Graphical Models in Loopy Distribution Grids: Topology estimation, change detection & limitations

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Abstract—Graphical models are a succinct way to represent the structure in probability distributions. This article analyzes nodal voltages in typical power distribution grids that can be non-radial using graphical models. Using algebraic and structural properties of graphical models, algorithms exactly determining topology and detecting line changes for distribution grids are presented along with their theoretical limitations. We show that if distribution grids have minimum cycle length greater than three, then nodal voltages are sufficient for efficient topology estimation without additional assumptions on system parameters. In contrast, line failure or change detection using nodal voltages does not require any structural assumption on grid. The performance of the designed algorithms is analyzed using linearized power flow samples and further validated with non-linear AC power flow samples generated by Matpower on test grids.

Keywords—Concentration matrix, Conditional independence, Distribution grids, Graphical lasso, Graphical models, Power flows, line outage.

I. INTRODUCTION

Power distribution grids comprise part of the power network that include low and medium voltage lines that connect distribution substations to the end users/consumers. Conventional power distribution networks had uni-directional flow of electricity from the substation generator to the end-users. However, due to residential solar generation, storage devices and demand response policies, current distribution grids may observe bi-directional flow of electricity. The presence of active controllable devices on the distribution grid and their potential to provide grid services has made estimation problems in distribution grids of paramount importance in recent years. In this paper we discuss issues in topology estimation and related topology change detection in distribution grids. This problem has applications in various areas such as fault detection and localization, and estimation of critical lines that affect local marginal prices. Structurally, a majority of distribution grids, in particular the U.S., are radial or tree-structured [1]. However, in European grids as well as in urban areas, the residential distribution grid is often loopy and has cycles. In either case, the true operational topology is determined by the current status of breakers/switches on an underlying set of permissible edges as shown in Fig.1. Topology estimation and change detection thus refers to identifying the status or change of status of each line in the set. In this paper, we synthesize methods for analyzing the status of the set of permissible edges, for the more general case of loopy distribution grids and present connections to the restricted case of radial network.

It is worth mentioning that the estimation problems in distribution grids are hampered by the sparsity of real-time meters, including on grid lines. New placement of line/breaker status monitors is further complicated when presence of underground lines are employed. As such we focus on using measurements of nodal voltages for our estimation goals. Such nodal measurements have become more accessible in recent years with usage of installed nodal high fidelity meters such as phasor measurement units (PMUs) [2], micro-PMUs [3], FNETs [4], and sensors on smart controllable devices. Samples collected from such meters can be used to generate statistics of state variables such as voltages. In this work we will use techniques from probabilistic graphical models to develop theoretical algorithms for our estimation goals.

A. Prior Work

Estimation problems, including topology estimation and change detection, in power distribution grids has attracted significant attention in recent years. Researchers have looked at multiple approaches, both active and passive, in learning using varying measurement type and availability. Example of such schemes include greedy methods [5], [6], voltage signature based methods [7], [8], probing schemes [9], imposing graph cycle constraints [10] and iterative schemes for addressing missing data [11], [12]. In contrast to the referred work that employ static voltage samples, learning schemes that exploit dynamic voltage measurements are reported in [13], [14].

In work related closest to this article, authors have addressed topology identification using properties of probabilistic graphical model of nodal voltages. [15] uses signs in inverse covariance matrix of voltage magnitudes for topology identification, but limited to radial topologies in grids with constant \( r/x \) (resistance to reactance) line ratio. [6], [16] uses voltage conditional independence tests for guaranteed topology identification, but limited to radial distribution grids in single and three-phase networks. [17] discusses topology change detection using an approximate graphical model (Markov random field). Under a similar approximate graphical model, topology reconstruction algorithms are proposed for radial and loopy distribution grids in [18] and [19] respectively under independent nodal current injections.

B. Contribution

In this work, we consider graphical model [20] based learning schemes for topology estimation and line failure detection...
using complex voltage measurements from an AC power flow model for loopy grids. Our work first discusses sparsity and algebraic properties of the voltage graphical model under uncorrelated nodal power injections. In particular, we show that the structure of the graphical model includes additional edges over and above the topology of the underlying power grid graph. Approximate schemes for topology learning have been discussed previously [17], [18], [19] by ignoring spurious edges that in practice have small values. In that approximate setting, the graphical model exactly corresponds to the true topology and hence makes learning the topology easy. In contrast, we take a principled theoretical approach in this paper and develop two topology learning algorithms and present conditions under which exact recovery is possible without ignoring the spurious edges. The first algorithm relies on local neighborhood counting within the estimated graphical model, while the second algorithm uses algebraic sums of terms in the inverse covariance matrix. We show that the first approach is able to estimate the true structure for grids with minimum cycle length greater than six, while the second approach only requires minimum cycle length to be greater than three (no triangles). It is worth noting that our algorithms do not require constant $r/x$ ratio or radial structure and hence generalizes prior work [15], [6]. Further knowledge of values of line impedances and nodal injection statistics are not required as input for the learning algorithms. Finally, we develop a graphical model based topology change detection algorithm that is able to estimate multiple simultaneous line failures in the grid using entries in the graphical model. The performance of the developed algorithms are calibrated using both linear and non-linear power flow samples generated using Matpower [21].

Fig. 1. Distribution grid with a substation (reference node). Operational edges are denoted by solid lines, while open switches are denoted by dotted lines. $P_k^1 \equiv \{(k,j),(j,i)\}$ is one path from $k$ to $i$.

