ON SOME EQUIVALENT DEFINITIONS OF $\rho$-CARLESON MEASURES ON THE UNIT BALL

BENOIT F. SEHBA

Abstract. We give in this paper some equivalent definitions of the so called $\rho$-Carleson measures when $\rho(t) = (\log(4/t))(\log \log(4t))^q$, $0 \leq p, q < \infty$. As applications, we characterize the pointwise multipliers on $LMOA(\mathbb{S}^n)$ and from this space to $BMOA(\mathbb{S}^n)$. Boundedness of the Cesàro type integral operators on $LMOA(\mathbb{S}^n)$ and from $LMOA(\mathbb{S}^n)$ to $BMOA(\mathbb{S}^n)$ is considered as well. This is Chapter 2 of [8] and some of results were already published in [3] and [7].

1. Holomorphic function spaces and Carleson measures in the unit ball

1.1. Some holomorphic function spaces in the unit ball of $\mathbb{C}^n$. We define here various holomorphic function spaces involved in this paper. We refer to the book [13] for the proof of different assertions stated below.

Recall that for $\alpha > -1$ the weighted Lebesgue measure $d\nu_\alpha$ is defined by

$$d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z),$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

is a normalizing constant so that $V_\alpha(\mathbb{B}^n) = 1$.

**Definition 1.1.** For $\alpha > -1$ and $0 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B}^n)$ consists of holomorphic functions $f$ in $L^p(\mathbb{B}^n, d\nu_\alpha)$, that is

$$A^p_\alpha(\mathbb{B}^n) = L^p(\mathbb{B}^n, d\nu_\alpha) \cap H(\mathbb{B}^n).$$

We use the notation

$$\|f\|_{p,\alpha}^p := \int_{\mathbb{B}^n} |f(z)|^p d\nu_\alpha(z)$$

for $f \in L^p(\mathbb{B}^n, d\nu_\alpha)$.

**Definition 1.2.** For $0 < p < \infty$ the Hardy space $H^p(\mathbb{B}^n)$ is the space of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_p^p := \sup_{0 < r < 1} \int_{\mathbb{B}^n} |f(r\xi)|^p d\sigma(\xi) < \infty.$$
and
\[ Q_\delta(\xi) = \{ z \in \mathbb{B}^n : |1 - \langle z, \xi \rangle| < \delta \}. \]

These are the higher dimension analogues of Carleson regions. For \( f \in \mathcal{H}^1(\mathbb{B}^n) \), we still denote \( f(\xi) \), for \( \xi \in \mathbb{S}^n \), the admissible limit at the boundary, which exists a.e.

**Definition 1.3.** The space of functions of bounded mean oscillation \( \text{BMOA} \) is the space of all \( f \in \mathcal{H}^1(\mathbb{B}^n) \) for which there exists a constant \( C > 0 \) so that
\[
\sup_{B=B_\delta(\xi), \delta \in [0, 1], \xi \in \mathbb{S}^n} \frac{1}{\sigma(B)} \int_B |f - f_B|d\sigma \leq C.
\]

Here and anywhere else, \( f_B \) denotes the mean-value of \( f \) on \( B \).

The space \( \text{BMOA} \) is Banach space when equipped with the norm
\[
||f||_{\text{BMOA}} = |f(0)| + \sup_{B=B_\delta(\xi), \delta \in [0, 1], \xi \in \mathbb{S}^n} \frac{1}{\sigma(B)} \int_B |f - f_B|d\sigma.
\]

We now define the space of functions of logarithmic mean oscillation \( \text{LMOA} \).

**Definition 1.4.** An analytic function \( f \) belongs to \( \text{LMOA} \) if \( f \in \mathcal{H}^1(\mathbb{B}^n) \) and there exists a constant \( C > 0 \) so that
\[
\sup_{B=B_\delta(\xi), \delta \in [0, 1], \xi \in \mathbb{S}^n} \log \frac{4}{\sigma(B)} \int_B |f - f_B|d\sigma \leq C.
\]

The space \( \text{LMOA} \) is Banach space when equipped with the norm
\[
||f||_{\text{LMOA}} = |f(0)| + \sup_{B=B_\delta(\xi), \delta \in [0, 1], \xi \in \mathbb{S}^n} \log \frac{4}{\sigma(B)} \int_B |f - f_B|d\sigma.
\]

**Definition 1.5.** The Bloch space \( \mathcal{B} \) consists of all \( f \in H(\mathbb{B}^n) \) such that
\[
||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in B_n} |Rf(z)|(1 - |z|^2) < \infty.
\]

We also recall the following definition of the logarithmic (weighted) Bloch space \( \text{LB} \).

**Definition 1.6.** An analytic function \( f \) belongs to \( \text{LB} \) if
\[
\sup_{z \in B_n} |Rf(z)|(1 - |z|^2) \log \frac{4}{1 - |z|^2} < \infty.
\]

The natural norm on \( \text{LB}(\mathbb{B}^n) \) is given by
\[
||f||_{\text{LB}} = |f(0)| + \sup_{z \in B_n} |Rf(z)|(1 - |z|^2) \log \frac{4}{1 - |z|^2} < \infty.
\]

Both \( \mathcal{B} \) and \( \text{LB} \) are also Banach when equipped with the norms \( |||\cdot|||_{\mathcal{B}} \) and \( |||\cdot|||_{\text{LB}} \) respectively. Moreover, \( \text{BMOA} \) continuously embeds in \( \mathcal{B} \) and \( \text{LMOA} \) embeds continuously in \( \text{LB} \).
1.2. **Carleson measures on the unit ball of** \(\mathbb{C}^n\). We recall here the definition of Carleson measures and their equivalent in the unit ball. We also introduce Carleson measures with weight.

**Definition 1.7.** Let \(\mu\) denote a positive Borel measure on \(\mathbb{B}^n\). Then for \(0 < s < \infty\), the measure \(\mu\) is called a \(s\)-Carleson measure, if there is a finite constant \(C > 0\) such that for any \(\xi \in \mathbb{S}^n\) and any \(0 < \delta < 1\),

\[
\mu(Q_\delta(\xi)) \leq C(\sigma(B_\delta(\xi)))^s.
\]

When \(s = 1\), \(\mu\) is just called Carleson measure. The infimum of all these constants \(C\) will be denoted \(||\mu||_s\). We will also use \(||\mu||\) to denote \(||\mu||_1\). The following theorem is the higher dimension version of the theorem of L.Carleson [4] and its reproducing kernel formulation.

**Theorem 1.8.** For a positive Borel measure \(\mu\) on \(\mathbb{B}^n\), and \(0 < p < \infty\), the following are equivalent

i) The measure \(\mu\) is a Carleson measure

ii) There is a constant \(C_1 > 0\) such that, for all \(f \in \mathcal{H}^p(\mathbb{B}^n)\),

\[
\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq C_1 ||f||_p^p.
\]

iii) There is a constant \(C_2 > 0\) such that, for all \(a \in \mathbb{B}^n\),

\[
\int_{\mathbb{B}^n} \frac{(1 - |a|^2)^n}{|1 - \langle a, w \rangle|^{2n}} d\mu(w) < C_2.
\]

We say two positive constant \(K_1\) and \(K_2\) are comparable, denoted by \(K_1 \approx K_2\), if there is a absolute positive constant \(M\) such that

\[
M^{-1} \leq \frac{K_1}{K_2} \leq M.
\]

We note that the constants \(C_1, C_2\) in Theorem 1.8 are both comparable to \(||\mu||_s\).

The proof of this theorem can be found in [13]. We also have the following theorem in [13] and [12].

**Theorem 1.9.** For a positive Borel measure \(\mu\) on \(\mathbb{B}^n\), \(s > 1\) and \(0 < p < \infty\), the following are equivalent

i) The measure \(\mu\) is a \(s\)-Carleson measure

ii) There is a constant \(K_1 > 0\) such that, for all \(f \in A_{ns-(n+1)}^p\),

\[
\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq K_1 ||f||_{p,ns-(n+1)}^p.
\]

iii) There is a constant \(K_2 > 0\) such that, for all \(a \in \mathbb{B}^n\),

\[
\int_{\mathbb{B}^n} \frac{(1 - |a|^2)^n}{|1 - \langle a, w \rangle|^{2ns}} d\mu(w) < K_2.
\]

Here both \(K_1\) and \(K_2\) are comparable to \(||\mu||_s\).

We consider here generalized Carleson type measures with additional logarithmic terms.
**Definition 1.10.** Let $\mu$ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. For $\rho$ a positive function defined on $(0, 1)$, we say $\mu$ is a $(\rho, s)$-Carleson measure if there is a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C \frac{(\sigma(B_\delta(\xi)))^s}{\rho(\delta)}.$$  

When $s = 1$, $\mu$ is called a $\rho$-Carleson measure. We are interested in the particular case $\rho(t) = (\log(4/t))^p \log \log(e^{4/t}))^q$ with $0 \leq p, q < \infty$. We remark that the case $\rho(t) = (\log(4/t))^p$ has been studied in [11]. The corresponding measures in the latter are called $p$-logarithmic $s$-Carleson measures and when $s = 1$ we just called them $p$-logarithmic Carleson measures and when $p = 2$ and $s = 1$ we call them logarithmic Carleson measures, using the vocabulary of [11].

