FREE GROUP $C^*$-ALGEBRAS ASSOCIATED WITH $\ell_p$

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Abstract. For every $p \geq 2$, we give a characterization of positive definite functions on a free group with finitely many generators, which can be extended to the positive linear functionals on the free group $C^*$-algebra associated with the ideal $\ell_p$. This is a generalization of Haagerup’s characterization for the case of the reduced free group $C^*$-algebra. As a consequence, the associated $C^*$-algebras are mutually non-isomorphic, and they have a unique tracial state.

1. Introduction

N. P. Brown and E. Guentner introduce new $C^*$-completion of the group ring of a countable discrete group $\Gamma$ in [2]. More precisely, for a given algebraic two-sided ideal in $\ell_\infty(\Gamma)$, they define the associated group $C^*$-algebra. These recover the full group $C^*$-algebra for $\ell_\infty(\Gamma)$ itself, and the reduced group $C^*$-algebra for $c_0(\Gamma)$, respectively. Hence if we take $c_0(\Gamma)$ or $\ell_p(\Gamma)$ with $p \in [1, \infty)$ for example, we may obtain new group $C^*$-algebra. We remark that a standard characterization of amenability implies that the associated $C^*$-algebras of an amenable group are all isomorphic for any ideals. In [2], they also give a characterization of the Haagerup property and Property (T) in terms of ideal completions.

In this paper, we study their $C^*$-algebra of a free group associated with $\ell_p$. By [2], for any $p \in [1, 2]$, the group $C^*$-algebra of $\Gamma$ associated with $\ell_p$ is isomorphic to the reduced group $C^*$-algebra. Therefore the case where $p \in (2, \infty)$ is essential. Our main result is a characterization of positive definite functions on a free group, which can be extended to the positive linear functionals on the free group $C^*$-algebra associated with $\ell_p$. In [3], U. Haagerup gives its characterization for the case of the reduced free group $C^*$-algebra. Thus our theorem is a generalization of his result. As a consequence, the free group $C^*$-algebras associated with $\ell_p$ are mutually non-isomorphic for every $p \in [2, \infty]$. We also obtain that for each $p \in [2, \infty)$, the free group $C^*$-algebra associated with $\ell_p$ has a unique tracial state. Moreover, we consider algebraic ideals

$$D_-^p(\Gamma) = \bigcup_{\varepsilon > 0} \ell_{p-\varepsilon}(\Gamma)$$

for $1 < p \leq \infty$, and

$$D_+^p(\Gamma) = \bigcap_{\varepsilon > 0} \ell_{p+\varepsilon}(\Gamma)$$

for $1 \leq p < \infty$. Then, by using our characterization, it is also shown that the free group $C^*$-algebra associated with $D_-^p$ coincides with the free group $C^*$-algebra associated with $\ell_p$.

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2. Preliminaries

In this section, we fix the notations for the convenience of the reader and recall some results in [2].

Let $\Gamma$ be a countable discrete group and $\pi$ be a unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, we denote the matrix coefficient of $\pi$ by

$$\pi_{\xi, \eta}(s) = \langle \pi(s)\xi, \eta \rangle.$$ 

Note that $\pi_{\xi, \eta} \in \ell_\infty(\Gamma)$, where $\ell_\infty(\Gamma)$ is the abelian $C^*$-algebra of all bounded functions on $\Gamma$.

Let $D$ be an algebraic two-sided ideal of $\ell_\infty(\Gamma)$. If there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\pi_{\xi, \eta} \in D$ for all $\xi, \eta \in \mathcal{H}_0$, then $\pi$ is called $D$-representation. If $D$ is invariant under the left and right translation of $\Gamma$ on $\ell_\infty(\Gamma)$, then it is said to be translation invariant.

Throughout this paper, we assume that $D$ is a non-zero translation invariant ideal of $\ell_\infty(\Gamma)$. For each $p \in [1, \infty)$, we denote the norm on $\ell_p(\Gamma)$ by

$$|f|_p = \left( \sum_{s \in \Gamma} |f(s)|^p \right)^{\frac{1}{p}} \text{ for } f \in \ell_p(\Gamma).$$

Note that $\ell_p(\Gamma)$ is a translation invariant ideal of $\ell_\infty(\Gamma)$. We denote by $c_0(\Gamma)$ the of functions on $\Gamma$, vanishing at infinity. It is the non-trivial closed translation invariant ideal of $\ell_\infty(\Gamma)$.

