FROBENIUS SPLITTING, POINT-COUNTING, AND DEGENERATION

ALLEN KNUTSON

Abstract. Let \( f \) be a polynomial of degree \( n \) in \( \mathbb{Z}[x_1, \ldots, x_n] \), typically reducible but squarefree. From the hypersurface \( \{ f = 0 \} \) one may construct a number of other subschemes \( \{ Y \} \) by extracting prime components, taking intersections, taking unions, and iterating this procedure.

We prove that if the number of solutions to \( f = 0 \) in \( \mathbb{F}_p^n \) is not a multiple of \( p \), then all these intersections in \( \mathbb{A}^n_{\mathbb{F}_p} \) just described are reduced. (If this holds for infinitely many \( p \), then it holds over \( \mathbb{Q} \) as well.) More specifically, there is a Frobenius splitting on \( \mathbb{A}^n_{\mathbb{F}_p} \) compatibly splitting all these subschemes \( \{ Y \} \).

We determine when a Gröbner degeneration \( f_0 = 0 \) of such a hypersurface \( f = 0 \) is again such a hypersurface. Under this condition, we prove that compatibly split subschemes degenerate to compatibly split subschemes, and in particular, stay reduced.

Together these suggest that the number of \( \mathbb{F}_p \)-points on the general fiber \( Y \) and special fiber \( Y' \) of a Gröbner degeneration should, in good cases, differ by a multiple of \( p \). Under very special Gröbner degenerations (“geometric vertex decompositions”), we give a discontinuous injection of \( Y \) into \( Y' \) that lets us compare the relate their classes in the Grothendieck group of varieties, and thereby demonstrate this.

Our results are strongest in the case that \( f \)'s lexicographically first term is \( \prod_{i=1}^n x_i \). Then for all large \( p \), there is a Frobenius splitting that compatibly splits \( f \)'s hypersurface and all the associated \( \{ Y \} \). The Gröbner degeneration \( Y' \) of each such \( Y \) is a reduced union of coordinate spaces (a Stanley-Reisner scheme), and we give a result to help compute its Gröbner basis. We exhibit an \( f \) whose associated \( \{ Y \} \) include Fulton’s matrix Schubert varieties, and recover much more easily the Gröbner basis theorem of [Knutson-Miller ’05]. We show that in Bott-Samelson coordinates on an opposite Bruhat cell \( X_v \) in \( G/B \), the \( f \) defining the complement of the big cell also has initial term \( \prod_{i=1}^n x_i \), and hence the Kazhdan-Lusztig subvarieties \( \{ X_{w_0} \} \) degenerate to Stanley-Reisner schemes. This recovers, in a weak form, the main result of [Knutson ’08].

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1. INTRODUCTION, AND STATEMENT OF RESULTS

A commutative ring $\mathbb{R}$ is reduced if it has no nilpotents, i.e., if for $m > 1$ the map $r \mapsto r^m$ takes only 0 to 0. It is tempting to write this as $\ker(r \mapsto r^m) = 0$, and one may indeed do so if $m$ is a prime $p$ and $\mathbb{R}$ contains the field $\mathbb{F}_p$ of $p$ elements. Then the Frobenius map $r \mapsto r^p$ is $\mathbb{F}_p$-linear, and the condition of $\mathbb{R}$ being reduced says that this map has a one-sided inverse. This, for us, motivates the study of these inverses.

Define a (Frobenius) splitting $[\text{BrKu05}]$ of a commutative $\mathbb{F}_p$-algebra $\mathbb{R}$ as a map $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying three conditions:

- $\varphi(a + b) = \varphi(a) + \varphi(b)$
- $\varphi(ab) = a \varphi(b)$
- $\varphi(1) = 1$.

If $\varphi$ only satisfies the first two conditions (e.g. $\varphi \equiv 0$), we will call it a near-splitting$^{1}$. In section 2 we will recall from $[\text{BrKu05}$ section 1.3.1] the classification of near-splittings of affine space.

If $\mathbb{R}$ is equipped with a splitting $\varphi$, we will say $\mathbb{R}$ is split (not just “splittable”; we care about the choice of $\varphi$). Call an ideal $I \leq \mathbb{R}$ of a ring with a Frobenius (near-)splitting $\varphi$ compatibly (near-)split if $\varphi(I) \subseteq I$. For the convenience of the reader we recapitulate the basic results of Frobenius splitting we will use:

**Theorem.** $[\text{BrKu05}$ section 1.2] Let $\mathbb{R}$ be a Frobenius split ring with ideals $I, J$.

1. $\mathbb{R}$ is reduced.
2. If $I$ is compatibly split, then $I$ is radical, and $\varphi(I) = I$.
3. If $I$ and $J$ are compatibly split ideals, then so are $I \cap J$ and $I + J$. Hence they are radical.
4. If $I$ is compatibly split, and $J$ is arbitrary, then $I : J$ is compatibly split. In particular the prime components of $I$ are compatibly split.

Note that the sum of radical ideals is frequently not radical; “compatibly split” is a much more robust notion.

**Proof.** (1) Assume not, and let $r$ be a nonzero nilpotent with $m$ chosen largest such that $r^m \neq 0$ but $r^{m+1} = 0$. Let $s = r^m$. Then $0 = s^p$, so $0 = \varphi(s^p) = s$, contradiction.

(2) If $I$ is compatibly split, then $\varphi$ descends to a splitting of $\mathbb{R}/I$, so $\mathbb{R}/I$ is reduced. Equivalently, $I$ is radical. Since $I$ contains $\{i^p : i \in I\}$, one always has $\varphi(I) \supseteq I$.

(3) $\varphi(I \cap J) \subseteq \varphi(I) \cap \varphi(J) \subseteq I \cap J$. $\varphi(I + J) \subseteq \varphi(I) + \varphi(J)$ because $\varphi$ is additive.

(4) $r \in I : J \iff \forall j \in J, rj \in I \iff \forall j \in J, rj^p \in I \iff \forall j \in J, \varphi(rj^p) \in I$ (since $I$ is compatibly split) $\iff \forall j \in J, \varphi(r)j \in I \iff \varphi(r) \in I : J$.

\[\square\]

If $\varphi$ is a near-splitting such that $\varphi(1)$ is not a zero divisor, then parts 1,3,4 of this theorem still hold. Unlike splitting and near-splitting, this notion does not always pass to $\mathbb{R}/I$; the induced near-splitting $\varphi'$ on $\mathbb{R}/I$ may have $\varphi'(1)$ being a zero divisor, in which case part 2 of the theorem can fail.

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$^{1}$One might object that 0 is not so near to being a splitting. Such maps have also been called “$p^{-1}$-linear” and probably many other things.
Theorem 1. Let \( \mathfrak{F} \) be a compatibly split ideal in a Frobenius split ring. From it we can construct many more ideals, by taking prime components, sums, and intersections, then iterating. All of these will be radical.

It was recently observed \([\text{Schw}] [\text{KuMe}]\), and only a little harder to prove (a few pages, rather than a few lines), that a Noetherian split ring \( R \) has only finitely many compatibly split ideals. In very special cases the algorithm suggested in corollary \( \square \) finds all of them.

As we recall in section \( 2 \) there is a near-splitting on \( \mathbb{F}_p[x_1, \ldots, x_n] \) called \( \text{Tr} (\bullet) \) uniquely characterized by its application to monomials \( m \):

\[
\text{Tr} (m) = \begin{cases} 
\sqrt{m \prod_{i} x_i} / \prod_{i} x_i & \text{if } m \prod_{i} x_i \text{ is a pth power} \\
0 & \text{otherwise.}
\end{cases}
\]

The standard splitting of \( \mathbb{F}_p[x_1, \ldots, x_n] \) is \( \varphi(g) := \text{Tr} \left( (\prod_{i=1}^{n} x_i)^{p-1} g \right) \).

Lemma 1. The standard splitting is a Frobenius splitting, and the ideals that it compatibly splits are exactly the Stanley-Reisner ideals (meaning, those generated by squarefree monomials).

Occasionally we will need the near-splittings \( \text{Tr} (\bullet) \) defined on the coordinate rings of different affine spaces at the same time; in this case we will use subscripts to avoid confusion, e.g. \( \text{Tr}_H \) vs. \( \text{Tr}_{H \times L} \).

1.1. Point-counting over \( \mathbb{F}_p \) and Frobenius splitting. Our first result relates these.

Theorem 1. Let \( f \in \mathbb{F}_p[x_1, \ldots, x_n] \) be a polynomial of degree at most \( n > 0 \). Then the number of points \( \tilde{\varphi} \in \mathbb{F}_p^n \) in the affine hypersurface defined by \( f = 0 \) is congruent to \((-1)^{n-1} \text{Tr} (f^{p-1}) \).

If this number is not a multiple of \( p \), then some multiple of \( \text{Tr} (f^{p-1} \bullet) \) defines a Frobenius splitting on \( \mathbb{F}_p[x_1, \ldots, x_n] \), with respect to which \( (f) \) is compatibly split. (In this case \( \deg f \) is indeed \( n \), not less than \( n \), by the Chevalley-Warning theorem.)

If \( f = \prod_{i=1}^{m} f_i \) where each \( \deg f_i > 0 \), then the number of points \( \tilde{\varphi} \in \mathbb{F}_p^n \) in the subvariety defined by \( f_1 = f_2 = \ldots = f_m = 0 \) is congruent to \((-1)^{n-m} \text{Tr} (f^{p-1}) \).

In particular, if the number of points in the \( f = 0 \) hypersurface (or the \( f_1 = f_2 = \ldots = f_m = 0 \) subscheme, if \( f \) factors) is not a multiple of \( p \), then we can run the algorithm in corollary \( \square \) starting with the ideal \( (f) \), and produce only radical ideals.

If \( \deg f < n \), then one can use the Chevalley-Warning theorem to show that no multiple of \( \text{Tr} (f^{p-1} \bullet) \) defines a Frobenius splitting on \( \mathbb{F}_p[x_1, \ldots, x_n] \). On the other hand, if the hypersurface defined by \( f = 0 \) is smooth – regardless of \( \deg f \) – then there is some splitting of affine space that compatibly splits \( (f) \) \([\text{BrKu05}] \) proposition 1.1.6].

While we think that theorem \( \square \) provides an interesting link between point-counting and reducedness, we don’t have any real examples where the point-counting is the easiest way to demonstrate the splitting. Theorem \( 2 \) part (2) and especially theorem \( 4 \) provide more checkable sufficient conditions.

1.2. Frobenius splitting and degeneration. Given a weighting \( \lambda : \{1, \ldots, n\} \to \mathbb{N} \) on our variables \( \{x_i\} \), we can define the leading form \( \text{init}(f) \) of any polynomial \( f \) as the sum of the terms \( c \prod_{i} x_i^{e_i} \) with maximum \( \sum_{i} \lambda_i e_i \). It has a nice interpretation in terms of the Newton polytope of \( f \), which is defined as the convex hull of the exponent vectors of

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the monomials in \( f \); the weighting \( \lambda \) defines a linear functional on the space of exponent vectors, and it is maximized on one face \( F \) of \( f \)'s Newton polytope. “Take the exponent vector” is a map from the set of \( f \)'s terms to the Newton polytope, and \( \text{init}(f) \) is the sum of the terms lying over \( F \).

One can also define \( \text{init}(I) \) for any ideal \( I \leq \mathbb{R}[x_1, \ldots, x_n] \) (where \( \mathbb{R} \) is any base ring) as the \( \mathbb{R} \)-span of \( \{ \text{init}(f) : f \in I \} \). We mention that if \( \{ f_1, \ldots, f_m \} \subseteq I \) have the property that \( \{ \text{init}(f_i) \} \) generate \( \text{init}(I) \), then \( \{ f_1, \ldots, f_m \} \) is called a Gröbner basis for the pair \( (I, \lambda) \). We will not use much of the theory of Gröbner bases, but direct the interested reader to [Stu96].

**Lemma 2.** For any polynomial \( g \) and weighting \( \lambda \), \( \text{Tr}(\text{init}(g)) \) is either 0 or \( \text{init}(\text{Tr}(g)) \).

**Lemma 3.** If \( f = \text{init} f \), then for any subvariety \( Y \) compatibly split by \( \text{Tr}(f^{p-1} \bullet) \), we have \( Y = \text{init} Y \). In particular, if \( f \) is homogeneous, then \( Y \) is the affine cone over a projective variety and has a well-defined degree.

**Theorem 2.** Let \( f \in \mathbb{F}_p[x_1, \ldots, x_n] \) be of degree at most \( n \).

If \( \prod_x x_i \) is not in \( f \)'s Newton polytope, e.g. if \( \deg f < n \), then \( \text{Tr}(f^{p-1}) = 0 \). Hence no multiple of \( \text{Tr}(f^{p-1} \bullet) \) is a Frobenius splitting. Hereafter let \( \lambda \) be a weighting such that \( \prod_x x_i \) (or some \( \mathbb{F}_p \)-multiple) lies in \( \text{init}(f) \). In particular \( (1,1,\ldots,1) \) lies in \( f \)'s Newton polytope.

1. \( \text{Tr}(f^{p-1}) = \text{Tr}(\text{init}(f)^{p-1}) \), so (a multiple of) \( \text{Tr}(f^{p-1} \bullet) \) defines a Frobenius splitting iff (the same multiple of) \( \text{Tr}(\text{init}(f)^{p-1} \bullet) \) does.
2. Assume hereafter that some multiple of \( \text{Tr}(f^{p-1} \bullet) \) and \( \text{Tr}(\text{init}(f)^{p-1} \bullet) \) do define splittings. Let \( I \) be an ideal compatibly split with respect to the first splitting. Then \( \text{init}(I) \) is compatibly split with respect to the second splitting.
3. Let \( \mathcal{Y}_i \) and \( \mathcal{Y}_{\text{init} f} \) denote the poset of irreducible varieties compatibly split by \( \text{Tr}(f^{p-1} \bullet) \) and \( \text{Tr}(\text{init}(f)^{p-1} \bullet) \) respectively, partially ordered by inclusion. Then the map \( \pi_{\text{f,init}} : \mathcal{Y}_{\text{init} f} \to \mathcal{Y}_i \), \( Y' \mapsto \) the unique minimal \( Y \) such that \( \text{init} Y \supseteq Y' \) is well-defined, order-preserving, and surjective. Moreover, if \( Y_1 \in \mathcal{Y}_{\text{init} f}, Y_2 \in \mathcal{Y}_i, \) and \( Y_1 = \pi_{\text{f,init}}(Y_1) \), then \( Y_2 \supseteq Y_1 \iff \exists Y_2' \supseteq Y_1', \pi_{\text{f,init}}(Y_2') = Y_2 \) where \( Y_2' \in \mathcal{Y}_{\text{init} f} \), so the partial order on \( \mathcal{Y}_i \) is determined by that on \( \mathcal{Y}_{\text{init} f} \) plus the map \( \pi_{\text{f,init}} \).
4. Assume \( f \) is homogeneous, and let \( \mathcal{Y}_{\text{init} f}^{=\dim} := \{ Y' \in \mathcal{Y}_{\text{init} f} : \dim \pi_{\text{f,init}}(Y') = \dim Y' \} \). Then for any \( Y \in \mathcal{Y}_{\text{init} f}^{=\dim} \),

\[
\sum_{Y' \in \mathcal{Y}_{\text{init} f}^{=\dim}} \deg Y' = \deg Y.
\]

Some examples of the poset maps are given in figure [1]. Note that conclusion (2) runs the opposite direction of a standard principle, which is that for any ideal \( I \), if \( \text{init}(I) \) is radical, then \( I \) is radical.

