BOUNDS FOR DEL PEZZO SURFACES OF DEGREE TWO

ARITRA GHOSH, SUMIT KUMAR, KUMMARI MALLESHAM AND SAURABH KUMAR SINGH

Abstract. In this article, we obtain an upper bound for the number of integral points on the del Pezzo surfaces of degree two.

Contents

1. Introduction 1
2. An application of square sieve 3
3. An applications of the Poisson summation formula 4
4. Analysis of Character sums 5
   4.1. Multiplicative character sums 5
   4.2. The character sum $C(m, x)$ 5
   4.3. The character sum $C(p, x)$ 7
5. Estimates for $S(Q, B)$ 7
   5.1. Estimation of $S^\flat(Q, B)$ 7
   5.2. Estimation of $S^\#(Q, B)$ 11
6. Conclusion: Proof of Theorem 12
Acknowledgements 13
References 13

1. Introduction

Let $F(x_1, x_2, x_3)$ be an irreducible homogeneous polynomial of degree four with integer coefficients. Then the equation

$$y^2 = F(x_1, x_2, x_3)$$

defines a del Pezzo surface of degree two (see, [2]). We are interested in the rational points on this surface. Let

$$N(B) = \sharp \{(x_1, x_2, x_3) \in [-B, B]^3 : F(x_1, x_2, x_3) = y^2 \text{ for some } y \in \mathbb{Z}\} .$$

One expects to show that

$$N(B) \ll_{F, \epsilon} B^{2+\epsilon} .$$

In general, this bound is optimal. For example, if we take $F(x_1, x_2, x_3) = x_1^4 + x_2^4 - x_3^4$ one would get the bound $N(B) \gg B^2$. In [4], N. Broberg obtained the bound

$$N(B) \ll_{F, \epsilon} B^{4+\epsilon}$$

using the p-adic determinant method (introduced by Heath-Brown [7] for hypersurfaces and extended to arbitrary varieties by N. Broberg and P. Salberger in...
In [11], R. Munshi obtained the bound

\[ N(B) \ll_F B^{\frac{2}{9}} (\log B)^{\frac{8}{9}} \]

using different methods than that of N. Broberg.

In fact, R. Munshi [11] obtained bound for an equation of the following type

\[ y^d = F(x_1, \ldots, x_n), \quad d \geq 2 \]

with \( F \) being an irreducible homogeneous polynomial of degree \( md \) with \( m \geq 1 \). To count integer solutions to above equations, he introduced the \( d \)-power sieve which is based on ideas of D. R. Heath-Brown [6]. Later, T. D Browning [3] developed the polynomial sieve which extends both the square sieve of D. R. Heath-Brown and \( d \)-power sieve of R. Munshi. In [8], D. R. Heath-Brown and L. Pierce improved results of R. Munshi [11] for \( n \geq 8 \).

**Remark 1.** D. Bonolis and L. Pierce spotted a gap in R. Munshi’s original arguments. Later following suggestions of R. Munshi (see Remark 1 of [1]) D. Bonolis filled the gap in his paper [1] getting a bound of the same strength.

As an application of the main result in [12] of P. Salberger (where he uses global determinant method through which one can look at many congruences modulo many primes simultaneously), one can easily obtain the following bound

\[ N(B) \ll B^{\frac{3}{\sqrt{2}} + \epsilon}. \]

Through email communication we learnt from P. Salberger that he also has an unpublished result which improves the above exponent \( 3/\sqrt{2} \) to \( 36/17 \).

The aim of this article is to improve the exponent \( 36/17 \) of P. Salberger using the ideas in [11] of R. Munshi.

**Theorem 1.** We have

\[ N(B) \ll_{F, \epsilon} B^{2+\frac{10}{17} + \epsilon}. \]

Note that \( y^2 = F(x_1, x_2, x_3) \) defines a variety \( V \) in the weighted projective space \( \mathbb{P}(2, 1, 1, 1) \), where \( y \) is given weight 2 and each \( x_i \) is given weight 1. This variety can also be viewed as a cyclic 2-sheeted cover of the projective plane \( \mathbb{P}^2 \), via the natural map

\[ g : V \to \mathbb{P}^2; \]

\[ (y, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3). \]

To the above map, we associate a counting function

\[ N(g, B) = \sharp \{ P \in V(\mathbb{Q}) : H(g(P)) \leq B \}. \]

Serre’s conjecture, in this case, predicts that

\[ N(g, B) \ll B^2 (\log B)^\gamma, \]

for some \( \gamma < 1 \). As a corollary to Theorem [1] we also obtain the following bound

**Corollary 1.** Let \( g : V \to \mathbb{P}^2 \) be a cover of degree 2. Then we have

\[ N(g, B) \ll B^{2+\frac{10}{17} + \epsilon}, \]

where \( V \) and \( g \) are defined as above.
We use a variant of square sieve to prove Theorem \(1\). Initially the square sieve was introduced by D. R. Heath-Brown in [7] with prime moduli. Later L. Pierce developed a variant of square sieve in [9] with composite moduli. In fact we use a version of square sieve by L. Pierce with composite moduli. These composite moduli play an important role in our argument (serve as conductor lowering after splitting out the moduli, see Lemma 5.2 and Lemma 5.3). In its proof we will assume that \(N(B) \ll B^{2+\epsilon} \) by noting the fact that any integer \(\ell\) can be represented in the form \(x^2 + cy^2\) with \(|x|, |y| \leq B\) in at most \(B^{2+\epsilon}\) ways.

