Lower Bounds for Boxicity

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Abstract. An axis-parallel $b$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$ where $R_i$ is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity $\text{box}(G)$ is the minimum dimension $b$ such that $G$ is representable as the intersection graph of boxes in $b$-dimensional space. Although boxicity was introduced in 1969 and studied extensively, there are no significant results on lower bounds for boxicity. In this paper, we develop two general methods for deriving lower bounds. Applying these methods we give several results, some of which are listed below:

1. The boxicity of a graph on $n$ vertices with no universal vertices and minimum degree $\delta$ is at least $n/2(n - \delta - 1)$.
2. Consider the $G(n, p)$ model of random graphs. Let $p$ be such that $c_1 n \leq p \leq 1 - c_2 \frac{\log n}{n}$, where $c_1$ and $c_2$ are predetermined constants. Then, for $G \in G(n, p)$, almost surely $\text{box}(G) = \Omega(np(1 - p))$. On setting $p = 1/2$ we immediately infer that almost all graphs have boxicity $\Omega(n)$. Another consequence of this result is as follows: Let $m$ be an integer such that $c_3 n \leq m \leq c_4 n^2$ ($c_4$ is a suitable constant). Then, there exists a constant $c_4$ such that almost all graphs on $n$ vertices and exactly $m$ edges have boxicity at least $c_4 m/n$.
3. Let $G$ be a connected $k$-regular graph on $n$ vertices. Let $\lambda$ be the second largest eigenvalue in absolute value of the adjacency matrix of $G$. Then, the boxicity of $G$ is at least $\left(\frac{k^2/\lambda^2}{\log(1 + k^2/\lambda^2)}\right)(\frac{n-k-1}{2n})$.
4. The boxicity of random $k$-regular graphs on $n$ vertices (where $k$ is fixed) is $\Omega(k/\log k)$.
5. Consider all balanced bipartite graphs with $2n$ vertices. Let $m$ be an integer such that $c_5 n \log n \leq m \leq n^2 - c_6 n$, where $c_5$ and $c_6$ are predetermined constants. There exists a positive constant $c_6$ such that almost all balanced bipartite graphs with exactly $m$ edges have boxicity at least $c_6 m/(n \log n)$. We can also show that almost all balanced bipartite graphs have boxicity $\Omega(n/\log n)$.

Key words: Boxicity, Cubicity, Eigenvalue, Vertex Isoperimetric Problem, Minimum interval supergraph, Random graphs.

1 Introduction

Let $G(V, E)$ be a simple undirected graph with vertex set $V$ and edge set $E$. Let $\mathcal{F} = S_x \subseteq U : x \in V$ be a family of subsets of a universe $U$, where $V$ is an index set. The intersection graph $\Omega(\mathcal{F})$ of $F$ has $V$ as vertex set, and two distinct vertices $x$ and $y$ are adjacent if and only if $S_x \cap S_y \neq \emptyset$. An axis-parallel $b$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$ where $R_i$ is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity $\text{box}(G)$ is the minimum dimension $b$, such that $G$ is representable as the intersection graph of boxes in $b$-dimensional space. Roberts [27] introduced the concept of boxicity. Its applications include niche overlap (competition) in ecology and problems of fleet maintenance in operations research. Cozzens [13] showed that computing the boxicity of a graph is NP-hard. This was later strengthened by Yannakakis [35] and finally, by Kratochvil [24], who showed that deciding whether boxicity of a graph is at most 2 itself is NP-complete.

The class of graphs with boxicity at most 1 is exactly the popular class of interval graphs. A graph $G(V, E)$ is an interval graph if there exists a mapping, say $f$, from $V$ to intervals on the real line such that two vertices in $G$ are adjacent if and only if the intervals to which they are mapped overlap. $f$ is called an interval representation of $G$. Given a graph $G(V, E)$, let $I = \{I_i(V, E_i), 1 \leq i \leq k\}$ be a set of $k$ interval graphs such that $E = E_1 \cap E_2 \cap \cdots \cap E_k$. Then, we say that $I$ is an interval graph representation of $G$. The concept of boxicity can be equivalently formulated using the interval graph representation as follows:

Theorem 1. (Roberts [27]) The minimum $k$ such that there exists an interval graph representation of $G$ using $k$ interval graphs $I = \{I_i(V, E_i), 1 \leq i \leq k\}$ is the same as $\text{box}(G)$.
1.1 Some Known Bounds on Boxicity

There have been several attempts to establish upper bounds on boxicity, especially for graphs with special structures. Roberts, in his seminal work [27], proved that the boxicity of a complete $k$-partite graph is $k$, thereby showing that boxicity of a graph with $n$ vertices cannot exceed $[n/2]$. Chandran and Sivadasan [10] showed that $\text{box}(G) \leq \text{tw}(G) + 2$, where $\text{tw}(G)$ is the treewidth of $G$. Chandran et al. [9] proved that $\text{box}(G) \leq \chi(G^2)$ where, $G^2$ is the supergraph of $G$ in which two vertices are adjacent if and only if they are at a distance of at most 2 in $G$, and $\chi(G^2)$ is the chromatic number of $G^2$. From this result, they inferred that $\text{box}(G) \leq 2\Delta^2$, where $\Delta$ is the maximum degree of $G$. Scheinerman [28] showed that the boxicity of outer planar graphs is at most 2. Thomassen [21] proved that the boxicity of planar graphs is at most 3. In [14], Cozzens and Roberts studied the boxicity of split graphs.

In contrast, the literature provides few results regarding lower bounds on boxicity. Even ad hoc constructions that achieve high boxicity are rare. In [27], to prove that a complete $k$-partite graph has boxicity at least $k$, Roberts uses the pigeon-hole principle in conjunction with the fact that an interval graph does not contain an induced $C_4$. As a consequence, he shows that the boxicity of a $[n/2]$-partite graph is at least $[n/2]$. In [32], Trotter has characterized all graphs with boxicity $[n/2]$. Similar arguments are used by Cozzens and Roberts [14] to show that the boxicity of the complement of a cycle or a path of length $n$, is at least $[n/3]$. Motivated by the concept of boxicity, McKee and Scheinerman [26] study a parameter called chordality of a graph $G$, $\text{chord}(G)$ (a better name would be chordal dimension). It is the minimum $k$ such that $G$ can be expressed as the intersection of $k$ chordal graphs. Since, every interval graph is chordal, it follows that $\text{box}(G) \geq \text{chord}(G)$. To obtain lower bounds for chordality, the authors use the property that if a graph has chordality at most $k$, then $G$ contains a vertex whose neighbours induce a subgraph of chordality at most $k - 1$. Using this, they have managed to show that the chordality and hence the boxicity of a bipyramid graph is at least 3. Unfortunately, it does not look like it is adequate to give strong and general lower bounds. In [10], Chandran and Sivadasan provide a specialized construction to show that for any integer $k \geq 1$, there are graphs with treewidth at most $t + \sqrt{t}$ whose boxicity is at least $t - \sqrt{t}$.

It is interesting to note that coloring problems on low boxicity graphs were considered as early as 1948 [4]. Kostochka [23] provides an extensive survey on colouring problems of intersection graphs. In [21], the complexity of finding the maximum independent set in bounded boxicity graphs is considered. Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [33], the rectangle number [12], grid dimension [3], circular dimension [15,29], and the boxicity of digraphs [11] are some examples.

1.2 Our Results

In this paper, we present two methods to obtain lower bounds on boxicity. The underlying idea of both the methods is to make use of the vertex isoperimetric properties of the given graph (sometimes modified to suit our purpose). The first method allows us to derive strong lower bounds, but has the drawback that it is not of much use when there is a large independent set in the graph (consider for example, the case of bipartite graphs). But the second method overcomes this difficulty and is much more general, though the lower bounds derivable using it seem to be relatively weaker for comparable cases. Most of the already known lower bounds can be re-derived using these methods. Applying these methods we derive several new results, some of which are listed below:

1. The boxicity of a graph on $n$ vertices with no universal vertices and minimum degree $\delta$ is at least $n/2(\delta - 1)$ (Theorem 9, Section 1.3).

   **Remark 1.** There are two parameters, namely cubicity and sphericity which are closely related to boxicity. There have been several instances in the literature when these two parameters were compared with each other. Cubicity (Sphericity) of $G$ denoted as $\text{cub}(G)$ ($\text{sph}(G)$) is the minimum dimension $b$ such that $G$ can be represented as the intersection graph of axis-parallel unit cubes (unit spheres) in $b$-dimensional space. Since cubicity is a stricter notion of boxicity, $\text{cub}(G) \geq \text{box}(G)$. Apparently, there is no such relationship between cubicity (or boxicity) and sphericity. Havel [20] showed that there are graphs with sphericity 2 and arbitrarily high cubicity while Fishburn [16] constructed some graphs of cubicity at most 3 with sphericity greater than their cubicity. Maehara et al. [25] proved that the sphericity of the complement of a tree is at most 3. Using our result mentioned above, we can
easily infer that if $G$ is the complement of a bounded degree tree, then $\text{box}(G) = \Omega(n)$. Thus we have a large number of graphs that have arbitrarily higher value of boxicity (not just cubicity!) than their sphericity. This is a much stronger result compared to that of Havel’s, who refers to the class of star graphs (Consider the five–pointed star graph $K_{1,5}$: $\text{cub}(K_{1,5}) = 3$ while $\text{sph}(K_{1,5}) = 2$). But the boxicity of any star graph is 1.

