A new integrable anisotropic oscillator on the two-dimensional sphere and the hyperbolic plane

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Abstract
A new integrable generalization to the two-dimensional (2D) sphere, $S^2$, and to the hyperbolic space, $H^2$, of the 2D Euclidean anisotropic oscillator Hamiltonian with Rosochatius (centrifugal) terms is presented, and its curved integral of motion is shown to be quadratic in the momenta. To construct such a new integrable Hamiltonian, $H_\kappa$, we will use a group theoretical approach in which the curvature, $\kappa$, of the underlying space will be treated as an additional (contraction) parameter, and we will make extensive use of projective coordinates and their associated phase spaces. When the oscillator parameters $\Omega_1$ and $\Omega_2$ are such that $\Omega_1 \Omega_2 = \frac{41}{21}$, the system turns out to be the well-known superintegrable 1: 2 oscillator on $S^2$ and $H^2$. Nevertheless, numerical integration of the trajectories of $H_\kappa$ suggests that for other values of the parameters $\Omega_1$ and $\Omega_2$, the system is not superintegrable. In this way, we support the conjecture that for each commensurate, and thus superintegrable, $m: n$ Euclidean oscillator there exists a two-parametric family of curved integrable oscillators that turns out to be superintegrable only when the parameters are tuned to the $m: n$ commensurability condition.

Keywords: anisotropic oscillator, integrable systems, Lie–Poisson algebras, curvature, Poincaré disk, integrable deformation, Higgs oscillator
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1. Introduction

In this paper, we present a new integrable generalization of the two-dimensional (2D) Euclidean anisotropic oscillator Hamiltonian with Rosochatius (centrifugal) terms

\[ H = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + \Omega_1 q_1^2 + \Omega_2 q_2^2 + \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2}, \]  

(1.1)
to the 2D sphere, \( S^2 \), and to the hyperbolic space, \( H^2 \). The generalization, \( H_\kappa \), is shown to be integrable for any value of the set of parameters \( \{\Omega_1 > 0, \Omega_2 > 0, \lambda_1, \lambda_2\} \), and the curved integral of the motion that provides the integrability of \( H_\kappa \) turns out to be quadratic in the momenta.

Evidently, for any value of the \( \Omega_1 \) and \( \Omega_2 \) parameters the Hamiltonian (1.1) is indeed separable and, as a consequence, integrable in the Liouville sense; note that the separability of the kinetic energy term will be lost when a nonzero curvature is introduced. On the other hand, the Euclidean system (1.1) is known to be superintegrable (i.e., possessing a third independent and globally defined constant of the motion) only when the two frequencies, \( \omega_1 \) and \( \omega_2 \), defined through \( \omega_1^2 = 2\Omega_1 \) and \( \omega_2^2 = 2\Omega_2 \), are commensurate. Moreover, the two integrals of the motion for a given commensurate Euclidean oscillator are quadratic in the momenta only in the 1:1 and 1:2 cases, being of higher-order for any other superintegrable case (see [1–3]).

To generalize these results to a constant curvature scenario [4], we will use a simultaneous construction of \( S^2 \) and \( H^2 \) as homogeneous spaces of the one-parametric \( \kappa \) SO(3) Lie group, in which the curvature, \( \kappa \), will be introduced as a contraction parameter [5–7]. Within this approach, an integrable system on \( S^2 \) and \( H^2 \) will be called an anisotropic curved oscillator with centrifugal terms if its Euclidean limit, \( \kappa \to 0 \), gives (1.1).

We have to recall that the classification of all possible superintegrable systems on \( S^2 \) and \( H^2 \) with quadratic integrals of the motion was presented in [8]. Among them, we find only two curved superintegrable oscillator potentials: the Higgs oscillator, whose Euclidean limit is the 1:1 isotropic oscillator \( (\Omega_1 = \Omega_2, \lambda_1, \lambda_2 \) arbitrary), and an anisotropic oscillator whose zero curvature limit is the superintegrable Euclidean 1:2 oscillator \( (\Omega_2 = 4\Omega_1, \lambda_2 = 0, \lambda_1 \) arbitrary).

To the best of our knowledge, no other commensurate Euclidean oscillator has thus far been generalized to the curved case. Moreover, only recently has a first integrable (but not superintegrable) generalization of (1.1) for arbitrary values of \( \{\Omega_1, \Omega_2, \lambda_1, \lambda_2\} \) been presented in [9]. This curved Hamiltonian gives rise to the superintegrable Higgs oscillator when \( \Omega_1 = \Omega_2 \), and therefore it can be interpreted as the ‘anisotropic generalization’ of the Higgs oscillator. Due to this fact, and also taking into account that the curved superintegrable 1:2 oscillator was shown in [9] to be another different system, we then conjectured that for each commensurate \( m:n \) Euclidean oscillator there should exist a different two-parametric \( (\Omega_1, \Omega_2) \) curved anisotropic oscillator system that should provide the corresponding curved superintegrable \( m:n \) system when appropriate \( (\Omega_1, \Omega_2) \) values are considered.

The aim of this paper is to support this conjecture by presenting a new curved anisotropic oscillator, \( H_\kappa \), that

- is Liouville integrable, but not superintegrable, for arbitrary values of \( \{\Omega_1, \Omega_2, \lambda_1, \lambda_2\} \),
- reduces to the curved superintegrable 1:2 oscillator when \( \Omega_2 = 4\Omega_1 \),
- and gives (1.1) under the Euclidean limit \( \kappa \to 0 \).
The fact that $\mathcal{H}_\kappa$ does not seem to be superintegrable whenever $\Omega_2 \neq 4 \Omega_1$ will be studied through numerical integration of its bounded trajectories, which turn out to be non-closed for all the cases studied so far. In this respect, the description of the system in terms of projective canonical variables will be helpful, and the plot on the Poincaré disk of the potential defining $\mathcal{H}_\kappa$ provides an unexpected geometric footprint of superintegrability.

The paper is structured as follows. In section 2, we present a Cayley–Klein approach to $\mathbb{S}^2, \mathbb{H}^2$, and $\mathbb{E}^2$ that will provide a simultaneous description of the dynamics on these three spaces in terms of the curvature parameter, $\kappa$. In particular, the expression of the kinetic energy will be presented in ambient and geodesic polar coordinates. In section 3, the Poincaré and Beltrami projective coordinates and their associated phase spaces will be introduced and related with the ambient and geodesic polar descriptions. In this way, sections 2 and 3 provide a self-contained approach to the geometric dynamics on $\mathbb{S}^2$ and $\mathbb{H}^2$ that completes the description already presented in [8, 9]. In section 4, the known superintegrable 1:1 and 1:2 oscillator systems on $\mathbb{S}^2$ and $\mathbb{H}^2$ are revisited, and the usefulness of the projective coordinates in this type of curved integrability problems are illustrated. Section 5 includes the main new finding of this paper: the explicit form of the integrable anisotropic oscillator, $\mathcal{H}_\kappa$, that leads to the superintegrable curved oscillator 1:2 in the specific $\Omega_2 = 4 \Omega_1$ case. The non superintegrability of $\mathcal{H}_\kappa$ for generic values of $\Omega_1$ and $\Omega_2$ is illustrated in section 6 through the numerical integration of a selected sample of trajectories of the system; the use of projective variables also turns out to be helpful from this numerical perspective. Finally, some comments and open problems close the paper.

