Two-step differentiator for delayed signal

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Abstract: This paper presents a high-order differentiator for delayed measurement signal. The proposed differentiator not only can correct the delay in signal, but also can estimate the undelayed derivatives. The differentiator consists of two-step algorithms with the delayed time instant. Conditions are given ensuring convergence of the estimation error for the given delay in the signals. The merits of method include its simple implementation and interesting application. Numerical simulations illustrate the effectiveness of the proposed differentiator.

Keywords: Differentiator, delayed measurement, two-step.

1. Introduction

This paper focuses on the problem of estimating derivatives for delayed signals. Usually measurement and communication of signals all exist time delays. Obtaining the velocities of tracked targets is crucial for several kinds of systems with correct and timely performances, such as the missile-interception systems [1] and underwater vehicle systems [2]. However, the delay phenomenon make it difficult to obtain the undelayed derivatives of signal. In recent years, researchers tried to apply alternate methodologies to design differentiators [3-12]. The popular high-gain differentiators [6, 7, 8] provide for an exact derivative when their gains tend to infinity. In [9, 10], a differentiator via second-order (or high-order) sliding modes algorithm was proposed. The information one needs to know on the signal is an upper bound for Lipschitz constant of the derivative of the signal. In [11, 12], a finite-time-convergent differentiator based on finite-time stability and singular perturbation technique was presented. In all of the aforementioned papers, the signals are required to be undelayed, otherwise, delayed phenomena happen in derivative outputs.

This paper provides a design of high-order differentiator for delayed signal. The proposed differentiator algorithm has a two-step structure. The differentiator algorithm is composed by two
step sub-differentiators, the first step sub-differentiator estimates the derivatives at the delay, and
the second step sub-differentiator estimates the present derivatives. It is shown that, under suitable
conditions, for a given delay there exists a differentiator of suitable dimension achieving error
decay.

2. Problem statement

For the normal high-gain differentiator [6, 7, 8]:

\[
\dot{x}_i = x_{i+1} + \frac{k_i}{\varepsilon} (v(t - \Delta) - x_1), i = 1, \cdots, n - 1
\]

\[
\dot{x}_n = \frac{k_n}{\varepsilon^n} (v(t - \Delta) - x_1)
\]

where \(x_i (i = 1, \cdots, n)\) are the states of differentiator (1), \(\varepsilon > 0\) is the perturbation parameter, and
\(v(t - \Delta)\) is a signal with time delay \(\Delta > 0\). The following conclusion can be obtained:

\[
\lim_{\varepsilon \to 0} x_i = v^{(i-1)}(t - \Delta), i = 1, \cdots, n
\]

From (2), we can find that the delay phenomena happen in the estimation outputs. And the delay
phenomena exist in other differentiators [9-12]. We are interested in designing a differentiator to
force the states of differentiator to approximate the undelayed derivatives with the delayed signal
\(v(t - \Delta)\).

3. Design of high-order two-step differentiator

Here, we design a differentiator for a delayed signal, and the undelayed signal tracking and
derivatives estimation are obtained from the delayed signal. The proposed differentiator algorithm
has a two-step structure. It is shown that for a given delay there exists a differentiator of suitable
dimension such that the differentiator states approximate the undelayed derivatives of signals in
spite of the existence of time delay.

We design a two-step high-order differentiator as follow:
\[ \dot{x}_{i,1} = x_{i+1,1} \frac{k_i}{\varepsilon^i} (v (t - \Delta) - x_{1,1}), \quad i = 1, \ldots, n-1 \]
\[ \dot{x}_{n,1} = \frac{k_n}{\varepsilon^n} (v (t - \Delta) - x_{1,1}) \] (3)

and

\[ \dot{x}_{i,2} = x_{i+1,2} + \left( \sum_{j=i}^{n-1} \frac{1}{(j-i)!} \frac{k_j}{\varepsilon^j} \Delta^{j-i} \right) (v (t - \Delta) - x_{1,1}), \quad i = 1, \ldots, n-1 \]
\[ \dot{x}_{n,2} = \frac{k_n}{\varepsilon^n} (v (t - \Delta) - x_{1,1}) \] (4)

where \( k_1, \ldots, k_n, \) are selected such that \( s^n + k_1 s^{n-1} + \cdots + k_{n-1} s + k_n = 0 \) is Hurwitz, and \( \varepsilon > 0 \) is the perturbation parameter.

**Theorem 1:** For two-step high-order differentiator (3)-(4) and delayed signal \( v (t - \Delta) \), there exist \( k_1, \ldots, k_n, \) such that:

\[ \lim_{\varepsilon \to 0} X_{i,2} (s) = s^{i-1} V (s), \quad i = 1, \ldots, n-1 \] (5)

approximately. Where \( X_{i,2} (s) (i = 1, \ldots, n-1) \) and \( V (s) \) are the Laplace transformations of \( x_{i,2} (t), (i = 1, \ldots, n-1) \) and \( v (t) \), respectively.

