THE EXTREMAL LIMIT OF D-DIMENSIONAL BLACK HOLES

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The extreme limit of a class of D-dimensional black holes is revisited. In the static case, it is shown that a well defined extremal limiting procedure exists and it leads to new solutions of the type $AdS^2 \times \Sigma^{D-2}$, $\Sigma^{D-2}$ being a $(D-2)$-dimensional constant curvature symmetric space.

In this contribution, we would like to revisit the extremal limit of a large class of black hole solutions. The interest in the study of extreme and nearly-extreme black holes has recently been increased, mainly due to the link with one of the puzzle which is still unsolved in the black hole physics: the issue related to the statistical interpretation of the Bekenstein-Hawking entropy.

To begin with, first we review some particular solution of Einstein Eqs. In classical paper, Bertotti and Robinson obtained solutions of the Einstein Eqs. which geometrically are the direct product of two 2-dimensional manifolds. If the cosmological constant is vanishing, these manifolds have necessarily constant curvature and opposite sign. Locally they are homeomorphic to $AdS_2 \times S_2$. The original Bertotti-Robinson solution is also globally coincident with $AdS_2 \times S_2$. Since these geometries are strictly related to the near-horizon geometry of a generic extremal black hole, one may wonder if in the extremal limit, the geometry is still non trivial in the sense that a non vanishing Hawking temperature is present. This indeed is the case of De Sitter-Schwarzschild black, for which an appropriate limiting procedure leads to the Nariai solution.

Let us consider the Einstein Eq. with zero cosmological constant

$$G^\nu_{\mu} = 8\pi G T^\nu_{\mu}. \quad (1)$$
In presence of a covariant constant electric field, one may write \( \text{diag}T = (-E^2, -E^2, E^2, E^2) \) and we may try to solve the Einstein Eq. with the Bertotti-Robinson static ansatz

\[
\begin{align*}
\text{ds}^2 &= -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r_0^2d\Omega^2_2, \\
\end{align*}
\]  

(2)

where \( r_0^2 \) is a constant and \( d\Omega^2_2 \) is metric tensor of \( S_2 \). A standard elementary calculation gives

\[
\begin{align*}
G_0^0 &= -\frac{1}{r_0^2}, & G_1^1 &= -\frac{1}{r_0^2}, \\
G_2^2 &= G_3^3 = \frac{1}{2}V''(r). \\
\end{align*}
\]  

(3)

Since \( T_\mu^\nu = 0 \), one gets \( G_\mu^\mu = 0 \). As a result

\[
-\frac{2}{r_0^2} + V''(r) = 0.
\]  

(5)

The general solution of this elementary differential eq. may be presented in the form

\[
V(r) = \frac{r^2 + c_1r + c_2}{r_0^2},
\]  

(6)

where \( c_1 \) and \( c_2 \) are two constant. The Einstein Eqs. (3) and (4) are satisfied when

\[
r_0^2 = \frac{1}{GE^2}.
\]  

(7)

In summary, we have found the 2-parameter family of solutions

\[
\begin{align*}
ds^2 &= -(\frac{r^2 + c_1r + c_2}{r_0^2})dt^2 + \frac{1}{\frac{r^2 + c_1r + c_2}{r_0^2}}dr^2 + r_0^2d\Omega^2_2. \\
\end{align*}
\]  

(8)

The original Bertotti-Robinson solution corresponds to \( c_1 = 0 \) and \( c_2 = r_0^2 \) namely

\[
ds^2 = -(\frac{r^2}{r_0^2} + 1)dt^2 + \frac{1}{(\frac{r^2}{r_0^2} + 1)}dr^2 + r_0^2d\Omega^2_2.
\]  

(9)

which is regular and globally homeomorphic to \( AdS_2 \times S_2 \).

