VERTICAL AND HORIZONTAL SQUARE FUNCTIONS ON A CLASS OF NON-DOUBLING MANIFOLDS

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ABSTRACT. We consider a class of non-doubling manifolds $\mathcal{M}$ that are the connected sum of a finite number of $N$-dimensional manifolds of the form $\mathbb{R}^{n_i} \times M_i$. Following on from the work of Hassell and the second author [19], a particular decomposition of the resolvent operators $(\Delta + k^2)^{-M}$, for $M \in \mathbb{N}^*$, will be used to demonstrate that the vertical square function operator

$$
Sf(x) := \left( \int_0^\infty \left| t \nabla (I + t^2 \Delta)^{-M} f(x) \right|^2 \frac{dt}{t} \right)^\frac{1}{2}
$$

is bounded on $L^p(\mathcal{M})$ for $1 < p < n_{\text{min}} = \min_{i} n_i$ and weak-type $(1,1)$. In addition, it will be proved that the reverse inequality $\|f\|_p \lesssim \|Sf\|_p$ holds for $p \in (n_{\text{min}}, n_{\text{min}})$ and that $S$ is unbounded for $p \geq n_{\text{min}}$ provided $2M < n_{\text{min}}$.

Similarly, for $M > 1$, this method of proof will also be used to ascertain that the horizontal square function operator

$$
sf(x) := \left( \int_0^\infty \left| t^2 \Delta (I + t^2 \Delta)^{-M} f(x) \right|^2 \frac{dt}{t} \right)^\frac{1}{2}
$$

is bounded on $L^p(\mathcal{M})$ for all $1 < p \leq \infty$ and weak-type $(1,1)$. Hence, for $p \geq n_{\text{min}}$, the vertical and horizontal square function operators are not equivalent and their corresponding Hardy spaces $H^p$ do not coincide.

1. Introduction

In any historical account of the development of harmonic analysis, the doubling condition will certainly appear as a central actor. In each step of its genesis, from the minds of many great mathematicians in the 1960’s and 1970’s, the doubling condition was interwoven, seemingly inextricably, throughout the entirety of the body of work that embodied harmonic analysis. In a metric measure space $(X, d, \mu)$, the doubling condition reads that there must exist a constant $C > 0$ such that

$$
\mu(B(x, 2r)) \leq C \mu(B(x, r))
$$

for all $x \in X$ and $r > 0$, where the notation $B(x, r)$ is used to denote the ball of radius $r$ and centered at the point $x$. If this condition is satisfied then $(X, d, \mu)$ is said to be a space of homogeneous type, in the sense of Coifman and Weiss [11], whilst any space that does not satisfy this condition is called non-homogeneous.

Although this condition aligns with our real world intuition, with the continued progress of mathematics as a whole there are now many different situations and applications that depart from this idealised world and demand the consideration of non-homogeneous spaces. Moreover, it has become increasingly apparent that the doubling condition is not quite as critically indispensable for many harmonic analytic results as previously believed. As such, and acting as a reversal to the assimilation of the doubling condition in the adolescence of the field, there is now an intensive research effort underway that aims to unthread the doubling strand from this body of work, where possible, and push
the boundaries of harmonic analysis beyond this condition. This process must often be approached with the utmost care since one is certain to encounter behaviours that depart very far from the prototypical doubling example of Euclidean space $\mathbb{R}^d$. Some examples of notable results in this area include: the extension of Calderón-Zygmund theory to non-homogeneous spaces through the work of Nazarov, Treil and Volberg \[22, 23, 24\] and Tolsa \[33, 34, 31\]; the generalisation of $BMO$ and $H^1$ theory by Bui and Duong \[5\] and Tolsa \[32\]; the consideration of weighted norm inequalities through the work of Orobitg and Pérez \[26\]; and non-homogeneous Tb type theorems by Hytönen and Martikainen \[20\].

The non-doubling spaces that are of interest to us in this article consist of a particular class of non-doubling manifolds formed as connected sums.

**Definition 1.1.** A manifold $\mathcal{V}$ is said to be formed as the connected sum of a finite number of complete and connected manifolds $\mathcal{V}_1, \cdots, \mathcal{V}_l$ of the same dimension, denoted

$$\mathcal{V} = \mathcal{V}_1 \# \cdots \# \mathcal{V}_l,$$

if there exists some compact subset with non-empty interior $K \subset \mathcal{V}$ for which $\mathcal{V} \setminus K$ can be expressed as the disjoint union of open subsets $E_i \subset \mathcal{V}$ for $i = 1, \cdots, l$, where each $E_i$ is isometric to $\mathcal{V}_i \setminus K_i$ for some compact $K_i \subset \mathcal{V}_i$.

Fix dimension $N \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $l \geq 1$ and let $\mathbb{R}^{n_i} \times \mathcal{M}_i$ for $i = 1, \cdots, l$ be a collection of manifolds with $\mathcal{M}_i$ compact and $n_i + \dim \mathcal{M}_i = N$. We will be interested in smooth Riemannian manifolds $\mathcal{M}$ that are of the form

$$\mathcal{M} := (\mathbb{R}^{n_1} \times \mathcal{M}_1) \# \cdots \# (\mathbb{R}^{n_l} \times \mathcal{M}_l).$$

As in the previous definition, it is possible to choose open subsets $E_i \subset \mathcal{M}$ and compact $K \subset \mathcal{M}$ with non-empty interior for which $\mathcal{M} \setminus K$ can be expressed as the disjoint union of the $E_i$. The subsets $E_i$ are referred to as the ends of $\mathcal{M}$, $K$ the center of $\mathcal{M}$ and the entire manifold $\mathcal{M}$ itself is fittingly said to be a manifold with ends.

The constituent manifolds $\mathbb{R}^{n_i} \times \mathcal{M}_i$ for $1 \leq i \leq l$ each have topological dimension $N$, but asymptotic dimension $n_i$ at infinity. That is, for any ball $B(x, r) \subset \mathbb{R}^{n_i} \times \mathcal{M}_i$, it will be true that the volume of the ball will satisfy

$$V(B(x, r)) \simeq \begin{cases} r^N & \text{for } r \leq 1, \\ r^{n_i} & \text{for } r > 1. \end{cases}$$

If the values of $n_i$ differ, then the manifold with ends $\mathcal{M}$ will have varying asymptotic dimension on these ends. This will lead to a violation of the doubling condition.

In the landmark article by Grigoryan and Saloff-Coste \[17\], the authors effectively computed, using probabilistic methods, two-sided estimates for the heat kernel generated by the Laplacian $\Delta$ on this prototypical collection of non-doubling spaces. Although not the first to study this remarkable class of manifolds in detail (see \[25\] for a detailed historical account), this article acted as an inflection point for interest in this class and had a pronounced effect on the non-homogeneous community. Indeed, it essentially designated this class of manifolds as a battlefront for the advancement of non-homogeneous harmonic analysis. In point of fact, the sustained interest in these model spaces has led to investigations into the boundedness of the heat maximal operators \[14\], the functional calculus of $\Delta$ \[6\] and Littlewood-Paley decompositions \[4\]. Recently, and of particular significance to our article, Hassell in collaboration with the second named author considered the $L^p(\mathcal{M})$-boundedness of the Riesz transforms operator $\nabla \Delta^{-\frac{1}{2}}$ on such manifolds \[19\]. This paper in turn is a generalisation to the non-doubling setting of the result obtained by Carron, Coulhon and Hassell in \[8\]. For other relevant results we refer the reader to \[7, 13\] and references therein.
In this article, it is our aim to extend the classical theory of square functions to this class of non-doubling manifolds. Consider a general complete Riemannian manifold \( M \) and let \( \Delta \) denote the Laplace-Beltrami operator for this manifold. For \( M \in \mathbb{N}^\ast \), the vertical square function operator is defined:

\[
Sf(x) := \left( \int_0^\infty |t\nabla (I + t^2 \Delta)^{-M} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

The notion of square functions forms an essential part of harmonic analysis and has numerous applications, from the definition of Hardy spaces \([27]\) to providing an equivalent characterisation of the bounded holomorphic functional calculus of a sectorial operator \([12]\). As such, the above operators have been extensively studied, either in this form or defined with the semigroup replacing the higher-order resolvent (c.f. Remark 1.1), since the formation of harmonic analysis, and a great deal is known about their behaviour when the manifold under consideration is doubling.

- For the classical case when \( M \) is simply Euclidean space \( \mathbb{R}^d \), \( S \) is bounded on \( L^p(\mathbb{R}^d) \) for any \( p \in (1, \infty) \) and weak-type \((1,1)\).
- For general complete Riemannian manifolds, the work of Coulhon, Duong and Li \([10]\) states that the semigroup variation of the vertical square function is bounded on \( L^p(M) \) for all \( p \in (1, 2] \), whether doubling or not, and weak-type \((1,1)\) if the manifold is doubling and if Gaussian upper bounds are satisfied by the heat kernel.
- For doubling manifolds \( M \) whose heat kernel satisfies Gaussian upper bounds, one can define

\[
q_+ := \sup \left\{ p \in (1, \infty) : \|\nabla \Delta^{-\frac{1}{2}} f\|_p \leq \|f\|_p \right\}.
\]

It is then known that \( q_+ \geq 2 \) \([9]\) and that the semigroup variation of \( S \) is bounded on \( L^p(M) \) for any \( p \in (1, q_+) \). A sparse proof of this was shown in \([3, \text{Prop. 3.8}]\).

Nothing is currently known about the \( L^p(M) \)-boundedness of the vertical square function for \( p > 2 \) on non-doubling manifolds. Our aim in this article is to prove the following theorem.

**Theorem 1.1.** Let \( M = (\mathbb{R}^{n_1} \times M_1) \# \cdots \# (\mathbb{R}^{n_l} \times M_l) \) be a manifold with ends. For \( M \in \mathbb{N}^\ast \), the vertical square function operator \( S \), as defined in (1), will satisfy the following properties:

(i) \( S \) is bounded on \( L^p(M) \) for all \( p \in (1, n_{\min}) \), where \( n_{\min} := \min_i n_i \), and weak-type \((1,1)\); 

(ii) If \( 2M < n_{\min} \) then \( S \) is unbounded on \( L^p(M) \) for \( p \geq n_{\min} \); and

(iii) For \( p \in (n'_\min, n_{\min}) \), there exists \( c > 0 \) for which

\[
\|f\|_p \leq c \|Sf\|_p
\]

for all \( f \in L^p(M) \).

Let us briefly discuss the proof of the \( L^p(M) \)-boundedness portion of this result. Notice that through a change of variables \( S \) has the representation

\[
Sf(x) = \left( \int_0^\infty |\nabla (t^{-2} + \Delta)^{-M} f(x)|^2 t^{1-4M} \, dt \right)^{\frac{1}{2}}
\]

\[
= \left( \int_0^\infty |\nabla (k^2 + \Delta)^{-M} f(x)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}.
\]
This can be controlled from above by the corresponding high and low energy parts of this square function,

\[ S_f(x) \leq S_{>} f(x) + S_{<} f(x), \]

where

\[ S_{>} f(x) := \left( \int_1^\infty \left| \nabla (k^2 + \Delta)^{-M} f(x) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}} \]

and

\[ S_{<} f(x) := \left( \int_0^1 \left| \nabla (k^2 + \Delta)^{-M} f(x) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}. \]

The \( L^p(M) \)-boundedness of these two parts will be proved separately. The high energy component is local in nature and does not see the large scale non-doubling character of the manifold. It can thus be treated in a manner analogous to the classical Calderón-Zygmund case. This will be accomplished in Section 5.

The low energy component, on the other hand, will prove to be more challenging since it involves extensive interaction between the different ends of the manifold. Indeed, ultimately it is the low energy component that will prove to be solely responsible for the unboundedness of the operator on the range \( p \geq n_{\text{min}} \). Another major difficulty that appears is that, in contrast to many classical cases, it will not be possible to treat this term using estimates for the spatial derivative of the heat kernel \( \nabla e^{-t\Delta} \). This is due to the pronounced absence of these estimates in the literature. Instead, we rely on a decomposition of the resolvent operator \((\Delta + k^2)^{-1}\) that was constructed in [19] using a parametrix style argument. In Section 3 it will be proved that this decomposition can be generalised to the higher-order resolvent operators \((\Delta + k^2)^{-M}\). Following this, in Section 4, this higher-order resolvent decomposition will be applied to the low energy component and boundedness will ensue.

For the unboundedness portion of Theorem 1.1, observe that a constraint on the order of the resolvent is required, \( 2M < n_{\text{min}} \). This constraint is a consequence of our method of proof. Indeed, our proof that \( S \) is unbounded on \( L^p(M) \) for \( p \geq n_{\text{min}} \) relies heavily on the use of the Riesz potential operators \( \Delta^{-M} \). In an analogous manner to classical theory on Euclidean space, these operators are not well-defined when the order \( M \) is too large in comparison to the dimension, namely \( 2M \geq n_{\text{min}} \). As a result, our method of proof will not be applicable for this range. We do not study the range \( 2M \geq n_{\text{min}} \) here. Therefore, we do not study the case \( n_{\text{min}} = 2 \) which was investigated in [18, 25].

**Remark 1.1.** Through a careful study of the literature, one will notice that the term vertical square function is often used to refer to the operator

\[ \left( \int_0^\infty |t\nabla e^{-t\Delta} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \]

with the semigroup \( e^{-t\Delta} \) taking on the role held by the higher-order resolvent operator for \( S \). Although the semigroup form is more frequently encountered, our consideration of square functions defined using higher-order resolvent operators is by no means rare. For instance, in the article [15] by Frey, McIntosh and Portal, resolvent based conical square function estimates were proved for perturbations of Dirac-type operators on \( L^p(\mathbb{R}^d) \).

Indeed, the two different forms of square function are seen to be morally equivalent. For if one can prove that \( S \) is bounded on \( L^p(M) \), together with a similar estimate for \( \nabla \text{div} \), then one can obtain the boundedness of the holomorphic functional calculus of the corresponding Dirac-type operator on \( L^p(M) \) (c.f. [12, Cor. 6.8]). The boundedness of the semigroup square function would then follow immediately as a corollary. This argument
can also be reversed in order to obtain the $L^p(\mathcal{M})$-boundedness of the resolvent square function from the $L^p(\mathcal{M})$-boundedness of the semigroup square function.

As an alternative to the vertical square function $S$, one can also consider the horizontal square function for $M > 1$:

\begin{equation}
sf(x) := \left( \int_0^\infty \left| t^2 \Delta (I + t^2 \Delta)^{-M} f(x) \right|^{2 \frac{2}{t}} \frac{dt}{t} \right)^{\frac{1}{2}}.
\end{equation}

For this operator, we will prove the following theorem.

**Theorem 1.2.** Let $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1) \# \cdots \# (\mathbb{R}^{n_t} \times \mathcal{M}_t)$ be a manifold with ends and fix $M > 1$. For any $p \in (1, \infty)$ the square function operator $s$, as defined in (2), is bounded on $L^p(\mathcal{M})$ and there exists $c, C > 0$ such that for all $f \in L^p(\mathcal{M})$

$$c \|f\|_p \leq \|sf\|_p \leq C \|f\|_p.$$ 

In addition, the operator $s$ is of weak-type $(1, 1)$.

This result will be proved in Section 8 and, similar to the operator $S$, it will be achieved by decomposing $s$ into low and high energy components and then by proving $L^p(\mathcal{M})$-boundedness for both of these components separately. For the low energy component, we will once again make extensive use of the decomposition for the higher-order resolvent operators $(\Delta + k^2)^{-M}$.

The boundedness of $s$ on $L^p(\mathcal{M})$ for $1 < p < \infty$, as stated in Theorem 1.2, is a result that already exists in the literature. Indeed, the $L^p(\mathcal{M})$-boundedness of the semigroup square function,

$$\left( \int_0^\infty \left| t^2 \Delta (I + t^2 \Delta)^{-M} f(x) \right|^{2 \frac{2}{t}} \frac{dt}{t} \right)^{\frac{1}{2}},$$

was proved to hold in the general symmetric Markov semigroup setting (see [28, pg. 111]). This implies that $\Delta$ possesses a bounded $H^\infty(S_\mu^\circ)$-functional calculus on $L^p(\mathcal{M})$ for any $\mu \in [0, \pi)$, which immediately leads to the boundedness of $s$ on $L^p(\mathcal{M})$ (c.f. [12, Cor. 6.8]).

A new proof of this result has been included here to illustrate the applicability of our methods to resolvent based operators and to obtain the weak-type $(1, 1)$ bounds for $s$ which, to the best of our knowledge, is a result that is new. It is also particularly illuminating to compare the proofs for the vertical and horizontal square functions in order to glean some intuition as to why one is bounded on the full reflexive range, while the other fails for $p \geq n_{\min}$.

**Remark 1.2.** Our results can also be viewed through the lens of Hardy spaces. In an analogous manner to the classical case $\mathcal{M} = \mathbb{R}^d$, one can define, for $p > 0$, Hardy spaces $H^p_\nabla$ and $H^p_\Delta$ associated with $s$ and $S$ through the norms

$$\|f\|_{H^p_\nabla} := \|sf\|_p \quad \text{and} \quad \|f\|_{H^p_\Delta} := \|sf\|_p.$$

Theorem 1.1 tells us that for $p \in (n'_{\min}, n_{\min})$ it will be true that $H^p_\nabla = L^p$, but $H^p_\nabla \neq L^p$ for $p \in [n_{\min}, \infty)$ when $2M < n_{\min}$. In contrast, $H^p_\Delta = L^p$ for all $p \in (1, \infty)$, and thus the two square functions define distinct Hardy spaces for $p \geq n_{\min}$. It remains an open problem to check that $H^p_\nabla = H^p_\Delta$ for $p \in (1, n'_{\min}]$.

**Remark 1.3.** Comparing Theorem 1.1 with the main result of [19] we note a posteriori that in the considered setting boundedness of the Riesz Transform and vertical square function are equivalent for any $L^p$ space, including weak type $(1, 1)$ estimates. It is an interesting question whether such equivalence could be verified in some more general setting in the form of the abstract statement. It seems that the implication from Riesz
Transform bounds to the square functions estimates can be approach using [21, Theorem 9.5.1 Section 9]. We do not know how to approach the opposite implication. We expect any result which includes weak type (1,1) estimates in any directions to be especially challenging. Here we are more interested with understanding the difference between horizontal and vertical square functions and we do not attempt to answer this question.

2. Preliminaries

Throughout this article, we fix a manifold with ends \( \mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1) \# \cdots \# (\mathbb{R}^{n_l} \times \mathcal{M}_l) \). The notation \( K \) will be used to denote the center of \( \mathcal{M} \) and the sets \( K_i \) and \( E_i \) will be as given in Definition 1.1. In particular, these sets will have the property that \( \mathcal{M} \setminus K \) can be expressed as the disjoint union of the ends \( E_i \) and \( K_i \). In this manner, the ends \( E_i \) will be identified with the sets \( \mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i \). So, when \( \mathcal{M} \) can be expressed as the disjoint union of the ends \( E_i \) can also be viewed as belonging to the space \( \mathbb{R}^{n_i} \times \mathcal{M}_i \).

