EXAMPLES OF DE BRANGES-ROVNYAK SPACES GENERATED BY NONEXTREME FUNCTIONS

BARTOSZ ŁANUCHA, MARIA T. NOWAK

Abstract. We describe de Branges-Rovnyak spaces $\mathcal{H}(b_{a})$, $\alpha > 0$, where the function $b_{a}$ is not extreme in the unit ball of $H^{\infty}$ on the unit disk $\mathbb{D}$, defined by the equality $b_{a}(z)/a_{\alpha}(z) = (1 - z)^{-\alpha}$, $z \in \mathbb{D}$, where $a_{\alpha}$ is the outer function such that $a_{\alpha}(0) > 0$ and $|a_{\alpha}|^{2} + |b_{a}|^{2} = 1$ a.e. on $\partial\mathbb{D}$.

1. Introduction

Let $H^{2}$ denote the standard Hardy space in the open unit disk $\mathbb{D}$ and let $T = \partial\mathbb{D}$. For $\chi \in L^{\infty}(T)$ let $T_{\chi}$ denote the bounded Toeplitz operator on $H^{2}$, that is, $T_{\chi}f = P_{\chi}(\chi f)$, where $P_{\chi}$ is the orthogonal projection of $L^{2}(T)$ onto $H^{2}$. In particular, $S = T_{\chi}$ is called the shift operator. We will denote by $M(\chi)$ the range of $T_{\chi}$ equipped with the range norm, that is, the norm that makes the operator $T_{\chi}$ a coisometry of $H^{2}$ onto $M(\chi)$.

Given a function $b$ in the unit ball of $H^{\infty}$, the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of $H^{2}$ under the operator $(I - T_{b}T_{b}^{*})^{1/2}$ with the corresponding range norm $\| \cdot \|_{b}$.

It is known that $\mathcal{H}(b)$ is a Hilbert space with reproducing kernel

$$k_{b}^{a}(z) = \frac{1 - b(w)b(z)}{1 - wz} \quad (z, w \in \mathbb{D}).$$

Here we are interested in the case when the function $b$ is not an extreme point of the unit ball of $H^{\infty}$. Then there exists an outer function $a \in H^{\infty}$ for which $|a|^{2} + |b|^{2} = 1$ a.e. on $T$. Moreover, if we suppose that $a(0) > 0$, then $a$ is uniquely determined, and, following Sarason, we say that $(b, a)$ is a pair. The function $a$ is sometimes called the Pythagorean mate associated with $b$.

It is known that both $M(a)$ and $M(\overline{a})$ are contained contractively in $\mathcal{H}(b)$ (see [10] p. 25)). Moreover, if $(b, a)$ is a corona pair, that is, $|a| + |b|$ is bounded away from 0 in $\mathbb{D}$, then $\mathcal{H}(b) = M(\overline{a})$ (see e.g. [10] p. 62)).

Let us recall that the Smirnov class $\mathcal{N}^{+}$ consists of those holomorphic functions in $\mathbb{D}$ that are quotients of functions in $H^{\infty}$ in which the denominators are outer functions. If $(b, a)$ is a pair, then the quotient $\varphi = b/a$ is in $\mathcal{N}^{+}$, and conversely, for every nonzero function $\varphi \in \mathcal{N}^{+}$ there exists a unique pair $(b, a)$ such that $\varphi = b/a$ ([11]).

Many properties of $\mathcal{H}(b)$ can be expressed in terms of the function $\varphi = b/a$ in the Smirnov class $\mathcal{N}^{+}$. It is worth noting here that if $\varphi$ is rational, then the functions $a$ and $b$ in the representation of $\varphi$ are also rational (see [11]) and in such a case $(b, a)$ is called a rational pair. Recently spaces $\mathcal{H}(b^{r})$ for rational pairs have been studied in [11], [2] and [6]. In [2] the authors described also the spaces $\mathcal{H}(b^{r})$, where $b$ is a rational outer function in the closed unit ball of $H^{\infty}$ and $r$ is a positive number.

2010 Mathematics Subject Classification. 47B32, 30H10, 30H15.

Key words and phrases. Hardy space, de Branges-Rovnyak space, Smirnov class, rigid function.
Here we describe the Branges-Rovnyak spaces \( \mathcal{H}(b_\alpha) \), \( \alpha > 0 \), where \((b_\alpha, a_\alpha)\) is such a pair that
\[
\varphi(\alpha)(z) = \frac{b_\alpha(z)}{a_\alpha(z)} = \frac{1}{(1 - z)^\alpha}.
\]
(principal branch).

