Weak scalar curvature lower bounds along Ricci flow

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Received October 22, 2021; accepted September 14, 2022; published online January 13, 2023

Abstract In this paper, we study Ricci flow on compact manifolds with a continuous initial metric. It was known from Simon (2002) that the Ricci flow exists for a short time. We prove that the scalar curvature lower bound is preserved along the Ricci flow if the initial metric has a scalar curvature lower bound in the distributional sense provided that the initial metric is $W^{1,p}$ for some $n<p \leq \infty$. As an application, we use this result to study the relation between the Yamabe invariant and Ricci flat metrics. We prove that if the Yamabe invariant is nonpositive and the scalar curvature is nonnegative in the distributional sense, then the manifold is isometric to a Ricci flat manifold.

Keywords Ricci flow, low-regularity metric, weak scalar curvature

MSC(2020) 53E20

Citation: Jiang W S, Sheng W M, Zhang H Y. Weak scalar curvature lower bounds along Ricci flow. Sci China Math, 2023, 66: 1141–1160, https://doi.org/10.1007/s11425-021-2037-7

1 Introduction

The study of low-regularity Riemannian metrics with some weak curvature conditions is an important theme in Riemannian geometry. For sectional curvature lower bounds and Ricci curvature lower bounds, many beautiful results have been established (for Alexandrov space theory, see, e.g., [1, 4]; for Ricci curvature lower bounds, see, e.g., [8–13, 19]; for an optimal transport approach, see, e.g., [29, 41–43]).

However, it has not been well understood for scalar curvature lower bounds. One of the fundamental properties for classical scalar curvature lower bounds is that they are preserved by Ricci flow. Ricci flow was firstly introduced by Hamilton [17], which comes to be a powerful tool in geometry and has been used to prove the Poincaré conjecture (see [6, 23, 32–35]). The lower-bound-preserving property is still true for scalar curvature lower bounds in some weak sense and it is quite useful in the study of low-regularity Riemannian metrics with some weak scalar curvature conditions. Gromov [15] introduced a scalar curvature lower bound in some weak sense, which could be defined for $C^0$ metrics. Bamler [2] and Burkhardt-Guim [5] considered it by a Ricci flow approach, and they elaborated on defining a weak-sense scalar curvature lower bound which is required to be preserved by Ricci flow. If a metric satisfies that its Ricci flow has a scalar curvature lower bound at each short positive time, then this metric must...
satisfy the definition of scalar curvature lower bounds in [15]. It remains an open problem whether the converse is true (see [5]). Schoen also considered scalar curvature on low-regularity metrics. He proposed a question: if the Yamabe invariant $\sigma(M)$ is nonpositive, the metric admits singularities in a subset and the scalar curvature is at least $\sigma(M)$ away from the singular set, can we prove that the metric is smooth and Ricci flat provided that the singular set is small (see [28])? Li and Mantoulidis [28] gave an answer for the skeleton metrics and for three-dimensional manifolds with metrics admitting point singularities. For more related results, see the survey of Sormani [40].

McFeron and Székelyhidi [30] proved that if a Lipschitz metric is smooth away from a hypersurface satisfying certain conditions on the mean curvature and the metric has nonnegative scalar curvature pointwisely away from the hypersurface, then the scalar curvature will be nonnegative pointwisely on the whole manifold under Ricci flow. This property appears to be quite useful in the study of the positive mass theorem, especially for the rigidity part of the positive mass theorem (see [30,31]). Shi and Tam [38] have proved the nonnegative scalar-curvature-preserving property for $W^{1,p}$ ($n < p \leq \infty$) metrics which are smooth away from a singular set with Minkowski dimension at most $n - 2$. Moreover, they applied this property to get a positive mass theorem for low-regularity metrics and established an answer of Schoen’s question for $W^{1,p}$ metrics. LeFloh and Mardare [26] defined a scalar curvature lower bound in the distributional sense for $W^{1,p}$ metrics. This was further studied by [25,27] in asymptotically flat manifolds. Lee and LeFloh [25] proved a positive mass theorem on spin manifolds with $W^{1,n}$ metrics which have nonnegative scalar curvature in the distributional sense. Lee and LeFloh’s definition (see [25]) of weak scalar curvature lower bounds recovers the case in [30,31] and [38], and thus the positive mass theorem in [25] recovers it in [30,31] and [38] in the spin case.

We improve some of the above results. The main theorem is as follows.

**Theorem 1.1.** Let $M^n$ be a compact manifold with a metric $g \in W^{1,p}(M)$ ($n < p \leq \infty$). Assume the scalar curvature $R_g \geq a$ for some constant $a$ in the distributional sense, and let $g(t)$ ($t \in (0,T_0]$) be the Ricci flow starting from the metric $g$. Then for any $t \in (0,T_0]$, it holds that $R_{g(t)} \geq a$ on $M$.

In [5], Burkhardt-Guim introduced a weak scalar curvature lower bound for a $C^0$-metric using Ricci flow. By comparing the definition in [5] and Theorem 1.1, we get the following remark.

**Remark 1.2.** Let $M^n$ be a compact manifold with a metric $g \in W^{1,p}(M)$ ($n < p \leq \infty$). Assume the scalar curvature $R_g \geq a$ for some constant $a$ in the distributional sense. Then the scalar curvature $R_g \geq a$ in the sense of [5].

**Remark 1.3.** In [24], Lamm and Simon proved a short time existence result for Ricci flow with initial $W^{2,2} \cap L^\infty$ metrics on four-dimensional manifolds. Furthermore, they showed that the weak scalar curvature lower bound is preserved along the flow. This result could be compared with ours in the critical case $p = n$, which is not considered by us.

As a consequence of Theorem 1.1, we can prove the following result.

**Theorem 1.4.** Let $(M^n,g)$ ($n \geq 3$) be a compact manifold with a metric $g \in W^{1,p}(M)$ ($n < p \leq \infty$). Assume the distributional scalar curvature $R_g \geq 0$. Then either $(M,g)$ is isometric to a Ricci flat manifold or there exists a smooth positive scalar curvature metric on $M$.

**Remark 1.5.** By Theorem 1.4, if $M$ is a topological $n$-torus with a metric $g \in W^{1,p}(M)$ for some $n < p \leq \infty$, then $(M,g)$ is isometric to a flat torus provided that $g$ has nonnegative distributional scalar curvature. More generally, if $M$ is a differential manifold with the Yamabe invariant $\sigma(M) \leq 0$ and with a metric $g$ satisfying the same condition as above, then $(M,g)$ is isometric to a Ricci flat manifold.

**Remark 1.6.** Bourguignon, Gromov, Lawson, Schoen and Yau have established relevant results in the smooth version (see [16,22,36]).

**Remark 1.7.** We can check that if $g$ is $C^2$ away from a closed subset $\Sigma$ with its Hausdorff measure $\mathcal{H}^{n-1}(\Sigma) < \infty$ when $n < p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ when $p = \infty$, and if $R_g \geq a$ pointwisely on $M \setminus \Sigma$, then $R_g \geq a$ in the distributional sense. Though it is proved in [20], for the sake of completeness, we give a proof in Appendix A. Also, if $g$ is $C^2$ away from a hypersurface whose mean curvature taken with respect to the metric in the interior is at least that taken with respect to the metric in the exterior and
\( R_g \geq a \) away from the hypersurface, then \( R_g \geq a \) in the distributional sense. It has been proved in [25, Proposition 5.1].

Therefore, as an application, we can get the similar results for metrics admitting singularities in a subset, which are extensions of some results in [38]. Specifically, if \( g \) is \( C^2 \) away from a closed subset \( \Sigma \) with \( \mathcal{H}^{n-p}(\Sigma) < \infty \) when \( n < p < \infty \) or \( \mathcal{H}^{n-1}(\Sigma) = 0 \) when \( p = \infty \), and if \( R_g \geq a \) or \( R_g \geq 0 \) pointwisely on \( M \setminus \Sigma \), then the above theorems still hold.

The rest of this paper is organized as follows. In Section 2, we mollify the metric by convolutions, and we also provide an estimate of the weak scalar curvature in this smooth approximation (see Lemma 2.2). In Section 3, we recall the definitions of the Ricci flow and the Ricci-DeTurck flow, and we obtain some estimates for them. The main estimate we need is Theorem 3.2. In Section 4, we recall the definition of the conjugate heat equation and prove some properties of the solution to this equation (see Proposition 4.1), which will be needed in Section 5. In Section 5, we prove Theorems 1.1 and 1.4. In Section 6, we state some further questions. In Appendix A, we give lower scalar curvature bounds on singular metrics.

\section{An approximation of singular metrics}

Let \( M^n \) be a compact smooth manifold with a metric \( g \in C^0(M) \cap W^{1,2}(M) \), and \( h \) be a smooth metric on \( M \) such that \( C^{-1} h \leq g \leq C h \) for some constant \( C > 1 \). Throughout this paper, \( \nabla \) and the dot product denote the Levi-Civita connection and the inner product taken with respect to some smooth metric, and the dot product also denotes the inner product taken with respect to \( g \), respectively. We define the distributional scalar curvature of \( g \) as in [25–27], i.e.,

\[
\langle R_g, \varphi \rangle := \int_M \left( -V \cdot \nabla \left( \varphi \frac{d\mu_g}{d\mu_h} \right) + F \varphi \frac{d\mu_g}{d\mu_h} \right) d\mu_h, \quad \forall \varphi \in C^\infty(M),
\]

(2.1)

where \( \mu_h \) is the Lebesgue measure taken with respect to \( h \), \( V \) is a vector field and \( F \) is a scalar field, defined as

\[
\Gamma^k_{ij} := \frac{1}{2} g^{kl} (\nabla_i g_{jl} + \nabla_j g_{il} - \nabla_l g_{ij}),
\]

(2.2)

\[
V^k := g^{ij} \Gamma^k_{ij} - g^{ik} \Gamma^j_{ij} = g^{ij} g^{kl} (\nabla_i g_{jl} - \nabla_l g_{ij}),
\]

(2.3)

\[
F := \text{tr}_g \tilde{\text{Ric}} - \nabla_k g^{ij} \Gamma^k_{ij} + \nabla_k g^{jk} \Gamma^i_{kj} + g^{ij} (\Gamma^k_{ij} \Gamma^l_{kl} - \Gamma^k_{il} \Gamma^l_{jk}),
\]

(2.4)

with \( \tilde{\text{Ric}} \) the Ricci tensor of \( h \). By [25], \( \langle R_g, \varphi \rangle \) coincides with the integral \( \int_M R_g \varphi d\mu_g \) in the classical sense if \( g \in C^2(M) \) and \( \langle R_g, \varphi \rangle \) is independent of \( h \) for any \( g \in C^0(M) \cap W^{1,2}(M) \). For more details about the distributional scalar curvature and related results, see [20, 25–27].

