Parameters identification method for breast biomechanical numerical model

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Abstract

Bio-mechanical breast simulations are based on a gravity free geometry as a reference domain and a nonlinear mechanical model parameterised by physical coefficients. As opposed to complex models proposed in the literature based on medical imagery, we propose a simple but yet realistic model that uses a basic set of measurements easy to realise in the context of routinely operations. Both the mechanical system and the geometry are controlled with parameters we shall identify in an optimisation procedure. We give a detailed presentation of the model together with the optimisation method and the associated discretisation. Sensitivity analysis is then carried out to evaluate the robustness of the method.

Keywords: Breast surgery, numerical simulation, parameter identification, finite element method

1. Introduction

Bio-mechanical breast modelling and simulation with the finite element method (7, 8) provides powerful analysis tools to predict 3-D breast deformations for various bio-medical applications. Breast tissue modelling is usually based on non-linear mechanical models such as the Neo-hookean model expressed in the stress-free unloaded reference state of the breast.

Both the parameters of the mechanical model and the initial gravity-free configuration are unknown (22) and have to be identified for each patient. To solve the difficulty in determining the material parameters of soft tissues for simulations and providing the reference configuration, optimisation procedures are usually carried out on the basis of MR image or radiological acquisitions such as mammographic plate compression.
Breast models have also been assessed using predicted location of anatomical landmarks and selected in breast images acquired before and after in vivo compression by visual comparison. Then a complex procedure of reverse problem is achieved (1, 8, 14) for determining the reference state and the associated coefficients.

The methods proposed in the literature (7, 8) require images acquisition, complex identification procedures with high computational cost (9, 13) that would be unnecessary or too expensive for routinely or low-cost practices. From a practical point of view, quality does not necessarily mean high accuracy or high amounts of detailed information like the models proposed in (2), but to manage relevant and satisfactory results for surgeons. In this context, we propose an alternative and simpler method that provides an effective numerical technology enable to accurately predict breast deformations and running on a mid range laptop within few minutes following the model proposed by (4). The key ideas are, on the one hand, the introduction of a simple set of in vivo breast measures easy to achieve, and on the other hand, a four-parameters nonlinear mechanical model defining on a simplified two-parameters reference configuration (the one without gravity). Measurements provide the data we use to fit the model parameters (both mechanical and geometrical). Given an a priori geometry with only two degrees of freedom reduces the inverse problem difficulty, i.e. the recovery of the stress-free unloaded reference state and provides a faster identification procedure.

Concerning related works, in (4), the authors treat a similar model but only consider a two-parameter problem, namely the elasticity coefficients of the core domain, while the present contribution treats both the geometrical and mechanical aspects including the skin effect leading to a more complete six parameters model. The use of nonlinear models to simulate the skin, the muscles, and the tissues, is well-developed in the bio-mechanics context (10, 11, 21) while the finite element method is a popular technique for the discretisation (15, 18, 23) using alternatively the weak formulation or the minimisation framework (3, 6, 19). The Neo-hookean is considered as a well-adapted mechanical representation of the breast (7) and more generally for human tissues (1, 14). For the particular case of breast, we differentiate the skin from the core assuming a couple of coefficients for each material.

The paper is organised as follows. We present the breast model with its mathematical formulation in Section 2, followed by the parameter identification method in Section 3. In Section 4 we show the discretisation of this model and parameter evaluation method while results obtained using synthetic measurements are presented in Section 5. Finally, in Section 6 conclusions are presented.
2. Breast Modelling

The breast is a complex structure constituted of a mass of glandular tissue encased in variable quantities of fat, that account for its characteristic round shape, connected to the skin through a series of ligaments. All these types of tissues possess different mechanical properties while the proportion of these materials varies with factors like genetics and age (20). Moreover, the breast skin plays a major role as a wrapper tissue enjoying specific mechanical properties.

Deformation evaluation of the breast over external actions is achieved by considering a simplified stress-free geometrical domain of the breast equipped with the Neo-hookean mechanical model (7) where we differentiate the breast core and skin. Additionally, we also introduce the Chassaignac space (5) to reproduce the breast mobility. The gravity-free reference breast domain $\Omega_g$ is a piece of spherical cap where the plane section is attached to the torso. The domain is parameterised with the radius $R$ of the sphere while $H < R$ represents the non-truncated length as displayed in Figure 1. The cap which is placed in the torso plane corresponds to a circle of radius $r$ which is smaller than $R$.

