Coherent States for Particle Beams in the Thermal Wave Model

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Abstract

In this paper, by using an analogy among quantum mechanics, electromagnetic beam optics in optical fibers, and charge particle beam dynamics, we introduce the concept of coherent states for charged particle beams in the framework of the Thermal Wave Model (TWM). We give a physical meaning of the Gaussian-like coherent structures of charged particle distribution that are both naturally and artificially produced in an accelerating machine in terms of the concept of coherent states widely used in quantum mechanics and in quantum optics. According to TWM, this can be done by using a Schrödinger-like equation for a complex function, the so-called beam wave function (BWF), whose squared modulus is proportional to the transverse beam density profile, where Planck’s constant and the time are replaced by the transverse beam emittance and by the propagation coordinate, respectively. The evolution of the particle beam, whose initial BWF is assumed to be the simplest coherent state (ground-like state) associated with the beam, in an infinite 1-D quadrupole-like device with small sextupole and octupole aberrations, is analytically and numerically investigated.
1 Introduction

It has been recently pointed out that the charged particle beam transport through a generic optical device can be described by means of the so-called Thermal Wave Model (TWM) for particle beam dynamics [1]. According to this approach the transverse (longitudinal) dynamics of a particle beam is described in terms of a complex function, called beam wave function (BWF), whose squared modulus gives the transverse (longitudinal) density profile of the beam [1]-[3], [6] ([4],[5],[7],[8]). The BWF satisfies a Schrödinger-like equation where Planck’s constant is replaced by the transverse (longitudinal) beam emittance [1]-[3],[6] ([4],[5],[7],[8]).

This model has been successfully applied to a number of linear and nonlinear problems of particle beam dynamics [1]-[8]. In particular, it has been used in the transverse dynamics for describing the optics of a charged particle beam in a thin quadrupole-like lens with small sextupole and octupole deviations [3], and for estimating the luminosity in linear colliders [3]. TWM has also been applied for describing the self-consistent interaction of a relativistic electron-positron beams with a collisionless, overdense plasma [2], reproducing the main results for the beam filamentation threshold and the self-bunching equilibrium [2].

TWM seems also useful for describing longitudinal particle bunch dynamics in both conventional accelerators and plasma-based accelerators for which coherent instability, in terms of modulational instability, and soliton formation has been investigated [1],[3],[7]. Recently, TWM has been used for describing the longitudinal dynamics of a relativistic charged particle bunch in a circular accelerating machine, when the radio-frequency (RF) potential well and the self-interaction (wake fields) are present [8] as well as synchrotron radiation damping and quantum excitation (photon noise) are taken into account [8].

The paper is organized as follows: in section 2 we briefly review the basic features of the Thermal Wave Model and introduce the concept of coherent state for charged particle beam dynamics in analogy with the light beam optics. In section 3, we extend the perturbative approach previously discussed for a thin lens [3], [8] to investigate the stationary propagation of a charged particle beam through an infinite 1-D quadrupole with small sextupole and octupole deviations. The initial BWF is assumed to be in the
ground-like state of the harmonic oscillator (pure Gaussian profile). Numerical results for the BWF obtained with this approach are presented and discussed. Finally, in section 4 we give our remarks and conclusions.

2 The concept of coherent state in particle beams

Let us consider a relativistic charged particle beam travelling along $z$-axis with velocity $\beta c$ ($\beta \approx 1$). Denoting by $x$ the transverse coordinate of a single particle in the beam and by $p$ its conjugate momentum, we consider the following single-particle hamiltonian

$$H = \frac{p^2}{2} + U(x, z) .$$

where $H$ has been made dimensionless dividing by the quantity $m\gamma\beta^2 c^2$ ($m$ and $\gamma$ being the particle rest mass and the relativistic factor, respectively). In (1) $U(x, z)$ stands for a dimensionless potential filled by the particle.

2.1 Main features of TWM

As stated before [1]-[8], in the Thermal Wave Model the collective behaviour of a charged particle beam is ruled by a Schrödinger-like equation, which can be obtained from the single-particle description by means of the following Thermal Quantization Rules (TQR) [1]-[3], [6],

$$x \rightarrow \hat{x} , \quad p \rightarrow \hat{p} \equiv -i \epsilon \frac{\partial}{\partial x} , \quad \text{and} \quad H \rightarrow \hat{H} \equiv i \epsilon \frac{\partial}{\partial z} .$$

where here $\epsilon$ stands for the transverse beam emittance. Consequently, prescriptions (2) give immediately the following evolution equation for the BWF

$$i \epsilon \frac{\partial}{\partial z} \Psi(x, z) = -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Psi(x, z) + U(x, z)\Psi(x, z) .$$

Provided that the normalization condition $\int_{-\infty}^{+\infty} |\Psi(x, z)|^2 \, dx = 1$ is satisfied, the transverse number density of the beam particles $\Lambda(x, z)$ is given by

$$\Lambda(x, z) = N |\Psi(x, z)|^2 ,$$

where $N$ is the total number of particles. Eq.(3) is formally identical to the paraxial wave equation which describes the optical beam propagation in inhomogeneous media with
refractive index profile given by $U(x, z)$. In this analogy the inverse of the wave number is replaced by the beam emittance.

