Generalization of the $h$-Deformation to Higher Dimensions\footnote{To be appear in J. Phys. A}

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Abstract

In this article we construct $GL_h(3)$ from $GL_q(3)$ by a singular map. We show that there exist two singular maps which map $GL_q(3)$ to new quantum groups. We also construct their $R$-matrices and will show although the maps are singular but their $R$-matrices are not. Then we generalize these singular maps to the case $GL(N)$ and for $C_n$ series.
There exist two types of $SL(2)$ quantum groups. One is the standard $SL_q(2)$, another one is the Jordanian quantum group which is also called the $h$-deformation of $SL(2)$. Quantum matrices in two dimensions, admitting left and right quantum spaces, are classified [1]. One is the $q$-deformation of $GL(2)$, the other is the $h$-deformation. The $q$-deformation of $GL(N)$ has been studied extensively but in the literature only the two dimensional case of $h$-deformation has been studied.[2-7]

In ref. 8 it is shown that $GL_h(2)$ can be obtained from $GL_q(2)$ by a singular limit of a similarity transformation. We will show this method can be used successfully, for construction of $GL_h(N)$. In other words, at first we will consider the $GL(3)$, and introduce two singular maps which convert $GL_q(3)$ to $GL_h(3)$. Then we generalize one of the singular maps to $N$-dimensional case. We will use $R$-matrix of $GL_q(N)$ which by this map, results to a new $R$-matrix. Also, by this map one can obtain $h$-deformation of $C_n$ series, but can not for $B_n$ and $D_n$ series.

In this article we denote $q$-deformed objects by primed quantities. Unprimed quantities represent transformed objects.

Consider Manin’s $q$-plane with the following quadratic relation between coordinates.

$$x'y' = qy'x'. \quad (1)$$

By the following linear transformation:

$$
\begin{pmatrix}
    x' \\
    y'
\end{pmatrix}
= 
\begin{pmatrix}
    1 & \frac{h}{q^2-1} \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\quad (2)
$$

the relation (1) changes to $xy - qyx = hy^2$. For the case of $q = 1$, one get the relation of two dimensional $h$-plane. In fact $g$ itself is singular in the $q = 1$ case, but the resulting relation for the plane is non-singular.

The above linear transformation on the plane induces the following similarity transformation on the $R$-matrix of $GL_q(2)$.

$$R_h = \lim_{q \to 1} (g \otimes g)^{-1}R_q(g \otimes g). \quad (3)$$

Although the above map is singular, the resulting $R$-matrix is non-singular and is the well known $R$-matrix of $GL_h(2)$.
Now consider 3-dimensional Manin’s quantum space:

\[ x'_i x'_j = q x'_j x'_i \quad i < j, \]  

and consider the following linear transformation:

\[ X = g^{-1} X', \]  

where

\[
g = \begin{pmatrix}
\lambda_1 & \alpha & \beta \\
0 & \lambda_2 & \gamma \\
0 & 0 & \lambda_3 
\end{pmatrix}.
\]  

Here \( \alpha, \beta \) and \( \gamma \) are parameters which can be singular at \( q = 1 \). So they can be written as \( \frac{1}{f(q)} \) where \( f(1) = 0 \). The Taylor expansion of \( f(q) \) about \( q = 1 \) is \( f(q) = \frac{1}{h}(q - 1) + O((q - 1)^2) \). We need only the first term, because we are only interested in the behaviour of \( f(q) \) in the neighbourhood of \( q = 1 \). The coefficient of first term in the Taylor expansion, \( h \), plays the role of the deformation parameter for the new quantum group. The \( \lambda_i \)s can be made equal to 1 by rescaling.

To obtain \( \alpha, \beta \) and \( \gamma \) we should apply this map to the \( q \)-deformed plane and its dual, and require that the mapped plane and its dual be non-singular at \( q = 1 \). The following are the only singular maps satisfying this condition:

\[
g_1 = \begin{pmatrix}
1 & \frac{h}{q-1} & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
1 & \alpha & \beta \\
0 & 1 & \frac{h}{q-1} \\
0 & 0 & 1
\end{pmatrix}, \quad g_3 = \begin{pmatrix}
1 & \alpha & \frac{h}{q-1} \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{pmatrix}.
\]  

Here \( \alpha, \beta \) and \( \gamma \) (in \( g_1, g_2, g_3, \)) are non-singular parameters. Note that the \( R \)-matrices obtained from these maps, solve the quantum Yang-Baxter equation and are non-singular for \( q = 1 \).