For a function \( \varphi \) that is holomorphic on \( \mathbb{D} \) we define \( T_\varphi \) to be the operator of multiplication by \( \varphi \) on the domain \( \mathcal{D}(T_\varphi) = \{ f \in H^2 : \varphi f \in H^2 \} \). It is well known that \( T_\varphi \) is bounded on \( H^2 \) if and only if \( \varphi \in H^\infty \). Moreover, it was proved in [11] that the domain \( \mathcal{D}(T_\varphi) \) is dense in \( H^2 \) if and only if \( \varphi \in \mathcal{N}^+ \). More precisely, if \( \varphi \) is a nonzero function in \( \mathcal{N}^+ \) with canonical representation \( \varphi = b/a \), then \( \mathcal{D}(T_\varphi) = aH^2 \). In this case \( T_\varphi \) has a unique, densely defined adjoint \( T_\varphi^* \). In what follows we denote \( T_\varphi = T_\varphi^* \) (see [11] p. 286) for more details. The next theorem says that the domain of \( T_\varphi \) coincides with the de Branges-Rovnyak space \( \mathcal{H}(b) \).

**Theorem 1.1** ([11]). Let \((b, a)\) be a pair and let \( \varphi = b/a \). Then the domain of \( T_\varphi \) is \( \mathcal{H}(b) \) and for \( f \in \mathcal{H}(b) \),
\[
\|f\|^2 = \|f\|^2 + \|T_\varphi f\|^2.
\]

The next proposition was also proved in [11].

**Proposition 1.2** ([11]). If \( \varphi \) is in \( \mathcal{N}^+ \), \( \psi \) is in \( H^\infty \), and \( f \) is in \( \mathcal{D}(T_\varphi) \), then
\[
T_\psi T_\varphi f = T_\varphi T_\psi f = T_{\varphi \psi} f.
\]

**Corollary 1.3.** Let \( \varphi_1, \varphi_2 \in \mathcal{N}^+ \) have canonical representations \( \varphi_i = b_i/a_i \), \( i = 1, 2 \). If \( \varphi_2/\varphi_1 \in H^\infty \), then \( \mathcal{H}(b_1) \subset \mathcal{H}(b_2) \).

**Proof.** Put \( \psi = \varphi_2/\varphi_1 \). It follows from Proposition [1.2] that \( \mathcal{D}(T_{\varphi_1}) \subset \mathcal{D}(T_{\varphi_1 \psi}) \), and so
\[
\mathcal{H}(b_1) \subset D(T_{\varphi_1}) \subset D(T_{\varphi_1 \psi}) = D(T_{\varphi_2}) = \mathcal{H}(b_2).
\]

In the proof of our main theorem we will use the following description of invertible Toeplitz operators with unimodular symbols.

**Devinatz-Widom Theorem** ([7], p. 250). Let \( \psi \in L^\infty(\partial \mathbb{D}) \) be such that \( |\psi| = 1 \) a.e. on \( \partial \mathbb{D} \).

**The following are equivalent.**

(a) \( T_\psi \) is invertible.

(b) \( \text{dist}(\psi, H^\infty) < 1 \) and \( \text{dist}(\overline{\psi}, H^\infty) < 1 \).

(c) There exists an outer function \( h \in H^\infty \) such that \( \|\psi - h\|_\infty < 1 \).

(d) There exist real valued bounded functions \( u, v \) and a constant \( c \in \mathbb{R} \) such that \( \psi = e^{i(u + v + c)} \) and \( \|u\|_\infty < \frac{\pi}{2} \), where \( \overline{v} \) denotes the conjugate function of \( v \).

We will need also the notion of a rigid function in \( H^1 \). A function in \( H^1 \) is called rigid if no other functions in \( H^1 \), except for positive scalar multiples of itself, have the same argument as almost everywhere on \( \partial \mathbb{D} \). As observed in [9], every rigid function is outer. It is known that the function \((1 - z)^\alpha\) is rigid if \( 0 < \alpha \leq 1 \) and is not rigid if \( \alpha > 1 \) (see e.g. [4] Section 6.8).

The next theorem shows a close connection between kernels of Toeplitz operators and rigid functions in \( H^1 \) ([10] p. 70).

**Theorem 1.4.** If \( f \) is an outer function in \( H^2 \), then \( f^2 \) is rigid if and only if the operator \( T_{f^2} \) has a trivial kernel.

