Lagrangian Formulation of a Magnetostatic Field in the Presence of a Minimal Length Scale Based on the Kempf Algebra

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Abstract

In the 1990s, Kempf and his collaborators Mangano and Mann introduced a $D$-dimensional ($\beta, \beta'$)-two-parameter deformed Heisenberg algebra which leads to an isotropic minimal length $(\Delta X_i)^{\text{min}} = \hbar \sqrt{D \beta + \beta'}$, $\forall i \in \{1, 2, \cdots, D\}$. In this work, the Lagrangian formulation of a magnetostatic field in three spatial dimensions ($D = 3$) described by Kempf algebra is presented in the special case of $\beta' = 2\beta$ up to the first order over $\beta$. We show that at the classical level there is a similarity between magnetostatics in the presence of a minimal length scale (modified magnetostatics) and the magnetostatic sector of the Abelian Lee-Wick model in three spatial dimensions. The integral form of Ampere’s law and the energy density of a magnetostatic field in the modified magnetostatics are obtained. Also, the Biot-Savart law in the modified magnetostatics is found. By studying the effect of minimal length corrections to the gyromagnetic moment of the muon, we conclude that the upper bound on the isotropic minimal length scale in three spatial dimensions is $4.42 \times 10^{-19} \text{m}$. The relationship between magnetostatics with a minimal length and the Gaete-Spallucci non-local magnetostatics (J. Phys. A: Math. Theor. \textbf{45}, 065401 (2012)) is investigated.

Keywords: Phenomenology of quantum gravity; Generalized uncertainty principle; Minimal length; Classical field theories; Classical electromagnetism; Quantum electrodynamics; Noncommutative field theory

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1 Introduction

One of the most important problems in theoretical physics is the unification between the Einstein’s general theory of relativity and the Standard Model of particle physics [1]. According to Ref. [1], two important predictions of this unification are the following: (i) the existence of extra dimensions; and (ii) the existence of a minimal length scale on the order of the Planck length. Studies in string theory and loop quantum gravity emphasize that there is a minimal length scale in nature. Today’s theoretical physicists know that the existence of a minimal length scale leads to a modification of Heisenberg uncertainty principle. This modified uncertainty principle can be written as

\[ \Delta X \geq \frac{\hbar}{2\Delta P} + \frac{a_1}{2} \frac{\ell_P^2}{\hbar} \Delta P + \frac{a_2}{2} \frac{\ell_P^4}{\hbar^3} (\Delta P)^3 + \cdots, \]  

where \( \ell_P \) is the Planck length and \( a_i, \forall i \in \{1, 2, \cdots\} \), are positive numerical constants [2-4]. By keeping only the first two terms on the right-hand side of Eq. (1), we obtain the usual generalized uncertainty principle (GUP) as follows:

\[ \Delta X \geq \frac{\hbar}{2\Delta P} + \frac{a_1}{2} \frac{\ell_P^2}{\hbar} \Delta P. \]  

It is clear that in Eq. (2), \( \Delta X \) is always larger than \( (\Delta X)_{\text{min}} = \sqrt{a_1} \ell_P \). At the present time, theoretical physicists believe that reformulation of quantum field theory in the presence of a minimal length scale leads to a divergenceless quantum field theory [5-7]. During recent years, reformulation of quantum mechanics, gravity, and quantum field theory in the presence of a minimal length scale have been studied extensively [5-21]. H. S. Snyder was the first who formulated the electromagnetic field in quantized spacetime [22]. There are many papers about electrodynamics in the presence of a minimal length scale. For a review, we refer the reader to Refs. [12,13,14,15,16,19,20]. In our previous work [15], we studied formulation of electrodynamics with an external source in the presence of a minimal measurable length. In this work, we study formulation of a magnetostatic field with an external current density in the presence of a minimal length scale based on the Kempf algebra. This paper is organized as follows. In Section 2, the \( D \)-dimensional \((\beta, \beta')\)-two-parameter deformed Heisenberg algebra introduced by Kempf and his co-workers is studied and it is shown that the Kempf algebra leads to a minimal length scale [23-25]. In Section 3, the Lagrangian formulation of a magnetostatic field in three spatial dimensions described by Kempf algebra is introduced in the case of \( \beta' = 2\beta \), whereas the position operators commute to the first order in \( \beta \). It is shown that at the classical level there is a similarity between magnetostatics in the presence of a minimal length scale and the magnetostatic sector of the Abelian Lee-Wick model in three spatial dimensions. The Ampere’s law and the energy density of a magnetostatic field in the presence of a minimal length scale are obtained. In Section 4, the Biot-Savart law in the presence of a minimal length scale is found. We show that at large spatial distances the modified Biot-Savart law becomes the Biot-Savart law in usual magnetostatics. In
Section 5, we study the effect of minimal length corrections to the gyromagnetic moment of the muon. From this study we conclude that the upper bound on the isotropic minimal length scale in three spatial dimensions is \( 4.42 \times 10^{-19} \text{m} \). This value for the isotropic minimal length scale is close to the electroweak length scale (\( \ell_{\text{electroweak}} \sim 10^{-18} \text{m} \)). In Section 6, the relationship between magnetostatics in the presence of a minimal length scale and a particular class of non-local magnetostatic field is investigated. Our conclusions are presented in Section 7. We use SI units throughout this paper.