**Proof:** The laplace transformamtion of (3) is

\[ sX_{i,1} (s) = X_{i+1,1} (s) + \frac{k_i}{\varepsilon^i} \left( e^{-s\Delta} V (s) - X_{1,1} (s) \right), \quad i = 1, \ldots, n-1 \]
\[ sX_{n,1} (s) = \frac{k_n}{\varepsilon^n} \left( e^{-s\Delta} V (s) - X_{1,1} (s) \right) \] (6)

From (6), we have
\[ X_{1,1}(s) = \left( \sum_{i=1}^{n} \frac{k_i}{s^i \varepsilon^i} \right) (e^{-s\Delta V(s)} - X_{1,1}(s)) \]  

(7)

Therefore, we can get

\[ X_{1,1}(s) = \frac{s^{n-1} \varepsilon^{n-1} k_1 + \cdots + s \varepsilon k_{n-1} + k_n e^{-s\Delta V(s)}}{s^n \varepsilon^n + \cdots + s \varepsilon k_{n-1} + k_n} e^{-s\Delta V(s)} \]  

(8)

From (8), we have

\[ e^{-s\Delta V(s)} - X_{1,1}(s) = \frac{s^n \varepsilon^n}{s^n \varepsilon^n + \cdots + s \varepsilon k_{n-1} + k_n} e^{-s\Delta V(s)} \]  

(9)

The Laplace transformation of (4) is

\[ sX_{i,2}(s) = X_{i+1,2}(s) + \left( \sum_{j=1}^{n} \frac{1}{(j-i)! \varepsilon^j \Delta^{j-i}} k_j \right) (e^{-s\Delta V(s)} - X_{1,1}(s)) , \]

\[ i = 1, \cdots, n - 1 \]

\[ sX_{n,2}(s) = \frac{k_n}{\varepsilon^n} (e^{-s\Delta V(s)} - X_{1,1}(s)) \]  

(10)

From (10), we have

\[ X_{i,2}(s) = \sum_{m=i}^{n} \frac{1}{s^{m-i+1}} \left( \sum_{j=m}^{n} \frac{1}{(j-m)! \varepsilon^j \Delta^{j-m}} k_j \right) (e^{-s\Delta V(s)} - X_{1,1}(s)) \]

\[ i = 1, \cdots, n \]  

(11)

Therefore, from (11) and (9), we have equation

\[ X_{i,2}(s) = \frac{1}{s^{n-i+1} \varepsilon^n} \sum_{m=i}^{n} s^{n-m} \left( \sum_{j=m}^{n} \frac{1}{(j-m)! \varepsilon^{n-j} k_j \Delta^{j-m}} \right) (e^{-s\Delta V(s)} - X_{1,1}(s)) \]
\[
\sum_{m=i}^{n} s^{n-m} \left( \sum_{j=m}^{n-1} \frac{1}{(j-m)!} \varepsilon^{n-j} k_j \Delta^{j-m} \right) + \sum_{m=i}^{n} \frac{s^{n-m}}{(n-m)!} k_n \Delta^{n-m} s^{n-i+1}\varepsilon^n (e^{-s \Delta} V(s) - X_{1,1}(s))
\]

\[
= s^{i-1} \left( \sum_{m=i}^{n} s^{n-m} \left( \sum_{j=m}^{n-1} \frac{1}{(j-m)!} \varepsilon^{n-j} k_j \Delta^{j-m} \right) + k_n \left( 1 + s \Delta + \cdots + \frac{1}{(n-i)!} s^{n-i} \Delta^{n-i} \right) \right)
\]

\[
\times \frac{s^n \varepsilon^n}{s^n \varepsilon^n + s^{n-1} \varepsilon^{n-1} k_1 + \cdots + s \varepsilon k_{n-1} + k_n} e^{-s \Delta} V(s)
\]

(12)

for \( i = 1, \cdots, n \). Approximately, we can get

\[
e^{s \Delta} \approx 1 + s \Delta + \cdots + \frac{1}{(n-i)!} s^{n-i} \Delta^{n-i}, i = 1, \cdots, n - 1
\]

(13)

Therefore, from (12) and (13), approximately, we have

\[
\lim_{\varepsilon \to 0} X_{i,2}(s) = s^{i-1} V(s), i = 1, \cdots, n - 1
\]

(14)

i.e., we can get that

\[
\lim_{\varepsilon \to 0} (x_{i,2}(t) - v^{(i-1)}(t)) = o(\Delta^{n-i+1}), i = 1, \cdots, n - 1
\]

(15)

where \( O(\Delta^{n-i+1}) \) denotes the approximation of \( \Delta^{n-i+1} \) order \([13]\) between \( x_{i,2}(t) \) and \( v^{(i-1)}(t) \).

