On the $K$- and $L$-theory of hyperbolic and virtually finitely generated abelian groups

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May 2012

Abstract

We investigate the algebraic $K$- and $L$-theory of the group ring $RG$, where $G$ is a hyperbolic or virtually finitely generated abelian group and $R$ is an associative ring with unit.

Key words: Algebraic $K$- and $L$-theory, hyperbolic groups, virtually finitely generated abelian groups.

Mathematics Subject Classification 2000: 19D99, 19G24, 19A31, 19B28.

Introduction

The goal of this paper is to compute the algebraic $K$- and $L$-groups of group rings $RG$, where $G$ is a hyperbolic group or a virtually finitely generated abelian group and $R$ is an associative ring with unit (and involution).

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The Farrell-Jones Conjecture for a group $G$ and a ring $R$ predicts that the assembly maps

$$H_n^G\left(\mathbb{E}G; K_R\right) \cong \rightarrow H_n^G\left(\mathbb{G}/G; K_R\right) = K_n(RG);$$
$$H_n^G\left(\mathbb{E}G; L_R^{(-\infty)}\right) \cong \rightarrow H_n^G\left(\mathbb{G}/G; L_R^{(-\infty)}\right) = L_n^{(-\infty)}(RG),$$

induced by the projection $\mathbb{E}G \to \mathbb{G}/G$, are bijective for every integer $n$. This conjecture, introduced by Farrell and Jones in their groundbreaking paper [21], has many consequences. Knowing that it is true for a given group implies several other well-known conjectures for that group, such as the ones due to Bass, Borel, Kadison and Novikov. The Farrell-Jones Conjecture also helps to calculate the $K$- and $L$-theory of group rings, since homology groups are equipped with tools such as spectral sequences that can simplify computations.

Recently it has been established that the Farrell-Jones Conjecture is true for word hyperbolic groups and virtually finitely generated abelian groups. Using this fact, we are able to compute the $K$- and $L$-theory of group rings in several cases by analyzing the left-hand side of the assembly map. The key ingredients used to compute these groups are the induction structure that equivariant homology theories possess [26, Section 1] and the work of Lück-Weiermann [31], which investigates when a universal space for a given group $G$ and a given family of subgroups $\mathcal{F}$ can be constructed from a universal space for $G$ and a smaller family $\mathcal{F}' \subseteq \mathcal{F}$.

Even in basic situations determining the $K$- and $L$-groups is difficult. However, we are able to handle hyperbolic groups. The favorite situation is when the hyperbolic group $G$ is torsion-free and $R = \mathbb{Z}$, in which case $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$, the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ and the Whitehead group $\text{Wh}(G)$ vanish, and $K_n(\mathbb{Z}G)$ is computed by $H_n(BG; K(\mathbb{Z}))$, i.e., the homology with coefficients in the $K$-theory spectrum $K(\mathbb{Z})$ of the classifying space $BG$. Moreover, the $L$-theory $L_n^i(\mathbb{Z}G)$ is independent of the decoration $\langle i \rangle$, and is given by $H_n(BG; L(\mathbb{Z}))$, where $L(\mathbb{Z})$ is the (periodic) $L$-theory spectrum of $L(\mathbb{Z})$. Recall that $\pi_n(L(\mathbb{Z}))$ is $\mathbb{Z}$ if $n \equiv 0 \mod 4$, $\mathbb{Z}/2$ if $n \equiv 2 \mod 4$, and vanishes otherwise. We also give a complete answer for $G = \mathbb{Z}^d$. However, for a group $G$ appearing in an exact sequence $1 \to \mathbb{Z}^d \to G \to Q \to 1$ for a finite group $Q$, our computations can only be carried out under the additional assumption that the conjugation action of $Q$ on $\mathbb{Z}^d$ is free away from 0 or that $Q$ is cyclic of prime order.

In Section 1 precise statements of our results are given, as well as several examples. Section 2 contains the necessary background for the proofs, which are presented in Section 3.

The paper was supported by the Sonderforschungsbereich SFB 878 – Groups, Geometry and Actions –, the Leibniz-Preis of the first author, and the Fulbright Scholars award of the second author. The authors thank the referee for several useful comments.
1 Statement of Results

Let $K_n(RG)$ denote the algebraic $K$-groups of the group ring $RG$ in the sense of Quillen for $n \geq 0$ and in the sense of Bass for $n \leq -1$, and let $L_n^\infty(RG)$ be the ultimate lower quadratic $L$-groups of $RG$ (see Ranicki [37]). When considering $L$-theory, we will tacitly assume that $R$ is a ring with an involution. Let $NK_n(R)$ denote the Bass-Nil-groups of $R$, which are defined as the cokernel of the map $K_n(R) \to K_n(R[X])$. Recall that the Bass-Heller-Swan decomposition says

$$K_n(RZ) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \quad (1.1)$$

If $R$ is a regular ring, then $NK_n(R) = 0$ for every $n \in \mathbb{Z}$ (see for instance Rosenberg [38, Theorems 3.3.3 and 5.3.30]).

The $n$-th Whitehead group $Wh_n(G; R)$ is defined as

$$Wh_n(G; R) : = H^G_n(EG \to \{\bullet\}; K_R),$$

where $H^G_n(EG \to \{\bullet\}; K_R)$ is the relative term in the equivariant homology theory $H^G_*(\_; K_R)$ associated to the equivariant map $EG \to \{\bullet\}$ (see 2.4).
Thus, $\text{Wh}_n(G; R)$ fits into a long exact sequence
\[
\cdots \to H_n(BG; K(R)) \to K_n(RG) \to \text{Wh}_n(G; R) \\
\to H_{n-1}(BG; K(R)) \to K_{n-1}(RG) \to \cdots,
\]
where $K(R)$ is the non-connective $K$-theory spectrum associated to $R$ and $H_*(\cdot; K(R))$ is the associated homology theory. Notice that $\text{Wh}_1(G; Z)$ agrees with the classical Whitehead group $\text{Wh}(G)$. Whitehead groups arise naturally when studying $h$-cobordisms, pseudoisotopy, and Waldhausen’s $A$-theory. Their geometric significance can be reviewed, for example, in Dwyer-Weiss-Williams [20, Section 9] and Lück-Reich [29, Section 1.4.1], where additional references can also be found. When $G = Z$, it follows from (1.1) and the fact that $H_n(BZ; K(R)) \cong K_n(R) \oplus K_{n-1}(R)$ that there is an identification
\[
H^2_n(EZ \to \{\bullet\}; K_R) \cong NK_n(R) \oplus NK_{n-1}(R). \tag{1.2}
\]

Define the periodic $n$-th structure group with decoration $(-\infty)$ to be
\[
S^{\text{per},(-\infty)}_n(G; R) := \text{H}^D_n(EG \to \{\bullet\}; L^{(-\infty)}_R).
\]
These groups fit into the periodic version of the long exact surgery sequence with decoration $(-\infty)$,
\[
\cdots \to H_n(BG; L^{(-\infty)}(R)) \to L^{(-\infty)}_n(RG) \to S^{\text{per},(-\infty)}_n(G; R) \\
\to H_{n-1}(BG; L^{(-\infty)}(R)) \to L^{(-\infty)}_{n-1}(RG) \to \cdots,
\]
where $L^{(-\infty)}(R)$ is the spectrum whose homotopy groups are the ultimate lower quadratic $L$-groups. This periodic surgery sequence (with a different decoration) for $R = Z$ appears in the classification of ANR-homology manifolds in Bryant-Ferry-Mio-Weinberger [9, Main Theorem]. It is related to the algebraic surgery exact sequence and thus to the classical surgery sequence (see Ranicki [36, Section 18]). For $n \in Z$ define
\[
\text{UNil}^{(-\infty)}_n(D_\infty; R) := H^D_n(ED_\infty \to \{\bullet\}; L^{(-\infty)}_R). \tag{1.3}
\]
These groups are related to Cappell’s UNil-groups, as explained in Section 2.

Now we are able to state our main results.

1.1 Hyperbolic groups

Theorem 1.4 (Hyperbolic groups). Let $G$ be a hyperbolic group in the sense of Gromov [24], and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of $G$.

(i) For each $n \in Z$ there is an isomorphism
\[
H^G_n(EX; K_R) \oplus \bigoplus_{V \in \mathcal{M}} H^V_n(EX \to \{\bullet\}; K_R) \cong K_n(RG);
\]
(ii) For each \( n \in \mathbb{Z} \) there is an isomorphism
\[
H_n^G(EG; \mathbb{L}_{R}^{(-\infty)}) \oplus \bigoplus_{V \in \mathcal{M}} H_n^V(EG \rightarrow \{\bullet\}; \mathbb{L}_{R}^{(-\infty)}) \overset{\cong}{\rightarrow} L_n^{(-\infty)}(RG),
\]
provided that there exists \( n_0 \leq -2 \) such that \( K_n(RV) = 0 \) holds for all \( n \leq n_0 \) and all virtually cyclic subgroups \( V \subset G \). (The latter condition is satisfied if \( R = \mathbb{Z} \) or if \( R \) is regular with \( \mathbb{Q} \subseteq R \).)

A good model for \( EG \) is given by the Rips complex of \( G \) (see Meintrup-Schick [32]). Tools for computing \( H_n^G(EG; K_R) \) are the equivariant version of the Atiyah-Hirzebruch spectral sequence (see Davis-Lück [16, Theorem 4.7]), the \( p \)-chain spectral sequence (see Davis-Lück [17]) and equivariant Chern characters (see Lück [26]). More information about the groups \( H_n^V(EG \rightarrow \{\bullet\}; K_R) \) and \( H_n^V(EG \rightarrow \{\bullet\}; L_R^{(-\infty)}) \) is given in Section 2.

### 1.2 Torsion-free hyperbolic groups

The situation simplifies if \( G \) is assumed to be a torsion-free hyperbolic group.

**Theorem 1.5** (Torsion-free hyperbolic groups). Let \( G \) be a torsion-free hyperbolic group, and let \( \mathcal{M} \) be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of \( G \).

(i) For each \( n \in \mathbb{Z} \) there is an isomorphism
\[
H_n(BG; K(R)) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R) \oplus NK_n(R) \overset{\cong}{\rightarrow} K_n(RG);
\]

(ii) For each \( n \in \mathbb{Z} \) there is an isomorphism
\[
H_n(BG; L_{\infty}(R)) \overset{\cong}{\rightarrow} L_n^{(-\infty)}(RG).
\]

In particular, it follows that for a torsion-free hyperbolic group \( G \) and a regular ring \( R \),

\[
K_n(RG) = \{0\} \quad \text{for } n \leq 1
\]

and the obvious map
\[
K_0(R) \overset{\cong}{\rightarrow} K_0(RG)
\]

is bijective. Moreover, \( K_0(ZG) \), \( Wh(G) \) and \( K_n(ZG) \) for \( n \leq -1 \) all vanish.

**Example 1.6** (Finitely generated free groups). Let \( F_r \) be the finitely generated free group \( \ast_{i=1}^{r} \mathbb{Z} \) of rank \( r \). Since \( F_r \) acts freely on a tree it is hyperbolic. By Theorem 1.5

\[
K_n(RF_r) \cong K_n(R) \oplus K_{n-1}(R)^r \oplus \bigoplus_{V \in \mathcal{M}} (NK_n(R) \oplus NK_n(R))
\]

and
\[
L_n^{(-\infty)}(RF_r) \cong L_n^{(-\infty)}(R) \oplus L_{n-1}^{(-\infty)}(R)^r,
\]

where \( \mathcal{M} \) is a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of \( F_r \).
1.3 Finitely generated free abelian groups

Before tackling the virtually finitely generated abelian case, we first consider finitely generated free abelian groups. The $K$-theory case of Theorem 1.7 below is also proved in [14].

**Theorem 1.7** ($\mathbb{Z}^d$). Let $d \geq 1$ be an integer. Let $\text{MICY}$ be the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$. Then there are isomorphisms

\[
\text{Wh}_n(\mathbb{Z}^d; R) \cong \bigoplus_{C \in \text{MICY}} \bigoplus_{i=0}^{d-1} (NK_{n-i}(R) \oplus NK_{n-i}(R))^{(d-i-1)};
\]

\[
K_n(\mathbb{Z}^d) \cong \bigoplus_{i=0}^{d} K_{n-i}(\mathbb{R})^{(i)} \oplus \text{Wh}_n(\mathbb{Z}^d; R);
\]

\[
L_n^{(-\infty)}(\mathbb{Z}^d) \cong \bigoplus_{i=0}^{d} L_n^{(-\infty)}(\mathbb{R})^{(i)}.
\]

**Example 1.8** ($\mathbb{Z}^d \times G$). Let $G$ be a group. By Theorem 1.7

\[
K_n(\mathbb{Z}^d \times G) \cong K_n(\mathbb{Z}^d) \oplus K_n(\mathbb{R}^G) \oplus K_n(\mathbb{Z}^d; R) \oplus \bigoplus_{C \in \text{MICY}(\mathbb{Z}^d)} (NK_{n-i}(\mathbb{Z}^d) \oplus NK_{n-i}(\mathbb{Z}^d))^{(d-i-1)},
\]

where $\text{MICY}(\mathbb{Z}^d)$ is the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$. Since

\[
H_n(B(\mathbb{Z}^d; \text{K}(\mathbb{R}))) \cong \bigoplus_{i=0}^{d} H_n(BG; \text{K}(\mathbb{R}))^{(i)},
\]

this implies

\[
\text{Wh}_n(\mathbb{Z}^d \times G; R) \cong \bigoplus_{i=0}^{d} \text{Wh}_n(G; R)^{(i)} \oplus \bigoplus_{C \in \text{MICY}(\mathbb{Z}^d)} \bigoplus_{i=0}^{d-1} (NK_{n-i}(\mathbb{Z}^d) \oplus NK_{n-i}(\mathbb{Z}^d))^{(d-i-1)}.
\]

**Example 1.9** (Surface groups). Let $\Gamma_g$ be the fundamental group of the orientable closed surface of genus $g$, and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of $G$. If $g = 0$, then $\Gamma_g$ is trivial. If $g = 1$, then $\Gamma_g$ is $\mathbb{Z}^2$ and Theorem 1.7 implies

\[
K_n(\Gamma_1) \cong K_n(\mathbb{R}) \oplus K_{n-1}(\mathbb{R})^2 \oplus K_{n-2}(\mathbb{R}) \oplus \bigoplus_{V \in \mathcal{M}} (NK_n(\mathbb{R})^2 \oplus NK_{n-1}(\mathbb{R})^2)
\]

and

\[
L_n^{(-\infty)}(\Gamma_1) \cong L_n^{(-\infty)}(\mathbb{R}) \oplus L_n^{(-\infty)}(\mathbb{R})^2 \oplus L_n^{(-\infty)}(\mathbb{R}).
\]
If \( g \geq 2 \), then \( \Gamma_g \) is hyperbolic and torsion-free, so by Theorem 1.5

\[
K_n(R\Gamma_g) \cong H_n(B\Gamma_g; K(R)) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R)^2,
\]

and

\[
L_n^{(-\infty)}(R\Gamma_g) \cong H_n(B\Gamma_g; L^{(-\infty)}(R)).
\]

Since \( \Gamma_g \) is stably a product of spheres,

\[
H_n(B\Gamma_g; K(R)) \cong K_n(R) \oplus K_{n-1}(R)^{2g} \oplus K_{n-2}(R),
\]

and

\[
H_n(B\Gamma_g; L^{(-\infty)}(R)) \cong L_n^{(-\infty)}(R) \oplus L_{n-1}^{(-\infty)}(R)^{2g} \oplus L_{n-2}^{(-\infty)}(R).
\]

If \( R = \mathbb{Z} \), then for every \( i \in \{1, 0, -1, \ldots\} \cup \{-\infty\} \), \( L_n^{(i)}(\mathbb{Z}) \) is \( \mathbb{Z} \) if \( n \equiv 0 \) mod 4, \( \mathbb{Z}/2 \) if \( n \equiv 2 \) mod 4, and is trivial otherwise. Therefore,

\[
L_n^{(i)}(\mathbb{Z}\Gamma_g) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 \quad & \text{if } n \equiv 0, 2 \text{ mod } 4; \\ \mathbb{Z}^g \quad & \text{if } n \equiv 1 \text{ mod } 4; \\ (\mathbb{Z}/2)^g \quad & \text{if } n \equiv 3 \text{ mod } 4. \end{cases}
\]

More generally, cocompact planar groups (sometimes called cocompact non-Euclidean crystallographic groups), e.g., cocompact Fuchsian groups, are treated in Lück-Stamm [30] for \( R = \mathbb{Z} \). These computations can be carried over to arbitrary \( R \).

