Approximation of solutions of boundary value problems for differential equations with delayed arguments

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APPROXIMATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENTS

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Abstract. We discuss boundary value problems for first-order functional differential equations and investigate the approximation of extremal solutions of such problems. We use a monotone iterative method.

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1. INTRODUCTION

Let us consider the problem

\[
\begin{aligned}
&x'(t) = f(t, x(t), x(\alpha(t))), & t \in J := [0, T], \\
&x(0) = r x(T), & r \in (0, 1], 
\end{aligned}
\]

where \( f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and the function \( \alpha \in C(J, J) \) is such that \( \alpha(t) \leq t \) for all \( t \in J \).

In this paper we extend some results of paper [3] concerning the case where \( r = 1 \) and \( f \) satisfies a left-sided Lipshitz condition with suitable constants. We show the applicability of the monotone iterative method in obtaining monotone sequences approximating the extremal solutions of (1.1). We refer, e. g., to [2] for details about the monotone iterative method. Note that we use more general definition of lower and upper solution than the classical definition used, e. g., in [1].

We start by proving a comparison theorem used in the sequel. In Section 3, we prove the existence of a solution of the linear problem associated to (1.1). Finally, we prove the existence of monotone sequences approximating the extremal solutions of problem (1.1).
2. Comparison results

Lemma 1 ([11]). Suppose that \( \alpha \in C(J, J) \), \( \alpha(t) \leq t \) on \( J \), \( M \in C(J, \mathbb{R}) \), \( q \in C^1(J, \mathbb{R}) \) is such that
\[
q'(t) \leq -M(t)q(t) - N(t)q(\alpha(t)), \quad t \in J, \quad q(0) \leq 0,
\]
and
\[
\int_0^T N(t)e^{\int_0^t M(s)ds} dt \leq 1,
\]
where \( N \) is a nonnegative function integrable on \( J \). Then \( q(t) \leq 0 \) on \( J \).

Theorem 1. Assume that \( r \in (0, 1] \) and
\[
(1) \quad x \in C^1(J, \mathbb{R}), \quad \alpha \in C(J, J), \quad \alpha(t) \leq t, \quad t \in J, \quad M \in C(J, \mathbb{R}), \quad N \text{ is integrable on } J, \quad M(t) > 0, \quad t \in J \text{ and } N(t) \geq 0, \quad t \in J,
\]
\[\begin{align*}
(2) \quad x'(t) + M(t)x(t) + N(t)x(\alpha(t)) &\leq 0, \quad t \in J, \quad \text{if } x(0) \leq rx(T), \\
(3) \quad x'(t) + M(t)x(t) + N(t)x(\alpha(t)) + \frac{M(t)N(t)\alpha(t)+1}{rT} [x(0) - rx(T)] &\leq 0, \quad t \in J, \quad \text{if } x(0) > rx(T), \\
(4) \quad \int_0^T N(t)e^{\int_0^t M(s)ds} dt &\leq 1.
\end{align*}\]
Then \( x(t) \leq 0, \quad t \in J \).

Proof. In case (1), we assume that \( x \geq 0 \) on \( J \) and \( x \neq 0 \). Then \( x'(t) \leq 0, \quad t \in J \), so \( x \) is nonincreasing and \( x(t) \leq x(0) \quad t \in J \) so \( x(0) \geq x(T) \). On the other hand, we have \( x(0) \leq rx(T) \leq x(T) \). Thus \( x(0) = x(T) \) and \( x \) is a constant function, \( x(t) = C > 0 \). So \( x' \equiv 0 \) and \( [M(t) + N(t)]C \leq 0 \). Hence \( C = 0 \) and finally \( x(t) = 0, \quad t \in J \). It is a contradiction.

