ON THE INTEGRABILITY OF ORTHOGONAL DISTRIBUTIONS
IN POISSON MANIFOLDS

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ABSTRACT. In this article we study conditions for the integrability of the distribution defined on a regular Poisson manifold as the orthogonal complement (with respect to a pseudo-Riemannian metric) to the tangent spaces of the leaves of a symplectic foliation. Examples of integrability and non-integrability of this distribution are provided.

1. Introduction

Let \((M^n, P)\) be a regular Poisson manifold. Denote by \(S = \{S_m | m \in M\}\) the symplectic foliation of \(M\) by symplectic leaves (of constant dimension \(2 \leq k < n\) in the regular case). Denote by \(T(S)\) the sub-bundle of \(T(M)\) of tangent spaces to the symplectic leaves (the association \(x \to T_x(S)\) is an integrable distribution on \(M\) which we will also denote by \(T(S)\)). Let \(M\) be endowed with a pseudo-Riemannian metric \(g\) such that the restriction of \(g\) to each symplectic leaf is non-degenerate. By continuity, the signature of the restriction of \(g\) to \(T_m(S)\) is the same for all \(m \in M\).

Let \(\mathcal{N}_m = S_m^\perp\) be the subspace of \(T_m(M)\) that is \(g\)-orthogonal to \(S_m\). The association \(m \to \mathcal{N}_m\) defines a distribution \(\mathcal{N}\) which is transversal and complemental to the distribution \(T(S)\). The restriction of the metric \(g\) to \(\mathcal{N}\) is non-degenerate and has constant signature. In general, the distribution \(\mathcal{N}\) is not integrable.

If the metric \(g\) is Riemannian, and if the Poisson tensor is parallel with respect to the Levi-Civita connection \(\nabla = \nabla^g\) defined by \(g\), i.e. \(\nabla P = 0\), then it is a classical result of A. Lichnerowicz (see Vaisman 1994, Remark 3.11) that the distribution \(\mathcal{N}\) is integrable, and the restriction of the metric \(g\) to the symplectic leaves defines, together with the symplectic structure \(\omega_S = P|_{\mathcal{N}}\), a Kähler structure on symplectic leaves.

Integrability of the distribution \(\mathcal{N}\) depends strongly on the foliation \(S\) and its “transversal topology” (see Molino 1988, Reinhart 1983, Ch.4 for the Riemannian case). Thus, in general it is more a question of the theory of bundles with Ehresmann connections rather than that of Poisson geometry. Yet in some instances, it is useful to have integrability conditions in terms of the Poisson structure \(P\), and to relate integrability of the distribution \(\mathcal{N}\) with other structures of the Poisson manifold - Casimir functions, Poisson vector fields, etc.

Our interest in this question was influenced by our study of the representation of a dynamical system in metriplectic form, i.e. as a sum of a Hamiltonian vector field (with respect to a Poisson structure, (see Bloch et al 1996, Morrison 1986, Grmela 1990, Fish 2005) and a gradient one (with respect to a metric \(g\)). Integrability of the distribution \(\mathcal{N}\) guarantees that in the geometrical (local) splitting of the space \(M\) as a product of a symplectic leaf and a transversal submanifold with Casimir functions \(c^j\) as local coordinates (see Weinstein 1983), the transversal submanifold...
can be chosen to be invariant under the gradient flow (with respect to the metric $g$) of the functions $c^i$.

As a result one can separate observables of the system into Casimirs undergoing pure gradient (dissipative) evolution from those (along symplectic leaves) which undergo the mix of Hamiltonian and gradient evolutions. Such a separation leads to an essential simplification of the description of the transversal dynamics in metriplectic systems.

The structure of this article is as follows. In Section 2 we introduce necessary notions and notations. In Section 3 we obtain necessary and sufficient conditions on the metric $g$ and the tensor $P$ for the distribution $\mathcal{N}$ to be integrable. We derive these conditions in terms of covariant derivatives of the Poisson Tensor, in terms of covariant derivatives of Casimir covectors, and as conditions on the nullity of the Nijenhuis Torsion of the (1,1)-tensor $A^\mu_{\nu} = P^\mu_{\rho\sigma} g^\rho_{\nu\sigma}$. As a corollary we prove that the distribution $\mathcal{N}$ is integrable if parallel translation (via the Levi-Civita connection of the metric $g$) in the direction of $\mathcal{N}$ preserves the symplectic distribution $T(S)$.

In Section 4 we present integrability conditions in Darboux-Weinstein coordinates: the distribution $\mathcal{N}$ is integrable if and only if the following symmetry conditions are fulfilled for $\Gamma$

$$\Gamma_{IJs} = \Gamma_{JIs},$$

where $\Gamma_{I\beta\gamma} = g_{\alpha\sigma} \Gamma^\sigma_{\beta\gamma}$, and where capital Latin letters $I, J$ indicate the transversal coordinates, while small Latin letters indicate coordinates along symplectic leaves.

In Section 5 we describe some examples of non-integrability, describe a model example of a 4d Poisson manifold with Poisson structure of rank 2, where the distribution $\mathcal{N}$ is not integrable, and discuss nonintegrability in the case of a topologically nontrivial symplectic fibration.

In Section 6 we prove integrability of $\mathcal{N}$ for linear Poisson structures on dual spaces $g^*$ of real semi-simple Lie algebras $g$, with the metric $g$ induced by the Killing form, as well as for the dual $\mathfrak{se}(3)^*$ to the Lie algebra $\mathfrak{se}(3)$ of Euclidian motions with the simplest non-degenerate $ad^*$-invariant metric(s) (see Zefran et al. 1996).

2. Orthogonal distribution of Poisson manifold with a pseudo-Riemannian metric

Let $(M^n, P)$ be a regular Poisson manifold. We will be use local coordinates $x^\alpha$ in the domains $U \subset M$ with the corresponding local frame $\{\frac{\partial}{\partial x^\alpha}\}$ and the dual coframe $dx^\alpha$. Let $g$ be a pseudo-Riemannian metric on $M$ as above, and let $\Gamma$ denote the Levi-Civita connection associated with $g$. The tensor $P^{\tau\sigma}(x)$ defines a mapping

$$0 \rightarrow C(M) \rightarrow T^*(M) \xrightarrow{P} T(S) \rightarrow 0$$

where $C(M) \subset T^*(M)$ is the kernel of $P$ and $T(S)$ is (as defined above) the tangent distribution of the symplectic foliation $\{S^k\}$. The space $C(M)$ is a subbundle of the cotangent bundle $T^*M$ consisting of Casimir covectors. Locally, $C(M)$ is generated by differentials of functionally independent Casimir functions $c^i(x), \ i = 1, \ldots, n - k$ satisfying the condition $P^{\tau\sigma} dc^i_{\sigma} = 0$ (in this paper we assume the condition of summation by repeated indices).

We denote by $\mathcal{N}$ the distribution defined as the $g$-orthogonal complement $T(S)^\perp$ to $T(S)$ in $T(M)$. Then we have, at every point $x$ a decomposition into a
direct sum of distributions (sub-bundles)

\[ T_x M = T_x (S) \oplus N_x. \]

The assignment \( x \to N_x \) defines a transverse connection for the foliation \( S \), or, more exactly, for the bundle \((M, \pi, M/S)\) over the space of leaves \( M/S \), whenever one is defined (see below). We are interested in finding necessary and sufficient conditions on \( P \) and \( g \) under which the distribution \( N \) is integrable. By the Frobenius theorem, integrability of \( N \) is equivalent to the involutivity of the distribution \( N \) with respect to the Lie bracket of \( N \)-valued vector fields (sections of the sub-bundle \( N \subset T(M) \)).

Let \( \omega^i = \omega^i_{\mu} dx^\mu \) (\( i \leq d = n-k \)) be a local basis for \( C(M) \). For any \( \alpha \) in \( T^*(M) \), let \( \alpha^\sharp \) denote the image of \( \alpha \) under the bundle isomorphism \( \sharp: \alpha^* \to TM \) of index lifting induced by the metric \( g \). The inverse isomorphism (index lowering) will be conventionally denoted by \( \flat: T(M) \to T^*M \). We introduce the following vector fields: \( \xi_i = (\omega^i)^\sharp \in T(M) \).

**Lemma 1.** The vectors \( \xi_i \) form a (local) basis for \( N \).

**Proof.** Since \( g \) is non-degenerate, the vectors \( \xi_i \) are linearly independent and span a subspace of \( TM \) of dimension \( d \). For any vector \( \eta \in TM \),

\[ <\xi_i, \eta>_g = g_{\mu\nu} \xi^\mu_i \eta^{\nu} = g_{\mu\nu}g^{\mu\lambda} \omega^i_{\nu} \eta^\lambda = \omega^i_{\nu} \eta^\nu = \omega^i(\eta). \]

So the vector \( \eta \) is \( g \)-orthogonal to all \( \xi_i \) if and only if \( \eta \) is annihilated by each \( \omega^i \). That is, \( \eta \in \text{Ann}(C(M)) = \{ \lambda \in T(M) | \omega^j(\lambda) = 0, \forall j \leq d \} \). Since \( \text{Ann}(C(M)) = T(S) \), we see that the linear span of \( \{\xi_i\}^\perp \) is \( T(S) \).