Let \( a \) be some constant. We say that the distributional scalar curvature of \( g \) is at least \( a \) if \( \langle R_g, \varphi \rangle - a \int_M \varphi d\mu_g \geq 0 \) for any nonnegative function \( \varphi \in C^\infty(M) \). We abbreviate this inequality as \( \langle R_g - a, \varphi \rangle \geq 0 \).

The following mollification lemma has been proved in [14, Lemma 4.1], by a standard convolution mollifying procedure and use of partition of unity. Although their lemma is a \( W^{2,2} \) version, our version could be proved in the same way.

\textbf{Lemma 2.1 (See [14])}. Let \( M^n \) be a compact smooth manifold and \( g \) be a \( C^0 \cap W^{1,p} \ (1 \leq p \leq \infty) \) metric on \( M \). Then there exists a family of smooth metrics \( g_\delta \ (\delta > 0) \) such that \( g_\delta \) converge to \( g \) both in the \( C^0 \)-norm and in the \( W^{1,p} \)-norm as \( \delta \to 0^+ \).

Under this mollification, we have an estimate of the distributional scalar curvature. The following lemma has been essentially proved in [20]. Though [20] only gives a proof for the estimate of \( \langle R_{g_\delta}, \nu^2 \rangle \), the following lemma can be proved in the same way. In order to make this paper self-contained, we give a proof here.

\textbf{Lemma 2.2 (See [20])}. Let \( M^n \) be a compact smooth manifold and \( g \) be a \( C^0 \cap W^{1,1} \) metric on \( M \). Let \( g_\delta \) be the mollification in Lemma 2.1. Then we see that for any \( \epsilon > 0 \), there exists a \( \delta_0 = \delta_0(g) > 0 \) such that

\[
|\langle R_{g_\delta}, u \rangle - \langle R_g, u \rangle| \leq \epsilon \|u\|_{W^{1,\infty}(M)}, \quad \forall u \in C^\infty(M) \text{ and } \delta \in (0, \delta_0),
\]
where \( R_{g_\delta} \) is the scalar curvature of \( g_\delta \).

**Proof.** Let \( h \) be a smooth metric on \( M \) with \( C^{-1} h < g < Ch \). Here and below \( C \) denotes some positive constant which depends on \( n \) and the Sobolev constant of \( g \), but is independent of \( \delta \) and varies from line to line. Let \( V_\delta \) and \( F_\delta \) be the vector field and the scalar field in the definition of distributional scalar curvature of \( g_\delta \). Then we have

\[
\lim_{\delta \to 0^+} \left( \int_M |V_\delta - V|^n d\mu + \int_M |F_\delta - F|^{n/2} d\mu \right) = 0. \tag{2.5}
\]

We also have

\[
\lim_{\delta \to 0^+} \int_M \nabla \frac{d\mu_{g_\delta}}{d\mu_h} \nabla \frac{d\mu_g}{d\mu_h} = 0. \tag{2.6}
\]

Using the triangle inequality and Hölder’s inequality, we can calculate that

\[
\left| \int_M F_\delta u \frac{d\mu_{g_\delta}}{d\mu_h} - \int_M F u \frac{d\mu_g}{d\mu_h} \right| \\
\leq \int_M |F_\delta u - F u| \left| \frac{d\mu_{g_\delta}}{d\mu_h} \right| d\mu_h + \int_M |F u| \left| \frac{d\mu_{g_\delta}}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right| d\mu_h \\
\leq C \int_M |F_\delta u - F u| d\mu_h + \sup_M \left| \frac{d\mu_{g_\delta}}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right| \int_M |F u| d\mu_h \\
\leq C \left( \int_M |F_\delta - F|^{n/2} d\mu_h \right)^{2/n} \left( \int_M |u|^{n/(n-2)} d\mu_h \right)^{(n-2)/n} \\
+ \sup_M \left| \frac{d\mu_{g_\delta}}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right| \left( \int_M |F|^{n/2} d\mu_h \right)^{2/n} \left( \int_M |u|^{n/(n-2)} d\mu_h \right)^{(n-2)/n}.
\]

By the Sobolev inequality

\[
\left( \int_M |u|^{n/(n-2)} d\mu_h \right)^{(n-2)/n} \leq C \|u\|_{W^{1, \frac{n}{n-2}}(M)},
\]

we see that

\[
\left| \int_M F_\delta u \frac{d\mu_{g_\delta}}{d\mu_h} - \int_M F u \frac{d\mu_g}{d\mu_h} \right| \\
\leq C \left( \int_M |F_\delta - F|^{n/2} d\mu_h \right)^{2/n} \left( \int_M |F|^{n/2} d\mu_h \right)^{2/n} \|u\|_{W^{1, \frac{n}{n-2}}(M)}.
\]

Similarly, for the term involving \( V \), we can calculate that

\[
\left| \int_M V \cdot \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} \right) d\mu_h - \int_M V_\delta \cdot \nabla \left( u \frac{d\mu_g}{d\mu_h} \right) d\mu_h \right| \\
\leq \int_M |V - V_\delta| \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} \right) \right| d\mu_h + \int_M |V| \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} - u \frac{d\mu_g}{d\mu_h} \right) \right| d\mu_h \\
\leq \left( \int_M |V - V_\delta|^n d\mu_h \right)^{1/n} \left( \int_M \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} \right) \right|^{n/(n-1)} d\mu_h \right)^{(n-1)/n} \\
+ \left( \int_M |V|^n d\mu_h \right)^{1/n} \left( \int_M \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} - u \frac{d\mu_g}{d\mu_h} \right) \right|^{n/(n-1)} d\mu_h \right)^{(n-1)/n}. \tag{2.7}
\]

Using the Sobolev inequality and Hölder’s inequality again, we have

\[
\int_M \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} \right) \right|^{n/(n-1)} d\mu_h
\]

\[
\int_M \left| \nabla \left( u \frac{d\mu_{g_\delta}}{d\mu_h} - u \frac{d\mu_g}{d\mu_h} \right) \right|^{n/(n-1)} d\mu_h
\]
where \( C \) and we also have

\[
\text{Definition 3.1} \quad \text{The Ricci flow was introduced by Hamilton [17]. Its definition is as follows.}
\]

\[
\begin{align*}
\int_M \nabla u \frac{d\mu_g}{d\mu_h} \|u\|_W^{n/(n-1)} &\leq C(1 + \int_M \left| \nabla \frac{d\mu_g}{d\mu_h} \right|^n d\mu_h) \|u\|_W^{n/(n-1)} \quad (2.8) \\
\int_M \left( u \frac{d\mu_g}{d\mu_h} - u \frac{d\mu_g}{d\mu_h} \right)^{n/(n-1)} d\mu_h &\leq C \sup_M \left| \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right|^{n/(n-1)} \int_M \left| u \right|^{n/(n-1)} d\mu_h \\
&\quad + C \left( \int_M |u|^{n/(n-2)} d\mu_h \right)^{(n-2)/(n-1)} \left( \int_M \left| \nabla \left( \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right) \right|^n d\mu_h \right)^{1/(n-1)} \\
&\quad \leq C \left( \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right)^{n/(n-1)} + \left( \int_M \left| \nabla \left( \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right) \right|^n d\mu_h \right)^{1/(n-1)} \|u\|_W^{n/(n-1)} \quad (2.9)
\end{align*}
\]

We combine these estimates, and then get

\[
\begin{align*}
\left| \int_M V \cdot \nabla \left( u \frac{d\mu_g}{d\mu_h} \right) d\mu_h - \int_M V \cdot \nabla \left( u \frac{d\mu_g}{d\mu_h} \right) d\mu_h \right| &\leq C \left( \int_M |V| - V \right)^{1/n} \left( \int_M \left| \nabla \left( \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right) \right|^n d\mu_h \right)^{1/(n-1)} \|u\|_W^{n/(n-1)} \\
&\quad + C \left( \sup_M \left| \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right| + \left( \int_M \left| \nabla \left( \frac{d\mu_g}{d\mu_h} - \frac{d\mu_g}{d\mu_h} \right) \right|^n d\mu_h \right)^{1/(n-1)} \|u\|_W^{n/(n-1)} \quad (2.10)
\end{align*}
\]

Therefore, for any \( \epsilon > 0 \), there exists a \( \delta_0 > 0 \) small enough such that

\[
|\langle R_{g(t)}, u \rangle - \langle R_{g(t)}, u \rangle| \leq \epsilon \|u\|_W^{n/(n-1)} \quad \forall u \in C^\infty(M) \text{ and } \delta \in (0, \delta_0),
\]

which completes the proof of this lemma.

\[ \square \]

3 Estimates on Ricci flow

The Ricci flow was introduced by Hamilton [17]. Its definition is as follows.

**Definition 3.1 (Ricci flow).** The Ricci flow on \( M \) is a family of time-dependent metrics \( g(t) \) such that

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)},
\]

where \( \text{Ric}_g \) is the Ricci curvature tensor of \( g \).

The main theorem in this section is as follows.