### 1.4 Virtually finitely generated abelian groups

Consider the group extension

\[
1 \rightarrow A \rightarrow G \xrightarrow{\rho} Q \rightarrow 1,
\]

where \( A \) is isomorphic to \( \mathbb{Z}^d \) for some \( d \geq 0 \) and \( Q \) is a finite group. The conjugation action of \( G \) on the normal abelian subgroup \( A \) induces an action of \( Q \) on \( A \) via a group homomorphism which we denote by \( \rho: Q \rightarrow \text{aut}(A) \).

Let \( \text{MIC}_{\mathcal{U}}(A) \) be the set of maximal infinite cyclic subgroups of \( A \). Since any automorphism of \( A \) sends a maximal infinite cyclic subgroup to a maximal infinite cyclic subgroup, \( \rho \) induces a \( Q \)-action on \( \text{MIC}_{\mathcal{U}}(A) \). Fix a subset

\[
I \subseteq \text{MIC}_{\mathcal{U}}(A)
\]

such that the intersection of every \( Q \)-orbit in \( \text{MIC}_{\mathcal{U}}(A) \) with \( I \) consists of precisely one element.

For \( C \in I \), let

\[
Q_C \subseteq Q
\]
be the isotropy group of $C \in \text{MICY}(A)$ under the $Q$-action. Let

$$I_1 = \{ C \in I \mid Q_C = \{1\} \},$$

$$I_2 = \{ C \in I \mid Q_C = \mathbb{Z}/2 \},$$

and let $J$ be a complete system of representatives of maximal non-trivial finite subgroups of $G$.

### 1.4.1 $K$-theory in the case of a free conjugation action

**Theorem 1.11.** Consider the group extension $1 \rightarrow A \rightarrow G \xrightarrow{q} Q \rightarrow 1$, where $A$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and $Q$ is a finite group. Suppose that the $Q$-action on $A$ is free away from $0 \in A$.

(i) For each $n \in \mathbb{Z}$ there is an isomorphism induced by the various inclusions

$$\left( \bigoplus_{F \in J} \text{Wh}_n(F; R) \right) \oplus \left( \mathbb{Z} \otimes_{\mathbb{Q}} \text{Wh}_n(A; R) \right) \xrightarrow{\cong} \text{Wh}_n(G; R).$$

(ii) For each integer $n$, $\mathbb{Z} \otimes_{\mathbb{Q}} \text{Wh}_n(A; R)$ is isomorphic to

$$\left( \bigoplus_{C \in I_1} \bigoplus_{i=0}^{d-1} (NK_{n-i}(R) \oplus NK_{n-i}(R))^{(d-1)} \right) \oplus \left( \bigoplus_{C \in I_2} \bigoplus_{i=0}^{d-1} NK_{n-i}(R)^{(d-1)} \right).$$

If $Q = \{1\}$, then Theorem 1.11 reduces to Theorem 1.7 since $J = \emptyset$, $I_2 = \emptyset$ and $I_1$ is the set of maximal infinite cyclic subgroups of $G = \mathbb{Z}^d$.

**Theorem 1.12.** Under the assumptions of Theorem 1.11

(i) If $R$ is regular, then for each $n \in \mathbb{Z}$

$$\bigoplus_{F \in J} \text{Wh}_n(F; R) \cong \text{Wh}_n(G; R).$$

In particular,

$$\bigoplus_{F \in J} K_n(RF) \cong K_n(RG) \quad \text{for } n \leq -1$$

and

$$\bigoplus_{F \in J} \text{coker}(K_0(R) \rightarrow K_0(RF)) \cong \text{coker}(K_0(R) \rightarrow K_0(RG)).$$

(ii) If $R$ is a Dedekind ring of characteristic zero, then

$$K_n(RG) = \{0\} \quad \text{for } n \leq -2.$$
Applying Theorem 1.12 to the special case $R = \mathbb{Z}$, we recover the fact that if $G$ satisfies the assumptions of Theorem 1.11 then:

$$
\bigoplus_{F \in J} \text{Wh}(F) \xrightarrow{\cong} \text{Wh}(G);
$$

$$
\bigoplus_{F \in J} K_0(\mathbb{Z}F) \xrightarrow{\cong} K_0(\mathbb{Z}G);
$$

$$
\bigoplus_{F \in J} K_{-1}(\mathbb{Z}F) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G);
$$

$$
K_n(\mathbb{Z}G) \cong \{0\} \text{ for } n \leq -2.
$$

### 1.4.2 $L$-theory in the case of a free conjugation action

**Theorem 1.13.** Consider the group extension $1 \to A \to G \xrightarrow{\pi} Q \to 1$, where $A$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and $Q$ is a finite group. Suppose that the $Q$-action on $A$ is free away from $0 \in A$. Assume that there exists $n_0 \leq -2$ such that $K_n(RV) = 0$ for all $n \leq n_0$ and all virtually cyclic subgroups $V \subseteq G$. (By Theorem 1.12 (ii), this condition is satisfied if $R$ is a Dedekind ring of characteristic zero).

Then there is an isomorphism

$$
\left( \bigoplus_{F \in J} \text{S}_{n}^{\text{per},(-\infty)}(F; R) \right) \oplus \left( \bigoplus_{C \in J_C} \bigoplus_{H \in J_C} \text{UNil}_{n}^{(-\infty)}(D_{\infty}; R) \right) \xrightarrow{\cong} \text{S}_{n}^{\text{per},(-\infty)}(G; R),
$$

where $J_C$ is a complete system of representatives of the conjugacy classes of maximal finite subgroups of $W_G C = N_G C / C$.

If $Q$ has odd order, then

$$
\bigoplus_{F \in J} \text{S}_{n}^{\text{per},(-\infty)}(F; R) \xrightarrow{\cong} \text{S}_{n}^{\text{per},(-\infty)}(G; R).
$$

Dealing with groups extensions of the type described above when the conjugation action of $Q$ on $A$ is not free is considerably harder than the free case. However, we are able to say something when $Q$ is a cyclic group of prime order and $R$ is regular.

### 1.4.3 $K$-theory in the case $Q = \mathbb{Z}/p$ for a prime $p$ and regular $R$

**Theorem 1.14.** Consider the group extension $1 \to A \to G \xrightarrow{\pi} \mathbb{Z}/p \to 1$, where $A$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and $p$ is a prime number. Let $e$ be the natural number given by $A^e / p \cong \mathbb{Z}^e$, $J$ be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of $G$, and $\text{MICY}(A^e / p)$ be the set of maximal infinite cyclic subgroups of $A^e / p$. 
If $R$ is regular, then there is an isomorphism

$$\text{Wh}_n(G; R) \cong \bigoplus_{H \in J} \bigoplus_{i=0}^e \text{Wh}_{n-i}(H; R)^{(i)} + \bigoplus_{H \in \mathcal{M}_G} \bigoplus_{C \in \mathcal{M}_G(A; \mathbb{Z}/p)} \bigoplus_{i=0}^{e-1} \bigoplus_{H \in J} \bigoplus_{C \in \mathcal{M}_G(A; \mathbb{Z}/p)} \text{Wh}_{n-i}(R[\mathbb{Z}/p])^{(i)}.$$ 

**Remark 1.15** (Cardinality of $J$). Assume that the group $G$ appearing in Theorem 1.14 is not torsion-free, or, equivalently, that $J$ is non-empty. Consider $A$ as a $\mathbb{Z}/p$-module via the conjugation action $\rho: \mathbb{Z}/p \to \text{aut}(A)$. Then there is a bijection

$$H^1(\mathbb{Z}/p; A) \cong J,$$

defined as follows. Fix an element $t \in G$ of order $p$. Every $\mathfrak{a} \in H^1(\mathbb{Z}/p; A)$ is represented by an element $x$ in the kernel of $\sum_{i=0}^{p-1} \rho^i: A \to A$. Then $xt$ has order $p$. Send $\mathfrak{a}$ to the unique element of $J$ that is conjugate to $\langle xt \rangle$ in $G$.

If $p$ is prime and $R$ is regular, then Example 1.18 in the case $G = \mathbb{Z}/p$ is consistent with Theorem 1.14 and Remark 1.15. Namely, $\mathbb{Z}/p$ acts trivially on $\mathbb{Z}^d$, and so $A = A\mathbb{Z}/p, J = \{0\} \times \mathbb{Z}/p$, and $d = e$.

1.4.4 $L$-theory in the case $Q = \mathbb{Z}/p$ for an odd prime $p$.

**Theorem 1.16.** Consider the group extension $1 \to A \to G \to \mathbb{Z}/p \to 1$, where $A$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and $p$ is an odd prime number. Let $e$ be the natural number given by $A\mathbb{Z}/p \cong \mathbb{Z}^e$, and $J$ be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of $G$.

Then there is an isomorphism

$$\bigoplus_{H \in J} \bigoplus_{i=0}^e \mathcal{S}^\text{per,(-}\infty\rangle_{n-i}(H; R)^{(i)} \cong \mathcal{S}^\text{per,(-}\infty\rangle_{n}(G; R).$$

Two-dimensional crystallographic groups are treated in Pearson [33] and Lück-Stamm [30] for $R = \mathbb{Z}$. Some of the computations there can be carried over to arbitrary $R$. The Whitehead groups of three-dimensional crystallographic groups are computed in Alves-Ontaneda [1].

In the case $R = \mathbb{Z}$, we get a computation for all decorations.

**Theorem 1.17.** Consider the group extension $1 \to A \to G \to \mathbb{Z}/p \to 1$, where $A$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and $p$ is an odd prime number. Let $e$ be the natural number given by $A\mathbb{Z}/p \cong \mathbb{Z}^e$, and $J$ be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of $G$, and $\epsilon$ be any of the decorations $s, h, p$ or $\{j \mid j = 1, 0, -1, \ldots\} \times \{-\infty\}$. Define

$$\mathcal{S}^\text{per,\epsilon}_{n}(G; \mathbb{Z}) := H^G_n(EG \to \{\bullet\}; L^\epsilon_\mathbb{Z}).$$
Then there is an isomorphism

\[ \bigoplus_{H \in J} \bigoplus_{i=0}^{n-i} \mathcal{S}_{n-i}^{\text{per}, e}(H; \mathbb{Z})[i] \cong \mathcal{S}_n^{\text{per}, e}(G; \mathbb{Z}). \]

**Remark 1.18.** In general, the structure sets \( \mathcal{S}_{n-i}^{\text{per}, e}(H; \mathbb{Z}) \) depend on the decoration \( e \). We mention without proof that for an odd prime \( p \)

\[ \mathcal{S}_n^{\text{per}, s}(\mathbb{Z}/p; \mathbb{Z}) \cong \tilde{L}_n^s(\mathbb{Z}[\mathbb{Z}/p])[1/p] \cong \begin{cases} \mathbb{Z}[1/p][p^{-1}/2] & \text{n even;} \\ \{0\} & \text{n odd.} \end{cases} \]

**Remark 1.19.** The computation of the structure set \( \mathcal{S}_n^{\text{per}, e}(G; \mathbb{Z}) \) when the conjugation action of \( \mathbb{Z}/p \) on \( \mathbb{Z}^d \) is free plays a role in a forthcoming paper by Davis and Lück, in which this case is further analyzed to compute the geometric structure sets of certain manifolds that occur as total spaces of a bundle over lens spaces with \( d \)-dimensional tori as fibers.

**Remark 1.20** (Topological \( K \)-theory of reduced group \( C^* \)-algebras). All of the above computations also apply to the topological \( K \)-theory of the reduced group \( C^* \)-algebra, since the Baum-Connes Conjecture is true for these groups. In the Baum-Connes setting the situation simplifies considerably because one works with \( EG \) instead of \( EG \) and hence there are no Nil-phenomena. For instance, if \( G \) is a hyperbolic group, then there is an isomorphism

\[ K_n^G(EG) \cong K_n(C^*_r(G)) \]

from the equivariant topological \( K \)-theory of \( EG \) to the topological \( K \)-theory of the reduced group \( C^* \)-algebra \( C^*_r(G) \). In the case of a torsion-free hyperbolic group, this reduces to an isomorphism

\[ K_n(BG) \cong K_n(C^*_r(G)). \]

The case \( G = \mathbb{Z}^d \rtimes_p \mathbb{Z}/p \) for a free conjugation action has been carried out for both complex and real topological \( K \)-theory in detail in \cite{18} Theorem 0.3 and Theorem 0.6.

## 2 Background

In this section we give some background about the Farrell-Jones Conjecture and related topics.

