Diffusive limits for Adaptive MCMC for Normal Target densities

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Abstract

Adaptive MCMC are a new class of MCMC procedure that have been recently proposed in literature that accentuate the convergence of the chain to the target distribution. However, since the transition kernel is not same at each iteration, the convergence is more difficult to show. In an earlier paper by Basak et al, applying diffusion approximation, the authors arrived at a diffusion governing the dynamics of a suitably defined AMCMC for an arbitrary target density \( \psi(\cdot) \). The resulting diffusion was more easy to handle compared to its discrete counterpart. In this paper we study the diffusion when the target distribution is standard Normal. Although this is a degenerate one, it satisfies Hörmander’s hypoellipticity condition and hence it has positive density on its support. Next, under some assumptions, we show it has a unique invariant distribution whose marginal distribution is Normal.

Keywords and phrases: Adaptive MCMC, Diffusion approximation, Hörmander’s Hypoelliptic conditions, Ito’s Lemma, MCMC.

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1 Introduction

Markov chain monte Carlo (MCMC) methods are a class of algorithm used to simulate a sample from an arbitrary distribution known only upto a constant. One of

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the algorithms belonging to this class is the Random Walk Metropolis-Hastings (RW MH) sampler. The method involves choosing a Markov chain such that the (unique) invariant distribution is the target density of interest. This is done by choosing a proposal density, from which simulating a sample is possible, and then accepting the generated sample with a certain probability (called the MH acceptance probability). For more information see [8].

One disadvantage of this method is that the speed of convergence depends on the proposal density. Adaptive MCMC (AMCMC) have been devised to counter this problem. Here the parameter(s), which will typically be scaling constants in the proposal density, are a function of the previous samples that have been generated so far. Hence, the proposal density changes at each iteration. This should be done in such a way that the scaling constants involved in the proposal density are the best possible choices in some sense.

Although AMCMC is better suited for simulation purposes it is important that the limiting distribution is indeed the target distribution (also called the ergodic property in the MCMC literature, see for example [8]). Since the transition kernel changes at each iteration, verifying ergodicity sometimes becomes complicated. Rosenthal et al (9) proved some sufficient condition for ergodicity, viz., diminishing adaptation condition, where the difference between the total variation norm of the kernels at the $i^{th}$ and the $(i+1)^{th}$ iteration should go to zero, as $i$ goes to infinity, and the simultaneous uniform ergodicity condition where time to convergence starting from any point is uniformly bounded over the starting points. Very often, verifying this condition for a markov chain (even with simple target density, say Normal) can be quite involved.

This paper approaches the problem from a different standpoint. By applying the diffusion approximation scheme, together with an auxiliary variable, to the discrete chain we convert it to a continuous time two dimensional diffusion process. Our gain by such an enterprise is that we can then invoke results in literature for Markov processes to infer about its invariant distribution whose marginal can then be possibly identified with the target distribution of the MCMC. Sometimes this can be done easily when compared to the discrete time setting.

In a different context, Gelman et al, (see [4]), applied this technique to a MH algorithm, whose target distribution was multivariate normal with iid normal components. They used the resulting stochastic differential equation (SDE) to recommend
choices of the tuning parameter that result in fastest convergence of the diffusion to the target distribution. Later the results were extended by Bedard et al \[2\]. However all the above papers deal with standard MCMC, where the transition kernel was fixed at each iteration. To the best of our knowledge, this (along with its companion paper \[1\]) is the first instance where diffusion approximation is applied to AMCMC.

This paper is arranged as follows. Section 2 contains the definition of the AMCMC and briefly mentions the diffusion approximation procedure done in \[1\]. Section 3 contains the main result (Theorem 2) of this paper, i.e., existence of the invariant distribution of the process along with the identification of the target distribution. The various subsections of Section 3 contribute to the proof of Theorem 2. In 3.1 we show that the process is tight. This combined with the hypoelliptic condition in 3.2 shows that the process admits a smooth invariant distribution. After establishing moment conditions of the variables under consideration in Section 3.3 and Section 3.4 identification of the target distribution is proved in Section 3.5. We end with some concluding remarks in 4.

## 2 Definitions

We define the AMCMC in such that the scaling parameter in the Normal proposal density is a function of whether the previous sample was accepted or not (ideally it should not depend only on the previous sample but on the whole sequence of sample that has been generated, but computations become more extensive in that case). Here we formally define our algorithm:

1. Select arbitrary \( \{X_0, \theta_0\} \in \mathcal{R} \times [0, \infty) \) where \( \mathcal{R} \) is the state space. Set \( n = 1 \).

2. Propose a new move say \( Y \) where
   \[
   Y \sim N(X_{n-1}, \theta_{n-1})
   \]

3. Accept the new point with probability
   \[
   \alpha(X_{n-1}, Y) = \min\{1, \frac{\psi(Y)}{\psi(X_{n-1})}\}
   \]
   If the point is accepted set \( X_n = Y, \xi_i = 1 \); else \( X_i = X_{n-1}, \xi_i = 0 \)

4. \( \theta_n = \theta_{n-1} e^{\frac{1}{n}(\xi_n-p)} \quad p > 0 \)

5. \( n \leftarrow n + 1 \)
6. Goto step 2.

To apply the diffusion approximation to the AMCMC we define the continuous time process $X_n(t)$ for all $n \geq 1$ and for all $t > 0$ for any target distribution $\psi(\cdot)$:

$$X_n(0) = x_0 \in \mathbb{R};$$

$$X_n\left(\frac{i + 1}{n}\right) = X_n\left(\frac{i}{n}\right) + \frac{1}{\sqrt{n}} \theta_n\left(\frac{i}{n}\right) \xi_n\left(\frac{i + 1}{n}\right) \epsilon_n\left(\frac{i + 1}{n}\right), \quad i=0, 1, \ldots,$$

$$X_n(t) = X_n\left(\frac{i}{n}\right), \quad \text{if } i/n \leq t < (i + 1)/n \quad \text{for some integer } i. \quad (2.1)$$

Here, $\xi_n\left(\frac{i + 1}{n}\right)$ conditionally follows the Bernoulli distribution given by:

$$P\left(\xi_n\left(\frac{i + 1}{n}\right) = 1 \mid X_n\left(\frac{i}{n}\right), \theta_n\left(\frac{i}{n}\right), \epsilon_n\left(\frac{i + 1}{n}\right)\right) = \min\left\{ \frac{\psi(X_n\left(\frac{i}{n}\right) + \frac{1}{\sqrt{n}} \theta_n\left(\frac{i}{n}\right) \epsilon_n\left(\frac{i + 1}{n}\right))}{\psi(X_n\left(\frac{i}{n}\right))}, 1 \right\},$$

and $\{\epsilon_n\left(\frac{i}{n}\right)\}$ are all independent $N(0,1)$ random variables. The process $\theta_n(t)$ is defined as:

$$\theta_n(0) = \theta_0 \in \mathbb{R}^+$$

$$\theta_n\left(\frac{i + 1}{n}\right) = \theta_n\left(\frac{i}{n}\right) e^{\frac{1}{\sqrt{n}} (\xi_n\left(\frac{i + 1}{n}\right) - p_n\left(\frac{i}{n}\right))}, \quad i=0, 1, \ldots,$$

and $\theta_n(t) = \theta_n\left(\frac{i}{n}\right), \quad \text{if } i/n \leq t < (i + 1)/n \quad \text{for some integer } i. \quad (2.2)$

Where $p_n\left(\frac{i}{n}\right) \approx 1 - \frac{p}{\sqrt{n}}$ for some $p > 0$.

It has been observed in an earlier paper (see [1]) that the diffusion governing the dynamics of the limiting process is the following

**Theorem 1.** (from [1]) The limit of the process $Y_n(t) := (X_n(t), \theta_n(t))$, where $X_n(t)$ and $\theta_n(t)$ is given by (2.1) and (2.2) respectively, is governed by:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \quad \text{with } Y_t = (X_t, \theta_t)', \quad (2.3)$$

where,

$$b(Y_t) = \left( \frac{\theta_t^2 \psi'(X_t)}{2 \psi(X_t)}, \ \theta_t \left( p - \frac{\theta_t}{\sqrt{2\pi}} \frac{\psi'(X_t)}{\psi(X_t)} \right) \right),$$

$$\sigma(Y_t) = \begin{pmatrix} \theta_t & 0 \\ 0 & 0 \end{pmatrix}$$

and $W_t$ is a two dimensional Wiener process.
3 Main result

In this paper we concentrate on the case where the target density is standard Normal (ie, \( \psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \)). Then the diffusion looks like:

\[
    dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \quad \text{where,}
\]

\[
    b(Y_t) = \left( -\frac{\theta_t^2}{2}X_t, \theta_t \left( p - \frac{\theta_t}{\sqrt{2\pi}}|X_t| \right) \right). \tag{3.1}
\]

and \( \sigma(Y_t) \) remains the same.

**Theorem 2.** The \( X \)-marginal of the invariant distribution of (3.1) is \( N(0,1) \).

**Proof:** The proof of the above theorem is spread over various subsections. In Section 3.1 we show that the process \((X_t, \eta_t)\) where \( \eta_t = 1/\theta_t \) is tight. This combined with the hypoelliptic condition in Section 3.2 shows that the process admits an invariant distribution. The marginal of the invariant distribution is identified as the target distribution in Section 3.5.

3.1 Tightness of \((X_t, \eta_t)\)

We state a simple Lemma.

