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Accessibility
A discontinuous Galerkin method for solving the fluid and magnetohydrodynamic equations in astrophysical simulations

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ABSTRACT
A discontinuous Galerkin (DG) method suitable for large-scale astrophysical simulations on Cartesian meshes as well as arbitrary static and moving Voronoi meshes is presented. Most major astrophysical fluid dynamics codes use a finite volume (FV) approach. We demonstrate that the DG technique offers distinct advantages over FV formulations on both static and moving meshes. The DG method is also easily generalized to higher than second-order accuracy without requiring the use of extended stencils to estimate derivatives (thereby making the scheme highly parallelizable). We implement the technique in the AREPO code for solving the fluid and the magnetohydrodynamic (MHD) equations. By examining various test problems, we show that our new formulation provides improved accuracy over FV approaches of the same order and reduces post-shock oscillations and artificial diffusion of angular momentum. In addition, the DG method makes it possible to represent magnetic fields in a locally divergence-free way, improving the stability of MHD simulations and moderating global divergence errors, and is a viable alternative for solving the MHD equations on meshes where constrained transport cannot be applied. We find that the DG procedure on a moving mesh is more sensitive to the choice of slope limiter than is its FV method counterpart. Therefore, future work to improve the performance of the DG scheme even further will likely involve the design of optimal slope limiters. As presently constructed, our technique offers the potential of improved accuracy in astrophysical simulations using the moving mesh AREPO code as well as those employing adaptive mesh refinement.

Key words: MHD – methods: numerical.

1 INTRODUCTION
Discontinuous Galerkin (DG) methods have recently been implemented for solving systems of conservation laws to arbitrary orders of accuracy and have been shown to be competitive with more established and traditional finite volume (FV) approaches (Bassi & Rebay 1997a,b; Cockburn, Li & Shu 2004; Li & Shu 2005; Luo, Baum & Löhner 2008). In this paper, we develop a second-order DG formulation for arbitrary moving and static meshes that is appropriate for even the largest astrophysical simulations. DG techniques offer numerous advantages over FV methods, as summarized by Luo et al. (2008). In particular, DG procedures can be applied to arbitrary meshes [moving meshes, adaptive mesh refinement (AMR)] and the method is ‘compact’ in the sense that each cell is treated independently and elements communicate only with adjacent elements having a common face irrespective of the order of accuracy. The DG method is conservative and requires solving the Riemann problem across cell interfaces, similar to FV schemes. The main challenge with DG implementations lies in minimizing unphysical post-shock oscillations [e.g. with slope limiting, flux limiting, shock capturing or weighted essentially non-oscillatory (WENO) approaches] (Luo et al. 2008), which is also an issue for FV methods. Some DG formulations are found to be more sensitive to certain shock limiters than their FV counterparts, but techniques exist to prevent unphysical oscillatory solutions in high-order DG methods (Hoteit et al. 2004; Luo et al. 2008; Ghostine et al. 2009).

Our DG implementation falls into the class of centrosoidal Taylor basis procedures developed by Luo et al. (2008). This formulation of
DG is relatively new and is quite different from the more widespread approach that employs nodal basis value functions (which would not be generalizable to a moving Voronoi mesh where the number of faces per cell can change with time). The primary difference between our centroidal DG method and FV schemes is in the manner in which gradients (as well as higher order derivatives) are computed for the solution of fluid variables in a cell. FV methods require the use of an extended stencil (which becomes spatially broad for estimating higher order terms). DG techniques, on the other hand, evolve the coefficients of a set of basis functions that describe the solution local to a cell in the same way as cell averages are evolved in the FV approach. This localizes the solution within a given cell, which can lead to reduced numerical errors and makes codes highly parallelizable. The centroidal DG approach may thus be viewed as an extension of the FV method.

Moreover, the DG procedure allows for a locally divergence-free representation of the solution in a cell (Li & Shu 2005; Luo et al. 2008). This not only reduces the amount of memory required to store the result but is also useful for improving the accuracy of magnetohydrodynamic (MHD) simulations. The continuum equations of ideal MHD impose the condition $\nabla \cdot B = 0$, but discretized versions of the equations do not necessarily preserve the zero divergence constraint. The locally divergence-free DG method keeps divergences to zero within cell domains unlike FV schemes. However, it is important to point out that it does not guarantee a globally divergence-free solution (which would require continuous transverse magnetic field components across cell interfaces) due to the local discontinuous representation of the result (see Section 2.7). Globally divergence-free sets that represent face-averaged magnetic fields and higher order moments are possible for DG methods on logically rectangular grids and have been developed by Balsara (2004) and Balsara et al. (2009), but the generalizability to unstructured grids is non-trivial.

The strictest approach for preserving $\nabla \cdot B = 0$ at the discretized level to machine precision is the constrained-transport (CT) framework, developed for the MHD equations by Evans & Hawley (1988). The CT method uses Stoke’s theorem to represent the magnetic fields by face-averaged rather than cell-averaged quantities. However, while the CT method can be easily implemented on fixed rectangular grids when a single, fixed time step is used, it is presently not known whether CT can be adopted for meshes of arbitrary structure, moving meshes or general time-stepping schemes. The CT approach has been implemented in AMR codes by using synchronized time-stepping and restriction and prolongation operators (Balsara 2001; Fromang, Hennebelle & Teyssier 2006; Miniati & Martin 2011), although this makes the original AMR formulation significantly more complicated. In addition, sometimes CT schemes coupled with Godunov methods need to be modified to prevent pressures from becoming negative at the cost of maintaining conservation of energy to machine precision (Balsara & Spicer 1999a). A number of divergence cleaning schemes, such as the Dedner hyperbolic cleaning method and the Powell eight-wave technique, have been developed for controlling global divergence errors in situations where the CT algorithm cannot be employed (Powell et al. 1999; Tóth 2000; Dedner et al. 2002). The locally divergence-free DG implementation, either on its own or coupled to a cleaning scheme, may improve the divergence-free constraint.

Widely used grid-based codes for solving fluid flows in astrophysical systems, such as FLASH (Fryxell et al. 2000), ENZO (O’Shea et al. 2004), RAMSES (Fromang et al. 2006), ATHENA (Stone et al. 2008) and AREPO (Springel 2010), all employ the FV approach. These codes have had numerous successes in simulating cosmological structure formation, galaxy interactions, the interstellar medium and protoplanetary and accretion discs. Here, we investigate whether the DG procedure offers a viable alternative for designing future generations of simulation codes by directly comparing second-order DG and FV methods with the same time-integration scheme.

