Combined local search strategy for learning in networks of binary synapses

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Abstract – Learning in networks of binary synapses is known to be an NP-complete problem. A combined stochastic local search strategy in the synaptic weight space is constructed to further improve the learning performance of a single random walker. We apply two correlated random walkers guided by their Hamming distance and associated energy costs (the number of unlearned patterns) to learn a same large set of patterns. Each walker first learns a small part of the whole pattern set (partially different for both walkers but with the same amount of patterns) and then both walkers explore their respective weight spaces cooperatively to find a solution to classify the whole pattern set correctly. The desired solutions locate at the common parts of weight spaces explored by these two walkers. The efficiency of this combined strategy is supported by our extensive numerical simulations and the typical Hamming distance as well as energy cost is estimated by an annealed computation.

Models of associative memory have been well studied [1,2]. In these models, the binary synaptic weight is more robust to noise and much simpler for large-scale electronic implementations compared with its continuous counterpart. However, learning in networks of binary synapses is known to be an NP-complete problem [3]. For a single-layered feed forward network with binary weights (we call this network binary perceptron), the storage capacity was predicted to be $\alpha_s \simeq 0.83$ [4] provided that the number of weights $N$ tends to be infinity. Here we define the ratio of the number of patterns $P$ to $N$ as the constraint density $\alpha$ and the theoretical limit of $\alpha$ for the perceptron is termed the storage capacity. The binary perceptron attempts to perform a random classification of $\alpha N$ random input patterns [5]. Many efforts have been devoted to the nontrivial algorithmic issue of this difficult problem [6–12]. Amongst these, a message passing algorithm was developed to learn an extensive number of random patterns with $\alpha \simeq 0.7$ [9]. However, in this letter, we focus on the synaptic weight space random walking algorithm and propose a combined stochastic local search strategy to further improve the learning performance of a single random walker in the high-dimensional weight space. For all finite $\alpha$, a discontinuous ergodicity breaking transition for the binary perceptron at finite temperature is predicted by the dynamic mean-field theory [13]. When the transition temperature is approached, the traditional simulated annealing process is easily trapped by the suboptimal configurations where a finite fraction of patterns are still not learned [7]. The difficulty of local search heuristics for learning is likely to be connected to the fact that exponentially many small clusters coexist in the weight space with a more exponentially large number of suboptimal configurations [14]. Here, we define a connected component of the weight space as a cluster of solutions in which any two solutions are connected by a path of consecutive single-weight flips [15]. A configuration of synaptic weight is identified to be a solution if it is able to learn a prescribed set of patterns. Various stochastic local search strategies by virtue of random walks have

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been used to find solutions of constraint satisfaction problems [16–21]. In our previous study [12], we suggested a simple sequential learning mechanism, namely synaptic weight space random walking for the perceptronal learning problem. In this setting, $\alpha N$ patterns are presented in a randomly permuted sequential order and random walk of single- or double-weight flips is performed until each newly added pattern is correctly classified (learned). The previously learned patterns are not allowed to be misclassified in later stages of the learning process. This simple sequential learning rule was shown to have good performances on networks of $N \sim 10^3$ synapses or less. The mean achieved $\alpha$ is 0.57 for $N = 201$ and 0.41 for $N = 1001$.

In this work, we improve the learning performance by introducing a smart combined strategy. Instead of using a single random walker, we apply two correlated random walkers guided by their Hamming distance and associated energy costs. Both walkers expect to learn a same large set of patterns, but each walker first learns a small part of the whole pattern set (partially different for both walkers but with the same constraint density). Then both walkers explore their respective current weight spaces cooperatively until either of them finds a solution to classify correctly the whole pattern set. Therefore, each walker only needs to make the corresponding residual part of the whole pattern set learned. Notice that the weight spaces both walkers explore separately are actually different since the small set of patterns they have learned are not completely identical despite the same amount of learned patterns. If a solution exists, the found solution should belong to one of the common parts of the weight spaces explored by both walkers, therefore, the small Hamming distance between these two walkers and zero energy are favored during the multiple random walkings (see fig. 1(b)).

