Quasi-Excitations and Superconductivity in the t-J model on a Ladder

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Abstract

We study the t-J model on a ladder using the slave-fermion-CP$^1$ formalism which has been used successfully in studying lightly doped high-$T_C$ cuprates. Special attention is paid to the dynamics of composite gauge fields and the natures of quasi-excitations. The slave-fermion-CP$^1$ approach explains many aspects of the ladder system observed by experiments and numerical studies in a natural and coherent manner. We first obtain the low-energy effective model by integrating out half of the CP$^1$ variables (the Schwinger bosons) assuming a short-range antiferromagnetic order (SRAFO). The spin part of the effective model is the relativistic CP$^1$ model. In the single-chain case, there appears a topological $\theta$-term with $\theta = \pi$, as is well known. On the other hand, in the two-leg ladder case, we have $\theta = 2\pi$. The dynamics of the composite gauge boson strongly depends on the value of the coefficient of the $\theta$-term. This fact explains why the quasi-excitations in a chain and ladders of even legs are different. For a ladder, the gauge dynamics realizes in the confinement phase, so the quasi-excitations are charge-neutral objects like spin triplet and electrons, etc. The effective model reveals attractive force between holes, which generates superconductivity in SRAFO. The symmetry of hole-pair condensation should be of the $d$-wave type.

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1 Introduction

In the last few years, quasi-one-dimensional systems of strongly correlated electrons are of great theoretical and experimental interest in condensed matter physics[1]. Studies on these systems are expected to shed light on the mechanism of high-T\textsubscript{C} superconductivity. At present, it is known that spin excitations in an antiferromagnetic (AF) Heisenberg model put on an even-leg ladder have an energy gap, though those on an odd-leg ladder are gapless. Very recently, superconductivity has been observed in (weakly interacting) doped spin ladders with a spin gap[2], and the symmetry of the order parameter seems to be of a d-wave type as some theoretical studies predict[3].

The metallic phase of high-T\textsubscript{C} cuprates is anomalous, and a phenomenon called charge-spin separation (CSS) is expected to occur[4]. In mean-field theories (MFT) of the t-J model in slave-particle representations, the CSS is implemented naturally. However, the phase degrees of freedom of MF’s behave like (composite) gauge fields[5], so careful study is necessary in order to check the reliability of MFT. In the previous papers[6, 7], we argued that the CSS takes place at sufficiently low temperatures (T) in the two-dimensional (2D) t-J model. To derive the result, a gauge-theoretical treatment was essential.

In the present paper, we shall study the t-J model on a ladder of two legs by using the slave-fermion (SF) representation. The gauge-theoretical method is also applicable for this model straightforwardly. It clarifies why the quasi-excitations in chain and ladder systems are different so much; In a chain, the full CSS takes place and holons and spinons are quasi-excitations, while in a two-leg ladder, we shall see that the system is in a confinement phase and the gauge-charge-neutral objects like spin triplet and bound states of holons and spinons are low-energy excitations. This result explains the numerical studies in Ref.[8].

There are field-theoretical studies for the ladder systems, most of which use the bosonization techniques[9]. We want to stress here that the gauge-theoretical study is useful not only for the 2D systems of strongly-correlated electrons but also for the
quasi-one-dimensional systems. Furthermore, we believe that it gives us an universal and coherent understanding of a wide variety of strongly-correlated electron systems including the fractional quantum Hall effect.

This paper is organized as follows. In Sect.2, we shall introduce the t-J model in the SF-CP$^1$ representation. This procedure is closely related with our previous work which studied the 2D t-J model\cite{10}. In Sect.3, we shall obtain a low-energy effective model by integrating out half of the spin variables (CP$^1$ variables) assuming a short-range AF order (SRAFO). The effective model, as a result of the existence of SRAFO, contains important interactions among smooth spin variables and hole field. In Sect.4, we shall study the spin dynamics in the chain and ladder t-J model. The continuum limit of the spin part of the effective model is the CP$^1$ model with a topological term. Coefficient of the topological term depends on the number of legs and strongly influences the dynamics of composite gauge bosons. As we explained previously\cite{6, 7}, the mechanism of CSS in strongly-correlated electron systems is described by a (de)confinement phenomenon of composite gauge fields. In Sect.5 we shall study the dynamics of composite gauge bosons on a two-leg ladder using the method presented in \cite{7} and identify quasi-excitations. We find that the system is in the confinement phase down to $T = 0$. So the quasi-excitations should be gauge-charge-neutral bound states of holons and spinons, like a spin triplet (magnons) and electrons. In Sect.6, we shall study the possibility of superconductivity and its symmetry. The effective model involves an attractive force between holes in the SRAFO background which enhances a d-wave hole-pair condensation. In Sect.7 we present discussion.

2 The t-J model in the SF-CP$^1$ representation

We consider the t-J model on a two-leg ladder whose Hamiltonian is given by

\[ H = - \sum_{i, \sigma} \left( t \sum_{a=1}^{2} C_{i+1, a, \sigma}^\dagger C_{i, a, \sigma} + t' C_{i, 1, \sigma}^\dagger C_{i, 2, \sigma} + H.c. \right) \]
Here the suffix $a (=1,2)$ distinguishes one of two legs, $i$ labels sites along each leg. $C_{i,a,\sigma}$ is the electron operator with the spin $\sigma (=1,2)$, and $\vec{S}_{i,a}$ and $n_{i,a}$ are the spin and number operators on the site $(i,a)$. The physical states must satisfy

$$n_{i,a} < 2,$$  \hfill (2.2)

on each site. In the SF formalism, the electron operator is expressed in terms of the bosonic spinon operators $a_{i,a,\sigma}$ and the fermionic holon $\psi_{i,a}$ operators as

$$C_{i,a,\sigma} = \psi_{i,a}^\dagger a_{i,a,\sigma},$$  \hfill (2.3)

and the physical state condition (2.2) becomes

$$\sum_\sigma a_{i,a,\sigma}^\dagger a_{i,a,\sigma} + \psi_{i,a}^\dagger \psi_{i,a} = 1.$$  \hfill (2.4)

As in the previous paper \[10\], we solve the condition (2.4) by rewriting the operator $a_{i,a}$ in terms of CP$^1$ operator $z_{i,a}$ as

$$a_{i,a,\sigma} = (1 - \psi_{i,a}^\dagger \psi_{i,a})^{1/2} z_{i,a,\sigma} = (1 - \psi_{i,a}^\dagger \psi_{i,a}) z_{i,a,\sigma},$$  \hfill (2.5)

$$\sum_\sigma z_{i,a,\sigma}^\dagger z_{i,a,\sigma} = 1.$$  \hfill (2.6)

This representation is quite useful especially for the lightly doped case of the t-J model and the AF Heisenberg model\[10, 11\]. We shall treat $\psi_{i,a}$ and $z_{i,a,\sigma}$ as fundamental variables and employ the path-integral formalism (See Ref.[10] for detailed discussions on this formalism).

From (2.3) and (2.5), it is obvious that there appears a local gauge symmetry in the SF-CP$^1$ t-J model;

$$z_{i,a,\sigma} \rightarrow e^{i\phi_{i,a}} z_{i,a,\sigma},$$

$$\psi_{i,a} \rightarrow e^{i\phi_{i,a}} \psi_{i,a}.$$  \hfill (2.7)
The electron operator $C_{i,a,\sigma}$ is invariant under this transformation since it is a composite of a spinon and an anti-holon. One may expect that there appears composite gauge bosons with respect to (2.7). As we showed in the previous papers, these gauge bosons are introduced as auxiliary fields in path-integrals, and their dynamics specifies the nature of quasi-excitations.

3 Short-range AF and low-energy effective model

In this section we shall obtain the low-energy effective model by integrationg out half of the CP$^1$ variables $z_{i,a,\sigma}$ (e.g., the CP$^1$ variables on the odd sites) assuming a SRAFO.

