Partial decode-forward for quantum relay channels

Ivan Savov*, Mark M. Wilde* and Mai Vu†

* School of Computer Science and † Electrical and Computer Engineering Department, McGill University, Montréal, Canada

Abstract—A relay channel is one in which a Source and Destination use an intermediate Relay station in order to improve communication rates. We propose the study of relay channels with classical inputs and quantum outputs and prove that a “partial decode and forward” strategy is achievable. We divide the channel into many blocks and build codes in a randomized, block-Markov manner within each block. The Relay performs a standard Holevo-Schumacher-Westmoreland quantum measurement on each block in order to decode part of the Source’s message and then forwards this partial message in the next block. The Destination performs a novel “sliding-window” quantum measurement on two adjacent blocks in order to decode the Source’s message. This strategy achieves non-trivial rates for classical communication over a quantum relay channel.

I. INTRODUCTION

Suppose that a Source wishes to communicate with a remote Destination. Suppose further that a Relay is available that can decode the messages transmitted by the Source during one time slot and forward them to the Destination during the next time slot. With the Relay’s help, the Source and Destination can improve communication rates because the Destination can decode the intended messages in parallel from the channel outputs at two consecutive time slots. In this way, useful information is received both from the Source and the Relay.

The relay channel has been studied extensively in the context of classical information theory [1], [2], [3]. Theorem 1 The discrete memoryless relay channel is modelled as a conditional probability distribution \( p(y_1, y|x, x_1) \), where \( y_1 \) and \( y \) are the respective outputs at the Relay and Destination whenever the Source and Relay input symbols \( x \) and \( x_1 \). Two important families of coding strategies exist for relay channels: compress-and-forward and decode-and-forward [1], [3]. The partial decode-and-forward strategy differs from the decode-and-forward strategy in that it has the Relay decode only part of the message from the Source [1].

The study of quantum channels with information-theoretic techniques has been an active area for some time now [4]. Theoretical interest has focused on classical-quantum channels of the form \( (\mathcal{X}, \mathcal{N}^{X \rightarrow B}(x) \equiv \rho_x^B, \mathcal{H}^B) \), where, for each of the inputs \( x \in \mathcal{X} \), there corresponds an output quantum state, described by a density operator \( \rho_x^B \) in a finite-dimensional Hilbert space \( \mathcal{H}^B \). Classical-quantum channels are a useful abstraction for studying general quantum channels and correspond to the transmitters being restricted to classical encodings. In this setting, single-letter formulas characterize the capacity of point-to-point [5], [6] and multiple-access channels [7] and give achievable rates for other network channels [8], [9], [10].

The study of quantum channels finds practical applications in optical communications. Bosonic channels model the quantum aspects of optical communication channels, where information is encoded into continuous degrees of freedom. It is known that collective quantum measurements on bosonic-channel outputs outperform classical strategies, particularly in the low-photon-number regime [11]. In other words, quantum measurements are necessary to achieve their ultimate capacity. Ref. [11] also demonstrates that classical encoding is sufficient to achieve the Holevo capacity of the lossy bosonic channel, giving further motivation for the theoretical study of classical-quantum models.

In this paper, we develop a “partial decode and forward” strategy for classical-quantum relay channels. Our results here are the first extension of the quantum simultaneous decoding techniques used in [8], [9] to multi-hop networks. In the partial-decode-and-forward strategy given here, the Relay decodes part of the Source’s message in one block and forwards it in the next. The Destination performs a novel “sliding-window” quantum measurement to decode both parts of the Source’s message in two consecutive blocks [12], [2] and in doing so allows for the Source and Destination to achieve non-trivial communication rates. We state our main result in the Section II, introduce the necessary background on quantum systems and quantum decoding in Section III and give the proof in Section IV. We conclude and discuss open problems in Section V.

II. RESULTS

A classical-quantum relay channel \( \mathcal{N} \) is a map with two classical inputs \( x \) and \( x_1 \) and two output quantum systems \( B_1 \) and \( B \). For each pair of possible input symbols \((x, x_1) \in \mathcal{X} \times \mathcal{X}_1 \), the channel prepares a density operator \( \rho_{x,x_1}^{B_1B} \equiv \mathcal{N}^{X \otimes X_1 \rightarrow B_1B}(x, x_1) \),
\[(1)\]
where \( B_1 \) is the Relay output and \( B \) is the Destination output.

