SOME CONSIDERATIONS REGARDING
LORENTZ-VIOLATING THEORIES

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We investigate the compatibility of Lorentz-violating quantum field theories with the requirements of causality and stability. A general renormalizable model for free massive fermions indicates that these requirements are satisfied at low energies provided the couplings controlling the breaking are small. However, for high energies either microcausality or energy positivity or both are violated in some observer frame. We find evidence that this difficulty can be avoided if the model is interpreted as a sub-Planckian approximation originating from a nonlocal theory with spontaneous Lorentz violation. The present study thereby supports the validity of the standard-model extension as the low-energy limit of any realistic string theory that exhibits spontaneous Lorentz breaking.

1 Introduction

From a theoretical point of view, the minimal SU(3)×SU(2)×U(1) standard model leaves unresolved a variety of issues. It is therefore believed to be the low-energy limit of an underlying framework that also includes a quantum description of gravity. On the other hand, the standard model is phenomenologically successful, so observable effects from the presumed underlying physics must be minuscule. It then becomes an interesting challenge to identify possible experimental signals from such a fundamental theory accessible with present techniques.

A candidate signal of this type is the violation CPT and Lorentz invariance: In conventional renormalizable gauge theories including the standard model, these two symmetries are linked by CPT theorem and hold exactly. In contrast, attempts to construct an underlying framework often involve ingredients that bypass the CPT theorem. For example, string (M) theory is known to admit spontaneous Lorentz and CPT violation. Other frameworks can also lead to similar low-energy effects.

For the microscopic description of possible observable signals at presently accessible energy scales, an extension of the minimal standard model of particle physics has been developed. This standard-model extension has provided the basis for numerous experimental investigations discussed during this meeting and elsewhere, which constrain CPT and Lorentz violation.

In this talk, we study the fundamental properties of causality and stability...
in the context of the Lorentz-violating standard-model extension. These two properties appear essential for realistic theories, for it would be difficult to make meaningful experimental predictions without either causality or stability. In particular, it is of interest whether these two requirements constrain the parameter space and the range of validity of the standard-model extension, and whether insight into the underlying theory can be gained. Although the calculations presented here are carried out for free massive fermions, we expect that most of our results can be straightforwardly generalized to the other sectors of the standard-model extension.

2 Framework

The general Lorentz-violating Lagrangian for a single spin-$\frac{1}{2}$ fermion can be cast into a variety of forms. One such form reminiscent of the ordinary Dirac Lagrangian and emphasizing the derivative structure is

$$\mathcal{L} = \frac{1}{2} i \bar{\psi} \Gamma^\nu \partial_\nu \psi - \bar{\psi} M \psi ,$$

where

$$\Gamma^\nu := \gamma^\nu + \epsilon^{\mu\nu} \gamma_\mu + d^{\mu\nu} \gamma_5 \gamma_\mu + e^{\nu} + i f^{\nu} \gamma_5 + \frac{1}{2} g^{\lambda\mu\nu} \sigma_{\lambda\mu} ,$$

and

$$M := m + a_\mu \gamma^\mu + b_\mu \gamma_5 \gamma^\mu + \frac{1}{2} H^{\mu\nu} \sigma_{\mu\nu} .$$

The gamma matrices \(\{1, \gamma_5, \gamma^\mu, \gamma_5 \gamma^\mu, \sigma^{\mu\nu}\}\) have conventional properties, and the signature of the Minkowski metric \(\eta_{\mu\nu}\) is \(-2\). The extent of Lorentz violation is described by the parameters \(a_\mu, b_\mu, c_{\mu\nu}, \epsilon_{\mu\nu}, f_\mu, g_{\mu\nu\lambda},\) and \(H_{\mu\nu}\). As a consequence of the presumed hermiticity of the Lagrangian, all these coefficients are real, with \(c_{\mu\nu}\) and \(d_{\mu\nu}\) traceless, \(g_{\mu\nu\lambda}\) antisymmetric in its first two indices and \(H_{\mu\nu}\) antisymmetric. While all the parameters violate Lorentz invariance, only \(a_\mu, b_\mu, c_\mu, f_\mu, g_{\mu\nu\lambda}\) break CPT symmetry as well.

