Relativistic Feynman-Metropolis-Teller Treatment at Finite Temperatures and its Application to WDs

CSQCDIII
Sheyse Martins de Carvalho*
In collaboration with: J. Rueda and R. Ruffini
ICRANet, University of Nice Sophia-Antipolis, Sapienza University of Rome

*supported by the Erasmus Mundus Joint Doctorate Program by Grant Number 2010-1816 from the EACEA of the European Commission.
Introduction: Effects of temperature.
The equation of State:
  - Chandrasekhar Approach
  - Lattice Model
  - Salpeter Approach
  - Classical Feynman Metropolis Teller Treatment
  - Relativistic Feynman Metropolis Teller Treatment (T=0)

General Relativistic Equations of Equilibrium
Numerical Results: Application to White Dwarfs
Extending the Relativistic FMT Approach to finite temperatures
The Equation of State: Relativistic Feynman Metropolis Teller Treatment (T ≠ 0)
Numerical Results:
  - Carbon EoS
    - Mass and Radius of General Relativistic Carbon WD
Effects of Temperature

A crude estimate...

\[ P_{\text{ideal}} = P_{\text{deg}} \]

\[ P_{\text{ideal}} = n_e T \]

\[ P_{\text{deg}} = \frac{1}{3} \frac{2}{(2\pi\hbar)^3} \int_0^{P_e} \frac{c^2 p^2}{\sqrt{c^2 p^2 + m_e^2 c^4}} 4\pi p^2 dp \]
Equation of State: some known models

- Chandrasekhar: Uniform Approximation of Nucleons and Electrons
- Lattice Model: Point-like Nucleus + Uniform Electrons
- Salpeter: lattice + TF corrections
- Feynman-Metropolis-Teller approach based on the Thomas-Fermi model
- Relativistic Feynman-Metropolis-Teller: degenerate and finite temperatures
Chandrasekhar Approximation

S. Chandrasekhar, Astrophys. J. 74, 81 (1931)

→ No Coulomb Interaction

→ Electrons and Nucleons have a uniform distribution

→ Fully degenerate free-gas described by Fermi-Dirac statistics.

\[
\mathcal{E}_e = \frac{2}{(2\pi \hbar)^3} \int_0^{P_e^F} \sqrt{c^2 p^2 + m_e^2 c^4 4\pi p^2} \, dp
\]

\[
= \frac{m_e^4 c^5}{8\pi^2 \hbar^3} \left[ x_e \sqrt{1 + x_e^2 (1 + 2x_e^2)} - \text{arcsinh}(x_e) \right]
\]

\[
P_e = \frac{1}{3} \frac{2}{(2\pi \hbar)^3} \int_0^{P_e^F} \frac{c^2 p^2}{\sqrt{c^2 p^2 + m_e^2 c^4}} 4\pi p^2 \, dp
\]

\[
= \frac{m_e^4 c^5}{8\pi^2 \hbar^3} \left[ x_e \sqrt{1 + x_e^2 (2x_e^2/3 - 1)} + \text{arcsinh}(x_e) \right],
\]

\[
\mathcal{E}_{\text{unif}} = \mathcal{E}_{\text{N}} + \mathcal{E}_e \approx \frac{A_r}{Z} M_u c^2 n_e + \mathcal{E}_e ,
\]

\[
P_{\text{unif}} \approx P_e ,
\]
Lattice Model

(Baym, Bethe, Pethick, Nuclear Physics A, 1971)

→ Beyond electron-ion fluid approx.: point-like nucleus + background of deg. e-.
→ Introduces Wigner-Seitz cell.
→ Consider Coulomb Interaction.

Ne = Z non-relativistic electrons

\[ E_L = \mathcal{E}_{\text{unif}} V_{ws} + E_C, \]
\[ E_C = E_{e-N} + E_{e-e} = -\frac{9}{10} \frac{Z^2 e^2}{R_{ws}}, \]
\[ P_L = -\frac{\partial E_L}{\partial V_{ws}} = P_{\text{unif}} + \frac{1}{3} \frac{E_C}{V_{ws}}, \]
Salpeter Approach

E.E. Salpeter, Astrophys. J. 134, 669 (1961).

-The generalization of the lattice model came from Salpeter who studied the corrections due to the non-uniformity of the electron distribution inside a Wigner-Seitz cell.

