Hamiltonian reduction and Maurer-Cartan equations

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To Boris Feigin on the occasion of his 50-th Birthday

Abstract

We show that solving the Maurer-Cartan equations is, essentially, the same thing as performing the Hamiltonian reduction construction. In particular, any differential graded Lie algebra equipped with an even nondegenerate invariant bilinear form gives rise to modular stacks with symplectic structures.

This paper is our modest present to Boris Feigin from whom we learned the power of Homological Algebra.

1 From Maurer-Cartan equations to Moment maps

1.1 Let \( k \) denote the field of either real or complex numbers. Let \( \mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_- \) be a differential \( \mathbb{Z}/2\mathbb{Z} \)-graded Lie (super)-algebra (DGLA) over \( k \). Write \( d : \mathfrak{G}_\pm \to \mathfrak{G}_\mp \) for the differential, and \( H_\ast(\mathfrak{G}, d) := \text{Ker } d / \text{Im } d \) for the corresponding homology space, which is again \( \mathbb{Z}/2\mathbb{Z} \)-graded Lie super-algebra.

Depending on the problem, the space \( \mathfrak{G} \) may be either finite or infinite dimensional. To fix ideas, we shall assume below that \( \mathfrak{G} \) has finite dimension over \( \mathbb{C} \); moreover, it will be assumed that \( \mathfrak{G}_+ = \text{Lie } G_+ \) is the Lie algebra of a complex connected simply-connected linear algebraic group \( G_+ \). These assumptions, though certainly too restrictive, will allow us to make the main ideas more clear. In reality, the algebra \( \mathfrak{G} \) itself is typically infinite-dimensional while the homology algebra, \( H_\ast(\mathfrak{G}, d) \), is typically finite-dimensional. In such cases, the space \( \mathfrak{G} \) usually comes equipped with a natural topology. Various analytic issues (e.g., the closedness of the kernel and image of the differential \( d \)) that arise in such a topological framework require a considerable amount of machinery and are beyond the scope of this short paper.

1.2 The algebraic group \( G_+ \) acts on its Lie algebra \( \mathfrak{g}_+ \) via the adjoint action. Fix an \( \text{Ad } G_+ \)-orbit \( \mathcal{O} \subset \mathfrak{g}_+ \), and consider the following (locally closed) subscheme in \( \mathfrak{g}_- \):

\[
\text{MC}(\mathfrak{G}, \mathcal{O}) := \{ x \in \mathfrak{g}_- \mid dx + \frac{1}{2} [x, x] \in \mathcal{O} \}.
\]

The equation \( dx + \frac{1}{2} [x, x] = 0 \) is known as the Maurer-Cartan equation, so we call \( \text{MC}(\mathfrak{G}, \mathcal{O}) \) the Maurer-Cartan scheme \(^1\) associated to an orbit \( \mathcal{O} \). If \( \mathcal{O} = \{0\} \) is a one-point orbit, then our scheme \( \text{MC}(\mathfrak{G}, \mathcal{O}) \) reduces to the zero-scheme of the above-mentioned standard Maurer-Cartan equation.

\(^1\) It is actually a DG scheme, cf. [CFK], the fact that will be exploited later.
Given \( a \in \mathfrak{g}_+ \), let \( \xi_a \) denote an affine-linear algebraic vector field on \( \mathfrak{g}_- \) whose value at the point \( x \in \mathfrak{g}_- \) is \( \xi_a(x) := [a, x] - da \). The lemma below is well-known, see e.g. [GM, sect. 1.3].

**Lemma 1.2.1.** (i) The map \( a \mapsto \xi_a \) is a Lie algebra homomorphism.

(ii) For any orbit \( \mathcal{O} \subset \mathfrak{g}_+ \) and any \( a \in \mathfrak{g}_+ \), the vector field \( \xi_a \) is tangent to the Maurer-Cartan scheme \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) \subset \mathfrak{g}_- \).

It follows from part (i) of the Lemma that, exponentiating the vector fields \( \xi_a, a \in \mathfrak{g}_+ \), one obtains an action of the group \( G_+ \) on \( \mathfrak{g}_- \) by affine-linear transformations. This \( G_+ \)-action on \( \mathfrak{g}_- \) is known as the gauge action (it should not be confused with the ordinary Ad \( G_+ \)-action on \( \mathfrak{g}_- \)). We observe also that (by the Lemma) the map \( a \mapsto \xi_a \) intertwines the \( \text{Ad} G_+ \)-action on \( \mathfrak{g}_+ \) with \( G_+ \)-action on vector fields induced by the gauge action on \( \mathfrak{g}_- \). Further, part (ii) of the Lemma implies that the Maurer-Cartan scheme \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) \) is stable under the gauge action of \( G_+ \).