Since no departures from Lorentz symmetry have been observed to date, all Lorentz-breaking parameters must be minuscule in a certain class of observer inertial frames called \textit{concordant frames}, and the Earth must move nonrelativistically with respect to these frames. Throughout this talk, we shall work under the assumption that the size of the Lorentz violation is such that, in a concordant frame, a hermitian hamiltonian can be found and the dispersion relation still exhibits two positive- and two negative-valued roots, paralleling the conventional Dirac case. The Lagrangian can then be canonically quantized such that the energy is positive definite.
### 3 Microcausality and Stability

A quantum field theory is microcausal if any two local observables with spacelike separation can be measured independently. This is guaranteed if any two local, spatially separated operators commute. In the present case, such local operators are fermion bilinears and the above condition is satisfied if

\[ iS(x - x') = \{\psi(x), \overline{\psi}(x')\} = 0, \quad (x - x')^2 < 0 \tag{4} \]

holds. Note that the anticommutator function \( S(x - x') \) only depends on the coordinate differences due to translational invariance.

To determine an integral representation for the \( S(z) \), we insert the plane-wave expansion of the field operators into the anticommutator and use the generalization of the conventional spinor projectors. This gives the following expression:

\[ S(z) = \int_C \frac{d^4\lambda}{(2\pi)^4} e^{-i\lambda \cdot z} \frac{\text{cof}(\Gamma_\mu \lambda^\mu - M)}{\det(\Gamma_\mu \lambda^\mu - M)} \tag{5} \]

where \( C \) is the usual contour encircling all poles in clockwise direction, and \( \text{cof}(\cdot) \) denotes the matrix of cofactors. Notice that \( \lambda^\mu \) can be replaced by \( i\partial^\mu \) in the numerator of the integrand. It is then possible to pull the cofactor matrix outside the integral, because the contour \( C \) can be deformed such that the integrand is analytic in a neighborhood of \( C \). We obtain for the anticommutator function

\[ S(z) = \text{cof}(\Gamma_\mu i\partial^\mu - M) \int_C \frac{d^4\lambda}{(2\pi)^4} \frac{e^{-i\lambda \cdot z}}{\det(\Gamma_\mu \lambda^\mu - M)} \tag{6} \]

Next, we study \( S(z) \) outside the lightcone. We can take advantage of observer Lorentz invariance and boost to a frame such that \( z^\mu = (0, \vec{z}) \). To make further progress, it is necessary to investigate the pole structure of the integrand. Due to the above observer transformation we may no longer assume to be working in a concordant frame. In particular, it may not be possible to find a hermitian hamiltonian, so that complex eigenenergies may occur. Since the eigenenergies determine the location of the poles of the integrand, the contour \( C \) may fail to encircle them all. Thus, the case where a hermitian hamiltonian (and therefore real eigenenergies) exist in all frames has to be distinguished. We consider this case first.

A sufficient condition for the hermiticity of the hamiltonian in all observer frames is that the derivative structure of lagrangian \( \{\} \) is the conventional one, \( i.e., \Gamma^\mu = \gamma^\mu \). Then, all four roots \( E_{(j)}(\vec{\lambda}) \), \( j = 1, \ldots, 4 \), of the dispersion relation appearing in the denominator of the integrand in Eq. (5) are real. In this case, the contour integration can be directly performed. This argument confirms microcausality for the case \( \Gamma^\mu = \gamma^\mu \).
In cases when there exist observer frames that fail to admit the definition of a hermitian Hamiltonian, the above line of reasoning cannot be employed, and microcausality may break down. For example, consider a model with $c_{00}$ parameter only. The anticommutator function for this model is given explicitly by

$$S(z) = (i\zeta^0 \partial^0 - i\gamma^j \partial^j + m) \frac{1}{4\pi \zeta r} \frac{\partial}{\partial r} [\Theta(w^2) J_0(m\sqrt{w^2})]$$

where $\zeta = 1 + c_{00}$, $r = |\vec{z}|$, $w^2 = (z^0/\zeta)^2 - \vec{z}^2$, $\Theta$ denotes the Heaviside step function and $J_0(y)$ is the zeroth-order Bessel function. It follows that the anticommutator function $S(z)$ vanishes only in the region defined by $z^0 < (1 + c_{00})|\vec{z}|$. The propagation of signals therefore could occur with maximal speed $1/(1 + c_{00})$. For negative values of $c_{00}$, this exceeds 1 and hence violates microcausality.

