On the relation between weighted trees and tropical Grassmannians

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Abstract. — In this article, we will prove that the set of $4$-dissimilarity vectors of $n$-trees is contained in the tropical Grassmannian $\mathcal{G}_{4,n}$. We will also propose three equivalent conjectures related to the set of $m$-dissimilarity vectors of $n$-trees for the case $m \geq 5$. Using a computer algebra system, we can prove these conjectures for $m = 5$.

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1 Introduction

Let $T$ be a tree with $n$ leaves, which are numbered by the set $[n] := \{1, \ldots, n\}$. Such a tree is called an $n$-tree. We assume that $T$ is weighted, so each edge has a length. Denote by $D(i, j)$ the distance between the leaves $i$ and $j$ (i.e. the sum of the lengths of the edges of the unique path in $T$ from $i$ to $j$). We say that $D = (D(i, j))_{i,j} \in \mathbb{R}^{n \times n}$ is the dissimilarity matrix of $T$, or conversely, that $D$ is realized by $T$. The set of dissimilarity matrices of $n$-trees is fully described by the following theorem (see [2] or [3, Theorem 2.36]).

**Theorem 1.1** (Tree Metric Theorem). Let $D \in \mathbb{R}^{n \times n}$ be a symmetric matrix with zero entries on the main diagonal. Then $D$ is a dissimilarity matrix of an $n$-tree if and only if the four-point condition holds, i.e. for every four (not necessarily distinct) elements $i, j, k, l \in [n]$, the maximum of the three numbers $D(i, j) + D(k, l), D(i, k) + D(j, l)$ and $D(i, l) + D(j, k)$ is attained at least twice. Moreover, the $n$-tree $T$ that realizes $D$ is unique.

If $T$ is an $n$-tree, $(D(i, j))_{i<j} \in \mathbb{R}^{\binom{n}{2}}$ is called the dissimilarity vector of $T$.

We can reformulate the above theorem in the context of tropical geometry (see [1, Theorem 4.2]). For some background, I refer to section 2.

**Theorem 1.2.** The set $T_n$ of dissimilarity vectors of $n$-trees is equal to the tropical Grassmannian $\mathcal{G}_{2,n}$.

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We can generalize the definition of dissimilarity vectors of $n$-trees. Let $m$ be an integer with $2 \leq m < n$ and let $i_1, \ldots, i_m$ be pairwise distinct elements of $\{1, \ldots, n\}$. Denote by $D(i_1, \ldots, i_m)$ the length of the smallest subtree of $T$ containing the leaves $i_1, \ldots, i_m$. We say that the point $D = (D(i_1, \ldots, i_m))_{i_1 < \ldots < i_m} \in \mathbb{R}^{ \binom{n}{m} }$ is the $m$-dissimilarity vector of $T$.

The following result gives a formula for computing the $m$-subtree weights from the pairwise distances of the leafs of an $n$-tree (see [1, Theorem 3.2]).

**Theorem 1.3.** Let $n$ and $m$ be integers such that $2 \leq m < n$. Denote by $C_m \subset S_m$ the set of cyclic permutations of length $m$. Let

$$ \phi(m) : \mathbb{R}^{ \binom{n}{2} } \to \mathbb{R}^{ \binom{n}{m} } : X = (X_{i,j}) \mapsto (X_{i_1, \ldots, i_m}) $$

be the map with

$$ X_{i_1, \ldots, i_m} = \frac{1}{2} \min_{\sigma \in C_m} \{X_{i_1, i_{\sigma(1)}}, X_{i_{\sigma(1)}, i_{\sigma(2)}}, \ldots, X_{i_{\sigma(m-1)(1)}, i_{\sigma(m)(1)}}\}. $$

If $D \in T_n \subset \mathbb{R}^{ \binom{n}{2} }$ is the dissimilarity vector of an $n$-tree $T$, then the $m$-dissimilarity vector of $T$ is equal to $\phi(m)(D)$. So $\phi(m)(T_n)$ is the set of $m$-dissimilarity vectors of $n$-trees.