**Definition 1.2.2.** We write \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) := \mathcal{M}(\mathfrak{g}, \mathcal{O})/G_+ \) for the stack-quotient of \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) \) by the gauge \( G_+ \)-action, cf. [LMB], [To], and call \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) \) the modular stack attached to the orbit \( \mathcal{O} \subset \mathfrak{g}_+ \).

**Remark 1.2.3.** Usually, one has a natural \( \mathbb{Z} \)-grading \( \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \), making \( \mathfrak{g} \) a DGLA with differential \( \mathfrak{g} \to \mathfrak{g}_{*+1} \). We then put \( \mathfrak{g}_+ = \bigoplus_i \mathfrak{g}_{2i} \) and \( \mathfrak{g}_- = \bigoplus_i \mathfrak{g}_{2i+1} \). In such a case, one introduces a smaller Lie group \( G_0 \subset G_+ \) corresponding to the Lie subalgebra \( \mathfrak{g}_0 \subset \mathfrak{g}_+ \). This group acts naturally on \( \mathfrak{g}_2 \), and for any \( G_0 \)-orbit \( \mathcal{O}_2 \subset \mathfrak{g}_2 \) we may form the corresponding \( \mathfrak{g}_+ \)-orbit \( \mathcal{O} = \mathfrak{g}_+(\mathcal{O}_2) \). Define

\[
\mathcal{M}_1(\mathfrak{g}, \mathcal{O}) := \left\{ x \in \mathfrak{g}_1 \mid dx + \frac{1}{2}[x, x] \in \mathcal{O}_2 \right\}.
\]

It is clear that, for \( x \in \mathfrak{g}_1 \) we have: \( dx + \frac{1}{2}[x, x] \in \mathcal{O}_2 \) if and only if \( dx + \frac{1}{2}[x, x] \in \mathcal{O} \). This way the modular stack \( \mathcal{M}_1(\mathfrak{g}, \mathcal{O}) := \mathcal{M}_1(\mathfrak{g}, \mathcal{O})/G_0 \) becomes a (locally-closed) substack in \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) \).

1.3 Let \( E = \bigoplus_{i \in \mathbb{Z}} E^i \) be a DG vector space with differential \( d : E^* \to E^{*+1} \), and \((-,-) : E \times E \to \mathbb{C} \) a \( \mathbb{C} \)-bilinear form such that for any homogeneous \( x, y \in E \) we have

\[
(dx, y) + (-1)^{\deg x} (x, dy) = 0,
\]

and moreover \( E^i \perp E^j \) whenever \( i \neq j \).

Put \( E^*_i := (E^{-i})^* \), the dual of \( E^{-i} \), and let \( E^* := \bigoplus_{i \in \mathbb{Z}} E^*_i \). Dualizing the map \( d \) makes \( E^* \) a DG vector space. The assignment \( x \mapsto (x, -) \) gives rise to a morphism of DG vector spaces \( \kappa : E \to E^* \). We say that the form \((-,-)\) is non-degenerate provided the morphism \( \kappa \) induces an isomorphism \( H^*(\kappa) : H^*(E) \to H^*(E^*) \) on cohomology.
1.4 Now, let $G$ be a DGLA, as in §1.1-1.2, and let $\beta(-,-)$ be an even nondegenerate invariant bilinear form on $G$, that is, a $\mathbb{C}$-bilinear form $G \times G \to \mathbb{C}$ that restricts to a symmetric, resp. skew-symmetric, form on $G_+$, resp. on $G_-$, such that $G_+ \perp G_-$, (1.3.1) holds and for any homogeneous $x, y, z \in G$, one has:

$$\beta([x, y], z) = \beta(x, [y, z]).$$  \hspace{1cm} (1.4.1)

**Theorem 1.4.2.** Let $G$ be a DGLA with an even nondegenerate invariant bilinear form $\beta$, and $O \subset G_+$ an $\text{Ad} G_+$-orbit. Then, for any $x \in \mathcal{M}(G, O)$, the form $\beta$ induces a non-degenerate 2-form on $T_x \mathcal{M}(G, O)$, the tangent space (at $x$) to the modular stack. These 2-forms give rise to a symplectic structure on the stack $\mathcal{M}(G, O)$.

