Interacting Geodesics: Binary Systems around a Black Hole.

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We present a novel method to study interacting orbits in a fixed mean gravitational field associated with a solution of the Einstein field equations. The idea is to consider the Newton gravity among the orbiting particles in a geometry given by the main source. We apply the technique in the study of two and three self-gravitating particles moving around a black hole, i.e., in a Schwarzschild geometry. We also compare with the equivalent Newtonian problem and noted differences in the structural stability, e.g., binary systems were found only in the general relativistic approach.

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General relativistic effects are usually neglected in astrophysical systems in which many gravitational sources are considered. In most of the known examples these effects are insignificant. Also the n-body problem is intractable in Einstein theory without large approximations. Nevertheless the difference between the Newtonian and the General Relativistic approaches becomes important close to compact objects and around very massive structures. When non-interacting particles are been considered, the problem can be solved computing time-like geodesics. When more than one source is considered (in the non-stationary case) numerical techniques might be used to solve the Einstein field equations. However finding numerical solution to the simple two body problem is already a non-trivial problem, see for instance [1]. In some cases, analytical and numerical solutions to the n-body problem in GR are found as sequences of post-Newtonian approximations [2]. Following the post-Newtonian idea, there are some predictions of relativistic effects using a relativistic celestial mechanic [3]. Other approximation techniques are also available, the simplest consists in solving a typical Newtonian problem using pseudo-potentials that simulates the GR aspects relevant in the studied situation [4, 5].

We suggest in this article a novel technique that makes possible the study of small massive objects orbiting around a fixed structure with a large mass. The idea is to consider the Newtonian gravity among particles evolving in a geometry associated with a solution to the Einstein field equations.

We construct within the Newtonian gravity a quadri-force. Then this force is used to modify the geodesic equation to allow a “Newtonian” interaction among them. In this scenario the damping due to the emission of gravitational radiation is not considered.

In the context of Schwarzschild geometry, we study planar interacting orbits and compare the results with the equivalent situation in the usual Newtonian theory. We see a great difference in the structure stability of the studied systems. The most relevant result is that in our approach the generation of binary systems is favoured if compared with the equivalent pure Newtonian situation. In n-body simulations the production of binary systems is a key issue in the formation of astrophysical structures as galaxies and star clusters [6].

In our proposition the interacting particles do not change the mean gravitational field. In the general relativistic context this field is represented by a solution to the Einstein equations. Non-interacting particles move following geodesics, i.e., their world lines obey the geodesic equations,

\[ \dot{x}^{\lambda} \nabla_{\lambda} \dot{x}^{\mu} = \ddot{x}^{\mu} + \Gamma_{\alpha\beta}^{\mu} \dot{x}^{\alpha} \dot{x}^{\beta} = 0, \]  

(1)

where \( \Gamma_{\alpha\beta}^{\mu} = g^{\mu\gamma}(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}) \), and \( g_{\alpha\beta} \) is the metric tensor for the known solution. The dots represent ordinary derivation with respect to the proper time \( \tau \) and commas denote the usual partial derivation with respect to the coordinate.

The next step is to use Newton gravity to compute the interaction among orbiting particles. The Newtonian gravitational potential generated by a set of \( N \) massive particles is

\[ \Phi = - \sum_{a=1}^{N} \frac{Gm_a}{|r - r_a|}. \]  

(2)

In the rest of the article we shall use geometrical units: the Newton constant \( G = 1 \), the light speed \( c = 1 \), we also assume that the black hole mass is unit \( (M = 1) \).

Therefore the force \( f \) that acts on the particle \( b \) due to the other orbiting particles is

\[ f_{(b)} = -m_b \nabla_b \Phi(r_a \neq r_b). \]  

(3)
Now we look for a quadri-force that represents in some way the Newtonian interaction among the particles in a fixed pseudo-Riemannian geometry. The spatial components of the quadri-force $F^\mu$ is obtained from $f$ given in (3). From $f = d\mathbf{p}/dt$ and $F^i = d\mathbf{p}^i/dt$ we get $F^i = u_0 f^i$, where $u^a = dx^a/d\tau$. We want a quadri-force $F^\mu$ orthogonal to the quadri-velocity $u^\mu$. Hence, we use $u_\mu f^\mu = u_0 F_0 + u_0 u_i f^i = 0$ to obtain $F^0 = -u_i f^i$. We can write $F^\mu$ in a more adequate form as

$$F^i = m u_0 g^i_{\lambda(\mu)} \frac{\partial \Phi}{\partial x^\lambda}, \quad F^0 = -mg^{00} u^0 \frac{\partial \Phi}{\partial x^\mu}.$$  \hspace{1cm} (4)

The signature of $g_{\mu\nu}$ is taken as $-2$.

Finally, we introduce $F^\mu$ in the right side of the geodesic equations representing an external force to define the motion equations for a particle of mass $m_\alpha$,

$$m_\alpha \ddot{x}^\lambda_{(\alpha)} \nabla_\lambda \dot{x}^\mu_{(\alpha)} = F^\mu_{(\alpha)}.$$  \hspace{1cm} (5)

From now on we consider that all the particles moving around the center of attraction have the same mass $m$.

