Fermi acceleration in chaotic shape-preserving billiards

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We study theoretically and numerically the velocity dynamics of fully chaotic time-dependent shape-preserving billiards. The average velocity of an ensemble of initial conditions generally asymptotically follows the power law $\langle v \rangle \sim n^\beta$ with respect to the number of collisions $n$. If a shape of a fully chaotic time-dependent billiard is not preserved it is well known that the acceleration exponent is $\beta = 1/2$. We show, on the other hand, that if a shape of a fully chaotic time-dependent billiard is preserved then there are only three possible values of $\beta$ depending solely on the rotational properties of the billiard. In a special case when the only transformation is a uniform rotation there is no acceleration, $\beta = 0$. Excluding this special case, we show that if a time-dependent transformation of a billiard is such that the angular momentum of the billiard is preserved, then $\beta = 1/6$ while $\beta = 1/4$ otherwise. Our theory is centered around the detailed study of the energy fluctuations in the adiabatic limit. We show that there are three quantities, two scalars and one tensor, completely determine the energy fluctuations of the billiard for arbitrary time-dependent shape-preserving transformations. Finally we provide several interesting numerical examples all in a perfect agreement with the theory.

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I. INTRODUCTION

Because of their simplicity and generality, the billiards are one of the most important dynamical systems. They are used as a model system in various fields of research in classical and quantum mechanics. Billiards are especially convenient for numerical computation but they can be realized also experimentally, for example as a micro wave cavity, acoustic resonators, optical laser resonators and quantum dots [1], which is in the domain of quantum chaos, but the classical dynamics is important in the semiclassical picture.

A time-dependent billiard was first considered as a model of a cosmic ray particles acceleration process proposed by Fermi [2] and established by Ulam [3]. It was argued at the time that the moving wall accelerates the particle without limit. Such acceleration is called Fermi acceleration. Now it is well known that Fermi acceleration does not necessarily take place, for example in 1D system if the motion of the walls is sufficiently smooth [4]. However, Fermi acceleration exists in almost all 2D billiards.

Numerical result show that the average velocity of an ensemble of particles asymptotically follows the power law $\langle v \rangle \sim n^\beta$ where $\beta$ is the acceleration exponent. In billiards with fully chaotic dynamics one would intuitively expect that due to the loss of correlations between the successive velocity changes the acceleration exponent equals $\beta = 1/2$ in analogy with the random walk process. This intuitive result is theoretically well supported [5, 6]. In addition it is shown that there exist trajectories with measure zero which accelerate even exponentially in continuous time ($\beta = 1$). However, billiards can be transformed in such a special way that despite chaos, $\beta$ can be smaller than $1/2$ and even zero, as shown in this paper.

Various values of $\beta$ between 0 and 1 were found in systems with a coexisting regular and chaotic motion. There is a strong numerical and theoretical evidence that in such systems the exponential acceleration may become predominant [7, 8]. On the other hand the numerical studies of the time-dependent not shape-preserving elliptical billiard [10, 12], which is the integrable system as static, show that $\beta$ asymptotically equals $1/2$ while it passes a long transient regime where acceleration is sub-diffusive i.e. $\beta \leq 1/2$.

One of the basic assumptions of the theory of Gelfreich et al. [9], which predicts $\beta = 1/2$ for fully chaotic time dependent systems, is the existence of pairs of periodic orbits with a heteroclinic connection where their relative lengths undergo different time evolutions. However, in shape-preserving time-dependent billiards this assumption does not hold (at least not to the same order of magnitude), and as a consequence $\beta < 1/2$, as shown theoretically and numerically in this paper. A shape preserving transformation can be only a combination of rotation, translation and scaling. The scaling transformations alone were already studied in our previous work [13, 14], but here we provide a complete theory for a general shape preserving transformation.

