On sums of dependent random lifetimes under the Time Transformed Exponential model

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Abstract

Considered a pair of random lifetimes whose dependence is described by a Time Transformed Exponential model, we provide analytical expressions for the distribution of their sum. These expressions are obtained by using a representation of the joint distribution in terms of multivariate distortions, which is an alternative approach to the classical copula representation. Since this approach allows to obtain conditional distributions and their inverses in simple form, then it is also shown how it can be used to predict the value of the sum from the value of one of the variables (or vice versa) by using quantile regression techniques.

Keywords: Dependence models, C-convolution, distorted distributions, quantile regression, confidence bands.

1 Introduction

Let $X = (X_1, X_2)$ be a pair of dependent lifetimes. The vector $X$ is said to be described by a Time Transformed Exponential model (shortly, TTE model) if its joint survival function $\tilde{F}$ can be written as

$$\tilde{F}(x_1, x_2) = \tilde{G}(R_1(x_1) + R_2(x_2)), \quad x_1, x_2 \geq 0,$$

for a suitable one-dimensional, continuous, convex and strictly decreasing survival function $\tilde{G}$ and two suitable continuous and strictly increasing functions $R_i : [0, +\infty) \to [0, +\infty)$ such that $R_i(0) = 0$ and $\lim_{x \to \infty} R_i(x) = \infty$, for $i = 1, 2$. Clearly, the marginal survival functions for the lifetimes $X_i$ are given by $\tilde{F}_i(x_i) = \tilde{G}(R_i(x_i)), \quad x_i \geq 0, \quad i = 1, 2$.

TTE models have been considered in literature as an appropriate manner to describe bivariate lifetimes (see, e.g., [1, 8, 13] and references therein). Their main characteristic is that they “separate”, in a sense, aging of single lifetimes through the functions $R_i$, and dependence properties through $\tilde{G}$, the copula $\tilde{C}$ being a transformation of $\tilde{G}$ only (see Eq. (2) below and the reference above for details). This model is of interest in a variety of applicative fields since it is equivalent to the random frailty model, which assumes that the two lifetimes are conditionally independent given a random parameter that represents the risk due to a common environment; the well-known proportional hazard rate Cox model, where the proportional factor is not fixed but random, is an example. In this case, the
different choices for the function $G$ are obtained just by changing the distribution of random parameter.

For a number of applicative purposes, one can be interested in the sum $S = X_1 + X_2$. This happens, for example, in considering the total lifetime in stand-by systems, where a component is replaced by a new one under the same environmental stress after its failure, or in insurance theory, where the sum of two depended claims, due to common risks, may be evaluated. In this case, because of the dependence between $X_1$ and $X_2$, the classical convolution can not be applied to determine the distribution of $S$, and C-convolutions must be used (see, e.g., [2, 4, 17] for definition and examples of C-convolutions). That is, one can calculate the survival function of $S$ as

$$\bar{F}_S(s) = \Pr(S > s) = \int_{-\infty}^{\infty} f_1(x) \partial_1 \hat{C}(\bar{F}_1(x), \bar{F}_2(s-x)) dx,$$

where $F_1$ and $F_2$ are the marginal survival functions of $X_1$ and $X_2$, respectively, $f_1$ is the density function of $X_1$ (assuming its existence) and $\hat{C}$ is the survival copula of the vector $X$. Here, $\partial_1 \hat{C}$ means the partial derivative of $\hat{C}$ with respect to its first argument. Note that, in particular, for nonnegative random variables Eq. (1.2) reduces to

$$\bar{F}_S(s) = \bar{F}_1(s) + \int_{0}^{s} f_1(x) \partial_1 \hat{C}(\bar{F}_1(x), \bar{F}_2(s-x)) dx, \quad t \geq 0.$$

Since usually the integrals appearing in previous formulas are not easy to be solved, we describe in this paper an alternative tool to deal with the sum $S = X_1 + X_2$ that can be used when the joint distribution of $X$ is defined as in Eq. (1.1). This approach is based on an alternative representation for the survival function of $X$, which make use of the distortion representations of multivariate distributions recently introduced in [11], whose definition is provided in the next section. The advantage of this approach is twofold: it is particularly useful when the inverse of $\hat{G}$ is not available in closed form, thus also $\hat{C}$, and it also provides simple representations of the conditional distribution of $S$ given one of the $X_i$, and of its inverse, so that one can use it to predict the value of the sum from the value of one of the variables (or vice versa) by using quantile regression techniques. The purpose of this paper is to describe such an approach.

The rest of the paper is structured as follows. Basic definitions, notations and some preliminary results are introduced in Section 2. The main results for the representation of the distribution of the sum $S$ are provided in Section 3, while examples of their application in prediction are presented in Section 4 and Section 5.
Throughout the paper the notions increasing and decreasing are used in a wide sense, that is, they mean non-decreasing and non-increasing, respectively, and we say that \( f \) is increasing (decreasing) if \( f(x) \leq f(y) \) for all \( x \leq y \) (where this last inequality means that for every \( i \)-th component of the vectors one has \( x_i \leq y_i \)). Also, if \( f \) is a real valued function in more than one variables, then \( \partial_i f \) denotes the partial derivative of \( f \) with respect to its \( i \)-th variable. Analogously, \( \partial_{i,j} f = \partial_i \partial_j f \) and so on. Whenever we use a partial derivative we are tacitly assuming that it exists.

2 Notation and preliminary results

To simplify the notation we just consider here the bivariate case; the extension to the \( n \)-dimensional case is straightforward. Moreover, we just consider nonnegative random variables with absolutely continuous distributions even if many of the properties described below can be extended to arbitrary random variables.

Thus, let \( \mathbf{X} = (X_1, X_2) \) be a random vector with two possibly dependent nonnegative random variables having an absolutely continuous joint distribution function \( \mathbf{F} \) and marginal distributions \( F_1 \) and \( F_2 \). Let \( f \) be the joint probability density function (PDF) of \( (X_1, X_2) \) and let \( f_1 \) and \( f_2 \) be the PDFs of \( X_1 \) and \( X_2 \), respectively. Then it is well known (see, e.g., \cite{18}) that, from Sklar’s Theorem, there exists a unique absolutely continuous copula \( C \) such that \( \mathbf{F} \) can be written as

\[
\mathbf{F}(x_1, x_2) = \Pr(X_1 \leq x_1, X_2 \leq x_2) = C(F_1(x_1), F_2(x_2))
\] (2.1)

for all \( x_1, x_2 \). As a consequence, the PDF function can be obtained as

\[
f(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2)),
\]

where \( c := \partial_{1,2}C \) is the PDF of the copula \( C \). A similar representation holds for the joint survival function

\[
\tilde{\mathbf{F}}(x_1, x_2) = \Pr(X_1 > x_1, X_2 > x_2) = \tilde{C}(\tilde{F}_1(x_1), \tilde{F}_2(x_2))
\]

for all \( x_1, x_2 \), where \( \tilde{F}_1(x_1) = \Pr(X_1 > x_1) \) and \( \tilde{F}_2(x_2) = \Pr(X_2 > x_2) \) are the marginal survival functions and \( \tilde{C} \) is another suitable copula, called survival copula.

