Ordered covering arrays and upper bounds on covering codes

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Abstract
This work shows several direct and recursive constructions of ordered covering arrays (OCAs) using projection, fusion, column augmentation, derivation, concatenation, and Cartesian product. Upper bounds on covering codes in Niederreiter–Rosenbloom–Tsfasman (shorten by NRT) spaces are also obtained by improving a general upper bound. We explore the connection between ordered covering arrays and covering codes in NRT spaces, which generalize similar results for the Hamming metric. Combining the new upper bounds for covering codes in NRT spaces and ordered covering arrays, we improve upper bounds on covering codes in NRT spaces for larger alphabets. We give tables comparing the new upper bounds for covering codes to existing ones.

KEYWORDS
bounds on codes, covering array, covering code, Niederreiter–Rosenbloom–Tsfasman metric, ordered covering array, ordered orthogonal array
1 INTRODUCTION

Covering codes deal with the following question: given a metric space, what is the minimum number of balls of fixed radius necessary to cover the entire space? Several applications, such as data transmission, cellular telecommunications, decoding of errors, and football pool problem, have motivated the investigation of covering codes in Hamming spaces. Covering codes also have connections with other branches of mathematics and computer science, such as finite fields, linear algebra, graph theory, combinatorial optimization, mathematical programming, and metaheuristic search. We refer the reader to the book by Cohen et al. [10] for an overview of the topic.

Rosenbloom and Tsfasman [29] introduced a metric on linear spaces over finite fields, motivated by applications to interference in parallel channels of communication systems. This non-Hamming metric, implicitly posed by Niederreiter [24], is currently known as the Niederreiter–Rosenbloom–Tsfasman (NRT) metric (also RT metric, ρ metric, m-metric) and is an example of partially ordered set (poset) metric [16], see Section 2.1. Park and Barg [25] defined the ordered symmetric channel and the ordered erasure channel which generalize their Hamming counterpart and can be viewed as a set of dependent parallel channels. They show that linear codes under the NRT metric attain the capacity of these channels. Jain [18] introduced the notion of burst errors for this model of information transmission. Research on this metric is also motivated by the study of uniform distributions of points in the unit cube [1, 15, 30].

Central concepts about codes in Hamming spaces have been also investigated in NRT spaces, such as perfect codes, maximum distance separable (MDS) codes, linear codes, weight distribution, packing, and covering problems [3, 5, 6, 24, 26, 29, 31]. See the book by Firer et al. [16] for further concepts and applications of the NRT metric to coding theory.

The packing problem in NRT spaces, investigated by Quistorff [26], deals with bounds on the packing problem and some results concerning codes meeting the Singleton bound. Covering codes in the NRT metric has been less studied than packing codes. Brualdi et al. [3, Theorem 2.1] implicitly investigate covering codes in the metric space endowed with a poset metric, more specifically, when the poset is a chain. The same result is explicitly given by Yildiz et al. [31, Theorem 2.3]. The general problem of covering codes in the NRT space is proposed by the first two authors in [5], which deals mainly with upper bounds, recursive relations, and some sharp bounds as well as relations with MDS codes. More recently, the sphere covering bound in NRT spaces is improved in [6] under some conditions by generalizing the excess counting method. In the present work, we explore upper bounds and recursive relations for covering codes in NRT spaces using ordered covering arrays (OCAs).

Orthogonal arrays (OAs) play a central role in combinatorial designs with close connections to coding theory; see the book on orthogonal arrays by Hedayat et al. [17]. Covering arrays (CAs), also called t-surjective arrays, generalize orthogonal arrays and have received a lot of attention due to their applications to software testing and interesting connections with other combinatorial designs; see the survey paper by Colbourn [11].

Ordered orthogonal arrays (OOAs) are a generalization of orthogonal arrays introduced in 1996 independently by Lawrence [21] and Mullen and Schmid [23], motivated by their applications to numerical integration. OOAs are also related to the NRT metric, see [8]. A survey of constructions of ordered orthogonal arrays is in [20, Chap. 3], see also [13, Section VI.59.3]. For a survey of finite field constructions of OAs, CAs, and OOAs, see Moura et al. [22, Section 3].
Ordered covering arrays have been introduced by Krikorian in her master’s thesis [20], generalizing several of the mentioned designs (OAs, CAs, and OOAs). Krikorian [20] investigates recursive and Roux-type constructions of OCAs as well as other constructions using the columns of a covering array and discusses an application of OCAs to numerical integration (evaluating multidimensional integrals). In the present paper, we also give further results for OCAs.

In this work, we extend and build upon our results from the conference paper [7] in the following directions. We give several recursive constructions of OCAs based on projection, fusion, column augmentation, derivation, and concatenation. A new approach for the recursive construction of OCAs is presented based on column augmentation by adding a new chain to the NRT poset and adding new rows using a Cartesian product of suitable arrays (Theorem 8). As a consequence, we obtain a bound for the size of OCAs with strength $t$, alphabet $v$ a prime power, and NRT posets with $v + 2$ chains of length $t$ (Corollary 9). New upper bounds on covering codes in NRT spaces are obtained by modifying some of the code’s codewords that give the general upper bound (Proposition 1) to reduce the size of the covering code. We apply the newly found OCA bounds and covering code bounds to obtain new upper bounds on covering codes in NRT spaces for larger alphabets.

This work is organized as follows. We review the basics of the NRT poset, the NRT metric, and covering codes in NRT spaces in Section 2.1, and define OAs, CAs, OOAs, and OCAs in Section 2.2. Section 3 is devoted to new recursive constructions of OCAs yielding recursive relations for ordered covering array numbers (OCANs). We generalize and improve upper bounds on covering codes in Section 4. Finally, in Section 5, constructions of covering codes in NRT spaces are derived from OCAs. Tables 2 and 3 contrast some upper bounds obtained in this paper.

2 | PRELIMINARIES

2.1 | The NRT metric and covering codes

Any poset induces a metric, according to the seminal paper by Brualdi et al. [3]. Codes based on various poset metrics are presented in a systematic way in the book by Firer et al. [16], and we follow their notation. Here we include some basic definitions for the NRT poset and metric; we also introduce covering codes under the NRT metric, first studied in [5], which is one of the focal points of this paper.

Let $P$ be a finite partially ordered set (poset) and denote its partial order relation by $\preceq$. A poset is a chain when any two elements are comparable; a poset is an antichain when no two distinct elements are comparable. A subset $I$ of $P$ is an ideal of $P$ if $b \in I$ and $a \preceq b$, implies $a \in I$. The ideal generated by a subset $A$ of $P$ is the ideal of the smallest cardinality that contains $A$, denoted by $(A)$.

An element $a \in I$ is maximal in $I$ if $a \preceq b$ implies that $b = a$. Analogously, an element $a \in I$ is minimal in $I$ if $b \preceq a$ implies that $b = a$. A subset $J$ of $P$ is an anti-ideal of $P$ when it is the complement of an ideal of $P$. If an ideal $I$ has $t$ elements, then its corresponding anti-ideal has $n - t$ elements, where $n$ is the number of elements in $P$.

Given positive integers $m$ and $s$, let $\mathcal{R}[m \cdot s]$ be a set of $ms$ elements partitioned into $m$ blocks $B_i$ having $s$ elements each, where $B_i = \{b_{i+1}s, \ldots, b_{(i+1)s}\}$ for $i = 0, \ldots, m - 1$ and the elements of each block are ordered as $b_{i+1}s \preceq b_{i+2}s \preceq \cdots \preceq b_{(i+1)s}$. The set $\mathcal{R}[m \cdot s]$ is a poset consisting of the union of $m$ disjoint chains, each one having $s$ elements. This
poset is known as the Niederreiter–Rosenbloom–Tsfasman poset $\mathcal{R}[m \cdot s]$, or briefly the NRT poset $\mathcal{R}[m \cdot s]$. When $\mathcal{R}[m \cdot s] = [m \cdot s] = \{1, ..., ms\}$, the NRT poset $\mathcal{R}[m \cdot s]$ is denoted by NRT poset $[m \cdot s]$ and its blocks are $B_i = \{is + 1, ..., (i + 1)s\}$, for $i = 0, ..., m - 1$.

Given the NRT poset $[m \cdot s]$, the NRT distance between $x = (x_1, ..., x_{ms})$ and $y = (y_1, ..., y_{ms})$ in $\mathbb{Z}_q^{ms}$ is defined in [3] as

$$d_R(x, y) = |\text{supp}(x - y)| = |\{i : x_i \neq y_i\}|$$

A set $\mathbb{Z}_q^{ms}$ endowed with the distance $d_R$ is an Niederreiter–Rosenbloom–Tsfasman space, or simply, an NRT space. The notation $d_{\mathcal{R}[m \cdot s]}$ can be used to emphasize the structure of the NRT poset $[m \cdot s]$. The NRT space (metric or poset) is also called the RT space (metric or poset), see [5, 7, 31].

The NRT sphere centered at $x$ of radius $R$, denoted by $B(x, R) = \{y \in \mathbb{Z}_q^{ms} : d_R(x, y) \leq R\}$, has cardinality given by the formula

$$V_q^R(m, s, R) = 1 + \sum_{i=1}^{R} \sum_{j=1}^{\min\{m, i\}} q^{i-j}(q - 1)\Omega_j(i),$$

where, for $1 \leq i \leq ms$ and $1 \leq j \leq \min\{m, i\}$, the parameter $\Omega_j(i)$ denotes the number of ideals of the NRT poset $[m \cdot s]$ whose cardinality is $i$ with exactly $j$ maximal elements. The case $s = 1$ corresponds to the Hamming sphere $V_q(m, R) = V_q^R(m, 1, R)$.

**Definition 1.** Given an NRT poset $[m \cdot s]$, a subset $C$ of $\mathbb{Z}_q^{ms}$ is an $R$-covering of the NRT space $\mathbb{Z}_q^{ms}$ if for every $x \in \mathbb{Z}_q^{ms}$ there is a codeword $c \in C$ such that $d_R(x, c) \leq R$, or equivalently,

$$\bigcup_{c \in C} B_{d_R}(c, R) = \mathbb{Z}_q^{ms}.$$

The number $K_q^R(m, s, R)$ is the smallest cardinality of an $R$-covering of the NRT space $\mathbb{Z}_q^{ms}$.

