Inverse Perron values and connectivity of a uniform hypergraph

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Abstract

In this paper, we show that a uniform hypergraph $G$ is connected if and only if one of its inverse Perron values is larger than 0. We give some bounds on the bipartition width, isoperimetric number and eccentricities of $G$ in terms of inverse Perron values. By using the inverse Perron values, we give an estimation of the edge connectivity of a 2-design, and determine the explicit edge connectivity of a symmetric design. Moreover, relations between the inverse Perron values and resistance distance of a connected graph are presented.

Keywords: Hypergraph, Inverse Perron value, Laplacian tensor, Connectivity

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1. Introduction

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a hypergraph $G$, respectively. $G$ is $k$-uniform if $|e| = k$ for each $e \in E(G)$. In particular, 2-uniform hypergraphs are usual graphs. For $i \in V(G)$, $E_i(G)$ denotes the set of edges containing $i$, and $d_i = |E_i(G)|$ denotes the degree of $i$. The adjacency tensor $[8]$ of a $k$-uniform hypergraph $G$, denoted by $A_G$, is an order $k$ dimension $|V(G)|$ tensor with entries

$$a_{i_1i_2\cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The Laplacian tensor $[27]$ of $G$ is $L_G = D_G - A_G$, where $D_G$ is the diagonal tensor of vertex degrees of $G$. Recently, the research on spectral hypergraph theory via

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tensors has attracted much attention [7-10,14,19,24]. The spectral properties of the Laplacian tensor of hypergraphs are studied in [13,25,27,29,35].

The algebraic connectivity of a graph plays important roles in spectral graph theory [11]. Analogue to the algebraic connectivity of a graph, Qi [27] defined the analytic connectivity of a $k$-uniform hypergraph $G$ as

$$\alpha(G) = \min_{j=1,\ldots,n} \min \left\{ \mathcal{L}_G x^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x^k_i = 1, x_j = 0 \right\},$$

where $n = |V(G)|$, $\mathbb{R}^n_+$ denotes the set of nonnegative vectors of dimension $n$. Qi proved that $G$ is connected if and only if $\alpha(G) > 0$. In [20], some bounds on $\alpha(G)$ were presented in terms of degree, vertex connectivity, diameter and isoperimetric number. A feasible trust region algorithm of $\alpha(G)$ was given in [9].

For any vertex $j$ of uniform hypergraph $G$, we define the inverse Perron value of $j$ as

$$\alpha_j(G) = \min \left\{ \mathcal{L}_G x^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x^k_i = 1, x_j = 0 \right\}.$$

Clearly, the analytic connectivity $\alpha(G) = \min_{j \in V(G)} \alpha_j(G)$ is the minimum inverse Perron value. For a connected graph $G$, $\alpha_j(G)$ is the minimum eigenvalue of $\mathcal{L}_G(j)$, where $\mathcal{L}_G(j)$ is the principal submatrix of $\mathcal{L}_G$ obtained by deleting the row and column corresponding to $j$. $\mathcal{L}_G(j)$ is a nonsingular $M$-matrix, and its inverse $\mathcal{L}_G(j)^{-1}$ is a nonnegative matrix [16]. It is easy to see that $\alpha_j^{-1}(G)$ is the spectral radius of $\mathcal{L}_G(j)^{-1}$, which is called the Perron value of $G$. The Perron values have close relations with the Fielder vector of a tree [1, 15].

The resistance distance [17, 34] is a distance function on graphs. For two vertices $i,j$ in a connected graph $G$, the resistance distance between $i$ and $j$, denoted by $r_{ij}(G)$, is defined to be the effective resistance between them when unit resistors are placed on every edge of $G$. The Kirchhoff index [17, 33] of $G$, denoted by $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices in $G$, i.e., $Kf(G) = \sum_{\{i,j\} \subseteq V(G)} r_{ij}(G)$. $Kf(G)$ is a global robustness index. The resistance distance and Kirchhoff index in graphs have been investigated extensively in mathematical and chemical literatures [3-6,12,26,35,40].

