Multi-black-holes in three dimensions

Gérard Clément
Laboratoire de Gravitation et Cosmologie Relativistes
Université Pierre et Marie Curie, CNRS/URA769
Tour 22-12, Boîte 142
4 place Jussieu, 75252 Paris cedex 05, France

September 5, 2018

Abstract

We construct time-dependent multi-centre solutions to three-dimensional general relativity with zero or negative cosmological constant. These solutions correspond to dynamical systems of freely falling black holes and conical singularities, with a multiply connected spacetime topology. Stationary multi-black-hole solutions are possible only in the extreme black hole case.
In a now well-known paper [1], Bañados et al. gave a black hole solution to the three-dimensional Einstein equations with negative cosmological constant \( \Lambda = -l^{-2} \), and studied its properties. This regular solution, which has inspired a number of recent papers [2], is given by

\[
ds^2 = \nu^2 \left[ \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right] dt^2 - r^2 \left[ d\theta - \frac{\nu J}{2r^2} dt \right]^2 - \frac{l^2 dr^2}{r^2 - Ml^2 + \frac{J^2 l^2}{4r^2}},
\]

where \( \theta \) is periodic with period \( 2\pi \), the two parameters \( M \) and \( J \) (with \( M \geq 0, |J| \leq ML \)) are interpreted [1] as the mass and angular momentum of the black hole, and the constant \( \nu \) (\( \equiv N(\infty) \) in [1]) sets the scale of time.

The purpose of the present work is to construct exact dynamical multi-black-hole solutions to three-dimensional cosmological gravity. Some time ago, conformal techniques were used to construct static [3] and stationary [4] multi-center solutions to pure gravity (\( \Lambda = 0 \)) associated with configurations of massive and spinning point particles, as well as a class of static multi-particle solutions to cosmological gravity (\( \Lambda \neq 0 \)) [5]. Using similar methods, we shall construct what at first sight appear to be stationary multi-black-hole solutions. However these stationary solutions turn out to be inconsistent, being generically plagued by extra conical singularities, which are unphysical in the sense that their worldlines are not geodesics of the multi-black-hole space-time [5]. As we shall show, by taking the positions of the black-hole centers to be no longer constant but time-dependent, one can derive intrinsically dynamical solutions corresponding to systems of freely falling black holes together with —now physical— auxiliary conical singularities.

We first consider for simplicity the case of pure gravity which also admits a solution which, under certain circumstances, behaves as a black hole solution. This may be obtained from the \( J = 0 \) black hole of eq.(1) by putting \( \nu = \)
\[ \gamma l^2/c, \quad M = c^2l^{-2}, \quad r = c/\cos(l^{-1}X), \quad \theta = c^{-1}Y, \] leading to

\[ ds^2 = \gamma^2 l^2 \tan^2(l^{-1}X) dt^2 - \frac{1}{\cos^2(l^{-1}X)} (dX^2 + dY^2), \quad (2) \]

and taking the limit \( l \to \infty \), which yields

\[ ds^2 = \gamma^2 X^2 dt^2 - dX^2 - dY^2. \quad (3) \]

We recognize in (3) the well-known two-dimensional Rindler space-time \[ \mathbb{R}^2 \] with an extra compact spatial dimension \( Y \). As discussed in \[ \mathbb{R}^2 \], the transformation \( \tilde{t} = X \sinh(\gamma t), \quad \tilde{x} = X \cosh(\gamma t), \quad \tilde{y} = Y \) maps the metric (3) into the two disjoint regions \( I \ (\tilde{x}^2 > \tilde{t}^2) \) of the Minkowski cylinder \( ds^2 = dt^2 - d\tilde{x}^2 - d\tilde{y}^2 \) (with \( \tilde{y} \) periodic). The remaining two regions \( II \ (\tilde{t}^2 > \tilde{x}^2) \) of the Minkowski cylinder may be obtained by extending the metric (3) through the horizon \( X^2 = 0 \) (of perimeter \( 2\pi c \)) to \( X^2 = -\tilde{X}^2 < 0 \), and making the transformation \( \tilde{t} = X \cosh(\gamma t), \quad \tilde{x} = X \sinh(\gamma t), \quad \tilde{y} = Y \). The resulting Penrose diagram is shown in Fig. 1. Of course, this maximally extended Rindler cylinder is undistinguishable from the Minkowski cylinder. The distinction comes about if for instance the metric (3) arises as an interior solution generated by a ring of exotic matter \[ \mathbb{R}^2 \]: then the region inside the ring has its own points at timelike infinity, distinct from those of the exterior region (Fig. 2), and is thus a genuine black hole (this fact was not fully appreciated in \[ \mathbb{R}^2 \]). Let us here mention that the four-dimensional Rindler cylinder, \( ds^2 = \gamma^2 X^2 dt^2 - dX^2 - dY^2 - dZ^2 \) with \( Y \) periodic, may similarly arise as an interior solution generated by an infinite cylinder of exotic matter, leading to a black cosmic string \[ \mathbb{R}^2 \].