It has been shown in [1] that in the simplest case of a relativistic charged particle beam crossing a pure quadrupole (aberrationless lens) of focusing strength $k_1 > 0$, the BWF satisfies the following parabolic equation

$$
i c \frac{\partial}{\partial z} \Psi(x, z) = -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Psi(x, z) + \frac{1}{2} k_1 x^2 \Psi(x, z) \quad .$$

(5)

It is easy to show that Eq. (5) admits the following orthonormal discrete modes

$$\Psi_n(x, z) = \frac{1}{\sqrt{2^n n!}} H_n \left( \frac{x}{\sqrt{2} \sigma(z)} \right) \exp \left[ -\frac{x^2}{4 \sigma^2(z)} + i \frac{x^2}{2 \epsilon \rho(z)} + i (1 + 2n) \phi(z) \right] \quad \text{with} \quad n = 0, 1, 2, \ldots \quad ,$$

(6)

where $H_n$ are the Hermite polynomials, $\sigma(z)$ obeys to the following \textit{envelope equation}

$$\sigma'' + k_1 \sigma + \frac{\epsilon^3}{4 \sigma^2} = 0 \quad ,$$

(7)

and

$$\frac{1}{\rho} = \frac{\sigma'}{\sigma} \quad ,$$

(8)

$$\phi' = -\frac{\epsilon}{4 \sigma^2} \quad .$$

(9)

where each prime denotes the derivative with respect to $z$. Within the similarity with electromagnetic beams, $\Psi_n(x, z)$ play the role analogous to the one played by Hermite-Gauss electromagnetic modes, $\sigma(z)$ describes the \textit{beam caustic} (i.e. \textit{beam envelope}), and $\rho(z)$ represent the \textit{wavefront curvature radius}. By introducing the r.m.s. $\sigma_x(z)$ associated to a general BWF satisfying (5) as (the mean value of $x$ is assumed equal to zero)

$$\sigma_x(z) \equiv \langle x^2 \rangle^{1/2} = \left[ \int_{-\infty}^{+\infty} x^2 |\Psi(x, z)|^2 \, dx \right]^{1/2}$$

(10)

(the quantum-like expectation value), it is easy to see that $\sigma(z)$, appearing in (3)-(6), coincides with the r.m.s. of the fundamental mode $\Psi_0(x, z)$

$$\sigma(z) = \left[ \int_{-\infty}^{+\infty} x^2 |\Psi_0(x, z)|^2 \, dx \right]^{1/2} \quad .$$

(11)

In addition, we can also define the expectation value for the transverse linear momentum associated to $\Psi_0(x, z)$

$$\sigma_p(z) = \epsilon \left[ \int_{-\infty}^{+\infty} \left| \frac{\partial \Psi_0(x, z)}{\partial x} \right|^2 \, dx \right]^{1/2} \quad .$$

(12)
By following particle accelerator physics language, $\sigma(z)$, $\rho(z)$ and $\sigma_p(z)$ can be expressed in terms of some optical parameters, called Twiss parameters $\alpha(z)$, $\beta(z)$, and $\gamma(z)$:

\[
\sigma^2(z) = \epsilon \beta(z) \quad , \\
\frac{1}{\rho(z)} = -\frac{\alpha(z)}{\beta(z)} \quad , \\
\sigma_p^2(z) = \epsilon \gamma(z) \quad .
\]

It is suitable to introduce the following matrix

\[
\hat{T}(z) \equiv \begin{pmatrix} \gamma(z) & \alpha(z) \\
\alpha(z) & \beta(z) \end{pmatrix} \quad .
\]

whose determinant, as it is well-known, is constant with respect to $z$, namely

\[
d \left[ \det \hat{T}(z) \right] / dz = 0 \quad .
\]

Consequently, we choose for our convenience, without loss of generality, $\gamma \beta - \alpha^2 = 1/4$. Consequently from (13)-(15) follows that the determinant of the matrix

\[
\epsilon \hat{T}(z) \equiv \begin{pmatrix} \sigma^2(z) & -\sigma(z) \sigma'(z) \\
-\sigma(z) \sigma'(z) & \sigma^2(z) \end{pmatrix} \quad .
\]

is an invariant, namely

\[
\sigma_p^2 \sigma^2 - (\sigma \sigma')^2 = \frac{\epsilon^2}{4} = \text{const.} \quad .
\]

It is easy to prove that

\[
\sigma(z) \sigma'(z) = \int_{-\infty}^{+\infty} \Psi_0^* (x, z) \left( \frac{x \hat{p} + \hat{p} x}{2} \right) \Psi_0 (x, z) \, dx = \langle \frac{x \hat{p} + \hat{p} x}{2} \rangle \quad .
\]

