Parabolic Kazhdan-Lusztig polynomials and Schubert varieties

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Abstract

We shall give a description of the intersection cohomology groups of the Schubert varieties in partial flag manifolds over symmetrizable Kac-Moody Lie algebras in terms of parabolic Kazhdan-Lusztig polynomials introduced by Deodhar.

1 Introduction

For a Coxeter system \((W, S)\) Kazhdan-Lusztig \([5], [6]\) introduced polynomials

\[
P_{y,w}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k} q^k \in \mathbb{Z}[q], \quad Q_{y,w}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k} q^k \in \mathbb{Z}[q],
\]

called a Kazhdan-Lusztig polynomial and an inverse Kazhdan-Lusztig polynomial respectively. Here, \((y, w)\) is a pair of elements of \(W\) such that \(y \leq w\) with respect to the Bruhat order. These polynomials play important roles in various aspects of the representation theory of reductive algebraic groups.

In the case \(W\) is associated to a symmetrizable Kac-Moody Lie algebra \(g\), the polynomials have the following geometric meanings. Let \(X = G/B\) be the corresponding flag variety (see Kashiwara \([3]\)), and set \(X^w = B^{-w}B/B\) and \(X_w = BwB/B\) for \(w \in W\). Here \(B\) and \(B^{-}\) are the “Borel subgroups” corresponding to the standard Borel subalgebra \(b\) and its opposite \(b^{-}\) respectively. Then \(X^w\) (resp. \(X_w\)) is an \(\ell(w)\)-codimensional (resp. \(\ell(w)\)-dimensional) locally closed subscheme of the infinite-dimensional scheme \(X\).

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Here $\ell(w)$ denotes the length of $w$ as an element of the Coxeter group $W$. Set $X' = \bigcup_{w \in W} X_w$. Then $X'$ coincides with the flag variety considered by Kac-Peterson [2], Tits [10], et al. Moreover we have

$$X = \bigsqcup_{w \in W} X_w, \quad X' = \bigsqcup_{w \in W} X_w,$$

and

$$\overline{X^w} = \bigsqcup_{y \leq w} X^y, \quad \overline{X'_w} = \bigsqcup_{y \leq w} X_y$$

for any $w \in W$.

By Kazhdan-Lusztig [5] we have the following result (see also Kashiwara-Tanisaki [4]).

**Theorem 1.1.** (i) Let $w, y \in W$ satisfying $w \leq y$. Then we have

$$H^{2k+1}(\pi \overline{Q}^H_{X_w^y} y_{B/B}) = 0, \quad H^{2k}(\pi \overline{Q}^H_{X_w^y} y_{B/B}) = \overline{Q}^H(-k)^{\oplus P_{w,y,k}}$$

for any $k \in \mathbb{Z}$.

(ii) The multiplicity of the irreducible Hodge module $\pi \overline{Q}^H_{X_w^y}[-\ell(y)](-k)$ in the Jordan H"older series of the Hodge module $\overline{Q}^H_{X_w^y}[-\ell(w)]$ coincides with $P_{w,y,k}$.

**Theorem 1.2.** (i) Let $w, y \in W$ satisfying $w \leq y$. Then we have

$$H^{2k+1}(\pi \overline{Q}^H_{X_w^y} y_{B/B}) = 0, \quad H^{2k}(\pi \overline{Q}^H_{X_w^y} y_{B/B}) = \overline{Q}^H(-k)^{\oplus P_{y,w,k}}$$

for any $k \in \mathbb{Z}$.

(ii) The multiplicity of the irreducible Hodge module $\pi \overline{Q}^H_{X_w^y}[-\ell(y)](-k)$ in the Jordan H"older series of the Hodge module $\overline{Q}^H_{X_w^y}[-\ell(w)]$ coincides with $Q_{y,w,k}$.

Here $\pi \overline{Q}^H_{X_w^y}[-\ell(w)]$ and $\pi \overline{Q}^H_{X_w^y}[\ell(w)]$ denote the Hodge modules corresponding to the perverse sheaves $\pi \overline{Q}^H_{X_w^y}[-\ell(w)]$ and $\pi \overline{Q}^H_{X_w^y}[\ell(w)]$ respectively. In Theorem 1.1 we have used the convention so that $\pi \overline{Q}^H_{Z}[-\text{codim } Z]$ is a Hodge module for a locally closed finite-codimensional subvariety $Z$ since we deal with sheaves supported on finite-codimensional subvarieties, while in Theorem 1.2 we have used another convention so that $\pi \overline{Q}^H_{Z}[\text{dim } Z]$ is a Hodge module for a locally closed finite-dimensional subvariety $Z$ since we deal with sheaves supported on finite-dimensional subvarieties.
Let $J$ be a subset of $S$. Set $W_J = \langle J \rangle$ and denote by $W^J$ the set of elements $w \in W$ whose length is minimal in the coset $wW_J$. In [1] Deodhar introduced two generalizations of the Kazhdan-Lusztig polynomials to this relative situation. For $(y, w) \in W^J \times W^J$ such that $y \leq w$ we denote the parabolic Kazhdan-Lusztig polynomial for $u = -1$ by

$$P_{y,w}^{J,q}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,q} q^k \in \mathbb{Z}[q],$$

and that for $u = q$ by

$$P_{y,w}^{J,-1}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,-1} q^k \in \mathbb{Z}[q]$$

contrary to the original reference [1]. We can also define inverse parabolic Kazhdan-Lusztig polynomials

$$Q_{y,w}^{J,q}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k}^{J,q} q^k \in \mathbb{Z}[q], \quad Q_{y,w}^{J,-1}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k}^{J,-1} q^k \in \mathbb{Z}[q]$$

(see §2 below)

The aim of this paper is to extend Theorem 1.1 and Theorem 1.2 to this relative situation using the partial flag variety corresponding to $J$.

Let $Y$ be the partial flag variety corresponding to $J$. Let $1_Y$ be the origin of $Y$ and set $Y^w = B^{-w}1_Y$ and $Y_w = Bw1_Y$ for $w \in W^J$. Then $Y^w$ (resp. $Y_w$) is an $\ell(w)$-codimensional (resp. $\ell(w)$)-dimensional) locally closed subscheme of the infinite-dimensional scheme $Y$. Set $Y' = \bigcup_{w \in W^J} Y_w$. Then we have

$$Y = \bigsqcup_{w \in W^J} Y^w, \quad Y' = \bigsqcup_{w \in W^J} Y_w,$$

and

$$\overline{Y}^w = \bigsqcup_{y \geq w} Y^y, \quad \overline{Y}^w = \bigsqcup_{y \leq w} Y_y$$

for any $w \in W^J$.

