Synchronization Control of Complex Dynamical Networks Based on Uncertain Coupling

Qianqian Jia
Zhonghuan Information College, Tianjin University of Technology, Tianjin, China
99 Liu Kou Road Yangliuqing Xiqing District of Tianjin City, 022-86437560
Corresponding author, email: jqhb@163.com

Abstract
In recent years, the study of complex networks has been a focus subject of technology fields. In this paper, we consider the adaptive control and synchronization of uncertain complex networks. By using the adaptive control techniques with the linear feedback updated law and the well-known LaSalle invariant principle on dynamical system theory, some simple yet generic criteria are derived. Furthermore, the result is applied to typical chaotic cellular neural networks (CNN). Finally, numerical simulations are presented to demonstrate the feasibility and effectiveness of the proposed techniques.

Keywords: Uncertain complex networks, Adaptive control, Synchronization, LaSalle invariance principle

1. Introduction
Complexity and complex system is one of the key research topics in twenty-first Century. Complex network is an important tool for describing and understanding of complex systems, many individuals it will be summarized in the grounds of complex systems (node) network system, widely used in different fields of science, such as sociology, biology, computer science, in physics, engineering and so on, has become an important research topic in the field of complexity science. Synchronization is widespread in all kinds of complex network system, is a typical collective behavior on complex networks, and it is also one of the most important dynamic characteristics of complex network. The synchronization control of complex dynamical networks is a key link in the complex network of research and application, it is to study the nonlinear dynamics of the effective theoretical basis and tools, and in secure communication, network congestion control, the generation of harmonic oscillator Multi-Agent Consistent and other fields have great potential applications. In recent years, with the application of nonlinear control theory, the theory of the adaptive control has been developed. Due to advances in the theory, and effective calculation method is easy to get, the adaptive control method has been applied in many practical problems. Especially the small world network model and the scale free network model, has already caused the scholars interest [1, 2]. At present, the relationship between the network topology and the dynamic behavior has become one of the hot issues of research for each country in the world. Synchronization and control of complex dynamical network has become a very active research field. There are many important results for complex networks with different topologies [3-13].

This paper mainly studies adaptive control and synchronization problems of uncertain complex dynamic networks based on uncertain coupling. With the help of linear feedback adaptive control technology and famous LaSalle invariant principle in the dynamic system theory, a series of simple and practical guidelines of uncertain complex dynamic networks adaptive synchronization is given. Our results indicate that the method used in this paper does not extend the existing literature's ideas and techniques, and is simple and easy to implement in practice. Further, we apply these principles to the coupled cellular neural networks (CNN) models. Numerical simulation results show the correctness and effectiveness of this control method.
2. Model of Uncertain Complex Dynamic Network

Consider an uncertain complex network, which is coupled with N constant dynamic nodes, in which each node is a n-dimensional dynamical system. The state equation of the network can be described as follows [14-15]:

\[ \dot{x}_i = f(x_i) + g_i(x_{i1}, \ldots, x_{in}), \quad i = 1, 2, \ldots, N \]  

(1)

State equation of uncertain complex dynamic network using the adaptive controller is expressed as follows:

\[ \dot{x}_i = f(x_i) + g_i(x_{i1}, \ldots, x_{in}) + u, \quad i = 1, 2, \ldots, N \]  

(2)

Where \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n \) is the state variable of the node.

And \( u_i \in \mathbb{R}^n, f: \mathbb{R}^n \to \mathbb{R}^n, g_i: \mathbb{R}^m \to \mathbb{R}^n \) represent the adaptive controller, the nonlinear vector function, and the uncertain dissipative coupling function separately. The functions \( f(x_i) \) and \( g_i(x_{i1}, \ldots, x_{in}) \) are uncertain here.

Note that the coupling functions \( g_i(x_{i1}, \ldots, x_{in}) \) of system (1) and (2) can be divided into the following forms:

1. \( g_i \) is a linear combination of the nodal state variables, i.e., \( g_i = c \sum_{j=1}^{N} a_{ij} x_j \), where \( c \) is the coupling strength, \( \Gamma \) is the internal coupling matrix, and \( a_{ij} \) satisfies \( \sum_{j=1}^{N} a_{ij} = 0 \).

