Conceptual computation in artificial mathematical intelligence as a paradigm-shifting technique in physics and mathematics

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Abstract. We describe in a very concise way the multidisciplinary program of artificial mathematical intelligence and the need of a new and improved paradigm of formal computation based on conceptual generation, called conceptual computation, with the potential of being a global paradigm-shifting technique in theoretical physics and computational mathematics. Furthermore, we describe some of the most outstanding results regarding fundamental cognitive abilities needed for the materialization of conceptual computation such as analogical reasoning, metaphorical thinking and conceptual blending. Finally, we present a concrete example of the conceptual computation of the notion of binary continuous operation as a formal blend (i.e. many-sorted first order categorical colimit (pushout)) of the notions of continuous function between topological spaces and perfect square topological space.

1. Introduction

There is a quite natural and practical question within the foundations and origins of computer science (computational physics and mathematics) that needs a deeper answer: how much of current mathematics (i.e. mathematics described in contemporary mathematical journals) can be completely generated by a computer program? In other words, how near are we for constructing (programming) a machine which can simulate the way a modern working mathematician (resp. theoretical physicist) usually faces a solvable mathematical (resp. physical-mathematical) conjecture, works some time on it, and finally finds a formal solution for it?

Here it is important to clarify that the main purpose of the former question is to meta-model” and ‘meta-simulate’ the general modern mathematician (or related scientist e.g. theoretical physicist) in the specific intellectual activity of receiving a concrete conjecture (which can be solved within the mathematical framework), working on it and finally giving a clear answer, i.e., writing a (formalizable) solution to the conjecture in the form of either a proof or a counterexample. Note that questions of essentially the same kind were a source of inspiration for Alan Turing and John Von Neumann in order to settle the foundations of modern computer science.

In addition, since the former questions involve implicitly the mathematician’s (resp. theoretical physicist) mind, we should take inspiration from the current most relevant cognitive
theories concerning mathematical reasoning and closely related matters. The most successful theories that we currently have for understanding how our mind works are theories with a formal computational conceptual basis like the computational theory of mind [1]. This fact can be seen as a form of ‘heuristic’ support for the thesis that it is possible to meta-model, (formally and computationally) the intellectual job of a working mathematician (resp. theoretical physicist).

A second theoretical support is the fact that modern mathematics are essentially founded and conceptually bounded on Zermelo-Fraenkel set theory with choice (ZFC), proof, recursion, and model theory [2]. That means that the solution of a solvable conjecture should be precisely described as a formal (logical) consequence of the axioms of ZFC, using a finite (or recursively generated) number of inference rules and initial premises. More precisely, we can start with Newmann-Bernays-Gödel set theory (NBG), which is based on a finite axiomatization and, at the same time, is a conservative extension of ZFC [2].

In other words, when a mathematician (or a theoretical physicist) finally finds a correct solution of a conjecture, then the result of his/her research is simply a kind of computation of a theoretically-feasible computer program, which starts to run all the possible proofs of provable theorems of ZFC, starting from a finite sub-collection of axioms and following precise (logical) mechanical deduction rules. Here it is important to mention that we are focusing on solvable conjectures, i.e., on problems having an explicit formal proof or counterexample within the ZFC framework. These problems constitute mathematics (resp. theoretical physics) being studied by at most of the mathematicians today, and these are the ones producing most of the concrete applications.

Here, it is worth to mention that, in practical terms, the daily mathematics being develop today are mostly (i.e. quantitatively speaking) solvable mathematics, namely, the mathematical facts being published in the most prestigious mathematical journals are explicit proofs of theorems (resp. concrete counterexamples of conjectures). Therefore, we implicitly will restrict ourselves to these kind of mathematical sentences within this article.

The former problem or challenge is a very concise way of describing the multidisciplinary program of artificial mathematical intelligence (AMI), i.e., the construction (implementation) of a global interactive software being able to solve (conceptually) human solvable mathematical conjectures (in an interactive way) in less time than an average professional mathematician (resp. theoretical physicist) [3,4].