Thus, a DGLA with an even nondegenerate invariant bilinear form gives rise to symplectic stacks.

**Remark 1.4.3.** Morally, the main message of this paper is that all natural symplectic structures on moduli spaces ‘arising in nature’ come from an appropriate (even) nondegenerate invariant bilinear form on the DGLA that controls the moduli problem in question.

We will now show that the construction of the modular stack $\mathcal{M}(G, O)$ is a special case of the Hamiltonian reduction. That will yield the proof of Theorem 1.4.2.

1.5 Hamiltonian reduction construction. In the setup of Theorem 1.4.2, let $\omega := \beta|_{G_-}$ and consider $G_-$ as a symplectic vector space equipped with the symplectic form $\omega$. Consider also the gauge action of the group $G_+$ on $G_-$. This is a symplectic action, and we claim that it has moment map

$$\Phi : G_- \to G^*_+ \simeq G^*_+ : x \mapsto dx + \frac{1}{2} [x, x].$$

To this end, note that the differential of the map $\Phi$ at the point $x \in G_-$ is given by

$$\Phi'_x : y \mapsto \Phi'_x(y) = dy + [x, y] \quad \forall y \in G_- \cong T_x G_-.$$

A direct computation yields $\Phi'_x(a(x)) = [a, \Phi(x)] \quad \forall a \in G_+$. It follows that $\Phi$ is $G_+$-equivariant. Next, by (1.3.1), we have

$$\omega(a(x), y) = \beta(a, \Phi'_x(y)) \quad \forall a \in G_+, \ x, y \in G_-.$$  \hspace{1cm} (1.5.1)

It follows from the last equation combined with the $G_+$-equivariance of $\Phi$ that the $G_+$-action is Hamiltonian with moment map $\Phi$. Thus, $\mathcal{M}(G, O)$ is precisely the Hamiltonian reduction of $G_-$ with respect to the orbit $O \subset G_+ \simeq G^*_+$, see [AM].

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\textsuperscript{2}See explanation in §1.6 below.
1.6 In this subsection we explain the meaning of the word ‘nondegenerate’ in Theorem 1.4.2. We will see that this is closely related to the ‘self-dual nature’ of the Hamiltonian reduction construction.

In general, let \((X,\omega)\) be a smooth symplectic variety equipped with a Hamiltonian action of an algebraic group \(G\). Let \(\mathfrak{g} := \text{Lie}G\), fix a coadjoint orbit \(\mathfrak{O} \subset \mathfrak{g}^*\), and write \(\Phi : X \to \mathfrak{g}^*\) for the moment map.

We consider the scheme \(\Sigma := \Phi^{-1}(\mathfrak{O})\). If \(\Phi'\), the differential of \(\Phi\), is surjective at any point \(x \in X\) such that \(\Phi(x) \in \mathfrak{O}\), then \(\Sigma\) is a smooth (locally closed) subscheme of \(X\). In general, if \(\Phi'\) is not necessarily surjective, it is natural to view \(\Sigma\) as a DG scheme, which we denote \(\Sigma_{\text{DG}}\). The tangent space to this DG scheme at a closed point \(x \in \Sigma_{\text{DG}}\) is a DG vector space \(T_x\Sigma_{\text{DG}} = T_x^0\Sigma_{\text{DG}} \oplus T_x^1\Sigma_{\text{DG}}\) (concentrated in degrees 0 and 1), where

\[
T_x^0\Sigma_{\text{DG}} := T_xX, \quad T_x^1\Sigma_{\text{DG}} := \mathfrak{g}^*,
\]

with differential \(T_x^0\Sigma_{\text{DG}} \to T_x^1\Sigma_{\text{DG}}\), given by the map \(\Phi'_x : T_xX \to \mathfrak{g}^*\). We write \(H^*(T_x\Sigma_{\text{DG}})\) for the cohomology groups of the two-term complex above. We have \(H^0(T_x\Sigma_{\text{DG}}) = \ker \Phi'_x\), is the Zariski tangent space to \(\Sigma_{\text{DG}}\) (viewed as an ordinary scheme \(\Sigma\)), and \(H^1(T_x\Sigma_{\text{DG}}) = \mathfrak{g}^*/\ker \Phi'_x\), is the space measuring the failure of the differential of \(\Phi\) to be surjective.