2. Consider the $G(n, p)$ model of random graphs. Let $p$ be such that $c_1/n \leq p \leq 1 - c_2 \log n/n$, where $c_1$ and $c_2$ are predetermined constants. Then, for $G \in G(n, p)$, almost surely $\text{box}(G) = \Omega(np(1 - p))$ (Theorem 5 Section 5.3).

We can assume without loss of generality that for any interval graph $I$, $|\text{box}(I)| \leq k^2/\log(k + 2/\lambda)$ (Theorem 7 Section 5.3).

On setting $p = 1/2$ we immediately infer that almost all graphs have boxicity $\Omega(n)$. Another consequence of this result is as follows: Let $m$ be an integer such that $c_1 n \leq m \leq c_2 n^2$. Then, there exists a constant $c_4$ such that almost all graphs on $n$ vertices and exactly $m$ edges have boxicity at least $c_4 m/n$.

3. Let $G$ be a connected $k$-regular graph on $n$ vertices. Let $\lambda$ be the second largest eigenvalue in absolute value of the adjacency matrix of $G$. Then, the boxicity of $G$ is at least $\left(\frac{k^2/\lambda^2}{\log(1 + 2/\lambda)}\right) \left(\frac{n - \lambda - 1}{2n}\right)$ (Theorem 3 Section 4.3).

This lower bound should be compared with the known upper bound of $2k^2$ from [11]. Hence, we see that for $k$-regular graphs the lower bound and upper bound become comparable. It is conjectured in [9] that the correct lower bound is indeed $\Omega(k)$ and not $\Omega(k/\log k)$.

4. Let $\mathcal{G}(n, p)$ be such that almost all graphs on $n$ vertices and exactly $m$ edges have boxicity at least $c_4 m/(n \log n)$ (Corollary 4 Section 4.3). We can also show that almost all balanced bipartite graphs have boxicity $\Omega(n/\log n)$.

2 Preliminaries

For a graph $G(V, E)$, $|G|$ denotes the number of edges in $G$. The complement of $G$ is denoted by $\overline{G}$. A balanced $(A, B)$-bipartite graphs is a bipartite graph with $V = A \cup B$ and $|A| = |B|$. $\delta(G)$ and $\Delta(G)$ are the minimum degree and maximum degree of $G$ respectively.

2.1 Interval Graphs

Suppose $I(V, E)$ is an interval graph and $f$ is an interval representation of $I$. For a vertex $u$, let $l_f(u)$ and $r_f(u)$ denote the real numbers corresponding to the left end-point and right end-point respectively of the interval $f(u)$. When there is no ambiguity regarding the interval representation under consideration, we shall discard the subscript $f$ and use the abbreviated forms $l(u)$ and $r(u)$ respectively. Further, we refer to $l(u)$ as the “left end-point” and $r(u)$ as the “right end-point”.

We can assume without loss of generality that for any interval graph $G$, there is an interval representation such that all the interval end points map to distinct points on the real line.

**Induced vertex numbering:** Given an interval representation of $I$ with distinct end-points, we define the induced vertex numbering $\eta(·)$ as a numbering of the vertices in the increasing order of their right end-points, i.e. for any two distinct vertices $u$ and $v$, $\eta(u) < \eta(v) \iff r(u) < r(v)$.

**Definition 1.** A minimum interval supergraph of a graph $G(V, E)$, is an interval supergraph of $G$ with vertex set $V$ and with least number of edges among all interval supergraphs of $G$. 3
2.2 Neighbourhoods

Given a subset of vertices, say $X$, we define several vertex neighbourhoods of $X$.

1. **Vertex-boundary** $N(X, G) = \{ u \in V - X | \exists v \in X \text{ with } uv \in E \}$. The term vertex-boundary is borrowed from [19].
2. **Strong vertex-boundary** $N_S(X, G) = \{ u \in V - X | uv \in E, \forall v \in X \}$.
3. **Vertex neighbourhood** $N'(X, G) = \{ u | \exists v \in X \text{ with } uv \in E \}$.
4. **Closed vertex neighbourhood** $N[X, G] = X \cup N(X, G)$.

When $X$ is a singleton set, say $X = \{ v \}$, we will use the notations $N(v, G)$, $N_S(v, G)$, $N'(v, G)$ and $N[v, G]$ respectively. When there is no ambiguity regarding the graph under consideration, we discard the second argument and simply denote the neighbourhoods as $N(X)$, $N_S(X)$, $N'(X)$ and $N[X]$ respectively.

### Probability Spaces of Random Graphs

1. $\mathcal{G}(n, p)$: The probability space of graphs on $n$ vertices in which the edges are chosen independently with probability $p$.
2. $\mathcal{G}(n, m)$: The probability space of graphs on $n$ vertices having $m$ edges, in which the graphs have the same probability.
3. $\mathcal{G}_R(n, k)$: The probability space of all simple $k$-regular graphs on $n$ vertices with each graph having the same probability.
4. $\mathcal{G}_B(2n, p)$: The probability space of all balanced bipartite graphs on $2n$ vertices in which the edges are chosen independently with probability $p$.
5. $\mathcal{G}_B(2n, m)$: The probability space of all balanced bipartite graphs on $2n$ vertices and $m$ edges, in which the graphs have the same probability.

If $X$ is a random variable, $E(X)$ denotes the expectation of $X$.

### Standard Asymptotic Notations: (From [2])

For two functions $f$ and $g$, we write $f = O(g)$ if $f \leq c_1 g + c_2$ for all possible values of the variables of the two functions, where $c_1, c_2$ are absolute constants. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If the limit of the ratio $f/g$ tends to 0 as the variables of the functions tend to infinity, we write $f = o(g)$. We use $f = \omega(g)$ if $g = o(f)$.

3 The First Method: Based on Minimum Interval Supergraph

Given any positive integer $k$, the **vertex-isoperimetric problem** [19] is to minimize $|N(X, G)|$ over all $X \subseteq V$ such that $|X| = k$. Let

$$b_v(k, G) = \min_{X \subseteq V, |X| = k} |N(X, G)|.$$  

(1)

A brief introduction to isoperimetric problems can be found in [8]. Harper [19] gives a detailed treatment of the vertex-isoperimetric problem.

Let $c_v(k, G) = \max_{X \subseteq V, |X| = k} |N_S(X, G)|$. It is easy to see that $N_S(X, \overline{G}) = V - X - N(X, G)$. From this we infer the following:

$$c_v(k, \overline{G}) = \max_{X \subseteq V, |X| = k} |V - X - N(X, G)| = n - k - \min_{X \subseteq V, |X| = k} |N(X, G)| = n - k - b_v(k, G).$$  

(2)

**Observation 1.** Let $i$ and $j$ be two positive integers such that $i > j$. Then, $c_v(i, G) \leq c_v(j, G)$.

**Proof.** If $c_v(i, G) = k$, it implies that there exists a pair of sets $X, Y \subseteq V$ with $|X| = i$ and $|Y| = k$ such that $Y$ is the strong vertex boundary of $X$. For any set $A \subseteq X$ such that $|A| = j$, $Y \subseteq N_S(A, G)$. Therefore, $c_v(j, G) \geq k$. 


Proof. If $c_v(k, G) \geq i$, then it implies that there exists a pair of sets $X, Y \subseteq V$ with $|X| = k$ and $|Y| = i$, such that $Y = N_S(X, G)$. Then clearly $X \subseteq N_S(Y, G)$. It follows that $c_v(i, G) \geq c_v(|Y|, G) \geq k > j$, contradicting the assumption that $c_v(i, G) = j$.

3.1 Lower Bound for Boxicity

Lemma 1. Let $G(V, E)$ be any non-complete graph. Let $I_{\text{min}}$ be a minimum interval supergraph of $G$. Then,

$$\text{box}(G) \geq \frac{|G|}{\sum_{i=1}^{n-1} c_v(i, G)}.$$  \hspace{1cm} (3)

Proof. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ be an interval graph representation of $G$. Since each graph $I \in \mathcal{I}$ is an interval supergraph of $G$, $|I| \geq |I_{\text{min}}|, \forall I \in \mathcal{I}$. Hence, in any $I \in \mathcal{I}$, at most $|I_{\text{min}}|$ edges can be absent. In $G$, $|G|$ edges are absent. Since, an edge absent in $G$ should be absent in at least one interval graph in $\mathcal{I}$, there should be at least $|G|/|I_{\text{min}}|$ interval graphs in $\mathcal{I}$. Hence proved.

