ON THE STABILITY OF SOLUTIONS OF CERTAIN LINEAR SET DIFFERENTIAL EQUATIONS
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Abstract

New approaches to the study of stability of solutions of Set Differential Equations (SDEs) based on convex geometry and the theory of mixed volumes are proposed. The stability of the forms of program solutions of linear SDEs with a stable operator is proved. We consider the orbit of the action of homotheties group on the space of nonempty convex compacts (conv \( \mathbb{R}^n \)) as the form of a convex compact. For equations with periodic operator in the two-dimensional space the asymptotic stability conditions are established.

Keywords: Set Differential Equations, comparison method, direct Lyapunov method, Brunn–Minkowski inequality, Lyapunov stability, stable operator

MSC: 93D30; 52A39

1 Introduction

Differential equations with Hukuhara derivative (Set Differential Equations (SDEs)) were first considered in [1]. Further development of the theory of differential equations with Hukuhara derivative has been summarized in the monograph [2], where the conditions of existence and uniqueness of solutions of the Cauchy problem, the convergence of successive approximations including the principle of comparison and the theorems of Lyapunov’s
direct method have been formulated. In papers [5, 6] the results and methods of geometry of convex bodies, developed in the classical works of H. Minkowski and A.D. Alexandrov [3, 4], were used for the study of stability of solutions for dynamical systems in the space of convex compact sets in $\mathbb{R}^n$. In this paper, these ideas are applied to the study of stability of solutions for Set Differential Equations.

Let $(\text{conv } \mathbb{R}^n, d_H)$ be a metric space of nonempty convex compact sets in $\mathbb{R}^n$ and $d_H$ is the Hausdorff metric.

This work is devoted to the study of the properties of solutions for SDEs of the form

$$D_H X(t) = AX,$$ (1.1)

where $X(t) \in \text{conv } \mathbb{R}^n$, $A \in L(\mathbb{R}^n)$. Here and later, if $(X, \| \cdot \|_X)$ is a Banach space, then $L(X)$ is a Banach algebra of bounded linear operators on $X$.

We note that the dynamic properties of the differential equation (1.1) are significantly different from the properties of the similar ordinary differential equation (ODE)

$$\frac{dx}{dt} = Ax,$$ (1.2)

where $x \in \mathbb{R}^n$, $A \in L(\mathbb{R}^n)$.

Consider the following simple example [2]. Consider the ODE

$$\frac{dx}{dt} = -x,$$ (1.3)

$x \in \mathbb{R}$. A similar equation in $\text{conv } \mathbb{R}$ is of the form

$$D_H X(t) = JX(t),$$ (1.4)

where $X \in \text{conv } \mathbb{R}$, $J$ is a reflection operator, i.e.,

$$JX = \{-x \mid x \in X\}.$$

Let $X(t) = [x_1(t), x_2(t)] \in \text{conv } \mathbb{R}$, $x_1(t) \leq x_2(t)$ is a solution of differential equation (1.4) with the initial condition $X_0 = [x_{10}, x_{20}]$, $x_{10} \leq x_{20}$. Then
the Cauchy problem for (1.4) is equivalent to Cauchy problem for the system of differential equations

\[
\frac{dx_1}{dt} = -x_2, \quad x_1(0) = x_{10}, \quad \frac{dx_2}{dt} = -x_1, \quad x_2(0) = x_{20}.
\]

By integrating this system we obtain the solution of the Cauchy problem (1.4)

\[
X(t) = [x_{10} \cosh t - x_{20} \sinh t, x_{20} \cosh t - x_{10} \sinh t], \quad t \geq 0.
\]

We note that \(\text{diam } X(t) = e^t \text{diam } X_0\) and the solution \(x = 0\) of ODE (1.3) is asymptotically stable in the sense of Lyapunov.

Consider the stability problem of the solution \(X \equiv 0\) of (1.4) with respect to the Hausdorff metric. It is easy to see that \(d_H(X, \theta) = \max[|x_1|, |x_2|]\) and \(d_H(X(t), \theta) \to \infty\) for \(t \to \infty\), provided that \(X(0) = X_0\) is not a single point. Thus, the solution \(X = 0\) is unstable.

The use of Hausdorff metric as a measure does not lead to a meaningful problem statement about stability of solutions of SDEs. This is due to the fact that the nondecreasing of function diam \(X(t)\) is necessary condition for Hukuhara differentiability of mapping \(X(t)\).

This example shows that a meaningful problem statement about stability of solutions of SDEs is a non-trivial task and is concerned with an adequate choice of measures with respect to which we can consider the stability problem.

In this paper, the problem of choosing an appropriate measures with respect to which we investigate stability is solved on the basis of geometrical considerations.

We introduce the space of shapes of convex bodies as the quotient set of the space \(\text{conv } \mathbb{R}^n\) on action of group of homotheties in space \(\mathbb{R}^n\). Then in a conventional manner the quotient metric for Hausdorff metric \(d_H\) is introduced. The geometric meaning of quotient metric is the deviation of
the convex bodies shapes. The stability problem is considered with respect to this quotient metric.

We note that stability of shapes of attainable sets for linear impulsive systems was considered in [7]. Here, the space of shape of convex compact sets is considered as a quotient space of \( \text{conv } \mathbb{R}^n \) on the action of the general affine group.

2 Problem statement

Let \( \mathcal{G} \) be a certain affine group in the space \( \mathbb{R}^n \), then its action naturally extends to the space \( \text{conv } \mathbb{R}^n \)

\[ \mathcal{G} \mathcal{X} = \{ g x \mid x \in \mathcal{X} \}, \quad \mathcal{X} \in \text{conv } \mathbb{R}^n, g \in \mathcal{G}. \]

The orbit of \( X \), under the action of the group \( \mathcal{G} \) is defined as a subset

\[ \text{Or}_\mathcal{G}(X) = \{ \mathcal{G}X \mid \mathcal{G} \in \mathcal{G} \} \subset \text{conv } \mathbb{R}^n. \]

The set of all orbits is denoted by \( \text{conv } \mathbb{R}^n/\mathcal{G} \). This quotient space is endorsed with the following metric

\[ \rho[\text{Or}_\mathcal{G}(X), \text{Or}_\mathcal{G}(Y)] = \inf\{ d_H(g_1X, g_2Y) \mid g_i \in \mathcal{G}, i = 1, 2 \}. \]

Depending on the choice of the group \( \mathcal{G} \), we get different classification of elements of the space \( \text{conv } \mathbb{R}^n \). If, for example, \( \mathcal{G} = GL(\mathbb{R}^n) \setminus \mathbb{R}^n \) is a general affine group of the space \( \mathbb{R}^n \), then we obtain a more rough classification and if \( \mathcal{G} = \mathbb{R}^n \) is a group of translations of the space \( \mathbb{R}^n \) then we obtain a thinner classification.