Moreover, for a pair \((b, a)\) the following sufficient condition for density of \( \mathcal{M}(a) \) in \( \mathcal{H}(b) \) is known ([10] p. 72), ([4] vol. 2, p. 496).

**Theorem 1.5.** If the function \( a^2 \) is rigid, then \( \mathcal{M}(a) \) is dense in \( \mathcal{H}(b) \).
2. The spaces $\mathcal{H}(b_\alpha)$, $\alpha > 0$

Recall that for $\alpha > 0$ we define the pair $(b_\alpha, a_\alpha)$ by

$$\varphi_\alpha(z) = \frac{b_\alpha(z)}{a_\alpha(z)} = \frac{1}{(1-z)^\alpha}.$$ 

Consequently, the outer function $a_\alpha$ is given by

$$a_\alpha(z) = \exp \left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{|1 - e^{it}|^{2\alpha}}{1 + |1 - e^{it}|^{2\alpha}} dt \right\}. \quad (2.1)$$

Since both $a_\alpha$ and $(1-z)^\alpha$ are outer functions, the equality $(1-z)^\alpha b_\alpha(z) = a_\alpha(z)$ implies that $b_\alpha$ is also outer. Hence

$$b_\alpha(z) = a_\alpha(z) \varphi_\alpha(z) = \exp \left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{1}{1 + |1 - e^{it}|^{2\alpha}} dt \right\}. \quad (2.2)$$

This formula shows that $\log |b_\alpha(z)|$ is a function harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Moreover, $|b_\alpha(1)| = 1$. We now prove that actually $b_\alpha(1) = 1$. To this end, it is enough to note that $\arg b_\alpha(r) = 0$ for all $0 < r < 1$. Indeed,

$$\arg b_\alpha(r) = \frac{1}{4\pi} \int_0^{2\pi} \text{Im} \left( \frac{e^{it} + r}{e^{it} - r} \right) \log \frac{1}{1 + |1 - e^{it}|^{2\alpha}} dt$$

$$= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{2r}{|e^{it} - r|^2} \log \frac{1}{1 + |1 - e^{it}|^{2\alpha}} dt = 0,$$

because the integrand is an odd function.

The following proposition says for which $\alpha$ a nontangential limit at 1 of each function (and its derivatives up to a given order) from $\mathcal{H}(b_\alpha)$ exists.

**Proposition 2.1.** Let $n \in \mathbb{N}$. Every $f \in \mathcal{H}(b_\alpha)$ along with its derivatives up to order $n-1$ has a nontangential limit at the point 1 if and only if $\alpha > n-1/2$.

This is a consequence of Theorem 3.2 from [5] (see also [10] and [2]), which states that the following two conditions are equivalent:

(i) for every $f \in \mathcal{H}(b_\alpha)$ the functions $f(z), f'(z), \ldots, f^{(n-1)}(z)$ have finite limits as $z$ tends nontangentially to 1;

(ii) 

$$\int_0^{2\pi} \frac{\log |b_\alpha(e^{it})|}{|1 - e^{it}|^{2\alpha}} dt < +\infty.$$ 

Since

$$\log |b_\alpha(e^{it})|^2 = \log \frac{1}{1 + |1 - e^{it}|^{2\alpha}} = \log \left( 1 - \frac{|1 - e^{it}|^{2\alpha}}{1 + |1 - e^{it}|^{2\alpha}} \right)$$

and $|\log (1-x)| \approx |x|$ for $x$ sufficiently close to zero, we have

$$\log |b_\alpha(e^{it})| \approx \frac{|1 - e^{it}|^{2\alpha}}{1 + |1 - e^{it}|^{2\alpha}} \approx |1 - e^{it}|^{2\alpha}$$

whenever $t$ is sufficiently close to 0 or $2\pi$. This implies that

$$\int_0^{2\pi} \frac{\log |b_\alpha(e^{it})|}{|1 - e^{it}|^{2\alpha}} dt < \infty$$
if and only if

$$\int_0^{2\pi} \frac{1}{|1-e^{i t}|^{2n-2\alpha}} dt < \infty,$$

which holds only when \( \alpha > n - 1/2. \)

In particular, we see that every \( f \in \mathcal{H}(b_\alpha) \) has a nontangential limit at 1 if and only if \( \alpha > 1/2. \)

The next proposition is an immediate consequence of Corollary 1.3.

