A REVIEW ON SINGULAR PERTURBED DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, existing available literature on numerical solution of singularly perturbed and singular perturbed delay differential equations is reviewed. These equations arise in mathematical modeling of variational problems in control theory, physical and biological phenomena like optically bistable devices, description of human pupil reflex and a variety of models for physiological processes or diseases. The purpose of this work is to investigate the class of singular type problems that are solved till now using different numerical methods. The objective is to motivate the researchers to develop new methods for calculating efficient solution of such problems. There is a lot of work reported in literature to solve these types of equations both numerically and analytically. This paper limits its coverage to the work done by numerous researchers between 2002 and 2017.

INTRODUCTION

The peculiarity of detecting the relation between causes and effect emerge when the cause is small and effect is large. In theory of perturbation from mathematics and physical systems, the study of this relation got significance amount of attention in past and recent years. The work on finding the solution of singular perturbed differential equations (SPDE) started in year 1968. Various studies and surveys have been conducted by researchers on the development in this area. A survey on different asymptotic and numerical methods for solving singular perturbation problems done by Kadalbajoo and Reddy[1]. In 2002, Kadalbajoo and Patidar [2] presented another survey on the work done by numerous researchers in the area of singular perturbation from 1984 to 2000. Kadalbajoo and Patidar [3] presented a review of singularly perturbed partial differential equations in 2003. In [4] Kadalbajoo and Vikas gave brief survey on the computational techniques used to solve different classes of singular perturbation problems. In 2013, Sharma et al. [5] presented a review on singular perturbed differential equations with turning point and interior layers. From this discussion it can be concluded that this area has developed much in the past century. So it motivates us to write a review paper on work done by researchers for solving these equations.

In this survey we restrict our study mainly on singular perturbed delay differential equation which is a type of singular perturbed differential equations and very useful in wide range of applications in real life. In this survey we have given very brief review for the work done on singular perturbed delay differential equations and we have given detailed review of singular perturbed delay differential equations.

The paper is organized as follow: In section two, singular perturbed differential equations are introduced and applications of these equations discussed. In section three, survey of numerical solution on singular perturbed differential equation is given. In section four, singular perturbed delay differential equations are introduced and applications of these equations are given. In section five, methods used for solving singular perturbed delay differential equations are presented. Finally the paper is concluded with further scope of work in this area followed by references.

Introduction to singularly perturbed differential equations

A singularly perturbed differential equation is an ordinary differential equation in which the highest order derivative is multiplied by a small parameter known as perturbation parameter. The solution of singularly perturbed differential equations (SPDE) varies rapidly in the regions called as layers which may be apparent in the solution or its derivatives and often appear at the boundary of the domain. The study of solution of these equations is of great significance due to the formation of sharp boundary or interior layers when the perturbation parameter approaches to zero.

In real life, many problems in science and engineering, control theory, elasticity, fluid mechanics, biosciences are modeled by singular perturbed differential equations, for instance, in red cell system [6], variational problems in control
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theory [7,8,9], describing the motion of the sunflower [10],
signal transmission [11] and, depolarization in the Stein
model [12], thermo elasticity [13], Optics and physiology
[14] etc.

Work progress towards solution of SPDE
The general form of SPDE of second order solved is

\[ \varepsilon u' + f(r, u, u') = 0 \text{ subject to condition } u(a) = A \text{ and } u(b) = B. \]

To solve the perturbed differential equation following
techniques has been used by the researchers.

Spline based methods: The idea behind spline is fitting of
the domain is divided into
piecwise continuous polynomial passing through given
points. A function \( f(x) \) is a spline of degree \( k \) if it is \( k^{th} \) degree
polynomial \( p(x) \) and its first \( k-1 \) derivatives are also
continuous everywhere in domain \([x_0, x_n]\). There is
remarkable progress in the theory on numerical methods
and applications for solution of SPDE. The spline based
methods are easy and ready for computer implementation.

