FRT quantization theory for the nonsemisimple
Cayley-Klein groups

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Abstract. The quantization theory of the simple Lie groups and algebras was developed by Faddeev-Reshetikhin-Takhtadjan (FRT). In group theory there is a remarkable set of groups, namely the motion groups of n-dimensional spaces of constant curvature or the orthogonal Cayley-Klein (CK) groups. In some sense the CK groups are in the nearest neighborhood with the simple ones. The well known groups of physical interest such as Euclidean $E(n)$, Poincare $P(n)$, Galileian $G(n)$ and other nonsemisimple groups are in the set of CK groups. But many standard algebraical constructions are not suitable for the nonsemisimple groups and algebras, in particular Killing form is degenerate, Cartan matrix do not exist. Nevertheless it is possible to describe and to quantize all CK groups and algebras, as it was made for the simple ones. The principal proposal is to consider CK groups as the groups over an associative algebra $D$ with nilpotent commutative generators and the corresponding quantum CK groups as the algebra of noncommutative functions over $D$.

1 Introduction

It is well known [6], [8] that there are $3^n$ n–dimensional real spaces of constant curvature or Cayley–Klein spaces. These spaces are the most symmetric ones, i.e. their motion groups have the maximal dimension and for this reason are often used in physics. The simple group $SO(n+1)$ is the motion group of the n–dimensional spherical space. All other CK groups have the same dimension $n(n-1)/2$ and may be obtained from $SO(n+1)$ by the contractions and analytical continuations [2].

The notion of Lie group contraction was first introduced by E.İnönü and E.P.Wigner [9] as some limiting procedure and was later extend on new algebraical structures such as Lie bialgebra [18], Hopf algebra [11], graded contractions [16], [17], but the fundamental idea of degenerate transformations is presented in all cases (see [15] for detailes). On the other hand the degenerate transformation is something incorrect from mathematical point of view. So it is necessary to find instead of it an relevant mathematical construction. It seems that the consideration of the Lie groups as the groups over an associative algebra $D$ with nilpotent commutative generators is the appropriate tool at least.
in CK scheme. The validity of such approach is demonstrated for the FRT quantization theory [1]. It is possible to reformulate the quantum deformations of the simple groups in such a way to obtain the quantum deformations of all contracted CK groups.

This paper is organized as follows. In Section 2 the standard Inönü–Wigner contraction and our approach are compared for the simplest case of one-dimensional CK spaces and their motion group. The algebra \( D_n(\iota; \mathbb{C}) \) is introduced in Section 3. The orthogonal CK groups \( SO(N; j; \mathbb{R}) \) are described in Section 4 as the matrix groups over \( D(\iota; \mathbb{R}) \). Section 5 is devoted to the quantum orthogonal CK groups \( SO_v(N; j; \mathbb{C}) \) and the quantum algebras \( so_v(N; j; \mathbb{C}) \) are obtained as the dual object to the corresponding quantum groups in Section 6. The developed approach is illustrated in Sections 7 on the example of \( N = 3 \) quantum groups and algebras. The final remarks are given in Conclusion.

2 \ Inönü-Wigner contractions: traditional approach and suitable mathematical structure

Let us regard finite rotations on the angle \( \varphi \) for three planes: euclidean (Fig.1), galileian (Fig.2) and minkowskian (Fig.3).

Two dimensional vectors \( x^t = (x_0, x_1)^t \) in Cartesian basis are transformed under rotations as follows:

\[ x' = Ax, \]

where

\[
A = \begin{pmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi \\
\end{pmatrix} \in SO(2)
\]

is the ordinary rotation matrix,

\[
A = \begin{pmatrix}
1 & 0 \\
\varphi & 1 \\
\end{pmatrix} \in E(1)
\]
Figure 2: "Rotations" or Galilei transformations on the galileian plane

Figure 3: Hyperbolic rotations or Lorentz transformations on the minkowskian plane
is the Galilei transformation and

\[ A = \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix} \in SO(1, 1) \]

is the hyperbolic rotation or Lorentz transformation, respectively. The following quadratic forms are invariant under transformations (1):

\[ \text{inv}_e = x_0^2 + x_1^2, \quad \text{inv}_g = \sinh^2 \varphi - \cosh^2 \varphi, \quad \text{inv}_m = \sinh^2 \varphi - x_1^2. \]

The Lie algebras of these three groups may be written in a unified manner

\[ \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \in so(2; \omega), \quad \omega = 1, 0, -1, \tag{2} \]

where \( \omega = 1 \) correspond to Lie algebra of the simple rotation group \( SO(2) \), \( \omega = -1 \) correspond to the semisimple Lorentz group \( SO(1, 1) \) and \( \omega = 0 \) correspond to the nonsemisimple Galilei group \( E(1) \).

