Heavy quark masses from finite volume effects*

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I discuss the results of a new calculation of the charm and bottom quark masses in the quenched approximation and in the continuum limit of lattice QCD. The work has been done by the APE group at the “Tor Vergata” University \cite{1} making use of the step scaling method, previously introduced to deal with two scale problems, that allows to take the continuum limit of the lattice data. We have computed the RGI quark masses and then we have connected the results to the MS scheme. The continuum numbers are \( m_{\text{RGI}}^b = 6.73(16) \) GeV for the \( b \)-quark and \( m_{\text{RGI}}^c = 1.68(36) \) GeV for the \( c \)-quark, corresponding respectively to \( m_{\text{MS}}^b (m_{\text{RGI}}^b) = 4.33(10) \) GeV and \( m_{\text{MS}}^c (m_{\text{RGI}}^c) = 1.319(28) \) GeV. The latter result, in agreement with current estimates, is for us a check of the method.

1. Introduction

Quark masses are fundamental parameters of the QCD Lagrangian. Their accurate knowledge is required in order to give quantitative predictions of fundamental processes. A direct experimental measurement of quark masses is not possible because of confinement, and their determination can only be inferred from a theoretical understanding of the hadron phenomenology. The calculations can be numerically performed through different strategies, depending upon the quark flavor and the available computational facilities. Accurate non–perturbative measurements of the \( u, d, s \) and \( c \) quark masses have been obtained from a straightforward comparison of hadron spectroscopy and lattice QCD predictions.

The situation is different for the \( b \) quark, with present computers capabilities. In principle, the \( b \) quark mass could be extracted, similarly to lighter flavors, by looking at the heavy–light and heavy–heavy meson spectrum. However, heavy–light mesons are characterized by the presence of two different scales, i.e. \( \Lambda_{\text{QCD}} \), that sets the wavelengths of the light quark, and the heavy \( b \)-quark mass. Managing these two scales in a naive way would require a very large lattice (\( \mathcal{O}(100^4) \) points).

The task of determining the \( b \)-quark mass has been faced in literature by resorting to some approximations of the full theory, e.g. HQET on the lattice \cite{2}, lattice NRQCD \cite{3}, or QCD sum rules \cite{4,5}. A novel approach recently introduced is based on the non–perturbative renormalization of the static theory and its matching to QCD \cite{6,7}.

An alternative approach to the bottom quark physics, based on finite size scaling, has been proposed in a previous paper \cite{8}, where it has been applied to the heavy–light meson decay constants (see also \cite{9,10}). The main advantages of this \textit{step scaling method} (SSM) are that the entire computation is performed with the relativistic QCD Lagrangian and that the continuum limit can be taken, avoiding the unfeasible direct calculation. In order to implement the SSM, a finite size scheme is required, and we adopt the Schrödinger Functional (SF) as the most useful framework.

2. Step scaling functions and HQET

The SSM has been designed in order to deal with two scale problems in lattice QCD \cite{8} and it has been discussed in detail in \cite{8}. The main assumption of the method is that the finite size effects affecting the meson masses, \( M_{\{P,V\}} \) have a mild dependence upon variations of the heavy
quark mass and are controllable from a numerical point of view. Finite size effects can be obtained from

\[ \sigma_{P}(L, m, m_2) = \frac{M_P(m_1, m_2)|_{2L}}{M_P(m_1, m_2)|_{L}} \]

\[ \sigma_{V}(L, m, m_2) = \frac{M_V(m_1, m_2)|_{2L}}{M_V(m_1, m_2)|_{L}} \]  

(1)

for the pseudoscalar and vector meson masses respectively.

To validate the hypothesis of low sensitivity upon the high energy scale, we can make use of the HQET predictions on the heavy–light meson masses. In the infinite volume the pseudoscalar and vector meson masses have the following expansion in terms of the heavy–quark mass:

\[ M_X(m_h, m_l) = m_h + \bar{\Lambda}(m_l) + \frac{\alpha_X(m_l)}{m_h} + \ldots \]  

(2)

where \( X \in \{P, V\} \). Assuming the contribution of the \( 1/m_h^2 \) corrections to be negligible, at finite volume one has

\[ M_X(m_h, m_l, L) = m_h + \bar{\Lambda}_X(m_l, L) + \frac{\alpha_X(m_l, L)}{m_h} \]  

(3)

where \( \bar{\Lambda}_X(m_l, L) \) depends upon the spin of the meson state because of the contamination of the excited states to the finite volume correlations. Using eqs. 1 and 3 we obtain the HQET predictions for the step scaling functions of the heavy–light meson masses

\[ \sigma_X(L, m_h, m_l) = 1 + \frac{\sigma_X(0)(m_l, L)}{m_h} + \frac{\sigma_X(1)(m_l, L)}{m_h^2} \]  

(4)

This result requires some considerations. First we want to stress that in the infinite heavy–quark mass limit the step scaling functions have to be exactly equal to one, \( \sigma_X(L, m_l, m_h \to \infty) = 1^2 \). This represents a strong constraint for the fits of the heavy–quark mass dependence of the step scaling functions.