2. An application of square sieve

Let’s recall that

\[ N(B) = \# \{ (x_1, x_2, x_3) \in [-B, B]^3 : F(x_1, x_2, x_3) = \square \}. \]

Let \(P_1\) and \(P_2\) be two large positive real parameters such that

\[ P_2 \geq C \log B \quad \text{and} \quad 10P_2 \leq P_1 \]

for some large positive constant \(C\) which depends only on the form \(F\). Indeed, we choose \(P_1\) and \(P_2\) (see, Section 6) as follows:

\[ P_1 = B^{\frac{2}{3} - \epsilon} \quad \text{and} \quad P_2 = P_1^{\frac{3}{4} + \epsilon}. \]

We first detect squares in \(N(B)\) as follows:

\[ N(B) \ll \frac{1}{(P_1P_2)^2} \sum_{(x_1, x_2, x_3) \in [-B, B]^3} \left| \sum_{p_1 \sim P_1, p_2 \sim P_2} \frac{F(x_1, x_2, x_3)}{p_1p_2} \right|^2 \]

where \(\left( \frac{n}{m} \right)\) is the Jacobi symbol and \(p \sim P\) means that \(P \leq p \leq 2P\). From now on we write \(\chi_m(n) := \left( \frac{n}{m} \right)\) for any non-zero integer \(m\).

Let \(W : \mathbb{R}^3 \to \mathbb{R}\) be a non-negative compactly supported smooth function supported in \([-2, 2]^3\) and satisfying \(W(x_1, x_2, x_3) = 1\) whenever \((x_1, x_2, x_3) \in [-1, 1]^3\). Moreover

\[ \frac{\partial^{i_1+j_2+j_3}}{\partial x_1^{i_1} \partial x_2^{j_2} \partial x_3^{j_3}} W(x_1, x_2, x_3) \ll_{j_1, j_2, j_3} 1. \]

We now smooth out the sum over \(x_i\)’s in \((2)\) as follows:

\[ N(B) \ll \frac{1}{(P_1P_2)^2} \sum_{(x_1, x_2, x_3) \in \mathbb{Z}^3} \sum \left| \sum_{p_1 \sim P_1, p_2 \sim P_2} \chi_{p_1p_2}(F(x_1, x_2, x_3)) \right|^2. \]
Opening the absolute value square and letting $q = p_1p_2$, $q' = p_1'p_2'$ and $Q = P_1P_2$, we see that the right hand side of the above expression transforms into

$$\frac{1}{Q^2} \sum_{q, q' \sim Q} \sum_{x_1, x_2, x_3} \sum_{x_1, x_2, x_3} \chi_{qq'}(F(x_1, x_2, x_3)) W\left(\frac{x_1}{B}, \frac{x_2}{B}, \frac{x_3}{B}\right)$$

where $q \sim Q$ mean that $Q \leq q \leq 4Q$. We estimate the above sum by considering two cases when $q = q'$ and $q \neq q'$. Indeed, we have

$$N(B) \ll \frac{B^3}{Q} + S(Q, B),$$

where

$$S(Q, B) = \frac{1}{Q^2} \sum_{q, q' \sim Q} \sum_{q \neq q'} (qq')^3 \sum_{x=(x_1, x_2, x_3) \in \mathbb{Z}^3} \mathcal{C}(qq', x) \mathcal{I}(qq', x),$$

Lemma 3.1. Let $S(Q, B)$ be as in (3). Then we have

$$S(Q, B) = \frac{B^3}{Q^2} \sum_{q, q' \sim Q} \sum_{q \neq q'} 1 (qq')^3 \sum_{x=(x_1, x_2, x_3) \in \mathbb{Z}^3} \mathcal{C}(qq', x) \mathcal{I}(qq', x),$$

where

$$\mathcal{C}(qq', x) = \sum_{\beta_1, \beta_2, \beta_3(\mod qq')} \chi_{qq'}(F(\beta_1, \beta_2, \beta_3)) e\left(\frac{x_1\beta_1 + x_2\beta_2 + x_3\beta_3}{qq'}\right)$$

and

$$\mathcal{I}(qq', x) = \int \int \int_{\mathbb{R}^3} W(y_1, y_2, y_3) e\left(-\frac{x_1y_1B + x_2y_2B + x_3y_3B}{qq'}\right) dy_1 dy_2 dy_3,$$

with $x_i \ll (Q^2/B)B^i$ for all $i = 1, 2, 3$.

