Sparsity Pattern Recovery in Bernoulli-Gaussian Signal Model

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Abstract

In compressive sensing, sparse signals are recovered from underdetermined noisy linear observations. One of the interesting problems which attracted a lot of attention in recent times is the support recovery or sparsity pattern recovery problem. The aim is to identify the non-zero elements in the original sparse signal. In this article we consider the sparsity pattern recovery problem under a probabilistic signal model where the sparse support follows a Bernoulli distribution and the signal restricted to this support follows a Gaussian distribution. We show that the energy in the original signal restricted to the missed support of the MAP estimate is bounded above and this bound is of the order of energy in the projection of the noise signal to the subspace spanned by the active coefficients. We also derive sufficient conditions for no misdetection and no false alarm in support recovery.

1 Introduction

We consider the linear observation model

$$y = Ax + e,$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^N$ is the signal vector, $e \in \mathbb{R}^M$ is the noise vector, $A \in \mathbb{R}^{M \times N}$ is the measurement matrix, and $M \ll N$. In spite of this being an ill-posed problem, various algorithms have been proposed for estimation of the unknown signal $x$ and performance guarantees have been proven for them subject to sparsity of the signal $x$ and some coherence constraints on the measurement matrix $A$. This technique is known as compressive sensing or compressive sampling \cite{[1-5]} and it has received a lot of attention in recent past among researchers.

In this article we consider the problem of sparse support recovery, also known as sparsity pattern recovery, where the aim is to identify the indices of the non-zero elements of $x$. The main contribution of this article is non-asymptotic analysis of support recovery in terms of quality of the recovered support set. We analyze how much energy of the true signal remains in the missed coefficients under Bernoulli-Gaussian signal prior assumption. We also derive a sufficient condition for perfect support recovery under this signal model.

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In section 1.1 we discuss the most relevant prior work related to this article, in section 1.2 we briefly describe the contribution of this article. In section 2.1 we describe the probabilistic signal model for the variable \( x \), in section 2.2 the coherence property of the measurement matrix \( A \) is defined. Section 2.3 outlines the support recovery problem and defines the estimator for support set. Two theorems regarding the energy bound on the missed support and sufficiency condition for perfect support recovery are stated in sections 2.4 and 2.5 respectively. The proofs are given in section 3 and the results are discussed in section 4.

### 1.1 Related Work

Significant amount of work has been done in recent times on signal recovery in compressive sensing. The \( \ell_2 \)-norm of error in estimating the signal \( x \) is the most popular performance metric \([3, 4, 6]\), but in the noisy setting stability of the solution and boundedness of this performance metric do not give any direct guarantee about support recovery. Here we briefly describe the sparsity pattern recovery results most relevant to our work. Donoho et al. showed in their work \([3]\) that \( \ell_1 \)-constrained quadratic program with exaggerated noise level guarantees partial support recovery. They also derived the upper bound on the number of non-zero elements in the signal vector in terms of mutual coherence of the measurement matrix and minimum absolute value of the non-zero elements in the true signal for perfect support recovery using an orthogonal greedy algorithm. Candès et al. showed in \([7]\) that if the measurement matrix satisfies certain coherence properties and the signs of the non-zero elements of the signal are equally likely to be positive and negative then \( \ell_1 \)-regularized least squares solution recovers the signed support perfectly with very high probability when the regularization constant is chosen appropriately and the minimum absolute value of the non-zero elements of the signal is above certain threshold. Recovery of signed support means the support sets of true signal and the estimate are identical and the non-zero elements in the true signal and the estimate have the same signs. Zhao et al. showed in \([8]\) that the irrepresentable condition is almost necessary and sufficient for LASSO to select the true model both in the classical fixed \( N \) setting and in the large \( N \) setting as the observation size \( M \) gets large. At some special scenarios this irrepresentable condition coincides with the coherence condition used in the work of Donoho et al. A similar condition is used by Meinshausen et al. in \([9]\) to prove a model selection consistency result for Gaussian graphical model selection using the LASSO. Using replica method Guo et al. showed \([10]\) that the posterior distribution of estimating a single coefficient becomes asymptotically decoupled from estimation of other coefficients. Detecting a single coefficient is analogous to detecting this input coefficient with all other coefficients suppressed, but based on a noisier observation. They derived the maximum probability of making an error in detecting a single coefficient and the corresponding MMSE under the high SNR and large system limits. Rangan et al. \([11]\) use the same replica claim framework to obtain the mean squared error in estimation of the variable \( x \) under the large system limits for linear, LASSO, and zero-norm regularized estimators.

There is another class of papers where the minimum number of observations \( M \) needed for perfect support recovery or partial support recovery expressed as a fraction of the true support size is investigated \([12, 16]\). In these articles it is assumed that elements of the measurement matrix are i.i.d. Gaussian. Necessary and sufficient conditions for exhaustive search based decoders and \( \ell_1 \)-constrained least squares are derived in these articles.
1.2 Contributions

Our results are non-asymptotic with fixed model dimensions. Except [3] and [7] other support recovery results for linear observation model, discussed in section 1.1, are asymptotic analyses. Our first result is about partial support recovery. We characterize any support set in terms of the energy in the true signal restricted to this support set. More specifically, we explore the relationship between energy in the missed support and the noise energy under the probabilistic model where the signal prior is known. Most earlier partial support recovery results characterize the fraction of the support recovered \( i.e., \) they do not distinguish between missing the coefficient with the highest absolute value and the lowest absolute value but our performance metric captures that. To the best of our knowledge the only exception is the work by Akccakaya et al. [16]. They investigated the number of measurements needed for partial support recovery in terms of fraction of total energy in the true signal restricted to the recovered support. But their analysis is asymptotic whereas we have considered fixed model dimensions. Our second result is about sufficient conditions for guaranteeing no missed coefficient and no false detection for this Bernoulli-Gaussian signal model when the absolute value of any active coefficient is bounded below with a very high probability.

2 Problem Statement

2.1 Signal Model

We consider a probabilistic signal model for the sparse signal \( x \in \mathbb{R}^N \). Let \( S \) be a set whose entries are drawn from the set \( I = \{1, 2, \ldots, N\} \) in such a way that each entry of \( I \) is in the set \( S \) with probability \( p \ll 1 \) and their inclusion in \( S \) is independent of each other. Thus the probability that the cardinality of the support set \( S \) equals \( K \) is given by \( \mathbb{P}[|S| = K] = \binom{N}{K} p^K (1-p)^{N-K} \). To enforce sparsity we also assume that \( p < \frac{1}{2} \). Each element of \( x \) is identically zero if the corresponding index is not in the set \( S \), otherwise the element is Gaussian with mean \( \mu_1 \) and non-zero variance \( \sigma_1^2 \). The mean \( \mu_1 \) can be zero or non-zero. Elements of \( x \) are distributed independently given the support set. If \( x_S \) denotes the vector consisting of the elements of \( x \) whose indices are in the set \( S \), then the vector \( x_S \) follows i.i.d. Gaussian distribution \( i.e., \ x_S \sim \mathcal{N}(\mu_1 1_{|S|}, \sigma_1^2 I_{|S|}) \). Thus \( S \) is the support set of the signal vector \( x \) with expected cardinality \( \mathbb{E}[|S|] = Np \ll N \) and \( x \) is sparse with high probability. This Bernoulli-Gaussian model is quite popular in literature for a long time for modeling sparse vectors in diverse application areas and is also becoming increasingly popular in the compressive sensing research [10, 20, 21].

