On the first Robin eigenvalue of a class of anisotropic operators

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Abstract

The paper is devoted to the study of some properties of the first eigenvalue of the anisotropic $p$-Laplace operator with Robin boundary condition involving a function $\beta$ which in general is not constant. In particular we obtain sharp lower bounds in terms of the measure of the domain and we prove a monotonicity property of the eigenvalue with respect the set inclusion.

Keywords: Eigenvalue problems, nonlinear elliptic equations, Faber-Krahn inequality, Wulff shape, Robin boundary condition

Mathematics Subject Classifications (2010): 35P15, 35P30, 35J60

1 Introduction

Let $F$ be a norm in $\mathbb{R}^n$, that is a convex, even, $1$-homogeneous and non negative function defined in $\mathbb{R}^n$. Moreover we will assume that $F \in C^2(\mathbb{R}^n \setminus \{0\})$, and strongly convex that is for $1 < p < +\infty$, it holds

$$[F^p]_\xi(\xi) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}.$$ 

For $1 < p < +\infty$ the so-called anisotropic $p$-Laplacian is defined as follows

$$Q_p u := \text{div} \left( \frac{1}{p} \nabla F^p(\nabla u) \right).$$

The assumptions on $F$ ensure that the operator $Q_p$ is elliptic. The paper concerns the study of the following Robin eigenvalue problem for $Q_p$

$$\begin{cases}
- Q_p v = \ell_1(\beta, \Omega)|v|^{p-2}v & \text{ in } \Omega \\
F^{p-1}(\nabla v)F_\xi(\nabla v) \cdot \nu + \beta(x)F(\nu)|v|^{p-2}v = 0 & \text{ on } \partial\Omega,
\end{cases}$$

(1.1)

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with $C^{1,\alpha}$ boundary, $\alpha \in ]0,1[$, $\nu$ is the Euclidean unit outer normal to $\partial\Omega$ and the function $\beta: \partial\Omega \to ]0, +\infty[$ belongs to $L^1(\partial\Omega)$ and verifies

$$\int_{\partial\Omega} \beta(x)F(\nu) \, d\mathcal{H}^{n-1} = m > 0.$$

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Here $\ell_1(\beta, \Omega)$ is the first Robin eigenvalue of $Q_p$ and it has the following variational characterization

$$\ell_1(\beta, \Omega) = \inf_{v \in W^{1,p}(\Omega), v \neq 0} \frac{\int_{\Omega} F^p(\nabla v) dx + \int_{\partial \Omega} \beta(x)|v|^p F(\nu) dH^{n-1}}{\int_{\Omega} |v|^p dx},$$

and the minimizers of (1.2) are weak solutions to the problem (1.1) (see section 3 for the precise definition). When $F(\xi) = |\xi|$ is the Euclidean norm, this problem has been studied for instance in [9, 13, 8, 22, 13, 20]. In particular in [9] and [8] when $\beta(x) = \beta$ is a nonnegative constant and for any $p$, $1 < p < \infty$, the authors prove a sharp lower bound for $\ell_1(\beta, \Omega)$, keeping fixed the measure of the domain $\Omega$. More precisely, they prove the following Faber-Krahn type inequality

$$\ell_1(\beta, \Omega) \geq \ell_1(\beta, B_R),$$

where $B_R$ is a ball having the same measure than $\Omega$. To prove this result they need mainly two key properties of $\ell_1(\beta, \Omega)$, that is a level set representation formula and the decreasing monotonicity of $\ell_1(\beta, \Omega)$ with respect to the radius when $\Omega$ is a ball that is

$$\ell_1(\beta, B_r) \leq \ell_1(\beta, B_s), \quad r > s > 0.$$  

Despite to the Dirichlet eigenvalue, in general $\ell_1(\beta, \Omega)$ is not monotone decreasing with respect the set inclusion. For instance, in [20] when $p = 2$ and $\beta = \overline{\beta}$, the authors prove a sort of monotonicity property (1.4) for suitable convex domains which are not necessary balls and they prove that

$$\ell_1(\beta, \Omega_2) \leq \ell_1(\beta, \Omega_1),$$

where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are bounded, Lipschitz and convex domains such that $\Omega_1 \subset B_r \subset \Omega_2$. Our aim is to prove (1.3) and (1.5) in the anisotropic case for any $p$, $1 < p < \infty$ and for a suitable function $\beta$ which is in general, not necessary constant. In particular, regarding (1.3), we will prove the following anisotropic Faber-Krahn inequality

$$\ell_1(\beta, \Omega) \geq \ell_1(\beta, W_R),$$

where $W_R = \{ F^o(\xi) < R \}$, with $F^o$ polar norm of $F$, such that $|W_R| = |\Omega|$ and the function $\beta(x) = w(F^o(x))$ with $w$ non negative continuous function in $\mathbb{R}$ such that

$$w(t) \geq C(R)t,$$  

where $C$ is a suitable constant. To do this we need to establish a representation formula for $\ell_1(\beta, \Omega)$, for not constant $\beta$. As a consequence of this formula, we also obtain the following anisotropic weighted Cheeger inequality for $\ell_1(\beta, \Omega)$

$$\ell_1(\beta, \Omega) \geq h_\beta(\Omega) - (p - 1)\|\beta_1'(\Omega)\|_{L^\infty(\Omega)},$$  

where $p' = \frac{n}{n-1}$, $\beta_\Omega$ is a function defined in the whole $\Omega$ having trace on $\partial \Omega$ equals to $\beta$ and $h_\beta(\Omega)$ is the anisotropic weighted Cheeger constant defined in section 6. This result was proved in the Euclidean case in [22] for $p = 2$ and $\beta = \overline{\beta}$ constant.

Our paper has the following structure.

In section 2 we recall notation and preliminary results. In section 3 we prove some basic properties of $\ell_1(\beta, \Omega)$. In section 4 we prove some useful properties of the anisotropic radial problem. In section 5 we state and show the quoted monotonicity result for $\ell_1(\beta, \Omega)$ and finally in section 6 we prove the representation formula for level set in the general case of variable coefficients $\beta$ proving as applications the quoted Faber-Krahn inequality (1.6) and the anisotropic weighted Cheeger inequality (1.8).
2 Notation and preliminaries

2.1 Finsler norm

Let $F$ be a convex, even, 1-homogeneous and non negative function defined in $\mathbb{R}^n$. Then $F$ is a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n,$$  \hspace{1cm} (2.1)

and such that

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^n,$$  \hspace{1cm} (2.2)

for some constant $a > 0$. The hypotheses on $F$ imply there exists $b \geq a$ such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^n.$$  \hspace{1cm} (2.3)

Moreover, throughout the paper we will assume that $F \in C^2(\mathbb{R}^n \setminus \{0\})$, and $[F^p]_{\xi\xi}(\xi)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$, \hspace{1cm} (2.4)

with $1 < p < +\infty$.

The polar function $F^o: \mathbb{R}^n \to [0, +\infty]$ of $F$ is defined as

$$F^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F(\xi)}.$$  \hspace{1cm}

It is easy to verify that also $F^o$ is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F^o(\xi)}.$$  \hspace{1cm}

The above property implies the following anisotropic version of the Cauchy Schwartz inequality

$$|\langle \xi, \eta \rangle| \leq F(\xi)F^o(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n.$$  \hspace{1cm}

The set

$$\mathcal{W} = \{ \xi \in \mathbb{R}^n : F^o(\xi) < 1 \}$$

is the so-called Wulff shape centered at the origin. We put $\kappa_n = |\mathcal{W}|$, where $|\mathcal{W}|$ denotes the Lebesgue measure of $\mathcal{W}$. More generally, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at $x_0$ with measure $\kappa_n r^n$, and $\mathcal{W}_r(0) = \mathcal{W}_r$.

