Genus n Forms over Hyperbolic Groups

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Abstract

In 1962 M.J. Wicks [13] gave a list of forms for commutators in both free groups and free products. Since then similar lists have been constructed for elements of higher genus. In [11] A. Vdovina described a method for the construction of forms for elements of any genus in free products. We shall give a similar result for the construction of such forms in any hyperbolic group $H$ and from this we shall obtain a full list of forms for commutators in $H$.

1 Introduction

In 1962 M.J. Wicks [13] showed that any commutator in a free group or a free product of groups could always be reduced to a particular form.

Example 1.1. For any free group $F(X)$, a word $u \neq 1$ in $F(X)$ is a commutator in $F(X)$ if and only if $u$ is conjugate to a cyclically reduced word of the form $ABCA^{-1}B^{-1}C^{-1}$, with $A, B, C \in F(X)$, where at most one of $A, B$ and $C$ may be equal to the identity.

Example 1.2. For any free product $G = \ast_{i \in I} G_i$, if $v \in G$ is a commutator, either $v \in wG_iw^{-1}$ for some $w \in G$, $i \in I$, and $v$ is a commutator in $wG_iw^{-1}$, or some fully cyclically reduced conjugate of $v$ has one of the following forms.

1. $Xa_1X^{-1}a_2$ with $X \neq 1$, $a_1 \neq 1$, $a_1, a_2 \in G_i$ for some $i \in I$, and $a_1$ conjugate to $a_2^{-1}$ in $G_i$; or

2. $Xa_1Ya_2X^{-1}a_3Y^{-1}a_4$ with $X \neq 1$, $Y \neq 1$, $a_1, a_2, a_3, a_4 \in G_i$ for some $i \in I$, and $a_4a_3a_2a_1 = 1$; or

3. $Xa_1Yb_1Za_2X^{-1}b_2Y^{-1}a_3Z^{-1}b_3$ with $a_1, a_2, a_3 \in G_i$ for some $i \in I$ and $a_3a_2a_1 = 1$, $b_1, b_2, b_3 \in G_j$ for $j \in I$ and $b_3b_2b_1 = 1$, and either not all of $a_1, a_2, a_3, b_1, b_2, b_3$ are in any one free factor of $G$ or each of $X, Y, Z$ are nontrivial.
We call these ‘forms’ for commutators in such groups. Similar lists of forms have also been constructed in these settings for certain products of commutators, see [3] and [11], and products of squares, see [12].

In [11] A. Vdovina described a procedure for constructing forms for elements of any genus \( n \) in a free product. This involved an ‘extension’ over the free product of the graph associated with some orientable word. (These terms are explained below). In this paper we establish a similar method for constructing forms for elements of genus \( n \) in hyperbolic groups. We then use this to give a list of all the possible forms for commutators in hyperbolic groups as Wicks did for free groups and free products.

We begin by introducing a number of definitions including how we extend an orientable word over a hyperbolic group. This will put us in the position to state the main result of this paper (see Theorem 2.6), a method for constructing genus \( n \) forms in hyperbolic groups. We follow this by an example of how to implement this theorem. In Section 3 we give preliminary results which will be needed throughout the proof of Theorem 2.6 before moving on to the proof itself in Section 4. Finally we end this paper by proving Proposition 5.1 in Section 5 which states the possible forms for commutators in a hyperbolic group \( H \).

## 2 Definitions and Main results

### 2.1 Definitions

**Definition 2.1.** Let \( G \) be a group and \( g_1, \ldots, g_t \) be \( t \) elements in \( G \). We define the *genus* of \( (g_1, \ldots, g_t) \), denoted \( \text{genus}_G(g_1, \ldots, g_t) \), to be equal to \( k \), if \( k \) is the smallest integer such that there exist elements \( h_l, x_i, y_i \), for \( i = 1, \ldots, k \) and \( l = 1, \ldots, t \), with

\[
h_1 g_1 h_1^{-1} \ldots h_t g_t h_t^{-1} = [x_1, y_1] \ldots [x_k, y_k].
\]

Let \( H = \langle X | R \rangle \) be a hyperbolic group such that geodesic triangles in the Cayley graph \( \Gamma_X(H) \) are \( \delta \)-thin. Consider any word \( w \) in \( F(X) \). We denote the length of \( w \) by \( |w| \). If \( |v| \geq |w| \) for all words \( v \) in \( F(X) \) such that \( w =_H v \) then we say that the word \( w \) is *minimal* in \( H \). Clearly minimal words are represented by geodesic paths in \( \Gamma_X(H) \). If \( w \) is not minimal in \( H \) then we use the notation \( |w|_H \) to denote the length of a word minimal in \( H \) which is equal to \( w \) in \( H \). We can think of the \( \delta \)-thin condition as being equivalent to the following condition. Let \( w \) and \( z \) be any words in \( F(X) \) which are minimal in \( H \), with \( w = w_1 w_2 \) and \( z = z_1 z_2 \). If

\[
|w_2| = |z_1| \leq \frac{1}{2} (|w| + |z| - |wz|_H)
\]

then \( |w_2 z_1|_H \leq \delta \).
Let $A$ be an infinite countable alphabet. We shall define a word in $A^{\pm 1}$ to be orientable quadratic if each letter appears exactly twice, once with exponent 1 and once with exponent $-1$.

**Definition 2.2.** A orientable quadratic word $w$ is said to be redundant if there are letters $x$ and $y$ in $A^{\pm 1}$ which only occur in $w$ as subwords of the form $(xy)^{\pm 1}$. A word is called irredundant otherwise.

We use the term cyclic word to mean the equivalence class $[w]$ of a word $w$ under the relation which relates two words if one is a cyclic permutation of the other. When we talk about a word $w$ being cyclic we mean that $w$ is a representative of the equivalence class $[w]$. If a word $U$ is both cyclic and orientable quadratic then $U$ is called an orientable word in $A^{\pm 1}$.

**Definition 2.3** (Wicks Form). Let $W$ be an orientable word. $W$ is called a Wicks form if the following conditions hold.

1. $W$ is freely cyclically reduced and
2. $W$ is irredundant.

Consider an orientable word $U$ of genus $g$ in $A^{\pm 1}$, that is an orientable word such that $\text{genus}(U)_{F(A)} = g$. Take the disc $D^2$ and divide its boundary into $|U|$ segments. Write $U$ counterclockwise around the boundary of a disc, labelling each segment with a letter of $U$. Let the segments which are labelled by letters with exponent 1 be oriented counterclockwise and the segments which are labelled by letters with exponent $-1$ be oriented clockwise. Now identify the segments labelled by the same letters, respecting orientation. We obtain a closed compact surface of genus $g$. After identification the oriented boundary of the disc gives us an oriented graph embedded on this surface. We shall label this graph $\Gamma_U$ and call it the genus $g$ graph associated with $U$.

**Example 2.4.** Suppose that we have the orientable word $U = ABCA^{-1}B^{-1}C^{-1}$ we construct $\Gamma_U$ as shown in Figure 1.

Note that the graph $\Gamma_W$ associated to a Wicks form $W$ contains no vertices of degree 1 or 2 (if it did then rule 1 or 2 in the definition of a Wicks form would be violated).

Let $\Gamma$ be any oriented connected graph such that an Eulerian circuit exists in $\Gamma$. Here we are taking Eulerian circuit to be a circuit which traverses every edge exactly twice once in each direction. Let $v$ be a vertex of $\Gamma$ of degree $d$ and let the edges $e_1, \ldots, e_d$ be incident to $v$ and oriented away from $v$. Note that these are not necessarily distinct, we may have loops. We define $v$ to be regular if the edges can be renumbered such that the cyclic subwords $e_1^{-1}e_2, \ldots, e_{d-1}^{-1}e_d, e_d^{-1}e_1$ appear in an Eulerian circuit. See Figure 2. If every vertex of $\Gamma$ is regular then we say that it has a regular Eulerian circuit. The following result can easily be deduced from results found in [11].
Lemma 2.5. If $U$ is an orientable word in $\mathcal{A}$ then its associated graph has a regular Eulerian circuit labelled by $U$.

Let $\Gamma$ be any graph with a regular Eulerian circuit, $T$ say. Write $T$ around the boundary of a disc and identify the edges, respecting orientation, to obtain a surface $S$. Then the Euler characteristic $\chi(S)$ is given by the formula $v - e + 1$, where $v$ and $e$ are the number of vertices and edges respectively in $\Gamma$ and $\chi(S)$ is equal to $2 - 2\text{genus}(S)$. We define the genus$(\Gamma)$ to be equal to genus$(S)$. It follows that this is given by

$$\text{genus}(\Gamma) = \frac{1 - v + e}{2}. \quad (1)$$

2.2 Extension of an orientable word over a Hyperbolic group $H$

Let $U$ be an orientable word of genus $k$ in the infinitely countable alphabet $\mathcal{A}$ and $\Gamma_U$ its associated genus $k$ graph. We shall be thinking of edges of $\Gamma_U$ being labelled with letters of $U$ and $U$ itself as being a regular Eulerian circuit. Recall that $H = \langle X | R \rangle$ is a hyperbolic group in which geodesic triangles in $\Gamma_X(H)$ are $\delta$-thin. We shall now give a procedure which can be applied to $\Gamma_U$. This shall be called an extension of the orientable word $U$ over $H$. This involves the following three steps.

1. Let $e$ be a directed edge in $\Gamma_U$ with end points $u = \iota(e)$ and $v = \tau(e)$ (note that $u$ may be the same vertex as $v$). We replace $e$ by two new edges $e_1$ and $e_2$ with labels in $\mathcal{A}^{\pm 1}$ not in $U$ such that $\iota(e_1) = \iota(e_2) = u$ and $\tau(e_1) = \tau(e_2) = v$. We shall do this to every edge in $\Gamma_U$ and call the new graph $\Gamma_{U'}$ where $U'$ is the circuit
which reads $e_1$ wherever $U$ reads $e$ in $\Gamma_U$ and $e_2^{-1}$ wherever $U$ reads $e^{-1}$. Let $v$ be a vertex of $\Gamma_U$ of degree $d$ and assume that the edges $e^1, \ldots, e^d$ are oriented away from $v$. Since $v$ is regular, we can renumber the edges such that $U$ contains the cyclic subwords $(e^1)^{-1}e^2, \ldots, (e^{d-1})^{-1}e^d, (e^d)^{-1}e^1$. Thus in $\Gamma_U$, the circuit $U'$ contains the cyclic subwords $(e^1)^{-1}e^2, \ldots, (e^{d-1})^{-1}e^d, (e^d)^{-1}e^1$. See Figure 3.

Figure 3

2. Now we extend each vertex of $\Gamma_U$ by some cyclic word in $F(X)$. Let $v$ be a vertex of $\Gamma_U$ of degree $d \geq 2$ and assume that the edges $e^1_i, \ldots, e^d_i$, for $i = 1, 2$ are oriented away from $v$. Let $w$ be a cyclic word in $F(X)$ such that $w = w_1 \ldots w_d$. We have already shown that we can renumber the edges of $\Gamma_U$ such that $U'$ contains the cyclic subwords $(e^1_i)^{-1}e^2_i, \ldots, (e^{d-1}_i)^{-1}e^d_i, (e^d_i)^{-1}e^1_i$. We extend the vertex $v$ as follows. Remove the vertex $v$ from $\Gamma_U$ and add $d$ distinct vertices $v_1, \ldots, v_d$ such that

$$
i(e^2_d) = i(e^1_1) = v_1$$

and

$$
i(e^2_k) = i(e^1_{k+1}) = v_k, \quad \text{for } k = 1, \ldots, d - 1.$$

Now add an edge labelled by $w_j$ from $v_j$ to $v_{j+1}$ for all $j = 1, \ldots, d$. See Figure 4.

If $v$ is a vertex of degree one then we add a loop with the label $w$ to $v$. See Figure 4

Figure 4

After each vertex has been extended by some cyclic word in $F(X)$, we obtain a new graph which we shall call $\Gamma_{U''}$. We can still read $U'$ in this graph. In fact it is now a Hamiltonian cycle. We shall call $U'$ the Hamiltonian cycle associated with $U$. 

5
3. Consider a directed edge $e$ of the original graph $\Gamma_U$. In step 1 this is replaced by a pair of edges $(e_1, e_2)$. Then in step 2 we extend the end points of these edges such that we have a subgraph of $\Gamma_{U''}$ of the form shown in Figure 6, where $x$ and $y$ are subwords of the cyclic words in $F(X)$ used in the extension of the end points of $e$. We label the pair $(e_1, e_2)$ with a pair of words $(h_1, h_2)$ in $F(X)$ which are minimal in $H$ such that $h_1 =_H xh_2y$. We do this to every pair of edges. We then have an extension of $U$ over the hyperbolic group $H$.

2.3 Genus and Length of an Extension of $U$ over $H$

Suppose that we have an extension of an orientable word $U$ of genus $k$. We define the length of the extension to be the sum of the lengths in $F(X)$ of the cyclic words $w$ given in step 2 of the extension. That is, if $v_1, v_2, \ldots, v_m$ are the vertices of $\Gamma_U$, and these are extended by cyclic words $w_1, w_2, \ldots, w_m$ respectively, then the length of the extension is

$$\sum_{i=1}^{m} |w_i|.$$  

Let $v_1, \ldots, v_t$ be a subset of the vertices of $\Gamma_U$ that have been extended by cyclic words $w_1, \ldots, w_t$. Then we say that a genus $g$ joint extension has been constructed on these vertices if the genus of the $t$-tuple $(w_1, \ldots, w_t)$ is equal to $g - t + 1$ in $H$.

