Killing-Yano tensors in spaces admitting a hypersurface orthogonal Killing vector

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Methods are presented for finding Killing-Yano tensors, conformal Killing-Yano tensors, and conformal Killing vectors in spacetimes with a hypersurface orthogonal Killing vector. These methods are similar to a method developed by the authors for finding Killing tensors. In all cases one decomposes both the tensor and the equation it satisfies into pieces along the Killing vector and pieces orthogonal to the Killing vector. Solving the separate equations that result from this decomposition requires less computing than integrating the original equation. In each case, examples are given to illustrate the method.

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I. INTRODUCTION

Recently Garfinkle and Glass\textsuperscript{1} presented a method for finding Killing tensors in spaces with a hypersurface orthogonal Killing vector. The method involves a 3+1 (or more generally (n-1)+1) decomposition of the Killing tensor equation using the foliation orthogonal to the Killing vector. The approach of Garfinkle and Glass\textsuperscript{1} has been considered by Mirshekari and Will\textsuperscript{2} in showing that the Bach-Weyl metric does not admit a non-trivial Killing tensor. Since the Killing tensor equation is one of a class of similar tensor equations (Killing vector, conformal Killing vector, Killing-Yano, conformal Killing-Yano, etc.) it is natural to ask whether the approach of Garfinkle and Glass\textsuperscript{1} could be used on any of these other equations. In fact, the use of 3+1 decomposition to study the equations for a Killing vector has a long history in general relativity beginning with the work of Moncrief\textsuperscript{3} and Coll\textsuperscript{4} and continued \textit{e.g.} by Beig and Chruściel\textsuperscript{5}. More recently Gómez-Lobo and Valiente-Kroon\textsuperscript{6} considered this 3+1 decomposition in spinor formalism, and have also studied Killing spinor initial data sets. The main difference between these earlier works and the method of Garfinkle and Glass\textsuperscript{1} is the assumption of a hypersurface orthogonal Killing vector. This as-
sumption greatly restricts the cases to which the method applies; however it also provides a great simplification to the equations and thus makes them more tractable. A similar approach due to Bona and Collt treats the conformal Killing equation in static spacetimes, but then adds the further condition that the conformal Killing field is Lie derived by the static Killing field.

This paper generalizes the technique of by producing analogous methods for the Killing-Yano, conformal Killing, and conformal Killing-Yano equations. In each case the spacetime is assumed to possess a hypersurface orthogonal Killing vector, and the equations are decomposed with respect to the foliation orthogonal to the Killing vector. As a simple illustration of these techniques, we find the Killing-Yano tensors of the Bertotti-Robinson (BR) spacetime, and the conformal Killing-Yano tensors of a particular cylindrical vacuum metric due to Linet.8

**Notation:** Lower case Latin indices, $B^a$, range over n-dimensions. Greek indices, $B^\mu$, range over n–1 dimensions. For Killing vector $\xi^a$ an overdot will denote a Lie derivative, $\dot{A} := \mathcal{L}_\xi A$.

II. THE KILLING-YANO TENSOR METHOD

The Killing-Yano (KY) equation for antisymmetric tensor $A_{ab}$ can be written as

$$A_{a(bc)} = 0. \quad (1)$$

This generalizes Killing’s equation to antisymmetric tensors. There are at most 10 independent solutions of the KY equation on manifold $\mathcal{M}$. The maximum of 10 occurs if, and only if, $\mathcal{M}$ has constant curvature. There is an extensive literature covering KY tensors. In an early paper Collinson discussed the relationship between Killing vectors and KY tensors. He pointed out that all type D vacuum solutions which admit a Killing tensor also admit a KY tensor. Two works by Dietz and Rüdiger discuss the character of spacetimes admitting KY tensors. More recently, Ferrando and Sáez gave Rainich conditions for systems to admit KY tensors. Hall studied the existence of KY tensors in General Relativity, and Ibohal has used the Newman-Penrose formalism to integrate the KY equations and has found a number of spacetimes which contain KY tensors, including FRW, Kerr-Newman, and Bertotti-Robinson.9 Taxiarakis has proved that the only spacetimes which admit KY tensors have Petrov type D, N, or O.

