Non-trivial smooth families of $K3$ surfaces

David Baraglia

Received: 14 March 2021 / Revised: 1 November 2022 / Accepted: 2 November 2022 / Published online: 7 November 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
Let $X$ be a complex $K3$ surface, $\text{Diff}(X)$ the group of diffeomorphisms of $X$ and $\text{Diff}_0(X)$ the identity component. We prove that the fundamental group of $\text{Diff}_0(X)$ contains a free abelian group of countably infinite rank as a direct summand. The summand is detected using families Seiberg–Witten invariants. The moduli space of Einstein metrics on $X$ is used as a key ingredient in the proof.

1 Introduction
There is considerable interest in understanding the topology of diffeomorphism groups of 4-manifolds. While much remains unknown there has been some recent progress.

- Ruberman gave examples of simply-connected smooth 4-manifolds for which $\pi_0(\text{Diff}(X)) \to \pi_0(\text{Homeo}(X))$ is not injective [17, 18].
- Watanabe constructed many non-trivial homotopy classes in $\text{Diff}(S^4)$, thereby disproving the 4-dimensional Smale conjecture [21].
- Baraglia–Konno showed that $\pi_1(\text{Diff}(K3)) \to \pi_1(\text{Homeo}(K3))$ is not surjective [4].
- Smirnov showed that if $X$ is a hypersurface in $\mathbb{CP}^3$ of degree $d \neq 1, 4$, then $\pi_1(\text{Diff}(X))$ is non-trivial. Note that this result excludes $K3$, which corresponds to hypersurfaces of degree $d = 4$ [19].

The main result of this paper is the following.

Theorem 1.1 Let $X$ be a $K3$ surface. Then $\pi_1(\text{Diff}_0(X))$ contains a free abelian group of countably infinite rank as a direct summand.

In particular, $\pi_1(\text{Diff}_0(X))$ is not finitely generated. This contrasts with a recent theorem of Bustamante–Krannich–Kupers [6] who showed that if $M$ is a closed smooth
manifold of dimension $2n \geq 6$ with finite fundamental group, then the homotopy
groups of $\text{Diff}(M)$ are finitely generated.

The direct summand in the above theorem is detected using families Seiberg–Witten
invariants. The families Seiberg–Witten invariants were originally defined in [14]. In
this paper we consider a reformulation of the Seiberg–Witten invariants which we now
outline. Let $X$ be a compact, oriented smooth 4-manifold with $b^+(X) > 1$. Let $s$ be a
spin$^c$-structure on $X$ and let

$$d(X, s) = \frac{c_1(s)^2 - \sigma(X)}{4} - 1 + b_1(X) - b^+(X)$$

be the expected dimension of the Seiberg–Witten moduli space, where $\sigma(X)$ is the
signature of $X$. If $d(X, s) \leq -2$, then we construct a map

$$sw_s : \pi_{d(X, s) - 1}(\text{Diff}_0(X)) \to \mathbb{Z}.$$ 

The definition, roughly, is as follows. Let $f \in \pi_{d(X, s) - 1}(\text{Diff}_0(X))$. Then by the
clutching construction, $f$ defines a family $E_f \to S^{d(X, s)}$ over the sphere with fibres
diffeomorphic to $X$. The moduli space of solutions to the Seiberg–Witten equations on
the family $E_f$ with spin$^c$-structure $s$ is compact and has expected dimension $d(X, s) -
d(X, s) = 0$. For a generic perturbation the families moduli space is a compact oriented
0-manifold and $sw_s(f)$ is defined as a signed count of the points of this moduli space. If
$b^+(X) \leq -d(X, s) + 1$, then one has to deal with wall crossing phenomena. However,
we show that in the above situation there is a canonically defined chamber and we take
$sw_s(f)$ to be the Seiberg–Witten invariant defined with respect to this chamber. This
subtlety is crucial to this paper, since we will be concerned with the case $b^+(X) = 3$
and $d(X, s) = -2$.

We prove a number of results concerning the invariants $sw_s$.

**Theorem 1.2** Let $X$ be a compact, oriented, smooth 4-manifold with $b^+(X) > 1$. Then
for each spin$^c$-structure with $d(X, s) = -(n + 1) \leq -2$, the map

$$sw_s : \pi_n(\text{Diff}_0(X)) \to \mathbb{Z}$$

is a group homomorphism.

**Theorem 1.3** Assume that $b_1(X) = 0$. Then for any given $f \in \pi_n(\text{Diff}_0(X))$, $sw_s(f)$
is non-zero for only finitely many spin$^c$-structures with $d(X, s) = -(n + 1)$.

Theorem 1.3 is essentially a consequence of the compactness properties of the
Seiberg–Witten equations. However there is a subtlety due to the chamber structure
and wall crossing that requires some non-trivial arguments to overcome. From these
two theorems it follows that (for each $n \geq 1$) we can put the Seiberg–Witten invariants
together into a single homomorphism

$$sw : \pi_n(\text{Diff}_0(X)) \to \bigoplus_{s \mid d(X, s) = -(n+1)} \mathbb{Z}, \quad x \mapsto \bigoplus_{s \mid d(X, s) = -(n+1)} sw_s(x).$$
Now let $X$ be a $K3$ surface. Let
\[ \Delta = \{ \alpha \in H^2(X; \mathbb{Z}) \mid \alpha^2 = -2 \} \]
be the “roots” of $X$. For each $\alpha \in \Delta$ we get a unique spin$^c$-structure $s_\alpha$ characterised by $c_1(s_\alpha) = 2\alpha$ (since $X$ is simply connected and spin, the map $s \to c_1(s)$ is a bijection between spin$^c$-structures and elements of $H^2(X; \mathbb{Z})$ that are divisible by 2). Then $d(X, s_\alpha) = -2$ and so we have a homomorphism $sw_\alpha : \pi_1(\text{Diff}(X)) \to \mathbb{Z}$.

Choose an element $v \in H^2(X; \mathbb{R})$ such that $\langle v, \delta \rangle \neq 0$ for all $\delta \in \Delta$ and define $\Delta^\pm = \{ \delta \in \Delta \mid \pm \langle v, \delta \rangle > 0 \}$. Then
\[ \Delta = \Delta^+ \cup \Delta^- \]
and $\delta \in \Delta^+$ if and only if $-\delta \in \Delta^-$. The reason for splitting up $\Delta$ this way is that the invariants $sw$ and $sw_-$ are related to one another by the charge conjugation symmetry of the Seiberg–Witten equations. In fact, $sw_\alpha = -sw_{-\alpha}$ (see Proposition 2.8).

In Sect. 3 we recall the construction of the moduli space $Tein$ of Einstein metrics on $X$, which may be regarded as an analogue of Teichmüller space for $K3$ surfaces. Over $Tein$ is a universal family $Ein \to Tein$. For each $\delta \in \Delta^+$, we construct a homotopy class of map $g_\delta : S^2 \to Tein$. Let $E_\delta \to S^2$ be the family over $S^2$ obtained by pulling back the universal family under $g_\delta$. Using the geometry of $Tein$, we compute the families Seiberg–Witten invariant of $E_\delta$. This gives the following result.

**Theorem 1.4** Let $\alpha, \delta \in \Delta^+$. Then
\[ sw_\alpha(h_\delta) = \begin{cases} 1 & \text{if } \alpha = \delta, \\ 0 & \text{otherwise.} \end{cases} \]

Our main theorem follows directly from this. Note that $\pi_1(\text{Diff}_0(X))$ is abelian since $\text{Diff}_0(X)$ is a topological group.

A brief outline of the paper is as follows. In Sect. 2 we recall the construction of the families Seiberg–Witten invariants. We then show the invariants can be reformulated as maps $sw_\delta : \pi_n(\text{Diff}_0(X)) \to \mathbb{Z}$ and prove several properties of these invariants, in particular Theorems 2.6 and 2.9. In Sect. 3 we specialise to the case that $X$ is a $K3$ surface. We construct the family $Ein \to Tein$ over the Teichmüller space $Tein$ and use this to construct classes $h_\delta \in \pi_1(\text{Diff}_0(X))$. We then compute the Seiberg–Witten invariants of these classes and our main theorem follows.

**2 The families Seiberg–Witten invariant revisited**

In this section we will recall the definition of the families Seiberg–Witten invariant. We will also show that the definition of the invariant can be extended to situations where wall-crossing phenomena are present. We show that under certain conditions a distinguished chamber exists, hence we can still obtain a well-defined invariant.
Our approach to the families Seiberg–Witten invariant follows [14] but with some additional modifications as in [2]. Let $X$ be a compact smooth oriented 4-manifold and let $B$ be a compact smooth manifold. Suppose we have a smooth fibrewise oriented fibre bundle $\pi : E \to B$ whose fibres are diffeomorphic to $X$. Such a fibre bundle will be called a smooth family over $B$ with fibres diffeomorphic to $X$. We assume throughout that $B$ is connected. Choose a basepoint $p \in B$ and a diffeomorphism $X_p \cong X$, where $X_p = \pi^{-1}(p)$ denotes the fibre of $E$ over $p$. Then $\pi_1(B, p)$ acts by monodromy on the set of spin$^c$-structures on $X$. Suppose that $s$ is a monodromy invariant spin$^c$-structure on $X$. Then by monodromy invariance, $s$ can be uniquely extended to a continuously varying family of spin$^c$-structures $\tilde{s} = \{s_b\}_{b \in B}$ on the fibres of $E$ such that $\tilde{s}|_{X_p} \cong s$ (here continuously varying means that the family admits local trivialisations for which the spin$^c$-structure is constant). Note that the existence of the continuous family $\tilde{s}$ is in general a weaker condition than requiring the existence of a spin$^c$-structure on the vertical tangent bundle $T(E/B) = \text{Ker}(\pi_0)$ (because a spin$^c$-structure on the vertical tangent bundle determines a continuously varying family of spin$^c$-structures by taking the fibre-wise restriction, but not every continuously varying family of spin$^c$-structures arises this way). However, as explained in [1, 2], the existence of $\tilde{s}$ is sufficient to construct a families Seiberg–Witten moduli space.

Let

$$d(X, s) = \frac{c_1(s)^2 - \sigma(X)}{4} - 1 - b^+(X) + b_1(X)$$

be the virtual dimension of the ordinary Seiberg–Witten moduli space of $X$. Let $g = \{g_b\}_{b \in B}$ be a smoothly varying family of metrics on the fibres of $E$. Equivalently, $g$ is a metric on the vertical tangent bundle $T(E/B)$. Then we define $\mathcal{H}^+_g(X)$ to be the vector bundle on $B$ whose fibre over $b \in B$ is the space $H^+_g(X_b)$ of $g_b$-self-dual harmonic 2-forms. By a families perturbation $\eta$ we mean a smoothly varying family $\eta = \{\eta_b\}_{b \in B}$ of real 2-forms on the fibres of $E$, such that $\eta_b$ is $g_b$-self-dual. Let $[\eta_b] \in H^+_g(X_b)$ denote the $L^2$-orthogonal projection of $\eta_b$ to the space of self-dual harmonic forms (using the $L^2$-metric defined by $g_b$). The map $b \mapsto [\eta_b]$ defines a section of $\mathcal{H}^+_g(X)$, which we denote by $[\eta]$.

Recall that the Seiberg–Witten equations for $(X_b, s_b, g_b)$ with perturbation $\eta_b$ are:

$$D^+_A\psi = 0,$$

$$F^+_A + i\eta_b = \sigma(\psi),$$

where $A$ is a spin$^c$-connection, $\psi$ is a positive spinor for the spin$^c$-structure $s_b$ and $\sigma(\psi)$ denotes the imaginary self-dual 2-form corresponding to the trace-free part of $\psi^* \otimes \psi$ under Clifford multiplication. Let $w : B \to \mathcal{H}^+_g(X)$ be the section of $\mathcal{H}^+_g(X)$ sending $b$ to $2\pi c_1(s_b) + \eta$, the orthogonal projection of $2\pi c_1(s_b)$ to $H^+_g(X_b)$ using the $L^2$-metric defined by $g_b$. Then the $\eta_b$-perturbed Seiberg–Witten equations for $(X_b, s_b, g_b)$ admits reducible solutions if and only if $[\eta_b] = w$ (recall that a solution $(A, \psi)$ of the Seiberg–Witten equations is called reducible if $\psi = 0$ [16]). We refer
to \( w \) as the “wall” and we say that the families perturbation \( \eta \) does not lie on the wall if for all \( b \in B \), we have \( [\eta_b] \neq w_b \).

