On Population in a Polluted Patchy Environment

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Abstract. In this paper, the effect of diffusion on the permanence of a species in a polluted patchy environment is studied. We set up a single species diffusive system on a region composed of two patches, one of which is contaminated and the other is clear. When there is no diffusion and the exogenous toxicant concentration exceeds some a value, the population in the polluted patch will go to extinction. When the diffusion exists, we give suitable conditions for the permanence or extinction of the population in the system.

1. Introduction

It is well known that the pollution of the environment is a very serious problem in the world today. Organisms are often exposed to a polluted environment and take up toxicant. In order to protect environment and regulate the release of toxic substances wisely, we must assess the harm of the toxicant to the species exposed to it. Therefore, it is important to study the effect of toxicant on a population and establish protected regions to conserve the endangered species and ecosystems.

Since Hallam and his colleagues proposed a toxicant-population model in the early 1980s [1], mathematical models of single or multiple population with toxicant effect have been studied extensively. Recently, some authors study the influence of diffusion on the time-dependent single species dynamics [2-4]. Mahbuba and Chen [4] consider the following system:

\[
\begin{align*}
\dot{x}_i &= x_i [b_i(t) - a_i(t)x_i] + D_i(t)(x_j - x_i), \\
&= i, j = 1, 2.
\end{align*}
\]

(L)

If \( b_i(t) \) and \( a_i(t) \) (\( i = 1, 2 \)) are all positive periodic functions, then system (L) possesses a globally stable positive periodic solution for any positive diffusive rates \( D_i(t) \) and \( D_j(t) \). In [2], Cui and Chen study system (L) with \( n \) patches, assuming that all the coefficients are periodic functions with common period \( \omega > 0 \), and consider the effect of diffusion on the species that live in a changing patchy environment. In above papers, the authors always suppose that the species has positive intrinsic growth rates \( b_i(t) \) (\( i = 1, 2 \)) in each patch. But the most endangered species live in a polluted patchy environment, in the sense that in some of the isolated patches, the species will vanish without the contribution from other patches. This fact urges us to consider the effect of diffusion on the permanence and extinction of single species living in a polluted patchy environment. In [5-8], Ma, Hallam and Ahmad et al obtain some conditions for the persistence of population without diffusion. In [9], Wang and Chen study the global stability of system (L) under the assumption that the functions

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\(a_i(t)\) and \(b_i(t)\) \((i = 1, 2)\) are continuous and bounded above and below by positive constants, and \(D_i(t)\) \((i = 1, 2)\) are continuous, nonnegative, and bounded by positive constants.

In this paper, we consider a single species on a region composed of two patches, one of which is contaminated, and the other is unpolluted. The both patches are connected by diffusion. We will show that the species will be permanent for some diffusion rates and go to extinction for other diffusion rates. Thereby we can control the diffusion between two patches to conserve the species in the polluted environment. Then we give some conditions for the permanence or extinction of the population.

2. The model

The state variables of the model are \(x_1 = x_1(t)\) and \(x_2 = x_2(t)\), the biomass of the species in patch 1 and patch 2 at time \(t\), respectively, \(C_0(t)\), the concentration of toxicant in the organisms in the polluted patch at time \(t\) and \(C_e(t)\), the concentration of toxicant in the environment of polluted patch 1 at time \(t\).

Let us consider the following model:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[b_1(t) - a_1(t)x_1(t)] + D_1(t)(x_1(t) - x_1(t)); \\
\dot{x}_2(t) &= x_2(t)[b_2(t) - a_2(t)x_2(t)] + D_2(t)(x_2(t) - x_2(t)); \\
\dot{C}_0(t) &= k_0(C_0(t) - g(t)C_e(t) - m(t)C_0(t)); \\
\dot{C}_e(t) &= -k_e(C_e(t) + u(t)).
\end{align*}
\]

The initial conditions are \(x_1(0) > 0, x_2(0) > 0, C_0(0) > 0, C_e(0) > 0\), and all the coefficients in \((M_0)\) are continuous and bounded on \([0, +\infty)\) by positive constants. \(a_i(t)\) is the intrinsic growth rate for species \(x\) in patch \(i\) \((i=1,2)\); \(\alpha\), a positive constant, denotes the species response to the toxicant present in the organism; \(b_i(t)\) represents the self-inhibition coefficients; \(D_i(t)\) denotes the diffusion coefficient of species \(x\) from patch \(2\) to patch \(1\) and \(D_2(t)\) denotes the diffusion coefficient of species \(x\) from patch \(1\) to patch \(2\); \(k_0\) represents an organism’s net uptake rate of the toxicant from the polluted environment; \(g(t)\) and \(m(t)\) represent the egestion and depuration rates of the toxicant in an organism, respectively; \(h(t)\) denotes the loss rate of the toxicant in the environment due to natural degradation; \(u(t)\) represents the exogenous rate of toxicant input into the environment of patch 1. Given a function \(g(t)\) defined on \([c, +\infty)\), we set \(g^M = \sup\{g(t)| c \leq t < +\infty\}\), \(g^L = \inf\{g(t)| c \leq t < +\infty\}\), \(A(t,s) = \frac{1}{t-s} \int_s^t g(\tau) d\tau\) for \(c \leq t_1 < t_2\).

