In a number of previous studies, we have investigated the use of the volume element of the Bures (minimal monotone) metric — identically, one-fourth of the statistical distinguishability (SD) metric — as a natural measure over the $n^2 - 1$-dimensional convex set of $n \times n$ density matrices. This has led us for the cases $n = 4$ and 6 to estimates of the prior (Bures/SD) probabilities that qubit-qubit and qubit-qudit pairs are separable. Here, we extend this work from such bipartite systems to the tripartite “laboratory” quantum systems possessing $U \otimes U \otimes U$ symmetry recently constructed by Eggeling and Werner (Phys. Rev. A 63 [2001], 042324). We derive the associated SD metric tensors for the three-qubit and three-qutrit cases, and then obtain estimates of the various related Bures/SD probabilities using Monte Carlo methods.

PACS Numbers 03.65.Ta, 03.65.Ca, 89.70.+c, 02.40.Ky

Key words: Qubits, qudits, density matrix, Bures metric, statistical distinguishability metric, entanglement, Eggeling-Werner tripartite state, Monte Carlo simulation

I. INTRODUCTION

In a number of previous studies [1–4] — developing upon an initial idea of Życzkowski, Horodecki, Sanpera and Lewenstein [5] — we have investigated the use of the volume element of the Bures metric [6–8] as a natural measure on the quantum states. In particular, we have been interested in its use to evaluate the a priori probability that a (bipartite) state is classical in nature (separable) or, equivalently, the complementary probability that it is quantum (nonseparable/entangled). The Bures metric serves as the minimal monotone metric [9]. The probabilities of separability its volume element provide, appear to be larger than for any other member of the nondenumerable class of monotone metrics. (Classically, there is a unique monotone metric — the Fisher information metric [10,11].) In this sense, it also constitutes an upper bound on other (arguably plausible) prior probabilities of separability — while the maximal monotone metric [12,13] would provide a lower bound. (The Bures metric is associated with the symmetric logarithmic derivative and the maximal monotone metric with the right logarithmic derivative [14].)

In [2], we found for certain low-dimensional scenarios involving two qubits, analytically exact Bures probabilities of separability. This further led us in [3] to use numerical methods (quasi-Monte Carlo integration), employing a recently-developed parameterization of the $4 \times 4$ density matrices based on $SU(4)$ Euler angles [15], to investigate the conjecture that the Bures probability of separability of two arbitrarily coupled qubits — occupying a 15-dimensional convex set — might also be exact. In [16], we found numerical support for the conjecture that the Bures probability of separability of two qubits is $\frac{16807}{\pi^8} \approx 0.0733389$, where $\sigma_{Ag}$ is the silver mean, $\sqrt{2} - 1$. (Our primary conjecture there was that the volume — as measured in terms of (four times) the Bures metric, that is, the statistical distinguishability (SD) metric [8] — of the separable two-qubit states is $\frac{2\pi^4}{3}$. Certain other monotone metrics of interest also appear to give volumes of the separable states that are simple multiples of $\sigma_{Ag}$, such as $10\sigma_{Ag}$ for the Kubo-Mori metric [16].) These analyses, however, cast some doubt as to the naturalness of our earlier conjectures in [4] in regard to the Bures probability of separability of a qubit and qudit.

The development by Eggeling and Werner [17] of new (“laboratory”) systems of tripartite states — composed of three subsystems of equal but arbitrary finite Hilbert space dimension ($d$) — provides us with an opportunity to take this line of analysis to a finer level of description than previously, that is, by attaching Bures volumes to the sets of biseparable, triseparable states, as well as those having positive partial transpose with respect to the biseparable partition (cf. [18]). These states were devised by Eggeling and Werner to obtain a dimension-independent characterization of the separability properties of symmetric states. They possess an explicit parameterization as linear combinations of permutation operators, which is helpful in explicit computations. Also, there is a “twirl” operation, which brings an arbitrary tripartite state to this special subset.