**Proposition 2.2.** For every \( 0 < \alpha \leq \beta < \infty, \)

\( \mathcal{H}(b_\beta) \subset \mathcal{H}(b_\alpha). \)

Finally, we observe that

\[ |b_\alpha(z)| \geq \sqrt{\frac{1}{1 + 4\alpha}}, \]

which implies that \((b_\alpha, a_\alpha)\) is a corona pair for \( \alpha > 0. \)

**Corollary 2.3.** For \( \alpha > 0, \)

\[ M(a_\alpha) = M((1-z)^\alpha) \quad \text{and} \quad \mathcal{H}(b_\alpha) = M(p_\alpha) = M((1-z)^\alpha) \]

with equivalence of norms.

**Proof.** The equality of \( \mathcal{H}(b_\alpha) \) and \( M(p_\alpha) \) follows from the fact that \((b_\alpha, a_\alpha)\) is a corona pair, which in turn is a consequence of the fact that \( b_\alpha \) is bounded below. The latter implies that \( 1/b_\alpha \in H^\infty \) and so \( T_{b_\alpha} \) and \( T_{p_\alpha} \) are invertible. Hence

\[ M((1-z)^\alpha) = T_{b_\alpha} H^2 = T_{a_\alpha} H^2 \]

and

\[ M((1-z)^\alpha) = T_{p_\alpha} H^2 = T_{a_\alpha} H^2. \]

Both \( M(a_\alpha) \) and \( M((1-z)^\alpha) \) are boundedly contained in \( H^2. \) Hence, the Closed Graph Theorem implies equivalence of their norms. Similarly, one obtains the equivalence of norms in \( M(p_\alpha) \) and \( M((1-z)^\alpha). \) \( \square \)

3. MAIN RESULTS

We start with the following.

**Theorem 3.1.** For any \( n \in \mathbb{N} \) and \( n - 1/2 < \alpha < n + 1/2 \) we have

\[ M((1-z)^\alpha) = M((1-z)^\alpha) + \text{span}\{S^*(1-z)^\alpha, \ldots, S^n(1-z)^\alpha\}. \]

**Proof.** Let

\[ Q(z) = \frac{1 - z}{1 - z}, \quad z \in \mathbb{D}. \]

Then \( Q \) has a continuous extension to \( \mathbb{D} \setminus \{1\} \) and

\[ Q(e^{it}) = e^{(t-\pi)i}, \quad t \in (0, 2\pi), \]

which implies that

\[ T_{Q^n} = (-1)^n S^n \quad \text{for} \ n \geq 1. \]

Moreover, we observe that for \( n - 1/2 < \alpha < n + 1/2, \ n \geq 1, \) we have

\[ T_{Q^n} = T_{Q^{n-Q^n}} = (-1)^n T_{Q^{n-Q^n}} S^n. \]

Consequently,

\[ T_{(1-z)^\alpha} = T_{(1-z)^{\alpha-Q^n}} = (-1)^n T_{(1-z)^{\alpha-Q^n}} T_{Q^{n-Q^n}} S^n. \]

(3.1)
Observe now that the operator $T_{Q^\alpha-n}$ is invertible. This is an immediate consequence of the Devinatz-Widom Theorem.

Let $f \in \mathcal{M}((1 - z)^\alpha)$ and $f = T_{(1-z)^\alpha}g$ for a function $g \in H^2$. Since $T_{Q^\alpha-n}$ is invertible, there exists $g_0 \in H^2$ such that $(-1)^n g = T_{Q^\alpha-n}g_0$. Hence, using (3.1), we obtain

$$f = T_{(1-z)^\alpha}g = (-1)^n T_{(1-z)^\alpha}T_{Q^\alpha-n}g_0 = (-1)^n T_{(1-z)^\alpha}S^nS^*g_0 + \sum_{k=0}^{n-1} \langle g_0, z^k \rangle z^k \left(1 + \sum_{k=0}^{n-1} \langle g_0, z^k \rangle T_{(1-z)^\alpha}T_{Q^\alpha-n}z^k\right).$$

Since for $0 \leq k \leq n - 1$,

$$(-1)^n T_{(1-z)^\alpha}T_{Q^\alpha-n}z^k = (-1)^n T_{Q^\alpha-n}S^k 1 = S^{*(n-k)}T_{(1-z)^\alpha}1 = S^{*(n-k)}(1 - z)^\alpha,$$

we get

$$f = (1 - z)^n S^n g_0 + \sum_{k=0}^{n-1} \langle g_0, z^k \rangle S^{*(n-k)}(1 - z)^\alpha \in \mathcal{M}((1 - z)^\alpha) + \text{span}\{ S^*(1 - z)^\alpha, \ldots, S^n(1 - z)^\alpha \}.$$ 