\(^1\)The vector of ones of size \( |S| \times 1 \) is denoted by \( 1_{|S|} \). Similarly the vector of ones of size \( |S_1| \times 1 \) is denoted by \( 1_{|S_1|} \). It is also denoted by \( 1 \) when there is no ambiguity. The notations \( 1_{|S_0|} \) and \( 1_{01} \) are used interchangeably. The same applies to the subscripts used for the identity matrix \( I \).
2.2 Coherence of Measurement Matrix

Several conditions have been proposed which characterize coherence properties of the measurement matrix \( A \) and are used for deriving any performance guarantee for compressive sensing algorithms. Measurement matrix with entries drawn from i.i.d. Gaussian or Bernoulli distributions, and partial Fourier matrix are known to satisfy these properties. In [3], it is shown that if the mutual coherence, i.e., the magnitude of the maximum entry of the Gram matrix \( m(A) = \max_{i,j:i\neq j} |(A^T A)_{i,j}| \) is small then robust signal and support recovery is possible for sparse signals. Another condition known as restricted isometry property (RIP) is proposed in [4]. Here we assume that the measurement matrix \( A \) satisfies RIP with \((4Np, \varepsilon)\), i.e., for any sparse vector \( x \) with cardinality of support set \( \leq 4Np \),

\[
(1 - \varepsilon)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \varepsilon)\|x\|_2^2.
\]

(2)

Though determination of RIP of a given matrix is a NP-hard problem, it can be shown [22] that random matrices satisfy RIP properties with overwhelming probability. In contrary mutual coherence is a verifiable condition but it gives much weaker performance guarantee than RIP.

We note here that the constant 4 in the definition of RIP of \( A \) is arbitrary and a matter of convenience. In this article we also assume that \( \varepsilon \leq \frac{1}{3} \) in order to obtain simple expressions in our results. Leaving \( \varepsilon \) as a parameter makes the results difficult to interpret. We can always choose any other constant instead of 4 in definition of RIP for the measurement matrix and a different upper bound on \( \varepsilon \)-value. This will lead to different values of the constants appearing in our results.

2.3 Support Recovery

In this article we consider the problem of support recovery i.e., identifying the indices corresponding to the Gaussian with \( \sigma_1^2 \) variance. Assuming additive white Gaussian noise with variance \( \sigma_e^2 \), i.e., \( e \sim N(0, \sigma_e^2 I_M) \),

\[
y|S \sim N(\mu_1 A_S 1_{|S|}, \Phi(S)) ,
\]

(3)

where \( \Phi(S) \) is given by,

\[
\Phi(S) = \sigma_1^2 A_S A_T^S + \sigma_e^2 I_M .
\]

(4)

The maximum \textit{a posteriori} (MAP) estimate of the support set is given by,

\[
\hat{S}_{MAP} = \arg \max_S p(S|y) = \arg \max_S p(y|S)p(S)
\]

\[
= \arg \max_S \int_x p(y|x,S)p(x|S)dx \cdot p(S)
\]

\[
= \arg \min_S \frac{1}{2} \ln \det(\Phi(S)) + \frac{1}{2}(y - \mu_1 A_S 1_{|S|})^T \Phi(S)^{-1}(y - \mu_1 A_S 1_{|S|}) + |S| \ln \frac{1-p}{p} .
\]

(5)

We have adopted a probabilistic model for the number of active elements, the signal and the noise. Though the number of non-zero elements in \( x \) has mean \( Np \ll N \), it can be as large as \( N \) with
very small but non-zero probability. Similarly signal and noise energy can be arbitrarily large with vanishingly small but non-zero probability. Nevertheless the quantities like cardinality and energy are bounded with overwhelmingly high probability. Keeping this in mind we study the suboptimal estimator which minimizes the MAP cost function subject to the constraint $|S| \leq 2Np$:

$$
\hat{S} = \arg \min_{S: |S| \leq 2Np} \frac{1}{2} \ln \det(\Phi(S)) + \frac{1}{2} (y - \mu_1 A_S 1_{|S|})^T \Phi(S)^{-1} (y - \mu_1 A_S 1_{|S|}) + |S| \ln \frac{1-p}{p}.
$$

(6)

We define the event $E$ to be the cardinality of the true support being less than or equal to $2Np$. As we see later that event $E$ holds with high probability and the estimator defined in (6) satisfies certain performance criteria if event $E$ holds. Here we emphasize that instead of $2Np$ we can use $LNp$ for any other $L > 1$ in the definition of the event as $E$. Similarly we can use any other constraint $|S| \leq QNp$ in the definition of $\hat{S}$ in (6), where $Q > 1$. The choice of $L = Q = 2$ is arbitrary in the definition of the event $E$ and the definition of $\hat{S}$ but related to the constant used in the definition of RIP satisfied by the measurement matrix $A$. They are chosen in such a way that $L + Q \leq n$, when $A$ satisfies RIP with $(nNp, \varepsilon)$. As mentioned earlier we have arbitrarily chosen $n = 4$.

### 2.4 Energy in Missed Coefficients

Our first theorem, as stated below, shows that the total energy in the missed coefficients is of the order of the average energy in the projection of noise to the subspace spanned by the active columns of the $A$ matrix. Here we make no assumption about the mean of the Gaussian distribution $\mu_1$.

**Theorem 1** (Energy Bound on Missed Coefficients). For the signal and observation models under consideration, the $\ell_2$-norm of the signal restricted to the index set of missed coefficients is upper bounded by $K_1 \sqrt{Np} \sigma_e$ with probability exceeding $(1 - e^{-Np(2 \ln 2 - 1)}) (1 - 3e^{-Np(\beta - 1 - \ln \beta)})$, where $K_1 = 2 \left( \sqrt{\beta + C} + \sqrt{\beta} \right)$, $C = \ln \left( 1 + \frac{4\sigma_1^2}{3\sigma_e^2} \right) + 2 \ln \frac{1-p}{p}$ and $\beta > 1$.

Different values of the parameter $\beta$ give different values of the constant $K_1$ and also the probability with which the energy in the missed coefficients is bounded by $K_1^2 Np \sigma_e^2$. Both $K_1$ and the minimum probability are increasing function of $\beta$. This is natural since as we increase the bound, i.e., make it loose, the probability with which it is satisfied also increases. We also see that the constant $C$ is dependent on $p$ and the ratio $\sigma_1^2/\sigma_e^2$. Thus the constant $K_1$ increases as the signal model is known to be more sparse. The dependence of $K_1$ on $\sigma_1^2/\sigma_e^2$ is a bit counterintuitive. As we discuss in section 4 this bound becomes loose at high SNR. At very high value of this ratio there is no missed coefficient with a very high probability.

### 2.5 Perfect Support Recovery

It is hard to recover the support set perfectly for the zero-mean signal model since a significant number of coefficients are close to zero. Hence they are almost impossible to detect in the presence of noise. If the signal mean is high enough to ensure that all the coefficients are well above the
noise level then all of them are detected with a high probability. But even then ensuring that no false alarm happens is tough. It requires even higher value of the mean. The following theorem states these results.

**Theorem 2** (Sufficient Condition for Perfect Support Recovery). For the signal and observation models under consideration, all active coefficients are selected i.e., there is no missed coefficient with probability exceeding $(1 - e^{-Np(2\ln 2 - 1)})(1 - 3e^{-Np(\beta - 1 - \ln \beta)} - e^{-\frac{(\beta - 1 - \ln \beta)}{2}})$ if $|\mu_1| > K_2\sigma_1 + K_1\sqrt{Np}\sigma_e$ where $K_2 = \sqrt{\beta}$, and $\beta, \tilde{\beta} > 1$. $K_1$ and $C$ are as defined in theorem 1. Perfect support recovery happens with the same probability if $|\mu_1| > K_3\sigma_1 + K_4\sqrt{Np}\sigma_e$, where $K_3 = \max\{K_2, 6\sqrt{2\beta Np}\}$ and $K_4 = \max\{K_1, 3(\frac{1}{2} + \sqrt{3})\sqrt{2\beta}\}$.\n
Here the condition $|\mu_1| > K_2\sigma_1 + K_1\sqrt{Np}\sigma_e$ is needed for probabilistic guarantee for no misdetection. This condition implies that if the distribution of $x_S$ is such that with very high probability absolute values of all the elements are above the noise level in the subspace spanned by the active columns of the measurement matrix then with very high probability there is no active coefficient excluded from $\hat{S}$. In addition to this condition, we also need $|\mu_1| > 6\sqrt{2\beta Np}\sigma_1 + 3(\frac{1}{2} + \sqrt{3})\sqrt{2\beta}\sqrt{Np}\sigma_e$ for guarantee on no false alarm.