It has proved convenient in previous analyses to speak in terms of the SD metric rather than the Bures one, and we will follow such a course here. All the probabilities given below will, of course, be the same using either the Bures...
or SD metric. As a point of reference, for the bipartite (one-dimensional laboratory) Werner states [19] composed of two qubits, the Bures/SD probability of separability was found to be exactly $\frac{1}{4}$ [2, sec.II.A.7].

II. ANALYSES

Five parameters ($r_-, r_+, r_1, r_2, r_3$) characterize the Eggeling-Werner (EW) states, in general, but only four in the three-qubit case (where the parameter $r_-$ degenerates to identically zero). (Also, it is convenient to employ a “dummy” bound variable, $r_0 = 1 - r_- - r_+$, as in [17], as well as utilize the transformation to spherical coordinates, $r_1 = R \cos \theta, r_2 = R \sin \theta \cos \phi, r_3 = R \sin \theta \sin \phi$.) For an EW-density matrix, the parameters obey the relations [17, eq. (6)],

$$r_+, r_-, r_0 \geq 0, \quad r_+ + r_- + r_0 = 1, \quad r_1^2 + r_2^2 + r_3^2 \leq r_0^2. \quad (1)$$

The eigenvalues of an EW-density matrix are of the form,

$$\frac{r_+}{\nu_+} \text{ (multiplicity } \nu_+, \frac{r_-}{\nu_-} \text{ (m. } \nu_-), \quad \frac{1}{2r_0}(r_0 \pm \sqrt{r_1^2 + r_2^2 + r_3^2}) \text{ (m. } \nu_0), \quad \text{(2)}$$

where

$$\nu_+ = \frac{d^3 + 3d^2 + 2d}{6}, \quad \nu_- = \frac{d^3 - 3d^2 + 2d}{6}, \quad \nu_0 = \frac{d^3 - d}{3}. \quad \text{(3)}$$

(The multiplicity $\nu_+$ is equal to the dimension of the corresponding Bose subspace divided by the dimension — that is, 1 — of the respective [trivial] irrep, while $\nu_-$ similarly corresponds to the Fermi subspace and $\nu_0$ to the para-subspace.) So, for $d = 2$ there are, in general, three distinct eigenvalues, and for $d > 4$, four distinct eigenvalues.

Since we were able rather readily to determine, in addition, the corresponding eigenvectors of the EW states ($\rho$) we examined, we could directly implement the basic formula for the Bures metric,

$$d_B(\rho, \rho + d\rho)^2 = \frac{1}{2} \sum_{\alpha, \beta} | \langle \alpha | d\rho | \beta \rangle |^2 \frac{1}{\lambda_\alpha + \lambda_\beta}, \quad \text{(4)}$$

(where $|\alpha >, \alpha = 1, \ldots, n$, are eigenvectors of $\rho$ with eigenvalues $\lambda_\alpha$) without having to rely upon any of the interesting (indirect) methods Dittmann developed [20] — and we have applied elsewhere [2,4,21] — in order to avoid the (often problematical) computation of eigenvalues and eigenvectors of $\rho$. (For the SD metric we replace the coefficient $\frac{1}{2}$ in (4) with 2.) We also observe the relationship

$$d_B(\rho_1, \rho_2)^2 = 2 - 2\sqrt{F(\rho_1, \rho_2)}, \quad \text{(5)}$$

where $F(\rho_1, \rho_2)$ is the frequently-used Bures fidelity between density matrices $\rho_1$ and $\rho_2$. [6,22,23].)