Theorem 1 (Partial decode-forward inner bound). Let \( \{\rho_{x,x_1}\} \) be a cc-qq relay channel as in (1). Then a rate \( R \) is achievable, provided that the following inequality holds:
\[R \leq \max_{p(x,x_1)} \min \left\{ \frac{I(XX_1; B)_0}{I(U; B_1|X_1)_0 + I(X; B|X_1U)_0} \right\}, \quad (2)\]
where the information quantities are with respect to the classical-quantum state \( \rho^{UXX_1X_1X_1;}B_1B \equiv \rho_{x,x_1}^{B_1B} \)
\[\sum_{x,u,x_1} p(u, x, x_1) |u|u \otimes |x|\otimes |x_1| \otimes |x_1| \otimes |x_1| \otimes \rho_{x,x_1}^{B_1B}. \quad (3)\]
Our code construction employs codebooks \( \{x^n_1\}, \{u^n\}, \text{ and } \{x^n\} \) generated according to the distribution \( p(x_1)p(u|x_1)p(x|u,x_1) \). We split the message for each block into two parts \( (m, \ell) \in \mathcal{M} \times \mathcal{L} \) such that the rate \( R = R_m + R_\ell \). The Relay fully decodes the message \( \ell \) and re-encodes it directly in the next block (without using binning). The Destination exploits a “sliding-window” decoding strategy \([12, 2]\) by performing a collective measurement on two consecutive blocks. In this approach, the message pair \((m_j, \ell_j)\) sent during block \( j \) is decoded from the outputs of blocks \( j \) and \( j + 1 \), using an “AND-measurement.”

### III. Preliminaries

In this section, we introduce the notation used in our paper and some background information on quantum decoding.

1) Quantum systems: We denote quantum systems as \( B^1 \) and \( B \) and the corresponding Hilbert spaces as \( \mathcal{H}^{B^1} \) and \( \mathcal{H}^B \). We represent quantum states of a system and quantum measurement is a positive operator-valued measure \( \rho \). We denote quantum systems as \( B \).

2) Quantum measurement: Suppose we are given positive operators \( \rho \) by \( \mathcal{M} \) and \( \rho^B \), which is a positive semi-definite operator with trace distance between them. The trace distance between states \( \sigma \) and \( \rho \) is \( \|\sigma - \rho\|_1 = \text{Tr} [\sigma - \rho] \), where \( |X| = \sqrt{XX^*} \). Two states can substitute for one another up to a penalty proportional to the trace distance between them:

**Lemma 1.** Let \( 0 \leq \rho, \sigma, \Lambda \leq I \). Then

\[
\text{Tr} [\Lambda \rho] \leq \text{Tr} [\Lambda \sigma] + \|\rho - \sigma\|_1. \tag{4}
\]

**Proof:** This follows from a variational characterization of trace distance as the distinguishability of the states under an optimal measurement \( M \): \( \|\rho - \sigma\|_1 = 2 \max_{0 \leq M \leq I} \text{Tr} [M (\rho - \sigma)] \).

3) Quantum decoding: In a communication scenario, the decoding operations performed by the receivers correspond to quantum measurements on the outputs of the channel. A quantum measurement is a positive operator-valued measure (POVM) \( \{\Lambda_m\}_{m \in \mathcal{M}} \) on the system \( B^0 \). To be a valid POVM, the set \( \{\Lambda_m\} \) of \( \mathcal{M} \) operators should all be positive semi-definite and sum to the identity: \( \Lambda_m \geq 0, \sum_{m} \Lambda_m = I \).

