MONODROMY INVARIANTS AND POLARIZATION TYPES OF GENERALIZED KUMMER FIBRATIONS

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ABSTRACT. In this paper a monodromy invariant for isotropic classes on generalized Kummer type manifolds is constructed. This invariant is used to determine the polarization type of Lagrangian fibrations on such manifolds - a notion which was introduced in an earlier paper of the author. The result shows that the polarization type of a Lagrangian fibration of generalized Kummer type depends on the connected component of the moduli space.

CONTENTS

1. Introduction 1
2. Hyperkähler Manifolds and their fibrations 3
3. Polarization types of Lagrangian fibrations 10
4. An orbit of primitive isometric embeddings 11
5. Monodromy Invariants 13
6. Beauville–Mukai systems of generalized Kummer type 19
7. Polarization types of generalized Kummer fibrations 29
References 30

1. Introduction

In this paper we continue our study of the polarization type of Lagrangian fibrations on irreducible holomorphic symplectic manifolds which we started in [Wie16].

A Langrangian fibration \( f : X \to B \) is a holomorphic map from an irreducible holomorphic symplectic manifold \( X \) to a normal complex space of dimension \( \frac{1}{2} \dim X \) with connected fibers such that the restriction of the holomorphic symplectic form on \( X \) to the regular part of each fiber of \( f \) vanishes. It is well known that all smooth fibers are abelian varieties even if \( X \) is not projective. Given a smooth fiber \( F \) an immediate question is to ask for natural polarizations on it which is by definition the first Chern class \( H = c_1(L) \) of an ample line bundle \( L \) of \( F \).

It is known that for each smooth fiber \( F \) one can find a Kähler class \( \omega \) on \( X \) such that the restriction \( \omega|_F \) is integral and primitive and hence defines a polarization on \( F \), see Proposition 3.1. An ad–hoc definition of the polarization type of a Lagrangian fibration would be to set \( d(f) := d(\omega|_F) \) where the latter one is the polarization type of the polarization on \( F \) given by \( \omega|_F \). It follows that this does not depend on the chosen \( F \) and \( \omega \) and that the polarization type stays constant in families of Lagrangian fibrations. For a summary see also section 3.

In [Wie16] we proved that the polarization type of Lagrangian fibrations of \( K3^{[n]} \)–type is always principal. This motivates the speculation whether the polarization

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type only depends on the deformation class of the irreducible holomorphic symplectic manifold \( X \) itself, forgetting the Lagrangian fibration.

The purpose of this paper is to show that this is not the case, a fact which came as a surprise to us. Indeed, the following holds.

**Theorem 1.1** (Theorem 7.1, Proposition 6.29) Let \( f : X \to \mathbb{P}^n \) be a Lagrangian fibration of generalized Kummer type. If \( d = \text{Div}(\lambda) \) denotes the divisibility\(^1\) of \( \lambda = c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1)) \), then \( d^2 \) divides \( n + 1 \) and we have for the polarization type

\[
\mathfrak{d}(f) = \left( 1, \ldots, 1, \frac{n+1}{d} \right).
\]

Furthermore, for a fixed dimension \( \dim X = 2n \), the divisibilities of classes \( \lambda \) as above which can appear for the generalized Kummer type, are exactly the positive integers \( d \) such that \( d^2 \) divides \( n + 1 \).

The proof of Theorem 1.1 involves moduli theory of Lagrangian fibrations of generalized Kummer type, as for instance exploited in [Mar14]. The moduli theory appears in form of what is called a *monodromy invariant*.

Let \( X \) be an irreducible holomorphic symplectic manifold and consider the monodromy group \( \text{Mon}^2(X) \). A *faithful monodromy invariant*, see section 5 and [Mar13, Def. 5.16], is a \( \text{Mon}^2(X) \)-invariant map \( \vartheta : I(X) \to \Sigma \) where \( I(X) \subset H^2(X, \mathbb{Z}) \) is a \( \text{Mon}^2(X) \)-invariant subset and \( \Sigma \) is an arbitrary set, such that the induced map \( I(X)/\text{Mon}^2(X) \to \Sigma \) is injective.

The following is a generalized Kummer analogue of E. Markman’s monodromy invariant for the K3\(^n\) case, see [Mar14, 2.].

Let \( X \) be of generalized Kummer type. For a fixed positive integer \( d \), let us denote \( I_d(X) \subset H^2(X, \mathbb{Z}) \) the set of all primitive isotropic classes with divisibility \( d \). For the case that \( d^2 \) divides \( n + 1 \), let \( \Sigma_{n,d} \) denote the set of isometry classes of pairs \((H, w)\) such that \( H \) is a lattice isometric to the lattice \( L_{n,d} \) which is defined in (5.11) and \( w \in H \) is a primitive class with \( (w, w) = 2n + 2 \).

**Theorem 1.2** (Theorem 5.15) Let \( X \) be a generalized Kummer type manifold of dimension \( 2n \) and \( d \) a positive integer such that \( d^2 \) divides \( n + 1 \). There is a surjective faithful monodromy invariant

\[
\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}
\]

of the manifold \( X \).

A similar result as Theorem 1.2 was obtained independently by G. Mongardi und G. Pacienza in [MP16]. They also construct a faithful monodromy invariant function, see [MP16, Lem. 3.4], which requires a choice of an embedding of \( H^2(X, \mathbb{Z}) \) into the Mukai lattice, cf. 4.5. Also compare with [MP16, Rem. 3.10] and Theorem 4.9.

**Structure of the paper.** In section 2 we give a review of the theory of hyperkahler manifolds. Section 3 is a summary of the author’s paper [Wie16] about the definition of the polarization type of a Lagrangian fibration. In section 4 a canonical orbit of primitive isometric embeddings from the generalized Kummer lattice into the Mukai lattice (of torus type) is constructed which is a main ingredient for the construction of the monodromy invariant which is done in the next section 5. Section 6 has the purpose to recall the construction of Beauville–Mukai systems of generalized

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\(^1\)Here we mean with the divisibility \( k = \text{Div}(\lambda) \), the largest positive number \( k \), such that \((\lambda, \cdot)/k\) is an integral form.
Kummer type and to determine their polarization types. An excursion to the theory of Jacobians is needed, see subsection 6.7. Finally, we compute the polarization type of a Lagrangian fibration of generalized Kummer type in section 7.

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2. Hyperkähler Manifolds and their fibrations

In this section we recall the basic facts about irreducible holomorphic symplectic manifolds and their fibrations which are Lagrangian.

Definition 2.1 A compact Kähler manifold $X$ is called hyperkähler or irreducible holomorphic symplectic if $X$ is simply connected and $H^0(X, \Omega^2_X)$ is generated by a nowhere degenerate holomorphic two–form $\sigma$.

Note that $\sigma$ is automatically symplectic since every holomorphic form on a compact Kähler manifold is closed.

The most basic example is provided by the Douady space $S^{[n]}$ of $n$ points for a K3 surface $S$ which parametrizes zero–dimensional subspaces of $S$ of length $n$. A. Beauville [Bea84] showed that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$.

In this paper we are more interested in the following example. We start with a complex two–torus $S$ and take the Douady space $S^{[n+1]}$. This is holomorphic symplectic, but not simply connected. Then one uses the Douady–Barlet map $\rho : S^{[k]} \rightarrow S^{(k)}$, $Z \mapsto \sum_{z \in Z} (\dim \mathcal{O}_{Z,z})z$ which is a resolution of singularities of the symmetric product $S^{(k)} := (S \times \cdots \times S)/\Sigma_k$ to obtain a morphism $S^{[n+1]} \xrightarrow{\rho} S^{(n+1)} \rightarrow S$ where the last map is summation in $S$. By A. Beauville [Bea84] the fiber $S^{[[n]]} = K_n(S)$ over 0 is an irreducible holomorphic symplectic manifold of dimension $2n$, called generalized Kummer manifold as $S^{[[1]]}$ is the usual Kummer K3 surface.

An irreducible holomorphic symplectic manifold is of $K3^{[n]}$–type or of generalized Kummer type if it is deformation equivalent to $S^{[n]}$ for a K3 surface $S$ or to $S^{[[n]]}$ for a two–torus $S$, respectively.

The second cohomology $H^2(X, \mathbb{Z})$ of any irreducible holomorphic symplectic manifold $X$ admits the well known Beauville–Bogomolov–Fujiki quadratic form $q_X$ which is non–degenerate and of signature $(3, b_2(X) - 3)$, see [GHJ03, 23.3]. The associated bilinear form is denoted by $(\cdot, \cdot)$. On an abstract lattice we also denote the bilinear form by $(\cdot, \cdot)$. The lattice $H^2(X, \mathbb{Z})$ with the Beauville–Bogomolov–Fujiki form is a deformation invariant of the manifold $X$. For manifolds of generalized Kummer type this lattice is isometric to the abstract generalized Kummer lattice

$$\Lambda = \mathbb{U}^{\oplus 3} \oplus \langle -(2 + 2n) \rangle,$$

see [Bea84, Prop. 8] where $\langle -(2 + 2n) \rangle$ denotes the lattice of rank one with generator $l$ such that $(l, l) = -(2 + 2n)$ and $\mathbb{U}$ the unimodular rank two hyperbolic lattice.
A marking on an irreducible holomorphic manifold \( X \) is the choice of an isometry \( \eta : H^2(X, \mathbb{Z}) \to \Lambda \). The pair \((X, \eta)\) is then called a marked pair or a marked irreducible holomorphic symplectic manifold.

If \( X \) is a fixed irreducible holomorphic symplectic manifold, set \( \Lambda := H^2(X, \mathbb{Z}) \) and consider the Kuranishi family \( \pi : \mathcal{X} \to \text{Def}(X) \) with \( \mathcal{X}_0 := \pi^{-1}(0) = X \). We will view the base \( \text{Def}(X) \) sometimes as a germ but also as a representative which we choose small enough i.e. simply connected. Then by Ehresmann’s theorem, we can choose a trivialization \( \Sigma : R^2 \pi_* \mathbb{Z} \to \Lambda_{\text{Def}(X)} \) also called a marking and define the local period map by

\[
P : \text{Def}(X) \to \mathbb{P}(\Lambda_C), \quad t \mapsto [\Sigma_t(H^{2,0}(\mathcal{X}_t))] \]

where \( \Lambda_C := \Lambda \otimes \mathbb{C} \). It takes values in the period domain of type \( \Lambda \) [GHJ03, 22.3] namely

\[
\Omega_\Lambda := \{ p \in \mathbb{P}(\Lambda_C) \mid (p, p) = 0 \text{ and } (p, \bar{p}) > 0 \}
\]

which is connected since the signature of \( q_X \) is \((3, \text{rk}\Lambda - 3)\).

**Theorem 2.2 (Local Torelli, [Bea84], 8.)** The period map \( P : \text{Def}(X) \to \Omega_\Lambda \) is an open embedding.

Two marked pairs \((X_i, \eta_i), i = 1, 2, \) are called isomorphic if there is an isomorphism \( f : X_1 \to X_2 \) such that \( \eta_2 = \eta_1 \circ f^* \). There exists a moduli space of marked pairs \( \mathfrak{M}_\Lambda := \{(X, \eta) \text{ marked pair }\} / \cong \) which can be constructed by gluing all deformation spaces \( \text{Def}(X) \) of irreducible holomorphic symplectic manifolds \( X \) with \( H^2(X, \mathbb{Z}) \) isometric to \( \Lambda \). This gives a non-Hausdorff complex manifold of dimension \( \text{rk}\Lambda - 2 \).

The global period mapping

\[
P : \mathfrak{M}_\Lambda \to \Omega_\Lambda, \quad (X, \eta) \mapsto [\eta(H^{2,0}(X))] \]

is locally given by \( P : \text{Def}(X) \to \Omega_\Lambda \) and hence is again a local biholomorphism by the Local Torelli. If one takes an arbitrary connected component \( \mathfrak{M}_\Lambda^0 \) of \( \mathfrak{M}_\Lambda \) then by a result of D. Huybrechts [GHJ03, Prop. 25.12] the restriction \( P : \mathfrak{M}_\Lambda^0 \to \Omega_\Lambda \) is surjective.

If \( L \) denotes a line bundle on \( X \) by abuse of notation we also denote the universal family of the pair \((X, L)\) by \( \pi : \mathcal{X}_L \to \text{Def}(X, L) \) which comes with an universal line bundle \( L \) on \( \mathcal{X}_L \) such that \( (\mathcal{X}_L)_0 = X \) and \( L_0 = L \), see [Bea84, Cor. 1]. We consider again \( \text{Def}(X, L) \) as a germ but as well as a proper space. A representative of \( \text{Def}(X, L) \) is locally given by \( (c_1(L), \cdot) = 0 \) in \( \Omega_\Lambda \) hence it is a smooth hypersurface in \( \text{Def}(X) \), see [GHJ03, 26.1] and one defines \( \mathcal{X}_L \) as the preimage of it under \( \pi \). The family \( \pi : \mathcal{X}_L \to \text{Def}(X, L) \) is the restriction of the Kuranishi family \( \pi : \mathcal{X} \to \text{Def}(X) \) to \( \mathcal{X}_L \) and \( \text{Def}(X, L) \).

### 2.3. Lagrangian fibrations.

Due to D. Matsushita much is known about non-trivial fiber structures on irreducible holomorphic symplectic manifolds.

**Theorem 2.4 (Matsushita, [Mat99], [Mat00], [Mat01], [Mat03])** Let \( f : X \to B \) be a surjective holomorphic map with connected fibers from an irreducible holomorphic symplectic manifold \( X \) of dimension \( 2n \) to a normal complex space \( B \) such that \( 0 < \dim B < 2n \). Then the following statements hold.

(i) \( B \) is projective of dimension \( n \) and its Picard number is \( \rho(B) = 1 \).

(ii) For all \( t \in B \), the fiber \( X_t := f^{-1}(t) \) is a Lagrangian subspace i.e. \( \sigma|_{X_t^{\text{reg}}} = 0 \) where \( X_t^{\text{reg}} \) denotes the smooth part of \( X_t \).

(iii) If \( X_t \) is smooth then it is a projective complex torus i.e. an abelian variety.
Such a fibration $f : X \rightarrow B$ as in the Theorem is called a Lagrangian fibration. If $X$ is a $K3^{[n]}$–type manifold then we call $f : X \rightarrow B$ a $K3^{[n]}$–type fibration.

If the base of the Lagrangian fibration is smooth even more is known due to a deep result of J.-M. Hwang which was recently slightly generalized by C. Lehn and D. Greb to the non–projective case.

**Theorem 2.5** (Hwang, [Hwa08], [GL14]) Let $f : X \rightarrow B$ be a Lagrangian fibration such that $B$ is smooth and $\dim X = 2n$. Then $B \cong \mathbb{P}^n$.

If $f : X \rightarrow B$ is a $K3^{[n]}$–type fibration then E. Markman [Mar11, Thm. 1.3, Rem. 1.8] in combination with a result of D. Matsushita [Mat13, Thm. 1.2, Cor. 1.1] has shown that $B \cong \mathbb{P}^n$ without assuming smoothness of $B$. By [Yos12, Appendix] also in combination with [Mat13, Thm. 1.2, Cor. 1.1] this holds for Lagrangian fibrations of generalized Kummer type.

The basic example of a Lagrangian fibration on a generalized Kummer manifold can be obtained as follows. Let $f : S \rightarrow E$ be a surjective holomorphic map where $S$ is a two torus and $E$ is an elliptic curve. With the of the Douady–Barlet map we have a map

$$S[[n]] \hookrightarrow S^{[n+1]} \xrightarrow{\rho} S^{[n+1]} f \rightarrow E^{[n+1]} \cong \mathbb{P}^n \times E.$$ 

This map and the projection from $\mathbb{P}^n \times E$ to $\mathbb{P}^n$ defines a Lagrangian fibration $S[[n]] \rightarrow \mathbb{P}^n$ by Matsushita’s Theorem 2.4. Let $F$ denote a smooth fiber of $p$, then the fiber of the Lagrangian fibration $S[[n]] \rightarrow \mathbb{P}^n$ is isomorphic to the abelian subvariety of $F^{n+1}$ given by the equation $x_1 + \ldots + x_{n+1} = 0$ for $(x_1, \ldots, x_{n+1}) \in F^{n+1}$.