2. Geometry and geodesic dynamics on $\mathbb{S}^2$ and $\mathbb{H}^2$

To start with, we provide a complete group theoretical approach of $\mathbb{S}^2$ and $\mathbb{H}^2$, which leads to a very natural description of the (3D) ambient variables for these spaces. In particular, we present all the relevant dynamic quantities in terms of the latter variables, including the ambient momenta, which were omitted from previous works on the subject (see [8, 9] and references therein) and turn out to be useful. In particular, we explicitly provide the case of geodesic polar coordinates.

2.1. Vector model and ambient canonical variables

Let us consider a one-parametric family of 3D real Lie algebras (which is contained within the family of the so-called 2D orthogonal Cayley–Klein algebras [5–7]) that we denote collectively as $\mathfrak{s}\mathfrak{o}_\kappa(3)$, where $\kappa$ is a real contraction parameter. The Lie brackets of $\mathfrak{s}\mathfrak{o}_\kappa(3)$ in the basis spanned by $\langle \bar{J}_{01}, \bar{J}_{02}, \bar{J}_{12} \rangle$ are given by

$$[\bar{J}_{12}, \bar{J}_{01}] = \bar{J}_{02}, \quad [\bar{J}_{12}, \bar{J}_{02}] = -\bar{J}_{01}, \quad [\bar{J}_{01}, \bar{J}_{02}] = \kappa \bar{J}_{12}. \quad (2.1)$$

The Casimir invariant, coming from the Killing–Cartan form, reads

$$C = \bar{J}_{01}^2 + \bar{J}_{02}^2 + \kappa \bar{J}_{12}^2. \quad (2.2)$$

Notice that by appropriately redefining the Lie generators, the real parameter $\kappa$ can be rescaled to either $+1$, $0$ or $-1$. In particular, putting $\kappa = 0$ is equivalent to applying an İnönü–Wigner contraction [10]. In fact, the involutive automorphism defined by

$$\Theta: \bar{J}_i \to \bar{J}_i, \quad \bar{J}_0 \to -\bar{J}_0, \quad i = 1, 2,$$

generates a $\mathbb{Z}_2$-grading of $\mathfrak{s}\mathfrak{o}_\kappa(3)$ in such a manner that $\kappa$ is a graded contraction parameter [11]. This automorphism gives rise to the following Cartan decomposition:

$$\mathfrak{s}\mathfrak{o}_\kappa(3) = \mathfrak{s}\mathfrak{o}_0(3) \oplus \mathfrak{s}\mathfrak{o}_1(3) \oplus \mathfrak{s}\mathfrak{o}_2(3),$$

where $\mathfrak{s}\mathfrak{o}_0(3) = \mathfrak{s}\mathfrak{o}_0(3)$, $\mathfrak{s}\mathfrak{o}_1(3) = \mathfrak{s}\mathfrak{o}_1(3)$, and $\mathfrak{s}\mathfrak{o}_2(3) = \mathfrak{s}\mathfrak{o}_2(3)$.
\[ \mathfrak{so}_\kappa(3) = \mathfrak{h} \oplus \mathfrak{p}, \quad \mathfrak{h} = \langle J_{12} \rangle = \mathfrak{so}(2), \quad \mathfrak{p} = \langle J_{01}, J_{02} \rangle. \]

Now, the family of the three classical 2D Riemannian symmetric homogeneous spaces with constant Gaussian curvature, \( \kappa \), is defined by the quotient \( S^2_\kappa \equiv \text{SO}(3)/\text{SO}(2) \), where the Lie groups \( H = \text{SO}(2) \) and \( \text{SO}_\kappa(3) \) have \( \mathfrak{h} \) and \( \mathfrak{so}_\kappa(3) \), respectively, as Lie algebras. Namely,

\[ \kappa > 0: \text{2D Sphere} \quad \kappa = 0: \text{Euclidean plane} \quad \kappa < 0: \text{2D Hyperbolic space} \]

\[ S^2_\kappa \equiv \text{SO}(3)/\text{SO}(2) \quad S^0_0 \equiv \mathbb{E}^2 = \text{ISO}(2)/\text{SO}(2) \quad S^2_\kappa \equiv \mathbb{H}^2 = \text{SO}(2, 1)/\text{SO}(2). \]

The vector representation of \( \mathfrak{so}_\kappa(3) \) is given by the following 3 \times 3 real matrices:

\[ J_{01} = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_{02} = \begin{pmatrix} 1 & 0 & \kappa \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.3} \]

Their exponentials lead to the corresponding one-parametric subgroups of \( \text{SO}_\kappa(3) \):

\[ e^{\alpha J_{01}} = \begin{pmatrix} C_\kappa(\alpha) & -\kappa S_\kappa(\alpha) & 0 \\ S_\kappa(\alpha) & C_\kappa(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\gamma J_{02}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{pmatrix}, \]

\[ e^{\beta J_{12}} = \begin{pmatrix} C_\kappa(\beta) & -\kappa S_\kappa(\beta) & 0 \\ S_\kappa(\beta) & C_\kappa(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.4} \]

where we have introduced the \( \kappa \)-dependent cosine and sine functions defined by [7],

\[ C_\kappa(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l}}{(2l)!} = \begin{cases} \cos \sqrt{\kappa} x & \kappa > 0 \\ 1 & \kappa = 0 \end{cases}, \quad \cosh \sqrt{-\kappa} x & \kappa < 0 \]

\[ S_\kappa(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l+1}}{(2l + 1)!} = \begin{cases} \sin \sqrt{\kappa} x & \kappa > 0 \\ x & \kappa = 0 \end{cases}, \quad \sinh \sqrt{-\kappa} x & \kappa < 0 \]

The \( \kappa \)-tangent is defined as \( T_\kappa(x) = S_\kappa(x)/C_\kappa(x) \), and different useful relations involving these \( \kappa \)-functions can be found in [8, 12, 13]. For instance,

\[ C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1, \quad \frac{d}{dx} C_\kappa(x) = -\kappa S_\kappa(x), \quad \frac{d}{dx} S_\kappa(x) = C_\kappa(x), \quad \frac{d}{dx} T_\kappa(x) = \frac{1}{C_\kappa^2(x)}. \]

Under the matrix representations (2.3) and (2.4), the following relations hold:

\[ X^T I_+ + I_+ X = 0, \quad X \in \mathfrak{so}_\kappa(3), \quad Y^T I_+ Y = I_+, \quad Y \in \text{SO}_\kappa(3), \]

\( (X^T \) denotes the transpose of \( X \) \) with respect to the bilinear form

\[ I_+ = \text{diag}(+1, \kappa, \kappa). \tag{2.5} \]

Therefore, \( \text{SO}_\kappa(3) \) is a group of isometries of \( I_+ \) acting on a 3D ambient linear space, \( \mathbb{R}^3 = (x_0, x_1, x_2) \), through matrix multiplication. The origin, \( O \), in \( S^2_\kappa \) has ambient coordinates \( O = (1, 0, 0) \), and this point is invariant under the subgroup \( \text{SO}(2) = \langle J_{12} \rangle \) (see (2.4)). The orbit of \( O \) corresponds to the homogeneous space, \( S^2_\kappa \), which is contained in the ‘sphere’
\[ \Sigma_\kappa \equiv x_0^2 + \kappa \left( x_1^2 + x_2^2 \right) = 1, \quad (2.6) \]
determined by \( I_\kappa (2.5) \). Hence, when \( \kappa > 0 \), we recover a proper sphere, but when \( \kappa < 0 \) we find the two-sheeted hyperboloid. In fact, if we write \( \kappa = \pm 1/R^2 \) where \( R \) is the radius of the space, then the contraction \( \kappa \to 0 \) corresponds to the flat limit, \( R \to \infty \), which gives rise to two Euclidean planes, \( x_0 = \pm 1 \), with Cartesian coordinates \((x_1, x_2)\). Hereafter, when dealing with the hyperbolic and Euclidean spaces, we shall consider the upper sheet of the hyperboloid with \( x_0 \geq 1 \) and the Euclidean plane with \( x_0 = 1 \).