The binary perceptron realizes a random classification of $P$ random input patterns (see fig. 1(a)). To be more precise, the learning task is to seek for an optimal set of binary synaptic weights $\{J_i\}_{i=1}^N$ that could map correctly each random input pattern $\{\xi^\mu_i\} (\mu = 1, \ldots, P)$ to the desired output $\sigma^\mu$, the actual output $\sigma^\mu$ of the perceptron is $\sigma^\mu = \text{sgn}(\sum_i J_i \xi^\mu_i)$, where $J_i$ takes $\pm 1$ and $\xi^\mu_i$ takes $\pm 1$ with equal probability. If $\sigma^\mu = \sigma_0^\mu$, we say that the synaptic weight vector $\mathbf{J}$ has learned the $\mu$-th pattern. Therefore we define the number of patterns mapped incorrectly as the energy cost $E(\mathbf{J}) = \sum_\mu \Theta(-\sigma_0^\mu \sum_i J_i \xi^\mu_i)$ where $\Theta(x)$ is a step function with the convention that $\Theta(x) = 0$, if $x \leq 0$ and $\Theta(x) = 1$, otherwise. In the current setting, both $\{\xi^\mu_i\}$ and the desired output $\{\sigma_0^\mu\}$ are generated randomly independently. Without loss of generality, we assume $\sigma_0^\mu = +1$ for any input pattern in the remaining part of this letter, since one can perform a gauge transformation $\xi^\mu_i \rightarrow \xi^\mu_i \sigma_0^\mu$ to each input pattern without affecting the result.

Before introducing the combined learning strategy, we first briefly outline two simple local search strategies [12], i.e., single-weight flip (SWF) and double-weight flip (DWF). To learn a given set of random patterns, we first generate an initial weight configuration $\{J_0^1, J_0^2, \ldots, J_N^1\}$ at time $t = 0$. The first pattern $\xi^1$ is then presented to the perceptron. If this pattern is correctly learned by the initial weight configuration, then the second pattern $\xi^2$ is presented, otherwise the weight configuration is modified by a sequence of SWF or DWF until $\xi^1$ is correctly classified. All patterns are applied in a sequential order. Suppose at time $t$ the weight configuration

![Fig. 1: (Color online) The sketch of the binary perceptron and the multiple random walkings in the weight spaces. (a) $N$ input units (open circles) are connected directly to a single output unit (solid circle). A binary input pattern $\{\xi^1_1, \xi^1_2, \ldots, \xi^1_N\}$ of length $N$ is transferred through a sign function to a binary output $\sigma^\mu$, i.e., $\sigma^\mu = \text{sgn}(\sum_{i=1}^N J_i \xi^\mu_i)$. The set of $N$ binary synaptic weights $\{J_i\}$ is regarded as a solution of the perceptron problem if the output $\sigma^\mu = \sigma_0^\mu$ for each of the $P = \alpha N$ input patterns $\mu \in [1, P]$, where $\sigma_0^\mu$ is a preset binary value. (b) Mechanism of multiple random walkings. All configurations (represented by open or solid circles) are the solutions learning the initial small set of patterns for each random walker (A or B). The arrow indicates the movement for each walker in the weight space via SWF. The movement favors small Hamming distance between A and B and decreasing energy cost. The desired solutions (represented by solid circles) for the expected larger $\alpha$ locate at the common parts of weight spaces both walkers explore. Here two common parts are shown, but actually one or more than two are possible.]