On the other hand, if

$$f = (1 - z)^\alpha h + \sum_{k=1}^{n} c_k S^k(1 - z)^\alpha, \quad h \in H^2,$$

then, by (3.1) and (3.2),

$$f = T_{(1-z)^\alpha}h + \sum_{k=0}^{n-1} c_{n-k} S^{*(n-k)}(1 - z)^\alpha = (-1)^n T_{(1-z)^\alpha}T_{Q^\alpha-n}S^n h + (-1)^n \sum_{k=0}^{n-1} c_{n-k} T_{(1-z)^\alpha}T_{Q^\alpha-n}z^k$$

$$= T_{(1-z)^\alpha} \left( (-1)^n T_{Q^\alpha-n}S^n h + (-1)^n \sum_{k=0}^{n-1} c_{n-k} T_{Q^\alpha-n}z^k \right) \in \mathcal{M}((1 - z)^\alpha).$$

Now we prove our main result.

**Theorem 3.2.** Let $0 < \alpha < \infty$ and let $(b_\alpha, a_\alpha)$ be a pair, with the functions $b_\alpha$ and $a_\alpha$ given by (2.2) and (2.1), respectively. Then

(i) for $0 < \alpha < 1/2,$

$$\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) = (1 - z)^\alpha H^2,$$

(ii) for $n - 1/2 < \alpha < n + 1/2$, $n = 1, 2, \ldots,$

$$\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) + \mathcal{P}_n = (1 - z)^\alpha H^2 + \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all polynomials of degree at most $n - 1$, 

$\square$
(iii) \[ \mathcal{H}(b_{1/2}) = \overline{\mathcal{M}(a_{1/2})} = (1 - z)^{1/2} H^2, \]
where the closure is taken with respect to the \( \mathcal{H}(b_{1/2}) \)-norm,
(iv) for \( \alpha = n + 1/2, n = 1, 2, \ldots \),
\[ \mathcal{H}(b_{\alpha}) = \overline{\mathcal{M}(a_{\alpha}) + A_{\alpha}}, \]
where the closure is taken with respect to the \( \mathcal{H}(b_{\alpha}) \)-norm and \( A_{\alpha} \) is the \( n \)-dimensional subspace of \( \mathcal{H}(b_{\alpha}) \) defined by
\[ A_{\alpha} = \left\{ p_n \cdot P_+ \left( (1 - z)\alpha (1 - z)^{1/2} \right) + P_+ \left( p_n P_+ \left( (1 - z)\alpha (1 - z)^{1/2} \right) \right) : p_n \in \mathcal{P}_n \right\}, \]
where \( P_- = I - P_+ \).

Proof. (i) We know from Corollary 2.3 that for \( \alpha > 0 \),
\[ \mathcal{H}(b_{\alpha}) = \mathcal{M}(\pi_{\alpha}) = \mathcal{M}(1 - z)^\alpha. \]
We first observe that for \( 0 < \alpha < 1/2 \) the operator \( T_{(1 - z)^{\alpha} / (1 - z)^\alpha} \) is invertible. This follows from
\[ \frac{(1 - e^{it})^\alpha}{(1 - e^{it})} = e^{i\alpha(t - \pi)}, \quad t \in (0, 2\pi), \]
and the Devinzatz-Widom Theorem.
Consequently,
\[ \mathcal{M}(1 - z)^\alpha = T_{(1 - z)^{\alpha} / (1 - z)^\alpha} H^2 = T_{(1 - z)^{\alpha} / (1 - z)^\alpha} H^2 = (1 - z)^\alpha H^2. \]
(ii) Since \( \mathcal{H}(b_{\alpha}) \) contains \( \mathcal{M}(a_{\alpha}) = \mathcal{M}(1 - z)^\alpha \) and all polynomials (see e.g. [10, p. 25]), to prove (ii) it is enough to show that
\[ \mathcal{H}(b_{\alpha}) \subset \mathcal{P}_n + \mathcal{M}(1 - z)^\alpha. \]