We refer the reader to [5] for more details on covering codes in NRT spaces. In [5–7] the notation $K_q^R(m, s, R)$ is used instead of $K_q^R(m, s, R)$. In this work, we choose the latter from [16] to use a more current and simplified notation.

In the particular case of $s = 1$, an antichain $[m \cdot 1]$ induces covering codes in Hamming spaces and the numbers $K_q^R(1, 1, R) = K_q(m, R)$, Determining these numbers is a challenging problem in combinatorial coding theory [10]. On the other hand, the numbers $K_q^R(1, s, R)$ are induced by a chain $[1 \cdot s]$, and were completely evaluated in [3, 31]. The literature on $K_q^R(m, s, R)$ for a general NRT poset remains sparse [5–7].

One upper bound which we will improve is the general upper bound given in the next result.

**Proposition 1** (Castoldi and Monte Carmelo [5, Proposition 6]). Let $m$ and $s$ be positive integers. For every $q \geq 2$ and $R$ such that $0 < R < ms$,

$$K_q^R(m, s, R) \leq q^{ms-R}.$$
2.2 Ordered covering arrays

Ordered orthogonal arrays are related to classical combinatorial designs, namely, covering arrays (CAs) and orthogonal arrays (OAs). We refer the reader to a survey paper by Colbourn [11] on CAs, a book by Hedayat et al. [17] on OAs, and a survey paper by Moura et al. [22, Section 3] for their relations to ordered orthogonal arrays.

Let $t$, $v$, $\lambda$, $n$, $N$ be positive integers and $N \geq \lambda v^t$. Let $A$ be an $N \times n$ array over an alphabet $V$ of size $v$. An $N \times t$ subarray of $A$ is $\lambda$-covered if it has each $t$-tuple over $V$ as a row at least $\lambda$ times. A set of $t$ columns of $A$ is $\lambda$-covered if the $N \times t$ subarray of $A$ formed by them is $\lambda$-covered; when $\lambda = 1$, we say the set of columns is covered and often omit $\lambda$ from the notation.

**Definition 2** (CA and OA). Let $N$, $n$, $v$, and $\lambda$ be positive integers such that $t \leq n$. A covering array $CA_\lambda(N; t, n, v)$ is an $N \times n$ array $A$ with entries from a set $V$ of size $v$ such that any $t$-set of columns of $A$ is $\lambda$-covered. The parameter $t$ is the strength of the covering array. The covering array number $CAN_\lambda(t, n, v)$ is the smallest positive integer $N$ such that a $CA_\lambda(N; t, n, v)$ exists. An orthogonal array, denoted by $OA_\lambda(\lambda v^t; t, n, v)$ or simply by $OA_\lambda(t, n, v)$, has a similar definition with the additional requirement that the $\lambda$-coverage must be “exact,” and as a consequence, it is the same as a covering array with $N = \lambda v^t$.

Roughly speaking, each column in a CA or OA has the same “importance” towards coverage requirements. This happens because the set of columns is implicitly labeled by the antichain $R[n \cdot 1]$. In contrast, the importance of each column in an OCA and OOA depends on the anti-ideals of the NRT poset $R[m \cdot s]$. Ordered covering arrays are precisely defined as follows.

**Definition 3** (OCA and OOA). Let $t$, $m$, $s$, $v$, and $\lambda$ be positive integers such that $2 \leq t \leq ms$. An ordered covering array $OCA_\lambda(N; t, m, s, v)$ is an $N \times m s$ array $A$ with entries from an alphabet $V$ of size $v$, whose columns are labeled by an NRT poset $R[m \cdot s]$. For each anti-ideal $J$ of the NRT poset $R[m \cdot s]$ with $|J| = t$, the set of columns of $A$ labeled by $J$ is $\lambda$-covered. The parameter $t$ is the strength of the ordered covering array. The ordered covering array number $OCAN_\lambda(t, m, s, v)$ is the smallest positive integer $N$ such that there exists an $OCA_\lambda(N; t, m, s, v)$ exists. An ordered orthogonal array has the extra requirement of “exact” $\lambda$-coverage and thus it is the same as an ordered covering array with $N = \lambda v^t$, denoted by $OOA_\lambda(\lambda v^t; t, m, s, v)$ or simply by $OOA_\lambda(t, m, s, v)$. When $\lambda = 1$ we omit $\lambda$ from the notation.

Ordered covering arrays were first studied by Krikorian [20]. Ordered covering arrays are special cases of variable strength covering arrays [27, 28], where the sets of columns covered are specified by a general hypergraph.

**Example 1.** An OCA of strength 2 with 5 rows:

\[
OCA(5; 2, 4, 2, 2) = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]
The columns of this array are labeled by \([4 \cdot 2] = \{1, \ldots, 8\}\) and the blocks of the NRT poset \([4 \cdot 2]\) are \(B_0 = \{1, 2\}\), \(B_1 = \{3, 4\}\), \(B_2 = \{5, 6\}\) are \(B_3 = \{7, 8\}\). There are ten anti-ideals of size 2, namely,

\[\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{2, 4\}, \{2, 6\}, \{2, 8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}\]

The \(5 \times 2\) subarray constructed from each one of these anti-ideals covers all the pairs \((0, 0), (0, 1), (1, 0),\) and \((1, 1)\) at least once. The array \(OCA(5; 2, 4, 2, 2)\) is not a \(CA(5; 2, 8, 2)\) because many pairs of columns are not covered, \(\{1, 4\}\) for example.

According to [19], a \(CA(5; 2, 8, 2)\) does not exist since \(CAN(2, 8, 2) = 6\). Therefore the \(OCA(5; 2, 4, 2, 2)\) depicted gives an instance where an OCA exists, but no \(CA(5; 2, 8, 2)\) exists.

In an \(OCA(N; t, m, s, v)\) such that \(s > t\), each one of the first \(s - t\) elements of a block in the NRT poset \(R[m \cdot s]\) is not an element of any anti-ideal of size \(t\). Therefore, a column labeled by one of these elements is not part of any \(N \times t\) subarray that must be \(\lambda\)-covered in an OCA. So we assume \(s \leq t\) from now on.

Two trivial relationships between the ordered covering array number and the covering array number \(CAN_N(t, n, v)\) are:

1. \(\lambda v^t \leq CAN_N(t, m, s, v) \leq CAN_N(t, ms, v)\);
2. For \(1 \leq s \leq t \leq m\), \(CAN_N(t, m, v) \leq CAN_N(t, m, s, v)\).

We observe that if \(N = \lambda v^t\), an \(OCA_N(\lambda v^t; t, m, s, v)\) is an ordered orthogonal array \(OAO_N(\lambda v^t; t, m, s, v)\). When \(s = 1\), an \(OCA_N(N; t, m, 1, v)\) is a covering array \(CA_N(N; t, m, v)\).

The following OCA number is used throughout the paper. Let \(s \geq 3\) and \(v\) be a prime power; then according to [8, Theorem 3], there exists an \(OOA(v^s; s, v + 1, s, v)\). Therefore, for \(2 \leq m \leq v + 1\),

\[OCAN(s, m, s, v) = v^s.\] (1)

### 3 NEW RECURSIVE CONSTRUCTIONS FOR ORDERED COVERING ARRAYS

In this section, we show new recursive relations for ordered coverings arrays. Since the proofs of Propositions 2 and 3 and their consequences have already appeared in the conference paper [7] we only state these results. The others results in this section are new recursive constructions for OCAs. Proposition 2 gives constructions that show the size of a chain can be extended for \(s = t\) and OCAN monoticity for parameters \(s\) and \(m\).

**Proposition 2** (Castoldi et al. [7, Proposition 2]). Let \(N, t, m, s, v\) be positive integers.

1. The existence of an \(OCA_N(N; t, m, t - 1, v)\) implies the existence of an \(OCA_N(N; t, m, t, v)\).
2. The existence of an \(OCA_N(N; t, m, s + 1, v)\) implies the existence of an \(OCA_N(N; t, m, s, v)\).
(3) The existence of an $OCA_\lambda(N; t, m + 1, s, v)$ implies the existence of an $OCA_\lambda(N; t, m, s, v)$.

By Proposition 2 items (1) and (2), there exists an $OCA_\lambda(N; t, m, t, v)$ if and only if there exists an $OCA_\lambda(N; t, m, t - 1, v)$, for $t \geq 2$. By this equivalence when $t = 2$, the right-hand side OCA has $s = t - 1 = 1$ which corresponds to a covering array. This also shows us that the constraint $t > 2$ must hold to have ordered covering arrays essentially different from covering arrays.

To illustrate how $s = t = 2$ reduces to covering arrays, let us label the columns of a $CA_\lambda(N; 2, m, v)$ by the elements of $[m] = \{1, \ldots, m\}$. Make these columns of the $CA_\lambda(N; 2, m, v)$ correspond to columns of an $OCA_\lambda(2, m, 2, v)$ for the NRT poset given in Figure 1. We use the notation $\overline{a}$ to duplicate $a \in [m]$ in the NRT poset $\mathcal{R}[m \cdot 2]$ in such a way that $a$ and $\overline{a}$ are not comparable, but the columns of $OCA_\lambda(N; 2, m, 2, v)$ labeled by $a$ and $\overline{a}$ are equal.

This construction implies

$$OCAN_\lambda(2, m, 2, v) = CAN_\lambda(2, m, v).$$

(2)

In particular, since $CAN(2, m, 2)$ is determined [19], we get

$$OCAN(2, m, 2, 2) = \min \left\{ N : m \leq \left( \frac{N - 1}{2} \right) \right\}.$$ (3)

The next result shows a relationship for ordered covering arrays numbers over alphabets with different sizes. It is a generalization of [12, Lemma 3.1] and part of [14, Lemma 3.1], and indeed this is a fact that holds for the more general case of variable strength covering arrays [28].

