ON MULTITYPE RANDOM FORESTS WITH A GIVEN DEGREE SEQUENCE, THE TOTAL POPULATION OF BRANCHE
FORESTS AND ENUMERATIONS OF MULTITYPE FORESTS

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The degree sequence \((N_{i,j}(k), 1 \leq i, j \leq d, k \geq 0)\) of a multitype forest with \(d\) types, is the num-
ber of individuals type \(i\), having \(k\) children type \(j\). We construct a multitype forest sampled uniformly
from all multitype forest with a given degree sequence (MFGDS). For this, we use an extension of
the Ballot Theorem by [CL16], and generalize the Vervaat transform [Ver79] to multidimensional
discrete exchangeable increment processes. We prove that MFGDS are extensions of multitype
Galton-Watson (MGW) forests, since mixing the laws of the former, one obtains MGW forests with
fixed sizes by type (CMGW). We also obtain the law of the total population by types in a MGW
forest, generalizing Otter-Dwass formula [Ott49, Dwa69]. We apply this to obtain enumerations of
plane, labeled and binary multitype forests having fixed roots and individuals by types. We give an
algorithm to simulate certain CMGW forests, generalizing the unitype case of [Dev12].

1. INTRODUCTION

Bienaymé-Galton-Watson forests (GW forests) are a simplified model for the genealogy of pop-
ulations, where individuals have the same reproduction law. A natural generalization of such
model are the multitype Galton-Watson forest (MGW forests), applied when several types of in-
dividuals coexist (leading to different reproduction rates). Such MGW forest have applications
in biology, demography, genetics, medicine, epidemics, and language theory (see [Har63, San71,
Jag75, GP75, CKB+, AJ97, All11, Dur15, KA15]), and others. But also they have several ap-
plications for pure mathematics. Miermont [Mie08] has proved that under certain conditions,
MGW forests converge to Aldou’s Continuum Random Tree (CRT, see [Ald91a]). Conditioned
random forests also provides us with several applications. In the unitype setting, GW forests
conditioned on its population coincide with various combinatorial models, and also provide us
with the computation of characteristics in several branching processes conditioned to be large (see
[Ald91b, Pit98, Dev98, Dev12, Jan12] for applications and motivations). The scaling limit of such
trees towards the CRT, was proved by Le Gall [LG05], which, together with the work of Aldous,
opened the path to the study of random real trees.

In this paper, we work with MGW forests conditioned to have a total number \(n_i \in \mathbb{N}\) of indi-
viduals type \(i\), for \(i = 1, \ldots, d\), that is, conditioned with the total number of individuals by types
(CMGW forests). Such work is based on the law of the total population by types given in [CL16]
(see also [Wan14, ADG18]). Another way to condition a forest is by its degree sequence, that is,
the number of individuals having a fixed number of offspring, as done on [BM14b, Lei19] in the
unitype setting. With no doubt, this model helps to the study of invariance principles for random
graphs with a prescribed degree sequence, introduced as the configuration model by [BC78, Bol80]

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trees with given degree sequence to construct, from a discrete exchangeable increment (EI) process, a uniform tree from the set of Dwa69]. This formula says that the total number of individuals in a GW forest \( \tau \) forest, under certain conditions. The unitype case is known as the Otter-Dwass formula [Ott49, cyclical permutations lead to paths coding a multitype forest.\]

pely to them a generalized Vervaat transform. We use the results in [CL16] to know how many dxd define a multitype degree sequence, construct titype forest with a given degree sequence. In order to do this, we generalize the above algorithm: is that one has to carefully chose the cyclical permutations that lead bridges to excursions. Recover the desired tree. This algorithm was extended to unitype forests in [Lei19], the distinction minimum value (that is, apply the Vervaat transformation [Ver79]). From such excursion [Nak78, Mie08, multitype (conditioned) forests, is to prove its convergence towards a limiting object, such as the covergence of MGW forests to \textit{multitype Lévy forests}. We state the known results in the unidimensional case, and how we generalize them. Consider a unidimensional degree sequence \( S \), that is, a sequence of integers \( S = (N_i, i \geq 0) \) such that \( s := 1 + \sum iN_i = \sum N_i \). From such sequence we can obtain the child sequence \( c(S) := c = (c_1, \ldots, c_s) \), a vector with \( N_0 \) zeros, \( N_1 \) ones, and so on. In the paper [BM14b], the authors give an algorithm to construct, from a discrete exchangeable increment (EI) process, a uniform tree from the set of trees with given degree sequence \( S \) as follows: define \( W^b \) a walk with increments \( (c \circ \pi(j) - 1, j \in [s]) \), where \( \pi \) is a uniform random permutation on \([s]\), and let \( W \) be the walk with increments \( (c \circ \pi(i^* + j) - 1, j \in [s]) \), where \( i^* + j \) is considered modulo \( s \) and \( i^* \) is the first time \( W^b \) reaches its minimum value (that is, apply the Vervaat transformation [Ver79]). From such excursion \( W \) we can recover the desired tree. This algorithm was extended to unitype forests in [Lei19], the distinction is that one has to carefully chose the cyclical permutations that lead bridges to excursions.

We extend the previous construction to multitype forest, uniformly chosen from the set of multitype forest with a given degree sequence. In order to do this, we generalize the above algorithm: define a multitype degree sequence, construct \( dx \times dx \) exchangeable increment (EI) processes, and apply to them a generalized Vervaat transform. We use the results in [CL16] to know how many cyclical permutations lead to paths coding a multitype forest.

Also, using the results in [CL16], we obtain the law of the total population by types of a MGW forest, under certain conditions. The unitype case is known as the Otter-Dwass formula [Ott49, Dwa69]. This formula says that the total number of individuals in a GW forest \( \tau_k \) with \( k \) trees, say \( \#\tau_k \), having offspring distribution \( \nu \) is given by

\[
\mathbb{P}(\#\tau_k = n) = \frac{k}{n} \mathbb{P}(X_n = n - k),
\]

where \( X \) is a random walk with step law \( \nu \).

It turns out that, using the law of \( \#\tau_k \), it has been obtained the total number of plane, labeled and binary forests having \( k \) trees and \( n \) vertices, see [Pit98]. This paper generalizes those elementary connections between combinatorics and probability about enumerations of forests and lattice paths.
Algorithm is: generate a multinomial vector $S$ degree sequence; thus, we use both of our constructions to generalize such algorithm. Devroye’s have

**Definition.** Let $i$, for $i \in [d]$, and having offspring distribution $\nu = (v_1, \ldots, v_d)$. Indeed, an algorithm of Devroye [Dev12] simulates a GW tree conditioned to have size $n$, using a uniform tree with a given degree sequence; thus, we use both of our constructions to generalize such algorithm. Devroye’s algorithm is: generate a multinomial vector $S = (N_0, N_1, \ldots)$ with parameters $(n; v_0, v_1, \ldots)$, repeat until $1 + \sum iN_i = n$ and apply the algorithm to generate a uniform tree from the set of trees with degree sequence $S$. Our algorithm is analogous: generate $d \times d$ multinomial distributions with laws $(n; v_{ij}(0), v_{ij}(1), \ldots)$ until they form a multitype degree sequence, and apply the algorithm to generate a uniform multitype forest with such given degree sequence.

### 1.1. Preliminaries.

#### 1.1.1. Coding of unititype and multitype forests.

A rooted plane tree $T$ is a connected graph with no cycles having a distinguished vertex, together with a natural identification of each vertex by a finite sequence of non-negative integers (denoting its location on the tree). The root of $T$ will be denoted by $r(T)$, or simply $r$. A rooted plane forest is a directed planar graph whose connected components are rooted plane trees, those are ordered according to its roots. We will only consider finite rooted plane forests in the following.

We consider forests where each tree is labeled according to the breadth-first order (BFO), that is, from the initial individual to the top, traverse each tree generation by generation from left to right. We define the vector with $i$th component, the number of individuals having $i$ children, for any $i \geq 0$.

**Definition.** Let $T$ be a tree. The degree sequence $S = (N_0, N_1, \ldots)$ of $T$ is a vector with

$$N_i := N_i(T) = |\{u \in T : c(u) = i\}|,$$

where $c(u)$ is the number of children of individual $u$.

Let $(u_i)_1$ be the individuals in BFO of a plane forest. It is well known that the walk with increments $(c(u_i) - 1, i \geq 1)$ codes the branching forest, that is, determines its structure completely (see [Pit06, Lemma 6.2]). This is called the breadth-first walk (BFW) of the forest. Now, we briefly recall the analogous coding in the multitype case, following [CL16].

Define $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$. For a forest $F$, let $c_F : v(F) \mapsto [d]$ be an application from the set of vertices of $F$ to $[d]$, such that the children of each vertex are ordered by color, that is, if $u_i, u_{i+1}, \ldots, u_{i+j} \in v(F)$ have the same parent, then $c_F(u_i) \leq c_F(u_{i+1}) \leq \cdots \leq c_F(u_{i+j})$. The couple $(F, c_F)$ is a $d$-multitype forest. A subtree of type $i$ of $(F, c_F)$, denoted by $T^{(i)}$, is a maximal connected subgraph of $(F, c_F)$ whose all vertices are of type $i$. Subtrees of type $i$ are ranked according to the order of their roots, and with this ordering, we define the subforest of type $i$ of $(F, c_F)$ as $F^{(i)} = \{T_1^{(i)}, \ldots, T_k^{(i)}, \ldots\}$. For $u \in v(F)$, denote by $p_i(u)$ the number of children of type $i$ of $u$. Let $n_i \geq 0$ be the number of vertices in the subforest $F^{(i)}$ of $(F, c_F)$. The coding of the
We set $x(1)$ length $n \in q$ 
dimensional chain $x^{(i)} = (x^{(i), 1}, \ldots, x^{(i), d}) \in \mathbb{Z}^d$ with length $n_i \in \mathbb{N}$, defined for $0 \leq n \leq n_i - 1$ by
\begin{equation}
x^{(i), j}_{n + 1} - x^{(i), j}_n = p_j(u^{(i)}_{n + 1}) - 1 \{i = j\} \quad i, j \in [d].
\end{equation}
We set $x^{(i)}_0 = 0$. The set $(u^{(i)}_n; n \geq 1)$ is the labeling of the subforest $F^{(i)}$ in its own breadth-first order. In Figure 1 we show the BFO and

The cyclical permutations that we use are the following. For $n \in \mathbb{N}$, consider any application $y : [n]_0 \mapsto \mathbb{Z}^d$ with $y(0) = 0$. The $n$-cyclical permutations of $y$ are the $n$ applications $\theta_{q,n}(y)$, for $q \in [n - 1]_0$ given by
\begin{align*}
\theta_{q,n}(y) &= \begin{cases} 
y(j + q) - y(q) & j \leq n - q 
y(j + q - n) + y(n) - y(q) & n - q \leq j \leq n. 
\end{cases}
\end{align*}
We say that the path $y : \mathbb{N} \mapsto \mathbb{Z}$ is a downward skip-free chain, if $y_{k+1} - y_k \in \mathbb{Z}_+ \cup \{-1\}$. The possible paths that a coding of multitype forest can take are the following.

**Definition.** Fix any $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d_+$, and define $S_d$ as the set of $[\mathbb{Z}^d]^d$-valued sequences $x = (x^{(1)}, \ldots, x^{(d)})$ such that for all $i \in [d]$, $x^{(i)} = (x^{(i), 1}, \ldots, x^{(i), d})$ is a $\mathbb{Z}^d$-valued sequence starting at zero of length $n_i$, and where $x^{(i)} = (x^{(i), k}; k \in [n_i]_0)$ is non-decreasing when $i \neq j$, and a downward skip-free chain when $i = j$.

The $n$-cyclical permutations of $x \in S_d$ are given by
\begin{align*}
\theta_{q,n}(x) := (\theta_{q_1,n_1}(x^{(1)}), \ldots, \theta_{q_d,n_d}(x^{(d)})) \quad \forall q = (q_1, \ldots, q_d) \text{ such that } 0 \leq q \leq n - 1_d,
\end{align*}
with $1_d = (1, \ldots, 1)$ of length $d$. Each sequence $\theta_{q,n}(x)$ will be called a cyclical permutation of $x$.

For $m, n \in \mathbb{Z}^d_+$, write $m < n$ if $m < n$ (the inequality understood component-wise) and if there exists $i$ such that $m_i < n_i$. Sequences $x \in S_d$ will be denoted by $x = (x^{(i), k}; k \in [n_i]_0, i, j \in [d])$, and the vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d_+$, is called the length of $x$. Fix any such $x$ of length $n$, and
\[ r = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d \] with \( \sum r_i > 0 \). We say that the system \((r, x)\) admits a solution if there exists \( m \leq n \) such that

\[ r_j + \sum_{i=1}^{d} x^{i,j}(m_i) = 0 \quad \forall j \in [d]. \]

If there is no smaller solution \( m < n \) for the system \((r, \theta_{q,a}(x))\), then we call \( \theta_{q,a}(x) \) a good cyclical permutation. It is proved in [CL16] that only such good cyclical permutations code multitype forests, and the next lemma tells us how many there are.

**Lemma 1** (Multivariate Cyclic Lemma [CL16]). Let \( x \in S_d \) with \( x^{i,j}(n_i) \neq 0 \) for every \( i \in [d] \). Consider the system \((r, x)\) with solution \( n \) as above. Then, the number of good cyclical permutations of \( x \) is \( \det((-x^{i,j}(n_i))_{i,j \in [d]}) \).

Since in most of the cases, we fix the number of roots or number of individuals of each type, we need the following definition.

**Definition** (Root-type and individuals-type). We say a multitype plane forest with \( d \in \mathbb{N} \) types has root-type \( r = (r_1, \ldots, r_d) \in \mathbb{N}^d \), if it has \( r_i \) roots of type \( i \) for \( i \in [d] \), with \( r > 0 \) (that is, \( r_i > 0 \) for some \( i \)). Also, it has individuals-type \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) if it has \( n_i \) individuals of type \( i \), for \( i \in [d] \).

1.1.2. Multitype Galton-Watson forests. Consider a (unitype) branching forest with \( k \) trees and progeny distribution \( \nu \) on \( \mathbb{Z}_+ \), that is, each of the \( k \) individuals at generation 0 has offspring according to \( \nu \), and each of its children has offspring independently of the others and with the same law. Such forests are also called GW forests. A multitype Galton-Watson (MGW) forest in \( d \)-types, is a branching forest, where each individual has a type \( i \in [d] \), and has children independently of the others, according to a law \( \nu_i \) on \( \mathbb{Z}_+^d \). The progeny distribution of the forest is \( \nu = (\nu_1, \ldots, \nu_d) \).

The formal definition is the following.

**Definition.** A multitype Galton-Watson process is a Markov chain \( Z = \left( (Z_n^{(1)}, \ldots, Z_n^{(d)}) : n \geq 0 \right) \) on \( \mathbb{Z}_+^d \), with transition function

\[ \mathbb{P}(Z_{n+1} = (k_1, \ldots, k_d) | Z_n = (r_1, \ldots, r_d)) = \nu_1^{r_1} \ast \cdots \ast \nu_d^{r_d}(k_1, \ldots, k_d), \]

where \( \nu \) is the progeny distribution, and \( \nu_i^{r_i} \) is the \( j \)th iteration of the convolution product of \( \nu_i \) by itself, with \( \nu_i^0 = \delta_0 \).

For \( r \in \mathbb{Z}_+^d \), the probability measure \( \mathbb{P}_r \) is the law \( \mathbb{P}(\cdot | Z_0 = r) \). As in Theorem 1.2 in [CL16], we consider MGW trees satisfying the following. For \( i, j \in [d] \), let \( m_{i,j} = \sum_{z \in \mathbb{Z}_+^d} z_j v_i(z) \) be the mean number of children type \( j \) given by an individual type \( i \), and set \( M = (m_{i,j})_{i,j} \) as the mean matrix of the MGW tree. Whenever \( M \) is irreducible, by the Perron-Frobenius theorem (see [AN04, Chapter V.2]), it has a unique eigenvalue which is simple, positive and with maximal modulus. We say in such case that the MGW tree is irreducible. If the unique eigenvalue equals one (is less than one), then we say the tree is critical (subcritical). The tree is non-degenerate if individuals have exactly one offspring with probability different from one.

1.2. Statement of the results.

1.2.1. Multitype forests with a given degree sequence. To define uniform \( d \)-type forests with a given degree sequence, having root-type \( r > 0 \), we first define a multitype degree sequence. A multitype degree sequence \( S = (S_{i,j}, i, j \in [d]) \) is a sequence of sequences of non-negative integers \( S_{i,j} = (N_{i,j}(k) : k \in [m_{i,j}]) \), where \( m_{i,j} \in \mathbb{N} \), satisfying:

1. \( n_i = \sum_k N_{i,j}(k) \) for every \( i, j \in [d] \), with \( n_i > 0 \) for some \( i \),
(2) \( n_j = r_j + \sum_{k} k N_{i,j}(k) + \cdots + \sum_{k} k N_{d,j}(k) \), for every \( j \in [d] \), with \( 0 \leq r_i \leq n_i \) for every \( i \), and \( r > 0 \) for some \( i \).

(3) \( \det(-k_{i,j}) > 0 \) with \( k_{i,j} := \sum_{i} k N_{i,j}(k) - n_i \mathbf{1} \{ i = j \} \) and \( k_{i,j} < 0 \) for every \( i \in [d] \).