Proof. Reducing each $x_i$ modulo $qq'$, i.e., changing the variables $x_i$ to $\beta_i + x_iqq'$, we observe that $S(Q, B)$ transforms into

$$\frac{1}{Q^2} \sum_{q, q' \sim Q} \sum_{q \neq q'} 1 (qq')^3 \sum_{x=(x_1, x_2, x_3) \in \mathbb{Z}^3} W\left(\frac{\beta_1 + x_1qq'}{B}, \frac{\beta_2 + x_2qq'}{B}, \frac{\beta_3 + x_3qq'}{B}\right).$$

Here we used the fact that $F(x_1, x_2, x_3) \equiv F(\beta_1, \beta_2, \beta_3) \mod qq'$. Next, we apply the Poisson summation formula to the sum over $x_i$'s. Thus the sum over $x_i$ transforms into

$$\sum_{x=(x_1, x_2, x_3) \in \mathbb{Z}^3} \int \int \int_{\mathbb{R}^3} W\left(\frac{\beta_1 + y_1qq'}{B}, \frac{\beta_2 + y_2qq'}{B}, \frac{\beta_3 + y_3qq'}{B}\right)$$

$$\times e\left(-x_1y_1 - x_2y_2 - x_3y_3\right) dy_1 dy_2 dy_3.$$
Lastly, making a change of variable $\frac{\beta+yq'}{y} \rightarrow y_i$, we get the lemma. We get the range for $x_i$’s by using integration by parts on the integral $\mathcal{I}(qq', x)$.

4. Analysis of Character sums

In this section, we will analyze $C(qq', x)$ in (5). We start by recalling some results on bounds for these kind of character sums.

4.1. Multiplicative character sums. In this subsection, we collect some results on bounds for multiplicative character sums which are due to N. Katz [10]. Statement of the following lemma is a combination of the statements of Theorem 2.1 and Theorem 2.2 in [10].

Lemma 4.1. Let $k$ be a finite field of characteristic $p$ with cardinality $q$. Let $f \in k[t_1, t_2, \ldots, t_n]$ be a Deligne polynomial of degree $d \geq 1$ such that $f = 0$ defines a smooth hypersurface in $\mathbb{A}^n_k$. Let $\chi$ be a non-trivial multiplicative character on $k$. Then we have

$$\left| \sum_{t \in k^n} \chi(f(t)) \right| \leq (d - 1)q^{n/2}.$$

4.2. The character sum $C(m, x)$. In this subsection, we will analyse the following character sum

$$(7) \quad C(m, x) := \sum_{\beta_1, \beta_2, \beta_3 \mod m} \chi_m(F(\beta_1, \beta_2, \beta_3)) e\left(\frac{x_1\beta_1 + x_2\beta_2 + x_3\beta_3}{m}\right),$$

where $m \in \mathbb{N}$ and $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$. The above character sum satisfies multiplicative property. More precisely, we have the following lemma.

Lemma 4.2. Let $C(m, x)$ be as in (7). Then $C(m, x)$ is a multiplicative function, i.e., we have

$$C(mn, x) = C(m, x) C(n, x),$$

whenever $(m, n) = 1$.

Proof. We have

$$C(mn, x) = \sum_{\beta_1, \beta_2, \beta_3 \mod mn} \chi_{mn}(F(\beta_1, \beta_2, \beta_3)) e\left(\frac{x_1\beta_1 + x_2\beta_2 + x_3\beta_3}{mn}\right).$$

We write $\beta_i$ as

$$\beta_i = \beta'_i n + \beta''_i m, \quad \beta'_i \mod m \quad \text{and} \quad \beta''_i \mod n, \quad i = 1, 2, 3.$$

We observe that

$$\chi_{mn}(\ldots) = \chi_m(F(\beta'_1 n, \beta'_2 n, \beta'_3 n)) \chi_n(F(\beta''_1 m, \beta''_2 m, \beta''_3 m)) \chi_m(F(\beta'_1, \beta'_2, \beta'_3)) \chi_n(F(\beta''_1, \beta''_2, \beta''_3)).$$

and

$$e(\ldots) = e\left(\frac{x_1\beta'_1 + x_2\beta'_2 + x_3\beta'_3}{m}\right) e\left(\frac{x_1\beta''_1 + x_2\beta''_2 + x_3\beta''_3}{n}\right).$$

Hence the lemma follows. \qed
Thus it is sufficient to study $\mathcal{C}(m, x)$ whenever $m$ is a prime power. In fact, as we are ultimately interested in $\mathcal{C}(qq', x)$, it is enough to study $\mathcal{C}(p, x)$ for $p$ prime. The following lemma provides required bounds for such sums.

**Lemma 4.3.** Let $p$ a prime and $x = (x_1, x_2, x_3)$ be any triplet of integers. Then we have square root cancellations in $\mathcal{C}(p, x)$, i.e.,

$$\mathcal{C}(p, x) \ll p^{3/2}.$$  

*Proof.* Consider the character sum

$$\mathcal{C}(p, x) = \sum_{\beta_1, \beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(\beta_1, \beta_2, \beta_3)) e\left(\frac{x_1\beta_1 + x_2\beta_2 + x_3\beta_3}{p}\right).$$

First we consider the case when $p | x_i$ for all $i$. In this case the above sum looks like

$$\mathcal{C}(p, x) = \sum_{\beta_1, \beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(\beta_1, \beta_2, \beta_3)).$$

