We consider for two based graphs $G$ and $H$ the sequence of graphs $G_k$ given by the wedge sum of $G$ and $k$ copies of $H$. These graphs have an action of the symmetric group $\Sigma_k$ by permuting the $H$-summands. We show that the sequence of representations of the symmetric group $H_q(\text{Conf}_n(G); \mathbb{Q})$, the homology of the ordered configuration space of these spaces, is representation stable in the sense of Church and Farb. In the case where $G$ and $H$ are trees, we provide a similar result for glueing along arbitrary subtrees instead of the base point. Furthermore, we give similar stabilization results for configurations in spaces without any obvious action of the symmetric group.

1. Introduction

For a topological space $X$ and a finite set $S$ we define the ordered configuration space of $X$ with particles $S$ as

$$\text{Conf}_S(X) := \{ f : S \to X \text{ injective} \} \subset \text{map}(S, X).$$

For $n \in \mathbb{N}$ we write $n := \{1, 2, \ldots, n\}$ and $\text{Conf}_n(X) := \text{Conf}_n(X)$.

Let $G$ be a finite connected graph, then we are interested in the homology of $\text{Conf}_n(G)$, the ordered configuration space of $n$ particles in $G$. In [Lü14] we showed that at least one of $H^k(\text{Conf}_n(G); \mathbb{Q})$ cannot be representation stable. In this paper we show that by stabilizing the graph instead of the number of particles we get representation stability.

Let us first specify what we mean by stabilizing the graph. For three graphs $G_0, G_1$ and $H$ such that $H$ is a subgraph of $G_0$ as well as of $G_1$, denote by $G_0 \sqcup_H G_1$ the graph given by taking the disjoint union of $G_0$ and $G_1$ and glueing them together along $H$. Note that $H$ is still a subgraph of $G_0 \sqcup_H G_1$, so we can iterate this construction. We write $G_0 \sqcup_H G_1^{\sqcup k}$ for the $k$-fold iterated glueing. The symmetric group $\Sigma_k$ acts on this space by permuting the copies of $G_1$.

Let $G_0$ be a finite graph and $H_i \subset G_i$ be pairs of finite graphs for $1 \leq i \leq \ell$ such that each $H_i$ is also a subgraph of $G_0$. Denote by $H := \{H_1, \ldots, H_{\ell}\}$ and $G := \{G_0, \ldots, G_{\ell}\}$. Let $G[H, G] : \text{FI}^{\times \ell} \to \text{Top}$ be given by

$$G[H, G](j_1, \ldots, j_{\ell}) := G_0 \sqcup_{H_1} G_1^{\sqcup j_1} \sqcup_{H_2} \cdots \sqcup_{H_{\ell}} G_{\ell}^{\sqcup j_{\ell}}.$$ 

See Section 2 for more details. In this paper we prove that if we put certain restrictions on the graphs $G_i$ and $H_i$, then the $\text{FI}^{\times \ell}$-module

$$H^Z_{q,n}[H, G] := H_q(\text{Conf}_n(G[H, G]); \mathbb{Z})$$

is finitely generated.

Date: December 2016.
Theorem A. If each graph $H_i$ is a single point then $\mathbb{H}^Z_{q,n}[H, G]$ is finitely generated in degree $(3n, 3n, \ldots, 3n)$ for each $q, n \in \mathbb{N}$.

Theorem B. If all graphs $G_i$ and $H_i$ are trees, then $\mathbb{H}^Z_{q,n}[H, G]$ is finitely generated in degree $(2n, 2n, \ldots, 2n)$ for each $q, n \in \mathbb{N}$.

Corollary 1.1. In the same situation as in the theorems above, choose any non-decreasing (component-wise) functor $F: FI \to FI \times \ell$, then the $FI$-module $\mathbb{H}^Z_{q,n}[H, G] \circ F$ is finitely generated. In particular, the sequence $\mathbb{H}^Z_{q,n} := H_q(Conf_n(G[H, G] \circ F); \mathbb{Q})$ is representation stable and therefore the dimension of the sequence of vector spaces is eventually polynomial.

Corollary 1.2. Let $G, H$ be finite graphs with base point and define $G_k := G \vee H \vee \cdots \vee H$ $k$ times. Then the $FI$-module $H_q(Conf_n(G_k))$ is finitely generated in degree $3n$. In particular, the sequence $H_q(Conf_n(G_k); \mathbb{Q})$ is representation stable and therefore the dimension of the sequence of vector spaces is eventually polynomial in $k$.

Remark 1.3. The fact that the dimension of this sequence is bounded from above by a polynomial can be seen more easily if $G$ and $H$ are trees: from [Ghr01] Theorem 2.6, p. 8 we know that the rank of the first homology of star graphs is polynomial in the number of edges. This implies that the size of the generating set for $H_q(Conf_n(G_k))$ described in [CL16] Theorem 2, p. 2 is polynomial in $k$, giving a polynomial upper bound for the rank.