In the path-integral formalism, the partition function $Z$ is given as

$$Z = \int [Dz][D\psi] \exp \left[ \int d\tau A(\tau) \right],$$

$$A(\tau) = -\sum_{i,a} \left( \sum_{\sigma} \bar{z}_{i,a,\sigma} \dot{z}_{i,a,\sigma} + \bar{\psi}_{i,a} \dot{\psi}_{i,a} \right) - H. \quad (3.1)$$

The J-terms in $H$ is explicitly given in terms of $z_{i,a,\sigma}$ and $\psi_{i,a}$ as

$$A_J = -\frac{J}{2} \sum_{i,a} \rho_{i,a}^2 \rho_{i+1,a}^2 \left[ (\bar{z}_{i,a} z_{i+1,a})(\bar{z}_{i+1,a} z_{i,a}) - 1 \right],$$

$$A_{J'} = -\frac{J'}{2} \sum_{i} \rho_{i,1}^2 \rho_{i,2}^2 \left[ (\bar{z}_{i,1} z_{i,2})(\bar{z}_{i,2} z_{i,1}) - 1 \right], \quad (3.2)$$

where $\rho_{i,a}^2 = 1 - \bar{\psi}_{i,a} \psi_{i,a}$.

From (3.2), it is obvious that the SRAFO configurations, $\vec{S}_j \simeq -\vec{S}_i$ for a nearest-neighbor (NN) pair $(i,j)$, give dominant contributions to the path integral. One can express a spin $\vec{S}_i$ and its time-reversed spin $-\vec{S}_i$ as

$$\vec{S}_i = \bar{z}_i \vec{\sigma} z_i$$

$$-\vec{S}_i = \bar{\bar{z}}_i \vec{\sigma} \bar{z}_i$$

$$\bar{z}_\sigma \equiv \epsilon_{\sigma\sigma'} z_{\sigma'}, \quad \epsilon_{12} = -\epsilon_{21} = 1, \quad (3.3)$$

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where $\vec{\sigma}$ is the Pauli spin matrices. Therefore the assumption of SRAFO is written as
\begin{equation}
  z_i \simeq \tilde{z}_j, \quad \langle i, j \rangle = \text{NN sites}.
\end{equation}

To integrate out half of the CP\(^1\) variables, say, those on the odd sites, around the SRAF configurations (3.4), we pair every odd site with a NN even site and denote the CP\(^1\) variables in a pair as $z_o$ and $z_e$. For $J \geq J'$, we take $z_e = z_{i+1,a}$ for odd-site $z_o = z_{i,a}$, whereas for $J' \geq J$, $z_e = z_{i,a}$ for $z_o = z_{i,a}$ where $a = 2$ for $a = 1$ and $\bar{a} = 1$ for $a = 2$ (see Fig.1).

The odd site CP\(^1\) variable $z_o$ is parameterized as follows by using its reference coordinate $z_e$,
\begin{equation}
  z_o = p_{oe} z_e + q_{oe} \tilde{z}_e, \quad q_{oe} = (1 - \bar{p}_{oe} p_{oe})^{\frac{1}{2}} U_{oe}, \quad U_{oe} \in U(1).
\end{equation}

We substitute (3.5) into $A(\tau)$ and expand it in powers of $p_{oe}$ up to $O(p_{oe}^3)$;
\begin{equation}
  A = A_0 + A_p + O(p_{oe}^3),
\end{equation}
\begin{align}
A_0 &= \sum_{o \in \text{odd}} \left[ - \bar{\psi}_e \psi_e - \bar{\psi}_o \psi_o + (\rho_o^2 - \rho_e^2) \tilde{z} \tilde{z} + \mu_c (\rho_o^2 + \rho_e^2) \\
&\quad + \frac{1}{2} \rho_o^2 \sum_{\nu = u, d, s} J_\nu \rho_{o\nu} (\tilde{z}_z \rho_{o\nu})(\tilde{z}_o \rho_{\nu\nu}) - \sum_{\nu = u, d, s} t_{\nu} \rho_o \rho_{o\nu} [\bar{\psi}_o \psi_o U_{o\nu}(\tilde{z}_o \tilde{z}_e) \\
&\quad + \bar{U}_{o\nu}(\tilde{z}_e \tilde{z}_o) \bar{\psi}_o \psi_o] \right],
\end{align}
where the suffix $\nu(u, d, s)$ denotes the three NN directions from an odd site $o$, and we omit the suffix $e$ as $z = z_e$. $\mu_c$ is the chemical potential to enforce the hole concentration to be $\langle \bar{\psi}_i \psi_i \rangle = \delta$. $J_\nu, t_{\nu}$ denote
\begin{equation}
  J_\nu = \begin{cases} 
  J, \quad \nu = u, d \\
  J', \quad \nu = s
\end{cases}
\end{equation}
\begin{equation}
  t_{\nu} = \begin{cases} 
  t, \quad \nu = u, d \\
  t', \quad \nu = s.
\end{cases}
\end{equation}
\( A_p \) is given by
\[
A_p = \sum_{o \in \text{odd}} (-\bar{p}_o M_o p_o + \bar{p}_o k_o + \bar{l}_o p_o),
\]
(3.10)

\[
M_o = \frac{1}{2} \rho_o^2 \sigma^2 - \frac{1}{2} \sigma^2 \rho_o^2 + 2 \rho_o^2 \bar{z}\bar{z} \\
+ \frac{1}{2} \sum_\nu J_\nu \rho_o^2 \rho_\nu \left[ (\bar{z}_o \bar{z}) (\bar{z} z_o) - (\bar{z}_o \bar{z})(\bar{z} z_o) \right] \\
- \frac{1}{2} \sum_\nu t_\nu \rho_o \rho_\nu \left[ \bar{\psi}_o \psi_o U_o \bar{z}_o \bar{z} + \bar{z}_o \bar{z} \psi_o \psi_o \right],
\]
(3.11)

\[
k_o = -\rho_o^2 U_o \bar{z}\bar{z} - \sum_\nu \left[ \frac{J_\nu}{2} \rho_o^2 \rho_\nu (\bar{z}_o \bar{z})(\bar{z} z_o) U_o \right] \\
+ t_\nu \rho_o \rho_\nu \bar{\psi}_o \psi_o \bar{z}_o \bar{z},
\]
(3.12)

\[
l_o = -\rho_o^2 U_o \bar{z}\bar{z} - \sum_\nu \left[ \frac{J_\nu}{2} \rho_o^2 \rho_\nu (\bar{z}_o \bar{z})(\bar{z} z_o) U_o \right] \\
+ t_\nu \rho_o \rho_\nu \bar{\psi}_o \psi_o \bar{z}_o \bar{z}.
\]

We do Gaussian integration over \( p_o \)'s;
\[
\int [dp] \exp \left[ \int d\tau A_p \right] = \prod_{o \in \text{odd}} (\det M_o)^{-1} \cdot \exp \left[ \int d\tau A_1(\tau) \right].
\]
(3.13)

At low \( T, T < 2J + J' \), this reduces to [10]
\[
\int d\tau A_1(\tau) = \int d\tau d\tau' \sum_o \bar{l}_o(\tau) M_o^{-1}(\tau, \tau') k_o(\tau') \\
\sim \frac{2}{2J + J'} \int d\tau \sum_o \bar{l}_o(\tau) k_o(\tau).
\]
(3.14)