Now partition the vertices of $\Gamma_U$ into $p$ sets $V_1, \ldots, V_p$ such that a genus $g_i$ joint extension is constructed on the vertices in the set $V_i$, for all $i = 1, \ldots, p$. We say that $U$
has had a genus \( g \) extension over \( H \) if

\[
\sum_{i=1}^{p} g_i = g
\]

where \( g_i \geq 1 \) if \( |V_i| = 1 \) and its only element is a vertex of degree one or two (i.e. if \( g_i = 0 \) and hence \( t_i = 1 \) then \( V_i \)'s only element is a vertex of degree greater than or equal to 3).

2.4 Constructing genus \( n \) forms in \( H \)

We are now in a position to state the main result of this paper, a method for constructing forms for elements of genus \( n \) in \( H \). In the following theorem \( M \) is the number of elements of \( H \) represented by words of length at most \( 4\delta \) in \( F(X) \) and \( l = \delta(\log_2(12n - 6) + 1) \) where \( n \) is given in the hypothesis.

**Theorem 2.6.** Let \( h \) be a word in \( X \cup X^{-1} \) such that the genus \( H(h) = n \). Then \( h \) is conjugate in \( H \) to a word \( F \) in \( F(X) \) which is minimal in \( H \) such that \( F \) has one of the following forms.

1. \( |F| \leq (12n - 6)(12l + M + 4) \) and \( F =_H \theta(W) \) where \( W \) is a genus \( n \) Wicks form and \( \theta \) is a map from \( F(A) \) to \( F(X) \) such that \( |\theta(E)| \leq 12l + M + 4 \) for each letter \( E \) of \( W \).

2. \( F \) is the label on the Hamiltonian cycle \( U' \) obtained by a genus \( g \) extension of length at most \( 2(12n - 6)(12l + M + 4) \) on some orientable word \( U \) of genus \( k \), where \( n = g + k \).

The proof of this theorem runs from page 12 to page 38. It involves considering a genus \( n \) Wicks form \( W \) which can be mapped to a word in \( F(X) \) conjugate to \( h \) in \( H \) such that \( W \) has the property of being the shortest length Wicks form in \( F(X) \) which is conjugate to \( h \) in \( H \). From page 13 to page 31 we state and prove a number of preliminary lemmas involving the Wicks form \( W \) as well as showing that \( W \) either takes form 1 in our theorem or can be reduced to an orientable word \( U \) (obtained by setting certain letters of \( W \) to 1). Then from page 31 to page 38 we will show that if we don’t have form 1 then \( h \) is conjugate to a form given by an extension of \( U \) over \( H \) of bounded length and known genus, that is it takes form 2 in our theorem. Below we give an example of how to construct one of these genus \( n \) forms from an extension.

**Example 2.7.** (A possible form of an element of genus 3 in \( H \))

Let \( U = ABCC^{-1}B^{-1}A^{-1} \), an orientable word of genus 0. Let \( u_1, u_2, v_1, v_2 \) be the vertices of the associated graph \( \Gamma_U \). We shall do a joint genus 2 extension on the vertices \( u_1 \) and \( u_2 \), by words \( w_1 \) and \( w_2 \) respectively, and a joint genus 1 extension on the vertices \( v_1 \) and \( v_2 \), by words \( z_1 \) and \( z_2 \) respectively. See Figure 7. Here \( w_2 = w_{21}w_{22} \) and \( z_1 = z_{11}z_{12} \). Thus,
by Theorem 2.6, a possible form for an element of genus 3 is
\[ F = A_1B_1C_1^{-1}B_2^{-1}A_2^{-1} \]
where \( F \) is a word in \( F(X) \) which is minimal in \( H \) and
\[
\begin{align*}
A_1 &= H_{w_1^{-1}A_2w_1}, \\
B_1 &= H_{w_2B_2z_2}, \\
C_1 &= H_{z_1C_2z_2},
\end{align*}
\]
\( w_1w_2 \) is a commutator in \( H \), \( z_1z_2 = 1 \) and
\[ |w_1 + w_2 + z_1 + z_2| \leq 60(12l + M + 4). \]

3 Preliminary Results

Let \( H = \langle X | R \rangle \) be a finitely generated hyperbolic group. Then the following lemma by R. I. Grigorchuk and I. G. Lysionok in [5] shows that the conjugacy problem is solvable \( H \).

Lemma 3.1 (5). If minimal words \( h_1 \) and \( h_2 \) are conjugate in \( H \), then a word \( w \) can be found such that \( h_1 = H_{wh_2w^{-1}} \) and
\[
|w| \leq \frac{1}{2}(|h_1| + |h_2|) + M + 1,
\]
where \( M \) is the number of elements of \( H \) represented by words of length \( \leq 4\delta \).

The following lemma can be found in K.J. Friel [7], Lemma 2.2.4. A proof has been provided, to help understand Lemma 3.2 which follows it.

Lemma 3.2. Suppose that we have a closed path \( q = \gamma_0\gamma_1\ldots\gamma_n \) in \( \Gamma_X(H) \), where \( \gamma_i \) is a geodesic path, for \( i = 0, \ldots, n \). Let vertices \( \zeta_1 \) and \( \zeta_2 \) lie on \( \gamma_0 - \{ \nu(\gamma_0), \tau(\gamma_0) \} \) such that \( d(\nu(\gamma_0), \zeta_1) < d(\nu(\gamma_0), \zeta_2) \). Then vertices \( \eta_1, \eta_2 \notin \gamma_0 \) can be found on \( q \) such that
\[
\begin{enumerate}
\item \( d(\zeta_1, \eta_1), d(\zeta_2, \eta_2) \leq \delta(\log_2(n) + 1); \)
\item \( d_q(\nu(\gamma_0), \eta_1) > d_q(\nu(\gamma_0), \eta_2) \) and
\end{enumerate}
3. If \( \eta_1, \eta_2 \in \gamma_i \) for some \( i \neq 0 \) then \( d(\eta_1, \eta_2) = d(\zeta_1, \zeta_2) \).

Proof. Now for some integer \( k \) it follows that \( \log_2 n \leq k \leq \log_2(n) + 1 \). Therefore by adding paths of length zero between \( \tau(\gamma_n) \) and \( \iota(\gamma_0) \), we may assume that \( n = 2^k \) and replace the bound

\[
d(\zeta_1, \eta_1), d(\zeta_2, \eta_2) \leq \delta(\log_2(n) + 1)
\]

in part 1 by the bound

\[
d(\zeta_1, \eta_1), d(\zeta_2, \eta_2) \leq \delta k.
\]

We now carry out a subdivision on the closed path \( q \). Let \( q_0 \) and \( q_1 \) be geodesic paths from \( \tau(\gamma_0) \) to \( \tau(\gamma_{2^{k-1}}) \) and \( \iota(\gamma_0) \) respectively. Let \( b = b_1 \ldots b_m \) be a binary sequence of length \( m \leq k - 1 \), where \( b_i = 0 \) or \( 1 \) for all \( i = 1, \ldots, m \). We define \( q_{b_0} \) and \( q_{b_1} \) to be geodesic paths from \( \iota(q_b) \) to \( \tau(\gamma_{r(b)}) \) and \( \tau(\gamma_b) \) respectively, where

\[
r(b) = b_1 2^{k-1} + b_2 2^{k-2} + \ldots + b_m 2^{k-m} + 2^{k-(m+1)}.
\]

Note that, if \( m = k - 1 \) then we choose the geodesic paths \( q_{b_0} \) and \( q_{b_1} \) to be \( \gamma_{r(b)} \) and \( \gamma_{r(b)+1} \) respectively. This gives a subdivision of \( q \) into geodesic triangles. For example if \( k = 3 \) we have the subdivision shown in Figure 8.

![Figure 8](image)

Consider a geodesic triangle of the subdivision with sides \( q_b, q_{b_0} \) and \( q_{b_1} \). Let \( v \) be a vertex lying on \( q_b \). By the definition of \( \delta \)-thin triangles there exists a vertex \( v' \) on \( q_{b_0} \cup q_{b_1} \) such that \( d(v, v') \leq \delta \) and either

(i) \( v' \in q_{b_0} \) and \( d(\iota(q_b), v) = d(\iota(q_{b_0}), v') \), or

(ii) \( v' \in q_{b_1} \) and \( d(\tau(q_b), v) = d(\tau(q_{b_1}), v') \).
From this we see that for any vertex on a geodesic path \( q_b \) there always exists a vertex on a geodesic path \( q_{b'} \) where the length of the binary sequence \( b' \) is one greater than \( b \).

Now \( \zeta_1 \in \gamma_0 \) is within \( \delta \) of some vertex \( v_1 \) on either \( q_0 \) or \( q_1 \). From the above there exists a sequence of vertices \( v_1, \ldots, v_k \), where each \( v_i \) lies on a \( q_{b(v_i)} \) such that the length of the binary sequence \( b(v_{i+1}) \) is one greater than the length of the binary sequence \( b(v_i) \) and \( d(v_i, v_{i+1}) \leq \delta \) for all \( i = 1, \ldots, k - 1 \). This implies that \( d(\zeta_1, v_k) \leq \delta k \) and from the construction of the subdivision \( q_{b(v_k)} = \gamma_j \) for some \( j = 1, \ldots, n \). Therefore, let \( v_k = \eta_1 \).

We shall denote this path passing through the sequence of vertices by \( s_1 \), i.e. \( |s_1| \leq \delta k \).

For example if \( k = 3 \) we have a path as shown in figure 9. Clearly the same construction of a path \( s_2 \) of length at most \( \delta k \) applies to \( \zeta_2 \). Hence part 1 of the lemma holds.

Consider two vertices \( u_1 \) and \( u_2 \) lying on the geodesic \( q_b \) for some binary sequence \( b \) such that \( d(\iota(q_b), u_2) = d(\iota(q_b), u_1) + B \), for some positive constant \( B \). Then by the definition of \( \delta \)-thin triangle there exist vertices \( u'_1, u'_2 \) lying on \( q_{b_0} \cup q_{b_1} \) such that \( d(u_1, u'_1), d(u_2, u'_2) \leq \delta \) and either

(a) \( u'_1, u'_2 \in q_{b_0} \) and \( d(\iota(q_{b_0}), u'_2) = d(\iota(q_{b_0}), u'_1) + B \), or

(b) \( u'_1, u'_2 \in q_{b_1} \) and \( d(\tau(q_{b_1}), u'_2) = d(\tau(q_{b_0}), u'_1) - B \), or

(c) \( u'_2 \in q_{b_0} \) and \( u'_1 \in q_{b_1} \).

Now suppose that paths \( s_1 \) and \( s_2 \) follow the same sequence of \( q_b \)'s, until for some \( q_{b'} \), \( s_2 \) meets \( q_{b_0} \) and \( s_1 \) meets \( q_{b_1} \). Let \( s_1 \) and \( s_2 \) meet the geodesic path \( q_{b'} \) at the vertices \( x_1 \) and \( x_2 \) respectively. By (a) and (b) it follows that \( d(\zeta_1, \zeta_2) = d(x_1, x_2) \) and \( d(\iota(q_{b'}), x_2) < d(\iota(q_{b'}), x_1) \). Since \( s_2 \) meets \( q_{b_0} \) and \( s_1 \) meets \( q_{b_1} \), the construction of the \( q_{b'} \)'s implies that \( d_q(\iota(\gamma_0), \eta_1) > d_q(\iota(\gamma_0), \eta_2) \).

Figure 9
Finally, suppose that $s_1$ and $s_2$ meet exactly the same sequence of $q_i's$. Let $q_\nu$ be the last in this sequence. Again by our construction of the $q_i's$, $q_\nu = \gamma_l$ for some $i = 1, \ldots, n$ and by (a) and (b) it follows that $d(\xi_1, \xi_2) = d(\eta_1, \eta_2)$ and $d_q(\xi(\gamma_0), \eta_1) > d_q(\xi(\gamma_0), \eta_2)$. Hence both part 2 and 3 hold.

\begin{lemma}
Given $\xi_1$ and $\eta_1$ from above, if $\xi_3$ is vertex lying on $q$ such that

$$d_q(\xi(\gamma_0), \xi_1) < d_q(\xi(\gamma_0), \xi_3) < d_q(\xi(\gamma_0), \eta_1)$$

and $\xi_3 \neq \tau(\gamma_i)$ or $\tau(\gamma_i)$ for some $i = 1, \ldots, n$ then a vertex $\eta_3 \neq \xi_3$ can be found on $q$ such that

1. $d(\xi_3, \eta_3) \leq 2\delta(\log_2(n) + 1)$ and
2. $d_q(\xi(\gamma_0), \eta_1) > d_q(\xi(\gamma_0), \eta_3)$.

\end{lemma}

\begin{proof}
Again consider the subdivision constructed above with the path $s_1$ from $\xi_1$ to $\eta_1$. Let the terminal vertex of the path $s_1$ lie on $\gamma_l$ for some $l \leq n$. If $\xi_3$ lies on $\gamma_0$ then we have the hypothesis of the previous lemma, and the lemma holds. So assume that $\xi_3 \in \gamma_l$ for some $i = 1, \ldots, l$.

Consider a geodesic triangle $q_\nu q_\sigma q_\beta$ in the subdivision, for some binary sequence $b$. Let $v$ be a vertex on $q_\kappa$, where $\kappa = 0$ or 1. Then there exists a vertex $v'$ on $q_0 \cup q_\xi$, where $\xi = (\kappa + 1)$ mod 2, such that $d(v, v') \leq \delta$.