In this paper, \( \mathcal{G} \) is a group of homotheties namely the semidirect product of the group of dilations and group of translations of the space \( \mathbb{R}^n \).

We introduce a subset

\[ \mathcal{C} = \{ X \in \text{conv } \mathbb{R}^n \mid \text{int } X \neq \emptyset \}. \]
If $X \in \mathcal{C}$, then we set $\tilde{X} \overset{\text{def}}{=} \frac{X}{\sqrt[n]{V[X]}}$, where $V[X]$ is a volume of a convex compact $X$. Then $\text{Or}_\mathfrak{G}(X) = \text{Or}_\mathfrak{G}(\tilde{X})$ and if $Y \in \mathcal{C}$, then we get
\[
\rho[\text{Or}_\mathfrak{G}(X), \text{Or}_\mathfrak{G}(Y)] = \inf\{d_H(\tilde{X}, \tilde{Y} + x) | x \in \mathbb{R}^n\} = \rho[\text{Or}_{\mathbb{R}^n}(X), \text{Or}_{\mathbb{R}^n}(Y)].
\]

In the space $\text{conv} \mathbb{R}^n$ we consider the Cauchy problem for SDEs
\[
D_H X(t) = A X(t), \quad X(0) = X_0, \quad X_0 \in \mathcal{C}, \quad (2.1)
\]
where $D_H$ is a Hukuhara derivative operator, $X(t) \in \text{conv} \mathbb{R}^n$, $t \in \mathbb{R}_+$, $A \in L(\mathbb{R}^n)$ is an orthogonal operator, that is $A^* A^{-1} = I$, $A^*$ is the adjoint operator.

Since $X_0 \in \mathcal{C}$ we have $X(t) \in \mathcal{C}$ for all $t \geq 0$.

Let $X^*(t)$ be a program solution of the Cauchy problem (2.1) with the initial value $X^*(0) = X^*_0 \in \mathcal{C}$.

Next we give the definition of Lyapunov stability of solutions of the Cauchy problem (2.1).

**Definition 2.1** Program solution $X^*(t)$ is said to be

1) Lyapunov stable if for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon, X^*_0)$ such that, for all $X_0 \in \mathcal{C}$ the condition $\rho[\text{Or}_\mathfrak{G}(X_0), \text{Or}_\mathfrak{G}(X^*_0)] < \delta$ implies $\rho[\text{Or}_\mathfrak{G}(X(t)), \text{Or}_\mathfrak{G}(X^*(t))] < \varepsilon$ for all $t \geq 0$;

2) conditionally asymptotically stable with respect to the set $\mathfrak{M} \subset \mathcal{C}$, if it is stable and there exists a scalar $\sigma_0 > 0$ such that, for all $X_0 \in \mathfrak{M}$ if $\rho[\text{Or}_\mathfrak{G}(X_0), \text{Or}_\mathfrak{G}(X^*_0)] < \sigma_0$ then we have
\[
\lim_{t \to \infty} \rho[\text{Or}_\mathfrak{G}(X(t)), \text{Or}_\mathfrak{G}(X^*(t))] = 0.
\]

In this paper, we investigate the stability and asymptotic stability of solutions of the Cauchy problem (2.1) in the sense of the above definition.
3 Auxiliary results

For \(X, Y \in \mathcal{C}\) we define the functional

\[
\Delta[X,Y] = \frac{V_1^n[X,Y]}{V_{n-1}[X]V[Y]} - 1.
\]

Here \(V_1[X,Y]\) is the first mixed volume of convex compacts \(X\) and \(Y\). Based on Brunn–Minkowski inequality \([4]\), we get \(\Delta[X,Y] \geq 0\), and \(\Delta[X,Y] = 0\) if and only if \(Y \in \text{Or}_\mathfrak{G}(X)\). It is obvious that the functional \(\Delta[X,Y]\) depends only on the orbits \(\text{Or}_\mathfrak{G}(X)\) and \(\text{Or}_\mathfrak{G}(Y)\). This functional will be used as an analogue of the Lyapunov function.

The study of stability of solutions of the Cauchy problem (2.1) is based on Theorem 1.2, proved in \([8]\) on the basis of more accurate Brunn–Minkowski inequality established by V.I. Discant.

For \(X \in \mathcal{C}\), let \(R_X, r_X\) be the radii of the circumscribed and inscribed balls for convex compact \(X\) respectively. Let \(X(t)\) and \(Y(t)\) be one-parameter families of sets from \(\mathcal{C}\). It should be noted also that \(d_H((\bar{X})', \bar{Y}) = \rho(\text{Or}_\mathfrak{G}(X), \text{Or}_\mathfrak{G}(Y))\), where \(X'\) is a shift of set \(X \in \text{conv } \mathbb{R}^n\).

Lemma 3.1 Assume that

\[
R = \sup_{t \geq 0} \{R_{\bar{X}(t)}, R_{\bar{Y}(t)}\} < \infty, \quad r = \inf_{t \geq 0} \{r_{\bar{X}(t)}, r_{\bar{Y}(t)}\} > 0.
\]

Then there exist the positive constants \(\varepsilon_0, C_1\) and \(C_2\) which depend only on \(n, R\) and \(r\) such that

\[
\Delta[X,Y] \leq C_2\varepsilon[\text{Or}_\mathfrak{G}(X), \text{Or}_\mathfrak{G}(Y)] \tag{3.1}
\]

and if \(\Delta[X,Y] < \varepsilon_0\) we have

\[
C_1\varepsilon^n[\text{Or}_\mathfrak{G}(X), \text{Or}_\mathfrak{G}(Y)] \leq \Delta[X,Y]. \tag{3.2}
\]