We define a \textit{chamber} (of the families Seiberg–Witten equations) for \( (E, s) \) to be a connected component of the space of pairs \( (g, \eta) \), where \( g \) is a family of metrics and \( \eta \) is a family of perturbations not lying on the wall. In general, there are obstructions to the existence of chambers. For instance, if \( b^+(X) = 0 \), then there does not exist a chamber. On the other hand, if \( b^+(X) > \dim(B) + 1 \), then there exists a unique chamber [3].

Let \( C \) be a chamber of \( (E, s) \). Then for a sufficiently generic element \( (g, \eta) \in C \), the moduli space \( \mathcal{M}(E, s, g, \eta) \) of gauge equivalence classes of solutions to the Seiberg–Witten equations on the fibres of \( E \) (with respect to the spin\(^c\)-structure \( \tilde{s} \), metric \( g \) and perturbation \( \eta \)) is a smooth, compact manifold of dimension \( d(X, s_X) + \dim(B) \) (or is empty if this number is negative). See [14] for more details concerning the construction of the families Seiberg–Witten moduli space. Recall that a homology orientation for \( X \) is a choice of orientation of \( H^+(X) \oplus H^1(X; \mathbb{R}) \) (here \( H^+(X) \) denotes the space of harmonic self-dual 2-forms with respect to some metric on \( X \). It is straightforward to see that the homology orientations for different choices of metrics can be canonically identified with one another). We say that a homology orientation is monodromy invariant if it extends to a continuously varying orientation on the family \( \{H^+_g(X_b) \oplus H^1(X_b; \mathbb{R})\}_{b \in B} \).

Let \( \pi : \mathcal{M}(E, s, g, \eta) \to B \) be the projection to \( B \). A monodromy invariant homology orientation defines an orientation on \( T\mathcal{M}(E, s, g, \eta) \oplus \pi^*(TB) \), hence a Gysin homomorphism

\[
\pi_* : H^j(\mathcal{M}(E, s, g, \eta); \mathbb{Z}) \to H^{j-d(X,s)}(B; \mathbb{Z}).
\]

We define the \textit{families Seiberg–Witten invariant} \( SW(E, s, C, o) \in H^{-d(X,s)}(B; \mathbb{Z}) \) of \( E \) with respect to the monodromy invariant spin\(^c\)-structure \( s \), the chamber \( C \) and monodromy invariant homology orientation \( o \) to be

\[
SW(E, s, C, o) = \pi_*(1) \in H^{-d(X,s)}(B; \mathbb{Z}).
\]

The fact that \( SW(E, s, C, o) \) depends only on the chamber \( C \) and not on the particular choice of pair \( (g, \eta) \in C \) follows by much the same argument as in the unparametrised case. A generic path between pairs \( (g, \eta), (g', \eta') \) determines a cobordism (relative \( B \)) of the moduli spaces \( \mathcal{M}(E, s, g, \eta) \) and \( \mathcal{M}(E, s, g', \eta') \).

Let \( \pi : E \to B \) be a smooth family over \( B \) with fibres diffeomorphic to \( X \). Let \( \mathcal{H}^2(X) \) denote the local system over \( B \) whose fibre over \( b \in B \) is \( \mathcal{H}^2(X_b; \mathbb{R}) \). Assume that \( B \) is simply-connected and that \( b^+(X) > 1 \). Choose a basepoint \( p \in B \). Then parallel translation defines a trivialisation

\[
\tau : \mathcal{H}^2(X) \to B \times H^2(X_p; \mathbb{R}).
\]

Let \( H \subseteq H^2(X_p; \mathbb{R}) \) be a maximal positive definite subspace with respect to the intersection form and let \( H^+ \) denote the orthogonal complement of \( H \) (with respect
to the intersection form). This defines a decomposition \( H^2(X_p; \mathbb{R}) \cong H \oplus H^\perp \). Let \( \rho_H : H^2(X_p; \mathbb{R}) \to H \) denote the projection to the first factor.

Choose a smoothly varying family of metrics \( g = \{g_b\} \) and let \( \mathcal{H}_g^+(X) \) be defined as before. Let \( \iota_g : \mathcal{H}_g^+(X) \to \mathcal{H}^2(X) \) be the inclusion. Let \( \text{pr}_B : B \times H^2(X_p; \mathbb{R}) \to B \) be the projection to \( B \). Then the composition

\[
\varphi_g = (\text{pr}_B \times \rho_H) \circ \iota_g : \mathcal{H}_g^+(X) \to B \times H
\]

is an isomorphism of vector bundles. This follows since \( \tau(\iota_g(\mathcal{H}_g^+(X))) \) is a positive definite subbundle of \( B \times H^2(X_p; \mathbb{R}) \), so meets the negative definite subbundle \( B \times H^\perp \) in the zero section.

Let \( \mathfrak{s} \) be a spin^c-structure on \( X \). Since \( B \) is simply-connected, \( \mathfrak{s} \) is automatically monodromy invariant and so continuously extends to the fibres of \( E \).

Let \( w : B \to \mathcal{H}_g^+(X) \) be the wall with respect to the spin^c-structure \( \mathfrak{s} \). Then \( \varphi_g(w) \) is a section of the trivial bundle \( B \times H \). Let

\[
R_g = \sup_{b \in B} ||\varphi_g(w_b)||_H
\]

where \( || \cdot ||_H \) is the norm on \( H \) induced by the restriction to \( H \) of the intersection form on \( H^2(X_p; \mathbb{R}) \). Since \( B \) is compact, \( R_g \) is finite. Since \( b^+(X) > 1 \), \( H \) is a non-zero vector space and hence there exist elements of arbitrarily large norm. Let \( v \) be any element of \( H \) with \( ||v||_H > R_g \). Then the constant section \( b \mapsto b \cdot v \) is disjoint from \( \varphi_g(w) \). Therefore, the section \( v_g = \varphi_g^{-1}(v) \) of \( \mathcal{H}_g^+(X) \) is disjoint from the wall \( w \) and hence defines a chamber, depending only on \( g \) and \( v \) which we will denote by \( \mathcal{C}(g, v) \).

**Lemma 2.1** The chamber \( \mathcal{C}(g, v) \) does not depend on the choice of the pair \( (g, v) \).

**Proof** First we show that for fixed \( g \), the chamber \( \mathcal{C}(g, v) \) does not depend on the choice of \( v \). Let \( R_g = \sup_{b \in B} ||\varphi_g(w_b)||_H \) be defined as before and let \( v, v' \) be any two elements of \( H \) with \( ||v||_H, ||v'||_H > R_g \). The space \( \{x \in H \mid ||x||_H > R_g\} \) is homotopy equivalent to a sphere of dimension \( b^+(X) - 1 \) and is therefore connected, since we are assuming that \( b^+(X) > 1 \). Therefore we can find a continuous path \( \{v_t\}_{t \in [0, 1]} \) in \( \{x \in H \mid ||x||_H > R_g\} \) joining \( v \) to \( v' \). It follows that \( (g, (v_t)_g) \in \mathcal{C}(g, v) \) for all \( t \in [0, 1] \). Hence \( (g, (v')_g) \in \mathcal{C}(g, v) \) and \( \mathcal{C}(g, v) = \mathcal{C}(g, v') \).

Now let \( g, g' \) be two different families of metrics. We will show that there exists a \( v \in H \) for which \( \mathcal{C}(g, v) = \mathcal{C}(g', v) \). Together with the above shown independence of \( \mathcal{C}(g, v) \) on \( v \), this will show that \( \mathcal{C}(g, v) \) does not depend on the choice of pair \( (g, v) \).

Choose a continuous path \( \{g_t\}_{t \in [0, 1]} \) of families of metrics from \( g \) to \( g' \). Let

\[
R = \sup_{b \in B, t \in [0, 1]} ||\varphi_{g_t}(w_b)||_H.
\]

Compactness of \( B \times [0, 1] \) implies that \( R \) is finite. Now choose \( v \in H \) such that \( ||v||_H > R \). Then \( ||v||_H > \sup_{b \in B} ||\varphi_{g_t}(w_b)||_H \) for each \( t \in [0, 1] \). It follows that the pair \( (g_t, v) \) defines the same chamber for all \( t \in [0, 1] \). Hence \( \mathcal{C}(g, v) = \mathcal{C}(g', v) \). \( \square \)
Definition 2.2 Let $\pi : E \to B$ be a smooth family over $B$ with fibres diffeomorphic to $X$ and let $s$ be a spin$^c$-structure on $X$. Assume that $B$ is simply-connected and that $b^+(X) > 1$. For any pair $(g, v)$ with $v \in \{ x \in H \mid \|x\|_H > R_g \}$, we let $C_0(E, s)$ denote the chamber containing $(g, v)$. By Lemma 2.1, we see that $C_0(E, s)$ does not depend on the choice of the pair $(g, v)$. Furthermore, it is clear that $C_0(E, s)$ does not depend on the choice of maximal positive definite subspace $H \subseteq H^2(X; \mathbb{R})$, because the space of all such subspaces is connected (as it can be identified with the connected homogeneous space $O(3, 19)/O(3) \times O(19)$). We call $C_0(E, s)$ the canonical chamber for $(E, s)$.

Definition 2.3 Let $\pi : E \to B$ be a smooth family over $B$ with fibres diffeomorphic to $X$. Assume that $B$ is simply-connected and that $b^+(X) > 1$. Let $s$ be a spin$^c$-structure on $X$ and let $o$ be a homology orientation for $X$. Then we define the (canonical) families Seiberg–Witten invariant $SW(E, s, o)$ of $(E, s, o)$ to be the families Seiberg–Witten invariant of $(E, s, o)$ defined using the canonical chamber:

$$SW(E, s, o) = SW(E, s, C_0(E, s), o) \in H^{-d(X, s)}(B; \mathbb{Z}).$$

Let $X$ be a compact, oriented, smooth 4-manifold with $b^+(X) > 1$. Let $\text{Diff}(X)$ be the group of orientation preserving diffeomorphisms of $X$ with the $C^\infty$-topology and let $\text{Diff}_0(X)$ be the identity component. Let $n > 0$ be a positive integer. Consider an element $f \in \pi_n(\text{Diff}_0(X))$. Using the clutching construction, $f$ defines a topological fibre bundle $E_f \to S^{n+1}$ over $S^{n+1}$ with fibres homeomorphic to $X$ and structure group $\text{Diff}_0(X)$. From the main theorem of [15], it follows that $E_f$ can be made into a smooth fibre bundle with fibres diffeomorphic to $X$ in a unique way. Hence $E_f$ only depends on the homotopy class of $f$ up to isomorphism as a smooth fibre bundle with structure group $\text{Diff}_0(X)$.

We describe the clutching construction in detail in order to fix certain orientation conventions. Regard $S^{n+1}$ as the unit sphere in $\mathbb{R}^{n+2}$. The standard orientation on $\mathbb{R}^{n+2}$ induces an orientation on $S^{n+1}$ by the outer normal first convention. Let $S_{\pm}^{n+1} = \{(x_1, \ldots, x_{n+2}) \in S^{n+1} \mid \pm x_{n+2} > 0\}$ be the two hemispheres. Then $S^{n+1} = S_{+}^{n+1} \cup S_{-}^{n+1}$ and $S_{+}^{n+1} \cap S_{-}^{n+1} = S^n$. Given a map $f : S^n \to \text{Diff}_0(X)$, the fibre bundle $E_f \to S^{n+1}$ is given by taking the trivial bundles $S_{+}^{n+1} \times X, S_{-}^{n+1} \times X$ and identifying $(s, x) \in S^n \times X \subset S_{+}^{n+1} \times X$ with $(s, (f(s))(x)) \in S^n \times X \subset S_{+}^{n+1} \times X$.

Let $s$ be a spin$^c$-structure on $X$ and $o$ a homology orientation. Since $n > 0$, $S^{n+1}$ is simply-connected and $b^+(X) > 1$, the families Seiberg–Witten invariant

$$SW(E_f, s, o) \in H^{-d(X, s)}(S_{+}^{n+1}; \mathbb{Z})$$

is defined. If $d(X, s) = -(n + 1)$, then we can evaluate $SW(E_f, s, o)$ against the fundamental class of $S^{n+1}$ to obtain an integer invariant.

Definition 2.4 Let $X$ be a compact, oriented, smooth 4-manifold with $b^+(X) > 1$. Let $s$ be a spin$^c$-structure such that $d(X, s) = -(n + 1)$ for some $n \geq 0$. We define

$$sw_s : \pi_n(\text{Diff}_0(X)) \to \mathbb{Z}$$
by setting

\[ sw_\phi(f) = \int_{S^{n+1}} SW(E_f, s, \phi), \]

where we have chosen a homology orientation \( \phi \) and we have oriented \( S^{n+1} \) according to the convention described above. We have omitted from our notation the dependence of \( sw_\phi \) on the choice of homology orientation. Changing the homology orientation has the effect of changing \( sw_\phi \) by an overall sign.

