Cotangent bundle reduction

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Abstract

The general symplectic reduction theory presented in [3] becomes much richer and has many applications if the symplectic manifold is the cotangent bundle (T∗Q, ΩQ = −dΘQ) of a manifold Q. The canonical one-form ΘQ on T∗Q is given by ΘQ(αq) (Vαq) = αq (TαqπQ (Vαq))∗, for any q ∈ Q, αq ∈ T∗qQ, and tangent vector Vαq ∈ Tαq(T∗Q), where πQ : T∗Q → Q is the cotangent bundle projection and TαqπQ : Tαq(T∗Q) → TqQ is its tangent map (or derivative) at q. In natural cotangent bundle coordinates (q, p), we have ΘQ = p∗dq and ΩQ = dq ∧ dp.

Let Φ : G × Q → Q be a left smooth action of the Lie group G on the manifold and Q. Denote by g · q = Φ(g, q) the action of g ∈ G on the point q ∈ Q and by Φq : Q → TqG the diffeomorphism of G induced by g. The lifted left action G × T∗Q → T∗Q given by g · αq = TgαqΦg−1(αq) for g ∈ G and αq ∈ T∗qQ, preserves ΘQ and admits the equivariant momentum map J : T∗Q → g∗ whose expression is (J(αq), ξ) = αq(ξq(q)), where ξ ∈ g, the Lie algebra of G, (, ) : g∗ × g → R is the duality pairing between the dual g* and g, and ξq(q) = dΦ(exp tξ, q) ∈ TqG is the value of the infinitesimal generator vector field ξq of the G-action at q ∈ Q. Throughout this article it is assumed that the G-action on Q, and hence on T∗Q, is free and proper. Recall also that (T∗Q)µ, (ΩQ)µ denotes the reduced manifold at µ ∈ g∗ ⊆ Lie(G). Cotangent bundle reduction at zero is already quite interesting and has many applications. Let ρ : Q → Q/G be the G-principal bundle projection defined by the proper free action of G on Q, usually referred to as the shape space bundle. Zero is a regular value of J and the map ϕ0 : ((T∗Q)0, (ΩQ)0) → (T∗(Q/G), ΩQ/G) given by ϕ0([αq])((Tqρ)(vq)) := αq(vq), where αq ∈ J−1(0), [αq] ∈ (T∗Q)0, and vq ∈ TqQ is a well-defined symplectic diffeomorphism.

This theorem generalizes in two non-trivial ways when one reduces at a non-zero value of J: an embedding and a fibration theorem.

Embedding version of cotangent bundle reduction. Let µ ∈ g∗, Qµ := Q/Gµ, ρµ : Q → Qµ the projection onto the Gµ-orbit space, gµ := {ξ ∈ g | adξ µ = 0} the Lie algebra of the coadjoint isotropy subgroup Gµ, where adξ µ := [ξ, µ] for any ξ ∈ g, adξ µ : g → g∗ the dual map, µ′ := µ|gµ the restriction of µ to gµ, and (T∗Qµ, (ΩQ)µ) the reduced space at µ. The induced Gµ-action on T∗Q admits the equivariant momentum map Jµ : T∗Q → gµ∗ given by Jµ(αq) = J(αq)|gµ. Assume there is a Gµ-invariant one-form αµ on Q with values in (Jµ)−1(µ′). Then there is a unique closed two-form βµ on Qµ such that ρµ∗βµ = dαµ. Define the magmetic term Bµ := πQµ∗βµ, where πQµ : T∗Qµ → Qµ is the cotangent bundle projection, which is a closed two-form on T∗Qµ. Then the map ϕµ : ((T∗Q)µ, (ΩQ)µ) → (T∗Qµ, ΩQµ − Bµ) given by ϕµ([αq])((Tqρµ)(vq)) := (αq − µvq)(vq), for αq ∈ J−1(µ), [αq] ∈ (T∗Q)µ, and vq ∈ TqQ, is a symplectic embedding onto a submanifold of T∗Qµ covering the base Qµ. The embedding ϕµ is a diffeomorphism onto T∗Qµ if and only if g = gµ. If the

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one-form $\alpha_\mu$ takes values in the smaller set $J^{-1}(\mu)$ then the image of $\varphi_\mu$ is the the vector subbundle $[T\rho_\mu(VQ)]^o$ of $T^*Q_\mu$, where $VQ \subset TQ$ is the vertical vector subbundle consisting of vectors tangent to the $G$-orbits, that is, its fiber at $q \in Q$ equals $V_qQ = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$, and $^o$ denotes the annihilator relative to the natural duality pairing between $TQ_\mu$ and $T^*Q_\mu$. Note that if $\mathfrak{g}$ is Abelian or $\mu = 0$, the embedding $\varphi_\mu$ is always onto and thus the reduced space is again, topologically, a cotangent bundle.

It should be noted that there is a choice in this theorem, namely the one-form $\alpha_\mu$.

Connections. The one-form $\alpha_\mu$ is usually obtained from a left connection on the principal bundle $\rho_\mu : Q \to Q/G_\mu$ or $\rho : Q \to Q/G$. A left connection one-form $A \in \Omega^1(Q; \mathfrak{g})$ on the left principal $G$-bundle $\rho : Q \to Q/G$ is a Lie algebra valued one-form $A : TQ \to \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of $G$, satisfying the conditions $A(\xi_Q) = \xi$ for all $\xi \in \mathfrak{g}$ and $A(T_g\Phi_g(v)) = Ad_g(A(v))$ for all $g \in G$ and $v \in T_qQ$, where $Ad_g$ denotes the adjoint action of $G$ on $\mathfrak{g}$. The horizontal vector subbundle $HQ$ of the connection $A$ is defined as the kernel of $A$, that is, its fiber at $q \in Q$ is the subspace $H_q := \ker A(q)$. The map $v_q \mapsto \text{ver}_q(v_q) := [A(q)(v_q)]_{Q/G}(q)$ is called the vertical projection, while the map $v_q \mapsto \text{hor}_q(v_q) := v_q - \text{ver}_q(v_q)$ is called the horizontal projection. Since for any vector $v_q \in T_qQ$ we have $v_q = \text{ver}_q(v_q) + \text{hor}_q(v_q)$, it follows that $TQ = HQ \oplus VQ$ and the maps $\text{hor}_q : T_qQ \to H_qQ$ and $\text{ver}_q : T_qQ \to V_qQ$ are projections onto the horizontal and vertical subspaces at every $q \in Q$.

Connections can be equivalently defined by the choice of a subbundle $HQ \subset TQ$ complementary to the vertical subbundle $VQ$ satisfying the following $G$-invariance property: $H_{g\cdot q}Q = T_{g\cdot\Phi_g}(H_qQ)$ for every $g \in G$ and $q \in Q$. The subbundle $HQ$ is called, as before, the horizontal subbundle and a connection one-form $A$ is defined by setting $\alpha(q)(\xi_Q(q) + u_q) = \xi$, for any $\xi \in \mathfrak{g}$ and $u_q \in H_qQ$.

The curvature of the connection $A$ is the Lie algebra valued two-form on $Q$ defined by $B(u_q, v_q) = dA([u_q, v_q]) = [A(u_q), A(v_q)]$, where $\alpha$ is defined to be an exterior covariant derivative and the preceding formula for $B$ is often written as $B = DA$. Curvature measures the lack of integrability of the horizontal distribution, namely, $B(u, v) = -A([\text{hor}_u(v), \text{hor}_v(u)])$ for any two vector fields $u$ and $v$ on $Q$.

The Cartan Structure Equations state that $B(u, v) = dA(u, v) - [A(u), A(v)]$, where the bracket on the right hand side is the Lie bracket in $\mathfrak{g}$.

Since the connection $A$ is a Lie algebra valued one-form, for each $\mu \in \mathfrak{g}^*$ the formula $\alpha_\mu(q) := A^*(\mu)(q)$, where $A^* : \Omega^1(Q/G) \to \mathfrak{g}^*$ is the dual of the linear map $A(q) : T_qQ \to \mathfrak{g}$, defines a usual one-form on $Q$. This one-form $\alpha_\mu$ takes values in $J^{-1}(\mu)$ and is equivariant in the following sense: $\Phi_g^*\alpha_\mu = \alpha_{Ad_g^*\mu}$ for any $g \in G$.

