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On uniqueness for a rough transport-diffusion equation.

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Presented by

Abstract

In this Note, we study a transport-diffusion equation with rough coefficients and we prove that solutions are unique in a low-regularity class. To cite this article: G. Lévy, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Sur l'unicité pour une équation de transport-diffusion irrégulières. Dans cette Note, nous étudions une équation de transport-diffusion à coefficients irréguliers et nous prouvons l'unicité de sa solution dans une classe de fonctions peu régulières. Pour citer cet article : G. Lévy, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

1. Introduction

In this note, we address the problem of uniqueness for a transport-diffusion equation with rough coefficients. Our primary interest and motivation is a uniqueness result for an equation obeyed by the vorticity of a Leray-type solution of the Navier-Stokes equation in the full, three dimensional space. The main theorem of this note is the following.

Theorem 1.1 : Let \( \nu \) be a divergence free vector field in \( L^2(\mathbb{R}_+; H^1(\mathbb{R}^3)) \) and \( a \) a function in \( L^2(\mathbb{R}_+ \times \mathbb{R}^3) \). Assume that \( a \) is a distributional solution of the Cauchy problem

\[
\begin{align*}
\left( C \right) & \quad \partial_t a + \nabla \cdot (\nu a) - \Delta a = 0 \\
& \quad a(0) = 0,
\end{align*}
\]

where the initial condition is understood in the distributional sense. Then \( a \) is identically zero on \( \mathbb{R}_+ \times \mathbb{R}^3 \).
As a preliminary remark, the assumptions on both \( v \) and \( a \) entail that \( \partial_t a \) belongs to \( L^1_{loc}(\mathbb{R}^+, H^{-2}(\mathbb{R}^3)) \) and thus, in particular, \( a \) is also in \( C(\mathbb{R}^+, D'(\mathbb{R}^3)) \). In Theorem 1.1, \( a \) is to be thought of as a scalar component of the vorticity of \( v \), which is in the original problem a Leray solution of the Navier-Stokes equation. In particular, we only know that \( a \) belongs to \( L^2(\mathbb{R}^+ \times \mathbb{R}^3) \) and \( L^\infty(\mathbb{R}^+, H^{-1}(\mathbb{R}^3)) \), though we will not use the second assumption. The reader accustomed to three-dimensional fluid mechanics will notice that, comparing the above equation with the actual vorticity equations in 3D, a term of the type \( a \partial_t v \) is missing. In the original problem, where this Theorem first appeared, we actually rely on a double application of Theorem 1.1. For some technical reasons, only the second application of Theorem 1.1 takes in account the abovementioned term.

As opposed to the standard DiPerna-Lions theory, we cannot assume that \( a \) is in \( L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^3)) \) for some \( p \geq 1 \). However, our proof does bear a resemblance to the work of DiPerna and Lions; our result may thus be viewed as a generalization of their techniques. Because of the low regularity of both the vector field \( v \) and the scalar field \( a \), the use of energy-type estimates seems difficult. This is the main reason why we rely instead on a duality argument, embodied by the following theorem.

**Theorem 1.2:** Given \( v \) a divergence free vector field in \( L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \) and a smooth \( \varphi_0 \) in \( \mathcal{D}(\mathbb{R}^3) \), there exists a distributional solution of the Cauchy problem

\[
(C') \left\{ \begin{array}{l}
\partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\
\varphi(0) = \varphi_0
\end{array} \right.
\]

with the bounds

\[
\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}
\]

and

\[
\|\partial_j \varphi(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi(s)\|_{L^2(\mathbb{R}^3)}^2 ds \leq \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \|\partial_j v\|_{L^2(\mathbb{R}^3)}^2
\]

for \( j = 1, 2, 3 \) and any positive time \( t \).

By reversing the arrow of time, this amounts to build, for any strictly positive \( T \), a solution on \([0, T] \times \mathbb{R}^3 \) of the Cauchy problem

\[
(-C') \left\{ \begin{array}{l}
-\partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\
\varphi(T) = \varphi_T,
\end{array} \right.
\]

where we have set \( \varphi_T := \varphi_0 \) for the reader’s convenience.

2. **Proofs**

We begin with the dual existence result.

**Proof (of Theorem 1.2.)** Let us choose some mollifying kernel \( \rho = \rho(t, x) \) and denote \( v^\delta := \rho_\delta * v \), where \( \rho_\delta(t, x) := \delta^{-4} \rho(\frac{t}{\delta}, \frac{x}{\delta}) \). Let \((C')\) be the Cauchy problem \((C')\) where we replaced \( v \) by \( v^\delta \). The existence of a (smooth) solution \( \varphi^\delta \) to \((C'\)) is then easily obtained thanks to, for instance, a Friedrichs method combined with heat kernel estimates. We now turn to estimates uniform in the regularization parameter \( \delta \). The first one is a sequence of energy estimates done in \( L^p \) with \( p \geq 2 \), which yields the maximum principle in the limit. Multiplying the equation on \( \varphi^\delta \) by \( \varphi^\delta |\varphi^\delta|^{p-2} \) and integrating in space and time, we get