2. The case of $\rho_{p,q}$-Carleson measures

2.1. Some useful results. We give in this subsection some useful results for the proof of Theorem 2.9 and Theorem 3.1.

**Lemma 2.1.** Let $1 < N < \infty$ and $0 < \alpha < \infty$. The following assertions hold.

i) For any $0 \leq p < \infty$, there exists a positive constant $C_1$ not depending on $N$ so that

$$I_{N,\alpha,p} = \int_1^N \frac{e^{-\alpha t} dt}{(N - t + 2)^p} \leq \frac{C_1}{(N + 2)^p}.$$  

ii) If $\epsilon_1$ and $\epsilon_2$ are real so that $\log(2 + \epsilon_1) + \epsilon_2 > 1$, then for any $0 \leq p < \infty$, there exists a positive constant $C_2$ not depending on $N$ so that

$$J_{N,\alpha,p} = \int_1^N \frac{e^{-\alpha t} dt}{(\log(N - t + 2 + \epsilon_1) + \epsilon_2)^p} \leq \frac{C_2}{(\log(N + 2 + \epsilon_1) + \epsilon_2)^p}.$$  

**Proof:** i). A simple change of variables gives the following equalities

$$I_{N,\alpha,p} = \int_2^{N+1} \frac{e^{-\alpha x} dx}{x^p} = e^{-\alpha(N+2)} \int_2^{N+1} x^{-p} e^{\alpha x} dx.$$  

Thus i) can be written as

$$(2.11) \quad \int_2^{N+1} x^{-p} e^{\alpha x} dx \leq C_1(N + 2)^{-p} e^{\alpha(N+2)}.$$  

Let $f(x) = x^{-p} e^{\alpha x}$. Then $f'(x) = x^{-p-1} e^{\alpha x} (\alpha x - p) > 0$ if $x > \frac{p}{\alpha}$. Since $f(x)$ is obviously continuous on $[2, \infty)$ and increasing as $x > \frac{p}{\alpha}$, there is a positive constant $K$ such that for any $x \in [2, N + 1]$,

$$f(x) \leq K f(N + 1) = K(N + 1)^{-p} e^{\alpha(N+1)}.$$
Integrating by parts gives
\[
\int_2^{N+1} x^{-p}e^{\alpha x}dx = \left. \frac{1}{\alpha} x^{-p}e^{\alpha x} \right|_2^{N+1} + \frac{p}{\alpha} \int_2^{N+1} x^{-p-1}e^{\alpha x}dx
\]
\[
\leq \frac{K}{\alpha} (N+1)^{-p}e^{\alpha(N+1)} + \frac{Kp}{\alpha} (N+1)^{-p-1}e^{\alpha(N+1)} \int_2^{N+1} dx
\]
\[
\leq \frac{K(1+p)}{\alpha} (N+1)^{-p}e^{\alpha(N+1)}
\]
\[
\leq \frac{K'(1+p)}{\alpha} (N+2)^{-p}e^{\alpha(N+2)},
\]
where \(K'\) is another positive constant, independent of \(N\). Thus (2.11) is true, hence (i) is true.

(ii). The proof is similar to the proof of (i). Let \(x = N + 2 + \epsilon_1 - t\). Then \(t = N + 2 + \epsilon_1 - x\), and \(dt = -dx\). Thus
\[
J_{N,\alpha,p} = \int_{2+\epsilon_1}^{N+1+\epsilon_1} \frac{e^{-\alpha(N+2+\epsilon_1-x)}}{(\epsilon_2 + \log x)^p} = e^{-\alpha(N+2+\epsilon_1)} \int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2 + \log x)^{-p}e^{\alpha x}dx.
\]
Thus (ii) can be written as
\[(2.12) \int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2 + \log x)^{-p}e^{\alpha x}dx \leq C_2[\epsilon_2 + \log(N + 2 + \epsilon_1)]^{-p}e^{\alpha(N+2+\epsilon_1)}.
\]
Let \(g(x) = (\epsilon_2 + \log x)^{-p}e^{\alpha x}\). Then
\[
g'(x) = (\epsilon_2 + \log x)^{-p-1}e^{\alpha x} \left[ \alpha(\epsilon_2 + \log x) - \frac{p}{\alpha} \right].
\]
Since
\[
\lim_{x \to \infty} \left[ \alpha(\epsilon_2 + \log x) - \frac{p}{\alpha} \right] = \infty,
\]
we know that there exists a positive constant \(M\) such that \(g'(x) > 0\) for all \(x > M\). Thus \(g\) is continuous on \([2 + \epsilon_1, \infty)\) and increasing whenever \(x > M\). Therefore there is a positive constant \(K_1\) such that for any \(x \in [2 + \epsilon_1, N + 1 + \epsilon_1]\),
\[
g(x) \leq K_1 g(N + 1 + \epsilon_1) = K_1[\epsilon_2 + \log(N + 1 + \epsilon_1)]^{-p}e^{\alpha(N+1+\epsilon_1)}.
\]
Integrating by parts gives
\[
\int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2 + \log x)^{-p}e^{\alpha x}dx = \left. \frac{1}{\alpha} (\epsilon_2 + \log x)^{-p}e^{\alpha x} \right|_{2+\epsilon_1}^{N+1+\epsilon_1} + \frac{p}{\alpha} \int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2 + \log x)^{-p-1}e^{\alpha x}x^{-1}dx
\]
\[
\leq \frac{K_1}{\alpha} [\epsilon_2 + \log(N + 1 + \epsilon_1)]^{-p}e^{\alpha(N+1+\epsilon_1)}
\]
\[
+ \frac{K_1p}{\alpha} [\epsilon_2 + \log(N + 1 + \epsilon_1)]^{-p-1}e^{\alpha(N+1+\epsilon_1)} \int_{2+\epsilon_1}^{N+1+\epsilon_1} x^{-1}dx
\]
\[
\leq \frac{K_1(1+p)}{\alpha} [\epsilon_2 + \log(N + 1 + \epsilon_1)]^{-p}e^{\alpha(N+1+\epsilon_1)}
\]
\[
\leq \frac{K_2(1+p)}{\alpha} [\epsilon_2 + \log(N + 2 + \epsilon_1)]^{-p}e^{\alpha(N+2+\epsilon_1)},
\]
where \(K_2\) is another positive constant, independent of \(N\). Thus (2.12) is true, hence (ii) is true. The proof is complete. \(\square\)
Let set
\[ K_a(z) = \frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}}. \]

We have the following two theorems characterizing \((\rho_{p,q}, s)\)-Carleson measures in the unit ball.

**Theorem 2.2.** Let \(0 \leq p, q < \infty\) and \(0 < s < \infty\). Let \(\mu\) be a positive Borel measure on \(\mathbb{B}^n\). Then, \(\mu\) is a \((\rho_{p,q}, s)\)-Carleson measure with \(\rho_{p,q}(t) = (\log(4/t))^{p} (\log \log (e^4/t))^{q}\) if and only if

\[(2.13) \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{p} (\log \log \frac{e^4}{1 - |a|})^{q} \int_{\mathbb{B}^n} K_s^*(a) d\mu(z) \leq C < \infty.\]

**Proof:** The ideas of the proof are the same as in the proof of Theorem 2 of [11]. We first suppose that \(\mu\) is a \((\rho_{p,q}, s)\)-Carleson measure and prove (2.13). For \(|a| \leq \frac{3}{4}\), (2.13) is obvious since the measure is necessarily finite. Let \(\frac{3}{4} < |a| < 1\) and choose \(\xi = a/|a|\). For any nonnegative integer \(k\), let \(r_k = 2^{k-1}(1 - |a|), k = 1, 2, \ldots, N\) and \(N\) is the smallest integer such that \(2^{N-2}(1 - |a|) \geq 1\). Thus

\[(2.14) \log_2 \frac{4}{1 - |a|} \leq N \leq 1 + \log_2 \frac{4}{1 - |a|}.\]

Let \(E_1 = Q_{r_1}\) and \(E_k = Q_{r_k}(\xi) - Q_{r_{k-1}}(\xi), k \geq 2\). We have

\[ \mu(E_k) \leq \mu(Q_{r_k}(\xi)) \leq \frac{C 2^{(k-1)ns}(1 - |a|)^{ns}}{(\log \frac{4}{2^{k-1}(1 - |a|)})^{p} (\log \log \frac{e^4}{2^{k-1}(1 - |a|)})^{q}}.\]

Moreover, if \(k \geq 2\) and \(z \in E_k\), then

\[ |1 - \langle a, z \rangle| = |1 - |a| + |a|(1 - \langle \xi, z \rangle)| \]
\[ \geq -(1 - |a|) + |a| |1 - \langle \xi, z \rangle| \]
\[ \geq \frac{3}{4} 2^{k-1}(1 - |a|) - (1 - |a|) \]
\[ \geq 2^{k-2}(1 - |a|). \]

We also have for \(z \in E_1,\)

\[ |1 - \langle z, a \rangle| \geq 1 - |a| > \frac{1}{2}(1 - |a|). \]
Using the above estimates, Hölder’s inequality, the equivalence (2.14) and Lemma 2.1, we obtain

$$
\int_{B^n} K_\alpha^s(z) d\mu(z) \leq \frac{C}{(1 - |a|)^{\alpha s}} \sum_{k=1}^{N} \frac{1}{2^{knk} (\log \frac{4}{r_k})^p (\log \log \frac{e^t}{r_k})^q}
$$

$$
\leq C \int_{1}^{N} \frac{1}{2^{knk} (\log \frac{4}{r_k})^p (\log \log \frac{e^t}{r_k})^q} dt
$$

This prove that (2.13) holds.