Definition 2.1 The lower and upper averages of a function \(g\), denoted by \(m[g]\) and \(M[g]\), respectively, are defined by

\[
\begin{align*}
m[g] &= \liminf_{s \to +\infty} A(g, t_1, t_2) | t_1 - t_2 | \geq s \quad \text{and} \quad M[g] &= \limsup_{s \to +\infty} A(g, t_1, t_2) | t_1 - t_2 | \geq s.
\end{align*}
\]

Since the set \(\{A[g, t_1, t_2] | t_2 - t_1 \geq s\}\) gets smaller as \(s\) increases, the limits exists; and since \(g^L \leq A[g, t_1, t_2] \leq g^M\), it follows that \(g^L \geq M[g] \geq m[g] \geq g^L\).

Lemma 2.1 Both positive and nonnegative cones of \(R^2\) are invariant set with respect to system \((M_0)\).

Lemma 2.2 Each of \(C_0\) and \(C_e\) is a concentration, and thus, these variables satisfy the inequalities \(0 \leq C_0(t) \leq 1\) and \(0 \leq C_e(t) \leq 1\) for \(t \in R^+\). In order that \(C_0\), \(C_e\) remain less than one, it is necessary that \(k^M \leq g^L + m^L\) and \(u^M \leq h^L\). The specific computation is similar to Theorem 1 in paper [7].

The model \((M_0)\) discussed here is a four-dimensional system. Since the latter two equations in model \((M_0)\) are linear in \(C_0\), \(C_e\), respectively; they are explicitly solvable. Therefore, the key to our permanent-extinct problem is to find the sufficient conditions for permanence and extinction of species through \(C_0\) for the two-dimensional subsystem consisted of the first two equations in system \((M_0)\) in which \(C_0(t)\) is a bounded continuous function of \(u(t)\) and it’s a bounded continuous function.

Then for any \(\zeta > 0\) \((i = 1, 2)\), the system \((M_0)\) reduces to the following model \((M_1)\):

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[b_1(t) - a_1(t)x_1(t)] + D_1(t)(x_1(t) - x_1(t)); \\
\dot{x}_2(t) &= x_2(t)[b_2(t) - a_2(t)x_2(t)] + D_2(t)(x_2(t) - x_2(t)); \\
\dot{x}_0(t) &= \xi_1, \quad x_0(0) = \xi_2.
\end{align*}
\]
Definition 2.2 The system (M_1) is called permanence, if there exist positive constants \( \lambda, k \) and \( T \) such that \( \lambda \leq x_i(t) \leq k \) for any positive solution \( x(t) = (x_1(t), x_2(t)) \) of (M_1) as \( t \geq T \).

For the logistic equation

\[ \dot{N} = N \left\{ b(t) - a(t)N \right\}, \]

where \( b(t) \) is bounded above by a positive constant, and \( a(t) \) is bounded above and below by positive constants. In analogy with Theorem 1.1 in [8], we have the following lemma.

Lemma 2.3 (I) Suppose that \( M[b(t)] > 0 \), then any positive solutions of equation (1) defined on \([t_0, +\infty)\) for some \( t_0 \in [c, +\infty) \), are bounded above and below by positive constants.

(II) Suppose that \( M[b(t)] < 0 \), then all solutions \( N(t) \) of equation (1) with positive initial values satisfy

\[ \lim_{t \to +\infty} N(t) = 0. \]

Let us denote by \( N_0(t) \) any positive solution of equation (1). Then from Lemma 2.3 and the definitions of lower and upper averages of a function, \( M[N_0] \) and \( m[N_0] \) exist.

3. Main results

Theorem 3.1 When there is no diffusion between two patches, for system (M_1), we have the following conclusions:

(I) For patch 1, when \( M[b_1(t) - \alpha C_0(t)] < 0 \), \( \lim_{t \to +\infty} x_1(t) = 0 \); when \( M[b_1(t) - \alpha C_0(t)] > 0 \), the species \( x \) in patch 1 is permanent.

(II) For patch 2, when \( M[b_2(t)] > 0 \), the species in patch 2 is permanent.