The rest of the paper is organized as follows. The next section presents nomenclature and power flow relations in loopy power grids. Section III discusses properties of graphical model of power grid voltage measurements under a linearized AC power flow model. Section IV includes the first topology learning algorithm along with conditions for exact recovery. Section V describes the second learning algorithm. Topology change detection is discussed in Section VI. Section VII includes simulations results of our work on test cases. Conclusions and future work are included in Section VIII.

II. POWER GRID AND POWER FLOW

We consider a power grid and represent its structure by the graph, $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of $N + 1$ buses/nodes and $\mathcal{E}$ is the set of operational undirected lines/edge (see Fig. 1). Normally the operational topology is determined by closing switches/breakers within a set of permissible lines, $\mathcal{E}_\text{full}$. Such changes may be made hourly or daily depending on the load configuration served and other control needs, hence the need for grid topology estimation and change detection. We denote an edge between two nodes $i$ and $j$ by $(ij)$. Let $\mathcal{P}_i^1 \equiv \{(i k_1), (k_1 k_2), ... (k_n-1 j)\}$ be a set of $n$ distinct undirected edges that connect node $i$ and node $j$. We call $\mathcal{P}_i^1$ as a path of length $n$ from $i$ to $j$. If $i = j$ and path $\mathcal{P}_i^1$ has length greater than 2, we term it a ‘cycle’. The number of edges in the cycle is termed its ‘cycle length’. By definition, the minimum cycle length in a radial graph is considered to be infinite as it has no cycles. Note that for general loopy grids, there may be multiple paths between two nodes. The neighbors of a node are the set of nodes it shares an edge with. Nodes with the length of the shortest path connecting them equal to two are termed ‘two-hop neighbors’. Nodes with degree 1 are termed ‘leaves’ and their individual neighbors are termed ‘parents’. For power grid $G$, we use the following power flow model.

**Linear Coupled Power Flow (LC-PF) model:** Let $z_{ij} = r_{ij} + jx_{ij}$ denote the complex impedances of line $(ij)$ in the grid ($i^2 = -1$), where resistance and reactance are denoted by $r_{ij}$ and $x_{ij}$ respectively. The non-linear AC power flow equation for the injection at node $i$ is given by:

$$P_i = p_i + j q_i = \sum_{j: (ij) \in \mathcal{E}} V_i (V_j^* - V_j)/z_{ij}^*, \quad (1)$$

$$= \sum_{j: (ij) \in \mathcal{E}} \frac{v_i^2 - v_i v_j \exp(j \theta_i - j \theta_j)}{z_{ij}^*}, \quad (2)$$

where, the real valued scalars, $v_i$, $\theta_i$, $p_i$ and $q_i$ denote the voltage magnitude, voltage phase, active and reactive power injections respectively. The complex valued voltage and injection are given by $V_i (= v_i \exp(j \theta_i))$ and $P_i$ respectively. During normal operation, lossless models for power flow are employed [23], [5], [24] which can be obtained by linearizing the power flow equations in (1), assuming small deviations in both phase difference of neighboring nodes ($|\theta_i - \theta_j| << 1$ for edge $(ij)$), and voltage magnitude deviations from the reference bus ($|v_i - 1| << 1$ for node $i$).

$$P_i = p_i + j q_i = \sum_{j: (ij) \in \mathcal{E}} \frac{(v_i - v_j) - j (\theta_i - \theta_j)}{z_{ij}^*}, \quad (3)$$
Further one bus (normally the substation bus) is taken as reference and voltages at other buses are measured relative to it. Under the lossless model, the nodal injection at the reference bus is the negative sum of injections at all other nodes. Without a loss of generality, the reference bus is ignored and the power flow equations are restricted to the $N$ non-reference buses in the grid. Using $v, \theta, p, q$ to represent the vector of nodal voltage magnitude, phase, active and reactive injections respectively, we get the following Linear Coupled Power Flow (LC-PF) model, in matrix form:

$$
\begin{bmatrix}
v \\
\theta 
\end{bmatrix} =
\begin{bmatrix}
H_\beta & H_\beta \\
H_\beta & -H_\beta
\end{bmatrix}^{-1}
\begin{bmatrix}
p \\
q
\end{bmatrix},
$$

(4)

Here $H_\beta$ is the reduced weighted Laplacian matrix for the grid $\mathcal{G}$ with edge weights given by susceptances $\beta$ (conductances $g$). Here $g_{ij} + i\beta_{ij}$ equals $\frac{1}{\sqrt{v_i v_j}}$. The sparsity of $H_\beta$ and $H_\beta$ encodes the grid structure as noted below:

$$
H_\beta(i,j) = \begin{cases}
\sum_{k:(ik) \in \mathcal{E}} g_{ik} & \text{if } i = j \\
-g_{ij} & \text{if } (ij) \in \mathcal{E} \\
0 & \text{otherwise}
\end{cases}.
$$

(5)

The reduction is derived by removing the row and column corresponding to the reference bus from the weighted Laplacian matrix. We use $v, \theta, p, q$ for the $N$ dimensional reduced vectors to denote the voltage magnitude, phase, active and reactive injections at the non-substation nodes from this point onward. Note that (4) is a generalization of the LinDistFlow model \cite{25, 26, 23} used in radial networks to loopy grids. Further ignoring voltage magnitude deviations converts the model to the standard DC power flow model \cite{27} used in transmission grids. The accuracy of the LC-PF voltages with non-linear power flow samples are discussed in \cite{24, 16}. In the next section, we discuss the properties of the distribution of voltage fluctuations using only in the graphical model estimation step, the algorithm design for topology identification and change detection does not specifically depend on fluctuations being Gaussian.

