Polarization of Quantum Hall States, Skyrmions and Berry Phase

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We have discussed here the polarization of quantum Hall states in the framework of the hierarchical analysis of IQHE and FQHE in terms of Berry phase. It is observed that we have fully polarized states for the filling factor $\nu = 1$ as well as $\nu = \frac{1}{2m+1}$, $m$ being an integer. However, for $\nu = p$ as well as $\nu = \frac{p}{q}$, with $p > 1$ and odd, $q$ odd we have partially polarized states and for $\nu = \frac{p}{q}$, $p$ even,

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I. INTRODUCTION

The quantum Hall effect (integral and fractional) appears in two-dimensional electron systems in a strong magnetic field. In such a high magnetic field ($B \sim 10$ T) the spin of the electron has no dynamical role and we can assume the electrons to be spinless in these fully polarized quantum Hall states. But when $B$ has relatively smaller value i.e. when the Zeeman splitting is not so large, the corresponding system is not fully polarized. They are sometimes partially polarized and sometimes unpolarized. It is known from experiments that the QH states at filling factors $\nu = 4/3, 8/5, ...$ [1, 2] and at $2/3$ [3, 4] are unpolarized while the states at $\nu = 3/5$ [4] and $7/5$ [1] are partially polarized. From numerical computations it is known [5] that the states with $\nu = \frac{2}{2n+1}$ are unpolarized and with $\nu = \frac{1}{2n+1}$ are fully polarized in the vanishing Zeeman splitting limit.

Wu, Dev and Jain [6] have studied this problem and reported that all even numerator QH states are unpolarized and all these states with the numerator and denominator odd are partially polarized or fully polarized (in the vanishing Zeeman splitting limit). Later Mandal and Ravishankar [7] proposed a global model which accounts for all the observed quantum Hall states in terms of an abelian doublet of Chern-Simons gauge fields, with the strength of the Chern-Simons term given by a coupling matrix. Hansonn, Karlhede and Leinaas [8] proposed a new effective field theory for partially polarized quantum Hall states. They determined the density and polarization for the mean field ground states by couplings to two Chern-Simons gauge fields. They derived a sigma model covariantly coupled to the Chern-Simons field and found mean field solutions which describe partially polarized states.

It is also observed that the low energy excitations in various polarization states are quite different. It is now confirmed that the low energy excitation states for a fully polarized quantum Hall state is a skyrmion whereas for unpolarized states skyrmion excitations are not possible. For partially polarized states also, skyrmion excitations do not seem to occur as the skyrmionic solitons and are found not to be of usual ones. Wu and Sondhi [9] have shown that in higher Landau levels, skyrmions are not the low energy excitations even at small Zeeman energies. Thus it follows that skyrmions are the quasiparticles only at $\nu = 1, 1/3$ and $1/5$.

In some earlier papers [10, 11], we have analyzed the hierarchy of quantum Hall states from the viewpoint of chiral anomaly and Berry phase. It has been pointed out that this approach embraces in a unified way the whole spectrum of quantum Hall systems with their various characteristic features. Here we propose to study the various polarization states of quantum Hall skyrmions on the basis of this hierarchical model. Also, we shall study the low energy excitation states for various polarization states of quantum Hall fluid.

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II. POLARIZATION OF INTEGER AND FRACTIONAL QUANTUM HALL STATES

In some earlier papers, Basu and Bandyopadhyay have studied the whole spectrum of quantum Hall states (IQH) and (FQH) in a unified way from the chiral anomaly and Berry phase approach. It has been shown that the hierarchy of FQH states with filling factor \( \nu = p/q \) (\( p \) even or odd and \( q \) odd) is interpreted in terms of the fact that the Berry phase associated with even number of flux quanta can be removed to the dynamical phase. For \( p \) odd and \( p > 1 \), the corresponding state attains a higher Landau level whereas for \( p \) even, the system corresponds to particle-hole conjugate states.