For any polynomial \( f \) whose Newton polytope contains \( \prod_j x_j \), there is a unique minimal face of the polytope that contains it, and a corresponding minimal \( \text{init}(f) \) (minimal in number of terms). In this sense it is enough to study hypersurfaces \( f = 0 \) where \( \prod_j x_j \) lies in the interior of \( f \)'s Newton polytope.

One can also allow \( \lambda \) to take values in \( \mathbb{N}[\varepsilon], \) where \( \varepsilon \) is interpreted as infinitesimally positive (i.e. \( 1 > N_1 \varepsilon > N_2 \varepsilon^2 > \ldots > 0 \) for any \( N_1, N_2, \ldots \in \mathbb{N} \)) with which to break ties.
This doesn’t change any of the results; indeed, for any fixed $I$ and any such $\lambda$, there is a $\lambda'$ taking only $\mathbb{N}$-values with $\text{init}_\lambda(I) = \text{init}_{\lambda'}(I)$ [Stu96]. One sort of $\lambda$ that will often interest us is $\lambda = (0, \ldots, 0, 1, 0, \ldots, 0)$, which we may indicate by writing $\text{init}_i$ where the 1 is in the $i$th place.

In theorems [1] and [2] the interesting case is when $\deg f = n$, and there is little change if $f$ is replaced by its degree $n$ homogeneous component. (Indeed, this is the $\lambda = (1, 1, \ldots, 1)$ case of theorem [2].) Then $f = 0$ defines an anticanonical hypersurface of $\mathbb{P}^{n-1}$, so, when smooth, a Calabi-Yau hypersurface. In the $n = 3$ case, this is an elliptic curve, split for infinitely many $p$ (see e.g. [DaP99]). However, the hypersurfaces that interest us are typically highly reducible and in particular, singular.

Theorems [1] and [2] taken together show that certain Gröbner degenerations (meaning, replacements of $f$ by $\text{init}(f)$) of a hypersurface don’t change the number of $\mathbb{F}_p$-solutions, mod $p$. However, the number of solutions does indeed change. For example, $xy = 1$ has $p - 1$ solutions in $\mathbb{F}_p^2$, whereas $xy = 0$ has $2p - 1$.

**Figure 1.** The posets $Y_i$ (minus each one’s minimal element, $\{\vec{0}\}$) defined in theorem [2] part (3) for $f = xyz$ (left), $f = y(xz + y^2)$ (top), $f = z(xy + z^2)$ (bottom), and $f = xyz + y^3 + z^3$ (right) drawn as identifications of the lattice of faces of a 2-simplex. The maps between them come from evident choices of $\text{init}$.

### 1.3. Degeneration and point-counting over $\mathbb{F}_p$

We study a very special kind of degeneration in this section, that we called a geometric vertex decomposition in [KnMiY09, Kn].

Let $X \subseteq \mathbb{A}^n$ be reduced and irreducible, and split $\mathbb{A}^n$ as a Cartesian product $H \times L$, standing for Hyperplane and Line. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scaling the coordinate on $L$, i.e.

$$z \cdot (h, \ell) = (h, z\ell),$$

and define $X' := \lim_{t \to 0} t \cdot X$

using this action.

It is quite easy to determine the limit scheme $\overline{X}$ as a set. Let $\Pi \subseteq H$ be the closure of the image of the projection of $X$ to $H$, let $\overline{X}$ be the closure of $X$ inside $H \times (L \cup \{\infty\})$, and

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This is defined in the usual way, as the zero fiber of the closure of $\bigcup_{z \in \mathbb{G}_m} ((z) \times (z \cdot X)) \subseteq \mathbb{A}^1 \times (H \times L)$. One can also consider the limit branchvariety as in [Kn], but this will coincide with the limit scheme under conditions (1) or (2) in the theorem.
define \( \Lambda \subseteq H \) by \( \Lambda \times \{ \infty \} := \overline{X} \cap (H \times \{ \infty \}) = \overline{X} \setminus X \). Then
\[
X' = (\Pi \times \{0\}) \cup (\Lambda \times L)
\]
as a set [KnMiY09, theorem 2.2].

Though it was not pointed out in [KnMiY09], none of this changes if \( H \) is allowed to be an arbitrary scheme \( H' \) (though \( L \) must remain \( \mathbb{A}^1 \)). One can temporarily replace \( H' \) by an affine patch \( U \) embedded as a closed subset of an affine space \( H \), and \( X \) by \( X \cap (U \times L) \), then apply the theorems; the resulting statements then glue together to give the one for \( X \subseteq H' \times L \).

**Theorem 3.** Assume that \( X \subseteq H \times L, \Pi, \Lambda, \overline{X}, X' \) are as above. Assume one of the following:

1. \( X \) is irreducible, \( \Pi \) is normal, the projection \( X \to \Pi \) is degree 1, and \( \Lambda \) is reduced,
2. \( X' \) is reduced, or
3. the fibers of \( \overline{X} \to \Pi \) are connected.

(In fact (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) the projection \( \overline{X} \to \Pi \) is degree 1 or 0 to any component.) Let
\[
\Lambda' := \{ h \in H : \{ h \} \times L \subseteq X \}.
\]
Then \( \Pi \supseteq \Lambda \supseteq \Lambda' \), and the image of the projection \( X \to \Pi \) is \( \Pi \setminus \Lambda \cup \Lambda' \).

There is a decomposition \( X = (X \setminus (\Lambda' \times L)) \bigsqcup (\Lambda' \times L) \), where \( \pi \) gives an isomorphism of the first piece with \( \Pi \setminus \Lambda \). Consequently,
\[
[X'] = [X] + [\mathbb{A}^1] \cdot [\Lambda \setminus \Lambda']
\]
as elements of the Grothendieck ring of algebraic varieties. In particular, if \( X, H, L \) are defined over \( \mathbb{F}_p \), and \( |A| \) denotes the number of \( \mathbb{F}_p \)-rational points on \( A \), then
\[
|X'| = |X| + p |\Lambda \setminus \Lambda'| \equiv |X| \mod p.
\]

There is a constructible injection \( \iota : X \to X' \) defined by
\[
\iota(h, \ell) := \begin{cases} 
(h, \ell) & \text{if } h \in \Lambda' \\
(h, 0) & \text{if } h \notin \Lambda'
\end{cases}
\]
whose image is the complement of \( (\Lambda \setminus \Lambda') \times L \).

If \( X' \) is reduced, then the proper map \( \overline{X} \to \Pi \) takes \( [\mathcal{O}_X] \mapsto [\mathcal{O}_\Pi] \) as elements of K-theory (or even \( G \)-equivariant K-theory, if some group \( G \) acts on \( H \) and linearly on \( L \), preserving \( X \subseteq H \times L \) and hence also \( \Pi \subseteq H \)).

The \( \dim H = 1 \) example of \( X = \{ h^2 \ell = 1 \} \) with \( p - 1 \) points, degenerating to \( X' = \{ h^2 \ell = 0 \} \) with \( 2p - 1 \) points, satisfies (3) and shows that \( X' \) need not be reduced for this theorem. However, that is the condition we will make use of. In section 4 we give an example with \( \Lambda' \neq \emptyset \).

**Proposition 1.** Let \( f \in \mathbb{F}[h_1, \ldots, h_{n-1}, \ell] \) be of degree \( n \), of the form \( f = \ell g_1 + g_2 \) where \( \ell / g_1, g_2 \). Let \( (h_i, \ell) \) be the coordinates on \( H, L \). If \( \text{Tr}_{H(\mathbb{F}_p^{p-1})} \) defines a Frobenius splitting on \( H \), then \( \text{Tr}_{H \times L(\mathbb{F}_p^{p-1})} \) defines one on \( H \times L \). Assume this hereafter.

Let \( X \subseteq H \times L \) be a subscheme compatibly split by \( \text{Tr}_{H(\mathbb{F}_p^{p-1})} \), and let \( \Lambda' \subseteq \Lambda \subseteq \Pi \subseteq H \) be as above. Then theorem 3 applies. Moreover, \( \Pi \) and \( \Lambda \) are compatibly split by \( \text{Tr}_{H(\mathbb{F}_p^{p-1})} \), though \( \Lambda' \) may not be.
Proof. The first statement follows from theorem \(2\) part (1). Its part (3) implies that \(X'\) is compatibly split by \(\text{Tr}_{H \times L}((g_1)^{p-1} \bullet)\), so it is reduced, giving condition (2) of theorem \(3\).

Being reduced, \(X' = (\Pi \times \{0\}) \cup (\Lambda \times L)\), hence \(\Pi \times \{0\}\), \(\Lambda \times L\), and \(\Lambda \times \{0\} = (\Pi \times \{0\}) \cap (\Lambda \times L)\) are compatibly split too. It follows that \(\Pi, \Lambda\) are compatibly split by \(\text{Tr}_{H}((g_1)^{p-1} \bullet)\).

The first example in section \(5\) is of this type, and its \(\Lambda'\) is not compatibly split in \(H\).

Applying proposition \(1\) to \(X = \{f = 0\}\) itself, we see the interrelation between theorems \(1\), \(2\), and \(3\). Theorem \(3\) says that \(X\) and \(X'\) have the same number of solutions, mod \(p\). Applying theorem \(1\), we see that \(\text{Tr}(f^{p-1} \bullet)\) defines a Frobenius splitting iff \(\text{Tr}(\text{init}(f)^{p-1} \bullet)\) does. This gives an independent proof of theorem \(2\) part (1) in this special situation.

1.4. An important special case, which generalizes to schemes. In this section we ask that \(\text{init}(f) = \prod_i x_i\). This condition is a very restrictive one on degree \(n\) polynomials; for example the hypersurface defined by \(f = 0\) is necessarily singular. But there are some important examples that have this property, and our results are strongest here.

Theorem 4. Let \(f \in \mathbb{Z}[x_1, \ldots, x_n]\) be a degree \(k\) polynomial whose lexicographically first term is (a \(\mathbb{Z}\)-multiple of) a product of \(k\) distinct variables.

Let \(Y\) be one of the schemes constructed from the hypersurface \(f = 0\) by taking components, intersecting, taking unions, and repeating. (Or more generally, let \(Y\) be compatibly split with respect to the splitting \(\text{Tr}(f^{p-1} \bullet)\).) Then \(Y\) is reduced over all but finitely many \(p\), and over \(\mathbb{Q}\).

Let \(\lambda\) is the lexicographic weighting \((\epsilon, \epsilon^2, \ldots, \epsilon^n)\) on the variables. Let \(\text{init}Y\) be the initial scheme of \(Y\). Then (away from those \(p\) \(\text{init}Y\) is a Stanley-Reisner scheme. There is a bijective constructible map \(\Lambda^n\) to itself, taking each \(Y\) into its \(\text{init}Y\).

We thank Bernd Sturmfels for his guess that \(\text{init}Y\) might be a Stanley-Reisner ideal, as a way of understanding corollary \(4\) without direct reference to Frobenius splitting. It would be interesting to know if \(\text{init}Y\) is reduced over \(\mathbb{Z}\) not just \(\mathbb{Q}\), as holds \([KnMi05, Kn08]\) for the examples in section \(7\).

It is tempting to pull back the standard paving of \(\mathbb{A}^n\) by tori (one for each \(S \subseteq \{1, \ldots, n\}\), defined by the equations \(x_i = 0\) iff \(i \in S\)) to try to get a paving of each \(Y\), as in \([De85]\). This works as long as each \(\Lambda'\) occurring in this succession of degenerations is compatibly split, but as mentioned, they may not be.

Under this \(\text{init}f = \prod x_i\) condition, one can use lemma \(1\) and theorem \(2\) to bound the number of \(k\)-dimensional compatibly split subvarieties by \((\binom{n}{k})\), as in \([SchwT]\) (where they prove this bound without assuming \(\text{init}f = \prod x_i\)). If \(f\) is homogeneous, then theorem \(2\) part (4) lets one show that \((\binom{n}{k})\) also bounds the sum of the projective degrees of the \(k\)-dimensional compatibly split subvarieties.

In section \(7\) we apply theorem \(4\) to the general cases \(n = 2, n = 3\), and to two specific stratifications; the stratification of the space of matrices by matrix Schubert varieties, and of opposite Bruhat cells by Kazhdan-Lusztig varieties. To do this, we need the new result (theorem \(7\)) that with respect to Bott-Samelson coordinates on an opposite Bruhat cell, the complement of the big cell is given by an equation \(f\) with \(\text{init}f = \prod_i \epsilon_i\).

Part of this result has a generalization beyond affine space to schemes, where it is closely related to a result of \([BrKu05]\).
Theorem 5. Let $X$ be a normal variety of dimension $n$, with $\sigma \in H^0(X_{\text{reg}}, \omega^{-1})$ a section of the anticanonical bundle over the regular locus $X_{\text{reg}}$. Let $x \in X_{\text{reg}}$ have local coordinates $t_1, \ldots, t_n$, where the formal expansion of $\sigma$ at $x$ is

$$\sigma = f(t_1, \ldots, t_n) \left( dt_1 \wedge \cdots \wedge dt_n \right)^{-1}.$$  

1. (From the proof of [BrKu05, proposition 1.3.11].)  
If $X$ is complete, and the unique lowest-order term of $\sigma$ is $\prod_{i=1}^n t_i$, then there exists a unique Frobenius splitting of $X$ that compatibly splits the divisor $\{\sigma = 0\}$. In particular, if $\{\sigma = 0\}$ has $n$ components smooth at $x$ and meeting transversely there, the coordinates $\{t_i\}$ can be chosen to ensure this condition on $\sigma$.

