On the Post-linear Quadrupole-Quadrupole Metric

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June 24, 2016

Abstract

The Hartle-Thorne metric defines a reliable spacetime for most astrophysical purposes, for instance for the simulation of slowly rotating stars. Solving the Einstein field equations, we added terms of second order in the quadrupole moment to its post-linear version in order to compare it with solutions found by Blanchet in the frame of the multi-polar post-Minkowskian framework. We first derived the extended Hartle-Thorne metric in harmonic coordinates and then showed agreement with the corresponding post-linear metric from Blanchet.

We also found a coordinate transformation from the post-linear Erez-Rosen metric to our extended Hartle-Thorne spacetime. It is well known that the Hartle-Thorne solution can be smoothly matched with an interior perfect fluid solution with physically appropriate properties. A comparison among these solutions provides a validation of them. It is clear that in order to represent realistic solutions of self-gravitating (axially symmetric) matter distributions of perfect fluid, the quadrupole moment has to be included as a physical parameter.

Keywords: General Relativity; Post-Newtonian approximation

PACS: 04; 04.25.Nx

1 Introduction

In 1968, Hartle and Thorne (HT) [16, 26] proposed an approximate solution to the Einstein field equations (EFE) intended to represent the gravity field of neutron stars with mass, rotation and quadrupole moment as parameters. Berti et al. [1] compared the HT metric with the Manko [20, 21] (exact solutions) and Cook-Shapiro-Teukolsky metrics [6] (numerical solution), and showed that it is safe to use the HT metric, since it gives excellent results even for the innermost stable circular orbits with fast spin periods. Moreover, the exterior HT metric can be smoothly matched with a physically reasonable interior one. This provides realistic models of stars and for this reason, it is often used to validate exact and approximate solutions of EFE.

Stationary exact solutions of EFE in the vacuum case are characterized by two families of multipole-moments: mass- and spin-moments, see for example [18, 20, 22, 25]. Some of these solutions might be appropriate to represent stellar objects where the field moments can be related with corresponding body moments as integrals over the field generating sources. Moreover, the post-linear approximation of these metrics must be compatible with the HT solution. Quevedo et al. and Frutos et al. compared the HT solution with exact and approximate solutions of the EFE with...
a quadrupole moment $Q$ \cite{11, 12, 26} of first order. Comparisons with the second order in $Q$ of these solutions are still missing.

Geroch and Hansen (GH) defined a procedure to find the field multipole moments of static and stationary spacetimes \cite{13, 15}. Alternative definitions of relativistic multipole moments were given by Simon and Beig \cite{28} and by Thorne \cite{29}. It is important to mention that the GH multipole moments \cite{13, 15} are equivalent to the Thorne moments for stationary systems \cite{14}. Using the Ernst formalism \cite{5, 9}, Fodor et al. found an elegant method to find the multipole moments of a given spacetime \cite{10}. This method was later generalized by Hoenselaers and Perjéss \cite{19}. The relevance of taking the correct relativistic multipole moments of numerical spacetimes for modelling astrophysical objects such as neutron stars was discussed by Pappas and Apostolatos \cite{23}, who used a method due to Ryan \cite{27} to derive the multipole moments.

Nowadays, the use of harmonic coordinates is customary, since the form of the transformed metric tensor using harmonic coordinates has a special structure that allows to read off the Thorne moments directly even for the non-stationary case \cite{18, 29}. The multi-polar post-Minkowskian formalism (MPM) that was developed by Blanchet, Damour and Iyer \cite{2, 7} is also formulated in harmonic coordinates. Applying this formalism Blanchet found spacetimes containing mass-quadrupole and quadrupole-quadrupole terms \cite{3, 4}. The main goal of this paper is to compare these results with the ones we get from a static HT approximation with squared quadrupole moment.