In the last section, we provide stabilization results for stabilization along an interval and the circle $S^1$. For this, we take a based graph $G$ whose base point has valence at least two. For each $k \in \mathbb{N}$, define the space $V^I_k$ to be the interval with $k$ copies of $G$ wedged at $\frac{i}{k + 1} \in [0, 1]$. Let $V^{S^1}_k$ be given by the circle $S^1$ with $k$ copies of $G$ wedged at $\frac{i}{k + 1} 2\pi \in S^1$. Then $V^I_k$ and $V^{S^1}_k$ can be viewed as spaces over suitable categories, where morphisms induce maps mapping copies of $G$ via the identity to other copies of $G$. See Figure 4 and Figure 7 for illustrations of such continuous maps.

Theorem C. For each $n, q \in \mathbb{N}$, the $\mathbb{Z}[\hat{\Delta}]$-module $W^I_{q,n, \bullet} := H_q(Conf_n(V^I_k); \mathbb{Z})$ is finitely generated in degree $n$.

Theorem D. For each $n, q \in \mathbb{N}$, the $\mathbb{Z}[\hat{\Lambda}]$-module $W^{S^1}_{q,n, \bullet} := H_q(Conf_n(V^{S^1}_k))$ is finitely generated in degree $6n$.

For the precise definition of the categories $\hat{\Delta}$ and $\hat{\Lambda}$ as well as the proof see Section 5.
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2. **Representation stability and \( FI^{\ell} \)-modules**

In [CF13], Church and Farb introduced the concept of representation stability. We now recall the concept in the case of the symmetric group \( \Sigma_k \), for more details see [CF13, Section 2.3, p. 19].

Let \( \{V_k\}_{k \in \mathbb{N}} \) be a sequence of \( \Sigma_k \)-representations over \( \mathbb{Q} \) with linear maps
\[
\phi_k : V_k \to V_{k+1}
\]
which are homomorphisms of \( \mathbb{Q}\Sigma_k \)-modules. Here we consider \( V_{k+1} \) as \( \mathbb{Q}\Sigma_k \) module by the standard inclusion \( \Sigma_k \hookrightarrow \Sigma_{k+1} \).

To describe stability for such a sequence, we need to compare \( \Sigma_k \)-representations to \( \Sigma_{k'} \)-representations for \( k' > k \). Recall that the irreducible representations of \( \Sigma_k \) over the rational numbers are in one to one correspondence to partitions \( \lambda \) of \( k \).

Given a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0) \) of \( k \), define for \( k' - k \geq \lambda_1 \) the irreducible \( \Sigma_{k'} \)-representation \( V(\lambda)_{k'} \) to be the one corresponding to the partition \((k' - k, \lambda_1, \ldots, \lambda_\ell)\). Each irreducible representation can be written like this for a unique partition \( \lambda \). For more details, see [CF13, Section 2.1, p. 14] and [FH91].

**Definition 2.1** ([CF13, Definition 2.3, p.20]). The sequence \( \{V_k\} \) is (uniformly) representation stable if, for sufficiently large \( k \), each of the following conditions holds.

- \( \phi_k : V_k \to V_{k+1} \) is injective.
- The \( \mathbb{Q}\Sigma_{k+1} \) submodule generated by \( \phi_k(V_k) \) is equal to \( V_{k+1} \).
- Decompose each \( V_k \) into irreducible representations
  \[
  V_k = \bigoplus_\lambda c_{\lambda,k} V(\lambda)_k
  \]
  with multiplicities \( 0 \leq c_{\lambda,k} \leq \infty \). Then there exists an \( N \geq 0 \) such that for each \( \lambda \), the multiplicity \( c_{\lambda,k} \) is independent of \( k \geq N \).

This reduces the description of the infinite sequence of \( \Sigma_k \)-representations to a finite calculation.

In [CEF15], Church-Ellenberg-Farb introduced the notion of \( FI^{\ell} \)-modules, which we now recall. Let \( FI \) be the category with objects all finite sets and morphisms all injective maps. We often consider the skeleton of this category given by the restriction to the finite sets \( n := \{1, \ldots, n\} \) for \( n \geq 0 \).

**Definition 2.2.** Let \( R \) be a commutative ring. An \( R[FI] \)-module \( V_\bullet \) is a functor
\[
V_\bullet : FI \to RMod.
\]
It is said to be finitely generated in degree \( \ell \) if there exists a finite set \( X \) of elements in
\[
\bigcup_{S \subseteq FI, |S| \leq \ell} V_S,
\]
such that the smallest sub-\( FI \)-module containing all these elements is \( V_\bullet \). Here, \( |S| \) is the cardinality of \( S \).
For sequences of finite dimensional representations, the notion of a finitely generated FI-module is a generalization of representation stable sequences by the following result:

**Theorem 2.3** ([CEF15, Theorem 1.13, p. 8]). An FI-module $V_\bullet$ over a field of characteristic 0 is finitely generated if and only if the sequence $k \mapsto V_k$ is representation stable and each $V_k$ is finite dimensional.