### 2.1 Classifying Spaces for Families

Let \( G \) be a group. A *family of subgroups* of \( G \) is a collection of subgroups that is closed under conjugation and taking subgroups. Examples of such families
are

\[
\begin{align*}
\{1\} &= \{\text{trivial subgroup}\}; \\
\mathcal{FIN} &= \{\text{finite subgroups}\}; \\
\mathcal{VCY} &= \{\text{virtually cyclic subgroups}\}; \\
\mathcal{ALL} &= \{\text{all subgroups}\}.
\end{align*}
\]

Let \(\mathcal{F}\) be a family of subgroups of \(G\). A model for the universal space \(E_{\mathcal{F}}(G)\) for \(\mathcal{F}\) is a \(G\)-CW-complex \(X\) with isotropy groups in \(\mathcal{F}\) such that for any \(G\)-CW-complex \(Y\) with isotropy groups in \(\mathcal{F}\) there exists a \(G\)-map \(Y \to X\) that is unique up to \(G\)-homotopy. In other words, \(X\) is a terminal object in the \(G\)-homotopy category of \(G\)-CW-complexes whose isotropy groups belong to \(\mathcal{F}\). In particular, any two models for \(E_{\mathcal{F}}(G)\) are \(G\)-homotopy equivalent, and for two families \(\mathcal{F}_0 \subseteq \mathcal{F}_1\), there is precisely one \(G\)-map \(E_{\mathcal{F}_0}(G) \to E_{\mathcal{F}_1}(G)\) up to \(G\)-homotopy.

Let \(\mathcal{H}_c^c\) be an equivariant homology theory in the sense of Lück [26, Section 1]. Then, for every group \(G\) and every \(G\)-CW-pair \((X, A)\) there is a \(\mathbb{Z}\)-graded abelian group \(\mathcal{H}_c^{G}(X, A)\), and subsequently a \(G\)-homology theory \(\mathcal{H}_c^{G}\). For every group homomorphism \(\alpha: H \to G\), every \(H\)-CW-pair \((X, A)\) and every \(n \in \mathbb{Z}\), there is a natural homomorphism \(\text{ind}_\alpha: \mathcal{H}_c^{H}(X, A) \to \mathcal{H}_c^{G}(G \times_\alpha (X, A))\), known as the induction homomorphism. If the kernel of \(\alpha\) operates relatively freely on \((X, A)\), then \(\text{ind}_\alpha\) is an isomorphism.

Our main examples are the equivariant homology theories \(\mathcal{H}_c^c(\_; K_R)\) and \(\mathcal{H}_c^c(\_; L^{\langle -\infty \rangle}_R)\) appearing in the \(K\)-theoretic and \(L\)-theoretic Farrell-Jones Conjectures, where \(R\) is an associative ring with unit (and involution) (see Lück-Reich [29, Section 6]). The basic property of these two equivariant homology theories is that

\[
\begin{align*}
\mathcal{H}_c^c(G/H; K_R) &= \mathcal{H}_c^H(\{\bullet\}; K_R) = K_c^H(RH); \\
\mathcal{H}_c^c(G/H; L^{\langle -\infty \rangle}_R) &= \mathcal{H}_c^H(\{\bullet\}; L^{\langle -\infty \rangle}_R) = L^{\langle -\infty \rangle}_c(RH),
\end{align*}
\]

for every subgroup \(H \subseteq G\).
The Farrell-Jones Conjecture for a group $G$ and a ring $R$ states that the assembly maps

$$H_n^G(EG; K_R) \xrightarrow{\cong} H_n^G(G/G; K_R) = K_n(RG);$$

$$H_n^G(EG; L^{(-\infty)}_R) \xrightarrow{\cong} H_n^G(G/G; L^{(-\infty)}_R) = L_n^{(-\infty)}(RG),$$

induced by the projection $EG \to G/G$, are bijective for every $n \in \mathbb{Z}$ [21]. The Farrell-Jones Conjecture has been studied extensively because of its geometric significance. It implies, in dimensions $\geq 5$, both the Novikov Conjecture about the homotopy invariance of higher signatures and the Borel Conjecture about the rigidity of manifolds with fundamental group $G$. It also implies other well-known conjectures, such as the ones due to Bass and Kadison. For a survey and applications of the Farrell-Jones Conjecture, see, for example, Lück-Reich [29, Section 6] and Bartels-Lück-Reich [7].

**Theorem 2.1** (Farrell-Jones Conjecture for hyperbolic groups and virtually $\mathbb{Z}^d$-groups). The Farrell-Jones Conjecture is true if $G$ is a hyperbolic group or a virtually finitely generated abelian group, and $R$ is any ring.

The $K$-theoretic Farrell-Jones Conjecture with coefficients in any additive $G$-category for a hyperbolic group $G$ was proved by Bartels-Lück-Reich [6]. The version of the Farrell-Jones Conjecture with coefficients in an additive $G$-category encompasses the version with rings as coefficients. The $L$-theoretic Farrell-Jones Conjecture with coefficients in any additive $G$-category for hyperbolic groups and CAT(0)-groups was established by Bartels and Lück [5]. In that paper it is also shown that the $K$-theoretic assembly map is 1-connected for CAT(0)-groups. Note that a virtually finitely generated abelian group is CAT(0). Quinn [34, Theorem 1.2.2]) proved that the $K$-theoretic assembly map for virtually finitely generated abelian groups is bijective for every integer $n$ if $R$ is a commutative ring. However, the proof carries over to the non-commutative setting.

**Remark 2.2** (The Interplay of $K$- and $L$-Theory). $L^i(RG)$ can have various decorations for $i \in \{2, 1, 0, -1, -2, \ldots\} \bigcup \{-\infty\}$. One also finds $L^i_\epsilon(RG)$ for $\epsilon = p, h, s$ in the literature. The decoration (1) coincides with the decoration $h,s$ (0) with the decoration $p$, and (2) is related to the decoration $s$. For $j \leq 1$ there are forgetful maps $L^{(j+1)}_n(R) \to L^{(j)}_n(R)$ that fit into the so-called Rothenberg sequence (see Ranicki [35, Proposition 1.10.1 on page 104], [37, 17.2])

$$\cdots \to L^{(j+1)}_n(R) \to L^{(j)}_n(R) \to \tilde{H}^n(\mathbb{Z}/2; \tilde{K}_j(R))$$

$$\cdots \to L^{(j+1)}_{n-1}(R) \to L^{(j)}_{n-1}(R) \to \cdots .$$

$\tilde{H}^n(\mathbb{Z}/2; \tilde{K}_j(R))$ denotes the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$-module $\tilde{K}_j(R)$. The involution on $\tilde{K}_j(R)$ comes from the involution on $R$. There is a similar sequence relating $L^*_n(RG)$ and $L^h_n(RG)$, where the third term is the $\mathbb{Z}/2$-Tate-cohomology of $\text{Wh}_1^R(G)$. 

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For geometric applications the most important case is $R = \mathbb{Z}$ with the decoration $s$. If $K_0(\mathbb{Z}G)$, Wh$(G)$ and $K_n(\mathbb{Z}G)$ for $n \leq -1$ all vanish, then the Rothenberg sequence for $n \in \mathbb{Z}$ implies the bijectivity of the natural maps

$$L_n^h(\mathbb{Z}G) \xrightarrow{s} L_n^h(\mathbb{Z}G) \xrightarrow{s} L_n^p(\mathbb{Z}G) \xrightarrow{s} L_n^{(-\infty)}(\mathbb{Z}G).$$

In the formulation of the Farrell-Jones Conjecture (see Section 2), one must use the decoration $(-\infty)$ since the conjecture is false otherwise (see Farrell-Jones-Lück [23]).

2.3 Equivariant homology and relative assembly maps

Consider an equivariant homology theory $H^G_\ast$ in the sense of [26, Section 1]. Given a $G$-map $f : X \to Y$ of $G$-CW-complexes and $n \in \mathbb{Z}$, define

$$H^G_n(f) := H^G_n(\text{cyl}(f_0), X)$$

for any cellular $G$-map $f_0 : X \to Y$ that is $G$-homotopic to $f$. Here, cyl$(f_0)$ is the $G$-CW-complex given by the mapping cylinder of $f$. It contains $X$ as a $G$-CW-subcomplex. Such an $f_0$ exists by the Equivariant Cellular Approximation Theorem (see [39, Theorem II.2.1 on page 104]). The definition is independent of the choice of $f_0$, since two cellular $G$-homotopic $G$-maps $f_0, f_1 : X \to Y$ are cellularly $G$-homotopic by the Equivariant Cellular Approximation Theorem (see [39, Theorem II.2.1 on page 104]) and a cellular $G$-homotopy between $f_0$ and $f_1$ yields a $G$-homotopy equivalence $u : \text{cyl}(f_0) \to \text{cyl}(f_1)$ that is the identity on $X$. This implies that $H^G_n(f)$ depends only on the $G$-homotopy class of $f$. From the axioms of an equivariant homology theory, $H^G_n(f)$ fits into a long exact sequence

$$\cdots \to H^G_{n+1}(f) \to H^G_n(X) \to H^G_n(Y) \to H^G_n(f) \to H^G_{n-1}(X) \to \cdots \quad (2.5)$$

The following fact is proved in Bartels [3].

Lemma 2.6. For every group $G$, every ring $R$, and every $n \in \mathbb{Z}$:

(i) the relative assembly map

$$H^G_n(EG; K_R) \to H^G_n(EG; K_R)$$

is split-injective;

(ii) the relative assembly map

$$H^G_n(EG; L_R^{(-\infty)}) \to H^G_n(EG; L_R^{(-\infty)})$$

is split-injective, provided there is an $n_0 \leq -2$ such that $K_n(RV) = 0$ for every $n \leq n_0$ and every virtually cyclic subgroup $V \subseteq G$.

Remark 2.7. Notice that the condition appearing in assertion (ii) of the above lemma is automatically satisfied if one of the following stronger conditions holds:
(i) \( R = \mathbb{Z} \);

(ii) \( R \) is regular with \( \mathbb{Q} \subseteq R \).

If \( R = \mathbb{Z} \), this follows from [22, Theorem 2.1(a)]. If \( R \) is regular and \( \mathbb{Q} \subseteq R \), then for every virtually cyclic subgroup \( V \subseteq G \) the ring \( RV \) is regular and hence \( K_n(RV) = 0 \) for all \( n \leq -1 \).

Lemma 2.6 tells us that the source of the assembly map appearing in the Farrell-Jones Conjecture can be computed in two steps, the computation of \( H^G_n(EG; K_R) \) and the computation of the remaining term \( H^G_n(EG \to EG; K_R) \) defined in (2.4). Furthermore, Lemma 2.6 implies

\[
H^G_n(EG \to EG; K_R) \cong H^G_n(EG \to EG; K_R) \oplus H^G_n(EG \to EG; K_R); \tag{2.8}
\]

and

\[
H^G_n(EG \to EG; L^{(-\infty)}_R) \cong H^G_n(EG \to EG; L^{(-\infty)}_R) \oplus H^G_n(EG \to EG; L^{(-\infty)}_R), \tag{2.9}
\]

provided that there exists an \( n_0 \leq -2 \) such that \( K_n(RV) = 0 \) for every \( n \leq n_0 \) and every virtually cyclic subgroups \( V \subseteq G \). This is useful for calculating the Whitehead groups for a given group \( G \) and ring \( R \) which satisfy the Farrell-Jones Conjecture.

**Remark 2.10.** The induction structure of \( H^Z_*(-; K_R) \) can be used to define a \( \mathbb{Z}/2 \)-action on \( H^Z_n(\mathbb{E} \mathbb{Z} \to \{\bullet\}; K_R) \) that is compatible with the \( \mathbb{Z}/2 \)-action on \( NK_n(R) \oplus NK_n(R) \) given by flipping the two factors. The action is defined by the composition of isomorphisms

\[
\tau : H^Z_n(\mathbb{E} \mathbb{Z} \to \{\bullet\}; K_R) \xrightarrow{\text{ind}_{-\text{id}_\mathbb{Z}}} H^Z_n(\text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z} \to \{\bullet\}; K_R) \xrightarrow{\text{ind}_{-\text{id}_\mathbb{Z}}} H^Z_n(\mathbb{E} \mathbb{Z} \to \{\bullet\}; K_R),
\]

where the second map is induced by the unique (up to equivariant homotopy) equivariant map \( l : \text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z} \to \mathbb{E} \mathbb{Z} \); \( l \) is an equivariant homotopy equivalence since \( \text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z} \) is a model for \( \mathbb{E} \mathbb{Z} \).

To see that this corresponds to the flip action on \( NK_n(R) \oplus NK_n(R) \), consider the following diagram coming from the long exact sequence (2.5).

\[
\begin{array}{ccc}
H^Z_n(\mathbb{E} \mathbb{Z}; K_R) & \xrightarrow{\text{ind}_{-\text{id}_\mathbb{Z}}} & H^Z_n(\text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z}; K_R) \\
\downarrow \text{ind}_{-\text{id}_\mathbb{Z}} & & \downarrow \text{ind}_{-\text{id}_\mathbb{Z}} \\
H^Z_n(\mathbb{E} \mathbb{Z}; K_R) & \xrightarrow{l_*} & H^Z_n(\text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z}; K_R) \\
\downarrow l & & \downarrow l \\
H^Z_n(\mathbb{E} \mathbb{Z}; K_R) & \xrightarrow{l_*} & H^Z_n(\text{id}_{-\text{id}_\mathbb{Z}} \mathbb{E} \mathbb{Z}; K_R)
\end{array}
\]
Remark 2.11

proof of this fact can also be found in [19, Lemma 3.22]. The identifications come from the Five-Lemma and the fact that both groups are of type I, then is of type II, then it can be written as an amalgamated product \( D = \langle -\rangle \). Also recall that the two copies of \( NK_n(R) \) appearing in the decomposition of \( K_n(R[t, t^{-1}]) \) come from the embeddings \( R[t] \rightarrow R[t, t^{-1}] \) and \( R[t^{-1}] \rightarrow R[t, t^{-1}] \). By the definition of the induction structure for \( H^*(-; K_R) \),

\[
\text{ind}_- \text{id}_*: H^*_n(\{\bullet\}; K_R) \rightarrow H^*_n(\{\bullet\}; K_R)
\]
corresponds to the homomorphism \( K_n(R[t, t^{-1}]) \rightarrow K_n(R[t, t^{-1}]) \) induced by interchanging \( t \) and \( t^{-1} \) (see, for example, [29 Section 6]), which swaps the two copies of \( NK_n(R) \) in the decomposition of \( K_n(R[t, t^{-1}]) \). Therefore the above diagram implies that \( \tau \) coincides with the flip action on \( NK_n(R) \oplus NK_n(R) \). A proof of this fact can also be found in [19 Lemma 3.22].

Remark 2.11 (Relative assembly and Nil-terms).

Let \( V \) be an infinite virtually cyclic group. If \( V \) is of type I, then \( V \) can be written as a semi-direct product \( F \rtimes \mathbb{Z} \), and \( H^V(EV) \rightarrow \{\bullet\}; K_R \) can be identified with the non-connective version of Waldhausen's Nil-term associated to this semi-direct product (see [4 Sections 9 and 10]). If \( V \) is of type II, then it can be written as an amalgamated product \( V_1 \rtimes \mathbb{Z} V_2 \) of finite groups, where \( V_0 \) has index two in both \( V_1 \) and \( V_2 \). In this case, \( H^V(EV) \rightarrow \{\bullet\}; K_R \) can be identified with the non-connective version of Waldhausen's Nil-term associated to this amalgamated product \( [4] \).

