Bäcklund Transformations and Hierarchies of Exact Solutions for the Fourth Painlevé Equation and their Application to Discrete Equations

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Abstract. In this paper we describe Bäcklund transformations and hierarchies of exact solutions for the fourth Painlevé equation (PIV)

\[
\frac{d^2w}{dz^2} = \frac{1}{2w} \left( \frac{dw}{dz} \right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},
\]

with \(\alpha, \beta\) constants. Specifically, a nonlinear superposition principle for PIV, hierarchies of solutions expressible in terms of complementary error or parabolic cylinder functions as well as rational solutions will be derived. Included amongst these hierarchies are solutions of (1) for which \(\alpha = \pm \frac{1}{2}n\) and \(\beta = -\frac{1}{2}n^2\), with \(n\) an integer. These particular forms arise in quantum gravity and also satisfy a discrete analogue of the first Painlevé equation. We also obtain a number of exact solutions of the discrete fourth Painlevé equation

\[
x_{n+1}x_{n-1} + x_n(x_{n+1} + x_{n-1}) = \frac{-2z_nx_n^3 + (\eta - 3\delta^{-2} - z_n^2)x_n^2 + \mu^2}{(x_n + z_n + \gamma)(x_n + z_n - \gamma)},
\]

where \(z_n = n\delta\) and \(\eta, \delta, \mu\) and \(\gamma\) are constants, which, in an appropriate limit, reduces to PIV (1). A suitable factorisation of (2) facilitates the identification of a number of solutions which take the form of ratios of two polynomials in the variable \(z_n\) and the limits of these solutions yield rational solutions of (1).

1. Introduction

The six Painlevé equations (PI–PVI) were first derived around the turn of the century in an investigation by Painlevé and his colleagues into which second-order ordinary differential equations have the property that the singularities other than poles of any of the solutions are independent of the particular solution and so are dependent only upon the equation (cf., [20]); this property is now known as the Painlevé property. There has been considerable interest in Painlevé equations over the last few years primarily due to the fact that they arise as reductions of soliton equations solvable by inverse scattering as first demonstrated by Ablowitz & Segur [2]. Although first discovered from strictly mathematical considerations, the Painlevé equations have appeared in various of physical applications (cf., [1] and the references therein). The Painlevé equations may also be thought of as nonlinear analogues of the classical special functions though they are known to be transcendental since their solution is not expressible in terms of elementary functions. However rational solutions and one-parameter families of solutions of the Painlevé equations which can be written in terms of special functions are known to exist for particular values of the parameters. For example, there exist solutions of PII, PIII and PIV that are expressed in terms of Airy, Bessel and parabolic cylinder functions, respectively (cf., [18]).
Recently there has been considerable interest in integrable mappings and discrete systems, including discrete analogues of the Painlevé equations. Some of these mapping and discrete equations arise in physical applications. For example, a discrete analogue of PI (d-PI) arose in the study of the partition function in a two-dimensional model of quantum gravity [9,11,13,19]. Subsequently a discrete analogue of PII (d-PII) was derived in [25,27] and later discrete analogues of PIII–PV (d-PIII–d-PV) were obtained by Ramani et al. [28] using the method of singularity confinement [17]; for further details on the derivation of the discrete Painlevé equations see, for example, [12,16]. One important result of the investigations is that the form of the discrete Painlevé equations is not unique since there exist several possible discrete analogues of the Painlevé equations. Kajiwara et al. [21] and Grammaticos et al. [15] have derived exact solutions of d-PII and d-PIII in terms of discrete Airy and discrete Bessel functions, respectively, in analogue to the aforementioned results for the associated continuous Painlevé equations. We further remark that Lax pairs and isomonodromic deformation problems are known to exist for d-PI [13], d-PII [25] and d-PIII [26]. However, at present, there is no discrete analogue of PVI, nor are there Lax pairs for the versions of d-PIV and d-PV derived in [15].

In recent work [6,7], we have been concerned with the investigation of Bäcklund transformations and exact solutions for fourth Painlevé equation (PIV)

\[ w w'' = \frac{1}{2} (w')^2 + \frac{3}{2} w^4 + 4z w^3 + 2(z^2 - \alpha)w^2 + \beta, \quad ' = d/dz, \quad (1.1) \]

where \( \alpha \) and \( \beta \) are arbitrary constants, together with an examination of various applications of these solutions to several physically motivated nonlinear partial differential equations. In [6] we demonstrated how all known exact solutions of (1.1) can be categorised into one of three families; in two of these solutions can be determined in terms of the complementary error and parabolic cylinder functions whilst the third family consists of solutions which can be expressed as the ratio of two polynomials in \( z \).

In §2 we review the known Bäcklund transformations for PIV (1.1). In particular it is shown that there exist three-term recurrence relations, or nonlinear superposition formulae, for PIV, which can be effectively used for the derivation of solution hierarchies of PIV involving algebraic manipulations alone. In §3 we briefly discuss hierarchies of exact solutions for PIV including one-parameter families of solutions expressible in terms of parabolic cylinder functions, solutions expressible in terms of complementary error functions and rational solutions. In §4 we show that one family of solutions for PIV, the so-called half-integer hierarchy, generate solutions of the discrete first Painlevé equation and are relevant to problems in two-dimensional quantum gravity.

In §5 we discuss some new solutions of the discrete fourth Painlevé equation (d-PIV) given by the three-point, non-autonomous mapping

\[ x_{n+1} x_{n-1} + x_n (x_{n+1} + x_{n-1}) = \frac{-2z_n x_n^3 + (\eta - 3\delta^{-2} - z_n^2)x_n^2 + \mu^2}{(x_n + z_n + \gamma)(x_n + z_n - \gamma)}, \quad (1.2) \]

with the variable \( x_n \) to be found in terms of \( z_n \equiv n\delta + \zeta \). This mapping is identified as the discretised version of PIV. This is observed by taking the limit of (1.2) as \( \delta \to 0 \), with \( \gamma = 1/\delta \) and \( \eta \) and \( \mu \) finite. This process yields (1.1) with the parameters \( \alpha \) and \( \beta \) in that equation related to \( \eta \) and \( \mu \) according to \( \alpha = \frac{1}{4}\eta \) and \( \beta = -\frac{1}{2}\mu^2 \). Since d-PIV reduces to (1.1) in the appropriate limit then it can be expected that exact solutions of (1.2) exist which should tend to known continuous solutions in the same limit. Exact solutions of d-PIV additional to those described in §5 have been derived very recently and a discussion of these may be found in [4].
2. Bäcklund transformations for the fourth Painlevé equation

