GOODWILLIE TOWER OF THE NORM FUNCTOR

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Abstract. In this note we compare the fracture squares from genuine equivariant stable homotopy theory and the fracture squares which appear in the Goodwillie tower for the norm functor.

1. Introduction

Let $G$ be a finite group and $S^G$ be the simplicial model monoidal category of orthogonal $G$-spectra constructed by M. Mandell and P. May [MM02]. Recall that $S^G$ is a model of the genuine equivariant stable homotopy category, i.e. the homotopy category of $S^G$ is equivalent to the main construction of [LMSM86].

In [HHR16], the symmetric monoidal structure on the category $S^G$ was enriched with the structure of norms. Namely, let $H$ be a subgroup of $G$, then M. A. Hill, M. J. Hopkins, and D. C. Ravenel constructed the norm functor $N^G_H: S^H \to S^G$ which is symmetric monoidal and can be thought as a “genuine stabilization” of the coinduction. In this paper, we study the Goodwillie tower of $N^G_H$.

More precisely, the norm functor is a homotopy functor acting between stable simplicial model categories. For studying those, T. Goodwillie [Goo03] introduced the notion of $n$-excisive functors for a given natural number $n$ (1-excisive functors are also known as exact or linear.) He showed that any homotopy functor $F$ has the universal natural transformation into a $n$-excisive functor for any $n$ (see Theorem 2.1.5 for a precise statement.) The universal functor is denoted by $P_n F$.

Dually, R. McCarthy [McC01] showed that any homotopy functor has the universal natural transformation from a $n$-excisive functor (Corollary 2.1.6); we will denote this universal functor by $P^n F$. In this paper, we compute $P_n N^G_H$ and $P^n N^G_H$.

Theorem. Let $\mathcal{F}_n$ be the family of subgroups $K \subset G$ such that $|K\setminus G/H| > n$. Then $P_n N^G_H \simeq \tilde{E} \mathcal{F}_n \land N^G_H$ and $P^n N^G_H \simeq F(\tilde{E} \mathcal{F}_n, N^G_H)$.

Furthermore, N. Kuhn used the interplay between $P_n$ and $P^n$ to inscribe the map $P_n F \to P_{n-1} F$ into a homotopy cartesian square [Kuh04 Proposition 1.9] (see also Proposition 2.1.13). One can also use the interplay between different families of subgroups in $G$ to obtain fracture squares of localization functors acting on $S^G$.

The paper is organized as follows. In section 2.1 we recall fundamental theorems and constructions from calculus of homotopy functors acting between stable simplicial model categories. In Section 2.2 we list the properties for the norm functor $N^G_H: S^H \to S^G$ and show that the results of the previous section apply to $N^G_H$. We also construct the fracture squares both in Goodwillie’s setting and in the equivariant stable homotopy setting (Proposition 2.1.13 and Corollary 2.2.8).
In Section 3, we compute approximations \( P_n, P^n \) for the norm functors and as well as for \( LN_G^H \), where \( L : S^G \rightarrow S^G \) is a linear functor given by a composition of \( A \wedge - \) and \( F(A, -) \) for a \( G \)-CW complex \( A \). We state our main results in Theorem 3.2.17 and in Corollary 3.2.18.

In the last section, we compute the Goodwillie tower \( N_{\varepsilon}^{C,n} \), \( k \geq 1 \) using Theorem 3.2.17, and show possible consequences of that computations for the divided power functor \( \Gamma^n : S \rightarrow S, X \mapsto (X \wedge n)^H \Sigma_n \).

2. Background

2.1. Functor calculus. The idea of the functor calculus is established in the series of papers by T. Goodwillie \[Goo03\]. He mostly worked with categories of spaces or spectra, however, there were several attempts to extend his approach to more general categories. Informally, the main result of all those generalizations is that the original ideas of T. Goodwillie (and even the proofs) work in the new settings as well. The brief overview can be found in Kunh’s notes \[Kuh07\]. We also would like to point out the preprint by L. A. Pereira \[Per13\] for the case of functors between model categories, and the manuscript by J. Lurie \[Lur17, Section 6.1\] for functors between quasi-categories.

In this section, we recall several basic notions and a few main results concerning calculus for functors between simplicial model categories, which we will require further. We refer the reader to \[Qui67, Chapter II.2\] for the definition of simplicial model category (see also \[Hir03, Chapter 9\]).

Let \( S \) be a finite set, and let \( \mathcal{P}(S) \) be a poset of all subsets in \( S \). We set \( \mathcal{P}_0(S) = \mathcal{P}(S) - \{\emptyset\} \) and \( \mathcal{P}_1(S) = \mathcal{P}(S) - \{S\} \).

**Definition 2.1.1.** Let \( \mathcal{C} \) be a simplicial model category.

1. A \( d \)-cube \( \mathcal{X} \) is a functor \( \mathcal{X} : \mathcal{P}(S) \rightarrow \mathcal{C}, |S| = d \).
2. A \( d \)-cube \( \mathcal{X} \) is called homotopy cartesian if the natural map

\[
\mathcal{X}(\emptyset) \rightarrow \text{holim}_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)
\]

is a weak equivalence.
3. A \( d \)-cube \( \mathcal{X} \) is homotopy cocartesian if the natural map

\[
\text{hocolim}_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \rightarrow \mathcal{X}(S)
\]

is a weak equivalence.
4. A \( d \)-cube \( \mathcal{X} \) is called strongly cocartesian if \( \mathcal{X}|_{\mathcal{P}(T)} : \mathcal{P}(T) \rightarrow \mathcal{C} \) is homotopy cocartesian for all \( |T| \geq 2, T \subset S \).

**Remark 2.1.2.** We would like to point out the abuse of notation in the definition above. A homotopy (co)limit can be (and should be) defined in any relative category (i.e. a category \( \mathcal{C} \) equipped with a class of weak equivalences \( \mathcal{W} \), cf. \[DHKS04\] ). However, in that case it is naturally only an object of the homotopy category \( \text{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}] \), but not \( \mathcal{C} \) itself. Nevertheless, in the case of simplicial model category one can choose a functorial lift into \( \mathcal{C} \), e.g. see \[Hir03, Chapter 18\]. Everywhere in this paper by a homotopy (co)limit we would mean that particular lift.

**Remark 2.1.3.** The natural maps in [2, 3] factorise through the ordinary (co)limit.
Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between simplicial model categories. Recall that $F$ is called a homotopy functor, if $F$ preserves weak equivalences. Denote by $hF: \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})$ the functor between homotopy categories induced by $F$.

**Definition 2.1.4.** A homotopy functor $F$ is called $n$-excisive if it maps strongly cocartesian $(n+1)$-cubes into homotopy cartesian cubes.

One of the main result of [Goo03, Theorem 1.8] is that any homotopy functor $F: \mathcal{C} \to \mathcal{D}$ has an universal natural transformation $F \to P_n F$, where $P_n F$ is a $n$-excisive functor.