The following properties of $F$ and $F^o$ hold true:

$$\langle F_\xi(\xi), \eta \rangle = F(\xi), \quad \langle F^o_\xi(\xi), \xi \rangle = F^o(\xi), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

$$F(F^o_\xi(\xi)) = F^o(F_\xi(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

$$F^o(\xi)F^o_\xi(F_\xi(\xi)) = F(\xi)F^o_\xi(F^o_\xi(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$  \hspace{1cm}

2.2 Anisotropic perimeter

We recall the definition of anisotropic perimeter for a bounded, Lipschitz open set:

**Definition 2.1.** Let $K$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. The anisotropic perimeter of $K$ is:

$$P_F(K) = \int_{\partial K} F(\nu) \, d\mathcal{H}^{n-1}$$

where $\nu$ denotes the unit outer normal to $\partial K$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.
Clearly, the perimeter of $K$ is finite if and only if the usual Euclidean perimeter of $K$, $P_E(K)$ is finite. Indeed, by the quoted properties of $F$ we obtain that

$$aP_E(K) \leq P_F(K) \leq bP_E(K).$$

Furthermore, an isoperimetric inequality for the anisotropic perimeter holds (see for instance [3, 11, 17]). Namely let $K$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, then

$$P_F(K) \geq n \kappa_n \frac{1}{n} |K|^{1\frac{1}{n}},$$

where $\kappa_n$ is the Lebesgue measure of the unit Wulff shape. In particular, the equality in (2.5) holds if and only if the set $K$ is homothetic to a Wulff shape. We recall the following so-called weighted anisotropic isoperimetric inequality (see for instance [3] and [4])

$$\int_{\partial \Omega} f(F'(x)) F(\nu) dH^{n-1} \geq \int_{\partial W_R} f(F'(x)) F(\nu) dH^{n-1} = f(R) P_F(W_R),$$

(2.6)

where $W_R$ is a Wulff shape such that $|\Omega| = |W_R|$ and $f : [0, R] \to [0, +\infty]$ is a nondecreasing function such that $g(z) = f(z^{\frac{1}{n}})z^{1\frac{1}{n}}$, $0 \leq z \leq R^n$, is convex with respect to $z$.

If $\Omega \subset \mathbb{R}^n$ is a bounded open set, the anisotropic Cheeger constant of $\Omega$ is defined as follows

$$h_F(\Omega) = \inf_{U \subset \Omega} \frac{P_F(U)}{|U|},$$

(2.7)

In [10] the authors prove that

$$\frac{1}{R_F} \leq h_F(\Omega) \leq \frac{n}{R_F},$$

(2.8)

where $R_F$ is the anisotropic inradius that is the radius of the biggest Wulff shape contained in $\Omega$.

### 2.3 Anisotropic $p$-Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u \in W^{1,p}(\Omega)$. For $1 < p < +\infty$ the anisotropic $p$-Laplacian is defined as follows

$$Q_p u := \text{div} \left( \frac{1}{p} \nabla \xi [F^p](\nabla u) \right).$$

The hypothesis (2.4) on $F$ ensures that the operator is elliptic, hence there exists a positive constant $\gamma$ such that

$$\frac{1}{p} \sum_{i,j=1}^n \nabla \xi_{ij} [F^p](\eta) \xi_i \xi_j \geq \gamma |\eta|^{p-2}|\xi|^2,$$

for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$.

For $p = 2$, $Q_2$ is the so-called Finsler Laplacian, and when $F(\xi) = |\xi| = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean norm, $Q_p$ reduces to the well known $p$-Laplace operator.

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $1 < p < +\infty$, and consider the following eigenvalue problem with Dirichlet boundary conditions related to $Q_p$

$$\left\{ \begin{array}{ll}
-Q_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{array} \right.$$

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The smallest eigenvalue, denoted by $\lambda_D(\Omega)$, has the following well-known variational characterization:

$$\lambda_D(\Omega) = \min_{\varphi \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega F^p(\nabla \varphi) \, dx}{\int_\Omega |\varphi|^p \, dx}.$$

For the first eigenvalue of the anisotropic $p$-Laplacian with Dirichlet boundary conditions, the following isoperimetric inequality holds (see [2]).

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$, be a bonded domain with $n \geq 2$ then

$$|\Omega|^\frac{2}{n} \lambda_D(\Omega) \geq \kappa_n^\frac{2}{n} \lambda_D(W).$$

Moreover, the equality holds if and only if $\Omega$ is homothetic to a Wulff shape.

Finally we recall that for a given bounded open set in $\mathbb{R}^n$, the anisotropic Cheeger inequality states that (see for instance [7, 10, 21])

$$\lambda_D(\Omega) \geq \left( \frac{h_F(\Omega)}{p} \right)^p, \quad 1 < p < \infty. \quad (2.9)$$

## 3 The first Robin eigenvalue of $Q_p$

In this section we will investigate some properties of the first Robin eigenvalue related to $Q_p$, $1 < p < \infty$. From now on we assume that

$\Omega \subset \mathbb{R}^n$ is a bounded open set with $C^{1,\alpha}$ boundary and $\alpha \in ]0,1[$. \quad (3.1)

Let us consider the following Robin eigenvalue problem for $Q_p$

$$
\begin{aligned}
- Q_p u &= \ell |u|^{p-2} u & \text{in } \Omega \\
F^{p-1}(\nabla u) F_\xi(\nabla u) \cdot \nu + \beta(x) F(\nu) |u|^{p-2} u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

(3.2)

where $u \in W^{1,p}(\Omega)$, $\nu$ is the Euclidean unit outer normal to $\partial \Omega$ and the function $\beta : \partial \Omega \to [0, +\infty[$ belongs to $L^1(\partial \Omega)$ and verifies

$$
\int_{\partial \Omega} \beta(x) F(\nu) \, d\mathcal{H}^{n-1} = m > 0.
$$

(3.3)

From now on we will write $\bar{\beta}$ instead of $\beta$ when $\beta$ is a positive constant.

**Definition 3.1.** A function $u \in W^{1,p}(\Omega)$, $u \not\equiv 0$ is an eigenfunction to (3.2) if $\beta(\cdot)|u|^p \in L^1(\partial \Omega)$ and

$$
\int_{\Omega} F^{p-1}(\nabla v) F_\xi(\nabla v) \cdot \nabla \varphi \, dx + \int_{\partial \Omega} \beta(x) |u|^{p-2} u \varphi F(\nu) \, d\mathcal{H}^{n-1} = \ell \int_{\Omega} |u|^{p-2} u \varphi \, dx
$$

(3.4)

for any test function $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\partial \Omega)$. The corresponding number $\ell$, is called Robin eigenvalue.

The smallest eigenvalue of (3.2), $\ell_1(\beta, \Omega)$ has the following variational characterization

$$
\ell_1(\beta, \Omega) = \inf_{v \in W^{1,p}(\Omega) \setminus \{0\}} \inf_{v \not\equiv 0} J[\beta, v] := \inf_{v \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F^p(\nabla v) \, dx + \int_{\partial \Omega} \beta(x) |v|^p F(\nu) \, d\mathcal{H}^{n-1}}{\int_{\Omega} |v|^p \, dx}.
$$

(3.5)
By definition we have
\[ \ell_1(\beta, \Omega) \leq \lambda_D(\Omega), \]
where \( \lambda_D(\Omega) \) is the first Dirichlet eigenvalue of \( Q_p \). Indeed choosing as test function in \( (3.5) \), the first Dirichlet eigenfunction \( u_D \) of \( \lambda_D(\Omega) \) in the Reileigh quotient, we get
\[
\ell_1(\beta, \Omega) = \min_{v \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} [F(\nabla v)]^p \, dx + \int_{\partial \Omega} \beta |v|^p F(\nu) \, dH^{n-1}}{\int_{\Omega} |v|^p \, dx} \]
\[
\leq \frac{\int_{\Omega} [F(\nabla u_D)]^p \, dx + \int_{\partial \Omega} \beta |u_D|^p F(\nu) \, dH^{n-1}}{\int_{\Omega} |u_D|^p \, dx} = \lambda_D(\Omega). \]

The following existence result holds.