**Lemma 1.** \( \int_0^t \theta_u dW_u \) is a martingale with respect to \( \mathcal{F}_t = \sigma(X_s, \theta_s; 0 \leq s \leq t) \) and hence

\[
    E(\int_0^t \theta_u dW_u) = 0.
\]

**Proof:** It is sufficient to show that the martingale \( Y_t := \int_0^t \theta_s dW_s \) is \( L_2 \)-bounded. So using Iso’s isometry it is sufficient to show that

\[
    E\left( \int_0^t \theta_s^2 ds \right) < \infty
\]

But,

\[
    d\theta_t^2 = 2\theta_t d\theta_t
\]

\[
    = 2\theta_t^2 (p - \frac{1}{\sqrt{2\pi}}|X_t|) dt
\]

\[
    \leq 2\theta_t^2 p dt
\]

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Now writing \( Y_t = \theta^2_t \) we have that
\[
dY_t \leq 2pY_t \, dt
\]
Multiplying both sides by the integrating factor \( e^{-2pY_t} \) we get
\[
d(Y_t e^{-2pY_t}) \leq 0
\]
Integrating and taking expectations we get
\[
(Y_t e^{-2pY_t}) - (Y_0) \leq 0 \quad \Rightarrow \quad \theta^2_t \leq \theta^2_0 e^{2pt} \quad \text{and} \quad E(\theta^2_t) \leq E(\theta^2_0) e^{2pt}, \quad \forall t > 0.
\]
Also
\[
\int_0^t \theta^2_s ds \leq \theta^2_0 2p(e^{2pt} - 1)
\]
and hence,
\[
E(\int_0^t \theta^2_s ds) < \infty, \quad \forall t > 0. \quad (3.2)
\]

Here is the main result of this subsection.

**Lemma 2.** Assume \( E(X_0^2) \) and \( E(\theta_0^2) \) are finite. Then, for the coupled system (3.1)

1. \( \sup_{t \geq 0} E(X_t^2) < \infty \), whenever \( E(X_0^2) \) and \( E(\theta_0^2) \) are finite;
2. \( \theta_t > 0 \) almost surely;
3. \( \eta_t > 0 \) almost surely, where \( \eta_t = \frac{1}{\theta_t} \);
4. Joint distribution of \( \{(X_t, \eta_t) : t \geq 0\} \) is tight.

**Proof.**

1. \[
\sup_t E(X_t^2) < \infty
\]
Applying Ito’s Rule to \( Y_t = X_t^2 \) we get
\[
dY_t = 2X_t dX_t + (1/2)2\theta_t^2 dt
\]
\[
= -X_t \theta_t^2 dt + 2X_t \theta_t dW_t + \theta_t^2 dt
\]
\[
= (-X_t^2 \theta_t^2 + \theta_t^2) dt + 2X_t \theta_t dW_t
\]
\[
\Rightarrow dY_t + Y_t \theta_t^2 dt = \theta_t^2 dt + 2X_t \theta_t dW_t.
\]
Multiplying both sides by \( e^{\int_0^t \theta_s^2 ds} \) we get
\[ d(Y_t e^{\int_0^t \theta_s^2 du}) = e^{\int_0^t \theta_s^2 du} \theta_t^2 dt + 2e^{\int_0^t \theta_s^2 du} X_t \theta_t dW_t \]
\[ \Rightarrow Y_t e^{\int_0^t \theta_s^2 du} - Y_0 = \int_0^t \theta_s^2 e^{\int_0^s \theta_u^2 du} ds + 2 \int_0^t e^{\int_0^s \theta_u^2 du} X_s \theta_s dW_s \]
\[ \Rightarrow X_t^2 = e^{-\int_0^t \theta_s^2 du} X_0^2 + e^{-\int_0^t \theta_s^2 du} \int_0^t \theta_s^2 e^{\int_0^s \theta_u^2 du} ds + 2e^{-\int_0^t \theta_s^2 du} \int_0^t e^{\int_0^s \theta_u^2 du} X_s \theta_s dW_s. \]

Writing \( F_s = e^{\int_s^t \theta_u^2 du} \), (therefore \( F_s^t = \theta_s^2 F_s \)) we have:
\[ X_t^2 = \frac{X_0^2}{F_t} + \int_0^t \frac{F_s}{F_t} X_s \theta_s dW_s \]
\[ = \frac{X_0^2}{F_t} + \frac{F_t - 1}{F_t} + 2 \int_0^t \frac{F_s X_s \theta_s dW_s}{F_t} \]
\[ \Rightarrow E(X_t^2) \leq E(X_0^2) + 1 + 2E(\frac{1}{F_t} \int_0^t F_s X_s \theta_s dW_s), \quad \text{since } F_t \geq 1 \quad \forall \, t \geq 0. \] (3.3)

Write \( Y_t = \int_0^t F(s) X_s \theta_s dW_s \) and \( \overline{F}_t = \frac{1}{F_t} = e^{-\int_0^t \theta_u^2 du} \). So the term inside the expectation on the right side is \( Y_t \overline{F}_t := Z_t \). Applying Ito’s lemma to \( Z_t \) we get,
\[ dZ_t = Y_t d\overline{F}_t + \overline{F}_t dY_t + dY_t d\overline{F}_t, \]
\[ = -Y_t \theta_t^2 \overline{F}_t dt + \overline{F}_t X_t \theta_t F(t) dW_t + 0 \]
\[ = -Z_t \theta_t^2 dt + X_t \theta_t dW_t \quad \text{with } Z_0 = 0. \] (3.4)

Writing \( \tilde{Z}_t = -Z_t \) we have the SDE of \( \tilde{Z}_t \) as
\[ d\tilde{Z}_t = - \tilde{Z}_t \theta_t^2 + X_t \theta_t d\tilde{W}_t \quad \text{where } \tilde{W}_t = -W_t \]

Comparing the SDE of \( Z_t \) and \( \tilde{Z}_t \), and noticing that \( Z_0 = \tilde{Z}_0 = 0 \), we have that \( Z_t = \tilde{Z}_t \) in distribution. This implies, \( Z_t \) is symmetric about 0. Therefore, \( E(Z_t) = 0 \) if we show that \( E|Z_t| < \infty \). Note, if \( E(Z_t) = 0 \), for each \( t \geq 0 \), then from (3.3), for each \( t \geq 0 \),
\[ E(X_t^2) \leq E(X_0^2) + 1. \] (3.5)

Thus, it suffices to show that \( Z_t \) has finite expectation for each \( t \geq 0 \).
Now
\[ Z_t^2 = \frac{1}{F_t^2} \left( \int_0^t F_s \theta_s X_s dW_s \right)^2 \leq \left( \int_0^t F_s \theta_s X_s dW_s \right)^2 \text{ a.s.} \]
\[ \Rightarrow E(Z_t^2) \leq E \left( \int_0^t F_s \theta_s X_s dW_s \right)^2 \text{ from Ito's isometry} \]
\[ \leq E \left( \int_0^t (F_s \theta_s)^4 ds \int_0^t X_s^4 ds \right)^{\frac{1}{2}} \]
\[ \leq \left[ E \left( \int_0^t (F_s \theta_s)^4 ds \right) E \left( \int_0^t X_s^4 ds \right) \right]^{\frac{1}{2}} \text{ from Cauchy-Schwarz inequality (3.6)} \]

Let us define \( Y_t = \theta_t^n \) for \( n = 1, 2, \ldots \) we have
\[ dY_t = n\theta_t^{n-1} d\theta_t = n\theta_t^{n-1} \left( \theta_t (p - \frac{|X_t|}{\sqrt{2\pi}}) dt \right) \]
\[ < np\theta_t^n = npY_t \]

Multiplying by \( e^{-np\theta_t} \) we have
\[ d(e^{-np\theta_t} Y_t) < 0 \]
\[ \Rightarrow Y_t e^{-np\theta_t} - Y_0 < 0 \]
\[ \Rightarrow \theta_t^n < \theta_0^n e^{np\theta_t} \]
\[ \Rightarrow \theta_t^{n_{mps}} < \theta_0^n e^{p(m+n)t} \] for \( s \leq t \) and \( n = 1, 2, \ldots, m = 0, 1, \ldots \)(3.7)

Now since
\[ X_t = X_0 - \int_0^t \frac{X_t \theta_t^2}{2} ds + \int_0^t \theta_s dW_s \]
\[ \Rightarrow X_t^2 \leq 3(|X_0|^2 + \int_0^t \frac{|X_s|^2}{2} ds + (\int_0^t \theta_s dW_s)^2) \]
\[ \Rightarrow X_t^4 \leq 27(|X_0|^4 + (\int_0^t \frac{|X_s|^2}{2} ds + (\int_0^t \theta_s dW_s)|^4) \]

Writing \( M_t := |\int_0^t \theta_s dW_s| \) which is a (loc) submartingale. Now applying Burkholder-Davis-Gundy (BDG) inequality to \( M_t \) we have
\[ E(M_t^4) \leq E\left( \sup_{0 \leq u \leq t} M_u \right)^4 \leq C_4 E(|M_t|^2) = C_4 E\left( \int_0^t \theta_s^2 ds \right)^2 < C_4 E(\theta_0^2 2p(e^{2pt} - 1))^2 \]
\[ = C_4 \theta_0^2 2p(e^{2pt} - 1) \] (3.8)
Again, the equation, \( d\theta_t = \theta_t \left(p - \frac{|X_t|\theta_t}{\sqrt{2\pi}}\right) dt \), yields

\[
0 = d(e^{-pt}\theta_t) + e^{-pt} \frac{|X_t|\theta_t^2}{\sqrt{2\pi}} dt
\]

\[
\Rightarrow \theta_0 = e^{-pt}\theta_t + \int_0^t e^{-ps} \frac{|X_s|\theta_s^2}{\sqrt{2\pi}} ds
\]

\[
\Rightarrow e^{pt}\theta_0 - \theta_t = \int_0^t e^{p(t-s)} \frac{|X_s|\theta_s^2}{\sqrt{2\pi}} ds.
\]

Thus,

\[
\left| \int_0^t \frac{|X_s|\theta_s^2}{\sqrt{2\pi}} ds \right| \leq \left| \int_0^t e^{p(t-s)} \frac{|X_s|\theta_s^2}{\sqrt{2\pi}} ds \right| \leq |\theta_0 e^{pt} - \theta_t| \leq \theta_0 e^{pt} + |\theta_t|
\]

\[
\Rightarrow \left( \int_0^t \frac{|X_s|\theta_s^2}{\sqrt{2\pi}} ds \right)^4 \leq (\theta_0 e^{pt} + \theta_t)^4. \quad (3.9)
\]

