Gravitational thermodynamics and universal holographic duality in dynamical spacetimes

Shao-Feng Wu\textsuperscript{a,*}, Bin Wang\textsuperscript{b}, Xian-Hui Ge\textsuperscript{a}, Guo-Hong Yang\textsuperscript{a}

\textsuperscript{a}Department of Physics, Shanghai University, Shanghai 200444, P.R. China
\textsuperscript{b}INPAC and Department of Physics, Shanghai Jiaotong University, Shanghai 200240, P.R. China

Abstract
We construct a generalized Smarr formula which could provide a thermodynamic route to derive the covariant field equation of general theories of gravity in dynamic spacetimes. Combining some thermodynamic variables and a new chemical potential conjugated to the number of degree of freedom on the holographic screen, we find a universal Cardy-Verlinde formula and give its braneworld interpretation. We demonstrate that the associated AdS-Bekenstein bound is tighten than the previous expression for multi-charge black holes in the gauged supergravities. The Cardy-Verlinde formula and the AdS-Bekenstein bound are derived from the thermodynamics of bulk trapping horizons, which strongly suggests the underlying holographic duality between dynamical bulk spacetime and boundary field theory.

Keywords: Gravitational thermodynamics, Bulk/boundary connection, Smarr formula, Cardy-Verlinde formula, Bekenstein bound

1. Introduction
Since the discovery of the black hole (BH) entropy and the analogue between the laws of BH mechanics and thermodynamics [1], there are increased interest on the thermodynamic feature of gravity. It is generally believed that the puzzling feature should be clarified in an underlying quantum theory of gravity. Actually, gravitational thermodynamics has inspired some fascinating ideas about the spacetime or gravity, such as the holographic principle...
2 and the suggestion that gravity might not be a fundamental interaction but rather an emergent large scale/numbers phenomenon [3–5].

One of the intriguing outcomes of holographic principle was found by studying the entropy bounds in a radiation dominated universe [6]. Describing the radiation as a continuum conformal field theory (CFT), Verlinde revealed that the Friedmann equation knows a higher-dimensional version of Cardy formula for the entropy of a two-dimensional CFT [7]. The so called Cardy-Verlinde (CV) formula hence indicates that it shares a common origin with the Friedmann equation in a single underlying fundamental theory. The CV formula can be derived from the celebrated correspondence between a Schwarzschild BH in the Anti-de Sitter (AdS) space and strongly-interacting CFTs at high temperature [8, 9], which is the most manifest realization of the holographic principle. According to the duality, it can be believed that the CV formula holds in the high-temperature and strongly-interacting cases. But even in the free field theory, there is a hint that the CV formula agrees, up to a constant factor, in the high-temperature limit [10]. After the discovery of the CV formula, much effort has been made to understand the CV formula. For instance, the CV formula can be realized by a moving brane in the Schwarzschild-AdS background [11]. Other discussions, especially the check of the CV formula in different situations, can be found in [12–21] and references therein. However, it was found [22] that there seems no natural and universal modification of CV formula to encompass all AdS BHs, such as the multiple charged BHs in the maximally supersymmetric gauged supergravities. Moreover, the CV formula implies a normalized Bekenstein bound. Based on this bound, Verlinde suggested that the Casimir entropy of closed universe should be less than the entropy of a BH with the same size. This holographic bound developed the proposal of cosmological holographic bound which was first presented by Fischler and Susskind [23], see the review [24]. Although the universal formulation of CV formula has not been found for multiple charged BHs, it was accidentally found that an AdS-Bekenstein bound holds for many cases [22].

On the other hand, the theory of emergent gravity recently has also been promoted by Verlinde [25], who argued that space and inertia can be emergent, and gravity can be explained as entropic force that influences nonlocally the particle outside the holographic screen. This illuminating idea has attracted considerable interest in various aspect of physics, together with some debates on its theoretical [26, 27] and experimental viability [28, 29].

Among other things, Verlinde provided a new thermodynamic method
to derive the Einstein equation, based on the thermodynamic equipartition law of energy, where the Tolman mass [30] is taken as the total energy behind a static holographic screen and its surface area $A$ is identified with its number of microscopic degrees of freedom $N$, see the earlier discussion on the equipartition law [31, 32]. Compared with Jacobson’s pioneering work of the derivation of the field equation [33] and the following extension [34–38], Verlinde’s method manipulates the thermodynamic quantities on the screen but does not involve the variation of thermodynamic quantities along the horizon generator and the local condition of vanishing expansion for equilibrium surface. This seems to preserve some non-local aspects of gravity. The equipartition law is further extended to theories of general gravity in stationary spacetimes [39], where a generalized Komar mass was proposed as the source for gravity and the number of microscopic degrees of freedom is assumed to be proportional to the Wald entropy [40, 41].

If one does not assume the relation between the entropy and the degrees of freedom, the equipartition law still teaches us a generalized Smarr formula\(^1\) [43]. It should be noted that the generalized Smarr formula in itself might have the profound physical meaning. Actually, based on the Tolman mass and Unruh effect, Abreu and Visser [44] proposed a robust entropy bound for uncollapsed matter in 4-dimensional stationary systems. This bound is double to the holographic bound for collapsed matter and the factor 2 was conjectured to be an intrinsic feature for uncollapsed matter, ultimately arising from the difference of the usual Euler relation for uncollapsed matter and the Smarr formula for general relativity. We notice that this bound can be directly extended to higher dimension. Interestingly, when the spacetime dimension increases, the holographic entropy bound is approached and simultaneously the difference of factor 2 between Euler relation and Smarr formula declines as the mentioned conjecture, see Eq. (21) for the generalized Smarr formula in any dimension. This result suggests an insight that the generalized Smarr formula in the system with gravity could take role as the Euler relation in the system without gravity.

Although the equipartition law or the generalized Smarr formula is very interesting, most of the relative works were restricted on the stationary spacetimes. In Ref. [45], the equipartition law is used to derive the Friedmann

\(^1\)The original Smarr formula is built only for Kerr-Newman BHs [42] while the generalized version is compatible with a general stationary metric.
equation in the standard cosmology, where the thermal energy is assumed as a special active gravitational mass. However, it was pointed out that this active gravitational mass is negative for an accelerated expanding universe \[27\]. Other discussion on a dynamic Smarr-like formula for Einstein gravity can be found in \[46\].

In the first part of this paper, we will show that the generalized Smarr formula can be constructed for general gravity theories both in static and dynamic spacetimes. The key point is that we can find a Noether conserved charge as an extension of Tolman mass. As expected, it is shown that the new quasilocal gravitational mass can be always non-negative in the evolution of universe, contrary to the negative active gravitational mass used in \[27, 45\]. The generalized Smarr formula that we will build connects the new mass with the well-known Hayward temperature \[48\] and the Wald-Kodama entropy \[49–51\] on the dynamic trapping horizons \[48, 49\]. If taking this generalized Smarr formula as a prior, one can derive the covariant field equation of general gravity theories even in the dynamic spacetime, nicely extending Verlinde’s derivation of Einstein equation in the static spacetime based on the equipartition law of Tolman mass.

The second part of this paper is triggered by another issue about the equipartition law: Verlinde’s concise ansatz \( N = A \) seems very reasonable in itself from the holographic point of view, but what is the chemical potential conjugated to the number of degrees of freedom? See an attempt on this question \[47\]. In this paper, we will propose that the work term in the first law of thermodynamics \[48, 51, 52\] can be reinterpreted to extract a definition of chemical potential conjugated to \( N = A \).

Remarkably, combining the new chemical potential and some thermodynamic variables on the trapping horizon, we can derive a universal CV formula from the bulk/boundary duality. We say it is universal in the sense that the AdS BHs in the bulk can be arbitrary static or dynamic BHs with spherical symmetry, certainly including the aforementioned multiple charged BHs in the gauged supergravity. We further give the braneworld interpretation of the universal CV formula and obtain a universal AdS-Bekenstein bound. Interestingly enough, the bound is more stringent than the previous expression for multi-charge BHs \[22\]. We expect that the universal and more stringent entropy bound could be more useful to identify the boundary CFTs with dual gravity and to qualitatively explore the fundamental theory of quantum gravity \[24\]. Moreover, the CV formula and the AdS-Bekenstein bound, due to their derivation from the thermodynamics of dynamic trapping
horizons in the bulk, strongly suggest the underlying holographic duality between dynamical bulk spacetimes and boundary CFTs, see the works on the AdS/CFT correspondence for time-dependent backgrounds \[53\].

The rest of the paper is arranged as follows. In Sec. II, we review some thermodynamic variables on trapping horizons. In Sec. III, we propose the new mass and build the generalized Smarr formula in dynamic spacetimes. In Sec. IV, we present the new chemical potential. Then we derive a universal CV formula and give its braneworld interpretation. In Sec. V, we obtain an AdS-Bekenstein bound from the CV formula and compare the bound with the previous expression for charged BHs. The conclusion and discussion are given in the last section.

