NON-RELATIVISTIC SUPERSYMMETRY*

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Abstract The supersymmetric extensions of the Schrödinger algebra are reviewed.

1. Non-relativistic Chern-Simons theory.

The $c \to \infty$ limit of the Chern-Simons theory of a complex scalar field $\Phi$, interacting, through the Chern-Simons term, with a scalar and a vector potential, $A_0$ and $A$, leads to the non-relativistic Lagrangian [1]

$$\int d^2r \left\{ \frac{\kappa}{2c}(\partial_t A) \times A - A_0[\kappa B + e|\Phi|^2] + i\Phi^* \partial_t \Phi - \frac{1}{2m}|\nabla \Phi|^2 + \frac{e^2}{2mc\kappa}|\Phi|^4 \right\}. \quad (1.1)$$

This theory is invariant with respect to the Galilei group, and has the following conserved quantities:

$$\begin{align*}
\{ r \to r + a \} & \quad P = \int d^2r \frac{1}{2i} [\Phi^* \nabla \Phi - (\nabla \Phi)^* \Phi] \\
\{ r \to Ar \} & \quad J = \int d^2r \, r \times P \\
\{ r \to r + Vt \} & \quad G = tP - m \int d^2r \rho \\
\{ t \to t + e \} & \quad H = \int d^2r \left[ \frac{|\nabla \Phi|^2}{2m} - \lambda_1 \rho^2 \right]
\end{align*} \quad (1.2)$$

where $\rho = |\Phi|^2$ and $\lambda_1 = e^2/2mc\kappa$. It also has the non-relativistic ‘conformal’ symmetry discovered in the early seventies by Niederer and by Hagen [2], namely

* Walifest MRST15 : New directions in the application of symmetry principles to elementary particle physics. Syracuse (N.Y.) 1993. Ed. J. Schechter. p.109-118. Singapore : World Scientific (1994).
\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbf{r} \\
t
\end{array} \right) \to \left( \begin{array}{c}
d \cdot \mathbf{r} \\
d^2 \cdot t
\end{array} \right) \\
\left( \begin{array}{c}
\mathbf{r} \\
t
\end{array} \right) \to \left( \begin{array}{c}
\frac{\mathbf{r}}{1 - ft} \\
\frac{t}{1 - ft}
\end{array} \right)
\end{array} \right.
\end{align*}
\]

\[D = tH - \frac{1}{2} \int d^2 \mathbf{r} \mathbf{r} \cdot \mathbf{P}\]
\[K = -t^2 H + 2tD + \frac{1}{2} m \int d^2 \mathbf{r} \mathbf{r}^2 \rho\]

The 8 generators in Eq. (1.2-3) form the planar Schrödinger algebra denoted by \(\text{sch}(2)\); time translations, dilatations and expansions span an \(o(2, 1)\) subalgebra.

By taking the \(c \to \infty\) limit of the \(N = 2\) supersymmetric theory of Lee, Lee, and Weinberg [4], Leblanc, Lozano and Min [3] generalized (1.1) to

\[
\int d^2 \mathbf{r} \left\{ \frac{\kappa}{2c} \left( \partial_t \mathbf{A} \right) \times \mathbf{A} - A_0 [\kappa B + e(\Phi^2 + |\Psi|^2)] + i \Phi^* \partial_t \Phi + i \Psi^* \partial_t \Psi \right.
\]

\[
- \frac{1}{2m} \left( (D\Phi)^2 + (D\Psi)^2 \right) + \frac{e}{2mc} B|\Psi|^2 + \lambda_1 |\Phi|^4 + \lambda_2 |\Phi|^2 |\Psi|^2 \bigg\},
\]

where \(\Psi\) is a two-component fermion field, \(\lambda_1 = e^2/2mc \kappa\), and \(\lambda_2 = 3\lambda_1\). (Notice here the Pauli term). The extended system (1.4) is also Schrödinger-symmetric: Putting \(\rho_B = |\Phi|^2\) and \(\rho_F = |\Psi|^2\), one finds the conserved charges

\[
\begin{align*}
P &= \int d^2 \mathbf{r} \left[ \frac{1}{2i} \left( \Phi^* D\Phi - (D\Phi)^* \Phi + \Psi^* D\Psi - (D\Psi)^* \Psi \right) \right] \\
J &= \int d^2 \mathbf{r} \left[ \mathbf{r} \times \mathbf{P} + \frac{1}{2} \rho_F \right] \\
G &= tP - m \int d^2 \mathbf{r} \left[ \rho_B + \rho_F \right] \\
H &= \int d^2 \mathbf{r} \left[ \frac{1}{2m} \left( (D\Phi)^2 + (D\Psi)^2 \right) \right.
\]

