Quantum noise and self-sustained radiation of $\mathcal{PT}$-symmetric systems

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The observation that $\mathcal{PT}$-symmetric Hamiltonians can have real-valued energy levels even if they are non-Hermitian has triggered intense activities, with experiments in particular focusing on optical systems, where Hermiticity can be broken by absorption and amplification. For classical waves, absorption and amplification are related by time-reversal symmetry. This work shows that microreversibility-breaking quantum noise turns $\mathcal{PT}$-symmetric systems into self-sustained sources of radiation, which distinguishes them from ordinary, Hermitian quantum systems.

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A common factor in quantum systems with a non-Hermitian Hamiltonian is the non-conservation of particle number, either because the system is leaky, or because there is loss or gain in an absorbing or amplifying medium. Ignoring nonlinear effects such as the feedback in a laser, such systems ordinarily do not possess stationary states; instead, they only support decaying quasibound states with complex energy, where the imaginary part $\text{Im } E = -1/2\tau$ (setting $\hbar = 1$) accounts for particle loss with decay rate $1/\tau$ (particle gain is associated to a negative decay rate). A notable exception are non-Hermitian systems with loss and gain combined such that they are invariant under joint parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry \cite{1}. If there is no leakage, these $\mathcal{PT}$-symmetric systems generically possess a set of real-valued energy levels, as well as complex-conjugate pairs of complex energy levels. Systems with entirely real spectrum define a consistent unitary extension of quantum mechanics \cite{2, 3}. This observation has led to intense research efforts delivering a new theoretical perspective on systems as varied as quantum field theories and complex crystals \cite{4}, while experimental realizations in particular focus on optical systems where Hermiticity can be violated by absorption and amplification \cite{5}.

For classical waves, amplification and absorption are strictly related by time reversal. The existence of stationary states with real energy can therefore be seen as a consequence of the balance of amplification and absorption in parity-related regions of a $\mathcal{PT}$-symmetric system. At the heart of absorption and amplification, however, are noisy microscopic quantum processes (such as spontaneous and stimulated emission events, and stimulated absorption events) which effectively break time-reversal symmetry \cite{6}. The objective of this Letter is to show that the effects of this quantum noise distinguish $\mathcal{PT}$-symmetric systems from Hermitian quantum systems, and indeed suggest an alternative interpretation of the physics behind non-Hermitian $\mathcal{PT}$ symmetry: (i) Accounting for quantum noise, $\mathcal{PT}$-symmetric systems with stationary states are self-sustained sources of radiation, fed by the pumping in the amplifying parts of the system. (ii) That the energy of these states is real means that the system is stabilized at the lasing threshold. (iii) When the system is leaky, the emitted radiation breaks parity symmetry (i.e., the emission pattern is asymmetric). (iv) In the limit of a non-leaking system, the emitted radiation intensity approaches a constant value, and provides a direct measure of the non-Hermiticity of the system. The internal energy density of radiation then diverges, which entails a practical limitation for the implementation of $\mathcal{PT}$ symmetry in closed systems.

These conclusions are obtained by employing the quantum-optical input-output formalism \cite{7} in its scattering formulation \cite{8-10}. The scattering approach also provides insight into $\mathcal{PT}$ symmetry for classical waves \cite{11, 12}, which defines the starting point of this Letter.

Scattering approach to non-Hermitian $\mathcal{PT}$-symmetric systems.—Probing the internal dynamics of an optical system by external radiation naturally leads to the scattering scenario depicted in Fig. 1. The relation $a^\text{out} = Sa^\text{in}$ between incoming and outgoing wave amplitudes is
provided by the scattering matrix

$$S(E) = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},$$

(1)

which contains blocks describing reflection \((r, r')\) and transmission \((t, t')\) when probed from the left or right, respectively. Each block consists of an \(N \times N\)-dimensional matrix, where \(N\) is the number of modes at each entrance. The scattering matrix is generally energy dependent (which arises from the wave number dependence of constructive and destructive interference), and its poles determine the energies of quasibound states in the system \[14\].

In general, the scattering matrix fulfills the following two reciprocity relations: The Onsager relation

$$S(\gamma, -B, E) = S^T(\gamma, B, E),$$

(2)

and the relation

$$S(-\gamma, B, E) = [S^\dagger(\gamma, B, E^*)]^{-1}$$

of classical microreversibility. Here, \(\gamma\) and \(B\) characterize two possible sources of broken time-reversal symmetry: absorption or amplification (\(\gamma > 0\) or \(\gamma < 0\), which contribute an imaginary symmetric (non-Hermitian) term to the Hamiltonian \(H\), and magneto-optical effects \((B)\), which contribute an antisymmetric (but still Hermitian) term.