Under our assumption, $D$ contains $c_0(\Gamma)$, which is the ideal of all finitely supported functions on $\Gamma$. Moreover, if $\pi$ has a cyclic vector $\xi$ such that $\pi_{\xi, \xi} \in D$, then $\pi$ is a $D$-representation with respect to a dense subspace

$$\mathcal{H}_0 = \text{span}\{\pi(s)\xi : s \in \Gamma\}.$$ 

We denote by $\lambda$ the left regular representation of $\Gamma$. It is easy to see that $\lambda$ is a $c_0$-representation, or a $D$-representation for any $D$.

The $C^*$-algebra $C_D^*(\Gamma)$ is the $C^*$-completion of the group ring $\mathbb{C}\Gamma$ by $\| \cdot \|_D$, where

$$\|f\|_D = \sup\{\|\pi(f)\| : \pi \text{ is a } D\text{-representation}\} \text{ for } f \in \mathbb{C}\Gamma.$$ 

Note that if $D_1$ and $D_2$ are ideals of $\ell_\infty(\Gamma)$ with $D_1 \supseteq D_2$, then there exists the canonical quotient map from $C_{D_1}^*(\Gamma)$ onto $C_{D_2}^*(\Gamma)$. We denote by $C^*(\Gamma)$ the full group $C^*$-algebra, and by $C^*_\lambda(\Gamma)$ the reduced group $C^*$-algebra, respectively. In [2], the following results are obtained:

- $C^*(\Gamma) = C^*_{\ell_\infty}(\Gamma)$ and $C^*_\lambda(\Gamma) = C^*_{c_0}(\Gamma)$.
- $C^*_{\ell_p}(\Gamma) = C^*_\lambda(\Gamma)$ for every $p \in [1, 2]$.
- $C^*(\Gamma) = C^*_D(\Gamma)$ if and only if there exists a sequence $(h_n)$ of positive definite functions in $D$ such that $h_n \to 1$.
- If $C^*(\Gamma) = C^*_{\ell_p}(\Gamma)$ for some $p \in [1, \infty)$, then $\Gamma$ is amenable.
- $\Gamma$ has the Haagerup property if and only if $C^*(\Gamma) = C^*_{c_0}(\Gamma)$. 

3. Positive definite functions on a free group

Let $F_d$ be the free group on finitely many generators $a_1, \ldots, a_d$ with $d \geq 2$. We denote by $|s|$ the word length of $s \in F_d$ with respect to the canonical generating set $\{a_1, a_1^{-1}, \ldots, a_d, a_d^{-1}\}$. For $k \geq 0$, we put

$$W_k = \{s \in F_d : |s| = k\}.$$

We denote by $\chi_k$ the characteristic function for $W_k$.

In the following lemma, the case where $q = 2$ is given by Haagerup in [5, Lemma 1.3]. His proof also works for $q \in [1, 2]$ by using Hölder’s inequality, instead of Cauchy-Schwarz inequality. We remark that this is also appeared in [1].

**Lemma 3.1.** Let $q \in [1, 2]$. Let $k, \ell$ and $m$ be non negative integers. Let $f$ and $g$ be functions on $F_d$ such that $\text{supp}(f) \subset W_k$ and $\text{supp}(g) \subset W_\ell$, respectively. If $|k - \ell| \leq m \leq k + \ell$ and $k + \ell - m$ is even, then

$$|(f \ast g)\chi_m|_q \leq |f|_q|g|_q,$$

and if $m$ is any other value, then

$$|(f \ast g)\chi_m|_q = 0.$$

**Proof.** It is shown by an argument similar as in [5, Lemma 1.3]. However for convenience, we give the complete proof.

Note that

$$(f \ast g)(s) = \sum_{t,u \in F_d \atop |tu| = s} f(t)g(u) = \sum_{|t| = k \atop |u| = \ell} f(t)g(u).$$

Since the possible values of $|tu|$ are $|k - \ell|, |k - \ell| + 2, \ldots, k + \ell$, we have

$$|(f \ast g)\chi_m|_q = 0$$

for any other values of $m$.