2 Modified Commutation Relations with a Minimal Length Scale

Kempf and co-workers have introduced a modified Heisenberg algebra which describes a \( D \)-dimensional quantized space [23-25]. The Kempf algebra in a \( D \)-dimensional space is characterized by the following modified commutation relations

\[
[X^i, P^j] = i\hbar \left[ (1 + \beta P^2)\delta^{ij} + \beta' P^i P^j \right],
\]

\[
[X^i, X^j] = i\hbar \frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2} (P^i X^j - P^j X^i),
\]

\[
[P^i, P^j] = 0,
\]

where \( i, j = 1, 2, \ldots, D \) and \( \beta, \beta' \) are two non-negative deformation parameters (\( \beta, \beta' \geq 0 \)). In Eqs. (3) and (4), \( \beta \) and \( \beta' \) are constant parameters with dimension (\text{momentum})\(^{-2}\). Also, in the above equations \( X^i \) and \( P^i \) are position and momentum operators in the deformed space.

An immediate consequence of Eq. (3) is the appearance of an isotropic minimal length scale which is given by [26]

\[
(\Delta X^i)_{\text{min}} = \hbar \sqrt{D\beta + \beta'}, \quad \forall i \in \{1, 2, \ldots, D\}.
\]

In Ref. [27], Stetsko and Tkachuk introduced a representation which satisfies the modified Heisenberg algebra (3)-(5) up to the first order in deformation parameters \( \beta \) and \( \beta' \). The Stetsko-Tkachuk representations for the position and momentum operators in the deformed space can be written as follows:

\[
X^i = x^i + \frac{2\beta - \beta'}{4} (p^2 x^i + x^i p^2),
\]

\[
P^i = p^i (1 + \frac{\beta'}{2} p^2),
\]

where \( x^i \) and \( p^i = i\hbar \partial / \partial x^i \) are position and momentum operators in ordinary quantum mechanics, and \( p^2 = \sum_{i=1}^{D} p^i p^i \). In this article, we study the special case of \( \beta' = 2\beta \), in which the
position operators commute to the first order in deformation parameter $\beta$, i.e., $[X^i, X^j] = 0$ and thus a diagonal representation for the position operator in the deformed space can be obtained. For this linear approximation, the modified Heisenberg algebra (3)-(5) becomes

$$
[X^i, P^j] = i\hbar \left[(1 + \beta P^2)\delta^{ij} + 2\beta P^i P^j\right], \tag{9}
$$

$$
[X^i, X^j] = 0, \tag{10}
$$

$$
[P^i, P^j] = 0. \tag{11}
$$

In 1999, Brau [28] showed that the following representations satisfy (9)-(11), in the first order in $\beta$:

$$
X^i = x^i, \tag{12}
$$

$$
P^i = p^i(1 + \beta p^2). \tag{13}
$$

It is necessary to note that the Stetsko-Tkachuk representations (7),(8) and the Brau representations (12),(13) coincide when $\beta' = 2\beta$. Benczik has shown that the energy spectrum of some quantum systems in the deformed space with a minimal length are representation-independent [29]. It seems that the laws of physics in the presence of a minimal length must be representation-independent.