Suppose that a spacetime has a hypersurface orthogonal Killing vector $\xi^a$. Define $V$
such that
\[ \xi^a \xi_a = \epsilon V^2 \]  
(2)
where \( \epsilon = \pm 1 \). Then the metric in directions orthogonal to \( \xi^a \) is given by
\[ h_{ab} = g_{ab} - \epsilon V^{-2} \xi_a \xi_b \]  
(3)
One can use \( h^{ab} \) as a projection operator to project any tensor in directions orthogonal to \( \xi^a \). In particular, the KY tensor can be decomposed as
\[ A_{ab} = 2V^{-1} S_{[a} \xi_{b]} + Q_{ab} \]  
(4)
where \( S_a \) and antisymmetric \( Q_{ab} \) are orthogonal to \( \xi^a \).

Projecting the KY equation using all combinations of \( h^{ab} \) and \( \xi^a \) yields the following
\[ D_a Q_{bc} + D_b Q_{ac} = 0, \]  
(5)
\[ D_a S_b = 0, \]  
(6)
\[ L_\xi Q_{ab} = \epsilon V^3 D_{[a} V^{-2} S_{b]}, \]  
(7)
\[ L_\xi S_a = -Q_{ab} D^b V. \]  
(8)
Here \( L_\xi \) denotes the Lie derivative with respect to Killing vector \( \xi^a \) and \( D_a \) denotes the derivative operator on the space orthogonal to \( \xi^a \).

The first two equations say that \( Q_{ab} \) and \( S^a \) are respectively a Killing-Yano tensor and a Killing vector on the space orthogonal to \( \xi^a \). The last two equations are additional conditions that these tensors must satisfy. These last two equations are most easily implemented in a coordinate system adapted to the Killing vector. Choose a coordinate system \((y, x^\mu)\) such that \( x^\mu \) are coordinates on the surface orthogonal to the Killing vector and \( L_\xi \) is simply a partial derivative with respect to \( y \). Use \( \partial_\mu \) or a comma to denote a derivative with respect to the \( x^\mu \) coordinates. The Latin indices in this section are \( n \)-dimensional, and the method below projects objects and equations down to \( n-1 \) dimensions with Greek indices.

Equations (7,8) become
\[ \dot{Q}_{\mu\nu} = \epsilon V \partial_{[\mu} S_{\nu]} + 2 \epsilon S_{[\mu} \partial_{\nu]} V \]  
(9)
\[ \dot{S}_\mu = -Q_{\mu\alpha} h^{\alpha\nu} \partial_\nu V. \]  
(10)

Thus the method for finding Killing-Yano tensors on the \( n \)-dimensional space consists of two steps:

1. find all Killing-Yano tensors and all Killing vectors on the \( n-1 \) dimensional space
2. subject those Killing-Yano tensors and Killing vectors to the conditions of Eq.(9) and Eq.(10)

III. KILLING-YANO TENSORS OF THE BERTOTTI-ROBINSON METRIC

The BR spacetime (up to an overall scale) has line element
\[ ds^2 = \frac{1}{r^2}(-dt^2 + dr^2) + r^2 d\phi^2 + r^2 \sin^2 \theta d\psi^2 \]  
(11)
This spacetime is the direct product of the 2-sphere and 2-dimensional anti de-Sitter spacetime, i.e. $S^2 \otimes \text{AdS}_2$. Defining coordinate $w := -\ln r$ allows the BR line element to be transformed to the form

$$ds^2 = -e^{2w} dt^2 + dw^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2.$$  \hspace{1cm} (12)

For the convenience of the reader in following this section, additional properties of the BR spacetime are collected in Appendix A.

The method of the previous section will be used to work out the KY tensors of the BR metric. First the KY tensors of the 2-dimensional $w\vartheta$ surface will be found, then these will be used to find the KY tensors of the 3-dimensional $w\vartheta\varphi$ surface, and finally find the KY tensors of 4-dimensional BR spacetime.

$c_1, c_2$ etc. will denote constants, and $k_1, k_2$ etc. will denote quantities that depend only on the coordinate associated with the Killing vector.

