From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems

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We introduce a covariant Multipole Expansion for the scattering of a massive particle emitting photons or gravitons in $D$ dimensions. We find that these amplitudes exhibit very powerful features such as universality, soft exponentiation, orbit and spin multipoles, etc. Using SO($D$) representation theory we show that the photon and graviton amplitudes are related via a new double copy procedure for massive spinning states. All these features are then promoted to properties of the observables arising in the classical version of such theories. Focusing on radiation, we provide two main applications: 1) An exponential Soft Theorem relating conservative effects and gravitational radiation to all orders in $\omega$; whose leading order directly leads to the $D=4$ Memory Effect. 2) A classical double copy to evaluate gravitational radiation from QED Bremsstrahlung, matching previous classical computations and extending them to spin-quadrupole order.

With the advent of QFT it was observed that dynamics of massive objects subject to long-range forces could be described from the classical limit of Scattering Amplitudes [1–8]. This picture has seen renewed interest with the aim of providing more accurate templates for Gravitational Wave (GW) events, leading to remarkable Post-Minkowskian (PM) results [9–18]. Two key ingredients in this endeavour are the following amplitudes,

$$M_4 = \begin{array}{ccc}
\left< a \right. & \left| \begin{array}{c}
abla \nabla \nabla \nabla \\
q & p_1 & p_2 & p_3 & k
\end{array} \right| & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\end{array} \\
M_5 = \begin{array}{ccc}
\left< a \right. & \left| \begin{array}{c}
\nabla \nabla \nabla \\
q & p_1 & p_2 & p_3 & k
\end{array} \right| & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\end{array} \
$$

which are associated to conservative and non-conservative effects [19, 20]. The bodies $a$ and $b$ carrying internal structure are here understood as point particles with spin, which is especially relevant in the quest for better templates [21]. While $M_4$ has been studied to high PM orders, $M_5$ is much more complicated. It has only recently been introduced in this context by O’Connell et al. in the spinless case [20, 22]. Even though these objects control fundamental effects such as the Coulombian/Newtonian potentials, both $M_4$ and $M_5$ strongly depend on the matter content even if no contact interactions are allowed. We will argue that the reason for their classical piece, $\langle M_n \rangle : = \lim_{\hbar \to 0} M_n$, to be universal is that it is precisely identified with their decomposition into fundamental amplitudes. The main example we provide is that, at LO in the coupling,

$$\langle M_4 \rangle = \begin{array}{ccc}
\left< a \right. & \left| \begin{array}{c}
\nabla \nabla \nabla \nabla \\
q & p_1 & p_2 & p_3 & k
\end{array} \right| & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\end{array} \\
\langle M_5 \rangle = \begin{array}{ccc}
\left< a \right. & \left| \begin{array}{c}
\nabla \nabla \nabla \\
q & p_1 & p_2 & p_3 & k
\end{array} \right| & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\left. \right> a & \left< b \right. \\
\end{array} \
$$

(2)

In this work we denote by $A_{h.n}^\phi$ the transition amplitudes of a massive spin-$s$ state emitting $n-2$ massless particles. The case $h = 1$, i.e. photon emission, has a long history in QED, see for instance [23, 24]. We start by reconsidering these objects in light of recent developments and unveil several new features. As an introductory example, one can study the soft expansion and double copy of $A_3^\phi$ and $A_4^\phi$ for a scalar source. It was shown in [25] via direct computation that the double copy is realized in a massive version of the KLT formula [26, 27]:

$$A_{n}^{\text{ph},0} \times A_{n}^{\text{ph},0} = K_{n}A_{n}^{\text{gr},0}, \quad n = 3, 4. \quad (3)$$

with $K_3=1$ and $K_4=\frac{1}{2^{1/2}}\frac{k_1 k_2}{p_1 p_2 + k_1 k_2}$, where $p_1 + k_1 = p_2 + k_2$ and $k_i$ is massless. While $A_3$ corresponds to a classical on-shell current and can be used to evaluate conservative effects, it is not enough for the computation of radiative effects even at LO in the coupling [29, 30]. This case can be understood from the fact that it does not posses orbit multipoles, in contrast with $A_4$. Let us define orbit multipoles as each of the terms appearing in the soft-expansion of $A_n$ with respect to an external photon/graviton. Such expansion is trivial for $A_3$. For $A_4$, it truncates at subleading order for photons [31, 32]. It follows from (3) that it truncates at subsubleading order for gravitons. As a consequence, both amplitudes can be directly constructed via Soft Theorems without the need for a Lagrangian. The only seed is the amplitude $A_3^\phi(p_1, k_1) = (e \cdot p_1)^h$ which is fixed up to a constant using 3-pt. kinematics. Let us then write the soft expansion with respect to $k_2 \to 0$ as

$$A_4^{\text{ph}} = \frac{1}{2} \sum_{a=1,2} \epsilon_a^2 p_a \epsilon_2^2 p_a \epsilon_2^2 p_a \epsilon_2^2 p_a A_3^{\text{ph}} = \frac{1}{2} \left( \frac{p_1 \epsilon_1 F_k - p_1 \epsilon_2 F_k}{p_1 \cdot k_2} \right), \quad (4)$$

where $F_2 \cdot J_a = F_2^{\mu \nu} J_{a \mu \nu}$ is the action of the angular momentum operator [28] on the corresponding particle and

1 We restore units in the final results and redefine $-iJ_{CS} \to J_{	ext{here}}$ with respect to [28]. We work in mostly minus signature.
\[ F_k = p_1 \cdot F_2 \cdot k_1, \quad F_e = p_1 \cdot F_2 \cdot e_1. \]

Analogously
\[
A_4^{\mu_\lambda} = \sum_{p_a=p_1, p_2, k_1, k_2} \frac{1}{2} \left( \epsilon_{\mu \rho \sigma} \right)_{p_a} \epsilon_{\rho \tau \eta} \frac{2 p_{\alpha_1} \cdot k_{\lambda_2}}{k_1 \cdot k_2} A_3^{\mu \nu \rho \sigma} = \frac{1}{2 k_1 \cdot k_2} \times \left[ (p_1 \cdot e_1)^2 F_k - 2 p_1 \cdot e_1 F_k F_e + p_2 \cdot k_2 F_e^2 \right].
\]

Given that \( F_2 \cdot J_a \) truncates when acting on \( A_3 \), the exponential has been inserted to get the soft-expansion at the desired order. The result not only manifests the double copy (3) but, as we will show, it generates the frequency expansion of classical radiation in these theories. The first term of the soft expansion therefore determines the dipole radiation formula in EM and the Einstein's quadrupole radiation in GR, whereas the subleading orders contribute to electric/magnetic higher multipoles [33].