Thus we can consider that \( x \) has some negative value. Note that if \( x(0) \leq 0 \), then \( x(t) \leq 0, \quad t \in J \), by Lemma 1. Assume that \( x(0) > 0 \), then \( x(T) > 0 \). Let us consider the function \( v \) defined by
\[
v(t) = e^{\int_0^t M(s)ds} x(t), \quad t \in J.
\]
We have \( v(0) = x(0) > 0 \) and \( v(T) = e^{\int_0^T M(s)ds} x(T) > 0 \). Since \( x \) has some negative value, there exists \( t_* \in (0, T) \) such that
\[
v(t_*) = \min_{t \in [0, T]} v(t) < 0.
\]
Note that
\[
v'(t) = e^{\int_0^t M(s)ds} M(t)x(t) + e^{\int_0^t M(s)ds} x'(t) \leq -N(t)e^{\int_0^t M(s)ds} v(\alpha(t)).
\]
The integration of this from \( t_* \) to \( T \) yields
\[
-v(t_*) < v(T) - v(t_*) \leq -\int_{t_*}^T N(t)e^{\int_0^t M(s)ds} v(\alpha(t)) dt
\]
approximation of solutions of bvp for delay de

\[ \leq -v(t_*) \int_{t_*}^{T} N(t)e^{\int_{t_*}^{t} M(s) \, ds} \, dt \leq -v(t_*) \]

So \(-v(t_*) < -v(t_*)\), which is a contradiction. Thus \(x(0) \leq 0\) and, in view of Lemma 1, we get \(x(t) \leq 0, t \in J\).

In case (3), let us consider the function

\[ w(t) = x(t) + \frac{t}{rT} [x(0) - rx(T)], \quad t \in J. \]

It yields that \(w(0) = r w(T)\). Moreover,

\[ w'(t) + M(t)w(t) + N(t)w(\alpha(t)) = x'(t) + M(t)x(t) + N(t)x(\alpha(t)) \]

\[ + \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [x(0) - rx(T)] \leq 0. \]

This is included in case (2). Thus \(w(t) \leq 0, t \in J\) and finally \(x(t) \leq 0, t \in J\). \(\square\)

3. Linear problem

Now we are going to prove the existence of solution of linear problem associated to (1.1). We will need that result in succeeding section.

**Theorem 2.** Let \(\sigma \in C(J, \mathbb{R}), \alpha \in C(J, J), \alpha(t) \leq t, t \in J, M, N \in C(J, \mathbb{R})\), \(M(t) > 0, t \in J, N(t) \geq 0, t \in J\) and

\[ \begin{align*}
    x'(t) + M(t)x(t) + N(t)x(\alpha(t)) &= \sigma(t), \quad t \in J, \\
    x(0) &= rx(T), \quad r \in (0, 1].
\end{align*} \]

Moreover, assume that there exist functions \(y_0, z_0 \in C^1(J, \mathbb{R})\) such that

1. \(y_0 \leq z_0\) on \(J\),
2. \(y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \leq \sigma(t) - a(t), \quad t \in J,\)
3. \(z_0'(t) + M(t)z_0(t) + N(t)z_0(\alpha(t)) \geq \sigma(t) - b(t), \quad t \in J,\)

where

\[ a(t) = \begin{cases} 
    0 & \text{if } y_0(0) \leq ry_0(T), \\
    \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T),
\end{cases} \]

and

\[ b(t) = \begin{cases} 
    0 & \text{if } z_0(0) \geq rz_0(T), \\
    \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T).
\end{cases} \]

(3) \(\int_0^T N(t)e^{\int_{t_*}^{t} M(s) \, ds} \, dt \leq 1.\)

Then there exists, in sector \([y_0, z_0], = \{w \in C^1(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), t \in J\},\) a unique solution of (3.1).
Proof. First we prove the uniqueness of a solution. Let \( x_1, x_2 \) be solutions of (3.1). Put \( v_1 = x_1 - x_2 \) and \( v_2 = x_2 - x_1 \). Then

\[
\begin{align*}
v_1(0) &= x_1(0) - x_2(0) = r[x_1(T) - x_2(T)] = rv_1(T), \\
v_1'(t) + M(t)v_1(t) + N(t)v_1(\alpha(t)) &= \sigma(t) - \sigma(t) = 0, \quad t \in J,
\end{align*}
\]

and

\[
\begin{align*}
v_2(0) &= rv_2(T), \\
v_2'(t) + M(t)v_2(t) + N(t)v_2(\alpha(t)) &= 0, \quad t \in J.
\end{align*}
\]

In view of Theorem 1, we have \( v_1 \leq 0 \) and \( v_2 \leq 0 \). Hence \( x_1 = x_2 \).