Conventional time-reversal \(\mathcal{T} H = H^*\) transforms solutions according to \(\mathcal{T} \psi = \psi^*\), which interchanges incoming and outgoing states, and therefore transforms the scattering matrix according to

$$\mathcal{T} S(\gamma, B, E) = [S^\dagger(\gamma, B, E^*)]^{-1} = S(-\gamma, -B, E^*).$$

(4)

Assuming that energy is real, a system has \(\mathcal{T}\) symmetry, \(\mathcal{T} S = S\) hence \(S^\dagger = S^{-1}\), if \(S(\gamma, B) = S(-\gamma, -B)\), which requires \(\gamma = B = 0\) \[13\]. Parity \(\mathcal{P} H(x) = H(-x)\) transforms solutions according to \(\mathcal{P} \psi(x) = \psi(-x)\), which exchanges the left and right leads and yields

$$\mathcal{P} S(\gamma, B, E) = \sigma_x S(\gamma, B, E) \sigma_x,$$

(5)

where \(\sigma_x\) is a Pauli matrix operating on the blocks in Eq. \[1\]. The \(\mathcal{PT}\) operation on the scattering matrix is therefore given by

$$\mathcal{PT} S(\gamma, B, E) = \sigma_x S^\dagger(\gamma, B, E) \sigma_x^{-1} \sigma_x$$

$$= \sigma_x S(-\gamma, -B, E^*) \sigma_x.$$

(6)

For Hermitian systems, \(\mathcal{PT}\) symmetry implies \(S = \sigma_x S^T \sigma_x\) \[13\]. For non-Hermitian systems, \(\mathcal{PT}\) symmetry implies the additional condition \(\mathcal{P} \gamma = -\gamma \ [\gamma(x) = -\gamma(-x)]\), i.e., there is a balance of absorption and amplification in parity-related regions.

Let us now explore from the scattering perspective how real-energy bound states appear in \(\mathcal{PT}\)-symmetric systems. As shown in Fig. \[1\] such systems can be constructed by joining two regions, where the left region, with scattering matrix \(S_1 = \begin{pmatrix} r_1 & t'_1 \\ t_1 & r_1 \end{pmatrix}\), is \(\mathcal{PT}\)-symmetric to the right region, \(S_2 = \mathcal{PT} S_1\), which using standard block-inversion formulas can be written as

$$S_2 = \begin{pmatrix} \frac{1}{(r_1^{-1}t_1 - t_1')^*} & \frac{1}{(t_1' - r_1^*-1t_1)^*} \\ \frac{1}{(t_1' - r_1^*-1t_1)^*} & \frac{1}{(r_1 - t_1't_1'^*-1t_1)^*} \end{pmatrix}.$$

(8)

Bound states can be studied by closing the system off by mirrors with small transmission probability \(\Gamma \ll 1\), described by a scattering matrix

$$S_T = -\begin{pmatrix} \sqrt{1-\Gamma} & i\sqrt{\Gamma} \\ i\sqrt{\Gamma} & \sqrt{1-\Gamma} \end{pmatrix}.$$

(9)

The mirrors can be included by wave matching at their inward-facing faces, which amounts to an algebraic elimination of the amplitudes \(a^\text{in} = a^\text{out}\) in Fig. \[1\].

The scattering matrix of the left half of the system then reads

$$S_L = -\begin{pmatrix} \frac{r_1 + \sqrt{1-\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} & \frac{i t_1' \sqrt{\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} \\ \frac{it_1 \sqrt{\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} & \frac{t_1' \sqrt{1-\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} - r_1' \end{pmatrix},$$

(10)

while the scattering matrix \(S_R = \mathcal{PT} S_L\) of the right half again follows from symmetry. These scattering matrices relate amplitudes of ingoing and outgoing modes (defined in Fig. \[1\]) according to

$$\begin{pmatrix} a^\text{in}_L \\ a^\text{out}_L \end{pmatrix} = S_L \begin{pmatrix} a^\text{in}_R \\ a^\text{out}_R \end{pmatrix}, \quad \begin{pmatrix} a^\text{in}_R \\ a^\text{out}_R \end{pmatrix} = S_R \begin{pmatrix} a^\text{in}_L \\ a^\text{out}_L \end{pmatrix}.$$

(11)