Magnetic Terms and Curvature. There are two methods to construct the one-form $\alpha_\mu$ from a connection. The first is to start with a connection one-form $A^\mu \in \Omega^1(Q; \mathfrak{g}_\mu)$ on the principal $G_\mu$-bundle $\rho_\mu : Q \to Q/G_\mu$. Then the one-form $\alpha_\mu := \langle \mu|_{\mathfrak{g}_\mu}, A^\mu \rangle \in \Omega^1(Q)$ is $G_\mu$-invariant and has values in $(J^\mu)^{-1}(\mu|_{\mathfrak{g}_\mu})$. The magnetic term $B_\mu$ is the pull back to $T^*(Q/G_\mu)$ of the $\mu$-component $d\alpha_\mu$ of the curvature of $A^\mu$ thought of as a two-form on the base $Q/G_\mu$.

The second method is to start with a connection $A \in \Omega^1(Q, \mathfrak{g})$ on the principal bundle $\rho : Q \to Q/G$, to define $\alpha_\mu := \langle \mu|_{\mathfrak{g}}, A \rangle \in \Omega^1(Q)$, and to observe that this one-form is $G$-invariant and has values in $J^{-1}(\mu)$. The magnetic term $B_\mu$ is in this case the pull back to $T^*(Q/G)$ of the $\mu$-component $d\alpha_\mu$ of the curvature of $A$ thought of as a two-form on the base $Q/G$.

The Mechanical Connection. If $(Q, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold and $G$ acts by isometries, there is a natural connection on the bundle $\rho : Q \to Q/G$, namely, define the horizontal space at a point to be the metric orthogonal to the vertical space. This connection is called the mechanical connection and its horizontal bundle consists of all vectors $v_q \in TQ$ such that $J(\langle v_q, \cdot \rangle) = 0$.

To determine the Lie algebra valued one-form $A$ of this connection, the notion of locked inertia tensor needs to be introduced. This is the linear map $I(q) : [v_q] \to \mathfrak{g}^*$ depending smoothly on $q \in Q$ defined by the identity $\langle I(q)(\xi), \eta \rangle = \langle \xi_Q(q), \eta_Q(q) \rangle$ for any $\xi, \eta \in \mathfrak{g}$. Since the $G$-action is free, each $I(q)$ is invertible. The connection one-form whose horizontal space was defined above is given by $A(q)(v_q) = I(q)^{-1}(J(\langle v_q, \cdot \rangle))$. 

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Denote by $K : T^*Q \to \mathbb{R}$ the kinetic energy of the metric $\langle \cdot, \cdot \rangle$ on the cotangent bundle, that is, $K((v_q, \gamma)) := \frac{1}{2} |v_q|^2$. The one-form $\alpha_\mu = A(\cdot)^* \mu$ is characterized for the mechanical connection $A$ by the condition $\tilde{K}(\alpha_\mu(q)) = \inf \{ K(\beta_q) \mid \beta_q \in J^{-1}(\mu) \cap T_q^*Q \}$. 

**The amended potential.** A simple mechanical system is a Hamiltonian system on a cotangent bundle $T^*Q$ whose Hamiltonian function is the sum of the kinetic energy of a Riemannian metric on $Q$ and a potential function $V : Q \to \mathbb{R}$. If there is a Lie group $G$ acting on $Q$ by isometries and leaving the potential invariant, then we have a simple mechanical system with symmetry. The amended or effective potential $V_\mu : Q \to \mathbb{R}$ at $\mu \in \mathfrak{g}^*$ is defined by $V_\mu := \mathcal{H} \circ \alpha_\mu$, where $\alpha_\mu$ is the one-form associated to the mechanical connection. Its expression in terms of the locked moment of inertia tensor is given by $V_\mu(q) := V(q) + \frac{1}{2} \langle \mu, \mathbb{I}(q)^{-1} \rangle$. The amended potential naturally induces a smooth function $\tilde{V}_\mu \in \mathcal{C}^\infty(Q/G_\mu)$. 

The fundamental result about simple mechanical systems with symmetry is the following. The push-forward by the embedding $\varphi_\mu : (T^*Q)_\mu, (\Omega_Q)_\mu \to (T^*Q_{\mu}, \Omega_{Q_{\mu}} - B_\mu)$ of the reduced Hamiltonian $H_\mu \in \mathcal{C}^\infty((T^*Q)_\mu)$ of a simple mechanical system $H = K + V \circ \pi_Q \in \mathcal{C}^\infty(T^*Q)$ is the restriction to the vector subbundle $\varphi_\mu((T^*Q)_\mu) \subset T^*(Q/G_\mu)$, which is also a symplectic submanifold of $(T^*(Q/G_\mu), \Omega_{Q/G_\mu} - B_\mu)$, of the simple mechanical system on $T^*(Q/G_\mu)$ whose kinetic energy is given by the quotient Riemannian metric on $Q/G_\mu$ and whose potential is $\tilde{V}_\mu$. However, Hamilton’s equations on $T^*(Q/G_\mu)$ for this simple mechanical system are computed relative to the magnetic symplectic form $\Omega_{Q/G_\mu} - B_\mu$.

There is a wealth of applications starting from this classical theorem to mechanical systems, spanning such diverse areas as topological characterization of the level sets of the energy-momentum map to methods of proving nonlinear stability of relative equilibria (block diagonalization of the stability form in the application of the energy-momentum method).

**Fibration version of cotangent bundle reduction.** There is a second theorem that realizes the reduced space of a cotangent bundle as a locally trivial bundle over shape space $Q/G$. This version is particularly well suited in the study of quantization problems and in control theory. The result is the following. Assume that $G$ acts freely and properly on $Q$. Then the reduced symplectic manifold $(T^*Q)_\mu$ is a fiber bundle over $T^*(Q/G)$ with fiber the coadjoint orbit $\mathcal{O}_\mu$. How this is related to the Poisson structure of the quotient $(T^*Q)/G$ will be discussed later.

**The Kaluza-Klein construction.** The extra term in the symplectic form of the reduced space is called a magnetic term because it has this interpretation in electromagnetism. To understand why $B_\mu$ is called a magnetic term, consider the problem of a particle of mass $m$ and charge $e$ moving in $\mathbb{R}^3$ under the influence of a given magnetic field $B = B_x i + B_y j + B_z k$, $\operatorname{div} B = 0$. The Lorentz Force Law (written in the International System) gives the equations of motion

$$m \frac{dv}{dt} = e v \times B,$$

where $e$ is the charge and $v = (\dot{x}, \dot{y}, \dot{z}) = \dot{q}$ is the velocity of the particle. What is the Hamiltonian description of these equations?

There are two possible answers to this question. To formulate them, associate to the divergence free vector field $B$ the closed two-form $B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy$. Also, write $B = \text{curl} A$ for some other vector field $A = (A_x, A_y, A_z)$ on $\mathbb{R}^3$, called the magnetic potential.

**Answer 1.** Take on $T^*\mathbb{R}^3$ the symplectic form $\Omega_B = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z - eB$, where $(p_x, p_y, p_z) = p := mv$ is the momentum of the particle, and $h = m \|v\|^2/2 = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2$ is the Hamiltonian, the kinetic energy of the particle. A direct verification shows that $dh = \omega_B(X_{h^B}, \cdot)$, where

$$X_h = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + e(B_y \dot{y} - B_x \dot{z}) \frac{\partial}{\partial p_x} + e(B_z \dot{z} - B_y \dot{y}) \frac{\partial}{\partial p_y} + e(B_x \dot{x} - B_z \dot{z}) \frac{\partial}{\partial p_z},$$

which gives the equations of motion (0.1).

