THE MEAN VALUE PROBLEM OF SMALE’S PROBLEMS

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Abstract. We show two results of mean value problem, Smale’s mean value problem is comprehensively solved in this paper.

Introduction

In 1998, Smale proposed some mathematics problems which need to be solved[1] to a request from Vladimir Arnold, to propose a list of problems for the 21st century. The mean value problem on the list is still open in full generality. Early in 1981, Smale posed the mean value problem of complex polynomial[2]: Given a complex polynomial $f$ of degree $d \geq 2$ and a complex number $s$, is there a critical point $\theta$ of $f$ (ie, $f'(\theta) = 0$) such that $|\frac{f(s)-f(\theta)}{s-\theta}| \leq |f'(s)|$, $c = 1$. Mathematicians have obtained some partial results[3, 4, 5].

In this paper, through the root locus method established by us. We prove two results: Given a complex polynomial $f$ of degree $d \geq 2$ and a complex number $s$. For every critical point $\theta_i$ (ie, $f'(\theta_i) = 0$), $i = 1, 2, \cdots, n - 1$, there exists its adjacent domain $\Omega_i$. For all points in the extended complex plane except the adjacent domain $\Omega = \{\Omega_i\}, i = 1, 2, \cdots, n - 1$, such that $|\frac{f(s)-f(\theta_i)}{s-\theta_i}| \leq |f'(s)|$.

Namely, the inequality in Smale’s mean value problem is true.

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Namely, the inequality in Smale’s mean value problem is true, when symbol in inequality is converse.

1. Root Locus Of the meromorphic function $W(s)$ and Their Properties

In textbooks of automatic control theory, the factor at the left side of the root locus equation is the rational fraction function of the constant coefficient. The root locus equation is only concerning two degree of 0 and 180 degree numbers which obtains the real number values[6]. So, the root locus equations and the results of the root locus in automatic control theory are all very special and limited. The proofs of results in automatic control theory are not comprehensive and accurate.
We need to study the more general root locus equations of the meromorphic function $W(s)$. Let $K = \left| \prod_{j=1}^{m} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} \frac{G_{p_j}}{G(s)} \right|$. The $K$ is the reciprocal of the modulus of meromorphic function $W(s) = u(\sigma, t) + iv(\sigma, t)$. After $K$ and meromorphic function $W(s)$ are multiplied, the product result is the unit complex value of meromorphic function $W(s)$.

**Definition 1.1.** In the extended complex plane $\mathbb{C} \cup \{ \infty \}$, the equation (1.1) is called as the root locus equation.

$$K G(s) \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{l_j}(s) \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{j_p}(s) = a + ib$$

In which, $a + ib$ is the unit complex number value of meromorphic function $W(s)$. $\alpha = 2\pi + \arg(\frac{s}{\gamma})$. The zeros $z_l$ and poles $p_j$ are points in $\mathbb{C} \cup \{ \infty \}$, and may be no conjugate.

**Lemma 1.2.** After the coincident and finite zeros and poles of Eq(1.1) are cancelled, for any finite zero of Eq(1.1), its $K$ value is $K = +\infty$; for any finite pole of Eq(1.1), its $K$ value is $K = 0$.

**Proof.** The Eq(1.1) can be transformed to the following characteristic equation.

$$K G(s) \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{l_j}(s) = a + ib K \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{j_p}(s) = 0.$$ Obviously, if $K = 0$, all roots of the pole factor $\prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{j_p}(s)$ are the roots of characteristic equation. Conversely, all the roots of characteristic equation if $K = 0$ are all of the roots of the pole factor. So, at poles of Eq(1.1), $K = 0$.

The Eq(1.1) can be transformed to its another characteristic equation. $G(s) \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{l_j}(s) = a + ib K \prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{j_p}(s) = 0$. If $K = +\infty$, all roots of the zero factor $\prod_{j=1}^{n} \left( 1 - \frac{\sigma}{\gamma_j} \right)^{\beta_j} G_{l_j}(s)$ are the roots of the characteristic equation. Conversely, all roots of the characteristic equation when $K = +\infty$ are the roots of the zero factor. So, at zeros of Eq(1.1), $K = +\infty$. \(\square\)

According to the expression of $K$ of the root locus equation (1.1) and according to Lemma 1.2, it is obvious that the $K$ values are continuous concerning the complex variable $s$ in $\mathbb{C} \cup \{ \infty \}$. So, we can prove Theorem 1.3 simply.

**Theorem 1.3.** Let $s \in \mathbb{C} \cup \{ \infty \}$, the $K$ value of $s \in \mathbb{C} \cup \{ \infty \}$ takes all of the non-negative real number value from 0 to the $+\infty$.

In complex analysis, the phase angle is namely argument[2]. In automatic control theory, it is called as the phase angle. Here, we need to use name of the phase angle. In this paper, $K$ is called as the gain.

Let $\Delta = \{ s = \sigma + it \in \mathbb{C} \cup \{ \infty \}, s \text{ is not zero or pole of } W(s) \text{ and Eq(1.1)} \}$.

**Lemma 1.4.** For any point $s \in \Delta$, the phase angle of meromorphic function $W(s)$ can be written as the $\varphi(\sigma, t)$ function: $\varphi(\sigma, t) = \arg \left( \frac{\sum_{j=1}^{n} \gamma_j \arg \left( \frac{\sigma}{\gamma_j} \right) - \gamma_j \arg \left( \frac{\Delta_y}{\sigma} \right) + \arg \left( \frac{\sum_{j=1}^{n} \gamma_j \arg \left( \frac{\sigma}{\gamma_j} \right)}{G_{l_j}(\sigma, t)} \right) - \sum_{j=1}^{n} (\beta_j \arg \left( \frac{\Delta_y}{\sigma - \gamma_j} \right) - \beta_j \arg \left( \frac{\Delta_y}{\sigma} \right) + \arg \left( \frac{\gamma_j \arg \left( \frac{\sigma}{\gamma_j} \right)}{G_{p_j}(\sigma, t)} \right)}{\sum_{j=1}^{n} (\beta_j \arg \left( \frac{\Delta_y}{\sigma - \gamma_j} \right) - \beta_j \arg \left( \frac{\Delta_y}{\sigma} \right) + \arg \left( \frac{\gamma_j \arg \left( \frac{\sigma}{\gamma_j} \right)}{G_{p_j}(\sigma, t)} \right)} \right)$. Let $s = (\sigma, t) \in \Delta$, then the phase angle condition equation of Eq(1.1) on $s$ is:

$$\varphi(\sigma, t) = 2\pi + \arg(b/a)$$

In which, $q$ is an integer number.
Lemma 1.5. Let Eq(1.1) be the root locus equation of $2q\pi + \alpha$ degree, for any point $s = (\sigma, t) \in \Delta$, if the point $s$ satisfies the phase angle condition equation (1.2) of $2q\pi + \alpha$ degree, then it must be on the root locus of $2q\pi + \alpha$ degree of Eq(1.1).

Proof. Assuming that a point $s_1 = (\sigma_1, t_1) \in \Delta$ satisfies the phase angle condition equation (1.2), $\varphi(\sigma_1, t_1) = 2q\pi + \arg(b_1/a_1)$. The phase angle of $W(s)$ is $\varphi(\sigma, t)$, this phase angle expression is same as the expression of the left side of Eq(1.2).

Because the point $s_1$ satisfies the phase angle condition equation (1.2), when the point $s_1$ is substituted into the function $W(s)$, $W(s)$ obtains a complex value, according to the phase angle expression of Eq(1.2) of the point $s_1$ and the condition which a point $s_1$ satisfies the phase angle condition equation (1.2), this complex value can be written as: $K^*_1(a_1 + ib_1)$, and $\alpha_1 = \arg(b_1/a_1)$, $K^*_1$ is the modulus of function $W(s)$ of point $s_1$. $K^*_1$ is a non-zero positive real value. $\alpha_1$ is the phase angle of the function $W(s)$ of the point $s_1$, and it is the value of the right side of Eq(1.2).