Following is the work reported in literature on spline based
method: SPDE of different order extensively studied in
literature because of their multiple applications [6-14] by the
researchers. Aziz and Khan [15] in 2002 proposed the
solution of second order singularly perturbed boundary value
problem by using cubic spline in compression and
interpolatory condition. The considered equation was of form

\[ \varepsilon y'' = p(x)y' + q(x)y + r(x), \quad y(a) = a_0 \text{ and } y(b) = a_n. \]

Depending on the parameters involved in method,
scheme found to have second and fourth order of
convergence. In 2012, Jha [16] used finite difference method
for nonlinear singularly perturbed singular boundary value
problem of second order using non polynomial spline finite
difference method. Author claimed that method for fourth order
convergent and is easy to implement. He considered two test
examples to check efficiency of the scheme and calculate the
root mean square error. In 2013, El-Salam [17] used a second
and fourth order convergent parametric spline method for
second order singularly perturbed boundary-value problem.
He has shown that the order of convergence depends on
the choice of two parameters. Akram and Naheed [18] in 2013
solved fourth order singularly perturbed boundary-value
problem using septic spline. In this paper spline of septic
degree are substituted in the equation to obtain numerical
solution by applying the properties of continuity of
derivatives of second, fourth and sixth order. Author proved
the method to be of fourth order convergent and having better
results as compared to the existing solution.

Remarks: In spline based methods, the domain is divided into
subintervals by selecting uniform partition and spline
approximation methods such as collocation, finite element
etc. were used to approximate the solution. Spline based
methods provides accurate results but one needs skill to
handle the calculations.

Transformation Methods: In mathematics transformation is
an operation that converts one function into another. For
example Laplace is the integral transform used to find the
solution of differential equations. Following transformation
methods are reported in literature to find the solution of
SPDE: Dogan et al. [19] in 2012 numerically treated
singularly perturbed two point boundary value problem using
differential transformation method. This method is an iterative
method for procuring analytical Taylor series solutions of
differential equations and it can be used to solve both linear
and nonlinear initial value problems. In 2013, Mishra and
Saini [20] discussed the numerical solution of singularly
perturbed two-point boundary value problem via Liouville-
Green Transform. It is simple and easy to implement
analytical method for obtaining solution of differential
equations.

Remarks: Both transformation methods presented analytical
solution of SPDE as a function of \( x \) instead of giving
approximate value of solution at selected points. The solution
of equation could be obtained at any point in domain.

Methods using B-spline as basis function: A spline function
that has minimal support with respect to the given degree,
smoothness and domain partition is called B-spline. This basis
function has been used by various researchers to solve linear
and nonlinear differential equations. Gupta et al. [21] in 2011
applied cubic B-Spline scheme to find numerical solution of
system of singularly perturbed boundary value problems. In
2015 Mishra and Saini [22] discussed quartic B–Spline
basis function for solving singular singularly perturbed third-order
boundary value problems. In this paper basis of quartic B-
spline is derived and matrix form of the system is solved by
using MATLAB. The efficiency of the method is presented by
implementing on test examples and absolute error was
discussed.

Remarks: In above mentioned papers mathematicians used B–
Spline basis function of order two and three respectively. The
discussed technique is easy to apply and provides better
approximate solution due to compactly support properties of
B–Spline basis functions.

Introduction to singularly perturbed delay differential
equations

Singularly perturbed delay differential equation is an equation
in which evolution of system at a certain time depends on the
rate at an earlier time. The delay in process arises due to
requirement of definite time to sense the instruction and react
to it. The periodic oscillation of breathing frequency under
constant conditions shown by some person is an example of
delay equation. This delay is caused by cardiac insufficiency
in the physiological circuit controlling the carbon dioxide
level in blood [23]. For more applications of delay differential
equations in Biosciences refer [23]. The delay differential
equation in which the highest derivative is multiplied by
perturbation parameter is known as perturbed delay
differential equation. The delay differential equation can be
classified as retarded delay differential equation and neutral
differential equation. The applications of delay differential
equations arises in modeling of neural variable, variational
problems in control theory [24,25], description of human
pupil reflex [26], physical and biological phenomena like
optically bistable devices [27] and in numerical modeling in
biosciences [28], HIV infection [29] etc.