$w\vartheta$ and $w\vartheta\varphi$ surfaces

The 2-dimensional $w\vartheta$ space has line element

$$ds^2 = dw^2 + d\vartheta^2.$$  \hspace{1cm} (13)

For any 2-dimensional space, the unique solution (up to an overall scale) of Eq.(5) is the volume element. Since the $w\vartheta$ space is just ordinary 2-dimensional Euclidean space, it has the three Killing vectors of that space. Thus we have

$$Q_{\mu\nu} = k_1 \partial_{[\mu} w \partial_{\nu]} \vartheta$$  \hspace{1cm} (14)

$$S_{\mu} = k_2 \partial_{\mu} w + k_3 \partial_{\mu} \vartheta$$

$$+ k_4 (\partial \partial_{\mu} w - w \partial_{\mu} \vartheta)$$  \hspace{1cm} (15)

Using the $\varphi$ Killing vector of metric (12) and recalling that the Killing vector norm is $\epsilon V^2$, yields $V = \sin \vartheta$ and $\epsilon = 1$. Imposing Eq.(10) we find

$$\dot{k}_2 \partial_{\mu} w + \dot{k}_3 \partial_{\mu} \vartheta + \dot{k}_4 (\partial \partial_{\mu} w - w \partial_{\mu} \vartheta)$$

$$= -k_1 \cos \vartheta (\partial_{\mu} w)$$  \hspace{1cm} (16)

It then follows that the quantities $k_1, \dot{k}_2, \dot{k}_3$ and $\dot{k}_4$ all vanish. Thus we have $Q_{\mu\nu} = 0$ and

$$S_{\mu} = c_2 \partial_{\mu} w + c_3 \partial_{\mu} \vartheta + c_4 (\partial \partial_{\mu} w - w \partial_{\mu} \vartheta)$$  \hspace{1cm} (17)

Now, using Eq.(9) we find

$$0 = (c_4 \sin \vartheta + c_2 \cos \vartheta + c_4 \vartheta \cos \vartheta)$$

$$\times (\partial_{\mu} w \partial_{\nu} \vartheta - \partial_{\nu} w \partial_{\mu} \vartheta).$$  \hspace{1cm} (18)

This implies that $c_2$ and $c_4$ vanish. It follows that $S_{\mu} = c_3 \partial_{\mu} \vartheta$. Use of Eq.(14) results in

$$A_{\mu\nu} = c_3 \sin \vartheta \ 2 \partial_{[\mu} \vartheta \partial_{\nu]} \varphi$$  \hspace{1cm} (19)

the $w\vartheta\varphi$ surface and the BR spacetime

The 3-dimensional $w\vartheta\varphi$ space has line element

$$ds^2 = dw^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2$$  \hspace{1cm} (20)
Here the Killing vectors are \((\partial/\partial w)^{\alpha}\) and the three Killing vectors of the 2-sphere, which will be denoted by \(\xi^{1\alpha}, \xi^{2\alpha}, \xi^{3\alpha}\). We therefore have

\[
S_{\mu} = k_{2}\partial_{\mu} w + k_{3}\xi^{1}_{\mu} + k_{4}\xi^{2}_{\mu} + k_{5}\xi^{3}_{\mu}
\]  
(21)

From the results of the previous subsection it follows that

\[
Q_{\mu\nu} = c_{1}\sin \vartheta 2\partial_{[\mu} \partial_{\nu]} \varphi
\]  
(22)

The \(t\) Killing vector of metric (12) provides \(\epsilon = -1\) and \(V = e^{w}\). Using Eq. (10) we have

\[
0 = \dot{k}_{2}\partial_{\mu} w + \dot{k}_{3}\xi^{1}_{\mu} + \dot{k}_{4}\xi^{2}_{\mu} + \dot{k}_{5}\xi^{3}_{\mu}
\]  
(23)

Since the terms on the right hand side are linearly independent, the coefficient of each term vanishes. Thus \(\dot{k}_{2} = \dot{k}_{3} = \dot{k}_{4} = \dot{k}_{5} = 0\). Therefore one has \(k_{2} = c_{2}, k_{3} = c_{3}, k_{4} = c_{4}, k_{5} = c_{5}\). It then follows that \(S_{\mu}\) takes the form

\[
S_{\mu} = c_{2}\partial_{\mu} w + \ell_{\mu}
\]  
(24)

where \(\ell_{\mu}\) is the sum of 2-sphere Killing vectors, defined as

\[
\ell_{\mu} := c_{3}\xi^{1}_{\mu} + c_{4}\xi^{2}_{\mu} + c_{5}\xi^{3}_{\mu}.
\]  
(25)