This result reduces the uniform decomposition of the representations $V_k$ to finding a finite set of generators. Furthermore, Church-Ellenberg-Farb proved that the dimension of representation stable sequences grows polynomially:

**Theorem 2.4** ([CEF15, Theorem 1.5, p. 4]). Let $V_\bullet$ be an FI-module over a field of characteristic 0. If $V_\bullet$ is finitely generated then the sequence of characters $\chi_{V_\bullet}$ is eventually polynomial. In particular, $\dim V_k$ is eventually polynomial in $k$.

In order to describe stabilization in multiple directions we look at the product category $\text{FI} \times \ell$ consisting of $\ell \geq 1$ copies of the category FI. An $\text{FI} \times \ell$-module is then a functor $\text{FI} \times \ell \to \text{RMod}$, the notion of finite generation is defined analogously.

To define such a module, it is sufficient to define it on the skeleton consisting of the objects $(j_1, \ldots, j_\ell)$ for $j_i \in \mathbb{N}$ and the morphisms between them. In the introduction we defined $G[H, G]$ for those objects. To define the images of morphisms, notice that each summand of $G_i$ can be labeled by a number between 1 and $j_i$. For a map $\phi: j_i \hookrightarrow j'_i$ we define the induced map to send the summand with label $m \in j_i$ to the summand with label $\phi(m)$ via the identity.

Clearly, if $V$ is a finitely generated FI$\times\ell$-module and $F: \text{FI} \to \text{FI} \times \ell$ is any non-decreasing functor, then the FI-module $F^*V := V \circ F$ is finitely generated: since $F$ is non-decreasing, each component of $F$ is either eventually constant or unbounded.

### 3. Generators for the homology of configuration spaces of graphs

In this section, we give an overview over the results of [CL16] that we need in this paper. This includes a generating set for the homology of configuration spaces of trees and a description of how to turn those into generating sets for the non-tree case.

#### 3.1. Configurations in trees

We need the following definition from the mentioned paper.

**Definition 3.1.** A representative $S$ of a homology class $\sigma \in H_q(\text{Conf}_n(G))$ is called a product of the cycles $S_1$ and $S_2$ for $[S_1] \in \text{Conf}_{T_1}(G)$ and $[S_2] \in \text{Conf}_{T_2}(G)$ with $T_1 \sqcup T_2 = n$ if the image of $(S_1, S_2)$ under the product map

$$\text{Conf}_{T_1}(G) \times \text{Conf}_{T_2}(G) \to G^n$$

is equal to the image of $S$ under the inclusion $\text{Conf}_n(G) \subset G^n$.

For $k \geq 3$ let $\text{Star}_k$ be the star graph with $k$ leaves and $H$ be the tree with two vertices of valence three. We call an element $\sigma \in H_1(\text{Conf}_n(G))$ basic if there exists a piecewise linear embedding $\iota$ of $\text{Star}_k$ for some $k$ or $H$ into $G$ such that $\sigma$ is in the image of the induced map $H_1(\iota)$.

We will use the following result:
Theorem 3.2 ([CL16, Theorem 2, p. 2]). Let $G$ be a finite tree and $n$ a natural number. Then the homology of $\text{Conf}_n(G)$ in degree $q \geq 0$ is generated by products of basic cycles.

Remark 3.3. We will also use that the embeddings of $H$ can be chosen such that they contain precisely two essential vertices, which can be arranged by splitting an $H$-graph containing $k$ vertices into $k - 1$ of them, each containing exactly two vertices. Also, note that after fixing those two vertices, we can choose the edges of the embedded $H$-graph arbitrarily: in the proof of the theorem above we only needed that the valence of the vertices is at least three. The cycles given by different choices of edges differ by cycles in the stars of the corresponding vertices.

3.2. Glueing two leaves of a graph. In order to prove Theorem A, we need to understand how to turn a set of generators for $H_q(\text{Conf}_n(G))$ into a set of generators for $H_q(\text{Conf}_n(\overline{G}))$, where $\overline{G}$ is $G$ with two of its leaves glued together.

Definition 3.4. Let $G$ be a graph and $W$ a subset of the vertices. The configuration space of $G$ with sinks $W$ is defined as

$$\text{Conf}_{\text{sink}}^n(G, W) = \{(x_1, \ldots, x_n) | \text{for } i \neq j \text{ either } x_i \neq x_j \text{ or } x_i = x_j \in W\} \subset G^n.$$  

This definition is useful since it allows comparing configurations in $G$ to configurations in a quotient of $G$ by turning the collapsed part into a sink. This technique can be used to investigate the local geometry of the configuration space by collapsing large parts of the graph. For a typical example of a loop in $\text{Conf}_{\text{sink}}^n([0, 1], \{0, 1\})$, see Figure 1.

![Figure 1](image_url)  

**Figure 1.** A 1-cycle in the configuration space of the interval with two sinks.

In this paper, we will use the same combinatorial model of the configuration space of a graph (with or without sinks) as in [CL16]. We only briefly sketch the construction, for more details see [CL16] and [Lü14].

