SURFACES IN $S^4$ WITH NORMAL HARMONIC GAUSS MAP

EDUARDO HULETT

ABSTRACT. We consider conformal immersions of Riemann surfaces in $S^4$ and study their Gauss maps with values in the Grassmann bundle $\mathcal{F} = SO_5/T^2 \to S^4$. The energy of maps from Riemann surfaces into $\mathcal{F}$ is considered with respect to the normal metric on the target and immersions with harmonic Gauss maps are characterized. We also show that the normal-harmonic map equation for Gauss maps is a completely integrable system, thus giving a partial answer of a question posed by Y. Ohnita in [11]. Associated $S^1$-families of parallel mean curvature immersions in $S^4$ are considered. A lower bound of the normal energy of Gauss maps is obtained in terms of the genus of the surface.

1. Introduction

A conformally immersed Riemann surface $f : \Sigma \to S^4$ has a well defined Gauss map which assigns to a point $p \in \Sigma$ the oriented 2-plane $df(T_p\Sigma)$ viewed as a point in the Grassmann bundle $G_2(TS^4)$ of oriented 2-planes in $TS^4$, which is a homogeneous manifold diffeomorphic to the flag manifold $\mathcal{F} = SO_5/SO_2 \times SO_2$. An early result of Eells and Salamon [6] asserts that a smooth map $\phi : \Sigma \to \mathcal{F}$ of a Riemann surface is primitive if and only if it is the Gauss map of a minimal (i.e. conformal harmonic) immersion $\pi \circ \phi : \Sigma \to S^4$, where $\pi : \mathcal{F} \to S^4$ is the homogeneous projection. Primitive maps from Riemann surfaces in $\mathcal{F}$ are holomorphic maps with respect to the horizontal $F$-structure determined by the canonical 4th-order Cartan inner authomorphism of the Lie algebra $\mathfrak{so}_5(\mathbb{C})$ [3]. A remarkable property of primitive maps is their equi-harmonicity i.e. harmonic with respect to every invariant metric on the target $\mathcal{F}$ [1], [2], [3].

A related question is to study the geometry of immersions of orientable surfaces in $S^4$ for which their Gauss maps satisfy the harmonic map equation with respect to a specific $SO_5$-invariant metric on $\mathcal{F}$. In this article we consider conformal immersions $f : \Sigma \to S^4$ of Riemann surfaces whose Gauss maps are normal-harmonic or harmonic with respect to the so-called normal metric on the target $\mathcal{F}$. Since naturally reductive homogeneous spaces may be considered as generalizations of Riemannian symmetric spaces, the choice of the normal metric on $\mathcal{F}$ seems to be very natural.

Partially supported by ANPCyT, CONICET and SECYT-UNC, Argentina.
There are however other geometrically interesting possible choices of invariant metrics on $F$ which are considered in [10].

The first main result of the paper is Theorem 5.1 which is a generalization of the well-known Theorem of Ruh-Vilms [13]. It characterizes immersed surfaces in $S^4$ with normal-harmonic Gauss maps. Although this result is quite natural and elementary, to the best of this author’s knowledge it has not been reported before.

Theorem 6.2, the second main result in the paper, establishes the complete integrability of the normal-harmonic map equation for Gauss maps. In other words it asserts that the normal-harmonic map equation for Gauss maps $\Sigma \to F$ can be encoded in a loop of flat connections $S^1 \ni \lambda \mapsto d + \alpha_\lambda$ on the trivial principal $SO_5$ bundle over $\Sigma$, a manifestation of complete integrability. Its proof is consequence of identity (59) in Lemma 6.1 which is a special property of the Gauss map. This fact provides a partial answer to the following question posed by Y. Ohnita [11]: Are there examples of harmonic maps, other than super-horizontal and primitive maps from a Riemann surface $\Sigma$ to a $k$-symmetric ($k > 2$) manifold $G/K$ which satisfy condition of Lemma 6.1?

The paper is organized as follows. In section 2 we derive the structure equations of isometric immersions of orientable surfaces in $S^4$. As is well known, any such immersion determines a conformal or Riemann surface structure on an orientable surface for which the immersion results conformal. In section 3 we give a detailed construction of the Gauss map and some elementary facts on the geometry of the flag manifold $F \equiv SO_5/SO_2 \times SO_2$. We describe also the Maurer-Cartan one form or moment map $\beta$ of $F$ following [4]. In section 4 we closely follow [1] and [4] to derive the harmonic map equation for smooth maps into $F$ when the normal metric has been fixed, we call these maps normal-harmonic. When maps from Riemann surfaces are considered, the normal-harmonic map equation takes a very simple form. In section 5 we derive an explicit formula of the tension of the Gauss map of a conformal immersion in terms of the mean curvature vector of the immersion. A direct consequence is the proof of Theorem 5.1. Also we obtain Lemma 6.1 which establishes property (59), a distinctive algebraic-geometric property of Gauss maps. As a consequence of we show that the normal-harmonic map equation for Gauss maps is a completely integrable system. We use this information to describe the $S^1$-loop of conformal immersions of a simply connected Riemann surface determined by a normal-harmonic Gauss map. The last section deals with a computation of the normal energy of Gauss maps. We obtain a formula relating the normal energy...
of the Gauss map and the Willmore energy of the corresponding immersion. This allows to obtain a lower bound for the normal energy of Gauss maps which depends on the genus of the immersed Riemann surface.

2. Structure equations

On $\mathbb{R}^5$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$ the euclidean metric

$$\langle ., . \rangle = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2$$

induces the usual canonical Riemannian metric $\langle ., . \rangle$ with constant curvature one on the unit sphere $S^4 = \{ x \in \mathbb{R}^5 : \langle x, x \rangle = 1 \}$. The matrix Lie group $SO_5 = \{ A \in Gl_5(\mathbb{R}) : A^t A = I, detA = 1 \}$ acts transitively on $S^4$ by isometries.

An immersion $f : \Sigma \to S^4$ of a Riemann surface is conformal if $\langle f_z, f_z \rangle^c = 0$, for every local complex coordinate $z = x + iy$ on $M$, where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

are the complex partial derivatives and $\langle ., . \rangle^c$ is the complex bilinear extension of the riemannian metric to $\mathbb{C}^5$:

$$\langle z, w \rangle^c = z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 w_4 + z_5 w_5.$$ 

Thus $f$ is conformal if and only if on any local complex coordinate $z = x + iy$ it satisfies

(1) \hspace{1cm} $\langle f_x, f_y \rangle = 0, \quad \| f_x \|^2 = \| f_y \|^2.$

We fix on $\Sigma$ the induced Riemannian metric $g = f^* \langle ., . \rangle$ so that $f : (M, g) \to S^4$ is an isometric immersion. The 2nd Fundamental form of the surface $f : M \to S^4$ is defined by $II = d^2 f$ i.e.

$$II = -\langle df, dN_1 \rangle N_1 - \langle df, dN_2 \rangle N_2,$$

where $\{ N_1, N_2 \}$ is a local orthonormal frame. On any local chart $(U, z = x + iy)$ of $M$ we introduce a conformal parameter $u$ defined by $\langle f_z, f_z \rangle = e^{2u}$, so that $g|_U = 2e^{2u}(dx^2 + dy^2)$. The mean curvature vector of $f$ is defined by $H = \frac{1}{2} \text{trace}II$, which in terms of $f$ and $u$ is given by

$$H = e^{-2u}f_{\bar{z} z}^\perp.$$

Defining $h_i = \langle H, N_i \rangle = e^{-2u}\langle f_{\bar{z} z}, N_i \rangle$ we decompose $H = h_1 N_1 + h_2 N_2$.

Since $f$ is conformal

$$2\langle f_{\bar{z} z}, f_z \rangle^c = \frac{\partial}{\partial z} \langle f_z, f_z \rangle^c = 0, \quad 2\langle f_{\bar{z} z}, f_{\bar{z}} \rangle^c = \frac{\partial}{\partial \bar{z}} \langle f_{\bar{z}}, f_{\bar{z}} \rangle^c = 0,$$
thus \( f_{\bar{z}z} \) has no tangential component and so

\[
f_{\bar{z}z} = -e^{2u}f + e^{2u}H.
\]

Defining \( \xi_i := \langle f_{\bar{z}z}, N_i \rangle^c \), we decompose

\[
f_{\bar{z}z} = 2u_z f_z + \xi_1 N_1 + \xi_2 N_2.
\]

From this equation we obtain

\[
f_{z}^{\perp} = \frac{1}{4} \left[ II(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) - II(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) \right] - \frac{i}{2} II(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = II(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}).
\]

On the other hand from

\[
0 = \frac{\partial}{\partial z} \langle N_1, f_z \rangle = \langle \frac{\partial}{\partial z} N_1, f_z \rangle + \langle N_1, f_{\bar{z}z} \rangle,
\]

we obtain

\[
\langle \frac{\partial}{\partial z} N_1, f_z \rangle = -e^{2u}h_1.
\]

In an analogous way

\[
\langle \frac{\partial}{\partial z} N_2, f_z \rangle = -e^{2u}h_2,
\]

we obtain

\[
\langle \frac{\partial}{\partial z} N_2, f_z \rangle = -\langle N_2, f_{\bar{z}z} \rangle^c = -\xi_2 N_2.
\]

Defining \( \sigma := \langle \frac{\partial}{\partial z} N_2, N_1 \rangle \), we obtain the complex derivative of the normal fields

\[
\frac{\partial}{\partial z} N_1 = -h_1 f_z - e^{-2u} \xi_1 f_{\bar{z}} - \sigma N_2.
\]

\[
\frac{\partial}{\partial z} N_2 = -h_2 f_z - e^{-2u} \xi_2 f_{\bar{z}} + \sigma N_1.
\]