**Answer 2.** Take on $T^*\mathbb{R}^3$ the canonical symplectic form $\Omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$ and the Hamiltonian $h_A = \|p - eA\|^2/2m$. A direct verification shows that $dh_A = \Omega(X_{h^A}, \cdot)$, where $X_{h^A}$ has the same expression (0.2).
Next we show how the magnetic term in the symplectic form $\Omega_B$ is obtained by reduction from the Kaluza-Klein system. Let $Q = \mathbb{R}^3 \times S^1$ with the circle $G = S^1$ acting on $Q$, only on the second factor. Identify the Lie algebra $g$ of $S^1$ with $\mathbb{R}$. Since the infinitesimal generator of this action defined by $\xi \in g = \mathbb{R}$ has the expression $\xi_0(q, \theta) = (q, \theta, 0, \xi)$, if $T^*Q$ is trivialized as $S^1 \times \mathbb{R}$, a momentum map $J : T^*Q = \mathbb{R}^3 \times S^1 \times S^1 \times \mathbb{R} \to g^*$ is given by $J(q, \theta; p, \mu) = (p, \theta) \cdot (0, \xi) = p\mu$, that is, $J(q, \theta; p, \mu) = p$.

In this case the coadjoint action is trivial, so for any $\mu \in g^* = \mathbb{R}$, we have $G_\mu = S^1$, $g_\mu = \mathbb{R}$, and $\mu' = \mu$.

The one-form $\alpha_\mu = \mu(A_0 dq + A_1 dy + A_2 dz + d\theta) \in \Omega^1(Q)$, where $d\theta$ denotes the length one-form on $S^1$, is clearly $G_\mu = S^1$-invariant, has values in $J^{-1}(\mu) = \{(q, \theta; p, \mu) | q, p \in \mathbb{R}, \theta \in S^1\}$, and its exterior differential equals $d\alpha_\mu = \mu B$. Thus the closed two-form $\beta_\mu$ on the base $Q_\mu = Q/G_\mu = Q/S^1 = \mathbb{R}^3$ equals $\mu B$ and hence the magnetic term, that is, the closed two-form $B_\mu = \pi^*_Q \beta_\mu$ on $T^*Q_\mu = T^*\mathbb{R}^3$, is also $\mu B$ since $\pi_\mu : Q = \mathbb{R}^3 \times S^1 \to Q/G_\mu = \mathbb{R}^3$ is the projection. Therefore, the reduced space $(T^*Q)_\mu$ is symplectically diffeomorphic to $(T^*\mathbb{R}^3, dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z - \mu B)$, which coincides with the phase space in Answer 1 if we put $\mu = e$. This also gives the physical interpretation of the momentum map $J : T^*Q = \mathbb{R}^3 \times S^1 \times S^1 \times \mathbb{R} \to g^* = \mathbb{R}$, $J(q, \theta; p, \mu) = p$ and hence of the variable conjugate to the circle variable $\theta$: $p$ represents the charge. Moreover, the magnetic term in the symplectic form is, up to a charge factor, the magnetic field.

The kinetic energy Hamiltonian $h(q, \theta; p, \mu) := \frac{1}{2m}p^2 + \frac{1}{\mu}B$ of the Kaluza-Klein metric, that is, the Riemannian metric obtained by keeping the standard metrics on each factor and declaring $\mathbb{R}^3$ and $S^1$ orthogonal, induces the reduced Hamiltonian $h_\mu(q) = \frac{1}{2m}p^2 + \frac{1}{\mu}B$ which, up to the constant $\mu^2/2$, equals the kinetic energy Hamiltonian in Answer 1. Note that this reduced system is not the geodesic flow of the Euclidean metric because of the presence of the magnetic term in the symplectic form. However, the equations of motion of a charged particle in a magnetic field are obtained by reducing the geodesic flow of the Kaluza–Klein metric.

A similar construction is carried out in Yang–Mills theory where $A$ is a connection on a principal bundle and $B$ is its curvature. Magnetic terms appear also in classical mechanics. For example, in rotating systems the Coriolis force (up to a dimensional factor) plays the role of the magnetic term.

**Reconstruction of dynamics for cotangent bundles.** A general reconstruction method of the dynamics from the reduced dynamics was given in [13]. For cotangent bundles, using the mechanical connection, this method simplifies considerably.

Start with the following general situation. Let $G$ act freely on the configuration manifold $Q$, $h : T^*Q \to \mathbb{R}$ be a $G$-invariant Hamiltonian, $\mu \in g^*$, $\alpha_\mu \in J^{-1}(\mu)$, and $c_\mu(t)$ the integral curve of the reduced system with initial condition $[\alpha_\mu] \in (T^*Q)_\mu$ given by the reduced Hamiltonian function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$. In terms of a connection $A \in \Omega^1(J^{-1}(\mu); g_\mu)$ on the left $G_\mu$-principal bundle $J^{-1}(\mu) \to (T^*Q)_\mu$ the reconstruction procedure proceeds in four steps:

- **Step 1:** Horizontally lift the curve $c_\mu(t) \in (T^*Q)_\mu$ to a curve $d(t) \in J^{-1}(\mu)$ with $d(0) = \alpha_\mu$.
- **Step 2:** Set $\xi(t) = A(d(t)) (X_h(d(t)))) \in g_\mu$.
- **Step 3:** With $\xi(t) \in g_\mu$, determined in Step 2, solve the nonautonomous differential equation $\dot{q}(t) = T_{\xi(t)}L_{\xi(t)} \xi(t)$ with initial condition $g(0) = e$, where $L_{\xi(t)}$ denotes left translation on $G$; this is the step that involves “quadratures” and is the main obstacle to finding explicit formulas;
- **Step 4:** The curve $c(t) = g(t) \cdot d(t)$, with $d(t)$ found in Step 1 and $g(t)$ found in Step 3 is the integral curve of $X_\xi$ with initial condition $c(0) = \alpha_\mu$.

This method depends on the choice of the connection $A \in \Omega^1(J^{-1}(\mu); g_\mu)$. Here are several particular cases when this procedure simplifies.

(a) **One-dimensional coadjoint isotropy group.** If $G_\mu = S^1$ or $G_\mu = \mathbb{R}$, identify $g_\mu$ with $\mathbb{R}$ via the map $a \in \mathbb{R} \leftrightarrow a \zeta \in g_\mu$, where $\zeta \in g_\mu$, $\zeta \neq 0$, is a generator of $g_\mu$. Then a connection one-form on the $S^1$ (or $\mathbb{R}$) principal bundle $J^{-1}(\mu) \to (T^*Q)_\mu$ is the one-form $A = \frac{1}{\mu} \theta_\mu$, where $\theta_\mu$ is the pull-back of the canonical one-form $\theta \in \Omega^1(T^*Q)$ to the submanifold $J^{-1}(\mu)$. The curvature of this connection is the two-form on $(T^*Q)_\mu$ given by $\text{curv}(A) = -\frac{1}{\mu^2} \omega_\mu$, where $\omega_\mu$ is the reduced symplectic form on $(T^*Q)_\mu$. In this case, the curve $\xi(t) \in g_\mu$ in Step 2 is given by $\xi(t) = \Lambda[h](d(t))$, where

\[ \Lambda[h](d(t)) = \frac{1}{\mu} \omega_\mu(d(t)). \]
where $\Lambda \in \mathfrak{x}(T^*Q)$ is the **Liouville vector field** characterized by the property of being the unique vector field on $T^*Q$ that satisfies the relation $d\theta(\Lambda, \cdot) = \theta$. In canonical coordinates $(q^i, p_i)$ on $T^*Q$, $\Lambda = p_i \pi_i\frac{\partial}{\partial q^i}$.

(b) **Induced connection.** Any connection $A \in \Omega^1(Q; \mathfrak{g}_{\mu})$ on the left principal bundle $Q \to Q/G_{\mu}$ induces a connection $\tilde{A} \in \Omega^1(J^{-1}(\mu); \mathfrak{g}_{\mu})$ by $A(\alpha_q)(V_{\alpha_q}) := A(q)(T_{\alpha_q} \pi_Q(V_{\alpha_q}))$, where $q \in Q$, $\alpha_q \in T^*_q Q$, $V_{\alpha_q} \in T_{\alpha_q}(Q^* Q)$, and $\pi_Q : T^* Q \to Q$ is the cotangent bundle projection. In this case, the curve $\xi(t) \in \mathfrak{g}_{\mu}$ in Step 2 is given by $\xi(t) = A(\xi(t))(\mathcal{F}(dt(t)))$, where $q(t) := \pi_Q(dt(t))$ is the base integral curve and the vector bundle morphism $\mathcal{F} : T^* Q \to T Q$ is the fiber derivative of $h$ given by $\mathcal{F}(\alpha_q)(\beta_q) := \frac{d}{dt}h(\alpha_q + t\beta_q)$ for any $\alpha_q, \beta_q \in T^*_q Q$. Two particular instances of this situation are noteworthy.