**3 Proofs**

**3.1 Some Propositions**

Before proceeding further we provide the following propositions. The first proposition is a consequence of RIP. It shows near orthonormality of the columns of $A$ matrix i.e., the column spaces of any two submatrices $A_i$ and $A_j$ of the matrix $A$ are almost orthogonal to each other if $S_i \cap S_j = \emptyset$ and $|S_i| + |S_j| \leq 4Np$.

**Proposition 1.** If $S_i \subset \{1, 2, \ldots, N\}$, $S_j \subset \{1, 2, \ldots, N\}$, $S_i \cap S_j = \emptyset$, $A$ satisfies RIP with $(4Np, \varepsilon)$ and $|S_i| + |S_j| \leq 4Np$, then the vector induced norm $\|A_i^T A_j\|_2 \leq \varepsilon$.

**Proof.** This proof is due to [3]. Let $S = S_i \cup S_j$. Note that $A_i^T A_j$ is a submatrix of $A_S^T A_S - I_{|S|}$. Since the induced norm of a submatrix never exceeds the norm of the matrix,

$$\|A_i^T A_j\|_2 \leq \|A_S^T A_S - I_{|S|}\|_2 \leq \max\{(1 + \varepsilon) - 1, 1 - (1 - \varepsilon)\} = \varepsilon,$$  \hspace{1cm} (7)

since the singular values of the matrix $A_S^T A_S$ lie between $1 - \varepsilon$ and $1 + \varepsilon$. \hfill $\Box$

**Proposition 2.** Let $A_i = U_i \Sigma_i V_i^T$ be the Singular Value Decomposition (SVD) of $A_i$. Let $U_i$ be the submatrix formed by taking the first $|S_i|$ columns of $U_i$ and $U_j$ be the submatrix formed by taking the rest $M - |S_i|$ columns of $U_i$. If $x \in \mathbb{R}^{|S_i|}$, then $\|U_i^T A_j x\|_2 \leq \frac{\varepsilon}{\sqrt{1 - \varepsilon}}\|x\|_2$ and $\|U_i^T A_j x\|_2 \geq \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}\|x\|_2$. Also, if $v \in \mathbb{R}^{|S_j|}$, then $\|U_j^T U^T \tilde{U}_j v\|_2 \geq \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}\|v\|_2$ where $\tilde{U}_j$ is defined similar to $U_i$.\n

Proof. From proposition 1, \( \| A_j^T A_j x \|_2 \leq \varepsilon \| x \|_2 \) and \( \| A_j^T A_j x \|_2 = \| V_i \Sigma_i^T U_i^T A_j x \|_2 = \| \Sigma_i^T U_i^T A_j x \|_2 \) where \( \Sigma_i \) is the upper left \( |S_i| \times |S_i| \) diagonal submatrix of \( \Sigma \). Thus \( \| \Sigma_i^T U_i^T A_j x \|_2 \leq \varepsilon \| x \|_2 \). Since elements on \( \text{diag}(\Sigma_i) \geq \sqrt{1-\varepsilon} \), we conclude that \( \| U_i^T A_j x \|_2 \leq \frac{\varepsilon}{\sqrt{1-\varepsilon}} \| x \|_2 \). Now \( \| A_j x \|_2^2 \geq (1-\varepsilon) \| x \|_2^2 \). Thus \( \| U_j^T A_j x \|_2 \geq \sqrt{(1-\varepsilon) - \frac{2^2}{1+\varepsilon}} \| x \|_2 = \sqrt{\frac{1-2\varepsilon}{1-\varepsilon}} \| x \|_2 \). Now we can rewrite this as \( \| U_j^T \bar{U} \Sigma_j V_j^T x \|_2 \geq \sqrt{\frac{1-2\varepsilon}{1-\varepsilon}} \| x \|_2 \). Taking \( v = \Sigma_j V_j^T x \), we see that \( \| U_j^T \bar{U} v \|_2 \geq \sqrt{\frac{1-2\varepsilon}{1-\varepsilon}} \| v \|_2 \).

Corollary 2.1. If \( x \in \mathbb{R}^{\| S_i \|} \), then for the i.i.d. Gaussian signal model, \( x^T A_j^T \Phi(S_i)^{-1} A_j x \geq \frac{1-2\varepsilon}{1-\varepsilon} \frac{1}{\sigma^2} \). Also the singular values of \( A_j^T \Phi(S_i)^{-1} A_j \) are greater than or equal to \( \frac{1-2\varepsilon}{1-\varepsilon} \frac{1}{\sigma^2} \).

Proof. We note that,

\[
\Phi(S_j) = \sigma_j^2 A_j A_j^T + \sigma_e^2 I_{M_j} = U_j (\sigma_j^2 \Sigma_j \Sigma_j^T + \sigma_e^2 I_M) U_j^T,
\]

and \( A_j^T \Phi(S_j)^{-1} A_j \) is a symmetric and positive definite matrix. Thus

\[
x^T A_j^T \Phi(S_j)^{-1} A_j x = x^T A_j^T U_j (\sigma_j^2 \Sigma_j \Sigma_j^T + \sigma_e^2 I_M)^{-1} U_j^T A_j x \leq \frac{1}{\sigma_e^2} \| U_j^T A_j x \|_2^2 \geq \frac{1-2\varepsilon}{1-\varepsilon} \| x \|_2^2.
\]

The last inequality follows from proposition 2. Since \( A_j^T \Phi(S_i)^{-1} A_j \) is symmetric and positive definite, it has SVD \( A_j^T \Phi(S_i)^{-1} A_j = U \Sigma U^T \). Let the \( k \)-th singular value be \( \sigma_k \) and the singular vector corresponding to the singular value \( \sigma_k \) be \( u_k \in \mathbb{R}^{\| S_i \|} \). Then

\[
u_k^T A_j^T \Phi(S_i)^{-1} A_j u_k = u_k^T U \Sigma U^T u_k = \sigma_k.
\]

Since \( \| u_k \|_2^2 = 1 \), from (12) and (13) it follows that \( \sigma_k \geq \frac{1-2\varepsilon}{1-\varepsilon} \frac{1}{\sigma_e^2} \) and this is true for any \( k \). \( \square \)

The next proposition is about the tail probability bound of the Chi-squared distribution.

