TAUTOLOGICAL BUNDLES ON PARABOLIC MODULI SPACES: EULER CHARACTERISTICS AND HECKE CORRESPONDENCES

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ABSTRACT. We calculate the Euler characteristic of associated vector bundles over the moduli spaces of stable parabolic bundles on smooth curves. Our method is based on a wall-crossing technique from Geometric Invariant Theory, certain iterated residue calculus and the tautological Hecke correspondence. Our work was motivated by the results of Teleman and Woodward on the index of K-theory classes on moduli stacks.

1. Introduction

Let $C$ be a smooth, complex projective curve of genus $g ≥ 2$, and fix a point $p ∈ C$. Denote by $Δ$ the set of vectors $c = (c_1 > c_2 > ... > c_r) ∈ ℜ^r$ such that $∑ c_i = 0$ and $c_1 - c_r < 1$. We will call a vector $c ∈ Δ$ regular if no nontrivial subset of its coordinates sums to an integer. For such a $c ∈ Δ$, there exists a smooth projective moduli space $P_0(c)$ of dimension $(r^2 - 1)(g - 1) + (\ell)$ ([SzT, MS, B]), whose points are in one-to-one correspondence with equivalence classes of pairs $(W, F_s)$, where $W$ is a vector bundle of rank $r$ on $C$ with trivial determinant, $F_s$ is a full flag in the fiber $W_p$, and the pair satisfies a certain parabolic stability condition depending on a regular $c ∈ Δ$.

There is a natural way to associate to an integer $k > 0$ and a vector $λ ∈ ℤ^r$ satisfying $λ_1 + ... + λ_r = 0$ a line bundle $ℒ(λ; k)$ on $P_0(c)$ in such a way that if $c = λ/k$, then $ℒ(λ; k)$ is ample.

Notation: Let $V = ℜ^r/(1, ..., 1)ℜ$ thought as the Cartan subalgebra of the Lie algebra of $SU_r$; we denote by $ρ = \frac{1}{2}(r - 1, r - 3, ..., r + 1)$ the half-sum of positive roots of $SU_r$, and set $λ = λ + ρ$, $k = k + r$. We introduce the notation $w_Φ = ∏_{1 < j} (2\sinh(x_1 - x_i))$ for the Weyl denominator, a function on $V ⊗ ℜ C$.

In [SzT] we gave a new proof of the parabolic Verlinde formula for the Euler characteristic of the line bundle $ℒ(λ; k)$ on $P_0(c)$. In fact, we proved a more precise formula, which has the following form:

$$\chi(P_0(c), ℒ(λ; k)) = N · \sum_{B∈D} \int_{Z_B} w_Φ^{1-g} (x/ξ) \exp(λ, x) OS_{B,c},$$

where the sum runs over some finite set $D$, $N$ is a constant, which depends on $g$, $r$ and $k$, $Z_B$ is a cycle near the origin in $V ⊗ ℜ C \{w_Φ = 0\}$ and $OS_{B,c}$ is a differential form on $V ⊗ ℜ C \{w_Φ = 0\}$, which depends on $c$ and $B$ (for details see page 11).

In this paper, we present a formula for the Euler characteristic of a wider class of vector bundles on $P_0(c)$: we associate to a dominant weight $ν$ of $GL_r$, a tautological vector bundle $U_ν$ on $P_0(c) × C$ and calculate the Euler characteristic

$$\chi(P_0(c), ℒ(λ; k) ⊗ π_1(U_ν ⊗ X^c)), $$
where $\pi : P_0(c) \times C \to P_0(c)$ is the projection, and $K$ is the canonical bundle on $C$. Our main result is Theorem 3.3 (cf. Example 5 and (10) for some examples).

Our proof follows the strategy of [SzT], whose basic ideas we recall now. The simplex $\Delta$ of parabolic weights $c$ parametrizing stability conditions contains a finite number of hyperplanes (walls) on whose complement (the set of regular elements in $\Delta$) the stability condition is locally constant. This induces a chamber structure on $\Delta$, such that the left-hand side and the right-hand side of (1) manifestly polynomial in the variables $(k; \lambda)$ on each chamber. We introduce the notation $l_c(k; \lambda)$ and $r_c(k; \lambda)$ for these polynomials, where $c$ is any element of the corresponding chamber.

Using geometric invariant theory, we show that the wall-crossing terms, i.e. the differences between the two polynomials associated to neighbour chambers (specified by $c_+$ and $c_-$) for the left-hand side and the right-hand side coincide:

\[
l_c(k; \lambda) - l_{c'}(k; \lambda) = r_{c'}(k; \lambda) - r_c(k; \lambda).
\]

Next, we choose a pair of chambers adjacent to two special vertices of the simplex $\Delta$, and consider the corresponding pairs of polynomials:

\[
l_{c_+}(k; \lambda), l_{c_-}(k; \lambda) \quad \text{and} \quad r_{c_+}(k; \lambda), r_{c_-}(k; \lambda)
\]

from the left-hand side and the right-hand side of (1), respectively. Using Serre duality, we derive certain symmetry properties of $l_{c_+}(k; \lambda)$ and $l_{c_-}(k; \lambda)$, and then we prove that $r_{c_+}(k; \lambda)$ and $r_{c_-}(k; \lambda)$ satisfy the same symmetries.

Finally, we show that a set of polynomials parametrized by the chambers in $\Delta$ is uniquely determined by the wall-crossing terms (3) and our symmetry properties for the polynomials (4), and thus we obtain that $l_c(k; \lambda)$ and $r_c(k; \lambda)$ coincide.

In this paper, we follow a similar path. To demonstrate the technique, below, after our introductory remarks in §1.2, we will present our arguments for the case $\tau = 2$, when the formula for the Euler characteristic has a simple form (cf. (10)).

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1.1. Remarks

In this paragraph, we discuss the relationship of our paper with earlier works.

As mentioned above, we will follow the ideas of [SzT], where the case of the line bundles on the moduli spaces was treated. Let us highlight some of the new phenomena that we encounter in this work. The symmetry of Euler characteristics (2) on the moduli spaces $P_0(c_+)$ and $P_0(c_-)$ is only true after an affine transformation; in fact, they need to be shifted by a linear combination of Euler characteristics of line bundles, which then can be calculated using the results of [SzT] (cf. Propositions 5.3, 5.6 and 5.7). The appearance of Hessians (cf. [W, TW]) in our framework in the formulas for Euler characteristics (cf. Theorem 3.3) is remarkably simply explained by the relations in the cohomology ring of the curve (cf. page 6). The directional derivatives, on the other hand, appear from a comparison of the Chern characters of the corresponding vector bundles under the Hecke isomorphism (cf. Proposition 4.10).

Our work owes a lot to the paper of Teleman and Woodward [TW], where a similar formula is derived for Euler characteristics of vector bundles on stacks. The advantage of our approach is its technical simplicity, and much more explicit formulas when the invariants of moduli spaces are to be calculated. There are also subtle
differences in the final formulas, which are manifest, in particular, in the appearance of certain determinantal factors in our formalism.

The idea of the formulas for push-forwards in the cohomological setting, in particular, the Hessian, first appeared in the seminal paper of Witten [W]. Mathematically sound approaches in this cohomological/symplectic setting were employed by Jeffrey and Kirwan [JK] and Meinrenken [M]. In particular, the wall-crossing ideas, which play a major role in our work already appeared in [JK].

Finally, a few words on the significance of our paper: we show that the residue/wallcrossing methods of [SzT] may be successfully employed to describe the push-forward maps in the full K-theory of moduli spaces. The formulas we find, even though they are similar to the results of [JK] and [TW], are new, and, in fact, are the first explicit formulas for these quantities.

1.2. The residue formula for rank 2

In this case we need to consider the moduli space of vector bundles with parabolic structures at two points to calculate our wall-crossing terms. For the convenience of the reader, we recall below some notations from [SzT, §9]. We fix two points: \( p, q \in \mathbb{C} \), and consider the moduli space

\[
P_0(p, q) = \{ W \to \mathbb{C}, F_1 \subset W_p, G_1 \subset W_q \mid \text{rk}(W) = 2, \det(W) \cong 0 \}
\]

of rank-2 stable parabolic bundles \( W \) with fixed determinant isomorphic to \( 0 \), with parabolic structure given by a line \( F_1 \subset W_p \) with weight \( (c, -c) \), and a line \( G_1 \subset W_q \) with weight \( (a, -a) \).

The space of admissible parabolic weights is a square

\[
\square = \{(c, a) \mid 1 > 2c > 0, 1 > 2a > 0 \};
\]

it is easy to check that the set of isomorphism classes of parabolic bundles remains unchanged when we vary the parabolic weights in each of the two chambers defined by the conditions

\[ c > a \quad \text{and} \quad c < a. \]

There are thus two moduli spaces, which we denote by \( P_0(c > a) \) and \( P_0(c < a) \).

\[ \begin{array}{c}
(0, \frac{1}{2}) \\
(0, 0) \\
(\frac{1}{2}, 0)
\end{array} \quad \begin{array}{c}
P_0(c < a) \\
P_0(c > a)
\end{array} \]

\[ \text{Figure 1. The space of admissible weights in the case of rank } r = 2, \]

two points.

Denote by the same symbol \( U \) universal bundles over \( P_0(c > a) \times \mathbb{C} \) and \( P_0(c < a) \times \mathbb{C} \). Then \( U \) is endowed with two flags, \( F_1 \subset F_2 = U_p \) and \( G_1 \subset G_2 = U_q \); we choose a normalization of \( U \) such that \( F_1 \subset U_p \) is trivial. For \( \lambda, \mu \in \mathbb{Z} \), we introduce the line bundle

\[
L(k; \lambda, \mu) = \det(U_p)^{k(1-g)} \otimes \det(\pi_+(U))^{-k} \otimes (F_2/F_1)^{\lambda} \otimes (G_1)^{-\lambda} \otimes (G_2/G_1)^{\mu} \otimes (F_1)^{-\mu}
\]
on \( P_0(c > a) \times \mathbb{C} \) and \( P_0(c < a) \times \mathbb{C} \).
Let $\nu = (\nu_1 \geq \nu_2) \in \mathbb{Z}^2$ be a dominant weight of $GL_2$, denote by $\rho_\nu$ the irreducible representation of $GL_2$ with highest weight $\nu$, and by $\bar{\rho}_\nu$ its restriction to $SU_2 \subset GL_2$. We denote by $U_\nu \to P_0(c, a) \times \mathbb{C}$ the bundle associated to the representation $\rho_\nu$.

Our goal is to calculate Euler characteristics

\begin{equation}
\chi^\nu_\nu(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c > a), \mathcal{L}(k; \lambda, \mu) \otimes \pi_r(U_\nu \otimes \mathbb{C})) \quad \text{and} \quad \chi^\nu_\nu(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c < a), \mathcal{L}(k; \lambda, \mu) \otimes \pi_r(U_\nu \otimes \mathbb{C})).
\end{equation}

Let $\text{Exp} : \text{Lie}(SU_2) \to SU_2$ be the exponential map and let

$$\phi(x) = \text{trace}(\bar{\rho}_\nu \circ \text{Exp}(x/2)) = \frac{\sinh((\nu_1 - \nu_2 + 1)x/2)}{\sinh(x/2)}$$

be the character function on the Lie algebra of a maximal torus of $SU_2$.

We introduce the notation

$$\bar{\phi}(x) = 2 \frac{d}{dx} \phi(x) \quad \text{and} \quad \bar{\phi}(x) = 2 \frac{d}{dx} \phi(x),$$

(where the factors of 2 are introduced for convenience) and define two polynomials in $k, \lambda, \mu$ which as we will show, equal to $\phi$: Fact 1. The difference of these polynomials has the form:

\begin{align*}
R^\nu_\nu(k; \lambda, \mu) - R^\nu_\nu(k; \lambda, \mu) &= g(\nu_1 - \nu_2 + 1)x/2) \frac{\text{Res}}{u=0} \frac{e^{u(\lambda-\mu)} e^{u(\nu_1 + \nu_2)/2}}{(2 \sinh(u/2))^2 g(1 - e^{u(k+2)})} \bar{\phi}(u) \\
&= g \frac{\text{Res}}{u=0} \frac{e^{u(\lambda-\mu)} e^{u(\nu_1 + \nu_2)/2}}{(2 \sinh(u/2))^2 g(1 - e^{u(k+2)})} \bar{\phi}(u).
\end{align*}

Fact 2. An easy calculation via substitutions shows the following:

\begin{align*}
R^\nu_\nu(k; \lambda, \mu) &= -R^\nu_\nu(k; \lambda, -\mu, -1) = -R^\nu_\nu(k; -\lambda + k + 1 - (\nu_1 + \nu_2), \mu) - \frac{(-2k + 4)^g}{\text{Res}} \frac{e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu)} e^{u(\nu_1 + \nu_2)/2}}{\sinh(u/2)^2 g(1 - e^{u(k+2)})} \bar{\phi}(u) \\
&\quad \text{and} \quad \frac{(-2k + 4)^g}{\text{Res}} \frac{e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu+k+2)} e^{u(\nu_1 + \nu_2)/2}}{\sinh(u/2)^2 g(1 - e^{u(k+2)})} \bar{\phi}(u).
\end{align*}
1.3. Hecke correspondences, Serre duality and the symmetry argument

In this section, we prove that the polynomials $\chi_r^\vee(k; \lambda, \mu)$ and $\chi_r^\vee(k; \lambda, \mu)$ (cf. §1) satisfy the same antisymmetries as the polynomials $R_r^\vee$ and $R_r^\vee$ (cf. Fact 2).