Here, the injection vector $\begin{bmatrix} p \\ q \end{bmatrix}$ is modelled by the Gaussian random variable $P^{LC}(p, q) \equiv N(0, \Sigma_{(p, q)})$ where the covariance matrix of injections is given by

$$
\Sigma_{(p, q)} = \begin{bmatrix}
\Sigma_{pp} & \Sigma_{pq} \\
\Sigma_{qp} & \Sigma_{qq}
\end{bmatrix}.
$$

(6)

Under Assumption 1, each block in $\Sigma_{(p, q)}$ is a diagonal matrix. Given the voltages $v, \theta$ are related by a linear model to injections, their distribution is also a zero-mean Gaussian $\mathcal{N}(0, \Sigma_{v, \theta})$ \cite{28}. Using (4), the covariance matrix is given by

$$
\Sigma_{v, \theta} = \begin{bmatrix} H_\beta & H_\beta \\
H_\beta & -H_\beta
\end{bmatrix}^{-1}
\begin{bmatrix}
\Sigma_{pp} & \Sigma_{pq} \\
\Sigma_{qp} & \Sigma_{qq}
\end{bmatrix}
\begin{bmatrix} H_\beta & H_\beta \\
H_\beta & -H_\beta
\end{bmatrix}^{-1}.
$$

(7)

The following result describes the analytic form of $\Sigma_{v, \theta}^{-1}$ which will later be used for structure estimation.

**Theorem 1.** For LC-PF, the inverse covariance matrix $\Sigma_{v, \theta}^{-1}$ of nodal voltages satisfies

$$
\Sigma_{v, \theta}^{-1} = \begin{bmatrix} J_{vv} & J_{v\theta} \\
J_{\theta v} & J_{\theta \theta} \end{bmatrix}
$$

where

$$
J_{vv} = H_\beta D^{-1}(\Sigma_{pp} H_\beta - \Sigma_{pq} H_\beta),
J_{v\theta} = H_\beta D^{-1}(\Sigma_{pq} H_\beta + \Sigma_{pp} H_\beta) - H_\beta D^{-1}(\Sigma_{pq} H_\beta + \Sigma_{pp} H_\beta),
J_{\theta v} = H_\beta D^{-1}(\Sigma_{pq} H_\beta - \Sigma_{pp} H_\beta) + H_\beta D^{-1}(\Sigma_{pq} H_\beta - \Sigma_{pp} H_\beta),
J_{\theta \theta} = H_\beta D^{-1}(\Sigma_{pq} H_\beta + \Sigma_{pp} H_\beta) + H_\beta D^{-1}(\Sigma_{pq} H_\beta + \Sigma_{pp} H_\beta),
$$

$D(i,i) = |\Sigma_{pp}(i,i)\Sigma_{qq}(i,i) - \Sigma_{pq}^2(i,i)|$ for diagonal $D$.

**Proof:** In (7), each block in $\Sigma_{(p, q)}$ is a diagonal matrix with $\Sigma_{pp} = \Sigma_{qp}$. Thus the following holds

$$
\begin{bmatrix}
\Sigma_{pq} & \Sigma_{pq} \\
\Sigma_{pp} & \Sigma_{pp}
\end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} & -D^{-1} \\
-D^{-1} & D^{-1}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{pp} & \Sigma_{pq} \\
\Sigma_{qp} & \Sigma_{pp}
\end{bmatrix},
$$

where, $D$ is a diagonal matrix with $D(i,i)$ given by the determinant of $\Sigma_{pp}(i,i)\Sigma_{qq}(i,i) - \Sigma_{pq}^2(i,i)$. Inverting $\Sigma_{v, \theta}$ using (7) then proves the result.

Next we describe the graphical model of the distribution of nodal voltages.

**Graphical Model:** By definition, the probability distribution of a $n$ dimensional random vector $X = [X_1, X_2, ..., X_n]^T$ corresponds to an undirected graphical model $\mathcal{G}M$ \cite{20} with vertex set $\mathcal{V}_\mathcal{G}M$ representing variables and edges representing conditional dependence. For node node $i \in \mathcal{G}M$, its neighbors form the smallest set of nodes $N(i) \subset \mathcal{V}_\mathcal{G}M - \{i\}$ such that for any node $j \notin N(i)$, $i$ is conditionally independent of $j$ given the set $N(i)$, i.e., $\mathcal{P}(X_i | X_{N(i)}, X_j) = \mathcal{P}(X_i | X_{N(i)})$.

For a Gaussian graphical model, it is known that the edges in the graphical model correspond to non-zero terms in the inverse covariance matrix (also called ‘concentration’ matrix) \cite{20}. In our case, we determine the structure of the graphical model $\mathcal{G}M$ of voltages using properties of $\Sigma_{v, \theta}^{-1}$. Note that

III. PROBABILISTIC GRAPHICAL MODEL OF COMPLEX VOLTAGES

For a $N$ bus (ignoring the reference bus) system, the total number of scalar voltage variables is $2 \times N$, considering both magnitude and phase. We use graphical models to represent the structure within the distribution of voltages under ambient fluctuations of nodal injections. The following assumption states the model of nodal injections at the non-reference nodes.

**Assumption 1:** Fluctuations at non-reference nodal injections in the grid are uncorrelated zero-mean Gaussian random variables with non-zero covariances. Thus, $(p_i, q_i)$ is uncorrelated from $(p_j, q_j)$ if $i \neq j$, while each is a two-dimensional zero-mean Gaussian.

