HYPERBOLICITY IN TEICHMÜLLER SPACE

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Abstract. We review and organize some results describing the behavior of a Teichmüller geodesic and draw several applications: 1) We show that Teichmüller geodesics do not back track. 2) We show that a Teichmüller geodesic segment whose endpoints are in the thick part has the fellow travelling property. This fails when the endpoints are not necessarily in the thick part. 3) We show that if an edge of a Teichmüller geodesic triangle passes through the thick part, then it is close to one of the other edges.

1. Introduction

Two points in Teichmüller space determine a unique Teichmüller geodesic that connects them. One would like to understand the behavior of this geodesic and how the given data, two end points \( x, y \) in \( \mathcal{T}(S) \) Teichmüller of a surface \( S \), translate to concrete information about the geodesic segment \([x, y]\) connecting them. Much is known about this relationship. (See \cite{Raf05, Raf07a, CRS08, Raf07b}.) The first part of the paper is devoted to organizing and improving some of these results which are scattered through several papers. Accumulation of these results provides a complete (coarse) description of a Teichmüller geodesic. One can summarized this as follows:

**Theorem A.** Let \( \mathcal{G} : \mathbb{R} \to \mathcal{T}(S) \) be a Teichmüller geodesic. For every subsurface \( Y \), there is an interval of times \( I_Y \) (possibly empty) where \( Y \) is isolated at \( \mathcal{G}_t \), for \( t \in I_Y \). During this interval, the restriction of \( \mathcal{G} \) to \( Y \) behaves like a geodesic in \( \mathcal{T}(Y) \). Outside of \( I_Y \), the projection to the curve complex of \( Y \) moves by at most a bounded amount.

In fact, we know for which subsurfaces \( Y \) the interval \( I_Y \) is non-empty, and in what order these intervals appear along \( \mathbb{R} \). And applying the theorem inductively, we can describe the restriction of the geodesic to \( Y \) during \( I_Y \) (Section 5).

In the rest of the paper we consider some of the implications of the above theorem and we examine to what extend Teichmüller geodesics behave like geodesics in a hyperbolic space. It is known that the Teichmüller space is not hyperbolic; Masur showed that Teichmüller space
is not $\delta$–hyperbolic \cite{MW95} and Minsky showed that the thin part of Teichmüller space has a product like structure that resembles a space with positive curvature \cite{Min96}. However, there is a strong analogy between the geometry of Teichmüller space and that of a hyperbolic space. For example, the isometries of Teichmüller space are either hyperbolic, elliptic or parabolic \cite{Thu88, Ber78} and the geodesic fellow is exponentially mixing \cite{Mas82, Vee86}. There is also a sense that Teichmüller space is hyperbolic relative to its thin parts; Masur and Minsky showed that electrified Teichmüller space is $\delta$–hyperbolic \cite{MM99}.

Each application of Theorem A presented in this paper examines how the Teichmüller space equipped with the Teichmüller metric is similar to or different from a relatively hyperbolic space. Apart from their individual utility, these results also showcase how one can apply Theorem A to answer geometric problems in Teichmüller space.

As the first application, we show that Teichmüller geodesics do not backtrack. This is a generalization of a theorem of Masur and Minsky \cite{MM99} stating that the shadow of a Teichmüller geodesic to the curve complex is an un-parametrized quasi-geodesic. We show:

**Theorem B.** The projection of a Teichmüller geodesic to the complex of curves of any subsurface $Y$ of $S$ is an un-parametrized quasi-geodesic in the curve complex of $Y$.

This produces a sequence of markings, analogous to a resolution of a hierarchy \cite{MM99}, which is obtained directly from Teichmüller geodesic.

As the second application, we examine the fellow traveling properties of Teichmüller geodesics. We show:

**Theorem C.** Consider a Teichmüller geodesics segment $[x, y]$ with end points $x$ and $y$ in the thick part. Any other geodesic segment that starts near $x$ and ends near $y$ fellow travels $[x, y]$.

In contrast to above, we can provide examples where:

**Theorem D.** When the end points of a geodesic segment are allowed to be in the thin part, the above theorem does not hold.

As our third application, we prove that geodesic triangles are slim while they pass through the thick part of Teichmüller space, suggesting similarities between Teichmüller space and relatively hyperbolic groups.

**Theorem E.** For a geodesic triangle $\triangle(x, y, z)$ in Teichmüller space, if a large segment of $[x, y]$ is in the thick part, then it is either close to $[x, z]$ or $[y, z]$.
Organization of the paper. In Section 2 we make the notion of coarsely describing a point in Teichmüller space precise. This means to record enough information so that one can estimate the length of any curve on the surface and the distance between two points in Teichmüller space. It turns out that it is sufficient to keep track of which curves are short as well as the length and the twisting parameter of the short curves.

A Teichmüller geodesic is the image of a quadratic differential under the Teichmüller geodesic flow. In Section 3 we discuss how one can translate the information given by the flat structure of a quadratic differential to obtain the combinatorial information needed to describe a point in $\mathcal{T}(S)$.

The precise statement for the description of a Teichmüller geodesic and some related statements are given Section 5. Theorem B is proven in Section 6, Theorems C and D are proven in Section 7 and Theorem E is proven in Section 8.

Notation. The notation $A \asymp B$ means that the ratio $A/B$ is bounded both above and below by constants depending on the topology of $S$ only. When this is true we say $A$ is comparable with $B$ or $A$ and $B$ are comparable. The notation $A \lesssim B$ means that $A/B$ is bounded above by a constant depending on the topology of $S$. Similarly, $A \lesssim b$ means $|A - B|$ is uniformly bounded and $A \lesssim B$ means $(B - A)$ is uniformly bounded above in both cases by a constant that depend only on the topology of $S$.

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2. Combinatorial description of a point in Teichmüller space

In this section, we discuss the notion of a marking which provides a combinatorial description of a point in Teichmüller space (see Definition 2.2). Given a description of a point $x$ in Teichmüller space we are able to estimate the extremal length of any curve at $x$ (Theorem 3.1). Also, given the description of two points $x, y \in \mathcal{T}(S)$, we are able to estimate the Teichmüller distance between them (Theorem 2.4). We first establish terminology and the definitions of some basic concepts.

2.1. Teichmüller metric. Let $S$ be a compact surface of hyperbolic type possibly with boundary. The Teichmüller space $\mathcal{T}(S)$ is the space of all conformal structures on $S$ up to isotopy. In this paper, we consider
only the Teichmüller metric on \( \mathcal{T}(S) \). For two points \( x, y \in \mathcal{T}(S) \) the Teichmüller distance between them is defined to be
\[
d_T(x, y) = \frac{1}{2} \log \max_f K_f,
\]
where \( f: x \to y \) ranges over all quasi-conformal maps from \( x \) to \( y \) in the correct isotopy class and \( K_f \) is the quasi-conformal constant of the map \( f \). (See [GL00, Hub06] for background information.) A geodesic in this metric is called a Teichmüller geodesic.

**Arcs and curves.** By a curve in \( S \) we mean a free isotopy class of an essential simple closed curve and by an arc in \( S \) we mean a proper isotopy class of an essential simple arc. In both cases, essential means that the given curve or arc is neither isotopic to a point nor it can be isotoped to \( \partial S \). The definition of an arc is slightly different when \( S \) is an annulus. In this case, an arc is an isotopy class of a simple arc connecting the two boundaries of \( S \), relative to the endpoints of the arc.

We use \( i(\alpha, \beta) \) to denote the geometric intersection number between arcs or curves \( \alpha \) and \( \beta \) and we refer to it simply as the intersection number.

Define the arc and curve graph \( \mathcal{AC}(S) \) of \( S \) as follows: the vertices are essential arcs and curves in \( S \) and the edges are pairs of vertices that have representatives with disjoint interiors. Giving the edges length one turns \( \mathcal{AC}(S) \) into a connected metric space. The following is contained in [MM99, MM00, Kla99]

**Theorem 2.1.** The graph \( \mathcal{AC}(S) \) is locally infinite, has infinite diameter and is Gromov hyperbolic. Furthermore, its boundary at infinity can be identified with \( \mathcal{EL}(S) \), the space of ending laminations of \( S \).

Recall that, \( \mathcal{EL}(S) \) is the space of irrational laminations in \( \mathcal{PML}(S) \) after forgetting the measure. An irrational lamination is one that has non-zero intersection number with every curve.

**Measuring the twist.** It is often desirable to measure the number of times a curve \( \gamma \) twists around a curve \( \alpha \). This requires us to choose a notion of zero twisting. The key example is the case where \( S \) is an annulus with a core curve \( \alpha \). Then \( \mathcal{AC}(S) \) is quasi-isometric to \( \mathbb{Z} \).

Choose an arc \( \tau \in \mathcal{AC}(S) \) to serve as the origin. Then the twist of \( \gamma \in \mathcal{AC}(S) \) about \( \alpha \) is
\[
\text{twist}_\alpha(\gamma, \tau) = i(\gamma, \tau),
\]
relative to choice of origin \( \tau \).

In general, if \( \alpha \) is a curve in \( S \) let \( S^\alpha \) be the corresponding annular cover. A notion of zero twisting around \( \alpha \) is given by a choice of arc
\( \tau \in \mathcal{AC}(S^\alpha) \). Then, for every \( \gamma \in \mathcal{AC}(S) \) intersecting \( \alpha \) essentially, we define

\[
\text{twist}_\alpha(\gamma, \tau) = i(\tilde{\gamma}, \tau),
\]

where \( \tilde{\gamma} \) is any essential lift of \( \gamma \) to \( S^\alpha \). Since there may be several choices for \( \tilde{\gamma} \), this notion of twisting is well defined up to an additive error of at most one.

