On limitation of mass spectrum in non-Hermitian $\mathcal{PT}$-symmetric models with the $\gamma_5$-dependent mass term

V.N.Rodionov

Plekhanov Russian University, Moscow, Russia, E-mail vnrodionov@mtu-net.ru

Abstract

The modified Dirac equations for the massive particles with the replacement of the physical mass $m$ with the help of the relation $m \rightarrow m_1 + \gamma_5 m_2$ are investigated. It is shown that for a free fermion theory with a $\gamma_5$ mass term, the finiteness of the mass spectrum at the value $m_{\text{max}} = m_1^2/2m_2$ takes place. In this case the region of the unbroken $\mathcal{PT}$-symmetry may be expressed by means of the simple restriction of the physical mass $m \leq m_{\text{max}}$. Furthermore, we have that the areas of unbroken $\mathcal{PT}$-symmetry $m_1 \geq m_2 \geq 0$, which guarantees the reality values of the physical mass $m$, consists of three different parametric subregions: i) $0 \leq m_2 < m_1/\sqrt{2}$, ii) $m_2 = m_1/\sqrt{2} = m_{\text{max}}$, (iii)$m_1/\sqrt{2} < m_2 \leq m_1$. It is vary important, that only the first subregion (i) defined mass values $m_1, m_2$, which correspond to the description of traditional particles in the modified models, because this area contain the possibility transform the modified model to the ordinary Dirac theory. The second condition (ii) is defined the ”maximon” - the particle with maximal mass $m = m_{\text{max}}$. In the case (iii) we have to do with the unusual or ”exotic” particles for description of which Hamiltonians and equations of motion have no a Hermitian limit. The formulated criterions may be used as a major test in the process of the division of considered models into ordinary and ”exotic fermion theories”.

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1 Introductory remarks

As it is well-known the idea about existence of a maximal mass in a broad spectrum mass of elementary particles at the Planck mass was suggested by Moisey Markov in 1965 [1]

\[ m \leq m_{Planck} \cong 10^{19} \text{GeV}. \]

After that a more radical approach was developed by V.G. Kadyshevsky [2]. His model contained a limiting mass \( M \) as a new fundamental physical constant. Doing this condition of finiteness of the mass spectrum should be introduced by the relation:

\[ m \leq M, \quad (1) \]

where a new constant \( M \) was named by the fundamental mass.

Really in the papers [2]-[13] the existence of mass \( M \) has been understood as a new principle of Nature similar to the relativistic and quantum postulates, which was put into the ground of the new quantum field theory. At the same time the new constant \( M \) is introduced in a purely geometric way, like the velocity of light is the maximal velocity in the special relativity.

Indeed, if one chooses a geometrical formulation of the quantum field theory, the adequate realization of the limiting mass hypothesis is reduced to the choice of the de Sitter geometry as the geometry of the 4-momentum space of the constant curvature with a radius equal to \( M \) [2]. The detailed analysis of the different aspects of the construction of the modified quantum field theory with the maximal mass in the curved momentum de Sitter space has allowed to obtain a number of interesting results. In particular, it has been shown that non-Hermitian fermionic Hamiltonians with the \( \gamma_5 \)-dependent mass term must arise in the modified theory (see, for example, [8], [9]).

Now it is well-known fact the reality of the spectrum in models with a non-Hermitian Hamiltonian is a consequence of \( \mathcal{P}\mathcal{T} \)-invariance of the theory, i.e. a combination of spatial and temporary parity of the total Hamiltonian: \([H, \mathcal{P}\mathcal{T}]_\psi = 0\). When the \( \mathcal{P}\mathcal{T} \) symmetry is unbroken, the spectrum of the quantum theory is real. These results explain the growing interest in this problem. For the past a few years studied a lot of new non-Hermitian \( \mathcal{P}\mathcal{T} \)-invariant systems (see, for example, [14] - [35]). In the literature, which was devoted to the study of non-Hermitian operators there are examples, with the \( \gamma_5 \) mass extension.