And, from (3.2), \((\theta_0 e^{pt} + \theta_t)^4 \leq 8(\theta_0^4 e^{4pt} + \theta_t^4) \leq 8(\theta_0^4 e^{4pt} + \theta_t^4) = 16\theta_0^4 e^{4pt}\). Hence, from (3.8) and (3.9)

\[
E(X_t^4) \leq 27(E(X_0^4) + 16e^{4pt}E(\theta_t^4) + 4C_4p^2(e^{2pt} - 1)^2E(\theta_0^4))
\]

\[
\Rightarrow E(\int_0^t X_s^4 ds) \leq 27 \left(E(X_0^4) t + ((16 + C_4p^2)(e^{4pt} - 1)/4p - 2C_4p^2(e^{2pt} - 1)/2p + C_4p^2t)E(\theta_0^4)\right)
\]

\[
< \infty \quad (3.10)
\]

And

\[
E(\int_0^t F_s^4\theta_s^4 ds) \leq E(\theta_0 \int_0^t e^{4ps} e^s \int_0^s e^{46u} du ds)
\]

\[
\leq E(\theta_0^4 \int_0^t e^{4ps} e^s \int_0^s e^{46u} e^{2pu} du ds)
\]

\[
= E(\theta_0^4 \int_0^t e^{4ps} e^{46(e^{2ps}-1)/2p} ds)
\]

\[
\leq E(\theta_0^4 e^{46(e^{2pt}-1)/p} \int_0^t e^{4ps} ds) < \infty \quad (3.11)
\]

So, from (3.10) and (3.11) we have:

\[
E(Z_t^2) < \infty.
\]

This proves 1.
Remark 1. Although we have only proved that the second moment of $X_t$ is uniformly bounded, it is shown in Section 3.3 that all even ordered moments (and therefore all moments) are uniformly bounded.

2. Take $V_t = \frac{1}{\theta_t}$. Then

$$dV_t = -\frac{1}{\theta_t^2} d\theta_t \Rightarrow -\frac{1}{\theta_t^2} \theta_t (p - \frac{1}{\sqrt{2\pi}} |X_t| \theta_t) dt$$

$$= -V_t (p - \frac{|X_t|}{V \sqrt{2\pi}}) dt$$

$$= (-V_t p + \frac{|X_t|}{\sqrt{2\pi}}) dt$$

Multiplying by the integrating factor $e^{pt}$ we get

$$d(e^{pt} V_t) = \frac{e^{pt} |X_t|}{\sqrt{2\pi}} dt$$

$$\Rightarrow e^{pt} V_t - V_0 = \int_0^t \frac{1}{\sqrt{2\pi}} e^{pu} |X_u| du$$

$$\Rightarrow V_t = V_0 e^{-pt} + \int_0^t e^{-p(t-u)} \frac{|X_u|}{\sqrt{2\pi}} du$$

(3.12)

$$\Rightarrow E(V_t) = E(V_0) e^{-pt} + \int_0^t e^{-p(t-u)} E|X_u| \frac{1}{\sqrt{2\pi}} du$$

$$\leq E(V_0) e^{-pt} + \frac{M_0}{p \sqrt{2\pi}} [1 - e^{-pt}]$$

(since $\sup_{t>0} E(|X_t|) < M_0$, for some $M_0 > 0$, depending on $E(|X_0|)$)

$$\leq M_1$$

(3.12c) (depends on $E(1/\theta_0)$, and $E(|X_0|)$)

$$\Rightarrow \sup_{t \geq 0} E(1/\theta_t) \leq M_1$$

Therefore $\theta_t > 0$ almost surely.

3. Define, $\eta_t = 1/\theta_t$. From the proof of the second part, it follows that

$$\eta_t = \eta_0 e^{-pt} + \int_0^t e^{-p(t-u)} \frac{|X_u|}{\sqrt{2\pi}} du > 0, \text{ whenever } \eta_0 > 0.$$
for any $t > 0$, for each sample points, \[ \int_0^t e^{-p(t-u)|X_u|/\sqrt{2\pi}} du = 0 \] if only if $X_u = 0$ for almost all $u \in (0, t]$ w.r.t. Lebesgue measure. This is impossible, since $X_u = 0$ for almost all $u$ implies $dX_u = \theta u dW_u$. Then $X_u$ would be non-zero for almost all $u$, since as

\[ \theta_u = \frac{e^{\mu u}}{\eta_0 + \int_0^u (1/\sqrt{2\pi}) e^{ps} |X_s| ds} > 0 \]  

(3.13)

for all $u$. This contradicts our assumption that $X_u = 0$ for almost all $u$. Therefore $X_u \neq 0$ for $u$ on a set of positive Lebesgue measure. Hence \[ \int_0^t e^{-p(t-u)|X_u|/\sqrt{2\pi}} du > 0 \] almost all $u$. Thus, $\eta_t > 0$ a.s. Repeating the same argument will, in fact, give $X_u \neq 0$ for almost all $u \in (0, t]$, for any $t > 0$. This proves the third part.

4. This follows from the above where we have shown that $\sup_{t \geq 0} E(|X_t|) < \infty$ and $\sup_{t \geq 0} E(1/\theta_t) < \infty$. Let $R_1$ and $R_2$ be two positive numbers. Then

\[ P(|X_t| < R_1, |\eta_t| < R_2) = 1 - P(|X_t| > R_1) \cup |\eta_t| > R_2) \]
\[ > 1 - (P(|X_t| > R_1) + P(|\eta_t| > R_2)) \]
\[ > 1 - E(|X_t|)/R_1 - E(|\eta_t|)/R_2 \]

Hence given any $\epsilon > 0$ we can choose $R_1, R_2$ sufficiently large so that $P(|X_t| < R_1, |\eta_t| < R_2) > 1 - \epsilon$. This proves tightness of $(X_t, \eta_t)$.

3.2 Hypoelliptic condition

Here we show that the vector fields corresponding to 3.1 satisfies the Hormanders hypoelliptic condition (see the proposition for the statement of the condition).

Since the condition requires smooth vector fields, we convert the drift and diffusion coefficients in 2.3 into smooth vector fields.

For this purpose define

\[ b_\epsilon(x, \eta) = \left( -\frac{x}{2\eta^2}, -p \eta + \frac{g_\epsilon(x)}{\sqrt{2\pi}} \right), \]

where $g_\epsilon(x) \to |x|$ as $\epsilon \downarrow 0$ in the pointwise is a smooth function, and $\sigma(x, \eta) = \begin{pmatrix} 1/\eta & 0 \\ 0 & 0 \end{pmatrix}$ as the drift and the diffusion coefficient respectively of our new equation. Such function $g_\epsilon$ can be constructed by convoluting the function $|x|$ with a
mollifier. Defining in this fashion \( ||b_\epsilon(\cdot, \cdot) - b(\cdot, \cdot)|| \to 0 \) as \( \epsilon \downarrow 0 \).

Consider a SDE in the Stratonovich form:

\[
dX_t = A_0(X_t)dt + \sum_\alpha A_\alpha(X_t) \circ dB^\alpha_t. \tag{3.14}
\]

where \( f \) is a smooth function on \( M \) and \( \circ \) denotes Stratonovich integral. SDE in the Ito and the Stratonovich form are interchangeable: for a multidimensional SDE given in the Ito’s form

\[
dX_t = \tilde{b}(t, X_t) dX_t + \sigma(t, X_t) dW_t.
\]

can be readily converted into the Stratonovich form from the following equation:

\[
b_i(t, x) = \tilde{b}_i(t, x) - \frac{1}{2} \sum_{j=1}^p \sum_{k=1}^n \frac{\partial \sigma_{i,j}}{\partial x_k} \sigma_{k,j}; \quad 1 \leq i \leq n
\]

where \( b(t, x) = (b_i(t, x))' \) is the drift term for the Stratonovich form. From the form of \( \sigma(x, \eta) \) we can say that \( b' \) and \( \tilde{b}' \) are the same. We can identify the drift and diffusion coefficients \( A_0(X_t) \) and \( A_1(X_t) \) can be identified as vector fields in \( M \).

Here is the condition due to Hörmander, ([5]):

**Proposition 1.** Let \( \{A_0, A_1, \ldots, A_n\} \) be \( n \) smooth vector fields on \( \mathbb{R}^d \). Define the Lie Bracket \([V, W](x)\) between two vector fields \( V \) and \( W \) as

\[
[V, W](x) = DV(x)W(x) - DW(x)V(x),
\]

where \( D(V(x)) \) is the Frechet derivative. The (parabolic) Hörmanders hypoelliptic condition is satisfied if:

\[
A_{j_0}(y), \quad [A_{j_0}(y), A_{j_1}(y)] \quad [[A_{j_0}(y), A_{j_1}(y)], A_{j_2}(y)] \quad \cdots \quad [[[A_{j_0}(y), A_{j_1}(y)], A_{j_2}(y)], \ldots, A_{j_n}(y)] \quad \text{spans } \mathbb{R}^d \quad \text{for every point } y \in \mathbb{R}^d \quad \text{and any } \{j_0, j_1, \ldots, j_d\} \in \{1, 2, \ldots, n\} \quad j_0 \leq j_1, \ldots, j_n.
\]

**Lemma 3.** The vector fields \( A'_0(y) \) and \( A_1(y) \) satisfies Hörmanders hypoelliptic condition.

**Proof:** Identifying \( 3.14 \) with \( 3.1 \) we have (writing \( y = (x, \eta) \)):

\[
A'_0(y) = -\frac{x}{2\eta^2} \frac{\partial}{\partial x} + (-p\eta + \frac{g_\epsilon(x)}{\sqrt{2\pi}}) \frac{\partial}{\partial \eta}
\]

\[
A_1(y) = \frac{1}{\eta} \frac{\partial}{\partial x}
\]
Therefore vectors corresponding to \( A_0^*(y), A_1(y) \) and \([A_0^*(y), A_1(y)]\) are \((- \frac{x}{2n^2} - p\eta + \frac{g(x)}{\sqrt{2\pi}}, (1/\eta, 0)\) and \((- \frac{1}{2n^2} + p\eta - \frac{g'(x)}{\sqrt{2\pi}}, \frac{g(x)}{2\pi\eta})\).

Since \( \theta = 1/\eta > 0 \) and \( x \neq 0 \) almost surely, one obtains that these two vectors span \( \mathbb{R}^2 \), almost surely. (what about \( A_0 \)?)

