On the thermodynamic instability of LST

Alex Buchel

Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106-4030, U.S.A.

Abstract

The high energy thermodynamics of Little String Theory (LST) is known to be unstable. An unresolved question is whether the corresponding instability in LST holographic dual is of stringy or supergravity origin. We study UV thermodynamics of a large metric deformation of the LST dual realized (in the extremal case) by type IIB fivebranes wrapping a two-sphere of a resolved conifold, and demonstrate that the resulting black hole has negative specific heat. This explicitly shows that the high energy thermodynamic instability of the LST holographic dual is of the supergravity origin.

\textsuperscript{1}buchel@itp.ucsb.edu
1 Introduction

Little string theory (LST) is a non-local theory defined on the world volume of NS5-branes in the limit of vanishing string coupling $g_{str} \to 0$ where the string scale $\alpha'$ is kept fixed [1, 2] (for a review see [3]). This theory does not include dynamical gravity and in the IR flows to six-dimensional Yang-Mills theory. Nonetheless, LST is quite different from a local field theory: it exhibits T-duality in toroidal compactifications and a Hagedorn density of states at very high energy — both the intrinsic attributes of the “standard” string theories. The latter property in particular implies that the statistical mechanics of LST breaks down at a finite temperature, known as the Hagedorn temperature. The purpose of this paper is to better understand the ultraviolet thermodynamics of the LST.

A classical thermodynamics of the LST on the world volume of flat NS5 branes can be easily deduced from its holographic dual, realized as a near horizon geometry of non-extremal NS5 branes [4]. It was shown in [4] that as the number $N$ of the five-branes is large, $N \gg 1$, and the energy $\mu$ above extremality in string units satisfies $\mu \gg N$, the background geometry is smooth with small curvatures everywhere; in addition, the string perturbation theory is also good as the dilaton is bounded from above by its value at the horizon, $\frac{N}{\mu}$. Ignoring loop/stringy corrections, one finds that the Hawking temperature of the resulting black hole is independent of the energy $\mu$ and coincides precisely with the Hagedorn temperature $T_H$ of the LST

$$\beta_H = \frac{1}{T_H} = 2\pi \sqrt{N \alpha'}.$$  \hfill (1.1)

Since the can tune the energy and the temperature of the system independently, its equation of state (in the ultraviolet) is

$$S = \beta_H E,$$  \hfill (1.2)

which leads to an exponential growth of the density of states

$$\rho(E) \sim e^{\beta_H E}.$$  \hfill (1.3)
One-loop corrections to the Hagedorn density of states of LST were studied in \cite{5,6,7}. The finite energy corrections to the density of states \((1.3)\) were argued to be of the form

\[ \rho(E) \sim E^\alpha e^{\beta E} \left( 1 + O \left( \frac{1}{E} \right) \right). \]  

The sign of \(\alpha\) in \((1.4)\) is of uttermost importance as it determines the stability of the thermal ensemble representing LST at high energy. The explicit calculation of \cite{7} indicated that \(\alpha\) is negative implying the negative specific heat and thus the thermal instability of the system. According to authors of \cite{7} the instability in question is of stringy origin. It is represented by a (massless at tree level) string mode that winds once around the Euclidean time direction, but is supposed to become tachyonic at one-loop level.

An alternative explanation of the instability of the LST at high energy has been advocated in \cite{8}. Following the conjecture of \cite{9,10} that thermodynamic instabilities in field theories should correspond to classical instabilities of the dual spacetime geometry, Rangamani proposed \cite{8} that the supergravity dual to LST at Hagedorn temperature suffers from a Gregory-Laflamme (GL) \cite{11,12} like instability, thereby causing the thermal ensemble to be unstable. Though he did not manage to explicitly demonstrate that the metric fluctuations about the background of interest have indeed a zero frequency mode, that is capable of explaining the origin of the instability, Rangamani conjectured that the required mode is the one responsible for the GL instability in the near-extremal NS5-branes claimed in \cite{13}, that would survive the decoupling limit for the finite temperature LST.

