On the Forecast Combination Puzzle

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Abstract. It is often reported in forecast combination literature that a simple average of candidate forecasts is more robust than sophisticated combining methods. This phenomenon is usually referred to as the “forecast combination puzzle”. Motivated by this puzzle, we explore its possible explanations including estimation error, invalid weighting formulas and model screening. We show that existing understanding of the puzzle should be complemented by the distinction of different forecast combination scenarios known as combining for adaptation and combining for improvement. Applying combining methods without consideration of the underlying scenario can itself cause the puzzle. Based on our new understandings, both simulations and real data evaluations are conducted to illustrate the causes of the puzzle. We further propose a multi-level AFTER strategy that can integrate the strengths of different combining methods and adapt intelligently to the underlying scenario. In particular, by treating the simple average as a candidate forecast, the proposed strategy is shown to avoid the heavy cost of estimation error and, to a large extent, solve the forecast combination puzzle.

Key Words: combining for adaptation, combining for improvement, multi-level AFTER, model selection, structural break

1. Introduction

Since the seminal work of Bates and Granger (1969), both empirical and theoretical investigations support that when multiple candidate forecasts for a target variable are available to an analyst, forecast combination often provides more accurate and robust forecasting performance in terms of mean square forecast error (MSFE) than using a single candidate forecast. The benefits of forecast combination are attributable to the facts that individual forecasts often use different sets of information, are subject to model bias from different but unknown model misspecifications, and/or are varyingly affected by structural breaks. The review of Timmermann (2006) provides a comprehensive account of various forecast combination methods. In particular, one popular method is to

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combine forecasts by estimating a theoretically optimal weight through the minimization of mean square error (MSE). For example, Bates and Granger (1969) propose to find the optimal weight using error variance-covariance structure of the individual forecasts. Granger and Ramanathan (1984) construct the optimal weight under a linear regression framework.

Despite the ever-increasing popularity and sophistication of combining methods, it is repeatedly reported from past literature that the simple average (SA) is a very effective and robust forecast combination method that often outperforms more complicated combining methods (see Winkler and Makridakis (1983), Clemen and Winkler (1986) and Diebold and Pauly (1990) for some early examples). In a review and annotated bibliography on earlier studies, Clemen (1989) raises the question, “What is the explanation for the robustness of the simple average of forecasts?”. Specifically, he proposes two questions of interest, “(1) Why does the simple average work so well, and (2) under what conditions do other specific methods work better?” The robustness of SA is also echoed in more recent literature. For example, Stock and Watson (2004) build autoregressive models with univariate predictors (macroeconomic variables) as candidate forecasts for output growth of seven developed countries, and find that SA, together with other methods of least data adaptivity, is among the top-performing forecast combination methods. Stock and Watson (2004) further coin the term “Forecast Combination Puzzle” (for brevity, we refer to the puzzle as FCP hereafter ), which refers to “the repeated finding that simple combination forecasts outperform sophisticated adaptive combination methods in empirical applications”. In another recent example, Genre et al. (2013) use survey data from professional forecasters as the individual candidates to construct combined forecasts for three target variables. Despite some promising results of complicated methods, they further note that the observed improvement over SA is rather vague when a period of financial crisis is included in the analysis. The past empirical evidence appears to support the mysterious existence of FCP, which is also summarized in Timmermann (2006, section 7.1).

Many attempts have been made to demystify FCP. One popular and arguably the most well-studied explanation for FCP is the estimation error of the combining methods
that rely on the optimal weight estimation by MSE minimization. Smith and Wallis (2009) rigorously study the estimation error issue. Using the forecast error variance-covariance structure, they show both theoretically and numerically that the estimator targeting the optimal weight can have large variance and consequently, the estimated optimal weight can be very different from the true optimal weight, often even more so than simple equal weight. Elliott (2011) studies the theoretical maximal performance gain of the optimal weight over SA by optimizing the error variance-covariance structure, and points out that the gain is often small enough to be overshadowed by estimation error. Timmermann (2006) and Hsiao and Wan (2014) also illustrate conditions for the optimal weight to be close to the equal weight so that the relative gain of the optimal weight over SA is small. Claeskens et al. (2014) consider the random weight and show that when the weight variance is taken into account, SA can perform better than using the “optimal” weight. Under linear regression settings, Huang and Lee (2010) discuss the estimation error and the relative gain of the optimal weight.

In addition to estimation error, nonstationarity and structural breaks in the data generating process (DGP) are believed to contribute to the unstable performance of the estimated “optimal” weight. For example, Hendry and Clements (2004) demonstrate that when candidate forecasting models are all misspecified and breaks occur in the information variables, forecast combination methods that target the optimal weight may not perform as well as SA. Also, Huang and Lee (2010) propose that the candidate forecasts are often weak, that is, they have low predictive content on the target variable, making the optimal weight similar to simple equal weight.

While the aforementioned points are valid and valuable, they do not depict the complete picture of the puzzle. In this paper, we provide our perspectives on FCP to contribute to its settling. In our view, besides providing explanations of FCP, it is also very important to point out the potential danger of recommending SA for broad and indiscriminate use. Here, we focus on the mean squared error (MSE). It should be pointed out that the main points are expected to stand for other losses as well (e.g., absolute error) and that some combination approaches (e.g., AFTER) can handle general loss functions.
The rest of this article is organized as follows. In section 2, we list some aspects that have not been much addressed but are important towards the understanding of FCP in our view. We formally introduce the problem setup of the forecast combination problem we consider in section 3. Our understandings of FCP are elaborated in sections 4-8. In particular, section 5 proposes a multi-level AFTER approach to solve FCP. The performance of this approach is also evaluated in section 9 using a U.S. Survey of Professional Forecasters (SPF) data. A brief conclusion is given in section 10.

2. Additional Aspects of FCP

The previous work has nicely pointed out that estimation error is an important source of FCP and has characterized the impact of the estimation error in idealized settings. Indeed, in general, when the forecast combination weighting formula is valid in the sense that an optimal weight can be correctly estimated by minimizing MSE, insufficiently small sample size may not support reliable estimation of the weight, resulting in inflated variance of the combined forecast. The explanation with structural breaks also makes sense for certain situations. However, in our view, there are several additional aspects that need to be considered for understanding FCP.

1. A key factor missing in addressing the FCP is the true nature of improvability of the candidate forecasts. While we all strive for better forecast performance than the candidates, that may not be feasible (at least for the methods considered). Thus we have two scenarios (Yang, 2004): i) One of the candidates is pretty much the best we can hope for (within the considerations of course) and consequently any attempt to beat it will not succeed. We refer to this scenario as “Combining for Adaptation” (CFA), because the proper goal of a forecast combination method under this scenario should be targeting the performance of the best individual candidate forecast, which is unknown. ii) The other is that a significant gain of accuracy over all the individual candidates can be materialized. We refer to this scenario as “Combining for Improvement” (CFI), because the proper goal of a forecast combination method under this scenario should be targeting the performance of the best combination of the candidate forecasts to overcome
defects of the candidates. In our experience, both scenarios occur commonly in real problems. Without factoring in this aspect, comparison of different combination methods may be grossly misleading due to the well-known sin of comparing apples to oranges. In our view, empirical studies on forecast combinations in the future need to bring this lurking aspect into the analysis. With the above forecast combination scenarios spelled out, a natural question follows: Can we design a combination method to bridge the two camps of methods proposed for the two scenarios respectively, so as to help solve the FCP?

