An essay on some problems of approximation theory  

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Abstract

Several questions of approximation theory are discussed: 1) can one approximate stably in $L^\infty$ norm $f'$ given approximation $f_\delta$, $\|f_\delta - f\|_{L^\infty} < \delta$, of an unknown smooth function $f(x)$, such that $\|f'(x)\|_{L^\infty} \leq m_1$?  
2) can one approximate an arbitrary $f \in L^2(D), D \subset \mathbb{R}^n, n \geq 3$, is a bounded domain, by linear combinations of the products $u_1u_2$, where $u_m \in N(L_m), m = 1, 2, L_m$ is a formal linear partial differential operator and $N(L_m)$ is the null-space of $L_m$ in $D, N(L_m) := \{w : L_mw = 0 \text{ in } D\}$?  
3) can one approximate an arbitrary $L^2(D)$ function by an entire function of exponential type whose Fourier transform has support in an arbitrary small open set? Is there an analytic formula for such an approximation?

1 Introduction

In this essay I describe several problems of approximation theory which I have studied and which are of interest both because of their mathematical significance and because of their importance in applications.

1.1

The first question I have posed around 1966. The question is: suppose that $f(x)$ is a smooth function, say $f \in C^\infty(\mathbb{R})$, which is $T$-periodic (just to avoid a discussion of its behavior near the boundary of an interval), and which is not known; assume that its $\delta$-approximation $f_\delta \in L^\infty(\mathbb{R})$ is known, $\|f_\delta - f\|_{L^\infty} < \delta$, where $\|\cdot\|_{L^\infty}$ is the $L^\infty(\mathbb{R})$ norm. Assume also that $\|f'\|_{L^\infty} \leq m_1 < \infty$. Can one approximate stably in $L^\infty(\mathbb{R})$ the derivative $f'$, given the above data $\{\delta, f_\delta, m_1\}$?

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By a possibility of a stable approximation (estimation) I mean the existence of an operator $L_\delta$, linear or nonlinear, such that

$$
\sup_{f \in C^\infty(\mathbb{R})} \| L_\delta f - f' \|_\infty \leq \eta(\delta) \to 0 \text{ as } \delta \to 0, \tag{1.1}
$$

where $\eta(\delta) > 0$ is some continuous function, $\eta(0) = 0$. Without loss of generality one may assume that $\eta(\delta)$ is monotonically growing.

In 1962-1966 there was growing interest to ill-posed problems. Variational regularization was introduced by D.L. Phillips [2] in 1962 and a year later by A.N. Tikhonov [30] in 1963. It was applied in [1] in 1966 to the problem of stable numerical differentiation. I then proposed and published in 1968 [3] the idea to use a divided difference for stable differentiation and to use the stepsize $h = h(\delta)$ as a regularization parameter. If $\| f'' \| \leq m_2$, then $h(\delta) = \sqrt{\frac{2\delta}{m_2}}$, and if one defines (3):

$$
L_\delta f_\delta := \frac{f_\delta(x + h(\delta)) - f_\delta(x - h(\delta))}{2h(\delta)}, \quad h(\delta) = \sqrt{\frac{2\delta}{m_2}}, \tag{1.2}
$$

then

$$
\| L_\delta f_\delta - f'(x) \|_\infty \leq \sqrt{2m_2\delta} := \varepsilon(\delta). \tag{1.3}
$$

It turns out that the choice of $L_\delta$, made in [3], that is, $L_\delta$ defined in (1.2), is the best possible among all linear and nonlinear operators $T$ which approximate $f'(x)$ given the information $\{\delta, m_2, f_\delta\}$. Namely, if $\mathcal{K}(\delta, m_j) := \{ f : f \in C^j(\mathbb{R}), m_j < \infty, \| f - f_\delta \|_\infty \leq \delta \}$, and $m_j = \| f^{(j)} \|_\infty$, then

$$
\inf_T \sup_{f \in \mathcal{K}(\delta, m_2)} \| T f_\delta - f' \| \geq \sqrt{2m_2\delta}. \tag{1.4}
$$

One can find a proof of this result and more general ones in [4], [5], [8], [9] and various applications of these results in [4] - [7].

The idea of using the stepsize $h$ as a regularization parameter became quite popular after the publication of [3] and was used by many authors later.

In [27] formulas are given for a simultaneous approximation of $f$ and $f'$.

1.2

The second question, that I will discuss, is the following one: can one approximate, with an arbitrary accuracy, an arbitrary function $f(x) \in L^2(D)$, or in $L^p(D)$ with $p \geq 1$, by a linear combination of the products $u_1 u_2$, where $u_m \in N(L_m)$, $m = 1, 2$, $L_m$ is a formal linear partial differential operator, and $N(L_m)$ is the null-space of $L_m$ in $D$, $N(L_m) := \{ w : L_m w = 0 \text{ in } D \}$?
This question has led me to the notion of property $C$ for a pair of linear formal partial differential operators \{$L_1, L_2$\}.