In a spherical geometry we consider quantum Hall states in the two dimensional surface of a 3D sphere with a magnetic monopole of strength \( \mu \) at the centre. The angular momentum relation is given by

\[
J = r \times p - \mu \hat{r}, \quad \mu = 0, \pm 1/2, \pm 1, \pm 3/2 \ldots .
\] (1)

The spherical harmonics incorporating the term \( \mu \) have been extensively studied by Fierz \[12\] and Hurst \[13\]. Following them we write

\[
Y^{m,\mu}_\ell = (1 + x)^{-\nu/2}(1 - x)^{\nu/2} \left( \frac{\partial^{\ell-m}}{\partial z^{\ell-m}} \right) \left[ (1 + x)^{\ell-\mu}(1 - x)^{\ell+\mu} \right] e^{i\nu\phi} e^{-i\mu\chi}
\] (2)

where \( x = \cos \theta \) and the quantities \( m \) and \( \mu \) just represent the eigenvalues of the operators \( i\frac{\partial}{\partial \phi} \) and \( i\frac{\partial}{\partial \chi} \) respectively.

From the description of spherical harmonics we can construct a two-component spinor \( \theta = \begin{pmatrix} u \\ v \end{pmatrix} \) where

\[
u = Y^{1/2,1/2}_1 = \sin \frac{\theta}{2} \exp \left[ i(\phi - \chi) / 2 \right]
\] (3)

\[
u = Y^{-1/2,1/2}_1 = \cos \frac{\theta}{2} \exp \left[ -i(\phi + \chi) / 2 \right]
\] (3)

Then the \( N \)-particle wave function for the quantum Hall fluid can be written as

\[
\psi^{(m)}_N = \prod_{i<j} (u_i v_j - u_j v_i)^m
\] (4)

where \( \nu = \frac{1}{m} \), \( m \) being an odd integer.

It is noted that \( \psi^{(m)}_N \) is totally antisymmetric for odd \( m \) and symmetric for even \( m \). Following Haldane \[14\], we can identify \( m \) as \( m = J_i + J_j \) for an \( N \)-particle system where \( J_i \) is the angular momentum of the \( i \)-th particle. It is evident from eqn. (1) that with \( r \times p = 0 \) and \( \mu = \frac{1}{2} \) we have \( J_i(J_j) = \frac{1}{2} \) which gives us \( m = 1 \), the complete filling of the lowest Landau level. From the Dirac quantization condition \( e\mu = \frac{1}{2} \), we note that this state corresponds to \( e = 1 \) describing the IQH state with \( \nu = 1 \).

The next higher angular momentum state can be achieved either by taking \( r \times p = 1 \) and \( | \mu | = \frac{3}{2} \) (which implies the higher Landau level) or by taking \( r \times p = 0 \) and \( | \mu_{eff} | = \frac{3}{2} \) implying the ground state for the Landau level. However, with \( | \mu_{eff} | = \frac{3}{2} \), we find the filling fraction \( \nu = \frac{1}{4} \) which follows from the condition \( e\mu = \frac{1}{2} \) for \( \mu = \frac{3}{2} \). Generalizing this we can have \( \nu = \frac{1}{4} \) with \( | \mu_{eff} | = \frac{5}{2} \).

It is noted that when \( \mu \) is an integer, we can have a relation of the form

\[
J = r \times p - \mu \hat{r} = r' \times p'
\] (5)

This indicates that the Berry phase which is associated with \( \mu \) may be unitarily removed to the dynamical phase. Evidently, the average magnetic field may be considered to be vanishing in these states. The attachment of \( 2m \) vortices to an electron effectively leads to the removal of Berry phase to the dynamical phase. So, FQH states with \( 2\mu_{eff} + 2m + 1 \) (\( m \) an integer) can be viewed as if one vortex line is attached to the electron. Now we note that for a higher Landau level we can consider the Dirac quantization condition \( e\mu_{eff} = \frac{1}{2}n \), with \( n \) being a vortex of strength \( 2\ell + 1 \). This can generate FQH states having
the filling factor of the form \( \frac{n}{2\mu_{eff}} \) where both \( n \) and \( 2\mu_{eff} \) are odd integers. In this case, we can write \( 2\mu_{eff} = 2m' \pm 1 \) with \( 2m' = 2mn \). Indeed, we can write the filling factor as

\[
\nu = \frac{n}{2\mu_{eff}} = \frac{1}{2\mu_{eff} + \frac{1}{n}} = \frac{n}{2mn \pm 1}
\]

(6)

where \( 2\mu_{eff} \pm 1 \) is an even integer given by \( 2m' = 2mn \). This is essentially the Jain classification scheme with the constraint of \( n \) being an odd integer.