Proof. Inequality (3.2) is a direct consequence of the reasoning in the proof of Theorem 1.2 from \([8]\). Next, we will prove the inequality (3.1).
By definition of the metric, we get

\[(\tilde{Y}(t))' \subset \tilde{X}(t) + \varrho B_1(0), \quad \varrho = \varrho[\text{Or}_\varrho(X(t)), \text{Or}_\varrho(Y(t))] = d_H(X(t), (Y(t))'),\]

where \(B_1(0) \subseteq \mathbb{R}^n\) is an open unit ball with center at \(x = 0\). By monotony of functional \(V_1[X, Y]\) and using Brunn-Minkowski inequality, we obtain

\[1 \leq V_1[\tilde{X}(t), \tilde{Y}(t)] = V_1[\tilde{X}(t), (\tilde{Y}(t))'] \leq V_1[\tilde{X}(t), \tilde{X}(t) + \varrho B_1(0)] = 1 + \varrho V_1[\tilde{X}(t), \tilde{B}_1(0)] \leq 1 + \varrho V_1[R_{\tilde{X}(t)}(0)]R_v,\]

where \(v = V[\tilde{B}_1(0)]\). Thus, we get the equality

\[\Delta[X(t), Y(t)] = V_1[\tilde{X}(t), \tilde{Y}(t)] - 1 \leq C_2 \varrho[\text{Or}_\varrho(X(t)), \text{Or}_\varrho(Y(t))], \quad C_2 = R_v.\]

The lemma is proved.

In order to estimate the changes of functionals \(V[X]\) and \(V_1[X, X^*]\) along the solutions of the Cauchy problem, we shall use the comparison method [2].

Let \(k = \{k_1, \ldots, k_{n-1}\}\) be a certain unordered set of indices, where \(k_i \geq 0\), and \(K\) be the set of all such index sets.

Define the auxiliary functionals

\[\Xi_k[X, X^*] = V[A^{k_1}X, A^{k_2}X, \ldots, A^{k_{n-1}}X, X^*]\]

and the functions \(\zeta_k(t) = \Xi_k[X(t), X^*(t)]\).

Using the continuity of the functional of mixed volume, it is easy to show that for the sets \(k = (k_1, \ldots, k_p, 0, \ldots, 0), k_j \geq 1, j = 1, \ldots, p\) we have the formula

\[
\frac{d\zeta_k(t)}{dt} = \zeta_{k_1+1, k_2, \ldots, k_p, 0, \ldots, 0}(t) + \ldots + \zeta_{k_1, k_2, \ldots, k_{p+1}, 0, \ldots, 0}(t) + (n-p)\zeta_{k_1, k_2, \ldots, k_p, 1, 0, \ldots, 0}(t),
\]

(3.3)

and for the sets \(k \in K\) in which \(k_j \geq 1, j = 1, \ldots, n-1\), we have

\[
\frac{d\zeta_k(t)}{dt} = \zeta_{k_1+1, k_2, \ldots, k_{n-1}}(t) + \ldots + \zeta_{k_1, k_2, \ldots, k_{n-1}+1}(t) + \zeta_{k_1-1, k_2-1, \ldots, k_{n-1}-1}(t).
\]

(3.4)
Define the set
\[ l_\infty = \{ \{ x_k \}_{k \in \mathbb{K}} \mid \sup_{k \in \mathbb{K}} |x_k| < \infty \} \] (3.5)
and the norm \( \| x \|_{l_\infty} = \sup_{k \in \mathbb{K}} |x_k| \).

On the set \( l_\infty \) the operations of addition and nonnegative scalar multiplication are defined in a natural way.

It is easy to see that \( (l_\infty, \| . \|_\infty) \) is a Banach space. Let us show that \( \{ \Xi_k[X, X^*] \}_{k \in \mathbb{K}} \subset l_\infty \). In fact,
\[ X' \subset R_X \overline{B}_1(0), \quad (X^*)' \subset R_X \overline{B}_1(0), \]
\[ \Xi_k[X, X^*] = \Xi_k[X', (X^*)'] = V[A^{k_1}X', A^{k_2}X', \ldots, A^{k_{n-1}}X', (X^*)'] \]
\[ \leq R_{X'}^{n-1}R_XV[A^{k_1}\overline{B}_1(0), A^{k_2}\overline{B}_1(0), \ldots, A^{k_{n-1}}\overline{B}_1(0), \overline{B}_1(0)] \leq R_{X'}^{n-1}R_X. \]

Therefore, differential equations (3.3) and (3.4) can be represented in an abstract form
\[ \frac{d\zeta}{dt} = \Omega \zeta, \]
where \( \zeta \in l_\infty, \Omega: l_\infty \to l_\infty \) is a linear operator. It is obvious that \( \Omega \in L(l_\infty) \) and \( \| \Omega \|_{L(l_\infty)} = n. \)

Hence it follows that
\[ V_1[X(t), X^*(t)] = \sum_{k \in \mathbb{K}} a_k \Xi_k[X_0, X_0^*], \] (3.6)
where \( a_k \) are the coefficients that do not depend on the operator \( A \). Operator \( \Omega \) is positive relative to the cone \( l^+_\infty = \{ \{ x_k \}_{k \in \mathbb{K}} \in l_\infty \mid x_k \geq 0 \} \) and therefore the coefficients \( a_k \) are nonnegative.

Assuming that \( X_0^* = X_0 \) in formula (3.6), we get
\[ V[X(t)] = \sum_{k \in \mathbb{K}} a_k M_k[X_0], \] (3.7)
where \( M_k[X_0] = \Xi_k[X_0, X_0]. \)

Let us prove that
\[ \sum_{k \in \mathbb{K}} a_k = e^{nt}. \] (3.8)
Let $X_0 = \overline{B}_1(0)$, then as the linear operator $A$ is orthogonal, we get $A\overline{B}_1(0) = \overline{B}_1(0)$. It is obvious that $X(t) = e^{t\overline{B}_1(0)}$ is a unique solution of the Cauchy problem (2.1) with initial conditions $X(0) = \overline{B}_1(0)$ and $M_k[\overline{B}_1(0)] = V[\overline{B}_1(0)]$. From (3.7) we obtain

$$e^{nt}V[\overline{B}_1(0)] = V[e^{t\overline{B}_1(0)}] = \sum_{k\in K} a_k M_k[\overline{B}_1(0)] = \sum_{k\in K} a_k V[\overline{B}_1(0)],$$

and we get (3.8).

**Lemma 3.2** For the volume $V[X(t)]$ of solution $X(t)$, $X(0) = X_0$ of linear differential equation (2.1) the following estimate holds for all $t \geq 0$

$$V[X_0] e^{nt} \leq V[X(t)] \leq M[X_0] e^{nt},$$

where $M[X_0] = \max_{k\in K} M_k[X_0]$.