We note that the construction of the partial flag variety similar to the ordinary flag variety in Kashiwara [3] has not yet appeared in the literature. In the case where $W_J$ is a finite group (especially when $W$ is an affine Weyl group), we can construct the partial flag variety $Y = G/P$ and the properties of Schubert varieties in $Y$ stated above are established in exactly the same manner as in Kashiwara [3] and Kashiwara-Tanisaki [8]. In the case $W_J$ is an
infinite group we cannot define the “parabolic subgroup” $P$ corresponding to $J$ as a group scheme and hence the arguments in Kashiwara [3] are not directly generalized. We leave the necessary modification in the case $W_J$ is an infinite group to the future work.

Our main result is the following.

**Theorem 1.3.** (i) Let $w, y \in W^J$ satisfying $w \leq y$. Then we have

$$H^{2k+1}(\pi Q^H_{Y^w})_{y1Y} = 0, \quad H^{2k}(\pi Q^H_{Y^w})_{y1Y} = Q^H(-k)^{\oplus Q^{J,-1}_{w,y,k}}$$

for any $k \in \mathbb{Z}$.

(ii) The multiplicity of the irreducible Hodge module $\pi Q^H_{Y^w}[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $Q^H_{Y^w}[\ell(w)]$ coincides with $Q^{J,-1}_{y,w,k}$.

**Theorem 1.4.** (i) Let $w, y \in W^J$ satisfying $w \geq y$. Then we have

$$H^{2k+1}(\pi Q^H_{Y^w})_{y1Y} = 0, \quad H^{2k}(\pi Q^H_{Y^w})_{y1Y} = Q^H(-k)^{\oplus P^{J,q}_{y,w,k}}$$

for any $k \in \mathbb{Z}$.

(ii) The multiplicity of the irreducible Hodge module $\pi Q^H_{Y^w}[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $Q^H_{Y^w}[\ell(w)]$ coincides with $Q^{J,-1}_{y,w,k}$.

In Theorem 1.3 we have used the convention so that $\pi Q^H_Z[-\text{codim } Z]$ is a Hodge module for a locally closed finite-codimensional subvariety $Z$, and in Theorem 1.4 we have used another convention so that $\pi Q^H_Z[\dim Z]$ is a Hodge modules for a locally closed finite-dimensional subvariety $Z$.

We note that a result closely related to Theorem 1.4 was already obtained by Deodhar [4].

The above results imply that the coefficients of the four (ordinary or inverse) parabolic Kazhdan-Lusztig polynomials are all non-negative in the case $W$ is the Weyl group of a symmetrizable Kac-Moody Lie algebra.

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## 2 Kazhdan-Lusztig polynomials

Let $R$ be a commutative ring containing $\mathbb{Z}[q, q^{-1}]$ equipped with a direct sum decomposition $R = \bigoplus_{k \in \mathbb{Z}} R_k$ into $\mathbb{Z}$-submodules and an involutive ring
endomorphism $R \ni r \mapsto \overline{r} \in R$ satisfying the following conditions:

\[(2.1) \quad R_i R_j \subset R_{i+j}, \quad \overline{R_i} = R_{-i}, \quad 1 \in R_0, \quad q \in R_2, \quad \overline{q} = q^{-1}.
\]

Let $(W, S)$ be a Coxeter system. We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ and $\geq$ the length function and the Bruhat order respectively. The Hecke algebra $H = H(W)$ over $R$ is an $R$-algebra with free $R$-basis \(\{T_w\}_{w \in W}\) whose multiplication is determined by the following:

\[(2.2) \quad T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if} \quad \ell(w_1 w_2) = \ell(w_1) + \ell(w_2),
\]

\[(2.3) \quad (T_s + 1)(T_s - q) = 0 \quad \text{for} \quad s \in S.
\]

Note that $T_e = 1$ by (2.2).

We define involutive ring endomorphisms $H \ni h \mapsto \overline{h} \in H$ and $j : H \to H$ by

\[(2.4) \quad \sum_{w \in W} r_w T_w = \sum_{w \in W} \tau_w T_{w^{-1}}, \quad j(\sum_{w \in W} r_w T_w) = \sum_{w \in W} r_w (-q)^{\ell(w)} T_{w^{-1}}.
\]

Note that $j$ is an endomorphism of an $R$-algebra.

**Proposition 2.1 (Kazhdan-Lusztig [3]).** For any $w \in W$ there exists a unique $C_w \in H$ satisfying the following conditions:

\[(2.5) \quad C_w = \sum_{y \leq w} P_{y,w} T_y \quad \text{with} \quad P_{w,w} = 1 \quad \text{and} \quad P_{y,w} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i
\]

for $y < w$,

\[(2.6) \quad \overline{C_w} = q^{-\ell(w)} C_w.
\]

Moreover we have $P_{y,w} \in \mathbb{Z}[q]$ for any $y \leq w$.

Note that $\{C_w\}_{w \in W}$ is a basis of the $R$-module $H$. The polynomials $P_{y,w}$ for $y \leq w$ are called Kazhdan-Lusztig polynomials. We write

\[(2.7) \quad P_{y,w} = \sum_{k \in \mathbb{Z}} P_{y,w,k} q^k.
\]

Set $H^* = H^*(W) = \text{Hom}_R(H, R)$. We denote by $\langle \ , \ \rangle$ the coupling between $H^*$ and $H$. We define involutions $H^* \ni m \mapsto \overline{m} \in H^*$ and $j : H^* \to H^*$ by

\[(2.8) \quad \langle \overline{m}, h \rangle = \langle m, \overline{h} \rangle, \quad \langle j(m), h \rangle = \langle m, j(h) \rangle \quad \text{for} \quad m \in H^* \quad \text{and} \quad h \in H.
\]
Note that $j$ is an endomorphism of an $R$-module. For $w \in W$ we define elements $S_w, D_w \in H^*$ by

$$\langle S_w, T_x \rangle = (-1)^{\ell(w)} \delta_{w,x}, \quad \langle D_w, C_x \rangle = (-1)^{\ell(w)} \delta_{w,x}. \tag{2.9}$$

Then any element of $H^*$ is uniquely written as an infinite sum in two ways

$$\sum_{w \in W} r_w S_w \quad \text{and} \quad \sum_{w \in W} r'_w D_w$$

with $r_w, r'_w \in R$. Note that we have

$$S_w = \sum_{y \geq w} (-1)^{\ell(w)} - \ell(y) P_{w,y} D_y \tag{2.10}$$

by $C_w = \sum_{y \leq w} P_{y,w} T_y$. By (2.6), we have

$$D_w = q^{\ell(w)} D_w, \tag{2.11}$$

and we can write

$$D_w = \sum_{y \geq w} Q_{w,y} S_y, \tag{2.12}$$

where $Q_{w,y}$ are determined by

$$\sum_{w \leq y \leq z} (-1)^{\ell(y) - \ell(w)} Q_{w,y} P_{y,z} = \delta_{w,z}. \tag{2.13}$$

Note that (2.12) is equivalent to

$$T_w = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} Q_{y,w} C_y. \tag{2.14}$$