In particular the modified Dirac equations for the massive Thirring model in two-dimensional space-time with the purely algebraic replacement of the
physical mass $m$ by $m \to m_1 + \gamma_5 m_2$ (where $m_1 \geq 0$, and $m_2$ real) was investigated by Bender et al. \cite{25}. As the foundation of this study is assumed the a model with the density of the Hamiltonian, which is represented in the form:

$$H(x, t) = \bar{\psi}(x, t) \left(-i \gamma \cdot \nabla + m_1 + \gamma_5 m_2 \right) \psi(x, t). \quad (2)$$

The equation of motion following from the (2), may be expressed as

$$\left(i \partial_{\mu} \gamma^{\mu} - m_1 - \gamma_5 m_2 \right) \psi(x, t) = 0. \quad (3)$$

It is obvious that the Hamiltonian associated with the Hamiltonian density (2) is non-Hermitian due to the appearance in it the $\gamma_5$-dependent mass components ($H \neq H^+$).

A. Mustafazadeh identified the necessary and sufficient requirements of reality of eigenvalues for pseudo-Hermitian and $\mathcal{PT}$-symmetric Hamiltonians and formalized the use these Hamiltonians in his papers \cite{15}, \cite{16}. According to the recommendations of this works we can define Hermitian operator $\eta$, which transform non-Hermitian Hamiltonian by means of invertible transformation to the Hermitian-conjugated one. It is easy to see that with Hermitian operator

$$\eta = e^{\gamma_5 \vartheta} \quad (4)$$

we can obtain

$$\eta H \eta^{-1} = H^+, \quad (5)$$

$$H = \alpha p + \beta (m_1 + \gamma_5 m_2) \quad (6)$$

and

$$H^+ = \alpha p + \beta (m_1 - \gamma_5 m_2), \quad (7)$$

where matrices $\alpha_i = \gamma_0 \cdot \gamma_i$ and $\beta = \gamma_0$, and $\vartheta = \text{arctanh}(m_2/m_1)$

In addition, multiplying the Hamilton operator $H$ on the left on $e^{\vartheta \gamma_5 /2}$, we can obtain

$$e^{\gamma_5 \vartheta /2} H = H_0 e^{\gamma_5 \vartheta /2}, \quad (8)$$

where $H_0 = \alpha p + \beta m$ is a ordinary Hermitian Hamiltonian of a free particle.

The mathematical sense of the action of the operator (4) it turns out, if we notice that according to the properties of matrices $\gamma_5$, all the even degree of $\gamma_5$ are equal to 1, and all odd degree are equal to $\gamma_5$. Given that
\( \text{ch} \) decomposes on even and \( \text{sh} \) odd degrees, the expressions (5)-(8) can be obtained by representing exponential operator \( \eta \) in the form
\[
\eta = e^{\gamma_5 \vartheta} = \text{ch} \vartheta + \gamma_5 \text{sh} \vartheta,
\]
where
\[
\text{ch} \vartheta = \frac{m_1}{m}; \quad \text{sh} \vartheta = \frac{m_2}{m}, \quad (9)
\]
and taking into account that the matrices \( \gamma_5 \) commute with matrices \( \alpha_i \) and anti-commute with \( \beta \).

The region of the unbroken \( \mathcal{PT} \)-symmetry in the paper [25] has been found in the form
\[
m_1^2 \geq m_2^2. \quad (10)
\]

However, it is not apparent that the area with undisturbed \( \mathcal{PT} \)-symmetry (10) does not include the regions, corresponding to the some unusual particles, description of which radically distinguish from traditional one. This particles which was named ”exotic” in the frame of geometric approach to the construction of model with the Maximal Mass, should play a special role in the world of the elementary particles [8],[9].

The distinguishing feature of ”exotic” particles consists of the fact that Hamiltonians and equations of motion for their description have no the limit when \( M \to \infty \). Thus, they not to agree in this limit with the ordinary Dirac expressions and one can assume that in this case we deal with a description of some new particles, properties of which have not yet been studied. This fact for the first time has been fixed back in the early works on the development of the theory with a fundamental mass [2].

In the frame of purely algebraic approach, when take place the extending of mass parameter \( m \to m_1 + \gamma_5 m_2 \), there are not an explicit restrictions of masses. However, from (5),(6) one can easy obtain a number of the limiting values of the mass parameters. In this connection it is very interesting to learn could in algebraic model receive a description of such particles? Consequently the question arises: ”what precisely particles are considered: exotic or traditional, when the condition (10) is executed?”