The value \( N_{i,j}(k) \) represents the number of individuals of type \( i \) with \( k \) children of type \( j \), so \( n_i \) represents the total number of individuals of type \( i \). Thus, the total number of vertices is \( s := n_1 + \cdots + n_d = \sum_{i} k N_{i,j}(k) + \cdots + \sum_{i} k N_{d,j}(k) \) for \( j \in [d] \). The last condition is imposed to obtain a forest with the same degree sequence (see page 12). For simplicity, we will assume that our multitype degree sequences satisfy the third condition and we focus on the first two conditions. Table 1 summarizes the case \( d = 2 \). More explicitly, the tree in Figure 1 has multitype degree sequence \( S_{1,1} = (4, 1, 1) \), \( S_{1,2} = (3, 3) \), \( S_{2,1} = (3, 0, 1) \) and \( S_{2,2} = (3, 1) \).

| \( S_{1,1} \) | \( S_{1,2} \) | \( S_{2,1} \) | \( S_{2,2} \) |
|------|------|------|------|
| \( (N_{1,1}(0), \ldots, N_{1,1}(m_{1,1})) \) | \( (N_{1,2}(0), \ldots, N_{1,2}(m_{1,2})) \) | \( n_1 = \sum_{k} k N_{1,j}(k) \) | \( n_2 = \sum_{k} k N_{2,j}(k) \) |
| \( (N_{2,1}(0), \ldots, N_{2,1}(m_{2,1})) \) | \( (N_{2,2}(0), \ldots, N_{2,2}(m_{2,2})) \) | \( n_1 = n_2 + \sum_{i} k N_{1,j}(k) \) | \( n_1 + n_2 = s \) |

TABLE 1. Relations on the degree sequence of a 2-type forest.

As in the unitype case, we construct the canonical child sequence \( c = (c_{i,j}, i, j \in [d]) \) from the degree sequence, that is, let \( c_{i,j} \) be a sequence whose first \( N_{i,j}(0) \) entries are zeros, the next \( N_{i,j}(1) \) entries are ones, and so on. Let \( \sigma_{i,j} \) be any permutation on \([n_i]\), and construct \( w^b = \{ w_{i,j}^b; i, j \in [d] \} \), where

\[
\begin{align*}
  w_{i,j}^b(k) &= \sum_{l=1}^{k} (c_{i,j} \circ \sigma_{i,j}(l) - \mathbf{1} \{ i = j \}), \\
  &\quad k \in [n_i].
\end{align*}
\]

Remark 1. Note that \( k_{i,j} := w_{i,j}^b(n_i) \) does not depend on the permutation, so it is deterministic. Also, note that the system of equations \( (r, w^b) \) admits \( n \) as a solution, since by definition

\[
  r_j + \sum_{i=1}^{d} w_{i,j}^b(n_i) = r_j - n_j + \sum_{i=1}^{d} \sum_{k} k N_{i,j}(k) = 0 \quad \forall j \in [d].
\]

Finally, note that \( -k_{j,j} = r_j + \sum_{k} k N_{j,j}(k) \) since by definition, we have

\[
  -k_{j,j} = n_j - \sum_{k} k N_{j,j}(k) = r_j + \sum_{i=1}^{d} \sum_{k} k N_{i,j}(k) - \sum_{i=1}^{d} \sum_{k} k N_{i,j}(k) = r_j + \sum_{i=1}^{d} k_{i,j}.
\]

Trivially, whenever \( r_j > 0 \) we have \( -k_{j,j} > 0 \).

From the Multivariate Cyclic Lemma 1, we know that \( \det(-k_{i,j}) \) is the number of good cyclical permutations of \( w^b \), considering the system \( (r, w^b) \). We define a Vervaat-type transformation of \( w^b \), given by choosing uniformly at random a good-cyclical permutation from all the good-cyclical permutations. After that, the algorithm is similar to the unidimensional case.

Definition (Multidimensional Vervaat Transform). Fix any \( w^b \) as constructed above, with corresponding \( r \) and any \( u \in [\det(-k_{i,j})] \). Define \( V(w^b, u) \) as follows: enumerate the \( \det(-k_{i,j}) \) good cyclical permutations of \( w^b \), using the lexicographic order on the set of \( q \) such that \( \theta_{q,b}(w^b) \) codes a forest; then, let \( V(w^b, u) \) be the \( u \)-th good cyclical permutation.

Let \( \mathbb{F}_r S \) be the set of multitype plane forests with degree sequence \( S \), having root-type \( r \) and individuals-type \( n \). Let \( W = (W_{i,j}, i, j \in [d]) \) be a random process taking values on \( S_d \). Then, we denote by \( \mathbb{P}_r \) the law in which each \( W_{i,j} \) starts at \( W_{i,j}(0) = r_i \mathbf{1} \{ i = j \} \).

The next result gives us a simple way to obtain the BFW (as constructed in (1)) of a MFGDS (the proof is given on page 12).
**Theorem 1** (Uniform multitype forest with a given degree sequence). Fix the degree sequence $S$ of a multitype forest having root-type $r$ and individuals-type $n$. Let $W$ be the BFW coding a forest taken uniformly at random from $\mathbb{F}_{r,S}$. Let $\pi = (\pi_{i,j}, i,j \in [d])$ be independent random permutations, where $\pi_{i,j}$ takes values on $[n_i]$, and let $U$ be an independent uniform variable on $[\det(-k_{i,j})]$. Define the processes $W^b = (W^b_{i,j}, i,j \in [d])$ as

$$W^b_{i,j}(k) = \sum_{l=1}^{k} (c_{i,j} \circ \pi_{i,j}(l) - 1 \{i = j\}), \quad k \in [n_i],$$

where $c = (c_{i,j}, i,j \in [d])$ is the child sequence of $S$. Then

$$V(W^b, U) \overset{d}{=} W,$$

under the law $\mathbb{P}_r$ and the uniform law on $\mathbb{F}_{r,S}$, respectively.

From the proof, we obtain $|\mathbb{F}_{r,S}|$, the number of multitype forests with a given degree sequence $S$ (cf. Theorem 3.3.2 in [Ngu16])

$$|\mathbb{F}_{r,S}| = \frac{\det(-k_{i,j})}{\prod n_i} \prod \left( \binom{n_i}{S_{i,j}} \right).$$

In Figures 2 and 3 we depict two simulations of MFGDS.

**1.2.2. MGW forests conditioned by types.** Before turning to the joint law of the number of individuals of type $i \in [d]$, of a MGW forest, we prove that the latter model is a mixture of MFGDS in Section 4. This justifies the importance of MFGDS.

Let $X = (X^{(1)}, \ldots, X^{(d)})$ be $d$ random walks on $\mathbb{Z}^d$, where

$$\mathbb{P}\left(X^{(i)} = k\right) = v_i(k + e_i), \quad i = 1, \ldots, d.$$
k is a vector with entries in $\mathbb{Z}_+$ except at position $i$, which takes values on $\mathbb{Z}_+ \cup \{-1\}$, and $e_i$ is the vector with zeros except a one at position $i$. We will write $X^{(i)} = (X^{i,1}, \ldots, X^{i,d})$. Our hypotheses are the following:

**H1:** For every $i \in [d]$, the law $\nu_i$ has independent components, that is

$$v_i^{n_i}(k + e_i) = \prod_j \mathbb{P}(X^{i,j}_{n_i} = k_j) \quad k = (k_1, \ldots, k_d), k_j \in \mathbb{Z}_+ \text{ for } j \neq i \text{ and } k_i \in \mathbb{Z}_+ \cup \{-1\}.$$ 

**H2:** For every $i, j \in [d]$, with $i \neq j$

$$\mathbb{E}
\left( X^{i,j}_{n_i}; \sum_{t \in [d]} X^{t,j}_{n_t} = n_j - r_j \right) = \frac{n_i(n_j - r_j)}{n} \mathbb{P}
\left( \sum_{t \in [d]} X^{t,i}_{n_t} = n_j - r_j \right).$$

It is important to remark that we do not assume that the components $\nu_i$ have the same distribution.

The following result (see page 15 for the proof) is a generalization of the Otter-Dwass formula [Ott49, Dwa69] (Lemma 3.8 in [ADG18] also gives a formula for the joint law of the sizes of a MGW forest, but less explicit).

**Theorem 2.** Consider an irreducible, non-degenerate and (sub)critical MGW forest, and let $n_i > 0$ for every $i$ and $r > 0$. Suppose that H1 and H2 and are also satisfied. If $O_i$ is the number of type $i$ individuals, then

$$\mathbb{P}_r(O_i = n_i, i \in [d]) = \frac{r}{n} \prod_{i=1}^d \mathbb{P}
\left( \sum_{t \in [d]} X^{t,i}_{n_t} = n_i - r_i \right),$$

where $r = r_1 + \cdots + r_d$ and $n = n_1 + \cdots + n_d$, and $r_i < n_i$.

**Remark 2.** Under the assumption $n_i > 0$ for every $i$, the proof is simpler, but we think this hypothesis can be dropped as in [CL16].
Remark 3. On page 17, we obtain the case when \( n_i = r_i \) for some \( i \)'s. Since Theorem 2 has a different formula on such case, the law of \( \sum_{i\in[d]}O_i \) (computed on Corollary 1) does not have a nice expression.

For the next results denote by \( \mathbb{P}_{r,n}^{\text{plane}}, \mathbb{P}_{r,n}^{\text{labeled}} \) and \( \mathbb{P}_{r,n}^{\text{binary}} \), the set of \( d \)-type plane, labeled and binary forests having root-type \( r \) and individuals-type \( n \), where \( r_i < n_i \) for every \( i \) and \( r > 0 \). Our labeled multitype forests have labels on \([n]\), that is, for \( F \in \mathbb{P}_{r,n}^{\text{labeled}} \), each individual \( v \) has a unique label \( i \in [n] \) and a type \( \phi(v) \in [d] \); also, \( F \) has fixed root set \([r]\), that is, the \( r_1 \) type 1 roots have labels on \([1, \ldots, r_1]\), the \( r_2 \) type 2 roots have labels on \([r_1 + 1, \ldots, r_1 + r_2]\), and so on. Using Theorem 2, we give in Subsection 5.1 three examples of distributions were the law of a MGW forest conditioned by the number of individuals of each type can be computed explicitly. This generalizes the constructions given in [Pit98].

Proposition 1. For fixed \( p \in (0,1) \), let \( \mathcal{G}_{r,p} \) be a \( d \)-type GW forest with root-type \( r \), having geometric offspring distribution with parameter \( p \) independently for each type, that is, \( \nu_i(k_1, \ldots, k_d) = \prod_i p(1 - p)^{k_i} \) for \( k_i \geq 0 \). Let \( \#i\mathcal{G}_{r,p} \) be the number of type \( i \) individuals in \( \mathcal{G}_{r,p} \). Then

\[
\mathbb{P}(\mathcal{G}_{r,p} = F|\#i\mathcal{G}_{r,p} = n_i, i \in [d]) = \frac{1}{\prod_{i \in [d]}(n_i + n_i - r_i - 1)} \quad \forall F \in \mathbb{P}_{r,n}^{\text{plane}},
\]

thus, such conditioned forest is uniform on \( \mathbb{P}_{r,n}^{\text{plane}} \).

Proposition 2. Fix \( \mu \in \mathbb{R}^+ \), let \( \mathcal{P}_{r,\mu} \) be a \( d \)-type GW forest with root-type \( r \), having Poisson offspring distribution of parameter \( \mu \) independently for each type, that is, \( \nu_i(k_1, \ldots, k_d) = \prod_i \mu^{k_i}/k_i! \) for \( k_i \geq 0 \). Let \( \#i\mathcal{P}_{r,p} \) be the number of type \( i \) individuals in \( \mathcal{P}_{r,p} \). If \( \mathcal{P}_{r,n}^{\star} \) is \( \mathcal{P}_{r,n} \) relabeled by \( d \) uniform random permutations, one for each type, then

\[
\mathbb{P}(\mathcal{P}_{r,p}^{\star} = F|\#i\mathcal{P}_{r,p} = n_i, i \in [d]) = \frac{1}{n!^{n^d-r}} \quad \forall F \in \mathbb{P}_{r,n}^{\text{labeled}},
\]

thus, such conditioned forest is uniform on \( \mathbb{P}_{r,n}^{\text{labeled}} \).

Proposition 3. For \( 0 < p < 1 \), let \( \mathcal{B}_{r,p} \) be a \( d \)-type GW forest with root-type \( r \), having Bernoulli offspring distribution with parameter \( p \), for each vertex independently of the type, that is, \( \nu_i(k_1, \ldots, k_d) = \prod_i p^{k_i}(1 - p)^{1-k_i} \) with \( k_i \in \{0,1\} \). Assume that \( n_i - r_i \) is an even number for every \( i \in [d] \). Since any vertex \( v \) has zero or two children with probability \( p \) or \( 1 - p \) respectively, then \( \nu_i(c_1(v), \ldots, c_d(v)) = \prod_i p^{c_i(v)/2}(1 - p)^{(1-c_i(v))/2} \). Let \( \#i\mathcal{B}_{r,p} \) be the number of type \( i \) individuals in \( \mathcal{B}_{r,p} \). Then

\[
\mathbb{P}(\mathcal{B}_{r,p} = F|\#i\mathcal{B}_{r,p} = n_i, i \in [d]) = \frac{1}{n!^{(n^d-r)/2}} \quad \forall F \in \mathbb{P}_{r,n}^{\text{binary}},
\]

thus, such conditioned forest is uniform on \( \mathbb{P}_{r,n}^{\text{binary}} \).

As a simple consequence of our results, we obtain the following enumerations.

Lemma 2. The number of \( d \)-type plane, labeled, and binary forest, with root-type \( r \) and individuals-type \( n \) is given respectively by

\[
\frac{r}{n} \prod_{i \in [d]} \left( \frac{n + n_i - r_i - 1}{n_i - r_i} \right), \quad \frac{r}{n} n^{n-r} \text{ and } \frac{r}{n} \prod_{i \in [d]} \left( \frac{n}{n_i - r_i} \right)/2.
\]

Finally, we give an algorithm to simulate MGW processes conditioned by its types. This is done using the following proposition and an Accept-Reject method (see Algorithm 9).
**Proposition 4.** Let $W$ be the BFW of a CMGW($n$) forest satisfying the Hypotheses of Theorem 2, having offspring distribution $v$, and root-type $r$, with $0 < r_i < n_i$ for every $i$. Generate independent multinomial vectors $S_{i,j} = (N_{i,j}(0), N_{i,j}(1), \ldots)$ with parameters $(n_i, v_{i,j}(0), v_{i,j}(1), \ldots)$, and stop the first time that $r_j + \sum k_{i,j} n_{i,j}(k) = n_j$ for every $j$. Denote by $S$ the multitype degree sequence obtained, and let $V(W^b, U)$ be the BFW generated by Theorem 1 using the degree sequence $S$. Then,

$$
\mathbb{P}_r (V(W^b, U) = w) = \frac{1}{\det(-k_{i,j})} \mathbb{P}_r (F = F' | \tau, F = n_j, \forall j),
$$

for every multitype forest $F$ coded by $w$, with root-type $r$, individuals-type $n$, multitype degree sequence $(n_{i,j}, i, j \in [d])$ and with $k_{i,j} = \sum k_{i,j}(k) - n_i \mathbb{1} \{i = j\}$.

The paper is organized as follows: in Section 2 we construct uniform unitype forests with a given degree sequence. This will be used in Section 3, to construct MFGDS and prove Theorem 1. In Section 4 we prove that under an independence assumption, the CMGW forests are mixtures of MFGDS. Section 5 is devoted to prove the joint law of the number of individuals by types in a MGW forest, which is Theorem 2. In that section we also obtain in Corollary 1, the law of the total population in a MGW forest. Examples satisfying the hypotheses of Theorem 2 are given in Subsection 5.1. The algorithms are given in Section 6.

2. CONSTRUCTION OF UNITYPE RANDOM FORESTS WITH A GIVEN DEGREE SEQUENCE

A well-known encoding of forests by skip-free random walks is given as follows. Define the set of all bridges finishing in position $-m$ at time $s$, as

$$
\mathbb{B}^{s,m} = \{(y(1), \ldots, y(s)) \in \mathbb{Z}^s : y(j) - y(j-1) \geq -1 \text{ for } j \in [s], y(s) = -m\}.
$$

For $i \in [s] - 1$ and $y \in \mathbb{B}^{s,m}$, define $\theta_i(y)$ as the cyclic permutation of $y$ at $i$, that is

$$
(\theta_i(y))(j) = \begin{cases} y(j+i) - y(i) & j + i \leq s \\ y(j+i-s) + y(s) - y(i) & s - i \leq j \leq s. \end{cases}
$$

This transformation puts the last $s - i$ increments of $y$ as the first $s - i$ increments of $\theta_i(y)$, and the first $i$ increments of $y$ as the last $i$ increments of $\theta_i(y)$.

For any $u \in [m] - 1$ and $y \in \mathbb{B}^{s,m}$, let $\tau_u$ be the time that $y$ hits $\min(y) + u$ for the first time. The Vervaat-type transformation $V$ that we use is given by

$$
V(y, u) = \theta_{\tau_u}(y).
$$

Note that this transformation leads the set of bridges, to the set of excursions of size $s$ finishing at $-m$

$$
\mathbb{E}^{s,m} = \{(w(1), \ldots, w(s)) \in \mathbb{Z}^s : w(j) - w(j-1) \in \{-1, 0, 1, \ldots\} \text{ for } j \in [s], w \text{ first reaches } -m \text{ at time } s\}.
$$

Now, let $F$ be a forest with trees $T_1, \ldots, T_m$, for $m \in \mathbb{N}$. The degree sequence of $F$ is given by

$$
N_i(F) = \sum_{j=1}^{m} N_i(T_j).
$$

Note that, any finite sequence of non-negative integers $S = (N_i, i \geq 0)$, such that for some $m \in [\|S\|]$, we have

$$
s := \sum N_i = \sum iN_i + m,
$$

is the degree sequence of some forest with $m$ trees. In this case we call $S$ a degree sequence. The size of the forest $F$ will also be denoted by $|F|$.
Fix any degree sequence $S$ and obtain its child sequence $c := c(S)$. As before, we obtain an EI process using uniform permutations of the child sequence $c$. Let $\sigma$ be any permutation on $[s]$. Define the bridge

$$W^b(j) = \sum_{i=1}^{j} (c \circ \sigma(i) - 1) \quad j \in [s],$$

with $W^b(0) = 0$. Note that $W^b(s) = -m$. The set of paths taken by $W^b$ is

$$B_S = \{(y(1), \ldots, y(s)) \in \mathbb{B}^{s,m} : |j : y(j) - y(j-1) = i - 1| = N_i \text{ for every } i \geq 0\}.$$

From the excursions in $\mathbb{E}^{s,m}$, we consider those with fixed number of increments of given size:

$$\mathbb{E}_S = \{(w(1), \ldots, w(s)) \in \mathbb{E}^{s,m} : |j : w(j) - w(j-1) = i - 1| = N_i, i \geq 0\}.$$

Define $\mathbb{F}_S$ as the set of all forests with degree sequence $S$. Using Lemma 6.3 of [Pit06], it can be proved that there exists a bijection between $\mathbb{E}_S$ and $\mathbb{F}_S$, and we know that $|\mathbb{F}_S| = \frac{s^s}{s!} \prod_{i=1}^{s} \frac{1}{N_i^i}$.