Theorem 2. Let $G$ be a non-complete graph with $n$ vertices. Then,

$$\text{box}(G) \geq \frac{|G|}{\sum_{i=1}^{n-1} c_v(i, G)}.$$  \hspace{1cm} (4)

Proof. In view of Lemma 1, it is enough to show that $|I_{\text{min}}| \leq \sum_{i=1}^{n-1} c_v(i, G)$. Consider an interval representation with distinct end-points for $I_{\text{min}}$ and, let $\eta$ be the induced vertex numbering (see Section 2.1 for definition). Given an integer $k$, $1 \leq k \leq n$, let

$$S_k(\eta) = \eta^{-1} (\{1, 2, \ldots, k\}) = \{v \in V | \eta(v) \leq k\},$$

be the set of the first $k$ vertices numbered by $\eta$. We observe that, in any interval graph, for any three vertices $u, v$, and $w$ such that $\eta(u) < \eta(v) < \eta(w)$, if $w$ is adjacent to $u$, then $w$ is also adjacent to $v$ and therefore $N(S_k(\eta)) = \{u \in N(\eta^{-1}(k)) | \eta(u) > k\}$. Using this fact, we have

$$\sum_{k=1}^{n-1} |N(S_k(\eta))| = \sum_{u \in V} |\{v | uv \in E \text{ and } \eta(v) > \eta(u)\}| = |I_{\text{min}}|.$$  

By definition, $|N(S_k(\eta), G)| \geq b_v(k, G)$ and therefore, $|I_{\text{min}}| \geq \sum_{k=1}^{n-1} b_v(k, G)$. Now, the upper bound on $|I_{\text{min}}|$ follows by applying equation (2):

$$|I_{\text{min}}| = \left(\frac{n}{2}\right) - |I_{\text{min}}|$$

$$\leq \left(\frac{n}{2}\right) - \sum_{k=1}^{n-1} b_v(k, G) = \sum_{k=1}^{n-1} n - k - b_v(k, G) = \sum_{k=1}^{n-1} c_v(k, G).$$

3.2 Some Immediate Applications of Theorem 2

Theorem 3. Let $G(V, E)$ be a $(n - k - 1)$-regular graph for some positive integer $k$. Then, $\text{box}(G) \geq \frac{n}{2k}$.

Proof. Since $G$ is a $k$-regular graph, $|G| = nk/2$ and $c_v(1, G) = k$. For $2 \leq i \leq k$, $c_v(i, G) = k$ (by Observation 1) and for $i > k$, $c_v(i, G) = 0$ (by Observation 2). Hence, $\sum_{i=1}^{n-1} c_v(i, G) \leq k^2$. Applying Theorem 2, the result follows.
Tightness of Theorem 3. Consider the following co-bipartite graph \( G(V, E) \): \( V = A \uplus B \) and let \( n, k \) and \( l \) be integers such that, \( |A| = |B| = kl = n/2 \). Further, let \( A = A_1 \uplus A_2 \uplus \cdots \uplus A_l \) and \( B = B_1 \uplus B_2 \uplus \cdots \uplus B_l \), where, \( |A_i| = |B_i| = k \). For any two vertices, \( u \) and \( v \), let \( u \notin E \Longleftrightarrow u \in A_i \) and \( v \in B_i, 1 \leq i \leq l \). Clearly, \( G \) is an \((n - k - 1)\)-regular graph and therefore by Theorem 3, \( \text{box}(G) \geq n/2k = l \). Now, we present a set of \( l \) interval graphs \( \mathcal{I} = \{I_i|1 \leq i \leq l\} \) whose edge intersection gives \( G \). The interval representation \( f_i \) for each \( I_i \) is as follows: \( f_i(u) = [0, 1] \) if \( u \in A_i \), \( f_i(u) = [2, 3] \) if \( u \in B_i \), and \( f_i(u) = [1, 2] \) otherwise. Therefore, \( \text{box}(G) = l = n/2k \). In fact, we note that each \( I_i \) is a unit-interval graph by construction. Therefore, this graph acts as a tight example even if we were to replace \( \text{box}(G) \) with \( \text{cub}(G) \) in Theorem 3.

Theorem 4. Let \( k \geq 1 \) be an integer.
1. Let \( G \) be a \((n - k - 1)\)-regular co-planar graph. Then, \( \text{box}(G) \geq \frac{n}{k} \).
2. Let \( G \) be the complement of a \( k\)-regular \( C_4\)-free graph. Then, \( \text{box}(G) \geq \frac{n}{2k} \).
3. \( \text{box}(\overline{C_n}) \geq \frac{k}{4} \).

Proof. (1) Since \( \overline{G} \) is a planar graph, it does not contain \( K_{3,3} \) as a subgraph. Hence, \( c_v(i, \overline{G}) \leq 2 \) for \( 3 \leq i \leq k \). From Observation 2, \( c_v(i, \overline{G}) = 0 \) for \( i > k \). The result follows by applying Theorem 2 with \( c_v(i, \overline{G}) \leq k \), for \( i = 1, 2 \).

(2) Since \( \overline{G} \) is \( C_4 \) or \( K_{2,2} \) free, the result follows in the same way as for (1).

(3) We have \( ||C_n|| = n, c_v(1, C_n) = 2, c_v(2, C_n) = 1 \) and by Observation 2, \( c_v(i, C_n) = 0 \) for \( i > 2 \). Applying Theorem 2, the result follows. We recall that Cozzens and Roberts [14] showed that \( \text{box}(\overline{C_n}) = \lfloor n/3 \rfloor \) by different methods.

Remark 2. Theorems 3 and 4 are only indicative. We can bound the boxicity of various other graph classes using this method. If \( G \) is not regular we can give a lower bound in terms of the ratio \( \delta(\overline{G})/\Delta(\overline{G}) \), i.e. \( \text{box}(G) \geq \frac{n^3(\overline{G})}{2\Delta(\overline{G})^2} \).

Later, using the second method we will show an better lower bound of \( \frac{n^3(\overline{G})}{2\Delta(\overline{G})} \).

3.3 Lower Bound for Random Graphs: \( \mathcal{G}(n,p) \) and \( \mathcal{G}(n,m) \)

For the definitions of probability spaces such as \( \mathcal{G}(n,p) \) and \( \mathcal{G}(n,m) \) and asymptotic notations such as \( \mathcal{O} \), \( \omega \), etc, we refer to Section 2.

Lemma 2. Let \( n \) and \( i \) be positive integers. For \( n > 1 \) and \( 0 < i \leq n/e^2 \), \( \binom{n}{i} \leq \left( \frac{2e}{i} \right)^{2i} \).

Proof. We prove by induction on \( i \). For \( i = 1 \), the inequality is trivially satisfied. Let \( i = k + 1 \) and assume that the claim holds for \( i \leq k \).

\[
\binom{n}{k+1} = \binom{n}{k} \frac{n-k}{k+1} \leq \binom{n}{k} \left( \frac{n}{k} \right)^{2k} \frac{n}{k+1} = \left( \frac{n}{k} \right)^{2k+1} \left( \frac{k+1}{k+1} \right)^{2k} \leq \left( \frac{n}{k+1} \right)^{2k+1} e^{2i}.
\]

Since \( k+1 \leq n/e^2 \), it follows that \( \left( \frac{n}{k+1} \right)^{2k+1} e^{2i} \leq \left( \frac{n}{k+1} \right)^{2(k+1)}. \) Hence proved.

Theorem 5. Let \( G \in \mathcal{G}(n,p) \) and \( p \) be such that \( c_1/n \leq p \leq 1 - c_2 \frac{\log n}{n} \), where, \( c_1 = e^{c_2} \) and \( c_2 \) is some suitable positive constant. Then, almost surely, \( \text{box}(G) = \Omega(np(1-p)) \).

Proof. The outline of the proof is as follows. We will first show that for almost all graphs \( G \in \mathcal{G}(n,p) \), \( ||\overline{G}|| = \Omega(n^2(1-p)) \). Then, we show that \( \sum_{i=1}^{n-1} c_v(i, \overline{G}) = \mathcal{O}(n/p) \) for almost all graphs. The result follows by the application of Theorem 4.

Let \( f = np \). The probability that a particular edge is present in \( \overline{G} \) is \( q = 1 - f/n \leq e^{-f}/n \). The expected number of edges in \( \overline{G} \) is \( \binom{n}{2}q \). Consider the event \( \mathcal{E}_1 : ||\overline{G}|| < (1-\epsilon)\binom{n}{2}q \). For some constant \( \epsilon \in (0,1) \), we set \( c_2 = 10/e^2 \). Using Chernoff’s bound on Poisson trials, we have

\[
Pr(\mathcal{E}_1) \leq \exp \left( -\frac{c_2}{2} \frac{n}{2} q \right) \leq \frac{1}{n^\epsilon}.
\]

Next, we look at the following event: \( \mathcal{E}_2 : \sum_{i=1}^{n-1} c_v(i, \overline{G}) = \mathcal{O}(n^2/f) \). Our aim is to show that \( Pr(\mathcal{E}_2) \to 0 \) as \( n \to \infty \). To this end, we divide the summation in to five parts.
Range 1 \((1 \leq i \leq \frac{2n}{5} \log f)\): Let \(A \subseteq V, |A| = i\). Consider the random variable \(X = |N_S(A, \overline{G})|\) and the event \(\mathcal{E}_A : X > (n-i)e^{-fi/n}\) for some constant \(\epsilon > 6\). Note that \(q = \left(1 - \frac{6}{5}\right) \leq e^{-fi/n}\). Observing that \(\mathcal{E}_A\) is a monotone increasing event for ease of computation we will assume that \(\overline{G} \in \mathcal{G}(n, e^{-fi/n})\) instead of \(\mathcal{G}(n, q)\). Since \(E(X) = (n-i)e^{-fi/n}\) with respect to \(\mathcal{G}(n, e^{-fi/n})\), we bound \(Pr(\mathcal{E}_A)\) using Chernoff’s bound on Poisson trials: \(Pr(\mathcal{E}_A) \leq 2^{-\epsilon E(X)} \leq \exp\left(-\frac{\epsilon}{2}(n-i)e^{-fi/n}\right)\). Since \(c_v(i, \overline{G}) = \max_{A \subseteq V, |A| = i} |N_S(A, \overline{G})|\), using union bound and subsequent application of Lemma we get,

\[
Pr\left(c_v(i, \overline{G}) \geq (n-i)e^{-fi/n}\right) \leq \binom{n}{i} \exp\left(-\frac{\epsilon}{2}(n-i)e^{-fi/n}\right) \leq \exp\left(2i \log(n/i) - \frac{\epsilon}{2}(n-i)e^{-fi/n}\right). \tag{6}
\]

Then, we note that for any two integers \(a\) and \(b\), such that \(0 < a \leq b\),

\[
\sum_{i=a}^{b} \epsilon(n-i)e^{-fi/n} \leq \sum_{i=0}^{\infty} \epsilon ne^{-fi/n} = \frac{\epsilon n}{1 - e^{-f/n}} \leq \frac{2\epsilon n^2}{f}. \tag{7}
\]

The last expression is derived from the fact that \(e^{-f/n} \leq (1 - f/2n)\) when \(f/n < 1\). Let \(\mathcal{E}_{21} : \sum_{i=a}^{b} c_v(i, \overline{G}) \geq \frac{2\epsilon n^2}{f}\).