**Definition 1.** The curvature (Frobenius Tensor) of the “transversal connection” \( N \) is defined as the bilinear mapping

\[ \mathcal{R}_N: T(M) \times T(M) \to T(S) \]

defined by

\[ \mathcal{R}_N(\gamma, \eta) = v([h\eta, h\gamma]), \quad (1) \]

where \( h: T(M) \to N \) is \( g \)-orthogonal projection onto \( N \), and \( v: T(M) \to T(S) \) is \( g \)-orthogonal projection onto \( T(S) \).

It is known (see DeLeón, Rodrigues 1989, Sec.1.15) that \( N \) is integrable if and only if the curvature \( \mathcal{R}_N \) defined above is identically zero on \( TM \times TM \).

**Remark 1.** An equivalent way to characterize the integrability of \( N \) is to use the structure tensor of J. Martinet or the D. Bernard structure tensor of the annihilator \( N^* \subset T^*(M) \) of the distribution \( N \) (see Libermann 1976).

3. Integrability criteria.

The condition (1) is equivalent to \( v([\gamma, \eta]) = 0 \), for all \( \gamma, \eta \in N \). If we write the vectors \( \gamma, \eta \) in terms of the basis \( \{\xi_i\} \), then we have

\[ v([\gamma^i \xi_i, \eta^j \xi_j]) = v(\gamma^i(\xi_i \cdot \eta^j)\xi_j - \eta^j(\xi_j \cdot \gamma^i)\xi_i + \gamma^i\eta^j[\xi_i, \xi_j]) \]

\[ = \gamma^i\eta^j v([\xi_i, \xi_j]), \quad \text{since } v(\xi_k) = 0 \text{ for all } i, j \leq d. \]

Thus \( \mathcal{R} = 0 \) if and only if \( v([\xi_i, \xi_j]) = 0 \) for all \( i, j \leq d \).
Consider the linear operator \( A : T(M) \to T(M) \) defined by the (1,1)-tensor field \( A^\tau_\mu = P^\tau_\sigma g_\sigma\mu \). Since \( g \) is non-degenerate we have \( \text{Im} A = T(S) \). Since each basis vector \( \xi_i \in \mathcal{N} \) is of the form \( \xi_i^\mu = g^{\mu\nu}\omega^i_\nu \) with \( \omega^i \in \ker P \), we also have \( A^\tau_\mu \xi_i^\mu = P^\tau_\sigma g_\sigma\mu g^{\mu\nu}\omega^i_\nu = P^\tau_\nu \omega^i_\nu = 0 \).

Therefore \( \mathcal{N} \subset \ker A \), and by comparing dimensions we see that \( \mathcal{N} = \ker A \). Notice that operator \( A \) and the orthonormal projector \( v \) have the same image and kernel. We conclude that \( R = 0 \iff A[\xi_i, \xi_j] = 0, \ \forall i, j \leq d \).

We now prove the main result of this section.

**Theorem 1.** Let \( \omega^i, 0 \leq i \leq d \) be a local basis for \( C(M) \) and let \( (\omega^i)^2 = \xi_i \) be the corresponding local basis of \( \mathcal{N} \). Let \( \nabla \) be the Levi-Civita covariant derivative on \( TM \) corresponding to the metric \( g \). Then the following statements are equivalent:

1. The distribution \( \mathcal{N} \) is integrable.
2. For all \( i, j \leq d \), and all \( \tau \leq n \),

\[
P^{\tau\sigma}(\nabla_{\xi_i}\omega^j_\sigma - \nabla_{\xi_j}\omega^i_\sigma) = 0.
\]  

(2)

3. For all \( i, j \leq d \), and all \( \tau \leq n \),

\[
g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(\omega^j \wedge \omega^i)_{\sigma\alpha} = 0,
\]  

(3)

where \( \nabla_\lambda = \nabla_{\partial/\partial x^\lambda} \).

4. For all \( i, j \leq d \), and all \( \tau \leq n \),

\[
P^{\tau\sigma}g_{\sigma\lambda}(\nabla_{\xi_i}\xi^\lambda_j - \nabla_{\xi_j}\xi^\lambda_i) = 0.
\]  

(4)

5. The sub-bundle \( C(M) \) is invariant under the following skew-symmetric bracket on 1-forms generated by the bracket of vector fields:

\[
[\alpha, \beta]_g = [\alpha^\sharp, \beta^\sharp]^g
\]  

(5)

i.e. if \( \alpha, \beta \in \Gamma(C(M)) \), then \( [\alpha, \beta]_g \in \Gamma(C(M)) \).

**Proof.** Since the Levi-Civita connection of \( g \) is torsion-free, we know that

\[
[\xi_i, \xi_j] = \nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i.
\]

Therefore, in a local chart \( (x^\alpha) \),

\[
A^\lambda_\alpha[\xi_i, \xi_j] = A^\lambda_\alpha(\nabla_{\xi_i}\xi^\lambda_j - \nabla_{\xi_j}\xi^\lambda_i)
\]

\[
= P^{\tau\sigma}g_{\sigma\lambda}(\nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i)
\]

\[
= P^{\tau\sigma}(\nabla_{\xi_i}\omega^i_\sigma - \nabla_{\xi_j}\omega^j_\sigma)
\]

In the last step we have used the fact that lifting and lowering of indices by the metric \( g \) commutes with the covariant derivative \( \nabla \) defined by the Levi-Civita connection of \( g \).

Recalling from the discussion before the Theorem that the integrability of the distribution \( \mathcal{N} \) is equivalent to the nullity of \( A[\xi_i, \xi_j] \) for all \( i, j \leq d \), we see that statements (1), (2), and (4) are equivalent.
To prove the equivalence of these statements to (3) we notice that
\[ P^{\tau\sigma}(\nabla_{\xi_i} \omega^j_\alpha - \nabla_{\xi_j} \omega^i_\alpha) = P^{\tau\sigma}(\xi^\lambda_i \nabla_{\lambda} \omega^j_\alpha - \xi^\lambda_j \nabla_{\lambda} \omega^i_\alpha) \]
\[ = P^{\tau\sigma}(\omega^i_\alpha \nabla_{\lambda} \omega^j_\sigma - \omega^j_\sigma \nabla_{\lambda} \omega^i_\alpha) \]
\[ = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha) \]
\[ = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\sigma \omega^j_\alpha - \omega^j_\alpha \omega^i_\sigma) \]
\[ = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha). \]
Here, at the third step we have used the following equality
\[ P^{\tau\sigma}g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha) = -g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha), \]
since \( P^{\tau\sigma} \omega^j_\sigma = 0 \) (similarly for the second term).

To prove the equivalence the (5) with the other statements we act as follows. Let \( \alpha = \alpha_i \omega^i \) and \( \beta = \beta_j \omega^j \) be any two sections of sub-bundle \( C(M) \subset T^*(M) \). Then \( \alpha^2 = \sum_i \alpha_i \xi_i \) and \( \beta^2 = \sum_j \beta_j \xi_j \).

So we have
\[ [\alpha, \beta]_g = [\nabla_{\beta_j} \xi_i (\alpha_i \omega^i) - \nabla_{\alpha_i} \xi_i (\beta_j \omega^j)] \]
\[ = \beta_j \left[ \alpha_i \nabla_{\xi_i} \omega^i + \frac{\partial \alpha_i}{\partial x^k} \xi^k_i \omega^i \right] - \alpha_i \left[ \beta_j \nabla_{\xi_i} \omega^j + \frac{\partial \beta_j}{\partial x^k} \xi^k_i \omega^j \right] \]
\[ = \alpha_i \beta_j (\nabla_{\xi_i} \omega^i - \nabla_{\xi_i} \omega^j) + \beta^2(\alpha_i) \omega^i - \alpha^2(\beta_j) \omega^j \]
\[ = \alpha_i \beta_j (\nabla_{\xi_i} \omega^i - \nabla_{\xi_i} \omega^j) + \sum_j \beta_j \xi^k_i \frac{\partial \alpha_i}{\partial x^k} \omega^i - \sum_i \alpha_i \xi^k_i \frac{\partial \beta_j}{\partial x^k} \omega^j \]
\[ = \alpha_i \beta_j [\omega^i, \omega^j]_g + (\beta^2(\alpha_i) - \alpha^2(\beta_i)) \omega^i. \] (6)

At the last step we have used the following (recall that \( \omega^i = (\xi_i)^\sigma \))
\[ \nabla_{\xi_i} \omega^i - \nabla_{\xi_i} \omega^j = (\nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i)^\sigma = [\xi_i, \xi_j]^\sigma = [\omega^i, \omega^j]^\sigma = [\omega^i, \omega^j]_g. \]
The second term in (6) is always in the kernel \( C(M) \) of \( P \), thus applying \( P \) to both sides yields
\[ P^{\tau\sigma}([\alpha, \beta]_g)_\sigma = \alpha_i \beta_j P^{\tau\sigma}([\omega^i, \omega^j]_g)_\sigma = \alpha_i \beta_j P^{\tau\sigma}g_\sigma(\nabla_{\xi_j} \xi^\lambda_i - \nabla_{\xi_i} \xi^\lambda_j). \]
Therefore, condition (4) above holds if and only if the space of sections of the bundle \( C(M) \) of Casimir covectors is invariant under the bracket \([-,-]_g\).