The combinatorial model is a cube complex. We have a 0-cube for each configuration where for each edge $e$ which is occupied by $\ell \geq 1$ particles the positions of the particles are $\frac{i-1}{\ell} \in [0, 1] \cong e$ for $1 \leq i \leq \ell$. A $k$-cube for $k > 0$ is given by choosing such a 0-cell and $k$ particles of the corresponding configuration, each sitting at the first or last position of some edge. Moving along one of the axes of the cube then corresponds to moving the chosen particle linearly to the vertex. We only allow those $k$-cubes such that no two particles approach the same non-sink vertex.
Figure 2. The combinatorial model of $\text{Conf}^\text{sink}_2(I, \{0, 1\})$. It is homotopy equivalent to a 1-dimensional complex by pushing the two 2-cells towards the outer embedded $S^1$.

For an example, see Figure 2.

In the proof of the general case of Theorem 1 of [CL16], it was shown that the homology of $\text{Conf}_n(G)$ is generated by pairs $([S], \sigma)$ with $[S]$ a generator of the homology of $\text{Conf}_P(G)$ for $P \subset n$ and $\sigma$ a homology class of the configuration space of the particles $n - P$ in the interval with zero, one or two sinks or the circle $S^1$ with one sink, depending on the representatives of the class $[S]$. In the construction of these pairs, we started with the cycle $S$ and constructed pairs $([S], \sigma)$ via a spectral sequence argument. If $[S]$ has a representative avoiding a terminal vertex of the glued edge, we turn the corresponding vertex of the unit interval into a sink. This sink represents the ability to reorder the particles of $\sigma$ on the star of the corresponding vertex of $G$ 

We now make a case distinction on the type of the generator $\sigma$ to describe how to construct the homology classes in $H_*(\text{Conf}_n(G))$ corresponding to such pairs.

**$\sigma$ a standard generator in degree zero:** If $\sigma \neq 0$ is a zero cycle, then this pair just corresponds to the image of $[S]$ under

$$H_*(\text{Conf}_P(G)) \rightarrow H_*(\text{Conf}_P(G))$$

induced by the canonical inclusion $G \hookrightarrow \overline{G}$, where we add the remaining particles to the glued edge in the order given by any representative of $\sigma$.

**$\sigma$ a standard generator in degree 1 in the interval with two sinks:** Every such class can be represented by a cycle $\eta$ where all particles sit on the edge and reorder themselves in an alternating way on the left and right sinks, c.f. [Figure 1]. We think about such 1-cycles as having vertices where all particles sit on the interior of the interval and edges moving all of them to one of the sinks and back. The cycle corresponding to the pair $([S], \sigma)$ is constructed as follows: start on some vertex of $\eta$, then the corresponding configuration has all particles on the interior of the interval. For each 1-cell $T$ of $\eta$ describing a reordering on the sink corresponding to some vertex $v \in \overline{G}$, choose a cycle $\hat{S}$ avoiding $v$ with $[S] = [\hat{S}]$ and a chain $X$ with boundary $S - \hat{S}$. The reordering $T$ has two configurations $T_0$ and $T_1$ of particles on the interval, one before and one after the reordering. Denote by $X_T$, the chain $X$ where in each cell we add the particles $n - P$ to the glued edge in the order given
by $T_i$. Now glue together $X_{T_0}$, $X_{T_1}$, and the product of $\tilde{S}$ and a reordering path $\gamma$ of the particles $n - P$ in the star of $v$, c.f. Figure 3. Putting these together for all reorderings yields the homology class represented by this pair.

Figure 3. Constructing a piece of the cycle represented by a pair of cycles $[S]$ and $\sigma$.

$\sigma$ a standard generator in degree 1 in the circle with one sink: These 1-cycles are generated by cycles where all particles sit on the edge, then all but one of the outmost particles $p$ move to the sink, $p$ goes around the circle once and all other particles return to their initial position. To get the generator corresponding to this pair, we do the construction analogous to the previous case: represent $S$ by $\tilde{S}$ freeing one of the vertices $v$, move the particles off the glued edge onto the star of $v$, choose a chain moving $p$ to the other vertex of the glued edge and move all particles back to the glued edge. Then, piece together the chains in the obvious way.

To summarize, we can find a generating set for $H_\ast(\text{Conf}_n(G))$ by taking a generating set for $H_\ast(\text{Conf}_k(G))$ for all $k \leq n$, adding the remaining $n - k$ particles to the glued edge in different ways and choosing chains between homologous cycles of this form. For more details, see [CL16].

4. The Proof of Theorem A and Theorem B

Definition 4.1. Let $C$ be a subspace of $\text{Conf}_n(G)$ for a finite graph $G$ and $H$ a subspace of $G$. Then we say that $C$ is supported in $H$ if for each point $x \in C$ all particles $x_i$ are on $H$. We say that a homology class $\sigma \in H_q(\text{Conf}_n(G))$ is supported in $H$ if there exists a representative of $\sigma$ which is supported in $H$.

We first prove Corollary 1.2 for star graphs by hand.

Proposition 4.2. Corollary 1.2 is true for $G$ the point and $H$ the interval $[0, 1]$ with 0 as base point. In fact, the homology is generated by cycles meeting at most $n + 3$ many copies of $H$, so the FI-module is generated in degree $n + 3$.