We want to show that \(Pr(\mathcal{E}_{21}) \rightarrow 0\) as \(n \rightarrow \infty\) for \(a = 1\) and \(b = \frac{2n}{5} \log f\) in \(\mathcal{G}\). We recall that \(f \geq c_1 = e^e\). Hence, we have \(a \leq b \leq n/e^2\). In view of equation (7),

\[
Pr(\mathcal{E}_{21}) \leq Pr\left(\bigvee_{i=a}^{b} c_v(i, \overline{G}) \geq (n-i)e^{-fi/n}\right)
\]

\[
\leq \sum_{i=a}^{b} Pr\left(c_v(i, \overline{G}) \geq (n-i)e^{-fi/n}\right)
\]

(by (6)) \[
\leq \sum_{i=a}^{b} \exp\left(2i \log(n/i) - \frac{\epsilon}{2}(n-i)e^{-fi/n}\right) = \sum_{i=a}^{b} \exp(-h(i)),
\]

where \(h(i) = \frac{\epsilon}{2}(n-i)e^{-fi/n} - 2i \log(n/i)\). Clearly \(h(1) = \Theta(n)\). It is easy to verify that the slope of the function \(h(i)\) is negative and thus is a monotonically decreasing function of \(i\) for the range \(1 \leq i \leq \frac{2n}{5} \log f\). Therefore,

\[
\sum_{i=a}^{b} \exp(-h(i)) \leq b \exp(-h(b)) = \exp(-h(b) + \log b).
\]

Since \(b = \frac{2n}{5} \log f\),

\[
h(b) - \log b = \frac{\epsilon}{2} \frac{n}{f^{2/5}} - \frac{\epsilon}{5} \frac{n}{f^{3/5}} \log f - \frac{4n}{5} \log f \log \frac{5f}{2} \log f - \log \left(\frac{2n}{5} \log f\right) = \Theta\left(n f^{-2/5}\right), \tag{8}
\]

for \(\epsilon \geq 6\). Since \(f \geq e^e\),

\[
Pr(\mathcal{E}_{21}) \leq \exp\left(-\Theta\left(n f^{-2/5}\right)\right) \leq \frac{1}{n^2}. \tag{9}
\]

1 A property \(Q\) is a monotone increasing property if whenever \(G\) satisfies \(Q\), any supergraph of \(G\) on the same set of vertices also satisfies \(Q\). We make use of the property that, if \(0 \leq p_1 < p_2 \leq 1\), then, \(Pr(G \in \mathcal{G}(n, p_2), G\) satisfies \(Q) \leq Pr(G \in \mathcal{G}(n, p_1), G\) satisfies \(Q)\) (see [1] Theorem 1, Chapter 2).

2 We assume that \(\frac{2n}{5} \log f\) is an integer. If not, we can replace it with its floor, without affecting the proof. The same argument holds for the rest of the ranges.
Range 2 \((\frac{2n}{f} \log f \leq i \leq \frac{5n}{f} \log f)\): Let \(\mathcal{E}_{22}: \sum_{i=\frac{2n}{f} \log f}^{\frac{5n}{f} \log f} c_v(i, \overline{G}) > \frac{5n^2}{f}\). Suppose \(x\) and \(y\) are positive integers.
The event that \(c_v(x, \overline{G}) \geq y\) is equivalent to the following event
\[
\mathcal{E}_{x,y} : \exists A, B \subseteq V, |A| = x \text{ and } |B| = y, \text{ such that } A \cap B = \emptyset \text{ and } B \subseteq N_S(A, \overline{G}) \text{ and therefore, } \forall u, v \text{ such that } u \in A, v \in B \Rightarrow uv \in E(\overline{G}).
\]
Using this fact along with the union bound, we have
\[
\Pr(c_v(x, \overline{G}) \geq y) \leq \left(\frac{n}{x}\right) \left(\frac{n}{y}\right)^{eqy} \leq \left(\frac{n}{x}\right) \left(\frac{n}{y}\right)^{efxy/n}.
\] (10)
Let \(x = \frac{2n}{f} \log f\) and \(y = \frac{n}{\log f}\). Applying the above equation and subsequently Lemma 2
\[
\Pr\left(c_v\left(\frac{2n}{f} \log f, \overline{G}\right) \geq \frac{n}{\log f}\right) \leq \exp\left(\frac{4n}{5f} \log f \log\left(\frac{5f}{2 \log f}\right) + \frac{2n}{\log f} \log(\log f) - \frac{2n}{5}\right).
\] (11)
By Observation 1 it follows that if \(c_v\left(\frac{2n}{f} \log f, \overline{G}\right) \leq \frac{n}{\log f}\), then
\[
\sum_{i=\frac{2n}{f} \log f}^{\frac{5n}{f} \log f} c_v(i, \overline{G}) \leq \frac{5n}{f} \log f \times c_v\left(\frac{5n}{f} \log f, \overline{G}\right) \leq \frac{5n}{f} \log f \times \frac{n}{\log f} = \frac{5n^2}{f}.
\]
Using inequality (11) and the fact that \(f \geq c_1\),
\[
\Pr(\mathcal{E}_{22}) \leq \Pr\left(c_v\left(\frac{2n}{f} \log f, \overline{G}\right) \geq \frac{n}{\log f}\right) \leq \exp\left(-\Theta(n)\right) \leq \frac{1}{n^2}.
\] (12)
Range 3 \((\frac{5n}{f} \log f \leq i \leq \frac{5n}{f} \log f)\): Let \(\mathcal{E}_{23}: \sum_{i=\frac{5n}{f} \log f}^{\frac{5n}{f} \log f} c_v(i, \overline{G}) > \frac{5n^2}{f}\). We use the same arguments as in the case of Range 2 to show that \(\Pr(\mathcal{E}_{23}) \leq 1/n^2\). Let \(x = y = \frac{5n}{f} \log f\). Applying equation (11) and subsequently Lemma 2
\[
\Pr\left(c_v\left(\frac{5n}{f} \log f, \overline{G}\right) \geq \frac{5n}{f} \log f\right) \leq \exp\left(\frac{20n}{f} \log f \log\left(\frac{f}{5 \log f}\right) - \frac{25n}{f} \log^2 f\right).
\] (13)
Again using Observation 1 it follows that if \(c_v\left(\frac{5n}{f} \log f, \overline{G}\right) \leq \frac{5n}{f} \log f\), then,
\[
\sum_{i=\frac{5n}{f} \log f}^{\frac{5n}{f} \log f} c_v(i, \overline{G}) \leq \frac{n}{\log f} \times c_v\left(\frac{5n}{f} \log f, \overline{G}\right) \leq \frac{n}{\log f} \times \frac{5n}{f} \log f = \frac{5n^2}{f}.
\]
Using inequality (13) and the fact that \(f \geq c_1\),
\[
\Pr(\mathcal{E}_{23}) \leq \Pr\left(c_v\left(\frac{5n}{f} \log f, \overline{G}\right) \geq \frac{5n}{f} \log f\right) \leq \exp\left(-\frac{5n}{f} \log^2 f\right) \leq \frac{1}{n^2}.
\] (14)
Range 4 \((\frac{n}{\log f} \leq i \leq n/e^2)\): Let \(\mathcal{E}_{24}: \sum_{i=\frac{n}{\log f}}^{\frac{n}{e^2}} c_v(i, \overline{G}) \geq 3n^2/f\). Applying equation (10) with \(x = i \geq \frac{n}{\log f}\) and \(y = \frac{3n}{f} \log(n/i)\) and Lemma 2
\[
\Pr\left(c_v(i, \overline{G}) \geq \frac{3n}{f} \log(n/i)\right) \leq \binom{n}{i} \left(\frac{3n}{f} \log(n/i)\right) \exp\left(-3i \log(n/i)\right)
\]
\[
\leq \exp\left(\log(n/i) \left(\frac{6n}{f} \log\left(\frac{f}{3 \log(n/i)}\right) - i\right)\right)
\]
\[
\leq \exp\left(-\frac{i}{2} \log(n/i)\right).
\] (15)
The final expression comes from the fact that $f \geq c_1$. Next we observe that,
\[
\sum_{i=n/e}^{n/e^2} \frac{3n}{f} \log(n/i) \leq \int_{\frac{n}{e}}^{n/e^2} \frac{3n}{f} \log(n/z)dz \leq \frac{3n^2}{f}.
\]
Arguing as in the case of Range 1 and then applying (15),
\[
Pr(\mathcal{E}_{24}) \leq Pr\left(\bigvee_{i=n/e}^{n/e^2} c_v(i, G) \geq \frac{3n}{f} \log(n/i)\right) \leq \sum_{i=n/e}^{n/e^2} \exp\left(-\frac{i}{2} \log(n/i)\right).
\]
The function $i \log(n/i)$ increases monotonically from $n/\log f$ and has maxima at $n/e$. Therefore,
\[
Pr(\mathcal{E}_{24}) \leq \frac{n}{e^2} \exp\left(-\frac{n}{2} \log f \log \frac{n}{\log f}\right) = \frac{n}{e^2} \exp\left(-\frac{n \log \log f}{2 \log f}\right) \leq \frac{1}{n^2}.
\]
(16)