These equations together are the structure equations of the conformal immersion

\[
f : \Sigma \to S^4,
\]
The integrability condition of these equations are the Gauss, Codazzi and Ricci equations respectively,

\[
\begin{align*}
(G) & \quad 2u_{\bar{z}z} = e^{-2u}(|\xi_1|^2 + |\xi_2|^2) - e^{2u}(1 + \|H\|^2), \\
(C1) & \quad e^{2u}(\frac{\partial}{\partial \bar{z}}h_1 + h_2\sigma) = \frac{\partial}{\partial \bar{z}}\xi_1 + \xi_2\bar{\sigma}, \\
(C2) & \quad e^{2u}(\frac{\partial}{\partial \bar{z}}h_2 - h_1\sigma) = \frac{\partial}{\partial \bar{z}}\xi_2 - \xi_1\bar{\sigma}, \\
(R) & \quad -Im(\sigma) = e^{-2u}Im(\xi_1\bar{\xi}_2).
\end{align*}
\]

Let \(\nabla^\perp\) be the connection on the normal bundle \(\nu\) of the immersed surface. Then from the compatibility equations we obtain

\[
\nabla^\perp_\sigma H = (\frac{\partial}{\partial \bar{z}}h_1 + h_2\sigma)N_1 + (\frac{\partial}{\partial \bar{z}}h_2 - h_1\sigma)N_2.
\]

The Codazzi equations (C1) and (C2) in (9) may be expressed also in the following form

\[
\nabla^\perp_\bar{z} H = e^{-2u}(\frac{\partial}{\partial z}\xi_1 + \xi_2\bar{\sigma})N_1 + e^{-2u}(\frac{\partial}{\partial \bar{z}}\xi_2 - \xi_1\bar{\sigma})N_2
\]

The induced metric \(g = f^*(\langle , \rangle)\) on \(\Sigma\) is given in terms of the conformal parameter \(u\) by \(g = 2e^{2u}dz \otimes d\bar{z}\). The Gaussian curvature of the surface \((\Sigma, g)\) is just the curvature of the induced metric which is given by

\[
K = -\Delta_g u = -2e^{-2u}u_{\bar{z}z},
\]
where \(\Delta_g = 2e^{-2u}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\) is the Laplace operator on \(\Sigma\) determined by \(g\). Gauss equation (G) in (9) becomes

\[
K = (1 + \|H\|^2) - e^{-4u}(|\xi_1|^2 + |\xi_2|^2),
\]

Let \(\omega = \langle \nabla^\perp N_2, N_1 \rangle\) be the connection one form of the normal bundle \(\nu\). Then the normal curvature is defined by \(d\omega = K^\perp dA\), where \(dA\) is the area form of the induced metric \(g\). Thus in terms of \(\sigma\) we obtain \(\omega = 2Re(\sigma dz)\) and so

\[
d\omega = -4Im(\sigma)dx \wedge dy.
\]

Hence since \(d\omega = K^\perp dA_g\), and \(dA_g = 2e^{2u}dx \wedge dy\), the normal curvature function is given by

\[
K^\perp = -e^{-2u}Im(\sigma).
\]

Let \(F : \Sigma \rightarrow SO_5\) be a (local) adapted frame of \(f\) i.e.

\[
f(x) = F_0(x), \quad df(T_x M) = \text{span}\{F_1(x), F_2(x)\}, x \in M,
\]
where \( F_i, 0 \leq i \leq 4 \) are the columns of the orthogonal matrix \( F \). By a gauge transformation (rotating within the complex line generated by \( F_1 - iF_2 \)) we can assume that

\[
\frac{\partial}{\partial z} F = \frac{e^u}{\sqrt{2}} (F_1 - iF_2).
\]

Using (15) and routine computation we compute the complex derivative \( \frac{\partial}{\partial z} F \) of the frame \( F \) in terms of the frame itself.

\[
\begin{align*}
\frac{\partial}{\partial z} F_0 &= \frac{e^u}{\sqrt{2}} F_1 - i \frac{e^u}{\sqrt{2}} F_2, \\
\frac{\partial}{\partial z} F_1 &= -\frac{e^u}{\sqrt{2}} f - iu_z F_2 + (\frac{e^{-u}\xi_1 + e^u h_1}{\sqrt{2}}) N_1 + (\frac{e^{-u}\xi_2 + e^u h_2}{\sqrt{2}}) N_2, \\
\frac{\partial}{\partial z} F_2 &= i \frac{e^u}{\sqrt{2}} f + iu_z F_1 + i(\frac{e^{-u}\xi_1 - e^u h_1}{\sqrt{2}}) N_1 + i(\frac{e^{-u}\xi_2 - e^u h_2}{\sqrt{2}}) N_2, \\
\frac{\partial}{\partial z} N_1 &= -(\frac{e^{-u}\xi_1 + e^u h_1}{\sqrt{2}}) F_1 - i(\frac{e^{-u}\xi_1 - e^u h_1}{\sqrt{2}}) F_2 - \sigma N_2, \\
\frac{\partial}{\partial z} N_2 &= -(\frac{e^{-u}\xi_2 + e^u h_2}{\sqrt{2}}) F_1 - i(\frac{e^{-u}\xi_2 - e^u h_2}{\sqrt{2}}) F_2 + \sigma N_1.
\end{align*}
\]

Set

\[
a_i = \frac{e^{-u}\xi_i + e^u h_i}{\sqrt{2}}, \quad b_i = \frac{e^{-u}\xi_i - e^u h_i}{\sqrt{2}}, \quad i = 1, 2
\]

System (16) can be written in matrix form as \( F_z = FA \), where \( F = (F_1, F_2, F_3, F_4, F_5) \) in column notation and

\[
A = \begin{pmatrix}
0 & -\frac{e^u}{\sqrt{2}} & \frac{i e^u}{\sqrt{2}} & 0 & 0 \\
\frac{e^u}{\sqrt{2}} & 0 & iu_z & -a_1 & -a_2 \\
-\frac{i e^u}{\sqrt{2}} & -iu_z & 0 & -ib_1 & -ib_2 \\
0 & a_1 & ib_1 & 0 & \sigma \\
0 & a_2 & ib_2 & -\sigma & 0
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 & -\frac{e^u}{\sqrt{2}} & -i \frac{e^u}{\sqrt{2}} & 0 & 0 \\
\frac{e^u}{\sqrt{2}} & 0 & -iu_z & -\bar{a}_1 & -\bar{a}_2 \\
\frac{i e^u}{\sqrt{2}} & iu_z & 0 & \bar{b}_1 & \bar{b}_2 \\
0 & \bar{a}_1 & -\bar{b}_1 & 0 & \bar{\sigma} \\
0 & \bar{a}_2 & -\bar{b}_2 & -\bar{\sigma} & 0
\end{pmatrix}
\]
Since \( \bar{A} = B \) we get \( \bar{F}_z = FB \), so that in terms of matrices \( A, B \) the integrability condition \( F_{z\bar{z}} = F_{z\bar{z}} \), is equivalent to

\[
A_z - B_z = [A, B],
\]

which encodes the equations of Gauss, Codazzi and Ricci of the immersion given before. Let \( \Theta \) denote the left Maurer-Cartan form of the group \( SO_5 \), and consider the pullback \( \alpha = F^* \Theta = F^{-1} dF = Adz + Bd\bar{z} \). Then equation (20) is equivalent to

\[
d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.
\]

### 3. The Gauss map

A smooth map \( f : \Sigma \to S^4 \) determines a map \( \varphi : \Sigma \to \mathbb{CP}^4 \) defined by \( \varphi(x) = [f(x)] \). Let \( L \to \mathbb{CP}^4 \) be the tautological line bundle whose fiber over \( \ell \in \mathbb{CP}^5 \) is the complex line \( \ell \) itself. Then \( \varphi \) (and so \( f \)) determines a complex line subbundle \( \ell_0 \subset \Sigma \times \mathbb{C}^5 \) given by

\[
\varphi^*(L) = \ell_0 = \{(x, v) \in \Sigma \times \mathbb{C}^5 : v \in \mathbb{C}f(x)\}
\]

Any smooth subbundle \( E \subset \Sigma \times \mathbb{C}^5 \) can be equipped with a holomorphic structure for which a local section \( s \in \Gamma(E) \) is holomorphic if and only if \( \pi_E(\frac{\partial}{\partial z}s) = 0 \), where \( \pi_E \) is the orthogonal projection onto \( E \) and \( z \) is any local complex coordinate on \( \Sigma \). Now \( f \) is harmonic if and only if \( \varphi \) is harmonic since \( \varphi = \iota \circ f \), where \( \iota \) denotes the totally geodesic embedding \( S^4 \hookrightarrow \mathbb{CP}^4 \). In turn \( \varphi \) is harmonic if and only if the map

\[
d\varphi(\frac{\partial}{\partial z}) : \ell_0 \to \ell_0^\perp, \quad s \mapsto \pi_{\ell_0}^\perp(\frac{\partial}{\partial z}s)
\]

is holomorphic i.e. it sends holomorphic sections of \( \ell_0 \) to holomorphic sections of \( \ell_0^\perp \). Since \( \langle f_z, f_z \rangle = 0 \), \( f \) is a global holomorphic section of \( \ell_0 \) and \( \pi_{\ell_0}^\perp(\frac{\partial}{\partial z}f) = f_z \). Now if \( f \) is conformal, \( f_z \) is a section of \( \ell_0^\perp \) satisfying \( \pi_{\ell_0}(\frac{\partial}{\partial z}f_z) = e^{2u}H \) by equation (2). In particular \( \varphi \) (and hence \( f \)) is harmonic iif \( H = 0 \).