**Theorem 3.2.** There exists an \( \epsilon(n) > 0 \) such that for any compact \( n \)-manifold \( M \) with a \( W^{1,p} \) metric \( \hat{g} \) \( (n < p \leq +\infty) \), there exist a \( T_0 = T_0(n, g) > 0 \) and a family of metrics \( g(t) \in C^\infty(M \times (0, T_0), t \in (0, T_0] \) which solves Ricci flow for \( t \in (0, T_0] \) and satisfies

1. \( \lim_{t \to 0} \text{dGH}((M, g(t)), (M, \hat{g})) = 0; \)
2. \( |\text{Rm}(g(t))| dt \leq C(n, \hat{g}, p) \quad \forall t \in (0, T_0]; \)
3. \( \int_{T_0}^T M |\text{Rm}(g(t))|^2 d\mu_g(t) dt \leq C(n, \hat{g}, p), \)

where \( C(n, \hat{g}, p) \) is a positive constant independent of \( t \).
Remark 3.3. In this paper, we assume that $T_0 \leq 1$ for convenience.

Remark 3.4. The existence of $T_0$ and $g(t)$ and (1) have been proved by Simon (see [39, Theorem 1.1]). Shi and Tam [38] also got the similar estimates to (2). Here, we give a proof by using Moser’s iteration.

To prove Theorem 3.2, we consider the $h$-flow (see [39]).

Definition 3.5. Given a constant $\delta \geq 0$, a metric $h$ is called $(1+\delta)$-fair to $g$, if $h$ is $C^\infty$, $$\sup_M |\nabla^j \text{Rm}(h)| = k_j < \infty$$ and $$(1+\delta)^{-1}h \leq g \leq (1+\delta)h \quad \text{on } M.$$ Here and below, $\nabla\tilde{V}$ means the covariant derivative taken with respect to $h$.

Remark 3.6. Let $M$ be a compact manifold and $g$ be a $C^0$ metric on $M$. Then for any $0 < \epsilon < 1$, there exists a smooth metric $h$ which is $(1+\epsilon)$-fair to $g$. For a proof, see the remarks below [39, Definition 1.1].

Definition 3.7 ($h$-flow). Given a smooth metric $h$, the $h$-flow is a family of metrics $g$ satisfying $$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i V_j + \nabla_j V_i,$$ where the derivatives are taken with respect to $g$, $$V_j = g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}),$$ and $\Gamma$ and $\tilde{\Gamma}$ are the Christoffel symbols of $g$ and $h$, respectively.

The $h$-flow is equivalent to the Ricci flow modulo an action of diffeomorphisms (see [39]), and thus we only need to prove Theorem 3.2 for the $h$-flow. Firstly, we need the following theorem, which has been proved by Simon [39].

Theorem 3.8 (See [39, Theorem 1.1]). There exists an $\epsilon(n) > 0$ such that for any compact $n$-manifold $M$ with a complete $C^0$ metric $\hat{g}$ and a $C^\infty$ metric $h$ which is $(1+\epsilon(n))$-fair to $\hat{g}$, there exist a $T_0 = T_0(n,k_0)$ such that there is a $\hat{g}(t)$, $t \in (0,T_0)$ which solves $h$-flow for $t \in (0,T_0)$, with $h$ $(1+\epsilon(n))$-fair to $g(t)$, and satisfies

1. $\lim_{t \to 0^+} \sup_{x \in M} |g(x,t) - \hat{g}(x)| = 0$;
2. $\sup_M |\nabla g(t)| \leq \frac{C(n,k_0)}{t^{\frac{n}{2}}} \quad \forall t \in (0,T_0)$, $i \geq 1$,

where the derivatives and the norms are taken with respect to $h$.

Remark 3.9. In order to apply the flow, we let $h$ be $(1+\epsilon(n))$-fair to $\hat{g}$. By Remark 3.6, such a metric always exists.

Remark 3.10 (See [39]). Actually, take any family of smooth metrics $\{\hat{g}_\delta\}$ which converges to $\hat{g}$ uniformly on $M$ in the $C^0$-norm. Then $h$ is $(1+\epsilon(n))$-fair to $\hat{g}_\delta$ for $\delta$ small enough. Starting from a smooth metric $\hat{g}_\delta$, we get the $h$-flow $g_h(t)$, $t \in (0,T_0)$ with $T_0$ independent of $\delta$. Fix $t > 0$ and let $\delta \to 0^+$. By passing to a subsequence, we get $g(t)$, which is just the $h$-flow such that $g(t) = \hat{g}$ as in Theorem 3.8, and this convergence is smooth for each $t \in (0,T_0)$. For Ricci flow, the same procedure still works.

When the initial metric is $W^{1,p}$ ($n < p \leq +\infty$), we have the following estimate.

Theorem 3.11. In the condition of Theorem 3.8, if we further assume that $M$ is compact and $$\int_M |\nabla \hat{g}|^p d\mu = A$$ for some constant $A$ and $n < p \leq +\infty$, where the derivative and the norm are taken with respect to $h$, then there exists a $T_0 = T_0(n,h,A,p)$ such that $g(t)$ ($t \in (0,T_0)$) is the $h$-flow starting from the metric $\hat{g}$, and satisfies

1. $\int_M |\nabla g(t)|^p d\mu_h \leq 10A$, $\forall t \in (0,T_0)$;
2. $|\nabla g(t)| \leq \frac{C(n,h,A,p)}{t^{\frac{n}{2}}}$, $\forall t \in (0,T_0)$;
where the derivatives and norms are taken with respect to $t$. By (3.7), we have the evolution equation of $g$; thus, we have

$$\int_M |\tilde{\nabla} g(t)|^p d\mu_h \leq 10A, \quad \forall t \in (0, T],$$

then for the same $T$,

$$|\tilde{\nabla} g(t)| \leq \frac{C(n, h, A, p)}{t^{\frac{1}{n}}}, \quad \forall t \in (0, T]$$

also holds, where $C(n, h, A, p)$ is a positive constant independent of $t$.

**Proof.** By (3.7), we have the evolution equation of $|\tilde{\nabla} g_i|$, $i = 1, 2$,

$$\frac{\partial}{\partial t} |\tilde{\nabla} g_i|^2 = g^{\alpha \beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \|\tilde{\nabla} g_i\|^2 - 2g^{\alpha \beta} \tilde{\nabla}_\alpha (\tilde{\nabla} g_i) \cdot \tilde{\nabla}_\beta (\tilde{\nabla} g_i)$$

$$+ \tilde{Rm} \ast g^{-1} \ast \tilde{\nabla} g \ast \tilde{\nabla} g + \tilde{\nabla} \tilde{Rm} \ast \tilde{\nabla} g$$

$$+ g^{-1} \ast \tilde{\nabla} g \ast \tilde{\nabla} \tilde{Rm} \ast \tilde{\nabla} g + g^{-1} \ast \tilde{\nabla} g \ast \tilde{\nabla} \tilde{Rm} \ast \tilde{\nabla} g$$

where the derivatives and norms are taken with respect to $h$, and $\tilde{Rm}$ is the Riemannian curvature tensor of $h$. Thus,

$$\frac{\partial}{\partial t} |\tilde{\nabla} g_i|^2 \leq -C_1(n, h)|\tilde{\nabla}^2 g_i|^2 + C(n, h)|\tilde{\nabla} g|^2 + |\tilde{\nabla} g|$$

$$+ C(n)|\tilde{\nabla} g|^4 + C_2(n)|\tilde{\nabla} g|^2 |\tilde{\nabla}^2 g|$$

$$\leq -C_1(n, h)|\tilde{\nabla}^2 g_i|^2 + C(n, h)|\tilde{\nabla}^2 g|^2 + |\tilde{\nabla} g|^2 + |\tilde{\nabla} g|^4$$

$$+ C(n)|\tilde{\nabla} g|^4 + \frac{C_2(n)}{2\epsilon} |\tilde{\nabla} g|^2 + \frac{C_2(n)}{2\epsilon} |\tilde{\nabla}^2 g|^2, \quad \forall \epsilon > 0.$$

Here and below, $C$ and $C_i$'s are positive constants independent of $t$ and $C$ varies from line to line. Taking $\epsilon = \frac{C_2}{2}$, we have

$$\frac{\partial}{\partial t} |\tilde{\nabla} g_i|^2 \leq -C_1(n, h)|\tilde{\nabla}^2 g_i|^2 + C(n, h)|\tilde{\nabla} g|^2 + |\tilde{\nabla} g|^4$$

$$+ C(n)|\tilde{\nabla} g|^4 + \frac{C_2(n)}{2\epsilon} |\tilde{\nabla} g|^2 + |\tilde{\nabla} g|^4 \leq C(n, h)(|\tilde{\nabla} g| + |\tilde{\nabla} g|^2 + |\tilde{\nabla} g|^4) \quad (3.1)$$

Define $f = |\tilde{\nabla} g_i|^2 + 1$. Then we have

$$\frac{\partial}{\partial t} f - g^{\alpha \beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta f \leq C(n, h)(f + f^\frac{1}{2} + f^2) \leq C(n, h)f(1 + f).$$

Let $v = 1 + f$. Then

$$\frac{\partial}{\partial t} f - g^{\alpha \beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta f \leq C(n, h)f v. \quad (3.2)$$