## 3 Lagrangian Formulation of a Magnetostatic Field with an External Current Density in the Presence of a Minimal Length Scale Based on the Kempf Algebra

The Lagrangian density for a magnetostatic field with an external current density $J(x) = (J^1(x), J^2(x), J^3(x))$ in three spatial dimensions ($D = 3$) can be written as follows [30]:

$$
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) + J^i(x) A^i(x), \tag{14}
$$

where $i, j = 1, 2, 3$ , $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x)$ and $A(x) = (A^1(x), A^2(x), A^3(x))$ are the electromagnetic field tensor and the vector potential respectively.

The Euler-Lagrange equation for the components of the vector potential is

$$
\frac{\partial \mathcal{L}}{\partial A_k} - \partial_t \left( \frac{\partial \mathcal{L}}{\partial (\partial_t A_k)} \right) = 0. \tag{15}
$$

If we substitute (14) into (15), we will obtain the following field equation for the magnetostatic field

$$
\partial_t F^{tk}(x) = \mu_0 J^k(x). \tag{16}
$$
The electromagnetic field tensor $F_{ij}(x)$ satisfies the Bianchi identity
\[ \partial_i F_{jk}(x) + \partial_j F_{ki}(x) + \partial_k F_{ij}(x) = 0. \] (17)

The three-dimensional magnetic induction vector $B(x)$ is defined as follows [31]:
\[ F_{ij} = -\epsilon_{ijk} B^k, \quad F^{ij} = \epsilon^{ijk} B_k, \] (18)
where
\[ \{B^i\} = \{B_x, B_y, B_z\}, \quad \{B_i\} = \{-B_x, -B_y, -B_z\}. \] (19)

Using Eqs. (18) and (19), Eqs. (16) and (17) can be written in the vector form as follows:
\[ \nabla \times B(x) = \mu_0 J(x), \] (20)
\[ \nabla \cdot B(x) = 0. \] (21)

The above equations are the basic equations of magnetostatics [30].
An immediate consequence of Eq. (21) is that $B(x)$ can be written as follows:
\[ B(x) = \nabla \times A(x). \] (22)

Now, we want to obtain the Lagrangian density for a magnetostatic field in the presence of a minimal length scale based on the Kempf algebra. For this purpose, we must replace the ordinary position and derivative operators with the deformed position and derivative operators according to Eqs. (12) and (13), i.e.,
\[ x^i \rightarrow X^i = x^i, \] (23)
\[ \partial^i \rightarrow D^i := (1 - \beta \hbar^2 \nabla^2) \partial^i, \] (24)
where $\nabla^2 := \partial_i \partial_i$ is the Laplace operator. Using Eqs. (23) and (24) the electromagnetic field tensor in the presence of a minimal length scale becomes
\[ F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x) \rightarrow F_{ij}(X) = D_i A_j(X) - D_j A_i(X), \]
or
\[ F_{ij}(X) = F_{ij}(x) - \beta \hbar^2 \nabla^2 F_{ij}(x). \] (25)

It should be mentioned that the above modification of the electromagnetic field tensor has been introduced earlier by Hossenfelder and co-workers in order to study the minimal length effects in quantum electrodynamics in Ref. [16]. If we use Eqs. (23), (24), and (25), we obtain the
Lagrangian density for a magnetostatic field in the deformed space as follows:

\[
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}^i(x) F^{ij}(X) + J^i(X) A^i(X)
\]

\[
= -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) + \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{ij}(x) \nabla^2 F^{ij}(x) + J^i(x) A^i(x) + \mathcal{O} \left((\hbar \sqrt{2\beta})^4\right). \tag{26}
\]

The term \[\frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{ij}(x) \nabla^2 F^{ij}(x)\] in Eq. (26) can be considered as a minimal length effect.