Now applying Lemma 4.1, we see that the above sum is bounded by $p^{3/2}$. Now we consider the other case, i.e., $x_i \not\equiv 0 \mod p$ for some $i$. Without loss of generality, let’s assume $x_2 \not\equiv 0 \mod p$. In this case, we split $\mathcal{C}(p, x)$ as follows

$$(8) \quad \mathcal{C}(p, x) = \sum_{\beta_2, \beta_3 \equiv 0 \mod p} \ldots + \sum_{\beta_1, \beta_2, \beta_3 \equiv 0 \mod p} \ldots$$

We now consider the first term of the above expression which is given by

$$\sum_{\beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(0, \beta_2, \beta_3)) e\left(\frac{x_2\beta_2 + x_3\beta_3}{p}\right)$$

$$= \sum_{\beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(0, 1, \beta_3)) e\left(\frac{x_2 + x_3\beta_3}{p}\right)$$

$$= - \sum_{\beta_3 \equiv 0 \mod p} \chi_p(F(0, 1, \beta_3)) + p \sum_{\beta_3 \equiv 0 \mod p} \chi_p(F(0, 1, \beta_3)).$$

Note that when $p | \beta_2$, then $\chi_p(F(0, \beta_2, \beta_3)) = 0$ by the nature of the form $F$. The congruence condition determine $\beta_3$ uniquely. Hence the above expression is bounded by $p$. Now we consider the second sum of (8)

$$\sum_{\beta_1, \beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(\beta_1, \beta_2, \beta_3)) e\left(\frac{x_1\beta_1 + x_2\beta_2 + x_3\beta_3}{p}\right)$$

$$= \sum_{\beta_1, \beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(1, \beta_2, \beta_3)) e\left(\frac{(x_1 + x_2\beta_2 + x_3\beta_3)\beta_1}{p}\right)$$

$$= - \sum_{\beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(1, \beta_2, \beta_3)) + p \sum_{\beta_2, \beta_3 \equiv 0 \mod p} \chi_p(F(1, \beta_2, \beta_3)).$$
We apply Lemma 4.1 to the first term. Hence, it is bounded by \( p \). In the second term, given \( \beta_3 \), the congruence condition determine \( \beta_2 \) uniquely in terms of \( \beta_3 \). Finally apply Lemma 4.1 to sum over \( p \) bound by \( p^{3/2} \). Hence we have the lemma.

4.3. The character sum \( \mathcal{C}(p, x) \). In this subsection we will analyze the character sum

\[
\mathcal{C}(p, x) := \sum_{\alpha_1, \alpha_2, \alpha_3 \mod p} \mathcal{C}(p, \alpha) e \left( \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{p} \right).
\]

where \( \mathcal{C}(p, \alpha) \) is as given in (7) with \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \). We will encounter the above character sum later.

**Lemma 4.4.** We have

\[ \mathcal{C}(p, x) \ll p^3. \]

**Proof.** Plugging (7) into (9), we see that the character sum \( \mathcal{C}(p, x) \) is given by

\[
\sum_{\alpha_1, \alpha_2, \alpha_3 \mod p} \sum_{\beta_1, \beta_2, \beta_3 \mod p} \chi_p(F(\beta_1, \beta_2, \beta_3))
\times e \left( \frac{\alpha_1(x_1 - \beta_1) + \alpha_2(x_2 - \beta_2) + \alpha_3(x_3 - \beta_3)}{p} \right).
\]

Now executing the sum over \( \alpha_i \)'s we get congruence conditions \( \beta_i \equiv x_i \mod p \), for \( i = 1, 2 \) and \( 3 \), which determine \( \beta_i \)'s uniquely. Hence the above sum is bounded by \( p^3 \). \( \square \)

5. Estimates for \( S(Q, B) \)

We now decompose \( S(Q, B) \) in (4) as follows:

\[ S(Q, B) = S^3(Q, B) + S^2(Q, B), \]

where

\[ S^3(Q, B) = \sum_{q, q' \mod p} (...) \quad \text{and} \quad S^2(Q, B) = \sum_{q, q' \mod p} (...) \]

5.1. Estimation of \( S^2(Q, B) \). We recall that

\[
S^2(Q, B) = \frac{B^3}{Q^2} \sum_{q, q' \mod p} \frac{1}{(q, q')^3} \sum_{x_1, x_2, x_3 \in \mathbb{Z}^3} \mathcal{C}(qq', x) \mathcal{J}(qq', x).
\]

By the multiplicative property of \( \mathcal{C}(\ldots) \), as given in Lemma 4.2 it follows that

\[
S^2(Q, B) = \frac{B^3}{Q^2} \sum_{x_1, x_2, x_3 \in \mathbb{Z}^3} \mathcal{C}(p_1, x) \mathcal{C}(p_2, x) \times \sum_{p_1 \sim p_2} \frac{1}{(p_1, p_2)^3} \mathcal{C}(p_1', x) \mathcal{C}(p_2', x) \mathcal{J}(p_1 p_2 p_1' p_2', x).
\]
We further decompose the above sum as

\[ S^4(Q, B) = S^4_1(Q, B) - S^4_2(Q, B) - S^4_3(Q, B) + 2S^4_4(Q, B), \]

where

\[ S^4_1(Q, B) = \frac{B^3}{Q^2} \sum_{x_1, x_2, x_3 \leq \frac{Q^2}{B}} \sum_{p_1, p_2, p_3, p'_3} \sum_{p_1' \sim P_1, p_2' \sim P_2} \frac{1}{(p_1, x)(p_2, x)} \]

\[ \times \sum_{p'_1 \sim P_1, p'_2 \sim P_2} \frac{1}{(p'_1, x)(p'_2, x)} \mathcal{J}(p_1 p'_1, p_2 p'_2, x). \]

We now analyze \( S^4_2(Q, B) \) and \( S^4_3(Q, B) \) in the following lemma.