**Range 5** $(n/e^2 \leq i \leq n)$: Let $\mathcal{E}_{25} := \sum_{i=n/e^2}^{n} c_v(i, G) > 6n^2/f$. From (15), we note that $Pr(c_v(n/e^2, G) \geq 6n^2/f) \leq 1/n^2$. Again using Observation 4, we see that if $c_v(n/e^2, G) \leq 6n^2/f$, then, $\sum_{i=n/e^2}^{n} c_v(i, G) \leq n \times 6n^2/f = 6n^2/f$. Hence,
\[
Pr(\mathcal{E}_{25}) \leq Pr\left(c_v(n/e^2, G) \geq \frac{6n}{f}\right) \leq \frac{1}{n^2}.
\]
(17)

Let $Q$ be the union of the events $\mathcal{E}_1$ and $\mathcal{E}_2$. It is clear that $box(G) = \Omega(np(1-p))$ if the event $Q$ does not occur. From inequalities (8), (9), (12), (13), (16) and (17), it is evident that $Pr(Q) = O(1/n^2)$. Hence, almost surely $box(G) = \Omega(np(1-p))$.

**Corollary 1.** Let $G \in \mathcal{G}(n, p)$. If $p$ is such that $c_1/n \leq p \leq c_3 < 1$, where $c_3$ is a positive constant, then, almost surely, $box(G) = \Omega(np)$.

**Corollary 2.** Consider graphs with $n$ vertices and $m$ edges such that $c_1n/2 \leq m \leq c_2n^2/2$, where $c_1$ and $c_2$ are as defined in Corollary 1. Let $M = \binom{n}{2}$, be the total number of graphs on $n$ vertices and $m$ edges. There exists a positive constant $c$ such that, if $N_c$ is the number of graphs whose boxicity is at most $cm/n$, then,
\[
\frac{N_c}{M} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

**Proof.** We recall that, in the proof of Theorem 5, $box(G) = \Omega(np(1-p))$ if the event $Q$ does not occur. Let $p = \frac{m}{\binom{n}{2}}$. It can be shown that, given any property $\mathcal{P}$, $Pr(G \text{ satisfies } \mathcal{P}, G \in \mathcal{G}(n, m)) \leq 3\sqrt{m}Pr(G \text{ satisfies } \mathcal{P}, G \in \mathcal{G}(n, p))$ (see 2 Theorem 2 Chapter 2). Hence, $Pr(G \text{ satisfies } Q, G \in \mathcal{G}(n, m)) \leq 3\sqrt{m}O(1/n^2) = O(1/n)$. Therefore, if $G \in \mathcal{G}(n, m)$, then almost surely, $box(G) = \Omega(m/n)$. Hence proved.

For very dense graphs, i.e. with $p \rightarrow 1$ as $n \rightarrow \infty$, it is not possible to infer from Theorem 5 that $box(G) = \Omega(np)$ because of the additional factor $(1-p)$. For this case we use a different approach to bound the boxicity. A theorem follows in this regard.

**Theorem 6.** Let $G \in \mathcal{G}(n, p)$ with $p = 1 - g/n$, where $g(n)$ is any non-negative function such that $g(n) = o(n)$ and $g(n) = \omega(1/n)$. Then,
1. for $g \geq c \log n$ where $c$ is some constant, almost surely $box(G) = \Omega(n)$,
2. for $g < c \log n$, almost surely, $box(G) = \Omega\left(\frac{ng}{\log n}\right)$.

**Proof.** The flow of this proof is very similar to that of Theorem 5. The probability that a particular edge is present in $\overline{G}$ is $q = g/n$. The expected number of edges in $\overline{G}$ is $\binom{n}{2} g = \frac{(n-1)g}{2}$. Using Chernoff’s bound on Poisson trials, we have,
\[
Pr\left(||\overline{G}|| < (1 - \epsilon)\frac{(n-1)g}{2}\right) \leq e^{-\frac{\epsilon^2(n-2)g}{2}},
\]
(18)
for a constant $\epsilon \in (0, 1)$. Since $g = \omega(1/n)$, $Pr(||\overline{G}|| < (1 - \epsilon)(n-1)g/2) \rightarrow 0$ as $n \rightarrow \infty$.

Now we will give an upper bound for $\sum_{i=1}^{n} c_v(i, \overline{G})$. Let the expected degree of a vertex in $\overline{G}$ be $d_E = \frac{(n-1)g}{n}$. 9
Case 1 $g \geq c \log n$: First we consider $c_v(1, \overline{G})$. Using Chernoff’s bound on Poisson trials and subsequently union bound,

$$Pr(c_v(1, \overline{G}) \geq (1 + \epsilon') d_E = (1 + \epsilon) g) \leq ne^{-\frac{c}{2} g}. \quad (19)$$

for some constants $\epsilon', \epsilon \in (0, 1)$. Taking $c > 3/\epsilon^2$, the expression $\to 0$ as $n \to \infty$.

Case 2 $g < c \log n$: Here again we consider $c_v(1, \overline{G})$ and apply another variant of Chernoff’s bound on Poisson trials and subsequently the union bound. For $R > 6g$,

$$Pr(c_v(1, \overline{G}) \geq R) \leq n2^{-R}. \quad (20)$$

Taking $R = 2\log n$, we see that the expression $\to 0$ as $n \to \infty$. Therefore, almost surely, $c_v(1, \overline{G}) = O(\log n)$.

For $i > 1$, we use Lemma 2 and equation (10),

$$Pr (c_v(i, \overline{G}) \geq j) \leq \left( \frac{n}{i} \right) \left( \frac{n}{j} \right) \left( \frac{g}{n} \right)^{ij} \leq \exp (2i \log(n/i) + 2j \log(n/j) + ij \log g - ij \log n).$$

For $i > 2$, choosing $j > 6$, the right hand side for the above expression $\to 0$ as $n \to \infty$. Using Observation 2, we note that $Pr (c_v(2 \log n, \overline{G}) > 0) \to 0$ as $n \to \infty$. Therefore, almost surely,

$$\sum_{i=1}^{n-1} c_v(i, \overline{G}) = \sum_{i=1}^{c_v(1, \overline{G})} c_v(i, \overline{G}) = \Theta(c_v(1, \overline{G})).$$

Hence proved.

3.4 Spectral Lower Bounds

Consider a graph $G(V,E)$ and let $X \subseteq V$. Recalling the definition of vertex-boundary and vertex neighbourhood from Section 2.2 we have $N(X) \subseteq N'(X)$ and

$$|N(X)| \geq |N'(X)| - |X|. \quad (21)$$

Suppose $G(V,E)$ is a k-regular balanced $(A,B)$-bipartite graph with $2n$ vertices. For any $X \subseteq A$, Tanner [30] gives a lower bound for $|N'(X)|/|X|$ using spectral methods. Suppose the vertices in $A$ and $B$ are ordered separately, we define the bipartite incidence matrix $M$ as follows: $M = [m_{ij}]$, $m_{ij} = 1$ if the $i$th vertex in $A$ is connected to $j$th vertex in $B$. The largest eigenvalue of $MM^T$ is $k$. Let $\lambda'$ be the second largest eigenvalue of $MM^T$. Then,

$$|N'(X)| \geq \frac{k^2 |X|}{\lambda' + (k^2 - \lambda') \frac{2|X|}{n}}. \quad (22)$$

This result can be extended to any k-regular graph $G(V,E)$ (see [22]) as follows. Let $\lambda$ be the second largest eigenvalue in absolute value of the adjacency matrix of $G$. For any subset $X \subseteq V$,

$$|N'(X)| \geq \frac{k^2 |X|}{\lambda^2 + (k^2 - \lambda^2) \frac{|X|}{n}}. \quad (23)$$

Theorem 7. Let $G(V,E)$ be a connected k-regular graph on $n$ vertices. Let $\lambda$ be the second largest eigenvalue in absolute value of the adjacency matrix of $G$. Then,

$$\text{box}(G) \geq \left( \frac{k^2 / \lambda^2}{\log \left( \frac{k^2}{\lambda^2} + 1 \right)} \right) \left( \frac{n - k - 1}{2n} \right).$$
Proof. We first note that $|\overline{G}| = \frac{n(n-k-1)}{2}$. By the definition of $b_v(i)$ in Equation (1) (See Section 3) and by applying equations (21) and (23):

$$b_v(i, G) \geq \frac{k^2 i}{\lambda^2 + (k^2 - \lambda^2)a_n} - i. \quad (24)$$