The case where $q = 1$ is trivial. So we consider the case where $q \neq 1$.

First we assume that $m = k + \ell$. In this case, if $|s| = m$, then $s$ can be uniquely written as a product $tu$ with $|t| = k$ and $|u| = \ell$. Hence

$$(f \ast g)(s) = f(t)g(u).$$

Therefore

$$|(f \ast g)\chi_m|_q^q \sum_{|t| = k \atop |u| = \ell} |f(t)|^q|g(u)|^q \leq \sum_{|t| = k \atop |u| = \ell} |f(t)|^q|g(u)|^q = |f|_q^q|g|_q^q.$$  

Next we assume that $m = |k - \ell|, |k - \ell| + 2, \ldots, k + \ell - 2$. In these cases, we have $m = k + \ell - 2j$ for $1 \leq j \leq \min\{k, \ell\}$. Let $s = tu$ with $|s| = m$, $|t| = k$ and $|u| = \ell$. Then $s$ can be uniquely written as a product $t'u'$ such that $t = tv, u = v^{-1}u'$ with $|t'| = k - j, |u'| = \ell - j$ and $|v| = |v^{-1}| = j$. We define

$$f'(t) = \left( \sum_{|v| = j} |f(tv)|^q \right)^{\frac{1}{q}} \text{ if } |t| = k - j, \text{ and } f'(t) = 0 \text{ otherwise}.$$
We also define
\[ g'(u) = \left( \sum_{|v|=j} |g(v^{-1}u)|^q \right)^{\frac{1}{q}} \] if \(|u| = \ell - j\), and \(g'(u) = 0\) otherwise.

Note that \(\text{supp}(f') \subset W_{k-j}\) and \(\text{supp}(g') \subset W_{\ell-j}\). Moreover
\[ |f'|_q^q = \sum_{|t|=k-j} \left( \sum_{|v|=j} |f(tv)|^q \right) = |f|_q^q, \]
and similarly \(|g'|_q = |g|_q\). Take a real number \(p\) with \(1/p + 1/q = 1\). Since \(1 < q \leq 2\), we have \(2 \leq p < \infty\). In particular, \(q \leq p\). Thanks to Hölder’s inequality,
\[ |(f \ast g)(s)| = \left| \sum_{|t|=k \atop |u| = \ell \atop s = tu} f(t)g(u) \right| \]
\[ \leq \left( \sum_{|v|=j} |f(t'v)|^q \right)^{\frac{1}{q}} \left( \sum_{|v|=j} |g(v^{-1}u')|^p \right)^{\frac{1}{p}} \]
\[ \leq \left( \sum_{|v|=j} |f(t'v)|^q \right)^{\frac{1}{q}} \left( \sum_{|v|=j} |g(v^{-1}u')|^q \right)^{\frac{1}{q}} \]
\[ = (f'g')(u'). \]
Hence \(|(f \ast g)\chi_m| \leq (f' \ast g')\chi_m\). Since \((k-j) + (\ell-j) = m\), it follows from the first part of the proof that
\[ |(f \ast g)\chi_m|_q \leq |(f' \ast g')\chi_m|_q \leq |f'|_q |g'|_q = |f|_q |g|_q. \]

□

In the following lemma, the case where \(p = q = 2\) is given in the proof of [4, Theorem 1].

**Lemma 3.2.** Let \(1 \leq q \leq p \leq \infty\) with \(1/p + 1/q = 1\). Let \(\pi\) be a unitary representation of \(\Gamma\) on a Hilbert space \(\mathcal{H}\) with a cyclic vector \(\iota\) such that \(\pi_\xi \in \ell_p(\Gamma)\). Then
\[ \|\pi(f)\| \leq \liminf_{n \to \infty} \left| \left|(f \ast f^{(2n)})^{(2n)} \right|_q \right|^{\frac{1}{2n}} \]
for \(f \in c_c(\Gamma)\).
Therefore it follows that
\[ \|\pi(f)\| = \sup_{g \in c_c(\Gamma)} \lim_{n \to \infty} \left( \sum_{s \in \Gamma} (f^* \ast f)^{(s+2n)}(s)(\pi(s)\pi(g)\xi, \pi(g)\xi) \right)^{\frac{1}{p}}. \]