Let us denote the dependence of \( g_1, g_2 \) and \( g_3 \) on parameters explicitly:

\[
g_1 := g_1\left(\frac{h}{q-1}, \beta\right), \quad g_2 := g_2\left(\frac{h}{q-1}, \alpha, \beta\right), \quad g_3 := g_3\left(\frac{h}{q-1}, \alpha, \gamma\right).
\]  

It is easy to show that:

\[
g_1\left(\frac{h}{q-1}, \beta\right)g_1\left(0, -\beta\right) = g_1\left(\frac{h}{q-1}, 0\right)
\]
so all non-singular parameters in the above matrices can be set to zero. Moreover the 
$R$-matrices $R(g_1)$ and $R(g_2)$ which are obtained by formula (3) using $g_1\left(\frac{h}{q-1},0\right)$ and 
g_2\left(\frac{h}{q-1},0,0\right)$ respectively, are equivalent, because:

$$\left(s \otimes s\right)^{-1} R(g_2)(s \otimes s) = R(g_1).$$

where $s = e_{13} + e_{21} + e_{32}$. So, there are only two independent cases. The $R$-matrices 
corresponding to these transformations are non-singular and have been first obtained by 
Hietarinta [9]. The first case ( the trivial case) is $\beta = 0$ in $g_1$ (or $\alpha = \beta = 0$ in $g_2$) and 
the second case is $\alpha = \gamma = 0$ in $g_3$. The $h$-deformed quantum plane and its dual and 
$R$-matrices corresponding to these cases are:

**First case**

$$[x_1, x_2] = hx_2^2, \quad \eta_3^2 = \eta_2^2 = \{\eta_1, \eta_2\} = 0,$$

$$[x_1, x_3] = 0, \quad \{\eta_2, \eta_3\} = \{\eta_1, \eta_3\} = 0,$$

$$[x_2, x_3] = 0, \quad \eta_1^2 = -h \eta_2 \eta_1. \quad (11)$$

and the non-zero elements of $R$-matrix except for $R_{ijij} = 1$ are:

$$R_{1121} = R_{2122} = -R_{1112} = -R_{1222} = h,$$

$$R_{1122} = h^2. \quad (12)$$

**Second case**

$$[x_1, x_2] = 2hx_3x_2, \quad \{\eta_1, \eta_2\} = -2h \eta_3 \eta_2,$$

$$[x_1, x_3] = hx_3^2, \quad \eta_1^2 = -h \eta_3 \eta_1,$$

$$[x_2, x_3] = 0, \quad \eta_3^2 = \eta_2^2 = \{\eta_1, \eta_3\} = \{\eta_2, \eta_3\} = 0. \quad (13)$$

and the non-zero elements of $R$-matrix except for $R_{ijij} = 1$ are:

$$R_{1113} = R_{1333} = -h, \quad R_{1131} = R_{3133} = h,$$

$$R_{2132} = -R_{1223} = 2h \quad R_{1133} = h^2. \quad (14)$$
A linear transformation on the plane induces a similarity transformation on the quantum matrices acting upon it.

\[ M' = gMg^{-1}, \] 

(15)

The algebra of functions, \( GL_q(3) \), is obtained from the following relations:

\[ R'M'_1M'_2 = M'_2M'_1R'. \] 

(16)

Applying transformation (15) one easily obtains for the case of \( q = 1 \).

\[ RM_1M_2 = M_2M_1R. \] 

(17)

So the entries of the transformed quantum matrix \( M \) fulfill the commutation relations of the \( GL_h(3) \), for both \( g \)'s. It is easy to show that the \( h \)-deformed determinant is central, so it can be set to 1. A quantum group’s differential structure is completely determined by \( R \)-matrix [10]. One therefore expects that by these similarity transformations the differential structure of the \( h \)-deformation be obtained from that of the \( q \)-deformation.

\[
\begin{align*}
M_2dM_1 - R_{12}dM_1M_2R_{21} &= 0, \\
dM_2dM_1 + R_{12}dM_1dM_2R_{21} &= 0.
\end{align*}
\] 

(18)

Now, it is obvious that, defining \( dM := g^{-1}dMg \) and using the above relations the differential of \( GL_h(3) \) can be easily obtained from the corresponding differential structure of \( GL_q(3) \).

For the higher dimensions, there are several generalizations which depend on the position of singularity in \( g \). For example we consider the following generalization:

\[ g = \sum_{i=1}^{N} e_{ii} + \frac{h}{q-1} e_{1N} \] 

(19)

The general aspect of the contraction for arbitrary \( N \) can be obtained from this simple map. By inserting this map in (3) we will obtain the general form of the \( h \)-deformed \( R \)-matrix, which solves the quantum Yang-Baxter equation.

1- The series \( A_{n-1} \)
After applying this singular map, the corresponding $h$-deformed $R$-matrix will become:

\[
R_h = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2h \sum_{i>1} (e_{ii} \otimes e_{iN} - e_{iN} \otimes e_{ii})
- h(e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N}) - h(e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11})
+ h^2(e_{1N} \otimes e_{1N}).
\] (20)

Consider $N$-dimensional $q$-deformed quantum space

\[
x_i'x_j' = qx_j'x_i', \quad i < j.
\] (21)

Assume the following linear singular transformation

\[
x_i' = g_{ij}x_j.
\] (22)

By the above transformation and in the $q = 1$ case we obtain the $h$-deformed quantum plane as follows:

\[
x_ix_j = x_jx_i, \quad 1 < i < j \leq N,
\]

\[
[x_1, x_j] = 2hx_Nx_j, \quad [x_1, x_N] = h(x_N)^2.
\] (23)