From (3.7) and (3.14), the relevant terms of \( z \) and \( \psi \) are readily obtained. For the \( \text{CP}^1 \) field \( z \),
\[
A_0^z = -\frac{1}{2} \sum_o J_\nu (\bar{z}_o \bar{z})(\bar{z} z_o),
\]

\[
A_1^z = 2 \sum_o (2J + J')^{-1} \left[ (\bar{z} \bar{z})(\bar{z} \bar{z}) \right] \\
+ \sum_{\nu \nu'} \frac{J_\nu}{2} \frac{J_\nu'}{2} (\bar{z}_o \bar{z})(\bar{z} z_o)(\bar{z}_o \bar{z})(\bar{z} z_o) \\
+ (\bar{z} \bar{z}) \sum_\nu \frac{J_\nu}{2} (\bar{z}_o \bar{z})(\bar{z} z_o) + (\bar{z} \bar{z}) \sum_\nu \frac{J_\nu}{2} (\bar{z}_o \bar{z})(\bar{z} z_o). \]
(3.15)
Similarly, for the hole field $\psi$, we obtain

\[
A_{\psi}^{\text{rel}} = K + T_0 + T_1 + T_2,
\]

\[
K = -\sum_o \left[ \bar{\psi} (D_x + m) \psi + \bar{\eta} (D_x - m) \eta \right], \quad m = \mu_c + 2J + J',
\]

\[
T_0 = -\frac{1}{2} \sum_o \sum_o J_o \bar{\eta}_o \eta_o \bar{\psi}_o \psi_o (\bar{z} z_o)(\bar{z}_o z),
\]

\[
T_1 = \sum_o \sum_{\nu} t_{\nu}(b_{\nu} \bar{\psi}_o \bar{\eta}_o + c_{\nu} \eta_o \psi_o),
\]

\[
b_{\nu} = -(\bar{z} z_o) + \frac{(\bar{z} z_o)}{2J + J'} \left[ 2(\bar{z} \dot{z}) + \sum_{\nu'} J_{\nu'} (\bar{z}_{\nu'} z)(\bar{z} z_o) \right],
\]

\[
c_{\nu} = -(\bar{z} z_o \bar{z}) + \frac{(\bar{z} z_o \bar{z})}{2J + J'} \left[ 2(\bar{z} \dot{z}) + \sum_{\nu'} J_{\nu'} (\bar{z}_{\nu'} z)(\bar{z} z_o \bar{z}) \right],
\]

\[
T_2 = \frac{2}{2J + J'} \sum_o \sum_{\nu \nu'} t_{\nu} t_{\nu'} \eta_o \bar{\psi}_o \psi_o \bar{\eta}_o (\bar{z} z_o) (\bar{z} z_o),
\]

(3.16)

where

\[
D_x = \partial_x + iA_x = \partial_x - (\bar{z} \dot{z}).
\]

Above, we have defined

\[
\eta_o = U_{oe} \bar{\psi}_o,
\]

(3.17)

which transforms $\eta_o \to e^{i\phi_c} \eta_o$ under (2.7). The action of the effective lattice model is thus given as $A_{\text{eff}} = A_0^z + A_1^z + A_{\psi}^{\text{rel}}$.

$T_0$ and $T_2$ in the effective model show that there appear effective interactions between holes as a result of the SRAFO. One can expect that a superconducting phase appears when weak-inter-ladder interactions are included as recently observed by experiments [2]. The order parameter for superconductivity is the following hole-pair field,

\[
M_{o\nu} = \bar{\psi}_o \bar{\psi}_{o\nu} \eta_o,
\]

(3.19)

which has the electric charge $+2e$ and invariant under (2.7).
4 Spin dynamics in chain and ladder

In this section, we study the spin part of $A_{\text{eff}}$ by taking the continuum limit. An essential difference appears between the ladder and chain.

We shall first consider the case $J' \leq J$ and focus on the imaginary term in the effective model, i.e., the last two terms in (3.13).

\[
I^z = \frac{1}{2J + J'} \sum_{o\nu} J_{\nu} \left[ (\bar{z}\dot{z})(\bar{z}
abla_{o\nu} z)(\dot{\bar{z}} z_{o\nu}) + (\bar{z}\dot{\bar{z}})(\bar{z} z_{o\nu})\left(\bar{z}_{o\nu}\dot{z}\right) + \ldots \right] 
\]

(4.1)

We introduce (continuous) coordinate $x = ai$ ($a$ is the lattice spacing). For smooth configurations of $z$, we obtain

\[
\int d\tau I^z \simeq \frac{1}{2J + J'} a \sum_{o} (2J + J') \int d\tau \left( D_{x\bar{z}}D_{\tau\bar{z}} - D_{\tau\bar{z}}D_{x\bar{z}} \right)
\]

\[
\simeq \int dx d\tau \left( D_{x\bar{z}}D_{\tau\bar{z}} - D_{\tau\bar{z}}D_{x\bar{z}} \right)
\]

\[
\equiv 2\pi iQ,
\]

(4.2)

where $D_{\mu} = \partial_{\mu} + iA_{\mu} = \partial_{\mu} - (\bar{z}\partial_{\mu}z)$, and $Q$ is the topological charge which takes integer values for smooth configurations of $z$ since it is a wrapping number from $S_2$ of the $x - \tau$ space to $O(3)$ of the $\vec{S}$ space. On the other hand, for a single quantum spin chain,

\[
\int d\tau I^z_{\text{chain}} = \pi iQ.
\]

(4.3)

The reason why the coefficient of the topological term doubles in the ladder case is simply because there are twice as many degrees of freedom per unit length in the ladder case.

Before going into the detailed study on the effect of the topological term, let us obtain the continuum limit of the remaining terms in $A^z_0$ and $A^z_i$ in (3.13). By straightforward calculation, we obtain

\[
A^z_0 \simeq -\frac{1}{2} \sum_{o} \left( J(2a)^2 D_{x\bar{z}}D_{\tau\bar{z}} + J'(\sqrt{2a})^2 D_{x\bar{z}}D_{\tau\bar{z}} \right)
\]
\[\begin{align*}
\simeq & \ - (2J + J') a \int dx D_x \bar{z} D_x z, \\
A_z^i & \simeq \frac{2}{2J + J'} \int dx \left[ - a^{-1} D_\tau \bar{z} D_\tau z + (J + J'/\sqrt{2})^2 a D_x \bar{z} D_x z \right] + I^z. \quad (4.4)
\end{align*}\]

Therefore, the continuum limit of the spin part of \(A_z \equiv A_0^i + A_1^i\) is the relativistic \(\text{CP}^1\) model with the topological \(\theta\)-term.

\[\begin{align*}
A_{CP} & = \frac{1}{g^2} \int dx \sum_{\mu=\tau,x} D_\mu \bar{z} D_\mu z + I^z; \\
g^2 & = \frac{1 + J'/2J}{\sqrt{1 + (2 - \sqrt{2})J'/J}}. \quad (4.5)
\end{align*}\]

where we have rescaled the imaginary time as \(\tau \rightarrow v_z \tau\) and \(v_z = [J(J + (2 - \sqrt{2})J')/a]^{1/2}\) is the “speed of light” of the present system. Recently, the \(O(3)\) nonlinear-\(\sigma\) model with \(O(3)\) variables \(\vec{n}(x) \cdot \vec{n}(x) = 1\) was derived as an effective-low energy model of the AF Heisenberg models on a ladder\(^{12}\). This field theory model has essentially the same structure as (4.5). However, in order to discuss the hole-doped case, the use of \(\text{CP}^1\) variables is indispensable.

The nonlinear-\(\sigma\) model with the \(\theta\)-term has been studied both analytically and numerically as an effective field theory of the AF Heisenberg models\(^{13}\). Properties of the ground state and excitations strongly depend on the value of \(\theta\), i.e., the coefficient of the topological term, \(\int d\tau I^z(\theta) = i\theta Q\). It is expected that the model exhibits a phase transition at \(\theta = (2m + 1)\pi\) with \(m = \text{integer}\) as \(\theta\) varies. Here we explain this transition from the view point of gauge theory. It is well known\(^{13}\) that a composite gauge field \(A_\mu\) appears in the \(\text{CP}^{N-1}\) model, \(A_\mu = i(\bar{z} \partial_\mu z)\), which transforms as \(A_\mu \rightarrow A_\mu - \partial_\mu \phi\) under \(z \rightarrow e^{i\phi} z\). For the case of \(\theta = 0 \pmod{2\pi}\), the \(z\)-boson becomes massive due to the \(\text{CP}^1\) constraint\(^{14}\) and, after the \(z\)-integration, the composite gauge boson acquires the Maxwell term and so becomes dynamical. In the \((1+1)\) dimensions, the Maxwell gauge theory has only one phase, i.e., the confinement phase with a linear confining potential \(V(r) \propto r\) between a pair of charged particles. Therefore possible excitations in the \(\text{CP}^1\) model are the gauge-invariant triplet boson \(\vec{n} = (\bar{z} \vec{\sigma} z)\). These results are obtained for the \(\text{CP}^{N-1}\) model by the \(1/N\)-expansion.
[15], but expected to be correct for the $N = 2$ case also.

