq \phi_\beta \geq \phi_\alpha \) in the partial ordering for dilations, with the representations mapping into the operators on \( H_\delta, H_\beta \) and \( H_\alpha \), respectively. By assumption, we can write \( H_\delta = \)}
Let \( L_\delta \oplus H_\beta \oplus M_\delta \), where \( L_\delta \) and \( L_\delta \oplus H_\beta \) are invariant for \( \phi_\delta \). We say that \( \phi_\alpha \) is strongly non-trivial with respect to \( \phi_\alpha \) if there exists an \( a \in \mathcal{A} \) such that either

\[
P_{L_\delta} \pi_\beta(a)|_{H_\alpha} \neq 0 \quad \text{or} \quad P_{M_\delta} \pi_\beta(a)^*|_{H_\alpha} \neq 0.
\]

Otherwise, the dilation is said to be weakly trivial relative to \( \phi_\alpha \). Finally, \( \phi_\beta \) is a weak \( \partial \)-representation relative to \( \phi_\alpha \) if every lifting of \( \phi_\beta \) is weakly trivial relative to \( \phi_\alpha \).

### 3.1. Lemma

Each representation \( \phi_0 : \mathcal{A} \to B(H_0) \) in an extended family \( \mathcal{F}_\mathcal{A} \) dilates to a weak \( \partial \)-representation relative to \( \phi_0 \) which is also in \( \mathcal{F}_\mathcal{A} \).

**Proof.** The proof closely follows that of the existence of weak extremals, and is by contradiction. Hence we suppose \( \phi_0 \) does not lift to a weak \( \partial \)-representation relative to \( \phi_0 \). We define the ordinal \( \lambda \) as in the proof of (2.3).

Construct a chain of dilations in \( \mathcal{F}_\mathcal{A} \) where each of the representations by transfinite recursion on the ordinal \( \lambda \) as in (2.3): if \( \alpha \leq \lambda \) and \( \alpha \) is a limit ordinal, set \( \phi_\alpha \) to the spanning representation of \( \{ \phi_\delta \}_{\delta \prec \alpha} \) and if \( \alpha \) has a predecessor, let \( \phi_\alpha \) be a dilation to a strong (with respect to \( \phi_0 \)) nontrivial dilation of \( \phi_{\alpha-1} \), which exists by the assumption that \( \phi_0 \) does not lift to a weak \( \partial \)-representation. Then for any \( h \) in the unit sphere of \( H_0 \) and \( a \) in the unit ball of \( \mathcal{A} \), there are at most countably many \( \alpha \)'s with predecessors where \( P_{L_\delta} \pi_\beta(a)h \neq 0 \) or \( P_{M_\delta} \pi_\beta(a)^*h \neq 0 \). The same reasoning then gives a representation \( \phi_\beta \) in our chain dilating \( \phi_{\beta-1} \) which is weakly trivial with respect to \( \phi_0 \), a contradiction.

**Proof of (1.2).** This now follows the proof of (2.1). Construct a countably infinite chain of representations \( \{ \phi_i \} \) into the bounded operators on Hilbert spaces \( H_i \), where \( \phi_i \) is a weak \( \partial \)-representation with respect to \( \phi_{i-1} \) for each \( i \in \mathbb{N} \). Let \( \phi_\infty \) denote the spanning representation on the Hilbert space \( H_\infty \). Since a dilation of a weak \( \partial \)-representation with respect to a representation \( \phi \) is also a weak \( \partial \)-representation with respect to \( \phi \), \( \phi_\infty \) is a weak \( \partial \)-representation with respect to \( \phi_i \) for all \( i \). It easily follows that \( \phi_\infty \) is a \( \partial \)-representation.

Minimal \( \partial \)-representations dilating a given representation can be defined in the manner of minimal extremals.

### 4. The \( C^* \)-Envelope and the Šilov Ideal

The \( C^* \)-envelope of the operator algebra \( \mathcal{A} \), denoted \( C^*_e(\mathcal{A}) \), is a \( C^* \)-algebra which is determined by the property: there exists a completely isometric representation \( \gamma : \mathcal{A} \to C^*_e(\mathcal{A}) \) such that \( C^*(\gamma(\mathcal{A})) = C^*_e(\mathcal{A}) \) and if \( \rho : \mathcal{A} \to B(H) \) is any other completely contractive representation, then there exists an onto representation \( \pi : C^*(\rho(\mathcal{A})) \to C^*(\gamma(\mathcal{A})) \) such that \( \pi(\rho(a)) = \gamma(a) \) for all \( a \in \mathcal{A} \).

It is not hard to see that \( C^*_e(\mathcal{A}) \) is essentially unique, for if \( \rho \) also has the properties of \( \gamma \), then there exists an onto representation \( \sigma : C^*(\gamma(\mathcal{A})) \to C^*(\rho(\mathcal{A})) \) with \( \sigma(\gamma(a)) = \rho(a) \) for all \( a \in \mathcal{A} \). It follows that \( \sigma \) is the inverse of \( \pi \) and thus, as \( C^* \)-algebras, \( C^*(\gamma(\mathcal{A})) \) equals \( C^*(\rho(\mathcal{A})) \).

### 4.1. Theorem (1.2)

Every unital operator algebra has a \( C^* \)-envelope.

**Proof.** A proof follows directly from (1.2). Viewing \( \mathcal{A} \) as a subspace of \( B(K) \), the inclusion mapping \( \iota : \mathcal{A} \to B(K) \) is a completely isometric representation and thus, according to this
proposition, it dilates to a completely isometric representation $\gamma : A \to B(H)$ which is a $\partial$-representation.

To see that $C^*(\gamma(A))$ is the $C^*$-envelope, suppose $\psi : A \to B(H\psi)$ is also completely isometric. In this case $\sigma : \psi(A) \to B(H)$ given by $\sigma(\psi(a)) = \gamma(a)$ is completely contractive (and thus well-defined). By a theorem of Arveson, there exists a Hilbert space $K$ containing $H$ and a representation $\pi : C^*(\psi(A)) \to B(K)$ such that $\gamma(a) = \sigma(\psi(a)) = P_H \pi(\psi(a))|_H$ (10, Cor. 6.7). Since $a \mapsto P_H \pi(\psi(a))|_H$ is a representation of $A$ and $\gamma$ is a $\partial$-representation, $H$ reduces $\pi(\psi(A))$. Thus, $\sigma$ extends to an onto representation $C^*(\psi(A)) \to C^*(\gamma(A))$. □

Arveson says that $J$ is the Šilov boundary of the concrete operator algebra $A \subset B(K)$ if $J$ contains every ideal $I$ with the property that the restriction of the quotient $q : C^*(A) \to C^*(A)/I$ to $A$ is completely isometric. Since the inclusion of $A$ into $B(K)$ is completely isometric, there exists an onto representation $\pi : C^*(A) \to C^*_e(A) = C^*(\gamma(a))$ such that $\pi(a) = \gamma(a)$, where $\gamma$ is a representation as in (4.1) which generates the $C^*$-envelope of $A$. It is left to the interested reader to verify that the kernel of $\pi$ is the Šilov ideal of $A$.

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