Upon using Eq. (10) we find

\[
\dot{k}_{1}\sin \vartheta 2\partial_{[\mu} \partial_{\nu]} \varphi
\]
\[
= -e^{w}\partial_{[\mu}\ell_{\nu]} - e^{w} 2\ell_{[\mu} \partial_{\nu]} w
\]  
(26)

The last term on the right hand side is linearly independent of both the first curl term on the right hand side and the term on the left hand side. It then follows that this term must vanish. Therefore \(\ell_{\mu} = 0\), and the entire right hand side of this equation vanishes. The left hand side must therefore also vanish, and so \(\dot{k}_{1} = 0\). Thus \(k_{1} = c_{1}\). Finally we have

\[
Q_{\mu\nu} = c_{1}\sin \vartheta 2\partial_{[\mu} \partial_{\nu]} \varphi,
\]  
(27)

\[
S_{\mu} = c_{2}\partial_{\mu} w.
\]  
(28)

Applying this result in Eq. (11) we find that the general Killing-Yano tensor of the BR spacetime is

\[
A_{\mu\nu} = (c_{1}\sin \vartheta) 2\partial_{[\mu} \partial_{\nu]} \varphi
\]
\[
+ (c_{2}e^{w}) 2\partial_{[\mu} t \partial_{\nu]} w
\]  
(29)

Since the BR spacetime is the direct product \(S^2 \otimes AdS_2\), this result has a simple geometrical interpretation. The Killing-Yano tensors of the BR spacetime are the volume elements of \(S^2\) and \(AdS_2\).

IV. CONFORMAL KILLING-YANO TENSORS

The tensor version of the conformally covariant generalization of the KY equation, the CKY equation, was discovered by Tachibana. It can be written in the form

\[
\nabla_{a}A_{bc} + \nabla_{b}A_{ac}
\]
\[
= 2W_{c}g_{ab} - W_{a}g_{bc} - W_{b}g_{ac}
\]  
(30)
for some $W_a$. $A_{bc}$ is given in Eq.(4). It follows from Eq.(31) that
\[ W_a = \frac{1}{n-1} \nabla^b A_{ba}. \] (31)

In this paper, Tachibana showed that in a Ricci-flat space, for $A_{ab}$ a CKY bivector satisfying Eq.(30), $(1/3)\nabla^b A_{ab}$ is a Killing vector. It is well known that the Kerr metric admits a CKY bivector, and indeed all type D vacuum solutions and their charged counterparts have a CKY bivector.

In a manner just like the Killing-Yano case, we can decompose $A_{ab}$ as in Eq.(4). Similarly, $W_a$ can be decomposed as
\[ W_a = \gamma V^{-1} \xi_a + X_a \] (32)
where $X_a$ is orthogonal to $\xi^a$. Taking all projections of Eq.(30) we find the following:

\[ D_a Q_{bc} + D_b Q_{ac} = 2X_c h_{ab} - X_a h_{bc} - X_b h_{ac}; \] (33)
\[ D_{(a} S_{b)} = \gamma h_{ab}, \] (34)
\[ \mathcal{L}_\xi Q_{ab} = \epsilon V^3 D_{(a} V^{-2} S_{b),} \] (35)
\[ \mathcal{L}_\xi S_{a} = -Q_{ab} D^b V - V X_a. \] (36)

As in the Killing-Yano case, the first two equations have a simple geometrical interpretation. On the $n-1$ dimensional subspace orthogonal to the Killing vector $Q_{ab}$ is a CKY tensor and $S^a$ is a conformal Killing vector. The last two equations provide conditions that those tensors must satisfy.