For $\theta = \pi \, (\text{mod} \, 2\pi)$, it is expected that the behavior of the dynamical gauge field is quite different from the $\theta = 0$ case. In the present CP$^1$ model, the $\theta$-term is rewritten in terms of the gauge field (in the real-time formalism) as

$$
\int dt I^z(\theta) = i \frac{\theta}{2\pi} \int dx dt E,
$$

(4.6)

where $E$ is the electric field.

Coleman [16] gave the following semi-classical argument on the gauge dynamics of QED$_2$ with the above $\theta$-term. As $E = \partial_x A_0$, it is obvious that the effect of the $\theta$-term is interpreted as putting $\pm \theta/(2\pi)$ charges at the spatial infinities $x = \pm \infty$. In the case of $\theta = 0$, an electric flux appears through the Gauss’ law between a pair of oppositely charged sources (say, an electron and a positron), giving rise to a linear potential which confines an electron and a positron in one-spatial dimension. When a pair of $\pm \theta/2\pi$ charges are put in the spatial infinities, this confinement picture is not changed till the magnitude of charges increases up to $\theta/2\pi = 1/2$. In the case $\theta/2\pi = 1/2$, the “ground state” supports an electric flux of magnitude 1/2 lying along the entire space. As a pair of electron and positron are put into the system at $x_1, x_2$, there appear step-function-like jumps in the electric flux at $x_1, x_2$. The electric field is $E(x) = 1/2$ for $x < x_1$ and $x_2 < x$ and $E(x) = 1/2 - 1 = -1/2$, but its magnitude is still 1/2 for all points[17]. Therefore, the energy, proportional to $E(x)^2$, does not change as $x_1, x_2$ are varied, so there is no confining force between charges. Thus, at $\theta/2\pi = 1/2$, the system exhibits a phase transition from the confinement phase to the deconfinement phase. Similar behavior is expected also for the CP$^{N-1}$ model with the $\theta$-term. The unsolved problem is whether the phase transition at $\theta = \pi \, (\text{mod} \, 2\pi)$ is of first order or of second order. It may depend on the magnitude of the coupling constant[13]. However, it is known [18] that the $S = 1/2$ AF Heisenberg chain has gapless modes and spin-spin correlation functions have a power-law decay. These low-energy properties are described by the $k = 1$ Wess-Zumino-Witten model. Therefore it is correct that the CP$^1$ model with a $\theta$-term that corresponds to the
$S = 1/2$ AF Heisenberg chain must have a second-order phase transition at $\theta = \pi$.

The above discussion on the composite gauge boson reveals the essential difference between the spin chain and the spin ladder. As we showed, one has $\theta = \pi$ for the chain, which leads to the deconfinement phase of composite gauge boson, whereas one has $\theta = 2\pi$ for the two-leg ladder, which leads to the confinement phase. We expect that also for the hole-doped case the $\theta$-term is ineffective in the ladder t-J model and one may ignore its existence. Detailed study of the dynamics of the composite gauge boson and quasi-excitations in the ladder t-J model with doped holes will be given in the following section.

We have considered the case $J \geq J'$ so far. Similar results are obtained for the case $J' \geq J$. The continuum limit of the spin part is again the CP$^1$ model with $\theta = 0$ in (4.3), but the effective coupling constant $g^2$ is given as

$$g^2 = \sqrt{\frac{1}{2} + \frac{J'}{4J}},$$

and $v_z = \sqrt{2J(J + \frac{J'}{2})a}$. It is interesting to notice that for $\theta = 0 \pmod{2\pi}$ the CP$^{N-1}$ model is asymptotically free and $g^2$ becomes large at low energies. This fact and Eqs.(4.3) and (4.7) suggest $J'/J \to +\infty$ for the low-energy limit, as it is expected from the appearance of the energy gap for any finite value of $J' (\neq 0)$ in the ladder model. (The two points $J' = 0$ and $J' = \infty$ may be fixed points of renormalization group.)

The effect of doped holes on the dynamics of spins is examined by integrating out the hole field in the effective model. The relevant terms come from $K + T_1$ in (3.16),

$$\exp \left[ \int d\tau \Delta A^2 \right] = \int [D\psi][D\eta] \exp \left[ \int d\tau (K + T_1) \right].$$

(4.8)

To this end, the hole-hopping expansion is quite useful and reliable especially at sufficiently high $T$ [11]. In the hopping expansion, the bare propagator of holes is obtained from the $K$-term (3.16) as follows;

$$G_\psi(\tau_1 - \tau_2) = \langle \psi_{i,a}(\tau_1) \bar{\psi}_{i,a}(\tau_2) \rangle$$
\[
G_\eta(\tau_1 - \tau_2) = \langle \eta_{i,a}(\tau_1) \bar{\eta}_{i,a}(\tau_2) \rangle \\
= -G^*_{\bar{\psi}}(\tau_2 - \tau_1), \tag{4.9}
\]

where we employ the temporal gauge \( A_0 = 0 \) for simplicity. The hole concentration is expressed as

\[
\langle \bar{\psi}_i \psi_i \rangle \equiv \delta = \frac{e^{-\beta m}}{1 + e^{-\beta m}}. \tag{4.10}
\]

From (4.8) and (4.9), we obtain \( \Delta A^z \) for high \( T \) as follows;

\[
\int d\tau \Delta A^z = \int d\tau d\tau' \sum_{o,\nu} t^2_o b_{o\nu}(\tau) c_{o\nu}(\tau') \cdot \langle \bar{\psi}_o \eta_o(\tau) \bar{\eta}_o \psi_o(\tau') \rangle \\
\sim \delta (1 - \delta) \beta \int d\tau \sum_{o\nu} t^2_o b_{o\nu}(\tau) c_{o\nu}(\tau). \tag{4.11}
\]

It is straightforward to obtain the continuum limit of \( \Delta A^z \) (4.11) as

\[
\Delta A^z = \int dx \left[ C_{r} a^{-1} D_{r} D_{r} z + C_{x} a D_{x} z D_{x} z \right], \tag{4.12}
\]

where

\[
C_{r} = \frac{4(2t^2 + t^2 \beta)}{(2J + J')^2}, \quad C_{x} = 2t^2 \beta. \tag{4.13}
\]

Thus the effect of hole hoppings is incorporated in the form of renormalization of the relativistic CP\(^1\) model but the \( \theta \)-term is not generated. This renormalization increases the effective coupling \( g^2 \), hence increases the spin gap. This is expected since hoppings of holes should reduce the spin-spin correlations. However, as \( T \) is lowered, \( \Delta A^z \) in (4.12) becomes to overestimate the hole-hopping effects, not only because one needs to include higher-order effect of hopping expansion, but because of correlations among holes. As we explained before, there exists attractive force between holes sitting on the NN sites, as \( T_0 \) term in (3.16) indicates. This effective interaction between holes generates correlations among NN holes to hinder the single-hole hoppings. In some region, it may give rise to hole pairings and superconductivity as discussed in Sect.6.
5 Dynamics of composite gauge bosons and quasi-excitations in the confinement phase

In this section we shall study the gauge dynamics of the t-J model on a ladder. In Ref.[7] we studied the CSS of the t-J model on a 2D lattice both in the slave-boson and SF representations, and calculated the transition temperature $T_{CSS}(\delta)$ of the confinement-deconfinement (CD) transition. It was shown that the CSS takes place at low $T$'s below $T_{CSS}$, being compatible with experiments. The method presented in Sect.4 of Ref.[7] can be applied in a straightforward manner to the study of the CSS in the present ladder system. Therefore we present below the main steps and results. The reader who wants to know more details should refer Ref.[7].

