The asymptotic behavior of limit-periodic functions on primes and an application to $k$-free numbers

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We use the circle method to evaluate the behavior of limit-periodic functions on primes. For those limit-periodic functions that satisfy a kind of Barban-Davenport-Halberstam condition and whose singular series converge fast enough, we can evaluate their average value on primes. As an application, this result is used to show how tuples of different $k$-free numbers behave when linear shifts are applied.

1 Introduction

Limit-periodic functions are those arithmetical functions $f : \mathbb{N} \to \mathbb{C}$ which appear as limits of periodic functions with regards to the Besicovitch-seminorm defined via

$$||f||_2 := \left( \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^2 \right)^{1/2}$$

They have some wonderful properties, e.g., the mean-value as well as the mean-value in residue classes exist always. Limit-periodic functions are a special case of almost-periodic functions that have been explored by Harald Bohr und Abram Besicovitch in the 1920s. The distinction between those two lies in the approximation type: Almost-periodic functions appear as limits with regards to the Besicovitch-seminorm of linear combinations of the functions $k \mapsto e^{2\pi i \alpha k}$ with $\alpha \in \mathbb{R}$, whereas for limit-periodic functions, only $\alpha \in \mathbb{Q}$ is admissible. For further reference on limit-periodic functions, see [11].

The main result of this paper is a statement on the behavior of a limit-periodic function on primes on average. We prove in theorem 3.3 that under certain conditions the asymptotic relation

$$\sum_{p \leq x} f(p) = c_f \frac{x}{\log x} + o\left( \frac{x}{\log x} \right)$$

holds, with a constant $c_f$ explicitly given through an infinite series. Brüdern [2] has considered this result in a more general context.

As an application we show for arbitrary $\alpha_i \in \mathbb{N}_0$ and $r_i \in \mathbb{N}_{>1}$

$$\sum_{p \leq x} \mu_{r_1}(p + \alpha_1) \cdots \mu_{r_s}(p + \alpha_s) = \prod_p \left( 1 - \frac{D^*(p)}{\varphi(p^{r_i})} \right) \frac{x}{\log x} + o\left( \frac{x}{\log x} \right)$$

(1)

where $\mu_k$ denotes the characteristic function of the $k$-free numbers and $D^*(p)$ is a computable function of the prime $p$, depending on the choice of the numbers $\alpha_i$ and $r_i$. 

2 Some basic facts

We state some basic facts and notation for the later discourse.

**Definition 2.1 (k-free numbers)** For given $k \in \mathbb{N}_{>1}$ the function $\mu_k$ denotes the characteristic function of the set of $k$-free numbers, i.e.

$$\mu_k(n) := \begin{cases} 0 & \text{there is a } p \in \mathbb{P} \text{ with } p^k | n \\ 1 & \text{otherwise} \end{cases}$$

which is multiplicative. On prime powers it has the values

$$\mu_k(p^r) = 1 - \left\lfloor \frac{k}{r} \right\rfloor$$

where $\left\lfloor A \right\rfloor$ shall denote the *Iverson bracket* to the statement $A$, i.e., it equals 1 if $A$ is true, and 0 otherwise. As it is long known we also have

$$\mu_k(n) = \sum_{d | n} \mu(d)$$

**Lemma 2.2** For $x \in \mathbb{R}_{>1}$ we have the asymptotic relation

$$\sum_{n \leq x} \frac{\mu(n)^2}{\varphi(n)} = \log x + \gamma + \sum_p \frac{\log p}{p(p-1)} + O \left( \frac{\log x}{\sqrt{x}} \right)$$

with the Euler-Mascheroni-constant $\gamma$, see [1].

**Definition 2.3** We define the function $e$ with period 1 as usual through

$$e : \mathbb{R} \rightarrow \{ z \in \mathbb{C} : |z| = 1 \}, \ x \mapsto e^{2\pi ix}$$

We sometimes write $e^*_q$ for the function $n \mapsto e^*_{\frac{m}{q}}$. For $q \in \mathbb{N}$ Ramanujan’s sum $c_q$ is given by

$$c_q(n) := \sum_{a \leq q} e^*_q(n)$$

where the star on the sum shall denote the sum over all $a \leq q$ prime to $q$ only, i.e., their greatest common divisor equals 1.

**Lemma 2.4** With the geometric series we have for $a, b \in \mathbb{Z}$, $0 \leq a < b$, $\beta \in \mathbb{R}$ the inequality

$$\left| \sum_{a<n \leq b} e(\beta n) \right| \leq \min \left( b - a, \frac{1}{2||\beta||} \right)$$

where $||\beta||$ denotes the distance to the nearest integer. For a proof, see [8].

**The space $\mathcal{D}^2$ of limit-periodic functions**

For $q \in \mathbb{N}$, let $\mathcal{D}_q$ be the set of all $q$-periodic functions and $\mathcal{D} := \bigcup_{q=1}^{\infty} \mathcal{D}_q$. Write $\mathcal{D}^2$ for the closure of $\mathcal{D}$ with regards to the Besicovitch-seminorm $||.||_2$ which makes it a normed vector space in a canonical way. Limit-periodic functions are exactly the elements of this vector space.
Theorem 2.5  The vector spaces \( \mathcal{D}_q \) and \( \mathcal{D} \) possess the following bases, see [11],
\[
\mathcal{D}_q = \langle e_a^q : 1 \leq a \leq q \rangle_C \\
\mathcal{D} = \langle e_a^q : 1 \leq a \leq q, q \in \mathbb{N}, (a; q) = 1 \rangle_C
\]  (5)

Definition 2.6 (Besicovitch-seminorm)  The Besicovitch-seminorm of a function \( f \in \mathcal{D} \) is given through
\[
\|f\|_2 := \left( \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^2 \right)^{\frac{1}{2}}
\]

Note that for a \( q \)-periodic function we have the identity
\[
\sum_{n \leq x} |f(n)|^2 = \left( \left\lfloor \frac{x}{q} \right\rfloor + O(1) \right) \sum_{n \leq q} |f(n)|^2
\]

Comments 2.7  If \( f \) is a limit-periodic function, so is \( |f|, \text{Re}(f) \) and \( \text{Im}(f) \) as well as with \( a \in \mathbb{Z}, b \in \mathbb{N}, \)
\[
n \mapsto f(n + a) \\
n \mapsto f(bn)
\]

Furthermore, the mean-value
\[
M(f) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)
\]
exists for every \( f \in \mathcal{D}^2 \), as well as the mean-value in residue classes
\[
\eta_f(q, b) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x, n \equiv b(q)} f(n)
\]  (6)

for arbitrary \( b, q \in \mathbb{N} \). For the respective proofs, see [11].

Comment 2.8  For \( f \in \mathcal{D}^2 \) we have \( \eta_f(q, b) \ll q^{-\frac{1}{2}} \) which is easily seen with the Cauchy-Schwarz-inequality
\[
\frac{1}{x} \left| \sum_{n \leq x, n \equiv b(q)} f(n) \right| \leq \left( \frac{1}{x} \sum_{n \leq x} |f(n)| \right)^{\frac{1}{2}} \left( \frac{1}{x} \sum_{n \leq x, n \equiv b(q)} 1 \right)^{\frac{1}{2}} \ll \left( \frac{1}{x} \left( \frac{x}{q} + 1 \right) \right)^{\frac{1}{2}} \ll q^{-\frac{1}{2}}
\]

where \( \ll \) denotes as usual Vinogradov’s symbol.

