The time-energy certainty relation

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All energy measurements of a quantum system are prone to inaccuracies. In particular, if such measurements are carried over a finite period of time the accuracy of the result is affected by the length of that period. Here I show an upper bound on such inaccuracies and point out that the bound can be arbitrarily small if many copies of the system are available.

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INTRODUCTION

Since the onset of quantum mechanics the uncertainty principles have played the central role in attempts to comprehend this counterintuitive theory. The importance of these principles springs from the search for limitations of the theory. They tell us what is possible and what is fundamentally unachievable in the quantum world. Although the mathematical structure of quantum mechanics is well understood, its physical interpretation and applicability to the phenomena of the observed world is still debated. In particular, there is some confusion about the status of various time-energy uncertainty relations present in the textbooks on quantum theory [1, 2].

In this work I will focus on one important question that arises naturally in the context of the time-energy uncertainties: How accurate the energy measurement can be if it is carried over a small period of time? It has been noticed that there is no fundamental lower bound on such an accuracy no matter how short the measurement [3, 4]. In the course of this Letter I derive an upper bound on such inaccuracies and point out that the bound can be arbitrarily small if many copies of the system are available.

To find a matrix representation of the last equation it is convenient to arrange the first N states \(|\psi_n\rangle\) as columns of the matrix \(P\). Inserting unity \(PP^{-1}\) before states \(|k\rangle\) in Eq. (3), and multiplying by \(P^\dagger\) from the left results in the generalized eigenvalue problem

\[
U'k' = e^{-i\omega_k \delta t}S |k'\rangle,
\]

where \(U' = P^\dagger \hat{U} P\) and \(S = P^\dagger P\) are \(N \times N\) matrices with matrix elements expressed solely in terms of the autocorrelation function values, \(U'_{mn} = c_{n-m+1}\) and \(S_{mn} = c_{n-m}\). The new eigenvectors are \(|k'\rangle \equiv P^{-1} |k\rangle\).

At this point the problem of extracting all the energy related information of the system in the initial state \(|\psi_0\rangle\)
is solved. Given the measured values of the autocorrelation function \( c_n \), one finds the number \( K \) of eigenstates involved as a rank of the matrix \( S \), eigenfrequencies from the solution of Eq. \( \text{4} \) and coefficients \( d_k \) by solving linear set of equations \( \text{2} \) with now known \( K \) and \( \omega_k \). With known \( K \), \( d_k \) and \( \omega_k \) finding the expectation energy of the system or the corresponding standard deviation is trivial. The method presented above goes under the name of \( \text{9} \), \( \text{10} \), \( \text{11} \). You can find an introduction to the method in Ref. \( \text{10} \).

**INACCURACIES**

The analysis, so far, did not account for the inevitable measurement inaccuracies. The experimental values of the autocorrelation function differ from the unperturbed values that enter Eq. \( \text{2} \) and this difference will affect the computed values of the desired eigenfrequencies. The effect of the experimental imperfection will be further amplified by the requirement that the all measuring is done in a very short time interval \( T = N \delta t \), which satisfies \( T \ll 2\pi/\Delta \omega_{\text{max}} \), with \( \Delta \omega_{\text{max}} \equiv \max \omega_j, k \{ \omega_j - \omega_k \} \) being the largest possible frequency difference in Eq. \( \text{2} \). In other words the total measurement time \( T \) is much shorter than the shortest characteristic time interval of the quantum evolution.

Let denote the measured value of the autocorrelation function by \( \tilde{c}_n \equiv c_n + \eta_n \), where \( \eta_n \) contains all the statistical inaccuracies like noise and effect of finite number of the quantum system copies available. The systematic errors can be, in principle, eliminated and I assume they are absent. Also a time instant \( \eta \delta t \), at which the measurement of \( \tilde{c}_n \) is performed, can be estimated with only finite precision, but in this work it assumes a precise value \( \eta \delta t \), while the related inaccuracy is relegated to \( \eta_n \).

It is possible that even some states spanned over infinite number of energy eigenstates can be considered. One would pick \( K < \infty \) most important eigenstates i.e. with the greatest \( d_k \)'s and treat the remaining part as a noise contributing to \( \eta_n \). However, this contribution would have a systematic character rather than statistical and should account for only a negligible part of the total error \( \eta_n \).