Define \( \ell_1 := \pi_{\ell_0}^\perp(\frac{\partial}{\partial z}\ell_0) \subset \ell_0^\perp \subset \Sigma \times \mathbb{C}^5 \). Then \( f \) is conformal if and only if \( \ell_1 \) is an isotropic subbundle. A conformal map \( f \) is called isotropic if and only if the complex line subbundle \( \ell_2 := \pi_{\ell_1}^\perp(\frac{\partial}{\partial z}\ell_1) \subset \Sigma \times \mathbb{C}^5 \) is isotropic. This is clearly equivalent to the orthogonality of \( \ell_2 \) and \( \ell_2 \). It is easily seen that a conformal map \( f \) is isotropic if and only if \( \langle f_{zz}, f_{zz} \rangle^c \equiv 0 \) on any complex coordinate \( z \). On the other hand a conformal map \( f \) is called superconformal if \( \langle f_{zz}, f_{zz} \rangle^c \neq 0 \).

If \( f \) is isotropic then \( \ell_0, \ell_1, \ell_2 \) are mutually orthogonal line subbundles of the trivial bundle \( \Sigma \times \mathbb{C}^5 \), whereas \( \ell_1 \oplus \ell_2 \) is a maximal isotropic subbundle. The relevant geometric information of an isotropic immersion \( f : \Sigma \to S^4 \) is thus contained in the
ordered 3-uple \((\ell_0, \ell_1, \ell_2)\). This motivates introducing the manifold \(\mathcal{F}\) consisting of ordered 3-uples \((X_0, X_1, X_2)\) of mutually orthogonal complex lines in \(\mathbb{C}^5\) where \(X_0\) is the complexification of a real line in \(\mathbb{R}^5\) and \(X_1, X_2\) span a maximal complex isotropic subspace of \(\mathbb{C}^5\). Given \(l = (X_0, X_1, X_2) \in \mathcal{F}\), choose an ordered orthonormal basis \(\{u_0, u_1, \ldots u_4\}\) of \(\mathbb{R}^5\) so that

\[
X_0 = \mathbb{C}u_0, \quad X_1 = \mathbb{C}z_1, \quad X_2 = \mathbb{C}z_2,
\]

where the complex unit vectors \(z_1, z_2\) are given by

\[
z_j = \frac{u_{2j-1} - iu_{2j}}{\sqrt{2}}, \quad j = 1, 2
\]

Define the projection map \(\pi : \mathcal{F} \to S^4\) by demanding that the orthonormal basis \(\{\pi(l), u_1, \ldots u_4\}\) be positively oriented. According to this definition \(\pi(l) = \pm u_0 \in S^4\), depending on the orientation of the chosen orthonormal basis. Moreover every element \(l = (X_0, X_1, X_2) \in \mathcal{F}\) is uniquely determined by its projection \(\pi(l)\) and the complex isotropic line \(X_1\). This establishes a diffeomorphism between the Grassmann bundle \(G_2(TS^4)\) and \(\mathcal{F}\). In fact let \(p \in S^4\) and \(V\) an oriented 2-plane in \(T_pS^4\). Choose an oriented orthonormal base \(\{e_1, e_2\}\) in \(V\) and set \(X_0 := \mathbb{C}p, X_1 := \mathbb{C}(e_1 - ie_2)\). Thus the application sending \((p, V)\) to the (uniquely determined) element \(l = (X_0, X_1, X_2) \in \mathcal{F}\) is the desired diffeomorphism.

If \(f : \Sigma \to S^4\) is a conformal isotropic map we define its Gauss map \(\hat{f} : \Sigma \to \mathcal{F}\) by

\[
\hat{f} = (\ell_0, \ell_1, \ell_2)
\]

It is easily verified that with this definition \(\pi \circ \hat{f} = f\) holds. On the other hand if \(f\) is superconformal then \(\ell_2\) and \(\ell_2\) are no longer orthogonal. In this case we define \(\hat{f} : \Sigma \to \mathcal{F}\) by the condition \(\pi \circ \hat{f} = f\). That is, there is a uniquely defined isotropic complex line subbundle \(X_2 \subset \Sigma \times \mathbb{C}^5\) satisfying \(\pi(\ell_0, \ell_1, X_2) = f\). Note that \(\ell_2\) and \(\ell_2\) are contained in \(X_2 \oplus \overline{X}_2\) which coincides with the orthogonal complement bundle of \(\overline{\ell}_1 \oplus \overline{\ell}_0 \oplus \ell_1\). Define the Gauss map of a superconformal immersion \(f\) by

\[
\hat{f} = (\ell_0, \ell_1, X_2)
\]
Let \((X_0, X_1, X_2) \in \mathcal{F}\) with \(X_0 = C u_0\), and \(u_0 \in \mathbb{R}^5\) and \(X_j = C z_j\), \(j = 1, 2\) where the unit complex vectors \(z_1, z_2 \in \mathbb{C}^5\) are given by (24). If \(g \in SO_5\) then

\[
g \cdot z_j = \frac{g \cdot u_{2j-1} - ig \cdot u_{2j}}{\sqrt{2}}, \quad j = 1, 2
\]

are also complex unit vectors \(\in \mathbb{C}^5\). Therefore the following defines a transitive action of the Lie group \(SO_5\) on \(\mathcal{F}\)

\[
g \cdot (X_0, X_1, X_2) = (C g \cdot z_0, C g \cdot z_1, C g \cdot z_2).
\]

Hence \(\mathcal{F} \equiv SO_5 / SO_2 \times SO_2\), where the isotropy group \(SO_2 \times SO_2\) is the stabilizer of the basepoint

\[
o = (Ce_0, C(e_1 - ie_2), C(e_3 - ie_4)) \in \mathcal{F}.
\]

Let \(X \in \mathfrak{g} = \mathfrak{so}_5\), so that \(g_t = \exp(tX) \in SO_5\), \(\forall t\). Given \(l = (C u_0, C z_1, C z_2) \in \mathcal{F}\), with \(z_j\) give by (24). Then \(t \mapsto g_t \cdot l\) defines a curve in \(\mathcal{F}\) through \(l\). We compute its derivative at \(t = 0\),

\[
\frac{d}{dt}|_{t=0} \exp(tX). (C u_0, C z_1, C z_2) = \\
\frac{d}{dt}|_{t=0} \langle \exp(tX) u_0, \exp(tX) z_1, \exp(tX) z_2 \rangle = \\
\langle CX u_0, CX(u_1 - iu_2), CX(u_3 - iu_4) \rangle \in T_l \mathcal{F}.
\]

Since \(X + X^T = 0\) it follows that

\[
X u_0 = 0, \quad \langle X u_k, u_k \rangle = 0, \quad k = 1, 2, 3, 4.
\]

On the other hand taking derivative at \(t = 0\) of the identities

\[
\left\langle \exp(tX) \left( \frac{u_{2k-1} - iu_{2k}}{\sqrt{2}} \right), \exp(tX) \left( \frac{u_{2k-1} - iu_{2k}}{\sqrt{2}} \right) \right\rangle = 1, \quad \forall t \in \mathbb{R}, k = 1, 2,
\]

yields

\[
\langle X u_{2k-1}, u_{2k} \rangle = \langle X u_{2k}, u_{2k-1} \rangle = 0, \quad k = 1, 2.
\]

In particular any tangent vector at the basepoint \(o \in \mathcal{F}\) is an ordered 3-tuple of complex lines of the form

\[
(CX e_0, CX(e_1 - ie_2), CX(e_3 - ie_4)),
\]
where \( X \in \mathfrak{so}_5 \) varies in the subspace \( \mathfrak{p} \subset \mathfrak{so}_5 \) of matrices of the form
\[
\begin{pmatrix}
0 & a & b & c & d \\
-a & 0 & 0 & e & f \\
-b & 0 & 0 & g & h \\
-c & -e & -g & 0 & 0 \\
-d & -f & -h & 0 & 0
\end{pmatrix}
\] (27)

In this way \( T_o\mathcal{F} \) identifies with \( \mathfrak{p} \).

If \( g \in SO_5 \) let \( \tau_g : \mathcal{F} \rightarrow \mathcal{F} \) be the isometry sending \( g'.o \) to \( gg'.o \). It easily follows that the projection \( \pi : \mathcal{F} \rightarrow \mathbb{S}^4 \) satisfies
\[
\pi \circ \tau_g = \tau'_g \circ \pi, \quad \forall g \in SO_5,
\] (28)

where \( \tau'_g \) is the isometry of \( \mathbb{S}^4 \) induced by \( g \in SO_5 \).

3.1. The Maurer-Cartan form of \( \mathcal{F} \). Decompose \( \mathfrak{so}_5 = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{k} = \mathfrak{so}_2 \oplus \mathfrak{so}_2 \) is the Lie algebra of the maximal compact 2-torus \( T^2 = SO_2 \times SO_2 \) sitting within \( SO_5 \) according to the inclusion
\[
SO_2 \times SO_2 \ni (A, B) \mapsto diag(1, A, B) \in SO_5.
\] (29)

Let \( \mathfrak{p} \equiv T_o\mathcal{F} \) be the set of real skew-symmetric matrices (27). For \( A, B \in \mathfrak{p} \) define
\[
\langle A, B \rangle = -\frac{1}{2} \text{trace}(AB)
\] (30)

Then \( \langle \cdot, \cdot \rangle \) is an \( Ad(T^2) \)-invariant inner product on \( \mathfrak{p} \) which determines an \( SO_5 \)-invariant metric on \( \mathcal{F} \), the so-called normal metric.