The question arises, at which energy scale this breakdown of microcausality occurs. To this end, it is useful to introduce a definition of the velocity of a particle valid for arbitrary 3-momenta. Even in the conventional Lorentz- and CPT-symmetric case, the notion of a quantum velocity operator is nontrivial. The issue is further complicated in the present context. Here, we consider the group velocity defined for a monochromatic wave in terms of the dispersion relation. This choice seems appropriate for a variety of reasons. Insight about the scale $\tilde{M}$ of microcausality violations can then be gained by determining the value of the 3-momentum at which the group velocity reaches 1. Analyses for a variety of parameter combinations yield

$$\tilde{M} \gtrsim \mathcal{O}(M_P)$$

Here, we have assumed that the parameters $c_{\mu\nu}$, $d_{\mu\nu}$, $e_{\mu}$, $f_{\mu}$ and $g_{\mu\nu\lambda}$ are of order $m/M_P$, where $M_P$ denotes the Planck scale.

We mention in passing that the conclusion of microcausality breakdown at $\mathcal{O}(M_P)$ may be invalid if the $c_{\mu\nu}$ coefficient is nonzero. For example, in the above model with only a coupling $c_{00} < 0$, one can show that $\tilde{M} \gtrsim \mathcal{O}(\sqrt{mM_P})$. The effect of the $c_{\mu\nu}$ parameter on the dispersion relation is special for the following reason: The general spinorial and derivative structure of the associated quadratic field term is identical to the conventional Dirac kinetic term. Thus, it is a first-order correction to an existing zeroth-order term. None of the other Lorentz-violating couplings exhibits this feature.

The above analysis shows that the standard-model extension can develop problems when the symmetry-breaking scale is approached. This should not come as a surprise because the effects of the presumed underlying theory are likely to be no longer negligible at these energies. However, given the impracticality of achieving Planck-scale momenta in the laboratory, the issue of
microcausality breakdown is largely unimportant at the level of the standard-model extension.

Another important ingredient for realistic field theories is the requirement of stability. A field theory is stable if the energy is positive definite in all observer frames. This implies that the 4-momenta of all one-particle states in a particular frame must be timelike or lightlike with nonnegative 0th component. Only under this last condition, does energy positivity become an observer-invariant notion. This is satisfied in the conventional Dirac case.

In the present context, the energy is positive definite in concordant frames. In these frames, the dispersion relation has still two positive- and two negative-valued roots, which yield positive particle energies after the usual reinterpretation. However, contrary to the conventional case, these energies are in some instances 0th components of spacelike 4-momenta. As a result, energy positivity becomes observer-dependent.

As an example, consider a model that has all Lorentz-violating parameters except \( b_\mu \) set to zero. The dispersion relation for this model is given by

\[
(\lambda^2 - b^2 - m^2)^2 + 4b^2 \lambda^2 - 4(b \cdot \lambda)^2 = 0 .
\]  

(9)

It is straightforward to show that observer frames in which \( b_\mu = (b_0, 0, 0, b_3) \) and \( b_3^2 > m^2 + |b^\mu b_\mu| \) can always be chosen. In such a frame, the spacelike 4-vectors \( \lambda^\mu_\pm = (0, 0, 0, p_\pm) \) satisfy the dispersion relation (9). Here, the real quantities \( p_\pm \) are defined by

\[
p_\pm^2 = (2b_3^2 + b^2 - m^2) \pm \sqrt{(2b_3^2 + b^2 - m^2)^2 - (m^2 + b^2)^2} .
\]  

(10)

Moreover, the existence of these spacelike solution remains unaffected, when a nonzero \( a_\mu \) coefficient is included.