(b1) Assume that the Hamiltonian $h$ is that of a simple mechanical system with symmetry. Choosing $A$ to be the mechanical connection $A_{\text{mech}}$, the curve $\xi(t) \in \mathfrak{g}_{\mu}$ in Step 2 is given by $\xi(t) = A_{\text{mech}}(q(t))(\mathcal{F}(dt(t)))$.

(b2) If $Q = G$ is a Lie group, $\text{dim} G_{\mu} = 1$, and $\zeta$ is a generator of $\mathfrak{g}_{\mu}$, then the connection $A \in \Omega^1(G)$ can be chosen to equal $A(g) := \frac{1}{\langle \mu, \zeta \rangle} T g_{\gamma^{-1}}(\mu)$, where $\zeta$ is a generator of $\mathfrak{g}_{\mu}$ and $R_g$ is right translation on $G$.

(c) **Reconstruction of dynamics for simple mechanical systems with symmetry.** The case of simple mechanical systems with symmetry deserves special attention since several steps in the reconstruction method can be simplified. For simple mechanical systems the knowledge of the base integral curve $q(t)$ suffices to determine the entire integral curve on $T^* Q$. Indeed, if $h = K + V \circ \pi_Q$ is the Hamiltonian, the Legendre transformation $\mathcal{F} : T^* Q \to T Q$ determines the Lagrangian system on $T Q$ given by $\ell(u_q) = \frac{1}{2}\|u_q\|^2 - V(u_q)$ for $u_q \in T_q Q$. Lagrange’s equations are second order and thus the evolution of the velocities is given by the time derivative $\dot{q}(t)$ of the base integral curve. Since $\mathcal{F} h = (\mathcal{F} \ell)^{-1}$, the solution of the Hamiltonian system is given by $\mathcal{F}(\dot{q}(t))$. Using the explicit expression of the mechanical connection and the notation given in the general procedure, the method of reconstruction simplifies to the following steps.

To find the integral curve $c(t)$ of the simple mechanical system with $G$-symmetry $h = K + V \circ \pi_Q$ on $T^* Q$ with initial condition $c(0) = \alpha_q \in T^*_q Q$, knowing the integral curve $c_{\mu}(t)$ of the reduced Hamiltonian system on $(T^* Q)_\mu$ given by the reduced Hamiltonian function $h_{\mu} : (T^* Q)_\mu \to \mathbb{R}$ with initial condition $c_{\mu}(0) = [\alpha_q]_\mu$ one proceeds in the following manner. Recall the symplectic embedding $\varphi_{\mu} : ((T^* Q)_\mu, (\Omega_{Q, \mu})_\mu) \to (T^*(Q/G_{\mu}), \Omega_{Q/G_{\mu}} - B_{\mu})$. The curve $\varphi_{\mu}(c_{\mu}(t)) = T^*(Q/G_{\mu})$ is an integral curve of the Hamiltonian system on $(T^*(Q/G_{\mu}), \Omega_{Q/G_{\mu}} - B_{\mu})$ given by the function that is the sum of the kinetic energy of the quotient Riemannian metric and the quotient amended potential $\mathcal{V}_{\mu}$. Let $q_{\mu}(t) := \pi_{Q/G_{\mu}}(c_{\mu}(t))$ be the base integral curve of this system, where $\pi_{Q/G_{\mu}} : T^*(Q/G_{\mu}) \to Q/G_{\mu}$ is the cotangent bundle projection.

- **Step 1.** Relative to the mechanical connection $A_{\text{mech}} \in \Omega^1(Q; \mathfrak{g}_{\mu})$, horizontally lift $q_{\mu}(t) \in Q/G_{\mu}$ to a curve $q_h(t) \in Q$ passing through $q_h(0) = q$.

- **Step 2.** Determine $\xi(t) \in \mathfrak{g}_{\mu}$ from the algebraic system $\mathcal{S}(q_h(t), q_{\mu}(q_h(t))) = (\mu, \eta)$ for all $\eta \in \mathfrak{g}_{\mu}$, where $\mathcal{S}(\cdot, \cdot)$ is the $G$-invariant kinetic energy Riemannian metric on $Q$. This implies that $q_h(0)$ and $\xi(0)Q(q)$ are the horizontal and vertical components of the vector $\alpha_q \in T^*_q Q$ which is associated by the metric $\mathcal{S}(\cdot, \cdot)$ to the initial condition $\alpha_q$.

- **Step 3.** Solve $\dot{q}(t) = T_{q(t)} L_{\dot{g}(t)}(\xi(t))$ in $G_{\mu}$ with initial condition $q(0) = e$.

- **Step 4.** The curve $q(t) := g(t) \cdot q_{\mu}(t)$, with $q_{\mu}(t)$ and $q(t)$ determined in Steps 2 and 4 respectively, is the base integral curve of the simple mechanical system with symmetry defined by the function $h$ satisfying $q(0) = 0$. The curve $(\mathcal{F} h)^{-1}(\dot{q}(t)) \in T^* Q$ is the integral curve of this system with initial condition $c(0) = \alpha_q$. In addition, $q(t) = g(t) \cdot (\dot{q}_{h(t)} + \xi t Q(q_{h(t)}))$ is the horizontal plus vertical decomposition relative to the connection induced on $J^{-1}(\mu) \to (T^* Q)_\mu$ by the mechanical connection $A_{\text{mech}} \in \Omega^1(Q; \mathfrak{g}_{\mu})$.

There are several important situations when Step 3, the main obstruction to an explicit solution of the reconstruction problem, can be carried out. We shall review some of them below.
(c1) The case \(G_\mu = S^1\). If \(G_\mu\) is Abelian, the equation in Step 3 has the solution \(g(t) = \exp \int_0^t \xi(s)ds\). If, in addition, \(G_\mu = S^1\), then \(\xi(s)\) can be explicitly determined by Step 2. Indeed, if \(\xi \in \mathfrak{g}_\mu\) is a generator of \(\mathfrak{g}_\mu\), writing \(\xi(s) = a(s)\xi\) for some smooth real valued function \(a\) defined on some open interval around the origin, the algebraic equation in Step 2 implies that \(\langle a(s)\xi(t)\xi_q(q_h(t)), \xi_q(q_h(t))\rangle = \langle \mu, \xi \rangle\) which gives \(a(s) = \langle \mu, \xi \rangle / ||\xi_q(q_h(s))||^2\). Therefore, the base integral curve of the solution of the simple mechanical system with symmetry on \(T^*Q\) passing through \(q\) is

\[
q(t) = \exp \left( \langle \mu, \xi \rangle \int_0^t \frac{ds}{||\xi_q(q_h(s))||^2} \right) \cdot q_h(t)
\]

and

\[
\dot{q}(t) = \exp \left( \langle \mu, \xi \rangle \int_0^t \frac{ds}{||\xi_q(q_h(s))||^2} \right) \cdot \left( \dot{q}_h(t) + \frac{\langle \mu, \xi \rangle}{||\xi_q(q_h(s))||^2} \xi_q(q_h(t)) \right).
\]