The identifications come from the Five-Lemma and the fact that both groups fit into the same long exact sequence associated to the semi-direct product, or amalgamated product, respectively. This is analogous to the \( L \)-theory case which is explained below. If \( R \) is regular and \( \mathbb{Q} \subseteq R \), \( e.g. \), \( R \) is a field of characteristic zero, then \( H^V(EV) \rightarrow \{\bullet\}; K_R \) = 0 for every virtually cyclic group \( V \), and hence, for any group \( G \) the map

\[
H^G_n(EV; K_R) \xrightarrow{\cong} H^G_n(EV; K_R)
\]
is bijective (see [29 Proposition 2.6]).

Let \( \text{UNil}_n^h(R; R, R) \) denote the Cappell UNil-groups [10] associated to the amalgamated product \( D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 \). It is a direct summand in \( L^h_n(R[D_\infty]) \) and there is a Mayer-Vietoris sequence

\[
\cdots \rightarrow L^h_n(R) \rightarrow L^h_n(R[\mathbb{Z}/2]) \oplus L^h_n(R[\mathbb{Z}/2]) \rightarrow L^h_n(R[R[D_\infty]]) / \text{UNil}_n(R; R, R)
\]

\[
\rightarrow L^h_n(R) \rightarrow L^h_n(R[\mathbb{Z}/2]) \oplus L^h_n(R[R[D_\infty]]) / \text{UNil}_n(R; R, R)
\]

There is also a Mayer-Vietoris sequence that maps to the one above which comes from the model for \( ED_\infty \) given by the obvious \( D_\infty \)-action on \( R \):

\[
\cdots \rightarrow L^h_n(R) \rightarrow L^h_n(R[\mathbb{Z}/2]) \oplus L^h_n(R[\mathbb{Z}/2]) \rightarrow H^D_n(ED_\infty; L^h_R)
\]

\[
\rightarrow L^h_n(R) \rightarrow L^h_n(R[\mathbb{Z}/2]) \oplus L^h_n(R[R[D_\infty]]) / \text{UNil}_n(R; R, R)
\]

\[
\rightarrow L^h_n(R) \rightarrow L^h_n(R[\mathbb{Z}/2]) \oplus L^h_n(R[R[D_\infty]]) / \text{UNil}_n(R; R, R)
\]

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This implies
\[ \text{UNil}_n^h(R; R, R) \cong HD_\infty^D(ED_\infty \to \{\bullet\}; L_R^h). \]

**Lemma 2.12.** Assume that \( \tilde{K}_i(R) \cong \tilde{K}_i(R[\mathbb{Z}/2]) \cong \tilde{K}_i(R[D_\infty]) = 0 \) for all \( i \leq 0 \). (This condition is satisfied, for example, if \( R = \mathbb{Z} \).) Then:
\[ \text{UNil}_n^h(R; R, R) \cong \text{UNil}_n^{(-\infty)}(D_\infty; R). \]

**Proof.** Using the Rothenberg sequences (2.3), one obtains natural isomorphisms
\[
L_n^{(-\infty)}(R) \cong L_n^h(R);
L_n^{(-\infty)}(R[\mathbb{Z}/2]) \cong L_n^h(R[\mathbb{Z}/2]);
L_n^{(-\infty)}(R[D_\infty]) \cong L_n^h(R[D_\infty]).
\]
Thus, a comparison argument involving the Atiyah-Hirzebruch spectral sequences shows that the obvious map
\[ HD_\infty^D(ED_\infty \to \{\bullet\}; L_R^h) \cong HD_\infty^D(ED_\infty \to \{\bullet\}; L_R^{(-\infty)}) \]
is bijective for all \( n \in \mathbb{Z}. \)

Cappell’s UNil-terms [10] have been further investigated in [2, 12] and [13]. The Waldhausen Nil-terms have been analyzed in [15] and [25]. If 2 is inverted, the situation in \( L \)-theory simplifies. Namely, for every \( n \in \mathbb{Z} \) and every virtually cyclic group \( V \), \( H_V^\mathbb{Z}(EV \to \{\bullet\}; L_R^{(\infty)})[1/2] = 0 \). Therefore the map
\[ H_G^n(E_G \to \{\bullet\}; L_R^{(-\infty)})[1/2] \cong H_G^n(E_G \to \{\bullet\}; L_R^{(-\infty)})[1/2] \]
is an isomorphism for any group \( G \), and the decorations do not play a role (see [29, Proposition 2.10]).

**Remark 2.13** (Role of type I and II). Let \( \text{VCY}_I \) be the family of subgroups that are either finite or infinite virtually cyclic of type I, i.e., groups admitting an epimorphism onto \( \mathbb{Z} \) with finite kernel. Then the following maps are bijections
\[
H_n^G(E_{\text{VCY}_I}(G); K_R) \cong H_n^G(E_G; K_R);
H_n^G(E_G; L_R^{(-\infty)}) \cong H_n^G(E_{\text{VCY}_I}(G); L_R^{(-\infty)}).
\]
For the \( K \)-theory case, see, for instance, [15, 19]. The \( L \)-theory case is proven in Lück [27, Lemma 4.2]). In particular, for a torsion-free group \( G \), the map
\[ H_n^G(E_G; L_R^{(-\infty)}) \cong H_n^G(E_G; L_R^{(-\infty)}) \]
is a bijection.
3 Proofs of Results

We now prove the results stated in Section 1.

3.1 Hyperbolic groups

Proof of Theorem 1.4. By [31, Corollary 2.11, Theorem 3.1 and Example 3.5] there is a $G$-pushout

$$
\begin{array}{ccc}
\amalg_{V \in \mathcal{M}} G \times_{V} E_{V} & \xrightarrow{i} & EG \\
\downarrow \downarrow & & \downarrow \downarrow \\
\amalg_{V \in \mathcal{M}} G/V & \xrightarrow{} & EG
\end{array}
$$

where $i$ is an inclusion of $G$-CW-complexes, $p$ is the obvious projection and $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of $G$. Now Theorem 1.4 follows from Theorem 2.1 and Lemma 2.6.

Proof of Theorem 1.5.

(i) This follows from (1.2) and Theorem 1.4 (i).

(ii) Since any virtually cyclic subgroup of $G$ is trivial or infinite cyclic, the claim follows from Theorem 2.1 and Remark 2.13.

3.2 $K$ - and $L$-theory of $\mathbb{Z}^d$

As a warm-up for virtually free abelian groups, we compute the $K$ and $L$-theory of $R[\mathbb{Z}^d]$.

Proof of Theorem 1.7. Using the induction structure of the equivariant homology theory $H^*_{G}\left(-; K_R\right)$ and the fact that $B\mathbb{Z}^d$ is the $d$-dimensional torus, it follows that

$$
H^Z_n\left(\mathbb{Z}^d; K_R\right) \cong H_n^{(1)}\left(\mathbb{Z}^d; K_R\right) \\
\cong \bigoplus_{i=0}^d H_{n-i}\left(\{\bullet\}; K_R\right)^{(i)} \\
= \bigoplus_{i=0}^d K_{n-i}(R)^{(i)}.
$$

(3.1)

Similarly one shows that

$$
H^Z_n\left(\mathbb{Z}^d; L^{(-\infty)}_R\right) \cong \bigoplus_{i=0}^d L^{(-\infty)}_{n-i}(R)^{(i)}.
$$

(3.2)

Theorem 2.1 implies that

$$
\text{Wh}_n(\mathbb{Z}^d; R) := H^Z_n\left(\mathbb{Z}^d; \{\bullet\}; K_R\right) \cong H^Z_n\left(\mathbb{Z}^d; \mathbb{Z}^d; K_R\right).
$$
From [31 Corollary 2.10] it follows that
\[ \bigoplus_{C \in \text{MICY}} H_n^{zd}(EZ^d \rightarrow EZ^d/C; K_R) \cong H_n^{zd}(EZ^d \rightarrow EZ^d/K_R). \]

Since \( C \subseteq \mathbb{Z}^d \) is maximal infinite cyclic, \( \mathbb{Z}^d \cong C \oplus \mathbb{Z}^{d-1} \). Therefore the induction structure and (1.2) imply
\[ H_n^{zd}(EZ^d \rightarrow E(Z^d/C); K_R) \cong H_n^{zd}(EZ^{d-1} \rightarrow E(Z^{d-1}/C); K_R) \cong H_n^{zd}(EC \rightarrow \{\bullet\}; K_R)(^{d-1}_{i=0}) \cong (NK_{n-i}(R) \oplus NK_{n-i}(R))^{d-1}_{i=0}. \] (3.3)

Hence,
\[ Wh_n(Z^d; R) \cong \bigoplus_{C \in \text{MICY}} \bigoplus_{i=0}^{d-1} (NK_{n-i}(R) \oplus NK_{n-i}(R))^{d-1}_{i=0}. \]

From Theorem 2.1 and Lemma 2.6 (i)
\[ K_n(R[Z^d]) \cong H_n^{zd}(EZ^d; K_R) \oplus Wh_n(Z^d; R), \]
which, by (3.1), is isomorphic to
\[ \left( \bigoplus_{i=0}^{d} K_{n-i}(R)^{(i)} \right) \oplus Wh_n(Z^d; R). \]

Finally,
\[ L_n^{(-\infty)}(R[Z^d]) \cong H_n^{zd}(EZ^d; L_R^{(-\infty)}) \cong \bigoplus_{i=0}^{d} L_n^{(-\infty)}(R)^{(i)} \]

by Theorem 2.1 Remark 2.13 and 3.2.

3.3 Virtually finitely generated abelian groups

For the remainder of the paper we will use the notation introduced in Subsection 1.4 and the following notation. For \( C \in I \) there is an obvious extension
\[ 1 \rightarrow C \rightarrow N_GC \xrightarrow{pc} W_GC \rightarrow 1, \]
where \( p_C \) is the canonical projection. We also have the extension

\[
1 \to A/C \to W_G C \xrightarrow{\mathcal{W}} Q_C \to 1,
\]

which is induced by the given extension \( 1 \to A \to G \xrightarrow{q} Q \to 1 \). Since \( C \subseteq A \) is a maximal infinite cyclic subgroup, \( A/C \cong \mathbb{Z}^{d-1} \).

Notice that any infinite cyclic subgroup \( C \) of \( A \) is contained in a unique maximal infinite cyclic subgroup \( C_{\text{max}} \). In particular for two maximal infinite cyclic subgroups \( C, D \subseteq A \), either \( C \cap D = \{0\} \) or \( C = D \). Let

\[
N_G[C] := \{g \in G \mid |gCg^{-1} \cap C| = \infty\}.
\]

For every \( C \in I \)

\[
N_G[C] = N_G C = q^{-1}(Q_C).
\]

Consider the following equivalence relation on the set of infinite virtually cyclic subgroups of \( G \). We call \( V_1 \) and \( V_2 \) equivalent if and only if \( (A \cap V_1)_{\text{max}} = (A \cap V_2)_{\text{max}} \). Then for every infinite virtually cyclic subgroup \( V \) of \( G \) there is precisely one \( C \in I \) such that \( V \) is equivalent to \( gCg^{-1} \), for some \( g \in G \). We obtain from \([31, \text{Theorem 2.3}]\) isomorphisms

\[
\bigoplus_{C \in I} H_n^G(C) \to W_G C \cong H_n^G \left( \mathbb{Z}G; K_R \right); \quad (3.5)
\]

\[
\bigoplus_{C \in I} H_n^G(C) \to W_G C \cong H_n^G \left( \mathbb{Z}G; L_R^{(-\infty)} \right); \quad (3.6)
\]

where \( p_C : N_G C = q^{-1}(Q_C) \to W_G C = q^{-1}(Q_C)/C \) is the canonical projection.

**Lemma 3.7.** Let \( f : G_1 \to G_2 \) be a surjective group homomorphism. Consider a subgroup \( H \subseteq G_1 \). Let \( Y \) be a \( G_1 \)-space and \( Z \) be an \( H \)-space. Denote by \( f_H : f^{-1}(H) \to H \) the map induced by \( f \).

Then there is a natural \( G_1 \)-homeomorphism

\[
G_1 \times f^{-1}(H) \left( \text{res}^f_{G_1}(H) Y \times f_H Z \right) \cong Y \times f^*(G_2 \times_H Z),
\]

where \( f_H, f^* \) and \( \text{res}^f_{G_1}(H) \) denote restriction and the actions on products are the diagonal actions.