Now, suppose that (2.13) holds. For any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$, let $a = (1 - \delta)\xi$. Then $1 - |a| = \delta$ and for $z \in Q_\delta(\xi)$, we have $K_\alpha(z) \geq \frac{C}{\sigma(B_\delta(\xi))}$. Thus, we obtain easily that

$$
\infty > C \int_{B^n} K_\alpha^s(z) d\mu(z)
$$

$$
\geq C (\log \frac{4}{\delta})^p (\log \log \frac{e^t}{\delta})^q \int_{Q_\delta(\xi)} K_\alpha^s(z) d\mu(z)
$$

$$
\geq C (\log \frac{4}{\delta})^p (\log \log \frac{e^t}{\delta})^q \mu(Q_\delta(\xi)).
$$

We conclude that $\mu$ is a $(\rho_{p,q, \delta})$-Carleson measure. The proof is complete. \hfill \Box

The following is well-known (see also Lemma 2.4 below).

**Lemma 2.3.** The following assertions are satisfied.

i) There exists a constant $C > 0$ such that for any $f \in BMOA$,

$$
|f(z)| \leq C \log(\frac{4}{1 - |z|}) ||f||_{BMOA}, \quad z \in B^n.
$$

ii) Given $a \in B^n$, the function $f_a(z) = \log(\frac{e^t}{1 - (z,a)})$ is uniformly in $BMOA$.

**Lemma 2.4.** The following assertions are satisfied.

i) There exists a constant $C > 0$ such that for any $f \in LMOA$,

$$
|f(z)| \leq C \log(\frac{e^t}{1 - |z|}) ||f||_{LMOA}, \quad z \in B^n.
$$

ii) Given $a \in B^n$, the function $f_a(z) = \log(\frac{e^t}{1 - (z,a)})$ is uniformly in $LMOA$. 


Thus, we have to show that for any \( f \in \mathbb{L} \),
we set
\[
|f(z) - f(0)| = \left| \int_0^1 \frac{R f(tz)}{t} dt \right|
\]
for all \( z \in \mathbb{B}^n \). It follows that there exists a constant \( C > 0 \) such that for any \( f \in \mathbb{L} \) and any \( z \in \mathbb{B}^n \),
\[
|f(z) - f(0)| \leq C\|f\|_{\mathbb{L}} \int_0^1 \frac{|z|}{(1 - |tz|) \log(\frac{2}{1 - |z|})} dt
\]
This proves the pointwise estimate for all \( f \in \mathbb{L} \).

Let us now prove that the function \( f_\alpha(z) = \log \log(\frac{4}{1 - |z|}) \) is uniformly in \( LMOA \) or equivalently, by the characterization of \( [10] \) that the measure \( \mu_\alpha(z) = |\nabla f_\alpha(z)|^2 (1 - |z|^2) dV(z) \) is a logarithmic-Carleson measure (that is a \( \rho \)-Carleson measure with \( \rho(t) = \log^2(4/t) \)) with uniform bound. For any \( \xi \in \mathbb{S}^n \) and \( 0 < \delta < 1 \), we set
\[
I = \int_{|1 - (z, \xi)| < \delta} \frac{1 - |z|^2}{|1 - (a, \zeta)|^2} dV(z).
\]
We have to show that \( I \leq C \sigma(B_\delta(\xi)) \), where the constant \( C > 0 \) does not depend on the given \( \alpha \in \mathbb{B}^n \).

If \( |1 - (a, \zeta)| \geq 2\delta \) then, for any \( z \in \mathbb{B}^n \) such that \( |1 - (z, \xi)| < \delta, |1 - (a, z)| \geq \delta \). Thus,
\[
I \leq \delta^{-2} (\log \frac{e^4}{\delta})^{-2} \int_{|1 - (z, \xi)| < \delta} (1 - |z|^2) dV(z) \leq \frac{\sigma(B_\delta(\xi))}{(\log \frac{e^4}{\delta})^2}.
\]
If \( |1 - (a, \xi)| \leq 2\delta \), we obtain
\[
I \leq \sum_{j=0}^\infty \int_{2^{-j-1.3\delta} \leq |1 - (z, a)| \leq 2^{-j-1.3\delta}} (1 - |z|^2) dV(z)
\]
\[
\leq \delta^n (\log \frac{e^4}{\delta})^{-2} \int_{|1 - (a, \xi)| \leq 2^{-j-1.3\delta}} (1 - |z|^2) dV(z)
\]
\[
\leq \frac{\delta^n}{(\log \frac{e^4}{\delta})^2} \sum_{j=0}^\infty 2^{2(j+1)} \delta^{-2j(n+2)} \leq \frac{\sigma(B_\delta(\xi))}{(\log \frac{e^4}{\delta})^2}.
\]
The proof is complete. \( \square \)
2.2. $\rho_{p,q}$-Carleson measures. In this subsection, we give and prove several equivalent definitions of $\rho_{p,q}$-Carleson measures. We first establish a useful lemma. Let $\varphi_z$ be the involutive automorphism of $\mathbb{B}^n$ such that $\varphi_z(0) = z$ and $\varphi_z(z) = 0$, we remark that for any $a, b, c \in \mathbb{B}^n$,\[ K_a(z) \cdot K_b(\varphi_a(z)) = K_{\varphi_a(b)}(z) \]
and\[ K_a(\varphi_a(z)) \cdot K_a(z) = 1. \]

**Lemma 2.5.** Let $0 < s < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{B}^n$. Let\[ d\mu_a(z) = \frac{d\mu(\varphi_a(z))}{K^s_a(z)}. \]
Then\[ \sup_{a \in \mathbb{B}^n} ||\mu_a||_s \approx ||\mu||_s. \]

**Proof:** Using the previous remark, we obtain that\[ \int_{\mathbb{B}^n} K^s_b(z) \frac{d\mu(\varphi_a(z))}{K^s_a(z)} = \int_{\mathbb{B}^n} K^s_b(\varphi_a(w)) \frac{d\mu(w)}{K^s_a(\varphi_a(w))} = \int_{\mathbb{B}^n} K^s_a(w) K^s_b(\varphi_a(w)) d\mu(w) = \int_{\mathbb{B}^n} K^s_{\varphi_a(b)}(w) d\mu(w). \]
The conclusion follows by taking the supremum over $b \in \mathbb{B}^n$ and applying Theorem 2.2. \[\square\]

Let us now recall the following equivalence for the norm of elements of $\text{BMOA}$ space:
\[ ||f||_{\text{BMOA}} \approx \sup_{a \in \mathbb{B}^n} ||f \circ \varphi_a - f(a)||_p \]
for any $0 < p < \infty$ (see [13]).

**Lemma 2.6.** Let $0 \leq p, q < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.

i) There exists a positive constant $C_1$ such that for any $0 < \delta < 1$ and any $\xi \in \mathbb{S}^n$
\[ \mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{1-|\xi|})^p (\log \log \frac{e^4}{1-|\xi|})^q}. \]

ii) There exists a positive constant $C_2$ such that
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1-|a|})^p (\log \log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} K_a(z) d\mu(z) \leq C_2 < \infty. \]

iii) There exists a positive constant $C_3$ such that for any $f \in \text{BMOA}$,
\[ \sup_{a \in \mathbb{B}^n} (\log \log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(z)|^p K_a(z) d\mu(z) \leq C_3 ||f||_{\text{BMOA}}^p. \]
iv) There exists a constant $C_4 > 0$ such that for any $f \in \text{BMOA}$ and any $g \in \text{LMOA}$,

$$\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq C_4 ||f||^{p}_{\text{BMOA}} ||g||^{q}_{\text{LMOA}}.$$

**Proof:** The equivalence $i) \iff ii$ follows from Theorem 2.2. We show that $ii) \Rightarrow iii)$ follows from Theorem 2.2. We show that $ii) \Rightarrow iv) \Rightarrow i$).

$ii) \Rightarrow iii)$: We first remark that $ii)$ implies that $\mu$ is a Carleson measure and so $\frac{d\mu(\varphi_a(z))}{K_a(z)}$ for any fixed $a \in \mathbb{B}^n$ by Lemma 2.5.

