Nonlinear Fokker-Planck Equations Associated with Generalized Entropies: Dynamical Characterization and Stability Analyses

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Abstract. Dynamical characterization and behavior of two types of nonlinear Fokker-Planck equations (NFPEs) are studied within the framework of generalized thermostatistics. On the basis of generalized entropies NFPEs are shown to be constructed so that H-theorems hold with Lyapunov functionals that are given by free energy functionals associated with the generalized entropies. In the case of the ordinary type of NFPEs such H-theorems ensure convergence of solutions to their uniquely determined equilibrium solutions. In the case of mean-field type NFPEs (DNFPEs) that may exhibit bifurcation phenomena the H-theorems are shown to still hold to ensure global stability of solutions. Systematic description is given of local stability analysis based on the second-order variations of the free energy functionals.

1. Introduction
In recent years nonlinear Fokker-Planck equations (NFPEs) [1] have been attracting much attention in statistical physics as well as in its related fields of science such as computational neuroscience. While Fokker-Planck equations [2] are well known to arise from the so-called master equations of Markovian stochastic systems exhibiting diffusion processes and then take the form of linear equations in the probability density, NFPEs appear within different framework. The latter may be classified according to how the nonlinearity involved manifests itself, and also be differently classified into two types of categories according to whether bifurcations of solutions occur or not.

NFPEs without bifurcations are studied mostly in connection with the concept of non-additive entropies that are often called generalized entropies. One example of the simplest version of such NFPEs was first proposed by Plastino and Plastino [5] within the framework of Tsallis statistics [6, 7]. Their NFPE allows an H-theorem [4, 20, 21] to hold with the Lyapunov functional that is given by the free energy functional defined based on the use of the Tsallis entropy. The H-theorem there plays an important role and ensures convergence to a uniquely determined equilibrium solution (probability density), implying that the free energy continues to decrease until equilibrium state is reached [4, 20, 21]. Alternatively speaking, one can no longer expect the occurrence of bifurcations of solutions. The present author also studied previously the same kind of issue for the NFPE [21] associated with the Sharma-Mittal entropy [3] (see also [8, 22]). It is of interest to study systematically the relationship between generalized entropies and the
associated NFPEs in terms of free energies with which H-theorems hold. The result of the study undertaken from the viewpoint of dynamical characterization of NFPEs was briefly reported in [26]. The similar problem was also studied with the use of an alternative systematic method involving the irreversible thermodynamic theory and the probability current [1].

Another type of NFPEs are mean-field type NFPEs where the nonlinearity arises from the concept of mean-field couplings together with taking the thermodynamic limit [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1], a combination of which may lead to the occurrence of bifurcation phenomena. When the concept of free energy makes sense, an H-theorem also holds with the Lyapunov functional that is given by the free energy functional. The H-theorem, however, ensures global stability alone and local stability analysis is required to examine stability switching among bifurcated solutions, by computing the second-order variation of the free energy. A typical example was shown by the present author, within the framework of Gibbs-Boltzmann statistics, for an NFPE with a double well potential that can describe ferro-para magnetic phase transitions of mean-field type [11, 12].

From the viewpoint of both extending such an H-theorem as used for systems exhibiting bifurcations due to the mean-field concept to that for the case of NFPEs associated with generalized entropies and trying to observe the occurrence of bifurcations within the context of Tsallis thermostatistics, I have recently proposed double nonlinear Fokker-Planck equations (DNFPEs) of several types [23, 24, 25] and conducted global as well as local stability analyses. In this article I begin with reproducing the result of systematic study of the relationship between free energies based on generalized entropies and generalized NFPEs, that is described only briefly in a previous paper [26]. I present a general and systematic formulation of stability analyses of DNFPEs associated with a wide class of generalized entropies.

2. Generalized entropies together with free energies characterize nonlinear Fokker-Planck equations

We let the real-valued functions \( S \) and \( w \geq 0 \) satisfy the following conditions

\[
(1) \quad S'' \leq 0, \\
(2) \quad S' \geq 0, \quad w'' \leq 0, \quad \text{or} \quad S' \leq 0, \quad w'' \geq 0,
\]

where the prime and double prime represent respectively the first and second order derivative of the functions. Assuming a generalized entropy to take the form

\[
S = S \left[ \int w(p) dx \right],
\]

where \( p(x) \) denotes a probability density and the bracket \( [\cdot] \) the argument of the variable, we define the free energy as usual

\[
F = U - \gamma S = \int p(x) \varphi(x) \, dx - \gamma S \left[ \int w(p(x)) \, dx \right],
\]

with \( \gamma \) being a positive constant and \( U \) representing the energy given by the average of potential \( \varphi \):

\[
U = \int p \varphi(x) \, dx.
\]

