Abstract. For any $n \geq k \geq l \in \mathbb{N}$, let $S(n,k,l)$ be the set of all those non-negative definite matrices $a \in M_n(\mathbb{C})$ with $l \leq \text{rank } a \leq k$. Motivated by applications to $C^*$-algebra theory, we investigate the homotopy properties of continuous maps from a compact Hausdorff space $X$ into sets of the form $S(n,k,l)$. It is known that for any $n$, if $k - l$ is approximately 4 times the covering dimension of $X$ then there is only one homotopy class of maps from $X$ into $S(n,k,l)$, i.e. $C(X, S(n,k,l))$ is path connected. In our main Theorem we improve this bound by a factor of 8. By combining classical homotopy theory methods with $C^*$-algebraic techniques we also show that if $\pi_r(S(n,k,l))$ vanishes for all $r \leq d$ then $C(X, S(n,k,l))$ is path connected for any compact Hausdorff $X$ with covering dimension not greater than $d$.

1. Introduction.

For $n \in \mathbb{N}$, let $(M_n)_+$ stand for the $n \times n$ non-negative definite matrices over $\mathbb{C}$ and for fixed $k,l \in \mathbb{N}$ with $n \geq k \geq l$ define the space $S(n,k,l)$ by,
\[ S(n,k,l) = \{ b \in (M_n)_+ | l \leq \text{rank}(b) \leq k \}. \]

Let $S(n,k,l)$ be given the subspace topology induced by the norm topology of $M_n$. We focus on homotopy properties of $S(n,k,l)$. Our main theorem is the following;

**Theorem 3.5.** Let $X$ be a compact Hausdorff space $X$ with $\dim X \leq k - l$. Then there is only one homotopy class of functions $f : X \to S(n,k,l)$, i.e. the function space $C(X, S(n,k,l))$ is path connected.

Here, by $\dim X$ we mean the covering dimension of the space $X$ and $|m|$ stands for the largest integer which is less than or equal to $m \in \mathbb{R}$. An immediate consequence of Theorem 3.5 is that $\pi_r(S(n,k,l)) = 0, \forall r \leq 2(k - l) + 1$.

The spaces $S(n,k,l)$, or rather their homotopy properties have relevance in $C^*$-algebra theory and our motivation for this study is mainly due to this natural connection.

For a unital $C^*$-algebra $A$, let $T(A)$ denote the tracial state space. In other words $T(A)$ is the set of all normalized positive linear functionals $\tau$ on $A$ which satisfy the trace property $\tau(ab) = \tau(ba), \forall a,b \in A$. An important object of study for a $C^*$-algebra is the space $LAf_{f_0}(T(A))^{++}$ of all bounded, strictly positive, lower semi-continuous affine maps on $T(A)$. For a positive element $a$ in $A$, $\tau \mapsto \tau_a(\tau) = \lim_n \tau(a^{1/n})$ defines a positive lower semi-continuous affine map on $T(A)$. This definition extends to positive elements in $M_n(A), n \geq 2$, in which case $\tau$ in the right hand side stands for the non-normalized canonical extension of $\tau$ to $M_n(A)$. If $A$ is simple $\tau_a$ is strictly positive for all $a \in M_n(A)$ for any $n \in \mathbb{N}$. For sufficiently regular simple $C^*$-algebras (as in [2,13]), $\{ \tau_a: a \in M_{\infty}(A)_+ \}$ is dense in $LAf_{f_0}(T(A))^{++}$.
On the other hand, density of \( \{ \iota_a : a \in M_{\infty}(A)_+ \} \subseteq L^A f f(T(A))^{++} \) provides one with useful tools to work with to derive classification results and to establish regularity properties such as \( Z \)-stability \[3\,13\]. In \[13\, Theorem 3.4\], Toms shows that for unital simple ASH algebras with slow dimension growth \( \{ \iota_a : a \in M_{\infty}(A) \} \) is dense in \( L^A f f_b(T(A))^{++} \). He then uses this fact to show that such algebras are \( Z \)-stable and hence derives classification results. The proof of \[13\, Theorem 3.4\] heavily depends on homotopy properties of the spaces \( S(n,k,l) \) \[13\, Section 2\]. Our main result here is an improvement of \[13\, Proposition 2.5\], which is a key ingredient in proving that \( \{ \iota_a : a \in M_{\infty}(A)_+ \} \subseteq L^A f f(T(A))^{++} \) is dense for the class of algebras mentioned above. However, one should note that this improvement does not have a direct impact on the main result of \[13\].

**Spaces \( S(n,k,l) \) as a generalization of complex Grammarians.**

Let us consider the special case \( k = l \). Recall that \( G_k(\mathbb{C}^n) \) stands for the complex Grassmann variety of \( k \)-dimensional subspaces of \( \mathbb{C}^n \). Given a \( k \)-dimensional subspace \( V \) of \( \mathbb{C}^n \), one may identify it uniquely with the orthogonal projection of \( \mathbb{C}^n \) on \( V \). This identification leads to a natural homeomorphism from \( G_k(\mathbb{C}^n) \) to \( P_k(\mathbb{C}^n) \), where \( P_k(\mathbb{C}^n) \) is the set of all rank \( k \) projections in \( M_n(\mathbb{C}) \) equipped with the norm topology. It is not hard to see that the inclusion \( P_k(\mathbb{C}^n) \subseteq S(n,k,k) \) induces a homotopy equivalence. Hence, \( G_k(\mathbb{C}^n) \) is homotopy equivalent to \( S(n,k,k) \). Thus in a sense, the spaces \( S(n,k,l) \) can be viewed as a generalization of the Grassmann varieties, at least for homotopy interests. By setting \( k = l \) in Theorem 3.5 and using the homotopy equivalence of \( S(n,k,k) \) with \( G_k(\mathbb{C}^n) \) one can derive that \( G_k(\mathbb{C}^n) \) is simply connected (i.e. \( \pi_1(G_k(\mathbb{C}^n)) = 0 \)) for any pair of \( k,n \), which is a well known classical result. However, it should be noted that our proof of Theorem 3.5 does not provide an alternate proof of this fact, rather we use a stronger classical result in our proof.

**Rank varying bundles associated with the spaces \( S(n,k,l) \).**

Given a continuous map \( f : X \to G_k(\mathbb{C}^n) \), one has the associated locally trivial vector bundle \( f^*(\gamma) \) over \( X \), which is the pullback of the canonical \( k \)-dimensional vector bundle \( \gamma \) over \( G_k(\mathbb{C}^n) \) to \( X \).

In a similar vein, for each map \( a \in C(X,(M_n)_+) \) there is a naturally associated bundle (in the sense of \[7\, Chapter 2\]) \( \epsilon_a \) over \( X \) with the total space,

\[
E_a = \{(x,v) \in X \times \mathbb{C}^n | v \in a(x)(\mathbb{C}^n)\}
\]

and \( \pi_1 : E_a \to X \) being the natural coordinate projection. Admittedly, a typical bundle obtained in this nature is not necessarily a vector bundle in the classical sense \( (1,\,7) \). In fact such a bundle may not even have a constant fiber. Still, for a bundle \( \epsilon_a \) each fiber admits a natural vector space structure, which to an extent preserves local triviality. Moreover these bundles occur naturally in \( C^*\)-algebra theory (see \[14,15\]) and understanding this association would make the techniques used in section 3 more intuitive. Thus, before moving on to the next section we outline the structure of these bundles.
The structure of bundles associated with the spaces $S(n, k, l)$.