The geometry of the flag manifold \( (\mathcal{F}, \langle \cdot, \cdot \rangle) \) may be studied with aid of the so-called Maurer-Cartan one form \( \beta : T\mathcal{F} \rightarrow \mathfrak{so}_5 \) introduced by Burstall and Rawnsley in [4] which we describe below.

Recall the reductive decomposition \( \mathfrak{g} = \mathfrak{so}_5 = \mathfrak{k} \oplus \mathfrak{p} \). Consider the surjective application \( \xi_o : \mathfrak{g} \ni X \mapsto \frac{d}{dt}|_{t=0} \exp(tX).o \in T_o\mathcal{F} \). Thus \( \xi_o \) has kernel \( \mathfrak{k} \) and restricts to an isomorphism \( \mathfrak{p} \rightarrow T_o\mathcal{F} \). Form the associated vector bundle \([\mathfrak{p}] := SO_5 \times_K \mathfrak{p}\), then the map
\[
[(g, X)] \mapsto \frac{d}{dt}|_{t=0} \exp(tAd(g)X), \quad x = d\tau_g(\frac{d}{dt}|_{t=0} \exp(tX).o), \quad x = g.o,
\]
establishes an isomorphism of the associated bundle \([p]\) and the tangent bundle \(T\mathcal{F}\), where \(\tau_g\) is the isometry of \(\mathcal{F}\) sending \(g'.o\) to \(gg'.o\).

Since \(p\) is an \(Ad(K)\)-invariant subspace of \(\mathfrak{g}\), one has the inclusion \([p] \subset [\mathfrak{g}] := \mathcal{F} \times \mathfrak{g}\), given by \([p] \ni [(g, X)] \mapsto (g.o, Ad(g)X) \in [\mathfrak{g}]\). Note that the fiber of \([p] \rightarrow \mathcal{F}\) over the point \(g.o\) identifies with \(\{g.o\} \times Ad(g)p \subset [\mathfrak{g}]\). Hence there exists an identification of \(T\mathcal{F}\) with a subbundle of the trivial bundle \([\mathfrak{g}]\). This inclusion may be viewed as an \(\mathfrak{g}\)-valued one-form on \(\mathcal{F}\) which will be denoted by \(\beta\). Every \(X \in \mathfrak{g}\) determines a flow on \(\mathcal{F}\) defined by \(\varphi_t(x) = \exp(tX).x\), which is an isometry of \(\mathcal{F}\) for any \(t \in \mathbb{R}\). The vector field of the flow is then a Killing field denoted by \(X^*\) which is defined by

\[
X^*_x = \frac{d}{dt}|_{t=0}\exp(tX).x, \forall x \in \mathcal{F}.
\]

It is not difficult to show that

\[
\beta(x)(X^*) = Ad(g)[Ad(g^{-1})(X)]_p, \quad \forall X \in \mathfrak{p}, \quad x = g.o \in \mathcal{F}
\]

In particular at \(o \in \mathcal{F}\) we have \(\beta(X^*_o) = X\) for any \(X \in \mathfrak{p}\). From this formula it follows the equivariance of \(\beta\) which is expressed by

\[
\beta \circ d\tau_g = Ad(g)\beta, \quad \forall g \in SO_5.
\]

Equivalently, note that \(\xi\) in turn satisfies

\[
d\tau_g \circ \xi_o = \xi_{g.o} \circ Ad(g), \forall g \in SO_5.
\]

For \(x = g.o \in \mathcal{F}\) the application \(\xi_x : \mathfrak{g} \rightarrow T_x\mathcal{F}\) such that \(X \mapsto X^*_x\), maps \(\mathfrak{g}\) onto \(T_x\mathcal{F}\), and restricts to an isomorphism \(Ad(g)(\mathfrak{p}) \rightarrow T_x\mathcal{F}\) whose inverse coincides with \(\beta_x\). More details and properties of the one form \(\beta\) and proofs can be found in [4].

**Remark 3.1.** We fix the bundle metric on \([p]\) for which \(\beta_{F,o} : T_{F,o}\mathcal{F} \rightarrow Ad(F)(\mathfrak{p})\) an isometry for every \(F \in SO_5\).

### 4. Normal harmonic maps into \(\mathcal{F}\)

Here we study the harmonic map equation for smooth maps from a Riemann surface \(\Sigma\) into the flag manifold \(\mathcal{F} = SO_5/T^2\), on which we have fixed the normal metric \(\langle ., . \rangle\) defined by (30). Our approach is based on [1] (see also [9]) and is also
valid for a wider class of naturally reductive homogeneous spaces. Assume that \( \Sigma \) is compact and consider the energy of \( \phi \) by
\[
(32) \quad E(\phi) = \frac{1}{2} \int_{\Sigma} ||d\phi||^2 dA_g,
\]
where \( dA_g \) is the area form on \( \Sigma \) determined by a conformal metric \( g \), and \( ||d\phi||^2 \) is the Hilbert-Schmidt norm of \( d\phi \) defined by \( ||d\phi||^2 = \sum \langle d\phi(e_i), d\phi(e_i) \rangle \) for any orthonormal frame \( \{e_i\} \) on \( \Sigma \). By definition \( \phi \) is harmonic if it is an extreme of the energy functional \( \phi \mapsto E(\phi) \).

Denote by \( \nabla \) the Levi-Civita connection on \( \mathcal{F} \) determined by the normal metric \( \langle \cdot, \cdot \rangle \) and by \( \nabla^\phi \) the induced connection on the pull-back bundle \( \phi^*T\mathcal{F} \to \Sigma \). As is well known, by the first variation formula of the energy \([5]\) it follows that \( \phi \) is harmonic if and only if its tension vanishes,
\[
\text{trace}(\nabla d\phi) = 0.
\]
Being \( \Sigma \) a Riemann surface then \( \phi \) is harmonic if and only if on every local complex coordinate \( z \) on \( \Sigma \) the following equation holds
\[
(33) \quad \nabla^\phi_{\frac{\partial}{\partial z}} d\phi\left(\frac{\partial}{\partial z}\right) = 0,
\]
where the left hand of this equation is just a non-zero multiple of the tension field of \( \phi \) \([5]\).

For our purposes we need to reformulate equation \((33)\) in terms of the Maurer-Cartan form or Moment map \( \beta \) of \( \mathcal{F} \), see \([1]\), \([4]\).

Let \( D \) be the canonical connection of second kind i.e. the affine connection on \( \mathcal{F} \) determined by the condition that the \( D \)-parallel transport along the curve \( t \to \exp(tX).x \) is realized by \( d\exp(tX) \). At the basepoint \( o \in \mathcal{F} \) we have
\[
(34) \quad D_X \cdot Y^*(o) = \frac{d}{dt} \big|_{t=0} d\exp(-tX)^*Y^* = [X^*, Y^*](o) = -[X, Y]_p.
\]
Note also that \( D \) is determined by the condition \( (D_X \cdot X^*)_o = 0, \forall X \in \mathfrak{p} \). Since \( \nabla \neq D \) and \( D \) is metric i.e. \( D\langle \cdot, \cdot \rangle = 0 \), it follows that \( D \) has non-vanishing torsion. From \((34)\) we obtain
\[
(35) \quad T^D_o(X^*, Y^*) = -[X, Y]_p, \quad X, Y \in \mathfrak{p}.
\]
The following formula due to F. E. Burstall and J. Rawnsley \([4]\) allows to compute \( D \) in terms of \( \beta \) and the Lie algebra structure of \( \mathfrak{g} = \mathfrak{so}_5 \).
Lemma 4.1. \[\beta(D_X Y) = X\beta(Y) - [\beta(X), \beta(Y)], \quad X, Y \in \mathcal{X}(\mathcal{F}).\]

Let us now compute the Levi-Civita connection $\nabla$ of the normal metric on $\mathcal{F}$. Since $P: SO_5 \to \mathcal{F}$ is a riemannian submersion, $P$ sends $p$-horizontal geodesics in $SO_5$ onto $\nabla$-geodesics in $\mathcal{F}$. Hence $\nabla_X Y^* = 0$ for every $X \in p$ which implies $\nabla_X Y^* + \nabla_Y X^* = 0$, for any $X, Y \in p$. Since $\nabla$ is torsionless we deduce

$$\nabla_X Y^* = \frac{1}{2}[X^*, Y^*], \quad \forall X, Y \in p.$$  

Hence at $o \in \mathcal{F}$ we get

$$\nabla_X Y^* o = -\frac{1}{2}[X, Y]_p, \quad \forall X, Y \in p.$$  

We are now ready to obtain the following formula for the Levi-Civita connection $\nabla$ on $\mathcal{F}$ in terms of $\beta$ (for an equivalent formula see \[12\])

Lemma 4.2. \[\beta(\nabla_X Y) = X\beta(Y) - [\beta(X), \beta(Y)] + \frac{1}{2}\pi_p([\beta(X), \beta(Y)]), \quad X, Y \in \mathcal{X}(\mathcal{F}),\]

where $\pi_p: \mathcal{F} \times \mathfrak{so}(5) \to \mathcal{F} \times_K p = [p] \equiv T\mathcal{F}$ is the projection onto the the tangent bundle of $\mathcal{F}$.

Proof. Let $X^*, Y^*$ be fundamental (Killing) vector fields on $\mathcal{F}$ determined by $X, Y \in p$.