(c2) The case of compact Lie groups. An obvious situation when the differential equation in Step 3 can be solved is if \(\xi(t) = \xi\) for all \(t\), where \(\xi\) is a given element of \(\mathfrak{g}_\mu\). Then the solution is \(g(t) = \exp(\xi(t))\). However, Step 2 puts certain restrictions under this hypothesis, because it requires that \(\langle \xi(t)\xi_q(q_h(t)), \xi_q(q_h(t))\rangle = \langle \mu, \xi \rangle\) for any \(\xi \in \mathfrak{g}_\mu\). This is satisfied if there is a bilinear nondegenerate form \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{g}\) satisfying \(\langle \xi, \eta \rangle = \langle \xi_q(q), \xi_q(q) \rangle\) for all \(q \in Q\) and \(\xi, \eta \in \mathfrak{g}\). This implies that \(\langle \cdot, \cdot \rangle\) is positive definite and invariant under the adjoint action of \(G\) on \(\mathfrak{g}\) so semisimple Lie algebras of noncompact type are excluded. If \(G\) is compact, which ensures the existence of a positive adjoint invariant inner product on \(\mathfrak{g}\), and \(Q = G\), this condition implies that the kinetic energy metric is invariant under the adjoint action. There are examples in which such conditions are natural, such as in Kaluza–Klein theories. Concluding, if \(G\) is a compact Lie group and \(\langle \cdot, \cdot \rangle\) is a positive definite metric invariant under the adjoint action of \(G\) on \(\mathfrak{g}\) satisfying \(\langle \xi, \eta \rangle = \langle \xi_q(q), \xi_q(q) \rangle\) for all \(q \in Q\) and \(\xi, \eta \in \mathfrak{g}\), then the element \(\xi(t)\) in Step 2 can be chosen to be constant and is determined by the identity \(\langle \xi, v \rangle = \mu_{1\mathfrak{g}}\) on \(\mathfrak{g}_\mu\). The solution of the equation on Step 3 is then \(g(t) = \exp(\xi(t))\).

(c3) The case when \(\xi(t)\) is proportional to \(\xi(t)\). Try to find a real valued function \(f(t)\) such that \(g(t) = \exp(f(t)\xi(t))\) is a solution of the equation \(\dot{g}(t) = T_\xi L_{g(t)}(\xi(t)) = 0\). This gives, for small \(t\), the equation \(f(t_\xi(t)) + f(t_\xi(t)) = \xi(t)\), that is, it is necessary that \(\xi(t)\) and \(\xi(t)\) be proportional. So if \(\xi(t) = a(t)\xi(t)\) for some known smooth function \(a(t)\), then this gives \(f(t) = \int_0^t \mu_{1\mathfrak{g}} \alpha(r)dr\).

(c4) The case of \(G_\mu\) solvable. Write \(g(t) = \exp(f_1(t)\xi_1) \exp(f_2(t)\xi_2) \ldots \exp(f_n(t)\xi_n)\), for some basis \(\{\xi_1, \xi_2, \ldots, \xi_n\}\) of \(\mathfrak{g}_\mu\) and some smooth real valued functions \(f_i, i = 1, 2, \ldots, n\), defined around zero. It is known that if \(G_\mu\) is solvable, the equation in Step 3 can be solved by quadratures for the \(f_i\).

Reconstruction phases for simple mechanical systems with \(S^1\) symmetry. Consider a simple mechanical system with symmetry \(G\) on the Riemannian manifold \((Q, \langle \cdot, \cdot \rangle)\) with \(G\)-invariant potential \(V \in C^\infty(Q)\). If \(\mu \in \mathfrak{g}^*\) let \(\tilde{V}_{\mu}\) be the amended potential and \(\tilde{V}_\mu \in C^\infty(Q/G_\mu)\) the induced function on the base. Let \(c : [0, T] \to T^*Q\) be an integral curve of the system with Hamiltonian \(h = K + V \circ \pi_Q\) and suppose that its projection \(c_\mu : [0, T] \to (T^*Q)_\mu\) to the reduced space is a closed integral curve of the reduced system with Hamiltonian \(h_\mu\). The reconstruction phase associated to the loop \(c_\mu(t)\) is the group element \(g \in G_\mu\), satisfying the identity \(c(T) = g \cdot c(0)\). We shall present two explicit formulas of the reconstruction phase for the case when \(G_\mu = S^1\). Let \(\xi \in \mathfrak{g}_\mu = \mathbb{R}\) be a generator of the coadjoint isotropy algebra and write \(c(T) = \exp(\varphi \xi) \cdot c(0)\); in this case \(\varphi\) is identified with the reconstruction phase and, as we shall see in concrete mechanical examples, it truly represents an angle.

If \(G_\mu = S^1\), the \(G_\mu\)-principal bundle \(\pi_\mu : J^{-1}(\mu) \to (T^*Q)_\mu := J^{-1}(\mu)/G_\mu\) admits two natural connections: \(A = \frac{1}{\mu_\mu} \theta_\mu \in \Omega^1(J^{-1}(\mu))\), where \(\theta_\mu\) is the pull-back of the canonical one-form on the cotangent bundle to the momentum level submanifold \(J^{-1}(\mu)\), and \(\pi_\mu^* A_{\text{mech}} \in \Omega^1(J^{-1}(\mu))\). There is no reason to choose one connection over the other and thus there are two natural formulas for the reconstruction phase in this case. Let \(c_\mu(t)\) be a periodic orbit of period \(T\) of the reduced system and denote also by \(h_\mu\) the value of the Hamiltonian function on it. Assume that \(D\) is a two-dimensional surface in \((T^*Q)_\mu\) whose boundary is the loop \(c_\mu(t)\). Since the manifolds \((T^*Q)_\mu\) and \((T^*Q/S^1)\) are diffeomorphic (but not symplectomorphic), it makes sense to consider the base integral curve \(q_\mu(t)\) obtained by projecting \(c_\mu(t)\) to the base \(Q/S^1\), which is a closed curve of period \(T\). Denote by \(\langle \tilde{V}_\mu := \ldots\rangle\)
\[ \frac{1}{T} \int_0^T \dot{\nu}_\mu(q_\mu(t)) \, dt \] is the average of \( \dot{\nu}_\mu \) over the loop \( q_\mu(t) \). Let \( q_\mu(t) \in Q \) be the \( A_{\text{mech}} \)-horizontal lift of \( q_\mu(t) \) to \( Q \) and let \( \chi \) be the \( A_{\text{mech}} \)-holonomy of the loop \( q_\mu(t) \) measured from \( q(0) \), the base point of \( c(0) \); its expression is given by \( \exp \chi = \exp (- \int_\Omega B) \), where \( B \) is the curvature of the mechanical connection. Denote by \( \omega_\mu \) the reduced symplectic form on \((T^*Q)_\mu \). With these notations the phase \( \varphi \) is given by:

\[
\varphi = \frac{1}{\mu \zeta} \int_D \omega_\mu + \frac{2(h_\mu - \dot{\nu}_\mu)^T}{\mu \zeta} = \chi + \mu \zeta \int_0^T \frac{ds}{\| \zeta_Q(q_\mu(s)) \|^2}.
\] (0.3)

The first terms in both formulas are the so-called geometric phases because they carry only geometric information given by the connection, whereas the second terms are called the dynamic phases since they encapsulate information directly linked to the Hamiltonian. The expression of the total phase as a sum of a geometric and a dynamic phase is not intrinsic and is connection dependent. It can even happen that one of these summands vanishes. We shall consider now two concrete examples: the free rigid body and the heavy top.

**Reconstruction phases for the free rigid body.** The motion of the free rigid body is a geodesic with respect to a left invariant Riemannian metric on \( SO(3) \) given by the moment of inertia of the body. The phase space of the free rigid body motion is \( T^*SO(3) \) and a momentum map \( J : T^*SO(3) \to \mathbb{R}^3 \) of the lift of left translation to the cotangent bundle is given by right translation to the identity element. We have identified here \( \mathfrak{so}(3) \) with \( \mathbb{R}^3 \) by the Lie algebra isomorphism \( x \in (\mathbb{R}^3, \times) \mapsto \xi \in (\mathfrak{so}(3), [\cdot, \cdot]) \), where \( \hat{\xi}(y) = x \times y \), and \( \mathfrak{so}(3)^* \) with \( \mathbb{R}^3 \) by the inner product on \( \mathbb{R}^3 \). The reduced manifold \( J^{-1}(\mu)/G_\mu \) is identified with the sphere \( S_{\|\mu\|^2}^2 \) in \( \mathbb{R}^3 \) of radius \( \|\mu\| \) with the symplectic form \( \omega_\mu = -dS/\|\mu\| \), where \( dS \) is the standard area form on \( S_{\|\mu\|^2}^2 \) and where \( G_\mu \cong S^1 \) is the group of rotations around the axis \( \mu \).