**Corollary 1.** If \( (\nabla_{(\omega^i)})^\tau\sigma \omega^j_\alpha = 0 \) for all \( \sigma, i \) and \( j \), i.e. if \( \nabla_{(\omega^i)}P|_{C(M)} = 0 \) for all \( i \), then the distribution \( \mathcal{N} \) is integrable.

**Proof.** In the proof of the equivalence of statements (1) and (2) with statement (3) in the Theorem, it was shown that
\[ P^{\tau\sigma}(\nabla_{\xi_i} \omega^j_\alpha - \nabla_{\xi_j} \omega^i_\alpha) = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha) \]
\[ = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^j_\sigma \omega^i_\alpha - \omega^i_\alpha \omega^j_\sigma) \]
\[ = -g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\sigma \omega^i_\alpha) \]
\[ = g^{\lambda\alpha}(\nabla_{\lambda} P)^{\tau\sigma}(\omega^i_\sigma \omega^j_\alpha - \omega^j_\alpha \omega^i_\sigma). \]
Since \( \xi_i = (\omega^i)^\sigma \) for each \( i \), if \( \nabla_{(\omega^i)}P|_{C(M)} = 0 \), then condition (3) of the Theorem is fulfilled. \( \square \)

The following criteria specify the part of the A. Lichnerowicz condition that \( P \) is \( g \)-parallel (see Vaisman 1994) ensuring the integrability of the distribution \( \mathcal{N} \):
Corollary 2. If $\nabla_{\alpha^\sharp} : T(M) \to T(M)$ preserves the tangent sub-bundle $T(S)$ to the symplectic leaves for every $\alpha \in C(M)$, then $\mathcal{N}$ is integrable. 

Proof. If the parallel translation $\nabla_{\alpha^\sharp}$ along the trajectories of the vector field $\xi = \alpha^\sharp$ preserves $T(S)$, then it also preserves its $g$-orthogonal complement $\mathcal{N}$, and hence the dual to parallel translation in the cotangent bundle will preserve sub-bundle $C(M) = \mathcal{N}^\flat$ (see Lemma 1). That is, 

$$P_{\tau_\sigma} \nabla_{\alpha^\sharp} \beta_{\tau_\sigma} = 0$$

for any $\beta$ in $C(M)$. Writing this equality in the form $(\nabla_{\alpha^\sharp} P)^{\tau_\sigma} \beta_{\tau_\sigma} = 0$ and using the previous Corollary we get the result. \hfill \Box

Remark 2. Lichnerowicz’s condition, i.e. the requirement that $\nabla P = 0$, guarantees much more than the integrability of the distribution $\mathcal{N}$ and, therefore, the local splitting of $M$ into a product of a symplectic leaf $S$ and complemental manifold $\mathcal{N}$ with zero Poisson tensor. It also guarantees regularity of the Poisson structure, and reduction of the metric $g$ to the block diagonal form $g_S + g_N$, with the corresponding metrics on the symplectic leaves and maximal integral manifolds $N_m$ of $\mathcal{N}$ being independent on the complemental variables (i.e. the metric $g_S$ on the symplectic leaves is independent from the coordinates $y$ along $N_m$). Furthermore, the condition $\nabla P = 0$ also ensures the independence of the symplectic form $\omega_S$ from the transversal coordinates $y$ (see Vaisman 1994, Remark 3.11). Finally from $\nabla g^S \omega_S = 0$ follows the existence of a $g_S$-parallel Kahler metric on the symplectic leaves.

Corollary 3. Let $\nabla_\lambda \omega^i = 0$ for all $\lambda, i$ (i.e. the 1-forms $\omega^i = dc^i$ are $\nabla^g$-covariantly constant). Then 

i) The distribution $\mathcal{N}$ is integrable,

ii) the vector fields $\xi_i$ are Killing vector fields of the metric $g$, and 

iii) the Casimir functions $c^i$ are harmonic: $\Delta g^i = 0$.

Proof. The first statement is a special case of (3) in the Theorem above. To prove the second, we calculate the Lie derivative of $g$ in terms of the covariant derivative $\nabla \omega^i$,

$$(\mathcal{L}_\xi g)_{\sigma\lambda} = g_{\gamma\lambda} \nabla_{\sigma} \xi^\gamma_{\lambda} + g_{\sigma\tau} \nabla_{\lambda} \xi^\tau_{\lambda}$$

$$= \nabla_{\sigma} \omega^i_{\lambda} + \nabla_{\lambda} \omega^i_{\sigma}$$

$$= \frac{\partial \omega^i_{\lambda}}{\partial x^\nu} + \frac{\partial \omega^i_{\sigma}}{\partial x^\lambda} - \omega^i_{\gamma}(\Gamma^\gamma_{\sigma\lambda} + \Gamma^\gamma_{\sigma\lambda})$$

$$= \frac{\partial^2 c^i}{\partial x^\sigma x^\lambda} + \frac{\partial c^i}{\partial x^\lambda x^\sigma} - 2 \omega^i_{\gamma} \Gamma^\gamma_{\sigma\lambda}$$

$$= 2 \frac{\partial^2 c^i}{\partial x^\sigma x^\lambda} - 2 \omega^i_{\gamma} \Gamma^\gamma_{\sigma\lambda}$$

$$= 2 \nabla_{\lambda} \omega^i_{\sigma}.$$ 

Thus, if the condition of the Corollary is fulfilled, $\xi_i$ are Killing vector fields. The third statement follows from

$$\Delta g^i = div_g(\xi_i = (dc^i)^\sharp) = \frac{1}{2} Tr_g(\mathcal{L}_\xi g) = \frac{1}{2} g^{\lambda\mu}(\mathcal{L}_\xi g)_{\lambda\mu}.$$ 

\hfill \Box
3.1. **Nijenhuis Tensor.** Conventionally the integrability of different geometrical structures presented by a \((1,1)\)-tensor field can be characterized in terms of the corresponding Nijenhuis tensor. Thus, it is interesting to see the relation of our criteria presented above to the nullity of the corresponding Nijenhuis tensor.

**Definition 2.** Given any \((1,1)\) tensor field \(J\) on \(M\), there exists a tensor field \(N_J\) of type \((1,2)\) (called the Nijenhuis torsion of \(J\)) defined as follows (see DeLeon and Rodrigues 1989, Sec.1.10):

\[
N_J(\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - J[\xi, J\eta] + J^2[\xi, \eta]
\]

for all vector fields \(\xi, \eta\).

If \(J\) is an **almost product structure**, i.e. \(J^2 = Id\), then \(N_J = 0\) is equivalent to the integrability of \(J\). In fact, given such a structure on \(M\), we can define projectors \(v = (1/2)(Id + J)\) and \(h = (1/2)(Id - J)\) onto complementary distributions \(Im(v)\) and \(Im(h)\) in \(TM\) such that at each point \(x \in M\),

\[
T_xM = Im(v)_x \oplus Im(h)_x.
\]

It is known (see DeLeon and Rodrigues 1989, Sec.3.1) that \(J\) is integrable if and only if \(Im(v)\) and \(Im(h)\) are integrable, and that the following equivalences hold:

\[
N_J = 0 \iff N_h = 0 \iff N_v = 0.
\]

Consider now the two complementary distributions \(T(S)\) and \(N\) discussed above. Suppose that \(v\) is \(g\)-orthogonal projection onto the distribution \(T(S)\), and \(h\) is \(g\)-orthogonal projection onto \(N\). Applying these results in this setting we see that that the distribution \(N\) is integrable if and only if \(N_v = 0\).

Since \(v^2 = v\), and since any \(\xi \in T(M)\) can be expressed as \(\xi = v\xi + h\xi\), we have

\[
N_v(\xi, \eta) = [v\xi, v\eta] - v[v\xi, v\eta + h\eta] - v[v\xi + h\xi, v\eta] + v[v\xi + h\eta, v\eta + h\eta]
\]

\[
= (Id - v)[v\xi, v\eta] + v[h\xi, v\eta] + v[h\xi, v\eta]
\]

for all \(\xi\) and \(\eta\) in \(T(M)\). Since \(T(S)\) is integrable we have \([v\xi, v\eta] \in T(S)\), and so

\[
N_v(\xi, \eta) = v[h\xi, h\eta].
\]

As a result, we can restrict \(\xi\) and \(\eta\) to be sections of the distribution \(N\) to get the following integrability condition for \(N\) in terms of \((1,1)\)-tensor \(v\):

\(N\) is integrable \(\iff N_v(\xi, \eta) = v^\mu(\xi, \eta)^\mu = -\partial_\xi v^\mu(\xi \wedge \eta)^\mu = 0\),

for all \(\mu\) and all \(\xi, \eta \in \Gamma(N)\). The equality (*) on the right is proved in the same way as the similar result for the action of \(P^{\tau\sigma}\) in the proof of statement (3) of Theorem 1.