By (2.13) we see easily that

$$Q_{w,y} \in \mathbb{Z}[q], \tag{2.15}$$

$$Q_{w,w} = 1 \text{ and } \deg Q_{w,y} \leq (\ell(y) - \ell(w) - 1)/2 \text{ for } w < y. \tag{2.16}$$

The polynomials $Q_{w,y}$ for $w \leq y$ are called inverse Kazhdan-Lusztig polynomials (see Kazhdan-Lusztig [7]). We write

$$Q_{w,y} = \sum_{k \in \mathbb{Z}} Q_{w,y,k} q^k. \tag{2.17}$$

The following result is proved similarly to Proposition 2.1 (see Kashiwara-Tanisaki [4]).
Proposition 2.2. Let \( w \in W \). Assume that \( D \in H^* \) satisfies the following conditions:

\[
\begin{align*}
D &= \sum_{y \geq w} r_y S_y \text{ with } r_w = 1 \text{ and } r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i \\
\text{for } w < y; \\
\overline{D} &= q^{\ell(w)} D.
\end{align*}
\]

Then we have \( D = D_w \).

We fix a subset \( J \) of \( S \) and set

\[
W_J = \langle J \rangle, \quad W^J = \{ w \in W ; ws > w \text{ for any } s \in J \}.
\]

Then we have

\[
\begin{align*}
W &= \bigsqcup_{w \in W^J} w W_J, \\
\ell(wx) &= \ell(w) + \ell(x) \text{ for any } w \in W^J \text{ and } x \in W_J.
\end{align*}
\]

When \( W_J \) is a finite group, we denote the longest element of \( W_J \) by \( w_J \).

Let \( a \in \{ q, -1 \} \) and define \( a^\dagger \in \{ q, -1 \} \) by \( aa^\dagger = -q \). Define an algebra homomorphism \( \chi^a : H(W_J) \to R \) by \( \chi^a(T_w) = a^{\ell(w)} \), and denote the corresponding one-dimensional \( H(W_J) \)-module by \( R^a = R_1^a \). We define the induced module \( H^{J,a} \) by

\[
H^{J,a} = H \otimes_{H(W_J)} R^a,
\]

and define \( \varphi^{J,a} : H \to H^{J,a} \) by \( \varphi^{J,a}(h) = h \otimes 1^a \).

It is easily checked that \( H^{J,a} \ni k \mapsto \overline{k} \in H^{J,a} \) and \( j^a : H^{J,a} \to H^{J,a^\dagger} \) are well defined by

\[
\varphi^{J,a}(h) = \varphi^{J,a}(\overline{h}), \quad j^a(\varphi^{J,a}(h)) = \varphi^{J,a^\dagger}(j(h)) \quad \text{for } h \in H.
\]

Note that \( j^a \) is a homomorphism of \( R \)-modules and that

\[
\begin{align*}
\overline{rk} &= \overline{r} \overline{k} \quad \text{for } r \in R \text{ and } k \in H^{J,a}, \\
\overline{k} &= k \quad \text{for } k \in H^{J,a}, \\
j^{a^\dagger} \circ j^a &= \text{id}_{H^{J,a}}.
\end{align*}
\]

For \( w \in W^J \) set \( T_w^{J,a} = \varphi^{J,a}(T_w) \). It is easily seen that \( H^{J,a} \) is a free \( R \)-module with basis \( \{ T_w^{J,a} \}_{w \in W^J} \). Note that we have

\[
\varphi^{J,a}(T_{wx}) = a^{\ell(x)} T_w^{J,a} \quad \text{for } w \in W^J \text{ and } x \in W_J.
\]
Proposition 2.3 (Deodhar [1]). For any $w \in W^J$ there exists a unique $C^J_{w,a} \in H^J_{-1}$ satisfying the following conditions.

\[ C^J_{w,a} = \sum_{y \leq w} P^J_{y,w} T_{y} \text{ with } P^J_{y,w} = 1 \text{ and } P^J_{y,w} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i \]

for $y < w$.

(2.30) \[ C^J_{w,a} = q^{-\ell(w)} C^J_{w,a}. \]

Moreover we have $P^J_{y,w} \in \mathbb{Z}[q]$ for any $y \leq w$.

The polynomials $P^J_{y,w}$ for $y, w \in W^J$ with $y \leq w$ are called parabolic Kazhdan-Lusztig polynomials. We write

(2.31) \[ P^J_{y,w} = \sum_{k \in \mathbb{Z}} P^J_{y,w,k} q^k. \]

Remark 2.4. In the original reference [1] Deodhar uses

\[ (-1)^{\ell(w)} j^a (C^J_{w,a}) = \sum_{y \leq w} (-q)^{\ell(w)-\ell(y)} \overline{P^J_{y,w} T_{y}} \]

instead of $C^J_{w,a}$ to define the parabolic Kazhdan-Lusztig polynomials. Hence our $P^J_{y,w}$ is actually the parabolic Kazhdan-Lusztig polynomial $P^J_{y,w}$ for $u = a^j$ in the terminology of [1].

Proposition 2.5 (Deodhar [1]). Let $w, y \in W^J$ such that $w \geq y$.

(i) We have

\[ P^J_{y,w} = \sum_{x \in W^J, y \leq x \leq w} (-1)^{\ell(x)} P^J_{y,x,w}. \]

(ii) If $W_J$ is a finite group, then we have $P^J_{y,w} = P^J_{y,w,j}$.

Set

(2.32) \[ H^J_{-1,a,*} = \text{Hom}_R(H^J_{-1,a}, R), \]

and define $\varphi^J_{J_a} : H^J_{-1,a,*} \to H^*$ by

\[ \langle \varphi^J_a(n), h \rangle = \langle n, \varphi^J_a(h) \rangle \quad \text{for } n \in H^J_{-1,a,*} \text{ and } h \in H. \]
Then $t \varphi^{J,a}$ is an injective homomorphism of $R$-modules. We define an involution $-^*$ of $H^{J,a,*}$ similarly to (2.8). We can easily check that

\[(2.33)\quad t \varphi^{J,a}(n) = t \varphi^{J,a}(n) \quad \text{for any } n \in H^{J,a,*}.\]

For $w \in W^J$ we define $S_w^{J,a}, D_w^{J,a} \in H^{J,a,*}$ by

\[(2.34)\quad \langle S_w^{J,a}, T_x^{J,a} \rangle = (-1)^{\ell(w)} \delta_{w,x}, \quad \langle D_w^{J,a}, C_x^{J,a} \rangle = (-1)^{\ell(w)} \delta_{w,x}.\]