In [SzT, §7.1] (c.f. also §4.4) we describe the tautological variant of the Hecke correspondence which identifies the moduli spaces of parabolic bundles with different degrees and weights. Applying the Hecke correspondence at the point $p$ and $q$ to $P_0(c > a)$ and $P_0(c < a)$ respectively, we can identify these spaces as $\mathbb{P}^1 \times \mathbb{P}^1$-bundles over the moduli space $N_{-1}$ of stable bundles of degree $-1$ (cf. [SzT, Lemma 9.3]):

$$\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow P_0(c > a) \rightarrow N_{-1} \leftarrow P_0(c < a) \leftarrow \mathbb{P}^1 \times \mathbb{P}^1.$$ 

For each copy of $\mathbb{P}^1$, the moduli space $P_0(c > a)$ can be considered as a $\mathbb{P}^1$-bundle; applying Serre duality for families of curves as in [SzT, §5] we obtain the following two equalities:

$$\chi_r^\vee(k; \lambda, \mu) = -\chi_r^\vee(k; \lambda, -\mu - 1)$$

and

$$\chi_r^\vee(k; \lambda, \mu) = -\chi_r^\vee(k; -\lambda + k + 1 - (\nu_1 + \nu_2), \mu) + \sum_{i=0}^{\nu_1 - \nu_2} (\nu_1 - \nu_2 - 2i)\chi(P_0(c > a), \mathcal{L}(k; \lambda + \nu_1 - i, \mu)).$$

Similarly, for $P_0(c < a)$ we obtain that

$$\chi_r^\vee(k; \lambda, \mu) = -\chi_r^\vee(k; -\lambda - (\nu_1 + \nu_2) - 1, \mu) = -\chi_r^\vee(k; \lambda, -\mu + k + 1) + \sum_{i=0}^{\nu_1 - \nu_2} (\nu_1 - \nu_2 - 2i)\chi(P_0(c > a), \mathcal{L}(k; \lambda + \nu_1 - i, \mu)).$$

We showed in [SzT, §9] that

$$\sum_{i=0}^{\nu_1 - \nu_2} (\nu_1 - \nu_2 - 2i)\chi(P_0(c > a), \mathcal{L}(k; \lambda + \nu_1 - i, \mu)) = (-1)^{9-1}(2k + 4)^9 \text{Res}_{u=0} \left( \frac{e^{u(\lambda + \mu + 1)} - e^{u(\lambda - \mu)}}{(2\sinh(\frac{u}{2}))^2 1 - e^{u(k+2)}} \right) du$$

and

$$\sum_{i=0}^{\nu_1 - \nu_2} (\nu_1 - \nu_2 - 2i)\chi(P_0(c < a), \mathcal{L}(k; \lambda + \nu_1 - i, \mu)) = (-1)^{9-1}(2k + 4)^9 \text{Res}_{u=0} \left( \frac{e^{u(\lambda + \mu + 1)} - e^{u(\lambda - \mu)}e^{u(\nu_1 + \nu_2)}}{(2\sinh(\frac{u}{2}))^2 1 - e^{u(k+2)}} \right) du,$$

hence the polynomials $\chi_r^\vee$ and $\chi_r^\vee$ satisfy the same antisymmetries as $R_r^\vee$ and $R_r^\vee$ (cf. Fact 2 on page 4).

1.4. Wall-crossing in moduli spaces

Our next step is to compare the difference $\chi_r^\vee - \chi_r^\vee$ with the difference $R_r^\vee - R_r^\vee$ from Fact 1 on page 4.

In [SzT, §5] we presented a simple formula for the wall-crossing difference in Geometric Invariant Theory. The formula has the form of a residue of an equivariant
Using (23), we obtain that the wall-crossing difference (6) is equal to
\[
\pi \text{Note that that } c \text{ is defined through the Künneth decomposition of } 
\]
\[
\hat{\pi} \text{be the Poincare bundle over } \text{Jac}
\]
\[
U \text{A simple calculation shows that the restriction}
\]
\[
Z \text{§9), the space } Z^0 \text{ over which we integrate is isomorphic to the Jacobian of degree-0}
\]
\[
one bundles on } C:
\]
\[
Z^0 \simeq \{ V = L \oplus L^{-1} \mid L \in \text{Jac}^0, L_p = F_1, L_q^{-1} = G_1 \}.
\]
We thus obtain the following expression for the wall-crossing difference:
\[
(6) \quad \chi^>_i(k; \lambda, \mu) - \chi^<_i(k; \lambda, \mu) =
\]
\[
(-1)^g \text{Res}_{u=0} \frac{\exp(u(\lambda - \mu))}{(2\sinh(u/2))^{2g}} \int_{\text{Jac}} e^{n(2k+4)} \text{ch}(\pi_i(U_v \otimes K^1_i)|_{\text{Jac}}) \text{ du},
\]
where \( u \) plays the role of the equivariant parameter, the generator of \( H^*_C(\text{pt}) \); let \( J \)
be the Poincare bundle over \( \text{Jac} \times C \), satisfying \( c_1(J^p) = 0 \), then the class \( \eta \in H^2(\text{Jac}) \)
is defined through the Künneth decomposition of \( c_1(J)^2 \) (cf. page 16).

It follows from the Groethendieck-Riemann-Roch theorem that
\[
\text{ch}(\pi_i(U_v \otimes K^1_i)) = \pi_* \text{ch}(U_v).
\]
A simple calculation shows that the restriction \( U_v|_{Z^0} = J \oplus J^{-1} \) has \( C^* \)-weight 1, hence we have
\[
\text{ch}(U_v|_{Z^0}) = \bigoplus_{i=0}^{\nu_1-\nu_2} \text{ch}(\hat{J}^{\nu_1-\nu_2-2i}) \exp((\nu_1-i)u).
\]
Note that that \( \pi_* \text{ch}(J^n) = -n^2 \eta \), and thus
\[
\pi_* \text{ch}(U_v|_{Z^0}) = -\eta \sum_{i=0}^{\nu_1-\nu_2} \nu_1 - \nu_2 - 2i \exp((\nu_1-i)u) = -\eta \exp((\nu_1+\nu_2)u/2)\hat{\phi}(u).
\]
Using (23), we obtain that the wall-crossing difference (6) is equal to
\[
(7) \quad g(-2k+4)^{g-1} \text{Res}_{u=0} \frac{\exp(u(\lambda - \mu))e^{u(\nu_1+\nu_2)/2}\hat{\phi}(u)}{(2\sinh(u/2))^{2g}} \text{ du},
\]
and thus we have (cf. Fact 1 on page 4)
\[
(8) \quad R^>_i - R^<_i = \chi^>_i - \chi^<_i.
\]
Now we are ready for the final argument: we can rearrange equation (8) to describe the equality of wall-crossings as
\[
(9) \quad R^>_i(k; \lambda, \mu) - \chi^>_i(k; \lambda, \mu) = R^>_i(k; \lambda, \mu) - \chi^<_i(k; \lambda, \mu);
\]
we introduce the notation \( \Theta(k; \lambda, \mu) \) for this polynomial. Then \( \Theta(k; \lambda, \mu) \) satisfies 4
antisymmetries:
\[
\Theta(k; \lambda, \mu) = -\Theta(k; -\lambda, -\mu - 1) = -\Theta(k; -\lambda + k + 1, -\nu_1 - \nu_2, \mu) =
\]
\[
-\Theta(k; -\lambda - 1, -\nu_1 - \nu_2, \mu) = -\Theta(k; \lambda, -\mu + k + 1),
\]
hence it is anti-invariant with respect to the affine Weyl group action on \( \lambda \) and \( \mu \)
separately, and this implies \( \Theta = 0 \).

As \( P_0(c > a) \) is a \( \mathbb{P}^1 \)-bundle over the moduli space of rank-2 degree-0 stable
parabolic bundles \( P_0(c) \), substituting \( \mu = 0 \) in \( R^>_i \) and taking the derivative with
respect to $\delta$, we obtain the formula for rank 2:

\[
\chi(P_0(c), L(k; \lambda) \otimes \pi_i(U_\nu \otimes K^1)) = 
\exp\left(\frac{u (\lambda + 1 + 2)}{2 \sinh \left(\frac{u}{2}\right)} \right) \left( \frac{\varphi(\nu)}{2k + 4} + \frac{e^{(2k+4)u} \phi(\nu)}{(1 - e^{u(k+2)})} \right) \, du.
\]

1.5. Contents of the paper

The paper is organized as follows. We start with a quick overview of the theory of parabolic bundles in §2.1–§2.2; here we describe the vector bundles we are considering and introduce the chamber structure on the space of parabolic weights. In §2.3–§2.4 we briefly recall the notion of diagonal bases of hyperplane arrangements first introduced in [Sz1]. Using this object, in §3.1 we present our main result, Theorem 3.3. The proof of this theorem takes up the rest of the paper.

In §3.2 (cf. Corollary 3.8) we calculate the wall-crossing difference in residue formulas (general version of Fact 1 on page 4). In §4 we apply the formula for wall-crossings in GIT [SzT, Theorem 5.6] to the moduli space of parabolic bundles with different parabolic weights. We obtain Proposition 4.7, the higher rank version of formula (7) above.

In §5.1 we derive Weyl antisymmetries for the polynomials $\chi_\nu$, $\chi_\nu'$ and in §5.2–§5.3 we show the same antisymmetries for the polynomials $R_\nu$, $R_\nu'$. In §6.1 we finish the proof following the idea described above in §1.4. We end the paper with a mild generalizations (cf. §6.2) of our main result.

2. Preliminaries

2.1. Parabolic bundles

In this section, we briefly review the definition of parabolic bundles, repeat the basic facts about their moduli spaces from [SzT, §2] and describe the chamber structure on the space of the relevant parameters, known as parabolic weights.

Let $C$ be a smooth complex projective curve of genus $g \geq 2$, fix a point $p \in C$ and a positive integer $r$. A parabolic bundle on $C$ is a vector bundle $W$ of rank $r$ equipped with a full flag $F_*$ in the fiber over $p$:

\[
W_p = F_r \supseteq \ldots \supseteq F_1 \supseteq F_0 = 0,
\]

and parabolic weights $c = (c_1, ..., c_r)$ assigned to $F_r, F_{r-1}, ..., F_1$, satisfying the conditions

\[
c_1 > c_2 > \ldots > c_r \text{ and } c_1 - c_1 < 1.
\]

The parabolic degree of $W$ is defined as

\[
\text{pardeg}(W) = \deg(W) - \sum_{i=1}^r c_i.
\]

Any subbundle $W'$ of a parabolic bundle $W$ and the corresponding quotient $W/W'$ inherit a parabolic structure in a natural way (cf. [MS], definition 1.7).

The parabolic bundle $W$ is stable, if any proper parabolic subbundle $W' \subset W$ satisfies

\[
\frac{\text{pardeg}(W')}{\text{rk}(W')} < \frac{\text{pardeg}(W)}{\text{rk}(W)}.
\]
Note that the parabolic stability condition depends on parabolic weights only up to adding the same constant to all weights $c_i$, so we can assume that for fixed rank $r$ and degree $d$, the space of all values for the weights $c$ is the simplex

$$\Delta_d = \left\{ (c_1, c_2, ..., c_r) \mid c_1 > c_2 > ... > c_r, c_1 - c_r < 1, \sum_i c_i = d \right\}.$$

**Definition:** We will call a vector $c = (c_1, ..., c_r) \in \mathbb{R}^r$ such that $\sum_i c_i \in \mathbb{Z}$ regular if for any nontrivial subset $\Psi \subset \{1, 2, ..., r\}$, we have $\sum_{i \in \Psi} c_i \notin \mathbb{Z}$.

For fixed rank $r$, degree $d$ and regular $c = (c_1, ..., c_r) \in \Delta_d$, Mehta and Seshadri [MS] constructed a smooth projective moduli space of stable parabolic bundles $\tilde{P}_d(c)$, whose points are in one-to-one correspondence with the set of isomorphism classes of stable parabolic bundles of weight $c$.

Via the determinant map, the moduli space $\tilde{P}_d(c)$ fibers over the Jacobian of degree-$d$ line bundles on $C$ with isomorphic fibers. In this paper, we will focus on these fibers, the moduli space

$$P_d(c) = \{ W \in \tilde{P}_d(c) \mid \text{det} W \cong \mathcal{O}(dp) \},$$

which is smooth and projective of dimension $(r^2 - 1)(g - 1) + \binom{r}{2}$.

In [SZT] §2.4 we described a set of affine hyperplanes in $\Delta_d$, called walls, parametrized by a nontrivial partition $\Pi = (\Pi', \Pi'')$ of the first $r$ positive integers, and a pair of numbers $d', d''$, such that $d' + d'' = d$. We have shown that $\Delta_d$ is separated by these walls into a finite number of chambers, such that the moduli spaces $P_d(c)$ remain unchanged when varying $c$ within a chamber.

**Example 1.** Consider the case of rank-3 degree-0 stable parabolic bundles with parabolic weights $c = (c_1, c_2, c_3) \in \Delta$. As observed in [SZT] Example 1, in this case $\Delta$ is an open triangle (cf. Figure 2) and there exist only two different stability conditions. The wall separating the two chambers is given by the condition $c_2 = 0$. We write $P_0(\geq)$ for the moduli space $P_0(c_1, c_2, c_3)$ with $c_2 > 0$, and $P_0(\prec)$ for $P_0(c_1, c_2, c_3)$ with $c_2 < 0$.

2.2. **Vector bundles on the moduli space of parabolic bundles**

For a regular $c \in \Delta_d$, there exists a universal bundle $U$ over $P_d(c) \times C$, endowed with a flag $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = U_p$, and satisfying the obvious tautological properties. Such universal bundle $U$, and hence the flag line bundles $\mathcal{F}_i/\mathcal{F}_1$ are unique only up to tensoring by the pull-back of a line bundle from $P_d(c)$.

**Definition:** We will say that $U$ is normalized if the line subbundle $\mathcal{F}_1 \subset U_p$ is trivial.

For $k \in \mathbb{Z}$ and $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}^r$, such that $\sum_{i=1}^r \lambda_i = kd$, we define the line bundle

$$L_d(k; \lambda) = \text{det}(U_p)^{k(1-g)} \otimes \text{det}(\pi_\ast U)^{-k} \otimes (\mathcal{F}_r/\mathcal{F}_{r-1})^{\lambda_1} \otimes \cdots \otimes (\mathcal{F}_1)^{\lambda_r}.$$
on $P_d(c)$. It is easy to check that this line bundle is independent of the choice of the universal bundle $U$.

**Notation:** In this paper we will mostly consider degree-0 parabolic bundles, so for $d = 0$, we will omit the index $d$ from the line bundle $\mathcal{L}(k;\lambda)$ and the space of parabolic weights $\Delta$.

Let $\nu = (\nu_1, \ldots, \nu_r)$ be a dominant weight of $GL_r$, consider the irreducible representation $\rho_\nu$ with highest weight $\nu$, and denote by $\bar{\rho}_\nu$ its restriction to the subgroup $SU_r \subset GL_r$. We denote by $\phi^\nu$ the character $\phi^\nu = \text{trace}(\bar{\rho}_\nu \circ \text{Exp})$ on the Lie algebra $V$ of a maximal torus $T \subset SU_r$. We collect our maps on the following diagram.

$$
\begin{array}{c}
\text{GL}_r \\
\downarrow \\
\text{Exp}
\end{array}
\begin{array}{c}
\rho_\nu \\
\rho_\nu
\end{array}
\begin{array}{c}
\text{GL}(V_\nu) \\
T \subset SU_r
\end{array}
$$

Given a representation $\rho_\nu$ of $GL_r$, we denote by $U_\nu$ the vector bundle over $P_0(c) \times C$ associated to the principal $GL_r$-bundle.