To summarize the discussion of sources of the inaccuracies observe that all of them are contained in complex noise terms \( \eta_n \). In the absence of systematic errors, the noise level can be made arbitrarily small by increasing the number of copies of the quantum system undergoing the measurement. The time-energy uncertainty relation for a measurement on just one quantum system is introduced in \( \text{12} \). The largest magnitude of \( \eta_n \) is denoted by \( \eta_{\text{max}} \).

In order to see the influence of the noise on the extracted eigenfrequencies one must solve the generalized eigenvalue problem \( \text{4} \) with corrupted by noise matrices \( \tilde{U}' \) and \( \tilde{S} \). To do this lets first diagonalize the matrix \( S \). \( S \) is a Hermitian non-negative defined \( N \times N \) matrix. It has exactly \( K \) positive eigenvalues \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_K > \lambda_{K+1} = ... = \lambda_N = 0 \). The smallest positive eigenvalue will be denoted by \( \lambda_{\text{min}} \equiv \lambda_K \). The perturbed matrix \( \tilde{S} \) is still Hermitian and its eigenvalues differ from the unperturbed ones by at most \( N \eta_{\text{max}} \left| \lambda_k - \lambda_k \right| \leq N \eta_{\text{max}} \).

Thus, it is possible to tell the number of positive eigenvalues of \( S \) only for limited level of noise

\[
\eta_{\text{max}} < \frac{\lambda_{\text{min}}}{2N},
\]

where the factor \( 1/2 \) comes from the fact that zero eigenvalues are also perturbed up to \( N \eta_{\text{max}} \). This is the condition one must fulfill to extract all \( K \) frequencies that enter Eq. \( \text{2} \).

In the next step one has to truncate the space spanned by eigenvectors of \( \tilde{S} \) only to those that correspond to \( K \) positive eigenvalues, so that the inverse of the truncated matrix \( \tilde{S} \) could be found. The eigenvectors to positive eigenvalues of \( \tilde{S} \) are arranged as a columns of \( N \times K \) unitary matrix \( \tilde{Q} \). The diagonal \( K \times K \) matrix \( \tilde{D} \equiv \tilde{Q}^* \tilde{S} \tilde{Q} \) is nonsingular and can be inverted. Also the \( \tilde{U}' \) is truncated to \( \tilde{Q}^* \tilde{U}' \tilde{Q} \). The Eq. \( \text{4} \) takes the form an ordinary eigenvalue problem

\[
\tilde{D}^{-1} \tilde{Q}^* \tilde{U}' \tilde{Q} \tilde{k}' = e^{-i\omega_k \delta t} \tilde{Q} \tilde{k}'.
\]  

In general, for an eigenvalue equation \( AX = \lambda X \), where \( X \) is a matrix of eigenvectors of \( A \) and \( \lambda \) is the diagonal matrix of corresponding eigenvalues, the perturbation \( \delta A \) of \( A \) results in perturbation \( \delta \lambda \) of \( \lambda \) and \( X^{-1} (A + \delta A) X = \lambda + \delta \lambda \). It follows that \( \delta \lambda = X^{-1} \delta A X \) and using matrix 2-norm \( ||\delta \lambda|| \leq ||X^{-1}|| ||X|| ||\delta A|| \equiv \kappa(X) ||\delta A|| \), with \( \kappa(X) \) being the condition number of eigenvector matrix \( X \). \( \text{11} \).