A geometric structure on \( S \) often naturally defines a notion of zero twisting. For example, for a given point \( x \in T(S) \) and a curve \( \alpha \), we can define twisting around \( \alpha \) in \( x \) as follows: lift \( x \) to a the conformal structure \( x^\alpha \) on \( S^\alpha \). Consider the hyperbolic metric associated to \( x^\alpha \) and choose \( \tau \) in \( x^\alpha \) to be any hyperbolic geodesic perpendicular to \( \alpha \). Now, for every curve \( \gamma \) intersecting \( \alpha \) non-trivially, define

\[
\text{twist}_\alpha(\gamma, x) = \text{twist}_\alpha(\gamma, \tau) = i(\tilde{\gamma}, \tau).
\]

Similarly, for a quadratic differential \( q \) on \( S \) we can define \( \text{twist}_\alpha(\gamma, q) \); lift \( q \) to a singular Euclidean metric \( q^\alpha \) and choose \( \tau \) to be any Euclidean perpendicular arc to \( \alpha \). (See Section 3 for the definition of the Euclidean metric associated to \( q \).)

Similarly, any foliation, arc or curve \( \lambda \) intersecting \( \alpha \) essentially defines a notion of zero twisting. Since the intersection is essential the lift \( \lambda^\alpha \) of \( \lambda \) to \( S^\alpha \) contains an essential arc which we may use as \( \tau \). Anytime two geometric objects define notions of zero twisting, we can talk about the relative twisting between them. For example, for two quadratic differentials \( q_1 \) and \( q_2 \) and a curve \( \alpha \), let \( \tau_1 \) be the arc in \( q_1^\alpha \) that is perpendicular to \( \alpha \) and \( \tau_2 \) be the arc in \( q_2^\alpha \) that is perpendicular to \( \alpha \). Considering both these arcs in \( S^\alpha \), it makes sense to talk about their geometric intersection number. We define:

\[
\text{twist}_\alpha(q_1, q_2) = i(\tau_1, \tau_2).
\]

The expression \( \text{twist}_\alpha(x_1, x_2) \) for Riemann surfaces \( x_1 \) and \( x_2 \) is defined similarly.

**Marking.** Our definition of marking differs slightly from that of [MM00] and contains more information.

**Definition 2.2.** A marking on \( S \) is a triple \( \mu = (\mathcal{P}, \{l_\alpha\}_{\alpha \in \mathcal{P}}, \{\tau_\alpha\}_{\alpha \in \mathcal{P}}) \) where

- \( \mathcal{P} \) is a pants decomposition of \( S \).
- For \( \alpha \in \mathcal{P} \), \( l_\alpha \) is a positive real number which we think of as the length of \( \alpha \).
- For \( \alpha \in \mathcal{P} \), \( \tau_\alpha \) is an arc in the annular cover \( S^\alpha \) of \( S \) associated to \( \alpha \), establishing a notion of zero twisting around \( \alpha \).
For a curve \( \alpha \) in \( S \) and \( x \in \mathcal{T}(S) \), we define the extrema length of \( \alpha \) in \( x \) to be
\[
\Ext(x)(\alpha) = \sup_{\sigma \in [x]} \frac{\ell_2^2(\sigma)}{\text{area}(\sigma)}.
\]
Here, \( \sigma \) ranges over all metric in the conformal class \( x \) and \( \ell_2^2(\sigma) \) is the infimum of the \( \sigma \)-length of all representatives of the homotopy class of the curve \( \alpha \). Using the Extremal length, we define a map from \( \mathcal{T}(S) \) to the space of markings as follows: For any \( x \in \mathcal{T}(S) \), let \( \mathcal{P}_x \) be the pants decomposition with the shortest extremal length in \( x \) obtained using the greedy algorithm. For \( \alpha \in \mathcal{P}_x \), let \( l_\alpha = \Ext(x)(\alpha) \). As in the discussion of zero twist above, let \( \tau_\alpha \) be any geodesic in \( S^\alpha \) that is perpendicular to \( \alpha \) in \( x^\alpha \). We call this the short marking at \( x \) and denote it by \( \mu_x \).

As mentioned before, we can compute the extremal length of any curve in \( x \) from the information contained in \( \mu_x \) up to a multiplicative error. It follows from \[\text{[Min96]}\] that:

**Theorem 2.3.** For every curve \( \gamma \), we have
\[
\Ext(x)(\gamma) \leq \sum_{\alpha \in \mathcal{P}} \left( \frac{1}{l_\alpha} + l_\alpha \cdot \text{twist}_\alpha(\gamma, \tau_\alpha)^2 \right) i(\alpha, \gamma)^2.
\]

**Subsurface Projection.** To compute the distance between two points \( x, y \in \mathcal{T}(S) \) we need to introduce the concept of subsurface projection. We call a collection of vertices in \( \mathcal{AC}(S) \) having disjoint representatives a multicurve. For every proper subsurface \( Y \subset S \) and any multicurve \( \alpha \) in \( \mathcal{AC}(S) \) we can project \( \alpha \) to \( Y \) to obtain a multicurve in \( \mathcal{AC}(Y) \) as follows: let \( S^Y \) be the cover of \( S \) corresponding to \( \pi_1(Y) < \pi_1(S) \) and identify the Gromov compactification of \( S^Y \) with \( Y \). (To define the Gromov compactification, one needs first to pick a metric on \( S \). However, the resulting compactification is independent of the metric. Since \( S \) admits a hyperbolic metric, every essential curve in \( S \) lifts to an arc which has a well defined end points in the Gromov boundary of \( S^Y \).) Then for \( \alpha \in \mathcal{AC}(S) \), the projection \( \alpha_Y \) is defined to be the set of lifts of \( \alpha \) to \( S^Y \) that are essential curves or arcs. Note that \( \alpha_Y \) is a set of diameter one in \( \mathcal{AC}(Y) \) since all the lifts have disjoint interiors.

For markings \( \mu \) and \( \nu \), define
\[
d_Y(\mu, \nu) = \text{diam}_{\mathcal{AC}(Y)}(\mathcal{P}_Y \cup \mathcal{R}_Y)
\]
where \( \mathcal{P} \) and \( \mathcal{R} \) are the pants decompositions for \( \mu \) and \( \nu \) respectively.

**Distance Formula.** In what comes below, the function \( [a]_C \) is equal to \( a \) if \( a \geq C \) and it is zero otherwise. Also, we modify the \( \log(a) \) function to be one for \( a \leq e \). We can now state the distance formula:
Theorem 2.4 (Theorem 6.1, [Raf07a]). There is a constant $C > 0$ so that the following holds. For $x, y \in \mathcal{T}(S)$ let $\mu_x = (\mathcal{P}, \{l_\alpha\}, \{\tau_\alpha\})$ and $\mu_y = (\mathcal{R}, \{k_\beta\}, \{\sigma_\beta\})$ be the associated short markings. Then,
\[
 d_{\mathcal{T}}(x, y) \asymp \sum_Y \left[ d_Y(\mu_x, \mu_y) \right]_C + \sum_{\gamma \notin \mathcal{P} \cup \mathcal{R}} \left[ \log d_{\gamma}(\mu_x, \mu_y) \right]_C \\
 + \sum_{\alpha \in \mathcal{P} \setminus \mathcal{R}} \log \frac{1}{l_\alpha} + \sum_{\beta \in \mathcal{R} \setminus \mathcal{P}} \log \frac{1}{k_\beta} \\
 + \sum_{\gamma \in \mathcal{P} \cap \mathcal{R}} d_\mathbb{H}\left(\left(1/l_\gamma, \text{twist}_\gamma(x, y)\right), \left(1/k_\gamma, 0\right)\right).
\] (1)

Here, $d_\mathbb{H}$ is the distance in the hyperbolic plane.

Remark 2.5. In above theorem, $C$ can be taken to be as large an needed. However, increasing $C$ will increase the constants hidden inside $\asymp$. Let $\mathcal{L}$ be the left hand side of Equation (1) and $\mathcal{R}$ be the right hand side. Then, a stronger version of this theorem can be stated as follows: There is $C_0 > 0$, depending only on the topology of $S$, and for every $C \geq C_0$ there are constants $A$ and $B$ so that
\[
 \mathcal{L} \leq \mathcal{R} \leq A \mathcal{L} + B.
\]

As a corollary, we have the following criterion for showing two points in Teichmüller space are a bounded distance apart. Let $\epsilon_0 > \epsilon_1 > 0$, let $\mathcal{A}_x$ be a set of curves in $x$ that have extremal length less than $\epsilon_0$ and assume that every other curve in $x$ has a length larger than $\epsilon_1$. Let $\epsilon'_0, \epsilon'_1$ and $\mathcal{A}_y$ be similarly defined for $y$.

Corollary 2.6. Assume, for $x, y \in \mathcal{T}(S)$, that
\begin{enumerate}
  \item $\mathcal{A}_x = \mathcal{A}_y$
  \item For any subsurface $Y$ that is not an annulus with core curve in $\mathcal{A}_x$, $d_Y(\mu_x, \mu_y) = O(1)$.
  \item For $\alpha \in \mathcal{A}_x$, $\ell_x(\alpha) \asymp \ell_y(\alpha)$.
  \item For $\alpha \in \mathcal{A}_x$, $\text{twist}_\alpha(x, y) = O\left(1/\text{Ext}_x(\alpha)\right)$.
\end{enumerate}

Then, $d_{\mathcal{T}}(x, y) = O(1)$.