Note that two–dimensional Lagrangian fibrations are exactly the elliptic K3 surfaces.

**Definition 2.6**

(i) A family of Lagrangian fibrations over a connected complex space $S$ with finitely many irreducible components is an $S$–morphism

$$\mathcal{X} \xrightarrow{\phi} P \xleftarrow{\pi} S$$

where $\mathcal{X} \rightarrow S$ is a family of irreducible holomorphic symplectic manifolds and $P \rightarrow S$ is a family of projective varieties such that for every $s \in S$ the restriction $\phi|_{\mathcal{X}_s} : \mathcal{X}_s \rightarrow P_s$ to the irreducible holomorphic symplectic manifold $\mathcal{X}_s$ is a Lagrangian fibration.

(ii) Two Lagrangian fibrations $f_1$ and $f_2$ are deformation equivalent if there is a family of Lagrangian fibrations $\phi$ over a connected complex space $S$ containing $f_1$ and $f_2$ i.e. there are points $t_i \in S$ such that $\phi|_{\mathcal{X}_{t_i}} = f_i$, $i = 1, 2$.

**Definition 2.7** [Mar13, 5.2] Let $X_i$, $i = 1, 2$, denote two irreducible holomorphic symplectic manifolds, $L_i$ holomorphic line bundles on $X_i$ and $e_i$ classes in $H^2(X_i, \mathbb{Z})$.

(i) The pairs $(X_1, e_1)$ and $(X_2, e_2)$ are called deformation equivalent if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds over a connected complex space $S$ with finitely many irreducible components, a section $e$ of $R^2\pi_*\mathbb{Z}$, points $t_i$ in $S$ such that $X_{t_i} = X_i$ and $e_{t_i} = e_i$.

(ii) The pairs $(X_1, L_1)$ and $(X_2, L_2)$ are called deformation equivalent if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds over a connected complex space $S$ with finitely many irreducible components, a line bundle $\mathcal{L}$ on $\mathcal{X}$, points $t_i$ in $S$ such that $X_{t_i} = X_i$ and $\mathcal{L}_{X_{t_i}} = L_i$. 
Remark 2.8 Note that we can reformulate (ii) of Definition 2.7 as the following.

- The pairs \((X_1, L_1)\) and \((X_2, L_2)\) are called deformation equivalent if there exists a family \(\pi : X \to S\) of irreducible holomorphic symplectic manifolds over a connected complex space \(S\) with finitely many irreducible components, a section \(e\) of \(R^2 \pi_* Z\) which is everywhere of Hodge type \((1, 1)\), points \(t_i\) in \(S\) such that \(X_t = X_i\) and \(e_{t_i} = c_1(L_t)\).

Clearly, \(e_t := c_1(L_t)\) would give such a section. Conversely, given a section \(e\) as in the alternative definition, we get a line bundle \(L_t\) on \(X_t\) corresponding to \(e_t \in H^{1,1}(X_t, Z)\) with respect to the isomorphism \(\text{Pic}(X_t) \cong H^{1,1}(X_t, Z)\) since \(X_t\) is irreducible holomorphic symplectic. Then the Kuranishi family of the pair \((X_t, L_t)\) gives an universal line bundle on the respective total space for every \(t \in S\). Those line bundles glue to a line bundle \(\mathcal{L}\) on \(X\) with the property \(c_1(\mathcal{L}) = e_t\).

Proposition 2.9 Let \(f_i : X_i \to \mathbb{P}^n, i = 1, 2\), denote two Lagrangian fibrations of generalized Kummer or \(K3^{[n]}\)-type and set \(L_i := f_i^* \mathcal{O}_{\mathbb{P}^n}(1)\). The Lagrangian fibrations \(f_i\) are deformation equivalent in sense of Definition 2.6, if and only if the pairs \((X_i, L_i)\) are deformation equivalent.

Proof: The proof is exactly the same as in \([Wie16, \text{Prop. 3.9}]\). There everything is stated for the \(K3^{[n]}\)-type, but it carries over to the generalized Kummer type word by word. \(\square\)

Lemma 2.10 \([Wie16, \text{Lem. 3.5}]\) Let \(f : X \to B\) be a Lagrangian fibration and let \(L := f^* A\) be the pullback of a line bundle \(A\) on \(B\).

(i) \(L\) is isotropic with respect to the Beauville–Bogomolov quadratic form.

(ii) If \(A\) admits nontrivial sections then \(L\) is nef.

(iii) If \(X\) is of \(K3^{[n]}\) or generalized Kummer type and \(A\) is primitive, then \(L\) is primitive.

Proof: The first two statements are contained in \([Wie16, \text{Lem. 3.5}]\). The third statement is formulated for the \(K3^{[n]}\)-type, but the proof works also in the generalized Kummer case with use of \([Mat13, \text{Cor. 1.1}]\) and \(b_2(X) \geq 3\). \(\square\)

2.11. Orientation. We summarize section 4. of \([Mar11]\).

Let \(b_2 > 0\) a positive integer and \(\Lambda\) be an even lattice of signature \((3, b_2 - 3)\). Define

\[
\hat{C}_\Lambda := \{ x \in \Lambda_R \mid (x, x) > 0 \}.
\]

We have the following.

Lemma 2.12 \([Mar11, \text{Lem. 4.1}]\) If \(W \subset \Lambda_R\) is a three dimensional subspace such that the bilinear form of \(\Lambda\) is positive definite on it, then \(W \setminus \{0\}\) is a deformation retract of \(\hat{C}_\Lambda\). Therefore \(H^2(\hat{C}_\Lambda, Z) \cong \mathbb{Z}\) is a free abelian group of rank one. The reflection \(R_u\) for \(u \in \Lambda\) with \((u, u) \neq 0\) given by

\[
R_u(x) := (x, x) - 2\frac{(u, x)}{(u, u)}u,
\]

acts on \(H^2(\hat{C}_\Lambda, Z)\)

- as \(+1\) if \((e, e) < 0\) and
- as \(-1\) if \((e, e) > 0\),

therefore it defines a generator of \(H^2(\hat{C}_\Lambda, Z)\).
In particular, the Lemma implies that $\tilde{C}_\Lambda$ is connected, as $H_0(\tilde{C}_\Lambda, \mathbb{Z}) = H_0(W \setminus \{0\}, \mathbb{Z}) = \mathbb{Z}$.

**Definition 2.13** An orientation of $\tilde{C}_\Lambda$ is a choice of a generator of $H^2(\tilde{C}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$.

By speaking of oriented isometries of the lattice $\Lambda$, we mean isometries which preserve the orientation of $\tilde{C}_\Lambda$ in sense of the definition above: every isometry $g : \Lambda \to \Lambda$ induces a homeomorphism $g : \tilde{C}_\Lambda \to \tilde{C}_\Lambda$, therefore we have a morphism
\begin{equation}
O(\Lambda) \longrightarrow \text{Aut}(H^2(\tilde{C}_\Lambda, \mathbb{Z})) \cong \{\pm 1\}
\end{equation}

\begin{equation}
g \mapsto g^*.
\end{equation}

**Definition 2.15** The morphism in (2.14) above is also called spinor norm. Its kernel is denoted by $O^+(\Lambda)$ and isometries in it are called orientation preserving.

For a period $p \in \Omega_\Lambda$ let $\Lambda(p)$ denote the integral Hodge structure of weight two of $\Lambda$ determined by the period $p$, that is
\begin{equation}
\Lambda^{2,0}(p) = p, \quad \Lambda^{0,2}(p) = \bar{p} \quad \text{and} \quad \Lambda^{1,1}(p) = \{x \in \Lambda_\mathbb{C} \mid (x,p) = (x,\bar{p}) = 0\}.
\end{equation}

As in the geometric situation, we also set
\[\Lambda^{1,1}(p, R) := \{x \in \Lambda_R \mid (x,p) = 0\}\]
for $R \in \{\mathbb{Z}, \mathbb{R}\}$. Furthermore, consider the set
\begin{equation}
C'_p := \{x \in \Lambda^{1,1}(p, \mathbb{R}) \mid (x, x) > 0\}.
\end{equation}
The restriction of the bilinear form to $\Lambda^{1,1}(p, \mathbb{Z})$ has signature $(1, b_2 - 3)$. Therefore $C'_p$ has two connected components.

Let $x$ be in $C'_p$ with $p = \mathbb{C} \cdot \sigma$. We can define a subspace
\begin{equation}
W_x := \text{Re}(p) \oplus \text{Im}(p) \oplus \mathbb{R} \cdot x
\end{equation}
of $\Lambda_\mathbb{R}$ such that the bilinear form is positive definite on it. The subspace $W_x$ of $\Lambda_\mathbb{R}$ defines a generator of $H^2(\tilde{C}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ in the following way.

The subvector space $W_x$ has the canonical ordered basis
\begin{equation}
(\text{Re}(\sigma), \text{Im}(\sigma), x),
\end{equation}
which defines an orientation in the ordinary sense i.e. a volume form $\beta(\sigma) := \text{Re}(\sigma)^* \wedge \text{Im}(\sigma)^* \wedge x^*$ of the manifold $W_x \setminus \{0\}$. The orientation $\beta(\sigma)$ does not depend on the choice of $\sigma$, indeed we have $\beta(\lambda \sigma) = |\lambda|^{-1} \beta(\sigma)$ for any $\lambda \in \mathbb{C}$. Take the two sphere $S^2 \subset W_x \setminus \{0\}$ in $W_x$. It is well known, that the basis (2.19) gives a volume form on $S^2$ by restricting the two form
\[
x_1 \text{Im}(\sigma)^* \wedge x^* + x_2 x^* \wedge \text{Re}(\sigma)^* + x_3 \text{Re}(\sigma)^* \wedge \text{Im}(\sigma)^*
\]
to $S^2$, where $x_1, x_2, x_3$ are the standard coordinates with respect to the basis (2.19).

Use
\begin{equation}
H^2(S^2, \mathbb{Z}) = H^2(W_x \setminus \{0\}, \mathbb{Z}) = H^2(\tilde{C}_\Lambda, \mathbb{Z})
\end{equation}
to obtain a generator of $H^2(\tilde{C}_\Lambda, \mathbb{Z})$ i.e. an orientation in sense of Definition 2.13. Obviously we end up with the other generator, if we change the orientation of $W_x$ given by the basis (2.19).

**Principle 2.21** An element $x$ in $C'_p$ for a period $p \in \Omega_\Lambda$ determines a generator of $H^2(\tilde{C}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ i.e. an orientation of $\tilde{C}_\Lambda$. The two generators are distinguished by the two connected components of $C'_p$. Therefore a connected component of $C'_p$ determines an orientation of $\tilde{C}_\Lambda$. 

2.22. The geometric situation. Let $\mathfrak{M}_\Lambda$ denote the moduli space of isomorphism classes of marked pairs $(X, \eta)$ of type $\Lambda$ i.e. $X$ is an irreducible holomorphic symplectic manifold and $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ is a marking. Choose a connected component $\mathfrak{M}_\Lambda^0$ of $\mathfrak{M}_\Lambda$. Recall that for $(X, \eta) \in \mathfrak{M}_\Lambda^0$ there is a canonical choice for the connected component of

$$C'_X := \left\{ x \in H^{1,1}(X, \mathbb{R}) \mid (x, x) > 0 \right\}$$

namely the positive cone $C_X$ which contains the Kähler cone $K_X$ of $X$. Therefore, by Principle 2.21

$$\tilde{C}_X := \tilde{C}_{H^2(X, \mathbb{Z})} = \{ x \in H^2(X, \mathbb{R}) \mid (x, x) > 0 \}$$

has a natural orientation, which determines an orientation in sense of Definition 2.13 of $\tilde{C}_\Lambda$ via the homeomorphism $\eta : \tilde{C}_X \cong \tilde{C}_\Lambda$.

Definition 2.23 We will refer to the orientation of $\tilde{C}_\Lambda$ (in sense of Definition 2.13) which is induced by the marking $\eta$ and the natural orientation of $\tilde{C}_X$ for some (hence for all) marked pair $(X, \eta)$ in $\mathfrak{M}_\Lambda^0$, as the orientation compatible to the connected component $\mathfrak{M}_\Lambda^0$ of the moduli of marked pairs.

Consider the period map

$$\mathcal{P} : \mathfrak{M}_\Lambda^0 \longrightarrow \Omega_\Lambda, \quad (X, \eta) \mapsto [\eta(H^{2,0}(X))]$$

and set $p := \mathcal{P}(X, \eta)$. Then $\eta(H^{1,1}(X, \mathbb{R})) = \Lambda^{1,1}(p, \mathbb{R})$. An orientation of $\tilde{C}_\Lambda$ determines a connected component

$$(2.24) \quad \mathcal{C}_p \subset \mathcal{C}'_p$$

of $\mathcal{C}'_\Lambda$ by Principle 2.21. Equivalently, we can characterize the orientation compatible to $\mathfrak{M}_\Lambda^0$ by the condition $\eta(C_X) = \mathcal{C}_p$ for all $(X, \eta) \in \mathfrak{M}_\Lambda^0$ with $p = \mathcal{P}(X, \eta)$.

2.25. $\Omega^+_{\Lambda\perp}$ for an isotropic class. For the following see also [Mar14, 4.3]. We are still in the setting of 2.22. Let $\lambda \in \Lambda$ be a nontrivial isotropic class. We define a hyperplane section

$$(2.26) \quad \Omega_{\lambda\perp} := \Omega_\Lambda \cap \lambda^\perp = \{ p \in \Omega_\Lambda \mid (p, u) = 0 \}.$$

Note that the bilinear form on $\lambda^\perp \subset \Lambda_\mathbb{R}$ is degenerate since $\lambda$ is isotropic. The hyperplane section $\Omega_{\lambda\perp}$ has two connected components and we can still obtain a natural connected component of it from the geometrical situation in the following way.

For $p \in \Omega_{\lambda\perp}$, $\lambda$ belongs to $\Lambda^{1,1}(p, \mathbb{R})$ and is contained in the boundary of one of the connected components of $\mathcal{C}'_p$ since $\lambda$ is isotropic. For $(X, \eta) \in \mathfrak{M}_\Lambda^0$, either $\eta^{-1}(\lambda)$ or $\eta^{-1}(\lambda)$ belongs to $\partial \mathcal{C}_X$. We assume that the former is the case, otherwise take $-\lambda$. Then consider only periods $p$ in $\Omega_{\lambda\perp}$ such that $\lambda$ belongs to the closure of the distinguished connected component $\mathcal{C}_p$ in $\Lambda^{1,1}(p, \mathbb{R})$, see (2.24), determined by the orientation of $\tilde{C}_\Lambda$ compatible to $\mathfrak{M}_\Lambda^0$ i.e.

$$(2.27) \quad \Omega^+_{\lambda\perp} := \{ p \in \Omega_{\lambda\perp} \mid \lambda \in \partial \mathcal{C}_p \}$$

which is one of the connected components of $\Omega_{\lambda\perp}$. Note that the only common element of the closures of the connected components of $\Omega_{\lambda\perp}$ is the null vector, therefore $\Omega^+_{\lambda\perp}$ of (2.27) is indeed one of the connected components of $\Omega_{\lambda\perp}$. We refer to $\Omega^+_{\lambda\perp}$ as the compatible connected component of $\Omega_{\lambda\perp}$ with respect to the chosen connected component $\mathfrak{M}_\Lambda^0$ of the moduli of marked pairs.
2.28. Monodromy. We recall some basic definitions and state G. Mongardi’s monodromy result [Mon16].