**Proposition 3** (Castoldi et al. [7, Theorem 1]; Fusion). Let $t, m, s, v$ be positive integers. Then

$$OCAN_\lambda(t, m, s, v) \leq OCAN_\lambda(t, m, s, v + 1) - 2.$$ (4)

Thus for $s \geq 3$, $v$ a prime power, and $2 \leq m \leq v + 1$, Proposition 3 and Equation (1) give the following upper bound

$$OCAN(t, v + 1, t, v - 1) \leq v^t - 2.$$ (4)

We now show new recursive constructions for OCAs. Chateauneuf and Kreher [9, Construction D] develop a form of alphabet augmentation for $s = 1$ and strength 3, which yields

$$CAN(3, n, v) \leq CAN(3, n, v - 1) + n \cdot CAN(2, n - 1, v - 1) + n \cdot (v - 1).$$

**Figure 1** Blocks of the Niederreiter–Rosenbloom–Tsfasman poset $\mathcal{R}[m \cdot 2]$. 
Inspired by their construction, we show a form of alphabet augmentation for ordered covering arrays for \( s = 3 \) and strength 3.

**Theorem 4.** For \( m \geq 3 \) and \( v \geq 3 \), \( OCAN(3, m, 3, v) \) is less than or equal to \( OCAN(3, m, 2, v - 1) + mCAN(2, m - 1, v - 1) + CAN(2, m, v - 1) + m(v - 1)^2 + 1 \).

**Proof.** Let \( A \) be an \( OCA(M; 3, m, 2, v - 1) \), \( B \) a \( CA(M'; 2, m - 1, v - 1) \) and \( C \) a \( CA(M''; 2, m, v - 1) \) over the alphabet \( \{1, ..., v - 1\} \). Let \( N = M + mM' + M'' + m(v - 1)^2 + 1 \). It is sufficient to construct an \( OCA(N; 3, m, 2, v) \) according to Proposition 2 item (1). For this purpose, the proof is divided into two steps. We first construct an array \( B \) and an array \( C \) and a new array \( D \). Second, we prove that an array \( A \) constructed by vertically juxtaposing the arrays \( A, B, C, \) and \( D \) is an \( OCA(N; 3, m, 2, v) \).

**Step 1:** We can construct an \( OCA(M'; 2, m - 1, 2, v - 1) \) from the covering array \( B \) as shown in Equation (2). Let \( B^1, ..., B^{m-1} \) be the subarrays of two consecutive columns of \( OCA(M'; 2, m - 1, 2, v - 1) \) such that \( OCA(M'; 2, m - 1, 2, v - 1) = [B^1 B^2 ... B^{m-1}] \). We consider the following \((mM') \times 2m\) array

\[
B = \begin{bmatrix}
0 & B^1 & B^2 & \cdots & B^{m-3} & B^{m-2} & B^{m-1} \\
B^1 & 0 & B^2 & \cdots & B^{m-3} & B^{m-2} & B^{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
B^1 & B^2 & B^3 & \cdots & B^{m-2} & 0 & B^{m-1} \\
B^1 & B^2 & B^3 & \cdots & B^{m-2} & B^{m-1} & 0
\end{bmatrix}
\]

constructed by inserting \( M' \times 2 \) zero arrays in every possible position \( l, l = 1, ..., m \), with respect to the sequence of arrays \( B^1, ..., B^{m-1} \).

For \( i \in [m] \), let \( c_i \) be the column \( i \) of the covering array \( C \). From the covering array \( C = [c_1 c_2 ... c_m] \), we construct the following array:

\[
C = [0|c_1|0|c_2|0|c_j|\cdots|0|c_m].
\]

Consider the following two arrays \( D \) and \( E \) of order \((v - 1)^2 \times 2\) over the alphabet \( \{0, 1, ..., v - 1\} \):

\[
D = \begin{bmatrix}
0 & 1 \\
\vdots & \vdots \\
0 & v - 1 \\
\vdots & \vdots \\
0 & v - 1
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 \\
\vdots & \vdots \\
v - 1 & 0 \\
\vdots & \vdots \\
v - 1 & 0
\end{bmatrix}
\]
Observe that each nonzero element of \([1, ..., v - 1]\) appears exactly \(v - 1\) times in the nonzero column of \(D\) and \(E\). We define the array \(D\) of order \((m(v - 1)^2 + 1) \times 2m\) as:

\[
D = \begin{bmatrix}
D & E & E & \cdots & E & E \\
E & D & E & \cdots & E & E \\
E & E & D & \cdots & E & E \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
E & E & E & \cdots & D & E \\
E & E & E & \cdots & E & D \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

**Step 2:** Let \(N = M + mM' + M'' + m(v - 1)^2 + 1\). Let \(A\) be the array formed by vertically juxtaposing \(A, B, C,\) and \(D\). It remains to prove that \(A\) is an \(OCA(N; 3, m, 2, v)\).

Let \(B_i = [2i + 1, 2i + 2]\) be the blocks of the NRT poset \([m \cdot 2]\), for \(i = 0, ..., m - 1\). An anti-ideal of size 3 can be formed by choosing the maximal element of three distinct blocks or by choosing the two elements of one block and the maximal element of another block. Let \(x, y,\) and \(z\) be nonzero elements of the alphabet \([0, 1, ..., v - 1]\). We can represent the two types of anti-ideals of size 3 of the NRT poset \([m \cdot 2]\) by \(xylz\) and \(xyl\) (or \(xlyz\)). The patterns of the 3-tuples over \([0, 1, ..., v - 1]\) considering the two types of the anti-ideals of size 3 and whether 0 is a component of the 3-tuple are in Table 1, where Part I - IV are defined.

The patterns in Part I are covered by \(A\), the patterns in Part II are covered by \(B\), the pattern in Part III is covered by \(C\), and the patterns in Part IV are covered by \(D\). Therefore, the array \(A\) is an \(OCA(N; 3, m, s, v)\), where \(N = M + mM' + M'' + m(v - 1)^2 + 1\). 

We generalize in Proposition 5 the derived array bound:

\[
CAN(t - 1, n, v) \leq \frac{CAN(t, n + 1, v)}{v}
\]

established in [9]. Indeed, we construct an \(OCA(M; t, m, s, v)\) such that \(M \leq \left\lfloor \frac{N}{v} \right\rfloor\) by selecting rows of an \(OCA(N; t + 1, m + 1, s, v)\) and deleting \(s\) columns labeled by one block of the NRT poset \([(m + 1) \cdot s]\). We call this process a derivation on the number of blocks.

**Proposition 5.** \(OCAN(t, m, s, v) \leq \left\lfloor \frac{OCAN(t + 1, m + 1, s, v)}{v} \right\rfloor\).

**Table 1** Patterns of the 3-tuples for \(x, y, z \neq 0\).

| Part I   | xylz | xylz |
|----------|------|------|
| Part II  | xyl 0| xyl 0| x\(\neq 0\) |
| Part III | 0xly |      |
| Part IV  | x\(\neq 0\) | x\(\neq 0\) | 0x\(\neq 0\) | 0|0|0 | 0|0|0 |
Proof. Let $A$ be an OCA ($N; t + 1, m + 1, s, v$). For a fixed $\alpha \in \{0, 1, ..., v - 1\}$, let $A_{\alpha}$ be the array obtained by choosing the rows of $A$ such that each entry in the last column is equal to $\alpha$, and deleting the last $s$ columns. We choose $\alpha \in \{0, 1, ..., v - 1\}$ that occurs the least number of times in the last column of $A$, to ensure $A_{\alpha}$ has at most $\left\lceil \frac{N}{v} \right\rceil$ rows. We claim that the array $A_{\alpha}$ is an OCA ($M; t, m, s, v$) such that $M \leq \left\lceil \frac{N}{N} \right\rceil$. Indeed, let $J$ be an anti-ideal of the NRT poset $[m \cdot s]$ of size $t$. The set $J' = J \cup \{(m + 1)s\}$ is an anti-ideal of the NRT poset $[(m + 1) \cdot s]$ of size $t + 1$. The columns of $A$ labeled by $J'$ are covered. Now, looking at the rows such that the last entry is $\alpha$ in $A$, we have that the columns labeled by $J$ cover all the $t$-tuples over $\{0, 1, ..., v - 1\}$ at least once. Therefore, $A_{\alpha}$ is an OCA ($M; t, m, s, v$) such that $M \leq \left\lceil \frac{N}{v} \right\rceil$. □

In contrast to Proposition 5, we construct an OCA ($M; t, m, s, v$) such that $M \leq \left\lceil \frac{N}{v} \right\rceil$ by selecting rows of OCA ($N; t + 1, m, s + 1, v$) and by choosing one column in each block of the NRT poset $[m \cdot (s + 1)]$ to be deleted. We call this process a derivation on the size of the blocks, which is a specific property of ordered covering arrays.

**Proposition 6.** $OCAN(t, m, s, v) \leq \left\lceil OCAN(t + 1, m, s + 1, v) \right\rceil$.

**Proof.** Let $I = \bigcup_{i=0}^{m-2} \{i(s + 1) + 1\}$ be the ideal of the NRT poset $[m \cdot (s + 1)]$ formed by the $m - 1$ minimal elements of the first $m - 1$ blocks. Let $A$ be an OCA ($N; t + 1, m, s + 1, v$). For a fixed $\beta \in \{0, 1, ..., v - 1\}$, consider the array $A_{\beta}$ obtained by choosing the rows of $A$ such that each entry in the last column is equal to $\beta$, and deleting the $m - 1$ columns labeled by the minimal elements of the ideal $I$ as well as the last column (i.e., the one labeled by the maximal element $m(s + 1)$ of the last block). Then select $\beta \in \{0, 1, ..., v - 1\}$ that occurs the least number of times in the last column of $A$, to ensure $A_{\beta}$ has at most $\left\lceil \frac{N}{v} \right\rceil$ rows. We claim that the array $A_{\beta}$ is an OCA ($M; t, m, s, v$) such that $M \leq \left\lceil \frac{N}{v} \right\rceil$.