**Proposition 3.1.** Let \( \beta \in L^1(\partial \Omega) \), \( \beta \geq 0 \) be such that \( (3.3) \) holds. Then there exists a positive minimizer \( u \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega) \) of \( (3.5) \) which is a weak solution to \( (3.2) \) in \( \Omega \) with \( \ell = \ell_1(\beta, \Omega) \). Moreover \( \ell_1(\beta, \Omega) \) is positive and it is simple, that is the relative eigenfunction \( u \) is unique up to a multiplicative constant.

**Proof.** Let \( u_k \in W^{1,p}(\Omega) \) be a minimizing sequence of \( (3.5) \) such that \( \|u_k\|_{L^p(\Omega)} = 1 \). Then, being \( u_k \) bounded in \( W^{1,p}(\Omega) \) there exists a subsequence, still denoted by \( u_k \) and a function \( u \in W^{1,p}(\Omega) \) with \( \|u\|_{L^p(\Omega)} = 1 \), such that \( u_k \to u \) strongly in \( L^p(\Omega) \) and \( \nabla u_k \rightharpoonup \nabla u \) weakly in \( L^p(\Omega) \). Then \( u_k \) converges to \( u \) in \( L^p(\partial \Omega) \) and then almost everywhere on \( \partial \Omega \) to \( u \). Then by the weak lower semicontinuity and Fatou’s lemma we get
\[
\ell_1(\beta, \Omega) = \lim_{k \to +\infty} J[\beta, u_k] \geq J[\beta, u],
\]
then \( \beta(\cdot)|u|^p \in L^1(\partial \Omega) \) and \( u \) is an eigenfunction related to \( \ell_1(\beta, \Omega) \) by definition. Moreover \( u \in L^\infty(\Omega) \). To see that, we can argue exactly as in \[11\] in order to get that \( u \in L^\infty(\Omega) \).

Now the \( L^\infty \)-estimate, the hypothesis \( (2.4) \) and the properties of \( F \) allow to apply standard regularity results (see \[16\], \[27\]), in order to obtain that \( u \in C^{1,\alpha}(\Omega) \).

In order to prove that \( \ell_1(\beta, \Omega) > 0 \), we proceed by contradiction supposing that there exists \( \beta_0 \) which verifies \( (3.3) \) and such that \( \ell_1(\beta_0, \Omega) = 0 \). Then there exists \( u_{\beta_0} \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega) \) such that \( u_{\beta_0} \geq 0 \), \( \|u_{\beta_0}\|_{L^p(\Omega)} = 1 \) and
\[
0 = \ell(\beta_0, \Omega) = \int_{\Omega} F^p(\nabla u_{\beta_0}) \, dx + \int_{\partial \Omega} \beta_0 u_{\beta_0}^p \, F(\nu) \, dH^{n-1}.
\]
Then \( u_{\beta_0} \) has to be constant in \( \overline{\Omega} \) and then \( u_{\beta_0}^p \int_{\partial \Omega} \beta_0 F(\nu) = u_{\beta_0}^p m = 0 \). Being \( m > 0 \), then \( u_{\beta_0} = 0 \) in \( \overline{\Omega} \), and this is not true. Hence \( \ell_1(\beta_0, \Omega) > 0 \).

Finally to prove the simplicity of the eigenfunctions we can proceed exactly as in \[11\]. For completeness we recall the main steps. Let \( u, w \) be positive minimizers of the functional \( J \) defined in \( (3.5) \) such that \( \|u\|_p = \|w\|_p = 1 \), and let us consider the function \( \eta_t = (tu^p + (1 - t)w^p)^{1/p} \), with \( t \in [0,1] \). Obviously, \( \|\eta_t\|_p = 1 \). Clearly it holds:
\[
J[\beta, u] = \ell_1(\beta, \Omega) = J[\beta, w]. \tag{3.6}
\]
In order to compute \( J[\beta, \eta_t] \) we observe that by using the homogeneity and the convexity of \( F \) it is not hard to prove that (see for instance \[11\] for the precise computation)
\[
F^p(\nabla \eta_t) \leq t F^p(\nabla v) + (1 - t) F^p(\nabla w). \tag{3.7}
\]
Hence recalling (3.6), we obtain
\[
J[\beta, \eta] \leq tJ[\beta, u] + (1-t)J[\beta, w] = \ell_1(\beta, \Omega),
\]
and then \(\eta\) is a minimizer for \(J\). This implies that the equality holds in (3.7), and as showed in [11], this implies that \(u = w\) that is the uniqueness.

The following result characterizes the first eigenfunctions.

**Proposition 3.2.** Let \(\beta \in L^1(\partial \Omega), \beta \geq 0\) be such that (3.3) holds. Let \(\eta > 0\) and \(v \in W^{1,p}(\Omega)\), \(v \not\equiv 0\) and \(v \geq 0\) in \(\Omega\) such that
\[
\begin{align*}
- \frac{\eta}{p} v &= \eta v^{p-1} & \text{in } \Omega \\
F^{p-1}(\nabla v)F(v) \cdot \nu + \beta F(v)v^{p-1} &= 0 & \text{on } \partial \Omega
\end{align*}
\]
in the sense of Definition 3.7. Then \(v\) is a first eigenfunction of (3.1), and \(\eta = \ell_1(\beta, \Omega)\).

**Proof.** Let \(u \in W^{1,p}(\Omega)\) be a positive eigenfunction related to \(\ell_1(\beta, \Omega)\). Choosing \(u^p/(v+\varepsilon)^{p-1}\), with \(\varepsilon > 0\), as test function in the Definition 3.1 for the solution \(v\), and arguing exactly as in [11], we get the claim.

**Remark 3.1.** We observe that Propositions 3.1 and 3.2 generalize the results proved respectively in [13] for the Euclidean norm and in [11] when \(\beta(x) = \beta\) is a positive constant.

**Theorem 3.1.** Let \(\beta \in L^1(\partial \Omega), \beta \geq 0\) and such that (3.3) holds. The following properties hold for \(\ell_1(\beta, \Omega)\):

(i) \(\forall t > 0, \ell_1(\beta(t\nu), t\Omega) = t^{-p}\ell_1(\beta(y), \Omega), \quad x \in \partial(t\Omega), y \in \partial \Omega\);

(ii) \(\ell_1(\beta, \Omega) \leq \frac{m}{|\Omega|}\);

(iii) \(a^p\ell_\epsilon(a^{1-p}\beta, \Omega) \leq \ell_1(\beta, \Omega) \leq b^p\ell_\epsilon(b^{1-p}\beta, \Omega), \quad \) where \(a, b\) are defined in (2.2), (2.3) and \(\ell_\epsilon(a^{1-p}\beta, \Omega), \ell_\epsilon(b^{1-p}\beta, \Omega)\) are the first Robin eigenvalue for the Euclidean \(p\)-Laplacian corresponding respectively to the function \(a^{1-p}\beta\) and \(b^{1-p}\beta\);

(iv) If \(\beta(x) \geq \bar{\beta} > 0\), for almost \(x \in \partial \Omega\), then
\[
\sup_{|\Omega|=k} \ell_1(\beta, \Omega) = +\infty
\]

**Proof.** By the homogeneity of \(F\), we have:
\[
\ell_1\left(\beta \left(\frac{x}{t}\right), t\Omega\right) = \min_{\varphi \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega F_p(\nabla \varphi(x))dx + \int_{\partial(t\Omega)} \beta \left(\frac{x}{t}\right) \varphi(x) |F(\nu(x))d\mathcal{H}^{n-1}(x)}{\int_\Omega |\varphi(x)|^pdx}
\]
\[
= \min_{\varphi \in W^{1,p}(\Omega) \setminus \{0\}} \frac{t^{-p} \int_\Omega F_p(\nabla v(\frac{x}{t})) dy + \int_{\partial(t\Omega)} \beta \left(\frac{x}{t}\right) |v(\frac{x}{t})|^p F\left(\nu(\frac{x}{t})\right) d\mathcal{H}^{n-1}(y)}{t^n \int_\Omega |v(y)|^p dy}
\]
\[
= \min_{\varphi \in W^{1,p}(\Omega) \setminus \{0\}} \frac{t^{n-p} \int_\Omega F_p(\nabla v(y)) dy + t^{-n} \int_{\partial \Omega} \beta(y) |v(y)|^p F(\nu(y)) d\mathcal{H}^{n-1}(y)}{t^n \int_\Omega |v(y)|^p dy}
\]
\[
= t^{-p}\ell_1(\ell_1^{p-1}(\beta(y), \Omega)).
\]
In order to obtain the second property, it is sufficient to consider a non-zero constant as test function in (3.5).