Then any element of $H^{J,a,*}$ is written uniquely as an infinite sum in two ways

\[(2.35)\quad S_w^{J,a} = \sum_{y \in W^J, y \geq w} (-1)^{\ell(w) - \ell(y)} P_{w,y}^{J,a} D_y^{J,a}\]

by $C_w^{J,a} = \sum_{y \leq w} P_{y,w}^{J,a} T_y$. We see easily by (2.28) that

\[(2.36)\quad t \varphi^{J,a}(S_w^{J,a}) = \sum_{x \in W^J} (-1)^{\ell(x) - \ell(w)} S_{wx}^{J,a} \quad \text{for } w \in W^J.\]

By the definition we have

\[(2.37)\quad D_w^{J,a} = q^{\ell(w)} D_w^{J,a},\]

and we can write

\[(2.38)\quad D_w^{J,a} = \sum_{y \in W^J, y \geq w} Q_{w,y}^{J,a} S_y^{J,a}\]

where $Q_{w,y}^{J,a} \in R$ are determined by

\[(2.39)\quad \sum_{y \in W^J, w \leq y \leq z} (-1)^{\ell(y) - \ell(w)} Q_{w,y}^{J,a} P_{y,z}^{J,a} = \delta_{w,z}\]

for $w, z \in W^J$ satisfying $w \leq z$.

Note that (2.38) is equivalent to

\[(2.40)\quad T_w^{J,a} = \sum_{y \in W^J, y \leq w} (-1)^{\ell(w) - \ell(y)} Q_{y,w}^{J,a} C_y^{J,a}.\]

By (2.34) we have for $w, y \in W^J$

\[(2.41)\quad Q_{w,y}^{J,a} \in \mathbb{Z}[q],\]

\[(2.42)\quad Q_{w,w}^{J,a} = 1 \text{ and } \deg Q_{w,y}^{J,a} \leq (\ell(y) - \ell(w) - 1)/2 \text{ for } w < y.\]
We call the polynomials $Q_{w,y}^{J,a}$ for $w \leq y$ inverse parabolic Kazhdan-Lusztig polynomials. We write

$$Q_{w,y}^{J,a} = \sum_{k \in \mathbb{Z}} Q_{w,y,k}^{J,a} q^k.$$  \hfill (2.43)

Similarly to Propositions 2.1, 2.2, 2.3, we can prove the following.

**Proposition 2.6.** Let $w \in W^J$. Assume that $D \in H^{J,a,*}$ satisfies the following conditions:

$$D = \sum_{y \in \mathcal{W}^J, y \geq w} r_y S_y^{J,a} \text{ with } r_w = 1 \text{ and } r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i$$

for $y \in \mathcal{W}^J$ satisfying $w < y$.

Then we have $D = D_w^{J,a}$.

**Proposition 2.7 (Soergel [9]).** Let $w, y \in W^J$ such that $w \leq y$.

(i) We have $Q_{w,y}^{J,-1} = Q_{w,y}$.

(ii) If $W_J$ is a finite group, then we have

$$Q_{w,y}^{J,a} = \sum_{x \in \mathcal{W}^J, \ell(w) \leq \ell(x) + \ell(y)} (-1)^{\ell(x)+\ell(w)} Q_{ww,yx}.$$  \hfill (2.44)

3 Hodge modules

In this section we briefly recall the notation from the theory of Hodge modules due to M. Saito [8]. We denote by $\text{HS}$ the category of mixed Hodge structures and by $\text{HS}_k$ the category of pure Hodge structures with weight $k \in \mathbb{Z}$. Let $R$ and $R_k$ be the Grothendieck groups of $\text{HS}$ and $\text{HS}_k$ respectively. Then we have $R = \bigoplus_{k \in \mathbb{Z}} R_k$ and $R$ is endowed with a structure of a commutative ring via the tensor product of mixed Hodge structures. The identity element of $R$ is given by $[\mathbb{Q}^H]$, where $\mathbb{Q}^H$ is the trivial Hodge structure. We denote by $R \ni r \mapsto \tau \in R$ the involutive ring endomorphism induced by the duality functor $\mathbb{D} : \text{HS} \rightarrow \text{HS}^{\text{op}}$. Here $\text{HS}^{\text{op}}$ denotes the opposite category of $\text{HS}$. Let $\mathbb{Q}^H(1)$ and $\mathbb{Q}^H(-1)$ be the Hodge structure of Tate and its dual respectively, and set $\mathbb{Q}^H(\pm n) = \mathbb{Q}^H(\pm 1)^{\otimes n}$ for $n \in \mathbb{Z}_{\geq 0}$. We can regard $\mathbb{Z}[q, q^{-1}]$ as a subring of $R$ by $q^n = [\mathbb{Q}^H(-n)]$. Then the condition (2.1) is satisfied for this $R$. 

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Let $Z$ be a finite-dimensional algebraic variety over $\mathbb{C}$. There are two conventions for perverse sheaves on $Z$ according to whether $\mathbb{Q}_U[\dim U]$ is a perverse sheaf or $\mathbb{Q}_U[-\codim U]$ is a perverse sheaf for a closed smooth subvariety $U$ of $Z$. Correspondingly, we have two conventions for Hodge modules. When we use the convention so that $\mathbb{Q}_U[\dim U]$ is a perverse sheaf, we denote the category of Hodge modules on $Z$ by $\operatorname{HM}_d(Z)$, and when we use the other one we denote it by $\operatorname{HM}_c(Z)$. Let $D^b(\operatorname{HM}_d(Z))$ and $D^b(\operatorname{HM}_c(Z))$ denote the bounded derived categories of $\operatorname{HM}_d(Z)$ and $\operatorname{HM}_c(Z)$ respectively. Note that $d$ is for dimension and $c$ for codimension. Then the functor $\operatorname{HM}_d(Z) \to \operatorname{HM}_c(Z)$ given by $M \mapsto M[-\dim Z]$ gives the category equivalences

$$\operatorname{HM}_d(Z) \cong \operatorname{HM}_c(Z), \quad D^b(\operatorname{HM}_d(Z)) \cong D^b(\operatorname{HM}_c(Z)).$$

We shall identify $D^b(\operatorname{HM}_d(Z))$ with $D^b(\operatorname{HM}_c(Z))$ via this equivalence, and then we have

$$\text{(3.1) } \operatorname{HM}_c(Z) = \operatorname{HM}_d(Z)[−\dim Z].$$

Although there are no essential differences between $\operatorname{HM}_d(Z)$ and $\operatorname{HM}_c(Z)$, we have to be careful in extending the theory of Hodge modules to the infinite-dimensional situation. In dealing with sheaves supported on finite-dimensional subvarieties embedded into an infinite-dimensional manifold we have to use $\operatorname{HM}_d$, while we need to use $\operatorname{HM}_c$ when we treat sheaves supported on finite-codimensional subvariety of an infinite-dimensional manifold. In fact what we really need in the sequel is the results for infinite-dimensional situation; however, we shall only give below a brief explanation for the finite-dimensional case. The extension of $\operatorname{HM}_d$ to the infinite-dimensional situation dealing with sheaves supported on finite-dimensional subvarieties is easy, and as for the extension of $\operatorname{HM}_c$ to the infinite-dimensional situation dealing with sheaves supported on finite-codimensional subvarieties we refer the readers to Kashiwara-Tanisaki [I].