Suppose we are given positive operators \( \{P_m\}_{m \in \mathcal{M}} \) that are apt at detecting \( \text{Tr}[P_m \rho_m] \geq 1 - \epsilon \) and distinguishing \( \text{Tr}[P_m \rho_{m',\neq m}] \leq \epsilon \) the output states produced by each message. We can construct a valid POVM (known as the square-root measurement \([8, 6] \) by normalizing these operators:

\[
\Lambda_m \equiv \left( \sum_k P_k \right)^{-1/2} P_m \left( \sum_k P_k \right)^{-1/2}. \tag{5}
\]

The error analysis of a square-root measurement is greatly simplified by using the Hayashi-Nagaoka operator inequality.

**Lemma 2** (Hayashi-Nagaoka \([13]\)). If \( S \) and \( T \) are operators such that \( 0 \leq T \) and \( 0 \leq S \leq I \), then

\[
I - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \leq 2 (I - S) + 4T. \tag{6}
\]

3) Error analysis: In the context of our coding strategy, we analyze the average probability of error at the Relay:

\[
\tilde{p}_e^R = \frac{1}{|\mathcal{L}|} \sum_{\ell_j} \text{Tr} \left\{ \left( I - \Gamma_{\ell_j} \right) \rho_{\ell_j} \right\},
\]

and the average probability of error at the Destination:

\[
\tilde{p}_e^D = \frac{1}{|\mathcal{M}||\mathcal{L}|} \sum_{m_j, \ell_j} \text{Tr} \left[ \left( I - \Lambda_{m_j, \ell_j} \right) \rho_{m_j, \ell_j} \right]. \tag{7}
\]

The operators \( I - \Gamma_{\ell_j} \) and \( I - \Lambda_{m_j, \ell_j} \) correspond to the complements of the correct decoding outcomes.

**Definition 1.** An \((n, R, \epsilon)\) partial-decode-and-forward code for the quantum relay channel consists of two codebooks \( \{\rho^n_{m_j, \ell_j}\}_{m_j \in \mathcal{M}, \ell_j \in \mathcal{L}} \) and \( \{\sigma^n_\ell\}_{\ell \in \mathcal{L}} \) and decoding POVMs \( \{\Gamma_\ell\}_{\ell \in \mathcal{L}} \) and \( \{\Lambda_{m_j, \ell_j}\}_{m_j \in \mathcal{M}, \ell_j \in \mathcal{L}} \) such that the average probability of error is bounded from above as \( p_e = \tilde{p}_e^R + \tilde{p}_e^D \leq \epsilon \).

A rate \( R \) is achievable if there exists an \((n, R - \delta, \epsilon)\) quantum relay channel code for all \( \epsilon, \delta > 0 \) and sufficiently large \( n \).

### IV. Achievability Proof

The channel is used for \( b \) blocks, each indexed by \( j \in \{1, \ldots, b\} \). Our error analysis shows that:

- The Relay can decode the message \( \ell_j \) during block \( j \).
- The Destination can simultaneously decode \( (m_j, \ell_j) \) from a collective measurement on the output systems of blocks \( j \) and \( j + 1 \).

The error analysis at the Relay is similar to that of the Holevo-Schumacher-Westmoreland theorem \([5, 6]\). The message \( \ell_j \) can be decoded reliably, if the rate \( R_\ell \) obeys the following inequality:

\[
R_\ell \leq I (U; B_1 | X_1) \delta. \tag{8}
\]

We give a proof in the Appendix.

The decoding at the Destination is a variant of the quantum simultaneous decoder from \([8, 2]\). To decode the message \( (m_j, \ell_j) \), the Destination performs a “sliding-window” decoder, implemented as an “AND-measurement” on the outputs of blocks \( j \) and \( j + 1 \). This coding technique does not require binning at the Relay or backwards decoding at the Destination \([12, 2]\).

In this section, we give the details of the coding strategy and analyze the probability of error at the Destination.