Let $P$ be the NRT poset $R[m \cdot s]$ given in Figure 2. Let $J$ be an anti-ideal of $P$ of size $t$. The set $J' = J \cup \{m(s + 1)\}$ is an anti-ideal of the NRT poset $[m \cdot (s + 1)]$ of size $t + 1$. The columns of $A$ labeled by $J'$ cover all the $(t + 1)$-tuples over $\{0, 1, ..., v - 1\}$ at least once. Now, looking at the rows where the last entry is $\beta$ in $A$, the columns labeled

![Figure 2](image-url)
by $J$ cover all the $t$-tuples over $\{0, 1, ..., v - 1\}$ at least once. Therefore, $A_\beta$ is an
$OCA(M; t, m, s, v)$ such that $M \leq \left\lfloor \frac{N}{v} \right\rfloor$.

Theorem 8 gives a construction for an $OCA(M; t, m + 1, s, v)$ obtained by adding a block of
$s$ columns to an $OCA(N; t, m, s, v)$ and additional rows, in the same spirit as some existing
constructions of covering arrays which add a column. However, in the case of ordered
covering arrays, this is a more complicated task. Before we prove Theorem 8, we need to
construct an array and give a technical lemma showing the properties of this new array.

**Construction 1.** Let $P$ be an NRT poset $\mathcal{R}[2 \cdot s]$ with blocks $B_1 = \{b_1, ..., b_3\}$ and
$B_2 = \{b_{s+1}, ..., b_{2s}\}$, where $b_1 \leq b_2 \leq \cdots \leq b_s$ and $b_{s+1} \leq b_{s+2} \cdots \leq b_{2s}$. Let $j \leq 2s$, and define
an array $T^j$ with $2s$ columns labeled by $b_1, ..., b_{2s}$ in this order, and with rows specified as
follows. Each row of $T^j$ is indexed by each tuple $x = (x_1, ..., x_j) \in \{0, ..., v - 1\}^j$ such that
$(x_1, ..., x_{\frac{j}{2}}) \neq (x_j, x_{j-1}, ..., x_{\frac{j+1}{2}})$, and row $T^j_x = [a_1, a_2, ..., a_s, a_{s+1}, a_{s+2}, ..., a_{2s}]$ where:

1. $a_{s-i+1} = x_i$, if $1 \leq i \leq \min\{j, s\}$;
2. $a_{2s-i+1} = x_{j-i+1}$, if $1 \leq i \leq \min\{j, s\}$; and
3. $a_{s-j+1}, a_{s-j+1}$ as well as $a_{s+1}, ..., a_{2s-min\{j, s\}}$ are set arbitrarily.

**Lemma 7.** The array $T^j$ given in Construction 1 has $(v^j - v^j)\frac{j}{2}$ rows. Moreover, for every
anti-ideal $J$ of $P$ with $|J| = j$, letting $j_1 = |B_1 \cap J|$ and $j_2 = |B_2 \cap J|$ and $t = \min\{j_1, j_2\}$, we
have that the subarray of $T^j$ labeled by $J$ contains every tuple $(y, z) = (y_1, ..., y_{j_1}, z_1, ..., z_{j_2})$
of $\{0, ..., v - 1\}^j$ such that $(y_{j_1-t+1}, y_{j_2-t+2}, ..., y_{j_2}) \neq (z_{j_1-t+1}, z_{j_2-t+2}, ..., z_{j_2})$ as a row.

**Proof.** A tuple $(z_1, ..., z_n)$ is palindromic if $(z_1, ..., z_n) = (z_n, ..., z_1)$. The array $T^j$ has one
row per $x = (x_1, ..., x_j) \in \{0, ..., v - 1\}^j$, such that row $T^j_x$ is not a palindromic tuple.
There are $v^j\frac{j}{2}$ palindromic tuples, so the total number of rows in $T^j$ is $(v^j - v^j)\frac{j}{2}$. An
anti-ideal $J$ is always formed by “right-justified” subsets of block elements. Then
$J = \{b_{s-\min\{j_1, j_2\}+1}, ..., b_s\} \cup \{b_{2s-\min\{j_1, j_2\}+1}, ..., b_{2s}\}$, where $j_1 + j_2 = |J|$. By construction, for row
$T^j_x$, we assign different elements from the set $\{x_1, x_2, ..., x_j\}$ to these positions. Thus
every $j$-tuple of the form $(y_1, ..., y_{j_1}, z_1, ..., z_{j_2})$ where $y = (y_1, ..., y_{j_1})$ is not a suffix of
$z = (z_1, ..., z_{j_2})$ nor $z$ is a suffix of $y$ appear in these positions in some row of $T^j$.

We illustrate the construction from Lemma 7 in the following example.

**Example 2.** Take $v = 3$, $j = 3$, and the NRT poset $\mathcal{R}[2 \cdot 5]$ in Lemma 7. We obtain array
$T^3$, where * denotes the positions of the array filled arbitrarily. The anti-ideals of cardinality
$j = 3$ for the set of columns $\{1, ..., 10\}$ of $T^3$ are $\{3, 4, 5\}, \{4, 5, 10\}, \{5, 9, 10\}$, and $\{8, 9, 10\}$.

Take $v = 2, j = 4$, and the NRT poset $\mathcal{R}[2 \cdot 2]$ in Lemma 7. We obtain array $T^4$. There
is only one anti-ideal of cardinality $j = 4$ corresponding to columns of $T^4$, namely,
$\{1, 2, 3, 4\}$. 


For each of these arrays, we can verify that the subarray labeled by each anti-ideal visits every tuple in the alphabet, except for those tuples where one of the two parts is a suffix of the other part. For example, the subarray labeled by columns $\{4, 5, 10\}$ of $T^3$ does not contain tuple $(011)$ as a row, since 1 is a suffix of 01, but contains tuple $(010)$, since 0 is not a suffix of 0 and 01 is not a suffix of 0.

We are now ready to prove Theorem 8 that builds an $OCA(M; t, m + 1, s, v)$ obtained from an $OCA(N; t, m, s, v)$ by adding a block of $s$ columns and additional rows using the Cartesian product.

**Theorem 8.** For $s \leq t$ and $k = \min\{2s, t\}$, $OCAN(t, m + 1, s, v)$ is less than or equal to

$$OCAN(t, m, s, v) + \sum_{j=2}^{k} OCAN(t - j, m - 1, s, v) \cdot (v^{j} - v^{j+1}).$$

**Proof:** Let $B_0, B_1, ..., B_{m-1}, B_m$ be the blocks of the NRT poset $[(m + 1) \cdot s]$. Let $A$ be an $OCA(N; t, m, s, v)$ over $\{0, 1, ..., v - 1\}$. Let us consider the array $A'$ formed by concatenating $A$ with the subarray of $A$ formed by the last $s$ columns of $A$. We add rows of other arrays below $A'$ as described next. Let $k = \min\{2s, t\}$. For each $j \in \{2, ..., k\}$, we build $C^j$ which is the Cartesian product of rows of an $OCA(N_j; t - j, m - 1, s, v)$ with rows of the array $T^j$ constructed in Lemma 7. We note that if $j = t$, then the entries of an $OCA(1; 0, m - 1, s, v)$ can be set arbitrarily. Each array $C^j$ has $N_j \cdot (v^j - v^{j+1})$ rows.

Let $\mathcal{A}$ be the array obtained by vertically juxtaposing $A', C^2, ..., C^k$, and $M = N + \sum_{j=2}^{k} N_j \cdot (v^j - v^{j+1})$. We claim that $\mathcal{A}$ is an $OCA(M; t, m + 1, s, v)$. Let $J$ be an anti-ideal of size $t$ of the NRT poset $[(m + 1) \cdot s]$. We divide the proof into three cases.
1. If $J \cap B_{m-1} = \emptyset$ and $J \cap B_m = \emptyset$, then the $t$-tuples over $\{0, 1, \ldots, v - 1\}$ are covered by the columns of $A$ labeled by $J$, which are present in $A'$.

2. If $J \cap B_{m-1} = \emptyset$ or $J \cap B_m = \emptyset$, then the $t$-tuples over $\{0, 1, \ldots, v - 1\}$ are covered by the columns of $A'$ labeled by $J$ since the last $s$ columns of $A'$ contain all tuples in $A$ corresponding to block $B_{m-1}$ which together with other blocks already satisfied coverage in $A$.

3. If $J \cap B_{m-1} \neq \emptyset$ and $J \cap B_m \neq \emptyset$, then $\bigcup (B_{m-1} \cup B_m) \equiv j$ for $j \in \{2, \ldots, k\}$, where $k = \min \{2s, t\}$. For each $j \in \{2, \ldots, k\}$, we show that the juxtaposition of arrays $A'$ and $C^j$ cover the $t$-tuples over $\{0, 1, \ldots, v - 1\}$ in the subarray corresponding to the columns $J$. We regard each $t$-tuple over $\{0, 1, \ldots, v - 1\}$ as $(x, y_1, y_2)$, where $x$ is a $(t - j)$-tuple and $(y_1, y_2)$ is a $j$-tuple, with a component of $x$ corresponding to $J \setminus (B_{m-1} \cup B_m)$, the components of $y_1$ corresponding to $J \cap B_{m-1}$ and the components of $y_2$ corresponding to $J \cap B_m$. If $y_1$ is a suffix of $y_2$, since $(x, y_2)$ is a tuple covered in $A$ for columns corresponding to $J \setminus (B_{m-1} \cup B_m)$ and $B_{m-1}$, by the construction of $A'$ we have that $(x, y_1, y_2)$ is covered in $A'$ for columns corresponding to $J \setminus (B_{m-1} \cup B_m)$, $B_{m-1}$ and $B_m$. Similarly, if $y_2$ is a suffix of $y_1$, we get coverage in $A'$. On the contrary, if $Y_1$ and $Y_2$ are suffixes of the largest possible common size for $y_1$ and $y_2$ with the property that $Y_1 \neq Y_2$, then $(y_1, y_2)$ must be covered in $T^j$. Since $x$ is covered in columns corresponding to $J \setminus (B_{m-1} \cup B_m)$ of the OCA($N_j; t - j, m - 1, s, v$), then $(x, y_1, y_2)$ is covered in columns of $C^j$, which is the Cartesian product of an OCA($N_j; t - j, m - 1, s, v$) with $T^j$.

Therefore, $A$ is an OCA($M; t, m + 1, s, v$), where $M = N + \sum_{j=2}^{s} N_j \cdot (v^j - v^{j-1})$. \(\square\)

In the next example, we illustrate the construction given by Theorem 8.