After neglecting terms of order \((\hbar \sqrt{2\beta})^4\) and higher in Eq. (26) we obtain

\[
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) + \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{ij}(x) \nabla^2 F^{ij}(x) + J^i(x) A^i(x). \tag{27}
\]

The Lagrangian density (27) is similar to the magnetostatic sector of the Abelian Lee-Wick model which was introduced by Lee and Wick as a finite theory of quantum electrodynamics [32-36]. Eq. (27) can be written as

\[
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) - \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 \partial_n F_{ij}(x) \partial_n F^{ij}(x) + J^i(x) A^i(x) + \partial_n \Lambda_n(x), \tag{28}
\]

where

\[
\Lambda_n(x) := \frac{1}{4\mu_0} (\hbar \sqrt{2\beta})^2 F_{ij}(x) \partial_n F^{ij}(x). \tag{29}
\]

After dropping the total derivative term \(\partial_n \Lambda_n(x)\), the Lagrangian density (28) will be equivalent to the following Lagrangian density:

\[
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) - \frac{1}{4\mu_0} a^2 \partial_n F_{ij}(x) \partial_n F^{ij}(x) + J^i(x) A^i(x), \tag{30}
\]

where \(a := \hbar \sqrt{2\beta}\) is a constant parameter which is called Podolsky’s characteristic length [37-41].

The Euler-Lagrange equation for the Lagrangian density (30) is [42-44]

\[
\frac{\partial \mathcal{L}}{\partial A_k} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial (\partial_i A_k)}\right) + \partial_m \partial_k \left(\frac{\partial \mathcal{L}}{\partial (\partial_m \partial_k A_k)}\right) = 0. \tag{31}
\]

Using Eq. (23) together with the transformation rule for a contravariant vector, we obtain the following result to the first order in deformation parameter \(\beta\)

\[
J'^i(X) A'^i(X) = \frac{\partial X'^i}{\partial x^j} J^j(x) \frac{\partial X^j}{\partial x^k} A^k(x) = \delta^i_j \delta^i_k J^j(x) A^k(x) = J^i(x) A^i(x).
\]
If we substitute (30) into (31), we obtain the following field equation for the magnetostatic field in the deformed space \(^4\):

\[ \partial_l F^{lk}(x) - a^2 \nabla^2 \partial_l F^{lk}(x) = \mu_0 J^k(x). \]  

(32)

Using Eqs. (18) and (19), Eqs. (17) and (32) can be written in the vector form as follows:

\[ (1 - a^2 \nabla^2) \nabla \times B(x) = \mu_0 J(x), \]  

(33)

\[ \nabla \cdot B(x) = 0. \]  

(34)

Equations (33) and (34) are fundamental equations of Podolsky’s magnetostatics [45-48]. It should be noted that Eqs. (30), (33), and (34) can be obtained as the magnetostatic limit of Eqs. (20), (26), and (27) in our previous paper [15]. Using Stokes’s theorem the integral form of Eq. (33) can be written in the form:

\[ \oint_C [B(x) - (\hbar \sqrt{2}/\beta)^2 \nabla^2 B(x)] \cdot dl = \mu_0 I, \]  

(35)

where \( I \) is the total current passing though the closed curve \( C \). Equation (35) is Ampere’s law in the presence of a minimal length scale. It is clear that for \( \hbar \sqrt{2}/\beta \to 0 \), the modified Ampere’s law in Eq. (35) becomes the usual Ampere’s law.

Now, let us obtain the energy density of a magnetostatic field in the presence of a minimal length scale. The energy density of a magnetostatic field in the usual magnetostatics is given by [30]

\[ u_B = \frac{1}{2\mu_0} B(x) \cdot B(x) \]

\[ = \frac{1}{2\mu_0} (\nabla \times A(x)) \cdot (\nabla \times A(x)). \]  

(36)

Using Eqs. (23) and (24) the energy density of a magnetostatic field under the influence of a minimal length scale becomes

\[ u_B = \frac{1}{2\mu_0} (\nabla \times A(x)) \cdot (\nabla \times A(x)) \longrightarrow u_{B}^{ML} = \frac{1}{2\mu_0} (D \times A(X)) \cdot (D \times A(X)), \]

or

\[ \frac{\partial \phi_{i_1 \cdots i_k}}{\partial \phi_{j_1 \cdots j_k}} = \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k}, \]

where \( \phi_{i_1 \cdots i_k} := \partial_{i_1} \cdots \partial_{i_k} \phi \). This definition has been used by Moeller and Zwiebach in Ref. [44].
\[
\begin{align*}
\mathcal{u}^\text{ML}_B &= \frac{1}{2\mu_0}[(1 - \beta h^2 \nabla^2) \nabla \times A(x)] \cdot [(1 - \beta h^2 \nabla^2) \nabla \times A(x)] \\
&= \frac{1}{2\mu_0}B(x) \cdot B(x) - \frac{1}{2\mu_0} (\hbar \sqrt{2\beta})^2 B(x) \cdot \nabla^2 B(x) + \mathcal{O} \left( (\hbar \sqrt{2\beta})^4 \right),
\end{align*}
\]