Remark 4.3. For $n = 2$ the argument presented below is easily modified to show that $H_1(\text{Conf}_2(G))$ is generated in degree $n + 2 = 4$. Since $n + 3 \leq 2n$ for $n > 2$, this shows that $H_1(\text{Conf}_n(G))$ is generated in degree $2n$, which will be used in the proof of Theorem B.

Proof of Proposition 4.2. The combinatorial model of this configuration space is a graph (c.f. [Chr01, Theorem 2.6, p. 8], [Lü14]), so we only need to consider 1-cycles. Choose any subgraph $\text{Star}_3 \subset \text{Star}_k$. Assume that we have a connected 1-cycle $\sigma$ which visits each vertex of the configuration space at most once. The claim is that
we can write $\sigma$ as a sum of cycles where each particle uses at most one edge outside
of Star$_3$.

Let $p$ be a particle and choose a vertex $v$ of $\sigma$ where $p$ sits on the vertex of the
star. If this does not exist, then $p$ is fixed and therefore uses at most one edge. Now
follow the cycle until $p$ sits on the vertex again and the next edge would move $p$ onto the second leaf of Star$_k$ − Star$_3$, we call the corresponding vertex $w$. Now
choose the following path $\gamma$ back to $v$ during which $p$ always stays on Star$_3$: move $p$ onto an edge $e_1$ of Star$_3$ and keep it there. Follow $\sigma$ back ignoring the movement of
$p$ and using the connectedness of the configuration space of Star$_3$ to move $p$ out of
the way if other particles need to move along $e_1$.

This decomposes $\sigma$ into two parts: the segment of $\sigma$ between $v$ and $w$ followed
by $\gamma$, in which $p$ visits only one edge not in Star$_3$, and $\gamma$ followed by the remaining
segment of $\sigma$. Continuing this process, we get a sum decomposition of $\sigma$ where $p$
visits at most one additional edge in each summand.

Since we did not increase the number of edges visited by any other particle, we
can repeat this for every $p$ and get a sum decomposition of $\sigma$ of the required form.

Consequently, for each $N \geq n+3$ we can generate $H_1(\text{Conf}_n(G_N))$ by cycles such
that each one of them is supported in some subgraph Star$_{n+3} \rightarrow G_N$. Therefore,
the $\mathbb{Z}\Sigma_N$-span of the image of the map $H_1(\text{Conf}_n(G_{n+3})) \rightarrow H_1(\text{Conf}_n(G_N))$
is the whole module and the FI-module $H_1(\text{Conf}_n(G_n); \mathbb{Z})$ is finitely generated in
degree $n + 3$. \hfill \square

Proof of Theorem B. Let $n > 1$ and $(k_1, \ldots, k_l)$ be such that each $k_i$ is at least $2n$ and
assume that none of the $G_i$ for $i > 0$ is equal to the point. By Theorem 3.2, the homology of $\text{Conf}_n(G(k_1, \ldots, k_l))$ is generated by products of basic cycles, since
the graph is a tree. By Remark 3.3 we can assume that the embedded $H$-graphs contain exactly two vertices because $k_i > 3$ and therefore the valence of all vertices
is at least three.

If an $H$-generator involves a vertex in one of the $H_i$, then we can arrange that it
only involves edges of $G(3,3,\ldots,3) \subset G(k_1, \ldots, k_l)$ by Remark 3.3 again.

Each star generator with vertex on some $H_i$ with $k$ particles can be written
as a sum of generators each using only $2k$ different edges by Proposition 4.2 and
Remark 4.3. Therefore, each summand visits at most $2k$ distinct copies of each of
the $G_i$. Every star generator disjoint from all $H_i$ meets at most one of the copies of
one of the $G_i$.

Hence, we can generate the whole homology by generators which each meet at most
$2n$ different copies of each of the $G_i$. This implies that the $\mathbb{Z}[\Sigma_{k_i} \times \cdots \times \Sigma_{k_l}]$-span
of the image of $H_\ast(\text{Conf}_n(G(2n, \ldots, 2n))) \rightarrow H_\ast(\text{Conf}_n(G(k_1, \ldots, k_l)))$
is the whole module, finishing the proof. \hfill \square

Proof of Theorem A. The restriction that each $H_i$ is the point means that the glueing along $H_i$ corresponds to a wedge sum for the corresponding choices of base points. If all $G_i$ are trees this is true by Theorem B. We will now prove that the theorem remains true if we glue leaves together.

Let $(k_1, \ldots, k_l)$ be such that each $k_i$ is at least $3n$ and assume that none of the
$G_i$ for $i > 0$ is equal to the point. Now assume that all $G_i$ are trees, then we
We now describe how to turn \(
H^2_{q,n}[H, G](k_1, \ldots, k_t)
\) by cycles where the particles visit at most \(2n\) copies of each of the \(G_i\). If we now glue two leaves of one of the copies of \(G_i\) to some \(i_0\), then we know from Section 3.2 how to construct a generating set for the corresponding homology.