The first Bäcklund transformation for PIV was derived by Lukashevich [23] who wrote PIV (1.1) in terms of a system of Riccati equations. Rather than adopt the method given by Lukashevich, we follow Gromak [18] and consider the system

\[ w' = q + 2\varepsilon zw + \varepsilon w^2 + 2\varepsilon wv, \quad v' = p - 2\varepsilon zw - \varepsilon v^2 - 2\varepsilon wv, \quad (2.1a,b) \]

where \( \varepsilon^2 = 1, q^2 = -2\beta \) and \( p = -1 - \alpha\varepsilon - \frac{1}{2}q \). If \( v \) is eliminated between these equations then it is easily shown that \( w \) satisfies PIV (1.1). However, if \( w \) is eliminated then it transpires that \( v \) also is a solution of PIV though not with the parameters \( \alpha \) and \( \beta \) as in (1.1), but rather with \( \alpha_1 \) and \( \beta_1 \) where \( \alpha_1 = \frac{1}{4} \left( 2\varepsilon - 2\alpha + 3\varepsilon\sqrt{-2\beta} \right) \) and \( \beta_1 = -\frac{1}{2} \left( 1 + \alpha\varepsilon + \frac{1}{2}\sqrt{-2\beta} \right)^2 \). As equation (2.1a) may be rewritten in the form

\[ v = \frac{w' - 2\varepsilon zw - \varepsilon w^2 - \sqrt{-2\beta}}{2\varepsilon w}, \quad (2.2) \]

it is apparent that if \( w(z;\alpha,\beta) \) denotes a solution of PIV for parameters \( \alpha \) and \( \beta \) then setting \( \varepsilon = \pm 1 \) in (2.1–2.2) gives rise to two further solutions of PIV. If the transformations from \( w(z;\alpha,\beta) \) to these new solutions are denoted by \( \tilde{W}^\pm \) and \( \hat{W}^\pm \) then these Bäcklund formulae can be written as

\[ \tilde{W}^\pm(w(z;\alpha,\beta)) := w(z;\tilde{\alpha}^\pm,\tilde{\beta}^\pm) = \frac{w'(z;\alpha,\beta) - w^2(z;\alpha,\beta) - 2zw(z;\alpha,\beta) \mp \sqrt{-2\beta}}{2w(z;\alpha,\beta)}, \quad (2.3a) \]

\[ \tilde{\alpha}^\pm = \frac{1}{4} \left( 2 - 2\alpha \mp 3\sqrt{-2\beta} \right), \quad \tilde{\beta}^\pm = -\frac{1}{2} \left( 1 + \alpha \mp \frac{1}{2}\sqrt{-2\beta} \right)^2, \quad (2.3b,c) \]

\[ \hat{W}^\pm(w(z;\alpha,\beta)) := w(z;\hat{\alpha}^\pm,\hat{\beta}^\pm) = -\frac{w'(z;\alpha,\beta) + w^2(z;\alpha,\beta) + 2zw(z;\alpha,\beta) \mp \sqrt{-2\beta}}{2w(z;\alpha,\beta)}, \quad (2.4a) \]

\[ \hat{\alpha}^\pm = -\frac{1}{4} \left( 2 + 2\alpha \mp 3\sqrt{-2\beta} \right), \quad \hat{\beta}^\pm = -\frac{1}{2} \left( 1 - \alpha \mp \frac{1}{2}\sqrt{-2\beta} \right)^2, \quad (2.4b,c) \]

Thus given a solution \( w(z;\alpha,\beta) \neq 0 \) of PIV, the solutions \( w(z;\tilde{\alpha}^\pm,\tilde{\beta}^\pm) \) and \( w(z;\hat{\alpha}^\pm,\hat{\beta}^\pm) \) may be obtained using (2.3) and (2.4). Moreover, the transformations (2.3) and (2.4) are effectively nonlinear differentiation formulae for solutions of PIV since \( w(z;\tilde{\alpha}^\pm,\tilde{\beta}^\pm) \) and \( w(z;\hat{\alpha}^\pm,\hat{\beta}^\pm) \) are expressed in terms of \( w(z;\alpha,\beta) \) and its derivatives.

Kitaev [22] derived two further sets of Bäcklund transformations for PIV using the associated isomonodromy deformation representation, which has a regular singular point at \( \lambda = 0 \) and an irregular singular point of rank 2 at \( \lambda = \infty \) (cf., [14]). Kitaev showed that the Bäcklund transformations \( \tilde{W} \) and \( \hat{W} \) correspond to the regular point at \( \lambda = 0 \) and the transformations \( W^\dagger \) and \( W^\ddagger \), defined below, are associated with the irregular point at \( \lambda = \infty \). If \( w(z;\alpha,\beta) \) is a solution of PIV, then so also are \( W^\dagger(z;\alpha^\dagger,\beta^\dagger) \) and \( W^\ddagger(z;\alpha^\ddagger,\beta^\ddagger) \) where

\[ W^\dagger(w(z;\alpha,\beta)) := w^\dagger(z;\alpha^\dagger,\beta^\dagger) \]

\[ = w(z;\alpha,\beta) + \frac{2 \left( 1 - \alpha \mp \frac{1}{3}\sqrt{-2\beta} \right) w(z;\alpha,\beta)}{w'(z;\alpha,\beta) \mp \sqrt{-2\beta} + 2zw(z;\alpha,\beta) + w^2(z;\alpha,\beta)}, \quad (2.5a) \]

\[ \alpha^\dagger = \frac{3}{2} - \frac{1}{2}\alpha \mp \frac{3}{4}\sqrt{-2\beta}, \quad \beta^\dagger = -\frac{1}{2} \left( 1 - \alpha \mp \frac{1}{2}\sqrt{-2\beta} \right)^2, \quad (2.5b,c) \]

\[ W^\ddagger(w(z;\alpha,\beta)) := w^\ddagger(z;\alpha^\ddagger,\beta^\ddagger) \]

\[ = w(z;\alpha,\beta) + \frac{2 \left( 1 - \alpha \pm \frac{1}{3}\sqrt{-2\beta} \right) w(z;\alpha,\beta)}{w'(z;\alpha,\beta) \pm \sqrt{-2\beta} - 2zw(z;\alpha,\beta) - w^2(z;\alpha,\beta)}, \quad (2.6a) \]