In this paper we derive a differential equation for the velocity of a particle in a reference frame in which a billiard is at rest. This differential equation is used to derive a general formula for the time evolution of energy fluctuations on adiabatic time scales. We show that the order of magnitude of the energy fluctuations depend on whether a transformation is such that the angular momentum of a billiard (assuming a constant mass) is preserved or not. We derive also the corresponding asymptotic acceleration exponents. We show that there is no acceleration ($\beta = 0$) if the only transformation of a billiard is a uniform rotation, which is a counter example of the LRA conjecture [15], stated as follows: "Thus, on the basis of our investigations we can advance the following conjecture: chaotic
dynamics of a billiard with a fixed boundary is a sufficient condition for the Fermi acceleration in the system when a boundary perturbation is introduced. If this statement has to be understood as including the rigid/uniform rotation, which is a specific perturbation of the billiard boundary, then, in this sense, the result \( \beta = 0 \) in any at least partially chaotic billiard (when static) clearly violates the LRA conjecture. Excluding this special example we show that the value of the acceleration exponent is either \( \beta = 1/6 \) if the angular momentum of the billiard is preserved or \( \beta = 1/4 \) otherwise. Theoretical results are finally confirmed by the numerical results.

II. GENERALITIES

A billiard is a dynamical system in which a particle alternates between the force-free motion and instant reflections from a boundary. A boundary is a closed curve in the configuration space representing an infinite potential barrier.

Between collisions the particle velocity \( \mathbf{v} \) is constant while at collisions it changes instantly by a

\[
\Delta \mathbf{v} = R (\mathbf{v} - \mathbf{u}),
\]

(1)

where \( \mathbf{u} \) is a velocity of a boundary at a collision point and \( R \) is a reflection tensor

\[
R = -2 \mathbf{n} \mathbf{n}^T,
\]

(2)

where \( \mathbf{n} \) is a normal unit vector to the boundary at a collision point. The reflection tensor is symmetric and satisfies

\[
R^T R = -2R.
\]

(3)

Using the properties of \( R \) it is straightforward to show that the norm \( \| \mathbf{v} - \mathbf{u} \| \) is preserved at collisions. Thus, in general, if \( \mathbf{u} \neq 0 \), the norm of the velocity vector \( v = \| \mathbf{v} \| \) is not preserved. If \( \mathbf{u} \) is constant and zero for every point on a boundary, then the billiard is static. In a static billiard a magnitude of a particle velocity is constant.

III. PRIMED SPACE

Consider a coordinate system \( \mathbf{r}' = \mathbf{r}'(\mathbf{r},t) \) in which a billiard boundary is at rest and name it a primed space \( S' \). The velocity of a point in \( S' \) equals

\[
\mathbf{v}' = \frac{d\mathbf{r}'}{dt} = \frac{\partial \mathbf{r}'}{\partial t} + J \mathbf{v},
\]

(4)

where

\[
J = \begin{pmatrix} \partial \mathbf{r}'/\partial \mathbf{r} \end{pmatrix}
\]

(5)

is the Jacobian of the coordinate transformation and \( \mathbf{v} \) is the velocity in the physical (untransformed) space. By the definition of \( S' \), the points on the billiard boundary must satisfy \( \mathbf{v}' = 0 \), thus having the velocity

\[
\mathbf{u} = -J^{-1} \frac{\partial \mathbf{r}'}{\partial t},
\]

(6)

where \( \mathbf{r}' \) is a corresponding position of a boundary point in the primed space. At collisions the particle velocity \( \mathbf{v}' \) changes by a

\[
\Delta \mathbf{v}' = J \Delta \mathbf{v} = J R (\mathbf{v} - \mathbf{u}) = J R J^{-1} \mathbf{v}',
\]

(7)

where in the last equality we expressed \( \mathbf{v} \) from (4) and took into account (6).

IV. CONFORMAL TRANSFORMATIONS

If \( J \) is such that

\[
J R J^{-1} = R,
\]

(8)

then according to (7), \( \mathbf{v}' \) obeys the reflection law of a static billiard

\[
\Delta \mathbf{v}' = R \mathbf{v}',
\]

(9)

which implies that the norm \( \| \mathbf{v}' \| \) is preserved at collisions.

Using (5) in (3) gives the relation

\[
(J^T J)^{-1} R J^T (J^T J) R = -2R,
\]

(10)

which is satisfied only if

\[
J^T J = \alpha I,
\]

(11)

where \( I \) is the identity matrix and \( \alpha \) is a constant. We can determine \( \alpha \) by taking the determinant of both hand sides of (11). In 2D these gives \( \alpha^2 = |J|^2 \) and the relation

\[
J^T J = |J|.
\]

(12)

Transformations satisfying (12) are the angle-preserving or conformal transformations which are everywhere a combination of a scalar multiplication and a rotation.