**Proposition 3.** Suppose \( n \) independent and identically distributed variables \( X_i \sim N(0, \sigma^2) \). If Chi-squared distributed random variable \( Z = \sum_{i=1}^n X_i^2 \), then for any \( \beta > 1 \),

\[
\mathbb{P} [ Z > \beta n \sigma^2 ] \leq e^{-\frac{\beta}{2} (\beta - \ln \beta)}.
\]

Proof. Let \( \bar{X}_i = \frac{X_i}{\sigma} \). Then \( \bar{X}_i \sim N(0, 1) \) and are independently distributed. Let \( \bar{Z} = \sum_{i=1}^n \bar{X}_i^2 = \frac{Z}{\sigma^2} \) which is Chi-squared distributed with degree of freedom \( n \). Using Chernoff inequality,

\[
\mathbb{P} [ Z > \beta n \sigma^2 ] = \mathbb{P} \left[ e^{tZ} > e^{n \beta} \right], \quad \text{for any} \quad t > 0
\]

\[
\leq \mathbb{E} \left[ e^{tZ} \right] = \prod_{i=1}^n \mathbb{E} \left[ e^{tX_i^2} \right] = \frac{1-2t}{e^{nt}} \cdot \frac{1}{e^{nt}}, \quad \text{for} \quad t \in (0, 1/2)
\]

\[
= e^{-\frac{\beta}{2} (\ln(1-2t)+2\beta t)}.
\]
The minimum is attained at \( t = \frac{\beta - 1}{2\delta} \) which gives inequality (13). \( \square \)

We also use the following inequality at various places. If \( c, d > 0 \), then
\[
\frac{(a + b)^2}{c + d} \leq \frac{(a + b)^2 + (a - b)^2}{c + d} = \frac{2a^2}{c + d} + \frac{2b^2}{c + d} \leq \frac{2a^2}{c} + \frac{2b^2}{d}.
\] (17)

3.2 Proof of Theorem 1

Let us divide the indices for the columns of the \( A \) matrix into four disjoint subsets \( S_0, S_1, S_2 \) and \( S_3 \) such that \( S_0 \) denotes the columns which are in the true support and are correctly identified by the constrained MAP estimator \( \hat{S} \), \( S_1 \) denotes the missed columns, \( S_2 \) denotes the columns which are not in the true support but selected by \( \hat{S} \), and \( S_3 \) denotes the columns which are neither in true support nor in \( \hat{S} \). Define \( S_{ij} = S_i \cup S_j \). Let \( A_{ij} \) denote the matrix consisting of those columns of \( A \) which are indexed by the set \( S_{ij} \). Thus,
\[
y = A_{01}x_{01} + e = \mu_1 A_{01} 1_{01} + A_{01}z_{01} + e,
\] (18)
where \( z_{01} \sim \mathcal{N}(0, \sigma^2 I_{|S_{01}|}) \). For zero mean model, \( \mu_1 = 0 \) and \( z_{01} = x_{01} \).

We have defined the event \( E \) to be \( |S_{01}| \leq 2Np \). The mean value of \( |S_{01}| \) is \( \mathbb{E}[|S_{01}|] = Np \). Using Chernoff bound on upper tail of Binomial distribution [23, pp. 68],
\[
\mathbb{P}[|S_{01}| > (1 + \delta)\mathbb{E}[|S_{01}|]] < \left( \frac{e^\delta}{(1 + \delta)^{(1 + \delta)}} \right)^{\mathbb{E}[|S_{01}|]}.
\] (19)

Taking \( \delta = 1 \),
\[
\mathbb{P}[E] = \mathbb{P}[|S_{01}| \leq 2Np] > 1 - e^{\frac{-Np(2\ln 2 - 1)}{}}.
\] (20)

If \( E^c \) denotes the complement of \( E \) i.e., the event \( |S| > 2Np \), then for any event \( B \),
\[
\mathbb{P}[B] = \mathbb{P}[E]\mathbb{P}[B|E] + \mathbb{P}[E^c]\mathbb{P}[B|E^c] \geq \mathbb{P}[E]\mathbb{P}[B|E].
\] (21)

For the rest of the proof we assume that event \( E \) holds and all the subsequent probabilities are conditioned on event \( E \).

For convenience we define the function to be minimized in (6) as \( \gamma(S) \) i.e.,
\[
\gamma(S) = \frac{1}{2} \ln \det(\Phi(S)) + \frac{1}{2}(y - \mu_1 A_S 1_{|S|})^T \Phi(S)^{-1}(y - \mu_1 A_S 1_{|S|}) + |S| \ln \frac{1 - p}{p}
\] (22)
\[
= \frac{1}{2} \gamma_1(S) + \frac{1}{2} \gamma_2(S) + \gamma_3(S),
\] (23)
where \( \gamma_1(S) = \ln \det(\Phi(S)), \gamma_2(S) = (y - \mu_1 A_S 1_{|S|})^T \Phi(S)^{-1}(y - \mu_1 A_S 1_{|S|}) \) and \( \gamma_3(S) = |S| \ln \frac{1 - p}{p} \).

Let the SVD of \( A_0 \) be \( A_0 = U_0 \Sigma_0 V_0^T \). Let \( U_0 \) denote the submatrix of \( U \) consisting of the first \( |S_0| \) columns and \( U_0 \) denote the submatrix with the rest of the columns. Thus \( \bar{U}_0 \) forms an
orthonormal basis for the column space $\mathcal{A}_0$ of $\mathbf{A}_0$. $\mathbf{U}_0$ forms an orthonormal basis for the space $\mathbb{R}^M \setminus \mathcal{A}_0$. Let $\Sigma_0$ denote the $|\mathcal{S}_0| \times |\mathcal{S}_0|$ upper square submatrix of $\Sigma_0$. From (1),

$$\Phi(S_0) = \Phi(S_0) + \sigma_1^2 \mathbf{A}_1 \mathbf{A}_1^T.$$  \hspace{1cm} (24)

Hence applying matrix determinant lemma,

$$\gamma_1(S_0) = \ln \det(\Phi(S_0)) = \ln \det(\Phi(S_0)) + \ln \det(I_{|\mathcal{S}_1|} + \sigma_1^2 \mathbf{A}_1^T \Phi(S_0)^{-1} \mathbf{A}_1)$$  \hspace{1cm} (25)

$$= \ln \det(\Phi(S_0)) + \ln \det(I_{|\mathcal{S}_1|} + \sigma_1^2 \mathbf{A}_1^T \mathbf{U}_0(\sigma_1^2 \Sigma_0 \Sigma_0^T + \sigma_2^2 \mathbf{I}_M)^{-1} \mathbf{U}_0^T \mathbf{A}_1)$$  \hspace{1cm} (26)

$$\leq \ln \det(\Phi(S_0)) + |\mathcal{S}_1| \ln \left(1 + \frac{\sigma_1^2}{\sigma_2^2}(1 + \varepsilon)\right).$$  \hspace{1cm} (27)

The inequality in (27) follows from the facts that the maximum singular value of the matrix $\mathbf{A}_1$ is $\sqrt{1 + \varepsilon}$ and maximum value on the diagonal of the diagonal matrix $(\sigma_1^2 \Sigma_0 \Sigma_0^T + \sigma_2^2 \mathbf{I}_M)^{-1}$ is $\frac{1}{\sigma_2^2}$ and $\sigma_1^2 \mathbf{A}_1^T \mathbf{U}_0(\sigma_1^2 \Sigma_0 \Sigma_0^T + \sigma_2^2 \mathbf{I}_M)^{-1} \mathbf{U}_0^T \mathbf{A}_1$, being a symmetric and positive definite matrix, has SVD of the form $\mathbf{U} \Sigma \mathbf{U}^T$. A lower bound on $\gamma_1(S_0)$ can be obtained proceeding in a similar way as (27) was obtained but taking lower bound instead of upper bound. We note that from corollary 2.1 the minimum singular value of $\sigma_1^2 \mathbf{A}_2^T \Phi(S_0)^{-1} \mathbf{A}_2$ is at least $\frac{1 - 2\varepsilon}{1 - \varepsilon} \frac{1}{\sigma_2^2}$. Thus

$$\gamma_1(S_0) = \ln \det(\Phi(S_0)) \geq \ln \det(\Phi(S_0)) + |\mathcal{S}_2| \left(1 + \frac{\sigma_2^2}{\sigma_2^2} \left(1 - \frac{2\varepsilon}{1 - \varepsilon}\right)\right).$$  \hspace{1cm} (28)