The one dimensional geometries of constant curvature are realized on the rotation invariant surfaces (spheres) in these planes. According with Erlanger program due to F.Klein a geometry is completely determined by its motion group. In one dimensional case there is only one motion, namely translation. Let us introduce the intrinsic (Beltrami) coordinate on the spheres by relation

\[ \xi = x_1/x_0. \]

Then the translation operators \( T(a) : \xi \rightarrow \xi' = x_1'/x_0' \) looks as follows:

\[ \xi' = T(a)\xi = \frac{\xi + a}{1 - a\xi}, \quad a = \tan \varphi, \quad a \in \mathbb{R} \tag{3} \]

for the elliptic (spherical) geometry of constant positive curvature,

\[ \xi' = T(a)\xi = \xi + a, \quad a = \varphi, \quad a \in \mathbb{R} \tag{4} \]

for the euclidean flat geometry and

\[ \xi' = T(a)\xi = \frac{\xi + a}{1 + a\xi}, \quad a = \tanh \varphi, \quad a \in (-1, 1) \tag{5} \]

for the hyperbolic geometry of constant negative curvature. Translation operators may be written in the following unified manner

\[ \xi' = T(a; \omega)\xi = \frac{\xi + a}{1 - \omega a\xi}, \quad a = \frac{1}{\sqrt{\omega}} \tan \sqrt{\omega} \varphi, \tag{6} \]

with the same parameter \( \omega \) as in Eq.(2). The distance \( d_{AB} \) between points \( A \) and \( B \)

\[ \frac{1}{\sqrt{\omega}} \tan(\sqrt{\omega}d_{AB}) = \left| \frac{\xi_B - \xi_A}{1 + \omega \xi_B \xi_A} \right| \tag{7} \]

is invariant under translations.

The new parameter \( j = \sqrt{\omega} \) is appeared in Eqs.(3),(7), where \( j = 1 \) for \( \omega = 1 \), \( j = i \) for \( \omega = -1 \) and \( j = \sqrt{0} \) for \( \omega = 0 \). The solutions of equation
\[ j = \sqrt{0} \] depends on underlying mathematical structure, for example \( j = 0 \) over fields \( \mathbb{R} \) or \( \mathbb{C} \); \( j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0 \) over \( 2 \times 2 \) matrix ring; \( j_k = \theta_k, \theta_k^2 = 0, \theta_k \theta_m = -\theta_m \theta_k, k \neq m, k, m = 1, \ldots, N \) over Grassmann algebra. We take as the solution nilpotent commutative numbers \( j_k = \iota_k, \iota_k^2 = 0, \iota_k \iota_m = \iota_m \iota_k, k \neq m, k, m = 1, \ldots, N \).

For \( k = 1 \) the nilpotent number \( \iota, \iota^2 = 0 \), have been first introduced by W.K.Clifford [4] more then hundred years ago and was applied in geometry and mechanics [5]–[7]. In Russian publications its is named as dual number and in English is known as Study number. R.I.Pimenov [8] was the first who have introduce the different nilpotent numbers \( \iota_k, k = 1, \ldots, N \) with commutative law of multiplication, so it seems natural to call its as Pimenov numbers.

Now we are able to rewrite transformations \( x' = A(j)x \) in the form

\[
\begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cos j \varphi & -j \sin j \varphi \\ j^{-1} \sin j \varphi & \cos j \varphi \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix},
\]

where \( \det A(j) = \cos^2 j \varphi + \sin^2 j \varphi = 1 \), \( \text{inv}(j) = x'_0^2 + j^2 x'_1^2 \) and parameter \( j \) takes three values \( j = 1, \iota, i \). Matrices \( A(j) \) form the group \( \text{SO}(2; j) \). It is easily to check that \( \text{SO}(2; 1) \equiv \text{SO}(2) \), \( \text{SO}(2; i) \equiv \text{SO}(1,1) \) and \( \text{SO}(2; \iota) \equiv E(1) \). In the last case the properties of a functions of \( \iota \) arising from its Taylor expansion are exploited: \( \cos \iota \varphi = 1, \sin \iota \varphi = \iota \varphi \) and be definition \( \iota/\iota = 1 \). If we put \( j = \epsilon \in \mathbb{R}, \epsilon \to 0 \), then we obtain Inômi-Wigner contraction [9] on the group level. The remarkable property of Eq. (8) is that the matrix elements and vector components are real numbers for any value of the parameter \( j \).

There is other way to describe rotations (3), namely \( x'(j) = R(j)x(j) \) or more precisely

\[
\begin{pmatrix} x'_0 \\ jx'_1 \end{pmatrix} = \begin{pmatrix} \cos j \varphi & -\sin j \varphi \\ \sin j \varphi & \cos j \varphi \end{pmatrix} \begin{pmatrix} x_0 \\ jx_1 \end{pmatrix},
\]

where \( \det R(j) = \cos^2 j \varphi + \sin^2 j \varphi = 1 \) and \( \text{inv}(j) = x'_0^2 + j^2 x'_1^2 \). Matrices \( R(j) \) again form the same group \( \text{SO}(2; j) \). The remarkable property of Eq.(9) is that some matrix elements and vector components are nilpotent Pimenov numbers for \( j = \iota \)

\[
\begin{pmatrix} x'_0 \\ \iota x'_1 \end{pmatrix} = \begin{pmatrix} 1 & -\iota \varphi \\ \iota \varphi & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \iota x_1 \end{pmatrix},
\]

or complex numbers for \( j = i \)

\[
\begin{pmatrix} x'_0 \\ ix'_1 \end{pmatrix} = \begin{pmatrix} \cosh \varphi & -i \sinh \varphi \\ i \sinh \varphi & \cosh \varphi \end{pmatrix} \begin{pmatrix} x_0 \\ ix_1 \end{pmatrix}.
\]

This simple consideration suggests to regard Lie groups, Lie algebras and vector spaces not only over complex (or real) number field, but also over more
general algebraical structure (see next Section) which is generated by nilpotent commutative numbers.