Observe that if $a(x) \in S(n, k, k), \forall x \in X$, then the bundle $\epsilon_a$ is indeed a locally trivial vector bundle over $X$. However, if $a$ is not of constant rank the associated bundle $\epsilon_a$ is not locally trivial in the usual sense. Still, if we set $E_i = \{x \in X : \text{rank}(a(x)) = n_i\}$, then continuity of $a$ implies that the support projection of $a$ is continuous on $E_i$. Here, by support projection of $a$ we mean the function which assigns each $x$ to the orthogonal projection on the subspace $a(x)(\mathbb{C}^n)$. Hence, the restriction of $\epsilon_a$ to $E_i$ - denoted by $\epsilon_a \mid_{E_i}$, is a locally trivial vector bundle over $E_i$. In this manner we can partition $X$ into a finite collection of subsets, such that the restriction of $\epsilon_a$ to each subset is a locally trivial bundle of constant rank. One may now consider the possibility of applying classical vector bundle theory [1, 7] to establish the global structure of $\epsilon_a$. Hence, the restriction of $\epsilon_a$ to $E_i$ - denoted by $\epsilon_a \mid_{E_i}$, is a locally trivial vector bundle over $E_i$. In this manner we can partition $X$ into a finite collection of subsets, such that the restriction of $\epsilon_a$ to each subset is a locally trivial bundle of constant rank. One may now consider the possibility of applying classical vector bundle theory [1, 7] to establish the structure of $\epsilon_a$ in some local sense, i.e the structure of $\epsilon_a \mid_{E_i}$ for each $i$. But the issue with this is that the subsets $E_i$ formed here are highly non regular where as to apply classical theory of vector bundles the base spaces need to satisfy regularity properties such as compactness. We can overcome this if $q_i$- the support projection of $a$ on $E_i$, extends to a projection $p_i \in M_n(C(\overline{E}_i))$ for each $i$, where $\overline{E}_i$ is the closure $E_i$. If this is the case, then we can apply classical vector bundle theory (see [1, 7]) to understand the structure of bundles $\epsilon_{p_i}$. Moreover, if its also the case that for any two distinct $i, j$ with $\overline{E}_i \cap \overline{E}_j \neq \emptyset$ the extended projections $p_i$ and $p_j$ are comparable on $\overline{E}_i \cap \overline{E}_j$, then we can expect to use structural properties of the bundles $\epsilon_{p_i}$ (i.e local structure of $\epsilon_a$) to establish the global structure of $\epsilon_a$.

The above considerations would not hold true for arbitrary $a \in M_n(C(X))_+$. However, [15], c.f. [10] introduces a special class of elements in $M_n(C(X))_+$ called well supported positive elements, which are well behaved in the above sense. For an element $a$ in this class (Definition 2.2), each $q_i$ extends to $p_i \in M_n(C(\overline{E}_i))$ in such a way that for $i < j$ with $\overline{E}_i \cap \overline{E}_j \neq \emptyset$, $p_i$ is a sub-projection of $p_j$ of on $\overline{E}_i \cap \overline{E}_j$. From results in [14] it follows that for any $n, k, l \in \mathbb{N}$ the set of all well supported positive elements contained in $C(X, S(n, k, l))$ is homotopy equivalent to $C(X, S(n, k, l))$ [Lemma 3.1]. This fact plays a crucial role in the proof of our main theorem.

In section 2, we briefly recall some well known definitions and results from vector bundle theory [1, 7] as well as some other tools and notations that are necessary. In section 3, following the ideas from [10] and [13] we prove the main result. Section 4 (Theorem 4.6) shows that for any $n, k, l \in \mathbb{N}$ the path connectedness of $C(X, S(n, k, l))$ depends solely on homotopy of the space $S(n, k, l)$. In section 4 we do not assume that $k - l \geq \lfloor \frac{\dim X}{2} \rfloor$ and the proof of Theorem 4.6 is achieved mainly through applications of classical homotopy theory results (see [14]), to the space $S(n, k, l)$.

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2. Notations and Preliminaries.

Notations and Conventions. Unless stated otherwise we assume $X$ to be a compact Hausdorff space, $C(X,Y)$, $M_{n \times m}(C(X))$, $M_n(C(X))$ all have the usual meanings and $C(X) = C(X, \mathbb{C})$. We will often identify $M_n(C(X))$ with $C(X, M_n)$, where $M_n$ denotes $M_n(\mathbb{C})$. $(M_n)_+$ denotes all non-negative definite matrices in $M_n$ and its customary to call these as positive elements of $M_n$.

Given $x \in M_n$, $x^*$ denotes the conjugate transpose of $x$. By a projection $p$ in $M_n$ we mean a self adjoint idempotent, i.e $p = p^2 = p^*$. Note that the notions of the conjugate of an element, positive elements and projections carry over to $M_n(C(X))$ via point wise defined operations. We will use $M_n(C(X))_+$ to denote positive elements in $M_n(C(X))$.

The base field for all vector spaces and vector bundles that we will consider is the field of complex numbers.

Murray-von Neumann semi group of $C(X)$, semi group of isomorphism classes of vector bundles over $X$ and Serre-Swan Theorem.

Let $P_\infty(C(X)) = \bigcup_{m \in \mathbb{N}} P(M_m(C(X)))$, where $P(M_m(C(X)))$ denote the set of all projections in $M_m(C(X))$. A pair $p \in P(M_m(C(X)))$ and $q \in P(M_n(C(X)))$ are said to be Murray-von Neumann equivalent, denoted $p \sim q$, if there is some $v \in M_{m,n}(C(X))$ with $p = vv^*v$ and $q = vv^*$. The Murray-von Neumann equivalence class of $p \in P_\infty(C(X)))$ is denoted by $[p]_0$ and the Murray-von Neumann semigroup of $C(X)$ is the semigroup

$$D(C(X)) = (P_\infty(C(X))/\sim, +),$$

where $[p]_0 + [q]_0 = [p \oplus q]_0$.

Let $\text{Bun}(X)$ denote the set of locally trivial complex vector bundles over $X$, and let $\text{Bun}_k(X)$ denote the set of all $\epsilon \in \text{Bun}(X)$ of constant fiber dimension $k$. Write $\text{Vect}(X)$ to denote the semigroup of all isomorphism classes of bundles in $\text{Bun}(X)$ with addition being induced via the direct sum of bundles. The well known natural identification of $D(C(X))$ with $\text{Vect}(X)$, induced through the Serre-Swan Theorem [12] Theorem 2) (which states that $\text{Bun}(X)$ as a category with morphisms being the morphisms of vector bundles is equivalent to the category of finitely generated projective $C(X)$-modules), is central to our study. For the sake of completeness and to introduce some of the notations and terms that we will use, let us describe this identification.

For $p \in P(M_m(C(X))) \subset M_n(C(X))_+$, let $\epsilon_p = (E_p, \pi_1, X)$ be the bundle over $X$ defined as in the introduction. Since $p$ is a continuous projection, $\epsilon_p$ is a locally trivial vector bundle over $X$. Moreover, $p \sim q$ for some $q \in P_\infty(C(X))$, if and only if $\epsilon_p \cong \epsilon_q$. This gives a well defined injection $\psi : D(C(X)) \to \text{Vect}(X)$, which is a semigroup morphism and preserves dimensions. On the other hand, suppose $\epsilon$ is a locally trivial vector bundle over $X$, with total space $E$ and the fiber at $x \in X$ being $E_x$. From [12] Corollary 5, $\epsilon$ is a direct summand of a trivial bundle. That is, there is some $n \in \mathbb{N}$ and a bundle $\epsilon^n$ over $X$ such that $\epsilon \oplus \epsilon^n \cong \theta^n$, where $\theta^n$ is the $n$ dimensional product bundle. Bundle $\epsilon^n$ is called a complementary bundle for $\epsilon$. Let $F$ denote the total space of $\epsilon^n$ and let $F_x$ denote the fiber at $x \in X$. Then, we may assume that $E_x \oplus F_x = \mathbb{C}^n, \forall x \in X$. Let $p_x(x)$ denote the orthogonal
projection of \( \mathbb{C}^n \) on the fiber \( E_x \subset \mathbb{C}^n \). Then, the local triviality of \( \epsilon \) ensures that the map \( x \mapsto p_x(x) \) is continuous. Indeed, \( \psi(p_x) = \epsilon \) and thus \( \psi \) is a surjection.

Recall that a vector bundle \( \eta \) over \( X \) is called trivial if \( \eta \) is isomorphic to a product bundle over \( X \). We say that a projection \( p \) in \( M_n(C(X)) \) is a trivial projection if \( \epsilon_p \) is trivial, where \( \epsilon_p \) is defined as before. From the preceding paragraph it is clear that if \( p, q \in M_n(C(X)) \) with \( p \sim q \), then \( p \) is trivial iff \( q \) is trivial.

**Bounding the dimension of \( \epsilon^\perp \).**

From the preceding for each \( \epsilon \in \text{Bun}(C(X)) \) there is \( \epsilon^\perp \) so that \( \epsilon \oplus \epsilon^\perp \cong \theta^n \) for some \( n \in \mathbb{N} \). We would like to have a bound on the dimension on \( \epsilon^\perp \). Having such a bound will be useful for our work in section 3 as there we focus on \( S(n, k, l) \) for a fixed triple \( (n, k, l) \). Let us now address this concern and provide a bound for the dimension of \( \epsilon^\perp \) which depends only on the dimension of the base space \( X \).