This paper is organized as follows. In Section 2, some auxiliary lemmas are introduced. In Section 3, we show that a uniform hypergraph $G$ is connected if and only if one of its inverse Perron values is larger than 0, and some inequalities among
the inverse Perron values, bipartition width, isoperimetric number and eccentricities
of \( G \) are established. Partial results improve some bounds in \([20, 27]\). We also use the
inverse Perron values to estimate the edge connectivity of 2-designs. In Section 4,
some inequalities among the inverse Perron values, resistance distance and Kirchhoff
index of a connected graph are presented.

2. Preliminaries

For a positive integer \( n \), let \([n] = \{1, 2, \ldots, n\}\). An order \( m \) dimension \( n \) tensor
\( T = (t_{i_1 \ldots i_m}) \) consists of \( n^m \) entries, where \( i_j \in [n], j \in [m] \). When \( m = 2 \), \( T \) is an
\( n \times n \) matrix. Let \( \mathbb{R}^{[m,n]} \) denote the set of order \( m \) dimension \( n \) real tensors, and
let \( \mathbb{R}_+^n \) denote the \( n \)-dimensional nonnegative vector space. For \( T = (t_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]} \) and \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \), let \( T x^{m-1} \in \mathbb{R}^n \) denote the vector whose \( i \)-th component is
\[
(T x^{m-1})_i = \sum_{i_2, i_3, \ldots, i_m=1}^{n} t_{i_2 \ldots i_m} x_{i_2} x_{i_3} \cdots x_{i_m},
\]
and let \( T x^m \) denote the following polynomial
\[
T x^m = \sum_{i_1, \ldots, i_m=1}^{n} t_{i_1 i_2 \ldots i_m} x_{i_1} \cdots x_{i_m}.
\]

In 2005, Qi \([26]\) and Lim \([21]\) proposed the concept of eigenvalues of tensors, in-
dependently. For \( T = (t_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]} \), if there exists a number \( \lambda \in \mathbb{R} \) and a
nonzero vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) such that \( T x^{m-1} = \lambda x^{m-1} \), then \( \lambda \) is called
an \( H \)-eigenvalue of \( T \), \( x \) is called an \( H \)-eigenvector of \( T \) corresponding to \( \lambda \), where
\( x^{m-1} = (x_1^{m-1}, \ldots, x_n^{m-1})^T \).

For a vertex \( j \) of a \( k \)-uniform hypergraph \( G \), let \( L_G(j) \in \mathbb{R}^{[k,n-1]} \) denote the
principal subtensor of \( L_G \in \mathbb{R}^{[k,n]} \) with index set \( V(G) \setminus \{j\} \). By Lemma 2.3 in \([32]\),
we have the following lemma.

**Lemma 2.1.** Let \( G \) be a \( k \)-uniform hypergraph. For any \( j \in V(G) \), \( \alpha_j(G) \) is the
smallest \( H \)-eigenvalue of \( L_G(j) \).

A path \( P \) of a uniform hypergraph \( G \) is an alternating sequence of vertices and
edges \( v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l \), where \( v_0, \ldots, v_l \) are distinct vertices of \( G \), \( e_1, \ldots, e_l \) are
distinct edges of \( G \) and \( v_{i-1}, v_i \in e_i \), for \( i = 1, \ldots, l \). The number of edges in \( P \) is the
length of \( P \). For all \( u, v \in V(G) \), if there exists a path starting at \( u \) and terminating
at \( v \), then \( G \) is said to be connected \([2]\).
Lemma 2.2. [27] The uniform hypergraph $G$ is connected if and only if $\alpha(G) > 0$.

Let $G$ be a $k$-uniform hypergraph, $S$ be a proper nonempty subset of $V(G)$. Denote $\overline{S} = V(G) \setminus S$. The edge-cut set $E(S, \overline{S})$ consists of edges whose vertices are in both $S$ and $\overline{S}$. The minimum cardinality of such an edge-cut set is called edge connectivity of $G$, denote by $e(G)$.

Lemma 2.3. [27] Let $G$ be a $k$-uniform hypergraph with $n$ vertices. Then

$$e(G) \geq \frac{n}{k} \alpha(G).$$

The $\{1\}$-inverse of a matrix $M$ is a matrix $X$ such that $MXM = M$. Let $M^{(1)}$ denote any $\{1\}$-inverse of $M$, and let $(M)_{ij}$ denote the $(i, j)$-entry of $M$.