The effective lattice model of Sect.4 treats even and odd sites asymmetrically by integrating out the odd-site spins. To make the calculations and presentation below simpler and more transparent, we introduce another effective lattice model, which treats even and odd sites in a symmetric manner. The new symmetric model is very naturally obtained from the asymmetric model by adding odd-site spin variables so that it recovers the even-odd lattice symmetry. Both the previous asymmetric model and the present symmetric model have the same naive continuum limit in the spin part, i.e., the $\mathbb{C}P^1$ non-linear sigma model. Since the relation between the asymmetric model and the symmetric model are so intimate, these two models must fall into the same universality class; in particular the qualitative result on the CSS derived below should apply also to the asymmetric model.

To avoid confusion with the effective model in Sect.3 and to make the expressions more transparent, we change some notations; For example we use $x$ (and $y$) to denote sites of the ladder. The partition function of the symmetric model has the following path-integral representation:

\[
Z = \int [dz][d\zeta] \exp \left( \int_0^\beta d\tau A \right),
\]

\[
A = A_z + A_\zeta,
\]
\[
A_z = -\frac{1}{4J} \sum_x |D_x z_x|^2 + \sum_{(xy)} \frac{J_{xy}}{2} |\bar{z}_y z_x|^2,
\]
\[
A_\zeta = -\sum_x \bar{\zeta}_x (D_\tau + m_x) \zeta_x
+ \sum_{(xy)} t_{xy} (b_{xy} \bar{\zeta}_y \bar{\zeta}_x + c_{xy} \zeta_x \zeta_y),
\]
(5.1)

where \(\sum_{(xy)}\) denotes summation over NN pairs \((xy)\) on the ladder \([20]\). We write \(\tilde{J} \equiv (2J + J')/4\), and \(J_{xy} (t_{xy})\) implies \(J(t)\) for vertical pairs and \(J'(t')\) for horizontal pairs. \(\zeta_x\) (Grassmann number) denotes
\[
\zeta_x = \begin{cases} 
\psi_x & \text{hole at even site}, \\
\eta_x & \text{anti-hole at odd site}. 
\end{cases} \quad (5.2)
\]

Due to this rewriting, the fermion mass becomes staggered, i.e., \(m_x = m\) for even sites and \(m_x = -m\) for odd sites.

The spin part has been simplified to the CP\(^1\) lattice model by keeping in \(A_1^z\) only the first term. The hole part is simplified by (i) ignoring \(T_0\) and \(T_2\) focusing on the region out of the superconducting state, and (2) modifying \(b_\nu\) and \(c_\nu\) as
\[
b_{xy} = -\bar{\zeta}_x z_y + \frac{1}{2J} (\bar{z}_y \tilde{\zeta}_x),
\]
\[
c_{xy} = -\bar{\zeta}_x z_y + \frac{1}{2J} (\tilde{\zeta}_x \bar{z}_y). \quad (5.3)
\]

The action is invariant under the local \(U(1)\) gauge transformation
\[
z_{x\sigma} \rightarrow \exp(i\theta_x) z_{x\sigma},
\]
\[
\zeta_x \rightarrow \exp(i\theta_x) \zeta_x, \quad (5.4)
\]

with time-independent rotation angles \(\theta_x\).

Next, we introduce the following four composite gauge variables on the link \((x, y)\);
\[
B_{xy} \equiv (\bar{\zeta}_x z_y) \rightarrow e^{i\theta_x} B_{xy} e^{i\theta_y},
\]
\[
D_{xy} \equiv (\bar{\zeta}_x z_y) \rightarrow e^{-i\theta_x} D_{xy} e^{i\theta_y},
\]
\[
F_{xy} \equiv \zeta_x \zeta_y \rightarrow e^{i\theta_x} F_{xy} e^{i\theta_y},
\]
\[
G_{xy} \equiv \bar{\zeta}_x \zeta_y \rightarrow e^{-i\theta_x} G_{xy} e^{i\theta_y} \quad (5.5)
\]
where their transformation laws are also indicated. We enforced these relations strictly via delta functions like
\[ \delta(B_{xy} - \bar{\tilde{z}}_{x}z_{y}), \] etc., which amounts to insert into \( Z \) the following identity:

\[ 1 = \prod_{(xy)\tau} \int [dP][dQ][dR][dS] \int [dB][dD][dF][dG] \exp \int d\tau \sum_{(xy)} i[P(B - (\bar{\tilde{z}}_{x}z_{y})) + Q(D - (\bar{\tilde{z}}_{x}z_{y})) + R(F - \bar{\tilde{c}}_{x}\bar{\tilde{c}}_{y}) + S(G - \bar{\tilde{c}}_{x}\bar{\tilde{c}}_{y}) + h.c.]_{(xy)}. \quad (5.6) \]

Then by taking the temporal gauge, we have

\[ Z = \prod_{(xy)\tau} \int [dP][dQ][dR][dS] \int [dB][dD][dF][dG] \times I_{\xi}(R, S) I_{\zeta}(P, Q, F) \exp \int d\tau \sum_{(xy)} \tilde{A}_{xy}, \]

\[ I_{\xi}(R, S) = \int [d\xi] \exp \int d\tau \left[ -\sum_{x} \tilde{c}_{x}(D_{\tau} + m_{x})\zeta_{x} - i \sum_{(xy)} (R_{xy}\zeta_{x}\zeta_{y} + S_{xy}\tilde{c}_{x}\zeta_{y} + h.c.) \right], \]

\[ I_{\zeta}(P, Q, F) = \int [dz] \exp \int d\tau \left[ -\frac{1}{4J} \sum_{x} |\tilde{z}_{x}|^{2} + \frac{1}{2J} \sum_{(xy)} (t_{xy}F_{xy}\bar{\tilde{z}}_{y}\tilde{z}_{x} - h.c.) - i \sum_{(xy)} (-P_{xy}\bar{\tilde{z}}_{x}z_{y} + Q_{xy}\bar{\tilde{z}}_{x}z_{y} + h.c.) \right], \]

\[ \tilde{A}_{xy} = \frac{J_{xy}}{2} |D_{xy}|^{2} + t_{xy}(FB + h.c.)_{xy} + i(PB + QD + R\bar{\tilde{c}}_{x} + S\bar{\tilde{c}}_{x} + h.c.)_{xy} \quad (5.7) \]

Next, we integrate over \( \zeta_{x} \) and \( z_{x} \) by the hopping expansion; an expansion in powers of gauge fields. To do this we need the following on-site Green functions:

\[ \langle \zeta_{x}(\tau_{1})\bar{\tilde{c}}_{y}(\tau_{2}) \rangle = \delta_{xy}G_{x}(\tau_{1} - \tau_{2}), \]

\[ G_{x}(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \exp(i\omega_{n}\tau) = \theta[m_{x}] \sum_{L} (-1)^{L} \exp[-m_{x}(\tau + \beta L)] \theta[m_{x}(\tau + \beta L)] \]

\[ = \frac{\exp(-m_{x}\tau)}{1 + \exp(-m_{x}\beta)}[\theta(\tau) - \theta(-\tau) \exp(-m_{x}\beta)], \quad (5.8) \]
for $|\tau| < \beta$, and

$$
\langle \hat{z}_{x\sigma} (\tau_1) \hat{z}_{y\sigma'} (\tau_2) \rangle = \delta_{xy} \delta_{\sigma\sigma'} G(\tau_1 - \tau_2),
$$

$$
G(\tau) = \frac{4J}{\beta} \sum_{n=-\infty}^{\infty} \frac{\exp(i\omega_n \tau)}{\omega_n^2 + \sigma} (\omega_n \equiv 2\pi n/\beta)
$$

$$
= \frac{2J}{\sqrt{\sigma}} \sum_K \exp\left[-\sqrt{\sigma}(\tau + \beta K)\right]
$$

$$
= \begin{cases} 
\frac{4J}{\sigma} \delta(\tau) & \text{for } \sqrt{\sigma} \beta >> 1 \\
\frac{4J}{\beta \sigma} & \text{for } \sqrt{\sigma} \beta << 1
\end{cases}
$$

(5.9)

Value of $\sigma$ can be estimated in the large-$N$ approximation, etc. [6], but it is not necessary to derive the main conclusion below as long as it is finite.