To construct multi-black-hole solutions to pure three-dimensional gravity, we recall that the conformal map \( X + iY = Z(z) \) (with \( z = x + iy \)) generates from (3) the family of stationary flat metrics \[ \mathbb{R}^2 \]

\[ ds^2 = \gamma^2 X^2(z, \bar{z}) dt^2 - |Z'(z)|^2 dz d\bar{z} \quad (4) \]
Consider the multi-center map

\[ Z = \sum_{i=1}^{n} c_i \ln(z - a_i) + d \]  

(\(c_i\) and \(d\) real, \(a_i\) complex) of the region \(X(z, \bar{z}) > 0\) of the Euclidean \((x, y)\) plane into the spatial sections of the three-dimensional Rindler space-time; this map preserves spatial infinity \((X \to +\infty \iff |z| \to +\infty)\) if all the \(c_i\) are positive. For \(n = 1\) and \(a_1 = 0, c_1 = c > 0\), we recover the Rindler cylinder with \(X = c \ln r + d, Y = c \theta\), where \(z = re^{i\theta}, r > e^{-d/c}\). For \(n > 1\), we obtain what appears to be a system of \(p\) black holes, \(p \leq n\) being the number of connected components of the horizon \(X(x, y) = 0\), the total horizon perimeter being \(2\pi \sum_{i=1}^{n} c_i\). This solution may be maximally extended by taking two identical copies of the multiply connected \(X > 0\) region, which generalize the two regions \(I\) of the Rindler space-time, and connecting the corresponding \(p\) horizon components via \(p\) two-sided bridges made of two copies (past and future) of a region of type \(II\).

However, a serious problem with the above construction is that the metric \((4)\) has \(n - 1\) conical singularities associated with the zeroes of \(Z'(z)\). As conical singularities correspond to point particles, we must require for consistency \((5)\) that these follow geodesics of the multi-black-hole space-time. Now a point particle at rest in the geometry \((3)\) feels a static gravitational field

\[ -\Gamma_{00}^X = -\gamma^2 X \]  

which vanishes only on the horizon \(X = 0\), and the configurations such that the zeroes of \(Z'(z)\) sit on the horizon are obviously rather special. Of course, the problem is evaded for those zeroes of \(Z'(z)\) which lie behind the horizon and do not belong to the multi-black-hole space-time (the regions \(X < 0\) are cut out and replaced by connecting bridges). However, for all the zeroes of \(Z'(z)\) to lie behind the horizon this must be simply connected \((p = 1)\), in which case the space-time geometry reduces to that of the original Rindler cylinder (if the restriction to positive \(c_i\) is lifted, then static multi-black-hole
space-times with all the zeroes of $Z'(z)$ lying behind the horizon are possible, with several lines at spatial infinity). The conclusion is that the previously discussed static multi-black-hole solution is inconsistent. However, the preceding analysis hints strongly towards a dynamical solution. Consider for instance the map

$$Z = c \ln(z^2 - a^2) + d$$

(7)

($c > 0$); in the case $c \ln |a|^2 + d > 0$ this leads to a two-black-hole 'solution' with an unphysical conical singularity located at the 'center of mass' $z = 0$. The gravitational field (1) acting on this singularity, which tends to reduce $X(0)$ and thus the separation $2|a|$ between the black hole centers, pulls the two black holes together until they merge in a single black hole for $c \ln |a|^2 + d = 0$.