2. If the initial term of $\sigma$ is $\prod_{i=1}^n t_i$ for some term order, then there exists a Frobenius splitting of $X$ that compatibly splits the divisor $\{\sigma = 0\}$. If $X$ is complete, then the splitting is unique.

In proposition 4 we give an application of this to Brion’s “multiplicity-free subvarieties of $G/B$”: if $X$ is a multiplicity-free divisor, then $G/B$ possesses a Frobenius splitting compatibly splitting $X$.

1.5. Application to Gröbner bases. In a finite poset $P$, call an element $p$ basic if $p$ is not the unique greatest lower bound of $\{q \in P : q > p\}$. It is then trivial to prove ([LaSchü96, GK97], where they also determine the basic elements in Bruhat orders) that any $p$ is the greatest lower bound of $\{q \in P : q \geq p, q \text{ basic}\}$.

Theorem 6. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a degree $n$ polynomial with $\text{init } f = \prod_{i=1}^n x_i$. Let $\mathcal{Y}_i$ be a poset of compatibly split subvarieties with respect to the splitting $\text{Tr}(f^p - 1 \bullet)$ (here $p$ varies over some infinite set of primes), ordered by inclusion. (It need not be all of them.) Then over the rationals:

1. Any $\mathcal{Y} \in \mathcal{Y}_i$ has a Gröbner basis $(g_i)$ over whose initial terms $(\text{init } g_i)$ are squarefree monomials.
2. Any $\mathcal{Y} \in \mathcal{Y}_i$ is the scheme-theoretic intersection of $\{Z \in \mathcal{Y}_i : Z \geq \mathcal{Y}, Z \text{ basic in } \mathcal{Y}_i\}$, and of course it suffices to use only the minimal elements.
3. If we concatenate Gröbner bases of the minimal $\{Z \in \mathcal{Y}_i : Z \geq \mathcal{Y}, Z \text{ basic in } \mathcal{Y}_i\}$, we get a Gröbner basis of $\mathcal{Y}$.

Indeed (2) holds for $\mathcal{Y}$ a set of compatibly split subvarieties in any split scheme, and (3) holds whenever $\text{Tr}(\text{(init } f)^{p-1} \bullet)$ is a splitting.

As any single polynomial forms a Gröbner basis, we see that a concatenation of Gröbner bases is usually not a Gröbner basis. The special geometry of our situation is explained in lemma 6. In section 7.2 we use theorem 6 to recover the main results of [F92, KnMi05].

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2. Near-splittings of affine space and its ideals

Following [BrKu05, Section 1.3.1], we describe all the near-splittings on \( \mathbb{F}[x_1, \ldots, x_n] \), where \( \mathbb{F} \) is a perfect field over \( \mathbb{F}_p \). (In [BrKu05] they assume in general that \( \mathbb{F} \) is algebraically closed, but make no use of this in that section.)

**Proposition 2.** [BrKu05, section 1.3.1] Let \( \mathbb{F} \) be a perfect field over \( \mathbb{F}_p \). Then there exists a unique near-splitting \( \text{Tr}(\bullet) \) on \( \mathbb{F}[x_1, \ldots, x_n] \) such that for each monomial \( m = \prod_i x_i^{e_i} \),

\[
\text{Tr}(m) = \begin{cases} \sqrt{m} \prod_i x_i^{e_i} / \prod_i x_i & \text{if } m \prod_i x_i \text{ is a } p\text{th power} \\ 0 & \text{otherwise.} \end{cases}
\]

For each \( f \in \mathbb{F}[x_1, \ldots, x_n] \), the map \( \text{Tr}(f\bullet) : g \mapsto \text{Tr}(fg) \) is a near-splitting, and the association \( f \mapsto \text{Tr}(f\bullet) \) is a bijection from \( \mathbb{F}[x_1, \ldots, x_n] \) to the set of near-splittings.

Hereafter we will assume that \( \mathbb{F} \) is a perfect field over \( \mathbb{F}_p \), and the near-splittings we will consider will all be of the form \( c\text{Tr}(f^{p-1}\bullet) \) for some \( f \in \mathbb{F}[x_1, \ldots, x_n] \) and \( c \in \mathbb{F} \).

**Lemma 4.** Let \( f \in \mathbb{F}[x_1, \ldots, x_n] \).

1. If \( \text{Tr}(f^{p-1}) \) is a unit, and \( c \) is its inverse, then \( c \text{Tr}(f^{p-1}) \) is a splitting of \( \mathbb{F}[x_1, \ldots, x_n] \).
2. The principal ideal \( (f) \) is compatibly near-split by \( \text{Tr}(f^{p-1}\bullet) \), and compatibly split by \( c\text{Tr}(f^{p-1}\bullet) \) from part (1) if \( c \text{Tr}(f^{p-1}) = 1 \).

**Proof.**

1. \( c\text{Tr}(f^{p-1}\bullet) = \text{Tr}(c^p f^{p-1}\bullet) \), hence is a near-splitting, and the remaining condition that \( c \text{Tr}(f^{p-1}\bullet) = 1 \) is how we chose \( c \).
2. If \( rf \in (f) \), then \( \text{Tr}(f^{p-1}rf) = f\text{Tr}(r) \in (f) \); likewise \( c\text{Tr}(f^{p-1}rf) = fc\text{Tr}(r) \in (f) \).

We were tempted to generalize the definition of splitting by allowing \( \phi(1) \) to be a unit rather than actually 1. This would make some theorems nicer to state, but did not seem worth the confusion to people familiar with the usual definition.

We will later be interested in near-splittings \( \text{Tr}(f\bullet) \) on \( R[x_1, \ldots, x_n] \) where \( R \) is a certain perfect ring over \( \mathbb{F}_p \), meaning that the Frobenius map \( R \to R \) is bijective. (We won’t need to generalize the results of this section, though their proofs from [BrKu05, section 1.3.1] go through without change.) It is easy to show that such rings are Noetherian only when they are fields.

3. Proof of Theorem

**Lemma 5.** Let \( f \in \mathbb{F}[x_1, \ldots, x_n] \) be of degree at most \( n \) over a perfect field \( \mathbb{F} \). Then \( \text{Tr}(f^{p-1}) \) is the \( p \)-th root of the coefficient on \( \prod_i x_i^{p-1} \) in \( f^{p-1} \).

**Proof.** The only monomials noticed by \( \text{Tr}(\bullet) \) are of the form \( mp \prod_i x_i^{p-1} \), where \( m \) is itself a monomial, and so have degree \( p \deg m + (p-1)n \). By the assumption on \( \deg f \), its power \( f^{p-1} \) can’t contain such a monomial other than the one for \( m = 1 \).

For \( n = 3 \), the following is a standard argument from the theory of supersingular elliptic curves, and was studied for hypersurfaces in [Ka72, example 2.3.7.17].
Proof of theorem \( \square \) when \( f \) doesn’t factor. First observe that for any \( a \in \mathbb{F}_p \),

\[
1 - a^{p-1} \equiv \begin{cases} 
1 & \text{if } a = 0 \\
0 & \text{if } a \neq 0
\end{cases}
\]

by Fermat’s Little Theorem. If we let \( f^{p-1} = \sum c_e \prod_{i=1}^n x_i^{e_i} \), then

\[
\# \{ \bar{v} \in \mathbb{F}_p^n : f(\bar{v}) = 0 \} = \sum_{\bar{v} \in \mathbb{F}_p^n} (1 - f(\bar{v})^{p-1}) = p^n - \sum_{\bar{v} \in \mathbb{F}_p^n} f(\bar{v})^{p-1} \equiv - \sum_{\bar{v} \in \mathbb{F}_p^n} f(\bar{v})^{p-1} \mod p
\]

\[
= - \sum_{\bar{v} \in \mathbb{F}_p^n} \sum_{\bar{e}} c_e \prod_{i=1}^n v_i^{e_i} = - \sum_{\bar{e}} c_e \sum_{\bar{v} \in \mathbb{F}_p^n} \prod_{i=1}^n v_i^{e_i} = - \sum_{\bar{e}} c_e \sum_{i=1}^n \prod_{\bar{v} \in \mathbb{F}_p^n} v_i^{e_i}.
\]

Now consider the sum \( \sum_{\bar{v} \in \mathbb{F}_p^n} v^e \). If \( e = 0 \), this is \( p \cdot 1 \equiv 0 \). If \( (p-1)|e \) and \( e > 0 \), this is \( 0 + (p-1) \cdot 1 \equiv -1 \). Otherwise let \( b \) be a generator of \( \mathbb{F}_p^\times \) so \( b^e \neq 1 \), and observe that \( \sum_{\bar{v} \in \mathbb{F}_p^n} (bv)^e = b^e \sum_{\bar{v} \in \mathbb{F}_p^n} v^e \) is just a rearrangement of \( \sum_{\bar{v} \in \mathbb{F}_p^n} v^e \), so \( \sum_{\bar{v} \in \mathbb{F}_p^n} v^e = 0 \). Omitting zero terms from the sum, we have

\[
\# \{ \bar{v} \in \mathbb{F}_p^n : f(\bar{v}) = 0 \} \equiv - \sum_{\bar{e}, \forall i, e_i > 0, |e_i|} c_e \prod_{i=1}^n (-1)
\]

At this point the only terms entering have each \( e_i \geq p-1 \), so \( \sum_i e_i \geq (p-1)n \). But that is \( \geq \deg(f^{p-1}) \). So (as in the proof of lemma \( \square \)) the only \( \bar{e} \) has \( e_i = p-1 \forall i \).

On the other hand,

\[
\text{Tr} (f^{p-1}) = \text{Tr} \left( \sum_{\bar{e}} c_e \prod_{i=1}^n x_i^{e_i} \right) = \sum_{\bar{e}} \text{Tr} \left( c_e \prod_{i=1}^n x_i^{e_i} \right) = \sum_{\bar{e}} c_e \text{Tr} \left( \prod_{i=1}^n x_i^{e_i} \right)
\]

where the last step uses the fact that each \( c_e \) is in the prime field. The term \( \text{Tr} (\prod_{i=1}^n x_i^{e_i}) \) is 0 unless each \( e_i \geq p-1 \), so degree-counting as before, the only term that survives is \( c_{p-1,p-1,...,p-1} \). Combining, we get \( \# \{ \bar{v} \in \mathbb{F}_p^n : f(\bar{v}) = 0 \} \equiv (-1)^n \text{Tr} (f^{p-1}) \). \( \square \)

One can see from the proof that if \( \deg f < n \), then \( \text{Tr} (f^{p-1}) = 0 \), so \( \# \{ \bar{v} : f(\bar{v}) = 0 \} \) is a multiple of \( p \). This can be generalized (via a very similar proof) as follows:

**Theorem** (Chevalley-Warning). Let \( \{f_i\} \) be a set of polynomials in \( \mathbb{F}[x_1, \ldots, x_n] \), \( \mathbb{F} \) a finite field of characteristic \( p \), such that \( \deg(\prod_i f_i) < n \). Then \( \# \{ \bar{v} \in \mathbb{F}^n : f_i(\bar{v}) = 0 \forall i \} \) is a multiple of \( p \).

To have Frobenius splittings, we want \( \text{Tr} (f^{p-1}) \neq 0 \), so we will want our polynomial \( f \) to have degree \( n \). The Chevalley-Warning theorem is useful to us all the same, as in examples we will often want \( f \) to factor.

**Proof of the remainder of theorem \( \square \) when \( f \) factors.** Let \( X_i = \{ \bar{v} : f_i(\bar{v}) = 0 \} \), so the left side is \( \# (\bigcup_i X_i) \). Inclusion-exclusion says

\[
\# \left( \bigcup_i X_i \right) = \sum_{S \subseteq \{1, \ldots, m\}, S \neq \emptyset} (-1)^{|S|-1} \# \left( \bigcap_{i \in S} X_i \right) = \sum_{S \neq \emptyset} (-1)^{|S|-1} \# \{ \bar{v} \in \mathbb{F}^n : \forall i \in S, f_i(\bar{v}) = 0 \}.
\]
For each proper subset $S \subseteq \{1, \ldots, m\}$, and by the assumption that the $\{f_i\}$ are nonconstant, the Chevalley-Warning theorem applies to $\{\bar{v} \in \mathbb{F}^n : \forall i \in S, f_i(\bar{v}) = 0\}$. So mod $p$, only the $S = \{1, \ldots, m\}$ term survives:

$$\# (\bigcup_i X_i) \equiv (-1)^{m-1} \# \{\bar{v} \in \mathbb{F}^n : \forall i, f_i(\bar{v}) = 0\}. \quad \square$$

It is also easy to see from the proof of theorem 1 that there is no easy relation between point-counting and $\text{Tr} (f^{p^e})$ if $\deg f > n$. The point count draws focus on exponents $e_i > 0$ with $(p-1)|e_i$, whereas $\text{Tr} (f^{p^e})$ is concerned with exponents $e_i$ with $e_i \equiv p-1 \mod p$. These match up well only if degree considerations force $e_i = p-1$.

Put another way, if $\deg f > n$ and $\text{Tr} (f^{p^e})$ is a unit (which can only happen if $f$ is inhomogeneous), then the splendid geometric consequences of Frobenius splitting hold but are not detected by point-counting.

4. PROOFS OF LEMMA 2 AND THEOREM 2

Proof of theorem 2 part (1). If $\prod_i x_i$ is not in $f$’s Newton polytope, then $\prod_i x_i^{p-1}$ is not in $f^{p^e}$’s Newton polytope. Hence by lemma 5, $\text{Tr} (f^{p^e}) = 0$.