If we bring the point $s_1$ into the gain expression $\frac{\prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\beta G_{jp}(s)}{\prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\gamma G_{lz}(s)}$, according to the definition of the gain, we can obtain an unique gain $K_1$, $K_1 = \frac{\prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\beta G_{jp}(s_1)}{\prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\gamma G_{lz}(s_1)}$, the gain $K_1$ is a reciprocal of the modulus $K^*_1$ of $W(s)$ of the point $s_1$. Further obtain $K^*_1$. $K^*_1 = 1/K_1$. For that the gain $K_1$ multiplied the expression $G(s_1) \prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\gamma G_{lz}(s_1)/\prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\beta G_{jp}(s_1)$, we have, $K_1 G(s_1) \prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\gamma G_{lz}(s_1)/\prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\beta G_{jp}(s_1) = a_1 + ib_1$. This equation is a concrete situation which a point $s_1$ satisfies Eq(1.1). The previous results prove that the point $s_1$ is a root of Eq(1.1).

So, the point $s_1$ satisfies Eq(1.1). Therefore, it is proved that the point $s_1$ is on the root locus of Eq(1.1), and which its gain is $K_1$ and its degree is $2q\pi + \alpha_1$. Hence, if the point $s_1$ satisfies Eq(1.2), then the point $s_1$ is on the root locus of Eq(1.1), and which its gain is $K_1$ and its degree is $2q\pi + \alpha_1$. □

Lemma 1.6. Let Eq(1.1) be the root locus equation of $2q\pi + \alpha$ degree, for any point $s = (\sigma, t) \in \Delta$, if the point $s$ is on the root locus of $2q\pi + \alpha$ degree of Eq(1.1), then it must satisfy the phase angle condition equation of $2q\pi + \alpha$ degree of Eq(1.2).

Proof. Assume that the point $s_2 = (\sigma_2, t_2) \in \Delta$ is an arbitrary point on the root locus of Eq(1.1). The points which satisfy Eq(1.1) are all on the root locus of Eq(1.1). So, the point $s_2$ surely satisfies Eq(1.1). When $s_2$ is substituted into Eq(1.1), $K_2 G(s_2) \prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\gamma G_{lz}(s_2)/\prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\beta G_{jp}(s_2) = a_2 + ib_2$. According to Theorem 1.3, here, the gain $K_2$ is a positive real number, its phase angle of $K_2$ is $2q_1\pi$, the phase angle of the right side of the equation in this paragraph on the point $s_2$ is $2q_2\pi + \arg(b_2/a_2) = 2q_2\pi + \alpha_2$.

The factor of the left side of the equation in last paragraph on the point $s_2$ can be looked as two factors. One is $K_2$, another is $G(s_2) \prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\gamma G_{lz}(s_2)/\prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\beta G_{jp}(s_2)$. The phase angle of the factor of the left side of the equation in last paragraph on the point $s_2$ is equal to the summation of two phase angles of two factors of $K_2$ and $G(s_2) \prod_{j=1}^{m}(1-\frac{z_j}{z_1})^\gamma G_{lz}(s_2)/\prod_{j=1}^{n}(1-\frac{p_j}{p_1})^\beta G_{jp}(s_2)$. The phase angle of the factor of the left side of the equation in last paragraph on the point $s_2$ is equal to the phase angle of the right side of the equation in the last paragraph on the point $s_2$. The phase angle $2q_2\pi + \alpha_2$ subtracts the phase angle $2q_1\pi$ is equal to $2q_2\pi + \alpha_2 - 2q_1\pi = 2q_2\pi + \alpha_2$. 


The difference of the phase angle of the factor $a_2 + ib_2$ and the factor $K_2$ is: $2q\pi + \alpha_2$. So, the phase angle of expression $G(s_2) \prod_{j=1}^{n}(1 - \frac{a_j}{s_2}) = 2q\pi + \alpha_2 = 2q\pi + \text{arg}(b_2/a_2)$. We can obtain: the phase angle of $W(s)$ is $\varphi(\sigma, t) = \arg(G(s_2) \prod_{j=1}^{n}(1 - \frac{a_j}{s_2})e^{j\gamma_2}) = 2q\pi + \alpha_2$. Therefore, the phase angle of expression $G(s_2) \prod_{j=1}^{n}(1 - \frac{a_j}{s_2})$ is $2q\pi + \alpha_2 = 2q\pi + \text{arg}(b_2/a_2)$. We can obtain: the phase angle of $W(s)$ is $\varphi(\sigma, t) = 2q\pi + \text{arg}(b_2/a_2)$, the point $s_2$ satisfies the phase angle condition equation (1.2). Hence, if the point $s_2$ is on the root locus of Eq(1.1), which its gain is $K_2$ and its degree is $2q\pi + \alpha_2$, and satisfies Eq(1.1), then point $s_2$ satisfies Eq(1.2).

Lemma 1.5 gives the sufficient condition result of Theorem 1.7. Lemma 1.6 gives the necessary condition result of Theorem 1.7. So, sum up Lemma 1.5 and Lemma 1.6, we can obtain the following theorem.

**Theorem 1.7.** Let Eq(1.1) be $2q\pi + \alpha$ degree and $s = (\sigma, t) \in \Delta$ be an arbitrary point. A necessary and sufficient condition for that the point $s$ is on the root locus of $2q\pi + \alpha$ degree of Eq(1.1) is that point $s$ whether or not satisfies the phase angle condition equation (1.2).

**Definition 1.8.** Let $\Xi$ be a point set in $\mathbb{C} \cup \{\infty\}$, that are consist of all of points on the path of roots of Eq(1.1) traced out in $\mathbb{C} \cup \{\infty\}$ as $2q\pi + \alpha = 2q\pi + \text{arg}(\frac{b}{2})$ is a constant. That path of Eq(1.1) in $\mathbb{C} \cup \{\infty\}$ is called as the root locus of Eq(1.1). Namely, the set $\Xi$ is the root locus of the $2q\pi + \alpha$ degree.

The subset of $\Xi$ is part or entire root locus except zeros and poles. And the empty set $\emptyset$ is the case when zero and pole are coincident, the case has no root locus. Let $\tau$ be the collection of subsets of $\Xi$, the ordered pair $(\Xi, \tau)$ satisfying the following properties: $\emptyset$ and $\Xi$ itself are both open in $\tau$, the intersection of any two open sets is open in $\tau$, and the union of every collection of open sets is open in $\tau$. So, the collection $\tau$ is a topology on $\Xi$, and the ordered pair $(\Xi, \tau)$ is a topological space.

The meromorphic function $W(s)$ is the factor at Eq(1.1)’s left side. So, the $\varphi(\sigma, t)$ function which is in Lemma 1.4 can be used to compute the degree of the root locus of Eq(1.1). The factor at the right side of Eq(1.1) is the computing value.

**Definition 1.9.** When the phase angle of meromorphic function $W(s)$ at the left side of the root locus equation (1.1) is $2q\pi + \alpha$ degree, we call the root locus equation (1.1) as the $2q\pi + \alpha$ degree root locus equation.

In $\mathbb{C} \cup \{\infty\}$, the degree number of the phase angle of meromorphic function $W(s)$ at the left side of the root locus equation (1.1) is the degree number of the root locus of the root locus equation (1.1). So, we can give a definition of the degree of the root locus of Eq(1.1).

**Definition 1.10.** When the phase angle of meromorphic function $W(s)$ at the left side of the root locus equation (1.1) is $2q\pi + \alpha$ degree, we call the root locus of the root locus equation (1.1) as $2q\pi + \alpha$ degree root locus.