It is important to note that the previous works in pseudo-Hermitian quantum mechanics [25],[26] with \( \gamma_5 \)-dependent mass term have been carried out in such a way, that condition (10) is a single inequality for parameters of mass in the model. The main result obtained by us consists of that the region (10) of unbroken \( \mathcal{PT} \) symmetry in reality has an internal structure where
implicated as ordinary and "exotic" particles. This conclusion managed to get thanks to the existence of the restriction of the mass spectrum of particles \( m \leq m_{\text{max}} \) automatically arising in relativistic model with \( \gamma_5 \)-dependent mass term in the Hamiltonians (6) and (7).

This paper has the following structure. In section II the necessary and sufficient conditions of restrictions the mass spectrum are formulated in considered model. In the third section we study the basic characteristics of \( \mathcal{PT} \)-symmetric free fermionic models with \( \gamma_5 \) massive term.

2 Necessary and sufficient conditions of the mass spectrum restrictions in the model with \( \gamma_5 \) mass term.

Consider now the algebraic approach developed in the numerous papers on the study of quantum non-Hermitian mechanics. It was possible to expect, that the appearance of the models described by Hamiltonians type (6), (7) is the prerogative of a purely geometric approach to the construction of a modified theory with a maximal mass. However, in the paper [25] was considered the \( \mathcal{PT} \)-symmetric massive Thirring model with \( \gamma_5 \) mass extension in (1+1)-dimensional space.

Let us consider the relativistic quantum mechanics with the \( \gamma_5 \)-mass term in the case of 3+1 dimensional space-time. We introduce the following representation of a \( \gamma \)-matrices [36]:

\[
\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \tilde{\gamma} = \begin{pmatrix} 0 & \tilde{\sigma} \\ -\tilde{\sigma} & 0 \end{pmatrix}.
\]

According to these definitions, \( \gamma_0^2 = 1 \) and \( \gamma_i^2 = -1 \). We also have

\[
\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]

so that \( \gamma_5^2 = 1 \).

In this regard, there is interest in the study of the parameters, characterizing of the masses, included in the \( \gamma_5 \)-dependent mass model. Note that the \( \gamma_5 \) - extension of the mass in the Dirac equation consists of replacing

\( m \to m_1 + \gamma_5 m_2 \)

and after that two new mass parameters: \( m_1 \) and \( m_2 \) arises.
First-order equations (3) can be transformed into equations of second order by applying to (3) operator:

\[
\left(i\partial_\mu \gamma^\mu + m_1 - \gamma_5m_2\right).
\] (13)

In a result the Dirac equation converts to the Klein-Gordon equation:

\[
(\partial^2 + m^2) \psi(x, t) = 0
\] (14)

where

\[m^2 = m_1^2 - m_2^2.\] (15)

It is easy to see from (15) that the physical mass \(m\), appearing in the equation (14), is real when the inequality

\[m_1^2 \geq m_2^2.\] (16)

is accomplished.

Developed algebraic formalism in [25], no contains indications of the existence of other restrictions, in which participates the mass, in addition to (16). However, it is important, that this conditions should be obtained because they can help to establish the connection between algebraic and geometric approaches. Note also that the existence of such restrictions, in particular, may essentially modify the fundamental results obtained in the paper [25].

Consider the Hamiltonians of the type (6) and show that the algebraic approach allows one to set the sufficient condition of the limitations of the mass spectrum of particles in relativistic quantum mechanics with \(\gamma_5\)-mass component. As noted above, the inequality (16) was considered [25] as the requirement that determines the presence of broken or unbroken \(\mathcal{PT}\) -symmetry of the Hamiltonian. However, it is easy to see that (16) may not be as the single condition of this type.

Writing the following obvious inequality:

\[(m - m_2)^2 \geq 0\] (17)

and taking into account (15), we can obtain

\[m \leq \frac{m_1^2}{2m_2} = m_{\text{max}},\] (18)
that is direct indication of the existence of the restriction of the physical mass \( m \) in the considered model with \( \gamma_5 \)-dependent mass term.

It is interesting that (18) is obtained as the result of the simple algebraic transformation of relationships with subsidiary parameters of mass \( m_1 \) and \( m_2 \). It is quite natural that the value of the \( m_{\text{max}} \) is expressed through a combination of them. In particular, as the degree of deviation of the Hamiltonian \( H \) of a Hermitian forms is characterized by the mass \( m_2 \), then its value can be expressed from (18)

\[
m_2 = \frac{m_1^2}{2m_{\text{max}}}. \tag{19}
\]

To see whether the modified Hamiltonian (6) is Hermitian, we must check whether the contribution of the mass \( m_2 \) becomes vanishingly small? We do so by doing the parameter maximal mass very large (formally, when \( m_{\text{max}} \to \infty \)).