We are concerned with a nonlinear Fokker-Planck equation (NFPE) that reads

\[
\frac{\partial p(t,x)}{\partial t} = - \frac{\partial}{\partial x} \left( - \varphi'(x)p(t,x) \right) + \frac{\partial^2}{\partial x^2} K(p(t,x)),
\]
where the probability density \( p(t, x) \) is defined for \(-\infty < x < \infty \) and \( K(p) \) denotes a certain positive-valued nonlinear function of \( p \). The \( K(p) \) is to be determined so that the free energy \( F \) monotonically decreases with time when the time evolution equation of \( p(t, x) \) is given by the NFPE(6). Assuming the natural boundary condition so that the probability current

\[
j = -\varphi' (x)p - \frac{\partial}{\partial x} K(p)
\]

vanishes at \( x = \pm \infty \), the equilibrium probability density given by \( j = 0 \) satisfies

\[
\varphi' p_{eq} = - \frac{\partial}{\partial x} K(p_{eq}).
\]

Assuming \( p(t, x) \) to obey the above NFPE we compute the time derivative of the free energy \( F \):

\[
\frac{d F}{d t} = \int \varphi \frac{\partial p}{\partial t} dx - \gamma S' \left[ \int w(p) \ dx \right] \int w'(p) \frac{\partial p}{\partial t} dx
\]

\[
= \int dx \varphi \frac{\partial}{\partial x} \left( \varphi' p + \frac{\partial}{\partial x} K(p) \right) - \gamma S' \int dx w'(p) \frac{\partial}{\partial x} \left( \varphi' p + \frac{\partial}{\partial x} K(p) \right)
\]

\[
= - \int dx \varphi \left( \varphi' p + K'(p) \frac{\partial p}{\partial x} \right) + \gamma S' \int dx w''(p) \frac{\partial p}{\partial x} \frac{\partial}{\partial x} \left( \varphi' p + K'(p) \frac{\partial p}{\partial x} \right).
\]

Noting

\[
\varphi' = - \frac{K'(p_{eq})}{p_{eq}} \frac{\partial p_{eq}}{\partial x},
\]

we further rewrite \( \frac{d F}{d t} \) as

\[
\frac{d F}{d t} = \int I \left( \frac{\partial p}{\partial x}, \frac{\partial p_{eq}}{\partial x} \right) \ dx,
\]

with

\[
I \left( \frac{\partial p}{\partial x}, \frac{\partial p_{eq}}{\partial x} \right) = - \left( K'(p_{eq}) \right)^2 \frac{p_{eq}}{\partial x} \left( \frac{\partial p_{eq}}{\partial x} \right)^2 + \gamma S' w''(p) K'(p) \left( \frac{\partial p}{\partial x} \right)^2
\]

\[
+ \frac{K'(p_{eq})}{p_{eq}} \left( K'(p) - \gamma S' w''(p) p \right) \frac{\partial p_{eq}}{\partial x} \frac{\partial p}{\partial x} \frac{\partial}{\partial x}.
\]

The discriminant of the above quadratic form \( I \left( \frac{\partial p}{\partial x}, \frac{\partial p_{eq}}{\partial x} \right) \) is given by

\[
D_{isc} = \left[ K'(p_{eq}) \right]^2 \left( K'(p) + \gamma S' w''(p) p \right)^2.
\]

Then the condition for the quadratic form \( I \) to be nonpositive definite can be given by \( D_{isc} \leq 0 \) and thus one has

\[
K'(p) + \gamma S' w''(p) p = 0.
\]

Integrating this one obtains \( K(p) \):

\[
K(p) = - \gamma S' \int w''(p) dp.
\]
Substituting eq. (15) into eq. (12) one has

$$I = -p \left\{ \frac{K'(p_{eq})}{p_{eq}} \frac{\partial p_{eq}}{\partial x} - \frac{K'(p)}{p} \frac{\partial p}{\partial x} \right\}^2$$

$$= -\gamma^2 p \left\{ \frac{\partial}{\partial x} \left( S' w'(p) - S'_{eq} w'(p_{eq}) \right) \right\}^2.$$  (16)