It is clear from a picture that a bridge is sent to an excursion by the Vervaat transformation. Let us prove this is also the case for bridges in $B_S$, that is, bridges coming from a degree sequence $S$. The next three lemmas are inspired in [Lei19].

**Lemma 3.** For any $u \in [m] - 1$ and $y \in B_S$, the path $V(y,u)$ belongs to $\mathbb{E}_S$.

**Proof.** By definition of $T_u$, the minimum value that can take $y(T_u + j) - y(T_u)$ for $j + T_u \leq s$ is $-u > -m$. These are the first $s - T_u$ times of $V(y,u)$. On the remaining times $\{s - T_u, \ldots, s\}$, the minimum of $V(y,u)$ is attained for the first time at time $s$. This implies $V(y,u)(j) > -m$ for $j < s$, and hence $V(y,u) \in \mathbb{E}$. Since the Vervaat transformation only permutes the increments, it is clear that if $y \in B_S$ then $V(y,u) \in \mathbb{E}_S$. □

**Lemma 4.** Let $w \in \mathbb{E}_S$. Then, the number of different pairs $(y,u) \in B_S \times ([m] - 1)$ such that $V(y,u) = w$ is exactly $s$.

**Proof.** Consider any $i \in [s] - 1$ and the cyclical permutation $y' := \theta_i(w)$. It is clear that $\theta_{s-i}(y') = w$. In fact $V(y',u) = w$ for some $u \in [m] - 1$. This holds true since the last $s - i$ increments of $w$ are the first $s - i$ increments of $y'$, then $y'$ hits $y'(s - i) = w(s) - w(s - i)$ for the first time at time $s - i$. Hence, the Vervaat transform can be applied at $u = y'(s - i)$, giving us the path $w$.

Note that the path of $w$ can be partitioned in $m$ subexcursions, each one of the form $(w(j + T_i), j \in [T_{i+1} - T_i])$, with $i \in [m] - 1$ and $T_i$ the first hitting time to $-i$. First assume that $w$ can be partitioned in $k_w \in [m]$ identical subexcursions, each of length $l_w$. It follows $k_w l_w = s$. This is equivalent to say that there exists $k_w$ values $u$ such that $V(w,u) = w$. Those values are $w(j l_w) + m \in [m] - 1$ for $j \in [k_w]$. In this case, there are only $l_w$ different cyclic permutations $\theta_i(w)$ of $w$, each one having $k_w$ distinct values of $u$ such that $V(\theta_i(w),u) = w$. This proves that $w$ has exactly $s$ preimages. If $w$ cannot be partitioned in identical subexcursions, for every cyclical permutation $\theta_i(w)$ there is only one $u$ such that $V(\theta_i(w),u) = w$. This concludes the proof. □

Now, we construct a uniform forest on $\mathbb{F}_S$.

**Lemma 5.** Consider a degree sequence $S$ of a forest having $m$ trees and $s$ individuals, and let $\mathcal{F}$ be a forest taken uniformly at random from $\mathbb{F}_S$. Let $\pi$ be a uniform random permutation on $[s]$, $U$ an independent uniform variable on $[m] - 1$, and define the bridge

$$W^b(j) = \sum_{i=1}^{j} (c \circ \pi(j) - 1) \quad j \in [s],$$

where $c$ is the child sequence of $S$. Then

$$V(W^b, U) \overset{d}{=} \mathcal{F},$$
under the law \( \mathbb{P}_m \) that makes \( W^b \) start at \( m \) and the uniform law on \( \mathbb{F}_S \).

Proof. Fix any \( w \in \mathbb{E}_S \) and any of its cyclical permutations \( w^b \in \mathcal{B}_S \). From the \( s! \) possible values that \( \pi \) can take, only \( \prod N_i! \) give the same bridge \( w^b \). This is true since we can permute the labels of the \( N_i \) individuals having \( i \) children and obtain the same bridge. Hence

\[
\mathbb{P}_m \left( W^b = w^b \right) = \frac{1}{\left( \sum_{i=1}^s (N_i, i) \right)},
\]

which does not depend on \( w^b \).

By the previous lemma, there are \( s \) distinct pairs \((w^b, u)\) that are mapped to \( w \). Denote such pairs as \((w^b_k, u_k)\) \( \in \mathcal{B}_S \), for \( k \in [s] \). Then, using the independence of \( W^b \) and \( U \), and that the former is uniform, then

\[
\mathbb{P}_m \left( V(W^b, U) = w \right) = \sum_{k=1}^s \mathbb{P}_m \left( W^b = w^b_k, U = u_k \right) = \frac{s}{m} \mathbb{P}_m \left( W^b = w^b \right) = \frac{1}{|\mathbb{F}_S|}. \quad \square
\]

3. Construction of Multitype random forests with a given degree sequence

Recall that from a given multitype degree sequence, we can construct bridges \( w^b \) taking values on \( S_d \) such as before the remark 1, together with its multidimensional Vervaat transform \( \tilde{V}(w^b, u) \).

The set of possible paths of \( w^b \) constructed in this manner, will be denoted by \( \mathcal{B}_F \). Explicitly, this is the set of \( d \times d \) bridges \((w^b_{ij}, i, j \in [d])\) in \( S_d \), where \( w^b_{ij} \) finishes at \( k = \sum k_i N_i(k) - n_i \{ i = j \} \), and with \( N_i(k) \) increments of size \( k - 1 \), for each \( i, j \in [d] \) and \( k \). The set of plane forests with degree sequence \( S \) having root-type \( r \), will be denoted by \( \mathbb{F}_r \), and the set of BFWs coding such forests by \( \mathbb{E}_r \). Now we are ready to construct a forest taken uniformly at random from \( \mathbb{F}_r \).

Proof of Theorem 1. First we prove that from any multitype degree sequence, we can construct a multitype forest. From Remark 1, we can associate to the degree sequence a system of equations \((r, w^b)\) with solution \( n \). To such system we can associate a multitype forest using the Multivariate Cyclic Lemma 1 (note that \( k_i < 0 \) by definition of multitype degree sequence), since any good cyclical permutation codes a forest.

Now, define \( S_{ij} = (N_i, j(k); k \geq 0) \), and write

\[
\begin{pmatrix}
    n_i \\
    (S_{ij})
\end{pmatrix} := \begin{pmatrix}
    n_i \\
    N_i(0), N_i(1), \ldots
\end{pmatrix}.
\]

Fix any bridge \( w^b \in \mathcal{B}_F \). From the possible \( \prod (n_i)!^d \) values taken by the random permutations \((\pi_{ij}, i, j \in [d])\), exactly \( \prod B_i \prod N_i(k)! \) form the bridge \( w^b \). This is true since, permuting the labels of the \( N_i(k) \) individuals type \( i \) having \( k \) children type \( j \), we obtain the same bridge. This proves the assertion since this is true for every \( i, j, k \). Therefore

\[
\mathbb{P}_r \left( W^b = w^b \right) = \frac{1}{\prod (n_i)!}.
\]

Now, fix any \( w \in \mathbb{E}_F \) and \( i \in [d] \). We obtain the number of different pairs \((w^b, u)\) that can be mapped to \( w \) using the multidimensional Vervaat transform. The point is that such bridges can only be of the form \( \theta_q(w) \), that is, cyclical permutations of \( w \). Hence, each component \( w^{(i)} \) comes from a Vervaat transform \( \tilde{V}(\theta_j(w^{(i)}), u) \) for some \( j, u \). By Lemma 4, the number of pairs \((\theta_j(w^{(i)}), u)\) that can be mapped to \( w^{(i)} \) are exactly \( n_i \). This being true for every \( i \) implies there are \( \prod n_i \) unique pairs \((\theta_q(w), u)\) such that \( V((\theta_q(w), u)) = w \). Denote such pairs as

\[
A(w) = \left\{ (w^b_k, u_k) \in \mathcal{B}_F \times [\det(-k_{i,j})] : V((w^b_k, u_k)) = w, k \in \prod n_i \right\}.
\]
This implies
\[ P_r (V(W^b, U) = w) = P_r (W^b, U) \in A(w) \]
\[ = \sum_{k \in [1:n]} P_r (W^b, U) = (w_k^b, u_k) \]
\[ = \sum_{k \in [1:n]} \frac{1}{\prod \left( \frac{n_j}{s_{ij}} \right) \det(-k_{ij})} \frac{1}{\prod \left( \frac{n_j}{s_{ij}} \right)} \]
\[ = \frac{\det(-k_{ij})}{\prod n_i} \prod \left( \frac{n_j}{s_{ij}} \right). \]

This concludes the proof since the right-hand side is independent of \( w \), so \( V(W^b, U) \) is uniform. □

Remark 4. From this lemma we can conclude that the set of plane forests with degree sequence \( S \) having root-type \( r \) is
\[ |F_{r,s}| = \frac{\det(-k_{ij})}{\prod n_i} \prod \left( \frac{n_j}{s_{ij}} \right). \]

4. Relation between MFGDS and CMGW forests

Before turning to the results about conditioned MGW forests, let us prove that under an independence condition, a MGW conditioned by its degree sequence has the same law as a MFGDS. For any given \( n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \) and \( r = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d \) with \( \sum r_i > 0 \), define the set of all degree sequences having \( n_i \) individuals of type \( i \), as
\[ DS(r, n) = \{ S = (N_i, j), i, j \in [d], k \geq 0 \} : \]
\[ n_i = \sum k N_i, j, n_j = r_j + \sum k N_i, j, N_i, j(k) \geq 0 \text{ for } i, j \in [d] \} . \]

Also, for any given multitype forest \( F \), define its empirical multitype degree sequence \( \hat{N}(F) := \hat{N} = (\hat{N}_i, j, i, j \in [d], k \geq 0) \) as
\[ \hat{N}_i, j = \sum_{l : l \in F} 1 \{ c_i, j(l) = k \} , \]
where the product is taken over all vertices of subforest \( F \), and \( c_i, j(l) \) is the number of children type \( j \), that the \( l \)th individual of the subforest \( F \) of vertices type \( i \) has.

Lemma 6. Fix any \( n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \) and \( r = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d \) with \( \sum r_i > 0 \). Consider a multitype degree sequence \( S = (N_i, j, i, j \in [d], k \geq 0) \in DS(r, n) \). Consider a MGW forest with progeny distribution \( V = (V_1, \ldots, V_d) \) such that each \( V_i \) has independent components. Then, the law of a MGW forest conditioned to have multitype degree sequence \( S \) is the same as \( P_S \), the law of a MFGDS. Also, the law of a CMGW(\( n \)) forest is a finite mixture of the laws \( (P_S, S \in DS(r, n)) \).

Proof. Let \( \mathcal{F} \) be a MGW tree. By the assumption on \( V_i \), we can write \( \mathcal{V}(z) = \prod_i V_i(z_i) \) for any \( z = (z_1, \ldots, z_d) \in \mathbb{Z}_+^d \), any \( i \), and some laws \( v_i, j \) on \( \mathbb{Z}_+ \). Let \( (F_1, c(F_1)) \) and \( (F_2, c(F_2)) \) be two
multitype forests having degree sequence \( S \in DS(\mathbf{r}, n) \). Then

\[
\mathbb{P}_r (\mathcal{F} = F_1, \hat{N}(\mathcal{F}) = S) = \prod_{k \in k_1} \prod_{j} \prod_{i \in F_1^{(j)}} \nu_{i,j}(c_{i,j}(k))
\]

This implies the first assertion. Let \( \mathbf{O} = (O_1, \ldots, O_d) \) be the vector with the total number of individuals of each type in a MGW forest. To prove the second assertion, we sum over all the values in \( DS(\mathbf{r}, n) \), obtaining

\[
\mathbb{P}_r (\mathcal{F} \in \cdot | \mathbf{O} = \mathbf{n}) = \frac{1}{\mathbb{P}_r (\mathbf{O} = \mathbf{n})} \sum_{S \in DS(\mathbf{r}, n)} \mathbb{P}_r (\mathcal{F} \in \cdot, \hat{N}(\mathcal{F}) = S)
\]

where

\[
\lambda_S = \mathbb{P}_r (\hat{N}(\mathcal{F}) = S | \mathbf{O} = \mathbf{n})
\]

Note that trivially \( \sum_{S \in DS(\mathbf{r}, n)} \lambda_S = 1 \). \( \square \)

5. Law of the Number of Individuals by Types of a MGW Forest

The main result in [CL16] is the following.

**Theorem 3** (Theorem 1.2, [CL16]). Let \( Z \) be a \( d \)-type branching process, which is irreducible, non-degenerate and (sub)critical. For \( i \in [d] \), let \( O_i \) be the total number of individuals of type \( i \), up to the extinction time \( T \), and for \( i \neq j \), let \( A_{ij} \) be the total number of individuals of type \( j \) whose parent is of type \( i \), up to time \( T \). Then, for all integers \( r_i, n_i, k_{i,j} \) such that \( r_i \geq 0 \) with \( \mathbf{r} > 0 \), \( k_{i,j} \geq 0 \) for \( i \neq j \), \( -k_{j,i} = r_j + \sum_{i \neq j} k_{i,j} \), and \( n_i \geq -k_{i,i} \), we have

\[
\mathbb{P}_r (O_i = n_i, i \in [d], A_{ij} = k_{i,j}, i, j \in [d], i \neq j)
\]

(4)

\[
= \det(-k_{ij}) \prod_{i=1}^{d} \nu_{i}^{n_i} (k_i, \ldots, k_{i(i-1)}),
\]

where \( \mathbf{r} = (r_1, \ldots, r_d) \), \( \tilde{n}_i = n_i \lor 1 \) and \( (-k_{ij})_{i,j \in [d]} \) is the matrix to which we remove row \( i \) and column \( i \), for every \( i \) such that \( n_i = 0 \).

For simplicity, we use \( n_i > 0 \) for \( i \in [d] \) in the following. Let us give a hint on how to derive the law of the population by types for a 2-type GW forest, having \( r_i < n_i \) type \( i \) roots, for every \( i \). Recall the hypothesis \( \text{H1} \) about the independence in the components of \( \nu_i \) of Theorem 2. Recalling the definition of \( (X^{i,j}, i, j \in [d]) \) of Subsection 1.2.2, from Theorem 3, summing over all the possible
values of \((k_{i,j})\) we have

\[
(5) \\
\mathbb{P}_r (O_1 = n_1, O_2 = n_2) = \sum_{i=0}^{n_1-r_1} \sum_{j=0}^{n_2-r_2} r_1 r_2 + r_1 j + r_2 i \mathbb{P}(X_{n_1}^{1,1} = n_1 - r_1 - i, j) \mathbb{P}(X_{n_2}^{2,1} = (i, n_2 - r_2 - j)) \\
= \sum_{i=0}^{n_1-r_1} \sum_{j=0}^{n_2-r_2} \frac{r_1 r_2 + r_1 j + r_2 i}{n_1 n_2} \mathbb{P}(X_{n_1}^{1,1} = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^{2,2} = (i, n_2 - r_2 - j)) .
\]

We perform each summation in columns, obtaining three terms of the form

\[
\sum_{i=0}^{n_1-r_1} k_{i,1} \mathbb{P}(X_{n_1}^{1,1} = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^{2,2} = (i, n_2 - r_2 - j)),
\]

where \(k_{i,1}\) can be \(r_1\) or \(i\), and \(k_{i,2}\) can be \(r_2\) or \(j\). Hence, in order to perform the summation for any dimension, we need to expand the determinant \(\det(-k_{i,j})\), perform the summation in columns, and divide in cases: either a constant or a variable multiply the above probabilities. Note that, in the first case the summation is only a convolution. Is precisely the second case why we need Hypotheses H2. First, we describe explicitly \(\det(-k_{i,j})\).

**Definition.** An elementary forest is a forest of \(\mathcal{F}_d\) that contains exactly one vertex of each type. In particular, each elementary forest contains exactly \(d\) vertices and is coded by the \(d\) couples \((i_j, j)\) for \(j \in [d]\), where \(i_j\) is the type of the parent of vertex type \(j\). If the vertex of type \(j\) is a root, then we set \(i_j = 0\). We define the set \(D\) of vectors \((i_1, \ldots, i_d)\), with \(0 \leq i_j \leq d\) such that \((i_j, j), i \in [d]\) codes an elementary forest.

Recall Definition 1.1.1 of \(S_d\) of coding sequences of multitype forests.

**Definition.** For any \(r = (r_1, \ldots, r_d) \in \mathbb{Z}_d^d\) with \(r > 0\), let \(S_d^r\) be the subset of \(S_d\) of sequences \(x\) whose length belongs to \(\mathbb{N}^d\) and corresponds to the smallest solution of the system \((r, x)\).

Joining Lemmas 4.4, 4.5, 4.6 and 4.7 in [CL16], we obtain the following easy consequence, which is a precise description of the number of good cyclical permutations of \(x \in S_d^r\).

**Lemma 7.** Let \(x \in S_d^r\), where \(r > 0\). Assume that \(n = (n_1, \ldots, n_d) \in \mathbb{Z}_d^d\) is a solution of the system \((r, x)\). Then, the number of good cyclical permutations of \(x\) is

\[
\det(-k_{i,j}) = \sum_{(i_1, \ldots, i_d) \in D} \prod_{j=1}^{d} k_{i_j, j},
\]

where we set \(k_{0, j} = r_j\), and \(-k_{j,j} = r_j + \sum_{i \neq j} k_{i,j}, j \in [d]\), and \(k_{i,j} = x^{i,j}(n_i)\).

Now, we prove our theorem.