This is a consequence of the following statement, which involves the corresponding Euclidean sections.
If we start from the 2-dimensional hyperbolic space, with metric in the Poincaré form (\(R\) being the radius)

\[
ds^2 = \frac{R^2}{y^2}(dx^2 + dy^2),
\]

(10)

then the mapping (\(a, b\) suitable constant parameters)

\[
y + ix = \sqrt{\frac{r + b - \sqrt{r + b - 4aR^2e^{-i2a\tau}}}{r + b + \sqrt{r + b - 4aR^2e^{-i2a\tau}}}},
\]

(11)

reduces the Poincaré metric to

\[
ds^2 = \frac{(r + b)(r + b - 4aR^2)}{R^2}d\tau^2 + \frac{R^2}{(r + b)(r + b - 4aR^2)}dr^2.
\]

(12)

For example, choosing \(R = r_0, b = ir_0, 4aR^2 = 2ir_0\), one gets the BR solution, with unrestricted Euclidean time \(\tau\).

The situation changes for the particular solution which corresponds to the choice \(c_1 = -r^* < 0\) and \(c_2 = 0\) in Eq. (8), i.e.

\[
ds^2 = -(\frac{r^2 - r^*r}{r_0^2})dt^2 + \frac{1}{\frac{r^2 - r^*r}{r_0^2}}dr^2 + r_0^2d\Omega^2.
\]

(13)

This is a black hole solution with Hawking temperature

\[
\beta_H = \frac{4\pi r_0^2}{r^*}
\]

(14)

This solution is only locally homeomorphic to \(AdS_2 \times S_2\). In fact, if we choose \(b = 0, R = r_0\) and \(4a r_0^2 = r^*\), the Euclidean section of the metric (13) reduces to the Poincaré one, but now the mapping (11) reads

\[
y + ix = \sqrt{\frac{r - \sqrt{r - r^*e^{-i2a\tau}}}{r + \sqrt{r - r^*e^{-i2a\tau}}}}.
\]

(15)

Here one can see that \(\tau\) is defined modulo the period \(\beta = \frac{\pi}{\sqrt{\frac{4\pi r_0^2}{r^*}}}\), which coincides with the Hawking temperature computed requiring the absence of the conical singularity. This may be interpreted saying that solution (13) describes a manifold which is only a portion of \(H^2\).

A similar case is described by the choice \(c_1 = 0\) and \(c_2 = -b^2 < 0\), namely

\[
ds^2 = -(\frac{r^2 - b^2}{r_0^2})dt^2 + \frac{1}{\frac{r^2 - b^2}{r_0^2}}dr^2 + r_0^2d\Omega^2.
\]

(16)
Here, the Hawking temperature computed with the standard technique is
\[ \beta_H = \frac{2\pi r_0^2}{b}. \] (17)

However, if we choose \( R = r_0 \) and \( 2\pi r_0^2 = b \), the Euclidean section of the metric (16) reduces to the Poincare' one, and the mapping (11) reads
\[ y + ix = \frac{\sqrt{r + b} - \sqrt{r - b}e^{-i2a\tau}}{\sqrt{r + b} + \sqrt{r - b}e^{-i2a\tau}}. \] (18)
with \( 2a = \frac{b}{r_0} \). Again, \( \tau \) is defined modulo a period which coincides with (17).

It should be noted that these black hole solutions are similar to the Rindler space-times and the Hawking temperature is the Unruh one associated with the quantum fluctuations.

Now, let us consider a black hole solution corresponding to a D-dimensional charged or neutral black hole depending on parameters as the mass \( m \), charges \( Q_i \) and the cosmological constant \( \Lambda \). In the Schwarzschild static coordinates (with \( G = l_P^2 = 1 \) and \( D = d + 2 \)), it reads
\[ ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2d\Sigma_d^2. \] (19)

Here, \( d\Sigma_d^2 \) is the line element related to a constant curvature "horizon" \( d \)-dimensional manifold. The inner and outer horizons are positive simple roots of the shift function, i.e.
\[ V(r_H) = 0, \quad V'(r_H) \neq 0. \] (20)

The associated Hawking temperature is
\[ \beta_H = \frac{4\pi}{V'(r_H)}. \] (21)