Now we show that if \( x \) is a solution of (3.1), then \( y_0 \leq x \leq z_0 \). Put \( w_1 = y_0 - x \) and \( w_2 = x - z_0 \). Then we have

\[
\begin{align*}
w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) &\leq \sigma(t) - \sigma(t) = 0 \quad \text{if } w_1(0) \leq rw_1(T), \\
w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_1(0) - rw_1(T)] &\leq \sigma(t) - \sigma(t) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] \\
&= 0 \quad \text{if } w_1(0) > rw_1(T),
\end{align*}
\]

and

\[
\begin{align*}
w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) &\leq 0 \quad \text{if } w_2(0) \leq rw_2(T), \\
w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_2(0) - rw_2(T)] &\leq 0 \quad \text{if } w_2(0) > rw_2(T).
\end{align*}
\]

In view of Theorem 1, \( w_1 \leq 0, w_2 \leq 0 \) on \( J \). It shows that \( y_0(t) \leq x(t) \leq z_0(t), \quad t \in J \).

Finally, we prove that problem (3.1) has the solution \( x \). Let us consider two functions

\[
\begin{align*}
\bar{y}_0(t) &= \begin{cases} 
    y_0(t) & \text{if } y_0(0) \leq ry_0(T), \\
    y_0(t) + \frac{r}{rT}[y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T),
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\bar{z}_0(t) &= \begin{cases} 
    z_0(t) & \text{if } z_0(0) \geq rz_0(T), \\
    z_0(t) + \frac{r}{rT}[z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T).
\end{cases}
\end{align*}
\]

We have \( y_0(t) \leq \bar{y}_0(t) \) and \( \bar{z}_0(t) \leq z_0(t), \quad t \in J \). Moreover, \( \bar{y}_0(0) \leq r\bar{y}_0(T) \) and \( \bar{z}_0(0) \geq r\bar{z}_0(T) \). Note that if \( y_0(0) > ry_0(T) \), then \( \bar{y}_0(0) = r\bar{y}_0(T) \) and if \( z_0(0) < rz_0(T) \), then \( \bar{z}_0(0) = r\bar{z}_0(T) \).
We show that $\overline{y}_0$ and $\overline{z}_0$ are classical lower and upper solutions, respectively, of (3.1) and that $\overline{y}_0 \leq \overline{z}_0$. We have
\[
\overline{y}_0'(t) + M(t)\overline{y}_0(t) + N(t)\overline{y}_0(\alpha(t)) = y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \leq \sigma(t), \quad t \in J, \quad \text{if } y_0(0) \leq r y_0(T)
\]
and
\[
\overline{y}_0'(t) + M(t)\overline{y}_0(t) + N(t)\overline{y}_0(\alpha(t)) = y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) + \frac{M(t) + N(t)\alpha(t) + 1}{rT} [y_0(0) - r y_0(T)] \leq \sigma(t), \quad t \in J, \quad \text{if } y_0(0) \leq r y_0(T).
\]

Similarly,
\[
\overline{z}_0'(t) + M(t)\overline{z}_0(t) + N(t)\overline{z}_0(\alpha(t)) \geq \sigma(t), \quad t \in J.
\]

Thus $\overline{y}_0$ is a classical lower and $\overline{z}_0$ a classical upper solution of (3.1). Now consider the function $w = \overline{y}_0 - \overline{z}_0 \in C^1(J, \mathbb{R})$. We have
\[
w'(t) + M(t)w(t) + N(t)w(\alpha(t)) \leq 0, \quad t \in J
\]
and $w(0) \leq 0$. Lemma 1 yields $w \leq 0$ on $J$, hence $\overline{y}_0 \leq \overline{z}_0$.