V. SHAPE-PRESERVING TRANSFORMATIONS

In this paper we study only shape-preserving transformations which are linear conformal transformations. By definition, shape-preserving transformations preserve curvatures and angles. The Jacobian of a shape-preserving transformation is independent of position. A most general shape-preserving transformation has the form

\[
\mathbf{r}' = J (\mathbf{r} - \mathbf{z}),
\]

(13)
where \( \mathbf{z} \) is a translation vector, and \( J \) is the Jacobi of the form
\[
J = (q \, O)^{-1},
\]
where \( q \) is a scaling factor and
\[
O = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\]
is a rotation matrix.

Trajectories in \( S' \) are curved if a shape-preserving transformation depends on time. However, because shape-preserving transformations transform straight lines into straight lines, the curvature must vanish in the adiabatic limit. In this limit the curvature of the trajectories plays no role anymore, but the magnitude of the velocity is still governed by the transformation. A curvature radius of a trajectory in \( S' \) equals
\[
\mathcal{R} = \frac{ds'}{d\phi'} = \frac{v' \, dt}{\| \mathbf{v}' \times \mathbf{v}' \| / v'^2} = \frac{v'^3}{\| \mathbf{v}' \times \mathbf{v}' \|},
\]
where
\[
\mathbf{v}' = 2 \hat{J} J^{-1} \mathbf{v}' - 2 \hat{J} J^{-1} \frac{\partial \mathbf{v}'}{\partial t} + \frac{\partial^2 \mathbf{v}'}{\partial t^2}
\]
is the acceleration in \( S' \). The dot denotes the time derivative throughout this paper.

Vanishing curvatures and the law of reflection lead to the important conclusion that in the adiabatic limit the geometry of trajectories in \( S' \) of a shape-preserving time-dependent billiard approaches the velocity independent geometry of trajectories of the corresponding static billiard. This fact allows us to investigate the dynamics of a shape-preserving billiard much deeper then in a general time-dependent billiard. We can take the trajectories of a static billiard as an approximation for the trajectories in \( S' \) thus reducing the system of four differential equations to a single differential equation for \( v' \).

We construct the differential equation for \( v' \) from \( \mathbf{v}' \mathbf{v}' = \mathbf{v}' \cdot \mathbf{v}' \) using (13), (14) and (17),
\[
v' \mathbf{v}' = -\frac{2 \dot{q}}{q} v'^2 + \left( \omega^2 + \frac{\dot{q}}{q} \right) \mathbf{r}' \cdot \mathbf{v}' + \frac{\dot{\Gamma}}{q^2} \mathbf{r}' \wedge \mathbf{v}' - J \dot{\mathbf{z}} \cdot \mathbf{v}'
\]
where \( \omega = \dot{\phi} \) is the angular velocity of a billiard,
\[
\Gamma = \omega q^2
\]
is the quantity proportional to the angular momentum of the billiard and
\[
\mathbf{r}' \wedge \mathbf{v}' \equiv r'_x v'_y - r'_y v'_x \equiv v' \mathbf{r}' \sin \alpha'
\]
is the angular momentum of the particle in \( S' \).

We multiply (18) by \( q^4 \), move the first term from the right hand side to the left hand side and write the left hand side as a total derivative
\[
\frac{d}{dt} \left( \frac{q^4 \, v'^2}{2} \right) = B \mathbf{r}' \wedge \mathbf{v}' + A \mathbf{r}' \cdot \mathbf{v}' - \mathbf{a} \cdot \mathbf{v}',
\]
where we introduced
\[
B = q^2 \dot{\Gamma}, \quad A = \Gamma^2 - q^3 \dot{q}, \quad \mathbf{a} = q^3 \, O^{-1} \dot{\mathbf{z}}.
\]

The fact that \( \mathbf{r}' \cdot \mathbf{v}' \) and \( \mathbf{v}' \) are total time-derivatives of \( \mathbf{v}'^2 / 2 \) and \( \mathbf{r}' \) respectively allows us to rewrite (21) as
\[
\frac{d}{dt} \left( \frac{q^4 \, v'^2}{2} - \frac{A \, v'^2}{2} + \mathbf{a} \cdot \mathbf{v}' \right) = B \mathbf{r}' \wedge \mathbf{v}' - \dot{A} \mathbf{r}'^2 + \dot{\mathbf{a}} \cdot \mathbf{r}'.
\]

Note that \( \mathbf{r}' \wedge \mathbf{v}' \) is not a total time-derivative.