In the particular case of TTE models, i.e., in the case that the joint survival function \( \tilde{\mathbf{F}} \) is defined as in Eq. (1.1), then the corresponding survival copula \( \tilde{C} \) is the strict bivariate
Archimedean copula (see, e.g., [18], p. 112) defined as

$$\tilde{C}(u_1, u_2) = \tilde{G}(\tilde{G}^{-1}(u_1) + \tilde{G}^{-1}(u_2))$$

for all $u_1, u_2 \in [0,1]^2$. This model contains many families of copulas (see, [18], p. 117), thus it is a very general dependence model. The inverse function $\tilde{G}^{-1}$ is called the additive generator of the copula.

However, an alternative representation of $\tilde{F}$ based on distortion representations of multivariate distributions can be given. In some cases, and for specific applications, such an alternative representation can be preferable to classical copula approach.

For it first recall that a function $d : [0,1] \to [0,1]$ is said to be a distortion function if it is continuous, increasing and satisfies $d(0) = 0$ and $d(1) = 1$. If $G$ is a distribution function, we say that $F$ is a distorted distribution from $G$ if there exists a distortion function $d$ such that $F(x) = d(G(x))$ for all $x$, and similarly for the survival functions. This kind of representations were introduced in the theory of decision under risk (see e.g. [20, 21]) and they were also applied in the fields of coherent systems, order statistics and conditional distributions (see, e.g., [12, 16] and the references therein).

These representations were further extended to the multivariate case in the recent paper [11]. According to what defined there, and restricting to the bivariate case, a function $D : \mathbb{R}^2 \to \mathbb{R}$ is a bivariate distortion if it is a continuous 2-dimensional distribution with support included in $[0,1]^2$, and a bivariate distribution function $F$ is a distortion of the univariate distribution functions $H_1$ and $H_2$ if there exists a bivariate distortion $D$ such that

$$F(x_1, x_2) = D(H_1(x_1), H_2(x_2))$$

(2.2)

for all $x_1, x_2$.

This representation is similar to the copula representation, but here the $H_i$ are not necessarily the marginal distribution of $X$ and $D$ is not necessarily a copula. Actually, in some situations, we can choose a common univariate distribution $H = H_1 = H_2$, and some examples will be provided later (see also [10, 11]). Moreover, if $D$ has uniform univariate marginal distributions over the interval $(0,1)$, then $D$ is a copula, $H_1, H_2$ are the marginal distributions and (2.2) is the same as the copula representation (2.1) (but only in this case).

The main properties of model (2.2) were given in [11] and they are very similar to that of copulas. For example, if $D$ is a distortion function, then the right-hand side of (2.2)
defines a proper multivariate distribution function for any univariate distribution functions $H_1$ and $H_2$. Moreover, a similar representation holds for the joint survival function, that is, one can write

$$\bar{F}(x_1, x_2) = \bar{D}(\bar{H}_1(x_1), \bar{H}_2(x_2)), \quad (2.3)$$

where $\bar{F}(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2)$, $\bar{H}_i = 1 - H_i$ for $i = 1, 2$, and $\bar{D}$ is another suitable distortion function.

For TTE model note that, defining

$$\bar{H}_i(x_i) = \exp(-R_i(x_i)), \quad (2.4)$$

one has

$$\bar{F}(x_1, x_2) = \bar{G}(R_1(x_1) + R_2(x_2)) = \bar{G}(-\ln \bar{H}_1(x_1) - \ln \bar{H}_2(x_2)) = \bar{D}(\bar{H}_1(x_1), \bar{H}_2(x_2))$$

for all $x_1, x_2 \geq 0$, where

$$\bar{D}(u, v) = \bar{G}(-\ln(uv)), \quad u, v \in [0, 1]. \quad (2.5)$$

The function $\bar{D}$ satisfies the property to be a bivariate distortion if $\bar{G}$ satisfies the properties mentioned above, i.e., if it is an absolutely continuous strictly decreasing convex function in $[0, \infty)$ with $\bar{G}(0) = 1$ and $\bar{G}(\infty) = 0$. Note that if we add $\bar{G}(t) = 1$ for $t < 0$, then $\bar{G}$ is the survival function of a nonnegative random variable. Also note that $\bar{H}_1$ and $\bar{H}_2$ are two arbitrary survival functions satisfying $\bar{H}_1(0) = \bar{H}_2(0) = 1$. Thus, a representation through multivariate distortions as in $(2.2)$ holds for the TTE models, with $\bar{D}$ defined as in $(2.5)$.

It must be pointed out that with this representation the marginal survival functions $\bar{F}_i, i = 1, 2$, are not explicitly displayed, but can be obtained as

$$\bar{F}_i(x_i) = \bar{G}(-\ln \bar{H}_i(x_i)) = \bar{D}(\bar{H}_i(x_i), 1) = \bar{d}(\bar{H}_i(x_i)), \quad x_i \geq 0,$$

where $\bar{d}(u) = \bar{G}(-\ln u), u \in [0, 1]$, is a univariate distortion function. Finally, note that the representation through the multivariate distortion $(2.5)$ and the univariate survivals $(2.4)$ is a copula representation if and only if $\bar{D}(u, 1) = u$, that is, $\bar{G}(-\ln(u)) = u$ for $0 \leq u \leq 1$. This property leads to $\bar{G}(x) = \exp(-x)$ for $x \geq 0$ and $\bar{D}(u, v) = uv$ for $u, v \in [0, 1]$ which is the product copula that represents the independence case. For other (non-exponential) survival functions $\bar{G}$, we obtain models with dependent variables, whose dependence is described by $\bar{G}$. 
As an interesting particular case, this dependence model includes the one recently proposed in [5] for nonnegative random variables, which is characterized (see Proposition 3.1 in [5]) by the joint survival function