The paper is organized as follows. In the second section, we briefly describe the HT metric. We find a new expanded version of the HT metric with a squared quadrupole moment, in the third section. In the fourth section, the harmonic coordinates for this HT solution are obtained and the metric is expressed in these. In the fifth section, it is shown that our HT harmonic metric and the Blanchet metric coincide at our level of approximation. Finally, a coordinate transformation is found from the post-linear version of the Erez-Rosen metric \cite{5, 8, 30, 31, 32} to this HT solution with no rotation in the sixth section.

## 2 The Hartle-Thorne Metric

The Hartle-Thorne metric \cite{16, 26} is an approximate solution of vacuum EFE that describes the exterior of any slowly and rigidly rotating, stationary and axially symmetric body. The metric is given with accuracy up to the second order terms in the body’s angular momentum, and first order in its quadrupole moment. It therefore has three parameters: mass $M$, spin $S$ and quadrupole-moment $Q$. The HT solution is given by

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\theta\phi}d\theta d\phi + g_{t\phi}dtd\phi,$$

with metric components

$$
\begin{align*}
g_{tt} &= -(1 - 2U)[1 + 2K_1P_2(\cos \theta)] - 2\frac{J^2}{r^4}(2\cos^2 \theta - 1), \\
g_{t\phi} &= -2\frac{J}{r}\sin^2 \theta, \\
g_{rr} &= \frac{1}{1 - 2U}\left[1 - 2K_2P_2(\cos \theta) - \frac{2}{1 - 2U}\frac{J^2}{r^4}\right], \\
g_{\theta\theta} &= r^2[1 - 2K_3P_2(\cos \theta)], \\
g_{\phi\phi} &= g_{\theta\theta}\sin^2 \theta,
\end{align*}
$$

where
\[ K_1 = \frac{J^2}{mr^4}(1 + U) + \frac{5}{8} \left( \frac{q}{m^3} - \frac{J^2}{m^4} \right) Q_2^2 \left( \frac{r}{m} - 1 \right), \]
\[ K_2 = K_1 - \frac{6J^2}{r^4}, \]
\[ K_3 = \left( K_1 + \frac{J^2}{r^4} \right) + \frac{5}{4} \left( \frac{q}{m^3} - \frac{J^2}{m^4} \right) \frac{U}{\sqrt{1 - 2U}} Q_2^2 \left( \frac{r}{m} - 1 \right), \]
\[ m = \frac{GM}{c^2}, \quad J = \frac{GS}{c^3}, \quad q = \frac{GQ}{c^2}, \]
\[ U = \frac{m}{r} \quad \text{and} \quad P_2(\cos \theta) = \frac{1}{2} [3 \cos^2 \theta - 1]. \]

The functions \( Q_2^{1,2} \) are associated Legendre polynomials of the second kind
\[ Q_2^1 = \sqrt{x^2 - 1} \left( \frac{3}{2} x \ln \left( \frac{1 + x}{1 - x} \right) - \frac{(3x^2 - 2)}{(x^2 - 1)} \right), \]
\[ Q_2^2 = (x^2 - 1) \left( \frac{3}{2} \ln \left( \frac{1 + x}{1 - x} \right) - \frac{(3x^3 - 5x)}{(x^2 - 1)^2} \right). \]

3 The Post-linear Hartle-Thorne with \( Q^2 \) term

Neglecting \( m^3, J^2 \)-terms and changing \( q \rightarrow -q \) in the HT-metric one obtains
\[ g_{tt} = - \left( 1 - 2 \frac{m}{r} - \frac{q}{r^3} P_2 - \frac{2mq}{r^4} P_2 \right) \]
\[ g_{t\phi} = - \frac{2J}{r} \sin^2 \theta, \quad (3) \]
\[ g_{rr} = 1 + 2 \frac{m}{r} + \frac{q}{r^3} P_2 + 4 \frac{m^2}{r^4} + 10 \frac{mq}{r^4} P_2 \]
\[ g_{\theta\theta} = r^2 \left( 1 + \frac{q}{r^3} P_2 + 5 \frac{mq}{r^4} P_2 \right) \]
\[ g_{\phi\phi} = r^2 \sin^2 \theta \left( 1 + \frac{q}{r^3} P_2 + 5 \frac{mq}{r^4} P_2 \right). \]