2. Temperature, entropy and internal energy

In the stationary spacetime, many thermodynamic quantities are constructed based on the Killing time. In the dynamic spacetime with spherical symmetry, one can construct the corresponding quantities, since there is a preferred time direction which is analogue of the static Killing vector, namely the Kodama vector \[54\]. It should be stressed that the Kodama vector is also well-defined in the static spacetime with spherical symmetry. Hence all the quantities defined below are viable both to static and dynamic spacetimes. In static spacetimes, however, the Kodama vector is not exact but only along the Killing vector in general. This will lead to the subtle difference between two sets of thermodynamic quantities.

2.1. Hayward temperature

Let us introduce a \(d\)-dimensional spacetime \((M_d, g_{\mu\nu})\) as a warped product of a \((d - 2)\)-dimensional sphere \((\Omega_{d-2}, \gamma_{ij})\) and a two-dimensional orbit spacetime \((M_2, h_{ab})\). The line element can be written in the double-null coordinates

\[
 ds^2 = -2e^{-\varphi(u,v)}dudv + r^2(u,v)d^2\Omega_{d-2},
\]

where \(d^2\Omega_{d-2}\) denotes the line element of the \((d - 2)\)-dimensional sphere \(\Omega_{d-2}\) and \(r\) is its areal radius. The causal structure of this spacetime is convenient to be studied using null geodesics. The null expansions of two independent future-directed radial null geodesics are expressed as \(\theta_+ = (d - 2)r^{-1}r_u\) and \(\theta_- = (d - 2)r^{-1}r_v\). An \((d - 2)\)-dimensional surface is called as marginal if \(\theta_+\theta_- = 0\), trapped if \(\theta_+\theta_- > 0\), and untrapped if \(\theta_+\theta_- < 0\). The trapping
horizon [48, 49], which is a more general concept than the event horizon or the apparent horizon, is defined as the hypersurfaces foliated by marginal surfaces with $\theta_+ = 0$.

In this spacetime, the Kodama vector exists and can be given as

$$K^\mu = -\epsilon^{\mu\nu} \nabla_\nu r = (e^\varphi \partial_\varphi r, -e^\varphi \partial_\varphi r, 0, \cdots),$$

(2)

where $\epsilon_{\mu\nu} = \epsilon_{ab} (dx^a)_\mu (dx^b)_\nu$ and $\epsilon_{ab}$ is a volume element of $(M_2, h_{ab})$. Using the definition of the surface gravity associated with the trapping horizon, one can obtain the Hayward temperature [48]

$$T = \frac{\kappa}{2\pi} = -\frac{1}{2} e^{ab} \nabla_a K_b,$$

which was confirmed by the tunneling approach [55]. Hayward temperature is usually used in the dynamic spacetime, but as we have mentioned, it is well-defined in both static and dynamic spacetimes. Now we will restrict to the static spacetime and compare the Hayward temperature to the Hawking temperature. Suppose $(M_d, g_{\mu\nu})$ as a static spacetime with the line element

$$ds^2 = -h(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 d^2 \Omega_{d-2}.$$

(3)

The Kodama vector (2) is translated to $K^\mu = \sqrt{g/h}(1, 0, \cdots)$, which can be reduced to the Killing vector $\xi^\mu = (1, 0, \cdots)$ only when $g_{tt}g_{rr} = -h/g = -1$. The standard Hawking temperature on static Killing horizons is defined by

$$T_0 = \frac{\kappa_0}{2\pi} = -\frac{1}{4\pi} e^{\mu\nu} \nabla_\mu \xi_\nu = \frac{h'}{4\pi \sqrt{g/\hbar}}.$$

(4)

It is different with the Hayward temperature in static spacetimes

$$T = \frac{h g' + g h'}{8\pi \hbar}$$

(5)

if $g_{tt}g_{rr} \neq -1$. In [56], Hayward et al. pointed out that the operational meaning of $T$ is that the preferred Kodama observer just outside the horizon measures a thermal spectrum with the temperature $T/\|K\|$. Since $T/\|K\|$ is diverging at the horizon but $T$ is finite, he interpreted $T$ as a locally redshift-renormalized temperature, compared with $T_0$ that is usually regarded as the temperature measured by the static observer at infinity.
2.2. Wald-Kodama Entropy

It is well-known that the entropy of stationary horizons is well defined by Wald entropy [40, 41], which is a Noether charge associated with the Killing vector, but it is less understood for the horizon entropy in a dynamical spacetime, where the Killing vector can not be found in general. Iyer and Wald proposed that one can approximate the metric by its boost-invariant part to “create a new spacetime” where there is a Killing vector. However, the obtained dynamical entropy is not invariant under field redefinition in general [41]. Hayward have ever presented that the Wald entropy can be alternatively associated with the Kodama vector [49, 50]. For Einstein gravity, the dynamical horizon entropy, which has been called as Wald-Kodama entropy, has the same simple form of stationary BHs. Following Hayward’s proposal, we have given a general expression of Wald-Kodama entropy in generalized gravity theories [51]. It should be noted that, in static spacetimes, the Wald-Kodama entropy is exactly identified with the Wald entropy for several typical modified theories of gravity, including Gauss-Bonnet gravity, $f(R)$ gravity, and scalar-tensor gravity, even though the Kodama vector is not the exact Killing vector. In dynamic spacetimes, the Wald-Kodama entropy of Gauss-Bonnet gravity has the same form as the static case, but it has to be corrected for $f(R)$ gravity and scalar-tensor gravity. Interestingly, the nonequilibrium entropy production, which is usually invoked to interpret the extra term of the first law of $f(R)$ gravity and scalar-tensor gravity in the FRW spacetime with slowly varying horizon, is just identified with the corrected terms. Moreover, it has been proved that the Wald-Kodama entropy is satisfied with the second law of thermodynamics, which is an important assistant criterion supporting the Wald-Kodama entropy as a preferred definition.

We give the Wald-Kodama entropy [51]

\[ S = \frac{1}{8\kappa} \int_{B} Q_{\mu\nu} dB_{\mu\nu}, \]  

(6)

where

\[ Q_{\mu\nu} = -2X_{\mu\nu\lambda\rho}^{\lambda} \nabla_{\lambda} K_{\rho} + 4K_{\rho} \nabla_{\lambda} X_{\mu\nu\lambda\rho}^{\lambda}, \quad X_{\mu\nu\lambda\rho}\] = $\partial L / \partial R_{\mu\nu\lambda\rho}$,

$B$ denotes the section of horizon, and $L$ refers to any diffeomorphism-invariant Lagrangian involving no more than quadratic derivatives of metric $g_{\mu\nu}$ and the first order derivative of some scalar fields $\Phi_{(i)}$. 

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2.3. Misner-Sharp energy

The most simple definition of conserved gravitational energy in static spacetimes could be

\[ U = \int_{\Sigma} T^{\mu\nu} \xi_{\nu} d\Sigma_{\mu}, \]  

(7)

where \( T^{\mu\nu} \) denotes the energy-momentum tensor of matter and \( \Sigma \) is a spatial volume with boundary. This expression seems to have an inadequacy that a Schwarzschild BH has zero energy, see the textbook [57]. In fact, it can be avoided if one uses the Einstein equation and the energy can be obtained as the exact BH mass. However, when \( g_{tt}g_{rr} \neq -1 \), Eq. (7) can not always be written in an explicit quasi-local form. Interestingly, if one replaces the Killing vector with the Kodama vector, the expression

\[ U = \int_{\Sigma} T^{\mu\nu} K_{\nu} d\Sigma_{\mu} \]  

(8)

can be actually recast as an explicit quasi-local form. For Einstein gravity, Eq. (8) is just the well-known Misner-Sharp (MS) energy [58], which has all the expected physical limits for the active gravitational energy, including the “asymptotically flat, vacuum, small-sphere, Newtonian, test-particle and special-relativistic limits” [49, 50]. It hence has been commonly accepted as a “standard” expression of gravitational energy on round spheres [59]. The generalized MS energy for more general theories of gravity can be obtained if the current \( J^\mu_{\nu} = T^{\mu\nu} K_{\nu} \) is conserved and the explicit quasi-local form can be obtained, see Ref. [60] for the generalized MS energy of GB gravity and Ref. [61] for the energy of \( f(R) \) and scalar-tensor gravity. Using the Hayward temperature, surface entropy and MS energy, one can construct the first law [48]

\[ dU = TdS + WdV \]  

(9)

on trapping horizon for Einstein gravity, where \( W = -h_{ab}T^{ab}/2 \) is interpreted as the work density. The first law was further checked in more general theories of gravity [51], which involves the Wald-Kodama entropy and generalized MS energy. It should be noticed that the MS energy \( U \) takes the role as internal energy in the first law (9), instead of the usual BH mass.
3. Generalized Smarr formula