**Proof.** Applying the Brunn-Minkowski inequality, we obtain

$$\frac{dV[X(t)]}{dt} = nV_1[X(t), AX(t)] \geq nV[X(t)].$$

Hence the inequality $V[X(t)] \geq e^{nt}V[X_0]$ is valid for $t \geq 0$.

$$V[X(t)] \leq \|\zeta(t)\|_{L_\infty} \leq \|e^{t\Omega}\|_{L(L_\infty)} \|\zeta_0\|_{L_\infty} \leq e^{t\|\Omega\|_{L(L_\infty)}} \|M[X_0]\| = e^{nt} M[X_0].$$

The Lemma is proved.

**Lemma 3.3** Assume that $X(t)$, $X^*(t)$ are solutions of differential equation (2.1), $X(0) = X_0$, int $X_0 \neq \emptyset$, $X^*(0) = X_0^*$, int $X_0^* \neq \emptyset$. Then there exist the positive constants $\varepsilon_0$, $C_1$, $C_2$ that depend on $X_0^*$, such that, from inequality

$$g[\text{Or}_{\mathfrak{G}}(X_0), \text{Or}_{\mathfrak{G}}(X_0^*)] < \sigma_0$$

for all $t \geq 0$ we have

$$\Delta[X(t), X^*(t)] \leq C_2 g[\text{Or}_{\mathfrak{G}}(X(t)), \text{Or}_{\mathfrak{G}}(X^*(t))].$$

Moreover, for $t \geq 0$ if $\Delta[X(t), X^*(t)] \leq \varepsilon_0$, then we have

$$C_1 g^{n^2}[\text{Or}_{\mathfrak{G}}(X(t)), \text{Or}_{\mathfrak{G}}(X^*(t))] \leq \Delta[X(t), X^*(t)].$$
Proof. As a result of the assertion of Lemma 3.1, it suffices to prove that
\[ R = \sup_{t \geq 0} \{ R_{X(t)}, R_{X^*(t)} \} \leq R(X_0^*) < \infty, \quad r = \inf_{t \geq 0} \{ r_{X(t)}, r_{X^*(t)} \} \geq r(X_0^*) > 0, \]
for all \( X_0 \) for which \( \rho[\text{Or}_G(X_0), \text{Or}_G(X_0^*)] \leq \sigma_0 \), \( \sigma_0 \) is a sufficiently small positive constant.

Let \( T \) be any positive number. For the Cauchy problem (2.1), we consider the successive approximations for \( t \in [0, T] \)
\[ X_0(t) = X_0, \quad X_m(t) = X_0 + \int_0^t A X_{m-1}(s) \, ds. \]
Let \( h_{X(t)}(p) \) be a support function for convex compact set \( X(t) \). Without loss of generality, we can assume that the origin of coordinates is at the center of inscribed ball, then \( h_{X_0}(p) \geq r_{X_0} \) for \( p \in \partial B_1(0) \). Next, we prove by mathematical induction the inequality
\[ h_{X_m(t)}(p) \geq \sum_{k=0}^m \frac{t^k}{k!} r_{X_0}. \]
(3.10)
For \( m = 0 \) this inequality is obvious. Suppose that it is true for \( m = k - 1 \), then
\[
\begin{align*}
  h_{X_k(t)}(p) &= h_{X_0}(p) + \int_0^t h_{A X_{k-1}(s)}(p) \, ds = h_{X_0}(p) + \int_0^t h_{X_{k-1}(s)}(A^*p) \, ds \\
  &\geq r_{X_0} + \int_0^t \sum_{l=0}^{k-1} \frac{s^l}{l!} r_{X_0} \, ds = \sum_{l=0}^k \frac{t^k}{k!} r_{X_0}.
\end{align*}
\]
(3.11)
It is known [2], that the successive approximations \( X_k(t) \) converge uniformly with respect to \( t \in [0, T] \) to the solution \( X(t) \) of the Cauchy problem (2.1) and therefore \( \| h_{X_k(t)} - h_{X(t)} \|_{C(\partial B_1(0))} \to 0 \) for \( k \to \infty \). Thus, from inequality (3.10), it follows that \( h_{X(t)} \geq e^{t} r_{X_0} \) for all \( t \in \mathbb{R}_+ \). Hence,
\[
\inf_{t \geq 0} r_{\tilde{X}(t)} = \inf_{t \geq 0} \frac{r_{X(t)}}{\sqrt{V[X(t)]}} \geq \inf_{t \geq 0} \frac{e^{t} r_{X_0}}{\sqrt{V[X(t)]}} = \frac{r_{X_0}}{\sqrt{V[X_0]}}.
\]
Similarly, we can show that \( h_X(t) \leq e^t R_X(0) \) for all \( t \geq 0 \), and therefore

\[
\sup_{t \geq 0} R_{\tilde{X}}(t) = \sup_{t \geq 0} \frac{R_X(t)}{\sqrt{V[X(t)]}} \leq \sup_{t \geq 0} \frac{e^t R_X(0)}{\sqrt{V[X(0)]}} = \frac{R_X(0)}{\sqrt{V[X(0)]}}.
\]

Thus,

\[
r \geq \min \left[ \frac{r_X(0)}{\sqrt{M[X(0)]}}, \frac{r_X^*(0)}{\sqrt{M[X^*(0)]}} \right]
\]

and

\[
R \leq \max \left[ \frac{R_X(0)}{\sqrt{V[X(0)]}}, \frac{R_X^*(0)}{\sqrt{V[X^*(0)]}} \right].
\]

By the continuity of the functionals \( R_X, r_X, V[X] \) and \( M[X] \), there exists a positive constant \( \varepsilon_0 < \sigma_0 \) such that, the inequality \( \rho[\text{Or}_g(X_0), \text{Or}_g(X^*_0)] < \varepsilon_0 \) implies the estimates

\[
\left| \frac{R_X(0)}{\sqrt{V[X(0)]}} - \frac{R_X^*(0)}{\sqrt{V[X^*(0)]}} \right| < \frac{R_X^*(0)}{2 \sqrt{V[X^*(0)]}},
\]

\[
\left| \frac{r_X(0)}{\sqrt{M[X(0)]}} - \frac{r_X^*(0)}{\sqrt{M[X^*(0)]}} \right| < \frac{r_X^*(0)}{2 \sqrt{M[X^*(0)]}}.
\]

So, we get

\[
R \leq \frac{3R_X^*(0)}{2 \sqrt{V[X^*(0)]}}, \quad r \geq \frac{r_X^*(0)}{2 \sqrt{M[X^*(0)]}}.
\]

The Lemma is proved.