Fix \( g \in c_c(\Gamma) \) and we put \( \varphi(s) = \langle \pi(s)\pi(g)\xi, \pi(g)\xi \rangle \). Note that
\[ \varphi(s) = \langle \pi(s)\pi(g)\xi, \pi(g)\xi \rangle = \sum_{t,u \in \mathcal{F}_d} \overline{g(u)}\pi_{\xi,\xi}(u^{-1}st)g(t) = (\overline{g} \ast \pi_{\xi,\xi} \ast g^\vee)(s), \]
where \( g^\vee(s) = g(s^{-1}) \). Consequently, \( \pi_{\xi,\xi} \in \ell_p(\Gamma) \) implies \( \varphi \in \ell_p(\Gamma) \). Then by Hölder’s inequality,
\[ \left| \sum_{s \in \Gamma} (f^* \ast f)^{(2n)}(s)\varphi(s) \right| \leq (f^* \ast f)^{(2n)}_q \|\varphi\|_p. \]
Therefore it follows that
\[ \|\pi(f)\| \leq \liminf_{n \to \infty} \left( (f^* \ast f)^{(s+2n)} \right)^{\frac{1}{p}}. \]
□

By combining Lemma 3.1 and Lemma 3.2, we can prove the following.

**Lemma 3.3.** Let \( k \) be a non-negative integer. Let \( 1 \leq q \leq p \leq \infty \) with \( 1/p + 1/q = 1 \). If a unitary representation \( \pi \) of \( \mathcal{F}_d \) on a Hilbert space \( \mathcal{H} \) has a cyclic vector \( \xi \) such that \( \pi_{\xi,\xi} \in \ell_p(\mathcal{F}_d) \), then
\[ \|\pi(f)\| \leq (k + 1)\|f\|_q. \]
for \( f \in c_c(\mathcal{F}_d) \) with \( \text{supp}(f) \subset W_k \).

**Proof.** The case where \( q = 1 \) and \( p = \infty \) is trivial. So we may assume that \( 1 < q \leq 2 \) and \( 2 \leq p < \infty \) with \( 1/p + 1/q = 1 \). It is also shown by an argument similar as in [5, Lemma 1.4].

We consider the norm \( \|(f^* \ast f)^{(s+2n)}\|_q \). Write \( f_{2j-1} = f^* \) and \( f_{2j} = f \) for \( j = 1, 2, \ldots, 2n \). Then
\[ (f^* \ast f)^{(2n)} = f_1 \ast f_2 \ast \cdots \ast f_{4n}. \]
We also denote \( g = f_2 \ast \cdots \ast f_{4n} \). So we have
\[ (f^* \ast f)^{(s+2n)} = f_1 \ast g. \]
Note that \( \text{supp}(f_j) \subset W_k \) for \( j = 1, 2, \ldots, 4n \) and \( g \in c_c(\mathcal{F}_d) \). Put \( g_{\ell} = g\chi_{\ell} \). Then \( \text{supp}(g_{\ell}) \subset W_{\ell} \) and
\[ \|g\|_q = \sum_{\ell=0}^{\infty} \|g_{\ell}\|_{q}. \]
Here, remark that \( \|g_{\ell}\|_q = 0 \) for all but finitely many \( \ell \). Moreover set
\[ h = f_1 \ast g = \sum_{\ell=0}^{\infty} f_1 \ast g_{\ell} \]
and \( h_m = h\chi_m \). Then \( h \in c_c(\mathcal{F}_d) \) and
\[ \|h\|_q = \sum_{m=0}^{\infty} \|h_m\|_{q}. \]
Here, notice that $|h_m|_q = 0$ for all but finitely many $m$. By Lemma 3.1,

$$|(f_1 * g)_m|_q \leq |f_1|_q|g|_q$$

in the case where $|k - \ell| \leq m \leq k + \ell$ and $k + \ell - m$ is even. We also have

$$|(f_1 * g)_m|_q = 0$$

for any other values of $m$. Hence

$$|h_m|_q = \left| \sum_{\ell=0}^{\infty} (f_1 * g)_\ell \right|_q$$

in the case where $k - \ell \leq m \leq k + \ell$ and $k + \ell - m$ is even. We also have