The *ambient coordinates* \((x_0, x_1, x_2)\), subjected to the constraint \((2.6)\), are also called *Weierstrass coordinates*. In these variables, the metric on \( S^2_\kappa \) follows from the flat ambient metric in \( \mathbb{H}^3 \) divided by the curvature, \( \kappa \), and restricted to \( \Sigma_\kappa \):

\[ \text{d}s^2 = \frac{1}{\kappa} \left( \text{d}x_0^2 + \kappa \left( \text{d}x_1^2 + \text{d}x_2^2 \right) \right) = \frac{\kappa \left( x_1 \text{d}x_1 + x_2 \text{d}x_2 \right)^2}{1 - \kappa \left( x_1^2 + x_2^2 \right)} + \text{d}x_1^2 + \text{d}x_2^2. \quad (2.7) \]

A differential realization of the Lie algebra \( \mathfrak{so}_\kappa (3) \), in terms of first-order vector fields in the ambient coordinates, can be easily deduced from the vector representation \((2.3)\) and reads

\[ J_{01} = \kappa x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}, \quad J_{02} = \kappa x_2 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_2}, \quad J_{12} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}. \]

From it, a phase space (symplectic) realization of the Lie generators of \( \mathfrak{so}_\kappa (3) \), in terms of Weierstrass coordinates \( x_\mu \) and their conjugate momenta \( p_\mu \) \((\mu = 0, 1, 2)\), is obtained by setting \( \partial_\mu \to -p_\mu \) :

\[ J_{01} = x_0 p_1 - \kappa x_1 p_0, \quad J_{02} = x_0 p_2 - \kappa x_2 p_0, \quad J_{12} = x_1 p_2 - x_2 p_1. \quad (2.8) \]

As we will see in the sequel, these three functions will provide the essential building blocks for the integrals of motion of all the Hamiltonians we will deal with in this paper.

From the metric \((2.7)\), the free Lagrangian, \( L_\kappa \), defining the geodesic motion of a particle with unit mass in the 2D space, \( S^2_\kappa \), with ambient velocities, \( \dot{x}_\mu \), is obtained:

\[ L_\kappa = \frac{1}{2\kappa} \left( \dot{x}_0^2 + \kappa \left( \dot{x}_1^2 + \dot{x}_2^2 \right) \right) = \frac{\kappa \left( x_1 \dot{x}_1 + x_2 \dot{x}_2 \right)^2}{2 \left( 1 - \kappa \left( x_1^2 + x_2^2 \right) \right)} + \frac{1}{2} \left( \dot{x}_1^2 + \dot{x}_2^2 \right), \quad (2.9) \]

and the corresponding momenta, \( p_\mu = \partial L_\kappa / \partial \dot{x}_\mu \) \((\mu = 0, 1, 2)\), turn out to be

\[ p_0 = \dot{x}_0 / \kappa, \quad p_1 = \dot{x}_1, \quad p_2 = \dot{x}_2, \quad (2.10) \]
in a way consistent with \((2.6)\), which means that

\[ \Sigma_\kappa \equiv x_0 p_0 + x_1 p_1 + x_2 p_2 = 0. \]

Hence, the kinetic energy, \( T_\kappa \), in ambient coordinates reads

\[ T_\kappa = \frac{1}{2} \left( p_0^2 + p_1^2 + p_2^2 \right) = \frac{\kappa \left( x_1 p_1 + x_2 p_2 \right)^2}{2 \left( 1 - \kappa \left( x_1^2 + x_2^2 \right) \right)} + \frac{1}{2} (p_1^2 + p_2^2). \quad (2.11) \]

Evidently, the ambient coordinates, \( x_\mu \), can be parametrized in terms of two intrinsic quantities in different ways. For our purposes, we shall introduce the so-called geodesic polar coordinates (as a generalization of the usual Euclidean polar ones) and two sets of projective variables.
2.2. Geodesic polar variables

Let us now consider a point, \(Q \in S^2_\kappa\), with ambient coordinates \((x_0, x_1, x_2)\). The geodesic polar coordinates \((r, \phi)\) are defined \([13]\) through the following action of two of the one-parametric subgroups (2.4) onto the origin \(O = (1, 0, 0)\), namely

\[
Q(x_0, x_1, x_2) \equiv \exp \{ \phi J_{12} \} \exp \{ r J_{01} \} O,
\]

which means that

\[
\begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
C_\kappa(r) \\
S_\kappa(r) \cos \phi \\
S_\kappa(r) \sin \phi
\end{pmatrix}.
\]

Let now \(l_1\) and \(l_2\) be two base geodesics in \(S^2_\kappa\) which are orthogonal at the origin, \(O\). Then the ‘radial’ coordinate, \(r\), is the geodesic distance between \(Q\) and \(O\) measured along the geodesic, \(l\), that joins both points, while \(\phi\) is the angle which determines the orientation of \(l\) with respect to the base geodesic, \(l_i\) (see \([9]\) for details). Hence each, ‘translation’ generator, \(J_0\), moves \(O\) along the base geodesic, \(l_i\) (\(i = 1, 2\)), while the ‘rotation’ one, \(J_{12}\), leaves \(O\) invariant. According to each specific Riemannian space, \(S^2_\kappa\), we find that:

- In the sphere \(S^2 \equiv S^2\), we have \(\kappa = 1/R^2 > 0\), and the ‘radial’ coordinate, \(r\), has dimensions of length, \([r] = [R]\). Notice that the dimensionless coordinate, \(r/R\), which is an ordinary angle, is usually taken instead of \(r\) (see, e.g., \([14]\)). In this case, \(r \in [0, \pi/\sqrt{\kappa}]\) while \(\phi \in [0, 2\pi]\).

- In the hyperbolic or Lobachevski space \(S^2 \equiv H^2\) with \(\kappa = -1/R^2 < 0\), the ‘radial’ coordinate, \(r\), has also dimensions of length, but now \(r \in [0, +\infty)\) and \(\phi \in [0, 2\pi]\).

- Finally, in the flat (contracted) Euclidean plane, \(S^2_0 \equiv E^2\) with \(\kappa = 0\) (\(R \rightarrow \infty\)), we recover the usual polar coordinates such that \(r \in [0, +\infty)\) and \(\phi \in [0, 2\pi]\).