Similarly, to estimate the influence of the perturbations on the eigenvalues in Eq. \( \text{4} \) lets rewrite the matrix on the left hand side as follows

\[
(D^{-1} + \delta D^{-1}) \tilde{Q}^* \tilde{U}' + \delta \tilde{U}' \tilde{Q} \approx D^{-1} \tilde{Q}^* \tilde{U}' \tilde{Q} + \delta D^{-1} \tilde{Q}^* \tilde{U}' \tilde{Q} + D^{-1} \tilde{Q}^* \delta \tilde{U}' \tilde{Q},
\]

where \( \delta D^{-1} \equiv N \eta_{\text{max}} \text{diag}(1/\lambda_1^2, ..., 1/\lambda_K^2) \) and \( \delta \tilde{U}' \) has matrix elements \( \eta_{n-m+1} \). The first term on the right hand side of Eq. \( \text{7} \) has complex eigenvalues on the unit circle, which means that \( i \)-th row of \( \tilde{Q}^* \tilde{U}' \tilde{Q} \) must be of order of \( \lambda_i \) to cancel the factor \( 1/\lambda_i \) of \( D^{-1} \). The second term must then have matrix elements at most of order of \( N \eta_{\text{max}}/\lambda_{\text{min}} \). The matrix elements of the third term are clearly of order of \( \eta_{\text{max}}/\lambda_{\text{min}} \). The last two terms combined give an estimate of an error made when computing the eigenvalues of Eq. \( \text{4} \).
where $\kappa(P)$ is a condition number for matrix $P$ and the fraction is an estimate of the Frobenius norm, the upper bound for 2-norm, of the two last matrices in Eq. 8. The condition number for $\tilde{Q}$ matrix is 1, because $\tilde{Q}$ is unitary, and is skipped.

The last inequality gives an estimate of frequency error made when evolution time step $\delta t$ is very short i.e. $T \ll 2\pi/\Delta \omega_{\text{max}}$. Namely,

$$|\tilde{\omega}_k - \omega_k| \delta t \leq \kappa(P) \frac{K(N + 1)}{\lambda_{\text{min}}} \eta_{\text{max}}. \quad (8)$$

Multiplying both sides above by $N\hbar$ one gets something opposite to a time-energy uncertainty relation, the upper bound rather than lower bound for the measured energy inaccuracies. This is why I refer to it as the time-energy bound rather than lower bound for the measured energy. The procedure of finding unknown frequencies contains rather cryptic quantities $\kappa(P)$ and $\lambda_{\text{min}}$. The prescription for calculating $\lambda_{\text{min}}$ contains rather cryptic quantities $\kappa(P)$ and $\lambda_{\text{min}}$. The prescription for calculating $\kappa(P)$ is provided in the next paragraph.

The condition number as presented in Eq. 8 has little use, except perhaps a philosophical value, since it still contains rather cryptic quantities $\kappa(P)$ and $\lambda_{\text{min}}$. The prescription for calculating $\kappa(P)$ is provided in the next paragraph.

Let us take a closer look at the $K \times N$ matrix $P$. As mentioned in the introduction the matrix columns are made of components of vectors $|\psi_n\rangle$, $P_{kn} = \alpha_k e^{-i\omega_k n \delta t}$. The procedure of finding unknown frequencies $\omega_k$ and corresponding amplitudes $d_k = |\alpha_k|^2$ starts from measured values of the time autocorrelation function 2. In the autocorrelation function the phases of the coefficients $\alpha_k$ cancel out and only the modulai $d_k$ enter the Eq. 2. Thus, in the definition of $P$ one can safely use $\sqrt{d_k}$ instead of $\alpha_k$ and

$$P_{kn} = \sqrt{d_k} e^{-i\omega_k n \delta t}. \quad (9)$$

$P$ is a rectangular matrix, in general, and its condition number is expressed in terms of singular values rather then eigenvalues, $\kappa(P) = p_1/p_K$, where $p_1$ and $p_K$ are the largest and the smallest singular values of $P$, respectively. Notice that the matrix $P$ has exactly $K$ singular values because there are exactly $K$ linearly independent states $|k\rangle$ spanning the initial state in Eq. 11. The singular values $p_k$ are the square roots of the singular values of $P^\dagger P = S$. That is $p_k = \sqrt{\lambda_k}$. Finally,

$$\kappa(P) = \sqrt{\frac{1}{\lambda_{\text{min}}}} \leq \sqrt{\frac{\text{Tr}(S)}{\lambda_{\text{min}}}}. \quad (10)$$

At that point the inequality 8 becomes useful because the upper bound is expressed in terms of the trace and the smallest positive eigenvalue of the known matrix $S$. The eigenvalue is discussed in the following section.