Since $\prod_i x_i$ lies in $\text{init}(f)$, we know $\prod_i x_i^{p-1}$ lies in $\text{init}(f)^{p^e} = \text{init}(f^{p^e})$. So the coefficient on $\prod_i x_i^{p-1}$ in $f^{p^e}$ is the same as its coefficient in $\text{init}(f)^{p^e}$. Now apply lemma 5 to infer $\text{Tr} (f^{p^e}) = \text{Tr} (\text{init}(f)^{p^e})$. \quad \square$

To study the degeneration, it will be convenient to introduce the perfect base ring $R = \mathbb{F}_p[t^e : e \in \mathbb{Q}_+]$ of Puiseux polynomials. Define the ring endomorphism $h_\lambda$ of $R[x_1, \ldots, x_n]$ by

$$h_\lambda(x_i) = x_i t^{\lambda_i}, \quad \text{so} \quad (h_\lambda \cdot g)(x_1, \ldots, x_n) = g(x_1 t^{\lambda_1}, \ldots, x_n t^{\lambda_n}).$$

(where $h$ is for “homogenize”). Then $\text{init}(g)$ is the part of $h_\lambda(g)$ with highest $t$-degree.\footnote{Perhaps it would be more natural to take $x_i \mapsto x_i t^{\lambda_i}$, and clear denominators by multiplying by some $t^M$, so $\text{init}(g)$ would be $(t^M h_\lambda(g))|_{t=0}$. It didn’t seem to be worth keeping track of an extra sign, however.}

Note that $t$ is considered part of the base ring and not a new variable, for purposes of defining $\text{Tr} (\bullet)$ on $R[x_1, \ldots, x_n]$; in particular $\text{Tr} (f(t) g) = f(t^{1/p}) \text{Tr} (g)$ for any $f(t) \in R$.

Proof of lemma 2 First we prove

$$\text{Tr} (h_\lambda(g)) = h_\lambda(\text{Tr} (g)) t^{\frac{p-1}{p} \sum_1 \lambda_i}$$

for $g \in \mathbb{F}_p[x_1, \ldots, x_n]$, i.e. when $g$ has no $t$-dependence. Both sides are additive, so it is enough to check for $g = c \prod_i x_i^{e_i}$, $c \in \mathbb{F}_p$. If each $e_i \equiv -1 \mod p$, then

$$\text{Tr} \left( h_\lambda(c \prod_i x_i^{e_i}) \right) = \text{Tr} \left( c \prod_i (x_i t^{\lambda_i})^{e_i} \right) = \text{Tr} \left( t^{e_i \lambda_i} c \prod_i x_i^{e_i} \right) = t^{\frac{p-1}{p} \sum_1 \lambda_i e_i} \text{Tr} \left( \prod_i x_i^{e_i} \right)$$

$$= t^{\frac{p-1}{p} \sum_1 \lambda_i e_i} c \prod_i x_i^{(e_i+1)/p-1} = \left( c \prod_i x_i^{(e_i+1)/p-1} \right) = t^{\frac{p-1}{p} \sum_1 \lambda_i} h_\lambda \left( \text{Tr} \left( c \prod_i x_i^{e_i} \right) \right)$$

and both sides are zero otherwise. This proves the equation.
Now let $g$ be general, and consider $g$’s Newton polytope $P$. $\text{Tr}(\bullet)$ and $\text{init}$ are sensitive to different parts of $g$’s Newton polytope: $\text{Tr}(g)$ only depends on the terms lying on the intersection of $P$ with a coset $C$ of a lattice (namely, where all exponents are $\equiv -1 \mod p$), whereas $\text{init}(g)$ only depends on the terms lying over one face $F$ of $P$.

There are then two cases. If some terms of $g$ lie over $F \cap C$, then we can pick out the terms lying over $F$, and from those pick out the terms also lying over $C$, or do so in the opposite order. Either way we pick up the terms lying over $F \cap C$, and apply $\text{Tr}(\bullet)$ to them, obtaining $\text{Tr}(\text{init}(g)) = \text{init}(\text{Tr}(g))$.

The other possibility is that no terms lie over $F \cap C$ (e.g. if $F \cap C = \emptyset$). Then $\text{init}(g)$ picks out the terms lying over $F$, and $\text{Tr}(\text{init}(g)) = 0$. □

Proof of part (2). The ideal $\text{init}(I)$ is linearly generated by $\{\text{init}(g) : g \in I\}$. By lemma 2

$$\text{Tr}(\text{init}(f)^p - 1 \text{init}(g)) = \text{Tr}(\text{init}(f^{p-1}g)) = \left(\text{init}\left(\text{Tr}(f^{p-1}g)\right)\text{ or } 0\right) \in \text{init}(I)$$

so $\text{init}(I)$ is compatibly near-split by $\text{Tr}(\text{init}(f)^{p-1}\bullet)$.

□

We give now a criterion which may be of independent interest, guaranteeing that the limit of an intersection is the entire intersection of the limits.

**Lemma 6.** Let $A$ be a discrete valuation ring with parameter $t$, so $S = \text{Spec } A$ has one open point $S^\times$ and one closed point $S_0$, and let $F$ be a flat family over $\text{Spec } A$. Let $X, Y$ be two reduced flat subfamilies, and assume that the special fiber $(X \cup Y)_0$ of their union is reduced.

Then $X \cap Y$ is the closure of $X^\times \cap Y^\times$; it has no components lying entirely in the special fiber. In particular $(X^\times \cap Y^\times)_0 = X_0 \cap Y_0$.

**Proof.** Consider two gluings of $X$ to $Y$, along their common subschemes $\overline{X^\times \cap Y^\times} \hookrightarrow X \cap Y$:

$$(X \bigcup Y) / \overline{X^\times \cap Y^\times} \rightarrow (X \bigcup Y) / (X \cap Y) \cong X \cup Y \subseteq F.$$

Call this map $\pi : Z_1 \twoheadrightarrow Z_2$. It is finite, and an isomorphism away from $t = 0$, and $Z_1, Z_2$ are reduced, so Fun($Z_1$) is integral inside Fun($Z_2$)[t$^{-1}$].

If $X^\times \cap Y^\times \neq X \cap Y$, so Fun($Z_1$) $\neq$ Fun($Z_2$), then there exists $r \in$ Fun($Z_2$) such that $r/t \in$ Fun($Z_1$) \ Fun($Z_2$). By the integrality, $r/t$ satisfies a monic polynomial of degree $m$ with coefficients in Fun($Z_2$). Hence $r^m \equiv 0 \mod t$, but $r \not\equiv 0 \mod t$ (since $r/t \notin$ Fun($Z_2$)), so Fun($Z_2$)/⟨t⟩ = Fun((X $\cup$ Y)$_0$) has nilpotents, contrary to assumption. □

Using the branchvariety framework of [AKn09] (from whose lemma 2.1(1) this proof has been copied), one can analyze the situation when the “limit scheme” $(X \cup Y)$_0 of $X^\times \cap Y^\times$ is not assumed reduced. Then the zero fiber of $(X \bigcup Y) / \overline{X^\times \cap Y^\times}$ is the “limit branchvariety” of $X^\times \cap Y^\times$, which maps to the limit scheme $(X \cup Y)$_0. In [AKn09] we prove that limit branchvarieties are unique, hence this map is an isomorphism iff $(X \cup Y)$_0 is reduced.

**Corollary 2.** Let $S = \{I\}$ be a finite set of polynomial ideals in $\mathbb{F}[x_1, \ldots, x_n]$ such that for any $S' \subseteq S$, $\text{init}\bigcap_{I \in S} I$ is radical. Then $\text{init}(\sum_{I \in S} I) = \sum_{I \in S} \text{init} I$.

Put another way, for each $I \in S$, let $G_I$ be a Gröbner basis for $I$. Then $\bigcup_{I \in S} G_I$ is a Gröbner basis for $\sum_{I \in S} I$.  

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Proof. For any $S' \subseteq S$, the conditions apply to $S'$, so using induction we can reduce to the case $S = \{I_1, I_2\}$.

Let $F$ be the trivial family over $\mathbb{P}[[t]]$ with fiber $\mathbb{A}^n$, and $X, Y \subseteq F$ be the Gröbner families whose general fibers are defined by $I_1, I_2$ and special fibers by $\text{init} I_1, \text{init} I_2$. Then $\text{init}(I_1 + I_2)$ and $\text{init} I_1 + \text{init} I_2$ are the defining ideals of $(X^* \cap Y^*)_0$ and $X_0 \cap Y_0$ respectively. The condition $\text{init}(I_1 \cap I_2)$ radical allows us to invoke lemma [6] to infer these are equal.

A very similar result appears in [BoJSpStuT07, lemma 3.2].

Proof of part (3). Plainly any $Y' \in \mathcal{Y}_{\text{init}, f}$ has some $Y \in \mathcal{Y}_f$ such that $\text{init} Y \supseteq Y'$; take $Y = \mathbb{A}^n$. We first need to show there is a unique minimal such $Y$. By corollary [2] if $Y' \subseteq \text{init} Y_1$ and $Y' \subseteq \text{init} Y_2$, then $Y' \subseteq \text{init}(Y_1 \cap Y_2) = \text{init} \bigcup Z Z = \bigcup Z \text{init} Z$ where $Z$ ranges over the components (all compatibly split) of the $Y_1 \cap Y_2$. Hence $Y' \subseteq \text{init} Z$ for one of those $Z$. If $Y_1, Y_2$ were both minimal, then $Z = Y_1$ and $Z = Y_2$, showing the uniqueness.

Given $Y \in \mathcal{Y}_f$, let $Y'$ be a component of $\text{init} Y$, necessarily in $\mathcal{Y}_{\text{init}, f}$ by part (2). Since $\text{dim} \text{init} Y = \text{dim} Y$, there can be no $Z \subseteq Y, Z \in \mathcal{Y}_f$ with $Y' \subseteq \text{init} Z$. Therefore the map takes $Y' \mapsto Y$, proving the surjectivity.

Now $Y_1 = \pi_{f, \text{init}}(Y'_1)$, and $Y_2 \in \mathcal{Y}_f$.

- $\implies$: If $Y_1 \supseteq Y_2$, then $Y'_1 \subseteq \text{init} Y_1 \subseteq \text{init} Y_2$, so $Y'_1$ is contained in some component $Y'_2$ of $\text{init} Y_2$. Then $Y'_2 \subseteq \text{init} Y_2$.
  
  Since $Y_2$ is irreducible, $\text{init} Y_2$ is equidimensional of the same dimension as $Y_2$, so $\text{dim} Y'_2 = \text{dim} Y_2$, and also any $Z \subseteq Y_2$ has $\text{dim} Z < \text{dim} Y_2$. Hence $Y'_2 \not\subseteq \text{init} Z$.
  
  Together, this shows $\pi_{f, Y'_1} = Y_2$.

- $\impliedby$: Now say $\exists Y'_2 \supseteq Y'_1$. Then $Y'_1 \subseteq Y'_2 \subseteq \text{init} \pi_{f, \text{init}}(Y'_2)$. The definition of $\pi_{f, \text{init}}(Y'_1)$ is as the least $Y \in \mathcal{Y}_f$ such that $Y'_1 \subseteq \text{init} Y$, hence $\pi_{f, \text{init}}(Y'_1) \subseteq \pi_{f, \text{init}}(Y'_2)$. If $\pi_{f, Y'_2} = Y_2$, this says $Y_1 \subseteq Y_2$.

We conjecture that the equivalence relation induced on $\mathcal{Y}_{\text{init}, f}$ by $\pi_{f, \text{init}}$ can be determined from the $\mathbb{N}$-valued function $Y' \mapsto \text{dim} \pi_{f, \text{init}}(Y')$, as the symmetric, transitive extension of the relation “$Y'_1 \subseteq Y'_2$ and $\text{dim} \pi_{f, \text{init}}(Y'_1) = \text{dim} \pi_{f, \text{init}}(Y'_2)$”.

Proof of part (4). First we claim that $C$ is a component of $\text{init} Y$ iff $\text{dim} C = \text{dim} Y$ and $\pi_{f, \text{init}}(C) = Y$ and $\text{dim} C = \text{dim} Y$.

- $\implies$: $\text{init} Y$ is equidimensional of dimension $\text{dim} Y$, so $C$ has that same dimension.
  Also $\pi_{f, \text{init}}(C) \subseteq Y$, so $C \subseteq \text{init} \pi_{f, \text{init}}(C) \subseteq \text{init} Y$, hence $\pi_{f, \text{init}}(C)$ is contained in $Y$ and of the same dimension. Since $Y$ is irreducible, $\pi_{f, \text{init}}(C) = Y$.

- $\impliedby$: $C \subseteq \text{init} Y$, whose components have dimension $\text{dim} Y$, one of whom contains $C$. Hence $C$ equals that component, as above.

Since $f$ is homogeneous, by lemma [3] each $Y \in \mathcal{Y}_f$ is an affine cone and has a well-defined $\text{deg} Y$. Then

$$\text{deg} Y = \text{deg}(\text{init} Y) = \text{deg} \sum_{C \subseteq \text{comps}(\text{init} Y)} \text{deg} C = \text{deg} \sum_{C : \pi_{f, \text{init}}(C) = Y, \text{dim} C = \text{dim} Y} \text{deg} C$$

with the last equality by the claim above. \qed
One can extend this to a calculation of the Hilbert series, not just the degree, using the result of [Kn2].

5. PROOF OF THEOREM 3

Proof of theorem 3 (1) ⇒ (2). This is the Geometric Vertex Decomposition Lemma of [Kn].

(2) ⇒ (3). Let \( I_X \) be the ideal defining \( X \subseteq H \times L \). Working through the definition of the scheme \( X' \), we see its ideal of definition is \( \text{im}(I) \) with respect to the weighting on the variables \( \lambda(h_i) = 0, \lambda(\ell) = 1 \).

Let \( I_\Pi \subseteq I_\Lambda \) be the ideals defining \( \Pi, \Lambda \subseteq H \), with generators \( I_\Pi = \langle (p_i) \rangle, I_\Lambda = \langle (p_i), (q_i) \rangle \), where \( p_i, q_i \in \text{Fun}(H) \). Since \( X' \) is assumed reduced, and we know its support is \( (\Pi \times \{0\}) \cup (\Lambda \times L) \),

\[
I_{X'} = (I_{\Pi \times L} + (\ell)) \cap I_{\Lambda \times L} = \langle (p_i), (\ell q_i) \rangle.
\]

Lifting these to generators of \( I_X \), we learn

\[
I_X = \langle (p_i), (\ell q_i + q'_i) \rangle, \quad \text{for some } q'_i \in \text{Fun}(H).
\]

Projectively completing \( L \) to \( \text{Proj} \mathbb{P}[\ell, m] \), and closing up \( X \) to \( \overline{X} \subseteq H \times (L \cup \{\infty\}) \), we get

\[
I_{\overline{X}} = \langle (p_i), (\ell q_i + mq'_i) \rangle.
\]

To determine a fiber of \( \overline{X} \to H \), we specialize \( H \)’s coordinates to values. If all \( q_i, q'_i \to 0 \) under this specialization, then \( [\ell, m] \) is free, and the fiber is \( \mathbb{P}^1 \). Otherwise some equation \( \ell q_i + mq'_i = 0 \) uniquely determines \( [\ell, m] \in \mathbb{P}^1 \), making the fiber either empty or a point. This proves the fibers are connected.