In this paper we prove the proposal of \cite{8} and present yet another explicit example advocating the general philosophy of \cite{9,10}. We do not perform the stability analysis near the background representing finite temperature LST realized on flat NS5 branes as suggested in \cite{8}. Rather, we study large metric deformations and show that the resulting black hole geometries have negative specific heat at the classical level. Recall that in the flat case the temperature of the LST is (classically) independent of the energy. By changing the theory in the IR (say wrapping NS5 branes on a 2-cycle, so that
the theory is four dimensional macroscopically), while preserving the UV characteristics (having a 2-cycle of a finite size), we generically expect that temperature should become energy dependent. This must definitely be the case, if the deformation induces a phase transition in the IR, so that the deformed geometry is simply singular (and thus does not make sense) for sufficiently small energy. Then, by studying the classical thermodynamics of the deformed background at high energy we should be able to deduce $\alpha$ in (1.4). Specifically, we propose to study the stability issue of LST at high energy from the thermodynamics of large number of type IIB NS5 branes wrapping a two-cycle of a resolved conifold. The relevant extremal solution was discussed in [14, 15, 16]. It has $\mathcal{N} = 1$ SUSY in four dimensions and exhibits a naked singularity in the IR associated with the chiral symmetry breaking of the dual gauge theory at zero temperature. The nonextremal deformation of this background was constructed in [17]. It was argued there that the naked singularity (at the extremality) will be hidden beyond the black hole horizon for sufficiently large energy away from the extremality.

The paper is organized as follows. In the next section we describe the gravitational background representing a large number of type IIB NS5 branes wrapping a 2-cycle in the resolved conifold geometry at finite temperature. We study thermodynamics of this black hole solution, and show that at high energy the black hole has a negative specific heat. We analytically compute $\alpha$ in the density of states expression (1.4) for the discussed deformation. We briefly conclude in section 3. Technical details can be found in the appendix.

2 High energy thermodynamics of $\mathcal{N} = 1$ deformed LST

In the previous section addressing the origin of thermodynamic instability of LST at high energy, we proposed to study large metric deformation of its holographic dual realized by wrapping NS5 branes on a two-sphere of a resolved conifold. The motivation for choosing this particular background
came from the fact that the corresponding extremal solution had a naked
singularity in the IR associated with the zero temperature chiral symmetry
breaking phase transition of the dual gauge theory. As the result, the Hawk-
ing temperature of its nonextremal deformation should be energy dependent,
and so the classical analysis should be enough to extract $\alpha$ in the “corrected”
entropy-energy relation

$$S = \beta_H E + \alpha \log E + O(1/E).$$

(2.5)

We proceed by recalling the gravitational background holographically
dual to the $N = 1$ deformed LST at finite temperature, originally constructed
in [17]. From now on we set the string scale $\alpha' = 1$. The ten dimensional
string frame metric, NS-NS 3-form $H$, and the dilaton $\phi$ are given by

$$ds^2_{str} = \Delta^2 dt^2 - dx^2 - N\left[f^2(d[\ln \Delta])^2 + \frac{h}{4}(d\theta_1 + \sin^2 \theta_1 d\phi_1^2)
+ \frac{1}{4}(d\theta_2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{4}\left(d\psi + \sum_{i=1}^2 \cos \theta_i d\phi_i\right)^2\right],$$

$$H = \frac{N}{4}\left[(d\psi + \sum_{i=1}^2 \cos \theta_i d\phi_i) \wedge (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2)\right],$$

$$e^{-2\phi} = C \frac{f}{h\Delta},$$

(2.6)

where $N$ is the number of NS5 branes, $C$ is an integration constant related
to the asymptotic string coupling, and $\Delta$ is a new radial coordinate defined
in such a way that $\Delta = 0_+$ determines the black hole horizon, and $\Delta \to 1_-$
is the ultraviolet asymptotic.