Lemma 2.4. [3, 34] Let $G$ be a connected graph. Then

$$r_{ij}(G) = (L_G^{(1)})_{ii} + (L_G^{(1)})_{jj} - (L_G^{(1)})_{ij} - (L_G^{(1)})_{ji}.$$}

Let $\text{tr}(A)$ denote the trace of the square matrix $A$, and let $e$ denote an all-ones column vector.

Lemma 2.5. [30] Let $G$ be a connected graph of order $n$. Then

$$Kf(G) = n\text{tr}(L_G^{(1)}) - e^\top L_G^{(1)} e.$$}

Lemma 2.6. [36] Let $G$ be a connected graph of order $n$. Then $$(L_G(i)^{-1} - 0 \quad 0 \quad 0) \in \mathbb{R}^{n \times n}$$

is a symmetric $\{1\}$-inverse of $L_G$, where $i$ is the vertex corresponding to the last row of $L_G$.

3. Inverse Perron values of uniform hypergraphs

In the following theorem, the relationship between inverse Perron values and connectivity of a hypergraph is presented.

Theorem 3.1. Let $G$ be a $k$-uniform hypergraph. Then the following are equivalent:

1. $G$ is connected.
2. $\alpha_j(G) > 0$ for all $j \in V(G)$.
3. $\alpha_j(G) > 0$ for some $j \in V(G)$.
Proof. \((1)\implies(2)\). If \(G\) is connected, then by Lemma 2.2, we know that \(\alpha_j(G) > 0\) for all \(j \in V(G)\).

\((2)\implies(3)\). Obviously.

\((3)\implies(1)\). Suppose that \(G\) is disconnected. For any \(j \in V(G)\), let \(G_j\) be the component of \(G\) such that \(j \not\in V(G_j)\). Let \(x = (x_1, \ldots, x_{|V(G)|})^T\) be the vector satisfying

\[
x_i = \begin{cases} |V(G_1)|^{-\frac{1}{k}}, & \text{if } i \in V(G_1), \\ 0, & \text{otherwise.} \end{cases}
\]

Clearly, we have \(\sum_{i=1}^n x_i^k = 1\). Then we have \(0 \leq \alpha_j(G) \leq L_G x^k = 0\) for any \(j \in V(G)\), a contradiction to \((3)\). Hence \(G\) is connected if \((3)\) holds. \(\square\)

The bipartition width of a hypergraph \(G\) is defined as \([18, 28]\)

\[bw(G) = \min \left\{ |E(S, \overline{S})| : S \subseteq V(G), |S| = \left\lfloor \frac{n}{2} \right\rfloor \right\},\]

where \(\left\lfloor \frac{n}{2} \right\rfloor\) denotes the maximum integer not larger than \(\frac{n}{2}\). The computation of \(bw(G)\) is very difficult even for the graph case. In \([22]\), Mohar and Poljak use the algebraic connectivity to obtain a lower bound on the bipartition width of a graph. In the following, we use the inverse Perron values to obtain a lower bound on the bipartition width of a uniform hypergraph.

**Theorem 3.2.** Let \(G\) be a \(k\)-uniform hypergraph with \(n\) vertices. Then

\[bw(G) \geq \frac{n + (-1)^n}{k(n+1)} \sum_{j=1}^n \alpha_j(G).\]

*Proof.* Suppose that \(S_0 \subseteq V(G)\) satisfying \(|S_0| = \left\lfloor \frac{n}{2} \right\rfloor\) and \(|E(S_0, \overline{S_0})| = bw(G)\). Let \(x = (x_1, \ldots, x_n)^T\) be the vector satisfying

\[
x_i = \begin{cases} |S_0|^{-\frac{1}{k}}, & i \in S_0, \\ 0, & i \in \overline{S_0}. \end{cases}
\]

Then \(\sum_{i=1}^n x_i^k = 1\). For \(j \in \overline{S_0}\), we can obtain

\[
\alpha_j(G) \leq L_G x^k = \sum_{\{i_1, \ldots, i_k\} \in E(G)} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k})
\]

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\[ \alpha_j(G) \leq \sum_{\{i_1, \ldots, i_k\} \in E(S_0, \overline{S_0})} (x_{i_1}^k + \cdots + x_{i_k}^k - kx_{i_1} \cdots x_{i_k}) \]
\[ = \frac{1}{|S_0|} \sum_{e \in E(S_0, \overline{S_0})} |e \cap S_0| = \frac{t(S_0)bw(G)}{|S_0|}, \quad (3.1) \]

where \( t(S_0) = \frac{1}{|E(S_0, \overline{S_0})|} \sum_{e \in E(S_0, \overline{S_0})} |e \cap S_0| \).