For bodies with long range interactions as in (2), the leading term in the\( p \cdot k \) expansion of classical radiation in these theories.

\[ \exp(-D \cdot k) \]

The exponential has been inserted to get the soft-expansion at the transverse-traceless tensor in our results.

Spin-Multipoles

Our goal is to promote the above discussion for the case of spinning sources, which introduces a rich new set of structures. In fact, the seed \( A_3^{h,s} \) is not unique and contains a soft expansion encoding corrections to \( A_3^{h,0} \) [15, 23, 31, 34]. As the spin is the only quantum number available for the massive state, for any \( n \) we can write

\[
A_n^{h,s}(J) = \mathcal{H}_n \times \sum_{j=0}^{\infty} \omega(n, j=0) \cdot J_{\mu_1 \mu_2} \cdot \ldots J_{\mu_{2j-1} \mu_{2j}},
\]

where \( J_{\mu \lambda} \) acts on spin-\( s \) states. Products of \( J_{\mu \lambda} \) are symmetrized since \( \{J, J\} = 0 \) can be put in terms of lower multipoles. The sum is then guaranteed to truncate due to the Cayley-Hamilton theorem. For \( n = 3 \) we encode the helicity of the photon/graviton in the prefactor \( \mathcal{H}_3 \).

To begin, let us consider photon emission for \( s = \{1/2, 1\} \) and define its double copy. From two multipole operators \( X \) and \( X' \) acting on spin-\( s \) states, we introduce an operator \( X \odot X' \) acting on spin-\( 2s \) as

\[
X \odot X' = \begin{cases} 2^{-|D/2|} \text{tr}(X, X'), & 2s = 1, \\ \phi_{1 \mu_1 \nu_1} (X_{\mu_1 \nu_1}, X'_{\nu_1 \tau_1}) \phi_{2 \tau_1 \nu_2}, & 2s = 2, \end{cases}
\]

where \( \varepsilon \) and \( \phi \) are the respective massive polarizations and \( X \) denotes charge conjugation. We will show that these operations can be used to obtain scattering amplitudes in a gravity theory of a massive spin-2s field [36]. Here we will only need the following extension of (3):

\[
A_n^{h,s} \odot A_n^{h,s} = K_n A_n^{s,s}, \quad n = 3, 4.
\]

The case \( s = 0, \bar{s} = 0 \) was introduced by Holstein et al. [25, 37]. It was used to argue that the gyromagnetic ratios of both \( A_n^{h,1} \) and \( A_n^{h,1} \) must coincide, setting \( g = 2 \) as a natural value [16, 37]. We introduce the case \( s, \bar{s} \neq 0 \) as a further universality condition, and find it imposes strong restrictions on \( A_n^{h,s} \) for higher spins. More importantly, it can be used to directly obtain multipoles in the classical gravitational theory.

For (8) to hold we need to put \( A_n^{h,s} \) into the form (6) (although we will lift this restriction in [36]). The coefficients \( \omega(n, j=0) \) are universal once we consider minimal-coupling amplitudes, which are obtained from QED at \( s = 1/2 \) and from the \( W^\pm \) boson model at \( s = 1 \) [37]. The 3-pt. seeds in any dimension can be put as

\[
A_3^{ph} = \left( \frac{e \cdot p_1}{m} \right) (I + J), \quad J = \frac{e \mu q_{\mu \nu} J_{\mu \nu}}{e \cdot p_1},
\]

for \( q = p_1 - p_2 \). Denoting each operator by the corresponding \( \text{SO}(D - 1, 1) \) Young diagram, i.e. \( 1 = I \) and \( \mathbb{B} = J_{\mu \nu} \), the operation (7) gives the rules

\[
1_s \oplus 1_s = 1_{2s}, \quad 1_s \oplus \mathbb{B} = \frac{1}{2} \mathbb{B}_{2s},
\]

\[
\mathbb{B} \oplus \mathbb{B} = \mathbb{B}_{2s} + \mathbb{B}_{2s} + 1_{2s},
\]

which are a subset of the irreducible representations allowed by the Clebsch-Gordan decomposition. Rule (11) is explained in (19) below. The first term we denote by \( \Sigma_{\mu \rho \sigma} \) and has the symmetries of a Weyl tensor, i.e. is the traceless part of \( \{J_{\mu \lambda}, J_{\rho \sigma}\} \). For instance, the \( s = 2 \) amplitude as obtained from (8) is

\[
A_3^{ph,2} = \left( \frac{e \cdot p_1}{m} \right) \mathbb{B}_{2s} \left( I_{1\mu \nu} \mathbb{B}_{1\mu \nu} + \mathbb{B}_{1\mu \nu} W_{\mu \rho \sigma \beta} \Sigma_{\mu \rho \sigma \beta} \right) \mathbb{B}_{1\mu \nu},
\]

where \( W_{\mu \rho \sigma \beta} := q_{[\mu} e_{\nu]} q_{[\phi} e_{\bar{\beta}]} \) is the Weyl tensor of the graviton, reproducing the expected Weyl-quadrupole coupling [16, 38, 39], as shown in Appendix A.

To deepen understand these results, let us demand \( A_3^{ph,s} \) to be constructible from the double copy (8) for any spin:

\[
A_3^{ph,s} (J_{\mu \nu} + \bar{J}_{\mu \nu}) = A_3^{ph,s} (J_{\mu \nu}) \odot A_3^{ph,s} (\bar{J}_{\mu \nu}),
\]

where \( J_{\mu \nu} \oplus \bar{J}_{\mu \nu} \) is the generator acting on a spin \( s + \bar{s} \) representation. This relation yields the condition \( A_3^{1,1} A_3^{1,1} = A_3^{1,1} A_3^{1,1} A_3^{1,1} \) on the \( J_{\mu \nu} \) operators. Using that \( [J, \bar{J}] = 0 \)
and assuming the coefficients in (6) to be independent of the spin leads to

\[ A^{h,s}_3(J) = \left( \frac{e_p}{m} \right)^h \times e^{\omega_{\mu\nu} J^{\mu\nu}}, \quad h = 1, 2 \]  

(14)

with \( \omega_{\mu\nu} = \frac{e_\mu}{e_p} \) and \( H_3 = \left( \frac{e_p}{m} \right)^h \) fixed by the previous examples. This easily recovers such cases and matches the Lagrangian derivation \[40\] for \( s \in \{ \frac{1}{2}, 1, 2 \} \) in any dimension \( D \). After some algebra, (14) leads to the \( D=4 \) photon-current derived in \[41, 42\] for arbitrary spin via completely different arguments. On the gravity side, it matches the Kerr stress-energy tensor derived in \[43\] together with its spinor-helicity form recently found in \[15\], as we show in Appendix B. For \( s > h \) and \( D > 4 \), (14) contains a pole in \( \epsilon_p \) which reflects such interactions being non elementary \[44\]. In Appendix A we show such pole cancels for the classical multipoles and provide a local form of (14).