It follows from the subdivision that $\gamma_l = q_\nu$ for some binary sequence $b'$ of length $k$ (remember that $n = 2^k$). By the above paragraph we may choose a sequence of vertices $\xi_3 = v_1, v_2, \ldots$ through the $q_\kappa$'s such that each $v_j$ lies on $q_\kappa(v_j)$, where each binary sequence $b(v_j)$ is distinct. Let $v_r$ be the first vertex such that either

(i) the length of $b(v_r)$ is the same as the length of $b(v_{r+1})$, or
(ii) $v_r$ lies on $q_1 \cup q_2$.

(i) The length of the binary sequence decreases by one each time until $v_r$ is reached. Therefore there exists a path of length at most $\delta k$ from $\xi_3 = v_1$ to $v_r$. Now by the construction of the subdivision and the argument in the previous lemma, the length of the binary sequence associated to each vertex in the sequence $v_{r+1}, v_{r+2}, \ldots$ increases by one each time. Thus for some $m$, $v_m$ lies on $q$ and there exists a path from $v_{r+1}$ to $v_m$ of length at most $\delta k$. Let $\eta_3 = v_m$. Then $d(\xi_3, \eta_3) \leq 2\delta k$ and in this case the part 1 holds.

(ii) Without loss of generality let $v_r$ lie on $q_0$. Once again there is a path from $\xi_3 = v_1$ to $v_r$ of length at most $\delta k$. If $v_{r+1}$ lies on $\gamma_0$ then $\eta_3 = v_{r+1}$ and part 1 holds. If $v_{r+1}$ lies on $q_1$, then we use the same argument to as (i) to show that there is a $v_r$ on $q$ such that $d(v_{r+1}, v_m) \leq \delta k$. Hence part 1 holds in all cases.
Let \( s_3 \) be the path of length at most \( 2\delta k \) constructed above. From the previous lemma \( s_1 \) passes through \( k \) distinct \( q_b \)'s. Let the sequence of vertices which lie on these \( q_b \)'s be \( \zeta_1 = u_0, u_1, \ldots, u_k = \eta_1 \) such that \( u_j \) lies on \( q_b(u_j) \) for some binary sequence \( b(u_j) \). Now, if \( s_3 \) never passes through \( q_b(u_j) \) for all \( j = 1, \ldots, k \), then by the construction of the subdivision, part 2 clearly holds. Therefore, assume that \( s_3 \) passes through \( q_b(u_j) \) at the vertex \( x \), for some \( j \). Now, since \( d_q(\iota(\gamma_0), \zeta_1) < d_q(\iota(\gamma_0), \zeta_3) < d_q(\iota(\gamma_0), \eta_1) \), it follows from the construction of \( s_3 \) that \( d(\iota(q_b(u_j)), x) \leq d(\iota(q_b(u_j)), u_j) \). It clearly follows from the construction of the \( q_b \)'s and statement (a), (b) and (c) from the previous lemma that \( d_q(\iota(\gamma_0), \eta_1) > d_q(\iota(\gamma_0), \eta_3) \). See Figure 10. Hence the lemma holds. \( \square \)

![Figure 10](image-url)

### 4 Proof of Theorem 2.6

**Proof of Theorem 2.6.** We begin by considering all genus \( n \) Wicks forms in \( A^{\pm 1} \) (remember that this is an infinitely countable alphabet) and choosing one which when mapped to a word in \( F(X) \) which is conjugate to \( h \) in \( H \) has the property of being a word of shortest length over all such mapping of Wicks forms to words into \( F(X) \) which are conjugate to \( h \) in \( H \).

The genus of \( h \) is equal to \( n \) in \( H = \langle X | R \rangle \). Therefore, by definition, there exist words \( a_i, b_i \in X \cup X^{-1} \), for \( i = 1, \ldots, n \), such that

\[
h =_{H} [a_1, b_1][a_2, b_2] \cdots [a_n, b_n].
\]

In \( A^{\pm 1} \) the orientable word \( U = [A_1, B_1] \cdots [A_n, B_n] \) is a genus \( n \) Wicks form. Let \( L = \{A_1, \ldots, A_n, B_1, \ldots, B_n\} \) and let \( \phi : F(L) \to F(X) \) be a homomorphism defined by \( \phi(A_i) = a_i \) and \( \phi(B_j) = b_j \), for all \( i, j = 1, \ldots, n \). We shall call this a **labelling function**
for $U$. Note that

$$\phi(U) = [\phi(A_1), \phi(B_1)] \ldots [\phi(A_n), \phi(B_n)] = [a_1, b_1] \ldots [a_n, b_n].$$

Let $\mathcal{F}$ be the set of pairs $(U, \phi)$ where $U$ is a genus $n$ Wicks form and $\phi$ is a labelling function for $U$ such that $\phi(U)$ is conjugate to $h$ in $H$. Consider a pair $(W, \theta)$ in which $|\theta(W)|$ is minimal amongst all pairs in $\mathcal{F}$ (since we have shown that at least one pair exists in $\mathcal{F}$ this is always possible). Clearly $\theta(E)$ is minimal in $H$ for each letter $E$ of $W$ or our choice of minimal pair in $\mathcal{F}$ would be incorrect. We should note that there exists no genus $m$ Wicks form $V$, $m < n$, with a labelling function $\psi$ such that $\psi(V)$ is conjugate to $h$ in $H$, as this would contradict the genus of $h$ in $H$.

Now, for our minimal pair $(W, \theta)$ in $\mathcal{F}$ we are able to state a number of preliminary lemmas. The first lemma we state uses ideas from [9], Lemma 12. It displays bounded length properties of subwords of $\theta(A)$, where $A$ is any letter of $W$. This lemma will be used regularly in the proof of Lemma 4.3 which shows that in the Cayley graph $\Gamma_X(H)$ the path represented by the word $\theta(W)$ is ‘close’ to a geodesic path representing a word $F$ which is equal to $\theta(W)$ in $H$. Let $\Gamma_W$ be the genus $n$ graph associated to $W$.

**Lemma 4.1.** Let $E_1, \ldots, E_r, A, E_{r+1}, \ldots, E_s, A^{-1}, E_{s+1}, \ldots, E_t$ be the cyclic sequence of letters in the regular Eulerian circuit $W$ in $\Gamma_W$. Let $\theta(A) = a_1a_2$ where $a_1$ and $a_2$ are subwords of the word $\theta(A)$ which is minimal in $H$. Similarly, let $\theta(E_i) = e_{i1}e_{i2}$, for $1 \leq i \leq t$. Then

(i) $|a_1| \leq |e_1 a_2\theta(E_{i+1} \ldots E_r)a_1|_H$, for $1 \leq i \leq r$;

(ii) $|a_2| \leq |a_2\theta(E_{r+1} \ldots E_{j-1})e_{j1}|_H$, for $r + 1 \leq j \leq s$; and

(iii) $|a_1| \leq |a_2\theta(E_{r+1} \ldots E_s A^{-1} E_{s+1} \ldots E_{k-1})e_{k1}|_H$, for $s + 1 \leq k \leq t$.

**Proof.** First we shall prove statement (ii). Let $t_1$ be a word in $F(X)$ which is minimal in $H$ such that $t_1 =_H a_2\theta(E_{r+1} \ldots E_{j-1})e_{j1}$. Consider the edges labelled $A$ and $E_j$ in the graph $\Gamma_W$. Bisect $E_j$ into two new edges, the first new edge shall be denoted $E_{j1}$ and the second $E_{j2}$, where $E_{j1}, E_{j2}$ are elements of $A^{\pm 1}$ not occurring in $W$. Now remove edge $A$ and add a new edge, $A'$, joining $\iota(A)$ to $\tau(E_{j1})$. See Figure 11. Note that if $\tau(A)$ in $\Gamma_W$ has degree 3 then we also remove the edges $E_{r+1}$ and $E_s$ and add a new edge $E' \in A^{\pm 1}$ from $\iota(E_s)$ to $\tau(E_{r+1})$. We can see from the new graph that we have a regular Eulerian circuit $W'$. We know that $E_j$ occurs as $E_j^{-1}$ for some $i \neq j$, $1 \leq i \leq t$ in $W$ so $E_i$ is replaced by $E_{j2}^{-1}E_{j1}^{-1}$. But since these new edges remain together in the new Eulerian circuit $W'$ we shall ignore this as it does not effect the proof. Therefore the cyclic sequence of letters of $W'$ is either

$$E_1, \ldots, E_r, A', E_{j2}, \ldots, E_s, E_{r+1}, \ldots, E_{j1}, A'^{-1}, E_{s+1}, \ldots, E_t$$
if the degree of $\tau(A)$ in $\Gamma_W$ is greater than three or

$$E_1, \ldots, E_r, A', E_{j_2}, \ldots, E_{s-1}, E', E_{r+2} \ldots, E_{j_1}, A'^{-1}, E_{s+1}, \ldots, E_t,$$

otherwise. Suppose that the numbers of vertices and edges of $\Gamma_W$ are $v$ and $e$ respectively. If the degree of $\tau(A)$ in $\Gamma_W$ is greater than three then the numbers of vertices and edges in the new graph are $v + 1$ and $e + 1$ respectively. If the degree of $\tau(A)$ is three then the numbers of vertices and edges in the new graph are $v$ and $e$ respectively. In both cases it is easy to check, using equation (1), that the new graph also has genus $n$ and has no vertices of degree 1 or 2. Thus $W'$ is a genus $n$ Wicks form. Now we shall define a labelling function for $W'$. First consider the case where $\tau(A)$ has degree greater than 3. Let $L = \{E_1, \ldots, E_{j-1}, E_{j_1}, E_{j_2}, E_{j+1}, \ldots, E_t, A'\}$. We define a homomorphism $\psi: F(L) \rightarrow F(X)$ in the following way.

$$
\psi(E) = \begin{cases} 
  a_1 t_1 & \text{if } E = A' \\
  e_{j_1} & \text{if } E = E_{j_1} \\
  e_{j_2} & \text{if } E = E_{j_2} \\
  \theta(E) & \text{otherwise}
\end{cases}
$$
Therefore,

\[
\psi(W') =_H \psi(E_1 \ldots E_r A_1 A_2 \ldots E_s E_{r+1} \ldots E_{j_1} A_{r-1} E_{s+1} \ldots E_t)
\]

\[
=_H \psi(E_1 \ldots E_r) \psi(A_1) \psi(E_{j_2}) \psi(E_{j_1} \ldots E_s E_{r+1} \ldots E_{j-1})
\]

\[
\psi(E_{j_1}) \psi(A_1)^{-1} \psi(E_{s+1} \ldots E_t)
\]

\[
= H \theta(E_1 \ldots E_r) a_1 t_1 e_j \theta(E_{j_1} \ldots E_s E_{r+1} \ldots E_{j-1}) e_{j_1} t_1^{-1} a_1^{-1} \theta(E_{s+1} \ldots E_t)
\]

\[
= H \theta(E_1 \ldots E_r) a_1 a_2 \theta(E_{r+1} \ldots E_{j-1}) e_{j_1} e_{j_2} \theta(E_{j_1} \ldots E_s E_{r+1} \ldots E_{j-1})
\]

\[
e_{j_1} e_{j_1}^{-1} \theta(E_{r+1} \ldots E_{j-1})^{-1} a_2^{-1} a_1^{-1} \theta(E_{s+1} \ldots E_t)
\]

\[
= H \theta(E_1 \ldots E_r) a_1 a_2 \theta(E_{r+1} \ldots E_{j-1}) e_{j_1} e_{j_2} \theta(E_{j_1} \ldots E_s E_{r+1} \ldots E_{j-1}) a_2^{-1} a_1^{-1} \theta(E_{s+1} \ldots E_t)
\]

\[
= H \theta(E_1 \ldots E_r A_{r+1} \ldots E_s A_{r-1} E_{s+1} \ldots E_t)
\]

This implies that \(\psi(W')\) is conjugate to \(h\) in \(H\). Thus \((W', \psi)\) is an element of \(\mathcal{F}\). Now since \(|\theta(W)|\) was chosen to be minimal over all pairs in \(\mathcal{F}\), it follows that

\[
|\psi(W')| \geq |\theta(W)|. \tag{2}
\]

Also, we can see from Figure 11 that

\[
|\theta(W)| = |\psi(W')| - 2|\psi(A')| - 2|\psi(E_{j_1})| - 2|\psi(E_{j_2})| + 2|\theta(A)| + 2|\theta(E_j)|
\]

\[
= |\psi(W')| - 2|a_1 t_1| - 2|e_{j_1}| - 2|e_{j_2}| + 2|a_1 a_2| + |e_{j_1} e_{j_2}|
\]

\[
\geq |\psi(W')| - 2|t_1| + 2|a_2|. \tag{3}
\]

Therefore, by equations 2 and 3, it is clear that

\[
|a_2 \theta(E_{r+1} \ldots E_{j-1}) e_{j_1}|_H = |t_1|
\]

\[
\geq |a_2|.
\]

as required.