Our goal is to develop a DG algorithm for arbitrary meshes that is efficient and simple in its implementation and, to the extent possible, minimizes numerical errors, artificial diffusion of angular momentum and global inaccuracies in the magnetic field. We are particularly interested in the application of the method to moving mesh algorithms, such as the AREPO code written by Springel (2010). The moving mesh technique is a novel type of fluid solver that is essentially a hybrid of traditional static Eulerian codes and the pseudo-Lagrangian, mesh-free smoothed particle hydrodynamics (SPH) method. In the moving mesh approach, fluid elements move with the local velocity flow, rendering the method quasi-Lagrangian. This greatly reduces advection errors arising from large bulk velocity motions of the flow, making the code well suited for simulating galaxy collisions. AREPO has been generalized to solve the Navier-Stokes equations (Muñoz et al. 2013), as well the MHD equations using the Dedner and Powell divergence cleaning schemes (Pakmor, Bauer & Springel 2011; Pakmor & Springel 2013). We show in what follows that the DG scheme can improve the accuracy of the current version of AREPO as well as other FV codes.

Our paper is organized as follows. In Section 2, we describe the DG method with centroidal Taylor basis functions and demonstrate how it is a natural generalization of the FV method. In Section 3, we present the results of numerical tests in which we compare the DG and FV methods. In Section 4, we summarize the main findings of these tests. In Section 5, we discuss the advantages of DG methods for astrophysical applications. In Section 6, we briefly offer possible ways of refining the slope limiting technique, which could improve the accuracy of the DG method even further. In Section 7, we provide conclusions.

2 DISCONTINUOUS GALERKIN FORMULATION

2.1 Governing equations

The ideal MHD equations can be written in conservative form as

$$\frac{\partial U}{\partial t} + \nabla \cdot F = 0,$$

where $U$ is the conservative state vector and $F(U)$ is the flux function:

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho e \\ B \end{pmatrix}, \quad F = \begin{pmatrix} \rho v \\ \rho uv + p - BB^T \\ \rho ev + pu \cdot B - (v \cdot B) \\ Bv^T - vB^T \end{pmatrix}.$$

Here, $\rho$ is the gas density, $p = \rho e + \frac{1}{2}B^2$ is the total gas pressure, $e = \rho u + \frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}B^2$ is the total energy per unit mass and $u$ is the thermal energy per unit mass. In the numerical examples described in this paper, we consider an equation of state of the form $p_{\text{gas}} = (\gamma - 1)\rho u$, where $\gamma$ is the adiabatic index.

The above equations reduce to the Euler equations (which describe compressible, inviscid flows) in the case that $B = 0$. 
2.2 Discontinuous Galerkin method

The DG method is defined by first considering the weak formulation of the conservation equation (1) obtained by multiplying by a test function $\mathbf{W}$, integrating over the domain ($\Omega$) and performing integration by parts:

$$
\int_\Omega \frac{\partial \mathbf{U}}{\partial t} \mathbf{W} \, d\Omega + \int_\Gamma F \cdot \hat{n} \mathbf{W} \, d\Gamma + \int_\Omega \mathbf{F} \cdot \nabla \mathbf{W} \, d\Omega = 0,
$$

(3)

where $\Gamma = \partial \Omega$ is the boundary of $\Omega$ and $\hat{n}$ is the outward unit normal vector of the boundary.

We seek to discretize equation (3). We begin by writing the solution in each cell $e$ as a second-order accurate Taylor series expansion about the centroid $(x_e, y_e)$ and employing coordinates $(x, y)$ with origin at $(x_e, y_e)$. For example, in 2D

$$
U_e = \bar{U}_e + \frac{\partial U_e}{\partial x} |_{x} x + \frac{\partial U_e}{\partial y} |_{y} y,
$$

(4)

where $\bar{U}_e$ is the cell average of the fluid variables. Our local basis functions for each fluid variable are

$$
V_1 = 1, \ V_2 = x, \ V_3 = y,
$$

(5)

and the unknowns in this problem are the cell averages and the cell derivatives.

If we use the basis functions $V_i$ each as possible test functions $\mathbf{W}$, we obtain a set of evolution equations for the cell averages and derivatives [see also Luo et al. (2008) for a formal mathematical presentation of centred DG methods]:

$$
\frac{d}{dt} \int_{\Omega_e} \bar{U}_e \, d\Omega + \int_{\Gamma_e} F(U_e) \cdot \hat{n} \, d\Gamma = 0,
$$

(6)

$$
\frac{d}{dt} \int_{\Omega_e} \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial x} |_{x} x \right) \, d\Omega + \int_{\Gamma_e} F(U_e) \cdot \hat{n} \begin{pmatrix} x \\ y \end{pmatrix} \, d\Gamma

- \int_{\Omega_e} F(U_e) \cdot \nabla \begin{pmatrix} x \\ y \end{pmatrix} \, d\Omega = 0,
$$

(7)

where $U_{e,j}$ is a single component of $U_e$. Thus, the $x$ and $y$ derivatives are coupled and it is necessary to invert a 2 by 2 matrix to obtain the derivatives.

We see from equations (6) and (7) that the cell averages and cell derivatives decouple for the choice of the Taylor basis function. In fact, equation (6) is the same equation used to update an FV scheme. Thus, the centredial DG method is a natural higher order generalization of the FV approach if one asserts that cells are only allowed to communicate with their nearest neighbours.

The matrix

$$
M = \int_{\Gamma_e} \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \, d\Omega
$$

(8)

in equation (7) is called the mass matrix. It stores second-order moments of the cell (which have to be computed for every active cell at every time step in a moving mesh approach and every time a cell is refined in an AMR approach) and is symmetric. The moments for each cell are calculated exactly using Gaussian quadrature.

In the 3D case, using a Taylor basis, we have the following weak formulation of the Euler equations for a cell $e$ [again, coordinates $(x, y, z)$ in the notation below have origin $(x_e, y_e, z_e)$]:

$$
\frac{d}{dt} \int_{\Omega_e} \bar{U}_e \, d\Omega + \int_{\Gamma_e} F(U_e) \cdot \hat{n} \, d\Gamma = 0,
$$

(9)

$$
\frac{d}{dt} \int_{\Omega_e} \nabla \cdot \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial x} |_{x} x \right) \, d\Omega

+ \int_{\Gamma_e} \nabla \cdot \begin{pmatrix} F(U_e) \cdot \hat{n} x \\ F(U_e) \cdot \hat{n} y \\ F(U_e) \cdot \hat{n} z \end{pmatrix} \, d\Gamma - \int_{\Omega_e} \begin{pmatrix} F_x(U_e) \\ F_y(U_e) \\ F_z(U_e) \end{pmatrix} \, d\Omega = 0. \quad (10)
$$

Now if we define the volume- and moment-averaged quantities $Q_e$ and $R_e$ as

$$
Q_e = \int_{\Omega_e} U_e \, d\Omega,
$$

(11)

$$
R_{e,i} = \int_{\Omega_e} \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial x} |_{x} x \right) \, d\Omega,
$$

(12)

then we can write a second-order conservative discretization in time of equations (9) and (10):

$$
Q_e^{n+1} = Q_e^n - \Delta t \sum_f A_{ef} \hat{F}_{ef}^{(n+1/2)},
$$

(13)

$$
R_{e,i}^{n+1} = R_{e,i}^n - \Delta t \sum_f A_{ef} \hat{F}_{ef,i}^{(n+1/2)} + \Delta t \int_{\Omega_e} \begin{pmatrix} F_x(U_e) \ \nabla \cdot \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial x} |_{x} x \right) \, d\Omega \\ F_y(U_e) \ \nabla \cdot \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial y} |_{y} y \right) \, d\Omega \\ F_z(U_e) \ \nabla \cdot \begin{pmatrix} x^2 & xy & xy \\ xy & x^2 & xy \\ xy & xy & y^2 \end{pmatrix} \left( \frac{\partial U_e}{\partial z} |_{z} z \right) \, d\Omega \end{pmatrix} \, d\Omega,
$$

(14)

where $\hat{F}_{ef,i}^{(n+1/2)}$ is an appropriately time-averaged approximation to the true flux $F_{ef,i}$ across a cell face between cells $e$ and $f$. $A_{ef}$ is the area of the cell face and $(c_{ef,x}, c_{ef,y}, c_{ef,z})$ is the location of the centroid of the cell face in the coordinate system local to cell $e$. The volume integral in the interior of the cell is carried out with Gaussian quadrature.