We first examined the EW-states in the qubit ($d = 2$) and qutrit ($d = 3$) cases. It turned out that the Bures metric tensor in the qubit (four-parameter [$r_0 = 0$]) case is identical to the corresponding $4 \times 4$ submatrix of the Bures metric tensor in the qutrit (five-parameter) case. Also, numerical evidence we have adduced strongly indicates that the $5 \times 5$ metric tensor (and hence the volume element) in the case of three four-level ($d = 4$) quantum systems is identical to that in the three-qutrit case. Presumably, this holds true for tripartite EW-states with still higher-dimensional subsystems ($d > 4$). (In fact, T. Eggeling has been able to independently confirm this proposition, as well as the formulas (6)-(15) given immediately below by deriving a general formula — in terms of the subsystem dimension $d$ — for the Bures metric tensor of the EW tripartite states, on the basis of representation theory (App. I). He has also been able to show that the scalar curvature of the Bures metric is equal to $20 + 18/r_0$, and thus diverges for $r_0 = 0$ (cf. [24]).) “Surprisingly, it turns out that the separability sets we investigate are also independent of dimension” [17].

In terms of the spherical coordinates, the SD metric tensor elements ($g_{ij}$) which are not identically zero are the simple functions,

$$g_{r_- r_-} = \frac{1}{2} \left( \frac{2}{r_-} + \frac{1}{r_0 + R} - \frac{1}{-r_0 + R} \right), \quad \text{(6)}$$

$$g_{r_- r_+} = \frac{r_0}{(-r_0 - R)(-r_0 + R)}, \quad \text{(7)}$$
\[ g_{r-R} = -\frac{R}{(r_0 + R)(-r_0 + R)}, \quad (8) \]

\[ g_{r+r_+} = \frac{1}{2} \left( \frac{2}{r_+} + \frac{1}{r_0 + R} - \frac{1}{-r_0 + R} \right), \quad (9) \]

\[ g_{r+R} = -\frac{R}{(r_0 + R)(-r_0 + R)}, \quad (10) \]

\[ g_{RR} = \frac{r_0}{(-r_0 - R)(-r_0 + R)}, \quad (11) \]

\[ g_{\theta \theta} = \frac{R^2}{r_0}, \quad (12) \]

\[ g_{\phi \phi} = \frac{R^2 \sin^2 \theta}{r_0}. \quad (13) \]

The SD volume element (|\(g_{ij}|^{1/2}\)) in the qubit case (first, having set \(r_- = 0\) in the above and deleted the row and column of the 5 \(
\times 5\) metric tensor corresponding to \(r_-\) before computing |\(g_{ij}|^{1/2}\)) is (reverting to the original EW coordinates),

\[ |g_{ij}|^{1/2}_{\text{qubit}} = \frac{1}{r_0 \sqrt{r_+ \left( -r_1^2 - r_2^2 - (r_0 + r_3)(-r_0 + r_3) \right)}}, \quad (14) \]

and, quite similarly, in the qutrit case,

\[ |g_{ij}|^{1/2}_{\text{qutrit}} = \frac{1}{r_0 \sqrt{r_- r_+ \left( -r_1^2 - r_2^2 - (r_0 + r_3)(-r_0 + r_3) \right)}}, \quad (15) \]

To normalize (14) to a probability distribution over the EW qubit-qubit-qubit states, one must divide it by \(2\pi^2/3\), while to normalize (15) over the EW qutrit-qutrit-qutrit states, one divides it by \(\pi^3/2\).

Eggeling and Werner also gave explicit parameter ranges, so that \(\rho\) : (1) is biseparable with respect to the partition 1|23; (2) is triseparable; and (3) has a partial transpose that is positive with respect to the first tensor factor. We have used these explicit ranges in numerical (Monte Carlo) simulations to estimate the relative probabilities that an EW-state has any of these three special properties. In the qubit \((r_- = 0)\) case, however, biseparability simply implies that the corresponding partial transpose is positive [17].