For 4-dimensional Einstein gravity, one can define Komar energy \[ M_{Kom} = \frac{1}{4\pi} \int_\Sigma R^\mu\nu \xi_\nu d\Sigma_\mu \]
which can be related to Hawking temperature and surface entropy [42]:
\[ M_{Kom} = 2T_0 S. \]

Considering the Tolman mass [30]
\[ M_{Tol} = 2 \int_\Sigma \left(T^\mu\nu - \frac{1}{2} g^\mu\nu T\right) \xi_\nu d\Sigma_\mu, \quad (10) \]
and using Einstein equations, one can obtain the generalized Smarr formula
\[ M_{Tol} = 2T_0 S. \quad (11) \]

Verlinde [25] proposed that the total gravitational energy \( M \) behind the holographic screen is just the Tolman mass \( M_{Tol} \). In addition, following the spirit of holographic principle and in view of each fundamental bit occupying by definition one unit cell, he presented a concise ansatz about the number of degrees of freedom on the holographic screen and its area
\[ N = A. \quad (12) \]

Thus, the Smarr formula can be regarded as the equipartition rule of energy \( M = T_0 N/2 \) and the Einstein equation can be extracted from the equipartition rule by reversing the derivation of Smarr formula. The equipartition rule is further generalized to general theories of gravity in 4-dimensional stationary spacetimes, for which Padmanabhan [39] proposed the generalized Komar mass
\[ M = \frac{1}{4\pi} \int_\Sigma \left(X^\lambda_\mu_\rho_\sigma R_\nu_\lambda_\rho_\sigma - 2 \nabla^\lambda \nabla^\rho X_\mu_\lambda_\rho_\sigma\right) \xi_\mu d\Sigma_\nu \quad (13) \]
and assumed the number of microscopic degrees of freedom to be proportional to the Wald entropy.
Now we will demonstrate a generalized Smarr formula for general theories of gravity, which is applicable both to static and dynamic spacetimes. The key point is that we will propose a new gravitational mass

\[ M = \frac{d - 2}{d - 3} \frac{1}{16\pi} \int_{\Sigma} J_M^\mu d\Sigma_\mu, \tag{14} \]

where

\[ J_M^\mu = L_g K^\mu + 16\pi T^{\mu\nu} K_\nu - \Theta^\mu_g. \tag{15} \]

The boundary term \( \Theta_g^\mu \) arises from the variation of pure gravity Lagrangian \( L_g \) induced by the Kodama vector \( K \), i.e.

\[ \Theta_g^\mu = 2\nabla_\nu X_\lambda^{\mu\nu} \delta g^{\lambda\rho} - 2K_\nu \nabla_\nu \delta g^{\lambda\rho} + \omega^\mu_{(j)} K_\nu \nabla^\nu \Phi_{(j)}, \tag{16} \]

where

\[ \delta g^{\mu\nu} = -2\nabla^{(\nu} K^{\mu)}, \quad \omega^\mu_{(j)} = \frac{\partial L}{\partial \nabla_\mu \Phi_{(i)}}. \]

and \( \Phi_{(j)} \) denotes the scalar fields which are non-minimally coupled to gravity. The energy-momentum tensor is

\[ T^{\mu\nu} = \frac{1}{16\pi} \left[ L_m g^{\mu\nu} - \frac{\partial L_m}{\partial \nabla_\mu \Phi_{(i)}} \nabla^\nu \Phi_{(i)} \right], \quad i \neq j. \tag{17} \]

One can find that Eq. (14) is a Noether conserved charge since \( J_M^\mu \) can be identified with the Noether current of Wald-Kodama entropy \( J_S^\mu = K^\mu L - \Theta^\mu \) [40, 41, 51]. Here the total Lagrangian includes the contributions from gravity and matter, \( L = L_g + L_m \). The total boundary term \( \Theta^\mu = \Theta_g^\mu + \Theta_m^\mu \), where \( \Theta_m^\mu = \omega^\mu_{(i)} K_\nu \nabla^\nu \Phi_{(i)} \) (i \( \neq \ j)). Eq. (14) can be reduced to Eq. (13) when \( d = 4, \ \Phi_{(j)} = 0 \), the Kodama vector \( K \) is replaced with Killing vector \( \xi \) (then \( \Theta_g^\mu = 0 \)), and the field equation holds

\[ X^{\beta\mu\lambda} R_{\mu\nu\lambda}^{\alpha} - 2\nabla_\nu \nabla_\mu X^{\beta\mu\nu\alpha} - \frac{1}{2} L_g g^{\alpha\beta} + \frac{1}{2} \omega^\beta_{(j)} \nabla^\alpha \Phi_{(j)} = 8\pi T^{\beta\alpha}. \tag{18} \]

For the Einstein gravity with \( K = \xi \), Eq. (14) is reduced to the Tolman mass\(^3\). We hence refer Eq. (14) as the generalized Tolman mass.

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\(^2\)This indicates that, compared with Gauss-Bonnet gravity, there is more difference between Eq. (14) and Eq. (13) for scalar-tensor gravity.

\(^3\)Note that the coefficient \( \frac{d - 2}{d - 3} \) is necessary for the mass which can be reduced to the ADM mass of Schwarzschild BHs.
To construct a generalized Smarr formula, let us read the mass current (15) as
\[
J^\mu_M = K_\beta L_g + 16\pi T^{\beta\alpha} K_\alpha + 2X^{\alpha\beta}_{(\mu\nu)} \nabla_\alpha \delta g^{\mu\nu} \\
-2\nabla_\alpha X^{\alpha\beta}_{(\mu\nu)} \delta g^{\mu\nu} - \omega^{\beta}_{(j)} K_v \nabla^v \Phi^{(j)} \\
= K_\beta L_g + 16\pi T^{\beta\alpha} K_\alpha - 2X^{\alpha\beta}_{\mu\nu} \nabla_\alpha (\nabla^\nu K^\mu + \nabla^\mu K^\nu) \\
+ 2\nabla_\alpha X^{\alpha\beta}_{\mu\nu} (\nabla^\nu K^\mu + \nabla^\mu K^\nu) - \omega^{\beta}_{(j)} K_v \nabla^v \Phi^{(j)}. \tag{19}
\]

From the Wald-Kodama entropy (6), we can obtain another current
\[
\nabla_\alpha Q^{\alpha\beta} = -2\nabla_\alpha (X^{\alpha\beta\mu\nu} \nabla_\mu K_\nu - 2K_\nu \nabla_\mu X^{\alpha\beta\mu\nu}) \\
= -2\nabla_\alpha X^{\alpha\beta\mu\nu} \nabla_\mu K_\nu - 2X^{\alpha\beta\mu\nu} \nabla_\alpha \nabla_\mu K_\nu \\
+ 4K_\nu \nabla_\alpha \nabla_\mu X^{\alpha\beta\mu\nu} + 4\nabla_\alpha K_\nu \nabla_\mu X^{\alpha\beta\mu\nu}. \tag{20}
\]

Comparing Eq. (19) with Eq. (20), one can find that they are equal on-shell, since
\[
8\pi T^{\beta\alpha} K_\alpha = 2K_\nu \nabla_\alpha \nabla_\mu X^{\alpha\beta\mu\nu} - 2X^{\alpha\beta\mu\nu} \nabla_\alpha \nabla_\nu K_\mu \\
- \frac{1}{2} L_g g^{\alpha\beta} K_\alpha + \frac{1}{2} \omega^{\beta}_{(j)} K_v \nabla^v \Phi^{(j)} \\
= K_\alpha \left( X^{\beta\mu\lambda} P^{\alpha}_{\mu\nu\lambda} - 2\nabla_\nu \nabla_\mu X^{\beta\mu\lambda} - \frac{1}{2} L_g g^{\alpha\beta} + \frac{1}{2} \omega^{\beta}_{(j)} \nabla^\alpha \Phi^{(j)} \right),
\]
which is just the projection of field equation (18) along the vector \( K_\alpha \). Obviously, we have obtained a generalized Smarr formula
\[
M = \frac{d - 2}{d - 3} TS. \tag{21}
\]

Some remarks are in order. First, Eq. (21) is not restricted on the horizon, which is consistent with Verlinde’s spirit to associate the thermodynamics on a general holographic screen [25]. Note that the holographic screen has been interpreted as a minimal surface which relating to the entanglement entropy [63] and the thermodynamic laws have been constructed on the holographic screen [64]. Second, Eq. (21) is effective for any vector \( K_\alpha \). In other words, we have not apparently involved the property of the Kodama/Killing vector and then the spherical symmetry of spacetime in the above derivation. Respecting that in the general dynamic spacetime, a natural generalization

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of Kodama vector has been proposed as a preferred time direction [66], one can expect that Eq. (21) can be a meaningful thermodynamic realtion in the general dynamic spacetime. Moreover, even for more general vectors, Eq. (21) might be meaningful, see the recent proposal of local first law of thermodynamics [65], where the temperature and thermodynamic energy is actually not defined by the usual Killing vector but the four-velocity instead. Third, if one takes Eq. (21) as a prior identity which holding both for any screen and any vector $K_{\alpha}$, the full field equation (18) can be extracted. Even if the vector is required to be the Killing or Kodama time, one at least can get the field equation along the time direction. However, the full field equation still can be obtained if following the reasoning of Verlinde and Jacobson [25, 33]. Verlinde imposed the equipartition rule on a very small region and short time scale and consider the Killing vector that can be approached by approximate Killing vectors. Requiring that the equipartition rule remains valid for all these approximate Killing vectors, the full Einstein equation can be obtained. We note that the same reasoning also can be carried in the current situation, since the Kodama vector will be reduced to the Killing vector for the static spacetime with $g_{tt}g_{rr} = -1$ and certainly for the approximately flat spacetimes.