**Proof of Theorem 2.** By the independence imposed on \(v_i\), the product in Equation (4) can be expressed as follows:

\[
P(K) := \prod_{i=1}^{d} v_i^{x_{n_i}} (k_{i1}, \ldots, k_{i(i-1)}, n_i + k_{ii}, k_{ii(i+1)}, \ldots, k_{id}) \\
= \prod_{j=1}^{d} \mathbb{P}(X_{n_j}^{j,j} = n_j - r_j - \sum_{i \neq j} k_{i,j}) \prod_{i \neq j} \mathbb{P}(X_{n_i}^{i,j} = k_{i,j}) .
\]
We define the index set
\[ A(r, n) = \{ K = (k_{ij}) : k_{ij} \geq 0 \text{ for } i \neq j, 0 \leq -k_{jj} \leq n_i, -k_{jj} = r_j + \sum_{i \neq j} k_{ij}, \forall j \in [d] \}, \]
and use the notation \( \sum_{K \in A(r, n)} \) to denote the summation over all \( k_{ij} \) with \( i, j \in [d] \) and \( i \neq j \), such that \( K = (k_{ij}) \in A(r, n) \). Also, fix \( j \in [d] \) and define the index set
\[ A_j(r, n) = \{ K^j = (k_{1j}, \ldots, k_{dj}) : k_{ij} \geq 0 \text{ for } i \neq j, 0 \leq \sum_{i \neq j} k_{ij} \leq n_j - r_j \}. \]
Use the notation \( \sum_{K^j \in A_j(r, n)} \) to denote the summation over all \( k_{ij} \) with \( l \in [d] \) and \( l \neq j \), such that \( K^j = (k_{1j}, \ldots, k_{dj}) \in A_j(r, n) \). Then, we have
\[ \Pr_r(O_l = n_i, i \in [d]) = \frac{1}{\prod_{l=1}^{d} n_l} \sum_{K \in A(r, n)} \sum_{(i_1, \ldots, i_d) \in D} \prod_{j=1}^{d} k_{ij} P(K). \]

Denote by \( m \) the number of summands in \( \text{det}(-K) \) for any \( K = (k_{ij}) \in A(r, n) \). This is the number of elementary forests, hence, it does not depend on \( K \). Note that there exists \( m \) functions \( \sigma_l : [d] \to [d]_0 \) with \( \sigma_l(j) \neq j \) for every \( l, j \), such that
\[ \text{det}(-k_{ij}) = \sum_{l=1}^{m} \prod_{j=1}^{d} k_{\sigma_l(j), j}. \]
This means that for \( K^j = (k_{ij}) \in A(r, n) \), we have
\[ \text{det}(-k_{ij}) = \sum_{l=1}^{m} \prod_{j=1}^{d} k^j_{\sigma_l(j), j}. \]
Now, fix any of the \( m \) permutations \( \sigma_l \). We analyze the summation \( \sum_{K \in A(r, n)} \prod_{j=1}^{d} k_{\sigma_l(j), j} P(K) \). The determinant \( \text{det}(-k_{ij}) \) is the sum of \( m \) terms, each having one and only one term \( k_{\sigma_l(j), j} \) of each column \( j \). Hence, we can join such term with the corresponding probabilities (in the \( j \)th column) involving the walks \( X_{n_j}^{i,j} \) for \( i \in [d] \). This means that, for every \( j \in [d] \) we can join the terms \( K^j \in A_j(r, n) \) as follows
\[ \sum_{K^j \in A_j(r, n)} \prod_{j=1}^{d} k_{\sigma_l(j), j} P(K) \]
\[ = \prod_{j=1}^{d} \sum_{K^j \in A_j(r, n)} k_{\sigma_l(j), j} P(X_{n_j}^{i,j} = n_j - r_j - \sum_{l \neq j} k_{l,j}) \prod_{l \neq j} P(X_{n_j}^{i,j} = k_{l,j}). \]
Note that whenever \( \sigma_l(j) \neq 0 \) we have
\[ \sum_{K^j \in A_j(r, n)} k_{\sigma_l(j), j} P(X_{n_j}^{i,j} = n_j - r_j - \sum_{l \neq j} k_{l,j}) \prod_{l \neq j} P(X_{n_j}^{i,j} = k_{l,j}) = \mathbb{E} \left( X_{n_j}^{\sigma_l(j), j}, \sum_{l \in [d]} X_{n_j}^{l,j} = n_j - r_j \right), \]
which is related with Hypotheses H2. Thus, we define
\[ \tilde{k}_{\sigma_l(j), j} = \begin{cases} r_j & \sigma_l(j) = 0 \\ n_{\sigma_l(j)}(n_j - r_j)/n & \sigma_l(j) \neq 0. \end{cases} \]
The idea is that in Equation \( (6) \), when performing the summation with \( k_{\sigma_l(j), j} \neq r_j \) we use the Hypotheses H2, whereas when \( k_{\sigma_l(j), j} = r_j \) we simply use the convolution formula \( \sum_{l \in [d]} X_{n_j}^{l,j} \). This
Theorem 3 we have

\[ \sum_{K \in A(r, n)} \prod_{j=1}^{d} k_{\sigma(j), j} P(K) = \prod_{j=1}^{d} \overline{k}_{\sigma(j), j} P \left( \sum_{i \in [d]} X_{n_i}^{i,j} = n_j - r_j \right), \]

denotes neither there are children of type

Thus, from Theorem 3 we have

\[ P r \left( O_i = n_i, i \in [d] \right) \prod_{j=1}^{d} n_j = \prod_{j=1}^{d} \overline{k}_{j,j} P \left( \sum_{i \in [d]} X_{n_i}^{i,j} = n_j - r_j \right), \]

Define the matrix \( \overline{K} \) as a \( d \times d \) matrix with entries \( \overline{k}_{ij} = n \overline{k}_{ij} = n_i(n_j - r_j) \) for \( i \neq j \), and diagonal

\[ \overline{k}_{jj} = nr_j + \sum_{i \neq j} nk_{ij} = nr_j + \sum_{i \neq j} n_j(n_j - r_j) = nr_j + (n_j - r_j)(n - n_j) = n_j(n - n_j + r_j). \]

Then, using Lemma 4.5 of [CL16], which computes a determinant for integer valued matrices \( (k_{i,j}) \) satisfying our conditions, we have

\[ \sum_{(i_1, \ldots, i_d) \in D} \prod_{j=1}^{d} \overline{k}_{i,j} = n^{-d} \det(-\overline{K}). \]

To prove that \( \det(-\overline{K}) = n^d \frac{1}{n} \prod n_i \), factorize in row \( i \) the factor \( n_i \), obtaining

\[ \det(-\overline{K}) = \prod_{n_i} \begin{vmatrix} n - n_1 + r_1 & -(n_2 - r_2) & \cdots & -(n_d - r_d) \\ \vdots & \vdots & \ddots & \vdots \\ -(n_1 - r_1) & -(n_2 - r_2) & \cdots & n - n_d + r_d \end{vmatrix}. \]

Multiply the last row by minus one, and add it to every other row, to obtain

\[ \det(-\overline{K}) = \prod_{n_i} \begin{vmatrix} n & 0 & \cdots & -n \\ 0 & n & \cdots & -n \\ \vdots & \vdots & \ddots & \vdots \\ -(n_1 - r_1) & -(n_2 - r_2) & \cdots & n - n_d + r_d \end{vmatrix}. \]

Multiply by \( (n_i - r_i) / n \) each row \( i \in [d-1] \), and add it to the last row

\[ \det(-\overline{K}) = \prod_{n_i} \begin{vmatrix} n & 0 & \cdots & -n \\ 0 & n & \cdots & -n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n - \sum (n_i - r_i) \end{vmatrix}, \]

and, being a diagonal matrix, it follows that \( \det(-\overline{K}) = n^{d-1} r \prod n_i = n^{d-1} \frac{r}{n} \prod n_i \) as wanted. \( \square \)

Now we treat the case \( r_i = n_i \) for some \( i \). For simplicity assume \( r_i < n_i \) for \( i \in [d-1] \) and \( r_d = n_d \). This implies neither there are children of type \( d \) nor type \( d \) individuals have children. Thus, from Theorem 3 we have

\[ P r \left( O_i = n_i, i \in [d] \right) A_{ij} = k_{ij}, i, j \in [d-1], i \neq j, A_{id} = 0, A_{dl} = 0, l \neq d \)

\[ = \frac{\det(-k_{ij})}{n_1 \cdots n_d} \sum_{(k_1, \ldots, k_d) \in [k_1, \ldots, k_{d-1}]} \prod_{i=1}^{d-1} v_j^{n_i} (k_1, \ldots, k_{i-1}, n_i + k_{i+1}, k_{i+1}, \ldots, k_{(d-1)}, 0). \]
Since the matrix $(-k_{ij})$ has zeros in column and row $d$, except at $-k_{dd} = r_d = n_d$, we have $\det(-k_{ij}) = n_d\det(-k_{ij},i,j \in [d-1])$. Using the independence of Hypotheses H1, we reduce the problem to the joint law of the first $d-1$ components:

$$
\mathbb{P}_r(O_i = n_i, i \in [d], A_{ij} = k_{ij}, i,j \in [d-1], i \neq j, A_{id} = 0, A_{dl} = 0, l \neq d) = \mathbb{P}_{(r_1, \ldots, r_{d-1})}(O_i = n_i, i \in [d-1], A_{ij} = k_{ij}, i,j \in [d-1]) \prod_j \mathbb{P}(X_{n_d}^{d,j} = 0) \prod_{j \neq d} \mathbb{P}(X_{n_d}^{j,d} = 0).
$$

From this and the proof of Theorem 2, we obtain

$$
\mathbb{P}_r(O_i = n_i, i \in [d]) = \prod_j \mathbb{P}(X_{n_d}^{d,j} = 0) \prod_{j \neq d} \mathbb{P}(X_{n_d}^{j,d} = 0) \sum_{\sum_{i=1}^{d-1} r_i = \sum_{j \notin [d]} r_j} \prod_{i=1}^{d-1} \mathbb{P}(\sum_{j \notin [d]} X_{n_i}^{j,i} = n_i - r_i, X_{n_d}^{d,i} = 0).
$$

This shows that the formula of Theorem 2 does not work for the case $n_1 = r_1$ for some $i$. By the same reason, the next result, which is the law of the total number of individuals in a MGW forest, has four additional terms.

**Corollary 1.** Assume the hypotheses of Theorem 2 are satisfied, that $(X_{n}^{l,j}, i \in [d])$ are identically distributed for every $j$, and that $d = 2$. Let $O = \sum_{i \in [d]} O_i$, $n \in \mathbb{N}$ and $r < n$. Let $X_{n_i}^{(j)}$ have law $\sum_i X_{n_i}^{l,j}$ for any $n_1 + \cdots + n_d = n$. Then

$$
\mathbb{P}_r(O = n) = \frac{r}{n} \mathbb{P}(X_n^{(1)} + X_n^{(2)} = n - r)
+ \frac{r_1}{n-r_2} \mathbb{P}(X_n^{(1)} + X_n^{(2)} = n - r_1, X_n^{(1)} = 0) \mathbb{P}(X_n^{(2)} = 0)
+ \frac{r_2}{n-r_1} \mathbb{P}(X_n^{(1)} = 0, X_n^{(2)} = 0, X_n^{(2)} = n - r_1 = n - r)
- \frac{r}{n} \mathbb{P}(X_n^{(1)} = n - r) \mathbb{P}(X_n^{(2)} = 0) - \frac{r}{n} \mathbb{P}(X_n^{(1)} = 0) \mathbb{P}(X_n^{(2)} = n - r) .
$$

**Proof.** We have to sum over all $n_1, n_2$ such that $n_1 + n_2 = n$. Note that $n_1 \geq r_1$, and also $r_1 \leq n_1 \leq n - r_2$. It follows

$$
\mathbb{P}_r(O = n)
= \sum_{n_1 : r_1 \leq n_1 \leq n - r_2} \mathbb{P}_r(O_1 = n_1, O_2 = n - n_1) + \mathbb{P}_r(O_1 = r_1, O_2 = n - r_1) + \mathbb{P}_r(O_1 = n - r_2, O_2 = r_2)
= \frac{r}{n} \sum_{n_1 : r_1 \leq n_1 \leq n - r_2} \mathbb{P}(X_n^{1,1} + X_n^{2,1} = n_1 - r_1) \mathbb{P}(X_n^{1,2} + X_n^{2,2} = n - n_1 - r_2)
+ \mathbb{P}_r(O_1 = r_1, O_2 = n - r_1) + \mathbb{P}_r(O_1 = n - r_2, O_2 = r_2)
- \frac{r}{n} \mathbb{P}(X_n^{1,1} + X_n^{2,1} = n - r) \mathbb{P}(X_n^{1,2} + X_n^{2,2} = 0) - \frac{r}{n} \mathbb{P}(X_n^{1,1} = 0) \mathbb{P}(X_n^{1,2} + X_n^{2,2} = n - r) .
$$

Making the change of variables $n_1 - r_1 = l_1$, using the convolution formula and Theorem 2, gives the desired result.

5.1. **Application to the enumeration of plane, labeled and binary multitype forests with given roots and types sizes.** We provide three examples where Hypotheses H2 are satisfied, under the assumptions of Theorem 2. For simplicity, we consider $d = 2$, but the proofs also work for any $d \geq 3$. We perform the summation in Equation (5) explicitly in the next examples.
5.1.1. Geometric Offspring. Fix $\mathbf{r} = (r_1, r_2) \in \mathbb{N}^2$ with $r = r_1 + r_2 > 0$ and $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$. Denote by $\mathbb{F}_{r, n}^{\text{plane}}$ the set of two-type plane forests having $r_i$ roots and $n_i$ individuals of type $i$, for $i \in [d]$.

On the other hand, for any $p \in (0, 1)$ let $\mathcal{G}_{r, p}$ be a two-type Galton-Watson forest with $r_i$ roots of type $i$, having geometric offspring distribution with parameter $p$ independently for each individual, that is, $v_i(k_1, k_2) = p(1 - p)^{k_1}(1 - p)^{k_2}$. Recall that for any $F \in \mathbb{F}_{r, n}^{\text{plane}}$, we denote by $F^{(i)}$ the subforest of type $i$ of $F$. Suppose that $F \in \mathbb{F}_{r, n}^{\text{plane}}$ has $k_{1,2}$ type 2 individuals whose parent is of type 1, and $k_{2,1}$ type 1 individuals whose parent is of type 2. Hence, $n_1 - r_1 - k_{2,1}$ is the number of individuals type 1 whose parent is of type 1, and similarly for the type 2 individuals. Denoting by $c_i(v)$ the number of children type $i$ that vertex $v$ has, then

$$
\mathbb{P}(\mathcal{G}_{r, p} = F) = \prod_{v \in f(1)} v_1(c_1(v), c_2(v)) \prod_{v \in f(2)} v_2(c_1(v), c_2(v)) = p^{2n}(1 - p)^{n_1 - r_1 - k_{12}}(1 - p)^{k_{12}}(1 - p)^{n_2 - r_2 - k_{12}} = p^{2n}(1 - p)^{n - r},
$$

where $n = n_1 + n_2$ and $r = r_1 + r_2$.

Now, we compute the left-hand side of Hypotheses H2. Recall that the sum of $k$ independent geometric random variables with parameter $p$, has a negative binomial distribution $\text{NB}_{k,p}$ of parameters $k$ and $p$. From Equation (5), one obtains the sum

$$
\sum_{i=0}^{n_1 - r_1} i \mathbb{P}(X_{n_1}^{1,1} = n_1 - r_1 - i) \mathbb{P}(X_{n_1}^{2,1} = i) = \sum_{i=0}^{n_1 - r_1} i \binom{n_1 + n_1 - r_1 - i - 1}{n_1 - r_1 - i} \binom{n_2 + i - 1}{i} p^{n_1}(1 - p)^{n_1 - r_1 - i} p^{n_2}(1 - p)^{i} = p^n(1 - p)^{n_1 - r_1} \sum_{i=1}^{n_1 - r_1} \binom{n_1 + n_1 - r_1 - i - 1}{n_1 - r_1 - i} \binom{n_2 + i - 2}{i - 1} (n_2 + i - 1) = p^n(1 - p)^{n_1 - r_1} \sum_{i=0}^{n_1 - r_1 - 1} (n_2 + i) \binom{n_1 + n_1 - r_1 - 1 - i - 1}{n_1 - r_1 - i} \binom{n_2 + i - 1}{i},
$$

making a change of variable in the last step. For any $m, n_1, n_2 \in \mathbb{N}$, we use the equality

$$
\sum_{i=0}^{m} \binom{n_1 + m - i - 1}{m - i} \binom{n_2 + i - 1}{i} = \binom{n_1 + n_2 + m - 1}{m},
$$

which can be proved comparing the binomial coefficients in the convolution of two negative binomial random variables. Hence, if we define a function $f : [n_1 - r_1]_0 \to \mathbb{R}_+$, as

$$
f(k) = \sum_{i=0}^{n_1 - r_1 - k} i \binom{n_1 + n_1 - r_1 - k - i - 1}{n_1 - r_1 - k - i} \binom{n_2 + i - 1}{i},
$$
we obtain

\[ f(0) = n_2 \left( \frac{n + n_1 - r_1 - 2}{n_1 - r_1 - 1} \right) + \sum_{i=0}^{n_1-r_1-1} i \left( \frac{n_1 + n_1 - r_1 - 1 - i}{n_1 - r_1 - 1} \right) \left( \frac{n_2 + i - 1}{i} \right) \]

\[ = n_2 \left( \frac{n + n_1 - r_1 - 2}{n_1 - r_1 - 1} \right) + f(1) \]

\[ = n_2 \sum_{j=1}^{n_1-r_1} \left( \frac{n + n_1 - r_1 - 1 - j}{n_1 - r_1 - j} \right) \]

\[ = n_2 \sum_{j=0}^{n_1-r_1-1} \left( \frac{n - 1 + j}{j} \right), \]

making a change of variable. Now, for any \( m, r \in \mathbb{N} \) use the identity

\[ \sum_{j=0}^{m} \binom{r+j}{j} = \binom{r+m+1}{m}, \]

to deduce that

\[ \sum_{i=0}^{n_1-r_1} \omega_p(X_{n_1}^1 = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^1 = i) = n_2 \left( \frac{n + n_1 - r_1 - 1}{n_1 - r_1 - 1} \right) p^p (1-p)^{n_1-r_1}. \]

We compare this quantity with the right-hand side of Hypothesis H2:

\[ \frac{n_2}{n} (n_1 - r_1) \mathbb{P}(NB_{n,p} = n_1 - r_1) = \frac{n_2}{n} (n_1 - r_1) \left( \frac{n + n_1 - r_1 - 1}{n_1 - r_1} \right) p^p (1-p)^{n_1-r_1} \]

\[ = n_2 \frac{(n + n_1 - r_1 - 1)!}{n_1 - r_1 - 1)!} (1-p)^{n_1-r_1} \]

\[ = n_2 \left( \frac{n + n_1 - r_1 - 1}{n_1 - r_1 - 1} \right) p^p (1-p)^{n_1-r_1}, \]

which is identical to (7).