Consider the particular case of the Cauchy problem (2.1) when \( n = 2 \) and for some positive integer \( m \) the equality \( A^m = I \) is valid.

**Lemma 3.4** Assume that \( X(t) \) and \( X^*(t) \) are solutions of the Cauchy problem (2.1) with initial conditions \( X(0) = X_0, X^*(0) = X^*_0 \). Then for odd \( m, m \geq 3 \) we have the formula

\[
S[X(t), X^*(t)] = \frac{1}{m}(e^{2t} + 2 \sum_{q=1}^{[m/2]} e^{2t \cos \frac{2\pi q}{m}})S[X_0, X^*_0]
\]

\[
+ \frac{1}{m^2} \sum_{p=1}^{m-1} \left((m-p)e^{2t} + 2 \sum_{q=1}^{[m/2]} [(m-p) \cos \frac{2\pi pq}{m} \right)
\]

\[
+ 2t \sin \frac{2\pi pq}{m} \sin \frac{2\pi q}{m} e^{2t \cos \frac{2\pi q}{m}}) (S[X_0, A^p X^*_0] + S[X^*_0, A^p X_0]).
\]
For even $m$, $m \geq 4$ we have the formula

\[
S[X(t), X^*(t)] = \frac{1}{m}(e^{2t} + e^{-2t} + 2 \sum_{q=1}^{(m-2)/2} e^{2t \cos \frac{2\pi q}{m}})S[X_0] + \frac{1}{m^2} \sum_{p=1}^{m-1} ((m-p)(e^{2t} + (-1)^p e^{-2t}) + 2 \sum_{q=1}^{(m-2)/2} [(m-p) \cos \frac{2\pi pq}{m} + 2(\cos 2\pi \frac{pq}{m} + \sin 2\pi \frac{pq}{m})]e^{2t \cos \frac{2\pi q}{m}})(S[X_0, A^p X_0^*] + S[X_0^*, A^p X_0^*]).
\]

**Proof.** Let $S[X, Y]$ be a functional of Minkowski mixed area, then the auxiliary functions

\[
\xi_k(t) = \frac{1}{2}(S[X(t), A^k X^*(t)] + S[X^*(t), A^k X(t)]), \quad k = 0, \ldots, m - 1,
\]

satisfy the system of differential equations of $m$-th order (comparison system)

\[
\frac{d\xi}{dt} = \Omega \xi(t),
\]

where $\xi(t) \in \mathbb{R}^m$, $\Omega \in \mathbb{R}^{m \times m}$ is the matrix, the non-zero elements of which have the form $\omega_{12} = 2$, $\omega_{ij} = \omega_{m1} = 1$, $|i - j| = 1$, $(i, j) \neq (1, 2)$.

It is known [9] that the solution of comparison system has the form

\[
\xi(t) = -\frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} R_\Omega(\lambda) \, d\lambda \xi(0).
\] (3.12)

Here $R_\Omega(\lambda)$ is the resolvent of matrix $\Omega$, $\Gamma$ is the circuit consisting of a finite number of closed Jordan curves, oriented in the positive direction, covering the spectrum of $\sigma(\Omega)$ of matrix $\Omega$.

Next, we find the spectrum $\sigma(\Omega)$ and resolvent $R_\Omega(\lambda)$ of matrix $\Omega$. Let $f = (f_0, \ldots, f_{m-1}) \in \mathbb{C}^m$, $x = (x_0, \ldots, x_{m-1}) \in \mathbb{C}^m$ and consider the linear equation

\[
(\Omega - \lambda E)x = f.
\] (3.13)

Equation (3.13) is equivalent to the boundary value problem for finite-difference equation of second order

\[
x_{k-1} + x_{k+1} - \lambda x_k = f_k, \quad k = 1, \ldots, m - 1
\]
with the boundary conditions

\[ 2x_1 - \lambda x_0 = f_0, \quad x_m = x_0. \]

If \( f = 0 \) and equation (3.13) has only the trivial solution, then \( \lambda \in \varrho(\Omega) \), where \( \varrho(\Omega) \) is the resolvent set of matrix \( \Omega \). Moreover, \( \sigma(\Omega) = \mathbb{C} \setminus \varrho(\Omega) \).

The general solution of the homogeneous difference equation has the form

\[ x_k = c_1 q_1^k + c_2 q_2^k, \]

where \( q_1 \) and \( q_2 \) are the roots of a quadratic equation \( q^2 + 1 = \lambda q \), \( c_1 \) and \( c_2 \) are arbitrary constants. From the boundary conditions it follows that

\[ 2x_1 - \lambda x_0 = (c_1 - c_2)(q_1 - q_2) = 0, \]

\[ c_1 q_1^m + c_2 q_2^m = x_0. \]

There are two cases: \( c_1 = c_2 \) or \( q_1 = q_2 \). In the first case \( c_1(q_1^m + q_2^m - 2) = 0, \) and if \( (q_1^m + q_2^m - 2) \neq 0 \), then there is only the trivial solution of the equation (3.13). In the second case, the condition \( q_1 = q_2 \) implies that \( q_1 = q_2 = 1 \) or \( q_1 = q_2 = -1 \). If \( q_1 = q_2 = 1 \), then \( x_k = c \) is a solution of (3.13) for any \( c \), i.e., \( 2 \notin \varrho(\Omega) \). If \( q_1 = q_2 = -1 \), then \( x_k = (-1)^k(c_1 + c_2) \), and for \( k = m \) we get \( (1 + (-1)^{m+1})(c_1 + c_2) = 0 \), and if \( (1 + (-1)^{m+1}) \neq 0 \), then there is only the trivial solution of the equation (3.13). Thus, if \( m \) is the odd number then \(-2 \in \varrho(\Omega)\), otherwise it is obvious that \(-2 \notin \varrho(\Omega)\).