Following the (9) and taking advantage of the symbol

\[
m_2 = m_1 \tanh \vartheta, \tag{20}
\]

we can obtain

\[
\frac{m_1}{2m_{\text{max}}} = \tanh \vartheta \leq 1, \tag{21}
\]

Then we can establish the limits of change of parameters. At preset values of \( m \) and \( m_{\text{max}} \) as it follows from the (18)-(21) the limits of variation of parameters \( m_1 \) and \( m_2 \) are the following:

\[
m \leq m_1 \leq 2m_{\text{max}}; -2m_{\text{max}} \leq m_2 \leq 2m_{\text{max}}, \tag{22}
\]

where considering the possible change of the sign of \( m_2 \). In the areas of change of these parameters has a point in which \( \tanh \vartheta_0 = 1/\sqrt{2} \), where we have

\[
m_1 = \sqrt{2}m_{\text{max}}; m_2 = m_{\text{max}}. \tag{23}
\]

In this point the physical mass \( m \) (see (18)) reaches its maximum value \( m = m_{\text{max}} \).

Thus, we may rewrite (6) in the resulting form

\[
H = \tilde{\alpha} \vec{p} + \tilde{\beta} m_1 \left( 1 + \gamma_5 \tanh \vartheta \right). \tag{24}
\]
and taking into account (23) the areas of possible values of the parameter \( \vartheta \) can be presented in the form: (i) \( 0 \leq \vartheta < \vartheta_0 \) - corresponds to the theory having a Hermitian limit, i.e. the limit, when \( m_2 \to 0 \) (\( m_{\text{max}} \) tends to infinity) exists and (ii) \( \vartheta_0 < \vartheta < \infty \) - refers to the occasion when the such a limit is absent. The first condition corresponds to the description of ordinary particles and the second - to unusual or exotic particles. The limit value \( \vartheta = \vartheta_0 = \operatorname{arctanh}(1/\sqrt{2}) \) is responsible for the particles with the maximal mass, which was named by the maximons [1].

Thus, the limitation of the mass spectrum of particles (18), described by Hamiltonians (6), (7) and (24) is a simple consequence of the \( \gamma_5 \)-mass extension in (2), (3). Therefore the presence of the non-Hermitian Hamiltonian with a \( \gamma_5 \)-mass contribution, in essence, can be interpret as the sufficient condition of the limitation of the mass spectrum of particles in fermion models with \( \gamma_5 \) - massive term.

On the other hand, in the framework of geometrical approach, when the basis of the modified QFT with the Maximal Mass is a postulate (1) we need to get the appearance of non-Hermitian Hamiltonians with a \( \gamma_5 \)-contribution in its fermion sector (see for example [3], [14]). This consequence can be interpreted as a necessary condition of the finiteness of the mass spectrum [1].

Thus, we can see that the necessary and a sufficient conditions of the limitation of the physical mass of particles (1) and (18) in fermion sector are the using of the non-Hermitian \( \mathcal{PT} \) - symmetric quantum models with a \( \gamma_5 \) - massive contribution.

### 3 Free fermion models with \( \gamma_5 \)-massive contributions and the areas of unbroken \( \mathcal{PT} \) symmetry

Condition (16) is executed automatically, if one enter the parametrization (9). Moreover, from (18) we can express the values of \( m, m_1, m_2 \) with parameter \( \vartheta \).

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1In the case of geometric approach this limit was called a "flat limit" because when the parameter \( M \) is very large the anti de Sitter geometry does not differ from the Minkowski geometry in four dimensional pseudo-Euclidean p-space
Fig. 1. Dependence of $\nu = m/m_{\text{max}}$, $\nu_1 = m_1/m_{\text{max}}$, $\nu_2 = m_2/m_{\text{max}}$ from the parameter $\vartheta$.