Recalling that $c_v(i, \overline{G}) = n - i - b_v(i, G)$ and applying Theorem 2 we get,

$$\text{box}(G) \geq \frac{n(n-k-1)/2}{\sum_{i=1}^{n-1} \frac{n - i - \left( \frac{k^2 i}{\lambda^2 + (k^2 - \lambda^2)a_n} - i \right)}{(n-k-1)/2 \sum_{i=1}^{n-1} \frac{k^2 i}{\lambda^2 a_n + (k^2 - \lambda^2)i}}}$$

$$\geq \frac{(n-k-1)/2}{\sum_{i=1}^{n-1} \frac{k^2 i}{\lambda^2 a_n + k^2 i} \sum_{i=1}^{n-1} \frac{\lambda^2 n}{\lambda^2 a_n + k^2 i}}$$

$$= \frac{(n-k-1)/2}{k^2 \sum_{i=1}^{n-1} \frac{1}{\lambda^2 a_n + i}}$$

$$\geq \frac{(n-k-1)/2}{k^2 \left( \log \left( \frac{\lambda^2 n}{k^2} + n \right) - \log \left( \frac{\lambda^2 n}{k^2} \right) \right)}$$

$$= \frac{k^2 / \lambda^2}{\log \left( \frac{\lambda^2 n}{k^2} + 1 \right) \left( \frac{n-k-1}{2n} \right)}.$$ 

Hence proved. \( \square \)

Tightness of Theorem 7 Let $l$ and $p$ be integers $\geq 2$. Consider a complete $p$-partite graph $K_{l,l,...,l}$ where $n = lp$. $K_{l,l,...,l}$ is a strongly regular graph, i.e. it is a $k$-regular graph where each pair of adjacent vertices has the same number $a \geq 0$ of common neighbours, and each pair of non-adjacent vertices has the same number $c \geq 1$ of common neighbours. For a strongly regular graph, $\lambda$ can be obtained by solving the quadratic equation $x^2 + (c - a)x + (c - k)$ (See [3] Chapter 3). For $K_{l,l,...,l}$, noting that $k = c = (p-1)l$ and $a = (p-2)l$, we get $\lambda = l$. Using Theorem 2 we get $\text{box}(G) = \Omega \left( \frac{p}{\log(p)}\right)$. In [27], it is shown that $\text{box}(K_{l,l,...,l}) = p$. Hence, our result is tight up to a $O(\log(p)) = O \left( \log \left( \frac{\lambda^2 n}{k^2} + 1 \right) \right)$ factor. It would be interesting to see if we can improve this bound to $\text{box}(G) = \Omega \left( \frac{k^2}{\lambda^2} \left( \frac{n-k-1}{2n} \right) \right)$. For example, when $l = 2$ (This graph was considered by Roberts in [27] and can be obtained by removing a perfect matching from a complete graph), such an improved bound will evaluate to $\frac{n^2}{8}$. The boxicity of this graph was shown to be $n/2$ in [27].

Applying the Spectral Bound to Random Regular Graphs In 1986, Alon and Boppana [11] proved that for a fixed $k$, $\lambda \geq 2\sqrt{k-1}$ for almost all $k$-regular graphs. In the same paper it was conjectured that for any $k \geq 3$ and $e > 0$, $\lambda \leq 2\sqrt{k-1} + e$. Recently, Friedman [18] proved this conjecture albeit for a different model of regular graphs (involving multiple edges and loops). However, using contiguity theorems [34] it is easy to infer that the same result applies to the $G_R(n, k)$ model too. Hence, we can assume that $\lambda \approx 2\sqrt{k-1}$ for a random $k$-regular graph. We have the following corollary to Theorem 7.

Corollary 3. Let $G_R(n, k)$ be the probability space of random $k$-regular graphs, where $k$ is fixed. For $G \in G_R(n, k)$, almost surely, $\text{box}(G) = \Omega \left( \frac{k}{\log k} \right)$. 

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We conjecture that for almost all $k$-regular graphs $\text{box}(G) = \Omega(k)$. The reader may be interested to know that a comparable upper bound of $O(k)$ for boxicity was conjectured in [9]: In that paper it was shown that for any graph $G$, $\text{box}(G) \leq 2\Delta^2$ and it was conjectured that $\text{box}(G) = O(\Delta)$.

3.5 Limitations

The presence of large independent sets in $G$ puts a limitation on the lower bound that can be achieved using Theorem 2. For example, consider a bipartite graph on $2n$ vertices. It contains an independent set of size at least $n$, the best possible bound we can get by applying Theorem 2 is $(\binom{2n}{2}/2n) = 4$. In general, if $\alpha$ is the independence number of $G$, we cannot do better than $(n/\alpha)^2$. In view of this limitation, we seek to motivate the reader regarding the second method, where effective lower bounds are provided for bipartite graphs (see Theorem 10 onwards).

4 The Second Method

4.1 Preliminaries

Definition 2 (Cross expansion). Let $A, B \subseteq V$. Let $t$ be a fixed positive integer such that $t \leq |A|$. Let $n_t = \min_{S \subseteq A, |S| = t} \{|N[S, G] \cap B|\}$. The cross expansion coefficient $\beta_t(A, B)$ is defined as $n_t/|B|$. Usually, in the literature on expander graphs or vertex isoperimetric problems, to define the expansion coefficient, it is assumed that $A = B = V$ and the vertex boundary $N(S, G)$ is used instead of $N[S, G]$. We have modified the notion to suit our purpose.

Definition 3 (Co-expansion). Let $A, B \subseteq V$. Let $t$ be a fixed positive integer such that $t \leq |A|$. For $S \subseteq A$, let $m_t = \min_{S \subseteq A, |S| = t} \{|N'(S, G) \cap B|\}$. The co-expansion coefficient $\alpha_t(A, B)$ is defined as $m_t/t$. For $t \leq 0$, we define $\alpha_t(A, B) = \infty$.

We have used the word co-expansion to indicate that this is a certain type of expansion property in the complement of $G$. The reader may want to observe that the interpretation of the ratios in the co-expansion and cross expansion are somewhat different: In particular, note the difference in the denominator.

Observation 3. For any two positive integers $i$ and $j$ such that $i < j$, $i \cdot \alpha_t(A, B) \leq j \cdot \alpha_j(A, B)$. This property follows from the fact that $m_i \leq m_j$, by the definition of $N'(S, G)$.

The following facts regarding co-expansion are easy to verify.

Lemma 3. Let $G(V, E)$ be a non-complete graph on $n$ vertices with maximum degree $\Delta$ and minimum degree $\delta$, then for any $t \leq n$ we have:

1. Let $|V| = n$, $\alpha_t(V, V) \geq \frac{n - \Delta - t}{n - \Delta - 1}$.
2. If $G$ is a balanced $(A, B)$-bipartite on $2n$ vertices, then, $\alpha_t(A, B) = \frac{n - \Delta}{n - \delta}$.
3. If $G$ is $k$-regular, then, $\alpha_t(V, V) \geq 1$.
4. If $G$ is $k$-regular and $(A, B)$-bipartite, then, $\alpha_t(A, B) \geq 1$.

Definition 4 (t-expander). A graph $G(V, E)$ is called a t-expander if and only if the following holds for any $S \subseteq V$ with $|S| = t$,

$$\{|v \in V - S|uv \notin E \text{ for all } u \in S\}| < t.$$ 

It is easy to see that the above definition can be restated as follows: $G$ is a t-expander if and only if $G$ does not contain a $K_t,t$ as a subgraph. For example, a co-planar graph is a 3-expander.

Definition 5 (Bipartite t-expander). An $(A, B)$-bipartite graph $G = (V, E)$ is called a bipartite t-expander if and only if the following holds for any $S \subseteq A$ with $|S| = t$,

$$\{|v \in B \mid uv \notin E \text{ for all } u \in S\}| < t.$$ 

The following lemma on interval graphs will be used in the next section.

Lemma 4. Let $I$ be an interval graph. Two vertices $u$ and $v$ are adjacent in $I$ if and only if $l(u) < r(v)$ and $l(v) < r(u)$. Alternately, $u$ and $v$ are non adjacent in $I$ if and only if $r(u) < l(v)$ or $r(v) < l(u)$.
4.2 Lower Bound Theorem

**Theorem 8.** Let $S_1, S_2 \subseteq V$ such that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ and $S_1 \cup S_2 = V$. Let there be no vertex $u \in S_2$ such that $N[u] \cap S_1 = S_1$. Let $b^* = \text{box}(G)$. Let $t$ be a fixed positive integer such that $1 \leq t \leq |S_1|$. Let $\beta_t = \beta_t(S_1, S_2)$. Let $t^* = |S_2|(1 - 2b^*(1 - \beta_t))$. Let $\alpha^* = \alpha^*(S_2, S_1)$. Then

$$\text{box}(G) \geq \frac{1}{2\left(1 - \beta_t + \frac{t-1}{\alpha^*|S_2|}\right)}.$$ 

**Remark 3.** Before moving on to the proof, we would like to bring to notice certain observations regarding the lower bound.

1. $\alpha_t > 0$ and hence $\alpha^* > 0$. This follows from the fact that there is no $u \in S_2$ such that $N[u] \cap S_1 = S_1$.
2. $(1 - \beta_t) + \frac{t-1}{\alpha^*|S_2|} > 0$. If $t = 1$, then, $(1 - \beta_t) > 0$. This is because, if $\beta_t = 1$, it implies that every vertex in $S_1$ is connected to every vertex in $S_2$, which contradicts the assumption that there is no $u \in S_2$ such that $N[u] \cap S_1 = S_1$. If $t > 1$, then $\frac{t-1}{\alpha^*|S_2|} > 0$ only if $\alpha^* = \infty$, i.e. if $t^* \leq 0$. But, when $(1 - \beta_t) = 0$, $t^* = |S_2| > 0$.