Lemma 2.9 (Parseval’s identity)  As the basis (5) is an orthonormal basis of \( \mathcal{D}^2 \), Parseval’s identity holds as well
\[
\sum_{q=1}^{\infty} \sum_{a \leq q} |M(f \cdot e_a^q)|^2 = \|f\|_2^2
\]

The following example of a limit-periodic function is used in the application at the end of this paper.

Lemma 2.10  The function \( \mu_k \) is not periodic, but it is limit-periodic.
Proof. Assume we have a natural number $R$ with

$$\mu_k(n + R) = \mu_k(n)$$

for all $n \in \mathbb{N}$. Then we can deduce that for each $p \in \mathbb{P}$ and $m \in \mathbb{N}$ we have

$$p^k \nmid (1 + mR)$$

which is easily seen to be false with the theorem of Fermat-Euler. For the proof of the limit-periodic property, define for $k \in \mathbb{N}$, $k \geq 2$, and $\gamma \in \mathbb{R}_{>2}$ the arithmetical function $\nu_k^{(\gamma)}$ through

$$\nu_k^{(\gamma)}(n) := \begin{cases} 
\mu(s) \prod_{p \mid n} [p \leq \gamma] & \text{if } n = s^k \text{ with } s \in \mathbb{N} \\
0 & \text{otherwise}
\end{cases}$$

Then $\nu_k^{(\gamma)}$ is multiplicative. As a Dirichlet-convolution of multiplicative functions, the function

$$\mu_k^{(\gamma)} := \nu_k^{(\gamma)} * 1$$

is multiplicative as well. It is an approximation to $\mu_k$ as can be seen, when evaluated on prime powers:

$$\mu_k^{(\gamma)}(p^r) = 1 + \sum_{j \leq r} \left\{ \mu(p) [p \leq \gamma] \quad j = k \right\} = 1 - [p \leq \gamma] [k \leq r]$$

which means

$$\mu_k(n) = \sum_{d \mid n} \mu(d) \quad \mu_k^{(\gamma)}(n) = \sum_{\forall d \mid n \exists p \mid d : p \leq \gamma} \mu(d)$$

With the equations (2) and (9) we get for all $p \in \mathbb{P}$ and $r \in \mathbb{N}_0$

$$\mu_k^{(\gamma)}(p^r) \geq \mu_k(p^r)$$

As both functions are multiplicative and can only attain the values 0 or 1, we get directly for all $n \in \mathbb{N}$

$$\mu_k^{(\gamma)}(n) \geq \mu_k(n)$$

and

$$\left( \mu_k^{(\gamma)}(n) - \mu_k(n) \right) \in \{ 0, 1 \}$$

(10)

The function $\mu_k^{(\gamma)}$ is periodic with period $\tau := \prod_{p \leq \gamma} p^k$, as

$$\mu_k^{(\gamma)}(n) = \sum_{\forall d \mid n \exists p \mid d : p \leq \gamma} \mu(d) = \sum_{\forall d \mid n \exists p \mid d : p \leq \gamma} \mu(d) = \mu_k^{(\gamma)}(n + \tau)$$

From (10) we deduce

$$\left\| \mu_k^{(\gamma)} - \mu_k \right\|_2^2 = \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \left( \sum_{\exists p \mid d : p > \gamma} \mu(d) \right)^2 = \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \sum_{\exists p \mid d : p > \gamma} \mu(d)$$
and it follows
\[
\left\| \mu_k^{(y)} - \mu_k \right\|_2^2 \leq \limsup_{x \to \infty} \frac{1}{x} \sum_{d \mid x} \sum_{n \leq x, \exists p \mid d: p > y} 1 = \limsup_{x \to \infty} \frac{1}{x} \sum_{d \leq x} \sum_{n \leq x, \exists p \mid d: p > y} 1 \sum_{n \leq x} 1 \\
\leq \limsup_{x \to \infty} \frac{1}{x} \sum_{d \leq x} \frac{x}{d} = \limsup_{x \to \infty} \sum_{d \leq x} d^{-k} \\
\leq \sum_{d > y} d^{-k} \xrightarrow{y \to \infty} 0
\]

Thereby, we get
\[
\left\| \mu_k - \mu_k^{(y)} \right\|_2 \xrightarrow{y \to \infty} 0 \tag{11}
\]
which shows the limit-periodic property of \( \mu_k \). \( \square \)

**Lemma 2.11** For a limit-periodic function \( f \) that is bounded in addition, the function \( \mu_k f \) is limit-periodic as well, which can be easily seen.

**The Barban–Davenport–Halberstam theorem for \( D^2 \)**

The Barban-Davenport-Halberstam theorem in its original form for primes proves that the error term in the prime number theorem for arithmetic progressions is small in the quadratic mean, see [1], and for further references [5], [6]. We need a corresponding version for limit-periodic functions.

Define the error term in the sum over arithmetic progressions via
\[
E_f(x; q, b) := \sum_{n \leq x, n \equiv b(q)} f(n) - x\eta_f(q, b) \tag{12}
\]
Then the following lemma due to Hooley [7] holds.

**Lemma 2.12** If for all \( A \in \mathbb{R} \), \( b, q \in \mathbb{N} \)
\[
E_f(x; q, b) \ll \frac{x}{(\log x)^A}
\]
where the implicit constant in Vinogradov’s symbol is at most dependent on \( A \) or \( f \), then we have for all \( A \in \mathbb{R} \) and \( Q \in \mathbb{R}_{>0} \)
\[
\sum_{q \leq Q} \sum_{b \leq x} |E_f(x; q, b)|^2 \ll Qx + \frac{x^2}{(\log x)^A}
\]
3 Proof of the main theorem with the circle method

In this section we state and prove the main theorem with the circle method of Hardy and Littlewood \[12\]. Let \( f \in \mathcal{D}^2 \) be throughout this section a given function.

**Definition 3.1** We define for \( a, q \in \mathbb{N} \) the Gaußian sum of \( f \) via

\[
G_f(q, a) := \sum_{b \leq q} \eta_f(q, b) e\left(\frac{ab}{q}\right)
\]

**Comments 3.2** The Gaußian sum is a mean-value, as

\[
G_f(q, a) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} n \equiv b(q) f(n) e\left(\frac{an}{q}\right) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) e\left(\frac{an}{q}\right) = M(f \cdot e_\frac{a}{q})
\]

With Parseval’s identity we also have

\[
\mathcal{S}_f := \sum_{q=1}^{\infty} \sum_{a \leq q}^* |G_f(q, a)|^2 = \sum_{q=1}^{\infty} \sum_{a \leq q}^* \left| \sum_{b \leq q} \eta_f(q, b) e\left(\frac{ab}{q}\right) \right|^2
\]

\[
= \sum_{q=1}^{\infty} \sum_{a \leq q}^* \left| \sum_{b \leq q} \eta_f(q, b) e\left(-\frac{ab}{q}\right) \right|^2 = \sum_{q=1}^{\infty} \sum_{a \leq q}^* \left| M(f \cdot e_\frac{a}{q}) \right|^2 = ||f||_2^2
\]

(13)

Therefore, for limit-periodic functions the identity \( \mathcal{S}_f = ||f||_2^2 \) holds. The series \( \mathcal{S}_f \) is called *singular series* of \( f \).

We are now able to state the main theorem of this paper.

**Theorem 3.3** Let \( f \in \mathcal{D}^2 \) be an arithmetical function with

\[
\sum_{n \leq x} |f(n)|^2 = x ||f||_2^2 + o\left(\frac{x}{\log x}\right)
\]

and the remainder of the corresponding singular series \[13\] satisfies

\[
\sum_{q > w} \sum_{a \leq q}^* |G_f(q, a)|^2 = o\left(w^{-\frac{1}{r}}\right)
\]

(15)

with \( r \in \mathbb{R}_{>1} \). We then set \( Q = Q(x) := (\log x)^r \). Furthermore, we demand for all \( A \in \mathbb{R} \)

\[
\sum_{q \leq Q} \max_{1 \leq b \leq q} |E_f(x; q, b)| \ll \frac{x}{(\log x)^A}
\]

(16)

where the implicit constant in Vinogradov’s symbol is at most dependent on \( A \).