### A. Probabilities for qutrit case

For the three-qutrit case, based on some 131 million randomly generated points, an estimate of .0963689 was gotten for the SD/Bures probability of a positive partial transpose, .0694443 for the SD/Bures probability of biseparability, and .0165952 for triseparability. The sample was compiled in five roughly equal subsamples. (Actually, for each subsample, one billion random uniformly distributed points — \(r_-\) and \(r_+\) being drawn from [0,1] and \(r_1, r_2, r_3\) from [-1,1] — were tested to ascertain if they fulfilled the requirements (1) for an EW-state. Approximately 2.6 percent did in each subsample.) In Table I, we also show the results for each subsample, so one might gauge the stability of the overall results. (The probabilities were computed by taking the ratio of the SD volume assigned to the particular subset of states in question to the SD volume assigned to all those approximately 131 million states meeting the requirements for a general EW-state.) The standard deviations over the five samples are also given — assuming the nearly equal sample sizes are, in fact, strictly equal. (Since the true probabilities are unknown, but only estimated, we use the bias-adjusted standard deviation, \(\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}\), with \(n = 4\) rather than \(n = 5\).)
Based on the results in Table I, it would perhaps appear to be somewhat of an overstatement "that the two sets [biseparable states and states with positive partial transposes] come to be remarkably close (see. Fig. 11)" [17, p. 10].

| sample size | Bures/SD prob. of positive part. transp. | Bures/SD prob. of biseparable | Bures/SD prob. of triseparable |
|-------------|----------------------------------------|-----------------------------|-----------------------------|
| 26,176,260  | .0960822                               | .0692970                    | .0141430                    |
| 26,177,694  | .0962687                               | .0691823                    | .0142139                    |
| 26,179,742  | .0964395                               | .0696499                    | .0143551                    |
| 26,183,533  | .0963464                               | .0693287                    | .0141516                    |
| 26,173,004  | .0967076                               | .0697630                    | .0144027                    |
| 130,890,233 | .0963689                               | .0694443                    | .0142526                    |
| std. dev.   | .00023                                 | .000249                     | .000119                     |

TABLE I. Estimated Bures/SD probabilities, based on Monte Carlo simulations, that a three-qutrit EW-state: (1) has a positive partial transpose with respect to the partition $1|23$; (2) is biseparable with respect to the partition; (3) is triseparable; and (4) is biseparable with respect to the partition given that it is triseparable. For each of the five subsamples, one billion points were initially generated, from which those corresponding to EW-states were selected.

Życzkowski, Horodecki, Sanpera and Lewenstein, in the conclusion of their pioneering paper [5], had asked the question: "Has the set of separable states really a volume strictly smaller than the volume of the set of states with a positive partial transpose?" So, their question is answered affirmatively in the context here.

In our initial attempts to calculate the entries of Table I, however, we had been quite perplexed to find that roughly 14 percent of the probability mass assigned to the triseparable states appeared not to meet the EW-criteria for biseparability. (A randomly generated example of one such state had parameters $r_+ = .27, r_- = .1, r_1 = .589304, r_2 = .08100014, r_3 = -.138433$.) After correspondence with T. Eggeling regarding this clearly paradoxical situation, he concluded that it was necessary to incorporate — which we proceeded to do, as reflected in Table I exhibited here — the additional constraint,

$$r_1^2 + r_2^2 + r_3^2 \leq 4(r_+ - r_-)^2,$$

into the set of three already given in [17, sec. III]. The problem encountered, Eggeling indicated, stemmed from the fact that the third of the published constraints delimiting the domain of triseparability is of the third-degree. The added constraint (16) is necessary to ensure convexity. The plots in [17] other than Fig. 5 are themselves unaffected, he noted, since an appropriate numerical cutoff that took the “right” root of the third-degree polynomial was utilized in their generation. Fig. 1 is the same as Fig. 5 in [17], except that in addition to the central heart-shaped region of triseparable states displayed there, now there are also shown the three peripheral regions that the new fourth constraint (16) formally excludes.

