A Faster Algorithm Enumerating Relevant Features over Finite Fields

Mikito Nanashima

Department of Mathematical and Computing Science, Tokyo Institute of Technology, Tokyo, Japan
nanashima.m.aa@is.c.titech.ac.jp

Abstract.

A \(k\)-junta function is a function which depends on only \(k\) coordinates of the input. For relatively small \(k\) w.r.t. the input size \(n\), learning \(k\)-junta functions is one of fundamental problems both theoretically and practically in machine learning. For the last two decades, much effort has been made to design efficient learning algorithms for Boolean junta functions, and some novel techniques have been developed. However, in real world, multi-labeled data seem to be obtained in much more often than binary-labeled one. Thus, it is a natural question whether these techniques can be applied to more general cases about the alphabet size.

In this paper, we expand the Fourier detection techniques for the binary alphabet to any finite field \(\mathbb{F}_q\), and give, roughly speaking, an \(O(n^{0.8})\)-time learning algorithm for \(k\)-juntas over \(\mathbb{F}_q\). Note that our algorithm is the first non-trivial (i.e., non-brute force) algorithm for such a class even in the case where \(q = 3\) and we give an affirmative answer to the question posed in [MOS04].

Our algorithm consists of two reductions: (1) from learning juntas to LDME which is a variant of the learning with errors (LWE) problems introduced by [Reg05], and (2) from LDME to the light bulb problem (LBP) introduced by [Val88]. Since the reduced problem (i.e., LBP) is a kind of binary problem regardless of the alphabet size of the original problem (i.e., learning juntas), we can directly apply the techniques for the binary case in the previous work such as in [Val15,KKK18].

Keywords: learning juntas, PAC learning, computational learning theory, learning with errors, light bulb problem.

1 Introduction

1.1 Background and Motivation

In practical and theoretical sense, it is a fundamental challenge to separate relevant information from irrelevant information in data analysis. In many machine learning settings, collected data may contain many irrelevant features together with relevant features (e.g., DNA sequences and big data), and the efficient techniques for selecting relevant features are widely required. This problem is captured by learning juntas, which is one of most challenging and important issues in computational learning theory. Informally, we say an \(n\)-input function \(f : \mathcal{X}^n \to \mathcal{Y}\) is \(k\)-junta (\(k \leq n\)) iff \(f\) depends on only at most \(k\) coordinates of the input. Our task is to find the relevant coordinates (i.e., features) of a \(k\)-junta function \(f\), called a target function, from passively collected examples of the form \((x,f(x)) \in \mathcal{X}^n \times \mathcal{Y}\). An efficient learning algorithm for juntas will give us the techniques for finding indeed relevant features in the presence of many irrelevant information.

In the special case where the domain of a target function is binary, that is, \(\mathcal{X} = \mathbb{F}_2\), the learning junta problem has theoretically important meanings. For \(k = O(\log n)\), learning \(k\)-junta functions is a special case of learning polynomial-size DNF (disjunctive normal form) formulas and log-depth decision trees, which are also known as notorious open problems in computational learning theory, even in the uniform-distribution model (i.e., examples are distributed uniformly over \(\mathbb{F}_2^n\)). Therefore, for affirmative answer to such problems, finding an efficient learning algorithm for log-juntas is inevitable. Despite much effort by many researchers, efficient (i.e., polynomial-time) learning algorithms for log-juntas have not been found. From the other point of view (i.e., parameterized complexity introduced by [DF95]), learning juntas problem can be regarded as a parametrized learning problem for general Boolean functions, and in fact, a fixed parameter intractable results have been found in learning juntas under arbitrary
example distribution in [AKL09]. However, in the uniform-distribution model, any convincing argument on intractability of efficient learning algorithms for juntas has not been found until now. For further details about learning juntas, see the survey by Blum [Blu03].

On the positive side, some elegant techniques for learning Boolean juntas have been developed in the uniform-distribution model since the problem was posed in [Blu94,BL97]. Obviously, any $k$-junta functions can be learned in time $O(n^k)$ with high probability by brute-force search for all \( \binom{n}{k} \leq n^k \) patterns about relevant coordinates. The first polynomial factor improvement was found by Mossel et al. [MOS04], and the running time is reduced to $O(n^{\frac{3k}{2}})$, where $\omega$ denotes the exponential factor of the running time $O(n^\omega)$ of fast $n \times n$ matrix multiplication. Further improvement has been made by G. Valiant [Val15], and the faster learning algorithm in time $O(n^{\frac{3}{2}k})$ has been developed, which is the best learning algorithm at present. Their contribution is mainly to give a subquadratic algorithm for the light bulb problem which was posed in [Val88] and a reduction from learning Boolean juntas to the light bulb problem.

In real world, multi-labeled data such as questionnaires or DNA sequences (i.e., (A,T,G,C)) seem to be obtained in much more often than binary-labeled one. Then, it is a natural question whether the techniques for learning Boolean juntas can be modified to more general domains. Although the learning problem for $k$-juntas over the finite alphabet size $q \in \mathbb{N}$ was mentioned as a direction for future work in [MOS04], there are much less learnable results in the general case than in the binary case. Obviously, it can be solved in time $O(n^k)$ as in the case $F_2$. The subsequent work [Gop10] implicitly gave the non-trivial $O(n^{0.8k})$-time algorithm in the case where $q = 2^\ell$ for some $\ell \in \mathbb{N}$, by reducing the learning problem to $q - 1$ learning problems for junta functions of the range $\mathcal{Y} = F_2$. However, to the best of our knowledge, any non-trivial learning algorithm for juntas over more general domains has not been known, even in the case where $q = 3$. In this paper, we investigate the learnability of juntas over arbitrary finite fields, and explicitly give the first non-trivial learning algorithm for such classes.

1.2 Our Contributions

Let $F$ be arbitrary finite field of order $q$. Our learnable result mainly follows the framework of PAC (Provably Approximately Correct) learning which was first introduced by L. Valiant [Val84], and let us quickly overview PAC learning. For further details, see the textbook [KV94]. For a family of functions $\{\mathcal{C}_n\}$ where $\mathcal{C}_n \subseteq \{ f : \mathbb{F}^n \rightarrow \mathbb{F} \}$, called a concept class, we say that a concept class $\{\mathcal{C}_n\}$ is (uniformly and exactly) PAC learnable in time $T(n, \delta)$ if there exists a randomized oracle machine $L$ for $\{\mathcal{C}_n\}$, called a learning algorithm, satisfying that

- the algorithm $L$ takes $n \in \mathbb{N}$ and $\delta \in (0, 1)$ as input, and $L$ can access to an example oracle $\mathbb{O}(f)$ which returns an example $(x,f(x)) \in \mathbb{F}^n \times \mathbb{F}$, where $f \in \mathcal{C}_n$ is a fixed (unknown) target function and $x$ is selected uniformly at random over $\mathbb{F}^n$.
- For any $n \in \mathbb{N}$, $\delta \in (0, 1)$, and target function $f \in \mathcal{C}_n$, the algorithm $L$ outputs (some fixed representation of) $f$ in time $T(n, \delta)$ with probability at least $1 - \delta$.

In this paper, we focus on only the case where a concept class is $k$-juntas, which are formally defined as follows.

**Definition 1.** For a function $f : \mathbb{F}^n \rightarrow \mathbb{F}$, we say that a coordinate $i \in \{1, \ldots, n\}$ is relevant if there exist $x, y \in \mathbb{F}^n$ such that $x$ and $y$ differ only at the coordinate $i$ and $f(x) \neq f(y)$. For $k \leq n$, we say that a function $f$ is $k$-junta if $f$ has at most $k$ relevant coordinates.

Strictly speaking, the number of relevant coordinates is given in advance by some fixed function $k : \mathbb{N} \rightarrow \mathbb{N}$, and the concept class $\{\mathcal{C}_n\}$ is defined by $\mathcal{C}_n = \{ f : \mathbb{F}^n \rightarrow \mathbb{F} : f$ is $k(n)$-junta$\}$. Formally, our main result is the following.

1 In the usual definition of the uniform PAC learning model, the learner also takes $\epsilon \in (0, 1)$ as input and outputs a hypothesis $h : \mathbb{F}^n \rightarrow \mathbb{F}$ which is $\epsilon$-close to $f$, that is, $\Pr[f(x) \neq h(x)] \leq \epsilon$. However in learning $k$-juntas, the usual PAC learnability is equivalent to the exact learnability as mentioned in [Blu03]. Thus, for simplicity, we adopt the exact learnability for the definition.
Theorem 1. For any \( \epsilon > 0 \) and \( k = O(\log_2 n) \), \( k \)-juntas over any finite field \( \mathbb{F} \) is PAC learnable in time \( n^{\frac{1}{2} + \epsilon} \cdot \text{poly}(n, q^k, \ln \delta^{-1}) \).

Our learning algorithm mainly follows the line of work by [FGKP06] and consists of two reductions that generalize their reductions for the binary domain to any finite field \( \mathbb{F} \).

Now, we rephrase the learning juntas problem to make it easier to understand. For convenience, we regard an example oracle as input to a learning algorithm. We use the term “with high probability (w.h.p. for short)” to imply with some constant probability. For our purpose, it is enough to design a learning algorithm without taking an accuracy \( \delta \) that succeeds w.h.p., because by \( O(\ln \delta^{-1}) \) times repetition of the algorithm, we can also design a standard PAC learning algorithm taking an accuracy \( \delta \) as input. For the detail, see [HKLWSS]

Learning \( k \)-juntas (over finite field)

Input: \( n, k \in \mathbb{N} \), and an example oracle \( \mathcal{O}(f) \) where \( f : \mathbb{F}^n \to \mathbb{F} \) is \( k \)-junta

Goal: Find all (at most \( k \)) relevant coordinates w.h.p.

Note that if a learning algorithm succeeds in finding all relevant coordinates, then the algorithm can also get the table of the \( k \)-junta \( f \) in time \( \text{poly}(n, q^k) \) w.h.p., because each entry of the table of \( f \) is given by the example oracle with probability at least \( q^{-k} \). Therefore, the above definition is essentially equivalent to the usual PAC learning model for juntas.