(3) ⇒ the projection is degree 1 or 0 to any component. Recall that \( \Pi \), by definition, is closed in \( H \). The map \( \overline{X} \to \Pi \) is proper and its image contains an open set, so it is surjective. (Note that \( X \to \Pi \) itself is usually not surjective; part of our task is to describe the points missing from the image.) In particular it hits the generic point of each component of \( \Pi \), and being reduced has reduced generic fiber, so the connectedness of the fibers gives the claimed degree 1 over each component (or 0 if all the fibers are \( \mathbb{P}^1 \)’s).

So some fibers are points and some are \( \mathbb{P}^1 \)’s. The latter type are the ones lying over \( \Lambda' \). But even the point fibers come in two types: those in \( X \), and those in \( (\Lambda \setminus \Lambda') \times \{\infty\} \). This shows that \( X \) misses \( (\Lambda \setminus \Lambda') \times L \), and the same is true when we project out \( L \).

We wish to show that the map \( \pi : X \setminus (\Lambda' \times L) \to \Pi \setminus \Lambda \) is an isomorphism, not merely bijective. First we consider the case \( \Lambda = \emptyset \). Since \( I_\Lambda = \langle 1 \rangle \), we have \( I_X = \langle (p_i), (\ell + q'_i) \rangle \), where the \( (p_i) \) cut out \( \Pi \). If there is more than one relation \( \ell + q'_i \) on \( \Pi \) then the differences \( q_i' - q_i' \) are also satisfied on \( \Pi \), hence generated by the \( (p_i) \), so we may assume there is only one such relation. Then we can use it to eliminate \( \ell \) and determine that \( \pi : X \equiv \Pi \) is an isomorphism, in this restricted case \( \Lambda = \emptyset \). In the general case, we already know that \( \pi : X \setminus (\Lambda' \times L) \to \Pi \setminus \Lambda \) is bijective, so we need to check over open sets \( U = \text{Spec} \Lambda \) covering \( \Pi \setminus \Lambda \). Replacing \( \Pi \) by \( U \) (reimbedded as a closed subset of some new \( H \)) and \( X \) by \( \pi^{-1}(U) \), we can reduce to the already solved subcase that \( \Lambda = \emptyset \).

This gives an equation in the Grothendieck ring of algebraic varieties,

\[
[X] = [X \setminus (\Lambda \times L)] + [\Lambda' \times L] = [\Pi \setminus \Lambda] + [\Lambda' \times L] = [\Pi] - [\Lambda] + [\Lambda'][L]
\]
and hence
\[
[X'] = [\Pi \times \{0\}] + [\Lambda \times L] - [\Lambda \times \{0\}] \quad \text{since } X' = (\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times L)
\]
\[
= [\Pi] + [\Lambda][L] - [\Lambda]
\]
\[
=(X) + [\Lambda] - ([\Lambda][L]) + [\Lambda][L] - [\Lambda]
\]
\[
=[X] - [\Lambda'][L] + [\Lambda][L]
\]
\[
=[X] + [\Lambda'] \{ \Lambda \setminus \Lambda' \}.
\]
by the above

The map \( \iota : X \to X' \) defined by
\[
\iota(h, \ell) := \begin{cases} (h, \ell) & \text{if } h \in \Lambda' \\ (h, 0) & \text{if } h \notin \Lambda' \end{cases}
\]
is injective, and its image is the complement of \( (\Lambda \setminus \Lambda') \times L \).

For the statement on \( K \)-classes, first define \( \overline{X}' := \lim_{t \to 0} t \cdot \overline{X} \) by the same limiting procedure as \( X' \). Then note that the action of \( G \) on \( H \times L \) commutes with the \( G_m \)-action, and that \( K_G \)-classes are constant in locally free equivariant families such as the one defining \( \overline{X}' \). (For the nonequivariant statement one can take \( G = 1 \).) Hence \( [O_{\overline{X}}] = [O_{\overline{X}'}] \) as elements of \( K_G(H \times ([L \cup \{\infty\}])) \), and in turn
\[
[O_{\overline{X}'}] = [O_{\Pi \times \{0\}}] + [O_{\Lambda \times ([L \cup \{\infty\}])}] - [O_{\Lambda \times \{0\}}] \in K_G(H \times ([L \cup \{\infty\}])).
\]
Since the projection \( L \cup \{\infty\} \to pt \) takes \( O_{L \cup \{\infty\}} \) to \( O_{pt} \) with no higher direct images,
\[
\pi_1[O_{\overline{X}'}] = \pi_1[O_{\Pi \times \{0\}}] + \pi_1[O_{\Lambda \times ([L \cup \{\infty\}])}] - \pi_1[O_{\Lambda \times \{0\}}] = [O_H] + [O_{\Lambda}] - [O_{\Lambda}] = [O_H] \in K_G(\Pi).
\]

In fact the proof (2) \( \implies \) (3), and the statement about \( K \)-classes, did not use \( \Lambda \) reduced; it is enough to assume \( X' \) has no embedded components along \( \Pi \times \{0\} \). This covers the example of \( h^2 \ell = 0 \) given after the statement of theorem \( 3 \).

We give some examples in which \( \Lambda' \) appears. Let \( H = \{(x, y, 0)\} \) and \( L = \{(0, 0, \ell)\} \), and let \( X = \{(x, y, \ell) : x = \ell y\} \), with \( p^2 \) points. Its closure \( \overline{X} = \{(x, y, [\ell, m]) : xm = \ell y\} \) is the blowup of \( H \) at the origin (so \( \Pi = H \)). Hence
\[
\overline{X} \setminus X = \{(x, y, [\ell, 0]) : 0 = \ell y\} = \{(x, y, [1, 0]) : 0 = y\} = \{(x, 0, [1, 0])\}
\]
so \( \Lambda = \{(x, 0)\} \). The only point in \( \Lambda \) hit by the projection \( X \to H \) is \( (0, 0) \in H \), so \( \Lambda' = \{(0, 0)\} \). Thus
\[
X' = (H \times \{0\}) \bigcup_{\Lambda \times \{0\}} (\Lambda \times L)
\]
has \( p^2 + p^2 - p \) points, and \( \Lambda \setminus \Lambda' \) has \( p - 1 \), giving us the expected
\[
|X'| = p^2 + p^2 - p = p^2 + p(p - 1) = |X| + p |\Lambda \setminus \Lambda'|.
\]

If we take \( f = (\ell y - x)(x - 1) \in \mathbb{P}[x, y, \ell] \), then \( f = \ell y(x - 1) + x(1 - x) \) and \( \text{Tr}(f^{p - 1} \bullet) \) defines a splitting of \( H \times L \) that compatibly splits \( X \). But \( \Lambda' = \{(0, 0)\} \) is not compatibly split by \( \text{Tr}((y(x - 1))^{p - 1} \bullet) \); the only point that is compatibly split is \( (1, 0) \).

In the notation of the proof, the set \( \Lambda' \) is easily seen to be cut out by the ideal \( \langle (p_1, (q_1), (q'_{1})) \rangle \). In the following example, this ideal is not even radical, much less compatibly split. Let \( H, L \) be as above, with \( f = x(\ell y - x) = \ell xy - x^2 \), and \( X \) the hypersurface \( f = 0 \). Then
Proof. Let $(p_i), (q_i), (q'_i) = (q_i), (q'_i)$ = $\langle xy, -x^2 \rangle$, supported on $\Lambda' = \{(0, y)\}$ (which is compatibly split by $\text{Tr}(f^p - \bullet)$).

6. Proof of Lemmas 3 and 1, and Theorems 4, 6, and 5

Proof of lemma 3 Since the set of compatibly split ideals is finite [Schw, KuMe] hence discrete, if a connected group $G$ preserves the decomposition $R = R^p \oplus \ker \varphi$ on a ring $R$, it must preserve each compatibly split ideal. Here we take $G = \mathbb{G}_m$ acting by $z \cdot x_i = z^{\lambda_i} x_i$, for which $f$ is assumed to be a weight vector, and hence $G$ preserves $\ker \text{Tr}(f^p - \bullet)$. Together, we learn this $G$ preserves each compatibly split subscheme $Y$, which is equivalent to the statement $Y = \text{init} Y$. □

Proof of lemma 1 Trivially this $\varphi(1) = 1$, so $\varphi$ is a splitting. For $(\prod_i x_i)^r \in (\prod_i x_i)$,

$$\varphi \left( (\prod_i x_i)^r \right) = \text{Tr} \left( (\prod_i x_i)^{p-1}(\prod_i x_i)^r \right) = (\prod_i x_i)^{\text{Tr}(r)} \in (\prod_i x_i),$$

so $(\prod_i x_i)$ is compatibly split. The components (also compatibly split) of that ideal define the coordinate hyperplanes, whose intersections (also compatibly split) are the coordinate subspaces, whose unions (also compatibly split) are defined by squarefree monomial ideals.

For the converse, note that $\prod_i x_i = \text{init} \prod_i x_i$ for any weighting $\lambda$, hence by lemma 3 a compatibly split subscheme $Y$ must have $Y = \text{init} Y$ for any weighting $\lambda$, which forces $Y$ to be a coordinate subspace. (The same argument applies to the “standard splitting” of any toric variety.) □

For $X$ a Frobenius split scheme, let $\mathcal{Y}_X$ be its set of compatibly split subvarieties. Then there is an associated decomposition $X = \bigsqcup_{Y \in \mathcal{Y}_X} Y^o$, where $Y^o = Y \setminus \bigcup_{Z \subseteq Y, Z \not\subseteq Y} Z$. We point out that this is a stratification, meaning that every closed stratum $Y$ is the union $\bigcup_{Z \subseteq Y} Z^o$ of the open strata contained fully in it. Proof: $Y$ is certainly the union $\bigcup_{Z \subseteq Y} (Y \cap Z^o)$ of its intersection with the open strata meeting it. But the components of $Y \cap Z$ are again elements of $\mathcal{Y}_X$ by the properties of Frobenius splitting, so it suffices to use $Z \subseteq Y$. QED.

Within the geometric vertex decomposition context, we can define a sort of inverse to the map $\pi_{\ell, \text{init}}$ from theorem 2 part (3), in the following proposition.

Proposition 3. Let $H = \mathbb{A}^{n-1}_{\mathbb{F}_p}$ and $L = \mathbb{A}^1_{\mathbb{F}_p}$, with coordinates $h_1, \ldots, h_{n-1}$ and $\ell$. Let $f \in \text{Fun}(H \times L) = \mathbb{F}_p[h_1, \ldots, h_{n-1}, \ell]$ be of degree $n$, where $f = \ell g_1 + g_2$ and $g_1, g_2 \in \text{Fun}(H) = \mathbb{F}_p[h_1, \ldots, h_{n-1}]$.

Assume that $\text{Tr}_H(g_1^{p-1} \bullet)$ defines a Frobenius splitting on $\text{Fun}(H)$. Then $\text{Tr}_{H \times L}(f^{p-1} \bullet)$ defines a Frobenius splitting on $\text{Fun}(H \times L)$. Let $\mathcal{Y}_{H, g_1}, \mathcal{Y}_{H, f}$ denote the corresponding sets of compatibly split subschemes. Let $\pi : H \times L \to H$ denote the linear projection. Then the map

$$Y \in \mathcal{Y}_{H, f} \quad \mapsto \quad \pi(Y)$$

takes values in $\mathcal{Y}_{H, g_1}$, and is injective on $\{Y \in \mathcal{Y}_{H, f} : Y \text{ is not of the form } P \times L, P \subseteq H\}$. Let $\mathcal{Y}_{H, f}^{\mathbb{A}^n \times L}$ denote this subset of $\mathcal{Y}_{H, f}$.

Proof. Let $\lambda_{\ell}$ assign weight 1 to $\ell$ and weight 0 to any $h_i$, and let $\text{init}_{\ell} g, \text{init}_f$ denote the corresponding initial term or ideal, e.g. $\text{init}_f f = \ell g_1$. Then by theorem 2 $\text{Tr}_{H \times L}(f^{p-1}) = \ldots$
This splitting $\text{Tr}_{H \times L}((\ell g_1)^{p-1})$ compatibly splits the hyperplane $H \times \{0\}$, and the induced splitting on $H$ is $\text{Tr}(g_1^{p-1})$, under the identification of $\mathbb{P}_p[h_1, \ldots, h_{n-1}, \ell]/(\ell)$ with $\mathbb{P}_p[h_1, \ldots, h_{n-1}]$.

Let $Y \in \mathcal{Y}_{\Lambda^n, f}$, and $Y'$ its limit as studied in theorem 3. That is, if their ideals are $I_Y, I_{Y'}$, then $I_{Y'} = \text{init}_f I_Y$. So again by theorem 2, $Y'$ is compatibly split with respect to the splitting $\text{Tr}((\text{init}_f f)^{p-1}) = \text{Tr}((\ell g_1)^{p-1})$. Since $Y'$ is therefore reduced, $Y' = (\Pi \times \{0\}) \cup (A \times L)$ where $\Pi, A$ are also as in theorem 3. Since $H \times \{0\}$ is also compatibly split by $\text{Tr}((\ell g_1)^{p-1})$, we see that $(H \times \{0\}) \cap Y' = \Pi \times \{0\}$ is compatibly split by $\text{Tr}((\ell g_1)^{p-1})$. Since $Y$ is irreducible (being an element of $\mathcal{Y}_{\Lambda^n, f}$), so is $\Pi = \pi(Y)$. All together this shows that $\pi(Y) \in \mathcal{Y}_{H, g_1}$ as claimed.