58003-p2
is $J^t = (J_1^t, J_2^t, \ldots, J_N^t)$, and suppose this weight configuration correctly classifies the first $m$ input patterns $\xi^{(m)} (\mu = 1, \ldots, m)$ but not the $(m+1)$-th pattern $\xi^{m+1}$. The random walker will keep wandering in the weight space of the first $m$ patterns via SWF or DWF until a configuration that correctly classifies $\xi^{m+1}$ is reached. In the SWF protocol, a set $A(t)$ of allowed single-weight flips is constructed based on the current configuration $J^t$ and the learned patterns. $A(t)$ contains all integer indexes $i$ with the property that the single-weight flip $J_j^t \rightarrow -J_j^t$ does not make any barely learned patterns $\mu \in [1, m]$ being misclassified. At time $t' = t+1/N$, an index $j$ is chosen uniformly randomly from set $A(t)$ and the weight configuration is changed to $J^{t'}$ such that $J^{t'}_j = J^t_j$ if $i \neq j$ and $J^{t'}_j = -J^t_j$. The DWF protocol is very similar to the SWF protocol with the only difference that the set $A(t)$ contains pairs of integer indexes $(i, j)$ for allowed double-weight flips. This set can be constructed as follows. For the current configuration $J^t$, if there are no barely learned patterns ($h^t = +1$ or $\pm 1$ for double-weight flips) among the first $m$ learned patterns, $A(t)$ includes all the $N(N-1)/2$ pairs of integers $(i, j)$ with $1 \leq i < j \leq N$. Otherwise, randomly choose a barely learned pattern, say $m_1 \in [1, m]$ and for each integer $i \in [1, N]$ with the property $J_i^t \xi_i^{m_1} < 0$, put $i$ into another set $B(t)$, then do the following: 1) if $J_i^t \xi_i^{m_1} < 0$ for all the other barely learned patterns, then add all the pairs $(i, j)$ into the set $A(t)$; 2) otherwise, add all the pairs $(i, j)$ into the set $A(t)$, with the property that the integer $j \in B(t)$ satisfies $J_j^t \xi_j^{m_1} < 0$ for all those barely learned patterns $\mu \in [1, m]$ with $J_j^t \xi_j^\mu > 0$. In practice, when the number of added patterns is small, we use an alternative scheme where the set $A(t)$ is not pre-constructed and instead we randomly select a pair $(i, j)$ and flip them if the flip would not make any previously learned patterns misclassified. Once the number of flippable pairs of weights is substantially reduced, we will use the DWF protocol described above to keep learning proceeding. In this way, a learning of a relatively large set of patterns would be not very time consuming.

The combined strategy to improve the learning performance of single walker is illustrated in fig. 1(b). Before the learning starts, we divide the given set of patterns to be learned into three parts, namely $A, B$ and $C$ with the property that the number of patterns in $A \cup B$ equals to that in $A \cup C$. Then the first walker tries to learn $A \cup B$ by DWF protocol while the second walker tries to learn $A \cup C$ by the same protocol. After $A \cup B$ and $A \cup C$ have been learned by both walkers separately, each walker keeps wandering in its current weight space via SWF with the property that all previously learned patterns are still learned and no new pattern is added. In addition, they should communicate with each other by lowering down the sum of the Hamming distance and the associated energy costs. If the sum goes up, we accept the walking (both walkers modify their current configurations via SWF one time) with the probability $e^{-\beta N (\Delta H_d + \Delta c)}$ where $\Delta H_d$ is the change of Hamming distance with respect to the walking and $\Delta c$ the change of energy cost density $c(\pm E/N)$. $\beta$ serves as a control parameter to be optimized. One could also introduce another inverse temperature $\gamma$ (see eq. (1)) to control the decreasing rate of energy cost [22]. For simplicity, we set $\gamma = \beta$ in our simulations although they can be changed independently.

$H_d = \frac{1}{2} (1 - \frac{1}{2} \sum_i J_i^{(1)} J_i^{(2)})$, where $J_1^{(1)}$ is the current weight configuration of the first walker while $J_2^{(2)}$ the second walker. For the first walker, the energy cost is the number of patterns in $C$ classified by $J_1^{(1)}$ and the energy cost for the second walker is the number of patterns in $B$ misclassified by $J_2^{(2)}$. Once either of these two energy costs becomes zero, the whole learning process will be terminated and the whole set of patterns is learned. Otherwise, the learning process stops if the maximal number of attempts for multiple random walkings, namely $T_{\text{max}}$, is saturated. $T_{\text{max}}$ is a free parameter whose value should be chosen considering the trade-off between efficiency (a solution is found) and computational cost.

The combined local search strategy through multiple random walkings actually utilizes the smoothness of the weight space at the initial low constraint density to achieve the solution at expected high constraint density. In this process, the Hamming distance between both walkers and their associated energy costs play a key role in guiding either (or both) of them to the desired solution. To get a preliminary estimate of the optimal inverse temperature $\beta$, we perform an annealed computation which is able to give us a crude knowledge of the relation between the Hamming distance (or energy cost) and the inverse temperature. In our current setting, we can write the partition function as

$$Z = e^{-\frac{1}{2} \sum_j J_j^{(1)} J_j^{(2)} \prod_{\mu \in A \cup B} \Theta \left( \frac{1}{\sqrt{N}} \sum_i J_i^{(1)} \xi_i^{\mu} \right)} \cdot \prod_{\mu \in C} \Theta \left[ e^{-\gamma} + (1 - e^{-\gamma}) \Theta \left( \frac{1}{\sqrt{N}} \sum_i J_i^{(1)} \xi_i^{\mu} \right) \right]$$