From eq. (5.56) of [17], $f$ and $h$ satisfy second-order Toda-like system of
differential equations

$$[\ln f]' = \frac{4f^2}{1 + \frac{2}{h} - \frac{1}{h^2}},$$

(2.7)

\footnote{$\triangle$, $f(\Delta)$ used here are related to $c_1(r)$, $h(r)$, $a$ of [17] as follows $\triangle \equiv \triangle_1(r)$, $f(\Delta) = c_1^r(r)\triangle_1(r)h(r)/A$, $N = a^2$.}
supplemented by the first-order constraint

\[ 0 = \left( \frac{f'^2}{h^2} \right)'' + \frac{f'^2}{h^2} (h')^2 + 2h^2 \left( \left[ \frac{f'}{h} \right]' \right)^2 - 2f^2 h^2 \frac{f^4}{h^2} (h^2 + 2h - 1). \]  (2.9)

In eqs. (2.7), (2.8), (2.9) the derivatives are with respect to

\[ y \equiv \ln \Delta, \quad y \in (-\infty, 0). \]  (2.10)

As explained in [17], to have a regular Schwarzschild horizon we must have

\[ f = e^y \left( f_0 + \sum_{n=1}^{\infty} f_n e^{2yn} \right), \]
\[ h = h_0 + \sum_{n=1}^{\infty} h_n e^{2yn}, \quad \text{as } y \to -\infty, \]  (2.11)

for some positive constants\(^2\) \(f_0, h_0\). In the UV region we have

\[ h \to +\infty, \quad \text{as } y \to 0_-, \]
\[ f \to +\infty, \quad \text{as } h \to +\infty. \]  (2.12)

Furthermore, it was argued in [17] that both the IR and the UV asymptotics are compatible provided \(h_0 > 1\), where the inequality incorporates the physics of chiral symmetry restoration at finite temperature of the dual gauge theory.

We now study the thermodynamic properties of the black hole (2.6) keeping the dilaton at the horizon \(g_h\) fixed, for large \(h_0 \gg 1\). The first condition amounts to choosing

\[ C = \frac{h_0}{g_h f_0}, \]  (2.13)

and as we will see insures (for \(g_h \ll 1\)) that the dilaton is everywhere small. The second condition is the high energy limit of the thermodynamics. We

\(^2\)Only \(h_0\) is an independent parameter, \(f_0\) should be fixed by the boundary conditions [17].
don’t know how to solve (2.7), (2.8) in general. However, the $h_0 \to \infty$ asymptotic of the solution can be extracted with some work. We find

$$h(y) = -2y - 2 \log[-\sinh y] + h_0 + O(1),$$

$$f(y) = -\frac{1}{2 \sinh y} \left( 1 - \frac{1}{h(y)} + o \left( \frac{1}{h(y)} \right) \right).$$

(2.14)

With (2.14) we see that the dilaton is indeed bounded by its value at the horizon, also

$$f(y)e^{-y} \mid_{y \to -\infty} \to f_0 = 1 - 1/h_0 + o(1/h_0).$$

(2.15)

From the metric of (2.6) we can read off the Hawking temperature by identifying the periodicity of its Euclidean time direction with the inverse temperature

$$T^{-1} \equiv \beta = 2\pi N^{1/2} f_0 = \beta_H \left( 1 - \frac{1}{h_0} + o \left( \frac{1}{h_0} \right) \right).$$

(2.16)

Next, we compute the Bekenstein-Hawking entropy of the geometry (2.6). We find the 8-dimensional area of the event horizon $A_8$ of the black hole to be

$$A_8 = 2g_h^{-2} N^{5/2} \pi^3 h_0 V_3,$$

(2.17)

where $V_3$ is the 3-dimensional volume. The entropy of the black hole is then

$$S = \frac{A_8}{4G_N} = \frac{N^{5/2} h_0 V_3}{16\pi^3 g_h^4}.$$  

(2.18)

From the ordinary statistical mechanics we know that the energy is $dE = TdS$, so from (2.16), (2.18) we find the entropy-energy relation for our particular deformation of LST

$$S(E) = \beta_H E + \alpha \log E + o(\log E),$$

(2.19)

with

$$\alpha = -\frac{N^{5/2} V_3}{16\pi^3 g_h^4}.$$  

(2.20)

\(^3\text{See appendix for the details.}\)
Note that as in the flat LST case [6], $\alpha$ is an extensive quantity — it is proportional to the noncompact factor $V_3$ of the fivebrane world-volume. Since $\alpha$ in (2.20) is negative, the black hole (2.6) has negative specific heat and is thus thermodynamically unstable.