It is well known that if the vector fields \( A_0(y) \) and \( A_1(y) \) satisfy the above conditions then the solution of the SDE (3.14) admits a smooth density (see, [7]). Thus, satisfy (parabolic) Hörmander’s condition and hence even though the original diffusion is singular its transition probability has density (see [6]). Again, since the coupled diffusion is tight, it admits unique invariant probability by [6] and it admits a density.

**Remark 2.** Note that although we are interested in the distribution of \( X \) showing tightness of the process \( X \) only would not suffice since \( \theta_t \) may be a function of \( \{X_s; 0 \leq s \leq t\} \), so marginally \( X \) may not be a Markov process. Hence \( \sup_t E|X_t| < M \) would give the tightness of \( X \) but it would not be appropriate to tell about the existence of a unique distribution.

### 3.3 Uniform boundedness of moments of \( X_t \)

**Lemma 4.** For any \( k \geq 1 \), moment of order \( 2k \) of \( X_t \) is uniformly bounded, ie.,

\[
\sup_t E(X_t^{2k}) < \infty,
\]

whenever \( E(X_0^{2k}) \) is finite.

**Proof:** Applying Ito’s Lemma to \( Y_t = X_t^{2k} \) we get

\[
\begin{align*}
\frac{dX_t^{2k}}{dt} &= 2kX_t^{2k-1}dX_t + k(2k-1)X_t^{2k-2}\theta_t^2 dt \\
&= \left(-kX_t^{2k}\theta_t^2 + k(2k-1)X_t^{2k-2}\theta_t^2 \right)dt + 2kX_t^{2k-1}\theta_t dW_t \\
&\leq \left(-kX_t^{2k}\theta_t^2 + k(2k-1)(aX_t^{2k} + b}\theta_t^2 \right)dt + 2kX_t^{2k-1}\theta_t dW_t,
\end{align*}
\]

since for any small \( a > 0 \), there exists \( b \) large enough such that, \( x^{2k-2} < ax^{2k} + b \ \forall x \). Thus, for \( 0 < a < 1/(2k-1) \) we have

\[
\begin{align*}
\frac{dX_t^{2k}}{dt} &\leq -X_t^{2k}\theta_t^2 \left(k - k(2k-1)a\right) dt + k(2k-1)b\theta_t^2 dt + 2kX_t^{2k-1}\theta_t dW_t \\
\Rightarrow \frac{dX_t^{2k}}{dt} &+ C_kX_t^{2k}\theta_t^2 dt \leq k(2k-1)b\theta_t^2 dt + 2kX_t^{2k-1}\theta_t dW_t,
\end{align*}
\]
where $C_k = k(1 - (2k - 1)a) > 0$. Multiplying by the integrating factor $e^{C_k \int_0^t \theta^2 du} := F^C_t$ on both sides we get
\[
d\left(X_t^{2k} F^C_t \right) \leq k(2k - 1) b \theta^2 F^C_t dt + 2k F^C_t X_t^{2k-1} \theta_t dW_t
\]
\[\Rightarrow X_t^{2k} F^C_t \leq X_0^{2k} + k(2k - 1) b \int_0^t \theta^2 F^C_u du + 2k \int_0^t F^C_u X_u^{2k-1} \theta_u dW_u
\]
\[\Rightarrow X_t^{2k} \leq X_0^{2k} F_t^{-C_k} + k(2k - 1) b F_t^{-C_k} \int_0^t \theta^2 F^C_u du + 2k F_t^{-C_k} \int_0^t F^C_u X_u^{2k-1} \theta_u dW_u
\]
Now,
\[\int_0^t \theta^2 F^C_u du = (F^C_t - 1)/C_k
\]
\[\Rightarrow E\left(X_t^{2k}\right) \leq E\left(F_t^{-C_k} X_0^{2k}\right) + E\left(\frac{1}{C_k} (1 - F_t^{-C_k})\right) + 2k E\left(F_t^{-C_k} \int_0^t F^C_u X_u^{2k-1} \theta_u dW_u\right)
\]
Following the similar notation as before let $\overline{F}_{t,k} = F_t^{-C_k}$. Also define $Z_{t,k} = \overline{F}_{t,k} Y_{t,k}$ where $Y_{t,k} = \int_0^t F^C_u X_u^{2k-1} \theta_u dW_u$. Applying Ito’s lemma to $Z_{t,k}$ we have
\[dZ_{t,k} = Y_{t,k} d\overline{F}_{t,k} + \overline{F}_{t,k} dY_{t,k}
\]
\[= -C_k Y_{t,k} \theta^2 F_t^{-C_k} dt + \overline{F}_{t,k} X_t^{2k-1} \theta_t F_t^{-C_k} dW_t
\]
\[= -C_k Z_{t,k} \theta^2 dt + X_t^{2k-1} \theta_t d\bar{W}_t.
\]
Now, taking $\tilde{Z}_{t,k} = -Z_{t,k}$, yields
\[d\tilde{Z}_{t,k} = C_k Z_{t,k} \theta^2 dt - X_t^{2k-1} \theta_t d\bar{W}_t = -C_k \tilde{Z}_{t,k} \theta^2 dt + X_t^{2k-1} \theta_t d\bar{W}_t
\]
where $\bar{W}_t = -W_t$. Since $\tilde{Z}_{0,k} = -Z_{0,k} = 0 = Z_{0,k}$. Hence $Z_{t,k}$ and $\tilde{Z}_{t,k}$ has the same distribution, i.e., the distribution is symmetric around 0. Therefore to show that $E(Z_{t,k}) = 0$, $\forall t \geq 0$ one needs to show $Z_{t,k}$ has finite expectation $\forall t \geq 0$. Thus, as before, it is sufficient to show that $E(Z_{t,k}^2) < \infty$, $\forall t \geq 0$. Now,
\[Z_{t,k}^2 = \overline{F}_{t,k}^2 \left(\int_0^t F^C_s X_s^{2k-1} \theta_s dW_s\right)^2 \leq \left(\int_0^t F^C_s X_s^{2k-1} \theta_s dW_s\right)^2 a.s
\]
\[\Rightarrow E\left(Z_{t,k}^2\right) \leq E\left(\int_0^t F^C_s X_s^{2k-1} \theta_s dW_s\right)^2 = E\left(\int_0^t F^C_s \theta^2 X_s^{4k-2} ds\right)
\]
\[\leq E\left(\int_0^t F^C_s \theta^2 ds\right) E\left(\int_0^t X_s^{8k-4} \theta^2 ds\right) \leq \sqrt{E\left(\int_0^t F^C_s \theta^2 ds\right) E\left(\int_0^t X_s^{8k-4} \theta^2 ds\right)}
\]
by Cauchy-Schwarz inequality.
Again, \( E(\int_0^t F_s^{4C_k} \theta_s^2 ds) = E((F_t^{4C_k} - 1)/(4C_k)) \leq E(e^{4C_k \theta_0^2 e^{2p\theta}}/(4C_k)) < \infty \).

For \( \int_0^t X_s^{8k-4} \theta_s^2 ds \) we have

\[
X_t = X_0 - \int_0^t X_s \theta_s^2 ds + \int_0^t \theta_s dW_s
\]

\[
\Rightarrow X_t^{8k-4} \theta_t^2 \leq C_k^t \left(X_0^{8k-4} + \left(\int_0^t |X_s| \theta_s^2 ds\right)^{8k-4} + \left(\int_0^t \theta_s dW_s\right)^{8k-4}\right) \theta_0^2 e^{2pt}
\]

for some constant \( C_k^t > 0 \) independent of \( X_t \).

For the second term, from (3.9)

\[
|\int_0^t |X_s| \theta_s^2 ds| \leq |\int_0^t e^{p(t-s)} |X_s| \theta_s^2 ds| \leq \sqrt{2\pi} |\theta_0 e^{pt} + \theta_t| \leq \sqrt{2\pi} |\theta_0 e^{pt}
\]

\[
\Rightarrow \left(\int_0^t |X_s| \theta_s^2 ds\right)^{8k-4} \leq \left(\sqrt{2\pi}\right)^{8k-4} \left(|\theta_0 e^{pt}\right)^{8k-4}
\]

(3.15)

\[
\Rightarrow \int_0^t \left(\int_0^s |X_u| \theta_u^2 du\right)^{8k-4} ds \leq \left(2\pi\right)^{4k-2} \left(\theta_0^{8k-4} \int_0^t e^{(8k-4)p\theta} ds\right)^{8k-4}
\]

(3.16)

For the third term, write \( M_t := |\int_0^t \theta_s dW_s| \) and apply BDG inequality:

\[
E\left(M_t\right)^{8k-4} \leq E\left(\sup_{0 \leq s \leq t} M_s\right)^{8k-4} < C_{k0} E\left(\left|M_t\right|^{4k-2}\right)
\]

\[
= C_{k0} E\left(\int_0^t \theta_s^2 ds\right)^{4k-2} \leq E\left(\int_0^t e^{2p\theta} ds\right)^{4k-2}
\]

for some \( C_{k0} > 0 \) and for all \( t > 0 \)

(3.17)

(3.18)

Combining (3.16) and (3.18) we have

\[
E\left(\int_0^t X_s^{8k-4} \theta_s^2 ds\right) \leq \left(C_k^t E\left(X_0^{8k-4}\right) + C_k^t (2\pi)^{4k-2} E\left(\theta_0 \int_0^t e^{p\theta} ds\right)^{8k-4}\right)
\]

\[
+ C_k^t C_{k0} E\left(\int_0^t \theta_0^{8k-4} e^{2p\theta} ds\right)^{4k-2} E\left(\theta_0^2 \int_0^t e^{2p\theta} ds\right)^{4k-2}
\]

\[
< \infty
\]

This proves

\[
E(Z_{t,k}^2) < \infty
\]

and hence the lemma.
3.4 Finiteness of Time average of moments of $\theta$

In this section $C$ will stand for a generic constant that might take different values in different situations. We assume throughout that, for a fixed $k \geq 1$, $E(X_0^{2k})$, $E(\theta_0^{2k})$ and $E(\eta_0^{2k})$ are finite. For non-random initial data this is already assured.