Up to the second order in the gauge variables, we obtain

$$
Z = \int[dB][dD][dF][dG][dP][dQ][dR][dS] \exp \int d\tau \sum_{xy} [A_{\xi z} +
$$

$$
+ \frac{J_{xy}}{2} |D|^2 + t_{xy} (FB + h.c.)
$$

$$
+ i(PB + QD + RF + SG + H.c.)]_{xy},
$$

$$
A_{\xi z} = -c_1 \beta |R_0|^2 - c_2 |S|^2 + c_3 |\dot{S}|^2 - c_4 |F|^2 - c_5 |\dot{F}|^2 - c_6 |P|^2
$$

$$
+ c_7 |\dot{P}|^2 - c_8 |Q|^2 + c_9 |\dot{Q}|^2 - ic_{10}(\dot{F}\dot{P} - h.c.)
$$

(5.10)

where the coefficients $c_{1,2,...,10}$ are given by

$$
c_1 = g \exp(-\beta|m|), \quad g \equiv (1 + \exp(-\beta|m|))^{-2},
$$

$$
c_2 = \frac{g}{2|m|}, \quad c_3 = \frac{g}{8|m|^3}, \quad c_4 = \frac{2t_{xy}}{\sigma^{1/2}}, \quad c_5 = \frac{t_{xy}^2}{2\sigma^{3/2}},
$$

$$
c_6 = c_8 = \frac{8J^2}{\sigma^{3/2}}, \quad c_7 = c_9 = \frac{2J^2}{\sigma^{5/2}}, \quad c_{10} = \frac{t_{xy} J}{\sigma^{3/2}}
$$

(5.11)

$R_0$ is the $n = 0$ mode of the Fourier coefficients, $R_n \equiv \beta^{-1} \int_0^\beta d\tau R(\tau) \exp(-i\omega_n \tau)$.

The integrations over $P, Q, S$ are Gaussian, and the $R(\tau)$-integration is done in Fourier components $R_n$;

$$
\int \prod_n dR_n \exp[-c_1 \beta^2 |R_0|^2 + i\beta \sum_n (R_n F_{-n} + h.c.)]
$$

$$
\propto \exp\left(-\frac{1}{c_1} |F_0|^2\right) \prod_{n \neq 0} \delta(F_n)
$$

(5.12)
We further integrate over the pair fields $F$ and $G$ of fermions. To discuss the phase dynamics of residual gauge variables $B$ and $D$, we set their amplitudes to be the constants $\rho_{B,q}(\equiv |B_{xy}|)$ and $\rho_{D,q}(\equiv |D_{xy}|)$, which can be determined by a straightforward mean field theory ignoring phase fluctuations of mean fields. The suffix $q = (u,s)$ distinguishes whether the link $(x,y)$ is vertical, $(x(i,a), y(i+1,a))$ or horizontal $(x(i,1), y(i,2))$ \cite{20}. Thus we write

\begin{align*}
B_{xy} &= \rho_{B,q} U_{xy}, \quad U_{xy} \in U(1), \\
D_{xy} &= \rho_{D,q} V_{xy}, \quad V_{xy} \in U(1). \quad (5.13)
\end{align*}

Due to the completeness condition, $z_{\sigma} \bar{z}_{\sigma'} + \bar{z}_{\sigma} z_{\sigma'} = \delta_{\sigma\sigma'}$, we have the relation,

\begin{equation}
\rho_{B,q}^2 + \rho_{D,q}^2 = 1. \quad (5.14)
\end{equation}

After some calculations, we reach the following effective lattice gauge theory;

\begin{align*}
Z &= \int [dU][dV] \exp(\int d\tau \sum_{(xy)} A_{UV}) , \\
A_{UV} &= -\hat{c}_{D,q}|\dot{U}_{xy}|^2 - \hat{c}_{B,q}|\dot{V}_{xy}|^2 \\
\hat{c}_{D,q} &= \left(\frac{c_0}{c_8}\right)^2 \rho_{D,q}^2, \quad \hat{c}_{B,q} = \left[\frac{c_7}{c_6} + \frac{\beta^3}{(2\pi)^2} \frac{c_1 t_{xy}}{1 + \beta c_1 c_4} \right] \rho_{B,q}^2. \quad (5.15)
\end{align*}

Following the Polyakov-Susskind method \cite{21} for studying CD transitions in lattice gauge theories, this system can be mapped to the anisotropic XY model:\cite{7}.

\begin{align*}
Z_{XY} &= \int \prod_x \frac{d\alpha_x}{2\pi} \exp[\sum_{(xy)} J_{1,q} \cos(\alpha_y - \alpha_x) + \sum_{(xy)} J_{2,q} \cos(\alpha_y + \alpha_x)], \\
J_{1,q} &\equiv \beta^{-1} \hat{c}_{D,q}, \quad J_{2,q} \equiv \beta^{-1} \hat{c}_{B,q}. \quad (5.16)
\end{align*}

The $J_2$ term, coming from $|\dot{V}|^2$ term, expresses an anisotropy of XY spin couplings, and reduces the global spin symmetry from $U(1)$ (for $J_2 = 0$) down to $Z(2)$ (for $J_2 \neq 0$). A possible phase transition of this spin system describes the CD phase transition of the original $t$-$J$ model. For two and higher dimensional lattices, this spin system is known to have an order-disorder transition at $J_1 + J_2 \simeq 1$ \cite{7}, which
leads to existence of a finite transition temperature $T_{CSS}$. For $T > T_{CSS}$, the XY spin correlations decay exponentially, $f(|x|) \equiv \langle \vec{S}_x \vec{S}_0 \rangle \simeq \exp(-a|x|)$, which implies the potential energy among two gauge charges, $W(|x|) = -\ln f(|x|)$ (see Ref. [21] for this relation), to be a linear-rising confining potential, $W(|x|) \simeq a|x|$. So the gauge dynamics realizes here in the confining phase. For $T < T_{CSS}$, there is a long-range order, $f(|x|) = \text{const} + \exp(-c|x|)$, which implies a short-range potential $W(|x|) \simeq \exp(-c|x|)$. So the gauge dynamics realizes in the deconfining phase here [22].

In the present ladder system, the XY model (5.16) is put on a coupled two one-dimensional chains and its universality class is classified into that of a single one-dimensional chain, the partition function of which is given by

$$Z_{\text{chain}}^{\text{XY}} = \int \prod_i \frac{d\alpha_i}{2\pi} \exp[J_1 \sum_i \cos(\alpha_{i+1} - \alpha_i) + J_2 \sum_i \cos(\alpha_{i+1} + \alpha_i)].$$

One can solve its spin correlations exactly, which decay exponentially, $f(|x|) = A \exp(-a|x|)$. The coefficient $a$ is finite as long as $J_{1,2}$ are finite. Thus, we conclude that the gauge dynamics of the t-J ladder is always realized in the confining phase except for $T = 0$ at which $J_{1,2}$ diverge. This is the main conclusion of this section. The case of $T = 0$ is discussed later on.

Here we comment that the above conclusion is based on the hopping expansion. If $\sigma$, the mass of $z$, vanishes, the hopping expansion cannot be applicable. Our assumption $\sigma > 0$ is supported by the large-$N$ analysis of the CP$^{N-1}\sigma$ model in one spatial dimension [3]. This is in strong contrast with the case of a single chain. There, the topological term appears and it generates massless excitations, preventing us from making use of the hopping expansion and obtaining an effective lattice gauge theory. However, as discussed in Sect.4, we know that the spin chain system is realized in the deconfinement phase. We expect that the doped chain still supports this deconfinement phase, although a solid analysis is required. (The special case of supersymmetry $J/t = 2$ can be solved exactly by the Bethe ansatz and its result supports the CSS.)
How about the gauge dynamics on a ladder at $T = 0$? Since the system (5.17) develops a long-range order, one may conclude that the gauge system is in the deconfinement phase. However, the expression (5.17) itself is not adequate at $T = 0$. This can be seen as follows. At $T = 0$, $m = 0$ as one can see from (4.10). So the hopping expansion with respect to fermions is not applicable. In this case, we have a separate argument using the massless Schwinger model [23], the massless QED$_2$ without topological terms, which is obviously a relevant model for present argument. Here it is known [23] that the physical spectrum consists of a massive boson, a composite of fermion and antifermion. Since this is a gauge-invariant neutral object, its gauge dynamics is in the confinement phase. From this fact, we conclude that the t-J model on a ladder realizes its gauge dynamics in the confinement phase not only for $T > 0$ but also for $T = 0$.