![Fig. 1: Breast geometry configurations. Radius $R$ characterised the main sphere while $H$ identifies the truncated part following the negative y direction.](image)

The final and more realistic shape of the breast is obtained by determining the effect of gravity on the breast tissues as observed in figure 2.

To prescribe the boundary condition and compute the energy associated to the skin, we introduce the following notations, reproduced on Figure 2: $\Gamma_B$ is the back side plane of the breast which attaches to the torso, $\Gamma_F$ represents the surface associated to the skin of the breast and $\gamma_{13}$ is the arc of the infra-mammary
The hyperelastic neo-Hookean compressible relations (5) equipped with the so-called Chassaignac space represent a well-accepted model for biomechanical soft tissues. The Chassaignac space corresponds to a mobile zone $\Gamma_B$ located between the breast and the trunk which acts as a spring to maintain the breast close to the trunk. Under the gravity, the breast corresponds to a minimisation of the energy functional (3, 6, 19) associated with the neo-Hookean system. Such a functional aggregates the internal energy due to the bulk, the energy deriving from the skin displacement, the gravitational energy characterised by the gravitational field $a_g \in \mathbb{R}^3$ and the spring energy associated to Chassaignac space. Anisotropic model for skin (12) is also investigated to provide a more sophisticated description taking into account the mechanical behaviour of skin under large deformations.

For any generic point $p \in \Omega_g$, we seek the new position vector field $f : p \in \Omega_g \rightarrow f(p) \in \mathbb{R}^3$ which minimises the energy functional given by

$$J(f) = \int_{\Omega_g} W_{br}(\nabla f(p))dp + \int_{\Gamma_f} W_{sk}(\nabla| | f(p))d\!S_p - \int_{\Omega_g} \rho \cdot a_g \cdot f(p)dp + \int_{\Gamma_B} c||f(p) - p||d\!S_p$$

subject to the constraints

$$f(p) = p, \quad \text{in } \gamma_D,$$

$$f(p) \cdot \vec{N}_B = 0, \quad \text{in } \Gamma_B$$

Fig. 2: Notation and geometry of the gravity-free geometry (left) and configuration in the gravity field (right)
where \( c \) is the Chassaignac coefficient, and \( \rho \) stands for the bulk density. Constraint (2) represents the fixation of the infra-mammary fold on the torso, i.e., any point \( p \) in \( \gamma_D \) maintain their position while constraint (3) states that any point \( p \) on the breast plane attached to the torso (Chassaignac space) only moves inside that trunk plane.

The expressions for the volume strain-energy and the skin strain-energy densities, respectively represented by \( W_{br} \) and \( W_{sk} \), are given by

\[
W_{br}(F) = \frac{\mu_{br}}{2} \left( (FF^t)^2 - 3 - 2 \ln(\text{det}(F)) \right) + \frac{\lambda_{br}}{2} (\text{det}(F) - 1)^2,
\]

where \( F = \nabla f \) is the Jacobian matrix of \( f \) and \((\lambda_{br}, \mu_{br})\) are the Lamé parameters for the breast, and

\[
W_{sk}(F_\parallel) = \frac{\mu_{sk}}{2} \left( (F_\parallel F_\parallel^t)^2 - 2 - 2 \ln(\text{det}(F_\parallel)) \right) + \frac{\lambda_{sk}}{2} (\text{det}(F_\parallel) - 1)^2,
\]

where \((\lambda_{sk}, \mu_{sk})\) are the Lamé parameters for the skin while \( F_\parallel = \nabla_\parallel f_\parallel \) is the Jacobi matrix of the superficial (skin) displacement \( f_\parallel \) (\( f_\parallel \) is the restriction of \( f \) on \( \Gamma_F \) using local two-parameters representation since \( \Gamma_F \) is a surface).