In the first step, we reduce the learning juntas problem to another learning problem, learning with discrete memoryless errors (LDME). Simply speaking, the task of LDME is to learn a linear function \( \chi_\alpha : \mathbb{F}^n \to \mathbb{F} \) with \( \alpha \in \mathbb{F}^n \) under the condition that the label may be corrupted with noise, where \( \chi_\alpha(x) = a_1x_1 + \ldots + a_nx_n \). For simplicity, we regard a randomized function as a target function to capture the noise.

Learning with Discrete Memoryless Errors: LDME

Input: \( n, k \in \mathbb{N} \), \( \rho \in [0, 1] \), and an example oracle \( \mathcal{O}(f) \), where \( f : \mathbb{F}^n \to \mathbb{F} \) is randomized

The distribution of the value \( f(x) \) is determined by only a value of \( \chi_\alpha(x) \) (not \( x \) itself), where \( 1 \leq |\alpha| \leq k \). The target function has correlation with \( \chi_\alpha \) as follows:

\[
\text{Cor}(f, \chi_\alpha) := \mathbb{E}_{x,f}[e(f(x))e(\chi_\alpha(x))] \geq \rho.
\]

Goal: Find the coefficients \( a_\alpha \in \mathbb{F}^n \) for some \( a \in \mathbb{F} \setminus \{0, 1\} \) w.h.p.

For simplicity, we call the above function \( \chi_\alpha \) as a target linear function. The reason why we allow the algorithm to output \( a\alpha \) for some \( a \in \mathbb{F} \setminus \{0, 1\} \) is that the linear function \( \chi_{a\alpha} \) may also have large correlation in the above condition.

LDME, introduced first by [Gop10], is a variant of the well-known learning with errors problem (LWE) which has been known as one of most challenging problems in learning theory and even used as a hardness assumption in cryptography (see [Reg05,Reg10]). The difference between them is the noise setting. In LWE, the (unknown) distribution of noise is fixed in advance, while in LDME, the distribution is determined by the value of the target linear function, in other words, there exist \( q \) unknown distributions of the noise. Note that, in addition, we adopt slightly different condition about correlation from [Gop10]. In previous formulation, the probability that the (randomized) target function \( f \) agrees on the target linear function \( \chi_\alpha \) is bounded below. However, in our formulation by inner product, the probability is not always large. For example, even in the case where the subtraction \( f - \chi_\alpha \) is close to constant except for 0, our condition about large correlation may hold.

We first present the reduction from the learning juntas problem to LDME, which is a generalization of the binary case in [FGKP06]. The detail will be given in Section 3.

Theorem 2. If there exists a learning algorithm for solving LDME in time \( T(n, k, \rho) \), then there exists a learning algorithm for \( k \)-juntas over \( \mathbb{F} \) in time \( T(n, k, 1/q^{2k+2}) \cdot \text{poly}(n, q^k) \).

In the second step, we reduce LDME to the light bulb problem (LBP), which is a fundamental problem in machine learning and data analysis introduced by [Val15]. Roughly speaking, the task of LBP is to find a correlated pair from the other uncorrelated pairs. The formal definition is as follows:
**Light Bulb Problem: LBP**

Input: a set $S = \{x^1, \ldots, x^n\}$ of $n$ vectors, where $x^i \in \{\pm 1\}^d$ for each $i \in [n]$, and $\rho \in (0, 1]$

The instance $S$ contains one correlated pair $(x^{i^*}, x^{j^*})$ satisfying $\langle x^{i^*}, x^{j^*}\rangle \geq \rho d$, and the other pairs of vectors are selected independently and uniformly at random.

Goal: Find indices of the correlated pair $(i^*, j^*)$.

It is obvious that LBP is solved in time $O(n^2d)$ by calculating inner products of all pairs. As a breakthrough result, the first subquadratic algorithm for LBP has been found by [Val15]. Moreover, in the case where $\rho \geq n^{-\Theta(1)}$, a faster algorithm was presented by [KKK18]. Other subquadratic algorithms also have been proposed in [KKKC16, Alm18].

**Fact 1 ([KKK18, Corollary 2.2]).** For any $0 < \epsilon < \omega/3$ and $n^{-\Theta(1)} < \rho < 1$, if $d \geq 5\rho^{-\frac{3\omega}{2\epsilon}} - \frac{3\epsilon}{\omega}$, then there is a randomized algorithm for solving LBP with probability $1 - o(1)$ in time $\tilde{O}(n^{\frac{\omega}{3\epsilon}} + \rho^{-\frac{3\omega}{2\epsilon}} - \frac{3\epsilon}{\omega})$.

We present the second reduction from LDME to LBP. Note that the reduced problem is a kind of binary problem regardless of the alphabet size of the original problem. The detail will be given in Section 4.

**Theorem 3.** Assume that there exist $d \geq \Omega(\frac{\log N}{\rho})$ and an algorithm for solving LBP of degree $d$ in time $T(N, \rho)$ w.h.p., where $N$ is the number of vectors in LBP. Then for any target linear function $\chi_\alpha : \mathbb{F}^n \to \mathbb{F}$ (1 $\leq |\alpha| \leq k$) and any correlation $\rho$, LDME is solved w.h.p. in time

$$\text{poly}(n, \rho^{-1}d) \cdot T\left((qn)^{\frac{\omega}{3\epsilon}}, \frac{\rho}{2q^{\epsilon}}\right).$$

In our reduction, the size of data is stretched from $n$ to $O(n^{\frac{\omega}{3\epsilon}})$. Thus, the naive quadratic algorithm for LBP does not improve the trivial upper bound on the running time of LDME at all. However, by combining our reductions with the subquadratic algorithm for LBP, a non-trivial learnable result immediately holds over any finite field, and it is easily checked that Theorem 1 follows from Theorems 2 and 3 and Fact 1.

**2 Preliminaries**

We use log to denote logarithm of the base 2, and ln to denote natural logarithm. For any integer $n$, we define a set $[n] := \{1, 2, \ldots, n\}$. Let $\omega$ be the exponential factor of the running time $O(n^\omega)$ of fast $n \times n$ matrix multiplication, with best known bound of $\omega < 2.3728639$ in [Le 14].

In this paper, we focus on learnability over a finite field $\mathbb{F}$ of order $|\mathbb{F}| = q$. For $\alpha \in \mathbb{F}^n$, we define the weight of $\alpha$ by $|\alpha| = |\{i \in [n] : \alpha_i \neq 0\}|$. For $\alpha \neq 0^n$, we also define its initial $\text{init}(\alpha)$ by the first non-zero value of $\alpha$, that is, $\text{init}(\alpha) = v$ iff there exists $i \in [n]$ such that $\alpha_i = v$ and $\alpha_j = 0$ for each $1 \leq j < i$. It is easily checked that if $\alpha, \alpha' \in \mathbb{F}^n \setminus \{0^n\}$ ($\alpha \neq \alpha'$) satisfy $\text{init}(\alpha) = \text{init}(\alpha')$, then there is no $c \in \mathbb{F}$ such that $\alpha = c \alpha'$ (i.e., $\alpha$ and $\alpha'$ are linearly independent over $\mathbb{F}^n$). For $\alpha \in \mathbb{F}^n$, we define a linear function $\chi_\alpha$ by $\chi_\alpha(x) = \alpha_1x_1 + \cdots + \alpha_nx_n$ with arithmetic in $\mathbb{F}$.

For any $J \subseteq [n]$, we define a subspace $\mathbb{F}^J \leq \mathbb{F}^n$ by $\mathbb{F}^J = \{x \in \mathbb{F}^n : x_i = 0 \text{ for each } i \in J\}$, where $J = [n] \setminus J$. For any $\alpha \in \mathbb{F}^n$ and $J \subseteq [n]$, we also define $\alpha^J \in \mathbb{F}^J$ by $\alpha^J_i = \alpha_i$ if $i \in J$, otherwise (i.e., $i \in \bar{J}$), $\alpha^J_i = 0$.

For a subset $J \subseteq [n]$, we call a pair $(J, \bar{J})$ a partition of $[n]$. In addition, if $J$ consists of cyclically consecutive $\lfloor n/2 \rfloor$ coordinates, we say that the partition $(J, \bar{J})$ is consecutive. Obviously an index set $[n]$ has exactly $n$ consecutive partitions. Now we introduce the following useful lemma.

**Lemma 1.** For any $\alpha \in \mathbb{F}^n$ with $|\alpha| = k$, there exist at least one consecutive partition $(J, \bar{J})$ which satisfies that $|\alpha^J| = \lfloor k/2 \rfloor$ and $|\alpha^\bar{J}| = \lfloor k/2 \rfloor$. 4
Proof. For convenience, we say $i \in [n]$ is supportive if $\alpha_i \neq 0$. For $i \in [n]$, let $J_i \subset [n]$ be a subset which consists of cyclically consecutive $[n/2]$ coordinates from $i$, and $m_i$ be the number of supportive coordinates contained in $J_i$. For $J_1$, the remaining $[n/2]$ coordinates contain $k - m_1$ supportive coordinates, thus $k - m_1 \leq m_{[n/2]+1} \leq k - m_1 + 1$ (because $J_{[n/2]+1}$ also contains the first coordinate in the case where $n$ is odd). If $m_1 = [k/2]$, then $(J_1, \bar{J}_1)$ is a desired partition. So we assume that $m_1 \neq [k/2]$. If $m_1 \leq [k/2] - 1$, we have $m_{[n/2]+1} \geq k - m_1 \geq [k/2] + 1 \geq [k/2]$. Otherwise if $m_1 \geq [k/2] + 1$, we have $m_{[n/2]+1} \leq k - m_1 + 1 \leq [k/2]$. Since the difference between $m_i$ and $m_{i+1}$ must be 0 or $\pm 1$, there exist at least one coordinate $i$ satisfying $m_i = [k/2]$ in any cases.