Recall the following Theorem on locally trivial vector bundles over a compact Hausdorff space. Here, \( \lfloor x \rfloor \) stands for the smallest integer \( n \) with \( x \leq n \).

**Theorem 2.1.** Let \( X \) be a finite dimensional compact Hausdorff space. Let \( \epsilon, \gamma \in \text{Bun}(X) \) and \( \theta^n \) denote the product bundle of dimension \( n \). Let \( k \) be the dimension of \( \epsilon \) and write \( m = \lfloor \frac{\dim X}{2} \rfloor \). The following hold,

1. \( \epsilon \cong \eta \oplus \theta^{k-m} \) for some \( \eta \in \text{Bun}_m(X) \).
2. If \( k > \lfloor \frac{\dim X}{2} \rfloor \) and \( \epsilon \oplus \delta \cong \gamma \oplus \delta \), where \( \delta \) is a trivial bundle over \( X \), then \( \epsilon \cong \gamma \).

A proof of Theorem 2.1 in the case of \( X \) being a \( CW \)-complex is given by Husemoller [7, Chap. 9, Theorems 1.2 and 1.5]. Any compact Hausdorff space \( X \) of dimension \( d \) can be realized as the inverse limit of compact metric spaces \( X_\alpha \), with \( \dim X_\alpha \leq d \) for each \( \alpha \) [9, Chap 27, Theorem 8] and any compact metric space \( Y \) is homeomorphic to an inverse limit of finite simplicial complexes of dimension not greater than that of \( Y \) [9, Chap. 27, Theorem 12]). In [6, Theorem 2.5], Goodearl uses these identifications and Husemoller’s results [7, Chap. 9, Theorems 1.2 and 1.5] to prove the Theorem for arbitrary compact Hausdorff spaces.

One immediate consequence of Theorem 2.1 is that, given \( \epsilon \in \text{Bun}_k(X) \) we may choose a complementary bundle \( \epsilon^\perp \) so that the dimension of \( \epsilon^\perp \) is at most \( \lfloor \frac{\dim X}{2} \rfloor \).

To observe this, first choose some \( \delta \in \text{Bun}(X) \), say of dimension \( t \), so that \( \epsilon \oplus \delta \cong \theta^{k+t} \). If \( t > \lfloor \frac{\dim X}{2} \rfloor \), by part 1 of the Theorem, \( \delta = \gamma \oplus \theta^{k-t} \lfloor \frac{\dim X}{2} \rfloor \) for some \( \gamma \in \text{Bun}_k \lfloor \frac{\dim X}{2} \rfloor \). If dimension of \( \epsilon \oplus \gamma \) is \( k_1 \), then \( k_1 = k + \lfloor \frac{\dim X}{2} \rfloor \geq \lfloor \frac{\dim X}{2} \rfloor \).

Upshot of all this is that for fixed \( n, k \in \mathbb{N} \) with \( n \geq k + \lfloor \frac{\dim X}{2} \rfloor \), by following the methods discussed in the preceding subsection we can now construct a bijective correspondence between the isomorphism classes of \( \text{Bun}_k(X) \) and Murray-von Neumann equivalence classes of projections in \( P_k(M_n(X)) \). For convenience let us call this the Serre-Swan correspondence. In section 3, we will combine this correspondence with part 1 of Theorem 2.1 to construct continuous maps \( a : X \to (M_n)_+ \), which satisfy specified rank constraints.
Well supported positive elements.

Given \( a \in M_n(C(X))_+ \), the rank function of \( a \) - denoted \( r_a \), is the (lower semicontinuous) function defined on \( X \) by \( x \mapsto \text{rank}(a(x)) \). For \( a, b \in M_n(C(X)) \) if \( a - b \in M_n(C(X))_+ \), we write \( a \leq b \).

We now give the definition of a well supported positive element in \( M_n(C(X)) \), given in [15].

Definition 2.2. Let \( X \) be a compact Hausdorff space and let \( a \in M_n(C(X))_+ \). Suppose that \( n_1 < n_2 < \cdots < n_L \) denote all the values that \( r_a \) takes and set \( E_i = \{ x \in X : r_a(x) = n_i \} \). We say that \( a \) is well supported if, for each \( 1 \leq i \leq L \) there is a projection \( p_i \in M_n(C(\overline{E}_i)) \) such that \( \lim_{r \to \infty} a(x)^{1/r} = p_i(x), \forall x \in E_i \), and \( p_i(x) \leq p_j(x) \) whenever \( x \in \overline{E}_i \cap \overline{E}_j \), and \( i \leq j \).

Following Theorem can be used to replace arbitrary positive elements by well supported ones up to homotopy (see Lemma 3.1).

Theorem 2.3. [14 Theorem 3.9]. Let \( X \) be a compact Hausdorff space and let \( a \in M_n(C(X))_+ \). Then, for every \( \delta > 0 \), there exists a well supported element \( b \in M_n(C(X))_+ \) such that \( b \leq a \) and \( ||a - b|| < \delta \), with the range of \( r_b \) equal to the range of \( r_a \).

Remark 2.4. The Theorem stated above is only a part of Theorem 3.9 of [14]. There, in the hypothesis \( X \) is assumed to be a finite simplicial complex. However, the simplicial structure of \( X \) is required only for the second part of the Theorem, which guarantees that each \( \overline{E}_i \) corresponding to \( b \) (as defined in 2.2) can assumed to be a sub-complex of \( X \). For the Theorem as stated here, \( X \) being compact and Hausdorff suffices.

Some useful extension results.

Given a function \( f \) on \( X \) and a \( Z \subset X \), we use \( f \mid_Z \) to denote the restriction of \( f \) to \( Z \).

Let \( a \in M_n(C(X))_+ \) be well supported and fix a closed subset \( Y \subset X \). Let \( q \) be a trivial projection on \( Y \), majorized by the support projection of \( a \mid_Y \). In section 3 we would require to extend \( q \) to a trivial projection defined on \( X \), such that extension is also majorized by the support projection of \( a \). To accomplish this, we follow ideas developed by Toms [15] based on the work of Phillips [10].

Theorem 2.5. [10 Proposition 4.2 (1)]. Let \( X \) be a compact Hausdorff space of dimension \( d < \infty \), and let \( Y \subset X \) be closed. Let \( p, q \in M_n(C(X)) \) be projections with the property that \( \text{rank}(q(x)) + \left\lfloor \frac{d}{2} \right\rfloor \leq \text{rank}(p(x)), \forall x \in X \). Let \( s_0 \in M_n(C(Y)) \) be such that \( s_0^*s_0 = q \mid_Y \) and \( s_0s_0^* \leq p \mid_Y \). It follows that there is \( s \in M_n(C(X)) \) such that \( ts = q, ss^* \leq p \), and \( s_0 = s \mid_Y \).

Corollary 2.6. [15 Corollary 2.7. (ii)]. Let \( X \) be a compact Hausdorff space of dimension \( d < \infty \), and let \( E_1, \ldots, E_k \) be a cover of \( X \) by closed sets. Let \( q \in P(M_n(C(X))) \) and for each \( i = 1, \ldots, k \) let \( p_i \in M_n(C(E_i)) \) be a projection of constant rank \( n_i \). Assume that \( n_1 < n_2 < \cdots < n_k \) and \( p_i(x) \leq p_j(x) \) whenever \( x \in E_i \cap E_j \) and \( i \leq j \). Finally, suppose that \( n_i - \text{rank}(q) \geq \left\lfloor \frac{d}{2} \right\rfloor \) for every \( i \). Then the following hold:
If $Y \subset X$ is closed, $q \mid_Y$ is trivial and $\forall y \in Y,$

$$q(y) \leq \bigwedge_{\{i : y \in E_i\}} p_i(y),$$

then $q \mid_Y$ can be extended to a projection $\tilde{q}$ on $X$ which is also trivial and satisfies,

$$\tilde{q}(x) \leq \bigwedge_{\{i : x \in E_i\}} p_i(x), \forall x \in X.$$
Thus, for each \( y \in U_x \),
\[
\text{rank}[\chi_{(\eta, ||a||)}(a(y))] \geq m = \text{rank}[\chi_{(\eta, ||a||)}(a(x))].
\]
\[\square\]

3. Proof of the main result

Our first aim is to prove Lemma 3.2, which is the main technical result. Next we prove Proposition 3.4 which extends [7, Chap. 8, Theorem 7.2] to compact Hausdorff spaces. Theorem 3.5 follows immediately by combining the two Lemmas.