\[ \alpha^\ddagger = -\frac{3}{2} - \frac{1}{2}\alpha \pm \frac{3}{4}\sqrt{-2\beta}, \quad \beta^\ddagger = -\frac{1}{2} \left( 1 - \alpha \pm \frac{1}{2}\sqrt{-2\beta} \right)^2. \quad (2.6b,c) \]
These two transformations are valid for all solutions $w(z; \alpha, \beta)$ for which the numerators and denominators are non-zero. The transformation $W^{\pm\pm}$ is equivalent to the transformations $T_1$ (lower sign) and $T_2$ (upper sign) given by Murata [24]. In [6], it is shown that the Bäcklund transformations $W^{\pm\pm}$ are expressible in terms of $\tilde{W}^{\pm}$ and $\hat{W}^{\pm}$

$$w^{\pm\pm}(z; \alpha^{\pm\pm}, \beta^{\pm\pm}) = \begin{cases} \tilde{W}^+ \tilde{W}^+(w(z; \alpha, \beta)), & \text{if } 1 - \alpha - \frac{1}{2} \sqrt{-2\beta} > 0, \\ \tilde{W^-} \tilde{W}^+(w(z; \alpha, \beta)), & \text{if } 1 - \alpha - \frac{1}{2} \sqrt{-2\beta} < 0, \end{cases}$$

$$w^{\pm\pm}(z; \alpha^{\pm\pm}, \beta^{\pm\pm}) = \begin{cases} \hat{W}^+ \hat{W}^+(w(z; \alpha, \beta)), & \text{if } 1 + \alpha + \frac{1}{2} \sqrt{-2\beta} > 0, \\ \hat{W^-} \hat{W}^+(w(z; \alpha, \beta)), & \text{if } 1 + \alpha + \frac{1}{2} \sqrt{-2\beta} < 0. \end{cases}$$

Consequently the following Kitaev fractional transformations can be derived

$$w^{\pm\pm}(z; \alpha^{\pm\pm}, \beta^{\pm\pm}) = w(z; \alpha, \beta) \mp 1 - \frac{\alpha \pm \frac{1}{2} \sqrt{-2\beta}}{\hat{W}^+(w(z; \alpha, \beta))},$$

$$w^{\pm\pm}(z; \alpha^{\pm\pm}, \beta^{\pm\pm}) = w(z; \alpha, \beta) \pm \frac{1 + \alpha \pm \frac{1}{2} \sqrt{-2\beta}}{\hat{W}^+(w(z; \alpha, \beta))},$$

which are three-term recurrence relations, or nonlinear superposition formulae, for PIV (see [6] for further details. These results can be used for the efficient derivation of solution hierarchies of PIV by use of algebraic manipulations alone.

Fokas et al. [14] who, using a Schlesinger transformation formulation related to the isomonodromy method, also deduced a number of Bäcklund transformations for PIV. However, as for the Bäcklund transformations $W^{\pm\pm}$ and $W^{\pm\pm}$ above, these are expressible in terms of $\tilde{W}^{\pm}$ and $\hat{W}^{\pm}$ [6].

3. Solution Hierarchies for the fourth Painlevé equation

3.1. Introduction. In this section we shall discuss various one-parameter families of solutions and rational solutions for PIV. In order to characterise such families we start by recalling that the Riccati system associated with PIV is given by (2.1). Eliminating either $v$ or $w$ yields PIV. If $v \equiv 0$ in (2.1) then $p \equiv 0$ so that $q = -2(\alpha \varepsilon + 1)$ and hence

$$\beta = -2(1 + \alpha \varepsilon)^2, \quad (3.1)$$

which we shall refer to as the one-parameter family condition. One-parameter solutions are found by solving the Riccati equation,

$$w' - 2\varepsilon zw - \varepsilon w^2 - 2(1 + \alpha \varepsilon) = 0. \quad (3.2)$$

Applying the linearizing transformation $w = -\varepsilon u'/u$ to this form yields the Weber-Hermite equation

$$u'' - 2\varepsilon zu' - 2(\alpha + \varepsilon)u = 0. \quad (3.3)$$

Further, if we make the transformation $u(z) = \eta(\xi) \exp\left(\frac{1}{4} \varepsilon \xi^2\right)$, with $\xi = \sqrt{2} z$, then we obtain the parabolic cylinder function equation (cf., [3])

$$\frac{d^2 \eta}{d \xi^2} = \left(\frac{1}{4} \xi^2 - \nu - \frac{1}{2}\right) \eta, \quad (3.4)$$
with \( \nu = -\alpha - \frac{1}{2}(1 + \varepsilon) \). If \( \nu \notin \mathbb{Z}^+ \), then this equation has the general solution

\[
\eta(\xi) = AD_\nu(\xi) + BD_\nu(-\xi),
\]

where \( A \) and \( B \) are arbitrary constants and \( D_\nu(\xi) \) is the parabolic cylinder function (cf., [3]). If \( \nu \in \mathbb{Z}^+ \), then \( D_n(\xi) = \text{He}_n(\xi) \exp\left(-\frac{1}{4} \xi^2\right) \), where \( \text{He}_n(\xi) \) is the Hermite polynomial of degree \( n \) given by

\[
\text{He}_n(\xi) = (-1)^n \exp\left(\frac{1}{4} \xi^2\right) \frac{d^n}{d\xi^n} \left[ \exp\left(-\frac{1}{2} \xi^2\right) \right]. \tag{3.5}
\]

In general, the one-parameter families of solutions for PIV expressible in terms of parabolic cylinder functions are given by

\[
w(z;-(\nu+1),-2\nu^2) = -\frac{\phi'_\nu(z)}{\phi_\nu(z)}, \quad w(z;-(\nu+1)-2(\nu+1)^2) = -2z - \frac{\phi'_\nu(z)}{\phi_\nu(z)}, \tag{3.6a,b}
\]
\[
w(z;\nu+1,-2\nu^2) = \frac{\psi'_\nu(z)}{\psi_\nu(z)}, \quad w(z;\nu+1,-2(\nu+1)^2) = -2z - \frac{\psi'_\nu(z)}{\psi_\nu(z)}, \tag{3.6c,d}
\]

where \( \phi_\nu(z) \) satisfies the Weber-Hermite equation (3.3) with \( \varepsilon = 1 \) and \( \alpha = -\nu - 1 \) and \( \psi_\nu(z) \) satisfies (3.3) with \( \varepsilon = -1 \) and \( \alpha = \nu + 1 \) (cf., [10,23]). If \( \nu = n \), with \( n \) a positive integer, then \( \phi_n(z) \) and \( \psi_n(z) \) are polynomials of degree \( n \) expressible in terms of the Hermite polynomial \( \text{He}_n(\xi) \) (3.5).