We proceed sequentially by the following steps:

**Step 1:** We have

\[
\sup_{t>0} E(X_t^{2k}) < \infty.
\]

\[
\]

**Step 2:** We show that

\[
\sup_{t>0} E(\eta_t^{2k}) < \infty.
\]

Using

\[
\eta_t = \eta_0 e^{-pt} + \int_0^t e^{-p(t-u)} \frac{|X_u|}{\sqrt{2\pi}} du
\]

, to get,

\[
\eta_t^{2k} = \left( \eta_0 e^{-pt} + \int_0^t e^{-p(t-u)} \frac{|X_u|}{\sqrt{2\pi}} du \right)^{2k}
\]

\[
\leq 2^{2k-1} \left( \eta_0^{2k} e^{-2kpt} + \left( \int_0^t e^{-p(t-u)} \frac{|X_u|}{\sqrt{2\pi}} du \right)^{2k} \right)
\]

\[
\leq 2^{2k-1} \left( \eta_0^{2k} e^{-2kpt} + \left( \frac{1 - e^{-pt}}{p \sqrt{2\pi}} \right)^{2k} \left( \frac{p}{1 - e^{-pt}} \int_0^t e^{-p(t-u)} |X_u|^{2k} du \right) \right),
\]

where last inequality follows from Jensen’s inequality. Thus,

\[
\sup_{t>0} E(\eta_t^{2k}) \leq \sup_{t>0} 2^{2k-1} \left( E(\eta_0^{2k}) e^{-2kpt} + \left( \frac{1 - e^{-pt}}{p \sqrt{2\pi}} \right)^{2k} \left( \frac{p}{1 - e^{-pt}} \int_0^t e^{-p(t-u)} |X_u|^{2k} du \right) \right)
\]

\[
\leq 2^{2k-1} \left( E(\eta_0^{2k}) + \left( \frac{1}{p \sqrt{2\pi}} \right)^{2k} \sup_{t>0} \left( \frac{p}{1 - e^{-pt}} \int_0^t e^{-p(t-u)} |X_u|^{2k} du \right) \right)
\]

\[
\leq 2^{2k-1} \left( E(\eta_0^{2k}) + \left( \frac{1}{p \sqrt{2\pi}} \right)^{2k} \sup_{t>0} E(|X_t|^{2k}) \right) < \infty.
\]

Hence the result.

**Step 3:** We now prove

\[
\sup_{t>0} \frac{1}{t} \int_0^t E(|X_u| \theta_t) du < \infty.
\]
Note,
\[ d(1 + \theta_t) = d\theta_t = \theta_t(p - |X_t|\theta_t/\sqrt{2\pi})dt \]
\[ = (1 + \theta_t)(p - |X_t|\theta_t/\sqrt{2\pi})dt - (p - |X_t|\theta_t/\sqrt{2\pi})dt \]
\[ \Rightarrow d(1 + \theta_t) + (p - |X_t|\theta_t/\sqrt{2\pi})dt = (1 + \theta_t)(p - |X_t|\theta_t/\sqrt{2\pi})dt \]
\[ \Rightarrow \frac{d(1 + \theta_t)}{1 + \theta_t} + \frac{1}{1 + \theta_t}(p - |X_t|\theta_t/\sqrt{2\pi})dt = (p - |X_t|\theta_t/\sqrt{2\pi})dt \]
\[ \Rightarrow \frac{d(1 + \theta_t)}{1 + \theta_t} + \frac{1}{\sqrt{2\pi}}|X_t|\theta_t dt \leq p + \frac{|X_t|}{\sqrt{2\pi}} dt \]
\[ \Rightarrow \log \frac{1 + \theta_t}{1 + \theta_0} + \frac{1}{\sqrt{2\pi}} \int_0^t |X_u|\theta_u du \leq pt + \frac{1}{\sqrt{2\pi}} \int_0^t |X_u| du \]
\[ \Rightarrow \frac{1}{t} \int_0^t |X_u|\theta_u du \leq \sqrt{2\pi}p + \frac{1}{t} \int_0^t |X_u| du + \sqrt{2\pi} \frac{\log(1 + \theta_0)}{t}. \]

Thus, \( \frac{1}{t} \int_0^t E(|X_u|\theta_u) du \leq \sqrt{2\pi}p + \frac{1}{t} \int_0^t E(|X_u|) du + \sqrt{2\pi} \frac{E(\log(1 + \theta_0))}{t} \).

Therefore, using the moment bounds for \( X \),
\[ \sup_{t>0} \frac{1}{t} \int_0^t E(|X_u|\theta_u) du < C \]

\[ \square \]

Step 4: We now prove by induction, that for any \( k \geq 1 \),
\[ \sup_{t>0} \frac{1}{t} \int_0^t E(\theta_u^k) du < C. \tag{3.20} \]

Let, as before, \( \eta_t = \frac{1}{\theta_t} \) then \( d\eta_t = (-p\eta_t + |X_u|/\sqrt{2\pi})dt \).

Applying Ito’s lemma to \( Y_t = X_t^2\eta_t^{2-k/2} \), with \( k \geq 1 \) positive integer, we get
\[ dY_t = 2X_t\eta_t^{2-k/2}dX_t + (2 - k/2)X_t^2\eta_t^{1-k/2}d\eta_t + \frac{1}{2}2\eta_t^{2-k/2}(dX_t)^2 \]
\[ = 2X_t\eta_t^{2-k/2}(-\frac{X_t}{2\eta_t^2}dt + \frac{1}{\eta_t}dW_t) + (2 - k/2)X_t^2\eta_t^{1-k/2}(-p\eta_t dt + \frac{|X_t|}{\sqrt{2\pi}} dt) \]
\[ + \eta_t^{2-k/2}\eta_t^{-2}dt \]
\[ = \left(-X_t^2\eta_t^{-k/2} - p(2 - k/2)X_t^2\eta_t^{2-k/2} + \frac{2 - k/2}{\sqrt{2\pi}}|X_t|^3\eta_t^{1-k/2} + \eta_t^{-k/2}\right)dt \]
\[ + 2X_t\eta_t^{1-k/2}dW_t \tag{3.21} \]
Thus, integrating both side from 0 to \( t \) and rearranging we get,

\[
\int_0^t \theta_s^k dt = X_t^2 - X_0^2 + \int_0^t X_s^2 \theta_s^k dt + \frac{(4 - k)p}{2} \int_0^t X_s^2 \eta_t^\frac{2-k}{2} dt \]

\[
- \frac{2 - k/2}{\sqrt{2\pi}} \int_0^t |X_s|^3 \eta_t^{\frac{2-k}{2}} dt - 2 \int_0^t X_s \eta_s^{\frac{2-k}{2}} dW_s
\]

\[
= \frac{1}{t} \int_0^t \theta_s^k dt = \frac{1}{t} \left( X_t^2 - X_0^2 \right) + \frac{1}{t} \int_0^t X_s^2 \theta_s^k dt + \frac{(4 - k)p}{2t} \int_0^t X_s^2 \eta_t^\frac{2-k}{2} dt
\]

\[
- \frac{2 - k/2}{t\sqrt{2\pi}} \int_0^t |X_s|^3 \eta_t^{\frac{2-k}{2}} dt - \frac{2}{t} \int_0^t X_s \eta_t^{\frac{2-k}{2}} dW_s
\]

\[
\Rightarrow \frac{1}{t} \int_0^t \theta_s^k dt = \frac{1}{t} \left( X_t^2 - X_0^2 \right) + \frac{1}{t} \int_0^t X_s^2 \theta_s^k dt + \frac{(4 - k)p}{2t} \int_0^t X_s^2 \eta_t^\frac{2-k}{2} dt
\]

\[
- \frac{2 - k/2}{t\sqrt{2\pi}} \int_0^t |X_s|^3 \eta_t^{\frac{2-k}{2}} dt - \frac{2}{t} \int_0^t X_s \eta_t^{\frac{2-k}{2}} dW_s
\]  

(3.22)

First consider the first, third and fourth term in 3.22. We proceed in steps

Step 4a: \( 1 \leq k \leq 2 \): Since we have proved in the earlier steps that the \((2k)\)th moment of \( X_t \) and \( \eta_t \) is uniformly bounded we, by an application of C-S inequality, can say that

\[
\sup_{t>0} \frac{1}{t} E \left( X_t^2 \eta_t^{\frac{4}{2-k}} + X_0^2 \eta_0^{\frac{4}{2-k}} \right) < C, \quad (3.23)
\]

\[
\sup_{t>0} \frac{1}{t} \int_0^t E \left( X_s^2 \eta_s^{\frac{4}{2-k}} \right) dt < C, \quad (3.24)
\]

and

\[
\sup_{t>0} \frac{1}{t} \int_0^t |X_s|^3 \eta_s^{\frac{2-k}{2}} dt < C. \quad (3.25)
\]

For the fifth term \( \frac{1}{t} \int_0^t X_s \eta_s^{\frac{2-k}{2}} dW_s \) is a square integrable martingale (by Ito’s isometry) and hence

\[
\frac{1}{t} \int_0^t E \left( X_s \eta_s^{\frac{2-k}{2}} \right) dW_s = 0.
\]
Step 4b: Consider the second term in (3.22). For any $k \geq 1,$

$$\frac{1}{t} \int_0^t X_s^2 \theta_s^k ds = \frac{1}{t} \int_0^t (|X_s|^k \theta_s^k)(|X_s|^k) ds
\leq \left( \frac{1}{t} \int_0^t |X_s|^k \theta_s^k ds \right)^{\frac{k}{k+1}} \left( \frac{1}{t} \int_0^t |X_s|^{k+2} ds \right)^{\frac{1}{k+1}}$$

$$\Rightarrow E \left( \frac{1}{t} \int_0^t X_s^2 \theta_s^k \right) \leq E \left( \left( \frac{1}{t} \int_0^t |X_s|^k \theta_s^k ds \right)^{\frac{k}{k+1}} \left( \frac{1}{t} \int_0^t |X_s|^{k+2} ds \right)^{\frac{1}{k+1}} \right)$$

$$\leq \left( E \left( \frac{1}{t} \int_0^t |X_s|^k \theta_s^k ds \right) \right)^{\frac{k}{k+1}} \left( E \left( \frac{1}{t} \int_0^t |X_s|^{k+2} ds \right) \right)^{\frac{1}{k+1}}$$

$$= \left( \frac{1}{t} \int_0^t E(|X_s|^k) ds \right)^{\frac{k}{k+1}} \left( \frac{1}{t} \int_0^t E(|X_s|^{k+2}) ds \right)^{\frac{1}{k+1}}$$

$$\Rightarrow \sup_{t>0} \frac{1}{t} \int_0^t E(X_s^2 \theta_s^k) \leq \left( \sup_{t>0} \frac{1}{t} \int_0^t E(|X_s|^k) ds \right)^{\frac{k}{k+1}} \left( \sup_{t>0} \frac{1}{t} \int_0^t E(|X_s|^{k+2}) ds \right)^{\frac{1}{k+1}} < C,$$  \hspace{1cm} (3.26)