Proof. Condition (2) implies that the first two terms in Equation (1) are zero. Since $\mathcal{A}_x = \mathcal{A}_y$, curves in $\mathcal{P} \setminus \mathcal{R}$ and $\mathcal{R} \setminus \mathcal{P}$ have lengths that are bounded below. Hence the third and the forth terms of Equation (1) are uniformly bounded. The conditions on the lengths and twisting of curves in $\mathcal{A}_x$ imply that the last term is uniformly bounded; for points $p, q \in \mathbb{H}$, $p = (p_1, p_2)$, $q = (q_1, q_2)$, if $(p_1 - q_1) \asymp p_2 \asymp q_2$ then $d_\mathbb{H}(p, q) = O(1)$. $\square$
3. GEOMETRY OF QUADRATIC DIFFERENTIALS

A geodesic in Teichmüller space is the image of a quadratic differential under the Teichmüller geodesic flow. Quadratic differentials are naturally equipped with a singular Euclidean structure. We, however, often need to compute the extremal length of a curve. In this section, we review how the extremal length of a curve can be computed from the information provided by the flat structure and how the flat length and the twisting information around a curve change along a Teichmüller geodesic.

**Quadratic differentials.** Let $\mathcal{T}(S)$ be the Teichmüller space of $S$ and $\mathcal{Q}(S)$ be the space of unit area quadratic differentials on $S$. Recall that a quadratic differential $q$ on a Riemann surface $x$ can locally be represented as

$$q = q(z) \, dz^2,$$

where $q(z)$ is a meromorphic function on $x$ with all poles having a degree of at most one. All poles are required to occur at the punctures. In fact, away from zeros and poles, there is a change of coordinates so that $q = dz^2$. Here $|q|$ locally defines a Euclidean metric on $x$ and the expressions $\Im(\sqrt{q}) = 0$ and $\Re(\sqrt{q}) = 0$ define the horizontal and the vertical directions. Vertical trajectories foliate the surface except at the zeros and the poles. This foliation equipped with the transverse measure $|dx|$ is called the vertical foliation and is denoted by $\lambda_-$. The horizontal foliation is similarly defined and is denoted by $\lambda_+$. A neighborhood of a zero of order $k$ has the structure of the Euclidean cone with total angle $(k+2)\pi$ and a neighborhood of a degree one pole has the structure of the Euclidean cone with total angle $\pi$. In fact, this locally Euclidean structure and this choice of the vertical foliation completely determines $q$. We refer to this metric as the $q$–metric on $S$.

**Size of a subsurface.** For every curve $\alpha$, the geodesic representatives of $\alpha$ in the $q$–metric form a (possibly degenerate) flat cylinder $F_q(\alpha)$. For any proper subsurface $Y \subset S$, let $Y = Y_q$ be the representative of the homotopy class of $Y$ that has $q$–geodesic boundaries and that is disjoint from the interior of $F_q(\alpha)$ for every curve $\alpha \subset \partial Y$. When the subsurface is an annulus with core curve $\alpha$ we think of $F = F_q(\alpha)$ as its representative with geodesic boundary. Define $\text{size}_q(Y)$ to be the $q$–length of the shortest essential curve in $Y$ and for a curve $\alpha$ let $\text{size}_q(F)$ be the $q$–distance between the boundary components of $F$. When $Y$ is a pair of pants, $\text{size}_q(Y)$ is defined to be the diameter of $Y$. 
An estimate for lengths of curves. For every curve $\alpha$ in $S$, denote the extremal length of $\alpha$ in $x \in \mathcal{T}(S)$ by $\operatorname{Ext}_x(\alpha)$. For constants $\epsilon_0 > \epsilon_1 > 0$, the $(\epsilon_0, \epsilon_1)$-thick-thin decomposition of $x$ is the pair $(\mathcal{A}, \mathcal{Y})$, where $\mathcal{A}$ is the set of curves $\alpha$ in $x$ so that $\operatorname{Ext}_x(\alpha) \leq \epsilon_0$ and $\mathcal{Y}$ is the set of homotopy class of the components of $x$ cut along $\mathcal{A}$. We further assume that the extremal length of any essential curve $\gamma$ that is disjoint from $\mathcal{A}$ is larger than $\epsilon_1$.

Consider the quadratic differential $(x, q)$ and the thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ of $x$. Let $\alpha \in \mathcal{A}$ be the common boundary of subsurfaces $Y$ and $Z$ in $\mathcal{Y}$. Let $\alpha^*$ be the geodesic representative of $\alpha$ in the boundary of $Y$ and let $E = E_q(\alpha, Y)$ be the largest regular neighborhood of $\alpha^*$ in the direction of $Y$ that is still an embedded annulus. We call this annulus the expanding annulus with core curve $\alpha$ in the direction of $Y$. Define $M_q(\alpha, Y)$ to be $\operatorname{Mod}_x(E)$, where $\operatorname{Mod}_x(\cdot)$ is the modulus of an annulus in $x$. Recall from [Raf05, Lemma 3.6] that

$$\operatorname{Mod}_x(E) \gtrsim \log \frac{\operatorname{size}(Y)}{\ell_q(\alpha)} \quad \text{and} \quad \operatorname{Mod}_x(F) = \frac{\operatorname{size}(F)}{\ell_q(\alpha)}.$$ 

Let $G = E_q(\alpha, Z)$ and $M_q(\alpha, Z)$ be defined similarly.

The following statement relates the information about the flat lengths of curves to their extremal length. For a more general statement see [LR10] Lemma 3 and Theorem 7).

**Theorem 3.1.** Let $(x, q)$ be a quadratic differential and let $(\mathcal{Y}, \mathcal{A})$ be the thick-thin decomposition of $x$. Then

1. For $Y \in \mathcal{Y}$ and a curve $\gamma$ in $Y$

$$\operatorname{Ext}_x(\gamma) \gtrsim \frac{\ell_q(\gamma)^2}{\operatorname{size}(Y)^2}.$$

2. For $\alpha \in \mathcal{A}$ that is the common boundary of $Y, Z \in \mathcal{Y}$,

$$\frac{1}{\operatorname{Ext}_x(\alpha)} \gtrsim \log \frac{\operatorname{size}(Y)}{\ell_q(\alpha)} + \frac{\operatorname{size}(F_q(\alpha))}{\ell_q(\alpha)} + \log \frac{\operatorname{size}(Z)}{\ell_q(\alpha)}$$

$$\gtrsim \operatorname{Mod}_x(E) + \operatorname{Mod}_x(F) + \operatorname{Mod}_x(G).$$

Length and twisting along a Teichmüller geodesic. A matrix $A \in \text{SL}(2, \mathbb{R})$ acts on any $q \in \mathcal{Q}(S)$ locally by affine transformations. The total angle at a point does not change under this transformation. Thus the resulting singular Euclidean structure defines a quadratic differential that we denote by $Aq$. The Teichmüller geodesic
flow, \( g_t : Q \to Q \), is the action by the diagonal subgroup of \( \text{SL}(2, \mathbb{R}) \):

\[
g_t(q) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} q.
\]

The Teichmüller geodesic described by \( q \) is then a map

\[
\varphi : \mathbb{R} \to Q, \quad \varphi(t) = (x_t, q_t)
\]

where \( q_t = g_t(q) \) and \( x_t \) is the underlying Riemann surface for \( q_t \).

The flat length of a curve along a Teichmüller geodesic is well behaved. Let the horizontal length \( h_t(\alpha) \) of \( \alpha \) in \( q_t \) be the transverse measure of \( \alpha \) with respect to the vertical foliation of \( q_t \) and the vertical length \( v_t(\alpha) \) of \( \alpha \) be the transverse measure with respect to the horizontal foliation of \( q_t \). We have (see the discussion on [Raf05, Page 186])

\[
\ell_{q_t}(\alpha) \gtrsim h_t(\alpha) + v_t(\alpha).
\]

Since the vertical length decreases exponentially fast and the horizontal length increases exponentially fast, for every curve \( \alpha \), there are constants \( L_\alpha \) and \( t_\alpha \) so that

\[
(2) \quad \ell_{q_t}(\alpha) \gtrsim L_\alpha \cosh(t - t_\alpha).
\]

We call the time \( t_\alpha \) the balanced time for \( \alpha \) and the length \( L_\alpha \) the minimum flat length for \( \alpha \).