In general, when the extremal solution exists, namely
\[ V(r_{ex}) = 0, \quad V'(r_{ex}) = 0, \quad V''(r_{ex}) \neq 0, \] (22)
there exists a relationship between the parameters,
\[ F(m, g_i) = 0. \] (23)

When this condition is satisfied, it may happen that the original coordinates become inappropriate (for example when \( V(r) \) has a local maximum in \( r = r_{ex} \), i.e. \( V''(r_{ex}) < 0 \).
The extreme limit has been investigated in several places\(^3\)\(^4\)\(^5\)\(^6\). In order to investigate the extremal limit, we introduce the non-extremal parameter \(\epsilon\) and perform the following coordinate change

\[
r = r_{\text{ex}} + \epsilon r_1, \quad t = \frac{t_1}{\epsilon}.
\]

and parametrize the non-extremal limit by means of

\[
F(m, g_i) = k\epsilon^2, \quad (25)
\]

where the sign of constant \(k\) defines the physical range of the black hole parameters, namely the ones for which the horizon radius is non negative. In the near-extremal limit, we may make an expansion for \(\epsilon\) small. As a consequence

\[
V(r) = V(r_{\text{ex}}) + V'(r_{\text{ex}})r_1\epsilon + \frac{1}{2}V''(r_{\text{ex}})r_1^2\epsilon^2 + O(\epsilon^3). \quad (26)
\]

It is clear that

\[
V(r_{\text{ex}}) = k_1\epsilon^2, \quad V'(r_{\text{ex}}) = k_2\epsilon^2, \quad (27)
\]

where \(k_i\) are known constants.

Thus, the metric in the extremal limit becomes

\[
ds^2 = -dt_1^2(k_1 + \frac{1}{2}V''(r_{\text{ex}})r_1^2) + \frac{dr_1^2}{(k_1 + \frac{1}{2}V''(r_{\text{ex}})r_1^2)} + r_{\text{ex}}^2d\Sigma_2^2. \quad (28)
\]

As first example, let us consider the 4-dimensional charged RN black hole, where the horizon manifold is \(S^2\) and the shift function is given by

\[
V(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}. \quad (29)
\]

Here, the near-extremal condition reads

\[
F(m, Q) = \frac{Q^2}{m^2} - 1 = k\epsilon^2. \quad (30)
\]

When \(\epsilon = 0\), one has \(r_+ = r_- = r_{\text{ex}} = m\) and \(Q^2 = m^2\) and the physical range corresponds to \(k < 0\), for example we may take \(k = -a^2\). In this case, the shift function has a local minimum at \(r_{\text{ex}}\) and

\[
\frac{1}{2}V''(r_{\text{ex}}) = \frac{1}{m^2}, \quad (31)
\]
but \( V(r_{ex}) = -a^2c^2 < 0 \). As a result, the metric in the extremal limit is

\[
d s^2 = -d t_1^2 (-a^2 + \frac{r_1^2}{m^2}) + \frac{d r_1^2}{(-a^2 + \frac{r_1^2}{m^2})} + m^2 d \Omega_2^2 ,
\]

(32)

and this limiting metric describes the space-time locally \( \text{AdS}_2 \times S_2 \), we have previously discussed. This solution satisfies, in the extreme limit, a BPS-like condition, namely

\[
Q = m .
\]

(33)

We note that also the Bertotti-Robinson solution may be obtained in the limiting procedure, but assuming \( k > 0 \). Thus one has the local minimum for the shift function, and this corresponds to an extremal limit within the unphysical range of black hole parameters.

