A combinatorial characterization of second category subsets of $X^\omega$

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Abstract

Let a finite non-empty $X$ is equipped with discrete topology. We prove that $S \subseteq X^\omega$ is of second category if and only if for each $f : \omega \to \bigcup_{n \in \omega} X^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

Theorem 1 yields information about sets $S \subseteq X^\omega$ with the following property ($\Box$):

($\Box$) for each infinite $J \subseteq \omega$ and each $f : J \to \bigcup_{n \in \omega} X^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in J$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

Theorem 1. Assume that a non-empty $X$ is equipped with discrete topology. We claim that if $S \subseteq X^\omega$ is of second category then $S$ has the property ($\Box$).

Proof. Let us fix $f : J \to \bigcup_{n \in \omega} X^n$. Let $S_k(f)$ ($k \in \omega$) denote the set of all sequences $\{a_n\}_{n \in \omega}$ belonging to $X^\omega$ with the property that there exists $i \in J$ such that $i > k$ and the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends $f(i)$.
the finite sequence $f(i)$. Sets $S_k(f)$ ($k \in \omega$) are open and dense. In virtue of the Baire category theorem $\bigcap_{k \in \omega} S_k(f) \cap S$ is non-empty i.e. there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in J$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$. This completes the proof.

The proof of the following Observation is left as an exercise for the reader.

**Observation.** If $S \subseteq \{0,1\}^\omega$ has the property $(\Box)$ then for every open set $U \subseteq (0,\varepsilon)$ with $0 \in U$ there exists a $g \in S$ such that the sequence

$$\left\{ \sum_{k=n}^{\infty} \frac{g(k)}{2^k} \right\}_{n \in \omega}$$

has an infinite number of terms belonging to $U$.

**Corollary.** Assume that $f : (0, \varepsilon) \to \mathbb{R}$ is continuous and for each zero-one sequence $\{a_n\}_{n \in \omega}$ with an infinite number of ones the limit

$$\lim_{n \to \infty} f(\sum_{k=n}^{\infty} \frac{a_k}{2^k})$$

exists and equals 0. Then (cf. Proposition 1 in [5]) $\lim_{x \to 0^+} f(x) = 0$.

Theorem 2 yields information about sets $S \subseteq X^\omega$ with the following property $(\ast)$:

$(\ast)$ for each $f : \omega \to \bigcup_{n \in \omega} X^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

**Theorem 2.** Assume that a finite non-empty $X$ is equipped with discrete topology. We claim that if $S \subseteq X^\omega$ is of first category then $S$ does not have the property $(\ast)$. 

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Proof. Assume that $S \subseteq \bigcup_{i \in \omega} Y_i$, where sets $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq X^\omega$ are closed and nowhere dense. Let $\Pi_i : X^\omega \to X^\omega$ ($i \in \omega$) maps the sequence $(a_n)_{n \in \omega} \in X^\omega$ to the sequence $(a_{i+n})_{n \in \omega}$. Obviously $\Pi_i = \Pi_1 \circ \ldots \circ \Pi_1$ ($i \in \omega \setminus \{0\}$). Since $X$ is finite, $\Pi_1$ images of closed nowhere dense sets are closed and nowhere dense. Therefore each set $\Pi_i(Y_i)$ ($i \in \omega$) is closed and nowhere dense. Hence for each $i \in \omega$ there exists a sequence $b_i(0), b_i(1), \ldots, b_i(l(i))$ of elements of $X$ such that the set

$\{b_i(0)\} \times \{b_i(1)\} \times \ldots \times \{b_i(l(i))\} \times X \times X \times X \times \ldots$

is disjoint from $\Pi_i(Y_i)$. This gives:

(**) if $i \in \omega$ then each sequence $(a_n)_{n \in \omega} \in X^\omega$ which satisfies $a_i = b_i(0), a_{i+1} = b_i(1), \ldots, a_{i+l(i)} = b_i(l(i))$ does not belong to $Y_i$.