What is the meaning of the exponential \( e^{i\theta} \)? It corresponds to a finite Lorentz transformation induced by the massless emission. That is, \( p_2 = e^{i\theta} p_1 \), hence for generic spin it maps the state \( |p_1, \epsilon_1\rangle \) into \( |p_2, \epsilon_2\rangle \), where \( \epsilon_2 \neq \epsilon_1 \) is another polarization for \( p_2 \). This means it is composed both of a boost and a SO(\( D-1 \)) Wigner rotation. The boost can be removed in order to match SO(\( D-1 \)) multipoles in the classical theory, see Appendix A. Also, as \( e^{i\theta} \) is a Lorentz transformation, \( |\epsilon_2\rangle \) must live in the same irrep as \( |\epsilon_1\rangle \). This means that a projector is not needed when these objects are glued. A corollary of this is a simple formula for the full factorizations of \( A^{h,s}_3 \), e.g.

\[
\begin{array}{c}
\quad k_1 \quad k_2 \quad \ldots \quad k_{n-2} \quad k_n \quad \quad P_1 \quad P_2 \quad \ldots \quad P_{n-1} \\
\end{array}
\]

\[
= \prod_i (p_i, \epsilon_i)^h \langle \epsilon_2 | e^{J_{n-2} \ldots J_1} | \epsilon_1 \rangle = \prod_i (p_i, \epsilon_i)^h \langle \epsilon_2 | \epsilon_2 \rangle, 
\]

(15)

where \( p_i = p_1 + k_1 + \ldots + k_{i-1} \) and \( J_i = \frac{k_i \epsilon_i}{e_p} \). Each 3-pt. amplitude here maps \( P_i \) to \( P_{i+1} \) and their composition maps \( p_1 \) to \( p_2 \). The state \( |\epsilon_2\rangle \) depends on all \( \{k_i, \epsilon_i\}_{i=1}^n \) as well as their ordering. This factorization is enough to obtain the classical spin-multipoles of \( M_2 \) at least up to the quadrupole order we are interested in. To see this, we use the Baker-Campbell-Hausdorff formula in (15) and get the form

\[
A^{ph,s}_4 = \frac{1}{2} \left[ \frac{p_1 \cdot \epsilon_1 p_2 \cdot \epsilon_2}{p_1 \cdot k_1} \langle \epsilon_2 | e^{J_1 + J_2 - \frac{1}{2}[J_1, J_2] + \ldots} | \epsilon_1 \rangle + \frac{p_2 \cdot \epsilon_1 p_1 \cdot \epsilon_2}{p_2 \cdot k_1} \langle \epsilon_2 | e^{J_1 + J_2 + \frac{1}{2}[J_1, J_2] + \ldots} | \epsilon_1 \rangle + \text{c.t.} \right].
\]

(16)

This gives for \( s \leq 1 \)

\[
\omega^{(2)}_{\mu\nu} = \frac{p_1 \cdot F_1 p_2}{2} F_2^{\mu\nu} + \frac{p_1 \cdot F_2 p_2}{2} F_1^{\mu\nu} + \frac{p_1 \cdot (k_1 + k_2)}{4} [F_1, F_2]^{\mu\nu},
\]

(17)

\[
\omega^{(4)}_{\mu\nu\rho\sigma} = \frac{k_1 \cdot k_2}{16} \left( F_1^{\mu\nu} F_2^{\rho\sigma} + F_2^{\mu\nu} F_1^{\rho\sigma} \right).
\]

The role of the contact term in (16) is to restore gauge invariance. Here it is only needed for \( \omega^{(0)} \), thus by comparison with (4) one finds c.t. = \( \epsilon_1 \cdot \epsilon_2 \) and \( \omega^{(0)} = p_1 \cdot F_1 p_2 \cdot F_2 \). Already for spin-\( \frac{1}{2} \) it is clear that this decomposition of the Compton amplitude is not evident at all from a Feynman-diagram computation \[25, 45\] , whereas here it is direct. A key point of this splitting is that under the double soft deformation \( k_3 = \tau k_3, k_4 = \tau k_4 \), the multipole \( \omega^{(2)} \) is \( \mathcal{O}(\tau^2) \), whose leading order will be the classical contribution. It is now instructive to further decompose \( A^{ph,s} \) into irreps., which follows from

\[
\omega^{(4)}_{\mu\nu\rho\sigma,J} = \begin{cases}
\frac{1}{2} \left( \omega^{(4)}_{\mu\nu} + \omega^{(4)}_{\mu\nu} \right) Q^{\mu\nu}, & s = 1, \\
\left( \omega^{(4)}_{\mu\nu} + \omega^{(4)}_{\mu\nu} \right) \ell^{\mu\nu}, & s = \frac{1}{2},
\end{cases}
\]

where \( \ell^{\mu\nu} = \{J^{\mu\nu}, J^{\rho\sigma}\} \) and

\[
\tilde{1} = \frac{J^{\mu\nu} J^{\rho\sigma}}{2} Q^{\mu\nu} = \mathbb{E} = \{J^{\mu\nu}, J^{\nu\mu}\} + \frac{4}{D} \delta^{\mu\nu} \tilde{1}.
\]

(18)

The notation \( \omega^{(4)} \) denotes the corresponding projections. Among them we will be interested in the quadrupole operator \( Q^{\mu\nu} \), only present for \( s \geq 1 \).

Finally, \( A^{ph,s}_4 \) is found from (8) and matches the Lagrangian result for \( s \leq 2 \). We have used that (11) reads

\[
J_s^{\mu\nu} \otimes J_s^{\rho\sigma} = \frac{1}{4} \sum_{\nu \rho} \alpha_D D_D - 2 \delta^{[\nu \rho][\mu \sigma]} + \frac{\beta_D}{2D(D-1)} \delta^{[\nu \rho][\mu \sigma]} Q^{12s}.
\]

(19)

The normalizations \( \alpha_D, \beta_D \) depend solely on \( D \). However, it cancels out in the full computation and hence we set \( \alpha_D = \beta_D = 1 \) hereafter. Similarly, the condition \( A^{ph,\frac{1}{2}}_4 A^{ph,\frac{1}{2}} = A^{ph,0} A^{ph,1} \), as implied by (8), can be traced at this order to \( [\omega^{(2)}(2)]_{\mu\nu} = [\omega^{(4)}_{\mu\nu} - \omega^{(0)}] \), which holds up to terms subleading in the double soft limit.