The instabilities resulting from these spacelike 4-momenta are most transparent for sufficiently boosted observers: It is always possible to convert a spacelike vector with a positive 0th component to one with a negative 0th component by an appropriate observer Lorentz transformation. In the present case, this means that there exist otherwise acceptable observer frames in which a single root of the dispersion relation involves both positive and negative energies for varying 3-momenta. In such observer frames, the canonical quantization procedure fails.

In concordant frames, the energy is positive definite. However, the physics is independent of the observer, so the appearance of negative energies in a boosted frame must also lead to instabilities in the concordant frames. The above discussions implies that these instabilities can only be associated with the spacelike momenta satisfying the dispersion relation. To illustrate this,
let us introduce a U(1) gauge interaction for the moment because in the free fermion model the particle number is conserved. As an example, consider the following process in a concordant frame: A high-energy fermion emits a virtual photon, which then decays into a fermion-antifermion pair. We can write this as

$$f_{+1} \rightarrow f_{+1} + f_{+1} + \bar{f}_{-1},$$

(11)

where $f$ and $\bar{f}$ denote fermions and antifermions, respectively, and the subscript labels the helicity state. In ordinary QED, such a process is kinematically forbidden even though both the U(1) charge and angular momentum are conserved. However it can occur in the present context if the incoming fermion has an appropriate spacelike 4-momentum. Thus, there exist unstable single-particle states.

The scale $\tilde{M}$ of the 3-momentum at which spacelike 4-momenta occur can be calculated explicitly for various parameter combination. We find that

$$\tilde{M} \sim \mathcal{O}(M_P),$$

(12)

where we have assumed that the derivative-coupling coefficients have the same suppression as in the microcausality case, and the remaining parameters $a_\mu$, $b_\mu$ and $H_{\mu\nu}$ are of order $m^2/M_P$. This estimate shows that the instabilities appear only for Planck-scale 4-momenta in a concordant frame. The corresponding negative energies occur only for Planck-boosted observers relative to this frame. Since the Earth moves nonrelativistically with respect to concordant frames, the model maintains stability for all experimentally attainable physical momenta and in all experimentally attainable observer frames.

As for microcausality, the presence of a $c_{\mu\nu}$ parameter can invalidate (12). For example, a model with a positive $c_{00}$ coefficient only, exhibits spacelike momenta at a scale of order $\sqrt{mM_P}$. It follows that when microcausality and stability are imposed on a model with a $c_{\mu\nu}$ coupling, effects from the presumed underlying theory are likely to become non-negligible already at energies close to the geometric mean of $M_P$ and $m$. As this scale is within reach of some experiments, a theoretical analysis of such effects may require high-energy corrections to the standard-model extension. In the next section, we discuss a possible type of such corrections.

4 High-Energy Effects

The results from the previous section indicate that quantum field theories of massive fermions containing terms explicitly breaking Lorentz invariance can develop difficulties with microcausality or stability. However, in concordant frames, these difficulties primarily appear as the Planck scale is approached.
The question arises, whether there exist combinations of Lorentz-violating coefficients that maintain both causality and stability. Many parameter combinations eliminate one of the two difficulties. However, we are unaware of any set of values of the couplings $a_\mu$, $b_\mu$, ..., $H_{\mu\nu}$ that simultaneously guarantee microcausality and stability at all energy scales. Moreover, it has been shown rigorously that in conventional quantum field theory, such a set of parameters would have to yield the ordinary Lorentz-symmetric dispersion relation. It is then likely that these Lorentz-breaking parameters can be absorbed into a field redefinition and remain unobservable.

The Lorentz-violating standard-model extension was developed following a top-down approach. The original motivation was the possibility of spontaneous Lorentz-symmetry breakdown in an underlying framework such as strings. Indeed, the standard-model extension includes all Lorentz-violating, but observer-invariant, terms compatible with renormalizability and the usual gauge structure. It is thus the low-energy limit of any potential spontaneous Lorentz breaking in a more fundamental theory. It is therefore not surprising that difficulties develop as the Planck-scale is approached. One would expect higher-order nonrenormalizable operators to gain importance. On the other hand, the essential status of the requirements of causality and stability suggests to adopt the inverse line of reasoning. Such a bottom-up approach could provide valuable insights into the nature of the underlying theory at the Planck scale.