We use the term a truth table to denote a table of values of a function over $\mathbb{F}$ as in the binary case. For any function $f : \mathbb{F}^n \to \mathbb{F}$ and value $a \in \mathbb{F}$, we define a function $af : \mathbb{F}^n \to \mathbb{F}$ by $af(x) = a \cdot f(x)$. For a subset $J \subseteq [n]$, we define a restriction $\rho$ on $J$ as a partial assignment to $J$, and we use $f_{\rho} : \mathbb{F}^{|J|} \to \mathbb{F}$ to denote the restricted function of which variables are partially assigned $\rho$ on $J$. We use $|\rho|$ to denote the size of a restriction $\rho$, that is, $|\rho| = |J|$.

For a finite set $S$, we write $x \leftarrow u S$ for a random sampling of $x$ according to the uniform distribution over $S$. In the subsequent discussions, we assume the basic facts about probability theory, especially, pairwise independence and the union bound. We will make extensive use of the following tail bound.

**Fact 2** (Hoeffding inequality [Hoe63]). For real values $a, b \in \mathbb{R}$, let $X_1, \ldots, X_m$ be independent and identically distributed random variables with $X_i \in [a, b]$ and $\mathrm{E}[X_i] = \mu$ for each $i \in [m]$. Then for any $\epsilon > 0$, the following inequality holds:

$$\Pr \left[ \frac{1}{m} \sum_{i=1}^{m} X_i - \mu > \epsilon \right] < 2e^{-\frac{2\epsilon^2 m}{b^2}}.$$  

2.1 Fourier Analysis

We introduce some basics of Fourier analysis. For further details, see a textbook [OD14]. For each $a \in \mathbb{F}$, let $e(a) := e^{\frac{2\pi i a}{m}} \in \mathbb{C}$. For $a, b \in \mathbb{F}$, it is easy to see that $e(a + b) = e(a)e(b)$ and $e(-a) = e(a)$. For any two functions $f, g : \mathbb{F}^n \to \mathbb{C}$, we define their inner product by $(f, g) = \mathrm{E}_{x \in \mathbb{F}^n} f(x)g(x)$. Then a family \{$e(\chi_\alpha)$\}$_{\alpha \in \mathbb{F}^n}$ of $q^n$ functions forms an orthonormal basis, that is, $(e(\chi_\alpha), e(\chi_\beta)) = 1$ if $\alpha = \beta$, otherwise, $(e(\chi_\alpha), e(\chi_\beta)) = 0$. Therefore, for any function $f : \mathbb{F}^n \to \mathbb{F}$, a function $e(f)$ has a unique Fourier expansion form as $e(f(x)) = \sum_{\alpha} \hat{f}(\alpha)e(\chi_\alpha(x))$, where $\hat{f}(\alpha)$ is a Fourier coefficient given by $\hat{f}(\alpha) = (e(f), e(\chi_\alpha))$. We will use the following fact which can be easily checked.

**Fact 3.** If a function $f : \mathbb{F}^n \to \mathbb{F}$ satisfies that $\hat{f}(\alpha) \neq 0$ for some $\alpha \in \mathbb{F}^n$, then all coordinates $i \in [n]$ with $\alpha_i \neq 0$ are relevant.

2.2 $(a, A)$-Projection

We define a notion of $(a, A)$-projection which is a generalization of $A$-projection in $\mathbb{F}_2$ by [FGKP06].

**Definition 2** ($(a, A)$-projection). For $f : \mathbb{F}^n \to \mathbb{F}$, $A \in \mathbb{F}^{m \times n}$, and $a \in \mathbb{F}$, we define $f^a_A : \mathbb{F}^n \to \mathbb{C}$ by

$$f^a_A(x) = \sum_{\alpha : Aa = a^m} \hat{f}(\alpha)e(\chi_\alpha(x)) = \begin{cases} \sum_{\alpha : Aa = 1} \hat{f}(\alpha)e(a\chi_\alpha(x)) & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$$

**Lemma 2.** For $A \in \mathbb{F}^{m \times n}$ and $a \in \mathbb{F}$,

$$f^a_A(x) = \mathrm{E}_{p \sim \mathbb{F}^m}[e(a(f(x + A^Tp))e(\chi_{a^m}(p))].$$  \hspace{1cm} (1)

Moreover, if an example and its label are given by $(x, b) = (y - A^Tp, f(y) - \sum p_i)$ for $y \leftarrow u \mathbb{F}^n$ and $p \leftarrow u \mathbb{F}^m$, then for any $x \in \mathbb{F}^n$,

$$\mathrm{E}_b[e(abx)] = f^a_A(x),$$

where $b_x$ denotes a random variable according to the distribution of $b$ conditioned on the example $x$.  

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Proof. Let $g : \mathbb{F}^n \to \mathbb{C}$ be the right-hand side of $[\text{I}]$. It is enough to show that for any $\alpha \in \mathbb{F}^n$, \[
g(\alpha) = \hat{f_A}(\alpha).
\]
From the definition of $\hat{g}(\alpha)$, it follows that
\[
\hat{g}(\alpha) = E_x[g(x)e(\chi_\alpha(x))] = E_x[E_p[e(af(x + ATp))e(\chi_\alpha(p))]e(\chi_\alpha(x))]
\]
\[
= E_p[E_x[e(af(x + ATp))e(\chi_\alpha(x + ATp))]e(\chi_\alpha(Adp))e(\chi_\alpha(p))]
\]
\[
= \hat{af}(\alpha)E_p[e(\chi_{\alpha A}(p))e(\chi_{\alpha m}(p))]
\]
\[
= \hat{af}(\alpha)\mathbb{1}[A\alpha = a^m] = \hat{f_A}(\alpha).
\]
For the second part, notice that for any $x \in \mathbb{F}^n$ and $p \in \mathbb{F}^m$, exactly one element $y_p \in \mathbb{F}^n$ satisfying $y_p - ATp = x$ is determined. Therefore,
\[
E_{a_x}[e(ab_x)] = \sum_{p \in \mathbb{F}^m} q^{-m} \left( e(af(y_p) - a \sum p_i) \right)
\]
\[
= E_p[e(af(x + ATp) - \chi_{am}(p))]
\]
\[
= E_p[e(af(x + ATp))e(\chi_{am}(p))] = f_A^a(x) \quad \therefore \quad [\text{I}].
\]

2.3 Statistical Distance and Character Distance

We introduce the following two distances about random variables taking values in $\mathbb{F}$, which was introduced first in [BV10] and makes our analysis easier.

Definition 3 (statistical/character distance). For random variables $X, X'$ taking values in $\mathbb{F}$, we define their statistical distance $SD(X, X')$ by
\[
SD(X, X') = \frac{1}{2} \sum_{x \in \mathbb{F}} |Pr[X = x] - Pr[X' = x]|,
\]
and we also define their character distance $CD(X, X')$ by
\[
CD(X, X') = \max_{a \in \mathbb{F}} |E[e(aX)] - E[e(aX')]|.
\]

Fact 4 ([BV10, Claim 33]). For any random variables $X, X'$ taking values in $\mathbb{F},$
\[
CD(X, X') \leq 2 \cdot SD(X, X') \leq \sqrt{q-1} \cdot CD(X, X').
\]
In particular, $SD(X, X') = 0$ if and only if $CD(X, X') = 0$.

3 Reduction from Learning Junta to LDME

In this paper, for simplicity, we assume the following computational model:

- A learning algorithm can uniformly select an element in $\mathbb{F}$ with probability 1 in constant steps. In fact, a usual randomized model with binary coins may fail in selecting such random elements with exponentially small probability, but we can deal with this probability as a general confidence error. For the same reason, we allow algorithms to flip a biased coin which lands heads up with a rational probability (of the polynomial-time computable denominator).

- A learning algorithm with an example oracle $\mathbb{O}(f)$, where $f : \mathbb{F}^n \to \mathbb{F}$ is $k$-junta, can simulate an oracle $\mathbb{O}(f_{|\rho})$ w.r.t. any restriction $\rho$ of the size $|\rho| \leq k$. In fact, this simulation is done by taking several examples until getting an example consistent with $\rho$. Since the probability that an example consistent with $\rho$ is sampled is at least $q^{-k}$, by taking poly$(q^k)$ examples, the failure probability becomes exponentially small. We can also deal with this error probability as a general confidence error $\delta$, and the additional running time is at most poly$(n, q^k, \ln \delta^{-1})$.
3.1 Overview

Our learning algorithm (main1) has two phases, a checking phase and a detection phase, and repeats them alternately as the MOS algorithm by [MOS04]. The algorithm starts the checking phase with a set $R$ empty. If the algorithm finds relevant coordinates, then they will be put in $R$. In the checking phase, the algorithm verifies whether $R$ contains all relevant coordinates of the target function $f$ by examining that restricted functions $f|\rho$ are constant for all restrictions $\rho$ on $R$. If $R$ contains all relevant coordinates, then the algorithm outputs $R$ and halts, otherwise moves on to the detection phase. In the detection phase, the algorithm will find at least one relevant coordinate, add them to $R$, and will move on to the checking phase. Since the algorithm finds at least one relevant coordinate in each loop, the number of repetitions is at most $k$.

In the detection phase, we reduce the task of finding relevant coordinates to LDME in the subroutine addRC, where the target linear function $\chi_\alpha$ satisfies that $\hat{f}(\alpha a) \neq 0$ for some $a \in \mathbb{F} \setminus \{0\}$ (if there is not such an $\alpha$, $f$ must be constant and the algorithm must halt in the checking phase). Therefore, if the algorithm for LDME finds $\alpha$ (up to constant factor), then the learning algorithm can find at least one relevant coordinate $i$ such that $\alpha_i \neq 0$ by Fact 3.

3.2 Algorithms and Analysis

First we introduce two simple subroutines. For the proofs of their correctness (i.e., Lemmas 3 and 4), see Appendix A.