Now we prove the inequality in the right-hand side in (iii). The proof of the other inequality is similar. By using (3.5) and (2.3), we obtain that

$$\ell_1(\beta, \Omega) = \inf_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega F_p(\nabla v) dx + \int_{\partial\Omega} \beta(x)|v|^p F(\nu) d\mathcal{H}^{n-1}}{\int_\Omega |v|^p dx} \geq \ell_1(\bar{\beta}, \Omega) = \inf_{v \in \mathbb{R}} \frac{b^p \int_\Omega |\nabla v|^p dx + \int_{\partial\Omega} b^{1-p} \beta(x) |v|^p d\mathcal{H}^{n-1}}{\int_\Omega |v|^p dx} = b^p \ell_2(b^{1-p} \beta, \Omega),$$

where last equality follows, by definition of $\ell_2(b^{1-p} \beta, \Omega)$.

Finally we give the proof of (iv). Clearly $\ell_1(\beta, \Omega) \geq \ell_1(\bar{\beta}, \Omega)$, then by [11] Proposition 3.1, we know that

$$\ell_1(\bar{\beta}, \Omega) \geq \left( \frac{p-1}{p} \right)^p \frac{\bar{\beta}}{R_F \left( 1 + \bar{\beta}^{\frac{p-1}{p}} R_F \right)}$$

(3.8)

where $R_F$ is the anisotropic inradius of the subset $\Omega$. The claim follows constructing a sequence of convex sets $\Omega_k$ with $|\Omega_k| = 1$ and such that $R_F(\Omega_k) \to 0$, for $k \to \infty$. Let $k > 0$, proceeding as in [10] [12], it is possible to consider the $n$-rectangles $\Omega_k = \left[ -\frac{k^\frac{1}{p-1}}{2}, \frac{k^\frac{1}{p-1}}{2} \right]^{n-1}$ and suppose that $R_F(\Omega_k) = \frac{1}{2k} F^\circ(e_1)$. Then we obtain

$$\ell_1(\bar{\beta}, \Omega_k) \geq \left( \frac{p-1}{p} \right)^p \frac{4k^2 \bar{\beta}}{F^\circ(e_1) \left( 2k + \bar{\beta}^{\frac{p-1}{p}} F^\circ(e_1) \right)} \to +\infty \quad \text{for} \quad k \to \infty.$$
Theorem 4.1. Let \( v_p \in C^{1,\alpha}(W_R) \cap C(\overline{W_R}) \) be a positive solution to problem \((\ref{e1})\). Then \( v_p(x) = \varrho_p(F^o(x)) \), with \( x \in \overline{W_R} \), where \( \varrho_p(r), \ r \in [0,R], \) is a decreasing function such that \( \varrho_p \in C^{\infty}(0, R) \cap C^1([0, R]) \) and it verifies

\[
\begin{aligned}
-(p-1)(-\varrho'_p(r))^{p-2} \varrho''_p(r) + \frac{n-1}{r} (-\varrho'_p(r))^{p-1} &= \ell_1(\beta, W_R) \varrho_p(r)^{p-1}, \quad r \in ]0, R[,
\varrho_p(0) = 0,
-(\varrho''_p(R))^{p-1} + \beta(\varrho_p(R))^{p-1} = 0.
\end{aligned}
\]  

(4.3)

Remark 4.1. We observe that the first eigenvalue in the Wulff \( W_R = \{F^o(x) < R\} \) is the same for any norm \( F \). In particular it coincides with the first Robin eigenvalue in the Euclidean ball \( B_R \) for the \( p \)-Laplace operator. Finally we emphasize that in this case the eigenfunctions have more regularity because \( \beta \) is a positive constant.

Theorem 4.1 as in \([5, 11]\), suggests to consider, for every \( x \in W_R \), the following function

\[
f(r_x) = \frac{\left(-\varrho'_p(r_x)\right)^{p-1}}{(\varrho_p(r_x))^{p-1}} = \frac{[F(\nabla v_p(x))]^{p-1}}{v_p(x)^{p-1}} = \frac{[F(\nabla v_p(x))]^{p-1} F'(Dv_p(x)) \cdot \nu}{v_p(x)^{p-1} F(\nu)},
\]

(4.4)

where

\[r_x = F^o(x), \quad 0 \leq r_x \leq R.\]

Let us observe that \( f \) is nonnegative, \( f(0) = 0 \) and

\[
f(R) = \frac{(-\varrho'_p(R))^{p-1}}{\varrho_p(R)^{p-1}} = \frac{[F(\nabla v_p(x))]^{p-1}}{v_p(x)^{p-1}} = \frac{[F(\nabla v_p(x))]^{p-1} F'(\nabla v_p(x)) \cdot \nu}{v_p(x)^{p-1} F(\nu)} = \beta
\]

The following result proved in the Euclidean case in \([5]\) and in \([11]\) in the anisotropic case, states that the first Robin eigenvalue is monotone decreasing with respect the set inclusion in the class of Wulff shapes.

Lemma 4.1. The function \( r \to \ell_1(\beta, W_r) \) is strictly decreasing in \( ]0, \infty[ \).

In \([5]\) and \([11]\) the authors prove also the following monotonicity property for the function \( f \) defined in \((\ref{f})\).

Lemma 4.2. Let \( f \) be the function defined in \((\ref{f})\). Then \( f(r) \) is strictly increasing in \([0, R]\).

In the next result we prove a convex property for the function \( f \).

Theorem 4.2. Let \( f \) be the function defined in \((\ref{f})\). Then the function

\[
g(z) = f\left(z^{\frac{1}{n}}\right) z^{1-\frac{1}{n}}, \quad 0 \leq z \leq R^n,
\]

is convex with respect to \( z \).

Proof. We first observe that by \((\ref{f})\) it holds

\[
f'(r) = \frac{d}{dr} \left(-\frac{\varrho'_p(r)}{\varrho_p(r)} \right)^{p-1} = (p-1) f^{\frac{p-2}{p-1}} \left(\frac{-\varrho''_p}{\varrho_p} + \frac{\varrho'_p}{\varrho_p} \right)^2
\]

\[
= \ell_1(\beta, W_R) - \frac{(n-1)}{r} f + (p-1) f^{\frac{p-2}{p-1}} \quad \forall r \in ]0, R[.
\]

(4.5)
Then

\[ g'(z) = \frac{1}{n} f'(\frac{z}{n}) + \frac{(n-1)}{n} f(\frac{z}{n}) \]

\[ = \frac{1}{n} \left( \ell_1(\bar{\beta}, W_R) - \frac{(n-1)}{n} f\left(\frac{z}{n}\right) + (p-1) f^{\frac{n}{p-1}}\left(\frac{z}{n}\right) \right) + \frac{(n-1)}{n} f\left(\frac{z}{n}\right) \]

\[ = \frac{\ell_1(\bar{\beta}, W_R)}{n} + \frac{(p-1)}{n} f^{\frac{n}{p-1}}\left(\frac{z}{n}\right), \]

which is increasing and this implies the thesis. 