Fig. 1 shows the dependence of the relative values $\nu = m/m_{\text{max}}$, $\nu_1 = m_1/m_{\text{max}}$ and $\nu_2 = m_2/m_{\text{max}}$ on the parameter $\vartheta$. The values of the parameters $\nu_1$, $\nu_2$ are the following: $\nu \leq \nu_1 \leq 2$, $0 \leq \nu_2 \leq 2$. A particle mass $\nu$ may vary in a range of $0 \leq m \leq m_{\text{max}}$. When $\vartheta_0 = 0.881$ it reaches its maximum, which corresponds to the maximon $m = m_{\text{max}}$.

Using (18), (9) we can also obtain

$$\tanh(\vartheta) = \sqrt{\frac{1 \pm \sqrt{1 - \nu^2}}{2}}.$$  \hfill (25)

Two signs of the root in (25) are interpreted as two two-valued variables $\nu_1(\nu_3)$ and $\nu_2(\nu_4)$, which are multi-valued functions of $\nu$. Thus, we have

$$\nu_1(\nu_3) = \sqrt{2} \sqrt{1 \mp \sqrt{1 - \nu^2}};$$  \hfill (26)

$$\nu_2(\nu_4) = \left(1 \mp \sqrt{1 - \nu^2}\right).$$  \hfill (27)
It is easy to see, that between the symbols for the masses in geometric approach \[2\] and obtained here the values of (26), (27), after identification of the limiting masses $M$ and $m_{\text{max}}$, there are simple correlations.

Fig. 2 demonstrates the dependence of the parameters $\nu_1(\nu_3), \nu_2(\nu_4)$ on the variable $\nu$. Thus, the region of the existence of unbroken $\mathcal{PT}$ symmetry can be represented in the form $0 \leq \nu \leq 1$. For these values of $\nu$ parameters $\nu_1$ and $\nu_2$ determine the masses of the modified Dirac equation with a maximal mass $m_{\text{max}}$, describing the particles having the actual mass $m \leq m_{\text{max}}$. However, the new Dirac equations nonequivalent, because one of them describes ordinary particles ($\nu_1, \nu_2$), and the other corresponds to their exotic partners($\nu_3, \nu_4$). The special case of Hermiticity is on the line $\nu = 1$ ($m = m_{\text{max}}$ - the case of the maximon), which is the boundary of the unbroken $\mathcal{PT}$ - symmetry. In this point of the plot we have $\nu_1 = \nu_3 = \sqrt{2}$ and $\nu_2 = \nu_4 = 1$.

![Figure 2: The values of parameters $\nu_1, \nu_2, \nu_3, \nu_4$ as the function of $\nu$](image)

It is easy to verify that the new values of the mass parameters $m_3, m_4$ still satisfy the conditions (15) and (16). We emphasize once again that, from
Figure 3: The parametric areas of the unbroken $\mathcal{PT}$-symmetry $\nu_1^2 \geq \nu_2^2$ in plane $\nu_1, \nu_2$ for the Hamiltonian (24) consists of three specific subregions. Only the shaded area $II.$ meets the ordinary particles, and the bordering with it regions $I.$ and $III.$ correspond to the description of the exotic fermions.

In the frame of the condition (10) at Fig.3 we can see three specific sectors of unbroken $\mathcal{PT}$-symmetry of the Hamiltonian (24) in the plane $\nu_1, \nu_2$. The plane $\nu_1, \nu_2$ (there is considered the possible change of the sign of the parameter $m_2$) may be divided by the three groups of the inequalities:

\begin{align*}
I. & \quad \nu_1/\sqrt{2} \leq \nu_2 \leq \nu_1, \\
II. & \quad -\nu_1/\sqrt{2} < \nu_2 < \nu_1/\sqrt{2}, \\
III. & \quad -\nu_1 \leq \nu_2 \leq -\nu_1/\sqrt{2},
\end{align*}

\footnote{Note that this inequality was considered in the paper [25] as an expression, which is completely defines all possible consequences of unbroken $\mathcal{PT}$-symmetry in this model.}
Only the area II. corresponds to the description of ordinary particles, then I. and III. agree with the description of some as yet unknown particles. This conclusion is not trivial, because in contrast to the geometric approach, where the emergence of new unusual properties of particles associated with the presence in the theory a new degree of freedom (sign of the fifth component of the momentum \(\varepsilon = p_5/|p_5|\)), in the case of a simple extension of the free Dirac equation due to the additional \(\gamma_5\)-mass term, the satisfactory explanation is not there yet.