\[
\frac{1}{p} \|\varphi^\delta(t)\|_{L^p(\mathbb{R}^3)}^p + (p-1) \int_0^t \|\nabla \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 |\varphi^\delta(s)|^{\frac{p-2}{2}} ds = \frac{1}{p} \|\varphi_0\|_{L^p(\mathbb{R}^3)}^p.
\]
Discarding the gradient term, taking \( p \)-th root in both sides and letting \( p \) go to infinity gives
\[
\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}. \tag{7}
\]
To obtain the last estimate, let us derive for \( 1 \leq j \leq 3 \) the equation satisfied by \( \partial_j \varphi^\delta \). We have
\[
\partial_t \partial_j \varphi^\delta - \varphi^\delta \cdot \nabla \partial_j \varphi^\delta - \Delta \partial_j \varphi^\delta = \partial_j \varphi^\delta \cdot \nabla \varphi^\delta. \tag{8}
\]
Multiplying this new equation by \( \partial_j \varphi^\delta \) and integrating in space and time gives
\[
\frac{1}{2} \| \partial_j \varphi^\delta(t) \|^2_{L^2(\mathbb{R}^3)} + \int_0^t \| \nabla \partial_j \varphi^\delta(s) \|^2_{L^2(\mathbb{R}^3)} ds = \frac{1}{2} \| \partial_j \varphi_0 \|^2_{L^2(\mathbb{R}^3)}
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} \partial_j \varphi^\delta(s, x) \partial_j \varphi^\delta(s, x) \cdot \nabla \varphi^\delta(s, x) dx ds. \tag{9}
\]
Since \( \varphi \) is divergence free, the gradient term in the left-hand side does not contribute to Equation (9). Denote by \( I(t) \) the last integral written above. Integrating by parts and recalling that \( \varphi \) is divergence free, we have
\[
I(t) = - \int_0^t \int_{\mathbb{R}^3} \varphi^\delta(s, x) \partial_j \varphi^\delta(s, x) \cdot \nabla \partial_j \varphi^\delta(s, x) dx ds
\]
\[
\leq \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)} \int_0^t \| \partial_j \varphi^\delta(s) \|_{L^2(\mathbb{R}^3)} \| \nabla \partial_j \varphi^\delta(s) \|_{L^2(\mathbb{R}^3)} ds
\]
\[
\leq \frac{1}{2} \int_0^t \| \nabla \varphi^\delta(s) \|^2_{L^2(\mathbb{R}^3)} ds + \frac{1}{2} \| \varphi_0 \|^2_{L^\infty(\mathbb{R}^3)} \int_0^t \| \partial_j \varphi^\delta(s) \|^2_{L^2(\mathbb{R}^3)} ds.
\]
And finally, the energy estimate on \( \partial_j \varphi^\delta \) reads
\[
\| \partial_j \varphi^\delta(t) \|^2_{L^2(\mathbb{R}^3)} + \int_0^t \| \nabla \partial_j \varphi^\delta(s) \|^2_{L^2(\mathbb{R}^3)} ds \leq \| \varphi_0 \|^2_{L^\infty(\mathbb{R}^3)} \| \partial_j \varphi \|^2_{L^2(\mathbb{R}^3, \times \mathbb{R}^3)}. \tag{10}
\]
Thus, the family \( \{ \varphi^\delta \}_\delta \) is bounded in \( L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^3) \). Up to some extraction, we have the weak convergence of \( \{ \varphi^\delta \}_\delta \) in \( L^2(\mathbb{R}^+, H^2(\mathbb{R}^3)) \) and its weak-* convergence in \( L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^3) \) to some function \( \varphi \).

By interpolation, we also have \( \nabla \varphi^\delta \rightharpoonup \nabla \varphi \) weakly in \( L^4(\mathbb{R}^+, H^\frac{4}{3}(\mathbb{R}^3)) \) as \( \delta \to 0 \). As a consequence, because \( \varphi^\delta \rightharpoonup \varphi \) strongly in \( L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \) as \( \delta \to 0 \), the following convergences hold :
\[
\Delta \varphi^\delta \rightharpoonup \Delta \varphi \text{ in } L^2(\mathbb{R}^+ \times \mathbb{R}^3);
\]
\[
\varphi^\delta \cdot \nabla \varphi^\delta, \partial_j \varphi^\delta \rightharpoonup \varphi \cdot \nabla \varphi, \partial_j \varphi \text{ in } L^4(\mathbb{R}^+, L^2(\mathbb{R}^3)).
\]
In particular, such a \( \varphi \) is a distributional solution of \((C')\) with the desired regularity. \( \square \)

We now state a Lemma which will be useful in the final proof.

