Reconstruction of complex obstacles by farfield measurements

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Abstract. We are interested by the reconstruction of complex obstacles from the electromagnetic farfield map. The complex obstacle is characterized by its shape, its nature (penetrable or impenetrable), its type of boundary conditions and the values of the unknown coefficients distributed along its surface. In this paper, we focus on the particular case where the obstacle is given by an infinite cylinder and give an answer to the corresponding inverse scattering problem.

1. Introduction
1.1. Forward scattering problems for the electromagnetism
Let \( \Omega \) be a \( C^2 \) domain of \( \mathbb{R}^3 \) such that \( \mathbb{R}^3 \setminus \overline{\Omega} \) is connected. We set \( \eta(x) \) to be a real valued function of class \( C^1 \) such that \( \eta(x) > 0 \) and \( A(x) \) to be a real matrix valued function with \( C^1 \) entries such that for every \( x \), \( A(x) \) is invertible.

The two scattering problems we are interested with are the scattering by penetrable obstacles and impenetrable obstacles. We state them as follows:

1.1.1. The impenetrable obstacle case. The electromagnetic field \((E, H)\) satisfies the following scattering problem:

\[
\begin{align*}
\text{curl}E - i\kappa H &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
\text{curl}H + i\kappa E &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
\nu \times E &= 0 \text{ on } \partial\Omega_D (\text{ perfect conductor }), \\
\nu \times H - \eta(x)(\nu \times E) \times \nu &= 0 \text{ on } \partial\Omega_I (\text{ Surface impedance }), \\
\lim_{|x| \to \infty}(E^s \times x + |x|H^s) &= 0 (\text{ Silver-Müller RC }).
\end{align*}
\]

where \( \partial\Omega = \partial\Omega_D \cup \partial\Omega_I \) and \( \nu \) is the unit outward normal to \( \partial\Omega \). The total fields are given by \( E := E^i + E^s \) and \( H := H^i + H^s \) where the incident field \((E^i, H^i)\) is given by \( E^i := \frac{1}{2}\text{curl}pe^{i\kappa x \cdot d} \) and \( H^i := \text{curl}pe^{i\kappa x \cdot d} \) where \( p \in \mathbb{R}^3 \) is a polarization vector. The vector \((E^s, H^s)\) is the scattered field.
1.1.2. The penetrable obstacle case. In this case, the electromagnetic field \((E, H)\) satisfies the problem:

\[
\begin{aligned}
\text{curl} E - i\kappa H &= 0 \text{ in } R^3 \setminus \partial \Omega, \\
\text{curl} H + i\kappa A(x)E &= 0 \text{ in } R^3 \setminus \partial \Omega, \\
\nu \times E_+ - \nu \times E_- &= 0 \text{ in } \partial \Omega, \\
\nu \times H_+ - \nu \times H_- &= 0 \text{ on } \partial \Omega_D, \\
\nu \times H_+ - \nu \times H_- &= \eta(x)(\nu \times E) \times \nu \text{ on } \partial \Omega_I, \quad \text{(Surface impedance)} \\
\lim_{|x| \to \infty} (E^s \times x + |x|H^s) &= 0 \text{(Silver-Müller RC)}
\end{aligned}
\]

where \((E_+, H_+)\) and \((E_-, H_-)\) represent the total field \((E, H)\) in the exterior and in the interior of the obstacle.

The matrix \(A(x)\) is the electrical permittivity satisfying \(A(x) = I_2\) (Identity) for \(x \in R^3 \setminus \overline{\Omega}\) and \(A(x) > 0\) for \(x \in \Omega\) with \(A(x) \neq 1\) on \(\partial \Omega\) while \(\eta\) models the physical properties of the highly conductive coating.