**Proof.** The map sends \((g, (y, z))\) to \((gy, (f(g), z))\). Its inverse sends \((y, (k, z))\) to \((h, (h^{-1}y, z))\) for any \( h \in G_1 \) with \( f(h) = k \). \( \square \)

### 3.3.1 \( K \)-theory in the case of a free conjugation action

**Proof of Theorem 1.17** We prove assertions (i) and (ii) simultaneously by a direct computation.

By Theorem 2.1

\[
H_n^G \left( \mathbb{Z}G; K_R \right) \cong H_n^G \left( \{\bullet\}; K_R \right).
\]

Thus,

\[
\text{Wh}_n(G; R) := H_n^G \left( \mathbb{Z}G \to \{\bullet\}; K_R \right) \\
\cong H_n^G \left( \mathbb{Z}G \to \mathbb{Z}G; K_R \right) \\
\cong H_n^G \left( \mathbb{Z}G \to \mathbb{Z}G; K_R \right) \oplus H_n^G \left( \mathbb{Z}G \to \mathbb{Z}G; K_R \right),
\]

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using \([2,8]\). From \([30\text{ Lemma 6.3}]\) and \([31\text{ Corollary 2.11}]\), there is a \(G\)-pushout
\[
\begin{array}{ccc}
\prod_{F \in J} G \times_F EF & \xrightarrow{1} & EG \\
\| & \downarrow & \\
\prod_{F \in J} G / F & \xrightarrow{1} & EG \\
\end{array}
\]  
(3.8)
where \(J\) is a complete system of representatives of maximal finite subgroups of \(G\). This produces an isomorphism
\[
\bigoplus_{F \in J} \text{Wh}_n(F; R) := \bigoplus_{F \in J} H_n^F(\text{EF} \to \{\bullet\}; K_R) \xrightarrow{\cong} H_n^G(\text{EG} \to \text{EG}; K_R).
\]
By \([3.3]\),
\[
\bigoplus_{C \in \text{ICY}(A)} H_n^{N_AC}(EN_AC \to p_C^* EW_AC; K_R) \cong H_n^A(\text{EA} \to \text{EA}; K_R) = \text{Wh}_n(A; R),
\]
where \(p_C: N_AC = A \to W_AC = A/C\) denotes the projection. The conjugation action of \(Q\) on \(A\) induces an action of \(Q\) on \(H_n^A(\text{EA} \to \text{EA}; K_R)\). By the definition of the index set \(I\) and the subgroup \(Q_C \subseteq Q\), we obtain a \(\mathbb{Z}Q\)-isomorphism
\[
\bigoplus_{C \in I} \mathbb{Z}[Q_C] H_n^A(\text{EA} \to p_C^* E(A/C); K_R) \cong H_n^A(\text{EA} \to \text{EA}; K_R),
\]
and hence an isomorphism
\[
\bigoplus_{C \in I} \mathbb{Z}[Q_C] H_n^A(\text{EA} \to p_C^* E(A/C); K_R) \cong \mathbb{Z} \otimes \mathbb{Z}[Q] H_n^A(\text{EA} \to \text{EA}; K_R).
\]
If \(Q_C = \{1\}\), then \(N_GC = A \cong C \otimes \mathbb{Z}^{d-1}\) and \(W_GC \cong \mathbb{Z}^{d-1}\). Thus, by \([3.3]\),
\[
H_n^{N_GC}(\text{EN_GC} \to p_C^* \text{EW_GC}; K_R) \cong H_n^A(\text{EA} \to p_C^* E(A/C); K_R) \cong H_n^{C \otimes \mathbb{Z}^{d-1}}(EC \times EZ^{d-1} \to EZ^{d-1}; K_R) \cong \bigoplus_{i=0}^{d-1} (NK_{n-i}(R) \oplus NK_{n-i}(R)).
Therefore, the proof of the theorem will be finished once it is established that for \( C \in I_2, \)
\[
H^{N_G C}_n (\mathbb{E} N_G C \rightarrow p_C^* \mathbb{E} W_G C; \mathbb{K}_R)
\]
\[
\cong \mathbb{Z} \otimes \mathbb{Z}[Q_C] H^A_n (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R)^{(d-1)}.
\]

Assume that \( Q_C = \mathbb{Z}/2. \) Since the projection \( \mathbb{E} N_G C \times p_C^* \mathbb{E} W_G C \rightarrow \mathbb{E} N_G C \) is an \( N_G C \)-homotopy equivalence, \( H^{N_G C}_n (\mathbb{E} N_G C \rightarrow p_C^* \mathbb{E} W_G C; \mathbb{K}_R) \) is isomorphic to \( H^{N_G C}_n (\mathbb{E} N_G C \times p_C^* \mathbb{E} W_G C \rightarrow p_C^* \mathbb{E} W_G C; \mathbb{K}_R), \) which, by Lemma 3.9(ii), is isomorphic to
\[
H^{N_G C}_n (\mathbb{E} N_G C \times p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C}) \mathbb{E} Q C) \rightarrow p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C; \mathbb{K}_R).
\]

Sending a \( Q_C \)-CW-complex \( Y \) to the \( \mathbb{Z} \)-graded abelian group
\[
H^{N_G C}_n (\mathbb{E} N_G C \times p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} Y) \rightarrow p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} Y; \mathbb{K}_R)
\]
yields a \( Q_C \)-homology theory. Since \( \mathbb{E} Q C \) is a free \( Q_C \)-CW-complex, there is an equivariant Atiyah-Hirzebruch spectral sequence converging to
\[
H^{N_G C}_{n+j} (\mathbb{E} N_G C \times p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C) \rightarrow p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C; \mathbb{K}_R)
\]
whose \( E^2 \)-term is given by
\[
E^2_{i,j} = H^{Q C}_i (\mathbb{E} Q C; H^{N_G C}_{j} (\mathbb{E} N_G C \times p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C) \rightarrow p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C; \mathbb{K}_R);
\]

Lemma 3.9(ii) implies that:
\[
E^2_{i,j} \cong H^{Q C}_i (\mathbb{E} Q C; H^{N_G C}_j (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R))
\]
\[
\cong H^{Q C}_i (\mathbb{E} Q C; \mathbb{Z}[Q_C] \otimes \mathbb{Z} (Z \otimes \mathbb{Z}[Q_C] H^{N_G C}_j (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R))
\]
\[
\cong H^{[1]}_i (\mathbb{E} Q C; \mathbb{Z} \otimes \mathbb{Z}[Q_C] H^{N_G C}_j (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R))
\]
\[
\cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}[Q_C] H^{A}_j (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R) & i = 0 \\ \{0\} & i \neq 0. \end{cases}
\]

Hence, the Atiyah-Hirzebruch spectral sequence collapses at \( E^2 \). Therefore,
\[
H^{N_G C}_n (\mathbb{E} N_G C \times p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C) \rightarrow p_C^* (\mathbb{E} W_G C \times \mathbb{Q}_{C} \mathbb{E} Q C; \mathbb{K}_R)
\]
\[
\cong \mathbb{Z} \otimes \mathbb{Z}[Q_C] H^{A}_n (EA \rightarrow p_C^* E(A/C); \mathbb{K}_R) \cong \bigoplus_{i=0}^{d-1} NK_{n-i}(R)^{(d-1)}.
\]

This completes the proof of Theorem 1.1 once we have proved Lemma 3.9.
Lemma 3.9. Let $G$ be a group satisfying the assumptions of Theorem 1.11 and let $C$ be a maximal cyclic subgroup of $G$ such that $Q_C = \mathbb{Z}/2$. Then:

(i) For every $n \in \mathbb{Z}$, the projection $EW_GC \times \overline{\mathfrak{q}C} \to EW_GC$ induces a bijection,

$$H_n^{N_C}(\mathbb{Z}[N]C \times \overline{\mathfrak{q}C} \to EW_GC) \to p_{C}^{\ast}(EW_GC \times \overline{\mathfrak{q}C};\mathcal{K}_R) \cong H_n^{N_C}(\mathbb{Z}[N]C \times \overline{\mathfrak{q}C} \to EW_GC;\mathcal{K}_R).$$

(ii) There are isomorphisms of $\mathbb{Z}[Q_C]$-modules

$$H_j^{N_C}(\mathbb{Z}[N]C \times \overline{\mathfrak{q}C} \to EW_GC) \to p_{C}^{\ast}(EW_GC \times \overline{\mathfrak{q}C};\mathcal{K}_R) \cong H_j^1(\mathbb{Z}[Q]C \to EW_GC;\mathcal{K}_R) \cong \bigoplus_{l=0}^{d-1} (\mathbb{Z}[Q_C] \otimes \mathbb{Z}N_{j-l}(R))^{(d-l)}.$$

Proof. (i) Note that the conjugation action of $Q_C = \mathbb{Z}/2$ on $A/C$ is free away from $0 \in A/C$. To see this, let $q$ be a nontrivial element in $Q$, and let $aC \in A/C$ such that $aC$ is fixed under the conjugation action with $q$. Then there is a $c \in C$ such that $\rho(q)(a) = a + c$, where $\rho: Q \to \text{aut}(A)$ is given by the conjugation action of $Q$ in $A$. This implies $\rho(q)(2a) = 2a + 2c$. Thus,

$$\rho(q)(2a + c) = \rho(q)(2a) + \rho(q)(c) = \rho(q)(2a) - c = 2a + 2c - c = 2a + c.$$ 

By assumption, the $Q$-action on $A$ is free away from $0 \in A$, so $2a + c = 0$ and thus $2a \in C$. Since $C$ is maximal infinite cyclic, $aC = 0$ in $A/C$. Hence, the conjugation action of $Q_C$ on $A/C$ is free away from $0 \in A/C$.

Since $(A/C)^{Q_C} = \{0\}$, the extension (3.11) has a section, and $W_GC$ is isomorphic to the semidirect product $\mathbb{Z}^{d-1} \times \mathbb{Z}/2$ with respect to the involution $-id: \mathbb{Z}^{d-1} \to \mathbb{Z}^{d-1}$. Let $J_C$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of $W_GC$. Then every element in $J_C$ is isomorphic to $\mathbb{Z}/2$ and $J_C$ contains $2^{d-1}$ elements (see Remark 1.15). From [30] Lemma 6.3 and [31] Corollary 2.11, there is a $W_GC$-pushout

$$
\begin{align*}
\coprod_{H \in J_C} W_GC \times_H EH & \xrightarrow{i} EW_GC \\
\coprod_{H \in J_C} W_GC/H & \xrightarrow{p} EW_GC
\end{align*}
$$

(3.10)

Let $H \in J_C$. Given any $N_{G_C}$-space $Y$ and any $H$-space $Z$, Lemma 3.7 produces an $N_{G_C}$-homeomorphism

$$N_{G_C} \times_{p_{C}^{-1}(H)} (\text{res}_{N_{G_C}}^{p_{C}^{-1}(H)}(Y) \times p_{C}^{\ast}Z) \xrightarrow{\cong} Y \times p_{C}^{\ast}(W_GC \times_H Z),$$

(3.11)
where $p_C^*$ denotes restriction for both $p_C: N_G C \to W_G C$ and the homomorphism $p_C^{-1}(H) \to H$ induced by $p_C$. Since $\text{res}_{N_G C}^{p_C^{-1}(H)}(\mathcal{E}N_G C)$ is a $p_C^{-1}(H)$-CW-model for $\mathcal{E}p_C^{-1}(H)$, we obtain identifications of $N_G C$-spaces

$$\mathcal{E}N_G C \times p_C^* (W_G C \times_H EH) \cong N_G C \times_{p_C^{-1}(H)} (\mathcal{E}p_C^{-1}(H) \times p_C^* EH);$$

$$\mathcal{E}N_G C \times p_C^* (W_G C / H) \cong N_G C \times_{p_C^{-1}(H)} (\mathcal{E}p_C^{-1}(H));$$

$$p_C^* (W_G C \times_H EH) \cong N_G C \times_{p_C^{-1}(H)} p_C^* EH;$$

$$p_C^* (W_G C / H) \cong N_G C \times_{p_C^{-1}(H)} \{\bullet\},$$

by substituting $Y = \mathcal{E}N_G C$ or $Y = \{\bullet\}$, and $Z = EH$ or $Z = \{\bullet\}$ into (3.11). Using the induction structure of the equivariant homology theory $H^*_C(-; K_R)$ and the Five-Lemma, these identifications imply

$$H_n^{N_G C} (\mathcal{E}N_G C \times p_C^* (W_G C \times_H EH) \to p_C^* (W_G C \times_H EH); K_R) \cong H_n^{p_C^{-1}(H)} (\mathcal{E}p_C^{-1}(H) \times p_C^* EH \to p_C^* EH; K_R) \quad (3.12)$$

and

$$H_n^{p_C^{-1}(H)} (\mathcal{E}p_C^{-1}(H) \to \{\bullet\}; K_R) \cong H_n^{N_G C} (\mathcal{E}N_G C \times p_C^* (W_G C / H) \to p_C^* (W_G C / H); K_R) \quad (3.13)$$

are bijective for every $n \in \mathbb{Z}$.

Consider the following $p_C^{-1}(H)$-pushout, where the left vertical arrow is an inclusion of $p_C^{-1}(H)$-CW-complexes and the upper horizontal arrow is cellular.

$$\begin{array}{ccc}
\mathcal{E}p_C^{-1}(H) & \longrightarrow & p_C^* EH \\
\downarrow & & \downarrow \\
\mathcal{E}p_C^{-1}(H) & \longrightarrow & X
\end{array}$$

Then $X$ is a $p_C^{-1}(H)$-CW-complex, and for every subgroup $K \subseteq p_C^{-1}(H)$, the above $p_C^{-1}(H)$-pushout induces a pushout of CW-complexes:

$$\begin{array}{ccc}
\mathcal{E}p_C^{-1}(H)^K & \longrightarrow & (p_C^* EH)^K \\
\downarrow & & \downarrow \\
\mathcal{E}p_C^{-1}(H)^K & \longrightarrow & X^K
\end{array}$$

If $K = \{1\}$, then the spaces $\mathcal{E}p_C^{-1}(H)^K$, $(p_C^* EH)^K$ and $\mathcal{E}p_C^{-1}(H)^K$ are contractible, and thus $X^K$ is contractible. If $K$ is a non-trivial finite subgroup, then the spaces $\mathcal{E}p_C^{-1}(H)^K$ and $(p_C^* EH)^K$ are empty, and the space $\mathcal{E}p_C^{-1}(H)^K$ is contractible. Hence $X^K$ is contractible. If $K$ is an infinite subgroup of $C$,
then the spaces $E_{p'}^{-1}(H)^K$ and $E_{p'}^{-1}(H)^K$ are empty, and the space $(p'_C, EH)^K$ is contractible since $p'_C(K) = \{1\}$. Therefore, $X^K$ is contractible. If $K$ is infinite and not contained in $C$, then the spaces $E_{p'}^{-1}(H)^K$, $(p'_C, EH)^K$ and $E_{p'}^{-1}(H)^K$ are all empty, and so $X^K$ is also empty. Since $p'_C (H) \cong D_\infty$, an infinite virtually cyclic subgroup $K$ of $p'_C(H)$ is of type I if and only if it is an infinite subgroup of $C$. Hence, $X$ is a model for $E_{V CY_i}(p'_C(H))$. Notice that $E_{p'}^{-1}(H) \times p'_C, EH$ is a model for $E_{p'}^{-1}(H)$. Therefore, we have the following cellular $p'_C(H)$-pushout, where the left vertical arrow is an inclusion of $p'_C(H)$-CW-complexes and the upper horizontal arrow is cellular.

\[
\begin{array}{ccc}
E_{p'}^{-1}(H) \times p'_C, EH & \rightarrow & p'_C, EH \\
\downarrow & & \downarrow \\
E_{p'}^{-1}(H) & \rightarrow & E_{V CY_i}(p'_C(H))
\end{array}
\]

This induces an isomorphism

\[
H_{n}^{p'_C(H)}(E_{p'}^{-1}(H) \times p'_C, EH \rightarrow p'_C, EH; K_R) \\
\cong H_{n}^{p'_C(H)}(E_{p'}^{-1}(H) \rightarrow E_{V CY_i}(p'_C(H)); K_R), \quad (3.14)
\]

for every $n \in \mathbb{Z}$. Since $p'_C(H) \cong D_\infty$ is virtually cyclic, Remark 2.13 implies that the map

\[
H_{n}^{p'_C(H)}(E_{V CY_i}(p'_C(H)); K_R) \rightarrow H_{n}^{p'_C(H)}(\{\bullet\}; K_R)
\]

is bijective for every $n \in \mathbb{Z}$, and so by (3.14),

\[
H_{n}^{p'_C(H)}(E_{p'}^{-1}(H) \times p'_C, EH \rightarrow p'_C, EH; K_R) \\
\cong H_{n}^{p'_C(H)}(E_{p'}^{-1}(H) \rightarrow \{\bullet\}; K_R) \quad (3.15)
\]

is bijective for every $n \in \mathbb{Z}$. Therefore, (3.12), (3.13) and (3.15) imply that

\[
H_{n}^{N_{C}}(E_{N_{C}} \times p'_C, (W_{C} \times H, EH) \rightarrow p'_C, (W_{C} \times H, EH); K_R) \\
\cong H_{n}^{N_{C}}(E_{N_{C}} \times p'_C, (W_{C} \times H, EH) \rightarrow p'_C, (W_{C} \times H, EH); K_R) \quad (3.16)
\]

is an isomorphism for every $n \in \mathbb{Z}$.