3. Parameters identification

Let denote \( \Lambda = (\Lambda_g, \Lambda_m) \) the six parameters which represents the geometrical and the physical degrees of freedom respectively. First, for the two geometrical parameters \( \Lambda_g = (R, H) \), we create the stress-free configuration \( \Omega_g \), as shown in Figure 1, which defines the operator

\[
\Lambda_g \mapsto \Omega_g = \Omega_g(\Lambda_g).
\]

Secondly, for a set of physical parameters \( \Lambda_m = (\lambda_{br}, \mu_{br}, \lambda_{sk}, \mu_{sk}) \), minimisation of the energy \( J(f) = J(f; \Omega_g, \Lambda_m) = J(f; \Lambda) \) over the set of regular functions \( f \) defined on \( \Omega_g \) provides the solution \( f(p) = f(p; \Lambda) \). At last, applying the displacement over the reference domain provides the deformed domain \( \Omega(\Lambda) = f(\Omega_g) \) which, at the end of the day, defines the operator

\[
\Lambda \mapsto (\Lambda_m, \Omega_g) \mapsto \Omega(\Lambda) = f(\Omega_g) = \left\{ f(p, \Lambda); \ p \in \Omega_g \right\}.
\]
Third, to perform the parameters identification, we consider the following measurements: the volume, the surface area of the breast’s skin and three sizes, namely min_y, min_z and max_y, as displayed in Figure 3. Measures are performed for three patient positions, stand up, inclined (45 degrees) and horizontal, providing

![Figure 3: Size characteristic (min_y, max_y and min_z) of a breast (stand-up position)](image)

15 real values denoted \( \mathbf{M} = (M_1, \ldots, M_{15})^T \). On the other hand, given a set of parameters and solving three times the direct problem corresponding to the three positions provides the vectors \( M(\Lambda) = M(\Omega(\Lambda)) = (M_{k1}, \ldots, M_{k15})^T \). The problem consists in seeking the set of parameters such that \( M(\Lambda) \) is equal to \( \mathbf{M} \) in the least square sense. Introducing the cost function

\[
E(\Lambda) = \sum_{i=1}^{n} (M_i(\Lambda) - \mathbf{M}_i)^2,
\]  

(6)

we seek for \( \overline{\Lambda} = \arg \min_{\Lambda} E(\Lambda) \) which minimises the errors between the experimental data and the theoretical solution values.

To provide the optimal set of parameters, an iterative process is considered by evaluating the sequences \( \Lambda^{k+1} = \Lambda^k + \Delta \Lambda^k \) such that \( E(\Lambda^{k+1}) < E(\Lambda^k) \). We use the Gauss-Newton method to take advantage that function \( E \) casts in the specific form

\[
E(\Lambda) = (M_i(\Lambda) - \mathbf{M}_i)^T (M_i(\Lambda) - \mathbf{M}_i),
\]  

(7)
The first-order Taylor series expansion reads

\[ M(\lambda + \Delta \lambda) \approx M(\lambda) + \frac{\partial M}{\partial \lambda}(\lambda) \Delta \lambda = M + \frac{\partial M}{\partial \lambda} \Delta \lambda \]  

(8)

where \( \frac{\partial M}{\partial \lambda} \) is the Jacobian matrix. From relations 7 and 8 we deduce the first term of error for a perturbation of the parameters \( \Delta E = E(\lambda + \Delta \lambda) - E(\lambda) \)

\[ \Delta E = -\Delta \lambda^T \left[ 2 \left( \frac{\partial M}{\partial \lambda} \right)^T (M - M(\lambda)) + \left( \frac{\partial M}{\partial \lambda} \right)^T \frac{\partial M}{\partial \lambda} \Delta \lambda \right]. \]

Thus, by identification, we deduce

\[ \frac{\partial E}{\partial \lambda}(\lambda) \approx -2 \left( \frac{\partial M}{\partial \lambda} \right)^T (M - M(\lambda)) + \left( \frac{\partial M}{\partial \lambda} \right)^T \frac{\partial M}{\partial \lambda} \Delta \lambda \]  

(9)

We seek a perturbation \( \Delta \lambda \) that minimises \( E \), i.e., \( \frac{\partial E}{\partial \lambda} = 0 \). Therefore Equation 9 provides the expression

\[ \Delta \lambda = 2 \left[ \left( \frac{\partial M}{\partial \lambda} \right)^T \frac{\partial M}{\partial \lambda} \right]^{-1} \left( \frac{\partial M}{\partial \lambda} \right)^T (M - M(\lambda)) \]  

(10)

Since a direct computation of the jacobian matrix is not possible, we numerically evaluate each partial derivative setting

\[ \partial_i M(\lambda) \approx \frac{M(\lambda + \varepsilon e_i) - M(\lambda)}{\varepsilon} \in \mathbb{R}^n \]

where \( \varepsilon \) is fixed by the user in function of the minimisation problem and \( e_i = (\delta_{ij})_{j=1,...,n} \).