**Lemma 5.1.** Let \( S^4_2(Q, B) \) and \( S^4_3(Q, B) \) be as in (10). Then we have

\[ S^4_3(Q, B) \ll P_2^2, \]

and

\[ S^4_1(Q, B) \ll P_2^2. \]

**Proof.** Consider the sum \( S^4_2(Q, B) \), which is given by

\[ \frac{B^3}{Q^2} \sum_{x_1, x_2, x_3 \leq \frac{Q^2}{B}} \sum_{p_1, p_2, p_3, p'_3} \sum_{p_1' \sim P_1, p_2' \sim P_2} \frac{1}{(p_1, x)(p_2, x)} \]

\[ \times \sum_{p'_1 \sim P_1, p'_2 \sim P_2} \frac{1}{(p'_1, x)(p'_2, x)} \mathcal{J}(p_1 p'_1, p_2 p'_2, x). \]

Now using Lemma 4.3 to bound the character sums and executing all the sum trivially we get the required bound for \( S^4_2(Q, B) \). Analysing \( S^4_1(Q, B) \) in a similar way, we get bounds for this sum. Hence the lemma follows.

In the following lemma we will estimate \( S^4_1(Q, B) \) and \( S^4_2(Q, B) \). In fact we get better bounds for \( S^4_3(Q, B) \) compared to \( S^4_1(Q, B) \).

**Lemma 5.2.** Let \( S^4_1(Q, B) \) and \( S^4_2(Q, B) \) be as in (10). Then we have

\[ S^4_1(Q, B) \ll S^4_2(Q, B) \]

and

\[ S^4_2(Q, B) \ll \frac{S^4_1(Q, B)}{P_2}. \]
where $\mathcal{S}_1^{\delta}(Q,B)$ is given by

\[
\frac{B^{3+\epsilon}}{Q^2 P_2 P_3^6} \sum_{p_1,p_2 \sim P_1} \left| \sum_{x_1,x_2,x_3 \in \mathbb{Z}} \mathcal{C}(p_1, x) \mathcal{C}(p_1', x) V \left( \frac{B x_1}{Q^2 B^*}, \frac{B x_2}{Q^2 B^*}, \frac{B x_3}{Q^2 B^*} \right) \right|,
\]

where $V$ is a smooth bump function.

Proof. By making a change of variable $y_i \to p_1 p_1' y_i$ for $i = 1, 2, 3$ in (6) we see that $\mathcal{J}(p_1 p_1' p_2 p_2', x)$ transforms into

\[
(p_1 p_1')^3 \iint_{\mathbb{R}^3} W(p_1 p_1' y_1, p_1 p_1' y_2, p_1 p_1' y_3) e \left( -\frac{B(x_1 y_1 + x_2 y_2 + x_3 y_3)}{p_2 p_2'} \right) dy_1 dy_2 dy_3.
\]

Upon substituting this expression in place of $\mathcal{J}(p_1 p_1' p_2 p_2', x)$ in $\mathcal{S}_1^{\delta}(Q,B)$, we see that $\mathcal{S}_1^{\delta}(Q,B)$ is bounded by

\[
\frac{B^3}{Q^2} \iint_{\left[ T_1, \frac{2}{T_1} \right]^3} \sum_{p_1 \sim P_1, p_1' \sim P_1} \left| \sum_{x_1,x_2,x_3 \in \mathbb{Z}} \mathcal{C}(p_1, x) \mathcal{C}(p_1', x) W(p_1 p_1' y_1, \ldots, p_1 p_1' y_3) \right| \times \left| \sum_{p_2, p_2' \sim P_2} \frac{1}{(p_2 p_2')^3} \mathcal{C}(p_2, x) \mathcal{C}(p_2', x) e \left( -\frac{x_1 y_1 B + x_2 y_2 B + x_3 y_3 B}{p_2 p_2'} \right) \right| dy_1 dy_2 dy_3.
\]

We estimate the terms written after $\times$ in the above expression using

\[
\mathcal{C}(p_2, x), \mathcal{C}(p_2', x) \ll P_2^{3/2}
\]

from Lemma 4.3. Therefore we get

\[
\mathcal{S}_1^{\delta}(Q,B) \leq \frac{B^3}{Q^2 P_2} \iint_{\left[ T_1, \frac{2}{T_1} \right]^3} \sum_{p_1 \sim P_1, p_1' \sim P_1} \mathcal{C}(p_1, x) \mathcal{C}(p_1', x) W(p_1 p_1' y_1, p_1 p_1' y_2, p_1 p_1' y_3) \left| dy_1 dy_2 dy_3.\right.
\]