2. \( g_i \) is a combination of nonlinear functions, i.e., \( g_i = \sum_{j=1}^{N} a_{ij} H(x_j) \), where \( H(\cdot) \) is the internal coupling function.

In the above two cases, the internal coupling functions are consistent. But in reality, the internal coupling function \( g_i(i = 1, 2, \ldots, N) \) is not consistent. Then the internal coupling functions can be divided into two parts, namely, the linear and nonlinear functions. In the network (1) and (2), there are two integers \( i_0 \) and \( j_0 \) (\( i_0 \neq j_0 \)), which makes \( g_{i_0} = c \sum_{j=1}^{N} a_{i_0 j} x_j \) and \( g_{j_0} = c \sum_{i=1}^{N} a_{i j_0} H(x_i) \).

3. Synchronization of Uncertain Complex Dynamic Network

Definition 1: Let \( x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t)) \) be the solution of (1). Define the constant hyper plane.

\[ S = \{ x = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t)) : x_i(t) = s(t), i, j = 1, 2, \ldots, N \} \]  

(3)

As the synchronization manifold of uncertain complex dynamic network (1).

Where \( s(t) \in \mathbb{R}^n \) is known as synchronization state of the dynamic network, satisfying:

\[ s(t) = f[s(t)], \]  

(4)

Definition 2: The dynamic network (1) is synchronized, when:

\[ \text{Synchronization Control of Complex Dynamical Networks Based on Uncertain… (Qianqian Jia)} \]
We will analyze complete synchronization problem of uncertain dynamic network. The control objective is to design and implement a suitable controller $u_i$, so that to make the controlled network (2) reach complete synchronization. That means all solutions $x_i(t)(i = 1, 2, ..., N)$ of network (2) satisfy:

$$\lim_{t \to \infty} \| x_i(t) - s(t) \| = 0, \ i = 1, 2, ..., N$$

(5)

where $s(t) \in \mathbb{R}^n$ satisfy:

$$\dot{s}(t) = f[s(t)],$$

(7)

Let $f$ and $g_j$ represent the dynamic equation of the node and the internal coupling function separately. Suppose $f$ and $g_j$ are uncertain but satisfy the following conditions:

(a) There is a nonnegative constant $\gamma$ making:

$$| f(x_i - f(s)) - \gamma | x_i - s |, \ i = 1, 2, ..., N$$

(8)

(b) There is a nonnegative constant $\gamma_j$ making:

$$| g_j(x_{i1}, x_{i2}, ..., x_{in}) - g_j(s, s, ..., s) | \leq \sum_{j=1}^{N_j} \gamma_j | x_{ij} - s |, \ i = 1, 2, ..., N, \ j = 1, 2, ..., N$$

(9)

Theorem 1. Suppose that conditions (a) and (b) were established. And there is linear feedback controller.

$$u_i = \epsilon_i x_i - s,$$

(10)

where $\epsilon = \text{diag}(\epsilon_1, ..., \epsilon_N)$ is the intensity of feedback and has the self-tuning law as follows:

$$\dot{\epsilon}_i = -\alpha_i |\epsilon_i(t)| \exp(\mu t).$$

(11)

Where $\mu \geq 0$ is a real number which is optional and small enough, and $\alpha_i > 0 (i = 1, 2, ..., N)$ is an arbitrary constant. Synchronization error is expressed as follows:

$$e_i(t) = x_i(t) - s(t), \ i = 1, 2, ..., N$$

(12)

Let $Y = \max_{1 \leq i \leq N, 1 \leq j \leq N} \gamma_j$, if exist an optional positive constant $L_i$ satisfies:

$$L_i > \gamma + NY + \frac{\mu}{2}, \ i = 1, 2, ..., N$$

(13)

Then the complete synchronization of uncertain dynamic network (2) can be reached and we has:

$$\sum_{j=1}^{N} |e_j(t)| = \sum_{j=1}^{N} |x_j(t) - s(t)| = o(\exp(-\mu t)), \ i = 1, 2, ..., N$$