\[
- \frac{e}{2mc} B\rho_F - \lambda_1 \rho_B^2 - \lambda_2 \rho_B \rho_F \bigg]\]

\[D = tH - \frac{1}{2} \int d^2 \mathbf{r} \mathbf{r} \cdot \mathbf{P}\]
\[K = -t^2 H + 2tD + \frac{1}{2} m \int d^2 \mathbf{r} \mathbf{r}^2 (\rho_B + \rho_F)\]

\[(1.5)\]
The theory is also invariant with respect to global rotations of bosons and fermions, \( \Phi \rightarrow e^{i\alpha}\Phi \) and \( \Psi \rightarrow e^{i\beta}\Psi \), yielding the two central charges

\[
\begin{align*}
M_B &= m \int d^2 r \rho_B \quad \text{bosonic mass} \\
M_F &= m \int d^2 r \rho_F \quad \text{fermionic mass}
\end{align*}
\] (1.6)

Leblanc et al. [3] demonstrate furthermore that this system has an \( N = 2 \) supersymmetry:

\[
Q_2 = \frac{1}{\sqrt{2m}} \int d^2 r \Phi^* D_+ \Psi
\] (1.7)

(\( \text{where } D_+ = D_1 + iD_2 \)) and its complex conjugate, \( Q_2^* \), satisfy

\[
\{Q_2, Q_2^*\} = -iH.
\] (1.8)

Adding

\[
Q_1 = i\sqrt{2m} \int d^2 r \Phi^* \Psi
\] (1.9)

and its conjugate \( Q_1^* \) (which satisfy \( \{Q_1, Q_1^*\} = -2iM \)), as well as

\[
F = i[K, Q_2] \quad \text{and} \quad F^* = i[K, Q_2^*],
\] (1.10)

one gets a closed algebra:

\[ P, J, G, H, D, K, M_B, M_F, Q_1, Q_1^*, Q_2, Q_2^*, F, F^* \]

span a 16-dimensional supersymmetric extension of the Schrödinger algebra [3].

2. Schrödinger superalgebras.

Supersymmetric extensions of the Schrödinger algebra were considered previously [5]. For example, Beckers et al. consider an \( n \)-dimensional fermionic harmonic oscillator with the total Hamiltonian

\[
H_{\text{tot}} = H_B + H_F = \frac{1}{2m} \left( \frac{p^2}{m} + m\omega^2 r^2 \right) + \frac{1}{2} \omega \sum_{a=1}^{n} \left( \zeta^+_a \zeta^a - \zeta^-_a \zeta^+_a \right),
\] (2.1)

the \( \zeta^a_\pm (a = 1, \ldots, n) \) being the generators of a Clifford algebra. As pointed out by Niederer twenty years ago [6] for a bosonic oscillator and extended to the fermionic
case by Beckers et al., $H_{\text{tot}}$ admits the same Schrödinger symmetry as a free particle, with generators

$$
\begin{aligned}
J_{ab} &= r_a p_b - r_b p_a - (\zeta_{+a} \zeta_{-b} + \zeta_{-a} \zeta_{+b}) \\
H_B &= \frac{1}{2} \left( \frac{p^2}{m} + m \omega^2 r^2 \right) \\
C_\pm &= \pm \frac{i}{2m} e^{\mp 2i \omega t} (p \pm im\omega r)^2 \\
P_\pm &= \pm ie^{\mp i \omega t} (p \pm im\omega r) \\
M &= m
\end{aligned}
$$

Here $H_B$, $C_+$ and $C_-$ generate an $o(2,1)$ algebra; the angular momentum $J$ generates $o(n)$. The $P_\pm$ and $M$ span the $n$-dimensional Heisenberg algebra $h(n)$.