We define the twisting parameter of a curve along a Teichmüller geodesic to be the relative twisting of \( q_t \) with respect to the vertical foliation. That is, for any curve \( \alpha \) and time \( t \), let \( \tau_t \) be the arc in the annular cover of \( q_t \) that is perpendicular to \( \alpha \) and let \( \lambda_- \) be the vertical foliation of \( q_t \) (which is topologically the same foliation for every value of \( t \)). Define

\[
\text{twist}_t(\alpha) = \text{twist}_\alpha(\tau_t, \lambda_-).
\]

This is an increasing function that ranges from a minimum of zero to a maximum of \( T_\alpha = d_\alpha(\lambda_-, \lambda_+) \). That is, \( \tau_t \) looks like \( \lambda_- \) at the beginning and like \( \lambda_+ \) in the end. In fact, from [Raf07a, Equation 16] we have the following explicit formula:

\[
(3) \quad \text{twist}_t(\alpha) \doteq \frac{2T_\alpha e^{2(t-t_\alpha)}}{\cosh^2(t - t_\alpha)}.
\]

Also, [CRS08, Proposition 5.8] gives the following estimate on the modulus of \( F_t = F_{q_t}(\alpha) \):

\[
(4) \quad \text{Mod}_{q_t}(F_t) \doteq \frac{T_\alpha}{\cosh^2(t - t_\alpha)}.
\]
That is, the modulus of $F_t$ is maximum when $\alpha$ is balanced and goes to zero as $t$ goes to $\pm\infty$. The maximum modulus of $F_t$ is determined purely by the topological information $T_\alpha$, which is the relative twisting of $\lambda_-$ and $\lambda_+$ around $\alpha$. The size of $F_t$ at $q_t$ is equal to its modulus times the flat length of $\alpha$ at $q_t$. Hence,

$$\text{size}_{q_t}(F_t) = \frac{T_\alpha L_\alpha}{\cosh(t - t_\alpha)}$$

4. Projection of a quadratic differential to a subsurface

In this section, we introduce the notion of an isolated surface in a quadratic differential. Let $(x, q)$ be a quadratic differential, $Y \subset S$ be a proper subsurface and $Y$ be the representative of $Y$ with $q$-geodesic boundaries. Note that, when $Y$ is non-degenerate, it is itself a Riemann surface that inherits its conformal structure from $x$. In this case, for a curve $\gamma$ in $Y$, we use the expression $\text{Ext}^Y(\gamma)$ to denote the extremal length of $\gamma$ in the Riemann surface $Y$. The following lemma which is a consequence of ([Min96, Lemma 4.2]).

**Lemma 4.1** (Minsky). There exists a constant $m_0$ depending only on the topological type of $S$ so that, for every subsurface $Y$ with negative Euler characteristic the following holds. If $M_q(\alpha, Y) \geq m_0$ for every boundary component $\alpha$ of $Y$ then for any essential curve $\gamma$ in $Y$

$$\text{Ext}^Y(\gamma) \simeq \text{Ext}_x(\gamma).$$

Fixing $m_0$ as above, we say $Y$ is isolated in $q$ if, for every boundary component $\alpha$ of $Y$, $M_q(\alpha, Y) \geq m_0$. The large expanding annuli in the boundaries of $Y$ isolate it in the sense that one does not need any information about the rest of the surface to compute extremal lengths of curves in $Y$. As we shall see, when $Y$ is isolated, the restrictions of the hyperbolic metric of $x$ to $Y$ and the quadratic differential $q$ to $Y$ are at most a bounded distance apart in the Teichmüller space of $Y$.

For $x \in T(S)$ and $Y \subset S$ we define the Fenchel-Nielsen projection of $x$ to $Y$, a complete hyperbolic metric $x|_Y$ on $Y$, as follows: Extend the boundary curves of $Y$ to a pants decomposition $P$ of $S$. Then the Fenchel-Nielsen coordinates of $P|_Y$ defines a point $x|_Y$ of $T(Y)$ (see [Min96] for a detailed discussion).

Now, we construct a projection map from $q$ to $q|_Y$ by considering the representative with geodesic boundary $Y$ and capping off the boundaries with punctured disks. It turns out that the underlying conformal structure of $q|_Y$ and $x|_Y$ are not very different, but the quadratic differential restriction commutes with the action of $\text{SL}(2, \mathbb{R})$. When $Y$ is not isolated in $q$, the capping off process is not geometrically meaningful.
(or sometimes not possible). Hence, the process is restricted to the appropriate subset of $Q$.

**Theorem 4.2.** Let $Y$ be a subsurface of $S$ that is not an annulus and let $Q_Y(S)$ be the set of quadratic differentials $q$ so that $Y$ is isolated in $q$. There is a map $\pi_Y: Q_Y(S) \to Q(Y)$, with $\pi_Y(q) = q_Y$, so that

\[ d_{T(Y)}(q_Y, x_Y) = O(1). \]  

Furthermore, if, for $A \in \text{SL}(2, \mathbb{R})$, both $q$ and $Aq$ are in $Q_Y(S)$ then

\[ d_{T(Y)}((Aq)_Y, A(q_Y)) = O(1). \]

**Proof.** We first define the map $\pi_Y$. Let $(x, q)$ be a quadratic differential with $Y$ isolated in $q$. Let $Y$ be the representative of $Y$ with $q$–geodesic boundaries. Our plan, nearly identical to that of [Raf07b], is to fill all components of $\partial Y$ with locally flat once-punctured disks.

Fix $\alpha \subset \partial Y$ and recall that $E = E_q(\alpha, Y)$ is an embedded annulus and $\alpha^*$ is a boundary of $E$. Let $a_1, \ldots, a_n$ be the points on $\alpha^*$ which have angle $\theta_i > \pi$ in $E$. Note that this set is nonempty: if it is empty then $E$ meets the interior of the flat cylinder $F(\alpha)$, a contradiction. Let $E'$ be the double cover of $E$ and let $\alpha'$ be the pre-image of $\alpha^*$. Let $q'$ be the lift of $q|_E$ to $E'$. Along $\alpha'$ we attach a locally flat disk $D'$ with a well defined notion of a vertical direction, as follows.

Label the lifts of $a_i$ to $E'$ by $b_i$ and $c_i$. We will fill $\alpha'$ by symmetrically adding $2(n - 1)$ Euclidean triangles to obtain a flat disk $D'$ such that the total angle at each $b_i$ and $c_i$ is a multiple of $\pi$ and is at least $2\pi$.

![Figure 1. The filling of the annulus $E'$](image)

We start by attaching a Euclidean triangle to vertices $b_1, b_2, b_3$, which we denote by $\triangle(b_1, b_2, b_3)$ (see Figure 1). We choose the angle $\angle b_2$...
at the vertex $b_2$ so that $\theta_2 + \angle b_2$ is a multiple of $\pi$. Assuming $0 \leq \angle b_2 < \pi$, there is a unique such triangle. Attach an isometric triangle to $c_1, c_2, c_3$. Now consider the points $b_1, b_3, b_4$. Again, there exists a Euclidean triangle with one edge equal to the newly introduced segment $[b_1, b_3]$, another edge equal to the segment $[b_3, b_4]$ and an angle at $b_3$ that makes the total angle at $b_3$, including the contribution from the triangle $\Delta(b_1, b_2, b_3)$, a multiple of $\pi$. Attach this triangle to the vertices $b_1, b_3, b_4$ and an identical triangle to the vertices $c_1, c_3, c_4$. Continue in this fashion until finally adding triangles $\Delta(b_1, b_n, c_1)$ and $\Delta(c_1, c_n, b_1)$. Due to the symmetry, the two edges connecting $b_1$ and $c_1$ have equal length, and we can glue them together. We call the union of the added triangles $D'$. Notice that the involution on $E'$ extends to $D'$. Let $D = D(\alpha)$ be the quotient of $D'$, and note that $D$ is a punctured disk attached to $\alpha^*$ in the boundary of $E$.

For $i \neq 1$, the total angle at $b_i$ and at $c_i$ is a multiple of $\pi$ and is larger than $\theta_i > \pi$; therefore, it is at least $2\pi$. We have added $2(n - 1)$ triangles. Hence, the sum of the total angles of all vertices is $2 \sum_i \theta_i + 2(n - 1)\pi$, which is a multiple of $2\pi$. Therefore, the sum of the angles at $b_1$ and $c_1$ is also a multiple of $2\pi$. But they are equal to each other, and each one is larger than $\pi$. This implies that they are both at least $2\pi$. It follows that the quadratic differential $q'$ extends over $D'$ symmetrically with quotient an extension of $q$ to $D$.

Thus, attaching the disk $D(\alpha)$ to every boundary component $\alpha^*$ in $\partial Y$ gives a point $q_Y \in \mathcal{Q}(Y)$. This completes the construction of the map $\pi_Y$.

We now show that the distance in $\mathcal{T}(Y)$ between $q_Y$ and $x_Y$ is uniformly bounded. For this, we examine the extremal lengths of curves in two conformal structures. Since $Y$ is isolated in $q$, the boundaries of $Y$ are short in $x$. This implies, using [Min96] that, for any essential curve $\gamma$ in $Y$, the extremal lengths of $\gamma$ in $x$ and in $x_Y$ are comparable

(8) \[ \text{Ext}_{x_Y}(\gamma) \asymp \text{Ext}_x(\gamma) \]

(see the proof of Theorem 6.1 in [Min96, page 283, line 19]). We need to show that the extremal lengths of $\gamma$ in $q$ and in $q_Y$ are comparable as well. This obtain this after applying Lemma 4.1 twice. Once considering $Y$ as a subset of $q$ and once as a subset of $q_Y$, Lemma 4.1 implies:

\[ \text{Ext}_x(\gamma) \asymp \text{Ext}_Y(\gamma) \asymp \text{Ext}_{q_Y}(\gamma). \]

Since the extremal lengths of curves are comparable, the distance in the $\mathcal{T}(Y)$ between $x_Y$ and $q_Y$ is uniformly bounded above [Ker80, Theorem 4].
We note that defining the map $\pi_Y$ involved a choice of labeling of the points $\{a_i\}$. However, the above argument will work for any labeling. In fact, for any labeling of points in a boundary component of $Y$ in $q$, one can use the corresponding labeling $A(Y)$ in $(Aq)$ so that $A(q|_Y) = (Aq)|_Y$. Since all the different labeling result in points that are close in $T(Y)$ to $x|_Y$, Equation (7) holds independently of the choices made. This finishes the proof. □

5. PROJECTION OF A TEICHMÜLLER GEODESIC TO A SUBSURFACE

As mentioned before, a quadratic differential $q$ defines a Teichmüller geodesic $G: \mathbb{R} \to \mathcal{Q}(S)$ by taking

$$G(t) = (x_t, q_t), \quad q_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} q$$

where $x_t$ is the underlying Riemann surface for $q_t$. Let $\lambda_+$ and $\lambda_-$ be the horizontal and the vertical foliations of $q_t$.