Definition 2.29 Let $X_i$, $i = 1, 2$, be two irreducible holomorphic symplectic manifolds. An isometry $P : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is called a parallel transport operator if there exists a family $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds, points $t_i$ such that $\mathcal{X}_{t_i} = X_i$ and a continuous path $\gamma$ such that the parallel transport $P$, along $\gamma$ in the local system $R^2\pi_*\mathbb{Z}$ coincides with $P$. For $X := X_1 = X_2$ it is also called a monodromy operator and the subgroup $\text{Mon}^2(X)$ of $O(H^2(X, \mathbb{Z}))$ generated by monodromy operators is called the monodromy group.

Let $\Lambda$ denote a non-degenerate lattice of signature $(3, b_2 - 3)$.

Definition 2.30 Let $\mathcal{W}(\Lambda)$ denote the subgroup of $O^+(\Lambda)$ consisting of orientation preserving isometries acting as $\pm 1$ on the discriminant $\Lambda^\vee / \Lambda$. Denote by $$\chi : \mathcal{W}(\Lambda) \to \{\pm 1\}$$ the associated character. We also write $\mathcal{W}(X) := \mathcal{W}(H^2(X, \mathbb{Z}))$ for an irreducible holomorphic manifold $X$.

For a class $u \in \Lambda$ with $(u, u) \neq 0$ we have the rational reflection $R_u : \Lambda \to \Lambda$ defined by

$$(2.31) \quad R_u(x) := x - 2\frac{(u, x)}{(u, u)} u.$$ If $(u, u) < 0$, then by Lemma 2.12 the reflection $R_u$ is orientation preserving in sense of Definition 2.15 i.e. contained in $O^+(\Lambda_Q)$.

Definition 2.32 Let $\Lambda$ be a non-degenerate lattice of signature $(3, b_2 - 3)$. For a class $u \in \Lambda$ with $(u, u) \neq 0$, denote $\rho_u : \Lambda_Q \to \Lambda_Q \in O^+(\Lambda_Q)$ the orientation preserving isometry defined by

$$\rho_u := \left\{ \begin{array}{ll} R_u & \text{if } (u, u) < 0, \\ -R_u & \text{if } (u, u) > 0. \end{array} \right.$$ 

Remark 2.33

(i) If $(u, u) = \pm 2$, then $R_u$ and $\rho_u$ define honest integral isometries $\Lambda \to \Lambda$.

(ii) The action of $R_u$ on $\Lambda^\vee$ for a $h \in \Lambda^\vee$ is

$$R_u(h)(x) = h(R_u(x)) = h(x) - (2\frac{h(u)}{(u, u)} u, x),$$

i.e. $R_u(h) = h$ mod $\Lambda$, hence for $(u, u) = \pm 2$ the isometry $\rho_u$ is contained in $\mathcal{W}(\Lambda)$. More precisely we have

$$\chi(\rho_u) = \left\{ \begin{array}{ll} +1 & \text{if } (u, u) < 0, \\ -1 & \text{if } (u, u) > 0. \end{array} \right.$$ 

(iii) The isometry $R_u$ satisfies $R_u(u) = -u$ and $R_u|_{u, \perp} = \text{id}_{u, \perp}$, hence we have for the determinant $\det(R_u) = -1$. Therefore

$$\det(\rho_u) = \left\{ \begin{array}{ll} -1 & \text{if } (u, u) < 0, \\ (-1)^{b_2+1} & \text{if } (u, u) > 0. \end{array} \right.$$ 

Note that for the $K3^{[n]}$ and generalized Kummer case $b_2$ is odd and for the O’Grady examples $b_2$ is even.
Theorem 2.34 (Mongardi, [Mon16, Thm. 2.3]) Let $X$ be a generalized Kummer $n$–type manifold. Then $\text{Mon}^2(X)$ consists precisely of orientation preserving isometries $g \in \mathcal{W}(X)$ such that $\chi(g) \cdot \det(g) = 1$.

In particular, for a generalized Kummer manifold $X$, $\text{Mon}^2(X)$ is an index 2 sub group of $\mathcal{W}(X)$ as $|\mathcal{W}(X)/\text{Mon}^2(X)| = |\text{im} (\det \cdot \chi)| = 2$.

Corollary 2.35 For a generalized Kummer type manifold $X$, the monodromy group $\text{Mon}^2(X)$ is an index 2 sub group of $\mathcal{W}(X)$. The orientation preserving isometry $\rho_u \in \mathcal{W}(X)$ for a class $u \in H^2(X, \mathbb{Z})$ with $(u, u) = \pm 2$ defined in Definition 2.32 is never contained in $\text{Mon}^2(X)$.

Proof: The first statement we have just discussed. The second statement follows from Remark 2.33 (ii) and (iii). \hfill \square

3. Polarization types of Lagrangian fibrations

The author introduced the following notion in [Wie16]. Let $f : X \to B$ be a Lagrangian fibration. We known that all smooth fibers are abelian varieties by Theorem 2.4, even if $X$ is not projective. For an abelian variety $F$ of dimension $\dim F = n$, there is a well known classical notion of a polarization, cf. [BL03, p. 70], which is by definition the first Chern class $H = c_1(L)$ of an ample line bundle $L$ of $F$. Often one calls the ample line bundle $L$ a polarization. Furthermore, one can associate to such a polarization a type, which is a tuple

$$\text{d}(L) = (d_1, \ldots, d_n)$$

of positive integers such that $d_i$ divides $d_{i+1}$, cf. [BL03, p. 70].

Given a smooth fiber $F$ of the Lagrangian fibration $f$ we want to consider polarization on it induced from $X$. First of all, it is not clear, how to obtain a polarization on a smooth fiber $F$ of the Lagrangian fibration $f : X \to B$ if $X$ is not projective. However, due to the following statement, which is related to an observation of C. Voisin [Cam05, Prop. 2.1], it is always possible.

Proposition 3.1 [Wie16, Prop 4.3] For any smooth fiber $F$ there is a Kähler class $\omega$ on $X$ such that the restriction $\omega|_F$ is integral and primitive.

Such a class $\omega$ is called special Kähler class (with respect to $F$) and defines a polarization $\omega|_F$ on the abelian variety $F$ in the sense above. To this polarization one can associate its type $\text{d}(\omega|_F) := (d_1, \ldots, d_n)$ where again $d_i$ are positive integers such that $d_i$ divides $d_{i+1}$.

Definition 3.2 The polarization type of a Lagrangian fibration $f : X \to B$ is

$$\text{d}(f) := \text{d}(\omega|_F) = (d_1, \ldots, d_n).$$

This definition seems to be a bit ad–hoc, but it is convenient for a short introduction. The following statements were shown in [Wie16].

Theorem 3.3 [Wie16, Section 4] Let $f : X \to B$ be a Lagrangian fibration with $\dim X = 2n$. Then the following statements hold.

(i) [Wie16, Prop. 4.7] The polarization type $\text{d}(f)$ is well defined i.e. does not depend on the chosen smooth fiber and the chosen special Kähler class (with respect to this fiber) and is a primitive vector in $\mathbb{Z}^n$.

(ii) [Wie16, Thm. 4.9] The polarization type is a deformation invariant of the fibration i.e. if $f' : X' \to B'$ is a Lagrangian fibration deformation equivalent to $f$, then $\text{d}(f) = \text{d}(f')$. 

(iii) [Wie16, Prop. 4.6, Prop. 4.10] Let $B^o$ denote the subset of $B$ which parametrizes the smooth fibers. Then there exists a family of special Kähler classes, that is a map $\alpha : B^o \to \mathcal{H}$ where $\mathcal{H} \subset (\mathbb{R}^2 \pi_* \mathbb{Z} \otimes \mathcal{O}_B)|_{B^o}$ is a subbundle and $\alpha(t)$ is a special Kähler class with respect to the smooth fiber $X_t$ for every $t \in B^o$. In particular $d(\alpha(t)) = d(f)$ for every $t \in B^o$.

(iv) [Wie16, Prop. 4.6] The family of special Kähler classes $\alpha$ induces a holomorphic map, called moduli map,

$$\phi : B^o \to A_{d(f)},$$

$$t \mapsto (X_t, \alpha(t)),$$

where $A_{d(f)}$ denotes the moduli space of $d(f)$ polarized abelian varieties.

(v) [Wie16, Thm. 6.1] Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of $K3^{[n]}$--type. Then $d(f) = (1, \ldots, 1)$.

In this paper, we want to determine the polarization type of a Lagrangian fibration of generalized Kummer type.

4. An orbit of primitive isometric embeddings

The main ingredient for the construction of a monodromy invariant for isotropic classes in the second cohomology of a generalized Kummer manifold is a monodromy invariant orbit of primitive isometric embeddings of the Kummer–type lattice into the Mukai lattice.

The group of isometries $O(\tilde{\Lambda})$ of the Mukai lattice $\tilde{\Lambda} := \Lambda \oplus U$ (see also below) and $O(\Lambda)$ acts on the set $O(\Lambda, \tilde{\Lambda})$ of primitive isometric embeddings $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ of the lattice $\Lambda$ into $\tilde{\Lambda}$ by composition i.e. for $g \in O(\Lambda)$ and $\tilde{g} \in O(\tilde{\Lambda})$ one sets $g \cdot \iota := \iota \circ g$ and $\tilde{g} \cdot \iota := \tilde{g} \circ \iota$.

Definition 4.1 Let $\iota \in O(\Lambda, \tilde{\Lambda})$ be a primitive isometric embedding. An element $g \in O(\Lambda)$ leaves the $O(\tilde{\Lambda})$--orbit $[\iota] = O(\tilde{\Lambda}) \cdot \iota$ invariant if $g \cdot [\iota] := [\iota \circ g] = [\iota]$ i.e. if there exists $\tilde{g} \in O(\tilde{\Lambda})$ such that $\tilde{g} \circ \iota = \iota \circ g$. The orbit is called monodromy invariant if $\text{Mon}^2(X) \cdot [\iota] = [\iota]$ i.e. all elements in $\text{Mon}^2(X)$ leave the orbit $[\iota]$ invariant.

Remark 4.2 Let $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ denote a primitive isometric embedding. If $X$ is a generalized Kummer type manifold then $\iota(\Lambda)^{\perp} = (v)$ is of rank 1 since the Mukai lattice is of rank 8 and the Kummer type lattice is of rank 7. An isometry $\tilde{g} \in O(\tilde{\Lambda})$ with $\iota \circ g = \tilde{g} \circ \iota$ necessarily satisfies $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$ and $\tilde{g}(v) = \pm v$, otherwise $\tilde{g}$ cannot be an isometry.

The following Lemma is a special case of [Nik80, Cor. 1.5.2].

Lemma 4.3 Let $\Lambda$ be the generalized Kummer or $K3^{[n]}$ lattice. Write $\Lambda = w^{\perp} \subset \tilde{\Lambda}$ with $w$ primitive (cf. Remark 4.2). An isometry $g \in O(\Lambda)$ can be extended to an isometry $\tilde{g} \in O(\tilde{\Lambda})$ if and only if $g$ acts as $\pm 1$ on the discriminant $\Lambda^{\vee}/\Lambda$.

Proof: By [Nik80, Cor. 1.5.2] we can extend $g$ to such a $\tilde{g}$ if and only if we have an isometry $\varphi : \Lambda^{\perp} \to \Lambda^{\perp}$ with an additional property. Since $\Lambda^{\perp} = \langle w \rangle$ the only two isometries are $\varphi = \pm 1$. Following the exposition in [Nik80, 5. ff.], the additional property for $\varphi = \pm 1$ means that $g$ acts on $\Lambda^{\vee}/\Lambda$ as $\pm 1$. □
Corollary 4.4 Let $\Lambda = \omega^\perp \subset \tilde{\Lambda}$ be as in the Lemma above and let us denote $[i] = O(\tilde{\Lambda})$ an arbitrary invariant $O(\tilde{\Lambda})$--orbit of primitive isometric embeddings $\Lambda \hookrightarrow \tilde{\Lambda}$. Then the sub group $W(\Lambda) \subset O^+(\Lambda)$ defined in Definition 2.30 is equal to the sub group of all $g \in O^+(\Lambda)$ leaving the orbit $[i] = O(\tilde{\Lambda})$ invariant, i.e. there exists $\tilde{g}$ such that $i \circ g = \tilde{g} \circ i$.

**Proof:** An element $g \in O^+(\Lambda)$ leaves $O(\tilde{\Lambda})$ invariant if and only if it acts by $\pm 1$ on the discriminant $\Lambda^\perp/\Lambda$ by Lemma 4.3. \hfill \Box

In other words, $W(\Lambda) = \text{Stab}([i])$ is equal to the stabilizer of $[i]$ with respect to the action of $O^+(\Lambda)$ on the set of $O(\tilde{\Lambda})$--orbits of primitive isometric embeddings $O(\Lambda, \tilde{\Lambda})$.

With the knowledge of the monodromy group of a generalized Kummer manifold, see Theorem 2.34, one can construct an analogue of the monodromy invariant $O(\tilde{\Lambda})$--orbit as in [Mar10, Thm. 1.10].

Let $S$ be an abelian surface and let $H^\bullet(S)$ denote the even cohomology i.e.

$$H^\bullet(S) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

together with the bilinear form defined by $(v, w) := (v_2, w_2) - \int_S (v_0 \wedge w_4 + v_4 \wedge w_0)$ where $(v_2, w_2) = \int_S v_2 \wedge w_2$ denotes the intersection form on $H^2(S, \mathbb{Z})$ and $v = v_0 + v_2 + v_4$ with $v_i \in H^i(S, \mathbb{Z})$ the decomposition in $H^\bullet(S)$ and similarly for $w$. This lattice is even, unimodular, of rank 8 and isometric to the Mukai lattice

$$\tilde{\Lambda} := U^{\oplus 4}$$

where $U$ is the unimodular rank two hyperbolic lattice. We identify $H^4(S, \mathbb{Z}) = \mathbb{Z}$ where we use the Poincare dual to a point as a generator and similarly $H^0(S, \mathbb{Z}) = \mathbb{Z}$ by taking the Poincare dual of $S$.

**Definition 4.6** A Mukai vector is a triple $v = (r, c, s)$ in $H^0(S, \mathbb{Z}) \oplus H^1,1(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. It is called positive if one of the following cases are satisfied

(i) $r > 0$
(ii) $r = 0$, $c$ is effective and $s \neq 0$
(iii) $r = c = 0$ and $s < 0$

Let $v$ be a primitive Mukai vector on $S$. An ample divisor $H$ on $S$ is called $v$--generic if every $H$--semistable sheaf is $H$–stable. For a coherent sheaf $F \in \text{Coh}(S)$ set $v(F) := \text{ch}(F)^2$ which is a Mukai vector as easily verified. Choose a positive and primitive Mukai vector $v = (r, c, s)$ with $c \in \text{NS}(S)$ and $(v, v) \geq 6$ together with a $v$--generic ample class $H$. General results of S. Mukai [Muk84] imply that the moduli space $M_H(v)$ of $H$–stable sheaves $F$ with Mukai vector $v(F) = v$ is a projective holomorphic symplectic manifold but not irreducible. By [Yos01, Thm. 0.1] the Albanese torus of $M_H(v)$ is $S \times S^\vee$. Consider the Albanese map

$$\text{Alb}_v : M_H(v) \longrightarrow S \times S^\vee$$

and set $K_H(v) := \text{Alb}_v^{-1}(0, 0)$. Then we have dim $K_H(v) = (v, v) - 2 = 2n$ and by K. Yoshioka [Yos01, Thm. 0.2] this is an irreducible holomorphic symplectic manifold of Kummer type.