We now specialize to the case where $n = 4$ and the spacetime has a Lorentz signature. A vector $T^a$ exists such that
\[ Q_{ab} = \epsilon_{abc} T^c \] (37)
where $\epsilon_{abc}$ is the volume element of the 3-dimensional space orthogonal to the Killing vector. Then equations (33), (35), and (36) become

\[ D_{(a} T_{b)} = \psi h_{ab} \] (38)
\[ \mathcal{L}_\xi T^a = -\frac{1}{2} V^3 \epsilon_{abc} D_b (V^{-2} S_c) \] (39)
\[ \mathcal{L}_\xi S^a = \frac{1}{2} V^3 \epsilon_{abc} D_b (V^{-2} T_c) \] (40)

Thus $T^a$ is a conformal Killing vector of the 3-dimensional space. The additional conditions that $T^a$ and $S^a$ must satisfy are given by equations (39) and (40) respectively. In the adapted coordinate system these additional conditions take the form

\[ \dot{T}^\mu = -\frac{1}{2} V^3 \epsilon^{\mu\nu\alpha} \partial_\nu (V^{-3} S_\alpha) \] (41)
\[ \dot{S}^\mu = \frac{1}{2} V^3 \epsilon^{\mu\nu\alpha} \partial_\nu (V^{-3} T_\alpha) \] (42)

Thus to find the CKY tensors of the 4-dimensional spacetime, one does the following:

1. find all the conformal Killing fields $S^a$ and $T^a$ of the 3-dimensional surface orthogonal to the Killing vector.
2. subject those conformal Killing fields to the conditions of equations (41) and (42).
3. use Eq.(37) to find $Q_{ab}$ and then Eq.(4) to find $A_{ab}$.
V. CONFORMAL KILLING VECTORS

Since, as shown in the previous section, one step in finding CKY tensors involves finding conformal Killing vectors, we now apply the general method of this paper to finding conformal Killing vectors. Recall that a conformal Killing vector $K^a$ on an $n$-dimensional space is one for which

$$\nabla_aK_b + \nabla_bK_a = \frac{2}{n}(\nabla_cK^c)g_{ab}. \tag{43}$$

In a manner similar to the Killing-Yano case, we can decompose $K_a$ as

$$K_a = A\xi_a + B_a \tag{44}$$

where $B_a$ is orthogonal to $\xi^a$. Taking all projections of Eq.(43) it follows that

$$D_aB_b + D_bB_a = \frac{2}{n-1}(D_cB^c)h_{ab} \tag{45}$$

and

$$\mathcal{L}_\xi A = \frac{V^{n-1}}{n-1}D_a(V^{1-n}B^a) \tag{46}$$

and

$$D_aA = -\epsilon V^{-2}\mathcal{L}_\xi B_a \tag{47}$$

As in the Killing-Yano case, Eq.(45) has a simple geometrical interpretation. On the $n-1$ dimensional subspace orthogonal to Killing vector $\xi^a$, $B^a$ is a conformal Killing vector. However, this conformal Killing vector is also subject to additional conditions, the integrability conditions for $A$. Taking the curl of Eq.(47) we obtain

$$D_a(V^{-2}\mathcal{L}_\xi B_b) = 0. \tag{48}$$

Subtracting $D_a$ of Eq.(46) from $\mathcal{L}_\xi$ of Eq.(47) provides

$$\mathcal{L}_\xi \mathcal{L}_\xi B_a + \frac{\epsilon V^2}{n-1}D_a[V^{n-1}D_b(V^{1-n}B^b)] = 0. \tag{49}$$

In the adapted coordinate system, the additional conditions for $B^a$ become

$$\partial_\mu(V^{-2}\dot{B}_\mu) = 0 \tag{50}$$

$$\ddot{B}_\mu + \frac{\epsilon V^2}{n-1}\partial_\mu \left[\frac{V^{n-1}}{\sqrt{h}}\partial_\nu(\sqrt{h}V^{1-n}B^\nu)\right] = 0. \tag{51}$$

The equations for $A$ are

$$\dot{A} = \frac{V^{n-1}}{(n-1)\sqrt{h}}\partial_\nu(\sqrt{h}V^{1-n}B^\nu) \tag{52}$$

$$\partial_\mu A = -\epsilon V^{-2}\dot{B}_\mu \tag{53}$$

Thus the method for finding conformal Killing vectors on the $n$-dimensional space consists of three steps:

1. find all conformal Killing vectors on the $n-1$ dimensional space
2. subject those conformal Killing vectors to the conditions of Eq.(50) and Eq.(51)
3. solve Eq.(52) and Eq.(53) for $A$.