The tensor \(A_\nu^\mu = g_{\nu\sigma}P^{\sigma\mu}\) discussed above can be considered to be a linear mapping from \(T(M)\) to \(T(S)\), but since \(A\) is not idempotent, it does not define a projection. However, the tensors \(A\) and \(v\), having the same kernel and image are related in the sense that the integrability of \(N\) is also equivalent to

\[
A_\nu^\mu(\xi, \eta)^\nu = -\partial_\eta A_\nu^\mu(\xi \wedge \eta)^\mu = 0,
\]

for all sections \(\xi\) and \(\eta\) of the distribution \(N\) (using the same argument as for the tensor \(v^\mu\) above).

In fact, since the linear mappings \(A, v\) of \(T_m(M)\) have the same kernel and image for all \(m \in M\), there exists a (non-unique) pure gauge automorphism \(D : T(M) \to T(M)\).
T(M) of the tangent bundle (i.e. inducing the identity mapping of the base M and, therefore, defined by a smooth (1,1)-tensor field \( D_\nu^\mu \)) such that \( A_\sigma^\mu = D_\nu^\mu v_\sigma^\nu \). For any couple \( \xi, \eta \) of sections from \( \Gamma(N) \), we have

\[
A_\mu^\sigma [\xi, \eta] = -\partial_\nu A_\mu^\nu (\xi \wedge \eta)^\sigma = -\partial_\nu D_\kappa^\mu (v_\sigma^\kappa (\xi \wedge \eta)^\nu) + D_\kappa^\mu \partial_\nu v_\sigma^\kappa (\xi \wedge \eta)^\nu = -D_\kappa^\mu \partial_\nu v_\sigma^\kappa (\xi \wedge \eta)^\nu = -D_\kappa^\mu N_\nu (\xi, \eta) .
\]

This proves

**Theorem 2.** There exists a (not unique) invertible linear automorphism \( D \) of the bundle \( T(M) \) such that for all couples of vector fields \( \xi, \eta \in \Gamma(N) \)

\[
A[\xi, \eta] = D(N_\nu (\xi, \eta)).
\]

Thus, \( N_\nu|_{\mathcal{N} \times \mathcal{N}} \equiv 0 \) iff \( A[\xi, \eta] = 0 \) for all \( \xi, \eta \in \Gamma(N) \).

---

4. Local criteria for integrability

Since \( M \) is regular, any point in \( M \) has a neighborhood in which the Poisson tensor \( P \) has, in Darboux-Weinstein (DW) coordinates \((y^A, x^i)\), the following canonical form (see Weinstein 1983)

\[
P = \begin{pmatrix}
0_{p \times p} & 0_{p \times 2k} \\
0_{2k \times p} & I_k & -I_k \\
& 0_k & 0_k
\end{pmatrix}
\]

We will use Greek indices \( \lambda, \mu, \tau \) for general local coordinates, small Latin \( i, j, k \) for the canonical coordinates along symplectic leaves and capital Latin indices \( A, B, C \) for transversal coordinates. In these DW-coordinates we have, since \( P \) is constant,

\[
(\nabla_\lambda P)^\tau_\sigma = P^{ij}_\sigma \Gamma^T_{j \lambda} - P^{ij}_\tau \Gamma^T_{j \lambda} .
\]

Using the structure of the Poisson tensor we get, in matrix form,

\[
(\nabla_\lambda P)^\tau_\sigma = \begin{pmatrix}
0_{p \times p} & P^{ij}_\tau \Gamma^T_{j \lambda} \\
-P^{ij}_\mu \Gamma^s_{j \lambda} & P^{ij}_\mu \Gamma^s_{j \lambda} - P^{ij}_\mu \Gamma^s_{j \lambda}
\end{pmatrix},
\]

where the index \( \tau \) takes values \((T, t)\), and the index \( \sigma \) takes values \((S, s)\), transversally and along the symplectic leaf respectively.

In DW-coordinates we choose \( \omega^\tau = dy^\tau \) as a basis for the co-distribution \( C(M) \). Now we calculate (using the symmetry of the Levi-Civita connection \( \Gamma \))

\[
(\nabla_\lambda P)^\tau_\sigma (dy^I \wedge dy^J)_{\alpha \sigma} = -\delta^I_\alpha P^{ij}_\tau \Gamma^T_{j \lambda} + \delta^I_\alpha P^{ij}_\tau \Gamma^T_{j \lambda},
\]

so that

\[
g^{\lambda \alpha} (\nabla_\lambda P)^\tau_\sigma (dy^I \wedge dy^J)_{\alpha \sigma} = P^{ij}_\tau [g^{\lambda \lambda} \Gamma^T_{j \lambda} - g^{\lambda \lambda} \Gamma^T_{j \lambda}].
\]

This expression is zero if \( \tau = T \), so the summation goes by \( \tau = t \) only.

Substituting the Poisson Tensor in its canonical form we get the integrability criteria (3) of Theorem 1 in the form

\[
g^{\lambda \alpha} \Gamma^T_{J \lambda} - g^{\lambda \lambda} \Gamma^T_{J \lambda} = 0, \forall I, J, t.
\]

Using the metric \( g \) to lower indices, we finish the proof of the following
Theorem 3. Let \((y^I, x^i)\) be local DW-coordinates in \(M\). Use capital Latin indices for transversal coordinates \(y\) along \(N\) and small Latin indices for coordinates \(x\) along symplectic leaves. Then the distribution \(N\) is integrable if and only if
\[
\Gamma_{JIt} = \Gamma_{IJt}, \quad \forall \ I, J, t.
\]
(7)

5. Examples: Non-integrability

5.1. A Model 4d system. We now consider a (local) model example of the lowest possible dimension where the distribution \(N_g\) may not be integrable. This is the case of a 4-d Poisson manifold \((M = \mathbb{R}^4, P)\) where \(\text{rank}(P) = 2\) at all points of the manifold \(M\).

Let \(P_{ij}\) be the canonical Poisson tensor given in the global coordinates \(x^\alpha\) by the following 4 \(\times\) 4 matrix:
\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]
Let \(\omega^1 = dx^1\), and \(\omega^2 = dx^2\). Then \(\{\omega^1, \omega^2\}\) is a basis for the kernel \(C(M)\) of \(P\), and
\[
(\omega^1 \wedge \omega^2)_{\alpha\sigma} = \begin{cases}
1, & \alpha = 1, \sigma = 2 \\
-1, & \alpha = 2, \sigma = 1 \\
0, & \text{otherwise}.
\end{cases}
\]
(8)

If \(h\) is the Euclidian metric in \(\mathbb{R}^4\), then it is obvious that the \(h\)-orthogonal distribution \(N_h\) is generated by the basic vector fields \(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\) and is trivially integrable.

Let now \(g\) be an arbitrary pseudo-Riemannian metric defined on \(M = \mathbb{R}^4\) by a non-degenerate symmetric (0,2)-tensor \(g_{\lambda\mu}\). The corresponding \(g\)-orthogonal distribution is denoted by \(N\) and the Levi-Civita connection of the metric \(g\) by \(\nabla\).