**Remark 2.5** The invariant \( sw_\phi(f) \in \mathbb{Z} \) can be interpreted as follows. Choose a generic pair \((g, \eta) \in C_0(E_f, s)\). Then the moduli space \( \mathcal{M}(E, s, g, \eta) \) is a compact, oriented 0-manifold and \( sw_\phi(f) \) is simply the number of points of \( \mathcal{M}(E_f, s, g, \eta) \), counted with sign.

**Theorem 2.6** Let \( X \) be a compact, oriented, smooth 4-manifold with \( b^+(X) > 1 \). Then for each spin\(^c\)-structure with \( d(X, s) = -(n + 1) \leq -2 \), the map

\[ sw_\phi : \pi_n(Diff_0(X)) \to \mathbb{Z} \]

is a group homomorphism.

**Proof** Let \( f \in \pi_n(Diff_0(X)) \) and let \( E_f \to S^{n+1} \) be the corresponding family built from the clutching construction. Choose a generic pair \((g, \eta) \in C_0(E_f, s)\). The moduli space \( \mathcal{M}(E, s, g, \eta) \) is a finite set of points. Let \( p_1, \ldots, p_m \in S^{n+1} \) be the finitely many points over which \( \mathcal{M}(E_f, s, g, \eta) \) lies. Choose a point \( p \in S^{n+1} \) and an open disc \( D \subset S^{n+1} \) around \( p \) such that \( D \) is disjoint from \( p_1, \ldots, p_m \). Furthermore, we can assume that \( D \) is contained in the interior of \( S^{n+1}_+ \). Using cutoff functions it is possible to construct a smooth map \( \psi : S^{n+1} \to S^{n+1} \) such that \( \psi(D) = \{p\} \) and \( \psi : S^{n+1}_+ \setminus D \to S^{n+1}_- \setminus \{p\} \) is a diffeomorphism. Now consider the pullback \( \psi^*(E_f) \) of \( E_f \) under \( \psi \). The conditions on \( \psi \) ensures that it has degree 1 as a map of \( S^{n+1} \) to itself. Hence \( \psi \) is homotopic to the identity. It follows that \( \psi^*(E_f) \) is isomorphic to \( E_f \) as topological fibre bundles over \( S^{n+1} \) with structure group \( Diff(X) \). The main theorem of [15] then implies that \( \psi^*(E_f) = E_f \circ \psi \) and \( E_f \) are isomorphic as smooth fibre bundles. The pullback \((\psi^*(g), \psi^*(\eta)) \) is a generic pair in \( C_0(\psi^*(E_f), s) \) and the moduli space \( \mathcal{M}(\psi^*(E_f), s, \psi^*(g), \psi^*(\eta)) \) is obviously obtained by pulling back \( \mathcal{M}(E_f, s, g, \eta) \) by \( \psi \). Let \( X_p \) denote the fibre of \( E_f \) over \( p \) and fix a diffeomorphism \( X_p \cong X \). Since \( \psi \) takes the constant value \( p \) on \( D \), the restriction of \( \psi^*(E_f) \) to \( D \) is the constant family \( \psi^*(E_f)|_D \cong D \times X_p \cong D \times X \). Under this trivialisation of \( \psi^*(E_f)|_D \) we have that \( \psi^*(g), \psi^*(\eta) \) get sent to the constant pair \((g_p, \eta_p)\).

Now let \( f' \) be another element of \( \pi_n(Diff_0(X)) \) and let \( E_{f'} \to S^{n+1} \) be the family corresponding to \( f' \). Choose a generic pair \((g', \eta') \in C_0(E_{f'}, s)\). Let \( r : S^{n+1} \to S^{n+1} \) be the orientation reversing map given by \( r(x^1, \ldots, x^{n+2}) = (x^1, \ldots, x^{n+1}, -x^{n+2}) \). Observe that \( r \) exchanges the two hemispheres \( S^{n+1}_\pm \). The moduli space \( \mathcal{M}(E_{f'}, s, g', \eta') \) is a finite set of points \( p'_1, \ldots, p'_m \), hence we can further assume that \( D \) was chosen to be disjoint from \( r(p'_1), \ldots, r(p'_m) \). Equivalently, \( r(D) \) is disjoint from \( p'_1, \ldots, p'_m \). Let \( \psi' = r \circ \psi \circ r \). We then obtain the pullback
family $\psi'^* (E_f')$ with generic pair $(\psi'^*(g'), \psi'^*(\eta'))$. Moreover, we have a trivialisation of $\psi'^*(E_f')|_{r(D)}$ in which $\psi'^*(g'), \psi'^*(\eta')$ are sent to the constant pair $(g'_p, \eta'_p)$, where $p' = r(p)$.

Let $D_0 \subset D$ be a smaller open disc around $p$ whose closure is contained in $D$. Attach $S^{n+1} \setminus D_0$ and $S^{n+1} \setminus r(D_0)$ to each other using a neck $[0, 1] \times \partial D_0 \times [0, 1]$ (note that $\partial D_0$ is diffeomorphic to $S^n$). More precisely, consider the resulting space

$$(S^{n+1} \setminus D_0) \cup_{\partial D_0} ([0, 1] \times \partial D_0) \cup_{\partial D_0} (S^{n+1} \setminus r(D_0))$$

where we identify $(0, y) \in [0, 1] \times \partial D_0$ with $y \in \partial (S^{n+1} \setminus D_0) = \partial D_0$ and we identify $(1, y) \in [0, 1] \times \partial D_0$ with $y \in \partial (S^{n+1} \setminus r(D_0)) = \partial (r(D_0))$. Because $r$ is orientation reversing, this construction is easily seen to be the oriented connected sum of two copies of $S^{n+1}$. Of course the resulting space is just another copy of $S^{n+1}$. Using the trivialisations $\psi^*(E_f)|_D \cong D \times X$ and $\psi'^*(E_f')|_{r(D)} \cong r(D) \times X$, we can attach $\psi^*(E_f)|_{S^{n+1} \setminus D_0}$ to $\psi'^*(E_f')|_{S^{n+1} \setminus r(D_0)}$ by taking a constant family $[0, 1] \times \partial D_0 \times X$ along the neck.

Let $E$ denote the resulting family. Since $\psi^*(E_f)$ and $\psi'^*(E_f')$ are isomorphic to $E_f$ and $E_f'$ and since the neck $[0, 1] \times \partial D_0$ connects the lower hemisphere in $\psi^*(E_f)$ with the upper hemisphere in $\psi'^*(E_f')$, it is clear that $E$ is isomorphic to the family obtained by applying the clutching construction to $f + f'$, where $+$ denotes the group operation on $\pi_2(\text{Diff}_0(X))$.

Next, since $d(X, s) = -(n + 1) \leq -2$, it follows that the moduli space of solutions to the Seiberg–Witten equations for a 1-parameter family with fibres $(X, s)$ has expected dimension $d(X, s) + 1 = -n < 0$. Therefore, for a generic path $(g_t, \eta_t)$ from $(g(p), \eta(p))$ to $(g'(p'), \eta'(p'))$, there are no solutions to the Seiberg–Witten equations for $(X, s, g_t, \eta_t)$. Now we define a pair $(\tilde{g}, \tilde{\eta})$ for the family $E$ as follows. Restricted to $\psi^*(E_f)|_{S^{n+1} \setminus D_0}$, we take the pair to be $(\psi^*(g), \psi^*(\eta))$. Restricted to $\psi'^*(E_f')|_{S^{n+1} \setminus r(D_0)}$, we take the pair to be $(\psi'^*(g'), \psi'^*(\eta'))$. Restricted to the constant family on the neck $[0, 1] \times S^n$, we take the pair to be $(g_t, \eta_t)$, where $t \in [0, 1]$ is the coordinate for the $[0, 1]$ factor of the neck. Since there are no solutions to the Seiberg–Witten equations for $(g_t, \eta_t)$, it is clear that the moduli space for $(E, s, \tilde{g}, \tilde{\eta})$ is just the disjoint union

$$\mathcal{M}(\psi^*(E_f), s, \psi^*(g), \psi^*(\eta)) \cup \mathcal{M}(\psi'^*(E_f'), s, \psi'^*(g'), \psi'^*(\eta'))$$

of the corresponding moduli spaces for $\psi^*(E_f)$ and $\psi'^*(E_f')$. Since there are no solutions to the Seiberg–Witten equations of the glued family along the neck, and since the metrics and perturbations $(g, \eta)$ and $(g', \eta')$ of the original families were generic, it follows that $(\tilde{g}, \tilde{\eta})$ is also generic. It then follows that

$$\int_{S^{n+1}} SW(E, s, \tilde{g}, \tilde{\eta}) = sw_s(f) + sw_s(f').$$

To complete the proof, it remains to show that $\int_{S^{n+1}} SW(E, s, \tilde{g}, \tilde{\eta}) = sw_s(f + f')$. Since $E$ is isomorphic to the family obtained from applying the clutching construction
to \( f + f' \), we just need to show that the pair \((g, \tilde{\eta})\) lies in the canonical chamber in \( C_0(E, s) \).

Let \( \mathcal{H}^+_{\psi^s(g)}(X), \mathcal{H}^+_{\psi^s(g')}^r(X) \) and \( \mathcal{H}^+_{\tilde{g}}(X) \) be the bundles of harmonic self-dual 2-forms for the families of metrics \( \psi^s(g), \psi^s(g') \), and \( \tilde{g} \). Then from the construction of \( \tilde{g} \), we have that \( \mathcal{H}^+_{\tilde{g}}(X) \) is obtained by attaching \( \mathcal{H}^+_{\psi^s(g)}(X)|_{S^{n+1}\setminus D_0} \) and \( \mathcal{H}^+_{\psi^s(g')}^r(X)|_{S^{n+1}\setminus r(D_0)} \) to the bundle \( \mathcal{H}^+_{\tilde{g}}(X) \) over \([0, 1] \times S^n\) whose fibre over \((t, x) \in [0, 1] \times S^n\) is the space of \( g_t \)-self-dual harmonic 2-forms.

Let \( \mathcal{H}^2_E(X) \) denote the local system whose fibres are the degree 2 cohomology of the fibres of \( E \) (the subscript \( E \) is just to remind us which family \( \mathcal{H}^2_E(X) \) comes from). Similarly define the local systems \( \mathcal{H}^2_{\psi^s(E_f)}(X), \mathcal{H}^2_{\psi^s(E_f')}^r(X) \). Then \( \mathcal{H}^2_E(X) \) is obtained by attaching \( \mathcal{H}^2_{\psi^s(E_f)}(X)|_{S^{n+1}\setminus D_0} \) and \( \mathcal{H}^2_{\psi^s(E_f')}^r(X)|_{S^{n+1}\setminus r(D_0)} \) to the constant local system over \([0, 1] \times S^n\) with fibre \( H^2(X_p; \mathbb{R}) \). Choose a maximal positive definite subspace \( H \subseteq H^2(X_p; \mathbb{R}) \) and let \( \rho_H : H^2(X_p; \mathbb{R}) \to H \) be the projection. Taking the composition of inclusion and projection to \( H \), we obtain isomorphisms

\[
\varphi_{g} : \mathcal{H}^+_{\tilde{g}}(X) \to B \times H, \quad \varphi_{\psi^s(g)} : \mathcal{H}^+_{\psi^s(g)}(X) \to B \times H, \quad \varphi_{\psi^s(g')} : \mathcal{H}^+_{\psi^s(g')}^r(X) \to B \times H.
\]

Similarly, we obtain an isomorphism \( \varphi_{g_t} : \mathcal{H}^+_{g_t}(X) \to B \times H \). It is clear that the restriction of \( \varphi_{\tilde{g}} \) to \( S^{n+1}\setminus D_0 \) agrees with \( \varphi_{\psi^s(g)} \), the restriction of \( \varphi_{\tilde{g}} \) to \( S^{n+1}\setminus r(D_0) \) agrees with \( \varphi_{\psi^s(g')} \), and the restriction of \( \varphi_{\tilde{g}} \) to \([0, 1] \times S^n\) agrees with \( \varphi_{g_t} \).