The basic idea of the DG method is to update $Q_e$ and $R_e$ for each active cell during each time step. One can then obtain the cell averages of conserved fluid variables at the end of the step by dividing the $Q_e$ by the cell volume (and consequently translated to primitive variables). Derivative information is obtained by matrix inversion of the mass matrix applied to $R_e$. Derivatives of primitive variables may then be calculated by expanding the derivatives and solving for the primitive gradients, as, for example

$$
\frac{\partial (\rho v_i)}{\partial y} = \rho \frac{\partial (v_i)}{\partial y} + v_j \frac{\partial \rho}{\partial y}
$$

$$\Rightarrow \frac{\partial (v_i)}{\partial y} = \frac{1}{\rho} \left( \frac{\partial (\rho v_i)}{\partial y} - v_j \frac{\partial \rho}{\partial y} \right). \quad (15)
$$

Finally, in order to represent magnetic fields, we use a locally divergence-free basis (e.g. in 2D) for $(B_x, B_y)$ instead of a Taylor
basis, in particular:

\[
V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix},
\]

\[
V_4 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \quad V_5 = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix},
\]

\[
V_6 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \quad V_7 = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, \quad V_8 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \quad V_9 = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix},
\]

\[
V_{10} = \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}, \quad V_{11} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}.
\]

The number of required basis functions is reduced from 6 to 5 owing to the divergence-free constraint. We derive an equation for the evolution of the magnetic field gradients analogously to equation (7); namely

\[
\frac{d}{dt} \int_{\Omega_c} \begin{pmatrix} y^2 & 0 & xy \\ 0 & x^2 & -xy \\ xy & -xy & x^2 + y^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \, d\Omega
\]

\[
+ \int_{\Gamma_s} \begin{pmatrix} F(B_x) \cdot \hat{n} y \\ F(B_y) \cdot \hat{n} x \\ F(B_z) \cdot \hat{n} z \end{pmatrix} \, d\Gamma
\]

\[
- \int_{\Omega_c} \begin{pmatrix} F_x(B_x) \\ F_y(B_y) \\ F_z(B_z) \end{pmatrix} \cdot \begin{pmatrix} F_x(B_x) \\ F_y(B_y) \\ F_z(B_z) \end{pmatrix} \, d\Omega = 0,
\]

where

\[
\alpha_1 = \frac{\partial B_x}{\partial y},
\]

\[
\alpha_2 = \frac{\partial B_y}{\partial x},
\]

\[
\alpha_3 = \frac{\partial B_z}{\partial x} = -\frac{\partial B_y}{\partial x},
\]

and the coordinates \((x, y)\) have origin \((x_0, y_0)\). In this case, a 3 by 3 matrix has to be inverted to directly obtain all the magnetic field gradients.

In 3D, locally divergence-free basis functions may be chosen as

\[
\frac{d}{dr} \int_{\Omega_c} M_B \, d\Omega + \int_{\Gamma_s} \begin{pmatrix} F(B_x) \cdot \hat{n} y \\ F(B_y) \cdot \hat{n} z \\ F(B_z) \cdot \hat{n} x \end{pmatrix} \, d\Gamma
\]

\[- \int_{\Omega_c} \begin{pmatrix} F_x(B_x) \\ F_y(B_y) \\ F_z(B_z) \end{pmatrix} \cdot \begin{pmatrix} F_x(B_x) \\ F_y(B_y) \\ F_z(B_z) \end{pmatrix} \, d\Omega = 0,
\]

where

\[
M_B =
\begin{pmatrix}
y^2 & yz & 0 & 0 & 0 & 0 & xy & xy \\
yz & z^2 & 0 & 0 & 0 & 0 & xz & xz \\
0 & 0 & x^2 & xz & 0 & 0 & -xy & 0 \\
0 & 0 & xz & x^2 & 0 & 0 & -zy & 0 \\
0 & 0 & 0 & 0 & x^2 & xy & 0 & -xz \\
0 & 0 & 0 & 0 & xy & y^2 & 0 & -yz \\
xy & xz & -xy & yz & 0 & 0 & x^2 - y^2 & x^2 \\
xy & xz & 0 & 0 & -xz & -yz & x^2 & x^2 - z^2
\end{pmatrix}
\]

2.3 Fluid dynamics on a moving mesh

In the case of non-static meshes, the Euler and MHD equations need to be modified to account for the motion of the grid. The flux over an interface moving at velocity \(w\) or inside a cell moving at velocity \(v\) is a combination of the static flux and an advection step due to the movement:

\[
F_m(U) = F_s(U) - U w^T.
\]

All Riemann problems across cell interfaces are solved in the rest frame of the face, followed by adding appropriate terms to return to the lab frame, as described in detail in (Pakmor et al. 2011, see their equation 17). This approach retains a stable, upwind character. The Riemann problem is solved using an exact solver for the Euler equations and an HLLD solver (Miyoshi & Kusano 2005) for the MHD equations.
2.4 Time stepping

We use a second-order accurate in time MUSCL–Hancock scheme to update the fluid variables at the next time step, the same method as is used for AREPO’s FV solver, described in Springel (2010). In the MUSCL–Hancock procedure (van Leer 1974; Toro 1999), the essential idea is to use cell averages to predict the values of the primitive quantities at cell edges half a time step in advance [see equation 18 of Springel (2010)] and to use these predicted values to solve the Riemann problem and obtain \( \tilde{F}_{(n+1/2)} \) in order to finish updating the solution to the next time step. The same prediction equations are also used to calculate the flux in the interior of the cell in the volume integral term of equation (14). We find that this explicit time-updating scheme works very well for our DG method. Our approach is different from traditional DG formulations that typically use explicit or implicit Runge–Kutta techniques (Luo et al. 2008). A benefit of using the second-order MUSCL–Hancock integrator is that it can be coupled with a symplectic second-order gravity solver to treat fluids with self-gravity (Springel 2010).