By Theorem 3.1 we have
\[ \mathcal{H}(b_{\alpha}) = \mathcal{M}(1 - z)^\alpha = \mathcal{M}(1 - z)^\alpha + \text{span}\{ S^1(1 - z)^\alpha, \ldots, S^n(1 - z)^\alpha \}. \]
Therefore, we only need to show that
\[ \text{span}\{ S^1(1 - z)^\alpha, \ldots, S^n(1 - z)^\alpha \} \subset \mathcal{P}_n + \mathcal{M}(1 - z)^\alpha. \]
Clearly,
\[ S^1(1 - z)^\alpha = \frac{(1 - z)^\alpha - 1}{z} = \frac{(1 - z)^\alpha - (1 - z)^n + (1 - z)^n - 1}{z} \]
\[ = S^1(1 - z)^n - (1 - z)^\alpha S^1(1 - z)^{n - \alpha} \in \mathcal{P}_n + \mathcal{M}(1 - z)^\alpha \]
\((1 - z)^{n - \alpha} \in H^2 \) since \( n - \alpha > -1/2 \). Now assume that for any \( 1 \leq k < n \),
\[ S^k(1 - z)^\alpha \in \mathcal{P}_n + \mathcal{M}(1 - z)^\alpha, \]
or, in other words,
\[ S^k(1 - z)^\alpha = p_n + (1 - z)^\alpha h_k \text{ for some } p_n \in \mathcal{P}_n \text{ and } h_k \in H^2. \]
This completes the proof of (ii).

(iii) In view of Theorem 1.5, to prove (iii) it is enough to show that \( a_{1/2}^2 \) is a rigid function.

We actually prove that \( a_{\alpha}^2 \) is rigid for every \( 0 < \alpha \leq 1/2 \).

To this end, we observe that for \( \alpha > 0 \),

\[
\frac{1}{\sqrt{1 + 4\alpha^2}}|1 - z|^\alpha \leq |a_{\alpha}(z)| \leq |1 - z|^\alpha, \quad z \in \mathbb{D}.
\]

This follows from (3.3) and the representation of the outer function

\[
(1 - z)^\alpha = \exp \left\{ \frac{\alpha}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |1 - e^{it}| dt \right\}.
\]

Thus we have

\[
\frac{|a_{\alpha}(z)|}{|1 - z|^\alpha} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-it}|^2} \log \frac{1}{\sqrt{1 + |1 - e^{it}|^2\alpha}} dt \right\}
\]

which implies inequalities (3.3).

Now we use a reasoning analogous to that in [10] (X–5). If \( a_{\alpha}^2 \) is not rigid for some \( 0 < \alpha \leq 1/2 \), then by Theorem 1.4 there is a nonzero function \( g \) in the kernel of \( T_{a_{\alpha}} \). Then

\[
T_{\frac{1}{|1 - z|^{\alpha}}} \left( \frac{(1 - z)^\alpha g}{a_{\alpha}} \right) = P_+ \left( \frac{(1 - z)^\alpha g}{a_{\alpha}} \cdot \frac{\pi_a}{a_{\alpha}} \right) = T_{\frac{1}{|1 - z|^{\alpha}}} T_{\frac{1}{|1 - z|^{\alpha}}} g = 0,
\]

which means that \( (1 - z)^\alpha g/a_{\alpha} \) is a nonzero function in the kernel of \( T_{(1 - z)^\alpha/(1 - z)^\alpha} \), contrary to the fact that \( (1 - z)^\alpha g/a_{\alpha} \) is rigid for every \( 0 < \alpha \leq 1/2 \) (see, e.g., [4] Section 6.8).

(iv) We know that for every \( \alpha > 0 \),

\[
\mathcal{H}(b_{\alpha}) = \mathcal{M}(\pi_{\alpha}) = \mathcal{M}(1 - z)^\alpha = T_{(1 - z)^\alpha}H^2
\]

and \( \mathcal{M}(a_{\alpha}) = \mathcal{M}((1 - z)^\alpha) \) is the image under \( T_{(1 - z)^\alpha} \) of the range of \( T_{(1 - z)^\alpha/(1 - z)^\alpha} \), that is,

\[
\mathcal{M}((1 - z)^\alpha) = T_{(1 - z)^\alpha} T_{(1 - z)^\alpha}H^2.
\]

It follows that the orthogonal complement of \( \mathcal{M}((1 - z)^\alpha) \) in the space \( \mathcal{M}(1 - z)^\alpha \) is the image under \( T_{(1 - z)^\alpha} \) of \( \ker T_{(1 - z)^\alpha/(1 - z)^\alpha} \).