### 3.2. The Complementary Error Function Hierarchy

If the parameters in (3.2) are \( \alpha = 1 \) and \( \varepsilon = -1 \), then (3.3) may be integrated to yield

\[
u(z) = B - A \text{erfc}(z), \tag{3.7}
\]

where \( A \) and \( B \) are arbitrary constants and \( \text{erfc}(z) \) is the complementary error function given by

\[
\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) \, dt. \tag{3.8}
\]

Hence we obtain the exact solution of PIV

\[
w(z;1,0) = \Psi(z) \equiv \frac{2A \exp(-z^2)}{\sqrt{\pi} \left[ B - A \text{erfc}(z) \right]} \tag{3.9}
\]

(see also [18]).

Applying the Bäcklund transformations \( \tilde{W}^\pm, \tilde{W}^\pm, W^{\mp}, \text{and } W^{\mp} \) to (3.9) yields a hierarchy of solutions of PIV of the form

\[
w(z;m_1 + 1, -2m_2^2) = R(z, \Psi(z)),
\]

where \( m_1 \) and \( m_2 \) are integers such that \( \frac{1}{2} (m_1 + m_2) \in \mathbb{Z}^+ \) (i.e., either \( m_1 \) and \( m_2 \) are either both even integers or both odd integers), and \( R(z, \Psi(z)) \) is a rational function of its arguments; see Tables 3.2.2 and 3.2.3 in [6] for further details. For example, applying the transformation \( W^{\mp} \) to (3.9) yields

\[
W^{\mp} \left(w(z;1,0)\right) = w(z; -2, -2) = w(z;1,0) + \frac{2}{W^{\pm} \left(w(z;1,0)\right)}.
\]

and so since

\[
\tilde{W}^+ \left(w(z;1,0)\right) = w(z;0,-2) = -2z - w(z;1,0) \equiv -2z - \Psi(z)
\]
then
\[ w(z; -2, -2) = \Psi(z) - \frac{2}{2z + \Psi(z)}. \]

We remark that the subset of this hierarchy given by \( w(z; 2n + 1, 0) \) correspond to the bound state solutions discovered in [8] (see also [5]), which have the property that they decay exponentially as \( z \to \pm \infty \) and are analogues of the bound state solutions for the linear harmonic oscillator (cf., [29]).

### 3.3. The Half-integer Hierarchy.

If the parameters in (3.2) are \( \alpha = \pm \frac{1}{2} \) and \( \beta = -\frac{1}{2} \), then under the transformation \( w(z) = \pm \theta_\pm'(z)/\theta_\pm(z) \) with \( \theta_\pm(z) = \eta(\xi) \exp\left(\mp \frac{1}{2} \xi^2\right) \) and \( \xi = \sqrt{2}z \), it follows that \( \eta(\xi) \) satisfies (3.4) with \( \nu = -\frac{1}{2} \). Hence \( \eta(\xi) = AD_{-1/2}(\xi) + BD_{-1/2}(-\xi) \), with \( A \) and \( B \) arbitrary constants, and so we obtain the following solutions of PIV

\[ w(z; \pm \frac{1}{2}, -\frac{1}{2}) = \frac{\theta_\pm'(z)}{\theta_\pm(z)} = -z \pm \Theta(z), \]  \hspace{1cm} (3.10)

where
\[ \Theta(z) = -z + \frac{\sqrt{2} \left[ AD_{1/2}(\sqrt{2}z) - BD_{1/2}(-\sqrt{2}z) \right]}{AD_{-1/2}(\sqrt{2}z) + BD_{-1/2}(-\sqrt{2}z)}. \]  \hspace{1cm} (3.11)

Applying the Bäcklund transformations \( \tilde{W}^\pm \) and \( \tilde{W}^\pm \) (2.3, 2.4) to \( w(z; \pm \frac{1}{2}, -\frac{1}{2}) \) yields a hierarchy of solutions of PIV which have importance in connection with quantum gravity (see §4 below for further details).

### 3.4. Rational Solution Hierarchies.

It is easily verified that \( w = 1/z \) satisfies PIV (1.1) with the parameter choices \( \alpha = 2, \beta = -2 \) and this is a simple example of a rational solution of PIV. Two families of rational solutions for PIV take the forms

\[ w(z; \alpha, \beta) = \frac{P_n(z)}{Q_n(z)}, \hspace{1cm} w(z; \alpha, \beta) = -2z + \frac{P_n(z)}{Q_n(z)}, \]  \hspace{1cm} (3.12a,b)

where \( P_m(z) \) and \( Q_m(z) \) denote some polynomials of degree \( m \) consisting of either entirely even or else entirely odd powers of \( z \). Murata [24] has shown that (1.1) admits unique rational solutions of type (3.12a) or (3.12b) if the parameters are of the form

\[ (\alpha, \beta) = (\pm k, -2(1 + 2n + k)^2); \hspace{1cm} k, n \in \mathbb{Z}, \hspace{0.5cm} n \leq -1, \hspace{0.5cm} k \geq -2n, \]  \hspace{1cm} (3.13a)

\[ (\alpha, \beta) = (k, -2(1 + 2n + k)^2); \hspace{1cm} k, n \in \mathbb{Z}, \hspace{0.5cm} n \geq 0, \hspace{0.5cm} k \geq -n, \]  \hspace{1cm} (3.13b)

respectively. A third family of rational solutions of PIV is characterised by

\[ w(z; \alpha, \beta) = -\frac{2}{\sqrt{2}} z + \frac{P_n(z)}{Q_n(z)}, \]  \hspace{1cm} (3.14)

where \( (\alpha, \beta) = (n_1, -\frac{2}{\sqrt{2}}(1 + 3n_2)^2) \), with \( n_1 \) and \( n_2 \) either both even or both odd integers. An extensive discussion of the important properties of these rational solution families together with tables containing the first few solutions in each of these three hierarchies is given in [6].
4. Exact solutions of the discrete first Painlevé equation

It is well known that the Korteweg-de Vries (KdV) equation, which possesses a similarity reduction to PI, has several discrete versions; one of these is the Kac-Moerbeke equation

\[ w_{n,t} = -w_n(w_{n+1} - w_{n-1}), \tag{4.1} \]

which reduces to the KdV equation in the appropriate continuous limit. Fokas et al. [13] have demonstrated that in the case of (4.1) a similarity solution is characterised by solving it simultaneously with the discrete equation

\[ n = 2tw_n + w_n(w_{n+1} + w_n + w_{n-1}) \tag{4.2} \]

and the continuous limit \( w_n = -\frac{1}{2} t (1 - 2\varepsilon^2 \eta(z)), n = -\frac{1}{2} t^2 (1 + \varepsilon^4), \) as \( \varepsilon \to 0 \) maps (4.2) to PI (see [19]). Thus equation (4.2) is a version of discrete PI and it has arisen in the theory of two-dimensional quantum gravity (cf., [13,25]). We can use (4.1) to substitute for \( w_n w_{n+1} \) and \( w_n w_{n-1} \) in (4.2) and these two equations can be rewritten as

\[ w_{n+1} = \frac{n - w_{n,t} - 2tw_n - w_n^2}{2w_n}, \quad w_{n-1} = \frac{n + w_{n,t} - 2tw_n - w_n^2}{2w_n}. \tag{4.3a,b} \]

