LOW-LYING ZEROS OF A FAMILY OF QUADRATIC HECKE $L$-FUNCTIONS VIA RATIOS CONJECTURE

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Abstract. In this paper, we apply the ratio conjecture of $L$-functions to derive the lower order terms of the 1-level density of the low-lying zeros of a family quadratic Hecke $L$-functions in the Gaussian field. Up to the first lower order term, we show that our result is consistent with that obtained from previous work under the generalized Riemann hypothesis, when the Fourier transforms of the test functions are supported in $(-2, 2)$.

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1. Introduction

The density conjecture of N. Katz and P. Sarnak [22, 23] relates the low-lying zeros of a reasonable family of $L$-functions with the random matrices theory, predicting that the behavior of these zeros is the same as that of eigenvalues near 1 of a corresponding classical compact group. One important application of the density conjecture is to address the non-vanishing issue of the $L$-functions at the central point. As such issue has significant arithmetic consequences, many studies have been conducted to verify the density conjecture for various families of $L$-functions, we refer the reader to [1, 2, 6, 7, 10, 11, 16–19, 21, 26, 27, 29–31, 33, 34, 36], several examples of these investigations.

One may regard the density conjecture as asserting the main term behavior of the $n$-level density of low-lying zeros of families of $L$-functions. Therefore, the natural question to ask is what constitutes the lower order terms of these $n$-level densities. Researches in this direction can be found in [28, 32, 37].

When deriving the lower order terms, a powerful tool to deploy is the $L$-functions ratios conjecture of J. B. Conrey, D. W. Farmer and M. R. Zirnbauer in [3, Section 5]. This approach was applied by J. B. Conrey and N. C. Snaith in [4] to study the 1-level density function for zeros of quadratic Dirichlet $L$-functions. The general $n$-level density of the same family was carried out by A. M. Mason and N. C. Snaith in [25], and further enables them to show in [24] that the result agrees with the density conjecture when the Fourier transforms of test functions are supported in $(-2, 2)$.

It is then desirable to compare the expressions obtained for the $n$-level density functions conditional on the ratios conjecture to those that are not. For the family of quadratic Dirichlet $L$-functions, a result of S. J. Miller [27] matches the lower order terms of the 1-level density function obtained with or without assuming the ratios conjecture, when the Fourier transforms of test functions are supported in $(-1, 1)$. When the support is enlarged to $(-2, 2)$, D. Fiorilli, J. Parks and A. Södergren obtained the lower order terms of the 1-level density function in [8, 9] by applying either the ratios conjecture or otherwise. Their work assumes the generalized Riemann hypothesis (GRH) and the results obtained are further shown to match up to the first lower order term in [9].

In this paper, we are interested in another quadratic family of $L$-functions, namely, the family of quadratic Hecke $L$-functions in the Gaussian field. Previously, under GRH, we studied the low-lying zeros of such family to show that the density conjecture holds true for this family when the Fourier transforms of test functions are supported in $(-2, 2)$ in [12] and to obtain lower order terms of the 1-level density function in [13]. Motivated by the result in [9], our goal in this paper is to evaluate the 1-level density function using the ratios conjecture and to show that the result matches with the one from our previous work in [13] up to the first lower order term.

Throughout the paper, we let $K = \mathbb{Q}(i)$ be the Gaussian field and $\mathcal{O}_K = \mathbb{Z}[i]$ be the ring of integers of $K$. We write $\zeta_K(s)$ for the Dedekind zeta function of $K$ and $L(s, \chi)$ for the $L$-function associated to a Hecke character $\chi$ of $K$ of trivial infinite type, by which we mean that the component of $\chi$ at the infinite place of $K$ is trivial. We assume GRH.

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in this paper and we denote the non-trivial zeroes of $L(s, \chi)$ by $\frac{1}{2} + i\gamma_{X,j}$ so that $\gamma_{X,j} \in \mathbb{R}$. We order $\gamma_{X,j}$ by 

$$\ldots \leq \gamma_{X,-2} \leq \gamma_{X,-1} < 0 \leq \gamma_{X,1} \leq \gamma_{X,2} \leq \ldots.$$ 

We further normalize the zeros by defining

$$\tilde{\gamma}_{X,j} = \frac{\gamma_{X,j}}{2\pi},$$

where we denote $L = \log X$ with $X$ being a large number. For an even Schwartz class function $\phi$, the 1-level density for $L(s, \chi)$ with respect to $\phi$ is defined to be

$$S(\chi, \phi) = \sum_j \phi(\tilde{\gamma}_{X,j}).$$

We reserve the symbol $\chi_n$ for the quadratic Hecke character (4) defined in [12 Section 2.1] and we note that it is also shown there that $\chi_{i(1+i)c}$ is a primitive Hecke of trivial infinite type of modulus $(1+i)^5c$ when $c$ is square-free and $(c, 1+i) = 1$, where we say that any $c \in \mathcal{O}_K$ is square-free if the ideal $(c)$ is not divisible by the square of any prime ideal. The family of $L$-functions we aim to study is then given by

$$\mathcal{F} = \{L(s, \chi_{i(1+i)^5c}) : c \text{ square-free}, (c, 1+i) = 1\},$$

Let $w$ be another even Schwartz class function which is non-zero and non-negative, we define

$$W(X) = \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right),$$

where we use $\sum^*$ to denote a sum on square-free elements in $\mathcal{O}_K$ throughout the paper and write $N(n)$ for the norm of any element $n \in \mathcal{O}_K$.

We then define the 1-level density of the family $\mathcal{F}$ with respect to $w$ as the following sum

$$(1.1) \quad D(\phi; w, X) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) S(\chi_{i(1+i)^5c}, \phi).$$

Our purpose in the paper is to first evaluate $D(\phi; w, X)$ using the ratios conjecture. We shall formulate the appropriate version of the ratios conjecture concerning our family $\mathcal{F}$ in Conjecture 3.1 and use it to prove in Section 3 the following asymptotic expression of $D(\phi; w, X)$.

**Theorem 1.1.** Assuming the truth of GRH and Conjecture 3.1. Let $w(t)$ be an even, non-zero and non-negative Schwartz function and $\phi(x)$ an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support, then for any $\varepsilon > 0$,

$$(1.2) \quad D(\phi; w, X) = \frac{\hat{\phi}(0)}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) + \frac{\hat{\phi}(0)}{\mathcal{L}} \left(\frac{32N(c)}{\pi^2} + \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} - it\right) + \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it\right) - \frac{8}{\pi} Xc^{\frac{1}{2}+it} \zeta_K(1-2it)A(-it, it)\right) \phi \left(\frac{tX}{2\pi}\right) dt + O_c \left(X^{-1/2+\varepsilon}\right),$$

where $Xc$ and $A_\alpha$ are given in (3.3) and Lemma 3.2 respectively.