We have Mukai’s homomorphism of Hodge structures

$$\Theta_v : v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z})$$

\footnote{Note that $v(F) = (\text{rk}(F), c_1(F), c_2(F)/2 - c_2(F))$}
which can be defined as follows. Choose a quasi–universal family of sheaves \( \mathcal{E} \) on \( S \) of simplitude \( \rho \in \mathbb{N} \), cf. [Muk87, Thm. A.5]. That is a family of sheaves \( \mathcal{E} \in \text{Coh}(S \times M_H(v)) \) on \( S \) parametrized by \( M_H(v) \) (in particular, \( \mathcal{E} \) is flat over \( M_H(v) \)) and for every class \( F \in M_H(v) \) one has \( \mathcal{E}|_{F} = \mathcal{E}|_{S \times \{ F \}} \cong F^\oplus \rho \). Then set

\[
\Theta_v(x) := \frac{1}{\rho} \left[ \left( \text{pr}_{M_H(v)} \right)_! \left( \left( \text{ch}(\mathcal{E}) \left( \text{pr}_{S} \right)^* (\sqrt{\text{Td}(S)} x^\vee) \right) \right) \right],
\]

where \( x^\vee = -x_0 + x_2 + x_4 \) for \( x = x_0 + x_2 + x_4 \) and \([\cdot]\) denotes the part in \( H^2(S, \mathbb{Z}) \). Note that \( \sqrt{\text{Td}(S)} = 1 \) for an abelian surface \( S \). For the details see [Yos01, 1.2], [O’G97], [Muk87] and [Muk84].

By composing with the restriction map \( r : H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z}) \) we obtain a morphism

\[
\Theta_v : v^\perp \to H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z})
\]

which is an isometry of Hodge structures by [Yos01, Thm. 0.2] and which we also denote by \( \Theta_v \) by abuse of notation.

**Theorem 4.9** Let \( X \) be a manifold of generalized Kummer type of dimension \( 2n \geq 4 \). Then there exists a canonical monodromy invariant \( \text{O}(\tilde{\Lambda}) \)-orbit \( \iota_X \) of primitive isometric embeddings \( \Lambda = H^2(X, \mathbb{Z}) \to \tilde{\Lambda} \) into the Mukai lattice.

**Proof:** Let \( K_H(v) \) denote the manifold of generalized Kummer type described above such that \( \text{dim } X = \text{dim } K_H(v) \). Fix an isometry \( \varphi : H^*(S) \to \tilde{\Lambda} \) and let \( P : H^2(X, \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z}) \) be a parallel transport operator. Denote by \( \iota \) the primitive isometric embedding

\[
H^2(X, \mathbb{Z}) \xrightarrow{P} H^2(K_H(v), \mathbb{Z}) \xrightarrow{\Theta^{-1}_v v^\perp} \varphi \iota -\tilde{\Lambda}.
\]

Set \( \iota_X := \text{O}(\tilde{\Lambda}) \iota \). Let \( g \in \text{Mon}^2(X) \) denote a monodromy operator. By Theorem 2.34 \( g \) acts on \( H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z}) \) as \( \pm \text{id} \). By Lemma 4.3 \( g \) can be extended to an isometry \( \tilde{g} \) of \( \tilde{\Lambda} \) such that \( \iota \circ g = \tilde{g} \circ \iota \), i.e. the orbit \( \iota_X \) is monodromy invariant.

The orbit \( \iota_X \) is *canonical* in the following sense. We have made a choice of moduli spaces \( K_H(v) \subset M_H(v) \) of sheaves on an abelian surface \( S \) and therefore of Mukai’s homomorphism \( \Theta_v : v^\perp \to H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z}) \). It might be, that a different choice of moduli spaces and therefore of a different Mukai homomorphism could lead to another orbit of primitive isometric embeddings. With *canonical* we mean that we always end up with the same orbit.

This follows from K. Yoshioka’s method of proof of the main results in [Yos01, 4.3., Prop. 4.12., Proof of Thm. 0.1 and 0.2]. If we choose another irreducible holomorphic symplectic moduli space of dimension \( \text{dim } X \), then it is deformation equivalent to \( K_H(v) \) and Yoshioka’s proof for this statement uses deformations of moduli spaces of sheaves over families of surfaces [Yos01, Lem. 2.3], and Fourier–Mukai transforms for which the Mukai homomorphism varies continuously, see [Yos01, 2.2., Proof of Prop. 2.4.]. Therefore the \( \text{O}(\tilde{\Lambda}) \)-orbit does not change. \( \square \)

### 5. Monodromy Invariants

We start with basic facts about general monodromy invariants, as described in [Mar13, 5.3.]. In the next subsection the monodromy invariant for isotropic classes for the generalized Kummer case is constructed.

Let \( X \) be an irreducible holomorphic symplectic manifold. Let \( I(X) \subset H^2(X, \mathbb{Z}) \) denote a monodromy invariant subset, i.e. \( \text{Mon}^2(X) \cdot I(X) \subset I(X) \) and \( \Sigma \) a set.
Definition 5.1 [Mar13, Def. 5.16] A monodromy invariant of the pair \((X, e), e \in I(X)\), is a Mon\(^2\)(X)–invariant map \(\vartheta : I(X) \to \Sigma\) i.e. \(\vartheta(ge) = \vartheta(e)\) for all \(e \in I(X)\) and all \(g \in \text{Mon}\(^2\)(X)\). Further \(\vartheta\) is called faithful if the induced map \(\bar{\vartheta} : I(X)/\text{Mon}\(^2\)(X) \to \Sigma\) is injective.

5.2. Induced monodromy invariant subset. Let \(X'\) denote another irreducible holomorphic symplectic manifold deformation equivalent to \(X\). Let \(P : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})\) denote a parallel transport operator. Then we can define
\[
I(X') := P(I(X))
\]
to obtain a Mon\(^2\)(X') invariant subset \(I(X')\) of \(H^2(X', \mathbb{Z})\) induced by \(I(X)\). Indeed, this is well defined: if one has another parallel transport operator \(P' : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})\), then \(P'^{-1} \circ P\) is in Mon\(^2\)(X) hence \((P'^{-1} \circ P)(I(X)) = I(X)\) as \(I(X)\) is Mon\(^2\)(X) invariant. Hence \(P(I(X)) = P'(I(X))\).

Alternatively, we could define
\[
I(X') = \left\{ e' \in H^2(X', \mathbb{Z}) \mid \text{there exists } e \in I(X) \text{ such that } (X, e) \sim_{\text{def}} (X', e') \right\}.
\]
where the deformation equivalence of the pairs \((X, e)\) and \((X', e')\) is meant in the sense of Definition 2.7 as usual.

5.3. Induced monodromy invariant. Let \(X'\) be as above. If we have a monodromy invariant \(\vartheta : I(X) \to \Sigma\) then we can obtain an induced monodromy invariant on \(X'\) which we also denote by \(\vartheta : I(X') \to \Sigma\) by abuse of notation. If \(e' \in I(X')\) then there is a pair \((X, e)\) deformation equivalent to \((X', e')\) and we can define the induced monodromy invariant by
\[
\vartheta(e') := \vartheta(e).
\]
Note that this is well defined as \(\vartheta\) is Mon\(^2\)(X)–invariant.

The following is a very important statement for the computation of polarization types of Lagrangian fibrations and is based on the Global Torelli Theorem, see [Mar13, 5.2 ff.].

Proposition 5.4 [Mar13, Lem. 5.17] Let \(\vartheta : I(X) \to \Sigma\) be a faithful monodromy invariant and let \((X_i, e_i), i = 1, 2\), denote two pairs with \(X_i\) deformation equivalent to \(X\) and \(e_i \in I(X_i)\).

(i) \(\vartheta(e_1) = \vartheta(e_2)\) if and only if \((X_1, e_1)\) and \((X_2, e_2)\) are deformation equivalent.

(ii) If \(\vartheta(e_1) = \vartheta(e_2)\) and \(e_i = c_i(L_i)\) for holomorphic line bundles \(L_i\) on \(X_i\) and there exist Kähler classes \(\omega_i\) on \(X_i\) such that \((\omega_i, e_i) > 0\), then \((X_1, L_1)\) is deformation equivalent to \((X_2, L_2)\).

For effective isotropic classes, the requirements of the second statement of the Proposition above is always satisfied due to the following Lemma.

Lemma 5.5 [Wie16, Lem. 6.7] Let \(\lambda\) be a nontrivial isotropic class in the closure \(\bar{C}_X\) of the positive cone in \(H^{1,1}(X, \mathbb{R})\) with \(X\) an arbitrary irreducible holomorphic symplectic manifold. Then the Beauville–Bogomolov quadratic form satisfies \((x, \lambda) > 0\) for every class \(x\) in the positive cone \(\bar{C}_X\).

By definition the positive cone \(\mathcal{C}_X\) contains the Kähler cone \(\mathcal{K}_X\), therefore we always find Kähler classes as required in (ii) of Proposition 5.4, if the considered classes \(e_i\) are isotropic.
5.6. Monodromy invariants for isotropic classes. As one expects a close
relation between Lagrangian fibrations and isotropic line bundles, similar for K3 sur-
faces, we are interested in monodromy invariants defined on the subset of isotropic
classes of the second cohomology of an irreducible holomorphic symplectic manifold.
In this section a monodromy invariant for the isotropic classes on generalized Kummer
manifolds is constructed in analogy of [Mar11, 2.]

Let $X$ be a generalized Kummer type manifold of dimension $2n$. By Theorem
4.9 we have a canonical monodromy invariant $O(\tilde{\Lambda})$–orbit $\iota_X$ of primitive isometric
embeddings from $\Lambda := H^2(X, \mathbb{Z})$ into the Mukai lattice $\tilde{\Lambda}$ (4.5). Choose the following:

(i) A representative $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ in $\iota_X$.
(ii) A generator $v$ of the sublattice $\iota(\Lambda)^\perp = \langle v \rangle$, cf. Remark 4.2.

Remark 5.7 The Kummer type lattice $\Lambda$ has signature $(3,4)$, hence the orthogonal
complement $\iota(\Lambda)^\perp$ is positive definite of rank one as the Mukai lattice $\tilde{\Lambda} = U^{\oplus 4}$ has
signature $(4,4)$. Since the Gram discriminant of $\Lambda$ is $-(2n+2)$ the Gram discriminant
of $\iota(\Lambda)^\perp$ is $2n+2$, hence $\langle v, v \rangle = 2n+2$. Furthermore, by [Nik80, Thm 1.14.4] there
is a unique orbit of such primitive elements with square $2n+2$ (respectively $2n-2$
in the $K3^{[n]}$ case) in $\tilde{\Lambda}$. Since $\iota(\Lambda) = v^\perp$ we conclude that the action of $O(\Lambda) \times O(\tilde{\Lambda})$
on $O(\Lambda, \tilde{\Lambda})$ is transitive.

For a primitive and isotropic element $\alpha$ in the Kummer type lattice $\Lambda$ denote by
$H(\alpha, \iota)$ the lattice defined by

$$H(\alpha, \iota) := \text{sat} \langle \iota(\alpha), v \rangle = \text{sat} \langle \iota(\alpha), -v \rangle,$$

where sat denotes the saturation – the saturation of a sublattice $L$ is the maximal
sublattice of the same rank containing $L$.

Definition 5.9

(i) Let $\Lambda_1, \Lambda_2$ denote lattices and $e_i \in \Lambda_i$ elements. A morphism of the pairs $(\Lambda_i, e_i)$ is an isometry $g : \Lambda_1 \to \Lambda_2$ such that $g(e_1) = e_2$.
(ii) The divisibility or the divisor of an element $x \in \Lambda$ is defined as

$$\text{Div}(x) := \text{max} \{ k \in \mathbb{N} \mid \langle x, \cdot \rangle / k \text{ is an integral class in the dual } \Lambda^\vee \}.$$

Equivalently, $\text{Div}(x)$ is the unique positive generator of the ideal $\langle x, L \rangle = \text{Div}(x) \mathbb{Z} \subset \mathbb{Z}$. Note that if the lattice is unimodular, then $\text{Div}(x) = 1$ for
every primitive element $x$.

Denote by

$$\vartheta(\alpha) := [(H(\alpha, \iota), v)]$$

the isometry class of the pair $(H(\alpha, \iota), v)$.

Let $d$ be a positive number such that $d^2$ divides $2n+2$. Then define the lattice
$L_{n,d}$ as $\mathbb{Z}^2$ with form

$$\frac{2n + 2}{d^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The following Lemma is very similar to [Mar14, Lem. 2.5].

Lemma 5.12 Let $\alpha \in \Lambda$ be a primitive isotropic class and set $d := \text{Div}(\alpha)$.

(i) $\vartheta(\alpha)$ does not depend on the chosen representative $\iota \in \iota_X$.
(ii) For all $g \in \text{Mon}^2(X)$ we have $\vartheta(g(\alpha)) = \vartheta(\alpha)$. 
(iii) We can decompose $\alpha \in \Lambda \cong U^{3} \oplus \langle -2n - 2 \rangle$ as
$$\alpha = d\xi + b\delta$$
where $\xi \in U^{3}$ is primitive, $\delta$ is the generator of $\langle -2n - 2 \rangle$ and $\gcd(d,b) = 1$. Furthermore, $d^2$ divides $n + 1$.

(iv) The lattice $H(\alpha, \iota)$ is isometric to the lattice $L_{n,d}$ defined in (5.11).

(v) There is an integer $b$, namely the one in (iii), such that $(\iota(\alpha) - bv)/d$ is integral (i.e., contained in $H(\alpha, \iota)$). Also any integer $b$ with
- $\gcd(d,b) = 1$ and
- $(\iota(\alpha) - bv)/d$ is integral
satisfies $\vartheta(\alpha) = [(L_{n,d}, (d,b))]$.

Proof:  
(i) Let $\iota_{i} \in \iota_{X}$, $i = 1, 2$, be two representatives with $\iota_{i}(\Lambda) = \langle \nu_{i} \rangle$. Since the $\iota_{i}$ are in the same orbit $\iota_{X}$ there exists $\tilde{g} \in O(\Lambda)$ such that $\tilde{g} \circ \iota_{1} = \iota_{2}$. Hence $\tilde{g}(\iota_{1}(\nu_{1})) = \iota_{2}(\Lambda)$. We necessarily have $\tilde{g}(\nu_{1}) = \pm \nu_{2}$, otherwise we would have a contradiction to the bijectivity of $\tilde{g}$. We can assume $\tilde{g}(\nu_{1}) = \nu_{2}$ (otherwise take $-\tilde{g}$) then $\tilde{g}(\iota_{1}(\nu_{1})) = \langle \nu_{2} \rangle$ and the same holds for the saturation. Consequently $\tilde{g}$ gives the desired isometry of the pairs $(H(\alpha, \iota), \nu_{i})$ hence $\vartheta(\alpha)$ does not depend on the chosen $\iota$.

(ii) The orbit $\iota_{X} = O(\Lambda)\iota$ is monodromy invariant that means we have a $\tilde{g} \in O(\Lambda)$ such that $\tilde{g} \circ \iota = \iota \circ g$. With the same argument as in (i), we have $\tilde{g}(\nu) = \pm \nu$ (see Remark 4.2) and can assume $\tilde{g}(\nu) = \nu$. So $\tilde{g}$ defines an isometry between $(\iota(\alpha), \nu)$ and $(\iota(g(\alpha)), \nu)$ since $\tilde{g}(\iota(\alpha)) = \iota(g(\alpha))$ and in particular an isometry between the saturations $(H(\alpha, \iota), \nu)$ and $(H(g(\alpha), \iota), \nu)$, hence $\vartheta(\alpha) = \vartheta(g(\alpha))$.