First we consider the limits of (5) for large and small masses. When the mass of the interacting particles $m$ is much smaller than the fixed source mass $M$, we have nothing but the geodesic equations for a set of test-particles. For $m \gg M$ we get the special relativistic analogous to the Newton second law. And for small speeds, i.e., $v_0 \to 1$ and $u_i \to 0$, the Newtonian motion equations for a system of $N$ gravitating particles are recovered.

The method is “self-inconsistent” since some of the mathematical principles of a general relativistic framework are not satisfied. For instance, we are using an instantaneous interaction among the particles. The distance between two particles calculated in the Black Hole framework was used to compute the force of attraction without correction related to the different speed of the particles. The force is not covariant.

We must have these imperfections in mind to determine the limitation of the method and to check numerical results. Since the geometry will be kept fixed the mass of the orbiting particles must be much smaller than the main source. We have to systematically control variations of the total angular momentum and energy of the orbiting particles, they should be constant.

We use the above described approach to study particles moving around a spherical symmetric source, i.e, we numerically solved (5) when the field $g_{\mu\nu}$ is the Schwarzschild metric. The initial conditions are chosen to have for each particle a timelike worldline and that the spatial projection of these worldlines represent a bounded system of particles moving on the plane $\vartheta = \pi/2$.

In the first examples, three particles moving close to the black hole are studied. For this purpose we take: $L = 6.752, E_1 = 0.988, L_2 = 6.937, E_2 = 0.989$ and $L_3 = 5.895, E_3 = 0.984$. [In the Schwarzschild metric: $L = r^2 \dot{\varphi} \sin^2 \vartheta$ and $E = \dot{\varphi}/(1 - 2M/r)]$. Their initial distances from the center of attraction are, respectively, $r_1 = 42.5, r_2 = 45.0$ and $r_3 = 31.5$ in geometric units.

When we compute the orbits for three particles interacting only with a center of attraction ($F^\mu_{(1)} = F^\mu_{(2)} = F^\mu_{(3)} = 0$) this is equivalent to chose $m = 0$, as above mentioned we take the black hole mass as one ($M = 1$). For the already presented values of $L$ and $E$ we have bounded geodesic motion around the black hole Fig.1. In Fig.2 we present three particles with equal masses $m = 5 \times 10^{-4}$. The trajectories are quite different from the geodesics but the particles still have bounded motion. In Fig.3 we present the motion for a larger value of the masses, $m = 3 \times 10^{-3}$. In this case, one particle escapes (the dashed orbit) while the other couple form a binary system that falls into the black hole.

Now we consider only two particles farther from the black hole than the preceding ones. For this purpose, we consider $L_1 = 28.10, E_1 = 0.99937$ and $L_2 = 28.16, E_2 = 0.99937$. As in the previous examples, the non interacting orbiting particles follow bounded geodesic motions, Fig.4. In Fig.5 we present the trajectories for the pair of orbiting particles with $m = 10^{-5}$. In this case, we can clearly distinguish the motion of each particles. In the second case is considered $m = 10^{-4}$, now the particles move in the same region of the space in such way that their orbits fill a thick ring, Fig.6. Finally, in Fig.7 the particles mass are $m = 5 \times 10^{-4}$ and we have the formation of a stable binary system that remains in a confined motion around the black hole.

It is instructive to compare our results with the equivalent pure Newtonian three-body problem. The gravitational potential field of the system in this case is

$$\Phi_{\text{Newt}}(r) = -\frac{GM}{r} - \sum_{a=1}^{2} \frac{Gm_a}{|r - r_a|}.$$  \hspace{1cm} (6)

With $m_1 = m_2 = m$ the motion equations of the particles are,

$$\frac{d^2 x_a}{dt^2} = -\nabla_a \Phi_{\text{Newt}}.$$  \hspace{1cm} (7)
These equations are numerically solved by using parameters that confine the particles in a motion around the central mass. The Newtonian energy per unit of mass, $E_{\text{Newt}}$, is related to the relativistic specific energy by $E = \sqrt{1 - 2E_{\text{Newt}}}$. The specific angular momentum $L$ and energy $E$ used in the simulations are $L_1 = 6.55, E_1 = 0.988; L_2 = 6.75, E_2 = 0.989$, and $L_3 = 5.68, E_3 = 0.984$. These values are equivalent to the ones used in the first general relativistic simulations (Figs. 11, 2 and 3).

We start with the simulation of three test-particles (no interaction among them) moving around an attraction center in elliptical orbits, Fig. 11. We present in Fig. 2 interacting particles with $m = 5 \times 10^{-4}$. We see that one of them quickly unbound and the remaining couple stay in a bounded motion for some time. With a stronger interaction, $m = 3 \times 10^{-3}$ presented in Fig. 12 two particles quickly move far from the center of attraction while the remaining one stabilizes in a bounded elliptical motion closer to the black hole.