$$\bar{F}(x_1, x_2) = \bar{G}(\alpha x_1 + \beta x_2)$$  \hspace{1cm} (2.6)

for $x_1, x_2 \geq 0$, where $\alpha, \beta > 0$ are two positive scale parameters and $\bar{G}$ satisfies the above mentioned properties. This model, that from now on will be referred as $GK$-model (where the letters G and K indicates the initials of the authors Genev and Kolev of reference [5]) is an extension of the well known Schur-constant model which is obtained when $\alpha = \beta$ (see [3] and references therein). Properties of this model and of the corresponding sum $X_1 + X_2$ are studied also in [19]. It must be observed that the marginal survival functions are $\bar{F}_1(x_1) = \bar{G}(\alpha x_1)$ and $\bar{F}_2(x_2) = \bar{G}(\beta x_2)$ for $x_1, x_2 \geq 0$, and both of them belong to the scale parameter model defined by $\bar{G}$. Actually, this model is obtained by the distortion of univariate exponential distributions, i.e., if (2.6) holds, then

$$\bar{F}(x_1, x_2) = \hat{D}(\bar{H}_1(x_1), \bar{H}_2(x_2))$$

for all $x_1, x_2 \geq 0$, where $\bar{H}_1(x_1) = \exp(-\alpha x_1)$, $\bar{H}_2(x_2) = \exp(-\beta x_2)$ and $\hat{D}$ is defined as in Eq. (2.5).

3 Distribution and conditional distribution of the sum

In this section we use the distortion representation (2.3), with the multivariate distortion $\hat{D}$ defined as in (2.5), to study the sum $S = X_1 + X_2$ under the dependence model defined in the preceding section. As a consequence, we also obtain the analogous properties for the GK-model, i.e., the generalization (2.6) of the Schur-constant model.

**Proposition 3.1.** If (2.3) and (2.5) hold for $(X_1, X_2)$ and $S = X_1 + X_2$, then the joint PDF of $(X_1, S)$ is

$$g(x, s) = r_1(x) r_2(s - x) \bar{G}'' \left( - \ln \bar{H}_1(x) - \ln \bar{H}_2(s - x) \right)$$  \hspace{1cm} (3.1)

for $0 \leq x \leq s$ (zero elsewhere), where $r_i = (-\ln \bar{H}_i)'$ is the hazard rate function of $\bar{H}_i$ for $i = 1, 2$. 

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Proof. From (2.3), the joint PDF of \((X_1, X_2)\) is
\[
f(x_1, x_2) = h_1(x_1)h_2(x_2)d(\bar{H}_1(x_1), \bar{H}_2(x_2))
\]
for \(x_1, x_2 \geq 0\), where \(h_i = -\bar{H}_i'\) and \(d = \partial_{1,2} \bar{D}\). Then the joint PDF of \((X_1, S)\) is
\[
g(x, s) = f(x, s-x) = h_1(x)h_2(s-x)d(\bar{H}_1(x), \bar{H}_2(s-x))
\]
for \(0 \leq x \leq s\). The PDF of our specific distortion function \(\bar{D}\) is
\[
\bar{d}(u, v) = \frac{1}{uv} \bar{G}''(-\ln(uv))
\]
and
\[
g(x, s) = \frac{h_1(x)h_2(s-x)}{\bar{H}_1(x)\bar{H}_2(s-x)} \bar{G}''(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x))
\]
for \(0 \leq x \leq s\) which concludes the proof.

Remark 3.1. In particular, for the GK-model in (2.6), that is, with exponential survival functions \(H_1\) and \(H_2\) with shape parameters (hazard rates) \(\alpha\) and \(\beta\), the PDF reduces to
\[
g(x, s) = \alpha\beta \bar{G}''((\alpha - \beta)x + \beta s)
\]
for \(0 \leq x \leq s\) (zero elsewhere). Therefore its joint distribution function is
\[
G(x, s) = -\int_0^x \int_y^s \alpha\beta \bar{G}''((\alpha - \beta)y + \beta t)dt\,dy
\]
\[
= \int_0^x \alpha \bar{G}'(\alpha y)\,dy - \int_0^x \alpha \bar{G}'((\alpha - \beta)y + \beta s)\,dy
\]
where \(G = 1 - \bar{G}\). To solve this integral we consider two cases. If \(\alpha \neq \beta\), then
\[
G(x, s) = G(\alpha x) - \frac{\alpha}{\alpha - \beta} G((\alpha - \beta)x + \beta s) + \frac{\alpha}{\alpha - \beta} G(\beta s)
\]
(3.2)
while if \(\alpha = \beta\), then
\[
G(x, s) = G(\alpha x) - \alpha x G'(\alpha s)
\]
(3.3)
for \(0 \leq x \leq s\). In both cases, (3.2) and (3.3) can be represented as distorted distributions from \(G\) by replacing \(x\) with \(G^{-1}(G(x))\) and \(s\) with \(G^{-1}(G(s))\).

In particular, as immediate consequence one can obtain the distribution of the sum (C-convolution) for the GK-model as
\[
F_S(s) = \lim_{x \to \infty} G(x, s) = G(s, s).
\]
If \( \alpha \neq \beta \), then
\[
F_S(s) = \frac{\alpha}{\alpha - \beta} G(\beta s) - \frac{\beta}{\alpha - \beta} G(\alpha s)
\]
or if \( \alpha = \beta \), then
\[
F_S(s) = G(\alpha s) - \alpha s G'(\alpha s)
\]
for \( s \geq 0 \). Note that the first expression is a negative mixture (i.e. a linear combination with a negative weight) with PDF
\[
f_S(s) = \frac{\alpha \beta}{\beta - \alpha} [g(\alpha s) - g(\beta s)]
\]
for \( s \geq 0 \), where \( g = G' \). In the second case, one gets
\[
f_S(s) = -\alpha^2 s G''(\alpha s)
\]
for \( s \geq 0 \), which is the expression in Remark 2.7 of [3] (i.e. for the Schur-constant model).

The joint survival function of \((X_1, S)\) under the model (2.3) and (2.5) in the non-exponential case for the \( H_i \) is obtained in the following proposition. Unfortunately, an explicit expression can not be provided in general, but it is available in some cases, or easily available numerically (see the examples in the next sections).