$$\cdot \prod_{\mu \in A \cup C} \Theta \left( \frac{1}{\sqrt{N}} \sum_i J_i^{(2)} \xi_i^{\mu} \right) \cdot \prod_{\mu \in B} \Theta \left[ e^{-\gamma} + (1 - e^{-\gamma}) \Theta \left( \frac{1}{\sqrt{N}} \sum_i J_i^{(2)} \xi_i^{\mu} \right) \right] ,$$

where we still distinguish $\beta$ and $\gamma$ which are set to be equal in our simulations. Note that the added prefactor $N^{-1/2}$ makes the argument of $\Theta(\cdot)$ of order of unity for the sake of statistical mechanics analysis. In annealed approximation, we compute the disorder average of partition function $\langle Z \rangle$ where $\langle \cdot \rangle$ denotes the average over the input random patterns. We skip the detail of computation here and define the overlap between configurations $J_1^{(1)}$ and $J_2^{(2)}$ as $q = \frac{1}{N} \sum_i J_i^{(1)} J_i^{(2)}$, then the annealed approximated free

58003-p3
energy density $f_{ann}$ is given by

$$-\beta f_{ann} = \frac{\log(Z)}{N} = \max_q \left\{ -\beta \frac{q^2}{2} - q\hat{q} + \frac{\beta}{2} q + \log(4 \cosh \hat{q}) \right\}$$

$$+\alpha_c \log \left[ \int_0^\infty DtH \left( -\frac{qt}{\sqrt{1-q^2}} \right) \right]$$

$$+(\alpha - \alpha_c) \log \left[ e^{\gamma} H(0) + (1 - e^{-\gamma}) \right]$$

$$\cdot \int_0^\infty DtH \left( -\frac{qt}{\sqrt{1-q^2}} \right) \right] \right],$$

(2)

where $H(x) = \int_x^\infty Dt$ and $Dt \equiv \frac{dt}{2\pi} e^{-t^2/2}$. $\alpha_c$ is the common constraint density denoted by the ratio of the number of patterns in the set $A$ to $N$, and the set $A$ is the common part learned by both walkers at the initial stage. $\hat{q}$ is the conjugate counterpart of overlap $q$ and both of them are determined by the following recursive equations:

$$q = \tanh \hat{q},$$

(3a)

$$\hat{q} = \frac{\beta}{2} + \frac{\alpha_c}{\sqrt{1-q^2} \arccot \left( \frac{-q}{\sqrt{1-q^2}} \right)}$$

$$+ \frac{(\alpha - \alpha_c)(1 - e^{-\gamma})}{\sqrt{1-q^2} \left( \pi e^{-\gamma} + (1 - e^{-\gamma}) \arccot \left( \frac{-q}{\sqrt{1-q^2}} \right) \right)}.$$  

(3b)

After the solution of the above recursive equations is obtained, the annealed typical Hamming distance is calculated via $H_q^{\text{ann}} = \frac{1 - q^2}{2}$ and the annealed energy density is evaluated as

$$\epsilon_{\text{ann}} = \frac{(\alpha - \alpha_c) \left( \pi - \arccot \left( \frac{-q}{\sqrt{1-q^2}} \right) \right)}{\pi - \arccot \left( \frac{-q}{\sqrt{1-q^2}} \right) + e^\gamma \arccot \left( \frac{-q}{\sqrt{1-q^2}} \right)}.$$  

(4)

In practical learning of a single instance, we choose the optimal temperature where predicted $H_q^{\text{ann}}$ and $\epsilon_{\text{ann}}$ take small values (e.g., around 0.06 and 0.01, respectively) and these predicted values can also be compared with those obtained during the actual learning processes. Note that the learning performance is not very sensitive to small changes in the temperature as long as the used temperature yields relatively small predicted $H_q^{\text{ann}}$ and $\epsilon_{\text{ann}}$. In addition, we define an initial constraint density $\alpha_I$ as the number of patterns in $A \cup B$ or $A \cup C$ over $N$. Its value is chosen empirically to be (well) below the mean constraint density achieved by DWF such that each walker can learn the initial pattern set and the common parts of weight spaces both walkers explore exist. In practice, we choose $\alpha_I = 0.4, 0.35, 0.3$ for $N = 201, 501, 1001$ respectively.