(iii) Let $\delta$ be the generator of $\langle -2n - 2 \rangle \subset \Lambda$. Then $\delta_{\Lambda}^{\perp} = U^{3}$. Since $\alpha$ is primitive, we can write $\alpha = a\xi + b\delta$ such that $a > 0$ and $\xi \in \delta_{\Lambda}^{\perp} = U^{3}$ and $\gcd(a,b) = 1$. Then
$$0 = (\alpha, \alpha) = a^{2}(\xi, \xi) - (2n + 2)b^{2} \iff a^{2}(\xi, \xi) = (2n + 2)b^{2}.$$ 
As $(\xi, \xi)$ is even we get that $a^{2}$ divides $(n + 1)$. Since $\delta$ is primitive we have $\Div(\delta) = 2n + 2$ and $\Div(\xi) = 1$ as $\xi$ is primitive and $U^{3}$ is unimodular, hence
$$d = \Div(\alpha) = \gcd(\Div(a\xi), \Div(b\delta)) = \gcd(a, 2n + 2)b = a.$$

(iv),(v) We use the same notation as in (iii). The lattice $\iota(U^{3})^{\perp} \subset \tilde{A}$ is of rank 2 and contains $\iota(\delta)$ and $\nu$, hence it is the saturation of $\langle \iota(\delta), \nu \rangle$ as orthogonal complements are always saturated. As a complement of a unimodular lattice it is unimodular itself, hence it is the hyperbolic plane $U$. Consequently, we can assume that $\nu = (1, n + 1)$ and $\iota(\nu) = (1, -n - 1)$. We have $\iota(\delta) - \nu = (2n + 2)e$ where $e = (0, -1)$. Clearly $e$ is isotropic. Then set
$$u := \frac{1}{d}(bv - \iota(\alpha)) = -\iota(\xi) - \frac{b}{d}(2n + 2)e.$$ 
Hence, the existence of such an integer $b$ is proven.

As $\iota(\alpha) = -du + bv$ we have $(\nu, u) \subset H(\alpha, \iota) := \operatorname{sat}(\iota(\alpha), \nu)$. The complement $\delta_{\Lambda}^{\perp} = U^{3}$ is unimodular, hence we can find $\eta \in \delta_{\Lambda}^{\perp}$ such that $(\eta, \xi) = 1$ as $\xi \in U^{3}$ is primitive. For the intersection numbers we have
$$\begin{pmatrix}
(v, e) & (v, \iota(\eta)) \\
(u, e) & (u, \iota(\eta))
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}.$$
Therefore the sublattice \( \langle v, u \rangle \subset \tilde{\Lambda} \) must be saturated, otherwise the determinant of the matrix above must be divisible by a nontrivial square. Consequently we have \( H(\alpha, \iota) := \text{sat}(\iota(\alpha), v) = \langle v, u \rangle \).

Furthermore, \( (v, u) = b^{2n+2} \) and \( (u, u) = b^{2n+2} \). The Gram matrix \( G \) of \( H(\alpha, \iota) \) with respect to the basis \( v, u \) is therefore

\[
G = \frac{2n + 2}{d^2} \begin{pmatrix} d^2 & bd \\ bd & b^2 \end{pmatrix} = \frac{2n + 2}{d^2} \begin{pmatrix} d \\ b \end{pmatrix} \begin{pmatrix} d & b \end{pmatrix}.
\]

Since \( \gcd(d, b) = 1 \) there are integers \( i, j \in \mathbb{Z} \) with \( id + jb = 1 \). Set

\[
A := \begin{pmatrix} i & j \\ b & -d \end{pmatrix}.
\]

This is an integral matrix with \( A(d, b)^t = (1, 0)^t \) and determinant \(-1\), hence invertible over the integers. The Gram matrix with respect to the base change \( A \) is

\[
A^t GA = \frac{2n + 2}{d^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Therefore we have an isomorphism \( L_{n, d} \cong H(\alpha, \iota) \) of lattices via \( (x, y)^t :\to A(x, y)^t \), \( (v, u) \) where the product \( \cdot \) is seen as a formal euclidean product. In particular \( (d, b) \) is mapped to \( v \) i.e. \( \vartheta(\alpha) = [L_{n, d}, (d, b)] \).

Now let \( b' \) be any integer satisfying the assumptions in \( v \). We know that \( (d, b') \) is primitive and that

\[
u' := \frac{1}{d}(b'v - \iota(\alpha))
\]
is integral, therefore \( u' - u = \frac{b' - b}{d}v \) is also integral. Since \( v \) is primitive, \( d \) must divide \( b' - b \). Set \( c := \frac{b' - b}{d} \in \mathbb{Z} \). Then

\[
g_c := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in O(L_{n, d})
\]
is clearly an isometry of \( L_{n, d} \) with \( g_c(d, b)^t = (d, dc + b)^t = (d, b')^t \). Hence \( \vartheta(\alpha) = [L_{n, d}, (d, b)] = [L_{n, d}, (d, b')] \).

\[
\square
\]

**Lemma 5.13** The degenerate lattice \( L_{n, d} \) embeds primitively and isometrically into \( \tilde{\Lambda} = U^{\oplus 4} \), uniquely up to an isometry in \( O(\tilde{\Lambda}) \).

**Proof:** Follows by [BHPV03, Thm. 2.9].

**Lemma 5.14** Let \( \alpha \in \Lambda = U^{\oplus 3} \oplus \langle -2n - 2 \rangle \) be a primitive isotropic element in the Kummer type lattice. Then there exists a \( u \in \Lambda \) such that \( (u, \alpha) = 0 \) and \( (u, u) = \pm 2 \).

**Proof:** Write \( \alpha = \alpha_0 + \alpha_1 \) with \( \alpha_0 \in U^{\oplus 3} \) and \( \alpha \in \langle -2n - 2 \rangle \). The discriminant of \( U^{\oplus 3} \) is trivial since its unimodular, hence by Eichler’s criterion [Eic52, 10.1] the \( O(U^{\oplus 3}) \)-orbit of \( \alpha_0 \) is determined by it’s length \( (\alpha_0, \alpha_0) = 2n + 2 \). So there exists an isometry \( g \in O(U^{\oplus 3}) \) such that \( g(\alpha_0) = ((1, n + 1), 0, 0) \in U^{\oplus 3} \). Set \( u := g^{-1}(0, 0, (1, \pm 1)) \in U^{\oplus 3} \subset \Lambda \). Then \( (u, \alpha) = (u, \alpha_0) = 0 \) and \( (u, u) = ((1, \pm 1), (1, \pm 1)) = \pm 2 \).

\[
\square
\]
For a positive integer $d$ let $I_d(X) \subset \Lambda = H^2(X, \mathbb{Z})$ denote the subset of primitive isotropic elements $\alpha$ such that $\text{Div}(\alpha) = d$ which is clearly a $\text{Mon}^2(X)$–invariant subset. Let $\Sigma_{n,d}$ denote the set of isometry classes of pairs $(H, w)$ such that $H$ is isometric to $L_{n,d}$ and $w \in H$ is a primitive class with $(w, w) = 2n + 2$.

**Theorem 5.15** Let $X$ be a Kummer type manifold of dimension $2n$ and $d$ a positive integer such that $d^2$ divides $n + 1$. The map

$$\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}, \quad \alpha \mapsto \vartheta(\alpha) = [(H(\alpha, t), v)]$$

is a surjective faithful monodromy invariant of the manifold $X$.

**Proof:** By Lemma 5.12 $\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}$ is well defined and $\text{Mon}^2(X)$–invariant.

To show that $\vartheta$ is faithful i.e. that the induced map $\vartheta : I_d(X)/\text{Mon}^2(X) \longrightarrow \Sigma_{n,d}$ is injective, we assume $\alpha_1, \alpha_2 \in I_d(X)$ with $\vartheta(\alpha_1) = \vartheta(\alpha_2)$, that means we have an isometry $g : H(\alpha_1, t) \to H(\alpha_2, t)$ with $g(v) = v$ where $v$ is as usual a generator of $\iota(\Lambda)^{\perp}$.

We first show that both $\alpha_i$ lie in the same $\mathcal{W}(\Lambda)$–orbit, where the group $\mathcal{W}(\Lambda)$ was defined in Definition 2.30. We have $H(\alpha_1, t) \cong L_{n,d}$. By Lemma 5.13 there is up to an isometry in $O(\Lambda)$ a unique way to embed $H(\alpha_1, t)$ isometrically and primitively into $\hat{\Lambda}$, hence we can extend $g$ to an isometry $\hat{g} \in O(\hat{\Lambda})$. Since $v^{\perp} = \iota(\Lambda)$ we have in particular $\hat{g}(\iota(\Lambda)) = \iota(\Lambda)$, i.e it makes sense to set $h := \iota^{-1} \circ \hat{g} \circ \iota$ which is an isometry $h \in O(\Lambda)$ such that $\iota \circ h = \hat{g} \circ \iota$, hence $h$ leaves the orbit $\iota_X = O(\Lambda)t$ invariant and by Lemma 4.4 either $\mu = h$ or $\mu = -h$ is contained in the subgroup $\mathcal{W}(\Lambda)$ of orientation preserving isometries acting as $\pm 1$ on the discriminant $\Lambda^2/\Lambda$.

Choose $\mu$ such that it is in $\mathcal{W}(\Lambda)$. The null space of $H(\alpha_i, t) \subset \hat{\Lambda}$ is generated by $\iota(\alpha_i)$. Since $\hat{g} \in O(\hat{\Lambda})$ restricts to an isometry between $H(\alpha_1, t)$ and $H(\alpha_2, t)$ the null space of $H(\alpha_2, t)$ is generated by $\hat{g}(\iota(\alpha_1)) = \iota(\pm h(\alpha_1)) = \iota(\mu(\alpha_1))$. So we have $\iota(\mu(\alpha_1)) = \pm \iota(\alpha_2)$, hence $\mu(\alpha_1) = \pm \alpha_2$. By Lemma 5.14 we can choose a $u \in \Lambda$ with $(u, \alpha_2) = 0$ and $(u, u) = +2$. Then the isometry $\rho_u \in O(\Lambda)$ defined in Definition 2.32 i.e. $\rho_u(x) = -R_u(x) = -x + (u, x)u$ is contained in $\mathcal{W}(\Lambda)$, see Corollary 2.35 and Remark 2.33, and satisfies $\rho_u(\alpha_2) = -\alpha_2$, hence

$$\mathcal{W}(\Lambda)\alpha_1 = \mathcal{W}(\Lambda)(\pm \alpha_2) = \mathcal{W}(\Lambda)\alpha_2.$$ 

Now we show that $\mathcal{W}(\Lambda)\alpha = \text{Mon}^2(X)\alpha$ for every primitive isotropic element $\alpha \in \Lambda$. Since $\text{Mon}^2(X) \subset \mathcal{W}(\Lambda)$ is an index 2 subgroup by Corollary 2.35 we can write

$$\mathcal{W}(\Lambda) = \text{Mon}^2(X) \cup \text{Mon}^2(X)w$$

for every $w \in \mathcal{W}(\Lambda) \setminus \text{Mon}^2(X)$. By Lemma 5.14 we have an element $u \in \Lambda$ with $(u, \alpha) = 0$ and $(u, u) = -2$. Then the reflection $\rho_u(x) = R_u(x) = x + (u, x)u$ of $\Lambda$ (Definition 2.32) acts as $+1$ on the discriminant but has determinant $-1$, hence it is contained in $\mathcal{W}(\Lambda)$ but not in $\text{Mon}^2(X)$, see again Corollary 2.35 and Remark 2.33. In particular $\rho_u(\alpha) = \alpha$ therefore

$$\mathcal{W}(\Lambda)\alpha = \left(\text{Mon}^2(X) \cup \text{Mon}^2(X)\rho_u\right)\alpha$$

$$= \text{Mon}^2(X)\alpha \cup \text{Mon}^2(X)\rho_u(\alpha) = \text{Mon}^2(X)\alpha.$$ 

For surjectivity, assume we have a class $[(L_{n,d}, w)] \in \Sigma_{n,d}$, i.e. $w \in L_{n,d}$ is primitive such that $(w, w) = 2n + 2$. By Lemma 5.13 there exists a primitive isometric embedding $\iota_{n,d} : L_{n,d} \hookrightarrow \Lambda$. 


By Eichler’s criterion \[\text{Eic52, 10.}\] we can assume that \(t_{n,d}(w)\) is contained in a copy of \(U\) of \(\Lambda = U^\oplus 4\). Then the lattice \(t_{n,d}(w)\downarrow \subset \Lambda = U^\oplus 4\) is of signature \((3, 4)\) and since \((w, w) = 2n + 2\) the complement \(t_{n,d}(w)\downarrow\) is isomorphic to \(\Lambda \cong U^\oplus 3 \oplus \langle -2n - 2 \rangle\).

The action of \(O(\Lambda) \times O(\Lambda)\) on \(O(\Lambda, \Lambda)\) is transitive by Remark 5.7, hence the induced action of \(O(\Lambda)\) on the orbit set \(O(\Lambda, \Lambda)/O(\Lambda)\) is also transitive. Hence, we can choose an isometry \(g : t_{n,d}(w)\downarrow \to \Lambda\) such that

\[
\kappa : \Lambda \xrightarrow{g^{-1}} t_{n,d}(w)\downarrow \subset \Lambda
\]

belongs to the monodromy invariant orbit \(\nu_X = O(\Lambda)\). Recall from above that \((0, 1) \in \ker L_{n,d}\) is the generator of \(\ker L_{n,d}\). Clearly we have \((w, (0, 1)) = 0\) in \(L_{n,d}\) so we can set \(\alpha := g(t_{n,d}(0, 1))\). We can write

\[
\alpha = a\xi + b\delta
\]

where \(\xi \in U^\oplus 3\), \(\delta \in \langle -2n - 2 \rangle\), \(a > 0\) such that \(\gcd(a, b) = 1\). As in the proof of Lemma 5.12 (iv) it follows that \(a = \text{Div}(\alpha)\). We have \(\kappa(\Lambda) = (\kappa)\) and \(\kappa(\alpha) = t_{n,d}(0, 1)\) and from Lemma 5.12 again

\[
H(\alpha, \kappa) = \text{sat} \langle t_{n,d}(0, 1), t_{n,d}(w) \rangle \cong L_{n,a},
\]

where \(t_{n,d}(w)\) is mapped to \((a, b)\). The primitive element \(w \in \ker L_{n,d}\) is necessarily of the form \((\pm d, w_2)\) with \(\gcd(d, w_2) = 1\). Over the rational numbers we have clearly \(t_{n,d}(L_{n,d})_\mathbb{Q} = \langle t_{n,d}(0, 1), t_{n,d}(w) \rangle_\mathbb{Q} \). As \(t_{n,d}(L_{n,d})\) is saturated it follows that

\[
t_{n,d}(L_{n,d}) = \text{sat} \langle t_{n,d}(0, 1), t_{n,d}(w) \rangle = H(\alpha, \kappa).
\]

Now we have an isometry

\[
L_{n,d} \xrightarrow{t_{n,d}} H(\alpha, \kappa) \to L_{n,a}
\]

where \(w\) is mapped to \((a, b)\), hence \(\text{Div}(\alpha) = a = d\) i.e. \(\alpha \in L_d(X)\) and \(\vartheta(\alpha) = [(L_{n,d}, w)]\).

\[\square\]

### 6. Beauville–Mukai systems of generalized Kummer type

We define the notion of a **Beauville–Mukai system of generalized Kummer type**. It is similarly defined as in the \(K3^n\) case, see \[\text{Wie16}\]. The fibers of them are not Jacobian of curves anymore, but an abelian subvariety of a Jacobian. Therefore we dwell on some theory of complementary subvarieties in Jacobians, see subsection 6.4.

Let \(S\) be an abelian surface and \(v\) be a primitive Mukai vector on \(S\) of the form \(v = (0, c_1(D), s)\) where \(D\) is an ample divisor on \(S\) i.e. \(D\) is ample. We set \(2n := (D, D) - 2\). Note that we have \(h^0(S, D) = \frac{1}{2}(D, D) = n + 1\). Choose a \(v\)-generic ample class \(H\) on \(S\), hence \(M_H(v)\) is an holomorphic symplectic manifold as explained in section 4.

For simplicity we now fix a reference point \(F_0 \in M_H(v)\) such that \(\det(F_0) = O_S(D)\). By \[\text{Yos01}\] the Albanese map \(\text{Alb}_v : M_H(v) \to S \times S^\vee\) with respect the reference point \(F_0 \in M_H(v)\) can be written as

\[
(\text{Alb}_v)F_0 = \alpha \times \det F_0
\]

where \(\det F_0 : M_H(v) \to \text{Pic}^0(S) = S^\vee\) is defined as \(\det F_0(F) := \det(F) \otimes (\det(F_0))^{-1}\) and \(\alpha\) can be defined as

\[
\alpha(F) := \sum c_2(F) := \sum_i n_i x_i
\]

(6.1)
where we view $c_2(F)$ in the Chow ring represented by the cycle $[\sum_i n_i x_i]$, see [Yos01, 4.1 ff.] and [OG14, p. 11].

