Broué’s abelian defect group conjecture holds for the sporadic simple Conway group $\text{Co}_3$

Shigeo Koshitani$^a$,*, Jürgen Müller$^b$, Felix Noeske$^c$

$^a$Department of Mathematics, Graduate School of Science, Chiba University, Chiba, 263-8522, Japan
$^b$, $^c$Lehrstuhl D für Mathematik, RWTH Aachen University, 52062, Aachen, Germany

Abstract
In the representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime $p$, if a $p$-block $A$ of a finite group $G$ has an abelian defect group $P$, then $A$ and its Brauer corresponding block $A_N$ of the normaliser $N_G(P)$ of $P$ in $G$ are derived equivalent (Rickard equivalent). This conjecture is called Strong Version of Broué’s Abelian Defect Group Conjecture. In this paper, we prove that the strong version of Broué’s abelian defect group conjecture is true for the non-principal 2-block $A$ with an elementary abelian defect group $P$ of order 8 of the sporadic simple Conway group $\text{Co}_3$. This result completes the verification of the strong version of Broué’s abelian defect group conjecture for all primes $p$ and for all $p$-blocks of $\text{Co}_3$.

Keywords: Broué’s conjecture; abelian defect group; sporadic simple Conway group

1. Introduction and Notation
In the representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture which was introduced by Broué around 1988 [6], and is nowadays called Broué’s Abelian Defect Group Conjecture. He actually conjectures the following:

**Conjecture 1.1** (Strong version of Broué’s Abelian Defect Group Conjecture [6], [21]). Let $p$ be a prime, and let $(K, O, k)$ be a splitting $p$-modular system for all subgroups of a finite group $G$. Assume that $A$ is a block algebra of $O_{G}$ with a defect group $P$ and that $A_N$ is a block algebra of $O_{N_G(P)}$ such that $A_N$ is the Brauer correspondent of $A$, where $N_G(P)$ is the normaliser of $P$ in $G$. Then $A$ and $A_N$ should be derived equivalent (Rickard equivalent) provided $P$ is abelian.

In fact, a stronger conclusion than [1.1] is expected, namely that $A$ and $A_N$ are splendidly Rickard equivalent in the sense of Linckelmann ([34], [35]), which he calls splendidly derived equivalent, see [1.12]. Note that for principal block algebras, this notion coincides with the splendid equivalence given by Rickard in [49].

**Conjecture 1.2** (Rickard [49], [50]). Keeping the notation, we suppose that $P$ is abelian as in [1.1]. Then there should be a splendid Rickard equivalence between the block algebras $A$ of $O_{G}$ and $A_N$ of $O_{N_G(P)}$.

There are several cases where the conjectures [1.1] and [1.2] of Broué and Rickard, respectively, have been verified, albeit the general conjecture is widely open; for an overview, containing suitable references, see [12]. As for general results concerning blocks with a fixed defect group, by [30], [32], [34], and [35] the conjectures are proved for blocks with cyclic defect groups in arbitrary characteristic; in characteristic 2, by [31], [32], [49], and [56] they are known to hold for blocks with elementary abelian defect groups of order 4, but already the case of elementary abelian defect groups of order 8 is open in general. At least for principal blocks in characteristic...
2 it has been already known (at least for experts) that \(1.1\) and \(1.2\) hold by using a lifting method [41] 9.1.3(3), and recently a new lifting method was found [14] Theorem 4.33.

In the present paper we look at the case where a non-principal block \(A\) has an elementary abelian defect group \(P\) of order 8, namely, \(P = C_2 \times C_2 \times C_2\). The numbers of irreducible ordinary characters \(k(A)\) and of irreducible Brauer characters \(\ell(A)\), respectively, are important in block theory. For the principal 2-blocks they have been known for some time, see [22] and [27], for instance. However, only recently, the numbers of irreducible ordinary characters \(k(A)\) and of irreducible Brauer characters \(\ell(A)\) for non-principal 2-blocks have been determined in general, see [13]. In [13] it is proved with the help of the classification of finite simple groups, that Alperin’s weight conjecture and also the weak version (character theoretic version) of Broué’s abelian defect group conjecture for arbitrary 2-blocks with defect group \(C_2 \times C_2 \times C_2\) are both true. The strong version of Broué’s abelian defect group conjecture, namely, the existence of Rickard splendid equivalences between blocks corresponding via the Brauer correspondence for arbitrary 2-blocks with defect group \(C_2 \times C_2 \times C_2\), is still open. There are four cases for the inertial index \(e\) of \(A\) with the defect group \(P = C_2 \times C_2 \times C_2\). Namely, \(e = 1, 3, 7\) or 21, since \(\text{Aut}(P) \cong \text{GL}_3(2)\) has a unique maximal 2-subgroup, up to conjugacy, which is isomorphic to the Frobenius group \(F_{21} = C_7 \times C_3\) of order 21. For the cases where \(e = 1\) everything is known because the blocks are nilpotent, see Broué-Puig [10]. For the case \(e = 3\), there are results of Landrock [27] and Watanabe [50].

Our objective in this paper now is to investigate a non-principal 2-block with elementary abelian defect group \(P\) of order 8, which has inertial index 21. An interesting candidate for this endeavour is the non-principal 2-block of Conway’s third group \(\text{Co}_3\), for which we investigate whether the strong version of Broué’s abelian defect group conjecture holds; for previous results on \(\text{Co}_3\), its defect groups, and 2-modular characters confer [10] p.193 Table 6, [26] §7 p.1879, Theorems 3.10 and 3.11, and [57], for example. We remark that, as far as the quasi-simple groups related to the sporadic simple groups are concerned, this is the only 2-block for which Broué’s abelian defect group conjecture is not yet known to hold, since within this class of groups all other abelian 2-blocks are either cyclic or of Klein four defect, see [42].

Our main theorem of this paper is the following:

**Theorem 1.3.** Let \(G\) be the sporadic simple Conway group \(\text{Co}_3\), and let \((\mathcal{K}, \mathcal{O}, k)\) be a splitting 2-modular system for all subgroups of \(G\), see [1.1]. Suppose that \(A\) is a non-principal block algebra of \(O_G\) with a defect group \(P\) which is an elementary abelian group of order 8, and that \(A_N\) is a block algebra of \(\text{ON}_G(P)\) such that \(A_N\) is the Brauer correspondent of \(A\). Then \(A\) and \(A_N\) are splendidly Rickard equivalent, and hence the conjectures \(1.1\) and \(1.2\) of Broué and Rickard both hold.

Actually, \(1.3\) is the last tile in the mosaic proving both Broué’s abelian defect group conjecture and Rickard’s conjecture for \(\text{Co}_3\) in arbitrary characteristic. Since \(|G| = 2^{10} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23\), see [13] p.134, as the conjectures are proved for blocks with cyclic defect groups, it is sufficient to consider the primes \(p \in \{2, 3, 5\}\). For odd \(p\) the only block with defect at least 2 is the principal block, whose defect groups are not abelian. For \(p = 2\) there is precisely a unique block with a non-cyclic abelian defect group. Its defect group is isomorphic to \(C_2 \times C_2 \times C_2\) (see [63] \(\text{Co}_3\), [26] p.1879 and [57] p.494 §2). Therefore we may state the following immediate consequence of \(1.3\).

**Corollary 1.4.** The strong version of Broué’s abelian defect group conjecture \(1.1\) and even Rickard’s splendid equivalence conjecture \(1.2\) are true for all primes \(p\) and for all block algebras of \(O_G\) if \(G = \text{Co}_3\).

As a matter of fact, the main result \(1.3\) is obtained by proving the following:

**Theorem 1.5.** We keep the notation and the assumption as in \(1.3\) Let \(H\) be a maximal subgroup of \(G\) with \(H = R(3) \times S_3 \geq N_G(P)\), where \(R(3) = S_3 \cdot G_2(3) \cong \text{SL}_2(8) \times C_3\) is the smallest Ree group, \(S_3\) is the symmetric group on 3 letters, and \(C_3\) is the cyclic group of order 3. Let \(B\) be a block algebra of \(O_H\) such that \(B\) is the Brauer correspondent of \(A\), see [43] Chap.5 Theorem 3.8]. In addition, let \(\mathcal{J}\) denote the Green correspondence with respect to
(\(G \times G, \Delta P, G \times H\)), and let \(M = \mathfrak{f}(A)\). Then \(M\) induces a Morita equivalence between \(A\) and \(B\), and hence it is a Puig equivalence.

The following result is used to get 1.7 from our main result 1.5.

**Theorem 1.6** (Landrock-Michler \([29]\) and Okuyama \([45]\)). Let \(p = 2\), and let \(R(q) = 2^G_2(q)\) be a Ree group, where \(q = 3^{2n+1}\) for some \(n = 0, 1, 2, \ldots\). Let \((K, O, k)\) be a splitting 2-modular system for all subgroups of \(R(q)\), for all \(q\) at the same time, see \([60]\) Theorem 3.6], and let \(B_0(OR(q))\) be the principal block algebra of the group algebra \(OR(q)\). Then the block algebras \(B_0(OR(3))\) and \(B_0(OR(q))\) are Puig equivalent. In particular, Broué’s abelian defect group conjecture 1.1 and Rickard’s conjecture 1.2 hold for the principal block algebras of \(R(q)\) for any \(q\).

**Proof.** This follows from \([29]\) Theorem 5.3] and \([45]\) Example 3.3 and Remark 3.4].

**Corollary 1.7.** We keep the notation and the assumption as in 1.3. Let \(R(q) = 2^G_2(q)\) be a Ree group, where \(q = 3^{2n+1}\) for some \(n = 0, 1, 2, \ldots\). We may assume that \((K, O, k)\) also is a splitting 2-modular system for all subgroups of \(R(q)\), for all \(q\) at the same time. Let \(B_0(OR(q))\) be the principal block algebra of the group algebra \(OR(q)\). Then \(A\) and \(B_0(OR(q))\) are Puig equivalent.

**Strategy 1.8.** Our starting point for this work is the observation that the 2-decomposition matrix for the non-principal block \(A\) of \(C_{03}\) with an elementary abelian defect group of order 8, see \([57]\), is exactly the same as that for the principal 2-block \(B\) of \(R(3) \cong SL_2(8) \times C_3\), see \([29]\). Therefore it is natural to ask whether these two 2-block algebras are Morita equivalent not only over an algebraically closed field \(k\) of characteristic 2 but also over a complete discrete valuation ring \(O\) whose residue field is \(k\). Furthermore, one might even expect that they are Puig equivalent, see 1.12. If this is the case, since the two conjectures of Broué and Rickard 1.1 and 1.2 respectively have been shown to hold for the principal 2-block of \(R(3)\) in a paper of Okuyama \([45]\), it follows that these conjectures also hold for the non-principal 2-block of \(C_{03}\) with the same defect group \(P = C_2 \times C_2 \times C_2\).

The verification that \(A\) and \(B\) are indeed Morita equivalent relies on theorems by Linckelmann, Broué, Rickard and Rouquier. Linckelmann has shown in \([33]\) that a stable equivalence of Morita type between \(A\) and \(B\) which maps simple modules to simple modules is in fact a Morita equivalence, see \([2.4]\). To obtain an appropriate stable equivalence, we employ a variant of a “gluing” theorem, which is due to (originally Broué \([7]\) 6.3 Theorem]), Rickard \([19]\) Theorem 4.1], Rouquier \([56]\) Theorems 5.6 and 6.3, Remark 6.4], and Linckelmann, see \([34]\,\([36]\) and \([2.3]\). A stable equivalence between two blocks \(A\) and \(B\) may be deduced from Morita equivalences between unique blocks of the centralisers of non-trivial subgroups of \(P\) in \(C_{03}\) and \(R(3)\). Once we have obtained a stable equivalence of Morita type between \(A\) and \(B\), it remains to show that it preserves simplicity of modules as stated above. Usually this may be a very hard task.

**Contents 1.9.** The paper is structured as follows: In Section 2 we give the fundamental lemmas which are used to prove our main results. Furthermore, we establish some properties of the stable equivalences we consider, and collect some further results on Morita equivalences and Green correspondence for ease of reference. In Section 3 we investigate non-principal 2-blocks of the symmetric group \(S_5\) and the Mathieu group \(M_{12}\) whose structure will be used later on in order to get our main theorems. In Section 4 the main objective is to construct the stable equivalence of Morita type between the blocks \(A\) and \(B\) as outlined above. In order to apply gluing theorems of Rouquier and Linckelmann \([2.3]\) we begin by analysing the 2-local structure of \(C_{03}\) to identify the groups. Then, we combine this knowledge and what we get already in Section 3 to give a stable equivalence \(F\) as sought. Section 5 prepares the proof that \(F\) maps simple \(A\)-modules to simple \(B\)-modules. In order to prove this fact, we collect information on simple and indecomposable modules in the three blocks \(A\), \(B\), and \(A_N\). In Section 6 we determine the \(F\)-images of the simple \(A\)-modules, thus showing that they are indeed all simple. Finally, in Section 7 we combine the previous results to give complete proofs of our main theorems 1.3, 1.4, 1.5 and 1.7. At the end of the paper, we have collected several useful properties of the stable equivalences obtained through \([2.3]\).
Computations 1.10. A few words on computer calculations are in order. To find our results, next to theoretical reasoning we have to rely fairly heavily on computations. Of course, many of the data contained in explicit libraries and databases are of computational nature, and quite a few traces of further computer calculations are still left in the present exposition. But we would like to point out that we have found many of our intermediate results by explicit computations first, which have subsequently been replaced by more theoretical arguments.

As tools, we use the computer algebra system GAP [17], to calculate with permutation groups and tables of marks, as well as with ordinary and Brauer characters. We also make use of the data library [5], in particular allowing for easy access to the data compiled in [13], [18] and [63], and of the interface [62] to the data library [64]. Moreover, we use the computer algebra system MeatAxe [52] to handle matrix representations over finite fields, as well as its extensions to compute submodule lattices [37], radical and socle series [40], homomorphism spaces and endomorphism rings [39], and direct sum decompositions [38]. We give more comments later on where necessary.

Notation 1.11. Throughout this paper, we use the standard notation and terminology as is used in [13], [58] and [13].