## 3 Conclusion

The purpose of this paper was to argue that the instability of the ultraviolet LST thermodynamics in its holographic dual is of a supergravity origin, as suggested in [8]. To do so, we identified a large metric deformation of the gravity dual to LST where the (conjectured) threshold instability of the flat LST dual should be enhanced to classical instability of the nonextremal generalization of the deformed background. We argued that this should happen if the deformation in the (extremal case) induces a phase transition in the IR, but does not affect the UV physics. In our example, this large metric deformation is realized by a large number of NS5 branes wrapping a 2-cycle of a resolved conifold. By studying the UV thermodynamics of this background we explicitly showed that resulting black hole has a negative specific heat.

The nonextremal geometry of interest has small curvatures and small perturbative string loop corrections if both $1/N$ and $g_h$ are small. Thus our classical analysis is reliable.

## Acknowledgments

We wish to thank Anton Kapustin, Igor Klebanov and Arkady Tseytlin for valuable discussions. This work was supported in part by NSF grants PHY97-22022 and PHY99-07949.

## Appendix

In what follows we solve (2.7), (2.8) subject to the boundary conditions (2.11), (2.12) in the limit $h_0 \to \infty$. Note that from (2.7), (2.8) both $f$ and
$h$ are rapidly increasing functions, so in the zero order approximation in the limit $h_0 \to \infty$, we can set $1/h = 0$ in (2.7). This equation has then a general solution

$$f \approx f^{(0)} = -\frac{C_1}{2 \sinh(C_1 y + C_2)}, \quad (3.21)$$

where $C_i$ are integration constants. From the UV asymptotic we need $f \to \infty$ as $y \to 0_-$. This fixes $C_2 = 0$. Furthermore, the asymptotic at the horizon fixes $C_1 = 1$ so that

$$f^{(0)} = -\frac{1}{2 \sinh y}. \quad (3.22)$$

Note that satisfying the asymptotics uniquely fixes $f_0$,

$$f^{(0)}_0 = 1. \quad (3.23)$$

In the leading order eq. (2.8) reads

$$[\ln h^{(0)}]^\prime\prime = \frac{2}{h^{(0)} \sinh^2 y}. \quad (3.24)$$

Above equation is difficult to solve analytically. We claim however that in the limit $h_0 \to \infty$

$$[h^{(0)}]^\prime h^{(0)} \gg ([h^{(0)}]^\prime)^2, \quad (3.25)$$

uniformly for all $y$. In approximation (3.25), eq. (3.24) simplifies

$$[h^{(0)}]^\prime = \frac{2}{\sinh^2 y}. \quad (3.26)$$

The general solution of (3.26) is

$$h^{(0)} = C_1 y - 2 \log[\sinh y] + C_2, \quad (3.27)$$

where $C_i$ are integration constants. To satisfy asymptotic at the horizon we must choose $C_1 = -2$ and $C_2 = h_0 - 2 \ln 2$ so that

$$h^{(0)} = -2y - 2 \log[\sinh y] + h_0 - 2 \ln 2. \quad (3.28)$$

We can now go back to (3.23) to check the self-consistency of the approximation. Indeed, we find that $\frac{h^{(0)} f^{(0)}}{([h^{(0)}]^\prime)^2}$ with $h^{(0)}$ given by (3.28) has a
global minimum at $y = (-1/h_0 + O(h_0^{-2}))$, with the value at the minimum
$(h_0/2 - \log h_0 + O(1)) \to \infty$ as $h_0 \to \infty$.