It is straightforward to check that, by introducing (2.12) in the ambient metric (2.7) and in the free Lagrangian (2.9), we obtain

\[
ds^2 = dr^2 + S_\kappa^2(r) d\phi^2,
\]

and the conjugate momenta to the coordinates \((r, \phi)\) turn out to be

\[
p_\phi = S_\kappa(r) \dot{\phi}.
\]

Hence, the kinetic energy reads

\[
\mathcal{T}_\kappa = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa^2(r)} \right).
\]

Furthermore, by substituting (2.13) within (2.10), we obtain the relationships between the ambient and the geodesic polar momenta, \(P \equiv (p_r, p_\phi)\), which are summarized in table 1 with a symplectic realization of the \(so(3)\) generators (2.8) in terms of geodesic polar variables, \(J_{\mu} = (r, \phi, p_r, p_\phi)\). It is worth remarking that the kinetic energy, \(\mathcal{T}_\kappa \equiv \frac{1}{2} C\) (2.14), can also be recovered by computing the symplectic realization of the Casimir function, \(\mathcal{C}\) (2.2), of \(so(3)\).

We recall that these geodesic polar coordinates were the ones used (along with the so-called geodesic parallel coordinates) in the classification of the 2D superintegrable systems with quadratic integrals of motion in the momenta on the 2D sphere and hyperbolic space performed in \([8]\), in which both the curved isotropic oscillator (Higgs oscillator) and the 1: 2
Table 1. Expressions for the ambient variables \((x, \ P)\), the free Hamiltonian, \(T_x\), and the symplectic realization of the Lie–Poisson generators, \(J_{\mu}\), of \(so_3(3)\), in terms of geodesic polar, Poincaré, and Beltrami canonical variables. The specific expressions for \(S^2, H^2,\) and \(E^2\) are obtained when \(\kappa > 0, \ k < 0\) and \(\kappa = 0\), respectively.

| Polar variables \((r, \phi)\) | Poincaré variables \((\tilde{q}, \tilde{p})\) | Beltrami variables \((q, p)\) |
|-----------------------------|---------------------------------|-----------------------------|
| \(x_0\) \(= \ C_x(r)\)         | \(= \frac{1 - \eta q^2}{1 + \eta q^2}\) | \(= \frac{1}{(1 + \eta q^2)^{1/2}}\) |
| \(x_1\) \(= S_x(r) \cos \phi\)       | \(= \frac{\eta h}{1 + \eta q^2}\)       | \(= \frac{\eta h}{(1 + \eta q^2)^{1/2}}\) |
| \(x_2\) \(= S_x(r) \sin \phi\)        | \(= \frac{\eta l}{1 + \eta q^2}\) | \(= \frac{\eta l}{(1 + \eta q^2)^{1/2}}\) |
| \(p_0\) \(= -S_x(r) \tilde{p}\)       | \(= -q \tilde{p}\)                     | \(= \sqrt{1 + \kappa q^2} (q \cdot p)\) |
| \(p_1\) \(= C_x(r) \cos \phi \tilde{p} - \frac{\sin \phi}{S_x(r)} p_\phi\) | \(= \frac{1}{2} \left[ (1 + \kappa q^2) p_1 - 2 \kappa (q \cdot \tilde{p}) \tilde{q}_1 \right]\) | \(= \sqrt{1 + \kappa q^2} p_1\) |
| \(p_2\) \(= C_x(r) \sin \phi \tilde{p} + \frac{\cos \phi}{S_x(r)} p_\phi\) | \(= \frac{1}{2} \left[ (1 + \kappa q^2) p_2 - 2 \kappa (q \cdot \tilde{p}) \tilde{q}_2 \right]\) | \(= \sqrt{1 + \kappa q^2} p_2\) |

\[T_x = \frac{1}{2} \left( \tilde{p}^2 + \frac{\eta^2}{S_x(r)} \right)\]

\[= \frac{1}{2} \left( 1 + \kappa q^2 \right) \tilde{p}^2\]

\[= \frac{1}{2} \left( 1 + \kappa q^2 \right) (p^2 + \kappa (q \cdot p)^2)\]

\[J_{01} = \cos \phi \tilde{p} - \frac{\sin \phi}{S_x(r)} p_\phi\]

\[= \frac{1}{2} \left[ p_1 + \kappa (q \cdot \tilde{p}) \tilde{q}_1 + \kappa \tilde{q} p_1 J_{12} \right]\]

\[= p_1 + \kappa (q \cdot p) \tilde{q}_1\]

\[J_{02} = \sin \phi \tilde{p} + \frac{\cos \phi}{S_x(r)} p_\phi\]

\[= \frac{1}{2} \left[ \tilde{p}_2 + \kappa (q \cdot \tilde{p}) \tilde{q}_2 - \kappa \tilde{q} p_2 J_{12} \right]\]

\[= p_2 + \kappa (q \cdot p) \tilde{q}_2\]

\[J_{12} = p_\phi\]

\[= \tilde{q}_1 \tilde{p}_2 - \tilde{q}_2 \tilde{p}_1\]

\[= q_1 p_2 - q_2 p_1\]
superintegrable oscillator arose. Moreover, these coordinates were also used in the construction of the curved anisotropic 1:1 oscillator presented in [9].

3. Projective coordinates and phase spaces

In this section, we study in detail the two sets of canonical projective variables that are well adapted to both the sphere and, most specifically, to the hyperbolic or Lobachevsky plane: the Poincaré and Beltrami variables. Notice that the Poincaré variables were not considered in [9] and, moreover, we describe in detail, here the domain of both sets of variables according to the value of the curvature, \( \kappa \). As we will see in the sequel, such analysis is essential when computing, through numerical integration, the trajectories of the proposed Hamiltonians on these two curved spaces.

3.1. Poincaré canonical variables

Let us consider the stereographic projection [15] with ‘south’ pole \((-1, 0, 0)\) from the ambient coordinates, \((x_0, x_1, x_2) \in \mathbb{R}^3\), to the Poincaré coordinates, \((\tilde{q}_0, \tilde{q}_1) \in \mathbb{R}^2\). This projection maps any point, \((x_0, x_1, x_2) \in \Sigma_\kappa (2.6)\), through

\[
(x_0, x_1, x_2) \rightarrow (-1, 0, 0) + \lambda (1, \tilde{q}_1, \tilde{q}_2).
\]

Hence, we find that

\[
\lambda = \frac{2}{1 + \kappa \tilde{q}_1^2}, \quad x_0 = \lambda - 1 = \frac{1 - \kappa \tilde{q}_1^2}{1 + \kappa \tilde{q}_1^2}, \quad x = \lambda \tilde{q} = \frac{2\tilde{q}}{1 + \kappa \tilde{q}_1^2}, \quad (3.1)
\]

so that

\[
\tilde{q} = \frac{x}{1 + x_0}, \quad \tilde{q}_1^2 = \frac{1 - x_0}{\kappa(1 + x_0)},
\]

where hereafter, for any object with two components such as \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\), we denote

\[
a^2 = a_1^2 + a_2^2, \quad |a| = \sqrt{a_1^2 + a_2^2}, \quad a \cdot b = a_1b_1 + a_2b_2.
\]