$$|h_m|_q = \left| \sum_{\ell=0}^{\infty} (f_1 * g)_\ell \right|_q$$

for any other values of $m$. Hence

$$|h_m|_q = \left| \sum_{\ell=0}^{\infty} (f_1 * g)_\ell \right|_q$$

By writing $\ell = m + k - 2j$,

$$|h_m|_q \leq |f_1|_q \left| \sum_{j=0}^{\min\{m,k\}} |g_{m+k-2j}|_q \right|$$

By writing $\ell = m + k - 2j$,

$$|h_m|_q = \left| \sum_{\ell=0}^{\min\{m,k\}} |g_{m+k-2j}|_q \right|$$

Then

$$|h|_q^2 = \sum_{m=0}^{\infty} |h_m|_q^2$$

$$\leq (k + 1)^{\frac{2}{p}} |f_1|_q^2 \left( \sum_{m=0}^{\infty} \left| \sum_{j=0}^{\min\{m,k\}} |g_{m+k-2j}|_q \right| \right)$$

$$= (k + 1)^{\frac{2}{p}} |f_1|_q^2 \left( \sum_{j=0}^{\infty} |g_{j+k}|_q \right)$$

Hence $|f_1 * g|_q \leq (k + 1)|f_1|_q|g|_q$, i.e.,

$$|f_1 * (f_2 * \cdots * f_{4n})|_q \leq (k + 1)|f_1|_q|f_2|_q \cdots |f_{4n}|_q.$$
Moreover, we inductively get
\[ |(f^* f)^{(2n)}|_q \leq (k + 1)^{2n-1} |f|_q^{2n}. \]
Therefore it follows from Lemma 3.2 that
\[ \|\pi(f)\| \leq \liminf_{n \to \infty} \left| (f^* f)^{(2n)} \right|_q^{1/n} \leq (k + 1) |f|_q. \]
\[ \square \]

For a function \( \varphi \) on \( \Gamma \), we denote the corresponding linear functional on \( c_c(\Gamma) \) by
\[ \omega_{\varphi}(f) = \sum_{s \in \Gamma} f(s) \varphi(s) \quad \text{for } f \in c_c(\Gamma). \]
Note that \( \varphi \) is positive definite if and only if the functional \( \omega_{\varphi} \) extends to a positive linear functional on \( C^*(\Gamma) \) (see, e.g., [3] Theorem 2.5.11).

Now we can give a characterization of positive definite functions on \( \mathbb{F}_d \), which can be extended to the positive linear functionals on \( C^*_d(\mathbb{F}_d) \) for any \( p \in [2, \infty) \). The case of \( C^*_d(\mathbb{F}_d) \) is given in [5] Theorem 3.1. We remind the reader that \( C^*_d(\mathbb{F}_d) = C^*_{\ell_p}(\mathbb{F}_d) \). Hence the following theorem is a generalization to the case of \( C^*_d(\mathbb{F}_d) \) for any \( p \in [2, \infty) \).

For \( 0 < \alpha < 1 \), we set \( \varphi_{\alpha}(s) = \alpha^{|s|} \), and it is positive definite on \( \mathbb{F}_d \) by [5] Lemma 1.2.

**Theorem 3.4.** Let \( 2 \leq p < \infty \). Let \( \varphi \) be a positive definite function on \( \mathbb{F}_d \). Then the following conditions are equivalent:

1. \( \varphi \) can be extended to the positive linear functional on \( C^*_d(\mathbb{F}_d) \).
2. \( \sup_k |\varphi \chi_k|_p (k + 1)^{-1} < \infty \).
3. The function \( s \mapsto \varphi(s)(1 + |s|)^{-1 - \frac{1}{p}} \) belongs to \( \ell_p(\mathbb{F}_d) \).
4. For any \( \alpha \in (0, 1) \), the function \( s \mapsto \varphi(s)\alpha^{|s|} \) belongs to \( \ell_p(\mathbb{F}_d) \).

**Proof.** Without loss of generality, we may assume that \( \varphi(e) = 1 \). The proof is based on the one in [5] Theorem 3.1.