Setting
\[
\begin{cases}
\overline{y}_{n+1}'(t) = \sigma(t) - M(t)\overline{y}_{n+1}(t) - N(t)\overline{y}_{n+1}(\alpha(t)), & t \in J, \\
\overline{y}_{n+1}(0) = r \overline{y}_n(T),
\end{cases}
\]
and
\[
\begin{cases}
\overline{z}_{n+1}'(t) = \sigma(t) - M(t)\overline{z}_{n+1}(t) - N(t)\overline{z}_{n+1}(\alpha(t)), & t \in J, \\
\overline{z}_{n+1}(0) = r \overline{z}_n(T),
\end{cases}
\]
and arguing similarly to the proof of [1, Theorem 3.1], we show that there exists a solution of (3.1).

\[ \square \]

4. APPROXIMATION OF EXTREMAL SOLUTIONS OF (1.1)

In this section we develop monotone iterative technique for (1.1).

**Theorem 3.** Let $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $\alpha(t) \leq t$, $t \in J$, $M, N \in C(J, \mathbb{R})$, $M(t) > 0$, $t \in J$, $N(t) \geq 0$, $t \in J$. Moreover, assume that there exist functions $y_0, z_0 \in C^1(J, \mathbb{R})$ such that
\[
(1) \ y_0 \leq z_0 \text{ on } J,
\]
(2) \( y_0, z_0 \) are lower and upper solutions of (1.1), respectively, i.e.,
\[
y'_0(t) \leq f(t, y_0(t), y_0(\alpha(t))) - a(t), \quad t \in J,
\]
\[
z'_0(t) \geq f(t, z_0(t), z_0(\alpha(t))) - b(t), \quad t \in J,
\]
where
\[
a(t) = \begin{cases} 
0 & \text{if } y_0(0) \leq r y_0(T), \\
\frac{M(t) + N(t)\alpha(t) + 1}{N(t)} [y_0(0) - r y_0(T)] & \text{if } y_0(0) > r y_0(T),
\end{cases}
\]
and
\[
b(t) = \begin{cases} 
0 & \text{if } z_0(0) \geq r z_0(T), \\
\frac{M(t) + N(t)\alpha(t) + 1}{N(t)} [z_0(0) - r z_0(T)] & \text{if } z_0(0) < r z_0(T),
\end{cases}
\]
(3) \( f(t, u, v) - f(t, \bar{u}, \bar{v}) \leq M(t)[\bar{u} - u] + N(t)[\bar{v} - v], \) if \( y_0(t) \leq u \leq \bar{u} \leq z_0(t), \)
\( y_0(\alpha(t)) \leq u \leq \bar{u} \leq z_0(\alpha(t)), \)
(4) \( \int_0^T N(t) e^{\int_0^t M(s) ds} dt \leq 1. \)

Then there exist monotone sequences \( \{y_n\} \uparrow y \) and \( \{z_n\} \downarrow z \) uniformly on \( J \) with \( y_0 \leq y_n \leq z_n \leq z_0 \) for every \( n \in \mathbb{N} \) and \( y, z \in C^1(J, \mathbb{R}) \). Functions \( y \) and \( z \) are extremal solutions of (1.1).

**Proof.** Let us consider the problem
\[
\begin{aligned}
x'(t) + M(t)x(t) + N(t)x(\alpha(t)) &= \sigma_u(t), \quad t \in J, \\
x(0) &= rx(T), \quad r \in (0, 1],
\end{aligned}
\]
where \( \sigma_u(t) = f(t, u(t), u(\alpha(t))) + M(t)u(t) + N(t)u(\alpha(t)) \) for \( u \in C(J, \mathbb{R}), \)
\( y_0 \leq u \leq z_0 \) on \( J \). Note that, in view of Theorem 2, this problem has exactly one solution. Define operator \( A: [y_0, z_0] \to [y_0, z_0] \) as \( u \mapsto v \), where \( Au = v \) is the unique solution of (4.1).