Let \( w : S^{n+1} \to \mathcal{H}^+_{\tilde{g}}(X) \) denote the wall for the family \( E \) and set

\[
R = \sup_{b \in S^{n+1}} ||\varphi_{\tilde{g}}(w_b)||_H.
\]

Recall that we have assumed \((g, \eta) \in C_0(E_f, s)\). Fix an element \( v \in H \) such that \( ||v||_H > R \). Choose an \( \epsilon > 0 \) such that each \( u \) in the open ball \( B(v, \epsilon) = \{u \in H \mid ||u - v||_H < \epsilon\} \) has \( ||u||_H > R \) \( (\epsilon = (||v||_H - R)/2 \) would suffice). We will assume that \( \eta \) is chosen with \( \varphi_{g_t}|\eta_b \in B(v, \epsilon) \) for all \( b \in S^{n+1} \). Then we also have \( \varphi_{\psi^s(g)}|\psi^s(\eta)|_b \in B(v, \epsilon) \) for all \( b \in S^{n+1} \). Similarly, we can assume that \( \eta' \) was chosen so that \( \varphi_{g_t}|\eta'_b \in B(v, \epsilon) \) for all \( b \in S^{n+1} \) and hence \( \varphi_{\psi^s(g')}|\psi^s(\eta')|_b \in B(v, \epsilon) \) for all \( b \in S^{n+1} \) as well. Lastly, we can assume that the generic path \((g_t, \eta_t)\) joining \((g_p, \eta_p)\) to \((g'_p, \eta'_p)\) satisfies \( \varphi_{g_t}(\eta_t) \in B(v, \epsilon) \) for all \( t \in [0, 1] \). It follows that \( \varphi_{\tilde{g}}(\tilde{\eta})_b \in B(v, \epsilon) \) for all \( b \in S^{n+1} \). Therefore we can find a homotopy from \( \varphi_{\tilde{g}}(\tilde{\eta}) \) to the constant section \( v \) and hence \((\tilde{g}, \tilde{\eta})\) lies in \( C_0(E, s) \).

Let \( \text{Diff}(X) \) act on itself by conjugation. Since the identity element is fixed, this gives an action of \( \text{Diff}(X) \) on \( \pi_n(\text{Diff}_0(X)) \). We write this action as \( (f, h) \mapsto fhf^{-1} \).

**Proposition 2.7** Let \( X \) be a compact, oriented, smooth 4-manifold with \( b^+(X) > 1 \) and let \( f : X \to X \) be an orientation preserving diffeomorphism. Then for each \( \text{spin}^c \)-structure with \( d(X, s) = -(n + 1) \leq -2 \) we have

\[
sw_{\tilde{g}}(fhf^{-1}) = sw_{f^s(s)}(h).
\]
Proof Let $D_+, D_-$ be two copies of the unit disc in $\mathbb{R}^{n+1}$. Attaching $D_+$ and $D_-$ along their boundary gives $S^{n+1}$. Let $h \in \pi_1(\text{Diff}_0(X))$. The family $E_h$ is obtained by attaching $D_+ \times X$ to $D_- \times X$ using the attaching map $\partial D_+ \times X \to \partial D_+ \times X$, $(b, x) \mapsto a_h(b, x) = (b, (h(b))(x))$. Similarly $E_{f_{hf^{-1}}}$ is constructed using $(b, x) \mapsto a_{f_{hf^{-1}}}(b, x) = (b, (f(h(b))f^{-1})(x))$. Consider the maps $\tilde{f}^\pm : D_\pm \times X \to D_\pm \times X$ given by $\tilde{f}^\pm(b, x) = (b, f(x))$. One finds that

$$a_{f_{hf^{-1}}} \circ \tilde{f}^- = \tilde{f}^+ \circ a_h.$$

This says that the maps $\tilde{f}^\pm$ glue together to define a map $\tilde{f} : E_h \to E_{f_{hf^{-1}}}$. The map $\tilde{f}$ is an isomorphism of smooth families over $S^{n+1}$. Let $\tilde{s}$ be the continuous extension of $s$ to a family of spin$^c$-structures on the fibres of $E_{f_{hf^{-1}}}$. Then clearly $\tilde{f}^*(\tilde{s})$ is a continuous extension of $f^*(s)$ to a family of spin$^c$-structures on the fibres of $E_h$. Let $(g, \eta)$ be a generic pair for $(E_{f_{hf^{-1}}}, s)$ lying in the canonical chamber. Then $(\tilde{f}^*(g), \tilde{f}^*(\eta))$ is a generic pair for $(E_h, f^*(s))$ lying in the canonical chamber. Clearly $\tilde{f}$ induces an isomorphic between the corresponding moduli spaces for $(E_{f_{hf^{-1}}}, s, g, \eta)$ and $(E_h, f^*(s), \tilde{f}^*(g), \tilde{f}^*(\eta))$. Hence $sw_\tilde{s}(f_{hf^{-1}}) = sw_{f^*(s)}(h)$. \qed

Recall that there is an involution $s \mapsto \bar{s}$ on the set of spin$^c$-structure which we refer to as charge conjugation [16, Page 51]. Recall that $c_1(\bar{s}) = -c_1(s)$.

Proposition 2.8 Let $X$ be a compact, oriented, smooth 4-manifold with $b^+(X) > 1$ and let $s$ be a spin$^c$-structure with $d(X, s) = -(n+1) \leq -2$. Then

$$sw_{\bar{s}} = (-1)^{b_+(X) - b_1(X) - \eta}sw_s.$$

Proof Recall that charge conjugation gives rise to a bijection from the Seiberg–Witten equations for $(X, s, g, \eta)$ to the Seiberg–Witten equations for $(X, \bar{s}, g, -\eta)$. Fix a homology orientation for $X$, giving orientations on $\mathcal{M}(X, s, g, \eta)$ and $\mathcal{M}(X, \bar{s}, g, -\eta)$. The charge conjugation map is orientation preserving or reversing according to the sign of $(-1)^{d_s + 1 - b_1(X) + b^+(X)}$ where

$$d_s = \frac{c_1(s)^2 - \sigma(X)}{8},$$

see [16, Proposition 2.2.26]. Similarly, charge conjugation gives rise to a bijection of families moduli spaces. By a straightforward extension of [16, Proposition 2.2.26] to the families setting, we see that the charge conjugation isomorphism changes the orientation of the families moduli space by the same factor $(-1)^{d_s + 1 - b_1(X) + b^+(X)}$. Moreover, it is clear that if $(g, \eta)$ is in the canonical chamber for $(E, s)$, then $(g, -\eta)$ is in the canonical chamber for $(E, \bar{s})$. Hence

$$sw_{\bar{s}} = (-1)^su_{sw_s}.$$
where
\[ u = d_s + 1 - b_1(X) + b^+(X). \]

But since
\[ -n - 1 = d(X, s) = 2d_s + 1 - b_1(X) + b^+(X) \]
we see that
\[ d_s = \frac{-n - 2 + b_1(X) - b^+(X)}{2} \]

and hence
\[ u = \frac{-n - 2 + b_1(X) - b^+(X)}{2} + 1 - b_1(X) + b^+(X) = \frac{b^+(X) - b_1(X) - n}{2}. \]

\[ \square \]

**Theorem 2.9** Let \( X \) be a compact, oriented, smooth 4-manifold such that \( b^+(X) > 1 \) and \( b_1(X) = 0 \). For a given \( f \in \pi_n(\text{Diff}_0(X)) \), we have that \( sw_s(f) \) is non-zero for only finitely many spin\(^c\)-structures with \( d(X, s) = -(n + 1) \).

**Proof** Let \( g \) be a metric on \( X \) and consider the Seiberg–Witten equations on \( X \) with respect to the metric \( g \) and zero perturbation. Recall that the a priori estimates for solutions \((A, \psi)\) of the Seiberg–Witten equations (after gauge fixing) imply bounds on the norms of \( A, \psi \) in a suitable Sobolev space [16, §2.2]. A bound \( M(g) \) can be chosen which depends continuously on \( g \) and the topology of \( X \), but does not depend on the spin\(^c\)-structure \( s \). Hence for a smooth family \( E \to B \) over a compact base \( B \), we obtain compactness of the families Seiberg–Witten moduli space, taken over all spin\(^c\)-structures, with zero perturbation and a fixed family \( g = \{g_b\} \) of metrics. It follows that the families moduli space \( M(E, s, g, 0) \) is non-empty for only finitely many spin\(^c\)-structures, say, \( s_1, \ldots, s_m \). For any other spin\(^c\)-structure, \( s \), the moduli space \( M(E, s, g, 0) \) is empty. Hence \( \eta = 0 \) is a generic perturbation for \((E, s)\) and \((g, 0)\) defines a chamber for \((E, s)\).

Now let \( f \in \pi_n(\text{Diff}_0(X)) \) and take \( E \to B \) to be the family \( E_f \to S^{n+1} \) associated to \( f \). If \( b^+(X) > n + 1 \), then there is only one chamber and hence we deduce that \( sw_s(f) = 0 \) for all but finitely many spin\(^c\)-structures.

If \( b^+(X) \leq n + 2 \), then we have to consider chambers. Fix a family of metrics \( g \). Then we have shown that for all but finitely many spin\(^c\) structures \( s \), \((g, 0)\) is a generic perturbation and \( SW(E_f, s, g, 0) = 0 \). However, \((g, 0)\) might not lie in the canonical chamber. Hence we need to consider contributions to the Seiberg–Witten invariant from wall crossing.

For the rest of the proof, the family \( E_f \) and metric \( g \) will be fixed. To simplify notation we will write \( SW(s, \eta) \) instead of \( SW(E_f, s, g, \eta) \), whenever \( \eta \) is a generic perturbation for \((E, s, g)\). Fix a maximal positive definite subspace \( H \) of \( H^2(X; \mathbb{R}) \).

\[ \square \]
Let $\varphi : \mathcal{H}_g^+(X) \to H$ be the map which is the inclusion of $\mathcal{H}_g^+(X)$ into $H^2(X; \mathbb{R})$, followed by projection to $H$. For each spin$^c$-structure $s$, let $w(s)$ be the section of $H$ which sends $b \in B$ to $\varphi(2\pi c_1(s)^+ g_b)$. Given a perturbation $\eta$, let $w(\eta)$ be the section of $H$ given by $b \mapsto \varphi([\eta]_b)$. If $w(\eta)$ and $w(s)$ are disjoint, then $(g, \eta)$ defines a chamber for $(E, s)$ and hence the Seiberg–Witten invariant $SW(s, \eta)$ is defined.

Let $S(H)$ denote the unit sphere in $H$, which has dimension $b^+(X) - 1$. If $w(\eta)$ and $w(s)$ are disjoint, then $\phi = (w(\eta) - w(s)) / ||w(\eta) - w(s)||_H$ defines a section of $S(H)$. The wall crossing formula for the families Seiberg–Witten invariants [3, Corollary 5.5] (see also [14]) adapted to the present setting states that

$$SW(s, \eta_1) - SW(s, \eta_2) = Obs(\phi, \psi)s_{k-1}(D)$$  (2.1)

where $\phi, \psi : B \to S(H)$ are the sections of $S(H)$ given by

$$\phi = \frac{w(\eta_1) - w(s)}{||w(\eta_1) - w(s)||_H}, \quad \psi = \frac{w(\eta_2) - w(s)}{||w(\eta_2) - w(s)||_H},$$

$s_k(D)$ is the $k$-th Segre class of $D$, the families index of the family of spin$^c$ Dirac operators determined by $(E, s, g)$, $d = (c_1(s^2 - \sigma(X))/8$ and $Obs(\phi, \psi) \in H^{b^+(X)-1}(B; \mathbb{Z})$ is the primary difference class of $\phi, \psi$ ([20, §36]), the obstruction to constructing a homotopy of two maps $\phi, \psi : B \to S(H)$ over the $b^+(X) - 1$ skeleton of $B$. In our case $B = S^{n+1}$ and so $H^{b^+(X)-1}(B; \mathbb{Z}) = H^{b^+(X)-1}(S^{n+1}; \mathbb{Z})$ is zero unless $b^+(X) = n + 2$.

If $b^+(X) \neq n + 2$ then the primary obstruction vanishes implying that the value of $SW(s, \eta)$ does not depend on the choice of chamber. But we have already seen that for all but finitely many spin$^c$ structures $s$, $SW(s, 0) = 0$. Hence $sw_s = 0$ for all but finitely many $s$.