Consider \(\nabla_{\lambda}P^{\tau\sigma} = \partial_{\lambda}P^{\tau\sigma} + P^{\tau\rho}\Gamma_{\lambda}{}^{\sigma}_{\rho \mu} + P^{\rho\sigma}\Gamma_{\lambda}{}^{\tau}_{\rho \mu}\). Since \(P\) is constant, the first term of this expression is always zero. Furthermore, since each \(\omega^k\) is in the kernel of \(P\), we see that the third term in this expression will contract to zero with \((\omega^1 \wedge \omega^2)_{\alpha\sigma}\). Therefore,
\[
g^{\lambda\alpha}\nabla_{\lambda}P^{\tau\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda\alpha}P^{\tau\rho}\Gamma_{\lambda}{}^{\sigma}_{\rho \mu}(\omega^1 \wedge \omega^2)_{\alpha\sigma}.
\]
\[
= g^{\lambda 1}P^{\tau\mu}\Gamma_{\lambda}{}^{1}_{\mu} - g^{\lambda 2}P^{\tau\mu}\Gamma_{\lambda}{}^{1}_{\mu}, \quad \text{by } (8).
\]
The only values of \(\tau\) for which \(P^{\tau\mu} \neq 0\) are \(\tau = 3\) and \(\tau = 4\). We consider each case individually:

\(\tau = 3:\)
\[
g^{\lambda\alpha}\nabla_{\lambda}P^{3\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda 1}P^{34}\Gamma_{\lambda}{}^{2}_{4} - g^{\lambda 2}P^{34}\Gamma_{\lambda}{}^{1}_{4},
\]
\[
= g^{\lambda 1}\Gamma_{\lambda}{}^{2}_{4} - g^{\lambda 2}\Gamma_{\lambda}{}^{1}_{4},
\]
\[
= \frac{1}{2}(g^{\lambda 1}g^{2\delta} - g^{\lambda 2}g^{1\delta})(g_{\lambda\delta,4} + g_{4\delta,\lambda} - g_{\lambda 4,\delta}),
\]
\[
= g^{\lambda 1}g^{2\delta}(g_{4\delta,\lambda} - g_{4\lambda,\delta}).
\]
\[ \tau = 4: \]
\[ g^\lambda \nabla_\lambda P^\tau (\omega^1 \wedge \omega^2)_{\alpha \sigma} = g^{\lambda 1} P^{43} \Gamma^{2}_{\lambda 3} - g^{\lambda 2} P^{43} \Gamma^{1}_{\lambda 3}, \]
\[ = -g^{\lambda 1} \Gamma^{2}_{\lambda 3} + g^{\lambda 2} \Gamma^{1}_{\lambda 3}, \]
\[ = \frac{1}{2}(-g^{\lambda 1} g^{2 \delta} + g^{\lambda 2} g^{1 \delta})(g_{\lambda \delta, 3} + g_{3 \lambda, \delta} - g_{\lambda 3, \delta}), \]
\[ = g^{\lambda 1} g^{2 \delta}(g_{3 \lambda, \delta} - g_{3 \delta, \lambda}). \]

Thus, the integrability condition takes the form of the following system of equations
\[ g^{\lambda 1} g^{2 \delta}(g_{3 \lambda, \delta} - g_{3 \delta, \lambda}) = 0, \]
\[ g^{\lambda 1} g^{2 \delta}(g_{4 \delta, \lambda} - g_{4 \lambda, \delta}) = 0 \]
equivalent to the symmetry conditions (7).

Clearly both expressions are zero if \( g \) is diagonal. In fact, if \( g \) is block-diagonal, then both of the above terms will also vanish. For these special types of metric, the transversal distribution \( N \) is integrable. For more general metrics, however, \( N \) may not be integrable. For example, let
\[ g = \begin{pmatrix}
1 & 0 & f & 0 \\
0 & 1 & 0 & 0 \\
f & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \]
where \( f(x) \) satisfies to the condition \( \partial_2 f \neq 0 \). This symmetrical matrix has 1, 1, 1 + \( f, 1 - f \) as its eigenvalues. Thus \( g \) determines the Riemannian metric in the region \(|f| < 1\), and the condition
\[ g^\lambda \nabla_\lambda P^\tau (\omega^1 \wedge \omega^2)_{\alpha \sigma} = 0 \]
fails since, for \( \tau = 4 \) we have:
\[ g^\lambda \nabla_\lambda P^{4 \delta} (\omega^1 \wedge \omega^2)_{\alpha \sigma} = g^{\lambda 1} g^{2 \delta} (g_{3 \lambda, \delta} - g_{3 \delta, \lambda}), \]
\[ = g^{\lambda 1} (g_{3 \lambda, 2} - g_{3 2, \lambda}), \]
\[ = g^{\lambda 1} g_{3 \lambda, 2}, \]
\[ = g^{11} g_{31, 2}, \]
\[ = \partial_2 f \neq 0. \]

As an example of such a function \( f \) for which both conditions (i.e. conditions \(|f| < 1 \) and \( \partial_2 f \neq 0 \)) are fulfilled in the whole space \( \mathbb{R}^4 \) we can take the function \( f(x^1, \ldots, x^4) = \frac{1}{2} tan^{-1}(x^2) \) where the principal branch of \( tan^{-1}(x) \) is chosen (taking values between \(-\pi/2 \) and \( \pi/2 \)).

We can also see that the distribution \( N \) is not integrable by a direct computation. Observe that the local basis vectors for \( N \) are:
\[ \xi_1 = \partial_1 + f \partial_3, \quad \xi_2 = \partial_2. \]
Their Lie bracket is \([\xi_1, \xi_2] = \partial_2 f \partial_3\), which is not in the span of \( \{\xi_1, \xi_2\} \) (since \( \partial_2 f \neq 0 \)), hence \( N \) is not integrable.
5.2. Case of a symplectic fibration. Here we discuss a situation that demonstrates that the very possibility to choose a global metric $g$ such that the distribution $\mathcal{N}_g$ is integrable is determined mostly by the topological properties of the “bundle” of leaves of the symplectic foliation, i.e. the existence of a zero curvature Ehresmann connection.

A topologically simple (with regard to the transversal structure) example of a regular Poisson manifold is a symplectic fibration: a fiber bundle $(M, \pi, B)$ such that every fiber $S_b = \pi^{-1}(b)$ is endowed with a symplectic structure $\omega_S = \omega_b$. Each fiber is symplectomorphic to the model symplectic manifold $(F, \omega)$, and the transition functions of a trivialization of this bundle are symplectic isomorphisms on the fibers (see Guillemin et al 1996, Ch. 1). The inverse $P_b = \omega_b^{-1}$ of the symplectic form on each symplectic fiber defines, via the embedding $\bigwedge T(S_b) \to \bigwedge T(M)$, a smooth $(2,0)$-tensor field $P$, i.e. a regular Poisson structure on $M$.

The distribution $\mathcal{N}_g$, which is $g$-orthogonal to the fibers $S_b$ with respect to some (pseudo-)Riemannian metric on $M$ (with the condition that the restriction of $g$ to the fiber $S_b$ is non-degenerate), defines an Ehresmann connection $\Gamma_g$ on the bundle $(M, \pi, B)$. Integrability of the distribution $\mathcal{N}_g$ (i.e. integrability of the connection $\Gamma_g$) means that the curvature (Frobenius tensor) of the connection $\Gamma_g$ is zero.

On the bundle $(M, \pi, B)$ of a symplectic fibration there is a special class of symplectic connections $\Gamma$ distinguished by the condition that the holonomy mappings are symplectic diffeomorphisms of the fibers. It is proved in Guillemin et al 1996, Ch. I that if $F$ is compact, connected and simply connected, then for such a connection there exists a closed 2-form $\omega_\Gamma$ on $M$ whose restrictions to any fiber $S_b$ coincide with $\omega_b$, and such that the orthogonal complement $\omega_\Gamma$ of the tangent space $T_m(S)$ to the fiber passing through a point $m \in M$ is exactly the horizontal subspace $\text{Hor}_\Gamma(m)$ of the connection $\Gamma$ at the point $m$. The curvature of the connection $\Gamma$, which measures the degree of “non-integrability” of the distribution $\text{Hor}_\Gamma$, is determined by the form $\omega_\Gamma$ through the curvature identity proved in Guillemin et al 1996, Ch. I. Namely, let $v_1, v_2$ be two arbitrary vector fields on $B$ and denote by $v_1^1, v_2^1$ their horizontal lifts to vector fields in $M$. Then the curvature of $\Gamma$ is the vertical (i.e. restricted to the fiber) part of the 1-form $i_{[v_1^1, v_2^1]} \omega_\Gamma$, and one has the equality

$$-di_{[v_1^1, v_2^1]} \omega_\Gamma = i_{[v_1, v_2]} \omega_\Gamma \mod B$$

where $\mod B$ means “restricted to the fiber”. This restriction is zero if and only if the function $H = i_{v_1} i_{v_2} \omega_\Gamma$ is constant along the fibers $S_b$. Then $H = \pi^* h$ for some $h \in C^\infty(B)$ is a Casimir function for the Poisson structure on $M$ constructed as described above.

Having a connection $\Gamma$ (symplectic or not) available on the bundle $(M, \pi, B)$, one can define a whole class of (pseudo-)Riemannian metrics for which the orthogonal complement of $T(S)$ will coincide with $\text{Hor}_\Gamma$. Namely, we take a metric $g_{S,b}$ on $T_m(S)$ smoothly depending on the point $b$. Then we take an arbitrary metric $g_B$ on the base $B$ and lift it to the horizontal subspaces of $\Gamma$. The metric $g$ on the total space of the bundle $M$ is now defined by the condition of orthogonality of $T(S)$ and $\text{Hor}_\Gamma$. The projection $\pi : M \to B$ becomes a (pseudo-)Riemannian submersion (see Cheeger and Ebin 1975). There is a relation between the curvatures of $g_B, g$ and the curvature of the connection $\Gamma$ (O’Neill formula, see Cheeger and Ebin 1975, 3.20). Let $m \in M$, $b = \pi(m)$, $X, Y \in T_b(B)$ be two arbitrary tangent vectors at $b$, and let $\bar{X}, \bar{Y} \in \text{Hor}_\Gamma(m) \subset T_m(M)$ be their horizontal lifts at the point $m$. Then
for the sectional curvatures $K$ of the metric $g_B$ and $\bar{K}$ of $g$ one has

$$K_b(X, Y) = \bar{K}_m(\bar{X}, \bar{Y}) + \frac{3}{4} \| [\bar{X}, \bar{Y}]^{\text{vert}}(m) \|^2_{g_B},$$

where $^{\text{vert}}$ means taking the vertical component of the bracket of horizontal lifts to a neighborhood of $m$ of arbitrary vector fields in $B$ having values $X, Y$ at the point $b$. Thus, the curvature of the connection $\Gamma$ measures the difference between the sectional curvatures of the metric $g_B$ and its horizontal lift to the distribution $\text{Hor}_\Gamma$.