Finally the following comparison result for \( f \) holds

**Theorem 4.3.** Let \( f \) be the function defined in (4.4). Then there exists a positive constant \( C = C(R) \) such that

\[ f(0) \leq r C(R), \quad \text{for} \quad 0 \leq r \leq R \]

**Proof.** By (4.4) and by Lemma 4.2 we obtain that \( f \) verifies the following equation

\[ f'(r) = \ell_1(\bar{\beta}, W_R) - \frac{(n-1)}{r} f(r) + (p-1) f^{\frac{n}{p-1}}(r) \leq C(R) - \frac{(n-1)}{r} f(r) \] (4.6)

where

\[ C(R) = \ell_1(\bar{\beta}, W_R) + (p-1) f(R)^{\frac{n}{p-1}} = \ell_1(\bar{\beta}, W_R) + (p-1) \bar{\beta}^{\frac{n}{p-1}} \] (4.7)

Then by (4.6) multiplying both sides by \( r^{n-1} \) we get

\[ f'(r) r^{n-1} + (n-1) r^{n-2} f(r) \leq C(R) r^{n-1}, \]

and

\[ \frac{d}{dr} (r^{n-1} f(r)) \leq C(R) r^{n-1} \]

Then the claim follows integrating both sides between 0 and \( r \). 

**Remark 4.2.** The results contained in Lemma 4.2 and Theorem 4.2 ensures that \( f(r) \) is an admissible weight for the weighted anisotropic isoperimetric inequality quoted in (2.6)

5 **A monotonicity property for \( \ell_1(\bar{\beta}; \Omega) \)**

In this section we assume that \( \beta = \bar{\beta} \) is a positive constant. The first Robin eigenvalue \( \ell_1(\bar{\beta}, \Omega) \) has not, in general, a monotonicity property with respect the set inclusion. For instance in [15] in the Euclidean case, for the Laplace operator, the authors give a counterexample. More precisely, they construct a suitable sequence of sets \( \Omega_k \) such that \( P_1(\Omega_k) \to \infty, B_1(0) \subset \Omega_k \subset B_{1+\epsilon}(0) \) which verify

\[ \ell_1(\bar{\beta}, \Omega_k) > \ell_1(\bar{\beta}, B_1(0)) > \ell_1(\bar{\beta}, B_2(0)). \]

Here \( B_r(x_o) \) denotes the Euclidean ball with radius \( r \) and centered at the pint \( x_o \) and \( \lambda_1(B_{1+\epsilon}(0)) \) is the Euclidean first Dirichlet eigenvalue of the Laplacian of the ball \( B_{1+\epsilon}(0) \). In what follows we prove a monotonicity type property for the first Robin eigenvalue of the operator \( Q_p \) with respect the set inclusion. In the Euclidean case for the Laplace operator we refer the reader for instance to [20].
Theorem 5.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,\alpha}$ boundary, $\alpha \in ]0,1[$. Let $\mathcal{W}_R$ be a Wulff shape such that $\Omega \subset \mathcal{W}_R$ and $\beta$ a positive constant. Then

$$\ell_1(\beta, \mathcal{W}_R) \leq \ell_1(\beta, \Omega).$$

Proof. Let $v_p$ be the positive eigenfunction associated to $\ell_1(\beta, \mathcal{W}_R)$ and let $\Omega$ be a subset of $\mathcal{W}_R$.

Then for every $x \in \partial \Omega$, we can consider $f(x)$ as in (4.3) in order to get that the following Robin boundary condition on $\partial \Omega$ holds

$$[F(\nabla v_p(x))]^{p-1} F_\ell(\nabla v_p(x)) \cdot \nu + f(x) v_p(x)^{p-1} F(\nu) = 0.$$  \hspace{1cm} (5.1)

Having in mind that $\Omega \subset \mathcal{W}_R$ and using (5.1), we have that $v_p$ solves the following problem

$$\begin{cases}
- \mathcal{Q}_p v_p = \ell_1(\beta, \mathcal{W}_R)v_p^{p-1} & \text{in } \Omega \\
[F(\nabla v_p)]^{p-1} F_\ell(\nabla v_p) \cdot \nu + f(x) v_p^{p-1} F(\nu) = 0 & \text{on } \partial \Omega
\end{cases} \hspace{1cm} (5.2)$$

Using (5.2) and Lemma 4.2

$$\ell_1(\beta, \mathcal{W}_R) = \inf_{u \in \mathcal{W}_R} \frac{\int_{\Omega} |F(\nabla u)|^p dx + \int_{\partial \Omega} f(x)|v_p|^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\Omega} |v_p|^p dx} \leq \inf_{u \in \mathcal{W}_R} \frac{\int_{\Omega} |F(\nabla u)|^p dx + \int_{\partial \Omega} |v_p|^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\Omega} |u|^p dx} = \ell_1(\beta, \Omega)$$

When $\Omega$ contains a Wulff shape we have the following result

Theorem 5.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex open set with $C^{1,\alpha}$ boundary, $\alpha \in ]0,1[$. Let $\mathcal{W}_R$ be a Wulff shape such that $\mathcal{W}_R \subset \Omega$, then

$$\ell_1(\beta, \Omega) \leq \ell_1(\beta, \mathcal{W}_R).$$

Proof. First of all, we take the positive eigenfunction $v_p$ associated to $\ell_1(\beta, \mathcal{W}_R)$. By Theorem 4.1 $v_p(x) = g_p(F^p(x))$, and by (4.3) we can extend $g_p$ up to $+\infty$ and then $v_p$ in $\mathbb{R}^n$. Let us consider the super-level set

$$\mathcal{W}_+ = \{ x \in \mathbb{R}^n : v_p(x) > 0 \}.$$ 

By the property of $v_p$, $\mathcal{W}_+$ is a Wulff shape and clearly $\mathcal{W}_R \subset \mathcal{W}_+$.

Moreover, $v_p$ solves the following equation

$$- \mathcal{Q}_p v_p = \ell_1(\beta, \mathcal{W}_R)v_p^{p-1} \text{ in } \mathcal{W}_+.$$ 

To prove the Theorem we consider the set $\Omega = \Omega \cap \mathcal{W}_+$. Being $\Omega$ convex and due to the radially decreasing of the eigenfunction, three possible cases can occur.
Case 1: \( \partial \tilde{\Omega} = \partial \Omega \). Then in this case \( \mathcal{W}_R \subset \Omega \subset \mathcal{W}_+ \) and \( \tilde{\Omega} = \Omega \). Then for \( x \in \partial \Omega \) we put \( r_x = F^\alpha(x) \) and we can compute

\[
f(r_x) = \frac{(-\rho_p(r_x))^{p-1}}{(\rho_p(r_x))^{p-1}}.
\]

Then arguing as in the proof of Theorem 5.1 and recalling that by Lemma 4.2, \( f(r_x) \geq \bar{\beta} \), for any \( x \in \partial \Omega \) we get

\[
\ell_1(\bar{\beta}, \mathcal{W}_R) = \frac{\int_{\tilde{\Omega}} [F(\nabla v_p)]^p dx + \int_{\partial \tilde{\Omega}} f(r_x) |v_p|^p F(\nu) d\sigma}{\int_{\tilde{\Omega}} |v_p|^p dx} = \inf_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\tilde{\Omega}} [F(\nabla u)]^p dx + \int_{\partial \tilde{\Omega}} f(r_x) |u|^p F(\nu) d\sigma}{\int_{\tilde{\Omega}} |u|^p dx} \geq \inf_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\tilde{\Omega}} [F(\nabla u)]^p dx + \int_{\partial \tilde{\Omega}} \bar{\beta} |u|^p F(\nu) d\sigma}{\int_{\tilde{\Omega}} |u|^p dx} = \ell_1(\bar{\beta}, \Omega)
\]

and the first case is proved.

