Some Structural Properties of the Quantized Matrix Algebra $D_q(n)$

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Abstract. Let $D_q(n)$ be the quantized matrix algebra introduced by Dipper and Donkin. It is shown explicitly that the defining relations of $D_q(n)$ form a Gröbner-Shirshov basis. Consequently, several structural properties of $D_q(n)$ are derived.

Key words: quantized matrix algebra; Gröbner-Shirshov basis; PBW basis

1. Introduction

Let $K$ be a field of characteristic 0. The quantized matrix algebra $D_q(n)$, introduced in [2], has been widely studied and generalized in different contexts, for instance, [3], [4], [5], [6], and [7]. In this note, we show explicitly that the defining relations of $D_q(n)$ form a Gröbner-Shirshov basis. Consequently, this result enables us to derive several structural properties of $D_q(n)$, such as having a PBW $K$-basis, being of Hilbert series $$(1 - t)^n$$, of Gelfand-Kirillov dimension $n^2$, of global homological dimension $n^2$, being a classical Koszul algebra, and having elimination property for (one-sided) ideals in the sense of [9] (see also [10, A3]).

For classical Gröbner-Shirshov basis theory of noncommutative associative algebras, one is referred to, for instance [1].

Throughout this note, $K$ denotes a field of characteristic 0, $K^* = K - \{0\}$, and all $K$-algebras considered are associative with multiplicative identity 1. If $S$ is a nonempty subset of an algebra $A$, then we write $\langle S \rangle$ for the two-sided ideal of $A$ generated by $S$.

2. The defining relations of $D_q(n)$ form a Gröbner-Shirshov basis

In this section, all terminologies concerning Gröbner-Shirshov bases, such as composition, ambiguity, and normal word, etc., are referred to [1].

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Let $K$ be a field of characteristic 0, $I(n) = \{(i,j) \mid i, j = 1, 2, \ldots, n\}$ with $n \geq 2$, and let $D_q(n)$ be the quantized matrix algebra with the set of $n^2$ generators $d = \{d_{ij} \mid (i,j) \in I(n)\}$, in the sense of [2], namely, $D_q(n)$ is the associative $K$-algebra generated by the given $n^2$ generators subject to the relations:

$$d_{ij}d_{st} = qd_{st}d_{ij}, \quad \text{if } i > s \text{ and } j \leq t,$$

$$d_{ij}d_{st} = d_{qt}d_{ij} + (q - 1)d_{sj}d_{it}, \quad \text{if } i > s \text{ and } j > t,$$

$$d_{ij}d_{ik} = d_{ik}d_{ij}, \quad \text{for all } i, j, k,$$

where $i, j, k, s, t = 1, 2, \ldots, n$ and $q \in K^*$ is the quantum parameter.

Now, let $D = \{D_{ij} \mid (i,j) \in I(n)\}$, $K\langle D \rangle$ the free associative $K$-algebra generated by $D$, and let $S$ denote the set of defining relations of $D_q(n)$ in $K\langle D \rangle$, that is, $S$ consists of elements

$$(a) \ f_{ijst} = D_{ij}D_{st} - qD_{st}D_{ij}, \quad \text{if } i > s \text{ and } j \leq t,$$

$$(b) \ g_{ijst} = D_{ij}D_{st} - D_{st}D_{ij} - (q - 1)D_{sj}D_{it}, \quad \text{if } i > s \text{ and } j > t,$$

$$(c) \ h_{ijik} = D_{ij}D_{ik} - D_{ik}D_{ij}, \quad \text{for all } i, j, k.$$

Then, $D_q(n) \cong K\langle D \rangle / \langle S \rangle$ as $K$-algebras, where $\langle S \rangle$ denotes the (two-sided) ideal of $K\langle D \rangle$ generated by $S$, i.e., $D_q(n)$ is presented as a quotient of $K\langle D \rangle$. Our aim below is to show that $S$ forms a Gröbner-Shirshov basis with respect to a certain monomial ordering on $K\langle D \rangle$. To this end, let us take the deg-lex ordering $\prec_{d, \text{lex}}$ (i.e., the degree-preserving lexicographic ordering) on $D^*$, which is the set of all mono words in $D$, i.e., all words of finite length like $u = D_{ij}D_{kl} \cdots D_{ab}$. More precisely, we first take the lexicographic ordering $\prec_{\text{lex}}$ on $D^*$ which is the natural extension of the ordering on the set $D$ of generators of $K\langle D \rangle$: for $D_{ij}, D_{kl} \in D$,