Proposition 3.2. If (2.3) and (2.5) hold for \((X_1, X_2)\) and \( S = X_1 + X_2 \), then the joint survival function of \((X_1, S)\) is
\[
\bar{G}(x, s) = G(-\ln \bar{H}_1(s)) + \int_x^s r_1(y) g(-\ln \bar{H}_1(y) - \ln \bar{H}_2(s - y)) \, dy
\]
for \( 0 \leq x \leq s \), where \( g = -\bar{G}' \) is the PDF of \( \bar{G} \) and \( r_i = (-\ln \bar{H}_i)' \) is the hazard rate function of \( \bar{H}_i \) for \( i = 1, 2 \).

Proof. From Eq. (3.1) for the PDF of \((X_1, S)\) we get
\[
\bar{G}(x, s) = \int_x^s \int_s^\infty g(y, t) \, dt \, dy + \int_s^\infty \int_y^\infty g(y, t) \, dt \, dy
\]
\[
= \int_x^s \int_y^\infty r_1(y) r_2(t - y) \bar{G}''(-\ln \bar{H}_1(y) - \ln \bar{H}_2(t - y)) \, dt \, dy
\]
\[
+ \int_s^\infty \int_y^\infty r_1(y) r_2(t - y) \bar{G}''(-\ln \bar{H}_1(y) - \ln \bar{H}_2(t - y)) \, dt \, dy
\]
\[
= -\int_x^s r_1(y) \bar{G}'(-\ln \bar{H}_1(y) - \ln \bar{H}_2(s - y)) \, dy - \int_s^\infty r_1(y) \bar{G}'(-\ln \bar{H}_1(y)) \, dy
\]
\[
= G(-\ln \bar{H}_1(s)) + \int_x^s r_1(y) g(-\ln \bar{H}_1(y) - \ln \bar{H}_2(s - y)) \, dy
\]
which concludes the proof.
Therefore, the survival function of $S$ can be obtained as
\[
\tilde{F}_S(s) = \bar{G}(0, s) = \bar{G}(-\ln \bar{H}_1(s)) + \int_0^s r_1(y)g\left(-\ln \bar{H}_1(y) - \ln \bar{H}_2(s - y)\right)dy \quad (3.6)
\]
and its PDF as $f_S(s) = -\partial_2\bar{G}(0, s), s \geq 0$.

To get the explicit expression for $\tilde{F}_S$ we need to explicitate $\bar{G}$ and/or $\bar{H}_i$ and to solve this integral, eventually numerically. For example, if $\bar{H}_i(x) = \exp(-x)$ for $x \geq 0$, then
\[
\tilde{F}_S(s) = \bar{G}(0, s) = \bar{G}(s) + \int_0^s g(y + s - y)dy = \bar{G}(s) + sg(s)
\]
and $f_S(s) = sg'(s)$ for $s \geq 0$ which is the expression in Remark 2.7 of [3] for the Schur-constant model.

4 Predictions

The purpose of this section is to show how to predict the value of the sum $S = X_1 + X_2$ from $X_1 = x$ or vice versa by making use of the results in the previous section. To this purpose we need the conditional distribution of $(S|X_1 = x)$ in the TTE dependence model, that is obtained in the following proposition.

**Proposition 4.1.** If (2.3) and (2.5) hold for $(X_1, X_2)$ and $S = X_1 + X_2$, then the PDF of $(S|X_1 = x)$ is
\[
f_{S|X_1}(s|x) = -r_2(s - x) \frac{g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))}{g(-\ln \bar{H}_1(x))}
\]
and its distribution function is
\[
F_{S|X_1}(s|x) = 1 - \frac{g(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))}{g(-\ln \bar{H}_1(x))} \quad (4.1)
\]
for $0 \leq x \leq s$, where $g = -\bar{G}'$ and $r_2 = (-\ln \bar{H}_2)'$ is the hazard rate function of $\bar{H}_2$.

**Proof.** From [3.1], the PDF of $(X_1, S)$ is
\[
g(x, s) = -r_1(x)r_2(s - x)g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))
\]
for $0 \leq x \leq s$. Moreover, the first marginal survival function is
\[
\tilde{F}_1(x) = \Pr(X_1 > x) = G(x, 0) = \bar{G}(-\ln \bar{H}_1(x))
\]
and its PDF is $f_1(x) = r_1(x)g(-\ln \bar{H}_1(x))$, for $x \geq 0$. 
Hence, the PDF of \((S|X_1 = x)\) for \(x \geq 0\) such that \(g_1(x) > 0\), can be obtained as

\[
f_{S|X_1}(s|x) = \frac{g(x,s)}{f_1(x)} = -r_2(s-x)\frac{g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x))}{g(-\ln \bar{H}_1(x))}
\]

for \(s \geq x\) (zero elsewhere).

Then the associated distribution function is

\[
F_{S|X_1}(s|x) = \int_x^s f_{S|X_1}(t|x) dt
\]

\[
= -\int_x^s r_2(t-x)\frac{g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(t-x))}{g(-\ln \bar{H}_1(x))} dt
\]

\[
= \left[ \frac{g(-\ln \bar{H}_1(x) - \ln \bar{H}_2(t-x))}{g(-\ln \bar{H}_1(x))} \right]^{s}_{t=x}
\]

\[
= 1 - \frac{g(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x))}{g(-\ln \bar{H}_1(x))}
\]

for \(s \geq x \geq 0\) and we conclude the proof. \(\square\)

Hence, the conditional survival function is

\[
\bar{F}_{S|X_1}(s|x) = \frac{g(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s-x))}{g(-\ln \bar{H}_1(x))}
\]

Clearly, this is a distortion representation from \(\bar{H}_2(s-x)\), since

\[
\bar{F}_{S|X_1}(s|x) = d_{S|X_1}(\bar{H}_2(s-x)|\bar{H}_1(x))
\]

for \(s \geq x > 0\), where

\[
d_{S|X_1}(v|u) = \frac{g(-\ln uv)}{g(-\ln u)}
\]

for \(v \in [0,1]\) is a distortion function for all \(0 < u < 1\).

Note that the inverse function of \(\bar{F}_{S|X_1}(v|u)\) (i.e. its quantile function) can be obtained from the inverse functions of \(g\) and \(\bar{H}_2\) as

\[
\bar{F}_{S|X_1}^{-1}(v|u)(q|x) = x + \bar{H}_2^{-1}\left( \exp\left(-g^{-1}\left(qg(-\ln \bar{H}_1(x))\right)\right)\right)
\]

(4.2)

for \(0 < q < 1\). The inverse function of \(F_{S|X_1}\) can be obtained in a similar way.