**Classical Applications**

Very recently, Kosower et al. \[20\] have provided a QFT derivation of the following formulae

\[
\Delta p^\mu = \int \frac{d^D q}{(2\pi)^D-2} \delta(2q \cdot p_1) \delta(2q \cdot p_2) q^\mu e^{ip \cdot (M^4)} ,
\]

(20)

\[
\mathcal{J}_h(k) = \int \frac{d^D q_1}{(2\pi)^D-2} \delta(2q_1 \cdot p_1) \delta(2q_3 \cdot p_3) e^{iq_1b_1} e^{iq_3b_3} (M^4) ,
\]

(21)
encoding classical observables at LO in the coupling [19]. Here \( \Delta p^{\mu} = \frac{\partial}{\partial k^{\mu}} \) is the (conservative) momentum deflection of a massive body in a classical scattering setup, where the function \( \chi \) is the scattering angle [11]. The current \( J_{h}(k) \) reads \( \epsilon_{\mu}J^{\mu} (h = 1) \) and \( \epsilon_{\mu}T^{\mu\nu} (h = 2) \) and corresponds to the field radiated at \( r \rightarrow \infty \). Even though these were proven for \( D = 4 \), matching with classical results shows that they hold in any \( D \). As explained in [20], the classical limit \( \langle M \rangle \) is obtained by rescaling \( q_i = \hbar k_i \) and \( k = \hbar k \), after which we can extract the leading order in \( \hbar \). We extend this rule to include spin by scaling the angular-momentum as \( J = \hbar^{-1} J \), as in e.g. [15].

The \( \hbar \to 0 \) limit is captured by the cuts of \( M_4 \) and \( M_5 \) given in (2). For \( M_4 \), this was argued by one of the authors in [46], where the classical piece was identified as the singularity in \( q^2 \) up to 1-loop, see also [47]. For \( M_5 \), the key point is to introduce the average momentum transfer \( q = \frac{2n-2}{2} \), after which one expects the same construction to apply. In fact, noting that \( dDq_1 = dDq \) in (21) already shows that contact terms in \( q^2 \) appearing in \( \langle M_5^S \rangle \) will lead to local quantum contributions (details will be given somewhere else [36]).

To start with, consider \( \langle M_5^b \rangle = \frac{n_b}{q^2} \) where \( n_b \), a local numerator. Its scalar parts are \( n_{ph} = p_1 \cdot p_3 \) and

\[
\langle M_5^s \rangle = \frac{n_{s}}{q^2} = \frac{\sqrt{2\pi G}}{q^2} \left( (p_1 \cdot p_3)^2 - \frac{m_a^2 m_b^2}{D-2} \right),
\]

where the factor of \( D-2 \) arises from the graviton propagator. In \( D = 4 \) we can evaluate (20) to recover the 1PM scattering angle as in [11], first derived in the classical context by Portilla [48, 49]. See below for spin effects. Moving to \( \langle M_5^s \rangle \), the factorization of (2) together with the classical limit imply the form

\[
\langle M_5^s \rangle = \frac{1}{(q \cdot k)^{h-1}} \left[ \frac{n_{h}^{(a)}}{(q^2 - q \cdot k)(p_1 \cdot k)^2} \pm \frac{n_{h}^{(b)}}{(q^2 + q \cdot k)(p_3 \cdot k)^2} \right],
\]

where we pick \((-)\) for \( h = 2 \). The spurious pole \( q \cdot k \) arises from the \( t \)-channel of \( A_4^{Gr,s} \), and its cancellation provides a nice check of our formula. This further shows that the classical limits of \( M_4 \) and \( M_5 \) are universal and do not depend on the spin of the massive particles (nor the Lagrangian details if we assume \( A_{h,s}^s \) are constructible). This was emphasized in [25] at 4-pt. and is the first example of such universality at 5-pt.

**Exponentiated Soft Theorem**

As an application of orbit multipoles let us study \( \langle M_5^s \rangle \) for scalars. The numerators \( n^{(a)} \) can be read off directly from (5): Replacing \( \epsilon_1 \) by \( p_3 \), powers of the orbit multipole \( F \) translate to powers of \( F_{p} = p_1 \cdot F \cdot p_3 \), whereas \( F_k \) now becomes \( F_{q} = n_1 (p_1 \cdot F \cdot q) \), with \( n_1 = -1, n_3 = 1 \). The soft expansion (5) with respect to \( k_2 = k \) becomes

\[
n^{(a)}_{gr} = \frac{F_{q}^2}{2} e^{-\frac{F_{q}^2(p_1 \cdot k)}{\pi(p_1 \cdot p_3)^2} \left( (p_1 \cdot p_3)^2 - \frac{m_a^2 m_b^2}{D-2} \right)}. \tag{24}
\]

Further writing \( \frac{1}{q^2 + q \cdot k} = e^{\pm q \cdot k \frac{D-4}{D-2}} \) turns (23) into

\[
\langle M_5^s \rangle = \sum_{i=1,3} S_i e^{(F_{p} + \frac{q}{q^2 + q \cdot k} + q \cdot k)^2} \langle M_4^s \rangle \tag{25}
\]

where \( S_i = \frac{n_{s}(q \cdot k)}{(p_1 \cdot p_3)^2} \) for photons we find \( S_i = \frac{F_{q}^2}{2(p_1 \cdot p_3)^2} \).

This expression can be used to obtain \( \langle M_5^s \rangle \) from \( \langle M_4^s \rangle \) as an expansion in the graviton momenta \( k^a \) to any desired order (sub-subleading orders were studied in [50–52]). The spurious pole in \( S_i \) cancels out and one can check explicitly that \( S_1 + S_3 \) corresponds to the \( \hbar \to 0 \) limit of the Weinberg Soft Factor for the full \( M_5 [53] \). The first order of the exponential analogously corresponds to the \( h \to 0 \) limit of the subleading soft factor of Low [31, 34].