The scattering matrix of the composed system is obtained by algebraically eliminating the amplitudes \(a^\text{out}_R\) and \(a^\text{out}_L\) at the interface between both regions. For \(\Gamma \rightarrow 0\), these amplitudes become singular when

$$\det \text{Im}(r_1') = \det \left[\text{Im}(r_1' - \frac{t_1' t_1^*}{1 + r_1})\right] = 0,$$

(12)

which (due to the general connection of scattering poles and quasibound states) is the quantization condition of the closed system. Determinantal quantization conditions of this kind have been introduced for general systems in Ref. \[14\]; they also form the basis of exact numerical techniques as reviewed in \[16\]. The \(\mathcal{PT}\)-specific
version [12] of the quantization condition requires that the 
N real column vectors of \( \text{Im}(r'_L) \) be linearly depen-
dent, which generically can be achieved by varying a 
single real parameter such as \( E \) (identifying this as a 
problem of codimension one). Therefore, we recover that 
closed \( \mathcal{PT} \)-symmetric systems typically possess a num-
ber of bound states with real energy, even though the 
Hamiltonian is not Hermitian. Condition [12] also ad-
mits complex eigenvalues, which then appear in complex-
conjugated pairs.

Quantum noise.—The scattering approach can be ex-
tended to include quantum noise by passing from wave 
amplitudes \( a^\text{in}, a^\text{out} \) to bosonic annihilation operators 
\( \hat{a}^\text{in}, \hat{a}^\text{out} \), respectively. This defines the scattering vari-
ant of the input-output formalism [7, 8], which has been 
used to describe systems that are exclusively absorbing or 
amplifying. To adapt the approach to \( \mathcal{PT} \)-symmetric 
systems, where both effects are combined, we formally 
separate the absorbing regions from the amplifying re-
gions, and then join them together similar to the de-
scription of classical waves, given above.

For definiteness let us assume that the left half of the 
system is purely absorbing. For this part, the input-
output scattering relations then take the form

\[
\begin{pmatrix}
\hat{a}^\text{out}_L \\
\hat{a}^\text{in}_0
\end{pmatrix}
= S_L
\begin{pmatrix}
\hat{a}^\text{in}_L \\
\hat{a}^\text{out}_0
\end{pmatrix}
+ Q_L \hat{b}_L,
\]

which connects the ingoing and outgoing modes to 
bosonic operators \( \hat{b}_L \) and \( \hat{b}_R \) representing quantum fluc-
tuations in the left and right part of the medium. These 
operators can be defined microscopically in the frame-
work of system-and-bath approaches, or phenomenologi-
cally as operator-valued Langevin forces [7, 8]. The prop-
erties of these operators can be determined from the con-
dition that both \( \hat{a}^\text{in} \) and \( \hat{a}^\text{out} \) have to satisfy standard 
canonical commutation relations. This dictates that the 
coupling matrix \( Q_L \) satisfies the fluctuation-dissipation 
theorem [9]

\[
Q_L Q^\dagger_L = 1 - S_L S^\dagger_L.
\]

In the right half of the system, where the medium is am-
plifying, we have

\[
\begin{pmatrix}
\hat{a}^\text{in}_0 \\
\hat{a}^\text{out}_R
\end{pmatrix}
= S_R
\begin{pmatrix}
\hat{a}^\text{out}_L \\
\hat{a}^\text{in}_R
\end{pmatrix}
+ Q_R \hat{b}_R,
\]

where the commutation relations now dictate coupling to 
creation operators, with

\[
Q_R Q^\dagger_R = S_R S^\dagger_R - 1.
\]

By assumption, the operators \( \hat{b}_L \) and \( \hat{b}_R \) commute with 
\( \hat{a}^\text{in}_0 \); however, according to Eqs. [13] and [15] they do not 
commute with \( \hat{a}^\text{out}_0 \), which is a manifestation of broken 
micro-reversibility in quantum optics.

We can now describe the full \( \mathcal{PT} \)-symmetric system by
algebraically eliminating the interface operators \( \hat{a}^\text{left}_0 \) and
\( \hat{a}^\text{right}_0 \). In the absence of incoming radiation, the intensity 
emitted to the left and right is then given by

\[
I_L(E) = \frac{1}{2\pi} \langle \hat{a}^\text{out}_L \dagger \hat{a}^\text{out}_L \rangle, \quad I_R(E) = \frac{1}{2\pi} \langle \hat{a}^\text{out}_R \dagger \hat{a}^\text{out}_R \rangle,
\]

which can be evaluated assuming

\[
\langle \hat{b}_L \hat{b}_L \rangle = 0, \quad \langle \hat{b}_R \hat{b}_R \rangle = 0
\]

(ground-state population in the absorbing regions and to-
tal population inversion in the amplifying regions; these 
conditions minimize the quantum noise).