It is easy to construct examples of bundles which do not allow integrable Ehresmann connections using the following arguments. Suppose that a bundle $(M, \pi, B)$ with a simply-connected base $B$ allows an integrable Ehresmann connection $\Gamma$. The holonomy group of the connection $\Gamma$ is discrete (by the Ambrose-Singer Theorem, since the curvature is zero) and, therefore, any maximal integral submanifold (say, $V$) of $\Gamma$ is a covering of $B$. Since $B$ is simply-connected, the projection $\pi : V \rightarrow B$ is a diffeomorphism. Pick a point $b \in B$. Then every maximal integral manifold intersects the fiber $F_b$ at one point, defining in this way a smooth diffeomorphism $q : M \rightarrow F_b \simeq F$ smoothly depending on $b$. Together with the projection $\pi$ this mapping defines a trivialization $(\pi, q) : M \rightarrow B \times F$ of the bundle $(M, \pi, B)$.

This proves the following

**Proposition 1.** Let $(M, \pi, B; (F, \omega))$ be a symplectic fibration with the model symplectic fiber $(F, \omega)$, base $B$ and the total Poisson space $M$. If the bundle $(M, \pi, B)$ is topologically non-trivial, then the Poisson manifold $(M, P = \omega_b^{-1})$ cannot be endowed with a (global) pseudo-Riemannian metric $g$ such that the orthogonal distribution $\mathcal{N}_g$ would be integrable.

Thus, if we take an arbitrary nontrivial bundle over a simply-connected manifold $B$, it can not have a nonlinear connection of zero curvature. An example is the tangent bundle $(T(\mathbb{C}P(2)), \pi, \mathbb{C}P(2))$ over $B = \mathbb{C}P(2)$, where the standard symplectic structure on $B = \mathbb{C}P(2)$ determines a (constant) symplectic structure along the fibers.

6. **Examples of Integrability: Linear Poisson structure**

Let $\mathfrak{g}$ be a real $n$-dimensional Lie algebra with a basis $\{e_\mu\}$ and Lie bracket $[e_\mu, e_\nu] = c^\sigma_{\mu\nu} e_\sigma$. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. The Killing form $K$ on $\mathfrak{g}$ is the invariant, symmetric, bilinear form defined by

$$K(x, y) = Tr(ad(x) \circ ad(y)), \quad K_{\mu\nu} = Tr(ad_{e_\mu} \circ ad_{e_\nu}),$$

where $ad_v(X) = [v, X]$, $X \in \mathfrak{g}$ (see Bourbaki 1968, Ch.3). Let $\{f^\nu\}$ be the dual basis on the dual space $\mathfrak{g}^*$, and let $\lambda_\nu$ be coordinates for $\mathfrak{g}^*$ relative to this basis: $\lambda = \lambda_\nu f^\nu$, $\lambda_\nu = <\lambda, e_\nu>$. The dual space $\mathfrak{g}^*$ with its linear Lie-Poisson structure

$$P^{\mu\nu}(\lambda) = \{\lambda_\mu, \lambda_\nu\} = c^\sigma_{\mu\nu}\lambda_\sigma, \quad (9)$$

is a model example of a Poisson manifold. The adjoint action $Ad(g)$ of the corresponding Lie group $G$ on $\mathfrak{g}$ defines the linear co-adjoint action $Ad^*(g)$ of $G$ on $\mathfrak{g}^*$. The symplectic leaves of the Lie-Poisson structure (9) are co-adjoint orbits of $G$ (see Kirillov, 2004).

Casimir functions are exactly the $Ad^*(G)$-invariant functions on $\mathfrak{g}^*$. In many cases (for instance for real semi-simple Lie groups) one can choose $k = \text{rank}(G)$
polynomial Casimir functions \( c_i \) that are functionally independent on \( M = \mathfrak{g}^*_{\text{reg}} \), and any Casimir function is a function of the polynomials \( c_i \) (see Kirillov, 2004).

The tangent space to \( \mathfrak{g}^* \) at each point can be identified with the (vector) space \( \mathfrak{g}^* \) itself and, correspondingly, the tangent bundle \( T(\mathfrak{g}^*) \) splits \( Ad^*(G) \)-equivariantly: \( T(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g}^* \). The cotangent bundle \( T^*(\mathfrak{g}^*) \) takes the form \( T^*(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g} \).

The action of \( G \) induced by \( Ad^*(G) \) on \( T(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g}^* \) is \( Ad^*(g) \times Ad^*(g) \), while the dual action on the cotangent bundle \( T^*(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g} \) takes the form \( Ad^*(g) \times Ad(g) \). In particular, the action on the second factor coincides with the adjoint action of \( G \) on \( \mathfrak{g} \). Below we will be using these identifications without further comments.

If \( \mathfrak{g} \) is a real semi-simple Lie algebra, the Killing form \( K \) is non-degenerate and can be used to identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \). Under this identification, the adjoint action of \( G \) corresponds to the co-adjoint and, correspondingly, adjoint orbits correspond to co-adjoint ones. Thus, one can translate the linear Poisson structure to the Lie algebra \( \mathfrak{g} \) and use the available information about the (singular) foliation of \( \mathfrak{g} \) by the adjoint orbits (see Warner 1972, Sec.1.3) to study the \( K \)-orthogonal distribution \( \mathcal{N}_K \) defined by the (pseudo-Riemannian) Killing metric \( K \).

6.1. **Compact semi-simple Lie algebra.** Consider the case when \( \mathfrak{g} \) is a compact real semi-simple Lie algebra, i.e. the Lie algebra of a compact semi-simple Lie group. Then the Killing form \( K \) is negative definite, and, therefore, \( -K \) is an invariant Riemannian metric on \( \mathfrak{g} \) (see Knapp, 1996, Ch.4).

The canonical isomorphism \( T^*_x(\mathfrak{g}^*) \cong \mathfrak{g} \) defined above allows us to consider the Killing form \( -K_{\mu\nu} \) on \( \mathfrak{g} \) as a covariant metric \( g^{\mu\nu} \) on \( \mathfrak{g}^* \). Basic vectors \( e_\mu \in \mathfrak{g} \) are identified with the covectors \( d\lambda^\mu \) in \( \mathfrak{g}^* \). To be consistent with the upper/lower indices duality we will denote these basic covectors in \( \mathfrak{g}^* \) by \( e^\mu \).

Recall that we have the following condition (see (3) in Theorem 1) for the \( g \)-orthogonal space \( \mathcal{N} \) to be integrable.

\[
\mathcal{N} \text{ is integrable } \iff \ g^{\gamma\alpha} \nabla^\gamma P^{\tau\sigma} (\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha) = 0, \ \forall \tau,
\]

where \( \omega \) and \( \eta \) are any two elements in the kernel of \( P \). Using this condition we can prove the following

**Theorem 4.** Let \( \mathfrak{g} \) be a compact semi-simple Lie algebra and let \( M = \mathfrak{g}^*_{\text{reg}} \) with the standard linear Lie-Poisson structure. Let \( g \) be the inverse to the metric on \( \mathfrak{g} \) given by the restriction of the negative Killing form on \( \mathfrak{g} \). Then the distribution \( \mathcal{N} \) is integrable if and only if \( C(M) \) is an abelian subalgebra of \( T^*(M) \) endowed with the (pointwise on \( \mathfrak{g}^* \)) bracket induced from the Lie algebra \( \mathfrak{g} \) via the identification \( T^*(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g} \).