We use the Mellin inversion formula to separate $p_i$ from $p_i'$ in the weight function $W(p_1 p_1' y_1, p_1 p_1' y_2, p_1 p_1' y_3)$. Indeed, we may also assume that

\[
W(y_1, y_2, y_3) = W_1(y_1) W_2(y_2) W_3(y_3),
\]

and hence $W(p_1 p_1' y_1, p_1 p_1' y_2, p_1 p_1' y_3)$ can be rewritten as

\[
\frac{1}{(2\pi)^3} \iint_{\mathbb{R}^3} \tilde{W}_1(it_1) \tilde{W}_2(it_2) \tilde{W}_3(it_3) \times (p_1 p_1')^{-i(t_1 + t_2 + t_3)} y_1^{-i t_1} y_2^{-i t_2} y_3^{-i t_3} dt_1 dt_2 dt_3,
\]

where $\tilde{W}_j(s)$ is the Mellin transform of $W_j$ for $j = 1, 2, 3$. Since $W_j$’s are smooth bump functions, it can be seen easily (using integration by parts repeatedly) that $\tilde{W}_j(it_j)$’s are negligibly small if $t_j \gg B^*$ for some $j$. Therefore, after plugging (13) into (12), the expression inside $\left| \right|$ in (12) is dominated by

\[
\iint_{[-B^*, B^*]^3} \sum_{p_1, p_1' \sim P_1} \mathcal{C}(p_1, x) p_1^{-i(t_1 + t_2 + t_3)} \left| dt_1 dt_2 dt_3.\right.
\]
Hence, after estimating integral over $y_i$’s trivially, we see that $S_1^2(Q, B)$ is bounded by
\[
\frac{B^{3+\epsilon}}{Q^2 P_2^2 P_1^6} \int \int \int_{[-B^r, B^r]^3} \sum_{x_1, x_2, x_3 \in \frac{Q^2}{B^r} B^r} \sum_{p_1 \sim P_1} \left( \sum_{p_1, p_1' \sim P_1, p_1 \neq p_1'} \frac{1}{(p_1 p_1')^3} \right) \left| C(x_1, x_2, x_3; p_1, p_2) \right|^2 \ dt_1 dt_2 dt_3.
\]

Now we smooth out the sum over $x_i$ by introducing a bump function $V$. Then we open the absolute value square, interchanging $p_1, p_1'$ sums with the sums over $x_i$’s, and bounding integrals trivially, we get the first part of the lemma. Similar analysis gives the second part of the lemma. In fact the extra saving $P_2$ comes from the fact that the sum over $p_2'$ is absent in $S_1^2(Q, B)$.

\[\square\]

**Lemma 5.3.** We have
\[
S_1^{2*}(Q, B) \ll P_1^2 P_2^3 + \frac{B^3 B^r}{P_1^2} \sum_{p_1, p_1' \sim P_1, p_1 \neq p_1'} \frac{1}{(p_1 p_1')^3} \sum_{x_1, x_2, x_3 \in \frac{Q^2}{B^r} B^r} \left| C(x_1, x_2, x_3; p_1, p_2) \right|,
\]

where $C(x_1, x_2, x_3; p_1, p_2)$ is as given in (14).

**Proof.** Recall from (11) that the expression for $S_1^{2*}(Q, B)$ is given by
\[
\frac{B^{3+\epsilon}}{Q^2 P_2^2 P_1^6} \sum_{p_1, p_1' \sim P_1, p_1 \neq p_1'} \left| \left( \sum_{x_1, x_2, x_3 \in \mathbb{Z}} C(p_1, x) C(p_1', x) V \left( \frac{Bx_1}{Q^2 B^r}, \frac{Bx_2}{Q^2 B^r}, \frac{Bx_3}{Q^2 B^r} \right) \right) \right|,
\]

where $V(x, y, z)$ is a bump function. By Lemma 4.3, it follows that the diagonal $p_1 = p_1'$ contributes at most
\[
P_1^2 P_2^3
\]

to $S_1^{2*}(Q, B)$, thus getting the first term in the statement of the lemma. In the off-diagonal case, that is when $p_1 \neq p_1'$, we apply the Poisson summation formula to the sum over $x_i$’s. After an application of the Poisson summation formula, the sum over $x_i$ in $S_1^{2*}(Q, B)$ becomes
\[
\frac{Q^6 B^r}{B^3 (p_1 p_1')^3} \sum_{x_1, x_2, x_3 \in \mathbb{Z}} C(x_1, x_2, x_3, \gamma_3, \gamma_3') \mathcal{I}(x_1, x_2, x_3)
\]

where

\[\text{(14)} \quad C(...) = \sum_{\alpha_1, \alpha_2, \alpha_3 \text{ mod } p_1 p_1'} C(p_1, \alpha) C(p_1', \alpha) e \left( \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{p_1 p_1'} \right),
\]

and
\[
\mathcal{I}(...) = \int \int \int_{\mathbb{R}^3} V(y_1, y_2, y_3) e \left( -\frac{(x_1 y_1 + x_2 y_2 + x_3 y_3)(Q^2 B^r / B)}{p_1 p_1'} \right) \ dy_1 dy_2 dy_3.
\]