From the definition of $\beta$, (34) and (38) we have

$$\beta((\nabla_X Y^*)_o) - \beta((D_X Y^*)_o) = -\frac{1}{2}[X, Y]_p + [X, Y]_p = \frac{1}{2}[X, Y]_p, \quad \forall X, Y \in p.$$  

On the other hand the difference tensor $\nabla - D$ is $SO_5$-invariant, and so is $\beta(\nabla - D) = \beta(\nabla) - \beta(D)$ by formula (36). Hence it is determined by its value at the point $o \in \mathcal{F}$. For arbitrary vector fields on $\mathcal{F}$ formula (39) follows from a straightforward calculation. \[\square\]

Define the $D$-fundamental form of a smooth map $\phi: \Sigma \to \mathcal{F}$ by

$$Dd\phi(U, V) = D^\phi_U d\phi(V) - d\phi(\nabla^\Sigma_U V), \quad U, V \in \mathcal{X}(\Sigma),$$

in which $D^\phi$ is the connection on the pullback bundle $\phi^* T\mathcal{F} \to \Sigma$ determined by $D$ and $\nabla^\Sigma$ is the Levi-Civita connection on $M$ determined by a conformal metric. The
map $\phi : \Sigma \to F$ is called affine or $D$-harmonic if and only if $\text{trace}(Dd\phi) = 0$ holds, or equivalently

$$D_\phi^\beta d\phi(\frac{\partial}{\partial z}) = 0.$$  

**Lemma 4.3.** $\phi : \Sigma \to F$ is normal-harmonic if and only if it is $D$-harmonic. Hence $\phi$ is harmonic if and only if

$$\bar{\partial}(\phi^*\beta)' - [(\phi^\ast\beta)'' \land (\phi^\ast\beta)'] = 0,$$

where $\phi^*\beta = (\phi^\ast\beta)' + (\phi^\ast\beta)''$ is the decomposition of the (complex) one form $\phi^\ast\beta$ into its $(1,0)$ and $(0,1)$ parts.

**Proof.** Follows as consequence of the following formula for the tension of $\phi$

$$-\beta \left( \nabla_{\phi}^\beta d\phi(\frac{\partial}{\partial z}) \right) \, dz \wedge d\bar{z} = \bar{\partial}(\phi^*\beta)' - [(\phi^\ast\beta)'' \land (\phi^\ast\beta)'].$$

To obtain formula (42), first note that (39) implies $\beta(tr\nabla d\phi) = \beta(trDd\phi)$, which since $\Sigma$ is a Riemann surface is equivalent to

$$\beta(\nabla_{\phi}^\beta d\phi(\frac{\partial}{\partial z})) = \beta(D_\phi^\beta d\phi(\frac{\partial}{\partial z})).$$

On the other hand from formula (36) we obtain

$$\beta(D_\phi^\beta d\phi(\frac{\partial}{\partial z})) = \frac{\partial}{\partial z} \beta d\phi(\frac{\partial}{\partial z}) - [\beta d\phi(\frac{\partial}{\partial z}), \beta d\phi(\frac{\partial}{\partial \bar{z}})]$$

from which (42) follows. \[\square\]

Now we want to express the harmonic map equation in terms of the one form $\alpha = F^{-1}dF = \alpha_t + \alpha_p$, where $F$ is a frame of $\phi$. From the identity

$$\phi^\ast\beta = Ad(F)\alpha_p$$

we get

$$(\phi^\ast\beta)' = Ad(F)\alpha'_p, \quad (\phi^\ast\beta)' = Ad(F)\alpha''_p.$$  

Now we use the identity (see [3] pag. 241)

$$d[Ad(F)] = Ad(F) \circ ad\alpha,$$

to compute

$$d(\phi^\ast\beta) = d[Ad(F)\alpha_p] = Ad(F)\{d\alpha_p + [\alpha \land \alpha_p]\}.$$  

Thus

$$\bar{\partial}(\phi^\ast\beta)' = \bar{\partial}\{Ad(F)\alpha'_p\} = Ad(F)\{\bar{\partial}\alpha'_p + [\alpha''_p \land \alpha'_p] + [\alpha''_p \land \alpha'_p]\}.$$
Now since
\[ (\phi^* \beta'' \wedge (\phi^* \beta'))' = Ad(F)[\alpha''_p \wedge \alpha'_p] \]
We conclude
\[ \bar{\partial} (\phi^* \beta')' - [(\phi^* \beta'') \wedge (\phi^* \beta')'] = Ad(F)\{ \bar{\partial} \alpha'_p + [\alpha''_p \wedge \alpha'_p] \}. \]

Thus a direct consequence of Lemma 4.3 is the following

**Proposition 4.1.** Let \( \phi : \Sigma \to F \) be a smooth map. Then \( \phi \) is harmonic if and only if the \( so_5 \)-valued one form \( \alpha = F^{-1}dF \) satisfies
\[
(44) \quad \bar{\partial} \alpha'_p + [\alpha''_p \wedge \alpha'_p] = 0.
\]

Here note that \( [\alpha''_p \wedge \alpha'_p] = [\alpha_t \wedge \alpha'_p] \).

Set \( \alpha = Adz + Bd\bar{z} \), where \( A = F^{-1}F_t, B = F^{-1}F_p \), then decomposing \( A = A_t + A_p \) and \( B = B_t + B_p \) we have \( \alpha'_p = A_p dz \) and \( \alpha''_p = B_p d\bar{z} \). Thus respect to a complex local coordinate \( z \) equation (44) becomes
\[
(45) \quad \bar{\partial} \alpha'_p + [\alpha''_p \wedge \alpha'_p] = -(\frac{\partial}{\partial \bar{z}} A_p + [B_t, A_p]) dz \wedge d\bar{z}
\]

**Corollary 4.4.** \( \phi : \Sigma \to F \) is harmonic if and only if for every frame \( F \) of \( \phi \) and any complex coordinate \( z \) the complex matrices \( A_p, B_t \) satisfy
\[
(46) \quad \frac{\partial}{\partial \bar{z}} A_p + [B_t, A_p] = 0.
\]

**Remark 4.1.** From (45) and (42) we obtain the following formula for the tension of \( \phi \), where \( F \) is any frame of \( \phi \).
\[
(47) \quad \beta \left( \nabla^\phi \frac{d\phi}{dz} \left( \frac{\partial}{\partial \bar{z}} \right) \right) = Ad(F)\{ \frac{\partial}{\partial \bar{z}} A_p + [B_t, A_p] \},
\]

5. A FORMULA FOR THE TENSION OF THE GAUSS MAP

Here we obtain a formula for the tension of the Gauss map of a conformal immersion \( f : \Sigma \to S^4 \). As a by-product we characterize conformal immersions with normal-harmonic Gauss maps.

Let \( F = (F_0, F_1, F_2, N_1, N_2) \in SO_5 \) be a (local) frame in column notation of the Gauss map \( \hat{f} \) of a conformal immersion \( f \) i.e. \( \hat{f} = F.o \) on an open subset \( U \subseteq \Sigma \). If \( \Sigma \) is simply connected or contractible then there always exists a global frame
$F : \Sigma \to SO_5$. By the above considerations it follows that $F_1 - iF_2$ is a local section of $\ell_1$ and $N_1 - iN_2$ is a local section of $\ell_2$ (resp. $X_2$) if $f$ is isotropic (resp. superconformal). In either case the Gauss map of $f$ is given locally by

$$\hat{f} = (CF_0, C(F_1 - iF_2), C(N_1 - iN_2)) = F.o$$

In particular $F$ is also an adapted frame of $f$. Since

$$F' = \text{diag}(1, R(\theta_1), R(\theta_2)).F, \ R(\theta_i) = \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}, \ i = 1, 2$$

is also a frame of $\hat{f}$ and also an adapted frame of $f$, we may assume (after possibly applying a gauge) that (15) holds on a local complex chart $z$.

Set $\alpha = F^{-1}dF = Adz + Bd\bar{z}$ as before and decompose $A = A_t + A_p$ and $B = B_t + B_p$ according to the decomposition $g = \mathfrak{t} \oplus \mathfrak{p}$. Define the following complex one forms on $\Sigma$,

$$\alpha_t = A_t dz + B_t d\bar{z}, \ \alpha_p = A_p dz + B_p d\bar{z},$$

$$\alpha'_t = A_p dz, \ \alpha''_t = B_p d\bar{z},$$

$$\alpha'_t = A_t dz, \ \alpha''_t = B_t d\bar{z},$$

where

$$A_p = \begin{pmatrix} 0 & -\frac{iu}{\sqrt{2}} & \frac{iu}{\sqrt{2}} & 0 & 0 \\ \frac{iu}{\sqrt{2}} & 0 & 0 & -a_1 & -a_2 \\ -i\frac{iu}{\sqrt{2}} & 0 & 0 & -ib_1 & -ib_2 \\ 0 & a_1 & ib_1 & 0 & 0 \\ 0 & a_2 & ib_2 & 0 & 0 \end{pmatrix}, \ B_p = \begin{pmatrix} 0 & -\frac{iu}{\sqrt{2}} & -i\frac{iu}{\sqrt{2}} & 0 & 0 \\ \frac{iu}{\sqrt{2}} & 0 & 0 & -\bar{a}_1 & -\bar{a}_2 \\ i\frac{iu}{\sqrt{2}} & 0 & 0 & i\bar{b}_1 & i\bar{b}_2 \\ 0 & \bar{a}_1 & -i\bar{b}_1 & 0 & 0 \\ 0 & \bar{a}_2 & -i\bar{b}_2 & 0 & 0 \end{pmatrix}$$

$$A_t = \begin{pmatrix} 0 & 0 & iu_z \\ -iu_z & 0 & 0 \\ 0 & \sigma & 0 \\ -\sigma & 0 \end{pmatrix}, \ B_t = \begin{pmatrix} 0 & 0 & -iu_z \\ iu_z & 0 & 0 \\ 0 & \bar{\sigma} & 0 \\ -\bar{\sigma} & 0 \end{pmatrix}$$
Set

\[
M := \frac{\partial}{\partial \bar{z}} A_p + [B_t, A_p] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -A_1 & -A_2 \\
0 & 0 & 0 & -iB_1 & -iB_2 \\
0 & A_1 & iB_1 & 0 & 0 \\
0 & A_2 & iB_2 & 0 & 0 \\
\end{pmatrix},
\]

where the coefficients \(A_1, A_2, B_1, B_2\) are given respectively by

\[
\begin{align*}
A_1 &= \left( \frac{\partial}{\partial \bar{z}} a_1 + u \bar{b}_1 + a_2 \bar{\sigma} \right), \\
A_2 &= \left( \frac{\partial}{\partial \bar{z}} a_2 + u \bar{b}_2 - a_1 \bar{\sigma} \right), \\
B_1 &= \left( \frac{\partial}{\partial \bar{z}} b_1 + u \bar{a}_1 + b_2 \bar{\sigma} \right), \\
B_2 &= \left( \frac{\partial}{\partial \bar{z}} b_2 + u \bar{a}_2 - b_1 \bar{\sigma} \right).
\end{align*}
\]