Our next goal is then to compare the expression given for $D(\phi; w, X)$ in Theorem 1.1 with the one obtained from our previous work in [13]. For this, we recall that in [13 Lemma 2.10], the following expression is obtained for $D(\phi; w, X)$ when $\phi$ is an even Schwartz test function with compactly supported Fourier transform.

$$(1.3) \quad D(\phi; w, X) = \frac{\hat{\phi}(0)}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) + \frac{\hat{\phi}(0)}{\mathcal{L}} \left(\frac{32N(c)}{\pi^2} + \frac{2\Gamma'}{\Gamma} \left(1\right)\right)$$

$$- \frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{j \geq 1} S_j(\chi_{i(1+i)^5c}, \mathcal{L}; \hat{\phi}) + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-x/2}}{1-e^{-x}} \left(\hat{\phi}(0) - \hat{\phi} \left(\frac{x}{\mathcal{L}}\right)\right) dx,$$

where

$$S_j(\chi_{i(1+i)^5c}, \mathcal{L}; \hat{\phi}) = \sum_{\varpi \equiv 1 \mod (1+i)^3} \frac{\log N(\varpi)}{\sqrt{N(\varpi^{j})}} \chi_{i(1+i)^5c} \left(\varpi^j\right) \hat{\phi} \left(\frac{\log N(\varpi^j)}{\mathcal{L}}\right)$$

with the sum over $\varpi$ running over primes in $\mathcal{O}_K$. Here we note that in $\mathcal{O}_K$, every ideal co-prime to $(1+i)$ has a unique generator congruent to 1 modulo $(1+i)^3$ and such a generator is called primary. We shall use $\sum_{\varpi \equiv 1 \mod (1+i)^3}$ (or sum
Theorem 1.2. For any function $W$, we denote its Mellin transform by $\mathcal{M}W$, so that
\[ \mathcal{M}W(s) = \int_0^\infty W(t)t^{s-1}dt. \]
We further note that around $s = 1$,
\[ \zeta_K(s) = \frac{\pi}{4} \cdot \frac{1}{s-1} + \gamma_K + O(s-1), \]
where $\gamma_K$ is a constant. We write $\gamma = 0.57 \cdots$ for the Euler constant.

Our following result shows the agreement of the two expressions for $D(\phi;w,X)$ given in (1.2) and (1.3) up to the first lower order term.

Theorem 1.2. Assuming the truth of GRH and Conjecture 3.1. Let $w(t)$ be an even, non-zero and non-negative Schwartz function and $\phi(x)$ an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support. Then the expression (1.2) implies that
\[
D(\phi;w,X) = \hat{\phi}(0) + \int_1^\infty \hat{\phi}(\tau) \frac{d\tau}{\tau} - \frac{1}{1-e^{-\tau}} \left( \frac{\hat{\phi}(1)}{L} \right) d\tau + \frac{2}{L} \int_0^\infty \left( \frac{\hat{\phi}(0)}{\tau} - \frac{\hat{\phi}(0)}{1-e^{-\tau}} \left( \frac{\hat{\phi}(1)}{L} \right) \right) dt
\]
\[
= \frac{2}{L} \sum_{\omega \equiv 1 \mod (1+i)^3} \log N(\omega) \left( \frac{1 + \frac{1}{N(\omega)}}{N(\omega)^2} \right) \left( \frac{\pi^2}{27/3} + 2 \frac{\zeta_K(2)}{\zeta_K(2)} - 8 \frac{\pi}{\gamma_K} - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right)
\]
\[
+ \frac{1}{L} \log \left( \frac{\pi^2}{27/3} + 2 \frac{\zeta_K(2)}{\zeta_K(2)} - 8 \frac{\pi}{\gamma_K} - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) + O(L^{-2}).
\]
Also, when $\sup(\text{supp}(\hat{\phi}(u))) < 2$, the above expression agrees with that given in (1.3).

We give the proof of Theorem 1.2 in Section 4. Our approach is inspired by the proof of [9] Theorem 1.1, Theorem 1.4], although the computation in our case is more involved.

2. Lemmas

We include some lemmas that are needed in our proof of Theorems 1.1 and 1.2. Our first lemma evaluates certain sums under GRH, which is a combination of [13] Lemma 2.6, 2.7:

Lemma 2.1. Suppose that GRH is true. For any even, non-zero and non-negative Schwartz function $w$, we have for any primary $n \in \mathcal{O}_K$ and $\varepsilon > 0$,
\[
\sum_{(c,1+i)=1}^* \frac{w(N(c))}{X} \left( \frac{N(c)}{X} \right) = \delta_{\chi_n} \frac{\pi X}{3\zeta_K(2)} \hat{w}(0) \prod_{\pi \mid n} \left( 1 + \frac{1}{N(\chi^n)} \right)^{-1} + O \left( X^{(3/8)(1-\delta_{\chi_n})+\varepsilon} \right),
\]
\[
\sum_{(c,1+i)=1}^* \frac{w(N(c))}{X} \log N(c) = \log X + \frac{2}{\hat{w}(0)} \int_0^\infty w(x) \log x \, dx + O \left( X^{-1/2+\varepsilon} \right).
\]
In particular, the case $n = 1$ in the first equality above implies that
\[
W(X) = \frac{\pi X}{3\zeta_K(2)} \hat{w}(0) + O \left( X^{1/4+\varepsilon} \right).
\]

Our second lemma is a Poisson summation formula over $K$, which corresponds to the case $n = a = 1$ in [12] Lemma 2.7].
Lemma 2.2. For any Schwartz class function \( W \) and \( X > 0 \), we have
\[
\sum_{m \in \mathcal{O}_K} W \left( \frac{N(m)}{X} \right) = X \sum_{k \in \mathcal{O}_K} \widehat{W} \left( \sqrt{N(k)} X \right),
\]
where \( \widehat{c}(z) = \exp \left( 2\pi i \left( \frac{x}{2} - \frac{y^2}{4} \right) \right) \) and
\[
\widehat{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x+y)) \widehat{c}(-t(x+y)) \, dx \, dy, \quad t \geq 0.
\]

We further note that evaluating \( \widehat{W} \) in polar coordinates shows that \( \widehat{W} \) real and integration by parts further shows that
\[
(2.2) \quad \widehat{W}^{(\mu)}(t) \ll j \max \{ 1, |t|^{-j} \}
\]
for all integers \( \mu \geq 0, j \geq 1 \) and all real \( t \).

Now, let \( w(t) \) be an even, non-zero and non-negative Schwartz function as in either Theorem 1.1 or 1.2. We define
\[
(2.3) \quad g(y) = \overline{w} \left( \sqrt{2} y \right), \quad g_1(y) = g \left( \frac{y}{2} \right), \quad g_2(y) = \overline{g} \left( \sqrt{2} y \right).
\]

Our next lemma establishes a relation between the Mellin transforms of \( g \) and \( g_2 \).

Lemma 2.3. For any \( z \in \mathbb{C}, z \neq 0, -1 \), we have
\[
\zeta_K(z+1)Mg_2(z+1) = \zeta_K(-z)Mg(-z).
\]

Proof. Our proof of the lemma is motivated by Riemann’s proof of the functional equation of the Riemann zeta function \( \zeta(s) \) (see [8, §8]). We first note that, for \( \Re(z) > 1 \),
\[
\zeta_K(z)Mg(z) = \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} g(t) \left( \frac{t}{N(k)} \right)^z \frac{dt}{t} = \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} g(N(k)t)^z \frac{dt}{t} = \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} g(N(k)t)^z \frac{dt}{t} + \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} \bar{g}(\sqrt{N(k)}t)^z \frac{dt}{t} + \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} \bar{g}(\sqrt{N(k)}t)^z \frac{dt}{t}.
\]

Now applying Lemma 2.2 to the second of the above sums and another change of variables, we get
\[
\zeta_K(z)Mg(z) = \frac{1}{4} \frac{g(0)}{z} + \frac{1}{4} \frac{\bar{g}(0)}{z-1} + \frac{1}{4} \int_{1}^{\infty} \sum_{k \in \mathcal{O}_K} g(N(k)t)^z \frac{dt}{t} + \frac{1}{4} \int_{1}^{\infty} \sum_{k \in \mathcal{O}_K} \bar{g}(\sqrt{N(k)}t)^z \frac{dt}{t} + \frac{1}{4} \int_{1}^{\infty} \sum_{k \in \mathcal{O}_K} \bar{g}(\sqrt{N(k)}t)^z \frac{dt}{t}.
\]

Note that the last two integrals converge absolutely for all \( z \in \mathbb{C} \), by applying estimation 2.2 to both \( g \) and \( \bar{g} \). The last expression above thus gives an analytical extension of \( \zeta_K(z)Mg(z) \) to all \( z \in \mathbb{C}, z \neq 0, 1 \).