To translate this picture into an exact solution, we must introduce a time dependence in the multi-black-hole solution. This can be done by generalizing the conformal map $Z = Z(z)$ to the time-dependent map

$$Z = Z(z, t)$$

(8)

which leads from the static flat metric (3) to a dynamical flat metric. As we want to describe a system of moving black holes, we shall assume $Z(z, t)$ to be given by (5), where the positions $a_i$ of the centers are now time dependent, leading to the metric

$$ds^2 = (\gamma^2 X^2 - |A|^2)dt^2 + (AZ'dz + AZ'd\bar{z})dt - |Z'^2|dzd\bar{z},$$

(9)

where

$$A(z, t) = \sum_{i=1}^{n} \frac{c_i \partial_0 a_i(t)}{z - a_i(t)}$$

(10)

(such a transformation was previously used by Letelier and Gal’tsov [9] to construct multiple moving cosmic strings). The metric (9) has again a horizon at $X(z, t) = 0$, and $n - 1$ conical singularities following the worldlines $z_\alpha(t)$.
which solve the equation $Z'(z_\alpha, t) = 0$. For consistency, these worldlines must obey the geodesic equations

$$\ddot{x}_\alpha^\mu + \Gamma^\mu_{\nu\rho}(x_\alpha) \dot{x}_\alpha^\nu \dot{x}_\alpha^\rho = 0,$$

(11)

where $x_\alpha^0 \equiv t$ for all $\alpha$, and $\dot{} \equiv d/d\sigma_\alpha$, $\sigma_\alpha$ being the affine parameter on the $\alpha$th geodesic. Eliminating the $\sigma_\alpha$ in favour of coordinate time $t$, we are left with a system of $2(n-1)$ second-order differential equations for the $2n$ unknowns $(a_i(t), \bar{a}_i(t))$. The remaining two-fold arbitrariness is of course due to the possibility of arbitrary global time-dependent translations $z \to z + w(t)$; if we choose for instance the origin of the complex $z$-plane to coincide with the ‘center of mass’ of the multi-black-hole system, then the relative dynamics of the system are fully determined by integrating the consistency equations (11) with appropriate initial conditions.

We consider in more detail the symmetrical two-black-hole system with fixed conical singularity $z = 0$. The two-body problem may be reduced to that of the motion of one ‘black hole’ relative to the fixed point $\zeta = 0$ by the transformation $z^2 = \zeta$, which transforms (7) to

$$Z = c \ln(\zeta - \alpha(t)) + d$$

(12)

with $\alpha = a^2$. Then, this last motion may be transformed, by the global time-dependent translation $\zeta = \psi + \alpha(t)$, into that of a point particle following the worldline $\psi + \alpha(t) = 0$ relative to the fixed ‘black hole’ (the background Rindler space-time)

$$Z = c \ln \psi + d.$$  

(13)

These successive coordinate transformations mapping the geodesic $z = 0$ into a geodesic, it follows that the motion of the two black-hole centers is given by $\pm a = (-\psi)^{1/2}$, where $\psi = \psi(t)$ is a geodesic of the Rindler cylinder metric (13). These geodesics may easily be derived from those of Minkowski space-time by the Rindler transformation. Typical timelike and null geodesics are shown in Fig. 1. In the timelike case, as the conical singularity crosses the
horizon from region $I$ into region $II$, a single black hole bifurcates into two black holes which separate to a finite distance and merge again (the conical singularity falls back behind the horizon) after an infinite coordinate time. The global structure of the maximal extension of this space-time is schematized in Fig. 3; because of its multiply connected topology, this dynamical solution is clearly not equivalent to a stationary solution with point singularities. In the lightlike case (Fig. 4) the two black holes, infinitely separated at $t = -\infty$, fall upon each other and merge, again after an infinite coordinate time; the time-reversed evolution is equally possible. In all cases, the total horizon perimeter $4\pi c$ is a constant of the motion.

We now sketch how the above construction may be generalized to the case of $\Lambda < 0$ cosmological gravity (fuller details shall be given elsewhere). In the case $J = 0$, multi-black-holes may similarly be obtained from the one-black-hole (2) by the time-dependent conformal map $X + iY = Z(z, t)$, where now $X$ varies between $ml\pi$ (the horizon) and $(m + 1/2)l\pi$ (the line at spatial infinity, which may also be multiply connected) for a given integer $m$, and the functions $a_i(t)$ in (3) are determined by the condition that the zeroes of $Z'(z, t)$ follow geodesics. The resulting dynamical picture differs from that of the $\Lambda = 0$ case in several respects. All timelike geodesics of the $\Lambda < 0$ black hole cross periodically the horizon, leading to a pulsating system of two black holes periodically merging and coming again apart. A class of lightlike geodesics yield solutions describing the scattering of two black holes, with a one-black-hole intermediate state. Finally, we can now observe splitting and merging, not only of black holes, but also of universes (if we define a universe as a connected component of a spatial section of the full space-time).