In this scheme, the FQH states with \( \nu \) having the form

\[
\nu = \frac{n'}{2mn' \pm 1}
\]

(7)

with \( n' \) an even integer can be generated through particle-hole conjugate states

\[
\nu = 1 - \frac{n}{2mn \pm 1} = \frac{n(2m - 1) \pm 1}{2mn \pm 1} = \frac{n'}{2mn' \pm 1}
\]

(8)

where \( n(n') \) is an odd(even) integer. This leads some observed FQH states with

\[
\nu = \frac{p}{q}(p \text{ even, } q \text{ odd}) : \frac{2}{3}, \frac{2}{5}, \frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{6}{11}, \frac{6}{13}, \ldots
\]

(9)

Besides, for \( \nu = \frac{n}{q} \) with \( p > q \) we can think of these states as condensates of one IQH state with \( \nu \) an integer and one FQH state with \( \nu = \frac{n}{q} \) \( (p < q) \). In this category, we have even numerator states such as \( \nu = \frac{4}{3}, \frac{6}{5}, \text{ and } \frac{10}{7} \). [1][2]

Now to study the polarization states of various FQH systems, we observe that in the lowest Landau level, we have the filling factor \( \nu \) given by \( \nu = \frac{1}{\mu_{eff}} = \frac{1}{2m+1} \). As we have pointed out that the attachment of \( 2m \) vortices to an electron leads to the removal of Berry phase to the dynamical phase and this effectively corresponds to the attachment of one vortex(magnetic flux) to an electron, this electron will be a polarized one.

This is also true for \( \nu = 1 \) state, as in this case we have \( \mu = \frac{1}{2} \) implying one vortex line (magnetic flux) is attached to an electron.

However, in the higher Landau level, this scenario will change. Indeed, in this case we have \( \nu = \frac{n}{2mn+1} \) with \( n > 1 \) and an odd integer.

Now from the relation (6), for \( n > 1 \) and an odd integer

\[
\nu = \frac{n}{2\mu_{eff}} = \frac{1}{2\mu_{eff} + \frac{1}{n}} = \frac{1}{2mn \pm 1}
\]

(10)

where \( 2\mu_{eff} \pm 1 \) is an even integer. We note that as even number of flux units can be accommodated in the dynamical phase we may consider this as \( \frac{1}{n} \) flux unit is attached to an electron. This suggests that electrons will not be fully polarized as one full flux unit is not available to it. This will correspond to partially polarized states with \( \nu = p/q \) \( (p > 1 \text{ and odd, } q \text{ odd}) \) which represents the states like \( \frac{3}{5}, \frac{3}{7}, \frac{5}{9} \) and so on. It is observed here that IQH states like \( \nu = 3, 5 \) will also exhibit partially polarized states as from the Dirac quantization condition \( e\mu = \frac{2}{3} \), for \( n > 1 \) and odd, the filling factor \( \nu = n \) is achieved with \( \mu = 1/2 \). This suggest that one flux unit is shared by an electron in \( n \) number of Landau levels so that each electron is attached with \( \frac{1}{n} \) flux unit implying a partially polarized state.