The description of the set of $m$-dissimilarity vectors of $n$-trees as the image of $T_n$ under the map $\phi(m)$ is not useful to decide whether or not a given point in $\mathbb{R}^{ \binom{n}{m} }$ is an $m$-dissimilarity vector. So we are interested in finding a nice description of these sets as subsets of $\mathbb{R}^{ \binom{n}{m} }$. The case $m = 3$ is solved by the following result (see [1, Theorem 4.6]).

**Theorem 1.4.** $\phi(3)(T_n) = G_{3,n} \cap \phi(3)(\mathbb{R}^{ \binom{n}{2} })$.

In this article, we prove the following partial answer for the case $m = 4$.

**Theorem 1.5.** $\phi(4)(T_n) \subset G_{4,n} \cap \phi(4)(\mathbb{R}^{ \binom{n}{2} })$.

To finish the article, we propose three equivalent conjectures for the case $m \geq 5$. The case $m = 5$ is solved using a computer algebra system.

## 2 Tropical geometry

Consider the tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where the tropical sum is the maximum of two numbers and the tropical product is the usual sum of the numbers. Let $x_1, \ldots, x_k$ be real variables. Tropical monomials $x_1^{i_1} \ldots x_k^{i_k}$ represent linear forms $i_1x_1 + \ldots + i_kx_k$ and tropical polynomials $\oplus_{i \in I} a_ix_1^{i_1} \ldots x_k^{i_k}$ (with $I \subset \mathbb{N}^k$ finite) represent piece-wise linear forms

$$ \max_{i \in I} \{a_i + i_1x_1 + \ldots + i_kx_k\}. $$

(1)
If $F$ is such a tropical polynomial, we define the tropical hypersurface $\mathcal{H}(F)$ to be its corner locus, i.e. the points $x \in \mathbb{R}^k$ where the maximum is attained at least twice.

Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series, i.e. the field of formal sums $c = \sum c_q t^q$ in the variable $t$ such that the set $S_c = \{q | c_q \neq 0\}$ is bounded below and has a finite set of denominators. For each $c \in K^*$, the set $S_c$ has a minimum, which we call the valuation of $c$ and is denoted by $\text{val}(c)$.

A polynomial $f = \sum_{i \in I} f_i x_i^1 \ldots x_i^k$ over $K$ gives rise to a tropical polynomial $\text{trop}(f)$, defined by taking $a_i = -\text{val}(f_i)$ in (1).

**Theorem 2.1.** If $I \subset K[x_1, \ldots, x_k]$ is an ideal, the following two subsets of $\mathbb{R}^k$ coincide:

1. the intersection of all tropical hypersurfaces $\mathcal{T}(\text{trop}(f))$ with $f \in I$;
2. the closure in $\mathbb{R}^k$ of the set
   
   $$\{(-\text{val}(x_1), \ldots, -\text{val}(x_k)) | (x_1, \ldots, x_k) \in V(I)\} \subset \mathbb{Q}^k.$$  

**Proof.** See [4, Theorem 2.1].

For an ideal $I \subset K[x_1, \ldots, x_k]$, the set mentioned in Theorem 2.1 is called the tropical variety $\mathcal{T}(I) \subset \mathbb{R}^k$ of the ideal $I$.

We say that $\{f_1, \ldots, f_r\}$ is a tropical basis of $\mathcal{T}(I)$ if and only if $I = \langle f_1, \ldots, f_r \rangle$ and

$$\mathcal{T}(I) = \mathcal{T}(\text{trop}(f_1)) \cap \cdots \cap \mathcal{T}(\text{trop}(f_r)).$$

We are particularly interested in tropical Grassmannians $\mathcal{G}_{m,n} = \mathcal{T}(I_{m,n})$. In this case, the ideal

$$I_{m,n} \subset K[x_{i_1} \ldots i_m | 1 \leq i_1 < \ldots < i_m \leq n]$$

is the ideal of the affine Grassmannian $G(m,n) \subset K^\binom{n}{m}$ parameterizing linear subspaces of dimension $m$ in $K^n$. The ideal $I_{m,n}$ consists of all relations between the $(m \times m)$-minors of an $(m \times n)$-matrix.