Notation. For estimates concerning two quantities \( a, b \in \mathbb{R} \), the notation \( a \lesssim b \) will be employed to signify the existence of a constant \( C > 0 \) such that \( a \leq C \cdot b \). Similarly, \( a \simeq b \) will denote that \( a \lesssim b \) and \( b \lesssim a \) both hold. The dependence of the constant \( C \) on certain parameters should be clear from the context of the argument under consideration.

For \( x \in \mathbb{R}^d \), define \( \langle x \rangle := (1 + |x|^2)^{1/2} \). We employ the notation \( d(x, y) \) to denote the intrinsic distance between two points \( x \) and \( y \) in some ambient Riemannian manifold. When the space under consideration is the entire space \( \mathcal{M} \), we use the shorthand notation \( L^p \) to denote the Lebesgue space \( L^p(\mathcal{M}) \). Finally, for a function \( g(x, y) \) of two variables, the notation \( \nabla_x g(x, y) \) will be understood to denote the gradient with respect to the first variable.

Before delving into the substance of our proof, it will first be beneficial to record some useful estimates satisfied by the higher-order resolvent operators on the constituent manifolds \( \mathbb{R}^{n_i} \times \mathcal{M}_i \). Following this, we recall a vital decomposition of the first-order resolvent on the entire space \( \mathcal{M} \) that was introduced in [19]. This will form the foundation upon which our proofs of both Theorem 1.1 and 1.2 will be built.

2.1. Higher-Order Resolvents on \( \mathbb{R}^{n_i} \times \mathcal{M}_i \). Fix \( 1 \leq i \leq l \) and consider the higher-order resolvents \( (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j} \) for \( j \in \mathbb{N}^\ast \) and \( k > 0 \), where \( \Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} \) denotes the Laplacian on \( \mathbb{R}^{n_i} \times \mathcal{M}_i \). Recall the definition of the Bessel kernel \( G^d_a : \mathbb{R} \to (0, \infty) \) for \( a, d > 0 \),

\[
G^d_a(s) := \frac{1}{(4\pi)^{\frac{d+a}{2}} \Gamma(a/2)} \int_0^\infty e^{-\frac{t}{t+t^{-\frac{a}{2}}}} dt.
\]

The Bessel kernels are well-known to satisfy the estimates,

\[
G^d_a(s) \simeq \begin{cases} 
\frac{e^{-cs}}{s^{d-a}} & \text{if } a < d, \\
\max (1, \ln(s^{-1})) e^{-cs} & \text{if } d = a, \\
e^{-cs} & \text{if } a > d,
\end{cases}
\]

where the value of \( c \) is allowed to differ in each case and for the upper and lower estimates. So, when \( a < d \) for instance,

\[
\frac{e^{-cs_1}}{s^{d-a}} \lesssim G^d_a(s) \lesssim e^{-cs_2} \frac{e^{-cs_2}}{s^{d-a}},
\]

for some \( c_1 > c_2 > 0 \). Refer to [1] for a detailed proof.
Proposition 2.1. For \( j \in \mathbb{N}^* \), there exists \( c_1, c_2, c_3 > 0 \) for which
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \lesssim k^{n_1-2j} \cdot G_{2j}^n(c_1 d(x, y)k) + k^{N-2j} \cdot G_{2j}^N(c_1 d(x, y)k),
\]
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \gtrsim k^{n_1-2j} \cdot G_{2j}^n(c_2 d(x, y)k) + k^{N-2j} \cdot G_{2j}^N(c_2 d(x, y)k)
\]
and
\[
|\nabla_x (\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y)| \lesssim k^{n_1+1-2j} \cdot G_{2j-1}^n(c_3 d(x, y)k) + k^{N+1-2j} G_{2j-1}^N(c_3 d(x, y)k)
\]
for all \( x, y \in \mathbb{R}^n_1 \times M_1 \) and \( k > 0 \).

Proof. Recall that the higher-order resolvent can be expressed in terms of the heat operator through the integral relation
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j} = \frac{1}{(j-1)!} \int_0^\infty t^{j-1} e^{-tk^2} e^{-t \Delta_{\mathbb{R}^n_1 \times M_1}} \, dt.
\]
On applying the heat kernel estimate (11) of [19],
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) = \frac{1}{(j-1)!} \int_0^\infty t^{j-1} e^{-tk^2} e^{-t \Delta_{\mathbb{R}^n_1 \times M_1}(x, y)} \, dt
\]
\[
\lesssim \int_0^\infty t^{j-1} (t^{-\frac{n_1}{2}} + t^{-N/2}) e^{-tk^2} e^{-\pi \frac{d(x, y)^2}{t}} \, dt
\]
\[
= \int_0^\infty t^{1-\frac{n_1-2j}{2}} e^{-tk^2} e^{-\pi \frac{4c d(x, y)^2 k^2}{t}} \, dt + \int_0^\infty t^{1-\frac{N-2j}{2}} e^{-tk^2} e^{-\pi \frac{4c d(x, y)^2 k^2}{t}} \, dt.
\]
On applying a change of variables,
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \sim k^{n_1-2j} \int_0^\infty t^{1-\frac{n_1-2j}{2}} e^{-\frac{t}{4\pi} e^{-\pi \frac{4c d(x, y)^2 k^2}{t}}} \, dt
\]
\[
+ k^{N-2j} \int_0^\infty t^{1-\frac{N-2j}{2}} e^{-\frac{t}{4\pi} e^{-\pi \frac{4c d(x, y)^2 k^2}{t}}} \, dt
\]
\[
\sim k^{n_1-2j} \cdot G_{2j}^n(2\sqrt{c} d(x, y)k) + k^{N-2j} \cdot G_{2j}^N(2\sqrt{c} d(x, y)k).
\]
The lower estimate for \((\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y)\) and the upper estimate for \(\nabla_x (\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y)\) follow in an identical manner to the upper estimate through an application of (13) and (12) respectively of [19]. \( \square \)

Proposition 2.1, when combined with the Bessel kernel estimates (3), immediately yield the following corollary.

Corollary 2.1. For any \( j \in \mathbb{N}^* \) with \( j \neq \frac{n_1}{2} \) and \( j \neq \frac{N}{2} \), there exists \( c > 0 \) such that
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \lesssim \left( d(x, y)^{\min(2j-N,0)} k^{-\max(2j-N,0)} + d(x, y)^{\min(2j-n_1,0)} k^{-\max(2j-n_1,0)} \right) e^{-ckd(x,y)}.
\]
If \( j = \frac{n_1}{2} < \frac{N}{2} \),
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \lesssim \left( d(x, y)^{2j-N} + \max \left[ 1, \ln \left( \frac{1}{kd(x,y)} \right) \right] \right) e^{-ckd(x,y)}.
\]
If \( j = \frac{n_1}{2} = \frac{N}{2} \),
\[
(\Delta_{\mathbb{R}^n_1 \times M_1} + k^2)^{-j}(x, y) \lesssim \max \left[ 1, \ln \left( \frac{1}{kd(x,y)} \right) \right] e^{-ckd(x,y)}.
\]
Finally, if \( \frac{n}{2} < j = \frac{N}{2} \),
\[
(\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \lesssim \left( \max \left[ 1, \ln \left( \frac{1}{kd(x, y)} \right) \right] + k^{n_i-2j} \right) e^{-ckd(x, y)}.
\]

Similarly, for the operators \( \nabla (\Delta_{R^{n_i} \times M_i} + k^2)^{-j} \), the following estimates follow from Proposition 2.1.

**Corollary 2.2.** For any \( j \in \mathbb{N}^+ \) with \( j \neq \frac{n_i+1}{2} \) and \( j \neq \frac{N+1}{2} \), there exists \( c \) > 0 such that
\[
(10) \quad \left| \nabla_x (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \right| \lesssim \left( d(x, y)^{\min(2j-1-N,0)} k^{-\max(2j-1-N,0)} + d(x, y)^{\min(2j-1-n_i,0)} k^{-\max(2j-1-n_i,0)} \right) e^{-ckd(x, y)}.
\]

If \( j = \frac{n_i+1}{2} < \frac{N+1}{2} \),
\[
(11) \quad \left| \nabla_x (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \right| \lesssim \left( d(x, y)^{2j-1-N} + \max \left[ 1, \ln \left( \frac{1}{kd(x, y)} \right) \right] \right) e^{-ckd(x, y)}.
\]

If \( j = \frac{n_i+1}{2} = \frac{N+1}{2} \),
\[
(12) \quad \left| \nabla_x (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \right| \lesssim \max \left[ 1, \ln \left( \frac{1}{kd(x, y)} \right) \right] e^{-ckd(x, y)}.
\]

Finally, if \( \frac{n_i+1}{2} < j = \frac{N+1}{2} \),
\[
(13) \quad \left| \nabla_x (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \right| \lesssim \left( \max \left[ 1, \ln \left( \frac{1}{kd(x, y)} \right) \right] + k^{n_i+1-2j} \right) e^{-ckd(x, y)}.
\]

The below proposition follows almost immediately from the two previous corollaries. We provide an alternative proof, that follows directly from the first-order resolvent estimates, for the first estimate in the statement.

**Proposition 2.2.** Let \( j \in \mathbb{N}^+ \). There exists \( c > 0 \) such that
\[
(14) \quad (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \lesssim k^{-2(j-1)} \left( d(x, y)^{2-N} + d(x, y)^{2-n_i} \right) e^{-ckd(x, y)}
\]
and
\[
(15) \quad \left| \nabla_x (\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \right| \lesssim k^{-2(j-1)} \left( d(x, y)^{1-N} + d(x, y)^{1-n_i} \right) e^{-ckd(x, y)}
\]
for all \( x, y \in \mathbb{R}^{n_i} \times M_i \) and \( k > 0 \).

**Proof.** For any \( k > 0 \), the higher-order resolvent operator is given by the formula
\[
(\Delta_{R^{n_i} \times M_i} + k^2)^{-j} = \frac{1}{(j-1)!} \int_0^\infty t^{j-1} e^{-tk^2} e^{-t\Delta_{R^{n_i} \times M_i}} dt.
\]
Let \( e^{-t\Delta_{R^{n_i} \times M_i}}(x, y) \) denote the heat kernel on \( \mathbb{R}^{n_i} \times M_i \). Since \( x^{j-1} \lesssim e^{tx} \) for any \( 0 < \epsilon < 1 \),
\[
(\Delta_{R^{n_i} \times M_i} + k^2)^{-j}(x, y) \approx k^{-2(j-1)} \int_0^\infty (tk^2)(j-1) e^{-tk^2} e^{-t\Delta_{R^{n_i} \times M_i}}(x, y) dt
\]
\[
\lesssim k^{-2(j-1)} \int_0^\infty e^{-tk^2(1-\epsilon)} e^{-t\Delta_{R^{n_i} \times M_i}}(x, y) dt
\]
\[
= k^{-2(j-1)}(\Delta_{R^{n_i} \times M_i} + k^2(1-\epsilon))^{-1}(x, y).
\]
On applying the known functions estimates for the first-order resolvent, equation (17) from [19], we obtain
\[
(\Delta \mathbb{R}^n \times \mathcal{M}_i + k^2)^{-j}(x, y) \lesssim k^{-2(j-1)} (d(x, y)^{2-N} + d(x, y)^{2-n_i}) e^{-c'\sqrt{-cd(x, y)}},
\]
for some \( c' > 0 \). This proves (14) with \( c = \sqrt{1-cd} \).

Finally, when we come to consider the unboundedness of the square function \( S \) for \( p \geq n_{\min} \), the following lower bounds will prove useful. This lower bound follows directly from Proposition 2.1 and the lower bound for the Bessel kernel given in (3).

**Corollary 2.3.** For \( j < \frac{n}{2} \), there must exist \( c > 0 \) such that
\[
(\Delta \mathbb{R}^n \times \mathcal{M}_i + k^2)^{-j}(x, y) \gtrsim (d(x, y)^{2j-N} + d(x, y)^{2j-n_i}) e^{-ckd(x, y)},
\]
for all \( x, y \in \mathbb{R}^n \times \mathcal{M}_i \) and \( k > 0 \).

2.2. A Decomposition for the Resolvent. Recall from [19] that in order to prove the boundedness of the low energy part of the Riesz transforms operator, the resolvent was separated into four separate components. That is, for \( 0 < k \leq 1 \) the resolvent is given by
\[
(\Delta + k^2)^{-1} = \sum_{j=1}^{4} G_j(k).
\]
Let us recall the definitions of each of these components. For each \( i = 1, \ldots, l \), choose a point \( x_i^0 \) in the interior of \( K_i \). Let \( \phi_i \in C^\infty(\mathcal{M}) \) be a function with support entirely contained in \( \mathbb{R}^n \times \mathcal{M}_i \setminus K_i \) that is identically equal to 1 everywhere outside of a compact set. Define \( v_i := -\Delta \phi_i \) and let \( u_i \) be the function whose existence is asserted by [19, Lem. 2.7] for \( v_i \). The term \( G_1(k) \) is entirely supported on the diagonal ends and is defined through
\[
G_1(k)(x, y) := \sum_{i=1}^{l} (\Delta \mathbb{R}^n \times \mathcal{M}_i + k^2)^{-1}(x, y)\phi_i(x)\phi_i(y).
\]
Let \( G_{\text{int}}(k) \) be an interior parametrix for the resolvent that is supported close to the compact subset
\[
K_\Delta := \{ (x, x) : x \in K \} \subset \mathcal{M}^2,
\]
and agreeing with the resolvent of \( \Delta \mathbb{R}^n \times \mathcal{M}_i \) in a smaller neighbourhood of \( K_\Delta \) intersected with the support of \( \nabla \phi_i(x)\phi_i(y) \). \( G_2(k) \) is an operator with kernel that is compactly supported in \( \mathcal{M}^2 \),
\[
G_2(k)(x, y) := G_{\text{int}}(k)(x, y) \left( 1 - \sum_{i=1}^{l} \phi_i(x)\phi_i(y) \right).
\]
\( G_3(k) \) has the nice property that its kernel is multiplicatively separable into functions of \( x \) and \( y \),
\[
G_3(k)(x, y) = \sum_{i=1}^{l} (\Delta \mathbb{R}^n \times \mathcal{M}_i + k^2)^{-1}(x_i^0, y)u_i(x, k)\phi_i(y).
\]
For the final term \( G_4(k) \), first the error term is defined by
\[
(\Delta + k^2)(G_1(k) + G_2(k) + G_3(k)) = I + E(k).
\]
Then the operator \( G_4(k) \) is given by
\[
G_4(k)(x, y) := -(\Delta + k^2)^{-1}v_y(x),
\]
where $v_0(x) := E(k)(x, y)$. As computed in [19], it is useful to note that the error term $E(k)$ has the representation

$$E(k) = \sum_{i=1}^{l} (E_{i1}^1(k) + E_{i2}^2(k)) + E_3(k).$$

Here

$$E_{i1}^1(k)(x, y) := -2\nabla \phi_i(x) \phi_i(y) \left[ \nabla_x \left( \Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2 \right)^{-1}(x, y) - \nabla_x G_{\text{int}}(k)(x, y) \right],$$

$$E_{i2}^2(k)(x, y) := \phi_i(y)v_i(x) \left( -\left( \Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2 \right)^{-1}(x, y) + G_{\text{int}}(k)(x, y) + \left( \Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2 \right)^{-1}(x_i^0, y) \right)$$

for $i = 1, \ldots, l$, and

$$E_3(k)(x, y) := \left( (\Delta + k^2)G_{\text{int}}(k)(x, y) - \delta_y(x) \right) \left( 1 - \sum_{i=1}^{l} \phi_i(x)\phi_i(y) \right),$$

where $\delta_y$ is the Dirac-delta function centered at $y$.

### 3. Higher-Order Resolvents on $\mathcal{M}$

In this section, we investigate various properties of the higher-order resolvent operators $(\Delta + k^2)^{-j}$ for $\mathcal{M}$. For $a \in \mathbb{N}$ and $c > 0$, define the weight functions $\omega^c_a : \mathcal{M} \times [0, 1] \to (0, \infty)$ through

$$\omega^c_a(x, k) := \begin{cases} 1, & x \in K, \\ \langle d(x_i^0, x) \rangle^{-(n_i - a)} e^{-ckd(x_i^0, x)}, & x \in \mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i, \ 1 \leq i \leq l. \end{cases}$$

The dependence of the functions $\omega^c_a$ on the constant $c$ will often be kept implicit through the use of the shorthand notation $\omega_a$, and the value of $c$ will then be understood to change from line to line. Higher-order analogues of the key lemma from [19] will now be proved.

**Proposition 3.1.** Let $v \in L^\infty(\mathcal{M})$ be compactly supported in $K$. Let $u : \mathcal{M} \times \mathbb{R}^+ \to \mathbb{R}$ be a function, whose existence is asserted by [19, Lem. 2.7], that satisfies $(\Delta + k^2)u = v$,

$$|u(x, k)| \lesssim \|v\|_{L^\infty} \omega_2(x, k), \text{ and}$$

$$|\nabla u(x, k)| \lesssim \|v\|_{L^\infty} \omega_1(x, k),$$

for all $x \in \mathcal{M}$ and $0 \leq k \leq 1$. For $j \in \mathbb{N}$, define $u^{(j)} := \delta_{k^2}^{(j)}u$. Then

$$\left| u^{(j)}(x, k) \right| \lesssim k^{-2j} \|v\|_{L^\infty} \omega_2(x, k),$$

for all $x \in \mathcal{M}$ and $0 < k \leq 1$.

**Proof.** It will first be proved that for any $j \in \mathbb{N}$, the derivative

$$u^{(j)}(x, k) = \delta_{k^2}^{(j)}(\Delta + k^2)^{-1}v(x) = (-1)^j j!(\Delta + k^2)^{-j+1}v(x)$$

satisfies the integral identity

$$u^{(j)}(x, k) = (-1)^j \int_0^\infty t^je^{-tk^2}e^{-t\Delta}v(x) \, dt$$

for any $k > 0$ and $x \in \mathcal{M}$. This will be proved by induction. The base case,

$$u^{(0)}(x, k) = u(x, k) = \int_0^\infty e^{-tk^2}e^{-t\Delta}v(x) \, dt,$$
was shown to hold for any $k \geq 0$ in [19, Lem. 2.7]. For the inductive step, assume that (20) holds for a particular $j \in \mathbb{N}$. Then,
\begin{align*}
u^{(j+1)}(x, k) &= \partial_{k^2} \nu^{(j)}(x, k) \\
&= \partial_{k^2}(-1)^j \int_0^\infty t^j e^{-tk^2} e^{-t\Delta} v(x) \, dt \\
&= (-1)^j \lim_{h \to 0} \int_0^\infty t^j \left( \frac{e^{-t(k^2+h)} - e^{-tk^2}}{h} \right) e^{-t\Delta} v(x) \, dt. \tag{21}
\end{align*}

The mean value theorem tells us that there must exist some $c(h) \in [0, h]$ for which
\[\left| \frac{e^{-t(k^2+h)} - e^{-tk^2}}{h} \right| = te^{-t(k^2+c(h))} \leq te^{-tk^2}.\]

As the semigroup generated by the Laplace-Beltrami operator is Markovian, $\|e^{-t\Delta} v\|_\infty \lesssim \|v\|_\infty$ for all $t > 0$. In addition, from [17, Cor. 4.9], we know that $\|e^{-t\Delta} v\|_\infty \lesssim \|v\|_1 t^{-\frac{\omega}{2}} \lesssim \|v\|_1 t^{-\frac{\omega}{2}}$ for $t \geq 1$. Thus,
\begin{align*}
&\int_0^\infty t^{j+1} e^{-tk^2} |e^{-t\Delta} v(x)| \, dt = \int_0^1 t^{j+1} e^{-tk^2} |e^{-t\Delta} v(x)| \, dt + \int_1^\infty t^{j+1} e^{-tk^2} |e^{-t\Delta} v(x)| \, dt \\
&\quad \lesssim \|v\|_\infty \int_0^1 t^{j+1} e^{-tk^2} \, dt + \|v\|_1 \int_1^\infty t^{j+1} e^{-tk^2} \, dt < \infty.
\end{align*}

The dominated convergence theorem applied to (21) then yields
\[\nu^{(j+1)}(x, k) = (-1)^j \int_0^\infty t^{j+1} e^{-tk^2} e^{-t\Delta} v(x) \, dt,\]

thereby completing the inductive proof of (20) for general $j \in \mathbb{N}$, $x \in \mathcal{M}$ and $k > 0$.