4. Discretisation

To carry out numerical simulations, we introduce the discretisation of the breast model and the optimisation problem. We denote by \( T_{h,k} \) a mesh of the gravity free domain \( \Omega_g = \Omega_g(\Lambda_g) \) constituted of I non-overlapping tetrahedron cells \( \tau_i, i = 1, \ldots, I \), and N vertices \( P_n = (P_{nx}, P_{ny}, P_{nz}) \in \mathbb{R}^3, n = 1, \ldots, N \) while

\[ \Omega_{g,h} = \bigcup_{\tau_i \in T_{h,k}} \tau_i \]  

stands for the discrete domain. Moreover, \( T_k, k = 1, \ldots, K \), represents the faces of the tetrahedrons of the mesh that belong to \( \Gamma_f \). Quantities \( |\tau_i| \) and \( |T_k| \) represent the volume and the area of the cell and the triangle respectively. We also use a local indexation and denote by \( P_{ij} = (P_{ijx}, P_{ijy}, P_{ijz}) \in \mathbb{R}^3, j = 1, 2, 3, 4, \) the
vertices of $\tau_i$ and by $P_{kj} = (P_{kji}, P_{kjf}, P_{kjc}) \in \mathbb{R}^3$, $j = 1, 2, 3$, the vertices of $T_k$. To discretise function $f$, we associate to each node $P_n$ an approximation $f_n \approx f(P_n)$ and denote by $f_h$ the continuous, linear piecewise function while vector $\Phi_h = (f_{n\alpha}, f_{ny}, f_{nc}) p$ collects the $3N$ components of the new positions.

Vector $\Phi_h$ corresponds to the new configuration but not all the entries are necessarily unknowns of the problem since some of them are characterised by the boundary conditions. The sub-vector $X_h$ of $\Phi_h$ only contains the unknown values we shall use in the minimisation process while the boundary conditions define an operator

$$X_h \rightarrow \Phi_h = \mathcal{B}(X_h)$$

which provides the other entries to complete vector $\Phi_h$. In the present contribution, condition (2) yields that $f_n = P_n$ for any $P_n \in \gamma_D$ while relation (3) implies that for any $P_n \in \Gamma_B$, we set $f_{ny} = 0$ to maintain interface $\Gamma_B$ on the trunk plane $y = 0$. This two conditions completely define the vector of unknowns $X_h$ and operator $\mathcal{B}$.

4.1. The energy functional

Discrete version of the energy functional is given by

$$J_h(X_h) = J_h(\mathcal{B}(X_h)) = \hat{J}_h(\Phi_h) = J^1_h + J^2_h + J^3_h + J^4_h$$

where $J^1_h$, $J^2_h$, $J^3_h$ and $J^4_h$ represent the volume energy, the surface energy, the energy from the gravitational displacement, and the energy associated to the Chassaignac space respectively.

- For a new configuration characterised by the approximation $p \in \Omega_g \rightarrow f_h(p) \in \mathbb{R}^3$ and stored in vector $\Phi_h$, the internal energy on tetrahedron $\tau_i$ is given by

$$W_{\tau_i} = |\tau_i| \left( \frac{\mu}{2} \left[ \text{tr}(F_i F^T_i) - 3 + 2 \ln(\det(F_i)) \right] + \frac{4}{2} \left[ \det(F_i) - 1 \right]^2 \right),$$

where $F_i$ is the $3 \times 3$ matrix solution of the linear system

$$f_{i2} - f_{i1} = F_i(P_{i2} - P_{i1}), \quad f_{i3} - f_{i1} = F_i(P_{i3} - P_{i1}), \quad f_{i4} - f_{i1} = F_i(P_{i4} - P_{i1}),$$

finally having $J^1_h = \sum_{\tau_i \in T_h} W_{\tau_i}$. 