**Theorem 2.1.5.** Let $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor between simplicial model categories. Then there exists a homotopy $n$-excisive functor $P_n F: \mathcal{C} \to \mathcal{D}$ and a natural transformation $p_n: F \to P_n F$ that is universal up to weak equivalence among these. I.e. for any $n$-excisive functor $G$ and a natural transformation $p_n^G: F \to G$ there exists unique $\eta: hP_n F \to hG$ such that the following diagram commutes

\[
\begin{array}{ccc}
hF & \xrightarrow{p_n} & hP_n F \\
\downarrow & & \downarrow \eta \\
hG & \xleftarrow{p_n^G} & hF
\end{array}
\]

Recall that a category is pointed if it has an initial object $\emptyset$, a terminal object $\ast$, and the unique map $\emptyset \to \ast$ is an isomorphism. The simplicial model category $\mathcal{C}$ is called stable if $\mathcal{C}$ is pointed and a 2-cube $X$ in $\mathcal{C}$ is homotopy cocartesian if and only if $X$ is homotopy cartesian, cf. [Hov99]. R. McCarthy [McC01] discovered that the dual statement of Theorem 2.1.5 is also true provided $\mathcal{C}$ and $\mathcal{D}$ are stable. We summarize his result in the following corollary.

**Corollary 2.1.6.** Let $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor between simplicial stable model categories. There exists a homotopy $n$-excisive functor $P^n F: \mathcal{C} \to \mathcal{D}$ and a natural transformation $p^n: F \to P^n F$ that is universal up to weak equivalence among these. I.e. for any $n$-excisive functor $G$ and a natural transformation $p^n_G: G \to F$ there exists unique $\eta: hP^n F \to hG$ such that the following diagram commutes

\[
\begin{array}{ccc}
hG & \xrightarrow{p^n} & hF \\
\downarrow \eta & & \downarrow \\
hP^n F & \xleftarrow{p^n_G} & hG
\end{array}
\]

**Proof.** By taking the opposite categories, it follows from Theorem 2.1.5 and the fact that for stable categories a $n$-cube is homotopy cartesian if and only if it is homotopy cocartesian. \qed

**Definition 2.1.7.** Let $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor between simplicial stable model categories such that $F(\ast) = \ast$.

1. $F$ is $k$-reduced, if $P_k F \simeq \ast$;
2. $F$ is $k$-coreduced, if $P^{k-1} F \simeq \ast$;
3. $F$ is $k$-homogeneous, if $F$ is $k$-excisive and $k$-reduced;
4. $F$ is linear, if $F$ is 1-homogeneous (equivalently, 1-excisive).

Recall the T. Goodwillie classification of $k$-homogeneous functors [Goo03, Theorem 3.5].
**Theorem 2.1.9.** Let \( cr_k \) be the functor of \( \Delta \) in each variable. Then the composition \( k \) is an object of the diagram category \( D^{B \Sigma_k} \).

**Definition 2.1.8.** Let \( C \) be simplicial stable model categories, and let \( F: C \to D \) be a homotopy functor, \( F(*) = * \). We define \( cr_k F: C^{\times k} \to D \), the \( k \)-th cross-effect, to be the functor of \( k \) variables given by

\[
cr_k F(X_1, \ldots, X_k) = \text{hofib}(F\left( \coprod_{i \in [k]} X_i \right) \to \text{holim}_{T \in P_0(k)} F\left( \coprod_{i \in [k]-T} X_i \right)).
\]

Note that \( cr_k F(X, \ldots, X) \in D \) has a natural action of the symmetric group \( \Sigma_k \). Therefore, we can consider \( cr_k F(X, \ldots, X) \) as an object of the diagram category \( D^{B \Sigma_k} \).

**Theorem 2.1.9.** Let \( F: C \to D \) be a \( k \)-homogeneous functor between stable simplicial model categories. Then \( F(X) \cong cr_k F(X, \ldots, X)_{h \Sigma_k} \).

**Proof.** See [Kuh07, Theorem 5.12].

**Proposition 2.1.10.** Let \( H: C^{\times k} \to D \) be a \( k \)-multilinear functor (i.e. \( H \) is linear in each variable). Then the composition \( C \xrightarrow{\Delta} C^{\times k} \xrightarrow{H^\Delta} D \) is both a \( k \)-reduced and \( k \)-coreduced functor, where \( \Delta \) is given by \( X \mapsto (X, \ldots, X) \).

**Proof.** The composition \( H \circ \Delta \) is \( k \)-reduced by [Goo03, Lemma 3.2].

For a functor \( F: C \to D \) we set \( F^{op}: C^{op} \to D^{op} \) to be the same functor but acting between opposite categories. Since \( C, D \) are stable, \( F \) is \( k \)-coreduced if and only if \( F^{op} \) is \( k \)-reduced. Now, the statement follows from facts that \( H^{op} \) is also a \( k \)-multilinear functor and \( (H \circ \Delta)^{op} = H^{op} \circ \Delta \).

Recall that the forgetful functor \( D^{B \Sigma_k} \to D \) has the left adjoint \( D \to D^{B \Sigma_k} \) given by \( Y \mapsto \coprod_{\sigma \in \Sigma_k} Y \), where \( \Sigma_k \) acts on the right hand side by permuting components. An objects of \( D^{B \Sigma_k} \) is called \( \Sigma_k \)-induced, if it belongs to the essential image of the left adjoint functor.

**Corollary 2.1.11.** Let \( F: C \to D \) be a \( k \)-homogeneous functor. If the \( k \)-th cross-effect \( cr_k F(X, \ldots, X) \in D^{B \Sigma_k} \) is \( \Sigma_k \)-induced for all \( X \in C \), then \( F \) is also \( k \)-coreduced.

**Proof.** Since the functor \( cr_k F(X_1, \ldots, X_k) \) is \( k \)-multilinear, the \( k \)-th cross-effect \( cr_k F(X, \ldots, X) \) is \( k \)-coreduced by Proposition 2.1.10. Since \( cr_k F(X, \ldots, X) \) is \( \Sigma_k \)-induced, we have that

\[
F(X) \cong cr_k F(X, \ldots, X)_{h \Sigma_k} \cong cr_k F(X, \ldots, X)^{h \Sigma_k}.
\]

We finish the proof by the fact that \( k \)-coreduced functors are closed under passage to homotopy limits.

**Example 2.1.12.** Let \( C = D = S \) be the category of spectra, then \( X \mapsto (X \wedge n)^{h \Sigma_n} \) is \( n \)-excisive and \( n \)-reduced; \( X \mapsto (X \wedge n)^{h \Sigma_n} \) is \( n \)-excisive and \( n \)-reduced; and \( X \mapsto X \wedge n \) is \( n \)-excisive, and both \( n \)-reduced and \( n \)-coreduced.

Finally, we recall the fracture squares which appears in functor calculus. Set \( L_k = \text{cofib}(P^k \to \text{Id}) \) and let denote by \( L_I \) the composition \( L_{k-I} P_n \), where \( I = [k; n] \subset \mathbb{N}, k > 0 \). For consistence, we also denote \( P_n \) by \( L_I \), where \( I = [0; n] \).
Proposition 2.1.13. Let $\mathcal{C}, \mathcal{D}$ be simplicial stable model categories. Let $0 \leq k < m \leq n$, and set $I = [k; n], I_2 = [k; m - 1], I_1 = [m; n]$. Then for any homotopy functor $F: \mathcal{C} \to \mathcal{D}$ the following commutative diagram is homotopy cartesian:

$$
\begin{array}{c}
L_1 F \\
\downarrow \\
L_{I_2} F
\end{array}
\begin{array}{c}
L_1 F \\
\downarrow \\
L_{I_2} L_{I_1} F
\end{array}
$$

Proof. It is enough to show that the induced map between the homotopy fibers of the horizontal maps is an equivalence. The fiber of the upper one is

$$P^{m-1}L_{k-1}P_n F,$$

and the fiber the lower map is the composition

$$L_{k-1}P_{m-1}P^{m-1}L_{k-1}P_n F.$$

Now, the proposition follows by the fact that the map $P^{m-1}G \to P_{m-1}P^{m-1}G$ is an equivalence for any homotopy functor $G$. □

Example 2.1.14. Fix $n \in \mathbb{N}$. Let $F: S \to S, F(*) = *$ be a homotopy functor which commutes with filtered homotopy colimits. Set $D_n F = \text{fib}(P_n F \to P_{n-1} F)$ to be the $n$-th homogeneous layer of $F$. One can show that $D_n F(X) \simeq (\partial_n F \wedge X^{\wedge n})h\Sigma_n$ for some Borel $\Sigma_n$-equivariant spectrum $\partial_n F \in S^{B\Sigma_n}$.