Let us focus for simplicity on the leading order of (25). By considering bounded orbits with \( \omega \sim \frac{1}{\hbar} \) the GW frequency expansion becomes a non-relativistic expansion [54]. It can be checked that the LO in fact leads to Einstein’s Quadrupole Formula, see discussion below. For classical scattering we can use the LO to read the Memory Effect at \( r \to \infty \). Plugging (25) into (21) we get

\[
\int \frac{dDq}{(2\pi)^{D-2}} \delta(2q \cdot p_1) \delta(2q \cdot p_3) e^{iq(b_1 - b_3)} \sum_{i=1,3} S_i \langle M_4^s \rangle
\]

as \( k \to 0 \). Evaluating the sum and using (20) as a definition of \( \Delta p_1 = -\Delta p_3 \) we obtain

\[
\epsilon_{\mu \nu}T^{\mu \nu} = \frac{F_{p}/2}{p_1 \cdot k p_3 \cdot k} \left( \frac{p_1}{p_1 \cdot k} + \frac{p_3}{p_3 \cdot k} \right) \cdot F \cdot \Delta p + O(k^0), \tag{26}
\]

which at leading order in \( \Delta p \) (or \( G \), if restored) becomes

\[
T^{\mu \nu}(k) = \sqrt{8\pi G} \Delta \left[ \frac{p_1^\mu p_3^\nu}{p_1 \cdot k} + \frac{p_3^\mu p_1^\nu}{p_3 \cdot k} \right]^{TT}. \tag{27}
\]

In position space this gives the burst memory wave derived by Braginsky and Thorne [55] in \( D = 4 \) (a \( 4\pi r^{-1} \) factor arises from the retarded graviton propagator), see [56–58] for \( D > 4 \). Here we have provided a direct connection with the Soft Theorem (25), alternative to the expectation-value argument [59, 60]. This can also be seen as the Black Hole Bremsstrahlung of [61, 62] generalized to consistently include the dynamics of the sources.

**Classical Double Copy**

As the numerators in eqs. (22) and (23) correspond to \( A_{h,s}^s \) amplitudes, the multipole double copy can be directly promoted to \( \langle M_4 \rangle \) and \( \langle M_5 \rangle \). From a classical perspective, the factorization of (2) implies that the photon
Numerators can always be written as \( n_{ph} = t_a \mu_b^\mu \) where \( t_a \) and \( t_b \) only depend on particle 1 and 3 respectively. The simplest example is the scalar piece in \( \langle M_4 \rangle \), where \( t_a = p_1 \) and \( t_b = p_3 \). The KLT formula (8) translates to

\[
\frac{n_{gr} = n_{ph} \otimes n_{ph} - \text{tr}(n_{ph} \otimes n_{ph})}{(2\pi)^2 q^2} = \frac{1}{4} p_{3\mu} q_{\nu} p_{3\alpha} q_{\beta} \sum_{\mu \nu \alpha \beta} q^2,
\]

where we defined the trace operation as \( \text{tr}(n \otimes n) = \langle t_{a\mu} \otimes t_{b\mu} \rangle \). By combining (28) with eqs. (22) and (23), this establishes for the first time a classical double-copy formula that can be directly proved from the standard BCJ construction [36]. Moreover, up to this order it only requires as input Maxwell radiation as opposed to gluon color-radiation [19, 30] and contains no Dilaton/Axion states [22, 30, 63].

Let us start with \( \langle M_4 \rangle \) as example. To keep notation simple consider only particle \( a \) to have spin. From (9) we find that at the dipole level the numerator for \( \langle M_4 \rangle \) is \( n_{ph}^{(a)} = n_{0}^{(a)} + p_3 \cdot J_a \cdot q \). The gravity result follows from (28) by dropping contact terms in \( q^2 \). The rules (10) readily give the scalar and dipole parts, including (22). For the quadrupole part, rule (19) gives

\[
\frac{(p_3 \cdot J_a \cdot q) \otimes (p_3 \cdot J_a \cdot q) - \text{tr}(\cdots)}{q^2} = \frac{1}{4} p_{3\mu} q_{\nu} p_{3\alpha} q_{\beta} \sum_{\mu \nu \alpha \beta} q^2.
\]

Using (38), the SO(D − 1) quadrupole [39, 64] reads

\[
\frac{1}{4} p_{3\mu} q_{\nu} p_{3\alpha} q_{\beta} \sum_{\mu \nu \alpha \beta} q^2 \rightarrow \frac{p_1 p_3}{4(D - 3)} \times \frac{q \cdot Q_a \cdot q}{q^2 m_a^2}.
\]

Up to this order this agrees with the \( D = 4 \) computation [15, 43]. Agreement to all orders in spin is obtained from the formula (44) in Appendix A.

Moving to \( \langle M_5 \rangle \), in the examples that follow the numerators \( n_{ph} \) can be read either from classical results up to dipole order [20, 22, 30], from QED Bremstrhalung, or from (4), (9) and (17). They are all in agreement. For photons, the scalar part is

\[
n_{0}^{(a)} = 4e^2 p_1 \cdot R_3 \cdot F \cdot p_1, \quad n_{0}^{(b)} = 4e^2 p_3 \cdot R_1 \cdot F \cdot p_3,
\]

where \( R_1^{\mu \nu} = p_1^{[(\eta, 2q - k)^{\nu}]} \). For the spin part we have

\[
n_{1}^{(a)} = n_{0}^{(a)} - 2e^2 \left[p_1 \cdot R_3 \cdot kF \cdot J_a - F_{i1}R_3 \cdot J_a + p_1 \cdot k \cdot [F, R_3 \cdot J_a] \right],
\]

\[
n_{1}^{(b)} = n_{0}^{(b)} + 2e^2 \cdot p_3 \cdot F \cdot R_3 \cdot p_3,
\]

with \( R^{\mu \nu} = (2q + k)^{\nu} J_a \cdot (2q + k)_{a} \). Recall these numerators live in the support of \( \delta(p_1 \cdot q_i) \) in (21). Writing them as \( n_{1}^{(a)} = t_a \) one finds \( t_b^{(a)} = p_3 \) and \( t_b^{(b)} = p_1 + J_a \cdot (2q + k) \) as expected from their "3-pt. part". The scalar and dipole pieces obtained from (28) then recover the results of [22, 30, 65] for Pure and Fat Gravity (we obtain the latter as the limit \( D \to \infty \)). This provides a strong cross-check of our method. Using (19) we can also compute the quadrupole order. For instance, the \( Q^{\mu \nu} \) piece reads