1.2. Inverse scattering problems for the electromagnetism

It is well known that the scattered electrical field \(E^s\) and magnetic field \(H^s\) satisfy:

\[
E^s(x, d, p) = \frac{e^{i\kappa|x|}}{|x|} E^\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|^2}\right), (|x| \to \infty)
\]

\[
H^s(x, d, p) = \frac{e^{i\kappa|x|}}{|x|} H^\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|^2}\right), (|x| \to \infty)
\]

where \(\hat{x} := \frac{x}{|x|}\), see [3]. The vector valued functions \(E^\infty\) and \(H^\infty\) define analytic functions from \(S^2 \times S^2 \times R^3\) to the set of complex numbers \(C\). They are called respectively the electric and the magnetic farfield maps. It is easily seen from the Silver-Müller radiation conditions that \(H^\infty(\hat{x}, d, p) = \hat{x} \times E^\infty(\hat{x}, d, p)\). We denote by \(S^2\) the unit sphere.

We are interested by the following inverse scattering problem:

**The inverse problem for complex obstacles**

Knowing \(E^\infty(\hat{x}, d, p)\) for \((\hat{x}, d) \in S^2 \times S^2\) and for three independent directions of polarization \(p_1, p_2, p_3\), we want to

- **Reconstruct the shape of \(\Omega\).**
- **Decide if the obstacle is penetrable or impenetrable.**
- **Distinguish \(\partial \Omega_D\) from \(\partial \Omega_I\).**
- **Reconstruct the surface impedance \(\eta(x)\) on \(\partial \Omega_I\).**
- **Reconstruct the electric permittivity \(A(x)\) on \(\partial \Omega\) for the case of penetrable obstacle.**

We call \(\Omega\) with such properties a **complex obstacle**.

1.3. The acoustic cases

1.3.1. The conditions. We assume the following conditions on the complex obstacle:

- \(\Omega\) has the following form \(\Omega := D \times R\) where \(D\) is a bounded domain of class \(C^2\) of \(R^2\).
- \(\eta(x) := \eta(x_1, x_2), A(x) := A(x_1, x_2)\) and \(A(x)\) is given by a \(2 \times 2\) matrix (Orthotropic dielectric case).
1.4. The reduction
We take as a polarization direction \( p := (0, 0, 1) \). We set \( A(x) := (A_{i,j}(x))_{i,j=1,2} \). Hence \( E_3 =: u = e^{i\kappa x} + u^s \) satisfies the \textbf{impenetrable obstacle} problem for the Helmholtz equation:

\[
\begin{aligned}
\Delta u + \kappa^2 u &= 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
u &= 0 \text{ on } \partial D, \\
\frac{\partial u}{\partial \nu} + i\eta(x)u &= 0 \text{ on } \partial D_1, \quad \text{(Surface impedance)} \\
\lim_{|x| \to \infty} \sqrt{|x|}(\frac{\partial u}{\partial |x|} - i\kappa u^s) &= 0 \quad \text{(Sommerfeld R.C.)}
\end{aligned}
\]

while \( H_3 =: u = e^{i\kappa x} + u^s \) satisfies the \textbf{penetrable obstacle} problem for the Helmholtz case

\[
\begin{aligned}
\nabla \cdot A\nabla u + \kappa^2 u &= 0 \text{ in } \mathbb{R}^2 \setminus \partial D, \\
\frac{\partial u}{\partial \lambda} - \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial D, \\
u_+ - u_- &= 0 \text{ on } \partial D, \\
\lim_{|x| \to \infty} \sqrt{|x|}(\frac{\partial u}{\partial |x|} - i\kappa u^s) &= 0 \quad \text{(Sommerfeld R.C.)}
\end{aligned}
\]

where \( \frac{\partial u}{\partial \nu} := A\nabla u^s \cdot \nu \).

In both cases, the scattered wave \( u^s \) satisfies

\[
u^s(x, d, e) = \frac{e^{i\kappa|x|}}{\sqrt{|x|}} u^\infty(\hat{x}, d) + O(|x|^{-3/2})
\]

where \( u^\infty : S^1 \times S^1 \to C \) is the farfield map and \( S^1 \) is the unit circle.