Next, we perform the Hamiltonian reduction and consider the quotient \(\Sigma/G\) (where \(\Sigma\) is an ordinary scheme, not a DG scheme). If the \(G\)-action on \(\Sigma\) is free, then this quotient is a well-defined scheme again. In general, for a not necessarily free action, it is natural to consider \(\mathcal{M} := \Sigma/G\) as a stack, cf. [LM], [To]. Given \(x \in \Sigma\), write \(\bar{x}\) for the corresponding point in \(\mathcal{M}\). Then, the tangent space at \(\bar{x}\) to the stack \(\mathcal{M} = \Sigma/G\) is a DG vector space \(T_{\bar{x}}\mathcal{M} = T_{\bar{x}}^{-1}\mathcal{M} \oplus T_{\bar{x}}^0\mathcal{M}\) (concentrated in degrees \(-1\) and 0), where

\[
T_{\bar{x}}^{-1}\mathcal{M} := \mathfrak{g}, \quad T_{\bar{x}}^0\mathcal{M} := T_{\bar{x}}\Sigma,
\]

with differential \(T_{\bar{x}}^{-1}\mathcal{M} \to T_{\bar{x}}^0\mathcal{M}\), given by the derivative (at \(1 \in G\)) of the action-map \(g \mapsto g(x)\). Then, for the cohomology groups, we have \(H^0(T_{\bar{x}}\mathcal{M}) = T_{\bar{x}}\Sigma/\mathfrak{g} \cdot x\), is the normal space to the \(G\)-orbit through \(x \in \Sigma\), and \(H^{-1}(T_{\bar{x}}\mathcal{M}) = \mathfrak{g}^*\) is the Lie algebra of the isotropy group of the point \(x\), that measures the failure of the \(\mathfrak{g}\)-action on \(\Sigma\) to be infinitesimally-free.

Now, if we view (as has been explained earlier) \(\Sigma_{\text{DG}} := \Phi^{-1}(\mathfrak{O})\) as a DG scheme rather than an ordinary scheme, then the quotient \(\mathcal{M}_{\text{DG}} := \Sigma_{\text{DG}}/G\) becomes a DG stack rather than an ordinary stack. To get the tangent space of this DG stack, we must combine formulas (1.6.1) and (1.6.2) together. Thus, the tangent space to \(\mathcal{M}_{\text{DG}}\) is a DG vector space \(T_{\bar{x}}\mathcal{M}_{\text{DG}} = T_{\bar{x}}^{-1}\mathcal{M}_{\text{DG}} \oplus T_{\bar{x}}^0\mathcal{M}_{\text{DG}} \oplus T_{\bar{x}}^1\mathcal{M}_{\text{DG}}\), concentrated in degrees \(-1,0,1\), such that the corresponding 3-term complex \(T_{\bar{x}}^{-1}\mathcal{M}_{\text{DG}} \to T_{\bar{x}}^0\mathcal{M}_{\text{DG}} \to T_{\bar{x}}^1\mathcal{M}_{\text{DG}}\) reads:

\[
T_{\bar{x}}\mathcal{M}_{\text{DG}} : \mathfrak{g} \xrightarrow{\text{action}} T_{\bar{x}}X \xrightarrow{\Phi'_x} \mathfrak{g}^*.
\]
the isomorphism \( \omega : T_x X \xrightarrow{\sim} T_x^* X \), induced by the symplectic form \( \omega \), with the adjoint \( a^\top : T_x X \rightarrow \mathfrak{g}^* \) of the action-map \( a : \mathfrak{g} \rightarrow T_x X \), that is, one has:

\[
\Phi'_x = a^\top \circ \omega : T_x X \xrightarrow{\sim} T_x^* X \rightarrow \mathfrak{g}^*.
\]

This shows that the DG vector space \((T_x, \mathcal{M}_{DG})^*\) is canonically isomorphic to \(T_x, \mathcal{M}_{DG}\). The isomorphism \( T_x, \mathcal{M}_{DG} \xrightarrow{\sim} (T_x, \mathcal{M}_{DG})^* \) induces, of course, an isomorphism of cohomology groups, and the non-degeneracy of symplectic form on the DG stack \( \mathcal{M}_{DG} \) follows.

**1.7 Example: moduli of G-local systems.** A typical (infinite-dimensional) example of the Theorem above is the moduli space of bundles with flat connection on a compact \( C^\infty \)-manifold \( X \).

In more details, let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), and \( P \) a principal \( G \)-bundle on \( X \). Let \( \mathfrak{g}_P \) denote the associated vector bundle (with fiber \( \mathfrak{g} \)) corresponding to the adjoint representation \( \mathfrak{ad} : G \rightarrow \text{GL}(\mathfrak{g}) \). Thus, \( \mathfrak{g}_P \) is a bundle of Lie algebras.