Consequently this projection is well defined for any point \(Q \in \Sigma_\kappa\), except for the south pole \((-1, 0, 0)\) which goes to \(\infty\) in both the sphere and the hyperbolic space. Note that the ambient origin (‘north’ pole) \(O = (1, 0, 0) \in \Sigma_\kappa\) goes to the origin, \(\tilde{q} = (0, 0)\), of the 2D (projective) space, \(S^2_\kappa\). In particular, we find that:

- In the sphere with \(\kappa = 1/R^2 > 0\), \(\tilde{q} \in (-\infty, +\infty)\) and (3.1) map the equator with \(x_0 = 0\) (i.e., \(x^2 = 1/\kappa = R^2\)), onto the circle \(\tilde{q}_1^2 = 1/\kappa = R^2\)–the northern hemisphere with \(x_0 > 0\) onto the region inside that circle, \(\tilde{q}_1^2 < 1/\kappa\), and the southern hemisphere with \(x_0 < 0\) onto the outside region, \(\tilde{q}_1^2 > 1/\kappa\).
- In the hyperbolic space with \(\kappa = -1/R^2 < 0\) and such that \(x_0 \geq 1\) (the upper sheet of the hyperboloid), it is verified that \(\tilde{q} \in [-1/\sqrt{|\kappa|}, +1/\sqrt{|\kappa|}]\) and

\[
\tilde{q}_1^2 = \frac{1}{|\kappa|} \left( \frac{x_0 - 1}{x_0 + 1} \right) \leq R^2,
\]

which is just the Poincaré disk. The points at the infinity in the hyperboloid correspond to the circle, \(\tilde{q}_1^2 = 1/|\kappa| = R^2\) (take \(x_0 \rightarrow +\infty\)).
In the Euclidean plane with $\kappa = 0 (R \to \infty)$, the Poincaré coordinates are proportional to the Cartesian ones, $x \equiv q = 2\tilde{q}$.

If we substitute (3.1) in the ambient metric (2.7) and free Lagrangian (2.9), we get the conjugate Poncaré momenta, $\tilde{p}$, namely

$$\kappa \equiv \tilde{\kappa} = \tilde{\kappa} + \tilde{\kappa} = \tilde{\kappa} + \tilde{\kappa} = \tilde{\kappa} + \tilde{\kappa}.$$

We can compute the Poincaré symplectic realization of $so(3)$ by proceeding in a similar way as presented in section 2. The final result is summarized in table 1. Notice that the factor $1/8$ (instead of $1/2$) in $\kappa$ and in $\kappa$ is fully consistent with the fact that, under the $\kappa \to 0$ limit, Poincaré coordinates are twice the Euclidean coordinates.

It is also important to stress that Poincaré variables lead to a conformally flat diagonal metric with conformal factor, $f$, such that

$$dx^2 = f(|q|)^2 dq^2, \quad f(|q|) = \frac{2}{1 + \kappa q^2}.$$

Therefore, free motion will be described by the kinetic energy Hamiltonian

$$\mathcal{T}_\kappa = \frac{1}{2f(|q|)^2} \tilde{p}^2.$$

Once more, on the Euclidean plane with $\kappa = 0$ these expressions reduce to

$$f(|q|) = 2, \quad dx^2 = 4 dq^2 = d\tilde{q}^2 = d\tilde{x}^2, \quad \mathcal{T}_0 = \frac{1}{8} \tilde{p}^2 = \frac{1}{2} \tilde{p}^2 = \frac{1}{2} \tilde{p}^2,$$

since we have the relations $q \equiv x = 2\tilde{q}$ and $p \equiv \tilde{p} = \frac{1}{2}\tilde{p}$.

### 3.2. Beltrami canonical variables

The Beltrami projective coordinates, $(q_1, q_2) \in \mathbb{R}^2$, are defined through the central projection with pole $(0, 0, 0) \in \mathbb{R}^3$ of a point $(x_0, x_1, x_2) \in \Sigma_\kappa$. Specifically,

$$(x_0, x_1, x_2) \longrightarrow (0, 0, 0) + \mu \left(1, q_1, q_2\right).$$

This is tantamount to say

$$\mu = \frac{1}{\sqrt{1 + \kappa q^2}}, \quad x_0 = \mu, \quad x = \mu q = \frac{q}{\sqrt{1 + \kappa q^2}}. \quad (3.2)$$

Hence

$$q = \frac{x}{x_0}, \quad q = \frac{1 - x_0^2}{\kappa x_0^2}.$$

The ambient origin, $O = (1, 0, 0) \in \Sigma_\kappa$, goes to the origin, $q = (0, 0)$, in the projective $S^2$, as it should. However, the domain of $q$ depends on the value of the curvature, $\kappa$, as follows:

- In $S^2$ with $\kappa = 1/R^2 > 0$, $q \in (-\infty, +\infty)$. However, since the points in the equator with $x_0 = 0$ go to infinity, the projection (3.2) has to be separately defined for the two hemispheres with $x_0 > 0$ and $x_0 < 0$. 


In $\mathbb{H}^2$ with $\kappa = -1/R^2 < 0$ and $x_0 \geq 1$, $q \in [-R, +R]$ and

$$q^2 = \frac{x_0^2 - 1}{|k|^2} \leq R^2,$$

which is the Poincaré disk in Beltrami coordinates. Similar to the Poincaré coordinates, the points at the infinity in the hyperboloid go to the circle, $q^2 = 1/|k| = R^2$.

- In $\mathbb{E}^2 (\kappa = 0)$, the Beltrami coordinates are just the Cartesian ones, $x = q$.

In table 1 we also summarize the Beltrami symplectic realization of the Lie–Poisson algebra, $so_4(3)$. Note that in this case, the kinetic energy Hamiltonian is given by

$$\mathcal{T}_\kappa = \frac{1}{2} \left( 1 + \kappa q^2 \right) \left( p^2 + \kappa (q \cdot p)^2 \right),$$

which was used in [9] for the construction of the anisotropic Higgs oscillator. One can appreciate that the three sets of canonical variables presented in table 1 provide three very different expressions for the kinetic energy. In particular, the two projective coordinates provide polynomial quantities, which will simplify the search of algebraic invariants.

### 4. Two quadratically superintegrable curved oscillators

Now let us use the previously mentioned variables to review the only two curved superintegrable oscillators with integrals that are quadratic in the momenta, as shown in [8]. The first one is the so-called Higgs oscillator [8, 9, 16–22], which is a curved analogue of the Euclidean isotropic $1:1$ oscillator. In terms of ambient, geodesic polar, Poincaré and Beltrami canonical variables, the Higgs Hamiltonian with two Rosochatius potentials is written, respectively, as

$$\mathcal{H}^{1:1}_\kappa = \mathcal{T}_\kappa + \delta \frac{x_1^2}{1 - \kappa x_1^2} + \frac{\lambda_1}{x_1^2} + \frac{\lambda_2}{x_2^2}$$

$$= \mathcal{T}_\kappa + \delta \mathcal{T}_\kappa^2(r) + \frac{\lambda_1}{S_\kappa^2(r) \cos^2 \phi} + \frac{\lambda_2}{S_\kappa^2(r) \sin^2 \phi}$$

$$= \mathcal{T}_\kappa + \delta \left( 1 + \kappa q^2 \right)^2 + \frac{1}{4} \left( 1 + \kappa q^2 \right)^2 \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right)$$

$$= \mathcal{T}_\kappa + \delta q^2 + \left( 1 + \kappa q^2 \right) \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right), \quad (4.1)$$

where the corresponding expressions for the kinetic energy, $\mathcal{T}_\kappa$, are given in (2.11) and in table 1. The Euclidean limit leading to the $1:1$ oscillator with two Rosochatius terms is obtained when $\kappa \to 0$. The integrals of the motion for this Hamiltonian are explicitly given in [9]. A glimpse at (4.1) makes it evident that Beltrami coordinates are the most suitable for both symbolic and numerical computations concerning this system.