Similarly, we also deduce from Lemma 2.2 that for \( \Re(z) > 0 \),
\[
\zeta_K(z+1)Mg_2(z+1) = \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} \bar{g}(\sqrt{t}) \left( \frac{t}{N(k)} \right)^{z+1} \frac{dt}{t} + \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} \bar{g}(\sqrt{N(k)}t)^{z+1} \frac{dt}{t} + \frac{1}{4} \sum_{k \in \mathcal{O}_K} \int_{0}^{\infty} \bar{g}(\sqrt{N(k)}t)^{z+1} \frac{dt}{t}.
\]

Once again by applying estimation 2.2 to both \( g \) and \( \bar{g} \), we see that the last two integrals above converge absolutely for all \( z \in \mathbb{C} \), so the last expression above thus gives an analytical extension of \( \zeta_K(z+1)Mg_2(z+1) \) to all \( z \in \mathbb{C}, z \neq 0, 1 \). Now, by comparing the above expressions for \( \zeta_K(z)Mg(z) \) and \( \zeta_K(z+1)Mg_2(z+1) \), we readily deduce the assertion of the lemma. \( \square \)
We first follow the recipe given in [3] to derive a suitable version of the ratios conjecture for the family \( \mathcal{F} \). We start by considering the expression

\[
R(\alpha, \beta) = \frac{1}{W(X)} \sum_{n=1}^{\infty} \frac{w\left(\frac{N(c)}{X}\right)}{L\left(1/2 + \alpha, \chi_{(1+i)^{c}}\right)} L\left(1/2 + \beta, \chi_{(1+i)^{c}}\right).
\]

Similar to the treatment in [14, Section 4.1], we may approximate \( L(s, \chi_{(1+i)^{c}}) \) by

\[
L(s, \chi_{(1+i)^{c}}) \approx \sum_{n \neq 0} \frac{\chi_{(1+i)^{c}}(n)}{N(n)^s} + X_c(s) \sum_{n \neq 0} \frac{\chi_{(1+i)^{c}}(n)}{N(n)^{1-s}},
\]

where \( \sum_{n \neq 0} \) denotes a sum over non-zero integral ideals in \( \mathcal{O}_K \) and

\[
X_c(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \left( \frac{\pi^2}{32N(c)} \right)^{s-1/2}.
\]

Writing \( \mu_q \) for the Möbius function on \( K \), we obtain under GRH that

\[
\frac{1}{L(s, \chi_{(1+i)^{c}})} = \sum_{m \neq 0} \frac{\mu_q(m) \chi_{(1+i)^{c}}(m)}{N(m)^s}.
\]

Applying (3.2) and (3.3) to (3.1), we see that

\[
R(\alpha, \beta) \sim R_1(\alpha, \beta) + R_2(\alpha, \beta),
\]

where

\[
R_1(\alpha, \beta) = \frac{1}{W(X)} \sum_{n=1}^{\infty} \frac{w\left(\frac{N(c)}{X}\right)}{N(m)^{\frac{1}{2} + \alpha} N(n)^{\frac{1}{2} + \beta}},
\]

and

\[
R_2(\alpha, \beta) = \frac{1}{W(X)} \sum_{n=1}^{\infty} \frac{w\left(\frac{N(c)}{X}\right)}{N(m)^{\frac{1}{2} + \beta} N(n)^{\frac{1}{2} - \alpha}}.
\]

When \( nm \) is an odd square, we expect to gain a main contribution to both \( R_1 \) and \( R_2 \). Applying Lemma 2.1, we have in this case

\[
\frac{1}{W(X)} \sum_{n=1}^{\infty} \frac{w\left(\frac{N(c)}{X}\right)}{N(m)^{\frac{1}{2} + \beta} N(n)^{\frac{1}{2} - \alpha}} \sim \prod_{\varpi \equiv 1 \mod (1+i)^3} \left( 1 + \frac{1}{N(\varpi)} \right)^{-1}.
\]

We then deduce that, upon denoting \( \square \) for a perfect square,

\[
R_1(\alpha, \beta) \sim \tilde{R}_1(\alpha, \beta) = \sum_{nm=\text{odd}} \frac{\mu_q(m)}{N(m)^{\frac{1}{2} + \beta} N(n)^{\frac{1}{2} - \alpha}} \prod_{\varpi \equiv 1 \mod (1+i)^3} \left( 1 + \frac{1}{N(\varpi)} \right)^{-1}.
\]

A computation on the Euler product shows that

\[
\tilde{R}_1(\alpha, \beta) = \zeta_K(1+2\alpha) \zeta_K(1+\alpha+\beta) A(\alpha, \beta),
\]

where

\[
A(\alpha, \beta) = \left( \frac{2^{1+\alpha+\beta} - 2^{\beta-\alpha}}{2^{1+\alpha+\beta} - 1} \right) \prod_{\varpi \equiv 1 \mod (1+i)^3} \left( 1 - \frac{1}{N(\varpi)^{1+\alpha+\beta}} \right)^{-1}
\]

\[
\times \left( 1 - \frac{1}{(N(\varpi) + 1)N(\varpi)^{1+2\alpha}} - \frac{1}{(N(\varpi) + 1)N(\varpi)^{\alpha+\beta}} \right).
\]

Note that the product \( A(\alpha, \beta) \) is absolutely convergent for \( \Re(\alpha), \Re(\beta) > -1/4 \).

Similarly, we obtain

\[
R_2(\alpha, \beta) \sim \tilde{R}_2(\alpha, \beta) = \frac{1}{W(X)} \sum_{n=1}^{\infty} \frac{w\left(\frac{N(c)}{X}\right)}{N(m)^{\frac{1}{2} + \alpha} N(n)^{\frac{1}{2} + \beta}} \tilde{R}_1(-\alpha, \beta).
\]
Combining (3.5) with (3.6) and (3.8), we deduce the following appropriate version of the ratios conjecture for our family $F$.

**Conjecture 3.1.** Assuming GRH. Let $\varepsilon > 0$ and let $w$ be an even and nonnegative Schwartz test function on $\mathbb{R}$ which is not identically zero. For complex numbers $\alpha$ and $\beta$ satisfying $|\Re(\alpha)| < 1/4$, $(\log X)^{-1} \ll \Re(\beta) < 1/4$ and $\Im(\alpha), \Im(\beta) \ll X^{1-\varepsilon}$, we have that

$$
\frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) L\left(\frac{1}{2} + \alpha, \chi_{(1+i)^c}\right) \frac{L(1/2 + \alpha, \chi_{(1+i)^c})}{L(1/2 + \beta, \chi_{(1+i)^c})}
= \frac{\zeta_K(1+2\alpha)}{\zeta_K(1+\alpha+\beta)} A(\alpha, \beta) + \frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) X_c \left(\frac{1}{2} + \alpha\right) \frac{\zeta_K(1-2\alpha)}{\zeta_K(1-\alpha+\beta)} A(-\alpha, \beta) + O_c(X^{-1/2+\varepsilon}),
$$

where $A(\alpha, \beta)$ is defined in (3.5) and $X_c(s)$ is defined in (3.3).