**Codebook construction.** Fix a distribution \( p(u, x, x_1) \) and independently generate a different codebook for each block \( j \):

- Randomly and independently generate \( 2^{n R_\ell} \) sequences \( x^n_j (\ell_{j-1}), \ell_{j-1} \in [1 : 2^{n R_\ell}] \), according to \( \prod_{i=1}^n p(x_i) \).
- For each \( x^n_j (\ell_{j-1}) \), randomly and conditionally independently generate \( 2^{n R_m} \) sequences \( u^n_j (\ell_j|\ell_{j-1}), \ell_j \in [1 : 2^{n R_m}] \), according to \( \prod_{i=1}^n p(u_i|x_{i1}(\ell_{j-1})) \).
• For each $x^n_1(\ell_{j-1})$ and each corresponding $u^n(\ell_{j-1})$, randomly and conditionally independently generate $2^{nR_m}$ sequences $x^n(m_j,\ell_{j-1})$, $m_j \in [1:2^{nR_m}]$, according to the distribution: $\prod_{i=1}^{n} p(x_i|x_{i1}(\ell_{j-1}), u_i(\ell_{j-1})).$

Transmission. The transmission of $(\ell_j, m_j)$ to the Destination happens during blocks $j$ and $j+1$. At the beginning of block $j$, we assume that the Relay has correctly decoded the message $\ell_{j-1}$. During block $j$, the Source inputs the new messages $m_j$ and $\ell_j$, and the Relay forwards the old message $\ell_{j-1}$. That is, their inputs to the channel for block $j$ are the codewords $x^n(m_j,\ell_j,\ell_{j-1})$ and $x^n(\ell_{j-1})$, leading to the following state at the channel outputs:

$$\rho_j^{\ell_j} = B_{\ell_j}^{m_j} B_{\ell_{j-1}}^{x^n(\ell_{j-1})}. $$

During block $j+1$, the Source transmits $(m_{j+1},\ell_{j+1})$ given $\ell_j$, whereas the Relay sends $\ell_j$, leading to the state:

$$\rho_{m_{j+1},\ell_{j+1},\ell_j} = \rho_{x^n(m_{j+1},\ell_{j+1},\ell_j), x^n(\ell_{j-1})}. $$

Our shorthand notation is such that the states are identified by the messages that they encode, and the codewords are implicit.

Decoding at the Destination. We now determine a decoding POVM that the Destination can perform on the output systems spanning blocks $j$ and $j+1$. The Destination is trying to recover messages $\ell_j$ and $m_j$ given knowledge of $\ell_{j-1}$.

First let us consider forming decoding operators for block $j+1$. Consider the state obtained by tracing over the systems $X$, $U$, and $B_3$ in (3):

$$\theta_{X^jU} = \sum_{x_1} p(x_1) |x_1\rangle x_1^X \otimes \tau_{j,1},$$

where $\tau_{j,1} = \sum_{u,x} p(u|x_1) p(x|x_1,u) \rho_{u,x}^{\ell_j}$, Also, let $\bar{\tau}_{j,1}$ denote the following state: $\bar{\tau}_{j,1} = \sum_{x_1} p(x_1) \tau_{j,1}$. Corresponding to the above states are conditionally typical projectors (4) of the following form:

$$\Pi_{\ell_j}^{(j+1)} = \Pi_{x^n(\ell_{j-1})}^{B_{\ell_{j-1}}}, \quad \Pi_{x^n(\ell_{j-1})}^{(j+1)} = \Pi_{\ell_{j+1}}^{B_{\ell_{j+1}}} + 1,$$

which we combine to form the positive operator:

$$\Pi_{\ell_j}^{(j+1)} = \Pi_{x^n(\ell_{j-1})}^{B_{\ell_{j-1}}} \Pi_{x^n(\ell_{j-1})}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)},$$

that acts on the output systems $B_{\ell_{j+1}}$ of block $j+1$.

Let us now form decoding operators for block $j$. Define the conditional typical projector for the state $\rho_{m_j,\ell_j,\ell_{j-1}}$ using (4)

$$\Pi_{\ell_j}^{(j)} = \Pi_{x^n(\ell_{j-1})}^{B_{\ell_{j-1}}} \Pi_{x^n(\ell_{j-1})}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)}.$$