Suppose that $T \in (0, 1]$ is a constant such that $\int_M |\tilde{\nabla} g(t)|^p d\mu_h \leq 10A, \forall t \in (0, T]$. Then $v$ has a uniformly bounded $L^2(M)$-norm on $[0, T]$; in other words, we have

$$\int_M v^q d\mu_h \leq C(n, h)A + C(n, h), \quad \forall t \in [0, T].$$

For any $q > 0$, we multiply $f^q$ to the equation (3.2) and integrate it. Then we get

$$\int_M \left( \frac{\partial}{\partial t} f - g^{\alpha \beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta f \right) f^q d\mu_h \leq C(n, h) \int_M f^{q+1} v d\mu_h. \quad (3.3)$$
Thus we have
\[
\frac{1}{q} + \frac{1}{p} = \frac{1}{q} \frac{q}{q - 1} d\mu_h
\leq C(n, h) \left( - \int_M g^{\alpha \beta} (\nabla^{\alpha \beta} f) f \frac{q}{q - 1} d\mu_h \right)
\leq C(n, h) \int_M g^{\alpha \beta} (\nabla^{\alpha \beta} f) (\nabla^{\alpha \beta} f) + \int_M f^{q+1} v d\mu_h. \tag{3.4}
\]
By Hölder’s inequality, we have
\[
- \int_M (\nabla^{\alpha \beta} f) (\nabla^{\alpha \beta} f) f^{q-1} d\mu_h
\leq C(n, h) \int_M f^{q+1} |\nabla f| f^{q-1} d\mu_h
\leq C(n, h) \int_M f^{q+1} |\nabla f| f^{q-1} d\mu_h + C(n, h) \int_M f^{q+2} d\mu_h, \forall \epsilon > 0. \tag{3.5}
\]
We obtain
\[
- \int_M g^{\alpha \beta} (\nabla^{\alpha \beta} f) f^{q-1} d\mu_h = - \epsilon \int_M (g^{\alpha \beta} f f^{q-1} d\mu_h
\leq - C(n, h) \int_M f^{q+1} |\nabla f| f^{q-1} d\mu_h. \tag{3.6}
\]
Recall that \(v = f_1 + 1\) and hence \(f^{q+2} \leq f^{q+1} v\). Taking \(\epsilon = \frac{C(n, h)}{2C}\) in (3.5) and combining the inequalities (3.4)–(3.6), we get
\[
\frac{1}{q} + \frac{1}{p} = \frac{1}{q} \frac{q}{q - 1} d\mu_h + C(n, h) \int_M \frac{q}{q + 1} \left( \int_M \nabla f \right) f^{q+1} d\mu_h
\leq C(n, h) \int_M f^{q+1} |\nabla f| f^{q-1} d\mu_h \tag{3.7}
\]
Since
\[
\int_M \nabla f \frac{q+1}{p+1} \int M f^{q+1} d\mu_h = \left( \frac{q+1}{2} \right) \int M |\nabla f| f^{q+1} d\mu_h,
\]
we get
\[
\frac{1}{q} + \frac{1}{p} = \frac{1}{q} \frac{q}{q - 1} d\mu_h + C(n, h) \frac{q}{(q + 1)^2} \int M \nabla f \frac{q+1}{p+1} d\mu_h
\leq C(n, h) \frac{q+1}{q} \int M f^{q+1} d\mu_h. \tag{3.8}
\]
If we let \(q \geq \frac{p}{2} - 1 > 0\), then \(0 < \frac{p}{q+1} \leq \frac{q}{q+1} \leq 1\), and thus we have
\[
\frac{\partial}{\partial t} \int M f^{q+1} d\mu_h + \int M |\nabla f| f^{q+1} d\mu_h \leq C(n, h, p) \int M f^{q+1} d\mu_h. \tag{3.9}
\]
For the last term, using Hölder’s inequality and the interpolation inequality, then we get
\[
\int M f^{q+1} d\mu_h \leq \left( \int M \nabla f \right)^{\frac{q}{2}} \left( \int M (f^{q+1})^{\frac{p+1}{p}} d\mu_h \right)^{\frac{p}{p+1}}
\leq C_\epsilon(n, h, A) \left( \epsilon \left( \int M (f^{q+1})^{\frac{p+1}{p}} d\mu_h \right)^{\frac{p+1}{p}} + \epsilon^{-\mu} \int M f^{q+1} d\mu_h \right). \tag{3.10}
\]
where
\[
\mu = \left( \frac{n - 2}{n} - \frac{p - 2}{p} \right) \left( \frac{p - 2}{p} - 1 \right) = \frac{p - n}{n}. \]
We also have the Sobolev inequality
\[
\int M f^{q+1} d\mu_h + \int M |\nabla f| f^{q+1} d\mu_h \geq C_\gamma(n, h) \left( \int M (f^{q+1})^{\frac{p+1}{p}} d\mu_h \right)^{\frac{p+1}{p}}. \tag{3.11}
\]
Taking \( \epsilon = \frac{C}{2C_0Cq} \) and combining (3.9)–(3.11), we get
\[
\frac{\partial}{\partial t} \int_M f^{q+1} d\mu_t + \left( \int_M (f^{q+1})^{\frac{m}{n}} d\mu_t \right)^{\frac{n}{m}} \leq C(n, h, A, p)q^{1+\mu} \int_M f^{q+1} d\mu_t. \tag{3.12}
\]
For any \( 0 < t' < t'' < T' \leq T \leq 1 \), let
\[
\psi(t) = \begin{cases} 
0, & \text{if } 0 \leq t \leq t', \\
\frac{t - t'}{t'' - t'}, & \text{if } t' \leq t \leq t'', \\
1, & \text{if } t'' \leq t \leq T.
\end{cases}
\]
Multiplying (3.12) by \( \psi \), we get
\[
\frac{\partial}{\partial t} \int_M \psi f^{q+1} d\mu_t + \psi \left( \int_M (f^{q+1})^{\frac{m}{n}} d\mu_t \right)^{\frac{n}{m}} \leq [C(n, h, A, p)q^{1+\mu} + \psi] \int_M f^{q+1} d\mu_t. \tag{3.13}
\]
Integrating it with respect to \( t \), we get
\[
\sup_{t \in [t', T']} \int_M f^{q+1} d\mu_t + \int_{t'}^{T'} \left( \int_M (f^{q+1})^{\frac{m}{n}} d\mu_t \right)^{\frac{n}{m}} dt \\
\leq \left[ C(n, h, A, p)q^{1+\mu} + \frac{1}{t'' - t'} \right] \int_M f^{q+1} d\mu_t dt. \tag{3.14}
\]
Then we have
\[
\int_{t'}^{T'} \int_M f^{(q+1)(1+\frac{\mu}{2})} d\mu_t dt \\
\leq \int_{t'}^{T'} \left( \int_M f^{q+1} d\mu_t \right)^{\frac{n}{m}} \left( \int_M f^{(q+1)}^{\frac{m}{n}} d\mu_t \right)^{\frac{m}{n}} dt \\
\leq \sup_{t \in [t', T']} \left( \int_M f^{q+1} d\mu_t \right)^{\frac{n}{2}} \int_{t'}^{T'} \left( \int_M f^{(q+1)}^{\frac{m}{n}} d\mu_t \right)^{\frac{m}{n}} dt \\
\leq \left[ C(n, h, A, p)q^{1+\mu} + \frac{1}{t'' - t'} \right] \left( \int_{t'}^{T'} \int_M f^{q+1} d\mu_t dt \right)^{1+\frac{n}{2}}, \forall q \geq \frac{p}{2} - 1. \tag{3.15}
\]
Define
\[
H(q, \tau) = \left( \int_{\tau}^{T} \int_M f^q d\mu_t dt \right)^{\frac{1}{q}}, \forall q \geq \frac{p}{2}, 0 < \tau < T'.
\]
Then the equation (3.15) can be shortly written as
\[
H \left( q \left( 1 + \frac{2}{n}, t' \right) \right) \leq \left[ C(n, h, A, p)q^{1+\mu} + \frac{1}{t'' - t'} \right]^\frac{1}{q} H(q, t').
\]
Fix \( 0 < t_0 < t_1 < T' \leq 1 \) and \( q_0 = \frac{p}{2} \), and set \( \chi = 1 + \frac{2}{n}, q_k = q_0 \chi^k \) and \( \tau_k = t_0 + (1 - \frac{1}{\chi})(t_1 - t_0) \). Then we have (see also [18, 21])
\[
H(q_{k+1}, \tau_{k+1}) \leq \left[ C(n, h, A, p)q_0^{1+\mu} + \frac{1}{t_1 - t_0} \chi \lambda^k \right]^\frac{1}{q_k} H(q_k, \tau_k) \\
\leq \left[ C(n, h, A, p)q_0^{1+\mu} \chi(1+\mu) + \frac{1}{t_1 - t_0} \frac{n + 2}{2} \chi^k \lambda^k \right]^\frac{1}{q_k} H(q_k, \tau_k) \\
\leq \left[ \frac{C(n, h, A, p)q_0}{t_1 - t_0} \right]^\frac{1}{q_k} \chi^{\frac{(1+\mu)k}{q_k}} H(q_k, \tau_k).
\]
where in the last inequality we use \(0 < t_0 < t_1 < T\). By iteration, we get

\[
H(q_{n+1}, \tau_{n+1}) \leq \left[ \frac{C(n, h, A, p, q_0)}{t_1 - t_0} \right]^{\sum_{k=0}^{m} \frac{k}{q_k}} \chi_{\sum_{k=0}^{m} \frac{k+1}{q_k}} H(q_0, \tau_0) \leq C(n, h, A, p, q_0) \left( \frac{1}{t_1 - t_0} \right)^{\sum_{k=0}^{m} \frac{k+2}{q_k}} H(q_0, \tau_0),
\]

since \(\sum_{k=0}^{\infty} \frac{1}{q_k} = \frac{n+2}{2q_0}\) and \(\sum_{k=0}^{\infty} \frac{k+1}{q_k}\) converges. Letting \(m \to \infty\), we get

\[
H(p_\infty, \tau_\infty) \leq C(n, h, A, p, q_0) \left( \frac{1}{t_1 - t_0} \right)^{n+2} H(q_0, \tau_0), \quad \forall \ q_0 \geq \frac{p}{2},
\]

where \(p_\infty = +\infty\) and \(\tau_\infty = t_1\). Letting \(q_0 = \frac{p}{2}\), we have

\[
H(\infty, t_1) \leq C(n, h, A, p) \left( \frac{1}{t_1 - t_0} \right)^{n+2} H \left( \frac{p}{2}, t_0 \right).
\]

Thus we have

\[
\sup_{(y, t) \in M \times [t_1, T]} f(y, t) \leq C(n, h, A, p) \left( \frac{1}{t_1 - t_0} \right)^{n+2} \left( \int_{t_0}^{T} A^\frac{p}{2} dt \right)^{\frac{2}{p}}.
\]

Letting \(t_1 \to T'\) and \(t_0 = T'/2\), we get

\[
\sup_{y \in M} f(y, T') \leq C(n, h, A, p) \frac{1}{(T')^{\frac{p}{2}}}, \quad \forall T' \in [0, T].
\]

Thus we have

\[
|\nabla g(t)| \leq \frac{C(n, h, A, p)}{t^{\frac{p}{2}}}, \quad \forall t \in (0, T],
\]

which completes the proof of this lemma. \(\square\)

**Proof of Theorem 3.11.** The existence of the h-flow \(g(t)\) is claimed in Theorem 3.8, so we only need to prove that the conclusions (1)–(3) hold for some \(T_0(n, h, A, p)\).