Similar to the derivation of [14, Lemma 4.3], we deduce from Conjecture 3.1 the following result needed in the calculation of the 1-level density.

**Lemma 3.2.** Assuming GRH and Conjecture 3.1 we have for any $\varepsilon > 0$, $(\log X)^{-1} \ll \Re(r) < 1/4$ and $\Im(r) \ll X^{1-\varepsilon}$,

$$
\frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) L'\left(\frac{1}{2} + r, \chi_{(1+i)^c}\right)
= \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} A_\alpha(r, r) - \frac{4}{\pi} \frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) + O_c(X^{-1/2+\varepsilon}),
$$

where

$$
A_\alpha(r, r) = \frac{\partial}{\partial \alpha} A(\alpha, \beta) \bigg|_{\alpha=\beta=r}.
$$

We now proceed as in [14, Section 4.4] to see that, assuming GRH, it follows from Lemma 3.2 that

$$
D(\phi; w, X) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} + 2 A_\alpha(r, r) - \frac{8}{\pi} X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) \right) \phi \left( \frac{i\ell r}{2\pi} \right) dr + O_c\left( X^{-1/2+\varepsilon} \right),
$$

where $D(\phi; w, X)$ is defined in (3.5). Note that the integrand in (3.9) is analytic in the region $\Re(r) \geq 0$ (in particular it is analytical at $r = 0$), the assertion of Theorem 1.1 now follows by moving the contour of integration from $\Re(r) = a - 1/2$ to $\Re(r) = 0$.

4. **Proof of Theorem 1.2**

4.1. **Initial treatment.** In this section, we consider the expansions of the $D(\phi; w, X)$ given in (1.2) as powers of $1/\ell$ with $\ell = \log X$. Recall from (3.9) that up to an error term of size $O(\ell^{-2})$, we have

$$
D(\phi; w, X) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} + 2 A_\alpha(r, r) - \frac{8}{\pi} X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) \right) \phi \left( \frac{i\ell r}{2\pi} \right) dr,
$$

where $(\log X)^{-1} < a' < 1/4$.

We set

$$
I = -\frac{8}{\pi} \frac{1}{W(X)} \sum_{(c,1+i)=1}^\ast w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) \phi \left( \frac{i\ell r}{2\pi} \right) dr.
$$

We shall postpone the evaluation of $I$ in the next section and proceed here the treatment on the other terms on the right-hand side of (4.1).
In view of this, we can shift the contour of the last integration in (4.5) from (4.6).

Next note that, similar to [14, (4.11)], we have

\[ \frac{1}{W(X)} \sum_{(c,1+i)=1}^* \frac{N(c)}{X} \int \frac{32N(c)}{\pi^2} \phi \left( \frac{t\mathcal{L}}{2\pi} \right) dt = \frac{\tilde{\phi}(0)}{\mathcal{L}} \frac{1}{W(X)} \sum_{(c,1+i)=1}^* \frac{N(c)}{X} \log \left( \frac{32N(c)}{\pi^2} \right) \]

\[ = \frac{\tilde{\phi}(0)}{\mathcal{L}} \left( \log \frac{32}{\pi^2} + \mathcal{L} + \frac{2}{w(0)} \int_0^\infty w(x) \log x \, dx \right) + O \left( \mathcal{L}^{-2} \right). \]

Furthermore, we obtain via a direct calculation (via the observation that \( A(r,r) = 1 \)) that

\[ A_\alpha(r,r) + \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} = \frac{-1}{N(\varpi) + 1} \sum_{\varpi \equiv 1 \mod (1+i)^3} \frac{N(\varpi) \log N(\varpi)}{N(\varpi)^{1+2r}}. \]

It follows from this that we have, after the substitution \( u = -i\mathcal{L}r/(2\pi) \) and interchange summations the integral,

\[ \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} + 2A_\alpha(r,r) \right) \phi \left( \frac{i\mathcal{L}r}{2\pi} \right) dr \]

\[ = -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \mod (1+i)^3} \frac{N(\varpi) \log N(\varpi)}{N(\varpi) + 1} \sum_{j \geq 1} \frac{1}{N(\varpi)^j} \int_{\text{C}'} \phi(u) \exp \left( -2\pi i u \left( \frac{2j \log N(\varpi)}{\mathcal{L}} \right) \right) du. \]

where \( \text{C}' \) denotes the horizontal line \( \Im(u) = -\mathcal{L}a'/(2\pi) \).

As \( \hat{\phi} \) is compactly supported and \( \phi(z) = \int_R \hat{\phi}(x)e^{2\pi i xz} \, dx \), it follows from integration by parts that uniformly for \( \frac{-\mathcal{L}a'}{2\pi} \leq t \leq 0 \),

\[ |\phi(T + it)| \ll \frac{1}{|T| + 1}. \]

In view of this, we can shift the contour of the last integration in (4.5) from \( \text{C}' \) to \( \Im(u) = 0 \) to deduce that

\[ \frac{1}{2\pi i} \int_{(a')} \left( 2 \frac{\zeta_K(1+2r)}{\zeta_K(1+2r)} + 2A_\alpha(r,r) \right) \phi \left( \frac{i\mathcal{L}r}{2\pi} \right) dr \]

\[ = -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \mod (1+i)^3} \frac{\log N(\varpi)}{N(\varpi)} \left( 1 + \frac{1}{N(\varpi)^j} \right) \phi \left( \frac{2j \log N(\varpi)}{\mathcal{L}} \right). \]

### 4.2. Evaluation of \( I \)

In this section, we evaluate \( I \). Note that it follows from (4.4) that

\[ A(-\gamma, \gamma) = \frac{3(2 - 2^{2r})}{4 - 2^{2r}} \frac{\zeta_K(2)}{\zeta_K(2 - 2r)}. \]

Substituting the above in the right-hand side of (4.4), we deduce from the definitions of \( X_c \) given in (3.3) and a change of variable \( r = 2\pi i \tau / \mathcal{L} \) that

\[ I = -\frac{8}{\pi} \int_{\text{C}'} \left( \frac{\Gamma(1/2 - 2\pi i \tau / \mathcal{L})}{\Gamma(1/2 + 2\pi i \tau / \mathcal{L})} \right)^{2\pi i \tau / \mathcal{L}} \left( 1 + \frac{2 - 2^{4\pi i \tau / \mathcal{L} + 1}}{4 - 2^{4\pi i \tau / \mathcal{L}}} \right) \frac{\zeta_K(1 - 4\pi i \tau / \mathcal{L})}{\zeta_K(2 - 4\pi i \tau / \mathcal{L})} \phi(\tau) \]

\[ \times \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) N(c)^{-2\pi i \tau / \mathcal{L}} \, d\tau, \]
where we also denote $C'$ for the horizontal line $\Im(\tau) = -\frac{c_0'}{2\pi}$.