(1)\( \Rightarrow \) (2): It follows from (1) that \( \omega_{\varphi} \) extends to the state on \( C^*_d(\mathbb{F}_d) \). Hence for \( f \in c_c(\mathbb{F}_d) \), we have
\[ |\omega_{\varphi}(f)| \leq \|f\|_{\ell_p}. \]
Put
\[ f = |\varphi|^{p-2} \varphi \chi_k. \]
Then
\[ |\omega_{\varphi}(f)| = |\varphi \chi_k|_{\ell_p}^p. \]
Notice that
\[ \|f\|_{\ell_p} = \sup\{\|\pi(f)\| : \pi \text{ is an } \ell_p\text{-representation}\}. \]
Let \( \pi \) be an \( \ell_p\)-representation of \( \mathbb{F}_d \) on a Hilbert space \( \mathcal{H} \) with a dense subspace \( \mathcal{H}_0 \). Then
\[ \|\pi(f)\|^2 = \sup_{\xi \in \mathcal{H}_0 \atop \|\xi\| = 1} \langle \pi(f^* f)\xi, \xi \rangle_{\mathcal{H}}. \]
Fix \( \xi \in \mathcal{H}_0 \) with \( \|\xi\| = 1 \). We denote by \( \sigma \) the restriction of \( \pi \) onto the subspace \( \mathcal{H}_\sigma = \overline{\text{span}}\{\pi(s)\xi : s \in \mathbb{F}_d\} \subset \mathcal{H} \).
Then
\[ \langle \pi(f^*f)\xi,\xi \rangle_H = \langle \sigma(f^*f)\xi,\xi \rangle_{\mathcal{H}}. \]
Note that \( \xi \) is cyclic for \( \sigma \) such that \( \sigma_\xi \in \ell_p(F_d) \). Take a real number \( q \) with \( 1/p + 1/q = 1 \). Since \( 2 \leq p < \infty \), we have \( 1 < q \leq 2 \). By Lemma 3.3,
\[ \|\sigma(f)\| \leq (k+1)|f|_q. \]
Hence
\[ \|\sigma(f^*f)\| = \|\sigma(f)\|^2 \leq (k+1)^2|f|_q^2. \]
Therefore we obtain
\[ \|f\|_{\ell^p} \leq (k+1)|\varphi\chi_k|_p^{-1}. \]
Consequently,
\[ |\varphi\chi_\alpha|_p \leq k+1. \]
(2) \( \Rightarrow \) (3):
\[
\sum_{s \in F_d} |\varphi(s)|^p(1+|s|)^{-p-2} = \sum_{k=0}^{\infty} \sum_{|s|=k} |\varphi(s)|^p(1+k)^{-p-2} = \sum_{k=0}^{\infty} |\varphi\chi_k|_p^p(1+k)^{-p}(1+k)^{-2} \leq \left\{ \sup_k |\varphi\chi_k|_p(k+1)^{-1} \right\}^p \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty.
\]
(3) \( \Rightarrow \) (4): Easy.
(4) \( \Rightarrow \) (1): Note that \( \psi_\alpha(s) = \varphi(s)\alpha^{|s|} \) is also positive definite. By the GNS construction, we obtain the unitary representation \( \sigma_\alpha \) of \( F_d \) with the cyclic vector \( \xi_\alpha \) such that \( \psi_\alpha(s) = \langle \sigma_\alpha(s)\xi_\alpha,\xi_\alpha \rangle \).
Since \( \sigma_\alpha \) is an \( \ell_p \)-representation, \( \psi_\alpha \) can be seen as a state on \( C_{\ell_p}^*(F_d) \). By taking the weak-* limit of \( \psi_\alpha \) as \( \alpha \nearrow 1 \), we conclude that \( \varphi \) can be extended to the state on \( C_{\ell_p}^*(F_d) \).