Then we have

\[
\sum_{p \leq x} f(p) = c_f \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)
\]

(17)
with a constant $c_f$ that is represented through the infinite series

$$c_f := \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G_f(q, a)$$

Comments 3.4

1. The condition (16) implies

$$E_f(x; q, b) \ll \frac{x}{(\log x)^A}$$

for all $A \in \mathbb{R}$, $b, q \in \mathbb{N}$, and we can apply theorem 2.12.

2. The following identity can be verified easily

$$c_f = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} M(f \cdot c_q)$$

In what follows, we assume the conditions of theorem 3.3. For notational simplification, we write $G, E, \eta, \text{etc.}$ instead of $G_f, E_f, \eta_f, \text{etc.}$

3.1 Split in major and minor arcs

**Definition 3.5 (Major and minor arcs)** With the unit interval $\mathcal{U} := \left( \frac{Q}{x}, 1 + \frac{Q}{x} \right]$ we define for $a, q \in \mathbb{N}$ with $1 \leq a \leq q \leq Q, (a; q) = 1$ the major arcs through

$$\mathfrak{M}(q, a) := \left\{ \alpha \in \mathcal{U} : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{x} \right\}$$

Let the symbol $\mathfrak{M}$ denote the union of all major arcs

$$\mathfrak{M} := \bigcup_{q \leq Q} \bigcup_{a \leq q} \mathfrak{M}(q, a)$$

We define the minor arcs as usual as the complement in the unit interval

$$\mathfrak{m} := \mathcal{U} \setminus \mathfrak{M}$$

For sufficient large $x$ each pair of major arcs is disjunct.

**Definition 3.6** We define exponential sums $S$ and $T$ for $\alpha \in \mathbb{R}$ via

$$S(\alpha) := \sum_{n \leq x} f(n) e(\alpha n)$$

$$T(\alpha) := \sum_{p \leq x} e(-\alpha p)$$

and have then

$$\sum_{p \leq x} f(p) = \int_{0}^{1} S(\alpha) T(\alpha) d\alpha$$

$$= \int_{\mathcal{U}} S(\alpha) T(\alpha) d\alpha = \int_{\mathfrak{M}} S(\alpha) T(\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha) T(\alpha) d\alpha$$

(18)
3.2 The major arcs

On \( \mathfrak{M} \) we approximate \( S \) resp. \( T \) by the functions \( S^* \) resp. \( T^* \) that are defined for \( \alpha \in \mathfrak{M}(q, a) \), \( \alpha = \frac{a}{q} + \beta \), via

\[
S^*(\alpha) := G(q, a) \sum_{n \leq x} e(\beta n) \\
T^*(\alpha) := \frac{\mu(q)}{\varphi(q)} \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n}
\]

**Lemma 3.7** The function \( T^* \) satisfies on \( \mathfrak{M}(q, a) \) with \( \alpha = \frac{a}{q} + \beta \) the inequality

\[
T^*(\alpha) \ll \frac{\mu(q)^2}{\varphi(q) \log x} \frac{x}{1 + \|\beta\| x}
\]  

**Proof.** The case \( \beta \in \mathbb{Z} \) is trivial. For \( \beta \notin \mathbb{Z} \) the method of partial summation and estimate (14) can be applied:

\[
T^*(\alpha) \ll \frac{\mu(q)^2}{\varphi(q)} \left( \frac{1}{\log x} \sum_{2 \leq n \leq x} e(-\beta n) + \frac{1}{t (\log t)^2} \left| \sum_{2 \leq n \leq t} e(-\beta n) \right| dt \right)
\]

\[
\ll \frac{\mu(q)^2}{\varphi(q)} \left( \frac{1}{\|\beta\| \log x} + \frac{1}{\|\beta\|} \int_{t=2}^{x} \frac{1}{t (\log t)^2} dt \right) \ll \frac{\mu(q)^2}{\varphi(q)} \frac{1}{\|\beta\| \log x} \quad \square
\]

The next lemma makes the approximation through \( S^* \) and \( T^* \) on the major arcs more precise.

**Lemma 3.8** We have for \( \alpha \in \mathfrak{M}, \alpha = \frac{a}{q} + \beta \) and arbitrary \( A \in \mathbb{R} \)

\[
S(\alpha) = S^*(\alpha) + \sum_{b \leq q} e\left( \frac{ab}{q} \right) \Xi (x; q, b; \beta) \\
T(\alpha) = T^*(\alpha) + O\left( \frac{x}{(\log x)^A} \right)
\]

where

\[
\Xi (x; q, b; \beta) := e(\beta x) E([x] ; q, b) - 2\pi i \beta \int_{t=1}^{x} e(\beta t) E([t] ; q, b) dt
\]  

**Proof.** If we evaluate \( S \) at the rational number \( \frac{a}{q} \) we get with definition (12)

\[
\sum_{n \leq x} f(n) e\left( \frac{an}{q} \right) = \sum_{b \leq q} e\left( \frac{ab}{q} \right) \sum_{n \leq [x]} f(n) = \sum_{b \leq q} e\left( \frac{ab}{q} \right) \left( [x] \eta(q, b) + E([x] ; q, b) \right) \\
= [x] G(q, a) + \sum_{b \leq q} e\left( \frac{ab}{q} \right) E([x] ; q, b)
\]
Applying partial summation twice yields the stated claim:

\[
S(\alpha) = e(\beta x) S\left(\frac{\alpha}{q}\right) - 2\pi i \beta \sum_{t=1}^{x} e(\beta t) \left( [t] G(q, a) + \sum_{b \leq q} e\left(\frac{ab}{q}\right) E([t] ; q, b) \right) dt
\]

\[
= G(q, a) \left( [x] e(\beta x) - 2\pi i \beta \sum_{t=1}^{x} e(\beta t) dt \right) + \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta)
\]

\[
= S^*(\alpha) + \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta)
\]

For the second statement, we use partial summation another time

\[
T(\alpha) = \frac{1}{\log x} \sum_{p \leq x} e(-\alpha p) \log p + \int_{t=2}^{x} \frac{1}{t (\log t)^2} \sum_{p \leq t} e(-\alpha p) \log p dt
\]

and apply afterwards the estimate

\[
\sum_{p \leq v} e(-\alpha p) \log p = \frac{\mu(q)}{\varphi(q)} \sum_{n \leq v} e(-\beta n) + O\left(\frac{v}{(\log v)^4}\right)
\]

that is valid for all \(v, A \in \mathbb{R}_{>1}\), see [12, Lemma 3.1].