**Proof.** Let \( \omega = \omega_\alpha e^\alpha \) and \( \eta = \eta_\beta e^\beta \) be two (local) sections of \( C(M) \). Then we have

\[
g^{\gamma\alpha} \nabla^\gamma P^{\tau\sigma} (\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha) = g^{\gamma\alpha} \nabla^\gamma P^{\tau\sigma} (\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha),
\]

\[
= g^{\gamma\alpha} c^\gamma_{\tau\sigma} (\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha).
\]
Here the expression (9) for the Poisson tensor was used as well as the flatness of the metric $-K$ on $M = g_{\text{reg}}^\ast$. Observe that
\[
g^{\gamma\alpha}_c e^\gamma_{\tau\sigma} = g^{\gamma\alpha}_c [e^\tau, e^\sigma],
\]
\[
= -\langle e^\alpha, [e^\sigma, e^\tau]\rangle_g,
\]
\[
= -\langle [e^\alpha, e^\sigma], e^\tau\rangle_g,
\]
\[
= -g^{\gamma\tau} c^\gamma_{\alpha\sigma}.
\]
Hence,
\[
g^{\gamma\alpha}_c \nabla_\gamma P^{\tau\sigma}(\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha) = -g^{\gamma\tau} c^\gamma_{\alpha\sigma} (\omega_\alpha \eta_\sigma - \omega_\sigma \eta_\alpha),
\]
\[
= -g^{\gamma\tau} (c^\gamma_{\alpha\sigma} - c^\gamma_{\sigma\alpha}) \omega_\alpha \eta_\sigma,
\]
\[
= -2g^{\gamma\tau} c^\gamma_{\alpha\sigma} \omega_\alpha \eta_\sigma,
\]
\[
= -2g^{\gamma\tau} [\omega, \eta]_\gamma.
\]

Using the Killing form to identify $g$ and $g^\ast$, we can consider $M$ as $M = (g_{\text{reg}}, P)$. Thus, we have the following

**Corollary 4.** The distribution $\mathcal{N}$ on the manifold $g_{\text{reg}}^\ast$ for a compact semi-simple Lie algebra $g$ is integrable. Furthermore, via the identification of $g^\ast$ with $g$ as above, each connected component (Weyl Chamber) of the Lie algebra $t$ of a maximal torus $T \subset G$ is a maximal integral surface of the distribution $\mathcal{N}$ at each point $x$. 

**Proof.** Let $t$ be one of the maximal commutative subalgebras of $g$ (the Lie algebra of a maximal torus $T \subset G$).
Recall that the root decomposition delivers the $K$-orthogonal decomposition of $g$:
\[
g = t \oplus \sum_{\alpha \in \Sigma} g^\alpha,
\]
where $\Sigma$ is the root system of the couple $(g_c, t_c)$ and $g^\alpha = g \cap g^\alpha_c$.

Any connected component (Weyl Chamber) of $t$ is a maximal integral surface of the distribution $\mathcal{N}$ at each regular point $x$ since $t$ is $K$-orthogonal to the tangent space of each adjoint orbit (symplectic leaf) in $M = g_{\text{reg}}^\ast$ (see Knapp, 1996, Ch.4)
\[
T_x(\text{Ad}(G)X) = \left\{ x + \sum_{\alpha \in \Sigma} g^\alpha \right\}.
\]

Through each point $x \in g$ there passes at least one such subspace $t$, and a point $x$ is regular if and only if this $t$, containing $x$ is unique. This proves the statement. 

**6.2. Non-compact semi-simple Lie algebras.** Let $G$ be a connected real semi-simple Lie Group with Lie Algebra $g$. As in the compact case, symplectic orbits of $g$ with this structure are exactly adjoint orbits of $G$, but in contrast to the compact case, an adjoint orbit of $X \in g$ is closed iff $X$ is semi-simple, i.e $ad(X)$ is semi-simple (see Warner 1972, Prop.1.3.5.5). The subset $g'$ of regular elements, i.e. semi-simple elements $X$ with centralizer $X^0$ of minimal dimension (see Warner 1972, Sec.1.3.4) endowed with the induced Poisson structure is an open and dense
subset of \( g \). Its structure is as follows. Let \( \mathfrak{j} \) be a Cartan subalgebra of \( g \). Let \( \mathfrak{j}' = \mathfrak{j} \cap \mathfrak{g}' \). Put
\[
\mathfrak{g}(\mathfrak{j}) = \bigcup_{x \in G} Ad(x)\mathfrak{j}',
\]
where \( G = \text{Int}(\mathfrak{g}) \) is the adjoint group of \( \mathfrak{g} \). Then (see Warner, Prop. 1.3.4.1),
\[
\mathfrak{g}' = \bigcup_l \mathfrak{g}(\mathfrak{j}_l),
\]
where \( \mathfrak{j}_l \) for \( 1 \leq l \leq q \) are representatives of (a finite number of) conjugacy classes of Cartan subalgebras of \( \mathfrak{g} \).

Now, pick \( 1 \leq l \leq q \) and let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{\mu} \) be the Cartan decomposition of \( \mathfrak{g} \) such that \( \mathfrak{j}_l = \mathfrak{j}_l \mathfrak{k} \oplus \mathfrak{j}_l \mathfrak{\mu} = \mathfrak{j}_l \cap \mathfrak{k} \oplus \mathfrak{j}_l \cap \mathfrak{\mu} \) is the direct sum decomposition of the Cartan subalgebra \( \mathfrak{j}_l \) into compact and noncompact parts. It is known that the Killing form \( K \) is positive definite on \( \mathfrak{\mu} \) and negative definite on \( \mathfrak{k} \). Using the Cartan decomposition of \( \mathfrak{j}_l \) above, we see that the restriction of the Killing form to the subspace \( \mathfrak{X} + \mathfrak{j}_l \subset T_X(\mathfrak{g}) \), and all its conjugates, has constant signature and is non-degenerate at all points \( X \in \bigcup_{x \in G} Ad(x)\mathfrak{j}' \). Therefore, the same is true for its \( K \)-orthogonal complement. The restricted root decomposition
\[
\mathfrak{g} = \mathfrak{j}_l \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,
\]
where \( \Sigma \) is the system of (restricted) roots of the pair \( (\mathfrak{g}, \mathfrak{j}_l \mathfrak{\mu}) \) can be used to show that \( \mathfrak{X} + \mathfrak{j}_l \) is the \( K \)-orthogonal complement to \( T_X(\text{Ad}(G)X) \) in \( T_X(\mathfrak{g}) \).

We call an element \( \lambda \in \mathfrak{g}^* \) regular if the corresponding element \( X_\lambda + iK^{-1}\lambda \) of \( \mathfrak{g} \) is regular. Then, arguments similar to those in the previous subsection can be used to prove the following

**Proposition 2.** Let \( \mathfrak{g} \) be a real semi-simple Lie algebra. Consider the dual space \( (\mathfrak{g}^*, P) \) with its linear Poisson structure. Endow \( \mathfrak{g}^* \) with the (pseudo)-Riemannian metric \( K^* \) induced by the Killing form on \( \mathfrak{g} \). Let \( M = \mathfrak{g}^*_\text{reg} \) be the (open and dense) submanifold of \( \mathfrak{g}^* \) of co-adjoint orbits (symplectic leaves) of \( \ast \)-regular elements. Then the restriction of \( K^* \) to each orbit in \( M \) has constant signature and is non-degenerate. The \( K^* \)-orthogonal distribution \( \mathcal{N} \) to the symplectic leaves is integrable. Maximal integral submanifolds of \( \mathcal{N} \) are images under the identification \( i_K : \mathfrak{g} \equiv \mathfrak{g}^* \) of (the regular parts of) Cartan subalgebras of \( \mathfrak{g} \).

**Remark 3.** We had to add the condition of semi-simplicity of an element because of the presence in \( \mathfrak{g} \) of a principal nilpotent orbit of the same maximal dimension. The restriction of the Killing form to such orbits is degenerate. For example, consider the Lie algebra \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \). In addition to the closed semi-simple adjoint orbits of elliptic and hyperbolic elements, there is also the adjoint nilpotent orbit in \( \mathfrak{g} \) of the same dimension 2.

**6.3. Dual to the Euclidian Lie algebra \( \mathfrak{e}(3) \).** Let \( \mathfrak{g} \) be the Lie algebra \( \mathfrak{se}(3) \) of the group of proper Euclidian motions in \( \mathbb{R}^3 \), let \( \mathfrak{g}^* \) be its dual, and let \( M = \mathfrak{g}^*_{\text{reg}} \) be the (open, connected and dense) subspace of 4d co-adjoint orbits in \( \mathfrak{g}^* \).
The Killing form of the Lie algebra \( \mathfrak{so}(3) \) is degenerate, so we identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) via the Euclidean scalar product (see Marsden and Ratiu 1994, Ch.8). Elements of \( \mathfrak{g} \) can be represented as vectors \((\mathbf{x}, \mathbf{p})\) in \( \mathfrak{so}(3) \oplus \mathbb{R}^3 \) with the Lie bracket defined as

\[
[(\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}')] = (\mathbf{x} \times \mathbf{x}', \mathbf{x} \times \mathbf{p}' + \mathbf{p} \times \mathbf{x}').
\]

We can consider vectors \( \mathbf{x} \) in \( \mathfrak{so}(3) \) to be skew-symmetric matrices due to the isomorphism

\[
\mathbf{x} \mapsto \hat{\mathbf{x}} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.
\]

The canonical linear Poisson structure on \( M \subset \mathfrak{g}^* \cong \mathfrak{g} \) has, in this notation, the Poisson tensor has the form

\[
P(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{p}} \end{pmatrix}.
\]

Casimir functions of this structure are \( c_1 = \mathbf{x} \cdot \mathbf{p} \) and \( c_2 = \mathbf{p} \cdot \mathbf{p} \). The subspace of regular (4d) co-adjoint orbits is, in this notation, defined by the condition \( c_2(\mathbf{x}, \mathbf{p}) \neq 0 \).