The Albanese fiber $K_H(v) = (\text{Alb}_v)^{-1}(0,0)$ is an irreducible holomorphic symplectic manifold of dimension $2n$ see section 4 and for $F \in K_H(v)$ the fitting support $\text{supp}(F)$ is an element of the linear system $|D|$. This leads to the following commutative diagram

$$(6.2) \quad K_H(v) \xrightarrow{\pi} M_H(v) \xrightarrow{(\text{Alb}_v)_{F_0}} S \times S^\vee \xrightarrow{\text{pr}_{S^\vee}} \{D\} \xrightarrow{\Phi_L} S^\vee = \text{Pic}^0(S)$$

where $\{D\} \to S^\vee$ is the map $C \mapsto \mathcal{O}_S(C) \otimes \det(F_0)^{-1}$. The induced map $K_H(v) \to |D|$ is a Lagrangian fibration by Matsushita’s Theorem 2.4.

**Definition 6.3** In the setting as above, the Lagrangian fibration

$$\pi : K_H(v) \to |D|, \quad F \mapsto \text{supp}(F)$$

is called a Beauville–Mukai system of generalized Kummer type.

### 6.4. An excursion to the theory of Jacobians.
To consider the fibers of Beauville–Mukai systems of generalized Kummer type and polarizations on them, we deal with some theory of complementary abelian subvarieties.

If $M$ is an abelian variety, $A \subset M$ is an abelian subvariety and $L$ a polarization on $M$, then one can define a so called complementary subvariety $B$ to $A$ (with respect to $L$). We only consider the case when $L = \Theta$ is a principal polarization [BL03, 12.1], for the more general setting see [BL03, 5.3]. We denote the induced isogeny of $L$ by $\phi_L$.

We assume for this section, that $\Theta$ is a principal polarization, therefore we can identify $M$ with its dual $M^\vee$ via the homomorphism $\phi_{\Theta}$. By [BL03, Prop. 1.2.6] for any polarization $L$ the isogeny $\phi_L$ has always a $\mathbb{Q}$–inverse and we can define the $\mathbb{Q}$–endomorphism

$$g_A := \iota \circ \phi_{\Theta}^{-1} \circ \iota^\vee : M \otimes \mathbb{Q} \to M \otimes \mathbb{Q}$$

where $\iota = \iota_A : A \hookrightarrow M$ denotes the inclusion. Choose a positive number $m$ such that $mg_A$ is an endomorphism of $M$. By [BL03, Prop. 12.1.3] we have

$$B := (\ker(mg_A))_0 \subset M = \ker \iota^\vee \cong (A/B)^\vee,$$

where $(\ker(mg_A))_0$ denotes the identity component. Furthermore, $B$ is an abelian subvariety of $M$ called the complementary subvariety to $A$ (with respect to $L$). Conversely, $A$ is also the complementary subvariety to $B$ and $(A, B)$ is called a pair of complementary subvarieties in $M$.

**Proposition 6.6** [BL03, Cor. 12.1.5] Let $(A, B)$ be a pair of complementary abelian subvarieties in a principally polarized abelian variety $(M, \Theta)$ with $\dim A \geq \dim B = r$. Denote by $\iota_A$ and $\iota_B$ the inclusions of $A$ and $B$ into $M$ respectively and assume $\text{deg}(\iota_A) = (d_1, \ldots, d_r)$. Then $\text{deg}(\iota_B) = (1, \ldots, 1, d_1, \ldots, d_r)$.

### 6.7. The case of a Jacobian.
Let $\iota : C \to S$ be a smooth curve in an abelian surface $S$. Denote by $\Theta$ the principal polarization of the Jacobian $\text{Jac}(C)$ of $C$ and define $K(C) := \ker \text{Jac}(\iota)$ to be the kernel of the homomorphism $\text{Jac}(\iota)$ induced by
the inclusion \( \iota : C \hookrightarrow S \) and by the universal property of the Jacobian [BL03, 11.4.1.], i.e. we have an exact sequence
\[
K(C) \hookrightarrow \Jac(C) \xrightarrow{\Jac(\iota)} S.
\]
Using the several identifications of the dual and double dual, the dual of the pullback or the double pullback
\[
(\nu^*)^\vee = (\nu^*)^* : \Jac(C) = (\Jac(C))^\vee = \Pic^0(C) \longrightarrow S = (S^\vee)^\vee = \Pic^0(S^\vee)
\]
is nothing but the map \( \Jac(\iota) \). Of course, you can also see \( \Jac(\iota) \) as the Albanese map induced by \( \iota \)
\[
\Alb(\iota) : \Alb(C) \longrightarrow \Alb(S) = S
\]
if you identify \( \Alb(C) \) and \( \Jac(C) \), cf. [BL03, Prop. 11.11.6]. More concretely, the map \( \Jac(\iota) \) viewed as a map \( \Pic^0(C) \to S \) is
\[
(6.8) \quad \mathcal{O}_C \left( \sum n_i x_i \right) \longrightarrow \sum n_i x_i.
\]
Indeed, for a given point \( c \), denote by \( \alpha_c : C \hookrightarrow \Jac(C) \), \( x \mapsto \mathcal{O}_C(x - c) \) the Abel–Jacobi map. Then
\[
\Jac(\iota)(\alpha_c(x)) = \Jac(\iota)(\mathcal{O}_C(x - c)) = x - c = \iota^{-c}(x),
\]
therefore it satisfies exactly the property of the unique morphism as described in [BL03, 11.4.1.].

We can see the dual \( S^\vee \) as an abelian subvariety of \( \Jac(C) \) in the following sense.

**Lemma 6.9** The pullback morphism \( \iota^* : S^\vee \to \Jac(C) \) is an injection. Therefore \( K(C) \) is connected.

**Proof:** Let \( L \) be a line bundle on \( S \) with \( L|_C = \mathcal{O}_C \). We have the standard exact sequence
\[
(6.10) \quad 0 \longrightarrow L \otimes \mathcal{O}_S(-C) \longrightarrow L \longrightarrow L|_C = \mathcal{O}_C \longrightarrow 0.
\]
Since \( C \) is effective, the line bundle \( L(-C) = L \otimes \mathcal{O}_S(-C) \) has no holomorphic sections i.e. \( H^0(S, L(-C)) = 0 \). In particular, \( L(-C) \) cannot be ample (cf. [BL03, Prop. 4.5.2]), therefore the associated hermitian form of \( c_1(L(-C)) \) must have less then four positive eigenvalues. By [BL03, Lem. 3.5.1] we then have \( H^1(S, L^\vee(-C)) = 0 \). The long exact sequence of (6.10) shows that \( h^0(S, L) = h^0(S, \mathcal{O}_C) = 1 \) i.e. \( L \) has a holomorphic section \( s \). Since \( 0 = c_1(L|_C) = [V(s)] \), the zero set \( V(s) \) of \( s \) is empty i.e. \( L = \mathcal{O}_S(V(s)) = \mathcal{O}_S(0) = \mathcal{O}_S \).

For the second statement identify \( \Jac(C)^\vee = \Jac(C) \) via the principal polarization. We have the short exact sequence
\[
0 \longrightarrow S^\vee \longrightarrow \Jac(C) \longrightarrow \Jac(C)/S^\vee \longrightarrow 0
\]
and by [BL03, Prop. 2.4.2] the dual sequence
\[
0 \longrightarrow (\Jac(C)/S^\vee)^\vee \longrightarrow \Jac(C) \longrightarrow S \to 0
\]
is also exact. Hence \( K(C) = \ker(\Jac(C) \to S) \cong (\Jac(C)/S^\vee)^\vee \) i.e. \( K(C) \) is connected. \( \square \)

In other words we have the following.

**Lemma 6.11** The abelian subvarieties \( K(C) \) and \( S^\vee \hookrightarrow \Jac(C) \) are a pair complementary abelian subvarieties of \( \Jac(C) \).
Lemma 6.12 Let $L$ be a polarization on an abelian surface $S$ of type $d(L) = (d_1, d_2)$. Then $h^0(S, L) = d_1d_2$. If $C \subseteq |L|$ is a not necessarily smooth curve, then we have for its arithmetic genus $g_a = d_1d_2 + 1$. Furthermore, if $c_1(L)$ is primitive and $(L, L) = 2d$, then $d(L) = (1, d)$.

Proof: By the well known formula for the (arithmetic) genus, we have $g_a = 1 + \frac{1}{2}(C, C)$. By the geometric Riemann–Roch [BL03, 3.6 ff.] and since $L$ is ample, we have

$$d_1d_2 = \chi(L) = h^0(S, L) = \frac{1}{2}(C, C) = g_a - 1.$$ 

If $(L, L) = 2d$ and $c_1(L)$ is primitive, the equation above also shows $2d = (L, L) = 2d_1d_2$. Since $d_1$ divides $d_2$ and $(d_1, d_2)$ is primitive as $c_1(L)$ is primitive, we have $d_1 = 1$ i.e. $d(L) = (1, d)$.

Remark 6.13 If $L$ is a polarization on an abelian variety $S$ with $d(L) = (d_1, \ldots, d_n)$, then by [BL03, 14.4] there is a natural polarization $L_\delta$ on the dual $S^\vee$, called dual polarization, characterized by the following equivalent properties

(a) $\phi_L^* L_\delta$ is algebraically equivalent to $L^{d_1d_n}$, (b) $\phi_L^* L_\delta = d_1d_nL_S$.

Furthermore, the type is given by $d(L_\delta) = (d_1, d_1d_{n-1}/d_2, \ldots, d_1d_{n-1}/d_2, d_n)$. If we are on an abelian surface, then obviously $d(L) = d(L_\delta)$.

Lemma 6.14 Let $(S, L)$ denote a polarized abelian surface of type $d(L) = (d_1, d_2)$. Then for every smooth curve $\iota : C \to S$ with $C \subseteq |L|$ the restriction

$$\Theta|_{S^\vee} := (\iota^*)^* \Theta$$

is a polarization of type $(d_1, d_2)$, where $\Theta$ denotes the principal polarization on $\text{Jac}(C) = \text{Pic}^0(C)$ and $(\iota^*)^*$ is viewed as a map $\text{Pic}(\text{Jac}(C)) \to \text{Pic}(S^\vee)$. In particular, if the Picard number is $\rho(S) = 1$, then $\Theta|_{S^\vee} = L_\delta$ where the latter is the dual polarization on $S^\vee$ to $L$, cf. Remark 6.13.

Proof: The proof is divided in three steps. In the first, we assume for the Picard number $\rho(S) = 1$ and show the existence of such a curve in $|L|$. In the second and still under the assumption $\rho(S) = 1$ it is shown that it holds for every smooth curve in $|L|$. In the third step we drop the restriction on the Picard number. We set $d := d_1d_2$.

- We first assume $\rho(S) = 1$ for the Picard number and prove the existence of such a curve $C \subseteq |L|$. Since $\rho(S) = 1$ we have also $\rho(S^\vee) = 1$. Note that we have $d(L_\delta) = (d_1, d_2)$ by Remark 6.13.

Consider the isogeny $\phi_L : S \to S^\vee$. Then

$$\ker \phi_L \cong \left( \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_1\mathbb{Z} \right) \oplus \left( \mathbb{Z}/d_2\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \right)$$

by [BL03, Lem. 3.1.4]. On $\ker \phi_L$ we have the alternating Weil pairing

$$e : \ker \phi_L \times \ker \phi_L \to \mathbb{C}^*$$

3In [BL03] they use the notation $K(L)$ for $\ker \phi_L$. 

Proof: With the discussion above we have $K(C) = \ker(\iota^*)^\vee$ which is exactly the definition as in (6.5)
see [BL03, p. 160], for the special case of an abelian surface see also [BL03, Ex. 6.7.3]. For \([x] = ([x_i]), [y] = ([y_i]) \in \ker \phi_L\) with respect to the isomorphism in (6.15), the pairing \(e\) can be calculated as
\[
e([x], [y]) = \exp \left( \frac{2\pi i}{d_1} (x_3y_1 - x_1y_3) \right) \cdot \exp \left( \frac{2\pi i}{d_2} (x_4y_2 - x_2y_4) \right),
\]
see [BL03, Ex. 6.7.3].

Choose a subgroup \(G \subset \ker \phi_L\) such that \(G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}\) and which is isotropic with respect to the pairing above.

Then \(\phi_L\) factorizes as
\[
\begin{diagram}
S & \xrightarrow{\phi_L} & S^\vee \\
\downarrow{p} & & \downarrow{p^*} \\
S/G & \xrightarrow{\rho} & S/G
\end{diagram}
\]
where \(p\) is the canonical projection which is a \(d = d_1d_2\) to 1 map. As \(G\) is isotropic, by [BL03, Prop. 6.7.1] the action of \(G\) on \(S\) lifts to a free action of \(G\) on \(L\), in particular we can define \(L_0 := L/G \in \text{Pic}(S/G)\). Since \(p\) is of degree \(d\) and \(p^*L_0 = L\), we have
\[
2d = (L, L) = (p^*L, p^*L) = d(L_0, L_0),
\]
i.e. \((L_0, L_0) = 2\), therefore \(L_0\) is a principal polarization on \(S/G\). Hence, \(H^0(S/G, L_0) = \mathbb{C}\sigma\) for a nontrivial section \(\sigma\). Define the curve \(C_0 := V(\sigma)\).

Then \(C_0\) is an element of \(\vert L_0\vert\) and we claim that \(C_0\) is smooth and irreducible.

Indeed, assume \(C_0 = C_1 + C_2\). Since \(\rho(S/G) = 1\) we have \(C_1 = m_1L_0\) and \(C_2 = m_2L_0\) with positive integers \(m_i\). Then
\[
2 = C_0^2 = (C_1 + C_2)^2 = (m_1 + m_2)^2(L_0, L_0) = 2(m_1^2 + m_2^2 + 2m_1m_2) > 2,
\]
which is absurd.

If \(C_0\) is not smooth, then let \(\nu : \tilde{C}_0 \to C_0\) be its normalization. For its genus we have \(g(\tilde{C}_0) < g_\sigma(C_0) = 2\). If \(g(\tilde{C}_0) = 0\), then \(\tilde{C}_0 = \mathbb{P}^1\) which is absurd, since \(\nu : \mathbb{P}^1 \to C_0 \to S/G\) would be a non constant regular map which is not possible. If \(g(\tilde{C}_0) = 1\) then \(\tilde{C}_0\) would be an elliptic curve which can be seen as an abelian subvariety of \(S/G\) after a translation, if necessary. Then \(\tilde{C}_0\) has a complementary abelian subvariety in the sense as above. This would mean \(\rho(S/G) \geq 2\) which contradicts \(\rho(S/G) = 1\).

We conclude that \(C_0\) is irreducible and smooth. In particular, \(C_0\) is of genus 2 and by Lemma 6.9, \((S/G) \cong (S/G)^\vee\) embeds into \(\text{Jac}(C_0)\). Both have the same dimension, hence \(S/G \cong \text{Jac}(C_0)\).

Set \(C := p^{-1}(C_0)\). Then \(C\) is an element of \(\vert L\) as \(L = p^*L_0\) and is smooth as \(p\) is etale. It has to be connected with a similar argument as above. Assume \(C = C_1 \cup C_2\) is a disjoint union. As \(\rho(S) = 1\) we have \(C_i = m_iL_i\) for positive integers \(m_i\) where \(L_i\) is the primitive part of \(L\). Then
\[
0 = (C_1, C_2) = (m_1L', m_2L') = 2m_1m_2 \frac{d_2}{d_1} > 0,
\]
which is absurd.

Hence, \(C\) is a connected smooth curve.