(14)
Proof. Construct Lyapunov function as follows:

\[ V(t) = \sum_{i=1}^{N} |e_i(t)| \exp(\mu t) + \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\alpha_i} (\varepsilon_i + L_i)^2 \]  

(15)

Where \( L_i \) is an optional positive constant. Calculating the derivative of \( V(t) \) along (16), we have:

\[ \dot{V}(t) = \sum_{i=1}^{N} [\text{sgn} e_i(t) \dot{e}_i(t) \exp(\mu t) + |e_i(t)| \mu \exp(\mu t) + \frac{1}{\sigma_i} (\varepsilon_i + L_i) \dot{\varepsilon}_i] \]

\[ = \sum_{i=1}^{N} [\text{sgn} e_i(t) f(x_i) - f(s) + g_i(x_{s_1}, x_{s_2}, ..., x_{s_N}) - g_i(s, s, ..., s) + \varepsilon_i e_i] \exp(\mu t) \]

\[ + |e_i(t)| \mu - \varepsilon_i + L_i |e_i(t)| \exp(\mu t) \]

\[ \leq \sum_{i=1}^{N} [\text{sgn} e_i(t) f(x_i) - f(s) + g_i(x_{s_1}, x_{s_2}, ..., x_{s_N}) - g_i(s, s, ..., s)] \]

\[ + |e_i(t)| \mu - L_i e_i^2(t) \exp(\mu t) \]

\[ \leq \sum_{i=1}^{N} \left[ \gamma + \sum_{j=1}^{N} \gamma_{ij} \right] |e_i(t)| \exp(\mu t) \leq 0 \]

In the process of proof, \( \text{sgn} e_i(t) e_i(t) = |e_i(t)| \) is used.

Obviously, \( \dot{V} = 0 \) if and only if \( e_i = 0 \) \((i = 1, 2, ..., N)\). Using LaSalle invariance principle, any solution of the error system (15) asymptotically converge to the largest invariant set:

\[ E = \{ (e, \varepsilon) \mid e = 0, \varepsilon = \varepsilon_0 \} \]  

(16)

Where \( \varepsilon_0 \) is a constant. That means \( e \to 0, \varepsilon \to 0 \) as \( t \to \infty \), i.e., the complete synchronization of uncertain dynamic network can be reached. By proof, the error satisfies:

\[ \sum_{i=1}^{N} |e_i(t)| = \sum_{i=1}^{N} |x_i(t) - s(t)| = \alpha \exp(-\mu t), \quad i = 1, 2, ..., N \]  

(17)

Corollary 1. If the coupled part of the network is:

\[ g_i = \sum_{j=1}^{N} a_{ij} x_j \] \((i = 1, 2, ..., N)\),

(18)

i.e. the coupling part is linear. Just conditions (a) met, the network can achieve synchronization.

Proof. If the coupling part of the network is linear, we have:
\[|g_j(x_1, x_2, \ldots, x_N) - g_i(s, s, \ldots, s)| = \sum_{j=1}^{N} a_{ij} |x_j - s| = \sum_{j=1}^{N} a_{ij} |e_j| \]  

\[g_j = \sum_{i=1}^{N} a_{ij} H(x_i), \quad (i = 1, 2, \ldots, N), \] \hspace{1cm} \text{(19)}

i.e., condition (b) is established. So just conditions (A) met, the network can achieve synchronization.

**Corollary 2.** If the coupled part of the network is:

\[g_j = \sum_{i=1}^{N} a_{ij} H(x_i) \] \hspace{1cm} \text{(20)}

i.e. the coupling part is nonlinear. When \(|H(x_i) - H(s)| \leq \delta |x_i - s| = \delta |e_i| (\delta \text{ is positive constant})\), just condition (a) is met, the network can achieve synchronization.