To see the claimed injectivity, let $Y_1, Y_2 \in \mathcal{Y}_{\Lambda^n, f}$ with $\pi(Y_1) = \pi(Y_2) =: \Pi$, and neither $Y_1$ nor $Y_2$ contain components of the form $P \times L$. Let $Y = Y_1 \cup Y_2$, again compatibly split. Since their corresponding degenerations $Y'_1, Y'_2, Y'$ are compatibly split (with respect to $\text{Tr}((\ell g_1)^{p-1})$) hence reduced, the projections $Y_1, Y_2, Y \to \Pi$ are each degree 1 over every component of $\Pi$ by theorem 3. If $Y_1 \neq Y_2$ over some component of $\Pi$, the degree of $Y_1 \cup Y_2 \to \Pi$ would be the sum of the two degrees, so each 2 not 1. Since $Y_1, Y_2$ agree over each component of their common projection, and contain no components of the form $P \times L$, we obtain $Y_1 = Y_2$.

\begin{proof}[Proof of theorem 4]
If $\deg f < n$, we can multiply it by the product of the variables not appearing in $\text{init} f$, thereby only increasing the set of $\{Y\}$ obtained by the algorithm. We thereby reduce to the case $\deg f = n$ and $\text{init} f = \prod_{i=1}^n x_i$.

If $p \nmid c$, then $c \prod_i x_i$ is still a term in the reduction of $f \bmod p$, and still the initial term. Hence $\text{init}(f)^{p-1} \equiv c^{p-1} \prod_i x_i^{p-1} \equiv \prod_i x_i^{p-1} \bmod p$ by Fermat’s Little Theorem.

Since $\text{Tr}(\prod_i x_i^{p-1})$ defines a splitting (indeed, the standard splitting), so does $f \bmod p$, by theorem 2 part (1). Part (2) says that for any compatibly split $I \leq \mathbb{P}[x_1, \ldots, x_n]$, the initial ideal $\text{init}(I)$ is compatibly split by $\text{Tr}(\text{init}(f)^{p-1}) = \text{Tr}(\prod_i x_i^{p-1})$, hence is a Stanley-Reisner ideal by lemma 1.

Now let $I \leq \mathbb{Z}[x_1, \ldots, x_n]$ be one of the ideals constructed using the algorithm in corollary 1. Only finitely many quotient rings $R/I'$ are encountered on the way to $R/I$, each of which is flat over an open set in $\text{Spec} \mathbb{Z}$. Hence if we restrict to primes $p$ in this finite intersection of open sets, when we work the algorithm $\bmod p$ we encounter $I \bmod p$. Let $S$ be the set of primes we are avoiding so far.

Let $(g_i)$ be a Gröbner basis for $S^{-1}I \leq (S^{-1}\mathbb{Z})[x_1, \ldots, x_n]$. If we increase $S$ to include the primes dividing the coefficients on the initial terms (init $g_i$), then we can rescale the $(g_i)$ to make their initial coefficients 1, and insist that no init $g_i$ divides any init $g_j, j \neq i$. For any $p \not\in S$, these properties hold also for $(g_i \bmod p)$.

As observed above, init($I \bmod p$) is a Stanley-Reisner ideal, $p \not\in S$. Hence the initial monomials init $(g_i \bmod p)$ are squarefree. So the initial monomials init $g_i$ are themselves squarefree. This proves that away from the bad primes in $S$, init $I$ is a Stanley-Reisner ideal. In particular init $I$, and $I$ itself, are radical over $S^{-1}\mathbb{Z}$, as was to be shown.
Now take $\lambda$ to be the lexicographic weighting. In this case, initial $I = \text{init}_n \text{init}_{n-1} \cdots \text{init}_1 L$, where $\text{init}_j$ is defined using the weighting $\lambda_j = (0, \ldots, 0, 1, 0, \ldots, 0)$.

For each $j$ in turn, let $L$ be the $j$th coordinate line $\{(0, \ldots, 0, *, 0, \ldots, 0)\}$, and $H$ the complementary coordinate hyperplane. Define $\ell_f \colon H \to L$ by

$$\ell_f(h) := \begin{cases} 
\ell & \text{if a minimal } Y \subseteq \mathbb{A}^n \text{ meeting } \{h\} \times L \text{ meets it in the single point } (h, \ell) \\
0 & \text{otherwise}
\end{cases}$$

where $Y$ ranges over the subschemes created by the algorithm. As there are only finitely many $Y$, this map is constructible. Then define the bijection $\iota_j : \mathbb{A}^n \to \mathbb{A}^n$ by

$$\iota_j(h + \ell) := h + \ell - \ell_f(h).$$

We claim that $\ell_f$ (and thus $\iota_1$) is well-defined. First we need to be sure that it is defined everywhere. For every $h$, some $Y$ meets and even contains $\{h\} \times L$, namely $Y = \mathbb{A}^n$. If no $Y$ meets $\{h\} \times L$ in a finite scheme, then $\ell_f(h) = 0$. Otherwise let $Y_h$ be the union of the $Y$ that meet $\{h\} \times L$ in a finite scheme. (Indeed, it is the unique largest such $Y$.) Then since $\text{init}_j Y_h$ is reduced, by the $(2) \implies (3)$ part of theorem 3, $Y_h$ intersects $\{h\} \times L$ in a point, $(h, \ell_h)$. This also shows that the choice of minimal $Y$ does not matter, since any such $Y$ lies in $Y_h$.

Now we claim that $\iota_j$ takes $Y$ into $\text{init}_j Y$. This is because this $\iota$ agrees with the $\iota$ from theorem 3 except possibly on $\Lambda' \times L \subsetneq Y$, where it is merely shifted by an element of $L$.

When we then define $\iota := \iota_n \circ \cdots \circ \iota_1$, we learn inductively that $\iota(Y) \subseteq Y'$.

It is quite unfortunate that the $\{\Lambda'\}$ from theorem 3 are not always compatibly split in $H$, which is to say, one cannot stratify the varieties in $\mathcal{J}_{\mathbb{A}^n,f}^{2* \times L}$ using the varieties in $\mathcal{J}_{H,0}^{\text{irr}}$ (or more precisely, their $Y^o = Y \setminus \cup_{Z \subset Y} Z$). If this were true, one could use induction to give $\mathbb{A}^n$ a paving by tori, compatibly paving each $Y$, pulled back using $\iota$ from the standard paving (and with $\iota$ regular on each torus stratum). The well-known example [De85] of such a simultaneous paving suggests there should be a criterion guaranteeing that the $\{\Lambda'\}$ are split, that would apply to the Kazhdan-Lusztig case considered in section 3 and [De85]. However, the subtle example at the end of section 5 is of this type, making such a criterion difficult to imagine.

**Proof of theorem 3**

1. By theorem 4, initial $Y$ is Stanley-Reisner and reduced, hence its ideal can be generated by squarefree monomials. Lifting those to generators of $Y'$'s ideal, we get the desired Gröbner basis.

2. Set-theoretically, it is plain that the intersection of $\{Z \in \mathcal{J}_f : Z \geq Y, Z \text{ basic in } \mathcal{J}_f\}$ is $Y$. But since this intersection is compatibly split, it is reduced, so the equation holds scheme-theoretically.

3. Since any intersection of compatibly split ideals is compatibly split, and by theorem 4 their initial ideals are radical, corollary 2 from section 4 allows us to concatenate their Gröbner bases.

We mention a property of $\pi_{f,\text{init}} : \mathcal{J}_{\text{init}_f} \to \mathcal{J}_f$: for all $Y' \in \mathcal{J}_{\text{init}_f}$, $\dim \pi_{f,\text{init}}(Y') \geq \dim Y'$.

**Proof:** $Y' \subseteq \text{init} \pi_{f,\text{init}}(Y')$, and $\text{init}$ preserves dimension. The examples in figure 1 show that the inequality may be strict.
Proof of theorem\footnote{5}. As this is the only place in the paper that we consider Frobenius splittings on schemes rather than on affine space, we will not build up all the relevant definitions, but assume the reader is familiar with \cite[section 1.3]{BrKu05} and \cite{KuMe}.

(1) The proof is exactly the same as in \cite[proposition 1.3.11]{BrKu05}, once one includes \cite[remark 1.3.12]{BrKu05} to relax nonsingularity to normality. Completeness enters as follows: the scheme-theoretic analogue of $\text{Tr}(f^{p^{-1}})$ is a global function on $X$, and the condition on $f$ ensures that $\text{Tr}(f^{p^{-1}})$ is locally of the form $1 +$ higher order. But since $X$ is complete, normal, and irreducible (being a variety), any global function on $X_{\text{reg}}$ is constant, so the higher order terms vanish and $\text{Tr}(\sigma^{p^{-1}}) = 1$.

(2) Write $f = \prod_i t_i (1 + \sum_e c_e t^e)$, where the $\{e\}$ are exponent vectors, all entries $\geq -1$. The condition on $f$ says that $\lambda \cdot e < 0$ for each summand. Write $f^{p^{-1}} = \prod_i t_i^{p^{-1}} (1 + \sum_e d_s t^e)$. The exponent vectors $s$ are sums of $p - 1$ of the vectors $\{e\}$.

What can such an $s$ look like? It is integral, with all entries $\geq 1 - p$. Since $\lambda \cdot s < 0$, and each entry of $\lambda$ is $\geq 0$, there must be some entry of $s$ in $(-p, 0)$. Hence $s$ cannot be $p$ times another vector.

Consequently the only term of $\prod_i t_i^{p^{-1}} (1 + \sum_e d_s t^e)$ that contributes in $\text{Tr}(f^{p^{-1}})$ is the first one, so $\text{Tr}(f^{p^{-1}}) = 1$ locally. By irreducibility, $\text{Tr}(f^{p^{-1}}) = 1$.

In either case, the same argument from theorem\footnote{4} guarantees that the splitting defined by $\sigma^{p^{-1}}$ does compatibly split the divisor $\sigma = 0$. Normality lets us extend the splitting from $X_{\text{reg}}$ to $X$ \cite[Lemma 1.1.7(iii)]{BrKu05}.

To see the uniqueness of the splitting, the proof of \cite[prop. 2.1]{KuMe} shows that a section $\gamma$ of $\omega^{1-p}$ defining a Frobenius splitting on a nonsingular variety splits a divisor $\{\sigma = 0\}$ iff $\gamma$ is a multiple of $\sigma^{p^{-1}}$. By the assumption that $\{\sigma = 0\}$ is anticanonical, this multiple must be by a global function. Completeness and irreducibility ensures this global function is a constant. \hfill $\Box$

7. Examples, theorem\footnote{7} and proposition\footnote{4}.

In sections\footnote{7.1} we investigate the condition $\text{init}(f) = \prod_i x_i$ in examples, sometimes using Macaulay 2 \cite{M2}. The most important family of examples is the Kazhdan-Lusztig varieties in section\footnote{7.3}. In section\footnote{7.4} we mention a corollary about Brion’s “multiplicity-free” subvarieties of a flag manifold [Br03].

7.1. Small dimensions.\footnote{7.1} We begin with a general weighting $\lambda_1, \ldots, \lambda_n$ of the variables, so $\prod x_i^{\lambda_i}$ has weight $\sum \lambda_i e_i$. Write $\prod x_i^{e_i} \triangleright \prod x_i^{\lambda_i}$ if its weight is greater.

Without loss of generality we may assume the $(\lambda_i)$ to be in decreasing order. Then for any two monomials $\prod x_i^{e_i}$ and $\prod x_i^{\lambda_i}$, if $\sum_{i \leq j} e_i \geq \sum_{i \leq j} f_i$ for all $i = 1, \ldots, n$, we may be sure that $\prod x_i^{e_i} \triangleright \prod x_i^{\lambda_i}$.

Once we choose a specific $\lambda$ (generic enough that $\prod x_i$ does not have the same weight as any other monomial), the condition that $\text{init}(f) = \prod_i x_i$ forces us to put coefficient 0 on any $m$ with $m \triangleright \prod_i x_i$. Different choices of $\lambda$ will lead to different sets of allowed $m$, but as our interest is not in $(\lambda)$ but in the set of varieties to which theorem\footnote{4} applies, it is enough for us to consider the maximal allowed subsets of monomials.
7.1.1. $n=2$. In this case the set of permitted monomials is already uniquely specified: \( \{x_1x_2, x_2^3\} \). So the possible polynomials are \( f = x_1x_2 + c_{20}x_2^3 \), defining the two (distinct) points \([1,0], [-c_{20}, 1]\) in \( \mathbb{P}^1 \).

7.1.2. $n=3$. The order considerations so far already tell us which monomials \( m \) have \( m \gg x_1x_2x_3 \) and which have \( x_1x_2x_3 \gg m \), except for \( m = x_2^3 \). One can find decreasing \( \lambda \) for either condition \( x_2^3 \gg x_1x_2x_3 \) or \( x_1x_2x_3 \gg x_2^3 \), but since we only want to consider the maximal cases, we ask that \( x_1x_2x_3 \gg x_2^3 \), which is achieved by e.g. \( (\lambda_1, \lambda_2, \lambda_3) = (3,1,0) \).

Now we are considering plane curves of the form

\[
f = x_1x_2x_3 + c_{030}x_3^3 + c_{021}x_2^2x_3 + c_{102}x_1x_3^3 + c_{012}x_2x_3^2 + c_{003}x_3^3.
\]

Being cubics and Frobenius split, they are elliptic curves with at worst nodal singularities \([BrKu05]\) 1.2.4–1.2.5]. Each curve has \([1,0,0] \) as a singular point, so they are indeed nodal.

If we write \( f \) as \( f = x_1(x_2x_3 + c_{102}x_3^2) + c_{030}x_3^3 + c_{021}x_2^2x_3 + c_{102}x_2x_3^2 + c_{003}x_3^3 \) and degenerate as in proposition \([\text{1}]\) we see \( I_\Lambda = (x_3) \cap (x_2 + c_{102}x_3) \). For generic values of the coefficients, \( \Lambda' = \{[1,0,0]\} \).

Being cubics, these curves can only have two more nodes. To find them we decompose the ideal generated by \( f \) and its derivatives, using the Macaulay 2 command

\[
\text{decompose} ((\text{ideal } f) + \text{ideal } \text{diff(matrix } \{[x_1, x_2, x_3]\}, f))
\]

Then for each component \( c \) of the possible singular set, we \( \text{eliminate}(\{x_1, x_2, x_3\}, c) \) to see for what \( f \) the singularities arise. The components turn out to be \( \{c_{003} = 0\} \) and \( \{c_{300} + c_{210}c_{102} = c_{210}c_{201} + c_{210}c_{003}\} \).