Thus, using Theorem 2, denoting by \(#i\mathcal{G}_{r,p}\) the number of individuals of type \( i \), we obtain

\[ \mathbb{P}(#_1\mathcal{G}_{r,p} = n_1, #_2\mathcal{G}_{r,p} = n_2) = \frac{r}{n} \left( \frac{n + n_1 - r_1 - 1}{n_1 - r_1} \right) \left( \frac{n + n_2 - r_2 - 1}{n_2 - r_2} \right) p^{2p} (1-p)^{n-r}. \]

It follows that

\[ \mathbb{P}(\mathcal{G}_{r,p} = f | #_1\mathcal{G}_{r,p} = n_1, #_2\mathcal{G}_{r,p} = n_2) = \frac{1}{\frac{r}{n} \left( \frac{n + n_1 - r_1 - 1}{n_1 - r_1} \right) \left( \frac{n + n_2 - r_2 - 1}{n_2 - r_2} \right)} \quad \forall f \in \mathbb{P}_{r,n}^{plane}, \]

being uniform on the set of two-type plane forests with \( r_i \) roots type \( i \), and \( n_i \) vertices of type \( i \). Note that this implies that the denominator on the right-hand side is the number of two-type plane forests with root-type \( r \) and individuals-type \( n \). We also obtain the distributional equality

\[ \mathcal{G}_{r,n}^{plane} \overset{d}{=} (\mathcal{G}_{r,p} | #_1\mathcal{G}_{r,p} = n_1, #_2\mathcal{G}_{r,p} = n_2), \]

where \( \mathcal{G}_{r,n}^{plane} \) is uniform on \( \mathbb{P}_{r,n}^{plane} \).

**General case**

In the general case \( d \in \mathbb{N} \), using Theorem 2, denoting by \(#i\mathcal{G}_{r,p}\) the number of individuals of type \( i \) of \( \mathcal{G}_{r,p} \), we obtain

\[ \mathbb{P}(#_i\mathcal{G}_{r,p} = n_i, \forall i \in [d]) = \frac{r}{n} p^{dn} (1-p)^{n-r} \prod_{i \in [d]} \left( \frac{n + n_i - r_i - 1}{n_i - r_i} \right). \]
It follows that
\[ \mathbb{P}(\mathcal{G}_{r,p} = F | \#_i \mathcal{G}_{r,p} = n_i, \forall i \in [d]) = \frac{1}{\mathcal{I}} \prod_{i \in [d]} \binom{n + n_i - r_i - 1}{n_i - r_i} \quad \forall F \in \mathcal{F}^{\text{plane}}_{r,n}, \]
being uniform on $\mathcal{F}^{\text{plane}}_{r,n}$. Note that the previous agrees with the unidimensional case, see Formula (35) in [Pit98].

5.1.2. Poisson Offspring. Let $r = (r_1, r_2) \in \mathbb{N}^2$ with $r = r_1 + r_2 > 0$ and $n = (n_1, n_2) \in \mathbb{N}^2$. For $\mu \in \mathbb{R}^+$, let $\mathcal{P}_{r,\mu}$ be a two-type GW with $r_1$ roots type $i$, and Poisson offspring distribution of parameter $\mu$, for every individual, independently from anyone, that is, $v_i(k_1, k_2) = e^{-\mu} \mu^{k_1} e^{-\mu} \mu^{k_2} / (k_1!k_2!)$.

Similarly as in the previous example, consider any $F \in \mathcal{F}^{\text{plane}}_{r,n}$ having $r_i$ roots type $i$, $k_{1,2}$ type 2 individuals whose parent is of type 1, and $k_{2,1}$ type 1 individuals whose parent is of type 2. Then
\[
\mathbb{P}(\mathcal{P}_{r,\mu} = F) = e^{-2\mu n_1} \mu^{n_1 - r_1 - k_{2,1}} \mu^{k_{2,1}} \prod_{v \in f(1)} \frac{1}{c_1(v)!} \prod_{v \in f(2)} \frac{1}{c_2(v)!} e^{-2\mu n_2} \mu^{n_2 - r_2 - k_{2,2}} \mu^{k_{2,2}} \prod_{v \in f(3)} \frac{1}{c_1(v)!} \prod_{v \in f(4)} \frac{1}{c_2(v)!}
\]
where the product is over any enumeration of the $n$ vertices in $F$.

We compute the left-hand side of Hypotheses H2. Recall that the sum of $k$ independent Poisson random variables with parameter $\mu$, has a Poisson distribution with parameter $k\mu$. Then
\[
\sum_{i=0}^{n_1 - r_1} i \mathbb{P}(X_{n_1}^{11} = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^{21} = i) = e^{-\mu} \sum_{i=0}^{n_1 - r_1} \frac{(n_1 \mu)^{n_1 - r_1 - i} (n_2 \mu)^i}{(n_1 - r_1 - i)! i!}
\]
To simplify the sum, note that
\[
\sum_{i=0}^{n_1 - r_1} i \binom{n_1 - r_1}{i} \left( \frac{n_2}{n_1} \right)^i = (n_1 - r_1) \sum_{i=1}^{n_1 - r_1} \frac{(n_1 - r_1 - 1)! (n_2)}{i!(n_1 - r_1 - i)! (n_1)^i}
\]
\[
= (n_1 - r_1) \frac{n_2}{n_1} \sum_{i=0}^{n_1 - r_1 - 1} \frac{(n_1 - r_1 - 1)! (n_2)}{i!(n_1 - r_1 - i - 1)! (n_1)^i}
\]
\[
= (n_1 - r_1) \frac{n_2}{n_1} \left( 1 + \frac{n_2}{n_1} \right)^{n_1 - r_1 - 1}
\]
\[
= \frac{n_2}{n} (n_1 - r_1) \left( \frac{n}{n_1} \right)^{n_1 - r_1}.
\]
Hence, it follows that
\[
\sum_{i=0}^{n_1 - r_1} i \mathbb{P}(X_{n_1}^{11} = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^{21} = i) = e^{-\mu} \frac{\mu^{n_1 - r_1}}{(n_1 - r_1)!} \frac{n_2}{n} (n_1 - r_1)^{n_1 - r_1}.
\]
This is the same as the right-hand side of Hypotheses H2, since

\[ \frac{n_2}{n} (n_1 - r_1) \mathbb{P}(P_{\mu} = n_1 - r_1) = e^{-n\mu} \frac{n_2}{n} \frac{\mu^{n_1-r_1}}{(n_1-r_1)!} \frac{r_2}{n} \frac{(n_1 - r_1)^{n_1-r_1}}{(n_1 - r_1)!} \quad \frac{r_2}{n} \frac{(n_1 - r_1)^{n_1-r_1}}{(n_1 - r_1)!} \cdot \]

Denoting by \( \#_i \mathcal{P}_{r,p} \) the number of individuals type \( i \), using Theorem 2 we obtain

\[ \mathbb{P}_r (\#_1 \mathcal{P}_{r,\mu} = n_1, \#_2 \mathcal{P}_{r,\mu} = n_2) = \frac{e^{-2n\mu} (n\mu)^{n-r}}{(n_1-r_1)! (n_2-r_2)! n} \]

and hence

\[ \mathbb{P}(\mathcal{P}_{r,\mu} = F | \#_1 \mathcal{P}_{r,\mu} = n_1, \#_2 \mathcal{P}_{r,\mu} = n_2) = \frac{\prod_i c_i(v_i)^{n_i-r_i}}{\frac{r}{n} n^{n-r}} \quad \forall F \in \mathbb{P}_{r,n}^{plane} \]

**General case**

In the general case, from Theorem 2 we obtain

\[ \mathbb{P}(\#_i \mathcal{P}_{r,\mu} = n_i, \forall i \in [d]) = \frac{r}{n} e^{-d n \mu} (n \mu)^{n-r} \]

From which we have

\[ \mathbb{P}(\mathcal{P}_{r,\mu} = F | \#_i \mathcal{P}_{r,\mu} = n_i, \forall i \in [d]) = \frac{\prod_i c_i(v_i)^{n_i-r_i}}{\frac{r}{n} n^{n-r}} \quad \forall F \in \mathbb{P}_{r,n}^{plane} \]

Note that this agrees with the unidimensional case, as seen in Formula (39) of [Pit98]. Since the right-hand side depends on \( F \), it is not uniform on the set of plane forests. To obtain a uniform forest, we introduce a function as in [Pit98]. Define \( \Psi : \mathbb{P}_{r,n}^{labeled} \to \mathbb{P}_{r,n}^{plane} \) as follows:

1. Order the trees of the forest, according to the natural order of the labels in the roots of type 1, then order the type 2 roots, and so on.
2. For each vertex \( v_i \) of type \( i \), order its \( c_1(v_i) \) children of type 1 according to its labels, its \( c_2(v_i) \) children of type 2 according to its labels, and so on.
3. Erase the labels.

Now, we find the number of forests in \( \mathbb{P}_{r,n}^{labeled} \) that are sent to a given plane forest \( F \). For each \( i \), there are \( (n_i - r_i)! \) ways to label the type \( i \) vertices (recall that our rooted labeled forests have root set \([r]\)). But the permutation of the children of a fixed type of each vertex also lead to the same forest \( F \). That is, if vertex \( v \) has \( c_i(v) \) children type \( i \), there are \( c_i(v)! \) labelings of such children leading to \( F \). This being true for every type and every vertex, we have

\[ \#\Psi^{-1}(F) = \frac{\prod_i (n_i - r_i)!}{\prod_{v \in F} \prod_i c_i(v)!} \]

This is exactly the numerator in the formula obtained above. Thus, we have the following interpretation: let \( \mathcal{P}_{r,n}^{labeled} \) have uniform distribution over the set of all \( d \)-type labeled forests, where the roots are in \([r]\), with roots-type \( r \) and individuals-type \( n \), and let \( \mathcal{P}_{r,n} \) be \( \mathcal{P}_{r,n} \) relabeled by \( d \) uniform random permutations, one for each type, then

\[ \mathcal{P}_{r,n}^{labeled} \overset{d}{=} (\mathcal{P}_{r,p}^{labeled} | \#_i \mathcal{P}_{r,p} = n_i, \forall i \in [d]) \]

We note that the previous formulas coincide with the results in [Pit98, Section 7] for the unitype case. But our formulas also relate directly enumerations of multitype labeled forests with the unitype enumerations. Recall our labeling on page 9 for forests in \( \mathcal{P}_{r,n}^{labeled} \). The above formulas also imply that the number of multitype labeled forests in \( \mathcal{P}_{r,n}^{labeled} \) with root set \([r]\) coincides
with the number of unitype labeled forests on \([n]\) with root set \([r]\), which by Cayley’s formula is \((r/n)n^{n-r}\). This comes from the following bijection. Regard each multitype forest \(F \in \mathcal{F}_{\text{labeled}}^{r,n}\) as a unitype labeled forest on \([n]\), together with the following labeling: the roots retain their labels and, according with the order on \(F\), the remaining \(n_1 - r_1\) type 1 individuals now have the new labels \(\{r + 1, \ldots, r + n_1 - r_1\}\), the remaining \(n_2 - r_2\) type 2 individuals have the new labels \(\{r + n_1 - r_1 + 1, \ldots, r + n_1 - r_1 + n_2 - r_2\}\), and so on.

### 5.1.3. Bernoulli Offspring

Let \(r = (r_1, r_2) \in \mathbb{N}^2\) with \(r = r_1 + r_2 > 0\) and \(n = (n_1, n_2) \in \mathbb{N}^2\). For \(0 < p < 1\), let \(\mathcal{B}_{r,p}\) be a two-type GW with \(r\) roots type \(i\), and Bernoulli offspring distribution of parameter \(p\), for each vertex independently of the others, that is, \(\nu_i(k_1, k_2) = p^{k_1}(1 - p)^{1-k_1}p^{k_2}(1 - p)^{1-k_2}\) with \(k_1, k_2 \in \{0, 1\}\). Since any vertex \(v\) has zero or two children with probability \(p\) or \(1 - p\) respectively, then \(\nu_i(c_1(v), c_2(v)) = p^{c_1(v)/2}(1 - p)^{1-c_1(v)/2}p^{c_2(v)/2}(1 - p)^{1-c_2(v)/2}\).

As before, consider any \(F \in \mathcal{F}_{r,n}^{\text{binary}}\) having \(r_1\) roots type \(i\), \(k_{i,2}\) type 2 individuals whose parent is of type 1, and \(k_{2,1}\) type 1 individuals whose parent is of type 2. Note that \(k_{1,2}\) and \(k_{2,1}\) are even numbers, as well as \(n_i - r_i\) for \(i = 1, 2\). Hence

\[
\mathbb{P}(\mathcal{B}_{r,p} = F) = \left(\frac{p}{(1 - p)}\right)^{(n_1 - r_1 - k_{2,1})/2} (1 - p)^{n_1} (p/(1 - p))^{k_{1,2}/2} (1 - p)^{n_2} \times \left(\frac{p}{(1 - p)}\right)^{(n_2 - r_2 - k_{1,2})/2} (1 - p)^{n_2} (p/(1 - p))^{k_{2,1}/2} (1 - p)^{n_1} = p^{(n-r)/2}(1 - p)^{2n -(n-r)/2}.
\]

Recall that twice the sum of \(n\) independent Bernoulli random variables with parameter \(p\), has distribution two times the Binomial distribution \(B_{n,p}\) of parameters \(n\) and \(p\). If \(n\) is even, we denote the sum over the even numbers up to \(n\) as \(\{0, 2, \ldots, n\} = 2[n/2]_0\). The left-hand side of Hypotheses **H2** is

\[
\sum_{i \in \{0, 2, \ldots, n\}} i \mathbb{P}(X_{n_1}^{\text{r,1}} = n_1 - r_1 - i) \mathbb{P}(X_{n_2}^{\text{r,2}} = i)
= \sum_{i \in \{0, 2, \ldots, n\}} \binom{n_2}{i/2} p^{i/2}(1 - p)^{n_2 - i/2} \binom{n_1}{(n_1 - r_1 - i)/2} p^{(n_1 - r_1 - i)/2}(1 - p)^{n_1 -(n_1 - r_1 - i)/2}
= 2p^{(n_1 - r_1)/2}(1 - p)^{n -(n_1 - r_1)/2} \sum_{i \in \{0, 2, \ldots, n\}} \binom{n_2 - 1}{i/2} \binom{n_1}{(n_1 - r_1 - i)/2}
= 2p^{(n_1 - r_1)/2}(1 - p)^{n -(n_1 - r_1)/2} n \sum_{i \in \{0, 2, \ldots, n\}} \binom{n_2 - 1}{i/2} \binom{n_1}{(n_1 - r_1 - 2 - i)/2}.
\]

Note that, since there are \(n_2\) individuals type 2 and at most each can have 2 children, the number \(k_{2,1}\) of individuals type 2 having children type 1 is bounded by \(2n_2\). Nevertheless, we have \(\mathbb{P}(X_{n_2}^{\text{r,2}} = i) = 0\) for \(i > 2n_2\), which agrees with the definition of \(\binom{n}{i} = 0\) whenever \(n < k\) for positive integers. Using Vandermonde’s identity, and adding the term \((n_1 - r_1)/2\) in both the numerator and denominator, the above is equal to

\[
2p^{(n_1 - r_1)/2}(1 - p)^{n -(n_1 - r_1)/2} n \binom{n}{(n_1 - r_1)/2} p^{(n_1 - r_1)/2}(1 - p)^{n -(n_1 - r_1)/2}.
\]

The right-hand side of Hypotheses **H2** is

\[
\frac{n_2}{n}(n_1 - r_1) \mathbb{P}(2B_{n,p} = n_1 - r_1) = \frac{n_2}{n}(n_1 - r_1) \left(\frac{n}{(n_1 - r_1)/2}\right) p^{(n_1 - r_1)/2}(1 - p)^{n -(n_1 - r_1)/2}.
\]
Therefore, Hypotheses \( H2 \) are satisfied and
\[
\mathbb{P}(O_1 = n_1, O_2 = n_2) = \frac{p}{n} \binom{n}{(n_1 - r_1)/2} \binom{n}{(n - n_2)(n_2 - r_2)/2} p^{(n-r)/2} (1-p)^{2n-(n-r)/2}.
\]
Denoting by \( \#_i \mathcal{R}_{r,p} \) the number of individuals type \( i \), we have
\[
\mathbb{P}(\mathcal{R}_{r,p} = F | \#_1 \mathcal{R}_{r,p} = n_1, \#_2 \mathcal{R}_{r,p} = n_2) = \frac{1}{\binom{n}{(n_1 - r_1)/2} \binom{n}{(n_2 - r_2)/2}} \forall F \in \mathbb{P}^{\text{binary}}_{r,n},
\]
being uniform on \( \mathbb{P}^{\text{binary}}_{r,n} \).

**General case**

For \( d \in \mathbb{N} \), Theorem 2 implies
\[
\mathbb{P}(\#_i \mathcal{R}_{r,n} = n_i, \forall i \in [d]) = p^{(n-r)/2} (1-p)^{dn-(n-r)/2} \prod_{i} \binom{n}{(n_i - r_i)/2}.
\]
This implies
\[
\mathbb{P}(\mathcal{R}_{r,p} = F | \#_i \mathcal{R}_{r,n} = n_i, \forall i \in [d]) = \frac{1}{\prod \binom{n}{(n_i - r_i)/2}} \forall F \in \mathbb{P}^{\text{binary}}_{r,n},
\]
being uniform on the set of binary \( d \)-type plane forests with root-type \( r \) and individuals-type \( n \).