At last, we would like to point out that the generalized Tolman mass (14) is non-negative, if the temperature and entropy are non-negative. This property can be used to relieve one problem against the entropic force scenario [27]. Consider Einstein gravity and 4-dimensional FRW spacetimes. It was shown [27] that the active gravitational mass

$$M_{\text{active}} = 2 \int_{\Sigma} \left( T^{\mu \nu} - \frac{1}{2} g^{\mu \nu} T \right) u_{\mu} u_{\nu} d\Sigma = \rho + 3p$$

for an ideal fluid with $T^{\mu \nu} = (\rho + p) u^{\mu} u^{\nu} + g^{\mu \nu} p$ is negative for an accelerated expanding universe with the equation of state $w < -1/3$. Thus, the temperature is negative if the equipartition rule is imposed and the entropy is positive. It was proved that it is impossible to redefine a new positive temperature and mass to derive the Einstein equation from the equipartition rule. However, their proof relies on the Killing vector so we can go beyond it. Consider the Hayward temperature

$$T = -\frac{r}{4\pi} (2H^2 + \dot{H}) = -\frac{r}{4\pi} (2H^2 + \dot{H}) = \frac{1}{3} r (3p - \rho)$$

where $H$ is the Hubble parameter and the dot denotes the derivative with respect to time. Since $T \leq 0$ for all physically allowed region $w \leq 1/3$, one
can define $-T$ as the real temperature and

$$-M = -2TS$$

(23)

as the mass which is always non-negative in the evolution of universe. This result is nontrivial since one can not similarly define $-M_{\text{active}}$ as the mass which is still negative for the desired region $w > -1/3$. In addition, it is encouraging to see that the positivity of the Hayward temperature in the FRW spacetime could be naturally predicted by the tunneling method [67].

4. Universal CV formula

In this section, we will build a universal CV formula and reveal its relation to the Friedmann equation from the braneworld perspective. Before that, we need to define a new chemical potential.

4.1. Chemical potential

In the first law (9), Hayward interpreted the term $WdV$ as the work done by changing the BH volume [48], mainly because it is consistent with the electromagnetic work for the RN BH. However, this interpretation is not unassailable since it is well-known that the volume of BHs can not be well-defined [70].

On the other hand, Verlinde proposed $N = A$ as the number of degree of freedom on the holographic screen. But what is the conjugated chemical potential?

Here we present that the work term $WdV$ can be reinterpreted as a chemical work $WdV = \mu dN$, where

$$\mu = \frac{rW}{d-2} = \frac{-rh_{ab}T^{ab}}{2(d-2)}$$

(24)

can be understood as the chemical potential. The chemical potential can be written as a geometric expression (when the field equation is used) and is well-defined both in static and dynamic spacetimes.

4.2. CV formula from bulk spacetimes

In 1+1-dimensional CFT, it is well-known that the Cardy formula [7]

$$S = 2\pi \sqrt{\frac{c}{6}(L_0 - \frac{c}{24})}$$
relates the entropy $S$ to the product of energy and radius $L_0$, the central charge $c$, and the Casimir effect $c/24$.

Verlinde presented that the Cardy formula is valid for higher dimensions [6]. This is now commonly known as CV formula. According to Witten’s argument [9] that the thermodynamics of CFT at high temperature can be identified with the one of large AdS BH, the CV formula can be derived in terms of the thermodynamics of Schwarzschild-AdS BHs, which can be expressed as [6]

$$S = \frac{2\pi a}{d-2} \sqrt{\tilde{E}_c(2\tilde{E} - \tilde{E}_c)}. \quad (25)$$

Here the rescaling $\tilde{x} = x/l/a$ can be understood as the UV/IR connection between the bulk and the boundary, and $a$ is the radius of sphere $\Omega_{d-2}$ where the CFT lives. The gravitational energy $E$ is identified with the mass of AdS BHs, which is usually calculated by the boundary stress-tensor after the suitable renormalization [68]. The Casimir energy $\tilde{E}_c$ is the non-extensive part of boundary energy $\tilde{E}$, which can be given by the violation of Euler relation

$$\tilde{E}_c = (d-2) (\tilde{E} - \tilde{T}_0 S + \tilde{p}_0 V), \quad (26)$$

where the pressure $\tilde{p}_0 = -\partial\tilde{E}/\partial V|_{S}$ and the volume $V = \Omega_{d-2} a^{d-2}$. Much effort has been made to understand and check the CV formula for different BHs. It was found that for RN-AdS BHs, the CV formula should be modified a little [17, 22]

$$S = \frac{2\pi a}{d-2} \sqrt{\tilde{E}_c(2\tilde{E} - \tilde{E}_c - \tilde{\Phi} Q)}, \quad (27)$$

where $\Phi$ denotes the electric potential, the electric charge

$$Q = \frac{\Omega_{d-2} \sqrt{2} (d-2) (d-3) q}{8\pi},$$

and the Casimir energy

$$\tilde{E}_c = (d-2) (\tilde{E} - \tilde{T}_0 S + \tilde{p}_0 V - \tilde{\Phi} Q). \quad (28)$$

Unfortunately, such minimally modified CV formula does not hold for multi-charge BHs in maximally supersymmetric gauged supergravities, that is [22]

$$S \neq \frac{2\pi a}{d-2} \sqrt{\tilde{E}_c(2\tilde{E} - \tilde{E}_c - \sum_i \tilde{\Phi}_i Q_i)}, \quad (29)$$

\footnote{We also use $\Omega_{d-2}$ to denote the area of sphere $\Omega_{d-2}$ with unit radius.}
where
\[ \tilde{E}_c = (d - 2) (\tilde{E} - \tilde{T}_0 S + \tilde{p}_0 V - \sum_i \tilde{\Phi}_i Q_i). \] (30)

In fact, the difference between the left and right hands in Eq. (29) has been calculated in [18] for the STU model in the \( d = 5, N = 2 \) gauged supergravity:
\[ S = \frac{2\pi a}{\sqrt{3W_h W_h}} \sqrt{\tilde{E}_c (2 \tilde{E} - \tilde{E}_c - \sum_i \tilde{\Phi}_i Q_i)}, \] (31)

where the superpotential \( W_h \tilde{W}_h \neq 3 \) unless all the charges are equal. Moreover, it should be noted that there is a modified CV formula for multi-charge BHs, which is
\[ S = \frac{2\pi a}{d-2} \sqrt{\left( \tilde{E}_c - \tilde{E}_q \right) \left[ 2 \left( \tilde{E} - \tilde{E}_q \right) - \left( \tilde{E}_c - \tilde{E}_q \right) \right]}, \] (32)

where \( \tilde{E}_q = \frac{\Omega_{d-2}(d-3)}{16\pi a} \sum_i q_i \) and \( \tilde{E} - \tilde{E}_q \) are interpreted as the proper internal energy [13] or the thermal excitation energy above the BPS state [15]. The essential idea of Eq. (32) is that the electrostatic self-repulsion makes no contribution to the pressure. However, it was argued that such modification "appears to be a somewhat ad hoc" and the CV formula has "no natural and universal modification" [22].

In the following, we will present a natural and universal CV formula. Contrary to Eqs. (25), (27), and (32) which consist of the Hawking temperature, the BH mass, the electric charges and electric potential, we will invoke a set of different thermodynamic quantities on the trapping horizon.