We treat the last sum in (4.7) by applying Mellin inversion to see that for $0 \leq \Re(r) \leq 1/2$,

$$\sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} = \frac{2\pi}{3\zeta(2)} \int_{(2)} \frac{2^{s+r}}{2^{s+r} + 1} \zeta_K(s+r) X^s \mathcal{M}w(s) \, ds.$$

We shift the contour of integration to the line $\Re(s) = 1/2 - \Re(r) + \varepsilon$ to encounter a simple pole at $s = 1 - r$. On the new line of integration, the convexity bound (see [20, Exercise 3, p. 100]), together with the rapid decay of $\mathcal{M}w$, gives

$$\zeta_K(s) \ll (1 + |s|^{1/4+\varepsilon}).$$

With this and recalling that the reside of $\zeta_K(s)$ at $s = 1$ is $\pi/4$, we get

$$\sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} = \frac{2\pi}{3\zeta(2)} X^{1-r} \mathcal{M}w(1-r) + O_{\varepsilon,w}\left((|\Im(\tau)| + 1)^{1/2+\varepsilon} X^{1/2-\Re(r)+\varepsilon}\right).$$

Combining the above with (2.1), we deduce that for any $\varepsilon > 0$ and $0 \leq \Re(r) \leq 1/2$,

$$\frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} = \frac{2}{w(0)} X^{1-r} \mathcal{M}w(1-r) + O_{\varepsilon,w}\left((|\Im(\tau)| + 1)^{1/2+\varepsilon} X^{-1/2-\Re(r)+\varepsilon}\right).$$

For small $\varepsilon, \eta > 0$, we change the contour $C'$ in (4.7) to the path

$$C = C_0 \cup C_1 \cup C_2,$$

where

$$C_0 = \{\Im(\tau) = 0, |\Re(\tau)| \geq L\}, \quad C_1 = \{\Im(\tau) = 0, \eta \leq |\Re(\tau)| \leq L\}, \quad C_2 = \{|\tau| = \eta, \Im(\tau) \leq 0\}. $$

As $\phi$ decays rapidly, the integration of $I$ over $C_0$ can be shown to be negligible. We now apply the Taylor expansion to treat the integration of $I$ over $C_1 \cup C_2$ by noting that

$$\frac{\Gamma(1/2 - 2\pi i \tau/L)}{\Gamma(1/2 + 2\pi i \tau/L)} = 1 - 2 - \frac{\Gamma'(1/2)}{\Gamma(1/2)} \frac{2\pi i \tau}{L} + O\left(\frac{|\tau|^2}{L^2}\right),$$

and that (see [15, Formula 2, Section 8.366])

$$\frac{-\Gamma'(1/2)}{\Gamma(1/2)} = 2 \log 2 + \gamma.$$

Using Taylor expansion and (1.4), we get

$$\left(1 + \frac{2 - 2\frac{4\pi i \tau}{L}}{4 - 2\frac{4\pi i \tau}{L}}\right) \frac{1}{\zeta_K\left(2 - \frac{4\pi i \tau}{L}\right)} = \frac{1}{\zeta_K(2)} + \frac{1}{\zeta_K(2)} \left(\frac{2 \log 2}{3}\right) + \zeta_K(2) \left(\frac{2 \pi i \tau}{L}\right) + O\left(\frac{|\tau|^2}{L^2}\right),$$

and

$$\zeta_K\left(1 - \frac{4\pi i \tau}{L}\right) = -\frac{\pi}{4} \cdot \frac{L}{4\pi i \tau} + \gamma_K + O\left(\frac{|\tau|}{L}\right).$$

Using the above formulas, we get, after a short computation,

$$\begin{align*}
& -\frac{8}{\pi} \frac{\zeta_K(2)}{L} \left(\frac{1}{\Gamma\left(\frac{1}{2} + \frac{2\pi i \tau}{L}\right)}\right) \frac{\pi^2}{32} \frac{2^{2\pi i \tau}}{2^{2\pi i \tau} - 1} \frac{\zeta_K\left(1 - \frac{4\pi i \tau}{L}\right)}{\zeta_K(2)} \phi(\tau) \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-\frac{2\pi i \tau}{L}}
\end{align*}$$

$$\begin{align*}
= \frac{1}{2\pi i \tau} \left(1 + \frac{2\pi i \tau}{L}\right) \left(2\gamma + 2 \log 4 + \log\left(\frac{\pi^2}{32}\right) + 2 \frac{\zeta_K'}{\zeta_K(2)} - \frac{4}{3} \log 2 - \frac{8}{\pi} \gamma_K \frac{Mw'(1)}{Mw(1)}\right) + O\left(\frac{|\tau|^2}{L^2}\right) + \phi(\tau) e^{-2\pi i \tau}
\end{align*}$$

$$+ O_{\varepsilon,w}\left(X^{-1/2+\varepsilon}\right).$$

We then deduce that

$$I = \frac{1}{2\pi i} \int_{C_1 \cup C_2} \frac{\phi(\tau)}{\tau} e^{-2\pi i \tau} d\tau + I' + O_w(L^{-2}),$$

where we also denote $C'$ for the horizontal line $\Im(\tau) = -\frac{c_0'}{2\pi}$.
where, combining the logarithm terms,

\[
I' = \frac{1}{\mathcal{L}} \left( 2\gamma + \log \left( \frac{\pi^2}{2\sqrt{3}} \right) + 2 \frac{\zeta'(2)}{\zeta(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) \int_{C_1 \cup C_2} \phi(\tau) e^{-2\pi i \tau} d\tau \\
= \frac{1}{\mathcal{L}} \left( 2\gamma + \log \left( \frac{\pi^2}{2\sqrt{3}} \right) + 2 \frac{\zeta'(2)}{\zeta(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) \int \phi(\tau) e^{-2\pi i \tau} d\tau + O(\mathcal{L}^{-2}) \\
= \frac{\tilde{\phi}(1)}{\mathcal{L}} \left( 2\gamma + \log \left( \frac{\pi^2}{2\sqrt{3}} \right) + 2 \frac{\zeta'(2)}{\zeta(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) + O(\mathcal{L}^{-2}).
\]

Similar to the treatment of \( I_1 \) in the proof of [9] Lemma 4.6, we have

\[
\int_{C_1 \cup C_2} \frac{1}{2\pi i} \phi(\tau) e^{-2\pi i \tau} d\tau = \int_{\mathcal{L}} \tilde{\phi}(\tau) d\tau + O(\mathcal{L}^{-2}).
\]

Thus, we conclude that

\[
I = \int_{\mathcal{L}} \tilde{\phi}(\tau) d\tau + \frac{\tilde{\phi}(1)}{\mathcal{L}} \left( 2\gamma + \log \left( \frac{\pi^2}{2\sqrt{3}} \right) + 2 \frac{\zeta'(2)}{\zeta(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) + O(\mathcal{L}^{-2}).
\]

Combining the above expression for \( I \) and \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) and \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) together, we deduce that the expression \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) is valid.