The hypothesized estimate for $\nu^{(j)}$ can now be proved using our freshly minted integral identity. Utilising the bound $x^j \lesssim e^{\epsilon x}$ for any $0 < \epsilon < 1$ leads to
\begin{align*}
\left| \nu^{(j)}(x, k) \right| &\leq k^{-2j} \int_0^\infty (tk^2)^j e^{-tk^2} e^{-t\Delta} |v(x)| \, dt \\
&\lesssim k^{-2j} \int_0^\infty e^{-tk^2(1-\epsilon)} e^{-t\Delta} |v(x)| \, dt \\
&= k^{-2j} \left( \Delta + k^2(1-\epsilon) \right)^{-1} |v(x)|.
\end{align*}

Apply [19, Lem. 2.7] to obtain,
\[\left| \nu^{(j)}(x, k) \right| \lesssim k^{-2j} \|v\|_\infty \omega_c^\alpha(x, k\sqrt{1-\epsilon}),\]

for some $c' > 0$. Thus (18) holds with constant $c = c' \sqrt{1-\epsilon}$. \qed

Notice that in the previous proposition, the proof of the integral identity (20) relied upon the key condition $k > 0$. For $k = 0$, if $j$ is too large then the integral in (20) will no longer be guaranteed to converge absolutely and, consequently, the derivative will not exist. This is analogous to the well-known fact that on standard Euclidean space $\mathbb{R}^d$, the Riesz potentials $\Delta_{\mathbb{R}^d}^{-\alpha}$ are not defined for $\alpha \geq \frac{d}{2}$. Acting as a converse to this, the below
Proposition 3.2. Let \( v \in L^\infty(\mathcal{M}) \) be compactly supported in \( K \) and \( u \) be as given in [19, Lem. 2.7]. Then for any \( 0 \leq j < \frac{n_{\min}}{2} - 1 \),
\[
\begin{align*}
|u^{(j)}(x, k)| & \lesssim \|v\|_\infty \omega_{2j+2}(x, k), \text{ and} \\
|\nabla u^{(j)}(x, k)| & \lesssim \|v\|_\infty \omega_{2j+1}(x, k),
\end{align*}
\]
for all \( k \in [0, 1] \) and \( x \in \mathcal{M} \). Moreover, for \( k > 0 \),
\[
\begin{align*}
\|u^{(j)}(\cdot, k) - u^{(j)}(\cdot, 0)\|_{L^\infty(\mathcal{M})} & \lesssim k \|v\|_{L^\infty(\mathcal{M})} \\
\|\nabla u^{(j)}(\cdot, k) - \nabla u^{(j)}(\cdot, 0)\|_{L^\infty(\mathcal{M})} & \lesssim k^{1-2j} \|v\|_{L^\infty(\mathcal{M})},
\end{align*}
\]
Proof. From the previous proposition, we know that the derivative \( u^{(j)}(x, k) \) exists for \( k > 0 \) and is given by the integral identity (20). With the addition of the assumption \( j < \frac{n_{\min}}{2} - 1 \), it is clear that \( u^{(j)}(x, 0) \) will also exist and that (20) will be equally applicable. Indeed, this follows in an identical inductive manner as the proof for \( k > 0 \) from Proposition 3.1. However, in order to apply the dominated convergence theorem at the inductive step of the proof, it will be required that
\[
\int_0^\infty t^{i+1}e^{-t}0|e^{-t\Delta}v(x)| \, dt
\]
is absolutely convergent for when \( 1 \leq i + 1 \leq j \). This follows from
\[
\int_0^\infty t^{i+1} |e^{-t\Delta}v(x)| \, dt = \int_0^1 t^{i+1} |e^{-t\Delta}v(x)| \, dt + \int_1^\infty t^{i+1} |e^{-t\Delta}v(x)| \, dt
\]
\[
\lesssim \|v\|_\infty \int_0^1 t^{i+1} \, dt + \|v\|_1 \int_1^\infty t^{i+1-\frac{n_{\min}}{2}} \, dt.
\]
The second integral will converge if and only if \( i + 1 < \frac{n_{\min}}{2} - 1 \), which is implied by our assumption \( j < \frac{n_{\min}}{2} - 1 \).

Notice that the restriction \( 0 \leq j < \frac{n_{\min}}{2} - 1 \) not only implies the existence of \( u^{(j)}(x, 0) \), but it also tells us that \( \frac{(\Delta + k^2)^{j}u^{(j)}}{2^j} \) is a solution of the equation \( (\Delta + k^2)^{j+1}u = v \) with \( u^{(j)} \in L^\infty \) uniformly in \( k \in [0, 1] \). Also observe that from the identity (19),
\[
(\Delta + k^2)u^{(j)} = -ju^{(j-1)}.
\]
Recall from the proof of [19, Lem. 2.7] that \( \|\Delta^m u^{(0)}\|_{L^\infty(V)} \lesssim 1 \) for any compact \( V \subset \mathcal{M} \setminus K \) and \( m \in \mathbb{N} \). Then, for any \( 1 \leq j < \frac{n_{\min}}{2} - 1 \),
\[
\begin{align*}
\|\Delta u^{(j)}\|_{L^\infty(V)} & \lesssim \|(\Delta + k^2)u^{(j)}\|_{L^\infty(V)} + k^2 \|u^{(j)}\|_{L^\infty(V)} \\
& \lesssim \|u^{(j-1)}\|_{L^\infty(V)} + k^2 \|u^{(j)}\|_{L^\infty(V)} \\
& \lesssim 1.
\end{align*}
\]
We also have
\[ \left\| \Delta^2 u^{(j)} \right\|_{L^\infty(V)} \leq \left\| \Delta (\Delta + k^2) u^{(j)} \right\|_{L^\infty(V)} + k^2 \left\| \Delta u^{(j)} \right\|_{L^\infty(V)} \]
\[ \simeq \left\| \Delta u^{(j-1)} \right\|_{L^\infty(V)} + k^2 \left\| \Delta u^{(j)} \right\|_{L^\infty(V)} \]
\[ \lesssim 1, \]
for all \( 1 \leq j < \frac{n\min}{2} - 1 \). This process can be repeated to arbitrarily high order to obtain \( \left\| \Delta^m u^{(j)} \right\|_{L^\infty(V)} \lesssim 1 \) for any \( m, j \in \mathbb{N} \) and compact subset \( V \subset \mathcal{M} \setminus K \). This proves that \( u^{(j)} \in C^\infty \) uniformly in \( k \in [0, 1] \) on any compact subset outside of \( K \). The argument for (22) then follows in an identical manner to the corresponding estimates in [19, Lem. 2.7]. In particular, for \( \zeta_i \in C^\infty(\mathcal{M}) \) with support contained in \( \mathbb{R}^{n_i} \times \mathcal{M} \setminus K_i \), therefore satisfying \( \text{supp } v \subset (\text{supp } \zeta_i)^c \), and such that \( (1 - \zeta_i) \) is compactly supported when viewed as a function on \( \mathbb{R}^{n_i} \times \mathcal{M}_i \). Define
\[ \tilde{u}^{(j)}_i(x, k) := (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-\frac{j+1}{2}} \left( (\Delta + k^2)^{\frac{j+1}{2}} (\zeta_i u^{(j)}(x, k)) \right). \]
It can be reasoned that \( \tilde{u}^{(j)}_i = u^{(j)} \zeta_i \), at which point Corollaries 2.1 and 2.2 can be applied to produce (22).

Let’s now prove (23). From the relation (20) in combination with the estimates
\[ \| e^{-t\Delta} v \|_\infty \lesssim \| v \|_\infty \] for \( t \leq 1 \) and \[ \| e^{-t\Delta} v \|_\infty \lesssim \| v \|_1 t^{-\frac{n\min}{2}} \] for \( t > 1 \),
\[ \left| u^{(j)}(x, k) - u^{(j)}(x, 0) \right| \leq \| v \|_\infty \int_0^1 t^j (1 - e^{-tk^2}) dt + \| v \|_1 \int_1^\infty t^j (1 - e^{-tk^2}) t^{-\frac{n\min}{2}} dt. \]
For the first term, the estimate \( 1 - e^{-x} \leq x \) for \( x \in (0, 1) \) implies
\[ \int_0^1 t^j (1 - e^{-tk^2}) dt \leq k^2 \int_0^1 t^{j+1} dt \]
\[ \leq k^2. \]
For the second term, notice that since \( j < \frac{n\min}{2} - 1 \) we must have \( j \leq \frac{n\min}{2} - \frac{3}{2} \). A change of variables then leads to
\[ \int_1^\infty t^j (1 - e^{-tk^2}) t^{-\frac{n\min}{2}} dt \leq \int_1^\infty t^{-\frac{3}{2}} (1 - e^{-tk^2}) dt \]
\[ = k \int_1^\infty t^{-\frac{3}{2}} (1 - e^{-t}) dt \]
\[ \leq k \int_0^\infty t^{-\frac{3}{2}} (1 - e^{-t}) dt \]
\[ \lesssim k. \]

Finally, let us consider the validity of (24). This estimate has already been proved for the case \( j = 0 \) in [19, Lem. 2.7]. For \( 1 \leq j < \frac{n\min}{2} - 1 \), observe that
\[ (\Delta + k^2) u^{(j)}(x, k) = -j u^{(j-1)}(x, k) \quad \text{and} \quad \Delta u^{(j)}(x, 0) = -j u^{(j-1)}(x, 0). \]
Therefore,
\[ \left| \Delta u^{(j)}(x, k) - \Delta u^{(j)}(x, 0) \right| \lesssim k^2 \left| u^{(j)}(x, k) \right| + \left| u^{(j-1)}(x, k) - u^{(j-1)}(x, 0) \right|. \]
Proposition 3.1 and (23) then imply
\[
|\Delta u^{(j)}(x, k) - \Delta u^{(j)}(x, 0)| \lesssim (k^{2-2j} + k) \|v\|_\infty \\
\lesssim k^{1-2j} \|v\|_\infty.
\]

Remark 2.8 of [19] states that
\[
\|\nabla g\|_{L^\infty(M)} \lesssim \|\Delta g\|_{L^\infty(M)} + \|g\|_{L^\infty(M)}.
\]

Therefore,
\[
\left\| \nabla (u^{(j)}(\cdot, k) - u^{(j)}(\cdot, 0)) \right\|_\infty \lesssim \left\| \Delta (u^{(j)}(\cdot, k) - u^{(j)}(\cdot, 0)) \right\|_\infty + \left\| u^{(j)}(\cdot, k) - u^{(j)}(\cdot, 0) \right\|_\infty \\
\lesssim k^{1-2j} \|v\|_\infty,
\]
which completes our proof. \qed

**Proposition 3.3.** Let \( v \in L^\infty(M) \) be compactly supported in \( K \). Then, for any \( j \in \mathbb{N}^* \),
\[
|\nabla (\Delta + k^2)^{-j}v(x)| \lesssim k^{-2(j-1)} \|v\|_\infty \omega_1(x, k)
\]
for all \( x \in M \) and \( 0 < k \leq 1 \).

**Proof.** Before beginning the proof, note that in the notation of Propositions 3.1 and 3.2, \( \nabla (\Delta + k^2)^{-j}v = \nabla u^{(j-1)} \). We use resolvent notation in this proposition to simplify our computations.

This estimate has already been proved for \( j = 1 \) in [19, Lem. 2.7]. For \( j > 1 \),
\[
|\nabla (\Delta + k^2)^{-j}v(x)| = |\nabla (\Delta + k^2)^{-(j-1)}(\Delta + k^2)^{-1}(\Delta + k^2)^{-j}v(x)|
\]
\[
\leq \sum_{i=1}^{4} |\nabla G_i(k)(\Delta + k^2)^{-(j-1)}v(x)|
\]
\[
\leq \sum_{i=1}^{4} \int_M |\nabla x G_i(k)(x, y)| \left| (\Delta + k^2)^{-(j-1)}v(y) \right| dy
\]
\[
= \sum_{i=1}^{4} Y_i(k)(x).
\]

Our desired estimate will be proved for each of the above terms separately.

**The Term \( Y_1(k) \).**

The product rule implies that
\[
\nabla x G_1(k)(x, y) = \sum_{i=1}^{l} \nabla x (\Delta_{\mathbb{R}^n \times M_i} + k^2)^{-1}(x, y) \phi_i(x) \phi_i(y) + (\Delta_{\mathbb{R}^n \times M_i} + k^2)^{-1}(x, y) \nabla \phi_i(x) \phi_i(y).
\]

This leads to the splitting
\[
Y_1(k)(x) \leq Y_1^1(k)(x) + Y_1^2(k)(x),
\]
where
\[
Y_1^1(k)(x) = \int_M \left| \sum_{i=1}^{l} \nabla x (\Delta_{\mathbb{R}^n \times M_i} + k^2)^{-1}(x, y) \phi_i(x) \phi_i(y) \right| \left| (\Delta + k^2)^{-(j-1)}v(y) \right| dy
\]
and

\[ Y_1^2(k)(x) = \int_{\mathcal{M}} \left| \sum_{i=1}^{l} \left( \Delta_{\mathbb{R}^n_i \times M_i} + k^2 \right)^{-1}(x, y) \nabla \phi_i(x) \phi_i(y) \right| \left| (\Delta + k^2)^{-(j-1)}v(y) \right| \, dy. \]

If \( x \in K \), then both of these terms will clearly vanish, trivially leading to

\[ Y_1(k)(x) \lesssim k^{-2(j-1)} \|v\|_\infty \omega_1(x, k). \]

Let us consider \( x \in \mathbb{R}^n_i \times M_i \setminus K_i \) for some \( 1 \leq i \leq l \). For the first term, let \( D > 0 \) be such that \( d(x_i^n, y) \geq 2D \) for all \( y \in \mathbb{R}^n_i \times M_i \setminus K_i \). Corollary 2.2 and Proposition 3.1 then produce,

\[ Y_1^1(k)(x) \lesssim \int_{\mathbb{R}^n_i \times M_i \setminus K_i} \left| \nabla_x (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-1}(x, y) \right| \left| (\Delta + k^2)^{-(j-1)}v(y) \right| \, dy \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \int_{\mathbb{R}^n_i \times M_i \setminus K_i} \left[ d(x, y)^{1-n_i} + d(x, y)^{1-N} \right] \left( d(x_i^n, y) \right)^{2-n_i}e^{-ckd(x, y) + d(x_i^n, y))} \, dy \]

\[ =: I_1 + I_2, \]

where the above integral has been separated into two distinct integrals \( I_1 \) and \( I_2 \) over the respective regions

\[ A_1 := (\mathbb{R}^n_i \times M_i \setminus K_i) \cap B(x, d(x_i^n, x)/2) \]

and

\[ A_2 := (\mathbb{R}^n_i \times M_i \setminus K_i) \cap B(x, d(x_i^n, x)/2)^c. \]

For \( I_1 \), it is evident that \( d(x_i^n, y) \geq d(x_i^n, x) \) for any \( y \in A_1 \). Therefore,

\[ I_1 = k^{-2(j-2)} \|v\|_\infty \int_{A_1} \left[ d(x, y)^{1-n_i} + d(x, y)^{1-N} \right] \left( d(x_i^n, y) \right)^{2-n_i}e^{-ckd(x, y) + d(x_i^n, y)))} \, dy \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{2-n_i}e^{-ckd(x_i^n, x)} \int_{A_1} \left[ d(x, y)^{1-n_i} + d(x, y)^{1-N} \right] e^{-ckd(x, y))} \, dy \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{2-n_i}e^{-ckd(x_i^n, x)} \left( \int_{B(x,D)} d(x, y)^{1-N} \, dy + \int_{B(x,d(x_i^n, x)/2)\setminus B(x,D)} d(x, y)^{1-n_i} \, dy \right) \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{2-n_i}e^{-ckd(x_i^n, x)} \left( 1 + d(x_i^n, x) \right) \]

\[ \lesssim k^{-2(j-1)} \|v\|_\infty \left( d(x_i^n, x) \right)^{1-n_i}e^{-ckd(x_i^n, x)}, \]

where the last line follows on absorbing the term \( (kd(x_i^n, x))^2 \) into the exponential. For the integral \( I_2 \),

\[ I_2 = k^{-2(j-2)} \|v\|_\infty \int_{A_2} \left[ d(x, y)^{1-n_i} + d(x, y)^{1-N} \right] \left( d(x_i^n, y) \right)^{2-n_i}e^{-ckd(x, y) + d(x_i^n, y)))} \, dy \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{1-n_i}e^{-ckd(x_i^n, x)} \int_{B(x,d(x_i^n, x)/2)^c} \left( d(x_i^n, y) \right)^{2-n_i}e^{-ckd(x_i^n, y)} \, dy \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{1-n_i}e^{-ckd(x_i^n, x)} \int_{\mathbb{R}^n_i \setminus B(x,d(x_i^n, x)/2)^c} \frac{1}{|y_1|^m}e^{-ck|y_1|} \, dy_1 \]

\[ \lesssim k^{-2(j-2)} \|v\|_\infty \left( d(x_i^n, x) \right)^{1-n_i}e^{-ckd(x_i^n, x)} \int_0^\infty r^{m-2}e^{-ckr} \, dr \]

\[ \lesssim k^{-2(j-1)} \|v\|_\infty \left( d(x_i^n, x) \right)^{1-n_i}e^{-ckd(x_i^n, x)}. \]
For the term $Y_2^2(k)$, Corollary 2.1 and Proposition 3.1 imply
\[
Y_2^2(k)(x) = \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \left| (\Delta^{n_1} + k^2)^{-1}(x, y) \nabla \phi_i(x) \phi_i(y) \right| \left| (\Delta + k^2)^{-(j-1)} v(y) \right| dy
\]
\[
\lesssim |\nabla \phi_i(x)| k^{-2(j-2)} \| v \|_\infty \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \left[ d(x, y)^{2-n_i} + d(x, y)^{2-N} \right] \left( d(x_i^0, y) \right)^{2-n_i} e^{-ck(d(x, y) + d(x_i^0, y))} dy.
\]
Using the same argument as for the term $Y_1^1(k)$, we will obtain
\[
Y_2^2(k)(x) \lesssim |\nabla \phi_i(x)| k^{-2(j-2)} \| v \|_\infty \left( d(x_i^0, x) \right)^{2-n_i} e^{-ckd(x_i^0, x)}
\]
\[
\lesssim k^{-2(j-1)} \| v \|_\infty \omega_1(x, k),
\]
where we used the fact that $\nabla \phi_i$ is compactly supported to obtain the last line.