Let $f(i)$ ($i \in \omega$) denote the sequence $b_i(0), b_i(1), \ldots, b_i(l(i))$; formally $f : \omega \to \bigcup_{n \in \omega} X^n$. Let $(a_n)_{n \in \omega} \in S$ and $I := \{i \in \omega : (a_{i+n})_{n \in \omega} \text{ extends } f(i)\}$. Obviously $I \subseteq \omega$, it suffices to prove that $I$ is finite. From (**) we conclude that for each $i \in I$ $(a_n)_{n \in \omega} \notin Y_i$. Suppose, on the contrary, that $I$ is infinite. Thus $(a_n)_{n \in \omega} \notin \bigcup_{i \in I} Y_i = \bigcup_{i \in \omega} Y_i$. Since $S \subseteq \bigcup_{i \in \omega} Y_i$ we conclude that $(a_n)_{n \in \omega} \notin S$, which contradicts our assumption. We have proved that $S$ does not have the property ($\ast$).

Remark 1 ([3]). Tomek Bartoszyński constructed a closed nowhere dense set $S \subseteq \omega^\omega$ with the property ($\ast$).

Remark 2 (inspired by [3]). Let $X$ is infinite, $\psi : X \to \omega$ and $\psi(X)$ is infinite. Let $S \subseteq X^\omega$ denote the set of all sequences of the form

$a, \ldots, a, b, \ldots, b, c, \ldots, c, \ldots$

where $a, b, c, \ldots \in X$. It is easy to check that $S$ is closed, nowhere dense and has the property ($\square$).
Let $\forall^\infty$ abbreviate ”for all except finitely many”.

**Note.** If $S \subseteq \omega^\omega$ satisfies $\exists g \in \omega^\omega \forall f \in S \forall^\infty k \ g(k) \neq f(k)$ then $S$ does not have the property $(\ast)$.

Let $\mathcal{M}$ denote the ideal of first category subsets of $\mathbb{R}$ and $\text{non}(\mathcal{M}) := \min\{\text{card } S : S \subseteq \mathbb{R}, S \not\in \mathcal{M}\}$. It is known (see [1], [2] and also [4]) that:

$$\text{non}(\mathcal{M}) = \min\{\text{card } S : S \subseteq \omega^\omega \text{ and } \neg \exists g \in \omega^\omega \forall f \in S \forall^\infty k g(k) \neq f(k)\}.$$ 

From this, the Note and Theorem 1 we deduce that $\text{non}(\mathcal{M})$ is the smallest cardinality of a family $S \subseteq \omega^\omega$ with the property that for each $f : \omega \rightarrow \bigcup_{n \in \omega} \omega^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

Let $\mathcal{M}([0,1]^\omega)$ denote the ideal of first category subsets of the Cantor discontinuum $[0,1]^\omega$. Obviously:

$$\text{non}(\mathcal{M}) = \min\{\text{card } S : S \subseteq [0,1]^\omega, S \not\in \mathcal{M}([0,1]^\omega)\}$$

From $(\ast \ast \ast)$, Theorem 1 and Theorem 2 we deduce that $\text{non}(\mathcal{M})$ is the smallest cardinality of a family $S \subseteq [0,1]^\omega$ with the property that for each $f : \omega \rightarrow \bigcup_{n \in \omega} [0,1]^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$. Another combinatorial characterizations of $\text{non}(\mathcal{M})$ can be found in [5].

**Remark 3.** Errata to [5].

| Page, line | For                                      | Read                                      |
|------------|------------------------------------------|-------------------------------------------|
| 22          | or all but finitely many                 | for all but finitely many                 |
| 26          | $k \in \omega(A \cup B)$               | $k \in \omega \setminus (A \cup B)$     |
| 27          | $C \Leftrightarrow C_1 \Leftarrow C_2 \Leftarrow C_3 \Leftarrow \ldots$ | $C \Leftrightarrow C_1 \Leftarrow C_2 \Leftarrow C_3 \Leftarrow \ldots \{f(a_n) : \{a_n\} \in \Phi\}$ is unbounded |
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References

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