We only have to show that the first term on the RHS is finite. We proceed by induction: We have proved (Step 3 in this section) that

$$\sup_{t>0} \frac{1}{t} \int_0^t E(|X_s| \theta_s) ds < C$$

Therefore from (3.23, 3.24 and 3.25) since $k = 1,$ we have that

$$\sup_{t>0} \frac{1}{t} \int_0^t E(\theta_s^k) du < C.$$  \hspace{1cm} (3.27)

Assume that the hypothesis is true for $k = m - 1,$ ie.,

$$\sup_{t>0} \frac{1}{t} \int_0^t E(\theta_s^{m-1}) ds < C \hspace{1cm} (3.27)$$

$$\sup_{t>0} \frac{1}{t} \int_0^t E(|X_s| \theta_s^m) < C \hspace{1cm} (3.28)$$

This also imply,

$$\sup_{t>0} \frac{1}{t} \int_0^t \theta_s^{\frac{m-1}{2}} ds < C.$$  \hspace{1cm} (3.29)
Then, for \( k = m \),
\[
d\theta_t^{m-1} = \frac{m-1}{2} \theta_t^{m-2} d\theta_t = \frac{m-1}{2} \theta_t^{m-2} (p - |X_t|\theta_t/\sqrt{2\pi})dt
\]
\[
= \frac{p(m-1)}{2} \theta_t^{m-2} dt - |X_t|\theta_t^{m+1}/\sqrt{2\pi} dt
\]
\[
\Rightarrow \int_0^t |X_s|\theta_s^{m+1}/\sqrt{2\pi} ds = \int_0^t \frac{p(m-1)}{2} \theta_s^{m-2} ds - (\theta_t^{m-2} - \theta_0^{m-2})
\]
\[
\Rightarrow \sqrt{2\pi}E(\theta_t^{m-2}) + \int_0^t E(|X_s|\theta_s^{m+1})ds = \frac{p(m-1)}{2} \sqrt{2\pi} \int_0^t E(\theta_s^{m-2})ds + \sqrt{2\pi}E(\theta_0^{m-2})
\]
\[
\Rightarrow \sup_{t>0} \frac{1}{t} \int_0^t E\left(|X_s|\theta_s^{m+1}\right) \leq \sqrt{2\pi} \sup_{t>0} \frac{1}{t} E(\theta_t^{m-2}) + \sup_{t>0} \frac{1}{t} \int_0^t E\left(|X_s|\theta_s^{m+1}\right)ds
\]
\[
\leq \frac{p(m-1)}{2} \sqrt{2\pi} \sup_{t>0} \frac{1}{t} \int_0^t E\left(\theta_s^{m-2}\right) + \sqrt{2\pi}E\left(\theta_0^{m-2}\right)
\]

By the induction hypothesis 3.27 and 3.28. This proves the proposition.
Therefore the second term on 3.22 is finite, for any \( k \geq 1 \). \( \Box \)

Step 4c: \( k \geq 3 \).

For \( 3 \leq k \leq 4 \) we can claim similarly as above that:
\[
\sup_{t>0} \frac{1}{t} E\left(X_t^{4-k} \eta_t^{k-4}\right) < C
\]
\[
\sup_{t>0} \frac{1}{t} \int_0^t E\left(X_s^{4-k} \eta_s^{k-4}\right) dt < C.
\]

For \( 4 \leq k \):
\[
\sup_{t>0} \frac{1}{t} E\left(X_t^{4-k} \eta_t^{k-4}\right) = \sup_{t>0} \frac{1}{t} E\left(X_t^{k-4} \theta_t^{k-4}\right)
\]
\[
= \sqrt{\sup_{t>0} \frac{1}{t} E\left(X_t^{4}\right) \sup_{t>0} \frac{1}{t} E\left(\theta_t^{k-4}\right)} < C,
\]
and,
\[
\sup_{t>0} \frac{1}{t} \int_0^t E\left(X_s^{4-k} \eta_s^{k-4}\right) < C \text{ by Step 4b}.
\]

For the fourth term we apply the Holder’s inequality with \( p = k - 1 \) and \( q = (k-1)/(k-2) \) to get
\[
|X_s|^{3} \theta_s^{k-2} \leq (|X_s|^3)^{p}/p + (\theta_s^{k-2})^{q}/q = \frac{k-2}{k-1} |X_s|^{2(k-1)} + \frac{1}{k-1} \theta_s^{k-2},
\]
\[
\Rightarrow \sup_{t>0} \frac{1}{t} \int_0^t E\left(|X_s|^{3} \theta_s^{k-2}\right)ds \leq \frac{k-2}{k-1} \sup_{t>0} \frac{1}{t} \int_0^t E\left(|X_s|^{2(k-1)}\right)ds + \frac{1}{k-1} \sup_{t>0} \frac{1}{t} \int_0^t E\left(\theta_s^{k-2}\right)ds.
\]
Finiteness of the first term has been proved in Section 3.3. The second term is finite by virtue of Step 4b. The last step is to show that the fifth term in (3.22) is 0. Now,

\[
E\left( \int_0^t X_s \eta_s^{\frac{2-k}{2}} dW_s \right)^2 = E\left( \int_0^t X_s^2 \eta_s^{2-k} ds \right) = E\left( \int_0^t X_s^2 \theta_s^{k-2} ds \right) < C \text{ by Step 4b}
\]

Therefore \( \int_0^t X_s \eta_s^{\frac{2-k}{2}} dW_s \) is a square integrable martingale and hence

\[
E\left( \int_0^t X_s \eta_s^{\frac{2-k}{2}} dW_s \right) = 0
\]

for any \( t > 0 \).

Steps 4a, 4b and 4c together proves Step 4.

3.5 Identifying the limiting distribution

In this section we obtain the limiting moment of \( E(X_t^{2k}) \) and show that is equal to \( \frac{(2k)!}{2^k k!} \) for any \( k \in \mathcal{N} \). Hence by the uniqueness of the moment generating functions we can say that the limiting distribution of \( X_t \) is \( N(0, 1) \). We proceed by induction

1. \( \lim_{t \to \infty} E(X_t^2) = 1 \).

Applying Ito’s lemma to \( X_t^2 \) we have

\[
dx_t^2 = \left( -X_t^2 \theta_t^2 + \theta_t^2 \right) dt + 2X_t \theta_t dW_t.
\]

Writing \( F_k(t) = k \int_0^t \theta_s^2 ds \) and multiplying by the IF = \( e^{F_1(t)} \) on both sides we have

\[
d\left( X_t^2 e^{F_1(t)} \right) = e^{F_1(t)} \theta_t^2 dt + e^{F_1(t)} X_t \theta_t dW_t
\]

\[
\Rightarrow X_t^2 = e^{-F_1(t)} \left[ X_0^2 + \int_0^t d(e^{F_1(s)}) + \int_0^t e^{F_1(s)} X_s \theta_s dW_s \right]
\]

\[
= e^{-F_1(t)} \left[ X_0^2 + \int_0^t e^{F_1(s)} X_s \theta_s dW_s \right]
\]

\[
= e^{-F_1(t)} \left[ X_0^2 + e^{F_1(t)} - 1 + \int_0^t e^{F_1(s)} X_s \theta_s dW_s \right]
\]

\[
= X_0^2 e^{-F_1(t)} + 1 - e^{-F_1(t)} + \int_0^t e^{F_1(s)-F_1(t)} X_s \theta_s dW_s
\]

\[
\Rightarrow E(X_t^2) = E(e^{-F_1(t)}) E(X_0^2) + 1 - E\left( e^{-F_1(t)} \int_0^t e^{F_1(s)} X_s \theta_s dW_s \right)
\]
We have proved in Lemma 2 that the third expectation is zero. Therefore

\[ E(X_t^2) = E(e^{-F_1(t)})E(X_0^2) + 1 - \lim_{t \to \infty} E(e^{-F_1(t)}) \]

\[ \lim_{t \to \infty} E(X_t^2) = E(X_0^2) \lim_{t \to \infty} E(e^{-F_1(t)}) + 1 - \lim_{t \to \infty} E(e^{-F_1(t)}) \quad (3.31) \]

\[ \frac{F_1(t)}{t} = \frac{1}{t} \int_0^t \theta_s^2 ds \geq \frac{1}{t} \int_0^t \frac{1}{\theta_s^2} ds = \frac{1}{t} \int_0^t \eta_s^2 ds. \]

Therefore,

\[ e^{-F_1(t)} = \frac{1}{e^{F_1(t)}} \leq \frac{1}{F_1(t)} = \frac{1}{t F_1(t)} \quad \text{since } F_1(t) > 0 \]

\[ \leq \frac{1}{t} \int_0^t \eta_s^2 ds \]

\[ \Rightarrow E(e^{-F_1(t)}) \leq \frac{1}{t} E \left( \frac{1}{t} \int_0^t \eta_s^2 ds \right) \leq \frac{1}{t} C \leq \frac{1}{t} C. \]

where \( C = \sup_{t>0} E(\frac{1}{t} \int_0^t \eta_s^2 ds) < \infty \) from Step 2 of Section 3.4 So

\[ \lim_{t \to \infty} E(e^{-F_1(t)}) = 0 \Rightarrow \lim_{t \to \infty} E(X_t^2) = 1 \quad \text{from } (3.31) \]

2. Assume this to be true for \( k-1 \), i.e., \( \lim_{t \to \infty} E(X_t^{2m}) = \frac{(2m)!}{2^{m}m!} \) for \( 1 \leq m \leq k-1 \).