This system also has an $N = 2$ conformal supersymmetry, with supercharges

$$
\begin{aligned}
Q_\pm &= (p \mp im\omega r) \cdot \zeta_\mp \sqrt{m} \\
S_\pm &= e^{\mp 2i \omega t} (p \pm im\omega r) \cdot \zeta_\mp \sqrt{m} \\
T_\pm &= e^{\mp i \omega t} \sqrt{m} \zeta_\pm.
\end{aligned}
$$

Note that $H_B$ and $H_F$ are both bosonic and are separately conserved. In the plane ($n=2$) the Beckers et al. algebra has 18 generators; its global structure is

$$
\tilde{\text{sch}}(2/2) \cong \left( \text{o}(2) \times \text{osp}(1/2) \right) \otimes h(n/2).
$$

The relation with the superalgebra of Leblanc et al. is explained in our recent paper [7]:

**Theorem.** In any dimension $n$ and for any given integer $N$, the $n$-dimensional Schrödinger algebra $\text{sch}(n)$ admits an $N$-supersymmetric extension with generators
\[
\begin{align*}
\mathcal{J}^{ab} &= Q^a P^b - P^a Q^b - \sum_{j=1}^{N} \xi_j^a \xi_j^b, \\
\mathcal{H} &= \frac{1}{2} P^2, \\
\mathcal{D} &= P \cdot Q, \\
\mathcal{K} &= \frac{1}{2} Q^2, \\
\mathcal{G} &= Q, \\
\mathcal{P} &= P, \\
\mathcal{M} &= 1, \\
\mathcal{H}_{jk} &= \xi_j \cdot \xi_k,
\end{align*}
\]

where \( j, k = 1, \ldots, N \), \( Q, P \in \mathbb{R}^n \), and \( \xi \in \mathbb{R}_1^n \) is an \( n \)-dimensional Grassmann vector. The global structure of the algebra is that of

\[
\widetilde{\text{sch}}(n/N) \cong \left( \mathfrak{o}(n) \times \mathfrak{osp}(1/N) \right) \otimes \mathfrak{h}(n/N).
\]

For \( n \neq 2 \) the extension is unique. For \( n = 2 \), however, the system admits an additional ‘twisted’ extension: For any integer \( \nu \), this latter is given by the generators

\[
\begin{align*}
\mathcal{J} &= Q \times P, \\
\mathcal{H} &= \frac{1}{2} P^2, \\
\mathcal{D} &= P \cdot Q, \\
\mathcal{K} &= \frac{1}{2} Q^2, \\
\mathcal{G} &= Q, \\
\mathcal{P} &= P, \\
\mathcal{M} &= 1, \\
\mathcal{H}_{jk} &= \xi_j \cdot \xi_k, \\
\mathcal{L}_{jk} &= \xi_j \times \xi_k,
\end{align*}
\]

where \( j = 1, \ldots, \nu \). Note that, in the plane, the cross product of two vectors is a scalar, \( \mathbf{u} \times \mathbf{v} = \varepsilon_{ij} u^i v^j \) where \( \varepsilon_{ij} \) is the totally antisymmetric symbol, \( \varepsilon_{12} = 1 \).

The fermionic charges (with the exception of supertranslations) come in pairs, one for the scalar product and one for the cross product, respectively. Also, there are two rather than just one types of ‘fermionic Hamiltonians’, namely \( \mathcal{H}_{jk} \) and \( \mathcal{L}_{jk} \).
As explained in Ref. 7, such ‘twisted’ extensions can only arise in the plane. The clue is that that the internal rotations in osp have to commute with ordinary space rotations, and this only happens in two dimensions. Put in another way, it is only in the plane that we can have two ‘scalar’ products, one symmetric, the other antisymmetric.

For \( \nu = 1 \) there is no \( \mathcal{H}_{jk} \)-type charge, and our ‘twisted’ super-schrödinger algebra has the structure of

\[
\tilde{\text{sch}}_c(1) \cong \left( \text{o}(2) \times \text{osp}(1/2) \right) \oplus \mathfrak{h}(2/1),
\]

which is again an \( N = 2 \) supersymmetric extension of the planar Schrödinger algebra. The difference with (2.4) is that there is now just one, rather than two Grassmann variables and hence one, rather than two super-translations.

Physical examples of this supersymmetry are provided by the magnetic vortex [11] presented in the next Section, and by Chern-Simons theory [3]. The dictionary between the superalgebra of Leblanc et al. [3] and our twisted super-Schrödinger algebra \( \tilde{\text{sch}}_c(1) \) is presented in Table I. here below.