**The Term $Y_2^2(k)$**

The kernel $\nabla_x G_2(k)(x, y)$ is compactly supported in some neighbourhood of
\[
K_\Delta := \{(x, x) \in \mathcal{M}^2 : x \in K \}.
\]
There must then exist some compact set $V$ containing $K$ such that $\nabla_x G_2(k)(x, y)$ is supported in $V \times V$. This implies that $Y_2(k)$ must also be compactly supported in $V$. For $x \in V^c$, the estimate
\[
(26) \quad Y_2(k)(x) \lesssim k^{-2(j-1)} \| v \|_\infty \omega_1(x, k)
\]
will then be trivially satisfied. It remains to ascertain the validity of this estimate for $x \in V$. The operators $\{\nabla G_2(k)\}_{k \in (0, 1)}$ constitute a family of pseudodifferential operators of order $-1$. At the scale of an individual chart on $\mathcal{M}$, it is evident that in local coordinates the symbol of $\nabla G_2(k)$, denoted $a_2(k)(x, \xi)$, will satisfy
\[
|\partial_\xi^\alpha a_2(k)(x, \xi)| \lesssim (|\xi|^2 + k^2)^{-\frac{1-|\alpha|}{2}}
\]
for all multi-indices $\alpha \geq 0$ and $k \in (0, 1)$. From this, standard pseudodifferential operator theory (c.f. [29, Sec. 0.2] for instance) implies
\[
|\nabla_x G_2(k)(x, y)| \lesssim d(x, y)^{1-N},
\]
uniformly in $k \in (0, 1)$. This, in combination with Proposition 3.1, implies
\[
Y_2(k)(x) \lesssim k^{-2(j-2)} \| v \|_\infty \int_V d(x, y)^{1-N} \left( d(x_i^0, y) \right)^{2-n_i} \exp(-ckd(x_i^0, y)) dy
\]
\[
\lesssim k^{-2(j-1)} \| v \|_\infty \omega_1(x, k),
\]
which completes the proof of (26) for $x \in V$.

**The Term $Y_3^1(k)$**

Recall from [19] that the kernel $\nabla_x G_3(k)(x, y)$ satisfies the estimate
\[
|\nabla_x G_3(k)(x, y)| \lesssim \omega_1(x, k) \omega_2(y, k).
\]
Proposition 3.1 then implies,
\[
Y_3(k)(x) = \int_{\mathcal{M}} |\nabla_x G_3(k)(x, y)| \left( \Delta + k^2 \right)^{-(j-1)} v(y) dy
\]
\[
\lesssim k^{-2(j-2)} \| v \|_\infty \int_{\mathcal{M}} \omega_1(x, k) \omega_2(y, k) \omega_2(y, k) dy
\]
\[
= k^{-2(j-2)} \| v \|_\infty \omega_1(x, k) \int_{\mathcal{M}} \omega_2(y, k)^2 dy.
\]
Therefore, in order to prove the desired estimate for $Y_3(k)$, it is sufficient to show that
\[ \int_{\mathcal{M}} \omega_2(y, k)^2 \, dy \lesssim k^{-2}. \]

Considering the integral over the center $K$ first,
\[ \int_K \omega_2(y, k)^2 \, dy \lesssim \int_K 1 \, dy \lesssim 1 \leq k^{-2}. \]
On the ends $\mathbb{R}^{n_i} \times M_i \setminus K_i$ for $1 \leq i \leq l$ we have,
\[ \int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \omega_2(y, k)^2 \, dy \lesssim \int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \langle d(x_i^0, y) \rangle^{4-2n_i} \exp(-2ckd(x_i^0, y)) \, dy \]
\[ \lesssim \int_0^{\infty} \langle r \rangle^{4-2n_i} \exp(-2ckr)r^{n_i-1} \, dr \]
\[ \lesssim k^{-1} \leq k^{-2}. \]

The Term $Y_4(k)$.

This can be handled in an identical manner to the term $Y_3(k)$ since $\nabla x G_4(k)(x, y)$ satisfies even stronger estimates than $\nabla x G_3(k)(x, y)$ (c.f. [19, Sec. 3]).

In [19], the decomposition (16) allowed for the Riesz transform to be separated into four corresponding components. The $L^p$-boundedness of each part was then proved independently. In this article, we will also make use of this decomposition. Notice that
\[ (\Delta + k^2)^{-M} = \frac{(-1)^{M-1}}{(M-1)!} \partial_k^{(M-1)}(\Delta + k^2)^{-1}. \]

On combining this with the splitting (16), we have the relation
\begin{equation}
(\Delta + k^2)^{-M} = \sum_{i=1}^{4} H_i^{(M)}(k),
\end{equation}
where $H_i^{(M)}(k) := \frac{(-1)^{M-1}}{(M-1)!} \partial_k^{(M-1)} G_i(k)$. To simplify notation, when the integer $M > 0$ is understood, the shorthand notation $H_i(k)$ will be employed. For the remainder of this section we will investigate various properties of these operators and, in particular, obtain asymptotic estimates for the kernels of $H_3(k)$ and $H_4(k)$.

**Proposition 3.4.** The kernel of the operator $H_3(k)$ satisfies
\[ |H_3(k)(x, y)| \lesssim k^{-2(M-1)} \omega_2(x, k) \omega_2(y, k) \]
and
\[ |\nabla x H_3(k)(x, y)| \lesssim k^{-2(M-1)} \omega_1(x, k) \omega_2(y, k), \]
for all $x, y \in \mathcal{M}$ and $0 < k \leq 1$. 

then implies that \( \phi_i \) must be replaced with Proposition \( 2.2 \). \[ \frac{x, y}{\partial x y} \] of the interior of \( K \), leads to \( |H_3(k)(x, y)| \lesssim \omega_2(x, k) \omega_2(y, k) \sum_{j=0}^{M-1} k^{-2(M-1-j)} \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-(j+1)} (x_i^0, y) \phi_i(y). \]

Proposition 3.1 then implies that

\[
|H_3(k)(x, y)| \lesssim \omega_2(x, k) \omega_2(y, k) \sum_{j=0}^{M-1} k^{-2(M-1-j)} \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-(j+1)} (x_i^0, y) \phi_i(y).
\]

Proposition 2.2, when combined with the fact that \( \supp \phi_i \subset \mathbb{R}^{n_i} \times M_i \setminus K_i \) and \( x_i^0 \) is in the interior of \( K_i \), leads to

\[
|H_3(k)(x, y)| \lesssim \omega_2(x, k) \omega_2(y, k) \sum_{j=0}^{M-1} k^{-2(M-1-j)} \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-(j+1)} (x_i^0, y) \phi_i(y).
\]

which proves our claim. The estimate for \( \nabla x H_3(k)(x, y) \) follows via an identical argument, except the use of Proposition 3.1 must be replaced with Proposition 3.3. \( \square \)

**Lemma 3.1.** For any \( j \in \mathbb{N} \), all \( 0 < k \leq 1 \) and \( y \in M \),

\[
\left\| \frac{\partial^{(j)}}{k^2} E_1(k)(\cdot, y) \right\|_{\infty} \lesssim k^{-2j} \omega_1(y, k).
\]

**Proof.** This lemma has already been proved for the case \( j = 0 \) in [19], so it can be assumed that \( j > 0 \). From the splitting (17), it is sufficient to prove this bound for the kernels of each of the components \( E_1^i(k) \), \( E_2^i(k) \) for \( i = 1, \cdots, l \) and \( E_3(k) \).

The Error Term \( E_1^i(k) \).

On differentiating the expression for \( E_1^i(k) \) with respect to \( k^2 \),

\[
\partial^{(j)}_{k^2} E_1^i(k)(x, y) = -2 \nabla \phi_i(x) \phi_i(y) \left[ (-1)^j j! \nabla_x (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-(j+1)}(x, y) - \nabla_x \partial^{(j)}_{k^2} G_{int}(k)(x, y) \right].
\]

\( G_{int}(k) \) agrees with the resolvent \( (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-1} \) near the diagonal and on the support of \( \nabla \phi_i(x) \cdot \phi_i(y) \). Therefore \( E_1^i \) will vanish on this set.

For \( (x, y) \) away from the diagonal and on the support of \( \nabla \phi_i(x) \phi_i(y) \), we have

\[
\left| \frac{\partial^{(j)}}{k^2} E_1^i(k)(x, y) \right| \lesssim \left| \nabla_x (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-(j+1)}(x, y) \right| + \left| \nabla_x \partial^{(j)}_{k^2} G_{int}(k)(x, y) \right| \lesssim k^{-2j} \omega_1(y, k) + \left| \nabla_x \partial^{(j)}_{k^2} G_{int}(k)(x, y) \right|,
\]

where the last line follows from Proposition 2.2. The operators \( \left\{ \nabla \partial^{(j)}_{k^2} G_{int}(k) \right\}_{k \in (0,1)} \) constitute a family of pseudodifferential operators of order \(-1 - 2j\). At the scale of an
individual chart on \( \mathcal{M} \), it is evident that in local coordinates the symbol of \( \nabla \partial^{(j)} \mathcal{G}_{\text{int}}(k) \), denoted \( a_{\text{int}}(k)(x, \xi) \), will satisfy
\[
|\partial^{\alpha}_{\xi} a_{\text{int}}(k)(x, \xi)| \lesssim (|\xi|^2 + k^2)^{-1 - 2j - |\alpha|}
\]
for all multi-indices \( \alpha \geq 0 \) and \( k \in (0, 1) \). From this, standard pseudodifferential operator theory (c.f. \cite[Sec. 0.2]{[source]})) implies that for any \( b > -1 - 2j + N \),
\[
|\nabla_x \partial^{(j)} \mathcal{G}_{\text{int}}(k)(x, y)| \lesssim k^{N - 1 - 2j - b}d(x, y)^{-b}
\]
uniformly in \( k \in (0, 1) \), for all \( x, y \in \mathcal{M} \). Setting \( b = N - 1 \) gives
\[
|\nabla_x \partial^{(j)} \mathcal{G}_{\text{int}}(k)(x, y)| \lesssim k^{-2j}d(x, y)^{1-N}
\]
uniformly in \( k \in (0, 1) \), for all \( x, y \in \mathcal{M} \). As we are considering \( (x, y) \) away from the diagonal and \( \mathcal{G}_{\text{int}}(k)(x, y) \) is compactly supported in \( \mathcal{M}^2 \), this estimate implies
\[
|\nabla_x \partial^{(j)} \mathcal{G}_{\text{int}}(k)(x, y)| \lesssim k^{-2j}d(x, y)^{1-N} \lesssim k^{-2j}\omega_1(y, k),
\]
thereby implying our desired estimate.

**The Error Term \( E_2^j(k) \).**

For the second term,
\[
\partial^{(j)}_{k^2} E_2^j(k)(x, y) = \phi_i(y)v_i(x) \left[ (-1)^j j! (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j+1} (x, y) + \partial^{(j)}_{k^2} \mathcal{G}_{\text{int}}(k)(x, y) \right]
\]
\[
+ (-1)^j j! (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j+1} (x^\circ, y, k^2) \right].
\]

The first two terms will vanish on the diagonal and on the support of \( \phi_i(y)v_i(x) \), since this is where \( \mathcal{G}_{\text{int}}(k) \) coincides with the resolvent. The desired estimate would then follow from Proposition 2.2.

For \((x, y)\) away from the diagonal and on the support of \( \phi_i(y)v_i(x) \),
\[
|\partial^{(j)}_{k^2} E_2^j(k)(x, y)| \lesssim \left| (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j+1} (x^\circ, y) - (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j+1} (x, y) \right|
\]
\[
+ \left| \partial^{(j)}_{k^2} \mathcal{G}_{\text{int}}(k)(x, y) \right|
\]
\[
\lesssim |x^\circ - x| \left| \nabla_x (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-j+1} (x^\circ, y) \right| + \left| \partial^{(j)}_{k^2} \mathcal{G}_{\text{int}}(k)(x, y) \right|
\]
\[
\lesssim k^{-2j}\omega_1(y, k) + \left| \partial^{(j)}_{k^2} \mathcal{G}_{\text{int}}(k)(x, y) \right|,
\]
for some \( \tilde{x} \) close to the set \( K_i \), where the last line follows from Proposition 2.2. Similar pseudodifferential reasoning as for the term \( E_1^j(k) \) then implies \( |\partial^{(j)}_{k^2} \mathcal{G}_{\text{int}}(k)(x, y)| \lesssim k^{-2j}\omega_1(y, k) \), thereby proving the desired estimate.

**The Error Term \( E_3^j(k) \).**

Finally, from the form of \( \partial^{(j)}_{k^2} E_3^j(k) \), it is clear that in an analogous manner to the previous two terms, pseudodifferential reasoning can be applied to obtain the estimate
\[
\left\| \partial^{(j)}_{k^2} E_3^j(k)(\cdot, y) \right\|_\infty \lesssim k^{-2j}\omega_1(y, k).
\]

\(\square\)

**Proposition 3.5.** The kernel of the operator \( H_4(k) \) satisfies
\[
|H_4(k)(x, y)| \lesssim k^{-2(M-1)}\omega_2(x, k)\omega_1(y, k)
\]
and
\[ |\nabla_x H_f(k)(x, y)| \lesssim k^{-2(M-1)}\omega_1(x, k)\omega_1(y, k) \]
for all \( x, y \in \mathcal{M} \) and \( 0 < k \leq 1 \).

**Proof.** Recall that
\[ G_4(k)(x, y) = -(\Delta + k^2)^{-1}v_y(x) = -\int_0^\infty e^{-tk^2} e^{-t\Delta}v_y(x) \, dt, \]
where \( v_y(x) := E(k)(x, y) \). If we define \( v^{(j)}_y := \partial_{k^2}^j v_y \) for \( j \in \mathbb{N} \), then Lemma 3.1 implies
\[ \left\| v^{(j)}_y \right\|_\infty = \left\| \partial_{k^2}^j E(k)(\cdot, y) \right\|_\infty \lesssim k^{-2j}\omega_1(y, k). \]
Therefore \( e^{-t\Delta}v^{(j)}_y \) is well-defined and we have by the dominated convergence theorem
\[ \partial_{k^2}^j e^{-t\Delta}v_y(x) = e^{-t\Delta}v^{(j)}_y(x). \]
Thus
\[ \partial_{k^2}^{(M-1)}G_4(k)(x, y) = \partial_{k^2}^{(M-1)} \int_0^\infty e^{-tk^2} e^{-t\Delta}v_y(x) \, dt 
= \int_0^\infty \partial_{k^2}^{(M-1)} \left( e^{-tk^2} e^{-t\Delta}v_y(x) \right) \, dt 
= \sum_{j=0}^{M-1} \binom{M-1}{j} (-1)^j t^j e^{-tk^2} e^{-t\Delta}v^{(M-1-j)}_y(x) \, dt. \]
The integral identity (20) then implies
\[ \partial_{k^2}^{(M-1)}G_4(k)(x, y) = \sum_{j=0}^{M-1} \binom{M-1}{j} (\Delta + k^2)^{-(j+1)}v^{(M-1-j)}_y(x). \]

An application of Proposition 3.1 leads to,
\[ \left\| \partial_{k^2}^{(M-1)}G_4(k)(x, y) \right\|_\infty \lesssim \sum_{j=0}^{M-1} k^{-2j} \omega_2(x, k) \left\| v^{(M-1-j)}_y \right\|_\infty 
\lesssim \omega_2(x, k)\omega_1(y, k) \sum_{j=0}^{M-1} k^{-2j} k^{-2(M-1-j)} 
\simeq k^{-2(M-1)}\omega_2(x, k)\omega_1(y, k). \]
The definition \( H_4(k) := \frac{(-1)^{M-1}}{(M-1)!}\partial_{k^2}^{(M-1)}G_4(k) \) then allows us to immediately obtain
\[ |H_4(k)(x, y)| \lesssim k^{-2(M-1)}\omega_2(x, k)\omega_1(y, k). \]
Estimates for \(|\nabla_x H_4(k)(x, y)|\) follow in an identical manner from (29), except the use of Proposition 3.1 must be replaced with an application of Proposition 3.3. \( \square \)
4. The Low Energy Square Function

Recall from Section 3 that the higher-order resolvent has the splitting

\[(\Delta + k^2)^{-M} = \sum_{i=1}^{4} H_i(k),\]

where \(H_i(k) := \frac{(-1)^{M-1}}{(M-1)!} k^{(M-1)} G_i(k)\). The low energy square function can then be controlled from above by

\[S_f(x) \lesssim \sum_{i=1}^{4} S_i^2 f(x),\]

where \(S_i\) is the part of the low energy square function corresponding to \(H_i(k)\),

\[S_i^2 f(x) := \left( \int_{0}^{1} |\nabla H_i(k) f(x)|^2 k^{4M-3} dk \right)^{\frac{1}{2}}.\]

The \(L^p\)-boundedness for \(p \in (1, n_{\min})\) and weak-type \((1, 1)\) property of \(S_f\) will be proved by demonstrating that each component \(S_i^2\) is itself bounded on \(L^p\) and weak-type \((1, 1)\) for \(i = 1, \cdots, 4\).

4.1. The Operator \(S_i^2\). From the definition of \(H_1(k)\), we have

\[H_1(k)(x, y) = \frac{(-1)^{M-1}}{(M-1)!} k^{(M-1)} \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-1}(x, y) \phi_i(x) \phi_i(y)\]

\[= \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M}(x, y) \phi_i(x) \phi_i(y).\]

As \(H_1(k)\) consists of finitely many terms, in order to prove the boundedness of the operator \(S_i^2\), we need only prove boundedness of the operators

\[S_{i}^{1,i} f(x) := \left( \int_{0}^{1} \left| \int_{\mathcal{M}} \nabla_x [(\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M}(x, y) \phi_i(x) \phi_i(y)] f(y) dy \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}\]

for all \(1 \leq i \leq l\). From the product rule, this operator is in turn controlled by

\[S_{i}^{1,i} f(x) \leq \Lambda_i^1 f(x) + \Pi_i f(x),\]

where

\[\Lambda_i^1 f(x) := \left( \int_{0}^{1} \left| \int_{\mathcal{M}} \nabla_x (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M}(x, y) \phi_i(x) \phi_i(y) f(y) dy \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}\]

and

\[\Pi_i f(x) := \left( \int_{0}^{1} \left| \int_{\mathcal{M}} (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M}(x, y) \nabla \phi_i(x) \phi_i(y) f(y) dy \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}.\]

First consider the operator \(\Lambda_i^1\). It is well-known from classical theory that the square function

\[\left( \int_{0}^{\infty} |\nabla (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M} g(x)|^2 k^{4M-3} dk \right)^{\frac{1}{2}} = \left( \int_{0}^{\infty} |\nabla (\Delta_{\mathbb{R}^{n_i} \times M_i} + I)^{-M} g(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}\]
is bounded on \( L^p(\mathbb{R}^{n_i} \times \mathcal{M}_i) \) for all \( p \in (1, \infty) \) and weak-type \((1, 1)\). Therefore,

\[
\| \Lambda_i f \|_p = \left\| \phi_i \cdot \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (\phi_i \cdot f) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \\
\leq \left\| \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (\phi_i \cdot f) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \\
\lesssim \|f\|_p
\]

for any \( p \in (1, \infty) \). Weak-type \((1, 1)\) bounds for \( \Lambda_i \) follow in the same manner.