Let \( \Omega^\ast(X, \mathfrak{g}_P) \) be the vector space of \( C^\infty \)-differential \( \ast \)-forms on \( X \) with values in \( \mathfrak{g}_P \). The Lie bracket on \( \mathfrak{g}_P \), combined with wedge-product of differential forms makes the graded space \( \mathfrak{g}_P := \bigoplus_q \Omega^q(X, \mathfrak{g}_P) \) a Lie super-algebra.

We are interested in the moduli space (or stack) \( \mathcal{M}(P) \) of flat \( C^\infty \)-connections on \( P \), modulo gauge equivalence. So, assume that \( \mathcal{M}(P) \) is non-empty and choose some flat \( C^\infty \)-connection \( \nabla \) on \( P \). The connection induces a differential \( \nabla : \Omega^\ast(X, \mathfrak{g}_P) \rightarrow \Omega^{\ast+1}(X, \mathfrak{g}_P) \), thus gives \( \mathfrak{g}_P \) the structure of a DGLA.

Any connection on \( P \) can be written in the form \( \nabla' = \nabla + \gamma \), for some \( \gamma \in \Omega^1(X, \mathfrak{g}_P) \). The curvature of \( \nabla' \) is \( \nabla' \circ \nabla' = \nabla \gamma + \frac{1}{2}[\gamma, \gamma] \). Thus, \( \nabla' \) is flat if and only if \( \gamma \) satisfies the Maurer-Cartan equation \( \nabla \gamma + \frac{1}{2}[\gamma, \gamma] = 0 \). Thus, in the notation of Remark 1.2.3 (with \( d := \nabla \)), we have \( \mathcal{M}(P) \cong \text{MC}_0(\mathfrak{g}_P, \{0\})/G_0 \). Here, \( G_0 := \text{Aut}(P) \) is the infinite-dimensional group of gauge transformations.

To proceed further, we assume \( X \) to be compact oriented, and assume also that there is a non-degenerate invariant symmetric bilinear form \( \langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R} \). The form \( \langle -, - \rangle \) induces a nondegenerate pairing \( \langle -, - \rangle_P : \mathfrak{g}_P \times \mathfrak{g}_P \rightarrow C^\infty(X) \). It is straightforward to verify that the following formula \((i = 0, \ldots, n = \dim_{\mathbb{R}} X)\):

\[
\beta : \Omega^i(X, \mathfrak{g}_P) \times \Omega^{n-i}(X, \mathfrak{g}_P) \xrightarrow{\wedge} (\mathfrak{g}_P \otimes \mathfrak{g}_P) \otimes \Omega^n(X) \xrightarrow{\langle -, - \rangle_P} \Omega^n(X) \xrightarrow{\int_X} \mathbb{R}
\]

gives a nondegenerate symmetric (even) bilinear form \( \beta : \mathfrak{g}_P \times \mathfrak{g}_P \rightarrow \mathbb{R} \). The form \( \beta \) is invariant, provided the connection \( \nabla \) was chosen so that the paring \( \langle -, - \rangle_P \) is \( \nabla \)-horizontal, i.e., such that \( \langle \nabla x, y \rangle_P + \langle x, \nabla y \rangle_P = 0 \), for any sections \( x, y \in \mathfrak{g}_P \), c.f. 1.3.1.

Now if \( \dim_{\mathbb{R}} X = 2 \), then \( \Omega^i(X, \mathfrak{g}_P) = 0 \) for \( i > 2 \), and therefore we have \( \text{MC}(\mathfrak{g}_P, \{0\}) = \text{MC}_0(\mathfrak{g}_P, \{0\}) \). Hence the construction of 1.1.5 applied to the DGLA \( \mathfrak{g} = \mathfrak{g}_P \) gives a symplectic structure on the stack \( \mathcal{M}(P) = \text{MC}_0(\mathfrak{g}_P, \{0\})/G_0 = \text{MC}(\mathfrak{g}_P, \{0\})/G = \mathcal{M}(\mathfrak{g}_P, \{0\}) \).