It remains to consider the case $b^+(X) = n + 2$. In this case we have $-n - 1 = d(X, s) = 2d - n - 3$ and hence $d = 1$. But $s_0(D) = 1$ and so Equation (2.1) reduces to $SW(s, \eta_1) - SW(s, \eta_2) = Obs(\phi, \psi)$. Furthermore, the primary obstruction is valued in $H^{n+1}(S^{n+1}; \mathbb{Z}) \cong \mathbb{Z}$. Let $v \in H^{n+1}(S^{n+1}; \mathbb{Z})$ be the generator corresponding to our chosen orientation on $S^{n+1}$. Then from [3, Proposition 5.7], we have

$$Obs(\phi, \psi) = (-1)^{b^+(X)-1}(\phi^*(v) - \psi^*(v)).$$

Integrating over $S^{n+1}$, the wall crossing formula reduces to

$$\int_{S^{n+1}} SW(s, \eta_1) - \int_{S^{n+1}} SW(s, \eta_2) = (-1)^{b^+(X)-1} (\deg(-\phi_{s, \eta_1}) - \deg(-\phi_{s, \eta_2})), $$

where we define

$$\phi_{s, \eta} = \frac{w(s) - w(\eta)}{||w(s) - w(\eta)||_H}$$

for any perturbation $\eta$ such that $w(\eta)$ and $w(s)$ are disjoint.
Noting that \( \text{deg}(-\phi) = (-1)^{b^+(X)} \text{deg}(\phi) \), the wall crossing formula can be re-written as

\[
\int_{S^{n+1}} SW(s, \eta_1) - \int_{S^{n+1}} SW(s, \eta_2) = -\text{deg}(\phi_{s, \eta_1}) + \text{deg}(\phi_{s, \eta_2}).
\]

Now let us take \( \eta_1 = \eta \) to be arbitrary and choose \( \eta_2 \) such that \( \|w(\eta_2)\|_H > \sup_B \|w(s)\|_H \). Then \((g, \eta_2)\) lies in the canonical chamber. Now since \( \|w(\eta_2)\|_H > \|w(s)\|_H \) for all \( b \in B \), we obtain a homotopy

\[
t \mapsto \frac{(1 - t)w(s) - w(\eta_2)}{\|(1 - t)w(s) - w(\eta_2)\|_H}, \quad t \in [0, 1]
\]

from \( \phi_{s, \eta_2} \) to the constant \(-v/\|v\|_H\). It follows that \( \text{deg}(\phi_{s, \eta_2}) = 0 \) and therefore

\[
\int_{S^{n+1}} SW(s, \eta) - \int_{S^{n+1}} SW(s, \eta_2) = -\text{deg}(\phi_{s, \eta}).
\]

But \((g, \eta_2)\) lies in the canonical chamber, so \( \int_{S^{n+1}} SW(s, \eta_2) = sw_s(f) \). Hence the above formula reduces to

\[
\int_{S^{n+1}} SW(s, \eta) = sw_s(f) - \text{deg}(\phi_{s, \eta}). \tag{2.2}
\]

Now we set \( \eta = 0 \). Then for all but finitely many \( s \), we have that \( w(s) \) is non-vanishing and that \( SW(s, 0) = 0 \). Hence for all but finitely many \( s \), we find

\[ sw_s(f) = \text{deg}(w(s)/\|w(s)\|_H). \]

To finish the proof, it remains to show that when \( b^+(X) = n + 2 \), there are only finitely many \( s \) such that \( d(X, s) = -(n + 1) \), \( w(s) \) is non-vanishing and \( w(s)/\|w(s)\|_H : S^{n+1} \to S(H) \) has non-zero degree. For convenience, let us say that a spin\(^c\)-structure \( s \) is valid if \( d(X, s) = -(n + 1) \) and \( w(s) \) is non-vanishing and let us write \( \text{deg}(w(s)) \) for the degree of \( w(s)/\|w(s)\|_H \). Then we need to show that \( \text{deg}(w(s)) = 0 \) for all but finitely many valid \( s \).

We first show that there is a constant \( \kappa \) such that \( \text{deg}(w(s)) = \kappa \) for all but finitely many valid \( s \). We will then argue that \( \kappa = 0 \).

Note that if \( b^+(X) = n + 2 \) and \( d(X, s) = -(n + 1) \), then

\[
\frac{c_1(s)^2 - \sigma(X)}{4} - n - 3 = -n - 1
\]

(where we used \( b_1(X) = 0 \)) and hence if \( s \) is valid, then

\[ c_1(s)^2 = \sigma(X) + 8. \]
We set \( N = \sigma(X) + 8 \). If \( N \geq 0 \), then for every non-zero \( c \in H^2(X; \mathbb{R}) \) such that \( c^2 = N \), the orthogonal projection \( c^{+s} \) of \( c \) to \( H^2_{gb}(X) \) is non-zero. This is because
\[
(c^{+s})^2 \geq (c^{+s})^2 - |(c^{-s})^2| = c^2 = N \geq 0
\]
and equality \((c^{+s})^2 = 0\) can only occur if \( c = 0 \). So if \( N \geq 0 \), then every non-zero \( c \in H^2(X; \mathbb{R}) \) defines a non-zero map \( w(c) : S^{n+1} \to H \) by taking \( w(c) = \varphi(2\pi c^{+s}) \). The set \( \{ c \in H^2(X; \mathbb{R}) \mid c \neq 0, \ c^2 = N \} \) is clearly connected if \( N > 0 \), since \( b^+(X) > 1 \). Also if \( N = 0 \), then \( \sigma(X) = -8 \) and so \( b^+(X), b^-(X) > 1 \). It follows that \( \{ c \in H^2(X; \mathbb{R}) \mid c \neq 0, \ c^2 = 0 \} \) is connected. Therefore the degree of \( w(c)/\|w(c)\|_H \) is a constant \( \kappa \). Now there are only finitely many spin\(^c\) structures \( s \) for which \( c_1(s) = 0 \), hence \( c_1(s) \in \{ c \in H^2(X; \mathbb{R}) \mid c \neq N \} \) for all but finitely many valid \( s \). So \( \deg(w(s)) = \kappa \) for all but finitely many valid \( s \).

Now we suppose \( N < 0 \). So \( \sigma(X) < -8 \) and in particular \( b^-(X) > b^+(X) > 1 \).
Let us define
\[
C_N = \{ c \in H^2(X; \mathbb{R}) \mid c^2 = N \}
\]
Then \( C_N \) is homotopy equivalent to a sphere of dimension \( b^-(X) - 1 \). For each \( b \in B \), consider
\[
S_b = \{ c \in C_N \mid c^{+s_b} = 0 \}.
\]
The condition \( c^{+s_b} = 0 \) means that \( c \) lies in the negative definite subspace \( H^2_{gb}(X) \).
Therefore \( S_b \) is a sphere of dimension \( b^-(X) - 1 \). In particular \( S_b \) is compact. Similarly, let
\[
S = \bigcup_{b \in B} S_b = \{ c \in C_N \mid c^{+s_b} = 0 \text{ for some } b \in B \}.
\]
Then \( S \) is a compact subset of \( C_N \) (by compactness of \( B \)). Choose an isometry \( H^2(X; \mathbb{R}) \cong \mathbb{R}^r \otimes \mathbb{R}^s \) where \( r = b^+(X) > 1 \) and \( s = b^-(X) > 1 \). We can further identify \( \mathbb{R}^r \otimes \mathbb{R}^s \) with \( \mathbb{R}^r \oplus \mathbb{R}^s \) with bilinear form \((x_1, y_1, x_2, y_2))_{r,s} = \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle \), where \( \langle x_1, x_2 \rangle \) and \( \langle y_1, y_2 \rangle \) are the standard inner products on \( \mathbb{R}^r \) and \( \mathbb{R}^s \). Then if \( c = (x, y) \in H^2(X; \mathbb{R}) \cong \mathbb{R}^r \otimes \mathbb{R}^s \), we have \( c^2 = x^2 - y^2 \), where \( x^2 = \langle x, x \rangle \) and \( y^2 = \langle y, y \rangle \). It follows that \( C_N \cong \{(x, y) \in \mathbb{R}^r \otimes \mathbb{R}^s \mid x^2 - y^2 = N \} \). Define a Euclidean norm \( \|\cdot\|_E \) on \( H^2(X; \mathbb{R}) \) by setting \( \|c\|_E^2 = x^2 + y^2 \). By compactness of \( S \), we have that \( S \) is contained in some ball \( B_R = \{ x \in H^2(X; \mathbb{R}) \mid \|x\|_E \leq R \} \) of sufficiently large radius \( R > 0 \). Then \( C_N \setminus (C_N \cap B_R) \) may be identified with
\[
\{(x, y) \in \mathbb{R}^r \otimes \mathbb{R}^s \mid x^2 - y^2 = N, \ x^2 + y^2 > R^2 \}.
\]
Equivalently, this is the set of pairs \((x, y)\) such that \( x^2 > (R^2 + N)/2 \) and \( y^2 = x^2 - N \). Recall that \( N < 0 \). Hence if \( R^2 > -N \), we see that this space is homotopy equivalent to \( S^{b^+(X)-1} \times S^{b^-(X)-1} \), which is connected as \( b^+(X), b^-(X) > 1 \). For any \( c \in C_N \setminus (C_N \cap B_R) \), define \( w(c) : S^{n+1} \to H \) as \( w(c) = \varphi(2\pi c^{+s}) \). Then since
\[ c \notin B_R, \text{ we have that } w(c) \text{ is non-vanishing and hence the degree of } \frac{w(c)}{||w(c)||_H} \text{ is defined. Since } C_N \setminus (C_N \cap B_R) \text{ is connected, the degree of } \frac{w(c)}{||w(c)||_H} \text{ is equal to a constant, } \kappa, \text{ for every } c \in C_N \setminus (C_N \cap B_R). \]

Let \( s \) be a valid spin\(^c\)-structure. Then \( c_1(s) \in C_N \). If \( c_1(s) \notin B_R \), then it follows that 
\[ \text{deg}(w(s)) = \kappa. \]
Next, we note that if \( c_1(s) \in B_R \), then \( c_1(s) \in B_R \cap H^2(X; \mathbb{Z}) \). But 
\( B_R \cap H^2(X; \mathbb{Z}) \) is finite because \( B_R \) is compact and \( H^2(X; \mathbb{Z}) \) is discrete. It follows that for all but finitely many valid \( s \), we have 
\[ \text{deg}(w(s)) = \kappa. \]

Let \( \psi : H^2(X; \mathbb{R}) \to H^2(X; \mathbb{R}) \) be an isometry of the intersection form on \( H^2(X; \mathbb{R}) \) that sends \( H \) to itself and reverses orientation on \( H \). Then \( B_R \cup \psi(B_R) \) is compact so there exists a \( c \in C_N \) such that \( c \notin B_R \cup \psi(B_R) \). Hence \( c, \psi(c) \in C_N \setminus (C_N \cap B_R) \). Therefore
\[ \text{deg}(w(c)) = \text{deg}(w(\psi(c))) = \kappa. \]

On the other hand, since \( \psi \) reverses orientation on \( H \), we have
\[ \text{deg}(w(\psi(c))) = -\text{deg}(w(c)) = -\kappa. \]
This gives \( \kappa = -\kappa \), hence \( \kappa = 0 \). Thus \( \text{deg}(w(s)) = 0 \) for all but finitely many valid \( s \) and the proof is complete. \( \square \)

### 3 The Einstein family

Let \( X \) be the underlying oriented smooth 4-manifold of a complex \( K3 \) surface. We will use the moduli space of Einstein metrics on \( X \) to construct non-trivial families. The construction of this moduli space follows [4, 9, 10].

Let \( Ein \) denote the space of all Einstein metrics on \( X \) with unit volume given the \( C^\infty \)-topology. Every Einstein metric on \( X \) is a hyper-kähler metric [12] and in particular, Ricci flat. Then \( \text{Diff}(X) \) acts on \( Ein \) by pullback. That is, for each \( \varphi \in \text{Diff}(X) \) we define
\[ \varphi_* : Ein \to Ein, \quad \varphi_*(g) = (\varphi^{-1})^*(g). \]

Let \( \text{TDiff}(X) \) denote the subgroup of \( \text{Diff}(X) \) consisting of those diffeomorphisms which act trivially on \( H^2(X; \mathbb{Z}) \). Let \( \text{Aut}(H^2(X; \mathbb{Z})) \) be the group of automorphisms of the intersection form. Then we have a short exact sequence
\[ 1 \to \text{TDiff}(X) \to \text{Diff}(X) \to \Gamma \to 1, \]
where \( \Gamma \subset \text{Aut}(H^2(X; \mathbb{Z})) \) is the subgroup of automorphisms that are induced by diffeomorphisms of \( X \). Note that since \( \Gamma \) is discrete, the identity components of \( \text{TDiff}(X) \) and \( \text{Diff}(X) \) are the same
\[ \text{TDiff}_0(X) = \text{Diff}_0(X). \quad (3.1) \]
As a consequence of the global Torelli theorem for $K3$ surfaces, one finds that $\text{TDiff}(X)$ acts freely and properly on $\text{Ein}$ [9, §4]. Let

$$T_{\text{Ein}} = \text{Ein}/\text{TDiff}(X)$$

be the quotient. Over $\text{Ein}$ we have the constant family $X \times \text{Ein} \to \text{Ein}$. We can equip the vertical tangent space of $X \times \text{Ein}$ with the tautological metric, namely the metric on the fibre $X \times \{g\}$ is $g$ itself. The action of $\text{Diff}(X)$ on $\text{Ein}$ lifts to $X \times \text{Ein}$ by setting

$$\varphi_*(x, g) = (\varphi(x), \varphi_*(g)).$$

It is easily checked that this action preserves the fibrewise metric. Let $E_{\text{Ein}} = (X \times \text{Ein})/\text{TDiff}(X)$ be the quotient. Over $\text{Ein}$ we have the constant family $X \times \text{Ein} \to \text{Ein}$. We can equip the vertical tangent space of $X \times \text{Ein}$ with the tautological metric, namely the metric on the fibre $X \times \{g\}$ is $g$ itself. The action of $\text{Diff}(X)$ on $\text{Ein}$ lifts to $X \times \text{Ein}$ by setting

$$\varphi_*(x, g) = (\varphi(x), \varphi_*(g)).$$

It is easily checked that this action preserves the fibrewise metric. Let $E_{\text{Ein}} = (X \times \text{Ein})/\text{TDiff}(X)$ be the quotient of $X \times \text{Ein}$ by the action of $\text{TDiff}(X)$.