Let the SVD of $\mathbf{A}_{01}$ be $\mathbf{A}_{01} = \mathbf{U}_{01} \Sigma_{01} \mathbf{V}_{01}^T$. Let $\mathbf{U}_{01}$ denote the submatrix of $\mathbf{U}_{01}$ consisting of the first $|\mathcal{S}_{01}|$ columns and $\mathbf{U}_{01}$ denote the submatrix with the rest of the columns. Thus $\mathbf{U}_{01}$ forms an orthonormal basis for the column space $\mathcal{A}_{01}$ of $\mathbf{A}_{01}$. $\mathbf{U}_{01}$ forms an orthonormal basis for the space $\mathbb{R}^M \setminus \mathcal{A}_{01}$. The measured data $\mathbf{y}$ is noisy linear combination of the columns of $\mathbf{A}$ selected by $\mathcal{S}_0$.

$$\mathbf{y} - \mu_1 \mathbf{A}_{01} \mathbf{1}_{01} = \mathbf{A}_{01} \mathbf{z}_{01} + \mathbf{e} = \mathbf{U}_{01} \Sigma_{01} \mathbf{V}_{01}^T \mathbf{z}_{01} + \mathbf{e}$$  \hspace{1cm} (29)

Let $\mathbf{e} = \mathbf{U}_{01} \mathbf{e}_{01} + \mathbf{U}_{01} \mathbf{e}_{01}$. Thus from (9) and (29) and the fact that $\mathbf{U}_{01}^T \mathbf{A}_{01} \mathbf{z}_{01} = \mathbf{0}_{M - |\mathcal{S}_{01}|}$,

$$\gamma_2(S_0) = (\mathbf{y} - \mu_1 \mathbf{A}_{01} \mathbf{1}_{01})^T \Phi(S_0)^{-1} (\mathbf{y} - \mu_1 \mathbf{A}_{01} \mathbf{1}_{01})$$  \hspace{1cm} (30)

$$= (\mathbf{U}_{01} \Sigma_{01} \mathbf{V}_{01}^T \mathbf{z}_{01} + \mathbf{e})^T \mathbf{U}_{01} (\sigma_1^2 \Sigma_{01} \Sigma_{01}^T + \sigma_2^2 \mathbf{I}_M)^{-1} \mathbf{U}_{01}^T (\mathbf{U}_{01} \Sigma_{01} \mathbf{V}_{01}^T \mathbf{z}_{01} + \mathbf{e})$$  \hspace{1cm} (31)

$$= (\Sigma_{01} \mathbf{V}_{01}^T \mathbf{z}_{01} + \mathbf{e}_{01})^T (\Sigma_{01} \Sigma_{01}^T + \sigma_2^2 \mathbf{I}_{01})^{-1} (\Sigma_{01} \mathbf{V}_{01}^T \mathbf{z}_{01} + \mathbf{e}_{01}) + \frac{1}{\sigma_2^2} \|\mathbf{e}_{01}\|_2^2$$  \hspace{1cm} (32)

$$\leq \frac{\sqrt{1 + \varepsilon} \|\mathbf{z}_{01}\|_2 + \|\mathbf{e}_{01}\|_2}{1 - \varepsilon} \|\mathbf{z}_{01}\|_2 + \frac{\|\mathbf{e}_{01}\|_2^2}{\sigma_2^2}$$  \hspace{1cm} (33)

$$< \frac{2(1 + \varepsilon) \|\mathbf{z}_{01}\|_2 + 2 \|\mathbf{e}_{01}\|_2^2}{\sigma_1^2} + \frac{\|\mathbf{e}_{01}\|_2}{\sigma_2^2}.$$  \hspace{1cm} (34)

Now we obtain a lower bound on $\gamma_2(S_{02})$. Let the SVD of $\mathbf{A}_{02}$ be $\mathbf{A}_{02} = \mathbf{U}_{02} \Sigma_{02} \mathbf{V}_{02}^T$. Let $\mathbf{U}_{02}$, $\mathbf{U}_{02}$ and $\Sigma_{02}$ be defined similar to $\mathbf{U}_{01}$, $\mathbf{U}_{01}$ and $\Sigma_{01}$ respectively. Let $\mathbf{W}_{1/02}$ be an orthonormal basis for the subspace spanned by $\mathbf{U}_{02} \mathbf{U}_{02}^T \mathbf{U}_{1}$. Let us denote this subspace by $\mathcal{A}_{1/02}$. Also, let $\mathbf{U}_{012}$ be an orthonormal basis for the column space of $\mathbf{A}_{012}$ and $\mathbf{U}_{012}$ be an orthonormal basis for
the left null space $\mathbb{R}^M \setminus \mathcal{A}_{012}$. The two subspaces $\mathcal{A}_{1\setminus 02}$ and $\mathbb{R}^M \setminus \mathcal{A}_{012}$ are orthogonal and their union is the subspace $\mathbb{R}^M \setminus \mathcal{A}_{02}$. Now,

$$
\gamma_2(S_{02}) = (y - \mu_1 A_{02} 1_{02})^T \Phi(S_{02})^{-1} (y - \mu_1 A_{02} 1_{02}) \tag{35}
$$

$$
= (y - \mu_1 A_{02} 1_{02})^T U_{02}(\sigma_1^2 \Sigma_{02} \Sigma_{02}^T + \sigma_2^2 I_M)^{-1} U_{02}^T (y - \mu_1 A_{02} 1_{02}) \tag{36}
$$

$$
= (y - \mu_1 A_{02} 1_{02})^T \bar{U}_{02}(\bar{\sigma}_1^2 \bar{\Sigma}_{02} \bar{\Sigma}_{02}^T + \sigma_e^2 I_2)^{-1} \bar{U}_{02}^T (y - \mu_1 A_{02} 1_{02}) + \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T (y - \mu_1 A_{02} 1_{02})\|^2_2 \tag{37}
$$

$$
\geq \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T (y - \mu_1 A_{02} 1_{02})\|^2_2 \tag{38}
$$

$$
= \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T (A_0 x_0 + A_1 x_1 + e - \mu_1 A_{02} 1_{02})\|^2_2 = \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T (A_1 x_1 + e)\|^2_2 \tag{39}
$$

$$
= \frac{1}{\sigma_e^2} \|\tilde{W}_{1\setminus 02}^T (A_1 x_1 + e)\|^2_2 + \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T (A_1 x_1 + e)\|^2_2 \tag{40}
$$

$$
= \frac{1}{\sigma_e^2} \|\tilde{U}_{02}^T A_1 x_1 + \tilde{W}_{1\setminus 02}^T e\|^2_2 + \frac{1}{\sigma_e^2} \|\bar{U}_{02}^T e\|^2_2 \tag{41}
$$

Now from proposition $\mathbb{P}$ $\|\tilde{U}_{02}^T A_1 x_1\|_2 \geq \sqrt{\frac{1-2\epsilon}{1-\epsilon}} |x_1|_2$. We assume that $\sqrt{\frac{1-2\epsilon}{1-\epsilon}} |x_1|_2 > \|\tilde{W}_{1\setminus 02}^T e\|_2$. Otherwise there is nothing left to prove. We note that $\|\tilde{W}_{1\setminus 02}^T e\|_2 = \|\tilde{e}_{1\setminus 02}\|_2 \leq \|\tilde{e}_1\|_2$. Thus,

$$
\gamma_2(S_{02}) \geq \frac{1}{\sigma_e^2} \left(\sqrt{\frac{1-2\epsilon}{1-\epsilon}} |x_1|_2 - \|\tilde{e}_1\|_2\right)^2 + \frac{1}{\sigma_e^2} \|\tilde{e}_{012}\|_2^2. \tag{42}
$$

Also,

$$
\gamma_3(S_{01}) - \gamma_3(S_{02}) = (|S_1| - |S_2|) \ln \frac{1-p}{p}. \tag{43}
$$