We obtain a $W_{C}$-homology theory by assigning to a $W_{C}$-CW-complex $Z$ the $Z$-graded abelian group $H_{n}^{N_{C}}(E_{N_{C}} \times p'_C, Z \rightarrow p'_C, Z; K_R)$. From the Mayer-Vietoris sequence associated to the $W_{C}$-pushout (3.10) and the bijectivity of the map (3.16), there is an isomorphism

\[
H_{n}^{N_{C}}(E_{N_{C}} \times p'_C, EW_{C} \rightarrow p'_C, EW_{C}; K_R) \\
\cong H_{n}^{N_{C}}(E_{N_{C}} \times p'_C, EW_{C} \rightarrow p'_C, EW_{C}; K_R). \quad (3.17)
\]
Since the projection $EW_GC \times \underline{\underline{\mathit{qC}}} \to EQC \to EW_GC$ is a $W_GC$-homotopy equivalence, the desired isomorphism now follows from (3.17).

(ii) By Lemma 3.7

$$H^N_{jC}(E_{N_GC} \times p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC) \to p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC); K_R)$$

$$\cong H^N_{jC}(E_{N_GC} \times p_C(W_GC \times \underline{\underline{\mathit{A/C}}} \text{res}_{W_GC}^{A/C} EW_GC)$$

$$\to p_C(W_GC \times \underline{\underline{\mathit{A/C}}} \text{res}_{W_GC}^{A/C} EW_GC); K_R)$$

$$\cong H^N_{jC}(N_GC \times _A (\text{res}_{N_GC}^A E_{N_GC} \times p_C \text{res}_{W_GC}^{A/C} EW_GC))$$

$$\to N_GC \times _A (\underline{\underline{\mathit{p/C}}} \text{res}_{W_GC}^{A/C} EW_GC); K_R).$$

The generator $t$ of $Q_C = Z/2$ acts on

$$H^N_{jC}(E_{N_GC} \times p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC) \to p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC); K_R)$$

by the square of $N_GC$-maps

$$E_{N_GC} \times p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC) \xrightarrow{\text{id} \times p_C^*(\text{id} \times r_1)} p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC)$$

$$E_{N_GC} \times p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC) \xrightarrow{p_C^*(\text{id} \times r_1)} p_C^*(EW_GC \times \underline{\underline{\mathit{qC}}} QC)$$

where $r_1: Q_C \to Q_C$ is right multiplication with $t$.

To simplify notation, let $X = \text{res}_{N_GC}^A E_{N_GC}$ and $Y = \underline{\underline{\mathit{p/C}}} \text{res}_{W_GC}^{A/C} EW_GC$. Choose an element $\gamma \in N_GC$ that is mapped to $t$ by $q_C: N_GC \to Q_C$, and equip

$$H^N_{jC}(N_GC \times _A (X \times Y) \to N_GC \times _A Y; K_R)$$

with the $Q_C$-action coming from square of $N_GC$-maps

$$N_GC \times _A (X \times Y) \xrightarrow{r_\gamma \times l_{\gamma^{-1}}} N_GC \times _A Y$$

$$N_GC \times _A (X \times Y) \xrightarrow{r_\gamma \times l_{\gamma^{-1}}} N_GC \times _A Y$$

where $l_{\gamma^{-1}}$ is given by $(x, y) \mapsto (\gamma^{-1} \cdot x, p_C(\gamma^{-1}) \cdot y)$ for $x \in X$ and $y \in Y$ and $r_\gamma$ is right multiplication by $\gamma$. It is straightforward to check that the isomorphism (3.18) is compatible with this $Q_C$-action.

Recall that $Q_C$ acts freely on $A$ away from the identity. This implies that if $t \in Q_C$ is the generator and $a \in A$, then $t \cdot (a + ta) = a + ta$, and so $a + ta = 0$. Therefore the $Q_C$-action on $A$ is given by $-\text{id}_A: A \to A$. Since the action of $t$ on $A$ is defined by conjugation by $\gamma^{-1}$, the following diagram commutes.

$$\begin{array}{ccc}
N_GC & \xrightarrow{c(\gamma^{-1})} & N_GC \\
\downarrow & & \downarrow \\
A & \xrightarrow{-\text{id}_A} & A
\end{array}$$
For any $A$-CW complex $Z$, this yields the commutative square

\[
\begin{array}{ccc}
H^*_N(GC \times A Z; K_R) & \xrightarrow{f_* \circ \text{ind}_A^{(\gamma^{-1})}} & H^*_N(GC \times A (A \times \text{id}_A Z); K_R) \\
\text{ind}_A^* \cong & & \text{ind}_A^* \cong \\
H_*^A(Z; K_R) & \xrightarrow{\text{ind}_A - \text{id}_A} & H_*^A(A \times \text{id}_A Z; K_R),
\end{array}
\]

where $\text{ind}_A^{NC}$ denotes induction with respect to inclusion, and $f_*$ is induced by the map $f : N_G C \times (\gamma^{-1}) (N_G C \times A Z) \to N_G C \times A (A \times \text{id}_A Z)$, which sends $(k, (g, z))$ to $(k \circ (\gamma^{-1})(g), (e, z))$. From the axioms of an induction structure, we also have that $\text{ind}_A^{(\gamma^{-1})}$ is induced by the map $f_2 : N_G C \times A Z \to N_G C \times (\gamma^{-1}) (N_G C \times A Z)$, which sends $(k, z)$ to $(e, (\gamma k, z))$ (see Section 1).

When $Z = X \times Y$ or $Z = Y$, we can define a map $l : A \times \text{id}_A Z \to Z$ such that $l(a, z) = a \gamma^{-1} z$. Since $\text{ind}_A^{NC}$ is natural, this produces the commutative square

\[
\begin{array}{ccc}
H^*_N(GC \times A (A \times \text{id}_A Z); K_R) & \xrightarrow{l'_* \circ \text{ind}_A^{(\gamma^{-1})}} & H^*_N(GC \times A Z; K_R) \\
\text{ind}_A^* \cong & & \text{ind}_A^* \cong \\
H_*^A(A \times \text{id}_A Z; K_R) & \xrightarrow{l'_* \circ \text{id}_A - \text{id}_A} & H_*^A(Z; K_R),
\end{array}
\]

where $l' : N_G C \times A (A \times \text{id}_A Z) \to N_G C \times A Z$ maps $(k, (a, z))$ to $(k, a \gamma^{-1} z)$. Notice that the composition $l'_* \circ f_* \circ \text{ind}_A^{(\gamma^{-1})} = l'_* \circ f_* \circ (f_2)_*$ is precisely the map $r_* \times l_{\gamma^{-1}}$ used to define the $Q_C$-action on $H^*_N(GC \times A (X \times Y) \to N_G C \times A Y; K_R)$. Thus, the equivalence

\[
H^*_N(GC \times A (X \times Y) \to N_G C \times A Y; K_R) \xrightarrow{\text{ind}_A^{NC}} H_*^A(X \times Y \to Y; K_R)
\]

is compatible with the $Q_C$-action, where the action on $H_*^A(X \times Y \to Y; K_R)$ is induced by the composition $l_* \circ \text{id}_A - \text{id}_A$.

Notice that $X$ is a model for $EA$ and $Y$ is a model for $ED$. Therefore, the map $l$ defined above is unique up to $A$-homotopy. Since $C \subseteq A$ is maximal cyclic, there is a $D \subseteq A$ such that $C \oplus D \cong A$ and $D \cong \mathbb{Z}^{d-1}$. Thus, the identification

\[
H_*^A(X \times Y \to Y; K_R) \cong H_*^{C \oplus D}((EC \times ED) \times ED \to ED; K_R)
\]

agrees with the $Q_C$-action since the action on $H^*_{C \oplus D}(EC \times ED \to ED; K_R)$ is induced by $(l_*^C, l_*^D) \circ \text{id}_C \circ \text{id}_D$, where $l_*^C : \text{id}_C - \text{id}_D EC \to EC$ denotes the unique $C$-equivariant map (up to $C$-homotopy) and $l_*^D : \text{id}_D ED \to ED$. 

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denotes the unique $D$-equivariant map (up to $D$-homotopy). Note that this establishes the first desired isomorphism of $\mathbb{Z}[Q_C]$-modules:

\[
H^N_{j}(\mathcal{E}N\mathcal{C} \times \mathcal{P}_C(EW\mathcal{C} \times \mathcal{P}_C\mathcal{Q}_C); K_R) \cong H^A_j(EA \rightarrow p_C E(A/C); K_R).
\]

Using the freeness of the action of $D$,

\[
H^C_j(\mathcal{E}C \times ED \rightarrow ED; K_R) \cong H^C_j(\mathcal{E}C \times BD \rightarrow BD; K_R).
\]

As in (3.3), since $BD$ is the $(d-1)$-dimensional torus,

\[
H^C_j(\mathcal{E}C \times BD \rightarrow BD; K_R) \cong \bigoplus_{l=0}^{d-1} H^C_{j-l}(\mathcal{E}C \rightarrow \{\bullet\}; K_R)(d-l).
\]

By Remark 2.10, the $Q_C$-action on $H^C_j(\mathcal{E}C \rightarrow \{\bullet\}; K_R)$, induced by $(l_C)_* \circ \text{ind}_{id_C}$, corresponds to interchanging the two copies of $NK_j(R)$ in the decomposition

\[
H^C_j(\mathcal{E}C \rightarrow \{\bullet\}; K_R) \cong NK_j(R) \oplus NK_{j-1}(R).
\]

Thus, we obtain an isomorphism of $\mathbb{Z}[Q_C]$-modules

\[
\bigoplus_{l=0}^{d-1} (\mathbb{Z}[Q_C] \otimes \mathbb{Z} NK_{j-l}(R))^{(d-l)} \cong H^C_j(BD \times EC \rightarrow BD; K_R),
\]

which yields the desired result.

Proof of Theorem 1.12

(i) Since $R$ is a regular ring, $NK_j(R) = \{0\}$ for every $j \in \mathbb{Z}$. Therefore,

\[
\bigoplus_{F \in J} \text{Wh}_n(F; R) \cong \text{Wh}_n(G; R)
\]

for every $n \in \mathbb{Z}$ by Theorem 1.11. Furthermore, $K_n(R) = 0$ for every $n \leq -1$. Thus, for any group $\Gamma$, the Atiyah-Hirzebruch spectral sequence implies that

\[
H_n(\mathcal{B} \Gamma; K(R)) = 0 \quad \text{for } n \leq -1,
\]

and the edge homomorphism induces an isomorphism

\[
H_0(\mathcal{B} \Gamma; K(R)) \cong K_0(R).
\]

Hence,

\[
\text{Wh}_n(\Gamma; R) \cong K_n(R \Gamma) \quad \text{for } n \leq -1
\]

and

\[
\text{Wh}_0(\Gamma; R) \cong \text{coker}(K_0(R) \rightarrow K_0(R \Gamma)).
\]

(ii) By Carter 11, $K_n(RF) = \{0\}$ for $n \leq -2$

for every $F \in J$. Now apply assertion (ii) to complete the proof. □
3.3.2 $L$-theory in the case of a free conjugation action

Proof of Theorem [1.17] As in the $K$-theory proof, Theorem 2.1 and 2.9 imply:

\[
S_{n}^{\text{per},(-\infty)}(G; R) := H^{G}_{n}(EG \to \{\bullet\}; L_{R}^{(-\infty)}) \\
\cong H^{G}_{n}(EG \to EG; L_{R}^{(-\infty)}) \\
\cong H^{G}_{n}(EG \to EG; L_{R}^{(-\infty)}) \oplus H^{G}_{n}(EG \to EG; L_{R}^{(-\infty)}).
\]

From the $G$-pushout (3.8),

\[
\bigoplus_{F \in J} S_{n}^{\text{per},(-\infty)}(F; R) := \bigoplus_{F \in J} H^{F}_{n}(EF \to \{\bullet\}; L_{R}^{(-\infty)}) \cong H^{G}_{n}(EG \to EG; L_{R}^{(-\infty)}).
\]

By (3.6),

\[
\bigoplus_{C \in J} H^{N_{G}C}_{n}(ENG_{C} \to p_{C}^{*}W_{G}C; L_{R}^{(-\infty)}) \cong H^{G}_{n}(EG \to EG; L_{R}^{(-\infty)}).
\]

Recall from the proof of Theorem [1.11] that $I = I_{1} \amalg I_{2}$.

If $C \in I_{1}$, then $N_{G}C = A \cong C \oplus \mathbb{Z}^{d-1}$ and $W_{G}C \cong \mathbb{Z}^{d-1}$. Therefore, as in (33), $H^{N_{G}C}_{n}(ENG_{C} \to p_{C}^{*}W_{G}C; L_{R}^{(-\infty)})$ is isomorphic to:

\[
H^{C\oplus\mathbb{Z}^{d-1}}_{n}(EC \times EZ^{d-1} \to EZ^{d-1}; L_{R}^{(-\infty)}) \cong \bigoplus_{i=0}^{d-1} H^{n-i}_{n}(EC \to \{\bullet\}; L_{R}^{(-\infty)}) = 0
\]

by Remark 2.13.