**Lemma 3.1** $E_{\text{Ein}}$ is a locally trivial smooth family over $T_{\text{Ein}}$ with fibres diffeomorphic to $X$.

**Proof** Let $\text{Met}$ denote the space of all metrics on $X$ with the $C^\infty$ topology. The Ebin slice theorem [8, Theorem 7.4], [7, Slice Theorem] implies that $\text{Met}/\text{TDiff}(X)$ has the structure of an infinite dimensional orbifold. Namely if $g \in \text{Met}$ and $S_g \subset \text{Met}$ is a local slice around $g$ as given by the Ebin slice theorem, then the isometry group $I(g)$ of $g$ preserves $S_g$ and the quotient $S_g/(I(g) \cap \text{TDiff}(X))$ may be identified with a neighbourhood of $[g] \in \text{Met}/\text{TDiff}(X)$. In particular, the isotropy group of $[g]$ is $I(g) \cap \text{TDiff}(X)$. The quotient $U = (X \times \text{Met})/\text{TDiff}(X)$ is a smooth orbifold bundle over $\text{Met}/\text{TDiff}(X)$. Away from the singular points of $\text{Met}/\text{TDiff}(X)$, it is a smooth fibre bundle with fibre $X$. The inclusion $\text{Ein} \to \text{Met}$ descends to an inclusion $t : T_{\text{Ein}} \to \text{Met}/\text{TDiff}(X)$ whose image is disjoint from the singularities. This is because if $g$ is an Einstein metric on $X$, then $I(g) \cap \text{TDiff}(X) = 1$ [9, Lemma 4.4]. The deformation theory of Einstein metrics around Kähler–Einstein metrics [13, Theorem 10.5, Corollary 3.5] implies that $T_{\text{Ein}}$ is a finite-dimensional, smoothly embedded submanifold of $\text{Met}/\text{TDiff}(X)$. So the pullback $E_{\text{Ein}} = t^*U$, is a smooth, locally trivial fibre bundle over $T_{\text{Ein}}$ with fibres diffeomorphic to $X$. \[\square\]

The tautological metric on $X \times \text{Ein}$ descends to a metric $g_{\text{Ein}}$ on the vertical tangent bundle such that the restriction of $g_{\text{Ein}}$ to the fibre of $E_{\text{Ein}}$ over $[g] \in T_{\text{Ein}}$ is a representative of the isomorphism class of Einstein metrics $[g]$. Thus $E_{\text{Ein}}$ is a family of Einstein metrics on $X$.

Let $Gr_3(\mathbb{R}^{3,19})$ denote the Grassmannian of positive definite 3-planes in $\mathbb{R}^{3,19}$. There is a period map

$$P : T_{\text{Ein}} \to Gr_3(\mathbb{R}^{3,19})$$

defined as follows. Fix an isometry $H^2(X; \mathbb{R}) \cong \mathbb{R}^{3,19}$. Then $P$ sends an Einstein metric $g$ to the 3-plane $H^+_g(X)$. Let

$$\Delta = \{\delta \in H^2(X; \mathbb{Z}) \mid \delta^2 = -2\}$$

\[\square\] Springer
and set

\[ W = \{ H \in Gr_3(\mathbb{R}^{3,19}) \mid H^\perp \cap \Delta = \emptyset \}. \]

The Grassmannian \( Gr_3(\mathbb{R}^{3,19}) \) is a contractible manifold (it is a symmetric space of non-compact type) and for each \( \delta \in \Delta \), the subset

\[ A_\delta = \{ H \in Gr_3(\mathbb{R}^{3,19}) \mid \delta \in H^\perp \} \]

is a codimension 3 embedded submanifold and \( W = Gr_3(\mathbb{R}^{3,19}) \setminus \bigcup_{\delta \in \Delta} A_\delta \). It follows from the global Torelli theorem for \( K3 \) that the period map \( P \) is a homeomorphism of \( T_{Ein} \) to the set \( W \) [5, Chapter 12, K].

Let \( Gr_3^+(\mathbb{R}^{3,19}) \) denote the set of pairs \((H, \sigma)\) where \( H \in Gr_3(\mathbb{R}^{3,19}) \) is a positive definite 3-plane and \( \sigma \) is an orientation on \( H \). The forgetful map \( Gr_3^+(\mathbb{R}^{3,19}) \to Gr(\mathbb{R}^{3,19}) \) is a double covering space. However \( Gr_3(\mathbb{R}^{3,19}) \) is contractible and so \( Gr_3^+(\mathbb{R}^{3,19}) \) is the trivial double covering consisting of two copies of \( Gr_3(\mathbb{R}^{3,19}) \). Let \( g \) be an Einstein metric on \( X \). Then \( g \) is a hyper-Kähler metric with holonomy group equal to \( Sp(1) \). Let \( I, J, K \) be a hyper-Kähler triple of complex structures for \( g \). Let \( \omega_I, \omega_J, \omega_K \) be the corresponding Kähler forms. Then \( \{ \omega_I, \omega_J, \omega_K \} \) defined an oriented basis for \( H_g^+(X) \). Since \( g \) has holonomy \( Sp(1) \), the triple \( \omega_I, \omega_J, \omega_K \) is determined up to an \( SO(3) \) transformation. Hence we obtain a canonical orientation on \( H_g^+(X) \). Furthermore, if two Einstein metrics \( g, g' \) define the same point in the moduli space \( T_{Ein} \) so that \( H_g^+(X) = H_{g'}^+(X) \), then the induced orientations are the same [4, Lemma 2.1]. This means that the period map \( P : T_{Ein} \to Gr_3(\mathbb{R}^{3,19}) \) admits a canonical lift \( \tilde{P} : T_{Ein} \to Gr_3^+(\mathbb{R}^{3,19}) \). Furthermore since \( T_{Ein} \) is path-connected [4, Page 4], it follows that the image of \( \tilde{P} \) is contained in a single component of \( Gr_3^+(\mathbb{R}^{3,19}) \). Let us denote this component by \( Gr_3'(\mathbb{R}^{3,19}) \). The forgetful map \( Gr_3^+(\mathbb{R}^{3,19}) \to Gr(\mathbb{R}^{3,19}) \) restricted to \( Gr_3'(\mathbb{R}^{3,19}) \) gives an homeomorphism \( Gr_3'(\mathbb{R}^{3,19}) \cong Gr_3(\mathbb{R}^{3,19}) \) and in this way, every \( H \in Gr_3(\mathbb{R}^{3,19}) \) inherits an orientation.

Choose an element \( v \in H^2(X; \mathbb{R}) \) such that \( \langle v, \delta \rangle \neq 0 \) for all \( \delta \in \Delta \) and define \( \Delta^\pm = \{ \delta \in \Delta \mid \pm \langle v, \delta \rangle > 0 \} \). Then

\[ \Delta = \Delta^+ \cup \Delta^- \]

and \( \delta \in \Delta^+ \) if and only if \( -\delta \in \Delta^- \).

Let \( \delta \in \Delta^+ \) and choose a point \( p \in A_\delta \) such that \( p \) does not lie on any \( A_{\delta'} \) for \( \delta' \in \Delta^+ \) other than \( \delta \) (each \( A_{\delta'} \) intersects \( A_\delta \) in a closed embedded submanifold of positive codimension. The set \( \Delta \) is countable, so \( \{ A_{\delta'} \}_{\delta' \in \Delta \setminus \{ \pm \delta \} } \) does not cover \( A_\delta \). Let \( H \subset H^2(X; \mathbb{R}) \) be the positive definite 3-plane corresponding to \( p \). As explained above \( H \) can be given a canonical orientation. Choose an oriented basis \( \theta_1, \theta_2, \theta_3 \) for \( H \) satisfying \( \langle \theta_i, \theta_j \rangle = \delta_{ij} \). Since \( p \in A_\delta \), we have \( \langle \theta_j, \delta \rangle = 0 \) for \( j = 1, 2, 3 \). Moreover, since \( p \) does not lie on \( A_{\delta'} \) for any \( \delta' \in \Delta^+ \) not equal to \( \delta \), we have \( \langle \theta_j, \delta' \rangle \neq 0 \) for some \( j \).
Let $B = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$ be the unit 2-sphere and choose an $\epsilon \in (0, 1)$. Consider the map

$$f_\delta : S^2 \to Gr_3(\mathbb{R}^{3,19})$$

defined by

$$f_\delta(x_1, x_2, x_3) = \text{span}(\omega_1, \omega_2, \omega_3)$$

where for $i = 1, 2, 3$, we set

$$\omega_i = \theta_i - \epsilon x_i \delta / 2.$$ 

We choose $\epsilon$ sufficiently small so that $f_\delta(x_1, x_2, x_3)$ is a positive definite subspace of $H^2(X; \mathbb{R})$.

**Lemma 3.2** If $\epsilon$ is sufficiently small then $f_\delta(x_1, x_2, x_3)$ does not lie on $A_{\delta'}$ for any $\delta' \in \Delta^+ \setminus \{\delta\}$.

**Proof** Consider the decomposition $\mathbb{R}^{3,19} \cong H \oplus K$, where $K = H^\perp$. Then any $x \in \mathbb{R}^{3,19}$ uniquely decomposes as $x = x_H + x_K$, where $x_H \in H$, $x_K \in K$. Let $\alpha \in \Delta^+ \setminus \{\delta\}$. We first show that if $\epsilon^2 < 4(\alpha_H)^2 / (2(\alpha_H^2 + 2))$, then $f_\delta(x_1, x_2, x_3) \notin A_{\alpha}$ for any $(x_1, x_2, x_3) \in S^3$. To see this, note that $f_\delta(x_1, x_2, x_3) \in A_{\alpha}$ if and only if $\langle \omega_i, \alpha \rangle = 0$ for $i = 1, 2, 3$. But

$$\langle \omega_i, \alpha \rangle = \langle \theta_i, \alpha \rangle - \frac{1}{2} \epsilon x_i \langle \delta, \alpha \rangle.$$ 

Let $v, w \in \mathbb{R}^3$ be given by $v = (\langle \theta_1, \alpha \rangle, \langle \theta_2, \alpha \rangle, \langle \theta_3, \alpha \rangle)$ and $w = \frac{1}{2} \epsilon (\langle \delta, \alpha \rangle)(x_1, x_2, x_3)$. Then $f_\delta(x_1, x_2, x_3) \in A_{\alpha}$ if and only if $v = w$. Let $|| \cdot ||$ denote the standard norm on $\mathbb{R}^3$. Then if $||w|| < ||v||$, it follows that $f_\delta(x_1, x_2, x_3) \notin A_{\alpha}$. But

$$||v||^2 = \langle \theta_1, \alpha \rangle^2 + \langle \theta_2, \alpha \rangle^2 + \langle \theta_3, \alpha \rangle^2 = \alpha_H^2$$

and

$$||w||^2 = \frac{1}{4} \epsilon^2 (\delta, \alpha)^2 (x_1^2 + x_2^2 + x_3^2) = \frac{1}{4} \epsilon^2 (\delta, \alpha)^2.$$ 

Recall that $\delta \in H^\perp$, hence $\langle \alpha, \delta \rangle = \langle \alpha_K, \delta \rangle$. Now since $\alpha_K, \delta$ lie in the negative definite space $H^\perp$, we can apply Cauchy–Schwarz to $-\langle , \rangle$ on $H^\perp$ to deduce that

$$\langle \alpha, \delta \rangle^2 = \langle \alpha_K, \delta \rangle^2 \leq \sqrt{|\alpha_K^2||\delta^2|} = \sqrt{2|\alpha_K^2|}.$$ 

Then since $\alpha^2 = -2 = \alpha_H^2 - |\alpha_K^2|$, we get

$$\langle \alpha, \delta \rangle^2 \leq \sqrt{2(\alpha_H^2 + 2)}$$

and thus
\[
||w||^2 \leq \frac{1}{4} \epsilon^2 \sqrt{2(\alpha_H^2 + 2)}.
\]
Hence if \( \epsilon^2 < 4(\alpha_H^2)/\sqrt{2(\alpha_H^2 + 2)} \), then \( ||w||^2 < ||v||^2 \) and \( f_\delta(x_1, x_2, x_3) \not\in A_\alpha \).