Then it follows

$$\frac{dF}{dt} = -\gamma^2 \int p \left\{ \frac{\partial}{\partial x} \left( S' w'(p) - S'_{eq} w'(p_{eq}) \right) \right\}^2 dx \leq 0,$$  (17)

where equality holds when

$$S' w'(p) - S'_{eq} w'(p_{eq}) = c \ (const).$$  (18)

Inequality (17) shows that an H-theorem of the standard type holds with the Lyapunov functional taking the form of free energy $F$ of eq. (4). Substituting eq. (14) into eq. (10), we obtain

$$\varphi' = \gamma S'_{eq} w''(p_{eq}) \frac{\partial p_{eq}}{\partial x} = \gamma S'_{eq} \frac{\partial}{\partial x} w'(p_{eq}),$$  (19)

and thus the equilibrium probability density $p_{eq}(x)$ turns out to satisfy

$$\varphi = \gamma S'_{eq} w'(p_{eq}) + c_0(\gamma),$$  (20)

where the constant $c_0(\gamma)$ is determined by the normalization condition. We note that applying the MAX ENT principle leads to the above equation. Accordingly the free energy can be rewritten as

$$F = \gamma \int p S'_{eq} w'(p_{eq}) dx - S \left[ \int w(p) dx \right] + c_0(\gamma).$$  (21)

Using this expression we can easily show that the free energy is bounded from below by its value $F_{eq}$ attained for the equilibrium probability density $p_{eq}$:

$$F_{eq} = \gamma \int p_{eq} S'_{eq} w'(p_{eq}) dx - S \left[ \int w(p_{eq}) dx \right] + c_0(\gamma).$$  (22)

To see this we write

$$F(p) - F(p_{eq}) = \gamma \int S'_{eq} w'(p_{eq}) (p - p_{eq}) dx - \left\{ S \left[ \int w(p) dx \right] - S \left[ \int w(p_{eq}) dx \right] \right\}. \quad (23)$$

Noting the condition assumed for the functions $w$ and $S$ (eqs. (1,2)), we have

$$\int (w(p) - w(p_{eq})) dx \leq \int w'(p_{eq}) (p - p_{eq}) dx, \quad (w'' \leq 0),$$  (24)

$$\int (w(p) - w(p_{eq})) dx \geq \int w'(p_{eq}) (p - p_{eq}) dx, \quad (w'' \geq 0),$$  (25)

$$S \left[ \int w(p) dx \right] - S \left[ \int w(p_{eq}) dx \right] \leq S' \left[ \int w(p_{eq}) dx \right] \int (w(p) - w(p_{eq})) dx,$$  (26)

and hence

$$S \left[ \int w(p) dx \right] - S \left[ \int w(p_{eq}) dx \right] \leq S' \left[ \int w(p_{eq}) dx \right] \int (w(p) - w(p_{eq})) dx$$

$$\leq S'_{eq} \int w'(p_{eq}) (p - p_{eq}) dx.$$  (27)
Accordingly substituting eq.(27) into eq.(23) leads to

\[ F(p) - F(p_{eq}) \geq 0. \tag{28} \]

Thermodynamic properties of the equilibrium free energy

We can show that the equilibrium free energy \( F_{eq} \) exhibits the Legendre transform structure and the thermodynamic relations hold as usual, when \( \gamma \) is viewed as temperature:

\[ \frac{\partial F_{eq}}{\partial \gamma} = -S_{eq}, \tag{29} \]

\[ \frac{\partial F_{eq}}{\partial h} = -\langle x \rangle \equiv -\int x \, p_{eq} \, dx. \tag{30} \]

In eqs.(29) and (30) the potential \( \varphi \) has been replaced by \( \varphi - hx \), where \( h \) is viewed as the parameter conjugate to the variable \( x \), which is, for example, supposed to play the role of magnetic field.

Examples of application of the present scheme based on the generalized entropies (3)

We now provide two simple cases that are related to the Tsallis entropy. First we set \( S(z) = \frac{1}{q-1} \left( 1 - z^{\frac{1}{q}} \right) \) together with \( w(p) = p^q \) we then have the well-known Tsallis entropy

\[ S = \frac{1 - \int p^q \, dx}{q - 1}, \tag{31} \]

and the associated NFPE (6) with \( K(p) = \gamma p^q \) that was proposed by Plastino and Plastino [5]. The H-theorem for this NFPE was shown in [20, 21, 4], and is recovered in the present scheme. Second, we set \( S(z) = \frac{q}{1-q} \left( 1 - z^{-\frac{1}{q}} \right) \) together with \( w(p) = p^q \). We obtain the Sharma and Mittal entropy [3]

\[ S = \frac{q}{1-q} \left( 1 - \left( \int p^q \, dx \right)^{-\frac{1}{q}} \right), \tag{32} \]

for which \( K(p) = \gamma p^q (\int p^q \, dx)^{-\frac{q+1}{q}} \). The H-theorem for the NFPE with this \( K(p) \) was studied in [21] in connection with the so-called escort probability density associated with Tsallis thermostatistics of the third choice [21, 8]. We see that the H-theorem is also recovered in the present scheme.