We can write the formula of Salpeter energy as:

$E_S = E_{CH} + E_C + E_{TS}^F$

The third contribution is obtained assuming $n_e[1+\epsilon(r)]$

$n_e = \frac{3Z}{4\pi R_{WS}^3}$

$E_{TS}^F = \frac{162}{175} \left( \frac{4}{9\pi} \right)^{2/3} \alpha^2 Z^{7/3} \mu_e$

The pressure of the Wigner-Seitz cell is given by:

$P_S = P_L + P_{TS}^S$

Where

$P_{TS}^S = \frac{1}{3} \left( \frac{P_e^F}{\mu_e} \right)^2 \frac{E_{TS}^F}{V_{WS}}$
Feynman-Metropolis-Teller Treatment: Classic

R. P. Feynman, N. Metropolis, and E. Teller, Phys. Rev. 75, 1561 (1949)

- FMT showed how to derive the EOS using Thomas-Fermi model
- The profile of electrons changes with distance, is not uniform.

\[ N_e = Z \text{ non-relativistic electrons} \]

\[ E_{\text{ws}} = E_N + E_k + E_C \]

\[ E_k^{(e)} = \int_0^{R_{\text{ws}}} 4\pi r^2 e_e [n_e(r)] dr \]

\[ E_N = M_N(Z, A)c^2 \]

\[ E_C = E_{e-N} + E_{e-e} \]

\[ P_{\text{TF}} = \frac{2 \int n_e^\text{unif}(R_{\text{ws}})}{3 V_{\text{ws}}} \]
Relativistic Feynman-Metropolis-Teller treatment

M. Rotondo, J. A. Rueda, R. Ruffini, and S. S. Xue, Phys. Rev. C 83, 045805(2011), arXiv:0911.4622.

\[ E^F_e = \sqrt{c^2(P_e^F)^2 + m_e^2c^4} - m_e^2c^2 - eV(r) = \text{constant} > 0, \]

\[ \frac{1}{3x} \frac{d^2 \chi(x)}{dx^2} = -\frac{\alpha}{\Delta^3} \Theta(x_e - x) + \frac{4\alpha}{9\pi} \left[ \frac{\chi^2(x)}{x^2} + \frac{2m_e}{m_\pi} \frac{\chi(x)}{x} \right]^{\frac{3}{2}}, \]

\[ E_{w_s} = E_N + E_k + E_C \quad E_N = M_N(A, Z)c^2, \]

\[ E_C = \frac{1}{2} \int_{R_{w_s}}^{R_{w_s}} 4\pi r^2 \varepsilon_e [n_p(r) - n_e(r)] V(r) dr, \]

\[ E_k = \int_0^{R_{w_s}} 4\pi r^2 (\varepsilon_e - m_e n_e) dr, \quad \varepsilon_e = \frac{2}{(2\pi \hbar)^3} \int_0^{P_e^F} \sqrt{c^2 p^2 + m_e^2c^4} 4\pi p^2 dp, \]

\[ P_{FMT} = P_e[n_e(R_{w_s})], \quad P_e = \frac{1}{2} \frac{2}{(2\pi \hbar)^3} \int_0^{P_e^F} \frac{c^3 p^2}{\sqrt{c^2 p^2 + m_e^2c^4}} 4\pi p^2 dp. \]

![Graph showing the density profile](image)
General Relativistic Equations of Equilibrium

Newtonian $\rightarrow$

\[
\frac{dF(r)}{dr} = \frac{GM(r)}{r^2}
\]

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)
\]

\[
\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2} \rho(r)
\]

General Relativity $\rightarrow$

\[
\frac{dv(r)}{dr} = \frac{2G 4\pi r^3 P(r)/c^2 + M(r)}{c^2 \left[1 - \frac{2GM(r)}{c^2 r}\right]}
\]

\[
\frac{dM(r)}{dr} = 4\pi r^2 \frac{\varepsilon(r)}{c^2}
\]

\[
\frac{dP(r)}{dr} = -\frac{1}{2} \frac{dv(r)}{dr} \left[\varepsilon(r) + P(r)\right]
\]
Inverse $\beta$-decay instability

M. Rotondo, J. A. Rueda, R. Ruffini, and S. S. Xue, Phys. Rev. C 83, 045805(2011), arXiv:0911.4622.

It is known that white dwarfs may become unstable against the inverse $\beta$-decay process $(Z,A) \rightarrow (Z-1,A)$ through the capture of energetic electrons. In order to trigger such a process, the electron Fermi energy must be larger than the mass difference between the initial nucleus $(Z,A)$ and the final nucleus $(Z-1,A)$.