Now suppose that the degree of \(\tau(A)\) in \(\Gamma_W\) is three. Let

\[
L' = \{E_1, \ldots, E_r, E_{r+2}, \ldots, E_{j-1}, E_{j_1}, E_{j_2}, E_{j_1+1}, \ldots, E_{s-1}, E_{s+2}, \ldots, E_t, A', E'\}
\]

We define a homomorphism \(\psi' : F(L') \rightarrow F(X)\) in the following way.

\[
\psi'(E) = \begin{cases} 
  a_1 t_1 & \text{if } E = A' \\
  e_{j_1} & \text{if } E = E_{j_1} \\
  e_{j_2} & \text{if } E = E_{j_2} \\
  \theta(E_s E_{r+1}) & \text{if } E = E' \\
  \theta(E) & \text{otherwise}
\end{cases}
\]

We can use the same argument to show that \((W', \psi') \in \mathcal{F}\) and again, since \(|\theta(W)|\) was chosen to be minimal over all pairs in \(\mathcal{F}\), it follows that

\[
|\psi'(W')| \geq |\theta(W)|. \tag{4}
\]
Also, we know that

\[ |\theta(W)| = |\psi'(W')| - 2|\psi'(A')| - 2|\psi'(E')| - 2|\psi'(E_{j1})| - 2|\psi'(E_{j2})| \\
+ 2|\theta(A)| + 2|\theta(E_{r+1})| + 2|\theta(E_s)| + 2|\theta(E_j)| \\
= |\psi'(W')| - 2a_1t_1 - 2|\theta(E_sE_{r+1})| - 2|e_{j1}| - 2|e_{j2}| \\
+ 2a_1a_2 + 2|\theta(E_{r+1})| + 2|\theta(E_s)| + 2|e_{j1}e_{j2}| \\
\geq |\psi'(W')| - 2t_1 + 2a_2. \tag{5} \]

Therefore, by equations (4) and (5), it is clear that

\[ |a_2\theta(E_{r+1} \ldots E_{j-1})e_{j1}|_H = |t_1| \]

\[ \geq |a_2|. \]

as required. Hence in both cases (ii) holds.

The same argument, using \((W^{-1}, \theta) \in \mathcal{F}\), can be used to show that (i) holds. Therefore we only need to consider case (iii). Let \(t_2\) be a word in \(F(X)\) which is minimal in \(H\) such that

\[ t_2 = H a_2 \theta(E_{r+1} \ldots E_s)a_2^{-1}a_1^{-1}\theta(E_{s+1} \ldots E_{k-1})e_{k1}. \tag{6} \]

Again we follow the same method of altering the graph \(\Gamma_W\). Bisect the edge labelled \(E_k\).

The first half shall be labelled by \(E_{k1}\) and the second half labelled by \(E_{k2}\), where \(E_{k1}, E_{k2}\) are elements of \(A^{\pm 1}\) not occurring in \(W\). Again, we shall remove the edge \(A\) but now we add a new edge \(A''\) joining \(\tau(E_{k1})\) to \(\tau(A)\), with \(A'' \in A^{\pm 1}\). See Figure 12. If \(\iota(A)\) in \(\Gamma_W\) has degree 3 then we also remove the edges \(E_r\) and \(E_{s+1}\) and add a new edge \(E'' \in A^{\pm 1}\) from \(\iota(E_r)\) to \(\tau(E_{s+1})\). We can see from the new graph that we have a regular Eulerian circuit \(W''\). The cyclic sequence of letters of \(W''\) is either

\[ E_1, \ldots, E_r, E_{s+1}, \ldots, E_{k1}, A'', E_{r+1}, \ldots, E_s, A''^{-1}, E_{k2}, \ldots, E_t, \]

if the degree of \(\iota(A)\) in \(\Gamma_W\) is greater than three, or

\[ E_1, \ldots, E_{r-1}, E'', E_{s+2}, \ldots, E_{k1}, A'', E_{r+1}, \ldots, E_s, A''^{-1}, E_{k2}, \ldots, E_t, \]

otherwise. Once again, it is easy to check that the genus of the new graph is \(n\) and it contains no vertices of degree 1 or 2. Thus \(W''\) is a genus \(n\) Wicks form. We shall define a labelling function for \(W''\). First consider the case where \(\iota(A)\) has degree greater than 3. Let \(K = \{E_1, \ldots, E_{k-1}, E_{k1}, E_{k2}, E_{k+1}, \ldots, E_t, A''\}\). We define a homomorphism \(\phi : F(K) \rightarrow F(X)\) in the following way:

\[
\phi(E) = \begin{cases} 
    a_2^{-1}t_2 & \text{if } E = A'' \\
    e_{k1} & \text{if } E = E_{k1} \\
    e_{k2} & \text{if } E = E_{k2} \\
    \theta(E) & \text{otherwise}
\end{cases}
\]
Therefore,

\[
\phi(W') = H \phi(E_1 \ldots E_r E_{s+1} \ldots E_{k_1} A''^{-1} E_{r+1} \ldots E_s A'' E_{k_2} \ldots E_t) \\
= H \phi(E_1 \ldots E_r E_{s+1} \ldots E_{k-1}) \phi(E_{k1}) \phi(A'')^{-1} \phi(E_{r+1} \ldots E_s) \\
\phi(A'') \phi(E_{k2}) \phi(E_{k+1} \ldots E_t) \\
= H \theta(E_1 \ldots E_r E_{s+1} \ldots E_{k-1}) e_{k1} t_2^{-1} a_2 \theta(E_{r+1} \ldots E_s) a_2^{-1} t_2 e_{k2} \theta(E_{k+1} \ldots E_t) \\
= H \theta(E_1 \ldots E_r E_{s+1} \ldots E_{k-1}) e_{k1} \theta(E_{s+1} \ldots E_{k-1})^{-1} a_1 a_2 \theta(E_{r+1} \ldots E_s) a_2^{-1} a_2^{-1} a_2 \\
\theta(E_{r+1} \ldots E_s) a_2^{-1} a_2 \theta(E_{r+1} \ldots E_s) a_2^{-1} a_1^{-1} \theta(E_{s+1} \ldots E_{k-1}) e_{k1} e_{k2} \theta(E_{k+1} \ldots E_t) \\
= H \theta(E_1 \ldots E_r E_{s+1} \ldots E_s) a_2^{-1} a_1^{-1} \theta(E_{s+1} \ldots E_{k-1}) e_{k1} e_{k2} \theta(E_{k+1} \ldots E_t) \\
= H \theta(E_1 \ldots E_r A E_{r+1} \ldots E_s A^{-1} E_{s+1} \ldots E_t) \\
= H \theta(W).
\]

This implies that \( \phi(W'') \) is conjugate to \( h \) in \( H \). Thus \((W'', \phi)\) is an element of \( \mathcal{F} \). Now since \( |\theta(W)| \) was chosen to be minimal over all pairs in \( \mathcal{F} \), it follows that

\[
|\phi(W'')| \geq |\theta(W)|. \tag{7}
\]

Also, we can see from Figure 12 that

\[
|\theta(W)| = |\phi(W'')| - 2|\phi(A'')| - 2|\phi(E_{k1})| - 2|\phi(E_{k2})| + 2|\theta(A)| + 2|\theta(E_k)| \\
= |\phi(W'')| - 2|a_2^{-1} t_2| - 2|e_{k1}| - 2|e_{k2}| + 2|a_1 a_2| + |e_{k1} e_{k2}| \\
\geq |\phi(W'')| - 2|t_2| + 2|a_1|. \tag{8}
\]
Therefore, by equations (7) and (8), it is clear that
\[
|a_2\theta(E_{r+1} \ldots E_s)a_2^{-1}a_1^{-1}\theta(E_{s+1} \ldots E_{k-1})e_{k_1}|_H = |t_2| \\
\geq |a_1|.
\]
as required. Now Suppose that \(\iota(A)\) has degree three in \(\Gamma_W\). Let
\[
K' = \{E_1, \ldots, E_{r-1}, E_{r+1}, \ldots, E_s, E_{s+1}, \ldots, E_{k-1}, E_{k+1}, E_{k+2}, \ldots, E_t, A'', E''\}.
\]
We define a homomorphism \(\phi' : F(K') \to F(X)\) in the following way.
\[
\phi'(E) = \begin{cases} 
  a_2^{-1}t_2 & \text{if } E = A'' \\
  e_{k_1} & \text{if } E = E_{k_1} \\
  e_{k_2} & \text{if } E = E_{k_2} \\
  \theta(E, E_{s+1}) & \text{if } E = E'' \\
  \theta(E) & \text{otherwise}
\end{cases}
\]
Again, we can use the same argument to show that \((W'', \phi') \in \mathcal{F}\) and again, since \(|\theta(W)|\) was chosen to be minimal over all pairs in \(\mathcal{F}\), it follows that
\[
|\phi'(W'')| \geq |\theta(W)|. \tag{9}
\]
Also, we know that
\[
|\theta(W)| = |\phi'(W'')| - 2|\phi'(A'')| - 2|\phi'(E'')| - 2|\phi'(E_{k_1})| - 2|E_{k_2}| \\
+ 2|\theta(A)| + 2|\theta(E_r)| + 2|\theta(E_{s+1})| + 2|\theta(E_k)| \\
= |\phi'(W'')| - 2|a_2^{-1}t_2| - 2|\theta(E_r, E_{s+1})| - 2|e_{k_1}| - 2|e_{k_2}| \\
+ 2|a_1a_2| + 2|\theta(E_r)| + 2|\theta(E_{s+1})| + 2|e_{k_1}e_{k_2}| \\
\geq |\phi'(W'')| - 2|t_2| + 2|a_1|. \tag{10}
\]
Therefore, by equations (9) and (10), it is clear that
\[
|a_2\theta(E_{r+1} \ldots E_s)a_2^{-1}a_1^{-1}\theta(E_{s+1} \ldots E_{k-1})e_{k_1}|_H = |t_2| \\
\geq |a_1|.
\]
as required. Hence in both cases (iii) holds.

Suppose that for each letter \(E\) of the Wicks form \(W\) we have \(|\theta(E)| \leq 12l + M + 4\). By the following lemma the maximum length of a genus \(n\) Wicks form is \(12n - 6\). For a proof of this lemma see M. Culler [6] Theorem 3.1.

**Lemma 4.2.** Let \(V\) be a Wicks form in the alphabet \(A^{\pm 1}\) such that \(\text{genus}_{F(A)} = m\). Then the length of \(V\) is at most \(12m - 6\).
It clearly follows that we have part 1 of the Theorem. Therefore, we shall assume that there is at least one letter of $W$ which is labelled by a word of length greater than $12l + M + 4$ in $F(X)$ (This of course implies that there are two since each letter appears twice.) For convenience in the proof we shall take $\hat{W}$ to be a cyclic permutation of $W$ such that the last letter of $\hat{W}$ is labelled by a word of length greater than $12l + M + 4$ in $F(X)$ but one should note that the proof does go through using any cyclic permutation.

Consider $\theta(\hat{W})$ as a path in the Cayley graph $\Gamma_X(H)$. Let $F$ and $R$ be words in $F(X)$ which are minimal in $H$ such that $F =_H \theta(\hat{W})$ and $h =_H RFR^{-1}$. See Figure 13. Suppose that $\alpha$ is the label on one of the letters of $\hat{W}$ with $|\alpha| > 12l + M + 4$. Since the word $\alpha$ is a minimal in $H$, the path in the Cayley graph above labelled by $\theta(\hat{W})$ contains a geodesic subpath labelled by $\alpha$.

**Lemma 4.3.** There exists a vertex $v$ on the geodesic path labelled by $F$ such that $d(\tau(\alpha), v) \leq 5l + M + 3$.

**Proof.** We shall assume that the letter of $\hat{W}$ labelled by $\alpha$ in the $\Gamma_X(H)$ appears before its inverse i.e. $\hat{W} = \ldots A \ldots A^{-1} \ldots$ and $\theta(A) = \alpha$. It is easy to show that the same proof follows through for the converse. Let $p_1$ and $q_1$ be vertices on $\alpha$ such that $d(\iota(\alpha), p_1) = d(q_1, \tau(\alpha)) = 2l + 1$. (Note that we could use smaller segments of $\alpha$ of length $l + 1$ here to prove this lemma but we require this set up for Lemma 4.5 to hold.) See Figure 14. By Lemma 3.2 part 1, there are vertices $p_2$ and $q_2$, which lie either on the path labelled by $\theta(\hat{W}) - \{\alpha\}$ or on the geodesic path labelled by $F$, such that

$$d(p_1, p_2), d(q_1, q_2) \leq \delta(\log_2(|\hat{W}|)) \leq \delta(\log_2(12n - 6) + 1) \leq l,$$

Suppose that $p_2$ lies on some geodesic path $\beta$ which is the label of some letter $B$ in $\hat{W}$ which is different to $A$. Lemma 4.1 implies that $p_1$ is within $l$ of $\iota(\alpha) \cup \tau(\alpha)$ but we have...
chosen $p_1$ such that this is not the case. Therefore $p_2$ and similarly $q_2$ can only lie on $\alpha^{-1} \cup F$. Also, if $p_2$ lies on $\alpha^{-1}$ then by Lemma 3.2 part 2, $q_2$ lies on $\alpha^{-1}$. This leaves us with just three possibilities. See Figure 15.

(i) Suppose that both $p_2$ and $q_2$ lie on $F$. By the triangle inequality it immediately follows that

$$d(\tau(\alpha), q_2) \leq d(\tau(\alpha), q_1) + d(q_1, q_2) \leq 3l + 1.$$ 

Therefore in this case the lemma holds.

(ii) Suppose that both $p_2$ and $q_2$ lie on $\alpha^{-1}$. Let $q_3$ be the vertex lying on $\alpha^{-1}$ such that $d(\iota(\alpha^{-1}), q_3) = 2l + 1$. We need the following lemma.