**Lemma 1.11.** For all finite zeros of Eq(1.1), they are on the root locus of all degree numbers of Eq(1.1) from $2q\pi$ degree to $2q\pi + 2\gamma\pi$ degree.

**Proof.** Assuming that the point $zp$ is an arbitrary finite zeros of Eq(1.1). When the point $zp$ is substituted into meromorphic function $W(s)$ at the left side of Eq(1.1),
we have: \[G(pz) \sum_{z=1}^{n} \frac{(1 - \frac{z}{pz})^n G_{is}(pz)}{\prod_{z=1}^{m} (1 - \frac{z}{pz})^n G_{js}(pz)} = 0.\] This equation can also be expressed as:

\[G(pz) \sum_{z=1}^{n} \frac{(1 - \frac{z}{pz})^n G_{is}(pz)}{\prod_{z=1}^{m} (1 - \frac{z}{pz})^n G_{js}(pz)} = K_{pz} e^{i \theta_{pz}} = 0.\]

In this equation, \(K_{pz}\) is the modulus of the function \(W(s)\) and \(\theta_{pz}\) is the phase angle of the function \(W(s)\). So, \(K_{pz} = 0\), and \(\theta_{pz} = 2\pi + \alpha\), \(K_{pz} e^{i(2\pi + \alpha)} = 0 \ast \left(\cos(\theta_{pz}) + i \sin(\theta_{pz})\right) = 0\) is true, and \(\theta_{pz}\) is an arbitrary degree number from 2\(\pi\) degree to 2\(\pi\) + 2\(\gamma\)\(\pi\) degree.

For the point \(pz\) that lets the left side of Eq(1.1) obtain 0, no matter what phase angle it is, since its modulus is 0. \(\cos(\theta_{pz}) + i \sin(\theta_{pz})\) is the non-zero and non-infinity, the modulus of the factor \(\cos(\theta_{pz}) + i \sin(\theta_{pz})\) is 1. \(\theta_{pz}\) represents an arbitrary degree number, which shows: no matter what value \(\theta_{pz}\) is, there is \(K_{pz} e^{i(2\pi + \alpha)} = 0 \ast \left(\cos(\theta_{pz}) + i \sin(\theta_{pz})\right) = 0\).

This proves: For the finite zero \(pz\) of Eq(1.1), when it is substituted into the meromorphic function \(W(s)\), for the phase angle of the arbitrary 2\(\pi\) + \(\alpha\) degree number of the meromorphic function \(W(s)\), its values are all equal to 0. According to Definition 1.10, the phase angle of the meromorphic function \(W(s)\) is namely the phase angle of the root locus of Eq(1.1). Because 2\(\pi\) + \(\alpha\) is an arbitrary degree number, when it obtains all of degree numbers, this shows the finite zeros of Eq(1.1) are simultaneously on the root locus of all of degree numbers of Eq(1.1) from 2\(\pi\) degree to 2\(\pi\) + 2\(\gamma\)\(\pi\) degree.

\[\square\]

**Lemma 1.12.** For all finite poles of Eq(1.1), they are on the root locus of all degree numbers of Eq(1.1) from 2\(\pi\) degree to 2\(\pi\) + 2\(\beta\)\(\pi\) degree.

**Proof.** Assuming that the point \(pz\) is an arbitrary finite pole of Eq(1.1). When the point \(pz\) is substituted into the meromorphic function \(W(s)\), we can obtain a value \(G_{pz} = G(pz) \sum_{z=1}^{n} \frac{(1 - \frac{z}{pz})^n G_{is}(pz)}{\prod_{z=1}^{m} (1 - \frac{z}{pz})^n G_{js}(pz)} = \infty.\)

In complex analysis, 0 can be written as: \(0 \ast e^{i \theta} = 0 \ast \left(\cos(\theta) + i \sin(\theta)\right).\) The points of non-zero and non-infinity finite values can also be written as: \(k \ast e^{i \theta} = k \ast \left(\cos(\theta) + i \sin(\theta)\right).\) The infinity can also be written as: \((+\infty) \ast e^{i \theta} = (+\infty) \ast \left(\cos(\theta) + i \sin(\theta)\right).\) In which, \(\left(\cos(\theta) + i \sin(\theta)\right)\) is a non-zero and non-infinity, and its modulus is 1 of the unit complex number value. For \((+\infty) \ast e^{i \theta} = (+\infty) \ast \left(\cos(\theta) + i \sin(\theta)\right),\) no matter what value \(\theta = 2\pi + \alpha\) obtains on the unit circle in \(C \cup \{\infty\},\) the values of this expression all obtain the infinity.

The above equation can also be expressed as: \(G_{pz} = G(pz) \sum_{z=1}^{n} \frac{(1 - \frac{z}{pz})^n G_{is}(pz)}{\prod_{z=1}^{m} (1 - \frac{z}{pz})^n G_{js}(pz)} = K_{pz} e^{i \theta_{pz}} = \infty.\) In this equation, \(K_{pz}\) is the modulus of the meromorphic function \(W(s)\) and \(\theta_{pz}\) is the phase angle of the meromorphic function \(W(s)\). So, \(K_{pz} = +\infty,\) and \(\theta_{pz} = 2\pi + \alpha,\) \(K_{pz} e^{i \theta} = (+\infty) \ast \left(\cos(\theta) + i \sin(\theta)\right) = \infty\) is true, and \(\theta_{pz} = 2\pi + \alpha\) is an arbitrary degree number from 2\(\pi\) degree to 2\(\pi\) + 2\(\beta\)\(\pi\) degree.

For the finite pole \(pz\) of Eq(1.1), when it is substituted into the meromorphic function \(W(s)\), for the phase angle of the arbitrary 2\(\pi\) + \(\alpha\) degree number of the meromorphic function \(W(s)\), its values are all equal to infinity. According to Definition 1.10, the phase angle of the meromorphic function \(W(s)\) is namely the phase angle of the root locus of Eq(1.1). Because 2\(\pi\) + \(\alpha\) is an arbitrary degree number from 2\(\pi\) degree to 2\(\pi\) + 2\(\beta\)\(\pi\) degree, when it obtains all of degree numbers, this shows the finite poles of Eq(1.1) are simultaneously on the root locus of all of degree numbers of Eq(1.1) from 2\(\pi\) degree to 2\(\pi\) + 2\(\beta\)\(\pi\) degree. \[\square\]
Lemma 1.13. All of the root locus of arbitrary $2q\pi + \alpha$ degree number of Eq(1.1) are originated from poles of Eq(1.1), and are finally received at zeros of Eq(1.1).

Proof. The arbitrary $2q\pi + \alpha$ degree number root locus of Eq(1.1) are the curves in $\mathbb{C} \cup \{\infty\}$. So, they all need to have their own origination points and receiving points. According to the relationship between the points in $\mathbb{C} \cup \{\infty\}$ and the root locus of Eq(1.1), the points in $\mathbb{C} \cup \{\infty\}$ can be divided into four types, one is the poles of Eq(1.1), the other is the zero of Eq(1.1), the general finite zeros or poles of Eq(1.1), and the last type is the infinity points. According to the relationship between the points in $\mathbb{C} \cup \{\infty\}$ and the root locus of Eq(1.1), the points in $\mathbb{C} \cup \{\infty\}$ only has the phase angles of only one degree number and non-zero finite gain value. It can not be the originating point and receiving point of the root locus of Eq(1.1).

When the infinite point in $\mathbb{C} \cup \{\infty\}$ let the meromorphic function $W(s)$ obtain a constant $A$. The gain values of the infinite point in $\mathbb{C} \cup \{\infty\}$ are the constant $|A|$. The phase angles of the meromorphic function $W(s)$ are all equal to $2q\pi + \arg(A)$ degree. So, these infinite points in $\mathbb{C} \cup \{\infty\}$ are also the ordinary points of the finite values with the gain $|A|$ on the $2q\pi + \arg(a)$ degree root locus. They can not originate or receive the root locus.