The standard-model extension breaks Lorentz invariance explicitly. However, a desirable feature of the fundamental theory is spontaneous symmetry breaking. One immediate advantage of this mechanism is that the dynamics remains Lorentz covariant. Therefore it does not come as a surprise that such an underlying framework avoids at least some of the difficulties plaguing more general models involving Lorentz and CPT violation. For instance, one consequence of the spontaneous character of the Lorentz violation is that observer invariance is naturally maintained. In the previous section, this property has proved to be an important advantage. In contrast, if observer Lorentz invariance is imposed in a theory with explicit Lorentz breaking, an additional ad hoc choice is required.

Another effect of spontaneous Lorentz violation is that the parameters $a_\mu$, $b_\mu$, ..., $H_{\mu\nu}$ are only fixed at low energies. As the Planck scale is approached, they must be associated with dynamical fields. A natural question is, whether these fluctuations alone can simultaneously maintain microcausality and stability. This issue has been previously been discussed in the context of a toy model. It was shown that a satisfactory resolution within the context of ordinary point-particle field theory seems unlikely. This is consistent with other
As expected, ingredients beyond conventional quantum field theory appear necessary.

A class of theories with free-field terms maintaining causality and stability must contain terms beyond the ones in Eq. (1). The new terms have to be non-renormalizable, and in a realistic scenario with spontaneous Lorentz violation they would correspond to higher-order nonrenormalizable operators correcting the standard-model extension at energies determined by the Planck scale.

The first step is to investigate whether any type of dispersion relation can satisfy all the requirements for consistency. In a concordant frame, such a dispersion relation would reproduce the physics of Eq. (1) for small 3-momenta but would avoid group velocities exceeding 1 and spacelike 4-momenta for large 3-momenta. These requirements could be implemented by combining the Lorentz- and CPT-breaking parameters with a suitable factor suppressing them only at large 3-momenta. This factor must be essentially constant at small 3-momenta and must overwhelm polynomial powers at large 3-momenta. Since the size of 3-momenta is frame dependent, it is to be expected that a suitable factor would also be frame-dependent and hence involve Lorentz- and CPT-violating coefficients.

To make further progress, it is useful to consider explicit examples. To simplify the discussion, the masses and the Lorentz-violating parameters are taken to be of order 1 in appropriate units. This makes it possible to focus on resolving the problems of stability and causality at Planck-scale energies in a concordant frame without the complications introduced by the hierarchy of scales.

Consider a model with a negative $c_{00}$ parameter only. As discussed in the previous section, this model violates microcausality at high energies. The replacement $c_{00} \rightarrow c_{00} \exp(c_{00} \lambda_0^2)$ in the dispersion relation has been shown to result in subluminal group velocities for all 3-momenta without introducing instabilities. In an arbitrary frame, this modification takes the form

$$c_{\mu\nu} \rightarrow c_{\mu\nu} \exp(c_{\mu\nu} \lambda^\mu \lambda^\nu),$$

establishing observer invariance of the resulting dispersion relation. It can also be shown that introducing similar exponential suppression factors in models with instabilities can also resolve this problem while maintaining subluminal group velocities.

The above demonstrations prove that stable and causal dispersion relations violating Lorentz and CPT symmetry can exist. The occurrence of transcendental functions of the 4-momenta corresponds to derivative couplings of arbitrary order in the lagrangian. A satisfactory framework incorporating Lorentz and CPT violation appears necessarily to be nonlocal in this sense. Although
it is in principle conceivable that a model with explicit Lorentz breaking might satisfy the requirements of causality and stability, it would appear somewhat contrived to implement both the necessary observer Lorentz invariance and nonlocal couplings by hand. On the other hand, one can see that spontaneous Lorentz and CPT violation in a nonlocal theory can naturally yield the desired ingredients for stability and causality at all scales.