**Lemma 4.4.** Let $x_1$ and $x_2$ be any vertices on $\alpha$ and $\alpha^{-1}$ respectively such that $d(x_1, x_2) \leq k$ for some constant $k$. If $x_3$ is a vertex on $\alpha^{-1}$ such that $d(\iota(\alpha^{-1}), x_3) = d(\tau(\alpha), x_1)$ then $d(x_2, x_3) \leq k$.

**Proof.** The proof falls into the following two cases:

(a) $d(\iota(\alpha^{-1}), x_2) \leq d(\iota(\alpha^{-1}), x_3) = d(\tau(\alpha), x_1)$

(b) $d(\iota(\alpha^{-1}), x_2) > d(\iota(\alpha^{-1}), x_3) = d(\tau(\alpha), x_1)$

(a) By Lemma 4.1 $d(\tau(\alpha), x_1) \leq d(x_1, \iota(\alpha^{-1}))$ and from the hypothesis and the triangle inequality it follows that

$$d(x_1, \iota(\alpha^{-1})) \leq k + d(\iota(\alpha^{-1}), x_2). \quad (11)$$

Since $\alpha$ is a geodesic path we have

$$d(\tau(\alpha), x_1) = d(\iota(\alpha^{-1}), x_3) = d(\iota(\alpha^{-1}), x_2) + d(x_2, x_3). \quad (12)$$

It follows from equations (11) and (12) that $d(x_2, x_3) \leq k$. 

Figure 14
(b) By Lemma 4.1 \(d(\epsilon(\alpha^{-1}), x_2) \leq d(x_2, \tau(\alpha))\) and from the hypothesis and the triangle inequality it follows that

\[
d(x_2, \tau(\alpha)) \leq k + d(\tau(\alpha), x_1).
\]

(13)

Since \(\alpha\) is a geodesic path we have

\[
d(\tau(\alpha), x_1) = d(\epsilon(\alpha^{-1}), x_3) = d(\epsilon(\alpha^{-1}), x_2) - d(x_2, x_3).
\]

(14)

It follows from equations (13) and (14) that \(d(x_2, x_3) \leq k\). Hence the lemma holds.

Returning to case (ii), the lemma above implies that \(d(q_2, q_3) \leq l\). Thus \(d(q_1, q_3) \leq 2l\). Similarly, if \(p_3\) is the vertex on \(\alpha^{-1}\) such that \(d(\tau(\alpha^{-1}), p_3) = 2l + 1\), we can follow the same argument to show that \(d(p_1, p_3) \leq 2l\).

Let the segment of \(\alpha\) from \(p_1\) to \(q_1\) be labelled by \(\alpha_1\). Therefore the segment on \(\alpha^{-1}\) from \(q_3\) to \(p_3\) is labelled \(\alpha_1^{-1}\). Let \(s\) and \(t\) be geodesic paths from \(p_1\) and \(q_1\) to \(p_3\) and \(q_3\) respectively. We have shown such paths to have length at most \(2l\). See Figure 16.
\(\theta(W)\) was chosen to be of shortest length over all pairs in \(F\), we can use Lemma 3.1 to show that

\[
|\alpha_1| \leq \frac{1}{2}(|s| + |t|) + M + 1 \\
\leq \frac{1}{2}(2l + 2l) + M + 1 \\
\leq 2l + M + 1.
\]

Therefore it follows that

\[
|\alpha| = d(\iota(\alpha), p_1) + |\alpha_1| + d(q_1, \tau(\alpha)) \\
\leq 2l + 1 + 2l + M + 1 + 2l + 1 \\
\leq 6l + M + 3.
\]

We have a contradiction. Hence this case can’t occur.

(iii) Suppose that \(p_2\) lies on \(F\) and \(q_2\) lies on \(\alpha^{-1}\). See Figure 15. Again let \(q_3\) be the vertex on \(\alpha^{-1}\) such that \(d(\iota(\alpha^{-1}), q_3) = 2l + 1\). As in case (ii) we can show that \(d(q_1, q_3) \leq 2l\).

As before, by Lemma 3.2 and Lemma 4.4 for each vertex \(u\) of \(\alpha\) lying between \(p_1\) and \(q_1\), there exists a vertex \(v\) on \(F \cup \alpha^{-1}\) such that \(d(u, v) \leq l\). Let \(u_1\) be the first vertex along \(\alpha\) which is within \(l\) of a vertex \(v_1\) on \(\alpha^{-1}\). By Lemma 3.2 part 2 vertex \(u_1\) clearly lies between \(p_1\) and \(q_1\) on \(\alpha\). Let \(v_2\) be the vertex on \(\alpha^{-1}\) such that \(d(\iota(\alpha^{-1}), v_2) = d(\tau(\alpha), u_1)\). By Lemma 4.4 it follows that \(d(v_1, v_2) \leq l\). Thus \(d(u_1, v_2) \leq 2l\).

Let the segment of \(\alpha\) from \(u_1\) to \(q_1\) be labelled by \(\alpha_2\). Therefore the segment on \(\alpha^{-1}\) from \(q_3\) to \(v_2\) is labelled \(\alpha_2^{-1}\). Let \(s'\) and \(t'\) be geodesic paths from \(p_1\) and \(u_1\) to \(p_3\) and \(v_2\) respectively. We have shown such paths to have length at most \(2l\). Again, since \(\theta(W)\) was chosen to be of shortest length in \(F\), we can use Lemma 3.2 to show that

\[
|\alpha_2| \leq \frac{1}{2}(|s'| + |t'|) + M + 1 \\
\leq \frac{1}{2}(2l + 2l) + M + 1 \\
\leq 2l + M + 1.
\]
Let $u_2$ be the vertex on $\alpha$ such that $d(\iota(\alpha), u_1) = d(\iota(\alpha), u_2) + 1$. Since $u_1$ was chosen to be the first vertex along $\alpha$ which was within $l$ of a vertex on $\alpha^{-1}$, there exists a vertex $v_3$ on $F$ such that $d(u_2, v_3) \leq l$. See Figure 17. Clearly it follows that 

$$d(\tau(\alpha), v_3) \leq d(\tau(\alpha), q_1) + |\alpha_2| + 1 + d(u_2, v_3)$$

$$\leq 2l + 1 + 2l + M + 1 + 1 + l$$

$$\leq 5l + M + 3.$$

Hence the Lemma holds in all cases. 

Consider all letters of $\hat{W}$ which have labels of length greater than $12l + M + 4$ in $F(X)$. We shall call these the long edges of $\hat{W}$. All other letters shall be called short edges. The terminal vertex of each long edge in $\Gamma_X(H)$ has each been shown, in the previous lemma, to be within $5l + M + 3$ of some vertex on $F$. Let $B$ be a long edge of $\hat{W}$ which is not the first long edge in the sequence of letters. Since $\hat{W}$ is quadratic, $B$ appears twice, once with exponent 1 and once with exponent $-1$. First we shall consider the appearance of $B$ with exponent 1. In the sequence of letters of $\hat{W}$, let $A \pm 1$ be the long edge before $B$ in the sequence such that no long edge appears between $A \pm 1$ and $B$ (note that $A^{\pm 1}$ could be $B^{-1}$). By Lemma 4.3, there exist vertices $u$ and $v$ on $F$ such that $d(\tau(\theta(A^{\pm 1})), u), d(\tau(\theta(B)), v) \leq 5l + M + 3$. See Figure 18.

**Lemma 4.5.** We can choose $u$ and $v$ such that $d(\iota(F), u) < d(\iota(F), v)$.

**Proof.** Suppose that $d(\iota(F), u) \geq d(\iota(F), v)$. From the proof of Lemma 4.3, there exist vertices $u'$ and $v'$ lying on $\theta(A^{\pm 1})$ and $\theta(B)$ respectively such that we have the following inequalities.

1. $d(u', u), d(v', v) \leq l$;
2. $2l + 1 \leq d(u', \tau(\theta(A^{\pm 1}))), d(v', \tau(\theta(B))) \leq 4l + M + 3$. 

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In this proof \( u \) and \( v \) were chosen in \( \Gamma_X(H) \) from \( u' \) and \( v' \) using part 1. of Lemma 3.2. Let \( q \) be the closed path labelled by the cyclic word \( \theta(\hat{V})F^{-1} \) in \( F(X) \) starting at \( \iota(\theta(A^{\pm 1})) \). Now we can use Lemma 3.3 to show that there exists another vertex \( v'' \) on \( q \) such that \( d(v', v'') \leq 2l \) and

\[
d_q(\iota(\theta(A^{\pm 1})), u) > d_q(\iota(\theta(A^{\pm 1})), v'') > d_q(\iota(\theta(A^{\pm 1})), u').
\]

Since \( d(v', \tau(B)) \geq 2l + 1 \), \( v'' \) must lie on \( F \cup \theta(B^{-1}) \) or Lemma 4.1 would be violated.

We need to consider two cases.

**Case 1** (\( v'' \) lies on \( F \))

See Figure 19. Using the triangle inequality we have the following inequalities.

\[
d(v, u) + d(u, v'') \leq 3l; \quad (15)
\]

\[
d(u', v') \leq 2l + d(v, u); \quad (16)
\]

\[
d(u', v') \leq 3l + d(u, v''). \quad (17)
\]

We can combine these to give \( 2d(u', v') \leq 5l + d(v, u) + d(u, v'') \leq 8l \) which implies that \( d(u', v') \leq 4l \).

We need to consider two possibilities. First suppose that \( A^{\pm 1} \neq B^{-1} \). Then by Lemma 4.1 it follows that \( d(\iota(\theta(B)), v') \leq d(u', v') \leq 4l \) and from 2 above

\[
|\theta(B)| = d(\iota(\theta(B)), v') + d(v', \tau(\theta(B))) \leq 4l + 4l + M + 3 \leq 8l + M + 3.
\]
Since $B$ is a long edge this is a contradiction.

Now suppose that $A \pm 1 = B - 1$. We can now strengthen statement 2 above to

$$2l + 1 \leq d(u', \tau(\theta(B^{-1}))) \leq 4l + M + 3 \quad (18)$$

and

$$d(v', \tau(\theta(B))) = 2l + 1. \quad (19)$$

We can choose $v'$ to be this vertex since this vertex is within $l$ of a vertex on $F \cup \theta(B - 1)$ (again see Lemmas 3.2 and 4.1) and if it lies on $\theta(B - 1)$ we would have case (ii) of the proof of Lemma 4.3, which we have already shown cannot occur.

Now consider the vertex $u''$ on $\theta(B^{-1})$ such that $d(\iota(\theta(B)), u'') = d(\iota(\theta(B^{-1})), u') + 1$. If $u$ was chosen by case (iii) of Lemma 4.3 then $u''$ is within $l$ of a vertex $v'''$ on $\theta(B)$, otherwise $u$ was chosen using case (i) of Lemma 4.3 and $d(u'', \tau(\theta(B^{-1}))) = 2l$. First consider the latter case. Let $E$ be the first letter after $B^{-1}$ in $\hat{W}$. By Lemma 4.1, a geodesic path from $\tau(\theta(B))$ to $\iota(\theta(E)) = \tau(\theta(B^{-1}))$ has length greater than $|\theta(B)|$. But by the triangle inequality

$$d(\tau(\theta(B)), \tau(\theta(B^{-1}))) \leq d(\tau(\theta(B)), v') + d(v', u') + d(u', u'' + d(u'', \tau(\theta(B^{-1}))))$$

$$\leq 2l + 1 + 4l + 1 + 2l$$

$$\leq 8l + 2.$$

This implies that $|\theta(B)| \leq 8l + 2$, but $B$ is a long edge so this is a contradiction. Therefore assume that $u''$ is within $l$ of a vertex $v'''$ on $\theta(B)$. Let $p$ be the vertex on $\theta(B)$ such that $d(\iota(\theta(B)), p) = d(\iota(\theta(B^{-1})), u'')$. By Lemma 4.4 $d(p, v''') \leq l$. Therefore, $d(p, u'') \leq 2l$. Thus we have all the bounded distances shown in Figure 20. From this it is clear that

$$d(v', p) \leq 4l + 1 + 2l = 6l + 1.$$ Since $\theta(B)$ is a geodesic path, it follows that

$$|\theta(B)| = d(\iota(\theta(B)), p) + d(p, v') + d(v', \tau(\theta(B)))$$

$$\leq d(\tau(\theta(B^{-1})), u'') + 6l + 1 + 2l + 1$$

$$\leq 4l + M + 2 + 6l + 1 + 2l + 1$$

$$\leq 12l + M + 4.$$
But $B$ is a long edge so this cannot occur.

**Case 2** ($v''$ lies on $\theta(B^{-1})$)

We can see from Lemma 3.3 that this case falls into two subcases.

(i) $B^{-1} = A^{\pm 1}$ and $d(\iota(\theta(B^{-1})), v'') > d(\iota(\theta(B^{-1})), u')$

(ii) $B^{-1}$ lies after $B$ in the sequence of letters of $\hat{W}$.

We first consider subcase (i). See Figure 21. As in the previous case with $A^{\pm 1} = B^{-1}$ (see equations (18) and (19)) we can strengthen statement 2 on page 23 to

$$2l + 1 \leq d(u', \tau(\theta(B^{-1}))) \leq 4l + M + 3$$

and

$$d(v', \tau(\theta(B))) = 2l + 1.$$  

It follows that $d(v'', \tau(\theta(B^{-1}))) \leq 4l + M + 3$. Consider a geodesic path $w$ from $v'$ to $\tau(\theta(B^{-1}))$. Lemma 4.1 implies that $|w| \geq d(\iota(\theta(B)), v')$ and by the triangle inequality we have that

$$|w| \leq d(v', v'') + d(v'', \tau(\theta(B^{-1})))$$

$$\leq 2l + 4l + M + 3$$

$$= 6l + M + 3.$$  

It follows that

$$|\theta(B)| = d(\iota(\theta(B)), v') + d(v', \tau(\theta(B)))$$

$$\leq 6l + M + 3 + 2l + 1$$

$$\leq 8l + M + 4.$$  

But $B$ is a long letter so we have a contradiction. Therefore this subcase can’t occur.