**Remark 2.2.** In case $m = 2$, the Plücker relations

$$p_{ijkl} := x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}$$

(with $i < j < k < l$) generate the ideal $I_{2,n}$. One can show that these polynomials also form a tropical basis of $I_{2,n}$, hence $\mathcal{G}_{2,n}$ is the intersection of the tropical hypersurfaces $\mathcal{H}(\text{trop}(p_{ijkl}))$. Note that trop$(p_{ijkl})$ is equal to

$$(x_{ij} \otimes x_{kl}) \oplus (x_{ik} \otimes x_{jl}) \oplus (x_{il} \otimes x_{jk}) = \max\{x_{ij} + x_{kl}, x_{ik} + x_{jl}, x_{il} + x_{jk}\},$$

so we get Theorem 1.2 using Theorem 1.1.
3 The case $m = 4$ : the proof of the main theorem

Remark 3.1. Let $\phi^{(4)} : \mathbb{R}^\binom{n}{2} \rightarrow \mathbb{R}^\binom{4}{2}$ be the map sending $X = (X(i,j))_{i<j}$ to $(X(i,j,k,l))_{i<j<k<l}$, where $X(i,j,k,l)$ is the minimum of the three terms

$$X(i,j) + X(j,k) + X(k,l) + X(i,l),$$
$$X(i,j) + X(j,l) + X(k,l) + X(i,k),$$
$$X(i,k) + X(j,k) + X(j,l) + X(i,l),$$

divided by two. By Theorem 1.3, the map $\phi^{(4)}$ sends the dissimilarity vector $D$ of a tree $T$ to its 4-dissimilarity vector $(D(i,j,k,l))_{i<j<k<l}$.

We will now prove the main theorem.

Proof of Theorem 1.5. Since the inclusion $\phi^{(4)}(T_n) \subset \phi^{(4)}(\mathbb{R}^\binom{4}{2})$ is evident, we only have to prove $\phi^{(4)}(T_n) \subset \mathcal{G}_{4,n}$.

Let $T$ be a tree with 4-dissimilarity vector

$$D := (D(i,j,k,l))_{i<j<k<l} = \phi^{(4)}(\phi^{(4)}((D(i,j))_{i<j})) \in \phi^{(4)}(T_n) \subset \mathbb{R}^\binom{4}{2}.$$ 

If $M \in K^{4 \times n}$, we denote by $M(i,j,k,l)$ the 4×4-minor coming from the columns $i, j, k, l$ of $M$. The tropical Grassmannian is the closure in $\mathbb{R}^\binom{n}{2}$ of the set

$$S := \{(\det(M_{i,j,k,l}))_{i<j<k<l} | M \in K^{4 \times n}\} \subset \mathbb{Q}^\binom{4}{2}.$$

Assume first that all edges of $T$ have rational length, hence $D \in \mathbb{Q}^\binom{4}{2}$. We are going to show that $D \in S$.