Consequently, from (13) follows that

\[
\langle x^2 \rangle \langle \hat{p}^2 \rangle - \langle \frac{x \hat{p} + \hat{p} x}{2} \rangle^2 = \frac{\epsilon^2}{4} \quad ,
\]

which is formally identical to Robertson-Schrödinger uncertainty relation [10], [11] for partial cases when $\langle x \rangle = \langle p \rangle = 0$. Furthermore, within the framework of TWM, (21) is a quantum-like version of the well-known Courant-Snyder invariant [12]. Note that (13), or equivalently (21), gives immediately the usual form of the Heisenberg-like uncertainty principle [1] which is analogous to Heisenberg uncertainty relation in quantum mechanics [13] (again for $\langle x \rangle = \langle p \rangle = 0$)

\[
\sigma_p \sigma \geq \frac{\epsilon}{2} \quad .
\]
By introducing, for the general hamiltonian (1), the total averaged energy associated to the transverse motion of the particles

\[ E(z) \equiv \int_{-\infty}^{+\infty} \Psi^*(x, z) \hat{H} \Psi(x, z) \ dx = \frac{\epsilon^2}{2} \int_{-\infty}^{+\infty} | \frac{\partial \Psi(x, z)}{\partial x} |^2 \ dx + \int_{-\infty}^{+\infty} U |\Psi(x, z)|^2 \ dx \ , \tag{23} \]

the following virial equation

\[ \frac{d^2 \sigma^2(z)}{dz^2} = 4E - 2\langle x \frac{\partial U}{\partial x} \rangle \ , \tag{24} \]

and the following energy-variation equation

\[ \frac{dE}{dz} = \int_{-\infty}^{+\infty} \frac{\partial U}{\partial z} |\Psi(x, z)|^2 \ dx \tag{25} \]

hold. For a quadrupole-like potential (see Eq.(5)), Eqs.(24) and (25) show that \( E \) is a constant of motion \( (dE/dz = 0) \), and, in particular, for \( \Psi = \Psi_0 \) recover the envelope equation (7). The equilibrium solution of (7) \( (d^2 \sigma(z)/dz^2 = 0) \), namely

\[ \sigma^2_0 = \frac{\epsilon}{2\sqrt{k_1}} \ , \tag{26} \]

plays an interesting role. In fact, (26) prescribes that if we prepare the system, before entering the quadrupole-like device, with given \( \sigma_0, \epsilon \) and \( k_1 \) in such a way to satisfy (26), the evolution of the beam through this device is performed with the transverse size fixed at the equilibrium value \( (\sigma(z) = \sigma_0) \). In other words, (26) corresponds to an initial beam configuration with the Twiss parameter \( \alpha(z) = 0 \), i.e. zero-divergence of the beam, or equivalently to a BWF with a wavefront whose curvature radius is infinity. Thus, the beam divergence (the curvature radius of the BWF wavefront) is also zero (infinity) during the beam evolution.

Note that for a quadrupole-like potential and for the equilibrium solution (26), the set of Hermite-Gauss modes (3) reduces to the hamiltonian eigenstates of the harmonic oscillator

\[ \Psi^0_n(x, z) = \frac{1}{\sqrt{2\pi\sigma_0^22^n(n!)^2}} \exp \left( -\frac{x^2}{4\sigma_0^2} + i(1+2n)\phi_0(z) \right) H_n \left( \frac{x}{\sqrt{2}\sigma_0} \right) \ , \tag{27} \]

where \( n = 0, 1, 2, \ldots, \)

\[ \phi_0(z) = -\sqrt{k_1} \frac{z}{2} \ , \tag{28} \]

and the energy values \( \mathcal{E}^0_n \), given by (23) under the substitution of these eigenstates, coincide with the hamiltonian eigenvalues of the harmonic oscillator

\[ \mathcal{E}^0_n = \left( n + \frac{1}{2} \right) \epsilon \sqrt{k_1} \ . \tag{29} \]
In particular, for $n = 0$ (27), (28) and (29) give the ground-like state

$$\Psi_0^0(x, z) = \frac{1}{2\pi\sigma_0^0|1/4} \exp \left( -\frac{x^2}{4\sigma_0^2} + i\phi_0(z) \right),$$  

(30)

which is a pure real Gaussian and the lowest energy reachable by the beam is $E_0^0 = (1/2)\epsilon\sqrt{k_1}$. The means $\langle x \rangle$ and $\langle p \rangle$ are equal to zero at this state of beam. In these conditions the uncertainty relation is minimized as

$$\sigma_0 \sigma_{p0} = \frac{\epsilon}{2}.$$  

(31)

Eq. (31) holds also during the evolution of the beam, because, in addition to (26), we have $\sigma_{p}(z) = \sigma_{p0} = \text{const.}$. In summary, we conclude that if we initially prepare the beam according to the matching conditions (26), its evolution is ruled by a quantum-like behaviour in terms of BWF-ground-like state which minimizes the uncertainty relation and corresponds to the lowest accessible beam energy $(1/2)\epsilon\sqrt{k_1}$.

By introducing the connection between emittance and transverse beam temperature $T$, showed both in the conventional beam physics [14], and in TWM [1], $\epsilon^2/4\sigma_0^2 = k_BT/(m_0\gamma\beta^2c^2)$ ($k_B$ being the Boltzmann constant), we easily obtain

$$\sigma_0^0 = \frac{k_BT}{m_0\gamma\beta^2c^2},$$  

(32)

where (24) has been used. Consequently, the beam energy associated with the equilibrium solution corresponds to the thermal energy and this result is in agreement with the equipartition theorem.

As we will show in subsection 2.3, BWF (30) belongs to the infinite series of coherent state functions, labeled by a complex number $\alpha = \alpha_1 + i\alpha_2$, and widely used in quantum mechanics and quantum optics [15]-[17]. To further develop this point, in the next subsection we present the general formalism of coherent states and their main properties.