Now we present results of two particles moving farther from the center of attraction in a situation similar to the analogous relativistic presented in Fig. 11. For this aim, we use $L_1 = 28.10, E_1 = 0.99937$ and $L_2 = 28.12, E_2 = 0.99937$. The motion of test particles is presented in Fig. 13. In Fig. 12 we have $m = 10^{-4}$ the particles stabilize in quasi-elliptical orbits. One closer and other more distant to the center of mass if we compare with the non-interacting situation. For a stronger interaction, $m = 3 \times 10^{-3}$, we have a interesting situation as shown in Fig. 13. The particles alternate quasi-elliptical motions closer an farther to the center of attraction, i.e., while one of them are orbititating in a large ellipse the other is in a small one. We make some simulations considering that the central mass is not fixed and has the unit mass ($M = 1$). No significative difference was noticed.

Before analysing the results, it is important to explain some numerical cautions necessary in these general relativistic simulations. In the first examples (Fig. 2 and Fig. 3) the particles were very close to the black hole (remember that if the BH have 10M⊙ the distance of 50 in geometric units will be about 50km) therefore the system is more unstable, see for instance Figs. 3, 4, and the error due to the inexact approach is larger since the general relativistic effects are very important. We use, to control such problems, the numerical values of the total angular momentum and energy of the orbiting particles (that should be constant) and their physical time - if they were very different we would have problems in the synchronizaton of the Newtonian interaction. The numerical variation of the total energy was always smaller than 1%, the total angular momentum varied at most 10% and the time less than 5%. We consider that it is the worst situation that we still can apply this technique, in the other examples with the general relativistic approach, Fig. 4 and Fig. 5 the deviations of the total angular momentum, total energy and time were very small - always less than 1%.

By comparing the orbits obtained in our GR framework with the equivalent situation in Newtonian gravity we conclude that the structrual stability of the same system changes substanccially with different approaches. The most interesting difference is the formation of binary systems noticed in the Fig. 3 (a binary system that is captured by the black hole) and in the Fig. 4 (the binary bounded motion stabilizes around the black hole). An analogous situation is very hard to be obtained in the Newtonian approach so close to the center of attraction. When the orbiting particles are very far from the main gravitational source, the Newtonian and the relativistic approach lead to similar results. In these cases, the gravitational force exerced by the central mass varies much more slowly than the interaction between the two orbiting particles.

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FIG. 1: Geodesic motion of three particles around the black hole. The initial angular momentum and energy of each particle are, respectively, $L_1 = 6.752$, $E_1 = 0.988$; $L_2 = 6.937$, $E_2 = 0.989$ and $L_3 = 5.895$, $E_3 = 0.984$.

FIG. 2: Now we use the same values of angular momentum and energy of the first figure but now with interacting particles. They have the same mass, $m = 5 \times 10^{-4}$. We obtain three particles in a bounded motion around the black hole.

FIG. 3: Using the same values of angular momentum and energy of the preceding figure and $m = 3 \times 10^{-3}$ we have now that two of the particles form a binary system that falls into the black hole while the other stabilizes in a bounded motion around the center.

FIG. 4: Geodesic motion around a black hole of two particles with $L_1 = 28.10$, $E_1 = 0.99937$ and $L_2 = 28.16$, $E_2 = 0.99937$.

FIG. 5: We maintain the values of angular momentum and energy of the preceding figure with two interacting particles with $m = 10^{-5}$. They have a bounded motion close to the geodesics.

FIG. 6: Using the same $L$ and $E$ of the last figure but with $m = 10^{-4}$, we have that the motion of the particles is still bounded but now they are more irregular and we cannot easily distinguish them.
FIG. 7: Increasing the mass to $m = 5 \times 10^{-4}$ we obtain a binary system moving around the center of attraction.

FIG. 8: Newtonian analogous to Fig. 1. Three test-particles with $L_1 = 6.55, E_1 = 0.988; L_2 = 6.75, E_2 = 0.989$ and $L_3 = 5.68, E_3 = 0.984$.

FIG. 9: With the same values of $L$ and $E$ of the preceding figure, and $m = 5 \times 10^{-4}$ we get the Newtonian analogous to Fig. 8. We can see that one of the particles escapes.

FIG. 10: Increasing the mass, $m = 3 \times 10^{-3}$, we have that two of the particles quickly escapes from the system while the remaining one tends to stabilizes in a elliptical motion.

FIG. 11: The Newtonian analogous to the system presented in Fig. 4 is reproduced here. Two test-particles with $L_1 = 28.10, E_1 = 0.99937$ and $L_2 = 28.12, E_2 = 0.99937$. The Newtonian analogous to the system presented in Fig. 4 is reproduced.

FIG. 12: We use the same values of angular momentum and energy of the preceding figure with interacting particles with $m = 10^{-4}$. The Newtonian analogous to the system presented in Fig. 4 is reproduced. We obtain, as a final configuration, two quasi-elliptical motion.
FIG. 13: Using the same values of $L$ and $E$ as in the previous figure and increasing to $m = 10^{-3}$ we obtain that the motion of the particles are linked in such way that they alternate between a small and a large precessing ellipse.