2- The series $B_n, C_n$ and $D_n$

The corresponding $q$-deformed $R$-matrix has order $N^2 \times N^2$, where $N = 2n + 1$ for $B_n$ and $N = 2n$ for $D_n$ and $C_n$ and it is given by [11]:

\[
R_q = \sum_{i \neq i'}^N e_{ii} \otimes e_{ii} + e_{N+1,i+1} \otimes e_{N+1,i+1} + \sum_{i \neq j,j'} e_{ii} \otimes e_{jj}
+ q^{-1} \sum_{i \neq i'} e_{i'i'} \otimes e_{ii} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}
- (q - q^{-1}) \sum_{i>j} q^{\rho_i - \rho_j} e_{ij} e_{ij} \otimes e_{i'i'}.
\] (24)

The second term is present only for the series $B_n$. Here $i' = N+1-i, j' = N+1-j, \epsilon_i = 1, i = 1, ..., N$ for the series $B_n$ and $D_n$, $\epsilon_i = 1, i = 1, ..., N, \frac{N}{2}$, $\epsilon_i = -1, i = \frac{N}{2} + 1, ..., N$ for the series $C_n$ and $(\rho_1, ..., \rho_N)$ is:

\[
(n-n+1, \frac{1}{2}, 0, \frac{1}{2}, ..., -n + \frac{1}{2}) \quad \text{for } B_n
\]

\[
(n, n-1, ..., 1, -1, ..., -n) \quad \text{for } C_n
\]

\[
(n-1, ..., 1, 0, 0, -1, ..., -n+1) \quad \text{for } D_n
\] (25)
By inserting this $R$-matrix in (3), the coefficient of $e_{1N} \otimes e_{1N}$ will become:

$$\frac{h^2}{q-1}(q^{-1} + 1)(1 + \epsilon_N q^{2N-\rho_1}).$$  \hspace{1cm} (26)

This expression is non-singular only when $\epsilon_N = -1$ and for $q = 1$ it is equal to $2Nh^2$. We thus see that only the $C_n$ series remains non-singular. The corresponding $h$-deformed $R$-matrix is:

\[
R_h = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2Nh^2 e_{1N} \otimes e_{1N}
- \frac{2h}{N} \sum_{i=2}^{N-1} e_{1i} \otimes e_{iN} + \epsilon_i e_{iN} \otimes e_{i'N}
+ \frac{2h}{N} \sum_{i=1}^{N-1} e_{iN} \otimes e_{1i} - \epsilon_i e_{i1} \otimes e_{1i'}.
\]  \hspace{1cm} (27)

So by this method we can obtain $SP_h(2n)$. The algebra $SP_q^{2n}(c)$ with generators $x'_1, \ldots, x'_N$ and relations

\[
R'_q(x' \otimes x') = qx' \otimes x',
\]  \hspace{1cm} (28)

is called the algebra of functions on quantum $2n$-dimensional symplectic space. After applying the singular transformation (19) to (28) we obtain the relations between the generators of $SP_h^{2n}(c)$:

\[
x_i x_j = x_j x_i, \quad 1 < i < j \leq N, \quad j \neq j',
\]  \hspace{1cm} (29)

\[
x_1 x_j = x_j x_1 + 2hx_N x_j, \quad j \neq N,
x_{i'} x_i = x_{i'} x_i + 2h\epsilon_{i'} x_N^2, \quad 1 < i < i' \leq N.
\]  \hspace{1cm} (30)

In $SP_q^{2n}(c)$ the equality $x''C'x' = 0$ holds. By applying the singular map (29), $C'$ transforms to $C = g^t C' g$, where $C$ is given by:

\[
C = \sum_{i=1}^{N} \epsilon_i e_{ii} - Nh\epsilon_{NN}.
\]  \hspace{1cm} (31)

The Quantum group $SP_q(2n)$ acts on $SP_q^{2n}(c)$ and preserves $x''C''x' = 0$, so we have:

\[
M''C''M' = C',
\]  \hspace{1cm} (32)

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on the other hand:

\[ M = g M' g^{-1}, \quad M' = (g^{-1})^t M^t g^t. \tag{33} \]

It follows that:

\[ M^t C M = C, \tag{34} \]

So we conclude that the quantum group \( SP_h(2n) \) acts on \( SP_h^{2n}(c) \) and preserves \( x^t C x = 0 \). It is interesting to note that the expression \( x'^t C' x' \), which should be equal to 1 for \( SO(2n) \) and \( SO(2n+1) \) (\( B_n \) and \( D_n \) series), is singular. So we cannot obtain the \( h \)-deformation of \( B_n \) and \( D_n \) series by contraction of the \( q \)-deformation, at least by this form (upper triangular matrix) of singular transformation (\( g \)).

One of the interesting problems is to construct \( U_h(gl(3)) \), and its generalization to higher dimensions.

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