**Lemma 2.1 :** Let \( \varphi \) be a fixed, divergence free vector field in \( L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \). Let \( \{ \varphi^\delta \}_\delta \) be a bounded family in \( L^\infty(\mathbb{R}^+ \times \mathbb{R}^3) \). Let \( \rho = \rho(x) \) be some smooth function supported inside the unit ball of \( \mathbb{R}^3 \) and define \( \rho^e := \varepsilon^{-3} \rho \left( \frac{x}{\varepsilon} \right) \). Define the commutator \( C \delta, \varphi \) by
\[
C \delta, \varphi(x, s) := v(s, x) \cdot (\nabla \rho^e \ast \varphi^\delta(s))(x) - (\nabla \rho^e \ast (v(s) \varphi^\delta))(x).
\]
Then
\[
\|C \delta, \varphi\|_{L^2(\mathbb{R}^+, \times \mathbb{R}^3)} \leq \|\nabla \rho\|_{L^1(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}^+, H^1(\mathbb{R}^3))} \|\varphi^\delta\|_{L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3))}. \tag{11}
\]
This type of lemma is absolutely not new. Actually, it is strongly reminiscent of Lemma II.1 in [2] and serves the same purpose. We are now in position to prove the main theorem of this note.
Proof (of Theorem 1.1.) : Let $\rho = \rho(x)$ be a radial mollifying kernel and define $\rho_\varepsilon(x) := \varepsilon^{-3}\rho(\varepsilon x)$. Convolving the equation on $a$ by $\rho_\varepsilon$ gives, denoting $a_\varepsilon := \rho_\varepsilon * a$,

\[
(C_\varepsilon) \quad \partial_t a_\varepsilon + \nabla \cdot (a_\varepsilon v) - \Delta a_\varepsilon = \nabla \cdot (a_\varepsilon v) - \rho_\varepsilon * \nabla \cdot (av).
\] (12)

Notice that even without any smoothing in time, $a_\varepsilon$, $\partial_t a_\varepsilon$ are in $L^\infty(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$ and $L^1(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$ respectively, which is enough to make the upcoming computations rigorous. In what follows, we let $\varphi^\delta$ be a solution of the Cauchy problem $(-C^\varepsilon_\delta)$, with $(-C^\varepsilon_\delta)$ being $(-C^\varepsilon)$ with $v$ replaced by $v^\delta$. Let us now multiply, for $\delta, \varepsilon > 0$ the equation $(C_\varepsilon)$ by $\varphi^\delta$ and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

\[
\int_0^T \int_{\mathbb{R}^3} \partial_t a_\varepsilon(s,x) \varphi^\delta(s,x) dxds = \langle a_\varepsilon(T), \varphi_T \rangle_{D'(\mathbb{R}^3), D(\mathbb{R}^3)} - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s,x) \partial_t \varphi^\delta(s,x) dxds
\]

and

\[
\int_0^T \int_{\mathbb{R}^3} [\nabla \cdot (v(s,x) a_\varepsilon(s,x)) - \rho_\varepsilon(x) * \nabla \cdot (v(s,x) a(s,x))] \varphi^\delta(s,x) dxds
\]

\[= \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{\varepsilon, \delta}(s,x) dxds,
\]

where the commutator $C^{\varepsilon, \delta}$ has been defined in the Lemma. From these two identities, it follows that

\[
\langle a_\varepsilon(T), \varphi_T \rangle_{D'(\mathbb{R}^3), D(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{\varepsilon, \delta}(s,x) dxds
\]

\[- \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s,x) \left( -\partial_t \varphi^\delta(s,x) - v(s,x) \cdot \nabla \varphi^\delta(s,x) - \Delta \varphi^\delta(s,x) \right) dxds.
\]

From the Lemma, we know that $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$ is bounded in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$. Because $v \cdot \nabla \varphi^\delta \rightarrow v \cdot \nabla \varphi$ in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ as $\delta \rightarrow 0$, the only weak limit point in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ of the family $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$ as $\delta \rightarrow 0$ is $C^{0,0}$. Thanks to the smoothness of $a_\varepsilon$ for each fixed $\varepsilon$, we can take the limit $\delta \rightarrow 0$ in the last equation, which leads to

\[
\langle a_\varepsilon(T), \varphi_T \rangle_{D'(\mathbb{R}^3), D(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{0,0}(s,x) dxds.
\] (13)

Again, the family $(C^{0,0})_{\varepsilon}$ is bounded in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and its only limit point as $\varepsilon \rightarrow 0$ is 0, simply because $v \cdot \nabla \varphi^\delta - \rho_\varepsilon * (v \cdot \nabla \varphi) \rightarrow 0$ in $L^2(\mathbb{R}_+ \times L^2(\mathbb{R}^3))$. Taking the limit $\varepsilon \rightarrow 0$, we finally obtain

\[
\langle a(T), \varphi_T \rangle_{D'(\mathbb{R}^3), D(\mathbb{R}^3)} = 0.
\] (14)

This being true for any test function $\varphi_T$, $a(T)$ is the zero distribution and finally $a \equiv 0$. □

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