Now since $S_{02} = \tilde{S}$, $\gamma(S_{02}) \leq \gamma(S_{01})$. Thus, from (27), (28), (31), (12) and (13),

$$
|S_2| \left(\ln \left(1 + \frac{\sigma_2^2}{\sigma_e^2} \left(\frac{1-2\epsilon}{1-\epsilon}\right)\right) + 2 \ln \frac{1-p}{p} \right) + \left(\sqrt{\frac{1-2\epsilon}{1-\epsilon}} |x_1|_2 - \|\tilde{e}_1\|_2\right)^2 + \frac{\|\tilde{e}_{012}\|_2^2}{\sigma_e^2} \leq |S_1| \left(\ln \left(1 + \frac{\sigma_2^2}{\sigma_e^2} (1+\epsilon)\right) + 2 \ln \frac{1-p}{p} \right) + \frac{\|\tilde{e}_{01}\|_2^2}{\sigma_e^2} + \frac{2(1+\epsilon)}{\|z_{01}\|_2^2} + \frac{2\|\tilde{e}_{01}\|_2^2}{(1-\epsilon)\sigma_1^2}. \tag{44}
$$

Since $|S_2| \geq 0$ and $p < 1/2$ for sparse signals, the first term is non-negative. Hence,

$$
\left(\sqrt{\frac{1-2\epsilon}{1-\epsilon}} |x_1|_2 - \|\tilde{e}_1\|_2\right)^2 \leq |S_1| \left(\ln \left(1 + \frac{\sigma_2^2}{\sigma_e^2} (1+\epsilon)\right) + 2 \ln \frac{1-p}{p} \right) + \frac{2(1+\epsilon)}{(1-\epsilon)\sigma_1^2} + \frac{2\|\tilde{e}_{01}\|_2^2}{\sigma_e^2} + \|\tilde{e}_{012}\|_2^2. \tag{45}
$$

Now $\|\tilde{e}_{01}\|_2^2 - \|\tilde{e}_{012}\|_2^2 \leq \|\tilde{e}_{2\setminus 01}\|_2^2 \leq \|\tilde{e}_2\|_2^2$. Now consider the expression $\|\tilde{e}_{01}\|_2^2$. Note that $\tilde{e}_{01} = \tilde{U}_{01}^T e$ is the projection of $e$ onto the $|S_{01}|$-dimensional subspace $\mathcal{A}_{01}$. Thus $\tilde{e}_{01} \sim \mathcal{N}(0, \sigma_e^2 I_{01})$. 


Let $\tilde{e}_{01}^T = [\tilde{e}_{01}^T, \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{|2Np-|S_0|]}$ such that $\tilde{e}_{01} \sim \mathcal{N}(0, \sigma_e^2 I_{|2Np|})$. By proposition 31 $\|\tilde{e}_{01}\|^2 \leq \|\tilde{e}_{01}\|^2 \leq 2\beta Np\sigma_e^2$ with probability exceeding $1 - e^{-Np(\beta - 1 - \ln \beta)}$ for $\beta > 1$. Similarly $\|z_{01}\|^2 \leq 2\beta Np\sigma_e^2$ with probability exceeding $1 - e^{-Np(\beta - 1 - \ln \beta)}$ and $\|\tilde{e}_2\|^2 \leq 2\beta Np\sigma_e^2$ probability exceeding $1 - e^{-Np(\beta - 1 - \ln \beta)}$.

Therefore with probability exceeding $1 - 3e^{-Np(\beta - 1 - \ln \beta)}$,

$$
\frac{(\sqrt{1 - \tilde{e}_2^2} \|x_1\|^2 - \|\tilde{e}_1\|^2)}{\sigma_e^2} \leq C|S_1| + 4 \left(1 + \frac{1}{1 - \epsilon}\right) \beta Np + 4\beta Np + 2\beta Np (46)
$$

since $\epsilon \leq 1/3$. Thus,

$$\sqrt{1 - \tilde{e}_2^2} \|x_1\|^2 \leq \|\tilde{e}_1\|^2 + \sqrt{(14\beta + 2C)Np\sigma_e} \leq (\sqrt{2\beta + \sqrt{14\beta + 2C})\sqrt{Np}\sigma_e. (48)}$$

Since $\epsilon \leq 1/3$, we can write (48) as $\|x_1\|^2 \leq \sqrt{(2\sqrt{2\beta + \sqrt{14\beta + 2C})\sqrt{Np}\sigma_e$. This holds with overall probability exceeding $(1 - e^{-Np(2\ln^2 - 1)})(1 - 3e^{-Np(\beta - 1 - \ln \beta)})$ for $\beta > 1$.

### 3.3 Proof of Theorem 2

Similar to theorem 31 we assume that event $E$, $i.e., |S_0| \leq 2Np$ is true. This holds with probability exceeding $1 - e^{-Np(2\ln^2 - 1)}$. For the rest of the proof all events and probabilities are conditioned on this event.

Here we show that if $\mu_1$ and $\sigma_1$ satisfy the condition stated in theorem 2 then $\gamma(S_02)$ cannot be smaller or equal to $\gamma(S_01)$ unless $S_1 = S_2 = \emptyset$. We obtained upper bound on $\gamma(S_01)$ and lower bound on $\gamma(S_02)$ in the proof of theorem 31. If the lower bound is greater than the upper bound then we reach a contradiction that $\gamma(S_02)$ cannot be the estimate of $S$. This happens when the inequality in (41) is reversed, $i.e.,$

$$|S_2| \left( \ln \left( 1 + \frac{\sigma^2}{\sigma_e^2} \left( \frac{2}{1 - \epsilon} \right) \right) + 2 \ln \frac{1 - p}{p} \right) + \left( \sqrt{1 - \tilde{e}_2^2} \|x_1\|^2 - \|\tilde{e}_1\|^2 \right) \frac{2}{\sigma_e^2} + \frac{\|e_{01}\|^2}{\sigma_e^2} > |S_1| \left( \ln \left( 1 + \frac{\sigma^2}{\sigma_e^2} (1 + \epsilon) \right) + 2 \ln \frac{1 - p}{p} \right) + \frac{\|e_{01}\|^2}{\sigma_e^2} + \frac{2(1 + \epsilon)\|z_{01}\|^2}{(1 - \epsilon)\sigma_1^2} + \frac{2\|\tilde{e}_{01}\|^2}{\sigma_e^2}. (49)$$

Thus the following inequality is sufficient for (49) to be true.

$$\frac{(\sqrt{1 - \tilde{e}_2^2} \|x_1\|^2 - \|\tilde{e}_1\|^2)}{\sigma_e^2} + \frac{\|e_{01}\|^2}{\sigma_e^2} > |S_1| \left( \ln \left( 1 + \frac{\sigma^2}{\sigma_e^2} (1 + \epsilon) \right) + 2 \ln \frac{1 - p}{p} \right) + \frac{\|e_{01}\|^2}{\sigma_e^2} + \frac{2(1 + \epsilon)\|z_{01}\|^2}{(1 - \epsilon)\sigma_1^2} + \frac{2\|\tilde{e}_{01}\|^2}{\sigma_e^2}. (50)$$
This is equivalent to,
\[
\left( \frac{1-2\varepsilon}{\sigma_e^2} \|x_1\|_2 - \|\tilde{e}_1\|_2 \right)^2 > |S_1| \left( \ln \left( 1 + \frac{\sigma_e^2}{\sigma_1^2} (1 + \varepsilon) \right) + 2 \ln \frac{1 - p}{p} \right) + \frac{\|\epsilon_{01}\|_2^2 - \|\epsilon_{012}\|_2^2}{\sigma_e^2} + \frac{2(1 + \varepsilon)\|z_{01}\|_2^2}{\sigma_1^2} + \frac{2\|\tilde{e}_{01}\|_2^2}{\sigma_e^2}.
\] (51)