**Corollary 3.5.** Let \( p \in [2,\infty) \) and \( \alpha \in (0,1) \). The positive definite function \( \varphi_\alpha \) can be extended to the state on \( C_{\ell_p}^*(F_d) \) if and only if
\[ \alpha \leq (2d-1)^{-\frac{1}{p}}. \]

**Proof.** Note that
\[ \varphi_\alpha \in \ell_p(F_d) \iff \sum_{k=1}^{\infty} (2d-1)^{k-1}\alpha^{pk} < \infty \iff (2d-1)^{\alpha^p} < 1 \iff \alpha < (2d-1)^{-\frac{1}{p}}. \]
Hence the corollary follows from Theorem 3.4.
Remark 3.6. Let $\pi_\alpha$ be the GNS representation of $\varphi_\alpha$. Then $\pi_\alpha$ is weakly contained in $\lambda$ if and only if $\alpha \leq (2d-1)^{-\frac{1}{p}}$. For $\alpha > (2d-1)^{-\frac{1}{p}}$, we refer the reader to [7] and [8].

As a consequence, we can obtain the following result. See also [2 Proposition 4.4].

**Corollary 3.7.** For $2 \leq q < p \leq \infty$, the canonical quotient map from $C^*_\ell_p(\mathbb{F}_d)$ onto $C^*_\ell_q(\mathbb{F}_d)$ is not injective.

**Proof.** It suffices to consider the case where $p \neq \infty$, because $\mathbb{F}_d$ is not amenable.

Suppose that the canonical quotient map from $C^*_\ell_p(\mathbb{F}_d)$ onto $C^*_\ell_q(\mathbb{F}_d)$ is injective for some $q < p$. Take a real number $\alpha$ with

$$(2d-1)^{-\frac{1}{p}} < \alpha \leq (2d-1)^{-\frac{1}{q}}.$$  

By using Corollary [3.5]

$$|\omega_{\varphi_\alpha}(f)| \leq \|f\|_{\ell_p} = \|f\|_{\ell_q} \quad \text{for } f \in c_c(\mathbb{F}_d).$$

Therefore it follows that $\varphi_\alpha$ can be also extended to the state on $C^*_\ell_q(\mathbb{F}_d)$, but it contradicts to the choice of $\alpha$. $\Box$

**Remark 3.8.** The previous result has also shown by N. Higson and N. Ozawa, independently. See also [2 Remark 4.5].

In [6], Powers proves that $C^*_\ell(\mathbb{F}_2)$ has a unique tracial state. In [5], Haagerup gives another proof of the uniqueness. Thanks to Theorem 3.4, Haagerup’s argument also works for the case of $C^*_\ell_p(\mathbb{F}_d)$.

**Corollary 3.9.** For each $p \in [2, \infty)$, the $C^*$-algebra $C^*_\ell_p(\mathbb{F}_d)$ has a unique tracial state $\tau \circ \lambda_p$, where $\lambda_p$ is the canonical quotient map $C^*_\ell_p(\mathbb{F}_d)$ onto $C^*_\ell(\mathbb{F}_d)$ and $\tau$ is the unique tracial state on $C^*_\ell(\mathbb{F}_d)$.

**Proof.** Any tracial state on $C^*_\ell_p(\mathbb{F}_d)$ corresponds to a positive definite function on $\mathbb{F}_d$. Take such a positive definite function $\varphi$. Then $\varphi|_K$ is constant for any conjugacy class $K$. Take a conjugacy class $K$ in $\mathbb{F}_d$ such that $K \neq \{e\}$. Put $k = \min\{|s| : s \in K\}$. Then $|W_{k+2n} \cap K| \geq (2d-1)^{n-1}$ for $n \geq 1$. Hence if $\varphi|_K = C$ for some non-zero constant $C$, then

$$\sum_{s \in K} |\varphi(s)|^p \alpha^{|s|} = \sum_{n=0}^{\infty} \sum_{s \in W_{k+2n} \cap K} C^n \alpha^{2n} \geq C^n \sum_{n=0}^{\infty} (2d-1)^{n-1} \alpha^{2n} = \infty$$

for $\alpha \geq (2d-1)^{-\frac{1}{p}}$. This contradicts (4) in Theorem 3.4. $\Box$

We define two algebraic ideals by

$$D^-_p(\Gamma) = \bigcup_{\varepsilon > 0} \ell_{p^-}(\Gamma)$$

for $1 < p \leq \infty$, and

$$D^+_p(\Gamma) = \bigcap_{\varepsilon > 0} \ell_{p^+}(\Gamma)$$

for $1 \leq p < \infty$. Note that $\ell_q(\Gamma) \subseteq D^-_p(\Gamma) \subseteq \ell_p(\Gamma)$ for $1 \leq q < p \leq \infty$, and $\ell_p(\Gamma) \subseteq D^+_p(\Gamma) \subseteq \ell_q(\Gamma)$ for $1 \leq p < q \leq \infty$. Then we also obtain the following.
Corollary 3.10. (1) For $2 \leq p < \infty$, the $C^*$-algebra $C^*_f(F_d)$ is canonically isomorphic to $C^*_D(F_d)$. In particular, $C^*_f(F_d) = C^*_D(F_d)$.