The fact that we use the same time-integration scheme for the DG and FV methods allow us to compare the advantages of one over the other in a direct manner. The primary difference in the second-order DG technique compared to the FV approach is only in the way in which cell gradients are handled. In a time step, it is the quantity

\[
R_{e,i} = \begin{pmatrix}
\frac{\partial U_{e,i}}{\partial x}
\frac{\partial U_{e,i}}{\partial y}
\frac{\partial U_{e,i}}{\partial z}
\end{pmatrix}
\]

that is evolved by equation (14) in a quite similar manner as volume-averaged conserved quantities are evolved in an FV scheme. After each time step update, the matrix system of equations is then inverted to obtain the gradients of the conserved variables, which are then transformed to gradients of the primitive variables (just as cell volume-integrated conserved variables are converted to cell-averaged primitive variables) for the next half-time step prediction step in the MUSCL–Hancock scheme.

2.5 Time step criterion

The maximum allowed time step allowed for a cell \( i \) is

\[
\Delta t_i = \min\left( \frac{\Delta t_{\text{CFL}}}{c_i + |v_i'|}, 1 \right),
\]

where \( R \) is the effective radius of the cell (e.g. \( R_i = \left( \frac{3V_i}{4\pi} \right)^{1/3} \) in 3D), \( c_i \) is the maximum signal velocity in the cell (e.g. the sound speed for the Euler equations or the fast magnetosonic wave speed for the MHD equations), \( |v_i'| \) is the velocity of the gas relative to the motion of the grid, and \( C_{\text{CFL}} < 1 \) is the Courant–Friedrichs–Lewy (CFL) coefficient. For all our simulations (DG and FV), we chose \( C_{\text{CFL}} = 0.25 \). FV schemes are linearly stable for \( C_{\text{CFL}} \) up to a value of 1. DG schemes typically require stricter CFL coefficients. For example, the class of nodal Runge–Kutta DG methods with representation of the solution up to an order \( p \) requires \( C_{\text{CFL}} \leq \frac{1}{2p+1} \) (Cockburn & Shu 1989, 2001), meaning that \( C_{\text{CFL}} \leq \frac{1}{2} \) for a second-order scheme. Similar linear analysis for central DG methods shows that \( C_{\text{CFL}} \) is allowed to be greater than regular DG methods (Liu et al. 2008). Therefore, it is worth noting that FV schemes can be run stably with larger time steps than DG schemes, although larger time steps will increase errors. For the simulations we present, we used a global time step determined by the smallest allowed time step for all the cells at a given time. Our DG method is also compatible with the hierarchical time-stepping scheme of Springel (2010).

2.6 Slope limiter

For the static and moving FV method and the static DG method, we use the slope limiter designed for unstructured grids of Barth & Jespersen (1989), which is also used by AREPO (Springel 2010). This limiter requires that the linearly reconstructed quantities on face centroids do not exceed the maxima or minima among all neighbouring cells. Each gradient is replaced with a slope-limited gradient:

\[
\langle \nabla \phi \rangle_i = \alpha_i \langle \nabla \phi \rangle_i,
\]

where the slope limiter coefficient \( 0 \leq \alpha_i \leq 1 \) is computed as

\[
\alpha_i = \min\left(1, \psi_{ij} \right)
\]

\[
\psi_{ij} = \begin{cases}
\frac{\left( \phi_{ij}^{\text{max}} - \phi_i \right)}{\Delta \phi_{ij}} & \text{for } \Delta \phi_{ij} > 0 \\
\frac{\left( \phi_i - \phi_{ij}^{\text{min}} \right)}{\Delta \phi_{ij}} & \text{for } \Delta \phi_{ij} < 0 \\
1 & \text{for } \Delta \phi_{ij} = 0
\end{cases}
\]

Here, \( \Delta \phi_{ij} = \langle \nabla \phi \rangle_i \cdot (f_{ij} - s_i) \) is the estimated change between the centroid \( f_{ij} \) of the face and the centre of cell \( i \), and \( \phi_{ij}^{\text{max}} = \max(\phi_j) \) and \( \phi_{ij}^{\text{min}} = \min(\phi_j) \) are the maximum and minimum values occurring for \( \phi \) among all neighbouring cells of cell \( i \), including \( i \) itself.

We find that the DG method on a moving mesh is quite sensitive to the choice of slope limiter and the above form results in excessive post-shock oscillations in certain test problems, likely due to the fact that it is not total variation diminishing (TVD) and that it also limits local smooth extrema. There are simple modifications that can be made to the presented slope limiter to make it TVD [see, for example, Duffell & MacFadyen (2011)]; however, such modifications will still unnecessarily limit local smooth extrema. Identifying a robust limiter which is not too dissipative is still an open problem for the moving DG method. In what follows, we adopt a limiter which falls into the class of WENO methods (Luo, Baum & Löhner 2007). We find this limiter to be robust for all of our tests and use it as the best, default choice for all test problems. In future work, we plan to explore a wide range of limiters that are improvements upon this basic limiter. Scale-free, problem-independent, memory-efficient limiters that do not indiscriminately clip local extrema have, for example, been developed by Balsara et al. (2007) for DG methods.

For each cell \( i \), we consider two candidate values for the slope of a primitive fluid variable \( \phi \). One candidate is the slope-limited gradient obtained with a stencil, as in the FV method (in the case of the magnetic field gradient, we project it to a divergence-free space). The other candidate is the unlimited DG local gradient. For each candidate \( k \), we compute an oscillation factor:

\[
o_{ik} = \sum_j |\Delta \phi_{ij}| + |\phi_j - (\phi_i + \Delta \phi_{ij})|
\]

The slope is then a weighted sum of the candidates:

\[
\langle \nabla \phi \rangle_i = \sum_k \left( \frac{\phi^k - \phi_i}{\epsilon + \gamma n_k} \right)
\]

where \( \epsilon \) is machine epsilon and \( \gamma = 0.5 \). The slope limiting occurs at the beginning of each time step, before any prediction steps or updating take place. Following the limiting step, the quantities in equation (25) are also recalculated.

In all of our numerical tests, we use the slope limiter of Springel (2010) for the static and moving FV results and the static DG
results. However, we use the modified WENO-type limiter as the default choice for the moving DG results because we find it is much more robust and reduces non-physical oscillations to a minimum. In a couple of our numerical tests with the moving DG method, we present the results of the original limiting scheme (labelled as ‘limiter 2’) for comparison, although we favour the alternate limiter for the moving DG method.

2.7 Magnetic field divergence errors

When solving the Riemann problem across flux interfaces, a constant magnetic field perpendicular to the interface must be assumed. We use the average value of the perpendicular magnetic fields extrapolated from the left and right sides of the interface: \( B_\perp = \frac{1}{2}(B_{\perp, L} + B_{\perp, R}) \). This means that despite having a locally divergence-free representation of the magnetic field inside each cell in the DG formulation, there is still a divergence error estimated by Stokes theorem:

\[
\nabla \cdot B_i = \frac{1}{V_i} \sum_{\text{faces}} B \cdot \mathbf{n}_i,
\]

where \( V_i \) is the volume of cell \( i \), and we sum (over the faces) the outward normal values of the magnetic field multiplied by the area of the face. However, a locally divergence-free representation of the magnetic field is expected to reduce global divergence errors because the contribution to the divergence error from within each cell is exactly zero.