2. The methods being examined in the literature on FCP are mostly specific choices (e.g., least squares estimation). Can we do better with other methods (that may or may not have been invented yet) to avoid the heavy estimation price? Also, the currently investigated methods often assume the forecasts are unbiased and the forecast errors are stationary, which may not be proper for many applications. What happens when these assumptions fail?

3. It has been stated in the literature that the simple methods (e.g., SA) are robust based on empirical studies. We feel this is not necessarily true in the usual statistical sense (rigorously or loosely). In many published empirical results, the candidate forecasts were carefully selected/built and thus well-behaved. Therefore, the finding in favor of robustness of SA may be proper only for such situations that the data analyst has extensive expertise on the forecasting problem and has done quite a bit of work on screening out poor/un-useful candidates. We argue that it is much more desirable to investigate FCP broadly so as to allow the possibility of poor/redundant candidates for wider and more realistic applications. It should be added that in various situations, the screening of forecasts is far from an easy task and its complexity may well be at the same level as model selection/averaging. Therefore, even for top experts, the view that we can do a good job in screening the candidate forecasts and then simply recruit SA is overly optimistic. With the above, an important matter is to examine the robustness of SA in a broader context.
As is described in the first item, there are two distinct scenarios: CFA and CFI. The CFA scenario can happen if one of the candidate forecasts is based on a model sophisticated enough to capture the true DGP (yet still relatively simple), and/or the other candidate forecasts only add redundant information. The CFI scenario can often happen when different candidate forecasts use different information, and/or their underlying models have misspecifications in different ways.

There are different existing combining methods designed for the two scenarios. The methods for the CFI scenario typically seek to estimate the optimal weight aggressively, and their examples include variance-covariance based optimization (Bates and Granger, 1969) and linear regression (Granger and Ramanathan, 1984). These methods are likely to suffer from estimation error, causing unstable performance relative to SA. On the other hand, the combining methods for the CFA scenario should ideally perform similarly to the best individual candidate forecast and should not be subject as severely to estimation error as the methods for CFI. The typical methods suitable for the CFA scenario include AIC model averaging (Buckland et al., 1997) and Bayesian model averaging (e.g., Garratt et al., 2003), both in parametric settings. The method of AFTER (Yang, 2004) can be applied more broadly in parametric and non-parametric settings, regardless of the nature of the candidate forecasts. As one of the main contributions in this article, we show that the distinction between the two scenarios provides one of the keys to understanding the FCP. We will see in section 4 that an analyst who fails to understand and bring in the underlying scenarios and specific types of data when choosing the combining methods can incorrectly apply a combining method not designed for the underlying scenario and consequently deliver forecasting results worse than other methods (e.g., SA).

For the questions raised in the second item regarding whether we can avoid the estimation price, we cannot fully address them without a proper framework, because for any sensible method, one can always find a situation to favor it to its competitors. The framework we consider with sound theoretical support is through a minimax view: If one has a specific class of combination of the forecasts in mind and wants to target the best combination in this class, then without any restriction/assumption on unbiasedness of the candidate forecasts and stationarity of the forecast errors, the minimax view seeks
a clear understanding of the minimum price we have to pay no matter what method (existing or not) is used for combining. It turns out that the framework from the minimax view is closely related to the forecast combination scenarios discussed in the first item, and Yang (2004) provides a detailed theoretical exposition of the distinct forecast combination scenarios and associated minimax results.

Indeed, Yang (2004) shows that from a minimax perspective, because of the aggressive target set for the CFI scenario, we have to pay an unavoidably heavier cost than the target set under the CFA scenario. Specifically, if we let $K$ denote the number of forecasts and $T$ denote the forecasting horizon, Yang (2004) shows that when the target is to find the optimal weight to minimize the general empirical risk over a set of weights satisfying a convex constraint (which is appropriate under the CFI scenario), the estimation cost is $O(K\log(1+T/K)T)$ for relatively large $T$ ($T > K^2$), and $O(\log(K)/\sqrt{T\log T})$ for relatively small $T$ ($T \leq K^2$). In contrast, if the target is to match the performance of the best individual forecast (which is appropriate under the CFA scenario), the estimation cost is only $O(\log(K)/T)$.

Because of the unavoidable heavy cost under the CFI scenario, it is not always ideal to pursue the aggressive target of the optimal weight. Indeed, even if the optimal weight gives better performance than the best individual candidate, the improvement may not be enough to offset the additional estimation cost (i.e., increased variance) as precisely (in minimax rate) identified in Yang (2004) and Wang et al. (2014). As another contribution of our work, we show in section 6 that an appropriately constructed forecast combination strategy can perform in a smart way according to the underlying CFI or CFA scenario. If CFI is the correct scenario, the proposed strategy can behave both aggressively and conservatively so that it performs similar to SA when SA is much better than e.g., the linear regression method.

Besides the estimation error and the necessary distinction of underlying scenarios discussed in the first two items, the following three reasons can also contribute to FCP. First, the weighting derivation formula used by complicated methods is often not suitable for the situation. For example, under structural breaks, old historical data no longer hold support for a valid optimal weighting scheme, and the known justification of well-
established combining methods fails as a result. Indeed, Hendry and Clements (2004) demonstrate that when candidate forecasting models are all misspecified and breaks occur in the information variables, methods that estimate the optimal weight may not perform as well as SA. In section 7, our Monte Carlo examples also show that SA may dominate the complicated methods when breaks occur in DGP dynamics. Second, it is common practice that the candidate forecasts are already screened in some ways so that they are more or less on an equal footing. For example, Stock and Watson (1998) and Stock and Watson (2004) apply various model selection methods such as AIC and BIC to identify promising linear or nonlinear candidate forecast models. Recently, Bordignon et al. (2013) select models of different types (ARMAX, time-varying coefficients, etc.) and suggest that SA works well when combining a small number of well-performing forecasts. In studies using survey data of professional forecasters, it is also expected that each professional forecaster performs some model screening before satisfactorily settling down with their own forecast. In these cases, there may not be particularly poor candidate forecasts, and the candidates (at least the top ones) may tend to contribute more or less equally to the optimal combination, making SA a competitive method. In section 8, we use Monte Carlo examples to show that screening can be a source of FCP. Lastly, the puzzle can also be a result of publication bias; people do not tend to emphasize the performance of SA when SA does not work well.

With all our understandings of FCP discussed above, we address the issues raised in the third item and provide further information on robustness of SA in sections 6-8. In particular, we will see that SA is actually not robust in performance in several directions: its performance may change significantly or even substantially when i) an optimal, poor or redundant forecast is added; or ii) the degree of the screening of the candidate forecasts is done differently. In addition, the size of the rolling window to deal with structural breaks affects the relative performance of SA as well. Fortunately, as will be seen, some combination methods can largely avoid these defects.
3. Problem Setup

Suppose that an analyst is interested in forecasting a real-valued time series $y_1, y_2, \ldots$. Given each time point $t \geq 1$, let $x_t$ be the (possibly multivariate) information variable vector revealed prior to the observation of $y_t$. The $x_t$ may not be accessible to the analyst. Conditional on $x_t$ and $z_{t-1} =: \{(x_j, y_j), 1 \leq j \leq t-1\}$, $y_t$ is subsequently generated from some unknown distribution $p_t(\cdot | x_t, z_{t-1})$ with conditional mean $m_t = E(y_t | x_t, z_{t-1})$ and conditional variance $v_t = \text{Var}(y_t | x_t, z_{t-1})$. Then, $y_t$ can be represented as $y_t = m_t + \varepsilon_t$, where $\varepsilon_t$ is the random noise with the conditional mean and the conditional variance being 0 and $v_t$, respectively.