Next, consider the operator \( \Pi_i \). For \( r > 0 \), define the set \( D_r := \{(x, y) \in \mathcal{M}^2 : d(x, y) \leq r \} \). Minkowski’s integral inequality implies that

\[
\Pi_i^1 f(x) \leq \int_{\mathcal{M}} \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (x, y) \nabla \phi_i(x) \phi_i(y) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} |f(y)| \, dy \\
= \int_{\mathcal{M}} |\nabla \phi_i(x)| \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (x, y) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \phi_i(y) |f(y)| \, dy \\
\leq \int_{\mathcal{M}} \pi_1^i(x, y) |f(y)| \, dy + \int_{\mathcal{M}} \pi_2^i(x, y) |f(y)| \, dy \\
=: \Pi_1^i f(x) + \Pi_2^i f(x),
\]

where \( \pi_1^i(x, y) \) and \( \pi_2^i(x, y) \) are the kernels defined by

\[
\pi_1^i(x, y) := |\nabla \phi_i(x)\chi_{D_r}(x, y) \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (x, y) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \phi_i(y)
\]

and

\[
\pi_2^i(x, y) := |\nabla \phi_i(x)| |1 - \chi_{D_r}(x, y)| \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (x, y) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \phi_i(y).
\]

Let’s prove that \( \Pi_1^i \) is \( L^p \)-bounded for all \( p \in [1, \infty) \). From Proposition 2.2,

\[
|\nabla \phi_i(x)\chi_{D_r}(x, y) \left( \int_0^1 |(\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-M} (x, y) \right|^2 k^{4M-3} dk \right\|_p^{\frac{1}{2}} \phi_i(y) \\
\lesssim \chi_{D_r}(x, y) \left( \int_0^1 |d(x, y)|^{2-N} \exp (-ckd(x, y)) k \, dk \right)^{\frac{1}{2}} \\
= \chi_{D_r}(x, y) d(x, y)^{2-N} \left( \int_0^1 k \exp (-2ckd(x, y)) \, dk \right)^{\frac{1}{2}} \\
\lesssim \chi_{D_r}(x, y) d(x, y)^{1-N}.
\]

It is obvious that an operator with this kernel will be bounded on \( L^p \) for all \( p \in [1, \infty] \) since the local decay \( d(x, y)^{1-N} \) is stronger than \( d(x, y)^{-N} \), and globally it is cut off by the function \( \chi_{D_r} \). Therefore \( \Pi_1^i \) is bounded on \( L^p \) for all \( p \in [1, \infty] \).

Next, consider the operator \( \Pi_2^i \). From an application of Hölder’s inequality for \( p \geq 1 \),

\[
\| \Pi_2^i f \|_p^p = \int_{\mathcal{M}} \left( \int_{\mathcal{M}} \pi_2^i(x, y) |f(y)| \, dy \right)^p \, dx \\
\leq \int_{\mathcal{M}} \| \pi_2^i(x, \cdot) \|_{L^{p'}(\mathcal{M})}^p \|f\|_{L^p(\mathcal{M})}^p \, dx.
\]
On applying Proposition 2.2 for \( x, y \in \mathbb{R}^{n_i} \times M_i \setminus K_i \) with \( d(x, y) > r \),
\[
\left( \int_0^1 \left| (\Delta_{\mathbb{R}^{n_i} \times M_i} + k^2)^{-M}(x, y) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}} \lesssim d(x, y)^{2-n_i} \left( \int_0^1 k \exp(-2ckd(x, y)) dk \right)^{\frac{1}{2}} 
\lesssim d(x, y)^{1-n_i}.
\]

It then follows from the fact that \( \nabla \phi_i \) is compactly supported,
\[
\int_{M} \| \pi_2^i (x, \cdot) \|_{L^p(\mathcal{M})}^p \, dx \lesssim \int_{\text{supp} \, \nabla \phi_i} \| d(x, \cdot)^{1-n_i} \|_{L^p(B(x, r)^c)}^p \, dx 
\lesssim \sup_{x \in \text{supp} \, \nabla \phi_i} \| d(x, \cdot)^{1-n_i} \|_{L^p(B(x, r)^c)}.
\]

For \( p = 1 \), this quantity is obviously finite and thus \( \Pi_2^i \) is bounded on \( L^1 \). For \( p > 1 \),
\[
\| d(x, \cdot)^{1-n_i} \|_{L^p(B(x, r)^c)} = \left( \int_{d(x, y) > r} d(x, y)^{(1-n_i)p'} \, dy \right)^{\frac{1}{p'}}.
\]

This quantity will be bounded from above by a constant, uniformly in \( x \), provided that
\[
p' (1 - n_i) < -n_i \quad \Leftrightarrow \quad p' > \frac{n_i}{n_i - 1} = n_i' \quad \Leftrightarrow \quad p < n_i.
\]

This proves that \( \Pi_2^i \) is bounded on \( L^p \) for \( 1 \leq p < n_i \). This completes the proof of the \( L^p \)-boundedness of \( \Pi^i \) for all \( p \in [1,n_i) \), thereby demonstrating that the operator \( S^1_{<} \) is \( L^p \)-bounded for \( p \in (1, n_{\text{min}}) \) and weak-type \((1,1)\).

### 4.2. The Operator \( S^2_{<} \).

The operator \( S^2_{<} \) is given by
\[
S^2_{<} f(x) = \left( \int_0^1 \left| \int_{\mathcal{M}} \nabla_x H_2(k)(x, y) f(y) \, dy \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

Minkowski’s integral inequality implies that
\[
S^2_{<} f(x) \leq \int_{\mathcal{M}} \left( \int_0^1 |\nabla_x H_2(k)(x, y)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} |f(y)| \, dy
= \int_{\mathcal{M}} h_2(x, y) |f(y)| \, dy,
\]

where \( h_2(x, y) \) is the kernel
\[
h_2(x, y) := \left( \int_0^1 |\nabla_x H_2(k)(x, y)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}.
\]

First observe that the kernel \( h_2 \) is compactly supported in \( \mathcal{M}^2 \). Next, notice that the operators \( \{\nabla H_2(k)\}_{k \in (0,1)} \) constitute a family of pseudodifferential operators of order \( 1 - 2M \). At the scale of an individual chart on \( \mathcal{M} \), it is evident that in local coordinates the symbol of \( \nabla H_2(k) \), denoted \( a_2(k)(x, \xi) \), will satisfy
\[
|\partial^\alpha_x a_2(k)(x, \xi)| \lesssim (|\xi|^2 + k^2)^{-1-2M-2\beta + \beta} d(x, y)^{-b},
\]
for all multi-indices \( \alpha \geq 0 \) and \( k \in (0,1) \). From this, standard pseudodifferential operator theory (c.f. [29, Sec. 0.2] for instance) implies
\[
|\nabla_x H_2(k)(x, y)| \lesssim k^{N+1-2M-b} d(x, y)^{-b},
\]
for any \( b > N + 1 - 2M \). Setting \( b = N - \frac{1}{2} \) then yields
\[
|\nabla_x H_2(k)(x, y)| \lesssim k^{\frac{N}{2} - 2M} d(x, y)^{\frac{1}{2} - N}.
\]
Therefore,

\[
h_2(x, y) = \left( \int_0^1 |\nabla_x H_2(k(x, y))|^2 k^{4M-3} dk \right)^{\frac{1}{2}} \lesssim \left( \int_0^1 dk \right)^{\frac{1}{2}} d(x, y)^{\frac{1}{2}-N} = d(x, y)^{\frac{1}{2}-N}.
\]

Thus \( S^2_\prec \) is pointwise bounded from above by an operator with kernel that is compactly supported in \( M^2 \) and controlled by \( d(x, y)^{\frac{1}{2}-N} \). It follows immediately that \( S^2_\prec \) must be bounded on \( L^p \) for all \( p \in [1, \infty] \).

4.3. The Operator \( S^3_\prec \). In an identical manner to the operator \( S^2_\prec \), Minkowski’s integral inequality allows us to control \( S^3_\prec \) from above by

\[
S^3_\prec f(x) \leq \int_M h_3(x, y) |f(y)| \, dy,
\]

where \( h_3(x, y) \) is the kernel defined through

\[
h_3(x, y) := \left( \int_0^1 |\nabla_x H_3(k(x, y))|^2 k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

Hölder’s inequality implies that

\[
\|S^3_\prec f\|_p^p \leq \int_M \left( \int_M h_3(x, y) |f(y)| \, dy \right)^p \, dx \leq \int_M \left( \int_M h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx \cdot \|f\|_p^p.
\]

Thus if it can be proved that

\[
\int_M \left( \int_M h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx < \infty
\]

then the operator \( S^3_\prec \) will be bounded on \( L^p \). In order to prove (30) it is sufficient to prove the below four separate conditions,

\[
\int_K \left( \int_K h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx < \infty,
\]

(32) \[
\int_{\mathbb{R}^n \times M \setminus K_i} \left( \int_K h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx < \infty \quad \text{for any } 1 \leq i \leq l,
\]

(33) \[
\int_K \left( \int_{\mathbb{R}^n \times M \setminus K_j} h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx < \infty \quad \text{for any } 1 \leq j \leq l,
\]

and

(34) \[
\int_{\mathbb{R}^n \times M \setminus K_i} \left( \int_{\mathbb{R}^n \times M \setminus K_j} h_3(x, y)^p \, dy \right)^{\frac{p}{p'}} \, dx < \infty \quad \text{for any } 1 \leq i, j \leq l.
\]

For the first estimate (31), observe that Proposition 3.4 implies \( \nabla_x H_3(k(x, y)) \lesssim k^{2-2M} \) for all \( x, y \in M \). This estimate, when applied to the definition of \( h_3(x, y) \), gives

\[
h_3(x, y) \lesssim 1 \quad \forall \ x, y \in M,
\]
which immediately implies the validity of (31).

Let us consider (32). According to Proposition 3.4, for \( x \in \mathbb{R}^{n_i} \times M_i \setminus K_i \) and \( y \in K \),

\[
|\nabla_{x} H_{3}(k)(x, y)| \lesssim k^{2-2M} \langle d(x_i^0, x) \rangle^{1-n_i} \exp(-ckd(x_i^0, x)).
\]

This implies that

\[
\begin{aligned}
    h_3(x, y) &= \left( \int_{0}^{1} |\nabla_{x} H_{3}(k)(x, y)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} \\
    &\lesssim \langle d(x_i^0, x) \rangle^{1-n_i} \left( \int_{0}^{1} k \exp(-2ckd(x_i^0, x)) \, dk \right)^{\frac{1}{2}} \\
    &\simeq \langle d(x_i^0, x) \rangle^{1-n_i} \left( \frac{1}{d(x_i^0, x)^2} \right)^{\frac{1}{2}} \\
    &\simeq \langle d(x_i^0, x) \rangle^{1-n_i}.
\end{aligned}
\]

(36)

On applying this estimate to (32),

\[
\int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \left( \int_{K} h_3(x, y)^{p^*} \, dy \right)^{\frac{1}{p^*}} \, dx \lesssim \int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \langle d(x_i^0, x) \rangle^{-n_i p} \, dx.
\]

This will clearly be finite provided that \( p > 1 \).

Let us now consider (33). For \( x \in K \) and \( y \in \mathbb{R}^{n_j} \times M_j \setminus K_j \), Proposition 3.4 tells us that

\[
|\nabla_{x} H_{3}(k)(x, y)| \lesssim k^{2-2M} \langle d(x_j^0, y) \rangle^{2-n_j} \exp(-ckd(x_j^0, y)).
\]

This leads to the pointwise estimate

\[
\begin{aligned}
    h_3(x, y) &\lesssim \langle d(x_j^0, y) \rangle^{2-n_j} \left( \int_{0}^{1} k \exp(-2ckd(x_j^0, y)) \, dk \right)^{\frac{1}{2}} \\
    &\lesssim \langle d(x_j^0, y) \rangle^{1-n_j}.
\end{aligned}
\]

Applying this to (33),

\[
\int_{K} \left( \int_{\mathbb{R}^{n_j} \times M_j \setminus K_j} h_3(x, y)^{p^*} \, dy \right)^{\frac{1}{p^*}} \, dx \lesssim \int_{K} \left( \int_{\mathbb{R}^{n_j} \times M_j \setminus K_j} \langle d(x_j^0, y) \rangle^{(1-n_j)p^*} \, dy \right)^{\frac{1}{p^*}} \, dx.
\]

This will be finite provided that

\[
(n_j - 1)p^* > n_j \iff p^* > \frac{n_j}{n_j - 1} = n_j^* \iff p \prec n_j.
\]

Finally, it remains to prove estimate (34). For \( x \in \mathbb{R}^{n_i} \times M_i \setminus K_i \) and \( y \in \mathbb{R}^{n_j} \times M_j \setminus K_j \), Proposition 3.4 implies that

\[
|\nabla_{x} H_{3}(k)(x, y)| \lesssim k^{2-2M} \langle d(x_i^0, x) \rangle^{1-n_i} \langle d(x_j^0, y) \rangle^{2-n_j} \exp(-ck(d(x_i^0, x) + d(x_j^0, y))).
\]

This leads to the pointwise bound

(37)

\[
|h_3(x, y)| \lesssim \langle d(x_i^0, x) \rangle^{1-n_i} \langle d(x_j^0, y) \rangle^{2-n_j} \left( \int_{0}^{1} k \exp(-2ckd(x_i^0, x) + d(x_j^0, y)) \, dk \right)^{\frac{1}{2}} \\
\lesssim \langle d(x_i^0, x) \rangle^{1-n_i} \langle d(x_j^0, y) \rangle^{2-n_j} \left( \frac{1}{(d(x_i^0, x) + d(x_j^0, y))^2} \right)^{\frac{1}{2}} \\
\leq \min \left( \langle d(x_i^0, x) \rangle^{-n_i} \langle d(x_j^0, y) \rangle^{2-n_j}, \langle d(x_i^0, x) \rangle^{1-n_i} \langle d(x_j^0, y) \rangle^{1-n_j} \right).
\]
Applying this to (34),

\[ \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \left( \int_{\mathbb{R}^{n_2} \times \mathcal{M}_j \setminus K_j} h_3(x, y)^{p'} \, dy \right)^{\frac{1}{p'}} \, dx \]

\[ \lesssim \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \left( \int_{D_j^1(x)} (d(x_i^0, x))^{(1-n_i)p'} (d(x_j^0, y))^{(1-n_j)p'} \, dy \right)^{\frac{1}{p'}} \, dx \]

\[ + \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \left( \int_{D_j^2(x)} (d(x_i^0, x))^{-(n_i-p')} (d(x_j^0, y))^{(2-n_j)p'} \, dy \right)^{\frac{1}{p'}} \, dx \]

\[ =: I_1 + I_2, \]

where

\[ D_j^1(x) := \{ y \in \mathbb{R}^{n_2} \times \mathcal{M}_j \setminus K_j : d(x_j^0, y) \geq d(x_j^0, x) \}, \]

\[ D_j^2(x) := \{ y \in \mathbb{R}^{n_2} \times \mathcal{M}_j \setminus K_j : d(x_j^0, y) < d(x_j^0, x) \}. \]

For the first term,

\[ I_1 = \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \frac{1}{d(x_i^0, x)^{(n_i-1)p'}} \left( \int_{D_j^1(x)} \frac{dy}{d(x_j^0, y)^{(n_j-1)p'}} \right)^{\frac{1}{p'}} \, dx. \]

The interior integral is given by

\[ \int_{D_j^1(x)} \frac{dy}{(d(x_j^0, y))^{(n_j-1)p'}} = \int_{\mathbb{R}^{n_2} \times \mathcal{M}_j \setminus K_j} \mathbf{1}_{d(x_j^0, y) \geq d(x_j^0, x)} \, dy \]

\[ \lesssim \int_{\mathbb{R}^{n_2} \times \mathcal{M}_j} \left( d(x_j^0, y) + d(x_i^0, x) \right)^{(n_j-1)p'} \, dy \]

\[ \lesssim \int_{\mathbb{R}^{n_2}} \left( |y_1 - x_{j,1}^0| + d(x_i^0, x) \right)^{(n_j-1)p'}, \]

where the notation \( x_{j,1}^0 \) denotes the Euclidean component of \( x_j^0 \) in \( \mathbb{R}^{n_j} \times \mathcal{M}_j \). This will be integrable when

\[ (n_j - 1)p' > n_j \iff p' > \frac{n_j}{n_j - 1} = n_j' \iff p < n_j. \]

In which case,

\[ \int_{D_j^1(x)} \frac{dy}{(d(x_j^0, y))^{(n_j-1)p'}} \lesssim \int_{d(x_i^0, x)} r^{n_j - 1} r^{(n_j - 1)p'} \, dr \]

\[ \simeq \left[ -r^{-(n_j - 1)p' + n_j} \right]_{d(x_i^0, x)}^{\infty} \]

\[ \lesssim (d(x_i^0, x))^{-(n_j - 1)p' + n_j}, \]

where we used the fact that \( p < n_j \) to deduce that \( -(n_j - 1)p' + n_j < 0 \) when performing the integration. Applying this estimate to \( I_1 \) gives

\[ I_1 \lesssim \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \frac{1}{(d(x_i^0, x))^{(n_i-1)p'}} \cdot (d(x_i^0, x))^{-(n_j - 1)p' + n_j \frac{n_j}{p'}} \, dx \]

\[ = \int_{\mathbb{R}^{n_1} \times \mathcal{M}_i \setminus K_i} \frac{dx}{(d(x_i^0, x))^{(n_i-1)p' + (n_j - 1)p' - n_j(p-1)}}. \]
This will be finite provided that

\[(n_i - 1)p + (n_j - 1)p - n_j(p - 1) > n_i \iff p > \frac{n_i - n_j}{n_i - 2}.
\]

Since \(\frac{n_i - n_j}{n_i - 2} < \frac{n_i - 2}{n_i - 2} = 1\), this will be satisfied when \(p > 1\). It has therefore been proved that \(I_1\) is finite when \(1 < p < n_j\).

It remains to consider the term \(I_2\).

\[
I_2 = \int_{\mathbb{R}^{n_i} \times M_1 \setminus K_i} \frac{1}{d(x_i, x)^{n_j}} \left( \int_{D_j(x)} \frac{dy}{d(x_j, y)^{(n_j - 2)p'}} \right)^{\frac{p}{p'}} \, dx.
\]

It will be proved that this term is finite for all \(1 < p < \infty\). The interior integral is given by

\[
\int_{D_j(x)} \frac{dy}{d(x_j, y)^{(n_j - 2)p'}} = \int_{\mathbb{R}^{n_j} \times M_1 \setminus K_j} \frac{1}{d(x_j, y)^{(n_j - 2)p'}} \, dy
\]

\[
\leq \int_{\mathbb{R}^{n_j} \times M_1} \frac{1}{d(x_j, y)^{(n_j - 2)p'}} \left( d(x_j, y)^{n_j} + D_K^{(n_j - 2)p'} \right) \, dy
\]

\[
\lesssim \int_{\mathbb{R}^{n_j} \times M_1} \frac{1}{d(x_j, y)^{(n_j - 2)p'}} \left( |y_1 - x_j| + D_K^{(n_j - 2)p'} \right) \, dy
\]

where \(D_K > 0\) is some constant that satisfies \(d(x_j, z) > D_K\) for all \(z \in M \setminus K\) and \(k = 1, \ldots, l\). Let’s estimate this integral based on three distinct cases.