Let us introduce some notations. Let $D \subset \mathbb{R}^n, n \geq 3,$ be a bounded domain, $L_m u(x) := \sum_{|j| \leq J_m} a_{jm}(x) D^j u(x), m = 1, 2,$ $j$ is a multiindex, $J_m \geq 1$ is an integer, $a_{jm}(x)$ are some functions whose smoothness properties we do not specify at the moment, $D^j u = \frac{\partial^j u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, $|j| = j_1 + \cdots + j_n$. Define $N_m := N_m(L_m) := \{w : L_m w = 0 \text{ in } D\},$ where the equation is understood in the sense of distribution theory. Consider the set of products \{\$w_1 w_2\$\}, where $w_m \in N_m$ and we use all the products which are well-defined. If $L_m$ are elliptic operators and $a_{jm}(x) \in C^\gamma(\mathbb{R}^n)$, then by elliptic regularity the functions $w_m \in C^{\gamma + J_m}$ and therefore the products $w_1 w_2$ are well defined.

**Definition 1.1.** A pair \{\$L_1, L_2$\} has property $C$ if and only if the set \{\$w_1 w_2\$\} for all $w_m \in N_m$ is total in $L^p(D)$ for some $p \geq 1$.

In other words, if $f \in L^p(D)$, then

$$\left\{ \int_D f(x) w_1 w_2 dx = 0, \forall w_m \in N_m \right\} \Rightarrow f = 0,$$

(1.5)

where $\forall w_m \in N_m$ means for all $w_m$ for which the products $w_1 w_2$ are well defined.

**Definition 1.2.** If the pair \{\$L, L$\} has property $C$ then we say that the operator $L$ has this property.

From the point of view of approximation theory property $C$ means that any function $f \in L^p(D)$ can be approximated arbitrarily well in $L^p(D)$ norm by a linear combination of the set of products $w_1 w_2$ of the elements of the null-spaces $N_m$.

For example, if $L = \nabla^2$ then $N(\nabla^2)$ is the set of harmonic functions, and the Laplacian has property $C$ if the set of products $h_1 h_2$ of harmonic functions is total (complete) in $L^p(D)$.

The notion of property $C$ has been introduced in [10]. It was developed and widely used in [10] - [21]. It proved to be a very powerful tool for a study of inverse problems [15] - [18], [20] - [21].

Using property $C$ the author has proved in 1987 the uniqueness theorem for 3D inverse scattering problem with fixed-energy data [12], [13], [17], uniqueness theorems for inverse problems of geophysics [12], [16], [18], and for many other inverse problems [18]. The above problems have been open for several decades.

1.3

The third question that I will discuss, deals with approximation by entire functions of exponential type. This question is quite simple but the answer was not clear to engineers in the fifties, it helped to understand the problem of resolution ability of linear instruments [22], [23], and later it turned to be useful in tomography [25]. This question in applications is known as spectral extrapolation.
To formulate it, let us assume that $D \subset \mathbb{R}^n$ is a known bounded domain,
\[ \tilde{f}(\xi) := \int_D f(x)e^{i\xi \cdot x}dx := \mathcal{F}f, \quad f(x) \in L^2(D), \tag{1.6} \]
and assume that $\tilde{f}(\xi)$ is known for $\xi \in \tilde{D}$, where $\tilde{D}$ is a domain in $\mathbb{R}^n$. The question is:
\begin{center}
\text{can one recover } f(x) \text{ from the knowledge of } \tilde{f}(\xi) \text{ in } \tilde{D}?
\end{center}

Uniqueness of $f(x)$ with the data $\{\tilde{f}(\xi), \xi \in \tilde{D}\}$ is immediate: $\tilde{f}(\xi)$ is an entire function of exponential type and if $\tilde{f}(\xi) = 0$ in $\tilde{D}$, then, by the analytic continuation, $\tilde{f}(\xi) \equiv 0$, and therefore $f(x) = 0$. Is it possible to derive an analytic formula for the recovery of $f(x)$ from $\{\tilde{f}(\xi), \xi \in \tilde{D}\}$? It turns out that the answer is yes (24 - 26). Thus we give an analytic formula for inversion of the Fourier transform $\tilde{f}(\xi)$ of a compactly supported function $f(x)$ from a compact set $\tilde{D}$.

From the point of view of approximation theory this problem is closely related to the problem of approximation of a given function $h(\xi)$ by entire functions of exponential type whose Fourier transform has support inside a given convex region. This region is fixed but can be arbitrarily small.

In sections 2, 3 and 4 the above three questions of approximation theory are discussed in more detail, some of the results are formulated and some of them are proved.