To study the leading correction to the Hagedorn thermodynamics we actually need to know the first correction to (3.22). We take the following ansatz for the leading correction

$$f = -\frac{1}{2\sinh y} \left(1 + \frac{\gamma}{h^{(0)}} + o\left(\frac{1}{h^{(0)}}\right)\right),$$

(3.29)

where $h^{(0)}$ is the zero order solution (2.11). Substituting (3.29) into (2.7), using eq. (3.26) and the approximation (3.25), we find $\gamma = -1$. In a similar fashion we can compute the leading correction to $h^{(0)}$ and see that it is a fixed constant. From (3.29) it follows that

$$f_0 = 1 - \frac{1}{h_0} + o\left(\frac{1}{h_0}\right).$$

(3.30)

Since our conclusion about UV thermodynamic instability of LST hinges on the fact that $\gamma < 0$, we also did numerical analysis to confirm (3.30). In the remaining of this section we describe them and present the results. First, we rewrite (2.7) and (2.8) in terms of

$$f_1(t) \equiv \frac{e^y}{f(y)},$$

$$f_2(t) \equiv \frac{1}{h(y)},$$

(3.31)

where we introduced new variable

$$t \equiv e^{2y}, \quad t \in [0, 1].$$

(3.32)

We find

$$0 = tf_1(t)[f_1(t)]'' + f_1(t)[f_1(t)']' - t([f_1(t)'])' + 1 + 2f_2(t) - f_2(t)^2,$$

$$0 = tf_1(t)^2f_2(t)[f_2(t)]'' + f_1(t)^2f_2(t)[f_2(t)']' - tf_1(t)^2([f_2(t)'])' + 2f_2(t)^3 - 2f_2(t)^4,$$

(3.33)
Figure 1: Numerical determination of $f_0(h_0)$. Dots represent values of $\log[1-f_0(h_0)]$ evaluated as detailed in the appendix section. The slope is predicted to be $(-1)$ from (3.30).

where all the derivatives are with respect to $t$. Near $t = 0$ we have power series expansion

$$f_1 = \sum_{k=0}^{\infty} \alpha_k t^k, \quad f_2 = \sum_{k=0}^{\infty} \beta_k t^k,$$

(3.34)

with

$$\alpha_0 = \frac{1}{f_0}, \quad \alpha_1 = -\frac{f_0(h_0^2 + 2h_0 - 1)}{h_0^2}, \quad \alpha_2 = \cdots,$$

$$\beta_0 = \frac{1}{h_0}, \quad \beta_1 = -\frac{2f_0^2(h_0 - 1)}{h_0^3}, \quad \beta_2 = \cdots,$$

(3.35)

while at $t \to 1_-$ we have boundary condition

$$0 < f_1(t) \ll f_2(t) \to 0.$$

(3.36)

In practice we integrated (3.33) from $t = \delta t = 10^{-4}$ with initial conditions, determined by $f_0, h_0$, specified by the first six terms in the expansion (3.34). Notice that is if a set $\{f_1(t, [f_0, h_0]), f_2(t, [f_0, h_0])\}$ is a solution to (3.33), then
so does \( \{ f_1(t\lambda, [f_0/\sqrt{\lambda}, h_0]), f_2(t\lambda, [f_0/\sqrt{\lambda}, h_0]) \} \) for arbitrary \( \lambda \). The latter in particular implies that if \( f_1(t = 1, [f_0, h_0]) = 0 \) (as it should be for the boundary condition (3.30)), than \( f_1(t = \lambda, [f_0/\sqrt{\lambda}, h_0]) = 0 \). This suggested a practical “trick” of fixing \( f_0 \) in terms of \( h_0 \) from the integration: for a fixed \( h_0 \) we take \( f_0 = 1 \) and find (via numerical integration) \( t^* \equiv t^*(h_0) \) such that \( f_1(t^*(h_0), [1, h_0]) = 0 \); then, above considerations guarantee that \( f_1(1, [\sqrt{t^*(h_0)}, h_0]) = 0 \). That is

\[
f_0(h_0) = \sqrt{t^*(h_0)}. \tag{3.37}
\]

The results of the numerical computations are presented in Fig. 1, as a plot of \( \log[1 - \sqrt{t^*(h_0)}] \) versus \( \log(h_0) \). According to (3.30) we expect the large \( \log(h_0) \) slope to be \((-1)\). We see that this is indeed the case.
References

[1] M. Berkooz, M. Rozali and N. Seiberg, “Matrix Description of M-theory on $T^4$ and $T^5$,” *Phys.Lett.* **B408** (1997) 105, [hep-th/9704089](https://arxiv.org/abs/hep-th/9704089).