To prove that (1) and (2) hold for some \(T_0(n, h, A, p)\), let \(f = |\nabla g_{ij}|^2 + 1\). We define

\[
\phi(t) = \int_M f^\frac{p}{2} d\mu_h.
\]

Then we have

\[
\phi'(t) = \int_M \frac{\partial}{\partial t} \left( f^\frac{p}{2} \right) d\mu_h.
\]

We set

\[
\mathcal{T} = \left\{ T \in (0, 1) \left| \int_M |\nabla g(t)|^p d\mu_h \leq 10A, \forall x \in M, \forall t \in [0, T] \right. \right\}
\]

and \(T_{\max} = \sup \mathcal{T}\). Take \(q = p/2 - 1\), and then the equation (3.9) gives

\[
\phi'(t) \leq C(n, h, p) \int_M f^{p/2} v d\mu_h,
\]

where \(v = f + 1\). By Lemma 3.12, we have

\[
v = f + 1 \leq 1 + \frac{C(n, h, A, p)}{p^{n/p}}, \quad \forall t \in (0, T_{\max}].
\]

Thus we get

\[
\phi'(t) \leq \left( 1 + \frac{C(n, h, A, p)}{p^{n/p}} \right) \int_M f^{p/2} d\mu_h, \quad \forall t \in (0, T_{\max}],
\]

(3.17)
Since $T_{\text{max}} \leq 1$, we have
\[ \phi'(t) \leq \frac{C(n, h, A, p)}{t^{n/p}} \phi(t), \quad \forall t \in (0, T_{\text{max}}], \]
and thus we have
\[ (\log \phi(t))' \leq \frac{C(n, h, A, p)}{t^{n/p}}, \quad \forall t \in (0, T_{\text{max}}]. \]
By integration, we get
\[ \log \phi(t) \leq \log \phi(0) + C(n, h, A, p) t^{1-n/p}, \quad \forall t \in (0, T_{\text{max}}], \]
and thus we have
\[ \phi(t) \leq \phi(0) e^{C(n, h, A, p) t^{1-n/p}}, \quad \forall t \in (0, T_{\text{max}}]. \]
Since $\phi(0) \leq C(n, h, A)$, we have
\[ \int_M |\nabla g(t)|^p \, dp_h \leq \phi(t) \leq C(n, h) e^{C(n, h, A, p) t^{1-n/p} A}, \quad \forall t \in (0, T_{\text{max}}]. \quad (3.18) \]
Suppose that if we fix $n$, $h$, $p$ and $A$, there exists a sequence of initial metrics $\{g_m\}_{m=1}^\infty$ such that each of the metrics $g_m$ satisfies the condition of the theorem and the corresponding $T_{\text{max},m}$ tends to 0. But by (3.18), we know that if $T_{\text{max}}$ satisfies $C(n, h) e^{C(n, h, A, p) T_{\text{max}}^{1-n/p}} \leq 5$ and $T_{\text{max}} < 1$, then the maximal time interval $(0, T_{\text{max}}]$ could be extended to $(0, T_{\text{max}} + \delta]$ for some $\delta > 0$ small enough, which leads to a contradiction. Hence, $T_{\text{max}}$ only depends on $n$, $h$, $p$ and $A$. Thus we see that the conclusions (1) and (2) hold simultaneously for some $T_0(n, h, A, p)$.

To prove (3), recall that Simon’s result, Theorem 3.8, gives
\[ \sup_M |\nabla g(t)| \leq \frac{c(n, h)}{t^{1/2}}, \quad \forall t \in (0, T_0]. \]
We choose a finite atlas for $M$ such that $h$ is uniformly equivalent to the Euclidean metric of each chart. For any chart $(U, \Phi)$, we let $\tilde{f}$ denote any component function $\nabla g_{jk}(t)$ of $\nabla g(t)$, and let $h_0$ denote the Euclidean metric of $(U, \Phi)$. We can assume that $2^{-1} h_0 \leq h \leq 2 h_0$. We choose an arbitrary point $p \in U$ and let $\gamma(u)$ be a curve satisfying $\gamma(0) = p$ and $\gamma'(u) = \frac{\partial}{\partial x^i}$, where $\frac{\partial}{\partial x^i}$ is the coordinate vector field of $(\Phi, U)$. Then we have
\[ \int_0^s \frac{d}{du} \tilde{f} \gamma(u) \, du = \tilde{f} \gamma(s) - \tilde{f}(p), \quad \forall s \in (0, s_0), \]
where $s_0$ is a positive constant independent of the chart and $t$, and $s_0$ is small enough such that $\gamma(u) \in U$ for any $u \in (0, s_0)$ and for any coordinate neighborhood $U$.

On the one hand, we have
\[ \int_0^s \frac{d}{du} \tilde{f} \gamma(u) \, du = \int_0^s d\tilde{f} |\gamma(u)'(u)| \, du = \int_0^s \frac{\partial \tilde{f}}{\partial x^i}(\gamma(u)) \, du. \]
On the other hand, we have
\[ \tilde{f} \gamma(s) - \tilde{f}(p) \leq 2 \sup_M |\nabla g(t)| \leq \frac{C(n, h, A, p)}{t^{1/2}}. \]
Thus there exists a $u_0 \in (0, s)$ such that
\[ \frac{\partial \tilde{f}}{\partial x^i}(\gamma(u_0)) \leq \frac{C(n, h, A, p)}{s t^{1/2}}. \]
Since
\[ \frac{\partial \tilde{f}}{\partial x^i}(\gamma(u_0)) - \frac{\partial \tilde{f}}{\partial x^i}(\gamma(0)) = \int_0^{u_0} \frac{\partial^2 \tilde{f}}{(\partial x^i)^2}(\gamma(u)) \, du \]
Proof. We calculate as in the proof of Lemma 3.12, but this time we preserve the $\epsilon$

Taking the right-hand side of (3.19) attains its minimum for $s = C(n,h,A,p) t^{\frac{4}{p}}$. Note that

$$\lim_{t\rightarrow 0^+} t^{\frac{4}{p}} = 0,$$

and thus there exists a $T_0 = T_0(n,h,A,p)$ such that for any $t \in (0,T_0)$, we have $C(n,h,A,p) t^{\frac{4}{p}} \leq s_0$. Therefore, we have that for any $t \in (0,T_0)$, the right-hand side of (3.19) attains its minimum for some $s \in (0,s_0)$, and the minimum value is

$$C(n,h,A,p) t^2 \leq C(n,h,A,p) t^{\frac{4}{p}}.$$

Since $p$ is an arbitrary point on $M$ and $2^{-1} h_0 \leq h \leq 2h_0$, we get

$$|\nabla^2 g(t)| \leq C(n,h,p,A) t^2 \leq C(n,h,A,p).$$

Thus (3) holds for some $T_0(n,h,A,p)$, which completes the proof of this theorem.

Theorem 3.2(2) follows immediately from Theorem 3.11. Moreover, for Theorem 3.2(3), we just need to prove the following lemma.

**Lemma 3.13.** In the condition of Theorem 3.11, we have

$$\int_0^{T_0} \int_M |\nabla^2 g(t)|^2 d\mu_h dt \leq C(n,h,A,p).$$

**Proof.** We calculate as in the proof of Lemma 3.12, but this time we preserve the $|\nabla^2 g|^2$ term in (3.1). Integrating both sides of the inequality (3.1), we have

$$\frac{\partial}{\partial t} \int_M f d\mu_h + \int_M |\nabla^2 g|^2 d\mu_h \leq C(n,h) \int_M g^{\alpha\beta} \nabla_\alpha \nabla_\beta f d\mu_h + C(n,h) \int_M f d\mu_h,$$

where $f = |\nabla g_j|^2 + 1$ and $v = f + 1$. Here and below, $C$ and $C_1$’s denote positive constants and $C$ varies from line to line. Using integration by parts twice, we have

$$\frac{\partial}{\partial t} \int_M f d\mu_h + \int_M |\nabla^2 g|^2 d\mu_h$$

$$\leq C(n,h) \int_M \nabla_\beta \nabla_\alpha g^{\alpha\beta} f d\mu_h + C(n,h) \int_M f d\mu_h$$

$$\leq C_1(n,h) \int_M |\nabla^2 g| f d\mu_h + C(n,h) \int_M f d\mu_h$$

$$\leq C_1(n,h) \epsilon \int_M |\nabla^2 g|^2 d\mu_h + C_2(n,h) \epsilon^{-1} \int_M f^2 d\mu_h + C(n,h) \int_M f d\mu_h, \quad \forall \epsilon > 0.$$

Taking $\epsilon = \frac{C_2}{2}$, since $f \leq v$, we have

$$\frac{\partial}{\partial t} \int_M f d\mu_h + \int_M |\nabla^2 g|^2 d\mu_h \leq C_2(n,h) \int_M f d\mu_h.$$