2 Stable approximation of the derivative from noisy data.

In this section we formulate an answer to question 1.1. Denote $\|f^{(1+a)}\| := m_{1+a}$, where $0 < a \leq 1$, and
\[ \|f^{(1+a)}\| = \|f'\|_{\infty} + \sup_{x,y \in \mathbb{R}} \frac{|f'(x) - f'(y)|}{|x - y|^a}. \tag{2.1} \]

**Theorem 2.1.** There does not exist an operator $T$ such that
\[ \sup_{f \in K(\delta, m_j)} \|Tf_\delta - f'\|_{\infty} \leq \eta(\delta) \to 0 \text{ as } \delta \to 0, \tag{2.2} \]
if $j = 0$ or $j = 1$. There exists such an operator if $j > 1$. For example, one can take $T = L_{\delta,j}$ where
\[ L_{\delta,j}f_\delta := \frac{f_\delta(x + h_j(\delta)) - f_\delta(x - h_j(\delta))}{2h_j(\delta)}, \quad h_j(\delta) := \left( \frac{\delta}{m_j(j - 1)} \right)^{\frac{1}{j}}, \tag{2.3} \]
and then
\[ \sup_{f \in K(\delta, m_j)} \|L_{\delta,j}f_\delta - f'\|_{\infty} \leq c_j \delta^{\frac{j - 1}{j}}, \quad 1 < j \leq 2, \tag{2.4} \]
\[ c_j := \left( \frac{j}{(j - 1)^{\frac{j - 1}{j}}} m_j^j \right). \tag{2.5} \]
Proof. 1. Nonexistence of $T$ for $j = 0$ and $j = 1$.

Let $f_\delta(x) = 0, f_1(x) := -\frac{m(x-2h)}{2}, 0 \leq x \leq 2h$. Extend $f_1(x)$ on $\mathbb{R}$ so that $\|f_1^{(j)}\|_\infty = \sup_{0 \leq x \leq 2h} \|f_1^{(j)}\|, j = 0, 1, 2$, and set $f_2(x) := -f_1(x)$. Denote $(Tf_\delta)(0) := b$.

One has

$$\|Tf_\delta - f_1'\| \geq \|(Tf_\delta)(0) - f_1'(0)\| = |b - mh|,$$

and

$$\|Tf_\delta - f_2'\| \geq |b + mh|.$$  \hfill (2.6)

Thus, for $j = 0$ and $j = 1$, one has:

$$\gamma_j := \inf_T \sup_{f \in K(\delta,m_j)} \|Tf - f'\| \geq \inf_b \max \|b - mh\|, |b + mh\| = mh. \hfill (2.8)$$

Since $\|f_s - f_\delta\|_\infty \leq \delta, s = 1, 2$, and $f_\delta = 0$, one gets

$$\|f_s\|_\infty = \frac{m h^2}{2} \leq \delta, s = 1, 2.$$  \hfill (2.9)

Take

$$h = \sqrt{\frac{2\delta}{m}}.$$  \hfill (2.10)

Then $m_0 = \|f_s\|_\infty = \delta, m_1 = \|f_s'\|_\infty = \sqrt{2\delta m}$, so (2.8) yields

$$\gamma_0 = \sqrt{2\delta m} \to \infty \text{ as } m \to \infty,$$  \hfill (2.11)

and (2.2) does not hold if $j = 0$.

If $j = 1$, then (2.8) yields

$$\gamma_1 = \sqrt{2\delta m} = m_1 > 0,$$  \hfill (2.12)

and, again, (2.2) does not hold if $j = 1$.

2. Existence of $T$ for $j > 1$.

If $j > 1$, then the operator $T = L_{\delta,j}$, defined in (2.3) yields estimate (2.4), so (2.2) holds with $\eta(\delta) = c_j \frac{\delta^{j-1}}{h^{j-1}}$ and $c_j$ is defined in (2.5). Indeed,

$$\|L_{\delta,j} f_\delta - f'\|_\infty \leq \|L_{\delta,j}(f_\delta - f)\|_\infty + \|L_{\delta,j} f - f'\|_\infty \leq \frac{\delta}{h} + m_j h^{j-1}.$$  \hfill (2.13)

Minimizing the right-hand side of (2.13) with respect to $h > 0$ for a fixed $\delta > 0$, one gets (2.4) and (2.5).

Theorem 2.1 is proved. \hfill $\square$

Remark 2.1. If $j = 2$, then $m_2 = m$, where $m$ is the number introduced in the beginning of the proof of Theorem 2.1, and using the Taylor formula one can get a better estimate in the right-hand side of (2.13) for $j = 2$, namely $\|L_{\delta,2}f_\delta - f'\|_\infty \leq \frac{\delta}{h} + \frac{m h}{2}$. Minimizing with respect to $h$, one gets $h(\delta) = \sqrt{\frac{2\delta}{m_2}}$ and $\min_{h > 0} \left( \frac{\delta}{h} + \frac{m h}{2} \right) := \varepsilon(\delta) = \sqrt{2\delta m_2}$. Now (2.8), with $m = m_2$, yields

$$\gamma_2 \geq \sqrt{2\delta m_2} = \varepsilon(\delta).$$  \hfill (2.14)
Thus, we have obtained:

**Corollary 2.1.** Among all linear and nonlinear operators $T$, the operator $Tf = L_\delta f := \frac{f(x+h(\delta))-f(x-h(\delta))}{2h(\delta)}$, $h(\delta) = \sqrt{\frac{2\delta}{m_2}}$, yields the best approximation of $f'$, $f \in K(\delta,m_2)$, and

$$\gamma_2 := \inf_T \sup_{f \in K(\delta,m_2)} \|Tf - f'\|_{\infty} = \varepsilon(\delta) := \sqrt{2\delta m_2}. \quad (2.15)$$

**Proof.** We have proved that $\gamma_2 \geq \varepsilon(\delta)$. If $T = L_\delta$ then $\|L_\delta f_\delta - f'\|_{\infty} \leq \varepsilon(\delta)$, as follows from the Taylor’s formula: if $h = \sqrt{\frac{2\delta}{m_2}}$, then

$$\|L_\delta f_\delta - f'\| \leq \frac{\delta}{h} + \frac{m_2 h^2}{2} = \varepsilon(\delta), \quad (2.16)$$

so $\gamma_2 = \varepsilon(\delta)$.

\[\square\]

**3 Property $C$**

**3.1**

In the introduction we have defined property $C$ for PDE. Is this property generic or is it an exceptional one?

Let us show that this property is generic: a linear formal partial differential operator with constant coefficients, in general, has property $C$. In particular, the operators $L = \nabla^2$, $L = i\partial_t - \nabla^2$, $L = \partial_t - \nabla^2$, and $L = \partial_t^2 - \nabla^2$, all have property $C$.

A necessary and sufficient condition for a pair \{L_1, L_2\} of partial differential operators to have property $C$ was found in [10] and [28] (see also [18]).

Let us formulate this condition and use it to check that the four operators, mentioned above, have property $C$.

Let $L_m u := \sum_{|j| \leq J_m} a_{jm} D^j u(x)$, $m = 1, 2$, $a_{jm} = \text{const}$, $x \in \mathbb{R}^n$, $n \geq 2$, $J_m \geq 1$. Define the algebraic varieties

$$L_m := \{z : z \in \mathbb{C}^n, L_m(z) := \sum_{|j| \leq J_m} a_{jm} z^j = 0\}, \quad m = 1, 2.$$

**Definition 3.1.** Let us write $L_1 \parallel L_2$ if and only if there exist at least one point $z^{(1)} \in L_1$ and at least one point $z^{(2)} \in L_2$, such that the tangent spaces in $\mathbb{C}^n T_m$ to $L_m$ at the points $z^{(m)}$, $m = 1, 2$, are transversal, that is $T_1 \parallel T_2$.

**Remark 3.2.** A pair \{L_1, L_2\} fails to have the property $L_1 \parallel L_2$ if and only if $L_1 \cup L_2$ is a union of parallel hyperplanes in $\mathbb{C}^n$.

**Theorem 3.3.** A pair \{L_1, L_2\} of formal linear partial differential operators with constant coefficients has property $C$ if and only if $L_1 \parallel L_2$.
Example 3.4 Let \( L = \nabla^2 \), then \( \mathcal{L} = \{ z : z \in \mathbb{C}^n, z_1^2 + z_2^2 + \ldots + z_n^2 = 0 \} \). It is clear that the tangent spaces to \( \mathcal{L} \) at the points \((1,0,0)\) and \((0,1,0)\) are transversal. So the Laplacian \( L = \nabla^2 \) does have property \( C \) (see Definition 1.2 in section 1). In other words, given an arbitrary bounded domain \( D \subset \mathbb{R}^n \) and an arbitrary function \( f(x) \in L^p(D), \rho \geq 1 \), for example, \( p = 2 \), one can approximate \( f(x) \) in the norm of \( L^p(D) \) by linear combinations of the product \( h_1 h_2 \) of harmonic in \( L^2(D) \) functions.

Example 3.5 One can check similarly that the operators \( \partial_t - \nabla^2, i \partial_t - \nabla^2 \) and \( \partial_t^2 - \nabla^2 \) have property \( C \).

**Proof of Theorem 3.3** We prove only the sufficiency and refer to [18] for the necessity. Assume that \( \mathcal{L}_1 \| \mathcal{L}_2 \). Note that \( e^{x \cdot z} \in \mathcal{N}(L_m) := N_m \) if and only if \( L_m(z) = 0, z \in \mathbb{C}^n \), that is \( z \in \mathcal{L}_m \). Suppose

\[
\int_D f(x)w_1 w_2 dx = 0 \quad \forall w_m \in N_m,
\]

then

\[
F(z_1 + z_2) := \int_D f(x)e^{x \cdot (z_1 + z_2)} dx = 0 \quad \forall z_m \in \mathcal{L}_m.
\]