**Example 3.** For $t = s = 3$, $m = 3$, and $v = 2$,

$$\text{OCAN}(3, 4, 3, 2) \leq \text{OCAN}(3, 3, 3, 2) + \text{OCAN}(1, 2, 3, 2) \cdot 2 + \text{OCAN}(0, 2, 3, 2) \cdot 4.$$ 

The OCA($16; 3, 4, 3, 2$), $A$, constructed in Theorem 8 is shown in (5). We explain each part of the array $A$. Let $B_0 = \{1, 2, 3\}$, $B_1 = \{4, 5, 6\}$, $B_2 = \{7, 8, 9\}$, $B_3 = \{10, 11, 12\}$ be the blocks of the NRT poset $[4 \cdot 3]$. By Equation (1), OCA($3, 3, 3, 2$) = 8 and its corresponding array is given by Part I in the columns labeled by the blocks $B_0$, $B_1$, and $B_2$. In Part I, we have the array $A'$, and the columns labeled by the blocks $B_2$ and $B_3$ are repeated in $A'$. For $j \in \{2, 3\}$, we build $T^j$ as in Lemma 7. In Parts II and III, we have the array $C_2$ which is the Cartesian product of an OCA($2; 1, 2, 3, 2$) = $\begin{pmatrix} \text{000} & \text{000} \\ \text{111} & \text{111} \end{pmatrix}$ with $T^2 = \begin{pmatrix} *10 & *01 \\ *01 & *10 \end{pmatrix}$. In Part IV, we have the array $C_3$. The symbol * in columns labeled by $B_0$ and $B_1$ can be set arbitrarily since an OCA($N; 0, 2, 3, 2$) does not need to have any special property. Finally, the columns labeled by $B_2$ and $B_3$ in Part IV give the array $T^3$. 

As discussed before, for \( v \) a prime power and \( t \geq 3 \), Castoldi et al. [8] show a construction for an \( OOA(v^t; t, v + 1, t, v) \) (see Theorem 3), which yields Equation (1). Using this result and Corollary 2 to create ingredients for Theorem 8, we establish an upper bound for \( OCAN(t, v + 2, t, v) \).

**Corollary 9.** Let \( v \) be a prime power.

1. If \( t \) is odd, then \( OCAN(t, v + 2, t, v) \leq v^t \left( t - \frac{2}{1 - v} \left( \frac{1}{v^{(t-1)/2}} - 1 \right) \right) \).
2. If \( t \) is even, then \( OCAN(t, v + 2, t, v) \leq v^t \left( t - \frac{2}{1 - v} \left( \frac{1}{v^{(t-2)/2}} - \frac{1}{v^{t/2}} \right) \right) \).

**Proof.** For \( v \) a prime power and \( t \geq 3 \), \( OCAN(t, v + 1, t, v) = v^t \) by Equation (1). For \( j \in \{2, ..., t - 3\} \) and since \( t - j \leq t \), Proposition 2 item (2) yields \( OCAN(t - j, v, t, v) = OCAN(t - j, v, t - j, v) \). By Equation (1), we obtain \( OCAN(t - j, v, t - j, v) = v^{t-j} \). For \( j = t - 2 \), Corollary 2 implies that \( OCAN(2, v, t, v) = OCAN(2, v, 2, v) = CAN(2, v, v) \). Bush’s construction [17, Theorem 3.1] gives \( CAN(2, v, v) = v^2 \). For \( j = t - 1 \), \( OCAN(1, v, t, v) = v \), and for \( j = t \), \( OCAN(0, v, t, v) = 1 \). In summary, \( OCAN(t - j, v, t, v) = v^{t-j} \), for all \( j \in \{2, ..., t\} \). Using Theorem 8, we obtain

\[
OCAN(t, v + 2, t, v) \leq v^t + \sum_{j=2}^{t} v^{t-j} \left( v^j - v^{\frac{j}{2}} \right)
\]

\[
= tv^t - \sum_{j=2}^{t} v^{t-j} \frac{j}{2}
\]

\[
= v^t \left( t - \sum_{j=2}^{t} \frac{1}{v^{\frac{j}{2}}} \right).
\]
We now consider two cases.

1. If \( t \) is odd, then
   \[
   \sum_{j=2}^{t} \frac{1}{v^{j}} = 2 \cdot \sum_{j=1}^{(t-1)/2} \frac{1}{v^{j}} = \frac{2}{1 - v} \left( \frac{1}{v^{(t-1)/2}} - 1 \right).
   \]
   Therefore \( OCAN(t, v + 2, t, v) \leq v^{t} \left( t - \frac{2}{1 - v} \left( \frac{1}{v^{(t-1)/2}} - 1 \right) \right) \).

2. If \( t \) is even, then
   \[
   \sum_{j=2}^{t} \frac{1}{v^{j}} = \sum_{j=2}^{t-1} \frac{1}{v^{j}} + \frac{1}{v^{t/2}}.
   \]
   Since \( t - 1 \) is odd,
   \[
   \sum_{j=2}^{t-1} \frac{1}{v^{j}} = \frac{2}{1 - v} \left( \frac{1}{v^{(t-2)/2}} - 1 \right).
   \]
   Therefore \( OCAN(t, v + 2, t, v) \leq v^{t} \left( t - \frac{2}{1 - v} \left( \frac{1}{v^{(t-2)/2}} - 1 \right) - \frac{1}{v^{t/2}} \right) \). \qed

4 | NEW UPPER BOUNDS ON COVERING CODES IN NRT SPACES

In this section, we improve the general upper bound on \( K_{s}^{R}(m, s, R) \) given in Proposition 1 for suitable values of \( m \) and \( R \) by constructing new covering codes. For this purpose, we start from the covering code used to prove Proposition 1 and our strategy consists of modifying some of its codewords to reduce the size of the covering code. At the end of this section, a table compares the upper bounds obtained in this section with those in the tables in [5].

To facilitate the readability of the arguments, we represent a vector \( x = (x_1, \ldots, x_{ms}) \in \mathbb{Z}_{q}^{ms} \) as a matrix:

\[
\begin{bmatrix}
x_1 & x_2 & \cdots & x_{ms} \\
| & \vdots & \ddots & \vdots \\
x_{k-1} & x_{k+2} & \cdots & x_{(m-1)s+2} \\
x_{k} & x_{k+1} & \cdots & x_{(m-1)s+1}
\end{bmatrix}
\]

4.1 | New covering codes for \( m \) odd

Let \( k \) be a positive integer. We first focus on the case where \( m \geq 3 \) is odd, \( m = 2k + 1 \), and the radius is \( (k + 1)s - j \), for \( j = 1, \ldots, s \).
Theorem 10. Let $q$, $s$, and $k$ be positive integers such that $q \geq 2$, $s \geq 3$, and $k \geq 1$. For $j = 1, \ldots, s$, $K^R_q(2k + 1, s, (k + 1)s - j) \leq q^{ks+j} - q^{k(s-2)+j}(q^k - 1)$.

Proof. The general upper bound for $K^R_q(2k + 1, s, (k + 1)s - j)$ is $q^{ks+j}$, and a $(s-1)$-covering $C$ of the NRT space $Z_q^{(2k+1)s}$ of size $q^{ks+j}$ is formed by the codewords:

$$c = \begin{bmatrix}
0 & \cdots & 0 & * \\
0 & \cdots & 0 & * \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & *
\end{bmatrix},$$

where the column $*$ uses the $s$-tuple $(0, \ldots, 0, (j-1), \ldots, (k+1)s)$, filled from bottom to top, according to the proof of Proposition 1. We divide the proof into three steps.

Step 1: We partition the set $C$ into $q^{ks+j}$ parts indexed by the set $Z_q^{k(s-2)+j}$. For each $\mathbf{z} = (z_0, z_1, \ldots, z_{k(s-2)}) \in Z_q^{k(s-2)+j}$, where $z_0$ is a $j$-tuple over $Z_q$, let $C_\mathbf{z}$ be the subset of $C$ formed by the codewords:

$$c = \begin{bmatrix}
0 & \cdots & 0 & \mathbf{z}_{s-2} & \cdots & \mathbf{z}_{k(s-2)} \\
0 & \cdots & 0 & \mathbf{z}_{s-3} & \cdots & \mathbf{z}_{k(s-2)-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \mathbf{z}_3 & \cdots & \mathbf{z}_{(k-1)(s-2)+1} \\
0 & \cdots & 0 & \mathbf{c}_{(k+1)s+2} & \cdots & \mathbf{c}_{2ks+2} \\
0 & \cdots & 0 & \mathbf{c}_{(k+1)s+1} & \cdots & \mathbf{c}_{2ks+1}
\end{bmatrix},$$

where the column $*$ uses the $s$-tuple $(0, \ldots, 0, \mathbf{z}_0)$, filled from bottom to top. For each $\mathbf{z} \in Z_q^{k(s-2)+j}$, write

$$Z_\mathbf{z} = Z_q^{ks} \times (Z_q^{s-j} \times \mathbf{z}_0) \times \prod_{i=0}^{k-1} (Z_q^2 \times \{z_{i(s-2)+1}, \ldots, z_{i(s-2)+s-2}\}).$$

The following properties hold:

(a) $C_\mathbf{z} \cap C_{\mathbf{z}'} = \emptyset$ if and only if $\mathbf{z} \neq \mathbf{z}'$;
(b) $|C_\mathbf{z}| = q^{ks}$ for all $\mathbf{z} \in Z_q^{k(s-2)+j}$;
(c) $C = \cup_{\mathbf{z} \in Z_q^{k(s-2)+j}} C_\mathbf{z}$;
(d) $C_\mathbf{z}$ is a $((k + 1)s - j)$-covering of the NRT space $Z_\mathbf{z}$ over the NRT poset $[(2k + 1) \cdot s]$.

We observe that (a), (b), and (c) tell us the set $\{C_\mathbf{z} : \mathbf{z} \in Z_q^{k(s-2)+j}\}$ is a partition of $C$.