The vector bundle $U_\nu$ has the following explicit construction. Let $U$ be the normalized universal bundle on $P_0(c) \times C$, and consider the full flag bundle $\text{Flag}(U) \rightarrow P_0(c) \times C$. Denote by $L_1, \ldots, L_r$ the standard quotient line bundles on $\text{Flag}(U)$. Then

$$
U_\nu = f_*(L_1^{\nu_1} \otimes L_2^{\nu_2} \otimes \cdots \otimes L_r^{\nu_r}).
$$

**Remark 2.1.** Note that the vector bundles $\mathcal{F}_r, \ldots, \mathcal{F}_1$ on the moduli space $P_0(c)$ define a section of the flag bundle $\text{Flag}(U_p) \rightarrow P_0(c) \times \{p\}$.

### 2.3. Notation

Following [SZT], we set up some extra notation for the space of parabolic weights.

- We represent the Cartan subalgebra $V = \text{Lie}(T)$ of the Lie algebra $\text{Lie}(SU_r)$ as the quotient vector space
  $$
  V = \mathbb{R}^r/\mathbb{R}(1, 1, \ldots, 1).
  $$

  There is a natural pairing between $V$ and

  $$
  V^* = \{a = (a_1, \ldots, a_r) \in \mathbb{R}^r \mid a_1 + \cdots + a_r = 0\}.
  $$

  Let $x_1, x_2, \ldots, x_r$ be the coordinates on $\mathbb{R}^r$; given $a \in V^*$, we will write $\langle a, x \rangle$ for the linear function $\sum_i a_ix_i$ on $V$. We will sometimes denote this linear function simply by $a$.

- Let $\Lambda$ be the integer lattice in the vector space $V^*$:
  $$
  \Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 + \cdots + \lambda_r = 0\}.
  $$

For $1 \leq i \neq j \leq r$, we define the element $\alpha^{ij}_i = x_i - x_j$ in $\Lambda$. Let

$$
\Phi = \{ \pm \alpha^{ij} \mid 1 \leq i < j \leq r\}
$$

be the set of roots of the $A_{r-1}$ root system with the opposite roots identified. Note that the permutation group $\Sigma_r$ acts on the vector space $V^*$, permuting the coordinates $x_i$, and the elements of $\Phi$.

- A basic object of our approach is an ordered linear basis $B$ of $V^*$ consisting of the elements of $\Phi$. Let us denote the set of these objects by $\mathcal{B}$:

$$
\mathcal{B} = \left\{ B = \left( \beta^{[1]}, \ldots, \beta^{[r-1]} \right) \in \Phi^{r-1} \mid B \text{ basis of } V^* \right\}
$$
• Given a basis $\mathcal{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B}$ of $V^*$, and an element $a \in V^*$, we define $[a]_\mathcal{B} \in \Lambda$ to be the unique element of $V^*$ satisfying $[a]_\mathcal{B} = a - \{a\}_\mathcal{B}$, where $\{a\}_\mathcal{B} \in \sum_{i=1}^{r-1} (0, 1) \beta^{[i]}$.

• We call $a \in V^*$ regular, if $[a]_\mathcal{B} \in \sum_{i=1}^{r-1} (0, 1) \beta^{[i]}$. Then for regular elements $a$ and $b$, we define the equivalence relation $a \sim b$ when $[a]_\mathcal{B} = [b]_\mathcal{B} \forall \mathcal{B} \in \mathcal{B}$.

• Given a partition $\Pi$ of $\{1, 2, \ldots, r\}$ into two nonempty sets, we will think of it as an ordered partition $\Pi = (\Pi', \Pi'')$ such that $r \in \Pi''$, and we will call these objects nontrivial partitions.

• Then [SzT, Lemma 4.1] the equivalence classes of the relation $\sim$ are precisely the chambers in $V^*$ created by the walls parameterized by a nontrivial partition $\Pi = (\Pi', \Pi'')$ of the first $r$ positive integers, and an integer $l$:

$$ S_{\Pi, l} = \{c \in V^* | \sum_{j \in \Pi'} c_j = l\}. $$

If $d = 0$, the set of parabolic weights $\Delta$ defined on page 8 can be considered as an open simplex in $V^*$. Then the intersection of the walls given in (12) with $\Delta$ are precisely the walls separating the chambers of parabolic weights $c$, in which the moduli spaces $P_0(c)$ of parabolic bundles are naturally the same (cf. end of §2.1).

2.4. Diagonal bases

A key component of our approach is the notion of diagonal basis introduced in [Sz1]. We refer to [Sz1], [Sz2] and [SzT, §3] for basic examples and results on diagonal bases; now we will briefly recall the combinatorial definition of this object.

• If we consider $\Phi$ as the edges of the complete graph on $r$ vertices, then the set $\mathcal{B}$ is the set of spanning trees of this graph with edges enumerated from 1 to $r - 1$. We denote these ordered trees by

$$ \mathcal{B} \mapsto \text{Tree}(\mathcal{B}). $$

• Given Tree($\mathcal{B}$), we have a sequence of $r$ nested partitions of the vertices, which starts with the total partition into 1-element sets, ends with the trivial partition, and the $j$th partition is induced by the first $j - 1$ edges. A diagonal basis $\mathcal{D} \subset \mathcal{B}$ is then a set of $(r - 1)!$ ordered trees such that the $(r - 1)!$ partition sequences obtained by reordering the edges of any one of the ordered trees are different from $(r - 1)! - 1$ sequences of partitions obtained from the remaining elements of $\mathcal{D}$.

Example 2. For $r = 3$, $\mathcal{D} = \{(\alpha^{2.3}, \alpha^{1.2}), (\alpha^{3.2}, \alpha^{1.3})\}$ is a diagonal basis.

3. Main result and wall-crossing in residue formulas

3.1. Main result

In this section, we formulate our main result, Theorem 3.3.

We introduce the notation $\mathcal{F}_{\Theta}$ for the space of meromorphic functions defined in a neighborhood of 0 in $V \otimes_{\mathbb{R}} \mathbb{C}$ with poles on the union of hyperplanes

$$ \bigcup_{1 \leq i < j \leq r} \{x | \langle x^i, x^j \rangle = 0\}. $$
In particular, the inverse of
\[ w_\Phi \overset{\text{def}}{=} \prod_{i<j} (2\sinh(x_i - x_j)) \]
is a function in \( \mathcal{F}_\Phi \). Any basis \( \mathbf{B} \in \mathcal{B} \) induces an isomorphism \( V^* \cong V \), and we will write \( \alpha_\Phi \) for the image of \( \alpha \in V^* \) under this isomorphism. We will sometimes omit the index \( \mathbf{B} \) to simplify the notation. For a function \( Q \) on \( V \) and \( \alpha \in V^* \), we introduce the directional derivative \( Q_{\alpha_\Phi} \). Then, given \( Q \) and \( \mathbf{B} \in \mathcal{B} \), we fix a homomorphism \( \alpha \mapsto \exp(Q_{\alpha_\Phi}) \)
from the additive group \( V^* \) to the multiplicative group of non-vanishing holomorphic functions on \( V \otimes_R C \).

Let \( K = \frac{1}{2\pi} \sum_{i<j} \alpha_{ij}^2 \) be the normalized Killing form of \( SU_r \) and let \( \delta \in \mathbb{R} \) be a small parameter; given a basis \( \mathbf{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B} \) of \( V^* \), a function \( f \in \mathcal{F}_\Phi \) and a holomorphic function \( Q = \text{const} \cdot K - \delta \phi \), defined in a neighborhood of 0 in \( V \otimes_R C \), we define
\[ \text{iBer}_{\mathbf{B}, Q}[f(x)](\alpha) \overset{\text{def}}{=} \int_{Z_\mathbf{B}} \frac{f(x) \exp(Q_{\alpha_\Phi}) \prod dQ_{\beta^{[1]}} \cdots dQ_{\beta^{[r-1]}}}{(1 - \exp(Q_{\beta^{[1]}})) \cdots (1 - \exp(Q_{\beta^{[r-1]}}))}, \]
where the naturally oriented cycle \( Z_\mathbf{B} \) is given by
\[ Z_\mathbf{B} = \{ x \in V \otimes_R C : (\beta^{[1]}, x) = \varepsilon_j, j = 1, \ldots, r - 1 \} \subset V \otimes_R C \setminus \{ w_\Phi(x) = 0 \} \]
with sufficiently small fixed real constants \( \varepsilon_j \) satisfying \( 0 \leq \varepsilon_{r-1} \ll \cdots \ll \varepsilon_1 \).

Thus, \( \text{iBer}_{\mathbf{B}, Q} \) is a linear operator associating to a meromorphic function \( f \in \mathcal{F}_\Phi \) a polynomial on \( V^* \).

We introduce the notation \( \mathcal{H}^\Phi \) for the space of holomorphic functions of the form \( Q = \text{const} \cdot K - \delta \phi \), defined in a neighborhood of 0 in \( V \otimes_R C \). We will always assume that our parameter \( \delta \) is small enough, so that the cycle given by \( \{ x \in V \otimes_R C : (\beta^{[1]}, x) = \varepsilon_j, j = 1, \ldots, r - 1 \} \subset V \otimes_R C \setminus \{ w_\Phi(x) = 0 \} \) is homotopic to the cycle \( Z_\mathbf{B} \).

**Notation:** We will write \( \text{iBer} \) for \( \text{iBer}_{\mathbf{B}, K} \) to simplify the notation. Now we are ready to recall the residue formula proved in [SzT].

**Theorem 3.1.** Let \( c \in \Delta \) be a regular element, which thus specifies a chamber in \( \Delta \) and a parabolic moduli space \( P_0(c) \). Then for a diagonal basis \( \mathcal{D} \), an arbitrary element \( \lambda \in \Lambda \), and a positive integer \( k \), the Euler characteristic of the line bundle \( \mathbf{L}(k; \lambda) \) (cf. §2.1) is equal to
\[ \chi(P_0(c), \mathbf{L}(k; \lambda)) = N_{\tau, k} \cdot \sum_{B \in \mathcal{D}} \text{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x; \mathbf{k}) \exp(\hat{\lambda}/\mathbf{k}; x)](-[c]_\mathbf{B}), \]
where \( N_{\tau, k} = (-1)^r(2) \tau^r(k+r)^{r-1} \), \( \hat{\lambda} = \lambda + \rho \) and \( \mathbf{k} = k + r \).

We will need the following property of the operator \( \text{iBer}_{\mathbf{B}, Q} \).

**Lemma 3.2.** Let \( Q = (k + r)K - \delta \phi \in \mathcal{H}^\Phi \), then for any vector \( w \in \Lambda \) and a function \( f \in \mathcal{F}_\Phi \), which depends on \( \delta \), we have
\[ \frac{\partial}{\partial \delta} \bigg|_{\delta = 0} \text{iBer}_{\mathbf{B}, Q}[f(x)](a + w) = \]
\[ \frac{\partial}{\partial \delta} \bigg|_{\delta = 0} \text{iBer}_{\mathbf{B}, Q}[f(x) \exp((k + r)w)](a) - \text{iBer}_{\mathbf{B}, (k+r)}[f(x)](a). \]
Proof. Note that
\[
i\text{Ber}_B[f(x)](a + w) = i\text{Ber}_B[f(x)\exp(Q, w)](a) =
\]
\[
i\text{Ber}_B[f(x)\exp((k + r)w - \delta\phi(x))] (a);
\]
then taking the derivative with respect to \(\delta\) at zero, we obtain the result. \qed

We are now ready to give the formula for the Euler characteristic of other vector bundles on the moduli spaces.

**Theorem 3.3.** Let \(\mathcal{K}\) be the canonical class of the curve \(C, \lambda \in \Lambda, k \in \mathbb{Z}_{>0}, \nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r) \in \mathbb{Z}_+^r, \hat{\lambda} = \lambda + \rho, \nu_{\text{det}} = (1, \ldots, 1, 1 - r) \sum_{j=1}^r \nu_j, Q = (k + r)K - \delta\phi^\nu \in \mathcal{H}^\Phi\) and let \(c \in \Delta\) be a regular element (cf. page 32). Then for any diagonal basis \(D \in \mathcal{B}\), the following equality holds:

\[
\chi(P_0(c), \mathcal{L}(k; \nu) \otimes \pi_i(U, \mathcal{K}^{\frac{1}{2}})) = 
\]
\[
N_r \cdot \sum_{B \in \mathcal{D}} \frac{\partial}{\partial \delta} \mid_{\delta = 0} i\text{Ber}_B \left[ \text{Hess}(Q(x))^{g-1}w^{1-2g}(x) \exp((\hat{\lambda} + \nu_{\text{det}}, x)) \right] (-[c]_B),
\]
where \(N_r = (-1)^{\binom{r}{2}}(g-1)^r r^g\).

Taking the derivative with respect to \(\delta\), we obtain the following explicit formulas.

**Corollary 3.4.** Let \(\lambda, k, \nu, Q, c\) and \(N_{r,k}\) be as above, then

\[
\chi(P_0(c), \mathcal{L}(k; \nu) \otimes \pi_i(U, \mathcal{K}^{\frac{1}{2}})) =
\]
\[
N_{r,k} \sum_{B \in \mathcal{D}} i\text{Ber}_B \left[ w^{1-2g}(x/k) \exp((\hat{\lambda} + \nu_{\text{det}}, x/k)) \left( \frac{-g}{k + r} \text{tr}(\text{Hess}(\phi^\nu(x/k))) - \sum_{i} \phi^{\nu}_{\beta[i]}(x/k) \exp(\langle \beta[i], x \rangle) \frac{1}{1 - \exp(\langle \beta[i], x \rangle)} + \sum_{i} \langle [c], \beta[i] \rangle \phi^{\nu}_{\beta[i]}(x/k) \right) \right] (-[c]_B).
\]

**Remark 3.5.** As explained in [SZT, Remark 4.3], the operator \(i\text{Ber}_B\) may be written as an iterated residue: for \(i = 1, \ldots, r - 1\) we define \(y_i = \langle \beta[i], x \rangle\) and write \(f\) and \(a\) in these coordinates: \(f(x) = \hat{f}(y), \langle a, x \rangle = \langle \hat{a}, y \rangle\). Then

\[
i\text{Ber}_B[f(x)](a) = \text{Res}_{y_i=0} \cdots \text{Res}_{y_{r-1}=0} \hat{f}(y) \exp(\langle \hat{a}, y \rangle) \frac{\text{dy}_1 \wedge \cdots \wedge \text{dy}_{r-1}}{(1 - \exp(y_1)) \cdots (1 - \exp(y_{r-1}))},
\]
where iterating the residues means that at each step we keep the variables with lower indices as unknown constants.