The function \( F(z) \), defined in (3.1), is entire. It vanishes identically if it vanishes on an open set in \( \mathbb{C}^n \) (or \( \mathbb{R}^n \)). The set \( \{ z_1 + z_2 \}_{z_m \in N_m} \) contains a ball in \( \mathbb{C}^n \) if (and only if) \( \mathcal{L}_1 \| \mathcal{L}_2 \). Indeed, if \( z_1 \) runs though the set \( \mathcal{L}_1 \cap B(z^{(1)}, r) \), where \( B(z^{(1)}, r) := \{ z : z \in \mathbb{C}^n, |z^{(1)} - z| < r \} \), and \( z_2 \) runs through the set \( \mathcal{L}_2 \cap B(z^{(2)}, r) \), then, for all sufficiently small \( r > 0 \), the set \( \{ z_1 + z_2 \} \) contains a small ball \( B(\zeta, \rho) \), where \( \zeta = z^{(1)} + z^{(2)} \), and \( \rho > 0 \) is a sufficiently small number. To see this, note that \( T_1 \) has a basis \( h_1, \ldots, h_{n-1} \), which contains \( n - 1 \) linearly independent vectors of \( \mathbb{C}^n \), and \( T_2 \) has a basis such that at least one of its vectors, call it \( h_n \), has a non-zero projection onto the normal to \( T_1 \), so that \( \{ h_1, \ldots, h_n \} \) are \( n \) linearly independent vectors in \( \mathbb{C}^n \). Their linear combinations fill in a ball \( B(\zeta, \rho) \).

Since the vectors in \( T_m \) approximate well the vectors in \( \mathcal{L}_m \cap B(z^{(m)}, r) \) if \( r > 0 \) is sufficiently small, the set \( \{ z_1 + z_2 \}_{z_m \in \mathcal{L}_m} \) contains a ball \( B(\zeta, \rho) \) if \( \rho > 0 \) is sufficiently small. Therefore, condition (3.1) implies \( F(z) \equiv 0 \), and so \( f(x) = 0 \). This means that the pair \( \{ L_1, L_2 \} \) has property \( C \). We have proved: \( \{ \mathcal{L}_1 \| \mathcal{L}_2 \} \Rightarrow \{ \{ L_1, L_2 \} \) has property \( C \} \).

\[\square\]

How does one prove property \( C \) for a pair \( \{ L_1, L_2 \} := \{ \nabla^2 + k^2 - q_1(x), \nabla^2 + k^2 - q_2(x) \} \) of the Schrödinger operators, where \( k = \text{const} \geq 0 \) and \( q_m(x) \in L^2_{\text{loc}}(\mathbb{R}^n) \) are some real-valued, compactly supported functions?

One way to do it [18] is to use the existence of the elements \( \psi_m \in N_m := \{ w : [\nabla^2 + k^2 - q_m(x)]w = 0 \text{ in } \mathbb{R}^n \} \) which are of the form

\[
\psi_m(x, \theta) = e^{ik\theta \cdot x}[1 + R_m(x, \theta)], \quad k > 0,
\]

where \( \theta \in M := \{ \theta : \theta \in \mathbb{C}^n, \theta \cdot \theta = 1 \} \). Here \( \theta \cdot w := \sum_{j=1}^n \theta_j w_j \), (note that there is no complex conjugation above \( w_j \)), the variety \( M \) is noncompact, and [18]

\[
\| R_m(x, \theta) \|_{L^\infty(D)} \leq c \left( \frac{\ln |\theta|}{|\theta|} \right)^{\frac{1}{2}}, \quad \theta \in M, \quad |\theta| \to \infty, \quad m = 1, 2,
\]

(3.3)
where \( c = \text{const} > 0 \) does not depend on \( \theta \), \( c \) depends on \( D \) and on \( \|q\|_{L^\infty(B_a)} \), where \( q = \overline{q}, q = 0 \) for \( |x| > a \), and \( D \subset \mathbb{R}^n \) is an arbitrary bounded domain. Also

\[
\|R_m(x, \theta)\|_{L^2(D)} \leq \frac{c}{|\theta|}, \quad \theta \in M, \quad |\theta| \to \infty, \quad m = 1, 2. \tag{3.4}
\]

It is easy to check that for any \( \xi \in \mathbb{R}^n, n \geq 3 \), and any \( k > 0 \), one can find (many) \( \theta_1 \) and \( \theta_2 \) such that

\[
k(\theta_1 + \theta_2) = \xi, \quad |\theta_1| \to \infty, \quad \theta_1, \theta_2 \in M, \quad n \geq 3. \tag{3.5}
\]

Therefore, using (3.5) and (3.3), one gets:

\[
\lim_{|\theta_1| \to \infty} \psi_1 \psi_2 = e^{i\xi \cdot x}, \tag{3.6}
\]

Since the set \( \{e^{i\xi \cdot x}\}_{\xi \in \mathbb{R}^n} \) is total in \( L^p(D) \), it follows that the pair \( \{L_1, L_2\} \) of the Schrödinger operators under the above assumptions does have property \( C \).

### 3.2

Consider the following problem of approximation theory [21].

Let \( k = 1 \) (without loss of generality), \( \alpha \in S^2 \) (the unit sphere in \( \mathbb{R}^3 \)), and \( u := u(x, \alpha) \) be the scattering solution that is, an element of \( N(\nabla^2 + 1 - q(x)) \) which solves the problem:

\[
\left[ \nabla^2 + 1 - q(x) \right] u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{3.5}
\]

\[
u = \exp(i\alpha \cdot x) + A(\alpha', \alpha) \frac{e^{|x|}}{|x|} + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \quad \alpha' := \frac{x}{|x|}, \tag{3.6}
\]

where \( \alpha \in S^2 \) is given.