$$D_{kl} < D_{ij} \iff \left\{ \begin{array}{l}
  k < i, \\
  \text{or } k = i \text{ and } l < j,
  \end{array} \right.$$

and for two words $u = D_{k_1l_1}D_{k_2l_2} \cdots D_{k_al_a}, v = D_{i_1j_1}D_{i_2j_2} \cdots D_{i_bj_b} \in D^*$,

$$u \prec_{\text{lex}} v \iff \text{there exists an } m \text{ such that } D_{k_ml_m} < D_{i_mj_m} \text{ but } D_{k_{m-1}l_{m-1}} = D_{i_{m-1}j_{m-1}}$$

(note that conventionally the empty word $1 < D_{ij}$ for all $D_{ij} \in D$). For instance

$$D_{41}D_{21}D_{31} \prec_{\text{lex}} D_{42}D_{13}D_{43} \prec_{\text{lex}} D_{42}D_{23}D_{34}D_{41}.$$ 

And then, by assigning each $D_{ij}$ the degree 1, $1 \leq i, j \leq n$, and writing $|u|$ for the degree of a word $u \in D^*$, we take the deg-lex ordering $\prec_{d, \text{lex}}$ on the set $D^*$: for $u, v \in D^*$,

$$u \prec_{d, \text{lex}} v \iff \left\{ \begin{array}{l}
  |u| < |v|, \\
  \text{or } |u| = |v| \text{ and } u \prec_{\text{lex}} v.
  \end{array} \right.$$

For instance,

$$D_{24}D_{11} \prec_{d, \text{lex}} D_{32}D_{24} \prec_{d, \text{lex}} D_{32}D_{31} \prec_{d, \text{lex}} D_{11}D_{12}D_{13}.$$
It is straightforward to check that $\prec_{d,\text{lex}}$ is a monomial ordering on $K\langle D \rangle$, namely, $\prec_{d,\text{lex}}$ is a well-ordering and

$$u \prec_{d,\text{lex}} v \text{ implies } wuv \prec_{d,\text{lex}} wvr \text{ for all } u, v, w, r \in D^*.$$  

With this monomial ordering $\prec_{d,\text{lex}}$ in hand, we are ready to prove the following result.

**Theorem 2.1** With notation as fixed above, let $J = \langle S \rangle$ be the ideal of $D_q(n)$ generated by $S$. Then, with respect to the monomial ordering $\prec_{d,\text{lex}}$ on $K\langle D \rangle$, the set $S$ is a Gröbner-Shirshov basis of the ideal $J$, i.e., the defining relations of $D_q(n)$ form a Gröbner-Shirshov basis.

**Proof** By [1], it is sufficient to check that all compositions determined by elements in $S$ are trivial modulo $S$. In doing so, let us first fix two more notations. For an element $f \in K\langle D \rangle$, we write $\overline{f}$ for the leading mono word of $f$ with respect to $\prec_{d,\text{lex}}$, i.e., if $f = \sum_{i=1}^{s} \lambda_i u_i$ with $\lambda_i \in K$, $u_i \in D^*$, such that $u_1 \prec_{d,\text{lex}} u_2 \prec_{d,\text{lex}} \cdots \prec_{d,\text{lex}} u_s$, then $\overline{f} = u_s$. Thus, the set $S$ of defining relations of $D_q(n)$ has the set of leading mono words

$$S = \left\{ \overline{f}_{ijst} = D_{ij} D_{st}, \text{ if } s < i, \ j \leq t, \right\}$$

$$= \left\{ \overline{f}_{ijst} = D_{ij} D_{st}, \text{ if } s < i, \ t < j, \right\}$$

$$= \left\{ \overline{h}_{ijk} = D_{ik} D_{ij}, \text{ if } k < j. \right\}$$

(note that if $j < k$, then since $h_{ijk} = D_{ij} D_{ik} - D_{ik} D_{ij}$, we have $\overline{h}_{ijk} = D_{ik} D_{ij}$). Also let us write $(a \wedge b)$ for the composition determined by defining relations $(a)$ and $(b)$ in $S$. Similar notations are made for compositions of other pairs of defining relations in $S$.