We then get

\[
T(\alpha) = \frac{\mu(q)}{\varphi(q)} \left( \frac{1}{\log x} \sum_{2 \leq n \leq x} e(-\beta n) + \int_{t=2}^{x} \frac{1}{t (\log t)^2} \sum_{n \leq t} e(-\beta n) dt \right) + O\left(\frac{x}{(\log x)^4}\right)
\]

\[
= T^*(\alpha) + O\left(\frac{x}{(\log x)^4}\right)
\]

**Corollary 3.9** We have

\[
\int_{\mathbb{N}} |S(\alpha)|^2 d\alpha = x ||f||_2^2 + o\left(\frac{x}{\log x}\right)
\]

**Proof.** Set

\[
\int_{\mathbb{N}} |S(\alpha)|^2 d\alpha = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]

with

\[
\Sigma_1 = \int_{\mathbb{N}} |S^*(\alpha)|^2 d\alpha = \sum_{q \leq Q} \sum_{a \leq q} \left| G(q, a) \right|^2 \int_{|\beta| \leq \frac{Q}{x}} \left| \sum_{n \leq x} e(\beta n) \right|^2 d\beta
\]

\[
\Sigma_2 = \sum_{q \leq Q} \sum_{a \leq q} \left| [\beta] \leq \frac{Q}{x} \right| \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta) \left| \sum_{n \leq x} e(\beta n) \right|^2 d\beta
\]

\[
\Sigma_3 = \sum_{q \leq Q} \sum_{a \leq q} \left| [\beta] \leq \frac{Q}{x} \right| \sum_{b \leq q} S^*\left(\frac{a}{q} + \beta \right) \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta) d\beta
\]

\[
\Sigma_4 = \sum_{q \leq Q} \sum_{a \leq q} \left| [\beta] \leq \frac{Q}{x} \right| \sum_{b \leq q} S^*\left(\frac{a}{q} + \beta \right) \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta) d\beta
\]
To evaluate $\Sigma_1$ we complete the integration limits to $\left[ -\frac{1}{2}, \frac{1}{2} \right]$ and use properties of the exponential function. For the error term that occurred, we apply (11) and remark that $||\beta|| = |\beta|$ for $|\beta| \leq \frac{1}{2}$. The series converges with Parseval’s identity to the limit $||f||^2_2$.

$$
\Sigma_1 = \sum_{q \leq Q} \sum_{a \leq q}^* |G(q,a)|^2 \left( |x| - \int \frac{\sum e(\beta n)^2}{Q < |\beta| \leq \frac{1}{2}} \right)
= x ||f||^2_2 - x \sum_{q > Q} \sum_{a \leq q}^* |G(q,a)|^2 + O\left( \frac{x}{Q} \right)
$$

The condition (15) of theorem 3.3 implies now together with the definition of $Q$

$$
\Sigma_1 = x ||f||^2_2 + o\left( \frac{x}{\log x} \right)
$$

For $\Sigma_2$ we note that the function $\Xi$ defined in (20) satisfies the following inequality:

$$
\Xi(x; q, b; \beta) \ll (1 + |\beta|x) \max_{1 \leq k \leq x} |E(k; q, b)|
$$

(22)

By neglecting the condition on co-primality for the sum over $a$, we get

$$
\Sigma_2 \ll \sum_{q \leq Q} \sum_{b \leq q} \int \Xi(x; q, b; \beta) \Xi(x; q, b; \beta) \sum_{a \leq q} e\left( \frac{a}{q} (b_1 - b_2) \right) d\beta
$$

$$
= \sum_{q \leq Q} \sum_{b \leq q} \int |\Xi(x; q, b; \beta)|^2 d\beta
$$

$$
\ll Q \int_{|\beta| \leq \frac{1}{Q}} (1 + |\beta|x)^2 d\beta \max_{1 \leq k \leq x} \sum_{q \leq Q} \sum_{b \leq q} |E(k; q, b)|^2
$$

As $f$ fulfills condition (16), we can use the Barban-Davenport-Halberstam statement for limit-periodic functions, theorem 2.12, and get for all $A \in \mathbb{R}$

$$
\Sigma_2 \ll Q^4 \max_{1 \leq k \leq x} \left( Qk + \frac{k^2}{(\log k)^A} \right) \ll Q^5 + Q^4 \frac{x}{(\log x)^A}
$$

Therefore

$$
\Sigma_2 = o\left( \frac{x}{\log x} \right)
$$

Using the estimates above for $\Sigma_1$ and $\Sigma_2$ and applying the Cauchy-Schwarz-inequality, we have for all $A \in \mathbb{R}$

$$
\Sigma_3 + \Sigma_4 \ll \sum_{q \leq Q} \sum_{a \leq q}^* \int \left| S^* \left( \frac{a}{q} + \beta \right) \sum_{b \leq q} e\left( \frac{ab}{q} \right) \Xi(x; q, b; \beta) \right| d\beta
$$

$$
\ll (\Sigma_1)^{\frac{1}{2}} (\Sigma_2)^{\frac{1}{2}} \ll x^{\frac{1}{2}} \left( \frac{x}{(\log x)^A} \right)^{\frac{1}{2}} = \frac{x}{(\log x)^{\frac{A}{2}}}
$$

So

$$
\Sigma_3 + \Sigma_4 = o\left( \frac{x}{\log x} \right)
$$
Lemma 3.10 We have for all $A \in \mathbb{R}$

$$\int_{\mathbb{R}} S(\alpha) (T(\alpha) - T^*(\alpha)) \, d\alpha \ll Q^{\frac{3}{2}} \frac{x}{(\log x)^A} \quad (23)$$

and

$$\int_{\mathbb{R}} T^*(\alpha) (S(\alpha) - S^*(\alpha)) \, d\alpha \ll Q \frac{x}{(\log x)^A} \quad (24)$$

PROOF. The Cauchy-Schwarz-inequality can be applied to equation (23) and we get

$$\left| \int_{\mathbb{R}} S(\alpha) (T(\alpha) - T^*(\alpha)) \, d\alpha \right| \leq \left( \int_{\mathbb{R}} |S(\alpha)|^2 \, d\alpha \int_{\mathbb{R}} |T(\alpha) - T^*(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}}$$

For the first factor we use a trivial estimate from equation (21)

$$\int_{\mathbb{R}} |S(\alpha)|^2 \, d\alpha \ll x$$

For the second factor, we use lemma 3.3 and get for all $A \in \mathbb{R}$

$$\int_{\mathbb{R}} |T(\alpha) - T^*(\alpha)|^2 \, d\alpha \ll \sum_{q \leq Q} \sum_{\alpha \leq q} \sum_{|\beta| \leq \frac{Q}{x}} \frac{x}{(\log x)^A} \, d\beta \ll Q^2 \frac{x}{(\log x)^{2A}}$$

which proves (24).

For the second statement we use the approximation property from lemma 3.3. The left-hand side in (21) is then equal to

$$\sum_{q \leq Q} \sum_{a \leq q} \sum_{|\beta| \leq \frac{Q}{x}} \frac{\mu(q)}{\varphi(q)} \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n} \sum_{b \leq q} e\left(\frac{ab}{q}\right) \Xi(x; q, b; \beta) \, d\beta$$

$$= \sum_{q \leq Q} \sum_{b \leq q} c_q(b) \sum_{|\beta| \leq \frac{Q}{x}} \frac{\mu(q)}{\varphi(q)} \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n} \Xi(x; q, b; \beta) \, d\beta$$

$$\ll \sum_{q \leq Q} \sum_{b \leq q} |c_q(b)| \sum_{|\beta| \leq \frac{Q}{x}} \left| \frac{\mu(q)}{\varphi(q)} \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n} \right| \Xi(x; q, b; \beta) \, d\beta$$

Using the approximation (19) for $T^*$ and (22) for $\Xi$ we get

$$\ll \sum_{q \leq Q} \sum_{b \leq q} \frac{\mu(q)^2}{\varphi(q)} |c_q(b)| \sum_{|\beta| \leq \frac{Q}{x}} \frac{1}{\log x} \left(1 + |\beta| x\right) \frac{1}{1 + |\beta|} \max_{1 \leq k \leq x} |E(k; q, b)| \, d\beta$$

$$\ll \frac{Q}{\log x} \sum_{q \leq Q} \varphi(q) \sum_{b \leq q} \frac{\mu(q)^2}{\varphi(q)} |c_q(b)| \sum_{1 \leq k \leq x} |E(k; q, b)|$$