For the surface energy, the discrete piecewise linear function $f_h$ transforms a triangle $T_k$ with vertices $OAB$ into a triangle $T_k'$ with vertices $O'A'B'$. The method is similar to the surface variation using in (16) to evaluate the energy deriving from the displacement variations. Since the translation and the rotation do not change the stress due to the deformation, we assume that $O'B'$ is collinear to $OB$ and $A'$ belongs to the same plane as triangle $OAB$. Function $f_h$ is a two-dimensional function locally given by

$$f_h|_O = O, \quad f_h|_A = A', \quad f_h(B) = B'.$$

The Jacobian matrix of $f_h$ is the constant matrix

$$J_{f_h} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To determine the matrix, one writes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \|OB\| \\ 0 \end{bmatrix} = \begin{bmatrix} \|O'B'\| \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \|OA\| \cos(\alpha) \\ \|OA\| \sin(\alpha) \end{bmatrix} = \begin{bmatrix} \|O'A'\| \cos(\alpha') \\ \|O'A'\| \sin(\alpha') \end{bmatrix},$$

where $\alpha = \angle(OA, OB)$ and $\alpha' = \angle(O'A', O'B')$. The first linear system gives $c = 0$ and $a = \frac{\|O'B\|}{\|OB\|}$.

Substituting these expressions in the second linear system we obtain

$$d = \frac{\|O'A'\| \sin(\alpha')}{\|OA\| \sin(\alpha)}, \quad b = \frac{\|O'A'\| \cos(\alpha') - \frac{\|O'A\|}{\|OA\|} \|OA\| \cos(\alpha)}{\|OA\| \sin(\alpha)}.$$

The superficial energy on triangle $T$ for the skin ($W_T$) is then given by

$$|T| \left( \frac{\mu_k}{2} \left[ \text{tr}(J_{f_1} J_{f_1}^T) - 2 - 2 \ln(\det(J_{f_1})) \right] + \frac{\lambda_k}{2} \left[ \det(\text{det}(J_{f_1}) - 1) \right]^2 \right)$$

and the whole superficial energy is approximated by

$$J^2_h = \sum_{T \in \Gamma} W_T.$$

For $J^3_h$ and $J^4_h$ we have

$$J^3_h = -\sum_{\tau_i} \frac{\alpha_k}{4} \rho a_k \left( f(P_{i1}) + f(P_{i2}) + f(P_{i3}) + f(P_{i4}) \right)$$
and
\[ J_h^4 = \sum_{T_j \subset \Gamma_B} \frac{|T_j|}{3} (\|f(P_{i1}) - P_{i1}\| + \|f(P_{i2}) - P_{i2}\| + \|f(P_{i3}) - P_{i3}\|). \]

4.2. Numerical approximation of the discrete energy minimiser

For a given vector \( X_h \) and the boundary conditions, we deduce vector \( \Phi_h = B(X_h) \), hence the continuous linear piecewise function \( f_h \) we use to compute the discrete energy functional. We then build the operator
\[ X_h \rightarrow \Phi_h = B(X_h) \rightarrow J_h(X_h; \Lambda) = \hat{J}_h(\Phi_h; \Lambda) \in \mathbb{R} \]
where \( \Lambda \) is a given set of parameters. The numerical solution we seek provides the vector \( \Phi_h = \Phi_h(\Lambda) \) which minimises the energy of the discrete mechanical system. Conjugate gradients method is employed to determine the minimiser \( \bar{X}_h \) of the discrete functional \( J_h(X_h; \Lambda) \).

4.3. Cost function discretisation

Let \( \Lambda = (\Lambda_m, \Lambda_g) \) be a set of parameters. We deduce the discrete gravity free configuration \( \Omega_{g,h} \) which provides the operator \( \Lambda_g \rightarrow \Omega_{g,h} \). Then we compute the solution \( \Phi_h(\Lambda) \) minimising the discrete energy functional \( J_h(f_h) = J_h(f_h; \Omega_{g,h}, \Lambda_m) = J_h(f_h; \Lambda) \). We deduce the final discrete breast \( \Omega_h(\Lambda) \) after applying the deformation
\[ \Omega_h(\Lambda) = \{f_h(p, \Lambda); p \in \Omega_{g,h}\} \]
with \( f_h \) being the function that provides the new position. We assess the measures on \( \Omega_h(\Lambda) \) and produce vector \( M_h(\Lambda) \). In conclusion, we have define the discrete measures operator \( \Lambda \rightarrow M_h(\Lambda) \). On the other hand, the discrete cost function reads
\[ E_h(\Lambda) = \sum_{i=1}^{n} \left( M_{h,i}(\Lambda) - \bar{M}_i \right)^2 \]  \hspace{1cm} (11)
We apply the iterative process to the discrete versions to get the best parameter set \( \Lambda \) by calculating successive variation of the parameters (\( \Delta \Lambda^i \)) until \( E_h(\Lambda^i) - E_h(\Lambda^{i+1}) < \epsilon_m \) for a given threshold \( \epsilon_m \).