By means of $S$ above, we start by listing all possible ambiguities $w$ of compositions of intersections determined by elements in $S$, as follows:

\[(a \wedge a)\] $w = D_{ij} D_{st} D_{kl}$ if $i > s > k$, $j \leq t \leq l$,

\[(a \wedge b)\] $w_1 = D_{ij} D_{st} D_{kl}$ if $i > s > k$, $j \leq t$, $t > l$,

\[w_2 = D_{ij} D_{st} D_{kl}\] if $i > s > k$, $j \leq t$, $t > l$,

\[(a \wedge c)\] $w_1 = D_{ij} D_{st} D_{sk}$ if $i > s$, $j \leq t$, $t > k$,

\[w_2 = D_{ij} D_{st} D_{sk}\] if $i > s$, $j > k$, $k \leq t$,

\[(b \wedge b)\] $w = D_{ij} D_{st} D_{kl}$ if $i > s > k$, $j > t > l$,

\[(b \wedge c)\] $w_1 = D_{ij} D_{st} D_{sk}$ if $i > s$, $j > t > k$,

\[w_2 = D_{ij} D_{st} D_{sk}\] if $i > s$, $j > k > t$,

\[(c \wedge c)\] $w = D_{ij} D_{ik} D_{it}$ if $j > k > t$.

Instead of writing down all tedious verification processes, below we shall record only the verification processes of four typical cases:

\[(a \wedge b)\] $w_1 = D_{ij} D_{st} D_{kl}$, if $i > s > k$, $j \leq t$, $t > l$,

\[(a \wedge c)\] $w_1 = D_{ij} D_{st} D_{sk}$, if $i > s$, $j \leq t$, $t > k$,

\[(b \wedge b)\] $w = D_{ij} D_{st} D_{kl}$, if $i > s > k$, $j > t > l$,

\[(b \wedge c)\] $w_1 = D_{ij} D_{st} D_{sk}$, if $i > s$, $j > t > k$,
because other cases can be checked in a similar way (the interested reader may contact the author directly in order to see other verification processes).

- The case \((a \land b)\) with \(w_1 = D_{ij}D_{st}D_{kl}\), where \(i > s > k, j \leq t, t > l\).

Since in this case \(w_1 = D_{ij}D_{st}D_{kl} = f_{ijst}D_{kl} = D_{ij}g_{stkl}\) with \(f_{ijst} = D_{ij}D_{st} - qD_{st}D_{ij}\), where \(i > s\) and \(j \leq t\), and \(g_{stkl} = D_{st}D_{kl} - D_{kl}D_{st} - (q - 1)D_{kt}D_{st}\), where \(s > k\) and \(t > l\), we have two cases to deal with.

Case 1. If \(l \geq j\), then \(i > s > k, t > l \geq j\), and it follows that

\[
(f_{ijst}, g_{stkl})_{w_1} = f_{ijst}D_{kl} - D_{ij}g_{stkl}
= -qD_{st}D_{ij}D_{kl} + D_{ij}D_{kt}D_{st} + (q - 1)D_{ij}D_{kt}D_{sl}
\equiv -q^2D_{st}D_{kl}D_{ij} - q(q - 1)D_{st}D_{kt}D_{sl} + D_{kl}D_{ij}D_{st} + (q - 1)qD_{kt}D_{ij}D_{st}
\equiv -qD_{kl}D_{st}D_{ij} - q(q - 1)D_{kt}D_{st}D_{ij} - q(q - 1)D_{kj}D_{sl}D_{il}
+ q^2(q - 1)D_{kt}D_{sl}D_{ij} + q^2(q - 1)D_{st}D_{k}D_{ij} + q^2D_{kl}D_{st}D_{ij}
\equiv 0 \text{ mod}(S_1, w_1)
\]