In the general case $J \neq 0$ ($J^2 \leq M^2l^2$), we can write the one-black-hole solution (1) as

$$ds^2 = h^2(\nu dt + \frac{J}{2clh^2}dY)^2 - \frac{l^2}{c^2}(h^2 + M + \frac{J^2}{4l^2h^2})(dX^2 + dY^2),$$

(14)
where $h^2 = r^2/l^2 - M$ is related to $X$ by

$$\frac{dh}{dX} = c^{-1}[h^2 + M + \frac{J^2}{4l^2 h^2}], \quad (15)$$

and $Y = c \theta$. The construction then proceeds as before, except that the solution $Z(z,t)$ must be analytically continued beyond $h^2 = 0$ to the largest, negative root $h_+^2$ of the right-hand side of (15) (the outer horizon). Particularly interesting is the extreme case $J^2 = M^2 l^2$, in which we would expect that the gravitational attraction and the centrifugal repulsion may balance, resulting in stationary solutions. Indeed, we can show that the lines $r = a_0$, $d\theta = (\nu/l)dt$ are lightlike geodesics for arbitrary $a_0$, corresponding to stationary systems of two black holes orbiting around the conical singularity at the constant angular velocity $\nu/2l$.

We have studied the classical dynamics of black holes in three-dimensional cosmological gravity. Limiting cases of special interest are pure gravity ($\Lambda = 0$) where, despite the fact that space-time is (almost everywhere) flat, we have obtained dynamical systems of freely falling black holes and conical singularities with non trivial topology, and extreme black holes ($J^2 = -M^2/\Lambda$), which may interact together with conical singularities to form stationary planetary systems.

Acknowledgment

I wish to thank Bernard Linet for a critical reading of the manuscript.
References

[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.

[2] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. 71 (1993) (328); A. Achúcarro and M.E. Ortiz, Phys. Rev. D48 (1993) 3600; N. Kaloper, Phys. Rev. D48 (1993) 2598; D. Cangemi, M. Leblanc and R.B. Mann, Phys. Rev. D48 (1993) 3606; J. Gamboa and A.J. Seguí-Santonja, Class. Quant. Grav. 9 (1992) L111.

[3] S. Deser, R. Jackiw and G. ’t Hooft, Ann. Phys. (N.Y.) 152 (1984) 220.

[4] G. Clément, Int. Journ. Theor. Phys. 24 (1985) 267.

[5] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) 153 (1984) 405.

[6] W. Rindler, “Essential Relativity”, Springer-Verlag, New-York 1977.

[7] G. Clément, Ann. Phys. (N.Y.) 153 (1984) 405.

[8] G. Clément and I. Zouzou, in preparation.

[9] P.S. Letelier and D.V. Gal’tsov, Class. Quant. Grav. 10 (1993) L101.
Figure captions

**Fig.1:** Penrose diagrams for the $\Lambda = 0$ black hole (Rindler cylinder), with a timelike geodesic (dashed line) and a lightlike geodesic (wavy line).

**Fig.2:** Penrose diagram for the black-hole space-time generated by two mirror-symmetric rings (heavy vertical worldlines). The exterior regions are truncated conical space-times, while the interior region is a truncated Rindler cylinder.

**Fig.3:** Penrose diagram for the timelike two-black-hole space-time with $Y = c\pi$ ($\Lambda = 0$). The mirror-symmetric conical singularities are shown as heavy worldlines in the two regions $I$ (their dashed analytic extensions into the regions $II$ are not associated with conical singularities). The double lines result from the superposition, induced by the map $z^2 = \zeta$, of the two disjoint horizon components.

**Fig.4:** Penrose diagram for the lightlike two-black-hole space-time with $Y = c\pi$ ($\Lambda = 0$).
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9402013v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9402013v1