3. Mean-field type nonlinear Fokker-Planck equations and stability analyses

We consider another type of NFPE that involves, besides the nonlinearity of \( K(p) \) in eq.(6), nonlinearity associated with the so-called "mean-field type coupling" term. We may call such an NFPE double nonlinear Fokker-Planck equation (DNFPE) [23, 24, 25, 26] and are concerned with the following equation:

\[ \frac{\partial p(t, x)}{\partial t} = -\frac{\partial}{\partial x} \left( \left( - \varphi'(x) + J \int x p(t, x) \, dx \right) p(t, x) \right) + \frac{\partial^2}{\partial x^2} K(p(t, x)), \tag{33} \]

with

\[ K(p) = -DS' \int^p \omega''(p) \, dp, \tag{34} \]
where $J$ and $D$ represent positive constants and $J \int xp(t, x) dx$ is the mean-field type coupling term. The integration is assumed to be over $-\infty < x < \infty$. The above-mentioned mean-field type coupling term is well known to be responsible for the occurrence of bifurcation phenomena [9, 10, 11, 12], when the standard Boltzmann entropy is taken with $K(p) = Dp$ and the potential $\varphi(x)$, for example, double well type. We define the free energy as usual

$$F(p(\cdot)) = U - DS \int \varphi dx - J \frac{1}{2} \left( \int xpdx \right)^2 - DS \int w(p)dx,$$

where the bracket [ ] denotes the argument of the function. With this free energy functional taken as a Liapunov functional (H-functional) we can show an H-theorem for the DNFPE(33) [23, 24, 25, 26].

H-theorem :

Let $p(t, x)$ satisfy the DNFPE(33). Then we have

1. The free energy $F((p(t, \cdot))$ is bounded from below:

$$F((p(t, \cdot)) > c_F(const), \quad (36)$$

2. $F((p(t, \cdot))$ is decreasing with time :

$$\frac{dF(p(t,\cdot))}{dt} \leq 0. \quad (37)$$

The boundedness inequality (36) can be proved in the same way as in the case of [23]. The second inequality (37) can be easily obtained by a straightforward calculation by introducing $R_p(x)$ that is normalized and satisfies

$$\left( \varphi' - J \langle x \rangle_p \right) R_p = -\frac{\partial}{\partial x} K(R_p), \quad (38)$$

where

$$\langle x \rangle_p = \int xpdx. \quad (39)$$

Here we note that the $R_p(x)$ is uniquely determined when $p$ is given, as in [21]. Noting eq.(38) instead of eq.(10) and repeating the procedure of obtaining eq.(17) one has

$$\frac{dF}{dt} = \int \left( \varphi - J \langle x \rangle_p \right) \frac{\partial p}{\partial t} dx - DS' \int w'(p) \frac{\partial p}{\partial t} dx$$

$$= -D^2 \int p \left\{ \frac{\partial}{\partial x} \left( S'w'(p) - S'R_p w'(R_p) \right) \right\}^2 dx \leq 0, \quad (40)$$

where $S'R_p = S' [\int w(R_p)dx]$ with the bracket [ ] denoting the argument of the function, and equality holds for $S'w'(p) - S'R_p w'(R_p) = c (const)$. We note that equilibrium solutions correspond to $R_{peq} = p_{eq}$, which takes the form of a self-consistent equation for the equilibrium solutions. It is also noted that one can no longer expect uniqueness of the equilibrium density $P_{eq}(x)$, since the self-consistent equation, in general, admit multi-solutions. Which of those multi-solutions are relevant has to be determined by the stability condition. In other words,
there may occur bifurcation phenomena involving stability switches, as the control parameter $D$ is varied. The H-theorem here ensures global stability of the system such that uniquely existing equilibrium solution is stable.