Expansion in area of finding solution of singular perturbed
delay differential equations

A general form of SPDDE is
A tremendous advancement has been reported in past decades in the theory of Spline and B-spline based methods for finding numerical solution of differential equations. An optimized B-spline method to solve singularly perturbed differential difference equation with delay as well as advanced was proposed by Sharma [30]. The considered equation was:

\[ e^y''(x) + a(x)y'(x - \delta) + b(x)y(x - \delta) + c(x)y(x) = f(x), \quad 0 < x < l \]

subject to the conditions \( y(x) = \Phi(x) \) on \( -\delta \leq x < 0 \).

A lot of work has been reported in literature for finding the approximate solution of these equations. Following are the techniques used for finding solution of these equations:

**Spline and B-spline based methods**

In 2012, Kumar [31] treated singularly perturbed delay differential equation of second order using B-spline collocation method. In this article the author has selected piecewise uniform mesh known as shishkin mesh which is adequate to handle singularly perturbed problems. The mesh is constructed in such a way that more mesh points are created in the boundary layer region than outside these regions. The SPDE of second order was treated by the cubic B-spline scheme and the solution obtained by solving tri-diagonal matrix. Author discussed the uniform convergence of the method and to exhibit the coherence of the method two test examples were considered. As the exact solution of the considered problems is not known the maximum absolute nodal error is estimated by

\[ E^N = \max_{0 \leq i \leq N} |y_i - y^N_i|, \quad 0 \leq i \leq N. \]

Author represented the results by maximum absolute error obtained by using uniform and shishkin mesh. It was observed that the error with the shishkin mesh was less as compared to the uniform mesh value as \( N \) increases.

Chakravarthy et al. [32] solved these equations by using cubic spline in compression. In this paper the author considered the SPDDE of second order and with large delay. Uniform mesh was selected to partition the domain and method based on cubic spline in compression was used to approximate the solution of the problem.

The equation is of form:

\[ e^y''(x) + a(x)y'(x) + b(x)y(x - 1) = f(x), \quad 0 \leq x \leq 2 \]

with \( y(0) = \Phi(x) : -1 \leq x \leq 0 \) and \( y(2) = \beta \). Author claimed that the presented schemes provide accurate results for the linear SPDDE with large delay.

To demonstrate the efficiency of method author applied the discussed technique on five test examples and presented results in tables for different values of \( N \) (total number of partition points). Maximum absolute error was calculated with double mesh principle. It was observed that the absolute error decreases with increase in value of \( N \).

Remarks: Spline based methods provide more accurate results for SPDDE with small delay, large delay and for the SPDDE with delay as well as advance. These methods are easy to apply due to properties of spline functions.

**Exponential collocation method**

In 2013, Yuzbagi and Sezer [33] used an exponential collocation method for finding solution of second order SPDDE. Author discussed the solution of the problem by considering the exponential basis set \( \{ e^x, e^{2x}, \ldots, e^{N-1} \} \) and solution was of the form \( y(x) = \sum_{i=0}^{N} a_i e^{\eta_i} \). The boundary \( 0 \leq x \leq b \) was divided with constant mesh length \( b/N \). The approximate solution and its derivatives presented in matrix form and substituted in differential equation. Based on the residual function \( R_k(x) = L[y_k(x)] - g(x) \), error function was determined and corrected exponential solution was presented. Four test examples were solved to demonstrate the efficiency of the method. Absolute error in all the examples was calculated for different values of \( N \).