In terms of the parameters \(u, h_i, \xi_i\), the above coefficients are given by,

\[
\begin{align*}
A_1 &= e^{-u} \left( \frac{\partial}{\partial z} \xi_1 + \xi_2 \bar{\sigma} \right) + e^u \left( \frac{\partial}{\partial \bar{z}} h_1 + h_2 \bar{\sigma} \right), \\
A_2 &= e^{-u} \left( \frac{\partial}{\partial z} \xi_2 - \xi_1 \bar{\sigma} \right) + e^u \left( \frac{\partial}{\partial \bar{z}} h_2 - h_1 \bar{\sigma} \right), \\
B_1 &= e^{-u} \left( \frac{\partial}{\partial z} \xi_1 + \xi_2 \bar{\sigma} \right) - e^u \left( \frac{\partial}{\partial \bar{z}} h_1 + h_2 \bar{\sigma} \right), \\
B_2 &= e^{-u} \left( \frac{\partial}{\partial z} \xi_2 - \xi_1 \bar{\sigma} \right) - e^u \left( \frac{\partial}{\partial \bar{z}} h_2 - h_1 \bar{\sigma} \right).
\end{align*}
\]

Using Codazzi’s equations,

\[
\begin{align*}
e^u \left( \frac{\partial}{\partial z} h_1 + h_2 \bar{\sigma} \right) &= e^{-u} \left( \frac{\partial}{\partial \bar{z}} \xi_1 + \xi_2 \bar{\sigma} \right), \\
e^u \left( \frac{\partial}{\partial z} h_2 - h_1 \bar{\sigma} \right) &= e^{-u} \left( \frac{\partial}{\partial \bar{z}} \xi_2 - \xi_1 \bar{\sigma} \right),
\end{align*}
\]

the above coefficients become
A_1 = \frac{e^u}{\sqrt{2}} \left( \frac{\partial}{\partial z} h_1 + \frac{\partial}{\partial \bar{z}} h_1 + h_2 (\sigma + \bar{\sigma}) \right),

A_2 = \frac{e^u}{\sqrt{2}} \left( \frac{\partial}{\partial z} h_2 + \frac{\partial}{\partial \bar{z}} h_2 - h_1 (\sigma + \bar{\sigma}) \right),

B_1 = \frac{e^u}{\sqrt{2}} \left( \frac{\partial}{\partial z} h_1 - \frac{\partial}{\partial \bar{z}} h_1 + h_2 (\sigma - \bar{\sigma}) \right),

B_2 = \frac{e^u}{\sqrt{2}} \left( \frac{\partial}{\partial z} h_2 - \frac{\partial}{\partial \bar{z}} h_2 - h_1 (\sigma - \bar{\sigma}) \right).

(56)

Remark 5.1. From (56) it follows that \( A_1, A_2 \) are \( \mathbb{R} \)-valued functions while \( B_1, B_2 \) are \( i\mathbb{R} \)-valued.

Using formula (47) for the Gauss map we get

\[ \beta \left( \nabla_{\frac{\partial}{\partial x}} \hat{f} \left( \frac{\partial}{\partial z} \right) \right) = \text{Ad}(F)M, \]

where \( M \) is given by (52) and \( F \) is a frame of \( \hat{f} \), i.e. \( \hat{f} = F.o \in \mathcal{F} \). The tension of the Gauss map at a point \( x \in \Sigma \) is given by

\[ \nabla_{\frac{\partial}{\partial x}} \hat{f} \left( \frac{\partial}{\partial z} \right)(x) = (\text{Ad}(F(x))M(x))^* \right|_{F(x).o} = F(x)M(x)F(x)^T(F(x).o) \in T_{F(x).o}\mathcal{F}, \]

where \( (\text{Ad}(F(x))M(x))^* \) denotes the fundamental or Killing vector field on \( \mathcal{F} \) determined by \( \text{Ad}(F(x))M(x) \in \mathfrak{g} \).

Since the base point was chosen to be \( o = (C e_0, C(e_1 - ie_2), C(e_3 - ie_4)) \in \mathcal{F} \), a straightforward calculation shows (dropping the point \( x \in \Sigma \)) that

\[ \nabla_{\frac{\partial}{\partial x}} \hat{f} \left( \frac{\partial}{\partial z} \right) = FM.o = F.(C M_0, C(M_1 - iM_2), C(M_3 - iM_4)), \]

where \( M_j \) is the \( j \)-th column of the matrix \( M \). In terms of the columns \( F_0, F_1, F_2, N_1, N_2 \) of \( F \) we obtain

\[ \nabla_{\frac{\partial}{\partial x}} \hat{f} \left( \frac{\partial}{\partial z} \right) = \]

\[ (0, \mathbb{C}[(A_1 + B_1)N_1 + (A_2 + B_2)N_2], \mathbb{C}[(-A_1 + iA_2)F_1 + (-iB_1 + B_2)F_2]), \]

(57)

Now recall equation (10) giving the normal derivative of the mean curvature vector,

\[ \nabla_{\frac{\partial}{\partial x}} \frac{1}{2} H = \left( \frac{\partial}{\partial z} h_1 + h_2 \sigma \right) N_1 + \left( \frac{\partial}{\partial z} h_2 - h_1 \sigma \right) N_2. \]

From (56) above, we obtain

\[ \sqrt{2} e^u \nabla_{\frac{\partial}{\partial x}} \frac{1}{2} H = (A_1 + B_1)N_1 + (A_2 + B_2)N_2. \]
Thus from (57) we arrive at the following formula for (a non-zero factor of) the tension of \( \hat{f} \),

\[
\nabla_{\frac{\partial}{\partial \bar{z}}} \hat{f} \left( \frac{\partial}{\partial z} \right) = (0, \mathbb{C}[\nabla \nabla_{\frac{\partial}{\partial \bar{z}}} H], \mathbb{C}[\Psi]),
\]

where \( \Psi = (-A_1 + iA_2)F_1 + (-iB_1 + B_2)F_2 \).

From these identities we conclude that \( \hat{f} \) is harmonic if and only if \( \nabla_{\nabla_{\frac{\partial}{\partial \bar{z}}} H} = 0 \) and \( \Psi = 0 \). Both equations \( \nabla_{\nabla_{\frac{\partial}{\partial \bar{z}}} H} = 0 \) and \( \Psi = 0 \) are equivalent, since by (56) and Remark (5.1) each equation separately is equivalent to \( A_1 = A_2 = B_1 = B_2 = 0 \).

We have thus completed the proof of the following theorem,

**Theorem 5.1.** Let \( f : \Sigma \rightarrow \mathbb{S}^4 \) be a conformal immersion of a Riemann surface and \( \hat{f} : \Sigma \rightarrow \mathcal{F} \) its Gauss map. Then \( \hat{f} \) is harmonic with respect to the normal metric on \( \mathcal{F} \) if and only if the surface \( f \) has parallel mean curvature vector, \( \nabla_{\nabla_{\frac{\partial}{\partial \bar{z}}} H} = 0 \).

### 6. Complete integrability

We first turn our attention to the following property of the Gauss map,

**Lemma 6.1.** Let \( \hat{f} : \Sigma \rightarrow \mathcal{F} \) be the Gauss map of a conformal immersion \( f \). Then for any frame \( F \) of \( \hat{f} \) the complex one forms \( \alpha', \alpha'' \) defined by (49) satisfy the following condition

\[
[\alpha'_{\hat{p}} \wedge \alpha''_{\hat{p}}]_{\hat{p}} = 0.
\]

**Proof.** Recall that for \( g \)-valued one forms \( \alpha, \beta \) on \( \Sigma \) their wedge product is defined by

\[
[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] + [\beta(X), \alpha(Y)]
\]

Locally \( [\alpha'_p \wedge \alpha''_p]_p = [A_p, B_p]_p dz \wedge d\bar{z} \), thus in order to prove (59) it is enough to prove \([A_p, B_p]_p = 0\), for an arbitrary local complex coordinate \( z \) on \( \Sigma \). Since \([A_p, B_p]_p \) is a \( 5 \times 5 \) skew-symmetric matrix, we need only check that \( 8 = \dim p \) entries vanish.

We perform a direct computation using the structure equations (16) and the explicit form of matrices \( A_p, B_p \) (50).