3. \( \lim_{t \to \infty} E(X_t^{2k}) = \frac{(2k)!}{2^{k}k!} \).

We have that

\[ dX_t^{2k} = \left( -kX_t^{2k-1} \theta_t^2 + (2k - 1)X_t^{2k-2} \theta_t^2 \right) dt + kX_t^{2k-1} \theta_t dW_t. \]

Multiplying with the \( IF = e^{F_k(t)} \) we have that

\[ d\left( X_t^{2k}e^{F_k(t)} \right) = k(2k - 1)e^{F_k(t)} X_t^{2k-2} \theta_t^2 dt + k e^{F_k(t)} X_t^{2k-1} \theta_t dW_t \]

\[ \Rightarrow X_t^{2k} = e^{-F_k(t)} \left[ X_0^{2k} + (2k - 1) \int_0^t k e^{F_k(t)} X_s^{2k-2} \theta_s^2 ds + k \int_0^t e^{F_k(s)} X_s^{2k-1} \theta_s dW_s \right] \]

\[ \Rightarrow E(X_t^{2k}) = E(e^{-F_k(t)})E(X_0^{2k}) + (2k - 1)E \left( \int_0^t k e^{-F_k(t)} e^{F_k(s)} X_s^{2k-2} \theta_s^2 ds \right) \]

\[ + E \left( e^{F_k(t)} \int_0^t k e^{F_k(s)} X_s^{2k-1} \theta_s dW_s \right) \]
We have proved in Lemma 4 that the third expectation is zero. Therefore

\[ \lim_{t \to \infty} E(X_t^{2k}) = E(X_0^{2k}) \lim_{t \to \infty} E(e^{-F_k(t)}) \]

+ \( (2k - 1) \lim_{t \to \infty} E \left( \int_0^t ke^{-F_k(t)} e^{F_k(s)} X_s^{2k-2} \theta^2 ds \right) \)

Arguing in the same way as the case for \( F_1(t) \):

\[ e^{-F_k(t)} = \frac{1}{e^{F_k(t)}} \leq \frac{1}{t^{F_k(t)/t}} = \frac{1}{t^{F_k(t)/t}} \]

\[ \leq \frac{1}{k} \frac{1}{t} \int_0^t \eta_s^2 ds \Rightarrow \lim_{t \to \infty} E(e^{-F_k(t)}) = 0. \]

Define, for \( 0 \leq m \leq k - 1 \),

\[ A_{k,2m}(t) = E \left( \int_0^t ke^{-F_k(t)} e^{F_k(s)} \theta_s^2 X_s^{2m} ds \right) \]

\[ = E \left( e^{-F_k(t)} \int_0^t X_s^{2m} d(e^{F_k(s)}) \right). \]

The term inside the expectation is

\[ \int_0^t X_s^{2m} d(e^{F_k(s)}) = X_t^{2m} e^{F_k(t)} - X_0^{2m} - \int_0^t e^{F_k(s)} d(X_s^{2m}) \]

\[ = X_t^{2m} e^{F_k(t)} - X_0^{2m} - \int_0^t e^{F_k(s)} \left( (-mX_s^{2m} \theta_s^2 + m(2m - 1)X_s^{2m-2} \theta_s^2) ds \right) \]

\[ + \int_0^t 2mX_s^{2m-1} \theta_s dW_s \]

\[ \Rightarrow e^{-F_k(t)} \int_0^t X_s^{2m} d(e^{F_k(s)}) = X_t^{2m} - X_0^{2m} e^{-F_k(t)} + e^{-F_k(t)} \int_0^t me^{F_k(s)} X_s^{2m} \theta_s^2 ds \]

\[ - e^{-F_k(t)} \int_0^t m(2m - 1)X_s^{2m-2} \theta_s^2 ds + e^{-F_k(t)} \int_0^t 2mX_s^{2m-1} \theta_s dW_s \]

(3.32)

And so,

\[ \Rightarrow A_{k,2m}(t) = E(X_t^{2m}) - E(e^{-F_k(t)}) E(X_0^{2m}) + \frac{m}{k} A_{k,2m} \]

\[ - \frac{m(2m - 1)}{k} A_{k,2m-2} + 0. \]

(3.33)
Defining \( B_{k,m} = \lim_{t \to \infty} A_{k,m}(t) \) (the limit exists since the integrand is a non-negative quantity). Applying limits on both sides of \( 3.33 \) we get

\[
B_{k,2m} = \lim_{t \to \infty} E(X_t^{2m}) - 0 + \frac{m}{k} B_{k,2m} - \frac{m(2m-1)}{k} B_{k,2m-2}
\]

\[
\Rightarrow (1 - \frac{m}{k}) B_{k,2m} = \lim_{t \to \infty} E(X_t^{2m}) - \frac{m(2m-1)}{k} B_{k,2m-2}
\]

\[
\Rightarrow B_{k,2m} = \frac{k}{k - m} \lim_{t \to \infty} E(X_t^{2m}) - \frac{m(2m-1)}{k - m} B_{k,2m-2}
\]

Substituting different values of \( m = 0, 1, 2, \ldots, k - 1 \) we get

\[
B_{k,0} = 1
\]

\[
B_{k,2} = \frac{k}{k - 1} - \frac{1}{k - 1} = 1
\]

\[
B_{k,4} = \frac{k}{k - 2} - \frac{2.3}{k - 2} = 3
\]

\[
B_{k,6} = \frac{k}{k - 3} - \frac{3.5}{k - 3} = 5.3
\]

\[
B_{k,8} = \frac{k}{k - 4} - \frac{4.7}{k - 4} = 5.3
\]

\[
B_{k,2k-2} = k(2k-3)(2k-5) \ldots 3.1 - (k - 1)(2k - 1) B_{k,2k-4}
\]

\[
= k(2k-3)(2k-5) \ldots 3.1 - (k - 1)(2k - 3) \ldots 3.1 = (2k-3)(2k-5) \ldots 3.1
\]

\[
= \frac{(2k-2)!}{2^{k-1}(k-1)!}
\]

From equation \( 3.32 \):

\[
\lim_{t \to \infty} E(X_t^{2k}) = (2k - 1) B_{k,2k-2}
\]

\[
= (2k - 1) \frac{(2k - 2)!}{2^{k-1}(k-1)!} = \frac{2k(2k - 1)!}{2^k k!}
\]

\[
= \frac{(2k)!}{2^k k!}
\]

To find the odd moments of \( X_t \) we perform the same procedure as above. We have

\[
dX_t = -X_t \frac{\theta^2}{2} dt + \theta dW_t
\]
Define $G_1(t) = \frac{1}{2} \int_0^t \theta_s^2 ds$. Multiply by the IF=$e^{G_1(t)}$ on both sides we have

\[
d(e^{G_1(t)} X_t) = e^{F_1(t)} \theta_t dW_t
\]

\[
\Rightarrow X_t = X_0 e^{-G_1(t)} + e^{-G_1(t)} \int_0^t e^{F_1(s)} \theta_s dW_s
\]

\[
\Rightarrow E(X_t) = E(X_0) E(e^{-G_1(t)}) + E(e^{-G_1(t)} \int_0^t e^{F_1(s)} \theta_s dW_s)
\]

Applying the same argument as above we can show that

\[
\lim_{t \to \infty} E(e^{-G_1(t)}) = 0
\]

and so

\[
\lim_{t \to \infty} E(X_t) = 0.
\]

Now we apply Mathematical Induction. Let us assume that

\[
\lim_{t \to \infty} E(X_t^{2m-1}) = 0
\]

where $m = 1, 2, \ldots, k$ is any positive integer. Applying Ito’s lemma to $X_t^{2k+1}$ we get

\[
d(X_t^{2k+1}) = (2k + 1) X_t^{2k} dX_t + \frac{1}{2} (2k + 1) 2k X_t^{2k-1} \theta_t^2 dt
\]

\[
= (2k + 1) X_t^{2k} \left(-X_t \frac{\theta_t^2}{2} dt + \theta_t dW_t\right) + (2k + 1) k \theta_t^2 X_t^{2k-1} dt
\]

\[
= \left(-\frac{1}{2} (2k + 1) X_t^{2k+1} \theta_t^2 + (2k + 1) k X_t^{2k-1} \theta_t\right) dt + (2k + 1) \theta_t X_t^{2k} dW_t.
\]

Define $G_k(t) = \frac{2k+1}{2} \int_0^t \theta_s^2 ds$ and the integrating factor as $e^{G_k(t)}$. Multiplying by it on both sides of the above equation we get:

\[
d\left(X_t^{2k+1} e^{G_k(t)}\right) = k(2k + 1) e^{G_k(t)} \theta_t^2 X_t^{2k-1} dt + (2k + 1) e^{G_k(t)} \theta_t X_t^{2k} dW_t
\]

\[
\Rightarrow X_t^{2k+1} = e^{-G_k(t)} \left[X_0^{2k+1} + k(2k + 1) \int_0^t e^{G_k(s)} \theta_s^2 X_s^{2k-1} ds + (2k + 1) \int_0^t e^{G_k(s)} \theta_s X_s^{2k} dW_s\right]
\]

\[
E(X_t^{2k+1}) = E(e^{-G_k(t)}) E(X_0^{2k+1}) + E\left(k(2k + 1) e^{-G_k(t)} \int_0^t e^{G_k(s)} \theta_s^2 X_s^{2k-1} ds\right)
\]

\[
+ (2k + 1) E\left(e^{-G_k(t)} \int_0^t e^{G_k(s)} X_s^{2k} \theta_s dW_s\right)
\]