Compare this formula with the number of binary trees, which is related to the Catalan numbers, see Theorem 2.1 in [Drm09].

### 6. Algorithms

In this section we present algorithms for the simulation of the unitype random forests presented before. Then, those algorithms are generalized to the multidimensional case.

First, we construct a (deterministic) degree sequence which is close to the (random) empirical degree sequence of a CGW tree. After that, hubs are added to the algorithm to ensure individuals with a lot of children. Being able to obtain degree sequences, we repeat the algorithm to simulate a uniform tree with such given degree sequence. Using this we describe the simulation of CGW \((n)\) trees given in [Dev12]. The construction for forests was given in Lemma 5, but we explicitly write the algorithm. As a side note, a new algorithm is proposed to simulate a tree which has offspring distribution almost as a CGW \((n)\). This is done by fixing \( n_1, m \in \mathbb{N} \), and constructing a CGW \((n)\) tree, where \( |n - n_1| \leq m \).

After that, we present the counterparts in the multidimensional case. First, we give a construction of multitype degree sequences approaching a given offspring distribution. Then, we recall the simulation of uniformly sampled multitype forests with a given degree sequence, which is Theorem 1. Finally, we present an algorithm to obtain MGW forests conditioned by its number of individuals for each type, which is a generalization of Devroye’s algorithm.

#### 6.1. Construction of unitype degree distributions approaching a given distribution

Let \( T \) be a GW tree with critical offspring distribution \( \nu = (\nu_i; i \geq 0) \), and denote by \( (\hat{N}_i; i \geq 0) \) the empirical degree sequence of \( T \), that is
\[
\hat{N}_i = \sum_{j=1}^{T} 1(c_j = i),
\]
with \( c_j \) the number of children of individual \( j \). Define the normalized empirical degree sequence \( \hat{\nu} = (\hat{\nu}_i; i \geq 0) \), where \( \hat{\nu}_i = \hat{N}_i / |T| \). It turns out that for a CGW \((n)\) tree \( T^n \) having \( n \) vertices and offspring distribution \( \nu \), the convergence of the normalized empirical degree sequence has been proved in Lemma 11 of [BM14b] (under the assumption of a finite variance offspring distribution).
The latter means that if we want to construct degree sequences, we can construct them roughly as $N^n_i \approx |v_i s_n|$.

We generalize such lemma of [BM14b] by dropping the finite variance condition. Let $\nu$ be an offsping distribution that is in the domain of attraction of an $\alpha$-stable law, for short DA($\alpha$), with parameter $\alpha \in (1,2)$. This means that $\nu([j,\infty)) = j^{-\alpha}L(j)$ where $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a slowly varying function, that is $\lim_{x \to \infty} L(tx)/L(x) = 1$ for every $t > 0$. See [BGT89, Chapter 8.3] for more details. Denote by $\mathbb{P}_\nu$ the probability distribution of $T$. The law of CGW($\alpha$) is denoted by $\mathbb{P}_\nu^n(\cdot) = \mathbb{P}_\nu(\cdot | T = n)$, and we only consider $n$ for which this has sense.

**Lemma 8.** Let $\nu$ be any critical and aperiodic distribution in DA($\alpha$), for $\alpha \in (1,2)$. Then, under $\mathbb{P}_\nu^n$ we have

$$\hat{\nu} \stackrel{(d)}{\to} \nu,$$

The proof of the multidimensional case of this result is given in the Appendix, page 36.

This lemma, together with the following, justifies constructing a degree sequence $S_n = (N^n_i, i \geq 0)$ with $s_n = |S_n|$, using an approximation of an empirical degree sequence as

$$\frac{N^n_i}{s_n} \to v_i \quad \forall i \geq 0. \tag{9}$$

Thus, we can obtain uniform trees with a given degree sequence, behaving as trees having a Pareto distribution, which is the content of Algorithm 1.

**Algorithm 1** Generate a degree sequence from a given distribution

**Input:** Any critical distribution $\nu$ and a natural $a_n$.

**Output:** A degree sequence, which normalized, approaches $\nu$.

1. Let $M = \sup \{i : [a_n v_i] > 0\}$.
2. for $i = 1$ to $M$ do
3. \hspace{1em} $N_i = [a_n v_i]$
4. \hspace{1em} end for
5. Set $s_n = 1 + \sum_1^M i N_i$.
6. Set $N_0 = s_n - \sum_1^M N_i$.
7. Define the degree sequence $S_n = (N_0, \ldots, N_M)$.

**Lemma 9.** Let $S_n$ be defined as in Algorithm 1, and consider any sequence $a_n \uparrow \infty$. Assume the distribution $\nu$ satisfies the Hypotheses of Lemma 8. Then the convergence in (9) holds true.

**Proof.** We emphasize the dependence of the degree sequence in $n$ writing $N^n_i$, and also $M^n$. Since, for any $i \geq 1$ we have $0 \leq iN^n_i/a_n \leq iv_i$ for every $n$, then, by the Weierstrass test

$$\frac{s_n}{a_n} = \frac{1}{a_n} + \sum_1^M i \frac{[a_n v_i]}{a_n} = \frac{1}{a_n} + \sum_1^\infty i \frac{[a_n v_i]}{a_n} \to \sum_1^\infty i v_i,$$

which equals 1, since $\nu$ is critical. This easily implies $N^n_i/s_n \to v_i$ for every $i \geq 1$, and also

$$\frac{N^n_0}{s_n} = 1 - \sum_1^M \frac{[a_n v_i]}{s_n} \to 1 - \sum_1^\infty v_i = v_0. \quad \square$$

Next, we add hubs to the degree sequence, that is, individuals with many children. Those individuals will have $I_{\beta i+1} = [\beta_i b_{s_n}]$ children, for $\hat{M}$ fixed positive reals $\beta_1 > \cdots > \beta_{\hat{M}} > 0$ and where $b_{s_n}$ is the spacial scaling for the BFW to converge. If necessary, we choose $c_n$ big enough such that $[\beta_{i+1} b_{s_n}] < [\beta_i b_{s_n}]$ whenever $\beta_{i+1} < \beta_i$. This condition ensures there are no unnecessary
repetitions in the child sequence. We also impose $|\beta_M b_{n_i}| > M$, since $M$ is the maximum number of children obtained in Algorithm 1. This is given in Algorithm 2.

**Algorithm 2** Generate a degree sequence from a given distribution and having hubs

**Input:** A distribution $\nu$, a natural $a_n$, and $\bar{M}$ positive reals $\beta_1 > \cdots > \beta_{\bar{M}}$.

**Output:** A degree sequence, which normalized, approaches $\nu$, and has individuals with $\lfloor \beta_i b_{n_i} \rfloor$ children, for $i \in [\bar{M}]$.

1: Obtain a degree sequence $S_n = (N_0, \ldots, N_M)$ and $s_n$ from Algorithm 1.
2: Define the degree sequence $\bar{S}_n = (\bar{N}_0, \bar{N}_1, \ldots, \bar{N}_{\bar{M}})$, as
3: if $i \in \{1, \ldots, \bar{M}\}$ then
4: $\bar{N}_i = N_i$
5: else if $i \in \{I_1, I_2, \ldots, I_{\bar{M}}\}$ then
6: $\bar{N}_i = \#\{k : I_k = I_i\}$
7: else if $i = 0$ then
8: $\bar{N}_0 = N_0 + \sum_{i=1}^{\bar{M}} (I_i - 1)\bar{N}_i$
9: else
10: $\bar{N}_i = 0$
11: end if
12: Set $\bar{s}_n = \sum \bar{N}_i$.

The fact that this algorithm gives us a degree sequence, follows from

$$\sum \bar{N}_i = \bar{N}_0 + \sum_1^\bar{M} N_i + \sum_1^\bar{M} \bar{N}_i = s_n + \sum_1^\bar{M} (I_i - 1)\bar{N}_i + \sum_1^\bar{M} \bar{N}_i = s_n + \sum_1^\bar{M} I_i\bar{N}_i,$$

and

$$1 + \sum i\bar{N}_i = 1 + \sum_1^M N_i + \sum_1^\bar{M} I_i\bar{N}_i = s_n + \sum_1^\bar{M} I_i\bar{N}_i.$$

Note that the ratio of the number of individuals from the two algorithms is given by

$$\frac{\bar{s}_n}{s_n} = \frac{\sum_0^M N_i}{s_n} + \frac{\sum_1^\bar{M} \bar{N}_i}{s_n}.$$

In Lemma 9 we proved the first term goes to one, thus, it suffices to assume the second term goes to zero to ensure such new degree sequence also approaches to the given distribution $\nu$.

**Lemma 10.** Let $\bar{S}_n$ be defined as in Algorithm 2, and consider any sequence $a_n \uparrow \infty$. Assume the distribution $\nu$ satisfies the Hypotheses of Lemma 8. If

$$\sum_1^\bar{M} \frac{\bar{N}_i^n}{s_n} \rightarrow 0,$$

then, for every $i \geq 0$ the convergence

$$\frac{\bar{N}_i^n}{\bar{s}_n} \rightarrow \nu_i$$

holds true.
6.2. Constrained simulation of unitype random trees with a given degree sequence and GW with given size. The paper [Dev12] gives an algorithm to simulate unitype GW trees with offspring distribution $\nu$ conditioned to have size $n$. The idea is: simulate a multinomial vector $(N_0, \ldots, N_K)$ with parameters $(n, \nu_0, \nu_1, \ldots)$ such that

$$n = \sum N_i = 1 + \sum iN_i,$$

that is, simulate the degree sequence of a CGW($n$). Then, obtain a uniform tree with degree sequence $(N_0, \ldots, N_K)$. The resulting tree will have law $\mathbb{P}_n^\nu$.

First, we give an algorithm to simulate uniform trees with a given degree sequence $S = (N_i; i \geq 0)$ satisfying (10). Algorithm 3 is obtained from [BM14b] and was proved in Lemma 5.

**Algorithm 3** Generate uniformly sampled trees with a given degree sequence

**Input:** A degree sequence $S = (N_i; i \geq 0)$ with $\sum N_i = 1 + \sum iN_i = s$.

**Output:** A uniformly sampled tree with the given degree sequence.

1: Define the vector $c = (c_1, c_2, \ldots, c_s)$, with $N_0$ zeros, $N_1$ ones, etc.
2: Set $\pi = (\pi_1, \ldots, \pi_s)$ a uniform random permutation of $[s]$.
3: For $j \in [s]$ define the walk

$$W^b(j) = \sum_{i=1}^j (c(\pi_i) - 1),$$

satisfying $W^b(0) = 0$ and $W^b(s) = -1$.
4: Let $i^* = \min\{j \in [s]: W^b(j) = \min_{1 \leq i \leq s} W^b(i)\}$ be the first time the partial sums reaches its minimum.
5: For $j \in [s]$ define the walk $V(W^b)$ of length $s$ as

$$V(W^b)(j) = \sum_{i=1}^j (c(\pi_{i^*+j}) - 1),$$

with $i^* + j \mod s$.
6: Generate the tree with breadth-first walk $V(W^b)$.

Using Algorithm 3, we give and prove Algorithm 4, which was proposed in [Dev12].

**Algorithm 4** Generate a GW tree conditioned to have size $n$

**Input:** A distribution $\nu$ and $n \in \mathbb{N}$.

**Output:** A tree with law $\mathbb{P}_n^\nu$.

1: Generate a multinomial vector $S = (N_i; i \geq 0)$ with parameters $(n, \nu_0, \nu_1, \ldots)$.
2: Let $K$ be the last non-zero component of $S$, that is $N_j = 0$ for $j > K$ and $N_K > 0$.
3: Define $\Xi(S) = 1 + \sum iN_i$.
4: if $\Xi(S) = n$ then
5: go to step 9
6: else
7: repeat from step 1
8: end if
9: Apply Algorithm 1 to the degree sequence $(N_0, \ldots, N_K)$.

The following lemma proves Algorithm 4 gives us an CGW($n$) tree.

**Lemma 11.** Let $S(i)$ be the $i$th vector obtained by step 1 of Algorithm 4, and let $K = \inf\{i: \Xi(S(i)) = n\}$. If $\tau_K$ is the tree obtained in step 9, then $\tau_K$ has the same law as a CGW($n$) tree.
Proof. Let $S$ be a vector with the same distribution as $S_{(1)}$. For any vector $s = (n_0,\ldots,n_k)$ with $\sum n_i = n$, by definition we have

$$\mathbb{P}(S = s) = \binom{n}{n_0,\ldots,n_k} \prod \nu_i^{n_i}.$$ 

Denote by $W^b$ the bridge with increments $c(\pi_i) - 1$, where $\pi$ is a uniform permutation of the child sequence $c = (c_i)$, the latter obtained from $S_{(K)}$. Denote by $W$ its Vervaat transform, which codes the tree $\tau_{(K)}$. Then, for any bridge $w^b$ of size $n$, having Vervaat transform $w$ and degree sequence $s = (n_0,\ldots,n_k)$ we have

$$\mathbb{P}(W = w) = n \mathbb{P}(W^b = w^b) = \frac{n}{\binom{n}{n_0,\ldots,n_k}} \mathbb{P}(S_{(K)} = s),$$

since there are $n$ bridges mapped to $w$ by the Vervaat transform, and there are $\left(\binom{n}{n_0,\ldots,n_k}\right)$ different labelings of such bridges. For the last term, we sum over all possible values of $K$ and use independence between simulations

$$\mathbb{P}(S_{(K)} = s) = \sum_k \mathbb{P}(S_{(k)} = s, \Xi(S_{(k)}) = n, \Xi(S_{(j)}) \neq n, j < k)$$

$$= \mathbb{P}(S = s, \Xi(S) = n) \sum_{k \geq 1} \mathbb{P}(\Xi(S) \neq n)^{k-1}$$

$$= \mathbb{P}(S = s, \Xi(S) = n) \frac{1}{\mathbb{P}(\Xi(S) = n)}.$$

Note that

$$\mathbb{P}(\Xi(S) = n) = \sum_{s=(n_0,\ldots): \sum n_i = n} \mathbb{P}(S = s) = \sum_{s=(n_0,\ldots): \sum n_i = n} \binom{n}{n_0,\ldots,n_k} \prod \nu_i^{n_i},$$

where the sum is over all degree sequences of plane trees having size $n$. We relate this with $\nu^n$, the $n$-th convolution of the law $\nu$ with itself. Using the formula for the convolution

$$\nu^n(n-1) = \sum_{(i_1,\ldots,i_n): \sum i_k = n-1} \prod_{k=1}^n \nu_{i_k}.$$ 

Fix any degree sequence $(n_i, i \geq 0)$ with $\sum n_i = n$, and note that the number of vectors $(i_1,\ldots,i_n)$ with $\sum i_k = n-1$ such that

$$\prod_{k=1}^n \nu_{i_k} = \prod_{k \geq 0} \nu_{i_k}^{n_i},$$

is equal to the number of different bridges $w^b$ of size $n$, having degree sequence $(n_i, i \geq 0)$. This number is $\left(\binom{n}{n_0\ldots,n_k}\right)$, therefore

$$\nu^n(n-1) = \sum_{s=(n_0,\ldots): \sum n_i = n} \binom{n}{n_0,\ldots,n_k} \prod \nu_i^{n_i}. $$

The latter, together with the Otter-Dwass formula [Ott49, Dwa69] imply

$$\mathbb{P}(\Xi(S) = n) = \nu^n(n-1) = n \mathbb{P}(|\tau| = n).$$
If $w$ codes the tree $t$, then

$$P(\tau_K = t) = \frac{n}{n_{n_0, \ldots, n_k}} P(S = s, \Xi(S) = n) \frac{1}{n P(\lvert \tau \rvert = n)}$$

$$= \prod_i \frac{v_i^{n_i}}{P(\lvert \tau \rvert = n)}$$

$$= P(\tau = t \mid \lvert \tau \rvert = n),$$

proving the assertion. □

A fast way to generate the multinomial vector $(N_0, N_1, \ldots)$ is using the binomials

$$N_0 \sim \text{BIN}(n, v_0)$$

$$N_1 \sim \text{BIN}(n - N_0, v_1/(1 - v_0))$$

$$N_2 \sim \text{BIN}(n - N_0 - N_1, v_2/(1 - v_0 - v_1))$$

$$\vdots$$

Using this conditional construction, the vector $(N_0, N_1, \ldots)$ has the desired multinomial distribution (see [Dev12]).

Using this two results, we can relax step 4 of Algorithm 4. Fix the number of initial individuals $n_1 \in \mathbb{N}$. We find a random $n$, close enough to $n_1$, and generate an approximated CGW($n$) tree. Let $m \in \mathbb{N}$ be the error term allowed in the simulations. Define $A_{n_1, m} := A = \{k \in \mathbb{N} : n_1 - m \leq k \leq n_1 + m\}$. Algorithm 5 generates an almost CGW tree (see Lemma 12 below for a explicit description of this) by adjusting the number of leaves to obtain a tree.

**Algorithm 5** Generate an approximated GW tree conditioned to have almost $n_1$ individuals

**Input:** A distribution $\nu$ and natural numbers $n_1, m$.

**Output:** A tree with law almost $P_n^{\nu}$, where $n \in A$.

1: Generate a multinomial vector $(N_0, N_1, \ldots)$ with parameters $(n_1, v_0, v_1, \ldots)$.
2: Let $K$ be the last non-zero component of $(N_0, N_1, \ldots)$, that is $N_j = 0$ for $j > K$.
3: Define $n = 1 + \sum_i N_i$.
4: if $n \in A$ then
5: go to step 9
6: else
7: repeat from step 1.
8: end if
9: Redefine $N'_0 = n - \sum_1^K N_i$.
10: Apply Algorithm 3 to the degree sequence $(N'_0, N_1, \ldots, N_K)$.