Case 2: \( \partial \tilde{\Omega} \cap \partial \Omega \neq \emptyset \) and \( \partial \tilde{\Omega} \cap \partial \Omega \neq \partial \Omega \). Then \( \partial \tilde{\Omega} \cap \mathcal{W}_+ \neq \emptyset \). Moreover, on \( \partial \tilde{\Omega} \cap \partial \Omega \) the eigenfunction \( v_p \) is positive, while on \( \partial \tilde{\Omega} \cap \partial \mathcal{W}_+ \) it is equal to zero. In particular, for every \( x \in \partial \tilde{\Omega} \cap \partial \Omega \) we still have that \( f(r_x) \geq \bar{\beta} \) as in the Case 1. We define the following test function \( \varphi \in W^{1,p}(\Omega) \)

\[
\varphi(x) = \begin{cases} 
v_p(x) & \text{in } \tilde{\Omega} \\
0 & \text{in } \Omega \setminus \tilde{\Omega}.
\end{cases}
\]

Then

\[
\ell_1(\bar{\beta}, \mathcal{W}_R) = \frac{\int_{\tilde{\Omega}} [F(\nabla v_p)]^p dx + \int_{\partial \tilde{\Omega} \cap \partial \Omega} f(r_x) v_p^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\tilde{\Omega}} v_p^p dx} = \frac{\int_{\Omega} [F(\nabla \varphi)]^p dx + \int_{\partial \tilde{\Omega} \cap \partial \Omega} f(r_x) \varphi^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\Omega} \varphi^p dx} \geq \frac{\int_{\Omega} [F(\nabla \varphi)]^p dx + \int_{\partial \tilde{\Omega} \cap \partial \Omega} \bar{\beta} \varphi^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\Omega} \varphi^p dx} = \frac{\int_{\Omega} [F(\nabla v)]^p dx + \int_{\partial \Omega} \bar{\beta} \varphi^p F(\nu) d\mathcal{H}^{n-1}}{\int_{\Omega} \varphi^p dx}
\]
When $\beta$ in general a function defined on $\Omega$, following problem a function belonging to $L^1(\Omega)$ formula holds positive minimizer of (3.5) cases, after a finite number of steps we could be either in Case 1 or in Case 2.

By Theorems 5.1 and 5.2 we get the following monotonicity property for $\ell_1$ for constant $\beta$.

**Corollary 5.1.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be as in (3.1) and convex. Let $W_R$ be a Wulff shape such that $\Omega_1 \subset W_R \subset \Omega_2$. Then $\ell_1(\beta, \Omega_2) \leq \ell_1(\beta, \Omega_1)$.

### 6 A representation formula for $\ell_1(\beta, \Omega)$

In this section we prove a level set representation formula for the first eigenvalue $\ell_1(\beta, \Omega)$ of the following problem

$$
\begin{cases}
-\mathcal{Q}_p \nu = \ell |v|^{p-2}v & \text{in } \Omega \\
F^{p-1}(\nabla v)\xi(\nabla v) \cdot \nu + \beta F(\nu)|v|^{p-2}v = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(6.1)

When $\beta = \bar{\beta}$ is a nonnegative constant a similar result can be found in [5] in the Euclidean case and in [11] for the anisotropic case. Our aim is to extend the known results assuming that $\beta$ is in general a function defined on $\partial \Omega$. In the next we will use the following notation. Let $\tilde{u}_p$ be the first positive eigenfunction such that $\max \tilde{u}_p = 1$. Then, for $t \in [0, 1]$,

$$
U_t = \{ x \in \Omega : \tilde{u}_p > t \}, \\
S_t = \{ x \in \Omega : \tilde{u}_p = t \}, \\
\Gamma_t = \{ x \in \partial \Omega : \tilde{u}_p > t \}.
$$

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,\alpha}$ boundary and let $\alpha \in [0, 1]$. Let $\beta$ be a function belonging to $L^1(\partial \Omega)$, $\beta \geq 0$ and such that (3.3) holds. Let $\tilde{u}_p \in C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ be a positive minimizer of (3.5) with $\|\tilde{u}_p\|_\infty = 1$. Then for a.e. $t \in [0, 1]$ the following representation formula holds

$$
\ell_1(\beta, \Omega) = \mathcal{F}_\Omega \left( U_t, \frac{|F(\nabla \tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}} \right),
$$

(6.2)

where $\mathcal{F}_\Omega$ is defined as

$$
\mathcal{F}_\Omega(U_t, \varphi) = \frac{1}{|U_t|} \left( -(p-1) \int_{U_t} \varphi^p dx + \int_{S_t} \varphi F(\nu) d\sigma + \int_{\Gamma_t} \beta F(\nu) d\sigma \right).
$$

(6.3)

**Proof.** Let $0 < \epsilon < t < 1$ and we define

$$
\psi_\epsilon = \begin{cases}
0 & \text{if } \tilde{u}_p \leq t \\
\frac{\tilde{u}_p}{\epsilon} & \text{if } t < \tilde{u}_p < t + \epsilon \\
\frac{1}{\epsilon} & \text{if } \tilde{u}_p \geq t + \epsilon.
\end{cases}
$$
The functions \( \psi \) are in \( W^{1,p}(\Omega) \) and increasingly converge to \( \tilde{u}_p^{-(p-1)} \chi_{U_t} \) as \( \varepsilon \to 0 \). Moreover, we can obtain that

\[
\nabla \psi \varepsilon = \begin{cases} 
0 & \text{if } \tilde{u}_p < t \\
\frac{1}{\varepsilon} \left( (p-1) \frac{t}{\tilde{u}_p} + 2 - p \right) \tilde{u}_p^{-p} & \text{if } t < \tilde{u}_p < t + \varepsilon \\
-(p-1) \frac{\nabla \tilde{u}_p}{\tilde{u}_p} & \text{if } \tilde{u}_p > t + \varepsilon.
\end{cases}
\]

Then choosing \( \psi \varepsilon \) as test function in (3.4), we get that the first integral is

\[
-(p-1) \int_{U_{t+\varepsilon}} \frac{|F(\nabla \tilde{u}_p)|^p}{\tilde{u}_p^p} dx + \frac{1}{\varepsilon} \int_{U_{t+\varepsilon} U_{t+\varepsilon}} \frac{|F(\nabla \tilde{u}_p)|^p}{\tilde{u}_p^{-p-1}} \left( (p-1) \frac{t}{\tilde{u}_p} + 2 - p \right) dx =
\]

\[
-(p-1) \int_{U_{t+\varepsilon}} |F(\nabla \tilde{u}_p)|^p dx + \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} \left( (p-1) \frac{t}{\tau} + 2 - p \right) \int_{S_{r}} \frac{|F(\nabla \tilde{u}_p)|^p}{\tilde{u}_p^{-p-1}} F(\nu) dH^{n-1},
\]

where last equality follows by the coarea formula. Then, reasoning as in [11] and [11] we get that

\[
\int_{\Omega} |F(\nabla \tilde{u}_p)|^p F(\nabla \psi \varepsilon) dx \xrightarrow{\varepsilon \to 0} -(p-1) \int_{U_t} |F(\nabla \tilde{u}_p)|^p dx + \int_{S_t} \frac{|\nabla \tilde{u}_p|^p}{\tilde{u}_p^{-1}} F(\nu) dH^{n-1}.
\]

As regards the other two integrals in (3.4), we have

\[
\int_{\partial \Omega} \beta \tilde{u}_p^{-p-1} \psi \varepsilon F(\nu) dH^{n-1} = \int_{\Gamma_{t+\varepsilon}} \beta F(\nu) dH^{n-1} + \int_{\Gamma_{t+\varepsilon}} \frac{\beta \tilde{u}_p - t}{\varepsilon} F(\nu) dH^{n-1} \xrightarrow{\varepsilon \to 0} \int_{\Gamma_t} \beta F(\nu) dH^{n-1},
\]

by dominated convergence theorem and by monotone convergence theorem and the definition of \( \psi \varepsilon \),

\[
\ell_1(\beta, \Omega) \int_{\Omega} \tilde{u}_p^{-p-1} \psi \varepsilon dx \xrightarrow{\varepsilon \to 0} \ell_1(\beta, \Omega) |U_t|.
\]

Summing the three limits, we get (6.2).