For \( \dim_{\mathbb{R}} X > 2 \), our construction only gives a symplectic structure on \( \mathcal{M}(\mathfrak{g}_P, \{0\}) \supset \mathcal{M}(P) \). However, it is known (see e.g., [Kar]) that if the Hard Lefschetz theorem holds for \( X \), then \( \mathcal{M}(P) \) is in effect a symplectic substack in \( \mathcal{M}(\mathfrak{g}_P, \{0\}) \).
Remark 1.7.1. For a trivial $G$-bundle $P$, the space $\mathcal{M}(P)$ may be identified with $\text{Hom}(\pi_1(X), G)/\text{Ad} G$, the moduli space of $G$-local systems on $X$. The symplectic structure on this space has been studied by many authors, see e.g. [Go], [We], [Kar].

2 $L_\infty$-algebra version

2.1 Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $L_\infty$-algebra, see e.g. [LM], [MSS]. Let $\beta$ be an even nondegenerate invariant bilinear form on $\mathfrak{g}$. Thus, for any homogeneous $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, one has:

$$\beta([x_1, \ldots, x_n], x_{n+1}) = (-1)^{n(\text{deg} x_{n+1})} \beta(x_1, [x_2, \ldots, x_{n+1}]).$$

Define

$$\Phi : \mathfrak{g}_- \to \mathfrak{g}_+ : x \mapsto dx + \frac{1}{2!}[x, x] + \frac{1}{3!}[x, x, x] + \cdots.$$

The differential of the map $\Phi$ at the point $x \in \mathfrak{g}_-$ is given by

$$\Phi_x' : y \mapsto \Phi_x'(y) = dy + [x, y] + \frac{1}{2!}[x, x, y] + \cdots, \quad \forall y \in \mathfrak{g}_-.$$

For any $a \in \mathfrak{g}_+$ and $x \in \mathfrak{g}_-$, let

$$\xi_a(x) := -(da + [x, a] + \frac{1}{2!}[x, x, a] + \cdots). \quad (2.1.1)$$

Observe that we have

$$\beta(\xi_a(x), y) = \beta(a, \Phi_x'(y)) \quad \forall a \in \mathfrak{g}_+, x, y \in \mathfrak{g}_-. \quad (2.1.2)$$

By [La, Appendix B], we also have

$$\Phi_x'(\xi_a(x)) = [a, \Phi(x)] + [a, \Phi(x), x] + \frac{1}{2!}[a, \Phi(x), x, x] + \cdots \quad \forall a \in \mathfrak{g}_+, x \in \mathfrak{g}_-. \quad (2.1.3)$$

- We say that two elements $x_0, x_1 \in \mathfrak{g}_-$ are gauge equivalent if there exists a path $a(t) \in \mathfrak{g}_+$ and a path $x(t) \in \mathfrak{g}_-$ such that $x'(t) = \xi_{a(t)}(x(t))$ and $x(0) = x_0, x(1) = x_1$.
- We say that two elements $b_0, b_1 \in \mathfrak{g}_+$ are adjoint equivalent if there exists two paths $a(t), b(t) \in \mathfrak{g}_+$ and a path $x(t) \in \mathfrak{g}_-$ such that $b(0) = b_0, b(1) = b_1$ and

$$b'(t) = [a(t), b(t)] + [a(t), b(t), x(t)] + \frac{1}{2!}[a(t), b(t), x(t), x(t)] + \cdots. \quad (2.1.4)$$

Remark 2.1.5. If $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded $L_\infty$-algebra, then we say that $b_0, b_1 \in \mathfrak{g}_2$ are adjoint equivalent if there exists a path $a(t) \in \mathfrak{g}_0$, a path $b(t) \in \mathfrak{g}_2$ and a path $x(t) \in \mathfrak{g}_1$ such that $b(0) = b_0, b(1) = b_1$ and (2.1.4) is satisfied.
2.2 If $\mathcal{O} \subset \mathfrak{g}_+$ is an adjoint equivalence class, then we define

$$\text{MC}(\mathfrak{g}, \mathcal{O}) := \{ x \in \mathfrak{g}_- | \Phi(x) \in \mathcal{O} \}, \quad \mathcal{H}(\mathfrak{g}, \mathcal{O}) := \text{MC}(\mathfrak{g}, \mathcal{O})/\text{gauge equivalence}.$$ 

**Remark 2.2.1.** This definition of the objects $\text{MC}(\mathfrak{g}, \mathcal{O})$ and $\mathcal{H}(\mathfrak{g}, \mathcal{O})$ involves solving differential equations, as well as formulas (2.1.1) - (2.1.3) which contain infinite series. One way to make sense of the definition above is to use a topological setting and to prove the convergence of all the series that arise.