Let us observe that matrix $R(j)$ in Eq. (11) is obtained from rotation matrix $A \in SO(2)$ by substitution of the real group parameter $\varphi$ with new group parameter $\varepsilon \varphi$. For $j = i$ this means the analytical continuation of the real group parameter into the complex number field (Weyl’s unitary trick), which gives in result the noncompact group $SO(1, 1)$ from the simple group $SO(2)$. In the same line Inönü-Wigner contraction ($j = i$) may be regarded as the continuation of the real $\varphi$ to the nilpotent values $\varepsilon \varphi$.

In the traditional approach $j = \varepsilon$, Eq. (11) looks as follows

$$
\begin{align*}
x'_0 &= x_0 \cos \varepsilon \varphi - \varepsilon x_1 \sin \varepsilon \varphi \\
\varepsilon x'_1 &= \varepsilon x_1 \cos \varepsilon \varphi + x_0 \sin \varepsilon \varphi
\end{align*}
$$

and in the limit $\varepsilon \to 0$ provide the Galilei transformation if the infinitesimals of the first order are compared in both sides of the second equation. Let us stress that the group transformation $R(\varepsilon)$ and vector $x(\varepsilon)$ are regarded together in Eq. (12) and can not be separated. Otherwise for example $\lim_{\varepsilon \to 0} R(\varepsilon) = I$ instead of to be a general element of the Galilei group $E(1)$. Just the similar situation is appeared in the case of quantum groups (see for example [13]) and it seems that the nilpotent commutative numbers provide the relevant mathematical structure to avoid such troubles.

### 3 Pimenov algebra $D_n(\iota; \mathbb{C})$

Algebra $D_n(\iota; \mathbb{C})$ is defined as an associative algebra with unit and nilpotent generators $\iota_1, \ldots, \iota_n$, $\iota_i^2 = 0$, $k = 1, \ldots, n$ with commutative multiplication $\iota_k \iota_m = \iota_m \iota_k$, $k \neq m$. The general element of $D_n(\iota; \mathbb{C})$ has the form

$$
a = a_0 + \sum_{p=1}^{n} \sum_{k_1 < \ldots < k_p} a_{k_1 \ldots k_p} \iota_{k_1} \ldots \iota_{k_p}, \quad a_0, a_{k_1 \ldots k_p} \in \mathbb{C}. \quad (13)
$$

For $n = 1$ we have $D_1(\iota_1; \mathbb{C}) \ni a = a_0 + a_1 \iota_1$, i.e. dual (or Study) numbers, when $a_0, a_1 \in \mathbb{R}$. For $n = 2$ the general element of $D_2(\iota_1, \iota_2; \mathbb{C})$ is written as follows: $a = a_0 + a_1 \iota_1 + a_2 \iota_2 + a_{12} \iota_1 \iota_2$.

Two elements $a, b \in D_n(\iota; \mathbb{C})$ are equal when $a_0 = b_0$, $a_{k_1 \ldots k_p} = b_{k_1 \ldots k_p}$, $p = 1, \ldots, n$. If $a = a_{k \iota_k}$ and $b = b_{k \iota_k}$ then the condition $a = b$, which is equivalent to $a_{k \iota_k} = b_{k \iota_k}$ make possible to define consistently division of Pimenov unit $\iota_k$ by itself, namely $\iota_k/\iota_k = 1$. Divisions of a real or complex numbers by Pimenov units $z/\iota_k$, $z \in \mathbb{R}, \mathbb{C}$, and different Pimenov units $\iota_m/\iota_k$, $k \neq m$ are not defined. A function $f : D_n(\iota; \mathbb{C}) \to D_n(\iota; \mathbb{C})$ is defined by its Taylor expansion

$$
f(a) = f(a_0) + \sum_{p=1}^{n} \sum_{k_1 < \ldots < k_p} f_{k_1 \ldots k_p} \iota_{k_1} \ldots \iota_{k_p},
$$
where the summation on all possible partitions of the number set \((k_1, \ldots, k_p)\) on \(p\) nonempty subsets is understood in the last equation. For example,

\[
a \in D_1(t; C), \quad f(a) = f(a_0) + t_1 a_1 f'(a_0);
\]

\[
a \in D_2(t; C), \quad f(a) = f(a_0) + t_1 a_1 f'(a_0) + t_2 a_2 f''(a_0) + t_1 t_2 (a_1 a_2 f'(a_0) + a_1 a_2 f''(a_0)).
\]