For any \( \mathbf{y} \) we have \( \mathbf{x} \times \mathbf{y} = -\mathbf{y} \hat{\mathbf{x}} = \hat{\mathbf{xy}}^T \). This allows us to express the adjoint action on \( \mathfrak{so}(3) \) as

\[
ad(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p}) = -(\mathbf{xy}, \mathbf{p} \hat{\mathbf{y}} + \mathbf{x} \hat{\mathbf{q}}) = \begin{pmatrix} \hat{\mathbf{y}} & 0 \\ \hat{\mathbf{q}} & \hat{\mathbf{y}} \end{pmatrix} (\mathbf{x}, \mathbf{p})^T.
\]

We want to construct a symmetric \((2,0)\) tensor \( g \) that is invariant under the adjoint action of the Lie algebra that we can possibly use as a Riemannian metric. Suppose then, that we have a non-degenerate scalar product \( g_0 \) defined on \( \mathfrak{g} \) by a constant symmetric matrix:

\[
< (\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}') >_{g_0} = (\mathbf{x}, \mathbf{p}) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} (\mathbf{x}', \mathbf{p}')^T.
\]

We extend \( g_0 \) to a covariant metric \( g \) on \( M \) by setting \( g(\mathbf{x}, \mathbf{p}) = g_0 \). We would like to choose this metric to be invariant under the co-adjoint action. Thus, \( g \) should satisfy the equation (we use the identification of the tangent and cotangent bundles to \( \mathfrak{g}^* \) as above)

\[
< \ad(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}') >_g + < (\mathbf{x}, \mathbf{p}), \ad(\mathbf{y}, \mathbf{q})(\mathbf{x}', \mathbf{p}') >_{g_0} = 0.
\]

Since this must hold for arbitrary vectors \((\mathbf{x}, \mathbf{p})\) and \((\mathbf{x}', \mathbf{p}')\) in \( \mathfrak{g} \), we have the following condition on \( g \).

\[
\begin{pmatrix} \hat{\mathbf{y}}^T & \hat{\mathbf{q}}^T \\ 0 & \hat{\mathbf{y}}^T \end{pmatrix} \cdot g + g \cdot \begin{pmatrix} \hat{\mathbf{y}} & 0 \\ \hat{\mathbf{q}} & \hat{\mathbf{y}} \end{pmatrix} = 0.
\]

This condition is equivalent to the following system of equations:

1) \( A \hat{\mathbf{y}} - \hat{\mathbf{y}} A + B \hat{\mathbf{q}} - \hat{\mathbf{q}} B^T = 0 \)
2) \( B \hat{\mathbf{y}} - \hat{\mathbf{y}} B - \hat{\mathbf{q}} C = 0 \)
3) \( B^T \hat{\mathbf{y}} - \hat{\mathbf{y}} B^T + C \hat{\mathbf{q}} = 0 \)
4) \( C \hat{\mathbf{y}} - \hat{\mathbf{y}} C = 0 \).
Since these equations must be valid for arbitrary $\dot{\mathbf{y}}$ and $\dot{\mathbf{q}}$, it is easy to see that the metric $g$ must be of the form

$$g = \begin{pmatrix} \alpha I & \beta I \\ \beta I & 0 \end{pmatrix}. \quad (10)$$

Thus, the only $ad$-invariant (constant) metrics on $M$ are those having this special form. Note that such a metric cannot be Riemannian (see Zefran et al. 1996). In fact, the distinct eigenvalues of such a metric are $\lambda_i = (\alpha \pm \sqrt{\alpha^2 + 4\beta^2})/2$, $i = 1, 2$ (of multiplicity 3 each). The product of these eigenvalues is $\lambda_1\lambda_2 = -\beta^2 \leq 0$. Thus, if non-degenerate (i.e. $\beta \neq 0$), the metric $g$ has signature $(3, 3)$.

Let $T_{(x,p)}(S)$ be the space tangent to the symplectic leaf passing through the point $(x, p) \in M$. Then we can define its $g^{-1}$-orthogonal complement $\mathcal{N}(x, p)$, and a $g^{-1}$-orthogonal distribution $\mathcal{N}$ on $M$. The covectors $\omega^1 = dc_1 = (p, x)$ and $\omega^2 = dc_2 = (0, p)$ form a basis for the subspace $C_m(M) = \ker(P_m) \subset T^*_m(M) \equiv \mathfrak{g}$, and the tangent vectors $\xi_1 = (\omega^1)^T$ and $\xi_2 = (\omega^2)^T$ form, at each point $m = (x, p)$, a basis for $\mathcal{N}_m$. We have

$$\xi_1 = \begin{pmatrix} \alpha p + \beta x \\ \beta p \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \beta p \\ 0 \end{pmatrix}.$$ 

Consider the case $\alpha = 0$. It is easy to see that, in the basis $\xi_i$, the restriction of the metric $g$ to the subspace $\mathcal{N}_m$ of the distribution $\mathcal{N}$ at a point $m = (x, p)$ has the form

$$g = \begin{pmatrix} 2c_1(x, p) & c_2(x, p) \\ c_2(x, p) & 0 \end{pmatrix}.$$ 

Since $\det(g) = -(p \cdot p)^2$, this restriction is non-degenerate on the subset of regular (4d) co-adjoint orbits. On the distribution $\mathcal{N}$ the metric $g$ has signature $(1, 1)$. Thus, on the tangent spaces $T(S)$ of symplectic foliation, $g$ has signature $(2, 2)$ at all (regular) points, and results of Sec.3 are applicable here.

However, the methods from that section are not necessary in this case. Since we have explicit expressions for the vectors $\xi_i$, and for the tensors $P$ and $g$, it is easy to check the integrability of $\mathcal{N}$ directly. The distribution $\mathcal{N}$ will be integrable if and only if the Lie bracket $[\xi_1, \xi_2]$ of vector fields $\xi_i$ considered as $\mathfrak{g}$-values functions on $M$ remains in $\mathcal{N}$.

**Proposition 3.** For any choice of (constant) non-degenerate $ad$-invariant metric $g$ on the subspace $M = \mathfrak{g}_{reg}$ of 4d co-adjoint orbits of dual space $e(3)^*$ of the 3-d Euclidian lie algebra $e(3)$, the distribution $\mathcal{N}$ is integrable. For metrics $(8)$ with $\alpha = 0$, the maximal integral submanifold passing through a point $(x, p)$ is presented, in parametrical form as

$$(s, t) \rightarrow e^s \begin{pmatrix} x^T \\ p^T \end{pmatrix} + e^t \begin{pmatrix} x^T \\ p^T \end{pmatrix}. $$

**Proof.** We calculate the Lie bracket of the basis for $\mathcal{N}$.

$$[\xi_1, \xi_2] = \xi_2^T(\xi_2^T) - \xi_1^T(\xi_2^T)$$

$$= \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha p^T + \beta x^T \\ \beta p^T \end{pmatrix} - \begin{pmatrix} \beta & \alpha \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \beta p^T \\ 0 \end{pmatrix}$$

$$= 0.$$

The last statement is easily checked by direct calculation. \qed
7. Conclusion

In this work we discuss necessary and sufficient conditions for the distribution $\mathcal{N}$ on a regular Poisson manifold $(M, P)$ defined as orthogonal complement of tangent to symplectic leaves with respect to some (pseudo-)Riemannian metric $g$ on $M$ to be integrable. We present these conditions in different forms, including a condition in terms of a symmetry of Christoffel coefficients of the Levi-Civita connection of the metric $g$ and get some corollaries, one of which specifies the part of the Lichnerowicz ($\nabla P = 0$) condition ensuring integrability of $\mathcal{N}$ (see Vaisman 1994, 3.11). We present examples of non-integrable $\mathcal{N}$ (the model 4d case and the case of a nontrivial symplectic fibration). We prove integrability of $\mathcal{N}$ on the regular part of the dual space $g^*$ of a real semi-simple Lie algebra $g$ and the same in the case of the 3d Euclidian Lie algebra $\mathfrak{e}(3)$ with a linear Poisson structure.

As the case of a symplectic fibration shows, the integrability of $\mathcal{N}$ is possible only on a topologically trivial bundle (trivial transversal topology). Thus, it would be interesting to study maximal integral submanifolds of $\mathcal{N}$ in the case of nontrivial symplectic bundles. In particular, it would be interesting to get conditions on the metric $g$ under which these maximal integral submanifolds would have maximal possible dimension. Probably the methods of Cartan-Kahler theory (see Bryant et. al., 1991) can be employed to investigate these questions.

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