### The Smallest Positive Eigenvalue of $S$

The problem of the error estimates of the extracted frequencies is completed in Eq. 8 and Eq. 10. To better understand the relation between the possible error and the system parameters in Eq. 2, I will express the eigenvalue $\lambda_{\text{min}}$ in terms of $K$, $N$, $T$, $\omega_k$ and $d_k$.

It turns out that $\lambda_{\text{min}}$ can be analytically estimated in the regime of short time span of the autocorrelation function. First helpful observation is that $\lambda_{\text{min}}$ is a very small positive number in the limit where $\Delta \omega_{\text{max}} \delta t \ll 1$. The $S$ is Hermitian non-negative defined and it’s matrix elements $S_{mn} = e_{n-m}$ approach $\sum_{k=1}^N d_k = \text{Tr}(S)/N$ if the time step $\delta t$ is small. That means that the largest eigenvalue $\lambda_1$ approaches $\text{Tr}(S)$ and there is little room left for other eigenvalues since the sum of all eigenvalues is $\text{Tr}(S)$. Second hint comes from the property that the nonzero eigenvalues of the $N \times N$ matrix $S = P^\dagger P$ are equal to nonzero eigenvalues of the $K \times K$ matrix $P P^\dagger$ since both are squares of singular values of the $K \times N$ matrix $P$. The last is a direct consequence of the singular value decomposition theorem 13.

Having established the smallness of $\lambda_{\text{min}}$ and the equivalence of nonzero eigenvalues of $P^\dagger P$ and $P P^\dagger$, one sees that the characteristic equation for $S$

$$a_N \lambda^N + ... + a_1 \lambda + a_0 = 0 \quad (11)$$

gives the estimate $\lambda_{\text{min}} \approx -a_0$, where $a_0 = \det(P P^\dagger)$ and $a_1 = -\sum_{k=1}^K \text{Minor}_{kk}(P P^\dagger)$. The matrix $P$ has the following structure

$$P = \text{diag}(\sqrt{d_1}, ..., \sqrt{d_K}) \times V, \quad (12)$$

where the $n$-th column of the $K \times N$ Vandermonde matrix $V$ is given by the vector $[e^{-i\omega_1 (n-1)\delta t}, ..., e^{-i\omega_K (n-1)\delta t}]$. This leads to the determinant

$$\det(P P^\dagger) = \det(V V^\dagger) \prod_{k=1}^K d_k. \quad (13)$$

The determinant of the product $V V^\dagger$ of the Vandermonde matrices in the limit of short time steps $\delta t$ is given by

$$\det(V V^\dagger) \approx \left( \frac{N}{K} \right)^K \prod_{j=1}^{K-1} \left( \frac{N^2 - j^2}{K^2 - j^2} K^2 - j^2 \right) \times$$

$$\times \delta t^{K(K-1)} \prod_{i,j \geq i+1}(\omega_j - \omega_i)^2. \quad (14)$$

The proof of this approximation is beyond the scope of this Letter. It is easy, however, to numerically confirm correctness of Eq. 14.

A minor $\text{Minor}_{kk}(P P^\dagger)$ is also given by Eq. 10, where $k$-th amplitude is dropped and $k$-th frequency in Eq. 14.
is dropped and also $K$ is replaced with $K - 1$ in both Eq. (12) and Eq. (13).

Summarizing the calculations, the smallest eigenvalue of $S$ is expressed in terms of unknown frequencies and amplitudes as follows

$$
\lambda_{\text{min}} \approx \frac{(N + K - 1)!}{(N - K)!}(2K - 1) \left( \frac{(K - 1)!}{(2K - 2)!} \right)^2 \times \frac{1}{\delta^2(K-1)} \prod_{k=1}^{K} d_k \prod_{j=1,j \neq k}^{K} (\omega_k - \omega_j)^2 \quad (15)
$$

for $K > 2$, and

$$
\lambda_{\text{min}} \approx \frac{N - 1/N}{12} \frac{\omega_1 - \omega_2}{\Omega_1 + \Omega_2} [N \delta t]^2 \quad (16)
$$

for $K = 2$.