Let \( k \) be a field and assume that \( A \) and \( B \) are finite dimensional \( k \)-algebras. We denote by \( \text{mod-}A \), \( \text{A-mod} \) and \( \text{A-mod}-B \) the categories of finitely generated right \( A \)-modules, left \( A \)-modules and \( (A, B) \)-bimodules, respectively. We write \( M_A \), \( A \mathcal{M} \) and \( A \mathcal{M}_B \) when \( M \) is a right \( A \)-module, a left \( A \)-module and an \( (A, B) \)-bimodule. In this note, a module always refers to a finitely generated right module, unless stated otherwise. We let \( M^\vee = \text{Hom}_A(M, A) \) be the \( A \)-dual of \( M \), so that \( M^\vee \) becomes a left \( A \)-module via \( (a \phi)(m) = \phi(am) \) for \( a \in A \), \( \phi \in M^\vee \) and \( m \in M \), and we let \( M^\oplus = \text{Hom}_k(M, k) \) be the \( k \)-dual of \( M \), so that \( M^\oplus \) becomes a left \( A \)-module as well via \( (a \phi)(m) = \phi(ma) \) for \( a \in A \), \( \phi \in M^\oplus \) and \( m \in M \). For \( A \)-modules \( M \) and \( N \) we write \( [M, N]^A \) for \( \dim_k(\text{Hom}_A(M, N)) \). We fix for a while an \( A \)-module \( M \). Then, for a projective cover \( P(S) \) of a simple \( A \)-module \( S \), we write \( [P(S) \mid M]^A \) for the multiplicity of direct summands of \( M \) which are isomorphic to \( P(S) \). We denote by \( \text{soc}(M) \) and \( \text{rad}(M) \) the socle and the radical of \( M \), respectively, and hence \( \text{rad}(M) = M \cdot \text{rad}(A) \). For simple \( A \)-modules \( S_1, \cdots, S_n \), and positive integers \( a_1, \cdots, a_n \), we write that \( "M = a_1 \times S_1 + \cdots + a_n \times S_n \), as composition factors" when the set of all composition factors are \( a_1 \) times \( S_1 \), \( \cdots \), \( a_n \) times \( S_n \).

In order to avoid being ambiguous, we sometimes use convention such as \( M = a_1 \times [S_1] + \cdots + a_n \times [S_n] \). For another \( A \)-module \( L \), we write \( M \mid L \) when \( M \) is isomorphic to a direct summand of \( L \) as an \( A \)-module. If \( A \) is self-injective, the stable module category \( \text{mod}-A \) is the quotient category of \( \text{mod}-A \) with respect to the projective \( A \)-homomorphisms, that is those factoring through a projective module.

In this paper, \( G \) is always a finite group and we fix a prime number \( p \). Assume that \( (K, \mathcal{O}, k) \) is a splitting \( p \)-modular system for all subgroups of \( G \), that is to say, \( \mathcal{O} \) is a complete discrete valuation ring of rank one such that its quotient field is \( K \) which is of characteristic zero, and its residue field \( \mathcal{O} / \text{rad}(\mathcal{O}) \) is \( k \), which is of characteristic \( p \), and that \( K \) and \( k \) are splitting fields for all subgroups of \( G \). By an \( \mathcal{O} \)-lattice we mean a finitely generated right \( \mathcal{O} \)-module which is a free \( \mathcal{O} \)-module. We denote by \( k_G \) the trivial \( kG \)-module, and similarly by \( \mathcal{O}_G \) the trivial \( \mathcal{O}G \)-lattice. If \( X \) is a \( kG \)-module, then we write \( X^* = \text{Hom}_k(X, k) \) for the contragredient of \( X \), namely, \( X^* \) is again a right \( kG \)-module via \( (\varphi g)(x) = \varphi(xg^{-1}) \) for \( x \in X \), \( \varphi \in X^* \) and \( g \in G \); if no confusion may arise we also call this the dual of \( X \). Let \( H \) be a subgroup of \( G \), and let \( M \) and \( N \) be an \( \mathcal{O} \)-lattice and an \( \mathcal{O}H \)-lattice, respectively. Then let \( M^\mathcal{O}_H \) and \( N^\mathcal{O}_H \) be the restriction of \( M \) to \( H \), and let \( N^\mathcal{O}_H = N \cap \mathcal{O}G = (N \otimes_{\mathcal{O}H} \mathcal{O}G)_{\mathcal{O}G} \) be the induction (induced module) of \( N \) to \( G \). A similar definition holds for \( kG \)- and \( kH \)-modules. For a subgroup \( Q \) of \( G \) we write \( \text{Scott}(G, Q) \) for the (Alperin-)Scott module with respect to \( Q \) in \( G \), see [43, Chap. 4, p. 297].

We denote by \( \text{Irr}(G) \) and \( \text{IBr}(G) \) the sets of all irreducible ordinary and Brauer characters of \( G \), respectively. Since the character field \( \mathbb{Q}(\chi) := \{ \chi(g) : g \in G \} \subseteq K \) of any character \( \chi \in \text{Irr}(G) \) is contained in a cyclotomic field, we may identify \( \mathbb{Q}(\chi) \) with a subfield of the complex number field \( \mathbb{C} \), hence we may think of characters having values in \( \mathbb{C} \). In particular,
we write $\chi^*$ for the conjugate of $\chi$, where of course $\chi^*$ is the character of the $KG$-module contragredient to the $KG$-module affording $\chi$. For $\chi, \psi \in \text{Irr}(G)$ we denote by $\langle \chi, \psi \rangle^G$ the usual inner product. If $A$ is a block algebra ($p$-block) of $OG$, then we write $\text{Irr}(A)$ and $\text{IBr}(A)$ for the sets of all characters in $\text{Irr}(G)$ and $\text{IBr}(G)$ which belong to $A$, respectively. We denote by $B_0(kG)$ the principal block algebra of $kG$, we write $1_G$ for the trivial character of $G$.

Let $G'$ be another finite group, and let $V$ be an $(OG, OG')$-bimodule. Then we can regard $V$ as a right $O[G \times G']$-module. A similar definition holds for $(kG, kG')$-bimodules. We denote by $\Delta G = \{(g, g') \in G \times G \mid g = g'\}$ the diagonal copy of $G$ in $G \times G$. For an $(OG, OG')$-bimodule $V$ and a common subgroup $Q$ of $G$ and $G'$, we set $V^{\Delta Q} = \{v \in V \mid qv = vq\}$ for all $q \in Q$. If $Q$ is a $p$-group, the Brauer construction is defined to be the quotient $V(\Delta Q) = V^{\Delta Q}/[\sum_{R \in Q} \text{Tr}_R^Q(V^{\Delta R}) + \text{rad} O-V^{\Delta Q}]$, where $\text{Tr}_R^Q$ is the usual trace map. The Brauer homomorphism $\text{Br}_{\Delta Q} : (OG)^{\Delta Q} \to kC_G(Q)$ is obtained from composing the canonical epimorphism $(OG)^{\Delta Q} \to (OG)(\Delta Q)$ and the canonical isomorphism $(OG)(\Delta Q) \cong kC_G(Q)$.

Let $n$ be a positive integer. Then, $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote the alternating and the symmetric groups on $n$ letters. Also, $C_n$ and $D_{2n}$ denote the cyclic group of order $n$ and the dihedral group of order $2n$, respectively. Moreover, for $i \in \{10, 11, 12, 22, 23, 24\}$, $M_i$ denotes the Mathieu group of degree $i$. We denote by $\mathbb{Z}(G)$ the centre of $G$, and by $S^g$ a set $g^{-1}Sg$ for $g \in G$ and a subset $S$ of $G$.

**Equivalences 1.12.** Let $A$ and $A'$ be block algebras of $OG$ and $OG'$, respectively. Then we say that $A$ and $A'$ are Puig equivalent if $A$ and $A'$ have a common defect group $P$, and if there is a Morita equivalence between $A$ and $A'$ which is induced by an $(A, A')$-bimodule $M$ such that, as a right $O[G \times G']$-module, $M$ is a trivial source module and $\Delta P$-projective. A similar definition holds for blocks of $kG$ and $kG'$. Due to a result of Puig (and independently of Scott), see [4], Remark 7.5, this is equivalent to a condition that $A$ and $A'$ have source algebras which are isomorphic as interior $P$-algebras, see [3] Theorem 4.1.

We say that $A$ and $A'$ are stably equivalent of Morita type if there exists an $(A, A')$-bimodule $M$ such that $A(M \otimes_A M^\vee)_{A'} \cong AA' \otimes (\text{proj } (A, A')$-bimod) and $A'(M^\vee \otimes_A M)_{A'} \cong A' A'_{A'} \otimes (\text{proj } (A, A')$-bimod).

We say that $A$ and $A'$ are splendidly stably equivalent of Morita type if $A$ and $A'$ have a common defect group $P$ and the stable equivalence of Morita type is induced by an $(A, A')$-bimodule $M$ which is a trivial source module and is $\Delta P$-projective, see [3] Theorem 3.1.

We say that $A$ and $A'$ are derived equivalent (or Rickard equivalent) if $D(\text{mod-}A)$ and $D'(\text{mod-}A')$ are equivalent as triangulated categories, where $D(\text{mod-}A)$ is the bounded derived category of $\text{mod-}A$. In that case, there even is a Rickard complex $M^* \in \mathcal{C}(A, A')$, where the latter is the category of bounded complexes of finitely generated $(A, A')$-bimodules, all of whose terms are projective both as left $A$-modules and as right $A'$-modules, such that $M^* \otimes_{A'} (M^*)^\vee \cong A$ in $K^b(\text{mod-}A)$ and $(M^*)^\vee \otimes_A M^* \cong A'$ in $K^b(\text{mod-}A')$, where $K^b(\text{mod-}A)$ is the homotopy category associated with $\mathcal{C}(A, A')$. In other words, in that case we even have $K^b(\text{mod-}A) \cong K^b(\text{mod-}A')$.

We say that $A$ and $A'$ are splendidly Rickard equivalent if $K^b(\text{mod-}A)$ and $K^b(\text{mod-}A')$ are equivalent via a Rickard complex $M^* \in \mathcal{C}(A, A')$ as above, such that additionally each of its terms is a direct sum of $\Delta P$-projective trivial source modules as an $O[G \times G']$-module.

**2. Preliminaries**

In this section we give several theorems crucial to the later sections of this paper. We state these results in a more general context; in particular, $G$ is an arbitrary finite group and $(K, O, k)$ is a $p$-modular splitting system for $G$. As we draw upon these lemmas frequently in the sequel, we state these explicitly for the convenience of the reader and ease of reference.

As stated in the introduction, our approach centres around [2, 3] which allows us to verify that a stable equivalence of Morita type is in fact a Morita equivalence. The stable equivalences investigated are obtained with the help of [2, 3] and are realised by tensoring with a bimodule given through Green correspondence. We proceed to study several properties of these stable equivalences, and give some further results needed in the upcoming parts of this paper.
Lemma 2.1 (Linckelmann \[33\]). Let $A$ and $B$ be finite-dimensional $k$-algebras such that $A$ and $B$ are both self-injective and indecomposable as algebras, but not simple. Suppose that there is an $(A,B)$-bimodule $M$ such that $M$ induces a stable equivalence between the algebras $A$ and $B$.

(i) If $M$ is indecomposable then for any simple $A$-module $S$, the $B$-module $(S \otimes_A M)_B$ is non-projective and indecomposable.

(ii) If for all simple $A$-module $S$ the $B$-module $S \otimes_A M$ is simple then $M$ induces a Morita equivalence between $A$ and $B$.

(iii) If $(M,M^\vee)$ induces a stable equivalence of Morita type between $A$ and $B$ then there is a unique (up to isomorphism) non-projective indecomposable $(A,B)$-bimodule $M'$ such that $M' \mid M$, and $(M',M'^\vee)$ induces a stable equivalence of Morita type between the algebras $A$ and $B$.

Proof. (i) and (ii) respectively are given in \[33\] Theorem 2.1(ii) and (iii)]. Part (iii) follows by \[33\] Theorem 2.1(i) and Remark 2.7].

We obtain a suitable stable equivalence to apply \[2.1\] through a “gluing theorem” as given in \[2.3\].

Lemma 2.2 (Koshitani-Linckelmann \[24\]). Let $A$ be a block algebra of $kG$ with defect group $P$, and let $(P,e)$ be a maximal $A$-Brauer pair such that $H = N_G(P,e) = N_G(P)$. Let $B$ be a block algebra of $kH$ such that $B$ is the Brauer correspondent of $A$. Let $\dagger$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$, and set $M = \dagger(A)$, in particular $M$ is an indecomposable $(A,B)$-bimodule with vertex $\Delta P$.

Take any subgroup $Q$ of $Z(P)$, and set $G_Q = C_G(Q)$ and $H_Q = C_H(Q)$. Let $e_Q$ and $f_Q$ be block idempotents of $kG_Q$ and $kH_Q$ satisfying $(Q,e_Q) \subseteq (P,e)$ and $(Q,f_Q) \subseteq (P,e)$, respectively, see \[35\] (40.9 Corollary]. Let $\dagger_Q$ be the Green correspondence with respect to $(G_Q \times G_Q, \Delta P, G_Q \times H_Q)$. Then we have

$$e_Q M(\Delta Q) f_Q = \dagger_Q(e_Q kG_Q)$$

and this is a unique (up to isomorphism) indecomposable direct summand of $e_Q kG_Q \downarrow_{G_Q \times H_Q}$ with vertex $\Delta P$.

Proof. We know $M = \dagger(A) \mid A_{\Delta G \times \Delta H} \downarrow_{kGkH}$. Hence, $M(\Delta Q) \mid (kG)(\Delta Q) = kC_G(Q) = kG_Q$. Thus,

$$e_Q M(\Delta Q) f_Q \mid e_Q kG_Q f_Q \mid (e_Q kG_Q \downarrow_{G_Q \times H_Q}).$$

By \[24\] Theorem], $e_Q M(\Delta Q) f_Q$ is an indecomposable $k[G_Q \times H_Q]$-module with vertex $\Delta P$. Thus Green correspondence yields $e_Q M(\Delta Q) f_Q = \dagger_Q(e_Q kG_Q)$. 

Lemma 2.3 (Linckelmann \[35, 36\]). Let $A$ be a block algebra of $kG$ with a defect group $P$, and let $(P,e)$ be a maximal $A$-Brauer pair in $G$. Set $H = N_G(P,e)$. Assume that

1. $P$ is abelian,
2. for each $Q$ with $1 \neq Q \leq P$, $kC_G(Q)$ has a unique block algebra $A_Q$ with the defect group $P$,
3. for each $Q$ with $1 \neq Q \leq P$, $kC_H(Q)$ has a unique block algebra $B_Q$ with the defect group $P$.

Let $B$ a block algebra of $\Delta H$ which is the Brauer correspondent of $A$. For each subgroup $Q$ of $P$, let $e_Q$ and $f_Q$ be the block idempotents of $A_Q$ and $B_Q$, respectively, and hence $A_Q = kC_G(Q)e_Q$ and $B_Q = kC_H(Q)f_Q$. Note that $e_P = e = f_P$ and $A_P = B_P$. Let $\dagger$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$, and set $\dagger_M = \dagger(A)$, see \[2.4\]. Moreover, let $\dagger_Q$ be the Green correspondence with respect to $(C_G(Q) \times C_G(Q), \Delta P, C_G(Q) \times C_H(Q))$. Now, assume further that

4. for each non-trivial proper subgroup $Q$ of $P$, the $(A_Q,B_Q)$-bimodule $\dagger_Q(A_Q)$ induces a Morita equivalence between $A_Q$ and $B_Q$.