The following is a direct consequence of Theorem 2.3.

**Lemma 3.1.** Let \( X \) be a compact Hausdorff space and \( a \in M_n(C(X))_+ \). Then, there exists a continuous path \( t \mapsto a_t \) in \( M_n(C(X))_+ \) connecting \( a_0 = a \) to \( a_1 \), where \( a_1 \) is well supported in the sense of Definition 2.2 and has the same rank values as that of \( a \). The path is such that \( \text{rank}(a_t(x)) = \text{rank}(a(x)) \), \( \forall t \in (0,1), \forall x \in X \).

**Proof.** Applying Theorem 2.3, choose a well supported positive element \( b \leq a \) such that \( b \) has the same rank values as that of \( a \), let \( a_t = (1 - t)a + tb \). We only have to verify that \( \text{rank}(a_t(x)) = \text{rank}(a(x)) \) for every \( t \in (0,1) \) and \( x \in X \). But this is immediate.

Since \( b \leq a \), if \( 0 < t < 1 \),
\[(1 - t)a \leq a_t \leq (1 - t)a + ta = a.\]
Therefore, \( \text{rank}(a_t(x)) = \text{rank}(a(x)), \forall t \in (0,1), \forall x \in X \). \[\square\]

**Lemma 3.2.** Let \( X \) be a compact Hausdorff space with \( \dim X < \infty \). Suppose \( n, k, l \in \mathbb{N} \) are such that \( k \leq n \) and \( k - l \geq \left\lfloor \frac{\dim X}{2} \right\rfloor \) and let \( a \in C(X, S(n,k,l)) \). Then, there is a continuous path \( h : [0,1] \to C(X, S(n,k,l)) \) such that \( h(0) = a \) and \( h(1) \) is a trivial projection of rank \( l \).

**Proof.** Let \( X, n, k, l \) and \( a \) be as given in the hypothesis. By Lemma 3.1 we can clearly assume that \( a \) is well supported.

Let the rank values of \( a \) be \( n_1 < n_2 < \ldots < n_L \) and let \( E_1, E_2, \ldots, E_L \) and \( p_1, p_2, \ldots, p_L \) be as in Definition 2.2. For convenience we will write \( F_i = E_i \) and \( d = \dim X \).

We first consider the case \( n_L \leq \left\lfloor \frac{d}{2} \right\rfloor \). Then, choose \( p \in M_n(C(X)) \) to be any trivial projection of rank \( l \) and let
\[
h(t) = (1 - t)a + tp.
\]
Now for each \( t \in [0,1], x \in X \),
\[
\text{rank}[h(t)(x)] \leq \text{rank}(a(t)(x)) + \text{rank}p \leq n_L + l \leq \left\lfloor \frac{d}{2} \right\rfloor + l \leq k,
\]
and clearly \( \text{rank}[h(t)(x)] \geq l \). Thus, we get the required path.
Now let us assume $n_L > \left\lfloor \frac{d}{2} \right\rfloor$.

Fix $r$ such that $n_r > \left\lfloor \frac{d}{2} \right\rfloor$ and $n_{r-1} \leq \left\lfloor \frac{d}{2} \right\rfloor$, where we allow the possibility $r = 1$ and set $n_0 = 0$, $F_0 = \emptyset$.

In what proceeds, we will construct a trivial projection $R \in M_n(C(X))$ of rank $l$ such that,

$$\text{rank } (R + a)(x) \leq k, \forall x \in X.$$  

Once we have such $R$, we define $h : [0, 1] \to M_n(C(X))$ by,

$$h(t) = (1 - t)a + tR, \forall t \in [0, 1].$$

Then it's immediate that this path satisfies the said rank constraints (i.e. remains in side $C(S(n, k, l)$).

We focus on constructing $R$. To this end, we first define a trivial projection $q_L \in M_n(C(\bigcup_{r \leq j \leq L} F_j))$ such that,

$$\text{rank } q_L = n_L - \left\lfloor \frac{d}{2} \right\rfloor.$$

and

$$\text{rank } (a + q_L)(x) \leq n_L, \forall x \in \bigcup_{r \leq j \leq L} F_j.$$  

We follow an inductive argument to define $q_L$.

Since $F_r$ is compact Hausdorff with $\text{dim} F_r \leq d$ and $p_r \in M_n(C(F_r))$ is a projection of rank $n_r > \left\lfloor \frac{d}{2} \right\rfloor$, using Theorem 2.1 (1) and Serre-Swan correspondence we find a trivial projection $q_r \in M_n(C(F_r))$ such that,

$$\text{rank } q_r = n_r - \left\lfloor \frac{d}{2} \right\rfloor.$$

and $q_r \leq p_r$.

By the requirements for well supportedness of $a$, each $p_i \in M_n(C(F_i))$ is of constant rank $n_i$ and whenever $r \leq i \leq j$ with $F_i \cap F_j \neq \emptyset$,

$$p_i(x) \leq p_j(x), \forall x \in F_i \cap F_j. \quad (3.1)$$

Also for all $j \geq r$,

$$\text{rank } p_j - \text{rank } q_r \geq n_r - \text{rank } q_1 \geq \left\lfloor \frac{d}{2} \right\rfloor.$$

Hence, we apply 2.6 with $X = \bigcup_{r \leq j \leq L} F_j$, $Y = F_1$, $q = q_r$ (by the remark following 2.7, $q$ in 2.6 need not be defined on $X$ a priori) to extend $q_r$ to a trivial projection in $M_n(C(\bigcup_{r \leq j \leq L} F_j))$ - which is again called $q_r$, such that whenever $r \leq j$,

$$q_r(x) \leq p_j(x), \forall x \in F_j.$$

Then, for each $r \leq j \leq L$,

$$\text{rank } (q_r + a)(x) \leq n_j, \forall x \in F_j.$$
If $r = L$ then we are done (defining $q_L$).

Thus, let us assume $r < L$.

Suppose that for some $r \leq t < L$ we have defined a trivial projection $q_t \in M_n(C(\bigcup_{r \leq j \leq L} F_j))$ such that the following hold,

\begin{align*}
\text{(3.2)} \quad \text{rank} \ q_t &= n_t - \left\lfloor \frac{d^2}{2} \right\rfloor, \\
\text{(3.3)} \quad q_t(x) &\leq p_j(x), \forall x \in F_j, \forall t \leq j \leq L, \\
\text{(3.4)} \quad \text{rank} \ (q_t + p_j) &\leq n_t, \forall r \leq j \leq t.
\end{align*}

Now whenever $t + 1 \leq j \leq L$, $(p_j - q_t) |_{F_{t+1}, i} \in M_n(C(F_{t+1}))$ is a projection constant rank $(n_j - n_t) + \left\lfloor \frac{d^2}{2} \right\rfloor$.

Thus, since $\dim F_{t+1} \leq d$, by applying 2.1 we choose a trivial projection $q_{t,t+1} \in M_n(C(F_{t+1}))$ such that,

\begin{align*}
\text{rank} \ q_{t,t+1} &= n_{t+1} - n_t \\
\text{and } q_{t,t+1} &\leq p_{t+1} - q_t.
\end{align*}

Moreover, by applying 2.6 with $X = \bigcup_{t+1 \leq j \leq L} F_j$, $Y = F_{t+1}$ and $q = q_{t,t+1}$ we extend $q_{t,t+1}$ to a trivial projection in $M_n(C(\bigcup_{t+1 \leq j \leq L} F_j))$ (which we again name $q_{t,t+1}$) such that whenever $j \geq t + 1$,

\begin{align*}
\text{(3.5)} \quad q_{t,t+1}(x) &\leq p_j(x) - q_t(x), \forall x \in F_j.
\end{align*}

Set $q_{t+1} = q_t + q_{t,t+1}$.

Then, since $q_t, q_{t,t+1}$ are orthogonal trivial projections, $q_{t+1}$ is a trivial projection in $M_n(C(\bigcup_{t+1 \leq j \leq L} F_j))$.