One can thus predict \(S\) from \(X_1 = x\) by using the quantile (or median) regression curve with

\[
m_{S|X_1}(x) := \bar{F}_{S|X_1}^{-1}(0.5|x).
\]

Moreover, one can compute the centered \(p\) confidence bands for these estimations as

\[
I_p(x) := \left[ \bar{F}_{S|X_1}^{-1}\left(\frac{1+p}{2} \mid x\right), \bar{F}_{S|X_1}^{-1}\left(\frac{1-p}{2} \mid x\right) \right].
\]
For example, the $p = 90\%$ centered confidence band for $S$ is

$$I_{0.9}(x) := \left[ \bar{F}_{S|X_1}^{-1}(0.95|x), \bar{F}_{S|X_1}^{-1}(0.05|x) \right].$$

Such an interval is computed below in some illustrative examples.

**Remark 4.1.** In particular, for the GK-model in (2.6) we get

$$\bar{F}_{S|X_1}(s|x) = \frac{g(\alpha x + \beta (s - x))}{g(\alpha x)} = \frac{g((\alpha - \beta)x + \beta s)}{g(\alpha x)}$$ (4.3)

and

$$\bar{F}_{S|X_1}^{-1}(q|x) = \frac{\beta - \alpha}{\beta} x + \frac{1}{\beta} g^{-1}(q g(\alpha x))$$ (4.4)

for $s \geq x \geq 0$ and $0 < q < 1$. Note that these expressions hold both for $\alpha \neq \beta$ and for $\alpha = \beta$.

The other conditional distribution can be obtained in a similar manner. However, it is more difficult to get an explicit expression since we need the PDF $f_{S}(s)$ of $S$. It is stated in the following proposition.

**Proposition 4.2.** If (2.3) and (2.5) hold for $(X_1, X_2)$ and $S = X_1 + X_2$, then the PDF of $(X_1|S = s)$ is

$$f_{X_1|S}(x|s) = \frac{-r_1(x)r_2(s - x)g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))}{f_{S}(s)}$$

and its distribution function is

$$F_{X_1|S}(x|s) = -\int_0^x \frac{r_1(t)r_2(s - t)g'(-\ln \bar{H}_1(t) - \ln \bar{H}_2(s - t))}{f_{S}(s)} dt$$ (4.5)

for $0 \leq x \leq s$, where $g = -G'$ and $r_i = (-\ln \bar{H}_i)'$ is the hazard rate function of $\bar{H}_i$ for $i = 1, 2$, and

$$f_{S}(s) = -\int_0^s r_1(x)r_2(s - x)g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))dx.$$ (4.6)

**Proof.** From (3.1), the PDF of $(X_1, S)$ is

$$g(x, s) = -r_1(x)r_2(s - x)g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x))$$

for $0 \leq x \leq s$. Its second marginal survival function was obtained in (3.6). It can also be obtained as in (4.6).
Hence, the conditional PDF of \((X_1|S = s)\) is

\[
f_{S|X_1}(x|s) = \frac{g(x, s)}{f_S(s)} = -r_1(x)r_2(s - x)g'(\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x)) / f_S(s).
\]

Then the associated distribution function is the one given in (4.5) for \(0 \leq x \leq s\) and the assertion is proved.

In particular, for the GK-model we have the following explicit expressions.

**Proposition 4.3.** If (2.6) holds for \((X_1, X_2)\) and \(S = X_1 + X_2\), then the distribution function of \((X_1|S = s)\) is

\[
F_{X_1|S}(x|s) = \frac{g((\alpha - \beta)x + \beta s) - g(\beta s)}{g(\alpha s) - g(\beta s)}
\]

when \(\alpha \neq \beta\) and

\[
F_{X_1|S}(x|s) = \frac{x}{s}
\]

when \(\alpha = \beta\), for \(0 \leq x \leq s\), where \(g = -\bar{G}'\) and \(\alpha, \beta > 0\) are the scale parameters in (2.6).

**Proof.** From the preceding proposition we have

\[
g(x, s) = -r_1(x)r_2(s - x)g'(-\ln \bar{H}_1(x) - \ln \bar{H}_2(s - x)) = -\alpha \beta g'(\alpha - \beta)x + \beta s
\]

for \(0 \leq x \leq s\) (zero elsewhere). Its second marginal PDF function \(f_S\) was obtained in (3.4) \((\alpha \neq \beta)\) and in (3.5) \((\alpha = \beta)\).

In the first case we get

\[
f_{X_1|S}(x|s) = \frac{g(x, s)}{f_S(s)} = (\alpha - \beta)g'((\alpha - \beta)x + \beta s) / (g(\alpha s) - g(\beta s))
\]

and in the second

\[
f_{X_1|S}(x|s) = \frac{g(x, s)}{f_S(s)} = -\alpha^2 g'(\alpha s) / (\alpha^2 s g'(\alpha s)) = \frac{1}{s}
\]

for \(0 \leq x \leq s\).

Then the associated distribution functions are

\[
F_{X_1|S}(x|s) = \int_0^x (\alpha - \beta)g'(\alpha - \beta)(t + \beta s) / g(\alpha s) - g(\beta s) dt \cdot \Bigg[\frac{g'(\alpha - \beta)(t + \beta s)}{g(\alpha s) - g(\beta s)}\Bigg]_{t=0}^x
\]

\[
= \frac{g((\alpha - \beta)x + \beta s) - g(\beta s)}{g(\alpha s) - g(\beta s)}
\]
Note that the expression (4.8) was obtained previously in Proposition 2.3 of [3] for the Schur-constant model (which is equivalent to (2.6) with $\alpha = \beta$).

As in the preceding case, expressions (4.7) and (4.8) can be used to obtain quantile regression curves to predict $X_1$ from $S$. An illustrative example is given in the following section. In both cases, they can be represented as distorted distributions from $G$ by replacing $x$ with $G^{-1}(G(x))$ and $s$ with $G^{-1}(G(s))$.

In the second case ($\alpha = \beta$), the inverse function is

$$F_{X_1|S}(q,s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}(qg(\alpha s) + (1-q)g(\beta s))$$

(4.9)

for $0 < q < 1$ and $s > 0$. Then the median regression curve is

$$m_{X_1|S}(s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}\left(\frac{1}{2}g(\alpha s) + \frac{1}{2}g(\beta s)\right).$$

The confidence bands can be obtained in a similar manner from (4.9) (see Example 5.2).