Then the $(A,B)$-bimodule $M$ induces a stable equivalence of Morita type between $A$ and $B$. 

refer the reader also to the appendix for a more detailed discussion of further properties of stable equivalences obtained through \[2.8\].
Proof. First, note \( H = N_G(P) \). Secondly, it follows from \([2,2]\) that \( e_Q \cdot M(\Delta Q) \cdot f_Q = f_Q(A_Q) \) for each \( Q \leq P \) since \( P \) is abelian by (1). Then since \( A_P = B_P \) and since \( A_P = \{ f_P(A_P) = e \cdot M(\Delta P) \cdot e \} \) the \((A_P, B_P)\)-bimodule \( e_P \cdot M(\Delta P) \cdot e \) induces a Morita equivalence between \( A_P \) and \( B_P \).

Now, for each \( Q \leq P \), it follows from the uniqueness of \( e_Q \) and \( f_Q \) that
\[
(Q, e_Q) \subseteq (P, e) \quad \text{and} \quad (Q, f_Q) \subseteq (P, e).
\]

Next, we want to claim
\[
E_G((Q, e_Q), (R, e_R)) = E_H((Q, f_Q), (R, f_R)) \quad \text{for} \quad Q, R \leq P,
\]
where \( E_G((Q, e_Q), (R, e_R)) \) is the set \( \{ \varphi : Q \to R \mid \text{there is } g \in G \text{ with } \varphi(u) = u^g, \text{ for all } u \in Q, \text{ and } (Q, e_Q)^g \subseteq (R, e_R) \} \). This is known by using \([2, \text{Proposition 4.21 and Theorem 3.4}] \) and \([3, \text{Theorem 1.8(1)}] \) since \( P \) is abelian, see \([23, \text{The proof of 1.15, Lemma}] \) for details. Therefore we can apply Linckelmann’s result \([35, \text{Theorem 3.1}] \).

We remark that in \([35, \text{Theorem 3.1}] \) and \([36, \text{Theorem A.1}] \), Linckelmann proves more general theorems than \([2,3]\). However, we formulate with \([2,3]\) a version which is specifically tailored to our practical purposes, and use this ad hoc version in the sequel.

In the notation of \([2,3]\) we have that the bimodule \( M \) realising a stable equivalence between \( A \) and \( B \) is a Green correspondent of \( A \). In fact it is a direct summand of \( 1_A \cdot kG \cdot 1_B \) as the next lemma shows.

**Lemma 2.4.** Let \( A \) be a block algebra of \( kG \) with defect group \( P \). Assume that \( (P, e) \) is a maximal \( A \)-Brauer pair such that \( H = N_G(P, e) = N_G(P) \). Let \( B \) be a block algebra of \( kH \) such that \( B \) is the Brauer correspondent of \( A \). Let \( \{ f \} \) be the Green correspondence with respect to \((G \times G, \Delta P, \Delta \times H)\). Then we have \( f(A) | 1_A \cdot kG \cdot 1_B \).

**Proof.** It follows from \([3, \text{Theorem 5(i)}] \) that \((A_G^{G \times G})^{-1}B = 1_A \cdot kG \cdot 1_B \) has a unique (up to isomorphism) indecomposable direct summand with vertex \( \Delta P \). Clearly, \( 1_A \cdot kG \cdot 1_B \mid (A_G^{G \times G}) \), hence by Green correspondence we have \( f(A) | 1_A \cdot kG \cdot 1_B \).

We remark that a stable equivalence of Morita type induced by the Green correspondent \( f(A) \) in the context of \([2,4]\) preserves vertices and sources, and takes indecomposable modules to their Green correspondents, see (i) and (iii) in \([A,3]\).

**Lemma 2.5.** Let \( G, H, \) and \( L \) be finite groups, all of which have a common non-trivial \( p \)-subgroup \( P \), and assume that \( H \leq G \). Let \( A, B, \) and \( C \) be block algebras of \( kG \), \( kH \), and \( kL \), respectively, all of which have \( P \) as their defect group. In addition, suppose that a pair \((A_M, B_N)\) induces a stable equivalence between \( A \) and \( B \) such that \( A_M = k_{\Delta P}^{G \times H}, B_N = k_{\Delta P}^{H \times L} \) (and hence \( M \) and \( N \) preserve vertices and sources, see (i) and (iii) of \([A,3]\)). Similarly, suppose that a pair \((B_N, C_M)\) induces a stable equivalence between \( B \) and \( C \) such that \( B_N = k_{\Delta P}^{L \times H}, C_M = k_{\Delta P}^{H \times L} \) (and hence \( N \) and \( M \) preserve vertices and sources, see (i) and (iii) of \([A,3]\)). Then we have \((A, C)\) - and \((C, A)\)-bimodules \( M \) and \( M' \), respectively, which satisfy the following:

1. \( A_M \oplus (\text{proj } (A, C))\)-bimodule and \( C_M' \oplus (\text{proj } (C, A))\)-bimodule.
2. \( A_M \) and \( C_M' \) are both non-projective indecomposable.
3. The pair \((M, M')\) induces a stable equivalence between \( A \) and \( C \).
4. The functors
   \[
   - \otimes A M : \text{mod-}A \to \text{mod-}C
   \]
   and
   \[
   - \otimes C M' : \text{mod-}C \to \text{mod-}A
   \]
   preserve vertices and sources of indecomposable modules. That is, for non-projective indecomposable \( A \)- and \( C \)-modules \( X \) and \( Y \) corresponding via \( X \otimes A M = Y \oplus (\text{proj}) \) and \( Y \otimes C M' = X \oplus (\text{proj}) \), respectively, there is a non-trivial \( p \)-subgroup \( Q \) and an
indecomposable $kQ$-module $S$ such that $Q$ is a common vertex of $X$ and $Y$ and that $S$ is a common source of $X$ and $Y$.

(5) $A M_C | k_{\Delta P} |^{G \times L}$ and $C M'_A | k_{\Delta P} |^{L \times C}$.

(6) In particular, if a pair $(\mathfrak{M}, \mathfrak{M}')$ induces a stable equivalence of Morita type between $A$ and $B$, and if a pair $(\mathfrak{M}, \mathfrak{M}')$ induces a stable equivalence of Morita type between $B$ and $C$, then we can replace $M'$ above by $M''$ and we have that the pair $(M, M'')$ induces a stable equivalence of Morita type between $A$ and $C$.

Proof. Obviously, the pair $(A(\mathfrak{M} \otimes_B \mathfrak{M})_C, C(\mathfrak{M} \otimes_B \mathfrak{M}')_A)$ induces a stable equivalence between $A$ and $C$. Clearly, $A(\mathfrak{M} \otimes_B \mathfrak{M}), (\mathfrak{M} \otimes_B \mathfrak{M})_C, (\mathfrak{M} \otimes_B \mathfrak{M}')_A$ are all projective. Since $A$ and $C$ are symmetric algebras, it follows from (2.1(iii)) that there are $(A, C)$- and $(C, A)$-bimodules $M$ and $M'$ which satisfy the conditions (1)–(4).

Next we want to show (5). It follows from [43 Chap.5 Lemma 10.9(iii)] that

$$M | \mathfrak{M} \otimes_B \mathfrak{M} | (k_{\Delta P} |^{G \times H}) \otimes_{kH} (k_{\Delta P} |^{H \times L}) \cong (kG \otimes_{kP} kH) \otimes_{kH} (kH \otimes_{kP} kL) \cong kG \otimes_{kP} kH \otimes_{kP} kL \cong \bigoplus_{h \in [P \cap H/P]} k[PhP]^{P \times P}.$$ 

Since $A M_C$ is indecomposable, there is an element $h \in H$ such that $M | k[PhP]^{P \times P}$. Set $(P \times P)_h = \{(u, h^{-1}uh) \in P \times P | u \in P \cap h^{-1}Ph\}$. Then

$$(P \times P)_h = \{(uh^{-1}, u) \in P \times P | u \in P \cap h^{-1}Ph\} = (h, 1) \Delta [P \cap Ph] \cdot (h^{-1}, 1).$$

We get by [43 Chap.5 Lemma 10.9(iii)] that $k[PhP] \cong k_{(h, 1)} \delta [P \cap Ph] \cdot (h^{-1}, 1) \otimes_{P \times P} P \times P$, and hence $M | k_{(h, 1)} \delta [P \cap Ph] \cdot (h^{-1}, 1) \otimes_{P \times P} P \times P$. Now, since $(h^{-1}, 1) \in H \times L \leq G \times L$, we have that

$$M | k_{(h, 1)} \delta [P \cap Ph] \cdot (h^{-1}, 1) \otimes_{P \times P} P \times P \cong kG \otimes_{kQ} kL,$$

where $Q = P \cap Ph$. Then for any $X$ in mod-$A$ the module $X \otimes_A M$ has a vertex contained in $Q$. If $Q$ is a proper subgroup of $P$ then, since $(M, M')$ induces a stable equivalence between $A$ and $C$, any module in mod-$C$ has a vertex properly contained in $P$, a contradiction since $P$ is a defect group of $C$. Hence $Q = P$, so that $h \in N_H(P) \subseteq N_G(P)$. Therefore $M | k_{\Delta P} |^{G \times L}$. An analogous argument gives the claim for $M'$.

(6) Follows from (1)–(5) and (2.1(iii)). \hfill $\square$

Next, we give some results on Morita equivalences and tensor products, which will be useful in Section 4.

Lemma 2.6. The following hold:

(i) Let $A$, $B$, $C$ and $D$ be finite dimensional $k$-algebras. Assume that an $(A, B)$-bimodule $M$ realises a Morita equivalence between $A$ and $B$, and so does a $(C, D)$-bimodule $N$ between $C$ and $D$. Then the $(A \otimes C, B \otimes D)$-bimodule $M \otimes N$ induces a Morita equivalence between $A \otimes C$ and $B \otimes D$.

(ii) Keep the notation as in (i). Assume that $P$ is a common $p$-subgroup of finite groups $G$ and $H$, and that $Q$ is a subgroup of $P$. Suppose moreover that $A$ and $B$ respectively are block algebras of $kG$ and $kH$, $C = D = kQ$ and $N = kQkQ$. If a $(kG, kH)$-bimodule $M$ satisfies that $M | k_{\Delta P} |^{G \times H}$, then $(M \otimes N) | k_{\Delta [P \times Q]} |^{(G \times Q) \times (H \times Q)}$.

Proof. The proof of (i) is straightforward. For (ii) observe that $k_{\Delta P} |^{(G \times Q) \times (H \times Q)}$ is isomorphic to $k[G \times Q] \otimes_{k[P \times Q]} k[H \times Q]$, and hence to $(kG \otimes_{kP} kH) \otimes_{kP} kQ$. If a $(kG, kH)$-bimodule $M$ satisfies that $M | k_{\Delta P} |^{G \times H}$, then $(M \otimes N) | k_{\Delta [P \times Q]} |^{(G \times Q) \times (H \times Q)}$.

Note that we cannot replace the Morita equivalence in (2.6) by a stable equivalence in general, see [51 Question 3.8].

Lemma 2.7. Let $G$ and $H$ be finite groups, let $A$ and $B$, respectively, be block algebras of $kG$ and $kH$. Let $X$ be an indecomposable $kG$-module in $A$, and let $Y$ be an indecomposable $kH$-module in $B$. Then the following hold:
Lemma 3.1. Set equivalences which will be required to apply 2.3 correspondent in the normaliser between the non-principal 2-block of \( \text{Co} \).

Proof. (i) and (ii) are clear. (iii) These follow from [25 Proposition 1.2]. □

Finally, we collect a few facts about Green correspondence, its compatibility with Brauer correspondence, and its transitivity (see [43 Chap.4, §4], for example).

Lemma 2.8. Let \( P \) be a \( p \)-subgroup of a finite group \( G \), and let \( N \) and \( H \) be subgroups of \( G \) with \( N_G(P) \leq N \leq H \leq G \). Furthermore, assume that \( f \), \( f_1 \) and \( f_2 \) are the Green correspondences with respect to \( (G,P,H) \), \( (H,P,N) \) and \( (G,P,N) \), respectively. Then from the definition and properties of Green correspondence and the Krull-Schmidt Theorem we get the following:

(i) We have \( \mathfrak{A}(G,P,N) \subseteq \mathfrak{A}(G,P,H) \cap \mathfrak{A}(H,P,N) \), where \( \mathfrak{A}(G,P,N) \) and the others are defined as in [43 Chap.4, §4].

(ii) For any indecomposable \( kG \)-module \( X \) with vertex in \( \mathfrak{A}(G,P,N) \), the isomorphism \( f_1(X) \cong f_2(X) \) holds.

(iii) It follows from Green’s result [43 Chap.5 Corollary 3.11] and Brauer’s first main theorem that \( f_2(X) \) belongs to \( A_X \). The Green correspondent \( f(X) \) has a vertex in \( \mathfrak{A}(G,P,N) \), and hence in \( \mathfrak{A}(H,P,N) \). By (ii), \( f_2 = f_1 \circ f \). Hence \( f_2(X) = f_1 \circ f(X) \) lies in the Brauer correspondent of \( A \) which is \( A_X \). Therefore, by the above, the block of \( f(X) \) corresponds to \( A_X \), namely, it is \( B \). □

3. Non-principal 2-blocks of \( S_5 \) and \( M_{12} \)

By the "gluing" theorem given in [23] we want to obtain a stable equivalence of Morita type between the non-principal 2-block of \( \text{Co}_3 \) with a defect group \( P = C_2 \times C_2 \times C_2 \) and its Brauer correspondent in the normaliser \( N_{\text{Co}_3}(P) \). To this end, we need to consider non-trivial subgroups of \( P \) and establish Morita equivalences between unique blocks of the associated centralisers in \( \text{Co}_3 \) and \( N_{\text{Co}_3}(P) \). The objective of this section is to show the existence of various Morita equivalences which will be required to apply [23]. The relevance of the groups related to \( S_5 \) and \( M_{12} \), respectively, will be revealed in [12] in the next section.

For the remainder of this paper, we let the characteristic \( p \) of \( k \) be 2.

Lemma 3.1. Set \( G = S_5 \).

(i) There exists a unique block algebra \( A \) of \( kG \) with defect one. In fact, a defect group \( T \) of \( A \) is generated by a transposition.

(ii) Set \( H = N_G(T) \). Then \( H = C_G(T) \cong T \times S_3 \cong D_{12} \).