### 2.2 Formalism of coherent states

In fiber optics the complex ray formalism [18], [19] has been used to describe Gaussian wavepackets propagating along the fiber in frame of Fock-Leontovich parabolic equation [20] which is a Schrödinger-like equation. In Refs. [21], [22] it was proved that the complex rays are just the coherent states describing the electromagnetic beams in fibers. Within
Dirac’s bra and ket formalism, the coherent state $|\alpha\rangle$ is the state which represents the following series

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,$$

(33)

where the number state $|n\rangle$ corresponds to discrete modes of the electromagnetic beam and $\alpha = \alpha_1 + i \alpha_2$ is a complex number. Since the following orthonormal condition holds

$$\langle n|m \rangle = \delta_{nm} ,$$

(34)

we have the normalization condition for coherent states:

$$\langle \alpha|\alpha \rangle = 1 .$$

(35)

A coherent state may be obtained using the unitary shift operator

$$\mathcal{D}_\alpha = \exp\left(\alpha a^\dagger - \alpha^* a\right) ,$$

(36)

where the photon creation and annihilation operators $a^\dagger$ and $a$ satisfy the boson commutation relation

$$[a, a^\dagger] = 1 .$$

(37)

Thus, if one acts with this operator on the fundamental mode state $|0\rangle$, such that

$$a |0\rangle = 0 ,$$

(38)

we have the normalized coherent state

$$\mathcal{D}_\alpha |0\rangle = |\alpha\rangle .$$

(39)

Note that $\mathcal{D}_\alpha$ has the following shifting properties

$$\mathcal{D}^\dagger_\alpha a \mathcal{D}_\alpha = a + \alpha ,$$

(40)

$$\mathcal{D}^\dagger_\alpha a^\dagger \mathcal{D}_\alpha = a^* + \alpha^* .$$

(41)

From these formulas it follows, that the coherent state is the eigenstate of the annihilation operator

$$a |\alpha\rangle = \alpha |\alpha\rangle .$$

(43)
If one calculates in the coherent state the means of quadrature components $Q$ and $P$ one has

$$\langle \alpha | Q | \alpha \rangle = \frac{1}{\sqrt{2}} \langle \alpha | a + a^\dagger | \alpha \rangle = \sqrt{2} \alpha_1 ,$$

$$\langle \alpha | P | \alpha \rangle = \frac{i}{\sqrt{2}} \langle \alpha | a - a^\dagger | \alpha \rangle = \sqrt{2} \alpha_2 .$$

The shift operator has the multiplication property

$$D_\alpha D_\beta = D_{\alpha + \beta} \exp(\text{i} \text{Im}[\alpha \beta^*]) ,$$

where $\text{Im}[...]$ stands for the imaginary part. The distribution function of the electromagnetic modes $|n\rangle$ in the coherent state

$$P(n) = |\langle n|\alpha \rangle|^2 = \frac{\exp(-|\alpha|^2)}{n!} |\alpha|^{2n}$$

is the Poisson distribution for which the expectation value of mode number

$$\langle n \rangle = \sum_{n=0}^{\infty} n P(n) = |\alpha|^2 ,$$

and the dispersion of the mode number

$$\sigma_n = \langle n^2 \rangle - \langle n \rangle^2 = |\alpha|^2$$

is equal to the same value. For coherent states the dispersion of photon quadrature components

$$\langle \alpha | \left( \frac{a + a^\dagger}{\sqrt{2}} \right)^2 | \alpha \rangle - \left( \langle \alpha | \frac{a + a^\dagger}{\sqrt{2}} | \alpha \rangle \right)^2 = \frac{1}{2} ,$$

$$\langle \alpha | \left( \frac{a - a^\dagger}{i\sqrt{2}} \right)^2 | \alpha \rangle - \left( \langle \alpha | \frac{a - a^\dagger}{i\sqrt{2}} | \alpha \rangle \right)^2 = \frac{1}{2} ,$$

are equal and there is no correlation of these quadrature components

$$\langle \alpha | \left\{ a + a^\dagger, a - a^\dagger \right\} | \alpha \rangle - 2 \langle \alpha | a + a^\dagger | \alpha \rangle \langle \alpha | a - a^\dagger | \alpha \rangle = 0 ,$$

where $\{ , \}$ denotes the anticommutator of two operators. In the coherent state representation the wave function of discrete mode state has the simple form

$$\langle \alpha | n \rangle = \frac{\alpha^n}{\sqrt{n!}} \exp \left( -\frac{|\alpha|^2}{2} \right) .$$
As one sees from the decomposition (33), the state \( \exp(|\alpha|^2/2)|\alpha\rangle \) is the generating function for the number state \( |n\rangle \).

According to (50) and (51) the products of the dispersions satisfies the condition of minimizing the Heisenberg uncertainty relation. The evolution of the coherent state \( |\alpha\rangle \) along the beam path due to the Hamiltonian of the oscillator \( H = \omega (a^\dagger a + 1/2) \) produces again the coherent state with extra phase factor

\[
\exp \left[ -i\omega z \left( a^\dagger a + \frac{1}{2} \right) \right] |\alpha\rangle = \exp \left[ -i\frac{\omega z^2}{2} \right] |\alpha\rangle \exp (-i\omega z). \tag{54}
\]

From that point of view the coherent state is considered as the closest to the classical state of the oscillator since it represents the wavepackets whose center moves along the classical trajectory in the phase space and the width of the packet is minimal and consistent with the uncertainty relation [15]-[17], [23], [24].