This theorem is deduced from Lemma 2.3 easily. From above theorem, we know that when \( M[b_1(t) - \alpha C_0(t)] < 0 \), the species \( x \) in patch 1 will be extinct, so we should establish some protected patch and control the diffusion between the polluted patch and the protected patch to conserve the endangered species.

Denote \( \Psi(t) = \max \{b_1(t) - \alpha C_0(t) + D_2(t) - D_1(t), b_2(t) + D_1(t) - D_2(t)\} \).

Theorem 3.2 Suppose \( M[\Psi(t)] < 0 \), then the solutions of (M_1) satisfy \( \lim_{t \to +\infty} N(t) = 0 \).

Proof. Consider the function \( V(t) = x_1(t) + x_2(t) \), calculating the derivative of function \( V(t) \) along the solution of (M_1),

\[ \dot{V} = \left\{ b_1(t) - \alpha C_0(t) + D_2(t) - D_1(t) \right\} x_1(t) - a_1(t) x_1^2(t) + \left\{ b_2(t) + D_1(t) - D_2(t) \right\} x_2(t) - a_2(t) x_2^2(t) < 0. \]

Let \( u(t) \) be the solution of the equation \( \dot{u}(t) = \Psi(t) u \) with \( u(0) = V(0) \). Using the standard comparison theorem and Lemma 2.3, we have \( V(t) \leq u(t) \to 0 \). So \( x_i(t) \to 0 \) as \( t \to +\infty \).

It is obvious that system (M_1) has a unique solution \( x(t) = (x_1(t), x_2(t)) \), which exists for all \( t > 0 \).

Moreover, we have the following theorem.

Theorem 3.3 For system (M_1), there exist \( B > 0 \) and \( \tau > 0 \), such that \( x_i(t) \leq B \) for \( t > \tau \).

Proof. Define \( V(t) = \max \{x_1(t), x_2(t)\} \). Calculating the upper-right derivative of \( V(t) \) along the positive solution of (M_1), we have

\[ D^+ V \big|_{(M_1)} \leq V \max \{b_1^{M} - \alpha C_0^{M} - a_1^{M} V, b_2^{M} - a_2^{M} V\}. \]

Let \( B = \max \{(|b_1^{M} - \alpha C_0^{M}| + 1)/a_1^{M}, (|b_2^{M}| + 1)/a_2^{M}\} \). If \( V \geq B \), then \( D^+ V \leq -V \). Hence, there exists \( \tau = \tau(x_1(0), x_2(0)) > 0 \) such that \( V(t) \geq B \) for all \( t > \tau \), which means that \( x_i \leq B \) for all \( t > \tau \) if \( x(t) \) exists.

A consequence of Theorem 3.3 is that for \( \zeta > 0 \) \((i = 1, 2)\), the solution of (M_1) is ultimately bounded above. We will show that this solution is also ultimately bounded below, away from zero, provided that one of the following conditions is satisfied.
(C1) \( M[b_2(t)-D_2(t)] > 0 \);
(C2) \( M[\psi(t)] > 0 \), \( \phi(t) = \min\{b_1(t)-\alpha C_0(t)+D_2(t)-D_1(t), \ b_2(t)+D_1(t)-D_2(t)\} \).

Theorem 3.4 Suppose that (C1) or (C2) holds, then there exist \( b (0 < b < B) \) and \( \tau > 0 \), such that the solution of system \((\text{M}_1)\), \( x_i(t) \geq b \) (i = 1, 2) for \( t > \tau \), and \( \Omega = \{(x_1, x_2) | b \leq x_i(t) \leq B, \ i = 1, 2\} \) is a positive invariant set of system \((\text{M}_1)\), i.e., system \((\text{M}_1)\) is permanent.