(iii) \( A \) is a nilpotent block algebra, \( k(A) = 2 \), \( \ell(A) = 1 \), and we can write \( \text{Irr}(A) = \{ \chi_4, \chi'_4 \} \) and \( \text{IBr}(A) = \{ 4_{kG} \} \), where the number 4 denotes the degree (dimension).

(iv) The unique simple \( kG \)-module \( 4_{kG} \) is a trivial source module.

(v) Let \( B \) be a block algebra of \( kH \) such that \( B \) is the Brauer correspondent of \( A \). Then \( k(B) = 2 \), \( \ell(B) = 1 \), and we can write \( \text{Irr}(B) = \{ \theta_2, \theta'_2 \} \) and \( \text{IBr}(B) = \{ 2_{kH} \} \), where the number 2 again gives the degree (dimension).

(vi) Let \( f \) be the Green correspondence with respect to \( (G \times G, \Delta T, G \times H) \), and set \( M = f(A) \). Then \( A_{MB} = 1_A \cdot kG \cdot 1_B \) and \( M \) induces a Puig equivalence between \( A \) and \( B \).
Proof. (i)-(iii) and (v) are immediate by \[13\] p.2, and \[18\] A₅, 2 (mod 2)] or \[63\] A₅, 2 (mod 2)].

(iv) It follows from \[13\] p.2 that $1_{H}^{G} = 1_{G} + \chi_{4} + \chi_{5}$, where $\chi_{i} \in \text{Irr}(G)$ and $\chi_{i}(1) = i$ for $i = 4, 5$. Thus, by (ii), $1_{H}^{G} \cdot A = \chi_{4}$, and hence $k_{H}^{G} \cdot A = 4_{A}$. 

(vi) We first show that $1_{A} \cdot kG$ induces a Morita equivalence between $A$ and $B$. To this end let $1_{A} \cdot \text{OG}$ be its lift to $O$, which is projective both as a left $O\text{G}$-module and as a right $O\text{H}$-module. Moreover, it follows from (iii), (v), and \[13\] p.2 that $\chi_{A} \cdot 1_{H}^{G} \cdot 1_{B} = \theta_{2}$, $\chi_{A} \cdot 1_{H}^{G} \cdot 1_{B} = \theta_{2}$ by interchanging $\theta_{2}$ and $\theta_{2}$ if necessary. Therefore

$$
\chi_{A} \cdot 1_{H}^{G} \cdot 1_{B} = \theta_{2}, \quad \chi_{A} \cdot 1_{H}^{G} \cdot 1_{B} = \theta_{2}.
$$

Hence by \[6\] 0.2 Théorème, we get that $1_{A} \cdot \text{OG} \cdot 1_{B}$ induces a Morita equivalence between $A$ and $B$, and so does $1_{A} \cdot kG$ between $A$ and $B$. As $1_{A} \cdot kG$ is a trivial source $k[G \times H]$-module with vertex $\Delta P$, we infer that this even is a Puig equivalence.

Finally, let $(\tau, e)$ be a maximal $A$-Brauer pair. Then we know $N_{G}(\tau, e) = H$ by (ii). Hence \[2\] 4 implies that $M(1_{A} \cdot kG)$ induces a Morita equivalence between $A$ and $B$. It follows from Morita’s Theorem, see \[15\] Sect. 3D (3.54) that $1_{A} \cdot kG$ already is indecomposable as an $(A, B)$-bimodule, implying that $M = 1_{A} \cdot kG$.

\[\square\]

**Lemma 3.2.** Let $R$ be any finite 2-group. Consider a finite group $G = R \times S_{5}$, and let $T$ be as in \[3\] (vi) Set $Q = R \times T$ and $H = N_{G}(Q)$. Let $A$ be a unique non-principal block algebra of $kG$ with defect group $Q$, and let $B$ be a block algebra of $kH$ such that $B$ is the Brauer correspondent of $A$. Then we get the following:

(i) $H = C_{G}(Q) = Q \times S_{3}$.

(ii) Let $f$ be the Green correspondence with respect to $(G \times Q, \Delta Q, G \times H)$, and set $M = f(A)$. Then $M \cong 1_{A} \cdot kG$.

(iii) $M$ induces a Puig equivalence between $A$ and $B$.

\[\square\]

Proof. This follows from \[3\] (vi) and \[2\].

We next turn to the Mathieu group $M_{12}$.

**Lemma 3.3.** Let $G = M_{12}$.

(i) There exists a unique block algebra $A$ of $kG$ with defect group $Q = C_{2} \times C_{2}$.

(ii) We can write $\text{Irr}(A) = \{16, 16^{*}, 144\}$, where the numbers 16 and 144 denote dimensions (degrees). Moreover, all the simple $kG$-modules in $A$ are trivial source modules.

(iii) Let $H = N_{G}(Q)$. Then $H \cong A_{4} \times S_{3} \cong \langle Q \times C_{3} \rangle \times S_{3}$.

(iv) Let $B$ be a block algebra of $kH$ such that $B$ is the Brauer correspondent of $A$. Let $f$ be the Green correspondence with respect to $(G \times Q, \Delta Q, G \times H)$, and set $M = f(A)$. Then $M$ induces a Puig equivalence between $A$ and $B$.

\[\square\]

Proof. (i)-(iii) except the last part of (ii) are easy by \[13\] p.33, and \[18\] M₁₂ (mod 2)] or \[63\] M₁₂ (mod 2)]. Actually, using the character table of $G$, it turns out that the conjugacy class 3B of $G$ is a defect class of $A$. Hence $Q$ is a Sylow 2-subgroup of the centraliser $C_{G}(3B) = A_{4} \times C_{3}$, while the normaliser $N_{G}(3B) = A_{4} \times S_{3}$ is a maximal subgroup of $G$, containing $Q$ as normal subgroup.

It remains to show the last statement in (ii). By \[13\] p.33, $G$ has a maximal subgroup $L \cong PSL_{2}(11)$. Then again \[13\] p.33 yields that $1_{L} \cdot 1_{A} = \chi_{16} + \chi_{16}^{*}$, where $\chi_{16}(1) = \chi_{16}^{*}(1) = 16$. Set $X_{kG} = kL \cdot 1_{A}$. Then $X = 16 + 16^{*}$ as composition factors. Since $\chi_{16} \neq \chi_{16}^{*}$, we get by \[15\] Chap.4 Theorem 8.9(i)] that $[X, X]^{G} = 2$. Therefore $X = 16 \oplus 16^{*}$. Hence 16 and 16* are both trivial source $kG$-modules. Finally, we know that $kW \cdot 1_{A} = 144$, where $W$ is a maximal subgroup of $G$ with $W = 2_{+}^{1+4}$, $S_{3}$. This shows that 144 is also a trivial source $kG$-module.

(iv) All elements of $Q \setminus \{1\}$ are conjugate in $H$, hence the character table of $G$ \[13\] p.33] shows that they all belong to the conjugacy class 2A of $G$. Take any element $t \in Q \setminus \{1\}$, and set $R = \langle t \rangle$. Thus we have

$$
C_{G}(R) \cong R \times S_{5} \quad \text{and} \quad C_{H}(R) \cong Q \times S_{3} \cong R \times (C_{2} \times S_{3}).
$$

\[\square\]
The algebra $kC_G(R)$ has a unique block algebra $A_R$ with the defect group $Q$ since $kS_5$ has a unique block algebra with defect group $C_2$, and similarly $kC_H(R)$ has a unique block algebra $B_R$ with the defect group $Q$ since $kS_3$ has a unique block algebra of defect zero. Moreover, we know by [3, ii] that $f_R(A_R)$ induces a Morita equivalence between $A_R$ and $B_R$, where $f_R$ is the Green correspondence with respect to $(C_G(R) \times C_G(R), \Delta Q, C_G(R) \times C_H(R))$. Thus it follows from [2, 3] that $M$ induces a stable equivalence of Morita type between $A$ and $B$.

Now, let $f$ be the Green correspondence with respect to $(G, Q, H)$. Take any simple $kg$-module $S$ in $A$. It follows from (ii), [20, 3.7. Corollary], and [14, Lemma 2.2] that $f(S)$ is a simple $kH$-module. Hence from [A, 3 v] and [2, 1 i] we obtain that $S \otimes A M$ is a simple $kH$-module in $B$. We then finally know that $M$ realises a Morita equivalence between $A$ and $B$ by [2, 1 ii].

**Lemma 3.4.** Set $R = C_2$, and let $G = R \times M_{12}$.

(i) There exists a unique block algebra $A$ of $kG$ with defect group $P = R \times C_2 \times C_2$.

(ii) We can write $IBr(A) = \{ 16, 16^*, 144 \}$, where the numbers $16$ and $144$ give the dimensions (degrees). Moreover, all the simple $kG$-modules $16, 16^*, 144$ in $A$ are trivial source modules.

(iii) Let $H = N_G(P)$. Then $H = R \times A_4 \times S_3 \cong (P \times C_3) \times S_3$. Note that $P \times C_3 \cong R \times (Q \times C_3)$ and $Q \times C_3 \cong A_4$, where $Q = C_2 \times C_2$.

(iv) Let $B$ be a block algebra of $kH$ such that $B$ is the Brauer correspondent of $A$. Let $f$ be the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$. Then $f(A)$ induces a Puig equivalence between $A$ and $B$.

**Proof.** This follows from [3, 3 iv] and [2, 6].

4. Obtaining stable equivalences

In this section, by using the lemmas in §§2-3 we shall obtain a stable equivalence of Morita type between the principal 2-block of the smallest Ree group $R(3)$ and the non-principal 2-block of $S_3$ with defect group $C_2 \times C_2 \times C_2$ under consideration. The following hypothesis determines our standard setting which we fix here for future reference.

**Hypothesis 4.1.** Let $G$ be the sporadic group $S_3$, and let $A$ be the block algebra of $kG$ with defect group $P = C_2 \times C_2 \times C_2$, see [53, Co3], [26, p.1879] and [57, p.494 §2]. Set $N = N_G(P)$, and let $AN$ be the Brauer correspondent of $A$ in $kN$. Furthermore, let $(P,e)$ be a maximal $A$-Brauer pair in $G$.

Let $Q$ be a subgroup of $P$ isomorphic to $C_2 \times C_2$, and $R$ one which is cyclic of order 2. Let $e_Q$ and $f_Q$ be block idempotents of the block algebras of $kC_G(Q)$ and $kC_H(Q)$, respectively, such that $(Q, e_Q) \subseteq (P, e)$ and $(Q, f_Q) \subseteq (P, e)$, see [53, §10 p.346]. Similarly define $e_R$ and $f_R$ by replacing $Q$ with $R$. We denote by $F_{21}$ the Frobenius group of order 21, namely, $F_{21} \cong C_7 \times C_3$, which is a maximal subgroup of $GL_3(2)$. Also, let $R(3) \cong SL_2(8) \times C_3$ be the smallest Ree group, see [13, p.6].

We first collect information on the subgroups of $S_3$ to consider.

**Lemma 4.2.** Assume [14,1] Then the following hold:

(i) $N \cong (P \times F_{21}) \times S_3 \cong \left( (P \times C_7) \times C_3 \right) \times S_3$.

(ii) There is a maximal subgroup $H$ of $G$ such that $N \leq H \cong (SL_2(8) \times C_3) \times S_3$, and $P \times C_7$ is isomorphic to a Borel subgroup of $SL_2(8)$.

(iii) $C_G(P) = C_H(P) = C_N(P) \cong P \times S_3$.

(iv) There exists a unique block algebra $\beta$ of $kS_3$ such that $\beta$ has defect zero, $\beta \cong \text{Mat}_2(k)$ as $k$-algebras, and $ekC_G(P) \cong kP \otimes \beta$.

(v) $N_G(P,e) = N$.

(vi) The inertial quotient $N_G(P,e)/C_G(P)$ is isomorphic to $F_{21}$.

(vii) All elements of $P - \{ 1 \}$ are conjugate in $N$. That is, any subgroup of $P$ of order 2 is conjugate to $R$ in $N$.

(viii) $C_G(R) \cong R \times M_{12}$ and $C_H(R) = C_N(R) \cong R \times A_4 \times S_3 \cong (P \times C_3) \times S_3$. 
(ix) All subgroups of \( P \) of order 4 are conjugate in \( N \). That is, any subgroup of \( P \) of order 4 is conjugate to \( Q \) in \( N \).

(xi) \( C_G(Q) \cong Q \times \mathfrak{S}_5 \) and \( C_H(Q) = C_N(Q) = C_H(P) \cong P \times \mathfrak{S}_3 \).

Proof. This is verified easily using GAP [17], with the help of the smallest faithful permutation representation of \( G \) on 276 points, available in [62] in terms of so-called standard generators \( \mathfrak{S}_3 \). Since in [62] also representatives of the conjugacy classes of elements, as well as of the maximal subgroups of \( G \) are provided, all above-mentioned subgroups of \( G \) can be constructed explicitly.

To begin with, using the character table of \( G \) [13, p.135], it turns out that the conjugacy class 3C of \( G \) is a defect class of \( A \). Hence \( P \) is a Sylow 2-subgroup of the centraliser \( C_G(3C) \), where by [13, p.135] again we have \( C_G(3C) \cong (\text{SL}_2(8) \times C_3) \times C_3 \), while the normaliser \( H = N_G(3C) \cong (\text{SL}_2(8) \times C_3) \times \mathfrak{S}_3 \) is a maximal subgroup of \( G \).

Using the data on subgroup fusions available in [5], it follows that the elements of \( P \) belong to the 2B conjugacy class of \( G \), hence [13, p.134] shows that \( C_G(R) \cong R \times M_{12} \), which is another maximal subgroup of \( G \). Moreover, it follows that \( C_G(Q) \cong C_2 \times C_{M_{12}}(2A) \cong C_2 \times C_2 \times \mathfrak{S}_3 \), where by [13, p.33] \( C_2 \times \mathfrak{S}_3 \) is a maximal subgroup of \( M_{12} \). Finally, the structure of \( C_H(P) \), \( C_H(R) \), and \( C_H(Q) \) follows from a consideration of the action of \( F_{21} \leq \text{GL}_3(2) \) on the defect group \( P \).

(xii) This follows by [27].

**Notation 4.3.** We use the notation \( H \), \( \beta \) and \( B \) as in [4.2] (ii), (iv) and (xi), respectively. We denote the unique simple \( k \mathfrak{S}_3 \)-module in \( \beta \) by \( 2 \mathfrak{S}_3 \).

It is now time to harvest what we have sown in our analysis of the 2-local structure of \( G \). In [4.5] we use our previous results to obtain a stable equivalence of Morita type between the blocks \( A \) and \( A_N \) via [2.3]. Similarly in [4.4] we derive a stable equivalence between the blocks \( B \) and \( A_N \), which together with the first yields the stable equivalence sought between \( A \) and \( B \) in [4.6].