We now apply Proposition 2.1.13 with $k = 0$ and $m = n$. Then $L_I = P_n, L_{I_2} = P_{n-1}$, and $L_{I_1} F(X) \simeq L_{n-1}P_n F(X) \simeq L_{n-1}D_n F(X) \simeq (\partial_n F \wedge X^{\wedge n})h\Sigma_n$. As a result, we obtain Kuhn's homotopy cartesian square [Kuh04 Proposition 1.9]:

$$
\begin{array}{c}
P_n F \\
\downarrow \\
P_{n-1} F
\end{array}
\begin{array}{c}
(\partial_n F \wedge (-)^{\wedge n})h\Sigma_n \\
\downarrow \\
(\partial_n F \wedge (-)^{\wedge n})t\Sigma_n
\end{array}
$$

Here $(-)^{\Sigma_n}$ is the Tate fixed points under $\Sigma_n$-action (see ibid.)

2.2. Equivariant stable homotopy theory. Let $G$ be a finite group. Recall first several categories which appear naturally in studying topological spaces with $G$-action. Denote by $\text{Top}^G$ (resp. $\text{Top}_p^G$) the category of compactly generated weak Hausdorff (resp. pointed) $G$-spaces and $G$-equivariant (resp. pointed) continuous maps. We call a map $f: X \to Y \in \text{Top}^G$ a weak $G$-equivalence if $f^H: X^H \to Y^H$ is a weak equivalence on $H$-fixed points for all subgroups $H \subset G$.

We denote by $S^G$ the $\text{Top}^G$-enriched category of orthogonal $G$-spectra constructed by M. Mandell and P. May, cf. [MM02 Definition 2.6]. Similarly, $S^G$ is the topological category of orthogonal $G$-spectra and $G$-equivariant maps, i.e $S^G(X, Y) = S^G_0(X, Y)^G$. In [MM02, categories $S^G_0$, $S^G$ are denoted by $\mathcal{F}_{G, \mathcal{F}}, G \not\in \mathcal{F}$, respectively; in this paper, we use the notation of [HHR16 Definition A.14].

Recall that $S^G_0, S^G$ are closed symmetric monoidal categories with respect to the smash product $\wedge$, see [MM02 Corollary 3.2] and [HHR16 Proposition A.17]. Moreover, $S^G_0$ is tensored and cotensored over the category $\text{Top}^*_G$; we denote the mapping spectrum as $F(K, X), K \in \text{Top}^*_G, X \in S^G$. Finally, we note that $S^G_0, S^G$ are complete and cocomplete.
Let $H \subset G$ be a subgroup of $G$. Recall here several basic properties of the norm functor $\tilde{N}_H^G : S^H \to S^G$, cf. \cite[Section A.4]{HHR16}.

**Proposition 2.2.1.**

1. Let $X$ be a $H$-set, then $\tilde{N}_H^G \Sigma^\infty_+ X \cong \Sigma^\infty_+ \text{Coind}_H^G X$.
2. Let $V$ be a $H$-representation, then $\tilde{N}_H^G S^{-V} \cong S^{-\text{Ind}_H^G V}$.
3. $\tilde{N}_H^G$ commutes with sifted colimits.

**Proof.** The first part follows because $\Sigma^\infty_+ : \text{Top}^H \to S^H$ is a symmetric monoidal functor. The second part is \cite[Proposition A.59]{HHR16}. For the last part we refer to ibid, Proposition A.53. □

Recall that a map $f : X \to Y \in S^G$ is called *stable weak equivalence* if $f$ induces an isomorphism on stable homotopy groups $\pi_k$ for all $k \in \mathbb{Z}$ and all $H \subset G$, cf. \cite[Definition 2.17]{HHR16}. There are different approaches to enhance $S^G$ with a model structure; for studying the norm functor the most suitable one is the positive complete model structure of section B.4 in \cite{HHR16}. Namely, $\tilde{N}_H^G$ is not a homotopy functor (so we can not apply the theory of the previous section), but it will be after precomposing with a functorial cofibrant replacement.

**Proposition 2.2.2.** The norm functor $\tilde{N}_H^G$ preserves weak equivalences between cofibrant objects in positive complete model structure on $S^H$.

**Proof.** See \cite[Proposition B.103]{HHR16} □

Let us fix $Q : S^H \to S^H$ a functorial cofibrant replacement. We will denote by $N_H^G$ the composition $\tilde{N}_H^G \circ Q$. Proposition 2.2.1 still holds for $N_H^G$, but we have to replace isomorphisms with (chains of) weak equivalences.

Finally, we recall stratification phenomena in $S^G$. Let $F$ be a family of subgroups of $G$ that is a set of subgroups closed under passage to subgroups and conjugates. Then we define a $G$-CW complex $E F$ by the property:

$$E F^K \simeq \begin{cases} \emptyset, & K \notin F; \\ \{\text{pt}\}, & K \in F. \end{cases}$$

By the Elmendorf theorem \cite{Elm83}, a space $E F$ exists and it is unique up to a weak $G$-equivalence; and so by the equivariant Whitehead theorem it is unique up to homotopy equivalence. Let denote by $\tilde{E} F \in \text{Top}_G^*$ the mapping cone of the collapse map $E F_+ \to S^0$. Note that if $F$ is empty, then $E F \simeq \emptyset$, $\tilde{E} F \simeq S^0$; in opposite, if $F$ is the family of all subgroups, then $E F, \tilde{E} F$ are contractible.

Let $X, Y$ be transitive $G$-sets; we will say that $X \geq Y$ if $\text{Hom}_G(X, Y) \neq \emptyset$. Note that it defines the partial order on the set of isomorphism classes of transitive $G$-sets. We will denote the corresponding poset by $\mathcal{O}_G$. This poset has the maximal element $G/e$ and the minimal element $G/G$.

Recall that a subset $I \subset \mathcal{O}$ in a poset $\mathcal{O}$ is an *interval* if for any $x \geq z \geq y \in \mathcal{O}, x, y \in I$ the element $z$ belongs to $I$. If an interval $I \subset \mathcal{O}_G$ contains the maximal element $G/e$ one can associate a non-empty family of subgroups $\mathcal{F}_I$ by the following rule

$$H \in \mathcal{F}_I \iff G/H \in I.$$ 

We remark that this correspondence defines a bijections between the set of non-empty families of subgroups and the set of intervals in $\mathcal{O}_G$ which contain $G/e$. We
extend it to the set of all families by sending an empty interval $I \subset \mathcal{O}_G$ to the empty family $\mathcal{F}_\emptyset$.

Finally, note that any interval $I$ is a complement of two intervals $I = I' \setminus I''$, where $G/e \in I'$, and $I''$ either also contains $G/e$, or it is empty. Furthermore, such presentation is unique.