Moreover, by 3.2,

\begin{align*}
\text{rank} \ q_{t+1} &= (n_t - \left\lfloor \frac{d^2}{2} \right\rfloor) + (n_{t+1} - n_t) \\
&= n_{t+1} - \left\lfloor \frac{d^2}{2} \right\rfloor
\end{align*}

and whenever $j \geq t + 1$, $\forall x \in F_j$ (by 3.3 and 3.5),

\begin{align*}
q_{t+1}(x) &= [q_t(x) + q_{t,t+1}(x)] \\
&\leq p_j(x)
\end{align*}

and finally for each $r \leq j \leq t + 1$ (by 3.4),

\begin{align*}
\text{rank} \ (q_{t+1} + p_j) &\leq \text{rank} \ q_{t,t+1} + \text{rank} \ (q_t + p_j) \\
&\leq (n_{t+1} - n_t) + n_t \\
&= n_{t+1}.
\end{align*}
By proceeding in this manner we construct a trivial projection \( q_L \in M_n(C( \bigcup_{r \leq j \leq L} F_j)) \) of rank \( n_L - \lfloor \frac{d}{2} \rfloor \) such that,

\[
\text{rank}(q_L + p_j)(x) \leq n_L, \forall x \in F_j, \forall r \leq j \leq L
\]

and

\[
q_L(x) \leq p_L(x), \forall x \in F_L.
\]

Choose \( R_1 \in M_n(C(X)) \) to be any trivial projection (of rank \( n_L - \lfloor \frac{d}{2} \rfloor \)) which extends \( q_L \). Note that such \( R_1 \) exists by Corollary 2.7.

By the choice of \( r \), whenever \( j < r \), \( \forall x \in F_j \),

\[
\text{rank}(R_1 + a)(x) \leq (n_L - \lfloor \frac{d}{2} \rfloor) + \lfloor \frac{d}{2} \rfloor \leq n_L.
\]

Thus, since \( R_1 |_{F_j} = q_L |_{F_j} \) whenever \( j \geq r \) we conclude that,

\[
\text{rank}(R_1 + a)(x) \leq n_L, \forall x \in X.
\]

If \( n_L = k \), then

\[
\text{rank} R_1 = k - \lfloor \frac{d}{2} \rfloor \\
\geq l
\]

and we choose \( R \) to be any trivial sub-projection of rank \( l \).

Hence we are left with the case \( k > n_L \).

Then,

\[
\text{rank} (1_n - R_1) = n - (n_L - \lfloor \frac{d}{2} \rfloor) \\
\geq (k - n_L) + \lfloor \frac{d}{2} \rfloor
\]

and we apply Theorem 2.1 (1) and Serre-Swan for one last time to choose a trivial projection \( R_2 \in M_n(C(X)) \) of rank \( k - n_L \) with \( R_2 \leq (1_n - R_1) \).

Now \( R_1 + R_2 \) is a trivial projection of rank \( k - \lfloor \frac{d}{2} \rfloor \) and

\[
\text{rank} (R_1 + R_2 + a)(x) \leq k, \forall x \in X.
\]

To complete the proof we choose \( R \) to be a trivial sub-projection of \( R_1 + R_2 \) of rank \( l \).

Recall the following theorem for locally trivial vector bundles bundles over CW-complexes.

**Theorem 3.3.** [7 Chp. 8, Theorem 7.2] Let \( X \) be a CW-complex and \( n,l \) be non-negative integers. Then, the function that assigns to each homotopy class \([f]: X \to G_l(C^n)\) the isomorphism class of the \( k \)-dimensional vector bundle \( f^*(\gamma_k^n) \) over \( X \) is a bijection, if \( n \geq l + \left\lceil \frac{\dim X}{2} \right\rceil \).
Combining 3.3 with Lemma 3.2 proves Theorem 3.5 for all CW-complex. The fact that Theorem 3.3 extends to the case of $X$ being Compact Hausdorff is probably well known. However, we could not find a clear reference for this in the literature. Note that since $G_l(C^\infty)$ (the $l$-dimensional Grassmannian over $C^\infty$) is the classifying space for $l$-dimensional vector bundles over $\mathbb{R}^\infty$ spaces, if one replaces $G_l(C^n)$ by $G_l(C^\infty)$ the conclusion of 3.3 holds even for paracompact spaces. But the application we have in mind require the target space to be $G_l(C^n)$. In Proposition 3.4, based on the proof of [6] Theorem 2.5 we apply dimension theory results [5, Chap. X, Sec. 10] and few $C^*$-algebraic techniques to extend the conclusion of 3.3 to compact Hausdorff spaces.

If $A, B$ are $C^*$-algebras and $\phi : A \to B$ is a $*$-homomorphism, then for any $n \in \mathbb{N}$ we have an induced $*$-homomorphism from $M_n(A)$ to $M_n(B)$ given by $[a_{ij}] \mapsto [\phi(a_{ij})]$. We will use $\phi$ to denote this $*$-homomorphism as well.

For two projections $p, q \in M_n(C(X))$, we write $p \sim_h q$ if there is a projection valued continuous path in $M_n(C(X))$ which connects $p$ and $q$.

**Proposition 3.4.** Let $X$ be a compact Hausdorff space with $\dim X < \infty$ and suppose $n, k \in \mathbb{N}$ with $n - k \geq \lceil \frac{\dim X}{2} \rceil$. Then the isomorphism classes of $k$-dimensional locally trivial complex vector bundles over $X$ are in bijective correspondence with the homotopy classes of maps $p : X \to P_k(C^n)$, where $P_k(C^n)$ stands for rank $k$ projections in $M_n(C)$.

**Proof.** Let $[X, P_k(C^n)]$ stand for the homotopy classes of maps in $P_k(M_n(C(X)))$ and let $Vect_k(X)$ denote the set of all isomorphic classes of locally trivial $k$-dimensional vector bundles over $X$.

Define $\psi : [X, P_k(C^n)] \to Vect_k(X)$ by $\psi([p]) = [\epsilon_p], \forall p \in P_k(M_n(C(X)))$, where $\epsilon_p$ is defined as in section 2. Since the homotopy equivalence of projections implies Murray-von Neumann equivalence (see [5, Prop. 2.2.7]), map $\psi$ is well defined by the discussion in section 2. Moreover, the discussion following Theorem 2.1 shows that $\psi$ is surjective.

To complete the proof of 3.4, we only have to show that if the vector bundles associated with two projections in $P_k(M_n(C(X)))$ are isomorphic then the two projections are homotopic in $P_k(M_n(C(X))))$.

Let us first assume that $X$ is a compact metric space.

Then by [9] Chap. 27, Theorem 8.1 or [5] Chap. X, Sec. 10, $X$ is homeomorphic to an inverse limit of finite simplicial complexes $X_\alpha$, with $\dim X_\alpha \leq \dim X$ for each $\alpha$.

Let $\psi_\alpha : X \to X_\alpha$ be the canonical induced maps. We have the induced homomorphisms,

$$\psi^{T}_\alpha : C(X_\alpha) \to C(X)$$

given by $\psi^{T}_\alpha (f) = f \circ \psi_\alpha$.

Moreover, by the inverse limit structure

$$\bigcup_{\alpha} \psi^{T}_\alpha (C(X_\alpha)) = C(X),$$

Combining 3.3 with Lemma 3.2 proves Theorem 3.5 for all CW-complex. The fact that Theorem 3.3 extends to the case of $X$ being Compact Hausdorff is probably well known. However, we could not find a clear reference for this in the literature. Note that since $G_l(C^\infty)$ (the $l$-dimensional Grassmannian over $C^\infty$) is the classifying space for $l$-dimensional vector bundles over $\mathbb{R}^\infty$ spaces, if one replaces $G_l(C^n)$ by $G_l(C^\infty)$ the conclusion of 3.3 holds even for paracompact spaces. But the application we have in mind require the target space to be $G_l(C^n)$. In Proposition 3.4, based on the proof of [6] Theorem 2.5 we apply dimension theory results [5, Chap. X, Sec. 10] and few $C^*$-algebraic techniques to extend the conclusion of 3.3 to compact Hausdorff spaces.

If $A, B$ are $C^*$-algebras and $\phi : A \to B$ is a $*$-homomorphism, then for any $n \in \mathbb{N}$ we have an induced $*$-homomorphism from $M_n(A)$ to $M_n(B)$ given by $[a_{ij}] \mapsto [\phi(a_{ij})]$. We will use $\phi$ to denote this $*$-homomorphism as well.

For two projections $p, q \in M_n(C(X))$, we write $p \sim_h q$ if there is a projection valued continuous path in $M_n(C(X))$ which connects $p$ and $q$.

**Proposition 3.4.** Let $X$ be a compact Hausdorff space with $\dim X < \infty$ and suppose $n, k \in \mathbb{N}$ with $n - k \geq \lceil \frac{\dim X}{2} \rceil$. Then the isomorphism classes of $k$-dimensional locally trivial complex vector bundles over $X$ are in bijective correspondence with the homotopy classes of maps $p : X \to P_k(C^n)$, where $P_k(C^n)$ stands for rank $k$ projections in $M_n(C)$.