The second quadratically superintegrable oscillator found in [8] is a curved version of the anisotropic Euclidean $1:2$ oscillator. In this case, the Hamiltonian reads
As expected, the $\kappa \to 0$ limit leads to the Euclidean 1:2 oscillator with a single Rosochatius term. For further study and for the sake of completeness, let us summarize the full superintegrability properties of this curved 1:2 oscillator in terms of Beltrami variables.

**Proposition 1.** [9] The Hamiltonian (4.2), written in Beltrami coordinates, is endowed with three integrals of motion quadratic in the momenta, and, is given by

$$\mathcal{H}_{k_1}^{1:2} = T_\kappa + \delta \frac{x_1^2}{(1 - \kappa x_1^2)} + 4\delta \frac{x_0^2 x_2^2}{(x_0^2 + \kappa x_2^2)(x_0^2 - \kappa x_2^2)} + \frac{\lambda_1}{x_1^2},$$

$$= T_\kappa + \delta \frac{S_0^2(r) \cos^2\phi}{1 - \kappa S_0^2(r) \cos^2\phi} + 4\delta \frac{T_\kappa^2(r) \sin^2\phi}{1 - \kappa T_\kappa^2(r) \sin^2\phi} + \frac{\lambda_1}{S_\kappa^2(r) \cos^2\phi},$$

$$= T_\kappa + \frac{4q_0^2}{(1 + \kappa q_0^2)^2} - 4\kappa q_0^2 + 4\delta \frac{4q_1^2(1 + \kappa q_1^2)(1 - \kappa q_1^2)}{(1 - \kappa q_0^2)^2 + 4\kappa q_0^2}(1 - \kappa q_0^2)^2 - 4\kappa q_0^2} + \frac{\lambda_1}{4q_1^2}$$

$$= T_\kappa + \delta \frac{\delta q_2^2}{(1 + \kappa q_2^2)} + 4\delta \frac{q_1^2 + \kappa q_2^2}{(1 + \kappa q_2^2)(1 - \kappa q_2^2)} + \frac{\lambda_1}{q_2^2}. \quad (4.2)$$

5. A new integrable curved anisotropic oscillator

In [9], the problem of the construction of an integrable anisotropic generalization of the Higgs oscillator (4.1) was faced, and the following integrable Hamiltonian, $\mathcal{H}_H^{k_1, k_2}$, was found. This Hamiltonian can be rewritten in terms of the four types of canonical variables as

$$\mathcal{H}_H^{k_1, k_2} = T_\kappa + \frac{S_0^2(r) \cos^2\phi}{1 - \kappa S_0^2(r) \cos^2\phi} + 4\delta \frac{T_\kappa^2(r) \sin^2\phi}{1 - \kappa T_\kappa^2(r) \sin^2\phi} + \frac{\lambda_1}{S_\kappa^2(r) \cos^2\phi},$$

where $T_\kappa$, $J_{\mu}$ are provided in table 1 and $\mathcal{H}_H^{k_1, k_2} = T_\kappa + \frac{S_0^2(r) \cos^2\phi}{1 - \kappa S_0^2(r) \cos^2\phi} + 4\delta \frac{T_\kappa^2(r) \sin^2\phi}{1 - \kappa T_\kappa^2(r) \sin^2\phi} + \frac{\lambda_1}{S_\kappa^2(r) \cos^2\phi}$. The two sets, $(\mathcal{H}_H^{k_1, k_2}, L_\kappa)$ and $(\mathcal{H}_H^{k_1, k_2}, L_\kappa)$, are formed by three functionally independent functions.
This ‘anisotropic Higgs oscillator’ only differs from (4.1) in the term containing $\Omega$, and the Higgs system is recovered in the isotropic limit, $\Omega \to 0$. The Hamiltonian, $\mathcal{H}_k^{\delta, \Omega}$, has one independent integral of the motion, which is quadratic in the momenta; numerical integration for many different values of $\Omega$ and initial conditions seems to indicate that this Hamiltonian system is superintegrable only in the case $\Omega = 0$ (see [9] for a detailed discussion).

Moreover, if we take $\Omega = 3\delta$ in (5.1), the anisotropic 1:2 Hamiltonian (4.2) is not recovered. Indeed, for the latter system, all bounded trajectories are periodic (it is a superintegrable system), while for the former system that is not the case. Thus, both systems are completely different curved generalizations of the same Euclidean system with potential $\delta = +\delta$.

Consequently, we conjectured in [9] that each of the Euclidean superintegrable anisotropic oscillators with commensurate frequencies [2, 3] would be the $\kappa \to 0$ limit of a different family of integrable curved anisotropic oscillators. In particular, the first step needed to support this conjecture would consist of obtaining a new curved anisotropic oscillator that, being integrable for any value of the anisotropy parameter, would reduce to the superintegrable 1:2 oscillator when the anisotropy parameter takes the appropriate values.

Such a new integrable Hamiltonian system constitutes the main result of this paper, which is summarized in the following statement.

**Proposition 2.** Let $\mathcal{H}_k$ be the following Hamiltonian, written in Beltrami variables:

$$\mathcal{H}_k = 2 \kappa^2 + \Omega \kappa \lambda_1 \lambda_2 + \frac{\lambda_1}{\lambda_2} \left( \frac{q_1^2}{1 + \kappa q_1^2} + \frac{q_2^2}{1 + \kappa q_2^2} \right).$$

Then, for any value of the real constants $\{\kappa, \Omega_1, \Omega_2, \lambda_1, \lambda_2\}$, the Hamiltonian system, $\mathcal{H}_k$, is integrable, and its integrals of motion (which are quadratic in the momenta) are given by
\[ I_{1,e} = \frac{1}{2} J_{01}^2 + \kappa J_{12}^2 + \Omega_1 \frac{q_1^2 \left( 1 + \kappa q_1^2 \right)}{(1 - \kappa q_1^2)^2} + \kappa \left( \Omega_2 - 4\Omega_1 \right) \frac{q_1^2 q_2^2}{(1 + \kappa q_2^2)(1 - \kappa q_2^2)^2} + \lambda_1 \frac{1 + \kappa q_1^2}{q_1^2} + \lambda_2 \frac{q_1^2}{q_2^2}, \]

\[ I_{2,e} = \frac{1}{2} J_{02}^2 + \Omega_2 \frac{q_2^2}{(1 - \kappa q_2^2)^2} + \frac{\lambda_2}{q_2^2}, \]

and are such that \( \mathcal{H}_e = I_{1,e} + I_{2,e} \) where \( \mathcal{T}_e \) and \( J_{p\epsilon} \) are the functions given in table 1.