This assumption, similar to prior work in literature, arises from the fact that changes in loads in the ambient regime are small and typically uncorrelated. Further small trends in injections can be empirically de-trended the injections or consider deviations between consecutive injection measurements to induce Assumption 1. Note that active and reactive injections at the same node may be correlated, only those at distinct nodes are uncorrelated. The Gaussian assumption of the fluctuation is
there are twice as many nodes in the graphical model $\mathcal{G}$ as there are buses in the grid as $\mathcal{G}$ includes separate nodes for bus voltage magnitudes and phases.

**Theorem 2.** The graphical model $\mathcal{G}$ for nodal voltage magnitudes and phase angles in grid $\mathcal{G}$ includes edges between voltage magnitudes and phase angles only at the same bus, neighboring buses, and two-hop neighboring buses.

**Proof:** Edges in $\mathcal{G}$ correspond to non-zero terms in $\Sigma^{-1}_{(v, \theta)}$ with analytic form given in Theorem 1. We prove the statement first by showing that for voltage magnitude and/or phases at buses $i$ and $j$ three or more hops away, the corresponding entries in $\Sigma^{-1}_{(v, \theta)}$ are zero for each block $J_{uv}$, $J_{v\theta}$, $J_{\theta v}$, and $J_{\theta \theta}$. First consider the four terms in the expression for $J_{v \theta}$ in Theorem 1. While $D^{-1}$, $\Sigma_{qq}$, $\Sigma_{pp}$ are all diagonal matrices, matrices $H_y, H_\theta$ have non-zero values for diagonal terms and for neighboring nodes (see [3]). From direct multiplication it is clear that $J_{uv}(i,j) = 0$ if $i, j$ are not neighbors and do not have common neighbor. Similarly it follows for $J_{v \theta}$, $J_{\theta v}$, and $J_{\theta \theta}$, the statement follows.

Note that if $(ij) \in E$ or node $k$ exists such that $(ik), (kj) \in E$, then the corresponding entry $\Sigma^{-1}_{(v, \theta)}$ is non-zero unless for pathological cases that form a set of measure zero. Thus the statement holds.

Remark: In the remaining part of the manuscript, we will ignore such pathological cases that induce degeneracy as well; thus if there is a true link in $\mathcal{G}$ between nodes $i$ and $j$ then terms in $J_{uv}$, $J_{v\theta}$, $J_{\theta v}$, and $J_{\theta \theta}$ do not conspire to be zero. Fig. 2 depicts the structure of a loopy grid $\mathcal{G}$ and its associated graphical model $\mathcal{GM}$ of complex voltages. Note that if nodes pertaining to voltage magnitude and phase angles are combined into a ‘hybrid’ node, we get a similar sized network $\mathcal{GM}_{hd}$ as the original grid but with additional edges between two hops neighbors as shown in Fig. 2(c).

To complete this section, we describe the estimation of $\Sigma^{-1}_{(v, \theta)}$ from i.i.d complex voltage samples.

**A. Estimation of Voltage Graphical Model**

Consider i.i.d. samples of the measured vector $y^k = \begin{bmatrix} v^k \\ \theta^k \end{bmatrix}$ for $1 \leq k \leq n$ in the grid. If $n$ is large, we use direct inverse of the sample covariance matrix. However for scenarios with restricted number of samples, we use the maximum likelihood estimator of a Gaussian graphical model [20] with a sparsity constraint to get the estimate $\hat{\Sigma}^{-1}_{(v, \theta)}$ using

$$\hat{\Sigma}^{-1}_{(v, \theta)} = \arg \min_S \{ -\log \det S + \langle S, \hat{\Sigma}_{(v, \theta)} \rangle + \lambda ||S||_1 \}$$

where $\hat{\Sigma}_{(v, \theta)} = \frac{1}{n-1} \sum_{k=1}^{n} (y^k - \bar{y}) (y^k - \bar{y})^T$.

Further, we use a $\ell_1$-norm to enforce sparsity of the estimated inverse covariance matrix; the sparsity arises from the discussion below regarding the structure of $\Sigma^{-1}_{(v, \theta)}$. This optimization problem is termed Graphical Lasso [29] in literature. The computational complexity for it scales as $O(N^3)$ where $N$ is the number of variables in the system [30].

In the next section, we use Theorem 2 we present the first topology learning algorithm in the next section using separation rules.

**IV. TOPOLOGY ESTIMATION BASED ON LOCAL NEIGHBORHOODS**

The following result presents separation rules in $\mathcal{GM}_{hd}$ that enable identification of true edges between non-leaf nodes in the grid graph $\mathcal{G}$.

**Theorem 3.** Let the graphical model $\mathcal{G}$ for voltages in power grid $\mathcal{G}$ be estimated and the ‘hybrid’ graph $\mathcal{GM}_{hd}$ constructed by merging voltages and phase angles at the same bus. Let minimum cycle length in $\mathcal{G}$ be greater than 6. Consider edge $(ij)$ in $\mathcal{GM}_{hd}$. Then, $\mathcal{GM}_{hd}$ has nodes $k, l$ with paths $k - i - l$ and $k - j - l$ but $(kl)$ not present if $(ij)$ is a true edge in $\mathcal{G}$ between non-leaf nodes $i$ and $j$.