In the next subsection we will discuss the properties of the coherent states in the coordinate representation in order to describe the coherent charged particle beam transverse dynamics within the framework of TWM.

### 2.3 Physical meaning of coherent states for charged particle beams

One of the needs that we understand to introduce the notion of coherent states for charged particle beams is connected with the experimental possibility of producing, by acting with an external electromagnetic force, a shifting off-axis of the beam propagation coordinate. For example, in an accelerating machine this can be done by means of some devices, such as kickers, RF cavities, etc. [25]-[27], which are able to produce a displacement of the transverse beam distribution center from its stationary position (design orbit or synchronous particle orbit) to another neighbouring path. This means that the center of transverse beam space-distribution has a shift and this effect allows us to naturally introduce, within the framework of TWM, the coherent state representation. In fact, let us suppose that the beam is in the ground-like state \( \Psi_0(x, z) \) given by (27) for \( n = 0 \), which describes the equilibrium state of the particle beam travelling along the stationary orbit [28]. In order to take into account the displacement of the transverse distribution...
center we introduce the following modified potential well

$$U(x, z) = \frac{1}{2}k_1x^2 - F_1(z)x + F_2(z),$$

(55)

where $F_1(z)$ and $F_2(z)$ account for the external electromagnetic forces. Since $U(x, z)$ is still quadratic with respect to $x$, from general properties of Schrödinger equation, the initial ground-like state $\Psi_0(x, z)$ continues to be Gaussian, but its center (wavepacket center) is now shifted off-axis. In order to explicitly find this transformed BWF, let us consider the following transformations on (3) with $U$ given by (55):

$$\Psi(x, z) = \exp\left[-\frac{i}{\epsilon} \int_0^z F_2(z') dz'\right] \Theta(x, z),$$

(56)

which allows us to get

$$i\epsilon \frac{\partial}{\partial z} \Theta(x, z) = -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Theta(x, z) + \left(\frac{1}{2}k_1x^2 - F_1(z)x\right) \Theta(x, z).$$

(57)

(note that $|\Psi(x, z)| = |\Theta(x, z)|$). It is very easy to prove that (57) has the following Gaussian solution

$$\Theta_0(x, z) = \left[\frac{1}{2\pi \sigma_0^2}\right]^{1/4} \exp\left[-\frac{(x - x_0(z))^2}{4 \sigma_0^2} + \frac{i}{\epsilon} p_0(z)x - i\delta_0(z)\right].$$

(58)

where $\sigma_0$ still satisfies the equilibrium condition (26) and the functions $x_0(z)$, $p_0(z)$, and $\delta_0(z)$ satisfy the following differential equations

$$x_0'' + k_1 x_0 = F_1(z),$$

(59)

$$p_0'' + k_1 p_0 = F_1'(z).$$

(60)

and

$$\delta_0' = \frac{p_0^2}{2\epsilon} - \sqrt{k_1 \frac{x_0^2}{4\sigma_0^2}} + \sqrt{k_1} \frac{x_0}{2}.$$  

(61)

Some considerations are in order:

i). Eqs.(59) and (60) coincide with the constraints for the following canonical transformation $x, p \rightarrow \tilde{x}, \tilde{p}$, with

$$\tilde{x} = x - x_0(z)$$

(62)

$$\tilde{p} = p - p_0(z)$$

(63)

which transforms the hamiltonian $H = p^2 + \frac{1}{2}k_1 x^2 - F_1(z)x$ into $\tilde{H} = \tilde{p}^2 + \frac{1}{2}k_1 \tilde{x}^2$.

ii). Consequently, $x_0$ and $p_0$ represent the space-coordinate and the momentum-coordinate...
shift, respectively. For the following it is suitable to introduce the following complex dimensionless shift

\[ \alpha(z) \equiv \frac{x_0(z)}{2\sigma_0} + \frac{i\sigma_0 p_0(z)}{\epsilon} \equiv \alpha_1(z) + i\alpha_2(z) \quad . \] (64)

iii). Eq.(59) (Eq.(60)) can be thought as classical motion equation of unit mass particle which feels the restoring force \(-k_1x_0 (-k_1p_0)\) and the external force \(F_1(z) (F'_1(z))\).

iv). It is very easy to see that, by virtue of (26), solution (58) still minimizes the uncertainty relation, and, consequently, describes a coherent state associated to the shifts \(x_0\) and \(p_0\).

v). According to the definition (4), solution (58) gives the following coherent-state-particle-distribution \(\Lambda_0(x, z)\)

\[ \Lambda_0(x, z) = N |\Theta_0(x, z)|^2 = \frac{N}{\sqrt{2\pi \sigma_0^2}} \exp \left[ -\frac{(x - 2\sigma_0 \alpha_1(z))^2}{2\sigma_0^2} \right] \quad . \] (65)