Recall that a point $x \in T(S)$ has an associated shortest marking $\mu_x$. We similarly define, for any $(x, q) \in \mathcal{Q}(S)$, a shortest marking $\mu_q$. The marking $\mu_q$ has the same pants decomposition and the same set of lengths $\{l_\alpha\}$ as $\mu_x$. However, we use the flat metric of $q$ to define the transversals $\tau_\alpha$, as follows. Recall that $q^\alpha$ is the annular cover of $q$ with respect to $\alpha$. Define $\tau_\alpha$ to be any arc connecting the boundaries of $q^\alpha$ that is perpendicular to the geodesic representative of the core. That is, the transversal is the quadratic differential perpendicular instead of the hyperbolic perpendicular.

In what follows, we often replace $q_t$ subscripts simply with $t$. For example, $\ell_t(\alpha)$ is short for $\ell_{q_t}(\alpha)$, while $\mu_t$ is short for $\mu_{q_t}$ and $M_t(\alpha, Y)$ is short for $M_{q_t}(\alpha, Y)$. We let $t_\alpha$ be the time when $\alpha$ is balanced along $G$ (see Equation (2)). We need the following two statements. First we have a lemma that is contained in the proof of Theorem 3.1 of [Raf07a].

**Lemma 5.1.** There is a uniform constant $c \geq 0$ so that

$$M_s(\alpha, Y) \leq M_t(\alpha, Y) + c$$

for all $s \leq t \leq t_\alpha$ and for all $t_\alpha \leq t \leq s$. □

Second we have a theorem that follows from the proof of Theorem 5.5 of [Raf05].

**Theorem 5.2.** There are constants $M_0$ and $C$ so that, if $M_t(\alpha, Y) \leq M_0 + c$ for some boundary component $\alpha$, then either

$$d_Y(\mu_t, \lambda_-) \leq C \quad \text{or} \quad d_Y(\mu_t, \lambda_+) \leq C.$$
We now define $I_Y$, the interval of isolation for $Y$. Choose a large enough $M_0$ (we need $M_0 > m_0$ as in Lemma 4.1 and we need $M_0$ to satisfy Theorem 5.2). Define the interval $I_{\alpha,Y} \subset \mathbb{R}$ to be empty when $M_t(\alpha,Y) < M_0$ and otherwise to be the largest interval containing $t_\alpha$ so that $M_t(\alpha,Y) \geq M_0$ for all $t \in I_{\alpha,Y}$. Define

$$I_Y = \bigcap_{\alpha \subset \partial Y} I_{\alpha,Y}.$$ 

Note that, by Lemma 5.1, for any $t$ outside of $I_Y$, there is a boundary component $\alpha$ such that $M_t(\alpha,Y) \leq M_0 + c$.

**Theorem 5.3.** Let $G: \mathbb{R} \to Q(S)$ be a Teichmüller geodesic with $G(t) = (x_t,q_t)$. Let $Y$ be a subsurface with the interval of isolation $I_Y$. Then there exists a geodesic $F: I_Y \to Q(Y)$ with $F(t) = (y_t,p_t)$, so that

- If $[a,b] \cap I_Y = \emptyset$ then
  $$d_Y(\mu_a,\mu_b) = O(1).$$
- For $t \in I_Y$,
  $$d_{T(Y)}(x_t|Y, y_t) = O(1).$$

In fact, we may take $p_t = q_t|Y$.

![Figure 2](image_url). The projection of $G$ to $T(Y)$ fellow travels the geodesic $F$.

**Proof.** For every $t \in [a,b]$, there exists a boundary component $\alpha$ so that $M_t(\alpha,Y) \leq M_0 + c$. By Theorem 5.2

$$d_Y(\mu_t, \lambda_-) \leq C \quad \text{or} \quad d_Y(\mu_t, \lambda_+) \leq C.$$

Let $J_- \subset [a,b]$ be the set of times where former holds and $J_+ \subset [a,b]$ be the set of times where latter holds. If $J_-$ or $J_+$ is empty, we are done by the triangle inequality. Otherwise, we note that these intervals
are closed and have to intersect. This implies that \( d_Y(\lambda_, \lambda_+) \leq 2C \).
Again we are done after applying the triangle inequality; the bound on \( d_Y(\mu_a, \mu_b) \) is at most \( 4C \). This proves the first conclusion of Theorem 5.3.

To obtain the second conclusion, we construct the candidate geodesic arc \( F \) in \( T(Y) \). Let \( I_Y = [c, d] \). As suggested in the statement of the theorem, let \( p_c = q_{c|Y} \) and let \( F = (y_t, p_t) \) be the geodesic segment from \([c, d] \rightarrow Q(Y)\) defined by

\[
p_t = \begin{bmatrix} e^{t-c} & 0 \\ 0 & e^{-t+c} \end{bmatrix} p_c.
\]

In fact, if we make consistent choices in the construction of \( q_{t|Y} \) for different values of \( t \), we have \( p_t = q_{t|Y} \). Now Equation (7) in Theorem 4.2 implies

\[
d_{T(Y)}(x_{t|Y}, y_t) = O(1).
\]

This finishes the proof.

For a Teichmüller geodesic segment whose end points are in the thick part of the Teichmüller space, we can look at the short markings at the end points of the segment instead of the horizontal and the vertical foliations, to determine which subsurfaces are isolated along the geodesic segment. That is, the end invariants can be taken to be the short markings instead of the horizontal and the vertical foliations.

**Corollary 5.4.** Let \( G : \mathbb{R} \rightarrow T(S) \) be a Teichmüller geodesic. Suppose \( a < b \) are times so that \( G(a) \) and \( G(b) \) are in the thick part. Then, for every subsurface \( Y \) we have

- Either \( I_Y \subset [a, b] \),
  \[
i(\lambda_-|Y, \mu_{a|Y}) = O(1) \quad \text{and} \quad i(\lambda_+|Y, \mu_{b|Y}) = O(1).
\]
  In particular,
  \[
d_Y(\lambda_-, \lambda_+) \lesssim d_Y(\mu_a, \mu_b).
\]
- Or \( I_Y \cap [a, b] = \emptyset \) and
  \[
d_Y(\mu_a, \mu_b) = O(1).
\]

**Proof.** Since the endpoints lie in the thick part of Teichmüller space, the times \( a \) and \( b \) are not in any interval \( I_Y \). That is, \( I_Y \) is either contained in \([a, b]\) or it is disjoint from it. If \( I_Y = [c, d] \) then all markings \( \mu_t, t \in [-\infty, c] \) project to a bounded set in \( A\mathcal{C}(Y) \). In fact, from [Raf05, Theorem 5.5] we know that \( i(\mu_{t|Y}, \lambda_{+|Y}) = O(1) \). Therefore, \( d_Y(\lambda_+, \mu_a) = O(1) \). Similarly, for \( t \in [d, \infty] \), \( i(\mu_{t|Y}, \lambda_{-|Y}) = O(1) \) and \( d_Y(\lambda_-, \mu_b) = O(1) \). The corollary follows immediately.
Order of appearance of intervals $I_Y$. By examining the subsurface projections one can determine which curves $\alpha$ are short along a Teichmüller geodesic $\mathcal{G}$. The following is the restatement of results in [Raf05] in a way that is more suitable for our purposes. Let $\mathcal{G}$ be a Teichmüller geodesic with horizontal and vertical foliations $\lambda_{\pm}$ and, for a curve $\alpha$, let $\mathcal{Z}(\alpha, D)$ be the set of subsurface $Z$ that are disjoint from $\alpha$ and have $d_Z(\lambda_+, \lambda_-) \geq D$.

**Theorem 5.5.** A curve $\alpha$ is short at some point along $\mathcal{G}$ if and only if $\alpha$ is the boundary of a subsurface $Y$ so that $Y$ is filled with subsurfaces with large projections. That is, there are constants $\epsilon$, $D_0$ and $D_1$ so that

- If $\text{Ext}_t(\alpha) \leq \epsilon$ then $\alpha$ is a boundary component of some subsurface $Y$, where $Y$ is filled by subsurfaces in $\mathcal{Z}(\alpha, D_0)$.
- Suppose that $\alpha$ is a boundary component of $Y$ and that $Y$ is filled by elements of $\mathcal{Z}(\alpha, D_1)$. Then there is a time $t \in \mathbb{R}$ when $\text{Ext}_t(\alpha) \leq \epsilon$.

**Proof.** This is a restatement of [Raf05, Theorem 1.1] after the following: two curves or arcs in $\mathcal{AC}(Y)$ have large intersection number if and only if their projections to some subsurface $Z$ of $Y$ is large. (This assertion is well known and follows from [CR07, Corollary D].) We have just translated the condition about intersection numbers to a condition about subsurface projections. □

One consequence of the above theorem is that the order in which the intervals $I_Y$ appear in $\mathbb{R}$ is essentially determined by any geodesic $g$ in $\mathcal{AC}(S)$ connecting $\lambda_-$ to $\lambda_+$. 