1.5. The corresponding inverse problem
The inverse problem reduces then to the following problem:

Knowing \( u^\infty(\theta, d) \) for \( (\theta, d) \in S^1 \times S^1 \), we want to reconstruct the complex obstacle \( D \). Precisely, we want to

- Reconstruct the shape of \( D \).
- Decide if the obstacle \( D \) is penetrable or impenetrable.
- Distinguish \( \partial D \) from \( \partial D_1 \).
- Reconstruct the surface impedance \( \eta(x) \) on \( \partial D_1 \).
- Reconstruct the coefficient \( A(x) \) on \( \partial D \) for the case of penetrable obstacles.

The inverse problem for detecting shapes has a long history. We refer for instance to the monographs [3], [9] and [1] for more information. As we mentioned it above, in this paper, we are concerned by reconstructing complex obstacles. One of the first works devoted to the reconstruction of boundary terms are given in [10], [4], [2] and [1], where different formulas have been proposed to compute the pointwise values of \( A \) on \( \partial D \) [4] and the maximum norm of the surface impedance \( \eta \) in [1]. In a recent work [8], we gave a systematic way how to reconstruct both the shapes and the pointwise values of the surface impedance and we tested these results numerically in [7]. In this paper, we follow this way to show that we can reconstruct more complex obstacles by giving the shapes, deciding the nature of the obstacles (penetrable or impenetrable), localize the support of the coating \( \eta \) and reconstruct both \( \eta \) and \( A \) on \( \partial D_1 \) and \( \partial D_\partial D \) respectively. This is deduced from the obtained formulas which give direct links between the farfield data and the unknowns of the inverse problem. One importance of these results is to show to what extend these formulas can be used for numerical purposes.
2. Presentation of the results

2.1. The main Theorem

It is well known, see [3], that the scattered field associated with the Herglotz incident field $v_g := v_g$ defined by $v_g(x) := \int_{S^1} e^{i \kappa x \cdot d} g(d) \, ds(d)$, $x \in \mathbb{R}^2$ with $g \in L^2(S^1)$ is given by $v_g(x) := \int_{S^1} u^s(x, d) g(d) \, ds(d)$, $x \in \mathbb{R}^2 \setminus D$, and its far field is $v_g^\infty(\hat{x}) := \int_{S^1} u^\infty(\hat{x}, d) g(d) \, ds(d)$, $\hat{x} \in S^1$.

We introduce a constant $\gamma_2 := \frac{e^{i \pi / 4}}{\sqrt{8\pi \kappa}}$ and $\Phi(x, y) := \frac{i}{4} H_{0}^{(1)}(\kappa|x-y|)$, $x \neq y, x, y \in \mathbb{R}^2$, the fundamental solution to the Helmholtz equation in $\mathbb{R}^2$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

We need the following identity, see [3],

$$ u^\infty(\hat{x}, d) = -\gamma_2 \int_{\partial D} \left\{ \frac{\partial u^s(y, d)}{\partial \nu} e^{-i \kappa \hat{x} \cdot y} - \frac{\partial e^{-i \kappa \hat{x} \cdot y}}{\partial \nu} u^s(y, d) \right\} ds(y) $$

(5)

for $(\hat{x}, d) \in S^1 \times S^1$.

Assume that $D \subseteq \subset \Omega$ for some known $\Omega$ with smooth boundary. For $z_0 \in \Omega \setminus \overline{D}$, denote by $\{z_p\} \subset \Omega \setminus D$ a sequence tending to $z_0$. For any $z_p$, set $D_p$ to be a $C^2$- regular domain such that $D \subset D_p$ with $z_q \in \Omega \setminus D^p$ for every $q = 1, 2, \cdots, p$ and that the Dirichlet interior problem on $D_p$ for the Helmholtz equation is uniquely solvable. In this case, the Herglotz wave operator $H$ defined from $L^2(S^1)$ to $L^2(\partial D_p)$ by

$$ H[g](x) := v_g(x) = \int_{S^1} e^{i \kappa x \cdot d} g(d) \, ds(d) $$

(6)

is injective, compact with dense range, see [3]. Now, we consider the sequence of point sources $\Phi(\cdot, z_p)$. For every $p$ fixed, we construct two density sequences $\{g^n_p\}$ and $\{f^n_m\}$ in $L^2(S^1)$ by the Tikhonov regularization such that

$$ ||v_{g^n_p} - \Phi(\cdot, z_p)||_{L^2(\partial D_p)} \rightarrow 0, \ n \rightarrow \infty $$