Upon using integration by parts repeatedly we see that $\mathcal{I}(...)$ is negligibly small unless
\[
x_i \ll \frac{p_1 p_1' B}{Q^2 B^r}, \quad \text{for all } i = 1, 2, 3.
\]

Note that we have $|\mathcal{I}(...)| \leq 1$. Hence we obtain the lemma.

\[\square\]
Lemma 5.4. Let \( x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \) be any triplet of integers. For any distinct primes \( p_1, p'_1 \), we have

\[
\mathcal{C}(x_1, x_2, x_3; p_1, p'_1) = \mathcal{C}(p_1, \bar{p}_1 x) \mathcal{C}(p'_1, \bar{p}_1 x)
\]

where \( \ell x := (\ell x_1, \ell x_2, \ell x_3) \) and \( \mathcal{C}(p, x) \) is as given in \((14)\).

Proof. The lemma follows by noting the fact that any \( \alpha_i \mod (p_1p'_1) \) in \((11)\) can be written as

\[
\alpha_i = \alpha''_i p'_1 + \alpha'_i p_1, \quad \alpha'_i \mod p'_1, \quad \alpha''_i \mod p_1
\]

where \( \bar{p}_1 \) is the inverse of \( p_1 \) modulo \( p'_1 \) and \( \bar{p}'_1 \) is the inverse of \( p'_1 \) modulo \( p_1 \). \( \square \)

Lemma 5.5. We have

\[
\mathcal{S}^1(Q, B) \ll P_1^2 P_2^3 + \frac{B^3}{P_1^2} B'.
\]

Proof. From Lemma 5.2, Lemma 5.3 and Lemma 4.2, we infer that

\[
\mathcal{S}^1(Q, B) \ll P_1^2 P_2^3 + \frac{P_2^3 B'}{P_1^2} \sum \sum \sum_{p_1, p'_1 \sim p, p_1 \neq p'_1} \frac{1}{(p_1p'_1)^3} \sum \sum \sum_{x_1, x_2, x_3 \ll \frac{p_1 p'_1 B}{3^2}} \left| \mathcal{C}(p_1, \bar{p}_1 x) \mathcal{C}(p'_1, \bar{p}_1 x) \right|.
\]

By Lemma 4.3 we have

\[
\mathcal{C}(p_1, \bar{p}_1 x) \ll p_1^3.
\]

Similar bounds holds for \( \mathcal{C}(p'_1, \bar{p}_1 x) \) as well. Using these bounds we see that

\[
\sum \sum \sum_{x_1, x_2, x_3 \ll \frac{p_1 p'_1 B}{3^2}} \left| \mathcal{C}(p_1, \bar{p}_1 x) \mathcal{C}(p'_1, \bar{p}_1 x) \right| \ll \frac{P_1^6 B^3}{P_2^8}.
\]

Finally executing the sum over \( x_i \), we get the lemma. \( \square \)

5.2. Estimation of \( \mathcal{S}^2(Q, B) \). In this section we estimate \( \mathcal{S}^2(Q, B) \). For this we get better bounds than \( \mathcal{S}^1(Q, B) \). Indeed, we have the following lemma.

Lemma 5.6. We have

\[
\mathcal{S}^2(Q, B) \ll \frac{P_1^3}{P_2}.
\]

Proof. We have

\[
\mathcal{S}^2(Q, B) = \mathcal{S}^1_1(Q, B) + \mathcal{S}^1_2(Q, B)
\]

where

\[
\mathcal{S}^1_1(Q, B) = \sum \sum \sum \sum \sum \sum (\ldots)
\]

and

\[
\mathcal{S}^1_2(Q, B) = \sum \sum \sum \sum \sum \sum (\ldots).
\]
We only have to deal with sum $S_1^3(Q, B)$ as the treatment for the sum $S_2^3(Q, B)$ is similar (in fact easier and getting better bounds). The expression for $S_1^3(Q, B)$ is given by

$$\frac{B^3}{Q^2} \sum_{p_1, p'_1 \sim P_1} \sum_{p_2 \sim P_2} \sum_{x_i \in \mathbb{Z}} \mathcal{C}(p_1 p'_1 p_2^2, x) \mathcal{J}(p_1 p'_1 p_2^2, x).$$

Let us deal with $\mathcal{C}(p_1 p'_1 p_2^2, x)$. From (17), we recall

$$\mathcal{C}(p_1 p'_1 p_2^2, x) = \sum_{\beta_1, \beta_2, \beta_3 \bmod p_1 p'_1} \chi_{p_1 p'_1} \left( F(\beta_1, \beta_2, \beta_3) \right) e \left( \frac{x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3}{p_1 p'_1 p_2^2} \right).$$

By Lemma 4.2 we get

$$\mathcal{C}(p_1 p'_1 p_2^2, x) = \mathcal{C}(p_1, x) \mathcal{C}(p'_1, x) \mathcal{C}(p_2^2, x).$$