**Example 3.** Denote by \(U\) the normalized universal bundle on the moduli spaces of rank-3 parabolic bundles \(P_0(\rangle\) and \(P_0(\langle)\) defined in Example 1. We have \(U \simeq U_\nu\) for \(\nu = (1, 0, 0)\) and

\[
\phi(\alpha^{12}, \alpha^{23}) = e^{\frac{2\alpha^{12} \cdot \alpha^{23}}{3}} + e^{\frac{-\alpha^{23} \cdot \alpha^{12}}{3}} + e^{\frac{-\alpha^{12} \cdot \alpha^{23}}{3}}; \quad \phi_{\delta^{12}}(\alpha^{12}, \alpha^{23}) = e^{\frac{2\alpha^{12} + \alpha^{23}}{3}} - e^{\frac{-\alpha^{23} - \alpha^{12}}{3}};
\]
\[
\phi_{\delta^{23}}(\alpha^{12}, \alpha^{23}) = e^{\frac{-\alpha^{23} \cdot \alpha^{12}}{3}} - e^{\frac{-\alpha^{12} \cdot \alpha^{23}}{3}}; \quad \text{tr}(\text{Hess}(\phi(\alpha^{12}, \alpha^{23}))) = \frac{2}{3} \phi(\alpha^{12}, \alpha^{23}).
\]
Let $\mathcal{D}$ be the diagonal basis from Example 2, writing the operator $i\text{Ber}_B$ for $B \in \mathcal{D}$ in the variables $(x, y)$ as explained in Remark 3.3 and using [SJT] Remark 4.6, we obtain

$$
\begin{align*}
\chi(P_0(<), \mathcal{L}(k; \lambda) \otimes \tau_\chi(U \otimes \mathfrak{K}^\perp)) &= \frac{N \cdot \text{Res}}{y = 0, x = 0} \frac{e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x + y + y^{2g-1}} - e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x + y^{2g-1}}}{(1 - e^{x(k+3)})(1 - e^{y(k+3)})w_\Phi(x, y)^{2g-1}} \cdot \left( \frac{2g}{3(k+3)} \phi(x, y) + \frac{e^{(k+3)\phi_x}(x, y)}{(1 - e^{(k+3)x})} + \frac{e^{(k+3)y} \phi_y(x, y)}{(1 - e^{(k+3)y})} \right) dx dy \end{align*}
$$
and

$$
\begin{align*}
\chi(P_0(>), \mathcal{L}(k; \lambda) \otimes \tau_\chi(U \otimes \mathfrak{K}^\perp)) &= \frac{N \cdot \text{Res}}{y = 0, x = 0} \frac{e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x + y + y^{2g-1}} - e^{\lambda_1 x + (\lambda_1 + \lambda_2) y + x + y^{2g-1}}}{(1 - e^{x(k+3)})(1 - e^{y(k+3)})w_\Phi(x, y)^{2g-1}} \cdot \left( \frac{2g}{3(k+3)} \phi(x, y) + \frac{e^{(k+3)\phi_x}(x, y)}{(1 - e^{(k+3)x})} + \frac{e^{(k+3)y} \phi_y(x, y)}{(1 - e^{(k+3)y})} \right) dx dy \end{align*}
$$

where $w_\Phi(x, y) = 2 \sinh(\frac{2}{3}y)2 \sinh(\frac{2}{3}y)2 \sinh(\frac{2}{3}y)$ and $N = (-1)^g(3(k+3)^2)^g$. One can compare these formulas with the ones from [SJT] Example 4.

### 3.2. Wall-crossing in residue formulas

We start the proof of Theorem 3.3 following the strategy described in §1. Our first step is to calculate the wall-crossing terms of the residue expressions from Theorem 3.3. We choose two regular elements $c^+, c^- \in \Delta$ in two neighbouring chambers separated by the wall $S_{\Pi, \perp}$ (cf. (12)) such that

$$
[c^+_\Pi] = 1 \text{ and } [c^-_{\Pi}] = 1 - 1,
$$

where we use the notation $c_{\Pi'} := \sum_{c \in \Pi'} c_\lambda$ for $c \in \Delta$. We denote by

$$
R^\gamma_+(k, \lambda) = N_{r} \cdot \frac{\partial}{\partial \xi} \bigg|_{\xi = 0} \sum_{B \in \mathcal{D}} \sum_{B \in \mathcal{D}} i\text{Ber}_{B, Q} \left[ \text{Hess}(Q(x))^{-1} w_{\Phi}(x)^{-2g} \exp(\lambda + \nu_{\text{det}}, x) \right] \left( -[c^+ B] \right)
$$

the two polynomials in $(k, \lambda) \in \mathbb{Z}_{>0} \times \Delta$. Then the wall-crossing term in the residue formula is the difference

$$
R^\gamma_+(k, \lambda) - R^\gamma_-(k, \lambda).
$$

Using [SJT] Lemma 4.11, we obtain the following expression for this difference.

**Lemma 3.6.** Let $(\Pi, 1)$ and $c^+, c^- \in \Delta$ be as above, and fix a diagonal basis $\mathcal{D} \subset \mathcal{B}$. Denote by $\mathcal{D}\big|\Pi$ the subset of those elements $B$ of $\mathcal{D}$ for which Tree$(B)$ (cf. §2.3) is a union of a tree on $\Pi'$, a tree on $\Pi''$ and a single edge $\beta_{\text{link}}$ (which we will call the link) connecting $\Pi'$ and $\Pi''$. Then

$$
R^\gamma_+(k, \lambda) - R^\gamma_-(k, \lambda) = N_{r} \cdot \frac{\partial}{\partial \xi} \bigg|_{\xi = 0} \sum_{B \in \mathcal{D}\big|\Pi} i\text{Ber}_{B, Q} \left[ (1 - \exp(Q_{\beta_{\text{link}}}(x))) \text{Hess}(Q)^{-1} w_{\Phi}(x)^{-2g} \exp(\lambda + \nu_{\text{det}}, x) \right] \left( -[c^+ B] \right)
$$
Remark 3.7. Note that the multiplication by $(1 - \exp(Q \Phi_{\text{link}}(x)))$ in Lemma 3.6 has the effect of canceling one of the factors in the denominator in the definition (13) of the operation $iBer$.

As observed in [SzT], even though this difference does not depend on the choice of $D$, it is convenient to choose a particular diagonal basis (cf. [SzT] page 19). Introducing the notation $\Phi'$ and $\Phi''$ for the $A_{r'}$ and $A_{r''}$ root systems corresponding to $\Pi'$ and $\Pi''$, using [SzT] Lemma 4.15 and taking the derivative with respect to $\delta$ at $\delta = 0$, we arrive at the following statement.

Corollary 3.8. Let $D'$ and $D''$ be diagonal bases of $\Phi'$ and $\Phi''$ correspondingly. Then

$$R_\gamma(x, \lambda) - R_\gamma(x, \lambda) = (k + r)N_{r,k} \sum_{B' \in D'} \sum_{B'' \in D''} \text{Res}_{\beta_{\text{link}} = 0} \frac{\text{iBer}_{B'} \text{iBer}_{B''}}{B'}$$

$$\left[ w^{\gamma - 2g}(x, \hat{k}) \exp(\lambda + v_{\text{det}, x}(\hat{k})) \left( \frac{\beta - \text{Hess}(\phi(x, k))}{k + r} \right) + \frac{\phi(x, \hat{k})}{1 - \exp(\beta_{[1]}(x, \hat{k}))} \right] \frac{\sum_{i \in \Pi'} \langle c^+, \beta_{[1]}(x, \hat{k}) \rangle}{x} \text{d}\beta_{\text{link}},$$

where $\text{Res}_{\beta_{\text{link}} = 0} \text{iBer}_{B'} \text{iBer}_{B''} \text{d}\beta_{\text{link}}$ is simply $\text{iBer}_{B}$ (cf. (13)) with $B$ obtained by appending $B'$, and then $B''$ to $\beta_{\text{link}}$, and the factor $(1 - \exp(\beta_{\text{link}}, x))$ removed from the denominator.

Example 4. Calculating the difference of the two polynomials from Example 3, we obtain the wall-crossing term:

$$-N \cdot \text{Res}_{y = 0} \frac{e^{\lambda x + (\lambda_1 + \lambda_2)y + x + \frac{y^2}{2} - 1}}{1 - e^{x(k + 3)}} \left( \frac{2g}{3(k + 3)} \phi(x, y) + \frac{e^{(k + 3)x} \phi(x, y)}{(1 - e^{(k + 3)x})} \right) \text{d}x \text{d}y.$$

4. Wall-crossing in Euler characteristics

In this section, we calculate the changes in Euler characteristics of vector bundles when varying the moduli spaces of parabolic bundles. The main result is Proposition 4.7, where we present explicit formulas for the wall-crossing terms for the left-hand side of (16).

4.1. Wall-crossing in master space

Fix the wall $S_{\Pi, l}$ given by an ordered partition $\Pi = (\Pi', \Pi'')$ of the first $r$ integers and an integer $l$, and two regular elements $c^+, c^- \in \Delta$ in two neighbouring chambers separated by the wall $S_{\Pi, l}$. Let

$$c' = \sum_{i \in \Pi'} x_i \text{ and } c'' = \sum_{i \in \Pi''} x_i.$$

Following Thaddeus [Th1], one can construct the "master space" $Z$ whose quotients, under different linearizations, by a fixed $C^\ast$-action, are the moduli spaces of $c^\pm$-stable parabolic bundles. In [SzT] [5], we showed that the elements $c^\pm$ may be chosen within their chambers so that $Z$ is a smooth, projective variety with a $C^\ast$-action, and identified the connected components of the fixed locus:

$$Z^{C^\ast} \simeq P_0(c^+) \sqcup P_0(c^-) \sqcup Z^0,$$
where $Z^0$ is the set of points representing rank-$r$ vector bundles $W$ on $C$, such that $W$ splits as a direct sum $W' \oplus W''$, where $W'$ and $W''$ are, respectively, $c'$ and $c''$-stable parabolic bundles of degree $l$ and $-l$, rank $r' = |\Pi'|$ and $r'' = |\Pi''|$

$$Z^0 = \{W = W' \oplus W'' \mid W' \in P_1(c'); W'' \in \tilde{P}_{-1}(c''); \det(W) \simeq 0\}.$$  

**Remark 4.1.** Note that $Z^0$ is fibered over $\text{Jac}^1$ with fibre $P_1(c') \times P_{-1}(c'')$ by the determinant map $P_1(c') \to \text{Jac}^1$, and 

$H^*(Z^0, \mathbb{Q}) \simeq H^*(P_1(c') \times P_{-1}(c''), \mathbb{Q}) \otimes H^*(\text{Jac}^1, \mathbb{Q})$. 

Consider the polynomials

$$\chi^\nu_{\pm}(k; \lambda) = \chi(P_0(c^\pm), \mathcal{L}(k; \lambda) \otimes \pi_{\nu}(\mathcal{U}_\nu \otimes \mathcal{X}^\pm)).$$

Our goal is to calculate the difference $\chi^\nu_{+}(k; \lambda) - \chi^\nu_{-}(k; \lambda)$.

Applying the Atiyah-Bott fixed-point formula to the master space $Z$ with the $\mathbb{C}^*$-action, we showed [SZT Theorem 5.6] that the wall-crossing polynomial $\chi^\nu_{+}(k; \lambda) - \chi^\nu_{-}(k; \lambda)$ is equal to

$$\frac{\text{Res}_{u=0} \int \frac{\text{ch}((\mathcal{L}(k; \lambda) \otimes \pi_{\nu}(\mathcal{U}_\nu \otimes \mathcal{X}^\pm))(Z^0_\mathbb{Z})}{E(\mathbb{N}_{Z^0})}\text{Todd}(Z^0)\ du,}$$

where $E(\mathbb{N}_{Z^0})$ is the K-theoretical Euler class of the conormal bundle of $Z^0$ in $Z$ and $u$ is an equivariant parameter.

Before we calculate this integral, we need to introduce some extra notations.

### 4.2. Restriction. Representations

For any weight $\nu = (\nu_1, ..., \nu_r)$ of $\text{GL}_r$, we define

$$|\nu| \overset{\text{def}}{=} \sum_i \nu_i;$$

the irreducible representation $\rho_{\nu}$ of $\text{GL}_r \simeq (\text{SL}_r \times \mathbb{C}^*)/\mathbb{Z}_r$ with highest weight $\nu$ can be decomposed by restriction as a product of the irreducible representation $\rho_{\nu}$ of $\text{SU}_r$ and the one-dimensional representation $\rho(|\nu|) : t \mapsto t^{\nu_1}$ of the center $Z(\text{GL}_r) \simeq \mathbb{C}^*$.

Let $\text{GL}_{r'} \times \text{GL}_{r''}$ be the subgroup of $\text{GL}_r$ induced by an ordered partition $(\Pi', \Pi'')$ of the first $r$ positive integers. The restriction of the irreducible representation $\rho_{\nu}$ of $\text{GL}_r$ decomposes as a direct sum of irreducible representations of $\text{GL}_{r'} \times \text{GL}_{r''}$:

$$\rho_{\nu} = \sum_{(\nu', \nu'')} \rho_{\nu'} \otimes \rho_{\nu''}. $$

Similarly, the restriction of the representation $\bar{\rho}_{\nu}$ to $\text{SU}_r \cap (\text{GL}_{r'} \times \text{GL}_{r''}) \subset \text{GL}_r$ can be decomposed as a direct sum

$$\bar{\rho}_{\nu} = \sum_{(\nu', \nu'')} \bar{\rho}_{\nu'} \otimes \bar{\rho}_{\nu''} \otimes \rho[rs] $$

of products of irreducible representations of $\text{SU}_{r'}$, $\text{SU}_{r''}$ and the one-dimensional torus $\mathbb{C}^* \simeq (Z(\text{GL}_{r'}) \times Z(\text{GL}_{r''})) \cap \text{SU}_r$, where $s = \sum_{i \in \Pi'} (|\nu'_i| - |\nu_1|/r)$. Let

$$w \overset{\text{def}}{=} \frac{s}{r'} \sum_{i \in \Pi'} x_i - \frac{s}{r''} \sum_{i \in \Pi''} x_i \in \mathbb{V}^a,$$
then the corresponding decomposition of character functions (cf. end of §2.2) has the form

$$\phi^\gamma = \sum_{(\nu',\nu'')} \phi^{\nu'} \phi^{\nu''} \exp(w).$$

Recall (cf. Lemma 3.6) that the a key role in our wall-crossing terms is played by the bases $B$ of $V^*$, obtained by appending $B'$, and then $B''$ to $\beta_{\text{link}}$. Using expression (19), we arrive at the following equalities for the directional derivatives of $\phi^\gamma$.