We have seen in the proof of theorem 1 that the right hand side is bounded above by \((14\beta + 2C)Np\) with probability exceeding \(1 - 3e^{-Np(\beta - 1 - \ln \beta)}\). Thus if
\[
\|x_1\|_2 > \sqrt{2(\sqrt{2\beta} + \sqrt{14\beta + 2C})\sqrt{Np}\sigma_e},
\] (52)

(51) is satisfied with probability exceeding \(1 - 3e^{-Np(\beta - 1 - \ln \beta)}\). Now for sufficiently large \(|\mu_1|\), \(\|x_1\|_2 = \|\mu_1 1 + z_1\|_2 \geq \|\mu_1 1\|_2 - \|z_1\|_2 \geq (|\mu_1| - \sqrt{\beta}\sigma_1)\sqrt{|S_1|}\) with probability exceeding \(1 - e^{-\frac{|S_1|(\beta - 1 - \ln \beta)}{2}}\). If \(|S_1| \geq 1\), then \(\gamma(S_{02})\) becomes greater than \(\gamma(S_{01})\) with probability exceeding \(1 - 3e^{-Np(\beta - 1 - \ln \beta)} - e^{-\frac{(\beta - 1 - \ln \beta)}{2}}\) if,
\[
|\mu_1| > \sqrt{\beta}\sigma_1 + \sqrt{2(\sqrt{2\beta} + \sqrt{14\beta + 2C})\sqrt{Np}\sigma_e}.
\] (53)

Hence \(|S_1| = 0\) i.e., the set \(S_1\) is empty and \(S_{01} = S_0\). Thus (53) is a probabilistic sufficient condition that no active coefficient is missing. Now we assume that (53) is satisfied and we investigate what (additional) condition guarantees no false alarm with very high probability. We assume \(S_2\) is not empty and find out the condition on \(\mu_1\) and \(\sigma_1\) that contradicts this assumption.

\[
\gamma_1(S_{02}) \geq \gamma_1(S_{01}) + |S_2| \left( 1 + \frac{\sigma_e^2}{\sigma_1^2} \left( \frac{1 - 2\varepsilon}{1 - \varepsilon} \right) \right),
\] (54)

and, \(\gamma_3(S_{02}) = \gamma_3(S_{01}) + |S_2| \ln \frac{1 - p}{p}.
\) (55)

Since set \(S_1\) is empty,
\[
\gamma_1(S_{01}) \leq \frac{\|\tilde{e}_0 + A_0 z_0\|_2^2}{(1 - \varepsilon)\sigma_1^2 + \sigma_2^2} + \frac{\|\epsilon_0\|_2^2}{\sigma_e^2}.
\] (56)

In obtaining (58) from (37) we lower bounded the first term by zero. Now we use a tighter lower bound by explicitly using the condition that \(\mu_1 \neq 0\).

\[
\gamma_2(S_{02}) = (y - \mu_1 A_{02} 1_{02})^T \tilde{U}_{02} (\sigma_e^2 \tilde{\Sigma}_{02}^T I_{02} + \sigma_2^2 I_{02})^{-1} \tilde{U}_{02}^T (y - \mu_1 A_{02} 1_{02}) + \frac{1}{\sigma_e^2} \|\tilde{U}_{02} (y - \mu_1 A_{02} 1_{02})\|_2^2.
\] (57)

Here \(y = A_0 x_0 + e\). Thus \(\tilde{U}_{02}^T (y - \mu_1 A_{02} 1_{02}) = \tilde{U}_{02}^T e = \epsilon_{02}\). Let \(\tilde{W}_{0|2}\) and \(\tilde{W}_{2|0}\) be the
orthonormal bases for the orthogonal subspaces $A_0 \setminus A_2$ and $A_2 \setminus A_0$ respectively. Thus,

\[
\gamma_2(S_{02}) \geq \frac{\|U_{02}^T(y - \mu_1 A_0 1_{02})\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|e_{02}\|_2^2}{\sigma_e^2} 
\]

(58)

\[
\geq \frac{\|W_{02}^T(y - \mu_1 A_0 1_{02})\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|W_{21}^T(y - \mu_1 A_0 1_{02})\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|e_{02}\|_2^2}{\sigma_e^2} 
\]

(59)

\[
= \frac{\|W_{02}^T(A_0 z_0 + e_0)\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|W_{21}^T(A_0 z_0 + e_0)\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|e_{02}\|_2^2}{\sigma_e^2} 
\]

(60)

\[
= \frac{\|U_{02}^T(A_0 z_0 + \bar{e}_0)\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|U_{21}^T(A_0 z_0 + \bar{e}_0)\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} + \frac{\|e_{02}\|_2^2}{\sigma_e^2} 
\]

(61)

\[
\geq \left(1 - \frac{2\varepsilon}{1 - \varepsilon^2}\right) \left[\frac{\|A_0 z_0 + e_0\|_2^2}{(1 - \varepsilon)\sigma_1^2 + \sigma_e^2} - \frac{\|e_2\|_2^2}{\sigma_e^2}\right] 
\]

(62)

\[
= \frac{1}{(1 - \varepsilon)\sigma_1^2 + \sigma_e^2} - \frac{1}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} 
\]

(63)

The last inequality follows from proposition 2. Noting that $\|e_0\|_2^2 - \|e_{02}\|_2^2 = \|\bar{e}_{2,0}\|_2^2 \leq \|\bar{e}_2\|_2^2$,

\[
\gamma_2(S_{02}) - \gamma_2(S_{01}) \geq \left(1 - \frac{2\varepsilon}{1 - \varepsilon^2}\right) \left[\frac{\|e_2 - \mu_1 A_2 1_2\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} - \frac{\|\bar{e}_2\|_2^2}{\sigma_e^2}\right] 
\]

(64)

\[
= \frac{1}{(1 - \varepsilon)\sigma_1^2 + \sigma_e^2} - \frac{1}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} 
\]

(65)

\[
\geq \left(\frac{\varepsilon(4 + \varepsilon)}{(1 - \varepsilon)(1 + \varepsilon)^2}\right) \frac{\|A_0 z_0 + \bar{e}_0\|_2^2}{\sigma_1^2 + \sigma_e^2} \leq 4 \left[\frac{\varepsilon(4 + \varepsilon)}{(1 - \varepsilon)(1 + \varepsilon)^2}\right] \beta Np < 6\beta Np 
\]

(66)

since $\frac{\varepsilon(4 + \varepsilon)}{(1 - \varepsilon)(1 + \varepsilon)^2} < \frac{3}{2}$ for $\varepsilon \leq \frac{1}{3}$. Also, $\|\bar{e}_2\|_2^2/\sigma_e^2 \leq 2\beta Np$. Then from (63) and (66),

\[
\gamma_2(S_{02}) - \gamma_2(S_{01}) \geq \left(1 - \frac{2\varepsilon}{1 - \varepsilon^2}\right) \left[\frac{\|e_2 - \mu_1 A_2 1_2\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2} - 8\beta Np\right] 
\]

(67)

Thus from (54), (55), and (67),

\[
\gamma(S_{02}) - \gamma(S_{01}) \geq |S_2| \left[\frac{1}{2} \left(\frac{\sigma_1^2}{\sigma_e^2} + \frac{\|e_2 - \mu_1 A_2 1_2\|_2^2}{(1 + \varepsilon)\sigma_1^2 + \sigma_e^2}\right) + \ln \frac{1 - p}{p}\right] 
\]

(68)

\[
+ \frac{1}{2} \left(\frac{1 - 2\varepsilon}{1 - \varepsilon^2}\right) \|e_2 - \mu_1 A_2 1_2\|_2^2 - 4\beta Np. 
\]

The coefficient of the term $|S_2|$ is positive for sparse problems when $p \leq \frac{1}{2}$. Then for any positive value of $|S_2|$, we reach a contradiction to the assumption that $S_{02}$ is the estimate if

\[
\frac{1}{2} \left(\frac{1 - 2\varepsilon}{1 - \varepsilon^2}\right) \|e_2 - \mu_1 A_2 1_2\|_2^2 > 4\beta Np. 
\]