Let $Z$ be a finite-dimensional algebraic variety over $\mathbb{C}$. When $Z$ is smooth, one has a Hodge module $\mathbb{Q}^H_Z[\dim Z] \in \operatorname{Ob}(\operatorname{HM}_d(Z))$ corresponding to the perverse sheaf $\mathbb{Q}_Z[\dim Z]$. More generally, for a locally closed smooth subvariety $U$ of $Z$ one has a Hodge module $\mathbb{Q}^H_U[\dim U] \in \operatorname{Ob}(\operatorname{HM}_d(Z))$ corresponding to the perverse sheaf $\mathbb{Q}_U[\dim U]$. For $M \in \operatorname{Ob}(D^b(\operatorname{HM}_d(Z)))$ and $n \in \mathbb{Z}$ we set $M(n) = M \otimes \mathbb{Q}^H(n)$. One has the duality functor

$$\text{(3.2) } \mathbb{D}_d : \operatorname{HM}_d(Z) \to \operatorname{HM}_d(Z)^{\text{op}}, \quad \mathbb{D}_d : D^b(\operatorname{HM}_d(Z)) \to D^b(\operatorname{HM}_d(Z))^{\text{op}}$$

satisfying $\mathbb{D}_d \circ \mathbb{D}_d = \text{Id}$, and we have

$$\text{(3.3) } \mathbb{D}_d(\mathbb{Q}^H_U[\dim U]) = \mathbb{Q}^H_U[\dim U](\dim U).$$

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for a locally closed smooth subvariety $U$ of $Z$.

Let $f : Z \to Z'$ be a morphism of finite-dimensional algebraic varieties. Then one has the functors:

$$ f^* : D^b(HM_d(Z')) \to D^b(HM_d(Z)), \quad f! : D^b(HM_d(Z')) \to D^b(HM_d(Z)),$$

$$ f_* : D^b(HM_d(Z)) \to D^b(HM_d(Z')), \quad f_! : D^b(HM_d(Z)) \to D^b(HM_d(Z'))$$

satisfying

$$ f^* \circ \mathbb{D}_d = \mathbb{D}_d \circ f^!, \quad f_* \circ \mathbb{D}_d = \mathbb{D}_d \circ f_!.$$

We define the functors $f^*, f!, f_*, f_!$ for $D^b(HM_c)$ by identifying $D^b(HM_c)$ with $D^b(HM_d)$. For $HM_c$ we use the modified duality functor

$$ \mathbb{D}_c : HM_c(Z) \to HM_c(Z)^{op}, \quad \mathbb{D}_c : D^b(HM_d(Z)) \to D^b(HM_d(Z))^{op}$$

given by

$$ \mathbb{D}_c(M) = (\mathbb{D}_d(M))[-2 \dim Z](- \dim Z).$$

It also satisfies $\mathbb{D}_c \circ \mathbb{D}_c = \text{Id}$. For a locally closed smooth subvariety $U$ of $Z$ we have $\pi Q^H_U[- \codim U] \in \text{Ob}(HM_c(Z))$ and

$$ \mathbb{D}_c(\pi Q^H_U[- \codim U]) = \pi Q^H_U[- \codim U](- \codim U).$$

When $f : Z \to Z'$ is a proper morphism, we have $f_* = f_!$ and hence $f_! \circ \mathbb{D}_d = \mathbb{D}_d \circ f_!$. When $f$ is a smooth morphism, we have $f^! = f^*[2(\dim Z - \dim Z')|(\dim Z - \dim Z')$ and hence $f^* \circ \mathbb{D}_c = \mathbb{D}_c \circ f^*$.

### 4 Finite-codimensional Schubert varieties

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbb{C}$. We denote by $W$ its Weyl group and by $S$ the set of simple roots. Then $(W, S)$ is a Coxeter system. We shall consider the Hecke algebra $H = H(W)$ over the Grothendieck ring $R$ of the category $HS$ (see §3), and use the notation in §2.

Let $X = G/B$ be the flag manifold for $\mathfrak{g}$ constructed in Kashiwara [3]. Here $B$ is the “Borel subgroup” corresponding to the standard Borel subalgebra of $\mathfrak{g}$. Then $X$ is a scheme over $\mathbb{C}$ covered by open subsets isomorphic to

$$ A^\infty = \text{Spec} \mathbb{C}[x_k; k \in \mathbb{N}]$$

(unless dim $\mathfrak{g} < \infty$).

Let $1_X = eB \in X$ denote the origin of $X$. For $w \in W$ we have a point $w1_X = wB/B \in X$. Let $B^-$ be the “Borel subgroup” opposite to $B$, and set $X^w = B^-w1_X = B^-wB/B$ for $w \in W$. Then we have the following result.
Proposition 4.1 (Kashiwa [3]). (i) We have $X = \bigsqcup_{w \in W} X^w$.

(ii) For $w \in W$, $X^w$ is a locally closed subscheme of $X$ isomorphic to $\mathbb{A}^\infty$ (unless $\dim g < \infty$) with codimension $\ell(w)$.

(iii) For $w \in W$, we have $\overline{X^w} = \bigsqcup_{y \in W, y \geq w} X^y$.

We call $X^w$ for $w \in W$ a finite-codimensional Schubert cell, and $\overline{X^w}$ a finite-codimensional Schubert variety.

Let $J$ be a subset of $S$. We denote by $Y$ the partial flag manifold corresponding to $J$. Let $\pi : X \to Y$ be the canonical projection and set $1_Y = \pi(1_X)$. We have $\pi(w1_X) = 1_Y$ for any $w \in W_J$. For $w \in W_J$ we set $Y^w = B^-w1_Y = \pi(X^w)$. When $W_J$ is a finite group, we have $Y = G/P_J$ and $Y^w = B^-wP_J/P_J$, where $P_J$ is the “parabolic subgroup” corresponding to $J$ (we cannot define $P_J$ as a group scheme unless $W_J$ is a finite group).

Similarly to Proposition 4.1 we have the following.

Proposition 4.2. (i) We have $Y = \bigsqcup_{w \in W_J} Y^w$.

(ii) For $w \in W_J$, $Y^w$ is a locally closed subscheme of $Y$ isomorphic to $\mathbb{A}^\infty$ (unless $\dim Y < \infty$) with codimension $\ell(w)$.

(iii) For $w \in W_J$, we have $\overline{Y^w} = \bigsqcup_{y \in W_J, y \geq w} Y^y$.

(iv) For $w \in W_J$, we have $\pi^{-1}(Y^w) = \bigsqcup_{x \in W_J} X^{wx}$.