**Proof:** We first prove the if part. Let edge $(ij)$ exists in $\mathcal{G}$ between non-leaf nodes $i, j$. There exists nodes $k$ and $l$ that are neighbors of $i$ and $j$ respectively in $\mathcal{G}$ with path $k - i - j - l$ of length 3. Any other path between $k, l$ in $\mathcal{G}$ must be longer than 3 hops as minimum cycle length is 6. Using Theorem 2 edges $k - j - l - j - k - i - l - i \in \mathcal{GM}_{hd}$. Thus $k, l$ are two-hop neighbors in $\mathcal{GM}_{hd}$.

Next, we prove the only if part by contradiction. Suppose $(ij)$ is not an edge in $\mathcal{G}$. Then, neighbors $i, j$ in $\mathcal{GM}_{hd}$ are two-hop neighbors in $\mathcal{G}$. As minimum cycle length is 6, $i$ and $j$ can have exactly one common neighbor in $\mathcal{G}$, say node $c$. Now paths $k - i - l$ and $k - j - l$ in $\mathcal{GM}_{hd}$ exist, thus $k, l$ must be one or two-hop neighbors of both $i$ and $j$ in $\mathcal{G}$. As $k - l$ doesn’t exist in $\mathcal{GM}_{hd}$, $k, l$ are separated by more than two hops in $\mathcal{G}$ and hence not both connected to $c$. Hence, cycles $i - r_1 - (k/l) - r_2 - j - c - i$ or $i - r_1 - k - r_2 - j - l - (l = c) - i$ must exist in $\mathcal{G}$. However they form cycles of length 6 or less, hence it violates the minimum cycle length condition. Finally, we show that $i, j$ must be non leaf nodes in $\mathcal{G}$. For neighbors $i, j$ in $\mathcal{GM}_{hd}$, first consider the case where node $i$ (without loss of generality) is a leaf node in $\mathcal{G}$. All neighbors of $i$ in $\mathcal{GM}_{hd}$ are either its parent or neighbors of its parent in $\mathcal{G}$ (separated by two hops). Thus all neighbors $k, l$ of $i$ in $\mathcal{GM}_{hd}$ also have an edge $k - l$ in $\mathcal{GM}_{hd}$ and are not separated by
Theorem 4. Let the graphical model \( \mathcal{GM} \) for voltages in power grid \( \mathcal{G} \) be estimated and the ‘hybrid’ graph \( \mathcal{GM}_{hd} \) constructed by merging voltages and phase angles at the same bus. Let minimum cycle length in \( \mathcal{G} \) be greater than 6 and number of non-leaf nodes be at least 3. Consider non-leaf node \( i \) and leaf node \( j \) in \( \mathcal{G} \) such that \( (ij) \) is in \( \mathcal{GM}_{hd} \). Then, \( (ij) \) is a true edge in \( \mathcal{G} \) if and only if non-leaf neighbors of \( i \) in \( \mathcal{G} \) and non-leaf neighbors of node \( j \) in \( \mathcal{GM}_{hd} \) are the same.

Proof: We prove the if part using contradiction. Let the true and only neighbor of leaf node \( j \) in \( \mathcal{G} \) be node \( k \neq i \). Since, \( (ij) \in \mathcal{GM}_{hd} \), it follows that, \( i, j \) are two hops away in \( \mathcal{G} \). As there are at least 3 non-leaf nodes and cycle length is greater than 6, there exists some node \( r \) which is not a leaf node in \( \mathcal{G} \) in one of two configuration in \( \mathcal{G} \): (a) \( r \) is a neighbor of \( i \), two hops away from \( k \), and three hops away from \( j \). (b) \( r \) is a neighbor of \( k \), and two hops away from \( i \) and \( j \).

Note that in configuration (a), \( r \) is a neighbor of \( i \) in \( \mathcal{G} \), but not a neighbor of \( j \) in \( \mathcal{GM}_{hd} \). In configuration (b), \( r \) is a neighbor of \( j \) in \( \mathcal{GM}_{hd} \), but not \( i \)’s neighbor in \( \mathcal{G} \). This, thus, contradicts the statement of the sets of non-leaf neighbors of \( i \) in \( \mathcal{G} \) and \( j \) in \( \mathcal{GM}_{hd} \) being the same.

For the only if part, consider leaf node \( j \) with parent node \( i \) in \( \mathcal{G} \). As two hop neighbors in \( \mathcal{G} \) are neighbors in \( \mathcal{GM}_{hd} \), any non-leaf node that is a neighbor of \( i \) in \( \mathcal{G} \) is a neighbor of \( j \) in \( \mathcal{GM}_{hd} \) and vice versa.

We now have the necessary tools to develop topology learning algorithm for loopy grids provided the assumptions of minimum cycle length being greater than 3, and number of non-leaf nodes greater than 2 hold. Note that the first condition is necessary for identification of edges between non-leaf nodes while the second is needed to identify neighbors of leaf nodes. These assumptions are not restrictive in general as distribution grids, if loopy, have generally large loops and furthermore many (greater than 2) non-leaf nodes. The steps of the learning the grid from the estimated graphical model \( \mathcal{GM}_{hd} \) are listed in Algorithm 1. An example for the learning steps is depicted in Fig. [3].