Algorithm 1 checks whether the target function is constant or not. We will use this subroutine to determine the end of learning, because the algorithm knows all relevant coordinates if and only if the restricted functions must be constant for all restrictions to the set of found coordinates.

Algorithm 1 Check Constant (const)

Input: $n \in \mathbb{N}$, $k \in \mathbb{N}$, $\delta \in (0, 1)$, $\mathbb{O}(f)$, where $f: \mathbb{F}^n \rightarrow \mathbb{F}$ is $k$-junta
Output: $a \in \mathbb{F}$ if $f \equiv a$ (constant), otherwise $\perp$

1: $m := \lceil q^k \ln \frac{2}{\delta} \rceil$
2: $(x^{(1)}, b^{(1)}), \ldots, (x^{(m)}, b^{(m)}) \leftarrow \mathbb{O}(f)$
3: if $b^{(i)} = a$ for each $i \in [m]$ return($a$) otherwise return($\perp$)

Lemma 3. For any input $(n, k, \delta, \mathbb{O}(f))$, const outputs $a \in \mathbb{F}$ if $f \equiv a$, otherwise $\perp$ with probability at least $1 - \delta$.

Proof. See Appendix A.1

Algorithm 2 checks whether the given $\alpha$ has nonzero entry at an irrelevant coordinate. In the following detection step, our learning algorithm may find an undesirable candidate $\alpha$, thus we must check whether the found $\alpha$ consists of only a part of relevant coordinates by this subroutine.

Algorithm 2 Check Relevant Coordinates (checkRC)

Input: $n \in \mathbb{N}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{F}^n$, $\delta \in (0, 1)$, $\mathbb{O}(f)$, where $f: \mathbb{F}^n \rightarrow \mathbb{F}$ is $k$-junta
Output: $\hat{f}(\alpha) \neq 0 \Rightarrow true$; $\alpha_i \neq 0$ for some irrelevant $i \in [n] \Rightarrow false$

1: $m := \lceil 2q^k \ln \frac{3}{\delta} \rceil$
2: for all $a \in \mathbb{F}$ do
3: $(x^{(1)}, b^{(1)}), \ldots, (x^{(m)}, b^{(m)}) \leftarrow \mathbb{O}(f)$
4: if $\sum_i \mathbb{I}(b^{(i)} - \chi_\alpha(x^{(i)}) = a) \geq (\frac{1}{q} + \frac{1}{2q^k})m$ then return(true)
5: end for
6: return(false)
**Lemma 4.** For any input \((n, k, \alpha, \delta, \mathcal{O}(f))\), if \(\hat{f}(\alpha) \neq 0\), then checkRC outputs true with probability at least \(1 - \delta\). Otherwise if \(\alpha_i \neq 0\) for an irrelevant coordinate \(i \in [n]\), checkRC outputs false with probability at least \(1 - \delta\).

Note that, in general, the condition that \(\hat{f}(\alpha) = 0\) does not imply the existence of an irrelevant coordinate \(i \text{ s.t. } \alpha_i \neq 0\). In the above lemma, we do not say anything about such cases.

**Proof.** See Appendix A.2

The following algorithm (Algorithm 3) is a core part of our reduction, which reduces the task of finding relevant coordinates to LDME, checks whether the found coordinates are indeed relevant, and returns them to the main algorithm. Let LDME\((n, k, \rho)\) be the learning algorithm for solving LDME.

**Algorithm 3** Add Relevant Coordinates (addRC)

**Input:** \(n \in \mathbb{N}, k \in \mathbb{N}, \delta \in (0, 1), R \subseteq [n], \mathcal{O}(f)\), where \(f : \mathbb{F}^n \rightarrow \mathbb{F}\) is \(k\)-junta

1. **for all** \(\rho\) on \(R\) **do**
2. 2. **for** \(M := [\rho^{k+2} \ln \frac{1}{\delta}]\) **times** do
3. 3. \(A \leftarrow \mathbb{F}^{k+1} \times (\mathbb{F}^n \setminus R)\)
4. 4. **for all** \(a \leftarrow \mathbb{F} \setminus \{0\}\) **do**
5. 5. **execute** LDME\((n - |R|, k, 1/q^{2k+2})\) with accuracy \(\delta/4\) (by repetition)
   where the oracle is simulated as follows:
   1. **get an example** \((x, b) \leftarrow \mathcal{O}(f|_\rho)\)
   2. **select** \(p \leftarrow \mathbb{F}^{k+1}\)
   3. \((x', b') := (x - A^T p, a \cdot (b - \sum_j p_j))\) and return \((x', b')\)
6. 6. **if** LDME finds \(\alpha \in \mathbb{F}^n\) and checkRC\((n - |R|, k, \alpha, \delta/\sqrt{Mq^{k+2} \mathcal{O}(f|_\rho)})\) for some \(\alpha' \in \mathbb{F} \setminus \{0\}\)
7. 7. **then** add all \(i\) s.t. \(\alpha_i \neq 0\) to \(R\) and return\((R)\)
8. 8. **end** for
9. 9. **end** for
10. 10. **end** for

**Lemma 5.** If the algorithm LDME solves LDME in time \(T(n, k, \rho)\) w.h.p. and \(R\) does not contain all relevant coordinates, then the subroutine addRC finds at least one relevant coordinate with probability at least \(1 - \delta\), and its running time is bounded above by \(T(n, k, 1/q^{2k+2}) \cdot \text{poly}(n, k, \ln \delta^{-1})\).

The correctness of Algorithm 3 will be given in Section 3.3.

The Algorithm 4 is our learning algorithm. Now we prove the learnability of Algorithm 4 by Lemma 5.

**Theorem 2** immediately follows from Lemma 6.

**Algorithm 4 main1**

**Input:** \(n \in \mathbb{N}, k \in \mathbb{N}, \delta \in (0, 1), \mathcal{O}(f)\), where \(f : \mathbb{F}^n \rightarrow \mathbb{F}\) is \(k\)-junta

**Output:** \(R \subseteq [n]\) consisting of all relevant coordinates

1. \(R := \emptyset\)
2. **loop**
3. 3. **if** \(|R| > k\) **then** halt and output “error”
4. 4. **if** \(\bot \neq \text{const}(n - |R|, k, \delta/(k+1)q^k + k), \mathcal{O}(f|_\rho))\) for all restrictions \(\rho\) on \(R\) **then**
5. 5. **Halt** and output \(R\)
6. **else**
7. 7. \(R \leftarrow \text{addRC}(n, k, \delta/(k+1)q^k + k), R, \mathcal{O}(f))\)
8. **end if**
9. **end loop**
Lemma 6. If the algorithm LDME solves LDME in time $T(n, k, \rho)$ w.h.p., then the algorithm **main** outputs all relevant coordinates for any $k$-junta function $f : \mathbb{F}^n \to \mathbb{F}$ with probability at least $1 - \delta$, and its running time is bounded above by $T(n, k, 1/q^{2k+2}) \cdot \text{poly}(n, k, \ln \delta^{-1})$.

**Proof.** First we show that the algorithm halts at most $k + 1$ loops assuming that all subroutines succeed. If $R$ contains all relevant coordinates, then for all restrictions $\rho$ on $R$, the restricted functions must be constant, thus the algorithm halts and outputs $R$ in line 5. On the other hand, if $R$ does not contain some relevant coordinates, **addRC** adds at least one relevant coordinate to $R$ by Lemma 5. Since $f$ has at most $k$ relevant coordinates, **addRC** is executed at most $k$ times, and the main loop is repeated at most $k + 1$ times.

In fact, the algorithm may fail in executing **const** and **addRC**. The number of the executions is at most $(k + 1)q^k + k$. Thus if we set their accuracy parameter as $\delta/((k + 1)q^k + k)$, then by the union bound, the total failure probability is bounded above by $\delta$. By Lemma 5, the total running time is at most

$$(k + 1)q^k \cdot O \left( n \cdot q^k \ln \left( \frac{(k + 1)q^k + k}{\delta} \right) \right) + k \cdot T(n, k, 1/q^{2k+2}) \cdot \text{poly} \left( n, q^k, \ln \left( \frac{(k + 1)q^k + k}{\delta} \right) \right)$$

$$= T(n, k, 1/q^{2k+2}) \cdot \text{poly}(n, q^k, \ln \delta^{-1}).$$

\[ \square \]

3.3 Proof of Lemma 5

In this subsection, we show the correctness of the reduction. First, we introduce the following simple fact.

**Claim 1.** For any vectors $\alpha, \beta \in \mathbb{F}^n \setminus \{0^n\}$, the following holds:

(i) If $\beta \neq c\alpha$ for any $c \in \mathbb{F}$ (i.e., $\alpha$ and $\beta$ are linearly independent), then for any $a, b \in \mathbb{F}$,

$$\Pr_x[x^T \alpha = a \text{ and } x^T \beta = b] = \frac{1}{q^2}.$$  

(ii) If $\beta = c\alpha$ ($c \neq 0$), then for any $a, b \in \mathbb{F}$,

$$\Pr_x[x^T \alpha = a \text{ and } x^T \beta = b] = \begin{cases} 1/q & (\text{if } b = ca) \\ 0 & (\text{otherwise}). \end{cases}$$

In other words, if $\alpha, \beta(\neq 0^n)$ satisfies the condition (i), then $\chi_\alpha(x)$ and $\chi_\beta(x)$ are uniformly and pairwise independently distributed w.r.t. the uniform selection of $x \in \mathbb{F}^n$.

**Proof.** See Appendix 3.1 \[ \square \]

Next, we show that for small subspace $\mathbb{F}^D$, only one vector $\alpha \in \mathbb{F}^D$ satisfies $A\alpha = 1^m$ with non-negligible probability w.r.t. the uniform selection of $A$.