Hence, both $(1 - \beta_t)$ and $\frac{t-1}{\alpha^*|S_2|}$ cannot be 0 at the same time.

3. The maximum value of $\frac{1}{2\left(1 - \beta_t + \frac{t-1}{\alpha^*|S_2|}\right)}$ can be bounded above by $\max\left(\frac{|S_1|}{2}, \frac{|S_1|}{2}\right)$. To see this, consider the following two cases:

(a) Suppose $1 - \beta_t > 0$. The minimum value $1 - \beta_t$ can take is $\frac{1}{|S_2|}$.

(b) Suppose $1 - \beta_t = 0$. From the above discussion $t^* > 0$ and $t^* = |S_2|$. Since, $\alpha^* |S_1|/|S_2|$, $\frac{t-1}{\alpha^*|S_2|}$ achieves a minimum value at $t = 2$, and this value is equal to $1/|S_1|$. 

4. We can assume that $t^* > 0$ because if $t^* \leq 0$, the lemma easily holds as follows: From the definition of $t^*$, we have $|S_2|(1 - 2b^*(1 - \beta_t)) \leq 0$ and also, $(1 - \beta_t) \neq 0$ in this case. Therefore,

$$b^* \geq \frac{1}{2(1 - \beta_t)} \geq \frac{1}{2\left(1 - \beta_t + \frac{t-1}{\alpha^*|S_2|}\right)}.$$ 

**Proof (of Theorem 8).** Let $I_1, \ldots, I_{b^*}$ be $b^*$ interval graphs such that $E(G) = E(I_1) \cap \cdots \cap E(I_{b^*})$. For each interval graph $I_i$, let $f_i$ be an interval representation with distinct end-points. We denote

$$v_i^- = l_{f_i}(v) \text{ and } v_i^+ = r_{f_i}(v).$$

Let $x_i \in S_1$ be the vertex such that $|\{w \in S_1 | w_i^+ > x_i^-\}| = t - 1$. Similarly, let $y_i \in S_1$ be the vertex such that $|\{w \in S_1 | w_i^+ < y_i^+\}| = t - 1$. Define

$$M_i^- = x_i^- \text{ and } m_i^+ = y_i^+$$

and let

$$X_i^+ = \{v \in S_2 : v_i^- > m_i^+\} \text{ and } Y_i^- = \{v \in S_2 : v_i^+ < M_i^-\}.$$

**Claim 1.** For all $i \in \{1, \ldots, b^*\}$, $|X_i^+| \leq |S_2|(1 - \beta_t)$ and $|Y_i^-| \leq |S_2|(1 - \beta_t)$.

**Proof.** Let $T_i = \{v \in S_1 | v_i^+ \leq m_i^+\}$. It follows from Lemma [4] that for each $v \in N[T_i]$, $v_i^- < m_i^+$. Thus, if $u \in N[T_i] \cap S_2$ then $v \notin X_i^+$. Hence $X_i^+ \subseteq S_2 - (N[T_i] \cap S_2)$. Recalling the definition of $\beta_t = \beta_t(S_1, S_2)$, we have $|N[T_i] \cap S_2| \geq \beta_t|S_2|$. It follows that $|X_i^+| \leq |S_2| - \beta_t|S_2| = |S_2|(1 - \beta_t)$. Using similar arguments, it follows that $|Y_i^-| \leq |S_2|(1 - \beta_t)$.

Let $Z = S_2 - \bigcup_{i \in \{1, \ldots, b^*\}} X_i^+ \cup Y_i^-$, i.e.

$$Z = \{u \in S_2 | u_i^- < m_i^+ \text{ and } u_i^+ > M_i^- \text{ for all } i \in \{1, \ldots, b^*\}\}.$$

**Claim 2.** $|Z| \geq t^* > 0$. 


Proof. First, we recall the assumption that $t^* > 0$ (see Remark 3 point 4). Using the definition of $Z$ and Claim 1 it follows that $|Z| \geq |S_2| - \sum_{i \in \{1, \ldots, b^*\}} |X_i^+| - \sum_{i \in \{1, \ldots, b^*\}} |Y_i^-| \geq |S_2| - 2b^*|S_2|(1 - \beta_i) \geq |S_2|(1 - 2b^*(1 - \beta_i)) = t^*$.

Claim 3. For each interval graph $I_i, i \in \{1, \ldots, b^*\}$,

$$|\{v \in S_1 \mid \exists u \in Z \text{ and } uv \not\in E(I_i)\}| \leq 2(t - 1).$$

Proof. Fix any interval graph $I_i$. Let $R_i = \{u \in S_1 \mid u_i^+ \geq m_i^+\}$ and let $L_i = \{u \in S_1 \mid u_i^- \leq M_i^-\}$. Recalling the definition of $m_i^+$ and $M_i^-$, it follows that $|R_i| = |L_i| = |S_1| - (t - 1)$. From the definition of $Z$, we know that for each vertex $v \in Z, v_i^- < m_i^+$ and $v_i^+ > M_i^-$. If vertex $u \in L_i \cap R_i$ then $u_i^+ \geq m_i^+ > v_i^+$ and $u_i^- \leq M_i^- < v_i^-$. It follows from Lemma 4 that $w \in E(I_i)$ for all $v \in Z$. In other words, $\{v \in S_1 \mid \exists u \in Z \text{ and } uv \not\in E(I_i)\} \subseteq S_1 - (R_i \cap L_i) = (S_1 - R_i) \cup (S_1 - L_i)$. Since $|S_1 - R_i| = t - 1$ and $|S_1 - L_i| = t - 1$, we obtain the claimed bound.

From the definition of $\alpha_t$ it follows that $|N'(Z, \overline{G}) \cap S_1| \geq \alpha_{|Z|}(S_2, S_1)|Z|$. Recalling that $|Z| \geq t^*$ (from Claim 2) and using Observation 3 we obtain

$$|N'(Z, \overline{G}) \cap S_1| \geq \alpha_{|Z|}(S_2, S_1)|Z| \geq \alpha_t(S_2, S_1)t^* = \alpha^*t^*.$$  \hspace{1cm} (25)

Clearly $N'(Z, \overline{G}) \cap S_1 = \{v \in S_1 \mid \exists u \in Z \text{ and } uv \not\in E(G)\}$. Since each missing edge in $G$ should be missing in at least one of the $b^*$ interval graphs, it follows from Claim 3 that $|N'(Z, \overline{G}) \cap S_1| \leq 2b^*(t - 1)$. Combining this with inequality (25), we obtain $2b^*(t - 1) \geq |N'(Z, \overline{G}) \cap S_1| \geq \alpha^*t^*$. It follows that

$$2(t - 1)b^* \geq \alpha^*t^*.$$ \hspace{1cm} (26)

Substituting for $t^*$ with $|S_2|(1 - 2b^*(1 - \beta_i))$, inequality (26) implies that

$$2(t - 1)b^* \geq \alpha^*|S_2|(1 - 2b^*(1 - \beta_i)).$$

Rearranging the terms, it follows that

$$2b^* \left(\frac{\alpha^*(1 - \beta_i)}{|S_2|} + \frac{t - 1}{|S_2|}\right) \geq \alpha^*.$$ \hspace{1cm} (27)

Recalling that $\alpha^* > 0$ and $(1 - \beta_i) + \frac{t - 1}{\alpha^*|S_2|} > 0$ (see Remark 3 points 1 and 2), on rearranging the above inequality we get the required expression. \hspace{1cm} $\square$

4.3 Consequences

In a graph $G$, a vertex is called a universal vertex if it is adjacent to all other vertices.

Theorem 9. Let $G$ be a non-complete graph with minimum degree $\delta$. Let $n_u$ be the number of universal vertices in $G$. Then

$$\text{box}(G) \geq \frac{n - n_u}{2(n - \delta - 1)}.$$  

Proof. Let $U$ be the set of universal vertices in $G$. We have $|U| = n_u$. Let $G' = G - U$. Let $\delta'$ be the minimum degree of $G'$. It is easy to see that $\delta' = \delta - n_u$. We apply Theorem 8 on $G'$ as follows: take $S_1 = S_2 = V - U$ and fix $t = 1$. Note that, we can apply Theorem 8 since there are no universal vertices in $G'$. Clearly $\beta_i = \beta_1(S_1, S_2) = \frac{\delta' + 1}{n - n_u}$, and thus $1 - \beta_i = \frac{n - \delta - 1}{n - n_u}$. Hence, from Theorem 8 we obtain that $\text{box}(G') \geq \frac{1}{2(1 - \beta_i)} = \frac{n - n_u}{2(n - \delta - 1)}$. \hspace{1cm} $\square$

For a $(n - k - 1)$-regular graph, this result is same as Theorem 3. Note however that for general graphs, the above result is better than the one mentioned in Section 3.2 namely $\text{box}(G) \geq \frac{n\delta(G)}{2\Delta(G)^2}$.  