(3.20)
For the last term, using Hölder’s inequality and the interpolation inequality, we get
\[
\int_M f v d\mu_h \leq \left( \int_M v^\frac{2}{p} d\mu_h \right)^{\frac{p}{2}} \left( \int_M f^{\frac{p-2}{p}} d\mu_h \right)^{\frac{p-2}{p}}
\]
\[
\leq C_2(n, h, A) \left[ \epsilon \left( \int_M f^{\frac{p}{p-2}} d\mu_h \right)^{\frac{p-2}{p}} + \epsilon^{-p} \int_M f d\mu_h \right],
\]
(3.21)
where \( \mu = \left( \frac{n-2}{n} - \frac{p-2}{p} \right) / \left( \frac{p-2}{p} - 1 \right) = \frac{p-n}{n} \). We also have the Sobolev inequality
\[
\left( \int_M f^{\frac{p}{p-2}} d\mu_h \right)^{\frac{p-2}{p}} \leq C_4(n, h) \left( \int_M |\nabla f|^2 d\mu_h + \int_M f d\mu_h \right).
\]
(3.22)
Since
\[
|\nabla f|^2 = \nabla |\nabla g|^2 + 1 = \frac{\nabla (\nabla g, \nabla g)}{2(|\nabla g|^2 + 1)^2} = \frac{2(|\nabla g| \nabla g)}{2(|\nabla g|^2 + 1)^2} \leq \frac{|\nabla^2 g| |\nabla g|}{(|\nabla g|^2 + 1)^2} \leq |\nabla^2 g|,
\]
(3.23)
taking \( \epsilon = \frac{1}{2C_2C_4} \), by (3.20)–(3.23), we have
\[
\frac{\partial}{\partial t} \int_M f d\mu_h + \int_M |\nabla^2 g|^2 d\mu_h \leq C(n, h, A, p) \int_M f d\mu_h.
\]
By Theorem 3.11,
\[
\int_M |\nabla g(t)|^p d\mu_h \leq 10A, \quad \forall t \in (0, T_0).
\]
Since \( p > n \geq 2 \), we have
\[
\int_M f d\mu_h \leq C(n, h, A, p),
\]
and thus we have
\[
\frac{\partial}{\partial t} \int_M f d\mu_h + \int_M |\nabla^2 g|^2 d\mu_h \leq C(n, h, A, p).
\]
Integrating it, we get
\[
\int_M f(T_0) d\mu_h - \int_M f(0) d\mu_h + \int_0^{T_0} \int_M |\nabla^2 g|^2 d\mu_h dt \leq C(n, h, A, p).
\]
Since \( \int_M f(T_0) d\mu_h \geq 0 \) and \( \int_M f(0) d\mu_h \leq C(n, h, A, p) \), we get
\[
\int_0^{T_0} \int_M |\nabla^2 g|^2 d\mu_h dt \leq C(n, h, A, p),
\]
which completes the proof of this lemma. \( \square \)

Now Theorem 3.2 would follow without much effort.

**Proof of Theorem 3.2.** Let \( g(t) \) be the \( h \)-flow stated in Theorem 3.8. Then there is a family of diffeomorphisms \( \phi(t) : M \to M \) such that \( \phi(t)^* g(t) (t \in (0, T_0]) \) is a Ricci flow [39].

Since \( M \) is compact, by Theorem 3.8(1), we see that \((M, g(t)) \) converges to \((M, \hat{g}) \) in the Gromov-Hausdorff distance. Since the Ricci flow \( \phi(t)^* g(t) \) and the \( h \)-flow \( g(t) \) are isometric for each \( t \in (0, T_0] \), Theorem 3.2(1) holds.

To prove (2) and (3), note that \( R_m = \partial^2 g + \partial g * \partial g \) and \( p > n \geq 2 \). Since the Ricci flow \( \phi(t)^* g(t) \) and the \( h \)-flow \( g(t) \) are isometric for each \( t \in (0, T_0] \) and both \( g(t) \) and \( \phi(t)^* g(t) \) are uniformly equivalent to \( h \), (2) and (3) of Theorem 3.2 follow immediately from Theorem 3.11 and Lemma 3.13. \( \square \)
4 The conjugate heat equation

Let $(M^n, \hat{g})$ be a compact manifold, and $g(t)$ be the Ricci flow starting from the metric $\hat{g}$. In this section, we suppose that $\hat{g}$ is smooth. Let $\mu_{g(t)}$ be the Lebesgue measure taken with respect to $g(t)$, and let $R_{\hat{g}}$ and $R_{g(t)}$ denote the distributional scalar curvatures of $\hat{g}$ and $g(t)$, respectively. Let $\hat{\varphi}$ be an arbitrary nonnegative function in $C^\infty(M)$ and take any $T \in (0, T_0]$. We consider the following conjugate heat equation:

$$
\begin{cases}
\partial_t \varphi_t = -\Delta_{g(t)} \varphi_t + R_{g(t)} \varphi_t & \text{on } M \times [0, T], \\
\varphi_t \big|_{s=T} = \hat{\varphi},
\end{cases}
$$

(4.1)

where $\Delta_{g(t)}$ is the Laplacian taken with respect to $g(t)$.

For fixed $(x, t) \in M \times (0, T_0]$, the conjugate heat kernel on the Ricci flow background is the function $K(x, t; \cdot, \cdot)$, defined for $0 \leq s < t$ and $y \in M$ and satisfying

$$
(-\partial_s - \Delta_y + R(y, s))K(x, t; y, s) = 0 \quad \text{and} \quad \lim_{s \to t^-} K(x, t; y, s) = \delta_x(y),
$$

where the Laplacian is taken with respect to $g(s)$, and $\delta_x$ is the Dirac delta distribution supported on $\{x\}$. $K$ also satisfies $(\partial_t - \Delta_x)K(x, t; y, s) = 0$, where $\Delta_x$ is taken with respect to $g(t)$.

By direct calculation (see also [3]), we see that the equation (4.1) has a solution with the explicit expression

$$
\varphi_t(x) = \int_M K(y, T; x, t)\hat{\varphi}(y)d\mu_{g(T)}(y).
$$

(4.2)

By the maximum principle, we find that this solution is nonnegative and unique. Furthermore, by this expression we can see that $\varphi_t$ is uniformly bounded. Our main purpose in this section is proving the following estimates for $\varphi_t$, which will be used in the proof of our main theorem.

**Proposition 4.1.** Assume that $\varphi_t$ satisfies the above properties, and then $\varphi_t$ satisfies

1. $\varphi_t \in C(n, h, A, p; \|\hat{\varphi}\|_{L^\infty}), \forall t \in [0, T]$;
2. $\int_M |\nabla g(t)\varphi_t|^2_{g(t)}d\mu_{g(t)} \leq C(n, h, A, p, \hat{\varphi}), \forall t \in [0, T]$;
3. $\int_M (R_{g(t)} - a)\varphi_t d\mu_{g(t)}$ is monotonically increasing with respect to $t$.

**Proof.** To see (1), by the equation (4.2), we have

$$
\varphi_t(x) \leq \|\hat{\varphi}\|_{L^\infty} \int_M K(y, T; x, t)d\mu_{g(T)}(y).
$$

(4.3)

We define

$$
F(t, T) = \int_M K(y, T; x, t)d\mu_{g(T)}(y).
$$

Then we have $\lim_{T \to t^+} F(t, T) = 1$ and

$$
\partial_t F(t, T) = \int_M (\Delta_y K(y, T; x, t) - R_T K(y, T; x, t))d\mu_{g(T)}(y),
$$

where we have used the standard evolution equation $\partial_t d\mu_{g(T)} = -R_T d\mu_{g(T)}$. By the divergence theorem, we have

$$
\int_M \Delta_y K(y, T; x, t)d\mu_{g(T)}(y) = 0.
$$

Thus by Theorem 3.2, we have

$$
\partial_t F(t, T) \leq C(n, h, A, p)\frac{T_0 - T}{T_0} F(t, T)
$$

for some $\alpha \in (0, 1)$. Since $\lim_{T \to t^+} F(t, T) = 1$, by integration we have

$$
F(t, T) \leq C(n, h, A, p), \quad \forall 0 \leq t < T \leq T_0.
$$

(4.4)
Thus we complete the proof of (2).

By (4.3) and (4.4), we have
\[ \varphi_t \leq C(n, h, A, p, \|\bar{\varphi}\|_{L^\infty}), \quad \forall t \in [0, T], \]
which proves (1).

To prove (2), define \( E(t) = \int_M |\nabla g(t)\varphi_t|^2 g(t) d\mu_g(t) \). By direct calculation, we have
\[
\partial_t E(t) = \int_M (-R_{g(t)} |D\varphi_t g(t)|^2 + 2\text{Ric}_{g(t)}(\nabla g(t)\varphi_t, \nabla g(t)\varphi_t) + 2\{\nabla g(t)\partial_t \varphi_t, \nabla g(t)\varphi_t\} g(t)) d\mu_g(t). \tag{4.5}
\]

By (4.1) and the Bochner formula, we have
\[
\int_M \langle \nabla g(t)\partial_t \varphi_t, \nabla g(t)\varphi_t \rangle g(t) d\mu_g(t)
\]
\[
= \int_M (-\langle \nabla g(t)(\Delta g(t)\varphi_t - R_{g(t)}\varphi_t), \nabla g(t)\varphi_t \rangle g(t) d\mu_g(t)
\]
\[
= \int_M (-\langle \nabla g(t)\Delta g(t)\varphi_t, \nabla g(t)\varphi_t \rangle g(t) + \langle \nabla g(t)(R_{g(t)}\varphi_t), \nabla g(t)\varphi_t \rangle g(t)) d\mu_g(t)
\]
\[
= \int_M \left( -\frac{1}{2} \Delta g(t)|\nabla g(t)\varphi_t|^2 g(t) + |\nabla^2 g(t)\varphi_t|^2 g(t) + \text{Ric}_{g(t)}(\nabla g(t)\varphi_t, \nabla g(t)\varphi_t) + \langle \nabla g(t)(R_{g(t)}\varphi_t), \nabla g(t)\varphi_t \rangle \right) d\mu_g(t)
\]
\[
= \int_M \left( |\nabla^2 g(t)\varphi_t|^2 g(t) + \text{Ric}_{g(t)}(\nabla g(t)\varphi_t, \nabla g(t)\varphi_t) - R_{g(t)}|\nabla g(t)\varphi_t|^2 g(t) \right) d\mu_g(t). \tag{4.6}
\]

Since \( |\Delta g(t)\varphi_t|^2 g(t) \leq C(n)|\nabla^2 g(t)\varphi_t|^2 g(t) \), using the Cauchy inequality, we see that (4.6) gives
\[
\int_M \langle \nabla g(t)\partial_t \varphi_t, \nabla g(t)\varphi_t \rangle g(t) d\mu_g(t) \geq \int_M (\text{Ric}_{g(t)}(\nabla g(t)\varphi_t, \nabla g(t)\varphi_t) - C(n)|\nabla^2 g(t)\varphi_t|^2 g(t)) d\mu_g(t). \tag{4.7}
\]