We see that

$$\mathcal{C}(p_2^2, x) = \prod_{\beta \bmod p_2^2} e \left( \frac{x \beta}{p_2^2} \right) = \begin{cases} p_2^6 & \text{if } p_2^2 \mid x_i \text{ for all } i = 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get

$$S_1^3(Q, B) = \frac{B^3}{Q^2} \sum_{p_1, p'_1 \sim P_1} \sum_{p_2 \sim P_2} \sum_{x_i \in \mathbb{Z}} \mathcal{C}(p_1, x) \mathcal{C}(p'_1, x) \mathcal{C}(p_2^2, x).$$

Now we use Lemma 4.3 to bound character sums $\mathcal{C}(p'_1, x), \mathcal{C}(p_1, x)$, and by executing the remaining sums we infer that

$$S_1^3(Q, B) \ll \frac{P_1^3}{P_2}.$$

Thus the lemma follows. \hfill \square

6. Conclusion: Proof of Theorem 1

We recall that

$$S(Q, B) = S_1^3(Q, B) + S_2^3(Q, B)$$

$$= S_1^3(Q, B) - S_2^3(Q, B) - S_3^3(Q, B) + 2S_1^3(Q, B) + S^3(Q, B).$$

From Lemma 5.1, Lemma 5.2, Lemma 5.5 and Lemma 5.6 we conclude that

$$S(Q, B) \ll P_1^3 P_2^3 + \frac{B^3}{P_2^2} B^e + \frac{P_1^3}{P_2}.$$

Therefore we have

$$N(B) \ll \frac{B^3}{P_1 P_2} + \frac{P_1^3 P_2^3}{P_3^2} + \frac{B^3}{P_3^2} + \frac{P_1^3}{P_2}.$$
We choose $P_1$ and $P_2$ such that

$$\frac{B^3}{P_1P_2} > \frac{B^3}{P_3^2}$$

which is same as the condition $P_1 < P_2^2$, thus we take $P_1 = P_2^{2-\epsilon}$. For this choice of $P_1$ and $P_2$, the first two terms dominate the other terms in $[17]$.

We get the optimal choice for $P_2$ by equating

$$\frac{B^3}{P_1P_2} = P_1^2P_3^2.$$ 

Therefore the optimal choice is $P_2 = B^{3/10}$. Thus, we have

$$N(B) \ll B^{2+1/10+\epsilon}.$$ 

Hence the proof of Theorem 1 follows.

Acknowledgements

Authors are grateful to Prof. Ritabrata Munshi for sharing his ideas with them and for his support and encouragement. Authors are thankful to Prof. Satadal Ganguly for his constant support and encouragements. Authors express their thanks to Prof. T. D. Browning, Prof. L. Pierce and Prof. Per Salberger for showing their interest in authors work and for giving useful suggestions and comments. Finally, first three authors would like to thank Stat-Math unit, Indian Statistical Institute Kolkata and the last author would like to thank IIT Kanpur for providing excellent research environment.

References

[1] D. Bonolis, A polynomial sieve and sums of Deligne type, https://arxiv.org/abs/1811.10560
[2] T. D. Browning, An overview of Manin’s conjecture for del Pezzo surfaces. Analytic number theory- A Tribute to Gauss and Dirichlet (Goettingen, 20th June- 24th June, 2005), Clay Mathematics Proceedings 7 (2007), pg. 39–56.
[3] T. D. Browning, The polynomial sieve and equal sums of like polynomials, Int. Math. Res. Not. IMRN. 07 (2015), 1987-2019.
[4] N. Broberg, Rational points on finite covers of $\mathbb{P}^1$ and $\mathbb{P}^2$, J. Number Theory. 101 (2003), 195-207.
[5] N. Broberg and P. Salberger, Counting rational points on threefolds, (English summary) Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 105–120, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 2004.11G35 ( 14G05).
[6] D. R. Heath-Brown, The square sieve and consecutive square-free numbers, Math. Ann. 266 (1984), 251-259.
[7] D. R. Heath-Brown, The density of rational points on curves and surfaces, Ann. of Math. 2 155 (2002), no. 2, 553–595.
[8] D. R. Heath-Brown and L. Pierce, Counting rational points on smooth cyclic covers, J. Number Theory. 8 (2012), 1741-1757.
[9] L. Pierce, A bound for the 3-part of class numbers of quadratic fields by means of the square sieve, Forum Math. 18 (2006), 677-698.
[10] N. Katz, Estimates for non singular multiplicative character sums, Int. Math. Res. Not. , 7 (2002), 333-349.
[11] R. Munshi, Density of rational points on cyclic covers of $\mathbb{P}^n$, J. Théor. Nombres Bordeaux, 2 (2009), 335-341.
[12] P. Salberger, Counting rational points on projective varieties, preprint.
Aritra Ghosh, Sumit Kumar, Kummari Mallesham
Stat-Math Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700108, India;
Email: aritrajp30@gmail.com, sumitve95@gamil.com, iitm.mallesham@gmail.com

Saurabh Kumar Singh
Indian Institute of Technology, Kanpur, India;
Email: skumar.bhu12@gmail.com