(7)

$$ ||v_{f^n_m} - \frac{\partial}{\partial x_j} \Phi(\cdot, z_p)||_{L^2(\partial D_p)} \rightarrow 0, \ m \rightarrow \infty. $$

(8)

Since both $v_{g^n_p}$ and $\Phi(\cdot, z_p)$ satisfy the same Helmholtz equation in $D_p$, (7) implies that

$$ ||v_{g^n_p} - \Phi(\cdot, z_p)||_{H^{1/2}(\partial D)} \rightarrow 0, \ n \rightarrow \infty $$

(9)

and

$$ ||\frac{\partial}{\partial \nu} v_{g^n_p} - \frac{\partial}{\partial \nu} \Phi(\cdot, z_p)||_{H^{-1/2}(\partial D)} \rightarrow 0, \ n \rightarrow \infty $$

(10)

Similarly, it follows from (8) that

$$ ||v_{f^n_m} - \frac{\partial}{\partial x_j} \Phi(\cdot, z_p)||_{H^{1/2}(\partial D)} \rightarrow 0, \ m \rightarrow \infty $$

(11)

and

$$ ||\frac{\partial}{\partial \nu} v_{f^n_m} - \frac{\partial}{\partial \nu} (\frac{\partial}{\partial x_j} \Phi(\cdot, z_p))||_{H^{-1/2}(\partial D)} \rightarrow 0, \ m \rightarrow \infty $$

(12)
Multiplying (5) by $f_m^p(d) g_n^p(\hat{x})$ and integrating over $S^1 \times S^1$, we have
\[
-\int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) f_m^p(d) g_n^p(\hat{x}) ds(d) ds(d) = \gamma_2 \int_{\partial D} \left\{ \int_{S^1} \frac{\partial u^s(y, d)}{\partial \nu} f_m^p(d) ds(d) \cdot \int_{S^1} e^{i\hat{x} \cdot y} g_n^p(\hat{x}) ds(\hat{x}) - \int_{S^1} \frac{\partial e^{i\hat{x} \cdot y}}{\partial \nu} g_n^p(\hat{x}) ds(\hat{x}) \cdot \int_{S^1} u^s(y, d) f_m^p(d) ds(d) \right\} ds(y).
\]

From (11), (12) and (13), we have
\[
\lim_{n \to \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) f_m^p(d) g_n^p(\hat{x}) ds(d) ds(d) = \gamma_2 \int_{\partial D} \left\{ v^s_{f_m^p}(y, z_p) - \frac{\partial v^s_{g_m^p}(y)}{\partial \nu} \Phi(y, z_p) \right\} ds(y) = \gamma_2 v^s_{f_m^p}(z_p) \tag{14}
\]
from the Green formula, where $v^s_{f_m^p}(\cdot)$ is the scattered wave corresponding to incident wave $v^s_{f_m^p}(x) = H[f_m^p](x)$.

Denote by $E_j^s(x, z_p)$ the scattered wave corresponding to the incident wave $\frac{\partial \Phi(x, z_p)}{\partial x_j}$, which is well defined for every $x \in R^2 \setminus \overline{D}$. Then it follows from (11), (12), the well posedness of the direct scattering problem and the use of interior estimate that
\[
E_j^s(x, z_p) = \lim_{m \to \infty} v^s_{f_m^p}(x), \quad x \in R^2 \setminus \overline{D}. \tag{15}
\]

Finally, it follows from (14) that
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) f_m^p(d) g_n^p(\hat{x}) ds(d) ds(d) = \gamma_2 E_j^s(z_p, z_p). \tag{16}
\]
We set
\[
I_j(z_p) := \frac{1}{\gamma_2} \lim_{m \to \infty} \lim_{n \to \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) f_m^p(d) g_n^p(\hat{x}) ds(d) ds(d). \tag{17}
\]
Let us mention that the construction of $f_m^p$ and $g_n^p$ is independent on the unknown obstacle. Hence $I_j(z_p)$ is computable from our data only.