5 Examples

In this section we provide some examples to illustrate the theoretical findings described in previous sections. In the first one we consider the sum of two dependent random variables satisfying the GK-model proposed in [5], i.e., the model (2.6).

**Example 5.1.** Let us assume that $(X_1, X_2)$ satisfies (2.6) for $\alpha \neq \beta$ and $G(x) = (1 + x)^{-\gamma}$ for $x \geq 0$ (Pareto type II survival function) and $\gamma > 0$. This model is equivalent to consider an Archimedean Clayton copula with $\theta = 1/\gamma$ and Pareto type II marginals. Then, from (3.2), the joint distribution function of $(X_1, S)$ is

$$G(x, s) = G(\alpha x) - \frac{\alpha}{\alpha - \beta} G((\alpha - \beta)x + \beta s) + \frac{\alpha}{\alpha - \beta} G(\beta s)$$

$$= 1 - (1 + \alpha x)^{-\gamma} + \frac{\alpha}{\alpha - \beta} (1 + (\alpha - \beta)x + \beta s)^{-\gamma} - \frac{\alpha}{\alpha - \beta} (1 + \beta s)^{-\gamma}$$

for $x \geq 0$ and $s > 0$. The marginal distribution function of $S$ is

$$G(s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}(g(\alpha s) + g(\beta s))$$

and the conditional distribution function is

$$F_{X_1|S}(x|s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}(g(\alpha x) + g(\beta s))$$

for $0 \leq x \leq s$. 

In the second case ($\alpha = \beta$), the inverse function is

$$F_{X_1|S}(q,s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}(qg(\alpha s) + (1-q)g(\beta s))$$

(4.9)

for $0 < q < 1$ and $s > 0$. Then the median regression curve is

$$m_{X_1|S}(s) = \frac{\alpha - \beta}{\alpha - \beta} g^{-1}\left(\frac{1}{2}g(\alpha s) + \frac{1}{2}g(\beta s)\right).$$

The confidence bands can be obtained in a similar manner from (4.9) (see Example 5.2).
for $0 \leq x \leq s$. Hence, the distribution function $F_S$ of $S$ (i.e. the C-convolution) is

$$F_S(s) = G(s, s) = 1 + \frac{\beta}{\alpha - \beta} (1 + \alpha s)^{-\gamma} - \frac{\alpha}{\alpha - \beta} (1 + \beta s)^{-\gamma}$$

for $s \geq 0$. Its PDF is

$$f_S(s) = \frac{\alpha \beta \gamma}{\alpha - \beta} (1 + \beta s)^{-\gamma - 1} - \frac{\alpha \beta \gamma}{\alpha - \beta} (1 + \alpha s)^{-\gamma - 1}$$

for $s \geq 0$. The distribution of $S$ is a negative mixture of two Pareto type II distributions and so its hazard rate goes to zero when $s \to \infty$ (which is the limit of the hazard rates of the members of the C-convolution). They are plotted in Figure 1 (right) jointly with the associated PDF functions (left) for $\gamma = \alpha = 2$ and $\beta = 1$. Note that the hazard rates of $X_1$ and $X_2$ are decreasing while the one of $S$ is not monotone (showing that the IFR class is not preserved by the sum of dependent r.v.).

If we want to predict $X_1$ from $S = s$, we need the conditional distribution obtained from (4.7) as

$$F_{X_1|S}(x|s) = \frac{g((\alpha - \beta)x + \beta s) - g(\beta s)}{g(\alpha s) - g(\beta s)} = \frac{(1 + (\alpha - \beta)x + \beta s)^{-\gamma - 1} - (1 + \beta s)^{-\gamma - 1}}{(1 + \alpha s)^{-\gamma - 1} - (1 + \beta s)^{-\gamma - 1}}$$

for $0 \leq x \leq s$. Its inverse function is then

$$F^{-1}_{X_1|S}(q|s) = \frac{-1 - \beta s + (q(1 + \alpha s)^{-\gamma - 1} + (1 - q)(1 + \beta s)^{-\gamma - 1})^{-1/\gamma}}{\alpha - \beta}$$

for $0 < q < 1$. The median regression curve is obtained by replacing $q$ with $1/2$. It is plotted in Figure 2, jointly with a sample from $(X_1, S)$ and the associated 50% and 90% centered confidence bands. We also include there the parametric (left) and non-parametric (right) estimations for these curves. Here, non-parametric means that we use the linear quantile regression procedure in R.

To estimate the parameters in the model from the sample we use recall the Kendall’s tau coefficient of $(X_1, X_2)$ is

$$\tau = \frac{\theta}{2 + \theta} = \frac{1}{1 + 2\gamma}$$

(see [18], p. 163). Therefore, $\gamma$ is estimated by

$$\hat{\gamma} = \frac{1 - \hat{\tau}}{2\hat{\tau}} = \frac{1 - 0.158}{2 \cdot 0.158} = 2.664557.$$

Then we recall that $E(X_1) = 1/(\alpha(\gamma - 1))$ and $E(X_2) = 1/(\beta(\gamma - 1))$ to estimate $\alpha$ and $\beta$, obtaining

$$\hat{\alpha} = \frac{1}{(\hat{\gamma} - 1)X_1} = \frac{1}{1.664557 \cdot 0.3880776} = 1.548042$$
Figure 1: Probability density (left) and hazard rate (right) functions for $X_1$ (red), $X_2$ (blue) and $S = X_1 + X_2$ (black) under the dependence model (2.6) with Pareto type II marginals studied in Example 5.1.

and

$$\hat{\beta} = \frac{1}{(\hat{\gamma} - 1)X_2} = \frac{1}{1.664557 \cdot 0.8674393} = 0.6925677.$$

For the non-parametric linear estimators of the quantile regression curves, we used the R library quantreg (see [6, 7, 9]). The estimated median regression line to estimate $X_1$ from $S$ obtained from our sample is

$$\hat{m}_{X_1|S}(s) = 0.09752378 + 0.17721635s.$$

The procedure to predict $S$ from $X_1$ is analogous.

In the next example we consider the general dependence model defined in (2.3) and (2.5). In this case we show how to predict $S$ from $X_1$.