If \( c_{003} = 0 \), the cubic breaks into the \( x = 0 \) line and a conic. If \( c_{030}c_{102} + c_{012}c_{102} = c_{021}c_{102} + c_{003} \), the curve breaks into the \( x_2 + c_{102}x_3 = 0 \) line and a conic, and \( \Lambda' \) contains the new node. If both are true, the curve is a cycle of three lines.

7.2. Matrix Schubert varieties. The matrix Schubert variety \( \overline{X}_\pi \subseteq M_n \) is the closure of \( B_+ \pi B_+ \) inside the space \( M_n \) of all matrices, where \( B_+ \) (respectively \( B_- \)) denotes the Borel group of upper (respectively lower) triangular matrices, and \( \pi \) is a permutation matrix. These varieties were introduced in \([P92]\), where their corresponding radical ideals \( I_\pi \) were determined. They have a couple of relations to flag manifold Schubert varieties; in particular \( \overline{X}_\pi \cap \text{GL}(n) / B_+ \) is the usual Schubert variety \( X_\pi \subseteq \text{GL}(n) / B_+ \). Hence the codimension of \( \overline{X}_\pi \) in \( M_n \) is the Coxeter length \( \ell(\pi) \).

It is easy to give examples of weightings \( \lambda \) on the matrix coordinates \( (x_{ij}) \) such that for the determinant \( d \) of any submatrix, \( \text{init}(d) \) is the product of the entries on the antidiagonal (times a sign). In \([KnMi05]\) we called these \textbf{antidiagonal term orders}, and showed that each \( \text{init}(I_\pi) \) is Stanley-Reisner over \( \mathbb{Z} \). In this section we do this part of \([KnMi05]\) (though only over \( \mathbb{Q} \)) much more easily using theorem \([4]\).

Let \( M = (m_{ij})_{i,j=1,...,n} \) be an \( n \times n \) matrix of indeterminates, and let \( d_{[i,j]} \) denote the determinant of the submatrix consisting of rows and columns \( i, i+1, \ldots, j \) from \( M \). Let

\[
f := \prod_{i=1}^{n-1} d_{[i,i+1]} \prod_{j=1}^{n} d_{[j,n]},
\]
This is homogeneous of degree $1 + 2 + \ldots + (n - 1) + n + (n - 1) + \ldots + 2 + 1 = n^2$. Let
\[
d_i' := (-1)^{i-1} \prod_{j,k : j + k = i + 1} m_{jk}
\]
be the product of the $i$th antidiagonal of $M$ (with a not particularly important sign), so
\[
\text{init}(d_{[1,i]}) = d_i', \quad \text{init}(d_{[j,ni]}) = d_{n+i-j-1}'
\]
and since the set of matrix entries is the union of the antidiagonals,
\[
\text{init}(f) = \prod_{i=1}^{n-1} \text{init}(d_{[1,i]}) \prod_{j=1}^{n} \text{init}(d_{[j,ni]}) = \prod_{i=1}^{n-1} d_i' \prod_{j=1}^{n} d_{n+i-j-1}' = \prod_{i=1}^{2n-1} d_i' = \pm \prod_{i,j} m_{ij}.
\]
Hence theorem 4 applies.

Next we apply the algorithm from corollary 11 to the ideal $\langle f \rangle$. We will restrict to the components $\langle \langle a_{[1,i]} \rangle \rangle$, which define exactly the matrix Schubert varieties $\overline{X}_\pi$, associated to the simple reflections. Each of these is $B_- \times B_+$-invariant under the left/right action, and this invariance persists as we intersect, decompose, and repeat.

Copying [BruKu05, theorem 2.3.1], we claim that every matrix Schubert variety $\overline{X}_\pi$ is produced by this algorithm. The proof is by induction on the length of $\pi$; we are given the $\ell(\pi) = 1$ base case to start with. We need the combinatorial fact that for any $\pi$ with $\ell(\pi) > 1$, there exist at least two permutations $\rho \neq \rho'$ covered by $\pi$ in the Bruhat order [BeGG75, lemma 10.3]. We know by induction that their matrix Schubert varieties have already been already been produced. Now $\overline{X}_\rho \cap \overline{X}_{\rho'} \supseteq \overline{X}_\pi$, and $\dim (\overline{X}_\rho \cap \overline{X}_{\rho'}) \leq \dim \overline{X}_\rho = \dim \overline{X}_\pi + 1$, so $\overline{X}_\pi$ must be a component of $\overline{X}_\rho \cap \overline{X}_{\rho'}$. Therefore it too is produced by the algorithm.

To apply theorem 6, we need to compute the basic elements of $S_n$, shown in [LaSchü96] to be those $\pi$ such that $\pi, \pi^{-1}$ are each Grassmannian. For those $\pi$, Fulton’s theorem [F92] states that $\overline{X}_\pi$ is defined by the vanishing of all $a \times a$ determinants in the upper left $b \times c$ rectangle, for $a, b, c$ determined by $\pi$ (the “essential set” of $\pi$ has only the box $\{(b, c)\}$). These determinants are already known to form a Gröbner basis for any antidiagonal term order [Stu90]. Now part (2) of theorem 6 recovers Fulton’s presentation of the ideals defining general $X_\pi$, and part (3) recovers the main result of [KnMi05], that Fulton’s generators form a Gröbner basis.

Note that while we used only $\prod_{i=1}^{n-1} d_{[1,i]}$ to produce the matrix Schubert varieties, that polynomial wasn’t of high enough degree to give a splitting, and we needed to flesh it out to $f$. It is interesting to note that this function (not $f$) was already enough to construct a Frobenius splitting on $GL_n \times B$ in [MeVdk92, section 3.4], when pulled back along $GL_n \times B \to g\ell_n$.

Finally, we mention that the definition of $f$ generalizes easily to the case of rectangular matrices, say $k \times n$ with $k \leq n$. Let $d_{[i_1,j_1]}^{[i_2,j_2]}$ denote the determinant of the submatrix using

$^4$We only need one, easy, direction. If $\pi$ has descents at both $i$ and $j$, then $\pi \uparrow i, \pi \uparrow j$ both cover $\pi$ (in opposite Bruhat order), and it is easy to see that $\pi$ is their unique greatest lower bound. Also, $\pi \mapsto \pi^{-1}$ is an automorphism of Bruhat order. Hence $\pi$ basic implies $\pi, \pi^{-1}$ Grassmannian.

$^5$The reference [Stu90] uses diagonal term orders, but the ideal is symmetric in the rows, so we can reverse them.
rows \([i_1, i_2]\) and columns \([j_1, j_2]\) (so the previous \(d_{i,j}\) is this \(d_{i,j}^{[i,j]}\)). Then take
\[
f := \prod_{i=1}^{k-1} d_{i,i} \prod_{i=1}^{n-k} d_{i,i+k-1} \prod_{j=1}^{k} d_{n-j+1,n}
\]
and the antidiagonal terms again exactly cover the matrix.

7.3. Kazhdan-Lusztig varieties. This example requires a fair amount of standard Lie
theory, in particular a pinning \((G, B, B_\pm, T, \Delta_\pm, W)\) of a connected simply connected
reductive algebraic group \(G\). A typical simple root will be denoted \(\alpha \in \Delta_+\), with corre-
sponding simple reflection \(r_\alpha \in W\), subgroup \((SL_2)_\alpha\) one-parameter unipotent subgroup
\(e_\alpha : G_\alpha \rightarrow B \cap (SL_2)_\alpha\) and minimal parabolic \(P_\alpha = (SL_2)_\alpha B\). That group has the Bruhat
decomposition \(P_\alpha = B \sqcup B\tilde{r}_\alpha B = B \sqcup (im e_\alpha)\tilde{r}_\alpha B\), where \(\tilde{r}_\alpha\) is a lift of \(r_\alpha\) to an element of the corre-
sponding \((SL_2)_\alpha \leq G\), chosen so that
\[
e_\alpha(c) \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad \tilde{r}_\alpha \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
under some isomorphism of \((SL_2)_\alpha\) with \(SL_2\), taking \(B_\pm \cap (SL_2)_\alpha\) to upper/lower triangular
2 \(\times\) 2 matrices.

For \(w \in W\) a Weyl group element, let \(X_w := B_w B/B \subseteq G/B\) and \(X_w := \overline{X_w}\) be the asso-
ciated Bruhat cell and Schubert variety, of codimension \(\ell(w)\) (the length of \(w\) as a Coxeter element). Define also the opposite Bruhat cell \(X_v := BvB/ B\) and opposite Schu-
bert variety \(X^v := X_w \cap X_v\). It is known [BrKu05, theorem 2.3.1] that \(G/B\) possesses a Frobenius splitting (meaning, on its structure sheaf of rings) that compatibly
splits all its Schubert varieties and opposite Schubert varieties.

Consequently, each \(X_v\) is Frobenius split, compatibly splitting the Kazhdan-Lusztig
varieties \(X_w := X_w \cap X_v\). In [Kn08] we showed that each Kazhdan-Lusztig variety has a flat
degeneration to a Stanley-Reisner scheme, using a sequence of different coordinate
systems. In what follows we will derive both of these results from theorem 4, using a
single identification of \(X_v\) with \(A^{\ell(v)}\).

As in [Kn08], these coordinates will depend on a reduced word \(Q\) for \(v\), i.e. a list
\((\alpha_1, \ldots, \alpha_{\ell(v)})\) of simple roots such that \(r_{\alpha_1} \cdots r_{\alpha_{\ell(v)}} = v\). Associated to \(Q\) is a Bott-
Samelson manifold \(BS^Q := p_{\alpha_1} \times B \cdots \times B p_{\alpha_{\ell(v)}}/B\) and birational map \(\beta_Q : BS^Q \rightarrow X^v\),
taking \([p_1, \ldots, p_{\ell(v)}] \mapsto \left(\prod_{i=1}^{\ell(v)} p_i\right) B/B\). In particular, we can use \(\beta_Q\) to define an isomorphism
\[
A^{\ell(v)} \rightarrow X^v, \quad (c_1, \ldots, c_{\ell(v)}) \mapsto \left(\prod_{i=1}^{\ell(v)} (e_{\alpha_i}(c_i)\tilde{r}_{\alpha_i})\right) B/B.
\]

Define \(\tilde{\beta}_Q : A^{\ell(v)} \rightarrow G\) by \(\tilde{\beta}_Q(c_1, \ldots, c_{\ell(v)}) = \prod_{i=1}^{\ell(v)} (e_{\alpha_i}(c_i)\tilde{r}_{\alpha_i})\). For \(\lambda\) a dominant weight
of \(G\), pick \(\vec{v}_\lambda\) a high weight vector of the irrep \(V_\lambda\) (with highest weight \(\lambda\), and \(\vec{v}_{-\lambda}\) a low

\[\text{As our personal interest is most often in the cohomology classes associated to Schubert varieties, we}
\text{prefer to privilege the codimension over the dimension, dictating our convention of Schubert/opposite}
\text{Schubert. This will have the drawback later that containment order on Schubert varieties, relevant for}
\text{computing the “basic” elements, is opposite Bruhat order.} \]
Proof. (1) It is easy to see that the function determinant of the upper left $i$ of minors) for the defining ideals of \( I \) finds generators for \( \omega \).

(2) If \( \langle \alpha, \lambda \rangle = 0 \), then \( m_\lambda(e_\alpha(c) \bar{r}_\alpha g) = m_\lambda(g) \) for all \( g \in G, c \in \mathbb{G}_a \).

(3) If \( \langle \alpha, \lambda \rangle = 1 \), then \( m_\lambda(e_\alpha(c) \bar{r}_\alpha g) = cm_\lambda(g) + m_\lambda(\bar{r}_\alpha g) \) for all \( g \in G, c \in \mathbb{G}_a \).

Lemma 7. Let \( \alpha \) be a simple root and \( \omega \) the corresponding fundamental weight. Let \( \lambda \) be a dominant weight, so \( \langle \alpha, \lambda \rangle \geq 0 \).

(1) The divisor \( m_\omega = 0 \) in \( G \) is the preimage of the Schubert divisor \( X_{r_\alpha} \subseteq G/B \).

(2) If \( \langle \alpha, \lambda \rangle = 0 \), then \( m_\lambda(e_\alpha(c) \bar{r}_\alpha g) = m_\lambda(g) \) for all \( g \in G, c \in \mathbb{G}_a \).

(3) If \( \langle \alpha, \lambda \rangle = 1 \), then \( m_\lambda(e_\alpha(c) \bar{r}_\alpha g) = cm_\lambda(g) + m_\lambda(\bar{r}_\alpha g) \) for all \( g \in G, c \in \mathbb{G}_a \).

Proof. (1) It is easy to see that the function \( m_\lambda \) is invariant up to scale under the left/right action of \( B_- \times B \). So the divisor \( m_\lambda = 0 \) is the preimage of some \( B_- \)-invariant divisor \( D \) on \( G/B \), necessarily some linear combination of the Schubert divisors. The coefficient of \( X_{r_\alpha} \) in \( D \) can be determined by restricting the class of \( D \) to the opposite Schubert curve \( X'_{r_\alpha} \), and turns out to be \( \langle \alpha, \lambda \rangle \). In particular, we get \( D = X_{r_\alpha} \) exactly if \( \lambda = \omega \).