The reconstruction of the complex obstacle $D$ is established by analyzing the behavior of (16) when $z_p$ approaches $z_0$. For this, we need the $C^2$ smoothness assumption on the regularity of $D$. Precisely, for every point $z_0 \in \partial D$, there exists a rigid transformation of coordinates under which the image of $z_0$ is $0$ and a function $f \in C^2(-r, r)$ such that
\[
f(0) = \frac{df}{dx}(0) = 0, \quad D \cap B(0, r) = \{(x, y) \in B(0, r); y > f(x)\} \tag{18}
\]
in terms of the new coordinates where $B(0, r)$ is the 2-dimensional ball of center $0$ with radius $r$. For the points $z_0 \in \partial D$, we choose the sequence $\{z_p\}_{p \in N}$ included in $C_{z_0, \theta}$, where $C_{z_0, \theta}$ is a cone with center $z_0$, angle $\theta \in [0, \frac{\pi}{2})$ and axis $\nu(z_0)$. The answer to the inverse problem is based on the following theorem.
The formulas (19)-(20)-(21)-(22) can be used to provide the following informations on the impenetrable obstacles.

2.2. Comments

Assume that \( \partial D \) is of class \( C^2 \), \( \eta \) is a real valued functions of class \( C^1 \) with positive lower bound and \( A(x) = a(x)I_2 \) where \( a(x) \) is a scalar function of class \( C^1 \) in \( D \). Then we have the following formulas:

1. \( \Re(I_j(z_p)) = \)
   \[
   \begin{cases}
   \frac{a(z_0)+1}{a(z_0)-1} \frac{\nu_j(z_0)}{4\pi|z_p-z_0|} + O(\ln|s|), & z_p \text{ near } z_0 \in \partial D_D \\
   -\frac{\nu_j(z_0)}{4\pi|z_p-z_0|} + O(\ln|s|), & z_p \text{ near } z_0 \in \partial D_I.
   \end{cases}
   \]
   \( \Re(I_j(z_0)) = \)

and

\( \Im(I_j(z_0)) = \)

\[
\begin{cases}
O(1), & z_p \text{ near } z_0 \in \partial D_D \\
-\frac{\nu_j(z_0)}{\pi(\eta(z_0))} \ln|s| + O(1), & z_p \text{ near } z_0 \in \partial D_I.
\end{cases}
\]

for the penetrable obstacles, while

2. \( \Re(I_j(z_p)) = \)

\[
\begin{cases}
\frac{\nu_j(z_0)}{4\pi|z_p-z_0|} + O(\ln|s|), & z_p \text{ near } z_0 \in \partial D_D \\
-\frac{\nu_j(z_0)}{4\pi|z_p-z_0|} + O(\ln|s|), & z_p \text{ near } z_0 \in \partial D_I.
\end{cases}
\]

and

\( \Im(I_j(z_0)) = \)

\[
\begin{cases}
O(1), & z_p \text{ near } z_0 \in \partial D_D \\
-\frac{\nu_j(z_0)\eta(z_0)}{\pi} \ln|s| + O(1), & z_p \text{ near } z_0 \in \partial D_I.
\end{cases}
\]

for the impenetrable obstacles.

2.2. Comments

The formulas (19)-(20)-(21)-(22) can be used to provide the following informations on the complex obstacle:

- A sample of points on \( \partial D \) and the normals on these points. The points can be given by numerically solving \( |\Re I_j(z,z)| = C \) for constants \( C \) large. This is justified by the properties:

\[
|I_1(z_p)| + |I_2(z_p)| < \infty \text{ for } z_p \in \mathbb{R}^2 \setminus D
\]

and

\[
|I_1(z_p)| + |I_2(z_p)| \to \infty \text{ for } z_p \to \partial D.
\]