where we use the abbreviation ML for the minimal length. If we use the vector identities

\[
\begin{align*}
\nabla \times (\nabla \times a) &= \nabla (\nabla \cdot a) - \nabla^2 a, \\
\nabla \cdot (a \times b) &= b \cdot (\nabla \times a) - a \cdot (\nabla \times b),
\end{align*}
\]

together with Eq. (34), the modified energy density \( u^\text{ML}_B \) can be written in the form

\[
\begin{align*}
\mathcal{u}^\text{ML}_B &= \frac{1}{2\mu_0}B(x) \cdot B(x) + \frac{1}{2\mu_0} (\hbar \sqrt{2\beta})^2 (\nabla \times B(x)) \cdot (\nabla \times B(x)) \\
&\quad + \nabla \cdot \Omega(x) + \mathcal{O} \left( (\hbar \sqrt{2\beta})^4 \right),
\end{align*}
\]

where

\[
\Omega(x) := \frac{1}{2\mu_0} (\hbar \sqrt{2\beta})^2 (\nabla \times B(x)) \times B(x).
\]

After dropping the total divergence term \( \nabla \cdot \Omega(x) \), the modified energy density (40) will be equivalent to the following modified energy density:

\[
\begin{align*}
\mathcal{u}^\text{ML}_B &= \frac{1}{2\mu_0}B(x) \cdot B(x) + \frac{1}{2\mu_0} (\hbar \sqrt{2\beta})^2 (\nabla \times B(x)) \cdot (\nabla \times B(x)) \\
&\quad + \mathcal{O} \left( (\hbar \sqrt{2\beta})^4 \right).
\end{align*}
\]

The term \( \frac{1}{2\mu_0} (\hbar \sqrt{2\beta})^2 (\nabla \times B(x)) \cdot (\nabla \times B(x)) \) in Eq. (42) shows the effect of minimal length corrections.

### 4 Green’s Function for a Magnetostatic Field in the Presence of a Minimal Length Scale

Substituting Eq. (22) into Eq. (33) and using the vector identity (38) we obtain

\[
(1 - a^2 \nabla^2) [\nabla (\nabla \cdot A(x)) - \nabla^2 A(x)] = \mu_0 J(x).
\]


In the Coulomb gauge ($\nabla \cdot A(x) = 0$), Eq. (43) can be written as

$$(1 - a^2 \nabla^2) \nabla^2 A(x) = -\mu_0 J(x).$$

(44)

The solution of Eq. (44) in terms of the Green’s function, $G(x, x')$, is given by

$$A(x) = A_0(x) + \frac{\mu_0}{4\pi} \int G(x, x') J(x') d^3 x',$$

(45)

where $A_0(x)$ and $G(x, x')$ satisfy the equations

$$(1 - a^2 \nabla^2) \nabla^2 A_0(x) = 0,$$

(46)

and

$$(1 - a^2 \nabla_x^2) \nabla^2_x G(x, x') = -4\pi \delta(x - x').$$

(47)

Now, let us solve Eq. (47) by writing $G(x, x')$ and $\delta(x - x')$ in terms of Fourier integrals as follows:

$$G(x, x') = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k} \cdot (x - x')} \tilde{G}(\mathbf{k}) d^3 k,$$

(48)

$$\delta(x - x') = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k} \cdot (x - x')} d^3 k.$$  

(49)

If we substitute Eqs. (48) and (49) into Eq. (47), we obtain the functional form of $\tilde{G}(\mathbf{k})$ as follows:

$$\tilde{G}(\mathbf{k}) = \frac{4\pi}{\mathbf{k}^2 + a^2 (\mathbf{k}^2)^2} = 4\pi \left( \frac{1}{\mathbf{k}^2} - \frac{a^2}{1 + a^2 \mathbf{k}^2} \right).$$

(50)