| Decay | $\epsilon_\beta^Z$ | $\rho_{\text{crit}}^{\beta,\text{relFMT}}$ | $\rho_{\text{crit}}^{\beta,\text{uniF}}$ |
|-------|------------------|------------------|------------------|
| $^4\text{He} \rightarrow ^3\text{H} + n \rightarrow 4n$ | 20.596 | $1.39 \times 10^{11}$ | $1.37 \times 10^{11}$ |
| $^{12}\text{C} \rightarrow ^{12}\text{B} \rightarrow ^{12}\text{Be}$ | 13.370 | $3.97 \times 10^{10}$ | $3.88 \times 10^{10}$ |
| $^{16}\text{O} \rightarrow ^{16}\text{N} \rightarrow ^{16}\text{C}$ | 10.419 | $1.94 \times 10^{10}$ | $1.89 \times 10^{10}$ |
| $^{56}\text{Fe} \rightarrow ^{56}\text{Mn} \rightarrow ^{56}\text{Cr}$ | 3.695 | $1.18 \times 10^{9}$ | $1.14 \times 10^{9}$ |
Numerical Results: Application to White Dwarfs

M. Rotondo, J. A. Rueda, R. Ruffini, and S. S. Xue, Phys. Rev. D (2011).

Results for $^{12}$C WDs

![Graph showing results for $^{12}$C WDs](image-url)
Results for $^{12}$C WDs

M. Rotondo, J. A. Rueda, R. Ruffini, and S. S. Xue, Phys. Rev. D (2011).
Numerical Results: Application to White Dwarfs

\[ R_0 = 7.810 \times 10^8 \text{ cm} = 0.01 R_{\odot} \]
\[ M_0 = 2.9 M_{\odot} \]
(0.3 – 1.3 \( M_{\odot} \))
Extending the Relativistic FMT Approach to Finite Temperatures

Relativistic and non degenerate gas of electrons at temperature $T$ surrounding a degenerate finite sized and positively charged nucleus.

$$E_e = \tilde{\mu}_e - eV = \text{constant} \quad \mu_e = Mc^2 + \tilde{\mu}_e$$

The electron density in this case is given by:

$$n_e = \frac{2}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{\exp(\frac{\tilde{E}(p) - \tilde{\mu}_e(p)}{k_B T}) + 1},$$

$$F_k(\eta, \beta) = \int_0^\infty \frac{t^k \sqrt{1 + (\beta/2)t}}{1 + \exp(t - \eta)} dt.$$

Replacing the particle densities into the Poisson equation we obtain the relativistic Thomas-Fermi Equation:

$$\frac{d^2 \chi(x)}{dx^2} = -4\pi \alpha x \left[ \frac{3}{4\pi \Delta^3} \Theta(x_e - x) - \frac{\sqrt{2}}{\pi^2} \left( \frac{m_e}{m_\pi} \right)^3 \beta^{3/2} [F_{1/2}(\eta, \beta) + \beta F_{3/2}(\eta, \beta)] \right]$$
Extending the Relativistic FMT Approach to Finite Temperatures

For the case of finite temperature both, the nucleus energy and the pressure, take into account the contribution of the internal energy density.

Then, the total energy of the Wigner-Seitz cell can be written as:

\[ E_{ws} = E_N + E_k + E_C \]

\[ E_N = M_N(A, Z)c^2 + \frac{3}{2}Ak_BT, \]

\[ E_k = \int_0^{R_{ws}} 4\pi r^2(\xi_e - m_e n_e)dr, \quad \xi_e = c^2 m_e n_e + \frac{8\pi \sqrt{2}}{(2\pi \hbar)^3} m_e c^5\beta^{5/2} \left[F_{3/2}(\eta, \beta) + \beta F_{5/2}(\eta, \beta)\right], \]

\[ E_C = \frac{1}{2} \int_{R_C}^{R_{ws}} 4\pi r^2 [n_p(r) - n_e(r)]V(r)dr. \]

The total pressure of the Wigner-Seitz cell is given by the sum of the internal energy pressure and the pressure of the uniform model but computed at the boundary of the cell.

\[ P = n_N k_BT + P_e = n_N k_BT + \frac{2}{3(2\pi \hbar)^3} \int_0^\infty \frac{c^2 p^2}{\tilde{E}(p)} \exp\left(\frac{E(p) - \tilde{\mu}_e(p)}{k_BT}\right) + 1. \]
Numerical Results: Application to White Dwarfs

Results for $^{12}\text{C}$ WDs
Numerical Results: Application to White Dwarfs

EoS for $^{12}$C WDs
Numerical Results: Application to White Dwarfs

Results for $^{12}$C WDs
Numerical Results: Application to White Dwarfs

JD-NS binary: PSR1738+0333

\[ M = 0.18M_{\odot}, \quad R = 0.032R_{\odot} \text{ (theoretically inferred)} \]

J. Antoniadis et. al (2012)
Conclusions

- The radius of the Wigner-Seitz cell is not changing with increasing temperature;

- For high densities the effects of temperature are not so important;

- The critical density of inverse $\beta$-decay is not changing;

- For low densities we have important effects for temperatures over $T=10^7$ K;
- He, O, Fe white Dwarfs.

- Neutron Star Cooling.
Thank You!