To conduct local stability analysis [12, 23, 24, 25, 26], we expand $F$ around $p_{eq}$

$$\delta F \equiv F (p_{eq} + \delta p) - F (p_{eq}) = \delta^{(1)} F [\delta p] + \delta^{(2)} F [\delta p, \delta p] + \ldots$$  \hspace{1cm} (41)

Noting

$$\int \delta p dx = 0,$$  \hspace{1cm} (42)

and differentiating $F$ with respect to $p$ yields

$$\delta^{(1)} F = \int \left( DS' R_p w'(R_p) - DS' w'(p) \right) \delta p dx,$$  \hspace{1cm} (43)

where we noted

$$\varphi - J \langle x \rangle x = DS' R_p w'(R_p),$$  \hspace{1cm} (44)

that is derived from eqs.(34) and (38). $\delta^{(1)} F$ vanishes for $p = p_{eq}$. Differentiating $F$ twice with respect to $p$ yields

$$2\delta^2 F = D \int \left( S'' R_p w''(R_p) \delta R_p + w'(R_p) S'' R_p \int w'(R_p) \delta R_p dx - S' w'(p) \delta p - w'(p) S'' \int w'(p) \delta p dx \right) \delta p dx.$$  \hspace{1cm} (45)

Differentiating eq.(44) with respect to $p$ one obtains

$$-Jx \int x \delta p dx = DS'' R_p w''(R_p) \delta R_p + D w'(R_p) S'' R_p \int w'(R_p) \delta R_p dx.$$  \hspace{1cm} (46)

Multiplying the above equation by $\delta p$ and integrating the product yields

$$-J \left( \int x \delta p dx \right)^2 = D \int S'_ R p w''(R_p) \delta R_p \delta p dx + D S'' R_p \int w'(R_p) \delta p dx \int w'(R_p) \delta R_p dx.$$  \hspace{1cm} (47)

Substituting this into eq.(45) one obtains

$$2\delta^2 F = -J \left( \int x \delta p dx \right)^2 - DS' \int w''(p) (\delta p)^2 dx - D \left( \int w'(p) \delta p dx \right)^2 S''$$

$$= -J \left( \int x \delta p dx \right)^2 + D \left| S' \right| \left| \int w''(p) (\delta p)^2 dx + D \left( \int w'(p) \delta p dx \right)^2 \right| S'' |.$$  \hspace{1cm} (48)

The second line of R.H.S.of the above equation holds owing to the assumption for the functions $S'$, $S''$, $w''$ eqs.(1) and (2).

Since the free energy $F$ continues decreasing with time owing to the H-theorem, positive definiteness of $\delta^2 F$, i.e. $\delta^2 F > 0$ implies local stability of the $p_{eq}$. Following the standard procedure we evaluate the above second-order variation at $p = p_{eq}$. Assuming

$$\int x^2 w''(p_{eq})^{-1} dx < \infty,$$ $\int w'(p_{eq})^2 w''(p_{eq})^{-1} dx < \infty$, and $\xi(x) \in L^2$ (class of square integrable functions) we put \[11, 12, 23, 24, 1, 19\]

$$\delta p = \left| w''(p_{eq}) \right|^{-\frac{1}{2}} \xi(x).$$  \hspace{1cm} (49)
Applying the decomposition procedure in [11, 12, 23, 24, 1, 19] we write ξ as

\[
\xi(x) = \left( w''(p_{eq}) \right)^{-\frac{1}{2}} \left( \alpha_1 + \alpha_2 x + \alpha_3 w'(p_{eq}) \right) + \xi_\perp,
\]

where \( \xi_\perp \) is perpendicular to \( \xi - \xi_\perp \). Defining

\[
I_0 = \int x \left( w''(p_{eq}) \right)^{-1} dx, \quad I_1 = \int x \left( w''(p_{eq}) \right)^{-1} dx, \quad I_2 = \int x^2 \left( w''(p_{eq}) \right)^{-1} dx,
\]

\[
J_0 = \int w'(p_{eq}) \left( w''(p_{eq}) \right)^{-1} dx, \quad J_1 = \int w'(p_{eq})x \left( w''(p_{eq}) \right)^{-1} dx, \quad J_2 = \int w'(p_{eq})^2 \left( w''(p_{eq}) \right)^{-1} dx,
\]

one has from eq.(42)

\[
\alpha_1 I_0 + \alpha_2 I_1 + \alpha_3 J_0 = 0,
\]

and from eq.(48)

\[
2\delta^2 F = A \alpha_2^2 + 2B \alpha_2 \alpha_3 + C \alpha_3^2,
\]