One of the main techniques that is needed for the fulfillment of AMI is the development of a more robust theoretical and computational framework for being able to compute or generate (mathematical) concepts in an artificial-human-style manner. This new kind of methodological (and in development) approach can be called conceptual computation. In this paper, we will show an explicit example of what we mean more precisely in this context. Before that we need to give a very brief introduction of some of the most fundamental cognitive abilities that our mind uses for conceptual creation in mathematics and related fields.

2. Results and discussion

In the following sections we discuss three of the most fundamental cognitive abilities used by the human mind during formal creation, i.e. analogical reasoning, metaphorical thinking and conceptual blending. Furthermore, we present an explicit computational formalization of conceptual blending in a common algebraic specification language.

2.1. Some of the most fundamental cognitive abilities for mathematical reasoning

Recent work in AI, cognitive sciences, and computational creativity has shown that analogy making [5], metaphorical reasoning [6] and conceptual blending [7] are part of a bigger collection of fundamental cognitive processes being present in all-day mathematical reasoning and mathematical concept creation.
All these results were obtained by a careful meta-analysis of classical mathematical and physical concepts like the ones of the natural, rational numbers and the Rutherford atom model [5, 8].

2.1.1. Analogical reasoning. The ability to find commonalities between two objects living in multiple conceptual environments, seems to be a very omnipresent cognitive process within all the scientific disciplines. In particular, the invention of well-known results in physics as the Rutherford atom model and the discovery of the complex numbers in pure mathematics can be seen as emerging from an analogical process [5]. In general, outstanding mathematicians of the twentieth century like Andre Weil pointed out the prominent role that analogical reasoning played in the development of modern mathematics [9].

Usually, analogy making is modeled more from a perspective of transferring logical information from a source domain into a target domain in order to obtain new qualitative insights regarding the proof of a particular logical or mathematical statement described in the target [10]. Other approaches aim to find analogical matches for sentences through a restricted higher-order anti-unification framework by computing the least general generalization of pairs of terms defined by means of a fixed collection of basic (first- and higher-order) substitutions [5]. In addition, there are formalizations that use a resolution calculus for finding new models (resp. refutations) by means of analogies with models (resp. refutations) coming from a knowledge base [11].

2.1.2. Conceptual blending. By considering quite more sophisticated examples coming from commutative algebra and algebraic number theory like the ones of prime ideals and Dedekind domains, it has been possible to co-discover (with the help of a computer program) a new mathematical class of rings, i.e., the containment-division rings [12], and to intuit that the cognitive process of formal conceptual blending could go beyond the abstract reconstruction of some mathematical notions [3, 12].

Specifically, conceptual blending starts to be considered as a kind of formal “meta-generator” of mathematical concepts and theories [12]. In particular, in [3] it was shown explicitly how to generate fundamental notions of Fields and Galois theory through formal conceptual blending using a categorical formalization in terms of colimits of theories in many-sorted first-order logic. In addition, the explicit implementations were written in the common algebraic specification language (CASL) [13]. A fundamental aspect of this construction is that the five generating concepts used belong to several areas of mathematics like topology, group theory, and linear algebra. This fact shows implicitly a kind of “global and computationally-feasible soundness” of this specific formalization of conceptual blending in order to obtain suitable and implementable meta-mathematical models of concept creation and theorem proving.

In fact, one of the most prominent instances of current research projects in this direction is the European Union project, concept creation theory (COINVENT), which aims to construct a cognitive-inspired and computational-feasible framework for concept creation based on Fauconnier and Turner’s conceptual blending [14]. In fact, as suggested in [3], it is possible to show that, essentially the most basic notions of field Theory can be explicitly generated through conceptual blending (in a formulation of colimits) starting with 5 basic mathematical concepts coming explicitly and implicitly from several areas of modern mathematics.