**Proposition 5.6.** The boundary curves of any isolated surface are in a 2–neighborhood of a geodesic $g$ in the curve complex. The order of appearance of intervals of isolations in $\mathbb{R}$ is coarsely determined by the order in which the vertices $\partial Y$ appear along $g$.

The proof uses both the description of a Teichmüller geodesic as well as some hyperbolicity result for the curve complex $\mathcal{C}(S)$. Namely, we use Masur and Minsky’s bounded geodesic image theorem:

**Theorem 5.7** (Theorem 3.1 in [MM00]). If $Y$ is an essential subsurface of $S$ and $g$ is a geodesic in $\mathcal{AC}(S)$ all of whose vertices intersect $Y$ nontrivially, then the projected image of $g$ in $\mathcal{AC}(Y)$ has uniformly bounded diameter.

**Proof of Proposition 5.6.** By Theorem 5.5 a boundary curve $\alpha$ of any isolated subsurface $Y$ is disjoint from some subsurface $Z$ where the
projection distance \( d_Z(\lambda_+, \lambda_-) \) is large. By Theorem 5.7, the geodesic \( g \) has to miss \( Z \) as well. Hence \( \alpha \) has a distance of at most 2 from \( g \).

Write \( g = g_− \cup g_0 \cup g_+ \), where \( Z \) intersects every curve in \( g_− \) and \( g_+ \) and where \( g_0 \) has length 10 and \( \partial Z \) is disjoint from a curve at the middle of \( g_0 \). From Theorem 5.7 we have that the projection of \( g_− \) to \( \mathcal{AC}(Z) \) is in a bounded neighborhood of \( \lambda_-|_{\partial Z} \) and the projection of \( g_+ \) to \( \mathcal{AC}(Z) \) is in a bounded neighborhood of \( \lambda_+|_{\partial Z} \). Let \( Y' \) be another isolated surface. We claim that if the boundary of \( Y' \) is close to a point in \( g_- \), then the interval \( I_{Y'} \) appears after the interval \( I_Y \).

Let \( I_Y = [a, b] \) and let \( t \in I_{Y'} \). Then \( t \notin [a, b] \) because \( Y \) and \( Y' \) intersect (the distance between their boundaries is larger than 1) and their boundaries can not be short simultaneously. Note that \( \partial Y' \) are part of the short marking \( \mu_t \). By Corollary 5.4 if \( t < a \) then \( i(\mu_t|_Y, \lambda_+|_Y) = O(1) \). Hence,

\[
i(\mu_t|_Z, \lambda_+|_Z) = O(1) \quad \text{and} \quad d_Z(\mu_t, \lambda_+) = O(1).
\]

But this is a contradiction because \( \partial Y' \) is close to a point in \( g_- \) which projects to a point in \( \mathcal{AC}(Z) \) near \( \lambda_-|_{\partial Z} \). Therefore, \( t > b \). □

Remark 5.8. Note that, using Corollary 5.4 we can restate the above statements for Teichmüller geodesic segments \( G : [a, b] \rightarrow \mathcal{T}(S) \) where \( G(a) \) and \( G(b) \) are in the thick part. All statements hold after replacing \( \lambda_- \) and \( \lambda_+ \) with \( \mu_a \) and \( \mu_b \) respectively.

6. No Back-tracking

As before, let \( \mathcal{G} \) be a Teichmüller geodesic with \( \mathcal{G}(t) = (x_t, q_t) \) and let \( \mu_t \) be the short marking associated to \( q_t \). In this section we examine the projection of markings \( \mu_t \) to the curve complex of a subsurface.

Theorem 6.1. For every subsurface \( Y \) of \( S \), the shadow of \( \mathcal{G} \) in \( \mathcal{AC}(Y) \) is an un-parametrized quasi-geodesic. That is, for \( r \leq s \leq t \in \mathbb{R} \)

\[
d_Y(\mu_r, \mu_s) + d_Y(\mu_s, \mu_t) \leq d_Y(\mu_r, \mu_t).
\]

Remark 6.2. We observe that the projection of \( \mu_t \) to \( \mathcal{AC}(Y) \) is a coarsely continuous path. That is, there is a constant \( B \) so that for every \( t \in \mathbb{R} \) there is a \( \delta \) where

\[
i(\mu_t, \mu_{t+\delta}) \leq B \quad \text{and hence} \quad d_Y(\mu_t, \mu_{t+\delta}) = O(1).
\]

To see this, note that since lengths change continuously, \( x_t \) and \( x_{t+\delta} \)

have the same thick-thin decompositions and the intersection between moderate length curves in \( x_t \) and \( x_{t+\delta} \) is bounded. Also, twisting along the short curves changes coarsely continuously (see Equation 3).
Remark 6.3. The reverse triangle inequality for a path (as given in the statement of the theorem) is a stronger condition than being a unparametrized quasi-geodesic. However, in Gromov hyperbolic spaces such as $AC(Y)$ the two conditions are equivalent. (See [MM00, Section 7] and [MMS10, Section 2.1] for relevant discussions.)

Remark 6.4. This contrasts with the way geodesics behave in the Lipschitz metric on $T(S)$, studied by Thurston in [Thu98], where the projection of a geodesic to a subsurface can backtrack arbitrarily far. (Examples can easily be produced using Thurston’s construction of minimal stretch maps [Thu98] and the results in [CR07].)

Proof. If $Y = S$, the above is a theorem of Masur and Minsky [MM00, Theorem 3.3], that is, we already know that the shadow of $G$ to $AC(S)$ is an unparametrized quasi-geodesic. Let $Y$ be a proper subsurface and consider the interval of isolation $I_Y = [c, d]$. If $Y$ is not an annulus, by the first part of Theorem 5.3 the shadow of $G(-\infty, c]$ and $G[d, \infty)$ have bounded diameter in $AC(Y)$ and by the second part of Theorem 5.3 and again using [MM00, Theorem 3.3], the shadow of $G[c, d]$ is an unparametrized quasi-geodesic in $AC(Y)$. It remains to check the case of an annulus. But in this case $AC(Y)$ is quasi-isometric to $Z$ and we need only to show that the twisting around the core of $Y$ is an increasing up to an additive error. This follows from Equation (3). □

7. Fellow traveling

Theorem 7.1. There is a constant $D > 0$ so that, for points $x, \bar{x}, y$ and $\bar{y}$ in the thick part of $T(S)$ where
\[ d_T(x, \bar{x}) \leq 1 \quad \text{and} \quad d_T(y, \bar{y}) \leq 1, \]
the geodesic segments $[x, y]$ and $[\bar{x}, \bar{y}]$ $D$-fellow travel in a parametrized fashion.

Remark 7.2. The proof also works when either $x$ or $y$ is replaced with measured foliation in $PML(S)$ and $G$ and $\bar{G}$ are infinite rays.

Proof. After adjusting $x$ and $y$ along the geodesic extension through $[x, y]$ by a bounded amount, we may assume that $d_T(x, y) = d_T(\bar{x}, \bar{y})$. Let
\[ G : [0, l] \to T(S) \quad \text{and} \quad \bar{G} : [0, l] \to T(S) \]
be Teichmüller geodesics connecting $x$ to $y$ and $\bar{x}$ to $\bar{y}$ respectively; $G(t) = (x_t, q_t)$ and $\bar{G}(t) = (\bar{x}_t, \bar{q}_t)$.

We first show that, for any curve $\alpha$, $\ell_{q_t}(\alpha) \leq \ell_{\bar{q}_t}(\alpha)$.
Since \(x\) and \(\pi\) are both in the thick part, for every curve \(\alpha\) we have (part (1) of Theorem 3.1)
\[
\text{Ext}_x(\alpha) \gtrsim l_{q_0}(\alpha)^2 \quad \text{and} \quad \text{Ext}_{\pi}(\alpha) \gtrsim l_{q_0}(\alpha)^2.
\]
But \(d_T(x, \pi) = 1\). Therefore,
\[
\text{Ext}_x(\alpha) \gtrsim \text{Ext}_{\pi}(\alpha) \implies l_{q_0}(\alpha) \gtrsim l_{q_0}(\alpha).
\]
The same argument works to show that \(l_{q_1}(\alpha) \gtrsim l_{q_1}(\alpha)\). The flat length of a curve is essentially determined by two parameters. From Equation (2) we have
\[
\ell_{q_t}(\alpha) \gtrsim L_\alpha \cosh(t - t_\alpha) \quad \text{and} \quad \ell_{q_t}(\alpha) \gtrsim L_\alpha \cosh(t - t_\alpha).
\]
Since, the flat lengths of \(\alpha\) are comparable at the beginning and the end they are always comparable. That is, \(L_\alpha \gtrsim L_\alpha\) and \(t_\alpha \gtrsim t_\alpha\).

We use Corollary 2.6 to prove \(d_T(x_t, \pi_t) = O(1)\) by checking the four conditions.