4.3. Comparing terms. In this section we show that the expression given in \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) is in agreement with that given in \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) when \( \sigma = \text{sup}(\text{supp} \hat{\phi}) < 2 \). In fact, applying \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) and \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) in \( \mathcal{L} \) and comparing it with \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \), we see with the help of Lemma \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) that \( \frac{\tilde{\phi}(1)}{\mathcal{L}} \) suffices to show that

\[
\left( 4.8 \right) - \frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \sum_{j\geq 1} S_j(\chi_{(1+i)^{c},L};\tilde{\phi}) \\
= \frac{1}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \frac{1}{2\pi i} \int_{\mathcal{L}} \left( 2 \frac{\zeta'(1+2it)}{\zeta(1+2it)} + 2A_\alpha(it,it) - \frac{8}{\pi} X \left( \frac{1}{2} + it \right) \zeta_K \left( 1 - 2it \right) A(-it,it) \right) \phi \left( \frac{t\mathcal{L}}{2\pi} \right) dt \\
= \frac{1}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \frac{1}{2\pi i} \int_{(\alpha')} \left( 2 \frac{\zeta'(1+2r)}{\zeta(1+2r)} + 2A_\alpha(r,r) - \frac{8}{\pi} X \left( \frac{1}{2} + r \right) \zeta_K \left( 1 - 2r \right) A(-r,r) \right) \phi \left( \frac{t\mathcal{L}}{2\pi} \right) dt,
\]

where \( (\log X)^{-1} < a' < 1/4 \).

Now, similar to the treatment in [13] Section 3, we write

\[
\left( 4.9 \right) - \frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \sum_{j\geq 1} S_j(\chi_{(1+i)^{c},L};\tilde{\phi}) = S_{\text{odd}} + S_{\text{even}},
\]

where

\[
S_{\text{odd}} = -\frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \sum_{j\equiv 1 \text{ (mod 2)}} S_j(\chi_{(1+i)^{c},L};\tilde{\phi}),
\]

and

\[
S_{\text{even}} = -\frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w \left( \frac{N(c)}{X} \right) \sum_{j\equiv 0 \text{ (mod 2)}} S_j(\chi_{(1+i)^{c},L};\tilde{\phi}).
\]

We note that it follows from [13] (3.2) that

\[
\left( 4.10 \right) S_{\text{even}} = -\frac{2}{\mathcal{L}} \sum_{\|\omega\|=1 \text{ mod } (1+i)^3} \sum_{j\geq 1} \frac{\log N(\omega)}{N(\omega)^j} \left( 1 + \frac{1}{N(\omega)} \right)^{-1} \tilde{\phi} \left( \frac{2j \log N(\omega)}{\mathcal{L}} \right) + O(X^{-3/4+\epsilon}),
\]

We then deduce from \( 4.6 \), \( 4.8 \), \( 4.9 \) and \( 4.10 \) that it remains to show that

\[
\left( 4.11 \right) S_{\text{odd}} = I + O(\mathcal{L}^{-2}),
\]

where \( I \) is defined in \( 1.2 \).
4.4. Evaluation of $S_{\text{odd}}$. In this section, we evaluation $S_{\text{odd}}$ to the first lower order term. Our starting point is essentially [13, Lemmas 3.5, 3.9, 3.10], but both our treatment and notations are slightly different from those used in [13].

We first note that, for any $l \in O_K$, we have

$$
\sum_{k \in \mathbb{Z}[i], k \neq 0} (-1)^{N(k)} \bar{w} \left( N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right) = \sum_{k \in \mathbb{Z}[i], k \neq 0} \bar{w} \left( N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right) - \sum_{k \in \mathbb{Z}[i], k \neq 0, (1+i,k)=1} \bar{w} \left( N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right)
$$

$$
= 2 \sum_{k \in \mathbb{Z}[i], k \neq 0} \bar{w} \left( N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right) - \sum_{k \in \mathbb{Z}[i], k \neq 0} \bar{w} \left( N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right)
$$

Applying the above in [13, Lemma 3.5], we derive that, under GRH,

$$
S_{\text{odd}} = \frac{X}{W(X)} \sum_{l \equiv 1 \mod (1+i)^3, N(l) \leq Z} \frac{\mu_{[l]}(1)}{N(l^2)} \left( \frac{1}{2} I_{(1+i)l}(X) - I_l(X) \right) + O(\mathcal{L}^{-2}),
$$

where $Z = X^{2/3 - \sigma/3}$ with $\sigma = \text{sup}(\text{supp } \hat{\varphi})$ and where

$$
I_l(X) = \int_{0}^{\infty} \hat{\varphi}(u) \sum_{k \in \mathbb{Z}[i], k \neq 0} \bar{w} \left( 2N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right) \, du.
$$

As in the proof of [13, Lemma 3.10], we can extend the sum over $l$ in (4.12) to all $l \equiv 1 \mod (1+i)^3$ with a negligible error. Thus we can write

$$
S_{\text{odd}} = \frac{X}{W(X)} \sum_{l \equiv 1 \mod (1+i)^3} \frac{\mu_{[l]}(1)}{N(l^2)} \left( \frac{1}{2} I_{(1+i)l}(X) - I_l(X) \right) + O(\mathcal{L}^{-2}).
$$

Extending the integral in (4.13) to $\mathbb{R}$ and substituting $\tau = \mathcal{L}(u - 1)$, we obtain that

$$
I_l(X) = \frac{1}{\mathcal{L}} \int_{-\infty}^{\infty} \hat{\varphi} \left( 1 + \frac{\tau}{\mathcal{L}} \right) \sum_{k \in \mathbb{Z}[i], k \neq 0} \bar{w} \left( 2N(k) \sqrt{\frac{e^{-\tau}}{2N(l^2)}} \right) \, d\tau + O(\mathcal{L}^{-2}).
$$

Here we note that, similar to the proof of [13, Lemma 3.9], the error term introduced in the above process is of $O(\mathcal{L}^{-2})$. We then break the integral above into integrals over $(-\infty, 0]$ and $[0, \infty)$ and denote them respectively by $I_l^-(X)$ and $I_l^+(X)$. By applying Poisson summation given in Lemma 2.2, we see that

$$
I_l^+(X) = \frac{1}{\mathcal{L}} \int_{0}^{\infty} \hat{\varphi} \left( 1 + \frac{\tau}{\mathcal{L}} \right) \left( -\bar{w}(0) + \sum_{k \in \mathbb{Z}[i]} g \left( N(k) \sqrt{\frac{e^{-\tau}}{N(l^2)}} \right) \right) \, d\tau
$$

$$
= \frac{1}{\mathcal{L}} \int_{0}^{\infty} \hat{\varphi} \left( 1 + \frac{\tau}{\mathcal{L}} \right) \left( -\bar{w}(0) + C(l, \tau) \sum_{j \in \mathbb{Z}[i]} \tilde{g} \left( \sqrt{N(j)}C(l, \tau) \right) \right) \, d\tau
$$

$$
= \frac{1}{\mathcal{L}} \int_{0}^{\infty} \hat{\varphi} \left( 1 + \frac{\tau}{\mathcal{L}} \right) \left( -\bar{w}(0) + C(l, \tau) \tilde{g}(0) + C(l, \tau) \sum_{j \in \mathbb{Z}[i], j \neq 0} \tilde{g} \left( \sqrt{N(j)}C(l, \tau) \right) \right) \, d\tau,
$$

where $g(y)$ is given as in (2.3) and

$$
C(l, \tau) = \sqrt{e^\tau N(l^2)}.
$$
Moreover, substituting \( \tau \) by \(-\tau\) in \( I_l^-(X) \), we obtain that

\[
I_l^-(X) = \frac{1}{\mathcal{L}} \int_0^\infty \hat{\phi}(1 - \frac{\tau}{2}) \sum_{k \in \mathbb{Z}[i], \quad k \neq 0} g \left( N(k) \sqrt{\frac{e^\tau}{N(l^2)}} \right) \, d\tau.
\]