**Lemma 4.4.** Let \( f_Q \) be the Green correspondence with respect to \( (H \times H, \Delta P, H \times N) \), and set \( \mathfrak{M} = f_1(B) \). Then \( \mathfrak{M} \) induces a stable equivalence of Morita type between \( B \) and \( A_N \).

**Proof.** By [2.4] \( \mathfrak{M} | 1_B \cdot kH \cdot 1_{A_N} \). We know by [4.2] (viii) and [4.2] (x) that
\[
C_H(Q) = C_N(Q) = P \times \mathfrak{S}_3 \quad \text{and} \quad C_H(R) = C_N(R) = (P \times C_3) \times \mathfrak{S}_3.
\]
Let \( \mathcal{A}_Q \), \( \mathcal{A}_R \), \( \mathbb{B}_Q \) and \( \mathbb{B}_R \) be the block algebras of \( kC_H(Q) \), \( kC_H(R) \), \( kC_N(Q) \) and \( kC_N(R) \), respectively, such that they have \( P \) as a defect group. Then
\[
\mathcal{A}_Q = \mathbb{B}_Q = kP \otimes k\mathfrak{S}_3; \beta \cong \text{Mat}_2(kP) \quad \text{and} \quad \mathcal{A}_R = \mathbb{B}_R = k[P \times C_3] \otimes k\mathfrak{S}_3; \beta
\]
where the isomorphism is of \( k \)-algebras. Thus we obviously know that
\[
f_Q(\mathcal{A}_Q) = \mathcal{A}_Q \quad \text{and} \quad f_R(\mathcal{A}_R) = \mathcal{A}_R,
\]
where \( f_Q \) and \( f_R \) are the Green correspondences with respect to
\[
(C_H(Q) \times C_H(Q), \Delta P, C_H(Q) \times C_N(Q)) \quad \text{and} \quad (C_H(R) \times C_H(R), \Delta P, C_H(R) \times C_N(R)),
\]
respectively. Thus \( f_Q(\mathcal{A}_Q) \) induces a Morita equivalence between \( \mathcal{A}_Q \) and \( \mathbb{B}_Q \), and \( f_R(\mathcal{A}_R) \) induces a Morita equivalence between \( \mathcal{A}_R \) and \( \mathbb{B}_R \). Therefore we get the assertion by [2.3].

**Lemma 4.5.** Let \( f_2 \) be the Green correspondence with respect to \( (G \times G, \Delta P, G \times N) \), and set \( \mathfrak{M} = f_2(A) \). Then we get
(i) \( \mathfrak{M} | 1_A \cdot kG \cdot 1_{A_N} \).
(ii) The bimodule \( e_R \mathfrak{M}(\Delta R) f_R \) induces a Morita equivalence between the block algebras \( kC_G(R) e_R \) and \( kC_N(R) f_R \).
(iii) The bimodule \( e_Q \mathfrak{M}(\Delta Q) f_Q \) induces a Morita equivalence between the block algebras \( kC_G(Q) e_Q \) and \( kC_N(Q) f_Q \).
(iv) \(\mathfrak{M}\) induces a stable equivalence of Morita type between \(A\) and \(A_N\).

**Proof.** (i) This follows from 4.2(v) and 2.4.
(ii) Let \(f_R\) be the Green correspondence with respect to \((C_G(R) \times C_G(R), \Delta P, C_G(R) \times C_N(R))\). We get from (i) and 2.2 that \(f_R(e_RkC_G(R)) = e_R\mathfrak{M}(\Delta R)f_R\). Hence we obtain the assertion by 3.4.
(iii) Analogous to the proof of (ii) if we use 3.2 instead of 3.4.
(iv) This follows by 3.4 and 3.2 (ii)–(iii) and 2.3.

**Lemma 4.6.** There is an \((A,B)\)-bimodule \(M\) which satisfies the following:

1. \(A_M\) is indecomposable,
2. \((A_M, B M^\vee_A)\) induces a stable equivalence of Morita type between \(A\) and \(B\),
3. \(A_M | k_{\Delta P} \uparrow^{G \times H} \text{ and } B M^\vee_A | k_{\Delta P} \uparrow^{H \times G}\),
4. the stable equivalence of Morita type induced by \(A_M\) preserves vertices and sources,
5. for any indecomposable \(X \in \text{mod-A}\) with vertex in \(\mathfrak{A}(G, P, N)\), it holds \((X \otimes_A M)_B = f(X) \oplus (\text{proj})\), where \(f\) is the Green correspondence with respect to \((G, P, H)\) (recall that \(\mathfrak{A}(G, P, N) \subseteq \mathfrak{A}(G, P, H) \cap \mathfrak{A}(H, P, N)\) by 2.3).

Proof. Let \(f_2\) be the Green correspondence with respect to \((G \times G, \Delta P, G \times N)\), and set \(\mathfrak{M} = f_2(A)\). Let \(f_2\) be the Green correspondence with respect to \((G, P, N)\). Moreover, let \(f_1\) be the Green correspondence with respect to \((H \times H, \Delta P, H \times N)\), and set \(\mathfrak{N} = f_1(B)\). Let \(f_1\) be the Green correspondence with respect to \((H, P, N)\). Then by 4.4 and 4.5 the bimodules \(\mathfrak{M}\) and \(\mathfrak{N}\) induce stable equivalences, so by 2.3 ii), and 2.5 there is a bimodule \(A_M-B\) such that

\[(*) \quad A_M \otimes_A \mathfrak{M} \otimes_A \mathfrak{N}^\vee_B = A_M \oplus \text{(proj \((A, B)\)-bimodule)}\]

and (1)–(4) hold.

It remains to show (5). Take any indecomposable \(X \in \text{mod-A}\) with vertex which is in \(\mathfrak{A}(G, P, N)\). Then it follows from (ii) that

\[X \otimes_A (\mathfrak{M} \otimes_A \mathfrak{N}^\vee) = X \otimes_A (M \oplus \text{(proj \((A, B)\)-bimodule)})\]

On the other hand, by 2.3 ii) we get

\[(X \otimes_A \mathfrak{M}) \otimes_A \mathfrak{N}^\vee = [f_2(X) \oplus (\text{proj } A_N\text{-module})] \otimes_A \mathfrak{N}^\vee = (f_2(X) \otimes_A \mathfrak{N}^\vee)_B \oplus ((\text{proj } A_N\text{-module}) \otimes_A \mathfrak{N}^\vee)_B = (f_2(X) \otimes_A \mathfrak{N}^\vee)_B \oplus (\text{proj } B\text{-module}) = (f_1^{-1}(f_2(X)))_B \oplus (\text{proj } B\text{-module}) = f(X) \oplus (\text{proj } B\text{-module})\]

\[\square\]

5. Modules in \(A, B\) and \(A_N\)

In the previous section, we have shown that there is a stable equivalence of Morita type between the blocks \(A\) and \(B\). As outlined in the introduction, our aim now is to verify that this equivalence is in fact a Morita equivalence with the help of 2.1. In other words, we need to show that the associated tensor functor takes simple modules to simple modules. Therefore in this intermediate section we collect all the necessary information on the simple modules and some indecomposable modules lying in the three blocks we consider.

In addition to the notation of our standard hypothesis 4.1, we fix the following:

**Lemma 5.1 (Suleiman-Wilson [57]).** The 2-decomposition matrix of \(A\) is given in Table 1 where \(S_1, \cdots, S_5\) are non-isomorphic simple \(kG\)-modules in \(A\) whose degrees are 73600, 896, 896, 19712, 131584, respectively. The two simple modules \(S_2\) and \(S_3\) are dual to each other, while the remaining are self-dual. There are two pairs \((\chi_6, \chi_7)\) and \((\chi_{18}, \chi_{19})\) of complex conjugate characters. All other \(\chi\)’s are real-valued.

**Proof.** See [57] §6.

\[\square\]
Lemma 5.5. We get the following:

Proof.

Remark 5.2. The 2-blocks of $\mathfrak{Co}_3$ have been studied before by several other people, see [16 p.193 Table 6], [20 §7 p.1879] and [27] Theorems 3.10 and 3.11.

Notation 5.3. We use the notation $\chi_{29}, \chi_6, \chi_7, \chi_{32}, \chi_{18}, \chi_{19}, \chi_{38}, \chi_{39}$, and $S_1, \cdots, S_5$ as in 5.1.

Lemma 5.4. All simple $kG$-modules $S_i, \cdots, S_5$ in $A$ have $P$ as a vertex.

Proof. See [20 3.7.Corollary].

Lemma 5.5. We get the following:

(i) $A_N = k[P \rtimes F_{21}] \otimes \beta \cong \text{Mat}_2(k[P \rtimes F_{21}])$, as $k$-algebras.

(ii) We can write $\text{Irr}(F_{21}) = \{k, 1, 1^*, 3, 3^*\}$.

(iii) We can write

$$\text{IBr}(A_N) = \{2_0 = k_{P \rtimes F_{21}} \otimes 2_{\mathfrak{S}_3}, 2 = 1 \otimes 2_{\mathfrak{S}_3}, 2^* = 1^* \otimes 2_{\mathfrak{S}_3}, 6 = 3 \otimes 2_{\mathfrak{S}_3}, 6^* = 3^* \otimes 2_{\mathfrak{S}_3}\}.$$

Note that there exists a unique simple $2_0$ which is self-dual.

(iv) The trivial source $A_N$-modules with vertex $P$ are precisely the simple $A_N$-modules.

Proof. (i)–(iii) are easy by 4.2 and the definition of $A_N$.

(iv) This follows from (iii) and the Green correspondence [39 Chap.4 Problem 10].

Lemma 5.6. Set $H = R(3) \cong SL_2(8) \rtimes C_3$. We get the following:

(i) For the principal block of $kH$ we have

$$\text{Irr}(B_0(kH)) = \{1_{\mathfrak{H}}, \chi_1, \chi_1^*, \chi_7a, \chi_7b, \chi_7c, \chi_{21}, \chi_{27}\},$$

and

$$\text{IBr}(B_0(kH)) = \{k_{\mathfrak{H}}, 1, 1^*, 6, 12\},$$

where the indices give the degrees (dimensions). The simples $k_{\mathfrak{H}}, 6, 12$ are self-dual, and the simples $k_{\mathfrak{H}}, 1, 1^*$ are trivial source $kH$-modules.

(ii) For the block $B$ we have

$$\text{Irr}(B) = \{\chi_{2a}, \chi_2, \chi_2^*, \chi_{14a}, \chi_{14b}, \chi_{14c}, \chi_{42}, \chi_{54}\},$$

and

$$\text{IBr}(B) = \{2_0 = k_{\mathfrak{H}} \otimes 2_{\mathfrak{S}_3}, 2 = 1 \otimes 2_{\mathfrak{S}_3}, 2^* = 1^* \otimes 2_{\mathfrak{S}_3}, 12 = 6 \otimes 2_{\mathfrak{S}_3}, 24 = 12 \otimes 2_{\mathfrak{S}_3}\},$$

where the indices give the degrees (dimensions). The simple $kH$-modules $2_0, 2, 2^*$ in $B$ are trivial source modules, the simple $kH$-modules $2_0, 12, 24$ are self-dual, and all the simples in $B$ have $P$ as their vertices.

Proof. (i) It follows from [13 p.6], and [18] $L_2(8).3$ (mod 2) or [63] $L_2(8).3$ (mod 2), see 4.2(xi). Clearly, $k_{\mathfrak{H}}, 1, 1^*$ are trivial source $kH$-modules.

(ii) $2_{\mathfrak{S}_3}$ is a trivial source $k\mathfrak{S}_3$-module. Therefore the simples $2_0, 2, 2^*$ are trivial source $kH$-modules, by (i) and 4.2(xi). Finally, use [20] 3.7.Corollary].
Lemma 5.8 (Landrock-Michler [29]). The radical and socle series of projective indecomposable \(kH\)-modules in \(B\) are the following:

\[
\begin{array}{ccc}
2_0 & 2 & 2^* \\
12 & 12 & 12 \\
20 & 20 & 20
\end{array}
\]

\[
\begin{array}{ccc}
2 & 2 & 2^* \\
12 & 12 & 12 \\
20 & 20 & 20
\end{array}
\]

\[
\begin{array}{ccc}
2^* & 2^* & 2^* \\
12 & 12 & 12 \\
20 & 20 & 20
\end{array}
\]

\[
\begin{array}{ccc}
24 & 24 & 24 \\
12 & 12 & 12 \\
24 & 24 & 24
\end{array}
\]

Proof. This follows from [29 Theorem 3.9, Theorem 4.1] and [5.6] \(\square\)

Lemma 5.9. Recall that \(R\) is a subgroup of \(P\) with \(R \cong C_2\), see [4.1]

(i) The Scott module \(\text{Scott}(\mathfrak{R}, R)\) has the radical and socle series

\[
\begin{array}{ccc}
k & 6 & 11^* \\
12 & & 12
\end{array}
\]

\[
\begin{array}{ccc}
k & 6 & 11 \\
12 & & 12
\end{array}
\]

(ii) A \(kH\)-module \(\text{Scott}(\mathfrak{R}, R) \otimes 2^{c_3}\) has the radical and socle series

\[
\begin{array}{ccc}
2_0 & 2 & 2^* \\
12 & 12 & 12 \\
20 & 20 & 20
\end{array}
\]

\[
\begin{array}{ccc}
\chi_2^a & \chi_2^a & \chi_2^a \\
\chi_5^a & \chi_5^a & \chi_5^a
\end{array}
\]

Proof. By [5.6(ii)], it suffices to prove (i). [13 p.6] says that \(\mathfrak{R}\) has a maximal subgroup \(M\) such that \(M = C_9 \times C_6\), \(\mathfrak{R} : M = 28\) and \(1_M^{\mathfrak{R}} = 1 + \chi_2^a + \chi_27^a\). Set \(X = kM^{\mathfrak{R}}\). Then \(X = 2 \times [k] + 1 \times [1^*] + 2 \times [6^*] + [12]\), as composition factors by [18] \(L_3(8.3)\) (mod 2) and [63] \(L_3(8.3)\) (mod 2). It holds by [43] 4 Thm.8.9(i) that \([X, X]^{\mathfrak{R}} = 1, [X, k]^{\mathfrak{R}} = [k, X]^{\mathfrak{R}} = 1\) Thus, \(X/\text{rad}(X) \cong \text{soc}(X) \cong k\). Now, it follows from [29 Theorem 4.1] that \(P(k\mathfrak{R})\) has the following radical and socle series:

\[
\begin{array}{ccc}
k & 6 & 11^* \\
12 & & 12
\end{array}
\]

Since there is an epimorphism \(P(k\mathfrak{R}) \to X\), we infer \(\text{soc}(X) \cong k\mathfrak{R} \cong \text{soc}(X) \cong \text{rad}(X)\) and \(\text{rad}(X)/\text{rad}(X) \cong \text{soc}(X)/\text{soc}(X) \cong 6\). Thus \(X\) has the radical and socle series as asserted. By the definition of \(X\), it holds that \(X = \text{Scott}(\mathfrak{R}, C_2)\), see [43 Chap.4 Theorem 8.4 and Corollary 8.5]. \(\square\)

Lemma 5.10. Recall that \(Q\) is a subgroup of \(P\) with \(Q \cong C_2 \times C_2\), see [4.1] Set \(U = \text{Scott}(\mathfrak{R}, Q)\).