How is the effect of weak but finite couplings among NN ladders in a plane and couplings among interlayers? They put the system into three dimensional one. The calculation given above can be redone to reach the same expression (5.16) for the XY spin model, but the sum over $(xy)$ is extended to the entire lattice. One concludes that there is a finite critical temperature $T_{CSS}$ below which the deconfinement phase appears. The value $T_{CSS}$ is small and approaches zero as these three-dimensional $J$ and $t$ couplings tend to vanish. One can determine $T_{CSS}$ using the 3D XY model and the MF values of $\rho_{B,q}$, etc.

What are the quasi-excitations in the confinement phase? In this phase, due to the confining force, only charge-neutral objects can be physical excitations. The typical examples are the original electrons $\bar{z}_x \psi_x$ which carry the electric charge $-e$ and spin $S = 1/2$, the charge density fluctuation $\bar{\psi}_x \psi_x$, and the spin triplet excitation $\bar{z}_x \vec{\sigma} z_x$, i.e., charge-neutral magnons of $S = 1$. Together with these local combinations, one may conceive nonlocal gauge-invariant combinations like $\bar{z}_x U_{xy} \psi_y$, etc. Real low-energy excitations are linear combinations of these states with definite electric charge and spin.
Finally, we note that the analysis in this section can be repeated when one includes the \(T_0\) and \(T_2\) terms into the symmetric model. These terms are necessary to generate the superconducting phase as we shall see in the following section. The corresponding calculations of \(T_{CSS}\) for the 2D t-J model are found in Ref.[7]. After tracing these calculations, one reaches the same effective XY spin model of (5.16) with the modified coefficients \(J_{1,2}\). Thus we obtain the conclusion that the confinement phase obtained above continues to exist even when the system is superconducting.

6 Superconductivity

As we saw in Sect.4, there are effective interactions between holes which appear as a result of SRAFO. In terms of the composite hole-pair field \(M_{\sigma\nu}\), \(T_0\) and \(T_2\) terms in (3.16) are rewritten as

\[
T_0 = \frac{1}{2} \sum_{\sigma,\nu} J_{\nu} \bar{M}_{\sigma\nu} M_{\sigma\nu},
\]

\[
T_2 = -\frac{2}{2J + J'} \sum_{\sigma} \sum_{\nu'\nu} t_{\nu'} \bar{M}_{\sigma\nu} M_{\sigma\nu'}.
\]

(6.1)

It is obvious that \(T_0\) induces a condensation of the hole-pairs,

\[
\langle M_{\sigma\nu} \rangle \neq 0,
\]

(6.2)

and \(T_2\) favors the “d-wave” symmetry,

\[
\sum_{\nu'} t_{\nu} \langle M_{\sigma\nu} \rangle = 0.
\]

(6.3)

For \(t' = t\), Eq.(6.3) means \(\langle M_{os} \rangle = -2 \langle M_{ou} \rangle = -2 \langle M_{od} \rangle\) if we assume \(\langle M_{o,u} \rangle = \langle M_{o,d} \rangle\) and \(\langle M_{o,s(u,d)} \rangle\) are independent of the position \(o\) [24].

It is straightforward to introduce a hole-pair field \(\Delta_{\sigma\nu}\) by a Hubbard-Stratonovich transformation. We shall obtain a Ginzburg-Landau (GL) theory of the hole-pair field by integrating out the hole field. To this end we take the temporal gauge \(A_0 = 0\) for calculational simplicity [25]. The \(T_0\) term is rewritten by \(\Delta_{\sigma\nu}\) as

\[
\exp(\int d\tau T_0) = \exp \left( \int d\tau \sum_{\sigma\nu} \frac{J_{\nu}}{2} (M_{\sigma\nu} \Delta_{\sigma\nu} + \bar{M}_{\sigma\nu} \Delta_{\sigma\nu} - \Delta_{\sigma\nu} \Delta_{\sigma\nu}) \right),
\]

(6.4)
up to an irrelevant constant. From this, one gets the relation,
\[ \langle M_{ov} \rangle = \Delta_{ov}. \] (6.5)

The GL potential energy, \( V_{GL}(\Delta) \) of \( \Delta_{ov} \) (that appears in the action integral \( f d\tau A \) in the form of \( -\beta V_{GL} \)), is obtained by the hole-hopping expansion as
\[
V_{GL}(\Delta) = V_0 + V_1 + V_2 + O(\Delta^4),
\]
\[
V_0 = \frac{1}{2} \sum_{ov} J_\nu \bar{\Delta}_{ov} \Delta_{ov},
\]
\[
V_1 = -\frac{1}{\beta} \left( \frac{1}{2} \right)^2 \int d\tau d\tau' \sum_{ov} J_\nu^2 \bar{\Delta}_{ov} \Delta_{ov} \cdot \langle M_{ov}(\tau) \bar{M}_{ov}(\tau') \rangle,
\]
\[
V_2 = \frac{1}{\beta} \left( \frac{1}{2} \right)^2 \frac{1}{J + J'/2} \int d\tau d\tau' d\tau'' \sum_{ov} J_\nu J_\nu' t_\nu t_{\nu'} \bar{\Delta}_{ov} \Delta_{ov} \times \langle \bar{M}_{ov}(\tau) M_{ov'}(\tau') M_{ov}(\tau'') \bar{M}_{ov'}(\tau'') \rangle, \] (6.6)

where \( \Delta_{ov} \) are assumed to be time-independent. The above correlation functions of \( M_{ov} \)'s are calculated by the hole propagator (4.9) and the expectation value of \( z \)-pair,
\[ \langle \bar{z} z_{ov} \rangle = D_\nu. \] (6.7)

The actual value of \( D_\nu \) should be obtained by the MFT [26], but here we only needs that it is nonvanishing due to SRAFO. After some calculation, we obtain
\[
V_1 = \frac{1}{8m} \sum_{ov} J_\nu^2 |D_\nu|^2 \bar{\Delta}_{ov} \Delta_{ov},
\]
\[
V_2 = -\frac{1}{J + J'/2} \frac{1}{16m^2} \sum_o \left( \sum_\nu t_\nu J_\nu D_\nu \Delta_{ov} \right) \left( \sum_{\nu'} t_{\nu'} J_{\nu'} \bar{D}_{\nu'} \bar{\Delta}_{ov'} \right).
\] (6.8)

We are interested in the case of low hole concentrations \( \delta \ll 1 \) where we have the expression,
\[ m \simeq \frac{|\ln \delta|}{\beta}, \] (6.9)
which comes from (4.10). Since \( m \) is large for \( \delta \ll 1 \), we have neglected the terms of \( O(m^{-3}) \) in the above.
Let us find a solution that minimizes $V_{GL}(\Delta)$. It is easily found by minimizing $V_0 + V_1$ and $V_2$ separately. By writing $\Delta_{\nu\nu} = e_\nu \Delta$, we have

\begin{align*}
V_0 + V_1 &= N_{\text{link}} \times C_2 |\Delta|^2 \\
C_2 &= \sum_\nu \left( -\frac{1}{2} J_\nu + \frac{1}{8m} |D_\nu|^2 J_\nu^2 \right) e_\nu^2.
\end{align*}

(6.10)

$V_2$ is minimized by configurations satisfying

\[ \sum_\nu t_\nu J_\nu D_\nu \Delta_\nu = 0. \]