FIG. 1. Plot for the section $r_+ = 0.27$ and $r_- = 0.1$ of the (central) heart-shaped set of triseparable states, along with the three peripheral regions that are excluded by the new fourth triseparability constraint (16). Without these three peripheral regions, the figure is identical to Fig. 5 in [9].
B. Probabilities for qubit case

Based on some 654 million sampled points, we obtained in the three-qubit case, an estimate of .216769 that an EW-state is biseparable (with respect to the partition 1\|23) and .0630532 that it is triseparable. (As observed earlier, in this specific case, biseparability implies a positive partial transpose [17].) Table II is the qubit analogue of Table I. (Again, one billion random uniformly distributed points were examined in each randomly generated subsample to ascertain if they parameterized an EW-three-qubit density matrix. Approximately, 13.1 percent did in each subsample.)

| Sample Size | Bures/SD prob. of biseparable state | Bures/SD prob. of triseparable state |
|-------------|-------------------------------------|--------------------------------------|
| 130,866,541 | .216612                             | .0631270                             |
| 130,860,088 | .216746                             | .0630338                             |
| 130,871,093 | .216888                             | .0631249                             |
| 130,852,647 | .216821                             | .0630014                             |
| 130,867,972 | .216779                             | .0629789                             |
| 654,326,341 | .216769                             | .0630532                             |
| Std. dev.   | .0001026                            | .000069                              |

|TABLE II. Estimated Bures/SD probabilities, based on Monte Carlo simulations, that a three-qubit EW-state is: (1) biseparable with respect to the partition 1|23; and (2) triseparable. For each of the five subsamples, one billion points were initially generated.|
So, the Bures/SD probabilities of biseparability and triseparability are considerably greater in the three-qubit case than in the three-qutrit one. This is as would be expected on the basis of past analyses [1].

C. Supplementary Analyses

Among the constraints that must be satisfied for an EW-state to be triseparable are [17, p. 6],

\[ 0 \leq r_- \leq \frac{1}{6}; \quad \frac{1}{4} (1 - 2r_-) \leq r_+ \leq 1 - 5r_- \tag{17} \]

If we integrate the normalized form of (15) over the full domain of parameters (1), except for imposing these constraints, we obtain an exact upper bound on the SD/Bures probability that a qutrit-qutrit-qutrit EW-state is triseparable of

\[ \frac{-70 - 325\sqrt{2} \csc^{-1} \sqrt{3} + 224\sqrt{5} \sin^{-1} \sqrt{\frac{2}{5}}}{400\pi} \approx 0.177661 \tag{18} \]

If we act similarly, but now set \( r_- = 0 \) and use the normalized form of (14), we obtain an exact upper bound of \( 27/64 = 0.421875 \) on the SD/Bures probability that a qubit-qubit-qubit EW-state is triseparable.

We can also get an apparently quite weak exact upper bound on the SD/Bures probability that a three-qutrit EW-state is biseparable, by simply imposing the single constraint \( 0 \leq r_- \leq 1/3 \) [17, p. 9]. This gives

\[ \frac{26\sqrt{2} + 54 \csc^{-1} \sqrt{3}}{27\pi} \approx 0.825312 \tag{19} \]

(while our Monte Carlo simulations [Table I] indicate that the actual Bures/SD probability is more on the order of .069). Though, somewhat to our disappointment, we have not been able to obtain exact Bures probabilities of separability for the EW-states, T. Eggeling has indicated (App. II) that if one restricts one’s consideration to the permutation-invariant EW states (which are commutative), then the required integrations can be performed analytically.

As an additional investigative probe into the SD/Bures geometry of the EW-tripartite states, having \( U \otimes U \otimes U \) symmetry, we have estimated — first, for the three-qubit case — the ratio \( \approx 0.34398 \) of the SD area of the boundary of the triseparable states to the SD area of the boundary of the biseparable states. To obtain this estimate, we extracted the 3 \( \times \) 3 submatrix of the 5 \( \times \) 5 metric tensor ((6)-(13)) corresponding to the variables \( r_1, r_2, r_3, r_+ \) and then took the square root (\( h \)) of its determinant (having, of course, set \( r_- = 0 \), as is demanded in the \( d = 2 \) case). Then, we randomly generated values of \( r_1, r_2, r_3 \) and computed for each of the various defining inequalities, in turn, that value(s) of \( r_+ \) which would saturate each. We then evaluated \( h \) if and only if that set of \( r_1, r_2, r_3, r_+ \) also satisfied (without necessarily saturating) all the remaining required inequalities. The estimate of .34398 is based on some 21,000,000 such points. (For the first 10,000,000 points, the estimate was .342709.) We have not explicitly ruled out the possibility that the SD areas themselves of the boundaries of the triseparable and biseparable states in question are unbounded in value, as preliminary analyses appear to indicate is the case for the area of the EW-states in general.