Similarly, for \( j \in S_0 \), we can obtain
\[ \alpha_j(G) \leq \frac{(k - t(S_0))bw(G)}{|S_0|}. \quad (3.2) \]

Combining (3.1) and (3.2), we can get
\[ \sum_{j=1}^{n} \alpha_j(G) = \sum_{j \in S_0} \alpha_j(G) + \sum_{j \in \overline{S_0}} \alpha_j(G) \leq \frac{|S_0|(k - t(S_0))bw(G)}{|S_0|} + \frac{|S_0|t(S_0)bw(G)}{|S_0|}. \]

If \( n \) is even, then \( |S_0| = |\overline{S_0}| \) and \( bw(G) \geq \frac{1}{k} \sum_{j=1}^{n} \alpha_j(G) \). If \( n \) is odd, then \( |S_0| = |\overline{S_0}| - 1 = \frac{n-1}{2} \) and
\[ \sum_{j=1}^{n} \alpha_j(G) \leq k \frac{|S_0|}{|S_0|} bw(G) = \frac{k(n + 1)bw(G)}{n - 1}, \quad bw(G) \geq \frac{n - 1}{k(n + 1)} \sum_{j=1}^{n} \alpha_j(G). \]

The isoperimetric number of a \( k \)-uniform hypergraph \( G \) is defined as
\[ i(G) = \min \left\{ \frac{|E(S, \overline{S})|}{|S|} : S \subseteq V(G), 0 \leq S \leq \frac{|V(G)|}{2} \right\}. \]

Let \( \beta(G) = \max_{j \in V(G)} \alpha_j(G) \) denote the maximum inverse Perron value of \( G \). In [20], it is shown that \( i(G) \geq \frac{2}{k} \alpha(G) \). We improve it as follows.

**Theorem 3.3.** Let \( G \) be a \( k \)-uniform hypergraph. Then
\[ i(G) \geq \frac{\alpha(G) + \beta(G)}{k}. \]

**Proof.** Suppose that \( S_1 \subseteq V(G) \) satisfying \( 0 \leq S_1 \leq \frac{|V(G)|}{2} \) and \( \frac{|E(S_1, \overline{S_1})|}{|S_1|} = i(G) \). Let
The vector satisfying
\[ x_i = \begin{cases} |S_1|^{-\frac{1}{k}} & i \in S_1, \\ 0 & i \in \overline{S}_1. \end{cases} \]

Then \( \sum_{i=1}^{n} x_i^k = 1 \). For \( j \in \overline{S}_1 \), we can obtain
\[ \alpha_j(\mathcal{G}) \leq \mathcal{L}_G x^k = \frac{t(S_1)|E(S_1, \overline{S}_1)|}{|S_1|} = t(S_1)i(\mathcal{G}), \quad (3.3) \]
where \( t(S_1) = \frac{1}{|E(S_1, \overline{S}_1)|} \sum_{e \in E(S_1, \overline{S}_1)} |e \cap S_1| \).

Similarly, for \( j \in S_1 \), we can get
\[ \alpha_j(\mathcal{G}) \leq \frac{(k - t(S_1))|E(S_1, \overline{S}_1)|}{|S_1|} \leq (k - t(S_1))i(\mathcal{G}). \quad (3.4) \]
From (3.3) and (3.4), we have
\[ \alpha(\mathcal{G}) + \beta(\mathcal{G}) \leq t(S_1)i(\mathcal{G}) + (k - t(S_1))i(\mathcal{G}) = ki(\mathcal{G}), \quad i(\mathcal{G}) = \frac{\alpha(\mathcal{G}) + \beta(\mathcal{G})}{k}. \]

The distance \( d(u, v) \) between two distinct vertices \( u \) and \( v \) of \( \mathcal{G} \) is the length of the shortest path connecting them. The eccentricity of a vertex \( v \) is \( \text{ecc}(v) = \max\{d(u, v) : u \in V(\mathcal{G})\} \). The diameter and radius of \( \mathcal{G} \) are defined as \( \text{diam}(\mathcal{G}) = \max_{v \in V(\mathcal{G})} \text{ecc}(v) \) and \( \text{rad}(\mathcal{G}) = \min_{v \in V(\mathcal{G})} \text{ecc}(v) \), respectively.