When the infinite points in $\mathbb{C} \cup \{\infty\}$ are the infinite zeros or poles of Eq(1.1). On the infinite zero or pole, the gain value is $K = +\infty$ or $K = 0$ respectively.

The finite poles and zeros of Eq(1.1) are on the root locus of all of degree numbers of Eq(1.1), and the infinite number of the different degree numbers root locus are at the same one point, so, they satisfy the condition that the root locus can be originated or received. On the finite and infinite poles of Eq(1.1), there is $K = 0$. On the finite and infinite zeros of Eq(1.1), there is $K = +\infty$. Thus, we can let the finite and infinite poles of Eq(1.1) as the origination points of the root locus. The finite and infinite zero points of Eq(1.1) are the receiving points of the root locus. \qed

Definition 1.14. The origination points of the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) are the points on the $2q\pi + \alpha$ degree root locus that their corresponding gain values equal to zero in $\mathbb{C} \cup \{\infty\}$, $K = 0$.

Definition 1.15. The receiving points of the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) are the points on the $2q\pi + \alpha$ degree root locus that their corresponding gain values equal to infinity in $\mathbb{C} \cup \{\infty\}$, $K = +\infty$.

Lemma 1.16. (1). For any infinite point of $\mathbb{C} \cup \{\infty\}$, if it satisfies

$$K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{n} |1 - \frac{1}{s^{\alpha}}| G_{lp}(s)}{|G(s)| \prod_{j=1}^{n} |1 - \frac{1}{s^{\alpha}}| G_{z}(s)} = 0,$$

then it is the infinite pole of the root locus of Eq(1.1). These infinite points must be the origination points of $2q\pi + \alpha$ degrees root locus of Eq(1.1) in $\mathbb{C} \cup \{\infty\}$.

(2). For any infinite point of $\mathbb{C} \cup \{\infty\}$, if it satisfies

$$K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{n} |1 - \frac{1}{s^{\alpha}}| G_{lp}(s)}{|G(s)| \prod_{j=1}^{n} |1 - \frac{1}{s^{\alpha}}| G_{z}(s)} = A,$$

in which $A \neq 0$ is a finite value, then it is a general point that its origin is a finite value. These infinite points are the general points on the root locus of Eq(1.1) in $\mathbb{C} \cup \{\infty\}$.

(3). For any infinite point of $\mathbb{C} \cup \{\infty\}$, if it satisfies
\[ K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{\nu} |1 - \frac{s}{G_p(s)}|}{|G(s)||\prod_{j=1}^{\nu} |1 - \frac{s}{G_j(s)}|} = +\infty, \text{ then the infinite point is the infinite zero of the root locus of } Eq(1.1). \] These infinite points must be the receiving points of \( 2q\pi + \alpha \) degrees root locus of \( Eq(1.1) \) in \( C \cup \{\infty\} \).

**Proof.** (i). For any infinite points of \( C \cup \{\infty\} \), if they satisfy
\[ K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{\nu} |1 - \frac{s}{G_p(s)}|}{|G(s)||\prod_{j=1}^{\nu} |1 - \frac{s}{G_j(s)}|} = 0, \] according to the definition of the origination points of the arbitrary degree root locus of \( Eq(1.1) \), the infinity points on the root locus of the arbitrary \( 2q\pi + \alpha \) degree of \( Eq(1.1) \) are also origination points of the arbitrary \( 2q\pi + \alpha \) degree root locus of \( Eq(1.1) \).

(ii). For any infinite points of \( C \cup \{\infty\} \), if they satisfy
\[ K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{\nu} |1 - \frac{s}{G_p(s)}|}{|G(s)||\prod_{j=1}^{\nu} |1 - \frac{s}{G_j(s)}|} = A, \] in which \( A \neq 0 \) is a finite value. So, all of the infinity points in \( C \cup \{\infty\} \) that their gains are the finite values are all general points. The infinity points in \( C \cup \{\infty\} \) are the general points that their gains are the finite values \( A \).

(iii). For any infinite points of \( C \cup \{\infty\} \), if they satisfy
\[ K = \lim_{|s| \to +\infty} \frac{\prod_{j=1}^{\nu} |1 - \frac{s}{G_p(s)}|}{|G(s)||\prod_{j=1}^{\nu} |1 - \frac{s}{G_j(s)}|} = +\infty. \] According to the definition of the receiving points of the arbitrary degree root locus of \( Eq(1.1) \), the infinity points on the root locus of the arbitrary \( 2q\pi + \alpha \) degree of \( Eq(1.1) \) are also receiving points of the arbitrary \( 2q\pi + \alpha \) degree root locus of \( Eq(1.1) \).

Sum up Lemma 1.13 and Lemma 1.16, we can obtain Theorem 1.17.

**Theorem 1.17.** In \( C \cup \{\infty\} \), all root locus of the arbitrary \( 2q\pi + \alpha \) degree number of \( Eq(1.1) \) are originated from their finite poles or the infinite poles. And these root locus are received by their finite zeros or the infinite zeros.

## 2. Some Properties of Root Locus of \( W(s) \)

In this section, we show the expressions of the argument\(^{[7]}\) of meromorphic function \( W(s) \) in \( C \cup \{\infty\} \).

(1). When \( u(\sigma, t) > 0, v(\sigma, t) \geq 0, \cdot \varphi(\sigma, t) = arg\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) = \arctan\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) \).

(2). When \( u(\sigma, t) < 0, v(\sigma, t) \geq 0, \cdot \varphi(\sigma, t) = arg\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) = \pi - \arctan\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) \).

(3). When \( u(\sigma, t) < 0, v(\sigma, t) < 0, \cdot \varphi(\sigma, t) = arg\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) = \pi + \arctan\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) \).

(4). When \( u(\sigma, t) > 0, v(\sigma, t) < 0, \cdot \varphi(\sigma, t) = arg\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) = -\arctan\left(\frac{v(\sigma, t)}{u(\sigma, t)}\right) \).

We need to create a sub-set \( \Delta_u \) of set \( \Delta \). Let \( \Delta_u = \{ s = (\sigma + it) \in C \cup \{\infty\}, s \) is not zero or pole of \( W(s) \), and there is \( u(\sigma, t) \neq 0 \} \).

**Lemma 2.1.** Let \( s = (\sigma, t) \in \Delta_u \), then we have: \( \frac{\partial \varphi}{\partial t}(\sigma, t) = \frac{\varphi'(\sigma, t)u(\sigma, t) - \varphi(\sigma, t)v(\sigma, t)}{u(\sigma, t) + v(\sigma, t)} \).

This lemma can easily be obtained by partial derivative calculation.

Let \( W_3(s) = iW(s) \), so, \( W_3(s) = iu(\sigma, t) - v(\sigma, t) \). The phase angle of function \( W_3(s) \) is \( \varphi_3(\sigma, t) = \frac{\pi}{2} + \varphi(\sigma, t) \). \( \frac{\partial \varphi_3}{\partial t}(\sigma, t) = \frac{\partial \varphi}{\partial t}(\sigma, t) \). \( \frac{\partial \varphi_3}{\partial \sigma}(\sigma, t) = \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \). We need to create another sub-set \( \Delta_v \) of set \( \Delta \). \( \Delta_v = \{ s = (\sigma + it) \in C \cup \{\infty\}, s \) is not zero or pole of \( W(s) \), and there is \( v(\sigma, t) \neq 0 \} \). According to Lemma 2.1 and the result which we give in this segment, we can obtain Lemma 2.2.
Theorem 2.7. Let \( s = (\sigma, t) \in \Delta_v \), then we have:

\[
\frac{\partial \varphi}{\partial \sigma} (\sigma, t) = \frac{v'_u(\sigma, t)u(\sigma, t) - v'_u(\sigma, t)v(\sigma, t)}{u^2(\sigma, t) + v^2(\sigma, t)}.
\]

We need to create another sub-set \( \Delta_{vu} \) of set \( \Delta_v \). Let \( \Delta_{vu} = \{ s = (\sigma + it) \in \mathbb{C} \cup \{ \infty \}, s \) is not zero or pole of \( W(s) \), and there is \( u(\sigma, t) = 0 \}. \) According to Lemma 2.2, we can obtain Lemma 2.3.