Step 2: For each $\mathbf{z} \in Z_q^{k(s-2)+j}$, we construct a new set $A_\mathbf{z}$ from $C_\mathbf{z}$ such that $A_\mathbf{z}$ is a $((k + 1)s - j)$-covering of the NRT space $Z_\mathbf{z}$. For each $c \in C_\mathbf{z}$ such that $(c_{(k+1)s+1}, c_{(k+2)s+1}, \ldots, c_{2ks+1}) = (c_{(k+1)s+2}, c_{(k+2)s+2}, \ldots, c_{2ks+2}) = (0, \ldots, 0)$ or $(c_{(k+1)s+2}, c_{(k+2)s+2}, \ldots, c_{2ks+2}) \neq (0, 0, \ldots, 0)$, define
where the column \( \ast \) uses the \( s \)-tuple \((0, \ldots, 0, z_0)\), filled from bottom to top.

Let \( A_z \) be the set of codewords \( c' \) defined above. The types of codewords in \( C_z \) that we are not using to define \( A_z \) are those such that \( c(0, \ldots, 0, z_0) \neq (0, 0, \ldots, 0) \) and \( (c_{k+1}s+2, c_{k+2}s+2, \ldots, c_{2ks+2}) = (0, 0, \ldots, 0) \). There are \( q^k - 1 \) such codewords and \( A_z \) has size \( q^k - (q^k - 1) \).

**Step 3:** It remains to show that the set \( A_z \) is a \((k + 1)s - j\)-covering of the NRT space \( \mathcal{Z}_z \). The proof is divided into three cases.

Indeed, for \( x \in \mathcal{Z}_z \), we know that \( x \) and \( c' \in A_z \) coincide in those \( k(s - 2) + j \) positions that are equal to \( z \). We highlight in bold the \( 2k \) positions in each codeword that coincide with the respective positions in \( x \).

1. If \( (x_{k+1}s+2, x_{k+2}s+2, \ldots, x_{2ks+2}) \neq (0, 0, \ldots, 0) \), then \( x \) is covered by the following element of \( A_z \):

\[
\begin{bmatrix}
c\ast_{k+1}s+2 & \cdots & c_{2ks+2} & Z_{s-2} & \cdots & Z_{k(s-2)} \\
c_{k+1}s+1 & \cdots & c_{2ks+1} & Z_{s-3} & \cdots & Z_{k(s-2)-1} \\
0 & \cdots & 0 & Z_{s-4} & \cdots & Z_{k(s-2)-2} \\
0 & \cdots & 0 & c_{k+1}s+2 & \cdots & c_{2ks+2} \\
0 & \cdots & 0 & c_{k+1}s+1 & \cdots & c_{2ks+1}
\end{bmatrix}
\]

2. If \( (x_s, x_{2s}, \ldots, x_{ks}) \neq (0, 0, \ldots, 0) \), then \( x \) is covered by the following element of \( A_z \):

\[
\begin{bmatrix}
x_s & \cdots & x_{ks} & Z_{s-2} & \cdots & Z_{k(s-2)} \\
x_{s-1} & \cdots & x_{ks-1} & Z_{s-3} & \cdots & Z_{k(s-2)-1} \\
0 & \cdots & 0 & Z_{s-4} & \cdots & Z_{k(s-2)-2} \\
0 & \cdots & 0 & x_{s} & \cdots & x_{ks} \\
0 & \cdots & 0 & x_{s-1} & \cdots & x_{ks-1}
\end{bmatrix}
\]

3. If \( (x_{k+1}s+2, x_{k+2}s+2, \ldots, x_{2ks+2}) = (x_s, x_{2s}, \ldots, x_{ks}) = (0, 0, \ldots, 0) \), then \( x \) is covered by the following element of \( A_z \):

\[
\begin{bmatrix}
x_{k+1}s+2 & \cdots & x_{2ks+2} & Z_{s-2} & \cdots & Z_{k(s-2)} \\
x_{k+1}s+1 & \cdots & x_{2ks+1} & Z_{s-3} & \cdots & Z_{k(s-2)-1} \\
0 & \cdots & 0 & Z_{s-4} & \cdots & Z_{k(s-2)-2} \\
0 & \cdots & 0 & x_{s} & \cdots & x_{ks} \\
0 & \cdots & 0 & x_{s-1} & \cdots & x_{ks-1}
\end{bmatrix}
\]
Therefore, the set \( A = \bigcup_{z \in \mathbb{Z}_q^{(1-s_2)s} A_2} \) of size \( q^{k_s+1} - q^{k(s-2)+j}(q^k - 1) \) is a \(((k + 1)s - j)\)-covering of the NRT space \( \mathbb{Z}_q^{(2k+1)s} \).

In addition to Theorem 10 being a refinement of the bound given in Proposition 1, for \( k = j = 1 \), Theorem 10 improves the bound \( q^{s+1} - q \) given in [7, Theorem 4] item (2).

**Corollary 11.** For \( q \geq 2 \) and \( s \geq 3 \), \( K_q^R(3, s, 2s - 1) \leq q^{s+1} - q^{s-1}(q - 1) \).

For \( j = s \), we have the following particular case of Theorem 10 which also improves upper bounds given in [5], see Table 2.

**Corollary 12.** For \( q \geq 2 \) and \( s \geq 3 \), \( K_q^R(2k + 1, s, ks) \leq q^{(k+1)s} - q^{k(s-2)+s}(q^k - 1) \).

### 4.2 New covering codes for \( m \) even

Next, we show Theorem 13 which is a parallel result to Theorem 10 for \( m \) even, and generalizes [7, Theorem 4] item (1). The proof of Theorem 13 follows the same method applied to prove Theorem 10.

**Theorem 13.** For \( q \geq 2 \), \( s \geq 3 \) and \( k \geq 1 \), \( K_q^R(2k, s, ks) \leq q^{ks} - q^{k(s-2)}(q^k - 1) \).

**Proof.** The general upper bound for \( K_q^R(2k, s, ks) \) is \( q^{ks} \), and a \( ks \)-covering \( C \) of the NRT space \( \mathbb{Z}_q^{2ks} \) of size \( q^{ks} \) is formed by the codewords:

\[
\begin{bmatrix}
0 & \cdots & 0 & c_{(k+1)s} & \cdots & c_{2ks} \\
0 & \cdots & 0 & c_{ks+2} & \cdots & c_{(2k-1)s+2} \\
0 & \cdots & 0 & c_{ks+1} & \cdots & c_{(2k-1)s+1}
\end{bmatrix}
\]

according to the proof of Proposition 1. We divide the proof into three steps.
Step 1: We partition the set $C$ into $q^{k(s-2)}$ parts indexed by the set $\mathbb{Z}_q^{k(s-2)}$. For each $z = (z_0, ..., z_k(s-2)) \in \mathbb{Z}_q^{k(s-2)}$, let $C_z$ be the subset of $C$ formed by the codewords:

$$c = \begin{bmatrix}
0 & \cdots & 0 & z_{k-2} & \cdots & z_{k(s-2)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & z_{1} & \cdots & z_{(k-1)(s-2)+1} \\
0 & \cdots & 0 & c_{ks+2} & \cdots & c_{(2k-1)s+2} \\
0 & \cdots & 0 & c_{ks+1} & \cdots & c_{(2k-1)s+1}
\end{bmatrix}.$$

For each $z \in \mathbb{Z}_q^{k(s-2)}$, define

$$Z_z = \mathbb{Z}_q^{ks} \times \left( \mathbb{Z}_q^{2} \times \{(z_0, ..., z_{s-2})\}\right) \times \cdots \times \left( \mathbb{Z}_q^{2} \times \{(z_{(k-1)(s-2)+1}, ..., z_{k(s-2)})\}\right).$$

The following properties hold:

(a) $C_z \cap C_{z'} = \emptyset$ if and only if $z \neq z'$;
(b) $|C_z| = q^{2k}$ for all $z \in \mathbb{Z}_q^{k(s-2)}$;
(c) $C = \bigcup_{z \in \mathbb{Z}_q^{k(s-2)}} C_z$;
(d) $C_z$ is a $ks$-covering of the NRT space $Z_z$ over the NRT poset $[2k \cdot s]$.

We note that (a), (b), and (c) tell us the set $\{C_z : z \in \mathbb{Z}_q^{k(s-2)}\}$ is a partition of $C$.

Step 2: For each $z \in \mathbb{Z}_q^{k(s-2)}$, we construct a new set $A_z$ from $C_z$ such that $A_z$ is a $ks$-covering of the NRT space $Z_z$. For each $c \in C_z$ such that $(c_{ks+2}, c_{(k+1)s+2}, \ldots, c_{(2k-1)s+2}) \neq (0, 0, \ldots, 0)$ or $(c_{ks+1}, c_{(k+1)s+1}, \ldots, c_{(2k-1)s+1}) = (c_{ks+2}, c_{(k+1)s+2}, \ldots, c_{(2k-1)s+2}) = (0, 0, \ldots, 0)$, define

$$c' = \begin{bmatrix}
c_{ks+2} & \cdots & c_{(2k-1)s+2} & z_{s-2} & \cdots & z_{k(s-2)} \\
c_{ks+1} & \cdots & c_{(2k-1)s+1} & z_{s-3} & \cdots & z_{k(s-2)-1} \\
0 & \cdots & 0 & z_{s-4} & \cdots & z_{k(s-2)-2} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & z_{1} & \cdots & z_{(k-1)(s-2)+1} \\
0 & \cdots & 0 & c_{ks+2} & \cdots & c_{(2k-1)s+2} \\
0 & \cdots & 0 & c_{ks+1} & \cdots & c_{(2k-1)s+1}
\end{bmatrix}.$$

Let $A_z$ be the set of codewords $c'$ defined above. The types of codewords in $C_z$ that are not using to define $A_z$ are those such that $(c_{ks+1}, c_{(k+1)s+1}, \ldots, c_{(2k-1)s+1}) \neq (0, 0, \ldots, 0)$ and $(c_{ks+2}, c_{(k+1)s+2}, \ldots, c_{(2k-1)s+2}) = (0, 0, \ldots, 0)$, and there are $q^k - 1$ such codewords. Hence, $A_z$ has size $q^{2k} - (q^k - 1)$.