Let \( g(t) = 4t/\sqrt{2(t+2)} \). This is an increasing function on \([0, \infty)\). Suppose \( \epsilon^2 < g(1) = 4/\sqrt{6} \). Then \( f_\delta(x_1, x_2, x_3) \not\in A_\alpha \) for all \( \alpha \) such that \( \alpha_H^2 > 1 \). On the other hand, we claim that there are only finitely many \( \alpha \in \Delta^+ \backslash \{ \delta \} \) with \( \alpha_H^2 < 1 \). To see this, note that
\[
-2 = \alpha^2 = \alpha_H^2 - \alpha_K^2,
\]
so \( |\alpha_K^2| = \alpha_H^2 + 2 < 3 \) and \( \alpha_H^2 + |\alpha_K^2| < 4 \). But the map \( x \mapsto ||x|| = x_H^2 + |x_K^2| \) defines a norm on \( \mathbb{R}^3 \). So the set \( \{ \alpha \in \Delta^+ \backslash \{ \delta \} \mid \alpha_H^2 < 1 \} \) is closed and bounded, hence compact. It is also discrete, hence finite. Thus by choosing \( \epsilon \) such that \( \epsilon^2 < 4/\sqrt{6} \) and such that \( \epsilon^2 < g(\alpha_H^2) \) for the finitely many \( \alpha \in \Delta^+ \backslash \{ \delta \} \) with \( \alpha_H^2 < 1 \), we have that \( f_\delta(x_1, x_2, x_3) \) does not lie on \( A_\delta' \) for any \( \delta' \in \Delta^+ \backslash \{ \delta \} \).

By the Lemma, we may choose an \( \epsilon \) such that \( f_\delta(x_1, x_2, x_3) \) does not lie on \( A_\delta' \) for any \( \delta' \in \Delta^+ \) not equal to \( \delta \). Moreover, we have
\[
(\langle \omega_1, \delta \rangle, \langle \omega_2, \delta \rangle, \langle \omega_3, \delta \rangle) = (\epsilon(x_1, x_2, x_3)).
\]
Then since \( (x_1, x_2, x_3) \in S^2 \), we see that at least one of \( \langle \omega_1, \delta \rangle, \langle \omega_2, \delta \rangle, \) and \( \langle \omega_3, \delta \rangle \) must be non-zero. So \( f(x_1, x_2, x_3) \) does not lie on \( A_\delta \). It follows that \( f \) maps to \( W \).

So there is a map \( g_\delta : B \to T_{Ein} \) such that \( f_\delta = P \circ g_\delta \), namely \( g_\delta = P^{-1} \circ f_\delta \).

**Definition 3.3** Let \( E_\delta = g_\delta^*(E_{Ein}) \) be the pullback of the family \( E_{Ein} \to T_{Ein} \) by the map \( g_\delta : S^2 \to T_{Ein} \). This is a smooth family of \( K3 \) surfaces over \( S^2 \).

**Remark 3.4** In addition to \( \delta \), the map \( g_\delta : S^2 \to T_{Ein} \) depends on a choice of \( p \in A_\delta \backslash (A_\delta \cap (\cup_{\delta' \in \Delta^+ \backslash \{ \delta \}} A_{\delta'})) \), a sufficiently small \( \epsilon > 0 \) and a choice of oriented orthonormal basis \( \{ (\theta_1, \theta_2, \theta_3) \} \). We will show this space is path-connected. Since moving \( (p, \epsilon, (\theta_1, \theta_2, \theta_3)) \) along a path only changes \( g_\delta \) by isotopy, this will imply that that family \( E_\delta \) is well defined up to isomorphism. To see connectedness, first note that \( A_\delta \backslash (A_\delta \cap (\cup_{\delta' \in \Delta^+ \backslash \{ \delta \}} A_{\delta'})) \) is path-connected. Any two points \( p_0, p_1 \) can be joined by a smooth path \( p_t \) valued in \( A_\delta \). By \([11, \text{Theorem 2.5}]\) we can assume \( p_t \) is transverse to the countably many codimension 3 submanifolds \( \{ A_{\delta'} \cap A_{\delta'} \mid \delta' \in \Delta^+ \backslash \{ \delta \} \} \) in \( A_\delta \). Since \( p_t \) is 1-dimensional, transversality means \( p_t \) is disjoint from each \( A_\delta \cap A_{\delta'} \) and hence \( p_t \) is a path in \( p \in A_\delta \backslash (A_\delta \cap (\cup_{\delta' \in \Delta^+ \backslash \{ \delta \}} A_{\delta'})) \). Next, using compactness of \([0, 1]\) we can find an \( \epsilon \) such that \( (p_t, \epsilon) \) is in the space of pairs for all \( t \). Lastly, the space of all triples \( (p, \epsilon, (\theta_1, \theta_2, \theta_3)) \) is a principal \( SO(3) \)-bundle over the space of pairs \( (p, \epsilon) \), so it is also path-connected.

**Remark 3.5** By construction, the family \( E_{Ein} \to T_{Ein} \) has structure group \( \text{TDiff}(X) \), hence the same is true of the pullback family \( E_\delta \). To say that \( E_\delta \) has structure group \( \text{TDiff}(X) \) amounts to saying that \( E_\delta \) is equipped with a trivialisation \( \mathcal{H}^2(X) \cong H^2(X; \mathbb{R}) \times B \) of the local system \( \mathcal{H}^2(X) \).

For each \( \delta \in \Delta^+ \) we have constructed a homotopy class of map \( g_\delta : S^2 \to T_{Ein} \). As \( T_{Ein} \) is simply connected \([4, \text{Page 4}]\), there is a bijection between unbased homotopy classes of maps \( S^2 \to T_{Ein} \) and the homotopy group \( \pi_2(T_{Ein}) \). Hence \( g_\delta \) defines a

\[ Springer \]
Non-trivial smooth families of $K3$ surfaces

Now since $T_{Ein} = Ein/TDiff(X)$, the long exact sequence of homotopy groups gives a map

$$\partial : \pi_2(T_{Ein}) \to \pi_1(TDiff_0(X)) = \pi_1(Diff_0(X))$$

where the second equality is by Equation (3.1). In particular, we may define $h_\delta = \partial[g_\delta] \in \pi_1(Diff_0(X))$. Applying the clutching construction to $h_\delta$, we recover the family $E_\delta$.

Since $T_{Ein}$ is simply connected, the Hurwitz theorem gives an isomorphism $\pi_2(T_{Ein}) = H_2(T_{Ein}; \mathbb{Z})$. From [9, Lemma 5.3], we have

$$H_2(T_{Ein}; \mathbb{Z}) = \bigoplus_{\delta \in \Delta^+} \mathbb{Z}[g_\delta].$$

Hence $\pi_2(T_{Ein})$ is a free abelian group with generators the maps $\{[g_\delta]\}_{\delta \in \Delta^+}$.

Recall that a $K3$ surface satisfies

$$b^+(X) = 3, \sigma(X) = -16, b_1(X) = 0.$$  

Since $X$ is spin and simply-connected, for each $u \in H^2(X; \mathbb{Z})$, there is a uniquely determined spin$^c$-structure $s_u$ for which $c_1(s_u) = 2u$. Then

$$d(X, s_u) = \frac{(2u)^2 + 16}{4} - 1 - 3 = u^2.$$  

Let $\alpha \in \Delta$. Then $\alpha^2 = -2$ and hence $d(X, s_\alpha) = -2$. Therefore we have the Seiberg–Witten invariant

$$sw_{s_\alpha} : \pi_1(Diff_0(X)) \to \mathbb{Z}.$$  

To simplify notation we will write $sw_{s_\alpha}$ for $sw_{s_\alpha}$. If $\alpha \in \Delta$, then from Proposition 2.8, we have

$$sw_{-\alpha} = -sw_{\alpha}.$$  

For this reason, it suffices to only consider the homomorphisms $sw_{s_\alpha}$ for $\alpha \in \Delta^+$. From Theorem 2.9, we have that for each $f \in \pi_1(Diff(X))$, $sw_{s_\alpha}(f)$ is non-zero for only finitely many $\alpha \in \Delta^+$. Therefore, we obtain a homomorphism

$$sw : \pi_1(Diff_0(X)) \to \bigoplus_{\alpha \in \Delta^+} \mathbb{Z}, \quad sw(f) = \bigoplus_{\alpha \in \Delta^+} sw_{s_\alpha}(f).$$

Recall that we constructed classes $h_\delta = \partial[g_\delta] \in \pi_1(Diff_0(X))$ such that the family $E_\delta$ is obtained from the clutching construction applied to $h_\delta$.
Theorem 3.6 Let $\alpha, \delta \in \Delta^+$. Then

$$sw_{\alpha}(h_{\delta}) = \begin{cases} 1 & \text{if } \alpha = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof By definition, $sw_{\alpha}(h_{\alpha})$ is the Seiberg–Witten invariant of $(E_{\delta}, s_{\alpha})$ with respect to the canonical chamber, where $E_{\delta}$ is the family obtained by the clutching construction applied to $h_{\delta}$.

We recall the construction of $E_{\delta}$. Choose a point $p \in A_{\delta}$ such that $p$ does not lie on any $A_{\delta'}$ for $\delta' \in \Delta^+$ other than $\delta$. Let $H \subset H^2(X; \mathbb{R})$ be the positive definite 3-plane corresponding to $p$. Choose a basis $\theta_1, \theta_2, \theta_3$ for $H$ satisfying $\langle \theta_i, \theta_j \rangle = \delta_{ij}$. Since $p \in A_{\delta}$, we have $\langle \theta_j, \delta \rangle = 0$ for $j = 1, 2, 3$. Let $B = S^2$ be the unit sphere in $\mathbb{R}^3$. We take $E_{\delta} \to S^2$ to be the pullback of the family $E_{Ein} \to T_{Ein}$ by a map $g_{\delta} : S^2 \to T_{Ein}$. Let $P : T_{Ein} \to Gr_3(\mathbb{R}^3, 19)$ be the period map. Then $f_{\delta} = P \circ g_{\delta} : S^2 \to Gr_3(\mathbb{R}^3, 19)$ is defined as

$$f_{\delta}(x_1, x_2, x_3) = \text{span}(\omega_1, \omega_2, \omega_3),$$

where $\epsilon > 0$ is sufficiently small and

$$\omega_i = \theta_i - \epsilon x_i \delta / 2, \quad \text{for } i = 1, 2, 3.$$

Let $\rho_H : H^2(X; \mathbb{R}) \to H$ be the projection to $H$ with kernel $H^\perp$. Then since $\theta_1, \theta_2, \theta_3 \in H$ and $\delta \in H^\perp$, we have

$$\rho_H(\theta_i) = \theta_i, \quad \rho_H(\delta) = 0.$$

In particular, this gives

$$\rho_H(\omega_i) = \theta_i.$$

Let $H^+_g(X) \to S^2$ be the bundle of harmonic self-dual 2-forms of the family $E_{\delta}$. Then $\omega_1, \omega_2, \omega_3$ is a frame for $H^+_g(X)$. Let $\varphi : H^+_g(X) \to H$ be the inclusion into $H^2(X; \mathbb{R})$ followed by $\rho_H$. Define $w(\alpha)$ to be the section of $H$ given by

$$w(\alpha)(b) = \varphi(2\pi c_1(s_{\alpha})^{+s_b}) = \varphi(4\pi \alpha^{+s_b}).$$

Let $\omega_1^*, \omega_2^*, \omega_3^*$ be the dual frame of $H^+_g$, defined by the condition

$$\langle \omega_i, \omega_j^* \rangle = \delta_{ij},$$

where $\langle \ , \ \rangle$ is the intersection pairing on $H^2(X; \mathbb{R})$. From $\omega_i = \theta_i - \epsilon x_i \delta / 2$, one finds

$$\langle \omega_i, \omega_j \rangle = \delta_{ij} - \epsilon^2 x_i x_j / 2.$$

One can then directly check that the dual frame is given by
\[ \omega_i^* = \omega_i + \mu x_i (x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3), \quad (3.2) \]

where

\[ \mu = \frac{\epsilon^2/2}{1 - \epsilon^2/2}. \]

We have

\[ \alpha^+ = (\alpha, \omega_1^*) \omega_1 + (\alpha, \omega_2^*) \omega_2 + (\alpha, \omega_3^*) \omega_3. \]

Applying \( \rho_H \), we get

\[ w(\alpha) = 4\pi \left( (\alpha, \omega_1^*) \theta_1 + (\alpha, \omega_2^*) \theta_2 + (\alpha, \omega_3^*) \theta_3 \right). \]

We use the basis \( \theta_1, \theta_2, \theta_3 \) to identify \( H \) with \( \mathbb{R}^3 \). Then

\[ w(\alpha) = 4\pi \left( (\alpha, \omega_1^*), (\alpha, \omega_2^*), (\alpha, \omega_3^*) \right). \]

Suppose that \( \alpha \neq \delta \). Then since \( p \) does not lie on \( A_\alpha \), we have \( \langle \theta_j, \alpha \rangle \neq 0 \) for some \( j \). From (3.2), we get

\[ w(\alpha) = 4\pi \left( (\alpha, \theta_1), (\alpha, \theta_2), (\alpha, \theta_3) \right) + O(\epsilon) \]

where \( O(\epsilon) \) denotes terms of order \( \epsilon \). Then (for sufficiently small \( \epsilon \)) it follows that \( w(\alpha) \) is non-vanishing and that \( \text{deg}(w(\alpha)) = 0 \). More precisely, since \( w \) is non-vanishing, it defines a map \( w/|w| : S^3 \to S(H) \) where \( S(H) \) is the unit sphere in \( H \) and this map has degree 0. Here we give \( S(H) \) the induced orientation (recall that \( H \) has a canonical orientation). The map has degree 0 because there is a homotopy from \( w \) to a constant map, given by contracting the \( O(\epsilon) \) term to zero.