Now consider (ii). See Figure 22. Consider the vertex $p$ which lies on $\theta(B)$ such that $d(\iota(\theta(B)), p) = 2l + 1$. Let $q'$ be the closed path labelled by the cyclic word $\theta(\hat{W})^{-1}F$ in
$F(X)$ starting at $\tau(\theta(B))$. Lemma 3.2 part 1 and part 2 imply that there is a vertex $p'$ on the closed path $q'$ such that

$$d(p, p') \leq l$$

and

$$d_{q'}(\tau(\theta(B)), v) > d_{q'}(\tau(\theta(B)), p').$$  \hfill (20)

It follows from Lemma 4.1 that $p'$ must lie on $F$ (It can’t lie on $\theta(B^{-1})$ or equation (20) would not hold). This also means that we have part 3 of Lemma 3.2, i.e.

$$d(p', v) = d(p, v').$$ \hfill (21)

See Figure 23. Now consider the closed path $q$ again. Lemma 3.3 implies that there exists a vertex $p''$ on $q$ such that $d(p, p'') \leq 2l$ and

$$d_q(\iota(\theta(A^{pm1})), u) > d_q(\iota(\theta(A^{pm1})), p'') > d_q(\iota(\theta(A^{pm1})), u').$$

Since $d(p, \iota(\theta(B))) = 2l + 1$ and $d(p, \tau(\theta(B))) > 2l + 1$, it follows from Lemma 4.1 that $p''$ lies on either $F$ or $\theta(B^{-1})$. First consider the case where $p''$ lies on $F$. We have the bounded distances as shown in Figure 24. Using the triangle inequality we have the following equations.

$$d(p', v) + d(v, u) + d(u, p'') \leq 3l,$$

$$d(u', v') \leq d(v, u) + 2l,$$

$$d(u', v') \leq d(u, p'') + 3l + d(p, v').$$
It follows from these equations and equation (21) that
\[
2d(u', v') \leq d(u, p'') + 3l + d(p, v') + d(v, u) + 2l \\
= d(u, p'') + 3l + d(p', v) + d(v, u) + 2l \\
\leq 8l \\
\implies d(u', v') \leq 4l.
\]

Lemma \[4.1\] implies that \(d(v', \iota(\theta(B))) \leq d(v', u') \leq 4l\). Thus
\[
|\theta(B)| = d(\iota(\theta(B)), v') + d(v', \tau(\theta(B))) \leq 4l + 4l + M + 3.
\]

But \(B\) is a long letter so we have a contradiction.

Finally we need to consider the case where \(p''\) lies on \(\theta(B^{-1})\). We have the bounded distances as shown in Figure 25. We know by the choice of vertex \(p\) that \(d(\iota(\theta(B)), p) = 2l + 1\). The vertex \(v'\) was chosen from either case (i) or case (iii) of Lemma \[4.3\]. If it was chosen by case (i) then \(d(v', \tau(\theta(B))) = 2l + 1\). If it was chosen by case (iii) then there exists a vertex \(v'''\) on \(\theta(B)\) such that \(d(v''', \tau(\theta(B))) = 2l + 1\) and \(v'''\) is within a distance \(l\) of a vertex on \(\theta(B^{-1})\). Either way there is vertex \(x\) which is on the \(\theta(B)\) at a distance \(2l + 1\) from \(\tau(\theta(B))\) which is within \(2l\) of a vertex \(x'\) on \(\theta(B^{-1})\). See Figure 26. It follows from Lemma \[4.4\] that there exist vertices \(x''\) and \(p'''\) lying on \(\theta(B^{-1})\) such that we have...
the following equations.

\[
\begin{align*}
    d(x, \tau(\theta(B))) &= d(x'', \tau(\theta(B^{-1}))), \\
    d(p, \tau(\theta(B))) &= d(p'', \tau(\theta(B^{-1}))), \\
    d(x', x'') &\leq 2l, \\
    d(p'', p''') &\leq 2l.
\end{align*}
\]  

Equations (22) and (23) imply that \(d(x, x'') \leq 4l\) and \(d(p, p''') \leq 4l\). See Figure 27. By

\[
\begin{align*}
    \left|\theta(B)\right| &\leq 2l + 1 + 2l + 4l + M + 3.
\end{align*}
\]

It follows that \(\left|\theta(B)\right| \leq 2l + 1 + 2l + 1 + 4l + M + 1 \leq 8l + M + 3\). But \(B\) is a long letter so we have a contradiction. Therefore this case can’t occur.

Hence we have the required result. \(\square\)

We shall label the segment of \(F\) between \(u\) and \(v\) by \(B_1\) and a geodesic path from \(\tau(\theta(B))\) to \(v\) by \(b_1\). If we were considering \(B^{-1}\) we would label the appropriate paths \(B_2\) and \(b_2\) respectively. Remember the last letter of \(\hat{W}\) was chosen to be long, thus if \(B\) is the last letter then \(v = \tau(\theta(B))\) and \(|b_1| = 0\). We shall do this for every long edge in \(\hat{W}\).

**Example 4.6.** Suppose that \(\hat{W} = ABCA^{-1}B^{-1}C^{-1}\) where \(A\) and \(C\) are long edges and \(B\) is a short edge. Then we label paths in \(\Gamma_X(H)\) as shown in Figure 28. Where \(|a_1|, |a_2|\) and \(|c_1|\) are at most \(5l + M + 3\) and \(|c_2| = 0\).
Let us write $\hat{W}$ around the boundary of a disc (i.e. divide the boundary up into $|\hat{W}|$ segments, assigning a letter to each) and identify the long edges, respecting orientation. We obtain a surface $S$ of genus $k \leq n$ with $Q$ holes. The boundary of the disc becomes a graph on this surface, we shall denote this graph $\Gamma_S$. This graph consists of short edges all of which are written around the boundary components and long edges all of which are properly embedded on the surface.

Separate the cyclic words in $A^{\pm 1}$ written around the $Q$ boundary components into $p$ disjoint sets, $B_1, \ldots, B_p$, with $p$ as large as possible as follows. Each set $B_i$ contains some $t_i$-tuple of cyclic words $(W^i_1, \ldots, W^i_{t_i})$, where $W^i_j$ is the path written around a boundary component of $S$, such that if a small edge $A$ of $\hat{W}$ appears in $W^i_j$, for some $j = 1, \ldots, t_i$ then $A^{-1}$ appears in $W^i_k$ for some $k = 1, \ldots, t_i$. We shall define the genus of $B_i$ to be equal to $g_i - t_i + 1$, for $i = 1, \ldots, p$.

**Lemma 4.7.** Let $g = \sum_{i=1}^p g_i$. For each $B_i$ containing a $t_i$-tuple $(W^i_1, \ldots, W^i_{t_i})$ we have

$$\text{genus}_H(\theta(W^i_1), \theta(W^i_2), \ldots, \theta(W^i_{t_i})) = \text{genus}_{F(A)}(W^i_1, \ldots, W^i_{t_i}), \quad \text{for } i = 1, \ldots, p.$$ 

It follows that $g + k = n$.

**Proof.** Suppose that this is not the case. Clearly the genus of $(W^i_1, \ldots, W^i_{t_i})$ in $F(A)$, which we shall denote $h_i - t_i + 1$, is at least $g_i - t_i + 1$ for all $i = 1, \ldots, p$. Therefore assume that $h_j > g_j$ for some $1 \leq j \leq p$. Now, since $W$ has genus $n$ in $F(A)$, if we identify all the short edges on the genus $k$ surface $S$, respecting orientation, we obtain a closed compact surface of genus $n$. Therefore,

$$n = k + \sum_{i=1}^p h_i > k + \sum_{i=1}^p g_i.$$ 

But this implies that $\text{genus}_H(h) = \text{genus}_H(\theta(W)) \leq k + \sum_{i=1}^p g_i < n$, a contradiction. Thus $g_i = h_i$ for all $i = 1, \ldots, p$ and $n = k + \sum g_i = k + g$. Hence the lemma holds.

Consider the surface $S$ with $Q$ holes with the embedded graph $\Gamma_S$ consisting of long and short edges. If we paste a disc onto each of the $Q$ boundary components and contract
the cyclic word of short edges to a point, we obtain a graph of genus $k$ consisting of long edges only on a closed compact surface of genus $k$. The orientable word $U$ associated with this graph is obviously the word obtained by setting all the short edges of $W$ to 1. As usual we denote this graph $\Gamma_U$. We should note that, by the way the geodesic path labelled by $F$ in $\Gamma_X(H)$ has been cut up into segments, the cyclic sequence of letters of $U$ gives the cyclic sequence of segments of $F$ by replacing a letter $E$ of $U$ with $E_1$ for its occurrence with exponent 1 and $E_2^{-1}$ for its occurrence with exponent $-1$. We now show that an extension over $H$ may be carried out on the orientable word $U$, so that part 2 of the theorem holds.

4.1 Extension of $U$

First we shall construct $p+1$ sets, $C_0, C_1, \ldots, C_p$, of cyclic words in $F(X)$ such that each vertex of $\Gamma_U$ will be extended by a unique element from one of these sets.

Consider a cyclic word $W^j_i \in B_i$, $1 \leq j \leq t_i$ and $1 \leq i \leq p$, in the graph $\Gamma_S$. Thus, if we are thinking of $\Gamma_S$ as being embedded in $S$, $W^j_i$ is a word in the short edges written around a boundary component of the surface. Suppose that there are $d$ end points of long edges lying on this boundary component. Let $E^1, \ldots, E^d$ be the long edges. See Figure 29 Here $W^j_i = W^i_{j_1} \cdots W^i_{j_d}$, where each $W^i_{j_k}$ is a subword which may or may not have

\[
(\xi_1)1 W^i_{j_1} (\xi_2)2 W^i_{j_2} (\xi_3)3 W^i_{j_3} \cdots (\xi_{d-1})d-1 W^i_{j_{d-1}} (\xi_d)1 W^i_{j_d}
\]

length zero. Also, note that the $E^i$’s are not necessarily distinct (both end points may lie on the same boundary component).

Since $U$ is an orientable word, each vertex of $\Gamma_U$ is regular. Therefore, since $\Gamma_U$ is obtained from $\Gamma_S$ by pasting each boundary component of the surface $S$ with a disc and contracting the edges around the boundary components to a point, we may think of the boundary components also as being ‘regular’. That is, we may renumber the long edges, as shown in the diagram, such that the following cyclic subwords appear in $W$.

\[
(E^1)\xi_1 W^i_{j_1} (E^2)^{-\xi_2} W^i_{j_2} (E^3)^{-\xi_3} \cdots (E^{d-1})^{\xi_{d-1}} W^i_{j_{d-1}} (E^d)^{-\xi_d} (E^d)^{\xi_d} W^i_{j_d} (E^1)^{-\xi_1},
\]

Figure 29
where $\varepsilon_k = \pm 1$, for all $k = 1, \ldots, d$. Now we have already shown that, for each long edge $E$ in $\hat{W}$, there exists words, $e_1$ and $e_2$, in $F(X)$ of length at most $5l + M + 3$ such that, in the Cayley graph $\Gamma_X(H)$, $e_1$ and $e_2$ label geodesic paths from $\tau(\theta(E))$ and $\tau(\theta(E^{-1}))$ respectively to vertices on the geodesic path labelled by $F$.

Let $z^i_j$ be a cyclic word in $F(X)$ such that
\[
z^i_j = z^i_{j_1} \cdots z^i_{j_d},
\]
where
\[
z^i_{j_k+1} = (e^k_{\mu_k})^{-1}\theta(W^i_{j_k})e^k_{\mu_{k+1}} \quad \text{for } k = 1, \ldots, d - 1,
\]
and
\[
z^i_{j_1} = (e^d_{\mu_d})^{-1}\theta(W^i_{j_d})e^1_{\mu_1},
\]
with
\[
\mu_k = \begin{cases} 
1 & \text{if } \varepsilon_k = 1 \\
2 & \text{if } \varepsilon_k = -1 
\end{cases}
\]
From the definition of $z^i_j$, in the group $H$ we can see that
\[
z^i_j =_H z^i_{j_1} \cdots z^i_{j_d} =_H (e^1_{\mu_1})^{-1}\theta(W^i_{j_1})e^2_{\mu_2} (e^2_{\mu_2})^{-1}\theta(W^i_{j_2})e^3_{\mu_3} \cdots (e^k_{\mu_d})^{-1}\theta(W^i_{j_d})e^1_{\mu_1}
\]
\[
=_{H} (e^1_{\mu_1})^{-1}\theta(W^i_{j_1})\theta(W^i_{j_2}) \cdots \theta(W^i_{j_d})e^1_{\mu_1}
\]
\[
=_{H} (e^1_{\mu_1})^{-1}\theta(W^i_{j_1})e^1_{\mu_1}
\]
\[
=_{H} \theta(W^i_{j_1}).
\]
We construct $z^i_j$ for all $j = 1, \ldots, t_i$ and $i = 1, \ldots, p$. We shall denote the set $\{z^i_j, j = 1, \ldots, t_i\}$ by $C_i$. Now, since $z^i_j \sim_H \theta(W^i_{j})$ for all $j = 1, \ldots, t_i$, it follows that
\[
\text{genus}_H(z^i_1, \ldots, z^i_{t_i}) = \text{genus}_H(\theta(W^i_{t_1}), \ldots, \theta(W^i_{t_i})) = g_i - t_i + 1.
\]
We shall say that the genus of $C_i$ is $g_i$.