Exploiting standard properties of Ramanujan’s sum and the divisor function $d(q)$ results in

$$\frac{Q}{\log x} \sum_{q \leq Q} \varphi(q) \max_{1 \leq b \leq q} |E(k; q, b)| \ll Q \max_{1 \leq k \leq x} \sum_{q \leq Q} \max_{1 \leq b \leq q} |E(k; q, b)|$$

Finally, with applying condition (10) we get the desired result (24).
3.3 The main term

Lemma 3.11 On the major arcs we have
\[ \int_{\mathfrak{M}} S^* (\alpha) T^* (\alpha) \, d\alpha = c_f \frac{x}{\log x} + o \left( \frac{x}{\log x} \right) \]
with the absolute convergent series
\[ c_f := \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) \tag{25} \]

PROOF. Lemma 2.2 and the requirement \((\ref{15})\) imply the absolute convergence of the series, as we have for all \(v, w \in \mathbb{R}_0^+\) and \(v' := \frac{\log v}{\log 2}, w' := \frac{\log w}{\log 2} - 1\)
\[ \sum_{v < q \leq w} \left| \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) \right| \leq \sum_{v' \leq k \leq w'} \left( \sum_{2^k < q \leq 2^k + 1} \frac{\mu(q)^2}{\varphi(q)} \right) \left( \sum_{2^k < q \leq 2^k + 1} \sum_{a \leq q}^* |G(q, a)|^2 \right) \]
\[ \leq \sum_{k \geq v'} \left( \sum_{q \leq 2^{k+1}} \frac{\mu(q)^2}{\varphi(q)} \right) \left( \sum_{q \geq 2^k} \sum_{a \leq q}^* |G(q, a)|^2 \right) \]
\[ \ll \sum_{k \geq v'} \left( \left( \log 2^k \right) \left( 2^{-\frac{k}{2}} \right) \right) \left( 2^{-\frac{k}{2}} \right) \lim_{v' \to \infty} 0 \]
The number \(r \in \mathbb{R}_+\) exists as we require \((\ref{15})\) to be true and it can be seen easily that the implicit constants can be chosen independently of \(v\) and \(w\). Cauchy’s criterion implies the stated convergence.

To evaluate the integral
\[ \int_{\mathfrak{M}} S^* (\alpha) T^* (\alpha) \, d\alpha = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) \int_{|\beta| \leq \frac{1}{2}} \sum_{m \leq x} e(\beta m) \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n} \, d\beta \]
we complete the integration limits to \([-\frac{1}{2}, \frac{1}{2}]\) and get
\[ \int_{\mathfrak{M}} S^* (\alpha) T^* (\alpha) \, d\alpha = \frac{x}{\log x} \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) + \Sigma_5 \]
with
\[ \Sigma_5 \ll \left| \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) \right| \left( \frac{x}{(\log x)^2} + \int_{\frac{1}{2} < |\beta| \leq 1} \left| \sum_{m \leq x} e(\beta m) \right| \sum_{2 \leq n \leq x} \frac{e(-\beta n)}{\log n} \, d\beta \right) \]
With the convergence of the series over \(q\) and lemma 3.7 as well as with the approximation \((\ref{11})\), we get
\[ \int_{\mathfrak{M}} S^* (\alpha) T^* (\alpha) \, d\alpha = c_f \frac{x}{\log x} + O \left( \frac{x}{\log x} \left( \frac{1}{\log x} + \left| \sum_{q > Q} \frac{\mu(q)}{\varphi(q)} \sum_{a \leq q}^* G(q, a) \right| + \frac{1}{Q} \right) \right) \]
\[ = c_f \frac{x}{\log x} + o \left( \frac{x}{\log x} \right) \]
Corollary 3.12 On the major arcs we have

\[
\int_{\mathbb{N}} S(\alpha) T(\alpha) \, d\alpha = c f \frac{x}{\log x} + o \left( \frac{x}{\log x} \right)
\]

(26)

PROOF. Writing

\[
S(\alpha) T(\alpha) = S(\alpha) (T(\alpha) - T^*(\alpha)) + (S(\alpha) - S^*(\alpha)) T^*(\alpha) + S^*(\alpha) T^*(\alpha)
\]

and approximating the terms, yields the stated result. □

3.4 The minor arcs

Lemma 3.13 For the integral on the minor arcs, we have

\[
\int_{\mathbb{M}} S(\alpha) T(\alpha) \, d\alpha = o \left( \frac{x}{\log x} \right)
\]

PROOF. We get with the Cauchy-Schwarz-inequality

\[
\left| \int_{\mathbb{M}} S(\alpha) T(\alpha) \, d\alpha \right| \leq \left( \int_{\mathbb{M}} |S(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}} \left( \int_{\mathbb{U}} |T(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}}
\]

An application of the prime number theorem yields then

\[
\int_{\mathbb{U}} |T(\alpha)|^2 \, d\alpha = \sum_{p \leq x} 1 \ll \frac{x}{\log x}
\]

We get with

\[
\int_{\mathbb{M}} |S(\alpha)|^2 \, d\alpha = \int_{\mathbb{U}} |S(\alpha)|^2 \, d\alpha - \int_{\mathbb{N}} |S(\alpha)|^2 \, d\alpha = \sum_{n \leq x} |f(n)|^2 - \int_{\mathbb{N}} |S(\alpha)|^2 \, d\alpha
\]

and, luckily, as of condition (14), we get with corollary 3.9

\[
\int_{\mathbb{M}} |S(\alpha)|^2 \, d\alpha = o \left( \frac{x}{\log x} \right)
\]

Putting altogether: With equation (18), corollary 3.12 and lemma 3.13 we get the statement (17) and this completes the prove of theorem 3.3.
4 An application to $k$-free numbers

In this section we give an application of theorem 3.3. For this purpose, let $s \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}_0$, $r_1, \ldots, r_s \in \mathbb{N}$ with $2 \leq r_1 \leq \cdots \leq r_s$ be fixed.

**Definition 4.1** For $a, q \in \mathbb{N}$ we set the value of $E_a(d_1, \ldots, d_s, q)$ to 1 (resp. 0) if the following system of congruences

\[
\begin{align*}
    n &\equiv -\alpha_j (d_j) \quad (1 \leq j \leq s) \\
    n &\equiv a \quad (q)
\end{align*}
\]

has a solution in $n$ (resp. has no solution).

We choose our function $f$ to be

\[ f(n) := \prod_{i} r_i (n + \alpha_i) \]

and

\[ \mathcal{F} := \{ n \in \mathbb{N} : f(n) = 1 \} \]

The function $f$ is limit-periodic as is shown when using the lemmas 2.10 and 2.11 and the comments 2.7. It only takes values from the set $\{0, 1\}$ and satisfies the requirements from theorem 3.3 as will be shown below. We will apply similar methods as Brüdern et al. [3], [4] and Mirsky [9], [10].

To exclude the trivial case, we assume further the choice of the parameter $\alpha_1, \ldots, \alpha_s, r_1, \ldots, r_s$ in such a way, that $\mathcal{F} \neq \emptyset$. The following theorem characterizes exactly this case.

**Theorem 4.2 (Mirsky)** The set $\mathcal{F}$ is non-empty if and only if for every prime $p$ there exists a natural number $n$ with $n \not\equiv -\alpha_i (p^{r_i})$ for $1 \leq i \leq s$. In this case, the set $\mathcal{F}$ even has a positive density [10, theorem 6].