**Proof.** Let $[X, P_k(C^n)]$ stand for the homotopy classes of maps in $P_k(M_n(C(X)))$ and let $Vect_k(X)$ denote the set of all isomorphic classes of locally trivial $k$-dimensional vector bundles over $X$.

Define $\psi : [X, P_k(C^n)] \to Vect_k(X)$ by $\psi([p]) = [\epsilon_p], \forall p \in P_k(M_n(C(X)))$, where $\epsilon_p$ is defined as in section 2. Since the homotopy equivalence of projections implies Murray-von Neumann equivalence (see [5, Prop. 2.2.7]), map $\psi$ is well defined by the discussion in section 2. Moreover, the discussion following Theorem 2.1 shows that $\psi$ is surjective.

To complete the proof of 3.4, we only have to show that if the vector bundles associated with two projections in $P_k(M_n(C(X)))$ are isomorphic then the two projections are homotopic in $P_k(M_n(C(X))))$.

Let us first assume that $X$ is a compact metric space.

Then by [9] Chap. 27, Theorem 8.1 or [5] Chap. X, Sec. 10, $X$ is homeomorphic to an inverse limit of finite simplicial complexes $X_\alpha$, with $\dim X_\alpha \leq \dim X$ for each $\alpha$.

Let $\psi_\alpha : X \to X_\alpha$ be the canonical induced maps. We have the induced homomorphisms,

$$\psi^{T}_\alpha : C(X_\alpha) \to C(X)$$

given by $\psi^{T}_\alpha (f) = f \circ \psi_\alpha$.

Moreover, by the inverse limit structure

$$\bigcup_{\alpha} \psi^{T}_\alpha (C(X_\alpha)) = C(X),$$
Note that for all $h$ continuous path, Theorem 3.3 and the identification of $\psi$ and hence $\bigcup_{\alpha} \psi_\alpha^T(C(X_\alpha))$ is indeed a $*$-subalgebra.

Suppose $p, q \in M_n(C(X))$ are projections of constant rank $k$ such that the corresponding vector bundles are isomorphic. This means, $p = v^*v, q = vv^*$ for some $v \in M_n(C(X))$.

Fix $0 < \epsilon < 1/3$. By (3.6), there is some $\alpha$ and $\tilde{v}_\alpha \in M_n(C(X_\alpha))$ such that,

$$\|\psi_\alpha^T(\tilde{v}_\alpha) - v\| < \epsilon/2.$$

Write $Y_\alpha = \psi_\alpha(X)$ and $v_\alpha = \tilde{v}_\alpha |_{Y_\alpha}, a_\alpha = v_\alpha^*v_\alpha, b_\alpha = v_\alpha v_\alpha^*$.

Now $a_\alpha \in M_n(C(Y_\alpha))$ and moreover by the choice of $v_\alpha$ it follows that,

$$\text{spec}(a_\alpha) \subset [0, \epsilon) \cup (1 - \epsilon, 1].$$

Similarly,

$$\text{spec}(b) \subset [0, \epsilon) \cup (1 - \epsilon, 1].$$

Let $f: [0, 1] \to [0, 1]$ be the continuous function which vanishes on $[0, \epsilon]$, is equal to 1 on $[1 - \epsilon, 1]$ and is linear on $(\epsilon, 1 - \epsilon)$. Then $f(a_\alpha), f(b_\alpha)$ are projections in $M_n(C(Y_\alpha))$ with

$$\|a_\alpha - f(a_\alpha)\| < \epsilon, \|b_\alpha - f(b_\alpha)\| < \epsilon.$$

Furthermore, as $a, b$ are Murray-von Neumann equivalent, by Lemma 3.3 of [4] there is some $s_0 \in M_n(C(Y_\alpha))$ such that,

$$f(a_\alpha) = s_0^*s_0, f(b_\alpha) = s_0 s_0^*.$$

Since $Y_\alpha$ is a closed in $X_\alpha$, we may choose some open neighborhood $U$ of $Y_\alpha$ so that $f(a_\alpha), f(b_\alpha)$ extend to projections in $M_n(C(U))$. Let $p_\alpha, q_\alpha$ denote these extensions respectively. Moreover, since $f(a_\alpha) \sim f(b_\alpha)$, we may choose the extensions in such a way that $p_\alpha \sim q_\alpha$.

As $X_\alpha$ is a finite simplicial complex, after a finite simplicial refinement of $X_\alpha$ via barycentric subdivisions, choose a sub complex $Z$ of $X_\alpha$ with $Y \subset Z \subset U$. For convenience let us denote the restrictions of $p_\alpha, q_\alpha$ to $Z$ by $p_\alpha, q_\alpha$.

Note that $p_\alpha, q_\alpha$ generate isomorphic vector bundles over $Z$, each of rank $k$. Hence, form Theorem 3.3 and the identification of $G_k(\mathbb{C}^n)$ with $P_k(\mathbb{C}^n)$, there is a continuous path,

$$t \mapsto h_\alpha(t) \in P_k(M_n(C(Z))),$$

such that $h_\alpha(0) = p_\alpha, h(1) = q_\alpha$.

This gives a path $t \mapsto h(t) \in P_k(M_n(C(X))),$ given by $h(t)(x) = h_\alpha(t)\psi_\alpha(x))$.

Note that for all $x \in X$,

$$\|p(x) - h(0)(x)\| = \|p(x) - f(a_\alpha)(\psi_\alpha(x))\|$$

$$\leq \|v^*v(x) - (\tilde{v}_\alpha^*\tilde{v}_\alpha)(\psi_\alpha(x))\| + \|(\tilde{v}_\alpha^*\tilde{v}_\alpha)(\psi_\alpha(x)) - f(a_\alpha)(\psi_\alpha(x))\|$$

$$\leq 2\epsilon + \|a_\alpha(\psi_\alpha(x)) - f(a_\alpha)(\psi_\alpha(x))\|$$

$$\leq 2\epsilon + \epsilon$$
Thus, $||p - h(0)|| < 1$ and similarly $||q - h(1)|| < 1$.

Therefore, from [8, Proposition 2.2.4]

$$p \sim_h h(0) \sim_h h(1) \sim_h q.$$ 

This completes the proof for compact metric spaces.

The proof of the Proposition in the case of $X$ being an arbitrary compact Hausdorff space follows almost identically. In this case $X$ is the inverse limit of compact metric spaces $X_\lambda$, with $\dim X_\lambda \leq \dim X$ for each $\lambda$. By the preceding the result holds for each $X_\lambda$, and now we can argue as in the preceding case.

**Theorem 3.5.** Let $X$ be a compact Hausdorff space with $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$. There is only one homotopy class of functions $f : X \to S(n, k, l)$, i.e. the function space $C(X, S(n, k, l))$ is path connected.

**Proof.** Observe that the case $n = k$ is straightforward. For any $a \in C(X, S(n, k, l))$ we have the linear path $t \mapsto (1 - t)a + 1_n$ connecting $a$ to $1_n$.

So we assume $n > k$.

Let $a, b \in C(X, S(n, k, l))$. As $\dim X \leq k - l$, by applying Lemma 3.2 choose trivial projections $p, q$ of rank $l$ such that there are paths inside $C(X, S(n, k, l))$ connecting $a$ to $p$ and $b$ to $q$. Since $n > k$ and $k - l \geq \lfloor \frac{\dim X}{2} \rfloor$, we get $n - l \geq \lfloor \frac{\dim X}{2} \rfloor$. Therefore, as $p, q$ are both trivial, by Proposition 3.4 there is a path inside $C(X, S(n, k, l))$ connecting $p$ and $q$. Thus, there is a path between $a$ and $b$ in $C(X, S(n, k, l))$. $\square$

**Corollary 3.6.** For every $r \leq 2(k - l) + 1$, $\pi_r(S(n, k, l)) = 0$.

**Proof.** Follows directly from Theorem 3.5 as $\dim S^r = r$. $\square$

Even though we derived the above result as a Corollary to Theorem 3.5, from a topological view point it might seem more natural (and technically easier) to first prove the conclusion of 3.6 independently and then apply techniques from homotopy theory and dimension theory to prove Theorem 3.5. This is indeed possible and applies in a slightly wider scope. In section 4, we show that for compact Hausdorff spaces of finite covering dimension the path connectedness of $C(X, S(n, k, l))$ depends solely on homotopy groups $\pi_r(S(n, k, l))$, $r \leq \dim X$. We chose not to follow this method for the proof of 3.5, due to two reasons. Firstly, with the techniques we used, the proof of Lemma 3.2 would not be any simpler if we assumed $X$ to be a sphere instead of a general compact Hausdorff spaces. Secondly, if we used such an argument (i.e proving 3.6 independently and then showing 3.5) it would have avoided the use of Proposition 3.4, but we thought 3.4 could be of independent interest for some readers.