\[
\frac{n_{1}^{(a)}(Q)}{q-k} = \frac{(32\pi G)^2}{8(D-2)} \left[ (p_1 \cdot p_3 F_{i1} p_1 \cdot k F_{p}) \{ R_3 \cdot F \cdot Q_a + \frac{m_b^2}{(D-2)} (F_{i3} \{ F \cdot Y \cdot Q_a - 2p_1 \cdot k \cdot p_1 \cdot F \cdot Q_a \cdot F \cdot Q) \} \right],
\]

with \( Y^{\mu \nu} = p_1^{[(2q-k)^{\nu}]} \), whereas \( n_{1}^{(b)}(Q) = 0 \). As before, we have dropped contact terms in \( q^2 \) and used the support of \( \delta(p_1 \cdot q_i) \). This result can be shown to agree with a much more lengthy computation of the full \( M_5^{gr} \) using Feynman diagrams. At this order, \( M_5^{gr} \) contains classical quadrupole pieces and quantum scalar and dipole pieces. Interestingly, we have found that the latter can be cancelled by adding the spin-1 spin-0 interaction \( (B_\mu \partial^\mu \phi)^2 \) to the Lagrangian, which signals their quantum nature.

**Discussion**

We have shown that key techniques of Scattering Amplitudes such as soft theorems and double copy can be promoted directly to study classical phenomena arising in Gravitational Waves (GW). These techniques drastically streamline the computation of radiation and spin effects; both are phenomenologically important for Black Holes, which are believed to be extremely spinning in nature [66, 67]. In that direction, one could for instance apply our formalism to derive the hexadecapole \( s = 2 \) order in radiation [68, 69] to LO in \( G \) but all orders in \( 1/c \). We now outline some other directions:

**The \( A_5^{gr} \) series:** Let us emphasize that these constitute building blocks even at loop orders [17, 46, 70]. For \( s > 2 \) the amplitudes \( A_4^{gr} \) were studied in [16] in the context of the \( O(G^2) \) potential and were found to contain polynomial ambiguities. We expect our construction, including soft expansion and double copy, to be a criteria for resolving such ambiguities and lead to further classical predictions. In the scalar setup, we expect \( A_5^{gr} \) to be relevant even for \( n > 4 \). In fact, \( A_5^{gr} \) as a double copy has been recently pointed out as a key ingredient in the computation of the \( O(G^3) \) potential by Bern et al. [17]. All these results made strong use of the \( D = 4 \) spinor-helicity formalism. Specializing our treatment of radiation to \( D = 4 \) is also a natural future direction in the hunt of simplifications even at loop orders, as in [10, 46].

**Soft Theorem/Memory Effect:** It would be interesting to understand the meaning of the higher orders of (25), considering for instance the Spin Memory Effect [71, 72]. Motivated by the infinite soft theorems of [73, 74] one could expect the corrections are related to a hierarchy of symmetries. One may also incorporate spin contributions and study their interplay with such orders. In the applications side, it is desirable to further investigate (25) at loop level [75, 76], which could lead to a simple way of obtaining \( \langle M_5 \rangle \) from \( \langle M_4 \rangle \).
Generic Orbits: For orbits more general than scattering $J(k)$ does not have the support of $\delta(2p_1\cdot q_1)$ \cite{29, 54}. In fact, for bounded orbits it contains the subleading terms $p_1\cdot q_1 \sim \omega$. Very nicely, by keeping such terms in the classical calculation we have checked they match with eqs. (31),(32), which in turn arise from the form in (17) via a natural “$F\rightarrow R$ replacement”. One could then try to explore the gravity case by combining our results with the EFT treatment of bounded orbits and their EOMs \cite{77}.

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Appendix A: From SO(D − 1, 1) to SO(D − 1) multipoles

In order to compare with classical results for spinning bodies it is sometimes necessary to choose a frame through the Spin Supplementary Condition (SSC). Let us show how this arises from our setup.

We have shown that the spin multipoles correspond to finite SO(D − 1, 1) transformations which map $p_1 \rightarrow p_2$. Such Lorentz transformations are composed of both a boost and a SO(D − 1) Wigner rotation. Spin multipoles of a massive spinning body are defined with respect to a reference time-like direction and form irreps. of SO(D − 1) acting on the transverse directions \cite{39, 64}. Hence, it is natural to identify such action with Wigner rotations of the massive states entering our amplitude. A simple choice for the time-like direction is the average momentum $u = \frac{p_1 + p_2}{m}$. In this frame boosts are obtained as $K^\nu = u_\nu J^{\mu\nu}$ whereas Wigner rotations read $S^{\mu\nu} = J^{\mu\nu} - 2u^{\mu}K^\nu$. Adopting $S^{\mu\nu}$ as classical spin tensor then corresponds to the covariant SSC, i.e. $u_\nu S^{\mu\nu} = 0$ \cite{43} (other choice was used in \cite{16, 78})). The momenta $p_1$ and $p_2$ can be aligned canonically to $p$ through the boost,

\begin{equation}
{p_1 = e^{\frac{p_1}{m} \cdot K} p, 
{p_2 = e^{-\frac{p_2}{m} \cdot K} p}, \tag{33}
\end{equation}

which defines canonical polarization vectors $\epsilon, \tilde{\epsilon}$ for $p$ through (recall $p_2$ is outgoing):

\begin{equation}
{\epsilon_1 = e^{\frac{p_1}{m} \cdot K} \epsilon, 
{\tilde{\epsilon}_2 = \tilde{\epsilon} e^{-\frac{p_2}{m} \cdot K}}. \tag{34}
\end{equation}

This replacement can then be applied to the multipole expansion (6), yielding an extra power of $q$ for each power of $J$, hence preserving the $h$-scaling. We find

\begin{equation}
{\epsilon_1 \cdot \epsilon_2 = \epsilon \tilde{\epsilon} + \frac{1}{m} q_\mu \epsilon K^\mu \tilde{\epsilon} + O(K^2), \tag{35}}
\end{equation}

\begin{equation}
{\epsilon_1 J^{\mu\nu} \epsilon_2 = \epsilon S^{\mu\nu} \tilde{\epsilon} + 2 u^{[\mu} K^{\nu]} \tilde{\epsilon} + \frac{q_\mu}{m} \epsilon \{ K^\nu, S^{\mu\sigma} \} \tilde{\epsilon} + O(K^2), \tag{36}}
\end{equation}

\begin{equation}
{\epsilon_1 \{ J^{\mu\nu}, J^{\rho\sigma} \} \epsilon_2 = \epsilon \{ S^{\mu\nu}, S^{\rho\sigma} \} \tilde{\epsilon} + O(K), \tag{37}}
\end{equation}