It is clear from previous equations that a function over algebra \(D_n(t; C)\) is completely determined by its real part \(f(a_0)\) and \(n\) derivatives \(f^{(r)}(a_0), r = 1, \ldots, n\), for example over \(D_1(t; C)\)

\[
e^{a_0 + t_1 a_1} = e^{a_0} + t_1 a_1 e^{a_0} = e^{a_0}(1 + t_1 a_1),
\]

\[
sin(a_0 + t_1 a_1) = sin a_0 + t_1 a_1 \cos a_0,
\]  

(15)

and over \(D_2(t; C)\)

\[
e^a = e^{a_0}(1 + t_1 a_1 + t_2 a_2 + t_1 t_2 (a_1 a_2 + a_1 a_2)),
\]

\[
sin(a) = sin a_0 + (t_1 a_1 + t_2 a_2) \cos a_0 + t_1 t_2 (a_1 a_2 \cos a_0 - a_1 a_2 \sin a_0).
\]  

(16)

The well known Grassmann algebra \(\Gamma_n(\xi)\) is the algebra with nilpotent generators \(\xi_k^2 = 0, k = 1, \ldots, n\) and anticommutative multiplication \(\xi_k \xi_m = -\xi_m \xi_k, k \neq m\). It is easy to verify that the product of two generators of Grassmann algebra has the same algebraic properties as the generator of algebra \(D_n(t; C)\)

\[t_k = \xi_k \xi_{n+k}, \quad k = 1, \ldots, n.\]

This means that this algebra is the subalgebra of even part of Grassmann algebra \(D_n(t; C) \subset \Gamma_{2n}(\xi)\).

For our aims it is convenient to regard the set of algebras \(D_n(j; C)\) with \(j_k = 1, t_k, \quad k = 1, \ldots, n\). If some parameters are equal to Pimenov numbers \(j_{s} = t_s, \quad s = 1, \ldots, m\) and remaining ones are equal to 1, then we have the algebra \(D_m(t; C)\) from the set \(D_n(j; C)\).

4 Orthogonal CK groups

Let us regard according to R.I. Pimenov [8] a specific vector space \(R_N(j)\) over \(D_{N-1}(j; R)\) with Cartesian coordinates \(x(j) = (x_1, J_{12} x_2, \ldots, J_{1,N} x_N)^t, \quad x_k \in R, \quad k = 1, \ldots, N\) and quadratic form

\[
x^t(j) x(j) = x_1^2 + \sum_{k=2}^{N} J_{1k}^2 x_k^2,
\]  

(17)

where

\[
J_{\mu \nu} = \prod_{r=\mu}^{\nu-1} j_r, \quad \mu < \nu, \quad J_{\mu \nu} = 1, \quad \mu \geq \nu, \quad j_r = 1, t_r, i.
\]  

(18)
Orthogonal CK groups $SO(N; j; \mathbb{R})$ are defined as the set of transformations of $\mathbb{R}_N(j)$ leaving invariant (17) and are realized in the Cartesian basis as the matrix groups over $D_{N-1}(j; \mathbb{R})$ with the help of the *special* matrices

$$(A(j))_{kp} = \tilde{J}_{kp} a_{kp}, \quad a_{kp} \in \mathbb{R},$$

$\tilde{J}_{kp} = J_{kp}, \quad k < p, \quad \tilde{J}_{kp} = J_{pk}, \quad k \geq p.$

These matrices act on vectors $x(j) \in \mathbb{R}_N(j)$ by matrix multiplication and are satisfied the following $j$-orthogonality relations:

$$A(j)A^t(j) = A^t(j)A(j) = I.$$  

Let in euclidean vector space $\mathbb{R}_n y = Dx$ is the transformation from Cartesian basis $x$ to the new ("symplectic") basis $y$ with the following quadratic form: $y^tC_0y$, where $(C_0)_{ik} = \delta_{ik'}, \quad k' = N + 1 - k$. The matrix $D$ is obtained from the invariant condition for quadratic form, i.e. $y^tC_0y = x^tD^tC_0Dx = x^tx$ for any $x \in \mathbb{R}_n$. One of the solutions of matrix equation $D^tC_0D = I$ has the form

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I & \tilde{C}_0 \\ i\tilde{C}_0 & -iI \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 & \tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ i\tilde{C}_0 & 0 & -iI \end{pmatrix},$$

for $N = 2n$ and $N = 2n + 1$, respectively, where $\tilde{C}_0 \in M_n(\mathbb{C})$ is the matrix with the real units on the second diagonal.