Assume that prior to the observation of $y_t$, the analyst has access to $K$ real-valued candidate forecasts $\hat{y}_{t,i}$ ($i = 1, \cdots, K$). These forecasts may be constructed with different model structures, and/or with different components of the information variables, but the details regarding how each original forecast is created may not be available in practice and are not assumed to be known. The analyst’s objective in (linear) forecast combination is to construct a weight vector $w = (w_1, \cdots, w_K)^T \in \mathbb{R}^K$, based on the available information prior to the observation of $y_t$, to find a point forecast of $y_t$ by forecast combination $\hat{y}_{t,w} = \sum_{i=1}^K w_i \hat{y}_{t,i}$. The weight vector may be different at different time points.

To gauge the performance of a procedure that produces forecasts $\{\hat{y}_t, t = 1, 2, \ldots\}$ given time horizon $T$, we consider the average forecast risk

$$R_T = \frac{1}{T} \sum_{t=1}^T E(y_t - \hat{y}_t)^2$$

in our analysis and simulation studies. For real data evaluation, since the risk cannot be computed, we use the mean square forecast error (MSFE) as a substitute:

$$\text{MSFE}_T = \frac{1}{T} \sum_{t=1}^T (y_t - \hat{y}_t)^2.$$  

According to the FCP, simple methods with little or no time variation in weight $w$ (e.g., equal weighting) often outperform complicated methods with much time variation in terms of $R_T$ and MSFE$_T.$
4. CFA versus CFI: A Hidden Source of FCP

In this section, we study the performance of forecast combination methods under the two distinct scenarios. Failure to recognize these scenarios can itself result in the FCP. We use two simple but illustrative Monte Carlo examples under regression settings similar to those of Huang and Lee (2010) to demonstrate the CFA and CFI scenarios.

**Case 1.** Suppose \( y_t (t = 1, \cdots, T) \) is generated by the linear model

\[
y_t = x_t \beta + \varepsilon_t,
\]

where \( x_t \)'s are i.i.d. \( N(0, \sigma_X^2) \), and \( \varepsilon_t \)'s are independent of \( x_t \)'s and are i.i.d. \( N(0, \sigma^2) \). Consider the two candidate forecasts generated by

- Forecast 1: \( \hat{y}_{t,1} = x_t \hat{\beta}_t \);
- Forecast 2: \( \hat{y}_{t,2} = \hat{\alpha}_t \),

where \( \hat{\beta}_t \) and \( \hat{\alpha}_t \) are both obtained from the ordinary least square (OLS) estimation using historical data.

Given that Forecast 1 essentially represents the true model, its combining with Forecast 2 cannot improve over the performance of the best individual forecast asymptotically, thus giving an example of the CFA scenario. Let \( T_0 \) be a fixed start point of the evaluation period, and let \( T \) be the end point. Given the evaluation period from \( T_0 \) to \( T \), let \( R_{T,1} \), \( R_{T,2} \) and \( R_{T,w} \) be the average forecast risks of Forecast 1, Forecast 2 and the combined forecast, respectively. If we let \( R_{T,SA} \) be the average forecast risk at time \( T \) for SA, we expect that \( R_{T,SA} > R_{T,1} \). Indeed, Proposition 2 in the Appendix shows

\[
\frac{R_{T,1}}{R_{T,SA}} \rightarrow \frac{\sigma^2}{\sigma^2 + \beta^2 \sigma_X^2 / 4} \quad \text{as} \quad T \rightarrow \infty,
\]

and asymptotically, the optimal combination assigns all the weight on Forecast 1.

Under the CFA scenario, since the best candidate is unknown, the natural goal of forecast combination is to match the performance of the best candidate.
Case 2. Suppose $y_t$ ($t = 1, \cdots, T$) is generated by the linear model

$$y_t = (x_{t,1} + x_{t,2}) \beta + \varepsilon_t,$$

where the $\mathbf{x}_t = (x_{t,1}, x_{t,2})^T$ are i.i.d. following a bivariate normal distribution with mean $0$ and common variance $\sigma^2_X = \sigma^2_{X_1} = \sigma^2_{X_2}$. Let $\rho$ denote the correlation between $x_{t,1}$ and $x_{t,2}$. The random error $\varepsilon_t$’s are independent of $\mathbf{x}_t$’s and are i.i.d. $N(0, \sigma^2)$. Consider the two candidate forecasts generated by

Forecast 1: $\hat{y}_{t,1} = x_{t,1} \hat{\beta}_{t,1}$;

Forecast 2: $\hat{y}_{t,2} = x_{t,2} \hat{\beta}_{t,2},$

where $\hat{\beta}_{t,1}$ and $\hat{\beta}_{t,2}$ are both obtained from OLS estimation with historical data.

Different from Case 1, Case 2 presents a scenario where each candidate forecast employs only part of the information set. It is expected, to some extent, that combining the two forecasts works like pooling different sources of important information, resulting in performance better than either of the candidate forecasts. By defining the average forecast risks $R_{T,1}, R_{T,2}, R_{T,SA}$ the same way as in Case 1, we can see from Proposition 3 in the Appendix that

$$\frac{R_{T,1}}{R_{T,SA}} \rightarrow \frac{\sigma^2_X \beta^2 (1 - \rho^2) + \sigma^2}{\sigma^2_X \beta^2 (1 - \rho^2)(1 - \rho)/2 + \sigma^2} \quad \text{as} \quad T \rightarrow \infty. \quad (2)$$

Clearly, when the two information sets are not highly correlated, SA can improve the forecast performance over the best candidate. This case gives a typical example of the CFI scenario, and it is appropriate to seek the more aggressive goal of finding the best linear combination of candidate forecasts.

Our view is that discussion of the FCP should take into account the different combining scenarios. Next, we perform Monte Carlo studies on the two cases to provide an explanation of the puzzle. Combining methods suitable for the CFA scenario have been developed to target performance of the best individual candidate. In our numerical studies, we choose the AFTER method [Yang 2004] as the representative, and it is known that AFTER pays a smaller estimation price than methods that target the
optimal linear or convex weighting. In contrast, combining methods for the CFI scenario usually attempt to estimate the optimal weight. We choose linear regression of the response on the candidate forecasts (LinReg) as the representative. The method of Bates and Granger (1969) without estimating correlation (BG for brevity) is used as an additional benchmark.

For Case 1, we perform simulations as follows. Set $\sigma^2 = \sigma^2_X = 1$. Consider a sequence of 20 $\beta$’s such that the corresponding signal-to-noise (S/N) ratios are evenly spaced between 0.05 and 5 in the logarithmic scale. For each $\beta$, we conduct the following simulation 100 times to estimate the average forecast risk. A sample of 100 observations is generated. The first 60 observations are used to build the candidate forecast models, which are subsequently used to generate forecasts for the remaining 40 observations. Forecast combination methods including SA, BG, AFTER and LinReg methods are applied to combine the candidate forecasts, and the last 20 observations are used for performance evaluation. The average forecast risk of each forecast combination method is divided by that of SA to obtain the normalized average forecast risk (denoted by normalized $R_T$). The results are summarized in Figure 1. For Case 2, we set $\beta = \beta_1 = \beta_2$, $\rho = 0$ and $\sigma^2 = \sigma^2_{X_1} = \sigma^2_{X_2} = 1$. The remaining simulation settings are the same as Case 1. The normalized average forecast risks (relative to SA) are summarized in Figure 2.