Let \((|S|, \sigma)\) be a pair of cycles, where \(S\) consists of \(m\) particles and meets at most \(2m \leq n\) copies of each \(G_i\). If \(\sigma\) is a 0-cycle then the cycle represented by this pair meets at most one copy of \(G_i\) more than \(S\), so in particular it still meets in total at most \(2n\) copies of each \(G_i\) (if it meets one more copy of \(G_i\), then \(m < n\) and therefore \(2m + 1 \leq 2n - 1\)). It remains to handle the case where \(\sigma\) is a 1-cycle.

Choose for each \(i\) an edge \(\epsilon_i \subset G_i\) at the base point \(H_i\). For each \(i\) and each particle \(p\), let \(G^p_i\) be a copy of \(G_i\) such that \(S\) is supported outside of \(G^p_i - H_i\) and \(G^p_i \neq G^p_i'\) for \(p \neq p'\). Denote by \(\epsilon_i^p\) the edge in \(G^p_i\) corresponding to \(\epsilon_i \subset G_i\). In the construction of the generator corresponding to \((|S|, \sigma)\) we then choose for each 1-cell \(T\) of a representative of \(\sigma\) a \((q + 1)\)-chain \(Y_T\) bounding \(S_T\) and \(S_T\), see Figure 3. We now describe how to turn \(Y_T\) into a chain \(Y'_T\) which is supported in the union of \(G_0\), all \(G^p_i\) and the copies of the \(G_i\) in the support of \(S\).

For each particle \(p\) define the continuous map
\[
\phi_p^S: G[H, G](k_1, \ldots, k_t) \to G[H, G](k_1, \ldots, k_t)
\]
which is the identity on \(G_0\) and all copies of all \(G_i\) meeting the support of \(S\). On a copy of \(G_i\) outside the support of \(S\), define the map to be given by
\[
x \mapsto \min\{d(x, H_i), 1/2\} \in [0, 1] \equiv \epsilon_i^p \subset G[H, G](k_1, \ldots, k_t),
\]
where the isomorphism is chosen such that 0 corresponds to the vertex \(H_i\) and the path metric on the graph is such that each edge has length 1. The image of this map is by definition contained in the union of \(G_0\), all \(G^p_i\) and all copies of the \(G_i\) meeting the support of \(S\). Now define the continuous map
\[
\phi^S: \text{Conf}_n(G(k_1, \ldots, k_t)) \to \text{Conf}_n(G(k_1, \ldots, k_t))
\]
to be given by mapping each particle \(p\) via \(\phi_p\). Every particle \(p\) is either mapped via the identity or lands on \(G^p_i - H_i\), and since \(G^p_i - H_i\) is disjoint from \(G^p_i - H_i\) for \(p \neq p'\), the map is well-defined.

By definition, \(\phi^S(S_{T_j}) = S_{T_j}\) for \(j \in \{0, 1\}\), so \(Y'_T := \phi^S(Y_T)\) has the required properties. Glueing together these chains gives a cycle supported in the union of \(G_0\) and at most \(3n\) copies of each \(G_i\).

Since each of the generators we construct in this way already comes with a choice of copies \(G^p_i\) for each \(p\) and \(i\), we can repeat the same argument for any subsequent glueing.

By glueing together leaves of all \(G_i\), we see inductively that the module \(H^2_{q,n}[H, G](k_1, \ldots, k_t)\) is generated by cycles meeting at most \(3n\) copies of each of the \(G_i\), which proves the theorem. \(\square\)

Remark 4.4. In the proof above, we always replace the movement of a particle \(p\) inside any new copy of \(G_i\) by constant movement in \(\epsilon_i^p \subset G^p_i\) instead of the “same” movement in \(G^p_i\) since for \(i = i_0\) we don’t know whether the leaves of \(G^p_i\) are already glued or not, so the “same” movement may not be well-defined in \(G^p_i\).
5. Finite generation over similar categories

In this section we provide examples for spaces over categories different from \( \text{FI}^\times \ell \) whose homology groups stabilize in a similar sense.

5.1. Stabilization along the interval. Let \( \tilde{\Delta} \) be the category with objects finite totally ordered sets and morphisms injective order preserving maps. Functors from this category can be given by defining them on the skeleton having \( \mathbf{k} := \{1, \ldots, k\} \) with the canonical ordering as objects.

Let \((G, b_G)\) be a based finite graph with \( \text{val}(b_G) \geq 2 \) and \( I \) the unit interval. Then let \( \V^I \colon \tilde{\Delta} \to \text{Top} \) be the functor sending \( \mathbf{k} \) to \( \mathbf{k} \) copies of \( G \) wedged to the unit interval at the points \( \frac{i}{k+1} \in I \) for \( 1 \leq i \leq k \). The image of a morphism \( \phi \colon \mathbf{k} \to \mathbf{k}' \) is given by the canonical map stretching the parts of the interval between the copies of \( G \) and mapping the copies of \( G \) by the identity to the copies of \( G \) according to \( \phi \), c.f. Figure 4.