**Condition (1).** We need to show that \(q_t\) and \(\overline{q}_t\) have the same thick-thin decompositions. Fix an \(\epsilon\) and let \((A, Y)\) be the \((\epsilon, \epsilon)\)–thick-thin decomposition of \(x_t\). Let \(\alpha \in A\) and let \(E, F\) and \(G\) be as in Theorem 3.1. Since \(\alpha\) is short, one of \(E, F\) or \(G\) must have a large modulus. That is, for every curve \(\beta\) intersecting \(\alpha\), we have
\[
\ell_{q_t}(\beta) \gtrsim \frac{1}{\epsilon}.
\]
(In fact it may be larger than \(e^{1/\epsilon}\).) Since the flat length in \(q_t\) and \(\overline{q}_t\) are comparable, we also have
\[
\ell_{\overline{q}_t}(\beta) \gtrsim \frac{1}{\epsilon}.
\]
We show the extremal length of \(\alpha\) is small in \(\pi_t\). If not, \(\alpha\) would pass through some thick piece of \(\pi_t\) and it would intersect some curve \(\beta\) with \(\text{Ext}_{\pi_t}(\beta) \gtrsim 1\). That is, \(\text{Ext}_{\pi_t}(\beta) \gtrsim \text{Ext}_{\pi_t}(\alpha)\). Part (1) of Theorem 3.1 implies \(\ell_{\pi_t}(\beta) \gtrsim l_{\pi_t}(\alpha)\) which is a contradiction. That is, there is an \(\epsilon_0\) so that if \(\text{Ext}_{x_t}(\alpha) \leq \epsilon\) then \(\text{Ext}_{\pi_t}(\alpha) \leq \epsilon_0\).

Arguing in the other direction, we can find \(\epsilon_1\) so that if \(\text{Ext}_{\pi_t}(\alpha) \leq \epsilon_1\) then \(\text{Ext}_{x_t}(\alpha) \leq \epsilon\). That is, every curve not in \(A\) is \(\epsilon_1\)–thick in \(\pi_t\). This proves that \((A, Y)\) is a \((\epsilon_0, \epsilon_1)\)–thick-thin decomposition for \(\pi_t\).

**Condition (2).** The size of a surface \(Y \in \mathcal{Y}\) is the flat length of the shortest essential curve in \(Y\). Hence, we have size \(q_t(Y) \gtrsim \text{size}_{\pi_t}(Y)\). Now, Theorem 3.1 implies that, for every curve \(\gamma\) in \(Y\), if \(\text{Ext}_{x_t}(\gamma) \gtrsim 1\) then \(\text{Ext}_{\pi_t}(\gamma) \gtrsim 1\) as well. But two curves of length one have bounded intersection numbers. Hence, they have bounded projection to every subsurface \(Z\). This means \(d_Z(\mu, \overline{\mu}) = O(1)\).
Condition (3). For each $\alpha \in \mathcal{A}$, as we saw before, $L_\alpha \approx \overline{T}_\alpha$ and $t_\alpha \approx \overline{t}_\alpha$. We now show that $T_\alpha \approx \overline{T}_\alpha$. Since the end points of $\mathcal{G}$ and $\overline{\mathcal{G}}$ are close, we have

$$d_\alpha(\mu_0, \mu_t) = O(1) \quad \text{and} \quad d_\alpha(\mu_t, \overline{\mu}_t) = O(1).$$

Also, from Corollary 5.4 we have

$$d_\alpha(\mu_0, \lambda_-) = O(1), \quad d_\alpha(\mu_t, \lambda_+) = O(1),$$

$$d_\alpha(\overline{\mu}_0, \overline{\lambda}_-) = O(1) \quad \text{and} \quad d_\alpha(\overline{\mu}_t, \overline{\mu}_+) = O(1).$$

Hence, using the triangle inequality,

$$T_\alpha = d_\alpha(\lambda_-, \lambda_+) \approx d_\alpha(\mu_0, \mu_t) \approx d_\alpha(\overline{\mu}_0, \overline{\mu}_t) \approx d_\alpha(\overline{\lambda}_-, \overline{\lambda}_+) = \overline{T}_\alpha.$$ 

Now Equation (4) implies

$$\text{Mod}_{x_t}(F_t) \approx \text{Mod}_{x_t}(\overline{F}_t).$$

Also, as seen above, the size of all subsurfaces are comparable in $q_t$ and $\overline{q}_t$. Therefore, by Theorem 3.1 $\text{Ext}_{x_t}(\alpha) \approx \text{Ext}_{x_t}(\alpha)$.

Condition (4). We show that $\text{twist}_\alpha(q_t, \overline{q}_t) \text{Ext}_{x_t}(\alpha) \approx 1$. Note that, since $d_\alpha(\lambda_-, \overline{\lambda}_-) = O(1)$,

$$\text{twist}_\alpha(q_t, \overline{q}_t) \approx |\text{twist}_\alpha(q_t, \lambda_-) - \text{twist}_\alpha(\overline{q}_t, \overline{\lambda}_-)|.$$

Denote $\text{twist}_\alpha(q_t, \overline{q}_t)$ (as before) by $\text{twist}_t(\alpha)$ and denote $\text{twist}_\alpha(\overline{q}_t, \overline{\lambda}_-)$ by $\overline{\text{twist}}_t(\alpha)$. We use Equation (3) and the facts $|t_\alpha - \overline{t}_\alpha| = O(1)$ and $|T_\alpha - \overline{T}_\alpha| = O(1)$ to estimate the right hand side of Equation (10).

If $t \approx t_\alpha$ (and hence $t \approx \overline{t}_\alpha$), then

$$\text{twist}_t(\alpha) \approx \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \quad \text{and} \quad \overline{\text{twist}}_t(\alpha) \approx \frac{T_\alpha}{\cosh^2(t - t_\alpha)}.$$

But $\text{Ext}_t(\alpha) \approx \frac{1}{\text{Mod}(F_t)}$. Thus using Equation (4)

$$\left|\text{twist}_t(\alpha) - \overline{\text{twist}}_t(\alpha)\right| \text{Ext}_t(\alpha) \approx \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \frac{\cosh^2(t - t_\alpha)}{T_\alpha} \approx 1.$$

If $t \approx t_\alpha$, then

$$T_\alpha - \text{twist}_t(\alpha) \approx \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \quad \text{and} \quad T_\alpha - \overline{\text{twist}}_t(\alpha) \approx \frac{T_\alpha}{\cosh^2(t - t_\alpha)}.$$
Hence, as before,
\[
\left| \text{twist}_t(\alpha) - \text{twist}_t(\alpha) \right| \text{Ext}_t(\alpha) \overset{\sim}{\leq} \frac{\left| (T_\alpha - \text{twist}_t(\alpha)) - (T_\alpha - \text{twist}_t(\alpha)) \right|}{\text{Mod}_t(\alpha)}
\]
\[
\overset{\sim}{\leq} \frac{T_\alpha}{\cosh^2(t - t_\alpha)} \overset{\sim}{<} 1.
\]
That is, the last condition in Corollary 2.6 holds and \( d_{T}(q_t,\eta_t) = O(1) \). This finishes the proof.

We now construct the counterexample.

**Theorem 7.3.** For every constant \( d > 0 \), there are points \( x,y,\bar{x} \) and \( \bar{y} \) in \( T(S) \) so that
\[
d_{T}(x,\bar{x}) = O(1)
\]
and
\[
d_{T}(y,\bar{y}) = O(1),
\]
and
\[
d_{T}([x,y],[\bar{x},\bar{y}]) \overset{\sim}{>\sim} d.
\]

**Proof.** For a given \( d \), we construct quadratic differentials \( q_0 \) and \( \tilde{q}_0 \) with the following properties: Let \( q_t \) be the image of \( q_0 \) under the Teichmüller geodesic flow and let \( x_t \) be the underlying conformal structures of \( q_t \). Let \( \eta_t \) and \( \pi_t \) be defined similarly. We will show that
\[
d_{T}(x_0,\bar{x}_0) = O(1), \quad d_{T}(x_{2d},\bar{x}_{2d}) = O(1),
\]
and
\[
d_{T}(x_{d},\bar{x}_{d}) \overset{\sim}{>\sim} d.
\]
This is sufficient to show that \( d_{T}(x_{d},\bar{x}_{d}) \overset{\sim}{>\sim} d \) for any \( t \in [0,2d] \). To see this note that, for any \( 0 \leq t < d \), we have
\[
d_{T}(x_{d},\bar{x}_{t}) + d_{T}(\bar{x}_{t},\bar{x}_{0}) \overset{\sim}{>\sim} d \quad \text{and} \quad d_{T}(x_{d},\bar{x}_{t}) + d_{T}(\bar{x}_{t},\bar{x}_{d}) \overset{\sim}{>\sim} d_{T}(x_{d},\bar{x}_{d}).
\]
Summing up both sides, we get
\[
2d_{T}(x_{d},\bar{x}_{t}) + d_{T}(\bar{x}_{0},\bar{x}_{d}) \overset{\sim}{>\sim} 2d + d_{T}(x_{d},\bar{x}_{d}).
\]
Hence,
\[
2d_{T}(x_{d},\bar{x}_{t}) \overset{\sim}{>\sim} d_{T}(x_{d},\bar{x}_{d}).
\]
A similar argument works for \( d < t < d \).

Let \( S \) be a surface of genus 2, \( \gamma \) be a separating curve in \( S \) and \( Y \) and \( Z \) be the components of \( S \setminus \gamma \). Consider a pseudo-Anosov map \( \phi \) on a torus and choose a flat torus \( T \) on the axis of \( \phi \) so that the vertical direction in \( T \) matches the unstable foliation of \( \phi \). Cut open a slit in \( T \) of size \( \epsilon = ce^{-d/2} \) and of angle \( \pi/4 \) (The constant \( 0 < c < 1 \) is to be
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specified below). Fix a homeomorphism from Y to this slit torus and call this marked flat surface $T_0$. Define

$$T_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} T_0.$$  

Note that $T_t$ is still a marked surface. The length of the slit is minimum at $t = 0$ and grows exponentially as $t \to \pm \infty$. For $-d/2 \leq t \leq d/2$, the length of the slit is smaller than $c$ but the length of shortest essential curve in $T_t$ in this interval is comparable with 1. Hence, for $c$ small enough, $M_t(\gamma, Y) \geq m_0$ (see Section 4) and $T_t$ looks like an isolated subsurface.