**Theorem 3.4.** Let \( \mathcal{G} \) be a connected \( k \)-uniform hypergraph with \( n \) vertices. Then
\[ \text{ecc}(j) \geq \frac{k}{2(k-1)(n-1)\alpha_j(\mathcal{G})}, \quad j \in V(\mathcal{G}). \]

**Proof.** For \( j \in V(\mathcal{G}) \), let \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}_+^n \) satisfying \( x_j = 0, \sum_{i=1}^{n} x_i^k = 1 \) and \( \alpha_j(\mathcal{G}) = \mathcal{L}_G x^k \). Then
\[ \alpha_j(\mathcal{G}) = \mathcal{L}_G x^k = \sum_{\{i_1, \ldots, i_k\} \in E(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k}). \quad (3.5) \]
By Cauchy-Schwarz inequality, we obtain
\[
\sum_{1 \leq s < t \leq k} \frac{k}{x_s^k} \frac{k}{x_t^k} \geq \frac{k(k-1)}{2} \left( \prod_{1 \leq s < t \leq k} \frac{k}{x_s^k} \frac{k}{x_t^k} \right)^{2/\left(k^2 - 1\right)} = \frac{k(k-1)}{2} x_{i_1} \cdots x_{i_k}. \tag{3.6}
\]

By (3.5) and (3.6), we have
\[
\alpha_j(G) \geq \sum_{\{i_1, \ldots, i_k\} \in E(G)} \left( x_{i_1}^k + \cdots + x_{i_k}^k - \frac{2}{k-1} \sum_{1 \leq s < t \leq k} \frac{k}{x_s^k} \frac{k}{x_t^k} \right)
\]
\[
= \frac{1}{k-1} \sum_{\{i_1, \ldots, i_k\} \in E(G)} \sum_{1 \leq s < t \leq k} \left( x_s^k - x_t^k \right)^2 = \frac{1}{k-1} \sum_{s,t \in E(G)} \left( x_s^k - x_t^k \right)^2. \tag{3.7}
\]

Let \( P = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l \) be the shortest path from vertex \( v_0 \) to vertex \( v_l \), where \( x_{i_0} = \max_{i \in V(G)} \{x_i\} \), \( v_l = j \), \( x_{v_l} = 0 \). Then
\[
\sum_{s,t \in E(P)} \left( x_s^k - x_t^k \right)^2 \geq \sum_{s,t \in E(P)} \left( x_s^k - x_t^k \right)^2 \]
\[
= \sum_{i=1}^{l} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2 + \sum_{u_j \in e \setminus \{v_{i-1}, v_i\}} \left( \frac{k}{x_{u_{i-1}}^k} - \frac{k}{x_{u_i}^k} \right)^2 + \frac{1}{2} \sum_{u_j \in e \setminus \{v_{i-1}, v_i\}} \left( \frac{k}{x_{u_{i-1}}^k} - \frac{k}{x_{u_i}^k} \right)^2 \]
\[
= \sum_{i=1}^{l} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2 + \frac{k-2}{2} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2 \]
\[
= \frac{k}{2} \sum_{i=1}^{l} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2. \]