Now, for any $f \in \text{BMOA}$, using Hölder’s inequality we obtain

$$\int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z) \leq \left( \int_{\mathbb{B}^n} |f(z) - f(a)|^{p+q} K_a(z) d\mu(z) \right)^{\frac{p}{p+q}} \left( \int_{\mathbb{B}^n} K_a(z) d\mu(z) \right)^{\frac{q}{p+q}}$$

$$\approx \left( \int_{\mathbb{B}^n} |f \circ \varphi_a(z) - f(a)|^{p+q} \frac{d\mu(\varphi_a(z))}{K_a(z)} \right)^{\frac{p}{p+q}} \left( \int_{\mathbb{B}^n} K_a(z) d\mu(z) \right)^{\frac{q}{p+q}}$$

$$\leq C ||\mu||^{p/(p+q)} ||f||^{p}_{\text{BMOA}} \left( \int_{\mathbb{B}^n} K_a(z) d\mu(z) \right)^{\frac{q}{p+q}}.$$

It follows that

$$I_1 \leq C ||\mu||^{p/(p+q)} ||f||^{p}_{\text{BMOA}} \left( \log \log \frac{e^4}{1 - |a|} \right)^{p+q} \int_{\mathbb{B}^n} K_a(z) d\mu(z),$$

where

$$I_1 = \left( \log \log \frac{e^4}{1 - |a|} \right)^q \int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z).$$

It is also clear that $ii)$ implies that $\mu$ is a $\rho$-Carleson measure with

$$\rho(t) = \left( \log \log \frac{e^4}{t} \right)^{p+q}, \quad t \in (0, 1),$$

which is equivalent to saying there exists a constant $C > 0$ so that

$$\left( \log \log \frac{e^4}{1 - |a|} \right)^{p+q} \int_{\mathbb{B}^n} K_a(z) d\mu(z) \leq C < \infty.$$

We conclude that

$$(2.15) \quad I_1 = \left( \log \log \frac{e^4}{1 - |a|} \right)^q \int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z) \leq C ||f||^{p}_{\text{BMOA}}.$$

Since $f \in \text{BMOA}$, we already know that there exists $C > 0$ so that

$$|f(a)| \leq C \log \frac{4}{1 - |a|} ||f||_{\text{BMOA}}.$$

Thus, setting

$$I_2 = \left( \log \log \frac{e^4}{1 - |a|} \right)^q \int_{\mathbb{B}^n} |f(a)| K_a(z) d\mu(z),$$

we obtain

$$I_2 \leq C \left( \log \frac{4}{1 - |a|} \right)^p \left( \log \log \frac{e^4}{1 - |a|} \right)^q ||f||^{p}_{\text{BMOA}} \int_{\mathbb{B}^n} K_a(z) d\mu(z).$$
We conclude using Theorem 2.2 that
\[ I_2 = (\log \log \frac{e^4}{1 - |a|})^q \int_{B^n} |f(a)|^p K_a(z) d\mu(z) \leq C ||f||_{B^{\text{MOA}}}^p, \]
where \( C \) is a constant independent of \( a \). Finally, we obtain combining (2.15) and (2.16) that for any \( a \in \mathbb{B}^n \),
\[ (\log \log \frac{e^4}{1 - |a|})^q \int_{B^n} |f(z)|^p K_a(z) d\mu(z) \leq 2^q (I_1 + I_2) \leq C ||f||_{B^{\text{MOA}}}^p. \]

**iii) \Rightarrow iv):** For any \( f \in B^{\text{MOA}} \), let \( d\mu_f(z) = \frac{|f(z)|^p}{||f||_{B^{\text{MOA}}}^p} d\mu(z) \). We would like to show that **iii)** implies that there exists a positive constant \( C_4 \) such that for any \( f \in B^{\text{MOA}} \) and any \( g \in L^{\text{MOA}} \),
\[ \sup_{a \in B^n} \int_{B^n} |g(z)|^q K_a(z) d\mu_f(z) \leq C_4 ||g||_{L^{\text{MOA}}}^q. \]
We remark that **iii)** implies in particular that for any \( f \in B^{\text{MOA}} \), the measure \( d\mu_f \) is a Carleson measure with \( ||\mu_f|| \approx ||\mu|| \). It follows easily as before that
\[ \int_{B^n} |g(z) - g(a)|^q K_a(z) d\mu_f(z) \leq C ||\mu|| \times ||g||_{B^{\text{MOA}}}^q \leq C ||\mu|| \times ||g||_{L^{\text{MOA}}}^q. \]
Now, using the pointwise estimate for \( g \in L^{\text{MOA}} \), we obtain
\[ \int_{B^n} |g(a)|^q K_a(z) d\mu_f(z) \leq C ||g||_{L^{\text{MOA}}}^q (\log \log \frac{e^4}{1 - |a|})^q \int_{B^n} K_a(z) d\mu_f(z). \]
It follows using **iii)** that there exists \( C > 0 \) so that
\[ \int_{B^n} |g(a)|^q K_a(z) d\mu_f(z) \leq C ||g||_{L^{\text{MOA}}}^q. \]
Finally, using inequalities (2.17) and (2.18), we conclude that for any \( a \in \mathbb{B}^n \),
\[ \int_{B^n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq 2^q \int_{B^n} |f(z)|^p (|g(z)| - |g(a)|)^q K_a(z) d\mu(z) \leq C_2 ||f||_{B^{\text{MOA}}}^p ||g(z)||_{L^{\text{MOA}}}^q, \]
which is **iv)**.

**iv) \Rightarrow i):** For any \( 0 < \delta < 1 \) and \( \xi \in \mathbb{S}^n \), let \( a = (1 - \delta)\xi \). From **iv)**, we have in particular that there exists \( C > 0 \) so that for any \( f \in B^{\text{MOA}} \) and any \( g \in L^{\text{MOA}} \),
\[ \int_{Q_3(\xi)} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq C ||f||_{B^{\text{MOA}}}^p ||g(z)||_{L^{\text{MOA}}}^q. \]
We test the above inequality with \( f(z) = f_a(z) = \log \frac{4}{1 - |a,z|} \) and \( g(z) = g_a(z) = \log \log \frac{e^4}{1 - |a|} \), which are uniformly in \( B^{\text{MOA}} \) and \( L^{\text{MOA}} \) respectively. Noting that for \( z \in Q_3(\xi) \), \( K_a(z) \geq C \sigma(Z(\xi))^4 \), \( \log \frac{1}{\delta} \leq |f_a(z)| \) and \( \log \log \frac{e^4}{\sigma} \leq |g_a(z)| \), we obtain
\[ \frac{C}{\sigma(B_3(\xi))} (\log \frac{4}{1 - |a|})^p (\log \log \frac{e^4}{1 - |a|})^q \int_{Q_3(\xi)} d\mu(z) \leq \int_{Q_3(\xi)} |f_a(z)|^p |g_a(z)|^q K_a(z) d\mu(z) \leq C' < \infty. \]
That is,
\[ \mu(Q_\delta(\xi)) \leq C \frac{\sigma(B_\delta(\xi))}{(\log \frac{1}{\delta})^p (\log \log \frac{1}{\delta})^q}. \]

The proof is complete. \( \square \)

Taking \( q = 0 \) in the above lemma, we obtain the following corollary (see also [11]).

**Corollary 2.7.** Let \( 0 \leq p < \infty \) and let \( \mu \) be a positive Borel measure on \( \mathbb{B}^n \). Then the following conditions are equivalent.

i) There exists a positive constant \( C_1 \) such that for any \( 0 < \delta < 1 \) and any \( \xi \in \mathbb{S}^n \),
\[ \mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{1}{\delta})^p} \]

ii) There exists a positive constant \( C_2 \) such that for any \( f \in \text{BMOA}_p \),
\[ \sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p K_\alpha(z) d\mu(z) \leq C_2 \|f\|_{\text{BMOA}_p}^p. \]

**Lemma 2.8.** Let \( 0 \leq p, q < \infty \) and let \( \mu \) be a positive Borel measure on \( \mathbb{B}^n \). Then the following conditions are equivalent.

i) There exists a positive constant \( C_1 \) such that for any \( 0 < \delta < 1 \) and any \( \xi \in \mathbb{S}^n \),
\[ \mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{1}{\delta})^p (\log \log \frac{1}{\delta})^q}. \]

ii) There exists a positive constant \( C_2 \) such that for any \( g \in \text{LMOA}_q \),
\[ \sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^p \int_{\mathbb{B}^n} |g(z)|^q K_\alpha(z) d\mu(z) \leq C_2 \|g\|_{\text{LMOA}_q}^q. \]

**Proof:** By Lemma 2.6, the assertion i) is equivalent to saying there exists a constant \( C > 0 \) such that for any \( f \in \text{BMOA}_p \) and any \( g \in \text{LMOA}_q \),
\[ \sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_\alpha(z) d\mu(z) \leq C \|f\|_{\text{BMOA}_p}^p \|g\|_{\text{LMOA}_q}^q. \]