In Case 1, it is clear from Figure 1 that AFTER is the preferred method of choice under the CFA scenario. LinReg, on the other hand, consistently underperforms compared to AFTER. Interestingly, when S/N is relatively low (less than 0.35), we observe the “puzzle” that LinReg performs worse than SA, which is due to the weight estimation error. If the analyst correctly identifies that it is the CFA scenario and applies a corresponding method like AFTER, the “puzzle” disappears: AFTER can perform better than (or very close to) SA, while LinReg fails.

In Case 2, if the analyst applies AFTER without realizing the underlying CFI scenario, we observe the “puzzle” that SA outperforms AFTER. The “puzzle” is not entirely surprising since AFTER is designed to target the performance of the best individual forecast, while [2] shows that SA can improve over the best individual forecast. LinReg appears to be the correct method of choice when S/N ratio is relatively high. However,
similar to what is observed in Case 1, LinReg suffers from weight estimation error when S/N ratio is low, once again giving the “puzzle” that LinReg performs worse than SA.

Case 2 also shows the interesting observation that it is not always optimal to apply SA even when SA is the “optimal” weight in a restricted sense. Indeed, (A.2) and (A.3) in Proposition 3 imply that if we adopt the common restriction that the sum of all weights is 1, SA is the asymptotic optimal weight. However, if we impose no restriction on the weight range, the asymptotic optimal weight assigns a unit weight to each candidate forecast. This explains the advantage of LinReg over SA in Case 2 when the S/N ratio
Figure 2: (Case 2) Comparing the average forecast risk of different forecast combination methods (dashed line represents the SA baseline; x-axis is in logarithmic scale).

The observations above illustrate that different combining methods can have strikingly different performance depending on the underlying scenario. The FCP can appear when a combining method is not properly chosen according to the correct scenario. Without knowing the underlying scenario, comparing these methods may not provide a complete picture of FCP, and blindly applying SA may result in sub-optimal performance. We advocate the practice of trying to identify the underlying scenario (CFA or CFI) when considering forecast combination. It should be pointed out that when
the relevant information is limited, it may not be feasible to confidently identify the forecast combination scenario. In such a case, a forced selection, similar to the comparison of model selection and model combining (averaging) described in Yuan and Yang (2005), would induce enlarged variability of the resulting forecast. A better solution is an adaptive combination of forecasts as illustrated in the next section.

5. Multi-level AFTER

With the understanding in section 4, we see that when considering forecast combination methods, an effort should be made to understand whether there is much room for improvement over the best candidate. When this is difficult to decide or impractical to implement due to handling a large number of quantities to be forecast in real time, we may turn to the question: Can we find an adaptive (or universal) combining strategy that performs well in both CFA and CFI scenarios? Note that here adaptive refers to adaptation to the forecast combination scenario (instead of adaptation to achieving the best individual performance). Another question follows: Under the CFI scenario, can the adaptive combining strategy still perform as well as SA when the price of estimation error is high? As we have seen in Case 2 of section 4, using methods (e.g., LinReg) intended for the CFI scenario alone cannot successfully address the second question.

It turns out that the answers to these two questions are affirmative. The idea is related to a philosophical comment in Clemen et al. (1995):

“Any combination of forecasts yields a single forecast. As a result, a particular combination of a given set of forecasts can itself be thought of as a forecasting method that could compete...”

The use of combination of forecast (or procedure) combinations is a theoretically powerful tool to achieve adaptive minimax optimality (see, e.g., Yang (2004), Wang et al. (2014)). In the context of our discussion, combined forecasts such as SA, AFTER and LinReg can all be considered as the candidate forecasts and may be used as individual candidates in a forecast combination scheme.

Accordingly, we design a two-step combining strategy: first, we construct three new candidate forecasts using SA, AFTER and LinReg; second, we apply the AFTER al-
gorithm on these new candidate forecasts to generate a combined forecast. We refer to this two-step algorithm as multi-level AFTER (or mAFTER for short) because two layers of AFTER algorithms are involved. The key lies in the AFTER algorithm on the second step, which allows mAFTER to automatically target the performance of the best individual candidate among SA, AFTER and LinReg. Under the CFA scenario, mAFTER can perform as if we are using AFTER alone considering that AFTER is the proper method of choice. Under the CFI scenario, mAFTER can perform closely to the better of SA and LinReg. Thus, when LinReg suffers from severe estimation error, mAFTER will perform closely to SA and thereby avoid the high cost.

Indeed, if we denote the forecasts generated from SA, LinReg and mAFTER by \( \hat{y}_t^{(SA)} \), \( \hat{y}_t^{(LR)} \) and \( \hat{y}_t^{(M)} \), respectively, we have Proposition 1 as follows.

**Proposition 1.** Under the regularity conditions shown in the Appendix, the average forecast risk of the mAFTER strategy satisfies

\[
\frac{1}{T} \sum_{t=T_0}^{T} \mathbb{E}(y_t - \hat{y}_t^{(M)})^2 \leq \inf_{1 \leq i \leq K} \frac{1}{T} \sum_{t=T_0}^{T} \mathbb{E}(y_t - \hat{y}_t^{(i)})^2 + \frac{c_1 \log(K)}{T} + \frac{1}{T} \sum_{t=T_0}^{T} \mathbb{E}(y_t - \hat{y}_t^{(SA)})^2 + \frac{c_2}{T},
\]

where \( c_1 \) and \( c_2 \) are some positive constants not depending on the time horizon \( T \).

Proposition 1 is a consequence of Theorem 5 in Yang (2004). It indicates that, in terms of the average forecast risk, mAFTER can match the performance of the best original individual forecast, the SA forecast and the LinReg forecast (whichever is the best), with a relatively small price of order at most \( \log(K)/T \).

To confirm that the mAFTER strategy can solve the “puzzles” illustrated in the previous section, we repeat the simulation studies of Case 1 and Case 2 and summarize the results in Figure 3 and Figure 4, respectively. In Case 1, it suffices to see that mAFTER correctly tracks the performance of AFTER. In Case 2, when S/N is relatively large (> 0.5), mAFTER takes advantage of the opportunity to improve over the original individual forecasts and performs very closely to LinReg; when S/N is relatively small (< 0.5), mAFTER behaves very similarly to SA and successfully avoids the heavy
estimation error suffered by LinReg. Therefore, rather than relying on SA, a “sophisticated” combining strategy like mAFTER can be an appealingly safe method that avoids FCP.

Note that mAFTER is a rather general forecast combination strategy. In the first step of the strategy, the analyst can choose their own way of generating new candidate forecasts (not necessarily restricted to AFTER and LinReg), as long as they include SA, representative methods for the CFA scenario, and representative methods for the CFI scenario. AFTER and LinReg are simply chosen in our study as convenient representatives. We also demonstrate the performance of the mAFTER strategy in the real data example in section 9.