**Computational Techniques**

In 2013, File et al. [34] presented a computational method to solve these equations with negative shift whose solution has boundary layer. In this scheme author reduced the second order SPDDE to first order equation and then employed numerical integration and interpolations. Author claim that the available asymptotic expansion methods for solving singular perturbed problems are difficult to apply as it is not easy to find appropriate asymptotic expansions in the inner and outer regions and matching of the coefficients of the inner and outer solution expansions is also a process that need skills. The proposed method overcome above problems. The left ended and right ended problem was treated separately. Taylor series expansion of \( y(x) \) in neighborhood of \( x=0 \) and \( x=1 \) is used to reduce second order singularly perturbed delay differential equation in first order delay differential equation. The interval \( [0,1] \) is divided in equal subparts of constant length \( 1/N \). The new equation integrated over an interval and solved by Trapezoidal Rule and Taylor series. Author discussed the discrete invariant imbedding algorithm to solve three term recurrence relation. Method employed on four examples and the result of method compared by the exact in two examples.

In 2015, D. Kumar Swamy et al. [35] proposed computational method for singularly perturbed delay differential equation of second order with twin layers or oscillatory behavior. Layer or oscillation behavior of the delay differential equation discussed depending on sign of \( (a(x) + b(x)) \). The layer behavior of the solution diminish as the delay increases and the solution exhibit oscillation behavior. The interval \( [0, 1] \) is divided into even number of sub intervals with constant mesh length. The second order delay differential equation is transformed into an equivalent first order neutral type delay differential equation. The new equation integrated on the interval \( [0, 0.5] \) and \( [0.5, 1] \) using Trapezoidal Rule.

In 2013, Devendra Kumar [36] solved boundary-value problem for a singularly perturbed delay differential equation with a small parameter multiplying to the second derivative and containing negative shift in the first derivative term.

Taylor's series expansions of the terms that involve delay are used and central difference formula used for first order terms.
derivative term. The method employed on a test example to check the result.

Remarks: In above discussed methods original SPDDE reduced to first order equivalent equation and integrated numerically to obtain the solution.

**Other techniques Used**

Asymptotics – fitted method: In 2012, Andargie et al. [37] proposed an Asymptotics – equation method for solving singularly perturbed delay differential two point boundary value problem of second order. The equation of form:

\[ ey'(x) + a(x)y(x - \delta) + b(x)y(x) = f(x) \quad 0 \leq x \leq 1 \]

with \( y(0) = \alpha : -\delta \leq x \leq 0 \) and \( y(1) = \beta \) was considered. Taylor’s expansion is used to approximate the term containing negative shift. In the new equation a fitted parameter on the boundary layer derivative is introduced and its value is determined by using theory of Singular Perturbation. Uniform mesh was selected and the interval \([0, 1]\) is divided into \( N \) equal parts with constant mesh length. Thomas Algorithm was discussed to solve three term recurrence relation. The proposed method was implemented on five test examples to demonstrate the efficiency of the method. Out of the 5 examples 3 examples were linear and other 2 nonlinear. On nonlinear examples quasi-linearization method was used. The delay parameter \( \delta \) is chosen to reduce the coefficient of second derivative of the modified problem. The absolute error in all the five examples has been discussed. From the result it was observed that with the increase in \( N \) the absolute error decrease.

A fourth order finite difference scheme presented by Gemenchis File et al. [41] in 2017 for numerical solution of SPDDE. The authors considered Reaction-Diffusion Equation

\[ \varepsilon^2 y'(x) + a(x)y(x - \delta) + b(x)y(x) = f(x) \quad 0 \leq x \leq 1 \]

with \( y(x) = \emptyset(x) - \delta \leq x \leq 0 \) and \( y(1) = \beta \) and uniform mesh with constant mesh length is considered. A fourth order finite difference scheme is derived by using Taylor series expansion and the resultant tri-diagonal system solved by the method of Discrete Invariant Imbedding algorithm. Stability analysis of the above scheme was carried out and found that the method is of fourth order uniformly convergent.