Let \( w \in N(\nabla^2 + 1 - q(x)) := N(L) \) be arbitrary, \( w \in H^2_{loc}, H^1 \) is the Sobolev space. The problem is: is it possible to approximate \( w \) in \( L^2(D) \) with an arbitrary accuracy by a linear combination of the scattering solutions \( u(x, \alpha) \)? In other words, given an arbitrary small number \( \varepsilon > 0 \) and an arbitrary fixed, bounded, homeomorphic to a ball, Lipschitz domain \( D \subset \mathbb{R}^n \), can one find \( v_\varepsilon(\alpha) \in L^2(S^2) \) such that

\[
\|w - \int_{S^2} u(x, \alpha)v_\varepsilon(\alpha)d\alpha\|_{L^2(D)} \leq \varepsilon? \tag{3.7}
\]

If yes, what is the behavior of \( \|v_\varepsilon(\alpha)\|_{L^2(S^2)} \) as \( \varepsilon \to 0 \), if \( w = \psi(x, \theta) \) where \( \psi \) is the special solution (3.2) - (3.3), \( \theta \in M, \text{Im} \theta \neq 0 \)?

The answer to the first question is yes. A proof [18] can go as follows. If (3.7) is false, then one may assume that \( w \in N(L) \) is such that

\[
\int_D \overline{w} \left( \int_{S^2} u(x, \alpha)v(\alpha)d\alpha \right) dx = 0 \quad \forall v \in L^2(S^2), \tag{3.8}
\]
This implies
\[ \int_D w u(x, \alpha) \, dx = 0 \quad \forall \alpha \in S^2. \tag{3.9} \]
From (3.9) and formula (5) on p. 46 in [IS] one concludes:
\[ v(y) := \int_D w G(x, y) \, dx = 0 \quad \forall y \in D' := \mathbb{R}^3 \setminus D, \tag{3.10} \]
where \( G(x, y) \) is the Green function of \( L \):
\[ \left[ \nabla^2 + 1 - q(x) \right] G(x, y) = -\delta(x, y) \quad \text{in} \ \mathbb{R}^3, \tag{3.11} \]
\[ \lim_{r \to \infty} \int_{|x| = r} \left| \frac{\partial G}{\partial |x|} - iG \right|^2 \, ds = 0. \tag{3.12} \]
Note, that in [IS, p. 46] formula (5) is:
\[ G(x, y, k) = \frac{e^{i|y|}}{4\pi|y|} u(x, \alpha) + o \left( \frac{1}{|y|} \right), \quad |y| \to \infty, \quad \frac{y}{|y|} = -\alpha. \tag{3.13} \]
From (3.10) and (3.11) it follows that
\[ \left[ \nabla^2 + 1 - q(x) \right] v(x) = -\overline{w}(x) \quad \text{in} \ D, \tag{3.14} \]
\[ v = v_N = 0 \quad \text{on} \ S := \partial D, \tag{3.15} \]
where \( N \) is the outer unit normal to \( S \), and (3.15) holds because \( v = 0 \) in \( D' \) and \( v \in H^2_{\text{loc}} \) by elliptic regularity if \( q \in L^2_{\text{loc}} \). Multiply (3.14) by \( w \), integrate over \( D \) and then, on the left, by parts, using (3.15), and get:
\[ \int_D |w|^2 \, dx = 0. \tag{3.16} \]
Thus, \( w = 0 \) and (3.7) is proved. \( \square \)

Let us prove that \( \|v_\varepsilon(\alpha)\|_{L^2(S^2)} \to \infty \) as \( \varepsilon \to 0 \). Assuming \( \|v_\varepsilon(\alpha)\| \leq c \quad \forall \varepsilon \in (0, \varepsilon_0) \), where \( \varepsilon_0 > 0 \) is some number, one can select a weakly convergent in \( L^2(S^2) \) sequence \( v_n(\alpha) \to v(\alpha) \), \( n \to \infty \), and pass to the limit in (3.7) with \( w = \psi(x, \theta) \), to get
\[ \|\psi(x, \theta) - \int_{S^2} u(x, \alpha) \nu(\alpha) \, d\alpha\|_{L^2(D)} = 0. \tag{3.17} \]
The function \( U(x) := \int_{S^2} u(x, \alpha) \nu(\alpha) \, d\alpha \in N(L) \) and \( \|U(x)\|_{L^\infty(\mathbb{R}^n)} < \infty \) (since \( \sup_{x \in \mathbb{R}^n, \alpha \in S^2} |u(x, \alpha)| < \infty \)). Since (3.17) implies that \( \psi(x, \theta) = U(x) \) in \( D \), and both \( \psi(x, \theta) \) and \( U(x) \) solve equation (3.5), the unique continuation principle for the solutions of the elliptic equation (3.5) implies \( U(x) = \psi(x, \theta) \) in \( \mathbb{R}^3 \). This is a contradiction since \( \psi(x, \theta) \) grows exponentially as \( |x| \to \infty \) in certain directions because \( \Im \theta \neq 0 \) (see formula (3.2)).
We have proved that if \( w = \psi(x, \theta), \) \( Im\theta \neq 0, \) then \( \|\nu_\epsilon(\alpha)\|_{L^2(S^2)} \to \infty \) as \( \epsilon \to 0, \) where \( \nu_\epsilon(\alpha) \) is the function from (3.7).