Lemma 2.3. Let \( s = (\sigma, t) \in \Delta_{vu} \), then we have:

\[
\frac{\partial \varphi}{\partial t} (\sigma, t) = \frac{v'_u(\sigma, t)u(\sigma, t) - v'_u(\sigma, t)v(\sigma, t)}{u^2(\sigma, t) + v^2(\sigma, t)}.
\]

\( \Delta_u \cap \Delta_{vu} = \emptyset, \Delta_u \subseteq \Delta, \Delta_{vu} \subseteq \Delta, \) and \( \Delta = \Delta_u \cup \Delta_{vu} \). So, we can sum up Lemma 2.1 and Lemma 2.3, we can obtain Theorem 2.4.

Theorem 2.4. Let \( s = (\sigma, t) \in \Delta \), then we have:

\[
\frac{\partial \varphi}{\partial \sigma} (\sigma, t) = \frac{v'_u(\sigma, t)u(\sigma, t) - v'_u(\sigma, t)v(\sigma, t)}{u^2(\sigma, t) + v^2(\sigma, t)},
\]

\[
\frac{\partial \varphi}{\partial t} (\sigma, t) = \frac{v'_u(\sigma, t)u(\sigma, t) - v'_u(\sigma, t)v(\sigma, t)}{u^2(\sigma, t) + v^2(\sigma, t)}.
\]

In many textbooks, the complex variable function of one variable, the following results exist:

\[
\frac{dW(s)}{ds} = \frac{\partial u(\sigma, t)}{\partial \sigma} + i \frac{\partial v(\sigma, t)}{\partial \sigma} = \frac{\partial u(\sigma, t)}{\partial t} - i \frac{\partial v(\sigma, t)}{\partial t},
\]

\[
\frac{\partial v(\sigma, t)}{\partial \sigma} = -\frac{\partial u(\sigma, t)}{\partial \sigma}.
\]

Theorem 2.5. For any point \( s = (\sigma, t) \in \Delta \), we have:

\[
\frac{\partial \varphi}{\partial \sigma} (\sigma, t) = \text{Im} \left( \frac{dW(s)}{dW(s)} \right) = 0.
\]

Proof. \( \frac{dW(s)}{W(s)} = \frac{\partial u(\sigma, t)}{u(\sigma, t)} + \frac{i \partial v(\sigma, t)}{u(\sigma, t)}, \)

\[
= \frac{\partial u(\sigma, t)}{u(\sigma, t)} + \frac{i \partial v(\sigma, t)}{u(\sigma, t)} = \frac{\partial v(\sigma, t)}{v(\sigma, t)}. \]

Obviously, according to Theorem 2.4, we can obtain Theorem 2.5.

According to Theorem 2.5, obviously, we can obtain Theorem 2.6.

Theorem 2.6. For any point \( s = (\sigma, t) \in \Delta \), if \( \frac{\partial \varphi}{\partial \sigma} (\sigma, t) = 0 \), then \( \text{Re} \left( \frac{dW(s)}{W(s)} \right) = 0. \)

If \( \frac{\partial \varphi}{\partial t} (\sigma, t) = 0 \), then \( \text{Im} \left( \frac{dW(s)}{W(s)} \right) = 0. \) Conversely, if \( \text{Re} \left( \frac{dW(s)}{W(s)} \right) = 0 \), then \( \frac{\partial \varphi}{\partial \sigma} (\sigma, t) = 0. \)

If \( \text{Im} \left( \frac{dW(s)}{W(s)} \right) = 0 \), then \( \frac{\partial \varphi}{\partial t} (\sigma, t) = 0. \)

Theorem 2.7. Let \( s_0 \in \Delta \), except the situation which the point \( s_0 \) is an infinite point in \( \mathbb{C} \cup \{ \infty \} \). Then \( W(s) \) is the finite zero of \( W(s) \) if and only if \( \frac{\partial \varphi}{\partial \sigma} (\sigma, t) |_{s = s_0} = 0 \) and \( \frac{\partial \varphi}{\partial t} (\sigma, t) |_{s = s_0} = 0 \) hold simultaneously.

Proof. On the zeros \( s_0 \) of \( W'(s) \), \( \frac{dW(s)}{W(s)} |_{s = s_0} = 0 \), except the situation which the point \( s_0 \) is an infinite point in \( \mathbb{C} \cup \{ \infty \} \). So, the two equations \( \text{Re} \left( \frac{dW(s)}{W(s)} \right) |_{s = s_0} = 0 \) and \( \text{Im} \left( \frac{dW(s)}{W(s)} \right) |_{s = s_0} = 0 \) are true simultaneously, so, two partial derivative equations \( \frac{\partial \varphi}{\partial \sigma} (\sigma, t) |_{s = s_0} = 0 \) and \( \frac{\partial \varphi}{\partial t} (\sigma, t) |_{s = s_0} = 0 \) are true simultaneously.

Conversely, except zeros and poles of the meromorphic function \( W(s) \), and except the situation which the point \( s_0 \) is an infinite point in \( \mathbb{C} \cup \{ \infty \} \). When two partial derivative equations \( \frac{\partial \varphi}{\partial \sigma} (\sigma, t) |_{s = s_0} = 0 \) and \( \frac{\partial \varphi}{\partial t} (\sigma, t) |_{s = s_0} = 0 \) are true simultaneously. Then, two equations \( \text{Re} \left( \frac{dW(s)}{W(s)} \right) |_{s = s_0} = 0 \) and \( \text{Im} \left( \frac{dW(s)}{W(s)} \right) |_{s = s_0} = 0 \)
are true simultaneously, so, the equation \( \frac{dW(s)}{W(s)} \mid s = s_0 = 0 \) is true. So, when two partial derivative equations \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \mid s = s_0 = 0 \) and \( \frac{\partial \varphi}{\partial t}(\sigma, t) \mid s = s_0 = 0 \) are true simultaneously. Such points are the non-repeated zeros of derivative of the meromorphic function \( W(s) \).

On the infinite points in \( \mathbb{C} \cup \{ \infty \} \), \( \frac{dW(s)}{W(s)} \mid s = s_0 = 0 \), but, in most situations, two equations \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \mid s = s_0 = 0 \) and \( \frac{\partial \varphi}{\partial t}(\sigma, t) \mid s = s_0 = 0 \) can not be true simultaneously. So, we need to exclude the infinite zeros of \( W'(s) \) in this lemma.

Let \( \Delta_d = \{ s = (\sigma + it) \in \mathbb{C} \cup \{ \infty \}, s \) is not zero or pole of \( W(s) \), the point \( s \) is not an infinite point in \( \mathbb{C} \cup \{ \infty \}. s \) is not the finite zeros of derivative of the function \( W(s) \)\}.)

**Theorem 2.8.** For any point \( s = (\sigma, t) \in \Delta_d \), the partial derivative of the phase angle of \( W(s) \) concerning the variable \( \sigma \) and variable \( t \) cannot simultaneously equal to 0.