3 RESULTS OF NUMERICAL TESTS

We perform a series of numerical tests documented in the literature to compare the static and moving DG and FV methods. The results of these tests are presented in the following subsections.

3.1 1D acoustic wave

The first test we present is a simple 1D acoustic wave, discussed in Stone et al. (2008) and also described in Springel (2010). This setup serves as a sensitive test of the convergence rate of a code. A simple acoustic wave of unit wavelength is initialized with very small amplitude \( \Delta p/\rho = 10^{-6} \) and \( \rho = 1 \) in a periodic domain of unit length. The gas has pressure \( p = 3/5 \) and adiabatic index \( \gamma = 5/3 \). The L1 error norm is computed when the wave returns to its original position (the analytic solution here is identical to the initial state). Fig. 1 shows the L1 error norms for the moving and static DG and FV methods as a function of mesh resolution. Second-order convergence is achieved, as expected. The moving and static DG algorithms show an ~60 and ~80 per cent reduction of errors over their FV counterparts, respectively.

3.2 1D Sod shock tube

We continue our investigation by simulating a 1D Sod shock tube. We adopt initial conditions commonly employed by others in the literature (Sod 1978; Hawley, Smarr & Wilson 1984a,b; Stone & Norman 1992; Stone et al. 2008). The left-hand side (\( x < 0 \)) is described by \( p_L = 1, \rho_L = 1 \) and \( v_L = 0 \), the right-hand side (\( x \geq 0 \)) is described by \( p_R = 0.1, \rho_R = 0.125 \) and \( v_R = 0 \), and the gas as adiabatic index \( \gamma = 1.4 \). We evolve the system until \( t = 5.0 \). The solutions in the moving FV and DG formulations are shown in Fig. 2.

3.3 Gresho vortex

We move on to a 2D test for the conservation of vorticity and angular momentum. The problem proposed by Gresho & Chan (1990) considers a static ‘triangle vortex’. We adopt the initial conditions...
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Figure 2. Solution of the 1D shock tube test at $t = 5.0$ (resolution 64) with the moving FV method (left) and the moving DG method (right). The moving DG method reduces post-shock oscillations noticeably (especially in the velocity) while also being less diffusive.

Figure 3. Convergence of the 1D shock tube test in the L1 norm. First-order convergence is achieved, as expected due to the discontinuity in the solution. The DG method on a moving mesh produces the smallest errors, an $\sim 30$ per cent reduction over the moving FV method.

The gas has constant density $\rho = 1$ and adiabatic index $\gamma = 5/3$.

The pressure profile

$$p(r) = \begin{cases} 
5 + \frac{25}{2}r^2 & \text{for } 0 \leq r < 0.2 \\
9 + \frac{25}{2}r^2 - 20r + 4 \ln(r/0.2) & \text{for } 0.2 \leq r < 0.4 \\
3 + 4 \ln 2 & \text{for } r \geq 0.4
\end{cases}$$

balances the centrifugal force with the pressure gradient so that the vortex is a steady-state solution.

Developing a scheme that minimizes angular momentum diffusion for grid-based methods is important because lack of angular

v_\phi(r) = \begin{cases} 
5r & \text{for } 0 \leq r < 0.2 \\
2 - 5r & \text{for } 0.2 \leq r < 0.4 \\
0 & \text{for } r \geq 0.4
\end{cases} \quad (33)
DG methods exhibit reduced levels of angular momentum diffusion owing to treating the gradient of a cell in a completely local manner. Here, the static DG method shows an advantage over the moving DG method due to the differences in their slope limiters. Static DG, static FV and moving FV all have the same slope limiter, while for the moving DG method, we find that it is generally better to take a weighted average of the local slope of a cell and the one obtained with a stencil to prevent spurious oscillations.

Momentum conservation is one of the main disadvantages of grid-based methods relative to SPH, which conserves total angular momentum due to its pseudo-Lagrangian character (Price 2012). [We note, however, that this ‘advantage’ of SPH comes at the expense of inaccurate handling of the mass continuity equation, as discussed by Vogelsberger et al. (2012).] The solutions of the Gresho vortex problem are shown in Fig. 5 and an error convergence plot is presented in Fig. 6. The static DG approach performs significantly better than the static FV method, likely owing to the purely local manner in which it handles gradients, and the moving DG approach offers a small improvement over the moving FV method here. In this test, the static DG method shows an advantage over the moving DG scheme, attributable to the differences in their slope limiters. We used the same slope limiter for the static DG, static FV and moving FV methods, while for the moving DG approach, we find that it is generally better to take a weighted average of the local slope of a cell and the one obtained with a stencil to prevent spurious oscillations. This weighting step prevents the gradient from being treated purely locally, since we use a stencil-estimated slope in the stencil. Further refinement of the slope limiter could lead to improvements in the moving DG scheme to the level of the static DG method in this test.

We also verified with additional tests that the static DG approach maintains its strong advantage over the static FV approach on arbitrary meshes. The regularity of a Cartesian grid is not a necessary requirement for the DG method to work well.

### 3.4 2D implosion

Next, we perform a 2D implosion test (Hui, Li & Li 1999) with periodic boundary conditions, as in Sijacki et al. (2012). The domain is a box of side length 0.3. The initial pressure and density are $p = 1.0, \rho = 1.0 \text{ for } x + y > 0.15 \text{ and } p = 0.14, \rho = 0.125 \text{ otherwise.}$ The gas is initially at rest and has adiabatic index $\gamma = 1.4$. This test is well suited for studying interacting shocks, Richtmyer–Meshkov instabilities, diffusivity and ability of codes to maintain a symmetric solution.

The development of the implosion is presented in Fig. 7 at several resolutions. We see that the DG method produces less diffusive results than the FV method. In this particular example, the original limiter adopted by Springel (2010) works well for the moving DG method (labelled as ‘limiter 2’ in the figure), producing sharp-shock interfaces, and so we show the results of both limiters for the moving DG method. A point of interest to examine is the low-density region that develops in the bottom-left corner of the simulations. The further along diagonally (towards the centre) the structure has developed, the less numerical diffusion is present. In the cases of static DG and moving DG with limiter 2, the region obtained at a resolution of $128^2$ resembles more closely the solution obtained by the FV approach at twice the resolution $256^2$ rather than at the same resolution $128^2$. This suggests that a purely local treatment of derivatives (no stencils) as in the DG method allows for a better treatment of fluid instabilities and increases the effective resolution of the simulation. The moving FV and moving DG results with the WENO-type limiter are fairly similar, but the DG solution shows that the low-density feature has advanced further diagonally, indicating less numerical diffusion. Note that there are some asymmetries that develop in the moving mesh approach owing to the fact that in our implementation the fluxes across interfaces are added in an arbitrary order and slight differences can arise from finite-precision arithmetic which can be amplified by the additional degree of freedom of the motion of the mesh (we call this effect ‘mesh noise’). The appearance and the magnitude of these asymmetries are sensitive to the mesh regularization parameters, and one could obtain more symmetric results with careful fine-tuning of the regularization parameters.