For example, if \( q(x) = 0 \) and \( k = 1 \) then \( \psi(x, \theta) = e^{i\theta \cdot x} \) and \( u(x, \alpha) = e^{i\alpha \cdot x}. \) So, if \( Im\theta \neq 0, \) \( \theta \in M, \) and \( \|e^{i\theta \cdot x} - \int_{S^2} \nu_\epsilon(\alpha)e^{i\alpha \cdot x}d\alpha\|_{L^2(D)} < \epsilon, \) then \( \|\nu_\epsilon(\alpha)\|_{L^2(S^2)} \to \infty \) as \( \epsilon \to 0. \)

It is interesting to estimate the rate of growth of

\[
\inf_{\nu \in L^2(S^2)} \frac{\|\nu(\alpha)\|_{L^2(S^2)}}{\|e^{i\theta \cdot x} - \int_{S^2} \nu(\alpha)e^{i\alpha \cdot x}d\alpha\|_{L^2(D)} < \epsilon}
\]

This is done in [21] (and [29]).

Property \( C \) for ordinary differential equations is defined, proved and applied to many inverse problems in [19].

4 Approximation by entire functions of exponential type.

Let \( f(x) \in L^2(B_a), \) \( B_a := \{x; x \in \mathbb{R}^n, |x| \leq a\}, a > 0 \) is a fixed number, \( f(x) = 0 \) for \( |x| > a, \) and

\[
\tilde{f}(\xi) = \int_{B_a} f(x)e^{i\xi \cdot x}dx.
\]  (4.1)

Assume that \( \tilde{f}(\xi) \) is known for all \( \xi \in \tilde{D} \subset \mathbb{R}^n, \) where \( \tilde{D} \) is a (bounded) domain.

The problem is to find \( \tilde{f}(\xi) \) for all \( \xi \in \mathbb{R}^n, \) (this is called spectral extrapolation), or, equivalently, to find \( f(x) \) (this is called inversion of the Fourier transform \( \tilde{f}(\xi) \) of a compactly supported function \( f(x), \) \( \text{supp} \ f(x) \subset B_a, \)) from a compact \( \tilde{D}. \)

In applications the above problem is also of interest in the case when \( \tilde{D} \) is not necessarily bounded. For example, in tomography \( \tilde{D} \) may be a union of two infinite cones (the limited-angle data).

In the fifties and sixties there was an extensive discussion in the literature concerning the resolution ability of linear instruments. According to the theory of the formation of optical images, the image of a bright point, which one obtains when the light, issued by this point, is diffracted on a circular hole in a plane screen, is the Fourier transform \( \tilde{f}(\xi) \) of the function \( f(x) \) describing the light distribution on the circular hole \( B_a, \) the two-dimensional ball. This Fourier transform is an entire function of exponential type. One says that the resolution ability of a linear instrument (system) can be increased without a limit if the Fourier transform of the function \( f(x), \) describing the light distribution on \( B_a, \) can approximate the delta-function \( \delta(\xi) \) with an arbitrary accuracy. The above definition is not very precise, but it is usual in applications and can be made precise: it is sufficient to specify the metric in which the delta-function is approximated. For our purposes, let us take a delta-type sequence \( \delta_j(\xi) \) of continuous functions which is defined
by the requirements: $\int_{\hat{D}} \delta_j(\xi) d\xi \leq c$, where the constant $c$ does not depend on $\hat{D}$ and $j$, and

$$\lim_{j \to \infty} \int_{\hat{D}} \delta_j(\xi) d\xi = \begin{cases} 1 & \text{if } 0 \in \hat{D}, \\ 0 & \text{if } 0 \notin \hat{D}. \end{cases}$$

The approximation problem is: can one approximate an arbitrary continuous (or $L^1(\hat{D})$) function $g(\xi)$ in an arbitrary fixed bounded domain $\hat{D} \subset \mathbb{R}^n$ by an entire function of exponential type $\tilde{f}(\xi) = \int_{B_a} f(x) e^{i\xi \cdot x} dx$, where $a > 0$ is an arbitrary small number?

The engineers discussed this question in a different form: can one transmit with an arbitrary accuracy a high-frequency signal $g(\xi)$ by using low-frequency signals $\tilde{f}(\xi)$? The smallness of $a$ means that the "spectrum" $f(x)$ of the signal $\tilde{f}(\xi)$ contains only "low spatial frequencies".