The similarity transformation

$$B(j) = D^{-1}A(j)D$$

in the vector space $\mathbb{R}_n(j)$ gives the realization of $SO(N; j; \mathbb{R})$ in a new ("symplectic") basis $y(j) = Dx(j)$ with the invariant quadratic form

$$y^t(j)C_0y(j) = 2 \sum_{k=1}^n J_{1k}J_{1k'}y_ky_{k'} + \epsilon J^2_{1,n+1}y_{n+1}^2,$$

$\epsilon = 1$ for $N = 2n + 1$, $\epsilon = 0$ for $N = 2n$ and the additional relations of $j$-orthogonality

$$B(j)C_0B^t(j) = B^t(j)C_0B(j) = C_0.$$  

The solution $D$ is not unique. There are different solutions $\tilde{D}$ of the matrix equation $D^tC_0D = I$. The similarity transformations $\tilde{B}(j) = \tilde{D}^{-1}A(j)\tilde{D}$ provide the different realization of $SO(N; j; \mathbb{R})$ as the matrix group over $D_{N-1}(j; \mathbb{R})$. In the case of quantum groups $SO_q(N; j; \mathbb{C})$ this correspond to the different couplings of CK and Hopf structures on the level of quantum groups, which on the level of quantum algebras mean the different choice of the primitive elements of the Hopf algebra [15].
5 Quantum orthogonal CK groups $SO_v(N; j; C)$

According to FRT theory of quantum groups the starting point of quantization is an algebra $C\langle t_{ik}\rangle$ of noncommutative polynomials of $N^2$ variables $t_{ik}, i, k = 1, \ldots, N$ over complex number field $C$. For well known [1] lower triangular matrix $R_q \in M_{N^2}(C)$ the generators $T = (t_{ik})_{i,k=1}^{N} \in M_{N}(C\langle t_{ik}\rangle)$ have the following commutation relations

$$R_q T_1 T_2 = T_2 T_1 R_q,$$

(25)

where $T_1 = T \otimes I$, $T_2 = I \otimes T \in M_{N^2}(C\langle t_{ij}\rangle)$. There are additional relations of $q$-orthogonality

$$T C T^t = T^t C T = C,$$

(26)

where $C = C_0 q^\rho$, $\rho = diag(\rho_1, \ldots, \rho_N)$,

$$\rho_1, \ldots, \rho_N = \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}), & N = 2n + 1 \\ (n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1), & N = 2n. \end{cases}$$

(27)

The quantum orthogonal group $SO_q(N; C)$ is defined as the quotient

$$SO_q(N; C) = C\langle t_{ik}\rangle/(25), (26).$$

(28)

From the algebraic point of view $SO_q(N; C)$ is a Hopf algebra with the following coproduct $\Delta$, counit $\epsilon$ and antipode $S:

$$\Delta T = T \otimes T, \quad \epsilon(T) = I, \quad S(T) = CT^t C^{-1}.$$

(29)

We shall regard the quantum deformations of the contracted complex CK groups and in this case the parameters $j$ take only two values: $j_k = 1, i_k$. We start now with the $D\langle t_{ik}\rangle$ — the algebra of noncommutative polynomials of $N^2$ variables over the algebra $D_{N-1}(j)$. In addition we transform the deformation parameter $q = \exp z$ as follows:

$$z = J v, \quad J \equiv J_1 N = \prod_{k=1}^{N-1} j_k,$$

(30)

where $v$ is the new deformation parameter. The transformation of quantum deformation parameter was suggested by E.Celeghini et al. [11].

In “symplectic” basis the quantum CK group $SO_v(N; j; C)$ is produced by the generating matrix $T(j) \in M_{N}(D\langle t_{ik}\rangle)$ equal to $B(j)$ (22) for $q = 1$. The noncommutative entries of $T(j)$ obey the commutation relations

$$R_v(j) T_1(j) T_2(j) = T_2(j) T_1(j) R_v(j)$$

(31)

and the additional relations of $(v,j)$-orthogonality

$$T(j) C(j) T^t(j) = T^t(j) C(j) T(j) = C(j),$$

(32)
where lower triangular $R$–matrix $R_v(j)$ and $C(j)$ are obtained from $R_q$ and $C$, respectively, by substitution $Jv$ instead of $z$:

$$R_v(j) = R_q(z \rightarrow Jv), \quad C(j) = C(z \rightarrow Jv).$$  \hspace{1cm} (33)

Then the quotient

$$SO_v(N; j; C) = D(t_{\theta k})/(31), \hspace{1cm} (34)$$

is Hopf algebra with the coproduct $\Delta$, counit $\epsilon$ and antipode $S$:

$$\Delta T(j) = T(j) \otimes T(j), \quad \epsilon(T(j)) = I, \quad S(T(j)) = C(j)T^t(j)C^{-1}(j).$$  \hspace{1cm} (35)

The matrix $D$ \((31)\) in the similarity transformation \((22)\) and the factor $J = J_{1N}$ in the deformation parameter transformation \((30)\) are selected consistently to provide the existence of the Hopf algebra structure for all possible values of the parameters $j$, i.e. for all contracted CK groups. For some other solution $\tilde{D}$ in Eq.\((32)\) the consistent factor $J$ in Eq.\((31)\) may be equal to product only some parameters $j_k$. It turn out that for some choice of $D$ not all CK contrations are allowed.