### 3.5 Kelvin–Helmholtz instability

In the next test, we consider shear flow in 2D which produces Kelvin–Helmholz (KH) instabilities. The initial conditions are those of Springel (2010). In a periodic box of side length 1.0, gas is set up to have uniform pressure $p = 2.5$ and adiabatic index $\gamma = 5.3$. The density is stratified vertically and has value $\rho = 2$ in the central
Figure 7. Implosion test ($t = 1.0$, periodic boundary conditions) plots of density for moving and static FV and DG methods at resolutions of $64^2$, $128^2$ and $256^2$. We have also included results of the moving DG method with the same limiters as static FV, moving FV and static DG (limiter 2), which shows the best results in this test, although in general, we prefer the modified limiter. The presence of asymmetry in the moving mesh simulations is due to mesh noise. The static DG method and the moving DG method (limiter 2) resolve features of the instability that occur in the low-density region in the lower-left-hand corner which their FV counterparts resolve only at twice the resolution.
Figure 8. KH test ($t = 2.0$, plots of density) for moving and static FV and DG methods at resolution $512^2$. As in Fig. 7, we included the results of the moving DG method with the two different limiters. Variations in colour reflect the local fluid density, as indicated by the colour bars to the right of each frame.

The static DG method is better than the static FV method at resolving secondary KH instabilities that develop over the primary KH billows, which the moving mesh approaches resolve. The moving FV and moving DG methods develop qualitatively similar structures, with small-scale structures being well preserved rather than mixed. The DG method that uses the original limiter (‘limiter 2’) resolves small features but also exhibits more diffusive mixing.

**Equation (35)**

$$v_y(x, y) = w_0 \sin(4\pi x) \left( \exp \left[ -\frac{(y-0.25)^2}{2\sigma^2} \right] + \exp \left[ -\frac{(y-0.75)^2}{2\sigma^2} \right] \right)$$

with $w_0 = 0.1$ and $\sigma = 0.5/\sqrt{2}$ in order to excite a single mode of the instability with wavelength equal to half the box size. The results of the various solvers are shown in Fig. 8. The static DG method is better than the static FV method at resolving secondary KH instabilities that develop over the primary KH billows, which the moving mesh approaches resolve. The moving FV and moving DG methods develop qualitatively similar structures, with small-scale structures being well preserved rather than mixed. The DG method that uses the original limiter (‘limiter 2’) resolves small features but also exhibits more diffusive mixing.
3.6 3D-driven turbulence

For a 3D test, we consider isothermal gas in a periodic box being turbulently driven by external stochastic forcing on large scales, as examined by Bauer & Springel (2012). We use the driving routine and parameters for Mach number $M \sim 0.3$ (subsonic) turbulence listed in table 1 of Bauer & Springel (2012). We are interested in finding how well the static and moving DG methods can reproduce a Kolmogorov-like velocity power spectrum ($P(k) \propto k^{-5/3}$). In Fig. 9, we present plots of the velocity magnitudes with our various methods at $t = 25.6$, computed at a resolution of $128^3$. In Fig. 10, the accompanying velocity power spectra are presented. The results agree with the expectations for a Kolmogorov cascade on the largest spatial scales. The DG method does an improved job of resolving the power to smaller spatial scales with the same number of cells. The improvement of using the DG method over an FV method is greater than the improvement of using a moving mesh over a static one.

We also compare the FV and DG method for supersonic ($M \sim 3.0$) turbulence. Supersonic turbulence consists of a network of shocks described by Burgers and not Kolmogorov theory. The network of shocks means that many cells are slope limited in the time evolution, which reduces the advantage of the DG method over the FV method, since the method is only accurate to first order. We find that this is the case: the DG and FV methods produce very similar power spectra (DG’s power law is only very slightly extended). We plot the magnitudes of the velocity in Fig. 11 and the accompanying velocity power spectra in Fig. 12. The power spectra for supersonic turbulence are slightly steeper than the $k^{-5/3}$ power law expected from Kolmogorov theory (a bump in the power spectrum is also seen near the resolution limit of the mesh due to the so-called bottle-neck effect in turbulence studies on mesh-based methods). The DG method of the same number of cells shows the power law extended to just slightly smaller spatial scales than the corresponding FV method.
Figure 11. Plots of a slice of the velocity magnitudes $|v|$ in supersonic turbulently driven isothermal gas for static and moving FV and DG methods in a 3D periodic box (resolution $128^3$).

Figure 12. Velocity power spectra for the supersonic turbulently driven isothermal gas computed with the static and moving FV and DG methods. The second-order FV and DG methods show comparable results due to the slope limiting that takes place in the network of shocks.

3.7 Magnetic rotor

We now move on to testing the MHD part of the code. First, we consider the magnetic rotor test (Balsara & Spicer 1999b; Tóth 2000). The setup of this problem is as follows. A dense rotating disc of fluid is tapered off into the ambient fluid, which is at rest. The computational domain is a periodic box of side length 1. The adiabatic index of the gas is $\gamma = 5/3$. The initial conditions are given by $p = 0.5$, $B_x = 2.5/\sqrt{4\pi}$ and $B_y = 0$,

\[
\rho = \begin{cases} 
10 & \text{if } r \leq r_0 \\
1 + f & \text{if } r_0 < r \leq r_1 \\
1 & \text{if } r > r_1
\end{cases}
\]  

(36)

\[
v_x = \begin{cases} 
-(y - 0.5)/r_0 & \text{if } r \leq r_0 \\
-f(y - 0.5)/r & \text{if } r_0 < r \leq r_1 \\
0 & \text{if } r > r_1
\end{cases}
\]  

(37)

\[
v_y = \begin{cases} 
(x - 0.5)/r_0 & \text{if } r \leq r_0 \\
f(x - 0.5)/r & \text{if } r_0 < r \leq r_1 \\
0 & \text{if } r > r_1
\end{cases}
\]  

(38)

where $r_0 = 0.1$, $r_1 = 0.115$, $f = (r_1 - r)/(r_1 - r_0)$ and $r^2 = (x - 0.5)^2 + (y - 0.5)^2$. In the rotor problem, centrifugal forces are not balanced, so in the evolution the magnetic field confines the rotating dense fluid into an oblate rotor shape.

The rotor problem can be a sensitive test for unphysical features that occur if the global divergence of the magnetic field is not sufficiently well constrained. Balsara & Spicer (1999b), Tóth (2000) and Li & Shu (2005) find that the Mach number $M = |v|/c$, where
Figure 13. Contour plots of Mach number in magnetic rotor at $t = 0.295$ (resolution $512^2$) shown for static and moving FV methods with Powell cleaning, moving locally divergence-free DG method with Powell cleaning and static locally divergence-free DG method without cleaning. The simulations do not exhibit numerical artefacts typical of some MHD solvers that employ cleaning schemes. The static locally divergence-free DG method does not require Powell cleaning. Owing to our current choice of limiter for the moving DG approach, which takes a weighted average of local and stencil-determined gradients, we require Powell cleaning in this case.