**DISCUSSION OF THE RESULTS**

Despite the apparent complexity, the estimate of $\lambda_{\text{min}}$ has a simple structure. To see this, it is convenient to first define the effective frequency distance $\Delta$

$$
\frac{1}{\Delta^2(K-1)} = \frac{1}{K} \sum_{k=1}^{K} d_k \prod_{j=1,j \neq k}^{K} (\omega_k - \omega_j)^2 \quad (17)
$$

for $K > 2$ and

$$
\frac{1}{\Delta^2} = \frac{1}{2} \frac{d_1}{d_2} (\omega_1 - \omega_2)^2, \quad K = 2. \quad (18)
$$

For $d_k = 1$ the effective frequency distance

$$
\Delta \in \left[ \min_{k,j} (\omega_k - \omega_j), \max_{k,j} (\omega_k - \omega_j) \right]
$$

with the tendency to approach the left end of the interval, meaning that $\Delta$ characterizes the shortest frequency gaps. The dependence on the amplitudes $d_k$ suggests that $\Delta$ is dominated also by the weakest Fourier components of the autocorrelation function Eq. (2).

The most important feature of $\lambda_{\text{min}}$ is that $\lambda_{\text{min}} \propto (N \delta t \Delta)^2(K-1)$. Using the total time span $T = N \delta t$ one arrives at

$$
\lambda_{\text{min}} \propto (T \Delta)^2(K-1). \quad (19)
$$

Since only short measurements are considered $T \Delta \omega_{\text{max}} \ll 1$ it follows that $T \Delta \ll 1$ and $\lambda_{\text{min}}$ decreases exponentially fast with the dimensionality of the Hilbert space the state $| \psi \rangle$ lives in. That in turn implies that the upper bound on the measured frequency inaccuracy in Eq. (8) is exponentially sensitive to the number of Fourier components $K$ in Eq. (2). Thus, for very short measurement times $T \Delta \ll 1$, even a moderate number of frequencies $K$ can blow the corresponding inaccuracies to a very high level. And this is perhaps the intuition that is hidden behind many time-energy uncertainty relations: “One cannot measure frequencies accurately in a very short time.” This intuition does not reflect a fundamental principle. It merely states the practical difficulty in analyzing systems involving many frequencies.

Although there are time-energy uncertainty principles that are not based on an intuition but on a rigorous derivation from equations of quantum mechanics [14], their meaning is different from the one considered in this work.

In fact one could perform an experiment testing the absence of the lower bound on the energy measurement accuracy in a very short time, which is set by some time-energy uncertainty relations. Such an experiment would involve many copies of a two level quantum system prepared in the same state. Many copies are necessary to decrease the statistical noise $\eta_{\text{max}}$ and the two energy levels would allow to keep $\lambda_{\text{min}}$ at a reasonable value. At fixed time interval $T$ it is possible to adjust $\lambda_{\text{min}}$, as long as $\lambda_{\text{min}} \ll 1$, by varying the number of time steps $N$, see Eq. (16).

**SUMMARY**

In this work the energy measurement on an evolving quantum system is considered. The evolution is governed by a discrete Hamiltonian and the initial state is spanned on a finite number $K$ of significant energy eigenstates. The main result is the following upper bound

$$
|\tilde{\omega}_k - \omega_k | T \leq \frac{KN(N + 1) \sqrt{\text{Tr}(S)}}{\lambda_{\text{min}}^{3/2}} \eta_{\text{max}}. \quad (20)
$$

for inaccuracy $|\tilde{\omega}_k - \omega_k |$ of measured frequency $\tilde{\omega}_k$ with respect to the actual frequency $\omega_k$. The total measurement time $T$ is short as compared to the characteristic time scales of the system, $T \omega_{\text{max}} \ll 1$, and is divided into $N$ time intervals by the sampling measurements. If the noise $\eta_n$ is statistical then its maximal level $\eta_{\text{max}}$ can be reduced by simultaneous use of many copies of the system. The systematic errors can, in principle, be eliminated. Despite the short measurement time $T$ the upper bound can be made arbitrarily low by the reduction of $\eta_{\text{max}}$.

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