By (4.5) and (4.7), since \( |R_{g(t)}| g(t) \leq c(n)|\text{Ric}_{g(t)}| g(t) \), we get
\[
\partial_t E(t) \geq \int_M (4\text{Ric}_{g(t)}(\nabla g(t)\varphi_t, \nabla g(t)\varphi_t) - R_{g(t)}|\nabla g(t)\varphi_t|^2 g(t) - C(n)|\nabla^2 g(t)\varphi_t|^2 g(t)) d\mu_g(t)
\]
\[
\geq -C(n, h, A, p) \int_M |\text{Ric}_{g(t)}| |\nabla g(t)\varphi_t|^2 d\mu_g(t) - C(n, h, A, p) \int_M R_{g(t)}^2 |\nabla g(t)\varphi_t|^2 d\mu_g(t). \tag{4.8}
\]

By Theorem 3.2 and (1), we have for some \( \alpha \in (0, 1) \),
\[
\partial_t E(t) \geq -\frac{C(n, h, A, p)}{t^\alpha} \int_M |\nabla g(t)\varphi_t|^2 d\mu_g(t) - C(n, h, A, p, \tilde{\varphi}) \int_M R_{g(t)}^2 d\mu_g(t).
\]

Thus we have
\[
\partial_t (E(t) + 1) \geq -\frac{C(n, h, A, p)}{t^\alpha} (E(t) + 1) - C(n, h, A, p, \tilde{\varphi}) \int_M R_{g(t)}^2 d\mu_g(t).
\]

Dividing both sides by \( E(t) + 1 \), we get
\[
\partial_t \log (E(t) + 1) \geq -\frac{C(n, h, A, p)}{t^\alpha} - C(n, h, A, p, \tilde{\varphi}) \int_M R_{g(t)}^2 d\mu_g(t).
\]

By Lemma 3.13, \( \int_M R_{g(t)}^2 d\mu_g(t) \) is integrable on \((0, T)\). Since \( \varphi_T = \tilde{\varphi} \), \( E(T) \leq C(n, h, A, p, \tilde{\varphi}) \) and \( \frac{1}{t^\alpha} \) is integrable on \((0, T)\), we can integrate the inequality above and get
\[
E(t) \leq C(n, h, A, p, \tilde{\varphi}), \quad \forall t \in [0, T].
\]

Thus we complete the proof of (2).
To prove (3), we can directly calculate that
\[
\partial_t \int_M (R_{\hat{g}(t)} - a) \varphi d\mu_{\hat{g}(t)}
\]
\[
= \int_M [(\Delta_{\hat{g}(t)} R_{\hat{g}(t)} + 2|Ric_{\hat{g}(t)}|^2_{\hat{g}(t)}) \varphi_t + (R_{\hat{g}(t)} - a)(-\Delta_{\hat{g}(t)} \varphi_t + R_{\hat{g}(t)} \varphi_t)
\]
\[
+ (R_{\hat{g}(t)} - a) \varphi_t (-R_{\hat{g}(t)})] d\mu_{\hat{g}(t)}
\]
\[
= \int_M 2|Ric_{\hat{g}(t)}|^2_{\hat{g}(t)} \varphi_t d\mu_{\hat{g}(t)} + \int_M (\Delta_{\hat{g}(t)} R_{\hat{g}(t)} \varphi_t - R_{\hat{g}(t)} \Delta_{\hat{g}(t)} \varphi_t) d\mu_{\hat{g}(t)} + a \int_M \Delta_{\hat{g}(t)} \varphi_t d\mu_{\hat{g}(t)}. 
\]
By integration by parts, we have
\[
\int_M (\Delta_{\hat{g}(t)} R_{\hat{g}(t)} \varphi_t - R_{\hat{g}(t)} \Delta_{\hat{g}(t)} \varphi_t) d\mu_{\hat{g}(t)} = 0,
\]
\[
\int_M \Delta_{\hat{g}(t)} \varphi_t d\mu_{\hat{g}(t)} = 0.
\]
Thus we have
\[
\partial_t \int_M (R_{\hat{g}(t)} - a) \varphi_t d\mu_{\hat{g}(t)} \geq 0.
\]
Therefore, \(\int_M (R_{\hat{g}(t)} - a) \varphi_t d\mu_{\hat{g}(t)}\) is monotonically increasing. Hence, we finish the proof of (3), and thus the proof of this proposition is completed.

\[\square\]

5 The proof of the main theorem

In this section, we give the proof of our main theorem. Let us restate Theorem 1.1 as follows.

**Theorem 5.1.** Let \(M^n\) be a compact manifold with a metric \(\hat{g} \in W^{1,p}(M)\) \((n < p \leq \infty)\). Assume the distributional scalar curvature \(R_{\hat{g}} \geq a\) for some constant \(a\), and let \(g(t)\) \((t \in (0, T_0])\) be the Ricci flow starting from the metric \(\hat{g}\). Then for any \(t \in (0, T_0]\), it holds that \(R_{g(t)} \geq a\) on \(M\).

**Proof.** By Lemma 2.1, we get a family of smooth metrics \(\hat{g}_\delta\) which converges to \(\hat{g}\) both in the \(C^0\)-norm and in the \(W^{1,p}\)-norm. Then by Lemma 2.2, we have
\[
\left| \int_M R_{\hat{g}_\delta} \varphi d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}}, \varphi \rangle \right| \leq b_\delta \|\varphi\|_{W^{1, \frac{n}{n-2}}(M)}, \quad \forall \varphi \in C^\infty(M) \text{ and } \delta \in (0, \delta_0], \quad (5.1)
\]
where \(b_\delta\) is a positive function of \(\delta\) which only depends on \(\hat{g}\) and satisfies \(\lim_{\delta \to 0^+} b_\delta = 0\) and \(\delta_0\) is some positive constant small enough.

Moreover, we have
\[
\lim_{\delta \to 0^+} \left\| \frac{d\mu_{\hat{g}_\delta}}{d\hat{g}_\delta} - 1 \right\|_{C^0(M)} = 0,
\]
and thus by Hölder’s inequality, we have
\[
\left| \int_M \varphi d\mu_{\hat{g}_\delta} - \int_M \varphi d\mu_{\hat{g}} \right| = \left| \int_M \varphi \left( \frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1 \right) d\mu_{\hat{g}} \right|
\]
\[
\leq \left\| \frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1 \right\|_{C^0(M)} \int_M |\varphi| d\mu_{\hat{g}}
\]
\[
\leq C(n, \hat{g}) \left\| \frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1 \right\|_{C^0(M)} \|\varphi\|_{W^{1, \frac{n}{n-2}}(M)}, \quad (5.2)
\]
By the triangle inequality, combining (5.1) and (5.2), we have
\[
\left| \int_M (R_{\hat{g}_\delta} - a) \varphi d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}}, \varphi \rangle \right| \leq \left| \int_M R_{\hat{g}_\delta} \varphi d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}}, \varphi \rangle \right| + |a| \left| \int_M \varphi d\mu_{\hat{g}_\delta} - \int_M \varphi d\mu_{\hat{g}} \right|
\]
\[
\leq b_\delta \|\varphi\|_{W^{1, \frac{n}{n-2}}(M)}, \quad \forall \varphi \in C^\infty(M) \text{ and } \delta \in (0, \delta_0], \quad (5.3)
\]
where \( b_\delta \) is a positive function of \( \delta \) which only depends on \( a, n \) and \( \hat{g} \) and satisfies \( \lim_{\delta \to 0^+} b_\delta = 0 \).

By the condition on \( R_\delta \), we have
\[
(R_\delta - a, \varphi) \geq 0, \quad \forall \varphi \in C^\infty(M), \quad \varphi \geq 0. \tag{5.4}
\]

Let \( g_\delta(t) \) and \( g(t) \) be the Ricci flows starting from the metrics \( \hat{g}_\delta \) and \( \hat{g} \), respectively, and \( R_{g_\delta(t)}, R_{g(t)} \) and \( d\mu_{g_\delta(t)}, d\mu_{g(t)} \) be the scalar curvatures and the volume forms of \( g_\delta(t) \) and \( g(t) \), respectively. Take an arbitrary \( T \in (0, T_0] \) and an arbitrary nonnegative function \( \tilde{\varphi} \in C^\infty(M) \). For any \( \delta \in (0, \delta_0] \), we let \( \varphi_t \) be the solution to the conjugate heat equation taken with respect to the family of metrics \( g_\delta(t) \) with \( \varphi_t |_{t=T} = \tilde{\varphi} \). Then \( \varphi_0 \geq 0 \), and by Proposition 4.1(3), (5.3) and (5.4), we have
\[
\int_M (R_{g_\delta(T)} - a) \tilde{\varphi} d\mu_{g_\delta(T)} \geq \int_M (R_{\hat{g}_\delta} - a) \varphi_0 d\mu_{g_\delta} \geq -b_\delta \| \varphi_0 \|_{W^{1, \frac{n}{2}}(M)}. \tag{5.5}
\]

By Hölder’s inequality and (1) and (2) of Proposition 4.1, since \( g_\delta(t), (\delta, t) \in (0, \delta_0] \times [0, T_0] \) and \( h \) are uniformly equivalent, we have
\[
\| \varphi_0 \|_{W^{1, \frac{n}{2}}(M)} \leq C(n, h, p, \| \tilde{\varphi} \|_{L^\infty(M)} + \| \nabla \tilde{\varphi} \|_{L^2(M)} \leq C(n, h, A, p, \tilde{\varphi}),
\]
and as mentioned in Remark 3.10, it is known in [39] that for any fixed \( T \in (0, T_0] \), \( g_\delta(T) \) smoothly converges to \( g(T) \) as \( \delta \) tends to \( 0^+ \).