If Eq. (50) is inserted into Eq. (48), the Green’s function $G(x, x')$ becomes

$$G(x, x') = \frac{1}{2\pi^2} \int e^{-i\mathbf{k} \cdot (x - x')} \left( \frac{1}{\mathbf{k}^2} - \frac{a^2}{1 + a^2 \mathbf{k}^2} \right) d^3 k$$

$$= \frac{1 - e^{-\frac{|x - x'|}{a}}}{|x - x'|}.$$  

(51)

This type of Green’s function has been considered in electrodynamics to avoid divergences associated with point charges [38,45,49,50]. Using Eqs. (45) and (51) the particular solution of Eq. (44), which vanishes at infinity is

$$A(x) = \frac{\mu_0}{4\pi} \int \frac{1 - e^{-\frac{|x - x'|}{a}}}{|x - x'|} J(x') d^3 x'.$$

(52)
The vector potential \( \mathbf{A}(\mathbf{x}) \) satisfies the Coulomb gauge condition \( \nabla \cdot \mathbf{A}(\mathbf{x}) = 0 \). The expression (52) can be applied to current circuits by making the substitution: \( \mathbf{J}(\mathbf{x})d^3\mathbf{x}' \rightarrow I d\mathbf{l}' \). Thus

\[
\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{1 - e^{-\frac{||\mathbf{x} - \mathbf{x}'||}{a}}}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{l}',
\]

where \( C \) is the contour defined by the wire. If we use Eqs. (22) and (52), we obtain the magnetic induction vector \( \mathbf{B}(\mathbf{x}) \) as follows:

\[
\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} [1 - (1 + \frac{||\mathbf{x} - \mathbf{x}'||}{a}) e^{-\frac{||\mathbf{x} - \mathbf{x}'||}{a}}] d^3\mathbf{x}',
\]

or

\[
\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} [1 - (1 + \frac{||\mathbf{x} - \mathbf{x}'||}{a}) e^{-\frac{||\mathbf{x} - \mathbf{x}'||}{a}}].
\]

Equation (54) is the Biot-Savart law in the presence of a minimal length scale. In the limit \( a = \hbar \sqrt{2\beta} \rightarrow 0 \), the modified Biot-Savart law in (54) smoothly becomes the usual Biot-Savart law, i.e.,

\[
\lim_{a \rightarrow 0} \mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{l} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{55}
\]

5 Upper Bound Estimation of the Minimal Length Scale in Modified Magnetostatics

Now, let us estimate the upper bounds on the isotropic minimal length scale in modified magnetostatics. By putting \( \beta' = 2\beta \) into (6) the isotropic minimal length scale becomes

\[
(\Delta X^i)_{\text{min}} = \sqrt{\frac{D + 2}{2}} (\hbar \sqrt{2\beta}) \quad , \quad \forall i \in \{1,2,\ldots, D\}. \tag{56}
\]

The isotropic minimal length scale (56) in three spatial dimensions is given by

\[
(\Delta X^i)_{\text{min}} = \frac{\sqrt{10}}{2} a \quad , \quad \forall i \in \{1, 2, 3\}, \tag{57}
\]

where \( a = \hbar \sqrt{2\beta} \).

In a series of papers, Sprenger and co-workers [51,52] have concluded that the minimal length
scale \((\Delta X^i)_{\text{min}}\) in Eq. (57) might lie anywhere between the Planck length scale \((\ell_P \sim 10^{-35} \text{ m})\) and the electroweak length scale \((\ell_{\text{electroweak}} \sim 10^{-18} \text{ m})\), i.e.,

\[
10^{-35} \text{ m} < (\Delta X^i)_{\text{min}} < 10^{-18} \text{ m}.
\] (58)

According to above statements, the upper bound on the isotropic minimal length scale in three spatial dimensions becomes

\[
(\Delta X^i)_{\text{min}} < 10^{-18} \text{ m}.
\] (59)

Inserting (59) into (57), we find

\[
a < 0.63 \times 10^{-18} \text{ m}.
\] (60)

In a series of papers, Accioly et al. [34, 36, 37] have estimated an upper bound on Podolsky’s characteristic length \(a\) by computing the anomalous magnetic moment of the electron in the framework of Podolsky’s electrodynamics. This upper bound on \(a\) is

\[
a < 4.7 \times 10^{-18} \text{ m}.
\] (61)

Note that the upper bound on the Podolsky’s characteristic length \(a\) in Eq. (60) is near to the upper bound on the Podolsky’s characteristic length in Eq. (61).