**Remark 5.** Note that Algorithm 5 gives us a degree sequence of size $n$, since

$$N'_0 + \sum_i^K N_i = n - \sum_i^K N_i + \sum_i^K N_i = n \quad \text{and} \quad 1 + \sum_i N_i = n.$$

To prove that Algorithm 5 generates an approximate CGW($n$) tree, we use the Local Limit Theorem [BGT89][Theorem 8.4.1]. Recall from the proof of Lemma 11 that for a fixed tree $t$ with degree sequence $s = (n_0, \ldots, n_k)$ having size $n$, we have $P(\tau = t \mid \lvert \tau \rvert = n) = \prod_i v_i^{n_i} / P(\lvert \tau \rvert = n).$
Lemma 12. Let $\tau$ be a GW tree with offspring distribution $v$ in $DA(\alpha)$, for $\alpha \in (1, 2]$. Let $S(i)$ be the $i$th vector obtained by step 1 of Algorithm 5, and let $K = \inf \{ i : \Xi(S(i)) \in A \}$.

Let $\tau_{(K)}$ be the tree obtained in step 10. Fix $n \in A$ and a tree $t$ with degree sequence $s = (n_0, \ldots, n_k)$ having size $n$. Then, for every $n_1$ big enough

$$\frac{\prod v_i^{n_i}}{\mathbb{P}(|\tau| = n) + c/n^{1+1/\alpha}} \leq \frac{\prod v_i^{n_i}}{\mathbb{P}(|\tau| = n) + c'/n^{1+1/\alpha}}$$

for some constants $c, c'$.

Proof. Let $S$ be a vector with the same distribution as $S_{(1)}$, and denote by $W$ the BFW of $\tau_{(K)}$. Fix the tree $t$ of size $n$, with $w$ its BFW and its degree sequence $s$ as in the statement of the Lemma. By the definition of step 10 in Algorithm 5, we have as before

$$\mathbb{P}(W = w) = \frac{n}{(n_0, \ldots, n_k)} \mathbb{P}(S_{(K)} = s) = \frac{n}{(n_0, \ldots, n_k)} \mathbb{P}(S = s, \Xi(S) \in A) \frac{1}{\mathbb{P}(\Xi(S) \in A)} .$$

Hence $\mathbb{P}(W = w) = n \prod v_i^{n_i}/\mathbb{P}(\Xi(S) \in A)$. Now we prove that $\mathbb{P}(\Xi(S) \in A)/n \approx \mathbb{P}(|\tau| = n)$. Summing over all the values in $A$ and using Equation 11 we have

$$\mathbb{P}(\Xi(S) \in A) = \sum_{k \in A} \mathbb{P}(\Xi(S) = k) = \sum_{k \in A} \mathbb{P}(W_k' = -1) ,$$

were $W'$ is a random walk with law $(v(j), j \geq -1)$.

By the Local Limit Theorem, we know there exists some positive constants $c$ and $c'$ such that for every $n_1$ big enough

$$c \sum_{k \in A} \frac{1}{a_k} \leq \sum_{k \in A} \mathbb{P}(W_k' = -1) \leq c' \sum_{k \in A} \frac{1}{a_k} ,$$

with $a_k = k^{1/\alpha} L(k)$ and $L$ a slowly varying function. We will use repeatedly the Potter bounds (see [BGT89, Theorem 1.5.6]), saying that for $x, y$ big enough we have $L(y)/L(x) \leq 2 \max \{ y/x, x/y \}$. Thus for $k \in A$

$$\frac{1}{2} \frac{n - 2m}{k} L(n - 2m) \leq L(k) \leq 2 \frac{n + 2m}{k} L(n + 2m) .$$

We also have for the same $k$

$$(n - 2m)^{-1/\alpha} \geq k^{-1/\alpha} \geq (n + 2m)^{-1/\alpha} ,$$

and since $|A| = 2m + 1$ is bounded, then (using without distinction the constants $c$ and $c'$)

$$c \frac{1}{a_{n+2m}} \leq \sum_{k \in A} \frac{1}{a_k} \leq c' \frac{1}{a_{n-2m}} .$$

Consider any constants $b, b'$ such that

$$\frac{b}{n^{1/\alpha}} \leq \frac{1}{(n + 2m)^{1/\alpha}} \leq \frac{1}{(n - 2m)^{1/\alpha}} \leq \frac{b'}{n^{1/\alpha}} ,$$

and again by the Potter bounds we have

$$L^{-1}(n - 2m) \leq 2 \frac{n - 2m}{n} L^{-1}(n) \quad \text{and} \quad \frac{1}{2} \frac{n}{n + 2m} L^{-1}(n) \leq L^{-1}(n + 2m) .$$

Those inequalities imply

$$c \frac{1}{a_n} \leq \sum_{k \in A} \frac{1}{a_k} \leq c' \frac{1}{a_n} .$$
Since there exists \( d \) and \( d' \) such that \( da_n^{-1} \leq P(W_n' = -1) \leq d'a_n^{-1} \) by the Local Limit Theorem, it follows

\[
(c - d') \frac{1}{a_n} \leq \sum_{k \in A} \frac{1}{a_k} - P(W_n' = -1) \leq (c' - d) \frac{1}{a_n}.
\]

This proves the lemma, since \( P(W_n' = -1) / n = P(\| \tau \| = n) \) by the Otter-Dwass formula.

6.3. **Constrained simulation of unitype random forests.** Now we give a way to simulate uniformly sampled forests with a given degree distribution, this is Algorithm 6, and was proved in Lemma 5.

**Algorithm 6** Generate uniformly sampled forests with a given degree sequence

**Input:** A degree sequence \( S = (N_i; i \geq 0) \) and \( m \in \mathbb{N} \) with \( \sum N_i = m + \sum iN_i = s \).

**Output:** A uniformly sampled forest with \( m \) trees having the given degree sequence.

1. Define the vector \( e = (c_1, c_2, \ldots, c_s) \), with \( N_0 \) zeros, \( N_1 \) ones, etc.
2. Set \( \pi = (\pi_1, \ldots, \pi_s) \) a uniform random permutation of \([s]\).
3. For \( j \in [s] \) define the walk

   \[
   W^b(j) = \sum_{i=1}^{j} (e \circ \pi_i - 1),
   \]

   satisfying \( W^b(0) = 0 \) and \( W^b(s) = -m \).
4. Let \( i^* = \min \{j \in [s] : W^b(j) = \min_{1 \leq i \leq s} W^b(i) \} \) be the first time the partial sums reaches its minimum.
5. Define an independent uniform variable \( U \) on \([m] - 1\), and \( \tau_U = \min \{j : W^b(j) = W^b(i^*) + U\} \).
6. Define the process \( V(W^b, U) \) of length \( s \) whose \( j \)th term is \( e \circ \pi_{\tau_U + j - 1} \) with \( \tau_U + j \mod s \).
7. Generate the forest with breadth-first walk \( V(W^b, U) \).

6.4. **Construction of multitype degree distributions approaching a given distribution.** The objective now is to construct multitype degree sequences explicitly (thus, extending Algorithm 1).

Since, under certain conditions, the law of a CMGW(\( n \)) forest is a finite mixture of the laws of uniform multitype forests with a given degree sequence (see Lemma 6), we generalize the unidimensional case, to prove that under some conditions, the normalized empirical degree sequence of a CMGW(\( n \)) forest converges to the offspring distribution \( v \). Thus, we can also construct degree sequences roughly as \( N_{i,j}(k) \approx \lfloor n_j v_{i,j}(k) \rfloor \).

The normalized empirical degree sequence of the CMGW(\( n \)) is defined as

\[
\hat{\nu}_{i,j}^n(m) = \frac{1}{N_{i,j}(m)} = \frac{1}{m} \sum_{l=1}^{m} \mathbb{1}\{c_j(u'(l)) = m\} \quad i, j \in [d], m \geq 0,
\]

where \( c_j(u'(l)) \) is the number of children type \( j \) of the \( l \)th individual type \( i \) of the forest.

Recall from 3 the definition of \( X \) from the offspring distribution \( v \). For the following lemma, we consider sequences \( n^l = (n_i^l, i \in [d]) \), \( k^l = (k_{i,j}^l, i, j \in [d]) \) and \( r^l = (r_{i,j}, i, j \in [d]) \) for \( l \in \mathbb{N} \) such that \( n^l \to \infty(1, \ldots, 1), k_{i,j}^l / n_i^l \to y_{i,j} \) with \( y_{i,j} \geq 0 \) for \( i \neq j \) and \( y_{i,j} \leq 0 \), and also \( r_{i,j} / n_i^l \to x_i \geq 0 \). The justification for the election of such indexes in the scaling constants, comes from Corollary 1 in [CPGUB17]. The following result generalizes Lemma 8, and the proof is given in the Appendix, page 36.

**Lemma 13.** Consider \( n^l, k^l \) and \( r^l \) as above. Let \( v = (v_1, \ldots, v_d) \) be the offspring distribution of a non-degenerate and irreducible MGW forest. Assume that \( v_i \) has independent components for every \( i \), all of which are aperiodic, and that \( X^{i,j} \) has mean zero. Suppose that there exists positive
Lemma 14. Let \( S_n \) be defined as in Algorithm 7, and consider any sequence \( a_n \uparrow \infty \) and root-type \( r^j = (r_{1}^j, \ldots, r_{d}^j) \) with \( r_{j}^j/a_n \to 0 \) for every \( j \in [d] \). Assume the distribution \( \nu \) satisfies \( 1 = \sum k \nu_{i,j}(k) \) for every \( i \). Then, for every \( k \geq 0 \), every \( i, j \in [d] \), as \( a_n \to \infty \) we have

\[
\frac{N_{i,j}(k)}{s_n(i)} \to \nu_{i,j}(k).
\]

Proof. We emphasize the dependence of the degree sequence in \( n \) writing \( N_{i,j}^n \), and also \( M_{i,j}^n \). Since, for any \( k \geq 1 \) we have \( 0 \leq k N_{i,j}^n(k)/a_n \leq k \nu_{i,j}(k) \) for every \( n \), then, by the Weierstrass test

\[
\frac{s_n(j)}{a_n} = \frac{r^j_j}{a_n} + \sum_{k=1}^{M_{i,j}^n} \sum_i k \frac{a_n \nu_{i,j}(k)}{a_n} \to \sum_k k \nu_{i,j}(k),
\]

which equals 1 by hypothesis. This implies \( N_{i,j}^n(k)/s_n(i) \to \nu_{i,j}(k) \) for every \( k \geq 1 \), and also

\[
\frac{N_{i,j}^n(0)}{s_n(i)} = 1 - \sum_{k=1}^{M_{i,j}^n} \frac{a_n \nu_{i,j}(k)}{s_n(i)} \to 1 - \sum \nu_{i,j}(k) = \nu_{i,j}(0).
\]

6.5. Constrained simulation of multitype random forests with given degree sequence and MGW with given type sizes. Now we propose Algorithm 8, using the multidimensional Vervaat transform as defined in page 6. This algorithm is precisely Theorem 1.

Finally, for fixed \( r = (r_1, \ldots, r_d) \) and \( n = (n_1, \ldots, n_d) \), we consider the simulation of multitype GW forests conditioned to have individuals-type \( n \) and root-type \( r \). Using Devroye’s idea of Algorithm 4 we propose Algorithm 9. We denote by \( \mathbb{P}_r(\cdot | \#_j = n_j, \forall j) \) the law of a CMGW(\( n \)) with root-type \( r \), and by \( \nu_{i,j} \) the \( j \)th marginal of the distribution \( \nu_i \).

The following proposition, stated on the introduction as Proposition 4, proves that from Algorithm 9 we construct a CMGW(\( n \)).
Algorithm 8 Generate uniformly sampled multitype forests with a given degree sequence

**Input:** A degree sequence $S_{i,j} = (N_{i,j}(k); k \in [m_{i,j}])$ satisfying $n_i = \sum k N_{i,j}(k)$ for every $j$, and $n_j = r_j + \sum_k k N_{i,j}(k)$, for every $j$.

**Output:** A uniformly sampled multitype tree with the given degree sequence.

1. Generate the vectors $c_{i,j} = (c_{i,j}(1), c_{i,j}(2), \ldots, c_{i,j}(n_i))$, with $N_{i,j}(0)$ zeros, $N_{i,j}(1)$ ones, etc., ordered in non-decreasing order $c_{i,j}(k) \leq c_{i,j}(k+1)$.
2. Generate $\pi_{i,j} = (\pi_{i,j}(1), \ldots, \pi_{i,j}(n_i))$, a uniform random permutation of $[n_i]$, everything independent.
3. Define $W^b = (W^b_{i,j}; i, j \in [d])$, where
   \[ W^b_{i,j}(k) = \sum_{l=1}^k (c_{i,j} \circ \pi_{i,j}(l) - 1 \{ i = j \}) \]
   satisfying $W^b_{i,j}(0) = 0$ and $W^b_{i,j}(n_i) = -k_i$.
4. Generate an independent uniform random variable $U$ on $[\det(k_{i,j})]$, where $k_{i,j} := \sum k N_{i,j}(k) - n_1 \{ i = j \}$.
5. Construct the multidimensional Vervaat transform $V(W^b, U)$ of $W^b$.
6. Generate the multitype forest with breadth-first walk $V(W^b, U)$.

Algorithm 9 Generate a CMGW(n) forest $F$

**Input:** A distribution $\nu$, and natural numbers $n_i > r_i \geq 1$, for $i \in [d]$.

**Output:** A multitype forest with law $\mathbb{P}_r$.

1. Generate independent multinomial vectors $S_{i,j} = (N_{i,j}(0), N_{i,j}(1), \ldots)$ with parameters $(n_i, v_{i,j}(0), v_{i,j}(1), \ldots)$.
2. Let $K_{i,j}$ be the last non-zero component of $S_{i,j}$, that is $N_{i,j}(K_{i,j}) > 0$ and $N_{i,j}(j) = 0$ for $j > K_{i,j}$.
3. Define $\Xi_j := \Xi(S_{i,j}, i \in [d]) = r_j + \sum k N_{i,j}(k)$ for every $j$.
4. if $\Xi_j = n_j$ for every $j$ then
5. go to step 9
6. else
7. repeat from step 1.
8. end if
9. Apply Algorithm 8 to the degree sequence $((N_{i,j}(0), \ldots, N_{i,j}(K_{i,j})); i, j \in [d])$, obtaining a multitype forest $F_0$ with breadth-first walk distributed as $V(W^b, U)$.
10. Define $k_{i,j} := \sum k N_{i,j}(k) - n_1 \{ i = j \}$.
11. Generate an independent uniform variable $V$ on $[0, 1]$.
12. if $V \leq \frac{\det(-k_{i,j})}{(d+1)! \prod \pi_{i,j}}$ then
13. Accept $F = F_0$
14. else
15. repeat from step 1.
16. end if

Proposition 5. Let $W$ be the breadth-first walk of a CMGW(n) forest satisfying the Hypotheses of Theorem 2, having offspring distribution $\nu$, and root-type $r$ with $1 \leq r_i < n_i$ for every $i$. Generate independent multinomial vectors $S_{i,j} = (N_{i,j}(0), N_{i,j}(1), \ldots)$ with parameters $(n_i, v_{i,j}(0), v_{i,j}(1), \ldots)$, and stop the first time $K = \inf\{ k : \Xi(S_{i,j}, i \in [d]) = n_j \text{ for every } j \}$. Denote by $S_{i,(K)}$ the multitype degree sequence obtained, and let $V(W^b, U)$ be the breadth-first walk generated by Algorithm 8.
using the degree sequence $S_{(K)}$. Then,
\[ \mathbb{P}_r \left( V(W^b, U) = w \right) = \frac{1}{\text{det}(k_{i,j})} \prod_i n_i \mathbb{P}_r \left( S = (n_{i,j}, i, j \in [d]) \right), \]
for every multitype forest $F$ with root-type $r$ and individuals-type $n$, coded by $w$ and with $k_{i,j} = \sum kn_{i,j}(k) - n_1 \{i = j\}$.

**Proof.** We follow the same lines as in Lemma 11. Fix any $d$-type forest $F$, having $r_i$ roots and $n_i$ vertices of type $i$, and degree sequence $s = (n_{i,j}, i, j \in [d])$. Using the same notation as in Theorem 1, let $W^b$ be a multidimensional bridge in $\mathbb{P}_r s$, having multidimensional Vervaat transform $w = V(w^b, u)$ for some $u \in [\text{det}(-k_{i,j})]$, where $k_{i,j} := \sum kn_{i,j}(k) - n_1 \{i = j\}$. Using that $W^b$ has exchangeable increments, that $U$ is independent and uniform, and that there are $\prod n_i$ pairs $(\theta_{q,n}(w), u)$ that can be mapped to $w$ (as seen on page 12), then
\[ \mathbb{P}_r \left( V(W^b, U) = w \right) = \prod_i n_i \mathbb{P}_r \left( W^b = w^b, U = u \right) \]
\[ = \frac{1}{\text{det}(k_{i,j})} \prod_i n_i \mathbb{P}_r \left( S = (n_{i,j}, i, j \in [d]) \right) \]
\[ = \frac{1}{\text{det}(k_{i,j})} \prod_i n_i \mathbb{P}_r \left( \Xi(s_{i,j}, i \in [d]) = n_j, \forall j \right), \]
where $S$ has the same distribution as $S_{(1)}$. We compute explicitly the last fraction of the above equation. For the term $\mathbb{P}(S = s)$ we use the definition of the multinomial distribution
\[ \mathbb{P}_r \left( S = s \right) = \prod_i \prod_j \left( \begin{array}{c} n_i \\ n_{i,j} \end{array} \right) \prod_{l \geq 0} V_{i,j}^{n_{i,j}(l)}(l). \]
For the denominator we have
\[ \mathbb{P}_r \left( \Xi(s_{i,j}, i \in [d]) = n_j, \forall j \right) = \sum_{s=(n_{i,j})} \mathbb{P}_r \left( S = s \right) \]
\[ = \sum_{s=(n_{i,j})} \prod_i \prod_j \left( \begin{array}{c} n_i \\ n_{i,j} \end{array} \right) \prod_{l \geq 0} V_{i,j}^{n_{i,j}(l)}(l). \]
On the other hand, note that for fixed $j$, using the formula for the convolution,
\[ \mathbb{P} \left( \sum_{k=1}^d X_{i,j}^{k,j} = n_j - r_j \right) = \sum_{\sum k_{i,j} = n_j - r_j, \forall i} \prod_k n_k \mathbb{P}(k, j) \]
\[ = \sum_{\sum kn_{i,j}(k) = n_j - r_j, \forall i} \prod_i \left( \begin{array}{c} n_i \\ n_{i,j} \end{array} \right) \prod_{l \geq 0} V_{i,j}^{n_{i,j}(l)}(l), \]
where in the last equality, we used the fact that $\prod \left( \begin{array}{c} n_i \\ n_{i,j} \end{array} \right)$ is the number of different bridges having the same degree sequence $(n_{1,j}(0), n_{1,j}(1), \ldots, \ldots, n_{d,j}(0), n_{d,j}(1), \ldots)$. Note that the above sum only depends on the sequences $(n_{i,j}, i \in [d])$. Thus, multiplying for all $j$ we have
\[ \mathbb{P}_r \left( \Xi(s_{i,j}, i \in [d]) = n_j, \forall j \right) = \prod_j \mathbb{P} \left( \sum_{k=1}^d X_{i,j}^{k,j} = n_j - r_j \right). \]
Therefore, using Theorem 2 we obtain
\[
P_r\left(V(W^b, U) = w\right) = \frac{1}{\prod n_i} \frac{\prod l \geq 0 \nu_{i,l}^{m_i} \left(v_{i,l}^{\nu_{i,l}}(l) \right)}{\prod \mathbb{P}\left(\sum_k x_{i,k}^{k} = n_j - r_j \right)}
\]
\[
= \frac{1}{\prod n_i} \frac{n \prod \mathbb{P}(O_j = n_j, \forall j)}{\mathbb{P}_r(W = w)}
\]
\[
= \frac{1}{n \prod \mathbb{P}_r(W = w | O_j = n_j, \forall j)}
\]
with \(n = \sum n_i\) and \(r = \sum r_i\). We remark that \(\mathbb{P}_r(W = w | O_j = n_j, \forall j) = \mathbb{P}_r(\mathcal{F} = F | \#_j, \mathcal{F} = n_j, \forall j)\) is the law of the MGW forest conditioned by its sizes.