VI. CKY TENSOR OF LINET’S VACUUM METRIC

The Petrov type D cylindrical vacuum line element found by Linet\textsuperscript{8} is written as

$$ds^2 = r^4(-dt^2 + dr^2 + dz^2) + r^{-2}d\varphi^2. \tag{54}$$
This static metric has Killing vectors $\partial_t$, $\partial_z$, and $\partial_\varphi$. We will use the method of the previous two sections to find the CKY tensors of this spacetime.

**tr and tr$\varphi$ surfaces**

We begin by finding all the conformal Killing fields of the $tr$ surface and then using those to find all the conformal Killing fields of the $tr\varphi$ surface. The 2-dimensional $tr$ space has line element

$$ds^2 = r^4(-dt^2 + dr^2). \quad (55)$$

Like all 2-dimensional metrics, this metric is conformally flat and the conformal Killing fields are therefore those of the underlying flat spacetime. For our purposes, it will be convenient to use null coordinates $u = t - r$ and $v = t + r$. The line element then becomes

$$ds^2 = -r^4 dudv \quad (56)$$

where $r = (v - u)/2$. It follows from metric (56) that the conformal Killing field takes the form

$$\tilde{B}^a = \alpha(\partial_u)^a + \beta(\partial_v)^a \quad (57)$$

where $\alpha$ is independent of $v$, and $\beta$ is independent of $u$. We now use this conformal Killing field to work out the general conformal Killing field of the $tr\varphi$ surface. We have $g_{\varphi\varphi} = r^{-2}$, therefore $\epsilon = 1$ and $V = r^{-1}$. It then follows that

$$\partial_\mu (V^{-2}\tilde{B}_\nu) = \frac{1}{2}r^6[-\partial_u \dot{\alpha} + \partial_v \dot{\beta} + 3r^{-1}(\dot{\alpha} + \dot{\beta})] \partial_\mu u \partial_\nu v \quad (58)$$

and therefore from Eq.(55)

$$- \partial_u \dot{\alpha} + \partial_v \dot{\beta} + 3r^{-1}(\dot{\alpha} + \dot{\beta}) = 0. \quad (59)$$

Taking $\partial_u \partial_v$ of this equation, and using the fact that $\alpha$ is independent of $v$ and $\beta$ is independent of $u$, we find

$$- \partial_u \dot{\alpha} + \partial_v \dot{\beta} - r^{-1}(\dot{\alpha} + \dot{\beta}) = 0. \quad (60)$$

Subtracting Eq.(60) from Eq.(59) yields

$$\dot{\alpha} + \dot{\beta} = 0. \quad (61)$$

However, since $\alpha$ is independent of $v$ and $\beta$ is independent of $u$, there exists a function $k_1(\varphi)$ such that $\dot{\alpha} = -k_1$ and $\dot{\beta} = k_1$. It then follows from Eq.(57) that

$$\tilde{B}^a = -k_1(\partial_u)^a + k_1(\partial_v)^a = k_1(\partial_r)^a. \quad (62)$$

Now taking $\partial_\varphi$ of Eq.(61) for $\tilde{B}_\mu$, and using Eq.(62) we find

$$0 = \dot{\tilde{B}}_\mu + \frac{\epsilon V^2}{n-1} \partial_\mu \left[ \frac{V^{n-1}}{\sqrt{h}} \partial_\nu (\sqrt{h}V^{1-n}\tilde{B}^\nu) \right]$$

$$= (r^4 k_1 - 3r^{-4} k_1) \partial_\mu r \quad (63)$$

It then follows that $k_1 = 0$. Therefore $\dot{\tilde{B}}^\mu = 0$ and so $\alpha$ and $\beta$ are independent of $\varphi$. Thus $\alpha$ is a function of $u$, and $\beta$ is a function of $v$. It then follows from Eq.(61) that there is a constant $c_1$ such that

$$8c_1 = \frac{V^{n-1}}{\sqrt{h}} \partial_\nu (\sqrt{h}V^{1-n}B^\nu) = \partial_u \alpha + \partial_v \beta + 3r^{-1}(\beta - \alpha). \quad (64)$$
Differentiating Eq. (64) by $\partial_u \partial_v$ yields

$$0 = \partial_u \alpha + \partial_v \beta - r^{-1}(\beta - \alpha).$$ \hspace{1cm} (65)