**Proof.** (I) Suppose that (C1) holds, we get
\[
\dot{x}_i(t) > x_i(t)(b_i(t)-D_i(t)-a_i(t)x_i(t)).
\]
Since \( M[b_2(t)-D_2(t)] > 0 \), by Lemma 2.3, for any solution \( \bar{N}(t) \) with \( \bar{N}(0) > 0 \) of the logistic equation
\[
\bar{N}(t) = N(t)(b_i(t)-D_i(t)-a_i(t)x_i(t)),
\]
we have \( \bar{N}(t) \subset [\bar{N}(0) > 0] \). Let \( N(t) \) be the solution of (3) with \( N(0) = x_2(0) \), using the standard comparison theorem again, we obtain \( x_i(t) \geq N(t) \). Set \( \varepsilon = \bar{N} / 2 \), then there exists \( T > 0 \) such that
\[
x_i(t) \geq N(t) > \bar{N}(0) - \varepsilon = \bar{N} / 2 := \Delta > 0, \quad t > T.
\]
Moreover, for \( x_1(t) \), we have
\[
\dot{x}_1 \geq x_1[b_1(t)-D_1(t)-\alpha C_0(t)-a_1(t)] + D_1(t), \quad t \geq T,
\]
\[
= -a_1(t)x_1^2 + [b_1(t)-D_1(t)-\alpha C_0(t)]x_1 + D_1(t) \geq f(x_1), \quad t \geq T.
\]
The algebraic equation
\[
-a_1^2(t)x_1^2 + [b_1(t)-D_1(t)-\alpha C_0(t)]x_1 + D_1(t) = 0
\]
gives us one positive root \( \bar{x}_1 = (b_1(t)-D_1(t)-\alpha C_0(t)+\sqrt{(b_1(t)-D_1(t)-\alpha C_0(t))^2 + 4a_1(t)D_1(t)}} / 2a_1^2(t) \). Clearly, for any \( x_1 \) (0 \( \leq x_1 < \bar{x}_1 \)), we have \( f(x_1) > 0 \). Choose \( \delta' \) (0 < \( \delta' < \bar{x}_1 \)), we have
\[
\dot{x}_1 \mid_{x_1 \leq \Delta} > f(\delta') > 0.
\]
If \( x_1(T) \geq \delta' \), then it also holds for \( t \geq T \) that if \( x_1(T) \leq \delta' \), and there must exist \( T' \leq T \), such that \( x_1(t) \geq \delta' \) for \( t \geq T' \). Let \( \tau = T' \), \( b = \min(\delta, \delta') \), then \( x_1(t) \geq b \) (i = 1, 2) for \( t \geq \tau \). Moreover, by (4) and (6), we have
\[
\dot{x}_1 \mid_{x_1 > \Delta} > B(h(t)-\alpha C_0 - a_1^2(t)B) < 0, \quad \dot{x}_2 \mid_{x_2 > \Delta} > B(h(t)-\alpha C_0 - a_1^2(t)B) < 0.
\]
Hence, all solutions of \((\text{M}_1)\) initiating in the boundary of \( \Omega \) enter the region \( \Omega \) for \( t \geq 0 \), so \( \Omega \) is positively invariant with respect to system \((\text{M}_1)\).

(II) Suppose that (C2) is satisfied. Let \( V(t) = x_1(t) + x_2(t) \), calculating the derivative of \( V \) along the solution of \((\text{M}_1)\), we have
\[
\dot{V} \mid_{x_1} = x_1(b_1(t)-\alpha C_0(t)-D_1(t)) - a_1(t)x_1^2
\]
\[
+ x_1(b_1(t)+D_1(t)-D_2(t)) - a_1(t)x_1^2
\]
\[
\geq V(\phi(t)-a(t)\rho),
\]
where \( a(t) = \max\{a_1(t), a_2(t)\} \). By Lemma 2.3 and (C2), for the logistic equation
\[
\dot{\rho} = \rho(\phi(t)-a(t)\rho),
\]
any solution of (7) \( \tilde{p}(t) \subset [p, q] \) \((0 < p < q)\). Let \( \rho(t) \) be the solution of (7) with \( \rho(0) = x_1(0) + x_2(0) \), by the standard comparison theorem, \( V(t) \geq \rho(t) \) for \( t \geq 0 \). Let \( \varepsilon = p / 2 \), then there exists \( T = T(x_1(0), x_2(0)) \), such that \( V(t) = x_1(t) + x_2(t) > \rho(t) - \varepsilon \geq \rho(t) - p / 2 = \xi > 0 \) for \( t \geq T \).

Let \( D_0 = \min\{D_1^a, D_2^b\} > 0 \) for \( t \geq T \), we have
\[
\dot{x}_i = x_i [b_i(t) - D_i(t) - \alpha C_i(t) - a_i(t)x_i^\alpha + D_i(t)x_i > -a_i(t)x_i^\alpha + [b_i(t) - D_i(t)]x_i + D_i^\beta \xi].
\]
An entirely similar argument, as proof of (I) shows, that there exist \( \delta_1 > 0 \) and \( T_1 \geq T \) such that \( x_i(t) \geq \delta_1 \) for \( t \geq T \). And for \( t \geq T \), we have \( \dot{x}_i > -a_i(t)x_i^\alpha + (b_i(t) - D_i(t) - D_i^\beta \xi)x_i + D_i^\beta \xi \). By the same argument as above, we conclude that there exist \( \delta_2 > 0 \) and \( T_2 > T \) such that \( x_i(t) \geq \delta_2 \) for \( t \geq T_2 \). Set \( \tau = \max\{T_1, T_2\} \), \( b = \min\{\delta_1, \delta_2\} \), then \( x_i(t) \geq b \) \((i = 1, 2)\) for \( t > \tau \).

So we know that \( \Omega \) is a positive invariant set of system (M1) when condition (C2) holds.

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