6 \textbf{Quantum CK algebras $so_v(N; j; C)$ as a dual to $SO_v(N; j; C)$}

By FRT quantization theory \([1]\) the dual space $\text{Hom}(SO_v(N; j; C), C)$ is an algebra with the multiplication induced by coproduct $\Delta$ in $SO_v(N; j; C)$:

$$l_1 l_2(a) = (l_1 \otimes l_2)(\Delta(a)), \hspace{1cm} (36)$$

$l_1, l_2 \in \text{Hom}(SO_v(N; j; C), C), \quad a \in SO_v(N; j)$. Let us formally introduce $N \times N$ upper (+) and lower (−) triangular matrices $L^{(\pm)}(j)$ as follows: it is necessary to put $j_k^{-1}$ in the nondiagonal matrix elements of $L^{(\pm)}(j)$, if there is the parameter $j_k$ in the corresponding matrix element of $T(j)$. For example, if $(T(j))_{12} = j_1 l_{12} + j_2 l_{21}$, then $(L^{(\pm)}(j))_{12} = j_1^{-1} l_{12} + j_2^{-1} l_{21}$. Formally the matrices $L^{(\pm)}(j)$ are not defined for $j_k = \iota_k$, since $\iota_k^{-1}$ do not exist, but $L^{(\pm)}_k$ are functionals on $t_{pr}$, so if we set an action of the matrix functionals $L^{(\pm)}(j)$ on the elements of $SO_v(N; j; C)$ by the duality relation

$$\{L^{(\pm)}(j), T(j)\} = R^{(\pm)}(j), \hspace{1cm} (37)$$

where

$$R^{(+)}(j) = PR_v(j)P, \quad R^{(-)}(j) = R_v^{-1}(j), \quad Pu \otimes w = w \otimes u, \hspace{1cm} (38)$$

then we shall have have well defined expressions even for $j_k = \iota_k$. 

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The elements of $L^{(\pm)}(j)$ satisfy the commutation relations
\[
R^{(+)}(j)L_1^{(\sigma)}(j)L_2^{(\sigma)}(j) = L_2^{(\sigma)}(j)L_1^{(\sigma)}(j)R^{(+)}(j),
\]
\[
R^{(+)}(j)L_1^{(+)}(j)L_2^{(-)}(j) = L_2^{(-)}(j)L_1^{(+)}(j)R^{(+)}(j), \quad \sigma = \pm
\]  \hspace{1cm} (39)

and additional relations
\[
L^{(\pm)}(j)C^t(j)(L^{(\pm)}(j))^t = C^t(j),
\]
\[
(L^{(\pm)}(j))^t(C^t(j))^{-1}L^{(\pm)}(j) = (C^t(j))^{-1},
\]
\[
l_{kk}^{(+)}l_{kk}^{(+)} = l_{kk}^{(-)}l_{kk}^{(+)} = 1, \quad l_{11}^{(+)}\ldots l_{NN}^{(+)} = 1, \quad k = 1, \ldots, N. \quad (40)
\]

An algebra $so_v(N; j; \mathbb{C}) = \{I, L^{(\pm)}(j)\}$ is called quantum CK algebra and is Hopf algebra with the following coproduct $\Delta$, counit $\epsilon$ and antipode $S$:
\[
\Delta L^{(\pm)}(j) = L^{(\pm)}(j)\otimes L^{(\pm)}(j), \quad \epsilon(L^{(\pm)}(j)) = I,
\]
\[
S(L^{(\pm)}(j)) = C^t(j)(L^{(\pm)}(j))^t(C^t(j))^{-1}. \quad (41)
\]

It is possible to show that algebra $so_v(N; j; \mathbb{C})$ is isomorphic with the quantum deformation \cite{10} of the universal enveloping algebra of the CK algebra $so(N; j; \mathbb{C})$, which may be obtained from the orthogonal algebra $so(N; \mathbb{C})$ by contractions \cite{2}. So there are at least two ways for construction of quantum CK algebras.

7 Example: $SO_v(3; j; \mathbb{C})$ and $so_v(3; j; \mathbb{C})$

The generating matrix for the simplest quantum orthogonal group $SO_v(3; j; \mathbb{C})$, $j = (j_1, j_2)$ is in the form
\[
T(j) = \begin{pmatrix}
t_{11} + ij_1j_2\tilde{t}_{11} & j_1t_{12} - ij_2\tilde{t}_{12} & t_{13} - i(j_1j_2\tilde{t}_{13}) \\
(j_1t_{21} + ij_2\tilde{t}_{21}) & t_{22} & j_1t_{23} - ij_2\tilde{t}_{23} \\
t_{13} + i(j_1j_2\tilde{t}_{13}) & (j_1t_{12} + ij_2\tilde{t}_{12}) & t_{11} - i(j_1j_2\tilde{t}_{11})
\end{pmatrix}. \quad (42)
\]