Let us focus on the Einstein gravity with a negative cosmological constant \( \Lambda = -\frac{(d-1)(d-2)}{2l^2} \). In terms of the general metric (1), the important entities to be calculated are the Eqs. (8) and (24) on the trapping horizon:
\[ U = \frac{(d-2)\Omega_{d-2}}{16\pi} r^{d-3} \left[ 1 + \frac{r^2}{l^2} \right], \] (33)
\[ \mu = \frac{d-2}{16\pi} \left( \frac{d-1}{l^2} + \frac{d-3 - 4\pi Tr}{r^2} \right). \] (34)

Now we assume that the thermodynamic quantities of trapping horizons in the asymptotic AdS spacetimes can be used to describe certain CFTs on the
boundary. Thus, the corresponding pressure of boundary CFTs is

$$\tilde{p} = -\frac{\partial \tilde{E}}{\partial \tilde{V}} \bigg|_S = \frac{l_r d-3}{16\pi a^{d-1}} \left( 1 + \frac{r^2}{l^2} \right) = \frac{\tilde{U}}{(d - 2)V}. \tag{35}$$

In terms of the first law in the bulk (9) and the reinterpretation of the chemical work, one can readily prove the mapping on the boundary

$$d\tilde{U} = \tilde{T}dS - \tilde{p}dV + \tilde{\mu}dN.$$ 

Then we can naturally write down the Casimir energy of boundary CFTs, which characterizes the violation of Euler relation

$$\tilde{U}_c = (d - 2)(\tilde{U} - \tilde{T}S + \tilde{p}V - \tilde{\mu}N) = \frac{(d - 2)\Omega_{d-2}}{8\pi a}l_r d-3. \tag{36}$$

Furthermore, one can express the entropy as

$$S = \frac{2\pi a}{d - 2} \sqrt{\tilde{U}_c(2\tilde{U} - \tilde{U}_c)}. \tag{37}$$

Intriguingly, Eq. (37) is the exact CV formula that describes certain CFTs. We argue that this result is in favor of our assumption, that is, there is an underlying holographic duality between thermodynamic quantities of trapping horizons in the bulk and boundary CFTs. Moreover, the CV formula (37) is universal in the sense that it is independent with the concrete metric of asymptotic AdS spacetimes. Compared with Eq. (27) for RN-AdS BHs, one can find that $\tilde{U}$ just equals to $\tilde{E} - \tilde{\Phi}Q/2$ and Eq. (36) equals to Eq. (28). Compared with Eq. (31) for STU model, one can find that $\tilde{U}$ equals to $\tilde{E} - \sum_i \tilde{\Phi}_i Q_i/2$ and Eq. (36) equals to Eq. (30) if and only if all the charges are equal.

4.3. CV formula on the brane universe

The CV formula (25) can be realized by the braneworld scenario. Savonije and Verlinde have shown that the motion of the brane in Schwarzschild-AdS background is viewed by the brane observer as the evolvement of a radiation dominated FRW universe [11]. When the brane crosses the horizon of bulk BHs, the Friedmann equation on the brane can be recast exactly as the CV formula (25) of the CFT dual to the bulk BH. This surprising result indicates a common origin of both sets of equations in a single underlying fundamental
theory. Here we will show that in arbitrary asymptotic AdS background, the brane motion can be described by the standard Friedmann equation where the effective energy is just the rescaled MS energy. Furthermore, the Friedmann equation can be recast as the universal CV formula (37).

Consider a \((d - 1)\)-dimensional brane with a constant tensor and take it as the boundary of the bulk spacetime. The location and the metric on the boundary brane are, at least partly, dynamical. The movement of the brane is described by Israel junction conditions [71]

\[
K_{\mu\nu} = \frac{\lambda}{d - 2} \gamma_{\mu\nu},
\]

(38)

where \(K_{\mu\nu}\) is the extrinsic curvature of the brane, \(\gamma_{\mu\nu}\) is the induced metric on the brane, and \(\lambda\) is a parameter related to the brane tensor. Consider an asymptotic AdS background with the metric (3). In terms of a new time parameter \(\tau\) satisfied with

\[
\frac{1}{g} \left(\frac{dr}{d\tau}\right)^2 - h \left(\frac{dt}{d\tau}\right)^2 = -1,
\]

(39)

the induced metric on the brane takes the FRW form

\[
ds^2 = -d\tau^2 + r^2(\tau)d\Omega_{d-2}^2.
\]

The equation of motion (38) can be translated into

\[
\frac{dt}{d\tau} = \frac{\lambda r}{(d - 2)\sqrt{gh}}.
\]

(40)

Inserting Eq. (40) into Eq. (39), one can obtain

\[
H^2 = \frac{\lambda^2}{(d - 2)^2} - \frac{g}{r^2},
\]

(41)

where \(H = \frac{dr}{r d\tau}\). Notice that MS energy on any sphere can be read as

\[
U = \frac{(d - 2)\Omega_{d-2}r^{d-3}}{16\pi} \left[1 - g + \frac{r^2}{l^2}\right].
\]

(42)

Solving \(g\) from (42) and tuning the \(d\)-dimensional cosmological constant to zero by setting

\[
\frac{\lambda}{(d - 2)} = \frac{1}{l},
\]

17
we have an important result that Eq. (41) can be rewritten as

$$H^2 = -\frac{1}{r^2} + \frac{16\pi U G_d}{(d-2)\Omega_{d-2}r^{d-1}}, \quad (43)$$

where we have recovered the Newton constant $G_d$ in the bulk. Compared with the previous works [14, 17] for RN-AdS BHs, the term $\sim U/r^{d-1}$ in Eq. (43) has not been recognized from the combination of a term $\sim E/r^{d-1}$ and a term $\sim Q^2/r^{2(d-2)}$. It should be noticed that Eq. (43) is not really radiation-dominated as it apparently looks like, since $U$ is not constant in general. And it has been pointed out that the universe is filled with the radiation and stiff matter when the bulk is an RN-AdS BH [17].

In the braneworld scenario, $G_d$ relates to the Newton constant $G$ on the brane by

$$G_d = \frac{G l}{d - 3}.$$  

Now selecting $r$ just as the rescaled radius $a$, and inserting $U = \tilde{U} a/l$, we obtain

$$H^2 = -\frac{1}{a^2} + \frac{16\pi G}{(d-2)(d-3)} \tilde{U}, \quad (44)$$

which is the standard Friedmann equation where the effective energy is just the rescaled MS energy.

Furthermore, let us recover the Newton constant in the entropy of bulk BHs

$$S = \frac{A}{4G_d} = \frac{A(d-3)}{4Gl}. \quad (45)$$

Substituting Eq. (42) into Eq. (43), one can find that the Hubble constant at the horizon obeys

$$H^2 = \frac{1}{l^2}. \quad (46)$$

Combining Eqs. (36), (45) and (46), one can eliminate $l$ and solve $H$ and $G$

$$H = \frac{(d-2)S}{2\pi \tilde{U} a^2}, \quad G = \frac{(d-2)(d-3)\Omega_{d-2}a^{d-4}}{8\pi \tilde{U} c}, \quad (47)$$

where we have focused on the moment when the brane crosses the horizon in the bulk. Then inserting Eq. (47) into the Friedmann equation (44), we eventually find the exact CV formula (37). This result indicates that the Friedmann equation (44) and the universal CV formula (37) have a common origin. We stress that both of them are independent with the concrete matter in the bulk.
5. AdS-Bekenstein bound

It is well-known that the Bekenstein bound in itself suggests the intimate relation between the system with and without gravity, since it is derived based on the Geroch process involving the BH but the gravitational constant is absent in the bound at last [72]. This relation was further revealed by the CV formula (25), which can be derived from AdS/CFT correspondence and gives the named normalized Bekenstein bound [6, 73] (also called as Bekenstein-Verlinde bound [14]) for certain CFTs:

\[ S \leq \frac{2\pi a}{d-2} E, \]  

(48)

where the equality holds when \( \tilde{E} = \tilde{E}_c \). In terms of the bulk quantities, the entropy bound can be described by

\[ S \leq \frac{2\pi l}{d-2} E, \]  

(49)

which was referred as AdS-Bekenstein bound [22]. From the minimally modified CV formula (27), this bound has been extended to the RN-AdS BH [14, 22]:

\[ S \leq \frac{2\pi l}{d-2} (E - \frac{\Phi Q}{2}). \]  

(50)

Interestingly, it was pointed out [22] that this minimally modified Bekenstein bound is a consequence of the cosmic censorship bound [74, 75]

\[ E \geq \frac{(d-2)\Omega_{d-2}}{16\pi} \left[ q^2 \left( \frac{\Omega_{d-2}}{A} \right)^{\frac{d-3}{d-1}} + \left( \frac{A}{\Omega_{d-2}} \right)^{\frac{d-3}{d-1}} \right] + 1 \left( \frac{A}{\Omega_{d-2}} \right)^{\frac{d}{d-1}}, \]

with equality being attained in the case of the RN-AdS BH. For multiple charged BHs, although the minimally modified CV formula does not hold (see Eq. (29)), a direct generalization of Eq. (50) has been verified, that is

\[ S \leq \frac{2\pi l}{d-2} (E - \frac{1}{2} \sum_i \Phi_i Q_i), \]  

(51)

which was called as the electrostatic AdS-Bekenstein bound [22].
Now consider the bulk form of the universal CV formula (37), from which we can obtain a new form of the AdS-Bekenstein bound

\[ S \leq \frac{2\pi l}{d-2} U. \]  \hfill (52)

We stress that this new bound is highly nontrivial because of three points: First, Eq. (52) inherits the feature of the CV formula (37), which is universal and is applicable even to the dynamical bulk spacetime. Second, Eq. (52) reduces to Eq. (50) for RN-AdS BHs, just like Eq. (51), which means that the new bound also supports the conjectured cosmic censorship bound. The third, perhaps the most crucial one, is that the new bound is tightened than Eq. (51). We will demonstrate the third point by studying three multi-charge BHs in the \( d = 4, d = 5, \) and \( d = 7 \) maximally supersymmetric gauged supergravities, respectively [76].