$c = \sqrt{\gamma p/\rho}$ is the sound speed, can suffer serious unphysical distortions around the central rotating area.

We show a zoom-in of the Mach number at the centre of the rotor in Fig. 13, evolved with several of our schemes. No obvious unphysical artefacts are present in any of the simulations. The static mesh results show better resolved features with some finer structures. This is due to the fact that in the moving mesh simulations, the mesh generating points move with the flow and there is actually a below-average density of mesh generating points in the centre of the rotor, reducing the effective resolution in the zoom-in portion of the figure. In principle, this could be overcome by allowing local refinement of the mesh. In Fig. 14, we show the global divergence errors of the magnetic field, and the divergence errors in each cell are presented in Fig. 15. Even though the divergence errors did not have a drastic impact on the solution, methods that minimize them provide greater stability for solving arbitrary MHD problems. In the figure, we see that the DG methods more tightly constrain the global divergence errors. In particular, the static DG method is the most successful one at reducing the global divergence errors and, in fact, does not require a cleaning scheme. It is followed by the moving DG method (which currently does require a Powell cleaning algorithm owing to the choice of limiter). The moving FV Powell method is third best, followed by the static Powell approach, which

Figure 14. Analysis of the global divergence of the magnetic field for the four methods presented in Fig. 13 for the magnetic rotor problem. The DG methods demonstrate a better handling of the global divergence errors, owing to their locally divergence-free formulation.
Figure 15. Divergence errors in the magnetic field for magnetic rotor test corresponding to the plots in Fig. 13.

is considerably worse at constraining divergence errors compared to the static DG method which has no cleaning applied.

3.8 Orszag–Tang vortex

As a final test, we consider the Orszag–Tang vortex (Orszag & Tang 1979), which is an excellent test of supersonic MHD turbulence. We use the initial conditions as described by Picone & Dahlburg (1991):

\[ \rho = \frac{\gamma^2}{4\pi} \]
\[ p = \frac{\gamma}{4\pi} \]
\[ \mathbf{v} = (-\sin(2\pi y), \sin(2\pi x)) \]
\[ \mathbf{B} = (-\sin(2\pi y), \sin(4\pi x)) \]

The domain is a box of side length 1 with periodic boundaries. The gas has adiabatic index \( \gamma = 5/3 \). We show the results of the simulations (density distribution and local cell \( B \)-field divergence errors) in Figs 16 and 17. The static DG method (which uses no Powell cleaning) maintains minimal divergence errors which plateau quickly. This is followed by the moving DG method with Powell cleaning and thirdly the FV moving method with Powell cleaning.

We also ran a static FV simulation with the Powell cleaning scheme (not shown), which became unstable due to the growth of large magnetic field divergence errors which corrupted the solution.

4 SUMMARY OF COMPARISON TO THE FINITE VOLUME APPROACH

On a static mesh, the DG formulation has clear advantages over the FV formulation. Our tests show a significant reduction of errors and angular momentum diffusion, and increased effective resolution which better characterizes small-scale fluid instabilities and recovers a Kolmogorov-like power law for turbulent cascade to smaller scales. In addition, the locally divergence-free representation of the magnetic field allows the MHD equations to be solved in a robust and stable manner without large global divergence errors. No magnetic field cleaning scheme is required in this case.

The DG method on a moving mesh also shows improvement over its FV counterpart in every test we performed. Numerical errors, angular momentum diffusion and post-shock oscillations are reduced, and the capability of resolving turbulence on small scales is enhanced. The advantages of DG over FV in our moving mesh formulation are not always as great as for the static mesh case because we had to employ a modified slope limiter to prevent unphysical oscillations. We suspect that refined limiters would allow us to more fully exploit the advantages of DG over FV, which we leave for future work.
4.1 Memory consumption and CPU time

The DG method does require greater memory usage and more CPU time than the FV approach. In our own implementation, the memory allocated to store local cell variables (fluid variables, fluid variable gradients, volumes and moments) is increased by 40 per cent for 3D simulations (for both the static and moving cases) due to the fact that in the DG method we require the second-order moments of cells in addition to their volumes and we also store moment-averaged derivative quantities $\mathbf{R}$ in addition to primitive gradients. However, this results in only a small net increase in total memory consumption in our implementation ($<10$ per cent). A significant portion of the total memory is dedicated to storing mesh connectivity information, which is the same for the DG and FV formulations.

The CPU usage is increased by approximately 25 per cent for static 3D DG runs and 35 per cent for moving mesh 3D runs. This
is due to the fact that in the DG method, we are required to perform additional steps, specifically reading gradient information in the input, writing gradient information at every snapshot, calculating second-order cell moments with Gaussian quadrature in addition to cell volumes, calculating flux update terms for the moment-averaged gradient quantities at every time step and converting these gradients to primitive gradients. For astrophysical applications that require self-gravity, the actual penalty in CPU consumption of the DG approach relative to the FV method will be significantly less than this.

5 STRENGTHS OF THE DG METHOD IN ASTROPHYSICAL CONTEXTS

The second-order DG method developed here shows improvements in accuracy over the second-order FV MUSCL–Hancock approach without significant increase in computation time or memory consumption. The procedure is readily compatible with hierarchical time stepping and mesh refinement. The performance of our DG method on simple test problems suggests that it would improve simulations of cosmological structure formation such as those performed with AREPO (e.g. Kereš et al. 2012; Torrey et al. 2012; Nelson et al. 2013; Vogelsberger et al. 2013).

The locally divergence-free representation of the magnetic field in the DG method reduces global divergence errors in $\nabla \cdot B$. It is desirable to use this DG representation of the solution in cases where CT is not applicable. Such is the case for a moving mesh, where it is presently unclear whether the CT approach can be adapted to an evolving unstructured mesh. The locally divergence-free DG technique improves the current FV Powell scheme used in AREPO to solve the MHD equations (Pakmor & Springel 2013). The method would be useful in studying, for example, the potentially important role magnetic fields play in accretion processes and explosions (e.g. Duffell & MacFadyen 2011, 2012, 2013), where the moving mesh formulation is needed to minimize large advection errors from bulk flows (Genel et al. 2013).

The DG method can be generalized to provide higher order accuracy on arbitrary meshes and does not adversely affect the parallelizability of current fluid solvers because cells continue to communicate with only their nearest neighbours. In this case, a higher order time-stepping scheme, such as Runge–Kutta, would be preferable to maintain the same order of accuracy in both the space and time domain.

Finally, we note that the DG method that we have implemented shows significant reduction of errors and angular momentum diffusion compared to the FV method. It could play an important role in improving the current generation of AMR codes.