(i) We have \(U \leftrightarrow 1 + \chi_27^a + 2 \times \chi_27^a\), and \(U = 4 \times [k\mathfrak{R}] + 2 \times [1] + 2 \times [1^*] + 5 \times [6] + 2 \times [12]\) as composition factors.

(ii) Set \(V = U \otimes 2^{c_3}\). Then \(V\) is a trivial source \(kH\)-module in \(B\) with vertex \(Q, V \leftrightarrow \chi_2^a + \chi_14^a + 2 \times \chi_5^a\), and \(V = 4 \times [20] + 2 \times [2] + 2 \times [2^*] + 5 \times [12] + 2 \times [24]\), as composition factors.
Proof. (i) We know that \( R \) has a subgroup \( \mathfrak{A}_4 \), see [13] p.6. Clearly, \( \text{Irr}(\mathfrak{A}_4) = \{1_{\mathfrak{A}_4}, \psi_1, \psi_2 = \psi_1^7, \psi_3\} \) where \( \psi_3 \) has degree 3. It follows from computations with GAP [17] that

\[
\begin{align*}
(1) & \quad 1_{\mathfrak{A}_4}^\uparrow_R \cdot 1_{B_0(kR)} = 1_R + \chi_7\psi + \chi_21 + 3 \times \chi_{27}, \\
(2) & \quad \psi_1^\uparrow_R \cdot 1_{B_0(kR)} = \chi_1 + \chi_7\psi + \chi_21 + 3 \times \chi_{27}, \\
(3) & \quad \psi_1^\uparrow_R \cdot 1_{B_0(kR)} = \chi_1^\uparrow + \chi_7\psi + \chi_21 + 3 \times \chi_{27}.
\end{align*}
\]

Let \( X = k_{\mathfrak{A}_4}^\uparrow_R \cdot 1_{B_0(kR)} \). First, we want to claim that \( P(12) \mid X \), where \( P(12) \) is the projective cover 12.

Set \( S = \text{SL}_2(8) \). By Clifford theory, we have \( 12 \downarrow_S = 4_1 \oplus 4_2 \oplus 4_3 \), where \( 4_1, 4_2, 4_3 \) are non-isomorphic simple \( kS \)-modules in \( B_0(kS) \) of dimension 4, see [18] \( L_2(8) \) (mod 2) and [63] \( L_2(8) \) (mod 2). Let \( V_i \) be the tautological \( kS \)-module, which is simple of dimension 2, and let \( V_2 \) and \( V_3 \) be its images under the action of the Frobenius automorphism of \( \mathbb{F}_8 \). Then the \( V_i \) are pairwise non-isomorphic, and by [11] p.220 we may assume that

\[
\begin{align*}
4_1 &= V_1 \oplus V_2, & 4_2 &= V_2 \oplus V_3, & 4_3 &= V_3 \oplus V_1.
\end{align*}
\]

Set \( g_\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in S \) for all \( a \in \mathbb{F}_8 \). We may assume that \( P = \{g_\alpha \mid a \in \mathbb{F}_8\} \leq S \), namely, \( P \) is a Sylow 2-subgroup of \( S \) with \( P \cong C_2 \times C_2 \times C_2 \), and that \( Q = \{g_0, g_1, g_0, g_1+a\} \), where \( a \in \mathbb{F}_8^* \) is a fixed primitive root, hence \( Q \cong C_2 \times C_2 \times C_2 \). Now the action of \( g_0 + g_1 + g_0 + g_1+a = (1+g_1)(1+g_0) \) in \( kQ \) is easily described in terms of Kronecker products of matrices, and it turns out that this element does not annihilate any of the \( kQ \)-modules \( 4_i \). Therefore \( 4_i \downarrow Q \) has a projective indecomposable summand, and thus we infer that \( 4_i \downarrow Q = P(kQ) \).

We conclude \( 12 \downarrow Q = 12 \downarrow S \downarrow Q = (4_1 \oplus 4_2 \oplus 4_3) \downarrow Q \cong \oplus \chi_{21} + \chi_{27} \) and it follows from [53] Theorem 3] that

\[
\begin{align*}
3 &= \frac{[P(kQ) \mid 12]_{\downarrow Q}^Q}{[P(12) \mid kQ_{\uparrow R}]^Q} = \frac{[P(12) \mid k_{\mathfrak{A}_4} \uparrow_R + 1_{\mathfrak{A}_4}^\uparrow_R]_{\uparrow R}^Q}{[P(12) \mid (k_{\mathfrak{A}_4}^R \oplus 1_{\mathfrak{A}_4}^R)_{\uparrow R}]^Q} = \frac{[P(12) \mid (k_{\mathfrak{A}_4}^R \oplus 1_{\mathfrak{A}_4}^R)]_{\uparrow R}^Q}{[P(12) \mid (k_{\mathfrak{A}_4} \uparrow_R + 1_{\mathfrak{A}_4}^\uparrow_R)_{\uparrow R}]^Q}.
\end{align*}
\]

Suppose that \( P(12) \nmid k_{\mathfrak{A}_4}^\uparrow_R \). Then \( 3 \times P(12) \mid (1_{\mathfrak{A}_4}^\uparrow_R \oplus 1_{\mathfrak{A}_4}^\uparrow_R)_{\uparrow R} \). Since \( P(12) \nmid \chi_{21} + \chi_{27} \) by [18] \( L_2(8).3 \) (mod 2) and [63] \( L_2(8).3 \) (mod 2), we know by (2) and (3) that \( 3 \times \chi_{21} + 3 \times \chi_{27} \) is contained in \( (\chi_1 + \chi_7\psi + \chi_21 + 3 \times \chi_{27}) + (\chi_1^7 + \chi_7\psi + \chi_21 + 3 \times \chi_{27}) \), which contradicts the multiplicity of \( \chi_{21} \).

Therefore \( P(12) \mid k_{\mathfrak{A}_4}^\uparrow_R \). Since \( P(12) \nmid \chi_{21} + \chi_{27} \) as seen above, it follows from [11] that \( k_{\mathfrak{A}_4}^\uparrow R \cdot 1_{B_0(kR)} = X \cong P(12) \) for a \( kR \)-module \( X \) such that

\[
X \leftrightarrow 1_R + \chi_7\psi + 2 \times \chi_{27}.
\]

Now, let \( U = \text{Scott}(R, Q) \), and hence \( U \mid X \) since \( Q \) is a Sylow 2-subgroup of \( \mathfrak{A}_4 \), see [13] Chap.4 Corollary 8.5). By the definition of Scott modules and [13] 4 Thm.8.9(i)], we know \( (\chi_2)_{\uparrow R} = 1_R \). Clearly, \( \chi_2 \neq 1_R \) since \( Q \nmid P \). Since \( P \) is a Sylow 2-subgroup of \( R \), it follows from [13] Chap.4, Theorem 7.5) that \( \text{dim}_k(U) \) is even. This means that \( \chi_2 \neq 1_R + 2 \times \chi_{27} \) and that \( \chi_2 \neq 1_R + \chi_7\psi + \chi_{27} \). If \( \chi_2 = 1_R + \chi_7\psi \) then \( \chi_2(2A) = 1 \times (-1) = 0 \) by [13] p.6, contradicting [28] II Lemma 12.6] since \( 2A \in Q \). Suppose that \( \chi_2 \neq 1_R + \chi_{27} \). Then since \( U \) is a trivial source \( kR \)-module, we get that \( U \) has the same radical and socle series of Scott(\( R, Q) \) just by the same method as in [6.9] Since \( U \cong \text{Scott}(R, Q) \mid R \uparrow R = 2 \) by [13] 4 Thm.8.9(i)], we have \( U \cong \text{Scott}(R, R) \), and hence \( Q \cong R \) by [13] Chap.4, Corollary 8.5], again a contradiction.

Therefore we know that \( \chi_2 = 1_R + \chi_7\psi + 2 \times \chi_{27} \) and \( U = X \), so that \( U = 4 \times [kR] + 2 \times [1] + 2 \times [1^*] + 5 \times [6] + 2 \times [12] \), as composition factors.

(ii) This follows from (i) and [4.2 xi]. \( \square \)

Remark 5.11. We will not need the precise structure of \( U = \text{Scott}(R, Q) \). Still we would like to remark that using the table of marks library of GAP [17], and the facilities available in the MeatAxe [52] and its extensions, \( U \) can actually be constructed and analysed explicitly.
In particular, it turns out that $U$ has Loewy length 5, but its radical and socle series do not coincide; they are

$$
\begin{array}{c|c|c|c}
& k6 & & \\
& k1 & 1^* 12 & 6 6 6 \\
& k1 & 1^* 12 & 6 6 6 \\
& k1 & 1^* 12 & k 6 \\
\end{array}
\text{and}
\begin{array}{c|c|c|c}
& 6 & & \\
& k1 & 1^* 12 & 6 6 6 \\
& k1 & 1^* 12 & k 6 \\
\end{array}
$$

respectively.

6. Images of simples in $A$ via Green correspondence

In this section we prove that the crucial hypothesis of 2.1 is fulfilled for the stable equivalence of Morita type we have established in 4.6. Namely, we show that simple modules in $A$ are taken to simple modules in $B$. For the first four simples this is almost immediate, as this amounts to determining the Green correspondents with respect to $(G, P, H)$, and these are easily determined theoretically and computationally. The image of the last simple $A$-module however, is more difficult to determine, and we make use of our knowledge on the modules of the blocks $A$ and $B$ we have gained in Section 5.

**Notation 6.1.** We use the notation $\mathcal{A}M_B$, $f$, $f_1$ and $f_2$ as in 4.6. Let $F : \text{mod-}A \to \text{mod-}B$ denote the functor giving the stable equivalence of Morita type of 4.6; namely, in the notation of 4.6 we have $F(X) = X \otimes_A M$ for each $X \in \text{mod-}A$.

**Lemma 6.2.** The following hold:

(i) $S_4 = 22 \otimes S_2$, where 22 is a simple $kG$-module in $B_0(kG)$.

(ii) We have

$$22 \downarrow_H = (6 \otimes k_{\mathfrak{S}_3}) \oplus (\text{proj}), \quad S_2 \downarrow_H = 2 \oplus 110 \oplus (\text{proj}) \quad \text{and} \quad (6 \otimes k_{\mathfrak{S}_3}) \otimes 2 = 12,$$

where $6 \otimes k_{\mathfrak{S}_3}$ is a simple $kH$-module in $B_0(kH) = B_0(kR(3)) \otimes B_0(k\mathfrak{S}_3)$, and 110 is an indecomposable $kH$-module in $B_0(kH)$, hence $S_2 \downarrow_H \cdot 1_B = 2$ and $S_2 \downarrow_H \cdot 1_B = 2^*$.

(iii) $12 \mid S_4 \downarrow_H$.

**Proof.** (i) This is obtained by [37, p.502], see [63, $\text{Co}_3$ (mod 2)], and a direct computation with Brauer characters in GAP [17].

(ii) By [18, $L_3(8).3 \mod 2$] or [63, $L_3(8).3 \mod 2$], except for the principal 2-block $B_0(kR(3)) = B_0(k[\text{SL}_2(8) \times C_3]$ there are only three 2-blocks of defect zero, consisting of the extensions of the Steinberg character of $\text{SL}_2(8)$ to $R(3)$. Hence it is easy to write down the block idempotents of $kR(3)$, and similarly those of $k\mathfrak{S}_3$. Thus, $H$ being a small group of order 9 072, using GAP [17] the block idempotents of $kH$ can be explicitly evaluated in a given representation. This yields the block components, which are then further analysed using the MeatAxe [52] and its extensions.

(iii) It follows from (i) and (ii) that

$$S_4 \downarrow_H = (22 \otimes S_2) \downarrow_H = 22 \downarrow_H \otimes S_2 \downarrow_H = \left((6 \otimes k_{\mathfrak{S}_3}) \oplus (\text{proj})\right) \otimes \left(2 \oplus 110 \oplus (\text{proj})\right) = \left((6 \otimes k_{\mathfrak{S}_3}) \oplus 2\right) \oplus (\text{other}) = 12 \oplus (\text{other}).$$

**Lemma 6.3.** We have $f(S_2) = 2$, $f(S_2^*) = 2^*$, $f(S_4) = 12$, and hence that $F(S_2) = 2$, $F(S_2^*) = 2^*$ and $F(S_4) = 12$.

**Proof.** By [6.2(ii)] the Green correspondents of $S_2$ and $S_2^*$ are immediate. By [5.4] all simple $A$-modules have vertex $P \in \mathcal{A}(G, P, H)$, and by [6.2(ii)] the direct summands of $(6 \otimes k_{\mathfrak{S}_3}) \otimes 110$ lie in the principal block. Therefore by [6.2(iii)] and [5.6(ii)] the simple module 12 is the unique summand of $S_4 \downarrow_H$ in $B$ with vertex $P$. Hence $f(S_4) = 12$. By [4.6(5)] and [2.1(i)] the functor $F$ maps any simple $A$-module to its Green correspondent in $B$, and so the claim follows.
Lemma 6.4. The simples $S_2$ and $S_2^*$ are trivial source $kG$-modules with $S_2 \leftrightarrow \chi_6$ and $S_2^* \leftrightarrow \chi_5^*$. 

Proof. We know by Lemma 6.5 (ii) that 2 and $2^*$ are trivial source $kH$-modules. Hence, by the definition of Green correspondence, Lemma 6.3 and Lemma 5.1, we get the assertion. \hfill $\square$

Lemma 6.5. The simple $kG$-module $S_1$ in $A$ is a trivial source module with $S_1 \leftrightarrow \chi_{29}$. 

Proof. It follows from [13] p.143 that $G$ has a maximal subgroup $L$ with $L = 2^S \mathfrak{S}_6(2)$. Then using GAP [17], we know that $1_{L \uparrow G}.1_A = \chi_{29}$. Hence the assertion follows by Lemma 5.1. \hfill $\square$

Lemma 6.6. We have $f(S_1) = 2_0$, and hence $F(S_1) = 2_0$. 