From the mathematical point of view the answer is nearly obvious: yes. The proof is very simple: if an approximation with an arbitrary accuracy were impossible, then

$$0 = \int_{\hat{D}} g(\xi) \left( \int_{B_a} e^{i\xi \cdot x} f(x) dx \right) d\xi \quad \forall f \in L^2(B_a).$$

This implies the relation

$$0 = \int_{\hat{D}} g(\xi) e^{i\xi \cdot x} d\xi \quad \forall x \in B_a.$$
This conclusion contradicts the usual intuitive idea according to which one cannot resolve details smaller than the wavelength.

In fact, if there is no noise, one can, in principle, increase resolution ability without a limit (superdirectivity in the antenna theory), but since the noise is always present, in practice there is a limit to the possible increase of the resolution ability.

Let us turn to the analytic formula for the approximation by entire functions and for the inversion of the Fourier transform of a compactly supported function from a compact \( \tilde{D} \).

Multiply (4.1) by \((2\pi)^{-n} \tilde{\delta}_j(\xi)e^{-i\xi \cdot x}\) and integrate over \( \tilde{D} \) to get

\[
f_j(x) = \int_{B_a} f(y)\delta_j(x-y)dy = \frac{1}{(2\pi)^n} \int_{\tilde{D}} \tilde{f}(\xi)\tilde{\delta}_je^{-i\xi \cdot x}d\xi
\]

(4.2)

where \( \tilde{\delta}_j(\xi) \) is the Fourier transform of \( \delta_j(x) \).

Let us choose \( \delta_j(x) \) so that it will be a delta-type sequence (in the sense defined above). In this case \( f_j(x) \) approximates \( f(x) \) arbitrarily accurately:

\[
\|f - f_j(x)\| \to 0 \text{ as } j \to \infty,
\]

(4.3)

where the norm \( \| \cdot \| \) is \( L^2(B_a) \) norm if \( f \in L^2(B_a) \), and \( C(B_a) \)-norm if \( f \in C(B_a) \).

If \( \|f\|_{C^1(B_a)} \leq m_1 \), then

\[
\|f - f_j(x)\|_{C(B_a)} \leq cm_1j^{-\frac{1}{2}}.
\]

(4.4)

The conclusions (4.3) and (4.4) hold if, for example,

\[
\delta_j(x) := P_j(|x|^2) \left( \mathcal{F}^{-1}h \right)(x),
\]

(4.5)

where

\[
P_j(r) := \left( \frac{j}{4\pi a_1^2} \right)^{\frac{n}{2}} \left( 1 - \frac{r}{4a_1^2} \right)^j, \quad 0 \leq r \leq a, \quad a_1 > a,
\]

(4.6)

\[
\tilde{h}(\xi) \in C_0^\infty(\tilde{D}), \quad \frac{1}{(2\pi)^n} \int_{\tilde{D}} \tilde{h}(\xi)d\xi = 1.
\]

(4.7)

**Theorem 4.1.** If (4.5)-(4.7) hold, then the sequence \( f_j(x) \), defined in (4.2), satisfies (4.3) and (4.4).

Thus, formula (4.2):

\[
f(x) = \mathcal{F}^{-1} \left[ \tilde{f}(\xi)\tilde{\delta}_j(\xi) \right],
\]

(4.8)

where \( \tilde{\delta}_j(\xi) := \mathcal{F}\delta_j(x) \), and \( \delta_j(x) \) is defined by formulas (4.5)-(4.7), is an inversion formula for the Fourier transform \( \tilde{f}(\xi) \) of a compactly supported function \( f(x) \) from a compact \( \tilde{D} \) in the sense (4.3). A proof of a Theorem similar to 4.1 has been originally published in [24].

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In [22], [23] apodization theory and resolution ability are discussed. In [26] a one-dimensional analog of Theorem 4.1 is given. In this analog one can choose analytically explicitly a function similar to $\tilde{h}(\xi)$.

Proof of Theorem 4.1 is given in [25], pp. 260-263.

Let us explain a possible application of Theorem 4.1 to the limited-angle data in tomography. The problem is: let $\hat{f}(\alpha, p) := \int_{l_{\alpha,p}} f(x) ds$, where $l_{\alpha,p}$ is the Radon transform of $f(x)$. Given $\hat{f}(\alpha, p)$ for all $p \in \mathbb{R}$ and all $\alpha \in K$, where $K$ is an open proper subset of $S^2$, find $f(x)$, assuming $\text{supp} f \subset B_a$. It is well known [25], that

$$\int_{-\infty}^{\infty} \hat{f}(\alpha, p)e^{ipt} dp = \tilde{f}(t\alpha), \quad t \in \mathbb{R}, \quad t\alpha := \xi.$$

Therefore, if one knows $\hat{f}(\alpha, p)$ for all $\alpha \in K$ and all $p \in \mathbb{R}$, then one knows $\tilde{f}(\xi)$ for all $\xi$ in a cone $K \times \mathbb{R}$. Now Theorem 4.1 is applicable for finding $f(x)$ given $\hat{f}(\alpha, p)$ for $\alpha \in K$ and $p \in \mathbb{R}$.

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