We call a subset $\Omega$ of $W_J$ (resp. $W$) admissible if it satisfies

\begin{equation}
(4.1) \quad w, y \in W_J \text{ (resp. } W \text{)} , w \leq y , y \in \Omega \Rightarrow w \in \Omega.
\end{equation}

For a finite admissible subset $\Omega$ of $W_J$ we set $Y^\Omega = \bigcup_{w \in \Omega} Y^w$. It is a quasi-compact open subset of $Y$. Let $\text{HM}_c^{B^-}(Y^\Omega)$ be the category of $B^-$-equivariant Hodge modules on $Y^\Omega$ (see Kashiwara-Tanisaki [4] for the equivariant Hodge modules on infinite-dimensional manifolds), and denote its Grothendieck group by $K(\text{HM}_c^{B^-}(Y^\Omega))$. For $w \in W_J$ the Hodge modules $\mathbb{Q}_{Y^w}[-\ell(w)]$ and $\pi\mathbb{Q}_{Y^w}[-\ell(w)]$ are objects of $K(\text{HM}_c^{B^-}(Y^\Omega))$. Note that $\mathbb{Q}_{Y^w}[-\ell(w)]$ is a perverse sheaf on $Y$ because $Y^w$ is affine. Set

\begin{equation}
(4.2) \quad \text{HM}_c^{B^-}(Y) = \lim_{\Omega} \text{HM}_c^{B^-}(Y^\Omega) , K(\text{HM}_c^{B^-}(Y)) = \lim_{\Omega} K(\text{HM}_c^{B^-}(Y^\Omega)),
\end{equation}
where $\Omega$ runs through finite admissible subsets of $W^J$. By the tensor product, $K(HM_c^B(Y))$ is endowed with a structure of an $R$-module. Then any element of $K(HM_c^B(Y))$ is uniquely written as an infinite sum

$$\sum_{w \in W^J} r_w[Q^H_{Y^w}[-\ell(w)]]$$

with $r_w \in R$.

Denote by $K(HM_c^B(Y)) \ni m \mapsto m \in K(HM_c^B(X))$ the involution induced by the duality functor $D_c$. Then we have $r_m = r_m$ for any $r \in R$ and $m \in K(HM_c^B(Y))$.

We can similarly define $HM_c^B(X)$, $Q^H_{X^w}[-\ell(w)]$ and $\pi Q^H_{X^w}[-\ell(w)]$ for $w \in W$, $K(HM_c^B(X)) \ni m \mapsto \overline{m} \in K(HM_c^B(X))$ (for $J = \emptyset$).

Let $pt$ denote the algebraic variety consisting of a single point. For $w \in W$ (resp. $w \in W^J$) we denote by $i_{X,w} : pt \to X$ (resp. $i_{Y,w} : pt \to Y$) denote the morphism with image $\{w1_X\}$ (resp. $\{w1_Y\}$). We define homomorphisms

\begin{align*}
(4.3) \quad \Phi : K(HM_c^B(X)) &\to H^*, \quad \Phi^J : K(HM_c^B(Y)) \to H^{J,-1,*} \\
(4.4) \quad &\Phi([M]) = \sum_{w \in W} \left( \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{X,w}^*(M)] \right) S^w, \\
(4.5) \quad &\Phi^J([M]) = \sum_{w \in W^J} \left( \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y,w}^*(M)] \right) S^J_{w,-1}. 
\end{align*}

By the definition we have

\begin{align*}
(4.6) \quad &\Phi([Q^H_{X^w}[-\ell(w)]]]) = (-1)^{\ell(w)} S^w_w \quad \text{for } w \in W, \\
(4.7) \quad &\Phi^J([Q^H_{Y^w}[-\ell(w)]]]) = (-1)^{\ell(w)} S^J_{w,-1} \quad \text{for } w \in W^J, 
\end{align*}

and hence $\Phi$ and $\Phi^J$ are isomorphisms of $R$-modules.

The projection $\pi : X \to Y$ induces a homomorphism

$$\pi^* : K(HM_c^B(Y)) \to K(HM_c^B(X))$$

of $R$-modules.

**Lemma 4.3.** (i) The following diagram is commutative.

\[
\begin{array}{ccc}
K(HM_c^B(Y)) & \xrightarrow{\Phi^J} & H^{J,-1,*} \\
\downarrow{\pi^*} & & \downarrow{t_\varphi^{J,-1}} \\
K(HM_c^B(X)) & \xrightarrow{\Phi} & H^*
\end{array}
\]
(ii) $\pi^*(m) = \pi^*(\overline{m})$ for any $m \in K(HM_B^-(Y))$.

(iii) $\Phi(m) = \Phi(\overline{m})$ for any $m \in K(HM_B^-(X))$.

(iv) $\Phi^J(m) = \Phi^J(\overline{m})$ for any $m \in K(HM_B^-(Y))$.

Proof. For $w \in W^J$ we have $\pi^*(Q_H^Yw) = Q_H^{\pi^{-1}Y_w}$, and hence Proposition 4.2 (iv) implies

$$\pi^*([Q_H^Y]) = \sum_{x \in W_J} [Q_H^{X_{w,x}}].$$

Thus (i) follows from (4.6), (4.7) and (2.36). Locally on $X$ the morphism $\pi$ is a projection of the form $Z \times \mathbb{A}^\infty \to Z$, and thus $\pi^* \circ \mathbb{D}_c = \mathbb{D}_c \circ \pi^*$. Hence the statement (ii) holds.

The statement (iii) is already known (see Kashiwara-Tanisaki [4]). Then the statement (iv) follows from (i), (ii), (iii), (2.33) and the injectivity of $t^J_{\varphi^{J,-1}}$. $\square$

Theorem 4.4. Let $w, y \in W^J$ satisfying $w \leq y$. Then we have

$$H^{2k+1}i_{Y,y}^*(\pi Q_H^Yw) = 0, \quad H^{2k}i_{Y,y}^*(\pi Q_H^Yw) = Q^H(-k)\oplus Q_{w,y,k}^{-1}$$

for any $k \in \mathbb{Z}$. In particular, we have

$$\Phi^J([\pi Q_H^Yw[-\ell(w)]])) = (-1)^{\ell(w)}D_w^{J,-1}.$$

Proof. Let $w \in W^J$ and set

$$(-1)^{\ell(w)}\Phi^J([\pi Q_H^Yw[-\ell(w)]])) = D = \sum_{y \in W_J, y \geq w} r_y S_{J,-1}.$$