By Cauchy-Schwarz inequality, we obtain
\[
\sum_{s,t \in E(G)} \left( x_s^k - x_t^k \right)^2 \geq \frac{k}{2} \sum_{i=1}^{l} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2 \geq \frac{k}{2l} \left( \sum_{i=1}^{l} \left( \frac{k}{x_{v_{i-1}}^k} - \frac{k}{x_{v_i}^k} \right)^2 \right) \]
\[
= \frac{k}{2l} \left( v_{v_0} - x_{v_1} \right)^2 \geq \frac{k}{2 \text{ecc}(j)} \left( v_{v_0} - x_{v_1} \right)^2 \geq \frac{k}{2(n-1) \text{ecc}(j)}. \tag{3.8}
\]
From (3.7) and (3.8), it yields that
\[
\alpha_j(\mathcal{G}) \geq \frac{k}{2(k-1)(n-1)\text{ecc}(j)}, \quad \text{ecc}(j) \geq \frac{k}{2(k-1)(n-1)\alpha_j(\mathcal{G})}.
\]

For a connected \(k\)-uniform hypergraph \(\mathcal{G}\) with \(n\) vertices, [20] showed that
\[
\text{diam}(\mathcal{G}) \geq \frac{4}{n^2(k-1)\alpha(\mathcal{G})}.
\]

By Theorem 3.4, we obtain the following improved result.

**Corollary 3.5.** Let \(\mathcal{G}\) be a connected \(k\)-uniform hypergraph with \(n\) vertices. Then
\[
\text{diam}(\mathcal{G}) \geq \frac{k}{2(k-1)(n-1)\alpha(\mathcal{G})}, \quad \text{rad}(\mathcal{G}) \geq \frac{k}{2(k-1)(n-1)\beta(\mathcal{G})}.
\]

In [27], it is shown that \(\alpha(\mathcal{G}) \leq \delta\), where \(\delta\) is the minimum degree of \(\mathcal{G}\). We improve it as follows.

**Theorem 3.6.** Let \(\mathcal{G}\) be a \(k\)-uniform hypergraph with \(n\) vertices. Then
\[
\alpha_j(\mathcal{G}) \leq \frac{(k-1)d_j}{n-1}, \quad j \in V(\mathcal{G}).
\]

**Proof.** For \(j \in V(\mathcal{G})\), let \(x = (x_1, \ldots, x_n)^T\) be the vector satisfying
\[
x_i = \begin{cases} \frac{(n-1)^{-\frac{1}{k}}}{k}, & i \neq j, \\ 0, & i = j. \end{cases}
\]

Then \(\sum_{i=1}^n x_i^k = 1\), and we can get
\[
\alpha_j(\mathcal{G}) \leq L_{\mathcal{G}}x^k = \sum_{\{i_1, \ldots, i_k\} \in E(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k - kx_{i_1} \cdots x_{i_k})
\]
\[
= \sum_{\{i_1, \ldots, i_k\} \in E_j(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k) = \frac{(k-1)d_j}{n-1}.
\]
By Theorem 3.6 we obtain the following result.

**Corollary 3.7.** Let $G$ be a $k$-uniform hypergraph with $n$ vertices, $m$ edges. Then

$$
\sum_{j=1}^{n} \alpha_j(G) \leq \frac{(k-1)km}{n-1}, \quad j \in V(G).
$$

A 2-$(n, b, k, r, \lambda)$ design can be regarded as a $k$-uniform $r$-regular hypergraph $G$ on $n$ vertices, $b$ edges, and $c(x, y) = |\{e \in E(G) : x, y \in e\}| = \lambda$ for any pair of distinct $x, y \in V(G)$. A 2-design satisfying $n = b$ is called a symmetric design.

**Theorem 3.8.** Let $G$ be a connected $k$-uniform hypergraph with $n$ vertices. Then $G$ is a 2-design if and only if $\alpha_1(G) = \cdots = \alpha_n(G) = \frac{\Delta(k-1)}{n-1}$, where $\Delta$ is the maximum degree of $G$.

**Proof.** We first prove the necessity. If $G$ is a 2-$(n, b, k, r, \lambda)$ design, then $\lambda(n-1) = r(k-1)$ and $\Delta = r = d_1 = \cdots = d_n$. For any $j \in V(G)$, by Theorem 3.6, we have

$$
\alpha_j(G) \leq \frac{r(k-1)}{n-1} = \lambda. \quad (3.9)
$$

Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}_+^n$ satisfying $x_j = 0$, $\sum_{i=1}^{n} x_i^k = 1$ and $\alpha_j(G) = \mathcal{L}_G x^k$. Then we get

$$
\alpha_j(G) = \mathcal{L}_G x^k \geq \sum_{\{i_1, \ldots, i_k\} \in E_j(G)} (x_{i_1}^k + \cdots + x_{i_k}^k - kx_{i_1} \cdots x_{i_k}) = \lambda \sum_{i \neq j} x_i^k = \lambda. \quad (3.10)
$$