Case 2. If \(l < j\), then \(i > s > k, t > j > l\), and it follows that

\[
(f_{ijst}, g_{stkl})_{w_1} = f_{ijst}D_{kl} - D_{ij}g_{stkl}
= -qD_{st}D_{ij}D_{kl} + D_{ij}D_{kt}D_{st} + (q - 1)D_{ij}D_{kt}D_{sl}
\equiv -q^2D_{st}D_{kl}D_{ij} - q(q - 1)D_{st}D_{kt}D_{il} + D_{kl}D_{ij}D_{st} + (q - 1)qD_{kt}D_{ij}D_{st}
\equiv -qD_{kl}D_{st}D_{ij} - q(q - 1)D_{kt}D_{st}D_{ij} - q(q - 1)D_{kj}D_{sl}D_{il}
+ q(q - 1)^2D_{kt}D_{s}D_{ij} + qD_{kl}D_{st}D_{ij} + q(q - 1)^2D_{kt}D_{sl}D_{il}
\equiv 0 \text{ mod}(S_1, w_1)
\]

- The case \((a \land c)\) with \(w_1 = D_{ij}D_{st}D_{sk}\), where \(i > s, j \leq t, t > k\).

Since in this case \(w_1 = D_{ij}D_{st}D_{sk} = T_{ijst}D_{sk} = D_{ij}T_{stsk}\), where \(f_{ijst} = D_{ij}D_{st} - qD_{st}D_{ij}\) with \(i > s\) and \(j \leq t\), and \(h_{stsk} = D_{st}D_{sk} - D_{sk}D_{st}\) with \(t > k\), we have two cases to deal with.

Case 1. If \(j \leq k\), then \(i > s, j \leq t, j \leq k\), thereby

\[
(f_{ijst}, h_{stsk})_{w_1} = f_{ijst}D_{sk} - D_{ij}h_{stsk}
= -qD_{st}D_{ij}D_{sk} + D_{ij}D_{sk}D_{st}
\equiv -q^2D_{st}D_{sk}D_{ij} + qD_{sk}D_{ij}D_{st}
\equiv -q^2D_{st}D_{sk}D_{ij} + q^2D_{st}D_{sk}D_{ij}
\equiv 0 \text{ mod}(S, w_1).
\]

Case 2. If \(j > k\), then \(i > s\) and \(k < j \leq t\), thereby

\[
(f_{ijst}, h_{stsk})_{w_1} = f_{ijst}D_{sk} - D_{ij}h_{stsk}
= -qD_{st}D_{ij}D_{sk} + D_{ij}D_{sk}D_{st}
\equiv -qD_{st}D_{sk}D_{ij} - q(q - 1)D_{st}D_{sk}D_{ik}
+ D_{sk}D_{ij}D_{st} + (q - 1)D_{sj}D_{sk}D_{st}
\equiv -qD_{sk}D_{st}D_{ij} - q(q - 1)D_{sj}D_{st}D_{ik}
+ qD_{sk}D_{st}D_{ij} + q(q - 1)D_{sj}D_{st}D_{ik}
\equiv 0 \text{ mod}(S, w_1).
\]
• The case \((b \land b)\) with \(w = D_{ij}D_{st}D_{kl}\), where \(i > s > k, j > t > l\).

Since in this case \(w = D_{ij}D_{st}D_{kl} = g_{ijst}D_{st}\) with \(g_{ijst} = D_{ij}D_{st} - D_{st}D_{ij} - (q - 1)D_{sj}D_{it}\) and \(g_{stkl} = D_{st}D_{kl} - D_{kl}D_{st} - (q - 1)D_{kl}D_{st}\), where \(i > s > k\) and \(j > t > l\), we have

\[
(g_{ijst}, g_{stkl})_w = g_{ijst}D_{kl} - D_{ij}g_{stkl}
\]

\[
\equiv -D_{st}D_{ij}D_{kl} - (q - 1)D_{sj}D_{it}D_{kl} + D_{ij}D_{kl}D_{st} + (q - 1)D_{ij}D_{kl}D_{st}
\]

\[
\equiv -D_{st}D_{ij}D_{kl} - (q - 1)D_{st}D_{kj}D_{it} - (q - 1)D_{sj}D_{kl}D_{it} - (q - 1)^2D_{sj}D_{kl}D_{it} + D_{ij}D_{kl}D_{st} + (q - 1)D_{kl}D_{it}D_{st}
\]

\[
\equiv -D_{kl}D_{st}D_{ij} - (q - 1)D_{kl}D_{st}D_{ij} - (q - 1)D_{kj}D_{st}D_{it} - (q - 1)D_{kj}D_{st}D_{it} - (q - 1)^2D_{kj}D_{st}D_{it} + (q - 1)D_{kl}D_{st}D_{ij} + (q - 1)D_{kl}D_{st}D_{ij} + (q - 1)^2D_{kl}D_{st}D_{ij} + (q - 1)^2D_{kj}D_{st}D_{it}
\]

\[
\equiv 0 \mod(S, w).
\]

• The case \((b \land c)\) with \(w_1 = D_{ij}D_{st}D_{sk}\), where \(i > s, j > t > k\).