Fix a rational number $E$ with $E \geq D(i,n)$ for all $i$. Define a new metric $D'$ by

$$D'(i,j) = 2E + D(i,j) - D(i,n) - D(j,n)$$

for all different $i,j \in [n]$, in particular $D'(i,n) = 2E$ for $i \neq n$. Note that $D' \in T_n$ and that $D'$ an ultrametric on $\{1, \ldots, n-1\}$, so it can be realized by an equidistant $(n-1)$-tree $T''$ with root $r$. Each edge $e$ of $T''$ has a well-defined height $h(e)$, which is the distance from the top node of $e$ to each leaf below $e$. Pick random rational numbers $a(e)$ and $b(e)$ for every edge $e$ of $T''$. If $i \in \{1, \ldots, n-1\}$ is a leaf of $T''$, define the polynomial $x_i(t)$ resp. $y_i(t)$ as the sum of the monomials $a(e)t^{2h(e)}$ resp. $b(e)t^{2h(e)}$, where $e$ is an edge between $r$ and $i$. It is easy to see that

$$D'(i,j) = \deg(x_j(t) - x_i(t)) = \deg(y_j(t) - y_i(t))$$

for all $i,j \in \{1, \ldots, n-1\}$.

Denote the distance from $r$ to each leaf by $F$. Since

$$2F = \max\{D'(i,j) | 1 \leq i < j \leq n-1\} < 2E,$$
we have $F < E$. The metric $D'$ on $[n]$ can be realized by a tree $T'$, where $T'$ is the tree obtained from $T''$ by adding the leaf $n$ together with an edge $(r, n)$ of length $2E - F$. If we define $x_n(t) = y_n(t) = t^{2E}$, we get that $D'(i, j) = \deg(x_j(t) - x_i(t)) = \deg(y_j(t) - y_i(t))$ for all $i, j \in [n]$. Consider the matrix

$$M' := \begin{bmatrix} 1 & x_1(t) & x_2(t) & x_3(t) & x_4(t) & \cdots & x_n(t) \\ x_1(t)^2 & x_2(t)^2 & x_3(t)^2 & x_4(t)^2 & \cdots & x_n(t)^2 \\ y_1(t) & y_2(t) & y_3(t) & y_4(t) & \cdots & y_n(t) \end{bmatrix}.$$  

We claim that $\deg(\det(M'(i, j, k, l))) = 2D'(i, j, k, l)$ for all $i, j, k, l \in [n]$. After renumbering the leaves, we may assume that $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and that $D'(1, 2) \leq D'(1, 3) \leq D'(1, 4)$. In Figure 1 all combinatorial types of the subtrees are pictured. Every edge in this picture may consist of several edges of the tree $T'$. Note that types I and II are different, since the top node $v$ sits on a different edge of the subtree. The type III case is special, since $n \in \{i, j, k, l\}$ (before the renumbering).

![Figure 1: The combinatorial types of 4-subtrees](image)

The determinant of $M'(1, 2, 3, 4)$ is equal to

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ 0 & y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \end{vmatrix}$$

$$= (y_2 - y_1)(x_4 - x_1)(x_3 - x_1)(x_4 - x_3) - (y_3 - y_1)(x_4 - x_1)(x_2 - x_1)(x_4 - x_2) + (y_4 - y_1)(x_3 - x_1)(x_2 - x_1)(x_3 - x_2)$$

(2)

The degree of the term $(y_2 - y_1)(x_4 - x_1)(x_3 - x_1)(x_4 - x_3)$ in (2) is

$$D'(1, 2) + D'(1, 3) + D'(1, 4) + D'(3, 4),$$

which equals $2D'(1, 2, 3, 4)$ for each of the three types.

If $v$ and $w$ are nodes between $r$ and $i$, we will denote the sum of the monomials $a(e)t^{2b(e)}$ for $e$ between $v$ and $w$ by $x_{i,v,w}(t)$. Analogously, we define
We are going to take a look at the type I case. In Figure 2, the arrows stand for edges of $T'$. For example, the edge $e_v$ is adjacent to $v$ and goes into the direction of $w$.