**Proof.** According to Theorem 2.7, if two equations \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) = 0 \) and \( \frac{\partial \varphi}{\partial t}(\sigma, t) = 0 \) hold simultaneously, then the point \( s \) is surely a finite zero of the derivative \( W'(s) \) of function \( W(s) \). So, except the finite zeros and poles of function \( W(s) \), except the infinite points in \( \mathbb{C} \cup \{ \infty \} \), and also except the finite zeros of derivative of the function \( W(s) \), and on other finite points in the extended complex plane \( \mathbb{C} \cup \{ \infty \} \), the partial derivative of the phase angle of the function \( W(s) \) concerning the variable \( t \) and variable \( \sigma \) cannot simultaneously equal to 0. □

**Remark 2.9.** Let \( s_0 = (\sigma_0 + i\tau_0) \in \Delta_d \), then, if \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \mid s = s_0 \neq 0 \); or, if \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \mid s = s_0 = 0 \), we have \( \frac{\partial \varphi}{\partial t}(\sigma, t) \mid s = s_0 = 0 \).

When \( 2\pi + \arg(b/a) \) is a constant, \( \varphi(\sigma, t) = 2\pi \tau + \arg(b/a) \), it is a function equation which contain two real variables, and it is the phase angle condition equation of Eq(1.1). According to Theorem 1.7, except zeros and poles of Eq(1.1), if the roots of this function equation exist in \( \mathbb{C} \cup \{ \infty \} \), the roots of Eq(1.1) also exist in \( \mathbb{C} \cup \{ \infty \} \). If the roots of this function equation constitute the root locus in \( \mathbb{C} \cup \{ \infty \} \), the roots of Eq(1.1) also constitute the root locus in \( \mathbb{C} \cup \{ \infty \} \).

In order to prove the existence and continuity of the roots of the implicit function equation \( \varphi(\sigma, t) = 2\pi \tau + \arg(b/a) \), we need to cite the implicit function theorem and involved concepts[7].

**Definition 2.10.** Let \( f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m \) be a continuously differentiable function. We think of \( \mathbb{R}^{n+m} \) as the Cartesian product \( \mathbb{R}^n \times \mathbb{R}^m \), and we write a point of this product as \( (x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_m) \). Starting from the given function \( f \), our goal is to construct a function \( g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \) whose graph \( (x, g(x)) \) is precisely the set of all \( (x, y) \) such that \( f(x, y) = 0 \).

As noted above, this may not always be possible. We will therefore fix a point \((a, b) = (a_1, \cdots, a_n, b_1, \cdots, b_m)\) which satisfies \( f(a, b) = 0 \), and we will ask for a \( g \) that works near the point \((a, b)\). In other words, we want an open set \( U \) of \( \mathbb{R}^n \) containing \( a \), an open set \( V \) of \( \mathbb{R}^m \) containing \( b \), and a function \( g : U \longrightarrow V \) such that the graph of \( g \) satisfies the relation \( f(x, y) = 0 \) on \( U \times V \), and that no other points within \( U \times V \) do so. In symbols, \( ((x, g(x)) | x \in U) = ((x, y) \in U \times V | f(x, y) = 0) \).
To state the implicit function theorem, we need the Jacobian matrix of $f$, which is the matrix of the partial derivatives of $f$. Abbreviating $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ to $(a, b)$, the Jacobian matrix is
\[
(Df)(a, b) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(a, b) & \cdots & \frac{\partial f_1}{\partial x_m}(a, b) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1}(a, b) & \cdots & \frac{\partial f_m}{\partial x_m}(a, b)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial f_1}{\partial y_1}(a, b) \\
\vdots \\
\frac{\partial f_m}{\partial y_1}(a, b)
\end{bmatrix}
= [X]Y.
\]

Where $X$ is the matrix of partial derivatives in the variables $x_i$ and $Y$ is the matrix of partial derivatives in the variables $y_j$. The implicit function theorem says that if $Y$ is an invertible matrix, then there are $U$, $V$, and $g$ as desired. Writing all the hypotheses together gives the following statement.

The implicit function theorem concerning variable $x$.

**Theorem 2.11.** Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates $(x, y)$. Fix a point $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ with $f(a, b) = c$, where $c \in \mathbb{R}^m$. If the Jacobian matrix $J_f(a, b) = (\partial f_i/\partial y_j)(a, b)$ is invertible, then there exists an open set $U$ containing $a$, an open set $V$ containing $b$, and a unique continuously differentiable function $g : U \to V$ such that $(x, g(x)) | x \in U = (x, y) | x \in U \times V | f(x, y) = c$.

The implicit function theorem concerning variable $y$.

**Theorem 2.12.** Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates $(x, y)$. Fix a point $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ with $f(a, b) = c$, where $c \in \mathbb{R}^n$. If the Jacobian matrix $J_f(a, b) = (\partial f_i/\partial x_j)(a, b)$ is invertible, then there exists an open set $V$ containing $b$, an open set $U$ containing $a$, and a unique continuously differentiable function $h : V \to U$ such that $(h(y), y) | y \in V = (x, y) | x \in U \times V | f(x, y) = c$.

**Lemma 2.13.** For the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1). And the locus which the points that satisfy the implicit function equation $\varphi(\sigma, t) = 2q\pi + arg(b/a)$ are generated in the two-dimensional real plane. The two loci are totally the same, and their properties like continuity are equivalent.

*Proof.* When $2q\pi + \alpha$ is a constant, the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) are curves that are constituted by roots of Eq(1.1). Theorem 1.7 proves that the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) can also be sufficiently and necessary determined by the real function which contains two real variables, namely, when $2q\pi + \alpha$ is a constant, the roots locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) can also be sufficiently and necessary determined by the implicit function equation $\varphi(\sigma, t) = 2q\pi + arg(b/a)$.

So, essentially, Theorem 1.7 proves: if the extended complex plane $\mathbb{C} \cup \{\infty\}$ is as a two-dimensional plane which let two real variables $\sigma$, $t$ be as its two coordinates, then, the root locus of the arbitrary $2q\pi + \alpha$ degree of Eq(1.1) are constituted by points which are determined by two real variables $\sigma$, $t$ that satisfy the implicit function equation $\varphi(\sigma, t) = 2q\pi + arg(b/a)$ in two-dimensional real plane.

Therefore, roots of Eq(1.1) in $\mathbb{C} \cup \{\infty\}$ and points which are determined by two real variables that satisfy the implicit function equation $\varphi(\sigma, t) = 2q\pi + arg(b/a)$ in two-dimensional plane are same. Namely, if the extended complex plane $\mathbb{C} \cup \{\infty\}$ and two-dimensional real plane which are determined by two real coordinates are considered as same one, when the root locus of the arbitrary $2q\pi + \alpha$ degree of
Eq(1.1) constitute the locus in \( \mathbb{C} \cup \{ \infty \} \), then, the points which are determined by two real coordinates that satisfy the implicit function equation \( \varphi(\sigma, t) = 2q\pi + \arg(b/a) \) also constitute the locus in two-dimensional real plane, two kinds of locus are fully same.

Because two kinds of root locus are completely same, the properties like existence and continuity of the two kinds of locus are equivalent. \( \square \)

According to Lemma 2.13, the problems of the complex variable functions are transformed into the problems of the real variable functions. So, we can use theorems of the real variable functions and the definition of the continuity to prove the existence and continuity of points which are determined by two real variables that satisfy the implicit function equation \( \varphi(\sigma, t) = 2q\pi + \arg(b/a) \) in two-dimensional real plane.

**Lemma 2.14.** Let \( s_0 = (\sigma_0 + it_0) \in \Delta_d \), \( s_0 \) satisfies \( \varphi(\sigma_0, t_0) = 2q\pi + \arg(b/a) \). There exists an open set \( U \) containing \( \sigma_0 \), an open set \( V \) containing \( t_0 \), and a unique continuously differentiable function \( g : U \rightarrow V \) such that \((\sigma, g(\sigma)) |_{\sigma \in U} = ((\sigma, t) \in U \times V | \varphi(\sigma, t) = 2q\pi + \arg(b/a)) \). Or, there exists an open set \( V \) containing \( t_0 \), an open set \( U \) containing \( \sigma_0 \), and a unique continuously differentiable function \( h : V \rightarrow U \) such that \((h(t), t) \in V \) = \((\sigma, t) \in U \times V | \varphi(\sigma, t) = 2q\pi + \arg(b/a)\) and \( \varphi(\sigma, t) = 2q\pi + \arg(b/a) \).