By the definition of $\pi Q_H^Yw[-\ell(w)]$ we have

$$\mathbb{D}_c(\pi Q_H^Yw[-\ell(w)]) = \pi Q_H^Yw[-\ell(w)][-\ell(w)],$$

and hence we obtain

$$D = q^{\ell(w)}D$$

by Lemma 4.3 (iv). By the definition of $\Phi^J$ we have

$$r_y = \sum_{k \in \mathbb{Z}} (-1)^k[H^{k}i_{Y,y}^*(\pi Q_H^Yw)].$$
and by the definition of $\pi Q_{Y^w}^H[-\ell(w)]$ we have

\begin{align}
(4.10) & \quad r_w = 1, \\
(4.11) & \quad \text{for } y > w \text{ we have } H^k i^{*}_{Y,y}(\pi Q_{Y^w}^H) = 0 \text{ unless } 0 \leq k \leq (\ell(y) - \ell(w) - 1). 
\end{align}

By the argument similar to Kashiwara-Tanisaki [4] (see also Kazhdan-Lusztig [7]) we have

\begin{equation}
[H^k i^{*}_{Y,y}(\pi Q_{Y^w}^H)] \in R_k.
\end{equation}

In particular, we have

\begin{equation}
\text{for } y > w \text{ we have } r_y \in \bigoplus_{k=0}^{\ell(y)-\ell(w)-1} R_k.
\end{equation}

Thus we obtain $D = D^{L-1}_w$ by (1.8), (1.11), (4.13) and Proposition 2.6. Hence $r_y = Q^J_{y,w}$. By (1.9) and (4.12) we have $[H^{2k+1} i^{*}_{Y,y}(\pi Q_{Y^w}^H)] = 0$ and $[H^{2k} i^{*}_{Y,y}(\pi Q_{Y^w}^H)] = q^k Q_{w,y,k}$ for any $k \in \mathbb{Z}$. The proof is complete.

By (2.33) and Theorem 4.4 we obtain the following.

\textbf{Corollary 4.5.} We have

\begin{equation}
[Q_{Y^w}^H[-\ell(w)]] = \sum_{y \geq w} P^{J-1}_{w,y} [\pi Q_{Y^w}^H[-\ell(y)]]
\end{equation}

in the Grothendieck group $K(HM_{c}^{-}(Y))$. In particular, the coefficient $P^{J-1}_{w,y,k}$ of the parabolic Kazhdan-Lusztig polynomial $P^{J-1}_{w,y}$ is non-negative and equal to the multiplicity of the irreducible Hodge module $\pi Q_{Y^w}^H[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $Q_{Y^w}^H[-\ell(w)]$.

\section{5 Finite-dimensional Schubert varieties}

Set

\begin{equation}
X_w = Bw1_X = BwB/B \quad \text{for } w \in W.
\end{equation}

Then we have the following result.

\textbf{Proposition 5.1 (Kashiwara-Tanisaki [5]).} Set $X' = \bigcup_{w \in W} X_w$. Then $X'$ is the flag manifold considered by Kac-Peterson [2], Tits [10], et al. In particular, we have the following.
(i) We have $X' = \bigcup_{w \in W} X_w$.

(ii) For $w \in W$, $X_w$ is a locally closed subscheme of $X$ isomorphic to $\mathbb{A}^{\ell(w)}$.

(iii) For $w \in W$ we have $\overline{X}_w = \bigcup_{y \in W, y \leq w} X_y$.

We call $X_w$ for $w \in W$ a finite-dimensional Schubert cell and $\overline{X}_w$ a finite-dimensional Schubert variety. Note that $X'$ is not a scheme but an inductive limit of finite-dimensional projective schemes (an ind-scheme).

For $w \in W^J$, we set $Y_w = Bw1_Y = \pi(X_w)$. Similarly to Proposition 5.1 we have the following.

**Proposition 5.2.** Set $Y' = \bigcup_{w \in W^J} Y_w$. Then we have the following.

(i) We have $Y' = \bigcup_{w \in W^J} Y_w$.

(ii) For $w \in W^J$, $Y_w$ is a locally closed subscheme of $Y$ isomorphic to $\mathbb{A}^{\ell(w)}$.

(iii) For $w \in W^J$, we have $\overline{Y}_w = \bigcup_{y \in W^J, y \leq w} Y_y$.

(iv) For $w \in W^J$, we have $\pi^{-1}(Y_w) = \bigcup_{x \in W_J} X_{wx}$.

For a finite admissible subset $\Omega$ of $W^J$ we set $Y'_\Omega = \bigcup_{w \in \Omega} Y_w'$. It is a finite dimensional projective scheme.

Let $\text{HM}^B_d(Y'_\Omega)$ be the category of $B$-equivariant Hodge modules on $Y'_\Omega$. For $w \in W^J$ the Hodge modules $Q^H_{Y_w}[\ell(w)]$ and $\pi Q^H_{X_w}[\ell(w)]$ are objects of $\text{HM}^B_d(Y'_\Omega)$. Note that $Q_{Y_w}[\ell(w)]$ is a perverse sheaf because $Y_w$ is affine. Set

\[
\text{HM}^B_d(Y') = \lim_{\Omega} \text{HM}^B_d(Y'_\Omega), \quad K(\text{HM}^B_d(Y')) = \lim_{\Omega} K(\text{HM}^B_d(Y'_\Omega)),
\]

where $\Omega$ runs through finite admissible subsets of $W^J$. By the tensor product $K(\text{HM}^B_d(Y'))$ is endowed with a structure of an $R$-module. Then any element of $K(\text{HM}^B_d(Y'))$ is uniquely written as a finite sum in two ways

\[
\sum_{w \in W^J} r_w[Q^H_{Y_w}[\ell(w)]] \quad \text{and} \quad \sum_{w \in W^J} r_w[\pi Q^H_{X_w}[\ell(w)]] \quad \text{with} \quad r_w, r'_w \in R.
\]

Denote by $K(\text{HM}^B_d(Y')) \ni m \mapsto \overline{m} \in K(\text{HM}^B_d(Y'))$ the involution of an abelian group induced by the duality functor $\mathcal{D}_d$. Then we have $\overline{\overline{m}} = m$ for any $r \in R$ and $m \in K(\text{HM}^B_d(Y'))$.

We can similarly define $\text{HM}^B_d(X'), Q^H_{X_w}[\ell(w)]$ and $\pi Q^H_{X_w}[\ell(w)]$ for $w \in W$, $K(\text{HM}^B_d(X'))$, and $K(\text{HM}^B_d(X')) \ni m \mapsto \overline{m} \in K(\text{HM}^B_d(X'))$ (for $J = \emptyset$).
For \( w \in W \) (resp. \( w \in W^J \)) we denote by \( i_{X',w} : \text{pt} \to X' \) (resp. \( i_{Y',w} : \text{pt} \to Y' \)) denote the morphism with image \( \{ w1_X \} \) (resp. \( \{ w1_Y \} \)). We define homomorphisms

\[
(5.3) \quad \Psi : K(HM^B_d(X')) \to H, \quad \Psi^J : K(HM^B_d(Y')) \to H^{J,q}
\]

of \( R \)-modules by

\[
(5.4) \quad \Psi([M]) = \sum_{w \in W} \left( \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{X',w}^*(M)] \right) T_w,
\]

\[
(5.5) \quad \Psi^J([M]) = \sum_{w \in W^J} \left( \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y',w}^*(M)] \right) T_w^{J,q}.
\]

By the definition we have

\[
(5.6) \quad \Psi([Q^H_{X,w}[\ell(w)]]) = (-1)^{\ell(w)} T_w \quad \text{for} \quad w \in W,
\]

\[
(5.7) \quad \Psi^J([Q^H_{Y,u}[\ell(w)]])) = (-1)^{\ell(w)} T_w^{J,q} \quad \text{for} \quad w \in W^J,
\]

and hence \( \Psi \) and \( \Psi^J \) are isomorphisms.