**Claim 2.** For any subset $D \subseteq [n]$ ($|D| \leq k$), $\alpha \in \mathbb{F}^D \setminus \{0^n\}$, and $m \geq k$,

$$\Pr_{A \sim \mathbb{F}^{m \times n}} [A\alpha = 1^m \text{ and } A\beta \neq 1^m \text{ for each } \beta \in \mathbb{F}^D \setminus \{\alpha\}] \geq \frac{q^{m-k} - 1}{q^{2m-k}}$$

Especially, if the parameter $m$ is selected as $m = k + 1$, then

$$\Pr_{A \sim \mathbb{F}^{(k+1) \times k}} [A\alpha = 1^{k+1} \text{ and } A\beta \neq 1^{k+1} \text{ for each } \beta \in \mathbb{F}^D \setminus \{\alpha\}] \geq \frac{1}{q^{k+2}}$$
Proof. The second part immediately follows from the first one, thus we give only a proof of the first part. It is sufficient to show that
\[
\Pr_A[A\alpha = 1^m] = \frac{1}{q^m} \text{ and } \Pr_A[A\beta \neq 1^m \text{ for each } \beta \in F^D \setminus \{\alpha\}|A\alpha = 1^m] \geq 1 - \frac{1}{q^{m-k}}.
\] (2)
Since \(\alpha \neq 0^m\), \(\Pr_{x \sim F^n}[x^T \alpha = 1] = q^{-1}\) holds, thus we have \(\Pr_A[A\alpha = 1^m] = q^{-m}\). By Claim 1, for any \(\beta \neq \alpha\), we have
\[
\Pr_x[x^T \beta = 1 \text{ and } x^T \alpha = 1] \leq \frac{1}{q^2}.
\]
Therefore,
\[
\Pr_x[x^T \beta = 1|x^T \alpha = 1] = \frac{\Pr_x[x^T \beta = 1 \text{ and } x^T \alpha = 1]}{\Pr_x[x^T \alpha = 1]} \leq \frac{q}{q^2} = \frac{1}{q}
\]
and
\[
\Pr_A[A\beta = 1^m|A\alpha = 1^m] \leq \frac{1}{q^m}.
\]
Since \(|D| \leq k\), the number of vectors \(\beta \in F^D\) is at most \(q^k\). Hence, by the union bound,
\[
\Pr_A[\exists \beta \in F^D \setminus \{\alpha\} \text{ s.t. } A\beta = 1^m|A\alpha = 1^m] \leq \frac{q^k}{q^m},
\]
which is equivalent to the second part of the inequality (2). \(\Box\)

Let \(f : F^n \rightarrow F\) be \(k\)-junta and \(D \subseteq [n]\) be the set of relevant coordinates of \(f\). In the following claims, we assume that the event in Claim 2 occurs for \(D\) and \(\alpha \in F^D\) satisfying \(\hat{f}(\alpha) \neq 0\) with \(m = k + 1\). Note that, in this case, the projected function satisfies \(f_A^\alpha \equiv \hat{a}f(\alpha)e(a\chi_\alpha)\) for any \(a \in F \setminus \{0\}\), because \(af\) is also \(k\)-junta that has the same domain \(D\). In addition, we assume that the example \((x, b)\) is simulated as follows: for \(y \leftarrow F^n\) and \(p \leftarrow F^{k+1}\),
\[
(x, b) := \left( y - A^T p, f(y) - \sum_j p_j \right).
\]

Claim 3. Let \(a \in F^n\). If the \((a, A)\)-projected functions satisfy \(f_A^\alpha \equiv \hat{a}f(\alpha)e(a\chi_\alpha)\) for all \(a \in F \setminus \{0\}\), and the noisy example \((x, b)\) is simulated as the above, then the conditional distribution of \(b_x\) is determined by only the value of \(\chi_\alpha(x)\), that is, for \(x, x' \in F^n\), if \(\chi_\alpha(x) = \chi_\alpha(x')\), then \(SD(b_x, b_{x'}) = 0\).

Proof. By Lemma 2, \(E[e(ab_x)] = f_A^\alpha(x) = \hat{a}f(\alpha)e(a\chi_\alpha(x))\) for any \(a \in F\) \((a \neq 0)\). By Fact 4
\[
\chi_\alpha(x) = \chi_\alpha(x') \implies E[e(ab_x)] = \hat{a}f(\alpha)e(a\chi_\alpha(x)) = \hat{a}f(\alpha)e(a\chi_\alpha(x')) = E[e(ab_{x'})] \text{ for any } a \in F
\]
\[
\iff CD(b_x, b_{x'}) = \max_{a \in F} |E[e(ab_x)] - E[e(ab_{x'})]| = 0 \iff SD(b_x, b_{x'}) = 0.
\]
\(\Box\)

In the algorithm addRC, an example of LDWE is simulated as \((x, a \cdot b)\) for some \(a \in F \setminus \{0\}\). Obviously if the distribution of \(b_x\) is determined, then the distribution of \(a \cdot b_x\) is also determined. In addition, it is also obvious that the value of \(\chi_\alpha(x) = a^{-1}\chi_{a\alpha}(x)\) is determined by the value of \(\chi_{a\alpha}(x)\). Therefore, the above claim means that the simulated oracle in the algorithm addRC returns indeed an instance of LDME for the target linear function \(\chi_{a\alpha}(x)\). Finally, we show that the simulated instance has a large correlation with the linear function \(\chi_{a\alpha}\) if the algorithm addRC chooses a “good” \(a \in F \setminus \{0\}\).

Claim 4. We assume the same notations and conditions in Claim 3. In addition, if the \(k\)-junta function \(f\) satisfies \(\hat{f}(\alpha) \neq 0\) and the parameter \(m\) is selected as \(m = k + 1\), \((i.e., A \in G(F^{k+1}) \times F^n)\), then
\[
\max_{a \in F \setminus \{0\}} Cor(ab, \chi_{a\alpha}) = \max_{a \in F \setminus \{0\}} |E_x[E_{(x, b)}[e(ab - a\chi_\alpha(x))]| \geq \frac{1}{q^{2k+2}}.
\]
Proof. Let $U_q$ be a uniformly distributed random variable over $F$. By Fact 4,
\[
(CD(b - \chi_\alpha(x), U_q) = \max_{a \in F} |E_{(x,b)}[e(ab - a\chi_\alpha(x))]| - E[e(aU_q)]| \geq \frac{2}{\sqrt{q - 1}} SD(b - \chi_\alpha(x), U_q). \tag{3}
\]
Now we show the RHS of (3) is non-zero. For this, it is enough to show that $E_{(x,b)}[e(b - \chi_\alpha(x))] \neq 0$ because it implies that the distribution of $b - \chi_\alpha(x)$ is not statistically identical to the uniform distribution $U_q$. By Lemma 2 and the assumption in Claim 3, we have
\[
E_{(x,b)}[e(b - \chi_\alpha(x))] = E_x[E_b[e(b_x)e(\chi_\alpha(x)))] = f(\alpha) E_x[e(\chi_\alpha(x))e(\chi_\alpha(x))] = f(\alpha) \neq 0.
\]
Since the parameter $m$ was selected as $m = k + 1$ and $f$ is $k$-junta, the value of $b := f(y) - \sum_j p_j$ is determined only by $p \in F^{k+1}$ and at most $k$ coordinates of $y \in F^n$. Moreover since $\alpha$ does not have non-zero values at irrelevant coordinates by Fact 2, the value of $\chi_\alpha(x) = \chi_\alpha(y - A^T p)$ depends only on the same $k$ coordinates of $y$ and $k + 1$ coordinates of $p$. Because $b - \chi_\alpha(x)$ is determined by at most $2k + 1$ values which are selected at uniformly random, $SD(b - \chi_\alpha(x), U_q) \neq 0$ implies $2 \cdot SD(b - \chi_\alpha(x), U_q) \geq 1/q^{2k+1}$.

Since the RHS of (3) is non-zero, $a = 0$ cannot maximize the LHS of (3). Hence we have
\[
\max_{a \in F \setminus \{0\}} |E[e(ab - a\chi_\alpha(x))]| = \max_{a \in F \setminus \{0\}} |E[e(ab - a\chi_\alpha(x))]| - E[e(aU_q)]| \geq \frac{1}{\sqrt{q - 1}} \cdot 2 \cdot SD(b - \chi_\alpha(x), U_q) \geq \frac{1}{q} \cdot \frac{1}{q^{2k+1}}.
\]

Now we give the proof of Lemma 5.

Proof (Lemma 5). First, for simplicity, let us assume that execution of checkRC succeeds with probability 1. If $f$ is not constant and some relevant coordinates are not contained in $R$, then there exists a restriction $\rho$ on $R$ such that the restricted function $f|_\rho$ is not constant. In this case, there exists $\alpha \in F^{|R|}$ such that $|\alpha| \geq 1$ and $f|_\rho(\alpha) \neq 0$.

For convenience, we regard $f$ as the restricted function as $f := f|_\rho$. For the set $D$ of relevant coordinates of $f$, $|D| \leq k$. By Claim 2 and the argument following Claim 2 for all $a' \neq 0$, $f_{A'} = a'f(a'\alpha)e(a'\chi_\alpha)$ with probability at least $1/q^{k+2}$ w.r.t. the uniform selection of $A$. Since addRC tries to select $A$ more than $q^{k+2} \ln 4/\delta$ times, at least one selected $A$ satisfies this condition with probability at least $1 - \delta/4$. Thus in the following argument, we assume that the algorithm addRC succeeds in selecting such an $A$.

If the algorithm addRC succeeds in selecting the above matrix $A$, then by Claims 5 and 3 there exists $\alpha \neq 0$ such that the simulated noisy example in line 5 corresponds to the example from LDME of the correlation $\rho \geq 1/q^{2k+2}$. By the assumption, the repetition of LDME recovers $\alpha$ up to constant factor (i.e., finds $a'\alpha$ for some $a' \in F$) with probability at least $1 - \delta/4$. If LDME is solved successfully, then at least one relevant coordinate is added to $R$ in line 7.

If the algorithm addRC fails in selecting $A$ and $\alpha$, the subroutine LDME may return some undesirable candidate. In this case, the subroutine checkRC returns false in line 6 and irrelevant coordinates are not added to $R$. Therefore, by the union bound, the failure probability is at most $\delta/4 + \delta/4 = \delta/2$ under the condition that checkRC succeeds with probability 1.