**TABLE I.** The dictionary between our superalgebra (2.7) with that of Leblanc et al. [3].

| LLM | DH |
|-----|----|
| \( J \) | \( \mathcal{J} + \mathcal{M} - \frac{1}{4} \mathcal{L} \) |
| \( H \) | \( \mathcal{H} \) |
| \( K \) | \( \mathcal{K} \) |
| \( D \) | \( \frac{1}{2} \mathcal{D} \) |
| \( G_\pm \) | \( \mp G_1 + iG_2 \) |
| \( P_\pm \) | \( \mp P_1 + iP_2 \) |
| \( N_F \) | \( -\frac{1}{2} \mathcal{L} \) |
| \( N_B \) | \( \mathcal{M} + \frac{1}{2} \mathcal{L} \) |
| \( Q_1 \) | \( \Xi^1 + i\Xi^2 \) |
| \( Q_1^* \) | \( -i(\Xi^1 - i\Xi^2) \) |
| \( Q_2 \) | \( \frac{1}{2}(Q + iQ^*) \) |
| \( Q_2^* \) | \( -\frac{i}{2}(Q - iQ^*) \) |
| \( F \) | \( -\frac{i}{2}(S + iS^*) \) |
| \( F^* \) | \( \frac{1}{2}(S - iS^*) \). |
3. Supersymmetry of the magnetic vortex.

A couple of years ago Jackiw [8] pointed out that a spin-0 particle in a Dirac monopole field has an $o(2,1)$ dynamical symmetry, generated by the spin-0 Hamiltonian, $H_0 = \frac{\pi^2}{2m}$ where $\pi = p - eA$, by the dilatation and by the expansion,

$$D = tH_0 - \frac{1}{4}\{\pi, r\} \quad \text{and} \quad K = -t^2H_0 + 2tD + m\mathbf{r}^2/2,$$

(3.1)

respectively, to which angular momentum adds an $o(3)$. This allowed him to calculate the spectrum and the wave functions group-theoretically.

Jackiw’s result was extended to spin-$\frac{1}{2}$ particles by D’Hoker and Vinet [9] who have shown that for the Pauli Hamiltonian

$$H = \frac{1}{2m} [\pi^2 - e\mathbf{B} \cdot \sigma]$$

(3.2)

not only the conformal generators $D$ and $K$, but also the fermionic generators

$$Q = \frac{1}{\sqrt{2m}} \pi \cdot \sigma \quad \text{and} \quad S = \sqrt{m/2} \mathbf{r} \cdot \sigma - tQ$$

(3.3)

are conserved. Thus the spin system admits an $o(3) \times \text{osp}(1/1)$ conformal super-symmetry, providing us with an algebraic solution of the Pauli equation.

More recently, Jackiw [10] found that the $o(2,1)$ symmetry, generated by — formally — the same $D$ and $K$ as above, is also present for a magnetic vortex (an idealization for the Aharonov-Bohm experiment), allowing for a group-theoretic treatment of the problem.

Very recently [11] we pointed out that the $N=2$ supersymmetry of the Pauli Hamiltonian of a spin-$\frac{1}{2}$ particle (present for any magnetic field in the plane [12]) combines, for a magnetic vortex, with Jackiw’s $o(2) \times o(2,1)$ into an $o(2) \times \text{osp}(1/2)$ superalgebra, which is a sub-superalgebra of the twisted super-Schrödinger algebra (2.7). To see this, let us consider a spin-$\frac{1}{2}$ particle in a static magnetic field $\mathbf{B} = (0,0,B(x,y))$. Dropping the irrelevant $z$ variable, we can work in the plane. Then our model is described by the Pauli Hamiltonian which is the planar version of (3.2),

$$H = \frac{1}{2m} [\pi^2 - eB\sigma_3],$$

(3.4)

It is now easy to see that the Hamiltonian is a perfect square in two different ways: both operators

$$Q = \frac{1}{\sqrt{2m}} \pi \cdot \sigma \quad \text{and} \quad Q^* = \frac{1}{\sqrt{2m}} \pi \times \sigma$$

(3.5)

where $\sigma = (\sigma_1, \sigma_2)$, satisfy

$$\{Q, Q\} = \{Q^*, Q^*\} = 2H.$$

(3.6)
Thus for any static and purely magnetic field in the plane, $H$ is an $N=2$ super-symmetric Hamiltonian. The ‘twisted’ charge $Q^*$ was used, e.g., by Jackiw [13], to describe the Landau levels in a constant magnetic field — a classic example of supersymmetric quantum mechanics [12] — where the supercharge $Q$ is a standard object to be looked at.

Let us assume henceforth that $B$ is the field of a point-like magnetic vortex directed along the $z$-axis, $B = \Phi \delta(r)$, where $\Phi$ is the total magnetic flux. This can be viewed as an idealization of the spinning version of the Aharonov-Bohm experiment [14].