**Proof.** If the coupling part of the network is nonlinear, we have:

\[|g_j(x_1, x_2, \ldots, x_N) - g_i(s, s, \ldots, s)| = \sum_{j=1}^{N} a_{ij} |e_j| \leq \sum_{j=1}^{N} a_{ij} \delta |x_j - s| = \sum_{j=1}^{N} a_{ij} \delta |e_j| \]  

\[i.e., \text{condition (b) is naturally established. So just condition (a) is met, the network can achieve synchronization.} \]  

\[|g_j(x_1, x_2, \ldots, x_N) - g_i(s, s, \ldots, s)| \leq \sum_{j=1}^{N} a_{ij} \delta |x_j - s| = \sum_{j=1}^{N} a_{ij} \delta |e_j| \] \hspace{1cm} \text{(21)}

\[|g_j(x_1, x_2, \ldots, x_N) - g_i(s, s, \ldots, s)| = \sum_{j=1}^{N} a_{ij} |x_j - s| \leq \sum_{j=1}^{N} a_{ij} |e_j| \]

\[i.e., \text{condition (b) is naturally established. So just condition (a) is met, the network can achieve synchronization.} \]

**4. Numerical Simulations**

Considered a three-dimensional network model of CNN. The state equation is as follows:

\[\dot{x}(t) = -C x(t) + A f(x(t)) + A' f(x(t - \tau)) + u(t), \] \hspace{1cm} \text{(22)}

Where,

\[A = \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1.0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

And \(u(t) = (0, 0, 0)^T\). Then the network state equation is expressed as follows:

\[x(t) = -C x(t) + A f(x(t)) \]  

\[\text{(23)}\]
Where \( x(t) = (x_1(t), x_2(t), x_3(t))^T \) is the state variables of the network. And \( f(x(t)) = [f(x_1(t)), f(x_2(t)), f(x_3(t))]^T \). Function \( f_i(x) = f(x)(i=1,2,3) \) is a piecewise linear function expressed as follows:

\[
f(x) = \frac{1}{2}(|x+1|-|x-1|)
\]

(24)

Obviously, the above network is a typical neural network. There is a chaotic attractor in the model, as shown in Figure 1, Figure 2, Figure 3.

![Figure 1. Chaotic attractor of CNN model in X-Y plane](image1)

![Figure 2. Chaotic attractor of CNN model in Y-Z plane](image2)

![Figure 3. Chaotic attractor of CNN model in X-Z plane](image3)

Considered a dynamic system, which is linear dissipative coupled by three cellular neural networks (CNN). The state equation of the whole system is expressed as follows:

\[
\dot{x}_i(t) = -C_i \dot{x}_i(t) + A \dot{F}(x_i(t)) + \sum_{j=1}^{3} b_{ij} \dot{F}(x_j(t)), \quad i = 1,2,3
\]

(25)
Where the \( x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \) is the state variable of the \( i \)-th nervous system. The external coupling matrix is expressed as follows:

\[
B = \begin{pmatrix}-8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & 8\end{pmatrix}
\]

The controlled state equation of system is expressed as follows:

\[
\dot{x}_i(t) = -C x_i(t) + Af(x_i(t)) + \sum_{j=1}^{3} b_{ij} \Gamma x_j(t) + e_i, \quad i = 1, 2, 3
\]  

(26)

Where \( e_i, \quad (i = 1, 2, 3) \) is the feedback intensity.

The correction law of feedback intensity \( e_i, \quad (i = 1, 2, 3) \) under the control method is expressed as follows:

\[
\dot{e}_i = -2 \left| x_i(t) - s(t) \right| \exp(0.0025t), \quad i = 1, 2, 3
\]  

(27)

Digital simulation results show that the coupled cellular neural networks (CNN) can achieve synchronization, as shown in Figure 4.

Now the external coupling matrix is expressed as follows:

\[
B = \begin{pmatrix}-4 & 2 & 3 \\ 1 & -4 & 2 \\ 3 & 1 & -4\end{pmatrix}
\]

Numerical simulation results show that the coupled cellular neural networks (CNN) can achieve synchronization, as shown in Figure 5.

Figure 4. Adaptive synchronization based on self tuning law (27) in the time interval [0,1] of the coupled CNN cellular neural network.
Consider a dynamic system which is nonlinear dissipative coupled by three cellular neural networks (CNN). Numerical simulation results show that the coupled cellular neural networks (CNN) can achieve synchronization, as shown in Figure 6.