Thus, (58) can be cast in the following form

\[ \Theta_0(x, z) = \left[ \frac{1}{2\pi \sigma_0^2} \right]^{1/4} \exp \left[ -\frac{x^2}{4\sigma_0^2} \right] \exp \left[ \frac{\alpha(z) x}{\sigma_0} - \frac{|\alpha(z)|^2}{2} - \frac{\alpha^2(z)}{2} \right] \exp [i\theta(z)] \quad , \] (66)

where

\[ \theta(z) \equiv \int_{-\infty}^{z} \left[ \frac{\sigma_0}{\epsilon} \alpha_1(z') F_1(z') - \frac{\sqrt{k_1}}{2} \right] dz' \quad . \] (67)

Note that, according to the formalism used in subsection 2.2, the BWF \(\Theta_0(x, z)\) is a coherent state produced by the complex shift \(\alpha(z)\). In the Dirac’s formalism, let us both denote this coherent state as \(|\alpha\rangle\), and introduce the notation

\[ \langle x | \alpha \rangle \equiv \Theta_0(x, z) \quad , \] (68)

For the sake of simplicity we will consider now these states at \(z = 0\), namely

\[ \langle x | \alpha_0 \rangle \equiv \Theta_0(x, 0) \equiv \Phi_{\alpha_0}(x) = \left[ \frac{1}{2\pi \sigma_0^2} \right]^{1/4} \exp \left[ -\frac{x^2}{4\sigma_0^2} \right] \exp \left[ \frac{\alpha_0 x}{\sigma_0} - \frac{|\alpha_0|^2}{2} - \frac{\alpha_0^2}{2} \right] \quad , \] (69)

where \(\alpha_0 \equiv \alpha(0) \equiv \alpha_1(0) + i\alpha_2(0) \equiv \alpha_{10} + i\alpha_{20}\). These functions have the following properties \([13]-[17]\)

\[ \langle \alpha_0 | \beta_0 \rangle = \int_{-\infty}^{+\infty} \Phi_{\alpha_0}^\ast(x) \Phi_{\beta_0}(x) \, dx = \exp \left( -\frac{|\alpha_0|^2}{2} - \frac{|\beta_0|^2}{2} + \alpha_0^\ast \beta_0 \right) \quad , \] (70)
which means nonorthogonality, and
\[
\frac{1}{\pi} \int_{-\infty}^{+\infty} d\alpha_{10} \int_{-\infty}^{+\infty} d\alpha_{20} \langle \alpha_0 \rangle \langle \alpha_0 \rangle = \hat{1} ,
\] (71)
which implies the completeness condition, i.e. any state \( |\Phi \rangle \) with the wave function \( \Phi(x) \) may be represented as a superposition of the coherent states:
\[
|\Phi \rangle = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\alpha_{10} \int_{-\infty}^{+\infty} d\alpha_{20} \langle \alpha_0 | \Phi \rangle |\alpha_0 \rangle ,
\] (72)
where the BWF in coherent state representation \( \langle \alpha_0 | \Phi \rangle \) is given by the overlap integral
\[
\langle \alpha_0 | \Phi \rangle = \int_{-\infty}^{+\infty} \Phi^*_{\alpha_0}(x) \Phi(x) \, dx .
\] (73)
Since the overlap integral of two different coherent states is not equal to zero these states form an overcomplete set of functions. The particle beam coherent state \( \Phi_{\alpha_0}(x) \) is a normalized eigenstate of the annihilation operator
\[
\hat{a} = \frac{1}{\sqrt{2}} \left( \frac{x}{\sqrt{2}\sigma_0} + \sqrt{2}\sigma_0 \frac{\partial}{\partial x} \right) ,
\] (74)
i.e.
\[
\hat{a} \, \Phi_{\alpha_0}(x) = \alpha_0 \, \Phi_{\alpha_0}(x) .
\] (75)
Since TWM is described for small intensities by the linear Schrödinger-like equation, any BWF associated to a charged particle beam can be represented as a continuous superposition of coherent states, or discrete superposition of Fock mode states.

3 Charged particle beam dynamics through an infinite 1-D quadrupole with small sextupole and octupole deviations

In this section we describe the transverse dynamics of the charge particle beam while it is travelling through an infinite 1-D quadrupole with small sextupole and octupole deviations (aberrations). We consider here the same kinematic and geometrical assumptions made in the introduction of section 2.

Without aberrations the beam is assumed to satisfy the matching condition (26) (unperturbed beam). Thus, according to the results of section 2, if the initial \( (z = 0) \) transverse beam distribution is purely Gaussian, i.e.
\[
\Psi_0^0(x, 0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp \left( -\frac{x^2}{4\sigma_0^2} \right) ,
\] (76)
the unperturbed beam evolution at any \( z > 0 \) is described by a particular coherent state which is the ground-like state (30) that we can obtain from (58) for the special case \( x_0 = p_0 = 0 \) or, equivalently, from (69) for \( \alpha_0 = 0 : \)

\[
\Psi_0(x, z) = \frac{1}{[2\pi \sigma_0^2]^{1/4}} \exp \left( -\frac{x^2}{4\sigma_0^2} \right) \exp \left( -\frac{i}{2} \sqrt{k_1} z \right) .
\]

(77)

In order to take into account small sextupole and octupole aberrations, we perturb the (3) into the following equation:

\[
i\epsilon \frac{\partial}{\partial z} \Psi(x, z) = -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Psi(x, z) + \frac{1}{2} k_1 x^2 \Psi(x, z) + \hat{V} \Psi(x, z) ,
\]

(78)

with the initial condition (76), where the perturbation

\[
\hat{V}(x) = \frac{1}{3!} k_2 x^3 + \frac{1}{4!} k_3 x^4 ,
\]

(79)

accounts for the aberrations. In particular \( \frac{1}{3!} k_2 x^3 (\frac{1}{4!} k_3 x^4) \) corresponds to the 1-D sextupole (octupole) potential term. Note that \( \hat{V}(x) \) given by (79) can be considered a small perturbation provided that the following conditions

\[
k_2 \sigma_0 / 3 k_1 << 1 \quad \text{and} \quad k_3 \sigma_0^2 / 12 k_1 << 1
\]

(80)

hold. In the next subsection we will give an approximate solution of the problem (78) with the specifications (76), (79), and (80).