(for generic $K$ and $S$ are independent). In terms of irreducible representations this decomposition can be thought of as branching SO(D − 1, 1) into SO(D − 1) \cite{79}. For instance, the dipole branches as $\square \rightarrow \square + \square$, which is a transverse dipole plus a transverse vector irrep, $K^\nu$. In the same way, in general the $\square$ irrep. of SO(D − 1, 1) also contains a $\square$ piece for SO(D − 1). This is the reason we can extract a quadrupole from Weyl piece in (30), namely by combining (37) with the replacement rule

\begin{equation}
{\{ S^{\mu\nu}, S^{\rho\sigma} \} = \frac{2}{D-3} \left( \eta^{[\rho [\mu} \bar{Q}^{\nu] \sigma] \bar{Q}^{\nu [\rho]} \right) + \text{other irreps}} \tag{38}
\end{equation}

where $\bar{Q}^{\rho\sigma} = \eta^{\rho\sigma} - u^{[\rho} u^{\sigma]}$. Thus we have the identity (c.f. \cite{80, 81})

\begin{equation}
{\omega_{\mu\nu\rho\sigma} \Sigma^{\mu\nu\rho\sigma} = \frac{2}{D-3} \frac{|\omega|_{\mu\nu\rho\sigma} (\epsilon_1 \{ J^{\mu\nu}, J^{\rho\sigma} \} | \epsilon_2)}{3}, \tag{39}}
\end{equation}

For instance, we extract a quadrupole contribution from $A_{3}^{h,s}$ in (12):

\begin{equation}
{A_{3}^{h,s} | Q = \frac{1}{4} \left( \frac{\epsilon \cdot p_1}{m} \right)^h q \cdot \bar{Q} \cdot q, \tag{40}}
\end{equation}

Of course, the SO(D − 1, 1) quadrupole present in $A_{4}^{h,s}$ also contains a SO(D − 1) quadrupole. It follows from (37) that it can be read through

\begin{equation}
{Q^{\mu\nu} = \bar{Q}^{\mu\nu} - \frac{4}{D(D-1)} \bar{Q}^{\rho\sigma} S^{2} + O(K). \tag{41}}
\end{equation}

In general the SO(D − 1) multipoles defined through the covariant SSC are given directly from the SO(D − 1, 1) ones, up to $O(K)$ terms. Due to unitarity, one expects the latter to drop from the amplitude, at least for $A_3$. Let us show explicitly how this happens. Note that 3-pt. kinematics implies $[qK, qJ] = 0$ and hence the spin piece of the 3-pt. amplitude (14) reads

\begin{equation}
{\epsilon_1 \epsilon_2 \epsilon \epsilon_2 \epsilon = \frac{1}{2} \left( \frac{q_\mu \epsilon_1 J^{\mu\nu} + q_\mu K^{\nu}}{m} \right) \epsilon = \tilde{\epsilon} \epsilon S \epsilon, \tag{42}}
\end{equation}

where one can check that the sum truncates at order $2s$. Thus the boost (33) is effectively subtracted from the finite Lorentz transformation leading to the interpretation
of the 3-pt. formula as a little-group rotation induced via photon/graviton emission. We end with a comment on
the case $s > h$ and $D > 4$: Note that the pole $\epsilon - p$ cancels in (40) for any dimension. This means we can provide
a local form of the 3-pt. amplitude which contains the same multipole as the exponential. For instance,
$$A_{3}^{ph,2} = \frac{(\epsilon \cdot p)}{m} \phi_{2} \left( I + \frac{\epsilon_{\mu} q_{\nu} J_{\mu\nu}}{\epsilon \cdot p} + \frac{q_{\mu} q_{\nu}}{4m^{2} \epsilon \cdot p} \times \right) \left( \epsilon_{\sigma} p_{\sigma} + \epsilon_{\rho} p_{\rho} - \frac{\eta_{\sigma\rho}(\epsilon \cdot p)}{D-3} \right) \left\{ J_{\mu\nu}, J_{\rho\sigma} \right\} \phi_{1},$$

also yields (40) and reduces to (14) in $D = 4$. In general
the 2$^{n}$-poles [43, 64] of (12) are obtained by performing $[\frac{1}{2}]$ traces with the spatial metric $\hat{\eta}_{\mu\nu}$ appearing in (38).

The result takes the local form
$$A_{3}^{h,s} \bigg|_{2^{n}} = \left( \frac{\epsilon \cdot p}{m} \right)^{h} \sum_{n=0}^{\infty} \left( \alpha_{n} + \beta_{n} \frac{q_{\mu} \epsilon_{s} S_{\mu\nu}}{\epsilon \cdot p} \right) \times \tilde{Q}^{(n)} \xi_{\mu_{1} \ldots \mu_{n}},$$

where $\alpha_{n}$, $\beta_{n}$ depend on the dimension $D$, and $\tilde{Q}^{(n)}$ are the transverse multipoles. In four dimensions
we find $\tilde{Q}^{(n)}$ to be a tensor product of the Pauli-Lubanski vector $S_{\mu}$ [16, 64], and $\alpha_{n} = \frac{1}{(2n)!}$, $\beta_{n} = \frac{1}{(2n+1)!}$.

**Appendix B: Spinor-Helicity Formulæ**

Here we show the exponential forms presented here for spin-multipoles contain as particular cases the ones of [15],
which implemented massive spinor-helicity variables in $D = 4$ [44]. Consider first $A_{3}^{gr,s}$: For plus helicity of the
graviton, the expression derived in [15] reads
$$A_{3}^{gr,s} = \frac{(p \cdot \epsilon)^{2}}{m^{2s}} \left( 2^{2} e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} \right)^{1^{2s}},$$

where $\epsilon = \epsilon^{+}$ carries the graviton helicity and $|\lambda \rangle^{2s}$ stands for the product $|\lambda(\epsilon_{1})_{\alpha_{1}} \cdots |\lambda(\epsilon_{2s})_{\alpha_{2s}}\rangle$ of SL(2, $C$) spinors
associated to each massive particle. The generator $J_{\mu\nu}$ in the exponent acts on such chiral representation. The
labels $\alpha_{i}$ are completely symmetrized little-group indices. The explicit construction of the massive spinors is not needed here (c.f. [44]), but solely the fact that spin-$s$

polarsation tensors can be expressed compactly as
$$\epsilon_{1} = \frac{1}{m^{1}} [1]^{*} |1\rangle^{*}, \quad \epsilon_{2} = \frac{1}{m^{1}} [2]^{*} |2\rangle^{*},$$