It follows from Corollary 2.7 that the latter is equivalent to saying that there exists a positive constant \( C \) such that
\[ \sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^p \int_{\mathbb{B}^n} K_\alpha(z) d\mu_g(z) \leq C < \infty, \]
where \( d\mu_g(z) = \frac{|g(z)|^q}{\|g(z)\|_{\text{LMOA}_q}^q} d\mu(z) \). This proves ii). The proof is complete. \( \square \)

**Theorem 2.9.** Let \( 0 \leq p, q < \infty \) and let \( \mu \) be a positive Borel measure on \( \mathbb{B}^n \). Then the following conditions are equivalent.

i) There is \( C_1 > 0 \) such that for any \( \xi \in \mathbb{S}^n \) and \( 0 < \delta < 1 \),
\[ \mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{1}{\delta})^p (\log \log \frac{1}{\delta})^q}. \]

ii) There is \( C_2 > 0 \) such that
\[ \sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^p (\log \log \frac{e^4}{1 - |a|})^q \int_{\mathbb{B}^n} K_\alpha(z) d\mu(z) \leq C_2 < \infty. \]
There is $C_3 > 0$ such that for any $f \in BMOA$,
\[
\sup_{a \in \mathbb{B}^n} (\log \log \frac{e^4}{1 - |a|})^2 \int_{B_n} |f(z)|^p K_a(z) d\mu(z) \leq C_3 \|f\|_{BMOA}^p.
\]

iv) There is $C_4 > 0$ such that for any $g \in LMOA$,
\[
\sup_{a \in \mathbb{B}^n} (\log \log \frac{4}{1 - |a|})^p \int_{B_n} |g(z)|^q K_a(z) d\mu(z) \leq C_4 \|g\|_{LMOA}^q.
\]

v) There is $C_5 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$,
\[
\sup_{a \in \mathbb{B}^n} \int_{B_n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq C_5 \|f\|_{BMOA}^p \|g\|_{LMOA}^q.
\]

vi) For $0 < r < \infty$, there is $C_6 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$ and any $h \in H^r(\mathbb{B}^n)$,
\[
\int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q |h(z)|^r d\mu(z) \leq C_6 \|f\|_{BMOA}^p \|g\|_{LMOA}^q \|h\|_{L^r}^r.
\]

**Proof:** We already have from Lemma 2.6 and Lemma 2.8 that i) $\iff$ ii) $\iff$ iii) $\iff$ iv) $\iff$ v). Let
\[
d\mu_{f,g}(z) = \frac{|f(z)|^p |g(z)|^q}{\|f(z)\|^p_{BMOA} \|g(z)\|^q_{LMOA}} d\mu(z).
\]

Then v) is equivalent to saying that
\[
\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} K_a(z) d\mu_{f,g} < C_5.
\]

By Theorem 1.8, this is equivalent to vi). The proof is complete.

\[\square\]

### 2.3. Some applications of $\rho_{p,q}$-Carleson measures.

As first application of Theorem 2.9, we consider the Cesaro-type integral operator $T_b$ defined by
\[
T_b(f)(z) = \int_0^1 f(tz) Rb(tz) \frac{dt}{t}, \quad b, f \in H(\mathbb{B}^n).
\]

The characterization of the boundedness properties of $T_b$ has been considered in [1], [2], [9] and [11] for the case of the unit disc. We first prove the following result on the boundedness of $T_b$ on $LMOA$.

**Corollary 2.10.** For $b \in H(\mathbb{B}^n)$, $T_b$ is bounded on $LMOA$ if and only if
\[
\sup_{a \in \mathbb{B}^n} (\log \log \frac{4}{1 - |a|})^2 \left(\log \log \frac{e^4}{1 - |a|}\right)^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]

**Proof:** We know from [10] that, an analytic $b$ is in $LMOA$ if and only if $(1 - |z|^2) |Rb(z)|^2 dV(z)$ is a $\rho$-Carleson measure with $\rho(t) = (\log(4t))^2$, which by Lemma 2.2 is equivalent to
\[
\sup_{a \in \mathbb{B}^n} (\log \log \frac{4}{1 - |a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]

It is not hard to see that
\[
R[T_b(f)](z) = f(z) Rb(z).
\]
It follows that $T_b$ is bounded on $LMOA$ if and only if for any $f \in LMOA$,
\[
\sup_{a \in \mathbb{B}^n} \left( \log \left( \frac{4}{1 - |a|} \right) \right)^2 \int_{\mathbb{B}^n} |f(z)|^2 |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < C \|f\|_{LMOA}^2,
\]
which by Theorem 2.9 is equivalent to saying that the measure $|Rb(z)|^2 (1 - |z|^2) dV(z)$ satisfies
\[
\sup_{a \in \mathbb{B}^n} \left( \log \left( \frac{2}{1 - |a|} \right) \right)^2 \left( \log \log \left( \frac{e^4}{1 - |a|} \right) \right)^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]
The proof is complete. \hfill \Box

**Corollary 2.11.** For $b \in H(\mathbb{B}^n)$, $T_b$ is bounded from $LMOA$ to $BMOA$ if and only if
\[
(2.20) \quad \sup_{a \in \mathbb{B}^n} \left( \log \log \left( \frac{e^4}{1 - |a|} \right) \right)^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]

**Proof:** It is well-known that an analytic $b$ is in $BMOA$ if and only if $(1 - |z|^2) |Rb(z)|^2 dV(z)$ is a Carleson measure that is
\[
\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]
It follows that $T_b$ is bounded from $LMOA$ to $BMOA$ if and only if for any $f \in LMOA$,
\[
\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^2 |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < C \|f\|_{LMOA}^2,
\]
which by Theorem 2.9 is equivalent to saying that the measure $|Rb(z)|^2 (1 - |z|^2) dV(z)$ satisfies
\[
\sup_{a \in \mathbb{B}^n} \left( \log \log \left( \frac{e^4}{1 - |a|} \right) \right)^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]
The proof is complete. \hfill \Box

We also obtain in the same way the following result.

**Corollary 2.12.** For $b \in H(\mathbb{B}^n)$, $T_b$ is bounded on $BMOA$ if and only if
\[
(2.21) \quad \sup_{a \in \mathbb{B}^n} \left( \log \left( \frac{4}{1 - |a|} \right) \right)^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]

Our next application is about the pointwise multipliers on $LMOA$. Given two Banach spaces of analytic functions $X$ and $Y$, we denote by $\mathcal{M}(X,Y)$ the space of multipliers from $X$ to $Y$, that is
\[
\mathcal{M}(X,Y) = \{f \in H(\mathbb{B}^n) : f \cdot g \in Y \text{ for any } g \in X\}.
\]
When $X = Y$, we just write $\mathcal{M}(X,X) = \mathcal{M}(X)$. The following lemma is an easy adaptation of [13, Lemma 3.20].

**Lemma 2.13.** Suppose that $X$ and $Y$ are two Banach spaces of holomorphic functions. If $X$ contains constant functions and each point evaluation is a bounded linear functional on $Y$, then every pointwise multiplier from $X$ to $Y$ is in $\mathcal{H}^\infty(\mathbb{B}^n)$.

We have the following characterization of $\mathcal{M}(LMOA)$ for the unit ball of $\mathbb{C}^n$
**Corollary 2.14.** An analytic function \( f \) on \( \mathbb{B}^n \) belongs to \( \mathcal{M}(LMOA) \) if and only if \( f \in \mathcal{H}^\infty(\mathbb{B}^n) \) and satisfies (2.19).

**Proof:** Instead of using Lemma 2.13, we give a direct proof of the fact that any element in \( \mathcal{M}(LMOA) \) is necessarily bounded. For this, we recall that for any \( f \in LMOA, \)

\[
|f(z)| \leq C ||f||_{LMOA} \log \log \frac{e^4}{1 - |z|^2}.
\]

Now, for any \( a \in \mathbb{B}^n \), let \( f_a(z) = \log \log \left( \frac{e^4}{1 - (z,a)} \right) \). \( f_a \in LMOA \) and \( ||f_a||_{LMOA} \leq C < \infty \).

It follows from these two facts that, if \( f \in \mathcal{M}(LMOA) \), then \( f \cdot f_a \in LMOA \) and for any \( z \in \mathbb{B}^n, \)

\[
|f(z)f_a(z)| \leq C ||f \cdot f_a||_{LMOA} \log \log \frac{e^4}{1 - |z|^2}.
\]

Taking \( z = a \) in the above inequality, we obtain

\[
|f(a)| \leq C ||f \cdot f_a||_{LMOA} < \infty.
\]

That is \( f \in H^\infty(\mathbb{B}^n) \).