In the adiabatic regime of sufficiently large \( v' \) we can consider \( q^4 \, v'^2 / 2 \) as the dominant term in (25), thus neglecting all the other terms we conclude that
\[
q^4 \, v'^2 = \text{constant},
\]
which is actually a more accurate version of the well known adiabatic theorem for ergodic time-dependent billiards \[16, 17,\]
\[
v \sqrt{\mathcal{A}} = \text{constant},
\]
where \( \mathcal{A} \) is the billiard area. Equation (27) follows from (29) after the approximation \( v' \approx v / q \) and substitution \( q^2 \approx \mathcal{A} \). It is important to note that the adiabatic invariant (27) is valid for all shape-preserving billiards, regardless of whether they are ergodic or not. The adiabatic invariant describes the evolution of the average velocity of an ensemble on the adiabatic time scales. The width of the velocity distribution is spreading around its average and eventually results in the Fermi acceleration. This paper provides an accurate theoretical description of this process.

If \( B = \dot{A} = 0 \) and \( \dot{\mathbf{a}} = 0 \) then (25) can be integrated exactly. It follows from (22) that \( B = 0 \) if a transformation is such that the angular momentum of a billiard is preserved, which is when \( \Gamma = \omega q^2 = \text{constant} \). Using this in (23) we see that \( \dot{A} = 0 \) when \( B = 0 \) if \( q^3 \dot{q} = \text{constant} \), which is when \( q \) is of the form
\[
q = \sqrt{c_1 + c_2 \,(t + c_3)^2},
\]
where \( c_1, \, c_2 \) and \( c_3 \) are constants. Very interestingly though, if \( q \) is of the form (28) and \( \omega \propto 1/q^2 \) and \( \mathbf{a} = 0 \), then (25) can be integrated exactly, but this driving is not periodic.

The only periodic and infinitely smooth solution of the condition \( B = \dot{A} = 0 \) and \( \dot{\mathbf{a}} = 0 \) is a uniform rotation, i.e., \( \omega = \dot{q} = 0 \) and \( \mathbf{a} = 0 \). In this case
\[
v'^2 - \omega^2 \, \mathbf{r}'^2 = \text{constant},
\]
as follows from [25]. Thus \( v' \) is bounded if \( v' \) is bounded. Obviously, if \( v' \) is bounded then \( v \) is bounded as well, so there is no Fermi acceleration in uniformly rotating billiards. Thus we conclude that the acceleration exponent \( \beta = 0 \).

Before we proceed we have to define several different dynamical time-scales relevant for shape-preserving time-dependent billiards. In a static billiard the two relevant scales are the ergodic time-scale \( \tau_E = l_E/v \) on which the particle uniformly visits the whole accessible phase space and the time averages can be replaced by the phase-space averages, and the correlation time scale \( \tau_C = l_C/v \) on which the autocorrelations vanish. The characteris-
tical geometrical lengths \( l_E \) and \( l_C \) are independent of the particle velocity and the corresponding time-scales are vanishing in the limit \( v \to \infty \). The next relevant scale is the adiabatic time-scale \( \tau_A \), defined as a scale on which the variations of the adiabatic invariant \( I_A \) remain relatively small. Adiabatic time-scale \( \tau_A \) is an increasing function of \( v' \) and can be made arbitrarily large. We shall denote with \( \tau_B \) a time-scale of a billiard motion, proportional to \( 1/u_{\text{max}} \) where \( u_{\text{max}} \) is a maximal velocity of the boundary.

Suppose the observation time \( t \) is much smaller than \( \tau_A \) and \( \tau_E \ll \tau_B \). We introduce three quantities which have a zero mean by construction:

\[
\begin{align*}
\zeta &= v' \sin \alpha' - \langle v' \sin \alpha' \rangle, \\
\eta &= v'^2/2 - \langle v'^2/2 \rangle, \\
\xi &= v' - \langle v' \rangle.
\end{align*}
\]

Where \( \langle \cdot \rangle \) denotes a phase-space average. Using these quantities we write [25] in the form

\[
E - E_0 = \int_0^t dt \, P,
\]

where

\[
E = \frac{\dot{q}^2}{2} - \frac{q^4}{2} \omega \, v' A \langle v' \sin \alpha' \rangle - \frac{A}{\dot{q}} \left( v'^2 - \langle v'^2 \rangle \right) + A \cdot (v' - \langle v' \rangle)
\]

is interpreted as the energy, \( E_0 \) is its initial value, and

\[
P = B \, v'_A \zeta - \dot{\zeta} \, \eta + \dot{\eta} \cdot \xi
\]

is interpreted as the power. Here we have made the approximation by substituting \( v' \) in the low order term \( B \, v' \wedge v' = B \, v' \cdot v' \sin \alpha' \), with the adiabatic approximation

\[
\dot{v}'_A = \frac{\dot{q}_0^2}{\dot{q}^2} v'_0,
\]

which comes from [26].