We then added \( q^2 \)-terms and checked that the corresponding metric
\[ g_{tt} = - \left( 1 - 2 \frac{m}{r} - \frac{q}{r^3} P_2 - \frac{2mq}{r^4} P_2 - 2 \frac{q^2}{r^6} P_2^2 \right) \]
\[ g_{t\phi} = - \frac{2J}{r} \sin^2 \theta, \quad (5) \]
\[ g_{rr} = 1 + 2 \frac{m}{r} + \frac{q}{r^3} P_2 + 4 \frac{m^2}{r^4} + 10 \frac{mq}{r^4} P_2 - \frac{1}{12} \frac{q^2}{r^6} [-8P_2^2 + 16P_2 - 77] \]
\[ g_{\theta\theta} = r^2 \left( 1 + \frac{q}{r^3} P_2 + 5 \frac{mq}{r^4} P_2 - \frac{1}{36} \frac{q^2}{r^6} [-44P_2^2 - 8P_2 + 43] \right) \]
\[ g_{\phi\phi} = r^2 \sin^2 \theta \left( 1 + \frac{q}{r^3} P_2 + 5 \frac{mq}{r^4} P_2 - \frac{1}{36} \frac{q^2}{r^6} [-44P_2^2 - 8P_2 + 43] \right). \]

is solution of the EFE by means of a REDUCE program \cite{17}. At this point, it is possible to add rotation into this metric to first order without problems.
Harmonic coordinates \((c_T, X, Y, Z)\) are especially useful, because the form of the metric tensor in these coordinates has a special structure where the Thorne-moments can be read off directly also for the non-stationary case [18, 29]. The harmonic coordinate condition reads

\[
\Box X^\mu = g^{\alpha\beta} \frac{\partial^2 X^\mu}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \frac{\partial X^\mu}{\partial x^\nu} = 0.
\]  

(7)

The solution of this equation for the HT metric including \(q^2\)-terms reads

\[
T = t \\
X = f \sin \theta \cos \phi = R \sin \theta \cos \phi \\
Y = f \sin \theta \sin \phi = R \sin \theta \sin \phi \\
Z = h \cos \theta = R \cos \theta,
\]

where

\[
f = \left[ r - m + \frac{1}{2} \frac{mq}{r^3} \cos^2 \theta + \frac{1}{72} \frac{q^2}{r^5} (32 P_2^2 - 4 P_2 - 55) \right]
\]

\[
h = \left[ r - m - \frac{1}{2} \frac{mq}{r^3} \sin^2 \theta + \frac{1}{72} \frac{q^2}{r^5} (32 P_2^2 - 16 P_2 - 43) \right].
\]

\[
R^2 = X^2 + Y^2 + Z^2 \\
\simeq [r - m]^2 + \frac{1}{12} \frac{q^2}{r^4} [8 P_2^2 - 17]
\]

\[
r \simeq R + m - \frac{1}{24} \frac{q^2}{R^5} [8 P_2^2 - 17]
\]

(9)

\[
\tan \vartheta = \frac{\sqrt{X^2 + Y^2}}{Z} = \frac{f}{h} \tan \theta \\
\simeq \left[ 1 + \frac{1}{2} \frac{mq}{r^3} \cos^2 \theta - \frac{1}{4} \frac{q^2}{r^6} \sin^2 \theta \right] \tan \theta
\]

\[
\theta \simeq \vartheta - \frac{1}{2} \left[ \frac{mq}{R^4} - \frac{1}{2} \frac{q^2}{R^6} \sin^2 \theta \right].
\]

(10)