We observe that (4.3) is essentially the same Riccati system as (2.1) since (4.3a) becomes (2.1a) when \( \varepsilon = 1 \) and \( w_n, w_{n+1}, n \) and \( t \) are replaced by \( v, w, p \) and \( z \) respectively and (4.3b) becomes (2.1b) if \( w_n, w_{n-1}, n \) and \( t \) are identified with \( w, v, -q \) and \( z \), respectively. If \( n \) is replaced by \( n + 1 \) in (4.3b) and \( w_{n+1} \) eliminated by using (4.3a) it follows that \( w_n(t) \) satisfies

\[ w_n w_{n,tt} = \frac{1}{2} (w_{n,t})^2 + \frac{3}{2} w_n^4 + 4tw_n^3 + 2 \left( t^2 + \frac{1}{2} n \right) w_n^2 - \frac{1}{2} n^2, \tag{4.4} \]

which is precisely PIV with \( \alpha = -\frac{1}{2} n \) and \( \beta = -\frac{1}{2} n^2 \). Thus the solution of d-PI (4.4) can be expressed in terms of solutions of PIV with parameters \( \alpha = -\frac{1}{2} n, \beta = -\frac{1}{2} n^2 \) and \( n \in \mathbb{Z} \). In [6] we showed that the first few solutions of (4.4), and hence also (4.1) and (4.2), are

\[ w_{\pm 1}(t) = w(t; \pm \frac{1}{2}, -\frac{1}{2}) = -t \mp \Theta(t), \tag{4.5a} \]

\[ w_{\pm 2}(t) = w(t; \pm 1, -2) = \pm \frac{\Theta^2(t) - t^2 \pm 1}{\Theta(t) \mp t}, \tag{4.5b} \]

\[ w_{\pm 3}(t) = w(t; \pm \frac{3}{2}, -\frac{9}{2}) = -\frac{3\Theta^2(t) \pm 4t\Theta(t) \pm t^2 \pm 1}{[\Theta(t) \pm t][\Theta^2(t) - t^2 \pm 1]}, \tag{4.5c} \]

with \( \Theta(t) \) as given in (3.11). We remark that Fokas et al. [13] obtained the solution \( w_1(t) \) by solving an associated Riemann-Hilbert problem for PIV; the derivation given here is much simpler.
5. Rational solutions of the discrete fourth Painlevé equation

Tamizhmani et al. [30] noted that if we set \( \eta = 3\delta^{-2} + \gamma^2 - 2(a^2 + b^2) \) and \( \mu = a^2 - b^2 \), then the d-PIV equation (1.2) can be factorised as

\[
(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n + a + b)(x_n + a - b)(x_n - a + b)(x_n - a - b)}{(x_n + zn + \gamma)(x_n + zn - \gamma)},
\]

(5.1)

and the parameters \( \alpha = \frac{1}{4} \eta \) and \( \beta = -\frac{1}{2} \mu^2 \) are now given by

\[
\alpha = \frac{3}{4} \delta^{-2} + \frac{1}{4} \gamma^2 - \frac{1}{2}(a^2 + b^2), \quad \beta = -\frac{1}{2}(a^2 - b^2)^2.
\]

(5.2)

Simple rational solutions of (5.1) can be found with \( x_n \) proportional to \( zn \); these are

\[
\begin{align*}
x_n &= -2zn, \quad a = \frac{1}{2} \delta + \gamma, \quad b = \frac{1}{2} \delta - \gamma, \\
x_n &= -\frac{2}{3}zn, \quad a = \frac{1}{6} \delta + \gamma, \quad b = \frac{1}{6} \delta - \gamma.
\end{align*}
\]

(5.3, 5.4)

In the limit as \( \delta \to 0 \), with \( \gamma = 1/\delta \), these discrete solutions tend to \( w(z;0,-2) = -2z \) and \( w(z;0,-\frac{2}{3}) = -\frac{2}{3}z \), respectively, which are the first members of the PIV hierarchies typified by (3.12b) and (3.14). If solutions of (5.1) proportional to \( 1/zn \) are sought then it is found that

\[
x_n = -\delta(\delta \pm \gamma) / zn, \quad a = \frac{2}{3} \delta \pm \gamma, \quad b = \frac{1}{3} \delta \pm \gamma,
\]

(5.5)

and so, in the limit as \( \delta \to 0 \), with \( \gamma = 1/\delta \), we have \( x_n \to w(z;\pm 2, -2) = \pm 1/z \) and these continuous solutions are members of the family described by (3.12a).

More complicated rational solutions of d-PIV (1.2) can be deduced by rewriting (5.1) as the pair

\[
\begin{align*}
x_n + x_{n-1} &= (x_n + a + b)(x_n + a - b) / (x_n + zn + \gamma), \\
x_n + x_{n+1} &= (x_n - a + b)(x_n - a - b) / (x_n + zn - \gamma),
\end{align*}
\]

(5.6a,b)

which are discrete analogues of the Riccati equation (3.2). Tamizhmani et al. [30] speculated that such a formalism ought to lead to discrete rational solutions though they did not present any such solutions. It is a routine calculation to verify that equations (5.6a) and (5.6b) are compatible if and only if \( a = \frac{1}{2} \delta \pm \gamma \). [We remark at this stage that (5.6) is not the only possibility for the splitting of (5.1). Easy generalisations of (5.6) include the multiplication of the right hand sides of (5.6a) and (5.6b) by constants \( C \) and \( 1/C \) respectively or the taking of the factors in different pairings. However, it can be shown that in either of these cases we obtain an incompatible couple of equations so that the choice of factorisation (5.6) is not as specialised and restrictive as it might first appear.]