Similarly, for all the higher-dimensional three-qudit cases (\( d > 2 \)), where we now extract the 4 \( \times \) 4 submatrix corresponding to \( r_+, r_1, r_2, r_3 \) and solve for the values of \( r_- \) that saturate the various constraints, in turn, we obtained (on the basis of 18,000,000 points) an estimate of .0949602 for the ratio of the SD area of the boundary of triseparable states to the SD area of the boundary of biseparable states, and an estimate of .0000105263 for the ratio of the SD area of the boundary of triseparable states to the SD area of the boundary of states having positive partial transposes (with respect to the first tensor factor). (For the first 9,000,000 points, the estimates were, respectively, .0959525 and .000010494.)

III. CONCLUDING REMARKS

Let us indicate a number of papers of Batle et al [25–27]. In particular, these authors have investigated — following the review paper of Terhal [28] — the question of estimating the probabilities of (bi-)separability, using various separability criteria [27]. They have not employed the Bures measure in their various studies, as we have done here and elsewhere [1–4], but rather the (ZHS) one utilized by Życzkowski, Horodecki, Sanpera and Lewenstein in their pioneering paper [5]. However, we have observed in [29] that the ZHSL measure is not, in fact, proportional to
the volume element of any monotone metric, and, additionally, is based on an over-parametrization of the $n^2 - 1$-dimensional convex set of $n \times n$ density matrices. So, one might argue, that the ZHSL measure serves best as a heuristic device, easily computable, but lacking a suitably rigorous theoretical rationale.

Acín et al. [18] have introduced a classification of mixed three-qubit states, in which they define the classes of separable, biseparable, $Q$ and Greenberger-Horne-Zeilinger states. Among other things, they conclude in contrast to the pure $W$-type states, that the mixed $W$ class is not of measure zero since it is contained in a ball of finite radius (cf. [30,31]). Following the classification in [18], we have obtained in [4] — using quasi-Monte Carlo methods — estimates of lower bounds on the Bures/SD probabilities of being a GHZ state and of being a $W$ state.

Clifton and Halverson [32] gave an elementary proof that the set of separable density matrices for any bipartite quantum system in which either part is infinite-dimensional is trace-norm nowhere dense in the set of all density operators (cf. [33]). They assert that this result complements investigations [30] concluding that if both parts are finite-dimensional, then there is a (norm-dense) separable neighborhood of the maximally mixed state. However, in the case of $4 \times 4$ density matrices, if one adopts the maximal monotone metric, then it appears that the ratio of the volume of separable states to that of separable and nonseparable states is zero [4].
Computing the Bures distance of $U \otimes U \otimes U$-invariant states

I start by recalling some known and useful facts about the nature of $U \otimes U \otimes U$-invariant states.

It is well known that the commutant of the set $\{U \otimes U \otimes U\}$ is spanned by the permutation operators of the $S_3$. More precisely every such symmetric state can be seen as a state (positive linear functional) on the finite dimensional C*-algebra $A$ consisting of the formal linear combinations of the group elements of $G = S_3$ (the group algebra $L(G)$). To associate to such a state $\omega \in A$ a density matrix $\rho_\omega \in B(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ we need a representation $\pi_d: A \rightarrow B(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$. The most convenient is of course

$$\pi_d(g) := \sum_{i,j,k=1}^d |g(ijk)| (ijk) = \forall g \in G \quad (1)$$

which extends easily to $A$. If we take for example $d = 2$ and $g = (12)(3)$ we get the matrix representation