**Lemma 4.2.** Let $B = \beta_{\text{link}} B' B''$ be a basis of $V^*$ described above. Then:

(i) For any $\alpha \in B'$ and any $\beta \in B''$ we have

$$\phi^\alpha = \sum_{\nu'} \phi^{\nu'} \phi^{\nu''} \exp(w) \quad \text{and} \quad \phi^\beta = \sum_{\nu''} \phi^{\nu''} \phi^{\nu'} \exp(w);$$

(ii) $\hat{\beta}_{\text{link}} = \frac{\nu}{r} \sum_{i \in \Pi'} x_i - \frac{\nu}{r} \sum_{i \in \Pi''} x_i$, and thus

$$\phi^{\hat{\beta}_{\text{link}}} = \sum_{(\nu',\nu'')} \frac{\nu}{r} \frac{\nu}{r} \phi^{\nu'} \phi^{\nu''} \exp(w);$$

(iii) $\text{tr}(\text{Hess}(\phi^\gamma)) = \sum_{(\nu',\nu'')} (\text{tr}(\text{Hess}(\phi^{\nu'})) \phi^{\nu''} + \text{tr}(\text{Hess}(\phi^{\nu''})) \phi^{\nu'} + s^2 \left( \frac{r}{r' r''} \right) \phi^{\nu'} \phi^{\nu''} \exp(w).$

4.3. **Restriction. Bundles**

Recall that our goal is to calculate the integral (18); our first step is to identify the characteristic classes under this integral. We showed in [SzT, Theorem 6.11], that

$$\text{Res}_{\beta_{\text{link}} = 0} \iint_{B' B''} \text{Ber} [w \Phi(\lambda) \exp(-2 g \phi(\lambda, x))] (-[c^+] B) \text{d} \beta_{\text{link}},$$

where $\phi \in \Sigma_r$ is the unique permutation which sends $\{1, ..., r'\}$ to $\Pi'$ preserving the order of the first $r'$ and the last $r''$ elements. Now we study the restriction of the class $\text{ch}(\tau_r(U_v \otimes K^2))$ to $Z^0$.

Let $\omega \in H^2(C)$ be the fundamental class of our curve $C$, and $e_1, ..., e_{2g}$ a basis of $H^1(C)$, such that $e_i e_{i+g} = \omega$ for $1 \leq i \leq g$, and all other intersection numbers $e_i e_j$ equal 0. For a class $\gamma \in H^2(P \times C)$ of a product, we introduce the following notation for its Künneth components (cf. [VI]):

$$\gamma = \gamma(0) \otimes 1 + \sum_i \gamma(e_i) \otimes e_i + \gamma(2) \otimes \omega \in \bigoplus_{l=0}^2 H^{2-l}(P) \otimes H^l(C).$$

It follows from the Groethendieck-Riemann-Roch theorem that

$$\text{ch}(\tau_r(U_v \otimes K^2)) = \text{ch}(U_v)(2).$$

Recall our notation $\beta$ for the Poincare bundle over $\text{Jac} \times C$, such that $c_1(\beta)(0) = 0$ and an element $\eta \in H^2(\text{Jac})$ defined by $\left( \sum_i c_1(\beta)(e_i) \otimes e_i \right)^2 = -2 \eta \otimes \omega$; then (cf. [Z]) for any $m \in Z$

$$\int_{\text{Jac}} e^{\eta m} = m^g.$$
The following statement is straightforward.

**Lemma 4.3.** Denote by $U[l]'$ and by $U[-l]''$ the normalized universal bundles on the moduli spaces $P_l(c') \times C$ and $P_{-l}(c'') \times C$, correspondingly (cf. beginning of §3.2). Let $U[l]'$ and $U[-l]''$ be the associated vector bundles on $P_l(c') \times C$ and $P_{-l}(c'') \times C$ (cf. (11)). Then

$$\text{ch}(U_{\nu}|_{Z^0}) = \sum_{(\nu',\nu'')} \exp(\sum_U \nu'_i) \left( \text{ch}(U[l]'_{\nu'}) \boxtimes \text{ch}(U[-l]''_{\nu''}) \boxtimes \left( 1 + \left( \frac{\text{st}}{\tau'^r} - \left( \frac{\text{st}}{\tau'^r} \right)^2 \frac{\eta}{2} \right) \otimes \omega \right) \right).$$

Putting Lemma 4.3 and equation (22) together, we obtain the following statement.

**Lemma 4.4.** In the notation of Lemma 4.3

$$\text{ch}(\pi!([U_{\nu} \otimes \Omega^2]|_{Z^0})) = \sum_{(\nu',\nu'')} \exp(\sum_U \nu'_i) \left( \text{ch}(U[l]'_{\nu'})_{(2)} \boxtimes \text{ch}(U[-l]''_{\nu''})_{(0)} + \text{ch}(U[l]'_{\nu'})_{(0)} \boxtimes \text{ch}(U[-l]''_{\nu''})_{(2)} + \text{ch}(U[l]'_{\nu'})_{(0)} \boxtimes \text{ch}(U[-l]''_{\nu''})_{(0)} \boxtimes \left( 1 + \left( \frac{\text{st}}{\tau'^r} - \left( \frac{\text{st}}{\tau'^r} \right)^2 \frac{\eta}{2} \right) \otimes \omega \right) \right).$$

**Example 5.** It follows from Example 1 that in rank 3 case $\Pi' = \{1\}$, $\Pi'' = \{0\}$ and the fixed locus $Z^0$ is the set of vector bundles that split as a direct sum of rank-2 degree-0 stable parabolic bundle and a line bundle of degree 0. We denote by $\Pi''$ the normalized universal bundle on the moduli space $P_0$ of rank-2 stable parabolic bundles with trivial determinant. Then for the universal bundle $U$ from Example 3 the Chern character $\text{ch}(U_{\nu}|_{Z^0})$ has two summands:

- for $\nu' = (0), \nu'' = (1,0)$ we have $\text{ch}(U''(2)) = \frac{1}{2} \eta \text{ch}(U')(0)$;
- for $\nu' = (1), \nu'' = (0,0)$ we have $\text{ch}(U')(0)$.

**Remark 4.5.** Recall that in [SZT] §6.2 we identified the functions on $V$ with cohomology classes on $P_0(c)$ and an equivariant cohomology classes on $Z^0$. Under these identifications, $\text{ch}(U_{\nu'})_{(0)}$ corresponds to the function $\Phi'(\nu') \exp(\nu'_{\text{det},\nu'} \lambda)$ and $\text{ch}(U_{\nu''})_{(0)}$ corresponds to the function $\Phi''(\nu'') \exp(\nu''_{\text{det},\nu''} \nu)$.

Now our goal is to calculate the wall-crossing integral (18) applying induction by rank based on Theorem 3.3. Using (22), we can write the inductive hypothesis in the following form:

$$\int_{P_0(c)} \text{ch}(\mathcal{L}(k;\lambda)) \text{ch}(U_{\nu})_{(2)} \text{Todd}(P_0(c)) = \text{ch}(\mathcal{L}(k;\lambda)) \text{ch}(U_{\nu})_{(2)} \text{Todd}(P_0(c)).$$

Fixing $k$ and varying $\lambda$, we can extend this hypothesis by linearity to the following linear combinations of Chern characters of line bundles

$$\sum_{i} \text{ch}(\mathcal{L}(k;\lambda^i)) = \text{ch}(\mathcal{L}(k;0)) \cdot \sum_{i} \text{ch}(\mathcal{L}(0;\lambda^i)).$$

Since any polynomial on $V$, up to a fixed degree may be represented as a linear combination of exponential functions of the form $\exp(\nu \lambda)$, formula (24) may be generalized in the following way.
Lemma 4.6. Let \( G(x) \) be a formal power series on \( V \), and denote by \( G(z) \) the characteristic class in \( H^*(P_0(c)) \) obtained by the identification of functions on \( V \) and cohomology classes on \( P_0(c) \) (cf. Remark 4.5). Then

\[
\sum_{B \in \mathcal{D}} \text{iBer}_{\mathcal{B},Q} \left[ \text{Hess}(Q(x))^{g-1} G(x) w_{\phi}^{1-2g}(x) \exp\left( \tilde{\lambda} + \nu_{\text{det}}, x \right) \right] (-[c]_B).
\]

Armed with this statement and equality (20), we are ready to calculate the integral (18). We start with the case \( l = 0 \).

- Note that for \( l = 0 \), \([c^+] = [c'] + [c'']\).
- Then using the induction hypothesis (25) and Remark 4.5, we conclude that the first summand in Lemma 4.4 contributes

\[
(k + r)N_{r,k} \sum_{(\nu',\nu'')} \text{Res} \left( \sum_{B \in \mathcal{D}'} \sum_{B' \in \mathcal{D}''} \text{iBer}_{B',B''} \left[ \text{exp}(\nu_{\text{det}} + \nu''_{\text{det}}, x, \hat{k}) \right] \right)
\]

\[
\exp\left( \tilde{\lambda} + \nu_{\text{det}} + \nu''_{\text{det}}, x, \hat{k} \right) \left( \frac{-g}{k + r} \text{tr} \left( \text{Hess}(\phi^\nu(x, \hat{k})) \right) \phi^\nu(x, \hat{k}) - \sum_{\beta[1] \in B'} \phi^{\nu''}(x, \hat{k}) \right)
\]

\[
\frac{\phi_{\tilde{B}[1]}(x, \hat{k}) \text{exp}(\beta[1], x)}{1 - \text{exp}(\beta[1], x)} + \sum_{\beta[1] \in B'} \langle [c], \beta[1], \phi^\nu(x, \hat{k}) \rangle \right] (-[c^+]_B) d\nu,
\]

to the wall-crossing integral (18).

- Note that

\[
\langle \nu''_{\text{det}} + \nu''_{\text{det}}, x \rangle + (x_{\phi(r')} - x_r) \sum \nu_i = \nu_{\text{det}} + w_r,
\]

hence after the identification of \( u \) with \( x_{\phi(r')} - x_r \) justified in \([SzT\) equation (33)] (see also Remark 4.5), we can replace the factors

\[
\exp(\nu_{\text{det}} + \nu''_{\text{det}}, x, \hat{k}) \exp(u \sum \nu_i) = \exp(\nu_{\text{det}} + w_r x, \hat{k})
\]

in (26).
- The second summand in Lemma 4.4 has the same form as (26) with exchanged \( \nu' \) and \( \nu'' \), \( B' \) and \( B'' \).
- Since

\[
\int_{\text{lac}} \left( \exp(\eta \frac{(k + r)i}{r't''} - \frac{s}{r't''} \eta) - \left( \frac{s}{r't''} \right)^g \right) = -g \frac{s^2r}{(k + r)i} \frac{(k + r)i}{r't''} \left( \frac{(k + r)i}{r't''} \right)^{g-1},
\]

where the first summand comes from the restriction of \( \mathcal{L}(k; \lambda)/\mathcal{E}(N_{Z^0}) \) to \( Z^0 \) (cf. \([SzT\) Lemma 6.4, Proposition 6.7]), the third summand in Lemma 4.4 for \( l = 0 \) contributes

\[
-g \frac{s^2r}{r't''} \sum_{(\nu',\nu'')} \text{Res} \left( \sum_{B \in \mathcal{D}'} \sum_{B' \in \mathcal{D}''} \text{iBer}_{B',B''} \left[ \text{exp}(\nu_{\text{det}} + \nu''_{\text{det}}, x, \hat{k}) \phi^{\nu'}(x, \hat{k}) \phi^{\nu''}(x, \hat{k}) \right] \right) \left( -[c^+]_B \right) d\nu_{\text{link}},
\]

to the wall-crossing integral (18).
- Finally, using Lemma 4.2, we arrive at the following statement for \( l = 0 \).
Proposition 4.7. Let $\mathcal{D}'$ and $\mathcal{D}''$ be diagonal bases of $\Phi'$ and $\Phi''$ and let $\beta_{\text{link}}$ be the link edge (cf. page 22). Then

$$
\chi^\vee_{\mathbf{Y}}(k;\lambda) - \chi^\vee_{\mathbf{Y}}(k;\lambda) = (k+r)N_{r,k} \sum_{B' \in \mathcal{D}'} \sum_{B'' \in \mathcal{D}''} \text{Res}_{\beta_{\text{link}}=0} \text{iBer} \text{iBer} \cdot \left[ \nu_{\Phi}^{1-2g}(x/\bar{k}) \exp(\langle \hat{\lambda} + \nu_{\det}(x/\bar{k}) \rangle) \left( \frac{-g}{k+r} \text{tr}(\text{Hess}(\varphi^\vee(x/\bar{k}))) + 1 \varphi^\vee_{\beta_{\text{link}}}(x/\bar{k}) - \sum_{i \neq \text{link}} \frac{\varphi_{\beta_{\text{link}}}(x/\bar{k}) \exp(\langle \beta^{[i]} \rangle, x)}{1 - \exp(\langle \beta^{[i]} \rangle, x)} + \sum_{i \neq \text{link}} \langle [c^+], \beta^{[i]} \rangle \varphi_{\beta_{\text{link}}}(x/\bar{k}) \right) \right] [-[c^+]_B] \ d\beta_{\text{link}}.
$$

Remark 4.8. Note that this wall-crossing term coincides with the one from Corollary 3.8 and hence with the one from Lemma 3.6.

Example 6. Let $z = c_1(\mathcal{F}''_0/\mathcal{F}'_0 \otimes \mathcal{F}'_0^*) \in H^2(P_0)$, where $\mathcal{F}'_0$ are flag bundles on $P_0$ (cf. Example 5). In particular, we have $\text{ch}(\mathcal{L}'_0(0)) = e^z + 1$.