Now we consider the states with the filling factor \( \nu = p/q \) with \( p \text{ even and } q \text{ odd} \). As we have pointed out earlier, this is achieved when we have particle-hole conjugate states given by \( \nu = 1 - \frac{n}{2mn+1} \) with \( n \) an odd integer. A hole configuration is described by the complex conjugate of the particle state, the spin orientation of the particle and hole state will be opposite to each other. Thus this will represent an unpolarized state. This gives us the general relation of unpolarized FQH states with filling factor \( \nu = p/q, p \text{ even and } q \text{ odd} \).
III. SKYRMION EXCITATIONS IN ARBITRARY POLARIZED QUANTUM HALL STATES

It has been found that in quantum Hall systems deviations from the incompressible filling factor $\nu$ is accomplished by the degradation of the system’s spin polarization. This effect has been observed near $\nu = 1$ in several experiments that directly probe the spin density of the electron gas [13, 16, 17, 18]. By noticing that the dynamics of quantum Hall system with a spin polarized ground state will follow that of a quantum ferromagnet and that the skyrmion is a charged object of the system, Sondhi et al. [19, 20] proposed a phenomenological action which is valid for the long wave length and small frequency limit. In this scheme, the competition between the Zeeman and Coulomb terms sets the size and energy of the skyrmions. Recently, Basu, Dhar and Bandyopadhyay [21, 22] have proposed a pure sigma model formalism for skyrmions taking resort to spherical geometry where 2D electron gas resides on the surface of a 3D sphere with a magnetic monopole placed at the centre. To study the system in pure sigma model formalism an $O(4)$ nonlinear sigma model in 3 + 1 dimensional manifold was introduced. In this framework, the quartic stability term introduced by Skyrme, known as the Skyrme term, determines the size of the quantum Hall skyrmions and the effect of the Zeeman energy and Coulomb energy is encoded in a certain parameter such that the size and energy are determined from the sigma model Lagrangian with the Skyrme term. Besides, the introduction of the topological $\theta$-term in the Lagrangian helps us to determine the spin and statistics of the skyrmion. The four order parameter fields help us to construct two independent $SU(2)$ algebras which are associated with the two mutually opposite orientations of the magnetization vector which resides on the 2D surface of the sphere.

The generalized Lagrangian is taken to be of the form [21]

$$L = 2\pi J^S_\mu A_\mu - \frac{M^2}{16} Tr(\partial_\mu U^\dagger \partial_\nu U) - \frac{1}{32\eta^2} Tr[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2$$

(11)

Here the topological current is defined as

$$J^S_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} Tr(U^{-1} \partial_\nu U)(U^{-1} \partial_\alpha U)(U^{-1} \partial_\beta U)$$

(12)

where $A_\mu$ is a four-vector gauge field, $\theta = g/c^2$ with $g = \nu e^2/h$, the Hall conductivity and $*F_{\mu\nu}$ is a Hodge dual given by

$$*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}$$

(13)

$M$ is a constant of dimension of mass and $\eta$ is a dimensionless coupling parameter.

The $SU(2)$ matrix $U$ is here defined as

$$U = n_0 I + n_i \tau_i$$

(14)

where the chiral fields $n_0, n_1, n_2$ and $n_3$ satisfy the relation $\sum n_i^2 = 1$. It may be noted that in 2 + 1 dimension, we have the normalized 3-vector field $n$ with $\sum n_i^2 = 1$ where $n_i$ corresponds to the local spin direction. In the $O(4)$ model $n_i (i = 1, 2, 3)$ corresponds to this spin direction which live on the 2-dimensional surface of the sphere where the extra field $n_0$ helps us to consider three boost generators in $(n_0, n_i)$ planes. In view of this, we can consider two types of generators such that the generator $M_i$ rotates the 3-vector $n(x)$ to any chosen axis and the boost generators $N_k$ would mix $n_0$ with the components of $n$. We can now construct the following algebra

$$[M_i, M_j] = i\epsilon_{ijk} M_k$$

$$[M_i, N_j] = i\epsilon_{ijk} N_k$$

$$[N_i, N_j] = i\epsilon_{ijk} M_k$$

(15)
which is locally isomorphic to the Lie algebra of the $O(4)$ group. This helps us to introduce the left and right generators

$$L_i = \frac{1}{2}(M_i - N_i)$$

$$R_i = \frac{1}{2}(M_i + N_i)$$

which satisfy

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

$$[R_i, R_j] = i\epsilon_{ijk}R_k$$

$$[L_i, R_j] = 0$$

Thus the algebra has split into two independent subalgebras each isomorphic to a $SU(2)$ algebra and corresponds to the chiral group $SU(2)_L \otimes SU(2)_R$. The left and right chiral group can now be taken to be associated with two mutually opposite orientations of the magnetization vector which resides on the 2D surface of the sphere.