That \( f \in \mathcal{M}(LMOA) \) means that for any \( g \in LMOA \), the measure \( |R(fg)(z)|^2 (1 - |z|^2) dV(z) \) is a logarithmic Carleson measure or equivalently that

\[
I_f(g) \leq C ||g||_{LMOA}^2,
\]

where

\[
I_f(g) = \sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^2 \int_{\mathbb{B}^n} |g(z)Rf(z) + f(z)Rg(z)|^2 (1 - |z|^2) K_a(z) dV(z).
\]

Since \( f \in H^\infty(\mathbb{B}^n) \) and \( |Rg(z)|^2 (1 - |z|^2) dV(z) \) is a logarithmic Carleson measure,

\[
\sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^2 \int_{\mathbb{B}^n} |f(z)Rg(z)|^2 (1 - |z|^2) K_a(z) dV(z) \leq C ||f||_{LMOA}^2 ||g||_{LMOA}^2.
\]

We deduce that if \( f \in H^\infty(B_n) \), then (2.22) is equivalent to

\[
\sup_{a \in \mathbb{B}^n} \left( \log \frac{4}{1 - |a|} \right)^2 \int_{\mathbb{B}^n} |g(z)|^2 |Rf(z)|^2 (1 - |z|^2) K_a(z) dV(z) \leq C ||g||_{LMOA}^2,
\]

which by Theorem 2.9 is equivalent to saying that \( |Rf(z)|^2 (1 - |z|^2) dV(z) \) satisfies

\[
\sup_{a \in \mathbb{B}^n} \left( \log \frac{2}{1 - |a|} \right)^2 (\log \log \frac{e^4}{1 - |a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K_a(z) dV(z) < \infty.
\]

The proof is complete. \( \square \)

Similarly, we can prove the following results.

**Corollary 2.15.** An analytic function \( f \) on \( \mathbb{B}^n \) belongs to \( \mathcal{M}(LMOA, BMOA) \) if and only if \( f \in \mathcal{H}^\infty(\mathbb{B}^n) \) and satisfies (2.20).

**Corollary 2.16.** An analytic function \( f \) on \( \mathbb{B}^n \) belongs to \( \mathcal{M}(BMOA) \) if and only if \( f \in \mathcal{H}^\infty(\mathbb{B}^n) \) and satisfies (2.21).
The orthogonal projection of $L^2(\partial \mathbb{B}^n)$ onto $\mathcal{H}^2(\mathbb{B}^n)$ is called the Szegö projection and denoted $P$. It is given by

\begin{equation}
P(f)(z) = \int_{\partial \mathbb{B}^n} S(z, \xi) f(\xi) d\sigma(\xi),
\end{equation}

where $S(z, \xi) = \frac{1}{(1-\langle z, \xi \rangle)^n}$ is the Szegö kernel on $\partial \mathbb{B}^n$. We denote as well by $P$ its extension to $L^1(\partial \mathbb{B}^n)$.

For $b \in \mathcal{H}^2(\mathbb{B}^n)$, the small Hankel operator with symbol $b$ is defined for $f$ a bounded holomorphic function by $h_b(f) := P(b\overline{f})$. As last application, we prove that if $b \in LMOA$, then the Hankel operator $h_b$ is bounded on $\mathcal{H}^1(\mathbb{B}^n)$.

The following lemma can be proved using integration by parts (see [6]).

**Lemma 2.17.** Let $f, g$ be holomorphic polynomials on $\mathbb{B}^n$. Then the following identity holds

$$
\int_{\mathbb{B}^n} f(\xi) \overline{\gamma(\xi)} d\sigma(\xi) = C_1 \int_{\mathbb{B}^n} f(z) g(\overline{z}) dV(z) + C_2 \int_{\mathbb{B}^n} Rf(z) g(\overline{z})(1 - |z|^2) dV(z) + C_3 \int_{\mathbb{B}^n} f(z) Rg(z)(1 - |z|^2) dV(z)
$$

where $C_1$, $C_2$ and $C_3$ being constants independent of $f$ and $g$.

**Theorem 2.18.** The Hankel operator $h_b$ extends into a bounded operator on $\mathcal{H}^1(\mathbb{B}^n)$ if $b \in LMOA$.

**Proof:** Let $b \in LMOA$ or equivalently, such that $(1 - |z|^2) |\nabla b(z)|^2 dV(z)$ is a logarithmic Carleson measure. For $f \in \mathcal{H}^1(\mathbb{B}^n)$ and $g \in BMOA$, we want to estimate $| \langle h_b(f), g \rangle | = | \langle b, fg \rangle |$. Applying Lemma 2.17 to $\langle b, fg \rangle$, it comes that we only need to estimate the following three integrals:

\begin{align*}
I_1 & := \int_{\mathbb{B}^n} |f(z)||g(z)||b(z)| dV(z), \\
I_2 & := \int_{\mathbb{B}^n} |f(z)||g(z)| + |\nabla g(z)||\nabla b(z)|(1 - |z|^2) dV(z), \\
I_3 & := \int_{\mathbb{B}^n} |\nabla f(z)||\nabla g(z)| dV(z).
\end{align*}

For the first one, we observe that since $g$ and $b$ are in all $\mathcal{H}^p(\mathbb{B}^n)$, the estimate $|g(z)b(z)| \leq C(1 - |z|^2)^{-1/2}$ holds. It follows using the fact that the measure $(1 - |z|^2)^{-1/2} dV(z)$ is a Carleson measure that

$$
I_1 \leq C \int_{\mathbb{B}^n} |f(z)|(1 - |z|^2)^{-1/2} dV(z) \leq C||f||_1.
$$

For $I_2$, we use Schwarz inequality to obtain

$$
I_2^2 \leq C \int_{\mathbb{B}^n} |f(z)|^2 (|g(z)|^2 + |\nabla g(z)|^2)(1 - |z|^2) dV(z) \times \int_{\mathbb{B}^n} |f(z)||\nabla b(z)|^2 (1 - |z|^2) dV(z)
$$

We conclude by using the fact that $|\nabla g(z)|^2(1 - |z|^2) dV(z)$, $|\nabla b(z)|^2(1 - |z|^2) dV(z)$ and $|g(z)|^2(1 - |z|^2) dV(z)$ are Carleson measures.
The main point is the estimate of $I_3$. We first recall that, by the weak factorization theorem (see [5]), any $f \in H^1(\mathbb{B}^n)$ can be written as

$$f = \sum_j h_j l_j \text{ with } \sum_j ||h_j||_2 ||l_j||_2 \leq C ||f||_1.$$  
Replacing $f$ by this weak factorization, we are led to estimate a sum of terms as

$$J := \int_{\mathbb{B}^n} |g(z)||l(z)||\nabla h(z)||\nabla b(z)|(1 - |z|^2) dV(z)$$
for $l$ and $h$ in $H^2(\mathbb{B}^n)$. We recall that, for $h \in H^2(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |\nabla h(z)|^2 (1 - |z|^2) dV(z) \leq C \||h||_2.$$  
Using this last inequality, Schwarz Inequality and Theorem (referer au chapitre precedent), we obtain

$$J \leq C \||h||_2^{1/2} \left( \int_{\mathbb{B}^n} |g(z)||l(z)||\nabla b(z)|^2 (1 - |z|^2) dV(z) \right)^{1/2} \leq C \||g||_{BMOA} \||l||_2 \||h||_2.$$  
This completes the proof of the theorem. \qed

3. $(\rho, s)$-Carleson measures with $s > 1$

We consider in this section the case of $(\rho, s)$-Carleson measures when $s > 1$. Using Theorem 1.9 and the following equivalence for the norm of elements of the Bloch space $B$:

$$||f||_B \approx ||f \circ \varphi_a - f(a)||_{p, \alpha}, \quad 0 < p < \infty \text{ and } \alpha > -1$$
(see [13]) we can prove in the same way as Theorem 2.9, the following theorem.

**Theorem 3.1.** Let $0 \leq p, q < \infty$, $1 < s < \infty$. Let $\mu$ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.

i) There is $C_1 > 0$ such that for any $\xi \in S^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))^s}{(\log \frac{4}{\delta})^p (\log \log \frac{e^4}{\delta})^q}.$$  

ii) There is $C_2 > 0$ such that

$$\sup_{a \in \mathbb{B}^n} \left( \int_{\mathbb{B}^n} K_a(z)^s d\mu(z) \right) \leq C_2 < \infty.$$  

iii) There is $C_3 > 0$ such that for any $f \in B$,

$$\sup_{a \in \mathbb{B}^n} \left( \int_{\mathbb{B}^n} |f(z)|^p K_a(z)^s d\mu(z) \right) \leq C_3 ||f||_B^p.$$  

iv) There is $C_4 > 0$ such that for any $g \in LB$,

$$\sup_{a \in \mathbb{B}^n} \left( \int_{\mathbb{B}^n} |g(z)|^q K_a(z)^s d\mu(z) \right) \leq C_4 ||f||_{LB}^q.$$  

v) There is $C_5 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in \mathcal{LB}$,

$$\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z)^s d\mu(z) \leq C_5 \|f\|_p \|g\|_q^r \mathcal{L}_{\mathcal{LB}}.$$

vi) For $0 < r < \infty$, there is $C_6 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in \mathcal{LB}$ and any $h \in A^{\dot{r}}_{ns-(n+1)}(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q |h(z)|^r d\mu(z) \leq C_6 \|f\|_p \|g\|_q^r \mathcal{L}_{\mathcal{LB}} \|h\|_r^{\dot{r}}_{ns-(n+1),r}.$$

We now move to applications of Theorem 3.1. We begin by considering the boundedness of the operator $T_b$ on the logarithmic Bloch space $\mathcal{LB}$. It is not hard to see that a function $f \in H(\mathbb{B}^n)$ is in $\mathcal{LB}$ if and only if for any $s > 1$ the measure

$$(1 - |z|^2)^{n(s-1)+1} |Rf(z)|^2 dV(z)$$

is $(\rho, s)$-Carleson measure with $\rho(t) = (\log(4/t))^2$ (see also Lemma 4.7 below) or equivalently that

$$\sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^2 \int_{\mathbb{B}^n} K_a(z)(1 - |z|^2)^{n(s-1)+1} |Rf(z)|^2 dV(z) < \infty.$$  

We have the following corollary.