Step 3: We now show that the set $A_z$ is a $ks$-covering of the NRT space $Z_z$. We divide the proof into three cases. Indeed, for $x \in Z_z$, we know that $x$ and $c' \in A_z$ coincide in those $k(s-2)$ positions that are equal to $z$. We highlight in bold the $2k$ positions in each codeword that coincide with the respective positions in $x$.

1. If $(x_{ks+2}, x_{(k+1)s+2}, \ldots, x_{(2k-1)s+2}) \neq (0, 0, \ldots, 0)$, then $x$ is covered by the following element of $A_z$:
2. If \((x_s, x_{2s},...,x_{ks}) \neq (0, 0, ..., 0)\), then \(x\) is covered by the following element of \(A_z\):

\[
\begin{bmatrix}
  x_s & x_s & z_k(s-2) & z_k(s-2) \\
  x_{s-1} & x_{ks-1} & z_k(s-2) & z_k(s-2) \\
  0 & 0 & z_k(s-2) & z_k(s-2) \\
  \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & z_k(s-2) & z_k(s-2) \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

3. If \((x_{ks+2}, x_{(k+1)s+2},...,x_{(2k-1)s+2}) = (x_s, x_{2s},...,x_{ks}) = (0, 0, ..., 0)\), then \(x\) is covered by the following element of \(A_z\):

\[
\begin{bmatrix}
  0 & 0 & 0 & z_k(s-2) \\
  0 & 0 & 0 & z_k(s-2) \\
  \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Therefore, the set \(A = \bigcup_{z \in Z_q^{(k-2)}} A_z\) of size \(q^{ks} - q^{k(s-2)}(q^k - 1)\) is a \(ks\)-covering of the NRT space \(Z_q^{2ks}\). \(\square\)

The construction above is illustrated in the following example.

Example 4. Proposition 1 gives the upper bound \(K^R_2(2, 3, 3) \leq 8\). A 3-covering code of the NRT space \(Z_2^6\) (NRT poset [2 \cdot 3]) that gives the general upper bound for \(K^R_2(2, 3, 3)\) is \(C = C_0 \cup C_1\), where

\[
C_0 = \begin{bmatrix}
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
  0 & 1 & 0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}.
\]
Following the construction in Theorem 13, we delete the third codeword of $C_0$ and $C_1$, and using the remaining codewords, we construct $A_0$ and $A_1$, respectively. Let $A = A_0 \cup A_1$, where

$$A_0 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right\} \quad \text{and} \quad A_1 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right\}.$$ 

Therefore, $A$ is a 3-covering code of the NRT space $\mathbb{Z}_2^6$ with 6 codewords improving the previous upper bound 8. We conjecture that $K^{R_2}(2, 3, 3) = 6$.

For $k = 1$, Theorem 13 improves the bound $q^s - q^{s-2}$ given in [7, Theorem 4] item (1).

**Corollary 14.** For $q \geq 2$ and $s \geq 3$, $K^{R_q}(2, s, s) \leq q^s - q^{s-2}(q - 1)$.

As a consequence of Theorem 13 and [5, Proposition 17] we obtain Theorem 10 for $j = 0$, as shown in the next corollary. This result improves upper bounds given in [5] as displayed in Table 2.

**Corollary 15.** For $q \geq 2$ and $s \geq 3$, $K^{R_q}(2 + 1, s, (k + 1)s) \leq q^k s - q^k (s-2)(q^k - 1)$.

### 4.3 A table comparing new and old upper bounds

We finish this section by giving a table of upper bounds. We compare the upper bounds on covering codes in NRT spaces obtained in Sections 4.1 and 4.2 with the upper bounds in the tables given in [5]. The column “Result” gives the result in this section used to get the respective bound in the column “New bounds.” Table 2 shows that sometimes the results of this section improve upper bounds from [5]. These improvements are marked in bold.

### 5 CONSTRUCTIONS OF COVERING CODES USING ORDERED COVERING ARRAYS

In this section, ordered covering arrays are used to construct covering codes in NRT spaces improving upper bounds on their size for larger alphabets. Theorem 16 is a generalization of a result already shown for covering codes in Hamming spaces connected with surjective matrices and Maximum Distance Separable (MDS) codes [10], and its proof has already appeared in the conference paper [7].

MDS codes provide a useful tool to construct covering codes in Hamming spaces, see [2, 4, 10] for instance. MDS codes can be applied in NRT spaces as follows. Suppose that there is an MDS code in the NRT space $\mathbb{Z}_v^{ms}$ with minimum distance $d + 1$. For every $q \geq 2$, $K^{R_q}(m, s, d) \leq v^{ms-d}K^{R_q}(m, s, d)$ [5, Theorem 30]. An extension of this result based on ordered covering arrays is described below. Indeed, this is analogous to an equivalent result relating to covering codes and covering arrays for the Hamming metric [10, Theorem 3.7.10].
Theorem 16 (Castoldi et al. [7, Theorem 3]). Let $v, q, m, s, R$ be positive integers such that $0 < R < ms$. Then

$$K_{vq}^R(m, s, R) \leq \text{OCAN} (ms - R, m, s, v) K_{q}^R (m, s, R).$$

Theorem 16 is very flexible in the sense that if we know upper bounds for $\text{OCAN} (ms - R, m, s, v)$ or $K_{q}^R (m, s, R)$, we might improve upper bounds for covering codes in NRT spaces for larger alphabets. Indeed, let us illustrate how the previous results on OCAs and covering codes can be applied in connection with Theorem 16.

Let $m$ be a positive integer such that $(t - 1)q + 1 \leq m \leq (t - 1)vq$. Since $K_{q}^R (m, s, ms - t) = q$ (see [5, Theorem 13]), Theorem 16 implies

$$K_{vq}^R (m, s, ms - t) \leq \text{OCAN} (t, m, s, v) \cdot q. \quad (5)$$

In particular, Equation (5) when $v = 2$ and $t = 3$ yields $K_{2q}^R (2m, s, 2ms - 3) \leq \text{OCAN} (3, 2m, s, 2) \cdot q$. For $s = 2$ or $s = 3$, Krikorian [20, Theorem 4.2.3] shows that

$$\text{OCAN} (3, 2m, s, 2) \leq \text{OCAN} (3, m, s, 2) + \text{OCAN} (2, m, 2, 2).$$

By Equation (2), $\text{OCAN} (2, m, 2, 2) = \text{CAN} (2, m, 2)$. The combination of the facts above produces

$$K_{2q}^R (2m, s, 2ms - 3) \leq (\text{OCAN} (3, m, s, 2) + \text{CAN} (2, m, 2)) \cdot q.$$
The discussion above is based on Corollaries 5 and 6 in [7]. Further applications of Theorem 16 are available in [7].

Here we continue to explore new relationships in the same spirit as mentioned above. As the main goal, the results from Section 4 combined with Theorem 16 might improve upper bounds for covering codes in NRT spaces.

**Corollary 17.** Let \( v \geq 2, q \geq 2, s \geq 3, \) and \( k \geq 1 \) be positive integers.

1. For \( j = 0, \ldots, s, \) \( K^{R}_{vq}(2k + 1, s, (k + 1)s - j) \leq \text{OCAN}(ks + j, 2k + 1, s, v)[q^{ks+j} - q^{k(s-2+j)}(q^k - 1)]. \)
2. \( K^{R}_{vq}(2k, s, ks) \leq \text{OCAN}(ks, 2k, s, v) \cdot [q^{ks} - q^{k(s-2)}(q^k - 1)]. \)

**Proof.** Item (1) follows as a straightforward combination of Theorems 16 and 10. Analogously, item (2) is derived from Theorems 16 and 13.

For \( k = 1 \) and a prime power \( v, \) two consequences of Corollary 17 are described in the following result.

**Corollary 18.** Let \( v \) be a prime power, \( q \geq 2, \) and \( s \geq 3. \) The following bounds hold:

1. \( K^{R}_{vq}(3, s, 2s - 1) \leq v^{s+1} \cdot [q^{s+1} - q^{s-1}(q - 1)]. \)
2. \( K^{R}_{vq}(2, s, s) \leq v^{s} \cdot [q^{s} - q^{s-2}(q - 1)]. \)

**Proof.** (1) For \( v \) a prime power, \( \text{OCAN}(s + 1, v + 1, s + 1, v) = v^{s+1} \) by Equation (1). Proposition 2 item (3) implies that \( \text{OCAN}(s + 1, 3, s + 1, v) = v^{s+1}. \) This value combined with Corollary 17 item (1) (when \( k = 1 \) and \( j = 1 \)) derives the desired upper bound.

(2) Since \( \text{OCAN}(s, v + 1, s, v) = v^{s} \) holds by Equation (1), Proposition 2 item (3) implies that \( \text{OCAN}(s, 2, s, v) = v^{s}. \) An application of Corollary 17 item (2) (when \( k = 1 \)) concludes the upper bound.

We now discuss the upper bounds on \( K^{R}_{vq}(3, s, 2s - 1) \) for a prime power \( v. \) Corollary 11 gives the upper bound \( (vq)^{s+1} - (vq)^{s-1}(vq - 1). \) On the other hand, Corollary 18 item (1) yields the upper bound \( v^{s+1} \cdot [q^{s+1} - q^{s-1}(q - 1)] = (vq)^{s+1} - (vq)^{s-1}v^{2}(q - 1). \) A closer look reveals that \( v^{2}(q - 1) > vq - 1 \) if and only if \( q > \frac{v^2}{v+1}. \) Since \( 1 < \frac{v^2}{v+1} < 2 \leq q, \) the inequality \( v^{2}(q - 1) > vq - 1 \) holds. Therefore, the upper bound from Corollary 18 item (1) improves that in Corollary 11.

Similarly, let us analyze the upper bounds on \( K^{R}_{vq}(2, s, s) \) for \( v \) a prime power. Corollary 14 yields the upper bound \( (vq)^{s} - (vq)^{s-2}(vq - 1). \) In contrast, Corollary 18 item (2) implies the upper bound \( v^{s} \cdot [q^{s} - q^{s-2}(q - 1)] = (vq)^{s} - (vq)^{s-2}v^{2}(q - 1). \) By comparing \( vq - 1 \) and \( v^{2}(q - 1) \) as we did before, the upper bound from Corollary 18 item (2) improves that from Corollary 14.