Suppose instead that \( \alpha = \delta \). In this case we find

\[ w(\alpha) = w(\delta) = 4\pi \epsilon (x_1, x_2, x_3) + O(\epsilon^2). \]

So (for sufficiently small \( \epsilon \)) \( w(\alpha) \) is non-vanishing and \( \text{deg}(w(\alpha)) = 1 \) (since the map \( (x_1, x_2, x_3) \mapsto 4\pi \epsilon (x_1, x_2, x_3) \) has degree 1). In all cases, we see that \( w(\alpha) \) is non-vanishing.

The family \( E_{Ein} \) has a fibrewise metric \( g_{Ein} \) which is a Ricci flat Einstein metric on each fibre. Let \( g = g_\delta^* (g_{Ein}) \) be the fibrewise metric on \( E_\delta \) obtained by pullback. Then for each \( b \in S^2 \), the metric \( g_b \) is Ricci flat and in particular has zero scalar curvature. Now consider the families Seiberg–Witten moduli space \( \mathcal{M}(E_\delta, s_\alpha, g, 0) \) for the zero perturbation \( \eta = 0 \).

Suppose that \( (A, \psi) \in \mathcal{M}(E_\delta, s_\alpha, g, 0) \) is a solution to the Seiberg–Witten equations in the family. Then \( (A, \psi) \) is the solution of the Seiberg–Witten equations on some fibre \( X_b \) with metric \( g_b \) and zero perturbation. The Weitzenböck formula together
with the Seiberg–Witten equations and the fact that $g_\delta$ has zero scalar curvature implies that $\psi = 0$ ([16, Corollary 2.2.6]). So every solution in $\mathcal{M}(E_\delta, s_\alpha, g, 0)$ is reducible.

On the other hand, since $w(\alpha)$ is non-vanishing, the perturbation $\eta = 0$ does not lie on the wall, that is, there are no reducible solutions. So the moduli space $\mathcal{M}(E_\delta, s_\alpha, g, 0)$ is empty. Recall that in the proof of Theorem 2.9, we used the wall crossing formula to deduce the identity

$$\int_{S^{n+1}} SW(s, \eta) = sw_\delta(f) - \deg(\phi_{\delta, \eta})$$

(see Equation (2.2)). Taking $s = s_\alpha$, $f = h_\delta$ and $\eta = 0$, we obtain

$$\int_{S^{n+1}} SW(s_\alpha, 0) = sw_\alpha(h_\delta) - \deg(w(\alpha)).$$

But $\mathcal{M}(E_\delta, s_\alpha, g, 0)$ is empty, so $SW(s_\alpha, 0) = 0$ and hence

$$sw_\alpha(h_\delta) = \deg(w(\alpha)).$$

Further, we have already shown that

$$\deg(w(\alpha)) = \begin{cases} 1 & \text{if } \alpha = \delta, \\ 0 & \text{otherwise}. \end{cases}$$

$$\square$$

As an immediate consequence of Theorem, 3.6, we have:

**Theorem 3.7** The homomorphism

$$sw : \pi_1(\text{Diff}_0(X)) \to \bigoplus_{\alpha \in \Delta^+} \mathbb{Z}$$

is surjective, hence $\pi_1(\text{Diff}_0(X))$ contains $\bigoplus_{\alpha \in \Delta^+} \mathbb{Z}$ as a direct summand (recall that $\pi_1(\text{Diff}_0(X))$ is abelian).

**Theorem 3.8** The boundary map

$$\partial : \pi_2(T_{Ein}) \to \pi_1(\text{TDiff}_0(X)) = \pi_1(\text{Diff}_0(X))$$

induced by the fibration $Ein \to Ein/\text{TDiff}(X) = T_{Ein}$ admits a left inverse, given by

$$\pi_1(\text{Diff}_0(X)) \to \pi_2(T_{Ein}), \quad x \mapsto \bigoplus_{\alpha} sw_\alpha(x)[g_\alpha].$$
Proof  Recall that
\[ \pi_2(T_{Ein}) \cong H_2(T_{Ein}; \mathbb{Z}) \cong \bigoplus_{\alpha \in \Delta^+} \mathbb{Z}[g_{\alpha}] \]
and that \( \partial[g_{\alpha}] = h_{\alpha} \). Hence, if we define \( t : \pi_1(\text{Diff}_0(X)) \to \pi_2(T_{Ein}) \) to be given by \( t(x) = \bigoplus_{\alpha \in \Delta^+ + \mathbb{Z}} s_{\alpha}(x)[g_{\alpha}] \). Using Theorem 3.6, it follows that \( t \circ \partial = \text{id} \), so that \( t \) is a left inverse of \( \partial \), as claimed. \( \square \)

From the homeomorphism \( P : T_{Ein} \to W = \text{Gr}_3(\mathbb{R}^3) \setminus \bigcup_{\delta \in \Delta} A_\delta \), we see that \( T_{Ein} \) is connected. Then since \( T_{Ein} = Ein / \text{TDiff}(X) \), it follows that \( \text{TDiff}(X) \) acts transitively on the connected components of \( Ein \) and that the components of \( Ein \) are all homeomorphic to each other. Choose arbitrarily a basepoint \( p \in Ein \). Since the components of \( Ein \) are all homeomorphic, the isomorphism class of \( \pi_1(Ein, p) \) does not depend on the choice of \( p \) and we simply write \( \pi_1(Ein) \). From the long exact sequence in homotopy groups associated to \( Ein \to T_{Ein} \) we get an exact sequence
\[ \cdots \to \pi_2(T_{Ein}) \xrightarrow{\partial} \pi_1(\text{TDiff}_0(X)) \to \pi_1(Ein) \to \pi_1(T_{Ein}). \]
We have also seen that \( T_{Ein} \) is simply-connected and that \( \partial \) admits a left inverse, so we obtain an isomorphism
\[ \pi_1(\text{TDiff}_0(X)) \cong \pi_1(\text{Diff}_0(X)) \cong \left( \bigoplus_{\alpha \in \Delta^+ + \mathbb{Z}} \mathbb{Z} \right) \oplus \pi_1(Ein), \]
where the summand \( \bigoplus_{\alpha \in \Delta^+} \mathbb{Z} \) is detected by the Seiberg–Witten invariants \( s_{\alpha} \).

Remark 3.9  Smooth families over \( S^2 \) with fibres diffeomorphic to \( X \) correspond, via the clutching construction, to elements of \( \pi_1(\text{Diff}_0(X)) \) considered modulo the conjugation action of \( \text{Diff}(X) \). For this reason we are interested in the action of \( \text{Diff}(X) \) on \( \pi_1(\text{Diff}_0(X)) \). The Seiberg–Witten invariants are compatible with this action in the following sense. Let \( \{ e_\alpha \}_{\alpha \in \Delta^+} \) denote the standard basis for \( \bigoplus_{\alpha \in \Delta^+} \mathbb{Z} \). For \( f \in \text{Diff}(X) \) and \( x \in H^2(X; \mathbb{R}) \), let us write \( f_\ast(x) = (f^{-1})^\ast(x) \) so that \( (f, x) \mapsto f_\ast(x) \) is a left action. Let \( \text{Diff}(X) \) act on \( \bigoplus_{\alpha \in \Delta^+} \mathbb{Z} \) by setting
\[ f \cdot e_\alpha = \begin{cases} e_{f_\ast \alpha} & \text{if } f_\ast \alpha \in \Delta^+, \\ -e_{-f_\ast \alpha} & \text{if } f_\ast \alpha \in \Delta^- . \end{cases} \]
Then it follows easily from Propositions 2.7 and 2.8 that the map \( s_{\alpha} : \pi_1(\text{Diff}_0(X)) \to \bigoplus_{\alpha \in \Delta^+} \mathbb{Z} \) is \( \text{Diff}(X) \)-equivariant.

Data availability  Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest  The author states that there are no conflicts of interest.
References

1. Baraglia, D.: Obstructions to smooth group actions on 4-manifolds from families Seiberg–Witten theory. Adv. Math. 354, 106730 (2019)
2. Baraglia, D., Konno, H.: A gluing formula for families Seiberg–Witten invariants. Geom. Topol. 24(3), 1381–1456 (2020)
3. Baraglia, D., Konno, H.: On the Bauer–Furuta and Seiberg–Witten invariants of families of 4-manifolds. J. Topol. 15(2), 505–586 (2022)
4. Baraglia, D., Konno, H.: A note on the Nielsen realization problem for K3 surfaces. Proc. Am. Math. Soc. arXiv:1908.11613 (2019) (to appear)
5. Besse, A.L.: Einstein manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10. Springer, Berlin (1987). xii+510 pp
6. Bustamante, M., Kranich, M., Kupers, A.: Finiteness properties of automorphism spaces of manifolds with finite fundamental group. arXiv:2103.13468 (2021)
7. Corro, D., Kordass, J.-B.: Short survey on the existence of slices for the space of Riemannian metrics. In: Proceedings of the IV Meeting of Mexican Mathematicians Abroad 2018. arXiv:1904.07031 (2019) (to appear)
8. Ebin, D.G.: The manifold of Riemannian metrics. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 11–40. Amer. Math. Soc., Providence
9. Giansiracusa, J.: The diffeomorphism group of a K3 surface and Nielsen realization. J. Lond. Math. Soc. (2) 79(3), 701–718 (2009) [Corrigendum: https://doi.org/10.1112/jlms.12194]
10. Giansiracusa, J., Kupers, A., Tshishiku, B.: Characteristic classes of bundles of K3 manifolds and the Nielsen realization problem. Tunis. J. Math. 3(1), 75–92 (2021)
11. Hirsch, M.W.: Differential topology. Graduate Texts in Mathematics, vol. 33. Springer, New York (1976). x+221 pp
12. Hitchin, N.: Compact four-dimensional Einstein manifolds. J. Differ. Geom. 9, 435–441 (1974)
13. Kiso, N.: Einstein metrics and complex structures. Invent. Math. 73(1), 71–106 (1983)
14. Li, T.-J., Liu, A.-K.: Family Seiberg–Witten invariants and wall crossing formulas. Commun. Anal. Geom. 9(4), 777–823 (2001)
15. Müller, C., Wockel, C.: Equivalences of smooth and continuous principal bundles with infinite-dimensional structure group. Adv. Geom. 9(4), 605–626 (2009)
16. Nicolaescu, L.I.: Notes on Seiberg–Witten theory. Graduate Studies in Mathematics, vol. 28. American Mathematical Society, Providence (2000). xviii+484 pp
17. Ruberman, D.: An obstruction to smooth isotopy in dimension 4. Math. Res. Lett. 5(6), 743–758 (1998)
18. Ruberman, D.: A polynomial invariant of diffeomorphisms of 4-manifolds. In: Proceedings of the Kirbyfest (Berkley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry (1999), pp. 473–488 (electronic)
19. Smirnov, G.: From flops to diffeomorphism groups. arXiv:2002.01233 (2020)
20. Steenrod, N.: The Topology of Fibre Bundles. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton (1951). viii+224 pp
21. Watanabe, T.: Some exotic nontrivial elements of the rational homotopy groups of Diff(S^4). arXiv:1812.02448v3 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.