Another upper bound on the minimal length scale has been obtained in Ref. [53] by considering minimal length corrections to the gyromagnetic moment of electrons and muons. If we compare Eq. (13) in this work with Eq. (40) in Ref. [16], we obtain

\[
\hbar \sqrt{\beta} = L_f \frac{\sqrt{3}}{\sqrt{3}},
\] (62)

where \(L_f\) is the minimal length scale in Refs. [16,53]. If we substitute (62) into (56), we will obtain the isotropic minimal length in three spatial dimensions as follows:

\[
(\Delta X^i)_{\text{min}} = \sqrt{\frac{5}{3}} L_f, \quad \forall i \in \{1, 2, 3\}.
\] (63)

The minimal length scale \(L_f\) in Eqs. (62) and (63) can be written as

\[
L_f = \frac{\hbar}{M_f c},
\] (64)

where \(M_f\) is a new fundamental mass scale [16,53]. Inserting Eq. (64) into Eq. (63), we find

\[
(\Delta X^i)_{\text{min}} = \sqrt{\frac{5}{3}} \frac{\hbar}{M_f c}, \quad \forall i \in \{1, 2, 3\}.
\] (65)
In Ref. [53] it was shown that the effect of minimal length corrections to the gyromagnetic moment of the muon leads to the following lower bound on the fundamental mass scale of the theory:

\[ M_f \geq \frac{577 \text{GeV}}{c^2}. \quad (66) \]

Substituting Eq. (66) into Eq. (65), the isotropic minimal length scale in three spatial dimensions becomes

\[ (\Delta X^i)_{\text{min}} \leq 4.42 \times 10^{-19} \text{m}. \quad (67) \]

If we insert Eq. (67) into Eq. (57), we will find

\[ a \leq 2.79 \times 10^{-19} \text{m}. \quad (68) \]

It is interesting to note that the numerical value of the upper bound on \( a \) in Eq. (68) and the numerical value of the upper bound on \( a \) in Eq. (60) are close to each other.

6 The Equivalence between the Gaete-Spallucci Non-Local Magnetostatics and Magnetostatics in the Presence of a Minimal Length Scale

Smailagic and Spallucci have proposed an approach to formulate quantum field theory in the presence of a minimal length scale [54-56]. Using the Smailagic-Spallucci approach, Gaete and Spallucci have introduced a \( U(1) \) gauge field with a non-local kinetic term whose magnetostatic sector is

\[ \mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) \exp(-\theta \nabla^2) F^{ij}(x) + J^i(x) A^i(x), \quad (69) \]

where \( \theta \) is a constant parameter with dimension of \((\text{length})^2\) [57]. The function \( \exp(-\theta \nabla^2) \) in Eq. (69) can be expanded in a formal power series as follows:

\[ \exp(-\theta \nabla^2) = \sum_{l=0}^{+\infty} (-1)^l \frac{\theta^l}{l!} (\nabla^2)^l, \quad (70) \]

where \((\nabla^2)^l\) denotes the \( \nabla^2 \) operator applied \( l \) times [58].

After inserting Eq. (70) into Eq. (69), we obtain the following Lagrangian density:

\[
\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) + \frac{1}{4\mu_0} \theta F_{ij}(x) \nabla^2 F^{ij}(x) \\
+ \frac{1}{4\mu_0} \sum_{l=2}^{+\infty} (-1)^{l+1} \frac{\theta^l}{l!} F_{ij}(x)(\nabla^2)^l F^{ij}(x) + J^i(x) A^i(x). \quad (71)
\]
If we neglect terms of order $\theta^2$ and higher in Eq. (71), we find

$$L = -\frac{1}{4\mu_0} F_{ij}(x) F^{ij}(x) + \frac{1}{4\mu_0} \theta F_{ij}(x) \nabla^2 F^{ij}(x) + J^i(x) A^i(x).$$

(72)

A comparison between Eqs. (27) and (72) clearly shows that there is an equivalence between the Gaete-Spallucci non-local magnetostatics to the first order in $\theta$ and the magnetostatic sector of the Abelian Lee-Wick model (or magnetostatics in the presence of a minimal length scale). The relationship between the non-commutative constant parameter $\theta$ in Eq. (72) and $a = \hbar \sqrt{2\beta}$ in Eq. (27) is

$$\theta = a^2.$$  

(73)

According to Eq. (73), $a = \sqrt{\theta}$ plays the role of the minimal length in the Gaete-Spallucci non-local magnetostatics [57,59].