5. Synthetic Cases

To assess the model quality and robustness, several numerical experiences are carried out using a manufactured solution. We solve the direct problem by prescribing a given set of parameters and compute the associated measurements and test the method ability to recover the initial parameters independently of the
initial guess. To this end, we create a numerical breast setting the 6 parameters $\Lambda = (R = 0.562, H = 0.5, \lambda = 1000, \mu = 150, \lambda_p = 8000, \mu_p = 1600)^T$ and compute the three configurations in the gravitational fields (see Figure 4 for the case stand up) that provides the reference set of measurements

$$M(\Lambda) = M_\Lambda = (M_{h,1}, \ldots, M_{h,15})^T$$

depending on the mesh characteristic size $h$ that corresponds to the mesh with 2948 vertices and about 12000 tetrahedrons. We want to evaluate the importance of the initial guess in the final results with initial guesses that went from 10% (approximate guess), 30% (reasonable guess), 60% (bad guess) and over 60% from the reference parameters. The results (in mean) can be seen in table 1. These results show the effectiveness of the algorithm in terms of converging towards the reference parameters with a value of loss close and in some cases inferior to 1% with the exception of the $\mu_p$ parameter with a loss value around 2%.

![Fig. 4: Breast under gravity with the given set of defined parameters $\Lambda$.](image)

| Granularity & Parameters | Geometrical  | Mechanical  | Parameter Loss |
|-------------------------|--------------|-------------|----------------|
|                         | R $\lambda$  | $\mu$       | $\lambda_p$    |
| Coarse (N=226)          | 0.08%        | 0.39%       | 0.42%          | 0.89%          | 0.27%          | 2.77%          |
| Medium (N=572)          | 0.10%        | 0.47%       | 1.39%          | 0.79%          | 0.36%          | 2.40%          |
| Medium-Thin (N=1014)    | 0.05%        | 1.20%       | 1.12%          | 0.79%          | 1.16%          | 2.32%          |
| Thin (N=1404)           | 0.06%        | 0.98%       | 1.07%          | 0.76%          | 0.95%          | 2.28%          |
| Very-Thin (N=2127)      | 0.08%        | 0.68%       | 0.92%          | 0.65%          | 0.85%          | 2.14%          |
| Ultra-Thin (N=2498)     | 0.07%        | 0.72%       | 0.97%          | 0.62%          | 0.86%          | 2.13%          |

Table 1: Values of the parameter loss (in %) with different sized meshes after 25 iterations. For each mesh size, we selected 10 tests and the values shown are the means from those tests.
5.1. Robustness

We evaluate the computational effort of the method, particularly when increasing the number of vertices. We report the mean squared error (residual) between the reference measurements and the measurements obtained with different mesh sizes with the reference parameters (10 cases per mesh size). We also report the relative running time with respect to the coarser mesh, in function of the mesh size in table 2.

| Mesh size   | 226  | 572  | 1014 | 1404 | 2127 |
|-------------|------|------|------|------|------|
| Residual    | 1.52e-3 | 2.72e-4 | 7.26e-5 | 3.91e-5 | 9.13e-7 |
| running time| 1    | 3.4  | 7.8  | 11.2 | 20   |

Table 2: Measurement's difference residual between the reference measurements and the measurements obtained with coarser meshes with the same values for the breast parameters $\Lambda$ and the work effort which is spent by the optimisation process.

Notice that the measurements obtained with the reference parameters and a mesh with just 572 nodes are approximately 5% different when comparing with the reference measurements. This means that by using coarser meshes we are, synthetically, introducing error to the measures.