**Algorithm 1 Topology Learning: Neighborhood Search**

**Input:** ‘Hybrid’ graphical model \( \mathcal{GM}_{hd} \) of nodal voltages constructed using Sec. III-A

**Output:** Grid \( \mathcal{G} \)

1: for all Edge \( (ij) \in \mathcal{GM}_{hd} \) do
2: \hspace{1em} if Nodes \( k, l \) exist satisfying Theorem 3 then
3: \hspace{2em} Add nodes \( i, j \) as non-leaf nodes with true edge \( (ij) \) in \( \mathcal{G} \)
4: \hspace{1em} end if
5: end for
6: Add unmarked nodes as leaves in \( \mathcal{G} \).
7: for all Leaf node \( j \) and non-leaf node \( i \) in \( \mathcal{G} \) do
8: \hspace{1em} if Edge \( (ij) \in \mathcal{GM}_{hd} \) and \( i, j \) satisfy Theorem 3 then
9: \hspace{2em} Add edge \( (ij) \) in \( \mathcal{G} \).
10: \hspace{1em} end if
11: end for

Computational Complexity: Considering general grids without any restriction on nodal degree, determining edges between non-leaf nodes takes \( O(N^4) \) comparisons between each candidate edge \( (ij) \) and their neighbor sets. Determination of edges between leaves and their parents is \( O(N^3) \) and thus the overall complexity is \( O(N^4) \) in the worst case.

It is worth mentioning that prior work on learning radial distribution grids exactly using graphical models cannot be directly applied here as presence of multiple paths in loopy grids render conditional independence tests ineffective. On the other hand, approximate algorithms such as in [31] assume, based on realistic data, that the edges between two-hop neighbors in the graphical model are weaker than direct links and hence can be omitted. This results in the graphical model and the true grid having the same structure, making detection easier. Our effort in Algorithm 1 is to develop an theoretical learning approach where such assumptions are not needed, though it has topological restriction. In the next section, we move beyond topological rules and design an improved algorithm based on algebraic properties of the inverse covariance matrix.

V. TOPOLOGY ESTIMATION USING SIGN RULES

As discussed in previous section, extracting the true grid topology \( \mathcal{G} \) from the estimated graphical model \( \mathcal{GM} \) or \( \mathcal{GM}_{hd} \) requires separating the true edges from ones between two hop neighbors. The following result gives a sign based separation test between such edges.

**Theorem 5.** Consider the graphical model \( \mathcal{GM} \) of nodal voltages with inverse covariance matrix \( \Sigma^{-1}_{(w,\theta)} = \begin{bmatrix} J_{vv} & J_{v\theta} \\ J_{\theta v} & J_{\theta\theta} \end{bmatrix} \) in grid \( \mathcal{G} \) with minimum cycle length greater than three. \( J_{vv}(i,j) + J_{\theta\theta}(i,j) < 0 \) if and only if \( (ij) \) is a true edge in \( \mathcal{G} \).

Proof: Consider nodes \( i, j \) in grid \( \mathcal{G} \). Using the expression in Theorem 1, we have

\[
J_{vv} + J_{\theta\theta} = H_g D^{-1}(\Sigma_{qq} + \Sigma_{pp})H_g + H_\beta D^{-1}(\Sigma_{qq} + \Sigma_{pp})H_\beta
\]
where $D^{-1}(\Sigma_{pp} + \Sigma_{pp})$ is a diagonal matrix with positive entries. Note the structure of $H_{\beta}$ and $H_{\beta}$ from [5]. For the if part, consider the case where $i, j$ are neighbors with edge $(ij) \in G$. As minimum cycle length is greater than three, there are no common neighbors of $i, j$. Hence we have

$$J_{vv}(i, j) + J_{\theta \theta}(i, j) = 8$$

$$- D^{-1}(i, i)(\Sigma_{qq}(i, i) + \Sigma_{pp}(i, i))(H_{\beta}(i, i)g_{ij} + H_{\beta}(i, i)\beta_{ij})$$

$$- D^{-1}(j, j)(\Sigma_{qq}(j, j) + \Sigma_{pp}(j, j))(H_{\beta}(j, j)g_{ij} + H_{\beta}(j, j)\beta_{ij}) < 0$$

To prove the only if part, consider the case where $i, j$ are not neighbors. It follows from the proof of Theorem 2 that if $i, j$ are three or more hops away, then $J_{vv}(i, j) = 0$ and $J_{\theta \theta}(i, j) = 0$. Finally, consider the case where $i, j$ are two hop neighbors with common neighbor set $S$. We have

$$J_{vv}(i, j) + J_{\theta \theta}(i, j) = \sum_{k \in S} D^{-1}(k, k)(\Sigma_{qq}(k, k) + \Sigma_{pp}(k, k))(g_{ik}g_{kj} + \beta_{ik}\beta_{kj}) > 0. \quad (9)$$

Hence proved that the statement holds only if $i, j$ are neighbors.

Note that the minimum cycle length being greater than 3 is necessary to prove Theorem 3. Without it, the positive contribution in (9) from common neighbors of two nodes $i, j$ may outweigh the negative contribution in (8) due to edge $(ij)$ and make $i, j$ seem like two hop neighbors. However, absence of triangles (three node cycles) is not a restrictive assumption as urban networks indeed have grid-like layout. It is worth mentioning that a similar sign based rule for voltage magnitudes was proposed in learning radial distribution grids as urban networks indeed have grid-like layout. It was previously mentioned that such an event can be detected by comparing the change in the inverse covariance matrix of voltage fluctuations before and after the event.