We apply the proposed combined local search strategy to learn random patterns of length $N = 201$ and compare the result with that obtained by DWF. DWF is able to go on even if SWF does not work, i.e., all weights are frozen but flipping certain pairs of weights is still permitted. Hence DWF can achieve higher mean $\alpha$ than SWF. Furthermore, its learning time grows almost linearly with $\alpha$ up to the constraint density where DWF can not proceed any more [12]. Therefore we only consider the comparison between combined local search strategy and DWF. As shown in fig. 2, multiple random walkings do outperform DWF which is easily trapped if the current configuration is frozen with respect to double-weight flips, which occurs with high probability when the constraint density becomes large [12]. It is expected that the weight space will become rather rugged at high $\alpha$ in the sense that exponentially many small solution clusters appear and most of them can not be connected by simple single-weight or double-weight flip. However, multiple random walkings start to find the solution for a high $\alpha$ from the weight space at a relatively small $\alpha_I$ where a big connected component each walker will probe is expected. To achieve the desired solution in available time scales, the Hamming distance between both walkers and the associated energy costs are needed to guide both walkers, since the solutions for the high $\alpha$ locate at the common parts of weight spaces both walkers explore and these common parts will emerge as independent clusters when the high $\alpha$ is finally reached. As shown in fig. 2, the combined strategy still has a finite probability to learn 0.78N input random patterns while DWF is not able to learn patterns with $\alpha \geq 0.70$. Even at small constraint density $\alpha \approx 0.60$, the fraction of learning success by multiple random walkings can be nearly 100% with less fluctuations and very small walking steps (see fig. 4(a)). In fig. 3, we also show the median learning time (walking steps) of 16 random pattern sets. This is...
Fig. 3: (Color online) Median learning time $\tau_{\text{med}}$ vs. constraint density $\alpha$ using multiple random walkings for different $N$. Sixteen random pattern sets are generated and the corresponding learning times (walking steps) are ordered; $\tau_{\text{med}}$ is the median of this ordered sequence. Those cases where the learning fails within $T_{\text{max}}$ are put at the top of the ordered learning time sequence. $T_{\text{max}} = 5 \times 10^4 N, 2 \times 10^4 N, 10^4 N$ for $N = 201, 501, 1001$ respectively. We choose the optimal temperature according to eq. (4) with relatively small predicted $H^{\text{ann}}_d$ and $e_{\text{ann}}$. The solid lines are power-law fittings of the form $\tau_{\text{med}} \propto (\alpha_{\text{cr}} - \alpha)^{-\delta}$, where $\alpha_{\text{cr}} \simeq 0.72, 0.575, 0.475$, and $\delta \simeq 1.374, 1.257, 0.644$ for $N = 201, 501, 1001$, respectively. The dashed-dotted line indicates the mean constraint density achieved by DWF for $N = 1001$, dashed line for $N = 501$ and dotted line for $N = 201$ [12].

done by recording walking steps needed to learn each pattern set, then the learning times are ordered [23]. We define $\tau_{\text{med}}$ as the median value of this ordered sequence. Those cases where the learning fails within $T_{\text{max}}$ are put at the top of the ordered learning time sequence. As shown in fig. 3, the critical constraint density $\alpha_{\text{cr}}$ at which $50\%$ of the presented pattern sets are learned successfully is larger than the mean one achieved by DWF (even DWF plus relearning) [12]. $\alpha_{\text{cr}} \simeq 0.72, 0.575, 0.475$ for $N = 201, 501, 1001$, respectively, and $\tau_{\text{med}}$ grows with $\alpha$ roughly as a power law $\tau_{\text{med}} \propto (\alpha_{\text{cr}} - \alpha)^{-\delta}$. As $N$ increases, the critical value $\alpha_{\text{cr}}$ and $\delta$ decrease, which implies that the quality of the local search algorithm for the binary perceptron decreases with increasing system size, and the attained constraint density via local search strategy would vanish if $N$ tends to be infinity [7,13] although numerical extrapolation to verify this point is difficult with current available data. However, the combined local search strategy does improve the learning performance of a single random walker.