**Example 5.** We display in Table 3 some particular instances showing how much improvement the upper bounds obtained by Corollary 18 offer when comparing with Corollaries 11 and 14 of Section 4. These upper bounds are, as far as we know, new; there are no tables in the literature for such large alphabets. Given the parameters \( v \) a prime power, \( q \geq 2, m \in \{2, 3\}, \) and \( s \geq 3, \) the penultimate column in Table 3 presents the upper bounds on \( K^{R}_{vq}(m, s, R) \) obtained by Corollary 11 or Corollary 14. The last column in Table 3 indicates the upper bounds derived by Corollary 18 item (1) or item (2).
| $v$ | $q$ | $v \cdot q$ | $m = 3$ | $s$ | $R = 2s - 1$ | Corollary 11 | Corollary 18 (1) |
|-----|-----|-------------|--------|-----|-------------|------------|----------------|
| 2   | 2   | 4           | 3      | 3   | 5           | 208        | 192           |
| 2   | 2   | 4           | 3      | 4   | 7           | 832        | 768           |
| 2   | 2   | 4           | 3      | 5   | 9           | 3328       | 3072          |
| 2   | 3   | 6           | 3      | 3   | 5           | 1116       | 1008          |
| 2   | 3   | 6           | 3      | 4   | 7           | 6696       | 6048          |
| 2   | 3   | 6           | 3      | 5   | 9           | 40,176     | 36,288        |
| 3   | 3   | 9           | 3      | 3   | 5           | 5913       | 5103          |
| 3   | 3   | 9           | 3      | 4   | 7           | 53,217     | 45,927        |
| 3   | 3   | 9           | 3      | 5   | 9           | 478,953    | 413,343       |
| 4   | 2   | 8           | 3      | 3   | 5           | 3648       | 3072          |
| 4   | 2   | 8           | 3      | 4   | 7           | 29,184     | 24,576        |
| 4   | 2   | 8           | 3      | 5   | 9           | 233,472    | 196,608       |
| 4   | 3   | 12          | 3      | 3   | 5           | 19,152     | 16,128        |
| 4   | 3   | 12          | 3      | 4   | 7           | 229,824    | 193,536       |
| 4   | 3   | 12          | 3      | 5   | 9           | 2,757,888  | 2,322,432     |

| $v$ | $q$ | $v \cdot q$ | $m = 2$ | $s$ | $R = s$ | Corollary 14 | Corollary 18 (2) |
|-----|-----|-------------|--------|-----|--------|------------|----------------|
| 2   | 2   | 4           | 2      | 3   | 3      | 52         | 48            |
| 2   | 2   | 4           | 2      | 4   | 4      | 208        | 192           |
| 2   | 2   | 4           | 2      | 5   | 5      | 832        | 768           |
| 2   | 3   | 6           | 2      | 3   | 3      | 186        | 168           |
| 2   | 3   | 6           | 2      | 4   | 4      | 1116       | 1008          |
| 2   | 3   | 6           | 2      | 5   | 5      | 6696       | 6048          |
| 3   | 3   | 9           | 2      | 3   | 3      | 657        | 567           |
| 3   | 3   | 9           | 2      | 4   | 4      | 5913       | 5103          |
| 3   | 3   | 9           | 2      | 5   | 5      | 53,217     | 45,927        |
| 4   | 2   | 8           | 2      | 3   | 3      | 456        | 384           |
| 4   | 2   | 8           | 2      | 4   | 4      | 3648       | 3072          |
| 4   | 2   | 8           | 2      | 5   | 5      | 29,184     | 24,576        |
| 4   | 3   | 12          | 2      | 3   | 3      | 1596       | 1344          |
| 4   | 3   | 12          | 2      | 4   | 4      | 19,152     | 16,128        |
| 4   | 3   | 12          | 2      | 5   | 5      | 229,824    | 193,536       |
6 CONCLUSION AND FURTHER WORK

In this work, we obtain new recursive relations and upper bounds for two different but related combinatorial objects: ordered covering arrays (Section 3) and covering codes in NRT spaces (Section 4). In Section 5, we explore new connections between these two objects and improve several upper bounds for covering codes in NRT spaces as exemplified in Tables 2 and 3.

We believe that more results for ordered covering arrays can be explored with different approaches that build on the connections established here. In addition, results for covering codes can be investigated for any finite poset with analogous connections to appropriate variable strength covering arrays. It is possible that known results on variable strength covering arrays [27, 28] can be used to improve upper bounds in covering codes for other poset metrics. There is a lot to explore in this area as much less is known for covering codes than for minimum distance codes under the poset metric [16].

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REFERENCES

1. A. Barg and P. Purkayastha, Bounds on ordered codes and orthogonal arrays, Moscow Math. J. 9 (2009), 211–243.
2. A. Blokhuis and C. W. H. Lam, More coverings by rook domains, J. Combin. Theory Ser. A. 36 (1984), no. 2, 240–244.
3. R. A. Brualdi, J. S. Graves, and K. M. Lawrence, Codes with a poset metric, Discrete Math. 147 (1995), no. 1–3, 57–72.
4. W. A. Candielli, On covering and coloring problems for rook domains, Discrete Math. 57 (1985), no. 1–2, 9–16.
5. A. G. Castoldi and E. L. Monte Carmelo, The covering problem in Rosenbloom–Tsfasman spaces, Electron. J. Combin. 22 (2015), no. 3, 0–00, paper 3.30.
6. A. G. Castoldi, E. L. Monte Carmelo, and R. da Silva, Partial sums of binomials, intersecting numbers, and the excess bound in Rosenbloom–Tsfasman space, Comput. Appl. Math. 38 (2019), 55. https://doi.org/10.1007/s40314-019-0828-2
7. A. G. Castoldi, E. L. Monte Carmelo, L. Moura, D. Panario, and B Stevens, Bounds on covering codes in RT spaces using ordered covering arrays, Algebraic Informatics. CAI 2019. Lecture Notes in Computer Science (M. Ciric, M. Droste, and J. É. Pin, eds.), vol. 11545, Springer, Cham, 2019, pp. 102–111. https://doi.org/10.1007/978-3-030-21363-3_9
8. A. G. Castoldi, L. Moura, D. Panario, and B. Stevens, *Ordered orthogonal array construction using LFSR sequences*, IEEE Trans. Inform. Theory. **63** (2017), no. 2, 1336–1347.

9. M. Chateauneuf and D. L. Kreher, *On the state of strength-three covering arrays*, J. Combin. Des. **10** (2002), 217–238.

10. G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering codes*, North Holland Mathematical Library, vol. 54, Elsevier, Amsterdam, 1997.

11. C. J. Colbourn, *Combinatorial aspects of covering arrays*, Matematiche. **59** (2004), 125–172.

12. C. J. Colbourn, *Strength two covering arrays: Existence tables and projection*, Discrete Math. **308** (2008), no. 5–6, 772–786.

13. C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs* (B. A. Davey and H. A. Priestley, eds.), 2nd ed., Chapman & Hall, Boca Raton, FL, 2006.

14. C. J. Colbourn, G. Kéri, P. P. R. Soriano, and J.-C. Schlage-Puchta, *Covering and radius-covering arrays: Constructions and classification*, Discrete Appl. Math. **158** (2010), no. 11, 1158–1180.

15. S. T. Dougherty and K. Shiromoto, *Maximum distance codes in \( \text{Mat}_{m,n}(\mathbb{Z}_k) \) with a non-Hamming metric and uniform distributions*, Des. Codes Cryptogr. **33** (2004), 45–61.

16. M. Firer, M. M. S. Alves, J. A. Pinheiro, and L. Panek, *Poset codes: Partial orders, metrics and coding theory*, Briefs in Mathematics, Springer, Switzerland, 2018.

17. S. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal arrays: Theory and applications*, Springer, New York, 2012.

18. S. Jain, *Bursts in \( m \)-metric array codes*, Linear Algebra Appl. **418** (2006), 130–141.

19. D. J. Kleitman and J. Spencer, *Families of \( k \)-independent sets*, Discrete Math. **6** (1973), no. 3, 255–262.

20. T. Krikorian, *Combinatorial constructions of ordered orthogonal arrays and ordered covering arrays*, M.Sc. Thesis, Dept. Math., Ryerson Univ., Toronto, ON, Canada, 2011.

21. K. M. Lawrence, *A combinatorial characterization of \((t, m, s)\)-nets in base \( b \)*, J. Combin. Des. **4** (1996), no. 4, 275–293.

22. L. Moura, G. Mullen, and D. Panario, *Finite field constructions of combinatorial arrays*, Des. Codes Cryptogr. **78** (2016), 197–219.

23. G. Mullen and W. Schmid, *An equivalence between \((t, m, s)\)-nets and strongly orthogonal hypercubes*, J. Combin. Theory Ser. A. **76** (1996), no. 1, 164–174.

24. H. Niederreiter, *Point sets and sequences with small discrepancy*, Monatsh. Math. **104** (1987), no. 4, 273–377.

25. W. Park and A. Barg, *The ordered Hamming metric and ordered symmetric channels*, IEEE International Symposium on Information Theory Proceedings, 2011, pp. 2283–2287.

26. J. Quistorff, *On Rosenbloom and Tsfasman’s generalization of the Hamming space*, Discrete Math. **307** (2007), no. 21, 2514–2524.

27. S. Raaphorst, *Variable strength covering arrays*, Ph.D. Thesis, School Elect. Eng. Comput. Sci., Univ. Ottawa, Ottawa, ON, Canada, 2013.

28. S. Raaphorst, L. Moura, and B. Stevens, *Variable strength covering arrays*, J. Combin. Des. **26** (2018), no. 9, 417–438.

29. M. Y. Rosenbloom, M. A. Tsfasman, *Codes for the \( m \)-metric*, Probl. Inf. Transm. **33** (1997), no. 1, 45–52.

30. M. M. Skriganov, *Coding theory and uniform distributions*, St. Petersburg Math. J. **33** (2002), 301–337.

31. B. Yildiz, I. Siap, T. Bilgin, and G. Yesilot, *The covering problem for finite rings with respect to the RT-metric*, Appl. Math. Lett. **23** (2010), no. 9, 988–992.

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