We evaluate the precision for different size meshes (namely medium, medium-thin, thin and very-thin corresponding to a number of 572, 1014, 1404, and 2127 nodes respectively) and we report the values in figure 5. We observe the volatility for some parameters during the first estimation steps but we obtain stable approximations after iteration 15. Tables 3 and 4 give the mean and standard deviation value of each parameter for those tests in the interval between iteration 15 and 25.

| Granularity & Parameters | Geometrical | Mechanical |
|-------------------------|-------------|------------|
| Medium (N=572)          | $R$ 0.35%   | $H$ 6.26%  |
|                         | $\lambda$ 48.70% | $\mu$ 8.71% |
|                         | $\lambda_p$ 2.17% | $\mu_p$ 20.33% |
| Medium-Thin (N=1014)   | $R$ 0.24%   | $H$ 4.30%  |
|                         | $\lambda$ 30.18% | $\mu$ 6.82% |
|                         | $\lambda_p$ 2.19% | $\mu_p$ 14.74% |
| Thin (N=1404)           | $R$ 0.09%   | $H$ 4.25%  |
|                         | $\lambda$ 17.93% | $\mu$ 5.57% |
|                         | $\lambda_p$ 1.64% | $\mu_p$ 14.30% |
| Very-Thin (N=2127)     | $R$ 0.05%   | $H$ 2.53%  |
|                         | $\lambda$ 4.21% | $\mu$ 2.23% |
|                         | $\lambda_p$ 1.20% | $\mu_p$ 5.35% |

Table 3: Parameter value difference (in %) between the reference parameters ($\overline{\Lambda}$) and the parameter values obtained with coarser meshes after the optimisation process with an initial guess with a variance up to 60% from $\overline{\Lambda}$

| Granularity & Parameters | Geometrical | Mechanical |
|-------------------------|-------------|------------|
| Medium (N=572)          | $R$ 0.27%   | $H$ 1.83%  |
|                         | $\lambda$ 4.55% | $\mu$ 7.22% |
|                         | $\lambda_p$ 5.01% | $\mu_p$ 21.15% |
| Medium-Thin (N=1014)   | $R$ 0.14%   | $H$ 0.64%  |
|                         | $\lambda$ 3.66% | $\mu$ 4.49% |
|                         | $\lambda_p$ 3.53% | $\mu_p$ 12.19% |
| Thin (N=1404)           | $R$ 0.05%   | $H$ 0.31%  |
|                         | $\lambda$ 2.94% | $\mu$ 2.52% |
|                         | $\lambda_p$ 1.68% | $\mu_p$ 7.10% |
| Very-Thin (N=2127)     | $R$ 0.05%   | $H$ 0.23%  |
|                         | $\lambda$ 2.89% | $\mu$ 1.53% |
|                         | $\lambda_p$ 1.34% | $\mu_p$ 4.01% |

Table 4: Parameter value standard deviation (in %) between the reference parameters and the parameters estimated with coarser meshes.
Three levels/rates of convergence are highlighted:

- High convergence. The geometrical parameters $\Lambda_g = (R, H)$ quickly converge to the solution after few iterations, and present a very stable behaviour (standard deviation values inferior to 2%).

- Mild convergence. The estimation method presents good approximations for the physical parameters $\mu$ and $\lambda_p$.

- Rough convergence. Parameters $\lambda$ and $\mu_p$ approximation is far away when using coarse meshes and converge to a wrong solution after some few iterations.

5.2. Sensitivity Analysis

Sensitivity analysis is an important issue for assessing the robustness of the method and evaluating the measures that preponderantly influence the parameters. Moreover, potential errors might derive from the performed measurements in a practitioner’s office and one has to assess the consequences of such errors on
the parameters. At last, it is a good indicator of ill-posed inverse problem. Minimising the cost function $\Lambda \rightarrow E_h(\Lambda)$ provides the relation $\mathbf{M} \rightarrow \Lambda(\mathbf{M})$, i.e. the best parameters that fit the measurements. The sensitivity analysis aims at assessing the impact of the measure variations on the parameters evaluation. Low sensitivity of parameter $R$ with respect to measure $m_1$ means that the relative partial derivative

$$d_{R,m_1} = \frac{1}{R(\mathbf{M})} \frac{\partial R}{\partial m_1} \ll 1$$

while an high sensitivity is obtained if $d_{R,m_1} \gg 1$.