Taking into account properties of matrix \(\gamma_0, \tilde{\gamma}, \gamma_5\), we can write also complex-conjugate equation from equation (3)

\[
\left(-p_0\gamma_0 - \mathbf{p}\tilde{\gamma} - m_1 - \gamma_5m_2\right)\psi^* = 0, \tag{28}
\]

where \(\tilde{\gamma}_\mu\) are transpose matrix and here take place the replacement of

\[
\gamma p = p_0\gamma_0 - \mathbf{p}\gamma = i\gamma_0 \frac{\partial}{\partial t} + i\tilde{\gamma}\nabla,
\]

\(\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3)\).

Rearranging function \(\psi^*\) according to \(\tilde{\gamma}\mu^*\psi^* = \psi^*\gamma^\mu\) then multiply the equation (28) on the right of \(\gamma_0\). Noticing that

\[
\tilde{\gamma}\gamma_0 = -\gamma_0\tilde{\gamma}, \quad \gamma_5\gamma_0 = -\gamma_0\gamma_5
\]

and introducing new bespinor \(\tilde{\psi} = \psi^*\gamma_0\), we can obtain

\[
\tilde{\psi} \left(\gamma p + m_1 - \gamma_5m_2\right) = 0.
\]

(31)

The operator \(p\) is assumed here acts on the function, standing on the left of it. Using (9) we can write equation (3), (31) in the following form

\[
(p\gamma - m\eta) \psi = 0 \tag{32}
\]

\[
\tilde{\psi} \left(p\gamma + m\eta^{-1}\right) = 0 \tag{33}
\]

Repeating the generally accepted procedure of obtaining the continuity equation for a Hermitian case (see, for example [37]) we get similar results for a \(\gamma_5\)-modified non-Hermitian quantum mechanics. Multiply (32) on the left of the \(\tilde{\psi}e^{-\theta\gamma_5}\) and the equation (33) on the right of the \(e^{\theta\gamma_5}\psi\) and sum up the resulting expressions, one can obtain

\[
\tilde{\psi}e^{-\theta\gamma_5/2}\gamma_\mu e^{\theta\gamma_5/2}(p\psi) + (p\tilde{\psi})e^{-\theta\gamma_5/2}\gamma_\mu e^{\theta\gamma_5/2} = p_\mu \left(\tilde{\psi}e^{-\theta\gamma_5/2}\gamma_\mu e^{\theta\gamma_5/2}\psi\right) = 0
\]

(34)
Here brackets indicate which of the function are subjected to the action of the operator $p$. The obtained equation has the form of the continuity equation $\partial_\mu j_\mu = 0$. Thus, the value of

$$j_\mu = \bar{\psi} e^{-i\gamma_5/2} \gamma_\mu e^{i\gamma_5} \gamma_\mu \psi = (\psi^* e^{i\gamma_5} \psi, \psi^* \gamma_0 \gamma e^{i\gamma_5} \psi)$$ (35)

Thus here the value of $j_\mu$ is a 4-vector of current density of particles in the model with $\gamma_5$-mass extension. It is very important that its temporal component

$$j_0 = \psi^* e^{i\gamma_5} \psi$$ (36)

positively defined and does not change in time. It is easy to see from the following procedure.

Let us construct the norm of any state for considered model for arbitrary vector, taking into account the weight operator $\eta$:

$$\Psi = \begin{pmatrix} x + iy \\ u + iv \\ z + iw \\ t + ip \end{pmatrix}.$$  

Using (9), (11), (12) and (36), in a result we have

$$\Psi^* \eta = \left( \frac{m_1 + m_2}{m} (x - iy), \frac{m_1 + m_2}{m} (u - iv), \frac{m_1 - m_2}{m} (z - iw), \frac{m_1 - m_2}{m} (t - ip) \right).$$

Then

$$\langle \Psi^* \eta | \Psi \rangle = \frac{m_1 + m_2}{m} (x^2 + y^2) + \frac{m_1 + m_2}{m} (u^2 + v^2), \frac{m_1 - m_2}{m} (z^2 + w^2), \frac{m_1 - m_2}{m} (t^2 + p^2)$$ (37)

is explicitly non negative, because $m_1 \geq m_2$ in the area of unbroken $\mathcal{PT}$-symmetry (10). Given that

$$\int d^3x \psi^* \eta \psi = \int d^3x j_0$$ (38)

we have the conservation of probability directly connected with correlation (36) and weight operator $\eta$. 