If \( a = \frac{1}{2} \delta \pm \gamma \), then we can seek solutions of (1.2) by finding solutions of the simpler form (5.6b) (or (5.6a) would do equally well). If we write \( x_n = P_n/Q_n \) then (5.6b) can be recast as

\[
\begin{align*}
\kappa_{n+1}P_{n+1} &= (zn + \gamma + \delta)P_n - \mu Q_n, \\
\kappa_{n+1}Q_{n+1} &= -(zn + \gamma)Q_n - P_n,
\end{align*}
\]

(5.7a, 5.7b)

where the ‘separation’ parameter \( \kappa_{n+1} \) can depend both on \( n \) and \( zn \) (recall that \( \mu = a^2 - b^2 \)). However, if for the moment we take \( \kappa_{n+1} = 1 \), then eliminating \( P_n \) between (5.7a) and (5.7b), setting \( \gamma = 1/\delta \) and taking the limit as \( \delta \to 0 \) shows that \( Q_n \to q(z) \) where \( q(z) \) satisfies

\[
q'' = (z^2 + \mu - 1)q.
\]

(5.8)
If in this equation we set $q(z) = \eta(\xi)$ with $\xi = \sqrt{2}z$ then we obtain the parabolic cylinder equation (3.4) with $\nu = -\frac{3}{4} \mu$.

If the variable $Q_n$ is eliminated from equations (5.7) and the usual limit taken then it follows in a manner similar to that outlined above that rational solutions $x_n = P_n/Q_n$ of (1.2) exist which tend to a function of the form $-\text{He}''_m(\xi)/\text{He}_m(\xi)$ whenever $\mu = -2m$, with $m \in \mathbb{N}$. As illustrated in §3.1 above, solutions of PIV (1.1) can be expressed in terms of parabolic cylinder functions when the parameters $\alpha$ and $\beta$ take certain values. Motivated by these comments concerning the (continuous) PIV case, we return to equations (5.7) for the discrete situation. If we write

$$ (P_n, Q_n) = (A_n, B_n) \times (-\delta)^n \Gamma \left( \frac{z_n - m\delta \mp \gamma}{\delta} \right), $$(5.9)

where $\Gamma(z)$ denotes the usual Gamma function, and choose $\kappa_{n+1} \equiv 1$, then we obtain the pair

$$ (P_n, Q_n) = (A_n, B_n) \times (-\delta)^n \Gamma \left( \frac{z_n - m\delta \mp \gamma}{\delta} \right), $$(5.10a)

$$ (z_n - m\delta \mp \gamma)A_{n+1} + (z_n + \delta \mp \gamma)A_n = \mu B_n, $$(5.10b)

For $m \in \mathbb{N}$ we can find exact solutions of these equations with $A_n$ and $B_n$ taking the forms of polynomials in $z_n$, consisting of either only even or only odd powers, and of degrees $m - 1$ and $m$ respectively. The first few solutions in this hierarchy are

$$ m = 1, \quad x_n = -\frac{\delta(\delta \mp \gamma)}{z_n}, $$ (5.11a)

$$ m = 2, \quad x_n = -\frac{2\delta(3\delta \mp 2\gamma)z_n}{2z_n^2 - \delta(2\delta \mp \gamma)}, $$ (5.11b)

$$ m = 3, \quad x_n = -\frac{3\delta(2\delta \mp \gamma)}{z_n \left[ 2z_n^2 - \delta(8\delta \mp 3\gamma) \right]}, $$ (5.11c)

$$ m = 4, \quad x_n = -\frac{4\delta(5\delta \mp 2\gamma) \left[ 2z_n^2 - \delta(11\delta \mp 3\gamma) \right]}{4z_n^2 - 4\delta(10\delta \mp 3\gamma)z_n^2 + 3\delta^2(3\delta \mp \gamma)(4\delta \mp \gamma)}, $$ (5.11d)

$$ m = 5, \quad x_n = -\frac{5\delta(3\delta \mp \gamma) \left[ 4z_n^2 - 4\delta(13\delta \mp 3\gamma)z_n^2 + 3\delta^2(3\delta \mp \gamma)(4\delta \mp \gamma) \right]}{z_n \left[ 4z_n^4 - 20\delta(4\delta \mp \gamma)z_n^2 + \delta^2(256\delta^2 \mp 125\gamma\delta + 15\gamma^2) \right]}.$$ (5.11e)

In each case the corresponding value of $\mu$ in (5.10a) is $\mu = -\delta m \left[(m + 1)\delta \mp 2\gamma\right]$ and this leads to the respective values of parameters $\alpha$ and $\beta$, as defined by (5.2), given by

$$ \alpha = \frac{3}{4}(\delta^2 - \gamma^2) \mp (m + 1)\gamma \delta - \frac{1}{4}\delta^2 \left[ 2m(m + 1) + 1 \right], \quad \beta = -\frac{1}{2}m^2\delta^2 \left[ (m + 1)\delta \mp 2\gamma \right]^2.$$ (5.12)

We emphasise at this stage that the discrete solutions (5.11) are exact and are valid for any $\delta$ and $\gamma$. We can recover continuous solutions by letting $\gamma = 1/\delta$ and taking the limit as $\delta \to 0$ which yields $\alpha = \mp(m + 1)$ and $\beta = -2m^2$. Then from (5.11) we obtain

$$ w(z; \mp 2, -2) = \mp \frac{1}{z}, \quad w(z; \mp 3, -8) = \mp \frac{4z}{2z^2 \mp 1}, \quad w(z; \mp 4, -18) = \mp \frac{3(2z^2 \mp 1)}{z(2z^2 \mp 3)}, $$

$$ w(z; \mp 5, -32) = \mp \frac{8z(2z^2 \mp 3)}{4z^4 \mp 12z^2 + 3}, \quad w(z; \mp 6, -50) = \mp \frac{5(4z^4 \mp 12z^2 + 3)}{z(4z^4 \mp 20z^2 + 15)}.$$ (5.13)

It is then clear that solutions (5.11) can be thought of as discrete analogues of the (continuous) solutions of PIV taking forms given by (3.6a) and (3.6d), with $\nu = m$.