[2] N. Seiberg, “Matrix Description of M-theory on $T^5$ and $T^5/Z_2$,” *Phys.Lett.* **B408** (1997) 98, [hep-th/9705221](https://arxiv.org/abs/hep-th/9705221).

[3] O. Aharony, ”A brief review of little string theories,” *Class.Quant.Grav.* **17**, (2000) 929, [hep-th/9911147](https://arxiv.org/abs/hep-th/9911147).

[4] J. M. Maldacena and A. Strominger, ”Semiclassical decay of near extremal fivebranes,” *JHEP* **9712**, (1997) 008, [hep-th/9710014](https://arxiv.org/abs/hep-th/9710014).

[5] T. Harmark and N. A. Obers, ”Hagedorn Behaviour of Little String Theory from String Corrections to NS5-Branes,” *Phys.Lett.* **B485** (2000) 285, [hep-th/0005021](https://arxiv.org/abs/hep-th/0005021).

[6] M. Berkooz and M. Rozali, ”Near Hagedorn Dynamics of NS Fivebranes, or A New Universality Class of Coiled Strings,” *JHEP* **0005**, (2000) 040, [hep-th/0005047](https://arxiv.org/abs/hep-th/0005047).

[7] D. Kutasov and D. A. Sahakyan, ”Comments on the Thermodynamics of Little String Theory,” *JHEP* **0102**, (2001) 021, [hep-th/0012258](https://arxiv.org/abs/hep-th/0012258).

[8] M. Rangamani, ”Little String Thermodynamics,” *JHEP* **0106** (2001) 042, [hep-th/0104123](https://arxiv.org/abs/hep-th/0104123).

[9] S. S. Gubser and I. Mitra, ”Instability of charged black holes in anti-de Sitter space,” [hep-th/0009120](https://arxiv.org/abs/hep-th/0009120).

[10] S. S. Gubser and I. Mitra, ”The evolution of unstable black holes in anti-de Sitter space,” [hep-th/0011127](https://arxiv.org/abs/hep-th/0011127).

[11] R. Gregory and R. Laflamme, ”Black Strings and p-Branes are Unstable,” *Phys.Rev.Lett.* **70** (1993) 2837, [hep-th/9301052](https://arxiv.org/abs/hep-th/9301052).
[12] R. Gregory and R. Laflamme, ”The Instability of Charged Black Strings and p-Branes,” *Nucl. Phys.* B428 (1994) 399, [hep-th/9404071](https://arxiv.org/abs/hep-th/9404071).

[13] H. S. Reall, ”Classical and Thermodynamic Stability of Black Branes,” [hep-th/0104071](https://arxiv.org/abs/hep-th/0104071).

[14] Ali H. Chamseddine and Mikhail S. Volkov, ”Non-Abelian BPS Monopoles in N=4 Gauged Supergravity,” *Phys. Rev. Lett.* 79 (1997) 3343, [hep-th/9707176](https://arxiv.org/abs/hep-th/9707176).

[15] Ali H. Chamseddine and Mikhail S. Volkov, ”Non-Abelian Solitons in N=4 Gauged Supergravity and Leading Order String Theory,” *Phys. Rev.* D57 (1998) 6242, [hep-th/9711181](https://arxiv.org/abs/hep-th/9711181).

[16] J. Maldacena and C. Nunez, ”Towards the large N limit of pure N=1 super Yang Mills,” *Phys. Rev. Lett.* 86 (2001) 588, [hep-th/0008001](https://arxiv.org/abs/hep-th/0008001).

[17] A. Buchel and A. Frey, ”Comments on supergravity dual of pure N=1 Super Yang Mills theory with unbroken chiral symmetry,” [hep-th/0103022](https://arxiv.org/abs/hep-th/0103022).