The transformation of the metric from the \((r, \theta, \phi)\) coordinates to such harmonic coordinates \((R, \vartheta, \phi)\), was performed by means of the differentials 1-forms \((dX, dY, dZ)\). From these 1-forms, it is solved for the other 1-forms \((dr, d\theta, d\phi)\)

\[
dr = \left( 1 - \frac{1}{r} (\alpha_1 - \alpha_2) \cos^2 \theta - \frac{\partial \alpha_1}{\partial r} \sin^2 \theta - \frac{\partial \alpha_2}{\partial r} \cos^2 \theta \right) \\
\times \left( \sin \theta (\cos \phi dX + \sin \phi dY) + \cos \theta dZ \right) \\
+ \frac{1}{r} \left( \frac{\partial \alpha_1}{\partial \theta} \sin^2 \theta + \frac{\partial \alpha_1}{\partial \theta} \cos^2 \theta \right) (\sin \theta dZ - \cos \theta (\cos \phi dX + \sin \phi dY)) \\
+ \frac{1}{r} (\alpha_1 - \alpha_2) \cos \theta dZ
\]

(11)
\[
rd\theta = \left(1 + U + U^2 - \frac{1}{r}(\alpha_1 \cos^2 \theta + \alpha_2 \sin^2 \theta) - \left(\frac{\partial \alpha_1}{\partial r} - \frac{\partial \alpha_1}{\partial r}\right) \sin^2 \theta\right) \times
\left(\cos \theta (\cos \phi dX + \sin \phi dY) - \sin \theta dZ\right)
- \frac{1}{r} \sin \theta \left(\frac{\partial \alpha_1}{\partial \theta} - \frac{\partial \alpha_2}{\partial \theta}\right) (\cos^2 \theta (\cos \phi dX + \sin \phi dY) - \cos \theta \sin \theta dz)
- \frac{1}{r} \sin \theta \left(\frac{\partial \alpha_1}{\partial r} - \frac{\partial \alpha_1}{\partial r}\right) \sin \theta dZ
\]
(12)

\[
r\sin \theta d\phi = (1 + U + U^2 - \alpha_1) (\cos \phi dY - \sin \phi dX)
\]
(13)

where

\[
\alpha_1 = \frac{1}{2} \frac{mq}{r^3} \cos^2 \theta + \frac{1}{72 r^5} (32P_2^2 - 4P_2 - 55)
\]
\[
\alpha_2 = -\frac{1}{2} \frac{mq}{r^3} \sin^2 \theta + \frac{1}{72 r^5} (32P_2^2 - 16P_2 - 43).
\]

Substituting (11), (12), and (13) into the metric with metric components (5), the metric in Cartesian coordinates to post-linear order takes the form

\[
ds^2 = g_{tt} dt^2 + 2 g_{ti} dt dX^i + g_{ij} dX^i dX^j
\]
(14)