Combining the above expressions for \( I_l^-(X) \) and \( I_l^+(X) \), we derive that, after some changes of variables,

\[
I_l(X) = -\frac{\tilde{w}(0)}{2} \int_1^\infty \hat{\phi}(u) \, du + \frac{1}{\mathcal{L}} \int_0^\infty \hat{\phi}(1 + \frac{\tau}{2}) \left( C(l, \tau) \sum_{j \in \mathbb{Z}[i], \quad j \neq 0} \tilde{g} \left( \sqrt{N(j)} \right) \right) \, d\tau
\]

\[
+ N(l)\tilde{g}(0) \int_1^\infty X(u^{-1/2}) \hat{\phi}(u) \, du + \frac{1}{\mathcal{L}} \int_0^\infty \hat{\phi}(1 - \frac{\tau}{2}) \sum_{k \in \mathbb{Z}[i], \quad k \neq 0} g \left( N(k) \sqrt{\frac{e^\tau}{N(l^2)}} \right) \, d\tau.
\]

We note that the expression \( I_{(1+i)l}(X) \) can be obtained via the expression of \( I_l(X) \) with \( g \) being replaced by \( g_1 \) there, where \( g_1(y) \) is given in (2.3).

Note that

\[
(4.14) \quad \tilde{w}(0) = \frac{\pi}{2} \tilde{w}(0), \quad \sum_{l \equiv 1 \mod (1+i)^3} \frac{\mu_{l|l}(l)}{N(l)^2} = -\frac{4}{3\zeta(2)}.
\]

We then deduce from (2.11) and (4.14) that

\[
\frac{X}{2W(X)} \sum_{l \equiv 1 \mod (1+i)^3} \frac{\mu_{l|l}(l)}{N(l)^2} \tilde{w}(0) \int_1^\infty \hat{\phi}(u) \, du = \int_1^\infty \hat{\phi}(u) \, du + O(\mathcal{L}^{-2}).
\]

From this and the observation that

\[
\tilde{g}_1(t) = 2\tilde{g}(\sqrt{2}t),
\]

we deduce that we have

\[
(4.15) \quad S_{\text{odd}} = \int_1^\infty \hat{\phi}(u) \, du + J(X) + O(\mathcal{L}^{-2}),
\]

where

\[
J(X) = \frac{1}{\mathcal{L}} \int_0^\infty \left( \hat{\phi}(1 + \frac{\tau}{2}) e^{\frac{\tau}{2}} \sum_{j \in \mathbb{Z}[i], \quad j \neq 0} h_1(N(j)e^{\frac{\tau}{2}}) + \hat{\phi}(1 - \frac{\tau}{2}) \sum_{k \in \mathbb{Z}[i], \quad k \neq 0} h_2(N(k)e^{\frac{\tau}{2}}) \right) \, d\tau,
\]

with

\[
h_1(x) = \frac{3\zeta(2)}{\pi \tilde{w}(0)} \sum_{l \equiv 1 \mod (1+i)^3} \frac{\mu_{l|l}(l)}{N(l)} \left( g \left( \sqrt{2N(l)x} \right) - \tilde{g} \left( \sqrt{N(l)x} \right) \right),
\]

and

\[
h_2(x) = \frac{3\zeta(2)}{\pi \tilde{w}(0)} \sum_{l \equiv 1 \mod (1+i)^3} \frac{\mu_{l|l}(l)}{N(l)^2} \left( g \left( \frac{x}{2N(l)} \right) - g \left( \frac{x}{N(l)} \right) \right).
\]

We now evaluate \( h_1(x) \) by applying the Mellin inversion to recast it as

\[
h_1(x) = \frac{3\zeta(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(5/2)} \sum_{l \equiv 1 \mod (1+i)^3} (2^{-z} - 1) \frac{\mu_{l|l}(l)}{N(l)^{1+z}} M_{g_2}(z) \frac{dz}{x^z}.
\]

\[
= \frac{3\zeta(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(5/2)} \frac{2^{-z} - 1}{(1 - 2^{-1-z})\zeta(1+z)} M_{g_2}(z) \frac{dz}{x^z}.
\]
Similarly, we have, with a change of variables $z \rightarrow -z$,

$$h_2(x) = \frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(1/2)} (2^{z-1} - 1) \frac{\mu_{[1]}(l)}{N(l)^{2-z}} M_g(z) \frac{dz}{x^z}$$

$$= \frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(-1/2)} \frac{2^{z-1} - 1}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z)x^z dz$$

$$= \frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(-5/4)} \frac{2^{z-1} - 1}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z)x^z dz.$$ 

With the above expressions for $h_1(x)$ and $h_2(x)$, we can write $J(x)$ given in (4.16) as

$$J(X) = \frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \int_{(5/2)} \left( \phi(1 + \tilde{z}) \int_{(1/2)} \frac{4(2^{z-1} - 1)\zeta_K(z)}{(1 - 2^{z-2})\zeta_K(1 + z)} M_g(z) \frac{dz}{e^{(z-1)\pi/2}} + \phi(1 - \tilde{z}) \int_{(-5/4)} \frac{4(2^{z-1} - 1)\zeta_K(-z)}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z)e^{-\pi/2} dz \right) d\tau$$

(4.17)

$$= \frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{1}{2\pi i} \left( \int_{(3/2)} \frac{4(2^{z-1} - 1)\zeta_K(z)}{(1 - 2^{z-2})\zeta_K(1 + z)} M_g(z) \int_{0}^{\infty} \phi(1 + \tilde{z}) e^{-(z-1)\pi/2} d\tau dz + \int_{(-5/4)} \frac{4(2^{z-1} - 1)\zeta_K(-z)}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z) \int_{0}^{\infty} \phi(1 - \tilde{z}) e^{\pi/2} d\tau dz \right).$$

Now we consider the Taylor expansions, centered at 1, of $\phi(1 + \tilde{z})$ and $\phi(1 - \tilde{z})$ in (4.17). By keeping only the constant terms, we see that their contribution to $J(X)$ equals, with another change of variables $z \rightarrow z + 1$ in the first integral,

(4.18)

$$\frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{8\phi(1)}{2\pi i} \left( \int_{(1/2)} \frac{2^{z-1} - 1}{(1 - 2^{z-2})\zeta_K(1 + z)} M_g(z + 1) \frac{dz}{z} - \int_{(-5/4)} \frac{2^{z-1} - 1}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z) \frac{dz}{z} \right).$$

We now shift the contour of the last integration to the line $\Re(z) = 1/2$. We apply Lemma 2.2 to see that the quantity in (4.18) equals

$$\frac{3\zeta_K(2)}{\pi \tilde{w}(0)} \frac{8\phi(1)}{2\pi i} R,$$

where $R$ is the residue of the function

$$\frac{2^{z-1} - 1}{(1 - 2^{z-2})\zeta_K(2 + z)} M_g(-z) \frac{dz}{z}$$

at $z = 0$. We observe via integration by parts that

$$M_g(-z) = \int_{0}^{\infty} g(t) t^{-z} \frac{dt}{t} = \int_{0}^{\infty} g(t) d\left(\frac{t^{-z}}{-z}\right) = \frac{1}{z} \int_{0}^{\infty} t^{-z} g'(t) dt.$$