This result can be proven through straightforward computations. Several comments are in order:

- When \( \Omega_1 = \delta, \Omega_2 = 4\delta \) and \( \lambda_2 = 0 \), the maximally superintegrable 1: 2 oscillator given in proposition 1 is recovered together with two of its (non-independent) integrals of the motion, although the third independent integral, \( L_e \), ensures that the superintegrability of the system cannot be obtained from proposition 2. Therefore, the system (5.2) can be properly called the ‘anisotropic generalization’ of the system (4.2).

- However, when \( \Omega_1 = \Omega_2 \), the Hamiltonian (5.2) is by no means the Higgs oscillator (4.1) with \( \lambda_2 = 0 \). Indeed, \( \mathcal{H}_e \) does not coincide with the anisotropic Higgs oscillator (5.1) either. Therefore, for arbitrary values of \( \Omega_1 \) and \( \Omega_2 \), the system \( \mathcal{H}_e \) defines a new integrable curved generalization of the Euclidean anisotropic oscillator (1.1).

- The Rosochatius terms containing \( \lambda_1 \) and \( \lambda_2 \) potentials are proper centrifugal barriers when both \( \lambda_1 \) and \( \lambda_2 \) are positive. In that case, these terms can be interpreted (see [9] for a complete geometric discussion) as noncentral oscillators on \( S^2 (\kappa > 0) \) with centers (in ambient coordinates) located at \( O_1 = (0, 1, 0) \) and \( O_2 = (0, 0, 1) \), respectively. Recall that the Higgs oscillator (4.1) is a central oscillator with its center at the origin, \( O = (1, 0, 0) \), for any value of \( \kappa \).

By making use of table 1, proposition 2 can be indeed rewritten in terms of ambient, polar, and Poincaré variables. In particular, the potential, \( \mathcal{U}_e \), of (5.2) reads

\[
\mathcal{U}_e = \Omega_1 \frac{x_1^2}{(1 - \kappa x_1^2)} + \Omega_2 \frac{x_0^2 x_2^2}{(x_0^2 + \kappa x_2^2)(x_0^2 - \kappa x_2^2)^2} + \frac{\lambda_1}{x_1^2} + \frac{\lambda_2}{x_2^2},
\]

\[
\mathcal{U}_e = \Omega_1 \frac{S_k^2(r) \cos^2 \phi}{(1 - \kappa S_k^2(r) \cos^2 \phi)} + \Omega_2 \frac{T_k^2(r) \sin^2 \phi}{(1 - \kappa T_k^2(r) \sin^2 \phi)^2} + \frac{\lambda_1}{S_k^2(r) \cos^2 \phi} + \frac{\lambda_2}{S_k^2(r) \sin^2 \phi},
\]

\[
\mathcal{U}_e = \Omega_1 \frac{4 q_1^2}{(1 + \kappa q_1^2)^2 - 4 q_1^2} + \Omega_2 \frac{4 q_2^2 (1 + \kappa q_2^2)^2 (1 - \kappa q_2^2)^2}{((1 - \kappa q_2^2)^2 + 4 \kappa q_2^2)((1 - \kappa q_2^2)^2 - 4 \kappa q_2^2)^2} + \frac{1}{4}(1 + \kappa q_1^2)^2 \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right). \tag{5.3}
\]
All these expressions allow us to confirm that, to the best of our knowledge, this potential differs from other curved anisotropic oscillators given in the literature [9, 22–24]. On the other hand, it becomes evident that – again – Beltrami variables are the simplest ones in order to deal with these kinds of systems.

However, although the expression for the potential in terms of Poincaré variables (5.3) is quite cumbersome, these projective coordinates present an unexpected and interesting feature from the integrability viewpoint. If the potential $\kappa\mathcal{U}_\kappa$ for the hyperbolic space is represented on the Poincaré disk (figure 2), the superintegrable case, $\Omega_1 = 1$, $\Omega_2 = 4$, is neatly distinguished from any other value of the anisotropy parameters, while the same superintegrable potential on the sphere does not present any singular feature (figure 1). However, the same plot in Beltrami coordinates does not provide such a visual approach to the superintegrability properties of the potential (recall that the Beltrami coordinates do not cover the complete sphere as the projection of the points on the equator, $x_0 = 0$, goes to infinity). These facts seem to indicate that the use of projective phase spaces can be meaningful from a qualitative viewpoint and furthermore, as we will see in the sequel, for the numerical integration of the dynamics.

6. Numerical integration and superintegrability

So far, we have obtained a new family of anisotropic curved oscillators, $\mathcal{H}_\kappa$ (5.2), that generalize the curved superintegrable 1:2 system (4.2). We have also explicitly proven the complete integrability of $\mathcal{H}_\kappa$, but its superintegrability is only ensured for the particular case when $\Omega_2 = 4\Omega_1$ and $\lambda_2 = 0$, which is just (4.2). Due to [8], we know that this is the only possible superintegrable case of $\mathcal{H}_\kappa$ whose constants of motion are quadratic in the momenta.

However, this poses the question of whether other values of $\Omega_1$ and $\Omega_2$ exist for which $\mathcal{H}_\kappa$ is also superintegrable. If that were the case, the additional integral of the motion (which would play the role of $\mathcal{L}_\kappa$ (4.3) for the 1:2 case) should be of higher degree in the momenta and it should depend analytically on the curvature parameter, $\kappa$. Moreover, since the Euclidean limit $\kappa \to 0$ is always well defined, the only possibilities for the superintegrability of $\mathcal{H}_\kappa$ would be provided by the choices for the $\Omega_1$ and $\Omega_2$ parameters that lead to superintegrable systems in the Euclidean limit, (i.e., for values $(\Omega_1, \Omega_2)$ that give rise to commensurate frequencies when $\kappa = 0$).

We remark that there is no shortcut available to perform the explicit algebraic search of the possible additional integral for the $\mathcal{H}_\kappa$ with $(\Omega_1, \Omega_2)$ associated to a commensurate Euclidean oscillator. Nevertheless, since taking into account that the superintegrability of a system implies that all its bounded trajectories are periodic (closed), we have performed a systematic numerical integration of $\mathcal{H}_\kappa$ for many different values $(\Omega_1, \Omega_2)$, and several initial conditions for each value. As a result of this numerical investigation, we find that the only closed trajectories for $\mathcal{H}_\kappa$ are found for $\Omega_2 = 4\Omega_1$ and $\lambda_2 = 0$, which corresponds to the quadratically superintegrable system described in Proposition 1. Therefore, we conjecture that $\mathcal{H}_\kappa$ is only integrable for generic values $(\Omega_1, \Omega_2)$.