Figure 4 gives the evolution of Hamming distance and energy cost density for multiple random walkings to find a solution of single instance at high constraint density. For $N = 201$, one can see from fig. 4(a) that a solution for $\alpha \simeq 0.597$ can be found within 2000 walking steps. In addition, the annealed computation reproduces the plateau values of Hamming distance and energy cost with very good agreement. If both walkers are guided only by Hamming distance, the solution can also be found for $\alpha \simeq 0.70$ and $N = 201$, but the fraction of success is reduced to $33.8\% \pm 16.5\%$ with $\beta = 2.6$. As displayed in fig. 4(b), the evolution of $H_d$ seems to be highly correlated (almost synchronous) with that of the energy cost and $H^{\text{ann}}_d$ is consistent with the plateau value of Hamming distance. Without the guide

Fig. 4: (Color online) Evolution of Hamming distance and energy cost density for multiple random walkings to find a solution of single instance at high constraint density. The evolution corresponds to the walker finding the desired solution. (a) $N = 201, \alpha = 0.597$. The dash-dotted line stays for the annealed Hamming distance while the dotted line for the annealed energy cost density. The temperature is chosen according to eq. (4) with relatively small predicted $H^{\text{ann}}_d$ and $e_{\text{ann}}$. $\alpha_{\text{cr}} = 0.4, \beta = 1.8, T_{\text{max}} = 5 \times 10^4 N$. (b) The same as (a), but both walkers are guided only by Hamming distance and the temperature is chosen with relatively small predicted $H^{\text{ann}}_d$ and $e_{\text{ann}}$. $\alpha_{\text{cr}} = 0.697, \beta = 2.6$. (c) The same as (a), but for $\alpha = 0.776, \beta = 1.7$. (d) $N = 1001, \alpha = 0.50, \alpha_{\text{cr}} = 0.3, \beta = 1.9, T_{\text{max}} = 5 \times 10^3 N$.
of Hamming distance, both walkers are easily trapped by suboptimal configurations with a small finite energy. However, guided by both Hamming distance and energy, searching for a solution can be speeded up. The efficiency of correlated random walkers is closely related to the fact that the Hamming distance guides both walkers to the common part while the energy term reduces the time both walkers spend exploring configurations with high-lying energy, moreover the absolute value of $N\Delta H_d$ is usually comparable to that of $N\Delta e$ during the learning process and the interaction of these terms makes both correlated walkers easily find a path to one of the common parts. If we apply a single walker to explore its weight space after the initial stage to find a solution and this walker is guided only by energy, we find it easily gets stuck in local minima of energy landscape as well. Figure 4(c) plots the evolution of Hamming distance and energy cost density for a very high constraint density $\alpha \simeq 0.776$ for which a larger number of walking steps to achieve the desired solution are required. For $N = 1001$ and expected $\alpha = 0.50$, it is very difficult to find a solution through two correlated random walkers provided that the maximal number of walking attempts is limited. However, we still found a solution with $T_{\text{max}} = 5 \times 10^4 N$ and the evolution of $H_d$ and $e$ is presented in fig. 4(d). In figs. 4(c) and (d), our annealed estimates of $H_d$ and $e$ seem to be larger than the actual plateau values, however, they still provide us the information to select the optimal control parameter $\beta$. To more accurately predict the actual plateau values of $H_d$ and $e$, a quenched computation within replica symmetry approximation or one-step replica symmetry breaking approximation is needed [4], which we leave for future work.

In conclusion, we apply two correlated random walkers instead of single walker to improve learning performance of the binary perceptron. The solution for small $\alpha_T$ can be easily obtained by SWF or DWF [12]. Both walkers then explore their respective weight space cooperatively to reach one of the common parts to which the solution for high $\alpha$ belongs. The smart combined strategy through multiple random walkings makes the whole weight space at expected high constraint density ergodic for both walkers and the learning in the most hard phase becomes possible. The efficiency of our method depends on the choice of a suitable temperature to help the walker overcome energy or entropic barriers and also relies on the smoothness of the initial weight space at $\alpha_T$ whose value should ensure the common parts of weight spaces both walkers explore exist. To this end, we derive annealed estimations (eq. (3) and eq. (4)) to select the suitable temperature with small predicted $H_d^\text{ann}$ and $e^\text{ann}$. Interestingly, the Hamming distance is found to be important for guiding correlated walkers to find a solution (see fig. 4(b)). However, as $N$ (also expected $\alpha$) increases (see fig. 3), the learning time to reach the desired solution grows rapidly, or a much larger $T_{\text{max}}$ should be preset. This supports the computational difficulty to find solutions for binary perceptron by virtue of local search heuristics [7,12,13]. Future research is needed to acquire a full understanding of this point.

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