**Case 1.** Suppose \(n_j - 1 - (n_j - 2)p' > -1\), which itself is equivalent to the condition \(p > \frac{n_i}{2}\). Then

\[
\int_{D_K} r^{n_j - 1 - (n_j - 2)p'} \, dr \simeq \left[ r^{n_j - (n_j - 2)p'} \right]_{D_K}^{2d(x_j, x)} \leq (d(x_j, x))^{n_j - (n_j - 2)p'}.
\]

Substituting this back into \(I_2\),

\[
I_2 \lesssim \int_{\mathbb{R}^{n_i} \times M_1 \setminus K_i} \frac{1}{d(x_i, x)^{n_j}} \cdot (d(x_i, x))^{n_j \frac{p}{p'} - (n_j - 2)p} \, dx
\]

\[
= \int_{\mathbb{R}^{n_i} \times M_1 \setminus K_i} \frac{dx}{d(x_i, x)^{n_j + (n_j - 2)p - n_j \frac{p}{p'}}}.
\]

This will be integrable when

\[n_i p + (n_j - 2)p - n_j(p - 1) > n_i \iff p > \frac{n_i - n_j}{n_i - 2},\]

which will be satisfied when \(p > 1\) since \(\frac{n_i - n_j}{n_i - 2} < \frac{n_i - 2}{n_i - 2} = 1\).

**Case 2.** Suppose that \(n_j - 1 - (n_j - 2)p' < -1\), which itself is equivalent to the condition \(p < \frac{n_i}{2}\). We would then have

\[
\int_{D_K} r^{n_j - 1 - (n_j - 2)p'} \, dr \simeq \left[ r^{n_j - (n_j - 2)p'} \right]_{D_K}^{2d(x_j, x)} \lesssim 1.
\]
Thus
\[ I_2 \lesssim \int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \frac{1}{\langle d(x_i^0, x) \rangle^{n_j p}} \, dx, \]
which will be finite when \( p > 1 \).

**Case 2.** Suppose \( n_j - 1 - (n_j - 2)p' = -1 \), which itself is equivalent to the condition \( p = \frac{3}{2} \). We would then have
\[
\int_{D_K}^{2d(x_i^0, x)} r^{n_j - 1 - (n_j - 2)p'} \, dr = \int_{D_K}^{2d(x_i^0, x)} r^{-1} \, dr
\]
\[
= \frac{1}{D_K^{n_j - 1}} \int_{D_K}^{2d(x_i^0, x)} \frac{D_K^j}{r} \, dr
\]
\[
\lesssim \int_{D_K}^{2d(x_i^0, x)} r^{n_j - 1} \, dr
\]
\[
\lesssim \langle d(x_i^0, x) \rangle^\epsilon.
\]

Therefore,
\[
I_2 \lesssim \int_{\mathbb{R}^{n_i} \times M_i \setminus K_i} \frac{1}{\langle d(x_i^0, x) \rangle^{n_j p - \epsilon p}} \, dx.
\]
This will be finite when
\[
n_i p - \epsilon(p - 1) > n_i \quad \iff \quad p > 1.
\]
This completes our proof of estimate (34) and thus shows that the operator \( S_3^\epsilon \) is bounded on \( L^p \) for all \( p \in (1, \min_i n_i) \).

It remains to prove that \( S_3^\epsilon \) is weak-type \((1,1)\). On combining (35), (36) and (37), it is evident that
\[
\sup_{y \in M} |h_3(x, y)| \lesssim \begin{cases} 
\langle d(x_i^0, x) \rangle^{-n_i} & \text{for } x \in \mathbb{R}^{n_i} \times M_i \setminus K_i, \ 1 \leq i \leq l, \\
1 & \text{for } x \in K.
\end{cases}
\]
This implies that for \( f \in C^\infty_c \),
\[
|S_3^\epsilon f(x)| \lesssim \int_M h_3(x, y) |f(y)| \, dy
\]
\[
\lesssim \sup_{y \in M} |h_3(x, y)| \int_M |f(y)| \, dy
\]
\[
\lesssim \begin{cases} 
\langle d(x_i^0, x) \rangle^{-n_i} \|f\|_{L^1} & \text{for } x \in \mathbb{R}^{n_i} \times M_i \setminus K_i, \ 1 \leq i \leq l, \\
\|f\|_{L^1} & \text{for } x \in K.
\end{cases}
\]
This function is clearly in \( L^{1,\infty} \), and thus \( S_3^\epsilon \) is weak-type \((1,1)\).

### 4.4. The Operator \( S_4^\epsilon \)

The previous proof of the \( L^p \)-boundedness and weak-type \((1,1)\) property of the operator \( S_3^\epsilon \) was derived entirely from the pointwise estimates for the kernel of \( \nabla H_3(k) \) provided by Proposition 3.4. Since the kernels of \( \nabla H_4(k) \) satisfy even stronger pointwise estimates by Proposition 3.5, an identical argument can be applied to obtain the \( L^p \)-boundedness and weak-type \((1,1)\) property of the operator \( S_4^\epsilon \). Moreover, due to the increased strength of the pointwise estimates for \( \nabla H_4(k) \), one will obtain \( L^p \)-boundedness for all \( p \) in the full range \((1,\infty)\) instead of the restricted range \((1, n_{\min})\). This improved range of boundedness will prove to be an essential part of our proof of the unboundedness of \( S \) for \( p \geq n_{\min} \) in Section 7. As such, the reader is advised keep it in mind.
5. The High Energy Square Function

Recall that the high energy part of the square function is defined by

$$S_{>} f(x) := \left( \int_{1}^{\infty} |\nabla (k^2 + \Delta)^{-M} f(x)|^2 k^{4M-3} dk \right)^{\frac{1}{2}}.$$ 

The main aim of this section is to verify that the operator $S_{>}$ is bounded on all $L^p$ spaces for $1 < p < \infty$ and is weak type (1, 1). The argument we describe below has the same structure as in [19, Section 5] which we adapt to the square function setting.

**Proposition 5.1.** The high energy square function operator $S_{>}$ is bounded on $L^p(\mathcal{M})$ for any $p \in (1, \infty)$. That is, for every $1 < p < \infty$ there exists $c > 0$ such that

$$\|S_{>} f\|_p \leq c \|f\|_p$$

for all $f \in L^p(\mathcal{M})$. In addition the operator $S_{>}$ satisfies weak type (1, 1) estimates.

**Proof.** Denote by $F_M$ the Fourier transform of the function $\lambda \rightarrow (\lambda + 1)^{-M}$. Note that

$$\int_{-\infty}^{\infty} e^{-i \lambda t} \lambda^2 k^2 F_M(k|\xi|) \. d\lambda = k^{1-2M} F_M(k|\xi|).$$

In fact $F_1(|\xi|) = c e^{|\xi|}$ and $k^{1-2(M+1)} F_{M+1}(k|\xi|) = \partial_{k^2} k^{1-2M} F_M(k|\xi|)$. Note also that there exist positive constants $C, c > 0$ (depending on $M$) such that

$$F_M(|\xi|) \leq C e^{-c|\xi|}.$$

Now, let $\eta \in C_c^{\infty}(\mathbb{R})$ be an even compactly supported function such that $0 \leq \eta(\xi) \leq 1$, $\eta(\xi) = 1$ for all $|\xi| \leq 1/2$ and $\eta(\xi) = 0$ for $|\xi| \geq 1$. For any $r > 0$ we define the function $G^{M}_{r,k}$ as the inverse Fourier transform of

$$k^{1-2M} F_M(k|\xi|) \eta(\xi/r).$$

Then we set

$$H^{M}_{r,k}(\lambda) = (\lambda^2 + k^2)^{-M} - G^{M}_{r,k}(\lambda)$$

so that the Fourier transform of $H^{M}_{r,k}$ is equal to

$$k^{1-2M} F_M(k|\xi|) \left( 1 - \eta(\xi/r) \right).$$

Hence for some positive constants $C, c > 0$

$$\sup_{\lambda} |H^{M}_{r,k}(\lambda)| \leq C e^{-cr^2}$$

Proceeding as in [19, (42) p. 1089], we have (using the evenness of $\hat{G}^{M}_{r,k}(\xi)$)

$$G^{M}_{r,k}(\sqrt{\lambda}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\sqrt{\lambda}} \hat{G}^{M}_{r,k}(t) \. dt = \frac{1}{\pi} \int_{0}^{r} \cos(t\sqrt{\lambda}) \hat{G}^{M}_{r,k}(t) \. dt.$$

By (42) and by the finite speed of propagation of $\cos(t\sqrt{\lambda})$ it follows that $G^{M}_{r,k}(\sqrt{\lambda})$ has Schwartz kernel supported in the set $\{(z, z') \mid d(z, z') \leq r\}$. Next, set $r = \tilde{r} > 0$ to be half the injectivity radius of $\mathcal{M}$. Note that the injectivity radius of $\mathcal{M}$ is strictly positive and, without loss of generality, it can be assumed that $\tilde{r} = 1$.

The operator $\nabla G^{M}_{r,k}(\sqrt{\lambda}) = \nabla G^{M}_{1,k}(\sqrt{\lambda})$ is a pseudodifferential operator of order $1 - 2M$ since the function $G^{M}_{1,k}(\lambda)$ is a symbol of order $-2M$ see [29, Chapter XII, Section 1]. This implies that for any $b \geq N + 1 - 2M$, see [30, Proposition 2.2 p. 6]

$$|\nabla G^{M}_{1,k}(\sqrt{\lambda})(x, y)| \leq C k^{N-2M-b+1} d(x, y)^{-b}.$$
Hence, taking $b_1 = N - \frac{1}{2}$ and $b_2 = N + \frac{1}{2}$,
\[
\int_1^\infty |\nabla G_{1,k}^M(\sqrt{\Delta})(x,y)|^2 k^{4M-3} \leq \int_1^{d(x,y)^{-1}} Ck^{2N-4M-2b_1+2}d(x,y)^{-2b_1}k^{4M-3} dk
\]
\[+ \int_1^\infty Ck^{2N-4M-2b_2+2}d(x,y)^{-2b_2}k^{4M-3} \leq Cd(x,y)^{-2N}.
\]

Note also that if $X$ is $C^\infty$ vector field on $\mathcal{M}$, uniformly bounded in every $C^m$ norm then
\[
\int_1^\infty |X \nabla G_{1,k}^M(\sqrt{\Delta})(x,y)|^2 k^{4M-3} dk \leq Cd(x,y)^{-2(N+1)}.
\]

Hence we can use vector-valued standard Calderon-Zygmund approach (with respect to $L^2([1, \infty), k^{4M-3}dk)$ to conclude the argument for $\nabla G_{1,k}^M(\sqrt{\Delta})$ and $k \geq 1$, see for example [27, §6.4, p. 28].

Next we consider the family of operators $\nabla H_{r,k}^M(\sqrt{\Delta})$. By Minkowski’s integral inequality
\[
\left( \int_1^\infty \left( \int_{\mathcal{M}} K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y)f(y)dy \right)^2 k^{4M-3} dk \right)^{\frac{1}{2}}
\]
\[
\leq \int_{\mathcal{M}} \int_1^\infty \left( K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \right)^2 k^{4M-3} dk \cdot |f(y)| dy
\]
\[= \int_{\mathcal{M}} h_*(x,y) |f(y)| dy,
\]
where
\[
h_*(x,y) = \left( \int_1^\infty \left( K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \right)^2 k^{4M-3} dk \right)^{-\frac{1}{2}}
\]
By Schur’s integral test, to conclude the proof it is now enough to show that there exists a constant $C$ such that
\begin{equation}
\sup_y \int_{\mathcal{M}} h_*(x,y) dx \leq C
\end{equation}
and
\begin{equation}
\sup_x \int_{\mathcal{M}} h_*(x,y) dy \leq C.
\end{equation}

Here we again use the approach described in [19]. First we note that finite propagation speed implies that
\[
K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) = K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \text{ if } x \notin B(y,r_2), \quad r_2 \geq r_1 > 0.
\]

It follows that for every $r > 1$
\begin{equation}
\int_{x \notin B(y,r)} \left| K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \right|^2 dx = \int_{x \notin B(y,r)} \left| K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \right|^2 dx
\end{equation}
\[\leq \int_{x \in \mathcal{M}} \left| K_{\nabla H_{r,k}^M(\sqrt{\Delta})}(x,y) \right|^2 dx
\]
\[= \left\langle \nabla_\Delta K_{H_{r,k}^M(\sqrt{\Delta})}(\cdot,y), \nabla_\Delta K_{H_{r,k}^M(\sqrt{\Delta})}(\cdot,y) \right\rangle
\]
\[= \left\langle \Delta K_{H_{r,k}^M(\sqrt{\Delta})}(\cdot,y), K_{H_{r,k}^M(\sqrt{\Delta})}(\cdot,y) \right\rangle
\]
\[= \int_{x \in \mathcal{M}} \left| K_{\sqrt{\Delta} H_{r,k}^M(\sqrt{\Delta})}(x,y) \right|^2 dx.
\]
A straightforward argument, see [19, Proposition 2.4], shows that \( \| (I + \Delta)^{-n} \|^{2}_{2 \to \infty} < \infty \) for any \( n \geq [N/4] +1 \). Hence by (41) for some \( c > 0 \)

\[
\sup_{y} \int_{x \in M} |K_{\sqrt{\Delta}H_{r,k}(\sqrt{\Delta})}^M(x,y)|^2 dx = \| \sqrt{\Delta}H_{r,k}(\sqrt{\Delta}) \|^{2}_{1 \to 2} = \| \sqrt{\Delta}H_{r,k}(\sqrt{\Delta}) \|^{2}_{2 \to \infty} \\
\leq \| (I + \Delta)^n \sqrt{\Delta}H_{r,k}(\sqrt{\Delta}) \|^{2}_{2 \to 2} \| (I + \Delta)^{-n} \|^{2}_{2 \to \infty} \\
\leq C \sup_{\lambda > 0} |(1 + \lambda^2)^n \lambda H_{r,k}(\lambda)|^2 \\
\leq Ce^{-ckr}.
\]

Now we can conclude that

\[
(46) \quad \sup_{y} \int_{x \notin B(y,r)} \int_{1}^{\infty} |K_{\sqrt{\Delta}H_{r,k}(\sqrt{\Delta})}^M(x,y)|^2 \sqrt{M-3} dtk \leq e^{-cr}
\]

This proves (43) since it implies in particular that

\[
\sup_{y} \int_{r \leq d(x,y) \leq 2r} h_s(x,y)^2 dx \leq e^{-cr}.
\]

The measure of the set \( \{ x \in M \mid r \leq d(x,y) \leq 2r \} \) is bounded by \( Cr^N, r \geq \bar{r} = 1 \), where \( N \) is the dimension of \( M \), uniformly in \( y \in M \). So we can apply Hölder’s inequality to find that

\[
\sup_{y} \int_{r \leq d(x,y) \leq 2r} h_s(x,y)dx \leq e^{-cr}.
\]

Then the above estimates can be summed over a sequence of dyadic annuli to obtain (43).

To prove (44) we use the notion of the Hodge Laplacian. Recall that for any bounded Borel function \( F \) if we write \( \Delta_q \) for the Hodge Laplacian acting on \( q \)-forms then we can write \( dF(\sqrt{\Delta_0}) = F(\sqrt{\Delta_1})d. \) This means that to verify (44) it is enough to prove (43) for \( H_{r,k}(\sqrt{\Delta_1}) \). This can be achieved by essentially repeating the above argument for \( \Delta_0 = \Delta \). See [19] for more detailed calculations. Now the continuity of the off-diagonal part of high energy square function follows from (43), (44) and Minkowski integral inequality.

\[ \square \]

6. The Reverse Inequality

In this section, the reverse inequality portion of Theorem 1.1 will be proved. That is, it will be shown that \( \| f \|_p \lesssim \| Sf \|_p \) for any \( p \in (n'_{\text{min}}, n_{\text{min}}) \). This will be achieved using, by now, standard arguments that can be found in [2, Thm. 7.1]. By the functional calculus of \( \Delta \) on \( L^2 \), for \( f \in L^p \cap L^2 \) we have the resolution of the identity,

\[
f = c_M \int_{0}^{\infty} t^2 \Delta(1 + t^2\Delta)^{-2M} f \frac{dt}{t}
\]

for some constant \( c_M > 0 \) that is dependent on \( M \). Then for any \( g \in L^{p'} \), Fubini-Tonelli’s theorem, whose application will be justified retrospectively, implies

\[
\int_{\mathcal{M}} f \cdot \bar{g} \simeq \int_{\mathcal{M}} \left( \int_{0}^{\infty} t^2 \Delta(1 + t^2\Delta)^{-2M} f(x) \frac{dt}{t} \right) \overline{g(x)} dx \\
= \int_{0}^{\infty} \int_{\mathcal{M}} t^2 \Delta(1 + t^2\Delta)^{-2M} f(x) \overline{g(x)} dx \frac{dt}{t} \\
= \int_{0}^{\infty} \int_{\mathcal{M}} t \nabla(1 + t^2\Delta)^{-M} f(x) t \overline{\nabla(1 + t^2\Delta)^{-M} g(x)} dx \frac{dt}{t}.
\]
On applying Fubini-Tonelli once more, followed by Cauchy-Schwarz and Hölder’s inequality,
\[
\int_M f \cdot g = \int_M \int_0^\infty t \nabla (1 + t^2 \Delta)^{-M} f(x) t \nabla (1 + t^2 \Delta)^{-M} g(x) \frac{dt}{t} \, dx \\
\leq \int_M Sf(x) \cdot Sg(x) \, dx \\
\leq \|Sf\|_p \|Sg\|_{p'}.
\]
Since $S$ is bounded on both $L^p$ and $L^{p'}$ for $p \in (n_{\min}', n_{\min})$ it follows that this quantity is finite, thereby retrospectively justifying our two previous applications of the Fubini-Tonelli theorem. Moreover, the $L^{p'}$-boundedness of $S$ in particular implies
\[
\int_M f \cdot g \lesssim \|Sf\|_p \|g\|_{p'},
\]
and thus
\[
\|f\|_p \lesssim \|Sf\|_p
\]
for all $f \in L^p \cap L^2$. As $S$ is bounded on $L^p$, this estimate must also hold true for all $f \in L^p$ by density.

7. Unboundedness for $p \geq n_{\min}$

Throughout this section we impose the restriction $2M < n_{\min}$ and provide a proof of the unboundedness portion of Theorem 1.1. The upper restriction on the order of the resolvent will allow us to utilize Proposition 3.2, which will form a key component of the proof. Our proof will exploit some of the ideas utilized in the unboundedness argument of [19, Sec. 6]. However, a number of difficulties will arise due to the quadratic nature of the square function and the higher-order degree of the resolvent.