### 3.1 Analytical results: perturbative approach

By using the standard perturbation approach we write the following expansion of the \( \Psi(x, z) \) in terms of the eigenstates \( \Psi_m^0 \)

\[
\Psi(x, z) = \sum_{m=0}^{+\infty} c_m(z) \Psi_m^0(x, z) .
\]

(81)

The substitution of Eq. (81) into (78) yields the infinite set of equations

\[
i\epsilon \frac{dc_n}{dz} = \sum_{m=0}^{+\infty} c_m(z) \langle n|\hat{V}|m \rangle ,
\]

(82)

where \( |n\rangle \equiv \Psi_n^0 \). To solve the set of equation (82) we consider the first order correction to the BWF in the case in which \( c_m(0) = \delta_{m,0} \), according to the initial condition (76). If we
write \( c_m(z) = \delta_{m,0} + c_m^1(z) \) (conditions (80) here correspond to the condition \( |c_m^1| << 1 \)), Eqs. (82) give
\[
ici \frac{d}{dz} c_n^1(z) = \langle n| \hat{V}|0 \rangle ,
\]
where
\[
\langle n| \hat{V}|0 \rangle \equiv \int_{-\infty}^{+\infty} \Psi_{n}^0(x, z) \hat{V}(x) \Psi_0^0(x, z) \, dx .
\]
Hence, the coefficients \( c_n^1(z) \) are given by
\[
c_0^1(z) = -i \frac{3}{4} \mu \xi ,
\]
\[
c_1^1(z) = -i \frac{3}{2\sqrt{2}} \nu \exp \left[ \frac{i \xi}{2} \right] \sin \left( \frac{\xi}{2} \right) ,
\]
\[
c_2^1(z) = -i \frac{3}{4} \mu \exp [i\xi] \sin (\xi) ,
\]
\[
c_3^1(z) = -i \frac{1}{12\sqrt{2}} \nu \exp \left[ \frac{i \xi}{2} \right] \sin \left( \frac{3\xi}{2} \right) ,
\]
\[
c_4^1(z) = -i \frac{1}{32} \mu \exp [i2\xi] \sin (2\xi) ,
\]
where we have introduced the dimensionless parameters \( \nu \equiv k_2 \sigma_0 / 3k_1 \) and \( \mu \equiv k_3 \sigma_0^2 / 12k_1 \), and the dimensionless length \( \xi \equiv \sqrt{k_1} z \). The non-normalized perturbed BWF is then obtained by the superposition (81) for four terms only:
\[
\Psi(x, z) = \left[ (1 + c_0^1(z)) \Psi_0^0(x, z) + c_1^1(z) \Psi_1^0(x, z) + c_2^1(z) \Psi_2^0(x, z) + c_3^1(z) \Psi_3^0(x, z) + c_4^1(z) \Psi_4^0(x, z) \right] .
\]
Note that now the BWF \( \Psi(x, z) \) given by (90) is not Gaussian anymore and, consequently, does not represent a coherent state anymore. In fact, the aberrations introduce a defocusing of the particles which produces a distortion of the particle beam distribution with respect the unperturbed Gaussian profile.

In the next subsection we analyze more accurately this distortion on the basis of numerical results.

### 3.2 Numerical results

According to the perturbation theory results obtained in the previous subsection, the parameters \( \mu \) and \( \nu \) represent a measure of the distortion due to sextupole and octupole, respectively, once the multipole distortion is defined as the ratio of the sextupole (octupole) aberration strength to the quadrupole strength. Provided that \( \mu << 1 \) and
$\nu << 1$, the distribution in configuration space is proportional to $|\Psi|^2$, where $\Psi$ is given by (90). In Fig.1 we have plotted the normalized transverse density profile versus the dimensionless transverse coordinate $x/\sqrt{2}\sigma_0$ for increasing values of $\xi$ and for $\mu = .005$ and $\nu = .05$ (see Figs.1a–1f). The dashed lines give the starting distribution. We can clearly see a weak distortion of the particle distribution as the beam propagates throughout the optical device, which is quite evident in Figs.1b, 1d, and 1f: the solid lines represent the distorted distributions for $\xi = 15.5, 21.5, 27.5$, respectively. As $\xi$ increases monotonically, the distortion increases and decreases alternatively. In particular in Figs.1a, 1c, and 1e ($\xi = 12.5, 18.5, 24.5$, respectively), the distortion is negligible (negligible distortion), while Figs.1b, 1d, and 1f show significative distortion. This effect is due to the fact that, according to (90), the BWF is given by a superposition of five modes which interfere each other. To clarify this point in Fig.2 we have plotted the evolution of the normalized distortion of the transverse density profile as a function of the dimensionless longitudinal coordinate $\xi$. The normalized distortion is defined here by $\Delta\sigma/\sigma_0$, where $\Delta\sigma \equiv \sigma(\xi) - \sigma_0$ is the deviation of the effective transverse beam size (r.m.s.) evaluated at a generic $\xi$ from the initial one $\sigma_0$. In Fig.2 we have fixed $\nu = .05$ and plotted the distortion for $\mu = 0, .0015, .002$. When $\mu = 0$, the octupole gives no contribution and the BWF is a superposition of three modes only (see Eq.(90)). From Fig.2 is quite evident that the oscillating behaviour of the distortion increases if we increase $\mu$.