These multi-charge BHs can be described by the uniform metric

\[ ds^2 = -\Psi^{-\frac{d-3}{2}} f dt^2 + \Psi^{-\frac{1}{2}} \left( \frac{1}{f} dr^2 + r^2 d\Omega_{d-2}^2 \right), \]

where

\[ f = 1 - \frac{m}{r^{d-3}} + \frac{r^2}{l^2} \Psi, \]

and

\[ \Psi = \prod_{i}^{J} \Psi_i, \quad \Psi_i = 1 + \frac{q_i}{r^{d-3}}. \]

One should be careful that here the radius of sphere \( \Omega_{d-2} \) is not \( r \) but \( r \Psi^{\frac{1}{d-2}} \). Hence the MS energy (33) should read as

\[ U = \frac{(d-2)\Omega_{d-2}}{16\pi} r^{d-3} \Psi^{\frac{d-3}{d-2}} \left[ 1 + \frac{r^2}{l^2} \Psi^{\frac{1}{d-2}} \right]. \]  \hfill (53)

The mass, charge, and the associated potential can be written as

\[ E = \frac{(d-2)\Omega_{d-2}}{16\pi} (m + \frac{d-3}{d-2} \sum_{i}^{J} q_i), \]  \hfill (54)

\[ Q_i = \sqrt{q_i (m + q_i)}, \]  \hfill (55)
Φᵢ = \frac{(d - 3)Ω_{d-2}}{16\pi} \frac{Qᵢ}{r^{d-3} + qᵢ}. \hspace{1cm} (56)

For convenience, we will set \(qᵢ = xᵢ^J - r^{d-3}\) and \(l = 1\) below. Let us consider the BH solution with \(J = 4\) in \(d = 4\), \(N = 8\) gauged supergravity. To compare two bounds, we can use Eqs. (53)-(56) to calculate

\[
(E - \frac{1}{2} \sumᵢ \PhiᵢQᵢ) - U = \frac{1}{8} \left[ \sumᵢ xᵢ^4 + \sumᵢ \sumᵢ < j < k (xᵢxⱼxₖ)^4 - 4 \prodᵢ xᵢ - 4 \prodᵢ xᵢ^2 \right]. \tag{57}
\]

A direct numerical analysis demonstrates that Eq. (57) is non-negative, and it is vanishing only when all the charges are equal. As an explicit example, we set \(q₁ \neq q₂ = q₃ = q₄\), which will simplify Eq. (57) as

\[
(E - \frac{1}{2} \sumᵢ \PhiᵢQᵢ) - U = \frac{1}{8} (x₁ - x₂)^2 (x₁ + 2x₁x₂ + 3x₂^2 + 3x₁x₈ + 8 + 2x₁x₉ + x₁x₁₀) \geq 0.
\]

Next, we consider the STU model in \(d = 5\), \(N = 2\) gauged supergravity, where \(J = 3\). We calculate

\[
(E - \frac{1}{2} \sumᵢ \PhiᵢQᵢ) - U = \frac{π}{8} \left[ \sumᵢ xᵢ^3 + \sumᵢ \sumᵢ < j < k (xᵢxⱼ)^3 - 3 \prodᵢ xᵢ - 3 \prodᵢ xᵢ^2 \right]. \tag{58}
\]

Numerical analysis can prove that Eq. (58) is non-negative, and it is also vanishing if and only if all the charges are equal. As a case with \(q₁ \neq q₂ = q₃\), Eq. (58) is

\[
(E - \frac{1}{2} \sumᵢ \PhiᵢQᵢ) - U = \frac{π}{8} (x₁ - x₂)^2 (x₁ + 2x₂ + 2x₁x₃ + x₄) \geq 0.
\]

One can find that the inequalities for \(d = 4\) and \(d = 5\) are similar, but they are different with the following case \(d = 7\). The reason could be that the scalar fields in the \(d = 7\), \(N = 2\) gauged supergravity are not constant even when the charges are equal \cite{22}. For the BH solution with \(J = 2\), we have

\[
(E - \frac{1}{2} \sumᵢ \PhiᵢQᵢ) - U = \frac{π^2}{16r^2} \left[ r^6 + \sumᵢ \left( 2r^2xᵢ^2 + 2r^4xᵢ^2 - 5r^{14}xᵢ^4 - 5r^{16}xᵢ^5 \right) + \prodᵢ xᵢ^2 \right]. \tag{59}
\]
We have checked it as non-negative by numerical method. When \( q_1 = q_2 \), Eq. (59) is

\[
(E - \frac{1}{2} \sum_i \Phi_i Q_i) - U = \frac{\pi^2}{16r^2} \left( r^\frac{4}{3} - x_1^\frac{4}{3} \right)^2 \chi(r, x_1) \geq 0,
\]

where

\[
\chi(r, x_1) = r^{\frac{22}{9}} + 2r^{\frac{16}{9}} x_1^{\frac{2}{3}} + 3r^{\frac{14}{9}} x_1^{\frac{4}{9}} + 4r^{2} x_1^{\frac{6}{9}} + 4r^{\frac{12}{9}} x_1^{\frac{8}{9}} + 3r^{\frac{8}{9}} x_1^{\frac{10}{9}} + 2r^{\frac{4}{3}} x_1^{\frac{12}{9}} + x_1^{\frac{16}{9}}.
\]

6. Conclusion and discussion

In this paper, we have constructed a generalized Smarr formula (21), which further reveals the closed relation between the general theories of gravity and thermodynamics, especially in the dynamic spacetime. The generalized Smarr formula developed the equipartition law given in [25, 31, 32] for stationary spacetimes and might be taken as a thermodynamic prior to derive the field equation

The generalized Smarr formula is constructed based on the new gravitational mass (14), which could be useful in the study of the evaporation and collapse of BHs as well as the evolution of universe. Actually, we have shown that the mass in the standard cosmology (23) is non-negative, contrary to the previous unreasonable definition (22).

Verlinde proposed to take the surface area of holographic screen as the number of degrees of freedom. We have found its conjugated chemical potential. Our definition of chemical potential is reasonable because not only one can avoid to involve the naive definition of BH’s volume to interpret the work term in the first law, but also the chemical potential and number of degrees of freedom are necessary to give a universal CV formula. Here we will give another strong evidence. Evaluating Eq. (24) in the static spacetime (3), one can find that the chemical potential

\[
\mu = \frac{r W}{d-2} = \frac{r}{d-2} (T_t^t + T_r^r)
\]

As pointed out by Padmanabhan [35], one should be careful that there may be a logic problem in the derivation of field equations from the thermodynamics, that is, one needs the off-shell evidence to support the quantities, such as the Wald entropy, to be the meaningful thermodynamic quantities.
which is vanishing on the horizon if and only if both matter density and radial pressure vanish on the horizon too. Note that $T_t^t$ should be equal to $T_r^r$ on the horizon for Einstein gravity, since we are concerning about the usual BHs where $h$ and $g$ tend to zero on the horizon with the same speed. Now we consider a certain field theory on the boundary that can be described by the chemical potential of the bulk BH. The chemical potential indicates that the dual field theory can be the radiation only when the bulk spacetime satisfies the condition ($T_t^t=T_r^r = 0$ on the horizon), since the radiation has the vanishing chemical potential. Furthermore, as we pointed out in Sec. 4.3, Eq. (43) or Eq. (44) is not radiation-dominated, unless the MS energy (42) is a constant that means

$$g = 1 - \frac{16\pi}{(d-2)\Omega_{d-2}} \frac{U}{r^{d-3}} + \frac{r^2}{l^2},$$

where $U$ should be taken as a constant. In terms of Einstein equations, it is very satisfactory to see that Eq. (60) just imposes the condition $T_t^t = T_r^r = 0$ on the horizon. In this regard, we think that the holographic duality between the brane universe and bulk BHs strongly supports the definition of chemical potential and its conjugated number of degrees of freedom.

Observing Eq. (44), one can find that the effective energy on the $(d-1)$-dimensional brane universe is exactly the rescaled MS energy on the $(d-2)$-dimensional BH horizon of the $d$-dimensional bulk spacetime. On the other hand, we notice that the effective energy in a $(d-1)$-dimensional FRW spacetime can also be identified with the MS energy on the $(d-3)$-dimensional apparent horizon $[37, 38]$. We suspect that this direct relation between two horizons which has not been recognized before might imply a new aspect of holographic duality.