The state obtained from (3) by tracing over $X$ and $B_3$ is

$$\rho_{X^jU} = \sum_{u,x_1} p(u|x_1) p(x|x_1,u) (|u\rangle u^U \otimes |x_1\rangle x_1^X \otimes \rho_{u,x_1}^{\ell_j}),$$

where $\rho_{u,x_1}^{\ell_j} = \sum_x p(x|x_1,u) \rho_{u,x_1}^{\ell_j}$. Define also the doubly averaged state $\bar{\rho}_{x_1}^{\ell_j} = \sum_{u,x} p(x|x_1,u) \rho_{u,x_1}^{\ell_j}$. The following conditionally typical projectors will be useful in our decoding scheme:

$$\Pi_{\ell_j}^{(j)} = \Pi_{x^n(\ell_{j-1})}^{B_{\ell_{j-1}}} \Pi_{x^n(\ell_{j-1})}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)} \Pi_{\ell_{j+1}}^{(j+1)}.$$
The first term (I) is bounded as follows:
\[
\begin{align*}
\text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \\
= \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \\
\leq \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \\
+ \text{Tr}\left[ F_{m_j,\ell_j}^{(j)} \right] = 1
\end{align*}
\]
where the inequality follows from Lemma [3]

We proceed to bound the term \( \beta \) as follows:
\[
\beta = \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j+1)} \right] \\
= \text{Tr}\left[ \left( I - \Pi_{\ell_j}^{(j+1)} \right) \rho_{m_j,\ell_j}^{(j+1)} \right] \\
\leq 1 - \text{Tr}\left[ \Pi_{\ell_j}^{(j)} \rho_{m_j,\ell_j}^{(j+1)} \right] + \rho_{m_j,\ell_j}^{(j+1)} \rho_{m_j,\ell_j}^{(j+1)} - \rho_{m_j,\ell_j}^{(j+1)} \rho_{m_j,\ell_j}^{(j+1)} \right]
\]
where the inequality follows from Lemma [1]

By taking the expectation over the code randomness, we obtain the upper bound:
\[
\begin{align*}
\mathbb{E}_{U^n X^n | X^n_1} \left\{ \beta \right\} &= 1 - \mathbb{E}_{X^n_1} \left[ \Pi_{\ell_j}^{(j+1)} \mathbb{E}_{U^n X^n | X^n_1} \left\{ \rho_{m_j,\ell_j}^{(j+1)} \right\} \right] \\
&\leq 1 - (1 - \epsilon) + 2 \sqrt{\epsilon}.
\end{align*}
\]

The inequality follows from \( \mathbb{E}_{U^n X^n | X^n_1} \left\{ \rho_{m_j,\ell_j}^{(j+1)} \right\} = \tau_{\ell_j} \),
the properties of typical projectors \( \Pi_{\ell_j}^{(j)} \), and Lemma [4]

The error term \( \alpha \) is bounded in a similar fashion.

We can split the sum in the second type of error, (II), as
\[
\sum_{\ell_j,m_j} \mathbb{E}_{U^n X^n | X^n_1} \left\{ \rho_{m_j,\ell_j}^{(j+1)} \right\} = \sum_{\ell_j,m_j} \mathbb{E}_{U^n X^n | X^n_1} \left\{ \rho_{m_j,\ell_j}^{(j+1)} \right\} + \sum_{\ell_j,m_j} \mathbb{E}_{U^n X^n | X^n_1} \left\{ \rho_{m_j,\ell_j}^{(j+1)} \right\}.
\]

\[
\begin{align*}
\sum_{\ell_j,m_j} \left[ \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \right] \\
= \sum_{\ell_j,m_j} \left[ \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \right] \\
+ \sum_{\ell_j,m_j} \left[ \text{Tr}\left[ \left( I - P_{m_j,\ell_j}^{(j)} \right) \rho_{m_j,\ell_j}^{(j)} \otimes \rho_{m_j,\ell_j}^{(j+1)} \right] \right].
\end{align*}
\]

We now analyze the two terms (A) and (B) separately.
Consider the following term:

\[ (B) = \sum_{\ell' \neq \ell, \ell'} \text{Tr} \left[ \rho_{m' \ell'}^{(j)} \rho_{m'' \ell''}^{(j+1)} \otimes \rho_{m''' \ell''' + 1}^{(j+1)} \right] \]

We want to calculate the expectation of \((B)\) under the code randomness \(E_{U^n} X^n_1\). The random variables in different blocks are independent, and so we can analyze the expectations of the terms \((B1)\) and \((B2)\) separately.