When we consider a generic test function we have

**Theorem 6.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with \( C^{1,\alpha} \) boundary and let \( \alpha \in ]0, 1[ \). Let \( \varphi \) be a nonnegative function in \( \Omega \) such that \( \varphi \in L^{p'}(\Omega) \), where \( p' = \frac{p}{p-1} \). If \( \varphi \neq |F(\nabla \tilde{u}_p)|^p/\tilde{u}_p^{-1} \), where \( \tilde{u}_p \) is the eigenfunction given in Theorem 6.7, and \( F_{\Omega} \) is the functional defined in (6.3), then there exists a set \( S \subset ]0, 1[ \) with positive measure such that for every \( t \in S \) it holds that

\[
\ell_1(\beta, \Omega) > F_{\Omega}(U_t, \varphi).
\]

The proof is similar to that obtained in [13] and [11], and we only sketch it here. It can be divided in two main steps. First, we claim that, if

\[
w(x) := \varphi - \frac{|F(\nabla \tilde{u}_p)|^{p-1}}{\tilde{u}_p^{p-1}}, \quad I(t) := \int_{U_t} w \frac{|F(\nabla \tilde{u}_p)|^{p-1}}{\tilde{u}_p} dx,
\]

then the function \( I : ]0, 1[ \to \mathbb{R} \) is locally absolutely continuous and

\[
F_{\Omega}(U_t, \varphi) \leq \ell_1(\beta, \Omega) - \frac{1}{|U_t| t^{p-1}} \left( d \frac{dt}{dt} (p I(t)) \right)
\]

(6.5)
for almost every \( t \in [0,1]. \) Second, we show that the derivative \( \frac{d}{dt}(t^p I(t)) \) is positive in a subset of \([0,1]\) with nonzero measure.

In order to prove \((6.5)\), using the coarea formula we obtain that, for a.e. \( t \in [0,1], \)

\[
F_{\Omega}(U_t, \varphi) = \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left( \int_{S_t} w F(\nu) dH^{n-1} - (p - 1) \int_{U_t} \left( \varphi^{p'} - \frac{[F(\nabla \tilde{u}_p)]^p}{\tilde{u}_p} \right) dx \right)
\]

\[
\leq \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left( \int_{S_t} w F(\nu) dH^{n-1} - p \int_{U_t} w \frac{F(\nabla \tilde{u}_p)}{\tilde{u}_p} dx \right)
\]

\[
= \ell_1(\beta, \Omega) + \frac{1}{|U_t|} \left( \int_{S_t} w F(\nu) dH^{n-1} - p I(t) \right)
\]

(6.6)

where the inequality in \((6.5)\) follows from the inequality \( \varphi^{p'} \geq v^{p'} + p'v^{p'-1}(\varphi - v) \), with \( \varphi, v \geq 0 \).

Proceeding as in \([11]\) and using the coarea formula we obtain for a.e. \( t \in [0,1], \)

\[
- \frac{d}{dt}(t^p I(t)) = t^{p-1} \left( \int_{S_t} w F(\nu) dH^{n-1} - p I(t) \right).
\]

(6.7)

Substituting \((6.7)\) in \((6.6)\) we obtain \((6.5)\). We can conclude the proof, arguing by contradiction exactly as in \([6, \text{Theorem 3.2}]\), indeed is possible to see that the function \( t^p I(t) \) has positive derivative in a set of positive measure. This fact with \((6.5)\) give us the inequality \((6.4)\).

### 6.1 Applications

In this section we use the representation formula given in Theorem \(6.1\) in order to get some estimates for \( \ell_1(\beta, \Omega) \).

#### 6.1.1 A Faber-Krahn type inequality

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with \( C^{1,\alpha} \) boundary, \( \alpha \in [0,1] \) and let \( W_R \) be the Wulff shape centered at the origin with radius \( R \) such that \( |\Omega| = |W_R| \).

Let \( \tilde{\beta} \) be a positive constant and let us consider the following Robin eigenvalue problem in \( W_R \) for \( Q_p \)

\[
\begin{cases}
- Q_p v = \ell_1(\tilde{\beta}, W_R)|v|^{p-2}v & \text{in } W_R, \\
F^{p-1}(\nabla v)F_{\xi}(\nabla v) \cdot \nu + \tilde{\beta} F(\nu)|v|^{p-2}v = 0 & \text{on } \partial W_R.
\end{cases}
\]

(6.8)

Let \( w(t), t \in [0, +\infty[ \), be a non negative continuous function such that

\[
w(t) \geq C(R) t,
\]

(6.9)

where \( C(R) = \ell_1(\tilde{\beta}, W_R) + (p - 1) \tilde{\beta} \frac{R}{R^p} \) is the constant appearing in \((4.7)\).

Let us consider the following Robin eigenvalue problem

\[
\begin{cases}
- Q_p u = \ell_1(\beta, \Omega)|u|^{p-2}u & \text{in } \Omega, \\
F^{p-1}(\nabla u)F_{\xi}(\nabla u) \cdot \nu + \beta(x) F(\nu)|u|^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(6.10)

where

\[
\beta(x) = w(F^p(x)), \quad x \in \partial \Omega.
\]

(6.11)

As a consequence of the representation formula \((6.1)\) for \( \ell_1(\beta, \Omega) \) we get the following Faber-Krahn inequality.
Theorem 6.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,\alpha}$ boundary, let $\alpha \in [0, 1]$ and let $W_R$ be the Wulff shape such that $|\Omega| = |W_R|$. Let $w(t)$, $t \in [0, +\infty[$, be a nonnegative continuous function which verifies (6.9) and let $\beta(x)$ be the function defined in (6.11). Then,

$$\ell_1(\beta, W_R) \leq \ell_1(\beta, \Omega).$$

(6.12)

Proof. We construct a suitable test function in $\Omega$ for (6.4). Let $v_p$ be a positive eigenfunction of the radial problem (6.8) in $B_R$. By Theorem 4.1, $v_p$ is a function depending only by $F^o(x)$, $v_p = \rho_p(F^o(x))$, and then we can argue as in Section 4 defining the function

$$f(r_x) = \varphi_*(x) = \left[ F\left(\nabla v_p(x)\right) \right]^{p-1} F_\xi(Dv_p(x)) \cdot \nu = \frac{(-\rho_p(r_x))^{p-1}}{(\rho_p(r_x))^{p-1}},$$

where $r_x = F^o(x) \in [0, R]$.

Denoted by $W_s = \{ x \in W_R : v_p(x) > s \}$, $0 < s < R$, clearly $W_s$ is a Wulff shape centered at the origin and by Theorem (6.1) we get

$$\ell_1(\beta(R), W_R) = F_{W_R}(W_s, \varphi_*) = \frac{1}{|W_s|} \left( -(p-1) \int_{W_s} \varphi'^2 \, dx + \int_{\partial W_s} \varphi_*(\nu) \, d\mathcal{H}^{n-1} \right)$$

(6.13)

Let $u_\nu$ be the first eigenfunction of (6.10) in $\Omega$ such that $\|u_\nu\|_\infty = 1$. For $x \in \Omega$ we set $u_\nu(x) = t$, $0 < t < 1$. Then we consider the Wulff shape $W_{r(t)}$, centered at the origin, where $r(t)$ is the positive number such that $|U_t| = |W_{r(t)}|$. Then, we define the following test function

$$\varphi(x) := f(r(t)) = f(F^o(x)).$$

We stress that clearly $r(t) < R$. Our aim is to compare $\mathcal{F}_{\Omega}(U_t, \varphi)$ with $\mathcal{F}_{W_R}(W_{r(t)}, \varphi_*)$. Then by (6.13) with $s = r(t)$ we have to show that

$$\mathcal{F}_{\Omega}(U_t, \varphi) \geq \frac{1}{|W_{r(t)}|} \left( -(p-1) \int_{W_{r(t)}} \varphi'^2 \, dx + \int_{\partial W_{r(t)}} \varphi_*(\nu) \, d\mathcal{H}^{n-1} \right) = \mathcal{F}_{W_R}(W_{r(t)}, \varphi_*).$$