The R-matrix is obtained from the standard one by Eq.\eqref{43} and is as follows
\[
R_v(j) = R_q(z \rightarrow Jv) =
\[
\begin{pmatrix}
e^{Jv} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-Jv} & 0 & 0 & 0 & 0 & 0 \\
0 & 2\sinh Jv & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2e^{-Jv/2}\sinh Jv & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2(1 - e^{-Jv})\sinh Jv & 0 & -2e^{-Jv/2}\sinh Jv & 0 & e^{-Jv} & 0 \\
0 & 0 & 0 & 0 & 0 & 2\sinh Jv & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{Jv}
\end{pmatrix},
\]  \hspace{1cm} (43)
where $J = j_1j_2$. Over the algebras $\mathbf{D}_2(j_1, j_2)$, $j_1 = \iota_1, j_2 = 1$, or $j_1 = 1, j_2 = \iota_2$, or $j_1 = \iota_1, j_2 = \iota_2$ this R-matrix may be written in the form

$$R_v(j) = I + Jv\tilde{R},$$

(44)

where

$$(\tilde{R})_{11} = (\tilde{R})_{99} = 1, (\tilde{R})_{23} = (\tilde{R})_{77} = -1, (\tilde{R})_{42} = (\tilde{R})_{86} = 2, (\tilde{R})_{53} = (\tilde{R})_{75} = -2$$

(45)

and all other elements of the matrix $\tilde{R}$ are equal to zero. The commutation relations and additional relations of $(v, j)$-orthogonality may be obtained from Eqs. (41), (42) by straightforward calculations, so we shall concentrate our attention on the construction of quantum algebra $so_v(3; j; C)$.

The matrix functionals $L^\pm(j)$ have the form

$$L^+(j) = \begin{pmatrix} l_{11} & l_{12}^{-1}l_{13} - ij_1^{-1}j_2^{-1}l_{13} & l_{13}^{-1}j_1^{-1}j_2^{-1}l_{13} \\ 0 & 1 & j_1^{-1}l_{21} - ij_2^{-1}l_{21} \\ 0 & 0 & l_{11}^{-1} \end{pmatrix},$$

(46)

$$L^-(j) = \begin{pmatrix} l_{11}^{-1} & 0 & 0 \\ -(j_1^{-1}l_{21} + ij_2^{-1}l_{21}) & 1 & 0 \\ -(l_{13} + ij_1^{-1}j_2^{-1}l_{13}) & -(j_1^{-1}l_{12} + ij_2^{-1}l_{12}) & l_{11} \end{pmatrix}.$$  (47)

Their actions on the generators of quantum group $SO_v(3; j; C)$ are given by Eq. (43) and are as follows [3]:

$$l_{11}(t_{22}) = 1, \quad l_{11}(t_{11}) = \cosh Jv, \quad l_{11}(\tilde{t}_{11}) = -J^{-1}\sinh Jv,$$

$$l_{12}(\tilde{t}_{21}) = -ij_1^2J^{-1}\sinh Jv, \quad l_{12}(\tilde{t}_{12}) = ij_1^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2),$$

$$l_{12}(t_{12}) = (\cosh 3Jv/2 - \cosh Jv/2)/2 = \tilde{t}_{12}(t_{12}), \quad \tilde{l}_{12}(t_{21}) = ij_2^2J^{-1}\sinh Jv,$$

$$l_{12}(\tilde{t}_{12}) = -ij_2^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2), \quad l_{21}(\tilde{t}_{12}) = -ij_1^2J^{-1}\sinh Jv,$$

$$l_{21}(\tilde{t}_{21}) = ij_1^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2), \quad \tilde{l}_{21}(t_{12}) = ij_2^2J^{-1}\sinh Jv,$$

$$l_{21}(t_{21}) = (\cosh 3Jv/2 - \cosh Jv/2)/2 = \tilde{l}_{21}(\tilde{t}_{21}), \quad l_{13}(t_{13}) = \cosh 2Jv - 1)/2 = \tilde{l}_{13}(\tilde{t}_{13}),$$

$$l_{13}(\tilde{t}_{13}) = -iJ^{-1}(2\sinh Jv - \sinh 2Jv), \quad \tilde{l}_{13}(t_{13}) = iJ(2\sinh Jv - \sinh 2Jv).$$ 

(48)

Only nonzero expressions are written out above. According to the additional relations (44) there are three independent generators of $so_v(3; j; C)$, for example, $l_{11}, l_{12}, \tilde{l}_{12}$. Their commutation relations follow from Eq. (43)

$$l_{11}l_{12}\cosh Jv - l_{12}l_{11} = l_{11}\tilde{l}_{12}t_{12}^2J^{-1}\sinh Jv, \quad l_{11}\tilde{l}_{12}\cosh Jv - \tilde{l}_{12}l_{11} = -l_{11}l_{12}t_{12}^2J^{-1}\sinh Jv,$$