Consider first the calculation in block \(j\), which leads to the following bound on the expectation of \((B1)\):

\[ E_{U^n X^n_1} \{ (B1) \} = \mathbb{E}_{U^n X^n_1} \{ \text{Tr} \left[ \rho_{m' \ell'}^{(j)} \rho_{m'' \ell''}^{(j+1)} \rho_{m''' \ell''' + 1}^{(j+1)} \right] \} \]

We proved the achievability of the rates given by the partial decode and forward inner bound, thus extending the study of classical-quantum channels to multi-hop scenarios. Another avenue for research would be to consider quantum communication scenarios, and results here might have applications for the design of quantum repeaters [15].

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APPENDIX

A. Decoding at the Relay

In this section we give the details of the POVM construction and the error analysis for the Relay decoder.

POVM Construction. During block $j$, the Relay wants to decode the message $\ell_j$ encoded in $u^n(\ell_j, \ell_{j-1})$, given the knowledge of the message $\ell_{j-1}$ from the previous block. Consider the state obtained by tracing over the systems $X$ and $B$ in (3):

$$\theta^U_{X, B_1} = \sum_{u, x_1} p(u|x_1) p(x_1) \rho_{x_1}^u \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u},$$

where $\rho_{x_1}^{u} \equiv \sum_u p(x_1, u) \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u}$. Further tracing over the system $U$ leads to the state

$$\theta^X_{X, B_1} = \sum_{x_1} p(x_1) \rho_{x_1}^u \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u},$$

where $\rho_{x_1}^{u} \equiv \sum_u p(x_1, u) \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u} \rho_{x_1}^{u}$. Corresponding to the above conditional states are conditionally typical projectors of the following form

$$\Pi_{\sigma(\ell_{j-1})} = \Pi_{\sigma(\ell_{j-1})} \Pi_{\sigma(\ell_{j-1})}, \quad \Pi_{\sigma(\ell_{j-1})} = \Pi_{\sigma(\ell_{j-1})} \Pi_{\sigma(\ell_{j-1})}.$$  

The Relay constructs a square-root measurement $\{\ell_j\}$ using formula (5) and the following positive operators:

$$P_{\ell_j^m}^{\ell(\ell_{j-1})} = \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})}.$$  

Error analysis. In this section we show that during block $j$ the Relay will be able to decode $\ell_j$ from the state $\rho_{\ell_j^m}^{\ell(\ell_{j-1})}$ provided the rate $R_\ell < I(U; B_1 | X_1) = H(B_1 | X_1) - H(B_1 | U X_1) - \delta$. The bound follows from the following properties of typical projectors:

$$\Tr[\Pi_{\ell_j^m}^{\ell(\ell_{j-1})}] \leq 2^{-n[H(B_1 | U X_1) + \delta]}$$  

$$\Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \leq 2^{-n[H(B_1 | X_1) - \delta]}. \quad \Pi_{\ell_j^m}^{\ell(\ell_{j-1})}.$$  

The average probability of error at the Relay is given by:

$$\bar{\rho}_E \equiv \frac{1}{|\ell|} \sum_{\ell_j} \tr \left( I - \Gamma_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right).$$

We consider the probability of error for a single message $\ell_j$ and begin by applying the Hayashi-Nagaoka operator inequality (Lemma 2) to split the error into two terms:

$$\bar{\rho}_E \equiv \tr \left( I - \Gamma_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) \leq 2 \tr \left( I - P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) + 4 \sum_{\ell_j \neq \ell_j} \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right).$$

We will bound the expectation of the average probability of error by bounding the individual terms. We bound the first term as follows:

$$E(1) = \tr \left( I - P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) = \tr \left( I - \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) = 1 - \tr \left( \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) \leq 1 - \tr \left( \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right),$$

where the inequality follows from Lemma 1.