We assume that practitioners provide measurements $m_i$ with a relative error of 10%. To simulate the error impact, we define two new sets of parameter $\Lambda(m_1,+)$ and $\Lambda(m_1,-)$ corresponding to a positive and a negative variation of 10% of parameter $m_i$, the others being fixed. To calculate the sensibility value we approximate the relative derivative with the expression

$$\frac{\partial \Lambda_j}{\partial m_i} = \frac{\Lambda_j(m_1,+)-\Lambda_j(m_1,-)}{0.2 \cdot m_i \cdot \Lambda_j(\mathbf{M})}.$$  \hspace{1cm} (12)

It results a $6 \times 15$ matrix that approximate the Jacobi matrix. Computations are achieved with the very-thin mesh and we report the results in Table 6 with the parameters in row and the measures in columns.

Such data have to be handled and interpreted with caution since the variation derives from the ±10% we prescribe and computational errors due to the discretization may degrade the approximation accuracy. Therefore we have two variations to take into consideration

$$\delta \Lambda_j(m_i) = \Lambda_j(m_i,+)-\Lambda_j(m_i,-)$$

which corresponds to the error of the variation of the measure $m_i$ and

$$\Delta \Lambda_j = \Lambda_j(\mathbf{M}) - \overline{\Lambda}_j$$

which corresponds to the error of the discretization with respect to the exact parameters $\overline{\Lambda}$. We classify the data in three groups. The red groups are the data where the discretization error is preponderant, namely $\delta \Lambda_j(m_i) < 2 \Delta \Lambda_j$. It results that the derivative approximation is wrong since the errors are mainly due to the discretization. The yellow corresponds to the data where $2 \Delta \Lambda_j < \delta \Lambda_j(m_i) < 5 \Delta \Lambda_j$, i.e. the discretization errors and the variations are of same order. At last, the green group provides the data such that $5 \Delta \Lambda_j <$...
\[ \delta \Lambda_j(m_i) \] which guarantee that the derivative is the consequence of the 10% variation of parameter \( m_i \) and can be considered as a correct approximation of the partial derivative. The values in the table are mapped according to the colours of reliability.

| Parameter | Sensitivity |
|-----------|-------------|
| \( m_1 \) | 1 | 2 | 3 | 4 | 5 |
| \( m_2 \) | 1.02553132 | 2.05653969 | -0.63456873 | -2.31247258 | 3.31231518 |
| \( m_3 \) | -1.28994942 | 3.97082587 | -0.53201235 | -5.67988684 | 7.21505134 |
| \( \mu \) | 1.52031861 | 7.15803654 | -5.96097052 | -6.65881365 | 54.63450396 |
| \( \lambda_p \) | 2.07340448 | 16.54477091 | -14.31235176 | -7.64070055 | 2.38295078 |
| \( \mu_p \) | 0.79831615 | 18.58978267 | -16.37506465 | -3.36883017 | 131.65886616 |

Fig. 6: Table with the sensibility analysis of the parameters. (Red - unreliable values, yellow - slightly reliable values, green - reliable values)

Measures \( m_{11}, m_6 \) and \( m_{11} \) corresponding to the volume for the three positions have no significant impact on the evaluation of the parameters since the error deriving from the discretization are too large keeping from evaluating an approximation of the derivative. Such measurements should be discarded from the cost function. In the same way, the max \( y \) given by the measure \( m_{15} \) is not relevant and should be eliminated. The other measures are valid since they provide an admissible approximation of the partial derivatives. We sum up hereafter the main conclusions.

- Parameters \( R \) and \( H \) do not present a significant sensitivity to the measurements except with measurement \( m_9 \) corresponding to the \( \min z \) in the lying down position.
- We observe a great sensitivity for \( \lambda \) and \( \mu \) with respect to the measurements that act in opposite way since we have positive and negative derivatives.
- Parameter \( \lambda_p \) and \( \mu_p \) have a high sensitivity with respect to the measurements over the 3 positions with positive or negative derivative leading to antagonist effects.

### 6. Conclusion

We have proposed a simple model to simulate breast displacement and identify the mechanical parameters associated to the Neo-Hookean model. The main points are the introduction of a two-parameter
geometry for the gravity-free configuration and a user friendly set of measurements that does not require sophisticated equipment and turns to be adequate for routinely operations. The sensitivity study with respect to the measure and the mesh has been carried out in a synthetic context to demonstrate the robustness of the method and its capacity to retrieve the good parameters with a controlled range of error.

7. Conflict of Interest

The authors declare that there is no conflict of interest regarding the content of this article.

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