Figure 2: Type I

Denote $x := x_3, [v, w] - x_1, [v, w]$, $x_{12} := x_2, [w, 2] - x_1, [w, 1]$, $x_{13} := x_3, [u, 3] - x_1, [w, 1]$, etc. Analogously, we define $y, y_{12}, y_{13}, \ldots, y_{34}$. The determinant $\text{det}(2)$ equals

$$y_{12}x_{34}(x + x_{13})(x + x_{14}) - x_{12}(y + y_{13})(x + x_{14})(x + x_{24}) + x_{12}(y + y_{14})(x + x_{13})(x + x_{23}).$$

Since $\deg(x) = \deg(y)$ is bigger than $\deg(x_{ij}) = \deg(y_{ij})$ for all $i$ and $j$, we have that the degree of the last two terms is equal to

$$\deg(x_{12}yx^2) > 2D'(1, 2, 3, 4),$$

but the term $x_{12}yx^2$ vanishes in the determinant. So, the degree of the sum of the last two terms in (3) is equal to

$$\deg(x_{12}(x^2(y_{14} - y_{13}) + xy(x_{13} + x_{23} - x_{14} - x_{24})))
= \deg(x_{12}(y_{34}x^2 - 2x_{34}xy])
= 2D'(1, 2, 3, 4).$$

We conclude that the determinant of $M'(1, 2, 3, 4)$ has degree $2D'(1, 2, 3, 4)$. Indeed, the coefficient of $t^{2D'(1, 2, 3, 4)}$ is equal to

$$(b(e'_u) - b(e_w))(a(e'_u) - a(e_w))(a(e'_v) - a(e_v))^2
+ (b'(e_u) - b(e_u))(a(e'_u) - a(e_w))(a(e'_v) - a(e_v))^2
- 2(b(e'_v) - b(e_v))(a(e'_u) - a(e_w))(a(e'_v) - a(e_v))(a(e'_u) - a(e_u)) \neq 0.$$
for type II and
\[
(b(e'_w) - b(e_u))(a(e'_w) - a(e_w)) - (b(e'_w) - b(e_u))(a(e'_u) - a(e_u)) \neq 0
\]
for type III.

\[\begin{array}{c}
\text{Figure 3: Type II and III}
\end{array}\]

Let \(M\) be the matrix obtained from \(M'\) by multiplying, for each \(i\), the \(i\)-th column of \(M'\) by \((tD(i,n) - E)^2\). We have
\[
D(i,j) = D'(i,j) + (D(i,n) - E) + (D(j,n) - E) = \deg \left( tD(i,n) - E \cdot tD(j,n) - E \cdot (x_i(t) - x_j(t)) \right).
\]

Using Remark 3.1, we get that
\[
2D(i,j,k,l) = \deg(\det(M(i,j,k,l))).
\]
If we replace each \(t\) in \(M\) by \(t^{-1/2}\), we have
\[
D(i,j,k,l) = -\val(\det(M(i,j,k,l)));
\]

hence \(\mathcal{D} \in S\).

Now assume \(T\) has irrational edge weights. We can approximate \(T\) arbitrarily close by a tree \(\tilde{T}\) with rational edge weights. From the arguments above, it follows that the 4-dissimilarity vector \(\tilde{D}\) of \(\tilde{T}\) belongs to \(S\), hence \(\mathcal{D} \in \mathcal{G}_{4,n}\).

4 What about the case \(m \geq 5\)?

The proof of Theorem 1.5 does not give an obstruction for the following to be true for \(m \geq 5\).

\textbf{Conjecture 4.1.} \(\phi^{(m)}(T_n) \subset \mathcal{G}_{m,n} \cap \phi^{(m)}(\mathbb{R}^{(n)}_2)\)

Note that using the same arguments as in the proof of Theorem 1.5, it suffices to show the following.
Consider the matrix

\[
M = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
x_1^{(1)} & x_2^{(1)} & \ldots & x_n^{(1)} \\
x_1^{(2)} & x_2^{(2)} & \ldots & x_n^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{(m-2)} & x_2^{(m-2)} & \ldots & x_n^{(m-2)}
\end{bmatrix} \in K^{m \times n}.
\]

Let \(i_1, \ldots, i_m\) be pairwise disjoint elements in \(\{1, \ldots, n\}\). Then we have that \(D(i_1, \ldots, i_m) = \deg(\det(M(i_1, \ldots, i_m)))\).