with

\[
A = \frac{D |S'|}{I_0} \left( -I_1^2 + I_0 I_2 \right) - \frac{J}{I_0^2} \left( -I_1^2 + I_0 I_2 \right)^2 + \frac{D |S''|}{I_0} \left( I_0 J_1 - J_0 I_1 \right)^2,
\]

\[
B = \left( I_0 J_1 - J_0 I_1 \right) \left\{ \frac{D |S'|}{I_0} - \frac{J}{I_0^2} \left( -I_1^2 + I_0 I_2 \right) + \frac{D |S''|}{I_0^2} \left( I_0 J_2 - J_0 I_2 \right) \right\},
\]

\[
C = \frac{D |S'|}{I_0} \left( I_0 J_2 - J_0 I_2 \right) - \frac{J}{I_0^2} \left( I_0 J_1 - J_0 I_1 \right)^2 + \frac{D |S''|}{I_0^2} \left( I_0 J_2 - J_0 I_2 \right)^2,
\]

where eq.(53) was used.

The stability condition of the equilibrium density is given by that the second-order variation of the free energy \( \delta^2 F \) be positive definite. Hence the condition for this is the one that

\[
A > 0,
\]

and the discriminant of eq.(54) be non-positive:

\[
B^2 - AC \leq 0.
\]

Inequalities (58) and (59) yield the stability condition

\[
v < \frac{u(u + rZ)}{Yu + r(YZ - X^2)},
\]

where

\[
u = \left| S' \right|, \quad v = \frac{J}{DI_0}, \quad r = \frac{S''}{I_0},
\]

\[
X = I_0 J_1 - I_1 J_0, \quad Y = I_0 I_2 - I_1^2, \quad Z = I_0 J_2 - J_0^2.
\]

It is noted that stability switches among multi-solutions to the self-consistent equation for the order parameter take place when

\[
v = \frac{u(u + rZ)}{Yu + r(YZ - X^2)}.
\]
We show a simple example where the DNFPE is derived by adding the mean-field type coupling term to the Plastino and Plastino NFPE with \( K(p) = Dp^q, 0 < q < 1 \) that is described in the end of Sec. 2. In this case one has

\[
X = q|q(q - 1)|^{-2} \left( \int p_{eq}^{2-q} dx \int xp_{eq}dx - \int xp_{eq}^{2-q} dx \right),
\]

\[
Y = |q(q - 1)|^{-2} \left( \int p_{eq}^{2-q} dx \int x^2 p_{eq}^{2-q} dx - \left( \int xp_{eq}^{2-q} dx \right)^2 \right),
\]

\[
Z = |(q - 1)|^{-2} \left( \int p_{eq}^{2-q} dx \int p_{eq}^q dx - 1 \right),
\]

(63)

\[
u = \frac{1}{|q - 1|}, \quad v = \frac{J}{D|q(q - 1)|^{-1} \int p_{eq}^{2-q} dx}, \quad r = 0.
\]

(64)

The stability condition (60) with substitution of the above mentioned parameters reads

\[
1 - \frac{J}{Dq} \left( \int x^2 p_{eq}^{2-q} dx - \left( \int \frac{xp_{eq}^{2-q} dx}{p_{eq}^{2-q} dx} \right)^2 \right) > 0.
\]

(65)

We see that the above result recovers the one previously obtained in [23].

4. Conclusions

We have explored dynamical behaviors of NFPEs with and without bifurcations within the framework of generalized thermostatistics based on generalized entropies of any kind including Tsallis one and Sharma and Mittal one. H-theorems together with the concept of free energies playing the role of Lyapunov functionals are a key ingredient to construct NFPEs: starting with providing a generalized entropy to define the associated free energy in a standard manner we construct the corresponding NFPE that exhibits an H-theorem with its Lyapunov functional given by the free energy. The H-theorem ensures convergence of any probability density given as a solution of the NFPE to a uniquely determined equilibrium density. In this case the NFPE has nothing to do with bifurcation phenomena. On the other hand, incorporating a simple version of the so-called mean-field type term into the above-mentioned NFPE to construct a DNFPE, we can in general expect the occurrence of bifurcations of its solutions. In such a case we have shown an H-theorem to still hold to ensure global stability of solutions of the DNFPE. For the purpose of studying stability exchanges associated with the occurrence of bifurcations local stability analysis becomes to be required. We have evaluated the second-order variation of the free energy functional and obtained the stability condition explicitly.

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