**Definition 2.2.3.** Let $I \subset \mathcal{O}_G$ be an interval, $I = I' \setminus I''$, $G/e \in I'$, and $I''$ is either empty, or $G/e \in I''$. The functor $L_I: S^G \to S^G$ given by the rule:

$$L_I(X) = F((E\mathcal{F}_{I'})_+, \tilde{E}\mathcal{F}_{I''} \land X)$$

is called the localization functor associated with $I$.

**Remark 2.2.4.** Since $(E\mathcal{F}_{I'})_+, \tilde{E}\mathcal{F}_{I''}$ are $G$-CW complexes, the functor $L_I$ is homotopy, i.e. it preserves stable weak equivalences. Notice that the induced functor $hL_I: \text{Ho}(S^G) \to \text{Ho}(S^G)$ is indeed a localization functor, i.e. $hL_I$ applying to the natural transformation $\text{Id} \to hL_I$ is an isomorphism.

Let $K \subset G$ be a subgroup. Let denote by $\Phi^K: S^G \to S$ the geometric fixed points functor. Recall that it can be defined by the formula

$$\Phi^K(X) = (\tilde{E}\mathcal{P}_K \land \text{Res}_K^G X)^K,$$

where $\mathcal{P}_K$ is the family of all proper subgroups in $K$, and $(-)^K$ is the categorical fixed points (precomposed with the fixed functorial fibrant replacement). Recall that $\Phi^K$ preserves smash products up to weak equivalence, $\Phi^K \Sigma^\infty X \simeq \Sigma^\infty X^K$, and it commutes with finite homotopy limits and any homotopy colimits, cf. [HHR16, Section B.10].

**Proposition 2.2.5.** The map $f: X \to Y$ is a stable weak equivalence if and only if $\Phi^K f$ is a weak equivalence for all subgroups $K \subset G$.

**Proof.** See [Sch18, Proposition 3.3.10].

**Corollary 2.2.6.** Let $\mathcal{F}$ be a family of subgroups in $G$. Then for any $X \in S^G$ the map $E\mathcal{F}_+ \land X \to E\mathcal{F}_+ \land F(E\mathcal{F}_+, X)$ is an equivalence.

**Remark 2.2.7.** Notice that Corollary 2.2.6 can be reformulated as the square

$$\begin{array}{ccc}
X & \to & F((E\mathcal{F})_+, X) \\
\downarrow & & \downarrow \\
\tilde{E}\mathcal{F} \land X & \to & \tilde{E}\mathcal{F} \land F((E\mathcal{F})_+, X).
\end{array}$$

is homotopy cartesian, cf. [GM95]. In the proposition below, we show that similar squares consisting of more general functors $L_I$ are also homotopy cartesian.

**Proposition 2.2.8.** Let $I_1, I_2 \subset \mathcal{O}_G$ be intervals such that $x_1 > x_2$ for all $x_1 \in I_1, x_2 \in I_2$, and $I = I_1 \sqcup I_2$ is also an interval. Then the following commutative diagram in the category of endofunctors of $S^G$ is homotopy cartesian:

$$\begin{array}{ccc}
L_{I_2} \to L_{I_1} & \to & L_{I_1} \\
\downarrow & & \downarrow \\
L_{I_2} & \to & L_{I_2} L_{I_1}.
\end{array}$$
Proof. It is enough to show that the induced map between the homotopy fibers of the vertical arrows is an equivalence. Let decompose $I$ as the complement of two intervals $I = I' \setminus I''$, $G/e \in I'$, and $I''$ is either empty, or $G/e \in I''$. Then $I' \cup I''$ is also an interval, and we have $I_1 = I' \setminus (I_2 \cup I'')$, $I_2 = (I_2 \cup I'') \setminus I''$. Let $X \in S^G$, then the homotopy fiber of the left vertical map applied to $X$ is

$$F((E_{F_{I'}})_+ \wedge \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge X)$$

$$\simeq F((E_{F_{I'}})_+ \wedge \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge (E\mathcal{F}_{I'})_+ \wedge \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge X),$$

and the homotopy fiber on the right hand side is given by

$$F((E_{F_{I'}})_+ \wedge \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge (E\mathcal{F}_{I'})_+ \wedge F((E_{F_{I'}})_+, \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge X)).$$

Note that the last formula is equivalent to

$$F((E_{F_{I'}})_+ \wedge \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge (E\mathcal{F}_{I'})_+ \wedge F((E_{F_{I'}})_+, \tilde{E} \mathcal{F}_{(I_2 \cup I'')} \wedge X)).$$

Hence, it is sufficient to show that the map $E \mathcal{F}_+ \wedge X \to E \mathcal{F}_+ \wedge F(E \mathcal{F}_+, X)$ is an equivalence for all families $\mathcal{F}$ and all $X \in S^G$. But this is Corollary 2.2.10

3. Computations

3.1. Cross-effects of the norm functor. Let $G$ be a finite group, and let $H \subset G$ be a subgroup. In this section we show that the $i$-th cross-effect $\text{cr}_i N^G_H(X_1, \ldots, X_n)$ has a form $\bigwedge_{j=1}^m \text{Ind}^G_{K_j} Y_j$, where $K_j$ are subgroups of $G/K \mid G/H \geq i$, and $Y_j \in S^{K_j}$ for all $1 \leq j \leq m$. Moreover, we propose a functorial choice for the spectra $Y_j$. This is a consequence of distributive laws [HHR16, Proposition A.37], but here we present a slightly more direct way to see it.

Let $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$ be the set with $n$ elements, and let $\lambda \in \text{Hom}(G/H, [n])$ be a map of finite sets. Denote by $G_\lambda \subset G$ the stabilizer subgroup of $\lambda$ under the natural $G$-action on $\text{Hom}(G/H, [n])$.

Definition 3.1.1. Let $X_1, \ldots, X_n$ be $H$-sets. Let denote by $F_\lambda(X_1, \ldots, X_n)$ the preimage of $\lambda$ under the natural map induced by $X_1 \to \{1\}, \ldots, X_n \to \{n\}$:

$$\text{Coind}^G_H(X_1 \sqcup \ldots \sqcup X_n) \to \text{Hom}(G/H, [n]).$$

Proposition 3.1.2.

(1) $F_\lambda(X_1, \ldots, X_n) \subset \text{Coind}^G_H(X_1 \sqcup \ldots \sqcup X_n)$ is stable under $G_\lambda$-action for all $X_1, \ldots, X_n$.

(2) $F_\lambda(X_1, \ldots, X_n)$ defines a functor from the category of $n$-tuples of $H$-sets to the category of $G_\lambda$-sets.

(3) Set $\mathcal{P}_n = \text{Hom}(G/H, [n])/G$. Then

$$\text{Coind}^G_H(X_1 \sqcup \ldots \sqcup X_n) = \coprod_{[\lambda] \in \mathcal{P}_n} \text{Ind}^G_{G_\lambda} F_\lambda(X_1, \ldots, X_n).$$

Proof. Follows immediately from the definitions.

Example 3.1.3. Let $G = C_2$, $H = e$, and $n = 2$. Then $\mathcal{P}_n$ consists of three elements $[\lambda_1], [\lambda_2], [\lambda_3]$: $\lambda_1(C_2) = \{1\}, \lambda_2(C_2) = \{2\}$, and $\lambda_3$ is a bijection. One can see that $F_{\lambda_1}(X_1, X_2) = \text{Coind}^C_2(X_1), F_{\lambda_2}(X_1, X_2) = \text{Coind}^C_2(X_2)$, and $F_{\lambda_3}(X_1, X_2) \cong X_1 \times X_2$. The stabilizer groups are $G_{\lambda_1} = G_{\lambda_2} = C_2$, and $G_{\lambda_3} = \{e\}$. Therefore, $\text{Coind}^C_2(X_1 \sqcup X_2) = \text{Coind}^C_2(X_1) \sqcup \text{Coind}^C_2(X_2) \sqcup \text{Ind}^C_2 X_1 \times X_2$. 