**Corollary 3.2.** For $b \in H(\mathbb{B}^n)$, the operator $T_b$ is bounded on $\mathcal{LB}$ if and only for any $s > 1$,

$$(3.24) \quad I_b < \infty,$$

where

$$I_b = \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^2 (\log \log \frac{e^4}{1 - |a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2)^{n(s-1)+1} K_a(z) dV(z).$$

**Proof:** Let

$$J_b(f) = \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^2 \int_{\mathbb{B}^n} K_a(z)(1 - |z|^2)^{n(s-1)+1} |f(z)|^2 |Rb(z)|^2 dV(z).$$

That $T_b$ is bounded on $\mathcal{LB}$ is equivalent to saying there exists a constant $C > 0$ such that for any $s > 1$ and any $f \in \mathcal{LB}$,

$$J_b(f) < C \|f\|_{\mathcal{LB}}^2$$

which by Theorem 3.1 is equivalent to (3.24).

Using Theorem 3.1 and the fact that any holomorphic function $f$ belongs to $\mathcal{B}$ if and only if the measure $|Rf(z)|^2 (1 - |z|^2)^{n(s-1)+1} dV(z)$ is a $s$-Carleson measure for any $s > 1$, we can prove the following result in the same way.

**Corollary 3.3.** For $b \in H(\mathbb{B}^n)$, the operator $T_b$ is bounded from $\mathcal{LB}$ to $\mathcal{B}$ if and only for $s > 1$

$$(3.25) \quad \sup_{a \in \mathbb{B}^n} (\log \log \frac{e^4}{1 - |a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2)^{n(s-1)+1} K_a(z) dV(z) < \infty.$$ 

The following well-known result (see for example [?]) follows the same way.

**Corollary 3.4.** For $b \in H(\mathbb{B}^n)$, the operator $T_b$ is bounded on $\mathcal{B}$ if and only for $s > 1$

$$(3.26) \quad \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2)^{n(s-1)+1} K_a(z) dV(z) < \infty.$$
We also obtain as in the previous section the following characterization of multipliers of Bloch-type spaces.

**Corollary 3.5.** An analytic function $f$ on $\mathbb{B}^n$ belongs to $\mathcal{M}(LB)$ if and only if $f \in H^\infty(\mathbb{B}^n)$ and satisfies (3.24).

**Corollary 3.6.** An analytic function $f$ on $\mathbb{B}^n$ belongs to $\mathcal{M}(LB, B)$ if and only if $f \in H^\infty(\mathbb{B}^n)$ and satisfies (3.25).

**Corollary 3.7.** An analytic function $f$ on $\mathbb{B}^n$ belongs to $\mathcal{M}(B)$ if and only if $f \in H^\infty(\mathbb{B}^n)$ and satisfies (3.26).

4. Some generalizations

We give some generalizations and their applications. The proofs here follow the same steps as in the two previous sections.

**Theorem 4.1.** Let $0 \leq p_1, p_2, q_1, q_2 < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.

i) There is $C_1 > 0$ such that for any $\xi \in S^n$ and $0 < \delta < 1$,

$$
\mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{1}{\delta})^{p_1 + p_2} (\log \log \frac{1}{\delta})^{q_1 + q_2}}.
$$

ii) There is $C_2 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$

$$
I(f, g) = \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{p_2} (\log \log \frac{e^4}{1 - |a|})^{q_2} \int_{\mathbb{B}^n} |f(z)|^{p_1} |g_1(z)|^{q_1} K_a(z) d\mu(z).
$$

iii) There is $C_3 > 0$ such that for any $g \in LMOA$

$$
I(g) \leq C_3 ||g||_{LMOA}^{q_1}.
$$

iv) There is $C_4 > 0$ such that for any $f \in BMOA$

$$
I(f) \leq C_4 ||f||_{BMOA}^{p_1}.
$$

**Theorem 4.2.** Let $0 \leq p_1, p_2, q_1, q_2 < \infty$, let $1 < s < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.

i) There is $C_1 > 0$ such that for any $\xi \in S^n$ and $0 < \delta < 1$,

$$
\mu(Q_\delta(\xi)) \leq C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^{p_1 + p_2} (\log \log \frac{e^4}{\delta})^{q_1 + q_2}}.
$$

ii) There is $C_2 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in LB$,

$$
J(f, g) \leq C_2 ||f||_{\mathcal{B}}^{p_1} ||g||_{LB}^{q_1}.
$$

$$
J(f, g) = \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{p_2} (\log \log \frac{e^4}{1 - |a|})^{q_2} \int_{\mathbb{B}^n} |f_1(z)|^{p_1} |g_1(z)|^{q_1} K_a(z) d\mu(z).
$$
iii) There is $C_3 > 0$ such that for any $g \in LB$,
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{p_1 + p_2} (\log \log \frac{e^4}{1 - |a|})^{q_1} \int_{\mathbb{B}^n} |g(z)|^{q_1} K^\alpha_a(z) d\mu(z) \leq C_3 ||g||_{LB}^{p_1}. \]

iv) There is $C_4 > 0$ such that for any $f \in \mathcal{B}$
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{p_2} (\log \log \frac{e^4}{1 - |a|})^{q_1 - q_2} \int_{\mathbb{B}^n} |f(z)|^{p_2} K^\alpha_a(z) d\mu(z) \leq C_4 ||f||_{\mathcal{B}}^{p_1}. \]

Let $0 \leq p, q < \infty$. An analytic function $f$ belongs to $BMOA_{p, q}$ with $\rho_{p, q}(t) = (\log(4/t))^p (\log \log(4/t))^q$ if $f \in H^1(\mathbb{B}^n)$ and there exists a constant $C > 0$ so that
\[ \sup_{B = B_{\xi}(r)} \frac{\log(4/\delta)}{\sigma(B)} \int_{B} |f - f_B| d\sigma \leq C. \]

By [10], a function $f$ belongs to $BMOA_{p, q}$ if and only if $d\mu(z) = (1 - |z|^2)|\nabla f(z)|^2 dV(z)$ is a $\rho^2_{p, q}$-Carleson measure. The following corollaries can be proved as in the previous sections.

**Corollary 4.3.** Let $0 \leq p, q < \infty$. Given an analytic function $b$, the operator $T_b$ is bounded from $LMOA$ to $BMOA_{p, q}$ if and only if
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{2p} (\log \log \frac{e^4}{1 - |a|})^{2q} \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K^\alpha_d(z) dV(z) < \infty. \]

**Corollary 4.4.** Let $0 \leq p, q < \infty$. Given an analytic function $b$, the operator $T_b$ is bounded from $BMOA$ to $BMOA_{p, q}$ if and only if
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^{2p + 2} (\log \log \frac{e^4}{1 - |a|})^{2q} \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K^\alpha_d(z) dV(z) < \infty. \]

In particular, we have the following.

**Corollary 4.5.** Given an analytic function $b$, the operator $T_b$ is bounded from $BMOA$ to $LMOA$ if and only if
\[ \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^4 \int_{\mathbb{B}^n} |Rb(z)|^2 (1 - |z|^2) K^\alpha_d(z) dV(z) < \infty. \]

Let us now move to applications of Theorem 4.2. We first introduce the following generalized $\alpha$-logarithmic-type Bloch spaces.