The normals are obtained as follows

\[
\nu(z_0) = \pm (t \sqrt{\frac{1}{1+t^2}}, \sqrt{\frac{1}{1+t^2}}) \text{ where } t := \lim_{z_p \to z_0} \frac{\Re I_1(z_p)}{\Re I_2(z_p)}.
\]
• Distinguish the parts where we have Dirichlet or Impedance type of boundary conditions. This is a consequence of the following identities:

\[
\frac{|\sum_{j=1,2} \nu_j(z_0)|RJ_j(z_p)|}{\ln(|(z_p - z_0)\cdot \nu(z_0)|)} = \begin{cases} 
0, & z_p \text{ near } z_0 \in \partial D_D, \\
\infty, & z_p \text{ near } z_0 \in \partial D_I.
\end{cases}
\] (23)

for both penetrable and impenetrable obstacles, where \(0 < s < 1\).

• Decide if the obstacle is penetrable or impenetrable. This is based on the following properties:

\[
\frac{1}{4\pi |\sum_{j=1,2} \nu_j(z_0)|RJ_j(z_p)|(z - z_0)\cdot \nu(z_0)|} = \begin{cases} 
\frac{|a(z_0) - 1|}{a(z_0) + 1}, & z_p \text{ near } z_0 \in \partial D_D \\
(\text{for penetrable obstacles}) & \\
1, & z_p \text{ near } z_0 \in \partial D.
\end{cases}
\] (24)

• In addition, we can obtain the following informations on \(a\) and \(\eta\):

\[
\frac{1}{4\pi |\sum_{j=1,2} \nu_j(z_0)|RJ_j(z_p)|(z - z_0)\cdot \nu(z_0)|} = \frac{a(z_0) - 1}{a(z_0) + 1}, \quad z_0 \in \partial D_D
\] (25)

and

\[
-\pi \left( \sum_{j=1,2} \nu_j(z_0)|RJ_j(z_p)| \ln(|(z_p - z_0)\cdot \nu(z_0)|) = \frac{1}{\eta(z_0)}, \quad z_0 \in \partial D_I \right)
\] (26)

for the penetrable obstacle, while

\[
-\pi \left( \sum_{j=1,2} \nu_j(z_0)|RJ_j(z_p)| \ln(|(z_p - z_0)\cdot \nu(z_0)|) = \eta(z_0), \quad z_0 \in \partial D_I \right)
\] (27)

for the impenetrable obstacle.

3. Justification of the results

In this section, we give an outline of the proof of Theorem 2.1. We explain it for the case \(j = 2\). We start by some preparations. For any given point \(z_0 \in \Gamma\), we firstly take the rotation \(R_{z_0}\) and the translation \(M_{z_0}\) such that \(R_{z_0}(\nu(z_0)) = (0,1)\), \(R_{z_0}(z_0) + M_{z_0} = 0\) in the new coordinate system \(\tilde{x}\). Under the transform \(\tilde{x} := T(x) := R_{z_0}(x) + M_{z_0}\), it follows that \(T(\nu(z_0)) = (0,1)\), \(T(z_0) = 0\). The justification of the results is based on the following propositions which give the dominant part of \(E_2(x,z)\) for \(x, z\) near \(z_0\).

PROPOSITION 3.1

1. Impedance boundary condition case. Let \(z_0 \in \partial D_I\), then there exist \(\delta(z_0) > 0\) and \(C > 0\) such that

\[
|\Re(E_2^s(x,z) - w_{\eta(z_0)}(x,z))| = O(\ln(|x - z|)) \text{ and } |\Im(E_2^s(x,z) - w_{\eta(z_0)}(x,z))| \leq C,
\] (28)

for \((x,z) \in B_+(z_0, \delta(z_0)) \cap C_{z_0,\theta}\), where \(B_+(z_0, \delta(z_0)) := B(z_0, \delta(z_0)) \cap (R^2 \setminus D)\) and \(B(z_0, \delta(z_0))\) is the ball of center \(z_0\) and radius \(\delta(z_0)\).