From the structure equations (16) and taking (17) into account we obtain

\[
\langle f_{\bar{z}}, \left[ \frac{\partial}{\partial \bar{z}} N_i \right]^T \rangle^e = \langle f_{\bar{z}}, \left[ \frac{\partial}{\partial \bar{z}} N_i \right]^T \rangle^e = e^{2u_i} h_i, \quad i = 1, 2,
\]

where \([\frac{\partial}{\partial \bar{z}} N_i]^T \) (resp.\([\frac{\partial}{\partial \bar{z}} N_i]^T \)) denote projection of \( \frac{\partial}{\partial \bar{z}} N_i \) (resp. \( \frac{\partial}{\partial \bar{z}} N_i \)) onto the tangent bundle of the immersed surface. Both equations together imply the vanishing of entries (0, 3) and (0, 4).
Denote by $\nabla^\Sigma$ the Levi-Civita connection on $\Sigma$ determined by the induced conformal metric $g = f^*\langle ., . \rangle$. Using again the structure equations (16) we obtain

$$\langle f_z, \frac{\partial}{\partial z} F_i - \nabla^\Sigma \frac{\partial}{\partial z} F_i \rangle^c = 0, \quad i = 1, 2.$$  

$$\langle \frac{\partial}{\partial \bar{z}} F_i - \nabla^\Sigma \frac{\partial}{\partial \bar{z}} F_i, [\frac{\partial}{\partial \bar{z}} N_i]^T \rangle^c = \langle \frac{\partial}{\partial \bar{z}} F_i - \nabla^\Sigma \frac{\partial}{\partial \bar{z}} F_i, [\frac{\partial}{\partial \bar{z}} N_j]^T \rangle^c = 0, \quad 1 \leq i, j \leq 2.$$  

The first equation above implies the vanishing of entries $(0, 1)$ and $(0, 2)$, while the second equation implies the vanishing of entries $(1, 3), (1, 4), (2, 3)$ and $(2, 4)$. This completes the proof of the Lemma.

Let $\hat{f} : \Sigma \to F$ be the Gauss map of $f$. Take a local frame $F$ of $\hat{f}$ and consider the one form $\alpha = F^*\Theta = F^{-1}dF$. If $\Sigma$ is simply connected then there always exists a global frame $F : \Sigma \to SO_5$ of any smooth map into $F$. According to the splitting $g = so_5 = k \oplus p$ the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$, decomposes into the following equations

$$(60) \left\{ \begin{array}{l}
\tilde{\partial}\alpha_p' + [\alpha_t \wedge \alpha_p'] + \partial\alpha_p'' + [\alpha_t \wedge \alpha_p''] + [\alpha_p' \wedge \alpha_p'']_p = 0, \\
\tilde{\partial}\alpha_p + \frac{1}{2}[\alpha_t \wedge \alpha_t] + [\alpha_p' \wedge \alpha_p'']_p = 0.
\end{array} \right.$$  

Further, assume that $\hat{f}$ is harmonic, hence it satisfies

$$0 = \tilde{\partial}\alpha_p' + [\alpha_t \wedge \alpha_p'].$$  

Since by Lemma 6.1 condition $[\alpha_p' \wedge \alpha_p'']_p = 0$ holds, the pair of equations (60) reduces to

$$(a) \quad \tilde{\partial}\alpha_p'' + [\alpha_t \wedge \alpha_p''] = 0, \\
(b) \quad \tilde{\partial}\alpha_p + \frac{1}{2}[\alpha_t \wedge \alpha_t] + [\alpha_p' \wedge \alpha_p'] = 0.$$  

If $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ set

$$(61) \quad \lambda.\alpha := \alpha_\lambda = \lambda^{-1}\alpha_p' + \alpha_t + \lambda\alpha_p''.$$  

Due to $A_p = B_p$ and $A_t = B_t$, $\alpha_\lambda$ is $g$-valued for every $\lambda \in S^1$. Moreover $\lambda.\alpha = \alpha_\lambda$ defines an action of $S^1$ on $g^c$-valued 1-forms which leaves invariant the solution set of equations (a) and (b) above. Comparing coefficients of $\lambda$ it follows that equations (a) and (b) above hold for $\alpha$ if and only if $\alpha_\lambda$ satisfies

$$(62) \quad d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \forall \lambda \in S^1.$$  

Due to $\alpha_p' = \alpha_p''$ and $\alpha_t = \alpha_t$, the one forms $\alpha_\lambda$ satisfy

$$\overline{\alpha_\lambda} = \alpha_\lambda, \quad \forall \lambda \in S^1,$$
thus $\alpha_{\lambda}$ is $\mathfrak{g}$-valued. It determines a connection $d + \alpha_{\lambda}$ on the trivial $SO_5$ bundle over $\Sigma$ which by (62) is automatically flat. By this reason equation (62) is called zero curvature condition (ZCC) [3]. In this way the harmonic map equation for Gauss maps to $\mathcal{F}$ is encoded in a loop of zero curvature equations, a manifestation of complete integrability [8]. We summarize our discussion in the following

Theorem 6.2. The harmonic map equation for the Gauss maps $\hat{f} : \Sigma \to \mathcal{F}$ of a conformal immersion $f : \Sigma \to S^4$ can be expressed as a loop of zero-curvature equations.

Conversely, let us assume that $\Sigma$ is simply connected (otherwise transfer the whole situation to the universal covering surface $\hat{\Sigma}$). Let $\alpha_{\lambda}$ be the loop of $so_5$-valued 1-forms (61). Fixed a point $m_o \in \Sigma$ we integrate to solve

\begin{equation}
(63) \quad dF_{\lambda} = F_{\lambda}\alpha_{\lambda}, \quad F_{\lambda}(m_o) = Id \in SO_5.
\end{equation}

The solution $F_{\lambda} = (F_0(\lambda), F_1(\lambda), F_2(\lambda), N_1(\lambda), N_2(\lambda)) : \Sigma \to SO_5$ is called an extended frame [4] and satisfies

\begin{equation}
(64) \quad F_{\lambda}^{-1}(F_{\lambda})_{z} = \lambda^{-1}A_p + A_t, \quad F_{\lambda}^{-1}(F_{\lambda})_{\bar{z}} = \lambda B_p + B_t, \quad \forall \lambda \in S^1.
\end{equation}

Since

\begin{equation*}
(\alpha_{\lambda})_p = \lambda^{-1}\alpha_p' + \lambda\alpha_p'' = (\alpha_{\lambda})'_p + (\alpha_{\lambda})''_p, \quad (\alpha_{\lambda})_t = \alpha_t,
\end{equation*}

the one form $\alpha_{\lambda}$ satisfies equations (a) and (b) for every $\lambda \in S^1$. Thus if $P : SO_5 \to \mathcal{F}$ denotes the projection map $P(g) = g.o$, then $\hat{f}_\lambda = P \circ F_{\lambda} : \Sigma \to \mathcal{F}$ is harmonic $\forall \lambda \in S^1$. The family $\{\hat{f}_\lambda, \lambda \in S^1\}$ is called the associated family of the harmonic Gauss map $\hat{f}$ (see [3]). Note that $\hat{f}_{\{\lambda=1\}} = \hat{f}$, hence each $\hat{f}_\lambda$ is a deformation of $\hat{f}$.

6.1. One parameter families of parallel mean curvature immersions.

From (64) we derive the following set of equations which encode the dependence of $F_{\lambda}$ from the parameter $\lambda$,
Thus h_{22} EDUARDO HULETT (65) Note that this implies \( \| u \| \) consequently the same Gaussian curvature function i.e.

\[
\begin{align*}
\frac{\partial}{\partial z} F_0(\lambda) &= \lambda^{-1} \frac{e^u}{\sqrt{2}} \left[ (F_1(\lambda) - iF_2(\lambda)) \right], \\
\frac{\partial}{\partial z} F_1(\lambda) &= -\lambda^{-1} \frac{e^u}{\sqrt{2}} F_0(\lambda) - i u_z F_2(\lambda) + \lambda^{-1} a_1 N_1(\lambda) + \lambda^{-1} a_2 N_1(\lambda), \\
\frac{\partial}{\partial z} F_2(\lambda) &= i \lambda^{-1} \frac{e^u}{\sqrt{2}} F_0(\lambda) + i u_z F_1(\lambda) + i \lambda^{-1} b_1 N_1(\lambda) + i \lambda^{-1} b_2 N_2(\lambda), \\
\frac{\partial}{\partial z} N_1(\lambda) &= -\lambda^{-1} a_1 F_1(\lambda) - i \lambda^{-1} b_1 F_2(\lambda) - \sigma N_2(\lambda), \\
\frac{\partial}{\partial z} N_2(\lambda) &= -\lambda^{-1} a_2 F_1(\lambda) - i \lambda^{-1} b_2 F_2(\lambda) + \sigma N_1(\lambda),
\end{align*}
\]

where \( a_i, b_i \) are defined by (17). From the first equation above we obtain

\[
\langle (f_{\lambda})_z, (f_{\lambda})_z \rangle = \langle (f_{\lambda})_z, (f_{\lambda})_z \rangle^c = e^{2u}, \forall \lambda \in S^1.
\]

Thus \( \{ f_{\lambda}, \lambda \in S^1 \} \) is a family of conformal immersions of \( \Sigma \) into \( S^4 \), with a common conformal factor \( u \). In particular all \( f_{\lambda} \) induce the same metric for every \( \lambda \in S^1 \) and consequently the same Gaussian curvature function i.e. \( K_{\lambda} = K, \forall \lambda \in S^1 \).

Let \( K_{\lambda}^\perp \) denote the normal curvature of \( f_{\lambda} \). Then from the 4th and 5th equations in (65) we obtain

\[
\sigma^\lambda = \langle N_2(\lambda), N_1(\lambda) \rangle = \sigma,
\]

which implies \( K_{\lambda}^\perp = K^\perp \) for all \( \lambda \).