**Definition 4.6.** For $0 \leq p, q < \infty$ and $\alpha > 0$. Let denote $\mathcal{B}^{p, q}_\alpha$ the space of holomorphic functions $f$ such that
\[ \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^\alpha |Rf(z)| (\log \frac{4}{1 - |z|^2})^p (\log \log \frac{e^4}{1 - |z|^2})^q < \infty. \]

These can be seen as special case of the so called $\mu$-Bloch spaces (see for example [12]) and one have that $\mathcal{B}^{p, q}_\alpha$ are Banach spaces with the norm
\[ ||f||_{\mathcal{B}^{p, q}_\alpha} = ||f(0)|| + \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^\alpha |Rf(z)| (\log \frac{4}{1 - |z|^2})^p (\log \log \frac{e^4}{1 - |z|^2})^q < \infty. \]

The usual Bloch space $\mathcal{B}$ then corresponds to the case $\alpha = 1$ and $p = q = 0$ while $\mathcal{B}^{0, 0}_1 = B_0$ are the so called $\alpha$-Bloch spaces (see [13]) and $\mathcal{B}^{1, 1}_1 = LB$ is the usual logarithmic Bloch space.
Let \( \varphi_z \) be the involutive automorphism of \( \mathbb{B}^n \) that interchanges 0 and \( z \). The Bergman metric of \( \mathbb{B}^n \) is given by

\[
\beta(z, w) := \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}
\]

for all \( z, w \in \mathbb{B}^n \). For any \( R > 0 \) and any \( a \in \mathbb{B}^n \), we write

\[
D(a, R) = \{ z \in \mathbb{B}^n : \beta(z, a) < R \}
\]

for the Bergman ball centered at \( a \) with radius \( R \). We have the following characterization of elements of \( \mathcal{B}_{p,q}^\alpha \).

**Lemma 4.7.** Let \( 0 \leq p, q < \infty \) and \( \alpha > 0 \). A function \( f \in H(\mathbb{B}^n) \) is in \( \mathcal{B}_{p,q}^\alpha \) if and only if \( (1 - |z|^2)^{n(s-1)+2\alpha-1}|Rf(z)|^2dV(z) \) is a \((\rho_{p,q}, s)\)-Carleson measure for any \( s > 1 \), where \( \rho_{p,q}(t) = (\log \frac{t}{\delta})^{2p}(\log \log \frac{t}{\delta})^{2q} \).

**Proof:** We first suppose that \( f \) belongs to \( \mathcal{B}_{p,q}^\alpha \) and show that there exists a constant \( C > 0 \) such that for any \( \xi \in \mathbb{S}^n \), \( 0 < \delta < 1 \) and any \( s > 1 \), the following inequality holds

\[
I_f(\delta) \leq C\sigma(B_\delta(\xi))^s,
\]

where

\[
I_f(\delta) = (\log \frac{4}{\delta})^{2p}(\log \log \frac{e^4}{\delta})^{2q} \int_{Q_\delta(\xi)} |Rf(z)|^2(1 - |z|^2)^{n(s-1)+2\alpha-1}dV(z).
\]

Let \( h(x) = (\log \frac{4}{\delta})^{2p}(\log \log \frac{e^4}{\delta})^{2q} \), \( h \) is decreasing on \((0, 1)\) and moreover, for any \( z \in Q_\delta(\xi) \), \( 1 - |z|^2 < |1 - \xi, z| < |\delta \). It follows using the definition of \( \mathcal{B}_{p,q}^\alpha \) that there exists a constant \( C > 0 \) so that for all \( f \in \mathcal{B}_{p,q}^\alpha \),

\[
I_f(\delta) \leq C \int_{Q_\delta(\xi)} \frac{h(\delta)}{h(1 - |z|^2)}(1 - |z|^2)^{n(s-1)-1}dV(z)
\]

\[
\leq C \int_{Q_\delta(\xi)} (1 - |z|^2)^{n(s-1)-1}dV(z)
\]

\[
\leq C\sigma(B_\delta(\xi))^s.
\]

This shows the necessary part.

Conversely, let us suppose that the analytic function \( f \) is such that there exists \( C > 0 \) so that for any \( \xi \in \mathbb{S}^n \), \( 0 < \delta < 1 \) and any \( s > 1 \),

\[
I_f(\delta) \leq C(\sigma(B_\delta(\xi))^s
\]

and show that in this case \( f \) belongs to \( \mathcal{B}_{p,q}^\alpha \). We recall that for \( a \in \mathbb{B}^n \), letting \( \delta = 1 - |a| \), for \( R > 0 \), there exists \( \lambda \in (0, 1) \) such that \( D(a, R) \subset Q_\delta(\xi) \) with \( a = (1 - \lambda\delta)\xi \) (see Lemma 5.23 of [13]). Now, using the mean value property, we obtain that for any \( a \in \mathbb{B}^n \),

\[
|Rf(a)|^2 \leq \frac{C}{(1 - |a|^2)^{ns+2\alpha}} \int_{D(a, R)} |Rf(z)|^2(1 - |z|^2)^{n(s-1)+2\alpha-1}dV(z).
\]

It follows from the above inclusion and the hypotheses on the measure \(|Rf(z)|^2(1 - |z|^2)^{n(s-1)+2\alpha-1}dV(z)\) that

\[
(1 - |a|^2)^{2\alpha}|Rf(a)|^2(\log \frac{4}{1 - |a|^2})^{2p}(\log \log \frac{e^4}{1 - |a|^2})^{2q} \leq \frac{C}{\delta^{ns}} I_f(\delta) \leq C < \infty.
\]

The proof is complete. \( \square \)
The following two corollaries can be proved exactly as before.

**Corollary 4.8.** Let $0 \leq p, q < \infty$, $\alpha > 0$ and $b \in H(\mathbb{B}^n)$. Then the following conditions are equivalent.

(a) $T_b$ is bounded from $LB$ to $\mathcal{B}^p_{\alpha}$.

(b) For any $s > 1$

\begin{equation}
I_{b,p,q} < \infty,
\end{equation}

\[I_{b,p,q} = \sup_{a \in \mathbb{B}_n} \left( \log \frac{4}{1 - |a|} \right)^{2p+2} (\log \frac{e^4}{1 - |a|})^{2q+2} \int_{\mathbb{B}^n} (1 - |z|^2)^{n(s-1)+2\alpha-1} K_{\alpha}^s(z) dV(z).\]

**Corollary 4.9.** Let $0 \leq p, q < \infty$, $\alpha > 0$ and $b \in H(\mathbb{B}^n)$. Then the following conditions are equivalent.

(a) $T_b$ is bounded from $B$ to $\mathcal{B}^p_{\alpha}$.

(b) For any $s > 1$

\begin{equation}
J_{b,p,q} < \infty,
\end{equation}

\[J_{b,p,q} = \sup_{a \in \mathbb{B}_n} \left( \log \frac{4}{1 - |a|} \right)^{2p+2} (\log \frac{e^4}{1 - |a|})^{2q+2} \int_{\mathbb{B}^n} (1 - |z|^2)^{n(s-1)+2\alpha-1} K_{\alpha}^s(z) dV(z).\]

In particular, we have the following.

**Corollary 4.10.** Given $b \in H(\mathbb{B}^n)$, the operator $T_b$ is bounded from $B$ to $LB$ if and only if for any $s > 1$,

\[
\sup_{a \in \mathbb{B}_n} \left( \log \frac{4}{1 - |a|} \right)^4 \int_{\mathbb{B}^n} (1 - |z|^2)^{n(s-1)+2\alpha-1} K_{\alpha}^s(z) dV(z) < \infty.
\]

**References**

[1] A. Aleman, A. G. Siskakis, An integral operator on $H^p$, Complex Variables, 28 (1995), 149-158.

[2] A. Aleman, A. G. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997), 337-356.

[3] A. Bonami, S. Grellier and B. F. Sehba, Boundedness of Hankel Operators on $H^1(\mathbb{B}^n)$, C. R. Math. Acad. Sc. Paris 344 (2007), no. 12, 749-752.

[4] L. Carleson, Interpolation by bounded analytic functions and the Corona problem, Annals Math., 76(1962), 547-559.

[5] R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611-635.

[6] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241. Springer-Verlag, New York-Berlin, 1980. xiii+436 pp. ISBN: 0-387-90514-6

[7] B. F. Sehba, On some equivalent definitions of $\rho$-Carleson measures in the unit ball of $\mathbb{C}^n$, to appear in Acta Sci. Math. (Szeged).

[8] B. F. Sehba, Operators on some analytic function spaces and their dyadic counterparts. Ph. D thesis, University of Glasgow, 2009.

[9] A. G. Siskakis, R. Zhao, A Volterra type operator on spaces of analytic functions, Function Spaces, Contemp. Math. 232, Amer. Math. Soc., Providence, RI, 1999, 299-311.

[10] W. S. Smith, BMO($\rho$) and Carleson measures, trans. amer. Math. Soc., V.287, 1(1985) 107-126.

[11] R. Zhao, On logarithmic carleson measures, Acta Sci. Math. (Szeged), 69 (2003), 605-618.

[12] R. Zhao and K. Zhu, Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, M. Soc. Math. Fr. (N.S.) No. 115 (2008), vi+103 pp. (2009). ISBN: 978-2-85629-267-9.

[13] Zhu, K. Spaces of holomorphic functions in the unit ball. Springer-Verlag, New York, 2005.
Benoît F. Sehba, Department of Mathematics, University of Ghana, P. O. Box LG 62 Legon, Accra, Ghana

E-mail address: bfsehba@ug.edu.gh