If $\mathcal{O} = \{0\}$, then there is an alternative, purely algebraic, approach based on the language of *formal schemes*. Specifically, let $\mathfrak{g}_-$ be the formal completion of the vector space $\mathfrak{g}_-$ at the origin. The coordinate ring of $\mathfrak{g}_-$ is the ring $\mathbb{C}[[\mathfrak{g}_-]]$ of formal power series, so that the series for $\Phi$ gives a well-defined morphism $\mathfrak{g}_- \to \mathfrak{g}_+$. Thus, $\Phi^{-1}(0) \subset \mathfrak{g}_-$ is a well-defined closed subscheme. Further, gauge equivalence gives a closed adjoint equivalence class, then we define

$$\text{MC}(\mathfrak{g}, \mathcal{O}) := \Phi^{-1}(0)/\text{gauge equivalence},$$

a pro-algebraic stack.

One can extend the concept of Hamiltonian reduction from Hamiltonian group-actions to Hamiltonian groupoid-actions. This way, using formula (2.1.2), one derives the following

**Theorem 2.2.2.** Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be an $L_\infty$-algebra with an invariant nondegenerate even bilinear form $\beta$, and $\mathcal{O} \subset \mathfrak{g}_+$ an adjoint equivalence class. Then, for any $x \in \mathcal{H}(\mathfrak{g}, \mathcal{O})$, the form $\beta$ induces a non-degenerate 2-form on $T_x\mathcal{H}(\mathfrak{g}, \mathcal{O})$, the tangent space (at $x$) to the modular stack. These 2-forms give rise to a symplectic structure on $\mathcal{H}(\mathfrak{g}, \mathcal{O})$.

3 From moment map to Maurer-Cartan equations

3.1 Let $(V, \omega)$ be a finite dimensional symplectic vector space, and $\mathbb{C}[V]$ the algebra of polynomial functions on $V$ viewed as a Poisson algebra with Poisson bracket $\{-,-\}$ corresponding to the symplectic structure.

Fix a finite dimensional Lie algebra $\mathfrak{g}$. Let $H : \mathfrak{g} \to \mathbb{C}^{>0}[V] = \bigoplus_{i \geq 1} \mathbb{C}^i[V]$, $a \mapsto H_a$, be a Lie algebra homomorphism. We get, tautologically, a (non-linear) Hamiltonian $\mathfrak{g}$-action on the vector space $V$ with polynomial moment map $\Phi : V \to \mathfrak{g}^*$, $\Phi(v) : a \mapsto H_a(v)$.

3.2 Introduce a $\mathbb{Z}$-graded vector space

$$\mathfrak{g} = g \oplus V \oplus g^*,$$

and write $\mathfrak{g}_0 := g$, $\mathfrak{g}_1 := V$, $\mathfrak{g}_2 := g^*$.

and write $\mathfrak{g}_+ := \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_- := \mathfrak{g}_1$. The canonical pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{C}$ gives a non-degenerate symmetric bilinear form on $\mathfrak{g}_+$. Combined with the symplectic form $\omega$ on $V$ this gives an even non-degenerate bilinear form $\beta$ on the super-space $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$.

Further, let $\Phi = \Phi_1 + \Phi_2 + \ldots : V \to \mathfrak{g}^*$ be an expansion of the moment map $\Phi$ into homogeneous components (by assumption there is no constant term). For each $i \geq 1$, the map $\Phi_i$ gives rise, via the canonical isomorphism $\mathbb{C}^i[V] \simeq (\text{Sym}^i V)^*$, to a linear map $\tilde{\Phi}_i : \text{Sym}^i V \to \mathfrak{g}^*$, such that $\tilde{\Phi}_i(y^i) = \Phi_i(y)$, $\forall y \in V$. 


We now define the following maps:

- \( \mathfrak{g}_0 \otimes \mathfrak{g} \to \mathfrak{g} \) given, for any \( u = x \oplus v \oplus \lambda \in \mathfrak{g} \oplus \mathfrak{v} \oplus \mathfrak{g}^* \), by \( a \otimes u \mapsto [a, u] := \text{ad} a(x) \oplus a(v) \oplus \text{ad}^* a(\lambda) \);
- \( \text{Sym}^i(\mathfrak{g}_-) \to \mathfrak{g}_2 \subset \mathfrak{g}_+ \), \( y_1 y_2 \cdots y_i \mapsto \tilde{\Phi}_i(y_1 y_2 \cdots y_i) \), for each \( i \geq 2 \).

Further, let \( d := \Phi_1 : \mathfrak{g}_- \to \mathfrak{g}_+ \), and define \( d : \mathfrak{g}_+ \to \mathfrak{g}_- \) to be the zero-map.