![Figure 4](image)

**Figure 4.** The map induced by \( \phi \colon 2 \to 4 \) given by \( \phi(1) = 2 \) and \( \phi(2) = 4 \).

Now Theorem C says that the module
\[
W_{q,n}^I := H_q(\text{Conf}_n(\V^I) ; \mathbb{Z})
\]
is finitely generated. For the proof we need the following result.

**Lemma 5.1.** Let \( H \subset G \) be two graphs such that each vertex of \( H \) incident to an edge of \( G - H \) has valence at least three and no pair of vertices of \( H \) is connected via a path in \( G - H \). Let \( S \) be a \( q \)-cycle in \( \text{Conf}_n(G) \) representing the zero homology class. Then we can find a \((q+1)\)-chain \( X \) with boundary \( S \) such that every particle that stays on \( H \) for each cell of \( S \) also stays on \( H \) for each cell of \( X \).

**Proof.** First, we show this for the case where \( G = H \lor G_0 \) and denote the base point by \( b_G \). Let \( X \) be some chain with boundary \( S \). We now replace \( X \) by a chain with the properties described in the statement. Let \( P \) be the set of particles that stay on \( H \) for each cell of \( S \).

Define the continuous map
\[
\phi \colon \text{Conf}_n(H \lor G_0) \to \text{Conf}_n(H \lor G_0 \lor \bigvee_{p \in P} [0,1])
\]
to be given by the inclusion for the particles \( n - P \) and
\[
x \mapsto \begin{cases} 
x & \text{if } x \in H, \\
\min\{d(x, b_G), 1/2\} & \text{if } x \in [0,1] \end{cases}
\]
for each particle \( p \in P \), similar to the map \( \phi \) in the proof of Theorem A. This gives a chain \( X' := \phi(X) \) bounding \( \phi(S) \) such that each particle \( p \in P \) stays on \( H \lor [0,1] \).
Choose three edges $e_1, e_2, e_3$ at $b_G$ belonging to $H$. We now define a map

$$\psi: \text{Conf}_n \left( H \lor G_0 \lor \bigvee_{p \in P} [0,1]_p \right) \to \text{Conf}_n(H \lor G_0)$$

by giving the images of cells $C$ of the combinatorial model of the domain of $\psi$ via the following case distinction:

1. **All moving particles of $C$ are on $G$:** First, assume that no moving particle of $C$ moves from $e_1$ to $b_G$. Map $C$ to the cell $C'$ where we put the particles sitting on any $[0,1]_p$ onto the edge $e_1$ ordered alphabetically first by the natural order on $P \subseteq \mathbb{n}$ and second by the order of the particles on the corresponding intervals. Put these particles onto $e_1$ in such a way that there are no other particles between them and the base point.

2. If in $C$ a particle $p_0$ moves from $e_1$ to $b_G$, take the face $D$ of $C$ where $p_0$ is fixed on the interior of $[0,1]_p$ and map this face according to the previous case. To describe the map on the whole cube $C$, map the movement of $p_0$ to the movement given by precisely the same description as in the previous paragraph.

3. **The particle $p_0$ moves off the edge $[0,1]_p$:** take the face of $C$ where $p_0$ is fixed on the interior of $[0,1]_p$ and map this face according to the previous case. To describe the map on the whole configuration space.

It is straightforward to check that this definition of $\psi$ on individual cubes is compatible with taking faces, so this defines a continuous map on the whole configuration space.

Since $\psi(\phi(S)) = S$, this defines a chain $X'' := \psi(\phi(X))$ such that each particle of $P$ always stays in $H$.

For the general case, observe that the assumption that no pair of vertices of $H$ is connected via a path in $G - H$ implies that $G$ can be constructed from $H$ by performing wedge sums. Now construct maps $\phi$ and $\psi$ analogously for this simultaneous wedging of multiple graphs.

**Corollary 5.2.** Let $G, H$ be finite based graphs and $n \in \mathbb{N}$. If the base point $b_G$ of $G$ has valence at least three, then the map

$$H_q(\text{Conf}_n(G)) \to H_q(\text{Conf}_n(G \lor H))$$

induced by the inclusion $\iota: G \hookrightarrow G \lor H$ is injective for all $q \geq 0$.

**Proof of ** Theorem C. Consider $V_k^f$ for $k \geq 3n$. If $G$ is a tree, then by Theorem 3.2, the homology is generated by products of basic cycles. These can be chosen such that each such generator visits at most $n$ copies of $G - \{b_G\}$. Therefore, the result is true in this case.

We now perform induction over the rank of the whole graph $V_k^f$ (handling all $n$ at the same time), so we need to show that after glueing two leaves of one of the graphs wedged to the interval the homology is still generated by cycles supported in $2n$ wedge summands.
Figure 5. Mapping a cell where particle \( p = 3 \) moves to \( v \) along \( e_1 \) and the particles 1 and 2 are parked on \( e_1 \).