If in (5.7) we let

$$ (P_n, Q_n) = (A_n, B_n) \times \delta^n \Gamma \left( \frac{z_n - m\delta \mp \gamma}{\delta} \right). $$ (5.13)
then we obtain

\[(z_n - m\delta \pm \gamma)(A_{n+1} - A_n) = -\mu B_n, \quad (5.14a)\]
\[(z_n - m\delta \pm \gamma)B_{n+1} + (z_n \mp \gamma)B_n = -A_n. \quad (5.14b)\]

As previously, exact polynomial solutions of these can be derived for integer parameters \(a \sim \delta\) with \(\nu\) of the form (3.6b) or (3.6c), with \(1 = 4\). These are all members of the so-called ‘\(\gamma\)’ hierarchy of rational solutions for PIV (1.1) and are of the form (3.6b) or (3.6c), with \(\nu = m\). It is straightforward to obtain further exact rational solutions of d-PIV by solving the pairs (5.10) or (5.14) for higher values of \(\gamma\) and in the limit \(\delta \to 0\) with \(\gamma = 1/\delta\), solutions (5.15) reduce to the functions

\[w(z; 0, -2) = -2z, \quad w(z; \pm 1, -8) = -2z \mp \frac{1}{z}, \quad w(z; \pm 2, -18) = -2z \mp \frac{4z}{2z^2 \pm 1}, \quad w(z; \pm 3, -32) = -2z \mp \frac{3(2z^2 \pm 1)}{z(2z^2 \pm 3)}. \]
\[w(z; \pm 4, -50) = -2z \mp \frac{8z(2z^2 \pm 3)}{4z^4 \pm 12z^2 + 3}. \]

These are all members of the so-called ‘\(-2z\)’ hierarchy of rational solutions for PIV (1.1) and are of the form (3.6b) or (3.6c), with \(\nu = m\). It is straightforward to obtain further exact rational solutions of d-PIV by solving the pairs (5.10) or (5.14) for higher values of \(m\).

We note here that although we have found some rational solutions of d-PIV (1.2) it can be anticipated that further such solutions exist which are not derivable using the procedure described above. This deduction follows from the observation that rational solutions of PIV (1.1) are possible for all parameter values as described by (3.13) and that so far we have discrete solutions which, in the appropriate limit, tend to the continuous solutions of type (3.6). Therefore, there should be discrete counterparts to the remaining continuous rational solutions. These can be constructed by appealing to some Bäcklund transformations for d-PIV which were given by Tamizhmani et al. [30]. These authors presented a sequence of transformations which, given a solution of (5.1) with parameters \(a, b\) and \(\gamma\), yields a further solution of the same equation but now with parameters \(a + \delta, b\) and \(\gamma\). In short, suppose that \(x_n\) is a solution of d-PIV with parameters \(a, b\) and \(\gamma\). Then

\[\bar{x}_n = -\frac{x_nx_{n+1} + x_n(z_n + a) + x_n(z_n - a) + b^2 - a^2}{x_n + x_{n+1}}, \quad (5.17)\]

with \(\bar{z}_n = z_n + \frac{1}{2}\delta\) also satisfies d-PIV but now with parameters \(\bar{a} = a + \frac{1}{2}\delta, \bar{b} = \gamma\) and \(\bar{\gamma} = b\). The subtlety with transformation (5.17) is that \(\bar{x}_n\) is defined on a lattice of points which is offset by \(\frac{1}{2}\delta\) from the original. In order to obtain a solution valid at points coinciding with the initial
lattice it is therefore necessary to reapply (5.17) which has the overall effect of raising the value of $a$ by $\delta$ whilst keeping $b$ and $\gamma$ invariant. Clearly $M$ applications of this sequence will increment $a$ to $a + M\delta$ and leaves the other two parameters unchanged so that from a starting solution with parameters $a = \frac{3}{2}\delta \pm \gamma$, $b = (N + \frac{1}{2})\delta \pm \gamma$, for $N = 0, 1, 2, \ldots$ (which are the parameter values in (1.2) appropriate to the family of solutions whose first few members are listed in (5.11)) we can obtain a solution with $a = (M + \frac{1}{2})\delta \pm \gamma$ and $b = (N + \frac{1}{2})\delta \pm \gamma$, or, in terms of the parameters $\alpha$ and $\beta$ as given by (5.2),

$$\alpha = \frac{3}{4}(\delta^2 - \gamma^2) \mp \gamma\delta(M + N + 1) + \left[\frac{1}{2}(M - N)(M + N + 1) - (M + \frac{1}{2})^2\right]\delta^2,$$  

(5.18a)

$$\beta = -\frac{1}{2}(M - N)^2\delta^2 \left[(M + N + 1)\delta \pm 2\gamma\right]^2.$$  

(5.18b)

In the usual limit our discrete rational solutions will then tend to continuous ones with associated parameters $\alpha = \mp(M + N + 1)$, $\beta = -2(M - N)^2$. Now $M$ and $N$ can be chosen so as to force these parameters to coincide with the form (3.13a) for any $k$ and $n$ in the permissible ranges; in this way we have a mechanism for deducing discrete analogues of all those rational solutions of PIV (1.1) which take the form $P_{n-1}(z)/Q_n(z)$.

A similar argument can be applied to the discrete solutions (5.15) and this demonstrates that given these results then suitable application of the Bäcklund transformations contained in [30] will generate another set of exact solutions of (1.2). These comprise the discrete counterpart to the $-2z^i$ rational hierarchy of (1.1) which itself is characterised by the parameter values given in (3.13b).

We remark here that of the three hierarchies of rational solutions for PIV (1.1), the simplest procedures outlined above have yielded discrete analogues of only two of these; no solutions corresponding to the $-\frac{2}{3}z$ family (3.14) have been found. The reason for this is that although the full discrete equation (5.1) admits the solution $x_n = -\frac{2}{3}z_n$ with $a = \frac{1}{6}\delta \pm \gamma$, the splitting of this equation into the two Riccati-like forms (5.6) gives a pair of equations whose compatibility requires that $a = \frac{1}{2}\delta \pm \gamma$. Thus, as noted by Tamizhmani et al. [30], this solution is not linearizable through the splitting assumption and thus it is unsurprising that the discrete analogue of the $-\frac{2}{3}z$ hierarchy of solutions cannot be generated in this way. However, we shall now show how use of the Bäcklund transformation (5.17) discussed above can be used to increment the values of $a$ or $b$ by $\pm\delta$ and thus lead to some more exact solutions in the discrete $-\frac{2}{3}z$ hierarchy. (It is noted that d-PIV (5.1) is invariant by interchange of parameters $a$ and $b$ or by a sign change of either of these. Therefore, given that two applications of (5.17) increases $a$ by $\delta$ it is easy to see how a suitable change in sign of $a$ and $b$ or the interchanging of these parameters allows transformations to be found that increase or decrease the values of $a$ or $b$ by integral multiples of $\delta$.) A few examples of simpler solutions in this hierarchy are:

$$x_n = -\frac{2}{3}z_n, \quad a = \frac{1}{6}\delta \pm \gamma, \quad b = -\frac{1}{6}\delta \pm \gamma,$$  

(5.19a)

$$x_n = -\frac{2}{3}z_n - \frac{\delta(\delta + 3\gamma)}{3z_n}, \quad a = \pm\gamma - \frac{5}{6}\delta, \quad b = \mp\gamma + \frac{1}{6}\delta,$$  