Now choose $\delta \ll \epsilon$ (specified below) and let $q_0$ be the quadratic differential defined by gluing $T$ to $\delta T_{-d/2}$. What we mean by this is that we first scale down $T_{-d/2}$ by a factor $\delta$. Then we cut open a slit in $T$ of the same size and angle as the size of the slit in $\delta T_{-d/2}$ and then glue these two flat tori along this slit. Fixing a homeomorphism from $Z$ to $T$ slit open, we obtain a marking for $q_0$ that is well defined up to twisting around $\gamma$. Let $G : [0, 2d] \to \mathcal{T}(S)$ be the Teichmüller geodesic segment defined by $q_0$.

Construct $\overline{q}_0$ in the similar fashion by gluing $T$ to $\delta T_{-3d/2}$. Now choose the marking map from $S$ to $\overline{q}_0$ so that $q_0$ and $\overline{q}_0$ have bounded relative twisting around $\gamma$. Let $\overline{G} : [0, 2d] \to \mathcal{T}(S)$ be the Teichmüller geodesic segment defined by $\overline{q}_0$.

Recall that, for $-d/2 \leq t \leq d/2$, the subsurface $\delta T_t$ is isolated (scaling by $\delta$ does not change the value of $M_t(\alpha, Y)$) and by Theorem 4.2 the projection of $T_t$ to the Teichmüller space of $Y$ fellow travels a Teichmüller geodesic. However, for $t > d/2$ and $t < -d/2$, the projection to the curve complex of $Y$ changes by at most a bounded amount. That is, the interval of isolation for $Y$ along $\overline{G}$, $I_Y = [0, d]$ and along $G$, $I_Y = [d, 2d]$. In particular,

$$d_Y(q_0, \overline{q}_0) = O(1) \quad \text{and} \quad d_Y(q_{2d}, \overline{q}_{2d}) = O(1).$$

Also, since no curve in $Y$ or $Z$ is ever short (the vertical and the horizontal foliation in $Y$ and $Z$ are co-bounded), the twisting parameters around any curves inside $Y$ or $Z$ are uniformly bounded. Projections of $q_0$ and $\overline{q}_0$ to $Z$ are identical and $\gamma$ is short in both $q_0$ and $\overline{q}_0$. Therefore, to show $d_T(x_0, \overline{x}_0) = O(1)$, it remains to show (Corollary 2.6) that the extremal lengths of $\gamma$ in $x_0$ and $\overline{x}_0$ are comparable. We have (Theorem 3.1)

$$\text{Ext}_{x_0}(\gamma) \dot{\asymp} \log \frac{1}{\delta} \quad \text{and} \quad \text{Ext}_{\overline{x}_0}(\gamma) \dot{\asymp} \log \frac{1}{e^{d}\delta} = \log \frac{1}{\delta} - d.$$
But these quantities are comparable for \( \delta \) small enough. A similar argument shows that \( d_T(x_{2d}, \bar{x}_{2d}) = O(1) \). Since \( Y \) is isolated in \( q_t \) for \( 0 \leq t \leq d \) the shadow to the \( \mathcal{AC}(Y) \) is an unparametrized quasi-geodesic. In fact, since no curve is short in \( Y \) in that interval, the shadow is a parametrized quasi-geodesic (\cite{RS09} Lemma 4.4). That is
\[
d_T(Y)(x_o, x_d) \asymp d.
\]
But the interval of isolation for \( Y \) along the geodesic \( G \) is \([d, 2d]\). Therefore,
\[
d_Y(x_o, x_{2d}) = O(1).
\]
As before, we have \( d_Y(x_o, x_{2d}) = O(1) \). Hence
\[
d_Y(q_d, q_{2d}) \asymp d.
\]
Now, by Theorem 2.4 we have
\[
d_T(x_d, \bar{x}_{d}) \asymp d_T(x_d, \bar{x}_d) \asymp d.
\]
This finishes the proof. \( \square \)

8. Thin triangles

Let \( x, y \) and \( z \) be three points in \( T(S) \) and let \( G: [a, b] \to T(S) \) be the Teichmüller geodesic connecting \( x \) to \( y \). In this section we prove Theorem E from the introduction.

**Theorem 8.1.** For every \( \epsilon \), there are constants \( C \) and \( D \) so that the following holds. Let \( [c, d] \) be a subinterval of \( [a, b] \) with \( (d - c) > C \) so that for every \( t \in [c, d] \), \( G(t) \) is in the \( \epsilon \)–thick part of \( T(S) \). Then, there is a \( w \in [G(c), G(d)] \) where
\[
\min \left( d_T(w, [x, z]), d_T(w, [x, z]) \right) \leq D.
\]

**Proof.** Consider the shadow map from \( T(S) \) to the curve complex \( \mathcal{AC}(S) \) sending a point \( x \) to its short marking \( \mu_x \). The geodesic triangle \( \triangle(\mu_x, \mu_y, \mu_z) \) in the arc and curve complex \( \mathcal{AC}(S) \) is \( \delta \)–slim. Since the shadow of \( [x, y] \) is a quasi-geodesic (Theorem B) for any \( w \in [x, y] \), \( \mu_w \) is \( \delta \)–close to the geodesic \( [\mu_x, \mu_y] \) in \( \mathcal{AC}(S) \). That is, for every \( w \in [x, y] \), there is a Riemann surface \( u \) in either \([x, z]\) or \([y, z]\) so that
\[
d_S(\mu_w, \mu_u) \leq 3\delta.
\]
The projection of \([G(c), G(d)]\) to \( \mathcal{AC}(S) \) is in fact a parametrized quasi-geodesic (\cite{RS09} Lemma 4.4)). Hence, by making \( C \) large, we can assume that the shadow of \([G(c), G(d)]\) is as long as we like. Thus, we can choose \( w \in [G(c), G(d)] \) so that \( \mu_w \) is far from either the shadow of \([x, z]\) or the shadow of \([y, z]\). To summarize, without loss of generality, we can assume that there is a \( w \in [G(c), G(d)] \) and a \( u \in [x, z] \) so
that \( d_S(\mu_w, \mu_u) = O(1) \) and that neither \( \mu_u \) nor \( \mu_w \) is in the \((10\delta)\)-neighborhood of the geodesic \([\mu_y, \mu_z]\).

We claim that \( u \) is in the thick part of Teichmüller space. Using Theorem \ref{thm:thick_part_1} it is enough to show, for every subsurface \( Y \) whose boundaries are close to \( \mu_u \) in \( \mathcal{AC}(S) \), that \( d_Y(\mu_x, \mu_z) = O(1) \). Since \( \mu_u \) is far away from \([\mu_y, \mu_z]\), Theorem \ref{thm:thick_part_2} implies that \( d_Y(\mu_y, \mu_z) = O(1) \). To prove the claim, we need to show that

\[
(11) \quad d_Y(\mu_x, \mu_y) = O(1).
\]

We prove \eqref{eq:11} by contradiction. Assume \( d_Y(\mu_x, \mu_y) \) is large. By Theorem \ref{thm:thick_part_1} \( \partial Y \) is short at some point \( v \in [x, y] \). But the shadow of \([x, y]\) is a quasi-geodesic and the shadow of \([G(c), G(d)]\) is a parametrized quasi-geodesic. Hence, by choosing \( C \) large enough, we can conclude that, for any such subsurface, \( d_S(\partial Y, \mu_u) \succ d_S(\mu_v, \mu_w) \) is large. This contradicts the fact that

\[
d_S(\partial Y, \mu_u) = O(1) \quad \text{and} \quad d_S(\mu_u, \mu_w) = O(1).
\]

Hence, \eqref{eq:11} holds and thus \( u \) is in the thick part of Teichmüller space.

We now claim, for any subsurface \( Y \subset S \), that

\[
d_Y(\mu_u, \mu_w) = O(1).
\]

This is because any such subsurface \( Y \) should appear near the curve complex geodesic connecting \( \mu_u \) and \( \mu_w \) and hence \( \partial Y \) has a bounded distance from \( \mu_w \) in \( \mathcal{AC}(S) \). As before, assuming \( d_Y(\mu_x, \mu_y) \) is large will result in a contradiction. Thus, \( d_Y(\mu_x, \mu_y) = O(1) \). Since \( \mu_u \) is far from the geodesic \([\mu_y, \mu_z]\), the bounded projection theorem implies that \( d_Y(\mu_y, \mu_z) = O(1) \) and by the triangle inequality, \( d_Y(\mu_x, \mu_z) = O(1) \).

On the other hand, by Theorem \ref{thm:bounded_projections}

\[
d_Y(\mu_x, \mu_y) = O(1) \quad \implies \quad d_Y(\mu_x, \mu_w) = O(1)
\]

and

\[
d_Y(\mu_x, \mu_z) = O(1) \quad \implies \quad d_Y(\mu_x, \mu_u) = O(1).
\]

The triangle inequality implies \( d_Y(\mu_w, \mu_u) = O(1) \). This proves the claim.

We have \( w \) and \( u \) are both in the thick part and that all subsurface projections between \( \mu_u \) and \( \mu_w \) are uniformly bounded. Corollary \ref{cor:uniform_bound} implies that \( d_T(u, w) = O(1) \).

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