We are interested in the statistical properties of the energy fluctuations \( \delta E = E - E_0 \), in particular in the second moment \( \langle \delta E^2 \rangle \). We assume that the quantities \( \zeta \), \( \eta \) and \( \xi \) are mutually uncorrelated and that in a regime where \( \tau_C \ll \tau_B \) their autocorrelation functions can be approximated with the Dirac delta distributions,

\[
\begin{align*}
\langle \zeta(t_1) \zeta(t_2) \rangle &= \frac{\kappa_\zeta}{v'} \delta(t_2 - t_1), \\
\langle \eta(t_1) \eta(t_2) \rangle &= \frac{\kappa_\eta}{v'} \delta(t_2 - t_1), \\
\langle \xi(t_1) \xi^*(t_2) \rangle &= \frac{K}{v'} \delta(t_2 - t_1),
\end{align*}
\]

where we have introduced numbers \( \kappa_\zeta \) and \( \kappa_\eta \) and a tensor \( K \). We expect that in the adiabatic limit \( \kappa_\zeta, \kappa_\eta \) and \( K \) are the same as in the static billiard and thus independent of the velocity. Note that these three quantities are also independent of the driving.

Taking into account the autocorrelation functions we find the following formula for the time evolution of the second moment of the energy fluctuations,

\[
\langle \delta E^2 \rangle = \int_0^t dt_1 \int_0^t dt_2 \, P(t_1) \, P(t_2)
\]

\[
= \int_0^t \frac{dt}{v'_A} \left( v'^2 A^2 B^2 \kappa_\zeta + \dot{A}^2 \kappa_\eta + \dot{\eta} \, \dot{\xi} \right).
\]

Again, here we made the approximation \( v' \approx v'_A \). Since all the terms under the integral are non-negative, \( \langle \delta E^2 \rangle \) is a strictly increasing function of time except when the billiard boundary is at rest and the integrand is zero.

We distinguish cases when the angular momentum of the billiard is preserved \( B = 0 \) and when it is not \( B \neq 0 \). If \( B \neq 0 \), then after neglecting small terms \( \dot{A}^2 \kappa_\eta + \dot{\eta} \, \dot{\xi} \),

\[
\langle \delta E^2 \rangle \approx \int_0^t \frac{dt}{v'_A} \, B^2 \kappa_\zeta.
\]

On the other hand, if \( B = 0 \), [41] vanishes and we have to deal with the previously neglected terms only

\[
\langle \delta E^2 \rangle = \int_0^t \frac{dt}{v'_A} \left( \dot{A}^2 \kappa_\eta + \dot{\eta} \, \dot{\xi} \right).
\]

The basic difference between the two cases is that with the increasing \( v' \) the variance of the energy fluctuations \( \langle \delta E^2 \rangle \) grows faster with time if \( B \neq 0 \) and slower if \( B = 0 \). In a case where \( B = 0 \), \( \dot{A} = 0 \) and \( \dot{\eta} = 0 \) we see that \( \langle \delta E^2 \rangle \) is constant. For a uniform rotation with the conserved angular momentum \( B = 0 \) this again implies no Fermi acceleration and \( \beta = 0 \). Note that once the quantities \( \kappa_\zeta, \kappa_\eta \) and \( K \) are determined for a billiard, they can be used to derive \( \langle \delta E^2 \rangle \) for arbitrary drivings.