6. Is SA Really Robust?

The SA has been praised for being robustly among top performers relative to other forecast combination methods. It is obvious that SA cannot be robust in the traditional statistical sense: even a single really bad candidate can damage the performance of the combined forecast to an arbitrarily worse position. A more interesting question is to assess robustness of SA in practically relevant settings.

The previous two sections have shown that SA is not robust in terms of its relative performance when dealing with the two different scenarios. In this section, we show that SA is not robust even in the loose sense when new forecast candidates are added to the candidate pool, especially if the new candidates have only redundant information with respect to the original candidate pool. In contrast, the AFTER-type combining methods can be rather robust against adding poor or redundant candidate forecasts. Here, we consider the following three cases.

Case 3. Suppose a new information variable \( x_{t,3} \) has the same distribution as \( x_{t,1} \), and is independent of \( z_{t-1} \) and \((x_{t,1}, x_{t,2})\). A new candidate forecast \( \hat{y}_{t,3} = x_{t,3}\hat{\beta}_{t,3} \) joins the candidate pool in Case 2, where \( \hat{\beta}_{t,3} \) is obtained from OLS estimation with historical data.

Case 4. A new candidate forecast \( \hat{y}_{t,3} = x_{t,2}\hat{\beta}_{t,2} \) identical to Forecast 2 joins the candi-
Figure 3: (Case 1) Performance of mAFTER under adaptation scenario (dashed line represents the SA baseline; x-axis is in logarithmic scale).

Date pool in Case 2.

Case 5. A new candidate forecast $\hat{y}_{t,3} = \tilde{x}_{t,2}\tilde{\beta}_{t,2}$ is generated using a transformed information variable $\tilde{x}_{t,2} = \exp(x_{t,2})$, where $\tilde{\beta}_{t,2}$ is obtained from OLS estimation with historical data.

Note that the new candidate in Case 3 is a very poor forecast, while the new candidates in Case 4 and Case 5 contain a subset of the information variables. In all of the cases above, no new information is added to the candidate pool. Following the same simulation setting as Case 2, we focus on SA and AFTER and compute the ratio be-
Figure 4: (Case 2) Performance of mAFTER under improvement scenario (dashed line represents the SA baseline; x-axis is in logarithmic scale).

between the MSFE after adding the new candidate and the MSFE in Case 2. Figure 5 shows that the performance of AFTER remains almost the same, while the performance of SA worsens after adding the non-informative or redundant candidate forecasts.

7. Improper Weighting Formulas: A Source of the FCP Revisited

Generally speaking, the popular forecast combination methods often implicitly assume that the time series and/or the forecast errors are stationary. It is expected in theory that they should perform well if we have access to long enough historical data.
In practice, however, such derived weighting formulas can often be unsuitable when the DGP changes and the candidate forecasts cannot adjust quickly to the new reality. For example, it is often believed that structural breaks can unexpectedly happen, making the relative performance of the candidate forecasts unstable and giving us the impression that SA performs well.

Next, we use a Monte Carlo example to illustrate the FCP under structural breaks. Rather than assuming deterministic shifts in information variables (Hendry and Clements, 2004), we consider breaks in the DGP dynamics:

\[
y_t = \begin{cases} 
  \sum_{k=1}^{4} \beta_{1,k}y_{t-k} + \varepsilon_t & \text{if } 1 \leq t \leq 50, \\
  \beta_{2,1}y_{t-1} + \beta_{2,2}y_{t-2} + \varepsilon_t & \text{if } 51 \leq t \leq 100, \\
  \beta_{3,1}y_{t-1} + \varepsilon_t & \text{if } 101 \leq t \leq 150,
\end{cases}
\]

where the coefficients \( \beta_{j,k} \) \((j = 1, 2, 3)\) are randomly generated from the uniform distribution on \((0, 1)\), and \(\varepsilon_t\)'s are \text{i.i.d.} \(N(0, 1)\). Here, structural breaks happen at \(t = 50\) and

Figure 5: Studying the robustness of SA against adding new candidate forecasts.
The candidate forecast models are autoregressions from lag 1 to lag 6, and we apply SA, BG, LinReg and AFTER to generate the combined forecasts. The simulation is repeated 100 times, and the last 100 time points serve as the evaluation period to obtain the average forecast risk. For comparison, we consider BG, LinReg and AFTER methods with estimation rolling window size $rw = 20$ or 40, meaning only the most recent $rw$ observations are used to estimate the weights for each forecast. The results are summarized in Table 1. The average forecast risk is normalized with respect to SA, and numbers in parentheses are standard errors.

Table 1: Comparing the normalized average forecast risk of different combination methods under structural breaks.

|       | SA   | LinReg | BG    | AFTER |
|-------|------|--------|-------|-------|
| standard | 1.000 | 1.026 (0.011) | 1.005 (0.003) | 1.047 (0.010) |
| $rw = 40$ | 1.000 | 1.060 (0.033) | 0.992 (0.002) | 0.991 (0.009) |
| $rw = 20$ | 1.000 | 1.64 (0.42) | 0.980 (0.003) | 0.952 (0.007) |

We can see from Table 1 that all three standard combining methods, when finding weights using all historical data, underperform compared to SA due to the unstable relative performance of candidate forecasts. As we shrink the estimation window size to the most recent 40 and 20 time points, BG and AFTER achieve better performance than SA while the performance of LinReg worsens. This result can be understood by noting that there are two opposing factors when we shrink the weight estimation window. When using only the most recent forecasts, we decrease the bias of the weighting formula supported by the old data but simultaneously increase the variance of the estimated weight. Among the three methods considered, the estimation error factor dominates for LinReg. On the other hand, AFTER is not designed to aggressively target the optimal weight, thus benefiting the most from the shrinking rolling window.

Due to the complex impact of structural breaks on forecast combination methods, it is arguably true that the focus should be made on how to detect the problem (see, e.g.,
and how to come up with new combining forms accordingly (e.g., using the most recent observations to avoid an improper weighting formula). However, proper identification of structural breaks can be difficult to achieve in practice, and this example shows that in the presence of structural breaks, the relative performance of SA is not as robust as BG and AFTER with naïvely chosen rolling windows.

8. Linking Forecast Model Screening to FCP

In empirical studies, the candidate forecasting models are often screened/selected in some way to generate a smaller set of candidates for combining. As is demonstrated in Case 3 of section 6, the performance of SA is particularly susceptible to poor-performing candidate models. The common practice of model screening may contribute to improving the performance of SA.

Next, we illustrate the impact of screening with a Monte Carlo example. Let \( x_t \in \mathbb{R}^p \) \((p = 20)\) be the \( p \)-dimensional information variable vector randomly generated from a multivariate normal distribution with mean \( 0 \) and covariance \( \Sigma \), where \((\Sigma)_{i,j} = \rho^{|i-j|}\) and \( \rho = 0 \) or \( 0.5 \). Consider a DGP with linear model setting

\[
y_t = x_t^T \beta + \varepsilon_t,
\]

where coefficient \( \beta = (3, 3, 2, 1, 1, 1, 0, 0, \ldots, 0) \) and \( \varepsilon_t \) are i.i.d. \( N(0, \sigma^2) \) with \( \sigma = 2 \) or 4. Under this setting, only the first 7 variables in \( x_t \) are important for \( y_t \), while the remaining variables are redundant.