**Remark 4.3.** The matrix \(M\) arising in Conjecture [4.1] has a sort of asymmetry. However, if one would construct polynomials \(x_j^{(3)}\) as in the conjecture with \(j \in \{1, \ldots, m\}\) for each leaf \(i \in \{1, \ldots, n\}\), the statement fails for

\[
N = \begin{bmatrix}
x_1^{(1)} & x_2^{(1)} & \ldots & x_n^{(1)} \\
x_1^{(2)} & x_2^{(2)} & \ldots & x_n^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{(m)} & x_2^{(m)} & \ldots & x_n^{(m)}
\end{bmatrix} \in K^{m \times n},
\]

even for \(m = 3\). Indeed, if the minimal subtree \(\tilde{T}\) of the equidistant tree \(T'\) containing the three leaves \(i_1, i_2, i_3\) does not contain the root \(r\), the degree of the determinant of \(N(i_1, i_2, i_3)\) is not equal to the length of \(\tilde{T}\). Instead, it is equal to the length of the subtree of \(T'\) containing the leaves \(i_1, i_2, i_3\) and the root \(r\). The same happens for \(m = 4\). So it seems that the row consisting of ones in the matrix \(M\) is necessary to cancel the distance between the top node of \(\tilde{T}\) and the root \(r\). On the other hand, the determinant of a maximal minor has to be homogeneous in the variables \(x_j^{(3)}\) of degree \(m\) (see Theorem [1.3]), so once we put a row with ones in \(M\), there should be a row consisting of quadric forms in the variables \(x_j^{(3)}\), i.e. the third row of \(M\).

We can simplify Conjecture [4.2] Firstly, we can see that the tree \(T\) can be considered as an equidistant \(n\)-tree, if we pick the top node to be the node on the
edge \((r, n)\) at distance \((d' + d'')/2\) of \(n\). For example, in the proof of Theorem 1.5 the types II and III are in fact equivalent. Secondly, assume \(I = \{i_1, \ldots, i_m\}\) is an \(m\)-subset of \(\{1, \ldots, n\}\) and let \(T_I\) be the minimal subtree of \(T\) containing the leaves in \(I\). The edges between the top node \(r_I\) of \(T_I\) and the root \(r\) of \(T\) do not give a contribution in the determinant of \(M(I) = M(i_1, \ldots, i_m)\). Also, the edges of \(T_I\) with 2-valent top node different from \(r_I\) can be canceled out in the computation of \(\deg(\det(M(I)))\). So we see that Conjecture 4.2 is equivalent to the following.

Conjecture 4.4. Let \(T\) be an equidistant \(m\)-tree with root \(r\) such that all edges of \(T\) have rational length.

For each edge \(e\) of \(T\), pick random numbers \(a_1(e), \ldots, a_{m-2}(e) \in C\) and denote its height in \(T\) by \(h(e)\). Let \(x_i^{(j)}(t) \in K\) (with \(i \in \{1, \ldots, m\}\) and \(j \in \{1, \ldots, m-2\}\)) be the sum of the monomials \(a_j(e)^{h(e)}\), where \(e\) runs over all edges between \(r\) and \(i\). Then the degree of the determinant of

\[
M = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
x_1^{(1)} & x_2^{(1)} & \ldots & x_m^{(1)} \\
(x_1^{(1)})^2 & (x_2^{(1)})^2 & \ldots & (x_m^{(1)})^2 \\
x_1^{(2)} & x_2^{(2)} & \ldots & x_m^{(2)} \\
\vdots & \vdots & \vdots & \vdots \\
x_1^{(m-2)} & x_2^{(m-2)} & \ldots & x_m^{(m-2)}
\end{bmatrix}
\]

is equal to the length \(D\) of \(T\).