We present some of these trajectories for $\mathcal{H}_\kappa$ on $S^2$ and $H^2$ in figures 3 and 4, respectively. They are plotted in ambient coordinates $(x_0, x)$, and Rosochatius terms are neglected ($\lambda_1 = \lambda_2 = 0$). The numerical integration has been performed in Beltrami coordinates due to the simpler explicit form of (5.2) with respect to (5.3). The initial Beltrami data have been chosen in such a manner that each trajectory is always confined in the hemisphere with $x_0 > 0$ in $S^2$, thus avoiding the problems with the
equator, \( x_0 = 0 \). Furthermore, the same trajectories are plotted in figures 5 and 6, but they now consider the projective plane with Beltrami coordinates \((q_1, q_2)\), with the same initial data as before.

As expected, a closed Lissajous-type 1: 2 curve only appears in figures 3(c), 4(c), 5(c), and 6(c) as a trajectory of the superintegrable \( \mathcal{H}^{1:2} \) case with \( (\Omega_1 = 1, \Omega_2 = 4) \). The remaining values of \((\Omega_1, \Omega_2)\) always provide non-periodic curves, despite the fact that they include the \((\Omega_1 = 1, \Omega_2 = 1)\) and \((\Omega_1 = 1, \Omega_2 = 9)\) cases, which do correspond to commensurate oscillators in the Euclidean limit. Note also that the projective Beltrami plots included in figures 5 and 6 allow for a better numerical intuition about the non-superintegrability of the system.

Finally, we remark that the dynamical consequences of the Rosochatius \( \lambda_i \)-terms in (5.2) can be numerically tested through an analysis similar to the one carried out in [9].
for the Hamiltonian (4.1). In particular, we find that the addition of the Rosochatius $\lambda_1$-term in (5.2) does not alter the integrability properties of $\mathcal{H}_e$. When $\lambda_1 > 0$, this term provides a centrifugal barrier that restricts the configuration space, but the shape of the trajectories is the same and the only closed ones are those obtained in the superintegrable case. However, it is worth stressing that if we consider the second Rosochatius term ($\lambda_2 \neq 0$) in (5.2), we find numerically that even the trajectories for the $\Omega_2 = 1, \Omega_2 = 4$ case are nonclosed. This seems to indicate that the superintegrability of (5.2) for $\Omega_2 = 1, \Omega_2 = 4$ is broken when the second Rosochatius term is added.

We would like to stress that this is a new and somewhat unexpected result. In this respect, we recall that if the two Rosochatius terms are added to the 2D Higgs oscillator in the form (4.1), the quadratic superintegrability of the system is preserved [8, 18, 19]. In contrast, it is
well known that the quadratic superintegrability of the Kepler–Coulomb system, both in its (flat) Euclidean and curved versions on $S^2$ and $H^2$, is preserved only if a single Rosochatius term is added [8]. Nevertheless, when the second Rosochatius term is considered, the Euclidean and curved Kepler–Coulomb Hamiltonians have been shown to be superintegrable, but in this case with an ‘additional’ quartic integral [25, 26]. In this sense, one could expect that the full ($\Omega_1 = 1$, $\Omega_2 = 4$) Hamiltonian (5.2) with both Rosochatius potentials could be superintegrable with higher-order integrals, but this does not seem to be the case, and probably the anisotropy of the Hamiltonian plays a relevant role in this respect.

Figure 3. Some trajectories on the sphere, $S^2$, for the Hamiltonian $\mathcal{H}_\kappa$ (5.2) with $\kappa = +1$, $\Omega_1 = 1$, and without Rosochatius terms $\lambda_1 = \lambda_2 = 0$. They are plotted in $\mathbb{R}^3$ with ambient coordinates $(x_0, \mathbf{x})$ fulfilling $x_0^2 + x^T \mathbf{x}^2 = 1$. Time runs from $t=0$ to $t=8$ and the Hamilton equations are solved in terms of Beltrami variables for the initial data $q_1 = 1$, $q_2 = -0.5$, $\dot{q}_1 = 1$, $\dot{q}_2 = 2$: (a) $\Omega_2 = 1$ (the 1:1 case), (b) $\Omega_2 = 3$, (c) $\Omega_2 = 4$ (the superintegrable 1:2 case), and (d) $\Omega_2 = 9$ (the 1:3 case). The only closed Lissajous-type trajectories correspond to the superintegrable case (c).
7. Concluding remarks and open problems

The results presented here seem to strongly support the conjecture proposed in [9]: for each commensurate $mn$: Euclidean oscillator, there should exist a different integrable anisotropic oscillator, $\mathcal{H}_k^{\Omega_1, \Omega_2}$, on $S^2$ and $H^2$ with arbitrary parameters $(\Omega_1, \Omega_2)$, and such that $\mathcal{H}_k^{\Omega_1, \Omega_2}$ has the former Euclidean system as the zero curvature $\kappa \to 0$ limit when the $(\Omega_1, \Omega_2)$ parameters are appropriately tuned to the $mn$ commensurability condition. Moreover, when the $(\Omega_1, \Omega_2)$ parameters correspond to the $mn$ condition, the Hamiltonian $\mathcal{H}_k^{\Omega_1, \Omega_2}$ should be superintegrable, and its integrals of the motion should have the same degree in the momenta as the ones for the commensurate $mn$: Euclidean oscillator.

So far, this conjecture has been proven to be valid for the $1: 1$ system (the Higgs oscillator), whose anisotropic counterpart fulfilling the above conditions is (4.1), and for the

![Figure 4](image-url)
system (the superintegrable oscillator (4.2)), whose counterpart (5.2) has been studied throughout this paper. Obviously, in order to proceed with a rigorous proof of this statement, a generic expression for the curved superintegrable analog of the Euclidean oscillator is needed, and this remains a challenging open problem.

There are also two methodological aspects of our results that we think deserve some attention. On one hand, the use of projective coordinates in this kind of algebraic integrability problem should be explored in more detail, since, as we see in figure 2, it could provide a qualitative geometric insight into the specific features of superintegrable systems. On the other hand, the fact that by working appropriately with the curvature, $\kappa$, as a 'contraction' parameter, all the algebraic and geometric aspects of the transition between flat and curved dynamics become much more evident. In this framework, it is also worth mentioning that the Lorentzian geometry counterpart of the results presented here (i.e., the corresponding integrable systems on the (1+1) dimensional (anti-)de Sitter and Minkowskian spacetimes) could be obtained from the results presented here by appropriately introducing a new

Figure 5. Trajectories on $S^2$ corresponding to figure 3, but now plotted on the 2D projective plane in terms of Beltrami variables $(q_1, q_2)$: (a) $\Omega_2 = 1$ (the 1: 1 case), (b) $\Omega_2 = 3$, (c) $\Omega_2 = 4$ (the superintegrable 1: 2 case), and (d) $\Omega_2 = 9$ (the 1: 3 case). The projective dynamics makes it more apparent that the trajectories for the non-superintegrable cases are non-closed.
Figure 6. Trajectories on $H^2$ from figure 4, but plotted on the 2D projective plane in terms of Beltrami variables $(q_1, q_2)$: (a) $\Omega_2 = 1$ (the 1: 1 case), (b) $\Omega_2 = 3$, (c) $\Omega_2 = 4$ (the superintegrable 1: 2 case), and (d) $\Omega_2 = 9$ (the 1: 3 case). Again, the only closed trajectory is the one corresponding to the superintegrable (c) case.

A contraction parameter that would be related to the speed of light [13, 20]. Work on all these lines is in progress.

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