In fact, our algorithm checkRC may fail. Since the number of executions of checkRC is at most $Mq^{k+2}$, by the union bound, the probability that some executions of checkRC fail is at most $\delta/2$. Thus, the total failure probability is at most $\delta$. The total running time is bounded above by
\[
M \cdot \text{poly}(n, q^k, \ln 1/\delta) \cdot T(n, k, 1/q^{2k+2}) = \text{poly}(n, q^k, \ln 1/\delta) \cdot T(n, k, 1/q^{2k+2}).
\]
4 Reduction from LDME to LBP

Consider the case where we can access to noisy examples that have a large correlation with a certain linear function and the distribution of the noise is determined by only the value of the linear function. First we introduce the following simple lemmas and their corollaries as observations of LDME.

**Lemma 7.** Let \( X \) be a random variable taking values in \( \mathbb{F} \). For \( 0 \leq \rho \leq 1 \), if \( |\mathbb{E}[\varepsilon(X)]| \geq \rho \), then there exists \( a \in \mathbb{F} \) such that
\[
\Pr[X = a] \geq \frac{1}{q} + \frac{\rho}{q^2}.
\]

*Proof.* See Appendix B.2 ⊓⊔

**Lemma 8.** Let \( \alpha, \beta \in \mathbb{F}^n \setminus \{0^n\} \) and \( X \) be a random variable taking values in \( \mathbb{F} \). If the distribution of \( X \) is determined by only the value of \( \chi_\alpha(x) \) where \( x \leftarrow \mathbb{F}^n \), and \( \alpha \) and \( \beta \) are linearly independent over \( \mathbb{F}^n \), then for all \( a \in \mathbb{F} \),
\[
\Pr_{x,X}[X - \chi_\beta(x) = a] = \frac{1}{q}.
\]

*Proof.* The proof is immediate from Claim 1 and the details are omitted. ⊓⊔

As a corollary, we have the following facts about LDME. Let \( \alpha, \beta \in \mathbb{F}^n \setminus \{0^n\} \), \( \chi_\alpha \) be a target linear function, and \( f \) be the target (randomized) function, that is, \( \text{Cor}(f, \chi_\alpha) \geq \rho \). If \( \beta = \alpha \), then by Lemma 7 there exists some value \( a \in \mathbb{F} \) such that \( \Pr[f(x) - \chi_\beta(x) = a] \geq 1/q + \rho/q^2 \). On the other hand, if \( \beta \) and \( \alpha \) are linearly independent, then by Lemma 8 \( \Pr[f(x) - \chi_\beta(x) = a] = 1/q \) for all \( a \in \mathbb{F} \). We essentially use the difference in our reduction. Note that we do not say anything about the case where \( \beta \neq \alpha \) but they are linearly dependent (i.e., \( \beta = c\alpha \) for some \( c \in \mathbb{F} \setminus \{0, 1\} \)).

4.1 Overview

Our main idea is similar to the split-and-list idea in previous work [Val15, KKK18]. Let \( \alpha \in \mathbb{F}^n \) be the coefficients of a target linear function with \( |\alpha| \leq k \). First we select a consecutive partition that divides the nonzero entries of \( \alpha \) into half by brute-force search, then list the values of linear functions \( \chi_\beta \) of weight \( 1 \leq |\beta| \leq k/2 \) where \( \beta \) is contained in either \( \beta \in \mathbb{F}^J \) or \( \beta \in \mathbb{F}^J \). Not to contain linearly dependent linear functions, we fix an initial value of the coefficient vector for each partition. Since there are at most \((q - 1)^2\) patterns about the initial values, we can easily guess the pair of initial values consistent with \( \alpha \).

As the above, we stretch a noisy example to \( O(n^2) \) entries taking values in \( \mathbb{F} \). Then, we translate the stretched data into an instance of LBP, that is, a \( \{\pm 1\} \)-valued instance. As we will see in the subsequent sections, we can observe the following three facts. First, each entry takes values uniformly over \( \mathbb{F} \). Second, the pair of entries corresponding to \( \alpha \) (we may call it a target pair) has some correlation in the sense that they take a certain value \( a \in \mathbb{F} \) with relatively high probability, where we refer to such a value \( a \) as a concentrated value. Finally, other pairs are distributed pairwise independently.

Now we translate each entry \( a \in \mathbb{F} \) into 1 or \(-1\) as follows: (1) For the case where \( a \) is concentrated, we change the entry to 1, (2) for the case where \( a \) is not concentrated, we flip a biased coin with the head probability \( q/(2(q - 1)) \), and if it comes up with head, then we change the entry to \(-1\), otherwise to 1. Because each entry is uniformly distributed, the probability that the entry is changed to \(-1\) is exactly \( \frac{q-1}{2} \cdot \frac{2}{q} = \frac{1}{2} \), that is, uniformly distributed over \( \{\pm 1\} \). Moreover, by pairwise independence, all pairs except the target pair are also independently distributed. On the other hand, in the target pair, the correlation remains even in resulting binary instance. In other words, the reduced instance is just the one of LBP.
4.2 Algorithms and Analysis

First, we introduce the following simple subroutine Algorithm 5 which checks whether a candidate linear function found in the main routine is indeed a target linear function or not. In fact, it can be also implemented by the standard empirical estimation of the correlation. However, by using the conditions in Lemmas 7 and 8 we avoid calculations of complex values.

Algorithm 5 Check Correlation (checkCor)
Input: $n \in \mathbb{N}, \rho \in (0, 1), \gamma \in \mathbb{F}^n, \delta \in (0, 1), \mathcal{O}(f)$, where $f : \mathbb{F}^n \rightarrow \mathbb{F}$ is a randomized function
Output: $|E[(f(x) - \chi_\gamma(x))]| \geq \rho \Rightarrow \text{true}$: the coefficients of a target linear function and $\gamma$ are linearly independent $\Rightarrow \text{false}$
1: $m := \lceil \frac{2n^2}{\rho^2 \ln 3} \rceil$
2: for all $a \in \mathbb{F}$ do
3: $(x^{(i)}), b^{(i)}) \ldots, (x^{(m)}, b^{(m)}) \leftarrow \mathcal{O}(f)$
4: if $\sum_{i} 1_{b^{(i)} - \chi_\gamma(x^{(i)}) = a} \geq (\frac{1}{\rho} + \frac{\rho}{\ln m})m$ then return(true)
5: end for
6: return(false)

Lemma 9. Let $\alpha \in \mathbb{F}^n$ and $\chi_\alpha$ be a target linear function. Then the subroutine checkCor outputs true if the given $\gamma$ satisfies $|E[(f(x) - \chi_\gamma(x))]| \geq \rho$ with probability at least $1 - \delta$. On the other hand, if $\gamma$ and $\alpha$ are linearly independent, checkCor outputs false with probability at least $1 - \delta$ in time $\text{poly}(n, \rho^{-1}, \ln \delta^{-1})$.

Proof. The lemma follows from Lemmas 7 and 8. We omit the detail of the proof because it is almost the same with the proof of Lemma 4.

The Algorithm 6 is our main reduction from LDME to LBP. Let $\text{LBP}(S, \rho)$ be a subroutine for solving LBP (of the degree $d$) with high probability. W.l.o.g., we can assume the failure probability is at most $1/4$ by constant number of repetitions.

Algorithm 6 Learning with Discrete Memoryless Errors (main2)
Input: $n, k \in \mathbb{N}, \rho \in (0, 1), \delta \in (0, 1), \mathcal{O}(f)$, where $f : \mathbb{F}^n \rightarrow \mathbb{F}$ is a randomized function
1: for all $\gamma \in \mathbb{F}^n$ of the size $|\gamma| = 1$ do
2: if checkCor($n, k, \gamma, \frac{k}{\ln 2}, \mathcal{O}(f))$ then return($\gamma$)
3: end for
4: for all consecutive partitions $(J, \bar{J})$ of $[n], a_1, a_2 \in \mathbb{F}, s_1, s_2 \in \mathbb{F} \setminus \{0\}$ do
5: repeat $M = \lceil \log 2/\delta \rceil$ times do
6: make a light bulb instance $S$ (where the degree $d \geq \frac{kn}{\rho^2 \ln 4}$ is determined by the subroutine LBP)
7: 1: (for $i$-th row, get an example $(x, b) \leftarrow \mathcal{O}(f)$
8: 2: for all $a \in \mathbb{F}^d, \beta \in \mathbb{F}^d$ satisfying $1 \leq |a|, |\beta| \leq [k/2], \text{init}(a) = s_1$, and $\text{init}(\beta) = s_2$ do
9: list all values $b - \chi_\alpha(x) - a_1$ and $\chi_\beta(x)$ (we regard $\alpha, \beta$ as indices of columns) end for
10: change entries taking $a_2$ to 1, and other entries to $-1$ with probability $q/2(q - 1)$ (otherwise, 1)
11: execute $(\gamma_1, \gamma_2) \leftarrow \text{LBP}(S, \frac{d}{2 \sqrt{\delta}}) (\gamma_1, \gamma_2 \in \mathbb{F}^n$ are indices of the correlated pair)
12: if checkCor($n, k, \gamma_1 + \gamma_2, \frac{k}{\ln 2}, \mathcal{O}(f))$ then return($\gamma_1 + \gamma_2$)
13: end repeat
14: end for
15: end for

The proof of Lemma 10 is informally given as mentioned in Section 4.1, and we give the detail in the next section. Theorem 5 is immediately implied by Lemma 10.
**Lemma 10.** Assume that the subroutine LBP solves LBP for some $d \geq \Omega(\frac{\ln N}{\rho^2})$ in time $T(N, \rho)$ w.h.p., where $N$ is the number of the vectors. Then the algorithm main2$(n, k, \rho, \delta)$ solves LDME for any target linear function $\chi_\alpha$ ($1 \leq |\alpha| \leq k$) in time $\text{poly}(n, \rho^{-1}, \ln \delta^{-1}) \cdot d \cdot T((qn)^{2}, \frac{\rho}{2q^2})$ with probability at least $1 - \delta$.