Assume we are given the graph \( V_k^I \) with possibly some of the leaves already glued together. Denote by \( G_i \) for \( 1 \leq i \leq k \) the \( i \)-th wedge summand of this graph. Notice that \( G_i \) is not necessarily the same as \( G_j \) for \( i \neq j \). Now let \( i_0 \) be the index of the graph \( G_{i_0} \) whose leaves we glue next. We use the description of generators for the homology of configurations in the glued graph of Section 3.2 again.

Let \( ([S], \sigma) \) be a pair of cycles and let \( k \) be the number of particles of \( S \). Then, we can assume by induction that \( S \) meets at most \( 2k \) of the \( G_i \). If \( k = n \), then the pair just corresponds to the cycle \( S \) considered as a cycle in the glued graph and therefore meets at most \( 2n \) of the \( G_i \), so we can assume \( k < n \). If \( \sigma \) is a 0-cycle, then the corresponding cycle in the glued graph meets at most \( 2k + 1 \leq 2(n-1)+1 < 2n \) of the \( G_i \).

It remains to handle the case that \( \sigma \) is a 1-cycle. To construct the cycle corresponding to this pair we choose a representative of \( \sigma \) and realize each 1-cell \( T \) of this representative with a chain \( Y_T \) bounding \( S_T \), see Figure 3.

If \( S \) is supported outside of \( G_{i_0} \), then \( Y_T \) can obviously be chosen to use at most one \( G_i \) more than \( S \), namely \( G_{i_0} \). Otherwise, take the nearest two graphs \( G_{i_0}, G_{i_1} \) which intersect the support of \( S \) only in their base point or not at all. Take the smallest connected subgraph \( H_{S,i_0} \) containing all \( G_i \) for \( \ell \leq i \leq r \), see Figure 6.

Figure 6. The subgraph \( H_{S,i_0} \).
We can now write $V^I_k$ as $H_\ell \vee_{b_{G_\ell}} H_{S,i_0} \vee_{b_{G_r}} H_r$ for some graphs $H_\ell, H_r$. If $H_\ell$ contains at least one copy of $G$ then the valence of $H_{S,i_0}$ at $b_{G_\ell}$ is at least three, and the same holds for $H_r$ and $b_{G_r}$. Lemma 5.1 shows that we can now replace $Y_T$ by a chain such that all particles that start in $H_{S,i_0}$ also stay in $H_{S,i_0}$. Keeping the particles in $H_\ell$ and $H_r$ fixed, this yields a chain $Y'_T$ which meets at most two of the $G_i$ more than $S$, namely $G_\ell$ and $G_r$. Therefore, the pair corresponds to a cycle meeting at most $2k + 2 \leq 2(n - 1) + 2 = 2n$ of the $G_i$. \[\square\]

5.2. **Stabilization along the circle.** Let $\tilde{\Lambda}$ be the category with objects finite cyclically ordered sets and morphisms injective maps preserving the cyclic order.

Now we do the same construction as in the previous section, only this time we attach along the circle $S^1$. Define the functor $V^S_{S^1} : \tilde{\Lambda} \to \text{Top}$ to be given on the sets $k$ (with the standard cyclic ordering) by the circle $S^1$ with $k$ copies of $G$ wedged to it at the angles $\frac{i+1}{k+1}2\pi$ for $1 \leq i \leq k$. Here we chose an arbitrary order representing the cyclic ordering, two different choices differ only by a rotation. Morphisms get mapped to continuous maps stretching the interval pieces of the $S^1$ and map the copies of $G$ by the identity to each other, c.f. Figure 7.

![Figure 7](image-url)

**Figure 7.** The map induced by $\phi : 2 \to 5$ given by $\phi(1) = 2$ and $\phi(2) = 5$. The graph $G_i$ is $G$ with label $i$.

**Proof of Theorem D.** We prove this theorem by cutting open each $V^S_{S^1}_k$ yielding $V^I_k$.

To construct a generating set for $W^S_{q,n,k}$ we use Section 3.2 again for the gluing of 0 and 1 of the unit interval inside $V^I_k$. Thus, we need to consider pairs of cycles $([S], \sigma)$, where $S$ has $\ell$ particles. By the previous theorem, we can assume that $S$ meets at most $2\ell$ copies of $G$.

Now given a chain $Y_T$ as in the proof of Theorem C, choose for each copy of $G - \{b_{G_\ell}\}$ intersecting with the support of $S$ the nearest copies of $G - \{b_{G_\ell}\}$ to the left and right which are disjoint from the support of $S$. Then, use Lemma 5.1 again to show that we can arrange that the support of $Y_T$ is given by the copies of $G$ meeting the support of $S$ and these chosen copies of $G - \{b_{G_\ell}\}$.

If $\ell = n - 1$, then $T$ can be represented in a way that the single particle $p$ only moves along the interval, so the corresponding cycle meets at most $(1 + 2)2(n - 1) = 6n - 6 \leq 6n$ copies of $G$. 

For $\ell < n - 1$, we can represent $T$ such that its particles only meet the first and last copy of $G$, so that the cycle represented by the pair meets at most $(1 + 2)(n - 2) + 2 = 6n - 4 \leq 6n$ copies of $G$.

\[\square\]

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