**Theorem 6.** In grid $G$ with voltage covariance matrix $\Sigma_{(v, \theta)}$ and injection covariance matrix $\Sigma_{(p, q)}$, matrices $H_{\beta}$ and $H_{\beta}$ are given by the following

$$\begin{bmatrix} H_{\beta} \\ H_{\beta} - H_{\beta} \end{bmatrix} = \Sigma_{(p, q)}^{1/2} \sqrt{\Sigma_{(v, \theta)}^{-1/2} \Sigma_{(p, q)}^{-1/2} \Sigma_{(v, \theta)}^{-1/2}}. \quad (10)$$

**Proof:** Using inverse of (7), it is clear that $\Sigma_{(p, q)}^{-1/2} \Sigma_{(v, \theta)}^{-1/2}$ is a positive definite matrix and hence has a unique square root $\Sigma_{(p, q)}^{-1/2} \Sigma_{(v, \theta)}^{-1/2}$, $\Sigma_{(v, \theta)}^{-1/2}$. The result thus follows.

In the next section, we look at a related problem of topology change detection, instead of topology estimation and discuss properties of voltage graphical models that enable detection.

**VI. TOPOLOGY CHANGE DETECTION**

Topology change detection refers to the problem of identifying changes to the topology such as line failures. In radial distribution grids, line failure can be immediately estimated given by the sub-network that loses power. In loopy grids, due to presence of multiple paths, estimation based on connectivity loss may not be possible. In the next result, we consider a single change in topology (line addition or failure) and show that such an event can be detected by comparing the change in the inverse covariance matrix of voltage fluctuations before and after the event.

**Theorem 7.** For grid $G$ with inverse covariance matrix of voltage measurements $\Sigma_{(v, \theta)}^{-1} = \begin{bmatrix} J_{vv} & J_{v\theta} \\ J_{v\theta} & J_{\theta\theta} \end{bmatrix}$, let $\Delta J_{vv}$ be the change in sub-matrix $J_{vv}$ after a single line event (addition or removal). Similarly define $\Delta J_{v\theta}$ Then the following hold:

$$\Delta J_{vv}(i, i) + \Delta J_{v\theta}(i, i) \begin{cases} 0 \text{ no change in edge } (ij) \\ > 0 \text{ edge } (ij) \text{ is added} \\ < 0 \text{ edge } (ij) \text{ is removed} \end{cases}$$

**Proof:** Let the neighborhood of node $i$ be $N_i$. From Theorem 4 it is clear that the expression for the diagonal of $J_{vv}$ follows:

$$J_{vv}(i, i) + J_{v\theta}(i, i) = \sum_{k \in N_i \cup \{i\}} D^{-1}(k, k)(\Sigma_{qq}(k, k) + \Sigma_{pp}(k, k))(H_{\beta}^2(i, k) + H_{\beta}^2(i, k))$$

$$\quad + \sum_{k \in N_i \cup \{i\}} D^{-1}(k, k)(\Sigma_{qq}(k, k) + \Sigma_{pp}(k, k))(g_{ik}^2 + \beta_{ik}^2) > 0 \quad \text{ (11)}$$

Note that this stays the same if the neighbor set $N_i$ stays constant. Thus for nodes $i, j$, if the edge status doesn’t change, $J_{vv}(i, i) + J_{v\theta}(i, i)$ and $J_{vv}(j, j) + J_{v\theta}(j, j)$ equal zero. Next consider the case where edge $(ij)$ is added to the network. Comparing before and after the event, we have

$$\begin{align*}
\Delta J_{vv}(i, i) + \Delta J_{v\theta}(i, i) &= 2D^{-1}(i, i)(\Sigma_{qq}(i, i) + \Sigma_{pp}(i, i)) \sum_{k \in N_i \cup \{i\}} (g_{ik}^2 + \beta_{ik}^2) \\
&+ D^{-1}(j, j)(\Sigma_{qq}(j, j) + \Sigma_{pp}(j, j))(g_{kj}^2 + \beta_{kj}^2) \\
&+ D^{-1}(i, i)(\Sigma_{qq}(i, i) + \Sigma_{pp}(i, i))(g_{ij}^2 + \beta_{ij}^2) > 0 \quad \text{ (12)}
\end{align*}$$

Computational Complexity: Determining the edges from the entries of the inverse covariance matrix of voltages takes $O(N^2)$ operations as all possible edge pairs are checked for existence. Notice that both Algorithm 1 and Algorithm 2 use only voltage measurements to estimate the topology. In case additionally statistics of injections at the nodes is also available, both the topology as well as values of line impedances can be determined as described in the next result.
Similarly, we can show that $\Delta J_{vv}(i,j) + \Delta J_{hb}(i,j) > 0$. For edge $(ij)$ removal, it follows similarly that the opposite sign in (12) is derived. Hence the result holds.

The steps in topology change detection are listed in Algorithm 3.

**Algorithm 3 Topology Change Detection**

**Input:** Inverse covariance matrix of nodal voltages, $\Sigma^{-1}_{(v,\theta)} = [J_{vv} \ J_{vb} \ J_{b\theta} \ J_{bh}]$ before and after event

**Output:** Edge added/removed

1: for all buses $i \in \mathcal{G}$ do
2: if $\Delta J_{vv}(i,i) + \Delta J_{hb}(i,i) \neq 0$ then
3: Mark no $i$ as terminal node of changed edge.
4: end if
5: end for
6: Use terminal nodes $i,j$ and determine if $(ij)$ is added/removed using Theorem 7

**Computational Complexity:** Determining the possible changed edge takes $O(N)$ operations as it entails only checking the changes in the diagonal entries in the inverse covariance matrix.

Note that unlike the topology learning algorithms, topology change detection does not need any assumption of minimum cycle length. It is worth mentioning that in practical settings, line failure/change would entail dynamics in the nodal voltage measurements and other time-series techniques may be used for their detection. Our theoretical result is primarily directed at understanding if stable pre- and post event voltage graphical models can be used for change detection. If Algorithm 3 is used to correctly identify consecutive change events in the grid, such events need to be sufficiently separated in time to ensure accurate estimation of inverse voltage covariance matrices. For DC power flow model in transmission grids, voltage phase angle statistics have been proposed for change detection [32]. Our effort in effect generalizes it to distribution grids as an application of graphical model for AC power flow.