| Technique Used | SPDDE | Method |
|----------------|-------|--------|
| 1. \( e u'(x) + a(x)u(x) + a(x)u(x - \delta) + a(x)u(x) + \beta(x)u(x + \eta) = f(x) \) | Cubic B-spline collocation method with shishkin mesh [30]. |
| Spline and B-spline | 2. \( e^x y(x) + a(x)u(x) + a(x)u(x - \delta) + b(x)u(x) = f(x) \) | Cubic B-spline collocation method with shishkin mesh [31]. |
| Exponential Basis | 3. \( e^x y(x) + a(x)u(x) + b(x)u(x - 1) = f(x) \) | Cubic B-spline in compression with uniform mesh [32]. |
| Computational Techniques | \( e^x y(x) + p(x)y(x - \delta) + q(x)y(x) = g(x) \) | Exponential collocation method with uniform mesh [33]. |
| Other techniques Used | \( e^x y(x) + a(x)u(x) + a(x)u(x) + b(x)u(x) = f(x) \) | Equation reduced to asymptotic equivalent equation using Taylor’s series with uniform mesh [34]. |
| Hp Finite Element Method | \( e^x y(x) + a(x)u(x) + a(x)u(x - \delta) + b(x)u(x) = f(x) \) | Taylor’s series expansion used in neighborhood of \( x=0 \) or \( x=1 \) in case of Boundary layer on left or right side respectively with uniform mesh [35]. |
| Hp Finite Element Method | \( e^x y(x) + a(x)u(x) + a(x)u(x) = f(x) \) | Taylor’s series with uniform mesh [36]. |
| Asymptotics – fitted method, by introducing a fitting factor with uniform mesh [37]. | 2. \( e^x y(x) + a(x)u(x) + b(x)u(x - 1) = f(x) \) | Asymptotics – fitted method [38]. |
| Terminal-boundary-value technique with uniform mesh [39]. | 3. \( e^x y(x) + a(x)u(x) + a(x)u(x) + b(x)u(x) = f(x) \) | Hp Finite Element Method [38]. |
| A hybrid initial value method with shishkin mesh [40]. | 4. \( e^x y(x) + a(x)u(x) + b(x)u(x - 1) = f(x) \) | A hybrid initial value method with shishkin mesh [40]. |
| An Asymptotic-Numerical Hybrid Method [41]. | 5. \( e^x y(x) + p(x)y(x - \delta) + q(x)y(x) = r(x) \) | An Asymptotic-Numerical Hybrid Method [42]. |

The method implemented on four numerical examples and found that presented method improved the findings of Soujanya and Reddy and Swamy. In this paper author discussed the effect of delay on amplitude of oscillations.

An Asymptotic-Numerical Hybrid Method: In 2017, Suleyman Cengizci [42] worked for approximations of the solutions of singularly perturbed second-order linear delay differential equation. The author considered following equation for numerical solution:

\[ ey'(x) + p(x)y(x - \delta) + q(x)y(x) = r(x) \quad 0 \leq x \leq 1 \]

with \( y(0) = \emptyset(x) - \delta \leq x \leq 0 \) and \( y(1) = \gamma \). Taylor series expansion used to linearized the delay term and resultant singularly perturbed ordinary differential equation

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solved by asymptotic method known as Successive complementary expansion method (SCEM). To test the scheme a left boundary layer problem considered for numerical simulation and the obtained solution was compared with exact solution and highly accurate approximations found in few iterations only.

The following table 1 shows various methods and mesh strategies used to solve SPDDE

**DISCUSSION AND CONCLUSION**

The study of Singularly Perturbed delay differential equation in the field of computational mathematics is very striking. The study of many theories and applied science results in singularly perturbed delay differential equation. Numerical solutions are important as they are easy and computed efficiently by using computer. The numerical and spline techniques appear to popular and ultimate tool to achieve the goal. This survey paper presented the development of various methods for different class of singularly perturbed delay differential equation. This paper will be very helpful to the researchers working in this area to develop the new numerical methods for solving singularly perturbed delay differential equations.

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