**Proof.** The function \( \varphi(\sigma, t) : R^2 \rightarrow R \) is a continuously differentiable function, \( R^2 \) have two coordinates \((\sigma, t)\).

For an arbitrary point \( s_0 = (\sigma_0 + it_0) \in \Delta \), it satisfies the implicit function equation \( \varphi(\sigma_0, t_0) = 2q\pi + \arg(b/a) \), where \( \arg(b/a) \in R \).

The point \( s_0 = (\sigma_0 + it_0) \in \Delta \) satisfies the condition of Theorem 2.11, according to Theorem 2.8, we can obtain: if \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \big|_{s=s_0} = 0 \), we have \( \frac{\partial \varphi}{\partial t}(\sigma, t) \big|_{s=s_0} \neq 0 \); or, if \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \big|_{s=s_0} = 0 \), we have \( \frac{\partial \varphi}{\partial t}(\sigma, t) \big|_{s=s_0} \neq 0 \).

The implicit function \( \varphi(\sigma, t) = 2q\pi + \arg(b/a) \) satisfies the implicit function theorem, so, according to the implicit function theorem concerning variable \( x \), we can obtain: if \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \big|_{s=s_0} \neq 0 \). There exists an open set \( U \) containing \( \sigma_0 \), an open set \( V \) containing \( t_0 \), and a unique continuously differentiable function \( g : U \rightarrow V \) such that \((\sigma, g(\sigma)) |_{\sigma \in U} = ((\sigma, t) \in U \times V | \varphi(\sigma, t) = 2q\pi + \arg(b/a)) \).

According to the implicit function theorem concerning variable \( y \). If \( \frac{\partial \varphi}{\partial \sigma}(\sigma, t) \big|_{s=s_0} \neq 0 \). There exists an open set \( V \) containing \( t_0 \), an open set \( U \) containing \( \sigma_0 \), and a unique continuously differentiable function \( h : V \rightarrow U \) such that \((h(t), t) \in V \) = \((\sigma, t) \in U \times V | \varphi(\sigma, t) = 2q\pi + \arg(b/a)\) and \( \varphi(\sigma, t) = 2q\pi + \arg(b/a) \).

\( \square \)

**Lemma 2.15.** Let \( s_0 = (\sigma_0 + it_0) \in \Delta_d \), then the root locus of arbitrary \( 2q\pi + \alpha \) degree of Eq(1.1) are all continuous on the points \( s_0 \).

**Proof.** According to the results of Lemma 2.14, we have: \( t = g(\sigma) \) is a continuous function inside an open set \( U \). So, for any given \( \varepsilon > 0 \), there surely exists a \( \delta_{\sigma_1} < \frac{\varepsilon}{\sqrt{(2)}} \), when \( |\sigma - \sigma_0| < \delta_{\sigma_1} < \frac{\varepsilon}{\sqrt{(2)}} \). Then, \( |t - t_0| = |g(\sigma) - g(\sigma_0)| < \frac{\varepsilon}{\sqrt{(2)}} \) is true. So, for any given \( \varepsilon > 0 \), let \( \delta_{\sigma_2} = \delta_{\sigma_1} < \frac{\varepsilon}{\sqrt{(2)}} \); when \( |\sigma - \sigma_0| < \delta_{\sigma_1} < \frac{\varepsilon}{\sqrt{(2)}} \), then, \( |t - t_0| = |g(\sigma) - g(\sigma_0)| < \frac{\varepsilon}{\sqrt{(2)}} \), and obtain: \( |\sigma + it - \sigma_0 - it_0| = |\sigma + ig(\sigma) - \sigma_0 - ig(\sigma_0)| = \sqrt{(\sigma - \sigma_0)^2 + (g(\sigma) - g(\sigma_0))^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon. \)
According to Lemma 2.14, we have: \( \sigma = h(t) \) is a continuous function inside an open set \( V \). So, for any given \( \varepsilon > 0 \), there surely exists a \( \delta_{\sigma_1} < 1 \sqrt{2} \), when \( |t - t_0| < \delta_{\sigma_1} \), then, \( |\sigma - \sigma_0| = |h(t) - h(t_0)| < \varepsilon \) is true. So, for any given \( \varepsilon > 0 \), let \( \delta_{\sigma_2} = \delta_{\sigma_1} < 1 \sqrt{2} \), when \( |t - t_0| < \delta_{\sigma_1} \), then, \( |\sigma - \sigma_0| = |h(t) - h(t_0)| < \varepsilon \), and obtain: \( |\sigma + it - \sigma_0 - it_0| = |h(t) + it - h(t_0) + it_0| = \sqrt{(t - t_0)^2 + (h(t) - h(t_0))^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon \).

So, it is proved that the root locus of the arbitrary \( 2q\pi + \alpha \) degree phase angle of Eq(1.1) are continuous on the point \((\sigma_0, t_0)\).

Because the point \((\sigma_0, t_0)\) is an arbitrary point except zeros and poles of Eq(1.1), except the infinite points in \( \mathbb{C} \cup \{\infty\} \) and except the derivative zeros of the function \( W(s) \) in \( \mathbb{C} \cup \{\infty\} \). Based on that the root locus of the arbitrary \( 2q\pi + \alpha \) degree of Eq(1.1) are continuous on the point \((\sigma_0, t_0)\), so, it is proved that the root locus of the arbitrary \( 2q\pi + \alpha \) degree of Eq(1.1) can not have any break points except the derivative zeros of the function \( W(s) \) and zeros and poles of Eq(1.1). When \( s_0 = (\sigma_0 + it_0) \in \Delta_q \), the root locus of arbitrary \( 2q\pi + \alpha \) degree phase angle of Eq(1.1) must be continuous on the point \( s_0 \).

Because of the limitation of the implicit function theorem, the zeros, poles and the finite zeros of derivative of the function \( W(s) \) are excluded in the result of the continuity of the root locus. The root locus are the results in \( \mathbb{C} \cup \{\infty\} \). But, the proof of the continuity of the root locus are achieved through utilizing the phase angle function which has two real variables and real function values. In the previous results, except three categories points which discretely distributed in \( \mathbb{C} \cup \{\infty\} \), every one root loci is continuous. We have used the definition of the continuity to prove Lemma 2.15. Because the root locus are all continuous around the three categories points, obviously, we can use the continuity judging theorems about the real value function of the real variables to prove that the root locus are surely continuous on the finite zeros of derivative of the function \( W(s) \). On the zeros and poles of Eq(1.1), the root locus are surely the left continuous or the right continuous. The proof method is the same as the proof of Lemma 2.15. Here, we needn’t to give their proofs. And we directly give their results.

The situation of the infinite points is more complex, here, we don’t give the proof of its result. Using the same method which has already been given in the above segment, we can prove this problem. We only give the result which the root locus are continuous on the infinite points in \( \mathbb{C} \cup \{\infty\} \).

**Theorem 2.16.** Let \( s = (\sigma + it) \in \mathbb{C} \cup \{\infty\} \), then, the root locus of arbitrary \( 2q\pi + \alpha \) degree of Eq(1.1) are all continuous on the points \( s \) in \( \mathbb{C} \cup \{\infty\} \).

### 3. The results about the mean value problem

Specially, the meromorphic function \( W(s) \) in Eq(1.1) can be a rational function. \( W_r(s) = \prod_{l=1}^{m} \frac{s - z_l}{s - p_l} \). In this paper, let \( n \geq m \). \( n \) and \( m \) are both natural number.