(69)
Since $\varepsilon \leq \frac{1}{3}$, it is sufficient to reach a contradiction that,
\[
\|\hat{e}_2 - \mu_1 A_2 1_2\|_2^2 > 32\beta Np\sigma_1^2 + 24\beta Np\sigma_e^2.
\] (70)

We note that $32\beta Np\sigma_1^2 + 24\beta Np\sigma_e^2 < (4\sqrt{2\beta Np}\sigma_1 + 2\sqrt{6\beta Np}\sigma_e)^2$ and $\|e_2\|_2 \leq \sqrt{2\beta Np}\sigma_e$. Thus if $|S_2| \geq 1$, and
\[
|\mu_1|(1 - \varepsilon) > \sqrt{2\beta Np}\sigma_e + 4\sqrt{2\beta Np}\sigma_1 + 2\sqrt{6\beta Np}\sigma_e
\] (71)
\[
= 4\sqrt{2\beta Np}\sigma_1 + (1 + 2\sqrt{3})\sqrt{2\beta Np}\sigma_e
\] (72)

then $\gamma(S_{02})$ cannot be smaller than or equal to $\gamma(S_{01})$. Thus $S_2$ must be empty. Since $\varepsilon \leq \frac{1}{3}$, a probabilistic sufficient condition for no false alarm is
\[
|\mu_1| > 6\sqrt{2\beta Np}\sigma_1 + 3\left(\frac{1}{2} + \sqrt{3}\right)\sqrt{2\beta Np}\sigma_e,
\] (73)
which holds with probability exceeding $1 - 3e^{-Np(\beta - 1 - \ln \beta)}$.

4 Discussion

From theorem [1] we see that the energy of the true signal restricted to the missed coefficients is of the order of energy in the projection of noise onto the subspace spanned by the true signal. A natural question that arises is what can we say about the estimate of the signal $\hat{x}$ obtained by regressing with the measurement matrix restricted to the columns indexed by $\hat{S}$? We mention here that $\hat{x}$ is not an optimal estimate of $x$ like MAP or MMSE estimates obtained directly from the observed data. Now $\hat{x}$ is given by
\[
\hat{x} = \arg \min_{x \in \mathbb{R}^N} \|y - Ax\|_2^2.
\] (74)

and it can be easily shown that
\[
\hat{x}_{\hat{S}} = \hat{x}_{02} = (A_{02}^T A_{02})^{-1} A_{02}^T y = V_{02} \Sigma_{02}^{-1} U_{02}^T (A_0 x_0 + A_1 x_1 + e)
\] (75)
\[
= V_{02} \Sigma_{02}^{-1} U_{02}^T (A_0 x_0 + A_1 x_1 + e) = x_{02} + V_{02} \Sigma_{02}^{-1} U_{02}^T (A_1 x_1 + e).
\] (76)

Now $\|V_{02} \Sigma_{02}^{-1} U_{02} A_1 x_1\|_2 \leq \frac{1}{\sqrt{1 - \varepsilon}} \frac{\varepsilon}{\sqrt{1 - \varepsilon}} \|x_1\|_2 \leq \frac{\varepsilon}{1 - \varepsilon} K_1 \sqrt{Np\sigma_e}$ and $\|V_{02} \Sigma_{02}^{-1} U_{02}^T e\|_2 \leq \sqrt{\frac{2}{1 - \varepsilon}} \sqrt{Np\sigma_e}$.

Also $\|x_1 - \hat{x}_1\|_2 = \|x_1\|_2 \leq K_1 \sqrt{Np\sigma_e}$. Thus $\|\hat{x} - x\|_2 \leq \left(\frac{K_1}{1 - \varepsilon} + \sqrt{\frac{2}{1 - \varepsilon}}\right) \sqrt{Np\sigma_e}$ with probability exceeding $(1 - e^{-Np(2\ln 2 - 1)})(1 - 3e^{-Np(\beta - 1 - \ln \beta)})$. This is optimal in the sense that even if the true support was known it is not possible to do any better. This also shows that even if there is any coefficient $i$ falsely detected, due to the restricted isometry property, it’s estimate $\hat{x}_{i(\hat{S})}$ must be small.

Let us analyze the values of the constants appearing in the theorem statements. Consider the example where $N = 4096, p = 0.01, M = 256, \mu_1 = 0$ and nominal SNR $10\log \frac{Np\sigma_1^2}{M\sigma_e^2} = 20$ dB.
Figure 1: The plot in the upper panel shows the constant $K_1$ as a function of the parameter $\beta$. Here $N = 4096, p = 0.01, M = 256, \mu_1 = 0$ and nominal SNR $10\log_{10}\frac{Np\sigma^2}{M\sigma^2_e} = 20$ dB. The figure in the bottom panel shows the least probability with which the energy in the missed coefficients is upper bounded by $K_1^2 Np\sigma^2_e$.

Then for $\beta = 1.6$, $K_1 = 12.94$ and the probability is at least $0.9854$ and for $\beta = 2$, $K_1 = 13.77$ and the probability is at least $1 - 1.06 \times 10^{-5}$. So the constants are modest for reasonable values of the system parameters. Fig. 1 shows the plots of the constant $K_1$ and the lower bound of the probability as functions of the parameter $\beta$ for this example. For the same values of $N, M, p$ and $\sigma^2_1$, theorem 2 gives the value of $K_2$ needed to obtain the lower bound on the absolute value of the mean $\mu_1$ to probabilistically guarantee perfect support recovery. If $\beta = 1.6, \bar{\beta} = 16$, then $K_3 = 10.75\sqrt{Np}, K_4 = 12.94\sqrt{2Np}$ and the probability is at least $0.9832$ and if $\beta = 2, \bar{\beta} = 25$, then $K_3 = 12.01\sqrt{Np}, K_4 = 13.77\sqrt{2Np}$ and the probability is at least $1 - 4.13 \times 10^{-5}$.

From the statement of theorem 1 we see that the constant $K_1$ depends on $C = \ln(1 + \frac{\sigma^2_1}{\sigma^2_e})$. The term $\frac{\sigma^2_1}{\sigma^2_e}$ is related to SNR. We see from Fig. 1 that with SNR the constant $K_1$ increases. So if the SNR increases in an unbounded fashion keeping the noise energy constant then does the energy in the missed support grows unbounded? The answer is no. If $\sigma_1$ becomes very large then irrespective of the value of $\mu_1$, the probability that any element of $x$ is close to zero and suppressed by noise becomes very small and every element is detected with high probability. From (46) we can see that

$$\left(\frac{1-2\epsilon}{1-\epsilon}\|x_1\|_2 - \|\bar{e}_1\|_2\right)^2 \leq C|S_1| + 8\beta Np + 4\beta Np + 2\beta Np.$$  

(77)

If $|S_1| \neq 0$, the left hand side grows as $\frac{\sigma^2_1}{\sigma^2_e}|S_1|$ whereas the right hand side grows as $\ln(\frac{\sigma^2_1}{\sigma^2_e})|S_1|$. Thus as SNR grows very large, set $S_1$ has to be empty and there is no missed coefficient with very high
probability. Therefore the upper bound stated in theorem is loose in the very high SNR regime. For any practical value of the SNR the term \( \ln(1 + \frac{\sigma^2}{\epsilon^2}) \) has a moderate value. Hence the constant \( K_1 \) is a reasonably small constant.

In order to obtain simple expressions in the theorem statements we have used the inequality \( \epsilon \leq \frac{1}{3} \) instead of having \( \epsilon \) appearing in those expressions. As a consequence the constants in the results show the worst case scenarios when \( \epsilon = \frac{1}{3} \). Proceeding in a similar way, for other values of the RIP constant we can obtain tighter constant values in our results.

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