where $|1\rangle_{\alpha}$ and $|2\rangle_{\alpha}$ live in the antichiral representation of SL(2, $C$). Inserting them into (14) we obtain
$$\langle \epsilon_{2} | A_{3}^{gr,s} | \epsilon_{1} \rangle = \frac{(p \cdot \epsilon)^{2}}{m^{2s+2}} \left( 2^{2} e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} \right)^{1^{2s}} \langle \epsilon_{1} | 1^{2s} e^{\frac{k_{\mu_{1} \nu_{1}} J_{\mu_{1} \nu_{1}}}{p \cdot \epsilon}} | \epsilon_{2} \rangle^{*},$$

where $J_{\mu\nu}$ and $\tilde{J}_{\mu\nu}$ are given by
$$J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} \otimes \tilde{I}^{(s-1)} + \tilde{I} \otimes \frac{1}{2} \sigma_{\mu\nu} \otimes \tilde{I}^{(s-2)} + \ldots, \quad (48)$$

$$\tilde{J}_{\mu\nu} = \frac{1}{2} \tilde{\sigma}_{\mu\nu} \otimes \tilde{I}^{(s-1)} + \tilde{I} \otimes \frac{1}{2} \tilde{\sigma}_{\mu\nu} \otimes \tilde{I}^{(s-2)} + \ldots, \quad (49)$$

with $\sigma_{\mu\nu} = \sigma(\mu) \tilde{\sigma}(\nu)$ and $\tilde{\sigma}_{\mu\nu} = \tilde{\sigma}(\mu) \sigma(\nu)$. They satisfy the self-duality conditions
$$J_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} J_{\rho\sigma}, \quad \tilde{J}_{\mu\nu} = -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{J}_{\rho\sigma} \cdot \quad (50)$$

As it is well known, choosing the graviton to have plus helicity leads to a self-dual field strength tensor, which
in turn implies that $k_{\mu} \epsilon_{s}^{\mu} J_{\mu\nu} = 0$. Then (47) reads

$$\langle \epsilon_{2} | A_{3}^{gr,s} | \epsilon_{1} \rangle = \frac{(p \cdot \epsilon)^{2}}{m^{2s+2}} \left( 2^{2} e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} \right)^{1^{2s}} \langle \epsilon_{1} | 1^{2s} e^{\frac{k_{\mu_{1} \nu_{1}} J_{\mu_{1} \nu_{1}}}{p \cdot \epsilon}} | \epsilon_{2} \rangle^{*}. \quad (51)$$

We can now plug the identity $[21]^{s} = (2)^{s} e^{\frac{k_{\mu_{1} \nu_{1}} J_{\mu_{1} \nu_{1}}}{p \cdot \epsilon}} |1\rangle^{2s}$ from [15] to obtain:

$$\langle \epsilon_{2} | A_{3}^{gr,s} | \epsilon_{1} \rangle = \frac{(p \cdot \epsilon)^{2}}{m^{2s+2}} \left( 2^{2} e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} \right)^{1^{2s}} \langle \epsilon_{1} | 1^{2s} e^{\frac{k_{\mu_{1} \nu_{1}} J_{\mu_{1} \nu_{1}}}{p \cdot \epsilon}} | \epsilon_{2} \rangle^{*}, \quad (52)$$

which has the structure of our formula (13), now in "spinor space". Extending the generators $J_{\mu\nu}$ to act on 2s slots, i.e.,
$J_{\mu\nu} \otimes I^{s} \otimes I^{s} \otimes J_{\mu\nu} \rightarrow J_{\mu\nu}$, then recovers (45). Consider now $A_{4}^{gr,s}$ for $s \leq 2$ as given in [15], where $(- -)$ denotes the helicity of the gravitons
$k_{1} = [1][1]$ and $k_{2} = [2][2],$

$$A_{4}^{gr,s} = \frac{\langle \epsilon_{1} | P_{1}^{2} | \epsilon_{1} \rangle^{2}}{p_{1} \cdot k_{1} p_{2} \cdot k_{2}} \frac{2^{2} e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}}}{1^{2s}} \langle \epsilon_{1} | 1^{2s} e^{\frac{k_{\mu_{1} \nu_{1}} J_{\mu_{1} \nu_{1}}}{p \cdot \epsilon}} | \epsilon_{2} \rangle^{*}, \quad (53)$$

In order to match this we double copy our formula (16). The sum in (16) exponentiates if we impose $[J_{1}, J_{2}] = 0$, which in turn is only possible if the polarizations are aligned, i.e. $\epsilon_{1} \propto \epsilon_{2}$. When the states have opposite helicity this can be achieved via a gauge choice. This yields

$$k_{1 \mu} \epsilon_{1 \nu} J_{\mu\nu} + k_{2 \mu} \epsilon_{2 \nu} J_{\mu\nu} = \frac{k_{\mu} \epsilon_{1 \nu} J_{\mu\nu}}{p \cdot \epsilon_{1}}, \quad (54)$$

where $k = k_{1} + k_{2}$. Expression (16) thus becomes

$$A_{4}^{ph,s} \bigg|_{\epsilon_{1} \times \epsilon_{2}} = \frac{p_{1} \cdot \epsilon_{1} p_{2} \cdot \epsilon_{2} k_{1} \cdot k_{2}}{p_{1} \cdot k_{1} p_{2} \cdot k_{2}} \langle \epsilon_{\alpha} | e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} | \epsilon_{\beta} \rangle \langle \epsilon_{\alpha} | \epsilon_{\beta} \rangle, \quad (55)$$

(note that $ct = \epsilon_{1} \cdot \epsilon_{2}$ drops out). The formula (8) gives

$$A_{4}^{gr,s} \bigg|_{\epsilon_{1} \times \epsilon_{2}} = \frac{(p_{1} \cdot \epsilon_{1})^{2} (p_{2} \cdot \epsilon_{2})^{2}}{p_{1} \cdot k_{1} p_{2} \cdot k_{2}} \langle \epsilon_{\alpha} | e^{- \frac{\epsilon_{\mu} \epsilon_{\nu}}{p \cdot \epsilon}} | \epsilon_{\beta} \rangle \langle \epsilon_{\alpha} | \epsilon_{\beta} \rangle, \quad (56)$$

for $s \leq 2$. This can be shown to match (53) following the same derivation as before and fixing $\epsilon_{1} = \frac{1}{[2]}$, $\epsilon_{2} =$
Note finally that, even though in any dimension $D$ there is an helicity choice such that (16) becomes (56), the factorization of (2) requires to sum over all helicities of internal gravitons.