Let us first consider the case of a single-mode resonator 
(\( N = 1 \)) with purely ballistic internal dynamics and ab-
sorption in the left region [17], with scattering matrices

\[
S_1 = \begin{pmatrix} 0 & t_1 \\ t_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1/t_1^* \\ 1/t_1 & 0 \end{pmatrix},
\]

where \( |t_1| \leq 1 \). Including the mirrors, the total scattering 
matrix is [18]

\[
S = \begin{pmatrix}
\sqrt{1 - \Gamma} (t_1^2 - t_1^* t_1) & |t_1|^2 \Gamma \\
\frac{t_1^2 (1 - \Gamma) - t_1^* t_1^2}{t_1^2 (1 - \Gamma) - t_1^* t_1^2} & \sqrt{1 - \Gamma} (t_1^2 - t_1^* t_1)
\end{pmatrix},
\]

and the quantization condition [12] for the closed re-
sonator takes the form \( \text{Im} t_1^2 = 0 \) (indeed, the scattering 
matrix elements diverge under this condition and 
\( \Gamma \to 0 \)). Following the quantum-optical procedure de-
scribed above [and using, in particular, Eqs. [12], [16],
and [18]], we find that this resonator emits radiation of 
intensity

\[
I_L(E) = \frac{\Gamma}{2\pi |t_1|^2} \left( (1 - \Gamma + |t_1|^2)^2 - 1 + \Gamma^2 \right),
\]

\[
I_R(E) = \frac{\Gamma (1 - |t_1|^2)}{2\pi |t_1/t_1^*|^2 - 1 + \Gamma^2}.
\]

Since \( |t_1| < 1 \) this gives \( I_R > I_L \), the difference being

\[
\Delta I(E) = I_R(E) - I_L(E) = \frac{\Gamma^2 (|t_1|^2 - |t_1|^2)^2}{2\pi |t_1/t_1^*|^2 - 1 + \Gamma^2}.
\]

Therefore, the emission from the right exit, close to the 
amplifying region of the medium, is larger than the emis-
sion from the left exit, close to the absorbing region of 
the medium. The overall output intensity to both sides can 
be written as

\[
I(E) = I_L(E) + I_R(E) = \frac{\Gamma (2 - \Gamma) (|t_1|^2 - |t_1|^2)}{2\pi |t_1/t_1^*|^2 - 1 + \Gamma^2}.
\]
Close to quantization in the closed system \( |\Gamma| \ll 1 \), \( E \approx E_0 \), where \( E_0 \) fulfills the quantization condition \( \text{Im} t_1^2(E_0) = 0 \), the emission pattern becomes symmetric (as a consequence of \( \mathcal{PT} \) invariance) and approaches a Lorentzian of the form

\[
I_L(E) = I_R(E) = \frac{\Gamma (|t_0|^2 - |t_0|^2)}{2\pi|2i\tau (E - E_0) + \Gamma|^2}.
\]

Here \( t_0 = t_1(E_0) \), while

\[
\tau = 2\Im \frac{1}{t_1} \frac{dt_1}{dE} \bigg|_{E=E_0} \approx 2L/c
\]

is the transmission delay time of propagation between the two mirrors (with \( L \) the length of each region and \( c \) the speed of light). The full width at half maximum is given by \( \Delta E = \Gamma/\tau \). While this width shrinks to zero as the system is closed off, remarkably the total intensity

\[
I_{\text{tot}} = \int I(E) \, dE = \frac{|t_0|^2 - |t_0|^2}{2\tau}
\]

remains finite, and can be interpreted as a direct measure of the degree of non-Hermiticity of the system (for ballistic transport, Hermiticity implies \( |t_0| = 1 \)).

In the case of a single-mode resonator with backscattering (where \( r_1 \) and \( r'_1 \) are finite), compact expressions can still be obtained as long as the leakage remains small (\( \Gamma \ll 1 \)), implying according to Eq. (10) that \( |r_L + 1|, |t_L|, |t'_L| \ll 1 \). The emission pattern is then still symmetric, with intensity

\[
I_L(E) = I_R(E) = \frac{1}{2\pi} \frac{(1 - |r'_L|^2)|t'_L|^2}{(2 \Im r'_L)^2 - i t_L t'_L}. \tag{28}
\]

Linearization around the quantization condition again reveals a Lorentzian line shape, with line width

\[
\Delta E = \text{Re} \left( \frac{d[(\text{Im} r'_L)/t_L t'_L]}{dE} \right)^{-1}. \tag{29}
\]

Accounting for the scaling (10) of scattering coefficients with \( \Gamma \), the total intensity \( I_{\text{tot}} \propto (1 - |r'_L|^2) \) again remains finite as \( \Gamma \to 0 \). In the Hermitian case, this limit would imply \( |r'_L| = 1 \), so that the intensity vanishes. Therefore, the emitted radiation is still a direct measure of the degree of non-Hermiticity.