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Recall the Mackey double coset formula. Let $X$ be a $H$-set and let $K$ be another subgroup of $G$. Then

\[(3.1.4) \quad \text{Res}^G_K \text{Coind}^H_K(X) \cong \prod_{g \in K \setminus G/H} \text{Coind}^K_{H_g} \text{Res}^H_{H_g} X,\]

where $H_g = K \cap gHg^{-1}$ and $H_g$ acts on $X$ via $h \cdot x = (g^{-1}hg) \cdot x, h \in H, x \in X$.

**Lemma 3.1.5.** Let $\lambda \in \text{Hom}(G/H, [n])$. Under the isomorphism

\[\text{Res}^G_{G\lambda} \text{Coind}^G_{G\lambda}(X_1 \sqcup \ldots \sqcup X_n) \cong \prod_{g \in G \setminus G/H} \text{Coind}^K_{H_g} \text{Res}^H_{H_g}(X_1 \sqcup \ldots \sqcup X_n)\]

the $G_\lambda$-subset $F_\lambda(X_1, \ldots, X_n)$ is identified with

\[\prod_{g \in G \setminus G/H} \text{Coind}^K_{H_g} \text{Res}^H_{H_g} X_{\lambda(g)}\]

for all $H$-sets $X_1, \ldots, X_n$. □

The latter one is a subset of $\text{Coind}^G_{H}(X_1 \sqcup \ldots \sqcup X_n)$ through the inclusions $X_{\lambda(g)} \subset X_1 \sqcup \ldots \sqcup X_n$.

We would like to obtain an analogous decomposition of Proposition 3.1.2 for the norm functor $N^G_H : S^H \to S^G$ as well. For doing that we turn Lemma 3.1.5 into a definition.

**Definition 3.1.6.** Let $\lambda \in \text{Hom}(G/H, [n])$. We define a functor $\tilde{F}_\lambda : (S^H)^n \to S^G$ by the rule

\[\tilde{F}_\lambda(X_1, \ldots, X_n) = \bigwedge_{g \in G \setminus G/H} N^G_{H_g} \text{Res}^H_{H_g} X_{\lambda(g)}.\]

Note that for any $H$-representation $V$ we have

\[(3.1.7) \quad \tilde{F}_\lambda(S^V \wedge X_1, \ldots, S^V \wedge X_n) \cong S^{\text{Res}^G_{G\lambda} \text{Ind}^G_{G\lambda} V} \wedge \tilde{F}_\lambda(X_1, \ldots, X_n).\]

The norm functor has an analogue of formula 3.1.4. Namely, for $X \in S^H$ we have an equivalence

\[(3.1.8) \quad \text{Res}^G_K N^G_H(X) \cong \bigwedge_{g \in K \setminus G/H} N^K_{H_g} \text{Res}^H_{H_g} X.\]

In the same manner as before we have the natural transformation

\[\eta_\lambda : \tilde{F}_\lambda \to \text{Res}^G_{G\lambda} N^G_H (- \vee \ldots \vee -)\]

induced by the maps $X_{\lambda(g)} \to X_1 \vee \ldots \vee X_n$. Applying the adjunction, we get the natural transformations

\[\mu_\lambda : \text{Ind}^G_{G\lambda} \tilde{F}_\lambda \to N^G_H (- \vee \ldots \vee -)\]

for all $\lambda \in \text{Hom}(G/H, [n])$.

**Proposition 3.1.9.** The natural transformation

\[\mu = \bigvee_{\lambda \in \mathcal{P}_n} \mu_\lambda : \bigvee_{\lambda \in \mathcal{P}_n} \text{Ind}^G_{G\lambda} \tilde{F}_\lambda(X_1, \ldots, X_n) \to N^G_H(X_1 \vee \ldots \vee X_n)\]

is a stable weak equivalence.
follows immediately. Now suppose that we know part 1 for a family $F$. All three statements can be proven by induction on the size of $F$. Moreover, after twisting each $X_i$ on $S^V$ for some $H$-representation $V$, both sides will twist on $S^{\text{Ind}_H^G}V$ by the same proposition and formula \ref{3.1.1}. Therefore, it is enough to show that $\mu$ is equivalence only for the case when all $X_i$ are suspension spectra of finite $H$-sets. In other words, we can assume that $X_i = \Sigma^\infty Y_i$, $Y_i \in \text{Set}^H$ for all $1 \leq i \leq n$. Then the right hand side is $\Sigma^\infty \text{Coind}_H^G(Y_1 \sqcup \ldots \sqcup Y_n)$, and the left hand side is equal to

$$\Sigma^\infty \prod_{\lambda \in \mathcal{P}_n} \text{Ind}_{G_\lambda}^G F_\lambda(Y_1, \ldots, Y_n).$$

Now the result follows from Proposition \ref{3.1.2}.

Let denote by $\mathcal{P}_n^{\text{surj}}$ the subset of surjective maps in $\mathcal{P}_n = \text{Hom}(G/H, [n])$. Note that if $\lambda \in \mathcal{P}_n^{\text{surj}}$, then $|G_\lambda\setminus G/H| \geq n$.

**Corollary 3.1.10.** Let $n \in \mathbb{N}$. Then for all $X_1, \ldots, X_n \in S^H$:

$$c_{\text{cr}} n_{n, n} \text{Ind}_{G_\lambda}^G F_\lambda(X_1, \ldots, X_n) \simeq \bigvee_{\lambda \in \mathcal{P}_n^{\text{surj}}} \text{Ind}_{G_\lambda}^G F_\lambda(X_1, \ldots, X_n).$$

\[\square\]

### 3.2. Goodwillie tower

Let $G$ be a finite group, and let $H \subset G$ be a subgroup. Recall that $\mathcal{O}_G$ is the poset of transitive $G$-sets. Let denote by $\mathbb{N}$ the poset of natural numbers (with usual ordering).

**Notation 3.2.1.** We denote by $q_H : \mathcal{O}_G \rightarrow \mathbb{N}$ the map given by the rule $q_H(X) = |X/H|$.

Since $q_H$ preserves order, $q_H^{-1}(I) \subset \mathcal{O}_G$ is an interval for any interval $I \subset \mathbb{N}$. In this section, we give sufficient conditions to compare $L_1 F$ with $L_{q_H^{-1}(I)} \circ F$ for a functor $F : S^H \rightarrow S^G$.

**Definition 3.2.2.** Let $K \subset G$ be a subgroup. The pointed $G$-space $E_K \in \text{Top}_G^*$ is given by

$$(E_K)^{K_1} = \begin{cases} S^0, & K_1 \text{ is conjugate to } K; \\ \{ \text{pt} \}, & \text{otherwise}. \end{cases}$$

**Example 3.2.3.** $E_e = E G_+, E_G = \widetilde{E} \mathcal{P}_G$. In general, if $\mathcal{F}' \subset \mathcal{F}$ are two families of subgroups such that the complement $\mathcal{F}' \setminus \mathcal{F}$ consists only of the subgroups conjugate to $K$, then $E_K = \widetilde{E} \mathcal{F}' \wedge E \mathcal{F}_+$.  