Inserting $A_i(r) = -(\Phi/2\pi) \epsilon_{ij} r^j / r^2$ into the Pauli Hamiltonian $H$, it is straightforward to check that $D$ and $K$ in the planar version of (3.1) generate, along with $H$, the $o(2,1)$ Lie algebra:

$$[D, H] = -iH, \quad [D, K] = iK, \quad [H, K] = 2iD. \quad (3.7)$$

The angular momentum, $J = r \times \pi$, adds to this $o(2,1)$ an extra $o(2)$. (The correct definition of angular momentum requires boundary conditions [15]).

Commuting $Q$ and $Q^*$ with the expansion, $K$, yields two more generators, namely

$$S = i[Q, K] = \sqrt{\frac{m}{2}} \left( r - \frac{\pi t}{m} \right) \cdot \sigma, \quad (3.8)$$

$$S^* = i[Q^*, K] = \sqrt{\frac{m}{2}} \left( r - \frac{\pi t}{m} \right) \times \sigma.$$

It is now straightforward to see that both sets $Q, S$ and $Q^*, S^*$ extend the $o(2,1) \cong osp(1/0)$ into an $osp(1/1)$ superalgebra. However, these two algebras do not close yet: the ‘mixed’ anticommutators $\{Q, S^*\}$ and $\{Q^*, S\}$ bring in a new conserved charge, viz.

$$\{Q, S^*\} = -\{Q^*, S\} = J + 2\Sigma, \quad (3.9)$$

where $\Sigma = \frac{1}{2} \sigma_3$. $J$ satisfies non-trivial commutation relations with the supercharges,

$$[J, Q] = -iQ^*, \quad [J, Q^*] = iQ, \quad [J, S] = -iS^*, \quad [J, S^*] = iS. \quad (3.10)$$

Thus, setting

$$Y = J + 2\Sigma = r \times \pi + \sigma_3,$$
the generators $H, D, K, Y$ and $Q, Q^*, S, S^*$ satisfy

\[
\begin{align*}
\{Q, D\} &= \frac{i}{2}Q, & \{Q^*, D\} &= \frac{i}{2}Q^*, \\
\{Q, K\} &= -iS, & \{Q^*, K\} &= -iS^*, \\
\{Q, H\} &= 0, & \{Q^*, H\} &= 0, \\
\{Q, Y\} &= -iQ^*, & \{Q^*, Y\} &= iQ, \\
\{S, D\} &= -\frac{i}{2}S, & \{S^*, D\} &= -\frac{i}{2}S^*, \\
\{S, K\} &= 0, & \{S^*, K\} &= 0, \\
\{S, H\} &= iQ, & \{S^*, H\} &= iQ^*, \\
\{S, Y\} &= -iS^*, & \{S^*, Y\} &= iS, \\
\{Q, Q\} &= 2H, & \{Q^*, Q^*\} &= 2H, \\
\{S, S\} &= 2K, & \{S^*, S^*\} &= 2K, \\
\{Q, Q^*\} &= 0, & \{S, S^*\} &= 0, \\
\{Q, S\} &= -2D, & \{Q^*, S^*\} &= -2D, \\
\{Q, S^*\} &= Y, & \{Q^*, S\} &= -Y.
\end{align*}
\]

(3.11)

Added to the $o(2, 1)$ relations, this means that our generators span the $osp(1/2)$ superalgebra. On the other hand,

\[
Z = J + \Sigma = r \times \pi \mp \frac{1}{2}\sigma_3
\]

(3.12)

commutes with all generators of $osp(1/2)$, so that the full symmetry is the direct product $osp(1/2) \times o(2)$, generated by

\[
\begin{aligned}
Y &= r \times \pi + \sigma_3, & Q &= \frac{1}{\sqrt{2m}}\pi \cdot \sigma, \\
H &= \frac{1}{2m} \left[ \pi^2 - eB\sigma_3 \right], & Q^* &= \frac{1}{\sqrt{2m}}\pi \times \sigma, \\
D &= -\frac{1}{4} \{\pi, q\} - \frac{eB}{2m} \sigma_3, & S &= \sqrt{\frac{m}{2}} q \cdot \sigma, \\
K &= \frac{1}{2}mq^2, & S^* &= \sqrt{\frac{m}{2}} q \times \sigma, \\
Z &= r \times \pi + \frac{1}{2}\sigma_3.
\end{aligned}
\]

(3.13)
where we have put $q = r - (\pi/m)t$. Notice that (3.6) is a sub-superalgebra of the quantized version of the ‘twisted’ superalgebra (2.7) obtained by the substitutions $P \to \pi$, $Q \to q$ and $\xi \to \sigma$.

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