**Lemma 3.1.** On every one root loci of Eq(3.1), there don’t exist two points which their gain values are same.

\[
KW_r(s) = a_f + ib_f
\]
Proof. Because Eq(3.1) has \( n \) poles. Every one pole only emits a root loci of Eq(3.1) which its degree is \( \arg(b_f/a_f) \). Eq(3.1) emits \( n \) root locus which their degrees are \( \arg(b_f/a_f) \). According to Theorem 1.3, for every one root loci of \( n \) root locus which their degrees are \( \arg(b_f/a_f) \), the gains of points on these root loci obtain gain values from 0 on poles to the positive infinity on zeros. So, if there are two points which their gain values are both \( \frac{1}{|F F_f|} \) on a same root loci of Eq(3.1), then, on every one root loci of other \( n - 1 \) root locus, there at least exists one point which its gain value is \( \frac{1}{|F F_f|} \). So, there are more than \( n \) points which their phase angles are all \( \arg(b_f/a_f) \) and gain values are all \( \frac{1}{|F F_f|} \).

Namely, there are more than \( n \) points which they satisfy the equation: \( \frac{1}{|F F_f|} W_r(s) = a_f + i b_f \). Further, \( W_r(s) = |F F_f|(a_f + i b_f) \). Let \( FF = |F F_f|(a_f + i b_f) \). \( FF \) is a complex number. So, \( W_r(s) = FF \). \( FF \prod_{j=1}^{n}(s - p_j) - \prod_{z=1}^{r}(s - z_i) = 0 \). It is \( n \) order polynomial equation. It has and only has \( n \) roots. So, on every one root loci of Eq(3.1), there exist two points which their gain values can be same, this situation cannot be true. \( \square \)

Theorem 3.2. On every root loci of Eq(3.1), the gain of the root loci continuously and strictly monotonically obtains values from 0 on poles of Eq(3.1) to the positive infinity on zeros of Eq(3.1).

Proof. If the gain \( |K(s)| \) can not be monotonic on the root loci of Eq(3.1), then, on the root loci, there surely exist two points \( s_1, s_2 \). \( |K(s_1)| = |K(s_2)| \). It is contradict to Lemma 3.1. According to Theorem 1.3, we can obtain Theorem 3.2. \( \square \)

For the inequality in mean value problem

\[
|f(s) - f(\theta)| \leq |f'(s)|.
\]

Let \( f(s) = \sum_{k=0}^{n}(a_k s^k) \). \( f(\theta) = \sum_{k=0}^{n}(a_k \theta^k) \). \( f(s) - f(\theta) = \sum_{k=0}^{n}(a_k s^k - \sum_{k=0}^{n}(a_k \theta^k)) = \sum_{k=1}^{n}(a_k(s - \theta) \sum_{j=1}^{k} s^{j-1} \theta^{k-j}) \). \( \frac{|f(s) - f(\theta)|}{|s - \theta|} = |f(s) - f(\theta)| = \sum_{k=1}^{n} a_k \sum_{j=1}^{k} s^{j-1} \theta^{k-j} |. \)

Namely, \( \frac{f(s) - f(\theta)}{s - \theta} \) is a polynomial of one variable which the order is \( n - 1 \). \( f'(s) \) is the derivative polynomial of the polynomial \( f(s) \). Eq(3.2) can be equivalently transformed into the inequality

\[
\frac{f'(s)}{f(s) - f(\theta)} \geq 1.
\]

The left side \( \frac{f'(s)}{f(s) - f(\theta)} \) of Eq(3.3) is a rational fraction which the polynomial \( f'(s) \) is divided by the polynomial \( \frac{f(s) - f(\theta)}{s - \theta} \). The zeros of Eq(3.3) are zeros of the polynomial \( f'(s) \), and the zeros of Eq(3.3) are zeros of derivative of the polynomial \( f(s) \).

The poles of of Eq(3.3) are zeros of the polynomial \( \frac{f(s) - f(\theta)}{s - \theta} = \sum_{k=1}^{n}(a_k \sum_{j=1}^{k} s^{j-1} \theta^{k-j}) \).

Do the root locus equation:

\[
K \frac{f'(s)}{\sum_{k=1}^{n}(a_k \sum_{j=1}^{k} s^{j-1} \theta^{k-j})} = a + ib.
\]

According to the results of the root locus in section 1 and section 2, we can obtain:
The root locus of Eq(3.4) begin at the zeros of the polynomial \( \sum_{k=1}^{n}(a_k \sum_{j=1}^{k} s^{j-1} \theta^{k-j}) \).
And they end at the zeros of \( f'(s) \). From the zeros of the polynomial \( \sum_{k=1}^{n} (a_k) \sum_{j=1}^{k} s^{j-1} g^{k-j} \) to the zeros of \( f'(s) \), the gain \( K = \left| \frac{\sum_{k=1}^{n} (a_k) \sum_{j=1}^{k} s^{j-1} g^{k-j}}{f'(s)} \right| \) strictly and monotonically increase. The gain \( K \) and the modulus of the rational fraction \( \frac{f'(s)}{\sum_{k=1}^{n} (a_k) \sum_{j=1}^{k} s^{j-1} g^{k-j}} \) are the relationship of reciprocal. From the zeros of the polynomial \( \sum_{k=1}^{n} (a_k) \sum_{j=1}^{k} s^{j-1} g^{k-j} \) to the zeros of \( f'(s) \), the modulus of the rational fraction \( \left| \frac{\sum_{k=1}^{n} (a_k) \sum_{j=1}^{k} s^{j-1} g^{k-j}}{f'(s)} \right| \) strictly and monotonically decrease.

At every zero \( \theta_i \) of \( f'(s) \), \( f' (\theta_i) = 0 \). So, around every zero of \( f'(s) \), there surely exists its adjacent domain \( \Omega_i \), \( i = 1, 2, \ldots, n - 1 \). For all points in the adjacent domain of zero \( \theta_i \) of \( f'(s) \), they all satisfy: \( \frac{|f'(s)|}{|s - \theta_i|} < 1 \), namely, \( \frac{|f(s) - f(\theta_i)|}{|s - \theta_i|} = |f(s) - f(\theta_i)| > |f'(s)| \).

But, for all points in the extended complex plane except all adjacent domains \( \Omega = \Omega_i \), \( i = 1, 2, \ldots, n - 1 \), they all satisfy: \( \frac{|f'(s)|}{|s - \theta_i|} \geq 1 \), namely, \( \frac{|f(s) - f(\theta_i)|}{|s - \theta_i|} = |f(s) - f(\theta_i)| \leq |f'(s)| \).

So, we have proved the following two theorems.

**Theorem 3.3.** Given a complex polynomial \( f \) of degree \( d \geq 2 \) and a complex number \( s \). For every critical point \( \theta_i \) (ie, \( f'(\theta_i) = 0 \)), \( i = 1, 2, \ldots, n - 1 \), there exists its adjacent domain \( \Omega_i \). For all points in the adjacent domain \( \Omega_i \), \( i = 1, 2, \ldots, n - 1 \), such that \( \frac{|f(s) - f(\theta_i)|}{|s - \theta_i|} < |f'(s)| \).

**Theorem 3.4.** Given a complex polynomial \( f \) of degree \( d \geq 2 \) and a complex number \( s \). For every critical point \( \theta_i \) (ie, \( f'(\theta_i) = 0 \)), \( i = 1, 2, \ldots, n - 1 \), there exists its adjacent domain \( \Omega_i \). For all points in the extended complex plane except the adjacent domain \( \Omega = \{ \Omega_i \}, i = 1, 2, \ldots, n - 1 \), such that \( \frac{|f(s) - f(\theta_i)|}{|s - \theta_i|} \leq |f'(s)| \).

Conflict of Interest: Authors declare that they have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors.

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