**Definition 3.2.4.** Let $\mathcal{F}$ be a family of subgroups of $G$. Let denote by $\delta(\mathcal{F})$ the number $\min_{K \in \mathcal{F}} |K\setminus G/H| = \min_{K \in \mathcal{F}} q_H(G/K)$.

**Lemma 3.2.5.** Let $F : S^H \rightarrow S^G$ be a reduced functor such that $E_K \wedge F$ is $q_H(G/K)$-homogeneous and $q_H(G/K)$-coreduced for any $K \subset G$. Then for any family $\mathcal{F}$

1. The functor $\widetilde{E} \mathcal{F} \wedge F$ is $\delta(\mathcal{F})$-excisive;
2. $E \mathcal{F}_+ \wedge F$ is $\delta(\mathcal{F})$-reduced and $\delta(\mathcal{F})$-coreduced;
3. $F(E \mathcal{F}_+, F) \simeq F(E \mathcal{F}_+, E \mathcal{F}_+ \wedge F)$ is $\delta(\mathcal{F})$-coreduced.

**Proof.** All three statements can be proven by induction on the size of $\mathcal{F}$. For the first part, if $\mathcal{F}$ is the family of all subgroups, then $E \mathcal{F}$ is contractible and the claim follows immediately. Now suppose that we know part 1 for a family $\mathcal{F}$, let us show
it is also true for a smaller family $\mathcal{F}'$ such that $\mathcal{F} \setminus \mathcal{F}'$ consists only of the subgroups conjugate to a subgroup $K$. Consider the cofiber sequence:

$$
\tilde{E} \mathcal{F}' \wedge E \mathcal{F}'_+ \wedge F \to \tilde{E} \mathcal{F} \wedge F \to \tilde{E} \mathcal{F} \wedge F.
$$

The left term is $E_K \wedge F$. Hence, by the assumptions $\tilde{E} \mathcal{F}' \wedge F$ is $N$-excisive, where $N = \max(q_H(G/K), \delta(\mathcal{F}))$. However, $N \leq \delta(F')$.

For the second part, we induct in the opposite direction. Namely, we start with $\mathcal{F} = \{e\}$, and argue as before but using the cofiber sequence:

(3.2.6) \[ E \mathcal{F}'_+ \wedge F \to E \mathcal{F}'_+ \wedge F \to E \mathcal{F}'_+ \wedge F \wedge F. \]

Here $\mathcal{F} \subset \mathcal{F}'$, $\mathcal{F}' \setminus \mathcal{F}$ consists of only one conjugacy class of $K$. Then the product $E \mathcal{F}'_+ \wedge \tilde{E} \mathcal{F} = E_K$, and by induction $E \mathcal{F}'_+ \wedge F$ is $\min(\delta(\mathcal{F}), q_H(G/K))$-reduced and $\min(\delta(\mathcal{F}), q_H(G/K))$-coreduced. Finally, $\delta(F') = \min(\delta(\mathcal{F}), q_H(G/K))$.

For the last part, we use that $P^b F(A, F) \simeq F(A, P^b F)$ for any functor $F$ and any $G$-CW complex $A$. \hfill \Box

**Lemma 3.2.7.** A functor $F: S^H \to S^G$ is $n$-excisive if and only if $\Phi^K F: S^H \to S^G$ are $n$-excisive for all $K \subset G$.

**Proof.** The functors $\Phi^K$ preserve finite homotopy limits and jointly conservative, see Proposition 2.2.5. Therefore, a $(n+1)$-cube $X$ in $S^G$ is cartesian if and only if the cubes $\Phi^K X$ are cartesian. \hfill \Box

**Corollary 3.2.8.** Let $F$ be a functor as in Lemma 3.2.7, and let $\mathcal{F}_n$ be the largest family of subgroups in $G$ such that $\delta(\mathcal{F}_n) > n$ (i.e. $K \in \mathcal{F}_n \iff |K \setminus G/H| > n$.) Then $E \mathcal{F}_n \wedge F$ is $n$-excisive, and $(E \mathcal{F}_n)_+ \wedge F$ is $(n+1)$-reduced. In particular, the natural map $F \to E \mathcal{F}_n \wedge F$ is the $n$-th Goodwillie approximation.

**Proof.** We note that $\delta(\mathcal{F}_n) \geq n+1$, so $(E \mathcal{F}_n)_+ \wedge F$ is $(n+1)$-reduced by applying the second part of Lemma 3.2.6.

By Lemma 3.2.7 it is enough to show that $\Phi^K (E \mathcal{F}_n \wedge F)$ is $n$-excisive for all $K \subset G$. However, if $K \in \mathcal{F}_n$, then $\Phi^K(\tilde{E} \mathcal{F}_n \wedge F(X))$ is contractible for all $X \in S^H$, and if $K \notin \mathcal{F}_n$, then $\Phi^K(\tilde{E} \mathcal{F}_n \wedge F) \simeq \Phi^K(F) \simeq \Phi^K(\tilde{E} K \wedge F).$ By the assumption $E_K \wedge F$ is $|K \setminus G/H|$-excisive, and $|K \setminus G/H| \leq n$. Hence $\Phi^K (E \mathcal{F}_n \wedge F)$ is also $n$-excisive for $K \notin \mathcal{F}_n$. \hfill \Box

Unfortunately, the assumptions of Lemma 3.2.7 are not enough to show the dual statement. Namely, we can not show that in such generality the $k$-th coreduction $L_k F$ is $F((E \mathcal{k}_n)_+, F)$.

**Example 3.2.9.** The functor $F: S \to S^{C_2}$, $X \mapsto EC_{2+} \wedge \text{triv}_{C_2}^G(X^{\wedge 2})$ is homogeneous of degree 2, 2-coreduced, and it satisfies all properties in Lemma 3.2.5 but the natural map

$$EC_{2+} \wedge \text{triv}_{C_2}^G(X^{\wedge 2}) \to F(EC_{2+}, EC_{2+} \wedge \text{triv}_{C_2}^G(X^{\wedge 2}))$$

is not an equivalence.

**Definition 3.2.10.** A reduced functor $F: S^H \to S^G$ is called norm-like, if it satisfies three following conditions:

(1) For any $X_1, \ldots, X_i \in S^H$ the $i$-th crosseffect $cr_i F(X_1, \ldots, X_i)$ has a form $\forall_{j=1}^m \text{Ind}_{K_j}^G Y_j$, where $K_j$ are subgroups of $G, |K_j \setminus G/H| \geq i$, and $Y_j \in S^{K_j}$ for all $1 \leq j \leq m$. 

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(2) For any \( X \in S^H \) and for any \( K \subset G \), \( cr_{q_H(G/K)} \Phi^K F(X, \ldots, X) \) is naturally equivalent to \( (\Sigma_{q_H(G/K)} \Phi^K F(X, \ldots, X)) \wedge \Phi^K F(X) \) as Borel \( \Sigma_{q_H(G/K)} \)-spectra.

(3) \( F \) is \( |G/H| \)-excisive.

**Lemma 3.2.11.** Let \( F : S^H \to S^G \) be a norm-like functor, and let \( B \in S^G \). Then \( B \wedge F(-) \) and \( F(B, F(-)) \) are both norm-like

**Proof.** Follows straightforward by the projection formulas. \( \square \)

**Proposition 3.2.12.** Let \( F : S^H \to S^G \) be a norm-like functor. Then

1. For any \( K \subset G \) the functor \( \Phi^K F : S^H \to S \) is \( q_H(G/K) \)-homogeneous and \( q_H(G/K) \)-coreduced.
2. For any \( K \subset G \) the functor \( E_K \wedge F : S^H \to S^G \) is \( q_H(G/K) \)-homogeneous and \( q_H(G/K) \)-coreduced.