**4. Homotopy groups of $S(n, k, l)$ and Path Connectedness of $C(X, S(n, k, l))$.**

In Theorem 4.6, combining well known homotopy theory techniques and classical $C^*$-algebraic results we prove that for a fixed integer $d$ if $\pi_r(S(n, k, l)) = 0$, $\forall r \leq d$, then $C(X, S(n, k, l))$ is path connected for every compact Hausdorff space $X$ with $\dim X \leq d$. Note that in this section we do not assume $k - l \geq \lfloor \frac{\dim X}{2} \rfloor$. 


The proof of Lemma 4.5 is of the same flavor as that of Proposition 3.4. We need few more technical results.

The following is well known and was used in the proof of 3.4 as well. We state it here for convenience.

**Proposition 4.1.** Let $X$ be a finite simplicial complex. Let $Y \subset X$ be closed and $U$ be an open neighborhood of $Y$ in $X$. Then, after a finite simplicial refinement of $X$, there is a sub-complex $Z$ of $X$ such that $Y \subset Z \subset U$.

Lemma 4.2 is a simpler version of [13, Lemma 2.1]. For the sake of completeness, we provide a proof.

**Lemma 4.2.** [13, Lemma 2.1] Let $X$ be a compact Hausdorff space and suppose that $a \in M_n(C(X))_+$. Let $l \in \mathbb{N}$ be such that $\text{rank}(a(x)) \geq l$, $\forall x \in X$. Then, there is some $\eta > 0$ such that for each $x \in X$, the spectral projection $\chi_{(\eta, \infty)}(a(x))$ has rank at least $l$.

**Proof.** For each $x \in X$, let $\eta_x = \frac{1}{2} \min\{\lambda \in \text{spec } a(x); \lambda > 0\}$. Note that since $l > 0$, $\eta_x$ exists for each $x \in X$. Then,

$$\text{rank}(a(x)) = \text{rank}\left[\chi_{(\eta_x, \infty)}(a(x))\right], \forall x \in X.$$ 

By Proposition 2.8, for each $x \in X$, the map $y \mapsto \text{rank}\left[\chi_{(\eta_y, \infty)}(a(y))\right]$ is lower semi-continuous. So, for each $x \in X$, there exists an open neighborhood $U_x$ of $x$ such that,

$$\text{rank}\left[\chi_{(\eta_y, \infty)}(a(y))\right] \geq \text{rank}\left[\chi_{(\eta_x, \infty)}(a(x))\right], \forall y \in U_x.$$ 

By compactness of $X$, choose some finite set of points $\{x_1, x_2, \ldots, x_L\}$ such that $X = \bigcup_{1 \leq i \leq L} U_{x_i}$. By setting $\eta = \min_{1 \leq i \leq L}(\eta_{x_i}) > 0$, for every $y \in U_{x_i}$, we get,

$$\text{rank}\left[\chi_{(\eta, \infty)}(a(y))\right] \geq \text{rank}\left[\chi_{(\eta_x, \infty)}(a(x))\right] \geq \text{rank}\left[\chi_{(\eta_{x_i}, \infty)}(a(x))\right] = \text{rank}(a(x)) \geq l.$$ 

Since $X = \bigcup_{1 \leq i \leq L} U_{x_i}$, this completes the proof. □

Let $X = \varprojlim X_\alpha$, where $(X_\alpha, \psi_{\alpha\beta})$ is an inverse system of compact Hausdorff spaces and $\psi_\alpha : X \to X_\alpha$ be the natural maps. In the proof of Lemma 4.5, given $a \in C(X, S(n, k, l)) \subset M_n(C(X))_+$ we need to approximate $a$ in norm, by elements in $\psi_\alpha^* \left( C(X_\alpha, S(n, k, l)) \right)$. Here, $\psi_\alpha^*$ is defined as in the proof of Proposition 3.4. To achieve this, we will use the previous Lemma with two other results from $C^*$-algebra theory. First we prove Lemma 4.3, which follows as a special case of a well known fact in $C^*$-algebras. For the sake of completeness we include the proof for the special case.
Lemma 4.3. Let $X = \lim_{\alpha} X_\alpha$, for a inverse system of compact Hausdorff spaces $(X_\alpha, \psi_\alpha)$. Let $\psi_\alpha : X \to X_\alpha$ be the natural maps and $\epsilon > 0$. Given $a \in M_n(C(X))_+$, there is some index $\alpha$ and some $b \in M_n(C(X_\alpha))_+$ such that,

$$\|a - \psi_\alpha^T(b)\| < \epsilon$$

where $\psi_\alpha^T(b) = b \circ \psi_\alpha$.

Proof. We may assume $\|a\| = 1$ and $\epsilon < 1$. Since $a$ is positive, by the functional calculus of $a$ there is some $c \in M_n(C(X))_+$ such that $c^2 = a$. As in the proof of 3.4, pick some $\alpha$ and $d \in M_n(C(X_\alpha))$ such that,

$$\|c - \psi_\alpha^T(d)\| < \frac{\epsilon}{3}.$$ \hfill (4.1)

Since $c$ is positive, $c^* = c$ where $c^*$ is the conjugate transpose of $c$.

Then,

$$\|c - \psi_\alpha^T(d^*)\| = \|(c^* - \psi_\alpha^T(d^*))\|
= \|(c - \psi_\alpha^T(d))^*\|
= \|c - \psi_\alpha^T(d)\|$$ \hfill (4.2)

Put $b = d^*d$ then $b \in M_n(C(X_\alpha))_+$ and moreover,

$$\|a - \psi_\alpha^T(b)\| = \|c^2 - \psi_\alpha^T(d^*d)\|
= \|c^2 - \psi_\alpha^T(d^*)\psi_\alpha^T(d)\|
\leq \|c^2 - c \cdot \psi_\alpha^T(d)\| + \|c \cdot \psi_\alpha^T(d) - \psi_\alpha^T(d^*)\psi_\alpha^T(d)\|
\leq \|c\| \cdot \|c - \psi_\alpha^T(d)\| + \|c - \psi_\alpha^T(d^*)\| \cdot \|\psi_\alpha^T(d)\|
< \epsilon$$

The last inequality follows from (4.1), (4.2) and the fact that $\|\psi_\alpha^T(d)\| \leq 2$. \hfill $\Box$

For $\epsilon \geq 0$, let $f_\epsilon : [0, 1] \to [0, 1]$ be defined by,

$$f_\epsilon(t) = \max\{\epsilon, t\}, \forall t \in [0, 1].$$

For any $a \in M_n(C(X))_+$, let $(a - \epsilon)_+$ denote the element $f_\epsilon(a) \in M_n(C(X)_+$ given by the functional calculus of $a$. Recall that by support projection of $d \in M_n(C(X))$ we mean the function (not necessarily continuous) which maps $x$ to the orthogonal projection of $C^n$ onto $d(x)(C^n)$. Note that the support projection of $f_\epsilon(a)$ is the spectral projection $\chi_{(\epsilon,1]}(a)$. To prove 4.5, we need the following proposition.

Proposition 4.4. [11] Proposition 2.2 | Let $X$ be compact Hausdorff and $a,b \in M_n(C(X))_+$. Let $\epsilon > 0$ and suppose that $\|b - a\| < \epsilon$. Then there exists $c \in M_n(C(X))$ such that,

$$(a - \epsilon)_+ = c^*bc.$$ 

We now prove Lemma 4.5.

Lemma 4.5. Suppose for each finite simplicial complex $Z$ with $\dim Z \leq d$, the function space $C(Z,S(n,k,l))$ is path connected. Then, for every compact Hausdorff space $X$ of covering dimension $d$, space $C(X,S(n,k,l))$ is path connected.
Proof. Like in the proof of 3.4, we first prove the result for the case of \(X\) being a compact metric space. In this case \(X = \varprojlim X_\alpha\), where \((X_\alpha, \psi_{\alpha\beta})\) is a inverse system finite simplicial complexes with \(\dim X_\alpha \leq d\). Let \(\psi : X \rightarrow X_\alpha\) be the natural maps. 