Now assume $C \in I_{2}$. We obtain a $W_{G}C$-homology theory by assigning to a $W_{G}C$-CW-complex $Z$ the $\mathbb{Z}$-graded abelian group

\[
H^{N_{G}C}_{i}(ENG_{C} \times p_{C}^{*}Z \to p_{C}^{*}Z; L_{R}^{(-\infty)}).
\]

Consider any free $W_{G}C$-CW-complex $Y$. Then there is an equivariant Atiyah-Hirzebruch spectral sequence converging to

\[
H^{N_{G}C}_{i+j}(ENG_{C} \times p_{C}^{*}Y \to p_{C}^{*}Y; L_{R}^{(-\infty)}),
\]

whose $E^{2}$-term is given by

\[
E^{2}_{i,j} = H^{W_{G}C}_{i}(Y; H^{N_{G}C}_{j}(ENG_{C} \times p_{C}^{*}W_{G}C \to p_{C}^{*}W_{G}C; L_{R}^{(-\infty)})).
\]

Using Lemma 3.7

\[
H^{N_{G}C}_{j}(ENG_{C} \times p_{C}^{*}W_{G}C \to p_{C}^{*}W_{G}C; L_{R}^{(-\infty)}) \\
\cong H^{N_{G}C}_{j}(N_{G}C \times C \text{res}_{N_{G}C}^{C}ENG_{C} \to N_{G}C \times C \{\bullet\}; L_{R}^{(-\infty)}) \\
\cong H^{C}_{j}(\text{res}_{N_{G}C}^{C}ENG_{C} \to \{\bullet\}; L_{R}^{(-\infty)}) \\
\cong H^{C}_{j}(EC \to \{\bullet\}; L_{R}^{(-\infty)}).
\]

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But this is zero by Remark 2.13. Hence, \( E_{i,j}^2 = 0 \) for all \( i, j \in \mathbb{Z} \). This implies that for every free \( W_G \)-CW-complex \( Y \), \( H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} Y \to p_C^{-1} Y; L_R^{(-\infty)}) \) vanishes. In particular,\n
\[
H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} E W_G \to p_C^{-1} E W_G; L_R^{(-\infty)}) = 0
\]

and\n
\[
H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} (W_G \times_H EH) \to p_C^{-1} (W_G \times_H EH); L_R^{(-\infty)}) = 0.
\]

Therefore the Mayer-Vietoris sequence associated to the \( W_G \)-pushout (3.10) yields an isomorphism\n
\[
\bigoplus_{H \in J_C} H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} W_G \to p_C^{-1} W_G; L_R^{(-\infty)}) \xrightarrow{\cong} H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} E W_G \to p_C^{-1} E W_G; L_R^{(-\infty)}).
\]

Since \( p_C^{-1}(H) \cong D_\infty \), Lemma 3.3 implies:\n
\[
H^n_{\text{NG}}(E \text{NG} \times p_C^{-1} W_G \to p_C^{-1} W_G; L_R^{(-\infty)}) \cong H^n_{\text{NG}}(N_G \times p_C^{-1}(H)_{\text{res}}) E \text{NG} \to N_G \times p_C^{-1}(H) \{\bullet\}; L_R^{(-\infty)}) \\
\cong H^n_{\text{NG}}(p_C^{-1}(H)_{\text{res}} E \text{NG} \to \{\bullet\}; L_R^{(-\infty)}) \\
\cong H_n^{D_{\infty}}(E \text{NG} \to \{\bullet\}; L_R^{(-\infty)}) \\
= \text{UNil}_{\text{NG}}^{(-\infty)}(D_\infty; R).
\]

The projection \( E \text{NG} \times p_C^{-1} E W_G \to E \text{NG} \) is a \( \text{NG} \)-homotopy equivalence, thus\n
\[
\bigoplus_{H \in J_C} \text{UNil}_{\text{NG}}^{(-\infty)}(D_\infty; R) \xrightarrow{\cong} H^n_{\text{NG}}(E \text{NG} \to \{\bullet\}; L_R^{(-\infty)}).
\]

Therefore,\n
\[
\bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \text{UNil}_{\text{NG}}^{(-\infty)}(D_\infty; R) \cong H^n_{\text{NG}}(E \text{NG} \to \{\bullet\}; L_R^{(-\infty)}),
\]

which implies that\n
\[
\left( \bigoplus_{F \in J} S^n_{\text{per}}^{(-\infty)}(F; R) \right) \oplus \left( \bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \text{UNil}_{\text{NG}}^{(-\infty)}(D_\infty; R) \right) \xrightarrow{\cong} S^n_{\text{per}}^{(-\infty)}(G; R).
\]

Theorem 1.13 now follows from the fact that \( I_2 \) is empty if \( Q \) has odd order. \( \square \)
3.3.3  \(K\)-theory in the case \(Q = \mathbb{Z}/p\) for a prime \(p\) and regular \(R\)

**Lemma 3.19.** Let \(f : G \to G'\) be a group homomorphism, \(F\) be a family of subgroups of \(G\) and \(R\) be a ring. Let \(X\) be a \(G'\)-CW-complex such that for every isotropy group \(H' \subseteq G'\) of \(X\) and every \(n \in \mathbb{Z}\),

\[
H_n^{f^{-1}(H')} \left( E_{F \cap f^{-1}(H')} \left( f^{-1}(H') \to \{\bullet\} \right) ; K_R \right) = 0,
\]

where \(F \cap f^{-1}(H') := \{ K \cap f^{-1}(H') | K \in F \}\). Then, for every \(n \in \mathbb{Z}\),

\[
H_n^G \left( E_{F \times f^*X} \times f^*X \to f^*X; K_R \right) = 0.
\]

**Proof.** Sending \(X\) to \(H_n^G \left( E_{F \times f^*X} \times f^*X \to f^*X; K_R \right)\) defines a \(G'\)-homology theory. There is an equivariant version of the Atiyah-Hirzebruch spectral sequence converging to \(H_n^{G}_{F \times j} \left( E_{F \times f^*X} \times f^*X; K_R \right)\) (see Davis-Lück [16, Theorem 4.7]). Its \(E^2\)-term is the Bredon homology of \(X\)

\[
E_{i,j}^2 = H_i^{Or(G')} (X; V_j),
\]

where the coefficients are given by the covariant functor

\[
V_j : \text{Or}(G') \to \mathbb{Z}\text{-MODULES}
\]

defined by

\[
G'/H' \to H_j^G \left( E_{F \times f^*(G'/H')} \to f^*(G'/H'); K_R \right).
\]

It suffices to show that the \(E^2\)-term is trivial for all \(i, j\). We will do this by showing that for every subgroup \(H' \subseteq G'\) and every \(j \in \mathbb{Z}\),

\[
H_j^G \left( E_{F \times f^*(G'/H')} \to f^*(G'/H'); K_R \right) = 0.
\]

Using the induction structure and Lemma \[3.7\]

\[
H_j^G \left( E_{F \times f^*(G'/H')} \to f^*(G'/H'); K_R \right)
\]

\[
\cong H_j^G \left( G \times f^{-1}(H') \left( \text{res}_{G}^{f^{-1}(H')} E_{F \times f^*(G')} \to \{\bullet\} \right); K_R \right)
\]

\[
\cong H_j^G \left( G \times f^{-1}(H') \left( E_{F \times f^{-1}(H')} \left( f^{-1}(H') \to \{\bullet\} \right); K_R \right) \right)
\]

\[
\cong H_j^{f^{-1}(H')} \left( E_{F \times f^{-1}(H')} \left( f^{-1}(H') \to \{\bullet\}; K_R \right),
\]

which is zero by assumption. \(\square\)

The following general lemma will also be needed.

**Lemma 3.20.** Let \(G_1\) and \(G_2\) be two groups. Let \(X\) be a \(G_1\)-CW-complex, and let \(\text{pr} : G_1 \times G_2 \to G_1\) be projection. Then, for every \(n \in \mathbb{Z}\), there are natural isomorphisms

\[
H_n^{G_1 \times G_2} \left( \text{pr}^*X; K_R \right) \cong H_n^{G_1} \left( X; K_{R(G_2)} \right);
\]

\[
H_n^{G_1 \times G_2} \left( \text{pr}^*X; L_R^{(-\infty)} \right) \cong H_n^{G_1} \left( X; L_R^{(-\infty)} \right).
\]

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Proof. We use the notation from [29 Section 6]). Let pr: Or(G_1 \times G_2) \to Or(G_2) be the functor given by induction with pr. It sends (G_1 \times G_2)/H to G_1/pr(H). Because of the adjunction between induction and restriction (see [16 Lemma 1.9]) and [16 Lemma 4.6]), it suffices to construct a weak equivalence of covariant Or(G_1)-spectra

\[ \text{pr}_*(K_R \circ G^{G_1 \times G_2}) \cong K_{RG_2} \circ G^{G_1}. \]

By definition, pr_*(K_R \circ G^{G_1 \times G_2}) sends G_1/H to K_R(G^{G_1}(G_1) \times \tilde{G}_2), where \( \tilde{G}_2 \) is the groupoid with one object and \( G_2 \) as its automorphism group. The desired weak equivalence of covariant Or(G_1)-spectra is now obtained by comparing the definition of the category of free \( R(G^{G_1}(G_1) \times \tilde{G}_2) \)-modules with the definition of the category of free \( R[G_2]G^{G_1}(G_1) \)-modules.

Proof of Theorem 1.14 By Theorem 2.1 and (2.8),

\[ \text{Wh}_n(G;R) \cong H^G_n(EG \to \underline{EG};K_R) \oplus H^G_n(E_G \to \underline{EG};K_R). \]

We begin by analyzing \( H^G_n(EG \to \underline{EG};K_R) \).

Note that every non-trivial finite subgroup \( H \) of \( G \) is isomorphic to \( \mathbb{Z}/p \), and is maximal. Furthermore, the normalizer \( N_G H \) is isomorphic to \( A^{\mathbb{Z}/p} \times H \). This can be seen as follows. The homomorphism \( q: G \to Q = \mathbb{Z}/p \) induces an injection \( H \to \mathbb{Z}/p \), which must be an isomorphism for non-trivial \( H \). Clearly \( A^{\mathbb{Z}/p} \) and \( H \) belong to \( N_G H \), and the subgroup generated by \( A^{\mathbb{Z}/p} \) and \( H \) is isomorphic to \( A^{\mathbb{Z}/p} \times H \). Thus, it remains to show that \( N_G H \subseteq A^{\mathbb{Z}/p} \times H \). Let \( t \in H \) be a generator of \( H \). Then every element in \( G \) is of the form \( at^j \) for some \( i \in \{0,1,2,\ldots,p-1\} \) and some \( a \in A \). If \( at^j \in N_G H \), then \( at^j t(at^j)^{-1} = at^{-1} = t^j \) for some \( j \in \{0,1,2,\ldots,p-1\} \). Since \( q(t) = q(at^{-1}) = q(t^j) = q(t)^j \), \( j = 1 \). Thus \( at^{-1} = t \) and \( t^{-1}at = a \). This implies that \( a \in A^{\mathbb{Z}/p} \), and hence, \( at^j \in A^{\mathbb{Z}/p} \times H \).

From [31 Corollary 2.10], there is a \( G \)-pushout

\[
\begin{array}{ccc}
\bigoplus_{H \in J} H^G_n(G \times A^{\mathbb{Z}/p \times H} \times A^{\mathbb{Z}/p} \times EH) & \to & G \times A^{\mathbb{Z}/p \times H} \times A^{\mathbb{Z}/p} \times EH \\
\downarrow & & \downarrow \Phi \\
\bigoplus_{H \in J} H^G_n(G \times A^{\mathbb{Z}/p \times H} \times A^{\mathbb{Z}/p} \times EH) & \to & EG
\end{array}
\]

which induces an isomorphism

\[ \bigoplus_{H \in J} H^G_n(G \times A^{\mathbb{Z}/p \times H} \times A^{\mathbb{Z}/p} \times EH) \to G \times A^{\mathbb{Z}/p \times H} \times A^{\mathbb{Z}/p};K_R \]

\[ \cong H^G_n(EG \to \underline{EG};K_R), \]

where \( J \) is a complete system of representatives of the conjugacy classes of
maximal finite subgroups of $G$. Since $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$, the induction structure implies

$$H^n_G(G \times A^{\mathbb{Z}/p} \times H \to G \times A^{\mathbb{Z}/p} \times H, K_R)$$

$$\cong H^{\mathbb{Z}/p \times H}(E A^{\mathbb{Z}/p} \times EH \to E A^{\mathbb{Z}/p}, K_R)$$

$$\cong H^H(B A^{\mathbb{Z}/p} \times (EH \to \{\bullet\}); K_R)$$

$$\cong \bigoplus_{i=0}^e H^H_{n-i}(EH \to \{\bullet\}; K_R)^{(i)}$$

$$\cong \bigoplus_{i=0}^e \text{Wh}_{n-i}(H; R)^{(i)}.$$ 

Therefore,

$$\bigoplus_{H \in J} \bigoplus_{i=0}^e \text{Wh}_{n-i}(H; R)^{(i)} \xrightarrow{\cong} H^n_G(EG \to EG; K_R). \quad \text{(3.21)}$$

Now we turn our attention to $H^n_G(EG \to EG; K_R)$. Let $A := A/A^{\mathbb{Z}/p}$ and $G := G/A^{\mathbb{Z}/p}$. Then the exact sequence (1.10) induces the exact sequence

$$1 \to A \to G \xrightarrow{\varphi} Q \to 1. \quad \text{(3.22)}$$

Let $\mathcal{F}$ be the family of subgroups $K$ of $G$ for which $\varphi(K)$ is finite, where $\varphi : G \to G$ is the quotient homomorphism. Let $\mathcal{VCY}$ be the family of virtually cyclic subgroups of $G$, $\mathcal{VCY}_I$ be the family of virtually cyclic subgroups of type $I$, and $\mathcal{F}_I = \mathcal{F} \cap \mathcal{VCY}_I$. The Farrell-Jones Conjecture in algebraic $K$-theory is true for any group appearing in $\mathcal{F}$, since every element in $\mathcal{F}$ is virtually finitely generated abelian (see Theorem 2.1). It is straightforward to check that $\varphi^* EG$ is a model for $E_{\mathcal{F}}(G)$. By the Transitivity Principle (see [21, Theorem A.10] and [29, Theorem 65]) and Remark 2.13,

$$H^n_G(E_{\mathcal{F}_I}(G); K_R) \cong H^n_G(\varphi^* EG; K_R). \quad \text{(3.23)}$$

Since every virtually cyclic subgroup of type $I$ in $G$ is infinite cyclic, every element $K \in \mathcal{VCY}_I$ belongs to $\mathcal{F}_I$, or is infinite cyclic and $\{K' \mid K' \subseteq K \text{ and } K \in \mathcal{F}_I\}$ consists of just the trivial group. Since $R$ is assumed to be regular, the map $H^n_G(EZ, K_R) \to H^n_G(\{\bullet\}; K_R)$ is bijective for every $n \in \mathbb{Z}$. Therefore, the Transitivity Principle implies that

$$H^n_G(E_{\mathcal{F}_I}(G); K_R) \cong H^n_G(E_{\mathcal{VCY}_I}(G); K_R). \quad \text{(3.24)}$$

By Remark 2.13

$$H^n_G(E_{\mathcal{VCY}_I}(G); K_R) \cong H^n_G(EG; K_R) \quad \text{(3.25)}$$

is also a bijection. Thus, (3.23), (3.24), (3.25) and the Five-Lemma imply that, for every $n \in \mathbb{Z},$

$$H^n_G(EG \to EG; K_R) \cong H^n_G(EG \to \varphi^* EG; K_R). \quad \text{(3.26)}$$
Next we show that the conjugation action of $\mathbb{Z}/p$ on $\overline{A}$ is free away from $0 \in \overline{A}$. Let $t$ be an element of $G$ that is mapped by $q: G \to \mathbb{Z}/p$ to a generator of $\mathbb{Z}/p$. Let $a \in A$ be given such that $pr(t) pr(a) pr(t)^{-1} = pr(a)$. Thus, there is a $b \in A^{\mathbb{Z}/p}$ such that $tat^{-1} = ab$. If $\rho: A \to A$ denotes the conjugation action with $t$, then $\rho(a) - a = b$ in $A$, where the group operation in $A$ is written additively. Since

$$p \cdot b = \sum_{i=0}^{p-1} \rho^i(b) = \sum_{i=0}^{p-1} \rho^i(\rho(a) - a) = \sum_{i=0}^{p-1} \rho^{i+1}(a) - \rho^i(a) = \rho^p(a) - a = 0,$$

it follows that $\rho(a) - a = b = 0$. This implies that $a \in A^{\mathbb{Z}/p}$, and thus $\overline{a} = 0$ in $\overline{A}$. Therefore, the conjugation action of $\mathbb{Z}/p$ on $\overline{A}$ is free away from $0 \in \overline{A}$.