So, for matrix spectrum we have

\[ \sigma(\Omega) = \left\{ 2 \cos \frac{2\pi q}{m} \mid q = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \right\}. \]

Next, we consider the equation (3.13) in the general case when \( f \neq 0 \). It is easy to show that the general solution of the inhomogeneous finite-difference equation has the form

\[ x_k = \begin{cases} 
  c_1 + c_2, & k = 0, \\
  c_1 q_1^k + c_2 q_2^k + \sum_{p=0}^{k-1} \frac{q_1^{k-p} - q_2^{k-p}}{q_1 - q_2} f_p, & k \geq 1. 
\end{cases} \]
Taking into account the boundary conditions,

\[ 2x_1 - \lambda x_0 = f_0, \quad x_m = x_0 \]

we obtain:

\begin{align*}
  c_1 - c_2 &= \frac{f_0}{q_2 - q_1}, \\
  (1 - q_1^m) c_1 + (1 - q_2^m) c_2 &= \frac{q_1^m - q_2^m}{q_1 - q_2} f_0 + \sum_{p=1}^{m-1} \frac{q_1^{m-p} - q_2^{m-p}}{q_1 - q_2} f_p.
\end{align*}

Since \( x_0 = c_1 + c_2 \), we get

\[ x_0 = \frac{q_1^m - q_2^m}{(q_1 - q_2)(2 - q_1^m - q_2^m)} f_0 + 2 \sum_{p=1}^{m-1} \frac{q_1^{m-p} - q_2^{m-p}}{(q_1 - q_2)(2 - q_1^m - q_2^m)} f_p. \]

Similarly, we can find all \( x_k, k = 1, \ldots, m - 1 \).

For \( q_i = q_i(\lambda), i = 1, 2 \) from (3.12), we can get the equality for mixed area \( S[X(t), X^*(t)] \) of solution of the Cauchy problem (2.1)

\begin{align*}
  S[X(t), X^*(t)] &= -\frac{1}{2\pi i} \oint \frac{(q_1^m(\lambda) - q_2^m(\lambda)) e^{\lambda t}}{(q_1(\lambda) - q_2(\lambda))(2 - q_1^m(\lambda) - q_2^m(\lambda))} d\lambda S[X_0, X^*] \\
  &\quad - \sum_{p=1}^{m-1} \frac{1}{2\pi i} \oint \frac{(q_1^{m-p}(\lambda) - q_2^{m-p}(\lambda)) e^{\lambda t}}{(q_1(\lambda) - q_2(\lambda))(2 - q_1^m(\lambda) - q_2^m(\lambda))} d\lambda \left( S[X^*_0, A^p X_0] + S[X_0, A^p X^*_0] \right).
\end{align*}

Thus, further calculations are reduced to finding the corresponding integrals in formula (3.14). The contour of integration can be represented as

\[ \Gamma = \bigcup_{q=0}^{[m/2]} \Gamma_q, \quad \Gamma_q = \{ \lambda \in \mathbb{C} : \left| \lambda - 2 \cos \frac{2\pi q}{m} \right| = \varepsilon \}, q = 0, \ldots, \left[ \frac{m}{2} \right], \]

where \( \varepsilon \) is a sufficiently small positive number.

Consider the integral \( I_q = \oint_{\Gamma_q} \frac{(q_1^m(\lambda) - q_2^m(\lambda)) e^{\lambda t}}{(q_1(\lambda) - q_2(\lambda))(2 - q_1^m(\lambda) - q_2^m(\lambda))} d\lambda \) for \( q \neq 0, m/2 \).

Then \( \lambda = 2 \cos \frac{2\pi q}{m} + \varepsilon e^{i\varphi}, \varphi \in [0, 2\pi] \). In this case, for sufficiently small \( \varepsilon \)
the small parameter power series expansions are valid:

\[
2 - (q_1^m(\lambda) + q_2^m(\lambda)) = -\frac{m^2\varepsilon^2(\sin\varphi - i\cos\varphi)^2}{4\sin^2\frac{2\pi q}{m}} + o(\varepsilon^2),
\]

\[
q_1(\lambda) - q_2(\lambda) = 2i\sin\frac{2\pi q}{m} + o(1).
\]

So, we get

\[
I_q = \frac{2\pi}{2\sin\frac{2\pi q}{m}}\varepsilon\left(\frac{i e^{i\phi} e^{2t \cos\frac{2\pi q}{m}}}{\sin\frac{2\pi q}{m}}\right) d\phi + o(1) = -\frac{4\pi i}{m} e^{2t \cos\frac{2\pi q}{m}} + o(1).
\]

Similarly, we obtain

\[
J_q = \frac{2\pi}{m^2} \sin\frac{2\pi q}{m} \sin\frac{2\pi q}{m} e^{2t \cos\frac{2\pi q}{m}} + o(1), \quad q \neq 0, \quad q \neq m/2.
\]

If \( q = 0 \), then

\[
I_0 = e^{2t} \int_0^{2\pi} \frac{2m \varepsilon\sin\varphi - i\cos\varphi}{2\varepsilon e^{i\varphi}} d\varphi + o(1) = -\frac{2\pi i e^{2t}}{m} + o(1).
\]

Similarly, we obtain

\[
J_0 = e^{2t} \int_0^{2\pi} \frac{2(m - p) e^{i\varphi}}{2\varepsilon e^{i\varphi}} \sqrt{\varepsilon} e^{2t} + o(1).
\]

If the number \( m \) is even, then it is necessary to calculate the integrals \( I_{m/2} \) and \( J_{m/2} \). In this case we get

\[
I_{m/2} = -\frac{2\pi i}{m} e^{-2t} + o(1).
\]

\[
J_{m/2} = (-1)^p \frac{2\pi i (m - p)}{m^2} e^{-2t} + o(1).
\]

Substituting the calculated integrals (3.15) – (3.20) in (3.12), we obtain the assertion of Lemma. This completes the proof.
4 Main result

In this section, we aim to establish stability conditions of the program solutions of differential equation (2.1).

**Theorem 4.1** Assume that \( A \) is an orthogonal operator, then any program solution \( X^*(t) \), \( X^*(0) = X_0^* \), int \( X_0^* \neq \emptyset \) is Lyapunov stable.