If we assume that the analyst has full access to the information vector \( x_t \)'s, we may build linear models as the candidate forecasts with any subset of the information variables. It is known from Wang et al. (2014) that if we select the best subset model with the right size using the ABC criterion (Yang, 1999) or combine the subset regression models by proper adaptive combining methods (Yang, 2001), the prediction risk can adaptively achieve the minimax optimality over soft and hard sparse function classes.

Inspired by this result, we consider the following screening-and-combining approach. First, given the model size (that is, the number of information variables used in a
candidate linear model), choose the best OLS model based on estimation mean square error. Second, from the \( p \) models selected from the first step, find the top \( X\% \) (\( X = 10, 20, 40, 60, 80 \)) of the models based on the ABC criterion. Note that the ABC criterion for a subset model with size \( r \) is 

\[
ABC(r) = \sum_{t=1}^{n} (y_t - \hat{y}_{t,r})^2 + 2r\sigma^2 + \sigma^2 \log \left( \frac{p}{r} \right),
\]

where \( n \) is the estimation sample size, \( \hat{y}_{t,r} \) is the fitted response, and \( \sigma^2 \) can be replaced by the estimation mean square error. The remaining subset models after the two-step screening are used to build the candidate forecasts for combining. In simulation, the total time horizon is set to be 200. The screening procedures are applied to the first 100 observations, and the remaining models are used to build the candidate forecasts for the latter 100 time points. Different forecast combination methods are applied, and their performances are evaluated using the last 50 observations. The simulation is repeated 100 times, and the normalized average forecast risk (relative to SA) is summarized in Table 2.

Table 2 shows that AFTER outperforms all the other competitors, including SA. This is consistent with our understanding of a typical CFA scenario, under which AFTER is the proper choice of combining methods. However, as we decrease \( X \) and select smaller sets of candidate forecasts for combining, the performance of SA gradually approaches that of AFTER. Such a result is not entirely surprising considering that when only the top few models are selected, simply averaging them can perform similarly to the optimal results obtained by the proper subset selection or combination methods (Wang et al., 2014). LinReg, which is not a proper choice under the CFA scenario, appears to underperform compared to SA. As \( X \) decreases, LinReg becomes less subject to weight estimation error, and the performance of LinReg improves relative to SA.

From this example, we can see that the performance of SA is not robust to the degree of screening. Generally, it is a very challenging task to ensure an optimal screening to make SA perform well. As a result, although SA works relatively well in this particular example for aggressive screening (keeping very few candidates), SA should not be preferred in general. Without a good screening/selection rule, it leaves too much freedom for the analyst to make poor decisions. We note that a possible solution is to first create new candidate forecasts (e.g., forecasts generated by linear regression method...
Table 2: Comparing the normalized average forecast risk of different forecast combination methods after the screening procedure.

| Top X% | 10% | 20% | 40% | 60% | 80% |
|--------|-----|-----|-----|-----|-----|
|        | $\sigma = 2, \rho = 0$ |
| **AFTER** | 0.998 | 0.989 | 0.966 | 0.951 | 0.945 |
| **BG**  | 1.000 | 0.999 | 0.997 | 0.997 | 0.996 |
| **LinReg** | 1.017 | 1.024 | 1.056 | 1.098 | 1.151 |
|        | $\sigma = 2, \rho = 0.5$ |
| **AFTER** | 0.996 | 0.990 | 0.968 | 0.956 | 0.951 |
| **BG**  | 1.000 | 0.998 | 0.997 | 0.997 | 0.996 |
| **LinReg** | 1.013 | 1.024 | 1.043 | 1.095 | 1.159 |
|        | $\sigma = 4, \rho = 0.5$ |
| **AFTER** | 0.994 | 0.987 | 0.984 | 0.981 | 0.974 |
| **BG**  | 0.999 | 0.998 | 0.998 | 0.998 | 0.997 |
| **LinReg** | 1.002 | 1.012 | 1.056 | 1.101 | 1.163 |

ods) to utilize most or all of the important information, and then the roles of a good screening/selection rule can be played by applying the multi-level AFTER approach (introduced in section 5) on both the original forecasts and the combined forecasts to reduce the influence of the poor-performing or redundant forecasts.
9. Real Data Example

In this section, we study the U.S. SPF (Society of Professional Forecasters) dataset to evaluate SA and the mAFTER strategy. This dataset is a quarterly survey on macroeconomic forecasts in the United States. Lahiri et al. (2013) nicely handled the missing forecasts by adopting two missing forecast imputation strategies known as the regression imputation (REG-Imputed) and the simple average imputation (SA-Imputed) to generate the complete panels. As pointed out by Lahiri et al. (2013), the change of data administration agency in 1990 and the subsequently shifting missing data pattern make it difficult to use the entire data period for meaningful evaluation. Therefore, we inherit their missing forecast imputation as well as the forecast selection strategies, and focus on the period from 1968:Q4 to 1990:Q4 to evaluate the performance of the mAFTER strategy.

Three macroeconomic variables are considered: seasonally-adjusted annual rate of change for GDP price deflator (PGDP), growth rate of real GDP (RGDP) and quarterly average of monthly unemployment rate (UNEMP). The datasets for RGDP and PGDP have 14 candidate forecasts, and the datasets for UNEMP have 13 candidate forecasts. Each forecast provides $g$-quarter ($g = 1, 2, 3, 4$) ahead forecasting. We apply SA, AFTER, BG, LinReg and mAFTER to each SPF dataset of a macroeconomic variable with a given missing forecast imputation method. Each forecast combination method uses the first one fourth of the total time horizon to build up the initial weights, and the remaining time points are used to calculate the normalized MSFE of each method relative to SA. By taking the average over the four MSFEs that correspond to the 1,2,3,4-quarter ahead forecasting, we summarize the performance of different combining methods in Table 3.

From Table 3, although AFTER performs quite differently with different target macroeconomic variables, the mAFTER strategy delivers overall robust performance for all three variables. For PGDP, AFTER performs the best, and beats SA by as much as 10%. Using mAFTER successfully maintains this advantage over SA. For RGDP, while SA and BG beat AFTER by up to 13%, mAFTER successfully pulls the performance to
Table 3: Comparing the performance of forecast combination methods with SPF datasets (values shown are normalized MSFEs averaged over 1,2,3,4-quarter ahead forecasting).

| Target Variable | SA  | LinReg | BG  | AFTER | mAFTER |
|-----------------|-----|--------|-----|--------|--------|
| **REG-imputed** |     |        |     |        |        |
| PGDP            | 1.00| 1.88   | 0.95| 0.90   | 0.90   |
| RGDP            | 1.00| 1.64   | 1.00| 1.11   | 1.01   |
| UNEMP           | 1.00| 1.79   | 0.99| 0.98   | 0.98   |
| **SA-imputed**  |     |        |     |        |        |
| PGDP            | 1.00| 2.17   | 0.98| 0.95   | 0.95   |
| RGDP            | 1.00| 1.83   | 1.00| 1.13   | 1.03   |
| UNEMP           | 1.00| 1.69   | 0.99| 0.97   | 0.98   |

within 3% of SA. Finally, for the UNEMP variable, SA, BG and AFTER all perform very similarly with no more than a 3% difference, and the performance of mAFTER does not deviate much from either SA or AFTER. The LinReg method that aggressively pursues the optimal weight performs poorly for all three target variables. It is interesting to note from Figure 6 that for both PGDP and RGDP variables, the largest performance difference between SA and AFTER is found in the one-quarter ahead forecasting; in each case, mAFTER robustly matches the better of SA and AFTER.