We first observe that by [26] Section 1.2.3, being $|U_t| = |W_{r(t)}|$ for all $t \in [0, 1]$,

$$\int_{U_t} \varphi'^2 \, dx = \int_{W_{r(t)}} \varphi'^2 \, dx.$$

Moreover, from the weighted isoperimetric inequality quoted in Remark 4.2, Theorem 4.3 and the assumption (6.11) on $\beta$ we get

$$\begin{align*}
\int_{\partial W_{r(t)}} \varphi_*(\nu) \, d\mathcal{H}^{n-1} &\leq \int_{\partial U_t} f(r(t)) F(\nu) \, d\mathcal{H}^{n-1} \leq \int_{\partial U_t} F^o(x) F(\nu) \, d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} F^o(x) F(\nu) \, d\mathcal{H}^{n-1} + \int_{\Gamma_t} F^o(x) F(\nu) \, d\mathcal{H}^{n-1} \\
&= \int_{S_t} \varphi F(\nu) \, d\mathcal{H}^{n-1} + \int_{\Gamma_t} F^o(x) F(\nu) \, d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} \varphi F(\nu) \, d\mathcal{H}^{n-1} + C(R) \int_{\Gamma_t} F^o(x) F(\nu) \, d\mathcal{H}^{n-1} \\
&\leq \int_{S_t} \varphi F(\nu) \, d\mathcal{H}^{n-1} + \int_{\Gamma_t} w(F^o(x)) F(\nu) \, d\mathcal{H}^{n-1}
\end{align*}$$

and this concludes the proof. \(\square\)

Remark 6.1. When $\beta = \beta$ is a nonnegative constant (6.12) is proved in [11] in the anisotropic case and in [8] in the Euclidean case.
### 6.1.2 A Cheeger type inequality for $\ell_1(\beta, \Omega)$

In this part we introduce the anisotropic weighted Cheeger constant and, using the representation formula we prove an anisotropic weighted Cheeger inequality for $\ell_1(\beta, \Omega)$. Following [7] we give

**Definition 6.1.** Let $g : \overline{\Omega} \to ]0, \infty[$ be a continuous function the weighted anisotropic Cheeger constant is defined as follows

$$h_g(\Omega) = \inf_{U \subset \Omega} \frac{\int_{\partial U} gF(\nu) d\mathcal{H}^{n-1}}{|U|} = \inf_{U \subset \Omega} \frac{P_F(g,U)}{|U|}.$$  

We observe that when $g(x) = c$ is a constant then

$$h_g(\Omega) = c \inf_{U \subset \Omega} \frac{P_F(U)}{|U|} = c h(\Omega),$$

where $h(\Omega)$ is the anisotropic Cheeger constant defined in (2.7). In [7] it is proved that actually $h_g(\Omega)$ is a minimum that is there exists a set $C \subset \Omega$ such that

$$h_g(\Omega) = \frac{P_F(g,C)}{|C|},$$

and we refer to $C$ as a weighted Cheeger set.

We observe that for suitable weight $g$ the constant $h_g(\Omega)$ verifies an anisotropic isoperimetric inequality

**Theorem 6.4.** Let $g(x) = w(F^\alpha(x)) = w(r)$, $r \geq 0$ with $w$ a non negative and nondecreasing function such that

$$w(r^\frac{1}{\alpha})^{1-\frac{1}{\alpha}}, \quad 0 \leq r \leq R,$$

is convex with respect to $r$. Then

$$h_g(\Omega) \geq h_g(\mathcal{W}_R) = \frac{nw(R)}{R},$$

where $\mathcal{W}$ is a Wulff shape with the same measure as $\Omega$.

**Proof.** The proof follows immediately from Remark 3.2.

When $\beta = \bar{\beta}$ is a nonnegative constant and $p = 2$ in [22] the following Cheeger inequality is proved in the Robin eigenvalue case

$$\ell_1(\bar{\beta}, \Omega) \geq \begin{cases} h(\Omega)\bar{\beta} - \bar{\beta}^2 & \text{always} \\ \frac{1}{4} [h(\Omega)]^2 & \text{if } \bar{\beta} \geq \frac{1}{2} h(\Omega) \end{cases} \quad (6.14)$$

In the next result we extend (6.14) to the anisotropic case for any $1 < p < \infty$ considering $\beta$ not in general constant.

**Theorem 6.5.** Let us consider problem (6.1) with $\beta \in C(\overline{\Omega})$ such that $\beta \geq 0$. Then the following weighted anisotropic Cheeger inequality holds

$$\ell_1(\beta, \Omega) \geq h_\beta(\Omega) - (p-1)\|\beta^{p'}\|_{L^\infty(\overline{\Omega})},$$

where $p' = \frac{p}{p-1}$.
Proof. Using $\beta$ as test function in (6.4) we obtain
\[
\ell_1(\beta, \Omega) \geq \mathcal{F}(U_t, \beta) = \frac{1}{|U_t|} \left( -(p-1) \int_{U_t} \beta^p \, dx + \int_{S_t} \beta F(\nu) \, d\mathcal{H}^{n-1} + \int_{\Gamma_t} \beta F(\nu) \, d\mathcal{H}^{n-1} \right)
\]
\[
= \frac{1}{|U_t|} \left( -(p-1) \int_{U_t} \beta^p \, dx + \int_{\partial U_t} \beta F(\nu) \, d\mathcal{H}^{n-1} \right) \geq -(p-1) \|\beta^p\|_\infty + h_\beta(\Omega).
\]
\[\square\]

Remark 6.2. We observe that the previous result continues to hold if we take $\beta \in C(\partial \Omega)$. Indeed in this case from a classical result, see for instance [18, Theorem 4.1], we know that the function $\beta$ is the trace of a nonnegative function $\beta_\Omega \in C(\Omega)$. Then inequality (6.15) holds with $\beta = \beta_\Omega$.

We emphasize the inequality (6.15) in the particular case of $\beta = \bar{\beta}$ is a nonnegative constant.

Corollary 6.1. The first eigenvalue $\ell_1(\bar{\beta}, \Omega)$ of (6.1) on a fixed bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary satisfies
\[
\ell_1(\bar{\beta}, \Omega) \geq \begin{cases} 
 h(\Omega)\bar{\beta} - (p-1)\bar{\beta}^p \frac{p-1}{p} & \text{always} \\
 \frac{1}{p^p} [h(\Omega)]^p & \text{if } \bar{\beta} \geq \frac{1}{p^{p-1}} [h(\Omega)]^{p-1}
\end{cases}
\]  
(6.16)

Proof. From the Theorem 6.3 we obtain, using the constant function $\bar{\beta}$ as test, we obtain the first part of the inequality. For the second part, suitable using as test function in the functional $\mathcal{F}_{\Omega}(U_t, \cdot)$, the constant $\frac{1}{p^{p-1}} [h_F(\Omega)]^{p-1}$ under the assumption that the constant $\bar{\beta} \geq \frac{1}{p^{p-1}} [h_F(\Omega)]^{p-1}$.
\[\square\]

Remark 6.3. From the anisotropic Cheeger inequality for constant $\bar{\beta}$ we obtain immediately a lower bound for $\ell_1(\bar{\beta}, \Omega)$ in terms of the anisotropic inradius of $\Omega$ different from (3.8) by using (2.8).

Remark 6.4. By (ii) Theorem 3.1 and Corollary 6.1 we obtain for $\bar{\beta} \geq \frac{1}{p^{p-1}} [h(\Omega)]^{p-1}$ the anisotropic Cheeger inequality (6.16) for the first Dirichlet eigenvalue of $Q_p$
\[
\lambda_D(\Omega) \geq \ell_1(\bar{\beta}, \Omega) \geq \frac{1}{p^p} [h(\Omega)]^p.
\]

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