From the universal CV formula, we have found a universal AdS-Bekenstein bound (52). We note that there is another method to obtain this bound. Consider the boundary energy of the field theory dual to a Schwarzschild-AdS BH

$$\tilde{E} = \frac{(d-2)\Omega_{d-2}r^{d-4}l}{16\pi} \left[ 1 + \frac{r^2}{l^2} \right],$$

where we have set $a = r$. It was found $[73]$ that the normalized Bekenstein bound (48) on the boundary can be derived from the holographic bound in the bulk by minimizing the boundary energy (61) with respect to the AdS radius. This relation might meliorate some problems of the Bekenstein
bound, such as the species problem. It is obvious that the rescaled form of MS energy (33) has the same form as Eq. (61), so we can obtain the rescaled form of the universal bound (52) by minimizing the rescaled Eq. (33). Thus, as pointed out in [73], one can regard the rescaled form of Eq. (52) not as an upper bound on entropy but as a universal lower bound on the energy of a field theory with a given size and entropy.

We have demonstrated that the universal bound (52) is tighten than the previous electrostatic AdS-Bekenstein bound (51) for three multi-charge BHs. Here we point out that the modified CV formula (32) can also give an entropy bound

\[ S \leq \frac{2\pi l}{d-2} (E - E_q). \]  

(62)

One might wonder which one is stringent. A direct calculation shows that

\[ (E - E_q) - U = \frac{3\pi}{8r^2} (r - \prod_i x_i) (r^2 - \prod_i x_i) (r + \prod_i x_i) \]

for \( d = 5, J = 3 \),

\[ (E - E_q) - U = \frac{1}{2r} (r - \prod_i x_i) (r - \prod_i x_i^3) \]

for \( d = 4, J = 4 \), and

\[ (E - E_q) - U = \frac{5\pi^2}{16r^2} \left[ r^6 + (x_1 x_2)^2 - r^{14/5} (x_1 x_2)^{4/5} - r^{16/5} (x_1 x_2)^{6/5} \right] \]

for \( d = 7, J = 2 \). With the mind of \( q_i = x_i^J - r^{d-3} \geq 0 \), a simple algebra analysis can prove for all cases that, \( E - E_q \geq U \) if one of the charges is large enough or the BHs are large with \( r \gg l \). Furthermore, according to the AdS/CFT correspondence, the modified bound (62) for dual field theories can be effective only for very large BHs with \( r \gg l \). Consequently, the universal bound (52) is always tighten than the modified bound (62) for the field theory with dual gravity.

This paper is focused on the static and dynamic spacetimes with spherical symmetry. But the extension to stationary spacetimes is possible. Presumably it would involve the generalized Kodama vector and the Hawking energy [66, 77] to replace the Kodama vector and MS energy. It is interesting to see whether the universal CV formula can include the case of rotating BHs [21].
Finally, with the mind that the CV formula could hold for strongly-interacting field theories and the Bekenstein bound is supposed to be valid for the system with limited self-gravity, we would like to emphasize that the derivation of the universal CV formula and the AdS-Bekenstein bound from the thermodynamics of bulk trapping horizons sheds light on the holographic duality between dynamic bulk spacetime and boundary field theory.

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References

References

[1] J.D. Bekenstein, Phys.Rev. D 7 (1973) 2333; J.D. Bekenstein, Lett.Nuovo.Cim. 11 (1974) 467; S. Hawking, Commun.Math.Phys. 43 (1975) 199.
[2] G. ’t Hooft, gr-qc/9310026; L. Susskind, J.Math.Phys. 36 (1995) 6377, hep-th/9409089.
[3] A.D. Sakharov, Sov.Phys.Dokl. 12 (1968) 1040; G. Volovik, Phys.Rept. 351 (2001) 195, gr-qc/0005091; B. Hu, Int.J.Theor.Phys. 44 (2005) 1785, gr-qc/0503067.
[4] C. Barcelo, S. Liberati and M. Visser, Living Rev.Rel. 8 (2005) 12, gr-qc/0505065; L. Sindoni, SIGMA 8 (2011) 027, arXiv:1110.0686.
[5] T. Padmanabhan, Phys.Rept.406 (2005) 49, gr-qc/0311036; T. Padmanabhan, Rep.Prog.Phys.73 (2009) 046901, arXiv:0911.5004.
[6] E.P. Verlinde, hep-th/0008140.
[7] J.L. Cardy, Nucl.Phys. B270 (1986) 317.
[8] J.M. Maldacena, Adv.Theor.Math.Phys. 2 (1998) 231, hep-th/9711200.
[9] E. Witten, Adv.Theor.Math.Phys. 2 (1998) 253, hep-th/9802150.
[10] D. Kutasov and F. Larsen, JHEP 0101 (2001) 001, hep-th/0009244.

[11] I. Savonije and E.P. Verlinde, Phys.Lett. B507 (2001) 305, hep-th/0102042.

[12] B. Wang, E. Abdalla and R.K. Su, Phys.Lett. B503 (2001) 394, hep-th/0101073; B. Wang, E. Abdalla and R.K. Su, Mod.Phys.Lett. A17 (2002) 23, hep-th/0106086; K. Ke and M. Li, Phys.Lett. B606 (2005) 173, hep-th/0407056.

[13] R.G. Cai, Phys.Rev. D63 (2001) 124018, hep-th/0102113.

[14] R.G. Cai, Y.S. Myung and N. Ohta, Class.Quant.Grav. 18 (2001) 5429, hep-th/0105070.

[15] D. Klemm et al., Nucl.Phys. B620 (2002) 519, hep-th/0104141.

[16] L. Cappiello and W. Mueck, Phys.Lett. B522 (2001) 139, hep-th/0107238.

[17] A.K. Biswas and S. Mukherji, JHEP 0103 (2001) 046, hep-th/0102138; A.K. Biswas and S. Mukherji, Phys.Lett. B578 (2004) 425, hep-th/0310238.

[18] G. Cardoso and V. Grass, Nucl.Phys. B803 (2008) 209, arXiv:0803.2819.

[19] D. Youm, Phys.Lett. B515 (2001) 170, hep-th/0105093; I.H. Brevik et al., Phys.Rev. D70 (2004) 043520, hep-th/0401073.

[20] U.H. Danielsson, JHEP 0203 (2002) 020, hep-th/0110265; A. Medved, Class.Quant.Grav. 19 (2002) 919, hep-th/0111238; J. Jing, Phys.Rev. D66 (2002) 024002, hep-th/0201247.

[21] D. Klemm, A. Petkou and G. Siopsis, Nucl.Phys. B601 (2001) 380, hep-th/0101076; D. Birmingham and S. Mokhtari, Phys.Lett. B508 (2001) 365, hep-th/0103108; M.R. Setare and E.C. Vagenas, Phys.Lett. B584 (2004) 127, hep-th/0309092; R.G. Cai, L.M. Cao and D.W. Pang, Phys.Rev. D72 (2005) 044009, hep-th/0505133; C.O. Lee, Phys.Lett. B670 (2008) 146, arXiv:0807.2685; M. Setare and M. Jamil, Phys.Lett. B681 (2009) 469, arXiv:0912.0861.
[22] G. Gibbons, M. Perry and C. Pope, Phys.Rev. D72 (2005) 084028, hep-th/0506233.

[23] W. Fischler and L. Susskind, hep-th/9806039.

[24] R. Bousso, Rev.Mod.Phys. 74 (2002) 825, hep-th/0203101; R. Brustein, Lect.Notes.Phys. 737 (2008) 619, hep-th/0702108.

[25] E.P. Verlinde, JHEP 1104 (2011) 029, arXiv:1001.0785.

[26] M. Visser, JHEP 1110 (2011) 140, arXiv:1108.5240.

[27] M. Li and Y. Pang, Phys.Rev.D 82 (2010) 027501, arXiv:1004.0877.

[28] A. Kobakhidze, Phys.Rev.D 83 (2010) 021502, arXiv:1009.5414; A. Kobakhidze, (2011), arXiv:1108.4161.

[29] M. Chaichian, M. Oksanen and A. Tureanu, Phys.Lett. B702 (2011), arXiv:1104.4650; M. Chaichian, M. Oksanen and A. Tureanu, Phys.Lett. B712 (2012) 272, arXiv:1109.2794.

[30] R.C. Tolman, Relativity, Thermodynamics, and Cosmology (Dover, New York, 1987).

[31] T. Padmanabhan, Class.Quant.Grav. 21 (2004) 4485, gr-qc/0308070.

[32] T. Padmanabhan, Mod.Phys.Lett. A25 (2010) 1129, arXiv:0912.3165.

[33] T. Jacobson, Phys.Rev.Lett. 75 (1995) 1260, gr-qc/9504004.