**Proof.** Define the function

\[
\varphi(t) = \Delta[X(t), X^*(t)] = \frac{V^n[X(t), X^*(t)] - V^{n-1}[X(t)]V[X^*(t)]}{V^{n-1}[X(t)]V[X^*(t)]}.
\]

By formulas (3.6) and (3.7), the numerator of \( \varphi(t) \) can be represented as

\[
V^n[X(t), X^*(t)] - V^{n-1}[X(t)]V[X^*(t)]
= \left( \sum_{k \in \mathbb{K}} a_k \Xi_k[X_0, X_0^*] \right)^n - \left( \sum_{k \in \mathbb{K}} a_k M_k[X_0] \right)^{n-1} \sum_{k \in \mathbb{K}} a_k M_k[X_0^*]
= \sum_{k_1 \in \mathbb{K}} \ldots \sum_{k_n \in \mathbb{K}} (\Xi_{k_1}[X_0, X_0^*] \ldots \Xi_{k_n}[X_0, X_0^*] - M_{k_1}[X_0] \ldots M_{k_{n-1}}[X_0] M_{k_n}[X_0^*]) a_{k_1} \ldots a_{k_n}.
\]

Therefore, taking into account Lemma 3.2, we obtain the estimate

\[
\varphi(t) \leq e^{-nt} \sum_{k_1 \in \mathbb{K}} \ldots \sum_{k_n \in \mathbb{K}} (\Xi_{k_1}[\tilde{X}_0, \tilde{X}_0^*] \ldots \Xi_{k_n}[\tilde{X}_0, \tilde{X}_0^*] - M_{k_1}[\tilde{X}_0] \ldots M_{k_{n-1}}[\tilde{X}_0] M_{k_n}[\tilde{X}_0^*]) a_{k_1} \ldots a_{k_n}.
\]

By the definition of the metric,

\[
(\tilde{X}_0)' \subset \tilde{X}_0^* + \rho_0 B_1(0), \quad \rho_0 = \rho[Or_\mathcal{E}(\tilde{X}_0), Or_\mathcal{E}(\tilde{X}_0^*)].
\]

By the monotony of the mixed volume functional, we obtain

\[
\Xi_k[\tilde{X}_0, \tilde{X}_0^*] = \Xi_k[(\tilde{X}_0)', \tilde{X}_0^*] \leq \Xi_k[\tilde{X}_0^*, \tilde{X}_0^*] + \sum_{k=1}^{n-1} C_{n-1}^k \rho_0^k R_{\tilde{X}_0^*}^{n-k} \nu
= M_k[\tilde{X}_0^*] + \sum_{k=1}^{n-1} C_{n-1}^k \rho_0^k R_{\tilde{X}_0^*}^{n-k} \nu.
\]

From the inclusion

\[
(\tilde{X}_0^*)' \subset \tilde{X}_0 + \rho_0 \overline{B}_1(0)
\]
and the monotony of the mixed volume, it follows the inequality

\[ M_k[\tilde{X}_0^*] \leq M_k[\tilde{X}_0] + \sum_{k=1}^{n-1} C_{n-1}^k \rho_0^k (R_{\tilde{X}_0^*} + \rho_0)^{n-k} \nu. \]

Choose a positive number \( \varepsilon_1 \) so that for all \( \rho, 0 < \rho < \varepsilon_1 \) the following inequalities hold

\[
\sum_{k=1}^{n-1} C_{n-1}^k \rho^k (R_{\tilde{X}_0^*} + \rho)^{n-k} \nu \leq 2(n - 1) \rho R_{\tilde{X}_0^*}^{n-1} \nu,
\]

\[
\sum_{k=1}^{n-1} C_{n-1}^k \rho^k R_{\tilde{X}_0^*}^{n-k} \nu \leq 2(n - 1) \rho R_{\tilde{X}_0^*}^{n-1} \nu,
\]

\[
\varepsilon_1 < \frac{M[X_0^*]}{2(n - 1) R_{\tilde{X}_0^*}^{n-1} \nu}.
\]

Then for all \( X_0 \) such that \( \rho_0 < \varepsilon_1 \) the following inequality holds

\[
\varphi(t) \leq e^{-n^2 t} A \rho_0 \sum_{k_1 \in K} \ldots \sum_{k_n \in K} \left[ (M_k[\tilde{X}_0^*] + 2(n - 1) \rho_0 R_{\tilde{X}_0^*}^{n-1} \nu)^n - (M_k[\tilde{X}_0^*] - 2(n - 1) \rho_0 R_{\tilde{X}_0^*}^{n-1} \nu)^{n-1} M_k[\tilde{X}_0^*] \right] a_{k_1} \ldots a_{k_n}.
\]

Applying Lagrange’s theorem on finite increments for function

\[
f(\rho) = (M_k[\tilde{X}_0^*] + 2(n - 1) \rho R_{\tilde{X}_0^*}^{n-1} \nu)^n - (M_k[\tilde{X}_0^*] - 2(n - 1) \rho R_{\tilde{X}_0^*}^{n-1} \nu)^{n-1} M_k[\tilde{X}_0^*]
\]

we get the following estimate

\[
\varphi(t) \leq e^{-n^2 t} A \rho_0 \sum_{k_1 \in K} \ldots \sum_{k_n \in K} a_{k_1} \ldots a_{k_n} = e^{-n^2 t} A \rho_0 \left( \sum_{k \in K} a_k \right)^n,
\]

where

\[
A = 2(n - 1) R_{\tilde{X}_0^*} \nu (M[X_0^*] + 2(n - 1) R_{\tilde{X}_0^*}^{n-1} \nu \varepsilon_1)^{n-1} (M[X_0^*] + 2n(n - 1) R_{\tilde{X}_0^*}^{n-1} \nu \varepsilon_1).
\]

By formula (3.8), we get

\[
\varphi(t) \leq A \rho_0.
\]
From the assertion of Lemma 3.1 it follows that there exists $\varepsilon_0 > 0$ such that for $\rho[\text{Or}_A(X_0), \text{Or}_A(X^*_0)] < \varepsilon_m$, $\varepsilon_m = \min[\varepsilon_0, \varepsilon_1]$ the estimate holds

$$\rho[\text{Or}_A(X(t)), \text{Or}_A(X^*(t))] \leq \left(\frac{C_2 A}{C_1}\right)^{1/n^2} \rho^{1/n^2}[\text{Or}_A(X_0), \text{Or}_A(X^*_0)].$$

For $\varepsilon > 0$ we choose $\delta(\varepsilon) = \min \{\varepsilon_m, \varepsilon_n^2 \left(\frac{C_2 A}{C_1}\right)^{-1/n^2}\}$, then for all $t \geq 0$ the inequality $\rho[\text{Or}_A(X(t)), \text{Or}_A(X^*(t))] < \varepsilon$ is fulfilled. This completes the proof.