If we insert Eq. (73) into Eq. (57), we find

$$(\Delta X^i)_{\text{min}} = \frac{\sqrt{10 \theta}}{2}, \quad \forall i \in \{1, 2, 3\}.$$  

(74)

Using Eq. (68) in Eq. (73), we obtain the following upper bound for the non-commutative parameter $\theta$:

$$\theta_{\text{MLCGMM}} \leq 7.78 \times 10^{-38} \text{ m}^2,$$

(75)

where we use the abbreviation MLCGMM for the minimal length corrections to the gyromagnetic moment of the muon. Chaichian and his collaborators have investigated the Lamb shift in non-commutative quantum electrodynamics (NCQED) [60,61]. They found the following upper bound for the non-commutative parameter $\theta$:

$$\theta_{\text{NCQED}} \leq (10^4 \text{GeV})^{-2},$$

or

$$\theta_{\text{NCQED}} \leq 3.88 \times 10^{-40} \text{ m}^2.$$  

(76)

For a review of the phenomenology of non-commutative geometry see Ref. [62]. The upper bound (75) is about two orders of magnitude larger than the upper bound (76), i.e.,

$$\theta_{\text{MLCGMM}} \sim 10^2 \theta_{\text{NCQED}}.$$  

(77)

If we insert (61) into (73), we obtain the following upper bound for $\theta$:

$$\theta_{\text{MLCGME}} \leq 2.2 \times 10^{-35} \text{ m}^2,$$

(78)

where we use the abbreviation MLCGME for the minimal length corrections to the gyromagnetic moment of the electron. The upper bound (78) is about four orders of magnitude larger than the upper bound (76), i.e.,

$$\theta_{\text{MLCGME}} \sim 10^4 \theta_{\text{NCQED}}.$$  

(79)
A comparison between Eq. (77) and Eq. (79) shows that $\theta_{MLCGMM}$ is nearer to $\theta_{NCQED}$. It should be emphasized that the magnetostatics in the presence of a minimal length scale is only correct to the first order in the deformation parameter $\beta$, while the Gaete-Spallucci non-local magnetostatics is valid to all orders in the non-commutative parameter $\theta$.

7 Conclusions

After the appearance of quantum field theory many theoretical physicists have attempted to reformulate quantum field theory in the presence of a minimal length scale [63,64]. The hope was that the introduction of such a minimal length scale leads to a divergenceless quantum field theory [65]. Recent studies in perturbative string theory and quantum gravity suggest that there is a minimal length scale in nature [1]. Today’s we know that the existence of a minimal length scale leads to a generalization of Heisenberg uncertainty principle. An immediate consequence of the GUP is that the usual position and derivative operators must be replaced by the modified position and derivative operators according to Eqs. (23) and (24) for $\beta' = 2\beta$. We have formulated magnetostatics in the presence of a minimal length scale based on the Kempf algebra. It was shown that there is a similarity between magnetostatics in the presence of a minimal length scale and the magnetostatic sector of the Abelian Lee-Wick model. The integral form of Ampere’s law and the energy density of a magnetostatic field in the presence of a minimal length scale have been obtained. Also, the Biot-Savart law in the presence of a minimal length scale has been found. We have shown that in the limit $h\sqrt{2\beta} \rightarrow 0$, the modified Ampere and Biot-Savart laws become the usual Ampere and Biot-Savart laws. It is necessary to note that the upper bounds on the isotropic minimal length scale in Eqs. (59) and (67) are close to the electroweak length scale ($\ell_{\text{electroweak}} \sim 10^{-18} m$). We have demonstrated the equivalence between the Gaete-Spallucci non-local magnetostatics up to the first order over $\theta$ and magnetostatics with a minimal length up to the first order over the deformation parameter $\beta$. Recently, Romero and collaborators have formulated a higher-derivative electrodynamics [66]. In this work we have formulated a higher-derivative magnetostatics in the framework of Kempf algebra whereas the authors of [66] have studied an electrodynamics consistent with anisotropic transformations of spacetime with an arbitrary dynamic exponent $z$. 

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