We saw in [SzT, Example 6] that in rank-3 case the Chern character of the restriction of the line bundle $\mathcal{L}(k;\lambda)$ multiplied by the inverse of the $K$-theoretical Euler class of the conormal bundle of $Z^0$ is equal to

$$
\exp \left( \frac{3(k+3)\eta}{2} \right) e^{\lambda z} \text{ch} \left( \mathcal{L}''(k + 1; \lambda_1) \right) e^z (2\sinh(u/2)2\sinh((z - u)/2))^{2g-1},
$$

where $\mathcal{L}''(k;\lambda)$ is a line bundle $\mathcal{L}(k,(\lambda,-\lambda))$ on $P_0$. Using Example 5 and Theorem 3.7 we conclude (cf. (18)) the the wall-crossing term

$$
\chi(P_0(<),\mathcal{L}_0(k,\lambda) \otimes \pi_0(U \otimes \mathbb{K})^\vee) - \chi(P_0(>),\mathcal{L}_0(k,\lambda) \otimes \pi_0(U \otimes \mathbb{K}))
$$

is equal to

$$
- \left( \frac{3(k+3)}{2} \right)^g \sum_{u=0}^{\text{Res}} e^{\lambda z} \text{ch} \left( \mathcal{L}''(k + 1; \lambda_1) \otimes \pi_0(U'' \otimes \mathbb{K})^\vee \right) e^z \text{Todd}(P_0) du
$$

$$
- \frac{2g}{3(k+3)} \text{Res}_{u=0} \text{Res}_{z=0} e^{\lambda z + \lambda z + z} (e^z + 1) \frac{1}{2} \frac{1}{2} \frac{1}{2} \left( \frac{1}{1 - e^{(k+3)z}} \right) \text{Todd}(P_0) du,
$$

where $\tilde{w}_\Phi(z,u) = 2\sinh(\frac{z+u}{2})2\sinh(\frac{z}{2})2\sinh(\frac{z}{2})$ and $N = (-1)^g (3(k+3))^g$. This integral is the Euler characteristic of a vector bundle on the moduli space of degree-$0$ rank-$2$ stable parabolic bundles, so we can calculate it using the induction by rank (cf. formula (10)). A simple calculation shows that the wall-crossing term (29) is equal to

$$
- \frac{2g}{3(k+3)} \text{Res}_{u=0} \text{Res}_{z=0} e^{\lambda z + \lambda z + z} (1 + e^u + e^z) \text{Todd}(P_0) du
$$

$$
- \frac{2g}{3(k+3)} \text{Res}_{u=0} \text{Res}_{z=0} e^{\lambda z + \lambda z + z} (1 - e^z) \text{Todd}(P_0) du.
$$

Note that this is exactly the same polynomial as in Example 4 and after changing $(z,u)$ to $(x,-y)$.

If $l \neq 0$, we will need one more step to calculate the wall-crossing term (18), which uses the tautological Hecke correspondence.
4.4. Hecke correspondence

In [SzT Section 7] we defined the tautological Hecke operators between the moduli spaces of parabolic bundles with different degrees and parabolic weights as follows: given a vector bundle $W$ on $C$ with a full flag $F_s$ in the fibre $W_p$ at $p \in C$, we consider the associated sheaf of sections $W$ and define the subsheaf

$$W[-1] = \{ \gamma \in H^0(C, W) \mid \gamma(p) \subset F_{r-1} \} \subset W.$$ 

Then $W[-1]$ is locally free, and thus defines a vector bundle, which we denote by $W[-1]$. Considering the associated morphism of vector bundles $W[-1] \to W$, we defined the full flag $G_s$ in the fibre $W[-1]_p$ and denoted this operator by $\mathcal{H} : (W, F_s) \mapsto (W[-1], G_s)$. We proved that $\mathcal{H}$ induces an isomorphism of the moduli spaces

$$\mathcal{H} : P_d(c_1, c_2, ..., c_r) \cong P_{d-1}(c_2, ..., c_r, c_1 - 1).$$

Applying $\mathcal{H}$ to the universal bundle $U$ on the moduli space $P_0(c) \times C$ we obtain a short exact sequence for the corresponding sheaves of sections:

$$0 \to U[-1] \to U \to F_r/F_{r-1} \to 0.$$

Considering the associated vector bundles, we arrive at the following equality

$$(30) \quad \text{ch}(U) = \text{ch}(U[-1]) + \omega \cdot \text{ch}(L(0; (1, 0, ..., 0, -1))),$$

where $\omega \in H^2(C)$ is the fundamental class of the curve (cf. the beginning of §4.3).

**Remark 4.9.** Note that under the Hecke isomorphism $\mathcal{H}$, the normalized (cf. §2.2) universal bundle $U$ on the moduli space $P_0(c_1, c_2, ..., c_r) \times C$ corresponds to the universal bundle $U[-1]$ on the moduli space $P_{-1}(c_1+1, c_2, ..., c_r, c_1 - 1, ..., c_1 - 1) \times C$. The line bundle $F_2/F_1$ is trivial.

Similarly, applying the Hecke operator $\mathcal{H}^1$ to the normalized universal bundle $U$, we obtain the universal bundle $U[-1]$ on $P_{-1}(c_1+1, ..., c_r, c_1 - 1, ..., c_1 - 1) \times C$. The line bundle $F_2/F_1$ is trivial.

**Notation:** Given an irreducible representation $\rho_\nu : \text{GL}_r \to \text{GL}(V_\nu)$ of highest weight $\nu$, we consider its weight decomposition

$$V_\nu = \bigoplus_{\mu \in \mathbb{Z}^*} V_{[\mu]},$$

where $V_{[\mu]}$ is the weight space of the weight $\mu$, and we denote by $m_{[\mu]} = \dim(V_{[\mu]})$.

**Proposition 4.10.** Let $U[-1]_\nu$ be the vector bundle on $P_{-1}(c) \times C$ associated to the irreducible representation $\rho_\nu$ of $\text{GL}_r$ with highest weight $\nu$ and the normalized universal bundle $U[-1]$ (cf. §2.2). Then

$$\text{ch}(U_\nu) = \text{ch}(U[-1]_\nu) + \omega \sum_{\mu} m_{\mu}(\mu_1 + ... + \mu_1) \text{ch}(L(0; (\mu_1, ..., \mu_{r-1}, \mu_r - |\nu|))),$$

where $|\nu| = \sum_i \nu_i$ and the sum runs over the weights $\mu$ of $\rho_\nu$ with highest weight $\nu$.

**Proof.** Given a rank-$r$ vector bundle $V$ on $P_0(c) \times C$ and a symmetric polynomial $f \in \mathbb{C}[y_1, ..., y_r]^{\mathbb{Z}_r}$, denote by $f(V) \in H^*(P_0(c) \times C)$ the cohomology class obtained by evaluating $f$ at the Chern roots of $V$. The flag $F_1 \subset F_2 \subset ... \subset F_r = U_p$ defines the cohomology classes

$$\xi_i = c_1(F_r/F_{r-1} \otimes F_1^i) \in H^2(P_0(c)),$$

and thus we have

$$\text{ch}(U_p) = e^{\xi_1} + ... + e^{\xi_{r-1} + 1};$$
it follows from Remark 2.1 that the Chern character of an associated bundle $\mathcal{U}_\nu$ is given by
\[
\text{ch}(\mathcal{U}_\nu)_p = \sum_{\mu} m_\mu \exp(\mu_1 \xi_1 + \ldots + \mu_{r-1} \xi_{r-1}).
\]
We note that the cohomology class $f(\mathcal{U}_p)$ in $H^*(\mathcal{P}_0(c) \times \mathbb{C})$ is well-defined for any (not necessarily symmetric) polynomial $f \in \mathbb{C}[y_1, \ldots, y_r]$.

We introduce the notation $f_i(y_1, \ldots, y_r) = \frac{1}{\text{deg}}(y_1 + \ldots + y_i)$; in particular, for any vector bundle $V$ on $\mathcal{P}_0(c) \times \mathbb{C}$, we have $f_i(V) = \text{ch}_i(V)$. It follows from (30) that
\[
f_i(U) = f_i(U[-1]) + \omega \partial_{y_i} f_i(U_p),
\]
and thus
\[
f_i(U)f_j(U) = f_i(U[-1])f_j(U[-1]) + \omega \left( \partial_{y_i} f_i(U_p) f_j(U[-1]) + \partial_{y_j} f_i(U[-1]) f_j(U) \right) = f_i(U[-1])f_j(U[-1]) + \omega \partial_{y_i} (f_i f_j)(U_p).
\]
For the last equality, we used the facts that $\omega \text{ch}(U) = \omega \text{ch}(U_p)$ and that according to (30), $\text{ch}(U_p) = \text{ch}(U[-1])$.

Since any symmetric polynomial $f \in \mathbb{C}[y_1, \ldots, y_r]^S$ may be written as a polynomial in $f_i$’s, (31) implies that for any symmetric polynomial $f$ we have:
\[
f(U) = f(U[-1]) + \omega \partial_{y_i} f(U_p).
\]

Let
\[
g_\nu(y_1, \ldots, y_r) = \sum_{\mu} m_\mu \exp(\mu_1 y_1 + \ldots + \mu_r y_r);
\]
since $g_\nu(U) = \text{ch}(\mathcal{U}_\nu)$, we have
\[
\text{ch}(\mathcal{U}_\nu) = \text{ch}(U[-1]_\nu) + \omega \partial_{y_i} g_\nu(U_p),
\]
and thus
\[
\text{ch}(\mathcal{U}_\nu) = \text{ch}(U[-1]_\nu) + \omega \sum_{\mu} m_\mu \mu_1 \exp(\mu_1 \xi_1 + \ldots + \mu_{r-1} \xi_{r-1}).
\]
Finally, note that
\[
\exp(\mu_1 \xi_1 + \ldots + \mu_{r-1} \xi_{r-1}) = \text{ch}(L(0; (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|))),
\]
hence we obtain the proof for $l = 1$. Iterating this argument, we obtain the proof for the general case.

4.5. Wall-crossing for $l \neq 0$

Recall that our goal is to calculate the wall-crossing integral (18) for non-zero $l$, or, more precisely, to prove Proposition 4.7 for the case when $l \neq 0$. The treatment of this case follows the logic of [SzT §7.2], hence, in this section, we will only highlight the differences which arise in our, more general, situation. For simplicity, we assume that $l$ is positive (the other case is analogous).

- We first apply the Hecke operators $\mathcal{H}^l$ and $\mathcal{H}^{-l}$ to the moduli spaces $\mathcal{P}_l(c')$ and $\mathcal{P}_{-l}(c'')$ to obtain
\[
P_0' = P_0(c'_{i+1}, \ldots, c'_r, c'_i - 1, \ldots, c'_1 - 1) \simeq P_l(c')
\]
\[
P_0'' = P_0(c''_{r+1}, \ldots, c''_r, 1, \ldots, c''_1, \ldots, c''_{r-1}) \simeq P_{-l}(c'').
\]
Next, applying the Hecke operator $\mathcal{H}^1 \times \mathcal{H}^{-1}$ to the wall-crossing term (18), we recast it as an integral over the moduli spaces of degree-0 parabolic bundles $P_0^\tau \times P_0^{\tau'}$, and thus we can calculate this integral using the induction by rank as in [4.3].

As in [SZT, page 33], to arrive at Proposition 4.7 we will need to make additional transformations of the formulas we obtained. We perform this transformation by applying Lemma 3.2 with

$$w = \sum_{i=1}^{l} (x_{\phi(r'-1+i)} - x_{\phi(r'+i)}) \in \Lambda,$$

where $\phi \in \Sigma_r$ is the permutation which sends $\{1, ..., r'\}$ to $\Pi'$ preserving the order of the first $r'$ and the last $r''$ elements.

The first summand on the right-hand side of (15) coincides with the shift of $\lambda$ we treated in [SZT, page 34]. An easy calculation shows that the second summand on the right-hand side of (15) eliminates the changes (cf. Proposition 4.10 and equation (22)) of the Chern character of $\pi_!(\mathcal{U}_\lambda \otimes \mathcal{X})|_{\mathcal{Z}_0}$ under the Hecke transformations $\mathcal{H}^1$ and $\mathcal{H}^{-1}$.

This completes the proof of Proposition 4.7 for arbitrary $l \in \mathbb{Z}$.

5. Symmetry

The main result of this section is Proposition 5.3, where we prove certain symmetry for the Euler characteristics of our vector bundles on the moduli spaces of parabolic bundles.

5.1. Symmetries through Serre duality

Denote by $N_{\pm 1}$ the moduli spaces of rank-$r$ degree-$\pm 1$ stable vector bundles and by $\mathcal{UN}^\pm$ the universal bundle over $N_{\pm 1} \times C$, normalized in such a way that $\det(\mathcal{UN}_1^\pm) \cong \mathcal{L}_{-1}(-1, 0, 0, 1)$ and $\det(\mathcal{UN}_1^\pm) \cong \mathcal{L}_1(-1, 0, 0, 1)$.

In [SZT Lemma 8.3] we identified the moduli spaces $P_1(\rangle)$ and $P_{-1}(\langle)$, which are isomorphic to the flag bundles

$$P_1(\rangle) \cong \text{Flag}(\mathcal{UN}_1^+)^{\tau} N_1 \quad \text{and} \quad P_{-1}(\langle) \cong \text{Flag}(\mathcal{UN}_1^-)^{\tau} N_{-1}.$$ 

The following is easy to verify.

**Lemma 5.1.** Under the normalization described above, the line bundles $\mathcal{F}_1 \subset \mathcal{P}^*(\mathcal{UN}_1^\pm)$ are isomorphic to $\mathcal{L}_{-1}(-1; 0, 0, 0, 1)$ and $\mathcal{L}_1(1; 0, 0, 0, 1)$, respectively (cf. §4.2).

Applying the Hecke operators $\mathcal{H}^{-1}$ and $\mathcal{H}$ (cf. §4.4) to the moduli spaces $P_{-1}(\langle)$ and $P_1(\rangle)$ we obtain

$$P_0(\langle) \cong P_{-1}(\langle) \quad \text{and} \quad P_0(\rangle) \cong P_1(\rangle).$$

Let $\tau \in \Sigma_r$ be the cyclic permutation $\tau \cdot (c_1, ..., c_r) = (c_2, ..., c_r, c_1)$, and consider two points in $V^*$:

$$\theta_1[k] = \frac{k + r}{r} \cdot (1, 1, ..., 1) - (k + r)x_r - \rho = \tau \cdot \left( \frac{k}{r} - \frac{k}{r}, ..., \frac{k}{r} \right) - \tau \cdot \rho,$$

$$\theta_{-1}[k] = -\frac{k + r}{r} \cdot (1, 1, ..., 1) + (k + r)x_1 - \rho = \tau^{-1} \cdot \left( -\frac{k}{r}, ..., -\frac{k}{r}, -\frac{k}{r} + k \right) - \tau^{-1} \cdot \rho.$$
We have shown that the two polynomials
\[ \chi_-(k;\lambda) = \chi(P_0(<), \mathcal{L}(k;\lambda)) \quad \text{and} \quad \chi_+(k;\lambda) = \chi(P_0(>), \mathcal{L}(k;\lambda)) \]
satisfy the following properties.