As a second example, let us consider the Schwarzschild-DeSitter space-time. Here the horizon manifold is still \( S^2 \) and since

\[
V(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} ,
\]

(34)

the solution is not asymptotically flat. As well known, there exist an event horizon and a cosmological horizon and \( r_H < r < r_C \). The near extremal condition is

\[
F(m, \Lambda) = \frac{1}{3} - m \sqrt{\Lambda} = k e^2 , \quad k > 0 .
\]

(35)

In this case, the shift function has a local maximum at the extremal radius \( r_H = r_C = r_{ex} = (\Lambda)^{-1/2} \) and the original coordinates are totally inappropriate. Furthermore, \( V(r_{ex}) = 2k e^2 > 0 \) and we have

\[
d s^2 = -d t_1^2 (2k - \Lambda r_1^2) + \frac{d r_1^2}{2k - \Lambda r_1^2} \\
\quad + \frac{1}{\Lambda} d \Omega_2^2 .
\]

(36)

This solution is locally \( dS_2 \times S_2 \) and is equivalent to the Nariai solution, a cosmological solution with \( \Lambda > 0 \).

As further example, let us consider the 4-dimensional topological black hole solution for which the horizon manifold \( \Sigma_2^2 \) is a compact negative constant curvature Riemann surface and

\[
V(r) = -1 - \frac{C}{r} + \frac{r^2}{l^2} .
\]

(37)
Here, $l$ is related to the negative cosmological constant, namely $\Lambda = -\frac{3}{l^2}$ and the constant $C$ is given by

$$C = m - l^* , \quad l^* = \frac{2}{3\sqrt{3}} l$$

(38)
m being the mass of the black hole. The horizon radius is a positive solution of

$$- lr^2 - (m - l^*)l^2 + r^3 = 0.$$  

(39)
The extremal solution exists for $m = 0$, since we have

$$- lr^2 + l^*l^2 + r^3 = (r - \frac{l}{\sqrt{3}})^2(r + \frac{2l}{\sqrt{3}})$$

(40)
and is given by

$$r_{ex} = \frac{l}{\sqrt{3}}.$$  

(41)
As a consequence, in order to investigate the extremal limit, we may put

$$m = C + l^* = kr_{ex}^2 , \quad k > 0.$$  

(42)
Thus,

$$V(r_{ex}) = -ke^2 , \quad k_1 = -k < 0.$$  

(43)
and

$$V''(r_{ex}) = \frac{3}{r_{ex}^2} > 0.$$  

(44)
The limiting metric turns out to be

$$ds^2 = -dt_1^2 [-k + \frac{3r_{ex}^2}{2r_{ex}^2}] + \frac{dr_{ex}^2}{[-k + \frac{3r_{ex}^2}{2r_{ex}^2}]} + r_{ex}^2 d\Sigma_2^2 ,$$

namely we have $AdS_2 \times \Sigma_2$, $\Sigma_2$ being a Riemann surface.

Finally, we also report the result obtained starting from the Kerr-Newmann black hole solution in the standard Boyer-Linquist coordinates. Defining the nearly-extreme condition by means

$$m^2 - a^2 - Q^2 = km^2 e^2 , \quad k > 0 ,$$

(46)
m, $a$ and $Q$ being respectively the mass, the angular parameter and the charge of the black hole and making use of

$$r = r_{ex} + \alpha r_1 , \quad \phi = \phi_1 + \frac{a t_1}{e(m^2 + a^2)} , \quad t = t_1$$

(47)
the limiting metric turns out to be

\[ ds^2 = (m^2 + a^2 \cos^2 \theta) \left[ -\frac{r_1^2 - km^2}{(m^2 + a^2)^2} dt_1^2 ight. \\
+ \frac{dr_1^2}{r_1^2 - km^2} + d\theta^2 \left. \right] + \frac{(m^2 + a^2)^2 \sin^2 \theta}{(m^2 + a^2 \cos^2 \theta)} \left( d\phi_1 + \frac{2ma}{(m^2 + a^2)^3} r_1 dt_1 \right)^2 \]

(48)

A similar solution (corresponding to \( c = 0 \)) has been recently obtained in [9], where one can find a detailed discussion of the related geometry.

In conclusion, we have shown that a large class of black hole solutions admits a well defined extremal limit procedure. In the static case, this procedure gives rise to new solutions of the kind \( AdS_2 \times \Sigma_{D-2} \), \( \Sigma_{D-2} \) being a (D-2)-dimensional constant curvature symmetric space.

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