4.1 Proof of the requirements (14) and (16)

**Definition 4.3** We define $D(p)$ and $D^*(p)$ as the number of natural numbers $n \leq p^{r_s}$ that solve at least one of the congruences $n \equiv -\alpha_i (p^{r_i})$ ($1 \leq i \leq s$), whereas we demand for $D^*(p)$ in addition the condition $(n; p) = 1$, i.e.,

\[
D(p) := \sum_{n \leq p^{r_s}} 1 \quad \quad \quad D^*(p) := \sum_{n \leq p^{r_s}} [p \mid n]
\]

We set

\[ D := \prod_{p} \left( 1 - \frac{D(p)}{p^{r_s}} \right) \]

The convergence of this product follows from $D(p) < p^{r_s}$ for every $p$ which is being implied by $\mathcal{F} \neq \emptyset$, see [13], and

\[ D(p) \leq \sum_{i \leq s} \sum_{n \leq p^{r_s}} [n \equiv -\alpha_i (p^{r_i})] = \sum_{i \leq s} p^{r_s-r_i} \ll p^{r_s-2} \]
The mean-value of \( f \)

**Theorem 4.4 (Mirsky)** We have for all \( \epsilon > 0 \)
\[
\sum_{n \leq x} f(n) = \mathcal{O} x + O \left( \frac{x^{\frac{2}{r_1+1} + \epsilon}}{x} \right)
\]

See [10, theorem 5].

As the function \( f \) can only assume the values 0 or 1, we also have
\[
\sum_{n \leq x} |f(n)|^2 = \mathcal{O} x + O \left( \frac{x^{2} r_1 + 1 + \epsilon}{x} \right)
\]
and \( M(f) = ||f||_2^2 = \mathcal{O} \). Therewith the requirement (14) for \( f \) follows.

**Definition 4.5** Set \( g(q, a) \) as
\[
g(q, a) = \sum_{d_1, \ldots, d_s = 1}^{\infty} \mu(d_1) \cdots \mu(d_s) \frac{\mathcal{E}_a(d_1, \ldots, d_s, q)}{[d_1, \ldots, d_s]} \left( [d_1^r; \ldots ; d_s^r]; q \right)
\]

It can be seen easily that the series converge.

**Theorem 4.6** For \( a, q \in \mathbb{N} \) and \( \epsilon > 0 \) we have
\[
\sum_{n \leq x} f(n) = \frac{x}{q} g(q, a) + O \left( \frac{x^{\frac{2}{r_1+1} + \epsilon}}{x} \right)
\]
whereas the implicit constant can be chosen independently from \( a \) or \( q \). The proof works analogous to the one in [10, theorem 5]. It uses the identity (3) for \( \mu_k \) and
\[
\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{n \equiv a(q)} \sum_{d_1^r \equiv (n+\alpha_1)} \cdots \sum_{d_s^r \equiv (n+\alpha_s)} \mu(d_1) \cdots \mu(d_s) = \sum_{n \leq x} \sum_{n \equiv a(q)} \mu(d_1) \cdots \mu(d_s)
\]

Hence, the mean-value in residue classes is equal to \( \frac{x}{q} g(q, a) \) and with the error term in (30) the validity of (16) for \( f \) is shown.

This should be compared with the results of Brüdern et al. [4] and Brüdern [2].

4.2 The remainder of the singular series of \( f \)

The validity of condition (15) for \( f \) is still open and will be shown in the following. We start this section with an investigation of the function \( g(q, a) \).

If we write
\[
g(q, a) = \sum_{d_1, \ldots, d_s = 1}^{\infty} \theta_{a,q}(d_1, \ldots, d_s)
\]
with

\[ \theta_{a,q}(d_1, \ldots, d_s) := \mu(d_1) \cdot \ldots \cdot \mu(d_s) \left( \frac{[d_1^{r_1}; \ldots; d_s^{r_s}] ; q}{[d_1^{r_1}; \ldots; d_s^{r_s}]} \right) E_a(d_1^{r_1}, \ldots, d_s^{r_s}, q) \]

then \( \theta_{a,q}(d_1, \ldots, d_s) \) is a multiplicative function in \( d_1, \ldots, d_s \), which follows from the multiplicativity of the three factors

\[ \mu(d_1) \cdot \ldots \cdot \mu(d_s), \quad \left( \frac{[d_1^{r_1}; \ldots; d_s^{r_s}] ; q}{[d_1^{r_1}; \ldots; d_s^{r_s}]} \right), \quad E_a(d_1^{r_1}, \ldots, d_s^{r_s}, q) \]

We then have

\[ g(q, a) = \prod_p \chi^{(q)}_a(p) \]

\[ \chi^{(q)}_a(p) := \sum_{\delta_1, \ldots, \delta_s = 0}^{\infty} \theta_{a,q}(p^{\delta_1}, \ldots, p^{\delta_s}) \]

As \( \mu(p^k) = 0 \) for \( k \geq 2 \) it follows

\[ \chi^{(q)}_a(p) = \sum_{\delta_1, \ldots, \delta_s \in \{0, 1\}} (-1)^{\delta_1 + \cdots + \delta_s} \left( \frac{[p^{\delta_1 r_1}; \ldots; p^{\delta_s r_s}] ; q}{[p^{\delta_1 r_1}; \ldots; p^{\delta_s r_s}]} \right) E_a(p^{\delta_1 r_1}, \ldots, p^{\delta_s r_s}, q) \] (31)

In the case \( p \nmid q \) we can write

\[ \chi^{(q)}_a(p) = \left( 1 - \frac{D(p)}{p^{r_s}} \right) \]

with [13] theorem 5. If we set in addition

\[ z(q) := \prod_{p \mid q} \left( 1 - \frac{D(p)}{p^{r_s}} \right)^{-1} \]

\[ h(q, a) := \prod_{p \mid q} \chi^{(q)}_a(p) \] (32)

we get

\[ g(q, a) = D z(q) h(q, a) \]

and

\[ z(q) \ll 1 \] (33)

by the comments in definition [13.3]

For \( a \equiv b \ (q) \) we have \( h(q, a) = h(q, b) \).

**Lemma 4.7 (Quasi-multiplicativity of \( h \))** The function \( h(q, a) \) is quasi-multiplicative, which means for all \( q_1, q_2 \in \mathbb{N} \), \( (q_1; q_2) = 1 \) and all \( a_1, a_2 \in \mathbb{N} \) we have

\[ h(q_1 q_2, a_1 q_2 + a_2 q_1) = h(q_1, a_1 q_2) h(q_2, a_2 q_1) \]

The proof follows by elementary divisor relations.

**Definition 4.8** We set

\[ H(q, a) := \sum_{b \leq q} h(q, b) e \left( \frac{ab}{q} \right) \]

\[ H(q) := \sum_{a \leq q} |H(q, a)|^2 \]

With the Gaussian sum \( G(q, a) = \sum_{b \leq q} \frac{1}{q} g(q, b) e \left( \frac{ab}{q} \right) \) of \( f \) it follows

\[ \sum_{a \leq q} |G(q, a)|^2 = D^2 q^{-2} z(q)^2 H(q) \] (34)
Properties of the function $H$

**Lemma 4.9** The function $H$ has the following useful properties:

1. $H(q)$ is a multiplicative function.