Subtracting Eq. (65) from Eq. (64) provides

$$8c_1 = 4r^{-1}(\beta - \alpha)$$ \hspace{1cm} (66)

from which it follows that

$$\beta - \alpha = 2c_1 r = c_1(v - u).$$ \hspace{1cm} (67)

But $\alpha$ depends only on $u$ and $\beta$ depends only on $v$ and so there exists a constant $c_2$ such that

$$\alpha = c_1 u + c_2$$ \hspace{1cm} (68)

$$\beta = c_1 v + c_2$$ \hspace{1cm} (69)

We therefore have

$$B^a = c_1 [u(\partial_u)^a + v(\partial_v)^a]$$

$$+ c_2 [(\partial_u)^a + (\partial_v)^a]$$

$$= c_1 [t(\partial_t)^a + r(\partial_r)^a] + c_2 (\partial_t)^a$$ \hspace{1cm} (70)

It then follows from Eq. (52) and Eq. (53) that $\dot{A} = 4c_1$ and $\partial_\mu A = 0$. We therefore have

$$A = 4c_1 \varphi + c_3.$$ \hspace{1cm} (71)

Finally, using Eq. (44) we find that the general conformal Killing vector of the 3-dimensional $tr \varphi$ surface is

$$K^a = c_1 [t(\partial_t)^a + r(\partial_r)^a + 4\varphi(\partial_\varphi)^a]$$

$$+ c_2 (\partial_t)^a + c_3 (\partial_\varphi)^a$$ \hspace{1cm} (72)

the $tr \varphi$ surface and the Linet spacetime

The vector fields $T^a$ and $S^a$ are conformal Killing fields on the $tr \varphi$ surface, with $z$ dependent coefficients and therefore take the form

$$T^a = k_1 [t(\partial_t)^a + r(\partial_r)^a + 4\varphi(\partial_\varphi)^a]$$

$$+ k_2 (\partial_t)^a + k_3 (\partial_\varphi)^a$$ \hspace{1cm} (73)

$$S^a = k_4 [t(\partial_t)^a + r(\partial_r)^a + 4\varphi(\partial_\varphi)^a]$$

$$+ k_5 (\partial_t)^a + k_6 (\partial_\varphi)^a$$ \hspace{1cm} (74)

Since $g_{zz} = r^4$ it follows that $V = r^2$ and $\epsilon = 1$. We then find

$$\epsilon^{\mu\nu\lambda} \partial_\nu (V^{-2} T_\lambda) =$$

$$- 6r^{-7} \epsilon^{\mu\nu\varphi} (4k_1 \varphi + k_3)$$ \hspace{1cm} (75)

$$\epsilon^{\mu\nu\lambda} \partial_\nu (V^{-2} S_\lambda) =$$

$$- 6r^{-7} \epsilon^{\mu\nu\varphi} (4k_4 \varphi + k_6)$$ \hspace{1cm} (76)

From the $t$ component of Eq. (41) and Eq. (42) it follows that

$$\dot{k}_1 t + \dot{k}_2 = -3r^{-4}(4k_1 \varphi + k_3).$$ \hspace{1cm} (77)

$$\dot{k}_4 t + \dot{k}_5 = 3r^{-4}(4k_1 \varphi + k_3).$$ \hspace{1cm} (78)

Therefore $k_1$, $k_3$, $k_4$, and $k_6$ vanish, and $k_2$ and $k_5$ are constants. Thus the conformal Killing fields subject to restrictions (41) and (42) take the form

$$T^a = c_1 (\partial_t)^a$$ \hspace{1cm} (79)

$$S^a = c_2 (\partial_t)^a$$ \hspace{1cm} (80)
Finally, using Eq.(37) and Eq.(4) we find that the general CKY tensor of the spacetime is
\begin{equation}
A_{\mu\nu} = 2c_1 r^3 \partial_{[\mu} r \partial_{\nu]} \varphi + 2c_2 r^6 \partial_{[\mu} z \partial_{\nu]} t 
\end{equation}

\textbf{VII. SUMMARY}

In this work a method is developed which decomposes the Killing-Yano tensor into separate terms based on the surface geometry of metrics with a hypersurface orthogonal Killing vector, and which thereby simplifies the solution of the Killing-Yano equation. Using this method, we have shown that the Bertotti-Robinson spacetime has a general KY tensor which is the sum of volume bivectors. An enhancement of this method has also been applied to the conformal Killing-Yano equation. The general CKY tensor has been constructed for Linet’s cylindrical vacuum metric.