12
The quantum analogue of the universal enveloping algebra of CK algebra $so(3; j; C) = \{X_{01}, X_{02}, X_{12}\}$ with the rotation generator $X_{02}$ as the primitive element of the Hopf algebra has been given in [11],[12]. The Hopf algebra structure of $so_w(3; j; X_{02})$ is given by

$$\Delta X_{02} = I \otimes X_{02} + X_{02} \otimes I,$$

$$\Delta X = e^{-wX_{02}/2} \otimes X + X \otimes e^{wX_{02}/2}, \quad X = X_{01}, X_{12},$$

$$\epsilon(X_{01}) = \epsilon(X_{02}) = \epsilon(X_{12}) = 0, \quad S(X_{02}) = -X_{02},$$

$$S(X_{01}) = -X_{01} \cos Jw/2 + j_2^2 \cos^{-1} \sin Jw/2,$$

$$S(X_{12}) = -X_{12} \cos Jw/2 - j_2^2 \cos^{-1} \sin Jw/2,$$

$$[X_{01}, X_{02}] = j_2^2 X_{12}, \quad [X_{02}, X_{12}] = j_2^2 X_{01}, \quad [X_{12}, X_{01}] = \frac{\sinh w}{w} X_{02}. \quad (50)$$

The isomorphism of $so_w(3; j; X_{02})$ and quantum algebra $so_v(3; j; C)$ is easily established with the help of the following relations between generators and deformation parameters

$$l_{11} = e^{-wX_{02}}, \quad l_{12} = JEX_{01} e^{-wX_{02}/2}, \quad \tilde{l}_{12} = JEX_{12} e^{-wX_{02}/2},$$

$$v = -iw, \quad E = i(2wJ^{-1} \sin Jw)^{1/2}. \quad (51)$$

Now the quantum analogues of the nonsemisimple CK groups and algebras are obtained by specific values of the parameters $j_1, j_2$. In particular $j_1 = \tau_1, j_2 = \tau_2$ corresponds to Euclidean quantum group $E_v(2; C)$ (cf. [11],[13]) and $j_1 = \tau_1, j_2 = \tau_2$ corresponds to Galilean quantum group $G_v(2; C)$ (cf. [12],[14]). So the quantum orthogonal CK algebras may be constructed both as the dual to the quantum group and by the contractions of quantum orthogonal algebras.

For $j_1 = j_2 = 1$ we have the quantum group $SO_q(3; C)$ and the quantum algebra $so_q(3; C)$. Let us mark the elements of the generating matrix of $SO_q(3; C)$, represented in the form (42), and generators of $so_q(3; C)$, represented in the form (46), (47), with the prime. Then all formulas for $SO_v(3; j; C)$ and $so_v(3; j; C)$ may be obtained from the corresponding formulas for $SO_q(3; C)$ and $so_q(3; C)$ by the following transformations of generators and deformation parameter

$$\tilde{t}_{11} = t_{11}, \quad \tilde{t}_{11} = j_1j_2 \tilde{t}_{11}, \quad \tilde{t}_{12} = j_1 \tilde{t}_{12}, \quad \tilde{t}_{12} = j_2 \tilde{t}_{12},$$

$$\tilde{t}_{13} = t_{13}, \quad \tilde{t}_{13} = j_1j_2 \tilde{t}_{13}, \quad \tilde{t}_{21} = j_1 \tilde{t}_{21}, \quad \tilde{t}_{21} = j_2 \tilde{t}_{21},$$

$$z = Jv,$$

$$l_{11} = l'_{11}, \quad l_{12} = j_1l'_{12}, \quad \tilde{l}_{12} = j_2 \tilde{l}_{12}, \quad \tilde{l}_{12} = j_2 \tilde{l}_{12}, \quad \tilde{l}_{21} = j_1 \tilde{l}_{21}, \quad \tilde{l}_{21} = j_2 \tilde{l}_{21},$$

$$l_{13} = l'_{13}, \quad \tilde{l}_{13} = j_1j_2 \tilde{l}_{13}. \quad (52)$$
This is nothing else than the contraction transformation if one replace parameters \( j_k \) with new parameters \( \epsilon_k \), which tends to zero. It worth mention that Inönü-Wigner contractions [9] of groups at least in CK scheme are just the regarding of groups over algebras \( D_n(j) \) with all or some nilpotent parameters \( j_k \).

8 Conclusion

It was demonstrated in previous Sections that the orthogonal CK groups are naturally arrised as the matrix groups over the algebra \( D \) with nilpotent commutative generators. Their explicit realization depend on the choice of the basis in the corresponding CK vector space over the algebra \( D \). The Cartesian basis provide the most simple and well known representations of the CK groups. Then the realization in an arbitrary basis may be obtained with the help of the similiraty transformations. In particular, the realization in so-called symplectic basis is needed for quantum deformations of the orthogonal CK groups. We have shown that FRT quantization theory of simple (and semisimple) groups describe also the deformations of the nonsemisimple orthogonal CK groups and algebras, if apply it to the corresponding objects over Pimenov algebra \( D \). The ambiguity of the transformations from Cartesian to symplectic basis provide the different realization of the diagonal elements of the matrix \( T(j) \), as the elements of the algebra \( D \), which on the level of quantum algebras leads to the different choice of the primitive elements of the Hopf algebra.

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