We now try to prove that the third expectation is zero.
Following the similar notation as before let \( \overline{F}_k(t) = e^{-G_k(t)} \). Also define \( Z_k(t) = \overline{F}_k(t)Y_k(t) \) where \( Y_k(t) = \int_0^t e^{G_k(s)}X_s^2\theta_s dW_s \). Applying Ito’s lemma to \( Z_k(t) \) we have

\[
\begin{align*}
\text{d}Z_k(t) &= Y_k(t)\text{d}\overline{F}_k(t) + \overline{F}_k(t)\text{d}Y_k(t) \\
&= -\frac{2k+1}{2}Y_k(t)\overline{F}_k(t)\theta_t^2 dt + \overline{F}_k(t)F_k(t)X_t^{2k}\theta_t dW_t \\
&= -\frac{2k+1}{2}Z_k(t)\theta_t^2 dt + X_t^{2k}\theta_t dW_t
\end{align*}
\]

Now, taking \( \tilde{Z}_k(t) = -Z_k(t) \), yields

\[
\begin{align*}
\text{d}\tilde{Z}_k(t) &= \frac{2k+1}{2}\tilde{Z}_k(t)\theta_t^2 dt + X_t^{2k}dW_t \\
&= \frac{2k+1}{2}\tilde{Z}_k(t)\theta_t^2 dt - X_t^{2k}d\tilde{W}_t
\end{align*}
\]

where \( \tilde{W}_t = -W_t \). Since \( \tilde{Z}_{0,k} = -Z_{0,k} = 0 = Z_{0,k} \). Hence \( Z_k(t) \) and \( \tilde{Z}_k(t) \) has the same distribution, i.e., the distribution is symmetric around 0. Therefore to show that \( E(Z_{t,k}) = 0, \forall t \geq 0 \) one needs to show \( Z_{t,k} \) has finite expectation \( \forall t \geq 0 \). Thus, as before, it is sufficient to show that \( E(Z_k(t)^2) < \infty, \forall t \geq 0 \). Now,

\[
Z_k(t)^2 = \overline{F}_k(t)^2Y_k(t)^2 = \overline{F}_k(t)^2\left(\int_0^t e^{G_k(s)}X_s\theta_s dW_s\right)^2
\]

\[
= e^{-2G_k(t)}\left(\int_0^t e^{G_k(s)}X_s\theta_s dW_s\right)^2
\]

\[
\Rightarrow E(Z_k(t)^2) \leq E\left(\int_0^t e^{2G_k(s)}X_s^4\theta_s^2 ds\right)
\]

\[
\leq \sqrt{E\left(\int_0^t e^{4G_k(s)}\theta_s^2 ds\right)E\left(\int_0^t X_s^8\theta_s^2 ds\right)}
\]

Now, as in above

\[
\left(\int_0^t e^{4G_k(s)}\theta_s^2 ds\right) = \frac{1}{2(2k+1)}(e^{2(2k+1)\int_0^t F(s)} - 1)
\]

where \( F(t) = \int_0^t \theta_s^2 ds \)

\[
< C \Rightarrow E\left(\int_0^t e^{4G_k(s)}\theta_s^2 ds\right) < \infty
\]

Now, one can write

\[
X_t^{8k}\theta_t^2 \leq C_t\left(X_0^{8k} + \left(\int_0^t \frac{X_s^2}{2} ds\right)^{8k} + \left(\int_0^t \theta_s dW_s\right)^{8k}\right)\theta_0^2 e^{2pt}
\]
The second and the third term can be shown to be finite (the details are the same as in 3.3 expect that the exponent of $X_t$ is $8k$, so it is not repeated here). Therefore

$$E\left(\int_0^t X_s^{8k}\theta_s^2\,ds\right) < \infty \Rightarrow E(Z_k(t))^2 < C$$

This proves that $E(Z_k(t)) = 0$. Therefore from 3.35

$$\lim_{t \to \infty} E(X_t^{2k+1}) = \lim_{t \to \infty} E(e^{-G_k(t)}) E(X_0^{2k+1}) + \lim_{t \to \infty} E\left(k(2k+1)e^{-G_k(t)} \int_0^t e^{G_k(s)} \theta_s^2 X_s^{2k-1} d\theta_s\right)$$

Writing

$$B_{k,2m-1}(t) = E\left(2k(2k+1)/2e^{-G_k(t)} \int_0^t e^{G_k(s)} \theta_s^2 X_s^{2m-1} d\theta_s\right) = E\left(2ke^{-G_k(t)} \int_0^t X_s^{2m-1} d(e^{G_k(s)})\right).$$

we have

$$\lim_{t \to \infty} E(X_t^{2k+1}) = E(X_0^{2k+1}) \lim_{t \to \infty} E(e^{-G_k(t)}) + \lim_{t \to \infty} B_{k,2k-1}(t) = \lim_{t \to \infty} B_{k,2k-1}(t)$$

(3.35)

Now $\int_0^t X_s^{2m-1} d(e^{G_k(s)})$

$$= X_t^{2m-1} e^{G_k(t)} - X_0^{2m-1} - \int_0^t e^{G_k(s)} d(X_s^{2m-1})$$

$$= X_t^{2m-1} e^{G_k(t)} - X_0^{2m-1} - \int_0^t e^{G_k(s)} \left[(2m - 1)X_s^{2m-2} dX_s + \frac{1}{2} (2m-1)(2m-2)X_s^{2m-3}\theta_s^2 \, ds \right]$$

$$= X_t^{2m-1} e^{G_k(t)} - X_0^{2m-1} - \int_0^t e^{G_k(s)} \left[(2m - 1)X_s^{2m-2} \left(-X_s \frac{\theta_s^2}{2} ds + \theta_s dW_s\right)\right]$$

$$= X_t^{2m-1} e^{G_k(t)} - X_0^{2m-1} + \left(\int_0^t (2m - 1)e^{G_k(s)} X_s^{2m-2} \theta_s^2 \right) - \frac{1}{2} (2m-1)(2m-2) \int_0^t e^{G_k(s)} X_s^{2m-3}\theta_s^2 \, ds - \int_0^t (2m - 1)e^{G_k(s)} X_s^{2m-2}\theta_s dW_s.$$
Therefore \( B_{k,2m-1}(t) \)

\[
\begin{align*}
&= E(2kX_t^{2m-1}) - E(X_0^{2m-1})E(e^{-G_k(t)}) + \frac{2m-1}{2k+1} 2ke^{-G_k(t)} \int_0^t X_s^{2m-1} e^{G_k(s)} \theta_s ds \\
&\quad - \frac{(2m-1)(2m-2)}{2k+1} 2ke^{-G_k(t)} \int_0^t X_s^{2m-3} e^{G_k(s)} \theta_s ds \\
&= E(2kX_t^{2m-1}) - E(X_0^{2m-1})E(e^{-G_k(t)}) + \frac{2m-1}{2k+1} 2ke^{-G_k(t)} \int_0^t X_s^{2m-1} d(e^{G_k(s)}) \\
&\quad - \frac{(2m-1)(2m-2)}{2k+1} 2ke^{-G_k(t)} \int_0^t X_s^{2m-3} d(e^{G_k(s)}) \\
&= E(2kX_t^{2m-1}) - E(X_0^{2m-1})E(e^{-G_k(t)}) + \frac{2m-1}{2k+1} B_{k,2m-1}(t) \\
&\quad + \frac{(2m-1)(2m-2)}{2k+1} B_{k,2m-3}(t)
\end{align*}
\]

Writing \( B_{k,j} \) as the limit of \( B_{k,j}(t) \) we have

\[
B_{k,2m-1} = 0 - 0 + \frac{2m-1}{2k+1} B_{k,2m-1} - \frac{(2m-1)(2m-2)}{2k+1} B_{k,2m-3}
\]

\[
\Rightarrow (2k + 2 - 2m)B_{k,2m-1} = -(2m-1)(2m-2)B_{k,2m-3}. \tag{3.36}
\]

Now

\[
B_{k,1}(t) = E(2ke^{-G_k(t)} \int_0^t X_s d(e^{G_k(s)}) ds)
\]

\[
= E\left(2ke^{-G_k(t)} \left[ X_te^{G_k(t)} - X_0 - \int_0^t e^{G_k(s)} dX_s \right] \right)
\]

\[
= E(2kX_t) - 2kE(X_0e^{-G_k(t)}) - E\left(2ke^{-G_k(t)} \int_0^t e^{G_k(s)} \left[-X_s \theta_s \frac{\theta_s}{2} ds + \theta_s dW_s \right] \right)
\]

\[
= E(2kX_t) - 2kE(X_0e^{-G_k(t)}) + \frac{1}{2k+1} E\left(2ke^{-G_k(t)} \int_0^t X_s d(e^{G_k(s)}) \right)
\]

\[
\Rightarrow B_{k,1} = 0 - 0 + \frac{1}{2k+1} B_{k,1}
\]

\[
\Rightarrow B_{k,1} = 0
\]

This and Eqn(3.36) would imply that

\[
B_{k,2m-1} = 0 \quad \text{for} \quad m = 1, 2, \ldots, k
\]

Therefore from 3.35 we have

\[
\lim_{t \to \infty} E(X_t^{2k+1}) = 0
\]

Therefore combining the two results we have \( E(X_r^r) = \begin{cases} \frac{(2k)!}{2^k k!} & \text{when } r = 2k \\ 0 & \text{when } r = 2k + 1 \end{cases} \)
4 Conclusion

Verifying Roberts et al’s condition (see [9]) for checking the ergocity of an AMCMC can sometimes prove difficult. In the companion paper (see [1]), we considered an AMCMC with the proposal kernel dependent on the previously generated sample and an arbitrary target distribution. There we performed a diffusion approximation technique to look at the continuous time version of the discrete chain. In this paper we narrowed down to the case where the target distribution is standard Normal. We investigate whether the invariant distribution of the diffusion is indeed the target distribution. It turns that the resulting diffusion (which although singular) admits an invariant distribution. Then identifying the limiting moments of $2k$ order of $X_t$ we identify the limiting distribution to be $N(0,1)$.

The techniques applied here are specific only when the target distribution is Normal. We hope that this can also be extended to other target distributions, where a identification of the limiting moments is possible. Also more choices of the proposal distribution can be made, where the kernel is dependent on a finite (or possibly infinite) past. We plan to take up these issues in our future work.

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