2. On prime powers we have

$$H(p^l) = \begin{cases} 
1 & \text{for } l = 0 \\
\frac{1}{p^{3l-2r_s}} \sum_{n,m \leq p^{r_s} \atop n \equiv m \pmod{(p^l-1)}} \left( \left[ \frac{n}{m} \equiv \left( p^l \right)^{-1} \right] - \frac{1}{p^l} \right) & \text{for } 1 \leq l \leq r_s \\
0 & \text{for } l > r_s 
\end{cases}$$

3. We have the inequalities

$$0 \leq H(p^l) \leq \begin{cases} 
sp^{3l-2r_1} & \text{for } 1 \leq l \leq r_1 \\
sp^{2l-r_1} & \text{for } r_1 < l \leq r_s 
\end{cases}$$

**Proof.** To statement [1] Let $q_1, q_2 \in \mathbb{N}$, $(q_1; q_2) = 1$ be given. We then have

$$H(q_1q_2) = \sum_{a \leq q_1q_2} |H(q, a)|^2 = \sum_{a_1 \leq q_1} \sum_{a_2 \leq q_2} |H(q_1q_2, a_1a_2 + a_2q_1)|^2$$

$$= \sum_{a_1 \leq q_1} \sum_{a_2 \leq q_2} \left| \sum_{b \leq q_1q_2} h(q_1q_2, b) e \left( \frac{a_1q_2 + a_2q_1}{q_1q_2} \right) \right|^2$$

and by using the quasi-multiplicative property of $h$

$$H(q_1q_2) = \sum_{a_1 \leq q_1} \sum_{a_2 \leq q_2} \left| \sum_{b_1 \leq q_1} \sum_{b_2 \leq q_2} h(q_1q_2, b_1q_2 + b_2q_1) e \left( \frac{a_1}{q_1} b_1q_2 \right) e \left( \frac{a_2}{q_2} b_2q_1 \right) \right|^2$$

$$= \sum_{a_1 \leq q_1} \sum_{a_2 \leq q_2} \left| \sum_{b_1 \leq q_1} h(q_1, b_1q_2) e \left( \frac{a_1}{q_1} b_1q_2 \right) \right|^2 \left| \sum_{b_2 \leq q_2} h(q_2, b_2q_1) e \left( \frac{a_2}{q_2} b_2q_1 \right) \right|^2$$

$$= H(q_1) H(q_2)$$

To statement [2] We write

$$H(p^l) = \sum_{a \leq p^l} |H(p^l, a)|^2 - \sum_{a \leq p^l} |H(p^l, a)|^2$$

$$= p^l \sum_{b \leq p^l} h(p^l, b)^2 - p^l \sum_{b_1, b_2 \leq p^l \atop b_1 \equiv b_2 (p^{r_s-1})} h(p^l, b_1) h(p^l, b_2)$$

and $H(p^l) = 0$ for $l > r_s$ follows, as in this case $b_1 \equiv b_2 (p^{r_s})$ and with the definition [3] of $\mathcal{E}_a$, the truth of $h(p^l, b_1) = h(p^l, b_2)$ is implied. As $H$ is multiplicative, we have $H(1) = 1$. For the
case $1 \leq l \leq r_s$ we first evaluate $\chi_a^{(p^i)}(p)$:

$$\chi_a^{(p^i)}(p) = p^{l-r_s} \sum_{n \leq p^{r_s}} \sum_{\delta_i \leq \delta_s \{0,1\}} (-1)^{\delta_1+\cdots+\delta_s} \frac{E_a(p^{\delta_1 r_i}, \ldots, p^{\delta_s r_s}, p^i)}{[p^{\delta_1 r_i}; \ldots; p^{\delta_s r_s}; p^i]}$$

$$= p^{l-r_s} \sum_{n \leq p^{r_s}} \sum_{\delta_i \leq \delta_s \{0,1\}} (-1)^{\delta_1+\cdots+\delta_s} \prod_{i=1}^{\delta_s} \left[ n \equiv -\alpha_i \left( p^{\delta_i r_i} \right) \right]$$

The product has the value 0 (resp. 1) if $n \equiv -\alpha_i \left( p^{r_i} \right)$ for at least one $i$ (resp. for no $i$ at all).

Hence, we have for $1 \leq l \leq r_s$

$$\chi_a^{(p^i)}(p) = p^{l-r_s} \sum_{n \leq p^{r_s}} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} 1$$

(36)

and with equation (35) and definition (32) of $h$

$$H(p^i) = p^{3l-2r_s} \sum_{n,m \leq p^{r_s}} \sum_{b \leq p^l} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} 1 - p^{3l-2r_s-1} \sum_{b_1 \leq p^l} \sum_{b_2 \leq p^l} \sum_{b_1 \equiv n (p^l)} \sum_{b_2 \equiv m (p^l)} 1$$

$$= p^{3l-2r_s} \sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^l \right) \right] - p^{3l-2r_s-1} \sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^{l-1} \right) \right]$$

$$= p^{3l-2r_s} \sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^l \right) \right] - \frac{1}{p} \sum_{n \equiv m \left( p^{l-1} \right)} \left[ n \equiv m \left( p^{l-1} \right) \right]$$

(37)

After multiple usages of

$$\sum_{n \leq p^{r_s}} \left[ n \equiv m \left( p^u \right) \right] = p^{s-u} - \sum_{n \leq p^{r_s}} \sum_{\forall i: n \equiv -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^u \right) \right]$$

we have for $0 \leq u \leq r_s$:

$$\sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \neq -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^u \right) \right] = p^{2s-u} - 2p^{s-u} \sum_{n \leq p^{r_s}} 1 + \sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \equiv -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^u \right) \right]$$

If we specify $u = l$ and $u = (l-1)$ we get with equation (37):

$$H(p^i) = p^{3l-2r_s} \sum_{n,m \leq p^{r_s}} \sum_{\forall i: n \equiv -\alpha_i (p^{r_i})} \left[ n \equiv m \left( p^l \right) \right] - \frac{1}{p} \sum_{n \equiv m \left( p^{l-1} \right)} \left[ n \equiv m \left( p^{l-1} \right) \right]$$

(38)

The reader should compare the surprisingly similar representations of $H(p^i)$ in (37) and (38).
To statement 3. With the definition 4.8 of H we always have \( H(p^j) \geq 0 \) and with (33) we have
\[
H(p^j) \leq p^{3l-2s} \sum_{n,m \leq \alpha(p^j)} \sum_{n \equiv m \pmod{p^j}} \leq p^{3l-2s} \sum_{n,m \leq \alpha(p^j)} \sum_{n \equiv m \pmod{p^j}} \sum_{n \equiv m \pmod{p^j}}
\]
\[
\leq p^{3l} \sum_{v,w \leq s} p^{-\max(l,r_v-r_w)} \leq p^{3l-\max(l,r_1-r_1)} \sum_{v,w \leq s} 1 \leq s^2 p^{3l-\max(l,r_1-r_1)}
\]
which completes the proof. \( \square \)

**Corollary 4.10** For all \( \epsilon > 0 \) and \( U \in \mathbb{R}_{>0} \), we have
\[
\sum_{U < q \leq 2U} q^{-2} z(q)^2 H(q) \ll U^{-1+\epsilon}
\]