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\textbf{Appendix A: Bertotti-Robinson}

The static Bertotti-Robinson (BR) metric is
\begin{equation}
g_{\mu\nu}^{\text{BR}} \, dx^\mu \, dx^\nu = [(1 + \lambda^2 z^2) \, dt^2 - (1 + \lambda^2 z^2)^{-1} \, dz^2] \\
- [(1 - \lambda^2 y^2) \, dx^2 + (1 - \lambda^2 y^2)^{-1} \, dy^2].
\end{equation}

\lambda^2 characterizes the electromagnetic energy density. Since the Weyl tensor vanishes, the Petrov type is 0. The BR spacetime has a diagonal trace-free Ricci tensor (with rows and columns along \( t, x, y, z \))
\begin{equation}
[R^\alpha_\beta]^{\text{BR}} = \lambda^2 \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\end{equation}

The BR manifold is non-singular with Kretschmann scalar
\begin{equation}
R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 8 \lambda^4.
\end{equation}

The BR metric is spanned by the null tetrad
\begin{align}
l_\alpha \, dx^\alpha &= (1/\sqrt{2})[(1 + \lambda^2 z^2)^{1/2} \, dt \\
&+ (1 + \lambda^2 z^2)^{-1/2} \, dz] \\
n_\alpha \, dx^\alpha &= (1/\sqrt{2})[(1 + \lambda^2 z^2)^{1/2} \, dt \\
&- (1 + \lambda^2 z^2)^{-1/2} \, dz] \\
m_\alpha \, dx^\alpha &= (1/\sqrt{2})[(1 - \lambda^2 y^2)^{1/2} \, dx \\
&- i(1 - \lambda^2 y^2)^{-1/2} \, dy] \\
\bar{m}_\alpha \, dx^\alpha &= (1/\sqrt{2})[(1 - \lambda^2 y^2)^{1/2} \, dx \\
&+ i(1 + \lambda^2 y^2)^{-1/2} \, dy]
\end{align}

Eight Newman-Penrose spin coefficients vanish, \( \kappa = \sigma = \bar{\lambda} = \nu = \rho = \mu = \tau = \pi \). The remaining four are
\begin{align}
\epsilon &= \gamma = -\frac{1}{2\sqrt{2}} \frac{\lambda^2 z}{\sqrt{1 + \lambda^2 z^2}} \\
\alpha &= \beta = -i \frac{\lambda^2 y}{2\sqrt{2} \sqrt{1 - \lambda^2 y^2}}
\end{align}
The null vectors are all geodesic

\[ l_{\alpha;\beta} = 2\gamma l_{\alpha} l_{\beta} + 2\gamma l_{\alpha} n_{\beta} \]  
(A9)

\[ n_{\alpha;\beta} = -2\epsilon n_{\alpha} n_{\beta} - 2\gamma n_{\alpha} l_{\beta} \]  
(A10)

\[ m_{\alpha;\beta} = 2\bar{\alpha} m_{\alpha} \bar{m}_{\beta} - 2\alpha m_{\alpha} m_{\beta} \]  
(A11)

The BR manifold admits antisymmetric tensor \( A_{\alpha\beta} \) as covariant constant bivectors

\[ A_{\alpha\beta} = k_0 l_{[\alpha} n_{\beta]} + k_1 m_{[\alpha} \bar{m}_{\beta]} \]  
(A12)

\[ A_{\alpha\beta;\nu} = 0 \]  
(A13)

\( A_{\alpha\beta} \) is therefore a KY solution. Note that

\[ l_{[\alpha} n_{\beta]} \sim dt \wedge dz \]  
and  
\[ m_{[\alpha} \bar{m}_{\beta]} \sim dx \wedge dy. \]  
These are the volume elements of the BR manifold.

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