In order to prove unboundedness of the square function operator on $L^p$ for $p \geq n_{\min}$ it is clearly sufficient to prove unboundedness of the low energy square function. Recall from our proof of the boundedness portion of Theorem 1.1 that the components $S^2_<$ and $S^4_<$ are bounded on $L^p(M)$ for all $p \in (1, \infty)$. The unboundedness of the operator $S_<$ will thus follow directly from the unboundedness of the operator
\[
S_\leq^{1+3} \phi(x) \equiv \left( \int_0^1 \left| \nabla (H_1(k) + H_3(k)) f(x) \right|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} \\
= \left( \int_0^1 \left[ \int_M (\nabla H_1(k)(x, y) + \nabla H_3(k)(x, y)) f(y) \, dy \right]^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}.
\]
Recall from the proof of the boundedness portion of Theorem 1.1 that the operator $S^1_<$ divides into two parts, $\Lambda^i$ and $\Pi^i$, depending on whether the gradient hits the resolvent factor or the function $\phi_i$. The term where the gradient hits the resolvent $\Lambda_i$ was proved to be bounded on $L^p$ for all $p \in (1, \infty)$. It is therefore sufficient to prove unboundedness of the operator
\[
\left( \int_0^1 \left[ \sum_{i=1}^l \int_M \nabla \phi_i(x)(\Delta_{R^n_i \times M_i} + k^2)^{-M}(x, y) \phi_i(y)f(y) + \xi_i(k)(x, y)f(y) \\
+ \nabla u_i(x, k)(\Delta_{R^n_i \times M_i} + k^2)^{-M}(x_i^p, y) \phi_i(y)f(y) \, dy \right]^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}, \tag{47}
\]
where $\xi_i(k, x, y)$ is defined through
\[
\xi_i(k, x, y) := \sum_{j=0}^{M-2} \frac{(-1)^{M+j-1}}{(M-j-1)!} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-(j+1)}(x^0_i, y) \nabla u_i^{(M-j)}(x, k) \phi_i(y).
\]

Let $\tau$ be a nonnegative function, not identically zero, that is compactly supported on one of the ends. It is clear that the unboundedness of the operator (47) will be implied by the unboundedness of the operator
\[
\tau(x) \left( \int_0^1 \left[ \sum_{i=1}^l \int_M \nabla \phi_i(x) (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x, y) \phi_i(y) f(y) + \xi_i(k, x, y) f(y) \right. \\
+ \left. \nabla u_i(x, k) (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x^0_i, y) \phi_i(y) f(y) dy \right]^2 k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

Let $\xi_i(k)$ be the operator corresponding to the kernel $\xi_i(k, x, y)$ and define
\[
\Xi f(x) := \tau(x) \left( \int_0^1 \left[ \sum_{i=1}^l \xi_i(k) f(x) \right]^2 k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

If it can be proved that $\Xi$ is bounded on $L^p$ for any $p \in (2, \infty)$ then the task of proving the unboundedness of the operator (48) on $L^p$ for $p \geq n_{\min}$ will be reduced to proving the unboundedness of the operator
\[
\tau(x) \left( \int_0^1 \left[ \sum_{i=1}^l \int_M \nabla \phi_i(x) (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x, y) \phi_i(y) f(y) \right. \\
+ \left. \nabla u_i(x, k) (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x^0_i, y) \phi_i(y) f(y) dy \right]^2 k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

To this end, the triangle inequality tells us that the $L^p$-boundedness of $\Xi$ will follow from the $L^p$-boundedness of the operators
\[
\Xi_i^j f(x) := \left( \int_0^1 \left| \xi_i^j(k) f(x) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}},
\]
for each $1 \leq i \leq l$ and $0 \leq j \leq M - 2$, where
\[
\xi_i^j(k, x, y) := \tau(x) \left( (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-(j+1)}(x^0_i, y) \nabla u_i^{(M-j)}(x, k) \phi_i(y).\right.
\]

This operator can then itself be controlled from above by
\[
\Xi_i^j f(x) \leq \left( \int_0^1 \left| \xi_i^{j,1}(k) f(x) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}} + \left( \int_0^1 \left| \xi_i^{j,2}(k) f(x) \right|^2 k^{4M-3} dk \right)^{\frac{1}{2}}
\]

where
\[
\xi_i^{j,1}(k, x, y) := \tau(x) \left( (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-(j+1)}(x^0_i, y) \nabla u_i^{(M-j)}(x, k) - \nabla u_i^{(M-j)}(x, 0) \right) \phi_i(y)
\]
and
\[
\xi_i^{j,2}(k, x, y) := \tau(x) \left( (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-(j+1)}(x^0_i, y) \nabla u_i^{(M-j)}(x, 0) \right) \phi_i(y).
\]

**Lemma 7.1.** For any $0 \leq j \leq M - 1$, $1 \leq i \leq l$ and $p \in (2, \infty)$, the operator $\Xi_i^{j,1}$ is bounded on $L^p$. 

Proof. Minkowski's integral inequality, that can be applied since \( \frac{p}{2} > 1 \), implies that

\[
\left\| \Xi_i^{j,1} f \right\|_p = \left\| \int_0^1 \left| \xi_i^{j,1}(k) f \right|^2 k^{4M-3} dk \right\|_{\frac{p}{2}}^\frac{1}{2} 
\]

(50)

\[
\leq \left( \int_0^1 \left\| \xi_i^{j,1}(k) f \right\|^2 k^{4M-3} dk \right)^\frac{1}{2} = \left( \int_0^1 \left\| \xi_i^{j,1}(k) \right\|_p^2 k^{4M-3} dk \right)^\frac{1}{2} \left\| f \right\|_p.
\]

For fixed \( k \), \( \xi_i^{j,1}(k) \) is a rank one operator given by \( \xi_i^{j,1}(k) = a_j \cdot \langle b_j, \cdot \rangle \) with

\[
a_j(x) = \tau(x) \left| \nabla u_i^{(M-1-j)}(x, k) - \nabla u_i^{(M-1-j)}(x, 0) \right|
\]

and

\[
b_j(y) := (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-(j+1)} (x_i^0, y) \phi_i(y).
\]

From Proposition 3.2, it is clear that

\[
\left\| a_j \right\|_p \lesssim k^{3+2j-2M}.
\]

Let \( D > 0 \) be such that \( d(x_i^0, y) \geq D \) for all \( y \in \text{supp} \phi_i \). Then, in a similar manner to (52) of [19],

\[
\left\| b_j \right\|_{p'} = \left( \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i} \left| (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-(j+1)} (x_i^0, y) \phi_i(y) \right|^{p'} dy \right)^\frac{1}{p'}
\]

(51)

\[
\leq k^{-2j} \left( \int_{d(x_i^0, y) \geq D} \left| d(x_i^0, y)^{2-n_i} \exp(-ckd(x_i^0, y)) \right|^{p'} dy \right)^\frac{1}{p'}
\]

\[
\lesssim \left\{ \begin{array}{ll}
\frac{k^{n_i}}{p} - 2 - 2j & p > \frac{n_i}{2} \\
-2j \log k & p = \frac{n_i}{2} \\
-2j & p < \frac{n_i}{2}
\end{array} \right.
\]

We therefore have

\[
\left\| \xi_i^{j,1}(k) \right\|_p = \left\| a_j \right\|_p \cdot \left\| b_j \right\|_{p'}
\]

\[
\lesssim \left\{ \begin{array}{ll}
k^{n_i+1-2M} & p > \frac{n_i}{2} \\
k^{3-2M} \log k & p = \frac{n_i}{2} \\
k^{3-2M} & p < \frac{n_i}{2}
\end{array} \right.
\]

On applying this estimate to (50) we find that \( \left\| \Xi_i^{j,1} \right\|_p < \infty \) for any \( p \in (2, \infty) \). \( \square \)

Lemma 7.2. For any \( 0 \leq j \leq M - 2 \), \( 1 \leq i \leq l \) and \( p \in (2, \infty) \), the operator \( \Xi_i^{j,2} \) is bounded on \( L^p \).

Proof. As was the case for \( \xi_i^{j,1}(k) \), for fixed \( k \) the operator \( \xi_i^{j,2}(k) \) is of rank one. In particular, we have \( \xi_i^{j,2}(k) = a'_j \cdot \langle b'_j, \cdot \rangle \) where

\[
a'_j(x) := \tau(x) \left| \nabla u_i^{(M-1-j)}(x, 0) \right|
\]
and \(b_j\) is as defined previously for the operator \(\xi_i^{j,1}(k)\). It is obvious from Proposition 3.2 that \(\|a'_j\|_p \lesssim 1\) uniformly in \(k\). Estimate (51) then implies

\[
\|\xi_i^{j,2}(k)\|_p \simeq \|a'_j\|_p : \|b_j\|_{p'}
\]

\[
\lesssim \|b_j\|_{p'}
\]

\[
\lesssim \begin{cases}
  k^{\frac{n_j}{p} - 2 - 2j} & p > \frac{n_j}{2} \\
  k^{-2j} \log k & p = \frac{n_j}{2} \\
  k^{-2j} & p < \frac{n_j}{2}.
\end{cases}
\]

For \(p > \frac{n_j}{2}\), once again by Minkowski’s inequality,

\[
\|\Xi_i^{j,2}\|_p \leq \left( \int_0^1 \|\xi_i^{j,2}(k)\|_p^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_0^1 \frac{2n_j}{k^p} + 4(M-j) - 7 \, dk \right)^{\frac{1}{2}}.
\]

This will be finite provided that

\[
\frac{2n_j}{p} + 4(M-j) - 7 > -1,
\]

which is implied by the restriction \(j \leq M - 2\). The cases \(p = \frac{n_j}{2}\) and \(p < \frac{n_j}{2}\) proceed similarly thereby producing \(\|\Xi_i^{j,2}\|_p < \infty\). \(\square\)

Now that the \(L^p\)-boundedness of \(\Xi\) has been proved for \(p \in (2, \infty)\), it remains to consider the unboundedness of the operator given in (49). Notice that

\[
\left( \int_0^1 \left[ \int_{\mathcal{M}} \nabla \phi_i(x) \left( (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x,y) - (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x_i^0, y) \right) \phi_i(y) f(y) \, dy \right]^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}
\]

\[
\leq \int_{\mathcal{M}} \nabla \phi_i(x) \left[ \int_0^1 \left| (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x,y) - (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x_i^0, y) \right|^2 k^{4M-3} \, dk \right]^{\frac{1}{2}} \phi_i(y) f(y) \, dy
\]

\[
=: \int_{\mathcal{M}} p(x,y) f(y) \, dy =: Pf(x).
\]

**Lemma 7.3.** The operator \(P\) is bounded on \(L^p\) for all \(p \in (1, \infty)\).

**Proof.** Let \(V \subset \text{supp} \phi_i\) be an open subset, compactly supported, that contains \(\text{supp} \nabla \phi_i\). For \(y \in V\), Proposition 2.2 implies

\[
p(x,y) \lesssim \nabla \phi_i(x) \left( \int_0^1 \left| (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x,y) \right|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} \phi_i(y)
\]

\[
+ \nabla \phi_i(x) \left( \int_0^1 \left| (\Delta_{\mathbb{R}^n_i \times \mathcal{M}_i} + k^2)^{-M}(x_i^0, y) \right|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} \phi_i(y)
\]

\[
\lesssim \nabla \phi_i(x) d(x,y)^{2-N} \left( \int_0^1 k \exp(-2ckd(x,y)) \, dk \right)^{\frac{1}{2}} \phi_i(y)
\]

\[
+ \nabla \phi_i(x) d(x_i^0, y)^{2-n_i} \left( \int_0^1 k \exp(-2ckd(x_i^0, y)) \, dk \right)^{\frac{1}{2}} \phi_i(y)
\]

\[
\lesssim \nabla \phi_i(x) \left( (d(x,y)^{1-N} + d(x_i^0, y)^{1-n_i}) \right) \phi_i(y).
\]
Schur’s test then tells us that the operator with kernel $p(x, y) \mathbb{1}_V(y)$ is bounded on $L^p$ for all $p \in (1, \infty)$.

On the other hand, for $y \in \text{supp} \phi_i \setminus V$, Proposition 2.2 and the mean value theorem give

$$
\nabla \phi_i(x) \left( \int_0^1 \left| \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x, y) - \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x_i^0, y) \right|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}} \phi_i(y)
\lesssim \nabla \phi_i(x) \left( \int_0^1 \left( \int_{M} d(x_i^0, y)^2 \left[ k^{4-4M}(d(x_i^0, y))^{-2(n_i-1)} \exp(-2ckd(x_i^0, y)) \right] \right)^{\frac{1}{2}} \phi_i(y)
\lesssim \nabla \phi_i(x) \left( \int_0^1 k \exp(-2ckd(x_i^0, y)) \right)^{\frac{1}{2}} \phi_i(y)
\lesssim \nabla \phi_i(x) (d(x_i^0, y))^{-(n_i-1)} \left( \int_0^1 k \exp(-2ckd(x_i^0, y)) \right)^{\frac{1}{2}} \phi_i(y).
$$

The kernel $\mathbb{1}_{\text{supp} \phi_i \setminus V}(y)p(x, y)$ is therefore compactly supported in $x$ and it decays to order $n_i$ in $y$. From the argument used to prove the boundedness of $S_{<0}^2$, it is clear that the corresponding operator will be bounded on $L^p$ for all $p \in (1, \infty)$. This demonstrates that $P$ is bounded on $L^p$ for all $p \in (1, \infty)$. \hfill \Box

From the above lemma, it is clear that the unboundedness of the operator in (49) will follow from the unboundedness of the operator

$$
\tau(x) \left( \int_0^1 \left[ \sum_{i=1}^l \int_{M} (\nabla \phi_i(x) + \nabla u_i(x, k)) \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x_i^0, y) \phi_i(y) f(y) \, dy \right]^{2} k^{4M-3} \, dk \right)^{\frac{1}{2}}.
$$

Choose $i$ such that $n_i$ is minimal. Then, in an identical manner to [19], the unboundedness of the above operator can be further reduced to proving the unboundedness of the operator

$$
\tau(x) \left( \int_0^1 \left[ \int_{M} (\nabla \phi_i(x) + \nabla u_i(x, k)) \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x_i^0, y) \phi_i(y) f(y) \, dy \right]^{2} k^{4M-3} \, dk \right)^{\frac{1}{2}}.
$$

From an application of Lemma 7.1 for the case $j = M - 1$, it is evident that in order to deduce that the above operator is unbounded on $L^p$ for $p \geq n_{\text{min}}$ it is sufficient to show that

$$
Rf(x) := \left( \int_0^1 |r(k)f(x)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}
$$

is unbounded on $L^p$, where $r(k)$ is the operator with kernel defined through

$$
r(k)(x, y) := \tau(x)(\nabla \phi_i(x) + \nabla u_i(x, 0)) \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x_i^0, y) \phi_i(y).
$$

This final claim will be proved in the below lemma.

**Lemma 7.4.** The operator $R$ is unbounded on $L^p$ for any $p \geq \min_i n_i$.

**Proof.** From the definition of the kernel $r(k)$, for $f \in L^p$ non-negative and $x \in M$ we have

$$
Rf(x) = \left( \int_0^1 |r(k)f(x)|^2 k^{4M-3} \, dk \right)^{\frac{1}{2}}
$$

$$
= a(x) \left( \int_0^1 \left[ \int_{M} \left( \Delta_{\mathbb{R}^n_1 \times M_i} + k^2 \right)^{-M}(x_i^0, y) \phi_i(y) f(y) \, dy \right]^2 k^{4M-3} \, dk \right)^{\frac{1}{2}},
$$
where \(a(x) := \tau(x)|\nabla \phi_i(x) + \nabla u_i(x,0)|\). From Corollary 2.3,

\[
Rf(x) \gtrsim a(x) \left( \int_0^1 \left( \int_M d(x^0_i,y)^{2M-n_i} \exp(-ckd(x^0_i,y))\phi_i(y)f(y)dy \right)^2 k^{4M-3} dk \right)^{\frac{1}{2}}
\]

\[
= a(x) \left( \int_0^1 \left( \int_M d(x^0_i,y)^{2M-n_i} \exp(-ckd(x^0_i,y))\phi_i(y)f(y)dy \right) \cdot \left( \int_M d(x^0_i,z)^{2M-n_i} \exp(-ckd(x^0_i,z))\phi_i(z)f(z)dz \right) k^{4M-3} dk \right)^{\frac{1}{2}}.
\]

On applying Tonelli’s theorem, (52)

\[
Rf(x) \gtrsim a(x) \left( \int_M \int_M d(x^0_i,y)^{2M-n_i} d(x^0_i,z)^{2M-n_i} \eta(y,z)\phi_i(y)\phi_i(z)f(y)f(z)dydz \right)^{\frac{1}{2}},
\]

with

\[\eta(y,z) := \int_0^1 k^{4M-3} \exp(-ckd(x^0_i,y) + d(x^0_i,z))) dk.\]

For \(y, z \in \text{supp} \phi_i\) the quantity \(d := (d(x^0_i,y) + d(x^0_i,z))\) is bounded from below. Therefore, after a change of variables it can be said that

\[
\eta(y,z) = \frac{1}{d^{4M-2}} \int_0^d k^{4M-3} \exp(-ck) dk \\
\geq \frac{1}{d^{4M-2}} \int_0^D k^{4M-3} \exp(-ck) dk \\
\geq \frac{1}{d^{4M-2}},
\]

where \(D > 0\) is some constant independent of \(y\) and \(z\). Applying this to (52),

\[
Rf(x) \gtrsim a(x) \left( \int_M \int_M d(x^0_i,y)^{2M-n_i} d(x^0_i,z)^{2M-n_i} \frac{\phi_i(y)\phi_i(z)f(y)f(z)dydz}{(d(x^0_i,y) + d(x^0_i,z))^{4M-2}} \right)^{\frac{1}{2}}.
\]

Due to symmetry, we can then say that

\[
Rf(x) \gtrsim a(x) \left( \int_M \int_M d(x^0_i,y)^{2M-n_i} d(x^0_i,z)^{2M-n_i} \left[ \int_M d(x^0_i,y)^{2M-n_i} \mathbb{1}_{d(x^0_i,y) \leq d(x^0_i,z)} \phi_i(y)f(y)dy \mathbb{1}_{d(x^0_i,z) \leq d(x^0_i,y)} \phi_i(z)f(z)dz \right] \right)^{\frac{1}{2}}.
\]

Set \(f_\varepsilon(y) := d(x^0_i,y)^{-\frac{\mu}{(1+\varepsilon)}} \phi_i(y)\) for \(\varepsilon > 0\). It is obvious that \(f_\varepsilon \in L^p\). In the argument to follow, as is frequently the case with asymptotic arguments, we will need to keep careful track of the dependence of the constants on \(\varepsilon\). On applying the previous estimate for \(Rf\) to the function \(f_\varepsilon\),

(53)

\[
Rf_\varepsilon(x) \gtrsim a(x) \left( \int_M d(x^0_i,y)^{2M-n_i(1+\frac{1+\varepsilon}{p})} \phi_i(y) \int_M d(x^0_i,z)^{2M-n_i(1+\frac{1+\varepsilon}{p})} \mathbb{1}_{d(x^0_i,y) \leq d(x^0_i,z)} \phi_i(z)f(z)dz dy \right)^{\frac{1}{2}}.
\]
Evidently

\[
\int_{\mathcal{M}} d(x^0_i, z)^{2 - 2M - n_i (1 + \frac{(1 + \varepsilon)}{p})} 1_{d(x^0_i, y) \leq d(x^0_i, z)} \phi_i(z) \, dz \gtrsim \int_{d(x^0_i, y)}^{\infty} r^{1 - 2M - \frac{n_i (1 + \varepsilon)}{p}} \, dr \\
= \frac{1}{\frac{n_i (1 + \varepsilon)}{p} + 2M - 2} d(x^0_i, y)^{2 - 2M - \frac{n_i (1 + \varepsilon)}{p}} \\
\gtrsim d(x^0_i, y)^{2 - 2M - \frac{n_i (1 + \varepsilon)}{p}},
\]

where the implicit constant in the final line is independent of \(\varepsilon\) since for \(\varepsilon\) small \((1 + \varepsilon)^{-1} \gtrsim 1\). On applying this to (53),

\[
R_{f_\varepsilon}(x) \gtrsim a(x) \left( \int_{\mathcal{M}} d(x^0_i, y)^{2 - n_i (1 + \frac{2(1 + \varepsilon)}{p})} \phi_i(y) \, dy \right)^{\frac{1}{2}} \\
\approx a(x) \left( \int_D r^{1 - 2n_i \frac{(1 + \varepsilon)}{p}} \, dr \right)^{\frac{1}{2}},
\]

for some constant \(D > 0\) determined by the distance of \(\text{supp} \phi_i\) to \(x^0_i\). This will blow up everywhere on the non-zero support of \(a\) when \(p > (1 + \varepsilon)n_i\), thereby forcing the norm \(\|R_{f_\varepsilon}\|_p\) to become infinite. Since \(\varepsilon\) was arbitrary, this implies that \(R\) is an unbounded operator for any \(p > n_i\).