**Proposition 3.2.1.** The above defined maps give \( \mathfrak{g} \) an \( L_\infty \)-structure (with all brackets that were not specified above being set equal to zero), such that \( \beta \) becomes an invariant form.

**Proof.** Straightforward computation. \( \square \)

We observe that two elements in \( \mathfrak{g}_2 = \mathfrak{g}^* \) are adjoint equivalent (in the sense of §2) if and only if they belong to the same coadjoint orbit of the group \( G = G_0 \) (corresponding to the Lie algebra \( \mathfrak{g} \)) acting on \( \mathfrak{g}^* \). Given such an orbit \( \mathcal{O} \subset \mathfrak{g}_2 = \mathfrak{g}^* \), we see that \( \mathcal{MC}(\mathfrak{g}, \mathcal{O}) = \tilde{\Phi}^{-1}(\mathcal{O}) \). Hence, for the Maurer-Cartan stack we get \( \mathcal{M}(\mathfrak{g}, \mathcal{O}) = \tilde{\Phi}^{-1}(\mathcal{O})/\text{Ad} G_0 \), is the standard Hamiltonian reduction of \( V \) over \( \mathcal{O} \).

**Remarks.** (i) Note that the construction above may be thought of as a non-linear version of the construction of the tangent space as a DG vector space, given in [LM],

(ii) In the special case \( H : \mathfrak{g} \to \mathcal{O}^2[V] \) (quadratic Hamiltonians), we have \( \Phi = \Phi_2 \), and the \( L_\infty \)-structure above reduces to an ordinary Lie super-algebra structure on \( \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{v} \oplus \mathfrak{g}^* \). The symmetric bracket \( \mathfrak{g}_- \times \mathfrak{g}_- \to \mathfrak{g}_+ \) is in this case provided by the map \( \Phi \) viewed as a linear map: \( \text{Sym}^2(\mathfrak{g}_-) = \text{Sym}^2(V) \to \mathfrak{g}^* = \mathfrak{g}_2 \).

This special case was also implicit in [KM]. Notice that a Lie algebra homomorphism \( \mathfrak{g} \to \text{Sym}^2 V \simeq \mathfrak{sp}(V) \) has a natural extension to a Lie algebra homomorphism \( \nu : \mathfrak{g}_+ = \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{sp}(V) \) by sending \( \mathfrak{g}^* \) to zero. We observe that \( \nu \) is of super Lie type in Kostant’s terminology, i.e. the condition on the Casimir in [KM] Theorem 0.1 holds (trivially) for this \( \nu \).

(iii) J. Stasheff informed us that it is possible to define Hamiltonian reductions with respect to (infinitesimal) actions of an \( L_\infty \)-algebra, and to extend the construction of §3 to such an \( L_\infty \)-setup. He also pointed out to us the relationship of our construction to the classical BRST complex. Namely, an \( L_\infty \)-structure on \( \mathfrak{g} \) is equivalent to a square zero derivation on the free super-commutative algebra \( S^{>0}(\mathfrak{g}[1]^*) \) generated by \( \mathfrak{g}[1]^* \), see e.g. [LM]. Here, \( \mathfrak{g}[1] \) is the super vector space with \( \mathfrak{g}[1]_+ = \mathfrak{g}_- \) and \( \mathfrak{g}[1]_- = \mathfrak{g}_+ \). For the \( L_\infty \)-algebra \( \mathfrak{g} \) in Proposition 3.2.1, we thus obtained a differential on \( S^{>0}(\mathfrak{g}[1]^*) \) which turns out to be the classical BRST operator on \( \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g} \otimes \text{Sym} V^* \) defined in [KS] p. 57.

To see this, it suffices to note that the generators of \( S^{>0}(\mathfrak{g}[1]^*) \) are

\[
\mathfrak{g}[1]^*_- = \mathfrak{g}^* \oplus \mathfrak{g} \quad \text{and} \quad \mathfrak{g}[1]^*_+ = V^* \, ,
\]

and the differential is defined on these generators by taking the sum of the maps

\[ H : \mathfrak{g} \to \text{Sym}^{>0}V^* \quad \text{and} \quad J : \mathfrak{g}[1]^* \to \mathfrak{g}^* \otimes \mathfrak{g}[1]^* , \]

where \( J \) is obtained from dualizing the map \( \mathfrak{g}_0 \otimes \mathfrak{g} \to \mathfrak{g} : a \otimes u \mapsto [a, u] \).

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