(2) \[
m_\lambda(e_\alpha(c) \bar{r}_\alpha g) = \langle \bar{v}_- \lambda, e_\alpha(c) \bar{r}_\alpha g \cdot \bar{v}_- \lambda \rangle = \langle (e_\alpha(c) \bar{r}_\alpha)^{-1} \cdot \bar{v}_- \lambda, g \cdot \bar{v}_- \lambda \rangle
\]
The condition on \( \lambda \) says that \( \bar{v}_- \lambda \) is a weight vector not only for \( B_- \) but for the opposite minimal parabolic \( P_- \). Then use the fact that \( e_\alpha(c), \bar{r}_\alpha \) are elements of the commutator subgroup of \( P_- \) (indeed, of \( SL_2 \)) to see that \( (e_\alpha(c) \bar{r}_\alpha)^{-1} \cdot \bar{v}_- \lambda = \bar{v}_- \lambda \), hence
\[
\langle (e_\alpha(c) \bar{r}_\alpha)^{-1} \cdot \bar{v}_- \lambda, g \cdot \bar{v}_- \lambda \rangle = \langle \bar{v}_- \lambda, g \cdot \bar{v}_- \lambda \rangle = m_\lambda(g).
\]

(3) The condition on \( \lambda \) tells us that \( r_\alpha \cdot (-\lambda) = (-\lambda) + \alpha \), i.e. the \( \alpha \)-string through \(-\lambda\) in \( (V_\lambda)^* \) is \((-\lambda, -\lambda + \alpha)\). Hence the representation of \( SL_2 \) on the sum of these two extremal (hence 1-dimensional) weight spaces is isomorphic to the defining representation, in which
\[
e_\alpha(-c) \cdot \bar{v}_- \lambda = cm_\lambda(g) + m_\lambda(\bar{r}_\alpha g).
\]

\[\Box\]

Let \( I^Q_w \) denote the ideal in \( \mathbb{Q}[c_1, \ldots, c_{\ell(v)}] \) corresponding to \( X_{w_0} \), by pulling back the ideal defining \( B_- w B_+ \subseteq G \) along the map \( \hat{\beta}_Q : A_{\ell(v)} \rightarrow G \). In the \( G = GL_n \) case, it is easy to find generators for \( I^Q_w \) as follows. Fulton’s theorem \cite{F92} gives generators (a collection of minors) for the defining ideals of \((B_- \times B_+)-\)orbit closures in \( M_{\ell,0} \) and therefore also in \( GL_n \). Pulling these back along \( \hat{\beta}_Q \), we obtain generators for \( I^Q_w \). While Fulton’s generators
were shown to be a Gröbner basis in \cite{KnMi05}, it is not obvious that their images in $I_w^Q$ are again a Gröbner basis (a case of which is treated in \cite{WY}). In any case the following theorem does not assume $G = \text{GL}_n$.

**Theorem 7.** Fix $v \in W$, and a reduced word $Q$ for $v$. Then the function $f$ on $\mathbb{A}^{\ell(v)}$ defined by

$$f(c_1, \ldots, c_{\ell(v)}) := \prod_{\omega} m_\omega(\beta_Q(c_1, \ldots, c_{\ell(v)}))$$

$\omega$ ranging over $G$’s fundamental weights

is of degree $\ell(v)$, and its lex-initial term is $\prod_i c_i$.

Under the identification of $\mathbb{A}^{\ell(v)}$ with $X_v^\circ$, the divisor $f = 0$ is the preimage of $\bigcup_\alpha X_{r_\alpha}$. By decomposing and intersecting repeatedly, we can produce all the other $X_{w_\omega}^\circ$ from this divisor. If $I_w^Q$ is the ideal in $\mathbb{Q}[c_1, \ldots, c_{\ell(v)}]$ corresponding to $X_v^\circ$, then init $I_w^Q$ is Stanley-Reisner.

We can produce a Gröbner basis for $I_w^Q$ by concatenating Gröbner bases for $I_w^{Q'}$, with $w' \leq w$ in Bruhat order, and $w'$ basic in opposite Bruhat order on $W$. (The basic elements of Bruhat orders were computed in \cite{LaSch96, GKi97}.)

**Proof.** More specifically, we claim that for each $\omega$, init $m_\omega(\beta_Q(c_1, \ldots, c_{\ell(v)})) = \prod_{i : \langle \omega, \alpha_i \rangle = 1} c_i$.

(Recall that for each $\omega$, there exists a unique $\alpha$ with $\langle \omega, \alpha \rangle \neq 0$, and in that case $\langle \omega, \alpha \rangle = 1$. So the product of these $\prod_{i : \langle \omega, \alpha_i \rangle = 1} c_i$ will be the desired $\prod_i c_i$.)

This is proven by induction on $\ell(v)$, as follows. Let $Q = \alpha_1 Q'$, where $Q'$ is therefore a reduced word for $v' := r_{\alpha_1} v < v$. If $\langle \omega, \alpha_1 \rangle = 0$, then by lemma 7 part (2),

$$m_\omega(\beta_Q(c_1, c_2, \ldots, c_{\ell(v)})) = m_\omega(\beta_Q'(c_2, \ldots, c_{\ell(v)}))$$

If $\langle \omega, \alpha_1 \rangle = 1$, then by lemma 7 part (3),

$$\text{init } m_\omega(\beta_Q(c_1, c_2, \ldots, c_{\ell(v)})) = c_1 m_\omega(\beta_Q'(c_2, \ldots, c_{\ell(v)}))$$

(where init $c_1$ is defined as in the proof of proposition 3). So init $m_\omega(\beta_Q(c_1, c_2, \ldots, c_{\ell(v)}))$ is either init $m_\omega(\beta_Q'(c_2, \ldots, c_{\ell(v)}))$ or $c_1$ times that, and chaining these together, we get that init $m_\omega(\beta_Q(c_1, \ldots, c_{\ell(v)})) = \prod_{i : \langle \omega, \alpha_i \rangle = 1} c_i$.

The identification $\mathbb{A}^{\ell(v)} \to X_v^\circ$ is given by $(c_1, \ldots, c_{\ell(v)}) \mapsto \beta_Q(c_1, \ldots, c_{\ell(v)}) \cdot B/B$. By lemma 7 part (1), the preimage of $X_{r_\alpha}$ is given by $m_\omega(\beta_Q(c_1, \ldots, c_{\ell(v)})) = 0$. Hence the preimage of $\bigcup_\alpha X_{r_\alpha}$ is given by $f = 0$.

Just as in the case of matrix Schubert varieties, we can obtain all $X_w$ by intersecting/decomposing from $\bigcup_\alpha X_{\omega}$ because each $X_w$ is a component of $\bigcap_{w' < w} X_{w'}$. Nothing changes when we intersect with $X^v$ (essentially because $w \leq v$ and $w' < w$ so $w' < v$; the necessary $X_{w_0^v}$ are thus once again available by induction).

Now apply theorem 4 to see that each init $I_w^Q$ is Stanley-Reisner over $\mathbb{Q}$.

For the Gröbner basis statement, we use theorem 6 noting that opposite Bruhat order is the relevant one for containment of Schubert varieties. Every basic element of $\{w' : w' \leq v\}$ is basic in the opposite Bruhat order. Some of the basic $w'$ for the opposite Bruhat order may not be basic for this subposet, but adding them to the Gröbner basis does no harm.

Theorem 7 shows that $I_w^Q$ has a Gröbner basis whose leading terms are squarefree, but does not fully determine it (except for $v = r_{\alpha_1}$), nor does it even determine the leading
terms, which generate the initial ideal. We determined this initial ideal in \[\text{Kn08}\]; it is the Stanley-Reisner ideal of the “subword complex” \(\Delta(Q, w)\) of \[\text{KnMi04}\]. The map \(\pi_{f,\text{init}}\) defined in theorem 2 part (3), from the set of coordinate spaces in \(A^n\) to the set of compatibly split subvarieties of \(X^v\), is just the map taking a subword of \(Q\) to its Demazure/nil Hecke product. Then the order-preserving property of \(\pi_{f,\text{init}}\) is a standard characterization of the opposite Bruhat order in terms of existence of subwords.

As in \[\text{De85}\], this result and its proofs are the same if \(G\) is taken to be a Kac-Moody Lie group; even though \(G\) is infinite-dimensional, \(X^v\) is still only \(\ell(v)\)-dimensional. Unfortunately, it does not thereby apply to the varieties \(X_i^v \cap X^v\) when the big cell \(X_i^v\) is infinite-dimensional. This is a shame, as these varieties include nilpotent orbit closures \[\text{Lu81}\], as nicely recounted in \[\text{Ma}\].

We compute a sample \(f\), where \(Q = r_1 r_2 r_3 r_2\) is a reduced word for \(v = 2431\) in the Weyl group \(S_4\). Then \(m_{\omega_i}: G \rightarrow A^1\) is the upper left \(i \times i\) minor, and \(\beta_Q(c_1, c_2, c_3, c_4)\) is

\[
\begin{bmatrix}
c_1 & -1 & 0 & 0 \\
1 & c_2 & -1 & 0 \\
1 & 1 & c_3 & -1 \\
1 & 1 & 1 & c_4 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
1 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
c_1 & c_3 - c_2 c_4 & c_2 & -1 \\
1 & 0 & 0 & 0 \\
0 & c_4 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix},
\]

with \(m_{\omega_1} = c_1, m_{\omega_2} = c_2 c_4 - c_3,\) and \(m_{\omega_3} = c_3\). Note that they need not be homogeneous; homogeneity should only be expected when \(v\) is 321-avoiding, as that is the condition (more generally known as “\(\lambda\)-cominuscule”; see e.g. \[\text{Ste01}\]) for the \(T\)-action on \(X^v\) to contain the scaling action.\footnote{Proof. The \(T\)-weights on \(X^v\) are \(\{x_i - x_j : i < j, \pi(i) > \pi(j)\}\). \(\pi\) contains the pattern 321 iff \(\exists i < j < k\) with \(\pi(i) > \pi(j) > \pi(k)\). When that is the case, the weights on \(X^v\) include \(x_i - x_j, x_j - x_k, x_i - x_k\), and no linear functional on \(T^*\) can take value 1 on all three. If \(\pi\) is 321-avoiding, let \(G_m \rightarrow T\) take \(z\) to the diagonal matrix \(D\) with \(D_{ii} = z\) if \(\exists j > i, \pi(j) < \pi(i)\) and \(D_{ii} = 1\) otherwise; this \(G_m\) acts by dilation on \(X^v\).}

![Figure 2](image_url)

**Figure 2.** A graphical way to compute \(\beta_Q(c_1, c_2, c_3, c_4)\), using the wiring diagram of \(Q = 1232\). Each path from \(i\) to \(j\) contributes a term to the \((i, j)\) entry, with a factor of \(-1\) for each step down and a factor of \(c_i\) for each avoidance of \(x\) in favor of going through the \(c_i\) atop it. The path pictured contributes \(-1 \times c_2 \times c_4\) to the \((1, 2)\) entry. We invite the reader to redo the matrix calculation above using this diagram.

Matrix Schubert varieties in \(M_n\) are in fact Kazhdan-Lusztig varieties in \(\text{GL}_{2n}/B_{2n}\), a fact Fulton used in \[\text{F92}\] to show that matrix Schubert varieties are normal, Cohen-Macaulay, and have rational singularities. Taking \(Q\) to be the “square word”

\[
Q = (r_n r_{n-1} \cdots r_2 r_1)(r_{n+1} r_n \cdots r_3 r_2) \cdots (r_{2n-1} \cdots r_n)
\]
in $S_{2n}$ from [KnMi04, example 5.1], a straightforward computation yields
\[
\tilde{\beta}_Q(c_{n,1}, c_{n-1,1}, \ldots, c_{1,1}, c_{n,2}, \ldots, c_{1,2}, \ldots, c_{n,n}, \ldots, c_{1,n}) = \begin{bmatrix} C \cdot D & (-1)^n I_n \\ I_n & 0_n \end{bmatrix}
\]
where $C$ is the matrix of indeterminates $c_{ij}$, the matrices $I_n, 0_n$ are the identity and zero matrices of size $n$, and $D$ is the diagonal matrix with alternating signs $D_{ii} = (-1)^{i-1}$. Then the $2n-1$ minors $m_{\alpha}$ are, up to signs, the $(d_{i,1})$ and $(d_{i,n})$ considered in section 7.2 These are homogeneous (being determinants), and the corresponding $v = (n+1)(n+2) \ldots (2n)123 \ldots n$ is indeed 321-avoiding.

As stated, theorem [7] is about Kazhdan-Lusztig varieties $X_w \cap X^\vee_v$ in a full flag manifold $G/B$. If $P \supset B$ is a parabolic subgroup and $v$ is minimal in its $W/W_P$ coset, then the composite map $X_v^\vee \hookrightarrow G/B \twoheadrightarrow G/P$ is an isomorphism of opposite Schubert cells. If $w$ is also minimal in its coset, this restricts to an isomorphism of Kazhdan-Lusztig varieties. For example, to study a neighborhood on a Schubert variety $w \in W_P$ centered at the most singular point $w_0P/P$, we can apply the theorem with $v = w_0 w_P^\circ$ and $w$ minimal in its coset. The matrix Schubert variety case just described is almost an example of this, except that $w$ is not minimal in its coset.

7.4. Multiplicity-free divisors on $G/B$ are splittable. A reduced subscheme $X \subseteq G/B$ is called multiplicity-free in [Br03] if the expansion $[X] = \sum_{\pi} c_{\pi}[X_{\pi}]$ of its Chow class in the basis of Schubert classes $[X_{\pi}], \pi \in W$ has $c_\pi = 0, 1$. Brion proves many wonderful geometric facts about such subschemes, when they are irreducible. We use theorems [5] and [7] to relate this property to Frobenius splitting.

**Proposition 4.** Let $X$ be a multiplicity-free divisor on a flag manifold $G/B$. Then there is a Frobenius splitting of $G/B$ that compatibly splits $X$. If $X$ does not contain the Schubert point $w_0 B/B$, then the splitting can also be made to compatibly split all the Schubert varieties.

**Proof.** We may assume that $X$ does not contain the Schubert point, by using some $g \in G$ to move some point outside $X$ to $w_0B/B$. (This may require extending the base field so $G/B$ has closed points outside $X$.)

Write $\alpha \in [X]$ if the Schubert class $[X_\alpha]$ appears in the expansion of the Chow class $[X]$. Let $Y = X \cup (\bigcup_{\alpha \in [X]} X_\alpha) \cup (\bigcup_{\alpha \in [X]} w_0 \cdot X_\alpha)$. Since $X$ does not contain the Schubert point, it does not contain any of the Schubert divisors $X_\alpha$. If $X$ contains an opposite Schubert divisor $w_0 \cdot X_{\alpha'}$ then $\alpha \in [X]$. Hence no component listed in this union equals any other. The Chow class of this sum is therefore the sum of the classes of the terms, hence $2 \sum_\alpha [X_\alpha]$, the anticanonical class.

Now apply theorem [5] part (2), and theorem [7] with $v = w_0$, to see that $Y$ defines a Frobenius splitting on $G/B$ with respect to which $Y$ is compatibly split. Since $X$ is a union of components of $Y$, it is also split.

If we actually care about splitting the original $X$, rather than $g \cdot X$, we can split using $g^{-1} \cdot Y$ instead. If $X$ doesn’t contain the Schubert point, then we can take $g = 1$. □

This raises the question of whether this proposition holds for any multiplicity-free subscheme, not just divisors. Note that this proposition does not require $X$ to be irreducible, though Brion gives counterexamples showing that some of his results depend on irreducibility.
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E-mail address: allenk@math.cornell.edu