VI. FERMI ACCELERATION

When we follow the particle velocity on the long run, we observe that the average velocity follows the adiabatic law [27] or [36], but in addition we see diffusion in the
The evolution of the average velocity $\langle v \rangle$ with respect to the number of collisions $n$ of an ensemble of initial conditions asymptotically follows the power law

$$\langle v \rangle \propto n^\beta,$$

where $\beta$ is the acceleration exponent. This law is empirically well established [7,18–20]. The Fermi acceleration is directly linked to the velocity diffusion process. As we shall see, $\beta$ is determined by the way how the diffusion constant depends on the velocity, describing the inhomogeneous diffusion in the velocity space. The acceleration exponent $\beta$ can be deduced from the time evolution of a second moment of the velocity fluctuations $\delta v' = v' - v'_{\text{A}}$. From $\delta E \approx \delta T$ we have

$$\delta E \approx \frac{q^4}{2} \left( \frac{q_0^2 v_0'/q^2 + \delta v'}{2} - \frac{q_0^4 v_0^2}{2} \right) \approx (q_0^2) v_0' \delta v',$$

from which it follows

$$\langle \delta v'^2 \rangle = \frac{\langle \delta E^2 \rangle}{(q_0^2) v_0^2}.$$  \hspace{1cm} (44)

We see from (45) and (40) that the time average of $\langle \delta v'^2 \rangle$ on the intervals $\tau$ satisfying $\tau_B \ll \tau \ll \tau_A$, is a linearly increasing function of time $t$, since the integrands may be considered as approximately constant, which must be true for $\langle \delta v'^2 \rangle$ as well and must be of the form

$$\langle \delta v'^2 \rangle \tau = \frac{2 D t}{\mu v},$$  \hspace{1cm} (46)

where $D$ is some velocity independent constant and if the angular momentum of the billiard is preserved ($B = 0$) then according to (42) $\mu = 3$ and if it is not ($B \neq 0$) then according to (41) $\mu = 1$.

Now according to (46), the evolution of the velocity distribution $P(v)$ is described by the inhomogeneous diffusion equation

$$\frac{\partial}{\partial t} P(v) = D \frac{\partial}{\partial v} \left( v^{-\mu} \frac{\partial}{\partial v} P(v) \right).$$ \hspace{1cm} (47)

We assume that after a long enough time the shape of the velocity distribution $P(v)$ is independent of time. Let the shape of the velocity distribution $P(v)$ be the same as the shape of some function $F(x)$ with the first two moments equal to unity: $\int F(x) \, dx = 1$ and $\int x \, F(x) \, dx = 1$. In this case we can write

$$P(v) = u^{-1} F(\nu u^{-1}),$$ \hspace{1cm} (48)

where the average velocity $u \equiv \langle v \rangle$ is a function of time. Putting (48) in (47) gives

$$- \frac{u}{u^2} \frac{\partial}{\partial x} \left( x F(x) \right) = \frac{D}{u^{\mu+3}} \frac{\partial}{\partial x} \left( x^{-\mu} \frac{\partial}{\partial x} F(x) \right),$$ \hspace{1cm} (49)

where $x = v u^{-1}$. Because $F(x)$ does not depend on time, $u$ must satisfy the following differential equation

$$\dot{u} u^{\mu+1} = k D,$$ \hspace{1cm} (50)

where $k$ is a positive constant, which is found together with $F(x)$ by solving the differential equation

$$\frac{\partial}{\partial x} F(x) + k x^{\mu+1} F(x) = 0$$ \hspace{1cm} (51)
and imposing the normalization conditions. We find

\[ F(x) \propto e^{-k x^{\mu+3}} \]

and

\[ \langle v \rangle = u \propto (D t)^{\frac{1}{\mu+3}}. \]

For the evolution of the mean velocity \( \langle v \rangle \) with respect to the number of collisions \( n \) it follows from \( \Delta n \propto \langle v \rangle \Delta t \) after a straightforward manipulation

\[ \langle v \rangle \propto n^{1/(\mu+3)} \]

and thus

\[ \beta = \frac{1}{\mu+3}. \]

From (45) combined with (40) we see that the diffusion exponent \( \mu \) equals 1 if \( B \neq 0 \) or 3 if \( B = 0 \) and the corresponding acceleration exponents are \( \beta = 1/4 \) and \( \beta = 1/6 \). Together with the \( \beta = 0 \) for uniformly rotating billiards discussed before, these exhaust all possible values for \( \beta \) in time-dependent shape-preserving fully chaotic billiards. This is the central result of this paper.

VII. NUMERICAL RESULTS

For numerical computation we choose a completely chaotic Sinai billiard, defined as an area between the coordinate axes and the circle \((x-a)^2 + (y-a)^2 = 4\) where \(a = \sqrt{2} + \sqrt{3}\), shown as a shaded area in figure 2a.

We shall consider four different drivings and test the validity of (40), which describes the time evolution of the second moment of the energy fluctuations.