We give an example to illustrate Conjecture 4.4 for \(m = 5\).

Example 4.5. Consider the equidistant 5-tree \(T\) of Figure 4. In the boxes, the distances of the edges are mentioned. Note that \(D = 37\).

![Equidistant 5-tree](image)
Following the notations of Conjecture 4.4 we have

\[
\begin{align*}
x_1^{(j)}(t) &= a_j(r, v) t^{10} + a_j(v, w) t^7 + a_j(w, 1) t^4, \\
x_2^{(j)}(t) &= a_j(r, v) t^{10} + a_j(v, w) t^7 + a_j(w, 2) t^4, \\
x_3^{(j)}(t) &= a_j(r, v) t^{10} + a_j(v, 3) t^7, \\
x_4^{(j)}(t) &= a_j(r, u) t^{10} + a_j(u, 4) t^6, \\
x_5^{(j)}(t) &= a_j(r, u) t^{10} + a_j(u, 5) t^6.
\end{align*}
\]

Using a computer algebra system, one can see that the determinant of \( M \) is a polynomial of degree 37 in the variable \( t \). Each of its coefficients is homogeneous of degree 5 in the numbers \( a_j(e) \), with \( j \in \{1, 2, 3\} \) and \( e \) an edge of \( T \).

If we take the numbers \( a_j(e) \) to be the first 24 = 3 \times 8 prime numbers (i.e. \( a_1(r, v) = 2, \ldots, a_3(u, 5) = 89 \)), the determinant of \( M \) has leading coefficient 3344.

**Remark 4.6.** In order to prove Conjecture 4.4 for a fixed value of \( m \), one could follow the strategy of Theorem 1.5. Indeed, the number \( t(m) \) of combinatorial types of equidistant \( m \)-trees is finite and for each of these types, one can compute the determinant of \( M \) and check whether its degree equals \( D \).

In this way, we can prove Conjecture 4.4 for \( m = 5 \) using a computer algebra system. For each of the three combinatorial types of equidistant 5-trees, the determinant of \( M \) can be computed, leaving the random numbers \( a_j(e) \) and the lengths \( l(e) \) of the edges as variables. This determinant (considered as a polynomial in the variable \( t \)) has degree equal to the length \( D \) of the tree \( T \) and its leading coefficient is a homogeneous polynomial \( c_T \) of degree 5 in the numbers \( a_j(e) \). If the tree \( T \) is binary, the polynomial \( c_T \) has 272 terms for the type corresponding to Example 4.5, and 144 terms for the other two types.

Note that the numbers \( a_j(e) \) are sufficiently random if they don’t vanish for the polynomial \( c_T \). We can conclude that the inclusion

\[
\phi(5)(G_{2,n}) \subset G_{5,n} \cap \phi(5)(\mathbb{R}(\binom{n}{2}))
\]

holds, i.e. Conjecture 4.4 for \( m = 5 \).

On the other hand, the number \( t(m) \) grows exponentially, e.g.

\[
t(4) = 2, t(5) = 3, t(6) = 6, t(7) = 11, t(8) = 23, t(9) = 46, t(10) = 98, etc.,
\]

and for each of these types, the square matrix \( M \) is of size \( m \), hence the computation of its determinant gets more complicated when \( m \) grows. So this technique is not suited in order to prove Conjecture 4.4 for every \( m \). However, one can hope to find a proof by induction on \( m \).

**References**

[1] C. Bocci, F. Cools, *A tropical interpretation of m–dissimilarity maps*, preprint (2008). [arXiv:0803.2184]
[2] P. Buneman, A Note on the Metric Properties of Trees, J. Combinatorial Theory 17 (1974), 48-50.

[3] L. Pachter, B. Sturmfels, Algebraic statistics for computational biology, Cambridge University Press, New York 2005

[4] D. Speyer, B. Sturmfels, The Tropical Grassmannian, Adv. Geom. 4 (2004), 389-411.