Let $\overline{J}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of $\overline{G}$. From [30, Lemma 6.3] and [31, Corollary 2.11], there is a $\overline{G}$-pushout

$$\coprod_{\overline{J} \in \partial} H^G_*(EG \times pr^* \overline{X} / pr^* \overline{X}; K_R) \to H^n_*(EG \times pr^* \overline{G} / pr^* \overline{G}; K_R). \quad (3.28)$$

Sending a $\overline{G}$-CW-complex $\overline{X}$ to the $\mathbb{Z}$-graded abelian group $H^G_*(EG \times pr^* \overline{X} / pr^* \overline{X}; K_R)$ defines a $\overline{G}$-homology theory. Since $pr^{-1}({\{1\}}) = A^{\mathbb{Z}/p} \cong \mathbb{Z}^c$ and $R$ is regular, Theorem 1.7 and Lemma 3.19 imply that if $\overline{X}$ is a free $\overline{G}$-CW-complex, then $H^G_*(EG \times pr^* \overline{X} \to pr^* \overline{X}; K_R) = 0$. Therefore, because $\coprod_{\overline{J} \in \partial} H^G_*(EG \times pr^* \overline{G} \cap pr^* \overline{H} / pr^* \overline{H}; K_R)$ and $E \overline{G}$ are free $\overline{G}$-CW-complexes, the Mayer-Vietoris sequence associated to the $\overline{G}$-pushout yields an isomorphism

$$\coprod_{\overline{J} \in \partial} H^G_*(EG \times pr^* \overline{G} / pr^* \overline{G}; K_R) \to H^n_*(EG \times pr^* E \overline{G} / pr^* E \overline{G}; K_R). \quad (3.28)$$

Notice that if $pr^{-1}(\overline{H})$ is torsion-free, then it is isomorphic to $\mathbb{Z}^c$. If $\overline{H}$ is trivial, then this follows from $pr^{-1}({\{1\}}) = A^{\mathbb{Z}/p}$. If $\overline{H}$ is a non-trivial finite subgroup of $\overline{G}$, choose an element $t \in pr^{-1}(\overline{H})$ such that $pr(t)$ is a generator of $\overline{H}$. Then every element in $pr^{-1}(\overline{H})$ is of the form $at^u$ for $a \in A^{\mathbb{Z}/p}$ and $u \in \{0, 1, 2, \ldots, p-1\}$. For two such elements $at^u$ and $bt^v$

$$at^u bt^v = abt^u t^v = bat^v t^u = bt^v at^u.$$

Hence $pr^{-1}(\overline{H})$ is a torsion-free abelian group containing $\mathbb{Z}^c$ as subgroup of finite index, and so $pr^{-1}(\overline{H}) \cong \mathbb{Z}^c$. Thus, if $pr^{-1}(\overline{H})$ is torsion-free, then Theorem 1.7 and Lemma 3.19 imply that

$$H^n_*(EG \times pr^* \overline{G} / pr^* \overline{G}; K_R) = 0.$$
Therefore, if $J'$ is the subset of $J$ consisting of those elements $H \in J$ for which $pr^{-1}(H)$ is not torsion-free, then (3.28) becomes

$$\bigoplus_{\pi \in \mathcal{T}} H_n^G(EG \times pr^* \mathbb{C}/\mathbb{P} \rightarrow pr^* \mathbb{C}/\mathbb{P}; K_R)$$

$$\cong H_n^G(EG \times pr^* \mathbb{E}\mathbb{G} \rightarrow pr^* \mathbb{E}\mathbb{G}; K_R). \quad (3.29)$$

Let $\mathcal{H} \in \mathcal{T}$. Then $pr^{-1}(H) \cong A^{2/p} \times \mathcal{H}$. Since $A^{2/p} \cong \mathbb{Z}^e$, the induction structure, Lemma 3.7, Lemma 3.20 and Theorem 1.7 imply

$$H_n^G(EG \times pr^* \mathbb{C}/\mathbb{P} \rightarrow pr^* \mathbb{C}/\mathbb{P}; K_R)$$

$$\cong H_n^G(G \times A^{2/p} \times \mathcal{P} (\text{res}_{G} A^{2/p} \times \mathbb{C} \rightarrow \{\bullet\}; K_R)$$

$$\cong H_n^{A^{2/p}}(\text{res}_{G} A^{2/p} \times \mathbb{C} \rightarrow \{\bullet\}; K_R)$$

$$\cong H_n^{A^{2/p}}(EA^{2/p} \rightarrow \{\bullet\}; K_R)$$

$$\cong H_n^{A^{2/p}}(EA^{2/p} \rightarrow \{\bullet\}; K_R)$$

$$\cong Wh_n(A^{2/p}; R[\mathbb{Z}/p])$$

$$\cong \bigoplus_{C \in \mathcal{M}_{T}C Y(A^{2/p})} (NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p]))^{(e-1)}.$$ 

Since the projection $\mathbb{E}G \times pr^* \mathbb{E}\mathbb{G} \rightarrow \mathbb{E}G$ is a $G$-homotopy equivalence, (3.29) implies that

$$\bigoplus_{\mathcal{T} \in \mathcal{T}} \bigoplus_{C \in \mathcal{M}_{T}C Y(A^{2/p})} (NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p]))^{(e-1)}$$

$$\cong H_n^G(\mathbb{E}G \rightarrow pr^* \mathbb{E}\mathbb{G}; K_R).$$

Together with (3.20), this produces the isomorphism

$$\bigoplus_{\mathcal{T} \in \mathcal{T}} \bigoplus_{C \in \mathcal{M}_{T}C Y(A^{2/p})} (NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p]))^{(e-1)}$$

$$\cong H_n^G(\mathbb{E}G \rightarrow \mathbb{E}G; K_R).$$

To complete the proof of the theorem, we must show that there is a bijection between the sets $J$ and $J'$. Send an element $H \in J$ to the element in $H' \in J'$ that is uniquely determined by the property that $pr(H)$ and $H'$ are conjugated in $G$. This map is well-defined and injective because two non-trivial finite subgroups $H_1$ and $H_2$ of $G$ are conjugate if and only if $pr(H_1)$ and $pr(H_2)$ are conjugate in $G$. The map is surjective, since for any element $\mathcal{H} \in J'$ there exists a finite subgroup $H \subseteq G$ with $\mathcal{H} = \text{pr}(H)$. This finishes the proof of Theorem 1.14. \qed

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3.3.4 \(L\)-theory in the case \(Q = \mathbb{Z}/p\) for an odd prime \(p\)

Proof of Theorem 1.16

An argument completely analogous to the one that established \([5.21]\) in the proof of Theorem 1.14 shows that

\[
\bigoplus_{H \in \mathcal{J}} \bigoplus_{i=0}^{e} S_{n-i}^{\text{per},(-\infty)}(H; R)^{(i)} \cong H_n^G(EG \rightarrow EG; L_R^{(-\infty)}). \tag{3.30}
\]

Since \(p\) is odd, every infinite virtually cyclic subgroup of \(G\) is of type I. Thus the result follows from Theorem 2.1, Remark 2.13 and (3.30).

Proof of Theorem 1.17

There are natural maps \(L^{(j)}_\mathbb{Z} \rightarrow L^{(j+1)}_\mathbb{Z}\) of covariant functors from \textsc{Groupoids} to \textsc{Spectra} for \(j \in \{2, 1, 0, -1, \ldots\}\). Let \(L^{(j+1, j)}_\mathbb{Z}\) be the covariant functor from \textsc{Groupoids} to \textsc{Spectra} that assigns to a groupoid \(\mathcal{G}\) the homotopy cofiber of the map of spectra obtained by evaluating \(L^{(j)}_\mathbb{Z} \rightarrow L^{(j+1)}_\mathbb{Z}\) at \(\mathcal{G}\). We obtain equivariant homology theories \(H^*_G(-; L^{(j+1)}_\mathbb{Z})\), \(H^*_G(-; L^{(j)}_\mathbb{Z})\) and \(H^*_G(-; L^{(j+1, j)}_\mathbb{Z})\) (see Lück-Reich [29, Section 6]), such that for any group \(G\) and any \(G\)-\(\mathcal{C}W\)-complex \(X\), there is a long exact sequence

\[
\cdots \rightarrow H^G_n(X; L^{(j+1)}_\mathbb{Z}) \rightarrow H^G_n(X; L^{(j)}_\mathbb{Z}) \rightarrow H^G_n(X; L^{(j+1, j)}_\mathbb{Z})
\]

\[
\rightarrow H^G_{n-1}(X; L^{(j+1)}_\mathbb{Z}) \rightarrow H^G_{n-1}(X; L^{(j)}_\mathbb{Z}) \rightarrow \cdots. \tag{3.31}
\]

If \(X = \{\bullet\}\), then

\[
H^G_n(\{\bullet\}; L^{(j+1)}_\mathbb{Z}) \cong L^{(j+1)}_n(\mathbb{Z}G);
\]

\[
H^G_n(\{\bullet\}; L^{(j)}_\mathbb{Z}) \cong L^{(j)}_n(\mathbb{Z}G);
\]

\[
H^G_n(\{\bullet\}; L^{(j+1, j)}_\mathbb{Z}) \cong \hat{H}^n(\mathbb{Z}/2; \mathbb{K}_j(\mathbb{Z}G)),
\]

and the sequence (3.31) can be identified with the Rothenberg sequence (2.39).

Since \(L^{(j+1)}_n(\mathbb{Z}) \cong L^{(j)}_n(\mathbb{Z})\) is a bijection for every \(n \in \mathbb{Z}\) by (2.30), a spectral sequence argument shows that the map

\[
H^G_n(BG; L^{(j+1, j)}_\mathbb{Z}) \cong H^G_n(BG; L^{(j+1, j)}(\mathbb{Z}))
\]

is a bijection for every \(n \in \mathbb{Z}\). Thus, \(H^G_n(EG; L^{(j+1, j)}_\mathbb{Z}) \cong H_n(BG; L^{(j+1, j)}(\mathbb{Z}))\) is zero for all \(n \in \mathbb{Z}\). Hence, for every \(n \in \mathbb{Z}\), there is an isomorphism

\[
H^G_n(EG \rightarrow \{\bullet\}; L^{(j+1, j)}_\mathbb{Z}) \cong H^G_n(\{\bullet\}; L^{(j+1, j)}_\mathbb{Z}).
\]

This implies that there is also a Rothenberg sequence for the structure sets

\[
\cdots \rightarrow S_{n}^{\text{per},(j+1)}(G; \mathbb{Z}) \rightarrow S_{n-1}^{\text{per},(j+1)}(G; \mathbb{Z}) \rightarrow \hat{H}^n(\mathbb{Z}/2; \mathbb{K}_j(\mathbb{Z}G))
\]

\[
\rightarrow S_{n-1}^{\text{per},(j+1)}(G; \mathbb{Z}) \rightarrow S_{n-1}^{\text{per},(j+1)}(G; \mathbb{Z}) \rightarrow \cdots. \tag{3.32}
\]
Since \( \text{NK}_n(\mathbb{Z}[\mathbb{Z}/p]) = 0 \) for \( n \leq 1 \) (see Bass-Murthy [8]), Theorem 1.14 implies that, for every \( n \leq 1 \),

\[
\bigoplus_{H \in J} \bigoplus_{i=0}^e \text{Wh}_{n-i}(H; \mathbb{Z})^{(\ell)} \cong \text{Wh}_n(G; \mathbb{Z}).
\]  

(3.33)

For every \( H \in J \) and \( j \leq -2 \), \( K_j(\mathbb{Z}H) = 0 \) by Carter [11]. Hence \( K_j(\mathbb{Z}G) \) vanishes for \( j \leq -2 \) by (3.33). Theorem 1.16 and (3.32) imply that the map

\[
\bigoplus_{H \in J} \bigoplus_{i=0}^e \mathcal{S}_{n}^{\text{per},(j)}(H; \mathbb{Z})^{(\ell)} \to \mathcal{S}_{n}^{\text{per},(j)}(G; \mathbb{Z})
\]

is bijective for all \( n \in \mathbb{Z} \) and \( j \in \{-1, -2, \ldots\} \cup \{-\infty\} \).

Next we use an inductive argument to show that this is also true for \( j = 0, 1 \). By taking the direct sum of the Rothenberg sequences (3.32) for the various elements \( H \in J \) and mapping it to (3.32) for \( G \), one obtains the following commutative diagram.

\[
\text{Diagram}
\]

The middle horizontal arrow is an isomorphism since it is induced by the isomorphism (3.33), which is compatible with the involutions. The Five-Lemma

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implies that the maps

\[ \bigoplus_{H \in J} \bigoplus_{i=0}^e S_{n-i}^{\text{per,}(j)}(H; \mathbb{Z}) \xrightarrow{\cdot (j)} S_n^{\text{per,}(j)}(G; \mathbb{Z}) \]

are bijective for all \( n \in \mathbb{Z} \) if and only if the maps

\[ \bigoplus_{H \in J} \bigoplus_{i=0}^e S_{n-i}^{\text{per,}(j+1)}(H; \mathbb{Z}) \xrightarrow{\cdot (j+1)} S_n^{\text{per,}(j+1)}(G; \mathbb{Z}) \]

are bijective for all \( n \in \mathbb{Z} \). This takes care of the decorations \( j = 1 \) and \( j = 0 \). A similar argument, where \( \tilde{K}_j(ZH) \) and \( \tilde{K}_j(ZG) \) are replaced by \( \text{Wh}(H) \) and \( \text{Wh}(G) \) in the Rothenberg sequence, shows that this is also true for the decoration \( s \), since it is true for the decoration \( h \) which is the the same as \( (1) \). This completes the proof of the theorem. \( \square \)

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