From Algorithm 9, the first 9 steps are used to obtain a forest \(\mathcal{F}_0\) with law \(V(W^b, U)\). The remaining steps are a usual Accept-Reject method to obtain a sample from the law of the conditioned MGW forest. For each multitype forest with root-type \(r\) and individuals type \(n\) coded by \(w\), define
\[
c_w = \frac{n \det(k_{i,j}(w))}{r \prod n_i} = \frac{\mathbb{P}_r(W = w | O_j = n_j, \forall j)}{\mathbb{P}_r(V(W^b, U) = w)}.
\]
Recall Definition 5 and Lemma 7. For \(i \neq j\), since \(k_{i,j} \leq n_j\) because the maximum number of type \(j\) descendants that any type \(i\) can have is \(n_j\), then
\[
\det(-k_{i,j}) \leq \sum_{(i_1, \ldots, i_d)} d \prod n_j = (d + 1)^{d-1} \prod n_i,
\]
where the last inequality is true by the following bijection. We define a function between the set of elementary forests on \(d\) types and labeled trees on \([d + 1]\) vertices having root with label \(d + 1\). Regard an elementary forest \(F\) on \(d\) types as a unitype tree on \((d + 1)\) vertices by adding a root with label \(d + 1\) having children the roots of \(F\), and assigning label \(i\) to the type \(i\) vertex (cf. the paragraph before Lemma 4.5 in [CL16], the remark after Proposition 7 in [BM14a]). This implies that the number of elementary forests on \(d\) types is \((d + 1)^{d-1}\) by Cayley’s formula.

The previous paragraph gives us the bound \(c_w \leq \frac{\det(-k_{i,j})}{c \prod n_i} \leq \frac{1}{(d + 1)^{d-1}}\). Thus the Accept-Reject method (see [Law13, Section 8.2.4]) applies whenever the uniform \(V\) satisfies
\[
V \leq \frac{\mathbb{P}_r(W = w | O_j = n_j, \forall j)}{c \mathbb{P}_r(V(W^b, U) = w)} = \frac{c_w \det(-k_{i,j})}{(d + 1)^{d-1} \prod n_i} \leq 1.
\]
This concludes the proof.

\[\square\]

**APPENDIX**

*Convergence of the normalized empirical degree sequence of CMGW trees with offspring distribution in DA.* We prove Lemma 13. Recall that Lemma 8 is the unidimensional case, so its proof is omitted. The normalized empirical degree sequence of the CMGW(\(n\)) was defined as
\[
\hat{v}_{i,j}(m) = \frac{1}{n_i} \hat{N}_{i,j}(m) = \frac{1}{n_i} \sum_{l=1}^{n_i} 1\{c_j(u^l) = m\}
\]
where \(c_j(u^l)\) is the number of children type \(j\) of the \(l\)th individual type \(i\) of the forest. The following result proves the convergence of the normalized empirical degree sequence to the offspring distribution. This proof is based in Lemma 11 of [BM14b].
Proof of Lemma 13. Fix $n_i \geq r_i \geq 0$ with $r > 0$. Define $W$ as the BFW of a CMGW$(n)$ forest, where under the law $\mathbb{P}_r$ has root-type $r$. Also, let $X = (X^{(i)}, i \in [d])$ with $X^{(i)}$ a random walk having step distribution as in Equation (3), and using also $\mathbb{P}_r$ as the law in which each $X^{i,j}$ starts at $X^{i,j}(0) = r_i \mathbf{1}\{i = j\}$. We define under $\mathbb{P}_r$ the random time

$$T(X) := \min \left\{ m = (m_1, \ldots, m_d) : r_j + \sum_{i} X_{m_i}^{i,j} = 0, \forall i \right\}$$

which is the minimal solution of the system $(r, X)$ (recall Equation (2)). Then, we have

$$W^d = (X|T(X) = n),$$

under $\mathbb{P}_r$. For $B = (B^{i,j}_m, m \geq 0, i, j \in [d])$ and $B^{i,j}_m \subset \mathbb{Z}_+$. If $\mathbb{P}_r^n$ is the law of the CMGW$(n)$ forest with root-type $r$, on one hand

$$\mathbb{P}_r^n(\hat{N} \in B) = \mathbb{P}_r(\{l : c_j(u^i(l)) = m \} \in B^{i,j}_m, m \geq 0, i, j \in [d] | \#_{i,\mathcal{F}} = n_i, \forall i).$$

On the other hand, defining $\hat{M} = (\hat{M}_{i,j}, i, j \in [d])$ with $\hat{M}_{i,j}(m) = \sum_{i=1}^{n_i} \mathbf{1}\{\Delta X_i^{i,j} = m\}$, if $\mathbb{P}_r^n$ is the law of the random walk conditioned with $T(X) = n$, we have

$$\mathbb{P}_r^n(\hat{M} \in B) = \mathbb{P}_r(\#\{l : \Delta X_l^{i,j} = m - 1 \{i = j\} \} \in B^{i,j}_m, m \geq 0, i, j \in [d] \mid T(X) = n).$$

Let $F(X)$ be a function of the first $n$ increments of $X$, which is invariant under $n$-cyclical permutations, that is,

$$F(X) = F(\theta_{q,n}(X)) \quad \forall q \leq n - 1_d.$$

Let $K = (k_{i,j})_{i,j}$ with $k_{i,j} \geq 0$ for $i \neq j$, $-k_{j,j} = \sum_{i \neq j} k_{i,j}$, and $n_i \geq -k_{i,j}$. Define the sets of multidimensional bridges and good cyclical permutations (see Equation (2)) with fixed final values

$$B_{k,n} := \{x = (x^{i,j})_{i,j} \in S_d : x^{i,j}(n_i) = k_{i,j}\} \quad \text{and} \quad E_{k,n} := \{x \in B_{k,n} : (r, x) \text{ has minimal solution } n\}.$$

Note that under $\mathbb{P}_r$ we have $X \in B_{k,n}$ if and only if $X \in S_d, x^{i,j}_{n_i} = k_{i,j}$ for $i \neq j$ and $x^{j,j}_{n_j} = k_{j,j} - r_j = -r_j - \sum_{i \neq j} k_{i,j}$, in agreement with the definition of $T(X)$. Since each bridge in $B_{k,n}$ can be permuted cyclically using $\prod n_i$ cyclical permutations, then

$$\mathbb{E}_r(F(X); T(X) = n, X \in B_{k,n}) = \frac{1}{\prod n_i} \sum_{q \leq n-1_d} \mathbb{E}_r(F(\theta_{q,n}(X)); T(\theta_{q,n}(X)) = n, \theta_{q,n}(X) \in B_{k,n})$$

$$= \frac{1}{\prod n_i} \sum_{q \leq n-1_d} \mathbb{E}_r(F(X); T(\theta_{q,n}(X)) = n, X \in B_{k,n})$$

$$= \frac{1}{\prod n_i} \mathbb{E}_r \left(F(X) \mathbf{1}\{X \in B_{k,n}\} \sum_{q \leq n-1_d} \mathbf{1}\{T(\theta_{q,n}(X)) = n\} \right).$$

From the Multivariate Cyclic Lemma 1, the number of cyclical permutations that are actually good cyclical permutations is the deterministic quantity $\det(-K)$. Therefore

$$\mathbb{E}_r(F(X); T(X) = n, X \in B_{k,n}) = \frac{\det(-K)}{\prod n_i} \mathbb{E}_r(F(X); X \in B_{k,n}).$$
The later implies
\[
\mathbb{E}_r (F(X)|T(X) = n, X \in B_{k,n}) = \frac{\det(-K)}{\prod n_i} \mathbb{E}_r (F(X); X \in B_{k,n})
\]
\[
= \frac{\det(-K)}{\prod n_i} \mathbb{E}_r (F(X); X \in B_{k,n})
\]
\[
= \mathbb{E}_r (F(X)|X \in B_{k,n}),
\]
using Theorem 1.2 and Equation (3.18) in [CL16] (see 3).

Let \( \mathcal{F}_n \) be the \( \sigma \)-algebra generated by the first \( n_i \) increments of \( X^{(i)} \) for every \( i \in [d] \). We will prove that for \( A \in \mathcal{F}_{[n/2]} \), with \( n \) (and possibly \( r \) and \( k \)) big enough, we have
\[
\mathbb{P}_r (A|X \in B_{k,n}) \leq c \mathbb{P}_r (A)
\]
(c.f. inequality (24), in Lemma 11 of [BM14b]).

For \( k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d \) and \( m = (m_{i,j}, i, j \in [d]) \in \mathbb{Z}^d \), define \( X_k = X \) as \( X^{i,j} = m_{i,j} \) for \( i, j \in [d] \). Summing over all the possible values and using that \( (X^{(i)}, i \in [d]) \) has i.i.d. increments, for all \( A \in \mathcal{F}_{[n/2]} \)
\[
\mathbb{P}_r (A|X \in B_{k,n}) = \sum_{m \in \mathbb{Z}^d} \mathbb{P}_r (A, X_{[n/2]} = m) \frac{\mathbb{P}_0 (X_{n-[n/2]} = k - m)}{\mathbb{P}_r (X = k)}.
\]
We bound the numerator and denominator by a constant, using the Local Limit Theorem, and then sum over all \( m \). Note that by hypothesis, every random walk \( X^{i,j} \) is centered, aperiodic, and with law in DA(\( \alpha_{i,j} \)) for \( \alpha_{i,j} \in (1,2] \); while \( X^{i,j} \) is non-decreasing, aperiodic, and with law in DA(\( \alpha_{i,j} \)) for \( \alpha_{i,j} \in (0,1) \). Thus, we apply Theorem 8.4.1 of [BGT89] to the random walk \( X^{i,j} \) for \( i, j \in [d] \). Note that in the case \( i \neq j \), the first paragraph of page 353 [BGT89] and Theorem 3 XVII.5 [Fel71] imply that there is no need of centering constants \( b_i \) to obtain the convergence \( \frac{1}{a_i} X^{i,j} \to Y^{i,j} \), while in the case \( i = j \) the random walks are already centered.

Now, we consider sequences \( n^l = (n^l_1, \ldots, n^l_0) \), \( k^l = (k^{i,j}, i, j \in [d]) \) and \( r^l = (r_1^l, \ldots, r_d^l) \) for \( l \in \mathbb{N} \) such that \( n^l \to \infty \), \( \frac{1}{a_{n^l-[n/2]}^{i,j}} \to y_{i,j} \) and \( \frac{r_i}{a_{n^l}} \to x_i \). But for ease of notation, we will omit the superscript \( l \) in the following. By the independence assumption, for every \( \varepsilon > 0 \) there exists \( L \in \mathbb{N} \) such that for all \( l \geq L \) we have
\[
\mathbb{P}_0 (X_{n-[n/2]} = k - m) = \prod_{i,j \in [d]} \mathbb{P} (X^{i,j}_{n-[n/2]} = k_{i,j} - m_{i,j})
\]\[
\leq \prod_{i,j \in [d]} \left( \frac{1}{a_{n-[n/2]}^{i,j}} + \frac{1}{a_{n-[n/2]}^{i,j}} g_{i,j} \right) \left( \frac{1}{a_{n-[n/2]}^{i,j}} \right)
\]\[
\leq (\varepsilon + C) \prod_{i,j \in [d]} \frac{1}{a_{n-[n/2]}^{i,j}}
\]
where \( g_{i,j} \) is the density of an \( \alpha_{i,j} \)-stable distribution, which is bounded by \( C \). Similarly, if \( C' = \min g_{i,j} (y_{i,j} - x_i) \mathbb{1} \{i = j\} \) (which is positive since stable densities are positive on its domain), then for \( \varepsilon \in (0, C') \) we have
\[
\mathbb{P}_r (X_n = k) \geq \prod_{i,j \in [d]} \left( - \frac{1}{a_{n}^{i,j}} \varepsilon + \frac{1}{a_{n}^{i,j}} g_{i,j} \left( \frac{k_{i,j} - r_i \mathbb{1} \{i = j\}}{a_{n}^{i,j}} \right) \right)
\]\[
\geq (\varepsilon + C' + o(1)) \prod_{i,j \in [d]} \frac{1}{a_{n}^{i,j}},
\]
using the hypotheses for the convergence of the rescaled $k_{i,j}$ and $r_j$. Joining both inequalities

$$
\frac{P_n(X_n - \lfloor n/2 \rfloor = k - m)}{P_n(X_n = k)} \leq c_\varepsilon \prod_{i,j \in [d]} a_{i,j}^{d_{i,j} - \lfloor n/2 \rfloor} = \prod_{i,j \in [d]} \left( \frac{n_i}{n_i - \lfloor n_i/2 \rfloor} \right)^{1/a_{i,j}} \frac{L_{i,j}(n_i)}{L_{i,j}(n_i - \lfloor n_i/2 \rfloor)}.
$$

Using the Potter bounds (see [BGT89, Theorem 1.5.6]), there exists $L' \in \mathbb{N}$ depending only on $\varepsilon$ such that for every $l \geq L'$

$$
\frac{L(|n_i/2|)}{L(n_i - \lfloor n_i/2 \rfloor)} \leq 2 \frac{|n_i/2|}{n_i - \lfloor n_i/2 \rfloor} \leq 3,
$$

proving Equation (12).

To prove $\hat{v}^l \to v$, it is enough to prove $\frac{1}{n_i} \hat{N}_{i,j}(m) \overset{d}{=} \frac{1}{n_i} \hat{M}_{i,j}(m) \to v_{i,j}(m)$, for every $i, j \in [d]$ and $m \geq 0$. To this end, for $m \geq 0$ define

$$
\hat{M}_{i,j}^{(1)}(m) := \# \{ l \leq \lfloor n_i/2 \rfloor : \Delta X_{i,j}^l = m - 1 \{ i = j \} \}
$$

and

$$
\hat{M}_{i,j}^{(2)}(m) := \# \{ l \in \{ \lfloor n_i/2 \rfloor + 1, \ldots, n_i \} : \Delta X_{i,j}^l = m - 1 \{ i = j \} \}.
$$

Note that $\hat{M}_{i,j}^{(1)}(m)$ is $\mathcal{F}_{\lfloor n_i/2 \rfloor}$ measurable, and that

$$
\frac{1}{n_i/2} \hat{M}_{i,j}^{(1)}(m) = \frac{1}{n_i/2} \sum_{i,j} \mathbb{1} \{ \Delta X_{i,j}^l = m - 1 \{ i = j \} \} \to P_n(X_{i,j}^l = m - 1 \{ i = j \}) = v_{i,j}(m).
$$

Since $X_{i,j}^l$ has i.i.d. increments, we have that $\hat{M}_{i,j}^{(2)}(m)/n_i$ also converges to the same quantity.

Let

$$
\hat{M}_{i,j}^{(1,n)}(m) = \frac{n_i/2}{n_i} \hat{M}_{i,j}^{(1)}(m), \quad \hat{M}_{i,j}^{(2,n)}(m) = \frac{n_i - n_i/2}{n_i} \hat{M}_{i,j}^{(2)}(m).
$$

Notice that $\hat{M}_{i,j}^{(1,n)}(m) \to v_{i,j}(m)/2$ in probability under $P_n(X \in B_{k,n})$ by (12), for $l = 1, 2$. Using the triangle inequality, since $\hat{M}_{i,j}(m)/n_i = \hat{M}_{i,j}^{(1,n)}(m) + \hat{M}_{i,j}^{(2,n)}(m)$, then for $\varepsilon > 0$

$$
P_n \left( |\hat{M}_{i,j}(m)/n_i - v_{i,j}(m)/2| > \varepsilon, i, j \in [d] \mid X \in B_{k,n} \right)
\leq c P_n \left( |\hat{M}_{i,j}^{(1,n)}(m) - v_{i,j}(m)/2| > \varepsilon/2, i, j \in [d] \right) + c P_n \left( |\hat{M}_{i,j}^{(2,n)}(m) - v_{i,j}(m)/2| > \varepsilon/2, i, j \in [d] \right),
$$

which converges to zero, proving the lemma.

\[\square\]

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