Remark. The assertion of Theorem 4.1 is valid if the orthogonality condition for operator $A$ is replaced by the condition $\sup_{k \in \mathbb{Z}} \|A_k\| < \infty$. Indeed, in this case, by Theorem 6.1 about stable operators [9], there is an orthogonal operator $A_1$ and a nonsingular operator $T$ such that, $A_1 = T^{-1}AT$. In the Cauchy problem (2.1) we make the change of variables $X = TY$, then this problem is of the form

$$D_HY(t) = AY(t), \quad Y(0) = Y_0. \quad (4.1)$$

Thus, it is obvious that $\text{Or}_A(TX) = T\text{Or}_A(X)$ and the stability problem of solution $X^*(t)$ of the Cauchy problem (2.1) is equivalent to the stability problem of solution $Y^*(t) = T^{-1}X^*(t)$ of the Cauchy problem (4.1).

**Theorem 4.2** Assume that $n = 2$ and there exists a positive integer number $m$ such that, operator $A^m = I$, then any solution $X^*(t)$ is conditional Lyapunov asymptotically stable relative to the set

$$\mathcal{M} = \left\{X_0 \mid \text{int } X_0 \neq \emptyset, \sum_{p=0}^{m-1} A^p X_0 \in \text{Or}_A \left(\sum_{p=0}^{m-1} A^p X^*_0\right)\right\}.$$

**Proof.** Stability of solution $X^*(t)$ is the consequence of Theorem 4.1. Let us prove the condition of attraction of $X^*(t)$ relative to the set $\mathcal{M}$. Consider the function

$$\varphi(t) = \Delta[X(t), X^*(t)] = \frac{S^2[X(t), X^*(t)]}{S[X(t)]S[X^*(t)]} - 1.$$
As a result of the assertion of Lemma 3.4 we get

\[ S[X(t), X^*(t)] = e^{2t} \left( \frac{1}{m} S[X_0, X^*_0] + \frac{1}{m^2} \sum_{p=1}^{m-1} (m-p)(S[A^p X_0, X^*_0] + S[X_0, A^p X^*_0]) \right) + o(e^{2t}), \quad t \to \infty. \]

Then we have

\[ S[X(t)] = e^{2t} \left( \frac{1}{m} S[X_0] + \frac{2}{m^2} \sum_{p=1}^{m-1} (m-p)S[A^p X_0, X_0] \right) + o(e^{2t}) = e^{2t} \frac{1}{m^2} \sum_{k=0}^{m-1} A^k X_0 + o(e^{2t}), \quad t \to \infty. \]

By Lemma 3.1 \( \rho[\text{Or } \phi(X(t)), \text{Or } \phi(X^*(t))]] \to 0 \) for \( t \to \infty \) if and only if when \( \phi(t) \to 0 \) for \( t \to \infty \). It's obvious that \( \phi(t) \to 0 \) for \( t \to \infty \) if and only if when

\[ S^2 \left[ \sum_{k=0}^{m-1} A^k X_0, \sum_{k=0}^{m-1} A^k X^*_0 \right] - S \left[ \sum_{k=0}^{m-1} A^k X^*_0 \right] S \left[ \sum_{k=0}^{m-1} A^k X_0 \right] = 0. \]

By Brunn–Minkowski theorem, the last equality is valid if and only if when

\[ \sum_{k=0}^{m-1} A^k X_0 \in \text{Or } \phi \left( \sum_{k=0}^{m-1} A^k X^*_0 \right). \]

The Theorem is proved.

5 Example

Assume that the operator \( A \) is the rotation operator in the positive direction by the angle \( \frac{2\pi}{m} \). Then, the matrix \( A \) of the linear operator \( A \) in the
canonical basis has the form
\[ A = \begin{pmatrix} \cos \frac{2\pi}{m} & -\sin \frac{2\pi}{m} \\ \sin \frac{2\pi}{m} & \cos \frac{2\pi}{m} \end{pmatrix}. \]

Let \( h_X(p) \) be a support function of a convex compact \( X \in \text{conv} \mathbb{R}^2 \), \( H_X(\theta) = h_X(\cos \theta, \sin \theta) \). Then
\[ H_{m-1} \sum_{k=0}^{A^kX_0}(\theta) = \sum_{k=0}^{m-1} H_{A^kX_0}(\theta) = \sum_{k=0}^{m-1} H_X(\theta - \frac{2\pi k}{m}). \]

For function \( H_{X_0}(\theta) \) we can obtain the Fourier series expansion
\[ H_{X_0}(\theta) = \sum_{p=-\infty}^{\infty} H_p e^{ip\theta}. \]

Hence, we obtain
\[ H_{m-1} \sum_{k=0}^{A^kX_0}(\theta) = \sum_{k=0}^{m-1} \sum_{p=-\infty}^{\infty} H_p e^{ip(\theta - \frac{2\pi k}{m})} = \sum_{p=-\infty}^{\infty} H_p e^{ip\theta} \sum_{k=0}^{m-1} e^{\frac{2\pi ipk}{m}} = \sum_{p=-\infty}^{\infty} H_{pm} e^{ipm\theta}. \]

By Theorem 4.2 from conditions
\[ \int_0^{2\pi} (H_{X_0}(\theta) - H_{X_0^*}(\theta)) e^{-mp\theta} d\theta = 0, p \in \mathbb{Z}_+ \]
it follows that
\[ \lim_{t \to \infty} \rho[\text{Or}_G(X(t)), \text{Or}_G(X^*(t))] = 0, \]
provided that \( \rho[\text{Or}_G(X_0), \text{Or}_G(X_0^*)] \) is the sufficiently small positive number.

Thus, for each solution of the Cauchy problem \( X^*(t), X^*(0) \in \mathcal{C} \) there is an infinite dimensional variety of solutions \( X(t) \), that are attracted to the solution \( X^*(t) \).
6 Conclusion

By Theorem 4.1 we can conclude that the solution of the Cauchy problem (2.1) has a stable form. Theorem 4.2 strengthens this result for the case of two-dimensional space and a periodic operator. It suggests that for each solution there is an infinite-dimensional variety of solutions that are attracted to the program solution. For further study it is of interest to generalize the Theorem 4.2 for spaces of dimension greater than 2, and also for stable nonperiodic operators $A$. The main hypothesis concerning this case is that the forms of all solutions of the Cauchy problem (2.1) asymptotically tend to a ball shape.

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