**Proof.** By properties 1 and 3, the functor \( \Phi^K F \) is \( q_H(G/K) \)-excisive. Hence, \( cr_{q_H(G/K)} \Phi^K F \) is a \( \Sigma^K \)-functor, and by property 2, we have

\[
\Phi^K F(X) \simeq (cr_{n_K} \Phi^K F(X, \ldots, X))_{h\Sigma_{n_K}} \simeq (cr_{n_K} \Phi^K F(X, \ldots, X))_{\Sigma_{n_K}},
\]

where we set \( n_K = q_H(G/K) \), and \( X \in S^H \). Therefore, \( \Phi^K F \) is \( q_H(G/K) \)-homogeneous and also \( q_H(G/K) \)-coreduced.

For the last part, we note that \( E_K \wedge F \simeq E_K \wedge \text{triv}_e \Phi^K F \). Hence, \( E_K \wedge F \) is \( q_H(G/K) \)-homogeneous. By property 2, we also have that \( E_K \wedge cr_{n_K} F(X, \ldots, X) \) is a \( \Sigma_{n_K} \)-induced spectrum, where \( n_K = q_H(G/K) \). So, \( E_K \wedge F \) is also \( q_H(G/K) \)-coreduced. \( \square \)

**Remark 3.2.13.** Notice that if \( E_K \wedge F \) is coreduced, then \( \Phi^K F \) is also coreduced, but not otherwise in general. It happens because \( E_K \) is not a compact \( G \)-spectrum; therefore the smash product with \( E_K \) does not preserves homotopy limits, and so coreduced functors.

Recall that \( \mathcal{F}_n \) is the largest family of subgroups in \( G \) such that \( \delta(\mathcal{F}_n) > n \), i.e. \( K \in \mathcal{F}_n \iff |K/G/H| > n \).

**Corollary 3.2.14.** Let \( F \) be a norm-like functor. Then \( \tilde{E} \mathcal{F}_n \wedge F \) is \( n \)-excisive, and \( (E \mathcal{F}_n)_+ \wedge F \) is \( (n + 1) \)-reduced. In particular, the natural map \( F \to \tilde{E} \mathcal{F}_n \wedge F \) is the \( n \)-th Goodwillie approximation.

**Proof.** Follows by Corollary 3.2.8 and the last part of Proposition 3.2.12. \( \square \)

**Lemma 3.2.15.** Let \( F : S^H \to S^G \) be norm-like. Then \( F(\tilde{E} \mathcal{F}_k, F) \) is \( k \)-excisive. In particular, the natural transformation \( L_k F \to F(\tilde{E} \mathcal{F}_k, F) \) is an equivalence.

**Proof.** Let denote \( F(\tilde{E} \mathcal{F}_k, F) \) by \( \tilde{P}^k F \). It is enough to show that the cross-effects \( cr_i \tilde{P}^k F(X_1, \ldots, X_i) \) are all zeros for all \( X_1, \ldots, X_i \in S^H \) and \( i \geq k \). By the first condition in Definition 3.2.10 we have

\[
\begin{align*}
  cr_i \tilde{P}^k F(X_1, \ldots, X_i) &\simeq \bigvee_{j=1}^m F(\tilde{E} \mathcal{F}_k, \text{Ind}^G_{K_j} Y_j) \\
  &\simeq \bigvee_{j=1}^m \text{Ind}^G_{K_j} F(\text{Res}^G_{K_j} \tilde{E} \mathcal{F}_k, Y_j).
\end{align*}
\]

Since \( |K_j/G/H| \geq i > k \), \( K_j \in \mathcal{F}_k \), and \( \text{Res}^G_{K_j} \tilde{E} \mathcal{F}_k \) are contractible for all \( 1 \leq j \leq m \). The last part follows from the third part of Lemma 3.2.12. \( \square \)

**Proposition 3.2.16.** The norm functor \( N^G_H : S^H \to S^G \) is norm-like.
Proof. We will check three conditions of Definition \[ \text{3.2.10} \] The first one follows from Corollary \[ \text{3.1.10} \] The last one follows for \[ N^G_H \] from Corollary \[ \text{3.2.8} \]

Let us check the second condition. Let \( K \) be a subgroup of \( G \) and set \( n_K = q_H(G/K) = |K\backslash G/H| \). By formula \[ \text{3.1.8} \] we have

\[
\Phi^K N^G_H \simeq \Phi^K \text{Res}^G_K N^G_H \simeq \Phi^K \left( \bigwedge_{g \in K \backslash G/H} N^K_{H_g} \text{Res}^H_{H_g} \right).
\]

Let denote by \( \mathcal{I} \) the set of bijections between \( H \backslash G/K \) and \([n_K]\). By the formula above we get that the \( n_K \)-th cross-effect \( \text{cr}_{n_K} \Phi^K(N^G_H)(X_1, \ldots, X_{n_K}) \) is

\[
\bigvee_{\lambda \in \mathcal{I}} \Phi^K \left( \bigwedge_{g \in K \backslash G/H} N^K_{H_g} \text{Res}^H_{H_g} X_{\lambda(\theta)} \right).
\]

We substitute \( X_1 = \ldots = X_n = X \) and identify \( \mathcal{I} \) with \( \Sigma_{n_K} \). So, we obtain that \( \text{cr}_{n_K} \Phi^K(N^G_H)(X, \ldots, X) \) is equivalent to

\[
(\Sigma_{n_K})_+ \wedge \Phi^K \left( \bigwedge_{g \in K \backslash G/H} N^K_{H_g} \text{Res}^H_{H_g} X \right) \simeq (\Sigma_{n_K})_+ \wedge \Phi^K \text{Res}^G_K N^G_H X
\]

\[
\simeq (\Sigma_{n_K})_+ \wedge \Phi^K N^G_H X. \quad \square
\]

Now, we are ready to prove the main theorem.

**Theorem 3.2.17.** Let \( I = [k; n] \) be an interval in \( \mathbb{N} \).

1. The functor \( L_{q_H^{-1}(I)} \circ N^G_H \) is \( n \)-excisive and \( k \)-coreduced.
2. The natural transformation \( L_I N^G_H \rightarrow L_{q_H^{-1}(I)} \circ N^G_H \) is an equivalence.
3. Moreover, for intervals \( I_1, \ldots, I_j \subset \mathbb{N} \) the natural transformation

\[
L_{I_1} L_{I_2} \cdots L_{I_j} N^G_H \rightarrow L_{q_H^{-1}(I_1)} \circ \cdots \circ L_{q_H^{-1}(I_j)} \circ N^G_H
\]

is an equivalence.