6 IMPROVING THE SLOPE LIMITER IN FUTURE WORK

We find that our moving DG formulation is sensitive to the choice of slope limiter and we cannot use the same slope limiter as we do for the static DG and moving and static FV method described in Springel (2010). In addition to capturing shock discontinuities, this limiter identifies and limits smooth extrema and produces unphysical oscillations in the solution. Our proposed alternate limiter, designed as a very simple WENO-type limiter, works well and provides stable results. However, some of the advantages gained by DG over FV are not as great as in the static mesh case owing to our choice of limiter, and so an investigation for more refined limiters is clearly a priority for future efforts. WENO- and Hermite WENO-type approaches (Luo et al. 2007) seem to be a promising avenue to explore. These limiters replace solution polynomials with reconstructed polynomials of the same order of accuracy which also preserve cell averages (hence, are fully conservative). Balsara et al. (2007) have also defined a subcell-based indicator for identifying troubled cells to be limited that does not indiscriminately clip extrema (which would reduce accuracy to first order). Such a method distinguishes between smooth substrustructure and shocks at cell interfaces and may be combined with our DG method.

7 CONCLUDING REMARKS

We have presented a new numerical procedure for solving the fluid and MHD equations on moving and static meshes based on the DG method. The technique is an attractive and competitive alternative to the predominant FV approaches used in astrophysics. The DG scheme we developed, which is based on the centroidal Taylor basis expansion in each cell, is in fact a generalization of the FV method, where local gradients (and higher order derivatives in general) are evolved just like fluid variables, instead of using a stencil to estimate their values. In this way, gradients are purely local. The second-order DG procedure we developed does not significantly increase the runtime of the simulations compared to the FV approach. The DG method can also be readily extended to higher order while keeping interelement communications minimal (elements only communicate with adjacent elements with a common face), unlike FV schemes. This allows for higher order DG codes to be highly parallelizable. In addition, the DG formulation is well suited for unstructured meshes and mesh-refinement strategies, since the method is compact (the representation of the solution on each element is independent).

On static meshes, second-order DG techniques demonstrate superior accuracy over the same-order FV method. Particularly striking is the reduction of angular momentum diffusion. As a result, DG schemes could reduce the disadvantage that Eulerian codes have compared to SPH (which conserves angular momentum, but has
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references

balsara d. s., 2001, j. comput. phys., 174, 614
balsara d. s., 2004, apjs, 151, 149
balsara d. s., spicer d., 1999a, j. comput. phys., 148, 133
balsara d. s., spicer d. s., 1999b, j. comput. phys., 149, 270
balsara d. s., altmann c., munz c.-d., dumbsber m., 2007, j. comput. phys., 226, 586
balsara d. s., rampf t., dumbsber m., munz c.-d., 2009, j. comput. phys., 228, 2480

barth t. j., jespersen d. c., 1989, aaaa 89-0366, the design and application of upwind schemes on unstructured meshes. aaaa, reston

bassi f., rebay s., 1997a, j. comput. phys., 131, 267
bassi f., rebay s., 1997b, j. comput. phys., 138, 251
bauer a., springel v., 2012, mnras, 423, 2558
cockburn b., shu c.-w., 1989, math. comput., 52, 411
cockburn b., shu c.-w., 2001, j. sci. comput., 16, 173
cockburn b., li f., shu c.-w., 2004, j. comput. phys., 194, 588

dedner a., kemm f., kröner d., munz c.-d., schnitzer t., wesenberg m., 2002, j. comput. phys., 175, 645
duffell p. c., macfadyen a. i., 2011, apjs, 197, 15
duffell p. c., macfadyen a. i., 2012, apj, 755, 7

duffell p. c., macfadyen a. i., 2013, apj, 775, 87

evans c. r., hawley j. f., 1988, apj, 332, 659
fromang s., hennebelle p., teyssier r., 2006, a&a, 457, 371
fryxell b. et al., 2000, apjs, 131, 273

genel s., vogelsberger m., nelson d., sijacki d., springel v., hernquist l., 2013, mnras, 435, 1426

ghostine r., kesserwani g., mos r., vazquez j., ghenaia m., 2009, int. j. numer. methods fluids, 59, 423
gresho p. m., chan s. t., 1990, int. j. numer. methods fluids, 11, 621

hawley j. f., smarr l. l., wilson j. r., 1984a, apjs, 55, 211

hawley j. f., smarr l. l., wilson j. r., 1984b, apj, 277, 296

hoite h., ackerer p., mos r., erhel j., philippe b., 2004, int. j. numer. methods eng., 61, 2566

hui w. h., li p. y., li z. w., 1999, j. comput. phys., 153, 596

keres d., vogelsberger m., sijacki d., springel v., hernquist l., 2012, mnras, 425, 2027

li f., shu c.-w., 2005, j. sci. comput., 22-23, 413

lika r., wendroff b., 2003, sim j. sci. comput., 25, 995

lius y., shu c.-w., tadmor e., zhang m., 2008, esaaim: math. modelling numer. anal., 42, 593

luo h., baum j. d., löhrer r., 2007, j. comput. phys., 225, 686

luo h., baum j. d., löhrer r., 2008, j. comput. phys., 227, 8875

miñón f., martin d. f., 2011, apjs, 195, 5

miyoshi t., kusano k., 2005, j. comput. phys., 208, 315

muñoz d. j., springel v., marcus r., vogelsberger m., hernquist l., 2013, mnras, 428, 254

genel d., vogelsberger m., genel s., sijacki d., kereš d., springel v., hernquist l., 2012, mnras, 425, 2027

li f., shu c.-w., 2005, j. sci. comput., 22-23, 413

liska r., wendroff b., 2003, sim j. sci. comput., 25, 995

liu y., shu c.-w., tadmor e., zhang m., 2008, esaaim: math. modelling numer. anal., 42, 593

luo h., baum j. d., löhrer r., 2007, j. comput. phys., 225, 686

luo h., baum j. d., löhrer r., 2008, j. comput. phys., 227, 8875

miñón f., martin d. f., 2011, apjs, 195, 5

miyoshi t., kusano k., 2005, j. comput. phys., 208, 315

muñoz d. j., springel v., marcus r., vogelsberger m., hernquist l., 2013, mnras, 428, 254

nelson d., vogelsberger m., genel s., sijacki d., kereš d., springel v., hernquist l., 2013, mnras, 429, 3353

o’shea b. w., bryan g., bordner j., norman m. l., abel t., harkness r., kritsuk a., 2004, preprint (astro-ph/0403044)
o’razs a. a., tang c.-m., 1979, j. fluid mech., 90, 129

pakmor r., springel v., 2013, mnras, 432, 176

pakmor r., bauer a., springel v., 2011, mnras, 418, 1392

picone j. m., dahllburg r. b., 1991, phys. fluids b, 3, 29

powell k. g., roe p. l., linde t. j., gombosi t. i., de zeeuw d. l., 1999, j. comput. phys., 154, 284

price d. j., 2012, j. comput. phys., 231, 759

sijacki d., vogelsberger m., kereš d., springel v., hernquist l., 2012, mnras, 424, 2999

sod g. a., 1978, j. comput. phys., 27, 1

springel v., 2010, mnras, 401, 791

stone j. m., norman m. l., 1992, apjs, 80, 753

stone j. m., gardiner t. a., teuben p., hawley j. f., simon j. b., 2008, apjs, 178, 137