The first example of the driving is a nonuniform rotation where the angle of the billiard \( \phi \) changes with time as \( \phi = \sin t \). According to (13) in this case \( B = \dot{\omega} = -\sin t \), \( A = \cos^2 t \) and \( a = 0 \). For large velocities if \( B \neq 0 \) we can neglect terms involving \( A \) and the equation (41) gives for the time evolution of the variance of energy fluctuations

\[ \langle \delta E^2 \rangle = \kappa_\varsigma v_0^4 \int_0^t dt \sin^2 t = \kappa_\varsigma v_0^4 \left( \frac{t}{2} - \frac{\sin 2t}{4} \right). \]

We took \( 10^6 \) initial conditions at three different initial velocities \( v_0' = 50, 100, 200 \), at initial time \( t_0 = 0 \) and uniformly distributed on the remaining 3D phase space. Theoretical prediction (56) agrees with numerical results very well as shown in figure 1a. In all cases \( \kappa_\varsigma \) is approximately the same \( \kappa_\varsigma \approx 0.0135 \), as determined by the fitting procedure, thus it is indeed independent of the velocity. At large times we observe Fermi acceleration with \( \beta = 1/4 \) as discussed below.

The second example of the driving is a composition of scaling and rotation in such a way that that the angular momentum of the billiard is preserved \( (B = 0) \). The angular velocity is \( \omega = 1 + 0.2 \cos t \) and \( q = 1/\sqrt{\omega} \), such that \( \Gamma = \omega q^2 = 1 \) and \( B = q^2 \dot{\Gamma} = 0 \). In this case we use the equation (44). From (13) we find

\[ \dot{A} = -\frac{25}{8} \frac{(77 \sin t - 40 \sin 2t + 3 \sin 3t)}{(5 + \sin t)^3}. \]

FIG. 2. (Color online) A translating Sinai billiard. a) A tensor \( K \) in a polar representation \( \kappa(\phi) = a + b \sin 2\phi \), with \( a = 0.11014 \) and \( b = -0.09416 \) (red line through circles), as determined numerically (circles) by shaking the billiard in different directions \( \phi \) like \( z = 0.2 \cos (\cos \phi, \sin \phi) \) with \( v_0' = 100 \). For convenience, the billiard shape and its orientation with respect to the tensor is represented with the shaded area. b) A time evolution of \( \langle \delta E^2 \rangle \) for three different \( v_0' \) in a circularly translating Sinai billiard where the translation vector equals \( z = 0.5 (\cos t, \sin t) \). The solid lines are theoretical curves (no fitting) \( \langle \delta E^2 \rangle = 0.25 (a t - b \sin^2 t) / v_0' \).
which is used in (42) leading to
\[
\langle \delta E^2 \rangle = \frac{\kappa_v}{v_0^2} \int_0^t dt \frac{\dot{A}^2}{q^2}.
\]
(58)

The analytical expression of this integral is too complicated to be shown here, and it was evaluated numerically in practice as well. We took $10^6$ initial conditions at three different initial velocities $v_0' = 50, 100, 200$, at initial time $t_0 = 0$ and uniformly distributed on the remaining 3D phase space. A very good agreement with the theory is found as shown in figure 1b. In all cases $\kappa_\eta$ is approximately the same $\kappa_\eta \approx 0.0183$, as determined by the fitting procedure, thus it is indeed independent of the velocity. At large times we observe Fermi acceleration with $\beta = 1/6$ as discussed below.

We used the third driving to find the tensor $K$ defined in [39]. We translate the billiard back and forth as $z = (\cos \phi, \sin \phi) \cos t$ in 20 different directions $\phi$ spanning from $-\pi/4$ to $\pi/4$. According to [42] we have
\[
\langle \delta E^2 \rangle = \frac{\kappa(\phi)}{v_0^2} \int_0^t dt \sin^2 t = \frac{\kappa(\phi)}{v_0^2} \left( \frac{t}{2} - \sin \frac{2t}{4} \right),
\]
(59)

where
\[
\kappa(\phi) = (\cos \phi, \sin \phi)^T K (\cos \phi, \sin \phi).
\]
(60)

An ensemble of $10^6$ initial conditions at $t_0 = 0$ and $v_0' = 100$, uniformly distributed on the remaining phase space, evolved one period in time, was used to determine the evolution of $\langle \delta E^2 \rangle$. Measurements of $\kappa(\phi)$ are shown as small circles in a polar plot in figure 2a. Because of the symmetries of the billiard, tensor $K$ must be of the form
\[
K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}
\]
(61)

and
\[
\kappa(\phi) = a + b \sin 2\phi.
\]
(62)

Values of $a$ and $b$ were found by the best fit procedure. In figure 2a we see that this model describes data very well.