**Proposition 5.2.** [SZT Proposition 8.5] The polynomials
\[ \chi_-(k;\lambda + \theta_{-1}[k]) \quad \text{and} \quad \chi_+(k;\lambda + \theta_1[k]) \]
are anti-invariant under the action of the group of permutations of \(\lambda_1, \ldots, \lambda_r\).

Similarly, we define two polynomials
\[ \chi_-^\vee(k;\lambda) = \chi(P_0(<), \mathcal{L}(k;\lambda) \otimes \pi_i(U_v \otimes \mathcal{K}^\perp)) \]
\[ \chi_+^\vee(k;\lambda) = \chi(P_0(>), \mathcal{L}(k;\lambda) \otimes \pi_i(U_v \otimes \mathcal{K}^\perp)) \]
and establish the Weyl antisymmetry for the modified polynomials
\[ f_-^\vee(k;\lambda) = \chi_-^\vee(k;\lambda) - \sum m_{\mu,1} \chi(P_0(<), \mathcal{L}(k;\lambda + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |v|))) \]
and
\[ f_+^\vee(k;\lambda) = \chi_+^\vee(k;\lambda) + \sum m_{\mu,1} \chi(P_0(>), \mathcal{L}(k;\lambda + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |v|))) \]
where we sum over all weights \(\mu\) of the irreducible representation \(\rho_v\) and \(|v| = \sum_i v_i\) (cf. notation on page 20).

**Example 7.** In case of rank-3 parabolic bundles and \(v = (1,0,0)\) (cf. Example 5), we have
\[ f_-^\vee(k;\lambda) = \chi(P_0(<), \mathcal{L}(k;\lambda) \otimes \pi_i(U \otimes \mathcal{K}^\perp)) - \chi(P_0(<), \mathcal{L}(k;\lambda + (1,\lambda_2,\lambda_3 - 1))); \]
\[ f_+^\vee(k;\lambda) = \chi(P_0(>), \mathcal{L}(k;\lambda) \otimes \pi_i(U \otimes \mathcal{K}^\perp)) + \chi(P_0(>), \mathcal{L}(k;\lambda + 1,\lambda_2,\lambda_3)). \]

**Proposition 5.3.** Let \(v_{\text{det}} = \sum_i v_i(1,\ldots,1,1-r)\); then the polynomials
\[ f_-^\vee(k;\lambda + \theta_{-1}[k] - v_{\text{det}}) \quad \text{and} \quad f_+^\vee(k;\lambda + \theta_1[k] - v_{\text{det}}) \]
are anti-invariant under the action of the group of permutations of \(\lambda_1, \ldots, \lambda_r\).

**Proof.** First, we will show the anti-invariance of the Euler characteristics of vector bundles on the moduli spaces of degree \(\pm 1\) parabolic bundles \(P_1(>)\) and \(P_{-1}(<)\), as it is simpler. Let \(U[1]\) and \(U[-1]\) be the universal bundles on \(P_1(>) \times \mathbb{C}\) and \(P_{-1}(<) \times \mathbb{C}\) that correspond to the normalized (cf. §2.2) universal bundles on \(P_0(>\) and \(P_0(<)\), respectively, and let
\[ \tilde{\theta}_{-1} = -\tau \cdot v_{\text{det}} - \rho \quad \text{and} \quad \tilde{\theta}_1 = -\tau^{-1} \cdot v_{\text{det}} - \rho. \]
Applying Serre duality for family of curves to the associated vector bundles \(U[\pm 1]_v\) (cf. (11)) on the moduli spaces \(P_{-1}(<)\) and \(P_1(>\), we obtain the following.

**Lemma 5.4.** The Euler characteristics \(\chi(P_{-1}(<), \mathcal{L}_{-1}(k;\lambda + \tilde{\theta}_{-1}) \otimes \pi_i(U[-1]_v \otimes \mathcal{K}^\perp))\) and \(\chi(P_1(>), \mathcal{L}_{-1}(k;\lambda + \tilde{\theta}_1) \otimes \pi_i(U[1]_v \otimes \mathcal{K}^\perp))\) are anti-invariant under the permutations of \(\lambda_1, \ldots, \lambda_r\).
Proof. Note that $U[-1] \simeq p^*(UN^{-}) \otimes (\mathcal{F}_2/\mathcal{F}_1)^*$ (cf. Remark 4.9), hence
\[
U[-1]_{\nu} \simeq p^*(UN^{-}_{\nu}) \otimes (\mathcal{F}_2/\mathcal{F}_1)^{-\Sigma v_i},
\]
where $UN^{-}_{\nu}$ is a vector bundle on $N_{-1} \times C$ obtained by (11) from the universal bundle $UN^{-}$. Then
\[
\pi_i(U[-1]_{\nu} \otimes K_1) \simeq \pi_i(p^*(UN^{-}_{\nu}) \otimes K_1) \otimes \mathcal{L}_{-1}(1; (0, ..., 0, -1, 0))^\Sigma v_i
\]
by Lemma 5.1, and thus
\[
\chi(P_{-1}(<), \mathcal{L}_{-1}(k; \lambda + \bar{\theta}_{-1}) \otimes \pi_i(U[-1]_{\nu} \otimes K_1) = \chi(P_{-1}(<), \mathcal{L}_{-1}(k + \sum v_i; \lambda - \frac{1}{r}(1, ..., 1) - \rho) \otimes \pi_i(p^*(UN^{-}_{\nu}) \otimes K_1)).
\]
Since the line bundle $\mathcal{L}_{-1}(\tau; (-1, ..., -1))$ is a pullback of the ample generator of $\text{Pic}(N_{-1})$ [SZ1 Lemma 8.4], the statement follows from Serre duality for families of curves [SZ1 Proposition 8.1]. The proof for the Euler characteristic on the moduli space $P_1(\mathbb{S})$ is similar.

Recall that our goal is to show certain antisymmetries for the polynomials $f_{\xi}(k; \lambda)$, which are the linear combinations of the Euler characteristics of vector bundles on the moduli spaces $P_0(\mathbb{S})$. We will follow the argument for the polynomial $f_{\xi}$ (the proof for $f_{\xi}$ is analogous).

Under the isomorphism $\mathcal{H} : P_0(<) \cong P_{-1}(<)$, vector bundles on $P_0(<)$ correspond to vector bundles on $P_{-1}(<)$. Below, we will write this correspondence explicitly and then apply Lemma 5.4 to the vector bundles on $P_{-1}(<)$ to obtain antisymmetries for the Euler characteristics.

Note that trivially
\[
\chi(P_{-1}(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) = -\sum \frac{v_i}{r}(1, ..., 1) + \mu,
\]
and thus it follows from Proposition 4.10 that
\[
\chi(P_{-1}(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) = \chi(P_{-1}(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) + \sum \mu_1 \chi(P_{-1}(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r).
\]

Using Lemma 5.4 and equations (32) and (33), for any permutation $\sigma \in \Sigma_r$ we obtain
\[
f_{\xi}(k; \sigma \cdot \lambda + \theta_{-1}[k] - v_{det}) \xi_2(k) \sigma \cdot \lambda + \theta_{-1}[k] = \sum \mu_1 \chi(P_{-1}(<), \mathcal{L}(k; \sigma \cdot \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) + \sum \mu_1 \chi(P_{-1}(<), \mathcal{L}(k; \sigma \cdot \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) = \sum \mu_1 \chi(P_{-1}(<), \mathcal{L}(k; \sigma \cdot \lambda + \theta_{-1}[k] - v_{det} + (\mu_1, ..., \mu_r - 1, \mu_r) = 5.4
\]
\[ (-1)^{\sigma} \chi(P_{-1}(<), \mathcal{L}_{-1}(k; \tau \cdot \lambda - \frac{k}{r}(1, \ldots, 1) + \hat{\theta}_{-1}) \otimes \pi_1(U_{\nu}[1] \otimes \mathcal{X}^\perp)) \]

\[ (-1)^{\sigma} \chi(P_{0}(<), \mathcal{L}_{0}(k; \lambda + \theta_{-1}[k] - \nu_{\det}) \otimes \pi_1(U_{\nu}[1] \otimes \mathcal{X}^\perp)) - \]

\[ (-1)^{\sigma} \sum_{\mu} m_{\mu} \chi(P_{0}(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - \sum_{\nu} \nu_{\lambda}(1, \ldots, 1) + \mu)) \text{ def } \]

\[ (-1)^{\sigma} f_{\gamma}(k; \lambda + \theta_{-1}[k] - \nu_{\det}), \]

which completes the proof of Proposition 5.3 for \( f_{\gamma} \). The proof for \( f_1 \) is similar. \( \square \)

5.2. The Affine Weyl group

We define an action of the affine Weyl group \( \Sigma \rtimes \Lambda \) on \( \Lambda \times \mathbb{Z}_{\geq 0} \), which acts trivially on the second factor, the level, and the action at level \( k > 0 \) is given by

\[ \sigma \cdot \lambda = \sigma \cdot (\lambda + \rho + \nu_{\det}) - \rho - \nu_{\det} \]

and

\[ \gamma \cdot \lambda = \lambda + (k + \tau) \gamma \quad \text{for} \quad \sigma \in \Sigma, \gamma \in \Lambda. \]

We denote the resulting group of affine-linear transformations of \( V^* \) by \( \tilde{\Sigma}[k] \). It is easy to verify that the stabilizer subgroup

\[ \Sigma_+ \text{ def } \text{Stab}(\theta_{1}[k] - \nu_{\det}, \tilde{\Sigma}[k]) \subset \tilde{\Sigma}[k] \]

is generated by the transpositions \( s_{i,i+1}, \) \( 1 \leq i \leq \tau - 2 \) and the reflection \( \alpha_{i}^{1,r} \circ s_{r-1,r}; \) similarly,

\[ \Sigma_- \text{ def } \text{Stab}(\theta_{-1}[k] - \nu_{\det}, \tilde{\Sigma}[k]) \subset \tilde{\Sigma}[k] \]

is generated by \( s_{i,i+1}, \) \( 2 \leq i \leq \tau - 1 \) and the reflection \( \alpha_{i}^{1,2} \circ s_{1,2}. \)

Then Proposition 5.3 may be recast in the following form: the polynomial \( f_{\gamma}(k; \lambda) \) is anti-invariant with respect to the copy \( \Sigma_+ \) of the symmetric group \( \Sigma_r \), while \( f_{\lambda}(k; \lambda) \) is anti-invariant with respect to the copy \( \Sigma_- \) of the symmetric group \( \Sigma_r. \)

The following statement is straightforward:

**Lemma 5.5.** Both subgroups \( \Sigma_+ \) are isomorphic to \( \Sigma_r \) and for \( r > 2 \) the two subgroups generate the affine Weyl group \( \tilde{\Sigma}[k]. \)

5.3. Symmetries in residue formulas

The main result of this section is Proposition 5.6, where we show the antisymmetries for the residues formulas on the right-hand side of \( (16) \).

Recall that in \( (5.1) \) we defined a pair of polynomials \( \chi_{\Sigma}^\gamma \) corresponding to the Euler characteristics from the left-hand side of \( (16) \) and proved the Weyl antisymmetry for the modified polynomials \( f_{\gamma}^\Sigma \). Now we define the two polynomials corresponding to the residue expressions from the right-hand side of \( (16) \):

\[ R_{\gamma}(k; \lambda) = N_{\tau} \cdot \left. \frac{\partial}{\partial \delta} \right|_{\delta=0} \sum_{B \in \mathbb{D}} \text{iBer}_{B,Q} \left[ \text{Hess}(Q(x))^g \cdot w_\phi^{1-2g}(x) \exp(\hat{\lambda} + \nu_{\det} \chi) \right] (-[\theta_1]_B) \]

and

\[ R_{\lambda}(k; \lambda) = N_{\tau} \cdot \left. \frac{\partial}{\partial \delta} \right|_{\delta=0} \sum_{B \in \mathbb{D}} \text{iBer}_{B,Q} \left[ \text{Hess}(Q(x))^g \cdot w_\phi^{1-2g}(x) \exp(\hat{\lambda} + \nu_{\det} \chi) \right] (-[\theta_{-1}]_B), \]
Proposition 5.6. The polynomial

\[
F^\gamma_\nu(k; \lambda) = R^\gamma_\nu(k; \lambda) + N_{r,k} \cdot \sum_\mu m_{\mu} \sum_{B \in \mathcal{D}} \text{iBer}_{B, (k+r)K} \left[ w_\Phi^{1-2g}(x) \exp(\langle \hat{\lambda} + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|, x \rangle) \right] (-[\theta_1]_B)
\]

and

\[
F^\gamma_\nu(k; \lambda) = R^\gamma_\nu(k; \lambda) - N_{r,k} \cdot \sum_\mu m_{\mu} \mu_1 \sum_{B \in \mathcal{D}} \text{iBer}_{B, (k+r)K} \left[ w_\Phi^{1-2g}(x) \exp(\langle \hat{\lambda} + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|, x \rangle) \right] (-[\theta_-]_B),
\]

where, as usual, the sum runs over all weights \( \mu \) of the irreducible representation \( \rho_\nu \) and \( |\nu| = \sum_1 \nu_i \) (cf. notation on page 20).

**Proposition 5.6.** The polynomial \( F^\gamma_\nu(k; \lambda) \) is anti-invariant with respect to \( \Sigma^+ \), and \( F^\gamma_\nu(k; \lambda) \) is anti-invariant with respect to \( \Sigma^- \).

**Proof.** We first consider a generator of \( \Sigma^- \) of the type \( \sigma = s_{1,i+1}, 2 \leq i \leq r - 1 \). Note that

\[\sigma \lambda + \rho + \nu_{\text{det}} = \sigma(\lambda + \rho + \nu_{\text{det}}) \quad \text{and} \quad \sigma \lambda + \rho + \mu - |\nu|x_r = \sigma(\lambda + \rho - |\nu|x_r) + \mu.\]

Using \([SzT]\) Lemma 4.5] and the facts that

\[\sigma \cdot \text{Hess}(Q(x)) = \text{Hess}(Q(x)) \quad \text{and} \quad \sigma \cdot w_\Phi^{1-2g}(x) = -w_\Phi^{1-2g}(x),\]

we obtain

\[
F^\gamma_\nu(k; \sigma \lambda) = \sum_{B \in \mathcal{D}} \sum_{\delta=0} \text{iBer}_{B, Q} \left[ -\text{Hess}(Q(x)) \left. \right|_{\delta=0} \cdot \nu_{\text{det}} \cdot w_\Phi^{1-2g}(x) \exp(\langle \hat{\lambda} + \rho + \nu_{\text{det}}, x \rangle) \right] (-\sigma^{-1} \cdot [\theta_-]_B) - N_{r,k} \sum_\mu m_{\mu} \mu_1 \sum_{B \in \mathcal{D}} \text{iBer}_{B, (k+r)K} \left[ -w_\Phi^{1-2g}(x) \exp(\hat{\lambda} + \sigma^{-1} \cdot \mu - |\nu|x_r) \right] (-\sigma^{-1} \cdot [\theta_-]_B) = -F^\gamma_\nu(k; \lambda).
\]

For the last equality we used the Weyl-invariance of the multiplicities of weights \( \mu \) of the irreducible representation \( \rho_\nu \).