Based on the above numerical simulation results, we can see that when the external coupling function of the network is both linear and nonlinear function, the control method applied to the network can make the network achieve synchronization. And the numerical simulation results clearly show that, compared with the method in literature [16], the feedback strength and self tuning law in this paper can make the network achieve complete synchronization in a shorter time. Therefore, the development of the method to further expand the existing literature of ideas and techniques, the results and the theoretical results of this paper are completely consistent.

5. Conclusion
This paper mainly studies adaptive control and synchronization of uncertain complex networks. The main work is supposed as follows:
1. For the uncertain complex networks with adaptive controller, uncertain complex dynamical network model under the adaptive control.

Figure 5. Adaptive synchronization based on self tuning law (27) in the time interval [0,1] of the coupled CNN cellular neural network with coupling matrix B

Figure 6. Adaptive synchronization based on self tuning law (27) in the time interval [0,1] of the coupled CNN cellular neural network with nonlinear coupling
2. Design the controller to make the network achieve synchronization, and prove its rationality by LaSalle invariance principle.

3. The theoretical results obtained in this paper are verified by MATLAB numerical simulation. The results of this paper have important theoretical significance and reference value in the practical engineering design.

4. In contrast to previous work, the main contribution of this paper is to propose a new self tuning law which can make the complex dynamic networks achieve the synchronization state more quickly.

References

[1] Strogatz SH. Exploring complex networks. Nature. 2001; 410(6825): 24-27.

[2] Al Barabasi. Albert R. Emergence of scaling in random networks. Science. 1999; 286(5439): 509-512.

[3] X Wang. Complex networks: Topology, dynamics and synchronization. Int. J. Bifurcation Chaos. 2002; 12(5): 885-916.

[4] Chai Y, Chen L, Wu R, et al. Adaptive pinning synchronization in fractional-order complex dynamical networks. Physics Letters A. 2009; 373(17): 1553-1559.

[5] Li H, Yue D. Synchronization of Markovian jumping stochastic complex networks with distributed time delays and probabilistic interval discrete time-varying delays. Journal of Physics A Mathematical & Theoretical. 2010; 43(10): 641-648.

[6] Yu W, Chen G, Cao J. Adaptive synchronization of uncertain coupled stochastic complex networks. Asian Journal of Control. 2011; 13(3): 418-429.

[7] Ji K, Wei D. Resilient control for wireless networked control systems. International Journal of Control Automation & Systems. 2011; 9(9): 285-293.

[8] W Yu, W Zheng, G Chen, W Ren. Second-order consensus in multi-agent dynamical systems with sampled position data. Automatica. 2011; 47: 1496-1503.

[9] Wu X, Lai D, Lu H. Generalized synchronization of the fractional-order chaos in weighted complex dynamical networks with nonidentical nodes. Nonlinear Dynamics. 2012; 69(1-2): 667-683.

[10] W Wong WK, Li H, Leung SY. Robust synchronization of fractional-order complex dynamical networks with parametric uncertainties. Communications in Nonlinear Science & Numerical Simulation. 2012; 17(12): 4877-4890.

[11] Chen L, Chai Y, Wu R, et al. Cluster synchronization in fractional-order complex dynamical networks. Physics Letters A. 2012; 376(35): 2381-2388.

[12] Li H. H ∞ H ∞ mathContainer Loading Mathjax, clustering synchronization and state estimation for complex dynamical networks with mixed time delays. Applied Mathematical Modelling. 2013; 37(12-13): 7223-7244.

[13] Liang Y, Wang X. Synchronization in complex networks with non-delay and delay couplings via intermittent control with two switched periods. Physica A Statistical Mechanics & Its Applications. 2014; 395(4): 434-444.

[14] J Zhou, J Lu. Adaptive synchronization of an uncertain complex dynamical network. IEEE. Trans. Automat, Control. 2006; 51.

[15] M Chen, D Zhou. Synchronization in uncertain complex networks. Chaos. 2006; 16: 13-101.

[16] Jia QQ. Adaptive Control and Synchronization of Uncertain Complex Networks. Applied Mechanics & Materials; 2015; 734(1): 321-326.