**Example 5.2.** Recall that in (2.3) we assume that the joint survival function of $(X_1, X_2)$ can be written as

$$\bar{F}(x_1, x_2) = \bar{D}(\bar{H}_1(x_1), \bar{H}_2(x_2)).$$
Figure 2: Scatterplot of the simulated sample from $(S, X_1)$ in Example 5.1 jointly with the exact median regression curve (continuous red lines) and the exact 50% and 90% confidence bands (continuous blue lines). The dashed lines represent the estimated curves when the model is known and the parameters are estimated (left) and when the model is unknowns and we use a non-parametric linear quantile regression estimator (right) from these data.

where $\bar{H}_1$ and $\bar{H}_2$ are two absolutely continuous survival functions with $\bar{H}_1(0) = \bar{H}_2(0) = 1$, while (2.5) asserts that

$$\hat{D}(u, v) = \bar{G}(-\ln(uv))$$

(5.1)

for $u, v \in [0, 1]$, where $\bar{G}$ satisfies the properties stated after Eq. (2.5). This model is a bivariate distorted distribution, for which the marginal survival functions are $\bar{F}_i(x_i) = \bar{G}(-\ln(\bar{H}_i(x_i)))$, for $i = 1, 2$. Thus, we can use the expressions obtained in Section 4, (4.1) and (4.2), to predict $S$ from $X_1$.

For example, we can choose

$$\bar{G}(x) = \bar{H}_1(x) = \bar{H}_2(x) = c \cdot (1 - \Phi(1 + x)) = c \cdot \Phi(-1 - x)$$

for $x \geq 0$, where $\Phi$ is the standard normal distribution and $c = 1/\Phi(-1) = 6.302974$ (i.e. $G$ is a truncated Normal distribution). Hence $g(x) = c \cdot \phi(1 + x)$ where $\phi = \Phi'$ is the PDF of a standard normal distribution. Note that, in this case, the associated Archimedean copula (that we could call Gaussian Archimedean copula) does not have an explicit expression (since it depends on $\bar{G}$ and on $\bar{G}^{-1}$). Thus, this is a practical example where the distortion
representation (5.1) can be used as a proper alternative.

Its inverse functions are

\[
\bar{G}^{-1}(x) = -1 - \Phi^{-1}(x/c)
\]

and

\[
g^{-1}(x) = -1 + (2 \ln c - \ln(2\pi) - 2 \ln x)^{1/2}.
\]

By using these expressions we compute \(\bar{F}_{S|X_1}^{-1}\) as in (4.2), obtaining the quantile regression curve plotted in Figure 3, left. The same figure also includes a sample of \(n = 100\) points from \((X_1, S)\) and the exact centered 50\% and 90\% (blue) confidence bands. Moreover, it shows the plot of the non-parametric linear quantile estimate (dashed lines) obtained from this sample.

As we know that \(X_1 < S\), we could also provide bottom 50\% and 90\% confidence bands obtained as \([x, \bar{F}_{S|X_1}^{-1}(0.5|x)]\) and \([x, \bar{F}_{S|X_1}^{-1}(0.1|x)]\), respectively. They are plotted in Figure 3, right. In this case, the median regression curve is also the upper limit for the 50\% confidence band. In our sample we obtain 10 data above the upper (exact) limit and 46 above the median regression curve (i.e. 54 data in the exact bottom 50\% confidence band). The estimated median regression line obtained from our sample is

\[
\hat{m}_{S|X_1}(x) = 0.3159734 + 0.7284655x
\]

for \(x \geq 0\).

In the next example we show a case of model (2.6) that cannot be represented with an explicit Archimedean copula, thus for which the distortion representations consists in a useful alternative tool. In fact, in this example \(\bar{G}\) is convex and an explicit expression for its inverse is not available. For this model we compute the explicit expressions for the C-convolution and the two conditional survival functions.

**Example 5.3.** Consider (2.6) with \(\alpha \neq \beta\) and the survival function

\[
\bar{G}(x) = \frac{2 + x}{2} e^{-x}
\]

for \(x \geq 0\). Its PDF is

\[
g(x) = \frac{1 + x}{2} e^{-x}
\]
Figure 3: Scatterplot of the simulated sample from \((X_1, S)\) in Example 5.2 jointly with the median regression curve (red) and the centered (left) or bottom (right) 50% and 90% confidence bands (blue). The dashed lines represent the estimated values when we use a linear quantile regression estimator.

For \( x \geq 0 \), that is, it is a translated Gamma (Erlang) distribution. The joint survival function of \((X_1, X_2)\) is

\[
\bar{F}(x_1, x_2) = \bar{G}(\alpha x_1 + \beta x_2) = \frac{1 + \alpha x_1 + \beta x_2}{2} \exp(-\alpha x_1 - \beta x_2)
\]

for \( x_1, x_2 \geq 0 \). The marginals have also translated Gamma distributions.

The joint distribution of \((X_1, S)\) can be obtained from (3.2). From this expression, one can obtain the survival function of \( S \) (C-convolution) as

\[
\bar{F}_S(s) = \frac{\alpha}{\alpha - \beta} \bar{G}(\beta s) - \frac{\beta}{\alpha - \beta} \bar{G}(\alpha s)
\]

\[
= \frac{\alpha}{\alpha - \beta} e^{-\beta s} - \frac{\beta}{\alpha - \beta} e^{-\alpha s} + \frac{\alpha \beta s}{2(\alpha - \beta)} \left( e^{-\beta s} - e^{-\alpha s} \right)
\]

for \( s \geq 0 \). Note that it is a negative mixture of two translated Gamma distributions.

The conditional survival function of \((S|X_1 = x)\) can be obtained from (4.3) as

\[
\bar{F}_{S|X_1}(s|x) = \frac{g((\alpha - \beta)x + \beta s)}{g(\alpha x)} = \frac{1 + (\alpha - \beta)x + \beta s e^{-\beta(s-x)}}{1 + \alpha x}
\]

for \( s \geq x \). Analogously, from (4.7), the conditional survival function of \((X_1|S = s)\) is

\[
\bar{F}_{X_1|S}(x|s) = \frac{g(\alpha s) - g((\alpha - \beta)x + \beta s)}{g(\alpha s) - g(\beta s)} = \frac{1 + \alpha s - (1 + (\alpha - \beta)x + \beta s)e^{(\alpha-\beta)(s-x)}}{1 + \alpha s - (1 + \beta s)e^{(\alpha-\beta)s}}
\]
Figure 4: Probability density (left) and hazard rate (right) functions for $X_1$ (red), $X_2$ (blue) and $S = X_1 + X_2$ (black) under the dependence model (2.6) with translated Gamma marginals studied in Example 5.3. The dashed lines represent the limiting behaviour.

for $0 \leq x \leq s$.