First we show that \( A \) is well defined. Put \( w = y_0 - v \). If \( y_0(0) \leq r y_0(T) \) then we have
\[
w(0) = y_0(0) - v(0) \leq r y_0(T) - rv(T) = w(T)
\]
and
\[
w' + M(t)w(t) + N(t)w(\alpha(t))
\]
\[
= y'_0(t) + M(t)y_0(t) + N(t)y_0(\alpha(t))
\]
\[
- v'(t) - M(t)v(t) + N(t)v(\alpha(t))
\]
\[
\leq f(t, y_0(y), y_0(\alpha(t))) - f(t, u(t), u(\alpha(t)))
\]
\[
+ M(t)[y_0(t) - u(t)] + N(t)[u(\alpha(t)) - y_0(\alpha(t))]
\]
\[
\leq M(t)[y_0(t) - u(t)] + N(t)[u(\alpha(t)) - y_0(\alpha(t))]
\]
\[
+ M(t)[u(t) - y_0(t)] + N(t)[y_0(\alpha(t)) - u(\alpha(t))]
\]
\[
= 0.
\]
By Theorem 1, we have \( w(t) \leq 0, \ t \in J \). Similarly if \( y_0(0) > r y_0(T) \) then
\[
0 > r w(T)
\]
and
\[
w'(t) + M(t)w(t) + N(t)w(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [w(0) - r w(T)] \leq 0.
\]
Hence \( w \leq 0 \) on \( J \). Analogously, we can show that \( v \leq z_0 \). Thus \( A \) is well defined.

Now we prove that \( A \) is monotone increasing. Put \( w = v_1 - v_2 \), where \( v_1 = Au_1 \), \( v_2 = Au_2 \) and \( u_1 \leq u_2 \). We have \( w(0) = r w(T) \) and
\[
w'(t) + M(t)w(t) + N(t)w(\alpha(t)) = \begin{cases} f(t,u_1(\alpha(t)))+M(t)u_1(t)+N(t)u_1(\alpha(t)) - f(t,u_2(\alpha(t)))-M(t)u_2(t)-N(t)u_2(\alpha(t)) \leq M(t)[u_2(t)-u_1(t)]+N(t)[u_2(\alpha(t))-u_1(\alpha(t))] + M(t)[u_1(t)-u_2(t)]+N(t)[u_1(\alpha(t))-u_2(\alpha(t))] = 0. \end{cases}
\]
In view of Theorem 1, \( v_1 \leq v_2 \). Since \( v_1 \) and \( v_2 \) were arbitrary, \( A \) is monotone increasing.

Define the sequences \( \{y_n\} \) and \( \{z_n\} \) as follows:
\[
y_{n+1} = Ay_n, \quad z_{n+1} = Az_n, \quad n \geq 0.
\]
Using the mathematical induction we can show that these sequences have the properties
\[
y_0 \leq y_1 \leq \cdots \leq y_n \leq z_n \leq \cdots \leq z_1 \leq z_0, \quad n \geq 0,
\]
because \( A \) is monotone increasing. Thus the sequence \( \{y_n\} \) is increasing and \( y_n \leq z_0 \), \( n \geq 0 \). Hence, there exists \( \lim_{n \to \infty} y_n(t) = y(t) \) for \( t \in J \). The convergence is uniform since \( \{y_n\} \) is bounded in \( C^1(J, \mathbb{R}) \). Similarly \( \{z_n\} \downarrow z \) uniformly on \( J \). It is easy to see that \( y \) and \( z \) are extremal solutions of (1.1).

Example 1. Let us consider the problem
\[
\begin{align*}
x'(t) &= e^{-x(t)} - tx \left( \frac{1}{3} t \right) - \frac{1}{2}, \quad t \in [0,1], \\
x(0) &= \frac{1}{3} x(1).
\end{align*}
\]
Put \( y_0 = 0 \) and \( z_0 = 1 \). All assumptions of Theorem 3 are satisfied with \( M(t) = 1 \) and \( N(t) = t \). Thus there exist monotone sequences converging uniformly to the extremal solutions of above problem in the sector \([0,1]_*\).
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