2. Dirichlet boundary conditions. If \(z_0 \in \partial D_D\), we obtain (28) by replacing \(w_{\eta(z_0)}\) by \(w_D\).
The functions $w_{\eta}(x, z)$ and $w_D$ are given by

$w_{\eta}(x, z) := \tilde{w}_{\eta}(x, \tilde{z})$ and $w_D(x, z) := \tilde{w}_D(x, \tilde{z})$ where $\tilde{w}_{\sigma(a)}(x, \tilde{z})$ and $\tilde{w}_D(x, \tilde{z})$ satisfy the following properties.

**Proposition 3.2** The function $w_{\eta}(x, \tilde{z})$ has the following explicit form

I) Impenetrable obstacle case:

$$\tilde{w}_{\eta}(x, \tilde{z}) = \frac{\nu_2(\zeta)}{4\pi} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \frac{[\xi_1 + i\kappa \eta(\zeta)]}{[\xi_1] - i\kappa \eta(\zeta)} d\xi_1$$

$$-i \frac{\nu_1(\zeta)}{4\pi} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \xi_1 \frac{[\xi_1 + i\kappa \eta(\zeta)]}{[\xi_1] - i\kappa \eta(\zeta)} d\xi_1,$$  

(29)

while $\tilde{w}_D(x, \tilde{z})$ has the form

$$\tilde{w}_D(x, \tilde{z}) = -\frac{\nu_2(\zeta)}{4\pi} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \frac{\xi_1}{[\xi_1]} d\xi_1.$$  

(30)

II) Penetrable obstacle case:

$$\tilde{w}_{\eta}(x, \tilde{z}) = \frac{\nu_2(\zeta)}{4\pi} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \frac{(a(\zeta) + 1) + ia(\zeta)\eta(\zeta)[\xi_1]}{(1 - a(\zeta)) + ia(\zeta)\eta(\zeta)[\xi_1]} d\xi_1$$

$$-i \frac{\nu_1(\zeta)}{4\pi} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \xi_1 \frac{-a(\zeta)\eta(\zeta) + i(a(\zeta) + 1)[\xi_1]}{(1 - a(\zeta)) + ia(\zeta)\eta(\zeta)[\xi_1]} d\xi_1,$$  

(31)

while $\tilde{w}_D(x, \tilde{z})$ has the form

$$\tilde{w}_D(x, \tilde{z}) = \frac{\nu_2(\zeta) a(\zeta) + 1}{4\pi a(\zeta) - 1} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \frac{\xi_1}{[\xi_1]} d\xi_1$$

$$-i \frac{\nu_1(\zeta) a(\zeta) + 1}{4\pi a(\zeta) - 1} \int_R e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1 e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1}} \frac{\xi_1}{[\xi_1]} d\xi_1.$$  

(32)

In particular, for impenetrable obstacles, we have:

$$\tilde{w}_{\eta}^+(x, \tilde{z}) = \frac{\nu_2(\zeta)}{2\pi(\tilde{x}_2 + \tilde{z}_2)} - i \frac{\nu_2(\zeta)\eta(\zeta)}{\pi} \ln(\tilde{x}_2 + \tilde{z}_2) + O(1)$$  

(33)

and

$$\tilde{w}_D(x, \tilde{z}) = -\frac{\nu_2(\zeta)}{2\pi(\tilde{x}_2 + \tilde{z}_2)} + O(1)$$  

(34)

while for penetrable obstacles, we have:

$$\tilde{w}_{\eta}^+(x, \tilde{z}) = \frac{\nu_2(\zeta)}{2\pi(\tilde{x}_2 + \tilde{z}_2)} - i \frac{\nu_2(\zeta)\eta(\zeta)}{\pi\eta(\zeta)} \ln(\tilde{x}_2 + \tilde{z}_2) + O(1)$$  

(35)

and

$$\tilde{w}_D(x, \tilde{z}) = -\frac{(a(\zeta) + 1)\nu_2(\zeta)}{2\pi(a(\zeta) - 1)(\tilde{x}_2 + \tilde{z}_2)} + O(1).$$  

(36)
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