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Theorem 10. Let $G$ be a non-complete bipartite graph with two parts $A$ and $B$. Let $u_B$ be the number of vertices in $B$ that are adjacent to all vertices in $A$. Let $\delta_A$ be the minimum degree of the vertices in $A$. Then,

$$\text{box}(G) \geq \frac{|B| - u_B}{2(|B| - \delta_A)}.$$

Proof. Let $U_B \subseteq B$ be the set of vertices that are adjacent to all the vertices in $A$. We have $|U_B| = u_B$. Let $G'$ be the graph induced by $A \cup (B - U_B)$ on $G$. Apply Theorem 8 on $G'$ as follows: fix $S_1 = A$, $S_2 = B - U_B$ and $t = 1$. Note that Theorem 8 can be applied since there is no vertex $u \in B - U_B$ that is adjacent to all vertices in $A$. Clearly, the minimum degree of the vertices in $S_1$ is $\delta_A = \delta_A - u_B$. We have $\beta_t = \beta_1(S_1, S_2) = \frac{\delta_A}{|B| - u_B}$, and thus $1 - \beta_t = \frac{|B| - \delta_A}{|B| - u_B}$. Applying Theorem 8 we obtain that $\text{box}(G') \geq \frac{1}{2(1 - \beta_t)} = \frac{|B| - u_B}{2(|B| - \delta_A)}$. \hfill \Box

Tightness of Theorem 10: Let $G$ be a balanced $(A, B)$-bipartite graph on $2n$ vertices with the following structure: Let $n$, $k$, and $l$ be integers such that $kl = n$. Further, let $A = A_1 \cup A_2 \cup \cdots \cup A_k$ and $B = B_1 \cup B_2 \cup \cdots \cup B_l$, where, $|A_i| = |B_i| = k$. For any two vertices, $u \in A_i$ and $v \in B_j$, let $uv \in E \iff i \neq j$. Clearly, $G$ is an $(n - k)$-regular graph. From Theorem 10 box($G$) $\geq n/2k = l/2$. Now, we present a set of $l + 2$ interval graphs $I = \{I_i\}$, $1 \leq i \leq l + 2$ whose edge intersection gives $G$. The interval representation for each $I_i$, $1 \leq i \leq l$ is as follows: Vertices in $A_i$ are mapped to the interval $[0, 1]$ and vertices in $B_i$ is mapped to the interval $[2, 3]$. The rest of the vertices are mapped to the interval $[0, 3]$. For $i = l + 1$ (resp. $i + 2$), we have a one-to-one mapping of vertices in $A_{l+1}$ ($B_{l+1}$) to $n$ disjoint intervals $[2j, 2j+1]$, $j \in \{0, 1, \ldots, n-1\}$. The rest of the vertices are assigned the interval $[0, 2n-1]$. Thus we note that Theorem 10 is tight up to a constant factor.

Theorem 11. Let $G$ be a graph on $n$ vertices with minimum degree $\delta$ and maximum degree $\Delta < n - 1$. If $G$ is a $t$-expander then

$$\text{box}(G) \geq \frac{n(n - \Delta - 1)}{2(t - 1)[(n - \Delta - 1) + (n - \delta - 1)]}.$$

Proof. We apply Theorem 8 on $G$ as follows. Fix $S_1 = S_2 = V$. It is easy to see that if $G$ is a $t$-expander then $n(1 - \beta_t) \leq t - 1$. From Lemma 3 we have $\alpha^* \geq (n - \Delta - 1)/(n - \delta - 1)$. Substituting for $1 - \beta_t$ and $\alpha^*$ in Theorem 8 we obtain the result.

In Theorem 4 of Section 3.2 we obtained lower bounds for the boxicity of regular co-planar graphs and regular $C_4$-free graphs based on the absence of $K_{3,3}$ and $K_{2,2}$ respectively as subgraphs in their complements. We observe that these graphs are specific subclasses of regular $t$-expanders. The same results can be obtained using the following corollary to Theorem 11.

Corollary 4. Let $G$ be a regular non-complete $t$-expander graph. Then

$$\text{box}(G) \geq \frac{n}{4(t - 1)}.$$

Theorem 12. Let $G \in \mathcal{G}_{2n}(2n, p)$ be a random balanced $(A, B)$-bipartite graph on $2n$ vertices. Suppose $c_1 \log n \leq p \leq e^{-c_2}$, for some constant $c_1 > 4$ and any positive constant $c_2$, then almost surely, box($G$) $= \Omega\left(\frac{np}{\log n}\right)$.

Proof. Let $U_B$ be the random variable for the number of vertices in $B$ that are adjacent to all vertices in $A$. We easily see that $E(U_B) = np^n$. Let us consider the event $E : U_B > (1 + \epsilon)ne^{-c_2}$ where $\epsilon \in (0, 1)$ is a constant chosen such that $(1 + \epsilon)e^{-c_2} < 1$. For $p = \exp(-c_2/n)$, $E(U_B) = ne^{-c_2}$. Using Chernoff’s bounds on Poisson trials, $Pr(E) \leq \exp\left(-\frac{c_2}{n}ne^{-c_2}\right) \leq \frac{1}{n^2}$. Since $U_B$ is a monotonically increasing event (See Footnote 11), for any $p < e^{-c_2/n}$, $Pr(E) \leq \frac{1}{n^2}$. We use Theorem 8 with $S_1 = A$ and $S_2 = B'$, where $B'$ is the subset of $B$ without the vertices which are adjacent to all vertices in $A$. We will first show that, for $t = c_1 \log n/p$, almost surely $n(1 - \beta_t) < t$, where $c_1 > 4$. 15
This is equivalent to showing that there exists no pair of sets, say \( X \subseteq A \) and \( Y \subseteq B \), \(|X| = |Y| = t\), such that \( \forall x \in X \) and \( y \in Y \), \( xy \notin E(G) \). Applying union bound, we have

\[
Pr \left( (1 - \beta_t) \geq t/n \right) \leq \left( \begin{array}{c} n \\ t \end{array} \right) q^t,
\]

where, \( q = 1 - p \leq e^{-p} \). Applying Lemma 2,

\[
Pr \left( (1 - \beta_t) \geq t/n \right) \leq \exp \left( 4t \log(n/t) - t^2p \right),
\]

(Substituting \( t = c_1 \log n/p \)) \leq \exp \left( \frac{c_1^2 \log^2 n}{p} (4 - c_1') \right) \leq \frac{1}{n^2}.

Using Chernoff’s bound on Poisson trials and union bound, we have \( Pr(\delta(G) < (1 - \epsilon)np) \leq \exp \left( -\frac{\epsilon^2}{2} np \right) \leq \frac{1}{n^2} \)
and \( Pr(\Delta(G) > (1 + \epsilon)np) \leq \exp \left( -\frac{\epsilon^2}{2} np \right) \leq \frac{1}{n^2} \) for some constant \( \epsilon \in (0, 1) \). Applying Lemma 3 we have \( \alpha^* = \Omega(1) \), almost surely. Substituting these values in Theorem 8 we get the desired result.

\[\square\]

**Corollary 5.** Consider balanced bipartite graphs with \( 2n \) vertices and \( m \) edges such that \( c_1 n \log n \leq m \leq n^2 - c_2 n \), where \( c_1 \) and \( c_2 \) are as defined in Theorem 12. Let \( M \left( \begin{array}{c} n^2 \\ m \end{array} \right) \) be the total number of balanced bipartite graphs on \( 2n \) vertices with exactly \( m \) edges. There exists a positive constant \( c \) such that, if \( N^B_c \) is the number of balanced bipartite graphs whose boxicity is at most \( cm/(n \log n) \), then,

\[
\frac{N^B_c}{M} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

**Proof.** The proof is very similar to that of Corollary 2. In Theorem 12 we recall that for \( G \in \mathcal{G}_B(2n, p) \), where \( c_1' \frac{\log n}{p} \leq p \leq e^{-c_1''} \), \( \text{box}(G) = \Omega \left( \frac{n^2}{\log^2 m} \right) \) with probability \( O(1/n^2) \). With \( p = \frac{m}{n^2} \), we note that \( p \) is in the permissible range as required by Theorem 12. It can be shown that, given any property \( \mathcal{P} \), \( Pr(G \text{ satisfies } \mathcal{P}, G \in \mathcal{G}_B(2n, m)) \leq 3\sqrt{m} Pr(G \text{ satisfies } \mathcal{P}, G \in \mathcal{G}_B(2n, p)) \) (this is a variant of [7, Theorem 2 Chapter 2]). Hence, \( Pr(G \text{ satisfies } \mathcal{P}, G \in \mathcal{G}_B(2n, m)) \leq 3\sqrt{m} O(1/n^2) = O(1/n) \). Therefore, if \( G \in \mathcal{G}_B(2n, m) \), then almost surely, \( \text{box}(G) = \Omega(m/(n \log n)) \). Hence proved.

\[\square\]

**Theorem 13.** Let \( G \) be a random \( k \)-regular balanced \((A, B)\)-bipartite graph on \( 2n \) vertices. Let \( \lambda' \) be the second largest eigenvalue of \( MM^T \), where \( M \) is the bipartite incidence matrix of \( G \) (See Section 3.4 for definition). If \( \lambda' \neq 0 \), then, \( \text{box}(G) = \frac{k}{4\lambda'} \).

**Proof.** We use Theorem 8 with \( S_1 = A \) and \( S_2 = B \). Since \( G \) is regular, by Lemma 3 we have \( \alpha^* \geq 1 \). Using (22) from Section 3.4 for any set \( X \subseteq A \), \(|X| = t\), we have

\[
n\beta_t \geq |N[X, G] \cap B| = |N(X, G)| \geq \frac{k^2 t}{\lambda' + (k^2 - \lambda') t/n}.
\]

On substituting \( t = \frac{n}{\lambda' + (k^2 - \lambda') t/n} \) in the above inequality, we observe that \( (1 - \beta_t) \leq t/n \). Substituting in Theorem 8 we get the desired result.

\[\square\]

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