For the case \(p = n_i\), we must carefully consider the asymptotics of \(R_{f_\varepsilon}\) as \(\varepsilon \to 0\). From the previous estimate,

\[
R_{f_\varepsilon}(x) \gtrsim a(x) \left( \int_D r^{1 - 2(1 + \varepsilon)} \, dr \right)^{\frac{1}{2}} \\
\gtrsim a(x) \frac{1}{\varepsilon} D^{-2\varepsilon}. \tag{54}
\]

Also,

\[
\|f_\varepsilon\|_p = \left( \int_{\mathcal{M}} d(x^0_i, y)^{-n_i (1 + \varepsilon)} \phi_i(y) \, dy \right)^{\frac{1}{p}} \\
\lesssim \left( \int_D r^{-1 - n_i \varepsilon} \, dr \right)^{\frac{1}{p}} \\
\approx \frac{1}{\varepsilon^p} D^{-\varepsilon}. \tag{55}
\]

Suppose that \(\|Rf\|_{n_i} \lesssim \|f\|_{n_i}\). Then, from (54) and (55), we would have

\[
\frac{1}{\varepsilon} D^{-2\varepsilon} \lesssim \frac{1}{\varepsilon^p} D^{-\varepsilon} \quad \Leftrightarrow \quad \varepsilon \lesssim \varepsilon^p D^{\varepsilon p},
\]

which is clearly not true asymptotically as \(\varepsilon \to 0\). A contradiction has been reached and we can safely conclude that \(R\) is also unbounded for \(p = n_i\). \(\Box\)
8. The Horizontal Square Function

In this section, Theorem 1.2 will be proved. In an analogous manner to the argument from Section 4 for the operator $S$, $s$ can be expressed as

$$sf(x) = \left( \int_0^\infty |t^2 \Delta (I + t^2 \Delta)^{-M} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$= \left( \int_0^\infty \left| \Delta \left( \frac{1}{t^2} + \Delta \right)^{-M} f(x) \right|^2 t^{3-4M} dt \right)^{\frac{1}{2}}$$

$$= \left( \int_0^\infty \left| \Delta (k^2 + \Delta)^{-M} f(x) \right|^2 k^{4M-5} dk \right)^{\frac{1}{2}}.$$

This can be controlled from above by the sum of the high and low energy components,

$$s_{<}f(x) := \left( \int_0^1 \left| \Delta (k^2 + \Delta)^{-M} f(x) \right|^2 k^{4M-5} dk \right)^{\frac{1}{2}},$$

and

$$s_{>}f(x) := \left( \int_1^\infty \left| \Delta (k^2 + \Delta)^{-M} f(x) \right|^2 k^{4M-5} dk \right)^{\frac{1}{2}}.$$

The proof of the high energy part $s_{>}$ is quite similar to our discussion of the vertical square function. In fact it is simpler argument because the kernel of the operator $\Delta (k^2 + \Delta)^{-M}$ is symmetric. Hence it suffices to prove only (43) and (44) automatically follows. Thus one does not have to use the notion of Hodge Laplacian this time. Otherwise the proof is unchanged and we skip it. Here we discuss only details of the low energy component.

8.1. Low Energy. As a first step to proving the boundedness of the low energy square function, observe that

$$s_{<}f(x) = \left( \int_0^1 \left| \left[ (k^2 + \Delta)^{-(M-1)} - k^2 (k^2 + \Delta)^{-M} \right] f(x) \right|^2 k^{4M-5} dk \right)^{\frac{1}{2}}.$$

From the formula

$$(k^2 + \Delta)^{-j} = \sum_{i=1}^{4} H_i^{(j)}(k)$$

we then obtain

$$s_{<}f(x) \lesssim \sum_{i=1}^{4} s_{<}^i f(x),$$

where

$$s_{<}^i f(x) := \left( \int_0^\infty \left| H_i^{(M-1)}(k) - k^2 H_i^{(M)}(k) \right| f(x) \right|^2 k^{4M-5} dk \right)^{\frac{1}{2}}.$$

The $L^p$-boundedness and weak-type $(1,1)$ property of each of these operators will be proved separately.
8.1.1. The Operator $s_{<}^1$. Notice that
\[
H_1^{(M-1)}(k)(x, y) - k^2 H_1^{(M)}(k)(x, y) = \sum_{i=1}^{l} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{(M-1)}(x, y)\phi_i(x)\phi_i(y) \\
- k^2 (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x, y)\phi_i(x)\phi_i(y)
\]
\[
= \sum_{i=1}^{l} \Delta_{\mathbb{R}^n_i \times M_i} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x, y)\phi_i(x)\phi_i(y).
\]

As this kernel only consists of finitely many terms, in order to prove that $s_{<}^1$ is bounded on $L^p$ and weak-type $(1, 1)$ it is sufficient to prove for any $1 \leq i \leq l$ that the operator
\[
s_{<}^1 := \left( \int_0^1 \left| \int_{M} \Delta_{\mathbb{R}^n_i \times M_i} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(x, y)\phi_i(x)\phi_i(y)f(y) \, dy \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}}
\]
is bounded on $L^p$ and weak-type $(1, 1)$. Recall from classical theory that the operator
\[
\left( \int_0^{\infty} \left| \Delta_{\mathbb{R}^n_i \times M_i} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}g(x) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}}
\]
is bounded on $L^p(\mathbb{R}^n_i \times M_i)$ for all $p \in (1, \infty)$ and weak-type $(1, 1)$. Therefore,
\[
\left\| s_{<}^1 f \right\|_p = \left\| \phi_i \cdot \left( \int_0^1 \left| \Delta_{\mathbb{R}^n_i \times M_i} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(\phi_i \cdot f) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}}_p
\]
\[
\lesssim \left\| \left( \int_0^{\infty} \left| \Delta_{\mathbb{R}^n_i \times M_i} (\Delta_{\mathbb{R}^n_i \times M_i} + k^2)^{-M}(\phi_i \cdot f) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}}_p
\]
\[
\lesssim \left\| f \right\|_p
\]
for any $p \in (1, \infty)$. Weak-type $(1, 1)$ bounds follow in the same manner.

8.1.2. The Operator $s_{<}^2$. The operator family $\left\{ H_2^{(M-1)}(k) - k^2 H_2^{(M)}(k) \right\}_{k \in (0,1)}$ constitutes a family of pseudodifferential operators of order $2 - 2M$. In a similar manner to Section 4.2, standard pseudodifferential operator theory can be applied to yield the $L^p$-boundedness of $s_{<}^2$ for any $p \in [1, \infty]$.

8.1.3. The Operator $s_{<}^3$. From an application of the triangle inequality followed by Minkowski’s integral inequality, the operator $s_{<}^3$ can be estimated from above by
\[
s_{<}^3 f(x) \lesssim W_1 f(x) + W_2 f(x) = \sum_{i=1,2} \int_{M} w_i(x, y) |f(y)| \, dy,
\]
where $W_1$ and $W_2$ are operators with the respective kernels
\[
w_1(x, y) := \left( \int_0^1 \left| H_3^{(M-1)}(k)(x, y) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}}
\]
and
\[
w_2(x, y) := \left( \int_0^1 \left| H_3^{(M)}(k)(x, y) \right|^2 k^{4M-1} \, dk \right)^{\frac{1}{2}}.
\]
The $L^p$-boundedness and weak-type $(1,1)$ property of the operators $W_1$ and $W_2$ will be proved separately.

**Lemma 8.1.** The operator $W_1$ is bounded on $L^p$ for all $p \in (1, \infty)$ and weak-type $(1,1)$.

**Proof.** As in Section 4.3, in order to obtain the boundedness of $W_1$ on $L^p$ for $1 < p < \infty$ it suffices to demonstrate that the kernel satisfies the estimates (31), (32), (33) and (34).

For the condition (31), observe that Proposition 3.4 implies $|H_3^{(M-1)}(k)(x, y)| \lesssim k^{4-2M}$ for all $x, y \in \mathcal{M}$. This estimate, when applied to the definition of $w_1(x, y)$, gives

\[
w_1(x, y) \lesssim 1 \quad \forall \ x, y \in \mathcal{M},
\]

which immediately implies the validity of (31).

For (32), Proposition 3.4 implies that for $x \in \mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i$ and $y \in K$,

\[
|H_3^{(M-1)}(k)(x, y)| \lesssim k^{4-2M} \langle d(x_i^0, x) \rangle^{2-n_i} \exp(-ckd(x_i^0, x)).
\]

Therefore,

\[
w_1(x, y) = \left( \int_0^1 \left| H_3^{(M-1)}(k)(x, y) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}} \lesssim \langle d(x_i^0, x) \rangle^{2-n_i} \left( \int_0^1 k^3 \exp(-2ckd(x_i^0, x)) \, dk \right)^{\frac{1}{2}} \lesssim \langle d(x_i^0, x) \rangle^{-n_i}.
\]

This will clearly imply (32) when $p > 1$.

Let us next consider (33). For $x \in K$ and $y \in \mathbb{R}^{n_j} \times \mathcal{M}_j \setminus K_j$, Proposition 3.4 implies that

\[
|H_3(k)(x, y)| \lesssim k^{4-2M} \langle d(x_j^0, y) \rangle^{2-n_j} \exp(-ckd(x_j^0, y)).
\]

Therefore,

\[
w_1(x, y) = \left( \int_0^1 \left| H_3^{(M-1)}(k)(x, y) \right|^2 k^{4M-5} \, dk \right)^{\frac{1}{2}} \lesssim \langle d(x_j^0, y) \rangle^{2-n_j} \left( \int_0^1 k^3 \exp(-2ckd(x_i^0, x)) \, dk \right)^{\frac{1}{2}} \lesssim \langle d(x_j^0, y) \rangle^{-n_j}.
\]

It is easy to see that this implies (33) provided $p < \infty$.

Finally, it remains to validate the estimate (34). Proposition 3.4 once again implies that for $x \in \mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i$ and $y \in \mathbb{R}^{n_j} \times \mathcal{M}_j \setminus K_j$,

\[
w_1(x, y) \lesssim \langle d(x_i^0, x) \rangle^{2-n_i} \langle d(x_j^0, y) \rangle^{2-n_j} \left( \int_0^1 k^3 \exp(-2ckd(x_i^0, x) + d(x_j^0, y)) \, dk \right)^{\frac{1}{2}} \lesssim \langle d(x_i^0, x) \rangle^{2-n_i} \langle d(x_j^0, y) \rangle^{2-n_j} \left( \frac{1}{(d(x_i^0, x) + d(x_j^0, y))^4} \right)^{\frac{1}{2}} \leq \min \left( \langle d(x_i^0, x) \rangle^{-n_i} \langle d(x_j^0, y) \rangle^{-n_j}, \langle d(x_i^0, x) \rangle^{2-n_i} \langle d(x_j^0, y) \rangle^{-n_j} \right).
\]
Applying this to (34),

\[
\int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \left( \int_{\mathbb{R}^{n_j} \times \mathcal{M}_j \setminus K_j} w_1(x, y)^{p'} \, dy \right)^{\frac{1}{p'}} \, dx
\]

\[
\lesssim \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \left( \int_{D_j^1(x)} (d(x^0_i, x))^{(2-n_j)p'} (d(x^0_j, y))^{-n_jp'} \, dy \right)^{\frac{1}{p'}} \, dx
\]

\[
+ \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \left( \int_{D_j^2(x)} (d(x^0_i, x))^{-n_jp'} (d(x^0_j, y))^{(2-n_j)p'} \, dy \right)^{\frac{1}{p'}} \, dx
\]

\[=: J_1 + J_2,
\]

where \(D_j^1(x)\) and \(D_j^2(x)\) are as defined in Section 4.3. For the first term,

\[
J_1 = \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \frac{1}{(d(x^0_i, x))^{(n_i-2)p}} \left( \int_{D_j^1(x)} \frac{dy}{(d(x^0_j, y))^{n_jp'}} \right)^{\frac{1}{p'}} \, dx.
\]

For the interior integral, in an analogous manner to (38),

\[
\int_{D_j^1(x)} \frac{dy}{(d(x^0_j, y))^{n_jp'}} \lesssim \int_{\mathbb{R}^{n_j}} \frac{dy_1}{(|y_1 - x^0_j| + d(x^0_i, x))^{n_jp'}}.
\]

This will be integrable since \(p' > 1\), in which case

\[
\int_{D_j^1(x)} \frac{dy}{(d(x^0_j, y))^{n_jp'}} \lesssim \int_{d(x^0_i, x)}^{\infty} \frac{r^{n_j-1}}{r^{n_jp'}} \, dr
\]

\[
\approx \left[ -r^{n_j(1-p')} \right]_{d(x^0_i, x)}^{\infty}
\]

\[= (d(x^0_i, x))^{n_j(1-p')}.
\]

Applying this estimate to \(J_1\) gives

\[
J_1 \lesssim \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \frac{1}{(d(x^0_i, x))^{(n_i-2)p}} \cdot (d(x^0_i, x))^{-n_j} \, dx
\]

\[= \int_{\mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i} \frac{dx}{(d(x^0_i, x))^{(n_i-2)p+n_j}}.
\]

This will be finite provided that

\[
(n_i - 2)p + n_j > n_i \iff p > \frac{n_i - n_j}{n_i - 2}.
\]

Since \(\frac{n_i - n_j}{n_i - 2} < \frac{n_i - 2}{n_i - 2} = 1\), this will be satisfied when \(p > 1\). It has therefore been proved that \(J_1\) is finite for \(1 < p < \infty\).

It remains to consider the term \(J_2\). Looking back to our proof of the boundedness of the square function \(S\), the term \(J_2\) is identical to the term \(I_2\) from Section 4.3. In Section 4.3 it was proved that \(I_2\) is finite for all \(p \in (1, \infty)\). Thus it can be safely concluded that \(W_1\) is bounded on \(L^p\) for all \(p \in (1, \infty)\).

Let us now prove that \(W_1\) is weak-type \((1, 1)\). On combining (56), (57) and (58), it is evident that

\[
\sup_{y \in \mathcal{M}} |w_1(x, y)| \lesssim \left\{ \begin{array}{ll}
(d(x^0_i, x))^{-n_i} & \text{for } x \in \mathbb{R}^{n_i} \times \mathcal{M}_i \setminus K_i, 1 \leq i \leq l,
1 & \text{for } x \in K.
\end{array} \right.
\]
This implies that for \( f \in C_c^\infty \),
\[
|W_1 f(x)| \leq \int_M w_1(x, y) |f(y)| \, dy
\]
\[
\leq \sup_{y \in M} |w_1(x, y)| \int_M |f(y)| \, dy
\]
\[
\lesssim \left\{ \begin{array}{ll}
\langle d(x_i, x) \rangle^{-n_i} \|f\|_{L^1} & \text{for } x \in \mathbb{R}^{n_i} \setminus \mathcal{M}_i \setminus K_i, 1 \leq i \leq l, \\
\|f\|_{L^1} & \text{for } x \in K.
\end{array} \right.
\]

This function is clearly in \( L^{1,\infty} \) and thus \( W_1 \) is weak-type \((1, 1)\). \( \square \)

It remains to bound the operator \( W_2 \). This follows trivially once one notices that \( k^2 H_3^{(M)}(k)(x, y) \) will satisfy the same asymptotic estimates as \( H_3^{(M-1)}(k)(x, y) \). Indeed, from Proposition 3.4,
\[
k^2 H_3^{(M)}(k)(x, y) \lesssim k^{4-2M} \omega_2(x, k) \omega_2(y, k).
\]
The argument used for the operator \( W_1 \) is therefore equally applicable to \( W_2 \), and thus \( W_2 \) is \( L^p \)-bounded for \( 1 < p < \infty \) and weak-type \((1, 1)\). It can then be concluded that \( s_3^\infty \) is bounded on \( L^p \) for all \( p \in (1, \infty) \) and weak-type \((1, 1)\).

8.1.4. The Operator \( s_3^\infty \). The previous proof of the \( L^p \)-boundedness and weak-type \((1, 1)\) property of \( s_3^\infty \) was derived entirely from the pointwise estimates for \( H_3^{(M-1)}(k)(x, y) \) and \( H_3^{(M)}(k)(x, y) \) provided by Proposition 3.4. Since the kernels \( H_4^{(M-1)}(k)(x, y) \) and \( H_4^{(M)}(k)(x, y) \) satisfy even stronger size estimates, provided by Proposition 3.5, it follows that \( s_3^\infty \) must also be bounded on \( L^p \) for all \( p \in (1, \infty) \) and weak-type \((1, 1)\).

8.2. The Reverse Inequality. The proof of the reverse inequality for \( s \), \( \|f\|_p \lesssim \|sf\|_p \) for \( p \in (1, \infty) \), follows in an essentially identical manner to our proof of the reverse inequality for \( S \) from Section 6. The only two notable differences that one will encounter is that the proof must begin with the resolution of the identity
\[
f = c_M \int_0^\infty (t\Delta)^2(1 + t\Delta)^{-2M} f \frac{dt}{t},
\]
and subsequently the identity
\[
\int_M (t\Delta)^2(1 + t\Delta)^{-2M} f(x) \cdot g(x) \, dx = \int_M t\Delta(1 + t\Delta)^{-M} f(x) \cdot t\Delta(1 + t\Delta)g(x) \, dx
\]
must be utilised.

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