It follows that $M_g(-z)$ has a pole at $z = 0$ and we apply (4.14) to see that around $z = 0$,

$$M_g(-z) = \frac{-g(0)}{z} - \int_{0}^{\infty} (\log t) g'(t) dt + O(z^2) = \frac{-\tilde{w}(0)}{z} - \int_{0}^{\infty} (\log t) g'(t) dt + O(z^2)$$

(4.19)

$$= -\frac{\pi}{2} \frac{\tilde{w}(0)}{z} - \int_{0}^{\infty} (\log t) g'(t) dt + O(z^2).$$
Moreover, we have around \( z = 0 \),

\[
\frac{2^{-z-1} - 1}{1 - 2^{-2-z}} = -\frac{2}{3} + \frac{-\log 2(\frac{3}{2}) + \frac{1}{2}(\log 2)\frac{1}{2}}{(1 - 2^{-2})^2} z \approx -\frac{2}{3} - \frac{4}{9}(\log 2)z
\]

\[\zeta_K(-z) \approx \zeta_K(0) + \zeta_k(2) - \zeta_K(0)\zeta_K(2) z + O(z^2).\]

(4.20)

Together with (4.19) and (4.20), we get

\[R = -\frac{\pi}{2} \hat{\nu}(0) \left(-\frac{2}{3} - \frac{4}{9}(\log 2)\right) \zeta_K(0) + \zeta_k(2) - \zeta_K(0)\zeta_K(2) \]

(4.21)

\[\frac{\pi}{2} \hat{\omega}(0) \left(-\frac{4}{9}(\log 2)\right) \zeta_K(0) + \zeta_k(2) - \frac{2}{3} \zeta_K(2) \int_{0}^{\infty} (\log t) g'(t) dt.\]

To further simplify \( R \), we use the fact that \( s \Gamma(s) = 1 \) (see [5, §10]) when \( s = 0 \) and the functional equation for \( \zeta_K(s) \) (see [2])

\[\pi^{-s} \Gamma(s) \zeta_K(s) = \pi^{-(1-s)} \Gamma(1-s) \zeta_K(1-s)\]

to obtain that \( \zeta_K(0) = -1/4 \).

We further use the relation (see [5, §10])

\[\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}\]

to derive that

(4.22)

\[\zeta_K(1-s) = \pi^{-s} \Gamma(s)^2 \sin(\pi s) \zeta_K(s)\]

Applying (1.4), we see that around \( s = 1 \), we have

\[\zeta_K(s) \sin(\pi s) = -\frac{\pi^2}{4} - \pi \gamma_K(s-1) + O((s-1)^2).\]

Using the above expansion and the fact that \( \Gamma'(1) = -\gamma \) (see [15, Formula 1, Section 8.366]) by noting also that \( \Gamma(1) = 1 \), we take the derivative on both sides of (4.22) to see that

\[-\zeta_K'(0) = -\frac{1}{\pi} \gamma_K + \frac{\gamma}{2} + \frac{\log \pi}{2}.

Now, inserting the values of \( \zeta_K(0), -\zeta_K'(0) \) into (4.21), together with a short calculation, we obtain that

\[J(X) = \frac{3\zeta_K(2)}{\pi \hat{\nu}(0)} \frac{8\hat{\phi}(1)}{2\pi i} R + O(L^{-2})\]

(4.23)

\[= \frac{8\hat{\phi}(1)}{L} \left( \frac{\gamma_K}{\pi} + \frac{\gamma}{2} + \frac{\log \pi}{2} + \frac{1}{4} \zeta_k(2) + \left( \frac{2}{3} \log 2 \right) \zeta_K(0) - \frac{1}{2\pi \hat{\omega}(0)} \int_{0}^{\infty} (\log t) g'(t) dt \right)\]

We evaluate the last integral above by noticing that for small \( \eta > 0 \), we have

\[\int_{0}^{\infty} (\log x) g'(x) dx = \int_{0}^{\infty} \sqrt{2} \log x \hat{\omega}'(\sqrt{2}x) dx = \int_{0}^{\infty} \log \left( \sqrt{\frac{x}{2}} \right) \hat{\omega}'(x) dx\]

\[= \hat{\omega}(0) \log(\sqrt{2}) + \int_{\eta}^{\infty} (\log x) \hat{\omega}'(x) dx + O(\eta \log(\eta^{-1}))\]

Now the above expression becomes, after integration by parts,

\[\hat{\omega}(0) \log(\sqrt{2}) - \int_{\eta}^{\infty} \frac{\hat{\omega}(x) - \hat{\omega}(0)I_{[0,1]}(x)}{x} dx + O(\eta \log(\eta^{-1}))\]

where \( I_{[0,1]} \) is the characteristic function of the interval \([0, 1]\).
By evaluating \( \tilde{\omega}(x) \) in polar coordinates, we see that

\[
\tilde{\omega}(x) = 4 \int_0^{\pi/2} \int_0^\infty \cos(2\pi r x \sin \theta) w(r^2) \, rdrd\theta.
\]

It follows from this and by letting \( \eta \to 0^+ \) and using Example (c) on page 132, we obtain that

\[
\int_0^\infty (\log x) g'(x) dx = \tilde{\omega}(0) \log(\sqrt{2}) - \int_0^\infty \frac{\tilde{\omega}(x) - \tilde{\omega}(0) I_{[0,1]}(x)}{x} dx
\]

\[
= \tilde{\omega}(0) \log(\sqrt{2}) - 4 \int_0^\infty w(r^2) r \left( \int_0^{\pi/2} \frac{\cos(2\pi r x \sin \theta)}{x} d\theta - \int_0^\infty \frac{\cos(2\pi r x \sin \theta)}{x} d\theta \right) \, dr
\]

Now the inner-most integrals over \( x \) become

\[
= \int_0^{2\pi r \sin \theta} \frac{\cos(u) - 1}{u} du + \int_{2\pi r \sin \theta}^\infty \frac{\cos(u) - 1}{u} du = \gamma + \log(2\pi r \sin \theta).
\]

Hence the expression in (4.24) is

\[
\tilde{\omega}(0) \log(\sqrt{2}) + \frac{\pi \gamma \tilde{\omega}(0)}{2} + \frac{\pi \log \tilde{\omega}(0)}{2} + \frac{\pi \log 2 \tilde{\omega}(0)}{2} + \frac{\pi}{2} \int_0^\infty w(r) \log r \, dr + 2 \int_0^\infty w(r) dr \int_0^{\pi/2} \log(\sin \theta) d\theta.
\]

As we have (see [15, Formula 3, Section 4.24])

\[
\int_0^{\pi/2} \log(\sin \theta) d\theta = -\frac{\pi}{2} \log 2.
\]

We thus conclude that

\[
\int_0^\infty (\log x) g'(x) dx = \frac{\tilde{\omega}(0)}{4} \log 2 + \frac{\pi \gamma \tilde{\omega}(0)}{2} + \frac{\pi \log \tilde{\omega}(0)}{2} + \frac{\pi}{2} Mw(1).
\]

Applying this to (1.23), we see that

\[
J(X) = \frac{\tilde{\omega}(1)}{\mathcal{L}} \left( 2\gamma + 2 \log 4 + \log \left( \frac{\pi^2}{32} \right) + 2 \zeta(2) - 4 - \frac{8}{3} \log 2 - \frac{8}{3} \gamma - 8 \frac{Mw(1)}{\mathcal{L}} \right) + O(\mathcal{L}^{-2}).
\]

With the above expression for \( J(X) \) and (4.11), we see that the relation given in (4.11) is valid and this completes the proof of Theorem 1.2.

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