By taking the expectation over the code randomness we obtain the bound

$$E(U^n X^n X_1^n) (1) = 1 - E(U^n X^n X_1^n) \tr \left( \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) \leq 1 - \tr \left( \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) + 2 \sqrt{\epsilon} \leq 1 - (1 - \epsilon) + 2 \sqrt{\epsilon} = \epsilon + 2 \sqrt{\epsilon}.$$  

The first inequality follows from Lemma 4 and the property

$$\tr \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \geq 1 - \epsilon.$$  

The second inequality follows from:

$$\tr \Pi_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \geq 1 - \epsilon.$$  

To bound the second term we proceed as follows:

$$E(U^n X^n X_1^n) \left\{ \{I\} \right\} = E(U^n X^n X_1^n) \sum_{\ell_j \neq \ell_j} \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right)$$

$$= E(U^n X^n X_1^n) \sum_{\ell_j \neq \ell_j} \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right)$$

$$= E(U^n X^n X_1^n) \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right) \tr \left( P_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right)$$

The expectation can be broken up because $\ell_j \neq \ell_j$ and thus the $U^n$ codewords are independent. We have also used

$$E(U^n X^n X_1^n) \left\{ \rho_{\ell_j^m}^{\ell(\ell_{j-1})} \right\} = \sigma_{\ell_j^m}^{\ell(\ell_{j-1})}.$$  

(26)
We continue by expanding the operator $P_{\ell_j}^{B_{ij}}$ as follows:

$$
= \mathbb{E}_{U^nX^nX_1^n} \sum_{\ell_{j'} \neq \ell_j} \text{Tr} \left[ \Pi_{\bar{\sigma}_{\ell_{j-1}}} \Pi_{\sigma_{\ell'_{j-1}}} \Pi_{\bar{\sigma}_{\ell_{j}}} \right] \\
= \mathbb{E}_{U^nX^nX_1^n} \sum_{\ell_{j'} \neq \ell_j} \text{Tr} \left[ \Pi_{\sigma_{\ell'_{j-1}}} \Pi_{\bar{\sigma}_{\ell_{j}}} \bar{\sigma}_{\ell_{j-1}} \Pi_{\bar{\sigma}_{\ell_{j}}} \right] \\
\leq 2^{-n[H(B_1|X_1) - \delta]} \mathbb{E}_{U^nX^nX_1^n} \sum_{\ell_{j'} \neq \ell_j} \text{Tr} \left[ \Pi_{\sigma_{\ell'_{j}}} \right] \\
\leq 2^{-n[H(B_1|X_1) - \delta]} \sum_{\ell_{j'} \neq \ell_j} 2^n[L(B_1|UX_1) + \delta] \\
\leq |\mathcal{L}| \sum_{\ell_{j'} \neq \ell_j} 2^{-n[I(U;B_1|X_1) - 2\delta]}.
$$

The first inequality follows from using (23) on the expression $\otimes$. The second inequality follows from the fact that $\Pi_{\bar{\sigma}_{\ell_{j-1}}}$ is a positive operator less than the identity. More precisely we have

$$
\Pi_{\sigma_{\ell'_{j}}} \Pi_{\bar{\sigma}_{\ell_{j}}} \Pi_{\bar{\sigma}_{\ell_{j-1}}} = \Pi_{\sigma_{\ell'_{j}}} \Pi_{\bar{\sigma}_{\ell_{j}}} \Pi_{\bar{\sigma}_{\ell_{j}}} \\
\leq \Pi_{\sigma_{\ell'_{j}}} I \Pi_{\sigma_{\ell'_{j}}} \\
= \Pi_{\sigma_{\ell'_{j}}}.
$$

The penultimate inequality follows from (23).

Thus if we choose $R_\ell \leq I(U;B_1|X_1) - 3\delta$, we can make the expectation of the average probability of error vanish in the limit of many uses of the channel.

**Proof conclusion.** Note that the gentle operator lemma for ensembles is used several times in the proof to guarantee that the effect of acting with one of the projectors from the “measurement sandwich” does not disturb the state too much. Furthermore, because each of the outputs blocks is operated on twice, we again depend on the gentle operator lemma to guarantee that the state disturbance is asymptotically negligible.