The case of the last generator \( \sigma = \alpha^{1,2} \circ s_{1,2} \) requires some extra observations. Since

\[
\sigma \lambda + \rho + \mu - \sum \nu_i x_r = s_{1,2} \cdot (\lambda + \rho) + \mu - |\nu|x_r + (k + r)(x_1 - x_2)
\]

and

\[
(35) \quad s_{1,2} \cdot [\theta_-] = [\theta_-] - (x_1 - x_2),
\]

where \( \theta_1 = \frac{1}{r} \cdot (1, 1, \ldots, 1) - x_r \) and \( \theta_{-1} = -\frac{1}{r} \cdot (1, 1, \ldots, 1) + x_1 \), and establish the Weyl antisymmetry for the modified pair of polynomials:

\[
\sum_{\mu} m_{\mu} \mu_1 \mu_2 \sum_{B \in \mathcal{D}} \text{iBer}_{B, (k+r)K} \left[ w_\Phi^{1-2g}(x) \exp(\langle \hat{\lambda} + (\mu_1, \mu_2, \mu_{r-1}, \mu_r - |\nu|, x \rangle) \right] (-[\theta]_B)
\]
we have

\[ \sum_{\mu} m_{\mu} \mu_1 \sum_{B \in \mathcal{D}} \text{iBer}_{B(k+r)k} \left[ w_{1-2g}^\Phi (x) \exp (\sigma \lambda + \rho + \mu - |\nu| x_r) \right] (-[\omega_1]_B) = \]

\[ \sum_{\mu} m_{\mu} \mu_1 \sum_{B \in \mathcal{D}} \text{iBer}_{B(k+r)k} \left[ -w_{1-2g}^\Phi (x) \exp (\lambda + s_{1,2} \cdot \mu - |\nu| x_r - (k + r)(x_1 - x_2)) \right] \]

\[ (-s_{1,2} \cdot [\omega_1]_B) = - \sum_{\mu} m_{\mu} \mu_2 \sum_{B \in \mathcal{D}} \text{iBer}_{B(k+r)k} \left[ w_{1-2g}^\Phi (x) \exp (\lambda + \mu - |\nu| x_r) \right] (-[\omega_1]_B) . \]

Note that

\[ \sigma, \lambda + \rho + \nu_{\det} = s_{1,2} \cdot (\lambda + \rho) + \nu_{\det} + (k + r)(x_1 - x_2), \]

hence, using (36), we obtain

\[ F^\chi_x (k; \sigma, \lambda) = N_{r, [\omega_1]} \frac{\partial}{\partial \delta} \bigg|_{\delta = 0} \sum_{B \in \mathcal{D}} \text{iBer}_{B, Q} [-\text{Hess}(Q(x))_g^{-1} w_{1-2g}^\Phi (x) \exp (\lambda + \rho + \nu_{\det} - (k + r)(x_1 - x_2)) \right] (-[\omega_1]_B) + \]

\[ N_{r,k} \sum_{\mu} m_{\mu} \mu_2 \sum_{B \in \mathcal{D}} \text{iBer}_{B(1+k+r)k} \left[ w_{1-2g}^\Phi (x) \exp (\lambda + \mu - |\nu| x_r) \right] (-[\omega_1]_B) . \]

Now using (35) and applying Lemma 3.2 with \( w = x_1 - x_2 \) to (37), we calculate that

\[ F^\chi_x (k; \sigma, \lambda) = N_{r, [\omega_1]} \frac{\partial}{\partial \delta} \bigg|_{\delta = 0} \sum_{B \in \mathcal{D}} \text{iBer}_{B, Q} [-\text{Hess}(Q(x))_g^{-1} w_{1-2g}^\Phi (x) \exp (\lambda + \nu_{\det}) \right] (-[\omega_1]_B) - \]

\[ N_{r,k} \sum_{\mu} m_{\mu} \mu_2 \sum_{B \in \mathcal{D}} \text{iBer}_{B(1+k+r)k} \left[ w_{1-2g}^\Phi (x) \exp (\lambda + \nu_{\det}) \right] (-[\omega_1]_B) + \]

\[ N_{r,k} \sum_{\mu} m_{\mu} \mu_2 \sum_{B \in \mathcal{D}} \text{iBer}_{B(1+k+r)k} \left[ w_{1-2g}^\Phi (x) \exp (\lambda + \mu - |\nu| x_r) \right] (-[\omega_1]_B) . \]

Finally, applying the following trivial equality

\[ \phi_{k,12} (x) \exp (\nu_{\det}) = \sum_{\mu} m_{\mu} (\mu_1 - \mu_2) \exp (\mu(x)) \]

to the last two summands in our expression for polynomial \( F^\chi_x (k; \sigma, \lambda) \), we conclude that \( F^\chi_x (k; \sigma, \lambda) = -F^\chi_x (k; \lambda) \). This finishes the proof of the anti-invariance of the polynomial \( F^\chi_x (k; \lambda) \); the proof for \( F^\chi_x (k; \lambda) \) is similar.

Note that the two differences \( \chi^\chi_x - f^\chi_x \) and \( \chi^\chi_x - f^\chi_x \) (cf. page 23) have the form of a linear combination of the Euler characteristics of line bundles on the moduli spaces of parabolic bundles; while the differences \( R^\chi_x - F^\chi_x \) and \( R^\chi_x - F^\chi_x \) (cf. page 25) may be written as an iterated residue of a meromorphic functions. Then using the residue formula for the Euler characteristic of line bundles, Theorem 3.1 we arrive at the following statement.

**Proposition 5.7.** For polynomials \( R^\chi_x, R^\chi_x, \chi^\chi_x, \chi^\chi_x, F^\chi_x, F^\chi_x \) and \( f^\chi_x, f^\chi_x \) defined on pages 25 and 23 we have:

\[ \chi^\chi_x (k; \lambda) - f^\chi_x (k; \lambda) = R^\chi_x (k; \lambda) - F^\chi_x (k; \lambda); \]

\[ \chi^\chi_x (k; \lambda) - f^\chi_x (k; \lambda) = R^\chi_x (k; \lambda) - F^\chi_x (k; \lambda). \]
6. Proof of Theorem 3.3 and some generalizations of our result

In this section, we finish the proof of our main result and present some of its generalizations.

6.1. Proof of Theorem 3.3

The proof of Theorem 3.3 follows the logic of [SzT]. In this section, we repeat the argument with only minor changes.

Recall that in §2.3 we introduced a chamber structure on $\Delta \subset V^*$ created by the walls $S_{\Pi,l}$, where $\Pi = (\Pi', \Pi'')$ is a nontrivial partition, and $l \in \mathbb{Z}$. Denote by $\Delta$ the set of its regular points. Denote by $q$ and let

$$\tilde{\Delta} = \{(k; a) | a/k \in \Delta \} \subset \mathbb{R}_{>0} \times V^*$$

be the cone over $\Delta \subset V^*$, and let

$$\tilde{\Delta}_{\text{reg}} = \{(k; a) | a/k \in \Delta \text{ is regular} \} \subset \tilde{\Delta}$$

be the set of its regular points. Denote by $\tilde{S}_{\Pi,l} \subset \tilde{\Delta}$ the cone over the wall $S_{\Pi,l} \subset \Delta$; then $\tilde{\Delta}_{\text{reg}}$ is the complement of the union of walls $\tilde{S}_{\Pi,l}$ in $\tilde{\Delta}$. Finally, denote by $\tilde{\Delta}_\Lambda^{\text{reg}}$ the intersection of the lattice $\mathbb{Z}_{>0} \times \Lambda$ with $\tilde{\Delta}_{\text{reg}}$.

By substituting $c = \lambda/k$, we can consider the left-hand side and the right-hand side of the equation in Theorem 3.3 as functions in $(k; \lambda) \in \tilde{\Delta}_\Lambda^{\text{reg}}$. We denote by $\chi(k; \lambda)$ and $R(k; \lambda)$ the left-hand side and the right-hand side, correspondingly.

We showed that $\chi(k; \lambda)$ and $R(k; \lambda)$ are polynomials on the cone over each chamber in $\Delta$ (cf. §2.3 and §3.1). We proved that the wall-crossing terms, i.e. the differences between polynomials on neighbouring chambers, for $\chi(k; \lambda)$ (cf. Proposition 4.7) and for $R(k; \lambda)$ (cf. Corollary 5.8) coincide, hence there exists a polynomial $\Theta(k; \lambda)$ on $\mathbb{Z}_{>0} \times \Lambda$, such that the restriction of $\Theta(k; \lambda)$ to $\tilde{\Delta}_\Lambda^{\text{reg}}$ is equal to the difference $\chi(k; \lambda) - R(k; \lambda)$.

Now for $r > 2$, we can conclude that

$$\Theta(k; \lambda) = \chi^\vee(k; \lambda) - R^\vee(k; \lambda) = \chi^\wedge(k; \lambda) - R^\wedge(k; \lambda),$$

where $\chi^\vee(k; \lambda)$ and $R^\vee(k; \lambda)$ are the restrictions of $\chi(k; \lambda)$ and $R(k; \lambda)$ to two specific chambers defined in [SzT] Lemma 8.3. Then, according to Proposition 5.7

$$\Theta(k; \lambda) = f^\vee(k; \lambda) - F^\vee(k; \lambda) = f^\wedge(k; \lambda) - F^\wedge(k; \lambda).$$

It follows from Propositions 5.3 and 5.6 that the polynomial $\Theta(k; \lambda)$ is anti-invariant with respect to the action of the subgroups $\Sigma^+_r$ (cf. the end of §5.2), and hence by Lemma 5.5 it is anti-invariant under the action of the entire affine Weyl group $\tilde{\Sigma}[k]$. It is easy to see that any such polynomial function has to vanish, and thus $\chi(k; \lambda) = R(k; \lambda)$.

As marked above, the argument does not work for $r = 2$, since in this case the groups $\Sigma^+_r$ and $\Sigma^-_r$ (cf. §5.2) coincide, and thus they do not generate the entire affine Weyl group. A solution is to consider the 2-punctured case, treated in §1.2-§1.4, this finishes the proof of Theorem 3.3.

6.2. Generalization

Now we formulate a mild generalization of our result, Theorem 6.1 and explain, following an idea of Teleman and Woodward [TW], how our formulas can be used to calculate the Euler characteristic of a more general class of vector bundles on the moduli spaces of parabolic vector bundles.
Let $v[1],...,v[m]$ be dominant weights of $GL_r$. Replacing $Q$ and $v_{det}$ in Theorem 3.3 by the multi-parameter version

$$Q = (k + r)K - \sum_{j} \delta_j \cdot \phi^{v[j]}, \quad v_{det} = \sum_{j=1}^{m} (1,...,1,1 - r) \frac{\sum_{i} v[j]_i}{r},$$

we can deduce the following Theorem.

**Theorem 6.1.** Let $Q$ and $v_{det}$ be as above, let $\mathcal{X}$ be the canonical class of the curve $C$, $\lambda \in \Lambda$, $k \in \mathbb{Z}_{\geq 0}$, $v = (v_1 \geq v_2 \geq \cdots \geq v_r) \in \mathbb{Z}^r$, $\lambda = \lambda + \rho$, and let $c \in \Delta$ be a regular element (cf. page 8). Then for any diagonal basis $D \in B$, the following equality holds:

$$\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_1(U_{v[1]} \otimes \mathbb{K}^2) \otimes \pi_1(U_{v[2]} \otimes \mathbb{K}^{\frac{1}{2}}) \otimes \cdots \otimes \pi_1(U_{v[m]} \otimes \mathbb{K}^{\frac{1}{2}})) =$$

$$N_r \cdot \frac{e^m}{\partial \delta_1 \cdots \partial \delta_m} \sum_{\delta_1 = \cdots = \delta_m = 0}^{\infty} \text{Ber} \left[ \text{Hess}(Q(x)) \cdot \frac{1}{1^2 - 2g}(x) \exp(\lambda + v_{det},x) \right] (-[c]_B).$$

The proof of this theorem is analogous to our proof of Theorem 3.3. Using Theorem 6.1 one can also obtain formulas for the Euler characteristics of vector bundles, which involve the exterior powers $\wedge^1 \pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}}))$. Let us briefly explain the case

$$\chi \left( P_0(c), \mathcal{L}(k;\lambda) \otimes \wedge^2 \pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}}) \right).$$

Recall that the $n$-th Adams operator $\psi^n$ is defined by $\psi^n L = L^n$ for a line bundle $L$ and extends to K-theory additively by the splitting principle. It follows from the Grothendieck-Riemann-Roch theorem and equation (22) that

$$\text{ch}(\psi^n(\pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}}))) = \sum_{i \geq 0} n^i \cdot \text{ch}_1(\pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}})) =$$

$$\frac{1}{n} \sum_{i \geq 1} n^i \cdot \pi^*_v(\text{ch}_1(U_{v_1})) = \frac{1}{n} \pi^*_v(\text{ch}(\psi^n(U_{v_1})) = \frac{1}{n} \pi^*_v(\text{ch}(\psi^n(\pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}}))).$$

Since for any vector bundle $V$

$$\text{ch} \left( \wedge^2 V \right) = \frac{\text{ch}(V^\otimes 2) - \text{ch}(\psi^2 V)}{2},$$

the Euler characteristic (38) equals

$$\frac{1}{2} \chi(P_0(c), \mathcal{L}(k;\lambda) \otimes (\pi_1(U_{v_1} \otimes \mathbb{K}^{\frac{1}{2}}))^2) - \frac{1}{4} \chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_1(\psi^2(U_{v_1}) \otimes \mathbb{K}^{\frac{1}{2}})).$$

Finally, note that the character function (cf. page 3) for $\psi^n(\pi_1(U_{v_1}))$ is $\phi^v(x^n)$, hence using Theorem 6.1 we obtain the formula for the Euler characteristic (38).

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