Since in this case \(w_1 = D_{ij}D_{st}D_{sk} = g_{ijst}D_{sk}\) with \(g_{ijst} = D_{ij}D_{st} - D_{st}D_{ij} - (q - 1)D_{sj}D_{it}\) with \(i > s\) and \(j > t\), and \(h_{stsk} = D_{st}D_{sk} - D_{sk}D_{st}\) with \(t > k\), we have

\[
(g_{ijst}, h_{stsk})_{w_1} = g_{ijst}D_{sk} - D_{ij}h_{stsk}
\]

\[
\equiv -D_{st}D_{ij}D_{sk} - (q - 1)D_{sj}D_{it}D_{sk} + D_{ij}D_{sk}D_{st}
\]

\[
\equiv -D_{st}D_{sk}D_{ij} - (q - 1)D_{st}D_{sj}D_{it} - (q - 1)D_{sj}D_{sk}D_{it} - (q - 1)^2D_{sj}D_{sk}D_{it} + D_{sk}D_{st}D_{ij} + (q - 1)D_{sk}D_{st}D_{ij} + (q - 1)^2D_{sk}D_{st}D_{ij} + (q - 1)^2D_{sk}D_{st}D_{it}
\]

\[
\equiv 0 \mod(S, w_1).
\]

This finishes the proof of the theorem.

\[\square\]

3. Some applications of Theorem 2.1

By means of Theorem 2.1, we derive several structural properties of \(D_q(n)\) in this section. All notations used in Section 2 are maintained.

**Corollary 3.1** The quantized matrix algebra \(D_q(n) \cong K \langle D \rangle / J\) has the linear basis, or more precisely, the PBW basis

\[
B = \left\{ d_{11}^{k_{11}}d_{12}^{k_{12}}\cdots d_{1n_1}^{k_{1n_1}}d_{21}^{k_{21}}\cdots d_{2n_2}^{k_{2n_2}}\cdots d_{n_1}^{k_{n_1}}\cdots d_{nn}^{k_{nn}} \mid k_{ij} \in \mathbb{N}, (i, j) \in I(n) \right\}.
\]
Proof With respect to the monomial ordering $\prec_{d, \text{lex}}$ on the set $D^*$ of mono words of $K\langle D \rangle$, we note that
\[ D_{11} \prec_{d, \text{lex}} D_{12} \prec_{d, \text{lex}} \cdots \prec_{d, \text{lex}} D_{1n} \prec_{d, \text{lex}} D_{21} \prec_{d, \text{lex}} D_{22} \prec_{d, \text{lex}} \cdots \prec_{d, \text{lex}} D_{2n} \prec_{d, \text{lex}} \cdots \prec_{d, \text{lex}} D_{n1} \prec_{d, \text{lex}} D_{n2} \prec_{d, \text{lex}} \cdots \prec_{d, \text{lex}} D_{nn}, \]
and the Gröbner-Shirshov basis $S$ of the ideal $J = \langle S \rangle$ has the set of leading mono words consisting of
\[ \mathcal{T}_{ijst} = D_{ij}D_{st} \text{ with } D_{st} \prec_{d, \text{lex}} D_{ij} \text{ where } s < i, j \leq t, \]
\[ \mathcal{G}_{ijst} = D_{ij}D_{st} \text{ with } D_{st} \prec_{d, \text{lex}} D_{ij} \text{ where } s < i, t < j, \]
\[ \mathcal{H}_{ijik} = D_{ij}D_{ik} \text{ with } D_{ik} \prec_{d, \text{lex}} D_{ij}, \text{ if } k < j. \]
It follows from classical Gröbner-Shirshov basis theory that the set of normal forms of $D^*$ (mod $S$) is given as follows:
\[ \left\{ D_{11}^{k_{11}}D_{12}^{k_{12}} \cdots D_{1n}^{k_{1n}}D_{21}^{k_{21}} \cdots D_{2n}^{k_{2n}} \cdots D_{nn}^{k_{nn}}, \mid k_{ij} \in \mathbb{N}, (i, j) \in I(n) \right\}. \]
Therefore, $D_q(n)$ has the desired PBW basis. □

Before giving next result, we recall three results of [8] in one proposition below, for the reader’s convenience.