**Proof.** By Proposition \[ \text{3.2.10} \] the norm functor is norm-like and the first two parts follow from Corollary \[ \text{3.2.13} \] and Lemma \[ \text{3.2.15} \]

The last statement follows by induction on the number of intervals and the fact that \( L_{q_H^{-1}(I)} \) has the form \( F(A, B \wedge -) \), see Lemma \[ \text{3.2.11} \] \( \square \)

**Corollary 3.2.18.** Let \( k < m \leq n \in \mathbb{N} \), and set \( I = [k; n], I_2 = [k; m-1], I_1 = [m; n], I = I_1 \sqcup I_2 \). Then the fracture square of Proposition \[ \text{2.1.13} \] applied to the norm functor \( N^G_H : S^H \rightarrow S^G \) is naturally equivalent to the fracture square of Proposition \[ \text{2.2.8} \] postcomposed with \( N^G_H \):

\[
L_{I_2} N^G_H \quad \longrightarrow \quad L_{I_1} N^G_H
\]

\[
| \quad \downarrow \quad | \quad \downarrow \quad |
\]

\[
L_{I_1} N^G_H \quad \longrightarrow \quad L_{I_2} L_{I_1} N^G_H
\]

\[
L_{q_H^{-1}(I_2)} \circ N^G_H \quad \longrightarrow \quad L_{q_H^{-1}(I_1)} \circ N^G_H
\]

\[
| \quad \downarrow \quad | \quad \downarrow \quad |
\]

\[
L_{q_H^{-1}(I_2)} \circ N^G_H \quad \longrightarrow \quad L_{q_H^{-1}(I_2) L_{q_H^{-1}(I_1)} \circ N^G_H}.
\] \( \square \)
4. Examples

Example 4.1. Let \( G = C_p \) be a cyclic group on \( p \)-elements, \( p \) is a prime number, and let \( H = \{ e \} \) be the trivial subgroup. Then the norm functor \( N^C_p : S \to S^C_p \) is \( p \)-excisive, and by Theorem 3.2.17 we obtain \( P^\alpha N^C_p = P\tilde{E} \). There exists only one non-trivial fracture square, which looks as follows:

\[
\begin{align*}
N^C_p & \longrightarrow F((EC_p)_+, N^C_p) \\
\downarrow & \quad \downarrow \\
\tilde{E}C_p \wedge N^C_p & \longrightarrow \tilde{E}C_p \wedge F((EC_p)_+, N^C_p).
\end{align*}
\]

In this case, one can identify the spectrum \( \tilde{E}C_p \wedge N^C_p X \) with \( \tilde{E}C_p \wedge \text{triv}_{C_p} X \) and \( F((EC_p)_+, N^C_p X) \) with the cofree Borel equivariant \( C_p \)-spectrum isomorphic to \( X^{h\gamma} \) where \( C_p \) acts by permutations. In particular, we obtain that \( (N^C_p X)^{C_p} \simeq (X^{h\gamma} \times (\Sigma^{n} \wedge \gamma)^C_p X) \). Here, \((-)^{C_p} \) is the Tate fixed points, and \( X \to (X^{h\gamma})^{C_p} \) is the Tate diagonal.

Example 4.3. Let \( p \) be a prime number as before, and let \( G = C_p^k \) be a cyclic group with \( p \)-elements. The norm functor \( N^{C_p^k}_e \) is \( p \)-excisive, and by Theorem 3.2.17 \( P^\alpha N^{C_p^k}_e = P^\alpha \tilde{E}C_p \). The top fracture square (i.e. containing the map \( P^\alpha \to P^{\alpha-1} \)) in this case looks as follows:

\[
\begin{align*}
N^{C_p^k}_e & \longrightarrow F((EC_p)_+, N^{C_p^k}_e) \\
\downarrow & \quad \downarrow \\
\tilde{E}C_p^k \wedge N^{C_p^k}_e & \longrightarrow \tilde{E}C_p^k \wedge F((EC_p)_+, N^{C_p^k}_e).
\end{align*}
\]

Any other square can be obtain inductively from this one. Namely, the square which contains the map \( P^m \to P^{m-1} \) is square 4.4 with \( k = m \) postcomposed with \( \tilde{E}C_m \wedge \text{triv}_{C_p^k} \). Here, \( K \in C_m \) if and only if \( |K| < p^{k-m} \).

Example 4.5. Let denote by \( \Gamma^n : S \to S \) the functor given by \( X \mapsto (X^{\Sigma^n})^{h\Sigma^n} \). Suppose first that \( X \simeq \Sigma^n \wedge Y \), where \( Y \) is finite CW-complex. Using the Segal conjecture together with the tom Dieck splitting, one can obtain that \( \Gamma^n(X) \) splits as follows:

\[
\Gamma^n(X) \simeq \bigvee_{K \subset \Sigma^n} \left( X^{h\Sigma^n} / \langle K \rangle \right)_{hW_K} \wedge P\Sigma^n.
\]

Here the wedge is taken over all conjugacy classes of subgroups \( K \subset \Sigma^n \) such that \( |K| = p^l \) for some prime \( p \) and natural \( l \), and \( W_K = N_{\Sigma^n}(K) / K \) is the Weyl group of \( K \). Here, we set \( p(K) = p \), if \( |K| = p^l \), and \( p(\{e\}) = 1 \).

The splitting above shows that the Goodwillie tower for \( \Gamma^n \) is simple (even splits) after restriction on finite suspension spectra. However, in general, this tower is hard to describe. Theorem 3.2.17 allows us to present the (at least a part of) Goodwillie tower and the associated fracture squares for \( \Gamma^n : S \to S \).

The functor \( \Gamma^n \) can be factorized as follows:

\[
\Gamma^n(X) \simeq \left( F((E\Sigma^n)_+, N_{\Sigma^n-1} \wedge (\Sigma^n \wedge \gamma)^{h\Sigma^n-1} X) \right)_{h\Sigma^n}.
\]
Note that functors $(-)_{\Sigma n}$ and $\text{triv}^{\Sigma n - 1}_e$ are linear and commute with filtered homotopy limits and colimits. Hence, the Goodwillie tower and the associated fracture squares for the functor $\Gamma^n$ can be obtained from the Goodwillie tower and the associated fracture squares for the functor $F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1}(-))$ by postcomposing with $(-)_{\Sigma n}$ and precomposing with $\text{triv}^{\Sigma n - 1}_e$.

Recall that $F_k$ is a family of subgroups in $\Sigma_n$ such that $K \in F_k$ if and only if $|\{n\}/K| > k$, i.e., $K$-action on the $n$-element set by permutations has more than $k$ orbits. The functor $F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1}(-))$ is norm-like (and naturally equivalent to $L_{n-1}N^{\Sigma n}_{\Sigma n-1}$). Therefore, by Theorem 3.2.17

$$P_k F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1}(-)) \simeq \tilde{E} F_k \wedge F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1}(-)).$$

Hence, the Goodwillie truncations of $\Gamma^n$ can be described by the following long formula

$$P_k \Gamma^n \simeq \left( \tilde{E} F_k \wedge F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1} \text{triv}^{\Sigma n - 1}_e(-)) \right)^{\Sigma n}.$$

Let us denote the composition $F((E\Sigma_n)_+, N^{\Sigma n}_{\Sigma n-1} \text{triv}^{\Sigma n - 1}_e(-))$ by $G$: $S \rightarrow S^{\Sigma n}$. Notice that $G(X)$ is simply the cofree $\Sigma_n$-Borel equivariant spectrum $X^{\wedge n}$. Then by Theorem 3.2.17 the following diagram is homotopy cartesian:

$$\begin{array}{ccc}
P_k \Gamma^n & \longrightarrow & (F((E F_{k-1})_+, \tilde{E} F_k \wedge G))^{\Sigma n} \\
\downarrow & & \downarrow \\
P_{k-1} \Gamma^n & \longrightarrow & (\tilde{E} F_{k-1} \wedge F((E F_{k-1})_+, \tilde{E} F_k \wedge G))^{\Sigma n}.
\end{array}$$

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