The last example of the driving is a circular translation of the center of mass of the billiard where $z = 0.5 (\cos t, \sin t)$, $q = 1$ and $\omega = 0$. Although the centre of mass of the billiard is rotating around the origin of the coordinate system, the billiard plane is not rotating, so the angular momentum of the billiard is zero and thus $B = 0$ and we expect a slower diffusion. From [42] and [61] we have
\[
\langle \delta E^2 \rangle = \frac{(a t - b \sin^2 t)}{4 v_0^2}.
\]
(63)

We took $10^6$ initial conditions at three different initial velocities $v_0' = 50, 100, 200$, at initial time $t_0 = 0$ and uniformly distributed on the remaining 3D phase space.

FIG. 3. The mean velocity versus the number of collisions and the acceleration exponents for two different drivings: rotation (crosses) with $\phi(t) = \sin t$ and translations (circles) with $z = (1/\sqrt{2}, -1/\sqrt{2}) \sin t$. In both we took ensemble of $10^6$ initial conditions at $v_0 = 0.1$, $t_0 = 0$ and uniformly distributed on the remaining phase-space. For very small velocities we expect $\beta = 1/2$. This regime is clearly visible in translating billiard (circles). At bigger velocities $\beta = 1/6$ in both cases. However, because in the case of rotation $B \neq 0$ the asymptotic value of $\beta$ eventually becomes $1/4$. The slopes of the lines have exact theoretical values without fitting.

Now without any fit, using the values of $a$ and $b$ computed previously, we find a very good agreement with the numerical results as shown in figure 2b.

Finally we calculated the acceleration exponents. We took two different drivings. One is a nonuniform rotation with $\phi(t) = \sin t$ and the other is a translation with $z = (1/\sqrt{2}, -1/\sqrt{2}) \sin t$. In the case of rotation $B \neq 0$ and we expect asymptotically $\beta = 1/4$, while in the case of translations $B = 0$ and we expect asymptotically $\beta = 1/6$. Numerical results shown in figure 3 confirm the expectations very well. Additionally, we can see the transient regimes where $\beta$ is different from asymptotic values. For small enough velocities we observe the regime where $\beta = 1/2$. This is because when the velocity is small the billiard undergoes many oscillations between the collisions and the successive velocity changes are effectively uncorrelated, leading to the random walk like process. In the rotating case we see that after the random walk like phase the system enters the intermediate regime where $\beta = 1/6$, which is because the asymptotically big term $B \dot{v}\zeta$ is still much smaller than $\dot{A}\eta$, regarding the equation (35). Eventually, for $n > 10^4$, $\beta$ becomes $1/4$ as predicted by the theory.

Finally, we should add that numerical calculations have been performed for various uniformly rotating billiards (Sinai billiard, elliptical billiard, Robnik billiard [21], oval billiard [7]) and the conservation law [29] has
been confirmed in double precision accuracy, which implies \( \beta = 0 \).

**VIII. CONCLUSIONS**

In time-dependent fully chaotic shape-preserving billiards the velocity dynamics is determined by the rotational properties of the billiard. If the transformation is such that the angular momentum of the billiard is preserved then the acceleration exponent is \( \beta = 1/6 \), except in the case where the only transformation is a uniform rotation and there is no acceleration, \( \beta = 0 \). On the other hand, if the transformation is such that the angular momentum of the billiard is not preserved then the acceleration exponent equals \( \beta = 1/4 \). These three values of \( \beta \) exhaust all possible values of the acceleration exponent in fully chaotic time-dependent billiards. However, if the structure of the phase space is more complicated, with the coexisting islands of regular motion, the acceleration exponents may differ from the prediction of the theory presented in this paper, simply because the assumption for the autocorrelation functions (37), (38) and (39) are not fulfilled. In this work we address only the fully chaotic billiards, while the results for the integrable ellipse and mixed type billiards in this context will be published elsewhere. The theory presented in this work complements the other more general theories of the velocity dynamics in time-dependent billiards [5, 22].

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