(2) For $2 < p \leq \infty$, the $C^*$-algebra $C^*_f(F_d)$ is canonically isomorphic to $C^*_D(F_d)$. In particular, $C^*_f(F_d) = C^*_D(F_d)$.

Proof. (1) It suffices to show that if $\varphi$ is a positive definite function on $F_d$, which can be extended the positive linear functional on $C^*_D(F_d)$, then $\varphi$ can be also extended to the one on $C^*_f(F_d)$.

Now assume that $\varphi$ is a positive definite function on $F_d$, which can be extend to the positive linear functional on $C^*_D(F_d)$. Then for any $q \in (p, \infty)$, $\varphi$ can be also extended to the positive linear functional on $C^*_q(F_d)$. By Theorem 3.4, $\varphi \varphi_\alpha \in \ell_q(F_d)$ for any $\alpha \in (0, 1)$. We set

$$r = \frac{pq}{q-p}.$$ 

Then $1/p = 1/q + 1/r$. If we take $(2d-1) - 2/r < \beta < (2d-1) - 1/r$, then $\varphi \beta \in \ell_r(F_d)$. Since

$$|\varphi \varphi_\alpha \varphi_\beta|_p \leq |\varphi \varphi_\alpha|_q |\varphi_\beta|_r,$$

we have $\varphi \varphi_\alpha \varphi_\beta \in \ell_p(F_d)$. Namely, $(\varphi \varphi_\beta) \varphi_\alpha \in \ell_p(F_d)$ for any $\alpha \in (0, 1)$. Thus $\varphi \varphi_\beta$ can be extended to the positive linear functional on $C^*_D(F_d)$. If $q \land p$, then $r \not\sim \infty$ and $\beta \not\sim 1$. Hence $\varphi \varphi_\beta \to \varphi$ in the weak-* topology. Therefore $\varphi$ can be also extended to the positive linear functional on $C^*_f(F_d)$.

(2) The proof is quite similar as in (1). Assume that $\varphi$ is a positive definite function on $F_d$ which can be extended to the positive linear functional on $C^*_D(F_d)$. By Theorem 3.4 we have $\varphi \varphi_\alpha \in \ell_p(F_d)$ for any $\alpha \in (0, 1)$. For any $q \in [2, p)$, we set

$$r = \frac{pq}{p-q}.$$ 

Then $1/q = 1/p + 1/r$. If we take $(2d-1) - 2/r < \beta < (2d-1) - 1/r$, then $\varphi \beta \in \ell_r(F_d)$. Since

$$|\varphi \varphi_\alpha \varphi_\beta|_q \leq |\varphi \varphi_\alpha|_p |\varphi_\beta|_r,$$

we have $\varphi \varphi_\alpha \varphi_\beta \in \ell_q(F_d)$. Namely, $(\varphi \varphi_\beta) \varphi_\alpha \in \ell_q(F_d)$ for any $\alpha \in (0, 1)$. Thus $\varphi \varphi_\beta$ can be extended to the positive linear functional on $C^*_D(F_d)$, and so $\varphi \varphi_\beta$ can be extended to the one on $C^*_D(F_d)$. If $q \land p$, then $r \not\sim \infty$ and $\beta \not\sim 1$. Hence $\varphi \varphi_\beta \to \varphi$ in the weak-* topology. Therefore $\varphi$ can be also extended to the positive linear functional on $C^*_f(F_d)$. $\square$

Remark 3.11. The isomorphism $C^*_\lambda(\Gamma) = C^*_{D^\infty}(\Gamma)$ has been already obtained in [2] for any $\Gamma$.

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