Denote by \(i : C \hookrightarrow S\) the inclusion, by \(q := p|_C = p \circ i : C \to C_0\) the induced \(d\) to 1 cover and by \(\Theta_0\) the principal polarization on \(\text{Jac}(C_0)\). Since \(\rho(S) = 1\), also \(\rho(S') = 1\) and \(\rho(\text{Jac}(C_0)) = 1\), so we have for the pullback
\((p^*)^*L_\delta = k\Theta_0\) for some positive integer \(k\). As \(p^*\) is surjective of degree \(d\), taking the self intersection on both sides gives
\[
2k^2 = (k\Theta_0, k\Theta_0) = ((p^*)^*L_\delta, (p^*)^*L_\delta) = d(L_\delta, L_\delta) = 2d^2
\]
and hence \(k = d\) i.e.
\[
(p^*)^*L_\delta = d\Theta_0.
\]
As \(q = p \circ \iota\) we can split \(q^*\) as
\[
q^* : \text{Jac}(C_0) \xrightarrow{p^*} S^\vee \xrightarrow{\iota^*} \text{Jac}(C).
\]
Since \(\rho(\text{Jac}(C_0)) = 1\), we have \((q^*)^*\Theta = a\Theta_0\) for some positive integer \(a\). By [BL03, Lem. 12.3.1] \((q^*)^*\Theta\) is algebraically equivalent to \(d\Theta_0\). Therefore \(ac_1(\Theta_0) = c_1((q^*)^*\Theta) = d\Theta_0\), hence \(a = d\) i.e.
\[
(q^*)^*\Theta = d\Theta_0.
\]
Finally write \((\iota^*)^*\Theta = bL_\delta\) for some positive integer \(b\). We have
\[
d\Theta_0 = (q^*)^*\Theta = (i^* \circ p^*)^*\Theta = (p^*)^*(\iota^*)^*\Theta = (p^*)^*(bL_\delta) = bd\Theta_0,
\]
hence \(b = 1\) i.e. \((\iota^*)^*\Theta = L_\delta\).

- We show that the statement holds for every element in \(|L|\) but still assume \(\rho(S) = 1\) for the Picard number.

Consider the open and connected set \(U \subset |L| \cong \mathbb{P}^{d-1}\) such that every element in \(U\) corresponds to a smooth curve in \(S\). Let \(C \to U\) be the associated family of smooth curves. We can take the relative Jacobian
\[
\pi_k : X^k := \text{Pic}^k(C/U) \longrightarrow U
\]
of degree \(k \in \mathbb{Z}\) of it.

By Lemma 6.12 the genus of \(C_t\) is \(g = d + 1\). By considering the image of \((X^d)^{(d)} \to X^d, (x_1, \ldots, x_d) \mapsto \sum x_i\) which is a divisor in \(X^d\), we obtain a line bundle \(M \in \text{Pic}(X^d)\) such that \(M_t := M|_{X^d_t}\) is the natural polarization on \(X^d_t = \text{Pic}^d(C_t)\).

Locally we can identify \(X^d\) with \(X^0\), say \(X^0_\delta = \pi_\delta^{-1}(V) \cong X^0_\delta = \pi_0^{-1}(V)\) where \(V \subset U \subset |L|\) is chosen connected, by twisting with a line bundle \(Q^V\) on \(\pi_\delta^{-1}(V)\) which has degree \(-d\) on the fibers \(C_t\) for \(t \in V\). Then we obtain on a line bundle \(L^V = M \otimes Q^V\) on \(X^0_\delta\), such that \(L^V_t := L^V|_{X^0_\delta}\) is the principal polarization on \(X^0_t = \text{Jac}(C_t)\) for \(t \in V\). Let \(\iota_t : C_t \to S\) denote the inclusion. Then the self intersection \(m_V : V \to \mathbb{Z}\)
\[
m_V(t) := \left((\iota_t^*)^*L^V_t, (\iota_t^*)^*L^V_t\right)
\]
of \((\iota_t^*)^*L^V_t\) is a continuous and integer valued function, therefore must be constant as \(V\) is chosen connected.

By the first part we know that there is an element \(t_0 \in U\) such that the statement for the curve \(C_{t_0}\) holds. For arbitrary \(t_N \in U\), choose a path \(\gamma\) from \(t_0\) to \(t_N\) in \(U\). By the discussion above, we can cover \(\gamma\) with finitely many connected open sets \(V_0, \ldots, V_N\) such that \(t_0 \in V_0\) and \(t_N \in V_N\) and we have elements \(t_i \in V_i \cap V_{i+1}\) for \(i = 1, \ldots, N - 1\). Then the self intersections \(m_{V_i}\) and \(m_{V_{i+1}}\) must coincide on \(V_i \cap V_{i+1}\).

By assumption, we have
\[
m_{V_0}(t_0) = (L_\delta, L_\delta) = 2d
\]
i.e. $mV_{0} \equiv 2d$. Assume $(\iota_{N}^{*})^{*}\mathcal{L}_{tN} = k\mathcal{L}_{\delta}$ for some positive integer $k$. Then
\[ 2d = mV_{0}(t_{0}) = mV_{1}(t_{1}) = \ldots = mV_{N}(t_{N}) = k^{2}2d \]
i.e. $k = 1$.

- We now consider the general case i.e. let $S$ be with arbitrary Picard number. We have an universal family $p : \mathcal{X} \to h_{2}$ of $(d_{1}, d_{2})$–polarized abelian surfaces over Siegel’s upper half plane $h_{2}$, see [BL03, 8.7]. Let $\mathcal{N}$ denote the line bundle on $\mathcal{X}$ such that $\mathcal{N}_{s} := \mathcal{N}|_{X_{s}}$ is the $(d_{1}, d_{2})$ polarization on $X_{s}$ for $s \in h_{2}$.

For each $s \in h_{2}$ let $U_{s} \subset |\mathcal{N}_{s}| \cong \mathbb{P}^{d-1}$ be the open set such that all elements in $U_{s}$ corresponds to smooth curves in $X_{s}$. Let $U \subset \mathbb{P}^{d-1}$ denote the open and connected subset such that for every for every $(s, t) \in h_{2} \times U$ the point $t \in \mathbb{P}^{d-1} \cong |\mathcal{N}_{s}|$ corresponds to an element in $U_{s}$. In particular it corresponds to a smooth curve $C^{s}_{t}$ in $X_{s}$. Let $\iota_{s,t} : C^{s}_{t} \hookrightarrow X_{s}$ denote the inclusion.

From the second step of the proof we know that for each $(s, t) \in h_{2} \times \mathbb{P}^{d-1}$ we can find a neighbourhood $V_{s,t} \subset U_{s}$ of $t$ and a relative principal polarization $\mathcal{L}^{s,t}$ on $\text{Pic}^{0}(C^{s}_{t}/U_{s})|_{V_{s,t}}$, where $C^{s}_{t} \to U_{s}$ denotes the associated family of smooth curves to $U_{s}$.

We can define the map
\[ \varphi : h_{2} \times U \longrightarrow \mathbb{Z}^{2}, \quad (s, t) \longmapsto \mathfrak{d}\left((\iota_{s,t}^{*})^{*}\mathcal{L}^{s}_{t}\right) \]
for the case that $(s, t) \in h_{2} \times V_{s,t}$. This is well defined and continuous, therefore must be constant as $U$ is connected. It is well known, see [BL03, 8.11, (1)], that the generic abelian surface has endomorphism ring $\text{End} = \mathbb{Z}$ i.e. has Picard number $\rho = 1$, by Lemma 6.25. Therefore the statement proven in the second step applies for a generic element $(s_{0}, t_{0}) \in h_{2} \times U$ i.e. $\varphi(s_{0}, t_{0}) = (d_{1}, d_{2}) \equiv \varphi$.

For our original situation this means that the type of $\Theta|_{S^{\nu}} = (\iota^{*})^{*}\Theta$ is $\mathfrak{d}(\Theta|_{S^{\nu}}) = (d_{1}, d_{2})$ for arbitrary $(d_{1}, d_{2})$–polarized $(S, L)$.

An immediate consequence of Lemma 6.14 and Proposition 6.6 is the following.

**Proposition 6.19** Let $(S, L)$ denote a polarized abelian surface of type $\mathfrak{d}(L) = (d_{1}, d_{2})$. Then for every smooth curve $C \subset |L|$, the type of the restriction of the principal polarization $\Theta$ of $\text{Jac}(C)$ to $K(C)$ is $\mathfrak{d}(\Theta|_{K(C)}) = (1, \ldots, 1, d_{1}, d_{2})$.

**Proof:** By Lemma 6.14 the restriction $\Theta|_{S^{\nu}}$ is a polarization of type $\mathfrak{d}(\Theta|_{S^{\nu}}) = (d_{1}, d_{2})$. By Proposition 6.6, the type of $\Theta|_{K(C)}$ is $\mathfrak{d}(\Theta|_{K(C)}) = (1, \ldots, 1, d_{1}, d_{2})$. $\square$

**6.20. Fibers of Beauville–Mukai systems.** Let $\pi : K_{H}(v) \to |D|$ denote a Beauville–Mukai system of generalized Kummer type. Consider a smooth curve $C \subset |D|$, then the fiber of the support morphism $M_{H}(v) \to \{D\}$ is given by the Jacobian $\text{Jac}^{d}(C)$ of a certain degree $d$. The restriction of the Albanese map $(\text{Alb}_{v})_{F_{0}} = \alpha_{F_{0}} \times \text{det}_{F_{0}} \to \text{Jac}^{d}(C) \subset M_{H}(v)$ is in the second component constant 0. Therefore, if we denote by $K^{d}(C) \subset \text{Jac}^{d}(C)$ the fiber of $\pi : K_{H}(v) \to |D|$, we have an exact sequence
\[ 0 \longrightarrow K^{d}(C) \hookrightarrow \text{Jac}^{d}(C) \xrightarrow{\alpha} S \]
where $\alpha = \text{pr}_S \circ (\text{Alb}_v)_{F_0}$ and the diagram

\begin{equation}
\begin{array}{ccc}
K_H(v) & \longrightarrow & M_H(v) \\text{(Alb}_v)_{F_0} \times S^\vee \\
\uparrow & & \uparrow \\
K^d(C) & \longrightarrow & \text{Jac}^d(C) \alpha \longrightarrow S \\
\end{array}
\end{equation}

\textbf{Lemma 6.23} The map $\alpha = \text{Jac}(\iota)$ above is the map induced by the inclusion $\iota : C \hookrightarrow S$ by the universal property of the Jacobian. More precisely $\alpha$ is given by

$$O_C(\sum_i n_i x_i) \mapsto \sum_i n_i x_i.$$ 

In particular, $K^d(C)$ is the kernel of this map.

\textbf{Proof:} This follows from the definition of the map $\alpha$, see (6.1). If $F \in \text{Jac}^d(C) \subset M_H(v)$, then $\alpha$ takes on $\text{Jac}^d(C)$ the form $O_C(\sum_i n_i x_i) \mapsto \sum_i n_i x_i$ which is the map induced by $\iota$ and the universal property of the Jacobian, see subsection 6.7. The second statement is obvious. \qed

By Lemma 6.9 we know that we can see the dual $S^\vee = \text{Pic}^0(S)$ as an abelian subvariety of $\text{Jac}^d(C)$, as the pullback $\iota^* : \text{Pic}^0(S) \hookrightarrow \text{Jac}(C) \cong \text{Jac}^d(C)$ is an embedding. We conclude that we are in the situation of 6.7 and therefore have the following.

\textbf{Proposition 6.24} In the situation above, $K^d(C)$ and $S^\vee$ are a pair of complementary abelian subvarieties in the principally polarized abelian variety $\text{Jac}^d(C)$.

\textbf{Proof:} Follows immediately from Lemma 6.11. \qed

\textbf{Lemma 6.25} [Wie16, Lem. 5.4] Let $A$ be an abelian variety. If $\text{End}(A) = \mathbb{Z}$ then its Picard number is $\rho(A) = 1$.

We now consider Jacobians of curves which are contained in linear systems defined on abelian surfaces.

\textbf{Theorem 6.26} [CvdG92, 3.B.] Let $S$ be an abelian surface, $\iota : C \hookrightarrow S$ a smooth curve and let

$$K(C) := \ker(\text{Jac}(C) \to S) \subset \text{Jac}(C)$$

be the kernel of the map $\text{Jac}(\iota)$ induced by the inclusion and the universal property of the Jacobian, as described in subsection 6.7. Then $\text{End}(K(C)) = \mathbb{Z}$, therefore we have for the Picard number $\rho(K(C)) = 1$.

\textbf{Proof:} We know by Lemma 6.9 that $K(C)$ is connected i.e. a honest abelian subvariety of $\text{Jac}(C)$. The requirement in [CvdG92, 2.II.] that $|C|$ defines a birational map on its image can be dropped, since the authors only use this to conclude that the map $\iota^* : S^\vee \to \text{Jac}(C)$ has finite kernel. In our setting this is the case by Lemma 6.9. Then by [CvdG92, 3.B.] we have $\text{End}(K(C)) = \mathbb{Z}$, hence $\rho(K(C)) = 1$ for the Picard number by Lemma 6.25. \qed

Furthermore, we can compute the polarization types of Beauville–Mukai systems of generalized Kummer type.
Theorem 6.27 The Picard number of the generic smooth fiber of a Beauville–Mukai system \( \pi : X \to |D| \) of generalized Kummer \( n \)-type equals one. In particular we have for its polarization type

\[
\sigma(D) = (1, \ldots, 1, d_1, d_2)
\]

where \( \sigma(D) = (d_1, d_2) \) is the type of the polarization defined by \( D \).

**Proof:** Let us denote \( C \in |D| \) a generic smooth curve. The fiber \( F = K(C) = K^{d}(C) \) of \( \pi \) over \( C \) is given as the kernel of the map \( \text{Jac}(\iota) : \text{Jac}^{d}(C) \to S \), see (6.21), where \( \iota : C \to S \) is the inclusion. We are therefore precisely in the situation of Theorem 6.26 which states that \( \rho(K(C)) = 1 \) for the Picard number. Let \( \omega \in K_X \) denote a special Kähler class for the fiber \( K(C) \). We are in the case of subsection 6.7 and by Proposition 6.19 the abelian subvariety \( K(C) \) admits a polarization \( L \) of type \( \sigma(L) = (1, \ldots, 1, d_1, d_2) \). Since \( \rho(K(C)) = 1 \), we have \( L = \omega|_{K(C)} \) as both are primitive. Therefore \( \sigma(D) = \sigma(\omega|_{F}) = \sigma(L) = (1, \ldots, 1, d_1, d_2) \) by Proposition 3.1. \( \square \)

6.28. Beauville–Mukai systems in the moduli. In this section we show that there are Beauville–Mukai systems in each connected component of the moduli of Lagrangian fibrations of \( K3[n] \) and generalized Kummer type. We check this in terms of the monodromy invariant.

The proof of the following Proposition is similar to [Mar14, Ex. 3.1]. However, we give a detailed proof.

**Proposition 6.29** Let \( d \) be a positive integer such that \( d^2 \) divides \( n + 1 \) and let \( b \) an integer satisfying \( \gcd(d, b) = 1 \). Then there exists a Beauville–Mukai system \( \pi : K_H(v) \to \mathbb{P}^n \) of generalized Kummer type and a primitive isotropic class \( \alpha \in H^2(K_H(v), \mathbb{Z}) \) such that the following holds.

(i) \( \text{Div}(\alpha) = d \).

(ii) the monodromy invariant \( \vartheta(\alpha) \) is represented by \( (L_{n,d}, (d, b)) \).

(iii) \( c_1(\pi^*\mathcal{O}_{\mathbb{P}^n}(1)) = \alpha \).

(iv) Its polarization type is given by \( \sigma(\alpha) = (1, \ldots, 1, d, \frac{n+1}{2}) \).