where

\[
g_{tt} = -1 + 2 \frac{w}{c^2} - 2 \frac{w^2}{c^4}
\]
\[
g_{tx} = \frac{2}{R^3} Y,
\]
\[
g_{ty} = -\frac{2}{R^3} X,
\]
\[
g_{xx} = 1 + 2 \frac{w}{c^2} + 2 \frac{w^2}{c^4} + \left(\frac{X^2}{R^2} - 1\right) \frac{m^2}{R^2}
+ \frac{1}{2} \frac{mq}{R^4} \left(1 - \frac{X^2}{R^2} - 5 \frac{Z^2}{R^2} + 15 \frac{X^2 Z^2}{R^4}\right)
+ \frac{q^2}{4R^6} \left(-1 + 3 \frac{X^2}{R^2} + 12 \frac{Z^2}{R^2} - 54 \frac{X^2 Z^2}{R^4} - 15 \frac{Z^2}{R^4} + 75 \frac{X^2 Z^4}{R^6}\right)
\]
\[
g_{xy} = m^2 \frac{XY}{R^4} + \frac{1}{2} \frac{mq}{R^6} \frac{XY}{R^8} \left(-1 + 15 \frac{Z^2}{R^2}\right) + \frac{3}{4} \left(\frac{q^2}{R^8}\right) \left(1 - 18 \frac{Z^2}{R^2} + 25 \frac{Z^4}{R^4}\right)
\]
\[
g_{xz} = m^2 \frac{XZ}{R^4} + \frac{1}{2} \frac{mq}{R^6} \frac{XZ}{R^8} \left(-7 + 15 \frac{Z^2}{R^2}\right) + \frac{3}{4} \left(\frac{q^2}{R^8}\right) \left(5 - 26 \frac{Z^2}{R^2} + 25 \frac{Z^4}{R^4}\right)
\]
\[
g_{yx} = 1 + 2 \frac{w}{c^2} + 2 \frac{w^2}{c^4} + \left(\frac{Y^2}{R^2} - 1\right) \frac{m^2}{R^2}
+ \frac{1}{2} \frac{mq}{R^4} \left(1 - \frac{Y^2}{R^2} - 5 \frac{Z^2}{R^2} + 15 \frac{Y^2 Z^2}{R^4}\right)
+ \frac{q^2}{4R^6} \left(-1 + 3 \frac{Y^2}{R^2} + 12 \frac{Z^2}{R^2} - 54 \frac{Y^2 Z^2}{R^4} - 15 \frac{Z^2}{R^4} + 75 \frac{Y^2 Z^4}{R^6}\right)
\]
(17)
\[
g_{yz} = \frac{m^2 YZ}{R^4} + \frac{1}{2}mq YZ \left( -7 + 15 \frac{Z^2}{R^2} \right) + \frac{3}{4}q^2 YZ \left( 5 - 26 \frac{Z^2}{R^2} + 25 \frac{Z^4}{R^4} \right)
\]
\[
g_{zz} = 1 + 2 \frac{w}{c^2} + 2 \frac{w^2}{c^4} + \left( \frac{Z^2}{R^2} - 1 \right) \frac{m^2}{R^2} + \frac{3}{2}mq \left( 1 - 6 \frac{Z^2}{R^2} + \frac{5}{R^4} \right)
\]
\[
+ 3 \frac{q^2}{4R^6} \left( -1 + 15 \frac{Z^2}{R^2} - 39 \frac{Z^4}{R^4} + 25 \frac{Z^6}{R^6} \right),
\]

where
\[
w = \frac{GM}{R} + \frac{GQ}{R^3} P_2(\cos \vartheta),
\]
equation (18) was used, and
\[
P_2(\cos \vartheta) = \frac{1}{2} (3 \cos^2 \vartheta - 1) = \frac{1}{2} \left( 3 \frac{Z^2}{R^2} - 1 \right).
\]

5 Comparison with known results

Our metric (15) can directly be compared with the one derived in Blanchet [3, 4] that was derived within the MPM-formalism which works with a tensor field
\[
h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta},
\]
where
\[
g_{\alpha\beta} = \sqrt{-g} g_{\alpha\beta}, \quad g = \det g_{\alpha\beta}
\]
and uses the Landau-Lifshitz form of the field equations in harmonic gauge (see e.g., [4] for more details).

The post-linear metric components under this harmonic coordinates can be written in the following form
\[
g_{tt} = - \left( 1 - \frac{2}{c^2} \frac{w}{c^2} + \frac{2}{c^4} \frac{w^2}{c^4} \right)
\]
\[
g_{ti} = g_{it} = - \frac{4}{c^4} \frac{w_i}{c^4}
\]
\[
g_{ij} = \left( 1 + \frac{2}{c^2} \frac{w}{c^2} + \frac{2}{c^4} \frac{w^2}{c^4} \right) - \frac{4}{c^4} h_{ij},
\]
The potential \( w \) is defined as in [18], and
\[
w_i = \frac{2G}{c^2 \tau^2} \epsilon_{ijk} S_j n_k,
\]
where \( S_i \) is the total angular momentum of the object, and \( n_i \equiv x^i/r \) with \( r^2 = x^2 + y^2 + z^2 \). In our case, we only have the \( z \) component, i.e., \( S_z = ma/c \).