Denote by \( H_{\lambda} \) the mean curvature vector of \( f_{\lambda} \). Since \( u \) is the common conformal parameter of all \( f_{\lambda} \) we get

\[
H_{\lambda} = e^{-2u} (f_{\lambda})_z^\perp.
\]

Decomposing

\[
H_{\lambda} = h_1^\lambda N_1(\lambda) + h_2^\lambda N_2(\lambda), \quad h_i^\lambda := \langle H_{\lambda}, N_i(\lambda) \rangle, i = 1, 2
\]

and using (65) we obtain

\[
H_{\lambda} = e^{-2u} \langle (f_{\lambda})_z, N_1(\lambda) \rangle N_1(\lambda) + e^{-2u} \langle (f_{\lambda})_z, N_2(\lambda) \rangle N_2(\lambda) =
\]

\[
-e^{-2u} \langle (f_{\lambda})_z, \frac{\partial}{\partial z} N_1(\lambda) \rangle N_1(\lambda) - e^{-2u} \langle (f_{\lambda})_z, \frac{\partial}{\partial z} N_2(\lambda) \rangle N_2(\lambda) =
\]

\[
h_1 N_1(\lambda) + h_2 N_2(\lambda).
\]

Thus \( h_i^\lambda = h_i, i = 1, 2 \) and so the dependence of \( H_{\lambda} \) on \( \lambda \) is only through \( N_i(\lambda) \).

Note that this implies \( \| H_{\lambda} \| = \| H \|, \forall \lambda \in S^1 \).
Also, since each $\phi_\lambda$ is harmonic then by Theorem 5.1 it follows that each $f_\lambda$ has parallel mean curvature i.e. $\nabla^\perp_\lambda H_\lambda = 0$, where $\nabla^\perp_\lambda$ denotes the covariant derivative of the normal bundle of $f_\lambda : \Sigma \to S^4$.

Let $\xi_i^\lambda := \langle (f_\lambda)_{zz}, N_i(\lambda) \rangle^c$, $i = 1, 2$, then from (64) and (65) we get
\begin{equation}
\xi_i^\lambda = -\langle (f_\lambda)_{zz}, N_i(\lambda) \rangle^c = -\langle (f_\lambda)_z, \frac{\partial}{\partial z} N_i(\lambda) \rangle^c = \lambda^{-2} \xi_i, \quad i = 1, 2.
\end{equation}

Thus if $II^\lambda$ denotes the second fundamental form of $f_\lambda$ then
\[ II^\lambda(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \lambda^{-2} \xi_1 N_1(\lambda) + \lambda^{-2} \xi_2 N_2(\lambda) \]

We summarize the conclusions of this discussion in the following

**Proposition 6.1.** Let $f : \Sigma \to S^4$ be a conformal immersion of a simply connected Riemann surface $\Sigma$. If the Gauss map $\hat{f} : \Sigma \to F$ is normal-harmonic, then $f$ is part of an $S^1$-loop of conformal immersions $f_\lambda : \Sigma \to S^4$ with parallel mean curvature satisfying the following properties:

i) all $f_\lambda$ have the same common conformal factor hence they induce on $\Sigma$ the same metric for every $\lambda \in S^1$ and consequently the same Gaussian curvature function. Also all $f_\lambda$ have the same normal curvature function given by (13).

Let $F_\lambda := (F_0(\lambda), F_1(\lambda), F_2(\lambda), N_1(\lambda), N_2(\lambda))$ be a global extended frame of $\hat{f}_\lambda$, then

ii) The mean curvature vector of $f_\lambda$ is given by
\[ H_\lambda = h_1 N_1(\lambda) + h_2 N_2(\lambda), \]
where $H = h_1 N_1 + h_2 N_2$ is the mean curvature vector of $f = f_{\lambda=1}$.

iii) The second fundamental form of $f_\lambda$ satisfies
\begin{equation}
II^\lambda(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \lambda^{-2} \xi_1 N_1(\lambda) + \lambda^{-2} \xi_2 N_2(\lambda),
\end{equation}
where $II(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \xi_1 N_1 + \xi_2 N_2$ is the $(2, 0)$ part of the second fundamental form $II$ of $f$.

7. On the normal energy

Here we compute the energy of the Gauss map $\hat{f} : \Sigma \to F$ of a a conformal immersion $f : \Sigma \to S^4$. The energy density of a smooth map $\phi : \Sigma \to F$ is defined by the Hilber-Schmidt norm
\[ \|d\phi\|^2(p) = \langle d\phi(e_1), d\phi(e_1) \rangle + \langle d\phi(e_2), d\phi(e_2) \rangle, \]
where \( \{e_1, e_2\} \) is any orthonormal basis of \( T_p \Sigma \).

To compute \( \|d \hat{f}\|^2 \) we use the induced metric \( g = f^* \langle , \rangle \) which is conformal and locally given by \( g = 2e^{2u}dz \otimes d\bar{z} \), where \( u \) is the conformal factor. Thus

\[
\frac{1}{2} \|d \hat{f}\|^2 = e^{-2u} \langle \hat{f}_z, \hat{f}_{\bar{z}} \rangle^c.
\]

Let \( F \) be any frame of \( \hat{f} \) then we have the following identity

\[
\hat{f}^* \beta = \text{Ad}(F) \alpha_p,
\]

in which \( \alpha_p = A_p dz + B_p d\bar{z} \) and \( A_p \) is given by (18), and \( B_p = \overline{A_p} \). Using this and Remark 3.1 we obtain

\[
(68) \quad \frac{1}{2} \|d \hat{f}\|^2 = e^{-2u} \langle \hat{f}_z, \hat{f}_{\bar{z}} \rangle^c = e^{-2u} \langle \hat{f}^* \beta \frac{\partial}{\partial z}, \hat{f}^* \beta \frac{\partial}{\partial \bar{z}} \rangle^c =
\]

\[
(69) \quad E(\hat{f}) = \int_{\Sigma} [1 + \|H\|^2 + e^{-4u}(|\xi_1|^2 + |\xi_2|^2)] dA_g.
\]

From Gauss equation we have

\[
(70) \quad e^{-4u}(|\xi_1|^2 + |\xi_2|^2) = 1 + \|H\|^2 - K \geq 0,
\]

where \( K \) is the Gaussian curvature of the induced metric \( g = f^* \langle , \rangle \) on \( \Sigma \). Hence

\[
E(\hat{f}) = \int_{\Sigma} [2(1 + \|H\|^2) - K] dA_g.
\]

On the other hand recall that the Willmore energy of the immersion \( f \) is defined by

\[
(71) \quad W(f) = \frac{1}{2\pi} \int_{\Sigma} [1 + \|H\|^2 - K] dA_g \geq 0.
\]

From (70) it follows that \( W \) measures how far is \( f \) from satisfying the condition \( II(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = f_{zz}^1 = 0 \). From (11) every surface satisfying \( II(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = 0 \) has parallel mean curvature vector. Recall that surfaces satisfying \( f_{zz}^1 = 0 \) are exactly those for
which the ellipse of curvature at any point $p \in \Sigma$ reduces to $H(p) \in T^\perp_p \Sigma$.

Combining (70) and (71) we obtain the following formula relating both energies

\begin{equation}
E(\hat{f}) = 4\pi W(f) + \int_\Sigma KdA_g.
\end{equation}

By Gauss-Bonnet formula $\int_\Sigma KdA_g = 2\pi \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic, hence a topological invariant of the surface $\Sigma$. In terms of the genus $g(\Sigma)$ the Euler characteristic is given by $\chi(\Sigma) = 2 - 2g(\Sigma)$. Since $W(f) \geq 0$ for every conformal immersion $f$, we obtain a lower bound for the normal energy of Gauss maps,

\begin{equation}
E(\hat{f}) \geq 2\pi(2 - 2g(\Sigma))
\end{equation}

An interesting question is to find classes of immersions for which equality is attained in (73). For instance if $\Sigma$ is homeomorphic to a sphere $S^2$ then inequality (73) becomes $E(\hat{f}) \geq 4\pi$. Let $f : S^2 \to S^4$ be any great sphere or equatorial inclusion of the round unit sphere $S^2$, then it is totally geodesic, hence minimal with $K = 1$. Using formula (69) above we obtain $E(\hat{f}) = 4\pi$.

The situation for tori is different. For, let $\Sigma$ be homeomorphic to a 2-torus $S^1 \times S^1$ then (73) becomes $E(\hat{f}) \geq 0$. Thus $E(\hat{f}) = 0$ if and only if $W(f) = 0$. This is equivalent to the condition $f_{zz}^\perp \equiv 0$ which in turn implies that the Gauss equation reduces to $K = 1 + \|H\|^2$, and so integrating we get $0 = \int_\Sigma KdA_g \geq \int_\Sigma dA > 0$. This shows that there is no conformal immersion of a 2-torus into $S^4$ with zero normal energy Gauss map. The search for a positive lower bound for the normal energy of Gauss maps of conformally immersed tori is an interesting problem. Recently A. Gouberman and K. Leschke [7] constructed a $\mathbb{CP}^3$-family of Willmore tori conformally immersed in $S^4$ with zero spectral genus and Willmore energy $W = 2\pi^2 n$, with $n$ a positive integer.

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C.I.E.M. - FA.M.A.F. UNIVERSIDAD NACIONAL DE CÓRDOBA, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, ARGENTINA. -PHONE/FAX: +54 351 4334052/51

E-mail address: hulett@famaf.unc.edu.ar