10. Conclusions

Inspired by the seemingly mysterious FCP, we provide our explanations of why the puzzle often occurs and investigate when a sophisticated combining method can work well compared to the simple average (SA). Our study illustrates that the following reasons can contribute to the puzzle.

First, estimation error is known to be an important source of FCP. Both theoretical and empirical evidence show that a relatively small sample size may prevent some com-
Figure 6: Comparing normalized MSFEs of different forecast combination methods with REG-Imputed SPF datasets. Left panel: PGDP variable. Right panel: RGDP variable. For each method, the bars from left to right represents 1,2,3,4-quarter ahead forecasting results, respectively. The dashed line represents the SA baseline.

- Binning methods from reliably estimating the optimal weight.
- Second, FCP can appear if we apply a combining method without consideration of the underlying data scenarios. The relative performance of SA may depend heavily on which scenario is more proper for the data.
- Third, the weighting formula of the combining methods is not always appropriate for the data, because structural breaks and shocks can unexpectedly happen. The weighting formula obtained by sophisticated methods cannot adjust fast enough to the reality, resulting in performance less stable than SA.
- Fourth, candidate forecasts are often screened in some way so that the remaining forecasts used for combining tend to have similar performance, and SA may tend to work well in such cases. However, SA can be sensitive to the screening process, and enlarging the pool of candidates may benefit other combination methods; therefore, empirical observations that SA works well after model screening should be taken with a grain of salt.
- Fifth, there may be publication bias in that people tend to report the existence of FCP when SA gives good empirical results but may not emphasize the performance of SA when it gives mediocre results.
Regarding the first two reasons above, our study shows that it is not hard to find data and build candidate forecasts in a certain way to favor a sophisticated or simple method. Under the CFA scenario, we realize that the heavy estimation price can be avoided by applying combining methods designed to target the performance of the best candidate forecast. Under the CFI scenario, although past literature has properly pointed out the potentially high cost of estimation error when targeting the optimal weight, it turns out that we do not have to pay the high cost. Indeed, a carefully designed mAFTER strategy can perform aggressively to target the optimal weight when information is sufficient to support exploiting the optimal weighting and perform conservatively like SA when the degree of estimation error is high. mAFTER can also intelligently perform according to the underlying scenario (CFA or CFI), avoiding the puzzle caused by improperly choosing the combining methods.

SA certainly can be the best or among the top combining methods, as observed empirically and reported in the literature. It may be particularly useful when one can legitimately narrow the focus to just a few well-behaving candidate forecasts. However, since the uncertainty of the process used to reach the small set of candidates is not reflected in the showcase examples in the literature, the “conditional” results in favor of SA may not be replicable when one starts from scratch with inhomogeneous raw models/forecasts. For such problems, the performance of SA may span the whole spectrum, from terrible to on top of the chart. Also, when information is rich for a stable forecasting problem, SA may lose greatly to a model-based method (e.g., regression). In contrast, when the analyst has little confidence in basic modeling assumptions on the data or in the quality of the available forecasts, perhaps SA (or the like) would be the choice to take.

The repeatedly reported puzzle in literature tends to give the sentiment that sophisticated methods are not trustworthy and simple methods should be used. Based on our understanding and the numerical results, it seems fair to say that if the sophisticated methods in those studies do not perform well, it is actually because they are not sophisticated enough, not the other way around! In particular, when SA is considered by mAFTER as a candidate, the possible advantage of SA is retained while the
un-robustness of SA is avoided. To a large extent, the forecast combination puzzle no longer exists if we are able to move forward intelligently by integrating the strengths of different combining methods.

APPENDIX

A. Assumptions of Proposition 1

The following two assumptions are sufficient regularity conditions for Proposition 1. Note that Assumption A.1 is satisfied if we truncate the candidate forecasts to have certain lower and upper bounds. Assumption A.2 is satisfied if the conditional distributions of the random noise are sub-Gaussian.

**Assumption A.1.** There exists a positive constant $M$ such that the candidate forecasts satisfy with probability 1 that

$$
\sup_{1 \leq i \leq K, 1 \leq t \leq T} |m_t - \hat{y}_{t,i}| \leq M.
$$

**Assumption A.2.** There exists a constant $r_0 > 0$ and continuous functions $0 < h_1(r), h_2(r) < \infty$ on $[-r_0, r_0]$ such that for every $1 \leq t \leq T$ and $r \in [-r_0, r_0]$,

$$
\mathbb{E}(\epsilon_t^2 \exp(r\epsilon_t)|x_t, z_{t-1}) \leq h_1(r),
$$

$$
\mathbb{E}(\exp(r\epsilon_t)|x_t, z_{t-1}) \leq h_2(r)
$$

with probability 1.

B. Propositions and Proofs

**Proposition 2.** Under the settings of Case 1, the average forecast risk of Forecaster 1 relative to the SA satisfies

$$
\frac{R_{T,1}}{R_{T,SA}} \to \frac{\sigma^2}{\sigma^2 + \beta^2 \sigma_X^2/4} \text{ as } T \to \infty.
$$

In addition, if we consider the weight vectors in $\mathbb{R}^2$, the asymptotic optimal combination weight $w^*$ satisfies

$$
w^* =: \arg \min_{w \in \mathbb{R}^2} \left( \lim_{T \to \infty} R_{T,w} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
$$
Proposition 3. Under the settings of Case 2, if we assume that $\beta = \beta_1 = \beta_2$ and $\sigma_X = \sigma_{X_1} = \sigma_{X_2}$, the average forecast risk of Forecast $i$ ($i = 1, 2$) relative to the SA satisfies
\[
\frac{R_{T,i}}{R_{T,SA}} \to \frac{\sigma_X^2 \beta_1^2 (1 - \rho^2) + \sigma_X^2}{\sigma_X^2 \beta_2^2 (1 - \rho^2) (1 - \rho) / 2 + \sigma_X^2} \quad \text{as } T \to \infty. \tag{A.1}
\]
In addition, if we further assume $\rho = 0$, the asymptotic optimal combination weight $\tilde{w}^*$ under the restriction $\Theta = \{w : w_1 + w_2 = 1\}$ satisfies
\[
\tilde{w}^* =: \arg \min_{w \in \Theta} \left( \lim_{T \to \infty} R_{T,w} \right) = \left( \frac{1/2}{1/2} \right), \tag{A.2}
\]
and the asymptotic optimal combination weight $w^*$ without the restriction satisfies
\[
w^* =: \arg \min_{w \in \mathbb{R}^2} \left( \lim_{T \to \infty} R_{T,w} \right) = \left( \frac{1}{1} \right). \tag{A.3}
\]

The proof of Proposition [2] is similar to that of Proposition [3]. In the following, we provide a sketch for the proof of Proposition [3].