**Proof.** Using (33) we get \( z(q)^2 \ll 1 \). We start with
\[
\mathcal{J}_3(U) := \sum_{U < q \leq 2U} q^{-2} z(q)^2 H(q) \ll U^{1-1} \sum_{q \leq 2U} q^{-1/2} H(q)
\]
The lemma 4.9 shows that \( H(q) = 0 \) if \( q \) is not \( (r_s + 1) \)-free. It can be seen easily that every \( (r_s + 1) \)-free number \( q \) possesses a unique representation \( q = q_1 q_2^s \cdots q_r^s \) with pairwise co-prime and squarefree natural numbers \( q_i \). Using lemma 4.9 again, we get
\[
U^{1-1} \sum_{q \leq 2U} q^{-1/2} H(q) \ll U^{1-1+\epsilon} \sum_{q_1 q_2^s \cdots q_r^s \leq 2U} \prod_{l \leq r_1} q_l^{2l-\frac{l}{r} - 2r_1} \prod_{r_1 < l \leq r_s} q_l^{l-\frac{l}{r} - r_1}
\]
To simplify notations, we set \( \nu := r_s - 1 \) and \( \tau(l) := \frac{l}{r_s} \left( r_s - \frac{r_s}{r_1} - 1 + 1 \right) \), and get
\[
\mathcal{J}_3(U) \ll U^{1-1+\epsilon + \tau(1)} \sum_{q_1 q_2^s \cdots q_r^s \leq 2U} \prod_{l \leq r_1} q_l^{2l-\frac{l}{r} - 2r_1 - \tau(l)} \prod_{r_1 < l \leq \nu} q_l^{l-\frac{l}{r} - r_1 - \tau(l)}
\]
\[
\mathcal{J}_3(U) \ll U^{-\frac{r_s-1}{r_s} + \epsilon} \sum_{q_1 q_2^s \cdots q_r^s \leq 2U} \prod_{l \leq r_1} q_l^{l+\frac{1}{r} - 2r_1} \prod_{r_1 < l \leq \nu} q_l^{l-\frac{l}{r} - r_1}
\]
As the exponents can’t be larger than \( -1 \), the sums over \( q_1, \ldots, q_r \) are \( O(U^\epsilon) \). If \( r_1 < r_s \), they are even convergent, which completes the proof. \( \square \)

Now we are finally able to estimate the remainder of the singular series: The function \( f \) satisfies the requirement 15 with \( r := 2r_s \) as with 34 and corollary 4.10
\[
\sum_{q > w} \sum_{a \leq q}^a |G_f(q,a)|^2 = D^2 \sum_{q > w} q^{-2} z(q)^2 H(q) = D^2 \sum_{j=0}^\infty \mathcal{J}_3(2^j w)
\]
\[
\ll w^{-\frac{r_s-1}{r_s} + \epsilon} \sum_{j=0}^\infty \left( 2^{\frac{r_s-1}{r_s} + \epsilon} \right)^j = o \left( w^{-\frac{1}{2r_s}} \right)
\]
As all requirements of theorem 3.3 are fulfilled, we have
\[
\sum_{p \leq x} f(p) = c_f \frac{x}{\log x} + o \left( \frac{x}{\log x} \right)
\]
with the absolute convergent series

\[ c_f = \mathcal{D} \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \frac{z(q)}{q} \sum_{a \leq q} H(q, a) \]

### 4.3 Evaluation of the series \( c_f \)

The function \( q \rightarrow \sum_{a \leq q}^* H(q, a) \) is multiplicative. Let \( q_1, q_2 \in \mathbb{N}, (q_1; q_2) = 1 \) be given. Then

\[
\sum_{a \leq q_1q_2}^* H(q_1q_2, a) = \sum_{a \leq q_1}^* \sum_{b \leq q_2}^* h(q_1q_2, b) e\left(\frac{ab}{q_1q_2}\right)
\]

\[
= \sum_{a_1 \leq q_1}^* \sum_{a_2 \leq q_2}^* \sum_{b_1 \leq q_1} \sum_{b_2 \leq q_2} h(q_1, b_1q_2) h(q_2, b_2q_1) e\left(\frac{a_1}{q_1}\right) e\left(\frac{b_1}{q_2}\right) e\left(\frac{a_2}{q_2}\right) e\left(\frac{b_2}{q_1}\right)
\]

\[
= \left( \sum_{a \leq q_1}^* H(q_1, a) \right) \left( \sum_{a \leq q_2}^* H(q_2, a) \right)
\]

The other factors \( \frac{\mu(q)}{\varphi(q)} \) and \( \frac{z(q)}{q} \) in the representation of \( c_f \) are trivially multiplicative. We then can write the series as an Euler product:

\[
c_f = \mathcal{D} \prod_p \left( \sum_{k=0}^{\infty} \frac{\mu(p^k)}{\varphi(p^k)} \frac{z(p^k)}{p^k} \sum_{a \leq p^k}^* H(p^k, a) \right) = \mathcal{D} \prod_p \left( 1 - \frac{z(p)}{p(p - 1)} \sum_{a \leq p}^* H(p, a) \right)
\]

Using properties of Ramanujan’s sum, we have

\[
\sum_{a \leq p}^* H(p, a) = \sum_{b \leq p} h(p, b) \sum_{a \leq p}^* e\left(\frac{ab}{p}\right) = \sum_{b \leq p} h(p, b) c_p(b)
\]

\[
= \varphi(p) h(p, p) - \sum_{b < p} h(p, b)
\]

To evaluate \( h(p, p) \) and \( h(p, b) \) we can apply the identity of (36) with \( l = 1 \) and get

\[
h(p, p) = p^{1-r_s} \sum_{n \leq p^r_s} \frac{1}{1 - p^{1-r_s}} \sum_{n \leq p^r_s \equiv 0(p)} \sum_{n \equiv a_i(p^r_s) \forall i: n \equiv -a_i(p^r_s)} 1
\]

as well as

\[
\sum_{b < p} h(p, b) = p^{1-r_s} \sum_{b < p} \sum_{n \leq p^r_s} \frac{1}{1 - p^{1-r_s}} \left( p - 1 \right) - p^{1-r_s} \sum_{b < p} \sum_{n \leq p^r_s \equiv 0(b)} \sum_{n \equiv -a_i(b^r_s) \forall i: n \equiv -a_i(b^r_s)} 1
\]

\[
= (p - 1) - p^{1-r_s} \sum_{b < p} \sum_{n \leq p^r_s \equiv 0(b)} \sum_{n \equiv -a_i(b^r_s)} (1 - |p|n)
\]

\[
= (p - 1) - p^{1-r_s} \sum_{n \leq p^r_s} (1 - |p|n)
\]
Using these results in (39) and noting that \( \varphi(p) = (p - 1) \), then

\[
\sum_{a \leq p} s H(p, a) = p^{1-r_s} \left( D(p) - p \sum_{n \leq p^{r_s}} [p|n] \right)
\]

and

\[
c_f = \mathcal{D} \prod_p \left( 1 - \frac{z(p)}{p^{r_s} (p-1)} \left( D(p) - p \sum_{n \leq p^{r_s}} [p|n] \right) \right).
\]

Looking again on the definitions (28) and (32) of \( \mathcal{D} \) and \( z(p) \), we get with those and \( \varphi(p^{r_s}) = p^{r_s-1}(p - 1) \)

\[
 c_f = \prod_p \left( 1 - \frac{D(p)}{p^{r_s}} - \frac{1}{p^{r_s} (p-1)} \left( D(p) - p \sum_{n \leq p^{r_s}} [p|n] \right) \right)
\]

\[
= \prod_p \left( 1 - \frac{1}{\varphi(p^{r_s})} \left( D(p) - \sum_{n \leq p^{r_s}} [p|n] \right) \right)
\]

\[
= \prod_p \left( 1 - \frac{D^*(p)}{\varphi(p^{r_s})} \right)
\]

The product is non-zero if and only if for each prime \( p \) there exists a relatively prime natural number \( n \leq p^{r_s} \) with \( n \neq -\alpha_i (p^{r_s}) \) for all \( 1 \leq i \leq s \), see theorem 4.2. In this case we have \( D^*(p) < \varphi(p^{r_s}) \) and the convergence of the product is implied by \( D^*(p) \ll p^{r_s-2} \), see estimate (29).

Hence, we have proven the identity (1).

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