2.1.3. Metaphorical reasoning. Pure mathematics has seen a tremendous explosion of metaphorically-inspired creative results during the last seventy years. For instance, new whole areas have appeared based on the effort of trying to understand one specific mathematical sub-domain in terms of others. As illustrative examples we can mention the development of modern algebraic geometry mainly within the Grothendieck’s school by means of introducing categorical
and homological methods to the classic framework described in terms of classical varieties and their rings of coordinates [15]; the quite outstanding solution of a classic number theoretical problem as Fermat’s Last Theorem by means of the metaphorical usage of new conceptual frameworks coming from Iwasawa Theory, the theory of modular forms and the theory of elliptic curves [16]; and integration of seminal methods coming from algebraic topology to fundamental notions of modern algebraic geometry as the development of a homotopy theory for schemes [17], among others.

The former results suggest that we can identify, formalize, and classify the most fundamental cognitive processes, which allow us to discover new mathematical notions and to state and subsequently to prove novel mathematical conjectures. This has the purpose, among others, of getting more general and accurate meta-formalizations of mathematical reasoning in several domains, such as propositional logic, (basic) commutative algebra and elementary arithmetic.

2.2. An enlightening example of the conceptual computation/generation of the notion of binary continuous operation in terms of formal conceptual blending

In this section, we show the explicit result of a conceptual generation of the mathematical notion of binary continuous operation using a categorical formalization of conceptual blending in terms of colimits in the co-complete category of mathematical concepts specified in a many-sort first-order logic setting (see [3]), where the particular implementations are done in the heterogeneous tool set (HETS) [18]. Since, as far as the authors know, there are not previous mathematical concepts of this kind of complexity implemented in HETS we should describe the concepts in the most complete manner. Therefore, a lot of preliminary set-theoretical facts and notions were explicitly included into the specifications. The two initial concepts used were (1) continuous functions between topological spaces and (2) perfect square topological space. Due to technical reasons we add a ‘prime’ symbol to some sort because we sometimes we need to deal with a sort also as an element. Furthermore, we add extra symbols for Cartesian products (sometimes as elements) due to the fact that HETS do not allow to do that in a straightforward way. Moreover, the symbolic identifications between the two spaces were codified in a very elementary generic space $G$. So, let us present the explicit conceptual computation obtained in HETS after doing small style improvements for ease of the non-specialized reader (Algorithm 1).

**Algorithm 1.** Conceptual generation of the mathematical notion of binary continuous operation.

```plaintext
spec contBinFunc =
sorts PX, PX, Sets, TX, TXX, X, XX
sorts TX < PX; TXX < PXX; PX, PXX, X, XX < Sets
op EmpSet : Sets
op PX’ : Sets
op PXX’ : Sets
op TX’ : Sets
op TXX’ : Sets
op Uni__ : Sets → Sets
op X’ : Sets
op XX’ : Sets
op __inter__ : Sets × Sets → Sets
op __ordpair__ : Sets × Sets → Sets
op __prod__ : Sets × Sets → Sets
op f : XX → X
op inversef : TX → TXX
pred __el__ : Sets × Sets
```
3. Conclusions
Conceptual computation possesses a wide potential into the creation of new theoretical-computational formal frameworks for the fulfillment of new forms of artificial intelligence like, for instance, artificial mathematical intelligence, with a huge potential in (formal) mathematics and (theoretical) physics. As the reader could realize this new kind of computational paradigm requires a quite sophisticated background in pure mathematics, computer, theoretical physics and cognitive science. So, it have an intrinsic multidisciplinary nature. Finally, conceptual
computation has the great advantage of being able of simulating one of the most mysterious features of the human creative mind: formal conceptual creation in (pure and applied) mathematics and (theoretical) physics. Thus, it is a very promising emerging field involving meta-mathematics, computer, (theoretical) physics and cognitive science (among others), for modeling unsettled conceptual problems in these fields.

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