**4.3 Proof of Lemma 10**

In this section, we use $\alpha$ to denote the coefficients of the target linear function, that is, the distribution of the target randomized function $f(x)$ is determined only by $\chi_\alpha(x)$ for each $x \in \mathbb{F}^n$. We assume that a partition $(J, \bar{J})$ is consecutive and divides a nonzero part of $\alpha$ into half as in Lemma 1.

We begin with the analysis of non-target pairs for each row in the reduced instance.

**Claim 5.** If a partition $(J, \bar{J})$ and linearly independent vectors $\beta, \beta' \in \mathbb{F}^J \cup \mathbb{F}^{\bar{J}}$ satisfy that $\alpha^J \neq 0^J$, $\alpha^\bar{J} \neq 0^\bar{J}$, and for any $a \in \mathbb{F} \setminus \{1\}$, $\alpha^J \neq a\beta, \alpha^\bar{J} \neq a\beta'$, $\alpha^J \neq a\beta'$, $\alpha^\bar{J} \neq a\beta$ and $\beta + \beta' \neq \alpha$, then $\chi_\beta$ and $\chi_{\beta'}$ are uniformly and pairwise independently distributed under any condition about $\chi_\alpha$, i.e., for any $v_1, v_2, v_3 \in \mathbb{F}$,

$$\Pr[\chi_\beta(x) = v_1 \text{ and } \chi_{\beta'}(x) = v_2 | \chi_\alpha(x) = v_3] = \frac{1}{q^3}.$$

**Proof.** See Appendix B.3.

In the reduction, we assume that the initial values $s_1$ and $s_2$ are consistent with $\alpha$, that is, $s_1 = \text{init}(\alpha^J)$ and $s_2 = \text{init}(\alpha^\bar{J})$. Any pair of indices $(\beta, \beta')$ except for $(\alpha^J, \alpha^\bar{J})$ satisfies the conditions in Claim 5 because they are non-zero and their initial values are fixed. In addition, the value of $f(x)$ depends on only the value of $\chi_\alpha$. Therefore, by Claim 4, the pair of entries indexed by $(\beta, \beta')$ are also uniformly and independently distributed.

For an element $a \in \mathbb{F}$ and a random variable $X$ taking values in $\mathbb{F}$, we use $X_{\text{bin}}^a$ to denote a $\{\pm 1\}$-valued random variable given by operation in the line 6 of main2, i.e.,

1. if $X$ takes $a$, set as $X_{\text{bin}}^a = 1$,
2. otherwise, flip a biased coin with the head probability $p = q/(2(q - 1))$, and if it comes up with head (resp. tail), set as $X_{\text{bin}}^a = -1$, (resp. $X_{\text{bin}}^a = 1$).

For any $a \in \mathbb{F}$, if $X$ is uniformly distributed over $\mathbb{F}$, then $\Pr[X_{\text{bin}}^a = 1] = \frac{q-1}{q} \cdot \frac{a}{2(q-1)} = \frac{1}{2}$. Moreover, it is easy to see that if $X, Y$ are uniformly and pairwise independently distributed, then $X_{\text{bin}}^a$ and $Y_{\text{bin}}^a$ are also uniformly and pairwise independently distributed over $\{\pm 1\}$. Therefore, any pair of entries indexed by $(\beta, \beta') \neq (\alpha^J, \alpha^\bar{J})$ is selected uniformly and independently.

Now we move on to the analysis of the target pair, that is, the pair of entries corresponding to $(\alpha^J, \alpha^\bar{J})$.

**Claim 6.** Let $(\bar{J}, \bar{J})$ be any partition of $[n]$. If a randomized function $f : \mathbb{F}^n \to \mathbb{F}$ has a correlation with $\chi_\alpha$ as Cor$(f, \chi_\alpha) \geq \rho$, then there exist $a_1, a_2 \in \mathbb{F}$ such that

$$\Pr_{x, f}[f(x) - \chi_\alpha(x) - a_1 = a_2 \text{ and } \chi_\alpha(x) = a_2] \geq \frac{1}{q^2} + \frac{\rho}{q^2}.$$

**Proof.** By Lemma 7 Cor$(f, \chi_\alpha) \geq \rho$ implies that there exists $a_1 \in \mathbb{F}$ such that

$$\Pr_{x, f}[f(x) - \chi_\alpha(x) = a_1] \geq \frac{1}{q} + \frac{\rho}{q^2}.$$

Therefore,

$$\frac{1}{q} + \frac{\rho}{q^2} \leq \Pr_{x, f}[f(x) - \chi_\alpha(x) = a_1] = \Pr_{x, f}[f(x) - \chi_\alpha(x) - a_1 = \chi_\alpha(x)] \leq q \cdot \max_{a_2 \in \mathbb{F}} \Pr_{x, f}[f(x) - \chi_\alpha(x) - a_1 = \chi_\alpha(x) = a_2].$$

$\square$
Then we estimate the correlation between the target pair in the reduced instance.

**Claim 7.** Let \( a \in \mathbb{F} \) and \( \mu \in [0,1] \). If random variables \( X \) and \( Y \) in \( \mathbb{F} \) satisfies

\[
\Pr[X = a, Y = a] \geq \frac{1}{q^2} + \mu \quad \text{and} \quad \Pr[X = a] = \Pr[Y = a] = \frac{1}{q},
\]

then,

\[
\Pr[X_{bin}^a \cdot Y_{bin}^a = 1] \geq \frac{1}{2} + 2\rho^2 \mu,
\]

where \( p = \frac{q}{4(q-1)} \) as in the definition of \( X_{bin}^a \).

**Proof.** Let \( p_1, p_2, p_3, p_4 \) denote probabilities as

\[
p_1 = \Pr[X = a, Y = a], \quad p_2 = \Pr[X = a, Y \neq a],
\]

\[
p_3 = \Pr[X \neq a, Y = a], \quad p_4 = \Pr[X \neq a, Y \neq a].
\]

Then, it follows that \( p_1 + p_2 + p_3 + p_4 = 1 \), \( p_1 \geq \frac{1}{q^2} + \mu \), and

\[
p_4 = 1 - \Pr[X = a] - \Pr[Y = a] + \Pr[X = a, Y = a] \geq 1 - \frac{2}{q} + \frac{1}{q^2} + \mu = \left(1 - \frac{1}{q}\right)^2 + \mu.
\]

Therefore, the probability is bounded below by

\[
\Pr[X_{bin}^a \cdot Y_{bin}^a = 1] = \Pr[X_{bin}^a = Y_{bin}^a]
\]

\[
= p_1 \cdot 1 + (p_2 + p_3) \cdot (1 - p) + p_4 \cdot (p^2 + (1 - p)^2)
\]

\[
= (1 - p) + p_1 \cdot p + p_4 \cdot (2p^2 - p)
\]

\[
\geq (1 - p) + \frac{1}{q^2}p + \left(1 - \frac{1}{q}\right)^2 (2p^2 - p) + \mu \cdot (p + 2p^2 - p)
\]

\[
= \frac{1}{2} + 2\rho^2 \mu.
\]

\[\square\]

For our settings, take \( X = f(x) - \chi_{\alpha,j}(x) - a_1 \), \( Y = \chi_{\alpha,j}(x) \), and \( \mu = \rho/q^3 \). Then we have

\[
\Pr[X_{bin}^a \cdot Y_{bin}^a = 1] \geq \frac{1}{2} + 2\frac{q^2}{4q(q-1)^2} \frac{\rho}{q^4} \geq \frac{1}{2} + \frac{\rho}{2q^3},
\]

and

\[
E[X_{bin}^a \cdot Y_{bin}^a] = 2 \Pr[X_{bin}^a \cdot Y_{bin}^a = 1] - 1 \geq \frac{\rho}{q^3}.
\]

Therefore, if we take sufficiently many samples, then the target pair has a correlation at least \( \frac{\rho}{2q^3} \) w.h.p.

Now we give the proof of Lemma 10.

**Proof (Lemma 10).** As in the proof of Lemma 5, we assume that all executions of \textbf{checkCor} will succeed. Under the condition, even if an incorrect candidate is found in brute-force search in \((J,J), a_1, a_2, s_1, \) and \( s_2 \), the algorithm \textbf{main2} does not output such an incorrect answer by Lemma 9. In fact, it is easily checked that the number of executions of \textbf{checkCor} in lines 2 and 8 is at most \( n(q-1) \) and \( nq^2(q-1)^2 \cdot M \), respectively. Therefore, by the union bound, the probability that at least one execution fails is bounded above by

\[
n(q-1) \cdot \frac{\delta}{4n(q-1)} + nq^2(q-1)^2M \cdot \frac{\delta}{4Mnq^2(q-1)^2} \leq \frac{\delta}{2}.
\]

Let \( \alpha \in \mathbb{F}^n \) be the coefficients of the target linear function and \( f \) be the target randomized function corrupted with noise. If \( |\alpha| = 1 \), then by our assumption on \textbf{checkCor}, the target linear function must be found in line 2. Therefore, we assume that \( 2 \leq |\alpha| \leq k \). In this case, we show that the reduced binary instance is the one of LBP with the correlation \( \frac{\rho}{2q^3} \) w.h.p. We assume that, as mentioned in the definition of the algorithm \textbf{main2}, all columns are labeled by vectors in \( \mathbb{F}^n \). In addition, assume that the algorithm \textbf{main2} succeeds in selecting \((J,J), a_1, a_2, s_1, \) and \( s_2 \) satisfying that
- $1 \leq |\alpha| \leq |\beta| \leq |\kappa|$ (by Lemma 1 such a consecutive partition must exist)
- $\Pr_{\alpha_1}(f(x) - \chi_{\alpha_i} - a_1 = \chi_{\alpha_i} = a_2) \geq 1/q + \rho/q^3$ (by Claim 6 such values of $a_1, a_2$ must exist)
- $\init(\alpha) = s_1$ and $\init(\beta) = s_2$