**Proof:** Let \( S \) be an abelian surface together with primitive ample line bundle \( L \) on \( S \) with \( (L, L) = (2n + 2)/d^2 \). Set \( \beta := c_1(L) \) and let \( s \) be an integer such that \( sb \equiv 1 \mod d \). Then \( v := (0, \beta, s) \) is a Mukai vector. In particular \( v \) is primitive since \( \beta \) is primitive and \( \gcd(d, s) = 1 \). Choose a \( v \)-generic ample class \( H \). We have \( (v, v) = d^2(\beta, \beta) = 2n + 2 \) hence \( K_H(v) \subset M_H(v) \) is irreducible holomorphic symplectic of dimension \( 2n \) and we obtain a Beauville–Mukai system \( \pi : M_H(v) \to |L^d| \) as described in section 6. We have Mukai’s Hodge isometry

\[
\Theta : v^\perp \to H^2(M_H(v), \mathbb{Z}) \overset{r}{\to} H^2(K_H(v), \mathbb{Z})
\]

see (4.7) and (4.8). The map \( r : H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z}) \) is the restriction.

Recall that the definition of \( \Theta \) needs the choice of a quasi–universal family of sheaves \( \mathcal{E} \) on \( S \) of similitude \( \rho \in \mathbb{N} \).

Set \( \alpha := \Theta(0, 0, 1) \) which is clearly isotropic and define \( v : H^2(K_H(v), \mathbb{Z}) \to H^*(S, \mathbb{Z}) \) to be \( \Theta^{-1} \) composed with the inclusion \( v^\perp \hookrightarrow H^*(S, \mathbb{Z}) \). Note that \( v \) is a representative of the monodromy invariant orbit constructed in Theorem 4.9.

(i) An element \( (r, c, t) \) belongs to \( v^\perp \) if and only if

\[
0 = ((0, d\beta, s), (r, c, t)) = d(\beta, c) - rs \iff rs = d(\beta, c).
\]

Hence \( d \) divides \( r \) since \( \gcd(d, s) = 1 \). Furthermore, we have \( (0, 0, 1), (r, c, t) = r \) for all \( (r, c, t) \in v^\perp \) hence \( \text{Div}((0, 0, 1)) \geq d \). As the lattice of a two
torus is $U^{33}$ i.e. in particular unimodular, we have $\text{Div}_H^2(S,\mathbb{Z}) = 1$ in $H^2(S,\mathbb{Z})$. This implies that $\text{Div}(\beta) = 1$ in $v^\perp$, hence we can find an element $c \in H^2(S,\mathbb{Z})$ such that $s = (c, \beta)$. Then $(d, c, 0)$ is contained in $v^\perp$ and $((0, 0, 1), (d, c, 0)) = d$, hence

$$\text{Div}(\alpha) = \text{Div}(0, 0, 1) = d.$$

(ii) We have $i(\alpha) - bv = (0, 0, 1) - (0, bd\beta, bs) = (0, bd\beta, 1 - bs)$ which is divisible by $d$ since $sb \equiv 1 \mod d$ by assumption. By Lemma 5.12 (v) the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_n, (d, b))$.

(iii) Let $\omega = [p] \in H^4(S, \mathbb{Z})$ denote Poincare dual of a point $p \in S$. By our notation we have $\omega = (0, 0, 1) = \omega^v \in H^4(S)$. Since $S$ is an abelian surface, one has $\sqrt{\text{Td}(S)} = 1$, hence $\sqrt{\text{Td}(S)} \omega = \omega$. Note that $E$ is a sheaf of rank zero, hence $\text{ch}(E) = \rho c_1(E) + \xi = \rho[D] + \xi$ for some divisor $D$ in $S \times M_H(v)$ and for some terms $\xi$ of higher degree. Furthermore, $(\text{pr}_S)^* \omega = [p \times M_H(v)] \in H^4(S \times M_H(v), \mathbb{Z})$ and $[(\text{pr}_M(v))((\xi, [p \times M_H(v)])|_2 = 0$ due to degree reasons. Then we have

$$\Theta(0, 0, 1) = r \left( \left( \text{pr}_M(v) h (D \cdot [p \times M_H(v)]) \right) \right)$$

$$= r \left( \left( \left[ F \in M_H(v) \mid p \in \text{supp}(F) \right] \right) \right)$$

$$= \left[ F \in K_H(v) \mid p \in \text{supp}(F) \right]$$

$$= \pi^*[C \in |L^d| \mid p \in C]$$

$$= \pi^* c_1(\mathcal{O}|_{L^d}(1)) = c_1(\mathcal{O}|_{L^d}(1))$$

since $V := \{ C \in |L^d| \mid p \in C \}$ is a hyperplane in a projective space, hence $|V| = c_1(\mathcal{O}|_{L^d}(1))$.

(iv) This follows directly from Theorem 6.27 since $\vartheta(L) = (1, \frac{n+1}{2d})$ by Lemma 6.12 i.e. $\vartheta(dL) = (d, \frac{n+1}{d})$. 

\[\square\]

6.30. Geometric interpretation of the monodromy invariant. As in the $K3^{[n]}$-case we have the following connected component of the moduli of generalized Kummer fibrations.

Let $\Lambda$ denote a lattice of signature $(3, b_2 - 3)$ which is isometric to the second integral cohomology of an irreducible holomorphic symplectic manifold.

Let $\mathfrak{M}_\Lambda$ denote the corresponding moduli space of isomorphism classes of marked pairs $(X, \eta)$ i.e. $X$ is an irreducible holomorphic symplectic manifold of the fixed deformation type and $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ is a marking. Choose a connected component $\mathfrak{M}_\Lambda^\circ$ of $\mathfrak{M}_\Lambda$ and consider the period map

$$\mathcal{P} : \mathfrak{M}_\Lambda^\circ \to \Omega_\Lambda, \quad (X, \eta) \mapsto \eta([H^2,0(X)])$$

Choose the orientation of $\tilde{C}_\Lambda$ compatible to $\mathfrak{M}_\Lambda^\circ$ in sense of Definition 2.23.

Let $\lambda \in \Lambda$ be a nontrivial isotropic class. After a possible change of the sign of $\lambda$ (cf. 2.25), we have a distinguished and compatible connected component

$$\Omega_{\Lambda \perp}^\circ := \{ p \in \Omega_{\Lambda \perp} \mid \lambda \in \partial C_p \}$$

of the hyperplane section $\Omega_{\Lambda \perp} = \Omega_\Lambda \cap \lambda^\perp$, see 2.25. Then define

$$\mathfrak{M}_{\Lambda \perp}^\circ := \mathcal{P}^{-1} \left( \Omega_{\Lambda \perp}^\circ \right) = \left\{ (X, \eta) \in \mathfrak{M}_\Lambda^\circ \mid \eta^{-1}(\lambda) \text{ is of type (1, 1) and in } \partial C_X \right\}.$$
which is a connected hypersurface of $\mathcal{M}_c^\circ$ by [Mar14, Lem. 4.4] and [Mar13, Cor. 5.11]. Consider the nef subspace

$$\mathcal{U}_c^\circ := \left\{ (X, \eta) \in \mathcal{M}_c^\circ \mid \eta^{-1}(\lambda) \text{ is nef} \right\}.$$ 

As in the $K3^{[n]}$–type we have the following result for the generalized Kummer case, which can be proved exactly in the same way as in the $K3^{[b]}$ case with use of [Mat13, Cor. 1.1].

**Theorem 6.31** [Wie16, Thm. 3.7] Let $\lambda$ be a primitive and isotropic element in the $K3^{[b]}$ or generalized Kummer lattice. The space $\mathcal{U}_c^\circ$ in the corresponding connected component $\mathcal{M}_c^\circ$ of the moduli of marked pairs has the following properties.

(i) It parametrizes isomorphism classes of marked pairs $(X, \eta)$ of $\mathcal{M}_c^\circ$ with $X$ of $K3^{[b]}$ or generalized Kummer type, respectively, admitting a Lagrangian fibration $f : X \to \mathbb{P}^n$ such that

$$\eta(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))) = \lambda.$$

(ii) It is smooth of dimension 20 for the $K3^{[n]}$ and of dimension 4 for the generalized Kummer case. Furthermore, it is open in $\mathcal{M}_c^\circ$.

(iii) It is connected.

We refer to this space $\mathcal{U}_c^\circ$ as a connected component of the moduli of Lagrangian fibrations.

We can state the geometric interpretation of the monodromy invariant.

**Proposition 6.32** Let $f_i : X_i \to \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations of generalized Kummer type. Let $\Lambda$ denote the generalized Kummer lattice and set $L_i := f_i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then the following statements are equivalent.

(i) The Lagrangian fibrations $f_i$ are deformation equivalent.

(ii) There exist markings $\eta_i : H^2(X_i, \mathbb{Z}) \to \Lambda$ such that the marked pairs $(X_i, \eta_i)$ are contained in the same connected component $\mathcal{U}_c^\circ$ for a primitive isotropic element in the generalized Kummer lattice.

(iii) [Mar13, Lem. 5.17] We have $\text{Div}(c_1(L_1)) = \text{Div}(c_1(L_2))$ for the corresponding divisibilities and $\vartheta(c_1(L_1)) = \vartheta(c_1(L_2))$ for the monodromy invariant.

7. Polarization types of generalized Kummer fibrations

We have now gathered everything to compute the polarization type of Lagrangian fibrations of generalized Kummer type.

**Theorem 7.1** Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of generalized Kummer type. Then for the polarization type $d(f)$ we have

$$d(f) = \left(1, \ldots, 1, d, \frac{n+1}{d}\right)$$

where $d := \text{Div}(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1)))$ denotes the divisibility of the associated element in the lattice $H^2(X, \mathbb{Z})$.

**Proof:** Let $f : X \to \mathbb{P}^n$ denote a Lagrangian fibration of generalized Kummer type and set $L := f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then $\lambda := c_1(L)$ is primitive and isotropic by Lemma 2.10 with respect to the Beauville–Bogomolov quadratic form. Let $d := \text{Div}(\lambda)$ denote the divisibility of $\lambda$, note that by Lemma 5.12 $d^2$ divides $n+1$. Consider the monodromy invariant $\vartheta : H_d(X) \to \Sigma_{n,d}$ as in Theorem 5.15. By Lemma 5.12 (v) there exists an integer $b$ such that $\vartheta(\lambda)$ is represented by $(L_{n,d}, (d, b))$ and we have $\gcd(d, b) = 1$. 


By Theorem 6.29 we have a Beauville–Mukai system \( \pi : X' \to \mathbb{P}^n \) of generalized Kummer type, respectively, together with a primitive isotropic class \( \alpha \in H^2(X', \mathbb{Z}) \) such that \( \text{Div}(\alpha) = d, L' := \pi^*O_{\mathbb{P}^n}(1) \) satisfies \( c_1(L') = \alpha \) and \( \vartheta(\alpha) \) is represented also by \( (L_n,d, (d,b)) \) i.e. \( \vartheta(\alpha) = \vartheta(\lambda) \).

By Lemma 5.5 we have \( (\omega, L) > 0 \) and \( (\omega', L') > 0 \) for Kähler classes \( \omega \) on \( X \) and \( \omega' \) on \( X' \) as \( L \) and \( L' \) are isotropic and nef, therefore are contained in \( \bar{K}_X \subset \bar{C}_X \) and \( \bar{K}_{X'} \subset \bar{C}_{X'} \), respectively. Hence we can apply Lemma 5.4 to see that the pairs \( (X, L) \) and \( (X', L') \) are deformation equivalent in the sense of Definition 2.7. By Proposition 2.9 the Lagrangian fibrations \( \pi \) and \( f \) are deformation equivalent. By Theorem 3.3 and Theorem 6.29 we have

\[
d(f) = d(\pi) = \left( 1, \ldots, 1, d, \frac{n + 1}{d} \right)
\]

which concludes the proof.

\[\square\]

References

[Bea84] Arnaud Beauville. Variétés kählériennes dont la première classe de chern est nulle. J. Differential Geom., 18:755–782, 1984. 3, 4

[BHPV03] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius van de Ven. Compact Complex Surfaces. Second Enlarged Edition, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. Springer, 2003. 17

[BL03] Christina Birkenhake and Herbert Lange. Complex Abelian Varieties, volume 302 Second Edition of Grundlehren der mathematischen Wissenschaft. Springer, 2003. 10, 20, 21, 22, 23, 24, 25

[Cam05] Frédéric Campana. Isotrivialité de certaines familles kählériennes de variétés non projectives. Mathematische Zeitschrift, 252(1):147–156, 2005. 10

[CvdG92] Ciro Ciliberto and Gerard van der Geer. On the Jacobian of a hyperplane section of a surface. Classification of irregular varieties (Trento, 1990) Lecture Notes in Math., Springer, Berlin, 1515:33–40, 1992. 26

[EEC52] Martin Eichler. Quadratische Formen und orthogonale Gruppen. Springer, 1952. 17, 19

[GHJ03] Mark Gross, Daniel Huybrechts, and Dominic Joyce. Calabi–Yau Manifolds and Related Geometries. Springer, 2003. 3, 4

[GL14] Daniel Greb and Christian Lehn. Base manifolds for lagrangian fibrations on hyperkähler manifolds. Int. Math. Res. Notices, 19:5483–5487, 2014. 5

[Hwa08] Jun-Muk Hwang. Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds. Invent. Math. 174, 3:625–644, 2008. 5

[Mar10] Eyal Markman. Integral constraints on the monodromy group of the hyperkähler resolution of a symmetric product of a K3 surface. Internat. J. of Math. 21, 21:169–223, 2010. 12

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In Wolfgang Ebeling et. al., editor, Complex and Differential Geometry, volume 8, pages 257–323. Springer Proceedings in Math., 2011. 5, 6, 15

[Mar13] Eyal Markman. Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections. Kyoto J. Math., 53:345–403, No. 2, 2013. 2, 5, 13, 14, 29

[Mar14] Eyal Markman. Lagrangian fibrations of holomorphic-symplectic varieties of K3\(^{(1)}\)-type. In Anne Frühbis-Krüger et. al., editor, Algebraic and Complex Geometry, volume 71. Springer Proceedings in Math., 2014. 2, 8, 15, 27, 29

[Mats99] Daisuke Matsushita. On fibre space structures of a projective irreducible symplectic manifold. Topology, pages 38(1):79–83, 1999. 4

[Mats00] Daisuke Matsushita. Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds. Math. Res. Lett., 7:389–391, 2000. 4

[Mats01] Daisuke Matsushita. Addendum to: On fibre space structures of a projective irreducible symplectic manifold. Topology, pages 38(1):79–83, 2001. 4

[Mats03] Daisuke Matsushita. Holomorphic symplectic manifolds and lagrangian fibrations. Acta Appl. Math, pages 75(1–3):117–123, 2003. 4
[Mat13] Daisuke Matsushita. On isotropic divisors on irreducible symplectic manifolds. arXiv:1310.0896, 2013.

[Mon16] Giovanni Mongardi. On the monodromy of irreducible symplectic manifolds. *Algebraic Geometry*, 3(3):385–391, 2016.

[MP16] Giovanni Mongardi and Gianluca Pacienza. Polarized parallel transport and uniruled divisors on deformations of generalized kummer varieties. *International Mathematics Research Notices*, (rnw346), 2016.

[Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Invent. math.*, 77:101–116, 1984.

[Muk87] Shigeru Mukai. On the moduli space of bundles on K3 surfaces I. *Tata Institute for fundamental research studies in mathematics*, 11:341–413, 1987.

[Nik80] V. V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Math. USSR Izvestija*, 14(1), 19080.

[O’G97] Kieran O’Grady. The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface. *J. Algebraic Geom.*, 6:4599–644, 1997.

[O’G14] Kieran O’Grady. *Compact Hyperkähler Manifolds: Examples*. online lecture notes http://www.mimuw.edu.pl/~gael/Document/hk-examples.pdf, 2014.

[Wie16] Benjamin Wieneck. On polarization types of Lagrangian fibrations. *manuscripta mathematica*, 151(3-4):305–327, 2016.

[Yos01] Kota Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann. 321*, 4:817–884, 2001.

[Yos12] Kota Yoshioka. Bridgeland’s stability and the positive cone of the moduli spaces of stable objects on an abelian surface. arXiv:1206.4838v2, 2012.

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