$$V_{(12)(3)} = \sum_{i,j,k=1}^d |ijk| (ijk) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

i.e. those doubly stochastic matrices acting on the kets of the canonical basis by swapping the tensor factors. As one may suggest this representation is not irreducible, i.e. it can be decomposed into irreducible subrepresentations. Since $A$ is spanned by 6 elements it is clear that the squares of the dimensions of the irreducible subrepresentations (irrep) must sum up to 6. This is only possible as $1^2 + 1^2 + 2^2$ or as $1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$ but since $A$ is non-abelian there must be at least one irrep of dimension greater than 1. This is exactly what lead us to the $R$-operators.

In fact every element of $A$ can be represented via two numbers corresponding to the two onedimensional components and one 2x2 matrix corresponding to the twodimensional irrep:

$$\rho \equiv r_+ \otimes r_- \otimes \frac{1}{2} \begin{pmatrix} r_0 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & r_0 - r_3 \end{pmatrix} \quad (2)$$

The full reduction of $\pi$ is

$$\pi \cong \bigoplus_{\nu_+} \pi_+ \oplus \bigoplus_{\nu_-} \pi_- \oplus \bigoplus_{\nu_0} \pi_0 \quad (3)$$

where $\pi_+$ is the trivial irrep assigning to each element always the number 1, $\pi_-$ is the alternating irrep assigning to each element in $G$ its signum and $\pi_0$ is the remaining twodimensional irrep. The $\nu$s are the corresponding multiplicities which are equal to the dimensions of the corresponding subspaces $R_+(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ (Bose), $R_-(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ (Fermi) and $R_0(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ (para) divided by the dimension of the respective irrep (1 or 2).

To compute the Bures distance we now can make use of the properties of such a representation. In fact a representation is nothing but a *-homomorphism, i.e. a mapping respecting the *-operation (hermitian conjugation in our case) and products:

$$\pi(a^*) = \pi(a)^* \quad \pi(ab) = \pi(a)\pi(b).$$

For the Bures distance we get:

$$d_B(\rho, \sigma)^2 = 2 - 2 \text{tr} \left[ \pi_d \left( \sqrt[8]{\sqrt{r} \sqrt{r^*}} \right) \right]$$

for $\rho = \pi_d(r)$ and $\sigma = \pi_d(s)$,

$$= 2 - 2 \text{tr} [R]_+ \text{tr} \left[ \pi_+ \left( \sqrt[8]{\sqrt{r} \sqrt{r^*}} \right) \right] - 2 \text{tr} [R]_- \text{tr} \left[ \pi_- \left( \sqrt[8]{\sqrt{r} \sqrt{r^*}} \right) \right] - 2 \text{tr} [R]_0 \text{tr} \left[ \pi_0 \left( \sqrt[8]{\sqrt{r} \sqrt{r^*}} \right) \right] \quad (5)$$

$$= 2 - 2 \nu_+ \sqrt{r \sqrt{r}} - 2 \nu_- \sqrt{r^* \sqrt{r^*}} - 2 \nu_0 \text{tr} \left[ \pi_0 \left( \sqrt[8]{\sqrt{r} \sqrt{r^*}} \right) \right] \quad (6)$$

IV. APPENDIX I OF T. EGGELING
For the last summand we have to use the expression of Hübner for 2x2 matrices:
\[
\text{tr} \left[ \pi_0 \left( \sqrt{\pi} \sqrt{r} \right) \right] = \frac{1}{2} (r_0 s_0 + r_1 s_1 + r_2 s_2 + r_3 s_3) + \frac{1}{2} \sqrt{r_0^2 - r_1^2 - r_2^2 - r_3^2} \sqrt{s_0^2 - s_1^2 - s_2^2 - s_3^2}.
\] (7)