These concentric spheres are the coadjoint orbits of the Lie–Poisson space \( \mathfrak{so}(3)^* \) and represent the level sets of the Casimir functions that are all smooth functions of \( \|\Pi\|^2 \), where \( \Pi \in \mathbb{R}^3 \) denotes the body angular momentum.

The Hamiltonian of the rigid body on the Lie–Poisson space \( T^*SO(3)/SO(3) \cong \mathbb{R}^3 \) is given by

\[
h(\Pi) := \frac{1}{2} \left( \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3} \right)
\]
where \( I_1, I_2, I_3 > 0 \) are the principal moments of inertia of the body. Let \( I := \text{diag}(I_1, I_2, I_3) \) denote the moment of inertia tensor diagonalized in a principal axis body frame. The Lie–Poisson bracket on \( \mathbb{R}^3 \) is given by \( \{f, g\}(\Pi) = -\Pi \cdot (\nabla f(\Pi) \times \nabla g(\Pi)) \) and the equation of motions are \( \dot{\Pi} = \Omega \times \Pi \), where \( \Omega \in \mathbb{R}^3 \) is the body angular velocity given in terms of \( \Omega_i := \dot{I}_i/I_i \), for \( i = 1, 2, 3 \), that is, \( \Omega = \Omega^{-1} \Pi \). The trajectories of these equations are found by intersecting a family of homothetic energy ellipsoids with the angular momentum concentric spheres. If \( I_1 > I_2 > I_3 \), one immediately sees that all orbits are periodic with the exception of four centers (the two possible rotations about the long and the short moment of inertia axis of the body), two saddles (the two rotations about the middle moment of inertia axis of the body), and four heteroclinic orbits connecting the two saddles.

Suppose that \( \Pi(t) \) is a periodic orbit on the sphere \( S_{\|\mu\|^2}^2 \) with period \( T \). After time \( T \) by how much has the rigid body rotated in space? The answer to this question follows directly from (0.1). Taking \( \zeta = \mu/\|\mu\| \) and the potential \( v = 0 \) we get

\[
\varphi = -\Lambda + \frac{2h_\mu T}{\|\mu\|} = \int_D \frac{2\|\Pi(s)\|^2 - \langle \Pi(s), \Pi(s) \rangle (\text{tr} \Pi)}{(\Pi(s), \Pi(s))^2} ds + \|\mu\|^3 \int_0^T \frac{ds}{\|\Pi(s), \Pi(s)\|^2},
\]

where \( D \) is one of the two spherical caps on \( S_{\|\mu\|^2}^2 \) whose boundary is the periodic orbit \( \Pi(t) \), \( h_\mu \) is the value of the total energy on the solution \( \Pi(t) \), and \( \Lambda \) is the oriented solid angle, that is,

\[
\Lambda := -\frac{1}{\|\mu\|} \int_D \omega_\mu, \quad |\Lambda| = \frac{\text{area } D}{\|\mu\|^2}.
\]

**Reconstruction phases for the heavy top.** The heavy top is a simple mechanical systems with symmetry \( S^1 \) on \( T^*SO(3) \) whose Hamiltonian function is given by \( h(\alpha_{\mathbf{k}}) := \frac{1}{2} \| \alpha_{\mathbf{k}} \|^2 + Mg\ell \mathbf{k} \cdot \mathbf{h} \chi \), where \( h \in SO(3) \), \( \alpha_{\mathbf{k}} \in T^*_\mathbf{k} SO(3) \), \( \mathbf{k} \) is the unit vector of the spatial Oz axis (pointing in the opposite direction of the gravity force), \( M \in \mathbb{R} \) is the total mass of the body, \( g \in \mathbb{R} \) is the value of the gravitational acceleration, the fixed point about which the body moves is the origin, and \( \chi \) is the unit vector of
the straight line segment of length \( \ell \) connecting the origin to the center of mass of the body. This Hamiltonian is left invariant under rotations about the spatial \( Oz \) axis. A momentum map induced by this \( S^1 \)-action is given by \( J : T^* SO(3) \to \mathbb{R}, J(\alpha_k) = T^*_e L_h(\alpha_k) \cdot k; \) recall that \( T^*_e L_h(\alpha_k) := \Pi \in \mathbb{R}^3 \) is the body angular momentum. The reduced space \( J^{-1}(\mu)/S^1 \) is generically the cotangent bundle of the unit sphere endowed with the symplectic structure given by the sum of the canonical form plus a magnetic term; equivalently, this is the coadjoint orbit in the dual of the Euclidean Lie algebra \( \mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3 \) given by \( O_\mu = \{ (\Pi, \Gamma) \mid \Pi \cdot \Gamma = \mu, \| \Gamma \|^2 = 1 \} \). The projection map \( J^{-1}(\mu) \to O_\mu \) implementing the symplectic diffeomorphism between the reduced space and the coadjoint orbit in \( \mathfrak{se}(3)^* \) is given by \( \alpha_k \mapsto (\Pi(\mu), \Gamma(\mu)) := (T^*_e L_h(\alpha_k), h^{-1} k) \). The orbit symplectic form \( \omega_\mu \) on \( O_\mu \) has the expression 
\[
\omega_\mu(\Pi, \Gamma)_\mu((\Pi \times x + \Gamma \times y), (\Pi \times x' + \Gamma \times y')) = -\Pi \cdot (x \times x') - \Gamma \cdot (x \times y' - x' \times y)
\]
for any \( x, x', y, y' \in \mathbb{R}^3 \). The heavy top equations \( \dot{\Pi} = \Pi \times \Omega + M g t \Gamma \times \chi, \Gamma = \Gamma \times \Omega \) are Lie-Poisson equations on \( \mathfrak{se}(3)^* \) for the Hamiltonian \( h(\Pi, \Gamma) = \frac{1}{2} \Pi \cdot \Omega + M g t \Gamma \cdot \chi \). The Lie-Poisson bracket \( \{f, \mu\} = -\Pi \cdot (\nabla_H f \times \nabla_H g) - \Gamma \cdot (\nabla_H f \times \nabla_R g - \nabla_H g \times \nabla_R f) \), where \( \nabla_H \) and \( \nabla_R \) denote the partial gradients.

Let \( (\Pi(t), \Gamma(t)) \) be a periodic orbit of period \( T \) of the heavy top equations. After time \( T \) by how much has the heavy top rotated in space? The answer is provided by (3):

\[
\varphi = \frac{1}{\mu} \int_D \omega_\mu + \frac{1}{\mu} \left( 2 h_\mu T + 2 M g \ell \int_0^T \Pi(s) \cdot \chi ds \right) = \int_D \left( \frac{2}{\| \Pi(s) \|^2} \left( \frac{(\Gamma(s) \cdot \Pi(s)) (\Pi(s) \cdot \Pi(s))}{(\Pi(s) \cdot \Pi(s))^2} \right) \right) ds + \int_0^T \frac{ds}{\| \Pi(s) \|^2},
\]
where \( D \) is the spherical cap on the unit sphere whose boundary is the closed curve \( \Gamma(t) \) and \( D \) is a two-dimensional submanifold of the orbit \( O_\mu \) bounded by the closed integral curve \( (\Pi(t), \Gamma(t)) \). The first terms in each summand represent the geometric phase and the second terms the dynamic phase.

**Gauged Poisson structures.** If the Lie group \( G \) acts freely and properly on a smooth manifold \( Q \), then \( (T^* Q)/G \) is a quotient Poisson manifold (see [14]), where the quotient is taken relative to the (left) lifted cotangent action. The leaves of this Poisson manifold are the orbit reduced spaces \( J^{-1}(O_\mu)/G \), where \( O_\mu \subset \mathfrak{g}^* \) is the coadjoint \( G \)-orbit through \( \mu \in \mathfrak{g}^* \) (see [14]). Is there an explicit formula for this reduced Poisson bracket on a manifold diffeomorphic to \( (T^* Q)/G \)? It turns out that this question has two possible answers, once a connection on the principal bundle \( \pi : Q \to Q/G \) is introduced. The discussion below will also link to the fiberization version of cotangent bundle reduction.