Proof of Proposition [3]. Let $r_{T,1} = \mathbb{E}(y_T - \hat{y}_{T,1})^2$, $r_{T,2} = \mathbb{E}(y_T - \hat{y}_{T,2})^2$ and $r_{T,w} = \mathbb{E}(y_T - \hat{y}_{T,w})^2$ be the point-wise forecast risks at time $T$ for forecaster 1, forecaster 2 and the combined forecast, respectively. We will first verify that under the restriction $\Theta = \{w : w_1 + w_2 = 1\}$,
\[
\begin{align*}
r_{T+1,1} &= \sigma^2 \left( 1 + \frac{1}{T-2} \right) + \sigma_{X_2}^2 \beta_1^2 + \sigma_{X_1}^2 \beta_2^2 \mathbb{E} \left( \rho^2 \frac{\hat{\sigma}_{X_2}}{\hat{\sigma}_{X_1}} \right) - 2 \rho \sigma_{X_1} \sigma_{X_2} \beta_1 \beta_2 \mathbb{E} \left( \hat{\sigma}_{X_2} \frac{\hat{\sigma}_{X_1}}{\hat{\sigma}_{X_2}} \right), \\
r_{T+1,2} &= \sigma^2 \left( 1 + \frac{1}{T-2} \right) + \sigma_{X_1}^2 \beta_1^2 + \sigma_{X_2}^2 \beta_2^2 \mathbb{E} \left( \rho^2 \frac{\hat{\sigma}_{X_1}}{\hat{\sigma}_{X_2}} \right) - 2 \rho \sigma_{X_1} \sigma_{X_2} \beta_1 \beta_2 \mathbb{E} \left( \hat{\sigma}_{X_1} \frac{\hat{\sigma}_{X_2}}{\hat{\sigma}_{X_1}} \right), \text{ and} \\
r_{T+1,w} &= \sigma^2 \left( 1 - w_1^2 - w_2^2 \right) + w_1 r_{T+1,1} + w_2 r_{T+1,2} + 2w_1 w_2 \left( \rho \sigma_{X_1} \sigma_{X_2} \beta_1 \beta_2 (1 + \mathbb{E}(\hat{\rho})^2) \right) \\
&\quad - \sigma_{X_1}^2 \beta_1 \beta_2 \mathbb{E} \left( \frac{\hat{\sigma}_{X_1}}{\hat{\sigma}_{X_2}} \right) - \sigma_{X_2}^2 \beta_1 \beta_2 \mathbb{E} \left( \frac{\hat{\sigma}_{X_2}}{\hat{\sigma}_{X_1}} \right) + \rho \sigma_{X_1} \sigma_{X_2} \sigma^2 \mathbb{E} \left( \frac{\hat{\rho}}{\hat{\sigma}_{X_1} \hat{\sigma}_{X_2}} \right),
\end{align*}
\]
where $\hat{\sigma}_{X_i} = \sqrt{\sum_{t=1}^{T} x_{t,i}^2 / T}$ is the estimated covariate standard deviation ($i = 1, 2$) and $\hat{\rho} = \frac{\sum_{t=1}^{T} x_{t,1} x_{t,2}}{T \sigma_{X_1} \sigma_{X_2}}$ is the estimated covariate correlation.
First, we have

\[ r_{T+1,1} = \mathbb{E}(y_{T+1} - x_{T+1,1}\hat{\beta}_{T+1,1})^2 \]

\[ = \mathbb{E}\left(\varepsilon_{T+1} + x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - \frac{x_{T+1,1} \sum_{t=1}^{T} x_{t,1} y_t}{\sum_{t=1}^{T} x_{t,1}^2}\right)^2 \]

\[ = \sigma^2 + \mathbb{E}\left(x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - \frac{x_{T+1,1} \sum_{t=1}^{T} x_{t,1}(\beta_1 x_{t,1} + \beta_2 x_{t,2} + \varepsilon_t)}{\sum_{t=1}^{T} x_{t,1}^2}\right)^2 \]

\[ = \sigma^2 + \mathbb{E}(x_{T+1,2}\beta_2)^2 + \mathbb{E}\left((x_{T+1,1}\beta_2)^2\left(\sum_{t=1}^{T} x_{t,1} x_{t,2}\right)^2\right) + \mathbb{E}\left(\frac{x_{T+1,1}^2 (\sum_{t=1}^{T} x_{t,1} \varepsilon_t)^2}{(\sum_{t=1}^{T} x_{t,1}^2)^2}\right) \]

\[ - 2 \mathbb{E}\left(\frac{x_{T+1,1} x_{T+1,2}\beta_2^2 \sum_{t=1}^{T} x_{t,1} x_{t,2}}{\sum_{t=1}^{T} x_{t,1}^2}\right) \]

\[ = \sigma^2 + \sigma^2 X_2 \beta_2^2 + \sigma^2 X_1 \beta_2^2 \mathbb{E}\left(\hat{\rho} \frac{\hat{\sigma}_X}{\hat{\sigma}_X} \right) + \frac{\sigma^2}{T - 2} - 2 \rho \sigma_X \sigma_X \beta_2^2 \mathbb{E}\left(\frac{\hat{\rho}}{\hat{\sigma}_X}ight). \]

The expression for \( r_{T+1,2} \) can be derived similarly. For \( r_{T+1,w} \), we have

\[ r_{T+1,w} = \mathbb{E}(y_{T+1} - w_1 \hat{\gamma}_{T+1,1} - w_2 \hat{\gamma}_{T+1,2})^2 \]

\[ = \sigma^2 + \mathbb{E}\left(w_1 (x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - x_{T+1,1}\hat{\beta}_{T+1,1}) \right. \]

\[ + w_2 (x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - x_{T+1,2}\hat{\beta}_{T+1,2}) \right) \]

\[ = \sigma^2 (1 - w_1^2 - w_2^2) + w_1^2 r_{T+1,1} + w_2^2 r_{T+1,2} \]

\[ + 2 w_1 w_2 \mathbb{E}\left((x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - x_{T+1,1}\hat{\beta}_{T+1,1}) \right. \]

\[ \times (x_{T+1,1}\beta_1 + x_{T+1,2}\beta_2 - x_{T+1,2}\hat{\beta}_{T+1,2}) \right) \]

\[ = \sigma^2 (1 - w_1^2 - w_2^2) + w_1^2 r_{T+1,1} + w_2^2 r_{T+1,2} + 2 w_1 w_2 A_1. \]

With tedious algebra, it is not hard to show that

\[ A_1 = \rho \sigma_X \sigma_X \beta_1 \beta_2 \left(1 + \mathbb{E}(\hat{\rho})^2\right) - \sigma^2 X_1 \beta_1 \beta_2 \mathbb{E}\left(\frac{\hat{\sigma}_X}{\hat{\sigma}_X} \right) - \sigma^2 X_2 \beta_1 \beta_2 \mathbb{E}\left(\frac{\hat{\sigma}_X}{\hat{\sigma}_X} \right) \]

\[ + \frac{\rho \sigma_X \sigma_x \sigma^2}{T} \mathbb{E}\left(\frac{\hat{\rho}}{\hat{\sigma}_X \hat{\sigma}_X} \right). \]

Together with the previous display, we verify the formula for \( r_{T+1,w} \). The formulas \( (A.1) \) and \( (A.2) \) can be verified straightforwardly by noting that the \( x_t \)’s are normally distributed and that \( r_{T,i}/R_{T,i} \to 1 \) as \( T \to \infty \) \((i = 1, 2)\). When there is no restriction on \( \text{w} \), \( r_{T+1,w} \) can be derived similarly as above. Then, we can show that when \( \text{w} = (1, 1)^T \), \( \lim_{T \to \infty} R_{T,w} = \sigma^2 \), which implies \( (A.3) \).
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