The mass-quadrupole and quadrupole-quadrupole metric components as obtained by Blanchet [3, 4] take the form
\[
\begin{align*}
    h_{00}^{M_{ab}} &= -\frac{21 M}{r^4} n_{ab} M_{ab} \\
    h_{ij}^{M_{ab}} &= \frac{M}{r^4} \left( -\frac{15}{2} \delta_{ij} n_{ab} M_{ab} - \frac{1}{2} \delta_{ij} n_{ab} M_{ab} + 6 n_{a(i} M_{j)a} - M_{ij} \right) \\
    h_{00}^{M_{a,b}} &= \frac{1}{r^6} \left( a_0^6 \tilde{n}_{abcd} M_{ab} M_{cd} + b_0^6 \tilde{n}_{ab} M_{ac} M_{bc} + c_0^6 \delta_{ij} n_{ab} M_{ab} \right) \\
    h_{ij}^{M_{a,b}} &= \frac{1}{r^6} \left( p_0^6 \delta_{ij} n_{abcd} M_{ab} M_{cd} + q_0^6 \delta_{ij} n_{abcd} M_{ab} M_{cd} + r_0^6 \delta_{ij} \tilde{n}_{abcd} M_{ab} M_{cd} \right. \\
    &\quad + \left. s_0^6 \tilde{n}_{ij} M_{ab} M_{ab} + t_0^6 \delta_{ij} n_{abcd} M_{ab} M_{cd} + u_0^6 \tilde{n}_{abcd} M_{ab} M_{cd} + v_0^6 \tilde{n}_{abcd} M_{ab} M_{cd} \right) \\
    &\quad + w_0^6 \tilde{n}_{a(i} M_{j)b} M_{ab} + x_0^6 \tilde{n}_{ij} M_{ab} M_{ab} + y_0^6 M_{a(i} M_{j)b} + z_0^6 M_{a(i} M_{j)a} \\
\end{align*}
\]

where

\[
\begin{align*}
    a_0^6 &= -\frac{63}{4}, \quad b_0^6 = -9, \quad c_0^6 = -\frac{21}{10}, \\
    p_0^6 &= -\frac{75}{4}, \quad q_0^6 = \frac{90}{11}, \quad r_0^6 = -\frac{9}{44}, \\
    s_0^6 &= \frac{25}{84}, \quad t_0^6 = -\frac{29}{42}, \quad u_0^6 = -\frac{11}{70}, \\
    v_0^6 &= -\frac{18}{11}, \quad w_0^6 = \frac{5}{21}, \quad x_0^6 = -\frac{10}{21}, \\
    y_0^6 &= \frac{23}{42}, \quad z_0^6 = -\frac{6}{35},
\end{align*}
\]

and \( \tilde{n}_{i_1...i_l} \) are the symmetric and trace-free parts (e.g., \([24, 29]\)) of the Cartesian tensor

\[ n_{i_1...i_l} \equiv n_{i_1} \cdots n_{i_l}. \]

For our axially symmetric body the quadrupole-moment \( Q \) appears in the Cartesian quadrupole mass-tensor in the form

\[
M_{ij} = -\frac{Q}{3} (\delta_{ij} - 3 \delta_{i3} \delta_{j3}).
\]

Substituting these \( M_{ab} \) components into (21) and the resulting \( h_{\alpha \beta} \) into (20), we get the metric components (15).

## 6 Transformation from Erez-Rosen to Hartle-Thorne

The Erez-Rosen (ER) metric represents an static exact solution of EFE with axial symmetry and a quadrupole moment \([5, 8, 30, 31, 32]\). Keeping only \( q^2 \) terms the ER-metric in spherical coordinates \((ct, r, \theta, \phi)\) reads