In Fig.3 we have plotted the evolution of the distortion fixing $\mu = .005$ and for $\nu = 0, .05$. When $\nu = 0$, we have no contribution from the sextupole, but, when both sextupole and octupole deviation are taken into account, we clearly see the enhancement of the distortion of the beam profile.

4 Remarks and conclusions

In this paper we have introduced the concept of coherent state for charged particle beams within the framework of TWM [1]-[8]. We have shown that the coherent Gaussian-like structures for charged particle beams, naturally or artificially generated in particle accelerators [25]-[27], correspond to examples of beam wave functions which have a formal definition identical to those ones are widely used in the quantum mechanics and quantum optics [15], [17].
We have used the analogy among quantum mechanics, electromagnetic beam optics in optical fibers, and particle beam dynamics in the framework of TWM in order to introduce the concept and to understand the physical meaning of coherent state for a charged particle beam. We have shown that such a kind of state is associated with the complex parameter $\alpha$, introduced in section 2, which describes the shift of the beam distribution center from the optical axis. In particular, we have proved that when the charged particles of a purely Gaussian beam, travelling in an accelerating machine, are shifted from their path by the action of some external transverse electromagnetic forces [25]-[27], the state after this action is just the coherent state $|\alpha\rangle$ (the real part of $\alpha$ corresponds to the shift of the space coordinate $x$ and the imaginary part of $\alpha$ corresponds to the shift of the conjugate momentum $p$).

Physically, due to the action of an external electromagnetic force, the distribution center results to be shifted. This shift, in turn, transforms the initial BWF modulus $|\Psi_0(x, z)|$ into the following: $|\Psi(x, z)|=|\Psi_0(x-x_0, z)|$. In addition, since the shifting does not change the equilibrium condition (26), because it is an intrinsic relationship between the quantities $\sigma_0$, $\epsilon$, and $k_1$, and since the BWF profile is still Gaussian, the uncertainty relation initially fixed at its minimum value $\epsilon$ is preserved during this shift. But this picture, in terms of formalism and physics of charged particle beams, fully corresponds to the coherent state description given in subsection 2.2, and the transformed BWF describes a new coherent state with respect to the simplest one which corresponds to the ground-like state. From the experimental point of view the shift operator $D_\alpha (36)$, which produces these new coherent states for charged particle beam, in an accelerator can be realized by means of some devices such as kickers, RF cavities, etc. [25]-[27].

Furthermore, by taking the ground-like-state as the initial condition, we have studied the particle beam evolution through a infinite 1-D quadrupole-like in the presence of small sextupole and octupole deviations. We have shown that, in the framework of TWM, this evolution is governed by a 1-D Schrödinger-like equation for the BWF which have been solved by means of the standard time-dependent perturbation techniques. To the first-order, this perturbative treatment has shown that, during the evolution, the particle beam profile does not correspond to a coherent state anymore. This is due to the aberrations which produce a distortion of the initial Gaussian profile: during the beam evolution
through the optical device the BWF is a superposition of four Hermite-Gauss eigenstates only. The interference of these four states produces, during the evolution, *small oscillating distortions* of the particle beam profile around the ground-like state. Numerical estimates for this effect have also been given.

In a forthcoming paper we will consider the present perturbative treatment by taking, instead of the ground-like state ($\alpha_0 = 0$), a general coherent state ($\alpha_0 \neq 0$) as initial condition. In addition, since in quantum optics there were introduced other quantum states which are generalization of coherent ones, such as squeezed and correlated states\textsuperscript{29}-\textsuperscript{32}, we will discuss their analogs in particle beam physics.
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Figure captions

Fig.1 Normalized transverse profile of the beam density $\sqrt{2\pi}\sigma_0|\Psi|^2$ vs. $x/\sqrt{2}\sigma_0$ for increasing value of $\xi$ and for $\nu = .05$ and $\mu = .005$.

Dashed line = starting distribution, solid line = distorted distribution
(a) $\xi = 12.5$; (b) $\xi = 15.5$; (c) $\xi = 18.5$; (d) $\xi = 21.5$; (e) $\xi = 24.5$; (f) $\xi = 27.5$;

Fig.2 Normalized distortion $\Delta\sigma/\sigma_0$ vs. $\xi$ for fixed $\nu = .05$ and $\mu = 0., .0015, .002$.

Fig.3 Normalized distortion $\Delta\sigma/\sigma_0$ vs. $\xi$ for fixed $\mu = .005$ and $\nu = 0., .05$. 