Combining (3.9) and (3.10), we can get

$$
\alpha_1(G) = \cdots = \alpha_n(G) = \lambda = \frac{r(k-1)}{n-1} = \frac{\Delta(k-1)}{n-1}.
$$

Next we prove the sufficiency. If $\alpha_1(G) = \cdots = \alpha_n(G) = \frac{\Delta(k-1)}{n-1}$. From Theorem 3.6 we obtain $d_1 = \cdots = d_n = \Delta$. Let $z = \left((n-1)^{-\frac{k}{k}}, \ldots, (n-1)^{-\frac{k}{k}}\right)^T \in \mathbb{R}_+^{n-1}$, $y = (z^T, 0)^T \in \mathbb{R}_+^n$. Then

$$
\mathcal{L}_G y^k = \sum_{\{i_1, \ldots, i_k\} \in E(G)} (y_{i_1}^k + \cdots + y_{i_k}^k - ky_{i_1} \cdots y_{i_k}) = \sum_{\{i_1, \ldots, i_k\} \in E_n(G)} (y_{i_1}^k + \cdots + y_{i_k}^k) = \frac{\Delta(k-1)}{n-1} = \alpha_n(G) = \alpha(G).
$$
By Lemma 2.1 we know that $\alpha(\mathcal{G}) = \alpha_n(\mathcal{G})$ is the smallest H-eigenvalue of $L_\mathcal{G}(n)$. Since $L_\mathcal{G}(n)z^k = \alpha(\mathcal{G})y^k = \alpha(\mathcal{G})$, $z$ is an H-eigenvector corresponding to $\alpha(\mathcal{G})$, that is

$$\alpha(\mathcal{G})z^{[k-1]} = L_\mathcal{G}(n)z^{k-1}.$$ 

For all $i \in V(\mathcal{G})$, we have

$$\alpha(\mathcal{G}) = \frac{1}{z_i^{k-1}} (L_\mathcal{G}(n)z^{k-1})_i = \frac{1}{z_i^{k-1}} \sum_{i_2,\ldots,i_k=1}^{n-1} (L_\mathcal{G}(n))_{i_2\ldots i_k} z_{i_2} \cdots z_{i_k}$$

$$= \sum_{i_2,\ldots,i_k=1}^{n-1} (L_\mathcal{G})_{i_2\ldots i_k} = c(i, n).$$

Then we get

$$c(1, n) = c(2, n) = \cdots = c(n-1, n) = \alpha(\mathcal{G}).$$

Similarly, we can obtain

$$c(i, j) = \alpha(\mathcal{G}), \ i, j \in V(\mathcal{G}) \ and \ i \neq j,$$

which implies that $\mathcal{G}$ is a 2-design. 

we give an estimation of the edge connectivity of a 2-design as follows.

**Theorem 3.9.** Let $\mathcal{G}$ be a 2-$(n, b, k, r, \lambda)$ design. Then

$$\frac{n\lambda}{k} \leq e(\mathcal{G}) \leq \frac{(n-1)\lambda}{k-1}.$$ 

Moreover, if $\mathcal{G}$ is a symmetric design, then $e(\mathcal{G}) = k = r$.

**Proof.** Since $\mathcal{G}$ is a 2-$(n, b, k, r, \lambda)$ design, we have $\lambda(n-1) = r(k-1)$. By Theorem 3.8 we have

$$\alpha(\mathcal{G}) = \frac{r(k-1)}{n-1} = \lambda.$$ 

It follows from Lemma 2.3 that

$$\frac{n\lambda}{k} = \frac{n}{k} \alpha(\mathcal{G}) \leq e(\mathcal{G}) \leq r = \frac{(n-1)\lambda}{k-1}. \quad (3.11)$$

Moreover, if $\mathcal{G}$ is a symmetric design, then $n = b$. Since $nr = bk$, we have $r = k$. 