**Lemma 4.8.** With $g_i$ and $d$ defined as above, if $g_i = 0$ and hence $t_i = 1$ then $d \geq 3$.

**Proof.** Suppose that $d = 1$. Then $W^i_1 = W^i_{11}$. We know that $\text{genus}_H(z^i_1) = \text{genus}_H(\theta(W^i_1)) = 0$ and Lemma 4.7 implies that $\text{genus}_{F(A)}(W^i_{1}) = 0$. That is $W^i_1 = W^i_{11} = 1$. But $W^i_{11}$ is a cyclic subword of the Wicks form $W$ and by definition $W$ is cyclically reduced. Therefore this can’t occur.

Now suppose that $d = 2$. Then $W^i_2 = W^i_{11}W^i_{12}$. With a similar argument it follows that $\text{genus}_{F(A)}(W^i_{2}) = 0$ and thus $W^i_{11} = (W^i_{12})^{-1}$. But the cyclic subwords $(E^1)^{\varepsilon_1}W^i_{11}(E^2)^{-\varepsilon_2}$ and, $(E^2)^{\varepsilon_2}W^i_{12}(E^1)^{-\varepsilon_1}$ appear in the Wicks form $W$ which implies we that we have redundancy in $W$. Therefore this also can’t occur. 

\[\Box\]
Finally we need to construct a set of cyclic words in $F(X)$ which we shall denote $C_0$. We do this in the following way. Let $v_1, \ldots, v_t$ be the set of vertices of $\Gamma_S$ which do not lie on a boundary component of $S$ i.e. they lie on the endpoints of only long edges. Note that each of these vertices has degree at least 3 since $\Gamma_S$ is the graph obtained by identifying long letters of $W$ and $W$ is a Wicks form (recall that $\Gamma_W$ has no vertices of degree 1 or 2). Consider $v_j$, for some $1 \leq j \leq t$. Let the degree of this vertex be $r$. Thus there are long edges $F_1, \ldots, F_r$ which have an end point which is $v_j$. Note that these $F_i$'s are not necessarily distinct, that is we could have loops. See Figure 30.

Thus, since $v_j$ is regular, we can renumber the long edges which have an end point $v_j$ such that the following cyclic subwords appear in $W$.

$$(F^1)^{\varepsilon_1}(F^2)^{-\varepsilon_2}, (F^2)^{\varepsilon_2}(F^3)^{-\varepsilon_3}, \ldots, (F^{r-1})^{\varepsilon_{r-1}}(F^r)^{-\varepsilon_r}, (F^r)^{\varepsilon_r}(F^1)^{-\varepsilon_1},$$

where $\varepsilon_k = \pm 1$, for all $k = 1, \ldots, d$. As in the construction of $C_i$ we again use the fact that for each long edge $e$ in $\hat{W}$ there exists words $e_1$ and $e_2$, in $F(X)$, of length at most $5l + M + 3$, such that, in the Cayley graph $\Gamma_X(H)$, $e_1$ labels a geodesic path from $\tau(\theta(E))$ to a vertex on $F$ and $e_2$ labels a geodesic path from $\tau(\theta(E^{-1}))$ to a vertex on $F$.

Let $z^0_j$ be a cyclic word in $F(X)$ such that

$$z^0_j = z^0_{j1} \cdots z^0_{jd},$$

where

$$z^0_{jk+1} = (f^{k}_{\mu_k})^{-1} f^{k+1}_{\mu_{k+1}} \quad \text{for } k = 1, \ldots, r - 1,$$

and

$$z^0_{j1} = (f^{r}_{\mu_r})^{-1} f^{1}_{\mu_1},$$

again with

$$\mu_k = \begin{cases} 1 & \text{if } \varepsilon_k = 1 \\ 2 & \text{if } \varepsilon_k = -1 \end{cases}.$$ 

Clearly $z^0_j$ is equal to 1 in $H$ so has genus 0 in $H$. Let $C_0 = \{ z^0_j : 1 \leq j \leq t \}$.

We have constructed the required sets of cyclic words in $F(X)$ which shall be used in the extension of $U$ needed to obtain part 2 of the Theorem.
Write \( W \) around the outer boundary component of an annulus and label the inner boundary component with \( F \) from some fixed base point. See Figure 31. Let \( A \) be a long edge of \( W \). For the occurrence of \( A \) with exponent 1 we add a properly embedded path, labelled by \( a_1 \in F(X) \) (remember this has length at most \( 5l + M + 3 \) in \( H \), see Lemma 4.3), from \( \tau(A) \) to \( \tau(A_1) \) on \( F \) and similarly, for the occurrence of \( A \) with exponent \(-1\), we add a properly embedded path, labelled by \( a_2 \in F(X) \), from \( \tau(A^{-1}) \) to \( \tau(A_2^{-1}) \) on \( F \). See Figure 32.

We do this for each long edge of \( W \). Identify the long edges of \( W \), respecting orientation to obtain a surface \( S' \). The new surface \( S' \) is just the surface \( S \) with a disc removed. The boundary of the disc removed from \( S \) is labelled by \( F \). Now the graph \( \Gamma_S \) is embedded in \( S' \) and is a subgraph of a larger graph also embedded in \( S' \), which consists of the boundary of the annulus and the properly embedded paths on the annulus after identification. We denote this graph \( \Gamma_{S'} \). See Figure 33.

We again consider sections of the graph \( \Gamma_S \) where long edges either meet a cyclic word \( W_j^i \in B_i \), \( 1 \leq j \leq t_i \), \( 1 \leq i \leq p \) or one of the vertices \( v_j' \), \( 1 \leq j' \leq t \), which does not lie on a boundary component of \( S \). Let \( E^1, \ldots, E^d \) be the long edges which have an end point on \( W_j^i \) (resp. which have an end point which is \( v_j' \) ). Let \( W_j^i = W_{j_1}^i \cdots W_{j_d}^i \). Again these can be renumbered such that \( W \) contains the cyclic subwords

\[
(E^1)^{\varepsilon_1} W_{j_1}^i (E^2)^{-\varepsilon_2} (E^2)^{\varepsilon_2} W_{j_2}^i (E^3)^{-\varepsilon_3} \cdots, (E^{d-1})^{\varepsilon_{d-1}} W_{j_{d-1}}^i (E^d)^{-\varepsilon_d} (E^d)^{\varepsilon_d} W_{j_d}^i (E^1)^{-\varepsilon_1},
\]
where \( \varepsilon_k = \pm 1 \), for all \( k = 1, \ldots, d \). See Figure 29. The word \( W_j^d \) obviously has length zero if we are considering \( v_{j'} \). In our new extended graph \( \Gamma_{S'} \) this section of the graph now takes the form as shown in Figure 34 where

\[
\mu_k = \begin{cases} 
1 & \text{if } \varepsilon_k = 1 \\
2 & \text{if } \varepsilon_k = -1
\end{cases}
\]

and

\[
\nu_k = \begin{cases} 
1 & \text{if } \mu_k = 2 \\
2 & \text{if } \mu_k = 1
\end{cases}
\]

Remove the edges \( E^1, \ldots, E^d \) from \( \Gamma_{S'} \). Now, for each \( k = 1, \ldots, d - 1 \), there is an edge path from \( \tau(E_{\mu_k}^k) \) to \( \tau(E_{\mu_k+1}^{k+1}) \) labelled by \((e_{\mu_k}^k)^{-1}W_{jk}^i e_{\mu_k+1}^{k+1}\), where \( e_{\mu_k}^k, e_{\mu_k+1}^{k+1} \in F(X) \) and \( W_{jk}^i \) is sequence of short edges in \( W \). Note that if \( k = d \) then the edge path from \( \tau(E_{\mu_d}^d) \) to \( \tau(E_{\mu_1}^1) \) is labelled by \((e_{\mu_d}^d)^{-1}W_{jk}^i e_{\mu_1}^1\). We have already defined \( z_{jk}^i = (e_{\mu_k}^k)^{-1}\theta(W_{jk}^i)e_{\mu_k+1}^{k+1} \) for \( k = 1, \ldots, d - 1 \) and \((e_{\mu_d}^d)^{-1}\theta(W_{jk}^i)e_{\mu_1}^1 \) for \( k = d \). For all \( k = 1, \ldots, d \), we shall also remove the edges labelled by \( W_{jk}^i \) and \( e_{jk}^i \), add new edges from \( \tau(E_{\mu_d}^d) \) to \( \tau(E_{\mu_1}^1) \), labelled by \( z_{jk}^i \), for \( k = 1, \ldots, d - 1 \), and add a new edge from \( \tau(E_{\mu_d}^d) \) to \( \tau(E_{\mu_1}^1) \), labelled by \( z_{jd}^i \), for \( k = d \). See Figure 35.

We now do this for all \( W_j^i \in B_i, \ j = 1, \ldots, t_i, \ i = 1, \ldots, p \), and \( v_{j'}, j' = 1, \ldots, t \). We shall call this new graph \( \Gamma_{S''} \). It is clear, by the way we numbered \( E^1, \ldots, E^d \), that the following are cyclic subwords of the Hamiltonian cycle in \( \Gamma_{S''} \) labelled by \( F \).

\[
(E_{\mu_1}^1)^{\varepsilon_1} (E_{\mu_2}^2)^{-\varepsilon_2} (E_{\mu_3}^2)^{\varepsilon_2} (E_{\mu_4}^3)^{-\varepsilon_3} \ldots, (E_{\mu_{d-1}}^d)^{-\varepsilon_{d-1}} (E_{\mu_d}^d)^{\varepsilon_d} (E_{\mu_1}^1)^{-\varepsilon_1},
\]

where \( \varepsilon_k = \pm 1 \), for all \( k = 1, \ldots, d \). The cyclic sequence of letters of \( U \) gives the cyclic sequence of segments of \( F \) by replacing a letter \( E \) of \( U \) with \( E_1 \) for its occurrence with exponent 1 and \( E_{2^{-1}} \) for its occurrence with exponent \(-1\). Thus, to show that \( \Gamma_{S''} \) is an extension of \( U \) over \( H \) by the cyclic words in the sets \( C_0, \ldots, C_p \), it is clear that we must
show that step 3, of the construction of an extension over \( H \), holds. Let \( A \) be a long edge of \( \hat{W} \) and without loss of generality assume that \( A \) appears before \( A^{-1} \). Consider the two edges \( A_1 \) and \( A_2 \) in \( \Gamma_{S''} \). There exist an \( z_{ij} \) and \( z_{ij}' \) which we shall denote \( x \) and \( y \) respectively such that the edge path \( A_1, x^{-1}, A^{-1}_2, y^{-1} \) is a cycle in \( \Gamma_{S''} \). See Figure 36. By the construction (see definitions of the \( z_{ij}' \)'s and Figure 34) \( x \) and \( y \) can be split up into subwords, \( x = x_1 x_2 \) and \( y = y_1 y_2 \) respectively, such that in \( \Gamma_X(H) \) we have closed paths as shown in Figure 37. Note that \( x_2 \) and \( y_2 \) are \( a_1 \) and \( a_2 \) but \( x_1 \) and \( y_1 \) may include part of \( \theta(W) \) as indicated in the diagram. From \( \Gamma_X(H) \) we can see that

\[
\theta(A) = H \quad y_1^{-1} A_1 x_2^{-1} = H \quad y_2 A_2 x_1.
\]

It follows that \( A_1 x^{-1} A_2^{-1} y^{-1} = H 1 \). Hence step 3 holds and we have an extension of \( U \) over \( H \).

4.2 Length and Genus of the Extension on \( U \)

We have already shown that the genus of \( U \) is \( k \). Remember that the genus of the extension is given by the sum of the genus of the cyclic words used. But these are just the elements of the sets \( C_0, C_1, \ldots, C_p \). Now the genus of every element of \( C_0 \) is zero (Note also that
the vertices extended by elements of $C_0$ have degree of at least 3, see page 33] and the genus of the elements in each set $C_i$, for $i = 1, \ldots, p$ is equal to $g_i - t_i + 1$ (again note that if the genus of $C_i = 0$ then the vertex extended by the only element of $C_i$ is of degree at least 3, see Lemma 4.8). Therefore, it follows from the definition that the genus of the extension is equal to the sum of the genus of each set $C_i$, for $i = 1, \ldots, p$. We know that

$$g = \sum_{i=1}^{p} g_i,$$

and $n = k + g$, see Lemma 4.7. Thus the extension is of the required genus for part 2 of the Theorem to hold.