In order to present these answers we review two bundle constructions. Let \( G \) act freely and properly on the manifold \( P \) and consider the a (left) principal \( G \)-bundle \( \rho : P \to P/G := M \). Let \( \tau : N \to M \) be a surjective submersion. Then the pull-back bundle \( \bar{\rho} : (n, p) \in \bar{P} := \{(n, p) \in N \times P \mid \rho(p) = \tau(n) \} \to n \in N \) over \( N \) is a principal (left) \( G \)-bundle relative to the action \( g \cdot (n, p) := (n, g \cdot p) \).

If there is a (left) \( G \)-action a manifold \( V \), then the diagonal \( G \)-action on \( P \times V \) is also free and proper and one can form the associated bundle \( P \times_G V := (P \times \mathbb{R})/G \) which is a locally trivial fiber bundle \( \rho_E : [p, v] \in E := P \times \mathbb{R} \to \rho(E) \) over \( M \) with fibers diffeomorphic to \( V \). Analogously one can form the associated fiber bundle \( \rho_E : E := P \times \mathbb{R} \to N \). Summarizing, the associated bundle \( \bar{E} := \bar{P} \times \mathbb{R} \to N \) is obtained from the principal bundle \( \rho : P \to M \), the surjective submersion \( \tau : N \to M \), and the \( G \)-manifold \( V \) by pull-back and association, in this order.

These operations can be reversed. First form the associated bundle \( \rho_E : E := P \times \mathbb{R} \to M \) and then pull it back by the surjective submersion \( \tau : N \to M \) to \( N \) to get the pull-back bundle \( \bar{\rho}_E : \bar{E} \to N \). The map \( \Phi : \bar{P} \times \mathbb{R} \to \bar{E} \) defined by \( \Phi([n, p, v]) := (n, [p, v]) \) is an isomorphism of locally trivial fiber bundles.

These general considerations will be used now to realize the quotient Poisson manifold \( (T^* Q)/G \) in two different ways. Let \( Q \) be a manifold and \( G \) a Lie group (with Lie algebra \( \mathfrak{g} \)) acting freely and properly on it. Let \( A \in \Omega^1(Q; \mathfrak{g}) \) be a connection one-form on the left \( G \)-principal bundle \( \pi : Q \to Q/G \). Pull back the \( G \)-bundle \( \pi : Q \to Q/G \) by the cotangent bundle projection \( \pi_Q / G : T^*(Q/G) \to Q/G \) to \( T^*(Q/G) \) to obtain the \( G \)-principal bundle \( \tilde{\pi}_{Q / G} : (\alpha_{[q]} , q) \in \tilde{Q} := \{(\alpha_{[q]} , q) \mid [q] = \pi(q) , q \in Q \} \to \alpha_{[q]} \in T^*(Q/G) \). This bundle is isomorphic to the annihilator \( (VQ)^c \subset T^* Q \) of the vertical bundle.
bundle isomorphism over $S$. Next, form the **coadjoint bundle** $\rho_S: S := \tilde{Q} \times_G \mathfrak{g}^* \to T^*(Q/G)$ of $\tilde{Q}$, $\rho_S((\alpha|q), \mu) = \alpha|q$, that is, the associated vector bundle to the $G$-principal bundle $\tilde{Q} \to T^*(Q/G)$ given by the coadjoint representation of $G$ on $\mathfrak{g}^*$. The connection-dependent map $\Phi_A: S \to (T^*Q)/G$ defined by $\Phi_A((\alpha|q), \mu) := (T^q\pi^*(\alpha|q) + A(q)^*\mu)$, where $q \in Q$, $\alpha_q \in T^*_qQ$, and $\mu \in \mathfrak{g}^*$, is a vector bundle isomorphism over $Q/G$. The **Sternberg space** is the Poisson manifold $(S, \{\cdot, \cdot\}_S)$, where $\{\cdot, \cdot\}_S$ is the pull-back to $S$ by $\Phi_A$ of the quotient Poisson bracket on $(T^*Q)/G$.

Next, we proceed in the opposite order. Construct first the coadjoint bundle $\rho_{\mathfrak{g}^*} : [q, \mu] \in \mathfrak{g}^* \mapsto [q] \in Q/G$ associated to the principal bundle $\pi : Q \to Q/G$ and then pull it back by the cotangent bundle projection $\pi_{Q/G} : T^*(Q/G) \to Q/G$ to $T^*(Q/G)$ to obtain the vector bundle $\rho_{\mathfrak{g}^*} : W := \{(\alpha|q), \mu|q = [q] \}$, $\rho_{\mathfrak{g}^*}((\alpha|q), \mu) = \alpha_q$ over $T^*(Q/G)$. Note that $W = T^*(Q/G) \oplus \mathfrak{g}^*$ and hence $W$ is also a vector bundle over $Q/G$. Let $HQ$ be the horizontal subbundle defined by the connection $A$; thus $TQ = HQ \oplus VQ$, where $HQ := \ker A(q)$. For each $q \in Q$ the linear map $T_q\pi_{|H_qQ} : H_qQ \to T_q(Q/G)$ is an isomorphism. Let $\text{hor}_q := (T_q\pi_{|H_qQ})^{-1}: T_q(Q/G) \to H_qQ \subset T_qQ$ be the horizontal lift operator induced by the connection $A$. Thus $\text{hor}_q^* : T^*_qQ \to T^*_q(Q/G)$ is a linear surjective map whose kernel is the annihilator $(H_qQ)^\circ$ of the horizontal space. The connection-dependent map $\Psi_A : (T^*Q)/G \to W$ defined by $\Psi_A((\alpha|q)) := (\text{hor}_q^*(\alpha|q), [q], (J_q^A(q)))$, where $q \in Q$, $\alpha_q \in T^*_qQ$, and $J_q^A : T^*Q \to \mathfrak{g}^*$ is the momentum map of the lifted action, $(J_q^A(q), \xi) = \alpha_q(\xi(q q))$ for $\xi \in \mathfrak{g}$, is a vector bundle isomorphism over $Q/G$ and $\Psi_A \circ \Phi_A = \Phi$. The **Weinstein space** is the Poisson manifold $(W, \{\cdot, \cdot\}_W)$, where $\{\cdot, \cdot\}_W$ is the push-forward by $\Psi_A$ of the Poisson bracket of $(T^*Q)/G$. In particular, $\Phi : S \to W$ is a connection independent Poisson diffeomorphism. The Poisson brackets on $S$ and $W$ are called **gauged Poisson brackets**. They are expressed explicitly in terms of various covariant derivatives induced on $S$ and on $W$ by the connection $A \in \Omega^1(Q; \mathfrak{g})$.

Recall that the connection $A$ on the principal bundle $\pi : Q \to Q/G$ naturally induces connections on pull-back bundles and affine connections on associated vector bundles. Thus both $S$ and $W$ carry covariant derivatives induced by $A$. They are given, according to general definitions, in the cases under consideration, by:

- **If $f \in C^\infty(S)$, $s = ([\alpha], q, \mu) \in S$, and $v_{\alpha|q} \in T_{\alpha|q}T^*(Q/G)$, then $\delta^S_A f(s) \in T^*_s T^*(Q/G)$ is defined by $\delta^S_A f(s)(v_{\alpha|q}) := df(s)(T^q((\alpha|q), \mu) \pi_{Q \times G \mathfrak{g}^*} ((v_{\alpha|q}; \text{hor}_q^*(T_q(q(q(\alpha|q)))))])$, where $\pi_{Q \times G \mathfrak{g}^*} : \tilde{Q} \times \mathfrak{g}^* \to Q \times_G \mathfrak{g}^*$ is the orbit map. The symbol $\delta^S_A$ signifies that this is a covariant derivative on the associated bundle $S$ induced by the connection $A$ on the principal $G$-pull-back bundle $\tilde{Q} \to T^*(Q/G)$. This connection $A$ is the pull-back connection defined by $A$.