Proof. First, let $f_1'$ be the Green correspondence with respect to $(R(3), P, P \times F_{21})$. Clearly, $f_1'(k_{P \times F_{21}}) = k_{P \times F_{21}}$. Since $f_1$ is the Green correspondence with respect to $(H, P, N) = (R(3) \times \mathfrak{S}_3, P, (P \times F_{21}) \times \mathfrak{S}_3)$, we know that $f_1(k_{P \times F_{21}} \otimes 2_{\mathfrak{S}_3}) = k_{P \times F_{21}} \otimes 2_{\mathfrak{S}_3}$, namely, $f_1(2_0) = 2_0$. 

By Lemma 2.8 (ii), $f_1 \circ f = f_2$. Thus it follows from [43, 6.5 and 2.8 (iii)] that $f_1 \circ f(S_1)$ is a trivial source $kN$-module in $A_N$ with vertex $P$. Hence Lemma 5.5 (iv) implies that 

$$f_1 \circ f(S_1) \in \{2_0, \bar{2}, \bar{2}^*, 6, 6^*\}.$$ 

Then since $S_1$ is self-dual by Lemma 5.1, we know that $f_1 \circ f(S_1)$ is also self-dual. Therefore $f_1 \circ f(S_1) = 2_0$, giving $f_1 \circ f(S_1) = f_1(2_0)$. This implies that $f(S_1) = 2_0$. Hence we get the assertion from Lemma 4.6 (5) and Lemma 2.1 (i). \hfill $\square$

Lemma 6.7. The following hold: 

(i) $\text{Ext}^1_A(S_1, S_2) = \text{Ext}^1_A(S_2, S_1) = 0$. 

(ii) $\text{Ext}^1_A(S_2, S_2) = 0$. 

(iii) $\dim_k [\text{Ext}^1_A(S_1, S_1)] = \dim_k [\text{Ext}^1_A(S_4, S_1)] = 1$. 

Proof. By Lemma 6.6 and Lemma 6.3, we know the simple images of the simple modules given under the stable equivalence $F$ of $6.1$. Hence the results are immediate by looking at the $B$-PIMs in [43, X.2 Proposition 1.12] or [11, §5] for instance. \hfill $\square$

Lemma 6.8. All composition factors of $F(S_5)/\text{rad}(F(S_5))$ and $\text{soc}(F(S_5))$ are isomorphic to the simple module 24. 

Proof. Take any simple $kH$-module $T$ in $B$ such that $T \not\cong 24$. Then we know by Lemma 5.6, Lemma 6.3 and Lemma 6.6 that $T = F(S_5)$ for $i \in \{1, 2, 3, 4\}$, where $S_3 = S_2^*$. It then follows from [28, II Lemma 2.7 and Corollary 2.8] and Lemma 6.4 that $\text{Hom}_B(F(S_5), T) = \text{Hom}_B(F(S_5), F(S_5)) \cong \text{Hom}_A(S_5, S_1) = \text{Hom}_A(S_5, S_1) = 0$. Thus we get the assertion for the head of $F(S_5)$. The assertion for the socle follows by the same argument and considering $\text{Hom}_B(T, F(S_5))$ instead. \hfill $\square$

We can now finally prove that also the image of the last remaining simple $A$-module $S_5$ under $F$ is a simple $B$-module. 

Lemma 6.9. We have $F(S_5) = 24$. 

Proof. By [13] p.134, $G$ has a maximal subgroup $H = U_3(5) \rtimes \mathfrak{S}_3$. Set $X = k_{U \uparrow G}.1_A$. By calculations in GAP [17], we know that $1_{U \uparrow G}.1_A = \chi_{29} + \chi_{39}$, so that 

$$X \leftrightarrow \chi_{29} + \chi_{39}. \hfill (4)$$ 

Hence, by Lemma 5.1 

$$X = 2 \times S_1 + S_2 + S_2^* + 2 \times S_1 + S_5, \quad \text{as composition factors}. \hfill (5)$$ 

Since $S_1$, $S_2$ and $S_2^*$ are trivial source $kG$-modules by Lemma 6.3 and Lemma 6.4, it follows from [41, 5.1 and 43, 4 Thm.8.9(ii)] that 

$$[S_1, X] \cong [X, S_1^G] = 1, \quad [S_2, X] \cong [X, S_2^G] = [S_2^*, X] \cong [X, S_2^*] = 0.$$ 

If $[S_5, X] \neq 0$ or $[X, S_5] \neq 0$, then the self-duality of $X$ and $S_5$ implies that $S_5 \mid X$, and hence $S_5$ is a trivial source $kG$-module, so that $S_5$ is liftable to $O$ by [43, 4 Thm.8.9(iii)], which contradicts to Lemma 5.1. Hence 

$$[S_5, X] \cong [X, S_5^G] = 0.$$
Assume \([S_1, X]^G \neq 0\) or \([X, S_1]^G \neq 0\). Then again the self-dualities of \(X\) and \(S_1\) in (5.1) say that both are non-zero. Thus we have endomorphisms \(\psi_1, \psi_2\) and \(\psi_3\) of \(X\) such that \(\psi_1 = \text{id}_X\), \(\text{Im}(\psi_2) \cong S_1\) and \(\text{Im}(\psi_3) \cong S_4\). This means \([X, X]^G \geq 3\). But (4) 4 Thm.8.9(i) and (4) yield that \([X, X]^G = 2\), a contradiction. Thus \([S_4, X]^G = [X, S_1]^G = 0\). These imply that

\[(6) \quad X/\text{rad}(X) \cong \text{soc}(X) \cong S_1.\]

Hence \(X\) is indecomposable. Set \(X_0 = \text{rad}(X)/\text{soc}(X)\), the heart of \(X\). Thus (6) implies

\[(7) \quad X_0 = S_2 + S_2^* + 2 \times S_4 + S_5,\]

as composition factors.

By (6.7(i)), it holds

\[\text{Im}(\psi_1) = 2, \quad \text{and} \quad \text{im}(\chi_1) = \text{soc}(X) \cong S_1.\]

Moreover, (6.7(iii)) yields that \(X_0/\text{rad}(X_0) / (S_4 \oplus S_5)\). These imply that the radical and socle series of \(X\) is one of the following:

\[
X = \begin{cases}
S_1 \\
S_2 \bigoplus S_5 \\
S_1
\end{cases} \quad \text{or} \quad \begin{cases}
S_1 \\
S_4 \\
S_2 \\
S_5
\end{cases}
\]

Now, it follows from (6.1) and Corollary 2.8, (6.3) and (6) that

\[\text{Hom}_B(F(X), 2) = \text{Hom}_B(F(X), 2) = \text{Hom}_B(F(X), F(S_2)) \cong \text{Hom}_B(X, S_2) = \text{Hom}_A(X, S_2) = 0.\]

Hence \([F(X), 2]^B = 0\). Similarly we obtain \([F(X), 2^s]^B = 0\) and \([F(X), 12]^B = 0\) and \([F(X), 2_0]^B = 1\). Similar for \(\text{soc}(F(X))\), too. Thus, by (5.6) we know that

\[(9) \quad F(X)/\text{rad}(F(X)) \cong 2_0 \oplus (r \times 24)\]

and \(\text{soc}(F(X)) \cong 2_0 \oplus (r' \times 24)\) for some \(r, r' \geq 0\). By (6.1) we have

\[(10) \quad F(X) = Y \oplus (\text{proj} \; B\text{-module})\]

for a non-projective indecomposable \(kH\)-module \(Y\) in \(B\). Thus, by (6.6) and (A.1)(i)-(ii) we have

\[(11) \quad 2_0 \mid Y/\text{rad}(Y) \quad \text{and} \quad 2_0 \mid \text{soc}(Y).\]

Recall that \(2_0 = k_\mathfrak{R} \otimes 2_{\mathfrak{M}_{42}}\) in (5.6(ii)). Since \(B\) and \(B_0(k\mathfrak{R})\) are Puig equivalent by (4.2(xi)), and \(Y\) is a trivial source module by (4.6) it follows that \(Y \cong \text{Scott}(\mathfrak{R}, S) \otimes 2_{\mathfrak{M}_{42}}\) for a subgroup \(S\) of \(P\). Clearly \(S \neq 1\) since \(Y\) is non-projective indecomposable. If \(S = 0\) then (11) yields \(Y = 2_0\), so that \(F(X) = 2_0 \oplus (\text{proj})\) and \(F(S_1) = 2_0\) by (6.6). This is a contradiction since \(X\) is non-projective indecomposable and non-simple. Thus \(S \cong Q\) or \(S \cong R\).

Suppose that \(S \cong Q\), namely \(Y \cong \text{Scott}(\mathfrak{R}, Q) \otimes 2_{\mathfrak{M}_{42}}\). Then it follows by (5.10)(ii) that

\[Y \leftrightarrow \chi_{20} + \chi_{14a} + 2 \times \chi_{54},\]

and we have

\[(12) \quad Y = 4 \times [2_0] + 2 \times [2] + 2 \times [2^s] + 5 \times [12] + 2 \times [24],\]

as composition factors.

We know by (6.6) and (6.3) that

\[F(S_1) = 2_0, \; F(S_4) = 12, \; F(S_2) = 2, \; F(S_5^*) = 2^s.\]

Thus it follows by (5), (8) and (A.1)(i)-(ii) that we can strip off \(2 \times S_1, 2 \times S_4, S_2, \) and \(S_5^*\) from the top of \(X\) and from the bottom of \(X_0\), and also \(2 \times [2_0], 2 \times [12], [2^s]\) from the top of \(Y\) and from the bottom of \(Y\) sequentially, by looking at (8) and (12). Consequently by (2.4(i)), we have \(F(S_5) = Z\) for an indecomposable \(kH\)-module \(Z\) in \(B\) such that \(Z = 2 \times [2_0] + [2] + [2^s] + 3 \times [12] + 2 \times [24]\), as composition factors. Then (6.8) yields \(Z/\text{rad}(Z) \cong \text{soc}(Z) \cong 24\) and \(\text{rad}(Z)/\text{soc}(Z) = [2_0] + [2] + [2^s] + 3 \times [12]\) as composition factors, which contradicts (5.8).
Therefore $S \cong R$ and $Y \cong \text{Scott}(R, R) \otimes 2_{e_2}$. Hence we get by $[5.9](ii)$ that

$$F(X) = Y \oplus \text{proj}, \quad Y = \begin{pmatrix} 2_0 \\ 12 \\ 12 \\ 2_0 \end{pmatrix}.$$ 

Thus by the same stripping-off method $[A.1](i)-(ii)$ taken above, we can subsequently strip off $2 \times S_1, 2 \times S_4, S_2, \text{ and } S_2^*$ from the top of $X$ and the bottom of $X$, and also $2 \times [2_0], 2 \times [12], [2], \text{ and } [2^*]$ from the top of $Y$ and the bottom of $Y$, by looking at $[5]$ and $[13]$. Hence we arrive at $F(S_3) = 24 \oplus \text{proj}$, so that $\square$

Remark 6.10. We know by $[1.7]$ that the block $A$ of $G$ and the principal 2-block $B_0(kR(3))$ of $R(3)$ are Puig equivalent. Let $X$ be the same as in the proof of $6.9$. Thus it follows from $[5.9](i)-(ii)$ and the proof of $6.9$ that the radical and socle series of $X$ is actually the first one in $[5]$ in the proof of $6.9$ and that $X$ is a trivial source $kG$-module in $A$ with vertex $C_2$.

7. PROOF OF THE MAIN RESULTS

Proof of [1.5]. First of all, consider the blocks $A$ and $B$ over $k$, namely, $A$ and $B$ are block algebras of $kG$ and $kH$, respectively. Hence $M$ is a $(kG, kH)$-bimodule. We know by $[4.6](ii)$ and $[5.1]$ that the functor $F$ defined by $M$ realises a stable equivalence of Morita type between $A$ and $B$. It follows from $[5.1], [6.3], [6.6]$ and $[6.9]$ that, for any simple $kG$-module $S$ in $A$, $F(S)$ is a simple $kH$-module in $B$. Hence, $\square$ (ii) yields that $AM_B$ realises a Morita equivalence between $A$ and $B$. Since $M$ is a $\Delta P$-projective trivial source $k[G \times H]$-module, the Morita equivalence is a Puig equivalence by $[47]$ Remark 7.5 or $[39]$ Theorem 4.1 (note that this was independently observed by L. Scott). Moreover, by $[43]$ 4 Thm.8.9(i)], the Morita equivalence lifts from $k$ to $O$; see also $[58]$ (38.8)Proposition or $[46]$ 7.8.Lemma].

Proof of Corollary [1.7]. This follows by [1.5], [1.6] and [2.7]  

Proof of Theorem [1.3]. This follows from [1.7], [2.7] and [4.2](i).  

APPENDIX A. PROPERTIES OF THE STABLE EQUIVALENCES CONSIDERED

In this appendix we collect some fundamental properties of the stable equivalences which are found throughout this paper, and in particular of the stable equivalence $F$ of $[6.1]$ For the large part, these properties are used at several steps in this paper, but they are also of independent interest, as a referenceable collection with proofs is desirable. Also, in this section, we aim to supply more general hypotheses for clarity.

The first fundamental property we collect is the the following ”stripping off”-method, which enables us to reduce the problem of determining the image of a module under a stable equivalence to determining the images of its head and socle components; the proof of $[6.9]$ bears testimony of the utility of this lemma. See also $[23]$ in which $[A.1]$ is firstly conceived and applied.

Lemma A.1. Let $A$ and $B$ be finite dimensional $k$-algebras for a field $k$ such that $A$ and $B$ are both self-injective. Let $F$ be a covariant functor such that

1. $F$ is exact.
2. If $X$ is a projective $A$-module, then $F(X)$ is a projective $B$-module,
3. $F$ induces a stable equivalence from $\text{mod-}A$ to $\text{mod-}B$.

Then the following holds:
(i) (Stripping-off method, case of socle) Let \( X \) be a projective-free \( A \)-module, and write \( F(X) = Y \oplus \langle \text{proj} \rangle \) for a projective-free \( B \)-module \( Y \). Let \( S \) be a simple \( A \)-submodule of \( X \), and set \( T = F(S) \). Now, if \( T \) is a simple \( B \)-module, then we may assume that \( Y \) contains \( T \) and that
\[
F(X/S) = Y/T \oplus \langle \text{proj} \rangle.
\]

(ii) (Stripping-off method, case of radical) Similarly, let \( X \) be a projective-free \( A \)-module, and write \( F(X) = Y \oplus \langle \text{proj} \rangle \) for a projective-free \( B \)-module \( Y \). Let \( X' \) be an \( A \)-submodule of \( X \) such that \( X/X' \) is simple, and set \( T = F(X/X') \). Now, if \( T \) is a simple \( B \)-module, then we may assume that \( T \) is an epimorphic image of \( Y \) and that
\[
\ker(F(X) \to T) = \ker(Y \to T) \oplus \langle \text{proj} \rangle.
\]

Proof. (i)-(ii) The assertions are got from [28, II Lemma 2.7 and Corollary 2.8] and [23, 1.11.Lemma], just as in [23, 3.25.Lemma and 3.26.Lemma]. \( \square \)

Next, we want to show that the stable equivalence of Morita type also commutes with taking the contragredient module if \( A \) and \( B \) are blocks of group algebras. This is made precise in [A,2]iv, but first we place ourselves into a more general context.