It would be interesting to identify theories from which these dispersion relations emerge naturally. A promising candidate for this type of framework is string theory. It provided the original motivation for the construction of the standard-model extension. Moreover, strings are known to admit spontaneous Lorentz violation and they have nonlocal interaction. A complete treatment of this question would be desirable, but is hampered by the absence of a satisfactory realistic string theory. Instead, we consider the field theory of the open bosonic string as an example and show that its structure is compatible with dispersion relations of the desired type.

The open bosonic string has no fermion modes. We will therefore consider the scalar tachyon. The relevant quadratic terms of the lagrangian for the tachyon in the presence of Lorentz violation are given by

$$
\mathcal{L} \supset \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + (\alpha'^{-1} + k_0) \phi^2 + \ldots + k_1 \langle B_{\mu\nu} \rangle \partial^{\mu} \phi \partial^{\nu} \phi \\
+ \ldots + k_2 \langle D_{\mu\nu\rho\sigma} \rangle \partial^{\mu} \phi \partial^{\nu} \phi \partial^{\rho} \phi \partial^{\sigma} \phi + \ldots
$$

(14)

Here, the scalar parameters $k_0, k_1, k_2, \ldots$ are determined by the theory, but their specific values are irrelevant for the present considerations. Each ellipsis represents quadratic terms involving vacuum expectation values of other tensors and terms with powers of $\lambda^2$.

For a plane-wave tachyon solution, the structure of the dispersion relation resulting from lagrangian (14) indeed exhibits features needed to maintain causality and stability. For example, it contains all terms of the dispersion relation that results from the replacement (13) in the $c_{00}$ model, as the reader is invited to verify.

We emphasize that the purpose of the above discussion is only to provide an outline indicating how a satisfactory dispersion relation for Lorentz violation could emerge in the context of string theory. In particular, we do not claim that the tachyon itself must necessarily obey such a relation, although it is conceivable that it does. Here, the tachyon dispersion relation is used merely as an illustration to display explicitly the appearance of nonlocal couplings in string theory that could be appropriate for a stable and causal theory with spontaneous Lorentz violation. This type of coupling is generic both for other fields in the open bosonic string and for fields in other string theories, including
ones with fermions.

References

1. J.S. Bell, Birmingham University thesis (1954); G. Lüders, Det. Kong. Danske Videnskabernes Selskab Mat. Fysiske Meddelelser 28, 5 (1954); W. Pauli, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (McGraw-Hill, New York, 1955).
2. V.A. Kostelecký and S. Samuel, Phys. Rev. D 39, 683 (1989); V.A. Kostelecký and R. Potting, Nucl. Phys. B 359, 545 (1991); Phys. Lett. B 381, 89 (1996).
3. F.R. Klinkhamer, Nucl. Phys. B 578, 277 (2000).
4. S.M. Carroll *et al.*, Phys. Rev. Lett. 87, 141601 (2001).
5. D. Colladay and V.A. Kostelecký, Phys. Rev. D 55, 6760 (1997); Phys. Rev. D 58, 116002 (1998).
6. Reviews of various experimental approaches can be found, for example, in these proceedings and V.A. Kostelecký, ed., *CPT and Lorentz Symmetry* (World Scientific, Singapore, 1999).
7. V.A. Kostelecký and C.D. Lane Phys. Rev. D 60, 116010 (1999); J. Math. Phys. B 40, 6245 (1999).
8. V.A. Kostelecký and R. Lehnert, Phys. Rev. D 63, 65008 (2001).
9. V.A. Kostelecký and R. Lehnert, in preparation.
10. See, e.g., G. Sansone and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable, Vol.1*, Noordhoff, Groningen, 1960.
11. H.J. Borchers and D. Buchholz, Commun. Math. Phys. 97, 169 (1985); J. Bros, H. Epstein and V. Glaser, Nuovo Cimento 31, 1265 (1964).
12. V.A. Kostelecký and S. Samuel, Phys. Lett. B 207, 169 (1988); Nucl. Phys. B336, 263 (1990); Phys. Rev. D 42, 1289 (1990); Phys. Rev. Lett. 64, 2238 (1990); Phys. Rev. Lett. 66, 1811 (1991).