Proposition 3.2 Adopting notations used in [8], let $K\langle X \rangle = K\langle X_1, X_2, \ldots, X_n \rangle$ be the free $K$-algebra with the set of generators $X = \{X_1, X_2, \ldots, X_n\}$, and let $\prec$ be a monomial ordering on $K\langle X \rangle$. Suppose that $G$ is a Gröbner-Shirshov basis of the ideal $I = \langle G \rangle$ with respect to $\prec$, such that the set of leading monomials $\text{LM}(G) = \{X_jX_i \mid 1 \leq i < j \leq n\}$. Considering the algebra $A = K\langle X \rangle/I$, the following statements hold.

(i) [8, P.167, Example 3] The Gelfand-Kirillov dimension $\text{GK.dim} A = n$.

(ii) [8, P.185, Corollary 7.6] The global homological dimension $\text{gl.dim} A = n$, provided $G$ consists of homogeneous elements with respect to a certain $\mathbb{N}$-gradation of $K\langle X \rangle$. (Note that in this case $G^\mathbb{N}(A) = A$, with the notation used in loc. cit.)

(iii) [8, P.201, Corollary 3.2] $A$ is a classical quadratic Koszul algebra, provided $G$ consists of quadratic homogeneous elements with respect to the $\mathbb{N}$-gradation of $K\langle X \rangle$ such that each $X_i$ is assigned the degree 1, $1 \leq i \leq n$. (Note that in this case $G^\mathbb{N}(A) = A$, with the notation used in loc. cit.) □

Remark Let $j_1j_2 \cdots j_n$ be a permutation of $1, 2, \ldots, n$. One may notice from the references respectively quoted in Proposition 3.2 that if, in the case of Proposition 3.2, the monomial
ordering $<$ employed there is such that
\[ X_{j_1} < X_{j_2} < \cdots < X_{j_n}, \]
and
\[ LM(\mathcal{G}) = \{X_{j_k}X_{j_t} \mid X_{j_t} < X_{j_k}, 1 \leq j_k, j_t \leq n\}, \]
or
\[ LM(\mathcal{G}) = \{X_{j_k}X_{j_t} \mid X_{j_k} < X_{j_t}, 1 \leq j_k, j_t \leq n\}, \]
then all results still hold true.

Applying Proposition 3.2 and the above remark to \(D_q(n) \cong K\langle D\rangle / J\), we are able to derive the result below.

**Theorem 3.3** The quantized matrix algebra \(D_q(n)\) has the following structural properties.

(i) The Hilbert series of \(D_q(n)\) is \(\frac{1}{(1-t)^{n^2}}\).

(ii) The Gelfand-Kirillov dimension \(\text{GK.dim}D_q(n) = n^2\).

(iii) The global homological dimension \(\text{gl.dim}D_q(n) = n^2\).

(iv) \(D_q(n)\) is a classical quadratic Koszul algebra.

**Proof** Recalling from Section 2 that with respect to the monomial ordering \(\prec_{d-\text{lex}}\) on the set \(D^*\) of mono words of \(K\langle D\rangle\), we have
\[ D_{kl} \prec_{d-\text{lex}} D_{ij} \iff \begin{cases} k < i, l < j, \\ k < i, j < l, \\ k < i, j = l, \\ k = i, l < j. \end{cases} (i, j) \in I(n), \]
and thus, the Gröbner-Shirshov basis \(S\) of the ideal \(J = \langle S\rangle\) has the set of leading mono words consisting of
\[ F_{ijst} = D_{ij}D_{st} \text{ with } D_{st} \prec_{d-\text{lex}} D_{ij} \text{ where } s < i, j \leq t, \]
\[ G_{ijst} = D_{ij}D_{st} \text{ with } D_{st} \prec_{d-\text{lex}} D_{ij} \text{ where } s < i, t < j, \]
\[ H_{ijk} = D_{ij}D_{ik} \text{ with } D_{ik} \prec_{d-\text{lex}} D_{ij}, \text{ if } k < j. \]
This means that \(D_q(n)\) satisfies the conditions of Proposition 3.2. Therefore, the assertions (i) – (iv) are established as follows.