Following the general formalism described above, the observations for one-dimensional scattering can be extended to the general case of \( \mathcal{PT} \)-symmetric systems with many modes, for which compact expressions are no longer available. The emitted intensity generally remains finite even in the limit of a closed system. Because the expectation values \( \langle a_0^\dagger a_0 \rangle \propto \Gamma^{-1} \), \( \langle a_0^\dagger a_0 \rangle \propto \Gamma^{-1} \) of the internal operators formally diverge in this limit, this is accompanied by a diverging internal energy density, which can be interpreted as the source of this radiation. Deviations from perfect \( \mathcal{PT} \) symmetry can be incorporated into the scattering formalism by taking \( S_R \neq \mathcal{PT} S_L \), which results in an asymmetric emission pattern even in the limit of non-leaky mirrors.

In standard laser theory \([19]\), an active optical medium is below threshold (and stable) as long as all fundamental modes are damped, which is characterized by complex energies with a negative imaginary part. With increasing pumping, the loss of the modes within the amplification window is counteracted by gain, and as soon as one of the energies acquires a positive imaginary part the system becomes unstable: the intensity increases exponentially until saturation sets in. Therefore, the radiation features described above are characteristic of a laser stabilized at threshold, the threshold condition being that the energy of the lasing mode is real. The system then is marginally stable, and the internal radiation energy accumulates due to the quantum fluctuations. In practice, this leads to saturation in the amplifying parts of the system and therefore identifies an obstacle for the implementation of strict \( \mathcal{PT} \) symmetry in closed optical systems. When the system is slightly opened up, the threshold condition is no longer met and the internal energy density is finite, while the emitted intensity remains a direct measure of the non-Hermiticity of the system.

**Concluding remarks.**—In summary, the present Letter demonstrates that accounting for quantum noise, optical realizations of non-Hermitian \( \mathcal{PT} \)-symmetric systems emit radiation of an intensity that provides a direct measure of non-Hermiticity.

This has both practical as well as fundamental implications. The fundamental consequences arise because non-Hermitian \( \mathcal{PT} \)-symmetric systems with an entirely real spectrum define a consistent unitary theory of quantum mechanics \([2, 3]\), formalized by the concept of quasi-Hermiticity, which introduces a new scalar product based on a generalized conjugation operation \( \mathcal{C} \) that satisfies \( \mathcal{C}^2 = 1 \), \( [\mathcal{C}, H] = [\mathcal{C}, \mathcal{PT}] = 0 \). The self-sustained radiation identified here shows that accounting for quantum noise, non-Hermitian \( \mathcal{PT} \)-symmetric systems are physically distinct from ordinary Hermitian quantum systems: the canonical commutation relations for the input and output operators are only invariant under unitary transformations, which constraints the possibility to introduce alternative scalar products. This enforces classical distinctions based on the transparency of such systems \([12]\).

From a practical perspective, the self-sustained radiation can be used as an indicator of successfully implemented non-Hermitian \( \mathcal{PT} \) symmetry in leaky systems, while the accompanying marginal instability and diverging internal energy density signifies a practical obstacle for its implementation in the limit of no leakage. Furthermore, these findings identify a hitherto unexplored arena to study quantum fluctuations in active systems close to the lasing threshold.

Non-optical realizations of \( \mathcal{PT} \)-symmetric systems offer a wide range of connections to quantum field theories...
and while the results in the present paper are not directly transferable in detail, they suggest that quantum noise should provide fundamental insight into such systems, as well.

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$$S_L = \begin{pmatrix}
\sqrt{1 - \Gamma} & it_1 \sqrt{T} \\
t_1 \sqrt{T} & t_1^* \sqrt{1 - \Gamma}
\end{pmatrix},$$

$$S_R = \begin{pmatrix}
\sqrt{1 - \Gamma} & i \sqrt{T} \\
t_1^* \sqrt{T} & t_1 \sqrt{1 - \Gamma}
\end{pmatrix}.$$