The second observation concerns the number of terms to be considered in eq. 4. At order \( O(1/m_h) \) one has

\[ \sigma_X(0)(m_l, L) = \bar{\Lambda}_X(m_l, 2L) - \bar{\Lambda}_X(m_l, L) \]  

(5)

\[ \sigma_X(1)(m_l, L) = \bar{\Lambda}_X(m_l, 2L) - 2\bar{\Lambda}_X(m_l, L) \]

We thank A. Kronfeld and R. Sommer for having pointed out this property of \( \sigma_X \).

corresponding to the static approximation in eq. 5. By increasing the physical volume \( L \), the difference between \( \bar{\Lambda}_X(m_l, 2L) \) and \( \bar{\Lambda}_X(m_l, L) \) decreases because the two quantities have to be equal in the infinite volume limit, making the heavy–quark mass expansion of the finite volume effects rapidly convergent. The same arguments apply to the coefficient \( \sigma_X(1)(m_l, L) \) that has to be considered when in the expansion of the meson masses, eq. 4, the order \( O(1/m_h) \) is taken into account. In our calculation we have performed the fits of the step scaling functions considering the \( O(1/m_h^2) \) term, \( \sigma_X(1) \), beyond the so called static evolution (SE) that retains \( \sigma_X(0) \) only. We will also report for comparison the fits for the SE that are anyway compatible.

![Figure 1](image.png)

Figure 1. The figure shows the continuum extrapolated step scaling functions \( \sigma_{P}(L_0) \) as functions of \( 1/m_l^{RGI} \) with two possible fits. The heavy extrapolations are shown only for the heavy–strange (Hs) set of data.

3. Results and Discussion

The physical numbers for the heavy meson spectrum have been obtained combining the results of a small volume \( (L_0 = 0.4 \text{ fm}) \) calculation, where the simulations have been performed at the physical values of the heavy quark masses, with the results of the step scaling functions according to the identity:

\[ M_X(L_\infty) = M_X(L_0) \sigma_X(L_0) \sigma_X(2L_0) \ldots \]  

(6)

The step scaling functions at the values of the heavy quark masses simulated on the small vol-
ume have been obtained by interpolation in the heavy–light case using eq. (4) while, in the heavy–heavy case, the results are linearly extrapolated. The plots of \( \sigma_P(L_0) \) and \( \sigma_P(2L_0) \) as functions of the inverse quark mass are shown in figs. 1 and 2 respectively.

![Graph](image)

Figure 2. The figure shows the continuum extrapolated step scaling functions \( \sigma_P(2L_0) \) as functions of \( 1/m_\text{RGI} \) with two possible fits. The heavy extrapolations are shown only for the heavy–strange (Hs) set of data.

Comparing the numerical heavy meson spectrum with the experimental results [14] we have obtained different determinations of the \( b \)-quark mass, depending upon the physical state used as experimental input.

Within the quenched approximation, the determinations of the quark masses coming from the heavy–heavy or from the heavy–light spectrum in principle differ because the theory does not account for the fermion loops. The numbers we quote as final results are obtained by averaging the results for the heavy-heavy and the heavy-strange vector and pseudoscalar mesons and by keeping the typical error of a single case:

\[
\begin{align*}
    m_{b}^{RGI} &= 6.73(16) \text{ GeV} \\
    m_{b}^{MS}(m_{b}^{MS}) &= 4.33(10) \text{ GeV} \\
    m_{c}^{RGI} &= 1.681(36) \text{ GeV} \\
    m_{c}^{MS}(m_{c}^{MS}) &= 1.319(28) \text{ GeV}
\end{align*}
\]

for the \( b \)-quark and charm quark, respectively. The latter results compare favorably with the results of the direct computations [5–11].

Our error estimate includes both the statistical error from the Monte Carlo simulation as well as the systematic error coming from the uncertainty on the lattice spacing corresponding at a given \( \beta \) value (we have used \( r_0 = 0.5 \text{ fm} \) [11][13]) and from the uncertainty on the renormalization constants. The final errors on the continuum quantities, of the order of 2\% percent for the renormalization constants and of about 1\% percent for the scale, are added in quadrature and then linearly added to the statistical errors. The evolution to the \( \overline{MS} \) scheme has been done using four-loop renormalization group equations.

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