The topological charge of quantum Hall skyrmions is given by the winding number

$$Z = \frac{1}{24\pi^2} \int_{S^3} dS_\mu \epsilon^{\mu\nu\lambda\sigma}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\sigma U)]$$

which is related to the homotopy $\pi_3(S^3) = Z$ and the electric charge is given by $\nu eZ$. We observe here that the Pontryagin index given by

$$q = 2\mu = -\frac{1}{16\pi^2} \int Tr^*F_{\mu\nu}F_{\mu\nu}d^4x$$

introduces the Berry phase for quantum Hall states as $\mu$ represents magnetic monopole strength. When there is a monopole of strength $\mu$ at the centre, the flux through the sphere is $2\mu$ and the phase is given by $e^{2\mu}$

Now to study the relevant quasiparticles for arbitrarily polarized states we observe the following possibilities.

1) Fully Polarized States:
   In this case the spin orientation is fixed: up($\uparrow$) or down($\downarrow$). So for the chiral group $SU(2)_L \times SU(2)_R$ only one $SU(2)_L(SU(2)_R)$ group will be operative. From the homotopic relation $\pi_3(SU(2)) = Z$, we will have solitons (skyrmions) with charge $\nu eZ$.

2) Partially Polarized States:
   In this case it will not be possible to split the $O(4)$ group into two independent left and right chiral groups $SU(2)_L$ or $SU(2)_R$ as each of these represent sharp polarization states. Thus we cannot represent the relevant quasiparticles as skyrmions.

3) Unpolarized States:
   In this case, we will have equal number of up and down spin electrons. Now from the homotopic relation $\pi_3(SU(2) \times SU(2)) = \pi_3(SO(4)) = 0$, we note that there will be no skyrmions.

Thus we find that skyrmions are the relevant quasiparticles only at the filling factor given by $\nu = 1$ and $\nu = \frac{1}{2m+1}$ with $m = $ an integer.
IV. DISCUSSION

We have studied here the polarization states of quantum hall fluid at different filling factors from the hierarchical scheme in the framework of Berry phase. It is found that for IQH and FQH states with \( \nu = 1 \) and \( \nu = \frac{1}{2m+1} \), \( m \) being an integer represent fully polarized states. The states with \( \nu = p/q \), \( p \) odd, \( q \) odd with \( p > 1 \) represent partially polarized states and with \( \nu = p/q \), \( p \) even, \( q \) odd with \( p > 1 \) represent unpolarized states. It may be noted that the same result for unpolarized states has also been found from numerical computations [5] and an exact diagonalization study has revealed that at \( \nu = 3/5 \), the system is partially polarized. These are also in agreement with experiments. Mandal and Ravishankar [7] have studied the polarization states in terms of an Abelian doublet of Chern-Simons gauge fields where the strength of the Chern-Simons term is given by a coupling matrix in the framework of composite fermion model. Their findings are also consistent with these results.

Wu and Sondhi [9] have calculated the energies of quasiparticles for odd integer filling factors \( \nu = 2k+1 \), \( k \geq 1 \) and have observed that skyrmions are not the low energy charged excitations even at small Zeeman energies. Mandal and Ravishankar [23] have studied the quasiparticles in partially polarized states and have observed that skyrmions in this case are not the usual one whereas for unpolarized states skyrmions do not exist at all. Here from the homotopic analysis we have found that skyrmions are the relevant quasiparticles only for fully polarized states whereas for partially polarized and unpolarized states skyrmionic excitations do not exist at all.

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