We now observe that for \( \alpha = n + 1/2 \),

\[
\ker T_{\frac{1}{|1 - z|^{\alpha}}} = \ker T_{\frac{1}{(1 - z)^\alpha}} T_{\frac{1}{(1 - z)^\alpha}} = (1 - z)^{1/2}P_n,
\]

where \( P_n \) is the set of all polynomials of degree at most \( n - 1 \). Finally, note that if \( p_n \) is in \( P_n \), then

\[
T_{(1 - z)^\alpha} \left( \frac{(1 - z)^{1/2}p_n}{(1 - z)^{1/2}p_n} \right) = P_+ \left( \frac{(1 - z)^\alpha(1 - z)^{1/2}p_n}{p_n + P_+ \left( \frac{(1 - z)^\alpha(1 - z)^{1/2}p_n}{p_n} \right)} \right).
\]

Our claim follows.
The following corollary is just another statement of (ii) in Theorem 3.2.

**Corollary 3.3.** For any \( n \in \mathbb{N} \) and \( n - 1/2 < \alpha < n + 1/2 \) we have
\[
\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) + \mathcal{P}_n = \mathcal{M}(a_\alpha) + \text{span}\{T_{\mathbb{R}}^1, \ldots, T_{\mathbb{R}}^n z^{n-1}\}.
\]

**Remark 3.4.** We observe that since \( a_\alpha^2 \) is rigid for all \( 0 < \alpha \leq 1/2 \), Theorem 1.6 implies that the space \( \mathcal{M}(a_\alpha) \) is dense in \( \mathcal{H}(b_\alpha) \) for all such \( \alpha \). However, for \( 0 < \alpha < 1/2 \) we have \( \mathcal{M}(a_\alpha) = \mathcal{H}(b_\alpha) \), while \( \mathcal{M}(a_{1/2}) \not\subseteq \mathcal{H}(b_{1/2}) \). The latter follows from the fact that every \( h \in H^2 \) satisfies \( |h(z)| = o((1 - |z|)^{1/2}) \) as \( |z| \to 1^- \). Thus if \( f \in \mathcal{M}(a_{1/2}) \), then \( f(z) = (1 - z)^{1/2}h(z) \), \( h \in H^2 \), and
\[
|f(z)| = |1 - z|^{1/2}|h(z)| = \left(\frac{|1 - z|}{1 - |z|}\right)^{1/2} |h(z)|(1 - |z|)^{1/2}.
\]
This shows that the nontangential limit of \( f \) at 1 is 0. On the other hand, \( \mathcal{H}(b_{1/2}) \) contains nonzero constant functions, so \( \mathcal{M}(a_{1/2}) \) cannot be equal to \( \mathcal{H}(b_{1/2}) \).

**Corollary 3.5.** If \( n - 1/2 < \alpha < n + 1/2 \), \( n \in \mathbb{N} \), and \( f \in \mathcal{H}(b_\alpha) \), then there is a function \( h \) in \( H^2 \) such that
\[
f(z) = f(1) + f'(1)(z - 1) + \ldots + \frac{f^{(n-1)}(1)}{(n-1)!}(z - 1)^{n-1} + (1 - z)^\alpha h(z).
\]

**Proof.** It follows from Proposition 3.1 that \( f \) and its derivatives of order up to \( n - 1 \) have nontangential limits at 1, say \( f(1), f'(1), \ldots, f^{(n-1)}(1) \). By Theorem 3.2 (ii), \( f \) can be written as
\[
f(z) = p_n(z) + (1 - z)^\alpha h(z) = \sum_{k=0}^{n-1} a_k (z - 1)^k + (1 - z)^\alpha h(z), \quad h \in H^2.
\]
Since every \( h \) in \( H^2 \) satisfies
\[
|h^{(k)}(z)| \leq C_k (1 - |z|)^{k+\alpha},
\]
we find that
\[
a_k = \frac{p_n^{(k)}(1)}{k!} = \frac{f^{(k)}(1)}{k!} \quad \text{for } k = 0, 1, \ldots, n - 1.
\]

The next theorem describes the space \( \mathcal{H}(\hat{b}_\alpha) \) where \( \hat{b}_\alpha \) is an outer function from the unit ball of \( H^\infty \) whose Pythagorean mate is \( \left(\frac{1 - z}{2}\right)^\alpha \), \( \alpha > 0 \).