When \( 0 < \Delta < 1 \), the larger the dimension \( n \) of the differentiator is, the higher the estimation precision is.

It means that \( x_{i,2} \) approximates the undelayed derivative \( v^{(i-1)}(t) \) for \( i = 1, \cdots, n - 1 \). This concludes the proof. \( \blacksquare \)

**Remark 1:** Reducing peaking phenomena

In two-step differentiator (3)-(4), peaking phenomena happen due to the infinity \( \varepsilon \). In order to reduce peaking phenomena sufficiently, we choose \( \varepsilon \) as
\[
\frac{1}{\varepsilon} = R = \begin{cases} 
R_0 t^p, & t \leq t_{\text{max}} \\
R_0, & t > t_{\text{max}} \end{cases}
\]  

(16)

where \( R_0 > 0 \), \( t_{\text{max}} > 0 \) and \( p \geq 1 \) are chosen according to the desired maximum error that depends on the value of \( R_{\text{max}} = R_0 t_{\text{max}}^p \). The selection of \( p \geq 1 \) can make \( t^p \) more smaller in \( t \in (0, 1) \).

**Remark 2:** From (12), we know that \( s^n \varepsilon^n + s^{n-1} \varepsilon^{n-1} k_1 + \cdots + s \varepsilon k_{n-1} + k_n \), or

\[
s^n + \frac{k_1}{\varepsilon} s^{n-1} + \cdots + \frac{k_{n-1}}{\varepsilon^{n-1}} s + \frac{k_n}{\varepsilon^n}
\]

is a high-order integral chain, the outputs \( x_{1,2}(t), i = 1, \cdots, n-1 \), can be filtered through choosing suitable \( \varepsilon \) and \( k_i \) \( (i = 1, \cdots, n) \). Moreover, the output delay caused by high-order integral chain can be corrected by using \( \Delta + \Delta_g \) instead of \( \Delta \) in two-step differentiator, i.e.,

\[
\dot{x}_{i,1} = x_{i+1,1} + \frac{k_i}{\varepsilon^i} (v(t - \Delta) - x_{1,1}), \quad i = 1, \cdots, n-1
\]

\[
\dot{x}_{n,1} = \frac{k_n}{\varepsilon^n} (v(t - \Delta) - x_{1,1})
\]

(17)

and

\[
\dot{x}_{i,2} = x_{i+1,2} + \left( \sum_{j=i}^{n} \frac{1}{(j-i)!} \frac{k_j}{\varepsilon^j} (\Delta + \Delta_g)^{j-i} \right) (v(t - \Delta) - x_{1,1}),
\]

\[
i = 1, \cdots, n-1
\]

\[
\dot{x}_{n,2} = \frac{k_n}{\varepsilon^n} (v(t - \Delta) - x_{1,1})
\]

(18)

where \( \Delta_g \) is the delay caused by high-order integral chain.

**4. Simulations**

In the following simulations, we select the function of \( \sin(t) \) as the desired signal \( v(t) \), and the delayed signal \( v(t - \Delta) = \sin(t - \Delta) \) is taken as the measurement signal. By two-step differentiator, the undelayed signal tracking and derivatives estimation are obtained from the delayed signal \( v(t - \Delta) = \sin(t - \Delta) \). Moreover, we know that \( \frac{dv(t)}{dt} = \cos(t) \), and \( \frac{d^2v(t)}{dt^2} = -\sin(t) \).
We select fourth-order differentiators as simulation examples.

Parameters: $\Delta = 0.5s$, $k_1 = 4$, $k_2 = 6$, $k_3 = 4$, $k_4 = 1$, and

\[
\frac{1}{\varepsilon} = R = \begin{cases} 
100t^2, & t \leq 1 \\
100, & t > 1 
\end{cases}
\]

4.1 Fourth-order high-gain differentiator [6, 7, 8].

\[
\begin{align*}
\dot{x}_1 &= x_2 + \frac{k_1}{\varepsilon} (v(t - \Delta) - x_1) \\
\dot{x}_2 &= x_3 + \frac{k_2}{\varepsilon^2} (v(t - \Delta) - x_1) \\
\dot{x}_3 &= x_4 + \frac{k_3}{\varepsilon^3} (v(t - \Delta) - x_1) \\
\dot{x}_4 &= \frac{k_4}{\varepsilon^4} (v(t - \Delta) - x_1)
\end{align*}
\] (19)

Figures 1, 2 and 3 denote respectively signal tracking, estimations of first-order and second-order derivatives by fourth-order high-gain differentiator. $x_1$, $x_2$ and $x_3$ approximate respectively $v(t - \Delta)$, $dv(t - \Delta)/dt$ and $d^2v(t - \Delta)/dt^2$ but not $v(t)$, $dv(t)/dt$ and $d^2v(t)/dt^2$.