\[
g_{tt} = -\left( 1 - 2U - \frac{4}{15} q U^3 P_2 - \frac{4}{15} q U^4 P_2 + \frac{8}{225} q^2 U^6 P_2^2 \right)
\]
\[
g_{rr} = 1 + 2U + 4U^2 + \frac{4}{15}qU^3P_2 + \frac{4}{45}qU^4(5P_2^2 + 11P_2 - 1) \\
+ \frac{8}{2025}q^2U^6(25P_2^3 - 12P_2^2 - 6P_2 + 2) \\
g_{\theta\theta} = r^2 \left( 1 + \frac{4}{15}qU^3P_2 + \frac{4}{45}qU^4(5P_2^2 + 5P_2 - 1) \\
+ \frac{8}{2025}q^2U^6(25P_2^3 - 12P_2^2 - 6P_2 + 2) \right) \\
g_{\phi\phi} = r^2 \sin^2 \theta \left( 1 + \frac{4}{15}qU^3P_2 + \frac{4}{5}qU^4P_2 + \frac{8}{225}q^2U^6P_2^2 \right)
\]

where

\[
q = \frac{Q}{m^3} \quad \text{and} \quad U = \frac{m}{r}.
\]

The following transformation converts the ER truncated metric into the static HT metric at the same level of approximation

\[
r = R \left( 1 + \frac{mq}{R^4}f_1 + \frac{q^2}{R^6}f_2 \right) \\
\theta = \Theta + \frac{mq}{R^4}g_1 + \frac{q^2}{R^6}g_2
\]

where

\[
f_1 = \frac{1}{135}(10P_2^2 - 8P_2 - 2) \\
f_2 = \frac{1}{4050}(40P_2^3 - 24P_2^2 - 43) \\
g_1 = -\frac{1}{45}(5P_2 - 2) \cos \Theta \sin \Theta \\
g_2 = \frac{P_2}{2025}(12 - 30P_2) \cos \Theta \sin \Theta
\]

with \(P_2 = P_2(\cos \Theta)\).

The transformed metric components are given by

\[
g_{tt} = -\left( 1 - 2\mathcal{U} - \frac{12}{45}qU^3P_2 - \frac{12}{45}qU^4P_2 + \frac{8}{225}q^2U^6P_2^2 \right) \\
g_{R\Theta} \simeq 0 \\
g_{RR} = 1 + 2\mathcal{U} + 4\mathcal{U}^2 + \frac{12}{45}qU^3P_2 + \frac{4}{3}qU^4P_2 \\
+ \frac{1}{315}q^2U^6 \left( \frac{168}{45}P_2^2 - \frac{336}{45}P_2 + \frac{1617}{45} \right) \\
g_{\Theta\Theta} = R^2 \left( 1 + \frac{12}{45}qU^3P_2 + \frac{30}{45}qU^4P_2 \\
+ \frac{1}{315}q^2U^6 \left( \frac{308}{45}P_2^2 + \frac{56}{45}P_2 - \frac{301}{45} \right) \right)
\]
\[
g_{\phi\phi} = R^2 \sin^2 \Theta \left( 1 + \frac{12}{45} q U^3 P_2 + \frac{30}{45} q U^4 P_2 \right. \\
\left. + \frac{1}{315} q^2 U^6 \left( \frac{308}{45} P_2^2 + \frac{56}{45} P_2 - \frac{301}{45} \right) \right),
\]
where \( U = m/R \).
Changing \( q \rightarrow 15q/2 \), one obtains the static HT metric at this level of approximation.

7 Conclusions

We expanded the HT-metric and kept only linear terms in the rotation parameter and quadratic terms in the mass parameter. Then we included second order terms in the quadrupole parameter by solving the EFE in vacuum perturbatively. We then shows that this form of the metric agrees with a corresponding metric that was derived within the MPM-formalism by Blanchet at the same order of approximation.

A transformation linking our static HT solution with the ER metric expanded in Taylor series was also found. This provides a validation of all these metrics. The quadrupole moment is an important feature which is included as a physical parameter in all these solutions.

These spacetimes can be used to represent realistic solutions of self-gravitating (axially symmetric) mass distribution of perfect fluid. This is because the HT solution can be smoothly matched with interior perfect fluid solution with physically reasonable properties.

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