Lemma A.2. Let \( A \) and \( B \) be finite dimensional \( k \)-algebras for a field \( k \).

(i) Assume that \( X \in \text{mod-} A \), and \( M \in A\text{-mod-} B \), and that \( AM \) is projective. Then the correspondence
\[
\Phi : B(M^X \otimes_A X^\otimes) \to B((X \otimes_A M)^\otimes)
\]
defined by
\[
\Phi(\psi \otimes_A \theta)(x \otimes_A m) = \theta(x \cdot \psi(m))
\]
for \( \psi \in M^X \), \( \theta \in X^\otimes \) and \( m \in M \), is an isomorphism of left \( B \)-modules.

(ii) Assume that \( Y \in A\text{-mod-} \) and \( N \in B\text{-mod-} A \), and that \( NA \) is projective. Then the correspondence
\[
\Theta : (N^Y \otimes_A Y^\otimes)_B \to ((N \otimes_A Y)^\otimes)_B
\]
defined by
\[
\Theta(\theta \otimes_A \psi)(n \otimes_A y) = \theta(\psi(n) \cdot y)
\]
for \( \psi \in N^Y \), \( \theta \in Y^\otimes \) and \( n \in N \), is an isomorphism of right \( B \)-modules.

(iii) If \( A \) moreover is a symmetric algebra, with symmetrising form \( t \in \text{Hom}_k(A,k) \), then as \( (B,A) \)-bimodules we have
\[
B(M^X)_A \cong B(M^\otimes)_A \quad \text{via the correspondence} \quad t_* : f \mapsto t \circ f.
\]
Thus we have an isomorphism of left \( B \)-modules
\[
\Psi : B(M^X \otimes_A X^\otimes) \cong_B B(M^\otimes \otimes_A X^\otimes) \xrightarrow{t_*} B(X \otimes_A M)^\otimes
\]
given by
\[
t_*(\psi) \otimes_A \theta \mapsto \psi \otimes_A \theta \mapsto \Phi(\psi \otimes_A \theta).
\]

(iv) If finally \( A \) and \( B \) are block algebras of finite groups, and \( M \) is self-dual, namely, \( M^* \cong M \) as \( (A,B) \)-bimodules, then as right \( B \)-modules we have
\[
(X^* \otimes_A M)_B \cong ((X \otimes_A M)^*)^\otimes_B.
\]

Proof. (i) Assume first that \( B = k \). The map \( \Phi \) is \( k \)-linear and an isomorphism if \( M = A \) as a left \( A \)-module. Clearly \( \Phi \) is compatible with direct sums and direct summands. Thus, since \( M \) is finitely generated projective as a left \( A \)-module, we know that \( \Phi \) is an isomorphism of \( k \)-spaces. It is easy to see by the definition of \( \Phi \) that \( \Phi \) is a homomorphism of left \( B \)-modules, too. A similar argument works works for (ii).

(iii) It is easy to see that \( t_* \) is a homomorphism of \( (B,A) \)-bimodules, and that \( t_* \) is injective. Hence the first assertion follows from [8, Proposition 2.7]. The second assertion now follows from this together with (i). Now (iv) follows easily from (iii). \( \square \)
Lemma A.3. Let $H$ be a proper subgroup of $G$, and let $A$ and $B$ be block algebras of $kG$ and $kH$, respectively. Now, let $M$ and $M'$ be finitely generated $(A,B)$- and $(B,A)$-bimodules, respectively, which satisfy the following:

1. $A M_B \mid 1_A kG \cdot 1_B$ and $B M'_A \mid 1_B kG \cdot 1_A$.
2. The pair $(M, M')$ induces a stable equivalence between mod-$A$ and mod-$B$.

Then we get the following:

(i) Assume that $X$ is a non-projective indecomposable $kG$-module in $A$ with vertex $Q$. Then there exists a non-projective indecomposable $kH$-module $Y$ in $B$, unique up to isomorphism, such that $(X \otimes_A M)_B = Y \oplus (\text{proj})$, and $Q^\circ$ is a vertex of $Y$ for some element $g \in G$ (and hence $Q^\circ \subseteq H$). Since $Q^\circ$ is also a vertex of $X$, this means that $X$ and $Y$ have at least one vertex in common.

(ii) Assume that $Y$ is a non-projective indecomposable $kH$-module in $B$ with vertex $Q$. Then there exists a non-projective indecomposable $kG$-module $X$ in $A$, unique up to isomorphism, such that $(Y \otimes_B M')_A = X \oplus (\text{proj})$, and $Q$ is a vertex of $X$.

(iii) Let $X, Y$ and $Q \subseteq H$ be the same as in (i). Then there is an indecomposable $kQ$-module $L$ such that $L$ is a source of both $X$ and $Y$. This means that $X$ and $Y$ have at least one source in common.

(iv) Let $X, Y$ and $Q \subseteq H$ be the same as in (ii). Then there is an indecomposable $kQ$-module $L$ such that $L$ is a source of both $X$ and $Y$. This means that $X$ and $Y$ have at least one source in common.

(v) Let $X, Y, Q$ and $L$ be the same as in (iii). In addition, suppose that $A$ and $B$ have a common defect group $P$ (and hence $P \subseteq H$) and that $H \subseteq \text{N}_G(P)$. Let $f$ be the Green correspondence with respect to $(G, P, H)$. If $Q \in \mathfrak{A} = \mathfrak{A}(G, P, H)$, then we have $(X \otimes_A M)_B = f(X) \oplus (\text{proj})$.

(vi) Let $X, Y, Q$ and $L$ be the same as in (ii). Furthermore, as in (v), assume that $P$ is a common defect group of $A$ and $B$, and that $H \supseteq \text{N}_G(P)$, and let $f$ and $\mathfrak{A}$ be the same as in (v). Now, if $Q \in \mathfrak{A}$, then we have $(Y \otimes_B M')_A = f^{-1}(Y) \oplus (\text{proj})$.

Proof. (i) Clearly, $X \mid X \downarrow_Q^G$. By (2) there exists a non-projective indecomposable $kH$-module $Y$ in $B$, unique up to isomorphism, such that $(X \otimes_A M)_B = Y \oplus (\text{proj})$. Hence,

$$Y \mid X \otimes_A M = X \otimes_{kG} M \mid X \otimes_{kG} kG_{kH} = X \downarrow_H \mid X \downarrow_Q^G \downarrow_H = \bigoplus_{g \in [Q \cap G/H]} (X \downarrow_Q^g) \downarrow_{Q \cap H}^H.$$ 

The last equality follows from Mackey Decomposition. Since $Y_{kH}^g$ is indecomposable, the Krull-Schmidt Theorem yields $Y \mid (X \downarrow_Q^g) \downarrow_{Q \cap H}^H$ for some $g \in G$. That is, $Y$ is $(Q \cap H)$-projective, so that there is a vertex $R$ of $Y$ such that $R \subseteq Q^\circ \cap H$. Since $Y \mid Y \downarrow_R^H$, it holds as above that

$$X \mid Y \otimes_B M' = Y \otimes_{kH} M' \mid Y \otimes_{kH} kG_{kG} = Y \uparrow^G \mid (Y \downarrow_R^H)^G = Y \downarrow_R^G.$$ 

Hence, $X$ is $R$-projective, so that there is a vertex $S$ of $X$ with $S \subseteq R$. Since $Q$ is also a vertex of $X$, we have $S = Q^\circ$ for some $g^1 \in G$. Namely, $Q^\circ \subseteq R$. This implies that $Q^\circ = S \subseteq R \subseteq Q^\circ \cap H \subseteq Q^\circ$, and hence $Q^\circ = R = Q^\circ \cap H = Q^\circ$. This yields that $Q^\circ \subseteq H$.

(ii) Similar to (i).

(iii) By the assumption, $Q$ is a common vertex of $X$ and $Y$. Let $L_{kQ}$ be a source of $Y_{kH}$. Then by the proof of (i), $X \mid Y \uparrow^G \mid L^G \uparrow^G = L^G$. Hence, $X \mid L^G$. Since $X$ has vertex $Q$ and $L$ is an indecomposable $kQ$-module, it follows that $L$ is a source of $X$, too.

(iv) This follows from (iii).

(v) Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{A}$ be those with respect to $(G, P, H)$ as in [33 Chap.4 §4]. Now, let $X$ be an indecomposable $kG$-module in $A$ such that a vertex of $X$ is in $\mathfrak{X}$. Thus, we can assume that $Q \in \mathfrak{X}$. If $X$ is projective then $Q$ is trivial, so that the trivial group is not contained in $\mathfrak{X}$ by the definition of $\mathfrak{X}$, a contradiction, since $H \neq G$. 

Finally, a fundamental property of the stable equivalences obtained through [2.3] (see also [33]) is that it preserves vertices and sources, and takes indecomposable modules to their Green correspondents.
Hence, $X$ is non-projective. Thus, we get by (i) and (ii) that there is a non-projective indecomposable $kH$-module $Y$ in $B$ such that $X \otimes_A M = Y \oplus (\text{proj } B\text{-mod})$ and that $Y$ also has $Q$ as its vertex. On the other hand, we know $(X \otimes_A M) |_{kH} = f(X) \oplus (\mathcal{Q}\text{-proj } B\text{-mod})$. This implies that $f(X) \oplus (\mathcal{Q}\text{-proj } B\text{-mod}) = Y \oplus (\text{proj } B\text{-mod}) \oplus V$ for a $kH$-module $V$.

Assume that $Y$ is $\mathcal{Q}$-projective. Since $Q$ is a vertex of $Y$, we have $Q \in c_{H} \mathcal{Q}$. Hence, we get by [33, Chap.4 Lemma 4.1(ii)] that $Q \in \mathcal{X}$. Then we have $Q \notin \mathcal{A}$, a contradiction. Therefore, by the Krull-Schmidt Theorem, we have $Y \cong f(X)$.

(vi) We get this exactly as in (iii) just by replacing $X$, $M$, and $f$ by $Y$, $M'$, and $f^{-1}$, respectively. \hfill $\Box$

Acknowledgements

The authors thank the referee for useful comments and remarks on the first draft of the paper. The authors are grateful to Burkhard Külshammer for pointing out [33] to them. A part of this work was done while the first author was staying in RWTH Aachen University in 2009 and 2010. He is grateful to Gerhard Hiss for his kind hospitality. For this research the first author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)20540008, 2008–2010; and also (B)21340003, 2009–2011. The first author still remembers that in January 1985, in Bad Honnef Bonn, Germany, there was a conference and he was informed by Peter Landrock about the non-principal 2-block of $Co_2$ with defect group $C_2 \times C_2 \times C_2$, which was interesting because the block is non-principal and the inertial index is 21.

References

[1] J.L. Alperin, Projective modules for $SL(2,2^n)$, J. Pure Appl. Algebra 15 (1979), 219–234.
[2] J.L. Alperin, M. Broué, Local methods in block theory, Ann. of Math. 110 (1979), 143–157.
[3] J.L. Alperin, M. Linckelmann, R. Rouquier, Source algebras and source modules, J. Algebra 239 (2001), 262–271.
[4] M. Auslander, I. Reiten, S.O. Smalø, Representation theory of Artin algebras, Cambridge Univ. Press, Cambridge, 1997.
[5] T. Breuer, GAP package CTbllib — The GAP Character Table Library, Version 1.1.3, http://www.gap-system.org/Packages/ctbllib.html
[6] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque 181–182 (1990), 61–92.
[7] M. Broué, Equivalences of blocks of group algebras, In: Finite Dimensional Algebras and Related Topics, Dlab, V., Scott, L.L. (eds.) pp.1–26 Kluwer Acad. Pub., Dordrecht, 1994.
[8] M. Broué, Higman’s criterion revisited, Michigan Math. J. 58 (2009), 125–179.
[9] M. Broué, L. Puig, Characters and local structure in $G$-algebras, J. Algebra 73 (1980), 306–371.
[10] M. Broué, L. Puig, A Frobenius theorem for blocks, Invent. Math. 56 (1980), 117–128.
[11] J.F. Carlson, Modules and Group Algebras, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1996.
[12] J. Chuang, J. Rickard, Representations of finite groups and tilting, Handbook of tilting theory, 359–391, London Math. Soc. Lecture Note Ser. 332, Cambridge Univ. Press, Cambridge, 2007.
[13] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[14] D. Craven, R. Rouquier, Perverse equivalences and Broué’s conjecture, Preprint (2010), arXiv:1010.1378v1 [math.RT].
[15] C.W. Curtis, I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Vol.1, Wiley, New York, 1981.
[16] D. Fendel, A characterization of Conway’s group $3$, J. Algebra 24 (1973), 159–196.
[17] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4.12, http://www.gap-system.org
[18] C. Jansen, K. Lux, R. Parker, R. Wilson, An Atlas of Brauer Characters, Clarendon Press, Oxford, 1995.
[19] R. Kessar, S. Koshitani, M. Linckelmann, Alperin’s weight conjecture for 2-blocks with elementary abelian defect groups of order 8, to appear in J. reine angew. Math. (2011), 46 pages.
[20] R. Knörr, On the vertices of irreducible modules, Ann. of Math. 110 (1979), 487–499.
[21] S. König, A. Zimmermann, Derived Equivalences for Group Rings, Lecture Notes in Math., Vol.1685, Springer, Berlin, 1998.
[22] S. Koshitani, The principal 2-block of finite groups with abelian Sylow 2-subgroups, Tsukuba J. Math. 4 (1980), 21–66.
[23] S. Koshitani, N. Kunugi, K. Waki, Broué’s abelian defect group conjecture for the Held group and the sporadic Suzuki group, J. Algebra 279 (2004), 638–666.
[24] S. Koshitani, M. Linckelmann, The indecomposability of a certain bimodule given by the Brauer construction, J. Algebra 285 (2005), 726–729.
[25] B. Külshammer, Some indecomposable modules and their vertices, J. Pure Appl. Algebra 86 (1993), 65–73.
[63] R. Wilson, J. Thackray, R. Parker, F. Noeske, J. Müller, F. Lübeck, C. Jansen, G. Hiss, T. Breuer, The Modular Atlas Project, http://www.math.rwth-aachen.de/~MOC

[64] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, R. Abbott, Atlas of Finite Group Representations, http://brauer.maths.qmul.ac.uk/Atlas/v3