The length of the extension is given by the sum of the length of the words in the sets $C_0, C_1, \ldots, C_p$. Each element of a set $C_i$, for $i = 0, \ldots, p$, takes the form

$$z_j^i = z_{j1}^i \cdots z_{jd}^i,$$

where

$$z_{jk+1}^i = (e_{\mu_k}^k)^{-1} \theta(W_{j_k}^i) e_{\mu_{k+1}}^{k+1} \quad \text{for } k = 1, \ldots, d - 1,$$

and

$$z_{j1}^i = (e_{\mu_d}^d)^{-1} \theta(W_{j_d}^i) e_{\mu_1}^1.$$
with all the $\theta(W_{jk}^i)$’s having length zero if $i = 0$. Each short edge of $W$ appears in a unique $W_{jk}^i$ and these have labels in $F(X)$ which have length at most $12l + M + 4$. Each $e_{jk}^k$ is a word of length at most $5l + M + 3$ arising from a long edge of $W$, see Lemma 4.3 and these each appear twice in some $z_i^j$. By Lemma 4.2 it follows that the maximum number of letters in $W$ is $12n - 6$. Thus the sum of number of short letters and long letters is at most $12n - 6$. Therefore, if $S$ and $L$ are the numbers of short edges and long edges respectively, the length of the extension is given by

$$(12l + M + 4)S + 2(5l + M + 3)L \leq 2(12n - 6)(12l + M + 4).$$

Hence we have the required extension and the theorem holds.

5 Forms for Commutators in Hyperbolic Groups

We shall now use Theorem 2.6 to obtain a full list of all possible forms for commutators in $H = \langle X \mid R \rangle$. Now since in this case $n = 1$, it follows that $l = \delta(\log_2(6) + 1)$.

Proposition 5.1. If $h$ is a commutator in $H$ then there are words $R$ and $F$ in $F(X)$ which are minimal in $H$ such that $h =_H RFR^{-1}$, where $|R| \leq 59l + 8M + 28 + 2\delta + \frac{|h|n}{2}$ and $F$ takes one of the following forms.

1. $|F| \leq 6(12l + M + 4)$ with $F =_H X Y Z X^{-1} Y^{-1} Z^{-1}$ and $|X|, |Y|, |Z| \leq 12l + M + 4$.

2. $F = A_1A_2^{-1}$ with $A_1 =_H \xi_1^{-1}A_2\xi_2$. Where $|\xi_1| + |\xi_2| \leq 12(12l + M + 4)$ and $\xi_1$ is conjugate to $\xi_2$ in $H$.

3. $F = A_1B_1A_2^{-1}B_2^{-1}$ with $A_1 =_H \xi_1A_2\xi_3$, $B_1 =_H \xi_4B_2\xi_2$. Where $|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4| \leq 12(12l + M + 4)$ and $\xi_1 \xi_2 \xi_3 \xi_4 =_H 1$.

4. $F = A_1B_1C_1A_2^{-1}B_2^{-1}C_2^{-1}$ with $A_1 =_H \xi_1A_2\rho_1$, $B_1 =_H \rho_2B_2\xi_2$ and $C_1 =_H \xi_3C_2\rho_3$. Where $|\xi_1| + |\xi_2| + |\xi_3| + |\rho_1| + |\rho_2| + |\rho_3| \leq 12(12l + M + 4)$ and $\xi_1 \xi_2 \xi_3 =_H \rho_1\rho_2\rho_3 =_H 1$. 
Proof. By Theorem 2.6 and our knowledge of genus 1 Wicks forms (see beginning of the paper), \( h \) is conjugate to a minimal word \( F \) which either has form 1 above or is obtained by a genus \( g \) extension of length at most \( 12(12l + M + 4) \) on some orientable word of genus \( k \) such that \( g + k = 1 \). This implies that there are only three possible orientable words which can have a suitable extension (this is easy to check).

(i) An orientable word \( U = AA^{-1} \) of genus 0 with a joint genus 1 extension constructed on the two vertices of \( \Gamma_U \).

(ii) An orientable word \( V = ABA^{-1}B^{-1} \) of genus 1 with a genus 0 extension constructed on the only vertex of \( \Gamma_V \).

(iii) An orientable word \( W = ABCA^{-1}B^{-1}C^{-1} \) of genus 1 with a genus 0 extension constructed on both of the vertices of \( \Gamma_W \).

(i) We extend the graph \( \Gamma_U \) as shown in Figure 38. By Theorem 2.6, \( F \) takes the form of

\[
\begin{array}{c}
\xrightarrow{A} \\
\xrightarrow{\xi_1} \\
\xrightarrow{A_1} \\
\xrightarrow{\xi_2} \\
\xrightarrow{A_2}
\end{array}
\]

Figure 38

the Hamiltonian cycle \( A_1A_2^{-1} \) in the extended graph and from the nature of the extension constructed on \( U \), it is clear that we have form 2.

(ii) We extend the graph \( \Gamma_V \) as shown in Figure 39. By Theorem 2.6, \( F \) takes the form

\[
\begin{array}{c}
\xrightarrow{B} \\
\xrightarrow{\xi_1} \\
\xrightarrow{B_1} \\
\xrightarrow{\xi_2} \\
\xrightarrow{B_2}
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{A} \\
\xrightarrow{\xi_1} \\
\xrightarrow{A_1} \\
\xrightarrow{\xi_2} \\
\xrightarrow{A_2}
\end{array}
\]

Figure 39

of the Hamiltonian cycle \( A_1B_1A_2^{-1}B_2^{-1} \) in the extended graph and from the nature of the extension constructed on \( V \), it is clear that we have form 3.

(iii) Finally, we extend the graph \( \Gamma_W \) as shown in Figure 40. Again, by Theorem 2.6, \( F \)
Figure 40

takes the form of the Hamiltonian cycle $A_1B_1C_1A_2^{-1}B_2^{-1}C_2^{-1}$ in the extended graph and from the nature of the extension constructed on $W$, it is clear that we have form 4. Hence $h$ is conjugate to some $F$ which takes one of the required forms.

Let $R$ be the shortest word in $H$ such that $h = H R F R^{-1}$. If $F$ takes form 1 then, by Lemma 3.1, it follows that

$$R \leq \frac{1}{2}(|F| + |h|_H) + M + 1$$

$$\leq \frac{1}{2}(6(12l + M + 4) + |h|_H) + M + 1$$

$$\leq 36l + 4M + 13 + \frac{|h|_H}{2}.$$ 

Thus the proposition holds. Therefore, suppose that $F$ is obtained by an extension of some orientable word. In the proof of Theorem 2.6, $F$ was constructed from some Wicks form $W$ and a labelling function $\theta$ such that $\theta(W)$ was minimal over the set of pairs in $F$. In the proof we chose a cyclic permutation $\hat{W}$ of $W$ such that the last letter was a long edge. Now in the genus 1 case all Wicks forms take the form $XYZX^{-1}Y^{-1}Z^{-1}$ where at most one of these letters is set to 1, see [13], and obviously all cyclic permutations are of this form too. Thus, in this case, we shall choose the cyclic permutation $\hat{W}$ which ends in a long letter such that $R$ is the shortest over all cyclic permutations which end in a long letter. Let $\hat{W} = ABCA^{-1}B^{-1}C^{-1}$. It follows from the proof of Theorem 2.6 that $F = H \theta(\hat{W})$. See Figure 41.

We need the following lemma.

**Lemma 5.2.** Let $u_1$ be the terminal vertex of the label of a long edge of $\hat{W}$ in $\Gamma_X(H)$. Suppose that there exist a vertex $u_2$ lying on $F$ such that $d(u_1, u_2) \leq K$, for some constant $K$. If $d(u_2, \tau(F)) - L \leq d(\iota(F), u_2) \leq d(u_2, \tau(F)) + L$ for some constant $L$ then

$$|R| \leq \frac{|h|_H}{2} + K + \frac{L}{2} + M + 2\delta + 1.$$ 

**Proof.** Without loss of generality let $u_1 = \tau(\theta(A))$. Also we shall let $h'$ be a minimal word such that $h' = H h$. Consider the vertex $u_2$. By Lemma 3.2 part 1., there exists a vertex
for $u_3$ on $h' \cup R \cup R^{-1}$ such that $d(u_2, u_3) \leq 2\delta$. First suppose that $u_3$ lies on either $R$ or $R^{-1}$. Without loss of generality we shall assume this to be $R$. Now by Lemma 3.2 part 3 we can see that $d(\iota(F), u_2) = d(\iota(F), u_3)$. By the triangle inequality $d(u_1, u_3) \leq K + 2\delta$ so there exists a path $s$ in Cayley graph from $u_1$ to $u_3$ of length at most $K + 2\delta$. Let $R = R_1R_2$ such that $u_3 = \iota(R_1) = \tau(R_2)$. Cut and paste along $s$ as shown in Figure 42. It is easy to see from the Cayley graph that

Thus we have a new cyclic permutation of $W$, with the last letter $A$ being a long edge. Now, since $\hat{W}$ was chosen to be the cyclic permutation which end in a long letter such that $R$ is minimal over all cyclic permutations which ends in a long letter, it follows that $|R| = |R_1| + |R_2| \leq |s| + |R_1|$. This implies that $d(\iota(F), u_2) = |R_2| \leq |s| \leq K + \delta$. 

Figure 41

Figure 42: Cut and paste along $s$
Therefore, by the hypothesis
\[ |F| = d(\iota(F), u_2) + d(u_2, \tau(F)) \leq 2d(\iota(F), u_2) + L \leq 2K + 4\delta + L. \]

Thus by Lemma 3.1
\[ |R| \leq \frac{1}{2}(|F| + |h'|) + M + 1 \leq \frac{1}{2}(2K + 4\delta + L + |h|_H) + M + 1 \leq \frac{|h|_H}{2} + K + \frac{L}{2} + M + 2\delta + 1. \]

As required.

Now suppose that \( u_3 \) lies on \( h' \). It is easy to see that either \( d(u_3, \iota(h)) \leq \frac{|h|_H}{2} \) or \( d(u_3, \tau(h)) \leq \frac{|h|_H}{2} \). Without loss of generality we shall assume the former. Using the triangle inequality it follows that
\[ d(u_1, \iota(h)) \leq d(u_1, u_2) + d(u_2, u_3) + \frac{|h|_H}{2} \leq K + 2\delta + \frac{|h|_H}{2}. \]

Therefore, there exists a path \( t \) of length at most \( K + 2\delta + \frac{|h|_H}{2} \) from \( \iota(h) \) to \( u_1 \). Since \( t =_H R\theta(A) \), it is easy to see that
\[ h =_H h' =_H R\theta(A)\theta(B)\theta(C)\theta(A)^{-1}\theta(B)^{-1}\theta(C)^{-1}R^{-1} =_H t_{R''} =_H \theta(B)\theta(C)\theta(A)^{-1}\theta(B)^{-1}\theta(C)^{-1}\theta(A) =_H t_{(R'')^{-1}}. \]

Again, from our choice of cyclic permutation of \( W \), we know that \( |R| \leq |t| \leq K + 2\delta + \frac{|h|_H}{2} \). Hence the lemma holds.

Suppose that \( F \) has form 2, that is \( F = A_1A_2^{-1} \). By the triangle inequality it follows that
\[ |A_2| - |\xi_1| - |\xi_2| \leq |A_1| \leq |A_2| + |\xi_1| + |\xi_2| \implies |A_2| - 12(12l + M + 4) \leq |A_1| \leq |A_2| + 12(12l + M + 4). \]

Also, by Lemma 4.3, \( \tau(A_1) \) on \( F \) in \( \Gamma(X) \) is within \( 5l + M + 3 \) of a terminal vertex of some long edge of \( \theta(\hat{W}) \). Therefore, Lemma 5.2 implies that
\[ |R| \leq \frac{|h|_H}{2} + 5l + M + 3 + \frac{12(12l + M + 4)}{2} + M + 2\delta + 1 \leq \frac{|h|_H}{2} + 8M + 59l + 2\delta + 28. \]
Thus in this case the Proposition holds. Now if $F$ takes form 3 or 4 we follow the same procedure. For form 3, that is $F = A_1B_1A_2^{-1}B_2^{-1}$, we have

$$|A_2| + |B_2| - |\xi_1| - |\xi_2| - |\xi_3| - |\xi_4| \leq |A_1| + |B_1| \leq |A_2| + |B_2| + |\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|$$

$$\Rightarrow |A_2| + |B_2| - 12(12l + M + 4) \leq |A_1| + |B_1| \leq |A_2| + |B_2| + 12(12l + M + 4)$$

and we know that $\tau(B_1)$ on $F$ in $\Gamma_X(H)$ is within $5l + M + 3$ of a terminal vertex of some long edge of $\theta(\hat{W})$. For form 4, that is $F = A_1B_1C_1A_2^{-1}B_2^{-1}C_2^{-1}$, we have

$$|A_2| + |B_2| + |C_2| - |\xi_1| - |\xi_2| - |\xi_3| - |\rho_1| - |\rho_2| - |\rho_3| \leq |A_1| + |B_1| + |C_1|$$

and

$$|A_1| + |B_1| + |C_1| \leq |A_2| + |B_2| + |C_2| + |\xi_1| + |\xi_2| + |\xi_3| + |\rho_1| + |\rho_2| + |\rho_3|$$

$$\Rightarrow |A_2| + |B_2| + |C_2| - 12(12l + M + 4) \leq |A_1| + |B_1| + |C_1|$$

and

$$|A_1| + |B_1| + |C_1| \leq |A_2| + |B_2| + |C_2| + 12(12l + M + 4)$$

and we know that $\tau(C_1)$ on $F$ in $\Gamma_X(H)$ is within $5l + M + 3$ of a terminal vertex of some long edge of $\theta(\hat{W})$. Thus we may use Lemma 5.2 in both cases to get the required bound for $|R|$. Hence the proposition holds.

Similar lists of forms can of course be constructed for elements of higher genus. Although, the number of possible extensions increases dramatically with the increase of genus. A. Vdovina lists the number of maximal orientable Wicks forms up to genus 15, see [2]. This gives an idea of the number of extensions one would need to do.

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