Let \( \pi' : X' \to Y' \) denote the projection. Let \( \Omega \) be a finite admissible subset of \( W \) and set \( \Omega' = \{ w \in W^J : wW_J \cap \Omega \neq \emptyset \} \). Then \( \Omega' \) is a finite admissible subset of \( W^J \) and \( \pi' \) induces a surjective projective morphism \( X'_\Omega \to Y'_{\Omega'} \). Hence we can define a homomorphism \( \pi'_\Omega : K(HM^B(X')) \to K(HM^B(Y')) \) of \( R \)-modules by

\[
(5.8) \quad \pi'_\Omega([M]) = \sum_{k \in \mathbb{Z}} (-1)^k [H^k \pi'_\Omega(M)].
\]

**Lemma 5.3.** (i) The following diagram is commutative.

\[
\begin{array}{ccc}
K(HM^B_d(X')) & \xrightarrow{\Psi} & H \\
\pi'_\Omega \downarrow & & \downarrow \varphi_{J,q} \\
K(HM^B_d(Y')) & \xrightarrow{\Psi^J} & H^{J,q}
\end{array}
\]

(ii) \( \overline{\pi'_\Omega(m)} = \pi'_\Omega(\overline{m}) \) for any \( m \in K(HM^B_d(X')) \).

(iii) \( \overline{\Psi(m)} = \Psi(\overline{m}) \) for any \( m \in K(HM^B_d(X')) \).

(iv) \( \overline{\Psi^J(m)} = \Psi^J(\overline{m}) \) for any \( m \in K(HM^B_d(Y')) \).
Proof. Let \( w \in W^J \) and \( x \in W_J \). Since \( X \rightarrow Y \) is an \( A^{\ell(x)} \)-bundle, we have
\[
\pi'_!(\mathbb{Q}^H_{X^{\ell(x)}}) = \mathbb{Q}^H_{Y^w}[-2\ell(x)](-\ell(x)),
\]
and hence
\[
\pi'_!(\mathbb{Q}^H_{X^{\ell(x)}}[\ell(w)]) = (-q)^{\ell(x)}[\mathbb{Q}^H_{Y^w}[\ell(w)]].
\]
Thus (i) follows from (5.6), (5.7) and (2.28).

The statement (ii) follows from the fact that \( \pi' \) is an inductive limit of projective morphisms and hence \( \pi'_! \) commutes with the duality functor \( \mathbb{D}_d \).

The statement (iii) is proved similarly to Kashiwara-Tanisaki \([4]\), and we omit the details (see also Kazhdan-Lusztig \([7]\)). Then the statement (iv) follows from (i), (ii), (iii), (2.24) and surjectivity of \( \varphi^{J,q}_I \).

Theorem 5.4. Let \( w, y \in W^J \) such that \( w \geq y \). Then we have
\[
H^{2k+1}\iota_{y^*,y}^*(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)]) = 0,
\]
\[
H^{2k}\iota_{y^*,y}^*(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}) = \mathbb{Q}^H(-k)^{\oplus D^{J,q}_{y^*,y}} k
\]
for any \( k \in \mathbb{Z} \). In particular, we have
\[
\Psi^J(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)]) = (-1)^{\ell(w)}C^{J,q}_{w}.
\]

Proof. Let \( w \in W^J \) and set
\[
(-1)^{\ell(w)}\Psi^J(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)]) = C = \sum_{y \in W_J, y \leq w} r_y T^{J,q}.
\]

By the definition of \( \pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)] \) we have \( \mathbb{D}_d(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)]) = \pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)](\ell(w)) \).

Hence we obtain
\[
(5.9) \quad \overline{C} = q^{-\ell(w)}C
\]
by Lemma 5.3 (iv). By the definition of \( \Psi^J \) we have
\[
(5.10) \quad r_y = \sum_{k \in \mathbb{Z}} (-1)^k [H^{k}\iota_{y^*,y}^*(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w})],
\]
and by the definition of \( \pi_!^\bigcirc\mathbb{Q}_{Y_w^w}[\ell(w)] \) we have
\[
(5.11) \quad r_w = 1,
\]
\[
(5.12) \quad \text{for } y < w \text{ we have } H^{k}\iota_{y^*,y}^*(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w}) = 0 \text{ unless } 0 \leq k \leq (\ell(w) - \ell(y) - 1).
\]

Moreover, by the argument similar to Kazhdan-Lusztig \([3]\) and Kashiwara-Tanisaki \([4]\) we have
\[
(5.13) \quad [H^{k}\iota_{y^*,y}^*(\pi_!^\bigcirc\mathbb{Q}_{Y_w^w})] \in R_k.
\]
In particular, we have

\[(5.14) \quad \text{for } y < w \text{ we have } r_y \in \bigoplus_{k=0}^{\ell(w)-\ell(y)-1} R_k.\]

Thus we obtain \( C = C_{w}^{J,q} \) by (5.9), (5.11), (5.14) and Proposition 2.3. Hence \( r_y = P_{y,w}^{J,q} \). By (5.10) and (5.13) we have \([H^{2k+1}i_{Y',y}^* (\pi Q_H^w)] = 0\) and \([H^{2k}i_{Y',y}^* (\pi Q_{H}^y)] = q^k P_{y,w,k} \) for any \( k \in \mathbb{Z} \). The proof is complete. \( \square \)

We note that a result closely related to Theorem 5.4 above is already given in Deodhar [1].

By (2.40) and Theorem 5.4 we obtain the following.

**Corollary 5.5.** We have

\[ [Q_{Y_w}^H[\ell(w)]] = \sum_{y \leq w} Q_{y,w}^{J,q}[\pi Q_{Y_y}^H[\ell(y)]] \]

in \( K(HM^B_d(Y')) \). In particular, the coefficient \( Q_{y,w,k}^{J,q} \) of the inverse parabolic Kazhdan-Lusztig polynomial \( Q_{y,w}^{J,q} \) is non-negative and equal to the multiplicity of the irreducible Hodge module \( \pi Q_{Y_y}^H[\ell(y)](-k) \) in the Jordan Hölder series of the Hodge module \( Q_{Y_w}^H[\ell(w)] \).

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