In the next section, we discuss the accuracy of our methods and its dependence on number of voltage samples considered using numerical experiments.

**VII. NUMERICAL SIMULATIONS**

While the theoretical analysis for our designed algorithms is based on linearized power flow models, we demonstrate the practicality of our algorithms for topology estimation and change detection using non-linear AC samples for different test cases. Nodal injection fluctuations centered around the mean injection are modelled by uncorrelated zero-mean Gaussian random variables with covariance taken as 1 in injection are modelled by uncorrelated zero-mean Gaussian cases. Nodal injection fluctuations centered around the mean using numerical experiments.

While the theoretical analysis for our designed algorithms are based on linearized power flow models, we demonstrate the practicality of our algorithms for topology estimation and change detection using non-linear AC samples for different test cases. Nodal injection fluctuations centered around the mean injection are modelled by uncorrelated zero-mean Gaussian random variables with covariance taken as $1$ in injection are modelled by uncorrelated zero-mean Gaussian cases. Nodal injection fluctuations centered around the mean using numerical experiments.

**A. Topology Estimation**

We first discuss results for Algorithm 1 (neighborhood search) and Algorithm 2 (sign rule) in learning the operational network. We consider a modified case with 56 nodes [15] derived from the IEEE 123 bus test feeder [33]. We consider loopy extensions of this system with differing minimum cycle lengths and generate Matpower AC voltage samples for the system. First consider the system with 3 loops with minimum cycle length of 7 as shown in Fig. 4(a). We apply Algorithm 1 ($\tau_1 = 0.1$) and Algorithm 2 ($\tau_2 = -0.05$) for topology inference, and demonstrate the effect of number of samples on estimation accuracy in 10 independent realizations in Fig. 4(b). Error in each run is computed as the ratio of the sum of number of false links identified and true links missed to the total number of links in the underlying network. Observe that both algorithms give zero errors in the large sample limit. This is consistent with Theorems [34][5] which prove exact recovery of either algorithm for cycle lengths greater than 6. However,
in the limited sample regime, the sign rule based algorithm clearly outperforms the neighborhood search approach.

Next, we consider the case in Fig. 5(a) with cycle length of 4. The errors in topology estimation for Algorithms 1, 2 are shown in Fig. 5(b). Note that while Algorithm 2 (sign rules) gives exact recovery at high samples, Algorithm 1 fails to do so as the minimum cycle length here is lower than 7 as necessitated by Theorems 3 and 4. Next we consider a modified 33 node system [21] which is made loopy with 3 node cycles as shown in Fig. 6(a). As the cycles of length 3 exist in Fig. 6(a), the case violates the minimum cycle length necessary for exact theoretical recovery in both algorithms. The accuracy at high samples indeed does not decay to zero for both of them as shown in Fig. 6(b) for 10 independent realizations. These simulation results thus validate the theoretical consistency and restrictions for exact topology learning for loopy power grids.

B. Change Detection

We now consider Algorithm 3 for topology change detection and present simulation results for a modified 33 bus distribution system [21] with loops. First we consider the case in Fig. 7(b) where the edge (6, 26) is removed from the underlying case in Fig. 7(a). As before voltage samples generated by Matpower are used to estimate the voltage inverse covariance for change detection. Using a threshold $\tau_3 = 0.1$ in Algorithm 3, we demonstrate in Figure 8 the accuracy of detecting the correct removed edge for various sample sizes from 10 realizations. In each simulation run, an error of 1 is recorded if the line removal goes undetected or a wrong line is identified, while error equals 0 if the correct line is identified. We plot the average of the error for 10 realizations along with the standard deviation (error bar) against the number of samples used. Clearly in the large sample limit, the removal of the edge is detected for every realization and the detection error decays to zero. Finally, we consider the case in Fig. 7(c)

Fig. 7. (a) modified 33 bus network before change (b) 33 bus network with edge (5, 25) removed (c) 33 bus network with edge (7, 20) added

VIII. CONCLUSION

Learning the operational topology as well as detecting changes are important problems in distribution grid control and
security. In this paper, we discuss theoretical aspects of both these problems for general distribution grids that may be loopy and have cycles. We present two learning algorithms based on nodal voltage graphical models that are able to estimate the grid topology under varying topological restrictions that are not restrictive. Crucially our theoretical methodology does not depend on the knowledge of line impedances and injections statistics, and further does not assume approximate probabilistic models or parameter restrictions. We extend the analysis to the problem of topology change detection and show that complex voltage statistics can be used to identify both line addition and removal. Simulation results on non-linear AC power flow samples demonstrate the applicability of this work to realistic grid samples.

This work, while generalizing several approaches, opens multiple directions of future work. In particular, the work can be extended to the case of three phase power flow models of the type discussed in [16], [34]. While Algorithm 1 using neighborhood search directly applies for multi-phase networks, extension of Algorithm 2 using sign rules will involve additional analysis. Theoretical understanding of selection of optimal thresholds for proposed algorithms is a necessary direction of future work. We are also interested in exploring topology change detection to the case of multiple line failure estimation, and with missing nodal measurements. The graphical model in such a setting will have additional edges due to Kron reduction. Designing learning algorithms in such regimes will the topic of future research.

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