**Theorem 3.6.** For \( \alpha > 0 \) let \( \hat{a}_\alpha(z) = \left(\frac{1 - z}{2}\right)^\alpha \) and let \( \hat{b}_\alpha \) be the outer function such that \( (\hat{b}_\alpha, \hat{a}_\alpha) \) is a pair. Then
\[
\mathcal{H}(\hat{b}_\alpha) = \mathcal{H}(b_\alpha).
\]

**Proof.** It is enough to show that \( (\hat{b}_\alpha, \hat{a}_\alpha) \) is a corona pair. The function \( \hat{a}_\alpha \) is continuous on \( \overline{D} \) and vanishes only at 1. Since \( \hat{b}_\alpha(1) = \hat{a}_\alpha(1) = 1 \), there exist \( \delta > 0 \) such that \( |\hat{b}_\alpha(z)| > 1/2 \) on \( D_1 = \overline{D} \cap \{ z : |z - 1| < \delta \} \) and \( |\hat{a}_\alpha(z)| > 1/2 \) on \( D_2 = \overline{D} \cap \{ z : |z + 1| < \delta \} \). Then the continuous function \( |\hat{b}_\alpha|^2 + |\hat{a}_\alpha|^2 \) is positive on the compact set \( \overline{D} \setminus (D_1 \cup D_2) \), so it is bounded from below by a strictly positive number \( c > 0 \).

**Remark 3.7.** Since \( \frac{1 - z}{2} \) is the Pythagorean mate for \( \frac{1 + z}{2} \), we remark that it follows from [2] that for \( \alpha > 0 \),
\[
\mathcal{H}\left(\left(\frac{1 + z}{2}\right)^\alpha\right) = \mathcal{H}\left(\frac{1 - z}{2}\right) = c + (1 - z)H^2
\]
as sets.
Finally, we remark that if \( u \) is a finite Blaschke product and \( b_\alpha \) is given by (2.2), then
\[
H(ub_\alpha) = H(b_\alpha).
\]
Since every function in \( H(u) \) is holomorphic in \( \mathbb{D} \) (see, e.g. [4, Sec. 14.2]) and \( H(b_\alpha) \) is invariant under multiplication by functions holomorphic in \( \mathbb{D} \) (see, e.g. [10, (IV-6)]), (3.4) follows from the equality
\[
H(ub_\alpha) = H(u) + uH(b_\alpha).
\]

**Question 3.8.** Can one characterize all inner functions \( u \) for which equality (3.4) holds?

**References**

[1] C. Costara, T. Ransford, *Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)?* J. Funct. Anal. 265 (2013), no. 12, 3204–3218.
[2] E. Fricain, A. Hartmann, W. T. Ross, *Concrete examples of \( H(b) \) spaces*, Comput. Methods Func. Theory 16 (2016), no 2, 287–306.
[3] E. Fricain, A. Hartmann, W. T. Ross, *Range spaces of co-analytic Toeplitz operators*, arXiv:1508.03001v1.
[4] E. Fricain, J. Mashreghi, *The theory of \( H(b) \) spaces. Vol.1*, Cambridge University Press, Cambridge, 2016.
[5] E. Fricain, J. Mashreghi, *Boundary behavior of functions in the de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory 2 (2008), no. 1, 87–97.
[6] B. Łanucha, M. Nowak, *De Branges-Rovnyak spaces and generalized Dirichlet spaces*, Publ. Math. Debrecen 91(2017), 171–184.
[7] N. K. Nikolski, *Operators, functions and systems: an easy reading. Volume 1: Hardy, Hankel and Toeplitz*, translated from the French by Andreas Hartmann, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002.
[8] D. Sarason, *Doubly shift-invariant spaces in \( H^2 \)*, J. Operator Theory 16 (1986), 75–97.
[9] D. Sarason, *Kernels of Toeplitz operators*, Toeplitz operators and related topics (Santa Cruz, CA, 1992), 153–164, Oper. Theory Adv. Appl., 71, Birkhäuser, Basel, 1994.
[10] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disc*, John Wiley and Sons Inc., New York, 1994.
[11] D. Sarason, *Unbounded Toeplitz operators*, Integral Equations Operator Theory 61 (2008), 281–298.

**Bartosz Łanucha,**
**Institute of Mathematics,**
**Maria Curie-Skłodowska University,**
**pl. M. Curie-Skłodowskiej 1,**
**20-031 Lublin, Poland**
**E-mail address:** bartosz.lanucha@poczta.umcs.lublin.pl

**Maria T. Nowak,**
**Institute of Mathematics,**
**Maria Curie-Skłodowska University,**
**pl. M. Curie-Skłodowskiej 1,**
**20-031 Lublin, Poland**
**E-mail address:** mt.nowak@poczta.umcs.lublin.pl