In Figure 4 we plot the probability density (left) and hazard rate (right) functions of $X_1$ (red), $X_2$ (blue) and $S$ (black) when $\alpha = 2$ and $\beta = 1$. Note that both marginals are IFR and the same holds for $S$. Also note that the limiting behaviour of the hazard rate of $S$ coincides with that of the best component ($X_2$) in the sum. This is according with the results on mixtures obtained in Lemma 3.3 of [15] (or Lemma 4.6 in [17]) and that in Theorem 1 of [2] on usual convolutions.

In the last example we show a case dealing with the GK model (2.6) where the inverse of the conditional distribution function $F_{(X_1|S)}$ of $(X_1|S)$ cannot be obtained in a closed form.

Then we need to use numerical methods (or implicit functions plots). It also shows that the quantile (median) regression curve $m_{(X_1|S)}(s) = F_{(X_1|S)}(0.5|s)$ is not always increasing.

Example 5.4. Let us consider the model (2.6) which leads to a survival copula in the family of Gumbel-Barnett copulas (see (4.2.9) in [18], p. 116). In this case, the additive generator of the copula is $\bar{G}^{-1}(t) = \ln(1 - \theta \ln t)$ for $t \in (0,1]$ and $\theta \in (0,1]$. These copulas are strict Archimedean copulas and the independence (product) copula is obtained for $\theta \to 0$. Hence,

$$
\bar{G}(t) = \exp\left(\frac{1}{\theta} - \frac{1}{\theta}e^t\right)
$$
and
\[ g(t) = \frac{1}{\theta} \exp \left( t + \frac{1}{\theta} - \frac{1}{\theta} e^t \right) \]
for \( t \geq 0 \). Note that the inverse of \( g \) has not an explicit form, thus one cannot use \((4.9)\) to compute the quantile functions of \((X_1|S)\). The same happens in \((4.4)\) for the quantile functions of \((S|X_1)\).

However, it is possible to plot the level curves of the conditional distribution function by using \((4.7)\), obtaining
\[ F_{(X_1|S)}(s|x) = \frac{g((\alpha - \beta)x - \beta s) - g(\beta s)}{g(\alpha s) - g(\beta s)} \quad (5.2) \]
when \( \alpha \neq \beta \). For example, if we choose \( \alpha = 3, \beta = 1 \) and \( \theta = 1 \) in \((5.4)\), we get
\[ F_{(X_1|S)}(s|x) = \frac{\exp(2x + s) - g(s)}{\exp(3s) - g(s)} = \frac{\exp(2x + s + 1 - e^{2x+s}) - \exp(s + 1 - e^s)}{\exp(3s + 1 - e^{3s}) - \exp(s + 1 - e^s)} \]
for \( 0 \leq x \leq s \). These level curves for \( q = 0.05, 0.25, 0.5, 0.75, 0.95 \) are plotted in Figure 5 left. Note that the median regression curve \( m_{(X_1|S)}(s) = F^{-1}_{(X_1|S)}(0.5|s) \) (red line, left) is first increasing and then decreasing. To explain this surprising fact we plot \( F_{(X_1|S)}(x|s) \) in Figure 5 right, for different values of \( s \), where one can observe that these distribution functions are not ordered in \( s \), that is, \((X_1|S = s)\) is not stochastically increasing in \( s \). Here the greater values for \( X_1 \) are obtained when \( S \approx 0.6 \) (green line). Also note that \( E(X_2) = 3E(X_1) \) and that \( X_1 \) and \( X_2 \) are negatively correlated. Therefore, the greater values of \( S \) are mainly obtained from the greater values of \( X_2 \) and the smaller values of \( X_1 \). Thus \( m_{(X_1|S)} \) is decreasing at the end.

Also note that
\[ \text{Cov}(X_1, S) = \text{Var}(X_1) + \text{Cov}(X_1, X_2) = \text{Var}(X_1) + E(X_1X_2) - E(X_1)E(X_2). \]

Therefore, \( \text{Cov}(X_1, S) \geq 0 \) when \( \text{Cov}(X_1, X_2) \geq 0 \) and, in particular, when \( X_1 \) and \( X_2 \) are independent. However, the covariance \( \text{Cov}(X_1, S) \) will be negative if \( \text{Var}(X_1) < -\text{Cov}(X_1, X_2) \). In our case, the marginal reliability functions of \( X_1 \) and \( X_2 \) are \( \bar{F}_1(t) = \bar{G}(3t) \) and \( \bar{F}_2(t) = \bar{G}(t) \), respectively. Their means are \( E(X_1) = 0.198782 \) and \( E(X_2) = 0.596347 \), their variances \( \text{Var}(X_1) = 0.019589 \) and \( \text{Var}(X_2) = 0.176301 \) and their covariance \( \text{Cov}(X_1, X_2) = -0.029889 \). Hence
\[ \text{Cov}(X_1, S) = \text{Var}(X_1) + \text{Cov}(X_1, X_2) = 0.019589 - 0.029889 = -0.010299 < 0. \]
Figure 5: Median regression curve (red) and quantile regression curves (blue) for $q = 0.05, 0.25, 0.75, 0.95$ (left) for $(S, X_1)$ in Example 5.4. Conditional distribution functions $G_{1|2}(x|s)$ for $s = 0.2$ (red), 0.4 (blue), 0.6 (green), 0.8 (orange), 1 (black) and 2 (purple). The black line in the left plot represents the line $X_1 = S$.

6 Conclusions

We formulated the TTE dependence model by using a distortion representation based on a specific fixed distortion function $D$. This representation is useful to compute the joint distribution of $X_1$ and the sum $S = X_1 + X_2$, as well as to provide expressions for the survival function of $S$ and the conditional distributions of $S$ given $X_1$ or $X_1$ given $S$. They can be used also to predict one value from the other by using quantile regression. Some examples illustrate these facts, showing that sometimes the classical copula approach can not be applied.

This paper is a first step on applications of distortion representations for dependence models. Thus, there are several tasks for future research. The main one could be to get explicit models by choosing appropriate functions $G$, $H_1$ and $H_2$, to study their main properties and how they fit to real data sets, allowing for the use of the prediction techniques developed here for these data sets. Other interesting questions deal with dependence models for which the multivariate distortion function $\widehat{D}$ differs from the one in Eq. [2.5], or how to get explicit expressions for the multivariate case.
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