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4 Conclusion

Starting with the researches, presented in the previous sections, we have shown that Dirac Hamiltonian of a particle with $\gamma_5$ - dependent mass term is non-Hermitian, and has the unbroken $\mathcal{PT}$ - symmetry in the area $m_1^2 \geq m_2^2$, which has three of a subregion. Indeed with the help of the algebraic transformations we obtain a number of the consequence of the relation (15). In particular there is the restriction of the particle mass in this model: $m \leq m_{\text{max}}$, were $m_{\text{max}} = m_1^2 / 2m_2$. Outside of this area the $\mathcal{PT}$ - symmetry of the modified Dirac Hamiltonians is broken.

In addition, we have shown that the introduction of the postulate about the limitations of the mass spectrum, lying in the ground of the geometric approach to the development of the modified QFT (see, for example [8],[9]), leads to the appearance of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians in the fermion sector of the model with the Maximal Mass. Thus, it is shown that using of non-Hermitian $\mathcal{PT}$-symmetric quantum theory with $\gamma_5$ mass term may be considered as necessary and sufficient conditions of the appearance of the limitation of the mass particle (18) in a fermion sector of the model.

In particular, this applies to the modified Dirac equation in which produced the substitution $m \to m_1 + \gamma_5 m_2$. Into force of the ambiguity of the definition of parameters $m_1, m_2$ the inequality $m_1 \geq m_2 \geq 0$ describes a particle of two types. In the first case, it is about ordinary particles, when $m_1, m_2 \geq 0$ mass parameters are limited by the terms

$$0 \leq m_2 \leq m_1 / \sqrt{2}. \quad (39)$$

In the second area we are dealing with so-called exotic partners of ordinary particles, for which is still accomplished (16), but one can write

$$m_1 / \sqrt{2} \leq m_2 \leq m_1. \quad (40)$$

Intriguing difference between particles of the second type from traditional fermions is that they are described by the other modified Dirac equations. So, if in the first case (39), the equation of motion there has a limit transition when $m_{\text{max}} \to \infty$ that leads to the standard Dirac equation, however in the inequality (40) such a limit is not there.

Thus, it is proved that the main progress, obtained by us the in the algebraic way of the construction of the fermion model with $\gamma_5$ - dependent
mass term applies to the limitations of the mass spectrum. Furthermore, the possibility of describing of the exotic particles are turned out essentially the same as in the model with a maximal mass, which was investigated by V.G.Kadyshevsky with colleagues [2] - [13] on the basis of geometrical approach. It is also shown that the transition point at the scale of the mass from the ordinary particles to the exotic one this is mass of the maximon.

On the basis of (18) it also has been shown that the parameters $m_1$ and $m_2$ have the auxiliary nature. This fact is easily proved by means of the comparison of the ordinary and exotic fermion fields. Thus, it is the important conclusion that the description of exotic fields may be considered with the help of the algebraic approach and is not the prerogative of the geometric formalism. Note that the polarization properties of the exotic fermion fields fundamentally differ from the standard fields that with taking into consideration of interactions may be of interest in the future researches. For example, if for massless fermions in the case of ordinary particles we have the mass condition $m_1 = m_2 = 0$ and as a consequence of the Weyl equations

$$ (E + p\vec{\sigma})u_L = 0, \quad (41) $$

$$ (E - p\vec{\sigma})u_R = 0, \quad (42) $$

where $E = \pm p$, in the case of exotic massless particles we have the possibility $m = 0$, when $m_2 = m_1 = 2m_{max}$ (see Fig.2). This leads to the modified system of Weyl equations:

$$ (E + p\vec{\sigma})u_L = 0 \quad (43) $$

$$ (E - p\vec{\sigma})u_R - 4m_{max}u_L = 0, \quad (44) $$

which remains consistent when $E = \pm p$. It is easy to see that in this case the approximation $m_2 \to 0$ (that is equivalent to $m_1 \to 0$ and hence $m_{max} = m_1^2/m_2 \to 0$) lead to the ordinary Weil equations.

The presence of this possibilities lets hope for that in Nature indeed there are the Fundamental Mass Value and some exotic fermion fields. It is tempting to think that the quanta of the exotic fermion field have a relation to the structure of the dark matter.

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