[34] C. Eling, R. Guedens and T. Jacobson, Phys.Rev.Lett. 96 (2006) 121301, gr-qc/0602001; E. Elizalde and P.J. Silva, Phys.Rev. D78 (2008) 061501, arXiv:0804.3721; R. Brustein and M. Hadad, Phys.Rev.Lett. 103 (2009) 101301, arXiv:0903.0823; M.K. Parikh and S. Sarkar, (2009), arXiv:0903.1176; G. Chirco, C. Eling and S. Liberati, Phys.Rev. D83 (2011) 024032, arXiv:1011.1405; R. Guedens, T. Jacobson and S. Sarkar, arXiv:1112.6215.

[35] T. Padmanabhan, arXiv:0903.1254.

[36] A. Frolov and L. Kofman, JCAP 0305 (2003) 009, hep-th/0212327; U. Danielsson, Phys.Rev. D 71 (2005) 023516; R. Bousso, Phys.Rev. D 71 (2005) 064024; G. Calcagni, JHEP 0509 (2005) 060, hep-th/0507125.
[37] R.G. Cai and S.P. Kim, JHEP 0502 (2005) 050, hep-th/0501055; R.G. Cai and L.M. Cao, Phys.Rev. D75 (2007) 064008, gr-qc/0611071; M. Akbar and R.G. Cai, Phys.Rev. D75 (2007) 084003, hep-th/0609128; R.G. Cai and L.M. Cao, Nucl.Phys. B785 (2007) 135, hep-th/0612144; Y. Gong and A. Wang, Phys.Rev.Lett. 99 (2007) 211301, arXiv:0704.0793.

[38] A. Sheykhi, B. Wang and R.G. Cai, Phys.Rev. D76 (2007) 023515, hep-th/0701261; A. Sheykhi, B. Wang and R.G. Cai, Nucl.Phys. B779 (2007) 1, hep-th/0701198; X.H. Ge, Phys.Lett. B651 (2007) 49, hep-th/0703253; S.F. Wu, B. Wang and G.H. Yang, Nucl.Phys. B799 (2008) 330, arXiv:0711.1209; S.F. Wu et al., Class.Quant.Grav. 25 (2008) 235018, arXiv:0801.2688; S.F. Wu, G.H. Yang and P.M. Zhang, Gen.Rel.Grav. 42 (2010) 1601.

[39] T. Padmanabhan, Phys.Rev. D81 (2010) 124040, arXiv:1003.5665.

[40] R.M. Wald, Phys.Rev.D 48 (1993) 3427, gr-qc/9307038.

[41] V. Iyer and R.M. Wald, Phys.Rev. D 50 (1994) 846, gr-qc/9403028.

[42] L. Smarr, Phys.Rev.Lett. 30 (1973) 71.

[43] R. Banerjee and B.R. Majhi, Phys.Rev.D 81 (2010) 124006, arXiv:1003.2312; R. Banerjee et al., Phys.Rev.D 82 (2010) 124002, arXiv:1007.5204.

[44] G. Abreu and M. Visser, Phys.Rev.Lett. 105 (2010) 41302; G. Abreu and M. Visser, JHEP 1103 (2011) 140, arXiv:1012.2867.

[45] R.G. Cai, L.M. Cao and N. Ohta, Phys.Rev. D81 (2010) 061501, arXiv:1001.3470.

[46] R.G. Cai, L.M. Cao and N. Ohta, Phys.Rev. D81 (2010) 084012, 1002.1136.

[47] W. Gu, M. Li and R.X. Miao, arXiv:1011.3419; R.X. Miao, J. Meng and M. Li, arXiv:1102.1166.

[48] S.A. Hayward, Class.Quant.Grav. 15 (1998) 3147, gr-qc/9710089; S.A. Hayward, Phys.Rev.Lett. 81 (1998) 4557, gr-qc/9807003.
[49] S.A. Hayward, Phys.Rev. D 53 (1996) 1938, gr-qc/9408002; S.A. Hayward, Class.Quant.Grav. 11 (1994) 3025, gr-qc/9406033.

[50] S.A. Hayward, S. Mukohyama and M. Ashworth, Phys.Lett. A256 (1999) 347, gr-qc/9810006.

[51] S.F. Wu et al., Phys.Rev. D81 (2010) 044034, arXiv:0912.4633.

[52] T. Padmanabhan, Class.Quant.Grav. 19 (2002) 5387, gr-qc/0204019.

[53] A. Hashimoto and S. Sethi, Phys.Rev.Lett. 89 (2002) 261601, hep-th/0208126; J. Simon, JHEP 0210 (2002) 036, hep-th/0208165; M. Cvetic, S. Nojiri and S.D. Odintsov, Phys.Rev. D69 (2004) 023513, hep-th/0306031; D. Robbins and S. Sethi, JHEP 0307 (2003) 034, hep-th/0306193; T. Hertog and G.T. Horowitz, JHEP 0504 (2005) 005, hep-th/0503071; S. R. Das et al., Phys.Rev. D74 (2006) 026002, hep-th/0602107; C.S. Chu and P.M. Ho, JHEP 0604 (2006) 013, hep-th/0602054; A. Awad et al., Phys.Rev. D77 (2008) 046008, arXiv:0711.2994.

[54] H. Kodama, Prog.Theor.Phys. 63 (1980) 1217.

[55] R. Di Criscienzo et al., Phys.Lett. B657 (2007) 107, arXiv:0707.4425; A. Chatterjee, B. Chatterjee and A. Ghosh, (2012), arXiv:1204.1530.

[56] S.A. Hayward et al., Class.Quant.Grav. 26 (2009) 062001, arXiv:0806.0014; S.A. Hayward et al., AIP Conf.Proc. 1122 (2009) 145, arXiv:0812.2534.

[57] S. Carroll, Spacetime and geometry. An introduction to general relativity (Addison-Wesley, 2003).

[58] C. Misner and D. Sharp, Phys.Rev. 136 (1964) B571.

[59] L. Szabados, Living Rev.Rel. 12 (2009).

[60] H. Maeda, Phys.Rev. D73 (2006) 104004, gr-qc/0602109; H. Maeda and M. Nozawa, Phys.Rev. D77 (2008) 064031, arXiv:0709.1199.

[61] R.G. Cai et al., Phys.Rev. D80 (2009) 104016, arXiv:0910.2387.

[62] A. Komar, Phys. Rev. 129 (1963) 1873.
[63] D.V. Fursaev, Phys.Rev. D82 (2010) 064013, arXiv:1006.2623.

[64] F. Piazza, Phys.Rev. D82 (2010) 084004, arXiv:1005.5151; Y. Tian and X.N. Wu, Phys.Rev. D83 (2011) 021501, arXiv:1007.4331; Y. Tian and X.N. Wu, JHEP 1101 (2011) 150, arXiv:1012.0411; S.F. Wu et al., arXiv:1008.2072.

[65] A. Ghosh and A. Perez, Phys.Rev.Lett. 107 (2011) 241301, arXiv:1107.1320; E. Frodden, A. Ghosh and A. Perez, (2011), arXiv:1110.4055.

[66] S.A. Hayward, Phys.Rev.Lett. 93 (2004) 251101, gr-qc/0404077; S.A. Hayward, Phys.Rev. D70 (2004) 104027, gr-qc/0408008.

[67] R. Di Criscienzo, L. Vanzo and S. Zerbini, JHEP 1005 (2010) 092, arXiv:1001.4617; R. Di Criscienzo et al., (2009), arXiv:0906.1725.

[68] J.D. Brown and J.W. York, Phys.Rev. D 47 (1993) 1407; V. Balasubramanian and P. Kraus, Commun.Math.Phys. 208 (1999) 413, hep-th/9902121; A. Batrachenko et al., JHEP 0505 (2005) 034, hep-th/0408205.

[69] A. Chamblin et al., Phys.Rev. D 60 (1999) 064018.

[70] B.S. DiNunno and R.A. Matzner, Gen.Rel.Grav. 42 (2010) 63, arXiv:0801.1734.

[71] W. Israel, Nuovo Cim. B44 (1966) 1.

[72] J. D. Bekenstein, Phys. Rev. D 23 (1981) 287.

[73] E. Halyo, JHEP 1004 (2010) 097, arXiv:0906.2164.

[74] P. Jang, Phys.Rev. D 20 (1979) 834.

[75] G. Gibbons, Class.Quant.Grav. 16 (1999) 1677, hep-th/9809167.

[76] K. Behrndt, M. Cvetic and W. Sabra, Nucl.Phys. B553 (1999) 317, hep-th/9810227; M. Duff and J.T. Liu, Nucl.Phys. B554 (1999) 237, hep-th/9901149; M. Cvetic and S.S. Gubser, JHEP 9904 (1999) 024, hep-th/9902195; J.T. Liu and R. Minasian, Phys.Lett. B457 (1999) 39, hep-th/9903269.

[77] S. Hawking, J. Math. Phys. 9 (1968) 598.