Taking \( r_0 = 1 - r_+ - r_- \) into account we get for infinitesimal displacements:
\[
d_{B}(\rho, \rho + d\rho)^2 = 2 - 2\nu_+ \sqrt{r_+ (r_+ + dr_+)} - 2\nu_- \sqrt{r_- (r_- + dr_-)} - 2\nu_0 \left( (1 - r_+ - r_-) (1 - r_+ - r_- - dr_+ - dr_-) + r_1 (r_1 + dr_1) + r_2 (r_2 + dr_2) + r_3 (r_3 + dr_3) \right) + \frac{1}{2} \sqrt{1 - (1 - r_+ - r_-)^2 - (r_2 - r_2^3)^2} \left( \frac{r_2 - r_2^3}{1 - r_+ - r_-} \right)^{\frac{1}{2}}.
\] (8)

Differentiating \( d_{B}(\rho, \rho + t d\rho) \) twice with respect to \( t \) at \( t = 0 \) we get for the metric tensor:
\[
g_{ij}(\rho) = \frac{1}{4} \left( \frac{\nu_- dr_+^2}{r_-} + \frac{\nu_+ dr_-^2}{r_+} + \nu_0 \left( \frac{dr_1^2 + dr_2^2 + dr_3^2 + (r_1 + dr_1 + r_2 + dr_2 + r_3 + dr_3)^2}{1 - r_+ - r_-} \right) \right).
\] (9)

For the volume element we can thus derive
\[
\sqrt{\text{det} g(\rho)} = \frac{\nu_0}{32 (1 - r_+ - r_-)^{\frac{9}{2}}} \sqrt{\frac{\nu_0 (\nu_+ r_+ + \nu_- (\nu_0 r_+ - \nu_+ (1 - r_+ - r_-)))}{r_+ r_- ((1 - r_+ - r_-)^2 - r_2^2 - r_3^2)}}.
\] (10)

with
\[
\nu_+ = \frac{d^3 + 3d + 2d}{6}, \quad \nu_- = \frac{d^3 - 3d^2 + 2d}{6}, \quad \nu_0 = \frac{d^3 - d}{3}.
\]

Since for three Qubits (\( d=2 \)) no completely antisymmetric states exist (the projector is trivial) we have to take \( \nu_- = 0 \) and \( r_- = 0 \) for \( d=2 \).
V. APPENDIX II OF T. EGGELING

For commutative subsets of states (=abelian algebras) one can write down the Bures distance directly. Therefore, taking the subset of permutation invariant tripartite Werner states (which is commutative) we can compute the volume element and integrate over the triangles you see in the picture of my old PRA paper. This integration can be done analytically and leads to:

\[ p_{\text{sep,trip,perm}} = \frac{\pi}{40} (-16 + 6\sqrt{6} + 5\log \frac{3(6 - \sqrt{6})}{6 + \sqrt{6}}) = 0.170502, \]  

\[ p_{\text{bisep,trip,perm}} = \frac{\pi}{10} (1 - 5\sqrt{5} + 4\sqrt{6} - 10\log \frac{(5 + \sqrt{5})(6 - \sqrt{6})}{(5 - \sqrt{5})(6 + \sqrt{6})}) = 0.179607, \]

\( \text{ppt}=\text{biseparable for permutation invariant tripartite states, so there is nothing to calculate. These numbers together with the Monte-Carlo simulations correspond to the intuition that triseparable and separable states are concentrated in the vicinity of the subset of permutation invariant states This is intuitive if one looks at the figures describing these sets of states.} \)

ACKNOWLEDGMENTS

I would like to express appreciation to the Kavli Institute for Theoretical Physics for computational support in this research and to T. Eggeling for his sustained interest and involvement.
[30] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu and R. Schack, Phys. Rev. Lett. 83, 1054 (1999).
[31] S. Szarek, quant-ph/0310061.
[32] R. Clifton and H. Halverson, Phys. Rev. A 61, 012108 (2000).
[33] P. B. Slater, J. Opt. B: Quantum Semiclass. Opt. 2, L19 (2000).