Fixed \(a \in C(X, S(n, k, l))\), our first goal is to show that there is some index \(\alpha\) and \(c \in C(X_\alpha, (M_n)_+)\) such that \(\psi^T_\alpha(c) \in C(X, S(n, k, l))\) and there is a path in \(C(X, S(n, k, l))\) connecting \(a\) to \(\psi^T_\alpha(c)\). 

To do this, first note that we may assume \(\|a\| = 1\) and use Lemma 4.2 to pick \(\eta > 0\) such that,

\[
\text{rank} \left[ \chi_{(2\eta, 1)}(a(x)) \right] \geq l, \forall x \in X.
\]

By Lemma 4.3, pick \(\alpha\) and \(b \in M_n(C(X_\alpha)_+)\) such that,

\[
\|a - \psi^T_\alpha(b)\| < \eta.
\]

Let us write \(a_\alpha = \psi^T_\alpha(b)\). Note that from Proposition 4.4 and (4.4),

\[
(a_\alpha - \eta)_+ = d^*a_d,
\]

for some \(d \in M_n(C(X))\).

Therefore, for every \(x \in X\),

\[
\text{rank} \left[ (a_\alpha - \eta)_+(x) \right] \leq \text{rank} \left( a(x) \right) \leq k.
\]

(4.5)

From the functional calculus for \(a_\alpha\) and (4.4)

\[
\| (a_\alpha - \eta)_+ - a \| \leq \| (a_\alpha - \eta)_+ - a_\alpha \| + \| a_\alpha - a \| < \eta + \eta = 2\eta.
\]

Therefore, by Proposition 4.4 it follows that,

\[
\text{rank} \left[ (a - 2\eta)_+(x) \right] \leq \text{rank} \left[ (a_\alpha - \eta)_+(x) \right], \forall x \in X.
\]

Now, from (4.3) and the discussion preceding Proposition 4.4,

\[
\text{rank} \left[ (a_\alpha - \eta)_+(x) \right] \geq l, \forall x \in X.
\]

(4.6)

Put \(c = \psi^T_\alpha((b - \eta)_+)\).

Then,

\[
c = (b - \eta)_+ \circ \psi_\alpha = ((b \circ \psi_\alpha) - \eta)_+ = (a_\alpha - \eta)_+.
\]

Thus by (4.5) and (4.6), \(c \in C(X, S(n, k, l))\).

To complete our first goal consider \(h : [0, 1] \rightarrow C(X, S(n, k, l))\) given by,

\[
h(t) = \left[ ((1 - t)a + ta_\alpha) - \eta \right]_+.
\]

We have \(h(0) = (a - \eta)_+\) and \(h(1) = c\).
Note that,
\[ \|a - ((1 - t)a + ta_\alpha)\| = t\|a - a_\alpha\| < \eta, \forall t \in [0, 1]. \]
Thus, by following the same type of argument we used to show \( c \in C(X, S(n, k, l)) \),
it follows that
\[ h(t) = \left[ ((1 - t)a + ta_\alpha) - \eta \right]_+ \in C(X, S(n, k, l)), \forall t \in [0, 1]. \]
The continuity of \( h \) can be established using functional calculus arguments, see [8, Lemma 1.25].
First goal is now achieved once when we notice that \( a \) is homotopic to \( (a - \eta)_+ \)
as maps in \( C(X, S(n, k, l)) \). But this is immediate. Indeed, one can use the linear
path \( t \mapsto (1 - t)a + t(a - \eta)_+ \).
Now suppose \( a, b \in C(X, S(n, k, l)) \). We need to construct a path in \( C(X, S(n, k, l)) \)
joining \( a \) to \( b \). From the first part w.l.o.g. we may assume that \( a = \psi^T_\alpha(c), b = \psi^T_\beta(d) \), for some \( \alpha = \beta \) and \( c, d \in M_n(C(X_\alpha))_+ \).
Put \( Y = \psi_\alpha(X) \subset X_\alpha \). Then as \( X \) is compact so is \( Y \). Moreover, \( Y \) is closed as each \( X_\alpha \) is Hausdorff.
For each \( y = \psi_\alpha(x) \),
\[
\text{rank}(c(y)) = \text{rank}(c(\psi_\alpha(x))) = \text{rank}(a(x))
\]
Hence,
\[ l \leq \text{rank}(c(y)) \leq k, \forall y \in Y. \]
Similarly,
\[ l \leq \text{rank}(d(y)) \leq k, \forall y \in Y. \]
Therefore as \( Y \) is closed, by Lemma 2.7 of [13] there is some open neighborhood \( U \) of
\( Y \) in \( X_\alpha \) and \( \tilde{c}, \tilde{d} \in C(U, S(n, k, l)) \) such that \( \tilde{c}, \tilde{d} \) are extensions of \( c, d \) respectively.
By Proposition 4.1, after a refinement of the simplicial structure of \( X_\alpha \), we have a finite sub complex \( Z \) of \( X_\alpha \) such that, \( Y \subset Z \subset U \). Hence, we may view \( \tilde{c}, \tilde{d} \in C(Z, S(n, k, l)) \). Now, as \( Z \) is a finite simplicial complex with \( \text{dim} Z \leq d \), by the hypothesis there is a path \( \tilde{g} : [0, 1] \to C(Z, S(n, k, l)) \) such that \( \tilde{g}(0) = \tilde{c} \) and \( \tilde{g}(1) = \tilde{d} \).
Define a path \( g : [0, 1] \to C(X, M_n) \) by,
\[ g(t)(x) = \tilde{g}(t)(\psi_\alpha(x)). \]
By definition of \( \tilde{g} \) it is clear that \( g(0) = \psi_\alpha^T(c) = a, g(1) = \psi_\alpha^T(d) = b \) and moreover,
\( g(t) \in C(X, S(n, k, l)) \).
This proves the result in the case of \( X \) being a compact metric space such that \( \text{dim} X \leq d \). To get the result for \( X \) being compact Hausdorff, write \( X = \lim X_\alpha \) where now \( X_\alpha \) is compact metric with \( \text{dim} X_\alpha \leq d \). Since \( C(X_\alpha, S(n, k, l)) \) is path connected for each \( \alpha \) by the first part of the proof, following a similar argument as before give the result. (Note here that the argument is simpler than in the first step. Since \( \psi_\alpha(X) \subset X_\alpha \) is compact metric for any \( \alpha \), we do not require Lemma 2.7 of [13].
\[ \square\]
Theorem 4.6. Let $X$ be compact Hausdorff with $\dim X \leq d$. If $\pi_r(S(n,k,l)) = 0$ for each $r \leq d$ then, $C(X,S(n,k,l))$ is path connected.

Proof. From Lemma 4.5, it suffices to show that $C(K,S(n,k,l))$ is path connected for every finite simplicial complex $K$ with $\dim K \leq d$. We will use induction on the number of simplexes in the complex $K$ to show this.

If $K$ consists of a single simplex then result is true since $K$ is contractible.

Suppose now that result is true for every simplicial complex which contains $r$ number of simplexes.

To complete the inductive step, let $K = L \cup \{s\}$ where $L$ is a sub complex of $K$ containing $r$ number of simplexes and $s$ is a $n$-simplex for some $n \leq d$.

Observe that $\{K,L\}$ is a NDR pair in the sense of [16]. Since $S(n,k,l)$ is locally compact, $S(n,k,l)$ is compactly generated. By [16, Diagram 6.3], following sequence is exact in the category of sets with base points,

$$
(C(L,S(n,k,l))) \xleftarrow{i_*} (C(K,S(n,k,l))) \xleftarrow{p_*} (C((K/L),S(n,k,l))),
$$

where $K/L$ stands for the quotient space. For a space $Z$, by $[C(Z,S(n,k,l))]$ we mean the set of all homotopy classes of maps in $C(Z,S(n,k,l))$ and as the base point we may choose any constant map $z \mapsto a$, for a fixed $a \in S(n,k,l)$. The maps $i_*$ and $p_*$ are the maps induced by the inclusion $i : L \rightarrow K$ and the quotient map $p : K \rightarrow K/L$.

By the induction hypothesis $[C(L,S(n,k,l))]$ consists of a single point. Since $(K/L) \cong S^n$ and $n \leq d$, by assumption $[C((K/L),S(n,k,l))]$ is also a singleton. Thus, by exactness of (4.7), $[C(K,S(n,k,l))]$ contains only one point, i.e. $C(K,S(n,k,l))$ is path connected. \hfill \Box

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