From $\lambda(n - 1) = r(k - 1)$ and (3.11), we have

$$\frac{n(k - 1)}{n - 1} \leq e(G) \leq k.$$ 

Since $e(G)$ is a positive integer, we can get $e(G) = k = r$. 

4. Inverse Perron values and resistance distance of graphs

For a vertex $i$ of a connected graph $G$, we define its resistance eccentricity as $r_i(G) = \max_{j \in V(G)} r_{ij}$.

**Theorem 4.1.** Let $G$ be a connected graph. For any $i \in V(G)$, we have

$$r_i(G) \leq \frac{1}{\alpha_i(G)}.$$ 

**Proof.** Without loss of generality, assume that $i$ is the vertex corresponding to the last row of the Laplacian matrix $L_G$. Since $\alpha_i(G)$ is the minimum eigenvalue of the principal submatrix $L_G(i)$, $\alpha_i^{-1}(G)$ is the spectral radius of the symmetric nonnegative matrix $L_G(i)^{-1}$. So $\alpha_i^{-1}(G) \geq \max_{j \neq i} (L_G(i)^{-1})_{jj}$. 

By Lemma 2.6, $N = \begin{pmatrix} L_G(i)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is a symmetric $\{1\}$-inverse of $L_G$. 

From Lemma 2.4, we get $r_{ij}(G) = (L_G(i)^{-1})_{jj}$ for any $j \neq i$. Hence

$$\alpha_i^{-1}(G) \geq \max_{j \neq i} (L_G(i)^{-1})_{jj} = r_i(G),$$

$$r_i(G) \leq \frac{1}{\alpha_i(G)}.$$ 

For a vertex $i$ of a connected graph $G$, its resistance centrality is defined as $Kf_i(G) = \sum_{j \in V(G)} r_{ij}(G)$. It is a centrality index of networks [5].

**Theorem 4.2.** Let $G$ be a connected graph with $n$ vertices. For any $i \in V(G)$, we have

$$nKf_i(G) - Kf(G) \leq \frac{n - 1}{\alpha_i(G)}.$$
Proof. Without loss of generality, assume that $i$ is the vertex corresponding to the last row of the Laplacian matrix $L_G$. Then $\alpha_i^{-1}(G)$ is the maximum eigenvalue of the symmetric matrix $L_G(i)^{-1}$. Let $e$ be the all-ones column vector, then

$$ \alpha_i^{-1}(G) \geq \frac{e^\top L_G(i)^{-1}e}{e^\top e} = \frac{e^\top L_G(i)^{-1}e}{n-1}. $$

By Lemma 2.6, $N = \begin{pmatrix} L_G(i)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is a symmetric $\{1\}$-inverse of $L_G$. By Lemma 2.5, we have

$$ Kf(G) = n\text{tr}(N) - e^\top Ne = n\text{tr}(L_G(i)^{-1}) - e^\top L_G(i)^{-1}e. $$

From Lemma 2.4, we get $r_{ij}(G) = (L_G(i)^{-1})_{jj}$ for any $j \neq i$. Hence $\text{tr}(L_G(i)^{-1}) = Kf_i(G)$ and

$$ Kf(G) = nKf_i(G) - e^\top L_G(i)^{-1}e. $$

By $\alpha_i^{-1}(G) \geq \frac{e^\top L_G(i)^{-1}e}{n-1}$ we get

$$ \alpha_i^{-1}(G) \geq \frac{e^\top L_G(i)^{-1}e}{n-1} = \frac{nKf_i(G) - Kf(G)}{n-1}, $$

$$ nKf_i(G) - Kf(G) \leq \frac{n-1}{\alpha_i(G)}. $$

\[\Box\]

Corollary 4.3. Let $G$ be a connected graph with $n$ vertices. Then

$$ Kf(G) \leq \frac{n-1}{n} \sum_{i=1}^{n} \alpha_i^{-1}(G). $$

Proof. By Theorem 4.2, we have

$$ \sum_{i=1}^{n} \frac{n-1}{\alpha_i(G)} \geq \sum_{i=1}^{n} (nKf_i(G) - Kf(G)) = nKf(G), $$

$$ Kf(G) \leq \frac{n-1}{n} \sum_{i=1}^{n} \alpha_i^{-1}(G). $$
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