(5.19b)

$$x_n = -\frac{2z_n(2z_n^2 + 3\gamma\delta + \delta^2)}{3(2z_n^2 + 3\gamma\delta - 2\delta^2)}, \quad a = \pm\gamma - \frac{7}{6}\delta, \quad b = \mp\gamma + \frac{5}{6}\delta,$$  

(5.19c)

$$x_n = -\frac{2z_n(4z_n^2 - 45\gamma^2\delta^2 + \delta^4)}{3(2z_n^2 - 3\gamma\delta - \delta^2)(2z_n^2 + 3\gamma\delta - 2\delta^2)}, \quad a = \pm\gamma + \frac{5}{6}\delta, \quad b = \mp\gamma + \frac{5}{6}\delta,$$  

(5.19d)

$$x_n = -\frac{8z_n^6 - 48\delta(\delta + 3\gamma)z_n^4 - 2\delta^2(38\delta^2 + 69\gamma\delta + 27\gamma^2)z_n^2 + 9\gamma^3(4\delta^3 + 11\gamma\delta^2 + 10\gamma^2\delta + 3\gamma^3)}{3z_n[4z_n^4 - 4\delta(\delta + 3\gamma)z_n^2 + 9\gamma\delta^2(\delta \mp \gamma)]}, \quad a = \pm\gamma - \frac{7}{6}\delta, \quad b = \mp\gamma + \frac{11}{6}\delta.$$  

(5.19e)
In the limit as $\delta \to 0$, with $\gamma \delta = 1$, the solutions (5.19) reduce to

$$w(z; 0, -\frac{2}{9}) = -\frac{2}{3} z, \quad w(z; \pm 1, -\frac{8}{9}) = -\frac{2}{3} z \mp \frac{1}{z}, \quad w(z; \pm 2, -\frac{2}{9}) = -\frac{2}{3} z \pm \frac{4z}{2z^2 + 3},$$

$$w(z; 0, -\frac{50}{9}) = -\frac{2}{3} z \pm \frac{24z}{(2z^2 - 3)(2z^2 + 3)}, \quad w(z; \pm 3, -\frac{8}{9}) = -\frac{2}{3} z \pm \frac{3(4z^4 + 4z^2 + 3)}{z(4z^4 + 12z^2 - 9)},$$

which belong to the $-\frac{2}{3} z$ hierarchy of rational solutions of PIV (1.1); see Table 4.1.3 of [6] for a more extensive list of such solutions. The method of generating the $-\frac{2}{3} z_n$ discrete rational solutions given in (5.19) is intrinsically less satisfying than that employed for the evaluation of solutions (5.11) and (5.15) in the other two families; this feature is a direct consequence of the fact that the $-\frac{2}{3} z_n$ forms do not arise from a suitable factorisation of d-PIV into a pair of simple compatible equations akin to (5.6). Instead the present method is based on a direct implementation of the Bäcklund transformation (5.17), which, for the for more complicated solutions in the hierarchy, becomes an increasingly laborious task. However, we believe that the Bäcklund transformation (5.17) will inevitably be needed to make a complete evaluation of each of the hierarchies and provide a systematic and efficient procedure for doing this.

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References

1. M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, L.M.S. Lect. Notes Math., 149, C.U.P., Cambridge, 1991.
2. M.J. Ablowitz and H. Segur, Phys. Rev. Lett., 38 (1977), 1103–1106.
3. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.
4. A.P. Bassom and P.A. Clarkson, Phys. Lett. A, 194 (1994), 358–370.
5. A.P. Bassom, P.A. Clarkson and A.C. Hicks, IMA J. Appl. Math., 50 (1993), 167–193.
6. A.P. Bassom, P.A. Clarkson and A.C. Hicks, Stud. Appl. Math., (1995), to appear.
7. A.P. Bassom, P.A. Clarkson and A.C. Hicks, “On the application of solutions of the fourth Painlevé equation to various physically motivated nonlinear partial differential equations”, preprint M94/32, Department of Mathematics, University of Exeter (1994).
8. A.P. Bassom, P.A. Clarkson, A.C. Hicks and J.B. McLeod, Proc. R. Soc. Lond. A, 437 (1992), 1–24.
9. E. Brezin and V. Kazakov, Phys. Lett. B, 236 (1990), 144–150.
10. P.A. Clarkson, Europ. J. Appl. Math., 1 (1990), 279–300.
11. M.R. Douglas, Phys. Lett. B, 238 (1990), 176–180.
12. A.S. Fokas, B. Grammaticos and A. Ramani, J. Math. Anal. Appl., 180 (1993), 342–360.
13. A.S. Fokas, A.R. Its and A.V. Kitaev, Commun. Math. Phys., 142 (1991), 313–344.
14. A.S. Fokas, U. Mugan and M.J. Ablowitz, Physica, 30D (1988), 247–283.
15. B. Grammaticos, F.W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, *Phys. Lett. A*, **185** (1994), 446–452.
16. B. Grammaticos and A. Ramani, Applications of Analytic and Geometric Methods to Nonlinear Differential Equations (P.A. Clarkson, ed.), NATO ASI Series C, **413**, Kluwer, Dordrecht, 1993, pp. 299–313.
17. B. Grammaticos, A. Ramani and V. Papageorgiou, *Phys. Rev. Lett.*, **67** (1991), 1825–1828.
18. V.I. Gromak, *Diff. Eqns.*, **14** (1977), 1510–1513.
19. D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.*, **64** (1990), 127–130.
20. E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.
21. K. Kajiwara, Y. Ohta, J. Satsuma, B. Grammaticos and A. Ramani, *J. Phys. A: Math. Gen.*, **27** (1994), 915–922.
22. A.V. Kitaev, private communication (1991).
23. N.A. Lukashevich, *Diff. Eqns.*, **3** (1967), 395–399.
24. Y. Murata, *Funkcial. Ekvac.*, **28** (1985), 1–32.
25. F.W. Nijhoff and V. Papageorgiou, *Phys. Lett. A*, **153** (1991), 337–344.
26. V. Papageorgiou, F.W. Nijhoff, B. Grammaticos and A. Ramani, *Phys. Lett. A*, **164** (1992), 57–64.
27. V. Periwal and D. Shewitz, *Phys. Rev. Lett.*, **64** (1990), 1326–1329.
28. A. Ramani, B. Grammaticos and J. Hietarinta, *Phys. Rev. Lett.*, **67** (1991), 1829–1832.
29. L.I. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1955.
30. K.M. Tamizhmani, B. Grammaticos and A. Ramani, *Lett. Math. Phys.*, **29** (1993), 49–54.

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