Then, the reduced instance must contain the pair of columns indexed by $(\alpha, \beta)$, we call it the target pair. For any pair of columns except for the target pair, as mentioned in the observation following Claim 5, the probability that their inner product does not exceed $\rho/q$ by Claim 7. If we select the sample size $d$, the total failure probability is bounded above by

$$\exp \left( -\frac{2\rho^2 d}{4 \cdot 4q^6} \right) \leq \exp \left( -\frac{\rho^2}{8q^6} \cdot \frac{8q^6}{\rho^2} \ln 4 \right) = \frac{1}{4}.$$

In other words, with probability at least 3/4, the algorithm reduces LDME to LBP of the correlation $\rho/2q^3$. W.l.o.g., we can assume that the failure probability of LBP is at most 1/4, (otherwise, it is achieved by constant number of repetitions). Thus, for each trial in lines 6 and 7, the probability that their inner product does not exceed $\rho/2q^3$ is bounded above by

$$\exp \left( -\frac{2\rho^2 d}{4 \cdot 4q^6} \right) \leq \exp \left( -\frac{\rho^2}{8q^6} \cdot \frac{8q^6}{\rho^2} \ln 4 \right) = \frac{1}{4}.$$

The total running time is bounded above by $\delta/2 + \delta/2 = \delta$. The total failure probability is bounded above by $\delta/2 + \delta/2 = \delta$. The total running time is bounded above by

$$nq \cdot \text{poly}(n, \rho^{-1}, \ln \delta^{-1}) + O(nq^4 \cdot \ln \delta^{-1} \cdot d \cdot 2^{d} n^{-2/3} (\text{poly}(n, \rho^{-1}, \ln \delta^{-1}) \cdot T(\rho/2q^3) + \text{poly}(n, \rho^{-1}, \ln \delta^{-1}) \cdot d \cdot T(\rho/2q^3)).$$

\[\square\]

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A Proofs for Subroutines

A.1 Proof of Lemma 3

If \( f \) is constant, then the algorithm obviously outputs the value with probability 1. If \( f \) is not constant, then there are two entries which have different values in the truth table of \( f \), and the probability that each entry appears is at least \( q^{-k} \) because the size of the truth table is at most \( q^k \). If \( m \) examples contain these entries, then the algorithm will output \( \bot \). The probability that each entry does not appear in \( m \) examples is bounded above by \( (1 - q^{-k})^m \leq e^{-m/q^k} \leq \frac{\delta}{2} \). By the union bound, the failure probability is at most \( \delta \).

A.2 Proof of Lemma 4

First, we consider the case where \( \hat{f}(\alpha) \neq 0 \). Assume that \( \Pr[f(x) - \chi_\alpha(x) = a] < \frac{1}{q} + \frac{1}{q^k} \) for all \( a \in \mathbb{F} \). Since \( \alpha \) does not have nonzero value at irrelevant coordinates by Fact 3, the value \( \hat{f} - \chi_\alpha \) is determined by at most \( k \) coordinates of \( x \), and \( \Pr[f(x) - \chi_\alpha(x) = a] = \frac{1}{q} \) for all \( a \in \mathbb{F} \). This implies \( \hat{f}(\alpha) = 0 \), which is contradiction. Thus, there exists \( \alpha' \in \mathbb{F} \) such that \( \Pr[f(x) - \chi_\alpha(x) = \alpha'] \geq \frac{1}{q} + \frac{1}{q^k} \). By the Hoeffding inequality, the probability that the condition in line 4 does not hold w.r.t. \( \alpha' \) is bounded above by \( e^{-\frac{m}{q^k}} \leq \frac{\delta}{2} < \delta \).

On the other hand, if there exists \( i \in [n] \) such that \( i \) is irrelevant and \( \alpha_i \neq 0 \), then for any \( a \in \mathbb{F} \),

\[
\Pr[f(x) - \chi_\alpha(x) = a] = \sum_{v \in \mathbb{F}} \Pr[f(x) - \chi_{\alpha'}(x) = v] \Pr[\alpha_i x_i = a - v] = \frac{1}{q},
\]

where \( \alpha_i' = 0 \) and \( \alpha_j' = \alpha_j \) for \( j \neq i \). By the Hoeffding inequality, the probability that the condition in line 4 holds is bounded above by \( e^{-\frac{m}{q^k}} \leq \frac{\delta}{2} \). Therefore, by the union bound, the probability that the condition holds for some \( a \in \mathbb{F} \) (i.e., the failure probability) is at most \( \delta \).
B Proofs of Lemmas and Claims

B.1 Proof of Claim 5

(i) If \( \beta \neq c\alpha \) for any \( c \in \mathbb{F} \), there are two coordinates \( i, j \in [n] \) satisfying \( \beta_i = c\alpha_i, \beta_j = c'\alpha_j, \) \( c \neq c' \), and \( \alpha_i, \alpha_j \neq 0 \). First we select values in \( \mathbb{F}^{|n|\setminus\{i,j\}} \), and for any choice, the remaining condition takes the following form: for some \( v_1, v_2 \in \mathbb{F} \),

\[
\alpha_i x_i + \alpha_j x_j = v_1 \quad \text{and} \quad c\alpha_i x_i + c'\alpha_j x_j = v_2.
\]

Since \( \alpha_i c'\alpha_j - \alpha_j c\alpha_i = \alpha_i \alpha_j (c' - c) \neq 0 \), the above equations have a unique solution w.r.t. \((x_i, x_j)\). The probability that they take the values of the unique solution is exactly \( q^{-2} \).

(ii) If \( \beta = c\alpha \) \( (c \neq 0) \), the condition takes the following form:

\[
\begin{align*}
\alpha_1 x_1 + \cdots + \alpha_n x_n &= a \\
\alpha_1 x_1 + \cdots + \alpha_n x_n &= c^{-1}b
\end{align*}
\]

Obviously, the probability is \( q^{-1} \) if \( a = c^{-1}b \), otherwise, the probability is 0.

B.2 Proof of Claim 7

For simplicity, let \( p_a := \Pr[X = a] \) for \( a \in \mathbb{F} \). First we show that

\[
|E[e(X)]| \geq \rho \implies \exists a \in \mathbb{F} \text{ s.t. } p_a - \frac{1}{q} \geq \frac{\rho}{q}.
\]

By contraposition, we assume that \( |p_a - \frac{1}{q}| < \frac{\rho}{q} \) for any \( a \in \mathbb{F} \). Then,

\[
|E[e(X)]| = \left| \sum_{a \in \mathbb{F}} p_a e(a) \right| = \left| \sum_{a \in \mathbb{F}} (p_a - \frac{1}{q}) e(a) \right|
\]

\[
\leq \sum_{a \in \mathbb{F}} \left| p_a - \frac{1}{q} \right| |e(a)|
\]

\[
< \frac{\rho}{q} \cdot \sum_{a \in \mathbb{F}} |e(a)| = \rho
\]

where the second equality follows from the fact that \( \sum_{a \in \mathbb{F}} e(a) = 0 \).

Now we have that \( |p_a - \frac{1}{q}| \geq \frac{\rho}{q} \) for some \( a \in \mathbb{F} \). If \( p_a - \frac{1}{q} \geq \frac{\rho}{q} \), then \( p_a \geq \frac{1}{q} + \frac{\rho}{q} \geq \frac{1}{q} + \frac{\rho}{q^2} \). Therefore, the remaining case is that \( p_a \leq \frac{1}{q} - \frac{\rho}{q} \). In this case,

\[
(q-1) \max_{b \in \mathbb{F}\setminus\{a\}} p_b \geq \sum_{b \neq a} p_b = 1 - p_a \geq \frac{q-1}{q} + \frac{\rho}{q}
\]

Thus, there exists \( b \in \mathbb{F} \) such that \( p_b \geq \frac{1}{q} + \frac{\rho}{q(q-1)} \geq \frac{1}{q} + \frac{\rho}{q^2} \).

B.3 Proof of Claim 5

Since \( \alpha \neq 0^n \), it is enough to show that, for any \( v_1, v_2, v_3 \in \mathbb{F} \),

\[
\Pr_x[\chi_\beta(x) = v_1, \chi_{\beta'}(x) = v_2, \chi_\alpha(x) = v_3] = \frac{1}{q^3}.
\]

W.l.o.g., we can assume that \( \beta \in \mathbb{F}^j \) and \( \beta \neq \alpha^j \) (in this case, either \( \beta' = \alpha^j \) or \( \beta' = \alpha^j \) may hold). First consider the case where \( \beta' \in \mathbb{F}^j \). We select three coordinates \((i_1, i_2, i_3)\) as follows: by linearly independence
of $\beta$ and $\alpha^J$, we can select $(i_1, i_2)$ such that $(\alpha_{i_1}, \alpha_{i_2})$ and $(\beta_{i_1}, \beta_{i_2})$ are also linearly independent. Then, we select $i_3 \in J$ to satisfy that $\beta_{i_3}' \neq 0$. Now we have the three vectors $\{(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}), (\beta_{i_1}, \beta_{i_2}, 0), (0, 0, \beta_{i_3}')\}$. It is not so difficult to see that they are linearly independent.

Otherwise if $\beta' \in F^J$, we select $i_3$ satisfying $\alpha_{i_3} \neq 0$, and we can select $(i_1, i_2)$ such that $(\beta_{i_1}, \beta_{i_2})$ and $(\beta_{i_1}', \beta_{i_2}')$ are also linearly independent. Then we have three vectors $\{(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}), (\beta_{i_1}, \beta_{i_2}, 0), (\beta_{i_1}', \beta_{i_2}', 0)\}$ which are also linearly independent.

In any case, for any assignment to $[n] \setminus \{i_1, i_2, i_3\}$, the solution of the remaining linear system in $x_{i_1}, x_{i_2}, x_{i_3}$ is uniquely determined, and the claim holds as in the proof of Claim 1.