Thus we can let \( \delta \to 0^+ \) in the inequality (5.5), and get
\[
\int_M (R_{g(T)} - a) \tilde{\varphi} d\mu_{g(T)} \geq 0, \quad \forall T \in (0, T_0] \text{ and } \tilde{\varphi} \in C^\infty(M).
\]

Since \( g(t) \) is a smooth metric for \( t \in (0, T_0] \), we see that \( R_{g(t)} \geq a \) pointwisely on \( M \), which completes the proof of this theorem. \( \square \)

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Recall that \( R_{g(t)} \) satisfies the standard evolution equation
\[
\partial_t R_{g(t)} = \Delta R_{g(t)} + 2 |\text{Ric}_{g(t)}|^2. \tag{5.6}
\]

Noting that \( R_\delta \geq 0 \) in the distributional sense, by Theorem 1.1 we see that \( R_{g(t)} \geq 0 \) for any \( t \in (0, T_0] \). There are two cases needed to be handled. First, if \( R_{g(t)} \equiv 0 \) on \( M \times (0, T_0] \), then (5.6) shows that \( \text{Ric}_{g(t)} \equiv 0 \) on \( M \times (0, T_0] \). Then by the Ricci flow equation, we have \( g(t) = g(s) \), \( \forall t, s \in (0, T_0] \). By Theorem 3.2(1), we directly find that \((M, \hat{g})\) is isometric to \((M, g(t))\). On the other hand, if \( R_{g(t)} \not\equiv 0 \) on \( M \times (0, T_0] \), then by the maximum principle we see that \( R_{g(t)} > 0 \) on \( M \) for any \( t \in (0, T_0] \). Since \( g(t) \) is smooth and \( R_{g(t)} > 0 \) pointwisely on \( M \), this theorem holds. \( \square \)

### 6 Further questions

In this section, let us discuss some problems related to singular metrics. To study the scalar curvature of low-regularity metrics, Schoen proposed such a conjecture (see [28]).

**Conjecture 6.1** (See [28]). Let \( g \) be a \( C^0 \) metric on \( M \) which is smooth away from a submanifold \( \Sigma \subset M \) with \( \text{codim}(\Sigma \subset M) \geq 3 \), \( \sigma(M) \leq 0 \) and \( R_\Sigma \geq 0 \) on \( M \setminus \Sigma \). Then \( g \) smoothly extends to \( M \) and \( \text{Ric}_g \equiv 0 \).

Gromov [15] also considered scalar curvature with low-regularity metrics. For a Ricci flow approach, see also [2, 5]. Motivated by their results, we have the following natural question.

**Question.** Let \((M^n, g, \Sigma)\) be a compact manifold with \( g \in C^0(M) \) and \( g \) being smooth away from \( \Sigma \), and \( R_\Sigma \geq 0 \) on \( M \setminus \Sigma \). What is the condition on \( \Sigma \) such that there exists a smooth metric on \( M \) with nonnegative scalar curvature? What is the condition on \( \Sigma \) such that the Ricci flow starting from \( g \) has nonnegative scalar curvature for time \( t > 0 \)?
Remark 6.2. Comparing the conjecture of Schoen (see [28]) and the structure of nonnegative scalar curvature in [36], one may expect that $\Sigma$ is a codimension-3 submanifold for the above question.

Remark 6.3. From our result and the main result in [20], if we assume $g \in C^0 \cap W^{1,p}(M)$ ($n < p \leq \infty$), the condition for $\Sigma$ should be $H^{n-p}(\Sigma) < \infty$ when $n < p < \infty$ or $H^{n-1}(\Sigma) = 0$ when $p = \infty$.

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant Nos. 12125105 and 12071425) and the Fundamental Research Funds for the Central Universities. The second author was supported by National Natural Science Foundation of China (Grant Nos. 11971424 and 12031017). The third author was supported by National Natural Science Foundation of China (Grant No. 11971424). The authors thank Professor Dan Lee and Professor Christina Sormani for many helpful suggestions.

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Appendix A Lower scalar curvature bounds on singular metrics

In this appendix, we check that if \( g \) is \( C^2 \) away from a closed subset \( \Sigma \) with \( H^{n-\frac{2}{p}}(\Sigma) < \infty \) when \( n < p < \infty \) or \( H^{n-1}(\Sigma) = 0 \) when \( p = \infty \), and \( R_g \geq a \) pointwise on \( M \setminus \Sigma \), then \( R_g \geq a \) in the distributional sense. Though it is proved in [20], for the sake of completeness, we give a proof here.

**Lemma A.1.** Let \( M^n \) be a smooth manifold with \( g \in C^1 \cap W^{1,p}_{\text{loc}}(M) \) and \( n \leq p \leq \infty \). Assume that \( g \) is smooth away from a closed subset \( \Sigma \) with \( H^{n-\frac{2}{p}}(\Sigma) < \infty \) if \( n < p < \infty \) or \( H^{n-1}(\Sigma) = 0 \) if \( p = \infty \), and \( R_g \geq a \) on \( M \setminus \Sigma \) for some constant \( a \). Then \( \langle R_g - a, u \rangle \geq 0 \) for any nonnegative \( u \in C^{\infty}(M) \).

To prove Lemma A.1, we need a standard cut-off function lemma (see [20, Lemma A.1] and [7]).

**Lemma A.2.** Let \( (M^n, h) \) be a smooth manifold. Assume that \( \Sigma \subset M \) is a closed subset. Then there exists a sequence of cut-off functions \( \varphi_\epsilon \) of \( \Sigma \) such that the following statements hold:

1. \( 0 \leq \varphi_\epsilon \leq 1 \) in a neighborhood of \( \Sigma \) and \( \varphi_\epsilon \equiv 1 \) on \( M \setminus B_\epsilon(\Sigma) \).
2. If \( H^{n-1}(\Sigma) = 0 \), then \( \lim_{\epsilon \to 0} \int_M |\nabla \varphi_\epsilon|^p dx = 0 \).
3. If \( H^{n-p}(\Sigma) < \infty \) with \( p > 1 \), then \( \lim_{\epsilon \to 0} \int_M |\nabla \varphi_\epsilon|^p dx = 0 \).

**Proof of Lemma A.1.** Let \( \eta_\epsilon \geq 0 \) be a sequence of smooth cut-off functions of \( \Sigma \) as in Lemma A.2 such that

1. \( \eta_\epsilon \equiv 1 \) in a neighborhood of \( \Sigma \);
2. \( \supp \eta_\epsilon \subset B_\epsilon(\Sigma) \) and \( 0 \leq \eta_\epsilon \leq 1 \);
3. \( \lim_{\epsilon \to 0} \int_M |\nabla \eta_\epsilon|^{p/(p-1)} d\mu_h = 0 \).

For any \( u \in C^{\infty}(M) \) with \( u \geq 0 \), we have

\[
\langle R_g - a, u \rangle = \langle R_g - a, \eta_\epsilon u \rangle + \langle R_g - a, (1 - \eta_\epsilon)u \rangle.
\]

(A.1)

Note that for any \( \epsilon > 0 \), we have

\[
\langle R_g - a, (1 - \eta_\epsilon)u \rangle = \int_{M \setminus \Sigma} (R_g - a)(1 - \eta_\epsilon)ud\mu_g \geq 0.
\]

(A.2)
Thus to prove the lemma, we only need to show that
\[ \lim_{\epsilon \to 0} |\langle R_g - a, \eta_\epsilon u \rangle| = 0. \] (A.3)

Actually, note that \( u \in C^\infty(M) \), and by definition,
\[ \langle R_g, u \rangle = \int_M \left( -V \cdot \hat{\nabla} \left( u \frac{d\mu_g}{d\mu_h} \right) + Fu \frac{d\mu_g}{d\mu_h} \right) d\mu_h. \]

We can estimate
\[ |\langle R_g - a, \eta_\epsilon u \rangle| \leq \int_M |V| \cdot |\hat{\nabla} \left( \eta_\epsilon u \frac{d\mu_g}{d\mu_h} \right)| d\mu_h + \int_M |F - a| \cdot \eta_\epsilon u \frac{d\mu_g}{d\mu_h} d\mu_h \]
\[ \leq C \int_M |V| |\hat{\nabla} \eta_\epsilon| d\mu_h + C \int_M |\hat{\nabla} g| |\eta_\epsilon| d\mu_h + C \int_M |V| |\eta_\epsilon| d\mu_h + C \int_M |F - a| |\eta_\epsilon| d\mu_h \]
\[ \leq C \left( \int_M |\hat{\nabla} g|^p d\mu_h \right)^{1/p} \left( \int_M |\hat{\nabla} \eta_\epsilon|^{p/(p-1)} + |\eta_\epsilon|^{p/(p-1)} d\mu_h \right)^{(p-1)/p} \]
\[ + C \left( \int_M |\hat{\nabla} g|^{2/p} d\mu_h \right)^{2/p} + 1 \left( \int_M |\eta_\epsilon|^{p/(p-2)} d\mu_h \right)^{(p-2)/p}. \]

Here and below, \( C = C(n, p, h, g) \) denotes a positive constant independent of \( \epsilon \) and varies from line to line.

By Hölder’s inequality and the Sobolev inequality, we have
\[ \left( \int_M |\eta_\epsilon|^{p/(p-1)} d\mu_h \right)^{(p-1)/p} \leq C(h) \left( \int_M |\eta_\epsilon|^{p/(p-2)} d\mu_h \right)^{(p-2)/p} \leq C(h) \left( \int_M |\hat{\nabla} \eta_\epsilon|^{p/(p-1)} d\mu_h \right)^{(p-1)/p}. \]

Thus we have
\[ |\langle R_g - a, \eta_\epsilon u \rangle| \leq C \left( \int_M |\hat{\nabla} \eta_\epsilon|^{p/(p-1)} d\mu_h \right)^{(p-1)/p}. \]

Letting \( \epsilon \to 0 \), by the properties of \( \eta_\epsilon \) we get (A.3). Thus we finish the proof of this lemma for \( n < p < \infty \). When \( p = \infty \), the argument above still works (see also [25]). Thus this lemma holds. \( \Box \)