4.2 Two-step fourth-order differentiator. From two-step differentiator (3)-(4), let $n = 4$, we get the two-step fourth-order differentiator as:

\[
\begin{align*}
\dot{x}_{1,1} &= x_{2,1} + \frac{k_1}{\varepsilon} (v(t - \Delta) - x_{1,1}) \\
\dot{x}_{2,1} &= x_{3,1} + \frac{k_2}{\varepsilon^2} (v(t - \Delta) - x_{1,1}) \\
\dot{x}_{3,1} &= x_{4,1} + \frac{k_3}{\varepsilon^3} (v(t - \Delta) - x_{1,1}) \\
\dot{x}_{4,1} &= \frac{k_4}{\varepsilon^4} (v(t - \Delta) - x_{1,1})
\end{align*}
\] (20)

and
\[
\begin{align*}
\dot{x}_{1,2} &= x_{2,2} + \left(\frac{k_1}{\varepsilon} + \frac{k_2}{\varepsilon^2}\Delta + \frac{k_3}{2!\varepsilon^3}\Delta^2 + \frac{k_4}{3!\varepsilon^4}\Delta^3\right) \left(v(t) - \dot{x}_{1,1}\right) \\
\dot{x}_{2,2} &= x_{3,2} + \left(\frac{k_2}{\varepsilon^2} + \frac{k_3}{\varepsilon^3}\Delta + \frac{k_4}{2!\varepsilon^4}\Delta^2\right) \left(v(t) - \dot{x}_{1,1}\right) \\
\dot{x}_{3,2} &= x_{4,2} + \left(\frac{k_3}{\varepsilon^3} + \frac{k_4}{\varepsilon^4}\Delta\right) \left(v(t) - \dot{x}_{1,1}\right) \\
\dot{x}_{4,2} &= \frac{k_4}{\varepsilon^4} \left(v(t) - \dot{x}_{1,1}\right)
\end{align*}
\] (21)

Figures 4, 5 and 6 denote respectively signal tracking, estimations of first-order and second-order derivatives by two-step fourth-order differentiator. \(x_{1,2}, x_{2,2}\) and \(x_{3,2}\) approximate respectively \(v(t), dv(t)/dt\) and \(d^2v(t)/dt^2\) in spite of the delayed signal \(v(t - \Delta)\).

4.3 Two-step fourth-order differentiator for signal with noise. We select \(\frac{1}{\varepsilon} = 6, k_1 = 4, k_2 = 6, k_3 = 4, k_4 = 1\).

Figures 7, 8, 9 and 10 denote respectively input signal with noise, signal tracking, estimations of first-order and second-order derivatives by two-step fourth-order differentiator. \(x_{1,2}, x_{2,2}\) and \(x_{3,2}\) approximate respectively \(v(t), dv(t)/dt\) and \(d^2v(t)/dt^2\) in spite of the delayed signal \(v(t - \Delta)\) and noise \(\delta(t)\).

From the simulations above, we find that by the normal differentiators, the undelayed derivatives can’t be obtained from the delayed signals, and time delays exist in the outputs. By the proposed two-step linear differentiator, we can not only obtain approximately the undelayed derivatives from delayed signals but also estimate the future information from the present signals.

5. Conclusion

This paper develops a two-step linear differentiator to estimate the undelayed derivatives of delayed signal. The differentiator algorithm is composed by two step sub-differentiators. The first step sub-differentiator estimates the derivatives at the delay, and the second step sub-differentiator estimates the present derivatives.
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Fig. 1. Signal tracking by fourth-order high-gain differentiator (19)

Fig. 2. Estimation of first-order derivative by fourth-order high-gain differentiator (19)

Fig. 3. Estimation of second-order derivative by fourth-order high-gain differentiator (19)
Fig. 4. Signal tracking by two-step fourth-order differentiator (20)-(21)

Fig. 5. Estimation of first-order derivative by two-step fourth-order differentiator (20)-(21)

Fig. 6. Estimation of second-order derivative by two-step fourth-order differentiator (20)-(21)
Fig. 7. Input signal with noise

Fig. 8. Signal tracking by two-step fourth-order differentiator

Fig. 9. Estimation of first-order derivative by two-step fourth-order differentiator

Fig. 10. Estimation of second-order derivative by two-step fourth-order differentiator