(i) Since \(D_q(n)\) has the PBW \(K\)-basis as described in Corollary 3.1, it follows that the Hilbert series of \(D_q(n)\) is \(\frac{1}{(1-t)^{n^2}}\).

(ii) This follows from Theorem 2.1, Proposition 3.2(i), and the remark made above.

Note that \(D_q(n)\) is an \(\mathbb{N}\)-graded algebra defined by a quadratic homogeneous Gröbner basis (Theorem 2.1), where each generator \(D_{ij}\) is assigned the degree 1, \((i, j) \in I(n)\). The assertions (iii) and (iv) follow from Proposition 3.2(ii) and Proposition 3.2(iii), respectively. □
We end this section by concluding that the algebra $D_q(n)$ also has the elimination property for (one-sided) ideals in the sense of [9] (see also [10, A3]). To see this, let us first recall the Elimination Lemma given in [9]. Let $A = K[a_1, a_2, \ldots, a_n]$ be a finitely generated $K$-algebra with the PBW basis $B = \{a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n\}$ and, for a subset $U = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \subset \{a_1, a_2, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, let

$$T = \left\{a_{i_1}^{\alpha_1}a_{i_2}^{\alpha_2} \cdots a_{i_r}^{\alpha_r} \mid (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r\right\}, \quad V(T) = K\text{-span}T.$$

**Lemma 3.4** ([9, Lemma 3.1]) Let the algebra $A$ and the notations be as fixed above, and let $L$ be a nonzero left ideal of $A$ and $A/L$ the left $A$-module defined by $L$. If there is a subset $U = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\} \subset \{a_1, a_2, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_n$, such that $V(T) \cap L = \{0\}$, then

$$\text{GK.dim}(A/L) \geq r.$$

Consequently, if $A/L$ has finite GK dimension $\text{GK.dim}(A/L) = m < n$ (= the number of generators of $A$), then

$$V(T) \cap L \neq \{0\}$$

holds true for every subset $U = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \subset \{a_1, a_2, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_m$, in particular, for every $U = \{a_1, a_2, \ldots, a_s\}$ with $m + 1 \leq s \leq n - 1$, we have $V(T) \cap L \neq \{0\}$. \hfill $\square$

For convenience of deriving the next theorem, let us write the set of generators of $D_q(n)$ as $D = \{d_1, d_2, \ldots, d_{n^2}\}$, i.e., $D_q(n) = K[d_1, d_2, \ldots, d_{n^2}]$. Thus, for a subset $U = \{d_{i_1}, d_{i_2}, \ldots, d_{i_r}\} \subset \{d_1, d_2, \ldots, d_{n^2}\}$ with $i_1 < i_2 < \cdots < i_r$, we write

$$T = \left\{d_{i_1}^{\alpha_1}d_{i_2}^{\alpha_2} \cdots d_{i_r}^{\alpha_r} \mid (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r\right\}, \quad V(T) = K\text{-span}T.$$

**Theorem 3.5** With notation as fixed above, Let $L$ be a left ideal of $D_q(n)$. Then $\text{GK.dim}D_q(n)/L \leq n^2$. If furthermore $\text{GK.dim}D_q(n)/L = m < n^2$, then

$$V(T) \cap L \neq \{0\}$$

holds true for every subset $U = \{d_{i_1}, d_{i_2}, \ldots, d_{i_{m+1}}\} \subset D$ with $i_1 < i_2 < \cdots < i_{m+1}$, in particular, for every $U = \{d_1, d_2 \ldots, d_s\}$ with $m + 1 \leq s \leq n - 1$, we have $V(T) \cap L \neq \{0\}$.

**Proof** By Corollary 3.1, $D_q(n)$ has the PBW basis

$$B = \{d^\alpha = d_1^{\alpha_1}d_2^{\alpha_2} \cdots d_2^{\alpha_{n^2}} \mid \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \in \mathbb{N}^{n^2}\}.$$ 

Also by Theorem 3.3(ii), $\text{GK.dim}D_q(n) = n^2$, thereby $\text{GK.dim}D_q(n)/L \leq n^2$. If furthermore $\text{GK.dim}D_q(n)/L = d < n^2$, then the desired elimination property follows from Lemma 3.4 mentioned above. \hfill $\square$
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