- **If $f \in C^\infty(W)$, $w = ([\alpha], q, \mu) \in W$, and $v_{\alpha|q} \in T_{\alpha|q}T^*(Q/G)$, then $\nabla^W_A f(w) \in T^*_w T^*(Q/G)$ is defined by $\nabla^W_A f(w)(v_{\alpha|q}) := df(w)(v_{\alpha|q}; T_q(q(\mu) \pi_{Q \times G \mathfrak{g}^*} (\text{hor}_q^*(T_q(q(q(\alpha|q)))))))$, where $\pi_{Q \times G \mathfrak{g}^*} : Q \times \mathfrak{g}^* \to Q \times_G \mathfrak{g}^*$ is the orbit map. The symbol $\nabla^W_A$ signifies that this is a covariant derivative on the pull-back bundle $W$ induced by the covariant derivative $\nabla^A$ on the coadjoint bundle $\mathfrak{g}^*$. This covariant derivative $\nabla^A$ is induced on $\mathfrak{g}^*$ by the connection $A$.

For $f \in C^\infty(W)$, we have $\delta^S_A (f \circ \Phi) = (\nabla^W_A f) \circ \Phi$.

To write the two gauged Poisson brackets on $S$ and on $W$ explicitly, we denote by $\hat{\mathfrak{g}} = Q \times_G \mathfrak{g}$ the adjacent bundle of $\pi : Q \to Q/G$, by $\Omega_{Q/G}$ the canonical symplectic structure on $T^*(Q/G)$, by $B \in \Omega^2(Q; \hat{\mathfrak{g}})$ the curvature of $A$, and by $\tilde{\mathfrak{g}}$ the $\hat{\mathfrak{g}}$-valued two-form $B \in \Omega^2(Q/G; \hat{\mathfrak{g}})$ on the base $Q/G$ defined by $B([u], [v]) = [q, B(q)(u, v)]$, for any $u, v \in T_qQ$ that satisfy $T_q\pi(u_q) = u_q$ and $T_q\pi(v_q) = v_q$. Note that both $S^*$ and $W^*$ are Lie algebra bundles, that is, their fibers are Lie algebras and the fiberwise Lie bracket operation depends smoothly on the base point. If $f \in C^\infty(S)$, denote by $\delta f/\delta s \in S^* = Q \times_G \mathfrak{g}$ the usual fiber derivative of $f$. Similarly, if $f \in C^\infty(W)$ denote by $\delta f/\delta w \in W^*$ the usual fiber derivative of $f$. Finally, $\xi : T^*(T^*(Q/G)) \to T(T^*(Q/G))$ is the vector bundle isomorphism induced by
The Poisson bracket of \( f, g \in C^\infty(S) \) is given by

\[
\{ f, g \}_S(s) = \Omega_{Q/G}(\alpha_{[q]}) (d^S_A f(s))^\ast, d^S_A g(s)^\ast - \left\langle s, \left[ \frac{\delta f}{\delta s}, \frac{\delta g}{\delta s} \right] \right\rangle \\
+ \left\langle \nu, (\pi^*_{Q/G} B)(\alpha_{[q]}) \left( d^S_A f(s)^\ast, d^S_A g(s)^\ast \right) \right\rangle,
\]

where \( v = [q, \mu] \in \mathfrak{g}^* \). The Poisson bracket \( f, g \in C^\infty(W) \) is given by

\[
\{ f, g \}_W(w) = \Omega_{Q/G}(\alpha_{[q]}) \left( \nabla^W_A f(w)^\ast, \nabla^W_A g(w)^\ast \right) - \left\langle w, \left[ \frac{\delta f}{\delta w}, \frac{\delta g}{\delta w} \right] \right\rangle \\
+ \left\langle \nu, (\pi^*_{Q/G} B)(\alpha_{[q]}) \left( \nabla^W_A f(w)^\ast, \nabla^W_A g(w)^\ast \right) \right\rangle.
\]

Note that their structure is of the form: “canonical” bracket plus a (left) “Lie-Poisson” bracket plus a curvature coupling term.

The symplectic leaves of the Sternberg and Weinstein spaces. The map the map \( \varphi_A : Q \times g^* \to T^*Q \) given by \( \varphi_A ((\alpha_{[q]}, q), \mu) := T_q \pi(\alpha_{[q]} + A(q)^* \mu) \), where \( ((\alpha_{[q]}, q), \mu) \in Q \times g^* \), is a \( G \)-equivariant diffeomorphism; the \( G \)-action on \( T^*Q \) is by cotangent lift and on \( Q \times g^* \) is \( g \cdot ((\alpha_{[q]}, q), \mu) = ((\alpha_{[q]}, g \cdot q), Ad^{-1}_\mu) \). The pull back \( \tilde{A} \) of the momentum map to \( Q \times g^* \) has the expression \( \tilde{J}_A ((\alpha_{[q]}, q), \mu) = \mu \), so if \( O \subset g^* \) is a coadjoint orbit we have \( J^{-1}_A(O) = \tilde{Q} \times O \), and hence the orbit reduced manifold \( J^{-1}_A(O)/G \), whose connected components are the symplectic leaves of \( S \), equals \( \tilde{Q} \times G \). Its symplectic form is the *Sternberg minimal coupling form* \( \tilde{\omega}_G + \rho^*_S \Omega_{Q/G} \).

In this formula the two-form \( \tilde{\omega}_G \) has not been defined yet. It is uniquely defined by the identity \( \pi^*_{Q \times g^*} \tilde{\omega}_G = dA + \Pi_{O} \omega_G \), where \( \omega_G \) is the minus orbit symplectic form on \( O \) (see [13]). \( \Pi_O : \tilde{Q} \times O \to O \) is the projection on the second factor, and \( \tilde{A} \in \tilde{\Omega}^2(\tilde{Q} \times O) \) is the two-form given by \( \tilde{A} ((\alpha_{[q]}, q), \mu) (u_{\alpha_{[q]}}, v_Q, \nu) = -\langle \mu, A(q)(v_Q) \rangle \) for \( ((\alpha_{[q]}, q), \mu) \in \tilde{Q} \times O \), \( (u_{\alpha_{[q]}}, v_Q) \in T_{(\alpha_{[q]}, q)} \tilde{Q} \), and \( \nu \in g^* \).

The symplectic leaves of the Weinstein space \( W \) are obtained by pushing forward by \( \Phi \) the symplectic leaves of the Sternberg space. They are the connected components of the symplectic manifolds \( \left( T^*(Q/G) \oplus (Q \times G) \right) \Omega_{Q/G} + \Pi^*_{Q \times G} \omega_G \), where \( O \) is a coadjoint orbit in \( g^* \), \( \Omega_{Q/G} \) is the canonical symplectic form on \( T^*(Q/G) \), \( \omega_G \) is a closed two-form on \( Q \times G \) to be defined below, and \( \Pi^*_{T^*(Q/G)} : T^*(Q/G) \oplus (Q \times G) \to T^*(Q/G) \), \( \Pi^*_O : T^*(Q/G) \oplus (Q \times G) \to Q \times G \) are the projections. The closed two-form \( \omega_{Q \times G} \in \Omega^2(Q \times G) \) is uniquely determined by the identity \( \pi^*_{Q \times G} \omega_{Q \times G} = \tilde{\omega}_G \), where \( \pi_{Q \times G} : Q \times G \to Q \times G \) is the orbit space projection, \( \omega_{Q \times G} \in \Omega^2(Q \times G) \) is closed and given by \( \omega_{Q \times G}(q, \mu) (\langle u_q, -\text{ad}_\xi \mu \rangle, \langle v_q, -\text{ad}_\eta \mu \rangle) := -d(A \times \text{id}_O)(q, \mu) (\langle u_q, -\text{ad}_\xi \mu \rangle, \langle v_q, -\text{ad}_\eta \mu \rangle) + \omega_G(\mu) (\text{ad}_\xi \mu, \text{ad}_\eta \mu), \) and \( A \times \text{id}_O \in \Omega^1(Q \times g^*) \) is given by \( (A \times \text{id}_O)(q, \mu) (u_q, -\text{ad}_\xi \mu) = \langle \mu, A(q)(u_q) \rangle \), for \( q \in Q, \mu \in g^*, u_q, v_q \in T_q Q, \xi, \eta \in g \).

Thus on the Sternberg and Weinstein spaces both the Poisson bracket as well as the symplectic form on the leaves have explicit connection dependent formulas.

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