(6.11)

Eq. (6.10) implies that the order parameter $\Delta$ develops for $C_2 < 0$. So the critical temperature $T_c$ is calculated from $C_2 = 0$ as

\[ T_c = \frac{\sum_\nu J_\nu^2 e_\nu^2}{\sum_\nu J_\nu e_\nu^2} \frac{1}{4|\ln \delta|} \sim \frac{D^2 J}{4|\ln \delta|}. \]

(6.12)

The last expression is for the case of $J' = J$, $t' = t$ and $\nu$-independent $e_\nu$ and $D_\nu = D$. In this case, Eq. (6.11) gives rise to the $d$-wave solution [24],

\[ \Delta_{os} = -2\Delta_{ou} = -2\Delta_{od}. \]

(6.13)

The MFT using a Gutzwiller renormalization of matrix elements predicts similar behavior of the superconductivity correlations. If we take into account the fluctuations of the order parameter of the superconductivity $\Delta_{\nu\nu}$, weak but finite three-dimensional inter-ladder interactions are required to realize a genuine superconducting phase. From our discussion given so far, it is obvious that the existence of the SRAFO is essential for appearance of the superconductivity and its $d$-wave type symmetry.

7 Discussion

In this paper, we studied the $t$-$J$ model on the ladder by using the SF-CP$^1$ formalism. We obtained the effective low-energy model by integrating out the half of the CP$^1$ variables. At low $T$'s, there exists the SRAFO which makes it easy to integrate
over the $\text{CP}^1$ variables. The low-energy effective model shows why the low-energy excitations are different in the ladder and the chain systems, reflecting the coefficient of the topological term in the $\text{CP}^1$ model. In the ladder system, the confinement mechanism works at any $T$ and therefore excitations are spin triplet and spinon-holon bound state. On the other hand, in the chain case, the full CSS takes place because of the $\theta$-term of the composite gauge boson. Similar analysis works for the $2\text{D } t\text{-}J$ model, which is studied in the previous papers. We stressed that the gauge-theoretical point of view, especially the CD transition, is quite useful and universally applicable for such problems of separation of degrees of freedom.

For even-number-leg ladder Heisenberg models, it is expected that an energy gap appears for spin excitations for any finite value of $J'$. The continuum field theory model is the $\text{CP}^1$ model which is known to be asymptotically free. Therefore the coupling constant becomes large at low energies. Our derivation shows that this means $J'/J \to +\infty$ for low-energy limit as expected.

The effective model also shows that there are effective attractive interactions between holes in the SRAFO background. By these interactions, the superconducting phase is possible under weak but finite 3D inter-ladder interactions.

In conclusion, we observed that the SF-$\text{CP}^1$ formalism and the gauge-theoretical method provide a natural and coherent way to understand various important properties of the $t\text{-}J$ model on a ladder.
References

[1] For review, see E.Dagotto and T.M.Rice, Science271,618(1996).

[2] M.Uehara et al, J.Phys.Soc.Jpn.(in press).

[3] M.Sigrist, T.M.Rice and F.C.Zhang, Phys.Rev.B49,12058(1994).

[4] P.W.Anderson, Phys.Rev.Lett.64,1839(1990).

[5] L.B.Ioffe and I.Larkin, Phys.Rev.B39,8988(1989).

[6] I.Ichinose and T.Matsui, Nucl.Phys.B394,281(1993).

[7] I.Ichinose and T.Matsui, Phys.Rev.B51,11860(1995).

[8] H.Tsunetsugu, M.Troyer and T.M.Rice, Phys.Rev.B49,16078(1994); ibid. B53, 251(1996).

[9] D.V.Khveshchenko, Phys.Rev.B50,380(1994);
    D.G.Shelton, A.A.Nersesyan and A.M.Tsvelik, Phys.Rev.B53,8521(1996).

[10] I.Ichinose and T.Matsui, Phys.Rev.B45,9976(1992).

[11] H.Yamamoto, G.Tatara, I.Ichinose and T.Matsui, Phys.Rev.B44,7654(1991).

[12] D.Senechal, Phy.Rev.B52,15319(1995);
    G.Sierra, J.Phys.A29,3299(1996);
    S.Dell’Aringa, E.Ercolessi, G.Morandi, P.Pieri and M.Roncaglia,
    Phys.Rev.Lett.78,2457(1997). In these analyses, the idea of integrating out a half
    of the spin degrees of freedom was applied to the SU(2) spins $\vec{S}_i$ themselves.
    However, it can not be extended to the doped case, in contrast with our treatment
    on the CP$^1$ variables.
The constraint can be respected by adding the term $\int dx \lambda(x)(\bar{z}(x)z(x) - 1)$ in the action, where $\lambda(x)$ is the Lagrange multiplier field. For example, in the large-$N$ analysis, $\lambda(x)$ develops a nonvanishing expectation value $\lambda_0$ and fluctuations around it are suppressed by $1/N$. Thus this term reduces to the mass term of $z$ field with the mass square $\lambda_0$.

When $|\theta|/2\pi > 1$, spontaneous generation of electron-positron pair occurs to shield charges at infinities. Therefore the system must be periodic with respect to $\theta \mod 2\pi$.

For definiteness, we assign $y(i, a)$ for given $x(i, a)$ as follows; $y(i + 1, 1)$ or $y(i, 2)$ for $x(i, 1)$ and $y(i + 1, 2)$ for $x(i, 2)$.
[24] The condition (6.3) has appeared in Ref.[10] and been called the divergence-less condition. The pattern of order parameters $\Delta_{ov}$ in the SF t-J model on a 2D lattice has been studied by the gap equation in G.Tatara and T.Matsui Phys.Rev.44,2867(1991) and the flux phase formation of $\Delta_{ov}$ ($\Delta_{ov}/|\Delta_{ov}| = 1,i,-1,-i$ for four links emerging from an odd site $o$) is obtained, which corresponds to the $s+id$ symmetry. The product of $\Delta_{ov}/|\Delta_{ov}|$ on 4 links around a plaquette is $-1$ in the flux phase, while $+1$ in the d-wave symmetry. In the two-leg ladder, the flux phase can be realized, e.g., by a configuration similar to the above 2D case, $\Delta_{os}/|\Delta_{os}| = 1$ for $o(i,1), \Delta_{os}/|\Delta_{os}| = -1$ for $o(j,2), \Delta_{ou}/|\Delta_{ou}| = i, \Delta_{od}/|\Delta_{od}| = -i$. Such a configuration, however, does not minimize the $V_2$ term of $V_{GL}$, reflecting the difference of a ladder and a 2D lattice; there are 3 and 4 NN sites, respectively. Although one needs a separate numerical study to support the present assumption like $\Delta_{ou} = \Delta_{od}$ which rules out the flux phase, the d-wave symmetry is a simple and very plausible possibility for the two-leg-ladder superconductivity.

[25] A full gauge-theoretical study of the composite gauge bosons is of course possible. Details of that will be reported in a future publication.

[26] I.Ichinose, T.Matsui and K.Sakakibara, work in progress.
Fig. 1 Layout of a ladder. Two vertical lines denote two legs, $a = 1$ and $a = 2$. A site on the ladder is labelled by a set of suffices $(i, a)$, where $i$ counts the site along each leg. The filled circles denote odd sites, and the open circles denote even sites. For a fixed odd site $o$, there are three nearest neighbor even sites, $(o, \nu)$ with $\nu = u$ (up), $\nu = d$ (down), $\nu = s$ (side). For example, for $o = (i, 1)$, $(o, u) = (i + 1, 1)$, $(o, d) = (i - 1, 1)$, $(o, s) = (i, 2)$ as illustrated in the Figure. For $o = (j, 2)$, $(o, u) = (j + 1, 2)$, $(o, d) = (j - 1, 2)$, $(o, s) = (j, 1)$. The hopping amplitude and the exchange coupling are $t, J$, respectively, for vertical links $(i, a), (i + 1, a)$, and $t', J'$ for horizontal links $(i, 1), (i, 2)$.